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r^2-VARIÉTÉS À CONNEXION AFFINE

Par

F. SŐLER (Moncton)

1. Introduction. ROTER [1] a prouvé pour les r^2-variétés riemanniennes que le tenseur de récurrence B est symétrique et ceci est basé sur les propriétés de symétrie du tenseur de courbure.

Plusieurs auteurs (cf. [3], [5], [6]) ont étudié les r^2-variétés munies d'une métrique et les propriétés de B sont bien connues.

Dans la présente mémoire, nous allons donner certaines propriétés du tenseur B quand la connexion est affine, et donc toutes les symétries de R ne peuvent pas être employées, (voir aussi SŐLER [7]) et en particulier étudier un cas spécial de r^2-variété, celui qui admet un champ r-vectoriel et qui a le tenseur S décomposable. On obtient alors que la partie antisymétrique de B considérée comme un vecteur de $\Lambda_n^{(2)*}$ est autorécurrent pour la dérivation induite par la connexion affine sur l'espace $\Lambda_n^{(2)*}$.

La restriction du champ r-vectoriel, d'être parallèle, nous permet d'obtenir pour les r^2-variétés à connexion affine l'annulation de la partie antisymétrique de B.

2. r^2-variétés affines. Soit V_n une variété différentiable à n dimensions. Un ouvert de V_n est homéomorphe à R^n. Soit $\mathscr{F}(p)$ l'algèbre des fonctions différentiables de classe C^1 en un voisinage de p et $\mathscr{F}(V_n)$ l'algèbre des fonctions différentiables sur V_n.

Un champ vectoriel c'est l'assignation d'un vecteur à chaque point de V_n. Il est une dérivation de l'algèbre $\mathscr{F}(V_n)$. L'ensemble $\mathfrak{X}$ de tous les champs vectoriels sur V_n est un $\mathscr{F}(V_n)$-module.

Un champ tensoriel de type (r, s), $\mathfrak{T}_s^r$, est isomorphe au produit tensoriel de r-fois $\mathfrak{X}$ et s-fois son dual. Il peut être considéré comme une application s-linéaire de $\overbrace{\mathfrak{X}\times\ldots\times\mathfrak{X}}^{s\text{-fois}}\to\mathfrak{T}_0^r$.

Donnée une connexion affine Γ, que nous supposons symétrique dans tout ce qui suit, la différentiation covariante associée est une application bilinéaire $\nabla:\mathfrak{X}\times\times\mathfrak{X}\to\mathfrak{X}$ représentée par $(X, Y)\to\nabla_X Y$ où $X, Y\in\mathfrak{X}$. Elle est une dérivation sur l'algèbre des champs tensoriels qui préserve le type et qui commute avec chaque contraction.

Pour un champ tensoriel $K\in\mathfrak{T}_s^r$ la dérivée covariante ∇K est une application $\nabla:\mathfrak{T}_s^r\to\mathfrak{T}_{s+1}^r$ et nous posons

$$(\nabla K)(X_1, \ldots, X_s: V) = (\nabla_V K)(X_1, \ldots, X_s)$$

pour la première dérivée covariante et

$$(\nabla^2 K)(X_1, \ldots, X_s: V: W) = [\nabla_W(\nabla_V K)](X_1, \ldots, X_s: V) \tag{2-1}$$

pour la deuxième; étant $X_1, \ldots, X_s, V, W\in\mathfrak{X}$.

Les champs tensoriels, torsion et courbure sont définis respectivement:

(2-2) $$T(V, W) = \nabla_V W - \nabla_W V - [V, W]$$

(2-3) $$R(V, W, Z) = \mathscr{A}(\nabla^2 Z)(:W:V) - \nabla_{[V,W]} Z$$

où $V, W, Z \in \mathfrak{X}(V_n)$ sont des champs vectoriels sur V_n et $\mathscr{A}$ est l'opérateur d'antisymétrisation.

Ruse [2] a défini une variété à tenseur de courbure récurrent, ou simplement r-variété, par la condition

(2-4) $$(\nabla_V R)(X, Y, Z) = R(X, Y, Z)A(V)$$

où R est le tenseur de courbure et A est une 1-forme. Une variété est récurrente de second ordre, ou simplement une r^2-variété, si elle satisfait á la condition

(2-5) $$(\nabla_W(\nabla_V R))(X, Y, Z) = R(X, Y, Z)B(V, W)$$

où $B \in \mathfrak{T}_2^0$ et $R \neq 0$.

Cette notion a été introduite par A. Lichnerowicz [3]. Dans ce qui suit, nous allons nous limiter aux r^2-variétés avec connexion affine symétrique.

3. Cas spécial de r^2-variété affine. Dans chaque point $p \in V_n$, il y a un espace vectoriel tangent T_p. Soit $\beta = \{V_i\}$, $i = 1, \dots, n$ une base de T_p et $\beta^* = \{\theta^i\}$, $i = 1, \dots, n$ la base canoniquement associée à l'espace vectoriel dual T_p^*.

On définit un champ tensoriel S par

(3-1) $$S(X, Y) = \sum_i \theta^i(R(X, Y, V_i)).$$

Quand sur V_n se trouve définie une métrique riemannienne le tenseur S est le tenseur de Ricci.

Proposition 1. *Si V_n est une r^2-variété, c'est-à-dire*

$$(\nabla^2 R)(X, Y, Z: V: W) = R(X, Y, Z)B(V, W)$$

alors

$$(\nabla^2 S)(X, Y: V: W) = S(X, Y)B(V, W).$$

Démonstration. Elle découle de (2-5) et de la définition (3-1).

Supposons maintenant l'existence d'un champ r-vectoriel $L \in \mathfrak{X}$, c'est-à-dire

(3-2) $$\nabla_V L = L\alpha(V)$$

où α est une 1-forme non nulle et V est un champ vectoriel arbitraire.

On sait (cf. [4]) que le commutateur des dérivées covariantes d'un champ vectoriel est donné par

(3-3) $$\mathscr{A}(\nabla^2 L)(:V:W) = R(W, V, L) - \nabla_{T(W,V)} L.$$

D'autre part la seconde dérivée de L, tenant compte de (3-2) est

(3-4) $$(\nabla^2 L)(:V:W) = L(\alpha(W)\alpha(V) + \nabla\alpha(V, W))$$

et par antisymétrisation de (3-4) on obtient

(3-5) $$\mathscr{A}(\nabla^2 L)(:V:W) = L\mathscr{A}(\nabla\alpha)(V: W).$$

A partir de (3-3) et (3-5) on obtient la proposition suivante.

PROPOSITION 2. *Si L est un champ r-vectoriel et α est la 1-forme de récurrence alors*

$$R(W, V, L)-\nabla_{T(W,V)}L = L\mathscr{A}(\nabla\alpha)(V:W) \tag{3-6}$$

et en particulier si $\mathscr{A}\Gamma=0$ alors

$$R(W, V, L) = L\mathscr{A}(\nabla\alpha)(V:W). \tag{3-7}$$

Supposons maintenant que le tenseur S défini dans (3-1) soit décomposable:

$$S(X, Y) = \mu(X)\gamma(Y). \tag{3-8}$$

Par Proposition 1 et (3-8), on a, après antisymétrisation pa rapport à V et W,

$$\mathscr{A}(\nabla^2)(S)(X, Y\colon V\colon W) = \mu(X)\gamma(Y)\mathscr{A}B(V, W), \tag{3-9}$$

et d'autre part le calcul de la seconde dérivée du deuxième membre de (3-8) donne

$$\nabla^2(\mu\otimes\gamma)(X, Y\colon V\colon W) = (\nabla^2\mu)(X\colon V\colon W\colon)\gamma(Y)+ \tag{3-10}$$

$$+(\nabla\mu)(X\colon W)(\nabla\gamma)(Y\colon V)+(\nabla\mu)(X\colon V)\nabla\gamma(Y\colon W)+\mu(X)(\nabla^2\gamma)(Y\colon V\colon W),$$

et par antisymétrisation:

(3-11)

$$\mathscr{A}[\nabla^2](\mu\otimes\gamma)(X, Y\colon V\colon W) = \mathscr{A}(\nabla^2)\mu(X\colon V\colon W)\gamma(Y)+\mu(X)\mathscr{A}(\nabla^2)\gamma(Y\colon V\colon W),$$

et si on tient compte de (3-3) on a

$$\mathscr{A}(\nabla^2\mu)(X\colon V\colon W) = \mu\big(R(W, V, X)\big)-(\nabla_{T(W,V)}\mu)(X) \tag{3-12}$$

$$\mathscr{A}(\nabla^2\gamma)(Y\colon V\colon W) = \gamma\big(R(W, V, Y)\big)-(\nabla_{T(W,V)}\gamma)(Y). \tag{3-13}$$

On peut énoncer la proposition suivante.

PROPOSITION 3. *Pour toute r^2-variété avec tenseur S décomposable se vérifie la relation suivante:*

$$\mu(X)\gamma(Y)\mathscr{A}B(V, W) = [\mu\big(R(W, V, X)\big)-(\nabla_{T(W,V)}\mu(X)]\gamma(Y)+ \tag{3-14}$$

$$+\mu(X)[\gamma\big(R(W, V, Y)\big)-(\nabla_{T(W,V)}\gamma)(Y)],$$

et en particulier si $\mathscr{A}\Gamma=0$

$$\mu(X)\gamma(Y)\mathscr{A}B(V, W) = \mu\big(R(W, V, X)\big)\gamma(Y)+\mu(X)\gamma\big(R(W, V, Y)\big) \tag{3-15}$$

pour $\forall\ X, Y, V, W\in\mathfrak{X}$.

Donnons maintenant dans l'équation (3-15) à X et à Y la valeur L. Nous allons considérer deux cas:

a) $\mu(L)=0$ (ou $\gamma(L)=0$).

Dans ce cas, à partir de (3-15) on obtient:

$$\mu\big(R(W, V, L)\big) = 0 \quad (\text{ou} \quad \gamma\big(R(W, V, L)\big) = 0).$$

b) $\mu(L)\neq 0$ et $\gamma(L)\neq 0$ sur un ouvert $\Omega\subset V_n$.

Alors si on tient compte de la Proposition 2 (équation (3-7)), l'équation (3-15) devient:

$$\mathscr{A}B(V, W) = 2\mathscr{A}(\nabla\alpha)(V\colon W).$$

Le cas où $\mu(L)$ et $\gamma(L)$ s'annulent simultanément est trivial. Résumons ces résultats.

PROPOSITION 4. *Pour toute r^2-variété avec tenseur S décomposable dans le produit tensoriel $S=\mu\otimes\gamma$ et sous la condition de l'existence d'un champ r-vectoriel L, une des deux conditions est vraie localement:*

i) *Si* $\mu(L)=0$ *(respectivement* $\gamma(L)=0$*) alors* $\mu(R(W, V, L))=0$ *(respectivement* $\gamma(R(W, V, L))=0$*).*

ii) *Si* $\mu(L)\neq 0$ *et* $\gamma(L)\neq 0$ *alors la partie antisymétrique du tenseur de réccurence B vérifie* $\mathscr{A}B(V, W)=2\mathscr{A}(\nabla\alpha)(V\colon W)$ *où* α *est 1-forme de récurrence du champ r-vectoriel.*

La première identité de Bianchi s'écrit:

$$\mathscr{P}_{XYZ}|R(X, Y, Z) = \mathscr{P}_{XYZ}|T(T(X, Y), Z)+\nabla_X T(Y, Z)\}$$

où $\mathscr{P}_{XYZ}|$ indique permutation circulaire sur les indices X, Y, Z et sommation. Quand la connexion affine est symétrique l'identité de Bianchi se réduit à

$$\mathscr{P}_{XYZ}|R(X, Y, Z) = 0 \tag{3-16}$$

ou ce qui est équivalent:

$$\mathscr{P}_{XYZ}|\alpha(R(X, Y, Z)) = 0 \tag{3-17}$$

où α est une 1-forme.

Si on tient compte de la Proposition 2, formule (3-7), on a:

$$\mathscr{P}_{XYZ}|\alpha(L)\mathscr{A}(\nabla\alpha)(V\colon W) = 0. \tag{3-18}$$

D'autre part, á partir de la définition de S (équation (3-1)) et de la Proposition 2, on a: $S(X, L)=\mathscr{A}(\nabla\alpha)(X\colon L)$. Et puisque S est décomposable, nous avons

$$\mathscr{A}(\nabla\alpha)(X\colon L) = \mu(X)\gamma(L). \tag{3-19}$$

Si on remplace (3-19) dans (3-18) et on tient compte de l'antisymétrie de $\mathscr{A}(\nabla\alpha)$ on obtient:

(3-20)

$$\alpha(L)\mathscr{A}\nabla\alpha(V\colon W) = \alpha(W)\mu(V)\gamma(L)-\alpha(V)\mu(W)\gamma(L) = [\alpha(W)\mu(V)-\alpha(V)\mu(W)]\gamma(L).$$

A partir de (3-20), nous pouvons énoncer la proposition suivante:

PROPOSITION 5. *Si* V_n *est une* r^2*-variété qui admet un champ r-vectoriel L tel que* $\alpha(L)\neq 0$ *et le tenseur* $S=\mu\otimes\gamma$ *est décomposable, alors:*

i) *Si* $\gamma(L)=0$ *ou* $\mu(L)=0\Rightarrow\mathscr{A}(\nabla\alpha)(V\colon W)=0$;

ii) *Si* $\gamma(L)\neq 0$ *et* $\mu(L)\neq 0\Rightarrow\mathscr{A}(\nabla\alpha)=\varkappa\cdot\mu\wedge\gamma$ *où* $\varkappa=\gamma(L)/\alpha(L)$.

COROLLAIRE. *Si une variété différentielle vérifie*

i) $(\nabla^2 R)(X, Y, Z\colon V\colon W)=R(X, Y, Z)B(V, W)$,

ii) $(\nabla L)(\colon V)=L\alpha(V)$ *et* $\alpha(L)\neq 0$,

iii) $S=\mu\otimes\gamma$ *et* $\gamma(L)\neq 0$,

alors $\mathscr{A}S(X, Y)=\mathscr{A}(\nabla\alpha)(V\colon W)$.

4. La partie antisymétrique du tenseur de récurrence. On sait que

$$\mathscr{A}(\nabla^2)R(X, Y, Z: V: W) = H(X, Y, Z, V, W)$$

où H est le tenseur défini par CARTAN [8]. Et puisque le commutateur de deux dérivations est une dérivation [9], on a

$$(4\text{-}1) \qquad \mathscr{A}(\nabla^2)\mathscr{A}(\nabla^2)(R)(X, Y, Z: V: W: P: Q) =$$

$$= [\mathscr{A}(\nabla^2)B(V, W: P: Q)+\mathscr{A}B(V, W)\mathscr{A}B(P, Q)]R(X, Y, Z)$$

et si on tient compte du fait que c'est une r^2-variété et de la définition de H on doit avoir

$$(4\text{-}2) \qquad \mathscr{A}(\nabla^2)\mathscr{A}(\nabla^2)(R)(X, Y, Z: V: W: P: Q) =$$

$$= 2\mathscr{A}B(P, Q)H(X, Y, Z, V, W)=2\mathscr{A}B(P, Q)B(V, W)R(X, Y, Z).$$

Si on compare (4-1) et (4-2) on déduit:

$$\mathscr{A}(\nabla^2)(\mathscr{A}B)(V, W: P: Q) = \mathscr{A}B(V, W)\mathscr{A}B(P, Q).$$

On peut considérer $\mathscr{A}B$ comme un vecteur de l'espace $\Lambda_n^{(2)*}$ des 2-formes, sur lequel on considère $\mathscr{A}(\nabla^2)$ comme la dérivation induite par Γ sur $\Lambda_n^{(2)*}$.

PROPOSITION 6. *La partie antisymètrique de B est autorécurrente pour la dérivation induite par* ∇ *sur l'espace* $\Lambda_n^{(2)*}$ *des 2-formes.*

Considérons maintenant le cas où la partie antisymétrique $\mathscr{A}B$ s'annule.

Si on tient compte de ii) de la Proposition 4, on voit que sous la condition $\mu(L)\neq 0$ et $\gamma(L)\neq 0$, $\mathscr{A}B=0$ dans les quatre cas donnés dans la proposition suivante:

PROPOSITION 7. *Pour les r^2-variétés qui admettent un champ r-vectoriel L, soit* $(\nabla L)(:V)=L\alpha(V)$ *avec* $\alpha(L)\neq 0$, *et qui ont le tenseur S décomposable, la partie antisymétrique du tenseur de récurrence s'annule si:*

i) $\alpha=0\Leftrightarrow L$ *est parallèle;*

ii) $\alpha=d\theta$ *où* $\theta\in\mathscr{F}(V_n)$*;*

iii) $\nabla\alpha=0$, *soit la* 1*-forme de récurrence est parallèle;*

iv) $\mathscr{A}\nabla\alpha=0$.

En particulier quand la 1-forme α est parallèle et le tenseur S se décompose $S=\alpha\otimes\gamma$. Pour être une r^2-variété $\nabla^2S(X, Y)=\alpha(X)\nabla^2\mu(Y: V: W)$ et puisqu'on doit avoir $\nabla^2S(X, Y)=B(V, W)\alpha(X)\gamma(Y)$ aussi, on en tire pour un ouvert sur $\Omega\subset V_n$ où la 1-forme α n'est pas nulle, l'autre forme vérifie $\nabla^2\gamma(X: V: W)= =B(V, W)\gamma(X)$ $\forall X, V, W\in\mathfrak{X}$.

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ON THE FREQUENCY OF TITCHMARSH'S PHENOMENON FOR $\zeta(s)$. II

By
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§ 1. Introduction. In a previous paper with the same title [3], I showed that if we fix any c in $\frac{1}{2}<c<1$ and any function H_1 of X satisfying $(\log X)^\delta < H_1 < X (\delta>0,$ fixed) then, with every fixed $\varepsilon>0$, there holds

(1)
$$\max_{X \leq t \leq X+H_1} |\zeta(c+it)| > \operatorname{Exp}\left((\log H_1)^{1-c-\varepsilon}\right)$$

for all $X>X_0=X_0(c, \delta, \varepsilon)$. In this note, which may be looked upon as an addendum to [3], we are concerned with upper bounds for the left hand side of (1) under some restrictions. More specifically let H be a function depending only on Y such that $H>(\log Y)^\delta (\delta>0,$ fixed) and $(\log H)(\log Y)^{-1}$ does not exceed $(\log\log Y)^{-1}$ for $Y>20$. Then I prove that for every fixed $\varepsilon>0$,

(2)
$$\min_{Y \leq X \leq Y+Y^\theta} \Big(\max_{X \leq t \leq X+H} |\zeta(c+it)| \Big) < \operatorname{Exp}\left((\log H)^{1-c+\varepsilon}\right)$$

holds for all $Y>Y_0=Y_0(c, \delta, \varepsilon)$. *Here and in what follows* $\theta=5/12$.

Combining (1) and (2) we have the following

THEOREM 1. *As* $Y\to\infty$, *we have*

(3)
$$\log\log\Big\{\min_{Y \leq X \leq Y+Y^\theta} \Big(\max_{X \leq t \leq X+H} |\zeta(c+it)| \Big)\Big\} \sim (1-c)\log\log H.$$

COROLLARY. *If* $\frac{1}{2}<c<1$ *and* $H_2=H_2(X)$ *satisfies*

$$\liminf_{X\to\infty} \frac{\log H_2}{\log\log X} > 0 \quad \textit{and} \quad \lim_{X\to\infty} \frac{\log H_2}{\log X} = 0,$$

then

(4)
$$\liminf_{X\to\infty} \left(\frac{\log\log\Big(\max_{X \leq t \leq X+H_2} |\zeta(c+it)| \Big)}{\log\log H_2} \right) = 1-c.$$

REMARKS. The corollary includes Titchmarsh's theorem as is easily seen by taking $H_2=\operatorname{Exp}\left(\frac{\log X}{\log\log X}\right)$. Titchmarsh's Theorem referred to here is (see [5], page 172)

(5)
$$\limsup_{t\to\infty} \left(\frac{\log\log\left(|\zeta(c+it)|+20\right)}{\log\log t} \right) \geqq 1-c.$$

The corollary includes Titchmarsh's Theorem and says something more than Titchmarsh's Theorem. The Theorem is also true when $(\log\log Y)^{-1}$ is replaced by any function of Y which tends to zero. But then the constant Y_0 will depend also on the nature of this function. For other remarks see § 4.

Because of (1) we need only prove the upper bound $(1-c+\varepsilon)\log\log H$ for the left side of (3). The method of proving this involves only familiar ideas. It is interesting because it leads by combining it with the method of my paper [4] to the following

THEOREM 2. *Let $\chi(p)$ be complex numbers of absolute value 1 associated with each prime p and suppose that the function $F(s)=\prod_p (1-\chi(p)p^{-s})^{-1}$ $(s=\sigma+it, \sigma>1)$ admits of an analytic continuation in $\sigma\geqq\frac{1}{2}-\delta$ (for some $\delta>0$) with the following properties. $F(s)$ has finitely many poles in this region and for all t with $|t|>\tau$, we have $|F(s)<|t|^A$ where A is a constant. Let $N_0(\alpha, T)$ denote the number of zeros of $F(s)$ in $\sigma\geqq\alpha$, $|\mathrm{Im}\, s|\leqq T$. Then for every constant $\varepsilon>0$ we have*

$$\log N_0((1-\delta)/2, T) = \log T+O(\log T)^{\varepsilon}) \tag{6}$$

where the O-constant depends on $\varepsilon, \delta, \tau$ and A.

REMARK. One may compare it with my earlier result [4] that for $(1-\delta)/2\leqq\alpha<\frac{1}{2}$

$$\log N_0(\alpha, T) = \log T+O((\log T)^{(1+\varepsilon)(2-2\alpha)^{-1}}) \tag{7}$$

where the O-constant depends on similar constants as before. But the results of [4] dealt with Dirichlet series which need not have an Euler product.

§ 2. **Proof of Theorem 1.** We have only to prove the upper bound $(1-c+\varepsilon)\cdot$ $\cdot\log\log H$ for every fixed positive ε and all sufficiently large Y, for the left hand side of (3). This is the object of this section.

LEMMA 1. *If $\alpha>\frac{1}{2}$ and $N(\alpha, T)$ denotes the number of zeros of $\zeta(s)$ in $\{\sigma\geqq\alpha; 0\leqq t\leqq T\}$, then (with $\theta=5/12$)*

$$N(\alpha, T+T^{\theta})-N(\alpha, T) = O(T^{5(1-\alpha+\varepsilon)(9-6\alpha)^{-1}})$$

where $\varepsilon>0$ is an arbitrarily small constant and the O-constant depends only on ε and not on α.

PROOF. The lemma can be proved by the method of Montgomery (see [2]). We say one or two words about the proof. Write $P=T^{\theta}$ and consider $\psi(s)=$ $=\zeta(s)M_P(s)-1$ where $M_P(s)=\sum_{n<P}\mu(n)n^{-s}$. We have to use the result

$$\int_T^{T+P}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = O(P^{1+\varepsilon})$$

due to TITCHMARSH (see [5], page 123), the well-known result

$$\int_{T_0}^{T_0+P} M_P\left(\frac{1}{2}+it\right)^2 dt = O(P^{1+\varepsilon})$$

valid for arbitrary T_0, and finally the result $\int_T^{T+P} |\psi(1+it)|^2 = O(P^{\varepsilon})$ which is easily deducible from well-known results.

Note that $5(1-\alpha)(9-6\alpha)^{-1}<\theta$ if $\alpha>\frac{1}{2}$ and so by using the lemma above there exists a region $R_1:\{U\leqq t\leqq U+10Y_1, \sigma\geqq\alpha\}$ in $\{Y\leqq t\leqq Y+Y^{\theta}, \sigma>\alpha\}$ (where $Y_1=Y^{\delta_1}$; $\delta_1>0$ is a small constant) which is free from the zeros of $\zeta(s)$. This is true for every δ_1 which is sufficiently small. We fix one such δ_1 and also an α close to $\frac{1}{2}$. In R_1 the functions $\log\zeta(s)$ and $G(s)=\sum_p p^{-s}$ (both obtained from analytic continuations of the usual branch at $s=3$ first parallel to the t axis and then parallel to the σ axis) differ from a bounded quantity. Since in this region $\zeta(s)=O(t)$ it follows by Borel—Caratheodory theorem (see [5], page 282) that in the region

$$R_2: \{\sigma \geqq \alpha+\delta_1,\ U+Y_1 \leqq t \leqq U+9Y_1\}$$

we have $\log\zeta(s)=O(\log t)$. Let now s be any point of the region

$$R_3: \{\alpha+2\delta_1 \leqq \sigma \leqq \alpha+4\delta_1,\ U+2Y_1 \leqq t \leqq U+8Y_1\}.$$

We can obtain a short approximation to $G(s)$. Thus:

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\,W=2} G(s+W)\Gamma(W)J^W\,dW = \sum_p (c^{-p/J}p^{-s}) \tag{8}$$

with $J=(\log Y)^A$ where A is a large constant depending upon δ_1 and other constants. Breaking off the limits of the integral at two suitable points and then moving the line of integration to $\mathrm{Re}(W)$ given by $\mathrm{Re}(W+s)=\alpha+\delta_1$ we are led to

$$|G(s)| \ll \Big|\sum_p e^{-p/J}p^{-s}\Big|+1.$$

Next we break off the portion $p>J^2$ of the series with a small error and so for s in R_3 we have

$$|G(s)| \ll \Big|\sum_{p\leqq J^2} (c^{-p/J}p^{-s})\Big|+1. \tag{9}$$

From this by using an old well-known result on mean values (see [1], page 334), we have

$$\iint_{R_3} |G(s)|^{2k}\,d\sigma\,dt \leqq C_1^k(Y_1+J^{3k})\sum_{n=1}^{\infty}(a_n n^{-\alpha-2\delta_1})^2, \tag{10}$$

where C_1 is a constant and a_n are defined by $(G(W))^k = \sum a_n n^{-W}$ (Re $W \geqq 2$). Since trivially $a_n \leqq k^k$ we have

(11) $$\iint_{R_3} |G(s)|^{2k} \, d\sigma \, dt \leqq (kC_2)^k (Y_1 + J^{3k})$$

where C_2 is a constant. Remembering that $J = (\log Y)^A$ we impose $(\log Y)^{3Ak} \leqq Y_1$. This is secured if $k = o\left(\frac{\log Y}{\log\log Y}\right)$ and we get

(12) $$\iint_{R_3} |G(s)|^{2k} \, d\sigma \, dt \leqq (kC_3)^k Y_1$$

where C_3 is a constant. From this it is immediate that if $10H \leqq Y_1$ with $H \geqq 3$ there exists a rectangle $R_4: \{\alpha + 2\delta_1 \leqq \sigma \leqq \alpha + 4\delta_1,\ V \leqq t \leqq V + 10H\}$ where $U + 2Y_1 \leqq$ $\leqq V$ and $V + 10H \leqq V + 8Y_1$ such that

(13) $$\iint_{R_4} |G(s)|^{2k} \, d\sigma \, dt \leqq H(kC_4)^k,$$

where C_4 is a constant. Let now M denote the maximum of $|\log \zeta(s)|$ on $\sigma = \alpha + 3\delta_1$ subject to $V + H \leqq t \leqq V + 9H$. Then using $|G(s)|^{2k} \ll \int_{|W-s| \leqq \delta_1} |G(W)|^{2k} \, da$ (da = element of area) using (13) we are led to

(14) $$M^{2k} \leqq H(kC_5)^k \quad \text{i.e.} \quad M \ll H^{1/(2k)} k^{1/2},$$

where C_4 is a constant. Of course, for the validity of (14), we should have $k =$ $= o\left(\frac{\log Y}{\log\log Y}\right)$, $10H \leqq Y_1$, $H \geqq 3$. Let now $k = \left(\frac{D \log Y}{\log\log H}\right)$ where D is a large constant. This satisfies the condition on k imposed already since $\frac{\log H}{\log Y}$ tends to zero as Y tends to infinity. Thus we get

(15) $$M = O\big((\log H)^{1/2+\varepsilon}\big), \quad (\varepsilon \text{ depends on } D),$$

subject only to $10H \leqq Y_1$ and $H \geqq 3$.

If now $H \geqq (\log Y)^{\delta_1}$ we can apply maximum modulus principle to the function $G_1(W) = \big(\log \zeta(W)\big) e^{(W-z)^2} Q^{w-z}$ ($Q > 0$, suitably chosen) in the rectangle $R_5: \{V + H \leqq \text{Im } W \leqq V + 9H,\ \alpha + 3\delta_1 \leqq \text{Re } W \leqq 1 + \delta_1\}$ provided z lies on the segment $L_1: \{\text{Re } z = c,\ V + 2H \leqq \text{Im } z \leqq V + 8H\}$ and deduce that

(16) $$M_1 = O\big((\log H)^{1-c+\varepsilon}\big)$$

where M_1 is the maximum of $|\log \zeta(z)|$ on L_1 and ε is an arbitrarily small positive constant.

This proves Theorem 1 completely. We now remark about the condition $H \geqq (\log Y)^{\delta_1}$. This is used only in the application of maximum modulus principle. By slight changes in the argument one can prove that, for $3 \leqq H \leqq \operatorname{Exp}\left(\frac{\log Y}{\log\log Y}\right)$,

$$(17) \qquad \min_{Y \leqq X \leqq X+X^\theta} \max_{\substack{X \leqq t \leqq X+H \\ \sigma \geqq c}} |\zeta(\sigma+it)| < \operatorname{Exp}\left(C_6(\log H)^{1/2+\varepsilon}\right)$$

where C_6 is a constant depending on c and $\varepsilon>0$. From this one can deduce, by the application of maximum modulus principle, that the right hand side in (17) can be replaced by $\operatorname{Exp}\left(C_6(\log H)^{1-c+\varepsilon}\right)$. An alternative treatment will be sketched in § 4.

§ 3. Proof of Theorem 2. We shall briefly sketch the proof. Let $\delta>0$ and $\eta>0$ be small constants. Then we will show by reductio ad absurdum that every rectangle $R_6:\left\{\sigma>\frac{1}{2}-\delta,\ T\leqq t\leqq T+20T_1\right\}$, where $T_1=\operatorname{Exp}\left((\log T)^\eta\right)$, contains a zero of $F(s)$ for all $T>\tau_0$. This will prove the theorem since, for Dirichlet series of finite order we know that $N(\sigma, T)=O(T\log T)$ (see [6], page 310, § 9. 621). We assume that what we wish to prove is false. Then by Borel—Caratheodory theorem we have in $R_7:\left\{\sigma\geqq\frac{1}{2}-\delta_2,\ T+T_1\leqq t\leqq T+19T_1\right\}$, where $\delta_2=\delta/2$, the estimate $\log F(s)=O(\log T)$. As in the proof of Theorem 1, we can prove that in a certain region $R_8:\left\{\sigma\geqq\frac{1}{2}+\delta_3,\ V_1+T_2\leqq t\leqq V_1+19T_2\right\}$ (where the t interval is contained in that for R_7 and $T_2=\operatorname{Exp}\left(\frac{(\log T)^\eta}{\log\log T}\right)$) the estimate $\log F(s)=O\left((\log T)^{\frac{1}{2}\eta+\varepsilon}\right)$ holds where the O-constant depends on $\delta_3>0$, $\varepsilon>0$ and η. We apply maximum modulus principle to the function $G_2(W)=\left(\log F(W)\right)e^{(W-z)^2}Q_1^{W-z}$ (with a suitable choice of $Q_1>0$) in the rectangle $R_9:\left\{\frac{1}{2}-\delta_2\leqq \operatorname{Re} W\leqq\frac{1}{2}+\delta_3, V_1+2T_2\leqq \operatorname{Im} W\leqq V_1+18T_2\right\}$ where z is restricted to the segment $L_2:\left\{\sigma=\frac{1}{2}-\delta_4,\ V_1+3T_2\leqq t\leqq V_1+17T_2\right\}$. Here δ_4 is a suitable positive constant depending on η. We then get for the maximum M_2 of $F(z)$ for z on L_2 the estimate

$$(18) \qquad M_2 = O\left(\operatorname{Exp}(\log T)^{(3/4)\eta}\right).$$

Also the same estimate holds if M_2 is replaced by the maximum of $|F(z)|$ in $R_{10}:\left\{\sigma\geqq\frac{1}{2}-\delta_4,\ V_1+4T_2\leqq t\leqq V_1+16T_2\right\}$. This can be contradicted (by the method of [4]) by considering a suitable lower bound for $\int_{L_3}|F(s)|^2\,dt$ where L_3 is the segment $\left\{\sigma=\frac{1}{2}-\delta_5\right.$ with $\delta_5=\frac{1}{2}\delta_4,\ \left.V_1+5T_2\leqq t\leqq V_1+15T_2\right\}$. (Note that we can approximate to $F(s)$, for s in $R_{11}:\left\{\sigma\geqq\frac{1}{2}-\delta_5,\ V_1+5T_2\leqq t\leqq V_1+15T_2\right\}$ by "a short Dirichlet polynomial".) This completes the proof of Theorem 2.

§ 4. Concluding remarks. Both in [3] and in this note it would, perhaps, be more satisfying to consider the quantities

$$M_3 = \max_{\substack{X \leqq t \leqq X+H_1 \\ \sigma \geqq c}} |\zeta(\sigma+it)|$$

and

$$M_4 = \min_{Y \leqq X \leqq Y+Y^\theta} \Big(\max_{\substack{X \leqq t \leqq X+H \\ \sigma \geqq c}} |\zeta(\sigma+it)| \Big).$$

For these quantities one can prove by our method the following

THEOREM 3. *If* $3 \leqq H_1 \leqq X$ *then*

$$M_3 > \operatorname{Exp}\{C_7 (\log H_1)^{1-c-\varepsilon}\}$$

where C_7 *is a positive constant depending only on* c $\left(\frac{1}{2} < c < 1\right)$ *and* $\varepsilon > 0$. *Also if* $H = H(Y) \geqq 20$ *and* $(\log H)(\log Y)^{-1} \leqq (\log\log Y)^{-1}$ *then*

$$M_4 < \operatorname{Exp}\{C_8 (\log H)^{1-c+\varepsilon}\}$$

where C_8 *is a constant depending only on* c $\left(\frac{1}{2} < c < 1\right)$ *and* $\varepsilon > 0$.

REMARK. The quantity $(\log\log Y)^{-1}$ can be replaced by any function of Y which tends to zero as Y tends to infinity. But the constant C_8 depends also on the nature of this function.

To prove the first part of Theorem 3, only slight changes are necessary in the arguments of [3]. To prove the second part some obvious changes are necessary in the arguments of § 2. We indicate only one of them and state it as

LEMMA 2. *Let the numbers* a_n *be defined by* $\left(\sum_p p^{-W}\right)^k = \sum a_n n^{-W}$, $\operatorname{Re}(W) > 1$. *Then for any constant* $c > \frac{1}{2}$,

$$\sum_{n=1}^{\infty} (a_n n^{-c})^2 = O\big((kC_9)^{(2-2c+\varepsilon)k}\big)$$

where C_9 *depends only on* c *and* ε.

PROOF. It is plain that $\sum a_n n^{-1-\varepsilon} = O(C_{10}^k)$ where C_{10} depends only on ε. It is also plain that $a_n \leqq k^k$. So we have only to prove that

$$\max_n \min \left(\frac{a_n}{n^{2c-1-\varepsilon}}, \frac{k^k}{n^{2c-1-\varepsilon}} \right) = O\big((kC_{11})^{(2-2c+\varepsilon)k}\big)$$

where C_{11} depends only on c and ε. Now, for $n > k^k$ it is clear that $k^k n^{-(2c-1-\varepsilon)} = O(k^{(2-2c+\varepsilon)k})$ and so it suffices to prove that

$$\max_{n \leqq k^k} (a_n n^{-(2c-1-\varepsilon)}) = O\big((kC_{12})^{(2-2c+\varepsilon)k}\big)$$

where C_{12} depends only on c and ε. This is clear from the fact that if $d>0$, then

$$a_n n^{-d} \leqq \Big(\sum_{p|n} p^{-d}\Big)^k \quad \text{and} \quad \sum_{p|n} p^{-d} \leqq \sum_{p\leqq \log n} p^{-d} + (\log n)^{-d} \sum_{p|n} \log p = O\big((\log n)^{1-d}\big)$$

where the O-constant depends only on d.

In conclusion I wish to thank Professors N. Levinson and P. Turán for their interest in this work.

Added in proof (November 1, 1977). (a) R. Balasubramanian's results on the mean square of $\left|\zeta\left(\frac{1}{2}+it\right)\right|$ (to appear in *Proc. London Math. Soc.*) enables us to write $\theta = 1/3$ instead of 5/12. (b) In Theorem 2 the error $O\big((\log T)^{\varepsilon}\big)$ can be replaced by $O(\log\log T)$. These will appear in a forthcoming paper.

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LINKS AND PRODUCTS IN COUNTABLY COMPACT AND *M*-SPACES

By

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1. Introduction. *M* and $\mathscr{C}$-spaces

Suppose $\{A_n\}$ is a sequence of subsets of a topological space X. Then, following MAKKOUK [4], we can define Li A_n, the *link* of A_n, as the set of all points $x \in X$ such that every neighbourhood of x meets all but finitely many A_n.

An *M-space* is defined as follows: X is M if it has a normal sequence $\{U_n\}$ of open covers such that every point sequence $\{x_n\}$, of the form $x_n \in \mathrm{St}(x, U_n)$ for some fixed $x \in X$, has a cluster point. A point sequence of the form $\{x_n : x_n \in \mathrm{St}(x, U_n)$ for fixed $x \in X\}$ will be called an *M-sequence for* x. Assume all spaces Hausdorff in this paper.

PROPOSITION 1. *If* X *is an M-space, then* $\mathrm{Li}\,\mathrm{St}(x, U_n) = C(x)$ *is countably compact. Further, if* $\{x_n\}$ *is an M-sequence for* x, *then* $\{x_n\} \cup C(x)$ *is countably compact.*

PROOF. If $\{x_n\}$ is a point sequence such that $x_n \in C(x)$ for each n, then $\{x_n\}$ is an M-sequence for x, so $\{x_n\}$ clusters in $C(x)$. Thus $C(x)$ is countably compact. Similarly, if $S = C(x) \cup \{x_n\}$, every point sequence in S clusters, so S is countably compact.

PROPOSITION 2. *Among the topological spaces which are M-spaces are metric spaces, pseudometric spaces and countably compact spaces.*

PROOF. See the literature.

The previous propositions make clear the relationship between M-spaces and countably compact spaces. Similar results are obtainable concerning paracompact spaces and sequential compact spaces.

DEFINITION 1. A topological space X is a $\mathscr{C}$*-space* if there exists a normal sequence of open covers of X such that, for every $x \in X$, every M-sequence about x is sequentially compact. (See ISHII, TSUDA, KUNUGI [2]).

PROPOSITION 3. *If* X *is a* $\mathscr{C}$*-space,* $C(x)$ *is sequentially compact for every* $x \in X$.

PROPOSITION 4. *If* X *is a paracompact M-space,* $C(x)$ *is compact.*

PROOFS. Proposition 3 follows from the fact that every $\mathscr{C}$-space is M. On the other hand, if X is paracompact M, then $C(x)$ is paracompact and countably compact, hence compact.

Actually, the previous result holds as well for Lindelöf or metacompact spaces (i.e., Lindelöf M-spaces or metacompact M-spaces have compact links), since Lindelöf or metacompact plus countably compact equals compact.

Recall that not every M-space is paracompact, since $[0, \Omega)$ is countably compact but not paracompact.

2. Products of M-spaces

ISIWATA [3] has given an example of two M-spaces whose product is not M. This leads to the question: what additional hypotheses will yield products of M-spaces which are M?

The next three results have been published previously; proofs are in the referenced papers.

PROPOSITION 5 (ISHII—TSUDA—KUNUGI [2]). *If X is a $\mathscr{C}$-space and Y is M, then $X\times Y$ is M.*

PROPOSITION 6 (also [2]). *Among M-spaces which are $\mathscr{C}$-spaces are first countable, locally compact and paracompact spaces.*

PROPOSITION 7 (RISHEL [8]). *Among the M-spaces which are $\mathscr{C}$-spaces are sequential spaces, k-spaces and weakly-k spaces.*

The concepts of sequential and k-spaces are well known generalizations of first countable and locally compact spaces, respectively, but weakly-k spaces are less well known.

DEFINITION 2. A topological space X is *weakly-k* if: a set F is closed in X if $F\cap C$ is finite for every C compact in X.

PROPOSITION 8 ([8]). *If X is an M-space, X is $\mathscr{C}$ iff X is weakly-k.*

From Propositions 5 and 8 we get

COROLLARY 1. *If X and Y are M-spaces, $X\times Y$ is M if either X or Y is weakly-k.*

In a recent paper, N. NOBLE [6] has defined a type of space we will call a c^*-space.

DEFINITION 3. X is a *c^*-space* if every infinite set in X meets some compact set in an infinite set.

PROPOSITION 9. *If X and Y are M-spaces and X is c^*, then $X\times Y$ is M.*

PROOF. Take U_n, V_n normal sequences of open covers of X and Y, respectively. Then

$$W_n = U_n\times V_n$$

is easily seen to form a normal sequence of open covers of $X\times Y$. So let $\{z_n\}=\{(x_n, y_n)\}$ be an M-sequence about $z=(x, y)\in X\times Y$. Then $\{y_n\}$ is an M-sequence about y in Y; Cl $\{y_n\}$ is a countably compact subset of $\{y_n\}\cup C(y)$.

The point sequence $\{x_n\}$ has either finite or infinite cardinality. If $\{x_n\}$ is finite, choose an $x_0\in\{x_n\}$ such that x_0 appears infinitely often. Then $\{x_0\}\times \mathrm{Cl}\,\{y_n\}$ is countably compact; thus $\{(x_0, y_n)\}$ clusters inside $\mathrm{Li\,St}\,(z, W_n)$. So $X\times Y$ is an M-space.

Now, if $\{x_n\}$ is infinite, a compact K exists in X such that $K\cap\{x_n\}=\{x_{n(i)}\}$ and $\mathrm{Cl}\,\{x_{n(i)}\}\times \mathrm{Cl}\,\{y_n\}$ is countably compact (since the product of a countably compact space with a compact space is countably compact). Further, $\{(x_{n(i)}, y_{n(i)})\}$ clusters in $C(z)$. Thus, $X\times Y$ is an M-space.

A similar proof is required for the following result; the proof is clear and will be omitted.

PROPOSITION 10. *If X is c^* and Y is countably compact, then $X\times Y$ is countably compact.*

A logical question is that of the relationship between c^*-spaces and weakly-k spaces.

PROPOSITION 11. *Every countably compact k-space is c^*, and every c^*-space is weakly-k.*

PROOF. The proof of the first assertion is in Noble's paper. To show the latter, assume $F\cap C$ is finite for every compact C in a c^*-space X. Then F is finite; thus it is closed. But this last says that X is weakly-k.

A similar space has been defined by CHIBA [1].

DEFINITION 4. X is a *k_0-space* if every point sequence which accumulates in X has a subsequence whose closure is contained in a compact set.

The proof of the next result is clear from a comparison of the definitions of M and $\mathscr{C}$-spaces.

PROPOSITION 12 (CHIBA). *X is a $\mathscr{C}$-space iff X is M and k_0.*

COROLLARY 2. *Weakly-k spaces and k_0-spaces are equivalent in the class of all M-spaces.*

A further relationship between weakly-k and k_0-spaces may also be derived by use of a property studied separately by ARHANGELSKII and RISHEL (see [7]).

DEFINITION 5. A topological space X has *countable tightness* iff the following holds:

for $U\subset X$, if $S\cap U$ is open in S for all countable sets $S\subseteqq X$, then U is open in X.

PROPOSITION 13. *If X is a k_0-space and has countable tightness, then X is weakly-k. Further, every weakly-k space is k_0.*

PROOF. To prove the first assertion, suppose F is nonclosed in X. Then a countably infinite set $S\subseteqq F$ exists such that $\mathrm{Cl}\, S\cap(X-F)\neq\emptyset$. Choose a point sequence $\{a_n : a_n\in S\}$. Then there exists an $\{a_{n(i)}\}$ such that $K=\mathrm{Cl}\,\{a_{n(i)}\}$ is compact and infinite. But $K\cap F\supseteqq\{a_{n(i)}\}$, so X is weakly-k.

The proof of the second assertion is clear and will be omitted.

As a final result on products we present the following relation between M-spaces and countable compactness.

PROPOSITION 14. *Let X and Y be M-spaces. If, for every countably compact space S in X, and every countably compact T in Y, $S\times T$ is countably compact, then $X\times Y$ is an M-space.*

PROOF. Let X and Y be M-spaces such that $X\times Y$ is not M. Then there exists a point $z\in X\times Y$ and an M-sequence $\{z_n\}$ such that $\{z_n\}$ does not cluster inside $C(z)$.

Letting $\{z_n\}=\{(x_n, y_n)\}$ and $z=(x, y)$; $\{x_n\}$ and $\{y_n\}$ are thus M-sequences in X and Y, respectively. By Proposition 1, $\{x_n\}\cup C(x)=S$ is countably compact, as is $\{y_n\}\cup C(y)=T$. However, $S\times T$ is not countably compact. Hence our conclusion.

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CONCERNING JOINS OF EQUATIONAL CLASSES OF BURNSIDE GROUPS

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A group $\mathfrak{G}=(G, \cdot, 1)$ is called a Burnside group if it has an exponent $m\in\mathbf{N}$, i.e. if it satisfies $x^m=1$ for some $m\in\mathbf{N}=:\{1, 2, ...\}$. Let K_m be the equational class of all groups satisfying $x^m=1$ where $m=\min(n|x^n=1 \text{ in } K_m, n\in\mathbf{N})$. We denote by $K_p\vee K_q$ the join of the equational classes K_p and K_q, i.e. the smallest equational class containing K_p and K_q. Further we denote by $K_p\times K_q$ the class of all products $\mathfrak{G}_1\times\mathfrak{G}_2$ where $\mathfrak{G}_1\in K_p$, $\mathfrak{G}_2\in K_q$.

In [2] J. PŁONKA proved that if p and q are mutually prime then $K_p\vee K_q=K_p\times K_q$, and he posed the question whether the converse is true. In this paper we answer the question of Płonka in a positive way by the following

THEOREM. *If $K_p\wedge K_q$ contains one-element algebras only*[1] *and if $K_p\vee K_q=K_p\times K_q$ then p and q are mutually prime.*

PROOF. Let P and Q be arbitrary equational classes. It was shown in [1] that there exists a polynomial $x\circ y$ such that $x\circ y=x$ is satisfied in P and $x\circ y=y$ is satisfied in Q if P, Q have the following properties:

(i) $P\wedge Q$ contains one-element algebras only

(ii) $P\vee Q=P\times Q$

(iii) Every algebra $\mathfrak{A}\in P\vee Q$ has a modular congruence-lattice.

So in our case there exists such a term since the assumption of modularity is satisfied and we may assume

$$x\circ y = x^{u_1}y^{v_1}\dots x^{u_s}y^{v_s} \tag{1}$$

where all $u_\sigma, v_\sigma (0\leqq\sigma\leqq s)$ are non-negative integers, because $x\circ y$ is a term in $K_p\vee K_q$. Define

$$u = \sum_1^s u_\sigma. \tag{2}$$

It follows:

$$x\circ 1 = x \quad \text{in} \quad K_p, \quad x\circ 1 = 1 \quad \text{in} \quad K_q. \tag{3}$$

So we obtain (by 1, 2, 3):

$$x^{u-1}=1 \quad \text{in} \quad K_p, \quad x^u=1 \quad \text{in} \quad K_q$$

[1] This fact is obvious if $pr+qs=1$, since we get for every group $\mathfrak{G}\in K_p\wedge K_q$ the equation $x= =x^1=x^{pr+qs}=x^{pr}x^{qs}=1$.

from which results:

$$p|u-1 \wedge q|u \Rightarrow pr = u-1 \wedge qs = u \quad (r, s \in \mathbf{Z}) \Rightarrow qs - pr = 1.$$

Thus p and q are mutually prime.

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EXISTENCE AND REGULARITY PROBLEMS FOR NONLINEAR FUNCTIONAL EQUATIONS

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1. Introduction

The purpose of this paper is presenting the proofs of the results announced by the author in [6] and [8] and applying them in the proofs of some simple regularity criteria for locally integrable solutions $f:R^n \to R$ of the equations

$$\sum_{i=1}^{k} a_i(x, t) f(\Phi_i(x, t)) = F(x, f(\lambda_1(x)), \ldots, f(\lambda_s(x))) + b(x, t), \tag{1.1}$$

where $x \in R^n$, $t \in \Omega \subset R^r$, $n > 1$, $r \geqq 1$, $\Phi_i: R^n \times \Omega \to R^n$, $\lambda_j: R^n \to R^n$, $a_i: R^n \times \Omega \to R$, $b: R^n \times \Omega \to R$, $F: R^{n+s} \to R$.

The starting point is a general regularity criterion (see H. ŚWIATAK [6] and [7]) which gives the conditions assuring that all the locally integrable solutions f of (1.1) are equal to C^∞ functions almost everywhere. Two lemmas, which will be used to prove the main theorems of chapters 2 and 3, contain the results entirely independent of this regularity criterion.

Criteria for the existence of solutions are not considered. It is shown only that, under some general assumptions, the existence of locally integrable solutions f of (1.1) implies the existence of continuous or C^∞ solutions.

2. Existence of continuous and C^∞ solutions as a consequence of the existence of locally integrable solutions

LEMMA 2.1. *Suppose that*

1° a_i, b *and* Φ_i *are continuous with respect to* $x \in R^n$ *for every fixed* t *from an open set* $\Omega \subset R^r$, $i = 1, \ldots, k$,

2° F *is continuous in* R^{n+s},

3° λ_j *are continuous in* R^n, $j = 1, \ldots, s$,

4° *for every Lebesgue measurable set* $E \subset R^n$ *such that* $\mu(E) = 0$ *the sets*

$$\Phi_i^{-1}(E, t) = \{x: \exists y \in E \text{ such that } \Phi_i(x, t) = y\}$$

are measurable and $\mu(\Phi_i^{-1}(E, t)) = 0$, $i = 1, \ldots, k$,

5° *for every Lebesgue measurable set* $E \subset R^n$ *such that* $\mu(E) = 0$ *the sets* $\lambda_j^{-1}(E)$ *are measurable and* $\mu(\lambda_j^{-1}(E)) = 0$, $j = 1, \ldots, s$.

Then every continuous function $\tilde{f}$, *which is equal to a solution* f *of* (1.1) *almost everywhere, is also a solution of this equation.*

PROOF. Let us fix a point $t\in\Omega$ and a point $x^*\in R^n$ and let

$$K_\nu=\left\{x:\ \varrho(x,x^*)<\frac{1}{\nu}\right\}.$$

Let

$$A_{i\nu}=\{x:\ x\in K_\nu, f(\Phi_i(x,t))+\tilde f(\Phi_i(x,t))\};\quad i=1,\dots,k,$$

and

$$B_{j\nu}=\{x:\ x\in K, f(\lambda_j(x))=\tilde f(\lambda_j(x));\quad j=1,\dots,s.$$

We shall prove that $A_{i\nu}$ and $B_{j\nu}$ are measurable and

(2.1) $$\mu(A_{i\nu})=\mu(Kv)\quad\text{for}\quad i=1,\dots,k,$$

(2.2) $$\mu(B_{j\nu})=\mu(K_\nu)\quad\text{for}\quad j=1,\dots,s.$$

To prove (2.1) let us define the sets

$$G_{i\nu}=\{y:\ y\in\Phi_i(K_\nu,t),\quad f(y)\neq\tilde f(y)\},$$

where

$$\Phi_i(K_\nu,t)=\{y:\ y=\Phi_i(x,t),\ x\in K_\nu\}$$

for $i=1,\dots,k$ and

$$\Phi_i^{-1}(G_{i\nu},t)=\{x:\ \exists y\in G_{i\nu}\text{ such that }\Phi_i(x,t)=y\}.$$

Since $f(y)=\tilde f(y)$ almost everywhere in R^n and since the Lebesgue measure is complete $G_{i\nu}$ are measurable sets and $\mu(G_{i\nu})=0$ for $i=1,\dots,k$. Therefore 4° implies that $\Phi_i^{-1}(G_{i\nu},t)$ are measurable and $\mu(\Phi_i^{-1}(G_{i\nu},t))=0$ for $i=1,\dots,k$.

Let

$$C_{i\nu}=K_\nu\setminus\Phi_i^{-1}(G_{i\nu},t);\quad i=1,\dots,k.$$

Obviously $\mu(C_{i\nu})=0$ for $i=1,\dots,k$. Moreover,

(2.3) $$A_{i\nu}=K_\nu\setminus C_{i\nu}\quad\text{and}\quad C_{i\nu}\subset K_\nu;\quad i=1,\dots,k.$$

The measurability of the sets $C_{i\nu}$ and K_ν implies the measurability of the sets $A_{i\nu}$ and, by (2.3), $\mu(A_{i\nu})=\mu(K_\nu)-\mu(C_{i\nu})=\mu(K_\nu)$ for $i=1,\dots,k$. Thus (2.1) is proved.

The proof of (2.2) is similar: let

$$H_{j\nu}=\{y:\ y\in\lambda_j(K_\nu), f(y)\neq\tilde f(y)\}$$

and

$$\lambda_j^{-1}(H_{j\nu})=\{x:\ \exists y\in H_{j\nu}\text{ such that }\lambda_j(x)=y\}$$

for $j=1,\dots,s$. Since $f(y)=\tilde f(y)$ almost everywhere in R^n and since the Lebesgue measure is complete, $H_{j\nu}$ are measurable sets and $\mu(H_{j\nu})=0$ for $j=1,\dots,s$. Therefore 5° implies $\mu(\lambda_j^{-1}(H_{j\nu}))=0$ for $j=1,\dots,s$.

Now we set

$$D_{j\nu}=K_\nu\cap\lambda_j^{-1}(H_{j\nu})\quad(j=1,\dots,s)$$

and notice that $\mu(D_{j\nu})=0$, $B_{j\nu}=K_\nu\setminus D_{j\nu}$ and $D_{j\nu}\subset K_\nu$ for $j=1,\dots,s$. Hence $\mu(B_{j\nu})=\mu(K_\nu)-\mu(D_{j\nu})=\mu(K_\nu)$ for $j=1,\dots,s$.

Let

$$A_\nu = A_{1\nu} \cap \ldots \cap A_{k\nu}.$$

Since $A_{i\nu} \subset K_\nu$ for $i=1, \ldots, k$ and since (2.1) holds,

$$\mu(A_\nu) = \mu(K_\nu). \tag{2.4}$$

With

$$B_\nu = \{x\colon x \in K_\nu,\ F(x, f(\lambda_1(x)), \ldots, f(\lambda_s(x))) = F(x, \tilde{f}(\lambda_1(x)), \ldots, \tilde{f}(\lambda_s(x)))$$

it follows

$$B_{1\nu} \cap \ldots \cap B_{s\nu} \subset B_\nu \subset K_\nu. \tag{2.5}$$

Since, moreover, $B_{j\nu} \subset K_\nu$ for $j=1, \ldots, s$ and (2.2) holds, $B_{1\nu} \cap \ldots \cap B_{s\nu}$ is a measurable set and $\mu(B_{1\nu} \cap \ldots \cap B_{s\nu}) = \mu(K_\nu)$. Now, (2.5) implies that $\mu(K_\nu \setminus (B_{1\nu} \cap \ldots \cap B_{s\nu})) = = \mu(K_\nu) - \mu(B_{1\nu} \cap \ldots \cap B_{\ \nu}) = 0$. Since, by (2.5), $K_\nu \setminus B_\nu \subset K_\nu \setminus (B_{1\nu} \cap \ldots \cap B_{s\nu})$ and since the Lebesgue measure is complete, $K_\nu \setminus B_\nu$ is a measurable set and $\mu(K_\nu \setminus B_\nu) = 0$. Since $B_\nu \subset K_\nu$ we have $B_\nu = K_\nu \setminus (K_\nu \setminus B_\nu)$ and measurability of the sets K_ν and $K_\nu \setminus B_\nu$ implies the measurability of B_ν. Therefore, the equality $\mu(B_{1\nu} \cap \ldots \cap B_{s\nu}) = = \mu(K_\nu)$ and (2.5) imply

$$\mu(B_\nu) = \mu(K_\nu). \tag{2.6}$$

Now, let us define

$$V_\nu = A_\nu \cap B_\nu; \quad \nu = 1, 2, \ldots.$$

Since $A_\nu \subset K_\nu$ and $B_\nu \subset K_\nu$, (2.4) and (2.6) imply

$$\mu(V_\nu) = \mu(K_\nu); \quad \nu = 1, 2, \ldots. \tag{2.7}$$

It follows from (2.7) that none of the sets V_ν can be empty.

Let $\{x_\nu\}$ be an arbitrary sequence such that $x_\nu \in V_\nu$ for $\nu = 1, 2, \ldots$. It follows from the definitions of the sets V_ν, A_ν and B_ν that every such a sequence $\{x_\nu\}$ converges to x^*. Moreover,

$$f(\Phi_i(x_\nu, t)) = \tilde{f}(\Phi_i(x_\nu, t)) \quad (i = 1, \ldots, k;\ \nu = 1, 2, \ldots)$$

with t fixed at the beginning of the proof, and

$$F(x_\nu, f(\lambda_1(x_\nu)), \ldots, f(\lambda_s(x_\nu))) = F(x_\nu, \tilde{f}(\lambda_1(x_\nu)), \ldots, \tilde{f}(\lambda_s(x_\nu)))$$

for $\nu = 1, 2, \ldots$.

At the points (x_ν, t) (1.1) can be written as

$$\sum_{i=1}^{k} a_i(x_\nu, t) \tilde{f}(\Phi_i(x_\nu, t)) = F(x_\nu, \tilde{f}(\lambda_1(x_\nu)), \ldots, \tilde{f}(\lambda_s(x_\nu))) + b(x_\nu, t).$$

Since $x_\nu \to x^*$ when $\nu \to \infty$, assumptions 1°, 2°, 3° and the continuity of the function $\tilde{f}$ imply

$$\sum_{i=1}^{k} a_i(x^*, t) \tilde{f}(\Phi_i(x^*, t)) = F(x^*, \tilde{f}(\lambda_1(x^*)), \ldots, \tilde{f}(\lambda_s(x^*))) + b(x^*, t).$$

This finishes the proof because x^* and t were arbitrary fixed points.

THEOREM 2.1. *Suppose that*

1° a_i *and* b *are* C^∞ *functions with respect to* $x \in R^n$ *for every fixed* t *from an open set* $\Omega \subset R^r$, $i=1, \dots, k$,

2° $a_i \in C^m$ *and* $b \in C^m$ *in* $R^n \times \Omega$, $i=1, \dots, k$,

3° *the mappings* $x \mapsto y = \Phi_i(x, t)$ *are diffeomorphisms in* R^n *for every fixed* $t \in \Omega$, $i=1, \dots, k$,

4° Φ_i *and* $\Phi_i^{-1}: (y, t) \mapsto x$, *where, in view of* 3°, x *is determined uniquely by the condition* $y = \Phi_i(x, t)$, *are functions of the class* C^m *in* $R^n \times \Omega$, $i=1, \dots, k$,

5° F *is continuous in* R^{n+s},

6° λ_j *are continuous in* R^n, $j=1, \dots, s$,

7° *there exists an* $\alpha \in \Omega$ *such that* $\Phi^i(x, \alpha) \equiv x$ *for* $i=1, \dots, k$,

8° *for every Lebesgue measurable set* $E \subset R^n$ *such that* $\mu(E)=0$ *the sets*

$$\Phi_i^{-1}(E, t) = \{x\colon \exists y \in E \text{ such that } \Phi_i(x, t) = y\}$$

are measurable and $\mu(\Phi_i^{-1}(E, t)) = 0$, $i=1, \dots, k$,

9° *for every Lebesgue measurable set* $E \subset R^n$ *such that* $\mu(E)=0$ *the sets* $\lambda_j^{-1}(E)$ *are measurable and* $\mu(\lambda_j^{-1}(E)) = 0$, $j=1, \dots, s$,

10° *there exists an r-tuple* q ($|q| \leqq m$) *such that the equation*

$$D_t^q \left(\sum_{i=1}^{k} a_i(x, t) f(\Phi_i(x, t)) \right)_{t=\alpha} = 0, \tag{2.8}$$

where the left-hand side is obtained by formal differentiation of the left-hand side of (1.1) *and by setting* $t=\alpha$, *is of constant strength, hypoelliptic at an* x_0.

Then every locally integrable solution f *of* (1.1) *is equal to a solution of the class* C^∞ *almost everywhere.*

PROOF. Assumptions 1° —7°, 10° and the assumptions of Theorem I of [6] (see also [7, Theorem 5.1]) coincide. Therefore every locally integrable solution f of (1.1) is equal to a function of the class C^∞ almost everywhere. Denote such a function by $\tilde{f}$.

Assumptions 5°, 6°, 8°, 9° and the assumptions 2° —5° of Lemma 2.1 coincide. Assumptions 1° and 3° imply that also assumption 1° of Lemma 2.1 is satisfied. Since the function $\tilde{f}$ is a continuous function which is equal to a solution f of (1.1) almost everywhere, it is also a solution of this equation. But $\tilde{f} \in C^\infty$ and in this way we have proved that the existence of locally integrable solutions of (1.1) implies the existence of solutions in C^∞.

REMARK 2.1. Similar results were obtained by the author in [4, Lemma 2, Theorem 4] in a special case, namely, $\Phi_i(x, t) = x + \varphi_i(t)$ and $F(x, u_1, \dots, u_s) \equiv 0$.

3. Regularity of locally integrable solutions

LEMMA 3.1. *Suppose that* $r \geqq n$ *and*

1° a_i, b, *and* Φ_i *are continuous in an open set* $\Omega \subset R^r$ *for every fixed* $x \in R^n$, $i=1, \dots, k$,

2° *there exists an* $\alpha \in \Omega$ *such that* $\Phi_i(x, \alpha) \equiv x$ *in* R^n *for* $i=1, \dots, k$ *and* $\sum_{i=2}^{k} a_i(x, \alpha) \neq 0$ *in* R^n, 3° $\Phi_1(x, t) \equiv x$ *in a neighbourhood of* α,

4° *for any fixed point* $x \in R^n$ *there exists a neighbourhood* U_x *of this point such that for every measurable set E the following condition is satisfied:*

$$\{E \subset U_x, \mu_n(E) = 0\} \Rightarrow \{\mu_r(E_{\Phi_i, x}) = 0 \quad \textit{for} \quad i = 2, \ldots, k\},$$

where

$$E_{\Phi_i, x} = \{t\colon \varphi_i(x, t) \in E\}.$$

Then every solution of of (1.1) *has to be continuous if it is equal to a continuous function almost everywhere.*

PROOF. Let f be a solution of (1.1) and let $\tilde{f}$ be a continuous function such that $f(x)=\tilde{f}(x)$ almost everywhere. Denote by Z the set of points at which the functions f and $\tilde{f}$ may not be equal. We shall prove that $Z=\emptyset$.

To prove this it is enough to show that for every fixed point $x^* \in R^n$ there exists a sequence of points $t_\nu \in \Omega$, $t_\nu \to \alpha$ such that

$$f(\Phi_i(x^*, t_\nu)) = \tilde{f}(\Phi_i(x^*, t_\nu)) \quad \text{for} \quad i = 2, \ldots, k. \tag{3.1}$$

In fact, (1.1) then implies the equation

$$a_1(x^*, t_\nu)f(x^*) + \sum_{i=2}^{k} a_i(x^*, t_\nu)\tilde{f}(\Phi_i(x^*, t_\nu)) = F(x^*, f(\lambda_1(x^*)), \ldots, f(\lambda_s(x^*))) + b(x^*, t_\nu)$$

and, in view of the continuity of the functions a_i, b and φ_i assumed in 1°, one obtains

$$a_1(x^*, \alpha)f(x^*) + \sum_{i=2}^{k} a_i(x^*, \alpha)\tilde{f}(\varphi_i(x^*, \alpha)) = F(x^*, f(\lambda_1(x^*)), \ldots, f(\lambda_p(x^*))) + b(x^*, \alpha).$$

Since on the basis of 2°, $\Phi_i(x^*, \alpha)=x^*$ and $\sum_{i=2}^{k} a_i(x^*, \alpha) \neq 0$, it follows that

$$\tilde{f}(x^*) = \frac{F(x^*, f(\lambda_1(x^*)), \ldots, f(\lambda_s(x^*))) + b(x^*, \alpha) - a_1(x^*, \alpha)f(x^*)}{\sum_{i=2}^{k} a_i(x^*, \alpha)}.$$

On the other hand, setting $x=x^*$, $t=\alpha$ into (1.1) yields

$$f(x^*) = \frac{F(x^*, f(\lambda_1(x^*)), \ldots, f(\lambda_s(x^*))) + b(x^*, \alpha) - a_1(x^*, \alpha)f(x^*)}{\sum_{i=2}^{k} a_i(x^*, \alpha)}$$

and therefore $f(x^*)=\tilde{f}(x^*)$.

Now, we shall prove that there exists a sequence $\{t_\nu\}$ satisfying condition (3.1).

Suppose that $x^* \in Z$. Let V^* be an open neighbourhood of x^* determined by 3° and 4° and consider a decreasing sequence of open sets

$$\ldots \subset V_{\nu+1} \subset V_\nu \subset \ldots \subset V_1 \subset V^*$$

such that

$$\bigcap_{\nu=1}^{\infty} V_\nu = \{x^*\}.$$

Denote $U_\nu = V_\nu \setminus Z$ and $Z_\nu = V_\nu \cap Z$ for $\nu = 1, 2, \ldots$. It is easy to see that

$$\ldots \subset U_{\nu+1} \subset U_\nu \subset \ldots \subset U_1$$

and that $\mu_n(Z_\nu) = \mu_n(V_\nu \cap Z) = 0$; consequently

$$\mu_n(U_\nu) = \mu_n(V_\nu \setminus Z) = \mu_n(V_\nu \setminus Z_\nu) = \mu_n(V_\nu \setminus V_\nu \cap Z_\nu) = \mu_n(V_\nu) > 0$$

for $\nu = 1, 2, \ldots$.

We are going to prove that the sets

$$Q_\nu = \bigcap_{i=2}^{k} U_{\nu, \Phi_i, x^*},$$

where

$$U_{\nu, \Phi_i, x^*} = \{t\colon \Phi_i(x^*, t) \in U_\nu\},$$

are not empty and that $\alpha \in \overline{Q}_\nu$ for $\nu = 1, 2, \ldots$.

In fact,

$$Q_\nu = \bigcap_{i=2}^{k} U_{\nu, \Phi_i, x^*} = \bigcap_{i=2}^{k} (V_{\nu, \Phi_i, x^*} \setminus Z_{\nu, \Phi_i, x^*}) = \bigcap_{i=2}^{k} V_{\nu, \Phi_i, x^*} \setminus \left(\bigcap_{i=2}^{k} Z_{\nu, \Phi_i, x^*} \right), \tag{3.2}$$

where

$$V_{\nu, \Phi_i, x^*} = \{t\colon \Phi_i(x^*, t) \in V_\nu\}, \quad Z_{\nu, \Phi_i, x^*} = \{t\colon \Phi_i(x^*, t) \in Z_\nu\}.$$

Since $Z_\nu \subset V_\nu \subset V^*$ and since $\mu_n(Z_\nu) = 0$ for $\nu = 1, 2, \ldots$, condition 4° implies that $\mu_r(Z_{\nu, \Phi_i, x^*}) = 0$ for $i = 2, \ldots, k$; $\nu = 1, 2, \ldots$. Hence

$$\mu_r \left(\bigcup_{i=2}^{k} Z_{\nu, \Phi_i, x^*} \right) = 0 \quad \text{for} \quad \nu = 1, 2, \ldots. \tag{3.3}$$

The sets V_{ν, Φ_i, x^*} are open since the sets V_ν are open and since the functions φ_i are continuous. Therefore also the sets $\bigcap_{i=2}^{k} V_{\nu, \Phi_i, x^*}$ ($\nu = 1, 2, \ldots$) are open. They are non-void since, in view of 3°, $\alpha \in \bigcap_{i=2}^{k} V_{\nu, \Phi_i, x^*}$. This implies

$$\mu_r \left(\bigcap_{i=2}^{k} V_{\nu, \Phi_i, x^*} \right) > 0 \quad \text{for} \quad \nu = 1, 2, \ldots. \tag{3.4}$$

By (3.2), (3.3) and (3.4) we have $\mu_r(Q_\nu) > 0$ for $\nu = 1, 2, \ldots$ and therefore the sets Q_ν cannot be empty.

Since the sets $\bigcap_{i=2}^{k} V_{\nu, \Phi_i, x^*}$ are open in R^r and since $\mu_r \left(\bigcup_{i=2}^{k} Z_{\nu, \Phi_i, x^*} \right) = 0$, it follows that

$$\bigcap_{i=2}^{k} V_{\nu, \Phi_i, x^*} \subset \overline{\bigcap_{i=2}^{k} V_{\nu, \Phi_i, x^*} \setminus \left(\bigcup_{i=2}^{k} Z_{\nu, \Phi_i, x^*} \right)} = \overline{Q}_\nu.$$

But $\alpha \in \bigcap_{i=2}^{k} V_{\nu, \Phi_i, x^*}$ and therefore $\alpha \in \overline{Q}_\nu$.

The fact that $\alpha \in \overline{Q}_\nu$ for $\nu = 1, 2, \ldots$ and the fact that $\mu_r(Q_\nu) > 0$ for $\nu = 1, 2, \ldots$ imply the existence of the sequence $\{t_\nu\}$ such that $t_\nu \in Q_\nu$ and $t_\nu \to \alpha$. This finishes the proof since at the points of the sets Q_ν condition (3.1) is satisfied.

THEOREM 3.1. *Suppose that* $r \geqq n$, *and*

1° $a_i \in C^\infty$ *and* $b \in C^\infty$ *with respect to* $x \in R^n$ *for every fixed t from an open set* $\Omega \subset R^r$, $i=1, \ldots, k$,

2° $a_i \in C^m$ *and* $b \in C^m$ *in* $R^n \times \Omega$, $i=1, \ldots, k$,

3° *the mappings* $x \to y = \Phi_i(x, t)$ *are diffeomorphism for every fixed* $t \in \Omega$, $i=1, \ldots, k$,

4° Φ_i *and* $\Phi_i^{-1}: (y, t) \mapsto x$ $\big(x$ *determined by the condition* $y = \Phi_i(x, t)\big)$ *are functions of the class* C^m *in* $R^n \times \Omega$, $i=1, \ldots, k$,

5° *F is continuous in* R^{n+s},

6° λ_j *are continuous in* R^n, $j=1, \ldots, s$,

7° *there exists an* $\alpha \in \Omega$ *such that* $\varphi_1(x, t) \equiv x$ *in* R^n *for t from an open neighbourhood of* α, $\Phi_i(x, \alpha) \equiv x$ *in* R^n *for* $i=2, \ldots, k$ *and* $\sum_{i=2}^{k} a_i(x, \alpha) \neq 0$ *in* R^n,

8° *for any fixed point* $x \in R^n$ *there exists a neighbourhood* U_x *of this point such that for every measurable set E the following condition is satisfied:*

$$\{E \subset U_x, \mu_n(E) = 0\} \Rightarrow \{\mu_r(E_{\Phi_i, x}) = 0 \quad \textit{for} \quad i = 2, \ldots, k\},$$

where

$$E_{\Phi_i, x} = \{t \colon \Phi_i(x, t) \in E\},$$

9° *there exists an r-tuple q* $(|q|) \leqq m)$ *such that the equation*

$$D_t^q \left(\sum_{i=1}^{k} a_i(x, t) f(\Phi_i(x, t))_{t=\alpha} = 0, \right.$$

with the left-hand side obtained by formal differentiation of the left-hand side of (1.1) *at* $t=\alpha$, *is of constant strength, hypoelliptic at an* x_0.

Then every locally integrable solution f of equation (1.1) *has to be a function of class* C^∞.

PROOF. Assumptions 1°—7° and 9° guarantee that the assumptions of Theorem 1 of [6] are satisfied. Therefore every locally integrable solution f of (1.1) is equal to a function of the class C^∞ almost everywhere.

Assumptions 2°, 4°, 7° and 8° allow one to apply Lemma 3.1 and this finishes the proof.

REMARK 3.1. A similar theorem was proved in [4] in a special case when $\Phi_i(x, t) = x + \varphi_i(t)$ $(i=1, \ldots, k)$, $F(x, u_1, \ldots, u_s) \equiv 0$ and if a differential equation obtained by formal differentiation of (1.1) with respect to t is either elliptic or hypoelliptic at $t=\alpha$. (Hypoellipticity refers only to the case $a_i(x, t) = a_i(t)$, $i=1, \ldots, k$.)

4. Simple regularity and existence criteria

The results of this chapter exemplify the applications of Theorems 2.1 and 3.1. The regularity criteria presented here concern a special class of equations (1.1), namely,

$$\sum_{i=1}^{k} a_i(x, t) f(xA_i(t) + \varphi_i(t)) = F(x, f(x\Lambda_1), \ldots, f(x\Lambda_s)) + b(x, t) \tag{4.1}$$

where $x \in R^n$, $t \in \Omega \subset R^r$ (or $t \in \Delta \subset R$), $A_i(t) = [A_{i\varrho\sigma}(t)]$ $(i=1, \ldots, k)$ and $\Lambda_j = [\Lambda_{j\varrho\sigma}]$ $(j=1, \ldots, s)$ are n-dimensional matrices.

THEOREM 4.1. *Suppose that*

1° *a_i and b are C^∞ functions with respect to $x \in R^n$ for every fixed t from an open set $\Omega \subset R^r$, $i=1, \ldots, k$,*

2° *$a_i \in C^m$ and $b \in C^m$ in $R^n \times \Omega$, $i=1, \ldots, k$,*

3° *the functions $A_{i\varrho\sigma}: \Omega \to R$ $(i=1, \ldots, k;\ \varrho, \sigma=1, \ldots, n)$ are functions of the class C^m in Ω and the matrices $A_i(t)=[A_{i\varrho\sigma}(t)]$ are invertible for every fixed $t \in \Omega$, $i=1, \ldots, k$,*

4° *the functions $\varphi_i: \Omega \to R^n$ $(\varphi_i: t \mapsto (\varphi_{i1}(t), \ldots, \varphi_{in}(t)))$ are functions of the class C^m in Ω,*

5° *the matrices $\Lambda_j=[\Lambda_{j\varrho\sigma}]$ $(j=1, \ldots, s;\ \varrho, \sigma=1, \ldots, n)$ are invertible,*

6° *there exists an $\alpha \in \Omega$ such that $A_i(\alpha)=I$, where I is the n-dimensional unit matrix, and $\varphi_i(\alpha)=0$ for $i=1, \ldots, k$,*

7° *F is continuous in R^{n+s},*

8° *there exists an r-tuple q $(|q| \leqq m)$ such that the equation*

$$D_t^q\left(\sum_{i=1}^k a_i(x, t) f\big(xA_i(t)+\varphi_i(t)\big)\right)\Bigg|_{t=\alpha} = 0, \tag{4.2}$$

where the left-hand side is obtained by formal differentiation of the left-hand side of (4.1) *and by setting $t=\alpha$, is of constant strength, hypoelliptic at an x_0.*

Then every locally integrable solution f of (4.1) *is equal to a solution of the class C^∞ almost everywhere.*

PROOF. For every fixed $t \in \Omega$ the mappings

$$x \mapsto y = xA_i(t)+\varphi_i(t); \quad i = 1, \ldots, k$$

and

$$y \mapsto x = yA_i^{-1}(t)-\varphi_i(t)A_i^{-1}(t); \quad i = 1, \ldots, k,$$

where $A_i^{-1}(t)$ is the inverse of $A_i(t)$, are linear mappings of R^n onto R^n. Therefore

$$\mu\big(\Phi_i^{-1}(E, t)\big) = |\det [A_{i\varrho\sigma}^{-1}(t)]|\, \mu(E) \tag{4.3}$$

with

$$\Phi_i^{-1}(E, t) = \{x\colon x = [y-\varphi_i(t)]A_i^{-1}(t),\ y \in E\}$$

for every measurable set $E \subset R^n$ and $i=1, \ldots, k$. Equalities (4.3) guarantee that assumption 8° of Theorem 2.1 is satisfied.

Similarly, the mappings

$$\lambda_j\colon x \mapsto y = x\Lambda_j; \quad j = 1, \ldots, s$$

and

$$\lambda_j^{-1}\colon y \mapsto x = y\Lambda_j^{-1}; \quad j = 1, \ldots, s$$

are linear mappings of R^n onto R^n and

$$\mu\big(\lambda_j^{-1}(E)\big) = |\det [\Lambda_{j\varrho\sigma}^{-1}]|\, \mu(E) \tag{4.4}$$

for every measurable set $E \subset R^n$ and $j=1, \ldots, s$. Equalities (4.4) imply that assumption 9° of Theorem 2.1 is satisfied.

Assumptions 1°, 2°, 5° and 10° are the same as in Theorem 2.1.

Assumptions 3° and 4° guarantee that the mappings

$$\Phi_i\colon\ (x,t)\mapsto y=\Phi_i(x,t)=xA_i(t)+\varphi_i(t)$$

and

$$\Phi_i^{-1}\colon\ (y,t)\mapsto x=\Phi_i^{-1}(y,t)=yA_i^{-1}(t)-\varphi_i(t)A_i^{-1}(t),$$

where $A_i^{-1}(t)$ is the inverse of $A_i(t)$, are functions of the class C^m in R^n i.e. assumption 4° of Theorem 2.1 is satisfied.

Assumption 3° of Theorem 2.1 is automatically satisfied since the mappings

$$x\mapsto y=\Phi_i(x,t)=xA_i(t)+\varphi_i(t);\quad i=1,\ldots,k$$

are diffeomorphisms in R^n for every fixed $t\in\Omega$ as linear mappings of R^n onto R^n.

Similarly, assumption 6° of Theorem 2.1 is satisfied since $\lambda_j\colon x\to xA_j$ $(j=1,\ldots,s)$ are continuous in R^n as linear mappings.

This completes the proof.

THEOREM 4.2. *Suppose that assumptions* 1°—7° *of Theorem* 4.1 *are satisfied with* $\Omega=\Lambda$, *where* Δ *is an open interval in* R, *and, moreover,*

8° $a_i(x,\alpha)>0$ *for all* $x\in R^n$, $i=1,\ldots,k$,

9° *for every fixed* $x\in R^n$ *the vectors* $v_i(x)=xA'_{i_i}(\alpha)+\varphi'_i(\alpha)$ $(i=1,\ldots,k)$ *span* R^n.

Then every locally integrable solution f *of* (4.1) *is equal to a solution of the class* C^∞ *almost everywhere.*

PROOF. It suffices to prove that 8° and 9° imply that assumption 8° of Theorem 4.1 is satisfied. It will be shown that the equation

$$\frac{\partial^2}{\partial t^2}\left(\sum_{i=1}^k a_i(x,t)\,f\big(xA_i(t)+\varphi_i(t)\big)\right)\Bigg|_{t=\alpha}=0 \tag{4.5}$$

is elliptic.

The principal part of (4.5) has the form

$$\sum_{i=1}^k a_i(x,\alpha)\big[\big(xA_i'(\alpha)+\varphi_i'\alpha)\big)\cdot\operatorname{grad}_x\big]^2 f(x).$$

The corresponding quadratic form

$$P_2(x,\xi)=\sum_{i=1}^k a_i(x,\alpha)\big[\big(xA_i'(\alpha)+\varphi_i'(\alpha)\big)\cdot\xi\big]^2=\sum_{i=1}^k a_i(x,\alpha)[v_i(x)\cdot\xi]^2\geqq 0$$

since, by 8°, $a_i(x,\alpha)>0$ for all $x\in R^n$, $i=1,\ldots,k$. Of necessity $P_2(x,\xi)>0$ for $\xi\neq 0$ since $P_2(x,\xi)=0$ for a certain $x\in R$ and for a certain $\xi\neq 0$, $\xi\in R^n$, together with 8°, imply

$$v_i(x)\cdot\xi=0\quad\text{for}\quad i=1,\ldots,k$$

which means that ξ is orthogonal to every vector $v_i(x)$. Such a conclusion contradicts assumption 8° and this finishes the proof of the ellipticity of (4.5).

Applying Lemma 2.1 and repeating the considerations of the first part of the proof of Theorem 4.1 one obtains the following simple criterion for the existence of the continuous solutions f of (4.1):

THEOREM 4.3. *Suppose that*

$1°$ *a_i and b are continuous with respect to $x \in R^n$ for every fixed t from an open set $\Omega \subset R^r$, $i=1, \ldots, k$,*

$2°$ *F is continuous in R^{n+s},*

$3°$ *the matrices $A_i(t)=[A_{i\varrho\sigma}(t)]$ $(i=1, \ldots, k)$ and $\Lambda_j=[\Lambda_{j\varrho\sigma}]$ $(j=1, \ldots, s)$ are invertible.*

Then every continuous function $\tilde{f}$, which is equal to a solution f of (4.1) almost everywhere, is also a solution of this equation.

THEOREM 4.4. *Suppose that assumptions $1°$—$7°$ of Theorem 4.1 are satisfied with $m=4$, $\Omega=\Delta \subset R$, $a_i(x, t) \equiv \mu_i > 0$ and $A_i(t) \equiv I$ for $i=1, \ldots, k$.*

If there exists an $\alpha \in \Delta$ such that

$1°$ *$\varphi_i(\alpha)=0$ for $i=1, \ldots, k$,*

$2°$ *$\overset{\alpha}{\varphi}{}'_i$ and $\overset{\alpha}{\varphi}{}''_i$ $i=(1, \ldots, k;\ \overset{\alpha}{\varphi}{}_i^{(1)}=\varphi_i^{(1)}(\alpha))$ span the space R^n,*

$3°$ *$\sum_{i=1}^{k} \mu_i(\overset{\alpha}{\varphi}{}'_i \cdot \xi)^2(\overset{\alpha}{\varphi}{}''_i \cdot \xi) \equiv 0$ for all $\xi \in R^n$,*

then every locally integrable solution f of (4.1) is equal to a solution of the class C^∞ almost everywhere and every continuous solution of this equation is a C^∞ function.

PROOF. In view of Theorem 4.1 it suffices to show that a differential equation of the form (4.2) is of constant strength, hypoelliptic at an x_0. Since, in this case, $a_i(x, t) \equiv \mu_i$ for $i=1, \ldots, k$, the equations of the form (4.2) are linear partial differential equations with constant coefficients. Therefore it suffices to show that one of them is hypoelliptic. Differentiating (4.1) formally with respect to t and setting next $t=\alpha$ one obtains

$$\frac{d^4}{dt^4}\left(\sum_{i=1}^{k} \mu_i f\big(x+\varphi_i(t)\big)\right)\bigg|_{t=\alpha} = \sum_{i=1}^{k} \mu_i [(\overset{\alpha}{\varphi}{}'_i \cdot \mathrm{grad})^4 + 3(\overset{\alpha}{\varphi}{}''_i \cdot \mathrm{grad})^2 +$$

$$+4(\overset{\alpha}{\varphi}{}'_i \cdot \mathrm{grad})(\overset{\alpha}{\varphi}_i \cdot \mathrm{grad}) + (\overset{\alpha}{\varphi}{}_i^{(4)} \cdot \mathrm{grad})] f(x) = 0.$$

The proof of the hypoellipticity of the last equation can be found in [5, Theorem 7] which deals with the equations

$$\sum_{i=1}^{k} \mu_i f\big(x+\varphi_i(t)\big) = f(x). \tag{4.6}$$

THEOREM 4.5. *Suppose that $\varphi_i \in C^2$ in an open interval $\Delta \subset R$ and that there exists an $\alpha \in \Delta$ such that $\varphi_i(\alpha)=0$ and $\varphi'_i(\alpha) \neq 0$ for $i=1, \ldots, k$. If equation (4.6), where $\sum_{i=1}^{k} \mu_i=1$ and $\mu_i>0$ for $i=1, \ldots, k$, is satisfied by all the polynomials*

$$p\colon x \to \sum_{\substack{j,l=1\\ j\neq 1}}^{k} p_{j1} x_j x_1,$$

then every locally integrable solution $f\colon R^n \to R$ of (4.6) is equal to a solution of the class C^∞ almost everywhere and every continuous solution of this equation is a C^∞ function.

The proof can be done by showing that

$$\frac{d^2}{dt^2}\left(\sum_{i=1}^{k}\mu_i f(x+\varphi_i(t))\right)_{t=\alpha}=0 \tag{4.7}$$

is an elliptic differential equation and by applying Theorem 4.1. The detailed proof of the ellipticity of (4.7) can be found in [7, Theorem 6.3].

REMARK 4.1. Stronger results concerning (4.6) with $\varphi_i(t)=a_i t$ $(i=1, \dots, k)$ were obtained in [1] and [2].

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MAXIMAL ELEMENTS AND COMPLETIONS OF NOETHER LATTICES

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§ 1. Introduction

This paper deals with completions of Noetherian lattice modules, completions of Noether lattices, and maximal elements in the completion of a Noether lattice. The basic concepts are given in § 2. Let Λ be a Noether lattice, let a be an element of Λ such that the a-adic pseudometric on Λ is a metric, and let Λ^* be the a-adic completion of Λ. In § 3 we determine various relations concerning the maximal elements of Λ and the maximal elements of Λ^*. We show (Corollary 3.2) that $a \leqq \leqq \wedge \{m \in \Lambda | m \text{ is maximal}\}$ and establish a one-to-one correspondence between maximal elements of Λ and those of Λ^* (Theorem 3.5). In §§4—6 we consider the particular case where Ω is a Noetherian Λ-module, $[a, I]$ is finite dimensional, and Ω^* is the a-adic completion of Ω. We establish in §4 several relations between the various metrics on Ω^*. In particular we show that the $a\Lambda$-*adic pseudometric and the a-adic pseudometric on Ω^* are metrics and they are equal to the extended metric on Ω^*. Completions of intervals of Ω are investigated in §5 and in §6 we prove that Ω can be embedded in Ω^* as a sublattice.

§ 2. Notation and preliminary remarks

A multiplicative lattice is a complete lattice on which there is defined a commutative, associative, join distributive multiplication such that the unit element of the lattice is a multiplicative identity. Let Λ be a multiplicative lattice and let Ω be a a complete lattice. We will denote elements of Λ by $a, b, c, \ldots,$ with the exception that the null element and unit element of Λ will be denote by 0 and I, respectively. Elements of Ω will be denoted by $A, B, C, \ldots,$ with the exception that the null element and unit element of Ω will be denoted by 0 and M, respectively. Recall that Ω is said to be an Λ-module if there is a multiplication between elements of Λ and Ω, denoted by aA for a in Λ and A in Ω, which satisfies:

$$\text{(i) } (ab)A = a(bA); \text{ (ii) } \big(\bigvee_\alpha a_\alpha\big)\big(\bigvee_\beta B_\beta\big) = \bigvee_{\alpha,\beta} a_\alpha B_\beta; \text{ (iii) } IA = A; \text{ and (iv) } 0A = 0;$$

for all a, a_α, b in Λ and for all A, B_β in Ω.

Let Ω be an Λ-module. For a, b in Λ and for A, B in Ω, (i) $a:b$ denotes the largest c in Λ such that $cb \leqq a$; (ii) $A:B$ denotes the largest c in Λ such that $cB \leqq A$. An element A in Ω is said to be meet principal in case $(b \wedge (B:A))A = bA \wedge B$, for all b in Λ and for all B in Ω; A is said to be join principal in case $b \vee (B:A) = (bA \vee B):A$, for all b in Λ and for all B in Ω; and, A is said to be principal in case A is both meet and join principal. If each element of Ω is the join (finite or infinite) of principal elements, Ω is called principally generated. Ω is said to be Noetherian if Ω satisfies the ascending chain condition, is modular, and is principally generated. If Λ is a Noetherian Λ-module, Λ is called a Noether lattice. For other general properties and definitions concerning Noetherian lattice modules, the reader is referred to the references. Examples of Noether lattices and Noetherian lattice modules can be found in [1], [2], [3] and [8].

Let Λ be a multiplicative lattice and let Ω be a Noetherian Λ-module. For a in Λ and A in Ω let $T(a,A)$ be the collection of all sequences $\langle B_i\rangle$, $i=1, 2, \ldots$, of elements of Ω satisfying

$$a^i A \geqq B_i \geqq B_{i+1} \geqq aB_i, \tag{2.1}$$

for all integers $i\geqq 1$. For $\langle C_i\rangle$ and $\langle B_i\rangle$ in $T(a, A)$, define

$$\langle C_i\rangle \leqq \langle B_i\rangle \quad \text{if and only if} \quad C_i \leqq B_i, \quad \text{for all} \quad i \geqq 1 \tag{2.2}$$

$$\langle C_i\rangle \vee \langle B_i\rangle = \langle C_i \vee B_i\rangle \tag{2.3}$$

$$\langle C_i\rangle \wedge \langle B_i\rangle = \langle C_i \wedge B_i\rangle. \tag{2.4}$$

It is easily verified that $T(a, A)$ is a complete, modular lattice under the relation $\leqq$ with the resulting join and meet being given by (2.3) and (2.4) respectively. This lattice will be denoted by $R(a, A)$.

We will require the following result. The reader is referred to [5, Theorem 3.2, p. 190] for a proof.

THEOREM 2.1. *Let Λ be a multiplicative lattice, let Ω be a Noetherian Λ-module, let a be an element of Ω, and let $\langle B_i\rangle$, $i=1, 2, \ldots$, be an element $R(a, A)$. Then there exists a natural number n such that $B_{m+i}=a^i B_m$, for all integers $m\geqq n$ and for all integers $i\geqq 0$.*

Let Λ be a multiplicative lattice, let Ω be an Λ-module, and let a be an element of Λ. For each A and B in Ω, define

$$d_a(A, B) = 2^{-\delta(A,B)} \tag{2.5}$$

where $\delta(A, B)=\sup\{n|A\vee a^n M=B\vee a^n M\}$. It is easily seen that d_a is a pseudometric and it is called the a-adic pseudometric on Ω (cf. [6], § 3). A necessary and sufficient condition for d_a to be a metric on Ω is given by the following result (see [6], Theorem 3.10, p. 352, for a proof).

THEOREM 2.2. *Let Ω be an Λ-module and let a be an element of Λ. Then the a-adic pseudometric on Ω is a metric if and onyl if $C=\bigwedge_n (C\vee a^n M)$, for each C in Ω.*

If Λ is a Noether lattice, Ω is a Noetherian Λ-module, and a is an element of Λ such that the a-adic pseudometric is a metric on Ω, then the a-adic completion, Ω^*, of Ω can be constructed and Ω^* is itself an Λ-module. We refer the reader to [6, §1—§6] for details concerning this construction. Of course Λ is always a Λ-module, so what we have said also applies to Λ. If the a-adic pseudometric on Λ is also a metric, then Ω^* is in a natural manner an Λ^*-module.

Recall ([6], Definition 4.7, p. 354), that a Cauchy sequence $\langle B_i \rangle$ of elements of Ω is regular in case

$$B_i \vee a^i M = B_{i+1} \vee a^i M, \tag{2.6}$$

for all integers $i \geqq 1$, and completely regular in case

$$B_i = B_{i+1} \vee a^i M \tag{2.7}$$

for all integers $i \geqq 1$. The elements of Ω^* are equivalence classes of Cauchy sequences and each B in Ω^* has a uniquely determined completely regular representative (cf. [6], Theorem 4.14, p. 356). If B is an element of Ω^* with completely regular representative $\langle B_i \rangle$, the contraction of B to Λ, denoted $B \cap \Lambda$, is the element $\bigwedge_n B_n$ of Λ and is uniquely determined by B (see [6], §7). For A in Ω, $A\Omega^*$ is the element of Ω^* determined by the constant Cauchy sequence $\langle B_i \rangle$, where $B_i = A$, $i = 1, 2, \ldots$, and is called the extension of A to Ω^* (see [6], §5). The following facts are known concerning these two concepts (see [6], Proposition 7.2 and Proposition 7.4, p. 364):

(2.8) For each A in Λ, $A = A\Lambda^* \cap \Lambda$.

(2.9) For each B in Λ^*, $(B \cap \Lambda)\Lambda^* \leqq B$.

REMARK 2.3. *Let Λ be a Noether lattice, let Ω be a Noetherian Λ-module, let a be an element of Λ such that the a-adic pseudometrics on Ω and Λ are metrics, let Ω^* and Λ^* be the a-adic completions of Ω and Λ respectively (thus Ω^* is an Λ-module and Λ^*-module), let b be an alement of Λ and let B be an element of Ω^*. Then $b\Lambda^* \cdot B = bB$.*

PROOF. Suppose $\langle B_i \rangle$, $i = 1, 2, \ldots$, is the completely regular representative of B. Then, by [6, Definition 6.5] the sequence $\langle bB_i \rangle$ is a representative of bB. Hence, the sequence $\langle bB_i \vee a^i M \rangle$, $i = 1, 2, \ldots$, is a representative of bB ([6], Corollary 4.6, p. 354). Since $\langle b \vee a^i \rangle$, $i = 1, 2, \ldots$, is the completely regular representative of $b\Lambda^*$ ([6], Remark 5.2, p. 356), it follows that $\langle (b \vee a^i) B_i \rangle$, $i = 1, 2, \ldots$, is a representative of $b\Lambda^* \cdot B$ ([6], Proposition 5.14, p. 360), and so $\langle (b \vee a^i) B_i \vee a^i M \rangle$, $i = 1, 2, \ldots$, is also a representative of $b\Lambda^* \cdot B$. Since

$$(b \vee a^i) B_i \vee a^i M = bB_i \vee a^i M,$$

for all integers $i \geqq 1$, the proof is complete.

§ 3. Maximal elements

In this section we establish several relations between maximal elements in a lattice and maximal elements in its completion which will be required in the sequel. For a multiplicative lattice Λ, we define Max (Λ) to be the set of all maximal elements of Λ.

LEMMA 3.1. *Let a be an element of a multiplicative lattice Λ such that the a-adic pseudometric on Λ is a metric and let c be an element of Λ. If $c \vee a = I$, then $c = I$.*

PROOF. If $c \vee a = I$, then $c \vee a^n = I$, for each integer $n \geqq 1$, and thus by Theorem 2.2

$$c = \bigwedge_n (c \vee a^n) = I$$

as claimed.

COROLLARY 3.2. *Let a be an element of a multiplicative lattice Λ such that the a-adic pseudometric on Λ is a metric. Then* $a \leqq \wedge \operatorname{Max}(\Lambda)$.

PROOF. If p is an element of $\operatorname{Max}(\Lambda)$, then $p \neq I$ so that $p \vee a \neq I$ by Lemma 3.1 and thus $a \leqq p$. It follows that $a \leqq \wedge \operatorname{Max}(\Lambda)$.

COROLLARY 3.3. *Let Λ be a Noether lattice, let Ω be a Noetherian Λ-module and let a be an element of Λ. If the a-adic pseudometric on Λ is a metric, then the a-adic pseudometric on Ω is a metric.*

PROOF. This follows from Corollary 3.2 above and Corollary 3.4, p. 193, of [5].

LEMMA 3.4. *Let Λ be a Noether lattice, let a be an element of Λ such that the a-adic pseudometric on Λ is a metric, let* Λ^* *be the a-adic completion of Λ, let c be an element of* Λ^*, *and let* $\langle c_i \rangle$ *be the completely regular representative of c. If* $c \neq I\Lambda^*$, *then* $c_i < I$, *for each integer* $i \geqq 1$.

PROOF. Suppose there exists an integer i such that $c_i = I$. Then $c_1 = I$ since $\langle c_i \rangle$ is decreasing. Since $\langle c_i \rangle$ is the completely regular representative of c, it follows that

$$I = c_1 = c_n \vee a,$$

for each integer $n \geqq 1$, and thus by Lemma 3.1, $c_n = I$, for all integers $n \geqq 1$. Hence $c = I\Lambda^*$ which completes the proof.

THEOREM 3.5. *Let Λ be a Noether lattice, let a be an element of Λ such that the a-adic pseudometric on Λ is a metric, let* Λ^* *be the a-adic completion of Λ. Then*

(3.1) *for each* m *in* $\operatorname{Max}(\Lambda)$, $m\Lambda^*$ *is in* $\operatorname{Max}(\Lambda^*)$;

(3.2) *for each* b *in* $\operatorname{Max}(\Lambda^*)$, *there exists an* m *in* $\operatorname{Max}(\Lambda)$ *such that* $b = m\Lambda^*$.

PROOF. Let m be an element of $\operatorname{Max}(\Lambda)$. Since $m \neq I$, we have $m\Lambda^* \neq I\Lambda^*$. Let b be an element of Λ^* such that $m\Lambda^* \leqq b < I\Lambda^*$ and let $\langle b_i \rangle$ be the completely regular representative of b. Since the completely regular representative of $m\Lambda^*$ is $\langle m \vee a^i \rangle$, $i = 1, 2, \ldots$, we have, for each integer $i \geqq 1$,

$$m = m \vee a^i \leqq b_i < I$$

by Corollary 3.2, Lemma 3.4, and [6, Remark 5.2. p. 356, and Proposition 5.9, p. 358]. It follows that $m = b_i$, for each integer $i \geqq 1$. Thus $m\Lambda^* = b$ and so $m\Lambda^*$ is a maximal element of Λ^* which establishes (3.1).

Let b be an element of $\operatorname{Max}(\Lambda^*)$ and let $\langle b_i \rangle$ be the completely regular representative of b. Since $b \neq I\Lambda^*$, we have $b_1 < I$ (Lemma 3.4). Hence, there is an m in $\operatorname{Max}(\Lambda)$ such that $b_1 \leqq m$. Consequently, since $\langle b_i \rangle$ is decreasing, $b_i \leqq m$, for all integers $i \geqq 1$, and thus

$$b \leqq m\Lambda^* < I$$

so that $b = m\Lambda^*$, which completes the proof.

COROLLARY 3.6. *Let Λ be a Noether lattice, let a be an element of Λ such that the a-adic pseudometric on Λ is a metric, and let Λ^* be the a-adic completion of Λ. Then*

$$(\wedge \operatorname{Max}(\Lambda))\Lambda^* \leqq \wedge \operatorname{Max}(\Lambda^*) \tag{3.3}$$

$$(\wedge \operatorname{Max}(\Lambda^*)) \cap \Lambda = \wedge \operatorname{Max}(\Lambda). \tag{3.4}$$

PROOF. Let b be an element of $\operatorname{Max}(\Lambda^*)$. Then there exists an m in $\operatorname{Max}(\Lambda)$ such that $m\Lambda^*=b$. Since $\wedge \operatorname{Max}(\Lambda) \leqq m$, we have

$$(\wedge \operatorname{Max}(\Lambda))\Lambda^* \leqq m\Lambda^* = b$$

by [6, Corollary 5.11, p. 359] and (3.3) follows.

Let m be an element of $\operatorname{Max}(\Lambda)$. Then

$$(\wedge \operatorname{Max}(\Lambda))\,\Lambda^* \leqq \wedge \operatorname{Max}(\Lambda^*) \leqq m\Lambda^*$$

so that

$$\wedge \operatorname{Max}(\Lambda) = (\wedge \operatorname{Max}(\Lambda))\Lambda^* \cap \Lambda \leqq (\wedge \operatorname{Max}(\Lambda^*)) \cap \Lambda \leqq m\Lambda^* \cap \Lambda = m$$

by [6, Proposition 7.2, p. 364]. Thus

$$\wedge \operatorname{Max}(\Lambda) \leqq (\wedge \operatorname{Max}(\Lambda^*)) \cap \Lambda \leqq \wedge \operatorname{Max}(\Lambda)$$

which establishes (3.4) and completes the proof.

LEMMA 3.7. *Let Λ be a Noether lattice, let a be an element of Λ such that the a-adic pseudometric on Λ is a metric and $[a, I]$ is finite dimensional, let Λ^* be the a-adic completion of Λ, and let c be an element of Λ^*. If there is an integer $n \geqq 0$ such that $a^n \Lambda^* \leqq c$, then $(c \cap \Lambda)\Lambda^* = c$.*

PROOF. Let the sequence $\langle c_i \rangle$ be the completely regular representative of c and assume such an n exists. Then

$$a^n \leqq a^n \vee a^i \leqq c_i, \quad \text{for all integers } i,$$

by [6, Remark 5.2, p. 356, and Proposition 5.9, p. 358]. Since $[a, I]$ is finite dimensional, so is $[a^n, I]$ ([7], Corollary 2.5). Hence, since $\langle c_i \rangle$ is decreasing, there exists an integer $k \geqq 0$ such that

$$c_i = c_k = \bigwedge_j c_j = (c \cap \Lambda), \quad \text{for all integers } i \geqq k.$$

It follows that $c = (c \cap \Lambda)\Lambda^*$ as claimed.

COROLLARY 3.8. *Let Λ be a Noether lattice, let a be an element of Λ such that the a-adic pseudometric on Λ is a metric and $[a, I]$ is finite dimensional, and let Λ^* be the a-adic completion of Λ. Then*

$$(\wedge \operatorname{Max}(\Lambda))\Lambda^* = \wedge \operatorname{Max}(\Lambda^*), \tag{3.5}$$

PROOF. It follows from (3.3), (3.4), and Lemma 3.7 that

$$(\wedge \operatorname{Max}(\Lambda))\Lambda^* = ((\wedge \operatorname{Max}(\Lambda^*)) \cap \Lambda)\Lambda^* = \wedge \operatorname{Max}(\Lambda^*)$$

which completes the proof.

§ 4. Metrics

If A and B are elements of a lattice K with $A \leqq B$, the interval $\{D \text{ in } K | A \leqq D \leqq B\}$ is a sublattice of K which we denote by $[A, B]$.

REMARK 4.1. *Let Λ be a multiplicative lattice, let Ω be an Λ-module, let A and B be elements of Ω with $A \leqq B$, and let b be an element of Λ such that $bD \leqq A$, for all D in $[A, B]$. For each c, d in $[b, I]$ let $c \circ d = cd \vee b$. For each C in $[A, B]$ and e in $[b, I]$, let $e \circ C = eC \vee A$. With these definitions $[b, I]$ is a multiplicative lattice and $[A, B]$ is an $[b, I]$-module (cf. [6], Remarks 2.8 and 2.9, p. 350). If Ω is a Noetherian Λ-module, then $[A, B]$ is a Noetherian $[b, I]$-module.*

Throughout the remainder of this section Λ is a Noether lattice, Ω is a Noetherian Λ-module, a is an element of Λ such that $[a, I]$ is finite dimensional and the a-adic pseudometric on Ω is a metric, and Ω^* is the a-adic completion of Ω.

In this section we establish some relations concerning various metrics on Ω^*. We will need the following result. The reader is referred to [7, Theorem 3.1] for a proof.

THEOREM 4.2. *If $\langle B_i \rangle$, $i = 1, 2, \ldots,$ is a sequence of elements of Ω such that, given a positive integer n, $B_{i+1} \leqq B_i \vee a^n M$, for all integers $i \geqq n$, then $\langle B_i \rangle$ is Cauchy.*

THEOREM 4.3. *The following two statements are equivalent.*

(4.1) *Ω is a complete Λ-module with respect to the a-adic metric on Ω.*

(4.2) *For any decreasing sequence $\langle E_i \rangle$, $i = 1, 2, \ldots,$ of elements of Ω and for any positive integer n, $E_i \leqq (\bigwedge_j E_j) \vee a^n M$, for all sufficiently large integers i.*

PROOF. Suppose Ω is a complete Λ-module with respect to the a-adic metric and $\langle E_i \rangle$ is a decreasing sequence of elements of Ω. Since $\langle E_i \rangle$ is decreasing, $\langle E_i \rangle$ is Cauchy (Theorem 4.2), and hence there is a C in Ω such that $E_i \to C$ as $i \to +\infty$. It follows that, for each positive integer n, $C \vee a^n M = E_i \vee a^n M$, for sufficiently large integers i. Hence, for each integer $h \geqq 1$,

$$C = \bigwedge_n (C \vee a^n M) \leqq \bigwedge_n (E_h \vee a^n M) = E_h$$

by Theorem 2.2, so that $C \leqq \bigwedge_h E_h$. Hence, given any positive integer n,

$$E_i \leqq E_i \vee a^n M = C \vee a^n M \leqq \big(\bigwedge_h E_h\big) \vee a^n M,$$

for all sufficiently large integers i, and so (4.1) implies (4.2).

Assume (4.2) and let $\langle C_i \rangle$ be a Cauchy sequence of elements of Ω. We may assume without loss of generality that $\langle C_i \rangle$ is a completely regular Cauchy sequence (cf. [6], Lemma 4.11 and 4.12, p. 355). We shall show $C_i \to \bigwedge_j C_j$ as $i \to \infty$. Assume

$\delta > 0$. Let m be the least positive integer n such that $2^n < \delta$. Since $\langle C_i \rangle$ is completely regular, $\langle C_i \rangle$ is decreasing, so that by (4.2),

$$C_i \leqq (\bigwedge_j C_j) \vee a^m M,$$

for large integers i, and hence

$$C_i \vee a^m M = (\bigwedge_j C_j) \vee a^m M,$$

for large integers i. It follows that $d_a(C_i, \bigwedge_j C_j) \leqq 2^{-m} < \delta$, for all large integers i, and so $C_i \to \bigwedge_j C_j$ as $i \to +\infty$ in the a-adic metric which completes the proof.

For the remainder of this section we also assume that the a-adic pseudometric on Λ is a metric and Λ^* is the a-adic completion of Λ. The reader is referred to Theorem 4.8 of [7] for a proof of the following Theorem.

THEOREM 4.4. *Λ^* is a Noether lattice and Ω^* is a Noetherian Λ^*-module.*

Since $a \in \Lambda$, $a\Lambda^*$ is in Λ^*, and since Ω^* is an Λ^*-module, the $a\Lambda^*$-adic pseudometric on Ω^* is defined. This pseudometric is in fact a metric.

LEMMA 4.5. *The $a\Lambda^*$-adic pseudometric on Ω^* is a metric.*

PROOF. From Corollary 3.2 and (3.3) of Corollary 3.6 we have

$$a\Lambda^* \leqq (\wedge \operatorname{Max}(\Lambda))\Lambda^* \leqq \wedge \operatorname{Max}(\Lambda^*).$$

Hence, the $a\Lambda^*$-adic pseudometric on Ω^* is a metric by Theorem 4.4 and [5, Corollary 3. 4, p. 193] as claimed.

Since the a-adic pseudometric on Ω^* (considered as an Λ-module) and the $a\Lambda^*$-adic pseudometric on Ω^* (considered as an Λ^*-module) always are the same (cf. [6], Remark 7.10) we obtain the following result as a corollary.

COROLLARY 4.6. *The a-adic pseudometric on Ω^* is a metric.*

COROLLARY 4.7. *Let A and B be elements of Ω such that $A \leqq B$. Then, the map $D \to D\Lambda^*$ of the Noetherian Λ-module $[A, B]$ with the a-adic metric to the Λ-module $[A\Omega^*, B\Omega^*]$ with the a-adic metric is an isometry.*

PROOF. For each D and E in $[A, B]$, it is easily verified that, for each integer $n \geqq 0$,

$$D\Omega^* \vee a^n \circ (B\Omega^*) = E\Omega^* \vee a^n \circ (B\Omega^*)$$

if and only if

$$D \vee a^n \circ B = E \vee a^n \circ B,$$

which shows the map is an isometry.

LEMMA 4.8. *Let A be an element of Ω^* and let $\langle A_i \rangle$, $i = 1, 2, \ldots$, be a representative of A. Then $A_i \Omega^* \to A$ as $i \to +\infty$ in the a-adic metric on Ω^*.*

PROOF. Let δ be a positive real number. Choose k to be the least positive integer h such that $2^{-h} < \delta$. Since $\langle A_i \rangle$ is Cauchy, there is an integer $n > 0$ such that

$$A_i \vee a^k M = A_j \vee a^k M,$$

for all integers $i, j \geqq n$. The sequence $\langle A_j \vee a^k M \rangle$, $j=1, 2, \ldots$, is a representative of $A \vee (a^k M)\Omega^*$ and, for each i, the constant sequence $\langle A_i \vee a^k M \rangle, j=1, 2, \ldots$, is a representative of $A_i \Omega^* \vee (a^k M)\Omega^*$ ([6], (5.8), p. 357). Consequently, for each integer $i \geqq n$, we have

$$A_i\Omega^* \vee a^k(M\Omega^*) = A_i\Omega^* \vee (a^k M)\Omega^* = A \vee (a^k M)\Omega^* = A \vee a^k(M\Omega^*).$$

It follows that $d_m(A_i\Omega^*, A) \leqq 2^{-k} < \delta$, for all integers $i \geqq n$. Hence $A_i\Omega^* \to A$ as $i \to +\infty$ is the a-adic metric.

Recall (see [6], §5) that the extended metric, d_a^*, on Ω^* is defined as follows. Let A and B be elements of Ω^* with representatives $\langle A_i \rangle$ and $\langle B_i \rangle$, respectively, then

$$d_a^*(A, B) = \lim_{i \to \infty} d_a(A_i, B_i)$$

The metrics d_a and d_a^* are related as follows.

LEMMA 4.9. *Let A and B be elements of Ω^*. Then $d_a(A, B) = d_a(A, B)$.*

PROOF. Let $\langle A_i \rangle$ and $\langle B_i \rangle$ be representatives of A and B, respectively. From Lemma 4.8 we have that $A_i\Omega^* \to A$ and $B_i\Omega^* \to B$ in the a-adic metric. Hence,

$$d_a^*(A, B) = \lim_{i \to \infty} d_a(A_i, B_i) = \lim_{i \to \infty} d_a(A_i\Omega^*, B_i\Omega^*) = d_a(A, B),$$

by Corollary 4.7, which completes the proof.

COROLLARY 4.10. *The metrics d_a^*, d_a, and $d_{a\Omega^*}$ are equal on Ω^*.*

§ 5. Interval completions

In this section we examine completions of intervals. Throughout this section Λ is a Noether lattice, Ω is a Noetherian Λ-module, a is an element of Λ such that $[a, I]$ is finite dimensional and the a-adic pseudometrics on Λ and Ω are metrics, Λ^* is the a-adic completion of Λ, and Ω^* is the a-adic completion of Ω. See Theorem 4.1 of [7] for a proof of the following theorem which we will need shortly.

THEOREM 5.1. *Let B and C be elements of Ω^*. Let the sequences $\langle B_i \rangle$ and $\langle C_i \rangle$ be the completely regular representatives of B and C, respectively. Then the sequence $\langle B_i \wedge C_i \rangle$ is Cauchy and is a representative of $B \wedge C$.*

We will require the following two technical lemmas.

LEMMA 5.2. *For each A and B in Ω with $A \leqq B$, the set $[A, B]\Omega^*$ is dense in the Λ-module $[A\Omega^*, B\Omega^*]$ with the a-adic metric.*

PROOF. Let C be an element of $[A\Omega^*, B\Omega^*]$ and let $\langle C_i \rangle$, $i=1, 2, \ldots$, be the completely regular representative of C (considered as an element of Ω with the a-adic metric). Since

$$A \leqq A \vee a^i M \leqq C_i \leqq B \vee a^i M$$

by [6, Remark 5.2, p. 356 and Proposition 5.9, p. 358] it follows that $A \leqq C_i \wedge B \leqq B$, and so $C_i \wedge B$ is in $[A, B]$,for each i. Furthermore, $\langle C_i \wedge B \rangle$ is Cauchy (Theorem 4.2) and is a representative of C by Theorem 5.1 and [6, Corollary 4.6, p. 354]. Thus $(C_i \wedge B)\Omega^* \to C$ as $i \to \infty$ in the a-adic metric (Lemma 4.8).

Let $\varepsilon > 0$ be a real number and set $k = \inf\{n | 2^{-n} < \varepsilon\}$. The sequence $\langle (a\Lambda^*)^i (M\Omega^*) \wedge B\Omega^* \rangle$, $i = 1, 2, \dots$, satisfies the condition of Theorem 2.1 (see Theorem 4.4), so there exists an $n > 0$ such that

$$(5.1) \qquad a^{n+k}(M\Omega^*) \wedge B\Omega^* = a^k \big(a^n (M\Omega^*) \wedge B\Omega^*\big)$$

by Remark 2.3. Also, since $(C_i \wedge B)\Omega^* \to C$ there exists an $N > 0$ such that

$$(5.2) \qquad (C_i \wedge B)\Omega^* \vee a^{n+k}(M\Omega^*) = C \vee a^{n+k}(M\Omega^*)$$

for all integers $i \geqq N$. Hence, for each $i \geqq N$,

$$C \vee a^k \circ (B\Omega^*) =$$
$$= C \vee a^k [a^n (M\Omega^*) \wedge B\Omega^*] \vee a^k (B\Omega^*) = C \vee [a^{k+n}(M\Omega^*) \wedge B\Omega^*] \vee a^k (B\Omega^*) =$$
$$= \big(B\Omega^* \wedge [C \vee a^{k+n}(M\Omega^*)]\big) \vee a^k (B\Omega^*) = \big(B\Omega^* \wedge [(C_i \wedge B)\Omega^* \vee a^{k+n}(M\Omega^*)]\big) \vee a^k (B\Omega^*) =$$
$$= (C_i \wedge B)\Omega^* \vee [a^{k+n}(M\Omega^*) \wedge B\Omega^*] \vee a^k (B\Omega^*) =$$
$$= (C_i \wedge B)\Omega^* \vee a^k [a^n (M\Omega^*) \wedge B\Omega^*] \vee a^k (B\Omega^*) = (C_i \wedge B)\Omega^* \vee a^k \circ (B\Omega^*)$$

by (5.1) and (5.2). Hence, for $i \geqq N$, $d_a\big((C_i \wedge B)\Omega^*, C\big) \leqq 2^{-k} < \varepsilon$ which completes the proof.

LEMMA 5.3. *For each A and B in Ω with $A \leqq B$, the Λ-module $[A\Omega^*, B\Omega^*]$ is complete with respect to the a-adic metric.*

PROOF. $[a\Lambda^*, I\Lambda^*]$ is finite dimensional since $[a, I]$ is finite dimensional and $[a, I] \cong [a\Lambda^*, I\Lambda^*]$ under $b \to b\Lambda^*$ (Lemma 3.7). We will use Theorem 4.3 to show $[A\Omega^*, B\Omega^*]$ is complete. Thus, let $\langle C_i \rangle$ be a decreasing sequence in $[A\Omega^*, B\Omega^*]$ and let j be a positive integer. Consider the sequence $\langle (a\Lambda^*)^i (M\Omega^*) \wedge (B\Omega^*) \rangle$, $i = 1, 2, \dots$. By Theorem 2.1 there exists an $n > 0$ so that

$$(5.3) \qquad a^{j+n}(M\Omega^*) \wedge B\Omega^* = a^j \big(a^n (M\Omega^*) \wedge B\Omega^*\big)$$

by Remark 2.3. Since Ω^* is a complete Λ^*-module with the $a\Lambda^*$-adic metric and since $\langle C_i \rangle$ is decreasing, we have an integer $N > 0$ for which

$$(5.4) \qquad C_i \leqq \big(\bigwedge_m C_m\big) \vee a^{j+n}(M\Omega^*), \quad \text{for all} \quad i \geqq N$$

by Remark 2.3. It follows that

$$C_i = B\Omega^* \wedge C_i \leqq B\Omega^* \wedge \big(\big(\bigwedge_m C_m\big) \vee a^{j+n}(M\Omega^*)\big) =$$
$$= \big(\bigwedge_m C_m\big) \vee \big(a^{j+n}(M\Omega^*) \wedge B\Omega^*\big) = \big(\bigwedge_m C_m\big) \vee a^j \big(a^n (M\Omega^*) \wedge B\Omega^*\big) \leqq$$
$$\leqq \big(\bigwedge_m C_m\big) \vee a^j (B\Omega^*) = \big(\bigwedge_m C_m\big) \vee a^j \circ (B\Omega^*),$$

for each integer $i \geqq N$, by (5.3) and (5.4). Hence the Λ-module $[A\Omega^*, B\Omega^*]$ is complete in the a-adic metric by Theorem 4.3 and the proof is complete.

The uniqueness of the completion (up to isomorphism) together with Corollary 4.7, Lemmas 5.2 and 5.3 yields the following

THEOREM 5.4 *For each A and B in Ω with $A \leqq B$, the Λ-module $[A\Omega^*, B\Omega^*]$ with the a-adic metric is the a-adic completion of the Noetherian Λ-module $[A, B]$.*

§ 6. Embeddings

Throughout this section Λ is Noether lattice, Ω is a Noetherian Λ-module, a is an element of Λ such that the a-adic pseudometric on Ω is a metric and $[a, I]$ is finite dimensional. We will show that the map $A \to A\Omega^*$ is a meet-homomorphism and hence Ω can be embedded in its completion. We begin with a technical lemma.

LEMMA 6.1. *Let A be an element of Ω and let B be a principal element of Ω. Then for each positive integer k, there exists an integer $n \geqq k$ such that*

$$(6.1) \qquad (A \vee a^n M):(B \vee a^n M) \leqq (A:B) \vee a^k$$

$$(6.2) \qquad (A \vee a^n M) \wedge (B \vee a^n M) \leqq (A \wedge B) \vee a^k M.$$

PROOF. Let $k>0$ be an integer. Since the sequence $\langle((A \vee a^i M):B) \vee a^k\rangle$, $i=1, 2, \ldots$, is decreasing and $[a^k, I]$ is finite dimensional ([7], Corollary 2.5), there is an integer $n \geqq k$ such that

$$((A \vee a^n M) \wedge B) \vee a^k B = \left[((A \vee a^n M):B) \vee a^k\right] B =$$
$$= \bigwedge_i \left[((A \vee a^i M):B) \vee a^k\right] B \leqq \bigwedge_i [(A \vee a^k B) \vee a^i M] = A \vee a^k B$$

by Theorem 2.2. It follows that

$$(A \vee a^n M):(B \vee a^n M) \leqq ((A \vee a^n M):B) \vee a^k \leqq (A \vee a^k B):B = (A:B) \vee a^k$$

and

$$(A \vee a^n M) \wedge (B \vee a^n M) = a^n M \vee ((A \vee a^n M) \wedge B) \leqq a^n M \vee ((A \vee a^k B) \wedge B) \leqq (A \wedge B) \vee a^k M$$

which establishes (6.1) and (6.2).

THEOREM 6.2 *For each A and B in Ω,*

$$(A \wedge B)\Omega^* = A\Omega^* \wedge B\Omega^*.$$

PROOF. There exists, since Ω is Noetherian, principal elements $B_1, B_2, \ldots, B_n$ in Ω such that $B = B_1 \vee \ldots \vee B_n \vee (A \wedge B)$. Our proof is by induction on n. Suppose then $B = B_1 \vee (A \wedge B)$. Since $B_1 \vee (A \wedge B)$ is a principal element in the Noetherian Λ-module $[A \wedge B, M]$, it follows from (6.2) that, for each integer $k \geqq 1$, there is an integer $n \geqq k$ such that

$$(A \vee a^n \circ M) \wedge (B_1 \vee (A \wedge B) \vee a^n \circ M) \leqq A \wedge (B_1 \vee (A \wedge B)) \vee a^k \circ M$$

which implies

$$(A \vee a^n M) \wedge (B_1 \vee a^n M) \leqq (A \wedge B) \vee a^k M$$

and so $A\Omega^* \wedge B\Omega^* \leqq (A \wedge B)\Omega^*$ by Theorem 5.1 and [6, Proposition 5.10, p. 359].

Assume now that $B=B_1\vee\ldots\vee B_{k+1}\vee(A\wedge B)$ and put $P=A\vee B_{k+1}$. Then $A\wedge B=A\wedge(P\wedge B)$, $P\wedge B=B_{k+1}\vee(A\wedge(P\wedge B))$, and $B=B_1\vee\ldots\vee B_k\vee(P\wedge B)$. From the induction hypothesis we have $(P\wedge B)\Omega^*=P\Omega^*\wedge B\Omega^*$. The case $n=1$ applied to A and $P\wedge B$ yields

$$(A\wedge B)\Omega^* = (A\wedge(P\wedge B))\Omega^* = A\Omega^*\wedge(P\wedge B)\Omega^*.$$

Thus

$$(A\wedge B)\Omega^* = A\Omega^*\wedge(P\Omega^*\wedge B\Omega^*) = A\Omega^*\wedge B\Omega^*$$

which completes the induction and the proof.

The map $A\to A\Omega^*$ is a join-homomorphism by definition [6, Definition 5.4, p. 357] and is known to be injective [6, Proposition 5.3, p. 356]. These two facts combined with the theorem above immediately yield the following corollary.

COROLLARY 6.3 *The map* $A\to A\Omega^*$ *is a lattice isomorphism from* Ω *onto* $\Omega\Omega^*$.

If we further assume that the a-adic pseudometric on Λ is a metric we obtain a similar result for residuation.

COROLLARY 6.4. *For each A and B in* Ω, $(A:B)\Lambda^*=A\Omega^*:B\Omega^*$.

PROOF. If B is principal, then $(A:B)\Lambda^*\geqq A\Omega^*:B\Omega^*$ by (6.1) and the fact that the sequence $\langle(A\vee a^iM):(B\vee a^iM)\rangle$, $i=1,2,\ldots$, is a representative of $A\Omega^*:B\Omega^*$ [7, Theorem 4.4]. Since $(A:B)B\leqq A$, we have $(A:B)\Lambda^*\cdot B\Omega^*=((A:B)B)\Omega^*\leqq A\Omega^*$ and consequently $(A:B)\Lambda^*\leqq A\Omega^*:B\Omega^*$.

In the general case there are principal elements $B_1,\ldots,B_k$ in Ω such that $B=B_1\vee\ldots\vee B_k$. Using Theorem 6.2 (and a simple induction argument) we obtain

$$(A:B)\Lambda^* = (A:(B_1\vee\ldots\vee B_k))\Lambda^* = (A:B_1)\Lambda^*\wedge\ldots\wedge(A:B_k)\Lambda^* =$$
$$= A\Omega^*:(B_1\Omega^*\vee\ldots\vee B_k\Omega^*) = A\Omega^*:B\Omega^*$$

which completes the proof.

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TAUBERIAN THEOREMS FOR THE FAMILY $F(a, q)$ OF SUMMABILITY METHODS

By

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The family $F(a, q)$ of summability methods was introduced by MEIR [2]. The summability matrix (c_{pk}) belongs to $F(a, q)$ if it satisfies the following conditions: p is a discrete or continuous parameter; a is a positive constant; q is a positive increasing function which tends to ∞ as p tends to ∞; for every fixed δ, $1/2<\delta<2/3$

(i) $$c_{pk} = \left(\frac{a}{\pi q}\right)^{1/2} e^{\frac{-a(k-q)^2}{q}}\left\{1+O\left(\frac{|k-q|+1}{q}\right)+O\left(\frac{|k-q|^3}{q^2}\right)\right\}$$

as $p\to\infty$ uniformly in k for $|k-q|\leqq q^\delta$;

(ii) $$\sum_{|k-q|>q^\delta} kc_{pk} = O(e^{-q^\eta}) \quad (p\to\infty),$$

where η is some positive number independent of p; and

(iii) $$c_{pk} \geqq 0.$$

The aim of this note is to prove the following

THEOREM 1. *If* $s_n\to s(F(a, q))$ *and*

$$\liminf\,(s_n-s_m) \geqq 0$$

when

$$m\to\infty,\quad n>m,\quad \frac{n-m}{\sqrt{m}}\to 0,$$

then $s_n\to s$.

We need the following lemmas.

LEMMMA 1 (cf. [3], Theorem 9). *Suppose that*

$$c_n(x)\geqq 0,\quad c_n(x)\to 0 \quad as \quad x\to\infty,\quad \sum c_n(x)=1,\quad T(x)=\sum c_n(x)\,s_n,$$

$\varphi(u)$ *is positive and differentiable with positive bounded derivative,* $\varphi(u)\to\infty$ *as* $u\to\infty$ *and*

$$\Phi(u)=\int_1^u \frac{dt}{\varphi(t)},$$

$$\sum_{n=0}^{M} c_n(x)\to 0 \quad if \quad M\to\infty,\quad x\to\infty,\quad \Phi(x)-\Phi(M)\to\infty;$$

$$\sum_{n=N}^{\infty} c_n(x)[\Phi(n)-\Phi(N)]\to 0 \quad if \quad N\to\infty,\quad x\to\infty,\quad \Phi(N)-\Phi(x)\to\infty.$$

Suppose also that if $s(t)=s_n$ *for* $n\leqq t<n+1$, *then*

$$\liminf\,[s(t)-s(u)]\geqq 0 \quad \textit{when} \quad u\to\infty, \quad t>u, \quad t-u=o[\varphi(u)].$$

Then s_n *is bounded.*

LEMMA 2. *If* $s_n=o(n^{1/2})$, *then summability* $F(a, q)$ *is equivalent to summability* (e, a).

PROOF. We prove the implication $F(a,q)\to(e,a)$. Suppose that $s=0$. If $s_n\to\to o(F(a,q))$, then

$$\sum c_{pn}s_n=o(1) \quad (q\to\infty),$$

that is,

$$\sum_{|n-q|\leqq q^\delta}\left(\frac{a}{\pi q}\right)^{1/2} e^{\frac{-a(n-q)^2}{q}}\left\{1+O\left(\frac{|n-q|+1}{q}\right)+O\left(\frac{|n-q|^3}{q^2}\right)\right\}s_n+\sum_{|n-q|>q^\delta}c_{pn}s_n=o(1) \quad (q\to\infty),$$

or

$$\sum\nolimits_1+\sum\nolimits_2=o(1) \quad (q\to\infty),$$

say. Statement (ii) together with the hypothesis $s_n=o(n^{1/2})$ imply that $\sum_2=o(1)$ as $q\to\infty$. The second and third terms of $\sum_1$ are equal to

$$O\Big(\int_0^{aq^{2\delta-1}} e^{-z}\,dz\Big) \quad\text{and}\quad O\Big(\int_0^{aq^{2\delta-1}} ze^{-z}\,dz\Big),$$

respectively, and both tend to zero as $q\to\infty$. Thus the first term in $\sum_1=o(1)$ as $q\to\infty$, that is,

$$\sum_{-q^\delta}^{+q^\delta} e^{-ah^2/q}s_{q+h}=o(q^{1/2}) \quad (q\to\infty).$$

This is similar to equation (9.10.9) in Theorem 151 of [1]. Following the argument on the same lines as in [1], we finally have

$$\left(\frac{a}{\pi q}\right)^{1/2}\int e^{-at^2/q}s(q+t)\,dt\to 0 \quad (q\to\infty),$$

that is, $s_n\to 0$ (e,a). Our arguments are plainly reversible.

PROOF OF THEOREM 1. Write $\varphi(u)=2\sqrt{u}$ so that $\Phi(u)=\sqrt{u}-1$. If $0\leqq\sqrt{q}-\sqrt{M}==\mu$, then $M\leqq q-\mu\sqrt{q}$; and if $0\leqq\sqrt{N}-\sqrt{q}=v$, then $N\geqq q+v\sqrt{q}$. We have

$$\sum_0^M c_{pn}\to 0 \quad (q\to\infty)$$

since by (ii)

$$\sum_{n<q-q^\delta}c_{pn}=o(1) \quad (q\to\infty)$$

and in

$$\sum_{q-q^\delta\leqq n\leqq q-\mu\sqrt{q}}c_{pn}=\sum_{q-q^\delta\leqq n\leqq q-\mu\sqrt{q}}\left(\frac{a}{\pi q}\right)^{1/2}e^{\frac{-a(n-q)^2}{q}}\left\{1+O\left(\frac{|n-q|+1}{q}\right)+O\left(\frac{|n-q|^3}{q^2}\right)\right\}$$

the last two terms are $O(q^{-1/2})$, while the first is $O\Big(\int\limits_{\mu}^{\infty} e^{-aw^2}\,dw\Big)$, which is small for large μ. Further

$$\sum_{n=N}^{\infty} c_{pn}[\Phi(n)-\Phi(N)] = \sum_{n=N}^{\infty} (\sqrt{n}-\sqrt{N})c_{pn} \leqq q^{-1/2}\sum_{n=N}^{\infty}(n-N)c_{pn} \leqq$$

$$\leqq q^{-1/2}\sum_{n\geqq q+v\sqrt{q}}(n-q)c_{pn}.$$

It can be seen as above that this sum also tends to zero as $v, q\to\infty$. Thus the conditions of Lemma 1 are satisfied and s_n is bounded.

Suppose that $s_n\to 0\ \big(F(a, q)\big)$. By Lemma 2

$$q^{-1/2}\int\limits_0^{\infty} e^{\frac{-a(t-q)^2}{q}} s(t)\,dt \to 0 \quad (q\to\infty)$$

or

$$\int\limits_0^{\infty} \exp\left\{-\frac{a(u^2-y^2)^2}{y^2}\right\}\frac{u}{y}\,s(u^2)\,du \to 0 \quad (y\to\infty).$$

To complete the proof we use Wiener's theorems with $g(t)=e^{-4at^2}$ (cf. [1], Theorem 241).

The following result can be obtained using Theorem B of [4].

THEOREM 2. *The conditions*

$$F(q) = \sum_{n=0}^{\infty} c_{pn}s_n = O(1) \quad (q\to\infty)$$

$$\min_{n\leqq n'\leqq n+\delta\sqrt{n}} \{s_{n'}-s_n\} = o_L(1)\delta \quad (n\to\infty)$$

together imply

$$\operatorname*{osc}_{n\to\infty} s_n = \operatorname*{osc}_{q\to\infty} F(q).$$

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THE MAXIMAL k-FREE DIVISOR OF m WHICH IS PRIME TO n. I

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1. Introduction. Let k be a fixed integer $\geqq 2$. A positive integer m is called k-free, if it is not divisible by the k-th power of any prime. Let Q_k denote the set of all k-free integers. It is clear that the integer $1 \in Q_k$ and if $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ is the canonical representation of $m > 1$, then $m \in Q_k$ if and only if $\alpha_i < k$ for $i = 1, 2, \dots, r$. Let $q_k(m)$ be the characteristic function of the set Q_k, that is, $q_k(m) = 1$ or 0 according as $m \in Q_k$ or $m \notin Q_k$. A divisor $d > 0$ of the positive integer m is called a k-free divisor of m if $d \in Q_k$. Let n be a fixed positive integer and let $\gamma_k(m; n)$ denote the maximal k-free divisor of m which is prime to n. In other words, $\gamma_k(m; n)$ denotes the greatest among the k-free divisors d of m such that $(d, n) = 1$. Let $\gamma_k(m)$ and $\delta_k(m)$ denote the maximal k-free divisor of m and the maximal odd k-free divisor of m, respectively. It is clear that $\gamma_k(m; 1) = \gamma_k(m)$ and $\gamma_k(m; 2) = \delta_k(m)$. Further, if p is a given prime, then $\gamma_k(m; p)$ will be the maximal k-free divisor of m which is not divisible by p. In particular, when $k = 2$, $\gamma_2(m; n) = \gamma(m; n)$, the maximal square-free divisor of m which is prime to n. Also, $\gamma_2(m) = \gamma(m; 1) = \gamma(m)$, the maximal square-free divisor (or the core) of m and $\delta_2(m) = \gamma(m; 2) = \delta(m)$, the maximal odd square-free divisor of m.

In this paper we establish asymptotic formulae for $\sum_{\substack{m \leqq x \\ (m,n)=1}} q_k(m)\varphi(m)$ and $\sum_{m \leqq x} \gamma_k(m; n)$ with uniform O-estimates for the error terms (see § 4), where $\varphi(n)$ is the Euler totient function. Also, we improve the O-estimates of the error terms on the assumption of the Riemann hypothesis. As particular cases of these asymptotic formulae, we deduce asymptotic formulae for $\sum_{m \leqq x} \gamma_k(m)$, $\sum_{m \leqq x} \delta_k(m)$ and $\sum_{m \leqq x} \gamma_k(m; p)$, where p is a given prime. We also discuss the case $k = 2$ and make some remarks about the earlier work (if any) on the orders of the error terms at the end of each of these asymptotic formulae. In fact, the orders of the error terms obtained in this paper are improved ones over those existing in the literature.

In § 2 we prepare the necessary background and in §3 we prove some lemmas which are needed in establishing the asymptotic formulae of § 4.

2. Preliminaries. Let $\mu(n)$ denote the Möbius function. Then it is known (cf. [8], the proof of Lemma 2) that

$$q_k(m) = \sum_{d^k \delta = m} \mu(d). \tag{2.1}$$

Let $\psi(n)$ denote the Dedekind ψ-function (cf. [5], p. 123 or [3]) and $J_k(n)$ denote the Jordan totient function (cf. [5], p. 147 or [1]). The functions $\varphi(n)$, $\psi(n)$ and $J_k(n)$ have the following arithmetical forms:

$$\varphi(n) = \sum_{d\delta=n} \mu(d)\delta = n \prod_{p|n} \left(1-\frac{1}{p}\right) \tag{2.2}$$

$$\psi(n) = \sum_{d\delta=n} \mu^2(d)\delta = n \prod_{p|n} \left(1+\frac{1}{p}\right) \tag{2.3}$$

$$J_k(n) = \sum_{d\delta=n} \mu(d)\delta^k = n^k \prod_{p|n} \left(1-\frac{1}{p^k}\right). \tag{2.4}$$

REMARK 2.1. It is clear that $\varphi(n)\leqq n$, $\psi(n)\geqq n$ and $\frac{1}{J_k(n)}=O\left(\frac{1}{n^k}\right)$, since

$$J_k(n) > n^k \prod_{p} \left(1-\frac{1}{p^k}\right) = \frac{n^k}{\zeta(k)}$$

(cf. [6], Theorem 280), where $\zeta(k)$ is the Riemann zeta function defined by $\zeta(k)= \sum_{m=1}^{\infty} \frac{1}{m^k}$.

Let $H_k(n)$ be the arithmetical function defined by $H_k(1)=1$ and

$$H_k(n) = n^k \prod_{p|n} \left(1-\frac{1}{p^{k-1}(p+1)}\right) \quad \text{for} \quad n>1. \tag{2.5}$$

Let $\sigma_t^*(n)$ denote the sum of the t-th powers of the square-free divisors of n and let $\theta(n)$ denote the number of the square-free divisors of n. It is clear that $\sigma_0^*(n)= =\theta(n)=2^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of $n>1$, $\omega(1)=0$.

Throughout the paper x denotes a real variable and ε denotes a preassigned positive real number. All the O-estimates that appear in this paper are independent of x and n, but might depend on ε. We describe this situation by mentioning the word "uniformly" at the end of each asymptotic formula.

We need the following:

LEMMA 2.1 (cf. [11], Lemma 3.6, $s=k$). *For $x\geqq 3$ and $n\geqq 1$,*

$$\sum_{\substack{m\leqq x\\(m,n)=1}} \frac{\mu(m)}{m^k} = \frac{n^k}{\zeta(k)J_k(n)} + O\left(\frac{\sigma_{-1+\varepsilon}^*(n)\delta(x)}{x^{k-1}}\right) \tag{2.6}$$

uniformly, where $\delta(x)$ is defined by

$$\delta(x) = \begin{cases} \exp\{-A\log^{3/5} x\,(\log\log x)^{-1/5}\} & \textit{for} \quad x\geqq 3, \\ 1 \quad \textit{for} \quad 0<x<3; \end{cases} \tag{2.7}$$

A being a positive constant.

REMARK 2.2. If h is a positive constant, then it is clear that $\delta(x)\log^h x= =O(\exp\{-A'\log^{3/5}x(\log\log x)^{-1/5}\})$, where A' is a positive constant $(0<A'<A)$. So, we may replace $\delta(x)\log^h x$ appearing in any O-term by $\delta(x)$.

LEMMA 2.2 (cf. [11], Lemma 5.3, $s=k$). *If the Riemann hypothesis is true, then for* $x\geqq 3$ *and* $n\geqq 1$,

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)}{m^k}=\frac{n^k}{\zeta(k)J_k(n)}+O\left(\sigma^*_{-1/2+\varepsilon}(n)x^{1/2-k}\omega(x)\right) \tag{2.8}$$

uniformly, where $\omega(x)$ *is defined by*

$$\omega(x)=\begin{cases}\exp\{A\log x(\log\log x)^{-1}\} & \text{for}\quad x\geqq 3,\\ 1\quad \text{for}\quad 0<x<3;\end{cases} \tag{2.9}$$

A being a positive constant.

REMARK 2.3. Sometimes it is convenient to replace $\sigma^*_{-1+\varepsilon}(n)$ or $\sigma^*_{-1/2+\varepsilon}(n)$ appearing in any O-term by $\theta(n)$ or $\tau(n)$. Clearly, $\sigma^*_{-1+\varepsilon}(n)\leqq\sigma^*_{-1/2+\varepsilon}(n)\leqq\theta(n)\leqq\tau(n)$, where $\tau(n)$ is the number of all divisors of n.

LEMMA 2.3. *For* $n\geqq 1$,

$$\sum_{\substack{m=1\\(m,n)=1}}^{\infty}\frac{\mu^2(m)}{\psi(m)J_k(m)}=\frac{\zeta(k)J_k(n)\alpha_k}{H_k(n)}, \tag{2.10}$$

where α_k *is the constant given by*

$$\alpha_k=\prod_p\left(1-\frac{1}{p^{k-1}(p+1)}\right), \tag{2.11}$$

the product being extended over all primes p.

PROOF. The series is absolutely convergent, since

$$\frac{\mu^2(m)}{\psi(m)J_k(m)}=O\left(\frac{1}{m^{k+1}}\right)$$

by Remark 2.1, and the general term of the series is a multiplicative function of m, so that the series can be expanded into an infinite product of Euler type (cf. [5], Theorem 286). Hence we have by (2.3) and (2.4),

$$\sum_{\substack{m=1\\(m,n)=1}}^{\infty}\frac{\mu^2(m)}{\psi(m)J_k(m)}=\prod_{\substack{p\\p\nmid n}}\left\{1+\frac{1}{(p+1)p^k(1-p^{-k})}\right\}=$$

$$=\prod_{\substack{p\\p\nmid n}}\left\{1-\frac{1}{p^k}+\frac{1}{p^k(p+1)}\right\}\cdot\prod_{\substack{p\\p\nmid n}}\left(1-\frac{1}{p^k}\right)^{-1}=\prod_{\substack{p\\p\nmid n}}\left\{1-\frac{p}{p^k(p+1)}\right\}\cdot\prod_{\substack{p\\p\nmid n}}\left(1-\frac{1}{p^k}\right)^{-1}=$$

$$=\frac{\prod_p\left(1-\frac{1}{p^{k-1}(p+1)}\right)}{\prod_{p|n}\left(1-\frac{1}{p^{k-1}(p+1)}\right)}\cdot\frac{\prod_{p|n}\left(1-\frac{1}{p^k}\right)}{\prod_p\left(1-\frac{1}{p^k}\right)}=\frac{\alpha_k\cdot J_k(n)\zeta(k)}{H_k(n)},$$

by (2.11) and (2.5). Hence (2.10) follows.

LEMMA 2.4. *For* $x\geqq 2$ *and* $n\geqq 1$,

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu^2(m)}{\psi(m)J_k(m)}=\frac{\zeta(k)J_k(n)\alpha_k}{H_k(n)}+O\left(\frac{1}{x^k}\right)\tag{2.12}$$

uniformly.

PROOF. We have by Remark 2.1,

$$\sum_{\substack{m>x\\(m,n)=1}}\frac{\mu^2(m)}{\psi(m)J_k(m)}=O\left(\sum_{\substack{m>x\\(m,n)=1}}\frac{1}{m^{k+1}}\right)=O\left(\sum_{m>x}\frac{1}{m^{k+1}}\right)=O\left(\frac{1}{x^k}\right).$$

Since

$$\sum_{\substack{m\leqq x\\(m,n)=1}}^{\infty}\frac{\mu^2(m)}{\psi(m)J_k(m)}=\sum_{\substack{m=1\\(m,n)=1}}^{\infty}\frac{\mu^2(m)}{\psi(m)J_k(m)}-\sum_{\substack{m>x\\(m,n)=1}}\frac{\mu^2(m)}{\psi(m)J_k(m)},$$

Lemma 2.4 follows by Lemma 2.3.

LEMMA 2.5. *For* $x\geqq 3$ *and* $n\geqq 1$,

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)}{\psi(m)m^{k-1}}=\frac{\alpha_k n^k}{H_k(n)}+O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(x)}{x^{k-1}}\right)\tag{2.13}$$

uniformly, where $\delta(x)$ *is given by* (2.7).

PROOF. Since

$$\frac{1}{\psi(m)}=\frac{1}{m}\sum_{d\delta=m}\frac{\mu(d)}{\psi(d)}$$

(cf. [9], Lemma 3), we have

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)}{\psi(m)m^{k-1}}=\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)}{m^k}\sum_{d\delta=m}\frac{\mu(d)}{\psi(d)}=\sum_{\substack{d\delta\leqq x\\(d\delta,n)=1}}\frac{\mu(d\delta)\mu(d)}{\psi(d)d^k\delta^k}=$$

$$=\sum_{\substack{d\delta\leqq x\\(d\delta,n)=1\\(d,\delta)=1}}\frac{\mu^2(d)\mu(\delta)}{\psi(d)d^k\delta^k}=\sum_{\substack{d\leqq x\\(d,n)=1}}\frac{\mu^2(d)}{\psi(d)d^k}\sum_{\substack{\delta\leqq\frac{x}{d}\\(\delta,dn)=1}}\frac{\mu(\delta)}{\delta^k}.$$

Now, using Lemma 2.1, we obtain

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)}{\psi(m)m^{k-1}}=\sum_{\substack{d\leqq x\\(d,n)=1}}\frac{\mu^2(d)}{\psi(d)d^k}\left\{\frac{d^kn^k}{\zeta(k)J_k(dn)}+O\left(\frac{\sigma^*_{-1+\varepsilon}(dn)\delta\left(\frac{x}{d}\right)}{\left(\frac{x}{d}\right)^{k-1}}\right)\right\}=\tag{2.14}$$

$$=\frac{n^k}{\zeta(k)J_k(n)}\sum_{\substack{d\leqq x\\(d,n)=1}}\frac{\mu^2(d)}{\psi(d)J_k(d)}+O\left(\frac{\sigma^*_{-1+\varepsilon}(n)}{x^{k-1}}\sum_{\substack{d\leqq x\\(d,n)=1}}\frac{\sigma^*_{-1+\varepsilon}(d)\delta\left(\frac{x}{d}\right)}{\psi(d)d}\right).$$

By (2.7), it is clear that $x^{\varepsilon}\delta(x)$ is monotonically increasing for every $\varepsilon>0$, so that by Remarks 2.1 and 2.3, we have

$$\sum_{\substack{d\leq x\\(d,n)=1}}\frac{\sigma^*_{-1+\varepsilon}(d)\,\delta\left(\frac{x}{d}\right)}{\psi(d)\,d}=O\left(\sum_{\substack{d\leq x\\(d,n)=1}}\frac{\tau(d)\left(\frac{x}{d}\right)^{\varepsilon}\delta\left(\frac{x}{d}\right)}{\left(\frac{x}{d}\right)^{\varepsilon}d^2}\right)=O\left(\frac{x^{\varepsilon}\delta(x)}{x^{\varepsilon}}\sum_{\substack{d\leq x\\(d,n)=1}}\frac{\tau(d)}{d^{2-\varepsilon}}\right)=$$

$$=O\left(\delta(x)\sum_{d\leq x}\frac{1}{d^{2-2\varepsilon}}\right)=O\big(\delta(x)\big),$$

since $\tau(n)=O(n^{\varepsilon})$ (cf. [6], Theorem 315). Hence the O-term in (2.14) is $O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\,\delta(x)}{x^{k-1}}\right)$. By Lemma 2.4 and Remark 2.1, the first term on the right hand side of (2.14) is

$$\frac{n^k}{\zeta(k)J_k(n)}\left\{\frac{\zeta(k)J_k(n)\alpha_k}{H_k(n)}+O\left(\frac{1}{x^k}\right)\right\}=\frac{\alpha_k n^k}{H_k(n)}+O\left(\frac{1}{x^k}\right)=\frac{\alpha_k n^k}{H_k(n)}+O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\,\delta(x)}{x^{k-1}}\right).$$

Hence Lemma 2.5 follows.

LEMMA 2.6. *If the Riemann hypothesis is true, then for* $x\geqq 3$ *and* $n\geqq 1$,

$$\sum_{\substack{m\leq x\\(m,n)=1}}\frac{\mu(m)}{\psi(m)\,m^{k-1}}=\frac{\alpha_k n^k}{H_k(n)}+O\big(\sigma^*_{-1/2+\varepsilon}(n)\,x^{1/2-k}\omega(x)\big) \tag{2.15}$$

uniformly, where $\omega(x)$ *is given by* (2.9).

PROOF. Following the same argument adopted in the proof of Lemma 2.5 and making use of Lemma 2.2 instead of Lemma 2.1, and observing that $\omega(x)$ is monotonically increasing, we get Lemma 2.6. We have only to replace $\sigma^*_{-1+\varepsilon}(n)\,\delta(x)$ in Lemma 2.5 by $\sigma^*_{-1/2+\varepsilon}(n)\,x^{-1/2}\omega(x)$.

3. Auxiliary results. In this section we prove some lemmas concerning the Euler φ-function which are needed in our present discussion. We first prove

LEMMA 3.1. *For* $x\geqq 3$ *and* $n\geqq 1$,

$$\Phi_n(x)\equiv\sum_{mn\leq x}\varphi(mn)=\frac{3x^2}{\pi^2\psi(n)}+O\big(x\lambda(x/n)\big) \tag{3.1}$$

uniformly, where $\lambda(x)$ *is defined by*

$$\lambda(x)=\begin{cases}\log^{2/3}x\,(\log\log x)^{4/3} & \text{for}\quad x\geqq 3,\\ 1 & \text{for}\quad 0<x<3.\end{cases} \tag{3.2}$$

PROOF. For any prime p such that $(p,n)=1$ and any positive integer α, we have

$$\Phi_{np^{\alpha}}(x)=\sum_{mnp^{\alpha}\leq x}\varphi(mnp^{\alpha})=\sum_{\substack{mnp^{\alpha}\leq x\\(p,m)=1}}\varphi(mnp^{\alpha})+\sum_{\substack{mnp^{\alpha}\leq x\\p\mid m}}\varphi(mnp^{\alpha}).$$

We have for $(p, m)=1$, $\varphi(mnp^\alpha)=\varphi(mn)\varphi(p^\alpha)$ and for $p|m$,

$$\varphi(mnp^\alpha) = mnp^\alpha \prod_{q|mnp^\alpha}\left(1-\frac{1}{q}\right) = mnp^\alpha \prod_{q|mn}\left(1-\frac{1}{q}\right) = mnp^\alpha \cdot \frac{\varphi(mn)}{mn} = p^\alpha \varphi(mn).$$

Hence

$$\Phi_{np^\alpha}(x) = \varphi(p^\alpha) \sum_{\substack{mnp^\alpha \leqq x \\ p \nmid m}} \varphi(mn) + p^\alpha \sum_{\substack{mnp^\alpha \leqq x \\ p|m}} \varphi(mn) =$$

$$= \varphi(p^\alpha)\Big\{ \sum_{mnp^\alpha \leqq x} \varphi(mn) - \sum_{\substack{mnp^\alpha \leqq x \\ p|m}} \varphi(mn)\Big\} + p^\alpha \sum_{\substack{mnp^\alpha \leqq x \\ p|m}} \varphi(mn) =$$

$$= \varphi(p^\alpha) \sum_{mnp^\alpha \leqq x} \varphi(mn) + \big(p^\alpha - \varphi(p^\alpha)\big) \sum_{\substack{mn \leqq x/p^\alpha \\ p|m}} \varphi(mn) =$$

$$= \varphi(p^\alpha)\Phi_n\left(\frac{x}{p^\alpha}\right) + p^{\alpha-1} \sum_{tnp \leqq x/p^\alpha} (tnp),$$

so that we have

$$\Phi_{np^\alpha}(x) = \varphi(p^\alpha)\Phi_n\left(\frac{x}{p^\alpha}\right) + p^{\alpha-1}\Phi_{np}\left(\frac{x}{p^\alpha}\right). \tag{3.3}$$

Putting $\alpha=1$ in (3.3), we get

$$\Phi_{np}(x) = \varphi(p)\Phi_n\left(\frac{x}{p}\right) + \Phi_{np}\left(\frac{x}{p}\right). \tag{3.4}$$

Now, substituting $\frac{x}{p}, \frac{x}{p^2}, \ldots, \frac{x}{p^{c-1}}$ in (3.4) for x, where $c=\left[\frac{\log x}{\log p}\right]$ and simplifying we get

$$\Phi_{np}(x) = \varphi(p)\Phi_n\left(\frac{x}{p}\right) + \sum_{r=1}^{c-1} \varphi(p)\Phi_n\left(\frac{x}{p^{r+1}}\right) + \Phi_{np}\left(\frac{x}{p^c}\right) = \varphi(p)\sum_{r=0}^{c-1}\Phi_n\left(\frac{x}{p^{r+1}}\right), \tag{3.5}$$

since

$$\Phi_{np}\left(\frac{x}{p^c}\right) = \sum_{mnp \leqq \frac{x}{p^c}} \varphi(mnp) = \sum_{mn \leqq \frac{x}{p^{c+1}}} \varphi(mnp) = 0 \quad \text{for} \quad \frac{x}{p^{c+1}} < 1,$$

c being $= \left[\frac{\log x}{\log p}\right]$.

Substituting $\frac{x}{p^\alpha}$ for x in (3.5), we get

$$\Phi_{np}\left(\frac{x}{p^\alpha}\right) = \varphi(p)\sum_{r=0}^{c-1}\Phi_n\left(\frac{x}{p^{r+\alpha+1}}\right). \tag{3.6}$$

Now, from (3.3) and (3.6), we have

$$\Phi_{np^\alpha}(x) = \varphi(p^\alpha)\Phi_n\left(\frac{x}{p^\alpha}\right) + p^{\alpha-1}\varphi(p)\sum_{r=0}^{c-1}\Phi_n\left(\frac{x}{p^{r+\alpha+1}}\right) = \tag{3.7}$$

$$= \varphi(p^\alpha)\sum_{r=0}^{c-1}\Phi_n\left(\frac{x}{p^{r+\alpha}}\right) = \varphi(p^\alpha)\sum_{r=0}^{\infty}\Phi_n\left(\frac{x}{p^{r+\alpha}}\right),$$

since $\Phi_n\left(\frac{x}{p^{r+\alpha}}\right)=0$ for $r \geqq c$.

It is known (cf. [13], Satz 1, p. 144) that

$$\Phi_1(x) = \sum_{m\leqq x}\varphi(m) = \frac{3x^2}{\pi^2} + O(x\lambda(x)), \tag{3.8}$$

where $\lambda(x)$ is given by (3.2).

We prove (3.1) by induction on n. From (3.8), it is clear that (3.1) is true for $n=1$. Let us assume that (3.1) is true for 1, 2, ..., $n-1$, where $n>1$ and prove it for n.

Since $n>1$, there is a prime p such that $p|n$. Let $n=Np^\alpha$, where $(p, N)=1$; clearly $1\leqq N\leqq n-1$. Hence by our induction assumption we have

$$\Phi_N(x) = \frac{3x^2}{\pi^2\psi(N)} + O(x\lambda(x/n)), \tag{3.9}$$

where the O-estimate is uniform in x and N.

Now, by (3.7) and (3.9), we have

$$\varphi_{Np^\alpha}(x) = \varphi(p^\alpha)\sum_{r=0}^{\infty}\Phi_N\left(\frac{x}{p^{r+\alpha}}\right) = \varphi(p^\alpha)\sum_{r=0}^{\infty}\left\{\frac{3}{\pi^2\psi(N)}\cdot\frac{x^2}{p^{2r+2\alpha}} + O\left(\frac{x}{p^{r+\alpha}}\lambda\left(\frac{x}{Np^{r+\alpha}}\right)\right)\right\} =$$

$$= \frac{3x^2}{\pi^2\psi(N)}\cdot\frac{\varphi(p^\alpha)}{p^{2\alpha}}\sum_{r=0}^{\infty}\frac{1}{p^{2r}} + O\left(\frac{\varphi(p^\alpha)}{p^\alpha}x\lambda\left(\frac{x}{Np^\alpha}\right)\sum_{r=0}^{\infty}\frac{1}{p^r}\right),$$

since $\lambda(x)$ is monotonically increasing. Hence

$$\Phi_{Np^\alpha}(x) = \frac{3x^2}{\pi^2\psi(N)}\cdot\frac{\varphi(p^\alpha)}{p^{2\alpha}}\cdot\frac{1}{1-\frac{1}{p^2}} + O\left(\frac{\varphi(p^\alpha)x\lambda\left(\frac{x}{Np^\alpha}\right)}{p^\alpha}\cdot\frac{1}{1-\frac{1}{p}}\right) =$$

$$= \frac{3x^2}{\pi^2\psi(N)}\cdot\frac{1}{p^\alpha\left(1+\frac{1}{p}\right)} + O(x\lambda(x/n)) = \frac{3x^2}{\pi^2\psi(Np^\alpha)} + O(x\lambda(x/n)),$$

where the O-estimate is uniform in x, N and p. Since $n=Np^\alpha$, we have

$$\Phi_n(x) = \frac{3x^2}{\pi^2\psi(n)} + O(x\lambda(x)),$$

where the O-estimate is uniform in x and n. Thus Lemma 3.1 is proved.

REMARK 3.1. O. HÖLDER [7] and S. S. PILLAI [9] independently obtained asymptotic formula for the sum $\sum_{mn \leqq x} \varphi(mn)$ with error term $O(x \log x)$. However, it is not clear whether their O-estimate for the error term is independent of x and n. Subsequently, E. COHEN (cf. [4], Lemma 3.2) established an asymptotic formula for the sum $\sum_{mn \leqq x} \varphi_s(m, n)$, which when $s=1$ reduces to the formula for $\sum_{mn \leqq x} \varphi(mn)$ with error term $\sum_{mn \leqq x} (x \log x)$, the O-estimate being uniform in x and n. The formula we obtained in (3.1) above gives not only a uniform, but also a better O-estimate for the error term. Also, our method of approach is entirely different from theirs.

LEMMA 3.2. *For* $x \geqq 3$, $n \geqq 1$ *and* $u \geqq 1$ *such that* $(n, u)=1$,

$$\sum_{\substack{mn \leqq x \\ (m,u)=1}} \varphi(mn) = \frac{3x^2 u}{\pi^2 \psi(nu)} + O\big(\theta(u) x \lambda(x/n)\big), \tag{3.10}$$

where the O-estimate is uniform in x, n and u.

PROOF. We have by Lemma 3.1,

$$\sum_{\substack{mn \leqq x \\ (m,u)=1}} \varphi(mn) = \sum_{mn \leqq x} \varphi(mn) \sum_{\substack{d\delta = m \\ d|u}} \mu(d) = \sum_{d|u} \mu(d) \sum_{\delta d n \leqq x} \varphi(\delta d n) =$$

$$= \sum_{d|u} \mu(d) \Phi_{dn}(x) = \sum_{d|u} \mu(d) \left\{ \frac{3x^2}{\pi^2 \psi(dn)} + O\left(x \lambda\left(\frac{x}{dn}\right)\right)\right\}.$$

Since $(u, n)=1$, we have $(d, n)=1$ for $d|u$. Hence

$$\sum_{\substack{mn \leqq x \\ (m,u)=1}} \varphi(mn) = \frac{3x^2}{\pi^2 \psi(n)} \sum_{d|u} \frac{\mu(d)}{\psi(d)} + O\big(x\lambda(x/n) \sum_{d|u} \mu^2(d)\big) =$$

$$= \frac{3x^2}{\pi^2 \psi(n)} \cdot \frac{u}{\psi(u)} + O\big(\theta(u) x \lambda(x/n)\big).$$

Hence Lemma 3.2 follows.

LEMMA 3.3 (cf. [12], Lemma 2.3). *For* $x \geqq 3$ *and* $u \geqq 1$,

$$\sum_{\substack{m \leqq x \\ (m,u)=1}} \mu(m) \varphi(m) = O\big(\sigma^*_{-1+\varepsilon}(u) x^2 \delta(x)\big), \tag{3.11}$$

where the O-estimate is uniform in x and u and $\delta(x)$ *is given by* (2.7).

LEMMA 3.4. *For* $x \geqq 3$ *and* $u \geqq 1$,

$$F_1(x, u) \equiv \sum_{\substack{m \leqq x \\ (m,u)=1}} \mu(m) \varphi(m) m^{k-1} = O\big(\sigma^*_{-1+\varepsilon}(u) x^{k+1} \delta(x)\big), \tag{3.12}$$

where the O-estimate is uniform in x and u.

PROOF. This follows by Lemma 3.3 and partial summation.

LEMMA 3.5 (cf. [12], Lemma 2.13). *If the Riemann hypothesis is true, then for* $x\geqq 3$ *and* $u\geqq 1$,

$$\sum_{\substack{m\leqq x\\(m,u)=1}} \mu(m)\varphi(m) = O\big(\sigma^*_{-1/2+\varepsilon}(u)x^{3/2}\omega(x)\big), \tag{3.13}$$

where the O-estimate is uniform in x and u and $\omega(x)$ *is given by* (2.9).

LEMMA 3.6. *If the Riemann hypothesis is true, then for* $x\geqq 3$ *and* $u\geqq 1$,

$$F_1(x,u) \equiv \sum_{\substack{m\leqq x\\(m,u)=1}} \mu(m)\varphi(m)m^{k-1} = O\big(\sigma^*_{-1/2+\varepsilon}(u)x^{k+1/2}\omega(x)\big), \tag{3.14}$$

where the O-estimate is uniform in x and u.

PROOF. This follows by Lemma 3.5 and partial summation.

LEMMA 3.7. *For* $x\geqq 3$, $n\geqq 1$ *and* $u\geqq 1$ *such that* $(n,u)=1$,

$$F_n(x,u) \equiv \sum_{\substack{m\leqq x\\(m,u)=1}} \mu(m)\varphi(mn)m^{k-1} = O\big(\sigma^*_{-1+\varepsilon}(nu)\varphi(n)x^{k+1}\delta(x)\big), \tag{3.15}$$

where the O-estimate is uniform in x, n and u.

PROOF. For any prime p such that $(p,nu)=1$ and for any positive integer α, we have

$$F_{np^\alpha}(x,u) = \sum_{\substack{m\leqq x\\(m,u)=1}} \mu(m)\varphi(mnp^\alpha)m^{k-1} = \sum_{\substack{m\leqq x\\(m,u)=1\\p|m}} \mu(m)\varphi(mnp^\alpha)m^{k-1} + \sum_{\substack{m\leqq x\\(m,u)=1\\(p,m)=1}} \mu(m)\varphi(mnp^\alpha)m^{k-1}.$$

Since $\varphi(mnp^\alpha)=p^\alpha\varphi(mn)$ for $p|m$ and $\varphi(mnp^\alpha)=\varphi(mn)\varphi(p^\alpha)$ for $(p,m)=1$, we have

$$F_{np^\alpha}(x,u) = \sum_{\substack{m\leqq x\\(m,u)=1\\p|m}} \mu(m)p^\alpha\varphi(mn)m^{k-1} + \sum_{\substack{m\leqq x\\(m,u)=1\\(p,m)=1}} \mu(m)\varphi(p^\alpha)\varphi(mn)m^{k-1} =$$

$$= p^\alpha \sum_{\substack{m\leqq x\\(m,u)=1\\p|m}} \mu(m)\varphi(mn)m^{k-1} + \varphi(p^\alpha)\Big\{\sum_{\substack{m\leqq x\\(m,u)=1}} \mu(m)\varphi(mn)m^{k-1} - \sum_{\substack{m\leqq x\\(m,u)=1\\p|m}} \mu(m)\varphi(mn)m^{k-1}\Big\} =$$

$$= \varphi(p^\alpha)F_n(x,u) + \big(p^\alpha-\varphi(p^\alpha)\big) \sum_{\substack{pt\leqq x\\(pt,u)=1}} \mu(pt)\varphi(ptn)p^{k-1}t^{k-1} =$$

$$= \varphi(p^\alpha)F_n(x,u) + p^{\alpha-1}\cdot p^{k-1} \sum_{\substack{t\leqq x/p\\(pt,u)=1\\(t,p)=1}} \mu(p)\mu(t)\varphi(p)\varphi(tn)t^{k-1} =$$

$$= \varphi(p^\alpha)F_n(x,u) - p^{k-1}\varphi(p^\alpha) \sum_{\substack{t\leqq x/p\\(t,up)=1}} \mu(t)\varphi(tn)t^{k-1}.$$

Hence

$$F_{np^\alpha}(x,u)=\varphi(p^\alpha)F_n(x,u)-p^{k-1}\varphi(p^\alpha)F_n\left(\frac{x}{p},pu\right). \tag{3.16}$$

We prove (3.15) by induction on n. From (3.12), it is clear that (3.15) is true for $n=1$. Let us assume that (3.15) is true for $1, 2, \ldots, n-1$, where $n>1$ and prove it for n.

Since $n>1$, there is a prime p such that $p|n$. Let $n=Np^\alpha$, where $(p, N)=1$. Clearly $1\leqq N\leqq n-1$. Hence by our induction assumption, we have

$$F_N(x,u)=O\big(\sigma^*_{-1+\varepsilon}(Nu)\varphi(N)x^{k+1}\delta(x)\big), \tag{3.17}$$

where the O-estimate is uniform in x, N and u.

Now, by (3.16) and (3.17),

$$F_{Np^\alpha}(x,u)=\varphi(p^\alpha)F(x,u)-p^{k-1}\varphi(p^\alpha)F_N\left(\frac{x}{p},up\right)= \tag{3.18}$$

$$=O\big(\varphi(p^\alpha)\sigma^*_{-1+\varepsilon}(Nu)\varphi(N)x^{k+1}\delta(x)\big)+O\left(p^{k-1}\varphi(p^\alpha)\sigma^*_{-1+\varepsilon}(Nup)\varphi(N)\left(\frac{x}{p}\right)^{k+1}\delta\left(\frac{x}{p}\right)\right)=$$

$$=O\big(\sigma^*_{-1+\varepsilon}(Nu)\varphi(Np^\alpha)x^{k+1}\delta(x)\big)+O\left(\sigma^*_{-1+\varepsilon}(Nu)\varphi(Np^\alpha)\frac{\sigma^*_{-1+\varepsilon}(p)}{p^2}\cdot x^{k+1}\delta\left(\frac{x}{p}\right)\right).$$

By (2.7), it is clear that $x^\varepsilon\delta(x)$ is monotonically increasing for every $\varepsilon>0$, so that we have

$$\frac{\sigma^*_{-1+\varepsilon}(p)}{p^2}\cdot x^{k+1}\delta\left(\frac{x}{p}\right)=\frac{\sigma^*_{-1+\varepsilon}(p)}{p^{2-\varepsilon}}x^{k+1-\varepsilon}\left(\frac{x}{p}\right)^{\varepsilon}\delta\left(\frac{x}{p}\right)\leqq$$

$$\leqq\frac{\sigma^*_{-1+\varepsilon}(p)}{p^{2-\varepsilon}}\cdot x^{k+1-\varepsilon}x^\varepsilon\delta(x)=\frac{\sigma^*_{-1+\varepsilon}(p)}{p^{2-\varepsilon}}\cdot x^{k+1}\delta(x)=$$

$$=\frac{1}{p^{2-\varepsilon}}\left(1+\frac{1}{p^{1-\varepsilon}}\right)\cdot x^{k+1}\delta(x)\leqq\frac{1}{p^{1-\varepsilon}}\cdot x^{k+1}\delta(x),\quad p\text{ being}\geqq 2.$$

Hence by (3.18), we have

$$F_{Np^\alpha}(x,u)=O\left(\sigma^*_{-1+\varepsilon}(Nu)\left(1+\frac{1}{p^{1-\varepsilon}}\right)\varphi(Np^\alpha)x^{k+1}\delta(x)\right)=$$

$$=O\big(\sigma^*_{-1+\varepsilon}(Nup^\alpha)\varphi(Np^\alpha)x^{k+1}\delta(x)\big),$$

where the O-estimate is uniform in x, N, p and u. Since $n=Np^\alpha$, we have

$$F_n(x,u)=O\big(\sigma^*_{-1+\varepsilon}(nu)\varphi(n)x^{k+1}\,\delta(x)\big).$$

Thus Lemma 3.6 is proved.

LEMMA 3.8. *If the Riemann hypothesis is true, then for* $x\geqq 3$, $n\geqq 1$ *and* $u\geqq 1$ *such that* $(n, u)=1$,

$$(3.19)\quad F_n(x, u) \equiv \sum_{\substack{m\leqq x\\(m,u)=1}} \mu(m)\varphi(mn)m^{k-1} = O\big(\sigma^*_{-1/2+\varepsilon}(nu)\varphi(n)x^{k+1/2}\omega(x)\big),$$

where the O-estimate is uniform in x, n and u.

PROOF. Following the same argument adopted in the proof of Lemma 3.7 and making use of (3.14) instead of (3.12) to verify (3.19) for $n=1$, and observing that $\omega(x)$ is monotonically increasing and making use of induction on n, we get Lemma 3.8. We have only to replace $\sigma^*_{-1+\varepsilon}(nu)\delta(x)$ in Lemma 3.7 by $\sigma^*_{-1/2+\varepsilon}(nu)x^{-1/2}\omega(x)$.

4. Main results. First we prove the following:

THEOREM 4.1. *For* $x\geqq 3$ *and* $n\geqq 1$,

$$(4.1)\quad \sum_{\substack{m\leqq x\\(m,n)=1}} q_k(m)\varphi(m) = \frac{3\alpha_k n^{k+1}}{\pi^2\psi(n)H_k(n)}x^2+O\big(\theta(n)x^{1+1/k}\delta(x)\big)+O\big(\theta^2(n)x\lambda(x/n)\big)$$

uniformly, where $H_k(n)$, $\delta(x)$, $\lambda(x)$ *and* α_k *are given by* (2.5), (2.7), (3.2) *and* (2.11), *respectively.*

PROOF. We have by (2.1),

$$\sum_{\substack{m\leqq x\\(m,n)=1}} q_k(m)\varphi(m) = \sum_{\substack{m\leqq x\\(m,n)=1}} \varphi(m)\sum_{d^k\delta=m}\mu(d) = \sum_{\substack{d^k\delta\leqq x\\(d^k\delta,n)=1}} \mu(d)\varphi(d^k\delta) =$$

$$= \sum_{\substack{d^k\delta\leqq x\\(d,n)=(\delta,n)=1}}' \mu(d)\varphi(d\delta)d^{k-1},$$

where the summation is taken over all ordered pairs (d, δ) such that $d^k\delta\leqq x$ and $(d, n)=(\delta, n)=1$.

Let $z=x^{1/k}$ and $\varrho(x)$ be a function of x such that $0<\varrho=\varrho(x)<1$, which will be suitably chosen later. If $d^k\delta\leqq x$, then both $d>\varrho z$ and $\delta>\varrho^{-k}$ cannot simultaneously hold, so that we have

$$(4.2)\quad \sum_{\substack{m\leqq x\\(m,n)=1}} q_k(m)\varphi(m) = \sum_{\substack{d^k\delta\leqq x\\d\leqq\varrho z\\(d,n)=(\delta,n)=1}} \mu(d)\varphi(d^k\delta)+\sum_{\substack{d^k\delta\leqq x\\\delta\leqq\varrho^{-k}\\(d,n)=(\delta,n)=1}} \mu(d)\varphi(d^k\delta)-$$

$$-\sum_{\substack{d\leqq\varrho z\\\delta\leqq\varrho^{-k}\\(d,n)=(\delta,n)=1}} \mu(d)\varphi(d^k\delta) = S_1+S_2-S_3,\quad\text{say.}$$

We have by Lemma 3.2,

$$S_1 = \sum_{\substack{d \leq \varrho z \\ (d,n)=1}} \mu(d) \sum_{\substack{\delta d^k \leq x \\ (\delta,n)=1}} \varphi(\delta d^k) = \sum_{\substack{d \leq \varrho z \\ (d,n)=1}} \mu(d) \left\{ \frac{3x^2 n}{\pi^2 \psi(d^k n)} + O\big(\theta(n) x \lambda(x/n)\big) \right\} =$$

$$= \frac{3x^2 n}{\pi^2 \psi(n)} \sum_{\substack{d \leq \varrho z \\ (d,n)=1}} \frac{\mu(d)}{\psi(d^k)} + O\big(\theta(n) x \lambda(x/n) \sum_{\substack{d \leq \varrho z \\ (d,n)=1}} \mu^2(d)\big) = \frac{3x^2 n}{\pi^2 \psi(n)} \sum_{\substack{d \leq \varrho z \\ (d,n)=1}} \frac{\mu(d)}{\psi(d) d^{k-1}} +$$

$$+ O\big(\theta(n) z^{k+1} \varrho \lambda(x/n)\big) + O\big(\theta^2(n) x \lambda(x/n)\big)$$

since

$$\sum_{\substack{d \leq \varrho z \\ (d,n)=1}} \mu^2(d) \leq \sum_{\substack{d \leq \varrho z \\ (d,n)=1}} 1 = \varrho z \frac{\varphi(n)}{n} + O\big(\theta(n)\big)$$

(cf. [2], Lemma 3.4). Hence by Lemma 2.5 and Remark 2.3,

(4.3)

$$S_1 = \frac{3x^2 n}{\pi^2 \psi(n)} \left\{ \frac{\alpha_k n^k}{H_k(n)} + O\left(\frac{\theta(n) \delta(\varrho z)}{(\varrho z)^{k-1}} \right) \right\} + O\big(\theta(n) z^{k+1} \varrho \lambda(x)\big) + O\big(\theta^2(n) x \lambda(x/n)\big) =$$

$$= \frac{3\alpha_k n^{k+1} x^2}{\pi^2 \psi(n) H_k(n)} + O\big(\theta(n) z^{k+1} \varrho^{1-k} \delta(\varrho z)\big) + O\big(\theta(n) z^{k+1} \varrho \lambda(x)\big) + O\big(\theta^2(n) x \lambda(x/n)\big).$$

We have by Lemma 3.7 and Remark 2.3,

$$S_2 = \sum_{\substack{d^k \delta \leq x \\ \delta \leq \varrho^{-k} \\ (d,n)=(\delta,n)=1}} \mu(d) \varphi(d^k \delta) = \sum_{\substack{\delta \leq \varrho^{-k} \\ (\delta,n)=1}} \sum_{\substack{d \leq \sqrt[k]{\frac{x}{\delta}} \\ (d,n)=1}} \mu(d) \varphi(d\delta) d^{k-1} = \tag{4.4}$$

$$= O\left(\sum_{\substack{\delta \leq \varrho^{-k} \\ (\delta,n)=1}} \theta(\delta n) \varphi(\delta) \left(\sqrt[k]{\frac{x}{\delta}} \right)^{k+1} \delta\left(\sqrt[k]{\frac{x}{\delta}} \right) \right) = O\left(\theta(n) z^{k+1} \sum_{\substack{\delta \leq \varrho^{-k} \\ (\delta,n)=1}} \frac{\theta(\delta)}{\delta^{1/k}} \delta\left(\sqrt[k]{\frac{x}{\delta}} \right) \right).$$

Since $\delta(x)$ is monotonically decreasing and $\sqrt[k]{\frac{x}{\delta}} \geq \varrho z$, we have $\delta\left(\sqrt[k]{\frac{x}{\delta}} \right) \leq \delta(\varrho z)$.

Also, we have

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \theta(m) \leq \sum_{m \leq x} \tau(m) = O(x \log x) \tag{4.5}$$

(cf. [6], Theorem 320), so that by partial summation,

$$\sum_{\substack{\delta \leq \varrho^{-k} \\ (\delta,n)=1}} \frac{\theta(\delta)}{\delta^{1/k}} = O\left(\varrho^{1-k} \log \frac{1}{e} \right). \tag{4.6}$$

Hence by (4.4) and (4.6), we have

$$S_2 = O\left(\theta(n) z^{k+1} \varrho^{1-k} \delta(\varrho z) \log\left(\frac{1}{\varrho} \right) \right). \tag{4.7}$$

Again, by Lemma 3.7 and Remark 2.3,

$$S_3 = \sum_{\substack{d \leqq \varrho z \\ \delta \leqq \varrho^{-k} \\ (d,n)=(\delta,n)=1}} \mu(d)\varphi(d^k\delta) = \sum_{\substack{\delta \leqq \varrho^{-k} \\ (\delta,n)=1}} \sum_{\substack{d \leqq \varrho z \\ (d,n)=1}} \mu(d)\varphi(d\delta)d^{k-1} = \tag{4.8}$$

$$= O\Big(\sum_{\substack{\delta \leqq \varrho^{-k} \\ (\delta,n)=1}} \theta(\delta n)\varphi(\delta)(\varrho z)^{k+1}\delta(\varrho z)\Big) = O\big(\theta(n)\, z^{k+1}\varrho^{k+1}\delta(\varrho z) \sum_{\substack{\delta \leqq \varrho^{-k} \\ (\delta,n)=1}} \theta(\delta)\delta\big).$$

By (4.5) and partial summation, we have

$$\sum_{\substack{\delta \leqq \varrho^{-k} \\ (\delta,n)=1}} \theta(\delta)\delta = O\big((\varrho^{-k})^2 \log(\varrho^{-k})\big) = O\left(\varrho^{-k}\log\left(\frac{1}{\varrho}\right)\right). \tag{4.9}$$

Hence by (4.8) and (4.9), we have

$$S_3 = O\left(\theta(n) z^{k+1}\varrho^{1-k}\delta(\varrho z)\log\left(\frac{1}{\varrho}\right)\right). \tag{4.10}$$

Hence by (4.2), (4.3), (4.7) and (4.10), we have

$$\sum_{\substack{m \leqq x \\ (m,n)=1}} q_k(m)\varphi(m) = \tag{4.11}$$

$$= \frac{3\alpha_k n^{k+1}x^2}{\pi^2\psi(n)H_k(n)} + O\left(\theta(n)\, z^{k+1}\varrho^{1-k}\delta(\varrho z)\log\left(\frac{1}{\varrho}\right)\right) + O\left(\frac{\theta(n)n}{\psi(n)} z^{k+1}\varrho\lambda(x)\right) +$$

$$+ O\big(\theta^2(n)\, x\lambda(x/n)\big).$$

Now, we choose

$$\varrho = \varrho(x) = \{\delta(x^{1/2k})\}^{1/k} \tag{4.12}$$

and write

$$f(x) = \log^{3/5}(x^{1/2k})\{\log\log(x^{1/2k})\}^{-1/5} = \left(\frac{1}{2k}\right)^{3/5} U^{3/5}(V - \log 2k)^{-1/5}, \tag{4.13}$$

where $U = \log x$ and $V = \log\log x$.

(4.14) For $V \geqq 2\log 2k$, that is, $\log x \geqq 4k^2$, $x \geqq \exp(4k^2)$,

we have

$$V^{-1/5} \leqq (V - \log 2k)^{-1/5} \leqq \left(\frac{V}{2}\right)^{-1/5},$$

and therefore

$$\frac{1}{2} k^{-3/5} U^{3/5} V^{-1/5} \leqq f(x) \leqq k^{-3/5} U^{3/5} V^{-1/5}. \tag{4.15}$$

(4.16) We assume without loss of generality that the constant A in $\delta(x)$ of (2.7) is less than 1.

By (4.12), (2.7) and (4.13),

$$\varrho = \exp\left\{-\frac{A}{k} f(x)\right\}. \tag{4.17}$$

By (4.14), we have $k^{-8/5}U^{3/5}V^{-1/5} \leqq \dfrac{U}{2k}$. Hence by (4.15), (4.16) and (4.17) and the above,

$$\varrho \geqq \exp\{-Ak^{-8/5}U^{3/5}V^{-1/5}\} \geqq \exp\{-k^{-8/5}U^{3/5}V^{-1/5}\} \geqq \exp\left\{-\frac{U}{2k}\right\} \exp\left\{-\frac{\log x}{2k}\right\},$$

so that $\varrho \geqq x^{-1/2k}$. Hence

$$\log\left(\frac{1}{\varrho}\right) \leqq \log(\sqrt{z}) = O(\log x) \quad \text{and} \quad \varrho z \geqq x^{1/2k}. \tag{4.18}$$

Since $\delta(x)$ is monotonically decreasing, $\delta(\varrho z) \leqq \delta(x^{1/2k}) = \varrho^k$, by (4.12) and so, by (4.15) and (4.17), we have

$$\varrho^{1-k}\delta(\varrho z) \leqq \varrho \leqq \exp\left\{-\frac{A}{2}k^{-8/5}U^{3/5}V^{-1/5}\right\}. \tag{4.19}$$

By (3.2), we have $\lambda(x) = \log^{2/3} x\,(\log\log x)^{4/3} = O(\log x)$. Hence by (4.18) and (4.19), the first and second O-terms of (4.11) are both

$$O\left(\theta(n) z^{k+1} \exp\left\{-\frac{A}{2}k^{-8/5}U^{3/5}V^{-1/5}\right\}\log x\right) = O\left(\theta(n)x^{1+1/k}\delta(x)\right),$$

by Remark 2.2. Hence Theorem 4.1 follows.

COROLLARY 4.1.1 ($n=1$). *For* $x \geqq 3$ *and* $k \geqq 2$,

$$\sum_{m \leqq x} q_k(m)\varphi(m) = \frac{3\alpha_k x^2}{\pi^2} + O\left(x^{1+1/k}\delta(x)\right). \tag{4.20}$$

COROLLARY 4.1.2 ($n=1$, $k=2$). *For* $x \geqq 3$,

$$\sum_{m \leqq x} \mu^2(m)\varphi(m) = \frac{3\alpha x^2}{\pi^2} + O\left(x^{3/2}\delta(x)\right), \tag{4.21}$$

where α *is the constant given by*

$$\alpha = \prod_p \left(1 - \frac{1}{p(p+1)}\right). \tag{4.22}$$

REMARK 4.1. Formula (4.21) has been established by E. COHEN (cf. [2], Corollary 5.1.2) with error term $O(x^{3/2})$.

COROLLARY 4.1.3 ($k=2$). *For* $x \geqq 3$ *and* $n \geqq 1$,

$$\sum_{\substack{m \leqq x \\ (m,n)=1}} \mu^2(m)\varphi(m) = \frac{3\alpha n^3 x^3}{\pi^2 \psi(n) H_2(n)} + O\left(\theta(n)x^{3/2}\delta(x)\right) + O\left(\theta^2(n)x\lambda(x/n)\right). \tag{4.23}$$

uniformly.

THEOREM 4.2. *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and* $n \geqq 1$,

$$(4.24)\quad \sum_{\substack{m \leqq x \\ (m,n)=1}} q_k(m)\varphi(m) = \frac{3\alpha_k n^{k+1}}{\pi^2 \psi(n) H_k(n)} x^2 + O\big(\theta(n) x^{1+2/2k+1}\omega(x)\big) + O\big(\theta^2(n) x\lambda(x/n)\big)$$

uniformly, where $\omega(x)$ *is given by* (2.9).

PROOF. Following the same procedure adopted in the proof of Theorem 4.1 and making use of Lemmas 2.6 and 3.8 instead of Lemmas 2.5 and 3.7, we get the following instead of (4.11):

$$(4.25)\quad \sum_{\substack{m \leqq x \\ (m,n)=1}} q_k(m)\varphi(m) =$$

$$= \frac{3\alpha_k n^{k+1}}{\pi^2 \psi(n) H_k(n)} x^2 + O\left(\theta(n) z^{k+1/2} \varrho^{1/2-k} \omega(\varrho z) \log\left(\frac{1}{\varrho}\right)\right) + O\big(\theta(n) z^{k+1} \varrho \lambda(x)\big) +$$
$$+ O\big(\theta^2(n) x \lambda(x/n)\big).$$

Now, choosing $\varrho = z^{-1/2k+1}$, we see that $0 < \varrho < 1$, $\frac{1}{\varrho} < z$, so that $\log\left(\frac{1}{\varrho}\right) <$ $< \log z$ and $z^{k+1/2}\varrho^{1/2-k} = z^{k+1}\varrho = x^{1+2/2k+1}$.

Since $\omega(x)$ is monotonically increasing, we have

$$\omega(\varrho z)\log\left(\frac{1}{\varrho}\right) \leqq \omega(z)\log z = O\big(\omega(x^{1/k})\log x\big) = O\big(\omega(x)\big),$$

by (2.9). Also,

$$\lambda(x) = \log^{2/3} x\, (\log\log x)^{4/3} = O(\log x) = O\big(\omega(x)\big).$$

Hence the first and second O-terms of (4.25) are both $O\big(\theta(n) x^{1+2/2k+1}\omega(x)\big)$. Hence Theorem 4.2 follows.

COROLLARY 4.2.1 ($n=1$). *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and* $k \geqq 2$

$$(4.26)\quad \sum_{m \leqq x} q_k(m)\varphi(m) = \frac{3\alpha_k x^2}{\pi^2} + O\big(x^{1+2/2k+1}\omega(x)\big).$$

COROLLARY 4.2.2 ($n=1$, $k=2$). *If the Riemann hypothesis is true, then for* $x \geqq 3$,

$$(4.27)\quad \sum_{m \leqq x} \mu^2(m)\varphi(m) = \frac{3\alpha x^2}{2} + O\big(x^{7/5}\omega(x)\big),$$

where α *is given by* (4.22).

REMARK 4.2. Formula (4.27) has been established by the first author (cf. [9], Theorem 5) with error term $O(x^{7/5+\varepsilon})$.

COROLLARY 4.2.3 ($k=2$). *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and* $n \geqq 1$,

$$(4.28)\quad \sum_{\substack{m \leqq x \\ (m,n)=1}} \mu^2(m)\varphi(m) = \frac{3\alpha n^3 x^2}{\pi^2 \psi(n) H_2(n)} + O\big(\theta(n) x^{7/5}\omega(x)\big) + O\big(\theta^2(n) x\lambda(x/n)\big)$$

uniformly.

THEOREM 4.3. *For* $x\geqq 3$ *and* $n\geqq 1$,

$$(4.29)\quad \sum_{m\leqq x}\gamma_k(m;\,n)=\frac{\alpha_k n^{k+1}x^2}{2\psi(n)H_k(n)}+O\big(\theta(n)x^{1+1/k}\delta(x)\big)+O\big(\theta^2(n)x\lambda(x/n)\log x\big)$$

uniformly, where $\delta(x)$ *and* $\lambda(x)$ *are given by* (2.7) *and* (3.2), *respectively.*

PROOF. We have

$$\gamma_k(m;\,n)=\sum_{d\mid\gamma_k(m;\,n)}\varphi(d)=\sum_{\substack{d\mid m\\ d\in Q_k\\ (d,n)=1}}\varphi(d)=\sum_{\substack{d\delta=m\\ (d,n)=1}}q_k(d)\varphi(d).$$

Hence by Theorem 4.1, we have

$$(4.30)\quad \sum_{m\leqq x}\gamma_k(m;\,n)=\sum_{\substack{d\delta\leqq x\\ (d,n)=1}}q_k(d)\varphi(d)=\sum_{\delta\leqq x}\sum_{\substack{d\leqq x/\delta\\ (d,n)=1}}q_k(d)\varphi(d)=$$

$$=\sum_{\delta\leqq x}\left\{\frac{3\alpha_k n^{k+1}}{\pi^2\psi(n)H_k(n)}\left(\frac{x}{\delta}\right)^2+O\left(\theta(n)\left(\frac{x}{\delta}\right)^{1+1/k}\delta\left(\frac{x}{\delta}\right)\right)+O\left(\theta^2(n)\frac{x}{\delta}\lambda\left(\frac{x}{n\delta}\right)\right)\right\}=$$

$$=\frac{3\alpha_k n^{k+1}x^2}{\pi^2\psi(n)H_k(n)}\sum_{m\leqq x}\frac{1}{m^2}+O\left(\theta(n)\sum_{m\leqq x}\left(\frac{x}{m}\right)^{1+1/k}\delta\left(\frac{x}{m}\right)\right)+\left(\theta^2(n)x\lambda(x/n)\sum_{m\leqq x}\frac{1}{m}\right).$$

By (2.7), it is clear that $x^{\varepsilon}\delta(x)$ is monotonically increasing for every $\varepsilon>0$, so that

$$\sum_{m\leqq x}\left(\frac{x}{m}\right)^{1+1/k}\delta\left(\frac{x}{m}\right)=\sum_{m\leqq x}\left(\frac{x}{m}\right)^{1+1/k-\varepsilon}\left(\frac{x}{m}\right)^{\varepsilon}\delta\left(\frac{x}{m}\right)\leqq x^{\varepsilon}\delta(x)\sum_{m\leqq x}\left(\frac{x}{m}\right)^{1+1/k-\varepsilon}=$$

$$=x^{1+1/k}\delta(x)\sum_{m\leqq x}\frac{1}{\delta^{1+1/k-\varepsilon}}=O\big(x^{1+1/k}\delta(x)\big).$$

Hence the first O-term in (4.30) is $O\big(\theta(n)x^{1+1/k}\delta(x)\big)$, so that by (4.30), we have

$$(4.31)\quad \sum_{m\leqq x}\gamma_k(m;\,n)=$$

$$=\frac{3\alpha_k n^{k+1}x^2}{\pi^2\psi(n)H_k(n)}\left\{\frac{\pi^2}{6}+O\left(\frac{1}{x}\right)\right\}+O\big(\theta(n)x^{1+1/k}\delta(x)\big)+O\big(\theta^2(n)x\lambda(x/n)\log x\big)=$$

$$=\frac{\alpha_k n^{k+1}x^2}{2\psi(n)H_k(n)}+O\left(\frac{n^{k+1}x}{\psi(n)H_k(n)}\right)+O\big(\theta(n)x^{1+1/k}\delta(x)\big)+O\big(\theta^2(n)x\lambda(x/n)\log x\big).$$

By (2.5) and (2.4), we have

$$H_k(n)>n^k\prod_{p\mid n}\left(1-\frac{1}{p^k}\right)=J_k(n),$$

so that by Remark 2.1, the first O-term in (4.31) is $O(x)=O\big(\theta(n)x^{1+1/k}\delta(x)\big)$. Hence Theorem 4.3 follows.

COROLLARY 4.3.1 ($n=1$). *For* $x\geqq 3$ *and* $k\geqq 2$,

$$\sum_{m\leqq x}\gamma_k(m)=\frac{\alpha_k x^2}{2}+O\left(x^{1+1/k}\delta(x)\right), \tag{4.32}$$

where α_k *is given by* (2.11).

COROLLARY 4.3.2 ($n=2$). *For* $x\geqq 3$ *and* $k\geqq 2$,

$$\sum_{m\leqq x}\delta_k(m)=\frac{2^{k-1}\alpha_k x^2}{(2^k+2^{k-1}-1)}+O\left(x^{1+1/k}\delta(x)\right). \tag{4.33}$$

COROLLARY 4.3.3 ($n=p$). *If* p *is a given prime, then for* $x\geqq 3$ *and* $k\geqq 2$,

$$\sum_{m\leqq x}\gamma_k(m;\,p)=\frac{p^k\alpha_k x^2}{2(p^k+p^{k-1}-1)}+O\left(x^{1+1/k}\delta(x)\right). \tag{4.34}$$

COROLLARY 4.3.4 ($k=2$). *For* $x\geqq 3$ *and* $n\geqq 1$,

$$\sum_{m\leqq x}\gamma(m;\,n)=\frac{\alpha n^3x^2}{2\psi(n)H_2(n)}+O\left(\theta(n)x^{3/2}\delta(x)\right)+O\left(\theta^2(n)x\lambda(x/n)\log x\right) \tag{4.35}$$

uniformly, where α *is given by* (4.22).

COROLLARY 4.3.5 ($k=2$, $n=1$). *For* $x\geqq 3$,

$$\sum_{m\leqq x}\gamma(m)=\frac{\alpha x^2}{2}+O\left(x^{3/2}\delta(x)\right). \tag{4.36}$$

REMARK 4.3. Formula (4.36) has been established by E. COHEN (cf. [2], Theorem 5.2) with error term $O(x^{3/2})$.

COROLLARY 4.3.6 ($k=2$, $n=2$). *For* $x\geqq 3$,

$$\sum_{m\leqq x}\delta(m)=\frac{2\alpha x^2}{5}+O\left(x^{3/2}\delta(x)\right). \tag{4.37}$$

THEOREM 4.4. *If the Riemann hypothesis is true, the for* $x\geqq 3$ *and* $n\geqq 1$,

$$\sum_{m\leqq x}\gamma_k(m;\,n)=\frac{\alpha_k n^{k+1}x^2}{2\psi(n)H_k(n)}+O\left(\theta(n)x^{1+2/2k+1}\omega(x)\right)+O\left(\theta^2(n)x\lambda(x/n)\log x\right) \tag{4.38}$$

uniformly, where $\omega(x)$ *and* $\lambda(x)$ *are given by* (2.9) *and* (3.2), *respectively.*

PROOF. Following the same procedure adopted in the proof of Theorem 4.3 and making use of Theorem 4.2 instead of Theorem 4.1, we get Theorem 4.4.

COROLLARY 4.4.1 ($n=1$). *If the Riemann hypothesis is true, then for* $x\geqq 3$ *and* $k\geqq 2$,

$$\sum_{m\leqq x}\gamma_k(m)=\frac{\alpha_k x^2}{2}+O\left(x^{1+2/2k+1}\omega(x)\right). \tag{4.39}$$

COROLLARY 4.4.2 ($n=2$). *If the Riemann hypothesis is true, then for* $x\geqq 3$ *and* $k\geqq 2$,

$$\sum_{m\leqq x}\delta_k(m)=\frac{2^{k-1}\alpha_k x^2}{(2^k+2^{k-1}-1)}+O\big(x^{1+2/2k+1}\omega(x)\big). \tag{4.40}$$

COROLLARY 4.4.3 ($n=p$). *If the Riemann hypothesis is true, then for* $x\geqq 3$ *and* $k\geqq 2$,

$$\sum_{m\leqq x}\gamma_k(m;\,p)=\frac{p^k\alpha_k x^2}{(p^k+p^{k-1}-1)}+O\big(x^{1+2/2k+1}\omega(x)\big) \tag{4.41}$$

where p *is a given prime.*

COROLLARY 4.4.4 ($k=2$). *If the Riemann hypothesis is true, then for* $x\geqq 3$ *and* $n\geqq 1$,

$$\sum_{m\leqq x}\gamma(m;\,n)=\frac{\alpha n^3x^2}{2\psi(n)H_2(n)}+O\big(\theta(n)x^{1+2/2k+1}\omega(x)\big)+O\big(\theta^2(n)x\lambda(x/n)\log x\big) \tag{4.42}$$

uniformly, where α *is given by* (4.22).

COROLLARY 4.4.5 ($k=2$, $n=1$). *If the Riemann hypothesis is true, then for* $x\geqq 3$,

$$\sum_{m\leqq x}\gamma(m)=\frac{\alpha x^2}{2}+O\big(x^{7/5}\omega(x)\big). \tag{4.43}$$

REMARK 4.4. Formula (4.43) has been established by the first author (cf. [10], Theorem 4) with error term $O(x^{7/5+\varepsilon})$.

COROLLARY 4.4.6 ($k=2$, $n=2$). *If the Riemann hypothesis is true, then for* $x\geqq 3$,

$$\sum_{m\leqq x}\delta(m)=\frac{2\alpha x^2}{5}+O\big(x^{7/5}\omega(x)\big). \tag{4.44}$$

THEOREM 4.5. *For* $x\geqq 3$ *and* $n\geqq 1$,

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{q_k(m)\varphi(m)}{m}=\frac{6\alpha_k n^{k+1}x}{\pi^2\psi(n)H_k(n)}+O\big(\theta(n)x^{1/k}\delta(x)\big)+O\big(\theta^2(n)\lambda(x/n)\big) \tag{4.45}$$

uniformly.

PROOF. This follows by Theorem 4.1 and partial summation.

THEOREM 4.6. *If the Riemann hypothesis is true, then for* $x\geqq 3$ *and* $n\geqq 1$,

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{q_k(m)\varphi(m)}{m}=\frac{6\alpha_k n^{k+1}x}{\pi^2\psi(n)H_k(n)}+O\big(\theta(n)x^{2/2k+1}\delta(x)\big)+O\big(\theta^2(n)\lambda(x/n)\big) \tag{4.46}$$

uniformly.

PROOF. This follows by Theorem 4.2 and partial summation.

THEOREM 4.7. *For* $x \geqq 3$ *and* $n \geqq 1$,

$$(4.47)\quad \sum_{m \leqq x} \frac{\gamma_k(m; n)}{m} = \frac{\alpha_k n^{k+1} x}{\psi(n) H_k(n)} + O\big(\theta(n) x^{1/k} \delta(x)\big) + O\big(\theta^2(n) \lambda(x/n) \log x\big)$$

uniformly.

PROOF. This follows by Theorem 4.3 and partial summation.

THEOREM 4.8. *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and* $n \geqq 1$,

$$(4.48)\quad \sum_{m \leqq x} \frac{\gamma_k(m; n)}{m} = \frac{\alpha_k n^{k+1} x}{\psi(n) H_k(n)} + O\big(\theta(n) x^{2/2k+1} \omega(x)\big) + O\big(\theta^2(n) \lambda(x/n) \log x\big)$$

uniformly.

PROOF. This follows by Theorem 4.4 and partial summation.

In conclusion, we would like to remark that asymptotic formulae for

$$\sum_{\substack{m \leqq x \\ (m, n)=1}} \frac{q_k(m) \varphi(m)}{m^2} \quad \text{and} \quad \sum_{m \leqq x} \frac{\gamma_k(m; n)}{m^2}$$

will be established in a separate paper.

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ÜBER TRANSLATIONEN UND DEN SATZ VON MENGER IN UNENDLICHEN GRAPHEN

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Terminologie und Notation

Ist E eine Menge und $R\subseteq E\times E$ eine antireflexive, antisymmetrische Relation, so heißt $G=(E, R)$ ein *gerichteter Graph*. E heißt die *Eckenmenge* von (E, R) und R heißt die Menge der *gerichteten Kanten* aus (E, R). Eine Folge $(a_i)_{i<k\leqq\omega}$ von Ecken aus E heißt *Kantenzug*, falls für jedes i mit $i+1<k$ gilt: $(a_i, a_{i+1})\in R$ oder $(a_{i+1}, a_i)\in R$. Ist $(a_i)_{i<k\leqq\omega}$ ein Kantenzug und $k=\omega$, so heißt $(a_i)_{i<k}$ ein *unendlicher Kantenzug*; ist $k<\omega$, so heißt $(a_i)_{i<k}$ *endlicher Kantenzug*. Eine Folge $b=(a_i)_{i<k\leqq\omega}$ heißt *Bahn*, falls die Ecken a_i paarweise verscheiden sind und für jedes i mit $i+1<k$ gilt: $(a_i, a_{i+1})\in R$. a_0 heißt die *Anfangsecke* der Bahn $(a_i)_{i<k\leqq\omega}$. Ist $(a_i)_{i<k}$ eine endliche Bahn, so heißt a_{k-1} die *Endecke* der Bahn $(a_i)_{i<k}$. Bei Gelegenheit fassen wir eine Ecke als einelementige Bahn auf; in diesem Sinne verstehen wir eine Menge von Ecken als eine Menge von Bahnen. Sei L eine Menge von Bahnen. Anf (L) bezeichne die Menge der Anfangsecken von Bahnen aus L und End (L) bezeichne die Menge der Endecken von Bahnen aus L. A und B seien Teilmengen der Eckenmenge E. Es heißt $\vec{b}$ eine *Bahn aus A*, falls die Anfangsecke von b die einzige Ecke von $\vec{b}$ aus A ist. Es heißt $\vec{b}$ eine *Bahn von A nach B*, falls $\vec{b}$ eine Bahn aus A ist und die Endecke von $\vec{b}$ die einzige Ecke von $\vec{b}$ aus B ist. Ist L eine Menge von paarweise eckendisjunkten Bahnen aus A, so heißt L eine *Verbindung von A*, falls Anf $(L)=A$ ist. Ist L eine Verbindung von A und ist jede Bahn aus L unendlich, so heißt L eine *Ferverbindung von A*. Sagen wir, daß L eine *Verbindung* ist, so meinen wir, daß L eine Verbindung von Anf (L) ist. Ist L eine Verbindung von A, so ist ein *Menger'scher Kantenzug bzgl.* L eine Folge $(a_i)_{i<k}$ $(0<k\leqq\omega)$ von Ecken mit folgenden Eigenschaften:

1. $(a_i, a_{i+1})\in R$ oder $(a_{i+1}, a_i)\in R$.
2. Es ist $(a_{i+1}, a_i)\in R$ genau dann, wenn a_i, a_{i+1} auf einer Bahn aus L liegen.
3. Liegt a_i auf einer Bahn $\vec{b}$ aus L, so liegt a_{i-1} oder a_{i+1} auf $\vec{b}$.
4. Ist $a_i=a_j$ mit $i\neq j$, so liegt a_i auf einer Bahn aus L, und es ist $a_{i+1}\neq a_{j+1}$.

Die folgende Skizze möge den Begriff des Menger'schen Kantenzuges illustrieren.

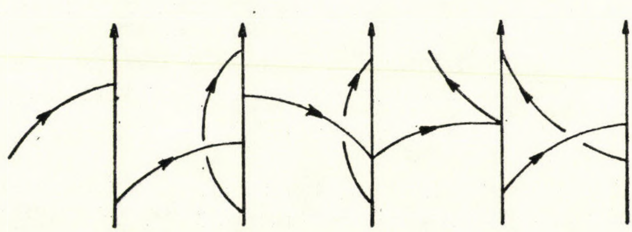


Fig. 1

Ist $\vec{b}=(a_i)_{i<k\leqq\omega}$ eine Bahn, so sei $V(\vec{b})=\{a_i|i<k\}$. Ist L eine Menge von Bahnen, so sei $V(L)=\bigcup\{V(\vec{b})|\vec{b}\in L\}$. Eine Ecke *a liegt auf einer Bahn* $\vec{b}$, falls $a\in V(\vec{b})$. Ist $\vec{b}=(a_i)_{i<k\leqq\omega}$ eine Bahn und sind $a, b\in V(\vec{b})$, so liegt *a vor b bzgl.* $\vec{b}$, falls Indizes $i, j<k$ existieren mit $i<j$, $a=a_i$ und $b=a_j$. Sind $(a_i)_{i<k}$, $(b_j)_{j<r\leqq\omega}$ zwei Kantenzüge, ist $k\neq 0$ eine natürliche Zahl und ist $a_{k-1}=b_0$, so sei

$$(a_i)_{i<k}{}^\frown(b_j)_{j<r\leqq\omega}$$

der Kantenzug $(c_l)_{l<s\leqq\omega}$ mit $c_l=a_l$ für $l<k$ und $c_l=b_{(l-k)+1}$ für $k\leqq l<s=(k+r)-1$. Es sein T, S zwei Verbindungen mit $V(T)\cap V(S)\subseteq \mathrm{Anf}(S)\cap \mathrm{End}(T)$. Dann sei

$$T^\frown S=\{\vec{t}^\frown\vec{s}|\vec{t}\in T,\ \vec{s}\in S,\ \mathrm{End}(\vec{t})=\mathrm{Anf}(\vec{s})\}\cup$$
$$\cup\{\vec{t}\in T|\mathrm{End}(\vec{t})\notin \mathrm{Anf}(S)\}\cup\{\vec{s}\in S|\mathrm{Anf}(\vec{s})\notin \mathrm{End}(T)\}.$$

Menger'sche Kantenzüge

Ist (E, R) ein gerichteter Graph und A eine abzählbare Teilmenge von E, so werden wir in dieser Arbeit ein notwendiges und hinreichendes Kriterium für die Existenz einer Fernverbindung von A angeben. Zum Beweis benötigen wir einige Lemmata.

LEMMA 1 (Klebelemma). *Sei L eine Verbindung von A und seien* $M=(a_i)_{i<k\leqq\omega}$, $N=(b_j)_{j<1\leqq\omega}$ *Menger'sche Kantenzüge bzgl. L. Angenommen, es existiert eine Bahn* $\vec{b}\in L$, *und es existieren Indizes* i_0, j_0 *mit* $a_{i_0}, a_{i_0+1}, b_{j_0}, b_{j_0+1}\in V(\vec{b})$ *und* $(a_{i_0+1}, a_{i_0})=(b_{j_0+1}, b_{j_0})$. *Dann existieren natürliche Zahlen r, s mit* $0=r<k$ *und* $s<1$ *derart daß*

$$(a_i)_{i<r}{}^\frown(b_j)_{s\leqq j<1\leqq\omega}$$

ein Menger'scher Kantenzug ist.

BEWEIS. Es sei $M_0=(a_i)_{i\leqq i_0}$ und $N_0=(b_j)_{j_0\leqq j<1}$. Tritt einer der folgenden Fälle ein, so verkürzen wir den Kantenzug $M_0{}^\frown N_0$.

1. Fall: Es existieren Ecken, die sowohl auf M_0 als auch auf N_0 liegen und die auf keiner Bahn aus L vorkommen.

Sei H die Menge dieser Ecken und sei $a_s=b_t$ die erste Ecke aus H, die man über M_0 erreicht. Dann ist

$$(a_i)_{i\leqq s}{}^\frown(b_j)_{t\leqq j<1}$$

ein Kantenzug, der keine Ecke aus H enthält.

2. *Fall:* M_0 und N_0 besitzen eine Kante gemeinsam, die auf einer Bahn aus L liegt.

Sei K die Menge dieser Kanten und sei $(a_{s+1}, a_s)=(b_{t+1}, b_t)$ die erste Kante aus K, die man über M_0 erreicht. Dann gehe man zu dem Kantenzug

$$(a_i)_{i\leqq s}{}^\frown(b_j)_{t\leqq j<1}$$

über.

Trifft auf $M_0{}^\frown N_0$ weder Fall 1 noch Fall 2 zu, so ist $M_0{}^\frown N_0$ bereits der gesuchte Menger'sche Kantenzug. Anderenfalls wird $M_0{}^\frown N_0$ nach Fallunterscheidung verkürzt.

DEFINITION. Ist L eine Verbindung von A, so sei E_L gleich A vereinigt mit der Menge aller Ecken $e\in E$, zu denen keine Verbindung K von $A\cup\{e\}$ existiert mit $\text{End}(K)\subseteq\text{End}(L)$. Also ist eine Ecke e nicht aus E_L genau dann, wenn $e\notin A$ und eine Verbindung K von $A\cup\{e\}$ existiert mit $\text{End}(K)\subseteq\text{End}(L)$.

LEMMA 2. *Ist L eine Verbindung von A und ist $e\in E_L$, so ist jeder Menger'sche Kantenzug von e aus bzgl. L endlich.*

BEWEIS. Angenommen, es gibt von e aus einen unendlichen Menger'schen Kantenzug M bzgl. L. Nach Bd 3. des Menger'schen Kantenzuges ist $e\notin A$. Wir definieren über einen Algorithmus eine Verbindung K von $A\cup\{e\}$ mit $\text{End}(K)\subseteq\text{End}(L)$. Also ist $e\notin E_L$. Widerspruch!

1. Schritt. a) Von jeder Ecke $a\in A$ laufe entlang der dort beginnenden Bahn $\vec{b}\in L$ bis ggfs. zur ersten Ecke b_a, auf der M die Bahn verläßt; sei $b_a\in L^{(1)}$.

b) Von der Ecke e laufe entlang M bis ggfs. zur ersten Ecke $b\in V(L)$, von dort laufe entlang der Bahn $\vec{b}\in L$ mit $b\in V(\vec{b})$ bis zur ggfs. ersten Ecke b_e, auf der M die Bahn $\vec{b}$ verläßt; sei $b_e\in L^{(1)}$.

$(n+1)$*-ter Schritt.* a) Von jeder Ecke $a\in L^{(n)}$ laufe entlang M bis ggfs. zur ersten Ecke $c_a\in V(L)$; sei $c_a\in M^{(n)}$.

b) Von jeder Ecke $a\in M^{(n)}$ laufe entlang der Bahn $\vec{b}\in L$ mit $a\in V(\vec{b})$ bis ggfs. zur ersten Ecke b_a, auf der M diese Bahn verläßt (eventuell $b_a=a$), sei $b_a\in L^{(n+1)}$.

Sei K die Menge der nach dieser Vorschrift gewonnenen Kantenzüge. Es ist $\text{Anf}(K)=A\cup\{e\}$. Wir beweisen induktiv, daß jeder Kantenzug aus K eine Bahn ist und daß je zwei verschiedene Bahnen aus K kreuzungsfrei sind.

Im ersten Schritt werden nur Bahnen konstruiert. Die im Schritt 1a) konstruierten Bahnen sind kreuzungsfrei. — Laufe entsprechend Schritt 1b) von der Ecke e entlang M bis zur ersten Ecke $b\in V(L)$. Es sei $\vec{b}$ diejenige Bahn aus L mit $b\in V(\vec{b})$. Laufe weiter entlang M (also längs der Bahn $\vec{b}$ entgegen der Richtung $\vec{b}$) bis zur ersten Ecke c, auf der M die Bahn $\vec{b}$ verläßt. Ist a die Anfangsecke von $\vec{b}$, so liegt b_a (Schritt 1a) vor c bzgl. $\vec{b}$. Also sind die im ersten Schritt konstruierten Bahnen kreuzungsfrei. Wir setzen daher voraus, daß die Kantenzüge, die man im n-ten Schritt erhält, kreuzungsfreie Bahnen sind.

a) Im Teil a) des $(n+1)$-ten Schrittes werden nur Bahnen durchlaufen, die auf keiner Bahn aus L liegen. Je zwei verschiedene Bahnen dieser Art sind nach der Bd. 4. eines Menger'schen Kantenzuges kreuzungsfrei. Es bleibt also nur noch zu zeigen, daß die Eigenschaft, eine Bahn zu sein, im $(n+1)$-ten Schritt, Teil a), nicht zerstört wird. Sei M' ein Teil von M, der im $(n+1)$-ten Schritt, Teil a), durchlaufen wurde und der eine Bahn $\vec{b}$ aus L in der Ecke $c_a\in V(\vec{b})$ kreuzt. Da M ein Menger'scher Kantenzug ist, läuft M auf $\vec{b}$ entgegen der Richtung von $\vec{b}$ und verläßt $\vec{b}$ an einer Ecke, die vor c_a bzgl. $\vec{b}$ liegt. Also kann nach Konstruktion c_a nicht auf solchen Bahnen liegen, die durch die ersten n-Schritte konstruiert wurden. Es wird daher die Eigenschaft, eine Bahn zu sein, im $(n+1)$-ten Schritt, Teil a), nicht zerstört.

b) Im Teil b) des $(n+1)$-ten Schrittes werden nur Bahnen durchlaufen, die auf einer Bahn aus L liegen. Aus den Eigenschaften eines Menger'schen Kantenzuges folgt, daß die Eigenschaft, eine Bahn zu sein, im $(n+1)$-ten Schritt, Teil b) nicht zerstört wird. Wir beweisen nun die Kreuzungsfreiheit der Bahnen. Sei $\vec{b}$ eine Bahn aus L, die im n-ten Schritt über M erreicht wird. Man betrachte alle Ecken auf $\vec{b}$, die in höchtens n-vielen Schritten über M erreicht werden. Sei b_1 die erste Ecke auf $\vec{b}$ dieser Art und sei b_2 die zweite Ecke auf $\vec{b}$ mit dieser Eigenschaft. Dann existiert nach Bd. 2), 4) eines Menger'schen Kantenzuges eine Ecke c, für die folgendes gilt:

1. b_1 liegt vor c bzgl. $\vec{b}$ oder $b_1=c$
2. c liegt vor b_2 bzgl. $\vec{b}$
3. M verläßt $\vec{b}$ an der Ecke c.

Alle Ecken, die auf der Bahn $\vec{b}$ zwischen b_1 und c liegen, kommen auf den Bahnen, die in den ersten n-Schritten konstruiert werden, nicht vor. Aufgrund dieser Überlegung sind die Bahnen, die im $(n+1)$-ten Schritt, Teil b), konstruiert werden, kreuzungsfrei.

LEMMA 3. *Sei L eine Verbindung von A. Ist $e\in V(L)$, so daß jeder Menger'sche Kantenzug von e aus bzg. L endlich ist, so ist $e\in E_L$.*

BEWEIS. Angenommen, $e\notin E_L$. Dann existiert eine Verbindung K von $A\cup\{e\}$ mit End $(K)\subseteq$ End (L). Nach Definition von E_L ist $e\notin A$. Wir definieren rekursiv einen unendlichen Menger'schen Kantenzug bzgl. L mit der Anfangsecke e. Sei $(x_i)_{i<t\leqq\omega}$ diejenige Bahn aus L mit $e\in V\big((x_i)_{i<t\leqq\omega}\big)$. Es sei $e=x_n$. Dann existiert eine größte Zahl $k<n$ mit x_k liegt auf einer Bahn aus K. Sei $a_i=x_{n-i}$ für $i\leqq n-k$ und sei $i_0=n-k$. Es sei $M_0^e=(a_i)_{i\leqq i_0}$. Angenommen, $M_n^e=(a_i)_{i\leqq i_n}$ ist bereits definiert, und a_{i_n} liegt auf einer Bahn $\vec{b}=(y_i)_{i<m\leqq\omega}$ aus K und auf einer Bahn $\vec{c}=(x_i)_{i<r\leqq\omega}$ aus L, und a_{i_n-1} liegt entweder auf $\vec{b}$ oder auf $\vec{c}$.

1. Fall: a_{i_n-1} liegt auf $\vec{c}$. Sei $y_1=a_{i_n}$. Dann existiert eine kleinste Zahl k mit $1<k<m$ derart, daß y_k auf einer Bahn aus L liegt. Denn nach Voraussetzung ist jeder Menger'sche Kantenzug von e aus bzgl. L endlich und nach Annahme ist End $(K)\subseteq$ End (L). Sei $a_{i_n+i}=y_{1+i}$ für $i\leqq k-1$, sei $i_{n+1}=i_n+(k-1)$ und sei $M_{n+1}^e=(a_i)_{i\leqq i_{n+1}}$.

2. Fall: a_{i_n-1} liegt auf $\vec{b}$. Sei $x_1=a_{i_n}$. Dann existiert eine größte Zahl k mit $k<1$ derart, daß x_k auf einer Bahn aus K liegt. Sei $a_{i_n+i}=x_{1-i}$ für $i\leqq 1-k$, sei $i_{n+1}=i_n+(1-k)$ und sei $M_{n+1}^e=(a_i)_{i\leqq i_{n+1}}$. Es sei $M^e=\bigcup_{n\in\omega} M_n^e$.

Behauptung. M^e ist ein Menger'scher Kantenzug bzgl. L.

Nach Konstruktion sind die Bdn. (1), (2) und (3) erfüllt. Um die Bd. (4) zu beweisen, nehmen wir an, daß Indizes i, j existieren mit $i\neq j$, $a_i=a_j$ und $a_{i+1}=a_{j+1}$. Seien die Indizes i_0, j_0 minimal gewählt bzgl. dieser Eigenschaft. Angenommen, $i_0=0$. Dann müßte, wegen $j_0\neq 0$, in die Ecke $e=a_0$ eine Bahn aus K hineingehen, d.h. es existiert eine Bahn $\vec{b}\in K$ mit $e\in V(\vec{b})$ und e ist keine Anfangsecke von $\vec{b}$. Dies ist aber nicht möglich, da genau eine Bahn $\vec{b}'\in K$ existiert, deren Anfangsecke e ist. Also ist $i_0\neq 0\neq j_0$. Wir werden nun beweisen, daß für alle $i, j\in\omega$ gilt: Ist $a_{i+1}=a_{j+1}$ und ist $a_{i+2}=a_{j+2}$, so ist $a_i=a_j$. Hieraus ergibt sich für $i=i_0-1$, $j=j_0-1$ sofort ein Widerspruch zur Minimalität von i_0 und j_0. Wir beweisen also die folgende Behauptung.

Behauptung. Für alle $i, j\in\omega$ gilt: Ist $a_{i+1}=a_{j+1}$ und ist $a_i\neq a_j$, so ist $a_{i+2}\neq a_{j+2}$.

Es sei $(a_{i_n}, \ldots, a_{i_{n+1}}]$ derjenige Teil des Kantenzuges M^e, der von a_{i_n+1} bis $a_{i_{n+1}}$ läuft, und es sei $[a_{i_{n+1}}, \ldots, a_{i_n})$ derjenige Teil von M^e, der von $a_{i_{n+1}}$ bis zur Ecke a_{i_n+1} läuft. Dann existieren Zahlen $i_n, i_m \in \omega$ mit $a_{i+1} \in (a_{i_n}, ..., a_{i_{n+1}}]$ und $a_{j+1} \in (a_{i_m}, ..., a_{i_{m+1}}]$.

1. Fall: Es existiert eine Bahn $\vec{c} \in L$, so daß $[a_{i_{n+1}}, ..., a_{i_n})$, $[a_{i_{m+1}}, ..., a_{i_m})$ Teilbahnen von $\vec{c}$ sind.

2. Fall: Es existiert eine Bahn $\vec{b} \in K$, so daß $(a_{i_n}, ..., a_{i_{n+1}}]$, $(a_{i_m}, ..., a_{i_{m+1}}]$ Teilbahnen von $\vec{b}$ sind.

Da die Elemente a_{i_n}, a_{i_m} aus $V(\vec{c})$ bzw. aus $V(\vec{b})$ sind, muß mit $a_{i+1} = a_{j+1}$ auch $a_i = a_j$ sein. Also ist für diese Fälle die Behauptung bewiesen.

3. Fall: $[a_{i_{m+1}}, ..., a_{i_m})$ ist Teilbahn einer Bahn $\vec{c} \in L$ und $(a_{i_n}, ..., a_{i_{n+1}}]$ ist Teilbahn einer Bahn $\vec{b} \in K$.

Nach Konstruktion von M ist $a_{i_{n+1}}$ die einzige Ecke aus $(a_{i_n}, ..., a_{i_{n+1}}]$, die zugleich auf $\vec{c}$ liegt. Andererseits liegt $a_{i+1} = a_{j+1}$ auf $\vec{c}$, also ist $a_{i+1} = a_{i_{n+1}}$. Sei b die Ecke, in der $\vec{b}$ die Bahn $\vec{c}$ verläßt. Es liegt a_{i+1} vor b bzgl. $\vec{c}$, oder es ist $a_{i+1} = b$, da $\vec{b}$ und $\vec{c}$ in gleicher Richtung laufen, solange sie einen Kantenzug gemeinsam verfolgen. Es liegt a_{j+1} vor a_{i_m} bzgl. $\vec{c}$. In a_{i_m} trifft eine Bahn $\vec{b}' \in K$ auf $\vec{c}$. $\vec{b}'$ und $\vec{b}$ dürfen sich nicht in a_{i_m} schneiden, denn $\vec{b}$ und $\vec{b}'$ sind kreuzungsfreie Bahnen, falls sie verschieden sind. Ist $\vec{b}$ gleich $\vec{b}'$, so können sie sich ohnehin nicht in einer Ecke kreuzen. Nach Wahl von $a_{i_{m+1}}$ liegt b vor $a_{i_{m+1}}$ bzgl $\vec{c}$ oder ist $b = a_{i_{m+1}}$. Wegen $a_{j+1} = a_{i+1} \in (a_{i_m}, ..., a_{i_{m+1}}]$ muß $b = a_{i_{m+1}} = a_{i+1} = a_{j+1} = a_{i_{n+1}}$ sein. Nach Wahl von a_{i+2} und a_{j+2} liegt dann a_{i+2} auf der Bahn $\vec{c}$, a_{j+2} auf der Bahn $\vec{b}$, und es ist $(a_{i+2}, a_{i+1}) \in R$ und $(a_{j+1}, a_{j+2}) \in R$. Da Mehrfachkanten nicht zugelassen sind, ist $a_{i+2} \neq a_{j+2}$.

Translationen

Definition. Eine Verbindung T von A heißt *Translation*, falls $V(T) \subseteqq E_T$ ist.

Beispiel.

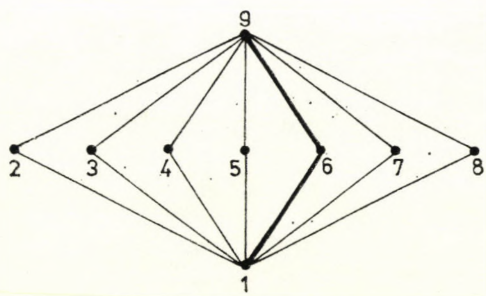


Fig. 1

Es ist $E = \{1, ..., 9\}$; die Kantenmenge ist durch Striche gekennzeichnet, jede Kante ist von unten nach oben gerichtet. Es sei $T = \{(1, 6), (6, 9)\}$, und es ist $E_T = E$.

Sind A, B zwei Teilmengen von E und ist $S \subseteqq E$, so *trennt S die Mengen A und B*, falls jede Bahn von A nach B eine Ecke aus S enthält. Sind L, K zwei Mengen von Bahnen und ist jede Bahn aus L ein Anfangsstück einer Bahn aus K, so schreiben wir $L \lesssim K$.

DEFINITION. Sind L, K Translationen von A, so *liege L vor K* (in Zeichen: $L \leqq K$) genau dann, wenn End (L) die Mengen A und End (K) trennt und $L \lesssim K$ ist.

LEMMA 4. *Ist $\mathfrak{K}$ eine $\leqq$-Kette von Translationen von A, so ist $\bigcup \mathfrak{K}$ eine Translation von A.*

BEWEIS. Sei $T = \bigcup \mathfrak{K}$. Angenommen, T ist keine Translation. Dann existiert eine Ecke e aus $V(T) \setminus A$ und eine Verbindung N von $A \cup \{e\}$ mit End $(N) \subseteqq$ $\subseteqq$ End (T). Da $e \in V(T)$ ist, existiert eine Verbindung $K \in \mathfrak{K}$ mit $e \in V(K)$.

Behauptung. Jede endliche Bahn aus N enthält eine Ecke aus End (K).

Sei $\vec{b}$ eine endliche Bahn aus N. Dann ist End $(\vec{b}) \subseteqq$ End (T). Also existiert eine Verbindung $L \in \mathfrak{K}$ mit $K \leqq L$ und End $(\vec{b}) \subseteqq$ End (L). Es trennt End (K) die Menge End (L) von A. Wegen $e \in V(K)$ trennt End (K) sogar die Mengen $A \cup \{e\}$ und End (L). Also enthält $\vec{b}$ eine Ecke aus End (K). Damit ist die Behauptung bewiesen.

Es sei N^* die Menge der Anfangsstücke von Bahnen aus N bis zur ersten Ecke aus End (K). Es ist N^* eine Verbindung von $A \cup \{e\}$ mit End $(N^*) \subseteqq$ End (K). Dies ist ein Widerspruch dazu, daß K eine Translation ist.

DEFINITION. Ist L eine Verbindung von $A, B \subseteqq E$ und $\vec{b} \in L$, so sei $\vec{b} \restriction B$ das längste Anfangsstück der Bahn $\vec{b}$, das nur Ecken aus B enthält. Es sei

$$L \restriction B = \{\vec{b} \restriction B \mid \vec{b} \in L\}.$$

LEMMA 5. *Ist L eine Verbindung von A, so ist $L \restriction E_L$ eine Translation.*

BEWEIS. Sei $T = L \restriction E_L$. Wir wollen $V(T) \subseteqq E_T$ beweisen. Es sei $e \in V(T)$. Nach Lemma 3 genügt es, zu beweisen, daß jeder Menger'sche Kantenzug von e aus bzgl. T endlich ist. Wegen $e \in V(T)$ und $T = L \restriction E_L$ ist $e \in E_L$ und folglich ist nach Lemma 2 jeder Menger'sche Kantenzug von e aus bzgl. L endlich. Also genügt es zu zeigen, daß jeder Menger'sche Kantenzug bzgl. T von e aus ein Menger'scher Kantenzug bzgl. L ist. Sei $(a_i)_{i<k\leqq\omega}$ ein Menger'scher Kantenzug bzgl. T mit $a_0 = e$. Wiederum genügt es zu zeigen, daß jede Ecke a_i, die auf einer Bahn aus L liegt, bereits aus E_L ist. Dann liegt a_i auf einer Bahn $\vec{b} \in L$ und ist $a_i \in E_L$, so ist nach dem Klebelemma und nach Lemma 3 jede Ecke aus $V(\vec{b})$ die vor a_i liegt bzgl. $\vec{b}$ auch aus E_L. Angenommen, es existiert eine Ecke a_i aus $V(L)$, die nicht aus E_L ist. Sei a_{i_0} die erste bzgl. $(a_i)_{i<k\leqq\omega}$ dieser Ecken. Dann ist $(a_i)_{i<i_0}$ ein Menger'scher Kantenzug bzgl. L. Nach Lemma 3 existiert von a_{i_0} aus ein unendlicher Menger'scher Kantenzug bzgl. L. Nach Lemma 1 existiert ein unendlicher Menger'scher Kantenzug von $a_0 = e$ aus bzgl. L. Dies steht im Widerspruch zu Lemma 2.

LEMMA 6. *Sei T eine Translation und sei L eine Verbindung von $A \supseteqq$ Anf (T). Trennt End (T) die Mengen Anf (T) und End (L), so existiert eine Verbindung K von A mit $T \lesssim K$ und End $(K) \subseteqq$ End (L).*

BEWEIS. Da End (T) die Mengen Anf (T) und End (L) trennt, liegt auf jeder endlichen Bahn $\vec{b}\in L$ mit Anf $(\vec{b})\subseteq$ Anf (T) eine Ecke aus End (T). Ist $\vec{b}\in L$, so sei $f(\vec{b})$ dasjenige Anfangsstück von $\vec{b}$, das von der Anfangsecke von $\vec{b}$ bis zur ersten Ecke aus End (T) läuft. $f(\vec{b})$ kann nach Definition eine unendliche Bahn aus L sein. Sei

$$N = \{f(\vec{b})|\vec{b}\in L\wedge \text{Anf}(\vec{b}) \subseteq \text{Anf}(T)\}.$$

Dann ist N eine Verbindung von Anf (T) mit End $(N)\subseteq$ End (T). Da T eine Translation ist, ist End $(N)=$ End (T). Daher muß für jede Bahn $\vec{b}\in L$ mit Anf $(\vec{b})\subseteq A\setminus$Anf (T) gelten: $V(\vec{b})\cap$ End $(T)=\emptyset$, d.h. $f(\vec{b})=\vec{b}$.

Behauptung 1. Ist $\vec{b}\in L$ mit Anf $(\vec{b})\subseteq A\setminus$Anf (T), so ist $V(\vec{b})\cap V(T)=\emptyset$.

Angenommen, es existiert eine Bahn $\vec{b}\in L$ mit Anf $(\vec{b})\subseteq A\setminus$Anf (T), und es existiert eine Bahn $\vec{t}\in T$ mit $V(\vec{b})\cap V(\vec{t})\neq\emptyset$. Sei $e\in V(\vec{b})\cap V(\vec{t})$ und $\vec{l}$ das Endstück von $\vec{b}$, das mit der Ecke e beginnt. Es ist $e\notin$ Anf (T) und $N\cup\{\vec{l}\}$ eine Verbindung von Anf $(T)\cup\{e\}$. Also ist T keine Translation. Widerspruch!

Es sei

$$K^* = \{\vec{c}|\exists\vec{a}\in N(\vec{a}^\frown\vec{c}\in L)\}\cup\{\vec{b}\in L|\text{Anf}(\vec{b}) \subseteq A\setminus\text{Anf}(T)\}.$$

Es ist Anf $(K^*)=(A\setminus$Anf $(T))\cup$ End (T).

Behauptung 2. $V(K^*)\cap V(T)=$ End (T).

Angenommen, es existiert eine Ecke $e\in E\setminus$End (T), es existiert eine Bahn $\vec{k}\in K^*$ und eine Bahn $\vec{t}\in T$ mit $e\in V(\vec{k})\cap V(\vec{t})$. Sei b die erste Ecke e bzgl. $\vec{t}$ mit $e\in V(\vec{k})\cap V(\vec{t})$. Nach Behauptung 1 ist genau die Anfangsecke der Bahn $\vec{k}$ aus End (T). Sei $\vec{l}$ die Bahn von b aus längs der Bahn $\vec{k}$. Dann liegt auf $\vec{l}$ keine Ecke aus End (T). Sei $\vec{s}$ die Bahn von Anf $(\vec{t})$ längs $\vec{t}$ bis zur Ecke b. Wegen $b\notin$ End (T) existiert auf $\vec{s}$ keine Ecke aus End (T). Also ist $\vec{r}=\vec{s}^\frown\vec{l}$ eine Bahn mit $V(\vec{r})\cap\cap$End$(T)=\emptyset$. Da nach Voraussetzung End(T) die Mengen Anf(T) und End(L) trennt, muß $\vec{r}$ unendlich sein. Es ist $N\cup\{\vec{l}\}$ eine Verbindung. Zum Beweis dieser Tatsache brauchen wir nur zu zeigen, daß $b\notin V(N)$ ist. Angenommen, $b\in V(N)$. Sei $\vec{c}$ diejenige Bahn aus L, die $\vec{k}$ als Endstück enthält. Dann ist $b\in V(\vec{c})$. Ist Anf$(\vec{k})=\{a\}$, so liegt wegen $b\in V(N)$ die Ecke b vor a bzgl. $\vec{c}$, oder es ist $b=a$. Andererseits liegt a vor b bzgl. $\vec{c}$, da $b\notin$ End (T) ist. Widerspruch! Folglich ist $N\cup\{\vec{l}\}$ eine Verbindung von $A\cup\{b\}$ mit $b\notin A$ und End $(N\cup\{\vec{l}\})\subseteq$ End (T). Also ist T keine Translation. Widerspruch!

Aus Behauptung 1. und 2. folgt, daß $T^\frown K^*$ eine Verbindung von A ist mit End $(T^\frown K^*)\subseteq$ End (L).

DEFINITION. Sei L eine Verbindung von A. Es sei $E(L)$ die Menge der Ecken $e\in E$, zu denen eine unendliche Bahn $(b_i)_{i<\omega}$ existiert mit $e=b_0$ und $\forall i>0$ $(b_i\notin V(L))$. Es sei $R(L)$ die Menge aller Kanten $(a, b)\in R$ mit $a, b\in E(L)$ und $b\notin V(L)$. Dann sei $G(L)=(E(L), R(L))$. Ist $a\in A$ und $(a, e)\in R$, so sei $L(a, e)=\{\{(0, b)\}|b\in A\setminus\{a\}\}\cup\{(0, a), (1, b)\}$ eine Verbindung von A. Es sei

$$G(a, e) = G(L(a, e)).$$

LEMMA 7. *Sei L eine Verbindung. Ist $\vec{b}$ eine Bahn in G mit* End $(\vec{b})\subseteq E(L)$, *die höchstens die Anfangsecke mit einer Ecke aus $V(L)$ gemein hat, so verläuft $\vec{b}$ ganz in $G(L)$.*

LEMMA 8. *Ist T eine Translation, so trennt* $\text{End}(T)\cap E(T)$ *die Mengen* $\text{Anf}(T)$ *und* $E(T)$ *in* G.

BEWEIS. Angenommen, es existiert eine Ecke $e\in E(T)$ und eine Bahn $\vec{b}$ aus $\text{Anf}(T)$ nach e, die keine Ecke aus $\text{End}(T)$ enthält. Da $e\in E(T)$ ist, existiert eine unendliche Bahn $\vec{c}$ in G von e ausgehend mit der Eigenschaft, daß höchstens die Anfangsecke von $\vec{c}$ eine Ecke aus T ist. Laufe von e aus die Bahn $\vec{b}$ zurück bis zum ersten Schnittpunkt b mit einer Bahn $\vec{d}$ aus T. Wie im Beweis zum Klebelemma zeigt man die Existenz einer unendlichen Bahn $\vec{l}$ von b aus, die als Ecke aus $V(T)$ nur die Ecke b enthält. Es ist $b\notin \text{End}(T)$, weil auf der Bahn $\vec{b}$ keine Ecke aus $\text{End}(T)$ liegt. Wegen $b\notin\text{End}(T)$ existiert ein unmittelbarer Nachfolger d von b bzgl. $\vec{d}$. Sei $\vec{d}_1$ die Bahn von $\text{Anf}(\vec{d})$ bis b längs $\vec{d}$ und $\vec{d}_2$ die Bahn von d längs $\vec{d}$. Es ist

$$L=(T-\{\vec{d}\})\cup\{\vec{d}_1{}^\frown\vec{l},\,\vec{d}_2\}$$

eine Verbindung von $\text{Anf}(T)\cup\{d\}$ mit $\text{End}(L)\subseteqq\text{End}(T)$ und $d\notin\text{Anf}(T)$. Folglich ist T keine Translation. Widerspruch!

Wir haben damit bewiesen, daß $\text{End}(T)$ die Mengen $\text{Anf}(T)$ und $E(T)$ trennt. Sei $\vec{b}$ eine Bahn von $\text{Anf}(T)$ bis $E(T)$ und a die letzte Ecke aus $V(\vec{b})\cap\cap\text{End}(T)$ bzgl. $\vec{b}$. Ist $\vec{k}$ die Bahn von a längs $\vec{b}$, so ist nach Lemma 7 $\vec{k}$ eine Bahn in $G(T)$. Insbesondere ist $a\in E(T)$. Also trennt sogar $\text{End}(T)\cap E(T)$ die Mengen $\text{Anf}(T)$ und $E(T)$.

LEMMA 9. *Sei T eine Translation in G und sei L eine Translation in $G(T)$ mit* $\text{Anf}(L)\subseteqq\text{End}(T)$. *Dann ist $T^\frown L$ eine Translation in G.*

BEWEIS. Sei $K=T^\frown L$. Angenommen, K ist keine Translation in G. Dann existiert eine Ecke $a\in V(K)$ mit $a\notin\text{Anf}(T)$ und es existiert eine Verbindung N von $\text{Anf}(T)\cup\{a\}$ mit $\text{End}(N)\subseteqq\text{End}(K)=\text{End}(L)\cup(\text{End}(T)\setminus\text{Anf}(L))$. Nach Lemma 8 trennt $\text{End}(T)$ in G die Mengen $\text{Anf}(T)$ und $\text{End}(L)\cup(\text{End}(T)-{}-\text{Anf}(L))$, also auch $\text{Anf}(T)$ und $\text{End}(N)$. Ist $\vec{b}\in N$, so sei $f(\vec{b})$ dasjenige Anfangsstück von $\vec{b}$, das von der Anfangsecke von $\vec{b}$ bis zur ersten Ecke aus $\text{End}(T)$ läuft. Sei

$$N^*=\{f(\vec{b})|\vec{b}\in N,\ \text{Anf}(\vec{b})\subseteqq\text{Anf}(T)\}.$$

Wie im Beweis zu Lemma 6 zeigt man, daß $\text{End}(N^*)=\text{End}(T)$ ist. Es sei

$$K^*=\{\vec{c}|\exists\vec{a}\in N^*(\vec{a}^\frown\vec{c}\in N\}\cup\{\vec{c}\in N|a\in V(\vec{c})\}.$$

Wie im Beweis zu Lemma 6 zeigt man, daß $T^\frown K^*$ eine Verbindung von $\text{Anf}(T)\cup\cup\{a\}$ ist mit $\text{End}(T^\frown K^*)\subseteqq\text{End}(N)\subseteqq\text{End}(L)\cup(\text{End}(T)\setminus\text{Anf}(L))$. Es sei $R^*=\{\vec{c}\in K^*|\text{Anf}(\vec{c})\subseteqq\text{Anf}(L)\}\cup\{\vec{c}\in N|a\in V(\vec{c})\}$. Dann ist R^* eine Verbindung von $\text{Anf}(L)\cup\{a\}$ mit $\text{End}(R^*)\subseteqq\text{End}(L)\subseteqq E(T)$. Da $T^\frown K^*$ eine Verbindung ist und wegen $\text{End}(R^*)\subseteqq E(T)$ ist nach Lemma 7 R^* eine Verbindung von $\text{Anf}(L)\cup\cup\{a\}$ in $G(T)$. Angenommen, $a\in V(T)$. Nach Lemma 8 ist $a\in\text{End}(T)$. Wegen $\text{End}(T)=\text{End}(N^*)$ existiert eine Bahn $\vec{b}\in N$ mit $a\in V(\vec{b})$ und $\text{Anf}(\vec{b})\subseteqq\text{Anf}(T)$. Wegen $a\notin\text{Anf}(T)$ steht dies im Widerspruch dazu, daß N eine Verbindung von $\text{Anf}(T)\cup\{a\}$ ist. Also ist $a\notin V(T)$. Daher ist $a\in V(L)$. Wegen $\text{Anf}(L)\subseteqq\text{End}(T)$ ist $a\notin\text{Anf}(L)$. Folglich ist $a\notin E_L$ in $G(T)$ und somit ist L keine Translation in $G(T)$. Widerspruch!

Lokal-verbindbar

DEFINITION. Eine Menge $A\subseteq E$ heißt *lokal-verbindbar*, falls zu jeder Verbindung L von A und zu jeder Ecke $a\in \mathrm{End}\,(L)$ eine Verbindung K von A existiert mit $\mathrm{End}\,(K)\subseteq \mathrm{End}\,(L)\setminus\{a\}$.

LEMMA 10. *Sei $A\subseteq E$. Dann sind folgende Aussagen äquivalent:*

(1) *A ist lokal-verbindbar.*

(2) *Zu jeder Translation T von A und zu jeder Ecke $a\in \mathrm{End}\,(T)$ existiert eine Verbindung K mit $T\lesssim K$ und a liegt auf einer unendlichen Bahn aus K.*

(3) *Für jede Translation T von A ist $\mathrm{End}\,(T)\subseteq E(T)$.*

(4) *Zu jeder Translation T von A und zu jeder Ecke $a\in \mathrm{End}\,(T)$ existiert eine Verbindung K von A mit $\mathrm{End}\,(K)\subseteq \mathrm{End}\,(T)\setminus\{a\}$.*

BEWEIS. (4)$\Rightarrow$(2). Sei T eine Translation von A und $a\in \mathrm{End}\,(T)$. Nach (4) existiert eine Verbindung L von A mit $\mathrm{End}\,(L)\subseteq \mathrm{End}\,(T)\setminus\{a\}$. Natürlich trennt $\mathrm{End}\,(T)$ die Mengen $\mathrm{Anf}\,(T)$ und $\mathrm{End}\,(L)$. Nach Lemma 6 existiert eine Verbindung K von A mit $T\lesssim K$ und $\mathrm{End}\,(K)\subseteq \mathrm{End}\,(L)\subseteq \mathrm{End}\,(T)\setminus\{a\}$. $\mathrm{End}\,(T)$ trennt die Mengen $\mathrm{Anf}\,(T)$ und $\mathrm{End}\,(K)$. Sei für $\vec{b}\in K$ die Bahn $f(\vec{b})$ das Anfangsstück von $\vec{b}$ bis zur ersten Ecke aus $\mathrm{End}\,(T)$. Dann ist $K^*=\{f(\vec{b})|\vec{b}\in K\}$ eine Verbindung von $\mathrm{Anf}\,(T)$ mit $\mathrm{End}\,(K^*)\subseteq \mathrm{End}\,(T)$. Da T eine Translation ist, muß $\mathrm{End}\,(K^*)=\mathrm{End}\,(T)$ sein. Also liegt a auf einer unendlichen Bahn aus K.

(2)$\Rightarrow$(3). Sei T eine Translation von A und $a\in \mathrm{End}\,(T)$. Nach (2) existiert eine Verbindung K von A mit $T\lesssim K$ und a liegt auf einer unendlichen Bahn $\vec{b}$ aus K. Sei $\vec{c}$ die Bahn von a längs $\vec{b}$. Dann ist $V(\vec{c})\cap V(T)=\{a\}$. Also ist $a\in E(T)$.

(3)$\Rightarrow$(4) Trivial.

(1)$\Rightarrow$(4). Trivial.

(4)$\Rightarrow$(1). Sei L eine Verbindung von A und $a\in \mathrm{End}\,(L)$. Ist $a\notin E_L$, so existiert eine Verbindung N von $A\cup\{a\}$ mit $\mathrm{End}\,(N)\subseteq \mathrm{End}\,(L)$. Es ist $K=\{\vec{b}\in N|a\notin V(\vec{b})\}$ eine Verbindung von A mit $\mathrm{End}\,(K)\subseteq \mathrm{End}\,(L)\setminus\{a\}$. Betrachten wir daher den Fall $a\in E_L$. Sei $T=L\restriction E_L$.

Behauptung. $a\in V(T)$. Ist $a\in A$, so ist $a\in V(T)$. Sei $a\notin A$, sei $\vec{b}\in L$ mit $a\in V(\vec{b})$ und sei c eine Ecke vor a bzgl. $\vec{b}$. Nach dem Klebelemma ist jeder Menger'sche Kantenzug von c aus bzgl. L endlich. Nach Lemma 3 ist $c\in E_L$. Also ist $a\in V(\vec{b}\restriction E_L)$. Damit ist die Behauptung bewiesen.

Da wir Bedingung (4) voraussetzen und (4) äquivalent zu (2) ist, existiert eine unendliche Bahn $\vec{d}$ mit der Anfangsecke a und $V(\vec{d})\cap V(T)=\{a\}$.

1. Fall: $V(\vec{d})\cap V(L)=\{a\}$. Ist $\vec{b}\in L$ mit $a\in V(\vec{b})$, so ist $K=(L\setminus\{b\})\cup\{\vec{b}\frown\vec{d}\}$ eine Verbindung von A mit $\mathrm{End}\,(K)\subseteq \mathrm{End}\,(L)\setminus\{a\}$.

2. Fall: $\{a\}\subsetneqq V(\vec{d})\cap V(L)$. Sei c die erste Schnittecke von $\vec{d}$ mit einer Bahn aus L. Da c keine Ecke aus E_L ist, existiert nach Lemma 3 von c aus ein unendlicher Menger'scher Kantenzug M bzgl. L. Angenommen, M hat mit einer Bahn aus T eine gemeinsame Ecke s, so daß die in M folgende Ecke ebenfalls aus T ist. Dann ist $s\in E_L$. Andererseits ist das Endstück von M mit der Anfangsecke s ein unendlicher Menger'scher Kantenzug bzgl. L. Nach Lemma 2 ist $s\notin E_L$. Widerspruch! Also hat M höchstens mit $\mathrm{End}\,(T)$ Ecken gemeinsam. Die Ecke a liegt nicht auf M. Ist (a) die Bahn, die aus der Ecke a besteht, so sei

$$N=\{\vec{b}\,|\,\exists\vec{t}\in T(\vec{t}\frown\vec{b}\in L)\}\setminus\{(a)\}.$$

Auf Grund der obigen Überlegung ist M ein Menger'scher Kantenzug bzgl. N. Es ist $a\notin\mathrm{Anf}\,(N)$. Wie im Beweis zu Lemma 2 definiert man eine Verbindung K^* von $\mathrm{Anf}\,(N)\cup\{a\}=\mathrm{End}\,(T)$ mit $\mathrm{End}\,(K^*)\subseteq\mathrm{End}\,(L)\setminus\{a\}$. Dann ist

$$K=\{\vec{t}\,|\,\vec{t}\in T \text{ unendlich}\}\cup\{\vec{t}\frown\vec{k}\,|\,\vec{t}\in T,\ \vec{k}\in K^*,\ \mathrm{End}\,(\vec{t})=\mathrm{Anf}\,(\vec{k})\}$$

eine Verbindung von A mit $\mathrm{End}\,(K)=\mathrm{End}\,(K^*)\subseteq\mathrm{End}\,(L)\setminus\{a\}$.

LEMMA 11. *Sei* $G=(E,R)$ *ein gerichteter Graph,* $A\subseteq E$, $A\subseteq E(A)$ *und sei* $E_L=A$ *für jede Verbindung* L *von* A. *Es sei* $e\in E\setminus A$, $a\in A$ *und* $(a,e)\in R$. *Dann ist* $(A\setminus\{a\})\cup\{e\}$ *in* $G(a,e)$ *lokal-verbindbar.*

BEWEIS. Sei T eine Translation von $(A\setminus\{a\})\cup\{e\}$ in $G(a,e)$ und sei $b\in\mathrm{End}\,(T)$.

Zu zeigen: Es gibt eine Verbindung K in $G(a,e)$ von $(A\setminus\{a\})\cup\{e\}$ mit $\mathrm{End}\,(K)\subseteq\mathrm{End}\,(T)\setminus\{b\}$.

Wir setzen $R=L(a,e)\frown T$. Es ist R eine Verbindung von A mit $\mathrm{End}\,(R)=\mathrm{End}\,(T)$. Nach Voraussetzung ist $A\subseteq E(A)$ und A ist die einzige Translation von A in G. Nach Lemma 10 (3) ist A lokal-verbindbar in G. Nach Definition von lokal-verbindbar existiert zu R eine Verbindung N von A mit $\mathrm{End}\,(N)\subseteq\mathrm{End}\,(T)\setminus\{b\}$. Wegen $E_N=A$ ist $e\notin E_N$. Also existiert eine Verbindung K^* von $A\cup\{e\}$ mit $\mathrm{End}\,(K^*)\subseteq\mathrm{End}\,(N)\subseteq\mathrm{End}\,(T)\setminus\{b\}$. Sei $\vec{b}_a\in K^*$ mit $a\in\mathrm{Anf}\,(\vec{b}_a)$. Es sei

$$K=K^*\setminus\{\vec{b}_a\}.$$

Behauptung. K ist eine Verbindung von $(A\setminus\{a\})\cup\{e\}$ in $G(a,e)$.

Wir sind fertig, wenn wir beweisen, daß von jeder Ecke, die auf einer endlichen Bahn aus K liegt, ein unendlicher Weg in $G(a,e)$ ausgeht. Sei $\vec{b}\in K$ eine endliche Bahn und $c\in V(\vec{b})$. Es ist $\mathrm{End}\,(\vec{b})\subseteq\mathrm{End}\,(T)\subseteq G(a,e)$. Also ist die Endecke von $\vec{b}$ die Anfangsecke einer unendlichen Bahn in $G(a,e)$. Analog zum Beweis des Klebelemmas existiert in $G(a,e)$ eine unendliche Bahn mit der Anfangsecke c. Es ist $\mathrm{End}\,(K)\subseteq\mathrm{End}\,(T)\setminus\{b\}$.

LEMMA 12. *Sei T eine $\leqq$-maximale Translation. Dann gilt in dem Graphen $G(T)$: Es ist* $E_L=\mathrm{Anf}\,(L)$ *für jede Verbindung L mit* $\mathrm{Anf}\,(L)\subseteq\mathrm{End}\,(T)$.

BEWEIS. Angenommen, in $G(T)$ existiert eine Verbindung L mit $\mathrm{Anf}\,(L)\subsetneqq E_L$ und $\mathrm{Anf}(L)\subseteq\mathrm{End}\,(T)$.

Behauptung. $(E_L\cap V(L))\setminus\mathrm{Anf}\,(L)\neq\emptyset$.

Sei $e\in E_L\setminus\mathrm{Anf}\,(L)$. Ist $e\in V(L)$, so ist die Behauptung bewiesen. Nehmen wir daher $e\notin V(L)$ an. Wegen $e\in E(T)$ existiert in $G(T)$ eine unendliche Bahn $\vec{b}$ von e aus mit $(V(T)\cap V(\vec{b}))\setminus\{e\}=\emptyset$. Angenommen, $V(b)\cap V(L)=\emptyset$. Dann ist $\{\vec{b}\}\cup L$ eine Verbindung in $G(T)$ mit $\mathrm{End}\,(\{\vec{b}\}\cup L)\subseteq\mathrm{End}\,(L)$. Also ist $e\notin E_L$. Widerspruch!

Daher ist $V(\vec{b})\cap V(L)\neq\emptyset$. Es sei a die erste Ecke auf $\vec{b}$, die auf einer Bahn aus L liegt. Wegen $\mathrm{Anf}\,(L)\subseteq\mathrm{End}\,(T)$ ist $a\notin\mathrm{Anf}\,(L)$. Angenommen, $a\notin E_L$. Dann existiert in $G(T)$ eine Verbindung N von $\mathrm{Anf}\,(L)\cup\{a\}$ mit $\mathrm{End}\,(N)\subseteq\mathrm{End}\,(L)$. Sei $\vec{d}\in N$ mit $a\in\mathrm{Anf}\,(\vec{d})$ und sei $\vec{c}$ die Bahn längs $\vec{b}$ von der Ecke e bis zur Ecke a. Wie im Beweis zum Klebelemma erhält man aus den Bahnen $\vec{d}$, $\vec{c}$ in $G(T)$ eine Bahn $\vec{k}$ von e aus mit $\mathrm{End}\,(\vec{k})=\mathrm{End}\,(\vec{d})$. Sei $N^*=(N\setminus\{\vec{d}\})\cup\{\vec{k}\}$. Dann ist in $G(T)$ N^* eine Verbindung von $\mathrm{Anf}\,(L)\cup\{e\}$ mit $\mathrm{End}\,(N^*)=\mathrm{End}\,(N)\subseteq\mathrm{End}\,(L)$. Also ist $e\notin E_L$. Widerspruch! Also ist $a\in(E_L\cap V(L))\setminus\mathrm{Anf}\,(L)$.

Es sei $K=L\upharpoonright E_L$. Aus obiger Behauptung folgt, daß $V(K)\neq\mathrm{Anf}\,(L)$ ist. Nach Lemma 5 ist K eine Translation von $\mathrm{Anf}\,(L)$ in $G(T)$. $T^\frown K$ ist nach Lemma 9 eine Translation in G. Nach Lemma 8 trennt $\mathrm{End}\,(T)$ die Mengen $\mathrm{Anf}\,(T)$ und $E(T)$ in G. Also trennt $\mathrm{End}\,(T)$ die Mengen $\mathrm{Anf}\,(T)$ und $\mathrm{End}\,(T^\frown K)$. Daher ist $T\leqq(T^\frown K)$ und $T\neq(T^\frown K)$. Also ist T nicht $\leqq$-maximal. Widerspruch!

KOROLLAR 13. *Ist T eine $\leqq$-maximale Translation in G und ist $A\subseteq\mathrm{End}\,(T)\cap\cap E(T)$, so ist A lokal-verbindbar in $G(T)$.*

BEWEIS. Da die Translation T maximal ist, ist nach Lemma 12 A die einzige Translation von A in $G(T)$. Die Behauptung folgt nun unmittelbar aus Lemma 10 (3).

SATZ 14. *Ist $G=(E, R)$ ein gerichteter Graph und A eine abzählbare Teilmenge von E, so besitzt A genau dann eine Fernverbindung, wenn A lokal-verbindbar ist.*

BEWEIS. Besitzt A eine Fernverbindung, so ist A lokal-verbindbar. Um die Umkehrung zu beweisen, nehmen wir an, daß A lokal-verbindbar ist. Sei das Element $a^*\in A$ fest gewählt. Es sei $(i_n, k_n)_{n\in\omega}$ eine Aufzählung aller Paare von natürlichen Zahlen, wobei $i_n\leqq n$ sein soll. Wir definieren rekursiv eine Folge $(G_n)_{n\in\omega}$ von Graphen, eine Folge $(K_n)_{n\in\omega}$ von Verbindungen von A und eine Folge $((a_{n,k})_{k\in\omega})_{n\in\omega}$ von Folgen von Elementen aus E mit den folgenden Eigenschaften:

1. $K_0=A$ und $K_n\lessapprox K_{n+1}$.
2. G_{n+1} ist ein Teilgraph von $G_n=(E_n, R_n)$.
3. $\mathrm{End}\,(K_n)\subseteq E_n$.
4. $\mathrm{End}\,(K_n)$ ist lokal-verbindbar in G_n.
5. $(a_{n,k})_{k\in\omega}$ ist eine Aufzählung von $\mathrm{End}\,(K_n)\cup\{a^*\}$.
6. Ist $(a, b)\in R_n$, so ist $b\notin V(K_n)$ für $n>0$.

Sei $G_0=G$, $K_0=A$ und $(a_{0,k})_{k\in\omega}$ eine Aufzählung von A. Seien G_n, K_n und $(a_{n,k})_{k\in\omega}$ mit obigen sechs Eigenschaften bereits definiert. Sei T_n eine $\leqq$-maximale Translation von $\mathrm{End}\,(K_n)$ in G_n. Nach Bedingung (6) ist $L_n=K_n{}^\frown T_n$ eine Verbindung von A.

1. Fall: $a_{i_n,k_n}\notin\mathrm{End}\,(L_n)$. Dann sei $G_{n+1}=G_n(T_n)$, $K_{n+1}=L_n$ und $(a_{n+1,k})_{k\in\omega}$ eine Aufzählung von $\mathrm{End}\,(K_{n+1})\cup\{a^*\}$.

2. Fall: $a_{i_n,k_n}\in\mathrm{End}\,(L_n)$. Also ist $a_{i_n,k_n}\in\mathrm{End}\,(T_n)$. Da nach Bedingung 4) $\mathrm{End}\,(K_n)$ lokal-verbindbar in G_n ist, muß nach Lemma 10 (3) $\mathrm{End}\,(T_n)\subseteq E_n(T_n)$ in G_n sein. Nach Lemma 12 gilt in $G_n(T_n)$: Für jede Verbindung L von $\mathrm{End}\,(T_n)$ ist $E_L=\mathrm{End}\,(T_n)$. Sei $e\in E_n(T_n)\setminus\mathrm{End}\,(T_n)$ mit $(a_{i_n,k_n}, e)\in R_n(T_n)$. Sei $K_{n+1}=L_n{}^\frown\{(a_{i_n,k_n}, e)\}$ und $G_{n+1}=G_n(T_n)\,(a_{i_n,k_n}, e)$. Es sei $(a_{n+1,k})_{k\in\omega}$ eine Aufzählung von $\mathrm{End}\,(K_{n+1})\cup\{a^*\}$. Nach Lemma 11 gilt Bedingung 4). Nach Definition sind die Bedingungen 1), 2), 3), 5) und 6) erfüllt.

Sei $K=\bigcup_{n\in\omega} K_n$. Aufgrund von Bedingung 1) und 2) ist K eine Verbindung von A. Ist $a\in\operatorname{End}(K)$, so existiert eine natürliche Zahl n mit $a\in\operatorname{End}(K_n)$. Es existiert ein $k\in\omega$ mit $a=a_{n,k}$, und es existiert eine Zahl $m\in\omega$ mit $n=i_m$ und $k=k_m$. Nach Konstruktion ist $a=a_{i_m,k_m}\notin\operatorname{End}(K_{m+1})$ und daher ist $a\notin\operatorname{End}(K)$. Widerspruch! Der folgende Satz ist eine Verallgemeinerung des Bernsteinschen Äquivalenzsatzes auf Fernverbindungen.

SATZ 15. *Sei $G=(E,R)$ ein Graph, $A\subseteqq E$ und seien L, N Fernverbindungen mit* $\operatorname{Anf}(L)\subseteqq A$ *und* $A\subseteqq\operatorname{Anf}(N)$. *Dann existiert eine Fernverbindung K mit* $\operatorname{Anf}(K)=A$, *so daß jede Bahn aus L unendlich viele Ecken mit Bahnen aus K gemeinsam hat.*

BEWEIS. Es sei $E^*=V(L)\cup V(N)$ und es sei $(a,b)\in R^*$ genau dann, wenn eine Bahn $\vec{b}=(a_i)_{i<\omega}\in(L\cup N)$ und ein Index $k<\omega$ existiert mit $a=a_k$ und $b=a_{k+1}$. Der Teilgraph (E^*, R^*) von (E, R) besitzt nur abzählbare Zusammenhangskomponenten. O.B.d.A. sei (E^*, R^*) zusammenhängend. Dann ist A abzählbar. Da eine Fernverbindung N mit $A\subseteqq\operatorname{Anf}(N)$ existiert, ist A lokal-verbindbar in (E^*, R^*). Die Konstruktion von K erfolgt in (E^*, R^*) wie in Satz 14, nur muß man im 2. Fall folgendes zusätzlich beachten: Liegt a_{i_n,k_n} auf einer Bahn $\vec{b}$ aus L und ist der unmittelbare Nachfolger d von a_{i_n,k_n} bzgl. $\vec{b}$ aus $E_n(T_n)$ in G_n, dann sei $e=d$ gesetzt (d.h. man wähle d aus). Sonst aber seien G_n, K_n, L_n und $(a_{n,k})_{k\in\omega}$ wie in Satz 14 ausgewählt.

Um einzusehen, daß die Konstruktion von K zum Ziele führt, sei $(b_i)_{i\in\omega}$ eine unendliche Bahn aus L. Angenommen, es gibt eine größte Zahl $j\in\omega$ mit $b_j\in V(K)$. Dann ist $(b_i)_{i\geqq j}$ eine unendliche Bahn in jedem Graphen G_n, die keine Bahn aus L_n schneidet. Es existiert eine natürliche Zahl n derart, daß T_n eine $\leqq$-maximale Translation von $\operatorname{End}(K_n)$ in G_n ist, mit $b_j\in V(T_n)$ und $L_n=K_n\frown T_n$.

1. Fall: $b_j\notin\operatorname{End}(T_n)$. Dann ist T_n keine Translation in G_n.

2. Fall: $b_j\in\operatorname{End}(T_n)$. Aufgrund der Maximalität von T_n geht durch die Ecke b_j eine Bahn $\vec{c}$ aus N, so daß der unmittelbare Nachfolger von b_j bzgl. $\vec{c}$ verschieden von b_{j+1} ist. Nach Fall 1 gilt für jede Zahl $k\geqq n$: Ist $b_j\in V(T_k)$, so ist $b_j\in\operatorname{End}(T_k)$. Nach Fall 2 gilt für jede Zahl $k>n$: Ist $b_j\in V(T_k)$, so ist $b_j\in\operatorname{End}(T_k)\cap\operatorname{Anf}(T_k)$. Also muß nach Konstruktion von K die Ecke b_{j+1} aus $V(K)$ sein.

Beide Fälle führen zu einem Widerspruch. Daher existiert keine größte Zahl $j\in\omega$ mit $b_j\in V(K)$.

Verbindungen mit einer Menge

Neben den Fernverbindungen von einer Menge A kann man auch Verbindungen von A mit einer Menge B betrachten.

DEFINITION. Eine Verbindung L heißt *Verbindung in Richtung B,* wenn $V(L)\cap B\subseteqq\operatorname{End}(L)$ ist. Eine Verbindung L heißt *Verbindung mit B,* wenn $V(L)\cap B=\operatorname{End}(L)$ ist.

Wie man aus der Definition ersieht, kann eine Verbindung mit B auch unendliche Bahnen enthalten. Dies ist unschön, doch wissen wir nicht, ob die folgenden Sätze und Lemmata bewiesen werden können, ohne unendliche Bahnen zuzulassen.

DEFINITION. Sei $G=(E, R)$ ein gerichteter Graph und seien $A, B\subseteq E$. A heißt *lokal mit B verbindbar,* wenn es zu jeder Verbindung L von A in Richtung B und zu jedem $a\in \mathrm{End}\,(L)$ eine Verbindung K von A in Richtung B gibt mit $\mathrm{End}\,(K)\setminus B\subseteq \subseteq \mathrm{End}\,(L)\setminus(B\cup\{a\})$.

Wir wollen beweisen, daß für abzählbare Graphen eine Verbindung von A mit B existiert, falls A mit B lokal-verbindbar ist. Hierzu definieren wir einen Graphen $G|B=(E|B, R|B)$ wie folgt: Ist $b\in B$, so sei

$$E_b = \{e_{bni} | n\in\omega,\ i\in\{0, 1\}\}$$

eine Menge von paarweise verschiedenen, „neuen" Ecken (nicht aus E) mit der einzigen Ausnahme: $e_{b00}=b=e_{b01}$. Weiter sei

$$E_{b_1}\cap E_{b_2} = \emptyset$$

für verschiedene Ecken b_1, b_2 aus B. Es sei

$$E|B = E\cup \bigcup_{b\in B} E_b,$$

$$R|B = \big(R\setminus\{(b, a)\in R | b\in B\}\big)\cup\big\{(e_{bni}, e_{b(n+1)i}) | b\in B,\ n\in\omega,\ i\in\{0, 1\}\big\}.$$

Hierdurch sei der Graph $G|B=(E|B, R|B)$ definiert. Wie unmittelbar aus der Definition von $G|B$ folgt, ist A genau dann mit B lokal verbindbar in G, wenn A in $G|B$ lokal verbindbar ist, und in G existiert eine Verbindung A mit B genau dann, wenn in $G|B$ eine Fernverbindung von A existiert. Also ergibt sich aus Satz 14:

KOROLLAR 16. *Ist $G=(E, R)$ ein gerichteter Graph, ist $B\subseteq E$ und ist A eine abzählbare Teilmenge von E, so besitzt A genau dann eine Verbindung mit B, wenn A lokal mit B verbindbar ist.*

Ganz entsprechend läßt sich auch Satz 15 übertragen. Ein ähnliches Resultat wurde in [1] bewiesen.

KOROLLAR 17. *Sei $G=(E, R)$ ein gerichteter Graph, seien $A, B\subseteq E$ und seien L, N Verbindungen mit B für die gilt:* $\mathrm{Anf}\,(N)=A$, $\mathrm{Anf}\,(L)\subseteq A$ *und* $\mathrm{End}\,(L)=B$. *Dann gibt es eine Verbindung K von A mit B, so daß* $\mathrm{End}\,(K)=B$ *ist.*

BEWEIS. Sei $L^*=L^\frown\{(e_{bn0})_{n\in\omega} | b\in B\}$ und sei $N^*=N^\frown\{(e_{bn1})_{n\in\omega} | b\in B\}$. Dann gibt es nach Satz 15 eine Fernverbindung K^* von A in $G|B$ mit der Eigenschaft, daß jede Bahn aus L^* unendlich viele Ecken mit Bahnen aus K^* gemeinsam hat. Daher ist $K=K^*\restriction E$ eine Verbindung von A mit B und $\mathrm{End}\,(K)=B$.

Der folgende Satz ist eine Verallgemeinerung des Satz von Menger für endliche Graphen.

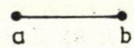
SATZ 18. *Sei $G=(E, R)$ ein gerichteter Graph, seien A, B Teilmengen von E und sei A abzählbar. Dann existiert eine A und B trennende Eckenmenge S und eine Verbindung L mit B folgenden Eigenschaften:*

1. $\mathrm{Anf}\,(L)\subseteq A$.
2. *An jeder Bahn aus L liegt genau eine Ecke aus S.*
3. *Jede Ecke aus S liegt auf einer Bahn aus L.*

Beweis. Sei T eine $\leqq$-maximale Translation von A in $G|B$. Dann ist nach Konstruktion von $G|B$ die Translation T eine Verbindung von A in Richtung B in G. Sei $S=\mathrm{End}\,(T)\cap(E|B)\,(T)$. Da $B\subseteq(E|B)\,(T)$ ist, trennt S, nach Lemma 8, die Eckenmenge A und B in $G|B$ und nach Konstruktion von $G|B$ dann auch in G. Bleibt noch die Verbindung L zu konstruieren. Da T $\leqq$-maximal in $G|B$ ist, ist nach Korollar 13 die Menge S lokal verbindbar in $(G|B)\,(T)$. Da nach Voraussetzung A und damit auch $\mathrm{End}\,(T)\cap(E|B)\,(T)=S$ abzählbar ist, existiert nach Satz 14 eine Fernverbindung K von S in $G|B$. Sei $K'=K\restriction E$ und K'' die Menge aller Bahnen aus K', deren Anfangsecke aus S ist. Es sei T' die Menge aller endlichen Bahnen aus T, deren Endecken aus S sind. Dann haben $L=K''{}^\frown T'$ und S die gewünschten Eigenschaften.

P. Erdős fragt: Ist es wahr, daß zu jedem Graphen (E, K) und zu je zwei disjunkten Teilmengen A, B von E eine A und B trennende Eckenmenge $S\subseteq E$ existiert und eine Menge $\mathfrak{W}$ von paarweise disjunkten Wegen von A nach B existiert, so daß jede Ecke $s\in S$ auf einem Weg aus $\mathfrak{W}$ liegt und jeder Weg aus $\mathfrak{W}$ genau ein Element aus S enthält.

Alle in dieser Arbeit beschriebenen Ergebnisse lassen sich auch auf ungerichtete Graphen übertragen, indem man für jede ungerichtete Kante



die Figur

einsetzt.

Satz 18 zeigt, daß die Frage von Erdős positiv beantwortet ist für abzählbare Graphen ohne unendliche Wege.

Definition. Ist $G=(E, R)$ ein gerichteter Graph, sind A, B Teilmengen von E und ist $\vec{b}=(e_i)_{i<k}$ eine Bahn von A nach B, so sei $l(\vec{b})=k$ die *Länge der Bahn* $\vec{b}$. $\vec{b}$ ist ein *kürzester Weg von A nach B,* falls

$$l(\vec{b}) = \min\{l(\vec{d})\,|\,\vec{d} \text{ ist Weg von } A \text{ nach } B \text{ und } \mathrm{Anf}\,(\vec{d}) = \mathrm{Anf}\,(\vec{b})\}$$

ist.

Korollar 19. *Sei $G=(E, R)$ ein gerichteter Graph, seien A, B Teilmengen von E und sei A abzählbar. Dann existiert eine Eckenmenge S und eine Verbindung L mit folgenden Eigenschaften:*

1. $\mathrm{Anf}\,(L)\subseteq A$ *und L ist eine Verbindung von kürzesten Bahnen von A nach B.*
2. *Auf jeder kürzesten Bahn von A nach B liegt eine Ecke aus S.*
3. *Auf jeder Bahn aus L liegt genau eine Ecke aus S.*
4. *Jede Ecke aus S liegt auf einer Bahn aus L.*

BEWEIS. Sei E^* die Menge aller Ecken aus E, die auf einer kürzesten Bahn von A nach B liegen und sei R^* die Menge aller Kanten aus R, die auf einer kürzesten Bahn von A nach B liegen. Dann ist jede kürzeste Bahn von A nach B in (E^*, R^*) eine kürzeste Bahn in (E, R) und umgekehrt. Mehr noch: Jede Bahn von A nach B ist kürzeste Bahn in (E^*, R^*). Die Behauptung folgt aus Satz 18.

Injektive Auswahlfunktionen

Ist R eine Relation, so sei $\mathrm{Vb}(R)$ der *Vorbereich von* R und $\mathrm{Nb}(R)$ der *Nachbereich von* R. Eine antireflexive, antisymmetrische Relation R heißt *bipartiter Graph,* falls $\mathrm{Vb}(R)\cap\mathrm{Nb}(R)=\emptyset$ ist. Ist $F=(F(i)|i\in I)$ eine Familie mit $I\cap\bigcap\bigcup_{i\in I} F(i)=\emptyset$, so sei $R^F=\{(i,x)|x\in F(i)\}$ der *zu* F *gehörende bipartite Graph.* Ist $F=(F(i)|i\in I)$ eine Familie, so heißt jedes Element f aus dem Cartesischen Produkt $\prod_{i\in I} F(i)$ *Auswahlfunktion bzgl.* F. Eine Abbildung f heißt *partielle Auswahlfunktion* bzgl. $F=(F(i)|i\in I)$, falls eine Teilmenge J von I existiert, so daß f eine Auswahlfunktion bzgl. der Beschränkung $F\restriction J$ von F auf J ist. Durch den Übergang von einer Familie F zu dem zugehörigen bipartiten Graphen R^F folgt aus Korollar 16 unmittelbar unser Ergebnis aus [3].

KOROLLAR 20. *Sei* $F=(F(i)|i\in I)$ *eine Familie und sei die Menge* I *abzählbar. Dann sind folgende Aussagen äquivalent:*

1. *F besitzt eine injektive Auswahlfunktion.*
2. *Zu jeder partiellen injektiven Auswahlfunktion g bzgl. F und zu jedem Element $i\in I$ existiert eine partielle injektive Auswahlfunktion f bzgl. F mit* $\mathrm{Vb}(f)=\mathrm{Vb}(g)\cup\{i\}$.

DEFINITION. Zwei Familien $F_1=(F_1(i)|i\in I_1)$, $F_2=(F_2(i)|i\in I_2)$ besitzen eine *gemeinsame Transversale* T genau dann, wenn injektive Auswahlfunktionen f_1 bzgl. F_1 und f_2 bzgl. F_2 existieren mit $\mathrm{Nb}(f_1)=T=\mathrm{Nb}(f_2)$.

SATZ 21. *Seien* $F_1=(F_1(i)|i\in I_1)$, $F_2=(F_2(i)|i\in I_2)$ *zwei Familien und seien die Mengen* I_1, I_2 *abzählbar. Dann sind folgende Bedingungen äquivalent:*

1. *F_1 und eine Subfamilie von F_2 besitzen eine gemeinsame Transversale.*
2. *Für alle partiellen injektiven Auswahlfunktionen g_1 bzw. g_2 von F_1 bzw. F_2, für alle Elemente $i\in\mathrm{Vb}(F_1)$ und für alle Elemente $a\in\mathrm{Nb}(F_1)$ gibt es partielle injektive Auswahlfunktionen f_1 bzw. f_2 von F_1 bzw. F_2 mit folgender Eingenschaft:*

$$\mathrm{Vb}(f_1)=\mathrm{Vb}(g_1)\cup\{i\}$$

und

$$\mathrm{Nb}(f_1)\setminus\mathrm{Nb}(f_2)\subseteqq(\mathrm{Nb}(g_1)\setminus\mathrm{Nb}(g_2))\setminus\{a\}.$$

BEWEIS. Man sieht unmittelbar ein, daß Bedingung 2) notwendig für Bedingung 1) ist. Bleibt zu zeigen, daß sie auch hinreichend ist. O.B.d.A. seien die Mengen $I_1, I_2, \bigcup_{i\in I_1} F_1(i)\cup\bigcup_{i\in I_2} F_2(i)$ paarweise disjunkt. Wir definieren einen tripartiten Graphen wie folgt: Sei

$$E=I_1\cup I_2\cup\Big(\bigcup_{i\in I_1} F_1(i)\cup\bigcup_{i\in I_2} F_2(i)\Big)$$

die Eckenmenge und

$$R=\{(i,x)|x\in F_1(i)\}\cup\{(x,i)|x\in F_2(i)\}$$

die Kantenmenge des Graphen $G=(E, R)$. Aus der Bedingung 2) folgt, daß I_1 mit I_2 lokal-verbindbar in G ist. Also existiert nach Korollar 16 eine Verbindung L von I_1 mit I_2. Hieraus folgt Bedingung 1). Aus der Konstruktion des Graphen G im Beweis zu Satz 21 und aus Korollar 17 ergibt sich als Korollar folgender bekannter Satz [2]; Seite 178.

KOROLLAR 22. *Es seien F_1, F_2 zwei Familien. Dann sind folgende Aussagen äquivalent:*

1) *F_1 und F_2 besitzen eine gemeinsame Transversale.*

2) *F_1 besitzt eine gemeinsame Transversale mit einer Subfamilie von F_2 und F_2 besitzt eine gemeinsame Transversale mit einer Subfamilie von F_1.*

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SUR LES SUITES TRANSFINIES

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Les suites transfinies de nombres réels et de fonctions réelles ont été introduites par SIERPIŃSKI dans sa note [12] et puis examininées dans les travaux [4]—[9] et [11].

Désignons par Ω le premier nombre ordinal non-dénombrable. On dit qu'une suite transfinie de nombres réels a_ξ, $\xi<\Omega$, est convergente vers un nombre réel $a(\lim_{\xi\to\Omega} a_\xi=a)$ lorsqu'il existe pour tout nombre $\varepsilon>0$ un nombre ordinal $\alpha<\Omega$ tel que $|a_\xi-a|<\varepsilon$ pour tout nombre ordinal $\xi>\alpha$ $(\xi<\Omega)$. Soit X un espace métrique, séparable. Désignons par R l'ensemble des nombres réels. On dit qu'une suite transfinie de fonctions $f_\xi: X\to R$, $\xi<\Omega$, est convergente vers une fonction f lorsque, quel que soit un point $x\in X$, $\lim_{\xi\to\Omega} f_\xi(x)=f(x)$. On sait que si toutes les fonctions f_ξ sont continues (de première classe de Baire) [semi-continues supérieurement] {ont les graphes fermés}, alors la fonction f l'est aussi ([11]). Des autres classes de fonctions ont été examinées de la même façon ([5]—[9]). Dans la note [6] LIPIŃSKI a même démontré une condition nécessaire et suffisante pour qu'une fonction f soit la limite d'une suite transfinie de fonctions de certaine classe. En particulier, en utilisant l'hypothèse du continu et cette condition LIPIŃSKI a démontré que chaque fonction $f: R\to R$ de première classe de Baire est la limite d'une suite transfinie de fonctions approximativement continues.

Le théorème 1 de cette communication implique comme ses corollaires que les classes des fonctions continues presque partout relativement à certaine mesure, des fonctions ponctuellement discontinues, des fonctions possédant la propriété (K) et les classes $G_i(X)$ $(i=1, 2, \ldots, 5)$ introduites par nous dans l'article [2] sont fermées relativement à la convergence des suites transfinies, c'est-à-dire les limites des suites transfinies de fonctions de ces classes appartiennent également à ces classes. Les théorèmes 2 et 3 de cette communication montrent que les classes des fonctions de propriété (G) et les fonctions step-like introduites par PEEK dans sa note [10] ont la même propriété. Les fonctions de propriété (K) et les fonctions de propriété (G) ont été introduites dans le travail [3] (la propriété (K) d'une fonction réelle d'une variable réelle a été définie dans la note [1]) et ont joué un rôle important dans certaines conditions suffisantes pour la mesurabilité des fonctions de deux variables. Enfin les théorèmes 4 et 5 montrent que la classe des fonctions approximativement continues et continues presque partout relativement à certaine mesure sur l'espace X et la classe des fonctions dérivées continues presque partout relativement à la même mesure sont fermées relativement à la convergence des suites transfinies et le théorème 6 montre que la classe des fonctions de première classe de Baire, ayant la propriété de Darboux et continues presque partout relativement à la mesure de Lebesgue sur l'intervalle $\langle -1, 1\rangle$ n'est pas de cette nature. Dans le théorème 6 on admet l'hypothèse du continu.

THÉORÈME 1. *Soient A et B deux ensembles de l'espace X tels que $\emptyset \neq A \subset B$. Soit une fonction $f: X \to R$ la limite d'une suite transfinie de fonctions $f_\xi: X \to R$, $\xi < \Omega$. Si la fonction réduite partielle f/B n'est continue en aucun point de l'ensemble A, alors il existe un nombre ordinal $\alpha < \Omega$ tel que la fonction réduite partielle f_α/B est aussi discontinue en tout point de A.*

DÉMONSTRATION. Soit $\{U_n\}$ une suite des ensembles ouverts d'une base de l'espace X qui coupent l'ensemble A. Posons

$$a_n = \operatorname*{osc}_{B \cap U_n} f.$$

Comme la fonction réduite f/B n'est continue en aucun point de l'ensemble A, tous les nombres a_n sont donc positifs. Dans tout ensemble $U_n \cap B$ fixons deux points x_1^n et x_2^n tels que $|f(x_1^n) - f(x_2^n)| > a_n/2$. Il existe pour tout point x_i^n ($i=1, 2$ et $n=1, 2, \dots$), un nombre ordinal $\alpha(i, n) < \Omega$ tel que $f_\xi(x_i^n) = f(x_i^n)$ pour tout nombre ordinal $\xi > \alpha(i, n)$ ($\xi < \Omega$). L'ensemble des nombres $\alpha(i, n)$ ($i=1, 2$ et $n=1, 2, \dots$) étant dénombrable, il existe donc un nombre ordinal $\alpha < \Omega$ qui est plus grand que tous les nombres $\alpha(i, n)$ ($i=1, 2$ et $n=1, 2, \dots$). Remarquons que la fonction f_α est telle que $f(x_i^n) = f_\alpha(x_i^n)$ pour tout point x_i^n ($i=1, 2$ et $n=1, 2, \dots$). Fixons un point $x \in A$. Désignons par a l'oscillation de la fonction réduite f/B au point x. Comme la fonction restreinte f/B n'est pas continue au point x, le nombre a est donc positif. Soit $\{U_{n_k}\}$ une partielle de la suite $\{U_n\}$ telle que le diamètre $d(U_{n_k}) < 1/k$, $U_{n_k} \supset$ $\supset U_{n_{k+1}}$ ($k=1, 2, \dots$) et $\{x\} = \bigcap_{k=1}^{\infty} U_{n_k}$. Remarquons que $\lim_{k \to \infty} a_{n_k} = a$. Dans la suite désignons par b_k l'oscillation de la fonction réduite f_α/B sur l'ensemble $U_{n_k} \cap$ $\cap B$ ($k=1, 2, \dots$). On vérifie facilement que $b_k \geqq a_{n_k}/2$ pour $k=1, 2, \dots$. Il en résulte que l'oscillation de la fonction réduite f_α/B au point x n'est pas plus petite que $a/2 > 0$. La fonction réduite f_α/B n'est pas donc continue au point x, d'où notre assertion.

COROLLAIRE 1. *La limite d'une suite transfinie de fonctions ponctuellement discontinues est également ponctuellement discontinue.*

Supposons dans la suite que dans l'espace X soit définie une mesure borelienne μ.

COROLLAIRE 2. *La limite d'une suite transfinie de fonctions définies sur l'espace X et continues presque partout relativement à la mesure μ est de la même nature.*

DÉFINITION 1 ([3]). On dit qu'une fonction $f: X \to R$ possède la propriété (K) lorsqu'elle est ponctuellement discontinue sur tout ensemble fermé $D \subset X$ ayant la propriété de Denjoy (c'est-à-dire tel que, quel que soit un ensemble ouvert $U \subset X$, si $U \cap D \neq \emptyset$, alors $\mu(U \cap D) > 0$).

COROLLAIRE 3. *La limite d'une suite transfinie de fonctions définies sur l'espace X et possédant la propriété (K) est aussi de propriété (K).*

REMARQUE. Il résulte du théorème 1 que les classes de fonctions $G_i(X)$ ($i=1, 2, \dots, 5$) définies dans la note [2] sont également fermées relativement à la convergence des suites transfinies.

Supposons dans la suite qu'il existe un couple $(F, \Rightarrow)$, où F soit une famille dénombrable d'ensembles de mesure μ positive et finie et $\Rightarrow$ désigne une relation de convergence des suites d'ensembles de la famille F vers les points $x \in X$, définie de manière que les deux conditions suivantes soient satisfaites:

(1) Il existe pour tout point $x \in X$ une suite d'ensembles $\{U_n\} \subset F$ telle que $U_n \Rightarrow x$.

(2) Toute sous-suite infinie d'une suite $\{U_n\}$ convergente vers un point $x \in X$ converge également vers ce point.

On dit qu'un point $x \in X$ est un point de densité d'un ensemble μ-mesurable $A \subset X$ relativement au couple $(F, \Rightarrow)$ lorsque, quelle que soit la suite $\{U_n\}$ d'ensembles de la famille F convergente au sens $\Rightarrow$ vers le point x, on a l'égalité

$$\lim_{n \to \infty} \mu(U_n \cap A)/\mu(U_n) = 1. \tag{3}$$

DÉFINITION 2 ([3]). On dit qu'une fonction $f: X \to R$ possède la propriété (G) relativement à la famille F lorsque, quels que soien un nombre $\varepsilon > 0$ et un ensemble $A \subset X$ de mesure μ positive, il existe un ensemble $U \in F$ tel que $\mu(U \cap A) > 0$ et $\operatorname{osc} f \leqq \varepsilon$ sur l'ensemble des points de densité (relativement au couple $(F, \Rightarrow)$) de l'ensemble $U \cap A$ appartenant à cet ensemble.

THÉORÈME 2. *La limite d'une suite transfinie de fonctions définie sur l'espace X et possédant la propriété (G) relativement à la famille F est de la même propriété.*

DÉMONSTRATION. Supposons, par contre, qu'il existe une suite transfinie de fonctions $f_\xi: X \to R$, $\xi < \Omega$, possédant la propriété (G) relativement à la famille F qui est convergente vers une fonction f qui ne possède pas de la propriété (G) relativement à cette famille. Il existe donc un ensemble A μ-mesurable, de mesure μ positive et un nombre $\varepsilon > 0$ tels que, quel que soit un ensemble $U \in F$, $\mu(A \cap U) = 0$ ou $\operatorname{osc} f > \varepsilon$ sur l'ensemble des points de densité (relativement au couple $(F, \Rightarrow)$) de l'ensemble $U \cap A$ appartenant à cet ensemble. Soit $\{U_n\}$ une suite des ensembles de la famille F pour lesquels $\mu(U_n \cap A) > 0$. Dans tout ensemble $U_n \cap A$ fixons deux points x_1^n et x_2^n qui sont des points de densité (relativement au couple $(F, \Rightarrow)$) de l'ensemble $U_n \cap A$ et tels que $|f(x_1^n) - f(x_2^n)| > \varepsilon$. On peut voir, de la même façon comme dans la démonstration du théorème 1, qu'il existe un nombre ordinal $\alpha < \Omega$, tel que $f_\alpha(x_i^n) = f(x_i^n)$ pour tout point x_i^n $(i = 1, 2$ et $n = 1, 2, \ldots)$. Remarquons, en outre, que la fonction f_α ne possède pas de la propriété (G) relativement à la famille F, comme $\operatorname{osc} f_\alpha > \varepsilon$ sur l'ensemble des points de densité (relativement au couple $(F, \Rightarrow)$) de l'ensemble $A \cap U$ appartenant à cet ensemble, quel que soit l'ensemble $U \in F$ tel que $\mu(A \cap U) > 0$. Cela contredit le fait que la fonction f_α possède la propriété (G) relativement à la famille F.

DÉFINITION 3 ([10]). Une fonction $f: X \to R$ est dite step-like lorsqu'il existe pour tout ensemble $A \neq \emptyset$ un ensemble ouvert $U \subset X$ tel que $U \cap A \neq \emptyset$ et la fonction réduite $f/A \cap U$ est constante.

THÉORÈME 3. *La limite d'une suite transfinie de fonctions step-like sur l'espace X est du même type.*

DÉMONSTRATION. Supposons, par contre, qu'il existe une suite transfinie de fonctions step-like $f_\xi: X \to R$, $\xi < \Omega$, convergente vers une fonction $f: X \to R$ qui ne soit pas step-like. Il existe donc un ensemble non-vide $A \subset X$ tel que la fonction réduite $f|A\cap U$ n'est pas constante pour tout ensemble ouvert $U\subset X$ tel que $U\cap A\neq\emptyset$. Soit $\{U_n\}$ une suite des ensembles ouverts de certaine base de l'espace X qui coupent l'ensemble A. Il existe donc pour tout indice n deux points x_1^n et x_2^n appartenant à l'ensemble $A\cap U_n$ et tels que $f(x_1^n)\neq f(x_2^n)$. Soit $\alpha<\Omega$ un nombre ordinal tel que $f_\alpha(x_i^n)=f(x_i^n)$ pour tout point x_i^n ($i=1, 2$ et $n=1, 2, \ldots$). Remarquons que, quel que soit l'ensemble ouvert $U\subset X$, si $U\cap A\neq\emptyset$, alors la fonction réduite $f_\alpha|A\cap U$ n'est pas constante, ce qui entraîne la contradiction, comme la fonction f_α est step-like.

Pour formuler le théorème 4, appelons une fonction $f:X\to R$ approximativement continue au point x relativement au couple $(F, \Rightarrow)$ lorsque, quel que soit le nombre a, le point x est, relativement à ce couple, un point de densité de celui des ensembles $\{t\in X; f(t)>a\}$ et $\{t\in X; f(t)<a\}$ qui contient ce point.

THÉORÈME 4. *Supposons que la mesure μ soit complète. La limite d'une suite transfinie de fonctions définies sur l'espace X, approximativement continues relativement au couple $(F, \Rightarrow)$ et continues presque partout relativement à la mesure μ a la même propriété.*

DÉMONSTRATION. Supposons qu'une suite transfinie de fonctions $f_\xi:X\to R$, $\xi<\Omega$, et une fonction $f:X\to R$ satisfassent à l'hypothèse de notre théorème. Fixons un point $x_0\in X$. Soit $\{x_n\}_{n=1}^\infty$ une suite dense dans l'espace X. Désignons par $\alpha<\Omega$ un nombre ordinal tel que $f_\alpha(x_i)=f(x_i)$ pour tout point x_i ($i=0, 1, \ldots$). Les fonctions f et f_α étant continues presque partout relativement à la mesure μ et la suite $\{x_n\}_{n=0}^\infty$ étant dense dans l'espace X, l'ensemble

$$A=\{t\in X; f_\alpha(t)\neq f(t)\}$$

est donc μ-mesurable et de μ-mesure zéro. Fixons un nombre a et supposons que $f(x_0)>a$. La fonction f_α étant approximativement continue au point x_0 relativement au couple $(F, \Rightarrow)$ et $f_\alpha(x_0)=f(x_0)>a$, x_0 est donc un point de densité (relativement au couple $(F, \Rightarrow)$) de l'ensemble $\{t\in X; f_\alpha(t)>a\}$ et, par conséquent, comme $\mu(A)=0$, il est également un point de densité (relativement à ce couple) de l'ensemble $\{t\in X; f(t)>a\}$, d'où notre théorème.

Une fonction $f:X\to R$ qui est intégrable relativement à la mesure μ sur tout ensemble de la famille F est dite fonction dérivée relative au couple $(F, \Rightarrow)$ lorsqu'on a l'égalité

$$\lim_{n\to\infty}\Big(\int_{U_n} f(t)\,d\mu\Big)/\mu(U_n)=f(x) \tag{4}$$

pour tout point $x\in X$ et pour toutes les suites $\{U_n\}\Rightarrow x$.

THÉORÈME 5. *La limite d'une suite transfinie de fonctions dérivées relative au couple $(F, \Rightarrow)$, continues presque partout relativement à la mesure μ est aussi la dérivée continue presque partout.*

DÉMONSTRATION. Supposons qu'une suite transfinie de fonctions $f_\xi:X\to R$, $\xi<\Omega$, et une fonction $f:X\to R$ satisfassent à l'hypothèse de notre théorème. Fixons un point $x_0\in X$. Soit $\{x_n\}_{n=1}^\infty$ une suite dense dans l'espace X. Désignons par $\alpha<\Omega$ un nombre ordinal tel que $f_\alpha(x_i)=f(x_i)$ pour tout point x_i ($i=0, 1, \ldots$). Remarquons que l'ensemble

$$A = \{t \in X;\ f_\alpha(t) \neq f(t)\}$$

est μ-mesurable et de μ-mesure zéro. On a donc, pour tout ensemble $U \in F$,

$$\int_U f_\alpha(t)\, d\mu = \int_U f(t)\, d\mu,$$

et, par conséquent,

$$f(x_0) = f_\alpha(x_0) = \lim_{n\to\infty} \Big(\int_{U_n} f_\alpha(t)\, d\mu \Big) / \mu(U_n) = \lim_{n\to\infty} \Big(\int_{U_n} f(t)\, d\mu \Big) / \mu(U_n)$$

pour toutes les suites $\{U_n\} \Rightarrow x_0$, ce qui termine la démonstration du théorème 5.

Le Professeur LIPIŃSKI m'a fait savoir la proposition suivante:

THÉORÈME 6. *Admettons l'hypothèse du continu. Soit* $f: \langle 0, 1\rangle \to R$ *une fonction de première classe de Baire, continue presque partout relativement à la mesure de Lebesgue. Alors f est la limite d'une suite transfinie de fonctions de première classe de Baire, continues presque partout relativement à la mesure de Lebesgue et ayant la propriété de Darboux.*

DÉMONSTRATION. D'après le théorème 2 du travail [6] il suffit de démontrer que, quel que soit l'ensemble dénombrable $A \subset \langle 0, 1\rangle$, il existe une fonction $f_1: \langle 0, 1\rangle \to R$ de première classe de Baire, continue presque partout, ayant la propriété de Darboux et telle que $f_1(x) = f(x)$ pour tout $x \in A$. Fixons un ensemble dénombrable $A \subset \langle 0, 1\rangle$. Soit $B \subset \langle 0, 1\rangle$ un ensemble du type F_σ, bilatéralement c-dense en soi, de mesure de Lebesgue zéro et tel que $B \supset A$. Il existe un homéomorphisme $h: \langle 0, 1\rangle \xrightarrow{\text{sur}} \langle 0, 1\rangle$ tel que $m(h(B)) = 1$ ($m(h(B))$ désigne la mesure de Lebesgue de l'ensemble $h(B)$). Posons $g(x) = f(h(x))$ pour $x \in \langle 0, 1\rangle$. Soit $C \subset \langle 0, 1\rangle$ un ensemble du type G_δ de mesure de Lebesgue zéro tel que $B \subset C$. Il existe une fonction $k: \langle 0, 1\rangle \to R$ approximativement continue et continue en tout point de continuité de la fonction réduite g/C et telle que $k(x) = g(x)$ pour tout point $x \in C$. (Dans le travail [14] est démontré que si $C \subset [0, 1]$ est un ensemble de mesure 0, du type G_δ et si g est une fonction de première classe de Baire sur [0, 1], alors il existe une fonction k approximativement continue sur [0, 1] telle que $g(x) = k(x)$ pour $x \in C$. Mais il résulte de la démonstration de ce théorème que k peut être continue en chaque point de continuité de la fonction réduite g/C.) Par conséquent la fonction $f_1(x) = k(h^{-1}(x))$ satisfait aux conditions exigées pour $x \in \langle 0, 1\rangle$.

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THE BIRKHOFF—EGERVÁRY—KŐNIG THEOREM FOR MATRICES OVER LATTICE ORDERED ABELIAN GROUPS

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1. Introduction. A *lattice ordered abelian* (additive) *group* (or abelian *l*-group) is an abelian group G which is also a lattice, and where the infimum and the group operation are related by

(1) $$\inf\{a+c, b+c\} = \inf\{a, b\}+c,$$

for $a, b, c\in G$, cf. CLIFFORD [4], BIRKHOFF [1], [3a, Ch. XIV], [3b, Ch. XIII], FUCHS [8, Ch. V]. If the order of G is full (linear), then G is called a *fully ordered abelian group* (abelian *o*-group) and we write $\min\{a, b\}$ in place of $\inf\{a, b\}$. If G is an abelian *l*-group we shall denote by G^{nn} the additive group of $n\times n$ matrices with elements in G. A matrix $A\in G^{nn}$ will be called a generalized doubly stochastic matrix (g.d.s. matrix) if

(2) $$a_{ij} \geqq 0, \quad i, j = 1, \ldots, n,$$

(3) $$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{jk}, \quad i, k = 1, \ldots, n$$

(i. e. all row and column sums are equal). Observe that the 0 matrix in G^{nn} is g.d.s. We denote by S_n the symmetric group on $\{1, \ldots, n\}$. If $\sigma\in S_n$ and $\varepsilon\in G$, the matrix $P_\sigma(\varepsilon)$ is defined by

(4) $$P_\sigma(\varepsilon)_{ij} = \begin{cases} \varepsilon & \text{if} \quad j = \sigma(i), \\ 0, & \text{otherwise.} \end{cases}$$

If $\varepsilon\geqq 0$, we call $P_\sigma(\varepsilon)$ a *generalized permutation* (g.p.) *matrix*. We shall prove:

THEOREM. *Let G be a lattice ordered abelian group. Every generalized doubly stochastic matrix with elements in G is the sum of generalized permutation matrices.*

In the case that $G=\mathbf{Z}$, the integers, this theorem is due to KŐNIG, 1916 [9, Theorem F], see also KŐNIG [10, p. 239, Theorem B], MIRSKY [15, p. 186, Theorem 11.1.5]. For $G=\mathbf{R}$, the real numbers, the theorem is due to BIRKHOFF 1946 [2], see also MARCUS—MINC [13, p. 97, Theorem 1.7], MIRSKY [15, p. 192, Theorem 11. 3.1] and MIRSKY [14], where many other references to Birkhoff's theorem and its proofs may be found. It should also be noted that in 1931 EGERVÁRY [5, Theorem II] proved a result for integral matrices which is more general than Kőnig's theorem. He observed that by continuity considerations his theorem may be shown to hold for real matrices. Thus in this way one obtains a result which contains Birkhoff's theorem.

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To obtain a result which contains both Birkhoff's theorem and Kőnig's theorem it is enough to prove our Theorem in the case that G is an abelian o-group. Though a proof of this case is essentially the same as the proof of Birkhoff's theorem given in MARCUS—MINC [13, pp. 97, 98], we have not found the result formulated for abelian o-groups, or in such a manner to contain both the case $G=\mathbf{R}$ and $G=\mathbf{Z}$. (BIRKHOFF [3a, p. 266, Ex. 4] remarks that the theorem is valid for matrices with elements in a fully ordered field, which is close.)

To extend the theorem to abelian l-groups we use an embedding result due to CLIFFORD [4, Theorem 2] who remarks that this theorem is a combination of Theorems 4 and 11 of LORENZEN [11], which, however, do not deal with l-groups explicitly. The Clifford—Lorenzen theorem states: Let G be an abelian l-group. Then there exists a family $G(\tau)$, $\tau\in T$, of abelian o-groups, and an isomorphism $g\to(g(\tau))_{\tau\in T}$ of G into the direct product $(G(\tau))_{\tau\in T}$ such that for $a, b\in G$ and $c=\inf\{a, b\}$ we have $c(\tau)=\min\{a(\tau), b(\tau)\}$. For this theorem, see BIRKHOFF [3b, p. 309, Theorem 22] and for a generalization, LORENZEN [12, Theorem 13]. Following ŠIK [18], and RIBENBOIM [16] we call a family of $G(\tau)$, $\tau\in T$, together with an isomorphism with the above property, a *realization* of G. (An example of an abelian l-group which is not the direct sum or direct product of o-groups is given in ŠIK [17].) With the above remarks the proof of our theorem is very straightforward.

2. Proofs. If σ is a permutation in S_n, and $A\in G^{nn}$ then we define the *diagonal* D_σ to be the set $\{a_{1\sigma(1)}, \dots, a_{n\sigma(n)}\}$. Part (a) of the proof of the Lemma below requires the Frobenius—Kőnig Theorem, see FROBENIUS [7], KŐNIG [10, p. 240, Theorem E], MARCUS—MINC [13, p. 97, 1.7.1], MIRSKY [15, p. 189, Cor. 11.2.6]: If every diagonal of $A\in G^{nn}$ has a zero element, then A contains a zero submatrix of order $p\times(n+1-p)$, for some p, $1\leqq p<n$.

LEMMA. *Let G be a lattice ordered abelian group and let $A\in G^{nn}$ be a generalized doubly stochastic matrix. If* $\inf D_\sigma=0$, *for every, $\sigma\in S_n$, then $A=0$.*

PROOF. (a) First assume that G is an abelian o-group. Since $\min D_\sigma=0$, for every $\sigma\in S_n$, every diagonal of A has a 0 member. Hence, by Frobenius—Kőnig, A has a 0 submatrix of order $r\times(n+1-r)$, for some r, $1\leqq r<n$. Without loss of generality, we may assume that $a_{ij}=0$, $i=1, \dots, r$, $j=r, \dots, n$. Let

$$\sum_{j=1}^{n} a_{ij} = s = \sum_{j=1}^{n} a_{ij}, \quad i=1, \dots, n.$$

Then

$$rs = \sum_{i=1}^{r}\sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{r}\sum_{j=1}^{r-1} a_{ij} \leqq \sum_{j=1}^{r-1}\sum_{i=1}^{n} a_{ij} = (r-1)s,$$

whence $s=0$. But then $a_{ij}=0$, $i, j=1, \dots, n$, and so $A=0$.

(b) Now suppose that G is an arbitrary abelian l-group. By the Clifford—Lorenzen theorem there exists a realization $G\to(G(\tau))_{\tau\in T}$, where the $G(\tau)$ are abelian o-groups. We put $A(\tau)=(a_{ij}(\tau))\in G(\tau)^{nn}$, and for $\sigma\in S_n$ we denote by $D_\sigma(\tau)$ the diagonal $\{a_{1\sigma(1)}(\tau), \dots, a_{n\sigma(n)}(\tau)\}$ of $A(\tau)$. Then, for each $\tau\in T$,

$$\min D_\sigma(\tau) = (\inf D_\sigma)(\tau) = 0,$$

whence by Part (a) of this proof $A(\tau)=0$, since $A(\tau)$ is a g.d.s. matrix in $G(\tau)^{nn}$. It follows that $A=0$.

PROOF OF THE THEOREM. We use induction on the number k of $\sigma\in S_n$ with $\inf D_\sigma>0$. If $k=0$, then by the Lemma, $A=0$ and the result holds. So suppose that $k>0$, and the result holds for matrices A' with fewer than k diagonals with non-zero infimum. Let $\sigma\in S_n$ be such that $d=\inf D_\sigma>0$, and put $A'=A-P_\sigma(d)$. Then A' is g.d.s., and for every $\pi\in S_n$, $\inf D'_\pi\leqq\inf D_\pi$. Also $\inf D'_\sigma=\inf D_\sigma-d=0$, where D'_π, $\pi\in S_n$, denotes the diagonal of A' corresponding to π. Hence A' has fewer than k diagonals with positive infimum, and by inductive assumption, A' is the sum of g.p. matrices. It follows that $A=A'+P_\sigma(d)$ is also a sum of g.p. matrices. The theorem is proved.

3. If $G=\mathbf{R}$, then it is well known that every g.d.s. matrix can be expressed as the sum of at most (n^2-2n+2) g.p. matrices $P_\sigma(\varepsilon_\sigma)$, with $\varepsilon_\sigma>0$, FARAHAT—MIRSKY [6], cf. MIRSKY [14], MARCUS—MINC [13, pp. 94—100], and (incidentally) this bound is best possible. We give an example of an abelian l-group G and a g.d.s. matrix $A\in G^{nn}$, which has a unique representation $A=\sum_{\sigma\in S_n} P_\sigma(\varepsilon_\sigma)$ and $\varepsilon_\sigma>0$, for all $n!$ permutations σ. We put $G=\mathbf{Z}^m$, where $m=n!$, the direct sum of $n!$ copies of the integers $\mathbf{Z}$, and we index the copies of $\mathbf{Z}$ by S_n. Thus the elements of G are the $n!$ tuples $(a(\tau))_{\tau\in S_n}$.

We next define $A\in G^{nn}$ by

$$a_{ij}(\tau)=\begin{cases}1, & \text{if } j=\tau(i),\\ 0, & \text{otherwise.}\end{cases}$$

Then A is a g.d.s. matrix for clearly $a_{ij}\geqq 0$, $i,j=1,\dots,n$ and, for each $\tau\in S_n$

$$\sum_{j=1}^{n} a_{ij}(\tau)=\sum_{j=1}^{n} a_{ji}(\tau)=1.$$

If, $\sigma,\tau\in S_n$, let $A=\sum_{\sigma\in S_n} P_\sigma(\varepsilon_\sigma)$. Then

$$\varepsilon_\sigma(\tau)\leqq\inf D_\sigma(\tau)=\inf\{a_{1\sigma(1)}(\tau),\dots,a_{n\sigma(n)}(\tau)\}=0,\quad\text{if}\quad\sigma\neq\tau$$

and it follows that

$$\varepsilon_\sigma(\sigma)=1,\quad \varepsilon_\sigma(\tau)=0,\quad\text{if}\quad\sigma\neq\tau.$$

We now write out our example in full, for the case that $n=3$ and the permutations in S_n are arranged in order

$$(1,2,3),\ (1,3,2),\ (2,1,3),\ (2,3,1),\ (3,1,2),\ (3,2,1),$$

where (i,j,k) is the permutation σ for which $\sigma(1)=i$, $\sigma(2)=j$, $\sigma(3)=k$. Then A is

$$\begin{pmatrix}(1,1,0,0,0,0) & (0,0,1,1,0,0) & (0,0,0,0,1,1)\\ (0,0,1,0,1,0) & (1,0,0,0,0,1) & (0,1,0,1,0,0)\\ (0,0,0,1,0,1) & (0,1,0,0,1,0) & (1,0,1,0,0,0)\end{pmatrix}.$$

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A SPECIAL EMBEDDING OF BOL LOOPS IN GROUPS

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§ 1. Introduction

A loop $(G, \cdot)$ is a *Bol loop* provided that the (right) Bol identity

(1) $$((x\cdot y)\cdot z)\cdot y = x\cdot((y\cdot z)\cdot y)$$

holds for all $x, y, z\in G$. Since their initial appearance in the algebra-geometry considerations of G. Bol [2], such loops (especially the di-associative ones[1]) have been the object of several algebraic studies. To trace the development of such investigations one may consult [3; Chapter II], [15], [4; Chapters VII and VIII], [5], [11], [17], [1], [12], [13], [18], [10], [16], [7], [8], [9], and [6]. Although this chronological list is long, it does not provide an exhaustive survey of the available literature on Bol loops. However, it is hoped that these references when supplemented with those papers cited in their bibliographies will provide any interested reader with a reasonably complete compilation of what is currently known about this specialized class of loops.

In this paper a general construction for Bol loops is presented — a construction which, in a sense, demonstrates that all Bol loops can be realized by the introduction of an appropriate binary operation on a judiciously selected subset of a group. In short, a Bol loop is specially embeddable (see § 3) in a group. This special embedding is then examined (see § 4) relative to isotopes, subloops, and homomorphic images. Finally, a universal embedding (see § 5) is obtained for finite Bol loops.

§ 2. Preliminary information

Let $(G, \cdot)$ be a loop. For each fixed $x\in G$ the translation maps $R(x)$ and $L(x)$ are defined by $yR(x)=y\cdot x$ and $yL(x)=x\cdot y$ for all $y\in G$. Since a loop is necessarily a quasigroup, $R(x)$ and $L(x)$ are one-to-one maps of G onto G and, hence, the inverse maps $R(x)^{-1}$ and $L(x)^{-1}$ exist. It follows easily from (1) that a loop $(G, \cdot)$ is a Bol loop if and only if

(2) $$L(x\cdot y)R(y) = L(y)R(y)L(x)$$

for all $x, y\in G$.

[1] Note that Moufang loops are precisely those Bol loops which are di-associative.

It is known (see [17; Theorem 2.2]) that a Bol loop $(G, \cdot)$ is power-associative and, more generally, satisfies the right power-alternative law

(3) $$x\cdot(y^m\cdot y^n) = x\cdot y^{m+n}$$

for all $x, y\in G$ and all integers m and n. In particular, the right and left inverses of each element in such a loop must coincide and

(4) $$R(x)^{-1} = R(x^{-1})$$

for all $x\in G$ where x^{-1} denotes the common right and left inverses of x. Since Bol loops need not satisfy the left inverse property, a similar result for $L(x)$ does not exist.

§ 3. A special embedding

Let $(G, \cdot)$ be a group whose identity element is denoted by e. Now let H be a non-empty subset of G. If

(A) H generates $(G, \cdot)$,

(B) for each $y\in H$ there is an automorphism $\theta[y]$ of $(G,\cdot)$ such that $(y^{-1}\cdot z)\theta[y]^{-1}\in H$ whenever $z\in H$,

(C) $\theta[e]=I$, the identity map on G,

then $x\circ y$ is defined by

(5) $$x\circ y = y\cdot x\theta[y]$$

for all $x, y\in H$.

THEOREM 3.1. *Under the above assumptions, $(H, \circ)$ is a Bol loop if and only if*

(D) $x\circ y\in H$ *for all* $x, y\in H$,

(E) $\theta[(y\circ z)\circ y]=\theta[y]\theta[z]\theta[y]$ *for all* $y, z\in H$.

PROOF. It is immediate from (5) that

(6) $$((x\circ y)\circ z)\circ y = x\circ((y\circ z)\circ y)$$

for all $x, y, z\in H$ if and only if

(7) $$x\theta[(y\circ z)\circ y] = x\theta[y]\theta[z]\theta[y]$$

for all $x, y, z\in H$. But then, since H generates the group $(G,\cdot)$ and each $\theta[u]$, $u\in H$, is an automorphism of $(G, \cdot)$, it follows from (7) that

(7′) $$x\theta[(y\circ z)\circ y] = x\theta[y]\theta[z]\theta[y]$$

for all $x\in G$ and all $y, z\in H$. Thus, (6) holds for all $x, y, z\in H$ if and only if (E) is satisfied.

If $(H,\circ)$ is a Bol loop, it is clear now that (D) and (E) are both satisfied.

In order to establish the sufficiency of Theorem 3.1 assume that (A), (B), (C), (D), and (E) hold. If $a\in H$ and e is the identity element of the group $(G,\cdot)$, it follows from (B) that $e=e\theta[a]^{-1}=(a^{-1}\cdot a)\theta[a]^{-1}\in H$. That is, e is a member of H. But then, employing (C) and the fact that every automorphism of $(G,\cdot)$ fixes e, one sees that $x\circ e=e\cdot x\theta[e]=e\cdot x=x$ and $e\circ x=x\cdot e\theta[x]=x\cdot e=x$ for all $x\in H$. Thus,

e is the identity element for $(H, \circ)$. Also, for $y, z \in H$, one sees from (B) and (5) that $(y^{-1} \cdot z)\theta[y]^{-1} \in H$ and that $(y^{-1} \cdot z)\theta[y]^{-1} \circ y = y \cdot (y^{-1} \cdot z)\theta[y]^{-1}\theta[y] = z$. Furthermore, for $x, y, z \in H$, it is observed that $x \circ y = z$ implies that $y \cdot x\theta[y] = z$ which, in turn, implies that $x = (y^{-1} \cdot z)\theta[y]^{-1}$. Thus, for $y, z \in H$, there is a unique $x \in H$ so that $x \circ y = z$. (It is now evident that $(H, \circ)$ *satisfies the right cancellation law*.) For $x \in H$, let $\bar{x}$ be that unique element in H such that $\bar{x} \circ x = e$. For $x, y, z \in H$, assume that $x \circ y = x \circ z$. It follows then that $\bar{x} \circ ((x \circ y) \circ x) = \bar{x} \circ ((x \circ z) \circ x)$. But, since (6) holds for all $x, y, z \in H$, one obtains now that $((\bar{x} \circ x) \circ y) \circ x = ((\bar{x} \circ x) \circ z) \circ x$. Thus, $y \circ x = z \circ x$ and, by right cancellation for $(H, \circ)$, it follows that $y = z$. Thus, $(H, \circ)$ *satisfies the left cancellation law*. Now,for $x, z \in H$, it follows from (D) that $\bar{x} \circ (z \circ x) \in H$. Then there is a unique $y \in H$ so that $y \circ x = \bar{x} \circ (z \circ x)$. Using this and (6), one sees that $\bar{x} \circ ((x \circ y) \circ x) = ((\bar{x} \circ x) \circ y) \circ x = (e \circ y) \circ x = y \circ x = \bar{x} \circ (z \circ x)$. From $\bar{x} \circ ((x \circ y) \circ x) = \bar{x} \circ (z \circ x)$ and the cancellation laws for $(H, \circ)$ one sees that $x \circ y = z$. So, for $x, z \in H$, there is a unique $y \in H$ so that $x \circ y = z$. Consequently, $(H, \circ)$ is a Bol loop and Theorem 3.1 is proved.

DEFINITION 3.1. A non-empty subset H of a set G is called a *special subset of a group* $(G, \cdot)$ whenever conditions (A), (B), (C), (D), and (E) hold.

In view of Theorem 3.1, special subsets of groups give rise to Bol loops. It will be seen in Theorem 3.2 that all Bol loops can be obtained in this manner.

DEFINITION 3.2. A Bol loop $(K, *)$ is *specially embeddable in* (or *specially embedded into*) a group $(G, \cdot)$ means that there is a special subset H of $(G, \cdot)$ (see Definition 3.1) so that $(K, *)$ and $(H, \circ)$ are isomorphic.

LEMMA 3.1. *Let $(K, \cdot)$ be a Bol loop and let G_λ be that subgroup of the symmetric group on the set K which is generated by all $L(x)$ for all $x \in K$. If $G_{\lambda,\varrho}$ is that subgroup of the symmetric group on K which is generated by all $L(x)$ and $R(x)$ for all $x \in K$, then G_λ is a normal subgroup of $G_{\lambda,\varrho}$.*

PROOF. Since G_λ is obviously a subgroup of the group $G_{\lambda,\varrho}$, normality is the only issue. For all $x, y \in K$ it follows from (2) that $R(y)L(x)R(y)^{-1} = L(y)^{-1}L(x \cdot y)$. That is $R(y)L(x)R(y)^{-1}$ is a member of G_λ for all $x, y \in K$. Replacing y by y^{-1} and using (4), one obtains $R(y)^{-1}L(x)R(y) = L(y^{-1})^{-1}L(x \cdot y^{-1}) \in G_\lambda$ for all $x, y \in K$. Taking inverses, one also deduces that $R(y)L(x)^{-1}R(y)^{-1} \in G_\lambda$ and $R(y)^{-1}L(x)^{-1}R(y) \in G_\lambda$ for all $x, y \in K$. This information combined with the obvious fact that $L(y)^{-1}L(x)^{\pm 1}L(y) \in G_\lambda$ and $L(y)L(x)^{\pm 1}L(y)^{-1} \in G_\lambda$ for all $x, y \in K$ indicates that G_λ is normal in $G_{\lambda,\varrho}$.

THEOREM 3.2. *If $(K, \cdot)$ is a Bol loop, then $(K, \cdot)$ is specially embeddable in a group.*

PROOF. Define G_λ and $G_{\lambda,\varrho}$ as in the preceding lemma. Let $H = \{L(x) |\text{ all } x \in K\}$ and let $G = G_\lambda$. For each $L(x) \in H$, define $\bar{\theta}[L(x)]$ by

$$Y\bar{\theta}[L(x)] = R(x)YR(x)^{-1}$$

for all $Y \in G_{\lambda,\varrho}$. Each $\bar{\theta}[L(x)]$ is an inner automorphism of the group $G_{\lambda,\varrho}$. For each $L(x) \in H$, let $\theta[L(x)]$ be the restriction of $\bar{\theta}[L(x)]$ to G. Clearly H generates the group G and, in view of Lemma 3.1, each $\theta[L(x)]$ is an automorphism of G. From (2) it follows that

$$L(x \cdot y) = L(y)R(y)L(x)R(y)^{-1} \tag{8}$$

for all $x, y \in K$. Setting $x = y^{-1}$ in (8) and rewriting, one sees that

$$L(y)^{-1} = R(y)L(y^{-1})R(y^{-1}) \tag{9}$$

for all $y \in K$. So $\{L(y)^{-1}L(z)\}\theta[L(y)]^{-1} = R(y)^{-1}L(y)^{-1}L(z)R(y) = L(z \cdot y^{-1}) \in H$ for all $y \in K$. Also $Y\theta[L(e)] = Y$ for all $Y \in G$. Therefore, conditions (A), (B), and (C) hold.

Now define φ by $x\varphi = L(x)$ for all $x \in K$. Then φ is a one-to-one map of K onto H. In view of (8) one sees that $x\varphi \circ y\varphi = L(x) \circ L(y) = L(y)\{L(x)\theta[L(y)]\} = L(y)R(y)L(x)R(y)^{-1} = L(x \cdot y) = (x \cdot y)\varphi$ for all $x, y \in K$. Hence, φ is an isomorphism of $(K, \cdot)$ onto $(H, \circ)$. Thus, $(H, \circ)$ is also a Bol loop and so, by Theorem 3.1, conditions (D) and (E) hold. Consequently, $(K, \cdot)$ is specially embedded into the group G and the proof is complete.

It should be pointed out that there is nothing unique about the group into which a Bol loop can be specially embedded. For instance, let p be any odd prime and let the Bol loop $(K, \cdot)$ be the cyclic group of order $p-1$. Trivially, $(K, \cdot)$ can be specially embedded into itself by choosing $\theta[x] = I$, the identity automorphism of $(K, \cdot)$, for all $x \in K$. But now let G be the cyclic group of order p, let a be any one of its $p-1$ generators, and let $H = \{a^0, a^1, \ldots, a^{p-2}\}$. Clearly H generates G. For each $a^i \in G$, define $\theta[a^i]$ by

$$a^j\theta[a^i] = a^{(i+1)j}$$

for $j = 0, 1, \ldots, p-1$. Then each $\theta[a^i]$ is an automorphism of G. In fact, the automorphism group $A(G)$ of G is cyclic of order $p-1$ and consists of $\theta[a^i]$, $i = 0, 1, \ldots, p-2$. Then φ is a one-to-one map of H onto $A(G)$ where $a^i\varphi = \theta[a^i]$ for $i = 0, 1, \ldots, p-2$. One should observe that $(a^i \circ a^j)\varphi = (a^j \cdot a^i\theta[a^j])\varphi = (a^{j+(j+1)i})\varphi$. Note also that one has $j + (j+1)i \not\equiv p-1 \pmod{p}$ whenever i and j are integers such that $0 \leqq i \leqq p-2$ and $0 \leqq j \leqq p-2$. Then it follows that $(a^i \circ a^j)\varphi = \theta[a^{(i+1)(j+1)-1}] = a^i\varphi \cdot a^j\varphi$ for all $a^i, a^j \in H$ and so φ is an isomorphism of $(H, \circ)$ onto $A(G)$. Then $(H, \circ)$ is a cyclic group of order $p-1$. Thus, $(K, \cdot)$ can, in addition to being specially embedded into itself, be also specially embedded into the cyclic group of order p.

§ 4. Isotopes, subloops, and homomorphic images

The special embedding of § 3 is now examined relative to isotopes, subloops, and homomorphic images.

A general discussion of isotopy theory is found in [4; Chapter III] and a discussion of isotopy theory for Bol loops is found in [17; Section 3]. Only one result (see [17; Lemma 3.4]) need be mentioned explicitly here. Namely,

LEMMA 4.1. *Let $(G, \cdot)$ be a Bol loop. Each loop isotopic to $(G, \cdot)$ is isomorphic to a principal isotope $(G, \circ)$ of $(G, \cdot)$ where $x \circ y = xR(f) \cdot yL(f)^{-1}$ for all $x, y \in G$ and some $f \in G$.*

THEOREM 4.1. *If a Bol loop $(K, *)$ is specially embeddable in a group $(G, \cdot)$, then each loop isotopic to $(K, *)$ is also specially embeddable in $(G, \cdot)$.*

PROOF. Let H be a special subset of $(G,\cdot)$ such that $(H,\circ)$ and $(K,*)$ are isomorphic. Let $f\in H$ and let $L=H\theta[f]$. For each $x\in L$, define $\bar\theta[x]$ by

$$\bar\theta[x]=\theta[f]^{-1}\theta[x\theta[f]^{-1}\circ f]. \tag{10}$$

Since $x\theta[f]^{-1}\circ f\in H$ for all $x\in L$, it is clear that $\bar\theta[x]$ is an automorphism of $(G,\cdot)$ for each $x\in L$. Just as "$\circ$" is defined for H (see (5)), define "$\otimes$" for L by

$$x\otimes y=y\cdot x\bar\theta[y]$$

for all $x, y\in L$. Now let $(H, \oplus)$ be that principal isotope of $(H, \circ)$ where $x\oplus y$ is defined by

$$x\oplus y=xR_0(f)\circ yL_0(f)^{-1}$$

for all $x, y\in H$. (Here $R_0(f)$ and $L_0(f)$ denote the translation maps for $(H,\circ)$ defined by $xR_0(f)=x\circ f$ and $xL_0(f)=f\circ x$ for all $x\in H$.)

It will now be established that $\theta[f]$ is an isomorphism of $(H, \oplus)$ onto $(L, \otimes)$. First, however, note the following: Let $f^{(-1)}$ denote the inverse of f in the Bol loop $(H, \circ)$ and let e denote the identity element of the group $(G, \cdot)$. Then e is also the identity element of $(H, \circ)$ and $e=f^{(-1)}\circ f=f\cdot f^{(-1)}\theta[f]$. So $f^{(-1)}=f^{-1}\theta[f]^{-1}$. Also it follows from condition (E) that

$$\theta[(f\circ f^{(-1)})\circ f]=\theta[f]\theta[f^{(-1)}]\theta[f]$$

So $\theta[f^{(-1)}]=\theta[f]^{-1}$. Thus, for use below, note that

$$f^{(-1)}=f^{-1}\theta[f]^{-1}, \tag{11}$$

$$\theta[f^{(-1)}]=\theta[f]^{-1}. \tag{12}$$

Adapting (9) to the present situation, one also sees that

$$L_0(f)^{-1}=R_0(f)L_0(f^{(-1)})R_0(f^{(-1)}). \tag{13}$$

Now, for all $x, y\in H$, one obtains

$$(x\oplus y)\theta[f]=\{xR_0(f)\circ yL_0(f)^{-1}\}\theta[f]=$$

$$=\{(x\circ f)\circ yR_0(f)L_0(f^{(-1)})R_0(f^{(-1)})\}\theta[f]\quad\text{(by (13))}$$

$$=\{(x\circ f)\circ\{(f^{(-1)}\circ(y\circ f))\circ f^{(-1)}\}\}\theta[f]=$$

$$=\{\{((x\circ f)\circ f^{(-1)})\circ(y\circ f)\}\circ f^{(-1)}\}\theta[f]\quad\text{(since } (H,\circ) \text{ is a Bol loop)}$$

$$=\{(x\circ(y\circ f))\circ f^{(-1)}\}\theta[f]\quad\text{(by (4) applied to the Bol loop } (H,\circ))$$

$$=\{f^{(-1)}\cdot(f\cdot y\theta[f]\cdot x\theta[y\circ f])\theta[f^{(-1)}]\}\theta[f]=$$

$$=\{(f^{-1}\theta[f]^{-1})\cdot(f\cdot y\theta[f]\cdot x\theta[y\circ f])\theta[f]^{-1}\}\theta[f]\quad\text{(by (11) and (12))}$$

$$=f^{-1}\cdot f\cdot y\theta[f]\cdot x\theta[y\circ f]\quad\text{(since } \theta[f] \text{ is an automorphism)}$$

$$=y\theta[f]\cdot x\theta[y\circ f].$$

But also, for all $x, y \in H$, one obtains

$$x\theta[f] \otimes y\theta[f] = y\theta[f] \cdot (x\theta[f])\bar{\theta}[y\theta[f]] =$$
$$= y\theta[f] \cdot x\theta[f]\theta[f]^{-1}\theta[y \circ f] \quad \text{(by (10) and (5))}$$
$$= y\theta[f] \cdot x\theta[y \circ f].$$

Hence, $(x \oplus y)\theta[f] = x\theta[f] \otimes y\theta[f]$ for all $x, y \in G$. This, along with the fact that $\theta[f]$ is a one-to-one map of H onto L, indicates that $\theta[f]$ is, indeed, an isomorphism of $(H, \oplus)$ onto $(L, \otimes)$.

Since $(L, \otimes)$ is isomorphic to the principal isotope $(H, \oplus)$ of $(H, \circ)$, it follows that $(L, \otimes)$ is also a Bol loop[2] and, since L generates $(G, \cdot)$, we conclude that L is a special subset of $(G, \cdot)$. But each loop isotopic to $(H, \circ)$ is isomorphic to a principal isotope of the form $(H, \oplus)$ for some $f \in H$ (see Lemma 4.1) and so the proof of Theorem 4.1 is complete.

THEOREM 4.2. *If a Bol loop $(K, *)$ is specially embeddable in a group $(G, \cdot)$, then each subloop of $(K, *)$ is specially embeddable in a subgroup of $(G, \cdot)$.*

PROOF. Let H be a special subset of $(G, \cdot)$ so that $(H, \circ)$ and $(K, *)$ are isomorphic. Let $\bar{H}$ be any subloop of $(H, \circ)$ and let $\bar{G}$ be the subgroup of $(G, \cdot)$ generated by $\bar{H}$. For each $x \in H$ (and, consequently, for each $x \in \bar{H}$) there corresponds an automorphism $\theta[x]$ of $(G, \cdot)$ so that conditions (A), (B), (C), (D), and (E) ar esatisfied. For $y, z \in \bar{H}$ note that

$$(y^{-1} \cdot z)\theta[y]^{-1} \circ y = z.$$

From this one deduces that $(y^{-1} \cdot z)\theta[y]^{-1} \in \bar{H}$ for all $y, z \in \bar{H}$. Consequently, to prove that $\bar{H}$ is a special subset of $\bar{G}$ it suffices to show that $\bar{G}\theta[x] = \bar{G}$ for each $x \in \bar{H}$.

Let x be an element of $\bar{H}$. Then, since $\bar{H}$ is a subloop of $(H, \circ)$, it follows that $y \circ x \in \bar{H}$ for all $y \in \bar{H}$. Also, in view of (5), one notes that $y\theta[x] = x^{-1} \cdot (y \circ x)$ for all $y \in \bar{H}$. But $\bar{H}$ generates $\bar{G}$ so $y\theta[x] \in \bar{G}$ for all $y \in \bar{H}$. Also it follows that $y^{-1}\theta[x] = (y\theta[x])^{-1}$ for all $y \in \bar{H}$ since $\theta[x]$ is an automorphism of $(G, \cdot)$. So $y^{-1}\theta[x] = (y\theta[x])^{-1} \in \bar{G}$ for all $y \in \bar{H}$. Hence, it is now evident that $y^{\pm 1}\theta[x] \in \bar{G}$ for all $y \in \bar{H}$. Furthermore, if $z \in \bar{G}$, then $z = y_1^{\varepsilon_1} \cdot y_2^{\varepsilon_2} \dots y_n^{\varepsilon_n}$ where the $y_i \in \bar{H}$ and $\varepsilon_i = 1$ or -1 and so $z\theta[x] = y_1^{\varepsilon_1}\theta[x] \cdot y_2^{\varepsilon_2}\theta[x] \dots y_n^{\varepsilon_n}\theta[x]$. But it has just been shown that each $y_i^{\varepsilon_i}\theta[x] \in \bar{G}$ and so $z\theta[x] \in \bar{G}$ for all $z \in \bar{G}$. Thus, it follows that $\bar{G}\theta[x] \subseteq \bar{G}$ for all $x \in \bar{H}$.

Now for $x \in \bar{H}$, let $x^{(-1)}$ be the inverse of x in the Bol loop $(\bar{H}, \circ)$. Recall see (12)) that $\theta[x^{(-1)}] = \theta[x]^{-1}$. But, if $x \in \bar{H}$, it follows that $x^{(-1)} \in \bar{H}$. This information yields $\bar{G}\theta[x]^{-1} = \bar{G}\theta[x^{(-1)}] \subseteq \bar{G}$. Hence, $\bar{G} \subseteq \bar{G}\theta[x]$ whenever $x \in \bar{H}$. Finally it follows that $\bar{G}\theta[x] = \bar{G}$ for all $x \in \bar{H}$ and the proof of Theorem 4.2 is complete.

THEOREM 4.3. *Let H be a special subset of a group $(G, \cdot)$. If α is a homomorphism of the group $(G, \cdot)$ onto the group $G\alpha$ with kernel K such that*

(a) $x = y$ *whenever* $x, y \in H$ *and* $x\alpha = y\alpha$,

(b) $K\theta[y] \subseteq K$ *for all* $y \in H$,

then the Bol loop $(H, \circ)$ is specially embeddable in $G\alpha$.

[2] Every loop isotopic to a Bol loop is a Bol loop (see [17; Section 3]).

PROOF. By (a) the restriction of α to H is a one-to-one map of H onto $H\alpha$. Now, for $y\in H$, define $\bar{\theta}[y\alpha]$ by

$$(x\alpha)\bar{\theta}[y\alpha] = (x\theta[y])\alpha$$

for all $x\in G$.

First it is necessary to show that each $\bar{\theta}[y\alpha]$, $y\in H$, is a well-defined map of $G\alpha$ into $G\alpha$. For this purpose, suppose that $x\alpha=z\alpha$ for $x, z\in G$. Then $z=x\cdot k$ for some $k\in K$ and

$$(z\alpha)\bar{\theta}[y\alpha] = (z\theta[y])\alpha = ((x\cdot k)\theta[y])\alpha = (x\theta[y]\cdot k\theta[y])\alpha = (x\theta[y])\alpha\cdot(k\theta[y])\alpha =$$
$$= (x\theta[y])\alpha \quad (\text{since } k\theta[y]\in K \text{ by (b)})$$
$$= (x\alpha)\bar{\theta}[y\alpha].$$

Hence, $\bar{\theta}[y\alpha]$ is well-defined for each $y\in H$.

Now, for $x, z\in G$ and $y\in H$, note that

$$(x\alpha\cdot z\alpha)\bar{\theta}[y\alpha] = ((x\cdot z)\alpha)\bar{\theta}[y\alpha] = ((x\cdot z)\theta[y])\alpha =$$
$$= (x\theta[y]\cdot z\theta[y])\alpha = (x\theta[y])\alpha\cdot(z\theta[y])\alpha = (x\alpha)\bar{\theta}[y\alpha]\cdot(z\alpha)\bar{\theta}[y\alpha].$$

Thus, for each $y\in H$, it follows that $\bar{\theta}[y\alpha]$ is a homomorphism of the group $G\alpha$ into itself.

Also for $x\in G$ and $y\in H$ observe that $((x\theta[y]^{-1})\alpha)\bar{\theta}[y\alpha]=(x\theta[y]^{-1}\theta[y])\alpha=x\alpha$. Thus, $\bar{\theta}[y\alpha]$ maps $G\alpha$ *onto* $G\alpha$. For $x, y\in H$ define $x\alpha\bar{\circ}y\alpha=y\alpha\cdot(x\alpha\bar{\theta}[y\alpha])$ and note that

$$(x\circ y)\alpha = (y\cdot x\theta[y])\alpha = y\alpha\cdot(x\theta[y])\alpha =$$
$$= y\alpha\cdot(x\alpha)\bar{\theta}[y\alpha] = x\alpha\bar{\circ}y\alpha.$$

Thus, $(H, \circ)$ and $(H\alpha, \bar{\circ})$ are isomorphic Bol loops.

The preceding information, combined with the fact that $H\alpha$ generates $G\alpha$, implies that $H\alpha$ is a special subset of $G\alpha$. Hence, $(H, \circ)$ is specially embeddable in $G\alpha$ and Theorem 4.3 is established.

§ 5. A universal embedding of finite Bol loops

THEOREM 5.1. *A finite Bol loop of order $n+1$ is specially embeddable in the free group on n generators.*

PROOF. Let $(K, \cdot)$ be a Bol loop of order $n+1$ and let $K=\{a_0, a_1, \dots, a_n\}$ with a_0 denoting the identity element of $(K, \cdot)$. Now let G be the free group on n free generators $x_1, x_2, \dots, x_n$ and let x_0 be the identity element of G. Now let $H= =\{x_0, x_1, \dots, x_n\}$. Clearly H generates G. Define a binary operation "$\circ$" on H by requiring that $x_i\circ x_j=x_k$ if and only if $a_i\cdot a_j=a_k$ for all $i, j, k=0, 1, \dots, n$. Then $(H, \circ)$ is a Bol loop isomorphic to $(K, \cdot)$.

For each fixed integer $j, 0\leqq j\leqq n$, define the map π_j by $x_i\pi_j=x_i\circ x_j$ for $i=0, 1, \dots, n$. Since $(H, \circ)$ is a Bol loop, π_j is a permutation of H. It will now be essential to establish that, for each $x_j\in H$, there exists an automorphism $\theta[x_j]$ of the group G with the property that

$$x_i\theta[x_j] = x_j^{-1}\cdot x_i\pi_j \tag{14}$$

for all $x_i \in H$. Once such automorphisms are available then $x_i \circ x_j = x_j \cdot x_i \theta[x_j]$ for all $x_i, x_j \in H$, it will follow that H is a special subset of G, and it will be evident that $(K, \cdot)$ is specially embeddable in G.

Let $\theta[x_0] = I$, the identity map, and note that (14) is satisfied for all $x_i \in H$ and $j=0$. Now for $p, q = 1, 2, \ldots, n$, with $p \neq q$, there exist automorphisms P_{pq}, V_p, and W_{pq} of G such that

$$P_{pq}: x_p \to x_q,\ x_q \to x_p,\ x_k \to x_k \quad \text{for} \quad k \neq p, q,$$

$$V_p: x_p \to x_p^{-1},\ x_k \to x_k \quad \text{for} \quad k \neq p,$$

$$W_{pq}: x_q \to x_p x_q,\ x_k \to x_k \quad \text{for} \quad k \neq q.$$

(See [14; p. 111].) For $x_j \in H$, $j \neq 0$, of order m in $(H, \circ)$, let $x_{j_1} = x_j \pi_j$, $x_{j_2} = x_{j_1} \pi_j, \ldots, x_{j_{m-1}} = x_0$. Since π_j is a permutation on the set

$$H - \{x_j, x_{j_1}, \ldots, x_{j_{m-1}}\},$$

there exists an automorphism θ_j of G (expressible as a finite product of automorphisms of G of type P_{pq}) such that $x_j \theta_j = x_{j_1}$, $x_{j_1} \theta_j = x_{j_2}, \ldots, x_{j_{m-3}} \theta_j = x_{j_{m-2}}$, $x_{j_{m-2}} \theta_j = x_j$, and $y \theta_j = y \pi_j$ for all $y \in H - \{x_j, x_{j_1}, \ldots, x_{j_{m-1}}\}$. Then θ_j followed by $n-1$ applications of automorphisms of type W_{jq}, $q \neq j$, and one application of V_j yields an automorphism $\theta[x_j]$ of G which satisfies (14). In view of the above remarks, the proof of Theorem 5.1 is complete.

It will now be shown (see Theorem 5.2) that the embedding effected in Theorem 5.1 is, in a sense, universal (see Definition 5.1). It is convenient, however, to first present a partial converse to Theorem 4.3.

LEMMA 5.1. *Let H be a special subset of a group $(G, \cdot)$ and let α be a homomorphism with kernel K of $(G, \cdot)$ onto a group $G\alpha$. If $H\alpha$ is a special subset of $G\alpha$ and if the restriction of α to H is a homomorphism of the Bol loop $(H, \circ)$ onto the Bol loop $(H\alpha, \circ)$, then $K\theta[y] \subseteq K$ for all $y \in H$.*

PROOF. For all $x, y \in H$, observe (in view of the hypotheses) that $y\alpha \cdot x\theta[y]\alpha = (y \cdot x\theta[y])\alpha = (x \circ y)\alpha = y\alpha \cdot (x\alpha)\theta[y\alpha]$. Consequently, $(x\theta[y])\alpha = (x\alpha)\theta[y\alpha]$ for all $x, y \in H$ and, since H generates G, it follows that

$$(z\theta[y])\alpha = (z\alpha)\theta[y\alpha] \tag{15}$$

for all $z \in G$ and all $y \in H$. From (15) with $z \in K$ it follows that $(z\theta[y])\alpha = (z\alpha)\theta[y\alpha] = (e\alpha)\theta[y\alpha] = e\alpha$ for all $y \in H$ where e is the identity element of $(G, \cdot)$. Hence, $K\theta[y] \subseteq K$ for all $y \in H$ and the proof is complete.

DEFINITION 5.1. A Bol loop $(K, *)$ is *universally embeddable* in a group $(G, \cdot)$ means that the following two conditions hold:

(c) $(K, *)$ is specially embeddable in $(G, \cdot)$.

(d) If $(K, *)$ is specially embeddable in a group $(\bar{G}, \cdot)$, then there is a homomorphism α of $(G, \cdot)$ onto $(\bar{G} \cdot)$ such that conditions (a) and (b) of Theorem 4.3 hold.

THEOREM 5.2. *A finite Bol loop of order $n+1$ is universally embeddable in the free group on n generators.*

PROOF. Let $(K, *)$ be a Bol loop of order $n+1$ and let $(G, \cdot)$ be the free group on n free generators. In view of Theorem 5.1, it is clear that condition (c) of Definition 5.1 holds.

Now assume that $(K, *)$ is specially embeddable in a group $(\bar{G}, \cdot)$. Then there is a special subset $\bar{H}$ of $(\bar{G}, \cdot)$ so that $(\bar{H}, \circ)$ and $(K, *)$ are isomorphic. Let $\bar{H}=\{a_0, a_1, ..., a_n\}$ with a_0 denoting the identity element of the Bol loop $(\bar{H}, \circ)$. Let $x_1, x_2, ..., x_n$ be free generators for the free group $(G, \cdot)$. Now let $H=\{x_0, x_1, ..., x_n\}$ where x_0 is the identity element of $(G, \cdot)$. Define "$\circ$" on H by requiring that $x_i \circ x_j = x_k$ if and only if $a_i \circ a_j = a_k$ for $i, j, k=0, 1, ..., n$. Then, just as in the proof of Theorem 5.1, it follows that H is a special subset of $(G, \cdot)$. Since $(\bar{G}, \cdot)$ is generated by $a_1, a_2, ..., a_n$, there is a homomorphism α of $(G, \cdot)$ onto $(\bar{G}, \cdot)$ so that $x_i\alpha = a_i$ for $i=0, 1, ..., n$. Furthermore, the restriction of α to H is an isomorphism of $(H, \circ)$ onto $(\bar{H}, \circ)$. Thus, by Lemma 5.1, condition (b) of Theorem 4.3 holds. Obviously, condition (a) of Theorem 4.3 holds and the proof is complete.

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ÜBER KOMPATIBLE FUNKTIONEN IN UNIVERSALEN ALGEBREN

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1. Einleitung

In seinem Buch über universale Algebra stellte G. GRÄTZER [3] folgendes Problem: Man beschreibe alle Algebren $\mathfrak{A}$ einer vorgegebenen Klasse, für die alle kompatiblen Funktionen Polynomfunktionen sind. Eine Funktion auf $\mathfrak{A}$ mit Werten in $\mathfrak{A}$ heißt dabei kompatibel, wenn sie mit sämtlichen Kongruenzrelationen auf $\mathfrak{A}$ verträglich ist.

In [12] führte R. WILLE den Begriff der zu einer universalen Algebra gehörenden Kongruenzklassengeometrie ein. In seinen Untersuchungen des so hergestellten Zusammenhanges zwischen Geometrie und Algebra spielen die kompatiblen Funktionen eine wesentliche Rolle. Es ist also von Interesse, jene Algebren zu kennen, für die alle kompatiblen Funktionen „algebraisch beschreibbar" sind. Dies ist z.B. der Fall, wenn alle kompatiblen Funktionen Polynomfunktionen sind. Daher nannte H. WERNER [11] Algebren mit dieser Eigenschaft „affin vollständig".

Affin vollständige Algebren stellen auch eine natürliche Verallgemeinerung polynomvollständiger Algebren dar (das sind Algebren $\mathfrak{A}$, für die jede Funktion auf $\mathfrak{A}$ mit Werten in $\mathfrak{A}$ eine Polynomfunktion ist). Polynomvollständige Algebren sind nämlich einfach, und somit ist jede Funktion kompatibel.

Das Problem von G. Grätzer ist bis jetzt nur in wenigen Fällen für Klassen, die auch nichteinfache Algebren enthalten, gelöst, so z.B. für endliche Boolesche Algebren (G. GRÄTZER [2]), für triviale Algebren, für Vektorräume (H. WERNER [11]), für endlich erzeugbare abelsche Gruppen (W. NÖBAUER [10]). Weitere Untersuchungen über kompatible Funktionen und Beispiele affin vollständiger Algebren finden sich in K. BAKER—A. PIXLEY [1], T. K. HU [4], A. ISKANDER [5], H. LAUSCH—W. NÖBAUER [8], W. NÖBAUER [9].

In dieser Arbeit werden kompatible Funktionen auf direkten Produkten von Algebren studiert, sowie für eine Klasse von endlichen Algebren charakterisiert. Unter anderem wird ein kurzer Beweis für die affine Vollständigkeit der endlichen Booleschen Algebren angegeben, sowie das Problem von Grätzer für die Klasse der symmetrischen Gruppen gelöst. Begriffe, die in dieser Arbeit wohl verwendet, aber nicht definiert werden, findet man in H. LAUSCH—W. NÖBAUER [7].

2. Grundlagen

Sei $\mathfrak{A}=\langle A, \Omega\rangle$ eine universale Algebra und k eine positive ganze Zahl. Auf der Menge $F_k(A)$ aller k-stelligen Funktionen über A erklären wir die Operationen von $\mathfrak{A}$ punktweise. Die so erhaltene Algebra $F_k(\mathfrak{A})=\langle F_k(A), \Omega\rangle$ heißt die volle k-stellige Funktionenalgebra über $\mathfrak{A}$. Die von den Projektionen ξ_i $(i=1, \ldots, k)$

und den konstanten Funktionen erzeugte Teilalgebra von $F_k(\mathfrak{A})$ nennen wir die Algebra der k-stelligen Polynomfunktionen über $\mathfrak{A}$ und bezeichnen sie mit $P_k(\mathfrak{A})$. Eine Algebra $\mathfrak{A}$ heißt k-polynomvollständig, wenn gilt $P_k(\mathfrak{A})=F_k(\mathfrak{A})$ und kurz polynomvollständig, wenn $\mathfrak{A}$ k-polynomvollständig für alle positiven ganzen Zahlen k ist.

Polynomvollständige Algebren sind stets einfach (siehe H. LAUSCH—W. NÖBAUER [7]). Daher definiert man: Eine Funktion $f\in F_k(\mathfrak{A})$ heißt kompatibel, wenn für jede Kongruenzrelation $\equiv$ auf $\mathfrak{A}$ gilt: Aus $a_i\equiv b_i$, $i=1, \dots, k$, folgt $f(a_1, \dots, a_k)\equiv f(b_1, \dots, b_k)$. Bezeichnet $C_k(A)$ die Menge aller kompatiblen Funktionen von $F_k(\mathfrak{A})$, so bildet $\langle C_k(A), \Omega\rangle$ eine Teilalgebra von $F_k(\mathfrak{A})$. Da die konstanten Funktionen und die k Projektionen auf jeder Algebra $\mathfrak{A}$ kompatibel sind, gilt $P_k(\mathfrak{A})\subset C_k(\mathfrak{A})$. Eine Algebra $\mathfrak{A}$ heißt k-affin vollständig, wenn $P_k(\mathfrak{A})= =C_k(\mathfrak{A})$ gilt. $\mathfrak{A}$ heißt affin vollständig, wenn $\mathfrak{A}$ k-affin vollständig für alle positiven ganzen Zahlen k ist.

Polynomvollständige Algebren mit höchstens abzählbarem Operationensystem sind stets endlich (siehe H. LAUSCH—W. NÖBAUER [7]). Um das Hindernis der Endlichkeit auszuschalten, erklärt man: Eine k-stellige lokale Polynomfunktion über $\mathfrak{A}$ ist ein Element $f\in F_k(\mathfrak{A})$ mit der Eigenschaft, daß für jede endliche Teilmenge $N\subset A^k$ eine Polynomfunktion $g\in P_k(\mathfrak{A})$ existiert, sodaß gilt: $f|N=g|N$. Die Menge aller k-stelligen lokalen Polynomfunktionen bildet klarerweise eine Teilalgebra von $F_k(\mathfrak{A})$ und wird mit $L_k(\mathfrak{A})$ bezeichnet. Eine Algebra $\mathfrak{A}$ heißt k-lokal polynomvollständig, wenn gilt $L_k(\mathfrak{A})=F_k(\mathfrak{A})$ und kurz lokal polynomvollständig, wenn $\mathfrak{A}$ k-lokalpolynomvollständig für alle positiven ganzen Zahlen k ist. Lokal polynomvollständige Algebren sind stets einfach.

Gilt $L_k(\mathfrak{A})=C_k(\mathfrak{A})$, so nennen wir $\mathfrak{A}$ k-lokal-affin vollständig und kurz lokal-affin vollständig, wenn $\mathfrak{A}$ k-lokal-affin vollständig für alle positiven ganzen Zahlen k ist.

Unmittelbar aus diesen Definitionen folgt:

LEMMA. *Ist $\mathfrak{A}$ eine endliche Algebra, so ist $\mathfrak{A}$ genau dann k-polynomvollständig (k-affin vollständig), wenn $\mathfrak{A}$ k-lokal polynomvollständig (k-lokal-affin vollständig) ist. Ist $\mathfrak{A}$ eine einfache Algebra, so ist $\mathfrak{A}$ genau dann k-affin vollständig (k-lokal-affin vollständig), wenn $\mathfrak{A}$ k-polynomvollständig (k-lokal polynomvollständig) ist.*

Ist $\mathfrak{A}$ k-affin vollständig, so ist $\mathfrak{A}$ auch n-affin vollständig für jedes $n\leqq k$ (siehe W. NÖBAUER [10]). Es ist noch ein offenes Problem, ob — wie im Falle der Polynomvollständigkeit — aus der 2-affinen Vollständigkeit von $\mathfrak{A}$ die affine Vollständigkeit folgt. In dieser Arbeit wird dieses Problem nur in einigen Spezialfällen gelöst.

LEMMA. *Die Klasse aller lokal-affin vollständigen Algebren umfaßt die Klasse aller affin vollständigen Algebren echt.*

BEWEIS. Jeder unendliche einfache Ring mit nichttrivialer Multiplikation ist zwar lokal-affin vollständig, aber nicht affin vollständig.

Bezeichne $\mathfrak{C}(\mathfrak{A})$ den Kongruenzverband von $\mathfrak{A}$ und seien θ_i, $i=1, \dots, n$, Teilmengen von $\mathfrak{C}(\mathfrak{A})$ mit $\bigcup_{i=1}^{n} \theta_i=\mathfrak{C}(\mathfrak{A})$.

LEMMA. *Sei T_i die Menge aller $f\in F_k(\mathfrak{A})$, die mit allen Kongruenzrelationen aus θ_i verträglich sind $(i=1, ..., n)$, so ist $C_k(\mathfrak{A})=\bigcap_{i=1}^{n} T_i$.*

BEWEIS. Folgt sofort aus der Definition der kompatiblen Funktion.

SATZ. *Eine Funktion $f\in F_k(\mathfrak{A})$ ist genau dann kompatibel, wenn für die Funktionen $f_{a_i}\in F_{k-1}(\mathfrak{A})$, $(a_i\in A)$, definiert durch: $f_{a_i}(x_1, ..., x_{k-1})=:f(x_1, ..., x_{k-1}, a_i)$ gilt:*
(1) $f_{a_i}\in C_{k-1}(\mathfrak{A})$ für alle $a_i\in A$.
(2) Für alle $(b_1, ..., b_{k-1})\in A^{k-1}$ und $(a_i, a_j)\in A^2$ gibt es jeweils eine endliche Folge von Elementen $f_{a_i}(b_1, ..., b_{k-1})=z_0, z_1, ..., z_n=f_{a_j}(b_1, ..., b_{k-1})$ und eine Folge von einstelligen Polynomfunktionen p_t, $t=0, ..., n-1$, sodaß gilt: $z_t=p_t(a_i)$, $z_{t+1}=p_t(a_j)$ oder $z_t=p_t(a_j)$, $z_{t+1}=p_t(a_i)$.

BEWEIS. Folgt unmittelbar aus den Eigenschaften der Kongruenzrelationen auf $\mathfrak{A}$ und einem Satz von A. Malcev (siehe E. T. SCHMIDT [13]).

3. Kompatible Funktionen in L-Algebren

DEFINITION. Unter einer L-Algebra verstehen wir eine Algebra $\mathfrak{A}=\langle A, +, /, 0\rangle$ mit zwei binären Operationen + (Addition) und / (Rechtssubtraktion), und einer nullären Operation 0, die folgende Gesetze für alle $a, b\in A$ erfüllt:

1. $(b/a)+a=b$
2. $(b+a)/a=b$
3. $(a/a)=0$
4. $a+0=0+a=a$.

DEFINITION. Unter einer normalen L-Unteralgebra $\mathfrak{N}$ von $\mathfrak{A}$ verstehen wir eine L-Unteralgebra von $\mathfrak{A}$, die folgende Eigenschaften für alle $a, b\in A$ und $n\in N$ besitzt:

(1) $\big((n+a)+b\big)/(a+b)\in N$
(2) $\big(a+(n+b)\big)/(a+b)\in N$
(3) $\big((n+a)/b\big)/(a/b)\in N$
(4) $a/(n+a)\in N$.

In [6] wurde gezeigt, daß eine umkehrbar eindeutige Beziehung zwischen den Kongruenzrelationen einer L-Algebra $\mathfrak{A}$ und ihren normalen L-Unteralgebren besteht. Dabei können wir eine Nebenklassenzerlegung analog zur Gruppentheorie durchführen. Wie eine einfache Rechnung zeigt, gilt:

LEMMA. *Sei $\mathfrak{A}$ eine L-Algebra, $\mathfrak{N}$ eine normale L-Unteralgebra von $\mathfrak{A}$. Dann ist die Kongruenzklasse $[a]$ modulo $N(a\in A)$ gegeben durch: $[a]=N+a=\{n+a|n\in N\}$.*

Sei $\mathfrak{A}$ eine endliche L-Algebra und $\mathfrak{N}$ eine nichttriviale normale L-Unteralgebra, die minimal ist. Bezeichne $A_N(\mathfrak{A})$ die Menge aller Funktionen $f\in F_k(\mathfrak{A})$, die mit allen jenen Kongruenzrelationen vertr äglich sind, die durch normale L-Unteralgebren, die $\mathfrak{N}$ umfassen, induziert werden. Sei $\{a_1, ..., a_r\}$ ein volles Vertretersystem für die Nebenklassen von $\mathfrak{A}$ nach $\mathfrak{N}$. Den Vertreter der Nebenklasse $N+a$ bezeichnen wir mit $\overline{N+a}$.

SATZ. *Ist $f\in A_N(\mathfrak{A})$, dann besitzt f eine Darstellung der Form:*

$$f(n_1+a_{i_1}, \dots, n_k+a_{i_k}) = g_{(i_1,\dots,i_k)}+(n_{(i_1,\dots,i_k)}+w(\overline{N+a_{i_1}, \dots, N+a_{i_k}})),$$

mit $i_1, \dots, i_k\in\{1, \dots, r\}$, $n_1, \dots, n_k, n_{(i_1,\dots,i_k)}\in N$, $w\in C_k(\mathfrak{A}/\mathfrak{N})$ *und* $g_{(i_1,\dots,i_k)}\in F_k(\mathfrak{N})$. *Umgekehrt ist jede Funktion $f\in F_k(\mathfrak{A})$, die so eine Darstellung besitzt aus $A_N(\mathfrak{A})$.*

BEWEIS. Sei $f\in A_N(\mathfrak{A})$, dann gibt es ein $w\in C_k(\mathfrak{A}/\mathfrak{N})$ mit $f(a_{i_1}, \dots, a_{i_k})= =n_{(i_1,\dots,i_k)}+w(\overline{N+a_{i_1}, \dots, N+a_{i_k}})$ mit $n_{(i_1,\dots,i_k)}\in N$. Es ist

$$f(n_1+a_{i_1}, \dots, n_k+a_{i_k}) = g_{(i_1,\dots,i_k)}(n_1, \dots, n_k)+f(a_{i_1}, \dots, a_{i_k})$$

mit $g_{(i_1,\dots,i_k)}\in F_k(\mathfrak{N})$.

Sei nun umgekehrt $f\in F_k(\mathfrak{A})$ und die Bedingung des Satzes erfüllt. Sei $\mathfrak{M}$ normale L-Unteralgebra verschieden von $\{0\}$ und $\mathfrak{A}$, mit $\mathfrak{N}\subset\mathfrak{M}$. Dann ergibt eine kurze Rechnung die Behauptung, wenn man benützt, daß $w\in C_k(\mathfrak{A}/\mathfrak{N})$ und daß $\mathfrak{M}/\mathfrak{N}$ normale L-Unteralgebra von $\mathfrak{A}/\mathfrak{N}$ ist.

FOLGERUNG. *Sei $\mathfrak{A}$ eine endliche L-Algebra und seien $\mathfrak{N}_1,\dots, \mathfrak{N}_s$ ihre nichttrivialen minimalen normalen L-Unteralgebren, so ist f genau dann kompatibel, wenn $f\in A_{N_i}(\mathfrak{A})$ für alle $i=1, \dots, s$.*

BEWEIS. Folgt aus obigem Satz und dem letzten Lemma des vorigen Paragraphen.

FOLGERUNG. *Besitzt die endliche L-Algebra $\mathfrak{A}$ genau eine nichttriviale minimale normale L-Unteralgebra $\mathfrak{N}$ und bezeichnet $\{a_1, \dots, a_r\}$ ein volles Vertretersystem für die Nebenklassen von $\mathfrak{A}$ nach $\mathfrak{N}$, dann ist eine Funktion $f\in F_k(\mathfrak{A})$ genau dann kompatibel, wenn sie eine Darstellung der folgenden Form besitzt:*

$$f(n_1+a_{i_1}, \dots, n_k+a_{i_k}) = g_{(i_1,\dots,i_k)}(n_1, \dots, n_k)+(n_{(i_1,\dots,i_k)}+w(\overline{N+a_{i_1}, \dots, N+a_{i_k}}))$$

mit $i_1, \dots, i_k\in\{1, \dots, r\}$, $n_1, \dots, n_k, n_{(i_1,\dots,i_k)}\in N$, $w\in C_k(\mathfrak{A}/\mathfrak{N})$ *und* $g_{(i_1,\dots,i_k)}\in F_k(\mathfrak{N})$.

BEWEIS. Ist $\mathfrak{N}$ einzige nichttriviale minimale normale L-Unteralgebra, so ist $A_N(\mathfrak{A})=C_k(\mathfrak{A})$.

SATZ. *Sei $\mathfrak{A}$ eine endliche Gruppe, die genau einen nichttrivialen minimalen Normalteiler besitzt, der von der Ordnung ungleich 2 ist. Dann ist $\mathfrak{A}$ genau dann k-affin vollständig, wenn $\mathfrak{N}$ k-polynomvollständig ist und $\mathfrak{A}/\mathfrak{N}$ k-affin vollständig ist.*

BEWEIS. Da $|\mathfrak{N}|\neq 2$, $\mathfrak{A}$ endlich und $\mathfrak{N}$ k-polynomvollständig, ist jede Funktion von der endlichen Menge A^k in N eine Polynomfunktion (siehe [6], Seite 163). Sei $\{g_1, \dots, g_r\}$ ein volles Vertretersystem für die Nebenklassen von $\mathfrak{A}$ nach $\mathfrak{N}$. Wir betrachten die Funktionen $f_t\colon A^k\to N$, definiert durch $f_t(g_{i_1}n_1, \dots, g_{i_k}n_k)=n_t^{-1}$ $(t=1, \dots, k)$. Die f_t sind also Polynomfunktionen. Wir bilden $g_t=\xi_t f_t$ (ξ_t bezeichne die t-te Projektion). Dann ist $g_t(g_{i_1}n_1, \dots, g_{i_k}n_k)=g_{i_t}$. Durch g_t und die konstanten Funktionen können wir also alle jene Funktionen von $F_k(\mathfrak{A})$ erzeugen, die auf jedem k-Tupel von Nebenklassen von $\mathfrak{N}$ konstant sind und kompatibel sind (da $\mathfrak{A}/\mathfrak{N}$ k-affin vollständig ist). Wir definieren nun Funktionen $h_{(i_1,\dots,i_k)}\colon A^k\to N$ durch:

$$h_{(i_1,\dots,i_k)}(x_1, \dots, x_k) = \begin{cases} g_{(i_1,\dots,i_k)}(n_1, \dots, n_k) & \text{wenn} \quad (x_1, \dots, x_k) = (g_{i_1}n_1, \dots, g_{i_k}n_k) \\ 1 & \text{sonst.} \end{cases}$$

Da $\mathfrak{N}$ k-polynomvollständig und $|\mathfrak{N}|\neq 2$, sind auch diese Funktionen Polynomfunktionen auf $\mathfrak{A}$. Durch geeignete Produktbildung der $h_{(i_1,\dots,i_k)}$ und der g_t kann man nach obiger Folgerung jede kompatible Funktion durch Polynomfunktionen darstellen, was wir zeigen wollten.

FOLGERUNG. *Bilden die Normalteiler einer endlichen Gruppe $\mathfrak{A}$ eine Kette: $\mathfrak{N}_0=\{0\}\subset\mathfrak{N}_1\subset\dots\subset\mathfrak{N}_{s-1}\subset\mathfrak{N}_s=\mathfrak{A}$, und ist $\mathfrak{N}_i/\mathfrak{N}_{i-1}$ für $i=1,\dots,s$ k-polynomvollständig, so ist $\mathfrak{A}$ k-affin vollständig, wenn $|\mathfrak{N}_i/\mathfrak{N}_{i-1}|\neq 2$ für $i=1,\dots,s-1$.*

BEWEIS. Wir führen obige Konstruktion schrittweise durch.

BEMERKUNG. Für Gruppen $\mathfrak{A}$, die den Voraussetzungen der obigen Folgerung genügen, folgt aus der 2-affinen Vollständigkeit die affine Vollständigkeit von $\mathfrak{A}$ (nach den bekannten Sätzen aus der Theorie der polynomvollständigen Algebren). Ist zusätzlich auch $|\mathfrak{A}/\mathfrak{N}_{s-1}|\neq 2$, so folgt schon aus der 1-affinen Volständigkeit von $\mathfrak{A}$ die affine Vollständigkeit (folgt aus einem Satz von J. SLUPECKI [14]).

4. Affine Vollständigkeit der symmetrischen Gruppen

SATZ. *Die symmetrischen Gruppen $\mathfrak{S}_n$ sind 1-affin vollständig genau für $n\neq 3,4$.*

BEWEIS. Wir verwenden Satz aus vorigem Paragraphen, also gilt unser Satz für $\mathfrak{S}_n$, $n\geqq 5$, unter Benützung des Satzes aus dem vorigen Paragraphen und der Tatsache, daß $\mathfrak{A}_n$ für $n\geqq 5$ und $\mathfrak{S}_n/\mathfrak{A}_n$ 1-polynomvollständig sind.

Für $n=3,4$ besitzt $\mathfrak{S}_n$ abelsche Gruppen der Ordnung größer 2 als nichttriviale minimale Normalteiler. Diese sind bekanntlich nicht 1-polynomvollständig. $\mathfrak{S}_2$ ist 1-polynomvollständig, für $\mathfrak{S}_1$ ist die Aussage trivial.

SATZ. *Die symmetrischen Gruppen $\mathfrak{S}_n$ mit $n>1$ sind nicht 2-affin vollständig und damit nicht affin vollständig.*

BEWEIS. Wir definieren $f\colon \mathfrak{S}_n^2\to\mathfrak{S}_n$ durch

$$f(x,y)=\begin{cases} x & \text{wenn } x\in\mathfrak{A}_n \text{ oder } x,y\notin\mathfrak{A}_n \\ y & \text{wenn } y\in\mathfrak{A}_n \text{ und } x\notin\mathfrak{A}_n. \end{cases}$$

Diese Funktion ist mit allen Kongruenzrelationen auf $\mathfrak{S}_n$ verträglich, also kompatibel. f ist aber keine Polynomfunktion, denn angenommen f wäre Polynomfunktion, d.h.

$$f(x,y)=a_{11}x^{s_{11}}a_{12}y^{s_{12}}a_{21}\dots y^{s_{r2}}a_{(r+1)1}.$$

Sei s die Gesamtanzahl der Transpositionen in der Zerlegung von $a_{11},\dots,a_{(r+1)1}$ in Transpositionen, t die Summe der Exponenten der Potenzen von x, z die Summe der Exponenten der Potenzen von y. Sind x,y gerade Permutationen, so ist $f(x,y)=x$, d.h. gerade, also muß s gerade sein. Sind x ungerade (gerade) und y gerade (ungerade), so müssen z und t gerade sein. Sind x und y ungerade Permutationen, so ist $f(x,y)=x$, d.h. ungerade, also müßte mindestens eine der drei Zahlen s, t, z ungerade sein. Widerspruch. Somit haben wir das Problem der Bestimmung aller affin vollständigen symmetrischen Gruppen gelöst:

Die symmetrische Gruppe $\mathfrak{S}_1$ ist affin vollständig.

Die symmetrischen Gruppen $\mathfrak{S}_2$, $\mathfrak{S}_n (n \geqq 5)$ sind 1-affin vollständig, aber nicht k-affin vollständig für $k>1$.

Die symmetrischen Gruppen $\mathfrak{S}_3$, $\mathfrak{S}_4$ sind für kein k k-affin vollständig.

5. Kompatible Funktionen auf direkten Produkten

LEMMA. *Sei eine Algebra $\mathfrak{A}$ direktes Produkt der Algebren $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ und $f \in C_k(\mathfrak{A})$. Dann gibt es eindeutig bestimmte $f_i \in C_k(\mathfrak{A}_i)$, $i=1, \ldots, n$, mit:*

$$f\big((a_1^1, \ldots, a_n^1), \ldots, (a_1^k, \ldots, a_n^k)\big) = \big(f_1(a_1^1, \ldots, a_1^k), \ldots, f_n(a_n^1, \ldots, a_n^k)\big).$$

BEWEIS. Folgt durch wiederholte Anwendung von Lemma 4 aus W. NÖBAUER [10].

SATZ. *Seien $\mathfrak{A}_i$, $i=1, \ldots, n$, k-polinomvollständige L-Algebren mit $|\mathfrak{A}_i| \neq 2$. Dann ist $\mathfrak{A} = \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_n$ k-affin vollständig.*

BEWEIS. Wir verwenden obiges Lemma. Jede der Funktionen $f_i \in C_k(\mathfrak{A}_i)$ kann man, da die $\mathfrak{A}_i$ polynomvollständige L-Algebren der Ordnung ungleich 2 sind, zu Polynomfunktionen aus $P_k(\mathfrak{A})$ fortsetzen mit:

$$f\big((a_1^1, \ldots, a_n^1), \ldots, (a_1^k, \ldots, a_n^k)\big) = \big(f_1(a_1^1, \ldots, a_1^k), 0, \ldots, 0)\big) + \ldots + \big(0, \ldots, 0, f_n(a_n^1, \ldots, a_n^k)\big),$$

wo die $f_i \in P_k(\mathfrak{A})$.

BEMERKUNG. Für L-Algebren, die obigem Satz genügen, folgt aus der 1-affinen Vollständigkeit die affine Vollständigkeit.

FOLGERUNG. *Das direkte Produkt von endlich vielen endlichen, einfachen, nichtabelschen Gruppen ist affin vollständig. Das direkte Produkt von endlich vielen einfachen, endlichen Ringen mit nichttrivialer Multiplikation ist affin vollständig (also ist insbesonders jeder halbeinfache Ring affin vollständig).*

BEWEIS. Folgt sofort aus den bekannten Sätzen der Theorie der polynomvollständigen Algebren und aus den Ergebnissen von [6].

SATZ. *Jede endliche Boolesche Algebra ist affin vollständig.*

BEWEIS. Jede endliche Boolesche Algebra $\mathfrak{A}$ ist direktes Produkt von Booleschen Algebren $\mathfrak{B}$ der Ordnung 2. Sei etwa $\mathfrak{A} = \mathfrak{B}^n$. Unter Benützung obigen Lemmas und analog zum Beweis obigen Satzes folgt die Behauptung, da ja $\mathfrak{B}$ polynomvollständig ist.

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SOLUTION D'UN PROBLÈME D'ERDŐS, GILLMAN ET HENRIKSEN ET APPLICATION A L'ÉTUDE DES HOMOMORPHISMES DE $\mathscr{C}(K)$

Par

J. ESTERLE (Talence)

§ 1 — Introduction

La motivation essentielle de ce travail est l'étude d'un problème classique d'Analyse Fonctionnelle, généralement attribué à I. Kaplansky: Soit K un compact infini. Existe-t-il un homomorphisme discontinu de $\mathscr{C}(K)$ dans une algèbre normée? (voir [1], [5], [10], [19], [27] et [28]). A. M. SINCLAIR a récemment montré dans [28] que pour qu'un tel homomorphisme existe il faut et il suffit qu'il existe un idéal premier non maximal I de $\mathscr{C}(K)$ tel que l'algèbre quotient $\mathscr{C}(K)/I$ possède une structure d'algèbre normée (ce résultat a été retrouvé indépendamment par l'auteur dans [10] par une méthode basée sur une extension de la première partie de [1]).

D'autre part B. E. Johnson a montré, entre autres résultats, dans [16], que si l'on admet l'hypothèse du continu le problème est équivalent pour tout compact infini à l'existence d'une norme d'algèbre sur l'algèbre des éléments bornés d'un corps ordonné maximal de type η_1 ayant la puissance du continu. Dans le présent article, qui est indépendant de [19], on retrouve et on précise ce résultat de Johnson. On construit un corps ordonné maximal $\mathscr{F}_{\omega_1}^{(1)}$ de type η_1 ayant la puissance du continu et on montre que s'il existe un homomorphisme discontinu de $\mathscr{C}(K)$ alors l'algèbre des éléments bornés de $\mathscr{F}_{\omega_1}^{(1)}$ possède une structure d'algèbre normée réelle, et ceci indépendamment de l'hypothèse du continu. La condition devient suffisante pour tout compact infini si l'on admet l'hypothèse du continu (théorème 4—2). Ces résultats améliorent ceux de Johnson même si l'on admet l'hypothèse du continu car le corps $\mathscr{F}_{\omega_1}^{(1)}$ est construit de manière purement formelle et on connaît avec précision sa structure. On montre en outre au théorème 5—3 que si l'algèbre des éléments infinitésimaux de $\mathscr{F}_{\omega_1}^{(1)}$ possède une structure d'algèbre normée elle possède une structure d'algèbre normée topologiquement simple.

Tout ce travail est basé sur les considérations algébriques des § 2 et 3 qui semblent présenter un certain intérêt en elles mêmes: on montre notamment (corollaire 2—4) que si on suppose $2^{\aleph_\beta}=\aleph_\alpha$ (α désignant un ordinal possédant un prédecesseur β) il existe des corps ordonnés maximaux de type η_α et de cardinal $2^{\aleph_\beta}$ qui ne sont pas isomorphes, ce qui résout un vieux problème d'ERDŐS, GILLMAN et HENRIKSEN (voir [9], problème 5—4; je dois dire qu'après avoir soumis la première version de cet article j'ai reçu une lettre de L. Gillman me signalant que son élève N. Bloch avait obtenu un résultat analogue dans sa thèse à l'Université de Rochester et qu'il avait mentionné ce fait dans sa conférence au Prague Topological Symposium de 1966, mais les travaux de N. Bloch n'ont jamais été publiés).

En fait on s'appuie ici sur deux théories fort anciennes, celle des ensembles totalement ordonnés de type η_α de HAUSDORFF (cf. [18]) et celle des groupes et des «corps de séries formelles» de HAHN (cf. [17]). L'utilisation des techniques de Hahn permet d'étendre aux groupes et corps totalement ordonnés les résultats obtenus pour les ensembles totalement ordonnés par Hausdorff, puis par SIERPINSKI et GILLMAN ([26] et [13]) ce qui ne semblait pas avoir été fait systématiquement jusqu'ici. On connaît en effet les résultats suivants pour les ensembles totalement ordonnés de type η_α, α désignant pour simplifier un ordinal possédant un prédecesseur β (la définition des ensembles ordonnés de type η_α est rappelée au début du § 2):

I) Il existe des ensembles totalement ordonnés de type η_α et de cardinal $2^{\aleph_\beta}$.

II) En particulier, il existe un ensemble totalement ordonné S_{ω_α} de type η_α et de cardinal $2^{\aleph_\beta}$ qui est réunion d'une famille $(S_\xi)_{\xi<\omega_\alpha}$ d'ensembles totalement ordonnés dont toute partie possède un sous-ensemble coinitial et cofinal de cardinal inférieur à $\aleph_\alpha$.

III) Tout ensemble totalement ordonné de type η_α contient une copie de tout ensemble totalement ordonné de cardinal $\aleph_\alpha$.

IV) Tout ensemble totalement ordonné de type η_α contient une copie de S_{ω_α} (en particulier tout intervalle de S_{ω_α} contient une copie de S_{ω_α}).

V) Les ensembles totalement ordonnés de type η_α et de cardinal $\aleph_\alpha$ sont isomorphes (mais il n'en existe que si $\aleph_\alpha = 2^{\aleph_\beta}$).

VI) Si $2^{\aleph_\beta} > \aleph_\alpha$, il existe des ensembles totalement ordonnés de type η_α et de cardinal $2^{\aleph_\beta}$ non isomorphes.

ERDŐS, GILLMAN et HENRIKSEN ont prouvé pour les corps ordonnés maximaux dans [9] l'analogue de III et V et le problème 5.4 de [9] solulerait une question analogue à VI. Je pensais que le problème de l'existence de corps ordonnés maximaux de type η_α était toujours ouvert pour $\alpha > 1$ (les corps hyper-réels donnent des exemples «naturels» de corps ordonnés maximaux de type η_1, voir [15]) mais L. Gillman m'a indiqué dans sa lettre que ce problème avait été résolu en 1962 par N. ALLING dans [3] (N. Alling utilise également les groupes et corps de Hahn, et on trouve dans [3] un énoncé voisin du lemme 2.1; Alling avait établi dans [2] l'analogue de I, III et V pour les groupes abéliens totalement ordonnés).

I et II sont établis au théorème 2.3, VI au corollaire 2.4, IV au théorème 3.2 (et en un certain sens au théorème 2.6). V et III sont donnés au théorème 3.2 et à la remarque 3.5 sous la forme améliorée établie récemment pour $\alpha = 1$ par B. E. JOHNSON dans [19] (cette amélioration figure également parmi les travaux non publiés de N. J. Bloch mentionnés par L. GILLMAN dans [14]). Certains résultats sont indirectement liés au célèbre « théorème d'immersion » des groupes totalement ordonnés de Hahn (voir la remarque 3—6).

Je dois remercier F. J. Rayner qui m'a aidé dans un domaine qui m'était fort peu familier et m'a suggéré d'utiliser les groupes et les « corps de séries formelles » de Hahn, ainsi que L. Gillman qui m'a signalé l'existence de la thèse non publiée de N. J. Bloch et m'a indiqué les références [2] et [3].

§ 2 — Solution d'un problème d'Erdős, Gillman et Henriksen

Pour tout ordinal α on notera $\aleph_\alpha$ l'aleph d'indice α et ω_α l'ordinal initial d'indice α (voir par exemple [6], ch. III, §. 6, ex. 10). Conformément à [18], on dira qu'un ensemble totalement ordonné T est de type η_α si et seulement si pour couple (A, B) de parties de T vérifiant $A > B$, card $A < \aleph_\alpha$, card $B < \aleph_\alpha$, il existe trois éléments c_1, c_2, c_3 de T tels que l'on ait: $c_1 > A > c_2 > B > c_3$ (la notation $A > B$ signifie que l'on a: $a > b$ pour tout $a \in A$ et tout $b \in B$). Pour tout ordinal $\xi \leqq \omega_\alpha$ on notera

S_ξ l'ensemble des suites transfinies $\{x_\theta\}_{\theta<\omega_\alpha}$ dont tous les termes sont égaux à 0 ou 1 et dont l'ensemble des rangs des termes égaux à 1 admet un plus grand élément inférieur à ξ. On munit S_ξ de l'ordre lexicographique. S_ξ est totalement ordonné, pour $\xi<\omega_\alpha$ toute partie de S_ξ possède un sous-ensemble coinitial et cofinal de cardinal $<\aleph_\alpha$, S_ξ est Dedekind-complet si ξ a un prédécesseur, S_{ω_α} est de type η_α et card $(S_{\omega_\alpha})=2^{\aleph_\beta}$ si α a un prédécesseur β (voir [15], ch. 13, [13] et [26]).

Si F est un corps totalement ordonné, on notera $\mathscr{R}(F)$ l'extension algébrique totalement ordonnée maximale de F (voir [4]). $\mathscr{R}(F)$ est un *corps totalement ordonné maximal* (voir [7], ch. 6, §. 2, déf. 4).

Pour tout ensemble totalement ordonné T on notera $\mathscr{G}(T, R)$ (resp. $\mathscr{G}_\alpha(T, R)$) l'espace vectoriel des fonctions de T dans R à support bien ordonné (resp. à support bien ordonné de cardinal $<\aleph_\alpha$) muni de son ordre total naturel: un élément $x\neq 0$ de $\mathscr{G}(T, R)$ est positif si et seulement si il prend une valeur positive sur le plus petit terme de son support (voir [11], ch. IV). Pour tout $x\in\mathscr{G}(T, R)$ on notera Ord (x) l'ordinal isomorphe au support de x. Pour tout groupe abélien totalement ordonné G on notera $\mathscr{F}(G, R)$ le «*corps des séries formelles à coefficients réels ayant G pour groupe d'exposants*» (voir [11], ch. VIII). Tout élément $x\neq 0$ de $\mathscr{F}(G, R)$ s'écrit «formellement»: $x=\sum_{i<\mathrm{Ord}(x)}\lambda_i X^{a_i}$ $(\lambda_i\in R-\{0\})$ où l'ensemble $(a_i)_{i<\mathrm{Ord}(x)}$ est une partie bien ordonnée de G que l'on appellera le *support de x*. Si α a un prédécesseur, on pose de même que plus haut:

$$\mathscr{F}_\alpha(G, R) = \{x\in\mathscr{F}(G, R)\colon \mathrm{card}\,[\mathrm{Supp}\,(x)] < \aleph_\alpha\}.$$

Soit $x'=\sum_{j<\mathrm{Ord}(x')}\lambda'_j X^{a'_j}$ un autre élément de $\mathscr{F}(G, R)$. Le produit xx' s'écrit:

$$xx' = \sum_{\nu<\mathrm{Ord}(xx')}\Big[\sum_{a_i+a'_j=a''_\nu}\lambda_i\lambda'_j X^{a''_\nu}\Big].$$

HAHN a prouvé (dès 1907!) dans [17] que $\mathscr{F}(G, R)$ et $\mathscr{F}_\alpha(G, R)$ sont bien des corps (voir [22] pour des généralisations au cas non commutatif. A. GLEYZAL a utilisé des méthodes analogues dans [12] pour construire ses «*corps réels transfinis*»). $\mathscr{F}(G, R)$ et $\mathscr{F}_\alpha(G, R)$ sont respectivement isomorphes en tant qu'espaces vectoriels réels à $\mathscr{G}(G, R)$ et $\mathscr{G}_\alpha(G, R)$ et on les munit de l'ordre induit par ces isomorphismes. Pour tout ordinal ξ, on pose $G_\xi=\mathscr{G}(S_\xi, R)$, $\mathscr{F}_\xi=\mathscr{F}(G_\xi, R)$ et pour tout ordinal α possédant un prédécesseur, on pose:

$$G^{(\alpha)}_{\omega_\alpha} = \mathscr{G}_\alpha(S_{\omega_\alpha}, R),\quad \mathscr{F}^{(\alpha)}_{\omega_\alpha} = \mathscr{F}_\alpha(G^{(\alpha)}_{\omega_\alpha}, R).$$

On va tout d'abord établir un lemme (on trouvera un résultat analogue dans [3], mais la démonstration donnée ici diffère sensiblement de celle de N. ALLING).

LEMME 2.1. — *Soit T un ensemble totalement ordonné et soit α un ordinal possédant un prédécesseur. Si T est de type η_α, $\mathscr{G}_\alpha(T, R)$ est de type η_α.*

Soit ξ un ordinal $\leqq\omega_\alpha$ et soit $x\in\mathscr{G}_\alpha(T, R)$. Si Ord $(x)\geqq\xi$ il existe un isomorphisme unique de ξ sur un segment B de Supp (x). On note alors $\psi_\xi(x)$ l'élément de $\mathscr{G}_\alpha(T, R)$ coincidant avec x sur B et nul hors de B. Si Ord $(x)<\xi$, on pose: $\psi_\xi(x)=x$. On voit aisément que l'on a les propriétés suivantes:

a) $\psi_\xi(x)\geqq\psi_\xi(y)$ si $x\geqq y$

b) $x>y$ si $\psi_\xi(x)>\psi_\xi(y)$

c) $\psi_\xi(x)=\psi_\xi(y)\Leftrightarrow\psi_\theta(x)=\psi_\theta(y)$ pour tout $\theta<\xi$ si ξ est un ordinal limite
d) $\psi_{\xi'}[\psi_\xi(x)]=\psi_{\xi'}(x)$ pour $x\in\mathscr{G}_\alpha(T,R)$ et $\xi'<\xi\leqq\omega_\alpha$.

Pour tout élément β de T on note $1\cdot\beta$ l'élément de $\mathscr{G}_\alpha(T,R)$ égal 1 en β et nul hors de $\{\beta\}$. Pour $x\in\mathscr{G}_\alpha(T,G)-\{0\}$, on note $\varphi_1(x)$ le plus petit terme du support de x et $\delta_1(x)$ la valeur de x en $\varphi_1(x)$. Pour toute partie A non vide de $\mathscr{G}_\alpha(T,R)$, on note A^+ l'ensemble des éléments positifs de A. Si A^+ est vide il est trivial de majorer A. Si A^+ est non vide, il est cofinal dans A. Si $\operatorname{card}(A)<\aleph_\alpha$, on a: $\operatorname{card}[\{\varphi_1(x)\}_{x\in A^+}]<\aleph_\alpha$ et il existe $\mu\in T$ tel que: $\mu<\varphi_1(x)\,(x\in A^+)$. On a: $\psi_1(1\cdot\mu)>\psi_1(x)\,(x\in A^+)$ d'où: $1\cdot\mu>A$. De même $-A$ est majorée et A est minorée. Soit maintenant B une autre partie non vide de $\mathscr{G}_\alpha(T,R)$ de cardinal $<\aleph_\alpha$ vérifiant: $A>B$. Si B^+ est non vide et si $\varphi_1(A)<\varphi_1(B^+)$, il existe $\mu\in T$ vérifiant: $\varphi_1(A)<<\mu<\varphi_1(B^+)$ et on a: $\psi_1(x)>1\cdot\mu>\psi_1(y)$ $(x\in A,\ y\in B^+)$ d'où d'après a): $A>>1\cdot\mu>B$. Si $\varphi_1(A)\cap\varphi_1(B^+)$ contient un élément (unique) γ, posons:

$$\lambda_1=\inf_{x\in\varphi_1^{-1}(\gamma)\cap A}\delta_1(x);\quad \lambda_2=\sup_{x\in\varphi_1^{-1}(\gamma)\cap B^+}\delta_1(x).$$

On a: $\lambda_1\geqq\lambda_2$. Si $\lambda_1<\lambda_2$ on a: $A>\frac{\lambda_1+\lambda_2}{2}\cdot\gamma>B$. Si $\lambda_1=\lambda_2$, l'un au moins des ensembles $\delta_1^{-1}(\lambda_1)\cap\varphi_1^{-1}(\gamma)\cap A$ ou $\delta_1^{-1}(\lambda_1)\cap\varphi_1^{-1}(\gamma)\cap B$, disons $\delta_1^{-1}(\lambda_1)\cap\cap\varphi_1^{-1}(\gamma)\cap A$, est vide si $\psi_1(A)>\psi_1(B)$. Si $\lambda_1\cdot\gamma$ majore B^+, soit $\mu>\gamma$. On a: $A>>\lambda_1\cdot\gamma+1\cdot\mu>B$. Sinon posons: $D=\{y\in B^+:y>\lambda_1\cdot\gamma\}$. Pour $y\in D$ on a: $\psi_1(y)==\lambda_1\cdot\gamma$ et $\varphi_1(y-\lambda_1\gamma)>\gamma$. Soit $\mu\in T$ tel que $\gamma<\mu<\varphi_1(y-\lambda_1\gamma)\,(y\in D)$. On a: $\psi_1[\lambda_1\cdot\gamma+1\cdot\mu]>\psi_1(y)\,(y\in D)$, d'où: $A>\lambda_1\cdot\gamma+1\cdot\mu>B$. On voit de même que si $\psi_1(A)>\psi_1(B)$ et si B^+ est vide, il existe $c\in\mathscr{G}_\alpha(T,R)$ tel que $A>c>B$.

Supposons maintenant que $\psi_1(A)\cap\psi_1(B)$ possède un (unique) élément. Comme α n'a pas de prédécesseur, $[0,\omega_\alpha[$ ne possède aucune partie cofinale de cardinal $<\aleph_\alpha$, il existe $\theta<\omega_\alpha$ vérifiant: $\theta>\operatorname{Ord}(x)\,(x\in A\cup B)$ et on a: $\psi_\theta(A)>>\psi_\theta(B)$. Soit θ_1 le plus petit ordinal possédant cette propriété. On a: $1<\theta_1<\omega_\alpha$. Pour $\xi<\theta_1$ soit c_ξ l'unique élément de $\psi_\xi(A)\cap\psi_\xi(B)$. On a pour $\xi'\leqq\xi:\psi_{\xi'}(c_\xi)=c_{\xi'}$. Si θ_1 est un ordinal limite l'un au moins des deux ensembles $\bigcap_{\xi<\theta_1}\psi_\xi^{-1}(c_\xi)\cap A$ et $\bigcap_{\xi<\theta_1}\psi_\xi^{-1}(c_\xi)\cap B$, disons $\bigcap_{\xi<\theta_1}\psi_\xi^{-1}(c_\xi)\cap A$, est vide. Soit c un élément de $\mathscr{G}_\alpha(T,R)$ vérifiant: $\psi_\xi(c)=c_\xi(\xi<\theta_1)$. Pour tout élément x de A il existe $\xi<\theta_1$ tel que l'on ait: $\psi_\xi(x)>c_\xi$ d'où, d'après b): $x>c$; posons: $D=\{y\in B;\ y>c\}$. Si D est vide, soit μ un élément de T majorant strictement le support de c. On a: $A>c+1\cdot\mu>B$. Si D est non vide, il est cofinal dans B et pour $y\in D$, $\varphi_1(y-c)$ majore $\bigcup_{\xi<\theta_1}\operatorname{Supp}(c_\xi)$ qui est de cardinal $<\aleph_\alpha$. Soit $\gamma\in T$ vérifiant: $\bigcup_{\xi<\theta_1}\operatorname{Supp}c_\xi<\gamma<\{\varphi_1(y-c)\}_{y\in D}$. On a: $1\cdot\gamma>y-c(y\in D)$ et: $c+1\cdot\gamma>B$. D'autre part pour $\xi<\theta_1$ on a $\psi_\xi(c++1\cdot\gamma)=\psi_\xi(c)=c_\xi$ d'où $A>c+1\cdot\gamma>B$. Si θ_1 possède un prédécesseur σ les ensembles non vides $D_1=\psi_\sigma^{-1}(c_\sigma)\cap A$ et $D_2=\psi_\sigma^{-1}(c_\sigma)\cap B$ sont respectivement coinitiaux et cofinaux dans A et B. Pour $x\in D_1$ et $y\in D_2$ on a: $\psi_1(x-c_\sigma)=\psi_{\theta_1}(x)-c_\sigma$, $\psi_1(y-c_\sigma)=\psi_{\theta_1}(y)-c_\sigma$ soit: $\psi_1(x-c_\sigma)>\psi_1(y-c_\sigma)$. Comme on l'a vu plus haut, il existe alors $d\in\mathscr{G}_\alpha(T,R)$ vérifiant $x-c_\sigma>d>y-c_\sigma(x\in D_1,\ y\in D_2)$ d'où: $A>c_\sigma+d>B$ et le lemme est démontré.

On a également le:

LEMME 2.2. — *Soit T un ensemble totalement ordonné et soit α un ordinal possédant un prédécesseur. Si toute partie de T possède un sous-ensemble coinitial et cofinal de cardinal $<\aleph_\alpha$, il en est de même de toute partie de $\mathscr{G}(T, R)$.*

Notons que si T vérifie les hypothèses du lemme, toute partie bien ordonnée de T est de cardinal $<\aleph_\alpha$ (toute partie cofinale de $[0, \omega_\alpha[$ a pour cardinal $\aleph_\alpha$) et que l'on a donc: $\mathscr{G}(T, R)=\mathscr{G}_\alpha(T, R)$. Soit A une partie non vide de $\mathscr{G}(T, R)$. Si $A^+\neq\emptyset$ et si $\varphi_1(A^+)$ n'a pas de plus petit élément, soit D une partie coinitiale de $\varphi_1(A^+)$ de cardinal $<\aleph_\alpha$ et pour tout $\gamma\in D$ soit x_γ un élément de $\varphi_1^{-1}(\gamma)\cap A^+$. On a: $\mathrm{card}\,[\{x_\gamma\}]_{\gamma\in D}<\aleph_\alpha$ et $\{x_\gamma\}_{\gamma\in D}$ est cofinal dans A. Si $\varphi_1(A^+)$ a un plus petit élément d, si A^+ est non vide et si $\psi_1(A)$ n'a pas de plus grand élément, $\delta_1[\varphi_1^{-1}(d)\cap A^+]$ n'a pas de plus grand élément et possède une partie dénombrable cofinale E. Pour tout élément λ de E soit x_λ un élément de $\psi_1^{-1}[\lambda\cdot d]\cap$ $\cap A^+$. L'ensemble dénombrable $\{x_\lambda\}_{\lambda\in E}$ est cofinal dans A^+ et donc dans A. Une vérification analogue montre que A possède une partie cofinale de cardinal $<\aleph_\alpha$ si A^+ est vide et si $\psi_1(A)$ n'a pas de plus grand élément.

Supposons maintenant que $\psi_1(A)$ a un plus grand élément et que A n'a pas de plus grand élément. Dans ce cas $\psi_{\omega_\alpha}(A)$ n'a pas de plus grand élément (on a : $\psi_{\omega_\alpha}(x)=$ $=x$ pour tout x). Soit θ le plus petit ordinal $\leqq\omega_\alpha$ tel que $\psi_\theta(A)$ n'ait pas de plus grand élément. On a : $\theta>1$. Si $\xi<\theta$, $\psi_\xi(A)$ a un plus grand élément c_ξ et on a: $c_{\xi'}=\psi_{\xi'}(c_\xi)$ si $\xi'<\xi$. Si $\theta=\sigma+1$, posons: $D=\psi_\sigma^{-1}(c_\sigma)\cap A$ et $D'=\{x-c_\sigma\}_{x\in D}$. D est non vide et cofinal dans A. On a: $\psi_\theta(x)-c_\sigma=\psi_1(x-c_\sigma)$ $(x\in D)$ et $\psi_1(D')$ n'a pas de plus grand élément. Par conséquent D' possède une partie cofinale M de cardinal $<\aleph_\alpha$ et l'ensemble $\{y+c_\sigma\}_{y\in M}$ est cofinal dans A.

Si θ est un ordinal limite, soit $\xi<\theta$. Si $c_{\xi+1}=c_\xi$ soit x un élément de $\psi_{\xi+1}^{-1}(c_{\xi+1})\cap A$. On aurait $\psi_{\xi+1}(x)=c_{\xi+1}=c_\xi=\psi_\xi(x)$, d'où: $x=\psi_\xi(x)=c_\xi$. x serait donc unique et d'après b) serait le plus grand élément de A, cas que nous avons écarté. On a donc: $c_{\xi+1}\neq c_\xi(\xi<\theta)$, d'où: $\mathrm{Ord}\,(c_\xi)=\xi$ $(\xi<\theta)$. Par conséquent l'ensemble bien ordonné $\bigcup_{\xi<\theta}\mathrm{Supp}\,(c_\xi)$ est isomorphe à $[0, \theta[$ et comme il a une partie cofinale de cardinal $<\aleph_\alpha$, on a: $\theta<\omega_\alpha$. Pour $\xi<\theta$, soit x_ξ un élément de $\psi_\xi^{-1}(c_\xi)\cap A$. L'ensemble $[\bigcap_{\xi<\theta}\psi_\xi^{-1}(c_\xi)]\cap A$ est vide ($\psi_\theta(A)$ aurait sinon un plus grand élément) et pour $x\in A$ il existe $\xi<\theta$ vérifiant $\psi_\xi(x)<c_\xi=\psi_\xi(x_\xi)$, ce qui montre que $\{x_\xi\}_{\xi<\theta}$ est cofinal dans A et achève la démonstration (si D est une partie cofinale de $-A$, $-D$ est une partie coinitiale de A). On a alors le théorème suivant:

THÉORÈME 2.3. — *Soit α un ordinal possédant un prédecesseur β.*

a) *$\mathscr{F}_{\omega_\alpha}^{(\alpha)}$ est un corps ordonné maximal ainsi que $\mathscr{F}_\xi$ pour tout ordinal $\xi\leqq\omega_\alpha$.*

b) *Pour $\xi<\omega_\alpha$, toute partie de $\mathscr{F}_\xi$ (resp. G_ξ) possède un sous-ensemble coinitial et cofinal de cardinal inférieur à $\aleph_\alpha$ et toute partie de $\mathscr{F}_{\omega_\alpha}^{(\alpha)}$ (resp. $G_{\omega_\alpha}^{(\alpha)}$) possède un sous-ensemble coinitial et cofinal de cardinal inférieur ou égal à $\aleph_\alpha$.*

c) *$\mathscr{F}_{\omega_\alpha}^{(\alpha)}$ (resp. $G_{\omega_\alpha}^{(\alpha)}$) est de type η_α et on a:*

$$\mathrm{card}\,[\mathscr{F}_{\omega_\alpha}^{(\alpha)}]=\mathrm{card}\,[G_{\omega_\alpha}^{(\alpha)}]=2^{\aleph_\beta};\quad \mathscr{F}_{\omega_\alpha}^{(\alpha)}=\bigcup_{\xi<\omega_\alpha}\mathscr{F}_\xi\,(\text{resp. } G_{\omega_\alpha}^{(\alpha)}=\bigcup_{\xi<\omega_\alpha}G_\xi).$$

$G_{\omega_\alpha}^{(\alpha)}$ est de type η_α (lemme 2.1) et $\mathscr{F}_{\omega_\alpha}^{(\alpha)}$ également (il est isomorphe pour l'ordre à $\mathscr{G}_\alpha(G_{\omega_\alpha}^{(\alpha)}, R)$). D'après le lemme 2.2 et la propriété des ensembles S_ξ rappelée plus haut, toute partie de $G_\xi=\mathscr{G}(S_\xi, R)$ possède un sous-ensemble coinitial et cofinal de cardinal $<\aleph_\alpha$ si $\xi<\omega_\alpha$ et il en est de même pour toute partie de $\mathscr{F}_\xi$ pour $\xi<\omega_\alpha$ ($\mathscr{F}_\xi$ est isomorphe en tant qu'espace vectoriel ordonné à $\mathscr{G}(G_\xi, R)$). On a d'autre part: $G^{(\alpha)}_{\omega_\alpha}\subseteqq\mathscr{G}(S_{\omega_\alpha}, R)=G_{\omega_\alpha}, \mathscr{F}^{(\alpha)}_{\omega_\alpha}\subseteqq\mathscr{F}(G^{(\alpha)}_{\omega_\alpha}, R)\subseteqq\mathscr{F}(G_{\omega_\alpha}, R)=\mathscr{F}_{\omega_\alpha}$ et ceci achève de démontrer b) ($\omega_\alpha<\omega_{\alpha+1}$!).

Soit maintenant A une partie de S_{ω_α} de cardinal $<\aleph_\alpha$, et pour tout élément x de A, soit $\lambda(x)$ un ordinal $<\omega_\alpha$ tel que: $x\in S_{\lambda(x)}$. On a card $\{\lambda(x)\}_{x\in A}<\aleph_\alpha$ et il existe $\nu<\omega_\alpha$ vérifiant: $A\subseteqq S_\nu$, d'où: $G^{(\alpha)}_{\omega_\alpha}=\mathscr{G}_\alpha(S_{\omega_\alpha}, R)=\bigcup_{\xi<\omega_\alpha}\mathscr{G}(S_\xi, R)=$ $=\bigcup_{\xi<\omega_\alpha} G_\xi$ et de même: $\mathscr{F}^{(\alpha)}_{\omega_\alpha}=\bigcup_{\xi<\omega_\alpha}\mathscr{F}_\xi$. Il est facile de voir que si card $(T)=2^{\aleph_\beta}$ on a: card $[\mathscr{G}_\alpha(T, R)]=2^{\aleph_\beta}$ d'où: card $(G^{(\alpha)}_{\omega_\alpha})=2^{\aleph_\beta}$, card $(\mathscr{F}^{(\alpha)}_{\omega_\alpha})=2^{\aleph_\beta}$ (on a: card $S_{\omega_\alpha}=2^{\aleph_\beta}$). Enfin pour tout groupe totalement ordonné G l'ensemble des parties bien ordonnées (resp. l'ensemble des parties bien ordonnées de cardinal inférieur à $\aleph_\alpha$ de G) vérifie les cinq conditions du théorème 2 de [22] et si G est divisible les corps $\mathscr{F}(G, C)$ et $\mathscr{F}_\alpha(G, C)$ (dont les définitions sont analogues à celles de $\mathscr{F}(G, R)$ et $\mathscr{F}_\alpha(G, R)$) sont algébriquement clos, ce qui entraîne que $\mathscr{F}(G, R)$ et $\mathscr{F}_\alpha(G, R)$) sont des corps ordonnés maximaux. Comme on a: $\mathscr{F}^{(\alpha)}_{\omega_\alpha}=\mathscr{F}_\alpha(G^{(\alpha)}_{\omega_\alpha}, R)$ et $\mathscr{F}_\xi=$ $=\mathscr{F}(G_\xi, R)$, ceci achève la démonstration du théorème.

On en déduit le corollaire suivant, qui répond à une des questions soulevées par Erdős, Gillman et Henriksen dans le problème 5.4 de [9]:

COROLLAIRE 2.4. — *Soit α un ordinal possédant un prédécesseur β. Si $2^{\aleph_\beta}$ est strictement supérieur à $\aleph_\alpha$, il existe des corps ordonnés maximaux de type η_α et de cardinal $2^{\aleph_\beta}$ qui ne sont pas isomorphes.*

Posons: $T=[0, \omega_{\alpha+1}[\times S_{\omega_\alpha}$. De même que dans [15], 13 K on voit que T, muni de l'ordre lexicographique, est un ensemble totalement ordonné de type η_α et on a: card $(T)=2^{\aleph_\beta}$. Il est facile de définir un isomorphisme d'ensemble ordonné de T sur une partie de $\mathscr{F}_\alpha[\mathscr{G}_\alpha(T, R), R]$. De même que plus haut on voit que $\mathscr{F}_\alpha(\mathscr{G}_\alpha(T, R), R)$ est un corps ordonné maximal de type η_α et de cardinal $2^{\aleph_\beta}$ qui n'est pas isomorphe à $\mathscr{F}^{(\alpha)}_{\omega_\alpha}$ puisque T ne possède aucune partie cofinale de cardinal $<\aleph_{\alpha+1}$ et ceci établit le corollaire. En particulier si on n'assume pas l'hypothèse du continu il existe des corps ordonnés maximaux de type η_1 équipotents à R et non isomorphes ce qui répond également à une question du problème 5.4 de [9].

Pour tout ordinal non-limite α, on pose:

$$B_{\omega_\alpha}=\{x\in\mathscr{F}^{(\alpha)}_{\omega_\alpha}/\exists n\in N: |x|<n\cdot 1\},\quad B'_{\omega_\alpha}=\{x\in\mathscr{F}^{(\alpha)}_{\omega_\alpha}/\forall n\in N: |x|<1/n\}.$$

On a le lemme suivant:

LEMME 2.5. — *Pour tout idéal premier J de B_{ω_α}, on a: $B_{\omega_\alpha}\simeq[B_{\omega_\alpha}/J]\oplus J$.*

Tout élément x de $\mathscr{F}^{(\alpha)}_{\omega_\alpha}$ s'écrit: $x=\sum_{i<\mathrm{Ord}(x)}\lambda_i X^{a_i}$.

Si $x\neq 0$, soit $v(x)$ le plus petit exposant intervenant dans l'écriture de x. L'application $x\to v(x)$ est la *valuation de l'ordre* sur $\mathscr{F}^{(\alpha)}_{\omega_\alpha}$ (voir [25], ch. D) et B_{ω_α} est l'anneau de valuation correspondant ($B_{\omega_\alpha}=\{x\in\mathscr{F}^{(\alpha)}_{\omega_\alpha}: v(x)\geqq 0\}$; $B'_{\omega_\alpha}=$ $=\{x\in\mathscr{F}^{(\alpha)}_{\omega_\alpha}: v(x)>0\}$). L'application $H\to v(H)$ est un isomorphisme de l'ensemble (totalement ordonné par inclusion) des idéaux de B_{ω_α} sur l'ensemble des parties D de $[G^{(\alpha)}_{\omega_\alpha}]^+\cup\{0\}$: vérifiant: $a\in D$, $b\geqq a\Rightarrow b\in D$. Posons: $V(H)=\{x\in[G^{(\alpha)}_{\omega_\alpha}]^+\cup$ $\cup\{0\}: x\notin v(H)\}$. Pour que H soit premier il faut et il suffit que $V(H)$ soit une partie de $[G_{\omega_\alpha}^{(\alpha)}]^+ \cup \{0\}$ stable par addition. Soit I un idéal premier. Pour tout élément $x = \sum_{i<\mathrm{Ord}(x)} \lambda_i X^{a_i}$ de I, posons

$$f_I(x) = \sum_{\substack{i<\mathrm{Ord}(x)\\ a_i \in V(I)}} \lambda_i X^{a_i}, \quad g_I(x) = \sum_{\substack{i<\mathrm{Ord}(x)\\ a_i \in v(I)}} \lambda_i X^{a_i}.$$

f_I et g_I sont linéaires et on a: $f_I(x)+g_I(x)=x (x \in B_{\omega_\alpha})$; $I \cap f_I(B_{\omega_\alpha}) = \{0\}$; $\mathrm{Ker} f_I = I$; $g_I(B_{\omega_\alpha}) = I$, $f_I \circ f_I = f_I$; $g_I \circ g_I = g_I$; soit $x' = \sum_{j<\mathrm{Ord}(x')} \lambda'_j X^{a'_j}$ un autre élément de B_{ω_α}. Posons:

$$\Gamma(x, x') = \{a_i + a'_j\}_{\substack{i<\mathrm{Ord}(x)\\ j<\mathrm{Ord}(x')}}.$$

$V(I)$ étant stable par addition, on a: $\Gamma(x, x') \cap V(I) = \Gamma[f_I(x), f_I(x')]$ et il résulte immédiatement de la définition du produit dans $\mathscr{F}_{\omega_\alpha}^{(\alpha)}$ que l'on a: $f_I(x) \cdot f_I(x') = f_I(xx')$; il existe un isomorphisme d'algèbre ordonnée θ_I de B_{ω_α}/I sur $f_I(B_{\omega_\alpha})$ vérifiant: $f_I = \theta_I \circ Q_I$ (Q_I désignant l'application canonique de B_{ω_α} sur B_{ω_α}/I) et l'application $x \to (\theta_I^{-1}[f_I(x)], g_I(x))$ donne l'isomorphisme voulu de B_{ω_α} sur $B_{\omega_\alpha}/I \oplus I$.

On dira que deux idéaux premiers I et $J (I \subset J, I \neq J)$ de B_{ω_α} sont successifs si et seulement si tout idéal premier contenant strictement I contient J. On a le:

THÉORÈME 2. 6. — *Soit α un ordinal non limite, soit I un idéal premier de B_{ω_α} et J un autre idéal premier le contenant strictement. Si I et J ne sont pas successifs, l'agèbre quotient J/I possède une sous-algèbre isomorphe à B'_{ω_α}.*

Posons: $D = v(J) \cap V(I)$. D est stable par addition et si $a \in D$, $b \in D$ $(a<b)$, on a: $[a, b] \subseteq D$. Soit φ_1 l'application de $[G_{\omega_\alpha}^{(\alpha)}]^+ = \mathscr{G}_\alpha^+(S_{\omega_\alpha}, R)$ dans S_{ω_α} définie au début de la démonstration du lemme 2—1. Soit $a \in D$, $m \in N$, $n \in N$; on a: $\frac{a}{m} \in v(J)$ [$V(J)$ est stable par addition] et $\frac{a}{m} \in V(I)$ $\left(\frac{a}{m} \leqq a!\right)$ d'où: $\frac{a}{m} \in D$ et de même: $\frac{na}{m} \in D$. Si on a: $\varphi_1(a) = \varphi_1(d)$ et si d appartient à $[G_{\omega_\alpha}^{(\alpha)}]^+$ il existe $r, r' \in Q^+$ tels que: $ra < b < r'a$ et d appartient à D. On a donc: $\varphi_1^{-1}[\varphi_1(D)] \cap [G_{\omega_\alpha}^{(\alpha)}]^+ = D$; il est facile de voir que pour que I et J soient successifs il faut et il suffit que $\varphi_1(D)$ soit réduit à un point. Dans le cas contraire soient β_1 et β_2 deux points distincts de $\varphi_1(D)$ $(\beta_1 < \beta_2)$. L'intervalle ouvert $]\beta_1, \beta_2[$ est un ensemble totalement ordonné de type η_α et il existe un morphisme d'ordre injectif t de S_{ω_α} dans $]\beta_1, \beta_2[$ ([26] th. 4). Pour $a \in G_{\omega_\alpha}^{(\alpha)}$, notons $T(a)$ l'unique élément de $G_{\omega_\alpha}^{(\alpha)}$ vérifiant: $T(a) \circ t = a$, $\mathrm{Supp}\,(T(a)) \subseteq t(S_{\omega_\alpha})$. L'application T est un morphisme injectif de groupe ordonné de $G_{\omega_\alpha}^{(\alpha)}$ dans lui-même et on a: $[T(G_{\omega_\alpha}^{(\alpha)})]^+ = T(G_{\omega_\alpha}^{+(\alpha)}) \subseteq D$. Pour tout élément $x = \sum_{i<\mathrm{Ord}(x)} \lambda_i X^{a_i}$ de B_{ω_α}, posons: $W(x) = \sum_{i<\mathrm{Ord}(x)} \lambda_i \cdot X^{T(a)_i}$ si $x \neq 0$, et $W(0) = 0$. W est évidemment un morphisme injectif d'algèbre ordonnée de B_{ω_α} dans lui-même et si $x \in B'_{\omega_\alpha}$ on a: $T(a_i) \in D = v(J) \cap V(I)$ $(i < \mathrm{Ord}\, x)$ si $x \neq 0$. f_I et θ_I étant les applications définies plus haut on a donc: $W(B'_{\omega_\alpha}) \subseteq f_I(J)$ d'où (en identifiant J/I avec $Q_I(J)$): $\theta_I^{-1}[W(B'_{\omega_\alpha})] \subseteq J/I$. Comme $\theta_I^{-1} \circ W$ est un morphisme injectif d'algèbre ordonnée, ceci établit le théorème.

§ 3 — Injection de $\mathscr{F}^{(\alpha)}_{\omega_\alpha}$ dans un corps ordonné maximal de type η_α

On a le lemme suivant:

LEMME 3.1. — *Soit α un ordinal et soit F un corps ordonné maximal dont toute partie possède un sous-ensemble coinitial de cardinal $<\aleph_\alpha$. Tout homomorphisme injectif φ respectant l'ordre d'un sous-corps H de F dans un corps ordonné maximal L de type η_α admet un prolongement à F. En particulier L contient une copie de F.*

Soit H' un sous-corps de F contenant H tel que φ soit prolongeable à H'. D'après un théorème bien connu d'ARTIN et SCHREIER (voir [4]) φ est prolongeable à $\mathscr{R}(H')$ que nous identifions à sa copie dans F. S'il existe un élément x de F n'appartenant pas à $\mathscr{R}(H')$, posons:

$$A = \{y\in\mathscr{R}(H'): y > x\}, \quad B = \{y\in\mathscr{R}(H'): y < x\}.$$

A et B possèdent respectivement une partie coinitiale A' et une partie cofinale B' de cardinal $<\aleph_\alpha$. Soit c un élément de L vérifiant: $\varphi(A')>c>\varphi(B')$. Comme x n'appartient pas à $\mathscr{R}(H')$ et que le corps $\mathscr{R}(H')(x)$ engendré par $\mathscr{R}(H')$ et x est une extension ordonnée de $\mathscr{R}(H')$, x est transcendant sur $\mathscr{R}(H')$. De même c est transcendant sur $\varphi[\mathscr{R}(H')]$ et $\mathscr{R}(H')(x)$ et $\varphi[\mathscr{R}(H')](c)$ s'identifient algébriquement au corps des fractions rationnelles à une indéterminée à coefficients dans $\mathscr{R}(H')$ et $\varphi[\mathscr{R}(H')]$ ce qui permet de prolonger canoniquement φ par un isomorphisme algébrique de $\mathscr{R}(H')(x)$ sur $\varphi[\mathscr{R}(H')](c)$ tel que $\varphi(x)=c$. φ est un isomorphisme d'ensemble ordonné de $\mathscr{R}(H')\cup\{x\}$ sur $\varphi[\mathscr{R}(H')]\cup\{c\}$ et il résulte du lemme 13.12 de [15] que φ est un isomorphisme de corps ordonné de $\mathscr{R}(H')(x)$ sur $\varphi[\mathscr{R}(H')](c)$. La première assertion résulte alors d'une application standard du lemme de Zorn, et la seconde s'en déduit (prendre $H=\{0\}$). En particulier tout corps ordonné maximal de type η_1 contient une copie de R.

On va maintenant établir le théorème suivant, qui achève d'éclaircir (sauf en ce qui concerne les corps hyper-réels) les questions soulevées dans le problème 5.4 de [9].

THÉORÈME 3.2. — *Soit α un ordinal possédant un prédecesseur β, soit $\mathscr{F}$ un corps totalement ordonné maximal de type η_α, soit R_1 une copie de R dans $\mathscr{F}$ et soient (x_1, x_2) un couple d'éléments positifs infiniment petits de $\mathscr{F}^{(\alpha)}_{\omega_\alpha}$ et (y_1, y_2) un couple d'éléments positifs infiniment petits de $\mathscr{F}$ vérifiant pour tout $n\in\mathbf{N}$, $x_1^n>x_2$; $y_1^n>y_2$.*

i) *Il existe un morphisme injectif φ de $\mathscr{F}^{(\alpha)}_{\omega_\alpha}$ dans $\mathscr{F}$ tel que $\varphi[\mathscr{F}_0]=R_1$, $\varphi(x_1)=y_1$, $\varphi(x_2)=y_2$.*

ii) *Si $\aleph_\alpha=2^{\aleph_\beta}$ et si* card $\mathscr{F}=2^{\aleph_\beta}$ *on peut choisir φ de façon à obtenir un isomorphisme.*

On munit $\mathscr{F}$ d'un bon ordre. $\mathscr{F}$ s'écrit: $\mathscr{F}=(t_\xi)_{\xi<\omega}$, ω étant un ordinal convenable. Soit θ le plus petit ordinal $<\omega_\alpha$ tel que x_1 et x_2 appartiennent à $\mathscr{F}_\theta$; x_1 est transcendant sur $\mathscr{F}_0$, y_1 est transcendant sur R_1 et en appliquant le lemme 13.12 de [15] de même que plus haut, on peut définir un isomorphisme de corps ordonné φ de $\mathscr{F}_0(x_1)$ sur $R_1(y_1)$ tel que $\varphi(\mathscr{F}_0)=R_1$, $\varphi(x_1)=y_1$ que l'on prolonge par un isomorphisme de $\mathscr{R}[\mathscr{F}_0(x_1)]$ sur $\mathscr{R}[R_1(y_1)]$ grâce au théorème d'Artin et Schreier déjà cité. Soit v la valuation de l'ordre sur $\mathscr{F}^{(\alpha)}_{\omega_\alpha}$. La restriction de v à $\mathscr{F}_0(x_1)$ est de rang 1 et il en est de même de la restriction de v à $\mathscr{R}[\mathscr{F}_0(x_1)]$ (cf. [25], ch. F, prop. 1). Pour tout élément positif x de $\mathcal{R}[\mathcal{F}_0(x_1)]$ il existe alors $n \in N$ vérifiant: $x \geqq x_1^n > x_2$. On a une propriété analogue pour $\mathcal{R}[R_1(y_1)]$ et y_2 et ceci permet d'appliquer à nouveau le lemme 13.12 de [15] pour prolonger φ à $\mathcal{R}[\mathcal{F}_0(x_1)]\,(x_2)$ de façon à avoir $\varphi(x_2)=y_2$. Comme $\mathcal{F}_\theta$ est un corps ordonné maximal, on peut prolonger ensuite φ à $\mathcal{F}_\theta$ (lemme 3.1). On va prolonger φ à $\mathcal{F}^{(\alpha)}_{\omega_\alpha} = \bigcup\limits_{\xi<\omega_\alpha} \mathcal{F}_\xi$ par récurrence transfinie. Soit $\gamma \in [\theta, \omega_\alpha[$. Supposons qu'on ait construit pour tout $\xi \in [0, \gamma[$ un prolongement φ_ξ de φ à $\mathcal{F}_\xi$ dont la restriction à $\mathcal{F}_{\xi'}$ coincide avec $\varphi_{\xi'}$ pour $\xi' < \xi$. Posons: $\mathcal{F}_\gamma' = \bigcup\limits_{\xi<\gamma} \mathcal{F}_\xi$; φ_γ est défini de manière naturelle sur $\mathcal{F}_\gamma'$ qui est un corps ordonné maximal. Si $\mathcal{F}_\gamma' \neq \mathcal{F}_\gamma$, posons, pour $s \in \mathcal{F}_\gamma$, $s \notin \mathcal{F}_\gamma'$: $A_s = \{u \in \mathcal{F}_\gamma' : u > s\}$; $B_s = \{u \in \mathcal{F}_\gamma' : u < s\}$; $D_s : \{t \in \mathcal{F} : \varphi[B_s] < t < \varphi[A_s]\}$; $D = \bigcup\limits_{\substack{s \in \mathcal{F}_\gamma \\ s \notin \mathcal{F}_\gamma'}} D_s$. Le théorème 2.4 b) et le fait que $\mathcal{F}$ est de type η_α entraînent que D_s n'est jamais vide. Soit δ le plus petit ordinal $<\omega_\alpha$ tel que t_δ appartienne à D et soit s un élément du complémentaire de $\mathcal{F}_\gamma'$ dans $\mathcal{F}_\gamma$ vérifiant: $t_\delta \in D_s$. Toujours d'après le lemme 13.12 de [15] on peut prolonger φ_ξ à $\mathcal{F}_\gamma'(s)$ par un morphisme de corps ordonné où t_δ est l'image de s et on étend ce prolongement à $\mathcal{F}_\gamma$ grâce au lemme 3.1. Ceci permet de définir φ_ξ pour tout $\xi < \omega_\alpha$ et établit i) (on a $\mathcal{F}^{(\alpha)}_{\omega_\alpha} = \bigcup\limits_{\xi<\omega_\alpha} \mathcal{F}_\xi$).

Supposons maintenant que l'on a: $\aleph_\alpha = 2^{\aleph_\beta}$, card $\mathcal{F} = 2^{\aleph_\beta}$. On peut prendre: $\omega = \omega_\alpha$. Soit $\varrho < \omega_\alpha$ et soit φ le morphisme de $\mathcal{F}^{(\alpha)}_{\omega_\alpha}$ dans $\mathcal{F}$ construit ci-dessus. Supposons que l'on ait: $\{t_\xi\}_{\xi<\varrho} \subseteqq \varphi[\mathcal{F}^{(\alpha)}_{\omega_\alpha}]$. Comme $[0, \omega_\alpha[$ n'a aucune partie cofinale de cardinal $<\aleph_\alpha$, il existe $\mu \in [\theta, \omega_\alpha[$ vérifiant: $\{t_\xi\}_{\xi<\varrho} \subseteqq \varphi[\mathcal{F}_\mu]$ (θ désignant l'ordinal introduit plus haut). Si $t_\varrho \notin \varphi[\mathcal{F}_\mu]$, posons: $U = \{z \in \varphi[\mathcal{F}_\mu] : z > t_\varrho\}$; $V = \{z \in \varphi[\mathcal{F}_\mu] : z < t_\varrho\}$. Soit γ le plus petit ordinal $<\omega_\alpha$ tel qu'il existe $s \in \mathcal{F}_\gamma$ vérifiant: $U > \varphi(s) > V$ (un tel ordinal existe car U et V possèdent respectivement des parties coinitiales et cofinales de cardinal $<\aleph_\alpha$ et $\varphi[\mathcal{F}^{(\alpha)}_{\omega_\alpha}]$ est de type η_α). On a évidemment: $s \notin \mathcal{F}_\gamma'$ et, A_s, B_s, D_s et D étant définis comme plus haut, on voit que $\varphi^{-1}(U)$ est coinitial dans A_s et $\varphi^{-1}(V)$ cofinal dans B_s, d'où: $t_\varrho \in D_s$. On a aussi: $\{t_\xi\}_{\xi<\varrho} \subseteqq \varphi[\mathcal{F}_\mu] \subseteqq \varphi[\mathcal{F}_\gamma']$. En outre on a: $\gamma > \theta$ et il résulte alors de la définition de φ que l'on a: $t_\varrho = \varphi(s)$. Comme ce raisonnement est valable pour $\varrho = 0$ (prendre simplement $\mu > \theta$) on a: $\varphi[\mathcal{F}^{(\alpha)}_{\omega_\alpha}] = \mathcal{F}$ et ceci achève la démonstration.

Remarque 3.3. — Dans l'énoncé du théorème 3.2, on peut pour tout $\xi < \omega_\alpha$ remplacer $\mathcal{F}_0$ et R_1 par $\mathcal{F}_\xi$ et une copie de $\mathcal{F}_\xi$ dans $\mathcal{F}$. D'autre part on peut imposer à φ des conditions plus strictes que $\varphi(x_1) = y_1$ et $\varphi(x_2) = y_2$.

Remarque 3.4. — Si E est un espace vectoriel totalement ordonné (sur Q ou R) et si F est un espace vectoriel totalement ordonné (sur le même corps) et de codimension 1 dans E, l'ordre de F est entièrement déterminé par l'ordre de $E \cup \{x\}$ où $x \in F$, $x \notin E$. On peut alors reprendre la méthode utilisée au théorème 3.2 pour prouver que tout espace vectoriel réel G de type η_α contient une copie de $G^{(\alpha)}_{\omega_\alpha}$ et que l'on a un isomorphisme si $\aleph_\alpha = 2^{\aleph_\beta} =$ card (G) (résultat analogue pour les groupes divisibles).

Remarque 3.5. — Soit F un corps totalement ordonné de cardinal $\leqq \aleph_\alpha$. Si F contient une copie R_1 de R soit $(x_\xi)_{\xi<\omega}$ un bon ordre du complémentaire de R_1 dans F (resp. un bon ordre du complémentaire de Q dans F si F ne contient aucune copie de R) ω désignant un ordinal convenable $\leqq \omega_\alpha$; pour $\theta \leqq \omega$ soit F_θ le corps engendré par $R_1 \cup \{x_\xi\}_{\xi<\theta}$ (resp. le corps engendré par $Q \cup \{x_\xi\}_{\xi<\theta}$). Il est alors facile de construire par récurrence transfinie une application croissante λ de $[0, \omega[$ dans $[0, \omega_\alpha[$ et une famille $(\varphi_\theta)_{\theta<\omega}$ de morphismes injectifs de F_θ dans $\mathscr{F}_{\lambda(\theta)}$ telles que la restriction de φ_θ à $F_{\theta'}$ coincide avec $\varphi_{\theta'}$ pour $\theta'<\theta$ et que l'on ait le cas échéant: $\varphi_0(R_1)=\mathscr{F}_0$. On définit ainsi un homomorphisme injectif φ de F dans $\mathscr{F}_{\omega_\alpha}^{(\alpha)}$ vérifiant: $\varphi(R_1)=\mathscr{F}_0$; on déduit alors de la première assertion du théorème 3—2 qu'il existe pour tout corps totalement ordonné maximal $\mathscr{F}$ de type η_α et pour toute copie R_2 de R dans $\mathscr{F}$ un homomorphisme injectif de F dans $\mathscr{F}$ appliquant R_1 sur R_2. On retrouve en particulier la résultat de la remarque 13—0 de [15] et l'amélioration qu'en donne JOHNSON dans [19] par une méthode directe qui évite le recours aux bases de transcendance. On a des résultats analogues pour les groupes et les espaces vectoriels réels totalement ordonnés.

REMARQUE 3.6. — a) Soit G un groupe totalement ordonné. On peut plonger G dans un groupe divisible G' de même cardinal ([11] ch. IV, § 5, lemme A). Soit α l'ordinal tel que $\aleph_\alpha = \text{Card}\, G$. Si α a un prédécesseur β on peut d'après la remarque précédente immerger G' (en tant que groupe ordonné) dans $G_{\omega_\alpha}^{(\alpha)}$. Le classique «théorème d'immersion» de Hahn montre que l'on peut injecter G dans le groupe $\mathscr{G}(T, R)$, T désignant le «squelette» de G, c'est à dire l'ensemble des classes archimédiennes de G muni de l'ordre induit par G (voir [17] pour la démonstration originale de HAHN et par exemple [11], [16] et [24] pour des démonstrations plus récentes et des références détaillées sur ce sujet). Le résultat de la remarque 3—5 présente donc des analogies avec le «theorème d'immersion» classique mais ne peut s'en déduire directement et vice-versa: supposons $2^{\aleph_\beta}=\aleph_1$. Pour $\xi<\omega_1$ le théorème d'immersion classique associe G_ξ à lui-même alors que la remarque 3—5 associe $G_{\omega_1}^{(1)}$ à G_ξ. Par contre la remarque 3—5 associe $G_{\omega_1}^{(1)}$ à lui-même alors que le théorème d'immersion classique associe G_{ω_1} à $G_{\omega_1}^{(1)}$.

b) Soit F un corps totalement ordonné commutatif. L. FUCHS indique dans [11], ch. VIII, § 5 que l'on peut immerger F dans le corps $\mathscr{F}(G, R)$ où G désigne le groupe (noté additivement) des classes archimédiennes de F pour le produit. (Ce résultat était déjà annoncé par A. GLEYZAL dans [12]. La seule démonstration écrite que j'en connaisse est celle donnée par F. J. RAYNER dans [23]). On voit de même que plus haut que ce «théorème d'immersion» des corps donne selon les cas des résultats plus fins ou moins fins que ceux obtenus grâce à la remarque 3—5.

§ 4 — Application a l'etude des homomorphismes de $\mathscr{C}(K)$

Dans ce paragraphe K désigne un compact infini et $\mathscr{C}_R(K)$ l'algèbre des fonctions continues de K dans R. Pour tout élément f de $\mathscr{C}_R(K)$ on pose: $Z(f)=f^{-1}(0)$. c_0 désigne l'algèbre des suites réelles convergeant vers 0. On a le lemme suivant:

LEMME 4.1. — *Soit I un idéal premier de $\mathscr{C}_R(K)$. Si I n'est pas maximal l'algèbre quotient $\mathscr{C}_R(K)/I$ possède une sous-algèbre isomorphe à B_{ω_1}.*

Soit L l'unique idéal maximal de $\mathscr{C}_R(K)$ contenant I, soit τ_0 l'élément de K définissant L et soit O_{τ_0} l'ensemble des éléments g de $\mathscr{C}_R(K)$ tels que $Z(g)$ soit un voisinage de τ_0. On va construire un morphisme non trivial de c_0 dans L/I (en utilisant une méthode voisine de celle du § 3 de [19]). D'après [15], 4 I on a l'inclusion: $O_{\tau_0} \subseteq$

$\subseteqq I \neq L$ et il existe $f \in L$, que l'on peut choisir $\geqq 0$ ([15], th. 5—5) n'appartenant pas à I. Pour tout $\varepsilon > 0$, l'ensemble $f^{-1}(]-\varepsilon, +\varepsilon[)$ est un voisinage de τ_0 sur lequel f n'est pas identiquement nulle, et on peut trouver une suite (t_n) d'éléments de K vérifiant: $f(t_n) > f(t_{n+1})$ $(n \in N)$; $f(t_n) \to 0$ $(n \to \infty)$. Soit M le plus grand élément de $f(K)$ et soient (r_n) et (s_n) deux suites de réels positifs vérifiant: $r_1 > s_1 > M$ et pour $n > 1: f(t_{n-1}) > r_n > s_n > f(t_n)$. Posons: $U = \bigcup_{m=1}^{\infty} f^{-1}(]s_{2m+1}, r_{2m}[)$, $V = \bigcup_{m=1}^{\infty} f^{-1}(]s_{2m}, r_{2m-1}[)$. On a: $U \cup V = K - Z(f)$; $K - Z(f)$ est un espace localement compact dénombrable à l'infini qui est donc normal ([8], ch. IX, § 4, ex. 6) et il existe deux fonctions continues α et β de $K - Z(f)$ dans R^+ vérifiant: $0 \leqq \alpha(t) \leqq 1$, $0 \leqq \beta(t) \leqq 1$ $(t \in K - Z(f))$, $\alpha + \beta = 1$, $\operatorname{Supp} \alpha \subseteqq U$, $\operatorname{Supp} \beta \subseteqq V$. En prolongeant $\alpha \cdot f$ et $\beta \cdot f$ par 0 sur $Z(f)$ on obtient deux éléments de $\mathscr{C}_R(K)$ et on a: $f = \alpha \cdot f + \beta \cdot f$. L'une au moins des fonctions $\alpha \cdot f$ et $\beta \cdot f$, disons $\alpha \cdot f$, n'appartient pas à I. Soient (l_m) et (l'_m) deux suites réelles vérifiant: $f(t_{2m-1}) > l_m > r_{2m}$ $(m \geqq 1)$; $s_{2m+1} > l'_m > f(t_{2m+1})$ $(m \geqq 1)$. Il existe une fonction continue γ de $K - Z(f)$ dans R vérifiant: $0 \leqq \gamma(t) \leqq 1$ $(t \in K - Z(f))$; $\operatorname{supp} \gamma \subseteqq \subseteqq \bigcup_{m=1}^{\infty} f^{-1}(]l'_m, l_m[)$; $\bigcup_{m=1}^{\infty} f^{-1}([s_{2m+1}, r_{2m}]) \subseteqq \gamma^{-1}(1)$. Pour tout élément $x = (x_m)$ de c_0, soit $\Psi(x)$ la fonction de K dans R définie da la façon suivante:

$$\Psi(x)(t) = 0 \quad \text{si} \quad t \in Z(f) \cup \left[\bigcup_{m=1}^{\infty} f^{-1}([l_{m+1}, l'_m]) \right] \cup f^{-1}([l_1, +\infty[),$$

$$\Psi(x)(t) = x_m \cdot \gamma(t) \quad \text{si} \quad t \in [l'_m, l_m].$$

$\Psi(x)$ est ainsi bien définie sur K, appartient à $\mathscr{C}_R(K)$ et on a: $\Psi(c_0) \subseteqq L$; Ψ est linéaire et pour tout couple (x, y) d'éléments de c_0 on voit que $\Psi(x) \cdot \Psi(y) - \Psi(xy)$ s'annule sur $Z(f) \cup \left[\bigcup_{m=1}^{\infty} f^{-1}([s_{2m+1}, r_{2m}]) \right]$ d'où:

$$[\Psi(x) \cdot \Psi(y) - \Psi(xy)] \cdot \alpha f = 0 \quad \text{et:} \quad \Psi(x) \cdot \Psi(y) - \Psi(xy) \in I.$$

Soit Q_I l'application canonique de $\mathscr{C}_R(K)$ sur $\mathscr{C}_R(K)/I$. $Q_I \circ \Psi$ est un morphisme d'algèbre de c_0 dans L/I. Posons: $J = \operatorname{Ker}(Q_I \circ \Psi)$. J est un idéal premier de c_0 (L/I) est intègre) et il existe un morphisme d'algèbre injectif θ_0 de c_0/J dans L/I vérifiant: $\theta_0 \circ Q_J = Q_I \circ \Psi$. Posons: $z_m = f(t_{2m-1})$ $(m \in N)$ et $z = (z_m)$. On a: $\Psi(z) \geqq \alpha \cdot f \geqq 0$ d'où: $\Psi(z) \notin I$ ([15], th. 5—5), $z \notin J$ et J est strictement inclus dans c_0.

Il est alors bien connu que l'on a la situation suivante ($\mathscr{C}(N)$ désignant l'algèbre de toutes les suites réelles et βN désignant le compactifié de Stone—Cêch de N): il existe un point τ de $\beta N - N$ tel que, H désignant l'ensemble $\{f \in \mathscr{C}(N) : \tau \in Z(f)\}$ on ait: $c_0 \cap H \subseteqq J$; H est un idéal maximal de $\mathscr{C}(N)$ et $\mathscr{F} = \mathscr{C}(N)/H$ est un corps hyperréel ([15], ch. 13). $\mathscr{F}$ est donc un corps ordonné maximal de type η_1 dont l'anneau de valuation (pour la valuation de l'ordre) s'indentifie à l^∞/H; c_0/H s'identifie à un idéal premier de l^∞/H (et le corps des fractions de c_0/H, muni de son ordre naturel, s'identifie à $\mathscr{F}$) de même que J/H et on a: $c_0/J \simeq c_0/H/J/H$. En outre c_0/H ne possède aucun ensemble cofinal dénombrable ([15], th. 14—16, b)) et on peut trouver deux éléments positifs y_1 et y_2 de c_0/H vérifiant pour tout $n \in N$: $y_1^n > y_2$ et: $y_2 \notin J/H$. Soient x_1 et x_2 deux éléments positifs de B'_{ω_1} vérifiant également $x_1^n > x_2$ $(n \in N)$. D'après le théorème 3—2 il existe un morphisme d'algèbre injectif φ de $\mathscr{F}_{\omega_1}^{(1)}$ dans $\mathscr{F}$ vérifiant: $\varphi(x_1) = y_1$, $\varphi(x_2) = y_2$; $\varphi^{-1}(c_0/H)$ et $\varphi^{-1}(J/H)$ sont des idéaux premiers de B_{ω_1} et on a: $x_1 \notin \varphi^{-1}(J/H)$, $x_2 \notin \varphi^{-1}(J/H)$, $x_1 \in \varphi^{-1}(c_0/H)$, $x_2 \in \in \varphi^{-1}(c_0/H)$; $\varphi^{-1}(J/H)$ et $\varphi^{-1}(c_0/H)$ ne sont donc pas consécutifs; d'après le théorème 2—6 il existe alors un morphisme d'algèbre ordonnée injectif θ_1 de B'_{ω_1} dans $\varphi^{-1}(c_0/H)/\varphi^{-1}(J/H)$. On déduit de l'existence de φ l'existence d'un morphisme injectif d'algèbre ordonnée θ_2 de $\varphi^{-1}(c_0/H)/\varphi^{-1}(J/H)$ dans c_0/J $(c_0/J \simeq c_0/H/J/H)$ et $\theta_0 \circ \theta_2 \circ \theta_1$ est un morphisme injectif d'algèbre ordonnée de B'_{ω_1} dans L/I. Il suffit alors d'«adjoindre l'unité» pour obtenir une injection de B_{ω_1} dans $\mathscr{C}_R(K)/I$ ce qui établit le lemme.

On a alors le:

THÉORÈME 4.2. — *Soit K un compact infini. S'il existe un homomorphisme discontinu de $\mathscr{C}_R(K)$, l'algèbre B_{ω_1} possède une structure d'algèbre normée. Réciproquement si on assume l'hypothèse du continu et si B_{ω_1} possède une structure d'algèbre normée, il existe un homomorphisme discontinu de $\mathscr{C}_R(K)$ (et tout idéal premier non maximal de $\mathscr{C}_R(K)$ est noyau d'un homomorphisme discontinu si K est séparable).*

Améliorant les résultats de [5] SINCLAIR a prouvé dans [28] que si $\mathscr{C}_R(K)$ possède un homomorphisme discontinu il existe un idéal premier non maximal I tel que $\mathscr{C}_R(K)/I$ soit normable (voir également [10]) et la première assertion se déduit immédiatement du lemme 4.1. Démontrons maintenant la seconde assertion. Tout d'abord si K est séparable (et infini) soit $(t_n)_{n\in N}$ une suite partout dense dans K. L'application $f \to \{f(t_n)\}_{n\in N}$ est une injection de $\mathscr{C}_R(K)$ dans l^∞ et on a: card $\mathscr{C}_R(K) = $ $=$ card $(l^\infty) =$ card (R). Soit I un idéal premier non maximal de $\mathscr{C}_R(K)$ (il en existe toujours d'après [15], 4J et 4K). Soit $\mathscr{F}$ son corps des fractions et soit B_I l'anneau de valuation de $\mathscr{F}$ pour la valuation de l'ordre. Si on assume l'hypothèse du continu on a: Card $(\mathscr{F}) =$ Card $(\mathscr{C}_R(K)/I) =$ Card $(\mathscr{C}_R(K)) =$ card $(R) = \aleph_1$ et d'après la remarque 3.5 il existe un morphisme injectif d'algèbre ordonnée φ de $\mathscr{F}$ dans $\mathscr{F}_{\omega_1}^{(1)}$. On a: $\varphi(B_I) \subseteq B_{\omega_1}$ d'où à fortiori: $\varphi(\mathscr{C}_R(K)/I) \subseteq B_{\omega_1}$ et I est noyau d'un homomorphisme (nécessairement discontinu) de $\mathscr{C}_R(K)$ si B_{ω_1} est normable.

Soit maintenant K un compact quelconque et soit $(t_n)_{n\in N}$ une suite d'éléments distincts de K. Posons: $K_1 = \overline{\{t_n\}}_{n\in N}$. Soit I un idéal premier non maximal de $\mathscr{C}_R(K_1)$, L l'unique idéal maximal le contenant et τ l'élément de K_1 définissant L. L'application $f \to f|K_1$, que nous noterons φ est comme bien connu une surjection de $\mathscr{C}_R(K)$ sur $\mathscr{C}_R(K_1)$; $\varphi^{-1}(L)$ est évidemment l'idéal maximal de $\mathscr{C}_R(K)$ défini par τ et $\varphi^{-1}(I)$ est un idéal premier de $\mathscr{C}_R(K)$ qui est strictement inclus dans $\varphi^{-1}(L)$ (φ est surjective). Soit Q_I l'application canonique de $\mathscr{C}_R(K_1)$ sur $\mathscr{C}_R(K_1)/I$. On a: Ker $[Q_I \circ \varphi] = \varphi^{-1}(I)$ et il existe un isomorphisme θ de $\mathscr{C}_R(K)/\varphi^{-1}(I)$ sur $\mathscr{C}_R(K_1)/I$. Ceci achève la démonstration (K_1 est séparable!).

REMARQUE 4.3. — On peut montrer que pour lever l'hypothèse du continu dans le thréoème 4.2 il suffirait de prouver que toute partie de $\mathscr{C}_R(K)/I$ possède un sous-ensemble cofinal de cardinal $\aleph_1$ si K est séparable (I désignant un idéal premier non maximal). Ceci renvoie au problème de l'isomorphisme des corps hyper-réels ayant la puissance du continu (dernière question du problème 5—4 de [9]). J'ignore si cela est vrai.

§ 5 — Semi-normes sur B_{ω_1}

B_{ω_1} est l'anneau de valuation de $\mathscr{F}_{\omega_1}^{(1)}$ pour la valuation de l'ordre et par conséquent tout idéal premier de B_{ω_1} est absolument convexe. Il suffit alors de reprendre la méthode utilisée au § 6 de [10] pour établir le:

THÉORÈME 5.1. — *Pour toute semi-norme vectorielle q sur B_{ω_1}, l'ensemble des idéaux q-fermés est bien ordonné pour l'inclusion. Pour toute semi-norme d'algèbre q sur B_{ω_1} tout idéal q-fermé est premier.*

Il serait hautement surprenant que B_{ω_1} (et par conséquent B'_{ω_1}) possède une structure d'algèbre de Banach, mais je ne sais pas prouver que c'est faux. On a dans ce domaine le (maigre) résultat suivant:

COROLLAIRE 5.2. — *Si B'_{ω_1} possède une structure d'algèbre de Banach, il s'agit d'une structure d'algèbre de Banach (radicale) topologiquement simple.*

B'_{ω_1} est évidemment une algèbre radicale. Si elle possède une structure d'algèbre de Banach soit I un idéal fermé de B'_{ω_1} $(I \neq \{0\})$. I est premier dans B_{ω_1} et par conséquent dans B'_{ω_1}. Si $I \neq B'_{\omega_1}$, soit α un élément de B'_{ω_1} n'appartenant pas à I. L'application $x \to \alpha \cdot x$, que nous noterons $\bar{\alpha}$, est évidemment un endomorphisme injectif de I qui est continu. D'autre part on a, pour tout $x \in I$ et tout $n \in N$: $\frac{|\alpha|}{n} > |x|$ et $\left|\frac{x}{\alpha}\right|$ est un élément infiniment petit de $\mathscr{F}_{\omega_1}^{(1)}$ qui appartient à B'_{ω_1} et par conséquent à I ce qui prouve que $\bar{\alpha}$ est en fait un isomorphisme de I sur lui-même. Soit p la norme considérée sur B'_{ω_1}. Posons: $\bar{p}(\bar{\alpha}) = \sup_{\substack{x \in I \\ x \neq 0}} \frac{p(\alpha x)}{p(x)}$. On a pour tout $n \in N$: $[p(\alpha^n)]^{1/n} \geqq$
$\geqq [\bar{p}(\bar{\alpha}^n)]^{1/n} \geqq \frac{1}{\bar{p}(\bar{\alpha}^{-1})}$ ce qui est absurde et le corollaire est démontré. On peut montrer de même qu'il n'existe aucun idéal premier de B_{ω_1} distinct de B'_{ω_1} et possédant une structure d'algèbre de Banach (utiliser le théorème du graphe fermé).

On a enfin le théorème suivant, qui complète le théorème 4.2:

THÉORÈME 5.3. — *Si B'_{ω_1} possède une structure d'algèbre normée, B'_{ω_1} possède une structure d'algèbre normée topologiquement simple.*

Il est clair que pour tout idéal premier J de B_{ω_1} il existe un idéal premier $L \neq \{0\}$ strictement inclus dans J et d'après le théorème 2—6 il existe un morphisme injectif φ de B'_{ω_1} dans J $(J \simeq J/\{0\}!)$. En outre si on construit φ comme au § 2 on voit que tout $x \in B'_{\omega_1}$ vérifiant: $\operatorname{Supp}(x) \subseteqq \bigcup_{y \in \varphi(B'_{\omega_1})} \operatorname{Supp}(y)$ appartient à $\varphi(B'_{\omega_1})$. Soit p une semi-norme vectorielle sur B'_{ω_1}; supposons que pour toute sous-algèbre A de B'_{ω_1} isomorphe à B'_{ω_1} il existe un idéal $J_A \neq \{0\}$ de A qui n'est pas p-partout dense dans A. Quitte à réduire J_A on peut le supposer premier $\left(\bigcap_{n=1}^{\infty} (J_A)^n\right.$ ne peut être réduit à $\{0\}$ car sinon A^+ possèderait une partie dénombrable coinitiale$\left.\right)$. On pourrait alors construire par récurrence deux suites (A_n) et (J_n) de sous-algèbres de B'_{ω_1} vérifiant pour tout n (la notation $\bar{D}^p$ désignant la p-adhérence d'une partie D de B'_{ω_1}): $A_1 = B'_{\omega_1}$; A_n est isomorphe à B'_{ω_1}; J_n est un idéal premier de $A_n \neq \{0\}$, $A_{n+1} \subseteqq J_n$; $A_n \not\subset \bar{J}_n^p$; $x \in A_n$ si $\operatorname{Supp}(x) \subseteqq \bigcup_{y \in A_n} \operatorname{Supp}(y)$. On pourrait alors construire une suite (x_n) d'éléments de B'_{ω_1} telle que l'on ait pour tout $n: x_n \in A_n$; $x_n \notin \bar{J}_n^p$ d'où à fortiori; $x_n \notin \bar{A}_n^p$. Pour tout n on associe à A_n et J_n l'application f_{J_n} de A_n dans lui même définie de la même façon que dans la démonstration du lemme 2.5. On a alors: $x_n - f_{J_n}(x_n) \in J_n$ d'où $f_{J_n}(x_n) \in A_n$, $f_{J_n}(x_n) \notin \bar{A}_{n+1}^p$. Posons: $y_n = f_{J_n}(x_n)$. Pour tout $\gamma \in \operatorname{Supp}(y_n)$ et pour tout $\delta \in \operatorname{Supp}(y_{n+1})$ on a évidemment: $\gamma < \delta$ et $\bigcup_{n=1}^{\infty} \operatorname{Supp}(y_n)$ est une partie bien ordonnée de $[G_{\omega_1}^{(1)}]^+$. $\mathscr{E}$ désignant de même qu'au §2 de [10] l'espace vectoriel de toutes les suites réelles et $\mathscr{E}_m$ le sous-espace de $\mathscr{E}$ formé des suites dont tous les termes de rang $\leqq m-1$ sont nuls, on associe à tout élément $\lambda = (\lambda_n)$ de $\mathscr{E}$ l'élément $\Psi(\lambda)$ de B'_{ω_1} défini de la façon suivante: $\Psi(\lambda)(a) = 0$ si $a \notin \bigcup_{n=1}^{\infty} \operatorname{Supp}(y_n)$; $\Psi(\lambda)(a) = \lambda_n y_n(a)$ si $a \in \operatorname{Supp}(y_n)$ ($\mathscr{F}_{\omega_1}^{(1)}$ étant identifié comme indiqué au début du §2 à un sous espace de l'espace des fonctions de $G_{\omega_1}^{(1)}$ dans R à support bien ordonné). Ψ est bien définie sur $\mathscr{E}$ et c'est évidemment un morphisme injectif d'espace vectoriel ordonné de $\mathscr{E}$ dans B'_{ω_1}. D'autre part si $\lambda = (\lambda_n)$ appartient à $\mathscr{E}_m$ on a: $\operatorname{Supp} \Psi(\lambda) \subseteqq \bigcup_{n=m}^{\infty} \operatorname{Supp}(y_n)$ d'où: $\Psi(\mathscr{E}_m) \subseteqq A_m$; e_m désignant pour tout m la suite dont le m^e terme est égal à 1 et dont tous les autres termes sont nuls on a: $\Psi(e_m) = y_m$. Posons $q = p \circ \Psi$. On voit alors que pour tout m l'espace $\mathscr{E}_m$ ne serait pas inclus dans la q-adhérence de $\mathscr{E}_{m+1}$ et ceci est impossible d'après le lemme 2.4 de [10]. Par conséquent il existe une sous-algèbre A de B'_{ω_1} isomorphe à B'_{ω_1} telle que tout idéal distinct de $\{0\}$ de A soit p-dense dans A. Soit p une norme d'algèbre sur A, A une sous-algèbre de B'_{ω_1} associée à p comme ci-dessus et φ un morphisme injectif de B'_{ω_1} sur A; $p \circ \varphi$ définit sur B'_{ω_1} une structure d'algèbre normée topologiquement simple, ce qui démontre le théorème.

Added in proof (November 1, 1977). Après avoir soumis cet article j'ai montré dans [31] et [32] que l'algèbre de convolution $L^1_*(0, 1)$ possède des sous-algèbres isomorphes à la «complexifiée» C'_{ω_1} de B'_{ω_1}. Compte-tenu du théorème 4—2 ceci permet d'obtenir moyennant l'hypothèse du continu des homomorphismes discontinus de $\mathscr{C}(K)$. Dales a également construit dans [29] des homomorphismes discontinus de $\mathscr{C}(K)$ par des méthodes très différentes (on trouvera un résumé succinct de nos deux constructions dans [30]). D'autre part R. Solovay nous a fait savoir q'uil a montré que tout homomorphisme de $\mathscr{C}(K)$ est continu pour certains modèles de théorie des ensembles (incluant l'axiome du choix) dans lesquels l'hypothèse du continu est fausse.

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GENERAL MOMENT AND PROBABILITY INEQUALITIES FOR THE MAXIMUM PARTIAL SUM

By

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1. Introduction. For random variables $X_1, \ldots, X_n$, not necessarily independent or identically distributed, put $S_n = \sum_1^n X_i$ $(S_0 = 0)$ and $M_n = \max_{0 \leqq k \leqq n} |S_k|$. The main result of this paper establishes effective bounds on $E(M_n^\nu)$ in terms of assumed bounds on $E\left|\sum_i^j X_k\right|^\nu$, for $\nu > 0$. The functions defining the bounds are required to satisfy certain mild structural constraints. The only dependence restrictions are those, if any, implied by the assumed bounds.

A secondary result provides an analogous maximal inequality involving exceedance probabilities $P\{M_n \geqq \lambda\}$ and $P\left\{\left|\sum_i^j X_k\right| \geqq \lambda\right\}$ instead of moments. For the majority of applications, this secondary form of maximal inequality suffices. Two exceptional situations are mentioned in Section 3.

2. Some general maximal inequalities. As a preliminary, we define for each pair $\nu > 0$ and $\gamma > 1$ a constant $A_{\nu,\gamma}$. First, given ν, we introduce the function

$$w_\nu(x) = \sum_{j=0}^{h-1} \binom{h}{j} x^{-(j+\varepsilon)/\nu} + \sum_{j=1}^{h} \binom{h}{j} x^{-j/\nu}, \quad x > 0, \tag{2.1}$$

where h is an integer defined to be $\nu - 1$ if ν is an integer and $[\nu]$ otherwise, and $\varepsilon = \nu - h$. This function will play a role later. For the present, we note that $w_\nu(x) \downarrow 0$ as $x \to \infty$, so that, since $\gamma > 1$ is assumed, there exists a minimum value $B = B_{\nu,\gamma}$ satisfying

$$2 + B^{-1} + w_\nu(B) \leqq 2^\gamma. \tag{2.2}$$

Define

$$A_{\nu,\gamma} = 2^\gamma B_{\nu,\gamma}. \tag{2.3}$$

In obtaining $A_{\nu,\gamma}$ from (2.2), the identity

$$1 + x^{-1} + w_\nu(x) = (1 + x^{-1/\nu})^h (1 + x^{-\varepsilon/\nu})$$

is useful. However, in most applications, what is important is not the actual value of $A_{\nu,\gamma}$ but rather the feature that it is a universal constant depending only on ν and γ and not upon properties of any random variables under consideration.

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THEOREM 1. *Let* $X_1, \ldots, X_n$ *be arbitrary random variables. Suppose that for constants* $\nu > 0$ *and* $\gamma > 1$,

$$E\left|\sum_{k=i}^{j} X_k\right|^{\nu} \leqq [g(i,j)]^{\gamma} \quad (\textit{all } 1 \leqq i \leqq j \leqq n), \tag{2.4}$$

where g satisfies either

$$g(i,j)+g(j+1,k) \leqq g(i,k) \quad (\textit{all } 1 \leqq i \leqq j \leqq k \leqq n) \tag{2.5}$$

or

$$g(i,j)/g(1,n) \leqq (j-i+1)/n \quad (\textit{all } 1 \leqq i \leqq j \leqq n). \tag{2.6}$$

Then

$$E(M_n^{\nu}) \leqq A_{\nu,\gamma}[g(1,n)]^{\gamma}, \tag{2.7}$$

where $A_{\nu,\gamma}$ *is given by* (2.3).

The choice of g may depend on the X_i's and typically does. The example $g(i,j)=c\sum_{i}^{j}\sigma_k^2$, where σ_k^2 is the variance of X_k, and c is some constant, satisfies (2.5) and arises in many applications. Varieties of such g functions are discussed in [8] and [5], and, for the important case of $g(i,j)$ of the form $\sum_{i}^{j} u_k$, in [1].

Theorem 1 is an immediate consequence of the following two lemmas. The first reduces the problem to the case of $g(i,j)$ of the form $\sum_{i}^{j} u_k$, and the second establishes the theorem for this special case.

LEMMA 1 (Reduction). *Let the nonnegative function* $g(i,j)$ *satisfy either* (2.5) *or* (2.6). *Then there exist nonnegative constants* $u_1, \ldots, u_n$ *such that*

$$g(1,n) = \sum_{k=1}^{n} u_k \tag{2.8a}$$

and

$$g(i,j) \leqq \sum_{k=i}^{j} u_k. \tag{2.8b}$$

PROOF. For the case (2.5), take $u_1=g(1,1)$ and $u_k=g(1,k)-g(1,k-1)$, $2 \leqq k \leqq n$. For the case (2.6), take $u_k=g(1,n)/n$, $1 \leqq k \leqq n$.

The first case of this result was originally noted and discussed in [5].

LEMMA 2 (Special case). *Let* $X_1, \ldots, X_n$ *be arbitrary random variables. Suppose that for constants* $\nu > 0$ *and* $\gamma > 1$ *and nonnegative constants* $u_1, \ldots, u_n$,

$$E\left|\sum_{k=i}^{j} X_k\right|^{\nu} \leqq \left(\sum_{k=i}^{j} u_k\right)^{\gamma} \quad (\textit{all } 1 \leqq i \leqq j \leqq n). \tag{2.9}$$

Then

$$E(M_n^\nu) \leqq A_{\nu,\gamma}\left(\sum_{k=1}^{n} u_k\right)^\gamma, \tag{2.10}$$

where $A_{\nu,\gamma}$ *is given by* (2.3).

PROOF. The theorem holds trivally for the case $n=1$. Make the induction hypothesis that the theorem has been established for all integers n satisfying $1\leqq n<N$. Let us show that the theorem follows also for $n=N$. Following the approach of [7], we deal with M_N by partitioning the X_i's into two sets, $\{X_1, \dots, X_m\}$ and $\{X_{m+1}, \dots, X_N\}$. In choosing m, we depart from [7] and incorporate a device from [1]. Put $u=u_1+\dots+u_N$ and define m to be the smallest integer such that $\frac{1}{2}u\leqq u_1+\dots+u_m$. Then

$$u_1+\dots+u_{m-1}\leqq \tfrac{1}{2}u \quad \text{and} \quad u_{m+1}+\dots+u_N\leqq \tfrac{1}{2}u. \tag{2.11}$$

Define

$$L_1 = M_{m-1} = \max_{0\leqq k\leqq m-1} |S_k|$$

and

$$L_2 = \max_{m\leqq k\leqq N} |S_k - S_m|.$$

For $1\leqq n\leqq m-1$, we have $|S_n|^\nu \leqq M_{m-1}^\nu = L_1^\nu$. For $m\leqq n\leqq N$, we have, recalling the representation $\nu=h+\varepsilon$ given with (2.1),

$$|S_n|^\nu \leqq (|S_m|+L_2)^\nu \leqq (|S_m|^\varepsilon + L_2^\varepsilon)(|S_m|+L_2)^h = |S_m|^\nu + L_2^\nu + K,$$

where

$$K=\sum_{j=0}^{h-1}\binom{h}{j}|S_m|^{j+\varepsilon}L_2^{h-j}+\sum_{j=1}^{h}\binom{h}{j}|S_m|^{j}L_2^{h-j+\varepsilon}.$$

It follows that

$$M_N^\nu \leqq L_1^\nu + L_2^\nu + |S_m|^\nu + K.$$

By (2.9), (2.11) and the induction hypothesis,

$$E(L_i^\nu)\leqq 2^{-\gamma}A_{\nu,\gamma}u^\gamma \quad (i=1,2) \quad \text{and} \quad E|S_m|^\nu \leqq u^\gamma.$$

Similarly, with the use of Hölder's inequality, we obtain for $r\geqq 0$, $s\geqq 0$ and $r+s=\nu$,

$$E|S_m|^r L_2^s \leqq (E|S_m|^\nu)^{r/\nu}(EL_2^\nu)^{s/\nu} \leqq \left(\sum_{k=1}^{m} u_k\right)^{\gamma r/\nu} A_{\nu,\gamma}^{s/\nu}\left(\sum_{k=m+1}^{N} u_k\right)^{\gamma s/\nu} \leqq$$

$$\leqq u^{\gamma r/\nu}A_{\nu,\gamma}^{s/\nu}(\tfrac{1}{2}u)^{\gamma s/\nu} = [2^{-\gamma}A_{\nu,\gamma}]^{s/\nu}u^\gamma.$$

Hence

$$E(K)\leqq 2^{-\gamma}A_{\nu,\gamma}u^\gamma w_\nu(2^{-\gamma}A_{\nu,\gamma}).$$

Thus

$$E(M_N^\nu)\leqq A_{\nu,\gamma}u^\gamma 2^{-\gamma}[2+2^\gamma A_{\nu,\gamma}^{-1}+w(2^{-\gamma}A_{\nu,\gamma})]. \tag{2.12}$$

By (2.2) and the definition of $A_{\nu,\gamma}$, (2.12) implies

$$E(M_N^\nu)\leqq A_{\nu,\gamma}u^\gamma,$$

completing the proof.

Let us briefly compare Theorem 1 (or Lemma 2) with previous results in the literature. Under the conditions of Lemma 2, and allowing $\gamma=1$, but restricting to $\nu\geqq 1$, BILLINGSLEY ([1], p. 102) gives

$$E(M_n^\nu) \leqq (\log_2 4n)^\nu \left(\sum_{k=1}^{n} u_k\right)^\gamma. \tag{2.13}$$

In the case $\gamma>1$ Lemma 2 improves (2.13) by replacing the factor $(\log_2 4n)^\nu$ by a constant not depending on n. (The case $\gamma=1$ is not subject to such an improvement, as is well-known.) In a similar vein, under the conditions of Lemma 2, for $\nu\geqq 2$ and $\gamma=\frac{1}{2}\nu$ (thus also allowing $\gamma=1$), SERFLING ([7], p. 1228) gives

$$E(M_n^\nu) \leqq (\log_2 2n)^\nu [g(1, n)]^{\nu/2}. \tag{2.14}$$

Theorem 1 improves (2.14) in the same fashion that Lemma 2 improves (2.13). Indeed, such an improvement was given by Theorem B of [7], but only under severe restrictions on g and for a less suitable type of constant replacing the log factor. The removal of the log factor is highly significant in connection with asymptotic applications of maximal inequalities.

In typical asymptotic applications (some exceptions are discussed in Section 3) of maximal inequalities, it suffices to have bounds on the exceedance probabilities $P\{M_n\geqq\lambda\}$, $\lambda>0$, rather than on the moments $E(M_n^\nu)$. But then one may start with mere probability inequalities for the partial sums, rather than the stronger moment inequalities (2.4). A fundamental result of this form, pertaining to the case $g(i,j)=\sum_i^j u_k$, has been developed by BILLINGSLEY ([1], p. 94). His theorem, in conjunction with Lemma 1, yields a generalization analogous to Theorem 1. Instead of the constant $A_{\nu,\gamma}$, however, the constant used is

$$C_{\nu,\gamma} = 2^\gamma\{1+[2^{-1/(\nu+1)}-2^{-\gamma/(\nu+1)}]^{-\nu-1}\}. \tag{2.15}$$

THEOREM 2. *Let* $X_1, \dots, X_n$ *be arbitrary random variables. Suppose that for constants* $\nu>0$ *and* $\gamma>1$, *and for all positive* λ,

$$P\left\{\left|\sum_{k=i}^{j} X_k\right| \geqq \lambda\right\} \leqq \lambda^{-\nu}[g(i,j)]^\gamma \quad (\text{all } 1\leqq i\leqq j\leqq n), \tag{2.16}$$

where g satisfies either (2.5) *or* (2.6). *Then for all positive* λ,

$$P\{M_n\geqq\lambda\} \leqq C_{\nu,\gamma}\lambda^{-\nu}[g(1, n)]^\gamma, \tag{2.17}$$

where $C_{\nu,\gamma}$ *is given by* (2.15).

(The case corresponding to (2.5) has been noted in [5].) A competitor to (2.17) follows immediately from Theorem 1. Namely,

COROLLARY 1. *Assume the conditions of Theorem* 1. *Then for all positive* λ,

$$P\{M_n\geqq\lambda\} \leqq A_{\nu,\gamma}\lambda^{-\nu}[g(1, n)]^\gamma, \tag{2.18}$$

where $A_{\nu,\gamma}$ *is given by* (2.3).

For asymptotic applications, (2.17) and (2.18) carry the same force. However, in the non-asymptotic case, the constant $A_{\nu,\gamma}$ is evidently better than $C_{\nu,\gamma}$; for example, for $v=4$ and $\gamma=2$, we have $A_{4,2}=401$ whereas $C_{4,2}=880{,}352$. In fact, even (2.14) can yield a competitive result for n sufficiently small.

Condition (2.4), which implies (2.16), is very broad in scope. Its most efficaceous form pertains to the case $\gamma=\frac{1}{2}v$, with $v>2$. Such results are developed and applied in [1], [4], [6], [7] and [8], under weak dependence restrictions on $X_1, \ldots, X_n$. Applications include tightness criteria apropos to weak convergence in metric spaces, laws of large numbers, the law of iterated logarithm, and the almost sure convergence of infinite series.

3. Further applications. Here we mention two applications in which a bound on $E(M_n^v)$ is needed and may not be pre-empted by implied bounds on $P\{M_n \geqq \lambda\}$. One such situation arises in the theory of *optimal stopping* (see [3]), wherein an important condition relative to the existence of an optimal stopping rule is that

$$E\{\sup_{n\geqq 1} |S_n|/a_n\} < \infty \tag{3.1}$$

hold for a certain increasing sequence of positive constants $\{a_n\}$. As noted in [3], it suffices for (3.1) to show that

$$\lim_{n\to\infty} E\left\{\max_{1\leqq k\leqq n}\left|\sum_{j=1}^{k} X_j/a_j\right|\right\} < \infty. \tag{3.2}$$

A second situation arises in the theory of *stochastic approximation* (see [2]), in connection with the question of almost sure convergence of certain multidimensional Robbins—Monroe procedures.

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ON APPLICATION OF THE DUALITY PRINCIPLE FOR THE APPROXIMATION BY SPLINE FUNCTIONS IN L_p-SPACES

By

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1. Let $\Delta_n=\{0=x_0<x_1<\ldots<x_n=1\}$ be a partition of the unit interval with the norm

$$\varrho_n = \max_{1\leq i\leq n} (x_i-x_{i-1}). \tag{1}$$

Let us denote by $S(0, \Delta_n)$ the set of step functions defined on $[0, 1]$ which have jumps at the points x_i $(i=0, \ldots, n)$. Let $S(l, \Delta_n)$ $(l=1, 2, \ldots)$ be the set of functions $f\in C[0, 1]$, restriction of which to (x_i, x_{i+1}) is a polynomial of degree at most l. We consider the approximation of functions $f(x)$ defined on $[0, 1]$ by the class $S(k, \Delta_n)$ in L_p-norm

$$\|f\|_p=\Big\{\int_0^1 |f(x)|^p\,dx\Big\}^{1/p}, \quad 1\leq p<\infty.$$

For $f\in L_p$ $(1\leq p<\infty)$, and $k=0, 1, \ldots,$ let

$$E^{(k)}_{\Delta_n}(f)_p = \inf_{s\in S(k, \Delta_n)} \|f-s\|_p. \tag{2}$$

We know, that every $f\in L_p$ $(1\leq p<\infty)$ can be approximated by functions of the class $S(k, \Delta_n)$, that is

$$E^{(k)}_{\Delta_n}(f)_p\to 0 \quad (|\Delta_n|\to 0);\ k=0, 1, \ldots \tag{3}$$

(see [2]).

In the present paper we investigate the degree of the best approximation in terms of the r^{th} modulus of continuity (smoothness), in other words we state a Jackson-type theorem. The proof of this theorem will be based on the duality principle.

2. Let L be a normed linear space, and let L^* be the conjugate (dual) space of L, that is L^* is the space of linear continuous functionals defined on L. Let G be an arbitrary linear subspace of L, and

$$G_\perp = \{g\in L^*:\ g(y)=0,\ y\in G\}.$$

For an arbitrary $x\in L$, we have (see [4])

$$\inf_{y\in G}\|x-y\|_L = \max_{\substack{g\in G_\perp\\ \|g\|_{L^*}\leq 1}} g(x).$$

In our case by the theorem of F. RIESZ we have (see [3])

(4) $$E_{\Delta_n}^{(k)}(f)_p = \max_{\substack{g\in S_\perp(k,\Delta_n,p)\\ \|g\|_q\leqq 1}} \int_0^1 f(x)g(y)\,dx$$

where

(5) $$S_\perp(k,\Delta_n,p) = \left\{g\in L_q\colon \int_0^1 s(x)g(x)\,dx = 0 \quad\text{if}\quad s\in S(k,\Delta_n) \quad \left(\frac{1}{p}+\frac{1}{q}=1\right)\right\}.$$

First of all let us prove the following

LEMMA 2.1. *For any* $g\in S_\perp(k,\Delta_n,p)$, $1\leqq p<\infty$, $k=0,1,\dots$ *we have*

(6) $$\left|\int_0^x g(t)\,dt\right| \leqq \varrho_n^{1/p}\|g\|_q \quad (x\in[0,1]).$$

PROOF. Let

(7) $$\Gamma_x(t) = \begin{cases}1 & 0\leqq t\leqq x\\ 0 & x<t\leqq 1\end{cases} \quad (x\in[0,1]),$$

and suppose $x_i<x\leqq x_{i+1}$ $(x_i, x_{i+1}\in\Delta_n)$. Let

$$s_{0,x}(t) = \begin{cases}1 & 0\leqq t\leqq x_i\\ 0 & x_i<t\leqq 1,\end{cases}$$

$$s_{1,x}(t) = \begin{cases}1 & 0\leqq t\leqq x_i\\ \dfrac{t-x_{i+1}}{x_i-x_{i+1}} & x_i<t\leqq x_{i+1}\\ 0 & x_{i+1}<t\leqq 1.\end{cases}$$

It is clear that $s_{0,x}(t)\in S(0,\Delta_n)$ and $s_{1,x}(t)\in S(k,\Delta_n)$ $(k=1,2,\dots)$. Furthermore we have

(8) $$\|\Gamma_x(t)-s_{j,x}(t)\|_p \leqq \varrho_n^{1/p} \quad (x\in[0,1],\ j=0,1).$$

Now, for $g\in S_\perp(k,\Delta_n,p)$ we have by (7), (8) and (5)

$$\left|\int_0^x g(t)\,dt\right| = \left|\int_0^1 \Gamma_x(t)g(t)\,dt\right| = \left|\int_0^1 [\Gamma_x(t)-s_{j_k,x}(t)]g(t)\,dt\right| \leqq$$
$$\leqq \|\Gamma_x(t)-s_{j_k,x}(t)\|_p\|g\|_q \leqq \varrho_n^{1/p}\|g\|_q$$

(where $j_k=0$ if $k=0$, $j_k=1$ if $k\geqq 1$), which was to be proved. (6) is an analogue of the Bohr's inequality (see [1]).

LEMMA 2.2. *Let* $k\geqq 0$, $m\geqq 1$ *be given integers satisfying* $k+1\geqq m$. *If* f *is the* m *times iterated integral function of* $f^{(m)}\in L_1$, *then*

(9) $$E_{\Delta_n}^{(k)}(f)_p \leqq \varrho_n^{m/p}\|f^m\|_1 \quad (1\leqq p<\infty)$$

and

(10) $$E_{\Delta_n}^{(k+1)}(f)_p \leqq \varrho_n^{m/p} E_{\Delta_n}^{(k+1-m)}(f)_1 \quad (1\leqq p<\infty).$$

PROOF. For $g\in S_\perp(k, \Delta_n, p)$ let

$$G(x) = \int_0^x g(t)\,dt \quad (x\in[0, 1]).$$

Since $g(t)$ is orthogonal to the function $s(t)\equiv 1$ we have

$$G(0) = G(1) = 0. \tag{11}$$

If f is an integral function of $f'\in L_1$ then we obtain by (11) and (6)

$$\left|\int_0^1 f(t)g(t)\,dt\right| \leqq |[Gf]_0^1| + \left|\int_0^1 G(t)f'(t)\,dt\right| \leqq$$

$$\leqq \varrho_n^{1/p}\|g\|_q\|f'\|_1.$$

Therefore using (4) we have

$$E_{\Delta_n}^{(k)}(f)_p \leqq \max_{\substack{g\in S_\perp(k,\Delta_n,p)\\ \|g\|_q\leqq 1}} \left|\int_0^1 g(t)f(t)\,dt\right| \leqq \varrho_n^{1/p}\|f'\|_1,$$

which proves the case $m=1$ of (9).

For an arbitrary $\varepsilon>0$, let $s_\varepsilon(t)\in S(k, \Delta_n)$ be a function satisfying

$$\|f'-s_\varepsilon\|_1 \leqq (1+\varepsilon)E_{\Delta_n}^{(k)}(f')_1.$$

Let

$$s_\varepsilon^*(t) = \int_0^t s_\varepsilon(u)\,du.$$

It is clear, that $s_\varepsilon^*(t)\in S(k+1, \Delta_n)$. Thus by the just proved case $m=1$ of (9) we have

$$E_{\Delta_n}^{(k+1)}(f)_p = E_{\Delta_n}^{(k+1)}(f-s_\varepsilon^*) \leqq \varrho_n^{1/p}\|f'-s_\varepsilon\|_1 \leqq \varrho_n^{1/p}(1+\varepsilon)E_{\Delta_n}^{(k)}(f')_1.$$

Since $\varepsilon>0$ is arbitrary, we obtain

$$E_{\Delta_n}^{(k+1)}(f)_p \leqq \varrho_n^{1/p}E_{\Delta_n}^{(k)}(f')_1.$$

Thus the case $m=1$ of (10) is proved.

It is easy to see that the case $m>1$ of (10) follows from the case $m=1$ by induction. Finally, the case $m>1$ of (9) is a consequence of (10) and the case $m=1$ of (9). This completes the proof of Lemma 2.2.

3. Let $f\in L_p$ $(1\leqq p<\infty)$. Denote $\Delta_h^r f(x)$ the r-th forward difference of f with step h, that is

$$\Delta_h^r f(x) = \sum_{j=0}^r (-1)^{r-j}\binom{r}{j} f(x+jh) \quad (h\geqq 0).$$

Define

$$\omega_{r,p}(f, \delta)_{[0,1]} = \sup_{0\leqq h\leqq\delta}\left\{\int_0^{1-rh} |\Delta_h^r f(x)|^p\,dx\right\}^{1/p} \quad \left(0\leqq\delta\leqq\frac{1}{r}\right).$$

THEOREM 3.1. *Let $k\geqq 0, r\geqq 1$ be given integers satisfying $k+1\geqq r$ and $f\in L_p$ $(1\leqq p<\infty)$. Then we have*

$$E^{(k)}_{\Delta_n}(f)_p \leqq c_r\omega_{r,p}(f, \varrho_n^{1/p})_{[0,1]}, \tag{12}$$

where

$$c_r = 1+\sum_{j=1}^{r}\binom{r}{j}\left(\frac{r}{j}\right)^r. \tag{13}$$

PROOF. By virtue of Theorem 2.5 of [2], there is a function $f^*\in L_p[0, 1+r]$ such that $f^*(x)=f(x)$ $x\in[0, 1]$ and

$$\omega_{r,p}(f^*, \delta)_{[0,1+r]} \leqq c\omega_{r,p}(f, \delta)_{[0,1]}, \quad \delta\geqq 0 \tag{14}$$

with a constant C depending only on r.

We introduce the following function:

$$f_{hr}(x) = \tag{15}$$

$$= (-1)^{r-1}h^{-r}\Big\{\int_0^h\Big\}^r\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f^*\left[x+\frac{j}{r}(t_1+\ldots+t_r)\right]dt_1\ldots dt_r,$$

$$h = |\Delta|^{1/p}, \quad x\in[0, 1].$$

It is clear, that

$$f_{hr}(x)-f(x) =$$

$$= h^{-r}\Big\{\int_0^h\Big\}^r(-1)^{r-1}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}f^*\left[x+\frac{j}{r}(t_1+\ldots+t_r)\right]dt_1\ldots dt_r,$$

from which

$$\|f_{hr}-f\|_p \leqq \omega_{r,p}(f^*, h)_{[0,r+1]} \tag{16}$$

follows. Furthermore using the formula

$$\Delta_h^r f^*(x) = \Big[\Big\{\int_0^h\Big\}^r f^*(x+t_1+\ldots+t_r)\,dt_1\ldots dt_r\Big]^{(r)}$$

we obtain

$$f_{hr}^{(r)}(x) = (-1)^{r-1}h^{-r}\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left(\frac{r}{j}\right)^r\Delta^r_{jh/r}f^*(x).$$

Therefore

$$\|f_{hr}^{(r)}\|_p \leqq h^{-r}\sum_{j=1}^{r}\binom{r}{j}\left(\frac{r}{j}\right)^r\|\Delta^r_{jh/r}f^*(x)\|_p \leqq \tag{17}$$

$$\leqq h^{-r}\left[\sum_{j=1}^{r}\binom{r}{j}\left(\frac{r}{j}\right)^r\right]\omega_{r,p}(f^*, h)_{[0,r+1]}.$$

Finally, using (9), (16), (17) we have

$$E^{(k)}_{\Delta_n}(f)_p \leqq \|f-f_{hr}\|_p+E^{(k)}_{\Delta_n}(f_{hr})_p \leqq \left[1+\sum_{j=1}^{r}\binom{r}{j}\left(\frac{r}{j}\right)^r\right]\omega_{r,p}(f^*, \varrho_n^{1/p}),$$

which, by (14), proves the theorem.

THEOREM 3.2. *Let* $k \geqq m$ *be natural numbers. If* f *is an* m *times iterated integral function* $f^{(m)} \in L_1$, *then we have for every* $1 \leqq r \leqq k+2-m$

$$E_{\Delta_n}^{(k)}(f)_p \leqq \varrho_n^{1/p} \omega_{r,1}(f^{(m)}, \varrho_n) \quad (1 \leqq p < \infty).$$

PROOF. This theorem is a consequence of the inequality (10) and Theorem 3.1.

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ON A CLASS OF LATTICE-ORDERED SEMIGROUPS

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§ 1. Introduction

In his paper [3], A. E. LAEMMEL has shown that some classes of lattice-ordered semigroups can be applied to the mathematical theory of codes and finite-state transducers. In Laemmel's paper, the set S of all join-irreducible elements of a lattice-ordered semigroup T is supposed to be a subsemigroup of T, and it plays an important role in the whole paper. In [4], O. STEINFELD mentioned that this semigroup S is a generalized Brandt semigroup. (See our Proposition 2.1.)

In [4] it is proved that there exists a matrix representation for generalized Brandt semigroups, which is similar to the Rees matrix representation of completely 0-simple semigroups. (See [4] Theorem 4.1 and [2] Theorem 3.5.) Hence the semigroup S mentioned above also has a matrix representation. (See Corollary 2.2.)

The main purpose of this paper is to prove a matrix representation, similar to Corollary 2.2, of the lattice-ordered semigroup T used by A. E. LAEMMEL [3]. We shall show that there exists a suitable lattice-ordered subsemigroup U of T such that T is isomorphic to a lattice-ordered matrix semigroup over U.

§ 2. Preliminaries

Let $\langle T; \cdot, 0, \wedge, \vee \rangle$ be a lattice-ordered semigroup with the following properties:

(1) 0 is the least element of the lattice $\langle T; \wedge, \vee \rangle$;

(2) 0 is the zero element of the semigroup $\langle T; \cdot \rangle$;

(3) $\langle T; \wedge, \vee \rangle$ is a distributive lattice.

(See postulates 1—6, 9—11 of LAEMMEL [3].)

T will always denote a lattice-ordered (l. o.) semigroup with the properties (1), (2), (3).

An element a of a lattice L is said to be *join-reducible* if there exist elements a_1, a_2 in L such that

$$a = a_1 \vee a_2 \quad (a_1, a_2 < a). \tag{2.1}$$

If an element a has no decomposition of the form (2.1), it is called *join-irreducible*.

A semigroup H with zero is called *zero-cancellative* if it has the following property: if a, b, c are elements of H such that $ac = bc \neq 0$ or $ca = cb \neq 0$, then $a = b$.

Henceforth let S denote the set of all join-irreducible elements of the l. o. semigroup T. We assume that S is a zero-cancellative subsemigroup of T. (See postulate 12 of LAEMMEL [3].)

Postulate 14 of LAEMMEL [3]: For each element a of S there exist elements e, f of S such that $ae=a$ and $fa=a$.

Postulate 15 of LAEMMEL [3]: For all pairs of non-zero idempotents e_i, e_j in S there exist elements Φ_{ij}, Φ_{ji} in S such that

$$e_i = \Phi_{ij} e_j \Phi_{ji}. \tag{2.2}$$

For convenience, Φ_{ii} can be defined as equal to e_i.

A *generalized Brandt semigroup* is a semigroup H with zero satisfying the following conditions:

(α) H is zero-cancellative;

(β) for each element a of H there exist elements e, f of H such that $ae=a$ and $fa=a$;

(γ) if e_i and e_j are idempotents of H then $e_i e_j = e_j e_i$;

(δ) for all pairs e_i, e_j of non-zero idempotents of H there exist elements q_{ij}, q_{ji} in H such that

$$q_{ij} q_{ji} = e_i \quad \text{and} \quad q_{ji} q_{ij} = e_j. \tag{2.3}$$

(See O. STEINFELD [4], where the following proposition is also taken from.)

PROPOSITION 2.1. *Consider the zero-cancellative subsemigroup S of all join-irreducible elements of the lattice-ordered semigroup T. If S satisfies postulates* 14 *and* 15 *of* LAEMMEL [3], *then S is a generalized Brandt semigroup.*

Let H be a semigroup with zero and with identity element e. Let $M^0(H; I, \Lambda; P)$ denote the Rees matrix semigroup over H with the sandwich matrix $P=(p_{\lambda i})$ $(\lambda \in \Lambda, i \in I, p_{\lambda i} \in H)$.

Let B be a semigroup with zero and I_1, I_2 be left ideals of B. By a *left translation* of I_1 into I_2 we mean a mapping φ of I_1 into I_2 such that $x\varphi \in I_2$, $s(x\varphi)=(sx)\varphi$ (for all $x \in I_1$ and $s \in B$).

We say that the left ideals I_1, I_2 of B are *left similar* if there exists a one-to-one left translation φ of I_1 onto I_2.

Dually we define the *right similarity* of right ideals.

By a *special similarly decomposable* semigroup we mean a semigroup B with 0 having the properties

(a) $$B = \bigcup_{i \in I} Be_i = \bigcup_{i \in I} e_i B \quad (e_i^2 = e_i;\ e_i e_j = 0 \text{ for } i \neq j;\ i, j \in I);$$

(b) for arbitrary $i, j \in I$, Be_i and Be_j are left similar;

(c) there exists at least one idempotent $e_k (k \in I)$ such that the semigroup $e_k B e_k$ is 0-cancellative.

Proposition 2.1 above and Theorem 5.1 of O. STEINFELD [4] imply

COROLLARY 2.2. *Suppose that the join-irreducible elements of the l.o. semigroup T form a zero-cancellative subsemigroup S of T. If S satisfies postulates* 14 *and* 15 *of* LAEMMEL [3], *then S is isomorphic with a Rees $I \times I$ matrix semigroup $M^0(e_1 S e_1; I, I; \Delta)$, where $e_1 S e_1$ $(0 \neq e_1 \neq e_1^2; e_1 \in S)$ is zero-cancellative.*

§ 3. Preparations for the main theorem

In §§ 3—4 we shall prove a matrix representation of the l.o. semigroup T, similar to Corollary 2.2. (See Theorem 4.1.)

Postulate 8 of LAEMMEL [3]: Every non-empty subset of T has a minimal element.

An element x_i is *redundant* in a join if

$$x_1 \vee x_2 \vee \ldots \vee x_n = x_1 \vee x_2 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_n.$$

If no element is redundant in a join, the join is called *irredundant*.

A theorem of G. BIRKHOFF [1], p. 142 implies now

PROPOSITION 3.1. *Let S be the zero-cancellative subsemigroup of all join-irreducible elements of the l.o. semigroup T. If T satisfies postulate* 8 *of* LAEMMEL [3], *then each element of T has one and only one representation as an irredundant join of some elements of S.*

Let e_1 be a non-zero idempotent element of S. Let $(e_1 S e_1)^{\vee}$ denote the intersection of all $\vee$-subsemilattices of T containing $e_1 S e_1$, that is, $(e_1 S e_1)^{\vee}$ is the $\vee$-subsemilattice of T generated by $e_1 S e_1$. Evidently, every element a of $(e_1 S e_1)^{\vee}$ has the form $a = \bigvee_{j=1}^{n} a_j$ $(a_j \in e_1 S e_1; j = 1, \ldots, n)$.

If S satisfies postulates 14 and 15 then, in view of Corollary 2.2 and Theorem 5.1 of [4], the subsemigroup S of all join-irreducible elements of T can be written in the form $S = \bigcup_{i \in I} e_i S = \bigcup_{i \in I} S e_i = \bigcup_{i \in I} \bigcup_{j \in I} e_i S e_j$ $(e_i e_j = 0$ if $i \neq j)$.

First we prove

THEOREM 3.2. *Let S be the zero-cancellative subsemigroup of all join-irreducible elements of the l.o. semigroup T. If T satisfies postulate* 8 *and S satisfies postulates* 14 *and* 15 *of* LAEMMEL [3], *then $(e_1 S e_1)^{\vee}$ is a sublattice and a subsemigroup of T.*

REMARK. This means that $(e_1 S e_1)^{\vee}$ is a l.o. subsemigroup of T.

To the proof we need

PROPOSITION 3.3. *Consider any two elements a, b of*

$$S = \bigcup_{i \in I} e_i S = \bigcup_{i \in I} S e_i = \bigcup_{i \in I} \bigcup_{j \in I} e_i S e_j.$$

Assume $a \in e_i S e_j$, $b \in e_k S e_l$ $(i, j, k, l \in I)$. If $i \neq k$ or $j \neq l$, then $a \wedge b = 0$; if $i = k$ and $j = l$, then $a \wedge b = \bigvee_{m=1}^{n} s_m$, where $s_m \in e_i S e_j$ $(m = 1, \ldots, n)$.

PROOF OF PROPOSITION 3.3. Since $a \wedge b \in T$, from Proposition 3.1 it follows $a \wedge b = \bigvee_{m=1}^{n} s_m$ $(s_m \in S;\ m = 1, \ldots, n)$. If $a \in e_i S$, $b \in e_k S$; $i, k \in I$; $i \neq k$, then, since $a = a \vee (a \wedge b) = a \vee \left(\bigvee_{m=1}^{n} s_m \right)$, for each j in I, $j \neq i$, we have $0 = e_j a = e_j a \vee \left(\bigvee_{m=1}^{n} e_j s_m \right) = \bigvee_{m=1}^{n} e_j s_m$, therefore $s_m \notin e_j S$ $(j \in I, i \neq j, m = 1, \ldots, n)$, that is, $s_m \in e_i S$ $(m = 1, \ldots, n)$.

Similarly, $b=b\vee(a\wedge b)$ implies $s_m\in e_k S$ $(m=1, \dots, n)$. Since $e_k S\cap e_i S=0$ if $k\neq i$, therefore $s_m=0$ $(m=1, \dots, n)$, whence $a\wedge b=0$.

Similarly, if $a\in Se_j$ and $b\in Se_l$ $(j, l\in I; j\neq l)$, then $a\wedge b=0$.

If $a, b\in e_i Se_j$, then $a\wedge b=\bigvee_{m=1}^{n} s_m$, where $s_m\in e_i Se_j$ $(m=1, \dots, n)$.

Proof of Theorem 3.2. By definition, $(e_1 Se_1)^\vee$ is a $\vee$-semilattice. Consider any two elements $a, b\in(e_1 Se_1)^\vee$. Since $a=\bigvee_{j=1}^{m} a_i$, $b=\bigvee_{j=1}^{n} b_j$ $(a_i\in e_1 Se_1;\ i=1, \dots, m$ and $b_j\in e_1 Se_1; j=1, \dots, n)$, by Proposition 3.3 we get

$$a\wedge b=\left(\bigvee_{i=1}^{m} a_i\right)\wedge\left(\bigvee_{j=1}^{n} b_j\right)=\bigvee_{i=1}^{m}\bigvee_{j=1}^{n}(a_i\wedge b_j)\in(e_1 Se_1)^\vee.$$

$e_1 Se_1$ is a subsemigroup of the semigroup S, therefore

$$a\cdot b=\left(\bigvee_{i=1}^{m} a_i\right)\cdot\left(\bigvee_{j=1}^{n} b_j\right)=\bigvee_{i=1}^{m}\bigvee_{j=1}^{n} a_i b_j\in(e_1 Se_1)^\vee.$$

By Corollary 2.2 $S\cong M^0(e_1 Se_1; I, I; \Delta)$ holds. Let $\mathcal{R}=R\big((e_1 Se_1)^\vee; I, I; \Delta\big)$ denote the set of all $I\times I$-matrices over $(e_1 Se_1)^\vee$ having only a finite number of non-zero entries. Consider any two elements $\mathbf{A}=(a_{ij})$, $\mathbf{B}=(b_{ij})$ $(a_{ij}, b_{ij}\in(e_1 Se_1)^\vee;$ $i, j\in I)$ of $\mathcal{R}$. We define the following three binary operatoins on $\mathcal{R}$:

$$\mathbf{A}\vee\mathbf{B}=(a_{ij}\vee b_{ij}); \tag{3.1}$$

$$\mathbf{A}\wedge\mathbf{B}=(a_{ij}\wedge b_{ij}); \tag{3.2}$$

$$\mathbf{A}\circ\mathbf{B}=\Big(\bigvee_{k} a_{ik} b_{kj}\Big). \tag{3.3}$$

Since $(e_1 Se_1)^\vee$ is a l.o. semigroup, from Theorem 3.2 we get the following

Corollary 3.4. *$\mathcal{R}=R\big((e_1 Se_1)^\vee; I, I; \Delta\big)$ is a lattice-ordered semigroup with respect to the operations* (3.1), (3.2), (3.3).

§ 4. The main theorem

Theorem 4.1. *Consider the zero-cancellative subsemigroup S of all join-irreducible elements of the lattice-ordered semigroup T. If T satisfies postulate* 8 *and S satisfies postulates* 14 *and* 15 *of* Laemmel [3], *then there exists an isomorphic mapping Φ of the l. o. semigroup T onto the l.o. matrix semigroup $\mathcal{R}$.*

For the proof we need the following

Lemma 4.2. *If $\mathbf{A}\in\mathcal{R}\big((e_1 Se_1)^\vee; I, I; \Delta\big)$, then the matrix $\mathbf{A}$ has one and only one representation as a join of Rees $I\times I$ matrices over the semigroup $e_1 Se_1$.*

Proof of Lemma 4.2. Consider the matrix $\mathbf{A}\in\mathcal{R}$. By definition, $\mathbf{A}$ has finitely many non-zero entries t_{ij} $\big(t_{ij}\in(e_1 Se_1)^\vee; i, j\in I\big)$. Let T_{ij} denote the $I\times I$ Rees matrix $(t_{ij})_{ij}$ over $(e_1 Se_1)^{\vee}$. Then, evidently, $\mathbf{A}$ can be uniquely written in the form $\mathbf{A}=\bigvee_{\text{finite}} T_{ij}=\bigvee_{\text{finite}} (t_{ij})_{ij}$, where the t_{ij} are all the non-zero entries of $\mathbf{A}$.

On the other hand, since $t_{ij}\in(e_1 Se_1)^{\vee}$, by Proposition 3.1 the element t_{ij} can be uniquely written in the form

$$t_{ij} = \bigvee_{k=1}^{n(i,j)} s_k^{i,j}$$

where $s_k^{i,j}\in e_1 Se_1$.

Hence $T_{ij}=\bigvee_{k=1}^{n(i,j)} \mathbf{A}_k^{i,j}$, where $\mathbf{A}_k^{i,j}\in M^0(e_1 Se_1; I, I; \Delta)$, therefore the matrix $\mathbf{A}$ has one and only one representation as a join of Rees matrices over the semigroup $e_1 Se_1$, that is, $\mathbf{A}=\bigvee_{\text{finite}} \bigvee_{k=1}^{n(i,j)} \mathbf{A}_k^{i,j}$.

Proof of Theorem 4.1. From Corollary 2.2 it follows $S\cong M^0(e_1 Se_1; I, I; \Delta)$. Let φ be an isomorphic mapping of the semigroup S onto the matrix semigroup $M^0(e_1 Se_1; I, I; \Delta)$.

By Proposition 3.1 every element t of T can be uniquely written in the form

$$t = \bigvee_{i=1}^{n} s_i \quad (s_i\in S). \tag{4.1}$$

We define a mapping Φ of T into $\mathcal{R}$ as follows:

$$t\Phi = \left(\bigvee_{i=1}^{n} s_i\right)\Phi = \bigvee_{i=1}^{n} (s_i\varphi) \quad (\in\mathcal{R}). \tag{4.2}$$

In view of Theorem 3.1, if $t\in S$, then $t\Phi=t\varphi$.

Now we show that the mapping Φ defined by 4.2 is one-to-one.

From Lemma 4.2 we get that if $\mathbf{B}\in R\big((e_1 Se_1)^{\vee}; I, I; \Delta\big)$, then $\mathbf{B}=\bigvee_{j=1}^{m} \mathbf{B}_j$ $\big(\mathbf{B}_j\in M^0(e_1 Se_1; I, I; \Delta);\ j=1, \dots, m\big)$. Let Ψ be a mapping of $\mathcal{R}$ into T such that

$$\mathbf{B}\Psi = \bigvee_{j=1}^{m} (\mathbf{B}_j\varphi^{-1}) = \bigvee_{j=1}^{m} s_j = u\in T. \tag{4.3}$$

From (4.3) and (4.2) we get

$$(\mathbf{B}\Psi)\Phi = \left(\bigvee_{j=1}^{m} (\mathbf{B}_j\varphi^{-1})\right)\Phi = \left(\bigvee_{j=1}^{m} s_j\right)\Phi = u\Phi =$$

$$= \bigvee_{j=1}^{m} (s_j\varphi) = \bigvee_{j=1}^{m} (\mathbf{B}_j\varphi^{-1}\varphi) = \bigvee_{j=1}^{m} \mathbf{B}_j = \mathbf{B}.$$

Similarly, we get $t\Phi\Psi=t$, therefore Ψ is the inverse mapping of Φ.

Therefore Φ is a one-to-one mapping of the l.o. semigroup T onto the l.o. matrix semigroup $\mathcal{R}=R\big((e_1 Se_1)^{\vee}; I, I; \Delta\big)$.

Now we show that Φ is a homomorphism with respect to the operations $\vee$ and $\circ$.

Consider any two elements t_1, t_2 of T. By Proposition 3.1 the elements t_1, t_2 can be written in the form: $t_1 = \bigvee_{i=1}^{n} s_i$, $t_2 = \bigvee_{i=n+1}^{m} s_i$ $(s_i \in S; i=1, ..., m)$.

Then from Corollary 3.4 we get

$$[t_1 \vee t_2]\Phi = \left[\left(\bigvee_{i=1}^{n} s_i\right) \vee \left(\bigvee_{j=n+1}^{m} s_j\right)\right]\Phi = \left[\bigvee_{i=1}^{m} s_i\right]\Phi = \bigvee_{i=1}^{m} s_i\varphi =$$

$$= \left[\bigvee_{i=1}^{n} (s_i\varphi)\right] \vee \left[\bigvee_{j=n+1}^{m} (s_j\varphi)\right] = t_1\Phi \vee t_2\Phi;$$

$$[t_1 \cdot t_2]\Phi = \left[\left(\bigvee_{i=1}^{n} s_i\right) \cdot \left(\bigvee_{j=n+1}^{m} s_j\right)\right]\Phi = \left[\bigvee_{i=n}^{n} \bigvee_{j=n+1}^{m} (s_i s_j)\right]\Phi = \bigvee_{i=1}^{n} \bigvee_{j=n+1}^{m} [(s_i s_j)\varphi] =$$

$$= \bigvee_{i=1}^{n} \bigvee_{j=n+1}^{m} [(s_i\varphi) \circ (s_j\varphi)] = \left[\bigvee_{i=1}^{n} (s_i\varphi)\right] \circ \left[\bigvee_{j=n+1}^{m} (s_j\varphi)\right] = t_1\Phi \circ t_2\Phi.$$

We have to prove that Φ is a homomorphism also with respect to the operation $\wedge$. For this end we need the following

LEMMA 4.3. *If* $a, b \in S$, *then* $a\Phi \wedge b\Phi = (a \wedge b)\Phi$.

PROOF. From the proof of Theorem 5.1 of [4] it follows that for each pair of indices $i, j \in I$ there exist elements $q_{1i}, q_{i1}, q_{1j}, q_{j1}$ such that $q_{i1}q_{1i} = e_i$, $q_{j1}q_{1j} = e_j$ and $q_{1i}q_{i1} = q_{1j}q_{j1} = e_1$, furthermore if $a \in e_i S e_j$ then $a\Phi = a\varphi = (q_{1i}aq_{j1})_{ij}$ is an isomorphic mapping of S onto $M^0(e_1 S e_1; I, I; \Delta)$.

If $a \in e_i S e_j$, $b \in e_k S e_l$ and $i \neq k$ or $j \neq l$, then $a \wedge b = 0$. (See Proposition 3.3.) Hence $(a \wedge b)\Phi = \mathbf{0}$ and $a\Phi = (q_{1i}aq_{j1})_{ij}$, $b\Phi = (q_{1k}bq_{l1})_{kl}$ $(i \neq k$ or $j \neq l)$ imply $a\Phi \wedge b\Phi = \mathbf{0}$. Therefore $(a \wedge b)\Phi = a\Phi \wedge b\Phi$.

If $a, b \in e_i S e_j$, then by Proposition 3.3 $a \wedge b = \bigvee_{m=1}^{n} s_m \in (e_i S e_j)^{\vee}$ and $a\Phi = (q_{1i}aq_{j1})_{ij}$, $b\Phi = (q_{1i}bq_{j1})_{ij}$ and

$$(a \wedge b)\Phi = \left(\bigvee_{m=1}^{n} s_m\right)\Phi = \bigvee_{m=1}^{n} (s_m\varphi) = \bigvee_{m=1}^{n} (q_{1i}s_m q_{j1})_{ij} =$$

$$= \left(q_{1i}\left(\bigvee_{m=1}^{n} s_m\right)q_{j1}\right)_{ij} = (q_{1i}(a \wedge b)q_{j1})_{ij}.$$

We shall show that $q_{1i}(a \wedge b)q_{j1} = q_{1i}aq_{j1} \wedge q_{1i}bq_{j1}$.

α) We know that S is a zero-cancellative semigroup, so if $q_{i1}c \neq 0$, then $c < d$ $(c, d \in e_1 S e_1)$ implies $q_{i1}c < q_{i1}d$ (because $q_{i1}c = q_{i1}d \neq 0$ implies $c = d$).

Similarly, $c < d$ implies $q_{i1}cq_{1j} < q_{i1}dq_{1j}$ (if $q_{i1}cq_{1j} \neq 0$).

β) $a \wedge b \leqq a$ implies $q_{1i}(a \wedge b)q_{j1} \leqq q_{1i}aq_{j1}$ and $a \wedge b \leqq b$ imply $q_{1i}(a \wedge b)q_{j1} \leqq q_{1i}bq_{j1}$, therefore

$$q_{1i}(a \wedge b)q_{j1} \leqq q_{1i}aq_{j1} \wedge q_{1i}bq_{j1}.$$

γ) $q_{i1}\{q_{1i}(a \wedge b)q_{j1}\}q_{1j} = q_{i1}q_{1i}\left(\bigvee_{m=1}^{n} s_m\right)q_{j1}q_{1j} = \bigvee_{m=1}^{n} (q_{i1}q_{1i}s_m q_{j1}q_{1j}) = \bigvee_{m=1}^{n} s_m = a \wedge b$.

δ) Assume $q_{1i}(a\wedge b)q_{j1} < q_{1i}aq_{j1}\wedge q_{1i}bq_{j1}$, then

$$a\wedge b = q_{i1}q_{1i}(a\wedge b)q_{j1}q_{1j} < q_{i1}\{q_{1i}aq_{j1}\wedge q_{1i}bq_{j1}\}q_{1j} \leqq$$
$$\leqq q_{i1}q_{1i}aq_{j1}q_{1j}\wedge q_{i1}q_{1i}bq_{j1}q_{1j} = a\wedge b.$$

This is a contradiction, therefore

$$q_{1i}(a\wedge b)q_{j1} = q_{1i}aq_{j1}\wedge q_{1i}bq_{j1}.$$

Hence

$$a\Phi\wedge b\Phi = (q_{1i}aq_{j1})_{ij}\wedge(q_{1i}bq_{j1})_{ij} = (q_{1i}aq_{j1}\wedge q_{1i}bq_{j1})_{ij} = \big(q_{1i}(a\wedge b)q_{j1}\big)_{ij} = (a\wedge b)\Phi.$$

Now we can finish the proof of Theorem 4.1:

$$(t_1\wedge t_2)\Phi = \left[\left(\bigvee_{i=1}^{n} s_i\right)\wedge\left(\bigvee_{j=n+1}^{m} s_j\right)\right]\Phi = \left[\bigvee_{i=1}^{n}\bigvee_{j=n+1}^{m}(s_i\wedge s_j)\right]\Phi =$$
$$= \bigvee_{i=1}^{n}\bigvee_{j=n+1}^{m}[(s_i\wedge s_j)\Phi] = \bigvee_{i=1}^{n}\bigvee_{j=n+1}^{m}[s_i\Phi\wedge s_j\Phi] = \bigvee_{i=1}^{n}\bigvee_{j=n+1}^{m}[s_i\varphi\wedge s_j\varphi] =$$
$$= \left[\bigvee_{i=1}^{n} s_i\varphi\right]\wedge\left]\bigvee_{j=n+1}^{m} s_j\varphi\right] = t_1\Phi\wedge t_2\Phi.$$

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ON THE DENSITY OF QUOTIENTS OF LACUNARY POLYNOMIALS

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Let $1=n_1<n_2<\dots$ and $1=m_1<m_2<\dots$ be two sequences of real numbers tending to $+\infty$. Let $[0,\infty]$ be the compactified positive half-line; $C[0,1]$ and $C[0,\infty]$ denote the spaces of complex-valued continuous functions on the corresponding intervals, supplied with supremum norm.

We proved in [1]

THEOREM 1. *The set of the quotients*

$$h(x)=\frac{\sum_{j=1}^{s}a_jx^{n_j}}{\sum_{j=1}^{s}b_jx^{n_j}} \tag{1}$$

(s arbitrary integer, a_j, b_j arbitrary complex constants) is dense in $C[0,1]$ considering only the $h(x)$'s bounded in $[0,1]$.

(In fact we stated this result for real-valued functions but the proof works in the general case too. Also we can restrict the b_j's to be positive real.)

Now we shall prove

THEOREM 2. *Assume that for an integer j_0 $\{n_j\}_{j>j_0}$ and $\{m_j\}_{j>j_0}$ are disjoint sets and their union (as a monotone increasing sequence) has Hadamard gaps. Then the set of quotients*

$$h(x)=\frac{\sum_{j=1}^{s}a_jx^{m_j}}{\sum_{j=1}^{s}b_jx^{n_j}} \tag{2}$$

(s, a_j, b_j arbitrary) is not dense in $C[0,1]$.

LEMMA 1. *Let $0<p_1<p_2<\dots$ be a sequence of real numbers satisfying the condition*

$$p_{k+1}/p_k \geqq q>1 \quad (k=1,2,\dots). \tag{3}$$

Then for each large enough integer r

$$\frac{1}{2}\sum_{j=1}^{s}|a_k|^2 \leqq \int_0^\infty \left|\sum_{k=1}^{s}a_kq_ke^{-p_ky}\right|^2 y^r\,dy \leqq \frac{3}{2}\sum_{k=1}^{s}|a_k|^2 \tag{4}$$

holds, where s and the complex a_k's are arbitrary; q_k is defined by

$$q_k = q_k(r) = \sqrt{\frac{(2p_k)^{r+1}}{r!}} \quad (k = 1, 2, \ldots). \tag{5}$$

PROOF OF LEMMA 1. We have by integrations by parts

$$\int_0^\infty e^{-py} y^r \, dy = \frac{r!}{p^{r+1}} \quad (p > 0). \tag{6}$$

(5) and (6) yield

$$c_{k,k'} \overset{\text{def}}{=} \int_0^\infty q_k e^{-p_k y} q_{k'} e^{-p_{k'} y} y^r \, dy = \left(\frac{2\sqrt{p_k p_{k'}}}{p_k + p_{k'}}\right)^{r+1} \tag{7}$$

(k, k' arbitrary). In particular

$$c_{k,k} = 1 \quad (k = 1, 2, \ldots) \tag{8}$$

is valid.

It follows from (3) and elementary considerations that

$$\frac{2\sqrt{p_k p_{k'}}}{p_k + p_{k'}} \leqq \frac{2\sqrt{q^{|k'-k|}}}{1 + q^{|k'-k|}} \quad (k, k' \text{ arbitrary}). \tag{9}$$

Applying Jensen's inequality with respect to the function $x^{|k'-k|}$ we obtain

$$\frac{1 + q^{|k'-k|}}{2} \geqq \left(\frac{1+q}{2}\right)^{|k'-k|}. \tag{10}$$

(7), (9) and (10) yield

$$c_{k,k'} \leqq \left(\frac{2\sqrt{q}}{1+q}\right)^{(r+1)|k'-k|} \quad (k, k' \text{ arbitrary}). \tag{11}$$

We show that the assumption

$$\left(\frac{2\sqrt{q}}{1+q}\right)^{r+1} \leqq \frac{1}{5} \tag{12}$$

implies (4). This will complete the proof of Lemma 1. We have by (7) and (8)

$$\int_0^\infty \left| \sum_{k=1}^s a_k q_k e^{-p_k y} \right|^2 y^r \, dy = \sum_{k=1}^s |a_k|^2 + \sum_{1 \leqq k < k' \leqq s} (a_k \bar{a}_{k'} + \bar{a}_k a_{k'}) c_{k,k'}. \tag{13}$$

Here $|a_k \bar{a}_{k'} + \bar{a}_k a_{k'}| \leqq |a_k|^2 + |a_{k'}|^2$, thus

$$\left| \sum_{1 \leqq k < k' \leqq s} c_{k,k'} (a_k \bar{a}_{k'} + \bar{a}_k a_{k'}) \right| \leqq \sum_{k=1}^s |a_k|^2 \sum_{k' \neq k} c_{k,k'} \tag{14}$$

holds. We obtain from (11) and (12) that $c_{k,k'} \leqq \left(\frac{1}{5}\right)^{|k'-k|}$ and so

$$\sum_{k' \neq k} c_{k,k'} \leqq 2\left(\frac{1}{5}+\frac{1}{5^2}+\dots\right) = \frac{1}{2} \quad (k = 1, 2, \dots, s). \tag{15}$$

(4) follows from (13), (14) and (15).

Proof of Theorem 2. Let $\lambda > 0$ be an arbitrary fixed number outside the set $\{m_j - n_k\}_{j,k=1}^{\infty}$. We state that

$$\max_{0 \leqq y \leqq \infty} \left| e^{-\lambda y} - \frac{\sum\limits_{j=1}^{s} a_j e^{-m_j y}}{\sum\limits_{j=1}^{s} b_j e^{-n_j y}} \right| \leqq \varepsilon \tag{16}$$

cannot be true if ε is small enough. Theorem 2 follows from this statement and the change of variable $x = e^{-y}$. The sequences $\{n_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty} \cup \{n_j + \lambda\}_{j=1}^{\infty}$ have Hadamard gaps, therefore we can choose an r satisfying the statement of Lemma 1 with both of these sequences. Suppose that (16) and the normalizing assumption

$$\int_0^{\infty} \left| \sum_{j=1}^{s} b_j e^{-n_j y} \right|^2 y^r \, dy = 1 \tag{17}$$

are true. It follows from (16) and (17) that

$$\int_0^{\infty} \left| \sum_{j=1}^{s} b_j e^{-(n_j+\lambda) y} - \sum_{j=1}^{s} a_j e^{-m_j y} \right|^2 y^r \, dy \leqq \varepsilon^2. \tag{18}$$

We have by (17), (5) and the second inequality in (4)

$$\sum_{j=1}^{s} \frac{|b\ |^2 r!}{2n_j)^{r+1}} \geqq \frac{2}{3}. \tag{19}$$

(18), (5) and the first inequality in (4) yield

$$\sum_{j=1}^{s} \frac{|b_j|^2 r!}{(2n_j + 2\lambda)^{r+1}} \leqq 2\varepsilon^2. \tag{20}$$

It is obvious by $0 < n_1 \leqq n_j$ $(j = 1, 2, \dots)$ that

$$\sum_{j=1}^{s} \frac{|b_j|^2 r!}{(2n_j)^{r+1}} \leqq \left(\frac{2n_1 + 2\lambda}{2n_1}\right)^{r+1} \sum_{j=1}^{s} \frac{|b_j|^2 r!}{(2n_j + 2\lambda)^{r+1}}. \tag{21}$$

We obtain from (19), (20) and (21)

$$\frac{1}{3} \leqq \left(\frac{2n_1 + 2\lambda}{2n_1}\right)^{r+1} \varepsilon^2 \tag{22}$$

which proves Theorem 2.

REMARK. Originally we could prove Theorem 2 only for sequences with Hadamard gaps large enough. G. Halász has suggested to use weight functions in the L^2 method. Also he has noticed that the L^1 method works too if we replace Lemma 1 by a theorem of ZYGMUND (see [2], p. 194).

P. TURÁN [3] raised the following question. Let $g(y)$ be a fixed real function for $0\leqq y<\infty$. Do the quotients

$$h(y)=\frac{\sum_{j=1}^{s} a_j e^{-n_j(y+ig(y))}}{\sum_{j=1}^{s} b_j e^{-n_j(y+ig(y))}} \tag{23}$$

(a_j, b_j complex constants) form a dense set in $C[0,\infty]$? J. KOREVAAR [4] investigated the corresponding problem in the case of the polynomial approximation.

We define for each $M>0$ the set

$$Y_M=\left\{t\colon t>0,\ \overline{\lim_{x\to t}}\left|\frac{g(x)-g(t)}{x-t}\right|<M\right\}. \tag{24}$$

THEOREM 3. *Suppose that for any $y>0$ there exists an $M=M_y$ such that Y_M is dense in the interval $[0,y]$. Then the set of functions $h(y)$ in (23) is dense in $C[0,\infty]$.*

PROOF. We make a modification of the argument in [1] and do not detail each step here. Let $f\in C[0,\infty]$ and $\varepsilon>0$ be arbitrary. We choose $y_1>0$ so that

$$+\infty\geqq y>y_1 \quad \text{implies} \quad |f(y)-f(y_1)|<\varepsilon. \tag{25}$$

The integer s and the nodes $y_1>y_2>\ldots>y_s=0$ can be chosen so that with $M=M_{y_1}$

$$y_q\in Y_M \quad (2\leqq q\leqq s-1), \tag{26}$$

and

$$|f(y_q)-f(y)|<\varepsilon(1-e^{-1/M}) \quad (y_{q+1}\leqq y\leqq y_q,\ 1\leqq q\leqq s-1). \tag{27}$$

LEMMA 2. *For suitably chosen integers $1=j_1<j_2<\ldots<j_{s-1}$ and complex numbers $1=c_1, c_2, \ldots, c_{s-1}$*

$$\left|f(y)-\frac{\sum_{k=1}^{q} f(y_k)c_k e_k(y)}{\sum_{k=1}^{q} c_k e_k(y)}\right|<\varepsilon \quad (y_{q+1}\leqq y\leqq +\infty;\ q=1,2,\ldots,s-1) \tag{28}$$

holds, where

$$e_k(y)=e^{-n_{j_k}(y-y_k+i(g(y)-g(y_k)))} \quad (1\leqq k\leqq s-1).$$

We introduce the notations

$$h_q(y)=\frac{P_q(y)}{Q_q(y)}=\frac{\sum_{k=1}^{q} f(y_k)c_k e_k(y)}{\sum_{k=1}^{q} c_k e_k(y)} \quad (1\leqq q\leqq s-1).$$

For $q=s-1$ (28) yields

$$\max_{0\leqq y\leqq+\infty}|f(y)-h_{s-1}(y)|<\varepsilon.$$

Here $h(y)=h_{s-1}(y)$ is a quotient considered in (23), thus Lemma 2 implies the theorem.

PROOF OF LEMMA 2. We shall define $c_1, j_1, c_2, j_2, \ldots, c_{s-1}, j_{s-1}$ successively so that for $q=1, 2, \ldots, s-1$ the relations (28),

(29) $$Q_q(y)\neq 0 \quad (0<y<+\infty)$$

and

(30) $$|f(y_{q+1})-h_q(y_{q+1})|<\varepsilon(1-e^{-1/M})$$

will be valid.

In the case $q=1$, $c_1\neq 0$ yields (29), furthermore (25), (27) and $h_1(y)\equiv f(y_1)$ yield (28) and (30).

Let us suppose that $1<q<s$, and c_k, j_k $(k=1, 2, \ldots q-1)$ have been already defined so that (28), (29), (30) are true with $q-1$ in place of q. We put $c_q=Q_{q-1}(y_q)$ and choose an n_{j_q} large enough. Then (28), (29) and (30) will be valid for q too.

Namely, suppose that δ $(0<\delta<y_q-y_{q+1})$ is fixed. Then $|e_q(y)|$ can be made small for $y\geqq y_q+\delta$ by choosing an n_{j_q} large enough. Thus the induction hypothesis will imply (29) and (28) for $y\geqq y_q+\delta$. $|e_q(y)|$ can be made large for $y\leqq y_q-\delta$ so that (29) will be true here. Moreover (27) will imply (28) for $y_{q+1}\leqq y\leqq y_q-\delta$ and (30).

It remains to prove that there exists a δ depending on f and h_{q-1} such that (28) and (29) are true for $y\in[y_q-\delta, y_q+\delta]$ independently of the choice of j_q.

PROPOSITION. *There is a* $\delta_1>0$ *with the property*

(31) $$|1+e^{-x((y-y_q)+i(g(y)-g(y_q)))}|>\delta_1+1-e^{-1/M}$$

(x arbitrary real number, $|y_q-y|\leqq\delta_1$*).*

PROOF. We may assume $\delta_1<e^{-1/M}$ and by (24), (26)

$$|g(y)-g(y_q)|<M|y-y_q| \quad (|y-y_q|\leqq\delta_1).$$

Thus the left side of (31) is greater than 1 when $|x|\leqq\dfrac{\pi}{2M|y-y_q|}$ and (31) holds in this case. In the opposite case

$$|1-e^{-x(y-y_q)}|>1-e^{-\pi/2M}>1-e^{-1/M}+\delta_1$$

is true for δ_1 sufficiently small, therefore (31) holds again.

Now we choose a δ_2, $0<\delta_2<\delta_1$ with the following properties:

(32) $$\left|\frac{Q_{q-1}(y)}{c_q}-1\right|<\delta_1 \quad (|y-y_q|\leqq\delta_2)$$

(33) $$\left|\frac{P_{q-1}(y)-Q_{q-1}(y)f(y_q)}{c_q}\right|<\frac{\varepsilon(1-e^{-1/M})}{1+\delta_2} \quad (|y-y_q|\leqq\delta_2).$$

By the induction hypothesis $c_q = Q_{q-1}(y_q) \neq 0$ so (32) is possible because of the continuity of Q_{q-1}. (33) follows from (30) applied with $q-1$ in place of q and from the continuity of the function considered.

The Proposition applied with $x = n_{j_q}$ implies

$$(34) \qquad |1+e_q(y)| > \delta_1 + 1 - e^{-1/M} \quad (|y-y_q| \leqq \delta_2).$$

We have by (32) and (34)

$$(35) \qquad \left|\frac{Q_q(y)}{c_q}\right| = \left|\frac{Q_{q-1}(y)}{c_q} + e_q(y)\right| \geqq |e_q(y)+1| - \left|\frac{Q_{q-1}(y)}{c_q} - 1\right| > 1 - e^{-1/M} \quad (|y-y_q| \leqq \delta_2),$$

therefore $Q_q(y) \neq 0$ for $|y-y_q| \leqq \delta_2$. According to the definitions

$$h_q(y) - f(y_q) = \frac{P_{q-1}(y) - Q_{q-1}(y) f(y_q)}{Q_q(y)} = \frac{\dfrac{P_{q-1}(y) - Q_{q-1}(y) f(y_q)}{c_q}}{\dfrac{Q_q(y)}{c_q}}$$

hence by (33) and (35) we obtain

$$(36) \qquad |h_q(y) - f(y_q)| < \frac{\varepsilon}{1+\delta_2} \quad (|y-y_q| \leqq \delta_2).$$

The existence of the required δ follows from (36) and from the continuity of f at y_q. Thus Lemma 2 is proved.

REMARK. The assumption in Theorem 3 does not imply the continuity of g. If g is continuous then Theorem 3 concerns the uniform approximation of continuous functions along the complex curve $\{z : z = y + ig(y), 0 \leqq y < +\infty\}$ by quotients of the form

$$\frac{\sum_{j=1}^{s} a_j e^{-n_j z}}{\sum_{j=1}^{s} b_j e^{-n_j z}}.$$

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SOME REMARKS CONCERNING IRREGULARITIES OF DISTRIBUTION OF SEQUENCES OF INTEGERS IN ARITHMETIC PROGRESSIONS. IV

By

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1. Throughout this paper, we shall use the notation $e(\alpha)=e^{2i\pi\alpha}$ where α is real. Furthermore, we shall write $f(N)\ll g(N)$ if there exists an absolute constant c such that $|f(N)|\leqq c|g(N)|$ for $N=1, 2, \ldots$.

In [4], K. F. Roth proved the following theorem:

Theorem 1 (K. F. Roth [4]). *Let N be a natural number and let $\mathcal{N}$ be a set of distinct natural numbers not exceeding N. Let us write*

$$\eta = N^{-1} \sum_{\substack{1\leqq n\leqq N\\ n\in\mathcal{N}}} 1$$

and for every natural numbers m, q, let

$$V_q(m) = \sum_{h=1}^{q}\Big(\sum_{\substack{1\leqq n\leqq N\\ n\equiv h\,(\mathrm{mod}\,q)\\ n\in\mathcal{N}}} 1-\eta \sum_{\substack{1\leqq n\leqq m\\ n\equiv h\,(\mathrm{mod}\,q)}} 1\Big)^2.$$

Then, for all natural numbers Q,

$$\sum_{q=1}^{Q} q^{-1} \sum_{m=1}^{N} V_q(m)+Q \sum_{q=1}^{Q} V_q(N) \gg \eta(1-\eta)Q^2N. \tag{1}$$

This theorem implies

Corollary 1 (K. F. Roth). *Let N, Q be natural numbers for which $Q\leqq\sqrt{N}$. For any sequence $\mathcal{N}(\subset\{1, 2, \ldots\})$, there exist positive integers q, $n(\leqq N)$ and integer h such that $1\leqq q\leqq Q$ and*

$$\Big| \sum_{\substack{1\leqq n\leqq N\\ n\equiv h\,(\mathrm{mod}\,q)\\ n\in\mathcal{N}}} 1-\eta \sum_{\substack{1\leqq n\leqq N\\ n\equiv h\,(\mathrm{mod}\,q)}} 1\Big| \gg \sqrt{\eta(1-\eta)Q}. \tag{2}$$

(In fact, (1) says that (2) holds on average.)

Choosing $Q=[\sqrt{N}]$, we obtain

Corollary 2 (K. F. Roth [4]). *For all natural numbers N and sequences $\mathcal{N}$, there exist natural numbers q, $n(\leqq N)$ and integer h such that $1\leqq q\leqq N$ and*

$$\Big| \sum_{\substack{1\leqq n\leqq N\\ n\equiv h\,(\mathrm{mod}\,q)\\ n\in\mathcal{N}}} 1-\eta \sum_{\substack{1\leqq n\leqq N\\ n\equiv h\,(\mathrm{mod}\,q)}} 1\Big| \gg \sqrt{\eta(1-\eta)}N^{1/4}.$$

K. F. Roth, S. L. G. Choi, M. N. Huxley, H. L. Montgomery and A. Sárközy have extended these results in various directions. In particular, K. F. Roth proved the following

THEOREM 2 (K. F. ROTH [5]). *Let k be a positive integer and suppose that the integer N satisfies*

$$N > (10k)^7. \tag{3}$$

Then for every set $s_1, s_2, \dots, s_N$ *of N real numbers, there exist integers n, q, satisfying*

$$1 \leqq n \leqq n+(k-1)q \leqq N, \tag{4}$$

such that

$$\left|\sum_{i=0}^{k-1} s_{n+iq}\right| \geqq \left(\frac{1}{10} kN^{-1} \sum_{j=1}^{N} s_j^2\right)^{1/2}. \tag{5}$$

(For further details and other references, see [6], [7] and [8].)

The aim of this paper is to elaborate the "modulo p analogue" of this theorem (i.e. a lower estimate of type (5) for periodic sequences with period length p) and to show that in this way, we may obtain a lower estimate for character sums. For this purpose, we need a slightly modified and more precise form of Theorem 2.

2. In this section, we shall prove the following

THEOREM 3. *Let N be a positive integer, Q a positive integer satisfying*

$$Q \geqq 2. \tag{6}$$

Let $s_1, s_2, \dots, s_N$ *be a set of N complex numbers. Let*

$$Q_1 = \left[\frac{Q}{2}\right]$$

and $s_i=0$ *for* $i=0, -1, -2, \dots$ *and* $i=N+1, N+2, \dots$. *For every integer n and positive integers q, k, let*

$$D(n, q, k) = s_n+s_{n+q}+s_{n+2q}+\dots+s_{n+(k-1)q}.$$

Then

$$\sum_{q=1}^{Q} \sum_{n=1-(Q_1-1)q}^{N} |D(n, q, Q_1)|^2 \geqq \left(\frac{2}{\pi} Q_1\right)^2 \sum_{m=1}^{N} |s_m|^2. \tag{7}$$

COROLLARY 3. *Having the assumptions and notations in Theorem 3, there exist an integer n and a positive integer q such that*

$$1 \leqq q \leqq Q \tag{8}$$

and

$$|D(n, q, Q_1)| \geqq \frac{2}{\pi}\left[\frac{Q}{2}\right] Q^{-1/2}\left(N+\frac{Q^2}{4}\right)^{-1/2}\left(\sum_{m=1}^{N} |s_m|^2\right)^{1/2}. \tag{9}$$

COROLLARY 4. *If* $\varepsilon>0$, $N>N_0(\varepsilon)$ *is a positive integer and* $s_1, s_2, \dots, s_N$ *is a set of* N *complex numbers then there exist integers* n, q *such that* $1 \leqq q \leqq \sqrt{N}$ *and*

$$|D(n, q, [\sqrt{N}/2])| \geqq \left(\frac{2}{\pi\sqrt{5}} - \varepsilon\right)\left\{\left(\sum_{m=1}^{N} |s_m|^2\right)\Big/ N\right\}^{1/2} N^{1/4}.$$

(Note that, by (3), Theorem 2 would give an estimate of this type only with $N^{1/14}$ instead of $N^{1/4}$ on the right hand side.)

PROOF OF THEOREM 3. Let us write

$$F(\beta) = \sum_{j=0}^{Q_1-1} e(j\beta)$$

and

$$S(\alpha) = \sum_{n=1}^{N} s_n e(n\alpha) = \sum_{-\infty}^{+\infty} s_n e(n\alpha)$$

(note that

(10) $$s_n = 0 \quad \text{for} \quad n<1 \quad \text{or} \quad n>N).$$

Following Roth's method, we start out from the integral

$$E = \int_0^1 \sum_{q=1}^{Q} |F(q\alpha) S(\alpha)|^2 \, d\alpha.$$

As Roth showed (see (11) in [4]),

$$\sum_{q=1}^{Q} |F(q\alpha)|^2 \geqq \left(\frac{2}{\pi} Q_1\right)^2.$$

Thus Parseval's formula yields that

(11) $$E = \int_0^1 |S(\alpha)|^2 \sum_{q=1}^{Q} |F(q\alpha)|^2 \, d\alpha \geqq \left(\frac{2}{\pi} Q_1\right)^2 \int_1^0 |S(\alpha)|^2 \, d\alpha = \left(\frac{2}{\pi} Q_1\right)^2 \sum_{m=1}^{N} |s_m|^2.$$

On the other hand, again by Parseval's formula (and with respect to (10)),

(12) $$E = \int_0^1 \sum_{q=1}^{Q} |F(q\alpha) S(\alpha)|^2 \, d\alpha = \sum_{q=1}^{Q} \int_0^1 \left| \sum_{j=0}^{Q_1-1} e(jq\alpha) \sum_{n=1}^{N} s_n e(n\alpha) \right|^2 d\alpha =$$

$$= \sum_{q=1}^{Q} \int_0^1 \left| \sum_{m=1}^{N+(Q_1-1)q} \left(\sum_{j=1}^{Q_1-1} s_{m-jq} \right) e(m\alpha) \right|^2 d\alpha =$$

$$= \sum_{q=1}^{Q} \int_0^1 \left| \sum_{m=1}^{N+(Q_1-1)q} D(m-(Q_1-1)q, q, Q_1) e(m\alpha) \right|^2 d\alpha =$$

$$= \sum_{q=1}^{Q} \sum_{m=1}^{N+(Q_1-1)q} |D(m-(Q_1-1)q, q, Q_1)|^2 = \sum_{q=1}^{Q} \sum_{n=1-(Q_1-1)q}^{N} |D(n, q, Q_1)|^2.$$

(11) and (12) yield (7).

PROOF OF COROLLARY 3. Let us write

$$D=\max_{\substack{1\le q\le Q\\ 1-(Q_1-1)q\le n\le N}}|D(n,q,Q_1)|.$$

Then for the left hand side of (7) we have

(13)
$$\sum_{q=1}^{Q}\sum_{n=1-(Q_1-1)q}^{N}|D(n,q,Q_1)|^2\le D^2\sum_{q=1}^{Q}\sum_{n=1-(Q_1-1)q}^{N}1=D^2\sum_{q=1}^{Q}(N+(Q_1-1)q)=$$
$$=D^2\left(NQ+(Q_1-1)\frac{Q(Q+1)}{2}\right)\le D^2Q\left(N+\left(\frac{Q}{2}-1\right)\frac{Q+1}{2}\right)\le D^2Q\left(N+\frac{Q^2}{4}\right).$$

Combining (7) with (13), we obtain that

$$D^2Q\left(N+\frac{Q^2}{4}\right)\ge\left(\frac{2}{\pi}Q_1\right)^2\sum_{m=1}^{M}|s_m|^2. \tag{14}$$

It follows from (14) that there exist n, q satisfying (8) and (9) which proves Corollary 3.

Corollary 4 can be obtained from Corollary 3 by choosing $Q=[\sqrt{N}]$.

3. The modulo p analogue of Theorem 3 is the following:

THEOREM 4. *Let p be any odd prime number and $\dots, t_{-2}, t_{-1}, t_0, t_1, t_2, \dots$ an infinite periodic sequence of complex numbers with period length p, i.e.*

$$t_u=t_v \quad \textit{for} \quad u\equiv v \pmod p. \tag{15}$$

For every integer n and positive integers q, k, let

$$E(n,q,k)=t_n+t_{n+q}+t_{n+2q}+\dots+t_{n+(k-1)q}.$$

Then

$$\sum_{q=1}^{p-1}\sum_{n=1}^{p}\left|E\left(n,q,\frac{p-1}{2}\right)\right|^2\ge\left(\frac{p-1}{\pi}\right)^2\sum_{m=1}^{p}|t_m|^2. \tag{16}$$

COROLLARY 5. *Having the assumptions and notations in Theorem 4, there exist an integer n and a positive integer q such that $1\le q\le p-1$ and*

$$\left|E\left(n,q,\frac{p-1}{2}\right)\right|=\left|t_n+t_{n+q}+t_{n+2q}+\dots+t_{n+\left(\frac{p-1}{2}-1\right)q}\right|\ge$$
$$\ge\frac{1}{\pi}\left\{\frac{p-1}{p}\sum_{m=1}^{p}|t_m|^2\right\}^{1/2}.$$

PROOF OF THEOREM 4. Let M be a large integer, and let us apply Theorem 3 with $N=Mp$, $Q=p-1$, $s_1=t_1, s_2=t_2, \dots, s_N=s_{Mp}=t_{Mp}$. (Then (6) holds trivially.) We obtain that

$$\sum_{q=1}^{p-1}\sum_{n=1-\frac{p-3}{2}q}^{Mp}\left|D\left(n,q,\frac{p-1}{2}\right)\right|^2\ge\left(\frac{p-1}{\pi}\right)^2\sum_{m=1}^{Mp}|t_m|^2. \tag{17}$$

Obviously,

$$D(n,q,k)=E(n,q,k) \quad \text{for} \quad 1\leqq n\leqq n+(k-1)q\leqq Mp,$$

and by (15),

$$E(n,q,k)=E(m,q,k) \quad \text{for} \quad n\equiv m \pmod p.$$

Finally, if we put

$$T=\max_{i=1,2,\dots,p}|t_i|$$

then

$$|E(n,q,k)|=|t_n+t_{n+q}+t_{n+2q}+\dots+t_{n+(k-1)q}|\leqq kT$$

and in the same way, $|D(n,q,k)|\leqq kT$ for every integer n and positive integers q, k.

Thus the left hand side of (17) can be estimated in the following way:

(18)
$$\sum_{q=1}^{p-1}\sum_{n=1-\frac{p-3}{2}q}^{Mp}\left|D\left(n,q,\frac{p-1}{2}\right)\right|^2=$$

$$=\sum_{q=1}^{p-1}\sum_{n=1-\frac{p-3}{2}q}^{0}\left|D\left(n,q,\frac{p-1}{2}\right)\right|^2+\sum_{q=1}^{p-1}\sum_{n=1}^{Mp-\frac{p-3}{2}q}\left|D\left(n,q,\frac{p-1}{2}\right)\right|^2+$$

$$+\sum_{q=1}^{p-1}\sum_{n=Mp-\frac{p-3}{2}q+1}^{Mp}\left|D\left(n,q,\frac{p-1}{2}\right)\right|^2\leqq$$

$$\leqq\sum_{q=1}^{p-1}\sum_{n=1-\frac{p-3}{2}q}^{0}\left(\frac{p-1}{2}\right)^2T^2+\sum_{q=1}^{p-1}\sum_{n=1}^{Mp-\frac{p-3}{2}q}\left|D\left(n,q,\frac{p-1}{2}\right)\right|^2+$$

$$+\sum_{q=1}^{p-1}\sum_{n=Mp-\frac{p-3}{2}q+1}^{Mp}\left(\frac{p-1}{2}\right)^2T^2=$$

$$=\sum_{q=1}^{p-1}\frac{p-3}{2}q\left(\frac{p-1}{2}\right)^2T^2+\sum_{q=1}^{p-1}\sum_{n=1}^{Mp-\frac{p-3}{2}q}\left|E\left(n,q,\frac{p-1}{2}\right)\right|^2+\sum_{q=1}^{p-1}\frac{p-3}{2}q\left(\frac{p-1}{2}\right)^2T^2\leqq$$

$$\leqq\sum_{q=1}^{p}p\cdot p\cdot p^2T^2+\sum_{q=1}^{p-1}\sum_{n=1}^{Mp}\left|E\left(n,q,\frac{p-1}{2}\right)\right|^2+\sum_{q=1}^{p}p\cdot p\cdot p^2T^2=$$

$$=\sum_{q=1}^{p-1}\sum_{m=0}^{M-1}\sum_{n=1}^{p}\left|E\left(mp+n,q,\frac{p-1}{2}\right)\right|^2+2p^5T^2=$$

$$=\sum_{q=1}^{p-1}\sum_{m=0}^{M-1}\sum_{n=1}^{p}\left|E\left(n,q,\frac{p-1}{2}\right)\right|^2+2p^5T^2=M\sum_{q=1}^{p-1}\sum_{n=1}^{p}\left|E\left(n,q,\frac{p-1}{2}\right)\right|^2+2p^5T^2.$$

The right hand side of (17):

$$\left(\frac{p-1}{\pi}\right)^2 \sum_{m=1}^{Mp} |t_m|^2 = \left(\frac{p-1}{\pi}\right)^2 \sum_{n=0}^{M-1} \sum_{m=1}^{p} |t_{np+m}|^2 = \tag{19}$$

$$= \left(\frac{p-1}{\pi}\right)^2 \sum_{n=0}^{M-1} \sum_{m=1}^{p} |t_m|^2 = M\left(\frac{p-1}{\pi}\right)^2 \sum_{m=1}^{p} |t_m|^2$$

(with respect to (15)).

By (18) and (19), (17) implies that

$$M \sum_{q=1}^{p-1} \sum_{n=1}^{p} \left| E\left(n, q, \frac{p-1}{2}\right)\right|^2 + 2p^5T^2 \geqq M\left(\frac{p-1}{\pi}\right)^2 \sum_{m=1}^{p} |t_m|^2.$$

Let us divide by M:

$$\sum_{q=1}^{p-1} \sum_{n=1}^{p} \left| E\left(n, q, \frac{p-1}{2}\right)\right|^2 + \frac{2p^5T^2}{M} \geqq \left(\frac{p-1}{\pi}\right)^2 \sum_{m=1}^{p} |t_m|^2.$$

This holds for M arbitrarily large which implies (16).

Proof of Corollary 5. Let us write

$$E = \max_{\substack{1 \leqq n \leqq p \\ 1 \leqq q \leqq p-1}} \left| E\left(n, q, \frac{p-1}{2}\right)\right|.$$

Then for the left hand side of (16) we get

$$\sum_{q=1}^{p-1} \sum_{n=1}^{p} \left| E\left(n, q, \frac{p-1}{2}\right)\right|^2 \leqq \sum_{q=1}^{p-1} \sum_{n=1}^{p} E^2 = p(p-1)E^2. \tag{20}$$

(16) and (20) yield that

$$p(p-1)E^2 \geqq \left(\frac{p-1}{\pi}\right)^2 \sum_{m=1}^{p} |t_m|^2.$$

Hence

$$E \geqq \frac{1}{\pi}\left\{\frac{p-1}{p} \sum_{m=1}^{p} |t_m|^2\right\}^{1/2}$$

which proves Corollary 5.

4. As A. Rényi remarked in [2], a lower estimate for character sums can be obtained by applying Parseval's formula for the Fourier expansion

$$\sum_{1 \leqq n \leqq xD} \chi(n) \sim \begin{cases} \dfrac{\varepsilon\sqrt{D}}{\pi} \displaystyle\sum_{n=1}^{+\infty} \frac{\bar{\chi}(n)}{n} \sin 2\pi n x & \text{for} \quad \chi(-1) = +1 \\ a_0 + \dfrac{\varepsilon\sqrt{D}}{\pi} \displaystyle\sum_{n=1}^{+\infty} \frac{\bar{\chi}(n)}{n} \cos 2\pi n x & \text{for} \quad \chi(-1) = -1 \end{cases}$$

where D is a fundamental discriminant, χ a character modulo D different from the principal character χ_0, and $a_0=-\frac{\varepsilon\sqrt{D}}{\pi}L(1,\chi)$ (see [1]). Rényi formulated this estimate in the following way:

THEOREM 5 (A. RÉNYI [2]; see also U. V. LINNIK [3]).

$$\sqrt{\frac{D}{12}} < \max_{1\le x\le D-1}\left|\sum_{n=1}^{x}\chi(n)\right|.$$

However, it seems that in this way, we may obtain only a slightly worse constant in place of $\sqrt{\frac{1}{12}}$. Namely, the correct form of this estimate seems to be

THEOREM 5′.

$$\sqrt{\frac{D}{2\pi^2}} \le \sqrt{\frac{D}{12}\prod_{p\mid D}\left(1-\frac{1}{p^2}\right)} < \max_{1\le x\le D-1}\left|\sum_{n=1}^{x}\chi(n)\right|. \tag{21}$$

We are going to show that in the most interesting case when p is a prime number, Corollary 5 yields a better estimate (better constant) for sums of type $\left|\sum_{x\le n\le y}\chi(n)\right|$ than the one in Theorem 5′.

THEOREM 6. *Let p be any odd prime number, χ any character modulo p. Then there exists an integer x for which*

$$\left|\sum_{n=x}^{x+\frac{p-1}{2}-1}\chi(n)\right| \ge \frac{1}{\pi}\left(\sqrt{p}-1/\sqrt{p}\right). \tag{22}$$

PROOF OF THEOREM 6. Let us apply Corollary 5 with $t_m=\chi(m)$ (where $m=0, \pm 1, \pm 2, \ldots$). We obtain that there exist an integer n and a positive integer q such that

$$1 \le q \le p-1 \tag{23}$$

and

$$\left|E\left(n, q, \frac{p-1}{2}\right)\right| = \left|\sum_{k=0}^{\frac{p-1}{2}-1}\chi(n+kq)\right| \ge \frac{1}{\pi}\left\{\frac{p-1}{p}\sum_{m=1}^{p}|\chi(m)|^2\right\}^{1/2}. \tag{24}$$

(23) implies that $(q,p)=1$. Thus there exists an integer q^* for which

$$qq^* \equiv 1 \pmod{p}.$$

Obviously, this integer q satisfies also $(q^*,p)=1$, thus $|\chi(q^*)|=1$. Hence

$$\left|\sum_{k=1}^{\frac{p-1}{2}-1}\chi(n+kq)\right| = \left|\chi(q^*)\sum_{k=0}^{\frac{p-1}{2}-1}\chi(n+kq)\right| = \tag{25}$$

$$= \left|\sum_{k=0}^{\frac{p-1}{2}-1}\chi(nq^*+kqq^*)\right| = \left|\sum_{k=0}^{\frac{p-1}{2}-1}\chi(nq^*+k)\right| = \left|\sum_{m=x}^{x+\frac{p-1}{2}-1}\chi(m)\right|$$

where $x=nq^*$.

Furthermore,

$$\sum_{m=1}^{p} |\chi(m)|^2 = \sum_{m=1}^{p-1} 1 = p-1. \tag{26}$$

(24), (25) and (26) yield (22).

We note that H. L. Montgomery and R. C. Vaughan have proved the following theorem:

For any positive integer N, there exists a prime number p such that $N<p\leqq 3N$ and for any integers x, y,

$$\left|\sum_{x\leqq n\leqq y}\left(\frac{n}{p}\right)\right| < c\sqrt{p}$$

where $\left(\frac{n}{p}\right)$ denotes the Legendre symbol and c is an absolute constant. (Montgomery's oral communication.)

Thus Theorem 5′ is best possible in the sense that the factor $\sqrt{\frac{1}{12}\prod_{p|D}\left(1-\frac{1}{p^2}\right)}$ can not be replaced by a large constant. (Consequently, also Theorem 4 is the best possible except at most the value of the constant factor on the right hand side.) Thus it is worth to determine the greatest constant c for which $\max_{x,y}\left|\sum_{x\leqq n\leqq y}\chi(n)\right| > (c-\varepsilon)\sqrt{p}$ holds for $p>p_0(\varepsilon)$ and each character modulo p. Theorem 5′ yields this estimate with $c=\frac{1}{\sqrt{12}}$ while Theorem 6 improves on this constant: it implies that also $c=\frac{1}{\pi}\left(>\frac{1}{\sqrt{12}}\right)$ can be chosen.

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ON CONSECUTIVE PRIMES

By

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1. Let

$$S(n) = \sum_{p<n} (n-p)^{-1}, \tag{1.1}$$

where p runs over the prime numbers. ERDŐS and DE BRUIJN [5] proved the following inequalities:

$$c_1 N \leqq \sum_{n \leqq N} S^2(n) \leqq c_2 N,$$

$$c_1 N/\log N \leqq \sum_{p \leqq N} S(p) \leqq c_2 N/\log N,$$

$$c_1 N/\log N \leqq \sum_{p \leqq N} S^2(p) \leqq c_2 N/\log N,$$

where c_1, c_2 are suitable constants. An easy computation yields:

$$\sum_{n \leqq N} S(N) = N + O(N/\log N).$$

Assuming the validity of the density hypothesis in the form

$$N(\sigma, T) < cT^{2(1-\sigma)} \log^2 T \tag{1.2}$$

when $1/2 \leqq \sigma \leqq 1$, $T > 0$, I have proved the following inequalities:

$$\sum_{n \leqq N} (S(n)-1)^2 = O\big(N \cdot (\log N)^{-1} \cdot \log\log N)^2\big), \tag{1.3}$$

$$\sum_{q<N} |S(q)-1| = O\big(N \cdot (\log N) \cdot (\log\log N)^{3/2}\big), \tag{1.4}$$

$$\sum_{q<N} (S(q)-1)^2 = O\big(N \cdot (\log N)^{-3/2} \cdot (\log\log N)\big), \tag{1.5}$$

where in the sum q runs over the primes. (See [1], [2], [3].) Now we prove the following

THEOREM 1. *For every integer $k \geqq 1$ we have*

$$\sum_{n \leqq N} S^k(n) = O(N), \tag{1.6}$$

$$\sum_{q \leqq N} S^k(q) = O(N/\log N). \tag{1.7}$$

Hence, providing (1.2) we can deduce stronger inequalities, namely the following assertion.

THEOREM 2. *Assuming* (1.2), *for every integer* $k \geqq 2$ *we have*

$$\sum_{n \leqq N} |S(n)-1|^k = O\big(N \cdot (\log N)^{-1+\varepsilon}\big), \tag{1.8}$$

$$\sum_{q \leqq N} |S(q)-1|^k = O\big(N \cdot (\log N)^{-2+\varepsilon}\big), \tag{1.9}$$

ε *being an arbitrary positive constant.*

We note that for $k=2$ (1.9) is better than (1.5).

2. Deduction of Theorem 2 from Theorem 1. Let $A \geqq 1$. Then we have

$$\sum_{n \leqq N} |S(n)-1|^k \leqq A^{k-2} \sum_{S(n) \leqq A+1} \big(S(n)-1\big)^2 + \sum_{S(n) > A+1} |S(n)-1|^k = A^{k-2} \textstyle\sum_1 + \sum_2 .$$

If l is an integer $\geqq 1$, then

$$\textstyle\sum_2 \leqq A^{-l} \sum_{n \leqq N} \big(S(n)\big)^{k+l},$$

and by (1.6)

$$\textstyle\sum_2 \leqq A^{-l} \cdot N \cdot c_{k+l},$$

c_{k+l} being a constant that depends on $k+l$. From (1.3) we have

$$\textstyle\sum_1 < cN \cdot (\log N)^{-1} \cdot (\log \log N)^2.$$

Now choose A so that $A^{l+k} = \log N$. For a suitable large l $\left(\text{say } l > \frac{2k}{\varepsilon}\right)$ we have

$$A^{-l} \ll (\log N)^{-1+\varepsilon/2}, \quad A^k \ll (\log N)^{\varepsilon/2}$$

and we get (1.8).

The proof of (1.9) is almost the same, but we start with the inequality

$$\sum_{q \leqq N} |S(q)-1|^k \leqq A^{k-1} \cdot \sum_{S(q) \leqq A+1} |S(q)-1| + A^{-l} \sum \big(S(q)\big)^{k+l}.$$

3. Proof of Theorem 1. We need some results on prime k-tuples which was achieved by Selberg's sieve method.

LEMMA 1. *Let g be a natural number, a_i, b_i $(i=1, \ldots, g)$ be integers satisfying*

$$E = \prod_{i=1}^{g} a_i \sum_{1 \leqq r < s \leqq g} (a_r b_s - a_s b_r) \neq 0. \tag{3.1}$$

Let $\varrho(p)$ denote the number of solutions of

$$\prod_{i=1}^{g} (a_i n + b_i) \equiv 0 \pmod p,$$

and suppose that $\varrho(p) < p$ for all p. Let $x > 1$ be real number, and $N(x; a_1, b_1, \ldots, a_g, b_g)$ denote the number of n in the interval $[1, x]$, for which all $a_i n + b_i$ are primes for $i = 1, \ldots, g$. Then

$$(3.2)\qquad N(x;\ a_1, b_1, \ldots, ag, bg) \leqq$$

$$\leqq 2^g \cdot g! \prod_p \left(1 - \frac{\varrho(p)-1}{p-1}\right)\left(1 - \frac{1}{p}\right)^{-g+1} \cdot \frac{x}{\log^g x} \cdot \left\{1 + O\left(\frac{\log\log 3x + \log\log 3|E|}{\log 3x}\right)\right\}$$

where the constant implied by the O-term depends at most on g.

LEMMA 2. *Suppose the conditions of the previous lemma; in addition, suppose that*

$$(3.3)\qquad b_0 \overset{\text{def}}{=} b_1 \ldots b_g \neq 0,$$

and $\varrho(p) < p-1$ if $p \nmid b_0$. Let $1 < x$, and $P(x; a_1, b_1, \ldots, a_g, b_g)$ denote the number of primes p in the interval $[1, x]$, for which all $a_i p + b_i$ $(i = 1, \ldots, g)$ are primes. Then

$$(3.4)\qquad P(x;\ a_1, b_1, \ldots, a_g, b_g) \leqq$$

$$\leqq 2^{2g+1}(g+1)! \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) \cdot \prod_{2<p\nmid b_0}\left(1 - \frac{\varrho(p)-1}{p-2}\right)\left(1 - \frac{1}{p}\right)^{-g+1} \times$$

$$\times \prod_{2<p|b_0}\left(1 - \frac{\varrho(p)-2}{p-2}\right)\left(1 - \frac{1}{p}\right)^{-g+1} \cdot \frac{x}{(\log x)^{g+1}}\left\{1 + O\left(\frac{\log\log 3|Eb_0|}{\log x}\right)\right\},$$

where the constant implied by the O-symbol depends at most on g.

For the proof of these Lemmas, see [4].

We take $\mathbf{l} = (l_1, \ldots, l_r)$. Let $N(x; \mathbf{l})$ denote the number of the integers n in $[1, x]$ for which $n - l_i$ $(i = 1, \ldots, r)$ are primes. From Lemma 1 we have

$$(3.5)\qquad N(x;\ \mathbf{l}) \leqq cA(\mathbf{l}) \cdot x \cdot (\log x)^{-r}, \quad A(\mathbf{l}) = \prod_p \left(1 - \frac{\varrho(p)-1}{p-1}\right)\left(1 - \frac{1}{p}\right)^{-r+1}$$

uniformly for $1 \leqq r \leqq k$, $1 \leqq l_i \leqq x$ $(i = 1, \ldots, r)$. Similarly, if $P(x; \mathbf{l})$ denotes the number of the primes p in $[1, x]$ for which $p - l_i$ $(i = 1, \ldots, r)$ are primes, then

$$(3.6)\qquad P(x;\ \mathbf{l}) \leqq cB(\mathbf{l})x \cdot (\log x)^{-r-1},$$

$$B(\mathbf{l}) = \prod_{2<p\nmid b_0}\left(1 - \frac{\varrho(p)-1}{p-2}\right)\left(1 - \frac{1}{p}\right)^{-r+1} \prod_{2<p|b_0}\left(1 - \frac{\varrho(p)-1}{p-2}\right)\left(1 - \frac{1}{p}\right), \quad b_0 = l_1 \ldots l_r$$

uniformly for $1 \leqq l_i \leqq x$, $i = 1, \ldots, r$; $r = 1, \ldots, k$. We have

$$\sum_{n \leqq x} S^k(n) = \sum_{p_1, \ldots, p_k < x}\ \sum_{n > \max(p_1, \ldots, p_k)} (n - p_1)^{-1} \cdot \ldots \cdot (n - p_k)^{-1} =$$

$$= \sum_{r=1}^{k}\ \sum_{\nu_1 + \ldots + \nu_r = k}\ \sum_{(l_1, \ldots, l_r)} \frac{N(x;\ \mathbf{l})}{l_1^{\nu_1} \ldots l_r^{\nu_r}},$$

where in the sum $\nu_1, \ldots, \nu_r$ are positive integers, $(l_1, \ldots, l_r)$ denotes an r-tuple of distinct numbers satisfying $1 \leqq l_i < x$. Similarly we have

$$\sum_{q \leqq x} S^k(q) = \sum_{r=1}^{k} \sum_{\nu_1+\ldots+\nu_r=k} \sum_{(l_1,\ldots,l_r)} \frac{P(x;\mathbf{l})}{l_1^{\nu_1}\ldots l_r^{\nu_r}}.$$

Now we prove that

$$\sum_{(l_1,\ldots,l_r)} \frac{N(x;\mathbf{l})}{l_1\ldots l_r} = O(x) \tag{3.7}$$

and that

$$\sum_{(l_1,\ldots,l_r)} \frac{P(x;\mathbf{l})}{l_1\ldots l_r} = O\left(\frac{x}{\log x}\right), \tag{3.8}$$

and this will give Theorem 1 immediately.

Let $D(\mathbf{l})$ denote for $\mathbf{l}=(l_1, \ldots, l_r)$ the product

$$D(\mathbf{l}) = \prod_{1\leqq i<j\leqq r} (l_j - l_i).$$

The p-th factor in (3.5) is

$$\left(1-\frac{\varrho(p)-1}{p-1}\right)\left(1-\frac{1}{p}\right)^{-r+1} = 1+h(p,\mathbf{l}).$$

Observing that $1 \leqq \varrho(p) \leqq r$, and that $\varrho(p)=r$, unless $p|D(\mathbf{l})$ we have

$$|h(p,\mathbf{l})| \leqq \begin{cases} \dfrac{b}{p} & \text{if} \quad \varrho(p) < r, \\ \dfrac{b}{p^2} & \text{if} \quad \varrho(p) = r, \end{cases}$$

b being a suitable positive constant. Let $f(d, \mathbf{l})$ be a multiplicative function defined for square-free integers by the relation

$$f(p,\mathbf{l}) = \begin{cases} \dfrac{b}{p^2}, & \text{if} \quad p \nmid D(\mathbf{l}), \\ \dfrac{b}{p}, & \text{if} \quad p|D(\mathbf{l}). \end{cases}$$

Then we have

$$|A(\mathbf{l})| \leqq \sum_{d=1}^{\infty} f(d,\mathbf{l}),$$

and consequently

$$A(\mathbf{l}) \leqq \left\{ \sum_{\substack{h=1 \\ (h,D(\mathbf{l}))=1}}^{\infty} f(h,\mathbf{l}) \right\} \cdot \left\{ \sum_{\delta|D(\mathbf{l})} f(\delta,\mathbf{l}) \right\}.$$

Evidently $f(d, \mathbf{l}) \leqq d^{-1} \cdot b^{\omega(d)}$ if $d | D(\mathbf{l})$ and $\leqq d^{-2} b^{\omega(d)}$ if $(d, D(\mathbf{l}))=1$, where $\omega(d)$ is the number of prime factors of d. Since $\omega(d) \ll \dfrac{\log d}{\log \log d}$, the first sum is bounded by

$$\sum_{h=1}^{\infty} \frac{b^{\omega(h)} |\mu(h)|}{h^2} < \infty,$$

and so

$$A(l) \ll \sum_{\delta | D(\mathbf{l})} f(\delta, \mathbf{l}).$$

Now we have

$$\sum_{(l_1, \dots, l_r)} \frac{A(\mathbf{l})}{l_1 \dots l_r} \ll \sum_{\delta < x^r} \frac{|\mu(\delta)| \cdot b^{\omega(\delta)}}{\delta} \cdot \sum_{D(\mathbf{l}) \equiv 0 \pmod{\delta}} (l_1 \dots l_r)^{-1}.$$

Let $\sum_\delta$ denote the last sum on the right. From the condition $D(\mathbf{l}) \equiv 0 \pmod{\delta}$ it follows that there exist coprime integers δ_{ij} $(i \neq j, i, j=1, \dots, r)$, so that $\delta = \prod_{i \neq j} d_{i,j}$, and $l_i - l_j \equiv 0 \pmod{\delta_{i,j}}$. The number of solutions of $\delta = \Pi \delta_{i,j}$ is majorized by $\tau_{r^2}(\delta) \ll \delta^{\varepsilon}$. ($\varepsilon$ being an arbitrary small positive constant.) Consequently

$$\sum\nolimits_\delta \ll \frac{\tau_{r^2}(\delta)}{\delta} \cdot (\log x)^r,$$

and

$$\sum_{(l_1, \dots, l_r)} \frac{A(\mathbf{l})}{l_1 \dots l_r} \ll (\log x)^r.$$

By this we proved (3.7), and so the first inequality of Theorem 1. The proof of the second inequality of it is almost the same and so we omit it.

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ON THE SUM OF DIGITS OF PRIMES

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1. Introduction

Let $q>1$ be a fixed integer. Then any positive integer n can be expressed in the form

$$n=\sum_{i=0}^{k} a_i q^i, \tag{1.1}$$

where each a_i is one of $0, 1, \ldots, q-1$. We put

$$\alpha(n)=\sum_{i=0}^{k} a_i. \tag{1.2}$$

In [1] I proved, assuming the validity of the density hypothesis — in the form: $N(\sigma, T) \ll T^{2(1-\sigma)} \log^2 T$ — for the Riemann zeta function that

$$\sum_{p \leqq x} \alpha(p)=\frac{q-1}{2} \frac{x}{\log q}+O\left(\frac{x}{(\log \log x)^{1/3}}\right), \tag{1.3}$$

where in $\sum$ we sum over the primes.

I. SHIOKOVA [2] proved this relation without any unsolved hypothesis, even with an improved remainder term, namely that

$$\sum_{p \leqq x} \alpha(p)=\frac{q-1}{2} \frac{x}{\log q}+O\left(x\left(\frac{\log \log x}{\log x}\right)^{1/2}\right). \tag{1.4}$$

E. HEPPNER [4] has proved the following assertion. Let $\mathcal{B}$ be a set of the natural numbers and $B(x)$ denote the number of its elements in the interval $[1, x]$. Assuming that

$$(\log B(x))/\log x \to 1 \quad (x \to \infty),$$

the relation

$$\sum_{\substack{n \leqq x \\ n \in \mathcal{B}}} \alpha(n)=\frac{q-1}{2} \cdot \frac{\log x}{\log q} \cdot B(x)\left(1+O\left(\left(\frac{\log \log x+\log \frac{x}{B(x)}}{\log x}\right)^{1/2}\right)\right), \tag{1.5}$$

holds. For the primes this gives the same remainder term as stated in (1.4).

For the set of primes we shall prove the stronger inequality

$$(1.6)\qquad \sum_{p\leqq x}\left|\alpha(p)-\frac{q-1}{2\log q}\cdot\log x\right|^k \ll x\cdot(\log x)^{k/2-1}\quad (k=1,2,\ldots).$$

This inequality seems to be optimal. In a paper written jointly by J. MOGYORÓDI [3] we proved that

$$\frac{\alpha(p)-M_p}{D_p}\quad\left(M_p=\frac{q-1}{2}\frac{\log p}{\log q},\quad D_p^2=\frac{q^2-1}{12}\cdot\frac{\log p}{\log q}\right)$$

has a limit distribution, namely the Gaussian law, if the density hypothesis is true. On this assumption by the same method we can prove that

$$x^{-1}\cdot(\log x)^{1-k/2}\sum_{p\leqq x}\left|\alpha(p)-\frac{q-1}{2\log q}\log x\right|^k \to C_k\quad (k=1,2,\ldots)$$

where

$$C_k=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|^k e^{-x^2/2}\,dx,$$

the kth absolute moment of the standard normal distribution.

2. Proof

Let for the sake of brevity $l_q=\frac{q-1}{2\log q}$, $l(x)=l_q\cdot\log x$, and introduce the notations

$$(2.1)\qquad A_k(x)=\sum_{p\leqq x}|\alpha(p)-l(x)|^k,$$

$$(2.2)\qquad B_k(x)=\sum_{n<x}|\alpha(n)-l(x)|^k.$$

Let $\xi_0,\xi_1,\ldots,\xi_{\mu-1}$ be completely independent random variables with the distribution

$$P(\xi_j=l)=\frac{1}{q}\quad (l=0,1,\ldots,q-1;\ j=0,1,\ldots,\mu-1).$$

Let $M=\frac{q-1}{2}$, $D^2=\frac{q^2-1}{12}$ be the mean value and the variance of ξ_j, respectively. Then, with the notation

$$\eta_\mu=\sum_{j=0}^{\mu-1}\xi_j,\quad \theta_\mu=\frac{\eta_\mu-\mu M}{D\cdot\sqrt{\mu}},$$

we have

$$\int|\theta_\mu|^k\,dP\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|^k e^{-x^2/2}\,dx\quad(\mu\to\infty),$$

which is a form of the central limit theorem. Furthermore

$$B_k(q^\mu) = (D\sqrt{\mu})^k \int |\theta_\mu|^k \, dP,$$

whence the inequality

$$B_k(x) \ll x (\log x)^{k/2} \tag{2.3}$$

immediately follows.

Let $N_k(x)$ denote the number of those primes $p \leqq x$, for which $p+k$ is prime.

LEMMA 1. *We have*

$$N_k(x) < c \frac{x}{\log^2 x} \prod_{p|k} \left(1-\frac{1}{p}\right)^{-1},$$

where c is an absolute constant and p in the product runs over the prime divisors of k.

For the proof see [5].

LEMMA 2. *For arbitrary real (or complex)* $a_1, \dots, a_r$ *the inequality*

$$(|a_1|+\dots+|a_r|)^k \leqq r^{-1} \cdot (|a_1|^k+\dots+|a_r|^k)$$

holds.

This is a special case of the well-known Hölder-inequality.

LEMMA 3. *Let* $\pi(x, k, l)$ *be the number of primes* $p \leqq x$ *in the arithmetical progression* $p \equiv l \pmod{k}$. *We have*

$$\pi(x, k, l) < c \frac{x}{\varphi(k) \log x}$$

uniformly for $k < \sqrt{x}$, $(l, k)=1$. (See [5].)

It is obvious that it is enough to prove (1.6) for the subsequence $x = x_n = q^n - 1$ only. Let $x = x_n$, $H = [\log x]$. We define $\delta(p, H)$ as

$$\alpha(p) - \frac{1}{H+1} (\alpha(p)+\alpha(p+1)+\dots+\alpha(p+H)).$$

Since

$$|\alpha(p)-l(x)| \leqq |\delta(p, H)| + \frac{1}{H+1} \sum_{j=0}^{H} |\alpha(p+j)-l(x)|,$$

therefore by Lemma 2, applying it twice, we get

$$|\alpha(p)-l(x)|^k \leqq 2^{k-1} \cdot |\delta(p, H)|^k + \frac{2^{k-1}}{(H+1)^k} \left\{ \sum_{j=0}^{H} |\alpha(p+j)-l(x)| \right\}^k \leqq$$

$$\leqq 2^{k-1} |\delta(p, H)|^k + \frac{2^{k-1}}{H+1} \cdot \sum_{j=0}^{H} |\alpha(p+j)-l(x)|^k.$$

So we have

$$A_k(x) \leqq 2^{k-1} \left(\Sigma_1 + \frac{1}{H+1} \cdot \Sigma_2 \right), \tag{2.4}$$

where

$$\sum_1 = \sum_{p \leqq x} |\delta(p, H)|^k, \quad \sum_2 = \sum_{p \leqq x} \sum_{j=0}^{H} |\alpha(p+j) - l(x)|^k. \tag{2.5}$$

First we estimate $\sum_2$. Let $e(n)$ denote the number of primes in the interval $[n-H, n]$. Then the sum

$$C = \sum_{n \leqq x} e^2(n)$$

is equal to the number of the solution of

$$p_1 - p_2 = j_2 - j_1, \quad p_1, p_2 \leqq x, \ j_1, j_2 = 0, 1, \dots, H.$$

By Lemma 1 we have

$$C \leqq H\pi(x) + \sum_{j_1 \neq j_2} N_{|j_1 - j_2|}'(x) \ll \frac{xH^2}{\log^2 x} + \frac{xH}{\log x} \ll x. \tag{2.6}$$

Furthermore

$$\sum_2 \leqq \sum_{n \leqq x+H} |\alpha(n) - l(x)|^k \cdot e(n) \leqq (B_{2k}(x+H))^{1/2} \cdot C^{1/2},$$

and so by (2.3) and (2.6) we get

$$\sum_2 \ll x \cdot (\log x)^{k/2}.$$

Now we estimate $\sum_1$. Let ν be an integer chosen so that $H^{3k} \leqq q^\nu \leqq x^{1/2}$. We observe that for a prime $p \leqq x$ satisfying the relation $p \equiv l (\mathrm{mod}\ q^\nu)$, $0 < l < q^\nu - H$ we have

$$\alpha(p) - \alpha(p+j) = \alpha(l) - \alpha(l+j)$$

for every $j = 0, 1, \dots, H$, consequently $\delta(p, H) = \delta(l, H)$. So we get

$$\sum_1 = \sum_{l=0}^{q^\nu - H - 1} \pi(x, q^\nu, l) \cdot |\delta(l, H)|^k + \sum_{l = q^\nu - H}^{q^\nu - 1} \sum_{p \equiv l (\mathrm{mod}\, q^\nu)} |\delta(p, H)|^k = \sum_A + \sum_B.$$

Observing that $\alpha(m) \ll \log m$, and consequnetly $\delta(p, H) \ll \log x$, by Lemma 3 we deduce that

$$\sum_B \ll \frac{xH}{\varphi(q^\nu)} (\log x)^{k-1} \ll \frac{x}{\log x}.$$

Here we used that $\varphi(q^\nu) = q^{\nu-1}\varphi(q) \gg q^\nu$ (q is fixed). Similarly we get

$$\sum_A \ll \frac{x}{\varphi(q^\nu) \log x} \cdot \sum_C,$$

where

$$\sum_C = \sum_{l=0}^{q^\nu - 1 - H} |\delta(l, H)|^k.$$

Since

$$|\delta(l, H)| \leqq |\alpha(l) - l(q^\nu)| + \frac{1}{H+1} \sum_{j=1}^{H} |\alpha(l+j) - l(q^\nu)|,$$

by the Hölder-inequality (Lemma 2) we get

$$|\delta(l, H)|^k \ll |\alpha(l)-l(q^\nu)|^k + \frac{1}{H}\sum_{j=1}^{H} |\alpha(l+j)-l(q^\nu)|^k,$$

and so

$$\sum\nolimits_C \ll B_k(q^\nu),$$

$$\sum\nolimits_A \ll \frac{x}{\log x}\cdot(\log q^\nu)^{k/2} \ll x(\log x)^{k/2-1}.$$

Collecting our inequalities we get (1.6).

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THE CONTINUITY OF BEST APPROXIMATIONS

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Introduction

Let X be a Banach space, M a subspace so that for any $x \in X$ there exists $p(x) \in M$ such that

$$\inf_{y \in M} \|x-y\| = \|x-p(x)\| = E(x).$$

In what follows under $p(x) \in M$ we shall mean the best approximations to $x \in X$ from M, under $E(x)$ the measure of best approximation.

It is known (see [1]), that if $p(x)$ is the unique best approximation to x, and the subspace M is approximatively compact with respect to x, i.e. for any sequence $\{y_n\} \in M$ such that

$$\|x-y_n\| \to E(x) \quad (n \to \infty),$$

the sequence $\{y_n\}$ is compact, then the best approximation $p(x)$ is continuous at the point x, i.e. for any sequence $\{x_n\} \subset X$ such that $\|x-x_n\| \to 0$ $(n \to \infty)$, we have

$$\sup_{p(x_n)} \|p(x)-p(x_n)\| \to 0 \quad (n \to \infty).$$

Let F be the set of elements of X, for which the best approximation is unique:

$$F = \{x \in X\colon\ p(x) \text{ is unique}\}$$

and take any $N \subseteqq F$.

In the first part of this paper we shall define those properties of the set N, which guarantee the uniform continuity of the best approximation on N.

Furthermore if $x \in F$, it is interesting to estimate the size of $\|p(x)-p(x_1)\|$ if the distance between x and x_1 is known, or the size of $\|p(x)-y\|$ for $y \in M$ if we know the measure of $\|x-y\|-E(x)$. These are the so-called correctness problems of best approximation.

For the case when X is the space of real valued continuous functions on $[a, b]$, M is a Chebysev system on $[a, b]$, G. FREUD [2] proved that for any $x \in X$ the operator of best approximation satisfies a Lipschitz condition at x i.e. for any $x_1 \in X$:

$$\|p(x)-p(x_1)\|_C \leqq C(x, M)\|x-x_1\|_C$$

where the constant $C(x, M)$ depends only on x and M.

In the second part of this note we shall deal with those sets $N \subseteqq X$ on which the operator of best approximation satisfies a uniform Lipschitz condition.

We shall discuss this question in details for the space of real valued continuous functions.

§ 1.

Take $N \subseteqq F$, $\varepsilon > 0$, $x \in X$, and introduce the following sets:

$$Y(x) = \{p(x) \in M \colon \|x - p(x)\| = E(x)\},$$

$$Y_\varepsilon^1(x) = \{y \in M \colon \text{there exists } x_1 \in X, \text{ such that } y \in Y(x_1) \text{ and } \|x - x_1\| \leqq \varepsilon\}.$$

Define

$$R_1(N, \varepsilon) = \sup_{x \in N} \sup_{y \in Y_\varepsilon^1(x)} \|p(x) - y\|$$

then the condition of uniform continuity of the operator $p(x)$ on N is equivalent to the condition

$$R_1(N, \varepsilon) \to 0 \quad (\varepsilon \to 0).$$

Let us introduce another characteristics of the operator of best approximation:

$$Y_\varepsilon^2(x) = \{y \in M \colon \|x - y\| \leqq E(x) + \varepsilon\}$$

and

$$R_2(N, \varepsilon) = \sup_{x \in N} \sup_{y \in Y_\varepsilon^2(x)} \|p(x) - y\|.$$

Here we are again interested in those properties of N which imply

$$R_2(N, \varepsilon) \to 0 \quad (\varepsilon \to 0).$$

From these definitions we can obtain some simple relations:

(1) $$R_1(N, \varepsilon) \leqq R_2(N, 2\varepsilon),$$

(2) $$R_i(N, \varepsilon) \geqq \varepsilon \quad (i = 1, 2)$$

(3) $$R_i(N, \varepsilon_1) \leqq R_i(N, \varepsilon_2) \quad (i = 1, 2)$$

for any $0 < \varepsilon_1 \leqq \varepsilon_2$.

Let us prove e.g. inequality (1).

If $y \in Y_\varepsilon^1(x)$, then there exists $x_1 \in X$ such that $y \in Y(x_1)$ and $\|x - x_1\| \leqq \varepsilon$. Using the inequality

(4) $$|E(x) - E(x_1)| \leqq \|x - x_1\|$$

we obtain

$$\|x - y\| \leqq \|x - x_1\| + \|x_1 - y\| \leqq \varepsilon + E(x_1) \leqq 2\varepsilon + E(x).$$

This means that $Y_\varepsilon^1(x) = Y_{2\varepsilon}^2(x)$, which implies (1). We omit the proof of the inequalities (2) and (3) because they are even more trivial.

Let now

$F_1 = \{x \in F \colon p(x) \text{ is continuous at } x\}$,

$F_2 = \{x \in F \colon M \text{ is approximatively compact with respect to } x\}$.

Then from Singer's theorem [1] mentioned already in the introduction we obtain:

$$F_2 \subseteqq F_1 \subseteqq F.$$

Let N be a subset of X. We shall call N approximatively dense in F_i if $N \subseteqq F_i$ and for any sequence $\{x_n\} \in N$ there exists a sequence $\{y_n\} \in M$ such that the sequence $\{x_n - y_n\}$ is compact in F_i $(i=1, 2)$.

THEOREM 1. *Let* $\varepsilon > 0$ *and* N *approximatively dense in* F_i. *Then* $R_i(N, \varepsilon)$ *monotonously converges to* 0 *as* $\varepsilon \to 0$ $(i=1, 2)$.

PROOF. We shall prove the theorem only in the case $i=2$ because in case $i=1$ the proof is rather similar and even easier than in the case $i=2$.

So let $i=2$. Assume that $R_2(N, \varepsilon)$ does not converge to zero as $\varepsilon \to 0$. Then we can find a positive constant Q such that for any $\varepsilon > 0$

$$R_2(N, \varepsilon) > Q. \tag{5}$$

(Here we used that $R_2(N, \varepsilon)$ is an increasing function of ε.)

Let $\alpha_n \downarrow 0$. Then by (5) we can construct two sequences $\{x_n\} \subset N$ and $\{y_n\} \subset M$ which satisfy the inequalities

$$\|x_n - y_n\| \leqq E(x_n) + \alpha_n \tag{6}$$

and

$$\|p(x_n) - y_n\| > Q. \tag{7}$$

Being N approximatively dense in F_2 we can find a sequence $\{\bar{y}_n\} \subset M$ such that the sequence $\{x_n - \bar{y}_n\}$ has a point of accumulation in F_2 and without loss of generality we may assume

$$x_n - \bar{y}_n \to x_0 \in F_2 \quad (n \to \infty); \tag{8}$$

moreover $N \subseteqq F_2 \subseteqq F_1$. Hence $p(x_n)$ is a unique best approximation to x_n and this implies that $x_n - \bar{y}_n$ has a unique best approximation $p(x_n) - \bar{y}_n$ $(n=1, 2, \ldots)$.

Then by (8) and continuity of the operator of best approximation at $x_0 \in F_2 \subseteqq F_1$ we have

$$p(x_n) - \bar{y}_n \to p(x_0) \quad (n \to \infty) \tag{9}$$

where $p(x_0)$ is the unique best approximation to x_0.

Now using (6) we have

$$\|x_0 - p(x_0)\| \leqq \|x_0 - (y_n - \bar{y}_n)\| \leqq \|x_0 - (x_n - \bar{y}_n)\| + \|x_n - y_n\| \leqq$$

$$\leqq \|x_0 - (x_n - \bar{y}_n)\| + E(x_n) + \alpha_n \leqq$$

$$\leqq \|x_0 - (x_n - \bar{y}_n)\| + \|x_0 - (x_n - \bar{y}_n)\| + \|x_0 - p(x_0)\| + \|p(x_0) - (p(x_n) - \bar{y}_n)\| + \alpha_n =$$

$$= \|x_0 - p(x_0)\| + 2\|x_0 - (x_n - \bar{y}_n)\| + \|p(x_0) - (p(x_n) - \bar{y}_n)\| + \alpha_n,$$

hence using (8) and (9) we have

$$\|x_0-p(x_0)\| \leqq \lim_{n\to\infty} \|x_0-(y_n-\bar{y}_n)\| \leqq \|x_0-p(x_0)\|,$$

i.e.

$$\lim_{n\to\infty} \|x_0-(y_n-\bar{y}_n)\| = \|x_0-p(x_0)\| = E(x_0). \tag{10}$$

By the approximative compactness of M with respect to $x_0 \in F_2$, we obtain that the sequence $\{y_n-\bar{y}_n\}$ is compact and without loss of generality we may assume

$$y_n-\bar{y}_n \to y_0 \in M \quad (n\to\infty). \tag{11}$$

Combining this with (10) we obtain

$$\|x_0-y_0\| = E(x_0).$$

But $p(x_0)$ is the unique best approximation to x_0, i.e.

$$y_0 = p(x_0). \tag{12}$$

On the other hand, using (7) we obtain

$$\|p(x_0)-y_0\| \geqq \|y_0-(p(x_n)-\bar{y}_n)\| - \|p(x_0)-(p(x_n)-\bar{y}_n)\| \geqq$$
$$\geqq \|y_n-p(x_n)\| - \|y_0-(y_n-\bar{y}_n)\| - \|p(x_0)-(p(x_n)-\bar{y}_n)\| \geqq$$
$$\geqq Q - \|y_0-(y_n-\bar{y}_n)\| - \|p(x_0)-(p(x_n)-\bar{y}_n)\|.$$

But this inequality, combined with (9) and (11), implies

$$\|p(x_0)-y_0\| \geqq Q > 0.$$

This contradicts (12); hence $R_2(N, \varepsilon)\to 0$ as $\varepsilon\to 0$.

EXAMPLE 1. It was mentioned that $F_2 \subseteqq F_1$, therefore the assumptions of Theorem 1 for $i=1$ are more general than for $i=2$. So it may be expected that $R_1(N, \varepsilon)\to 0$ in some cases when $R_2(N, \varepsilon)\not\to 0$. Let us give an example, when $R_1(N, \varepsilon)\to 0$ as $\varepsilon\to 0$ while $R_2(N, \varepsilon)\equiv\infty$.

Let $X=L_2=\{f(x: f(x)$ is 2π-periodic real valued function and $\int_0^{2\pi} |f(x)|^2 dx<\infty\}$ with norm $\|f(x)\| = \left(\int_0^{2\pi} |f(x)|^2 dx\right)^{\frac{1}{2}}$.

$$M = \left\{\sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx),\ a_k,\ b_k \in \mathbf{R}\right\}, \quad \text{where} \quad n\in \mathbf{Z}_+.$$

$$N = \left\{f(x)\in L_2\colon f(x) = \sum_{k=n+1}^{\infty} (a_k \cos kx + b_k \sin kx), \quad a_k,\ b_k\in\mathbf{R}\right\}.$$

If elements of the space L_2 are approximated by trigometric polynomials then the operator of best approximation is a linear operator of norm 1. Therefore in this case for any $\varepsilon>0$

$$R_1(N, \varepsilon) \leqq R_1(L_2, \varepsilon) \leqq \varepsilon.$$

On the other hand, for any $k>0$ we can find $f_k \in N$ such that $\|f_k\|=k$. For any $\varepsilon>0$, $q\in M$, $q\neq 0$, take $q_{k,\varepsilon}=\frac{q}{\|q\|}\sqrt{2k\varepsilon}$. Then

$$\|f_k-q_{k,\varepsilon}\| = \sqrt{\|f_k\|^2+\|q_{k,\varepsilon}\|^2} = \sqrt{k^2+2k\varepsilon} \leqq k+\varepsilon \doteq \|f_k\|+\varepsilon = E(f_k)+\varepsilon,$$

hence $q_{k,\varepsilon}\in Y_\varepsilon^2(f_k)$. Therefore $R_2(N,\varepsilon)\geqq \|q_k\|=\sqrt{2k\varepsilon}$ for any $k>0$ and it means that $R_2(N,\varepsilon)=\infty$ for any $\varepsilon>0$.

If we know that $R_1(N,\varepsilon)$ (or $R_2(N,\varepsilon)$) converges to zero and know the order of this convergence we can measure the distance between $p(x)$ and $p(x_1)$ (or between $p(x)$ and y, $y\in M$) by the known value of $\|x-x_1\|$ (or $\|x-y\|-E(x)$), independently from x and x_1 ($x\in N$, $x_1\in X$). Such problems were solved for some subsets in $C[0,1]$ (real valued), see [3], [4], and in $L_p[0,1]$, $p>1$. The following examples show that the operator of best approximation is uniformly continuous on some subsets of $L[0,1]$ and $C[0,1]$ (complex valued) so in these spaces it is also interesting to determine the order of $R_1(N,\varepsilon)$ and $R_2(N,\varepsilon)$ as $\varepsilon\to 0$.

EXAMPLE 2. Let $n\in \mathbf{Z}_+$, $0<\alpha\leqq 1$, $X=L[0,1]=\left\{f(x) \text{ real valued: } \int_0^1 |f(x)|\,dx<\infty\right\}$ with norm $\|f(x)\|_L=\int_0^1 |f(x)|\,dx$, $M=P_n$ the set of algebraic polynomials of degree at most n,

$$N=\{f\in C[0,1]\colon \omega(f,\delta)\leqq \delta^\alpha\} \quad \text{where} \quad \omega(f,\delta)=\sup_{\substack{x_1,x_2\in[0,1]\\ |x_1-x_2|\leqq\delta}} |f(x_1)-f(x_2)|.$$

Then for any $f\in N$ the best approximation is unique and what is more M is finite-dimensional; hence $N\subseteqq F_2$. Now we shall prove that $R_2(N,\varepsilon)\to 0$ as $\varepsilon\to 0$. (Then from (1) we obtain $R_1(N,\varepsilon)\to 0$.)

Let $f\in N$ and $\tilde f(x)=f(x)-f(0)$.

$$\|p(\tilde f)\|_L \leqq 2\|\tilde f\|_L = 2\int_0^1 |\tilde f(x)|\,dx \leqq 2\sup_{x\in[0,1]} |\tilde f(x)| \leqq 2\{\tilde f(0)+\omega(\tilde f,1)\} \leqq 2,$$

and using an inequality proved in [5] we obtain

$$\sup_{x\in[0,1]} |p(\tilde f,x)| \leqq C(n)\|p(\tilde f)\|_L \leqq 2C(n),$$

where the constant $C(n)$ depnds only on n. Further by Markov's inequality

$$\sup_{x\in[0,1]} |p'(\tilde f,x)| \leqq 2n^2 \sup_{x\in[0,1]} |p(\tilde f,x)| \leqq 4n^2C(n)=C_1(n).$$

Hence

$$\omega(p(f),\delta)\equiv \omega(p(\tilde f),\delta)\leqq C_1(n)\delta \leqq C_1(n)\delta^\alpha,$$

and we get for $f\in N$

$$\omega(f-p(f),\delta)\leqq C_2(n)\delta^\alpha. \tag{13}$$

On the other hand, $f-p(f)$ must have at least one zero in $[0, 1]$ and this implies $\sup_{x\in[0,1]} |f(x)-p(f, x)| \leqq C_2(n)$. Combining this with (13) we obtain that for any $\{f_k\}\in N$, the sequence $\{f_k-p(f_k)\}$ is uniformly bounded and the sequence of its modulus of continuity is bounded by the same modulus of continuity. So by the Arzela's theorem we obtain that $\{f_k-p(f_k)\}$ has a point of accumulation f_0 and it is evident that $f_0\in F_2$. This means that N is approximatively dense in F_2 (so in F_1 also) and by Theorem 1 we may conclude that $R_1(N, \varepsilon)$ and $R_2(N, \varepsilon)$ converge to zero as $\varepsilon\to 0$.

EXAMPLE 3. Let $n\in \mathbf{Z}_+$, $0<\alpha\leqq 1$, $X=\{f=\varphi+ig: \varphi, g\in C[0, 1]\}$, with the norm $\|f\|=\sup_{x\in[0,1]}\sqrt{\varphi(x)^2+g(x)^2}$, $M=\{r_n+it_n\}$, where r_n and t_n are algebraic polynomials of degree at most n.

$$N=\{f\in X: f=\varphi+ig, \omega(\varphi, \delta)\leqq\delta^\alpha, \omega(g, \delta)\leqq\delta\}.$$

In this case for any $f\in X$ the best approximation $p(f)$ is unique, M is finite-dimensional so $F_2=X$.

For the sequence $\{f_k\}\in N$ we construct the sequence $\{f_k-q_n(f_k)\}$, where $q_n(f_k)\in M$ and $q_n(f_k)$ interpolates $f_k(x)$ at the points $x_i=\frac{i}{n}$, $(i=0, 1, \ldots, n)$ and using the ideas of Example 2 we obtain that $R_1(N, \varepsilon)$ and $R_2(N, \varepsilon)$ converge to zero as $\varepsilon\to 0$.

§ 2.

Let us discuss now the characteristics of those sets $N\subseteqq X$ on which the operator of best approximation satisfies a uniform Lipschitz condition i.e. $R_1(N, \varepsilon)\leqq C\cdot\varepsilon$, where the constant C depends only on N. All estimates of this section can be analogously applied to $R_2(N, \varepsilon)$.

Let us give two definitions. We call the set $N\subseteqq X$ a *cone* if for any $x\in N$ and $\alpha>0$, $\alpha\cdot x\in N$ holds. For the set $S\subseteqq X$ we define $T(S)$ as the smallest cone containing S, so

$$T(S)=\{x\in X: \text{there exists } \alpha>0 \text{ such that } \alpha x\in S\}.$$

At first we prove two lemmas.

LEMMA 1. *Let* $\varepsilon>0$. *If N is a cone and there exists* $\varepsilon_0>0$ *for which* $R_1(\varepsilon_0, N)<\infty$, *then* $R_1(\varepsilon, N)\equiv C\varepsilon$ *where the constant C depends only on N.*

PROOF. Take any $\varepsilon_1>0$ and prove that if $R_1(\varepsilon_0, N)<\infty$ then $R_1(N, \varepsilon_1)<\infty$. Assume the contrary, i.e. $R_1(N, \varepsilon_0)<\infty$, $R_1(N, \varepsilon_1)=\infty$. Then exist sequences $\{x_k\}\in N$, $\{x_k^1\}\in X$, such that $\|x_k-x_k^1\|\leqq\varepsilon_1$ and for some $p(x_k)\in Y(x_k)$, $p(x_k^1)\in Y(x_k^1)$:

$$\|p(x_k)-p(x_k^1)\|>k. \tag{14}$$

Let $\bar{x}_k = \frac{\varepsilon_0}{\varepsilon_1} x_k \in N$, because of N is a cone, $\bar{x}_k^1 = \frac{\varepsilon_0}{\varepsilon_1} x_k^1 \in X$. Then $\|\bar{x}_k - \bar{x}_k^1\| \leqq \varepsilon_0$ and $\frac{\varepsilon_0}{\varepsilon_1} p(x_k) \in Y(\bar{x}_k)$, $\frac{\varepsilon_0}{\varepsilon_1} p(x_k^1) \in Y(\bar{x}_k^1)$. Hence, using (14) we obtain:

$$\frac{\varepsilon_0}{\varepsilon_1} k \leqq \frac{\varepsilon_0}{\varepsilon_1} \|p(x_k) - p(x_k^1)\| = \|p(\bar{x}_k) - p(\bar{x}_k^1)\| \leqq R_1(N, \varepsilon_0) < \infty.$$

But the left hand side of this inequality can be made as large as we like and the right hand side is independent of k. This is a contradiction which implies that $R_1(N, \varepsilon) < \infty$.

Let $\varepsilon_1, \varepsilon_2 > 0$, and let $R_1(N, \varepsilon_0) < \infty$ for some $\varepsilon_0 > 0$. Then as it was proved $R_1(N, \varepsilon_1) < \infty$, $R_1(N, \varepsilon_2) < \infty$. Let $\delta_k \downarrow 0$, then we can choose sequences $\{x_k\} \in N$, $\{x_k^1\} \in X$ with the following properties: $\|x_k - x_k^1\| \leqq \varepsilon_1$ and for some $p(x_k) \in Y(x_k)$, $p(x_k^1) \in Y(x_k^1)$,

$$\|p(x_k) - p(x_k^1)\| \geqq R_1(N, \varepsilon_1) - \delta_k. \tag{15}$$

Let $\bar{x}_k = \frac{\varepsilon_2}{\varepsilon_1} x_k$, $\bar{x}_k^1 = \frac{\varepsilon_2}{\varepsilon_1} x_k^1$ then $\bar{x}_k \in N$, $\bar{x}_k^1 \in X$, $\|\bar{x}_k - \bar{x}_k^1\| \leqq \varepsilon_2$, $\frac{\varepsilon_2}{\varepsilon_1} p(x_k) \in$ $\in Y(\bar{x}_k)$, $\frac{\varepsilon_2}{\varepsilon_1} p(x_k^1) \in Y(\bar{x}_k^1)$, hence

$$\left\| \frac{\varepsilon_2}{\varepsilon_1} p(x_k) - \frac{\varepsilon_2}{\varepsilon_1} p(x_k^1) \right\| \leqq R_1(N, \varepsilon_2)$$

and combining this inequality with (15) we obtain

$$\frac{\varepsilon_2}{\varepsilon_1} \big(R_1(N, \varepsilon_1) - \delta_k\big) \leqq R_1(N, \varepsilon_2).$$

But δ_k converges to zero as $k \to \infty$, so

$$\frac{\varepsilon_2}{\varepsilon_1} R_1(N, \varepsilon_1) \leqq R_1(N, \varepsilon_2)$$

or

$$\frac{R_1(N, \varepsilon_1)}{\varepsilon_1} \leqq \frac{R_1(N, \varepsilon_2)}{\varepsilon_2}.$$

It is evident that analogously we can obtain the opposite inequality, hence

$$\frac{R_1(N, \varepsilon_1)}{\varepsilon_1} = \frac{R_1(N, \varepsilon_2)}{\varepsilon_2}$$

for any $\varepsilon_1, \varepsilon_2 > 0$. This means that $\frac{R_1(N, \varepsilon)}{\varepsilon} = C$, a constant. Q.E.D.

LEMMA 2. *Let $N \subseteqq X$ and $R_1(N, \varepsilon) \leqq C\varepsilon$, then $R_1\big(T(N), \varepsilon\big) \leqq C\varepsilon$.*

PROOF. Take any $\bar{x}\in T(N)$ and $\bar{x}_1\in X$ such that $\|\bar{x}-\bar{x}_1\|\leqq\varepsilon$. Then there exists $\alpha>0$ such that $\alpha\bar{x}\in N$, further $\|\alpha\bar{x}-\alpha\bar{x}_1\|\leqq\alpha\varepsilon$. By assumption of the lemma, for any $p(\alpha\bar{x})\in Y(\alpha\bar{x})$ and $p(\alpha\bar{x}_1)\in Y(\alpha\bar{x}_1)$ we have

$$\|p(\alpha\bar{x})-p(\alpha\bar{x}_1)\|\leqq C\alpha\varepsilon. \tag{16}$$

Take any $p(\bar{x})\in Y(\bar{x})$ and $p(\bar{x}_1)\in Y(\bar{x}_1)$, then $\alpha p(\bar{x})\in Y(\alpha\bar{x})$ and $\alpha p(\bar{x}_1)\in \in Y(\alpha\bar{x}_1)$ and by (15) we obtain

$$\|p(\bar{x})-p(\bar{x}_1)\| = \frac{1}{\alpha}\|\alpha p(\bar{x})-\alpha p(\bar{x}_1)\| \leqq \frac{1}{\alpha}C\alpha\varepsilon = C\varepsilon.$$

This implies $R_1(T(N),\varepsilon)\leqq C\varepsilon$, Q.E.D.

From Lemmas 1 and 2 obviously follows

THEOREM 2. *Let* $\varepsilon>0$, $N\subseteqq X$. *Then* $R_1(N,\varepsilon)\leqq C\varepsilon$ *if and only if for some* $\varepsilon_0>0$, $R_1(T(N),\varepsilon_0)<\infty$.

PROOF. If $R_1(T(N),\varepsilon_0)<\infty$ for some $\varepsilon_0>0$ then by Lemma 1 (using that $T(N)$ is a cone) we obtain $R_1(T(N),\varepsilon)\equiv C\varepsilon$. But $N\subseteqq T(N)$, hence $R_1(N,\varepsilon)\leqq \leqq R_1(T(N),\varepsilon)=C\varepsilon$.

On the other hand, if $R_1(N,\varepsilon)\leqq C\varepsilon$ then by Lemma 2 $R_1(T(N),\varepsilon)\leqq C\varepsilon$ and in particular $R(T(N),\varepsilon_0)<\infty$ for any $\varepsilon_0>0$, Q.E.D.

Let us consider now some cones in the space of real valued functions continuous on an interval. At first settle the periodic case.

Let $C_0[0,2\pi]$ the space of 2π-periodic real valued functions with norm $\|f\|= =\sup|f(x)|$, $n\in\mathbf{Z}_+$, $r\in\mathbf{N}$, $C_0^r[0,2\pi]=\{f(x)\in C_0[0,2\pi], f^{(r)}\in C_0[0,2\pi]\}$, T_n the set of trigonometric polynomials of degree at most n.

THEOREM 3. *Let* $n\in\mathbf{Z}_+$, $r\in\mathbf{N}, f\in C_0^r[0,2\pi]\setminus T_n$. *Then for any* $f_1\in C_0[0,2\pi]$ *we have*

$$\|p_n(f)-p_n(f_1)\|\leqq C(n,r)\left(\frac{E_n(f^{(r)})}{E_n(f)}\right)^{2n/r}\|f-f_1\|.$$

PROOF. Take $f\in C_0^r[0,2\pi]\setminus T_n$ and define $\tilde{f}=p_n(f)-f$, analogously for $f_1\in \in C_0[0,2\pi]$ let $\tilde{f}_1=p_n(f)-f_1$. Then $p_n(\tilde{f})\equiv 0$, $p_n(\tilde{f}_1)=p_n(f)-p_n(f_1)$ and

$$\|\tilde{f}-p_n(\tilde{f}_1)\|\leqq\|\tilde{f}-f_1\|+E_n(\tilde{f}_1)\leqq E_n(\tilde{f})+2\|\tilde{f}-\tilde{f}_1\| = \|\tilde{f}\|+2\|\tilde{f}-\tilde{f}_1\|. \tag{17}$$

For $\tilde{f}(x)$ there exists a system of $2n+2$ points $0\leqq x_1<x_2<\ldots<x_{2n+2}<2\pi$ such that

$$\tilde{f}(x_i)=\gamma(-1)^i\|\tilde{f}\| \quad (i=1,2,\ldots,2n+2) \tag{18}$$

where $\gamma=\pm1$. For $m\in\mathbf{Z}_+$, $m\leqq 2n+1$ define

$$\omega_{im}(x)=\prod_{k=0}^{m}(x-x_{i+k}),\quad (i=1,2,\ldots,2n+2-m).$$

(17) and (18) imply

$$\gamma(-1)^{i+1}p_n(\tilde{f}_1,x_i)\leqq 2\|\tilde{f}-\tilde{f}_1\|$$

hence (see [4])

$$\|p_n(\tilde{f}_1)\|\leqq C_1(n)\max\left\{\sum_{i=1}^{2n+1}\frac{1}{\omega'_{1,2n}(x_i)},\ \sum_{i=2}^{2n+2}\frac{1}{\omega'_{2,2n}(x_i)}\right\}\|\tilde{f}-\tilde{f}_1\|. \tag{19}$$

Without loss of generality we may assume that

$$\sum_{i=1}^{2n+1} \frac{1}{\omega'_{1,2n}(x_i)} \geqq \sum_{i=2}^{2n+2} \frac{1}{\omega'_{2,2n}(x_i)}.$$

Let at first $r<2n$ and for some $1\leqq i\leqq 2n+1-r$ consider the system of points $x_i<x_{i+1}<\ldots<x_{i+r}$. Then by $\tilde{f}\in C_0^r[0, 2\pi]$ we get from the Lagrange interpolation formula

$$\sum_{k=0}^{r} \frac{\tilde{f}(x_{k+i})}{\omega'_{i,r}(x_{k+i})} = \frac{\tilde{f}^{(r)}(\xi_i)}{r!} \tag{20}$$

where $x_i<\xi_i<x_{i+r}$. Then from (18) we obtain

$$\sum_{k=0}^{r} \frac{1}{|\omega'_{i,r}(x_{k+i})|} \leqq \frac{\|\tilde{f}^{(r)}\|}{r!\|\tilde{f}\|} \tag{21}$$

hence

$$\frac{r+1}{(x_{r+i}-x_i)^r} \leqq \frac{\|\tilde{f}^{(r)}\|}{r!\|\tilde{f}\|}$$

or

$$x_{r+i}-x_i \geqq ((r+1)r!)^{1/r}\left(\frac{\|\tilde{f}\|}{\|\tilde{f}^{(r)}\|}\right)^{1/r} = C_2(r)\left(\frac{\|\tilde{f}\|}{\|\tilde{f}^{(r)}\|}\right)^{1/r}. \tag{22}$$

In [6] it is proved that

$$\sum_{i=1}^{2n+1} \frac{1}{|\omega'_{1,2n}(x_i)|} = \sum_{i}\left(\sum_{k=0}^{r} \frac{1}{|\omega'_{i,r}(x_{i+k})|}\right)\frac{\alpha_k}{\prod_{l=1}^{2n-r}(x_{j_l}-x_{i_l})}$$

where number of terms in $\sum_i$ is finite and depends on n and r; $\alpha_k=\pm 1$, and $[x_{j_l}, x_{i_l}]\supseteqq [x_i, x_{i+r}]$. Then combining (19), (21) and (22) we have for $r<2n$

$$\|p_n(\tilde{f}_1)\| \leqq C_3(n, r)\frac{\|\tilde{f}^{(r)}\|}{\|\tilde{f}\|}\frac{1}{\left(\frac{\|\tilde{f}\|}{\|\tilde{f}^{(r)}\|}\right)^{(2n-r)/r}}\|\tilde{f}-\tilde{f}_1\| = \tag{23}$$

$$= C_3(n, r)\left(\frac{\|\tilde{f}^{(r)}\|}{\|\tilde{f}\|}\right)^{2n/r}\|\tilde{f}-\tilde{f}_1\|.$$

Let now $r\geqq 2n$, then by setting $i=1, r=2n$ we obtain from (21)

$$\sum_{i=1}^{2n+1} \frac{1}{|\omega'_{1,2n}(x_i)|} = \sum_{k=0}^{2n} \frac{1}{|\omega'_{1,2n}(x_{k+1})|} \leqq \frac{\|\tilde{f}^{(2n)}\|}{(2n)!\|\tilde{f}\|}. \tag{24}$$

By Kolmogorov's inequality (see [7]) when $r\geqq 2n$, we have

$$\|\tilde{f}^{(2n)}\| \leqq C_4(n, r)\|\tilde{f}\|^{(r-2n)/r}\|\tilde{f}\|^{2n/r}.$$

Combining this inequality with (19) and (24) we obtain

$$\|p_n(\tilde{f}_1)\| \leqq C_5(n, r)\frac{\|\tilde{f}\|^{(r-2n)/r}\|\tilde{f}^{(r)}\|^{2n/r}}{\|\tilde{f}\|}\|\tilde{f}-\tilde{f}_1\| = $$

$$= C_5(n, r)\left(\frac{\|\tilde{f}^{(r)}\|}{\|\tilde{f}\|}\right)^{2n/r}\|\tilde{f}-\tilde{f}_1\|,$$

and using (23) we get this inequality for any $r\in\mathbf{N}$ with the constant $C_6(n, r)=$ $=\max\{C_3, C_5\}$.

On the other hand, $p_n(\tilde{f}_1)=p_n(f)-p_n(f_1)$, $\tilde{f}-\tilde{f}_1=f_1-f$, $\|\tilde{f}\|=\|f-p_n(f)\|=$ $=E_n(f)$, $\tilde{f}^{(r)}=p_n^{(r)}(f)-f^{(r)}$. So we have

$$\|p_n(f)-p_n(f_1)\| \leqq C_6\left(\frac{\|f^{(r)}-p_n^{(r)}(f)\|}{E_n(f)}\right)^{2n/r}\|f-f_1\|. \tag{25}$$

In [8] it was proved that if $f\in C_0^r[0, 2\pi]$ and the polynomial $t_n\in T_n$ satisfies the inequality

$$\|f-t_n\| \leqq AE_n(f)$$

with some $A>0$, then

$$\|f^{(r)}-t_n^{(r)}\| \leqq \left(1+\frac{\pi}{2}\right)\{L_{nr}(f)+A\}\,E_n(f^{(r)}) \tag{26}$$

where

$$L_{nr}(f) \leqq \frac{4}{\pi^2}\ln(p+1)+O(1) \tag{27}$$

with $p=\min(n, r)$. Using (26) and (27) with $t_n=p_n(f)$ we obtain

$$\|f^{(r)}-p_n^{(r)}(f)\| \leqq C_7(n, r)E_n(f^{(r)})$$

and combining this with (25) we obtain Theorem 3.

REMARK. Theorem 3 characterizes the periodic functions and it is evident that even in the case when continuous functions are approximated by algebraic polynomials, this theorem is not true $\left(E_n(f^{(r)})\right.$ can be zero while $\left.E_n(f)\neq 0\right)$.

APPLICATION. By Favard's inequality for periodic functions we have

$$E_n(f) \leqq \frac{M_r}{n^r}E_n(f^{(r)})$$

where $1<M_r<\dfrac{\pi}{2}$. Let $K>n^r$ and define the cone (which is not empty) by

$$N_k = \{f\in C_0^r[0, 2\pi]: E_n(f^{(r)}) \leqq KE_n(f)\}.$$

Then using Theorem 3 we obtain

$$R_1(N_k, \varepsilon) \leqq C(n, r, K)\,\varepsilon.$$

Let $f_0\in C_0^r[0, 2\pi]\setminus T_n$. Then $E_n(f_0)\neq 0$ and denote $C_0=\dfrac{E_n(f_0^{(r)})}{E_0(f_0)}$. In the space $C_0^r[0, 2\pi]$ define the following norm, for $g\in C_0^r[0, 2\pi]$:

$$\|g\|_1 = \|g\|_C+\|g^{(r)}\|_C.$$

Let $0<\lambda<1$ and

$$M_\lambda(f_0) = \{f_1 \in C_0^{(r)}[0, 2\pi]: \|f_0-f_1\|_1 \leqq \lambda E_n(f_0)\}.$$

Now, if $f\in M_\lambda(f_1)$, then

$$E_n(f) \geqq E_n(f_0)-\|f-f_0\|_C \geqq E_n(f_0)-\|f-f_0\|_1 \geqq E_n(f_0)-\lambda E_n(f_0) =$$
$$= (1-\lambda)E_n(f_0),$$
$$E_n(f^{(r)}) \leqq E_n(f_0^{(r)})+\|f_0^{(r)}-f^{(r)}\|_C \leqq E_n(f_0^{(r)})+\|f_0^{(r)}+f^{(r)}\|_1 \leqq E_n(f_0^{(r)})+\lambda E_n(f_0),$$

hence

$$\frac{E_n(f^{(r)})}{E_n(f)} \leqq \frac{E_n(f_0^{(r)})+\lambda E_n(f_0)}{(1-\lambda)E_n(f_0)} = \frac{C_0}{1-\lambda}+\frac{\lambda}{1-\lambda} = \frac{C_0+\lambda}{1-\lambda}.$$

By this inequality and Theorem 3 we obtain

$$R_1\big(M_\lambda(f_0), \varepsilon\big) \leqq C(n, r, \lambda, C_0)\varepsilon,$$

further by Lemma 2

$$R_1(T\big(M(f_0)\big), \varepsilon) \leqq C(n, r, \lambda, C_0)\varepsilon.$$

Now we shall prove a theorem about cones in the space $C[0, 1]$, i.e. in the space of continuous real valued functions with supremum norm. We shall approximate the elements of this space by polynomials of a Chebysev system $\{\varphi_i\}_{i=0}^n$ defined on $[0, 1]$. Furthermore if N is a cone in $C[0, 1]$ we define $\overline{N}=\{f\in N: \|f\|=1\}$. Then we have

THEOREM 4. *Let N be a cone in $C[0, 1]$ satisfying the following conditions:*
(i) *$\overline{N}$ is compact,*
(ii) $\inf_{f\in\overline{N}} E_n(f)=\delta_n>0.$

Then $R_1(N, \varepsilon) \equiv C\varepsilon$ where the constant C depends only on the cone N and the Chebysev system $\{\varphi_i\}_{i=0}^n$.

PROOF. Let $f\in N, f\neq 0$ and $p_n(f)$ be the polynomial of best approximation to f with respect to the system $\{\varphi_i\}_{i=0}^n$. Then by assumption (ii), $\|f-p_n(f)\| = E_n(f)>0$. For f we can find a system of $n+2$ points $0\leqq x_0<x_1<\ldots<x_{n+1}\leqq 1$ and a system of $n+2$ positive real numbers $t_0>0, \ldots, t_{n+1}>0$ such that

$$f(x_i)-p_n(f, x_i)=\gamma(-1)^i E_n(f) \tag{28}$$

$$\begin{cases} \sum (-1)^i t_i \varphi_k(x_i) = 0 \quad (k = 0, 1, \ldots, n) \\ \sum_{i=0}^{n+1} t_i = 1 \end{cases} \tag{29}$$

where $\gamma=\pm 1$. Let $f_1\in C[0, 1]$ and $\|f-f_1\|\leqq\varepsilon$, then

$$\|f-p_n(f_1)\| \leqq \varepsilon+E_n(f_1) \leqq E_n(f)+2\varepsilon. \tag{30}$$

In [9] it is proved that if the polynomial q_n satisfies $\|f-q_n\| \leqq E_n(f)+\varepsilon$ and $\{x_i\}_{i=0}^{n+1}$, $\{t_i\}_{i=0}^{n+1}$ satisfy (28) and (29) then

(31) $$|p_n(f, x_i)-q_n(x_i)| \leqq (\min_{i=0,1,\dots,n+1} t_i)^{-1}\varepsilon \quad (i=0,1,\dots,n+1).$$

So using (30) we can set $q_n = p_n(f_1)$ in (31), hence

(32) $$|p_n(f, x_i)-p_n(f_1, x_i)| \leqq 2(\min p_i)^{-1}\varepsilon.$$

Solving the system (29) we have

$$t_i = \frac{\Delta_i}{\sum_{j=0}^{n+1} \Delta_j} \quad (i=0,1,\dots,n+1)$$

where Δ_i is the following determinant:

$$\Delta_i = \begin{vmatrix} \varphi_0(x_0) & \varphi_0(x_1) & \dots & \varphi_0(x_{i-1}) & \varphi_0(x_{i+1}) & \dots & \varphi_0(x_{n+1}) \\ \varphi_1(x_0) & \varphi_1(x_1) & \dots & \varphi_1(x_{i-1}) & \varphi_1(x_{i+1}) & \dots & \varphi_1(x_{n+1}) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \varphi_{n-1}(x_0) & \varphi_{n-1}(x_1) & \dots & \varphi_{n-1}(x_{i-1}) & \varphi_{n-1}(x_{i+1}) & \dots & \varphi_n(x_{n+1}) \\ \varphi_n(x_0) & \varphi_n(x_1) & \dots & \varphi_n(x_{i-1}) & \varphi_n(x_{i+1}) & \dots & \varphi_n(x_{n+1}) \end{vmatrix}$$

$$(i=0,1,\dots,n+1).$$

Let $\Delta_{i_0} = \min_{i=0,1,\dots,n+1} \Delta_i$, then from (32) we obtain

(33) $$|p_n(f, x_i)-p_n(f_1, x_i)| \leqq 2\frac{\sum_{j=0}^{n+1} \Delta_j}{\Delta_{i_0}}\varepsilon \leqq \frac{C_1\varepsilon}{\Delta_{i_0}}$$

where $C_1 = 2 \sup_{x_0,\dots,x_{n+1}} \sum_{j=0}^{n+1} \Delta_j$ depends only on the Chebysev system $\{\varphi_i\}_{i=0}^n$.

By the Lagrange interpolation formula we have

(34)
$$p_n(f)-p_n(f_1) = \sum_{i=0}^{n} \frac{\{p_n(f, x_i)-p_n(f_1, x_i)\}\, U(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n, x)\}}{\Delta_{n+1}}$$

where

$$U(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n, x) = \begin{vmatrix} \varphi_0(x_0) & \dots & \varphi_0(x_{i-1}) & \varphi_0(x_{i+1}) & \dots & \varphi_0(x_n) & \varphi_0(x) \\ \varphi_1(x_0) & \dots & \varphi_1(x_{i-1}) & \varphi_1(x_{i+1}) & \dots & \varphi_1(x_n) & \varphi_1(x) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \varphi_n(x_0) & \dots & \varphi_n(x_{i-1}) & \varphi_n(x_{i+1}) & \dots & \varphi_n(x_n) & \varphi_n(x) \end{vmatrix}.$$

Being $U(x_0 \dots x_{i-1}, x_{i+1}, \dots x_n, x)$ uniformly bounded, we have from (33) and (34)

(35) $$\|p_n(f)-p_n(f_1)\| \leqq C_2 \frac{\varepsilon}{\Delta_{i_0}\Delta_{n+1}}$$

where the constant C_2 depends only on $\{\varphi_i\}_{i=0}^n$.

Let $\tilde{f}=\frac{f}{E_n(f)}\in N$, then $E_n(\tilde{f})=1$, $p_n(\tilde{f})=\frac{p_n(f)}{E_n(f)}$ and from (28) we obtain

$$\tilde{f}(x_i)-p_n(\tilde{f}, x_i) = \gamma(-1)^i. \tag{36}$$

Further by condition (ii) we have $E_n(f) \geqq \delta_n \|f\|$ for any $f\in N$, or

$$\|f\| \leqq \frac{E_n(f)}{\delta_n}. \tag{37}$$

From this inequality we obtain

$$\|\tilde{f}\| \leqq \frac{1}{\delta_n} \tag{38}$$

for $\tilde{f}\in N$, therefore

$$\|p_n(\tilde{f})\| \leqq 2\|\tilde{f}\| \leqq \frac{2}{\delta_n}. \tag{39}$$

Condition (i) and Arzela's theorem imply that

$$\omega(\varphi, \delta) \leqq \overline{\omega}_1(\delta) \tag{40}$$

for any $\varphi\in\overline{N}$, where $\omega(\varphi, \delta)$ is the modulus of continuity of φ and $\overline{\omega}_1(\delta)$ is some fixed modulus of continuity. So (40) and (38) yield

$$\omega(\tilde{f}, \delta) \leqq \|\tilde{f}\|\overline{\omega}_1(\delta) \leqq \frac{\overline{\omega}_1(\delta)}{\delta_n}. \tag{41}$$

On the other hand, in a finite-dimensional space any bounded set is compact, therefore using again Arzela's theorem and inequality (39) we get

$$\omega\big(p_n(\tilde{f}), \delta\big) \leqq \overline{\omega}_2(\delta) \tag{42}$$

where $\overline{\omega}_2(\delta)$ is another fixed modulus of continuity which depends only on δ_n and $\{\varphi_i\}_{i=0}^n$.

Then from (36), (41) and (42) we have

$$2 = |\{\tilde{f}(x_i)-p_n(\tilde{f}, x_i)\}-\{\tilde{f}(x_{i+1})-p_n(\tilde{f}, x_{i+1})\}| \leqq \tag{43}$$

$$\leqq |\tilde{f}(x_i)-\tilde{f}(x_{i+1})|+|p_n(\tilde{f}, x_i)-p_n(\tilde{f}, x_{i+1})| \leqq \omega(\tilde{f}, x_{i+1}-x_i)+\omega\big(p_n(\tilde{f}), x_{i+1}-x_i)\big) \leqq$$

$$\leqq \frac{\overline{\omega}_1(x_{i+1}-x_i)}{\delta_n}+\overline{\omega}_2\,(x_{i+1}-x_i) \leqq \omega^*(x_{i+1}-x_i)$$

where without loss of generality we may assume that $\omega^*(\delta)$ is s convex modulus of continuity. Then from (43) we obtain

$$x_{i+1}-x_i \geqq (\omega^*)^{-1}(2)=C_3 > 0,$$

where the constant C_3 depends only on the cone N and $\{\varphi_i\}_{i=0}^n$. Then using the Chebysev property of the system $\{\varphi_i\}_{i=0}^n$ it is easy to see that

$$\varDelta_i \geqq C_4 > 0 \quad (i = 0, 1, \ldots, n+1),$$

where the constant C_4 depends only on N and $\{\varphi_i\}_{i=0}^n$. From (35) we obtain that

$$\|p_n(f)-p_n(f_1)\| \leqq C_5\varepsilon$$

hence and by Lemma 1 $R_1(N, \varepsilon)\equiv C_3\varepsilon$. Q.E.D.

Added in proof (September 28, 1977.) The author has learned that M. S. Henry and D. Schmidt published a result (with an entirely different method of proof), which is essentially the same as Theorem 4 above (Continuity theorems for the product approximation operator, in: *Theory of Approximation with Applications*, Academic Press (New York, 1976), Theorem 3, p. 31).

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COMPOSITION PROPERTIES IN UNIFORM SPACES

By

M. D. RICE (Fairfax)

In this paper we describe the algebraic operators derived from the inversion-closed and regular ring properties in terms of the metric-fine and measurable operators, respectively. In discussing the above relationships, the generalized and separable composition properties are also introduced. These concepts have roughly the same relationship to the sub-M-fine and locally sub-M-fine operators, respectively, as each of the above algebraic operators has to its respective uniform operator. This analogy is formalized by Theorems 2.3 and 2.4, which characterize the algebraic operators in terms of the least upper bound operation U, the real-valued operator c, and the respective uniform operators mentioned above. These results show (cf. 2.7) that the countable, separable, and generalized composition properties coincide for spaces generated by uniform real-valued functions and that the separable composition property is equivalent to the countable composition property and the real extension (RE) property for spaces with a basis of finite dimensional uniform covers. The analogy previously mentioned is further strengthened by Theorem 2.8, which shows that the sub-inversion-closed spaces are precisely the spaces closed under generalized composition. Finally, in Theorem 2.9 we show that the finite dimensional operator f is an isomorphism from the full subcategory of respectively, locally sub-M-fine spaces with a basis of point-finite uniform covers, sub-M-fine, or metric-fine spaces onto the full subcategory of spaces with a basis of finite dimensional uniform covers which are (i) closed under separable composition, generalized composition, or inversion, respectively, and (ii) are locally sub-M-fine in the following restricted sense: each cover two-dimensionally uniform on each member of some one-dimensional Euclidean uniform cover is a uniform cover.

1. Preliminaries

Throughout this paper we will refer to $[R]_{1-4}$ for notation, basic definitions, and results pertaining to the locally sub-M-fine, sub-M-fine, metric-fine (M-fine), and measurable operators. These are denoted by $m_0, m_1, m,$ and m_*, respectively. (We note that the notation $m_* = M$ is used in $[Fr]_2$ and $m_* = b$ (for separable spaces) is used in $[H]_2$, while m has the same meaning in all cases.) The reader is referred to $[Fr]_{1-3}$ and $[H]_{1-3}$ for further information on the metric-fine and measurable operators.

We will also need the following definitions in the context of the vector lattice of all uniformly continuous real-valued functions $C(uX)$ on a separated uniform space uX. $C(uX)$ is *closed under inversion* if $f \in C(uX)$, never vanishing, implies

$1/f\in C(uX)$, *closed under countable composition* if $f_1, f_2, \ldots$, members of $C(uX)$ and $R^{\aleph_0} \xrightarrow{g} R$ continuous imply $g(f_1, f_2, \ldots)\in C(uX)$, and a *regular ring* if for each $f\in C(uX)$ there exists $h\in C(uX)$ such that $f^2h=f$. For a detailed discussion of the countable composition and inversion properties, the reader is referred to [CI], [HIJ], and $[I_2]$, and for the inversion and regular ring properties to $[H]_{1-4}$. To each uniformity u on X and algebraic property mentioned above, one may associate a smallest uniformity containing u, denoted by u_i, u_{cc}, and u_{reg}, respectively, which has the appropriate algebraic property. The existence of these uniformities follows from the fact that in each case, the associated class of uniform spaces is closed under uniform sums and quotients (see $[H]_3$).

Finally, we use the following notation. For each uniformity u on the set X, cu is the uniformity generated by $C(uX)$, eu (resp. pu) is the uniformity with basis of countable (resp. finite) u-covers, and fu is the uniformity with the basis of finite dimensional u-covers. Given uniformities u and v on the set X, u/v has the basis of covers of the form $\{V_s\cap U_t^s\}$, where $\{V_s\}\in v$ and $\{U_t^s\}\in u$, for each s. u/v need not be a uniformity ([P]); it is a uniformity if vX has a basis of point finite uniform covers ($[I]_1$, 7.5). We say that uX is *locally fine* if $u=u/u$. Given any uniformity u on the set X, there exists a smallest locally fine uniformity λu containing u ([GI]). If uX is separable ($u=eu$), then $m_0u=m_1u=\lambda u$ (essentially by $[R]_1$, 2.2) and in general each locally fine space is sub-M-fine (essentially [GI], 4.2), so for each uniformity u, $m_1u\subset\lambda u$.

2. Main results

The first result summarizes the special descriptions of the algebraic operators on separable spaces

THEOREM 2.1. *Let uX be separable.*

(i) *If uX has a basis of finite dimensional uniform covers,* $u_i=cmu$.
(ii) *If uX is generated by $C(uX)$, then* $u_{cc}=c\lambda u$.
(iii) $u_{reg}=m_*u$.

The assumption in (i) guarantees that $u=fu\subset fmu$. Since muX is a separable RE space ([GI], 4.12), it follows from ([CI], 3.4) that $cmu=fmu$. From ($[H]_1$, 4.1) there exists a separable metric-fine uniformity v on X such that $c(u_i)=cv$, so we obtain $mc(u_i)=me(u_i)=mu_i=mu$ since $u_i\subset mu$, and $mc(u_i)=mcv=mev=v$. This establishes that $cmu=cv\subset u_i$. Since $u\subset cmu$ and $C(muX)$ is closed under inversion, it follows that $u_i\subset cmu$.

In case (ii), we have $u=cu\subset c(u_{cc})$; hence $c(u_{cc})=u_{cc}$ since u_{cc} is the smallest uniformity containing u with the countable composition property. For the same reason $u_{cc}\subset c\lambda u$. From ($[H]_1$, 4.1) there exists a separable locally fine uniformity v on X such that $u_{cc}=cv$; hence $\lambda u=\lambda(u_{cc})=\lambda cv=v$ (the last equality coming from [GI], 4.8—4.9), so we obtain $c\lambda u=cv=u_{cc}$.

For case (iii), first note that $cm_*u=m_*u$ since m_*u has a basis of countable partitions ($[H]_2$, 4.1), while $u_{reg}\subset m_*u$ since $C(m_*uX)$ is a regular ring. The proof of part (i), using ($[H]_1$, 4.1) with m replaced by m_*, shows that $cm_*u\subset u_{reg}$ and completes the proof.

LEMMA 2.2. *Let v be a separable locally fine uniformity and let u be any uniformity which satisfies $eu\subset v$. Then $e(u\cup v)=v$.*

It is clear that $v\subset e(u\cup v)$. Conversely, assume the countable cover $\{B_n\}$ belongs to $u\cup v$. Then there exists a member of u, $\{U_s\}$, and a countable member of v, $\{V_m\}$, such that $\{U_s\cap V_m\}<\{B_n\}$. Define $S_{m,n}=\cup\{U_s: U_s\cap V_m\subset B_n\}$ for each pair m, n and let $\mathcal{S}_m=\{S_{m,n}: n\in N\}$. Clearly $\{U_s\}<\mathcal{S}_m$, so $eu\subset v$ implies $\mathcal{S}_m\in v$, $m=1, 2, \ldots$; hence vX locally fine implies $\{V_m\cap S_{m,n}\}\in v/v=v$. The latter cover refines $\{B_n\}$, so the proof is complete.

THEOREM 2.3. *For each uniform space uX,*
(i) $u_i=u\cup cmu$, *and*
(ii) $u_{\text{reg}}=u\cup m_* eu$.

The proofs of (i) and (ii) are essentially the same (based on 2.1), so we will establish only (i) in detail. By 2.1 (i), $(cu)_i=cmcu=cmeu$. By ([R]$_1$, 4.4), $meu=emu$, so $cmeu=cmu\subset u_i$. From 2.2, one has $e(u\cup cmu)\subset e(u\cup emu)=emu$, so $cmu=c(u\cup cmu)$; hence $C(u\cup cmu)X$ has the inversion property, so $u_i\subset u\cup cmu$. Since $m_* eu=em_* u$ ([R]$_1$, 4.6) and $m_* cu=m_* eu$, the proof of 2.3 (ii) is analogous to the above.

We remark that a characterization of the countable composition operator analogous to the ones found in 2.3 has not been found. The following is a weak result in this direction.

THEOREM 2.4. *For each uniform space uX,*
(i) $u=u_{\text{cc}}$ *if and only if* $cu=c\lambda cu$.
(ii) *If uX is an* RE *space, then* $u_{\text{cc}}=u\cup c\lambda cu$.

Part (i) is easily established using 2.1 (ii) and the definition of the countable composition property. To establish (ii), first note that $(cu)_{\text{cc}}=c\lambda cu$, so in general one has $u\cup c\lambda cu\subset u_{\text{cc}}$. To establish the converse inclusion, we show that if uX is an RE space, then $c(u\cup c\lambda cu)=c\lambda cu$. Using the proof technique of Lemma 2.2, let $\{B_n\}$ be a Euclidean member of $u\cup c\lambda cu$, and choose $\{U_s\}\in u$, $\{V_m\}\in c\lambda cu$, such that $\{U_s\cap V_m\}<\{B_n\}$. Define $\mathcal{S}_m$, $m=1, 2, \ldots$ as in 2.2. Now $\mathcal{S}_m/V_m$ is a finite dimensional uniform cover of V_m, so by ([I]$_1$, 4.22) there exists a countable finite dimensional $\mathcal{A}_m\in u$ such that $\mathcal{A}_m/V_m=\mathcal{S}_m/V_m$. Then $\{V_m\cap A: A\in\mathcal{A}_m\}$ is a member of $efu/c\lambda cu$ that refines $\{B_n\}$. By ([CI], 3.4), $efu=cu$ since uX is an RE space; hence $\{B_n\}\in cu/c\lambda cu\subset \lambda cu$, so $\{B_n\}\in c\lambda cu$, which completes the proof. Therefore $C(u\cup c\lambda cu)X$ has the countable composition property, so $u_{\text{cc}}\subset u\cup c\lambda cu$.

Theorem 2.3 shows that the inversion-closed (resp. regular ring) property is a very strong composition property: for each uniformly continuous function $X\xrightarrow{f} M$ to a metric space M and continuous (resp. Borel measurable) real-valued function on M, $g\circ f$ is uniformly continuous. This observation suggests a number of questions of the form: which operators correspond to composition properties such as those mentioned above, when we restrict consideration to the family of complete metric spaces, separable metric spaces, or complete separable metric spaces? One easily answers the question for separable metric spaces (or for $R-\{o\}$) — the composition property is still the inversion-closed (resp. regular ring) property. One also notes that the composition property obtained from the class of complete separable metric spaces, whose uniform structure is generated by uniformly continuous real-valued functions, is still countable composition (for each such metric space is a closed uniform subspace of a countable product of real lines). The following result treats the remaining two cases for complete metric spaces.

THEOREM 2.5. *For each uniform space uX, the smallest uniformity containing u which has the composition property associated with the family of complete metric (resp. complete separable metric) spaces is* $u\cup cm_1u$ *(resp.* $u\cup cm_0u$*).*

For convenience, call the composition property derived from the family of complete metric (resp. complete separable metric) space the *generalized* (resp. *separable*) *composition property* and denote the corresponding uniformity associated with u by u_{gc} (resp. u_{sc}). Also, recall that m_1 (resp. m_0) is the sub-M-fine (resp. locally sub-M-fine) operator defined in $[R]_1$ (resp. $[R]_2$).

We will first show that $u_{gc}=u\cup cm_1u$. Using 2.2 and the proof technique of 2.3, one shows that $cm_1u=c(u\cup cm_1u)$; since $C(m_1uX)$ has the generalized composition property, it follows that $u_{gc}\subset u\cup cm_1u$. Now assume that $\{V_n\}\in cm_1u$ is a countable star-bounded cover. From ($[R]_1$, 3.4) there exists a uniformly continuous mapping $uX\xrightarrow{f}M$ onto a metric space M and an open cover $\mathcal{O}$ of the completion of M, $\overline{M}$, such that $f^{-1}(\mathcal{O})<\{V_n\}$. Clearly one may assume that $\mathcal{O}=\{O_n\}$, where $f^{-1}(O_n)\subset V_n$. Now $\{O_n\cap M\}=\mathcal{U}$ is a star-bounded cover of M, belonging to α/M (α the fine uniformity on $\overline{M}$), so by ($[I]_1$, 4.21) there exists an isomorphic extension $\{Z_n\}$ of $\mathcal{U}$ over $\alpha\overline{M}$. Hence $\{\text{Int } Z_n\}$ is a countable star-bounded open cover of $\overline{M}$, so by ([CI], 3.4), $\{Z_n\}\in c\alpha$. Let $\overline{M}\xrightarrow{g}R^n$ be a continuous function to Euclidean n-space such that $g^{-1}\mathcal{S}(\varepsilon)<\{Z_n\}$ for some $\varepsilon>0$. By definition $g\circ f\in C(u_{gc}X, R^n)$ and $(g\circ f)^{-1}\mathcal{S}(\varepsilon)<\{V_n\}$; hence $\{V_n\}\in u_{gc}$, which completes the proof.

The proof of the other case is essentially the one given above, with m_1 replaced by m_0, except for the following modification. In the previous notation, if $\{V_n\}\in cm_0u$, then $\{V_n\}\in cm_1eu$ (for by $[R]_2$, proof of Theorem 1, $em_0u=m_0eu=m_1eu$). Hence there exists a uniformly continuous onto mapping $euX\xrightarrow{f}M$ satisfying the conditions of the above proof with the image M separable. The preceding proof now establishes that $\{V_n\}\in u_{sc}$ and completes the proof of 2.5.

COROLLARY 2.6. (i) *For each uniform space* uX, $u_{gc}=u_{sc}$ *if and only if* $m_1u=m_0u$.
(ii) *For each* RE *space* uX, $u_{sc}=u_{cc}$ *if and only if* $\lambda eu=\lambda cu$.

The proof techniques of (i) and (ii) are similar, so we will only indicate the proof of (i). If $u_{gc}=u_{sc}$, then 2.2 and 2.5 imply that $cm_1u=cm_0u$ (see proof of 2.5). Since em_0u and em_1u are separable locally fine uniformities, ($[H]_1$, 4.2) implies that $\lambda cm_0u=em_0u=\lambda cm_1u=em_1u$. Now from ($[R]_1$, 4.3) $m_1u=u/em_1u$ and from ($[R]_2$, Theorem 1) $m_0u=u/em_0u$, so we obtain $m_0u=m_1u$.

COROLLARY 2.7. (i) *If uX is an inverse limit of fine spaces, then* $u_{sc}=u_{cc}$.
(ii) *If uX is an* RE *space with a basis of finite dimensional uniform covers, then* $u_{sc}=u_{cc}$.
(iii) *If uX is an* RE *space with a basis of countable star-finite uniform covers, then* $u_{sc}=u_{cc}$.

Part (i) follows from the fact that the property $\lambda eu=\lambda cu$ (equivalently $eu\subset\lambda cu$) is preserved by inverse limits and $e\alpha=\lambda c\alpha$ for a fine uniformity α ($[H]_1$, 4.2). Part (ii) follows from the equation $efu=cu$, which is valid for RE spaces ([CI], 3.4).

Part (iii) follows from ([GI], 4.9), where $eu \subset cu/cu$ is established for spaces with a basis of countable star-finite uniform covers.

The reader should be aware that a space may have the countable composition property but not have the separable composition property. Let M be the hedgehog constructed by identifying a countably infinite number of unit intervals at zero, with the usual metric ϱ. M is a complete one-dimensional separable metric space having each uniformly continuous real-valued function bounded, so ϱM has the countable composition property. Since ϱ_{sc} contains $cm_0\varrho = c\alpha$ (by 2.5) and continuous real-valued functions on M need not be bounded, it follows that $\varrho_{sc} \neq \varrho_{cc} = \varrho$.

The preceding space also supplies a counter-example to the conjecture that countable composition is a hereditary property. For each $n=1, 2, \ldots,$ let $I_n = [1/2, 1/2+1/n^2]$ be the designated subset of the n^{th} spine of M and let $S = \bigcup I_n$. Define $S \xrightarrow{f} R$ by $f(x)=nx$ if $x \in I_n$; then f is uniformly continuous, but f^2 is not uniformly continuous, so $C(\varrho S)$ is not closed under finite composition. We note that the separable composition property is hereditary (see the proof of the following theorem) and implies the RE property; this is also the case for the generalized composition property. In view of 2.7 (ii) it is a reasonable conjecture that the separable composition property is equivalent to the countable composition property and the [PR] property. Recently it has been brought to the author's attention that J. Pelant has constructed an example of a locally sub-M-fine space which is not sub-M-fine; hence by 2.6 the generalized and separable composition properties do not coincide.

THEOREM 2.8. *The uniform spaces which have the generalized composition property are precisely the subspaces of the inversion-closed spaces.*

We will first show that the generalized composition property is hereditary. Suppose X has this property and $A \subset X$. Consider the following diagram, where f is uniformly continuous, M is a complete metric space, and g is continuous.

$$\begin{array}{ccccc} A & \xrightarrow{f} & M & \xrightarrow{g} & R \\ {\scriptstyle i}\downarrow & & & & \\ X & & & & \end{array}$$

By $([\mathrm{I}]_1$, 3.15) we may uniformly embed (in a necessarily closed manner) M in an injective metric space $\overline{M}$: $M \xrightarrow{e} \overline{M}$. By $([\mathrm{I}]_1$, 3.8) $\overline{M}$ is complete, so there exists a uniformly continuous function $X \xrightarrow{h} \overline{M}$ such that $h \circ i = e \circ f$. The Tietze Extension Theorem implies the existence of a continuous function $\overline{M} \xrightarrow{k} R$ such that $k \circ e = g$, so we obtain the following diagram:

$$\begin{array}{ccccc} A & \xrightarrow{f} & M & \xrightarrow{g} & R \\ {\scriptstyle i}\downarrow & & {\scriptstyle e}\downarrow & & \uparrow \\ X & \xrightarrow{h} & \overline{M} & \longrightarrow & {\scriptstyle k} \end{array}$$

By assumption on X, $k \circ h$ is uniformly continuous, so $k \circ h \circ i = (k \circ e) \circ f = g \circ f$ is also uniformly continuous, which completes the proof.

Now assume that uX has the generalized composition property and embed it as a uniform subspace of an injective uniform space vY. From $([\mathrm{R}]_1$, 3.2) $m_1 v = mv$, while from 2.3 (i) $v_i = v \cup cmv$. Then $v_i/X = v/X \cup (cmv)/X = u \cup (cm_1 v)/X = u \cup cm_1 u =$

$=u$ (by 2.5) since the operator m_1 preserves subspaces and sub-M-fine spaces are RE spaces ([GI], 4.12). Hence uX is a uniform subspace of $v_i Y$, which completes the proof.

NOTE. In connection with 2.8, we note that one may characterize those subspaces of an inversion-closed space which are also inversion-closed: $A \subset X$ inherits the inversion property if and only if A is completely separated (by a uniformly continuous real-valued function) from each disjoint uniform zero set. The proof of this fact is essentially found in $[H]_3$ (a direct proof was also supplied by the referee).

THEOREM 2.9. *The finite dimensional operator f is a categorical isomorphism from the full subcategory of, respectively, locally sub-M-fine spaces with bases of point finite uniform covers, sub-M-fine, or M-fine spaces onto the full subcategory of spaces with a basis of finite dimensional uniform covers which are*

(i) *closed under separable composition, generalized composition, or inversion, respectively, and*

(ii) *are locally sub-M-fine in the following restricted sense: if $\{C_m\}$ is a one-dimensional Euclidean uniform cover and $\mathcal{U}/C_m$ is a two-dimensional uniform cover for each m, then $\mathcal{U}$ is a uniform cover.*

Before proving 2.9 we comment that our result is modeled on the work done by A. W. Hager (in particular $[H]_1$, 4.2), which established the real-valued operator c as a categorical isomorphism from the full subcategory of separable locally fine (resp. separable metric-fine) spaces onto the full subcategory of spaces generated by uniformly continuous real-valued functions which have the countable composition (resp. inversion) property. Based on the work in $[R]_4$, the content of 2.9 is the characterization of the image categories by properties (i) and (ii).

It is clear that the image categories satisfy properties (i) and (ii). Assuming that uX satisfies condition (ii) we will show that the separable composition property implies $fu=fm_0u$. By ($[I]_1$, 4.25) each member of fm_0u may be refined by one of the form $\mathcal{V}=\bigcup_{i=1}^{n}\mathcal{V}_i$, where each $\mathcal{V}_i$ is uniformly discrete with respect to m_0u. For each i, define $B_i=\bigcup\{V\in\mathcal{V}_i\}$; then $\{B_i\}\in pm_0u$, so by 2.5 $\{B_i\}\in pu$. By ($[I]_1$, 4.22) we may choose a one-dimensional m_0u-uniform cover $\mathcal{V}_i'$ such that $\mathcal{V}_i'/B_i=\mathcal{V}_i$ (since $\mathcal{V}_i$ is uniformly discrete, it is a uniform cover of B_i, so this is possible). If we can show that each $\mathcal{V}_i'\in u$, then $\mathcal{V}=\{B_i\cap V\colon V\in\mathcal{V}_i',\ i=1,2,\dots,n\}\in u/pu=u$ (by $[I]_1$, 5.3 or an easy argument), which would conclude the proof.

By ($[R]_2$, Theorem 1) $m_0u=u/m_0eu$ and ([GI], 4.8 and proof of 4.9) show that each separable locally fine uniformity v satisfies $v\subset pv/cv$, so we obtain $m_0eu=$ $=em_0u\subset pm_0u/cm_0u$. Once again by 2.5, $cm_0u=cu$, so we now have $m_0u\subset$ $\subset u/(pu/cu)$. By a straightforward argument $u/(pu/cu)=(u/pu)/cu=u/cu$, so each $\mathcal{V}_i'$ is a one-dimensional member of u/cu. Suppose that $\{U_t\colon t<\tau\}$ is a one-dimensional member of u/cu. Choose $\{A_n\}\in cu$ and $\{C_s^n\}\in u$, $n=1,2,\dots$, such that $\{A_n\cap C_s^n\}<\{U_t\}$. Inductively define $D_0^n=\bigcup\mathcal{F}_0^n$, where $\mathcal{F}_0^n=\{C_s^n\colon A_n\cap C_s^n\subset U_0\},\dots,$ $D_t^n=\bigcup\mathcal{F}_t^n$, where $\mathcal{F}_t^n=\{C_s^n\colon A_n\cap C_s^n\subset U_t,\ C_s^n\notin\bigcup_{p<t}\mathcal{F}_p^n\}$, for $t<\tau$, $n=1,2,\dots$. Define $\mathcal{D}_n=\{D_t^n\colon t<\tau\}$, $n=1,2,\dots$. Since $\{C_s^n\}<\mathcal{D}_n$, $\mathcal{D}_n\in u$, while from the construction each $\mathcal{D}_{n/A_n}$ is a one-dimensional cover; hence by ($[I]_1$, 4.22) there exists a two-dimensional uniform cover $\mathcal{E}_n$ such that $\mathcal{E}_{n/A_n}=\mathcal{D}_{n/A_n}$. Finally, $\{A_n\cap E\colon E\in\mathcal{E}_n,\ n=1,2,\dots\}<\{U_t\}$, so in short hand notation $\{U_t\}\in 2-\dim u/cu$.

By ([I]$_1$, 4.25) $cu \subset (1-\dim cu)/pu$, so we obtain $\mathscr{V}_i' \in (2-\dim u/1-\dim cu)/pu$. Thus by assumption (ii), $\mathscr{V}_i' \in u/pu=u$.

We have now established that $u=fm_0u$ if uX is a member of the image category; hence by ([R]$_4$, 2.3), $m_0u=m_0(fm_0u)$ has a basis of point-finite uniform covers, so the operator f is onto (with respect to objects). Furthermore, f is one-to-one (with resp. to objects), for if $fu=fv$, where u, v are locally sub-M-fine uniformities with a basis of point-finite uniform covers, then by ([R]$_4$, 1.3) $m_0fu=u=m_0fv=v$. The reasoning used in the previous statements also shows that f is full and faithful; hence it is a categorical isomorphism.

Using results from [R]$_1$ the proofs of the other two cases are analogous to the previous proof.

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WACHSTUMSEIGENSCHAFTEN UND KONVERGENZFAKTOREN FÜR VERALLGEMEINERTE POTENZREIHENVERFAHREN DER LIMITIERUNG

Von

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Zu gegebenen komplexen Zahlen p_n $(n=0, 1, 2, \dots)$ und reellen Zahlen λ_n $(n=0, 1, 2, \dots)$ mit $0=\lambda_0<\lambda_1<\lambda_2<\dots$ bilden wir Limitierungsverfahren $V\{p_n, \lambda_n\}$, die das Abelverfahren sowie das Borelverfahren verallgemeinern und in folgender Weise definiert sind:

Die Dirichletsche Reihe $P(x)=\sum_{n=0}^{\infty} p_n x^{\lambda_n}$ sei konvergent für alle $x\in[0, \varrho)$ mit $0<\varrho\leqq\infty$. Ferner gelte $P(x)\neq 0$ für $\alpha\leqq x<\varrho$ mit einer Konstanten $\alpha<\varrho$. Wir nennen eine komplexwertige Folge $\{s_n\}$ dann $V\{p_n, \lambda_n\}$-limitierbar zum Wert c, wenn $\sum_{n=0}^{\infty} p_n s_n x^{\lambda_n}$ für alle $x\in[0, \varrho)$ konvergiert mit

$$\lim_{x\to\varrho-0}\frac{1}{P(x)}\sum_{n=0}^{\infty} p_n s_n x^{\lambda_n}=c.$$

Diese Verfahren werden im Falle $\lambda_n=n$ mitunter auch als Valironsche Verfahren bezeichnet [2; S.223], und wir erhalten für $p_n=1$, $\lambda_n=n$, $\varrho=1$ das Abelverfahren sowie für $p_n=\frac{1}{n!}$, $\lambda_n=n$, $\varrho=\infty$ das Borelverfahren.

Zur Summation unendlicher Reihen betrachten wir in Verallgemeinerung des Abelverfahrens noch eine weitere Klasse von Limitierungsverfahren $A\{\lambda_n\}$, wobei zu gegebenen reellen Zahlen λ_n $(n=0, 1, 2, \dots)$ mit $0=\lambda_0<\lambda_1<\lambda_2<\dots$ die Reihe $\sum a_n$ durch das Verfahren $A\{\lambda_n\}$ summierbar zum Wert c heißt, falls $\sum_{n=0}^{\infty} a_n x^{\lambda_n}$ für alle $x\in[0, 1)$ konvergiert mit

$$\lim_{x\to 1-0}\sum_{n=0}^{\infty} a_n x^{\lambda_n}=c.$$

Für $\lambda_n=n$ wurde bezüglich der Wirkfelder der Verfahren $V\{p_n, n\}$ in [1] und [6] die Frage untersucht, wie stark $V\{p_n, n\}$-limitierbare Folgen $\{s_n\}$ wachsen können, mit dem (kurz zusammengefaßten) Ergebnis, daß es keine eigentliche Wachstumsbeschränkung für diese s_n gibt, sondern daß eine Einschränkung der Größenordnung zulässiger s_n lediglich durch den Konvergenzradius von $\sum_{n=0}^{\infty} p_n x^n$ bedingt ist.

Zunächst zeigen wir in dieser Arbeit, daß die Ergebnisse aus [1] und [6] sowohl im Falle $\lambda_n=n$ wesentlich verschärft als auch für nichtganzzahlige λ_n verallgemeinert werden können. Das wichtigste Hilfsmittel dazu sind Ergebnisse des Verfassers über die asymptotische Approximation stetiger Funktionen durch Dirichletsche Reihen.

Wir betrachten neben der oben definierten Limitierung noch zusätzlich die Geschwindigkeit, mit der die Folgen $\{s_n\}$ durch ein Verfahren $V\{p_n, \lambda_n\}$ limitiert werden können und bezeichnen zu einer auf $[0, \varrho)$ positiven und stetigen Funktion h mit $C_V^{(h)}$ die Klasse aller Folgen $\{s_n\}$ mit

$$\frac{1}{P(x)} \sum_{n=0}^{\infty} p_n s_n x^{\lambda_n} = O(h(x)) \quad (x \to \varrho - 0).$$

Wir beweisen

SATZ 1. *Es sei ein Verfahren* $V\{p_n, \lambda_n\}$ *gegeben mit* $p_n \neq 0$ $(n=0, 1, 2, \dots)$ *und* $0=\lambda_0<\lambda_1<\dots<\lambda_n \to \infty$ $(n\to\infty)$, $\lambda_{n+1}-\lambda_n \geqq q > 0$, $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$. *Ferner seien* n_i $(i=1, 2, 3, \dots)$ *natürliche Zahlen mit*

$$\sum_{1 \leqq n \neq n_i (i=1,2,\dots)} \frac{1}{\lambda_n} = \infty, \quad n_{i+1} > n_i. \tag{1}$$

Dann gibt es zu jeder auf $[0, \varrho)$ *positiven, stetigen Funktion h und zu beliebig gegebenen komplexen Zahlen* w_n $(n=1, 2, 3, \dots)$ *mit*

$$\overline{\lim_{n\to\infty}} |p_n w_n|^{1/\lambda_n} \leqq \frac{1}{\varrho} \tag{2}$$

eine Folge $\{s_n\} \in C_V^{(h)}$ *so, daß für alle* $i=1, 2, 3, \dots$ *gilt*

$$s_{n_i} = w_{n_i}. \tag{3}$$

Es sei darauf hingewiesen, daß die Bedingung (1) aus Satz 1 im Falle $\lambda_n = n$ bereits erfüllt ist, wenn $n_{i+1} > n_i + 1$ gilt. Für die Verfahren $A\{\lambda_n\}$ erhalten wir einen entsprechenden Satz, wobei $C_A^{(h)}$ zu einer auf $[0, 1)$ stetigen Funktion h die Klasse aller $A\{\lambda_n\}$-summierbaren Reihen $\sum a_n$ mit

$$\sum_{n=0}^{\infty} a_n x^{\lambda_n} = O(h(x)) \quad (x \to 1-0)$$

bezeichne.

SATZ 2. *Es sei ein Verfahren* $A\{\lambda_n\}$ *gegeben mit Zahlen* λ_n, *die die Voraussetzungen von Satz* 1 *erfüllen. Ferner seien* n_i $(i=1, 2, 3, \dots)$ *natürliche Zahlen mit der Eigenschaft* (1). *Dann gibt es zu jeder auf* $[0, 1)$ *positiven, stetigen Funktion h und zu beliebig gegebenen komplexen Zahlen* w_n $(n=1, 2, 3, \dots)$ *mit*

$$\overline{\lim_{n\to\infty}} |w_n|^{1/\lambda_n} \leqq 1$$

eine Reihe $\sum a_n \in C_A^{(h)}$ *so, daß für alle* $i=1, 2, 3, \dots$ *gilt*

$$a_{n_i} = w_{n_i}.$$

Als Anwendung von Satz 1 und Satz 2 lassen sich die Konvergenzfaktoren der Verfahren $V\{p_n, \lambda_n\}$ und $A\{\lambda_n\}$ genau charakterisieren. Dabei nennen wir die Zahlen α_n $(n=0, 1, 2, \dots)$ Konvergenzfaktoren der Verfahren $V\{p_n, \lambda_n\}$ bzw.

der Verfahren $A\{\lambda_n\}$, falls für alle $V\{p_n, \lambda_n\}$-limitierbaren Folgen $\{s_n\}$ bzw. für alle $A\{\lambda_n\}$-summierbaren Reihen $\sum a_n$ stets die Reihe $\sum \alpha_n s_n$ bzw. die Reihe $\sum \alpha_n a_n$ konvergiert.

Im Falle $\lambda_n=n$ wurde in [6] der folgende Satz bewiesen, wobei $C_V^{(0)}$ die Klass e aller Folgen $\{s_n\}$ bezeichne, die $V\{p_n, n\}$-limitierbar zum Wert 0 sind.

SATZ 3. *Für das Verfahren* $V\{p_n, n\}$ *sei* $p_n \neq 0$ $(n=0, 1, 2, \dots)$, *und es gelte* $\lim\limits_{x\to\varrho-0} \frac{(\varrho-x)^\beta}{P(x)}=0$ *im Falle* $\varrho<\infty$ *bzw.* $\lim\limits_{x\to\infty} \frac{e^{-x\beta}}{P(x)}=0$ *im Falle* $\varrho=\infty$ *mit einer reellen Konstanten* β. *Ist dann die Folge* $\{\alpha_n s_n\}$ *für jede Folge* $\{s_n\}\in C_V^{(0)}$ *beschränkt, so gilt*

$$|\alpha_n| \leqq R|p_n| r^n \quad (n = 0, 1, 2, \dots)$$

mit Konstanten $R<\infty$, $r<\varrho$.

Bezüglich der Klasse $A\{\lambda_n\}$ wurde in [5] im Falle $\lambda_n=n$, d.h. speziell für das Abelverfahren A der folgende Satz bewiesen.

SATZ 4. *Ist die Reihe* $\sum \alpha_n a_n$ *konvergent für alle A-summierbaren Reihen* $\sum a_n$, *so gilt*

$$|\alpha_n| \leqq Rr^n \quad (n = 0, 1, 2, \dots)$$

mit Konstanten $R<\infty$, $r<1$.

Zur Verschärfung und Verallgemeinerung von Satz 3 beweisen wir

SATZ 5. *Es sei ein Verfahren* $V\{p_n, \lambda_n\}$ *gegeben, wobei* $p_n\neq 0$ $(n=0, 1, 2, \dots)$ *und* $0=\lambda_0<\lambda_1<\lambda_2<\dots<\lambda_n\to\infty$ $(n\to\infty)$, $\lambda_{n+1}-\lambda_n\geqq q>0$, $\sum\limits_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty$ *gelte. Ferner sei h eine auf* $[0, \varrho)$ *positive und stetige Funktion. Ist die Folge* $\{\alpha_n s_n\}$ *für alle* $\{s_n\}\in C_V^{(h)}$ *beschränkt, dann gilt*

$$|\alpha_n| \leqq R|p_n| r^{\lambda_n} \quad (n = 0, 1, 2, \dots) \tag{4}$$

mit Konstanten $R<\infty$, $r<\varrho$.

Entsprechend ergibt sich für die Klasse $A\{\lambda_n\}$ als Verschärfung von Satz 4 der folgende

SATZ 6. *Für das Verfahren* $A\{\lambda_n\}$ *gelte* $0=\lambda_0<\lambda_1<\lambda_2<\dots<\lambda_n\to\infty$ $(n\to\infty)$ *mit* $\lambda_{n+1}-\lambda_n\geqq q>0$ *und* $\sum\limits_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty$. *Ferner sei h eine auf* $[0, 1)$ *positive und stetige Funktion. Ist die Folge* $\{\alpha_n a_n\}$ *für alle Reihen* $\sum a_n\in C_A^{(h)}$ *beschränkt, dann gilt*

$$|\alpha_n| \leqq Rr^{\lambda_n} \quad (n = 0, 1, 2, \dots)$$

mit Konstanten $R<\infty$, $r<1$.

Zum Beweis von Satz 1 und Satz 2 benutzen wir als Hilfsmittel

SATZ 7. *Es sei* $0=\lambda_0<\lambda_1<\lambda_2<\dots<\lambda_n\to\infty$ $(n\to\infty)$ *mit* $\lambda_{n+1}-\lambda_n\geqq q>0$ *und* $\sum\limits_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty$. *Dann gibt es zu jeder auf* $[0, \varrho)$ $(0<\varrho\leqq\infty)$ *stetigen Funktion f und zu* *jeder auf* $[0, \varrho)$ *positiven, stetigen Funktion h eine für alle* $x \in [0, \varrho)$ *absolut konvergente Dirichletsche Reihe* $\sum a_n x^{\lambda_n}$ *so, daß gilt*

$$\left| f(x) - \sum_{n=0}^{\infty} a_n x^{\lambda_n} \right| < h(x) \quad (x \in [0, \varrho)).$$

Satz 7 wurde für $\varrho < \infty$ in [3; S.17] und für den Fall $\varrho = \infty$ in [4] bewiesen.

Beweis zu Satz 1. Zu gegebenen Zahlen w_n mit der Eigenschaft (2) setzen wir

$$f(x) = \sum_{i=1}^{\infty} p_{n_i} w_{n_i} x^{\lambda_{n_i}}.$$

Diese Reihe konvergiert dann wegen (2) für alle $x \in [0, \varrho)$ absolut, da die Bedingung $\lambda_{n+1} - \lambda_n \geqq q > 0$ die Abschätzung $\lambda_n \geqq kn$ mit positivem k impliziert.

Nach Satz 7 können wir bei Beachtung von (1) eine absolut konvergente Dirichletsche Reihe der Gestalt $g(x) = \sum_{0 < n \neq n_i}^{\infty} a_n x^{\lambda_n}$ so wählen, daß

$$f(x) - g(x) = O\big(P(x) h(x)\big) \tag{5}$$

für $x \to \varrho - 0$ ist. Die Folge $\{s_n\}$ mit

$$s_n = \begin{cases} w_{n_i} & (n = n_i, \ i = 1, 2, 3, \ldots) \\ -\dfrac{a_n}{p_n} & (n \neq n_i, \ i = 1, 2, 3, \ldots) \end{cases}$$

erfüllt offensichtlich (3), und es gilt $\{s_n\} \in C_V^{(h)}$ nach (5), womit Satz 1 bewiesen ist.

Beweis zu Satz 5. Gilt (4) unter den Voraussetzungen von Satz 5 nicht, dann können wir zu beliebigen Zahlen r_i $(i = 1, 2, 3, \ldots)$ mit $0 < r_i < \varrho$ und $r_i \to \varrho$ $(i \to \infty)$ natürliche Zahlen n_i so bestimmen, daß

$$|\alpha_{n_i}| \geqq |p_{n_i}| r_i^{\lambda_{n_i}} \tag{6}$$

für $i = 1, 2, 3, \ldots$ ist. Hierbei können wir die n_i so schnell wachsend wählen, daß die Bedingung (1) erfüllt ist. Wir setzen

$$w_n = \begin{cases} \dfrac{n}{|p_n| r_i^{\lambda_n}} & (n = n_i; \ i = 1, 2, 3, \ldots) \\ 0 \quad (n \neq n_i; \ i = 1, 2, 3, \ldots). \end{cases} \tag{7}$$

Diese Zahlen w_n erfüllen die Bedingung (2), da $\lambda_n \geqq kn$ mit einem $k > 0$ ist. Daher gibt es nach Satz 1 eine Folge $\{s_n\} \in C_V^{(h)}$ mit $s_n = w_n$ $(n = n_i; \ i = 1, 2, 3, \ldots)$, und wir erhalten aus (6) und (7)

$$\sup_n |\alpha_n s_n| \geqq \sup_i |p_{n_i} r_i^{\lambda_{n_i}} w_{n_i}| = \sup_i n_i = \infty,$$

was der in Satz 5 vorausgesetzten Beschränktheit von $\{\alpha_n s_n\}$ widerspricht. Damit ist Satz 5 bewiesen.

Die Beweise zu den Sätzen 2 und 6 verlaufen ganz analog zu den Beweisen von Satz 1 und Satz 5.

Es sei noch darauf hingewiesen, daß sich mit Hilfe von Satz 7 ganz entsprechende Sätze für die absolute Limitierung dieser Verfahren herleiten lassen.

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DECOMPOSITIONS OF INJECTIVE MODULES OVER NON-NOETHERIAN RINGS

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About the decompositions of injective modules, E. MATLIS [9], R. B. WARFIELD, JR [14] and I. BECK [2] have given many interesting results. Among others, E. MATLIS has completely determined in [9] the structure of an indecomposable injective module and the indecomposable decomposition of an injective module over a noetherian ring. Moreover, E. MATLIS [9], J. FORT [5] and R. B. WARFIELD, JR [14] proved that if the indecomposable decomposition of an injective module exists, then it is uniquely determined up to an isomorphism. In this paper, we are mainly interested in some full subcategories of the category of unitary modules over a non-noetherian commutative ring with unity such that every injective module in those categories is decomposable into a direct sum of indecomposable injective modules.

Let R be a commutative ring with unity, Spec (R) the set of all prime ideals in R and let $F(R)$ be the set of all prime ideals P in R such that the localization R_P of R by P is noetherian.[1] Let P be an element of $F(R)$. We shall denote by $G(P)$ the *generalization* of P, i.e., the set of all prime ideals Q of R such that $Q \subseteqq P$. A non-empty subset of $F(R)$ is said to be *of open type* if for any P in X we have $G(P) \subseteqq X$. Let $\mathfrak{M}$ be the category of unitary R-modules and let X be a subset of $F(R)$ of open type. We shall denote by $\mathfrak{N}'[X]$ the full subcategory of $\mathfrak{M}$ consisting of M in $\mathfrak{M}$ such that M_P $(=M \otimes_R R_P)=0$ for every P in X. $\mathfrak{N}[X]$ stands for the full subcategory of $\mathfrak{M}$ consisting of M in $\mathfrak{M}$ such that $\mathrm{Hom}_R(N, M)=0$ for every N in $\mathfrak{N}'[X]$. We shall say that R satisfies the condition $H[X]$ if for any proper ideal A of R such that

$$A = \bigcap_{\substack{P \in X \\ P \supseteqq A}} (AR_P \cap R),$$

we have that $\mathrm{Ass}\,(R/A) \cap X \neq \emptyset$.[2]

If R is a noetherian ring, R satisfies the condition $H[X]$ for any $X \subseteqq F(R)$ of open type. In particular, R satisfies the condition $H[\mathrm{Spec}\,(R)]$.

In §1, we first study the structures of injective modules in $\mathfrak{N}[X]$ when R satisfies the condition $H[X]$, and we shall also give a condition for R to satisfy the condition $H[X]$.

[1] For any multiplicatively closed set C in a commutative ring R, we shall denote by R_C the ring of quotients of R with respect to C and for each prime ideal P of R, we shall use R_P instead of $R_{(R-P)}$ (this is called the localization of R by P).

[2] For any R-module M, we shall denote by $\mathrm{Ass}\,(M)$ the set of all prime ideals P in R such that there is an $x \in M$ with $\mathrm{Ann}_R(x)=P$, or equivalently, there is a monomorphism of the quotient module R/P of R by P into M.

Let R be a commutative ring with unity and X a subset of $F(R)$ of open type. Let us consider the following conditions for $\mathfrak{N}[X]$.[3]

(I) Every injective R-module in $\mathfrak{N}[X]$ is decomposable into a direct sum of indecomposable injective R-modules in $\mathfrak{N}[X]$ and every indecomposable injective R-module in $\mathfrak{N}[X]$ is isomorphic to the injective hull of R/P for some P in X.

(II) Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be any family of injective R-modules in $\mathfrak{N}[X]$. Then, $\sum_{\lambda \in \Lambda} \oplus M_\lambda$ is injective as an R-module.

In §2, we shall investigate a pair of a ring R and a subset X of $F(R)$ of open type such that the conditions (I) and (II) hold in $\mathfrak{N}[X]$.

1. The basic properties

Throughout this paper, we assume that a ring is commutative and has a unit, a module is unitary and assume that $F(R)$ is not empty for any ring R considered in this paper. Let R be a ring and M an R-module. Then, it is well known that there exists a minimal injective R-module containing M and it is an essential extension of M. This is uniquely determined by M up to isomorphisms, and is called *the injective hull* of M (denoted by $E_R(M)$) (see [4], [7], [13] for details).

Let R be a ring and X a subset of $F(R)$ of open type. Then it is easy to see that $\mathfrak{N}[X]$ contains a non-zero R-module. Moreover, if an R-module M belongs to $\mathfrak{N}[X]$, then every submodule of M belongs to $\mathfrak{N}[X]$. In this section, let us denote by X an arbitrary subset of $F(R)$ of open type.

PROPOSITION 1.1. *Let R be a ring and M an R-module. Then M belongs to $\mathfrak{N}[X]$ if and only if $E_R(M)$ belongs to $\mathfrak{N}[X]$. Therefore, if $M \in \mathfrak{N}[X]$, every essential extension of M is also in $\mathfrak{N}[X]$.*

PROOF. If $E_R(M)$ belongs to $\mathfrak{N}[X]$, then its submodule M belongs to $\mathfrak{N}[X]$. Conversely, assume that M belongs to $\mathfrak{N}[X]$. For any submodule $M' \neq 0$ of $E_R(M)$, $M' \cap M \neq 0$. Furthermore, if $N \in \mathfrak{N}'[X]$, every submodule of N belongs to $\mathfrak{N}'[X]$. Thus any non-zero submodule of $E_R(M)$ does not belong to $\mathfrak{N}'[X]$, and so $E_R(M)$ belongs to $\mathfrak{N}[X]$.

LEMMA 1.1 (Lemma 1.2 in [1] or Corollary 1 of Proposition 3 in [6]). *Let R be a ring and C a multiplicatively closed set in R. Then for any R_C-module M, M is injective as an R-module if and only if M is injective as an R_C-module, where R_C is the ring of quotients of R with respect to C.*

PROPOSITION 1.2. *Let R be a ring, P an element in $F(R)$ and let $\{E_\lambda\}_{\lambda \in \Lambda}$ be any copies of injective R-modules of the form $E_\lambda = E_R(R/P)$ for each λ in Λ. Then $\sum_{\lambda \in \Lambda} \oplus E_\lambda$ is injective as an R-module.*

PROOF. By Lemma 2 of [4], $E_R(R/P) \cong E_{R_P}(R_P/PR_P)$ as both an R-module and an R_P-module. On the other hand, R_P is a noetherian ring and hence $\sum_{\lambda \in \Lambda} \oplus E_\lambda$

[3] The conditions (I) and (II) hold in $\mathfrak{N}[X]$ for every subset X of $F(R)$ of open type when R is a noetherian ring.

is injective as an R_P-module. Thus, by Lemma 1.1 $\sum_{\lambda\in\Lambda}\oplus E_\lambda$ is injective as an R-module.

LEMMA 1.2. *Let R be a ring satisfying the condition $H[X]$ and M a non-zero R-module belonging to $\mathfrak{N}[X]$. Then* $\mathrm{Ass}(M)\cap X\neq\emptyset$.

PROOF. Let $0\neq x$ be an element of M. Let $\mathrm{Ann}_R(x)=A$. Then, it is easily seen that

$$A=\bigcap_{\substack{P\in X\\ P\supseteqq A}}(AR_P\cap R).$$

Since R satisfies the condition $H[X]$, $\mathrm{Ass}(R/A)\cap X\neq\emptyset$. As $R/A\cong Rx\subseteqq M$, we have $\mathrm{Ass}(R/A)=\mathrm{Ass}(Rx)\subseteqq\mathrm{Ass}(M)$. Thus $\mathrm{Ass}(M)\cap X\neq\emptyset$.

THEOREM 1.1. *Let R be a ring. Then, the following conditions are equivalent.*
(1) *R satisfies the condition $H[X]$.*
(2) *Any non-zero injective R-module belonging to $\mathfrak{N}[X]$ contains an indecomposable injective R-module isomorphic to $E_R(R/P)$ for some $P\in X$.*

PROOF. (1)→(2). Let M be a non-zero injective R-module in $\mathfrak{N}[X]$. Then, by Lemma 1.2 there exists an element P in X such that $M\supseteqq M'\cong R/P$. Since M is injective, $M\supseteqq E_R(M')\cong E_R(R/P)$, which is an indecomposable injective R-module by Theorem 2.4 of [9]. Hence M contains an indecomposable injective R-module isomorphic to $E_R(R/P)$, $P\in X$.
(2)→(1). Let A be any proper ideal in R such that

$$A=\bigcap_{\substack{P\in X\\ P\supseteqq A}}(AR_P\cap R).$$

Then, it is easy to see that $R/A\in\mathfrak{N}[X]$ and by Proposition 1.1 $E_R(R/A)\in\mathfrak{N}[X]$. Thus, by the assumption, $E_R(R/A)$ contains a submodule E' isomorphic to $E_R(R/P)$ for some $P\in X$. Thus, as $\mathrm{Ass}(R/A)=\mathrm{Ass}(E_R(R/A))$, we have that $\mathrm{Ass}(R/A)=$ $=\mathrm{Ass}(E_R(R/A))\supseteqq\mathrm{Ass}(E')=\mathrm{Ass}(E_R(R/P))\ni P$, and so, $\mathrm{Ass}(R/A)\cap X\neq\emptyset$.

PROPOSITION 1.3.[4] *Let R be a ring with the condition $H[X]$. Then, any injective R-module belonging to $\mathfrak{N}[X]$ is the injective hull of a direct sum of indecomposable injective R-modules in $\mathfrak{N}[X]$, each of which is isomorphic to $E_R(R/P)$ for some P in X.*

PROOF. Let M be any injective R-module belonging to $\mathfrak{N}[X]$ and $\{E_\alpha\}_{\alpha\in\Omega}$ the set of all submodules of M which are indecomposable injective R-modules. Now, set $K=\{L\subseteqq\Omega\mid\sum_{\alpha\in\Omega}E_\alpha$ is a direct sum$\}$. Then, by Theorem 1.1, K is not empty if M is not zero. By Zorn's Lemma, there exists a maximal element L_0 in K with respect to the order by the canonical inclusion. Then, we infer that $M=E_R(M_0)$, where $M_0=\sum_{\alpha\in L_0}\oplus E_\alpha$. For, if $E_R(M_0)\subsetneqq M$, then $E_R(M_0)$ is a proper direct summand of M, that is, $M=E_R(M_0)\oplus M'$ and M' is a non-zero injective R-module belonging to $\mathfrak{N}[X]$. Thus by Theorem 1.1, M' contains an indecomposable injective R-module, and this contradicts the maximality of L_0. Hence $M=E_R(M_0)$.

[4] This follows also from Theorem 3 of [14] using Theorem 1.1.

LEMMA 1.3. *Let P be a prime ideal in a ring R. Then, for any element r in $R-P$, T_r: $E_R(R/P)\to E_R(R/P)$ defined by $T_r(x)=rx$ for x in $E_R(R/P)$ is an automorphism of $E_R(R/P)$.*

PROOF. Since T_r: $R/P\to R/P$ is a monomorphism and $E_R(R/P)$ is an essential extension of R/P, T_r is monomorphic. On the other hand, $E_R(R/P)$ is an indecomposable injective R-module by Theorem 2.4 of [9], and so T_r is an automorphism.

LEMMA 1.4. *Let P and P' be prime ideals in a ring R such that $P\subseteqq P'$. Then, we can regard $E_R(R/P)$ as an $R_{P'}$-module and it is an indecomposable injective $R_{P'}$-module.*

PROOF. By Lemma 1.3, for each element r in $R-P'$ T_r: $E_R(R/P)\to E_R(R/P)$ is an automorphism and hence $E_R(R/P)$ can be regarded as an $R_{P'}$-module. Moreover, by Lemma 1.1 $E_R(R/P)$ is injective as an $R_{P'}$-module and the indecomposablity of $E_R(R/P)$ as an $R_{P'}$-module follows from that of $E_R(R/P)$ as an R-module.

Let R be a ring satisfying the condition $H[X]$ and M an injective R-module belonging to $\mathfrak{N}[X]$. Then by Theorem 1.1 Ass $(M)\neq\emptyset$ if $M\neq 0$ and by Proposition 1.3, M can be expressed as follows:

$$\text{(a)}\qquad M=E_R\Big(\sum_{\alpha\in\Omega}\oplus E_\alpha\Big),$$

where each of $E_\alpha(\alpha\in\Omega)$ is isomorphic to $E_R(R/P_\alpha)$ for some prime P_α in X; let us call such an expression of M, simply, an *expression* of M. For each P in Ass (M), let us set

$$M(P)=E_R\Big(\sum_{\substack{\alpha\in\Omega\\ P_\alpha=P}}\oplus E_\alpha\Big),\quad M[P]=E_R\Big(\sum_{\substack{P'\in \mathrm{Ass}(M)\\ P'\subseteqq P}}\oplus M(P')\Big).$$

We shall then call $M(P)$ the *P-component* of M and $M[P]$ the *local component of M at P*, with respect to the expression (a) of M.[5]

By Proposition 1.2,

$$M(P)=\sum_{\substack{\alpha\in\Omega\\ P_\alpha=P}}\oplus E_\alpha$$

and it is a direct summand of M. By Lemma 1.4,

$$M'=\sum_{\substack{P'\in \mathrm{Ass}(M)\\ P'\subseteqq P}}\oplus M(P')$$

can be regarded as an R_P-module and for each $\alpha\in\Omega$, E_α, which appears in M', is injective as an R_P-module. Thus M' is injective as an R_P-module because R_P is a noetherian ring. Therefore, by Lemma 1.1, M' is injective as an R-module, that is,

$$M[P]=\sum_{\substack{P'\in \mathrm{Ass}(M)\\ P'\subseteqq P}}\oplus M(P')=\sum_{\substack{\alpha\in\Omega\\ P_\alpha\subseteqq P}}\oplus E_\alpha.$$

[5] For convenience, let us set $M(P)=0$ and $M[P]=0$ when $P\in X-\mathrm{Ass}(M)$ and $G(P)\cap \mathrm{Ass}(M)=\emptyset$, respectively.

By Corollary 4.2 of [14] or by Theorem 6 of [4], the P-components of M with respect to any two expressions of M are isomorphic and the local components of M at P with respect to any two expressions of M are isomorphic. From these facts, we yield the following proposition.

PROPOSITION 1.4. *Let R be a ring satisfying the condition $H[X]$ and M an injective R-module belonging to $\mathfrak{N}[X]$. Then we obtain the followings:*

(1) *For each element P in* $\operatorname{Ass}(M)$, *the P-component of M and the local component of M at P, with respect to an expression of M, can be written as direct sums of indecomposable injective R-modules belonging to $\mathfrak{N}[X]$.*

(2) *For each element P in* $\operatorname{Ass}(M)$, *the P-components of M with respect to any two expressions of M are isomorphic and the local components of M at P with respect to any two expressions of M are isomorphic.*

(3) *Let $M(P')$ $(P' \in \operatorname{Ass}(M))$ be the P'-components of M with respect to an expression of M. We then obtain*

$$M = E_R\Big(\sum_{P' \in \operatorname{Ass}(M)} \oplus M(P')\Big).$$

Let R be a ring with the condition $H[X]$ and M an injective R-module belonging to $\mathfrak{N}[X]$. For an element P in X, let us set $\{M_\alpha\}_{\alpha\in\Omega}$ be the set of all submodules of M each of which is isomorphic to $E_R(R/P')$ for some P' in X which is contained in P and set $V=\big\{L \mid L\subseteq\Omega,\ \sum_{\alpha\in L} M_\alpha$ is a direct sum$\big\}$. By Zorn's Lemma, there is a maximal element L_0 in V with respect to the order by the canonical inclusion. Then, let us call $M'=\sum_{\alpha\in L_0}\oplus M_\alpha$ a *local component of M at P*. By Lemma 1.4 and by Lemma 1.1, M' is injective as an R-module. Furthermore, it is easy to see that $G(P)\cap \operatorname{Ass}(M'')= =\varnothing$, where $M=M'\oplus M''$. Since M'' is injective as an R-module and belongs to $\mathfrak{N}[X]$, by Proposition 1.3, M'' can be expressed as

$$M'' = E_R\Big(\sum_{\lambda\in\Lambda}\oplus E_\lambda\Big),$$

where E_λ $(\lambda\in\Lambda)$ are indecomposable injective R-modules belonging to $\mathfrak{N}[X]$. Thus

$$M = \Big(\sum_{\alpha\in L_0}\oplus M_\alpha\Big)\oplus E_R\Big(\sum_{\lambda\in\Lambda}\oplus E_\lambda\Big)$$

is an expression of M and the local component $M[P]$ of M at P with respect to this expression of M is equal to M'. Thus, by Proposition 1.4 any two local components of M at P are isomorphic (from now on, we also denote by $M[P]$ a local component of M at P). From these facts, we yield the following proposition.

PROPOSITION 1.5. *Let R be a ring satisfying the condition $H[X]$ and M an injective R-module in $\mathfrak{N}[X]$. Then*

(1) *for each element P in* $\operatorname{Ass}(M)$, *a local component of M at P is a direct summand of M and it can be expressed as a direct sum of indecomposable injective R-modules each of which is isomorphic to $E_R(R/P')$ for some $P'\in X$ which is contained in P, and*

(2) *any two local components of M at P are isomorphic.*

PROPOSITION 1.6. *Let R be a ring. Then,* $\mathfrak{N}[X]$ *is closed under the localization in X.*

PROOF. Let M be any R-module in $\mathfrak{N}[X]$ and assume that the localization M_P of M by P does not belong to $\mathfrak{N}[X]$ for some P in X. Then there exists at least one R-submodule $N\neq 0$ in M_P which belongs to $\mathfrak{N}'[X]$. This implies that $N_{P'}=0$ for all P' in X. In particular, $N_P=0$ and this is impossible because N is a non-zero subset of an R_P-module M_P. Therefore, M_P also belongs to $\mathfrak{N}[X]$ for all $P\in X$.

REMARK. In general, $\mathfrak{N}[X]$ is not closed under the localization in $F(R)$.

PROPOSITION 1.7. *Let R be a ring and P an element in* Spec (R). *Then the following statements are equivalent.*

(1) R_P *is a noetherian ring.*

(2) *Let* $\{E_\lambda\}_{\lambda\in\Lambda}$ *be any copies of* $E_R(R/P)$. *Then,* $\sum_{\lambda\in\Lambda}\oplus E_\lambda$ *is also injective as an R-module.*

(3) *Let* $\{E_i\}_{i=1,2,3,\dots}$ *be countably infinite copies of* $E_R(R/P)$. *Then,* $\sum_i\oplus E_i$ *is injective as an R-module.*

PROOF. (1)→(2) follows from Proposition 1.2 and (2)→(3) is trivial. Let us show (3)→(1). Assume that R_P is not a noetherian ring. Then there exists an ideal A in R_P such that $A=(a_1, a_2, a_3, \dots)$ and $a_j\notin(a_1, a_2, \dots, a_{j-1})$ for all $j>1$. Let f_i be an R_P-homomorphism of $(a_1, a_2, \dots, a_i)$ into E_i defined by $f_i(a_i)=\bar{1}_i$ (= the canonical image of 1 in $(R_P/PR_P)\subseteqq E_i$) and $f_i(a_j)=0$ for all $j<i$ (this is well defined because $(a_1, a_2, \dots, a_{i-1})\ni ra_i$ $(r\in R_P)$ implies $r\in PR_P$), and let g_i be an R_P-homomorphism of R_P into E_i which is an extension of f_i and set $g_i(1)=x_i$ for all i. Let f be a mapping of A into $E=\sum_i\oplus E_i$ defined by

$$f(r)=\sum_{j=1}^{n}\left(\sum_{i=j}^{n} r_i a_i\right)x_j \quad \text{for} \quad r=\sum_{i=1}^{n} r_i a_i\in A.$$

Then, it is easy to see that f can be regarded as an R_P-homomorphism of A into E because $a_i x_j=0$ for all $i<j$ and $\sum_{j=1}^{n}\left(\sum_{i=j}^{n} r_i a_i\right)x_j=r\left(\sum_{i=1}^{n} x_i\right)$. But, by construction, f can not be extended to an R_P-homomorphism of R_P into E. Thus, E is not injective as an R-module by Lemma 1.1. This contradicts the hypothesis. The proof is complete. ∎

REMARK. Let R be a ring, X a subset of $F(R)$ of open type and let P be a prime ideal of R. Then, by Proposition 1.7, $\sum_{\alpha\in\Omega}\oplus E_\alpha$ $(E_\alpha=E_R(R/P)$ and Ω: any index set), being injective, implies that R_P is noetherian, that is, $P\in F(R)$. Hence, for decompositions of injective R-modules into indecomposable injective R-modules of the form $E_R(R/P')$ $(P'\in\operatorname{Spec}(R))$, it is natural to restrict our observation to injective R-modules belonging to $\mathfrak{N}[X]$, setting up with R satisfying the condition $H[X]$ by Theorem 1.1.

2. Some conditions for the decomposablity

Let R be a ring and X a non-empty subset of $F(R)$. The we denote by $\bar{X}$ the set of all elements P in $F(R)$ such that P is contained in some element in X, i.e., $\bar{X}=\bigcup_{P\in X} G(P)$, and let us call it the *open closure* of X. If $X=\{P_1, P_2, ..., P_n\}$, we denote $\bar{X}$ by $[P_1, P_2, ..., P_n]$. Let P_0 be an element in a subset X of $F(R)$. Then, we call P_0 a *maximal element* in X in case if $P\supseteqq P_0$ for $P\in X$, then $P=P_0$. Now, if X is a finite set, $\bar{X}$ contains only a finite number of maximal elements in $\bar{X}$ and any element of $\bar{X}$ is contained in some maximal element in $\bar{X}$. A subset X of $F(R)$ is said to be *of finite type* if there exists a finite subset X_0 in $F(R)$ such that $\bar{X}=\bar{X}_0$.

THEOREM 2.1. *Let R be a ring satisfying the condition $H[\bar{X}]$ for a subset X of $F(R)$ and M an injective R-module belonging to $\mathfrak{N}[\bar{X}]$. Then, if* Ass (M) *is contained in a subset of $\bar{X}$ being of finite type, M can be expressed as a direct sum of indecomposable injective R-modules belonging to $\mathfrak{N}[\bar{X}]$ each of which is isomorphic to $E_R(R/P)$ for some P in $\bar{X}$.*

PROOF. Let Y be a subset of $\bar{X}$ of finite type such that $Y\supseteqq \text{Ass}(M)$ and let $\{P_1, P_2, ..., P_n\}$ be the set of all maximal elements in Y. Then, by Proposition 1.5 a local component $M[P_1]$ of M at P_1 is injective as an R-module and it can be written as a direct sum of indecomposable injective R-modules each of which is isomorphic to $E_R(R/P)$ for some P in $\bar{X}$. Now, set $M=M[P_1]\oplus M_1$. Then, M_1 is an injective R-module and belongs to $\mathfrak{N}[\bar{X}]$. Moreover, Ass (M_1) is contained in the open closure $[P_2, P_3, ..., P_n]$ of $\{P_2, P_3, ..., P_n\}$. Thus, by Proposition 1.5 a local component $M_1[P_2]$ of M_1 at P_2 is injective as an R-module and we yield that

$$M = M[P_1]\oplus M_1[P_2]\oplus M_2.$$

Continuing as above, finally we obtain that

$$M = M[P_1]\oplus M_1[P_2]\oplus \ldots \oplus M_{n-1}[P_n]$$

and $M_i[P_{i+1}]$ can be expressed as a direct sum of indecomposable injective R-modules belonging to $\mathfrak{N}[\bar{X}]$ each of which is isomorphic to $E_R(R/P)$ for some P in $\bar{X}$. The proof is complete.

COROLLARY 1. *Let R be a ring satisfying the condition $H[F(R)]$, M an injective R-module belonging to $\mathfrak{N}[F(R)]$ and suppose that* Ass (M) *is of finite type. Then, M can be expressed as a direct sum of indecomposable injective R-modules in $\mathfrak{N}[F(R)]$ each of which is isomorphic to $E_R(R/P)$ for some P in $F(R)$.*

COROLLARY 2. *Let R be a ring with the condition $H[F(R)]$ and M an injective R-module belonging to $\mathfrak{N}[F(R)]$. Suppose that* Ass (M) *satisfies the maximal condition and has only a finite number of maximal elements in* Ass(M). *Then, M is decomposable into a direct sum of indecomposable injective R-modules in $\mathfrak{N}[F(R)]$, each of which is isomorphic to one of the $E_R(R/P)$, $P\in F(R)$.*

PROOF. Since Ass (M) satisfies the maximal condition, each element of Ass (M) is contained in some maximal element in Ass (M). Thus, Ass (M) is of finite type and hence we have the result by Theorem 2.1.

LEMMA 2.1. *Let R be a ring and X a subset of $F(R)$ of finite type. Let $\{P_1, P_2, \ldots, P_s\}$ be the set of all maximal elements in X, and set $C = R - \bigcup_i P_i$, $A = \{r \in R \mid cr = 0$ for some c in $C\}$. Then,* (1) *every R-module in $\mathfrak{N}[\bar{X}]$ can be regarded as an R/A-module, and* (2) *any injective R-module in $\mathfrak{N}[\bar{X}]$ can be regarded as an R_C-module.*

PROOF. (1) Let M be an arbitrary R-module in $\mathfrak{N}[\bar{X}]$, c an arbitrary element in C, and let $T_c\colon M \to M$ be an R-homomorphism defined by $T_c(x) = cx$ for $x \in M$. Then, $(\mathrm{Ker}\,(T_c))_{P_i} = 0$ for all i. This means $\mathrm{Ker}\,(T_c) = 0$. Therefore, it is easily seen that M can be regarded as an R/A-module.

(2) Let E be any injective R-module in $\mathfrak{N}[\bar{X}]$. In order to prove (2), it is sufficient to show that $T_c\colon E \to E$ defined by $T_c(x) = cx$ is an automorphism. f, being monomorphic, was obtained in the proof of (1). Let x be any element in E. Since $T'_c\colon R/A \to \to R/A$ defined by $T'_c(y) = cy$ is a monomorphism, for the homomorphism f of R/A into E defined by $f(\bar{1}) = x$ ($\bar{1}$ = the canonical image of 1 in R/A), there exists a homomorphism g of R/A into E such that $f = gT'_c$. Set $g(\bar{1}) = z$. Then $x = f(\bar{1}) = = g(T'_c(\bar{1})) = g(c\bar{1}) = cz$ and this implies that T_c is an automorphism.

PROPOSITION 2.1. *Let R be a ring and X a subset of $F(R)$ of finite type. Then we yield the followings:*

(1) *R satisfies the condition $H[\bar{X}]$.*

(2) *Every injective R-module in $\mathfrak{N}[\bar{X}]$ is decomposable into a direct sum of indecomposable injective R-modules belonging to $\mathfrak{N}[\bar{X}]$ each of which is isomorphic to $E_R(R/P)$ for some element P in $\bar{X}$.*

PROOF. As $\bar{X}$ is of finite type, there is only a finite number $\{P_1, P_2, \ldots, P_s\}$ of maximal elements P_i in $\bar{X}$ such that any element of $\bar{X}$ is contained in some P_i. Now, set $C = R - \bigcup_i P_i$. Then, C is a multiplicatively closed set in R and R_C is a ring with just maximal ideals $P_1R_C, P_2R_C, \ldots, P_sR_C$. Since $(R_C)_{P_iR_C} = R_{P_i}$ is a noetherian ring by the assumption for all i, by (E 1.2) of Appendix of [10], R_C is a noetherian ring.

Let M be an injective R-module in $\mathfrak{N}[\bar{X}]$. Then, by Lemma 2.1 M can be regarded as an R_C-module and by Lemma 1.1, M is injective as an R_C-module. Since R_C is a noetherian ring by the above remark, M contains a submodule E which is an indecomposable injective R-module and isomorphic to $E_{R_C}(R_C/P')$ for some prime ideal P' in R_C. Now, set $P = P' \cap R$. Then, by the same way as in the proof of Lemma 2 of [5], $E_{R_C}(R_C/P') \cong E_R(R/P)$ as an R-module and as an R_C-module. Furthermore, since R_P is a localization of a noetherian ring R_C, R_P is noetherian, and hence P belongs to $F(R)$. Thus, M contains an indecomposable injective R-module isomorphic to $E_R(R/P)$, $P \in \bar{X}$ and by Theorem 1.1, R satisfies the condition $H[\bar{X}]$. Thus we yield (1), and (2) follows from Theorem 2.1.

THEOREM 2.2. *Let R be a ring and X a subset of $F(R)$ of finite type. Then, for any family $\{M_\alpha\}_{\alpha \in \Omega}$ of injective R-modules in $\mathfrak{N}[\bar{X}]$, $\sum_{\alpha \in \Omega} \oplus M_\alpha$ is injective as an R-module and belongs to $\mathfrak{N}[\bar{X}]$.*

PROOF. Let $\{P_1, P_2, \ldots, P_n\}$ be the set of all maximal elements in $\bar{X}$. Then, since R satisfies the condition $H[\bar{X}]$ by Proposition 2.1, by the same way as in the proof of Theorem 2.1, we have the followings for each $\alpha \in \Omega$;

$$M_\alpha = M_\alpha[P_1] \oplus M_{\alpha 1}[P_2] \oplus \ldots \oplus M_{\alpha n-1}[P_n],$$

where $M_\alpha[P_1]$ and $M_{\alpha i}[P_{i+1}]$ are local components of M_α and $M_{\alpha i}$ at P_1 and P_{i+1}, respectively, and $M_\alpha = M_\alpha[P_1] \oplus M_{\alpha 1}$ and $M_{\alpha i} = M_{\alpha i}[P_{i+1}] \oplus M_{\alpha i+1}$, for $i = 1, 2, \ldots, n-1$. Now, since R_{P_1} and R_{P_i} are noetherian rings, $\sum_{\alpha \in \Omega} \oplus M_\alpha[P_1]$ and $\sum_{\alpha \in \Omega} \oplus M_{\alpha i}[P_{i+1}]$ are injective as an R_{P_1}-module and as an R_{P_i}-module, respectively. Thus by Lemma 1.1, $\sum_{\alpha \in \Omega} \oplus M_\alpha[P_1]$ and $\sum_{\alpha \in \Omega} \oplus M_{\alpha i}[P_{i+1}]$ are injective as an R-module. Hence we obtain that

$$\Big(\sum_{\alpha \in \Omega} \oplus M_\alpha[P_1]\Big) \oplus \Big(\sum_{\alpha \in \Omega} \oplus M_{\alpha 1}[P_2]\Big) \oplus \ldots \oplus \Big(\sum_{\alpha \in \Omega} \oplus M_{\alpha n-1}[P_n]\Big)$$

is injective as an R-module. Furthermore, it is easily seen that this is isomorphic to $\sum_{\alpha \in \Omega} \oplus (M_\alpha[P_1] \oplus \ldots \oplus M_{\alpha n-1}[P_n])$. Thus $\sum_{\alpha \in \Omega} \oplus M_\alpha$ is injective as an R-module.

REMARK. Let R be a ring and X a subset of $F(R)$ o ffinite type. Using then Proposition 2.1, R satisfies the condition $H[\bar{X}]$. Thus by Theorem 2.1 and Theorem 2.2 the conditions (I) and (II) described in the introduction, hold in $\mathfrak{N}[\bar{X}]$.

LEMMA 2.2. *Let A be an ideal in a ring R and f an R-homomorphism of A into $E_R(R/P)$ $(P \in \operatorname{Spec}(R))$, and assume that $A \subseteqq \sigma^{-1}(0)$, where $\sigma\colon R \to R_P$ is the canonical homomorphism. Then f is trivial.*

PROOF. Let a be any element in A. Since $a \in \sigma^{-1}(0)$, there is an element r in $R-P$ such that $ra=0$. On the other hand, by Lemma 1.3, $T_r\colon E_R(R/P) \to E_R(R/P)$ defined by $T_r(x) = rx$ for x in $E_R(R/P)$ is an automorphism. Thus $rf(a) = f(ra) = f(0) = 0$ and this implies that f is trivial.

THEOREM 2.3. *Let R be a ring, X a non-empty subset of $F(R)$ and assume that R satisfies the condition $H[\bar{X}]$. Then, if the set $\{P \in \bar{X} \mid AR_P \neq 0\}$ is a finite set for any proper ideal A in R, we yield the following facts:*

(1) *Every injective R-module belonging to $\mathfrak{N}[\bar{X}]$ is decomposable into a direct sum of indecomposable injective R-modules belonging to $\mathfrak{N}[\bar{X}]$ each of which is isomorphic to $E_R(R/P)$ for some P in $\bar{X}$.*

(2) *Let $\{M_\alpha\}_{\alpha \in \Omega}$ be any family of injective R-modules belonging to $\mathfrak{N}[\bar{X}]$. Then, $\sum_{\alpha \in \Omega} \oplus M_\alpha$ is injective as an R-module and belongs to $\mathfrak{N}[\bar{X}]$.*

PROOF. (1) Let M be any injective R-module belonging to $\mathfrak{N}[\bar{X}]$ and $M(P)$ $(P \in \operatorname{Ass}(M))$ the P-components of M with respect to an expression of M. Then by Proposition 1.4, M can be expressed as

$$M = E_R\Big(\sum_{P \in \operatorname{Ass}(M)} \oplus M(P)\Big).$$

Thus it is sufficient to show that $\sum_{P \in \operatorname{Ass}(M)} \oplus M(P)$ is injective as an R-module. Let us show this. Let A be any ideal of R, f an R-homomorphism of A into $\sum_{P \in \operatorname{Ass}(M)} \oplus M(P)$

and let P' be any element in $\bar{X}$ such that $AR_{P'}=0$. Then the canonical projection of $f(A)$ into $M(P')$ is zero by Lemma 2.2. Thus $f(A)\subseteqq \sum_{i=1}^{n} \oplus M(P_i)$, where P_i is an element in $\bar{X}$ such that $AR_{P_i}\neq 0$ for $i=1, 2, ..., n$. Since $\sum_{i=1}^{n} \oplus M(P_i)$ is injective as an R-module, f can be extended to an R-homomorphism of R into $\sum_{i=1}^{n} \oplus M(P_i)$. Therefore, $\sum_{P\in \mathrm{Ass}(M)} \oplus M(P)$ is injective as an R-module.

(2) By (1), for each $\alpha\in\Omega$, $M_\alpha = \sum_{P\in \mathrm{Ass}(M_\alpha)} \oplus M_\alpha(P) = \sum_{P\in \bar{X}} \oplus M_\alpha(P)$, where $M_\alpha(P)$ $(P\in\bar{X})$ are the P-components of M_α with respect to an expression of M_α. Moreover, for each P in $\bar{X}$, $\sum_{\alpha\in\Omega} \oplus M_\alpha(P)$ is injective as an R-module. For, since $M_\alpha(P)$ is injective as an R_P-module and since R_P is a noetherian ring, $\sum_{\alpha\in\Omega} \oplus M_\alpha(P)$ is injective as an R_P-module. Thus, by Lemma 1.1, $\sum_{\alpha\in\Omega} \oplus M_\alpha(P)$ is injective as an R-module. By the fact in the proof of (1), we have that $\sum_{\alpha\in X} \oplus (\sum_{\alpha\in\Omega} \oplus M_\alpha(P))$ is injective as an R-module. Furthermore, it is easily seen that $\sum_{\alpha\in\Omega} \oplus M_\alpha$ is isomorphic to $\sum_{P\in X} \oplus \oplus (\sum_{\alpha\in\Omega} \oplus M_\alpha(P))$. Thus $\sum_{\alpha\in\Omega} \oplus M_\alpha$ is injective as an R-module.

LEMMA 2.3. *Let R be a ring, P a prime ideal of R, A an ideal of R and let f be an R-homomorphism of A into $E_R(R/P)$. Then, f is trivial when* $\mathrm{Ker}(f)\subsetneqq P$.

PROOF. Take an element r in $\mathrm{Ker}(f)-P$. Then, $T_r\colon E_R(R/P)\to E_R(R/P)$ defined by $T_r(x)=rx$ for x in $E_R(R/P)$ is an automorphism by Lemma 1.3. For any element a in A, $rf(a)=f(ra)=af(r)=0$, and this means that f is trivial.

THEOREM 2.4. *Let R be a domain and X a subset of $F(R)$ of open type. Suppose that X satisfies the maximal condition and every non-zero ideal in R is contained in only a finite number of maximal elements in X. Then, we yield the following facts:*

(1) *R satisfies the condition $H[X]$.*

(2) *For any injective R-module M belonging to $\mathfrak{N}[X]$,*

$$M = \sum_{P\in \mathrm{Ass}(M)} \oplus M(P),$$

where $M(P)$ $(P\in \mathrm{Ass}(M))$ are the P-components of M with respect to an expression of M. In this case, M is a direct sum of its P-components $(P\in \mathrm{Ass}(M))$ with respect to any expression of M.

(3) *Any injective R-module belonging to $\mathfrak{N}[X]$ can be written as a direct sum of indecomposable injective R-modules in $\mathfrak{N}[X]$ each of which is isomorphic to $E_R(R/P)$ for some P in X.*

(4) *Let $\{M_\alpha\}_{\alpha\in\Omega}$ be any family of injective R-modules belonging to $\mathfrak{N}[X]$. Then, $\sum_{\alpha\in\Omega} \oplus M_\alpha$ is injective as an R-module.*

PROOF. (1) Let A be a non-zero proper ideal of R such that $A=\bigcap_{P\in X}(AR_P\cap R)$. By the assumption, there is a finite subset X' of X consisting of all P which are maximal and contain A. It is easy to see that $\mathfrak{N}[\bar{X}']\subseteqq\mathfrak{N}[X]$. Since R satisfies the condition $H[X']$ by Proposition 2.1 and $A=\bigcap_{P\in X'}(AR_P\cap R)$, we have that $\emptyset\neq \neq X'\cap \mathrm{Ass}\,(R/A)\subseteqq X\cap \mathrm{Ass}\,(R/A)$, and this shows that R satisfies the condition $H[X]$.

(2) By (1) and Proposition 1.4, M can be expressed as

$$M=E_R\Big(\sum_{P\in \mathrm{Ass}(M)}\oplus M(P)\Big).$$

Thus, it is sufficient to show that $\sum_{P\in \mathrm{Ass}(M)}\oplus M(P)$ is injective as an R-module. Let $B\neq 0$ be any ideal in R and f any R-homomorphism of B into $\sum_{P\in \mathrm{Ass}(M)}\oplus M(P)$. Now, we have that

$$\sum_{P\in \mathrm{Ass}(M)}\oplus M(P)=M(0)\oplus\Big(\sum_{\substack{P\in \mathrm{Ass}(M)\\ P\neq 0}}\oplus M(P)\Big).$$

Let p_0 be the canonical projection of $M(0)\oplus\Big(\sum_{\substack{P\in \mathrm{Ass}(M)\\ P\neq 0}}\oplus M(P)\Big)$ onto $\sum_{\substack{P\in \mathrm{Ass}(M)\\ P\neq 0}}\oplus M(P)$. Since $\sum_{\substack{P\in \mathrm{Ass}(M)\\ P\neq 0}}\oplus M(P)$ is a torsion R-module, $\mathrm{Ker}\,(f)=A$ is not zero when $p_0f\neq 0$. If $p_0f=0$, $f(B)\subseteqq M(0)$. Assume that $p_0f\neq 0$, i.e., $A=\mathrm{Ker}\,(f)\neq 0$. Then, by the hypothesis, there is only a finite number of maximal elements $\{P_1, P_2, \dots, P_n\}$ in X containing A. Let P be any element in X such that $P\subsetneqq P_i$ for all i. Then, by Lemma 2.3 the projection of $(p_0f)(B)$ to $M(P)$ is zero. Therefore, we have that

$$p_0f(B)\subseteqq M[P_1]\oplus M_1[P_2]\oplus\dots\oplus M_{n-1}[P_n],$$

where $M_i[P_{i+1}]$ is a local component of M_i at P_{i+1} and $M_i=M_i[P_{i+1}]\oplus M_{i+1}$ $(M_0=M)$ for $i=0,1,\dots,n-1$ (this decomposition is aquired by the same way as in the proof of Theorem 2.1). By Proposition 1.5, $M[P_1]\oplus M_1[P_2]\oplus\dots\oplus M_{n-1}[P_n]$ is injective as an R-module. Thus, f can be extended to an R-homomorphism of R into $M[P_1]\oplus M_1[P_2]\oplus\dots\oplus M_{n-1}[P_n]\subseteqq\sum_{P\in \mathrm{Ass}(M)}\oplus M(P)$ because $M(0)$ is contained in $M[P_1]$. Hence $\sum_{P\in \mathrm{Ass}(M)}\oplus M(P)$ is injective as an R-module. Furthermore, by Proposition 1.4 for each element P in $\mathrm{Ass}\,(M)$ the P-components of M with respect to any two expressions of M are isomorphic. Therefore, $M=\sum_{P\in \mathrm{Ass}(M)}\oplus M(P)$ implies that M is a direct sum of the P-components $(P\in \mathrm{Ass}\,(M))$ of M with respect to any expression of M.

(3) Let M be any injective R-module belonging to $\mathfrak{N}[X]$ and $M=E_R\Big(\sum_{\lambda\in\Lambda}\oplus E_\lambda\Big)$ an expression of M. Let $M(P)$ $(P\in \mathrm{Ass}\,(M))$ be the P-components of M with respect to this expression of M. Then by (1) $M=\sum_{P\in \mathrm{Ass}(M)}\oplus M(P)$ and by Proposition 1.4, for each P in $\mathrm{Ass}\,(M)$, $M(P)$ can be expressed as a direct sum of indecomposable injective R-modules in $\mathfrak{N}[X]$ each of which is isomorphic to $E_R(R/P')$ for some P' in X. Thus we infer the result.

(4) By (1), we have the following isomorphisms:

$$\sum_{\alpha\in\Omega}\oplus M_\alpha=\sum_{\alpha\in\Omega}\oplus\Big(\sum_{P\in \mathrm{Ass}(M_\alpha)}\oplus M_\alpha(P)\Big)\cong$$

$$\cong\sum_{\alpha\in\Omega}\oplus\Big(\sum_{P\in X}\oplus M_\alpha(P)\Big)\cong\sum_{P\in X}\oplus\Big(\sum_{\alpha\in\Omega}\oplus M_\alpha(P)\Big),$$

where for each $\alpha\in\Omega$, $M_\alpha(P)$ $(P\in X)$ are the P-components of M_α with respect to an expression of M_α. Since $\sum_{\alpha\in\Omega}\oplus M_\alpha$ belongs to $\mathfrak{N}[X]$, by Proposition 1.1, $M'=E_R(\sum_{\alpha\in\Omega}\oplus M_\alpha)$ also belongs to $\mathfrak{N}[X]$. Thus by (2), M' can be expressed as

$$M'=\sum_{P\in X}\oplus M'(P)=\sum_{P\in X}\oplus\bigl(\sum_{\alpha\in\Omega}\oplus M_\alpha(P)\bigr)$$

because $\operatorname{Ass}(M')=\operatorname{Ass}(\sum_{\alpha\in\Omega}\oplus M_\alpha)$, where $M'(P)$ $(P\in X)$ are the P-components of M' with respect to the expression $M'=E_R(\sum_{\alpha\in\Omega}\oplus(\sum_{P\in\operatorname{Ass}(M_\alpha)}\oplus M_\alpha(P)))$. Thus $\sum_{\alpha\in\Omega}\oplus M_\alpha$ is injective as an R-module.

Let R be a Krull domain and put $X=\{P\in\operatorname{Spec}(R)\mid$ the height of $P\leqq 1\}$. Then, X is of open type. I. BECK investigated in [2] the structures and the decomposabilities of injective modules in $\mathfrak{N}[X]$, and completely determined these. We can yield his results as the corollary of Theorem 2.4 because R and X satisfy the conditions of Theorem 2.4.

COROLLARY. *Let R be a Krull domain and $X=\{P\in\operatorname{Spec}(R)\mid$the height of $P\leqq 1\}$. Then we have the following facts:*

(1) *R satisfies the condition $H[X]$.*

(2) *Any injective R-module belonging to $\mathfrak{N}[X]$ contains an indecomposable injective R-module which belongs to $\mathfrak{N}[X]$ and every indecomposable injective R-module in $\mathfrak{N}[X]$ is isomorphic to $E_R(R/P)$ for some $P\in X$.*

(3) *Any injective R-module in $\mathfrak{N}[X]$ can be expressed as a direct sum of indecomposable injective R-modules in $\mathfrak{N}[X]$ each of which is isomorphic to $E_R(R/P)$ for some P in X.*

(4) *Let $\{M_\gamma\}_{\gamma\in\Gamma}$ be any family of injective R-modules in $\mathfrak{N}[X]$. Then, $\sum_{\gamma\in\Gamma}\oplus M_\gamma$ is also injective as an R-module and belongs to $\mathfrak{N}[X]$.*

LEMMA 2.4. *Let P_1 and P_2 be prime ideals in a ring R such that $P_1\subseteqq P_2$. Then, for any non-zero element x in $E_R(R/P_1)$ there is a non-trivial R-homomorphism f of $E_R(R/P_1)$ into $E_R(R/P_2)$ such that $f(x)\neq 0$.*

PROOF. Set $g(x)=\bar{1}$, where $\bar{1}$ is the canonical image of 1 in R/P_2. Then, g is well defined as an R-homomorphism of Rx into $E_R(R/P_2)$ because $\operatorname{Ann}_R(x)\subseteqq P_1\subseteqq P_2$. Since $E_R(R/P_2)$ is injective as an R-module, there is an R-homomorphism f of $E_R(R/P_1)$ into $E_R(R/P_2)$ such that $f=g$ on Rx and $f(x)\neq 0$.

THEOREM 2.5. *Let R be a ring, X a subset of $F(R)$ of open type and assume that R satisfies the condition $H[X]$ and that X satisfies the maximal condition. Let M be an injective R-module belonging to $\mathfrak{N}[X]$, S the set of all maximal elements in $\operatorname{Ass}(M)$ and let $M(P)$ $(P\in X)$ be the P-components of M with respect to an expression of M. Then, the following statements are equivalent:*

(1) $$M=\sum_{P\in\operatorname{Ass}(M)}\oplus M(P).$$

In this case, M is a direct sum of the P-components $(P\in\operatorname{Ass}(M))$ of M with respect to any expression of M.

(2) $\sum_{P\in S}\oplus M(P)$ *is injective as an R-module.*

(3) *M is decomposable into a direct sum of indecomposable injective R-modules belonging to* $\mathfrak{N}[X]$ *each of which is isomorphic to* $E_R(R/P)$ *for some P in X.*

(4) $\sum_{P\in S}\oplus E_R(R/P)$ *is injective as an R-module.*

(5) *Let* $\{M_\alpha\}_{\alpha\in\Omega}$ *be any family of injective R-modules* M_α *belonging to* $\mathfrak{N}[X]$ *such that* $\mathrm{Ass}(M_\alpha)\subseteq\bar{S}$. *Then* $\sum_{\alpha\in\Omega}\oplus M_\alpha$ *is injective as an R-module.*

PROOF. (1)$\to$(2). Since $\sum_{P\in S}\oplus M(P)$ is a direct summand of $\sum_{P\in \mathrm{Ass}(M)}\oplus M(P)$, it is injective as an R-module.

(2)$\to$(4). It is easily seen that $\sum_{P\in S}\oplus M(P)$ contains a submodule which is isomorphic to $\sum_{P\in S}\oplus E_R(R/P)$ and is a direct summand. This implies that $\sum_{P\in S}\oplus E_R(R/P)$ is injective as an R-module.

(1)$\to$(3). Since, by Proposition 1.4, $M(P)$ can be written as a direct sum of indecomposable injective R-modules each of which is isomorphic to $E_R(R/P')$ for some P' in X for all P in $\mathrm{Ass}(M)$, $\sum_{P\in \mathrm{Ass}(M)}\oplus M(P)$ can be expressed as the required form and this is equal to M.

(3)$\to$(1). Let $M=\sum_{\lambda\in\Lambda}\oplus E_\lambda$, where E_λ $(\lambda\in\Lambda)$ are indecomposable injective R-modules each of which is isomorphic to $E_R(R/P')$ for some P' in X. Then, by Proposition 1.4, M is a direct sum of the P-components $(P\in\mathrm{Ass}(M))$ of M with respect to this expression of M and for each P in $\mathrm{Ass}(M)$ the P-component of M with respect to this expression of M, is isomorphic to $M(P)$. Thus $M=\sum_{P\in\mathrm{Ass}(M)}\oplus M(P)$.

(4)$\to$(1). If S is a finite set, then by Corollary 1 of Theorem 2.1, (4) implies (3) and so (4) implies (1). Now, assume that S is an infinite set. Let $S=\{P_\gamma\}_{\gamma\in\Gamma}$ and $\{M[P_\gamma]\}_{\gamma\in\Gamma}$ the set of all the local components of M at P_γ $(\gamma\in\Gamma)$ with respect to an expression of M, and let

$$M'=\sum_{\gamma\in\Gamma}\oplus M[P_\gamma]$$

be the (external) direct sum of $\{M[P_\gamma]\}_{\gamma\in\Gamma}$. Then, it is easy to see that M is isomorphic to a submodule of M', which is a direct summand. Therefore, in order to prove the injectivity of M, it is sufficient to show the injectivity of M'. Let us show the injectivity of M'. Now, assume that there are an ideal A in R and an R-homomorphism $f\colon A\to\sum_{\gamma\in\Gamma}\oplus M[P_\gamma]$ such that

$$f(A)\nsubseteq\sum_{\gamma:\,\text{finite}}\oplus M[P_\gamma]$$

(let us regard, canonically, $M[P_\gamma]$ as a submodule of $\sum_{\gamma\in\Gamma}\oplus M[P_\gamma]$). Then, there exists an infinite set $\{\gamma_1,\gamma_2,\dots,\gamma_n,\dots\}$ consisting of γ_i in Γ such that $p_{\gamma_i}f(A)\neq 0$, where $p_{\gamma_i}\colon M'\to M[P_{\gamma_i}]$ is the canonical projection for $i=1,2,3,\dots$. For each i, there exists a submodule N_{γ_i} in $M[P_{\gamma_i}]$, isomorphic to $E_R(R/P'_{\gamma_i})$, $P'_{\gamma_i}\subseteq P_{\gamma_i}$ such that $q_{\gamma_i}p_{\gamma_i}f(A)\neq 0$, where $q_{\gamma_i}\colon M[P_{\gamma_i}]\to N_{\gamma_i}$ is a canonical projection. By Lemma 2.4, there is an R-homomorphism $h_{\gamma_i}\colon N_{\gamma_i}\to E_R(R/P_{\gamma_i})$ such that $h_{\gamma_i}q_{\gamma_i}p_{\gamma_i}f(A)\neq 0$.

For any element a in A, $h_{\gamma_i} q_{\gamma_i} p_{\gamma_i} f(a)=0$ for almost all i. Hence we have that

$$\sum_i h_{\gamma_i} q_{\gamma_i} p_{\gamma_i} f: A \to \sum_i \oplus E_R(R/P_\gamma), \quad \sum_i \oplus E_R(R/P_{\gamma_i}) \subseteqq \sum_{P\in S} \oplus E_R(R/P),$$

is an R-homomorphism, but this homomorphism can not be extended to R because its image is not contained in a direct sum of any finite number of $E_R(R/P_{\gamma_i})$ by the construction. This contradicts the injectivity of $\sum_{P\in S} \oplus E_R(R/P)$. Thus $f(A)$ is contained in a direct sum of some finite number of $M[P_{\gamma_i}]$. That is, M' is injective as an R-module.

(5)$\to$(4) is immediate.

(4)$\to$(5) may be proved by the same way as in the proof of (4)$\to$(1) and we omit it.

COROLLARY. *Let R be a ring and X a non-empty subset of $F(R)$. Assume that $\bar{X}$ satisfies the maximal condition and that R satisfies the condition $H[\bar{X}]$. Then the following statements are equivalent.*

(1) *For any injective R-module M belonging to $\mathfrak{N}[\bar{X}]$, M can be expressed as a direct sum of the P-components $M(P)$ $(P\in \operatorname{Ass}(M))$ with respect to any expression of M.*

(2) *Every injective R-module belonging to $\mathfrak{N}[\bar{X}]$ is decomposable into a direct sum of indecomposable injective R-modules in $\mathfrak{N}[\bar{X}]$ each of which is isomorphic to $E_R(R/P)$ for some P in $\bar{X}$.*

(3) *Let M be any injective R-module belonging to $\mathfrak{N}[\bar{X}]$ and $M(P)$ $(P\in \operatorname{Ass}(M))$ the P-components of M with respect to an expression of M. Then, $\sum_{P\in J} \oplus M(P)$ is injective as an R-module, where J is the set of all maximal elements in $\bar{X}$ belonging to* Ass (M).

(4) *$\sum_{P\in S} \oplus E_R(R/P)$ is injective as an R-module, where S is the set of all maximal elements in $\bar{X}$.*

(5) *Let $\{M_\alpha\}_{\alpha\in\Omega}$ be any family of injective R-modules belonging to $\mathfrak{N}[\bar{X}]$. Then, $\sum_{\alpha\in\Omega} \oplus M_\alpha$ is also injective as an R-module.*

PROOF. (1)$\leftrightarrow$(2)$\leftrightarrow$(3) follow from Theorem 2.5 and (4)$\leftrightarrow$(5) follows Theorem 2.5.

(4)$\to$(3). Since

$$\sum_{P\in S} \oplus E_R(R/P) = \Big(\sum_{P\in J} \oplus E_R(R/P)\Big) \oplus \Big(\sum_{P\in S-J} \oplus E_R(R/P)\Big),$$

$\sum_{P\in J} \oplus E_R(R/P)$ is injective as an R-module. Thus by Theorem 2.5, $\sum_{P\in J} \oplus M(P)$ is injective as an R-module.

(3)$\to$(4). As $\sum_{P\in S} \oplus E_R(R/P)$ belongs to $\mathfrak{N}[\bar{X}]$, by Proposition 1.1, $E_R\big(\sum_{P\in S} \oplus E_R(R/P)\big)$ also belongs to $\mathfrak{N}[\bar{X}]$. Thus, by the hypothesis, $E_R\big(\sum_{P\in S} \oplus E_R(R/P)\big)$ can be expressed as a direct sum of its components with respect to this expression. On the other hand, $\operatorname{Ass}\big(\sum_{P\in S} \oplus E_R(R/P)\big)$ is equal to S. Thus, $\sum_{P\in S} \oplus E_R(R/P)$ is injective as an R-module.

REMARKS. 1. Let R be a ring and X a subset of $F(R)$ of open type. Assume that R satisfies the condition $H[X]$ and that X satisfies the maximal condition. Then, when $\sum_{P\in S} \oplus E_R(R/P)$ is injective as an R-module, the conditions (I) and (II) described in the introduction hold in $\mathfrak{N}[\bar{X}]$, where S is the set of all maximal elements in X.

2. We give here, without the proof, an example of a ring R wich does not satisfy the condition $H[F(R)]$.

Let $R=\prod_{i=1}^{\infty} K_i$ be a direct product of fields K_i, $i=1, 2, 3, \ldots$. Then, $F(R)= =\operatorname{Spec}(R)$ and R does not satisfy the condition $H[F(R)]$.

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SPLINE FUNCTIONS AND THE CAUCHY PROBLEMS. IV

ON THE STABILITY OF THE METHOD

By

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Introduction

In the recent papers [2] and [3] we have introduced a method for approximating the solution of the non-linear ordinary differential equations $y'=f(x, y)$ and $y''=f(x, y, y')$, with the initial conditions $y(x_0)=y_0$, and $y(x_0)=y_0$, $y'(x_0)=y_0'$, respectively.

In this paper we are going to prove the stability of the method of approximating the solution of the first and the second order differential equations, with initial conditions, presented in [2] and [3], respectively. This paper is also considered to be an extension of the previous papers [2] and [3].

1. Stability of the method for the first order differential equation

The method for approximating the solution of the Cauchy problem in the non-linear first order differential equation was described, in details, in [3].

A change in one of the calculated values from $\bar{y}_k$ to $\bar{z}_k$ will lead us to solve

$$\bar{z}_{m+1}=\bar{z}_m+\int_{x_m}^{x_{m+1}} f\big(t, z_m^*(t)\big)\,dt$$

instead of (2.1.5) in [3]. Subtracting (2.1.5) of [3], from this and setting

$$\varepsilon_k=|\bar{z}_k-\bar{y}_k|,$$

we get

$$\varepsilon_{m+1}\leqq \varepsilon_m+c_6h\varepsilon_m\leqq (1+c_6h)^{m-k}\varepsilon_k\leqq c\varepsilon_k$$

where c is a constant, independent of h and $m=k, k+1, \dots, n-1$, or

$$\varepsilon_n\leqq (1+c_6h)^{n-k}\varepsilon_k\leqq e^{c_6(b-a)}\varepsilon_k\leqq c\varepsilon_k$$

which is a bounded multiple of the introduced error ε_k and is independent of h.

Also, for $q=0, 1, \dots, r$,

$$|\bar{z}_m^{(q+1)}-\bar{y}_m^{(q+1)}|=|f^{(q)}(x_m, \bar{z}_m)-f^{(q)}(x_m, \bar{y}_m)|$$

applying the Lipschitz condition, it becomes

$$\leqq L|\bar{z}_m-\bar{y}_m|\leqq Le^{c_6(b-a)}\varepsilon_k\leqq c_7\varepsilon_k$$

which is also a bounded multiple of the introduced error ε_k.

Hence, we have proved

THEOREM 1.1. *If any of the calculated values $\bar{y}_k$ is changed to $\bar{z}_k$, then the inequality*

$$|\bar{z}_m^{(t)} - \bar{y}_m^{(t)}| \leqq c_8 \varepsilon_k$$

holds for all $m=k+1, \ldots, n$ and all $t=0, 1, \ldots, r+1$ where $\varepsilon_k = |\bar{z}_k - \bar{y}_k|$ is the introduced error.

THEOREM 1.2. *If any of the calculated values $\bar{y}_k$ is changed to $\bar{z}_k$, and consequently, the spline function approximating the solution of* (1.1) *in* [3], *and constructed in theorem* 3.1 *in* [3], *is also changed from $S(x)$ to $s(x)$, then for any $x \in [x_m, x_{m+1}]$, $m=k, k+1, \ldots$ $\ldots, n-1$, the inequality*

$$|s_m(x) - S_m(x)| \leqq c_{11} \varepsilon_k$$

holds, where $\varepsilon_k = |\bar{z}_k - \bar{y}_k|$ is the introduced error.

PROOF. Consider the interval $[x_m, x_{m+1}]$ where $m=k, k+1, \ldots, n-1$. Then, analogously to the spline function $S_\Delta(x)$ introduced in theorem 3.1 in [3], the new spline function, due to the variation from $\bar{y}_k$ to $\bar{z}_k$, will be

$$s_m(x) = \sum_{j=0}^{r+1} \frac{\bar{z}_m^{(j)}}{j!} (x - x_m)^j + \sum_{p=1}^{r+2} b_p^{(m)} (x - x_m)^{p+r+1} \tag{1.1}$$

and satisfying the conditions, as of (3.1.4) in [3],

$$s_m^{(t)}(x_{m+1}) = s_{m+1}^{(t)}(x_{m+1}) = \bar{z}_{m+1}^{(t)}, \quad s_{n-1}^{(t)}(x_n) = \bar{z}_n^{(t)} \tag{1.2}$$

where $m=k, k+1, \ldots, n-2$. I.e. for all $m=k, k+1, \ldots, n-1$

$$\bar{z}_{m+1}^{(t)} = \sum_{j=0}^{r+1-t} \frac{\bar{z}_m^{(j+t)}}{j!} h^j + \sum_{p=1}^{r+2} t! \binom{p+r+1}{t} b_p^{(m)} h^{p+r+1-t}$$

which could be obtained by the combination of (1.1) and (1.2).

The system of equations corresponding to (3.1.5) in [3] will be

$$\sum_{p=1}^{r+2} t! \binom{p+r+1}{t} b_p^{(m)} h^{p-1} = G_t^{(m)} \tag{1.3}$$

where $t=0, 1, \ldots, r+1$ and

$$G_t^{(m)} = \frac{1}{h^{r+2-t}} \left(\bar{z}_{m+1}^{(t)} - \sum_{j=0}^{r+1-t} \frac{\bar{z}_m^{(j+t)}}{j!} h^j \right) \tag{1.5}$$

and corresponding to (3.1.7) in [3], we get

$$b_p^{(m)} = \frac{1}{h^{p-1}} \sum_{t=0}^{r+1} c_{pt} G_t^{(m)}. \tag{1.6}$$

Now, from equations (1.1) and (3.1.3) in [3], we get

$$|s_m(x)-S_m(x)| = \left| \sum_{j=0}^{r+1} \frac{\bar{z}_m^{(j)}}{j!}(x-x_m)^j + \right.$$

$$\left. + \sum_{p=1}^{r+2} b_p^{(m)}(x-x_m)^{p+r+1} - \sum_{j=0}^{r+1} \frac{\bar{y}_m^{(j)}}{j!}(x-x_m)^j - \sum_{p=1}^{r+2} a_p^{(m)}(x-x_m)^{p+r+1} \right| \leqq$$

$$\leqq \sum_{j=0}^{r+1} \frac{|\bar{z}_m^{(j)}-\bar{y}_m^{(j)}|}{j!} h^j + \sum_{p=1}^{r+2} |b_p^{(m)}-a_p^{(m)}| h^{p+r+1}.$$

Now, (1.6) and (3.1.7) in [3], imply

$$|b_p^{(m)}-a_p^{(m)}| = \frac{1}{h^{p-1}} \left| \sum_{r=0}^{r+1} c_{pt} G_t^{(m)} - \sum_{t=0}^{r+1} c_{pt} F_t^{(m)} \right| \leqq \frac{1}{h^{p-1}} \sum_{t=0}^{r+1} c_{pt} |G_t^{(m)} - F_t^{(m)}|$$

and from (1.5) and (3.1.5) in [3], we have

$$|G_t^{(m)}-F_t^{(m)}| = \frac{1}{h^{r+2-t}} \left| \bar{z}_{m+1}^{(t)} - \sum_{j=0}^{r+1-t} \frac{\bar{z}_m^{(j+t)}}{j!} h^j - \bar{y}_{m+1}^{(t)} + \sum_{j=0}^{r+1-t} \frac{\bar{y}_m^{(j+t)}}{j!} h^j \right| \leqq$$

$$\leqq \frac{1}{h^{r+2-t}} \left(\bar{z}_{m+1}^{(t)} - \bar{y}_{m+1}^{(t)} + \sum_{j=0}^{r+1-t} |\bar{z}_m^{(j+t)} - \bar{y}_m^{(j+t)}| \frac{h^j}{j!} \right).$$

Applying theorem 1.1, it becomes

$$\leqq \frac{1}{h^{r+2-t}} \left(c_8 \varepsilon_k + \sum_{j=0}^{r+1-t} c_8 \varepsilon_k \frac{h^j}{j!} \right) \leqq c_9 \varepsilon_k h^{t-r-2}$$

and so, we get

$$|b_p^{(m)}-a_p^{(m)}| \leqq \frac{1}{h^{p-1}} \sum_{t=0}^{r+1} c_9 c_{pt} \varepsilon_k h^{t-r-2}$$

and thus, from theorem 1.1, and the above inequality, we get

$$|s_m(x)-S_m(x)| \leqq \sum_{j=0}^{r+1} c_8 \varepsilon_k \frac{h^j}{j!} + \sum_{p=1}^{r+2} h^{p+r+1} \frac{1}{h^{p-1}} \sum_{t=0}^{r+1} c_9 c_{pt} \varepsilon_k h^{t-r-2} =$$

$$= c_8 \varepsilon_k \sum_{j=0}^{r+1} \frac{h^j}{j!} + c_9 \varepsilon_k \sum_{p=1}^{r+2} \sum_{t=0}^{r+1} c_{pt} h^t \leqq c_{11} \varepsilon_k$$

which is a bounded multiple of the introduced error ε_k. Hence the proposition.

THEOREM 1.3. *Under the assumptions of theorem* 1.2, *the inequality*

$$|s_m^{(t)}(x) - S_m^{(t)}(x)| \leqq c_{12} \varepsilon_k$$

holds for all $t=0, 1, \ldots, r+1$ *and* $m=k, k+1, \ldots, n-1$.

PROOF. Following the same procedure as in theorem 1.2, it is easy to prove this theorem.

CONSLUSION. As it has been shown in the above theorems, any variation in the calculated values is a bounded multiple of the introduced error ε_k. Hence the method is stable.

2. Stability of the method for the second order differential equation

The method for approximating the solution of the Cauchy problem in the non-linear second order differential equation was described, in details, in [2].

A change in any of the calculated values from $\bar{y}_k$ to $\bar{z}_k$ will impose a change in the calculated value of the first derivative at $x=x_{k+1}$ from $\bar{y}'_{k+1}$ to $\bar{z}'_{k+1}$. In general we assume that the change occured simultaneously in both of $\bar{y}_k$ and $\bar{z}'_k$. This leads us to solve

$$\bar{z}_{m+1} = \bar{z}_m + \bar{z}'_m h + \int_{x_m}^{x_{m+1}} \int_{x_m}^{t} f(u, z_m^*(u), z_m^{**}(u))\, du\, dt \tag{2.1}$$

and

$$\bar{z}'_{m+1} = \bar{z}'_m + \int_{x_m}^{x_{m+1}} f(t, z_m^*(t), z_m^{**\prime}(t))\, dt \tag{2.2}$$

instead of (2.1.11) in [2], and (2.1.12) in [2], respectively. Subtracting (2.1) from (2.1.11) of [2], (2.2) from (2.1.12) of [2] and denoting $\varepsilon_m = |\bar{z}_m - \bar{y}_m|$ and $\varepsilon'_m = |\bar{z}'_m - \bar{y}'_m|$ it is easy to get

$$\varepsilon_{m+1} \leqq \varepsilon_m(1+c_{23}h^2) + c_{24}h\varepsilon'_m, \tag{2.3}$$

which could be obtained in the same manner as shown in lemma 2.2.3 in [2], and

$$\varepsilon'_{m+1} \leqq \varepsilon'_m(1+c_{25}h) + c_{26}h\varepsilon_m, \tag{2.4}$$

which is obtained in the same manner as shown in lemma 2.2.1 of [2]. We proceed to prove the stability and we begin with:

LEMMA 2.1. *The inequality*

$$\varepsilon'_{m+1} \leqq c_{27}\varepsilon'_k + c_{28}\varepsilon_{r_0}$$

is true for all $m=k, k+1, \ldots, n-1$, *where,*

$$\varepsilon_{r_0} = \max(\varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_m), \quad k \leqq r_0 \leqq m.$$

PROOF. Starting with (2.4), the principle of the successive substitution implies

$$\begin{aligned}
\varepsilon'_{m+1} &\leqq \varepsilon'_m(1+c_{25}h) + c_{26}h\varepsilon_m,\\
\varepsilon'_m(1+c_{25}h) &\leqq \varepsilon'_{m-1}(1+c_{25}h)^2 + c_{26}h\varepsilon_{m-1}(1+c_{25}h),\\
\varepsilon'_{m-1}(1+c_{25}h)^2 &\leqq \varepsilon'_{m-2}(1+c_{25}h)^3 + c_{26}h\varepsilon_{m-2}(1+c_{25}h)^2,\\
&\vdots\\
\varepsilon'_{k+1}(1+c_{25}h)^{m-k} &\leqq \varepsilon'_k(1+c_{25}h)^{m-k+1} + c_{26}h\varepsilon_k(1+c_{25}h)^{m-k}
\end{aligned}$$

and it is easy to get

$$\varepsilon'_{m+1} \leqq \varepsilon'_k(1+c_{25}h)^{m-k+1}+c_{26}h \sum_{j=k}^{m} \varepsilon_j(1+c_{25}h)^{m-j} \leqq$$

$$\leqq \varepsilon'_k(1+c_{25}h)^{m-k+1}+c_{26}h \max_j \varepsilon_j \sum_{i=0}^{m-k} (1+c_{25}h)^i \leqq$$

$$\leqq \varepsilon'_k\left(1+c_{25}\frac{(b-a)}{n}\right)^n + c_{26}h\varepsilon_{r_0}\frac{(1+c_{25}h)^{m-k+1}-1}{c_{25}h} \leqq$$

$$\leqq \varepsilon'_k e^{c_{25}(b-a)}+\frac{c_{26}}{c_{25}}\varepsilon_{r_0}(e^{c_{25}(b-a)}-1) \leqq c_{27}\varepsilon'_k+c_{28}\varepsilon_{r_0}$$

where

$$\varepsilon_{r_0} = \max(\varepsilon_k, \varepsilon_{k+1}, \dots, \varepsilon_m), \quad k \leqq r_0 \leqq m.$$

Hence the proposition.

LEMMA 2.2. *The inequality*

$$\varepsilon_{m+1} \leqq \varepsilon_{r_0}(1+c_{29}h)+c_{30}h\varepsilon'_k$$

is true for all $m=k, k+1, \dots, n-1$, *where,*

$$\varepsilon_{r_0} = \max(\varepsilon_k, \varepsilon_{k+1}, \dots, \varepsilon_m), \quad k \leqq r_0 \leqq m.$$

PROOF. From lemma 2.1. we get

$$\varepsilon'_m \leqq c_{27}\varepsilon'_k+c_{28}\varepsilon_{r_1^*}$$

where

$$\varepsilon_{r_1^*} = \max(\varepsilon_k, \varepsilon_{k+1}, \dots, \varepsilon_{m-1}), \quad k \leqq r_1^* \leqq m-1$$

and since $\varepsilon_{r_1^*} \leqq \varepsilon_{r_0}$, we get

$$\varepsilon'_m \leqq c_{27}\varepsilon'_k+c_{28}\varepsilon_{r_0}.$$

Using the above inequality in 2.3, we get

$$\varepsilon_{m+1} \leqq \varepsilon_m(1+c_{23}h^2)+c_{24}h(c_{27}\varepsilon'_k+c_{28}\varepsilon_{r_0}) \leqq$$

$$\leqq \varepsilon_{r_0}(1+c_{23}h^2)+c_{24}h(c_{27}\varepsilon'_k+c_{28}\varepsilon_{r_0}) \leqq \varepsilon_{r_0}(1+c_{29}h)+c_{30}h\varepsilon'_k.$$

Hence the proposition.

THEOREM 2.1. *The inequality*

$$\varepsilon_{m+1} \leqq c_{31}\varepsilon_k+c_{32}\varepsilon'_k$$

is true for all $m=k, k+1, \dots, n-1$.

PROOF. We have from Lemma 2.2,

$$\varepsilon_{m+1} \leqq \varepsilon_{r_0}(1+c_{29}h)+c_{30}h\varepsilon'_k$$

where

$$\varepsilon_{r_0} = \max(\varepsilon_k, \varepsilon_{k+1}, \dots, \varepsilon_m), \quad k \leqq r_0 \leqq m.$$

The same procedure, as of Lemma 2.2, when repeated, but the interval $[x_m, x_{m+1}]$ is replaced by the interval $[x_{r_0-1}, x_{r_0}]$, implies

$$\varepsilon_{r_0} \leqq \varepsilon_{r_1}(1+c_{29}h)+c_{30}h\varepsilon_k'$$

where

$$\varepsilon_{r_1} = \max(\varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_{r_0-1}), \quad k \leqq r_1 \leqq r_0-1.$$

Going on and repeating these procedures, but only with intervals exchanged, we get

$$\varepsilon_{r_1} \leqq \varepsilon_{r_2}(1+c_{29}h)+c_{30}h\varepsilon_k'$$

where

$$\varepsilon_{r_2} = \max(\varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_{r_1-1}) \leqq \varepsilon_{r_3}(1+c_{29}h)+c_{30}h\varepsilon_k' \quad k \leqq r_2 \leqq r_1-1,$$

where

$$\varepsilon_{r_3} = \max(\varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_{r_2-1}), \quad k \leqq r_3 \leqq r_2-1,$$

and at the end we get the inequality

$$\varepsilon_{r_s} \leqq \varepsilon_{r_{s+1}}(1+c_{29}h)+c_{30}h\varepsilon_k',$$

where

$$\varepsilon_{r_{s+1}} = \max(\varepsilon_k) = \varepsilon_k, \quad r_{s+1} = k.$$

By rearranging the above inequalities, the principle of successive substitution implies

$$\begin{aligned}
\varepsilon_{m+1} &\leqq \varepsilon_{r_0}(1+c_{29}h) + c_{30}h\varepsilon_k', \\
\varepsilon_{r_0}(1+c_{29}h) &\leqq \varepsilon_{r_1}(1+c_{29}h)^2 + c_{30}h\varepsilon_k'(1+c_{29}h), \\
\varepsilon_{r_1}(1+c_{29}h)^2 &\leqq \varepsilon_{r_2}(1+c_{29}h)^3 + c_{30}h\varepsilon_k'(1+c_{29}h)^2, \\
&\vdots \\
\varepsilon_{r_s}(1+c_{29}h)^{s+1} &\leqq \varepsilon_k(1+c_{29}h)^{s+2}+c_{30}h\varepsilon_k'(1+c_{29}h)^{s+1}
\end{aligned}$$

and obviously we get

$$\varepsilon_{m+1} \leqq \varepsilon_k(1+c_{29}h)^{s+2}+c_{30}h\varepsilon_k' \sum_{j=0}^{s+1}(1+c_{29}h)^j \leqq$$

$$\leqq \varepsilon_k e^{c_{29}(b-a)}+\frac{c_{30}}{c_{29}}\varepsilon_k'(e^{c_{29}(b-a)}-1) \leqq c_{31}\varepsilon_k+c_{32}\varepsilon_k'.$$

Hence the proposition.

THEOREM 2.2. *The inequality*

$$\varepsilon_{m+1}' \leqq c_{33}\varepsilon_k'+c_{34}\varepsilon_k$$

is true for all $m=k, k+1, \ldots, n-1$.

PROOF. From lemma 2.1 we have

$$\varepsilon_{m+1}' \leqq c_{27}\varepsilon_k'+c_{28}\varepsilon_{r_0},$$

where

$$\varepsilon_{r_0} = \max(\varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_m).$$

Theorem 2.1 implies

$$\varepsilon_{r_0} \leqq c_{31}\varepsilon_k + c_{32}\varepsilon'_k.$$

Hence, we obviously get

$$\varepsilon'_{m+1} \leqq c_{27}\varepsilon'_k + c_{28}(c_{31}\varepsilon_k + c_{32}\varepsilon'_k) \leqq c_{33}\varepsilon'_k + c_{34}\varepsilon_k.$$

Thus the proof is complete.

DEFINITION 2.2. Let

$$\varepsilon_{m+1}^{(q+2)} = |\bar{z}_{m+1}^{(q+2)} - \bar{y}_{m+1}^{(q+2)}|,$$

where

$$\bar{z}_{m+1}^{(q+2)} = f^{(q)}(x_{m+1}, \bar{z}_{m+1}, \bar{z}'_{m+1})$$

and $q=0, 1, \ldots, r$.

THEOREM 2.3. *The inequality*

$$\varepsilon_{m+1}^{(q+2)} \leqq c_{35}\varepsilon_k + c_{36}\varepsilon'_k$$

is true for all $m=k, k+1, \ldots, n-1$ *and all* $q=0, 1, \ldots, r$.

PROOF. By definition 2.1 and equation (2.1.13-b) in [2], we have

$$\varepsilon_{m+1}^{(q+2)} = |f^{(q)}(x_{m+1}, \bar{z}_{m+1}, \bar{z}'_{m+1}) - f^{(q)}(x_{m+1}, \bar{y}_{m+1}, \bar{y}'_{m+1})|.$$

Applying the Lipschitz condition (2.1.2) of [2], it becomes

$$\leqq K(|\bar{z}_{m+1} - \bar{y}_{m+1}| + |\bar{z}'_{m+1} - \bar{y}'_{m+1}|).$$

Using theorems 2.1 and 2.2, we get

$$\varepsilon_{m+1}^{(q+2)} \leqq K(c_{31}\varepsilon_k + c_{32}\varepsilon'_k) + K(c_{33}\varepsilon'_k + c_{34}\varepsilon_k) \leqq c_{35}\varepsilon_k + c_{36}\varepsilon'_k.$$

Hence the proposition.

Theorems 2.1, 2.2 and 2.3 imply

$$|\bar{z}_m^{(t)} - \bar{y}_m^{(t)}| \leqq c_{37}\varepsilon_k + c_{38}\varepsilon'_k \leqq c_{39} \tag{2.5}$$

which holds for all $m=k+1, \ldots, n$ and all $t=0, 1, \ldots, r+2$.

THEOREM 2.4. *If any of the calculated values* $\bar{y}_k$ *is changed to* $\bar{z}_k$ *and* $\bar{y}'_k$ *is changed to* $\bar{z}'_k$, *and, consequently, the spline function approximating the solution of* (1.1) *in* [2], *and constructed in theorem* 3.1 *in* [2], *is also changed from* $S_\Delta(x)$ *to* $s_\Delta(x)$, *then for any* $x \in [x_m, x_{m+1}]$, $m=k, \ldots, n-1$, *the inequality*

$$|s_m^{(t)}(x) - S_m^{(t)}(x)| \leqq c_{40}\varepsilon_k + c_{41}\varepsilon'_k \leqq c_{42}$$

holds for all $t=0, 1, \ldots, r+2$, *where* ε_k *and* ε'_k *are the introduced errors.*

PROOF. Following the same procedure as of theorems 1.2 and 1.3 of Part I, it is easy to prove this theorem.

CONCLUSION. Theorems 2.1, 2.2, 2.3 and 2.4 ensure the stability of the method, since any variation in any of the calculated values is a bounded multiple of the introduced errors ε_k and ε'_k.

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A CLASS OF CONTINUOUS FUNCTIONS AND THEIR DEGREE OF APPROXIMATION

By

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1. Introduction. The problem of approximation of continuous functions by trigonometric polynomials has been discussed by ALEXITS [1]. GOEL and SAHNEY [5] have unified the above result. In this paper a wider class of functions and a more general method of approximation has been considered, which generalises the above results. This also answers one of the questions raised by HOLLAND, SAHNEY and TZIMBALARIO [3].

1.1. A series $\sum_{n=0}^{\infty} c_n$ with the sequence of partial sums $\{s_n\}$ is said to be summable to s by a regular triangular matrix method (Λ), defined by HARDY [2], such that $\Lambda_{n,k} \geqq 0$ for all $k \leqq n$ and $\Lambda_{n,k}=0$, $k>n$, also $\sum_k \Lambda_{n,k} \to 1$ as $n \to \infty$. Write

$$t_n = \sum_{k=0}^{n} \Lambda_{n,k} s_k \to s \quad \text{as} \quad n \to \infty.$$

We call t_n the (Λ)-means of $\sum_{n=0}^{\infty} c_n$.

1.2. Let $f(x)$ be periodic with period 2π, and integrable in the sense of Lebesgue. The Fourier series associated with $f(x)$ is given by

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1.2.1}$$

2. Let $C^*[0,2\pi]$ denote the class of all continuous functions on $[0,2\pi]$, periodic and of period 2π. Following LORENTZ [4] we define the degree of approximation of $f(x)$ by trigonometric polynomials as

$$E_n(f) \equiv \|f(x)-T_n(x)\| = \operatorname*{Max}_x |f(x)-T_n(x)|$$

where $T_n(x)$ is a trigonometric polynomial of degree n.

2.1. The following results are known.

THEOREM A [5]. *The degree of approximation of a periodic function f with period 2π and belonging to the class* Lip α, $0<\alpha\leqq 1$, *is given by*

$$\operatorname*{Max}_{0\leqq x \leqq 2\pi} |f(x)-T_n(x)| = O\left\{\frac{1}{P_n}\sum_{k=1}^{n} \frac{P_k}{k^{1+\alpha}}\right\} \tag{2.1.1}$$

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where $T_n(x)$ *are the* (N, p_n)*-means of its Fourier series* (1.2.1) *provided the sequence* $\{p_n\}$ *is positive and non-increasing.*

THEOREM B [3]. *If* $\omega(t)$ *is the modulus of continuity of* $f \in C^*[0,2\pi]$, *then the degree of approximation of f by the Nörlund means of the Fourier series for f is given by*

$$E_n \equiv \max_{0 \leqq t \leqq 2\pi} |f(t) - T_n(t)| = O\left\{\frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k \omega(1/k)}{k}\right\} \tag{2.1.2}$$

where T_n *are the* (N, p_n)*-means.*

The following theorem is proved here.

THEOREM. *If* $\omega(t)$ *is the modulus of continuity of* $f \in C^*[0,2\pi]$, *then the degree of approximation by the triangular matrix-means of the Fourier series of f is given by:*

$$E_n \equiv \max_{0 \leqq x \leqq 2\pi} |f(x) - t_n(x)| = O\left\{\sum_{k=1}^{n} \frac{D_{n,k}\omega(1/k)}{k}\right\} \tag{2.1.3}$$

where t_n *are the* (Λ)*-means, such that*

$$D_{n,k} = \sum_{r=0}^{k} \Lambda_{n,r}. \tag{2.1.4}$$

(2.1.5) Also, we define the sequence $\Lambda_{n,(u)}$ in terms of $\{\Lambda_{n,k}\}$, so that $\Lambda_{n,(u)}$ is monotonic decreasing for all $u \geqq 0$, i.e. $\Lambda_{n,u} \equiv \Lambda_{n,(u)}$.

3. We shall need the following lemmas.

LEMMA 1. *If the sequence* $\{\Lambda_{n,k}\}$ *is defined as in* (2.1.5) *then*

$$\omega(1/n) \leqq M \sum_{k=1}^{n} \frac{D_{n,k}\omega(1/k)}{k}$$

for M an arbitrary constant, not the same at each occurrence.

PROOF. Consider

$$\sum_{k=1}^{n} \frac{D_{n,k}\omega(1/k)}{k} \geqq \omega(1/n) \sum_{k=1}^{n} \left[\frac{1}{k}\left\{\sum_{r=0}^{k} \Lambda_{n,r}\right\}\right] \geqq$$

$$\geqq \omega(1/n) \sum_{k=1}^{n} \frac{1}{k}(k\Lambda_{n,k}) = \omega(1/n) \sum_{k=1}^{n} \Lambda_{n,k},$$

by (2.1.5). By regularity of the method of summation the latter is $M\omega(1/n)$, where M is some positive constant.

Consequently we get,

$$\omega(1/n) \leqq M\left\{\sum_{k=1}^{n} \frac{D_{n,k}\omega(1/k)}{k}\right\}$$

which proves the lemma.

LEMMA 2. *If the sequence* $\{\Lambda_{n,k}\}$ *is defined as in* (2.1.5) *then*

$$\left|\sum_{k=1}^{n} \frac{\Lambda_{n,k}\sin(k+1/2)u}{\sin(u/2)}\right| < c\,\frac{D_{n,(1/u)}}{u}$$

where c is a constant, not the same at each occurrence, and $\frac{1}{n} \leqq u \leqq \delta < \pi$.

PROOF. Choose m=integral part of $(1/u)$ and suppose that $\frac{1}{n} \leqq u \leqq \delta < \pi$. We observe that $m \sin\frac{u}{2} > \frac{mu}{\pi}$ for $0<u<\pi$, consequently for $u>0$ and $m \leqq n$ we have

$$\left|\frac{\sum_{k=1}^{n} \Lambda_{n,k}\sin(k+1/2)u}{\sin u/2}\right| \leqq$$

$$\leqq \frac{1}{\sin u/2}\left[\left|\sum_{k=1}^{m} \Lambda_{n,k}\sin(k+1/2)u\right| + \left|\sum_{k=m+1}^{n} \Lambda_{n,k}\sin(k+1/2)u\right|\right] <$$

$$< \frac{1}{\sin u/2}\left[\sum_{k=1}^{m} \Lambda_{n,k}\,|\sin(k+1/2)u| + \Lambda_{n,m}\left|\sum_{k=0}^{n}\sin(k+1/2)u\right|\right] \leqq$$

(by monotonic decreasing property of $\Lambda_{n,m}$)

$$\leqq \frac{D_{n,m}}{\sin u/2} + \Lambda_{n,m}\frac{|1/2\{1-\cos(n+1)u\}|}{(\sin u/2)^2} \leqq \frac{D_{n,m}}{\sin u/2} + \frac{m\Lambda_{n,m}}{m(\sin u/2)^2} <$$

$$< \frac{D_{n,m}}{\sin u/2} + \frac{cD_{n,m}}{m(\sin u/2)^2} = \frac{D_{n,(1/u)}}{\sin u/2} + \frac{cD_{n(1/u)}}{m(\sin u/2)^2} < \frac{D_{n,(1/u)}}{\sin u/2} + \frac{cD_{n,(1/u)}}{\sin(u/2)} < \frac{cD_{n,(1/u)}}{u}$$

which proves the Lemma.

4. PROOF OF THE THEOREM. We see, [6], that the n^{th} partial sum of the Fourier series is given by

$$s_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx) =$$

$$= \frac{1}{2\pi}\int_0^{\pi}\{f(x+t)+f(x-t)\}\frac{\sin(n+1/2)t}{\sin t/2}\,dt.$$

Thus we have,

$$t_n(x)-f(x) = \frac{1}{2\pi}\int_0^{\pi}\{f(x+t)+f(x-t)-2f(x)\}\sum_{k=0}^{n}\Lambda_{n,k}\frac{\sin(k+1/2)t}{\sin t/2}\,dt.$$

If we write

$$\varphi(t) = \left|\frac{1}{2}f(x+t)+\frac{1}{2}f(x-t)-f(x)\right|$$

we get $\varphi(t) \leqq \omega(t)$ and then

$$|f(x)-t_n(x)| \leqq \frac{1}{\pi}\int_0^{\pi/n} \frac{\omega(t)}{\sin t/2}\left|\sum_{k=0}^{n} \Lambda_{n,k}\sin(k+1/2)t\right| dt +$$

$$+\frac{1}{\pi}\int_{\pi/n}^{\pi}\omega(t)\left|\sum_{k=0}^{n}\frac{\Lambda_{n,k}\sin(k+1/2)t}{\sin t/2}\right| dt = I_1+I_2 \quad \text{(say)}.$$

Now

$$I_1 \leqq \frac{1}{\pi}\int_0^{\pi/n}\frac{\omega(t)}{t/2}\left|\sum_{k=0}^{n}\Lambda_{n,k}\sin(k+1/2)t\right| dt + o(1) = O(n)\int_0^{\pi/n}\omega(t)\,dt =$$

$$= O\left[n\omega(1/n)\int_0^{\pi/n} dt\right] = O[\omega(1/n)] = O\left[\sum_{k=1}^{n}\frac{D_{n,k}\omega(1/k)}{k}\right]$$

(by Lemma 1). Moreover

$$I_2 = \frac{1}{\pi}\int_{\pi/n}^{\pi}\omega(t)\left|\sum_{k=0}^{n}\Lambda_{n,k}\frac{\sin(k+1/2)t}{\sin t/2}\right| dt = O\left[\int_{\pi/n}^{\pi}\frac{\omega(t)}{t}D_{n,(1/t)}\,dt\right]$$

(by Lemma 2). Changing the variable, we get

$$I_2 = O\left[\int_{n/\pi}^{1/\pi}\omega(1/u)D_{n,(u)}u(-1/u^2)\,du\right] = O\left[\int_1^n \frac{D_{n,(u)}}{u}\omega(1/u)\,du\right] =$$

$$= O\left[\sum_{k=1}^{n}\frac{D_{n,k}\omega(1/k)}{k}\right].$$

Combining the estimates of I_1 and I_2 we have

$$E_n \equiv \max_{0\leqq x\leqq 2\pi}|f(x)-t_n(x)| = O\left[\sum_{k=1}^{n}\frac{D_{n,k}\omega(1/k)}{k}\right]$$

which proves the theorem.

5. The following corollaries can be derived from the theorem.

COROLLARY 1. *The degree of approximation of* $f\in C^*[0,2\pi]$ *and belonging to* Lip α, $0<\alpha\leqq 1$, *by the triangular matrix means* (Λ) *of its Fourier series, is given by:*

$$E_n \equiv \underset{0\leqq x\leqq 2\pi}{\mathrm{Max}}|f(x)-t_n(x)| = O\left\{\sum_{k=1}^{n}\frac{D_{n,k}}{k^{1+\alpha}}\right\} \tag{5.1}$$

where $t_n(x)$ *are the matrix means and the sequence* $\{\Lambda_{n,k}\}$ *is the same as defined in the theorem.*

PROOF. From (2.1.3)

$$E_n \equiv \max_{0 \leq x \leq 2\pi} |f(x) - t_n(x)| = O\left\{\sum_{k=1}^{n} \frac{D_{n,k}\omega(1/k)}{k}\right\} =$$

$$= O\left\{\sum_{k=1}^{n} \frac{D_{n,k}(1/k^\alpha)}{k}\right\} = O\left\{\sum_{k=1}^{n} \frac{D_{n,k}}{k^{1+\alpha}}\right\},$$

which completes the proof.

COROLLARY 2. *If* $\Lambda_{n,k} = \dfrac{p_{n-k}}{P_n}$ *in the theorem, then*

$$E_n = O\left[\frac{1}{P_n}\sum_{k=1}^{n} \frac{P_k\,\omega(1/k)}{k}\right],$$

and we have Theorem **B**.

PROOF.

$$E_n \equiv O\left[\sum_{k=1}^{n} \frac{D_{n,k}\omega(1/k)}{k}\right] = O\left[\sum_{k=1}^{n} \frac{\left\{\sum_{r=1}^{k} \Lambda_{n,r}\right\}\omega(1/k)}{k}\right] =$$

$$= O\left[\sum_{k=1}^{n} \frac{\left\{\sum_{r=1}^{k} \frac{p_{n-r}}{P_n}\right\}\omega(1/k)}{k}\right] = O\left[\frac{1}{P_n}\sum_{k=1}^{n} \frac{P_k\omega(1/k)}{k}\right].$$

REMARK. An independent proof of Corollary 1 can be developed along the same lines as the theorem.

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CONDITIONS FOR A RING TO BE FISSILE

By

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If the maximal torsion ideal of a ring A is a ring-theoretic summand of A then A is called fissile. If the ring A has minimum condition on principal right ideals then its additive group A^+ is the direct sum of a reduced torsion group and a divisible group. In this note we investigate under what conditions a ring with this additive structure is fissile (German: spaltbar). We obtain that if A has minimum condition on principal right ideals then A is fissile. This answers the question posed by DINH VAN HUYNH in two recent papers ([1] and [2]) and by F. SZÁSZ [8] in his Problem 74 (raised in 1968), and generalizes the theorem of F. SZÁSZ that an Artinian ring is fissile (cf. [7]).*

The following symbols and terminology will be used here:

Min-PR ring: ring with minimum condition on principal right ideals.

Max-PR ring: ring with maximum condition on principal right ideals.

$J \lhd A : J$ is an ideal of the ring A.

$H \leqq G : H$ is a subgroup of the group G.

$\simeq$: ring-theoretic isomorphism.

$\dotplus$: group-theoretic direct sum.

$\oplus$: ring-theoretic direct sum.

A^+ : the additive group of the ring A.

$M \cdot N$: set of all products mn, for $m \in M$, $n \in N$.

$T = \{a \in A \mid na = 0$ for some positive integer $n\}$ is the torsion ideal of the ring A (or the torsion subgroup of the group A^+).

THEOREM 1 (F. SZÁSZ [6]). *Let A be a Min-PR ring. Then its additive group $A^+ = B \dotplus D$, where B is a reduced torsion group and D is a divisible group.*

A proof of this theorem is given in [6] p. 424.

DEFINITION. The ring A is fissile if, and only if, for some $J \lhd A$, $A = J \oplus T$, where T is the torsion ideal of A.

THEOREM 2. *Let A be a ring and assume $A^+ = B \dotplus D$, where B is a reduced torsion group and D is divisible. Then:*

(1) $D \lhd A$ *and* $B \cdot D = D \cdot B = 0$.

* *Editorial remark.* Independently from the author, Dinh Van Huynh also has proved (May 1976) that every Min PR-ring is fissile. See *Bull. Acad. Polon. Sci. Classe III* (1976).

(2) *If T is the torsion subgroup of D, then the torsion subgroup of* A^+, $T_A=B\dotplus T$.

(3) $D/T\simeq A/T_A$.

(4) *For* $J\leqq A^+$, $D=T\oplus J$ *if, and only if,* $A=T_A\oplus J$. *Thus, in particular, A is fissile if, and only if, D is fissile.*

PROOF. (1) Since D is the maximal divisible subgroup of A^+, it is fully invariant and hence $D\lhd A$.

Let $d\in D$, $b\in B$ and suppose b has order n. Then $d=nd_1$ for some $d_1\in D$; hence $db=(nd_1)b=d_1(nb)=0$ and similarly $bd=0$.

(2) Since B is a torsion group, $B\leqq T_A=$ the torsion subgroup of A. Thus $T_A=$ $=B\dotplus(D\cap T_A)$ but clearly $D\cap T_A$ is the torsion subgroup of D.

(3) Consider the mapping

$$T+d \rightarrow T_A+d \quad (d\in D)$$

from D/T to A/T_A. Note that

$$T_A+d_1 = T_A+d_2 \quad (d_1, d_2\in D) \quad \Leftrightarrow \quad d_1-d_2\in T_A \Leftrightarrow d_1-d_2\in T_A\cap D = T \quad \Leftrightarrow$$
$$\Leftrightarrow \quad T+d_1 = T+d_2.$$

Thus the mapping is well-defined and injective. Since $A^+=B+D=B+T+D=$ $=T_A+D$, it is also surjective. It is clear that it preserves sums and products and hence it is an isomorphism.

(4) Suppose $D=T\oplus J$ so that $J\lhd D$ and $T\cap J=0$. Then $A^+=B\dotplus D=$ $=B\dotplus T\dotplus J=T_A\dotplus J$, since $T_A=B\dotplus T$. T_A is the torsion ideal of A and J (as an ideal of D) is a ring.

Now $J\simeq D/T$ and hence J is divisible. Since T_A is a torsion group $T_A\cdot J=$ $=J\cdot T_A=0$. Thus $A=T_A\oplus J$.

Assume conversely $A=T_A\oplus J$. Then $A^+=B\dotplus T\dotplus J$ and $T\dotplus J$ is the maximal divisible subgroup of A^+. Thus $D^+=T\dotplus J$ but $T\lhd D$ and $J\lhd D$ so that $D=T\oplus J$.

This theorem allows us to consider only the case where A is a divisible ring. In this case $A^+=T\dotplus F$, where T is a torsion divisible group (and hence the sum of groups of type p^∞) and F is a torsion-free divisible group (and hence a sum of groups isomorphic to the rational numbers). We have: $T\lhd A$ and $TA=AT=0$ and the question reduces to when can F be chosen so that it is closed under multiplication.

THEOREM 3. *Let A be a divisible ring and T its torsion ideal. Then A is fissile if, and only if* $A^2\cap T=0$.

PROOF. Assume A is fissile so that for some ideal J, $A=T\oplus J$. Clearly $J^2\subseteqq A^2$. On the other hand, suppose $a_i\in A$ $(i=1,2)$; writing $a_i=t_i+j_i$ with $t_i\in T$ and $j_i\in J$ (for $i=1,2$) we have $a_1a_2=j_1j_2$ since $AT=TA=0$. Thus $A^2\subseteqq J^2$ and so $A^2=J^2$. Hence $A^2\cap T=J^2\cap T\subseteqq J\cap T=0$ so that $A^2\cap T=0$.

Conversely suppose $A^2\cap T=0$ so that $T+A^2=T\dotplus A^2$. Since $T+A^2$ is a divisible subgroup of A^+, for some $C\leqq A^+$

$$A^+ = (T\dotplus A^2)\dotplus C = T\dotplus(A^2+C), \quad (A^2+C)(A^2+C) \subseteqq AA = A^2 \subseteqq A^2+C$$

and

$$T(A^2+C) = (A^2+C)T = 0.$$

Therefore $A^2+C\lhd A$ so that $A=T\oplus(A^2+C)$ and A is fissile.

THEOREM 4. *Let A be a ring and T an ideal such that $AT=TA=0$. Assume that for $\bar{a}\in\bar{A}=A/T$, $\bar{a}\in\bar{a}\bar{A}$. Then $A=T\oplus A^2$.*

PROOF. For $a\in A$, there are elements $e\in A$, $t\in T$ such that $a=ae+t$. Hence $A=A^2+T$. Since $T\lhd A$ and $A^2\lhd A$ it is sufficient to show that $A^2\cap T=0$.

If x is an element of A^2,

$$x=\sum_{i=1}^{n} a_i' a_i, \quad \text{where} \quad a_i', a_i\in A, \quad 1\leqq i\leqq n.$$

Calling n the length of this representation for x, we prove by induction on n that if x has a representation of length n and $x\in T$, then $x=0$.

Note first that if $a', a\in A$, there is an element $e\in A$ with $a'a=a'ae$. This is the case since $a=ae+t$ with $t\in T$ and therefore, $a'a=a'ae+a't=a'ae$ since $AT=0$.

Thus if $x\in T$ has a representation of length 1, $x=a_1'a_1$. Choosing e_1 so that $a_1'a_1e_1=0$, we have $0=xe_1=a_1'a_1e_1=a_1'a_1$ and hence $x=0$.

Suppose now that if $x\in T$ has a representation of length $<n, x=0$. Let $x=\sum_{i=1}^{n} a_1'a_i\in T$. Then if e is chosen so that $a_n'a_ne=a_n'a_n$,

$$0=xe=\sum_{i=1}^{n} a_i'a_ie=\sum_{i=1}^{n-1} a_i'a_ie+a_n'a_n.$$

Subtracting we get $x=\sum_{i=1}^{n-1} a_i'(a_i-a_ie)$ and hence x has a representation of length $n-1$. This implies $x=0$ and the proof is complete.

COROLLARY. *If A is a divisible ring and T its torsion ideal, then $\bar{A}=A/T$ is torsion-free and divisible and if $\bar{a}\in\bar{a}\bar{A}$, $A=T\oplus A^2$.*

REMARK. Theorem 4 is a generalization of the following

THEOREM OF KERTÉSZ ([4], Satz). *Let T be an ideal of the ring A and assume $A^+=T^+\dotplus B$ for some subgroup B. If $TB=BT=0$ and if A has a right identity* mod *T then T is a ring-theoretic direct summand of A.*

THEOREM 5. *Let R be a torsion-free divisible ring. If R is a* Min-*PR or a* Max-*PR ring, then for $r\in R$, $r\in rR$.*

REMARK. F. HANSEN [3] has proved this result for divisible right Noetherian rings and has shown that, in fact, this implies that such rings have a right identity.

PROOF. For $r\in R$, let $(r)=\{ir+rx|i\in Z, x\in R\}$ denote the principal right ideal generated by r.

Assuming that R satisfies Min-PR, we deduce from $(r)\geqq(2r)\geqq\ldots\geqq(2^nr)\geqq$ $\geqq\ldots$ that for some n, $(2^nr)=(2^{n+1}r)=2^nr=i(2^{n+1}r)+(2^{n+1}r)x$ for some $i\in Z$, $x\in R\Rightarrow$ there is a $k\neq0$ in Z and a $y\in R$ with $kr=ry$. Since R is divisible, $y=kz$ for some $z\in R$. Hence $kr=krz\Rightarrow r=rz$ since R is torsion-free. Thus if R is a Min-PR ring, the theorem is true.

Now assume R is a Max-PR ring. Note that if $r\in R$, $r=2^nr_n$ uniquely for each natural number n, since R is torsion-free, divisible. Then $(r_1)\subseteqq(r_2)\subseteqq\ldots\subseteqq(r_n)\subseteqq\ldots$

and so $(r_n)=(r_{n+1})$ for some n. Therefore, $r_{n+1}=ir_n+r_nx$ which implies $r=2ir+2rx$. From this we deduce (as above) that $r\in rR$.

Summarizing these results we obtain:

THEOREM 6. *Let A be a ring and assume $A^+=B\dotplus D$, where B is a reduced torsion group, and D a divisible group. Assume that if $\bar{a}\in\bar{A}=A/T_A$, then $\bar{a}\in\bar{a}\bar{A}$, where T_A is the torsion ideal of A. Then $A=T_A\oplus D^2$ and D^2 is the unique ring-theoretic complement of T_A.*

PROOF. By Theorem 2 (3) $A/T_A\approx D/T$, where T is the torsion ideal of D. Then $D=T\dotplus F$ and $TA=AT=0$ so that by Theorem 4, $D=T\oplus D^2$. We show next that if $D=T\oplus J$, then $J=D^2$. But $J^2=D^2$ (as shown above) so that $D^2=J^2\subseteqq J$ and this implies that $J=D^2$.

Finally by Theorem 2 (4), $A=T_A\oplus D^2$ and $A=T_A\oplus J\Rightarrow J=D^2$ again by Theorem 2 (4) and the uniqueness of the ring-theoretic complement of T in D.

COROLLARY 1. *Let A be a* Min-*PR ring. Then $A=T_A\oplus D^2$, where T_A is the torsion ideal of A and D the maximal divisible ideal. D^2 is the unique ring-theoretic complement of A.*

COROLLARY 2. *Let A be a* Max-*PR ring. If $A^+=B\dotplus D$, where B is a reduced torsion group and D is divisible, then $A=T_A\oplus D^2$, where T_A is the torsion ideal of A. D^2 is the unique ring-theoretic complement of A.*

We now show that if G is a mixed additive (abelian) group which is divisible, then there is a ring on G which is not fissile.

LEMMA (Everett). *Let G be an additive (abelian) group and assume $G=F\dotplus T$, where F is a ring (under $+$ and $\cdot$). Assume $\langle\ ,\ \rangle$ is a balanced map from $F\times F$ to T, i.e. $\langle a_1, a_2\rangle\in T$ for $a_1, a_2\in F$ and for $a_1, a_1', a_2, a_2'\in F$,*

(i) $$\langle a_1+a_1', a_2\rangle = \langle a_1, a_2\rangle+\langle a_1', a_2\rangle$$

(ii) $$\langle a_1, a_2+a_2'\rangle = \langle a_1, a_2\rangle+\langle a_1, a_2'\rangle$$

(iii) $$\langle a_1, a_2a_2'\rangle = \langle a_1\cdot a_2, a_2'\rangle.$$

Moreover, in G we define $*$ by

$$(a_1+t_1)*(a_2+t_2) = a_1\cdot a_2+\langle a_1, a_2\rangle \quad \textit{for} \quad a_1, a_2\in F \quad \textit{and} \quad t_1, t_2\in T.$$

Then G is a ring under $+$ and $*$.

PROOF. The result follows from Everett's fundamental theorem on ring extensions (cf., e.g., [5] p. 188). It is also easily verified directly; (i) and (ii) guarantee that the distributive laws hold and (iii) gives the associative law.

THEOREM 7. *Let G be a divisible additive (abelian) and mixed group. Then there is a ring A on G which is not fissile.*

PROOF. First let $H=Q\dotplus T$, where Q is the additive group of rational numbers, and $T(\neq 0)$ is a divisible torsion group; and let $f\neq 0$ be a homomorphism from Q to T; one exists since we can map $Q\to Q/Z$ and then map Q/Z onto a (non-zero) primary component of T. For $a, b \in Q$ define $\langle a, b\rangle = f(ab)$, where ab denotes the (ordinary) product of the rational numbers a and b. It is easily verified that $\langle\ ,\ \rangle$ satisfies (i), (ii) and (iii) of the Lemma. Thus (by the Lemma) there is a ring C with $C^+ = H$ and multiplication $*$ defined by $(a_1+t_1)*(a_2+t_2)=\langle a_1, a_2\rangle = f(a_1 a_2)$ for $a_1, a_2 \in F$, $t_1, t_2 \in T$. If $0 \neq t \in \mathrm{Im}\,(f)$, let $f(a)=t$, where $a \in Q$; then $a*1=\langle a, 1\rangle = f(a)=t\neq 0$. Thus $0\neq a*1 \in C^2 \cap T$. By Theorem 3, C is not fissile.

Now let $G=F\dotplus T$, where $F\neq 0$ is a divisible, torsion-free group and $T\neq 0$ is a divisible torsion group. Then $G=F_1\dotplus H$, where F_1 is divisible and torsion-free and H is as defined above. Construct a ring H_1 on F_1 (e.g. we can take A_1 to be a field), construct a ring C on H as above. Let A be the ring on G given by $A=A_1\oplus C$. Since $C^2\cap T\neq 0$, $A^2 \cap T \neq 0$ and again by Theorem 3, A is not fissile.

Corollary. *If $G=B\dotplus D$, where B is a reduced torsion group and D is a divisible mixed group, then there is a ring on G which is not fissile.*

Proof. Let R be a ring on B and S a ring on D which is not fissile (one exists by the theorem). Then by Theorem 2, $A=R\oplus S$ is a ring on G which is not fissile.

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A REMARK ON A PAPER OF FUCHS AND SZELE

By

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FUCHS and SZELE [1] have determined those associative rings which enjoy the property that every subring has a multiplicative identity. Their result is an application of the Artin-Wedderburn structure theory and a commutativity theorem of Jacobson. In this note we extend their work to alternative rings. Our result depends on a recent associativity theorem of SLATER [3] analogous to a commutativity theorem of HERSTEIN [2].

THEOREM (Herstein). *If R is an associative ring with centre Z such that for every $d\in R$ there exists a polynomial $P_d(t)$ with integer coefficients such that $d^2P_d(d)-d\in Z$, then R is commutative.*

If $u, v, w\in R$, R an alternative ring, then we call the element $(u, v, w)=(uv)w-u(vw)$ the associator of u, v and w.

THEOREM (Slater). *Suppose R is an alternative ring, and for each associator $d=(u, v, w)\in R$ there exists a polynomial $P_d(t)$ with integer coefficients such that $d=d^2P_d(d)$. Then R is associative.*

We turn now to our extension of the theorem of Fuchs and Szele.

THEOREM. *Every subring of an alternative ring R has a right identity if and only if R is a finite direct sum of (commutative and associative) fields of finite characteristic each of which is algebraic over its prime subfield.*

PROOF. Assume every subring of the alternative ring R has a right identity. Let $x\in R$; we show that there exists a polynomial $P_x(t)$ with integer coefficients such that $x^2P_x(x)=x$. Let $S=\left\{\sum_{i=1}^{n} a_i x^i \mid n\in \mathbf{Z}^+, a_i\in \mathbf{Z}\right\}$. Clearly S is an associative subring of R and thus by hypothesis there exists a right identity for S, say $e_s=\sum_{i=1}^{n} a_i x_i$. Then $xe_s=x$, and if we set $P_x(t)=\sum_{i=1}^{n} a_i t^{i-1}$ we have $x^2P_x(x)=x$.

For each associator $d=(u, v, w)\in R$ there exists a polynomial $P_d(t)$ such that $d=d^2P_d(d)$ and Slater's theorem implies that R is associative. At this point we could apply the result of Fuchs and Szele to finish; however we take a somewhat different route by next applying Herstein's theorem to see that R is commutative.

Any ideal of R is of the form Re for some idempotent e, since it has an identity as a subring. Thus if 1 is the identity of R, $R=Re\oplus R(1-e)$ and R is completely reducible and hence a finite direct sum of fields $F_1, \ldots, F_n$. No F_i is of characteristic zero for then it would contain the even integers. Applying the first paragraph again shows that every element of F_i is algebraic over its prime field.

The converse follows as in Fuchs and Szele.

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A HOMOLOGY THEORY FOR SPANNING TREES OF A GRAPH

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1. Results. The following conjecture has been formulated and partially solved by A. FRANK [2]:

THEOREM 1. *Given a k-connected graph G, k points $v_1, \dots, v_k \in V(G)$, and k positive integers $n_1, \dots, n_k$ such that $n_1 + \dots + n_k = |V(G)|$, there exists a partition $\{V_1, \dots, V_k\}$ of $V(G)$ such that $v_i \in V_i$, $|V_i| = n_i$ and each V_i spans a connected subgraph.*

A very closely related conjecture has been made by S. MAURER. A. FRANK [2] proved the above statement for the case $k=2$ and also when $n_1, \dots, n_{k-1} \leqq 3$. His proof also provides an efficient algorithm to find this partition. K. MILLIKEN proved the case $k=3$ and a generalization to infinite graphs (private correspondence). Theorem 1 has been proved, independently of the author, by E. GYŐRY. His proof uses more elementary methods [3].

The case $k=2$ also follows from an unpublished lemma of A. J. BONDY and the author:

THEOREM 2. *Given a 2-connected graph G on n points, with a specified point a. Call two spanning trees of G neighbouring if they have a common subtree with $n-1$ points, including a. Then any two spanning trees of G can be connected by a chain of spanning trees, in which any two consecutive members are neighbouring.*

This theorem is a strengthening of a well-known theorem of Whitney. It was used to show the following

THEOREM 3. (A. J. BONDY, L. LOVÁSZ, unpublished). *If G is a 2-connected, non-bipartite graph on $2n$ vertices then $V(G)$ can be partitioned into two n-element classes such that the edges connecting these classes form a connected spanning subgraph. In other words, there is a spanning tree T such that the two colour-classes in the (unique) 2-colouration of F have both n elements.*

Theorem 2 above gives the idea that to attack the general problem formulated in Theorem 1, one might try to generalize its contents to k-connected graphs. In order to do so we introduce a few notions.

Let G be a digraph and $a \in V(G)$. Assume there are spanning arborescences of G rooted at a (since all arborescences considered will be rooted at a, we shall not say this explicitly in the sequel). Set $m = |E(G)|$. Then each spanning arborescence of G can be regarded as a vertex of the m-dimensional hypercube. It will cause no confusion if we denote a spanning arborescence and its representing point by the same letter.

We define certain convex cells on this set of vertices. Take an arborescence $A\subset G$, and let $V(G)-V(A)=\{x_1, \dots, x_r\}$. Assume that each point x_i can be reached on at least two edges from A, and take, for each x_i, a set N_{x_i} of at least two edges connecting A to x_i. Denote by $C(A; N_{x_1}, \dots, N_{x_r})$ the convex hull of all spanning arborescences which arise from A by adding one line of each N_{x_i}. The set of these arborescences can be written formally as

$$\{A\}\times N_{x_1}\times\dots\times N_{x_r}$$

and therefore, $C(A; N_{x_1}, \dots, N_{x_r})$ is the cartesian product of r simplices of dimension $|N_{x_1}|-1, \dots, |N_{x_r}|-1$, respectively. This also implies that the faces of these convex cells are of the same form.

The 1-dimensional skeleton of the arborescence complex is the graph whose vertices are the spanning arborescences, two of them being adjacent iff they are "neighbouring" in the sense of Theorem 2. The 2-dimensional cells of $\mathcal{K}$ are of two kinds: triangles spanned by three spanning arborescences containing a common $(n-1)$-point arborescence, and parallelograms, whose vertices are obtained as follows: we take an $(n-2)$-point arborescence A and two edges connecting it to each of the remaining two points, and select one of the two edges at each of these two points in all possible ways.

Also, if $C=C(A, N_{x_1}, \dots, N_{x_r})$ and $C'=C(A', N'_{y_1}, \dots, N'_{y_p})$ are two of these convex cells then so is their intersection, if non-empty. Since each point of C must have 1 for each edge of A and similarly, each point of C' must have 1 for each edge of A', it follows that each point $\omega\in C\cap C'$ must have 1 for each coordinate of $A\cup A'$. Hence $A\cup A'$ must be an arborescence. Let $V(G)-V(A\cup A')=\{z_1, \dots, z_t\}$. For each z_i, at least one edge of G with head in z_i must occur in ω with positive weight. This edge belongs then to $N''_{z_i}=N_{z_i}\cap N'_{zi}$. Without loss of generality we may assume therefore that

$$|N''_{z_1}|, \dots, |N''_{z_q}|>1, \quad |N''_{z_{q+1}}|=\dots=|N''_{z_t}|=1.$$

Denote by A'' the arborescence $A\cup A'\cup N''_{z_{q+1}}\cup\dots\cup N''_{z_t}$, and set

$$C''=C(A''; N''_{z_1}, \dots, N''_{z_q}).$$

Then it is easy to see that

$$C\cap C'=C''.$$

This proves that the collection of all convex cells of the form $C(A; N_{x_1}, \dots, N_{x_r})$ form a cellular complex $\mathcal{K}$ (see e.g. ALEXANDROFF—HOPF [1]). We call $\mathcal{K}$ the *arborescence complex* of G (relative to a).

We shall set

$$\tau(C)=A \quad \text{if} \quad C=C(A; N_{x_1}, \dots, N_{x_r}).$$

We shall use the following notations: $L^r(\mathcal{K})$ denotes the group of r-dimensional chains of the complex $\mathcal{K}$, $H^r(\mathcal{K})$ is the r-dimensional reduced homology group. $\sim$ denotes homology of chains.

If C is a convex polytop then $[C]$ denotes its combinatorial hull, i.e. the cellular complex formed by its faces. $|\mathcal{K}|$ is the body of the complex $\mathcal{K}$.

Also recall one definition from graph theory: a set X of points is said to *separate* point b from point a if $a, b\notin X$ and every directed path from a to b meets X.

Now we are able to formulate the main result of this paper:

THEOREM 4. *Let G be a digraph,* $a \in V(G)$ *and assume that no point can be separated from a by less than k points* $(k \geqq 2)$. *Then the arborescence complex* $\mathcal{K}$ *of G relative to a satisfies* $H^0(\mathcal{K}) = \ldots = H^{k-2}(\mathcal{K}) = 0$.

We shall give a separate, elementary proof of the fact that $H^0(\mathcal{K})=0$ (this is essentially Theorem 2). Also, instead of $H^1(\mathcal{K})=0$ we prove the following, somewhat stronger result:

THEOREM 5. *The fundamental group of* $\mathcal{K}$ *is* 0.

It is hoped that the proofs in these two low-dimensional cases will provide motivation for the rather technical proof of Theorem 4. For similar reasons we shall sketch the proof of Theorem 1 in case $k=3$ separately.

From Theorem 4 we shall deduce a digraph analogue of Theorem 1:

THEOREM 6. *Let G be a digraph,* $v_1, \ldots, v_k \in V(G)$ *and assume that for each point* $x \neq v_1, \ldots, v_k$, *there are k openly disjoint paths connecting* $v_1, \ldots, v_k$ *to x. Let, furthermore, k positive integers* $n_1, \ldots, n_k$ *be given whose sum is* $|V(G)|$. *Then G contains k vertex-disjoint arborescences* $A_1, \ldots, A_k$, *such that* A_i *is rooted at* v_i *and* $|V(A_i)| = n_i$.

In fact, Theorem 5 implies Theorem 1: one just has to replace each unoriented edge by two, oppositely directed oriented edges. So we shall only deal with the proof of Theorem 5. Note that a similar trick would enable us to deduce Theorem 2 from Theorem 4, case $k=2$.

2. Proof of the connectivity of $\mathcal{K}$. Let B, B' be two spanning arborescences of the graph G; we are going to prove that there is a chain of spanning arborescences, which starts with B and ends with B' and in which any two consecutive members have an $(n-1)$-point common arborescence. Let A denote the largest common arborescence of B and B'; we use induction of $|V(G)-V(A)|$. If A has n or $n-1$ points the assertion is trivial, so suppose A has at most $n-2$ points.

Let $e=(u, v)$ and $e'=(u', v')$ be edges of B and B', respectively, such that $u, u' \in V(A)$ but $v, v' \notin V(A)$. We distinguish two cases.

Case 1. $v \neq v'$. In this case $A+e+e'$ is an arborescence, which can be completed to a spanning arborescence B''. Now by the induction hypothesis B and B'', as well as B'' and B', can be connected by chains of arborescences, which together yield a chain connecting B to B' as desired.

Case 2. $v=v'$. Since by hypothesis there is at least one more point outside A and also by the hypothesis of the theorem this cannot be separated from the root by v, there must be an edge $f=(x, y)$ with $x \in V(A)$ and $y \in V(G)-V(A)-\{v\}$. Consider the arborescences $A+e+f$ and $A+e'+f$. These can be completed to get two spanning arborescences B_1 and B_2. Then by the induction hypothesis we find chains of arborescences connecting B to B_1 to B_2 to B', which completes the proof.

3. Proof of Theorem 5. 1° Let $(B_0, B_1, \ldots, B_p=B_0)$ be a sequence of spanning arborescences, such that B_{i-1} and B_i are equal or adjacent in $\mathcal{K}$, for all $i=1, \ldots, p$. We are going to prove that there exists a cellular complex homomorphic to the 2-cell, bounded by a p-gon $(X_0, X_1, \ldots, X_p=X_0)$, and with 3- and 4-lateral 2-dimensional cells, and a mapping of it into $\mathcal{K}$ such that X_i is mapped onto B_i and cell onto cell. If this is the case we say shortly that the cycle $(B_0, \ldots, B_p)$ is *contractible*.

Let A denote the largest arborescence which is contained in all B_i; we use induction on $|V(G)-V(A)|$. If this number is 0, 1 or 2, the assertion is trivial.

2° Note that there is no common edge of $B_0, \ldots, B_p$ except those in A. Assume e is such an edge. Let us consider the path P in B_0 connecting a to e and let f be the edge of P leaving A. Then if e is contained in all B_i, the whole path P must be contained in all B_i. But then $A+f$ is a larger common sub-arborescence than A.

3° Suppose now that there is an edge $e=(x, y)\in E(B_i)$ $(1\leqq i\leqq p)$ such that $x, y\notin V(A)$ and y can be reached from A on an edge $e'=(z, y)$ $\big(z\in V(A)\big)$. Replace e by e' in each B_i which contains e. Let $B_0', B_1', \ldots, B_p'$ denote the resulting sequence. It is straightforward to see that B_{i-1}' and B_i' are adjacent or equal for all i. Using induction e.g. on the sum of distances from a of all points and in all B_i, we may assume the cycle $(B_0', B_1', \ldots, B_p')$ is contractible.

We may assume e.g. $e\notin B_0$. Consider all pairs $i\leqq j$ of indices such that $e\notin B_{i-1}$, $e\in B_i$, $e\in B_{i+1}, \ldots, e\in B_j$, $e\notin B_{j+1}$. As in 2° it follows that there is an edge f, connecting A to $V(G)-V(A)$, which is contained in $B_i, B_{i+1}, \ldots, B_j$. Then $A+f$ is a common subtree of the members of the cycle $(B_i, B_{i+1}, \ldots, B_j, B_j', \ldots, B_i', B_i)$ and therefore this cycle is contractible by induction. This implies that the cycle $(B_0, \ldots, B_p)$ is contractible (see Fig. 1).

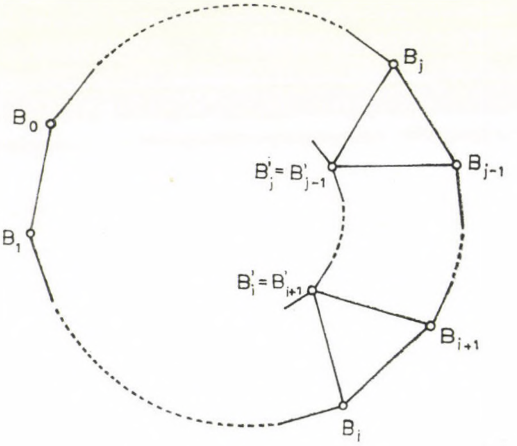

Fig. 1

4° So we may assume that if v is any point accessible from A on an edge then all B_i contain an edge connecting A to v (of course, different B_i may contain different edges of this type). Note that the connectivity assumption implies that there are at least three such points v.

Let $e=(u, v)$ be any edge with $u\in V(A)$, $v\notin V(A)$. Replace the (unique) edge of B_i entering v by e, to get a spanning arborescence B_i'. Then trivially B_{i-1}' and B_i' are adjacent vertices in $\mathscr{K}$. Moreover, the cycle $(B_0', B_1', \ldots, B_p')$ is contractible, since its members contain the common arborescence $A+e$.

Let A_i be the largest common arborescence in B_i and B_i'. Let $C_{i,0}=B_i, C_{i,1}, \ldots$ $\ldots, C_{i,r_i}=B_i'$ be a chain of spanning arborescences, such that $C_{i,j-1}$ and $C_{i,j}$ are adjacent for all $1 \leqq j \leqq r_i$, and $A_i \subseteqq C_{i,j}$. Then the cycle $(C_{i,0}, C_{i,1}, \dots, C_{i,r_i}, C_{i-1,r_{i-1}}, \dots, C_{i-1,0}, C_{i,0})$ is contractible, since its members contain the common arborescence $A_i \cap A_{i-1}$, which is clearly still larger than A. But this proves that $(B_0, \dots, B_p)$ is contractible (Fig. 2).

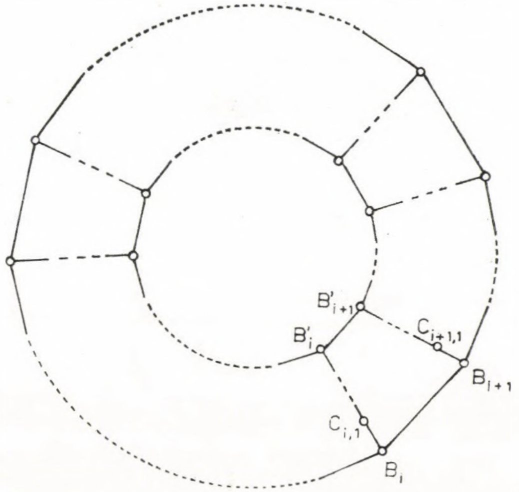

Fig. 2

4. Proof of Theorem 4. Let A be a (not necessarily spanning) arborescence rooted at a. Denote by $\mathscr{K}_A$ the subcomplex of $\mathscr{K}$ spanned by those vertices of $\mathscr{K}$ (spanning arborescences) which contain A. We prove by induction on $d=|V(G)-V(A)|$ and m that the m-dimensional homology group of $\mathscr{K}_A$ is 0 for $m=0, \dots, k-2$. For $d=0$ and $d=1$ the complexes in consideration consist of a single point and a single cell, respectively, so the assertion is true. So suppose $d \geqq 2$.

Let M be the set of vertices accessible from A on an edge. For each $x \in M$, let e_x be one of these edges.

Let B be any spanning arborescence containing A. For each $x \in M$, if the edge of B entering x does not start from A, replace it by e_x. Denote by $\omega(B)$ the resulting arborescence. It is easy to see that

$$B \mapsto \omega(B)$$

is an affine mapping of the n-dimensional space (it is simply the addition of all coordinates corresponding to edges entering x from a point outside A to the coordinate e_x, and then annulating these coordinates, for all $x \in M$). Also ω maps cells of $\mathscr{K}_A$ onto cells of $\mathscr{K}_A$. Thus we can define a homomorphism $\omega^r: L^r(\mathscr{K}_A) \to L^r(\mathscr{K}_A)$ for all r by

$$\omega^r(C^r) = \begin{cases} \omega(C^r) & \text{if } \omega \text{ does not degenerate on } C^r, \\ 0 & \text{otherwise.} \end{cases}$$

We need a

LEMMA. *Let*

$$\alpha: |\mathscr{K}_A| \to |\mathscr{K}_A|$$

be a continuous map which is affine on the cells. Define

$$\alpha^r\colon L^r(\mathcal{K}_A) \to L^r(\mathcal{K}_A),$$

as usual, by

$$\alpha^r(C^r) = \begin{cases} C^r, & \text{if } \alpha \text{ is not degenerate on } C^r, \\ 0, & \text{otherwise.} \end{cases}$$

Then there exist homomorphisms

$$\beta^r\colon L^r(\mathcal{K}_A) \to L^{r+1}(\mathcal{K}_A)$$

for all $r \leqq m-1$ *such that*

(1) *β is a deformation, i.e.*

$$\beta^{r-1}\delta^r x^r + \delta^{r+1}\beta^r x^r = x^r - \alpha^r x^r \quad (r \leqq m-1);$$

(2) *if C^r is a cell such that both $\tau(C^r)$ and $\tau(\alpha(C^r))$ contain an arborescence $A' \supseteqq A$ then so does each cell of $\beta^r C^r$;*

(3) *if both C^r and $\alpha(C^r)$ are faces of a cell C^q then each cell in $\beta^r C^r$ is a face of C^q. In particular, $\beta^r C^r = 0$ if $\alpha(C^r) \subseteqq C^r$ $(r \leqq m-1)$.*

PROOF OF THE LEMMA. For $r<0$ we set $\beta^r=0$. Suppose β^{r-1} is defined, we define β^r.

We consider first a cell C^r. Set

$$d^r = C^r - \alpha^r C^r - \beta^{r-1}\delta^r C^r, \quad b^{r-1} = \delta^r C^r.$$

Then

$$\delta^r d^r = b^{r-1} - \alpha^{r-1} b^{r-1} - \delta^r \beta^{r-1} b^{r-1}.$$

Since β^{r-1} fulfills (1) by the induction hypothesis, it follows that this equals to

$$\beta^{r-2}\delta^{r-1} b^{r-1} = \beta^{r-2}\delta^{r-1}\delta^r C^r = 0.$$

Thus d^r is a cycle. We are going to define $\beta^r C^r$ as an $(r+1)$-chain with

$$\delta^{r+1}\beta^r C^r = d^r;$$

then (1) will be satisfied for C^r. Such a chain $\beta^r C^r$ exists since $H^r(\mathcal{K}_A)=0$ by assumption.

To take care of (2) and (3) we need only to add a few remarks. Suppose both C^r and $\alpha(C^r)$ are contained in a cell C^q. Then, trivially, there is a unique minimal C^q with this property. By the induction hypothesis, also

$$\beta^{r-1}\delta^r C^r \in L^r([C^q]),$$

and hence

$$C^r - \alpha^r C^r - \beta^{r-1}\delta^r C^r \in L^r([C^q]).$$

Since $[C^q]$ is homologically trivial, the chain $\beta^r C^r$ can be chosen from $L^{r+1}([C^q])$. If we do so, condition (3) above will be satisfied.

Secondly, assume that both $\tau(C^r)$ and $\tau(\alpha(C^r))$ contain some arborescence $A' \supseteqq A$. There is a unique maximal arborescence A' with this property.

If C^r and $\alpha(C^r)$ have both been contained in some cell C^q then the minimal C^q with this property must belong to $\mathcal{K}_{A'}$. Then, however, we have already guaranteed that

$$\beta^r C^r \in L^{r+1}([C^q]) \subseteqq L^{r+1}(\mathscr{K}_{A'}).$$

So in this case (2) is automatically satisfied.

If this is not the case then, since we already know that $H^r(\mathscr{K}_{A'})=0$, the chain $\beta^r C^r$ can be chosen from $L^{r+1}(\mathscr{K}_{A'})$. This completes the proof of the Lemma.

Now we consider any chain $b^m \in L^m(\mathscr{K}_A)$ with $\delta^m b^m=0$. Set

$$b^m = \sum_{i=1}^{N} a_i C_i^m$$

where the C_i^m are cells. Let φ^r be the map β^r belonging to $\alpha=\omega$ in the Lemma. Also let

$$d_i^m = C_i^m - \omega^m C_i^m - \varphi^{m-1}\delta^m C_i^m.$$

Then $\delta^m d_i^m = 0$ follows as in the proof of the Lemma. We show d_i^m is, in fact, a boundary.

We distinguish two cases:

Case 1. $\tau(C_i^m)=A$. Then ω is the identity map on C_i^m and hence, $d_i^m=0$ by (3) in the Lemma.

Case 2. $\tau(C_i^m)\supset A$. Let z be any point of $\tau(C_i^m)$ accessible from A on an edge $(w, z)\in\tau(C_i^m)$. Then the edge (w, z) is not altered by ω and hence, $A'=A+(w, z)$ is contained in both $\tau(C_i^m)$ and $\tau(\omega(C_i^m))$. But then each cell in $\varphi^{m-1}\delta^m C_i^m$ is contained in $\mathscr{K}_{A'}$, by property (3) in Lemma 2. Then, however, all cells of d_i^m are contained in $\mathscr{K}_{A'}$. By the induction hypothesis on A, we know $H^m(\mathscr{K}_{A'})=0$ and hence, d_i^m is a boundary in $\mathscr{K}_{A'}$; therefore it is a boundary in $\mathscr{K}_{A'}$.

So $d_i^m \sim 0$, and

$$0 \sim \sum_{i=1}^{N} a_i d_i^m = b^m - \omega^m b^m - \varphi^{m-1}\delta^m b^m = b^m - \omega^m b^m,$$

whence $\omega^m b^m \sim b^m$. So it suffices to prove that $\omega^m b^m \sim 0$.

Note that each spanning tree of the form $\omega(B)$ necessarily contains at least one edge e_y entering y from A, for each $y\in M$. (Recall that M is the set of points accessible from A on an edge.) So if this edge e_y is unique, $\omega^m b^m$ is an m-chain in $\mathscr{K}_{A+e_y}$ and $\omega^m b^m \sim 0$ follows by the induction hypothesis.

So suppose that there are at least two edges connecting A to y for each $y\in M$.

If $M=V(G)-V(A)$ then all spanning trees of the form $\omega(B)$ belong to the same cell

$$C(A; S_y\colon y\in M)$$

where S_y is the set of edges connecting A to y. Thus $\omega^m b^m \sim 0$ follows from the fact that each convex cell is homologically trivial. Thus we may suppose that $M\neq \neq V(G)-V(A)$. Since M separates a from any point of $V(G)-V(A)-M$, this implies $|M|\geqq k$.

We consider a map σ as follows. We select a point $x\in M$, and consider an edge e_x entering x from A. Given any spanning arborescence B, replace its (unique) edge entering x by the edge e_x. Denote by $\sigma(B)$ the resulting arborescence. Again, this mapping is affine when spanning arborescences are considered as points of the

m-dimensional space and maps cells onto cells. Let σ^r be defined similarly as ω^r and denote by ψ^r the deformation map β^r constructed in the Lemma when $\alpha=\sigma$. Let

$$\omega^m b^m = \sum_{j=1}^{M} e_j C_j^m,$$

where the C_j^m are cells. Then

$$f_j^m = C_j^m - \sigma^m C_j^m - \psi^{m-1}\delta^m C_j^m$$

satisfies $\delta^m f_j^m = 0$. We show that $f_j^m \sim 0$. Trivially

$$|V(\tau(C_j^m))| \geqq |V(G)| - m \geqq |V(G)| - k + 2 \leqq |A| + 2,$$

whence there are at least two edges f, f' leaving A which are common in all vertices of C_j^m. We may assume $f \neq e_x$. But then both C_j^m and $\sigma(C_j^m)$ are cells of $\mathscr{K}_{A+f}$. By property (2) of ψ, this implies that f_j^m is a chain in $\mathscr{K}_{A+f}$, and so $f_j^m \sim 0$ follows by the induction hypothesis.

Thus

$$0 \sim \sum_{j=1}^{m} f_j^m = \omega^m b^m - \sigma^m \omega^m b^m - \psi^{m-1}\delta^m \omega^m b^m = \omega^m b^m - \sigma^m \omega^m b^m.$$

Now $\sigma^m \omega^m b^m$ is a chain in $\mathscr{K}_{A+e_x}$. Thus

$$b^m \sim \omega^m b^m \sim \sigma^m \omega^m b^m \sim 0.$$

This completes the proof of Theorem 4.

5. The proof of Theorem 6 in the cases $k \leqq 3$. *Case* $k=2$. Take a new point a and connect it to v_1 and v_2. Let B_i be a spanning arborescence of the resulting graph G', rooted at a and not containing the edge (a, v_i). Consider a chain of spanning arborescences, linking B_1 and B_2 as constructed in Section 2. Consider the number of points of these arborescences on the branch over (a, v_1). This number changes by at most 1 at a time, and for B_1 it is 0 while for B_2 it is n. So there is a spanning arborescence for which it is n_1. Deleting a we obtain the pair of spanning arborescences, rooted at v_1 and v_2 as desired.

Case $k=3$ (sketch). Take a new point a and connect it to v_1, v_2, v_3. For each spanning arborescence B denote by $x_i(B)$ the number of points on the branch over (a, v_i), and let $\mathbf{x}(B)=(x_1(B), x_2(B))$.

Let H be the triangle $\{(x, y): x \geqq 0,\ y \geqq 0,\ x+y \leqq n\}$. Dissect each 4-lateral 2-dimensional cell of $\mathscr{K}$ into two triangles, to get a 2-dimensional simplical complex $\overline{\mathscr{K}}$. Then it is easily seen that $\mathbf{x}$ maps the vertices of a triangle of $\overline{\mathscr{K}}$ onto vertices of some empty lattice triangle. Extend $\mathbf{x}$ affinely over all simplices of $\overline{\mathscr{K}}$.

Let now B_i be a spanning arborescence (rooted at a) such that every point is above (a, v_i). Let $B_1 = C_1, C_2, \ldots, C_{k_1} = B_2$ be a chain of spanning arborescences in $G' - (a, v_3)$; let us define the chains $B_2 = D_1, D_2, \ldots, D_{k_2} = B_3$, $B_3 = E_1, E_2, \ldots$ $\ldots, E_{k_3} = B_1$ analogously. The images of these chains by $\mathbf{x}$ give continuous curves connecting B_1 to B_2 to B_3 to B_1, which remain on the corresponding side of H. Therefore if we span a surface on the cycle in $\overline{\mathscr{K}}$ (which is possible by Theorem 5), the image of this surface covers the interior of H, in particular it contains the point (n_1, n_2). But the construction of the mapping $\mathbf{x}$ is such that if (n_1, n_2) is the image of something, it is necessarily the image of a vertex. This vertex is a spanning arborescence, from which we obtain the desired arborescences rooted at v_1, v_2 and v_3 by deleting a.

6. Proof of Theorem 6. Take a new point a and join it to $v_1, \dots, v_k$ by edges. It is clear that no point of the resulting digraph G can be separated from a by less than k points. Let $\mathcal{K}$ be the arborescence complex of G relative to a, and $\mathcal{K}'$ its $(k-1)$-dimensional skeleton. Subdividing each cell of $\mathcal{K}'$ into simplices whose vertices are also vertices of $\mathcal{K}$, we get a simplicial complex $\bar{\mathcal{K}}$.

For each spanning arborescence B, let $m_i(B)$ denote the number of vertices accessible from a on the edge (a, v_i). We want to show that there is a spanning arborescence B such that

$$m_i(B) = n_i \quad (i = 1, \dots, n). \tag{1}$$

Once such a spanning arborescence is found we are done since then the k components of $B-a$ are k disjoint arborescences rooted at $v_1, \dots, v_k$ and having the desired cardinalities.

Suppose indirectly no spanning arborescence satisfying (1) exists. Then for each spanning arborescence B there is an index $i(B)$, $1 \leqq i(B) \leqq k$ such that

$$m_{i(B)}(B) > n_{i(B)}. \tag{2}$$

Let $x_1, \dots, x_k$ be k affinely independent points in the $(k-1)$-dimensional space. Denote by S their convex hull and set $\mathcal{S} = [S]$. Define

$$\xi(B) = x_{i(B)}$$

and extend this to a simplicial map of $\bar{\mathcal{K}}$ into $\mathcal{S}$. This induces then homomorphisms $\xi^r : L^r(\bar{\mathcal{K}}) \to L^r(\mathcal{S})$ for each r.

We claim $\xi^{k-1} = 0$. It suffices to show that no simplex of $\bar{\mathcal{K}}$ can be mapped onto S. Suppose indirectly $(B_1, \dots, B_k)$ were a simplex in $\bar{\mathcal{K}}$ with $\xi(B_i) = x_i$ $(i = 1, \dots, k)$. Then $(B_1, \dots, B_k)$ is contained in a $(k-1)$-dimensional cell $C^{k-1} = C(A, N_{y_1}, \dots, N_{y_r})$ of $\mathcal{K}$. Let A_i be the branch of A above v_i $(i = 1, \dots, k)$ and denote by a_i the number of edges in $N_{y_1} \cup \dots \cup N_{y_r}$, ending in A_i. Thus

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{r} |N_{y_i}| = k + r - 1.$$

But $\xi(B_i) = x_i$ implies

$$n_i < m_i(B_i) \leqq |A_i| + a_i$$

and hence

$$\sum_{i=1}^{k} a_i \geqq \sum_{i=1}^{k} (n_i - |A_i| + 1) = (n-1) - (n-r-1) + k = k + r,$$

a contradiction. So $\xi^{k-1} = 0$.

Let C^r be a face of S, say

$$C^r = \text{convex hull of } (x_{i_0}, \dots, x_{i_r}).$$

Denote by $\mathscr{K}_{C^r}$ the subcomplex of $\mathscr{K}$ spanned by those vertices B in which each edge leaving a is one of $(a, v_{i_0}), \ldots, (a, v_{i_r})$, and let $\overline{\mathscr{K}}_{C^r}$ be obtained from $\mathscr{K}_{C^r}$ by the same subdivision as $\overline{\mathscr{K}}$ is obtained from $\mathscr{K}$. Then $\mathscr{K}_{C^r}$ is the arborescence complex of the $(r+1)$-connected graph $G+(a, v_{i_0})+\ldots+(a, v_{i_r})$, and therefore it satisfies

$$H^0(\mathscr{K}_{C^r}) = \ldots = H^{r-1}(\mathscr{K}_{C^r}) = 0$$

by Theorem 4. Thus also

$$H^0(\overline{\mathscr{K}}_{C^r}) = \ldots = H^{r-1}(\overline{\mathscr{K}}_{C^r}) = 0.$$

Let us define a homomorphism

$$\eta^r \colon L^r(\mathscr{S}) \to L^r(\overline{\mathscr{K}}) \quad (r \leqq k-1)$$

such that

$$\delta^r \eta^r = \eta^{r-1} \delta^r$$

and for each face C^r of S, $\eta^r C^r \in L^r(\overline{\mathscr{K}}_{C^r})$. We do this by induction on r. For $r<0$ let $\eta^r=0$. Suppose η^{r-1} is defined. Then for any face C^r of S,

$$b^{r-1} = \eta^{r-1} \delta^r C^r$$

is defined. Observe that

$$\delta^{r-1} b^{r-1} = \delta^{r-1} \eta^{r-1} \delta^r C^r = \eta^{r-2} \delta^{r-1} \delta^r C^r = 0.$$

Also $b^{r-1} \in L^{r-1}(\overline{\mathscr{K}}_{C^r})$. Since $H^{r-1}(\overline{\mathscr{K}}_{C^r})=0$, there is a chain in $L^r(\overline{\mathscr{K}}_{C^r})$ with boundary b^{r-1}; let this chain be $\eta^r C^r$. It is easy to check that this defines a homomorphism with the desired property.

Now take the composition map $\xi^r \eta^r$. We prove by induction on r that this is the identity map. This clearly holds for $r \leqq 0$. Since by the definition of η^r,

$$\eta^r C^r \in L^r(\overline{\mathscr{K}}_{C^r})$$

for each r-face C^r=convex hull of $(x_{i_0}, \ldots, x_{i_r})$, it follows that each vertex of any cell in $\eta^r C^r$ is a spanning arborescence containing no edge leaving a different from $(a, v_{i_0}), \ldots, (a, v_{i_r})$. Hence by the definition of ξ^r,

$$\xi^r \eta^r C^r = pC^r$$

with some integer p. Now

$$p\delta^r C^r = \delta^r \xi^r \eta^r C^r = \xi^{r-1} \delta^r \eta^r C^r = \xi^{r-1} \eta^{r-1} \delta^r C^r = \delta^r C^r$$

by the induction hypothesis. Hence $p=1$, i.e. $\xi^r \eta^r = \mathrm{id}$ is proved.

In particular $\xi^{k-1} \eta^{k-1} = \mathrm{id}$, which contradicts $\xi^{k-1}=0$. This completes the proof of Theorem 5.

7. We conclude with the much more elementary proof of Theorem 2. Let G be a non-bipartite 2-connected graph on n vertices, $(x_0, \ldots, x_{2p})$ an odd circuit in G and T a spanning arborescence of $G-x_0$ containing the path $(x_1, \ldots, x_{2p})$. Set $T=T+(x_0, x_1)$, $T''=T+(x_0, x_{2p})$.

By Theorem 1, there exists a sequence

$$T_0 = T, T_1, \ldots, T_N = T''$$

of spanning trees, such that T_i and T_{i+1} have a common subtree with $n-1$ points containing x_0. Let us 2-colour each T_i with red and blue such that x_0 is red. Let $f(i)$ be the number of red points. Then

$$|f(i)-f(i+1)| \leqq 1$$

and

$$f(N) = 2n+1-f(0).$$

Hence $f(i)$ must take the value n somewhere for $0 \leqq i \leqq N$, which proves the assertion.

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EMBEDDING OF REARRANGEMENT INVARIANT SPACES IN LORENTZ SPACES

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1. Introduction. Let $X(0, 1)$, $Y(0, 1)$ be function spaces of measurable functions. Let $f \in Y$, and consider $\omega_Y(f^*, t)$ "the modulus of continuity" of f^* with respect to the norm of Y. (See Definition 3.1 below.) We consider the following problem: under what conditions on $\omega_Y(f^*, t)$ can we assert that $f \in X$?

The results we obtain extend and simplify previous works by UL'JANOV [8], LEINDLER [4], STOROŽENKO [7]. The crucial part of our work is contained in §3, where we refine estimates by Ul'janov and Storoženko; then we prove our main results in §4 using Hardy type inequalities.

2. Preliminaries. In this paper we consider function spaces X of Lebesgue measurable functions on [0, 1] such that $\|f\|_X = \|f^*\|_X$, where f^* denotes the non-increasing rearrangement of f. These spaces are usually referred to as rearrangement invariant spaces (r.i. spaces) or symmetric spaces. The fundamental function associated with a r.i. space is defined by $\varphi_X(t) = \|\chi_{(0,t)}\|_X$, and it is easily seen that X can be renormed in such a way that φ_X is a concave function.

The Lorentz spaces associated with a r.i. space are defined by

$$\Lambda_\alpha(X) = \left\{ f \in M(0, 1) : \|f\|_{\Lambda_\alpha(X)} = \left\{ \int_0^1 [f^*(t)\varphi_X(t)]^{1/\alpha} \frac{dt}{t} \right\}^\alpha < \infty \right\}$$

for $0 < \alpha \leqq 1$. We shall also consider the Marcinkiewicz spaces $M(X)$ defined by

$$M(X) = \{ f \in M(0, 1) : \|f\|_{M(X)} = \sup_{t \in [0,1]} \{ f^{**}(t) \varphi_X(t) \} < \infty \}$$

where $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds$.

We shall assume that

$$\int_0^t \varphi_X(u) \frac{du}{u} = O(\varphi_X(t)) \quad \text{and} \quad \int_t^\infty \frac{du}{\varphi_X(u)} = O\left(\frac{1}{\varphi_X(t)} \right).$$

We refer the reader to [5] for more information concerning r.i. spaces, and [6] for a detailed study of $\Lambda_\alpha(X)$ spaces.

3. Estimates of generalized modules of continuity. In this section we obtain estimates of $f^*(t)$ in terms of the module of continuity $\omega_X(f^*, t)$, defined for a r.i. norm $\| \; \|_X$. Although similar results hold if we replace $\omega_X(f^*, t)$ by $\omega_X(f, t)$, we do not consider this type of results in the present paper. In fact, as it has been pointed out recently by JOHANSSON and WIK [2], $\omega_p(f^*, t) \leqq c\omega_p(f, t)$ always holds for $1 \leqq p < \infty$. (This result is based on work by Garsia and Rodemich (cf. [2] for the precise references).)

Inequalities of a similar type to the ones we obtain in this section have been proved by UL'JANOV [8], LEINDLER [4] and STOROŽENKO [7] (cf. also [2]). These results are, however, restricted to $\omega_p(f, t)$ modules of continuity.

DEFINITION 3.1. Let $f \in L^1$, and define, for $0 \leqq h \leqq 1$,

$$\Delta_h f(x) = [f(x+h) - f(x)]\chi_{(0,1-h)}(x).$$

Given a r.i. norm, we put $\omega_X(f, t) = \sup_{0 \leqq h \leqq t} \|\Delta_h f\|_X$, for $0 \leqq t \leqq 1$.

LEMMA 3.2. *Let $f \in X$, then*

$$\frac{1}{\varphi_X(1/2n)} \cdot \omega_X(f^*, 1/n) \geqq f^*(1/2n) - f^*(1/n), \quad n \geqq 2. \tag{3.1}$$

PROOF. Let $n \geqq 2$, then

$$\|\Delta_{1/n} f^*\|_X = \|(f^* - f^*_{1/n})\chi_{(0,1-1/n)}\|_X \geqq \|(f^* - f^*_{1/n})\chi_{(0,1/2n)}\|_X \geqq$$
$$\geqq [f^*(1/2n) - f^*(1/n)]\varphi_X(1/2n).$$

Therefore (3.1) follows, since $\omega_X(f, 1/n) \geqq \|\Delta_{1/n} f^*\|_X$.

LEMMA 3.3. *Let $f \in X$, then*

$$f^*\left(\frac{1}{2^{n+1}}\right) \leqq 8 \int_{1/2^n}^{1} \frac{\omega_X(f^*, t)}{\varphi_X(t)} \frac{dt}{t} + \frac{1}{\varphi_X(1/2)} \|f\|_X, \quad n = 1, 2, \ldots.$$

PROOF. By (3.1),

$$f^*\left(\frac{1}{2^{k+1}}\right) - f^*\left(\frac{1}{2^k}\right) \leqq \frac{1}{\varphi_X\left(\frac{1}{2^{k+1}}\right)} \omega_X\left(f^*, \frac{1}{2^k}\right), \quad k = 1, 2, \ldots.$$

Hence,

$$\sum_{k=1}^{n} \left[f^*\left(\frac{1}{2^{k+1}}\right) - f^*\left(\frac{1}{2^k}\right)\right] \leqq \sum_{k=1}^{n} \frac{1}{\varphi_X\left(\frac{1}{2^{k+1}}\right)} \omega_X\left(f^*, \frac{1}{2}\right),$$

$$f^*\left(\frac{1}{2^{n+1}}\right) - f^*\left(\frac{1}{2}\right) \leqq \sum_{k=1}^{n} \frac{1}{\varphi_X\left(\frac{1}{2^{k+1}}\right)} \omega_X\left(f^*, \frac{1}{2^k}\right).$$

Now, since $f^*(t) \leqq \frac{1}{\varphi_X(t)} \|f\|_X$ and $\varphi_X(t)\, t^{-1} \searrow$, we get

$$f^*\left(\frac{1}{2^{n+1}}\right) \leqq \sum_{k=1}^{n} \frac{4}{\varphi_X\left(\frac{1}{2^{k-1}}\right)} \cdot \omega_X\left(f^*, \frac{1}{2^k}\right) + \frac{1}{\varphi_X\left(\frac{1}{2}\right)} \|f\|_X, \tag{3.2}$$

but

$$\frac{1}{\varphi_X\left(\frac{1}{2^{k-1}}\right)}\omega_X\left(f^*,\frac{1}{2^k}\right) \leqq 2\int\limits_{1/2^k}^{1/2^{k-1}} \frac{\omega_X(f^*,t)}{\varphi_X(t)}\frac{dt}{t}$$

which combined with (3.2), yields

$$f^*\left(\frac{1}{2^{n+1}}\right) \leqq 8\sum_{k=1}^{n}\int\limits_{1/2^k}^{1/2^{k-1}} \frac{\omega_X(f^*,t)}{\varphi_X(t)}\frac{dt}{t} + \frac{1}{\varphi_X\left(\frac{1}{2}\right)}\|f\|_X \leqq$$

$$\leqq 8\int\limits_{1/2^n}^{1} \frac{\omega_X(f^*,t)}{\varphi_X(t)}\frac{dt}{t} + \frac{1}{\varphi_X\left(\frac{1}{2}\right)}\|f\|_X$$

as required.

COROLLARY 3.4. *Let* $0\leqq s\leqq 1/2$, *then*

$$f^*(s) \leqq 8\int\limits_{s}^{1} \frac{\omega_X(f^*,t)}{\varphi_X(t)}\frac{dt}{t} + \frac{1}{\varphi_X\left(\frac{1}{2}\right)}\|f\|_X. \tag{3.3}$$

We shall also need the following result which is of independent interest.

LEMMA 3.5. *Let X be a separable r.i. space, and let* $f\in X$, *then*

$$\omega_X(f^*,t) \leqq \|H\|_{X\to X}\|f^*\chi_{(0,t)}\|_X \tag{3.4}$$

where $H(f)(t)=\frac{1}{t}\int\limits_0^t f(s)\,ds$.

PROOF. Let $|f|=c\cdot\chi_E$, with $|E|=r$, then $f^*=c\chi_{(0,r)}$. We shall compute $(\Delta_h f^*)^{**}$. We consider several cases: (I) $r-h>0$, $1-h>r$; (II) $r-h>0$, $1-h<r$; (III) $r-h<0$, $1-h>r$; (IV) $r-h<0$, $1-h<r$. It is easy to verify that in all cases we have $(\Delta_h f^*)^{**}(t)\leqq H(f^*\chi_{(0,h)})(t)$. For example consider (I), then

$$|\Delta_h f^*(t)| = c\chi_{(r-h,r)}(t)$$

so that

$$(\Delta_h f^*)^*(t) \leqq c\chi_{(0,h)}(t),$$

$$(\Delta_h f^*)^{**}(t) = \frac{1}{t}\int\limits_0^t (\Delta_h f^*)(s)\,ds \leqq \frac{1}{t}c\chi_{(0,r)}(s)\chi_{(0,h)}(s)\,ds \leqq H(f^*\chi_{(0,h)}(t).$$

The other cases are treated similarly and we omit the details. Therefore, if $f=c\chi_E$, we have

$$(\Delta_h f^*)^{**}(t) \leqq H(f^*\chi_{(0,h)})(t), \quad \|\Delta_h f^*\|_X \leqq \|(\Delta_h f^*)^{**}\|_X \leqq \|H\|_{X\to X}\|f^*\chi_{(0,h)}\|_X$$

and (3.4) follows in this case. Let $f=\sum_{i=1}^{n} c_i\chi_{E_i}$, then we can find simple functions

f_i, $1 \leqq i \leqq n$, such that each f_i takes one value, and moreover $f^* = \sum_{i=1}^{n} f_i^*$. Hence,

$$\Delta_h f^* = \sum_{i=1}^{n} \Delta_h f_i^*,$$

$$(\Delta_h f^*)^{**}(t) \leqq \sum_{i=1}^{n} (\Delta_h f_i^*)^{**}(t) \leqq \sum_{i=1}^{n} H(f_i^* \chi_{(0,h)})(t) \leqq H(f^* \chi_{(0,h)})(t)$$

and (3.4) follows in this case as well. Finally let $f \in X$ be arbitrary, and let $\{f_i\}_{i=1}^{\infty}$ be a sequence of simple functions such that $\|f^* - f_i^*\|_X \to 0$, then

$$\|\Delta_h f^*\|_X \leqq 2\|f^* - f_i^*\|_X + \|\Delta_h f_i^*\|_X, \quad i = 1, \ldots,$$

and (3.4) follows. The Lemma is established.

4. In this section, we obtain sufficient conditions for embedding of r.i. spaces in terms of our generalized modulus of continuity.

The results in this section extend works by UL'JANOV [8] and STOROŽENKO [7]. (See also LEINDLER [3].)

We start with the following

THEOREM 4.1. *Let* $f \in Y$ *and*

$$\int_t^1 \frac{\omega_Y(f^*, u)}{\varphi_Y(u)} \frac{du}{u} = O\left(\frac{1}{\varphi_X(t)}\right), \quad 0 \leqq t \leqq 1/2.$$

Then $f \in M(X)$.

PROOF. We compute $\|f\|_{M(X)}$:

$$\|f\|_{M(X)} \leqq \sup_{0 \leqq t \leqq 1/2} \{\varphi_X(t) f^*(t)\} + \sup_{1/2 < t \leqq 1} \{\varphi_X(t) f^*(t)\} = I_1 + I_2.$$

To estimate I_1 we use (3.3) to obtain

$$I_1 \leqq c \cdot \left| \sup_{0 \leqq t \leqq 1/2} \left\{\varphi_X(t) \int_t^1 \frac{\omega_Y(f^*, u)}{\varphi_Y(u)} \frac{du}{u}\right\} + \sup_{0 \leqq t \leqq 1/2} \{\varphi_X(t) \|f\|_Y\} \right| \leqq c_1 + c_2 \|f\|_Y.$$

The estimate for I_2 is even simpler:

$$I_2 \leqq \sup_{1/2 \leqq t \leqq 1} \frac{\varphi_X(t)}{t} \int_0^1 f^*(u) \chi_{(0,t)}(u)\, du \leqq$$

$$\leqq \sup_{1/2 \leqq t \leqq 1} \left\{\frac{\varphi_X(t)}{t} \cdot \|f\|_Y \cdot \varphi_{Y'}(t)\right\} \leqq \sup_{1/2 \leqq t \leqq 1} \left\{\frac{\varphi_X(t)}{\varphi_Y(t)}\right\} \|f\|_Y \leqq c_3 \|f\|_Y$$

and the result follows.

REMARK 4.2. Observe that by Lemma 3.5, we have

$$\int_t^1 \frac{\omega_Y(f^*, u)}{\varphi_Y(u)} \frac{du}{u} = O\left(\frac{1}{\varphi_Y(t)}\right), \quad 0 \leqq t \leqq 1/2,$$

whenever Y satisfies the conditions of Lemma 3.5.

We now extend this result for the $\Lambda_\alpha(X)$ scale.

THEOREM 4.3. *Let* $f\in Y$, $0<\beta\leqq 1$, *and*

$$\int_0^1 \left|\frac{\omega_Y(f^*,t)}{\varphi_Y(t)}\varphi_X(t)\right|^{1/\beta}\frac{dt}{t}<\infty. \tag{4.1}$$

Then $f\in\Lambda_\beta(X)$.

PROOF.

$$\|f\|^{1/\beta}_{\Lambda_\beta(X)}\leqq\|f\cdot\chi_{(0,1/2)}\|^{1/\beta}_{\Lambda_\beta(X)}+\|f\cdot\chi_{(1/2,1)}\|^{1/\beta}_{\Lambda_\beta(X)}\leqq I_1+I_2.$$

As usual, the estimate of I_2 is much easier to derive, and we obtain $I_2\leqq c_1\|f\|_X$. Now,

$$I_1^\beta\leqq\left\{\int_0^{1/2}\left\{\int_t^1\frac{\omega_Y(f^*,u)}{\varphi_Y(u)}\frac{du}{u}+c\cdot\|f\|_Y\right\}^{1/\beta}\varphi_X^{1/\beta}(t)\frac{dt}{t}\right\}^\beta\leqq$$

$$\leqq\left\{\int_0^{1/2}\left\{\int_t^1\frac{\omega_Y(f^*,u)}{\varphi_Y(u)}\frac{du}{u}\right\}^{1/\beta}\varphi_Y(t)^{1/\beta}\frac{dt}{t}\right\}^\beta+c\|f\|_Y$$

so using Hardy's inequality, we get

$$I_1^\beta\leqq\left\{\int_0^1\left[\frac{\omega_Y(f^*,u)}{\varphi_Y(u)}\varphi_X(u)\right]^{1/\beta}\frac{du}{u}\right\}^\beta+c_2\|f\|_Y\leqq a+c_2\|f\|_Y.$$

Collecting these inequalities, the theorem follows.

REMARK 4.4. A similar result, where $Y=L^p$, $X=L^q$ was obtained by STOROŽENKO [7]. His result can be obtained from Theorem 4.5. In fact, let $Y=L^p$ and $X=L^q$, $\beta=\frac{1}{p}$, then (4.1) becomes

$$\int_0^1\frac{\omega_p^q(f^*,z)}{z^{q/p}}z^{q/q}\cdot\frac{dz}{z}=\int_0^1\omega_p^q(f^*,t)t^{-q/p}\,dt<\infty,$$

which is Storoženko's condition.

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REMARK TO THE EMBEDDING OF AN OPERATIONAL CALCULUS BASED ON THE $\mathscr{L}^{(\vartheta)}$-TRANSFORMATION IN A FIELD OF TRANSFORMABLE OPERATORS

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1. Introduction

Let $\mathscr{M}$ be the operator field of MIKUSIŃSKI [5] and $\mathfrak{Q}$ the sub-field of $\mathscr{M}$, which consists of all operators represented by convolution quotients φ/ψ, where φ, ψ $(\psi \not\equiv 0)$ are continuous complex-valued functions in $0 \leqq t < \infty$, whose Laplace transforms $\bar{\varphi}(z)$, $\bar{\psi}(z)$ are absolute convergent (see [4]). The operators $\varphi/\psi \in \mathfrak{Q}$ have generalized Laplace transforms $\mathscr{L}[\varphi/\psi] = \bar{\varphi}(z)/\bar{\psi}(z)$ ($\bar{\varphi}(z)/\bar{\psi}(z)$ is a pointwise quotient), which are meromorphic in some right half-planes of the complex plane.

We denote by $\mathfrak{M}$ the family of all functions $f(z)$ that are meromorphic in some right half-planes $\Delta = \{z: \operatorname{Re}(z) > \sigma\}$ (σ depends on $f(z)$). We define the equality and the pointwise operations in $\mathfrak{M}$ as usual. Then $\mathfrak{M}$ is a field.

Obviously the operators $\varphi/\psi \in \mathfrak{Q}$ have generalized Laplace transforms $\mathscr{L}[\varphi/\psi]$ in $\mathfrak{M}$. But there are functions in $\mathfrak{M}$ which are not generalized Laplace transforms of operators $\varphi/\psi \in \mathfrak{Q}$ (for example $f(z) = e^{\alpha z}$ if α is a complex number). Therefore in [6] an operator field $\mathfrak{A}$ is constructed, which is algebraically isomorphic to the field $\mathfrak{M}$.

In the present note an operational calculus based on the $\mathscr{L}^{(\vartheta)}$-transformation (see [3]) and the ultradistributions having compact supports (see [8]) will be embedded in the operator field $\mathfrak{A}$.

2. The operator field $\mathfrak{A}$

Let $\mathfrak{H}$ be the subalgebra of $\mathfrak{M}$, which consists of all functions $h(z)$ that are holomorphic in some right half-planes Δ. Suppose that $(h_n(z))$ is a sequence in $\mathfrak{H}$ and $h(z) \in \mathfrak{H}$. By definition, $\lim h_n(z) = h(z)$, if there exists a right half-plane Δ such that $h(z)$, $h_n(z)$ $(n = 1, 2, \dots)$ are holomorphic in Δ and if the sequence $(h_n(z))$ converges to $h(z)$ uniformly on every compact subdomain of Δ.

We call a sequence $(\varphi_n/\psi_n) \subset \mathfrak{Q}$ a fundamental sequence if there are functions $h(z)$, $g(z) \in \mathfrak{H}$ $(g \not\equiv 0)$ such that $\lim \bar{\varphi}_n(z) = h(z)$ and $\lim \bar{\psi}_n(z) = g(z)$ in $\mathfrak{H}$. It is easy to see that the function

$$\mathfrak{L}[\varphi_n/\psi_n] \overset{\text{def}}{=} h(z)/g(z)$$

belongs to $\mathfrak{M}$. Two fundamental sequences (φ_n/ψ_n) and (η_n/ζ_n) are equivalent if $\mathfrak{L}[\varphi_n/\psi_n] = \mathfrak{L}[\eta_n/\zeta_n]$ in the sense of $\mathfrak{M}$. The equivalence classes determined by this equivalence relation are called operators. Let $\mathfrak{A}$ be the set of all operators. An operator A represented by a fundamental sequence (φ_n/ψ_n) will be denoted by $A = \langle \varphi_n/\psi_n \rangle$. Two operators are equal if their representatives are equivalent. Let $A = \langle \varphi_n/\psi_n \rangle$ and $B = \langle \eta_n/\zeta_n \rangle$ belong to $\mathfrak{A}$. We define

$$A + B = \langle (\varphi_n * \zeta_n + \eta_n * \psi_n)/(\psi_n * \zeta_n) \rangle; \quad AB = \langle (\varphi_n * \eta_n)/(\psi_n * \zeta_n) \rangle,$$

where $*$ is the symbol for the well known convolution product of functions $\varphi(t)$, $0 \leqq t < \infty$.

THEOREM 1. *Under these operations $\mathfrak{A}$ is a field and the mapping* $\mathfrak{L}[A] = \bar{A}(z) :=$ $:= \mathfrak{L}[\varphi_n/\psi_n]$ *is an algebraic isomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{M}$.

(For the proof see [6].)

The subfield $\mathfrak{Q} \subset \mathcal{M}$ is a subfield of $\mathfrak{A}$ too. The operators $\varphi/\psi \in \mathfrak{Q}$ are the only operators in $\mathfrak{A}$ which have representatives (φ_n/ψ_n), where $\varphi_n = \varphi$ and $\psi_n = \psi$ are independent of n. In this case we write $\langle \varphi/\psi \rangle$ or φ/ψ. If $\varphi \in \mathfrak{Q}$ is a function then we write φ in $\mathfrak{A}$ too, and we obtain $\mathfrak{L}[\varphi] = \bar{\varphi}(z)$. Therefore the function $\mathfrak{L}[A] =$ $= \bar{A}(z) \in \mathfrak{M}$ will be called the (generalized) Laplace transform of the operator $A \in \mathfrak{A}$.

The differential operator $s = \langle \varphi/\varphi^{(-1)} \rangle$ $\left(\varphi^{(-1)} := \int_0^t \varphi(u)\,du;\quad \varphi \not\equiv 0\right)$ has the Laplace transform $\mathfrak{L}[s] = z$. For that reason we write formally $A = f(s)$ if $A \in \mathfrak{A}$ has the Laplace transform $\bar{A}(z) = f(z) \in \mathfrak{M}$.

There exist operators in $\mathfrak{A}$ which are not in $\mathcal{M}$. For example, $e^{\alpha s} \in \mathfrak{A}$ for all complex α, but $e^{\alpha s} \notin \mathcal{M}$ if α is not a real number. It is well known that the operator $e^{\alpha s}$ (α real) is a shift operator for the functions $\varphi(t)$. In the case when α is a complex number we are allowed to explain the operator $e^{\alpha s}$ to be a shift operator on a certain ring of holomorphic functions (see Section 3).

The convergence in $\mathfrak{A}$ is defined as follows:

A sequence $(f_n(s)) \subset \mathfrak{A}$ converges to $f(s) \in \mathfrak{A}$ if there exists quotients $f_n(z) =$ $= h_n(z)/g_n(z)$ and $f(z) = h(z)/g(z)$ $(h_n, g_n, h, g \in \mathfrak{H};\ g_n, g \not\equiv 0)$ such that $\lim h_n(z) =$ $= h(z)$ and $\lim g_n(z) = g(z)$.

In this sense of convergence the operator $e^{\alpha s}$ (α complex) has the representation

$$e^{\alpha s} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} s^k.$$

REMARK. It is possible to embed in $\mathfrak{A}$ also all functions $\varphi(t)$, $-\infty < t < \infty$, having two-sided Laplace transforms, which are absolute convergent in some right half-planes Δ. This means that the operational calculus $\mathfrak{A}$ is a certain two-sided operational calculus too. For example, the operator $e^{s^2} \in \mathfrak{A}$ can be identified with the function $\frac{1}{2\sqrt{\pi}} e^{-t^2/4}$, $-\infty < t < \infty$.

3. The operator field $\mathfrak{A}$ and an operational calculus based on the $\mathcal{L}^{(9)}$-transformation

Let $\mathfrak{G}$ be the subalgebra of $\mathfrak{H}$ consisting of all entire functions $g(z)$, which fulfil exponential estimations $|g(z)| < ce^{a|z|}$ for all z, where c and a are positive constants depending on $g(z)$. By use of the map $\mathfrak{L}$ in Theorem 1 $\mathfrak{G}$ is isomorphic to a subalgebra $\mathfrak{P}$ of the operator field $\mathfrak{A}$. Hence every operator $g(s) \in \mathfrak{P}$ has a representation

$$g(s) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} s^k \quad \left(\alpha_k = g^{(k)}(0)\right), \tag{3.1}$$

where the series converges in the sense of $\mathfrak{A}$. We denote by $\mathfrak{Q}(\mathfrak{P})$ the quotient field of $\mathfrak{P}$ in $\mathfrak{A}$.

Let $\mathfrak{P}^*$ be the class of all series

$$F(w) = \sum_{k=0}^{\infty} \beta_k / w^{k+1}, \tag{3.2}$$

which have a radius of convergence $\varrho < \infty$ (ϱ depends on F). Every series of the form (3.2) defines a function which is holomorphic for all $w \in \{w : \varrho < |w| \leqq \infty\}$. Moreover $F(\infty) = 0$ follows. We define the equality and the pointwise addition in $\mathfrak{P}^*$ as usual, and as multiplication we use the complex convolution ([3], p. 399)

$$F_1(w) \circ F_2(w) := \frac{1}{2\pi i} \oint_{|v|=\varrho} F_1(v) F_2(w-v)\, dv, \tag{3.3}$$

where $|w| > \varrho_1 + \varrho_2$ and $\varrho_1 < \varrho < |w| - \varrho_2$ (ϱ_k is the radius of convergence of $F_k(w)$ $(k=1, 2)$).

The $\mathscr{L}^{(\vartheta)}$-transformation

$$\mathscr{L}^{(\vartheta)}[g] = \int_0^{\infty(\vartheta)} e^{-zw} g(z)\, dz,$$

where $g(z) \in \mathfrak{G}$ and we have to integrate along a straight line, whose polar angle is ϑ $(0 \leqq \vartheta < 2\pi)$, from 0 to ∞, and the inversion

$$\mathscr{L}^{(\vartheta)-1}[F] = \frac{1}{2\pi i} \oint_{|w|=\varrho_1} e^{zw} F(w)\, dw,$$

where $F(w) \in \mathfrak{P}^*$ has the radius of convergence ϱ and $\varrho_1 > \varrho$, define a bijection of $\mathfrak{P}^*$ onto $\mathfrak{G}$ ([3], p. 375). Suppose that $\mathscr{L}^{(\vartheta)}[g_k] = F_k(w)$, then it follows from ([3], p. 399) that

$$\mathscr{L}^{(\vartheta)}[g_1 g_2] = F_1(w) \circ F_2(w).$$

We summarize these facts (in other form in [3]) as a

COROLLARY. *The $\mathscr{L}^{(\vartheta)}$-transformation defines an algebraic isomorphism of $\mathfrak{P}^*$ onto $\mathfrak{G}$.*

(Therefore $\mathfrak{P}^*$ is an integral domain under the convolution $\circ$.)

By use of Theorem 1 and the Corollary we obtain

THEOREM 2. *The mapping $\mathscr{L}^{(\vartheta)}\mathfrak{L}$: $A \to \mathscr{L}^{(\vartheta)}[\mathfrak{L}[A]]$, $A \in \mathfrak{P}$, defines an algebraic isomorphism of $\mathfrak{P}^*$ onto $\mathfrak{P}$.*

This isomorphsim can be extended to an isomorphism of the quotient field $\mathfrak{Q}^*(\mathfrak{P})^*$ of $\mathfrak{P}^*$ onto $\mathfrak{Q}(\mathfrak{P})$:

$$\mathfrak{Q}(\mathfrak{P}) \ni A/B \to \mathscr{L}^{(\vartheta)}[\mathfrak{L}[A]] // \mathscr{L}^{(\vartheta)}[\mathfrak{L}[B]] \in \mathfrak{Q}^*(\mathfrak{P}),$$

where $A, B \in \mathfrak{P}$ and the symbol // denotes the inverse operation of the convolution (3.3). Therefore we identify $\mathfrak{Q}^*(\mathfrak{P}^*)$ with $\mathfrak{Q}(\mathfrak{P})$. If $F(w)$ belongs to $\mathfrak{P}^*$ then we write $\{F(w)\}$ or F or $g(s)$ (in the case $F(w) = \mathscr{L}^{(\vartheta)}[g(z)]$) as element of $\mathfrak{A}$. Moreover we have for $F_1, F_2 \in \mathfrak{P}$ and all complex α, β

$$\alpha F_1 + \beta F_2 = \{\alpha F_1(w) + \beta F_2(w)\} \quad \text{and} \quad F_1 F_2 = \{F_1(w) \circ F_2(w)\}.$$

Suppose that $g(s) \in \mathfrak{P}$ has representation (3.1), then we obtain

$$g(s) = \left\{ \sum_{k=0}^{\infty} \alpha_k / w^{k+1} \right\}.$$

Now we consider some operators belonging to $\mathfrak{Q}(\mathfrak{P})$. Obviously, all rational operators $r(s)$ belong to $\mathfrak{Q}(\mathfrak{P})$. For example, the zero operator $0 = \langle \varphi / \psi \rangle$ $(\varphi \equiv 0, \psi \not\equiv 0)$, the unit operator $1 = \langle \psi / \psi \rangle$ and the differential operator s belong to $\mathfrak{P}$, whereas the integral operator $s^{-1} \in \mathfrak{Q}(\mathfrak{P})$ does not belong to $\mathfrak{P}$. By use of the $\mathscr{L}^{(3)}$-transformation we obtain $0 = \{0\}$, $1 = \{1/w\}$ and $s = \{1/w^2\}$. If $F = \{F(w)\} \in \mathfrak{P}$ then the differential theorem is given by $s^k F = \{(-1)^k F^{(k)}(w)\}$ $(k = 1, 2, \ldots)$, and $s^k F$ belongs to $\mathfrak{P}$ for all $F \in \mathfrak{P}$. On the other hand, $s^{-1} F$ is an element of $\mathfrak{P}$ only if $\beta_0 = 0$ in (3.2).

Finally, we consider the operators $e^{\alpha s} \in \mathfrak{A}$, where α is any complex number. It is easy to see that $e^{\alpha s} \in \mathfrak{P}$ for all complex α and $e^{\alpha s} = \{1/(w - \alpha)\}$. Let $F = \{F(w)\}$ be an arbitrary element of $\mathfrak{P}$. By use of the $\mathscr{L}^{(3)}$-transformation or the convolution $e^{\alpha s} F = \left\{ \frac{1}{w - \alpha} \circ F(w) \right\}$ it follows

$$e^{\alpha s}\{F(w)\} = \{F(w - \alpha)\}.$$

Hence we obtain

THEOREM 3. *Let α be an arbitrary complex number, then the operator $e^{\alpha s} \in \mathfrak{A}$ is a shift operator for the functions $\{F(w)\} \in \mathfrak{P}$.*

REMARK. This is one possibility of the interpretation of the operators $e^{\alpha s} \in \mathfrak{A}$. Another possibility to explain the operators $e^{\alpha s}$ to be shift operators for complex arguments can be given by use of such functions $\varphi(t) \in \mathfrak{A}$ which are holomorphic in a strip containing the t-axis, where these functions fulfil certain conditions. We omit the details. Another operator algebra, which contains the operators $e^{\alpha s}$ for complex α too, is constructed in [1].

It is well known that $\mathfrak{Q}$ contains especially the distributions (in the sense of L.Schwartz) having compact supports. Every such a distribution d has a representation (see [2])

$$d = \sum_{k=0}^{m} s^k \varphi_k, \tag{3.4}$$

where the functions φ_k $(k = 0, 1, \ldots, m)$ are continuous for all real t and have compact supports. Every function φ_k is an element of $\mathfrak{A}$ and has a Laplace transform $\mathfrak{L}[\varphi]$ which is a finite two-sided Laplace integral [7]. It is easy to show that $\mathfrak{L}[\varphi]$ is a function in $\mathfrak{G}$. Therefore from (3.4) it follows that every Schwartz-distribution with a compact support belongs to $\mathfrak{P}$.

4. Remark to ultradistributions and operators

In connection with the Fourier transformation of the Schwartz-distributions of exponential type we obtain the linear space of the ultradistributions. Let $\mathcal{U}_c$ be the linear subspace of the ultradistributions having compact supports [8]. Every element $\Phi \in \mathcal{U}_c$ has a representation

$$\Phi = \sum_{k=0}^{\infty} \alpha_k \delta^{(k)},$$

where the coefficients α_k fulfil the condition $\overline{\lim_k} \sqrt[k]{k!\,|\alpha_k|} < \infty$. The derivative $\delta^{(k)}$ $(k=0, 1, \ldots)$ of the delta-ultradistribution δ is determined uniquely by the class of functions

$$f_{\delta^{(k)}}(\zeta) = (-1)^{k+1} \frac{k!}{2\pi i \zeta^{k+1}} + p(\zeta),$$

where $p(\zeta)$ is an arbitrary polynomial in ζ. The convolution of two elements $\Phi = \sum_{k=0}^{\infty} \alpha_k \delta^{(k)}$ and Ψ of $\mathcal{U}_c$ is defined by

$$\Phi * \Psi = \sum_{k=0}^{\infty} \alpha_k \Psi^{(k)},$$

where the $\Psi^{(k)}$ are the derivatives of Ψ in the sense of the ultradistributions. Under this convolution $\mathcal{U}_c$ is an integral domain. From [8] it follows that $\mathcal{U}_c$ and $\mathfrak{G}$ are isomorphic algebras, where the isomorphism is defined by

$$g_{\Phi}(z) = \int_{\partial C_n} e^{-z\zeta} f_{\Phi}(\zeta)\, d\zeta. \tag{4.1}$$

∂C_n is the boundary of a domain of the form

$$C_n = \{\zeta : |\mathrm{Im}\,(\zeta)| \geqq n > 0 \,(\text{if } \mathrm{Re}\,(\zeta) \geqq 0);\ |\zeta| \geqq n \,(\text{if } \mathrm{Re}\,(\zeta) \leqq 0)\}$$

and $f_{\Phi}(\zeta)$ is a representative of $\Phi \in \mathcal{U}_c$. Therefore we are allowed to explain the operators $g(s) \in \mathfrak{P}$ to be ultradistributions having compact supports. By use of (4.1) we get

$$s^k \leftrightarrow \delta^{(k)} \quad (k = 0, 1, \ldots) \quad \text{and} \quad e^{-\alpha s} \leftrightarrow \delta_\alpha \quad (\alpha \text{ complex}),$$

where δ_α is the shifted δ in $\mathcal{U}_c$, which is determined by the representative

$$f_{\delta_\alpha}(\zeta) = -\frac{1}{2\pi i(\zeta - \alpha)} + p(\zeta).$$

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SETS OF UNIQUENESS FOR HAAR SERIES*

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§ 1. Introduction

Let $\chi_0, \chi_1, \ldots$ represent the Haar functions (see [1], [6] or [12] for the standard definition). A set $E \subseteqq [0, 1]$ is called a *set of uniqueness* for Haar series if the only sequence $a_0, a_1, \ldots$ of real numbers which satisfy

(1) $$\lim_{n\to\infty} \sum_{k=0}^{n} a_k \chi_k(x) = 0 \quad \text{for} \quad x \in [0, 1] \setminus E$$

is the sequence

(2) $$a_k = 0 \quad \text{for} \quad k = 0, 1, \ldots.$$

It is known that $E = \emptyset$ is the *only* set of uniqueness for Haar series [7; p. 626]. Hence any further study of sets of uniqueness must be carried out for some restricted class of Haar series. Since Haar functions are defined in terms of square roots of powers of 2, we introduce the following restriction:

Let p be a finite real number. A Haar series $T(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$ satisfies *condition* $G(p)$ if

(3) $$a_k = o([k]^{(p-1)/2}) \quad \text{as} \quad k \to \infty,$$

where $[k]$ represents the largest power of 2 in k; i.e., $[k] = 2^n$ if and only if $2^n \leqq k < 2^{n+1}$. We shall call a set $E \subseteqq [0, 1]$ a $G(p)$ *U-set* if the only sequence $a_0, a_1, \ldots$ of real numbers satisfying (1) *and* (3) is the sequence (2).

Let $p > q$. Then every Haar series which satisfies condition $G(q)$ also satisfies condition $G(p)$. Hence every $G(p)$ U-set is a $G(q)$ U-set. In particular, $E = \emptyset$ is always a $G(p)$ U-set. On the other hand, since the examples of McLaughlin and Price [7] satisfy condition $G(p)$ for each $p > 2$ we conclude that the empty set is the only $G(p)$ U-set when $p > 2$.

$G(2)$ U-sets have also been characterized. Indeed, in [8] it is shown that a Borel set is a $G(2)$ U-set if and only if it is countable.

The only other known result about $G(p)$ U-sets is also due to Mushegjan [8]. He has constructed a perfect $G(0)$ U-set of measure zero.

A brief outline of the remainder of this paper follows.

In section 2 we recall that Walsh functions and Haar functions are linear combinations of each other, and use that fact to move the formal product theory of Walsh functions over to Haar functions. This is especially interesting since the product of two Haar functions is not, in general, a Haar function.

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In section 3 we show that a closed set is a $G(p)$ U-set if and only if it is a $G(p)$ U^*-set (Theorem 1). This result is a consequence of the first integral of Haar series as its trigonometric analogue is a consequence of the second integral of trigonometric series.

The lemmas of section 4 begin to display the power of condition $G(p)$ when $p \leqq 2$ and the peculiarities of the Haar functions themselves. It is especially interesting to note that localization for Haar series introduces discontinuous limits unlike Rajchman's theory of localization for trigonometric series. This forced us to introduce $G(p)$ U^+-sets (see Lemma 4.1 and the definition that follows) and dyadic step functions.

The main results of this paper are found in sections 5 and 6. Concerning the size of $G(p)$ U-sets we show that each $G(p)$ U-set has measure zero when $p>0$ (Theorem 2), but that for each $p<0$ there are $G(p)$ U-sets with positive measure (Theorem 4). Concerning combinations of $G(p)$ U-sets we show that the union of a sequence of closed $G(p)$ U-sets is a $G(p)$ U-set when $p>0$ (Theorem 3). Theorem 3 is also true in case $p \leqq 0$ if we require that union to have Lebesgue measure zero. The proof is the same. However, by the Weierstrass M-test any Haar series satisfying condition $G(p)$ for some $p<0$ converges *uniformly*. Hence when $p<0$, *any* set of measure zero is a $G(p)$ U-set. The extension of Theorem 3 is of interest, then, only when $p=0$.

We close this section by pointing out that although the study of $G(p)$ U-sets is interesting in its own right, our study has been guided by a larger design. Indeed, it is most likely that the theory of $G(0)\,U$-sets closely parallels that of sets of uniqueness for trigonometric series. Hence a characterization of $G(0)$ U-sets, say in terms of certain arithmetic properties of the sets, would represent a major breakthrough for that same problem on the trigonometric series. And characterizing the necessarily less complicated $G(p)$ U-sets for $p>0$ seems like a good first approximation.

§ 2. The Walsh functions

Let $\psi_0, \psi_1, \dots$ represent the Walsh functions as enumerated by R.E.A.C. Payley (see [1], [3], [4] or [5]). Both the Haar functions and the Walsh functions form a complete orthonormal system in the Hilbert space $L^2[0, 1]$.

But these functions are more closely related than that. Indeed, except at points of discontinuity, each Walsh function ψ_k is a linear combination of the Haar functions $\chi_{[k]}, \chi_{[k]+1}, \dots \chi_{2[k]-1}$ [1; p. 62]. Specifically,

$$\psi_0(x) = \chi_0(x) \equiv 1 \quad (x \in (0, 1)) \tag{4}$$

and if k is a positive integer then

$$\psi_k(x) = [k]^{-1/2} \sum_{j=[k]}^{2[k]-1} \varepsilon_{kj} \chi_j(x) \tag{5}$$

for $x \in [0, 1] \setminus D$ where D represents the dyadic rationals and the $\varepsilon_{kj} = \pm 1$ form a symmetric matrix $[\varepsilon_{kj}]$ which has orthogonal rows. The special properties of the matrix $[\varepsilon_{kj}]$ allow us to express each Haar function as the same linear combination of Walsh functions (see [2] or [2]):

(6) $$\chi_k(x) = [k]^{-1/2} \sum_{j=[k]}^{2[k]-1} \varepsilon_{kj}\psi_j(x)$$

for $x \in [0, 1] \setminus D$.

The upshot of all this is that 2^nth partial sums of Haar series and 2^nth partial sums of Walsh series are essentially the same. Specifically, given a Haar series $T = \sum a_k \chi_k$ there is an *associated Walsh series* $T^* = \sum a_k^* \psi_k$ such that

(7) $$T_{2^n}^*(x) = T_{2^n}(x) \equiv \sum_{k=0}^{2^n-1} a_k \chi_k(x) \quad \text{for} \quad x \in [0, 1] \setminus D$$

and $n=0, 1, \ldots$; indeed we need only set $a_0^* = a_0$ and

(8) $$a_k^* = [k]^{-1/2} \sum_{j=[k]}^{2[k]-1} \varepsilon_{kj} a_j, \quad k = 1, 2, \ldots$$

and apply equation (6). Conversely, given a Walsh series $S = \sum A_k \psi_k$ there is an associated Haar series $S^* = \sum A_k^* \psi_k$ such that

(9) $$S_{2^n}^*(x) = S_{2^n}(x) \equiv \sum_{k=0}^{2^n-1} A_k \psi_k(x)$$

for $x \in [0, 1] \setminus D$ and $n=0, 1, \ldots$; indeed we need only set $A_0^* = A_0$ and

(10) $$A_k^* = [k]^{-1/2} \sum_{j=[k]}^{2[k]-1} \varepsilon_{kj} A_j, \quad k = 1, 2, \ldots$$

and apply equation (5).

Again, by the special properties of the matrix $[\varepsilon_{kj}]$ equation (8) ((respectively (10)) holds with the roles of a_k and a_j^* (respectively A_k and A_j^*) reversed. In particular, a series is the zero series if and only if its associated series is the zero series. We shall use this fact in the proof of Theorem 4.

We now match up growth conditions for associated series.

A Walsh series $S = \sum A_k \psi_k$ satisfies *condition* $F(p)$ $(-\infty < p < \infty)$ if

(11) $$\sum_{k=2^n}^{2^{n+1}} A_k^2 = o(2^{pn}) \quad \text{as} \quad n \to \infty.$$

Arutunjan and Talaljan [2] have shown that if S is a Walsh series whose coefficients tend to zero then its associated Haar series satisfies condition $G(2)$. Our first lemma shows that this conclusion is valid even if S only satisfies condition $F(1)$.

Lemma 2.1. *Let p be a finite real number. If a Haar series satisfies condition $G(p)$ then its associated Walsh series satisfies condition $F(p+1)$. On the other hand, if a Walsh series satisfies condition $F(p)$ then its associated Haar series satisfies condition $G(p+1)$.*

Proof. Let a_k satisfy (3) and a_k^* be defined by (8). Then

$$|a_k^*| \leqq [k]^{-1/2}[k] \cdot \max\{|a_j| : [k] \leqq j < 2[k]\} = [k]^{1/2} \cdot o([k]^{(p-1)/2}) = o([k]^{p/2})$$

by (3). In particular, since $2^n \leqq k < 2^{n+1}$ implies $[k]=2^n$,

$$\sum_{k=2^n}^{2^{n+1}-1} (a_k^*)^2 = 2^n o(2^{pn}) = o(2^{(p+1)n}).$$

On the other hand, if A_k satisfies (11) and A_k^* is defined by (10) then Schwarz' inequality yields

$$|A_k^*| \leqq [k]^{-1/2}[k]^{1/2} \left(\sum_{i=[k]}^{2[k]-1} A_i^2 \right)^{1/2} = \left(o([k]^p)\right)^{1/2} = o([k]^{p/2})$$

as was to be shown.

Let f be Lebesgue integrable over [0, 1]. A Haar series $T=\sum a_k \chi_k$ is the *Haar Fourier series* of f if

$$a_k = \int_0^1 f(x)\chi_k(x)\,dx, \quad k = 0, 1, \dots \tag{12}$$

and a Walsh series $S=\sum A_k \psi_k$ is the *Walsh Fourier series* of f if

$$A_k = \int_0^1 f(x)\psi_k(x)\,dx, \quad k = 0, 1, \dots.$$

LEMMA 2.2. *Let $f \in L^1[0, 1]$. If T is the Haar Fourier series of f then its associated Walsh series is the Walsh Fourier series of f. Conversely, if S is the Walsh Fourier series of f then its associated Haar series is the Haar Fourier series of f.*

PROOF. By symmetry we consider only the first statement. Write $T=\sum a_k \chi_k$ and let a_k^* be defined by (8). Then by (12) and (5)

$$a_k^* = [k]^{-1/2} \sum_{i=[k]}^{2[k]-1} \varepsilon_{ki} a_i = [k]^{-1/2} \sum_{i=[k]}^{2[k]-1} \varepsilon_{ki} \int_0^1 f(x)\chi_i(x)\,dx =$$

$$= [k]^{-1/2} \int_0^1 f(x) \sum_{i=[k]}^{2[k]-1} \varepsilon_{ki}\chi_i(x)\,dx = \int_0^1 f(x)\psi_k(x)\,dx.$$

In particular, T^* is the Walsh Fourier series of f.

A set $E \subseteqq [0, 1]$ is a $G(2)$ U^**-set* if any Haar series satisfying condition $G(2)$ which also satisfies

$$\lim_{n\to\infty} T_n(x) = f(x), \quad x \in [0, 1]/E,$$

for some $f \in L^1[0, 1]$, is the Haar Fourier series of f.

Since the Haar Fourier series of any integrable function f satisfies condition $G(2)$ [2; p. 1395] we define a $G(p)$ U^*-set as follows. A set $E \subseteqq [0, 1]$ is a $G(p)$ U^**-set* if given a Haar series T satisfying condition $G(p)$ and a function f whose Haar Fourier series satisfies condition $G(p)$ then

$$\lim_{n\to\infty} T_n(x) = f(x), \quad x \in [0, 1] \setminus E$$

implies that T is the Haar Fourier series of f.

ARUTUNJAN and TALALJAN [2] have shown that any countable set is a $G(2)$ U^*-set. The following slightly stronger result is a consequence of theorem 1 in [12].

LEMMA 2.3. *Let* $p\leqq 2$, *T be a Haar Fourier series satisfying condition* $G(p)$ *and Z be some countable subset of* [0, 1]. *Suppose there is an integrable function f such that*

$$\lim_{n\to\infty} T_{2^n}(x) = f(x) \quad a.e.\ x\in[0,1]$$

and

$$\limsup_{n\to\infty} |T_{2^n}(x)| < \infty \quad for\ x\in[0,1]\setminus Z.$$

Then T is the Haar Fourier series of f.

We close this section by combining associated Walsh series and the formal product of Walsh series to obtain a pseudoformal product theory for Haar series (see Lemma 2.4).

Reviewing the Walsh theory, given a Walsh polynomial $\lambda=\sum_{i=0}^{N} C_i\psi_i$ and a Walsh series $S=\sum_{i=0}^{\infty} A_i\psi_i$ their formal product is the Walsh series

$$(\lambda\circ S)(x) \equiv \sum_{i=0}^{\infty}\left\{\sum_{k=0}^{N} G_k A_{i*k}\right\}\psi_i(x)$$

where $i*k$ is that integer defined by $\psi_{i*k}(x)=\psi_i(x)\cdot\psi_k(x)$.

SNEIDER has shown [8; p. 287] that

$$\text{(13)}\qquad \begin{cases} 0\leqq j<2^M \quad\text{and}\quad q2^M\leqq k<(q+1)2^M \\ \text{imply}\quad q2^M\leqq j*k<(q+1)2^M \end{cases}$$

where q, M, j and k are integers. He has also shown that if the coefficients A_k of the Walsh series S tend to zero as $k\to\infty$ then

$$\text{(14)}\qquad \lim_{n\to\infty}\{(\lambda\circ S)_n(x)-\lambda(x)\cdot S_n(x)\}=0$$

uniformly for $x\in[0,1]$. Furthermore, the series $\lambda\circ S$ also has coefficients which tend to zero.

LEMMA 2.4. *Let* $\mu=\sum_{i=0}^{N} c_i\chi_i$ *be a Haar polynomial and* $T=\sum_{i=0}^{\infty} a_i\chi_i$ *be a Haar series satisfying condition* $G(p)$ *for some* $p\leqq 0$. *Then there is a Haar series* $\mu\circ T$ *satisfying condition* $G(2)$ *such that*

$$\text{(15)}\qquad \lim_{n\to\infty}\{(\mu\circ T)_{2^n}(x_0)-\mu(x_0)\cdot T_{2^n}(x_0)\}=0$$

for each dyadic irrational $x_0\in[0,1]$.

PROOF. Let μ^* and T^* be the associated Walsh series of μ and T respectively. Note that μ^* is therefore a Walsh polynomial. To apply (14) to μ^* and T^* we must first verify that the coefficients a_k^* of T^* tend to zero as $k\to\infty$:

$$|a_k^*| \leqq [k]^{-1/2} \sum_{j=[k]}^{2[k]-1} |a_j| = [k]^{1/2} o([k]^{(p-1)/2}) = o([k]^{p/2}) = o(1) \quad \text{as} \quad k\to\infty$$

by (8) and (3) since $p\leqq 0$.

Hence by (14),

$$\lim_{n\to\infty} \{(\mu^*\circ T^*)_n(x) - \mu^*(x)T_n^*(x)\} = 0 \tag{16}$$

uniformly for $x\in[0, 1]$. Also $\mu^*\circ T^*$ satisfies condition $F(1)$ since its coefficients tend to zero. Hence by Lemma 2.1, $(\mu^*\circ T^*)^*$ satisfies condition $G(2)$. We complete the proof by letting $\mu\circ T\equiv(\mu^*\circ T^*)^*$. Indeed (15) holds for $x_0\notin D$ by (9) and (16).

We shall see in section 6 that Lemma 2.4 is false when $p>0$.

§ 3. The first integral of Haar series

In this section we shall show that a closed set is a $G(p)$ U-set if and only if it is a $G(p)$ U^*-set. The preliminary lemmas deal with the first integral of Haar series.

By $\alpha_n=\alpha_n(x)$ and $\beta_n=\beta_n(x)$ we shall mean

$$\alpha_n \equiv p2^{-n} \leqq x < (p+1)2^{-n} \equiv \beta_n;$$

also $\alpha_n'(x)=\alpha_n(x)$ if $x\notin D$ and $\alpha_n'(x)=\alpha_n(x)-2^{-n}$ otherwise.

Given a Haar series $T=\sum a_k\chi_k$ we define its first integral by

$$L(T; x) \equiv \lim_{n\to\infty} \int_0^x T_{2^n}(t)\,dt \tag{17}$$

whenever this limit exists.

The following two lemmas appeared in [12].

LEMMA 3.1. *If T is any Haar series and n any nonnegative integer then $L(T;\alpha_n(x))$ and $L(T;\beta_n(x))$ exist and are finite for $x\in[0, 1]$. Furthermore,*

$$2^{-n}T_{2^n}(x) = L(T;\beta_n(x)) - L(T;\alpha_n(x)) \tag{18}$$

for each $x\in[0, 1]$.

LEMMA 3.2. *Let T be a Haar series satisfying condition $G(p)$ for some $p\leqq 2$. Suppose further that $L(T; x_0)$ exists and is finite. Then*

$$\lim_{n\to\infty} L(T;\alpha_n'(x_0)) = L(T; x_0)$$

and

$$\lim_{n\to\infty} L(T;\beta_n(x_0)) = L(T; x_0).$$

The following result was the main theorem in [5].

LEMMA 3.3. *Let $a<b$ be real numbers and G be a real valued function defined on $(a, b)\cap D$, where*

$$D = \{x : x \text{ is a dyadic rational}\}. \tag{19}$$

Suppose further that G satisfies three conditions:

(i) $\limsup_{n\to\infty} G(\alpha_n'(x)) \geqq G(x)$ *for* $x \in (a, b) \cap D$,

(ii) $\liminf_{n\to\infty} [G(\beta_n(x)) - G(\alpha_n(x))] \leqq 0$ *for* $x \in (a, b)$,

(iii) $\lim_{n\to\infty} 2^n [G(\beta_n(x) - G(\alpha_n(x))] \leqq 0$ *for all but countably many* $x \in (a, b)$.

Then G is monotone decreasing in $(a, b) \cap D$.

We combine these three lemmas to show that by adding a constant to the right hand side of (17) we can bring the limit sign inside.

LEMMA 3.4. *Let T be a Haar series which satisfies condition* $G(p)$ *for some* $p \leqq 2$. *Suppose that* $0 \leqq a < b \leqq 1$ *and that*

$$\lim_{n\to\infty} T_{2^n}(x) = f(x) \tag{20}$$

for all but countably may $x \in (a, b)$, *where* $f \in L^1[0, 1]$.
Then there is a constant C_1 *such that*

$$L(T; x) - \int_0^x f(t)\,dt = C_1 \tag{21}$$

at every point $x \in (a, b)$ *where* $L(T; x) < \infty$. *In particular,* (21) *holds for each* $x \in$ $\in (a, b) \cap D$.

PROOF. We follow [5; p. 352] by setting $F(x) = \int_0^x f(t)\,dt$, fixing $\varepsilon > 0$ and the Vitali-Caratheodory Theorem [9; p. 75] to select absolutely continuous functions applying φ_ε and ψ_ε on [0, 1] such that

$$|\varphi_\varepsilon(x) - F(x)| < \varepsilon, \quad |\psi_\varepsilon(x) - F(x)| < \varepsilon, \quad x \in [0, 1]$$

and the derivates of $\varphi_\varepsilon(x)$ (respectively $\psi_\varepsilon(x)$) are less than (respectively greater than) $f(x)$ whenever $f(x) \neq -\infty$ (respectively $+\infty$).

For each $x \in (a, b) \cap D$ set $G_\varepsilon(x) = \varphi_\varepsilon(x) - L(T; x)$ and $H_\varepsilon(x) = L(T; x) - \psi_\varepsilon(x)$. By Lemmas 3.1 and 3.2, G_ε and H_ε satisfy hypothesis (i) and (ii) of Lemma 3.3. By (18), (20) and the choices of φ_ε and ψ_ε, the functions G_ε and H_ε satisfy hypothesis (iii) of Lemma 3.3. By that lemma then, G_ε and H_ε are monotone decreasing on $(a, b) \cap D$. Letting $\varepsilon \to \infty$ we conclude that $F(x) - L(T; x)$ and $L(T; x) - F(x)$ are monotone decreasing on $(a, b) \cap D$, hence constant there; i.e., (21) holds for each $x \in (a, b) \cap D$. Finally, if $x_0 \in (a, b)$ and $L(T; x_0) < \infty$ then Lemma 3.2 and the continuity of F show (21) holds for x_0 also.

N. J. FINE [4; p. 403] proved that the first integral of the Walsh *Fourier* series of a function $f \in L^1[0, 1]$ and $\int_0^x f(t)\,dt$ differed by a constant throughout [0, 1]. Using Lemma 2.2 and (7) we have the same result for Haar Fourier series:

LEMMA 3.5. *Let $f \in L^1[0,1]$ and $T[f]$ represent the Haar Fourier series of f. Then there is a constant C_2 such that*

$$L(T[f]; x) - \int_0^x f(t)\,dt = C_2 \quad \textit{for all} \quad x \in [0,1].$$

Before showing that the class of closed $G(p)$ U-sets coincides with the class of closed $G(p)$ U^*-sets we cite one more known result [14]. An alternate proof of this fact is provided by Lemma 5.1 of this paper.

LEMMA 3.6. *Let $p \leqq 2$, E be a closed $G(p)$ U-set and Z be any countable set. Then $E \cup Z$ is a $G(p)$ U-set.*

THEOREM 1. *Let $p \leqq 2$ and E be a closed set. Then E is a $G(p)$ U-set if and only if E is a $G(p)$ U^*-set.*

PROOF. Clearly, *every* $G(p)$ U^*-set is a $G(p)$ U-set.

On the other hand, let f be a function whose Haar Fourier series $T[f]$ satisfies condition $G(p)$ and suppose that $T(x)$ is a Haar series satisfying condition $G(p)$ which converges to $f(x)$ for each x outside the closed $G(p)$ U-set E. Then by Lemma 3.4

$$L(T; x) - \int_0^x f(t)\,dt = C_1 \quad \text{for} \quad x \in I \cap D$$

where I is any subinterval of $[0,1] \setminus E$. By Lemma 3.5,

$$L(T[f]; x) - \int_0^x f(t)\,dt = C_2 \quad \text{for} \quad x \in I \cap D.$$

Subtracting these two equations we have

$$L(T - T[f]; x) = C_1 - C_2 \quad \text{for} \quad x \in I \cap D. \tag{22}$$

Combining (22) and (18) we conclude that the 2^nth partial sums of the Haar series $T - T[f]$ converge to zero throughout the interval I. Since E is closed and $I \subset [0,1] \setminus E$ was arbitrary we conclude that

$$\lim_{n\to\infty} (T - T[f])_{2^n}(x) = 0 \quad \text{for} \quad x \in [0,1] \setminus E.$$

But the 2^nth partial sums of any Haar series converge at a dyadic irrational x_0 if and only if the nth partial sums of that Haar series converge at x_0. Hence

$$\lim_{n\to\infty} (T - T[f])_n(x) = 0 \quad \text{for} \quad x \in [0,1] \setminus E \cup D. \tag{23}$$

Now T and $T[f]$ satisfy condition $G(p)$; hence so does $T - T[f]$. By Lemma 3.6 and hypothesis, $E \cup D$ is a $G(p)$ U-set. Hence (23) implies that $T - T[f]$ is the zero series; i.e., $T \equiv T[f]$ *is* the Haar Fourier series of f.

§ 4. Special properties of Haar functions

If E is a set let E^* represent its topological closure.

Recall that if $k=2^n+p$ where $2^n=[k]$ then the Haar function χ_k is identically $+\sqrt{2^n}$ on the open interval

$$\Delta(1, k) \equiv (p/2^n, (2p+1)2^{n+1}) \tag{24}$$

and identically $-\sqrt{2^n}$ on the open interval

$$\Delta(2, k) \equiv ((2p+1)/2^{n+1}, (p+1)/2^n). \tag{25}$$

Furthermore, the support of χ_k is precisely $\Delta^*(1, k)\cup\Delta^*(2, k)$.

LEMMA 4.1. *Let $T=\sum a_k\chi_k$ be a Haar series satisfying condition $G(p)$ for some $p\in(-\infty, \infty)$ and J be an open subinterval of* [0, 1] *with dyadic rational end points. Then there is a Haar series τ which also satisfies condition $G(p)$ and a step function g whose jumps occur only at dyadic rationals such that*

$$\tau_n(x) = 0 \quad \text{for} \quad x\in[0, 1]\setminus J^* \tag{26}$$

and $n=1, 2, \ldots,$ and

$$\lim_{n\to\infty} [T_n(x)-\tau_n(x)] = g(x) \quad \text{for} \quad x\in J. \tag{27}$$

PROOF. Since the end points for J are dyadic rationals, we can choose N so large that $k\geqq N$ implies the support of χ_k is a subset of J^* or disjoint from J. Let $n_1=N$ be the smallest such integer and choose a sequence $n_1<n_2<\ldots$ by insisting that

$$\Delta(1, n_k)\cup\Delta(2, n_k)\subset J \quad \text{for} \quad k = 1, 2, \ldots. \tag{28}$$

Then set

$$\tau(x) = \sum_{k=1}^{\infty} a_{n_k}\chi_{n_k}(x) \quad \text{for} \quad x\in[0, 1].$$

By (28), τ satisfies (26). Since each Haar function is a step function whose jumps occur only at dyadic rationals, so is $g(x)\equiv T_{n_1}(x)$. This choice for g surely satisfies (27).

Notice that even in the case T converges to zero in Lemma 4.1, the limit of τ may in general be discontinuous at a finite number of points. For this reason we shall introduce $G(p)$ $U^\dagger$-sets.

A *dyadic step function* is a step function whose jumps occur at diadic rationals. A set $E\subseteqq[0, 1]$ is a $G(p)$ $U^\dagger$*-set* if any Haar series T satisfying condition $G(p)$ which also satisfies

$$\lim_{n\to\infty} T_n(x) = f(x), \quad x\in[0, 1]\setminus E$$

for some dyadic step function f, must be the Haar Fourier series of f.

Clearly any $G(p)$ $U^\dagger$-set is a $G(p)$ U-set. A consequence of the following observation is that each $G(p)$ U^*-set is a $G(p)$ $U^\dagger$-set. In particular, Theorem 1 shows us that any closed $G(p)$ U-set is also a $G(p)$ $U^{\dagger}$-set. (This fact can also be obtained from Lemma 5.1.)

LEMMA 4.2. *If f is a dyadic step function then its Haar Fourier series satisfies condition $G(p)$ for every $p\in(-\infty, \infty)$.*

PROOF. Since f is a step function with jumps at dyadic rationals, there is an integer N so large that $k\geqq N$ implies f is constant on $\Delta(1, k)\cup\Delta(2, k)$. Hence all Haar Fourier coefficients of order than N are zero:

$$\int_0^1 f(x)\chi_k(x)\,dx = \text{const.} \int \chi_k(x)\,dx \equiv 0$$

if $k\geqq N$.

The following lemma appeared in [2].

LEMMA 4.3. *Let K_0 be a positive integer, $i_0=1$ or 2 and τ be a Haar series satisfying condition $G(p)$ for some $p\leqq 2$ such that*

$$\lim_{n\to\infty} \tau_n(x) = 0$$

for all but countably many $x\in\Delta(i_0, K_0)$. Suppose further that

$$\tau_{K_0+1}(x) \neq 0 \quad \textit{for any} \quad x\in\Delta(i_0, K_0).$$

Then given any $x_0\in[0, 1]$, $A>0$ and any positive integer M we can find a pair of natural numbers n and p and an interval $\Delta(i_p, p)$ of the form (24) or (25) such that $n>M$; $\Delta^*(i_p, p)\subset\Delta(i_0, K_0)$;

(29) $$x_0\notin\Delta^*(i_p; p);$$

(30) $$|\tau_n(x)| > A \quad \textit{for} \quad x\in\Delta(i_p, p);$$

and

(31) $$\tau_{p+1}(x) \neq 0 \quad \textit{for any} \quad x\in\Delta(i_p, p).$$

We close this section with another indication of the power condition $G(p)$ wields when $p\leqq 2$.

LEMMA 4.4. *Let $T=\sum a_k\chi_k$ be a Haar series satisfying condition $G(p)$ for some $p\leqq 2$ and $\Delta(i_0, N_0)$, $\Delta(i_1, N_1)$, ... be a sequence of intervals of the form (24) or (25) such that*

(32) $$[N_k] = 2[N_{k-1}], \quad k = 1, 2, \dots.$$

Suppose further that $i_k'=1$ or 2, $i_1'\neq i_k$ for $k=1, 2, \dots$ and that $\Delta(i_0, N_0)$, $\Delta(i_1', N_1)$, ... forms a nested sequence of intervals.

Then $T_{N_1+k}(x)\equiv 0$ for $x\in\Delta(i_k', N_k)$ and $k=1, 2, \dots$ implies

(33) $$T_{N_0+1}(x) \equiv 0 \quad \textit{for} \quad x\in\Delta(i_0, N_0).$$

PROOF. Suppose (33) does not hold. Since for each $k\leqq N_0$, χ_k is constant on $\Delta(i_0, N_0)$ we are supposing that there is a *nonzero* constant d such that

(34) $$T_{N_0+1}(x) \equiv d \quad \text{for} \quad x\in\Delta(i_0, N_0).$$

By (32) and the hypothesis concerning nestedness, we then have

$$T_{N_k+1}(x) = d + \sum_{j=1}^{k} a_{N_j}\chi_{N_j}(x)$$

for $x \in \Delta(i_k', N_k)$. In particular, if T_{N_1+1} is zero on $\Delta(i_1', N_1)$ then $\alpha_{N_1}\chi_{N_1} \equiv -d$ on $\Delta(i_1', N_1)$. Since $|\chi_{N_1}|$ is constant on $\Delta(i_1, N_1) \cup \Delta(i_1', N_1)$ we conclude that

$$\|a_{N_1}\chi_{N_1}\|_\infty = |d|.$$

Proceeding by induction we have

(35)
$$\|a_{N_k}\chi_{N_k}\|_\infty = 2^{k-1}|d|$$

for $k=1, 2, \ldots$.

Let m_0 be that integer determined by

$$\|\chi_{N_1}\|_\infty = 2^{m_0/2}.$$

Then

(36)
$$\|\chi_{N_k}\|_\infty = 2^{(m_0+k-1)/2} \equiv [N_k]^{1/2}$$

for $k=1, 2, \ldots$. Combining (35) and (36),

$$|a_{N_k}|[N_k]^{-1/2} = |d|2^{-m_0}$$

for $k=1, 2, \ldots$. Since T satisfies condition $G(p)$ for $p=2$ it must then be the case that $d=0$. This together with (34) establishes (33).

§ 5. The union of closed $G(p)$ U-sets

We have noted that if $p>2$ then $G(p)$ U-sets are empty and if $p=2$ then Borel $G(p)$ U-sets are countable. In particular, if $p \geqq 2$ then the countable union of closed $G(p)$ U-sets is a $G(p)$ U-set and the measure of a $G(p)$ U-set is zero. In this section we extend these results to $p>0$. The second result is trivial.

THEOREM 2. *Let $p>0$. Then every $G(p)$ U-set has measure zero.*

PROOF. We shall show that any set E with positive measure is *not* a $G(p)$ U-set.

Indeed, let $P \subset E$ be a perfect set such that $m(P)>0$ and let T be the Haar Fourier series of the characteristic function of P; i.e., $T = \sum a_k \chi_k$ where

$$a_k = \int_P \chi_k(x)\,dx, \quad k = 0, 1, \ldots.$$

Then $a_0 = m(P) > 0$ shows us that T is not the zero series. Since

$$\left|\int_P \chi_k(x)\,dx\right| \leqq m\big(\Delta(1, k) \cup \Delta(2, k)\big) \cdot [k]^{1/2} = [k]^{-1/2}$$

it is clear that T satisfies condition $G(p)$ for any $p>0$.

Finally, since Haar Fourier series converge at points of continuity [1; p. 472] it is clear that T converges to zero in every interval contiguous to P. In particular, T converges to zero off E.

NOTE. By using Lemma 4.2 and the above procedure one can show that no $G(p)$ U-set can contain an interval regardless of how large *negatively* p gets.

To obtain the result about the union of closed $G(p)$ U-sets we must first obtain three preliminary results. The first one is a special case of the general result.

LEMMA 5.1. *Let $p\leqq 2$, E be a closed $G(p)$ U-set and Z be any countable set of points. Then $E\cup Z$ is a $G(p)$ $U^{\dagger}$-set.*

PROOF. Let $T(x)=\sum a_k\chi_k(x)$ be a Haar series satisfying condition $G(p)$ which converges to a dyadic step function $f(x)$ for each $x\in[0,1]\setminus(E\cup Z)$. Let $T[f]=\sum c_k\chi_k$ represent the Haar Fourier series of f. We must show that $a_k=c_k$ for $k=0,1,\ldots$. Since E is a $G(p)$ U-set and $\tau\equiv T-T[f]$ is a Haar series satisfying condition $G(p)$, it suffices to show that τ converges to zero off E. We shall actually show that τ is identically zero off E.

Suppose that τ is *not* identically zero off E. Then there is a $z_0\in[0,1]\setminus E$ and an integer N such that $z_0\in\operatorname{supp}\chi_N$ but $a_N\neq c_N$. If we choose $N_0=N$ to be the least such integer then $\tau_{N_0+1}(x)\neq 0$ for any $x\in\Delta(i_0,N_0)$, where $i_0=1$ or 2.

Suppose for the moment that there is an interval $\Delta(i_L',N_L)\subset\Delta(i_0,N_0)$ disjoint from E such that

$$\tau_{N_L+1}(x)\neq 0 \quad \text{for any} \quad x\in\Delta(i_L',N_L). \tag{37}$$

Let $z_1,z_2,\ldots$ be an enumeration of the set $Z\cup\Pi$ where Π is the set of discontinuities of f. Since $\Delta(i_L',N_L)\cap E=\emptyset$ T converges to f in $\Delta(i_L',N_L)\setminus Z$. $T[f]$ converges to f at every point of continuity, hence in $\Delta(i_L',N_L)\setminus\Pi$. Thus τ satisfies all the hypotheses of Lemma 4.3. Applying that lemma countably many times we choose sequences $n_1<n_2<\ldots, j_1,j_2,\ldots$ and $k_1,k_2,\ldots$ of integers such that

$$\Delta(j_l,k_l)\subset\Delta(i_L',N_L),\quad l=1,2,\ldots,$$

$$z_l\notin\Delta^*(j_l,k_l),\quad l=1,2,\ldots, \tag{38}$$

$$\Delta^*(j_l,k_l)\subset\Delta(j_l,k_{l-1}),\quad l=2,3,\ldots \tag{39}$$

and

$$|\tau_{n_l}(x)|>l \quad \text{for} \quad x\in\Delta(j_l,K_l) \tag{40}$$

and $l=1,2,\ldots$. Now by (39) there is at least one point w_0 common to all the intervals $\Delta(j_l,k_l)$. Yet this will lead us to a contradiction. Indeed, by (38) $w_0\notin Z\cup\Pi$ and thus $\tau(w_0)\equiv 0$ *does* converge. On the other hand, w_0 satisfies (40) for each $l\geqq 1$ so $\tau(w_0)$ *does not* converge.

It suffices, then, to show (37) holds for some interval $\Delta(i_L',N_L)$ disjoint from E. We shall accomplish this by constructing a sequence of intervals satisfying the hypotheses of Lemma 4.4.

Indeed, set $i_1'=i_0$ and $N_1=N_0$. Suppose that intervals $\Delta(i_1,N_1),\ldots,\Delta(i_k,N_k)$ and $\Delta(i_1',N_1),\ldots,\Delta(i_1',N_k)$ have been chosen so that $i_m'\neq i_m$, $[N_m]=2[N_{m-1}]$, $z_0\in\Delta^*(i_m',N_m)$ for $m=1,\ldots,k$, and so that the second sequence is nested. Choose N_{k+1} so that $\Delta(i_k',N_k)=\Delta^*(1,N_{k+1})\cup\Delta^*(2,N_{k+1})$; clearly $[N_{k+1}]=2[N_k]$. Now $z_0\in\Delta^*(i_k',N_k)$ so let $i_{k+1}'=1$ if $z_0\in\Delta^*(1,N_{k+1})\sim\Delta^*(2,N_{k+1})$ and let $i_{k+1}'=2$ if $z_0\in\Delta^*(2,N_{k+1})$.

Continuing in this manner we can generate a nested sequence of intervals $\Delta(i_1', N_1)$, $\Delta(i_1', N_2)$, ... such that

$$z_0 = \bigcap_{k=1}^{\infty} \Delta^*(i_k', N_k). \tag{41}$$

Since $z_0 \in [0, 1] \setminus E$ and E is closed we use (41) to find a large k_0 so that $\Delta(i_{k_0}', N_{k_0})$ is disjoint from E. It thus suffices to show that (37) holds for some $L \geqq k_0$. However, if (37) failed to hold for any $L \geqq k_0$ then by applying Lemma 4.4 $k_0 - 1$ times we would eventually conclude that $\tau_{N_0+1}(x) \equiv 0$ for $x \in \Delta(i_0, N_0)$. Since this contradicts the choice of N_0 our proof is complete.

LEMMA 5.2. *Let $-\infty < p < \infty$, E be a closed $G(p)$ U-set, T be a Haar series satisfying condition $G(p)$ and f be a dyadic step function. Suppose further that I is an open interval such that*

$$\lim_{n\to\infty} T_n(x) = f(x) \quad for \quad x \in I \setminus E.$$

Then T converges to f at all but finitely many points of I.

PROOF. If $p > 2$ then E is empty and the result follows immediately.

Otherwise, let $J = (\alpha, \beta) \subseteqq I$ be an open interval with dyadic rational end points. By Lemma 4.1 there is a dyadic step function g and a Haar series τ satisfying condition $G(p)$ such that

$$\lim_{n\to\infty} \tau_n(x) = \begin{cases} f(x) - g(x) & \text{for} \quad x \in J \setminus E \\ 0 \quad \text{for} \quad x \in [0, 1] \setminus J^*. \end{cases} \tag{42}$$

Hence τ converges to a dyadic step function except on $E \cup \{\alpha, \beta\}$. But by Lemma 5.1, $E \cup \{\alpha, \beta\}$ is a $G(p)$ $U^\dagger$-set. Hence τ is the Haar Fourier series of the right hand side of (42). In particular, τ converges to $f - g$ at every point of continuity in J. Since the discontinuities of f and g are finite in number, we complete the proof by applying (27) and recalling that $J \subseteqq I$ was arbitrary.

LEMMA 5.3. *Let $p \leqq 2$, I be an open interval and $E \subseteqq I$ be a closed $G(p)$ U-set. Suppose that T is a Haar series satisfying condition $G(p)$ such that*

(i) $\limsup_{n\to\infty} |T_n(x)| < \infty$ *for* $x \in I \setminus E$

and

(ii) $\lim_{n\to\infty} T_n(x) = f(x)$ *a.e.* $x \in I$

where f is a dyadic step function. Then T converges to f at all but finitely many points in I.

PROOF. Let $J_1 \subseteqq I$ be an open interval disjoint from E and $J \subseteqq J_1$ be an open interval with dyadic rational end points. Apply Lemma 4.1 to choose a dyadic step function g and a Haar series τ satisfying condition $G(p)$ such that (26) and (27) hold. Let Π represent the discontinuity points of f and g.

Combining (27), hypothesis (i) and the fact that $J \cap E = \emptyset$ we see that

$$\limsup_{n\to\infty} |\tau_n(x)| < \infty \quad \text{for} \quad x \in J.$$

This fact together with hypothesis (ii) and (26) allow us to apply Lemma 2.3 to τ. Hence τ must converge to $f-g$ on $J\setminus\Pi$. Since $J\subseteq J_1\subseteq I$ were arbitrary subject to the condition $J_1\cap E=\emptyset$ we apply (27) to conclude that

$$\lim_{n\to\infty} T_n(x)=f(x) \quad \text{for} \quad x\in I\setminus E\cup\Pi.$$

By Lemma 5.1 $E\cup\Pi$ is a $G(p)$ $U^{\dagger}$-set hence $G(p)$ U-set. Since Π is finite hence closed, $E\cup\Pi$ is a *closed* $G(p)$ U-set. The proof is completed by applying Lemma 5.2 $E\cup\Pi$, T and f.

THEOREM 3. *Let $0\leqq p\leqq 2$ and $E_1, E_2, \dots$ be a sequence of closed $G(p)$ U-sets. Suppose further that in the case $p=0$ each E_i has Lebesgue measure zero. Then $\bigcup_{i=1}^{\infty} E_i$ is a $G(p)$ U-set.*

PROOF. It suffices to show $\bigcup_{i=1}^{\infty} E_i$ is a $G(p)$ $U^{\dagger}$-set. Let $T(x)$ be a Haar series satisfying condition $G(p)$ which converges to a dyadic step function $f(x)$ at each point $x\in[0,1]\setminus\bigcup_{i=1}^{\infty} E_i$. By Theorem 2 or hypothesis, $m(\bigcup E_i)=0$ so

$$\lim_{n\to\infty} T_n(x)=f(x) \quad \text{a.e.} \quad x\in[0,1]. \tag{43}$$

Let $N=\{x\colon \limsup_{n\to\infty}|T_n(x)|=\infty\}$. By (43) and Lemma 2.3 it suffices to show that N is at most countable. By Lemma 3 of [14] N is either at most countable or N is of the second category on itself. We shall complete this proof by showing N cannot be of the second category on itself. It is a standard argument [16; p. 350].

Indeed, suppose that N is of the second category on itself and uncountable. Then by defining $N_i=N\cap E_i$ we conclude that there is an open interval J and an index i_0 such that $N_{i_0}\cap J$ is dense in $N\cap J$ and that $N\cap J$ is infinite. $N\cap J=E_{i_0}\cap N\cap J\subseteq E_{i_0}\cap J$ since E_{i_0} is closed.

We may assume the end points of J are not in E_{i_0} and thus that $E_{i_0}\cap J$ is a closed $G(p)$ U-set contained in J. Furthermore, if $x\notin E_{i_0}\cap J$ then $x\notin N\cap J$ so the partial sums of T are bounded in $J\setminus E_{i_0}\cap J$. Hence by Lemma 5.3 T converges to f at all but finitely many points of J. In particular, $N\cap J$ is finite. Since this statement contradicts the choice of J we have shown that N must be countable.

We close this section by noting that in the case $0<p\leqq 2$, Lemma 5.3 is true when f is any function continuous except at a finite number of points. Since we do not need this larger class of functions, we leave verification to the reader.

§ 6. $U(\varepsilon)$-sets

We have seen that the Lebesgue measure of a $G(p)$ U-set is zero if $0<p<\infty$ and that *any* set of measure zero is a $G(p)$ U-set when $-\infty<p<0$. In this section we shall show that for any $-\infty<p<0$ there is a $G(p)$ U-set whose measure is "arbitrarily" close to 1. We begin by reformulating the problem using the $U(\varepsilon)$-set terminology of ZYGMUND [16].

Let $\varepsilon: \eta_0 \geqq \eta_1 \geqq \ldots \geqq \eta_k \to 0$ be a sequence of real numbers. A set $E \subset [0, 1]$ is a *$U(\varepsilon)$-set* for Haar series if the only Haar series $T = \sum a_k \chi_k$ converging to zero outside E whose coefficients satisfy

(44) $$|a_k [k]^{1/2}| \leqq \eta_k, \quad k = 0, 1, \ldots$$

is the zero series. Clearly if T satisfies condition $G(p)$ for some $p<0$ it will satisfy (44) for some choice of $\varepsilon: \eta_0, \eta_1, \ldots$. We need to show, then, that given any sequence ε there is a $U(\varepsilon)$-set with measure "arbitrarily" close to 1.

THEOREM 4. *Let $\varepsilon: \eta_0 \geqq \eta_1 \geqq \ldots \geqq \eta \to 0$ be a sequence of real numbers and $\delta > 0$. Then there is a $U(\varepsilon)$-set E' such that $m(E') > 1 - \delta$.*

PROOF. Select positive integers $m_1 \leqq m_2 \leqq \ldots$ such that $m_n \to \infty$ and

(45) $$2^{6m_n} \eta_{2^n} \to 0 \quad \text{as} \quad n \to \infty.$$

Let $\langle x \rangle = x - [[x]]$, where $[[\ ]]$ denotes the greatest integer function, and define the set $E_n \subseteqq [0, 1]$ by

$$E_n = \{x : x \in [0, 1] \quad \text{and} \quad \langle 2^n x \rangle \geqq 2^{-2m_n}\}.$$

Then $m(E_n) > 1 - 2^{-2m_n}$ and since $m_n \to \infty$,

$$\lim_{n \to \infty} m(E_n) = 1.$$

Hence we can choose an integer n_1 such that $m(E_{n_1}) > 1 - \delta/2$ and a real number p such that

$$(1 - \delta/2) \prod_{i=1}^{\infty} (1 - p^{-i}) > 1 - \delta.$$

Let $F_1 = E_{n_1}$ and continue, generating a sequence $n_1 < n_2 < \ldots$ of integers such that

$$m(F_{i+1}) \geqq (1 - p^{-i}) m(F_i)$$

where

$$F_i = \bigcap_{k=1}^{i} E_{n_k}.$$

If we let $E' = \bigcap_{i=1}^{\infty} E_{n_i}$ then the choice of p implies $m(E') > 1 - \delta$. It suffices to show that E' is a $U(\varepsilon)$-set.

Suppose, then, that $T = \sum a_k \chi_k$ is a Haar series satisfying (44) such that

(46) $$\lim_{n \to \infty} T_n(x) = 0 \quad \text{for each} \quad x \in [0, 1] \setminus E'.$$

We need to show that the associated Walsh series T^* of the Haar series T is the zero series.

To simplify what follows set $p_l = 2^{n_i}$ and $q_i = 2^{2m_{n_i}}$. Also set

$$\lambda_i(x) = \begin{cases} 1 & \text{if} \quad p_i x \in [0, q_i^{-1}] \\ 0 & \text{otherwise.} \end{cases}$$

It is known [10; p. 869] that λ_i is a Walsh polynomial:

(47) $$\lambda_i(x) = (q_i)^{-1} \sum_{j=0}^{q_i-1} \psi_{j\cdot p_j}(x).$$

Let λ_i^* represent its associated Haar polynomial. Since T satisfies condition $G(0)$ we apply Lemma 2.4 to choose a Haar series τ satisfying condition $G(2)$ such that

(48) $$\lim_{n\to\infty} \{\tau_n(x) - \lambda_i^*(x) T_n(x)\} = 0$$

for each dyadic irrational x. By (46) and (48),

(49) $$\lim_{n\to\infty} \tau_n(x) = 0 \quad \text{for} \quad x \in [0, 1] \setminus E' \cup D$$

(see (19)). On the other hand, if $x \in E'$ then $x \in E_{n_i}$ which by the construction indicated above means that $p_i x \in [0, q_i^{-1}]$; i.e., $\lambda_i(x) = 0$. Combining this fact with (48) we conclude

(50) $$\lim_{n\to\infty} \tau_n(x) = 0 \quad \text{for} \quad x \in E' \setminus D.$$

By (49), (50) and Lemma 2.3 we conclude that τ is the zero series. What remains to be seen, then, is that $\tau \equiv 0$ implies $T^* \equiv 0$.

By the proof of Lemma 2.4, $\tau = (\lambda \circ T^*)^*$ so we already have $\lambda \circ T^* \equiv 0$. Using (8), (47) and the definition of the Walsh formal product we conclude that for each pair of nonnegative integers i and j,

(51) $$0 \equiv q_i^{-1} \sum_{k=0}^{q_i-1} a^*_{j*kp_i}.$$

We pause to write down three facts.

By the definitions of p_i, q_i and (45) we have

(52) $$\lim_{i\to\infty} q_i^3 \eta_{p_i} = 0.$$

By (8), (44) and the monotonicity of $\eta_0, \eta_1, \dots$ we have

(53) $$|a_l^*| \leqq \eta_{p_i} q_i^{1/2} \quad \text{for} \quad l \geqq p_i$$

when $i = 1, 2, \dots$.

Finally, if $1 \leqq k < q_i$ then $p_i \leqq kp_i < q_i p_i$. In particular, if for each fixed j we choose i so large that $p_i > j$ we have by (13) that

(54) $$p_i \leqq j * kp_i < q_i^2 \quad \text{if} \quad 1 \leqq k < q_i.$$

We now complete the proof by solving (51) for $a_j^* \equiv a_{j*0}^*$ and using these facts:

$$|a_j^*| = \left| \sum_{k=1}^{q_i-1} a^*_{j*k\,p_i} \right| \leqq \sum_{l=p_i}^{q_i^2} |a_l^*| \leqq q_i^2 \eta_{p_i} q_i^{1/2} \leqq q_i^3 \eta_{p_i}.$$

by (53) and (54) for i sufficiently large. Letting $i \to \infty$ for each fixed j we have by (52) that $a_j^* = 0$. Since j was arbitrary we have $T^* \equiv 0$ as was to be shown.

We close the paper by noting that the restriction $p \leqq 0$ was crucial to Lemma 2.4. Indeed, if Lemma 2.4. held for any $p_0 > 0$ then the proof of Theorem 4 would establish the existence of a $G(p_0/2)$ U-set with positive measure. This is impossible by Theorem 2.

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ETALEMENTS CRISTALLOGRAPHIQUES

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0. Introduction

Si dans $\mathbf{R}^2$ nous considérons l'animal carré T composé de cinq carrés et formant la lettre T (Fig. 1) ou l'animal triangulaire K heptagone équilatéral composé de cinq triangles équilatéraux (Fig. 2), on voit que ni T ni K ne peuvent paver régulière-

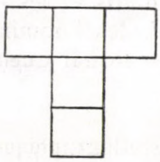


Fig. 1

Fig. 2

ment le plan. Les figures 3 et 4 montrent que les densités régulières d'empilements sont $\eta(T) \geqq 5/6$ et $\eta(K) \geqq 5/6$ et on peut prouver qu'on a égalité dans les inégalités précédentes.

Par contre si l'on considère des pavements éventuels de $\mathbf{R}^2$ par des transformés de T (ou K) à l'aide d'un groupe cristallographique d'isométries les figures 5 et 6

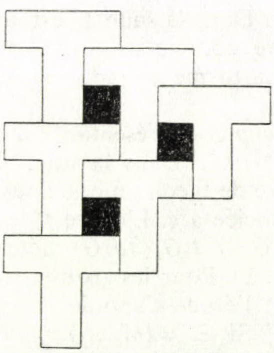


Fig. 3

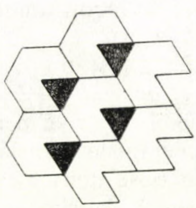


Fig. 4

montrent que ces pavements existent. Le but de ce papier est de systématiser en partie la situation précédente.

Pour ce, au § 1, on considère les notions d'étalements cristallographiques et semi-cristallographiques de $\mathbf{R}^n$ et leurs propriétés immédiates. Au § 2 on définit les

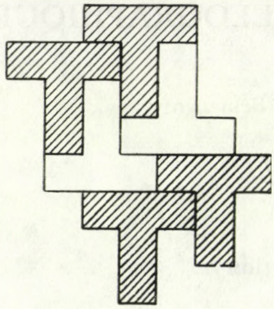


Fig. 5

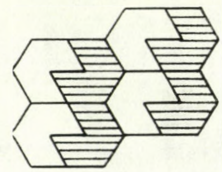


Fig. 6

animaux strictement cristallographiques et au § 3 on envisage le cas particulier du plan. On a alors à considérer les k-iamonds, les k-iominos et les k-hexes. Nous avons étudié systématiquement les k-iamonds pour $k \leqq 7$, les k-iominos pour $k \leqq 6$ et les k-hexes pour $k \leqq 5$. Nous nous contentons ici d'indiquer quelques résultats.

1. Etalements cristallographiques et semi-cristallographiques

Soit dans $\mathbf{R}^n$ le groupe $E(n)$ des isométries affines, $T(n) \approx \mathbf{R}^n$ le groupe des translations, $O(n)$ le groupe orthogonal: on a $T(n) \lhd E(n)$, $O(n) = E(n)/T(n)$ et $E(n)$ est produit semi-direct de $T(n)$ et de $O(n)$.

Un groupe $\Sigma \subset E(n)$ est dit *cristallographique* si

(i) *il admet comme sous-groupe distingué un réseau G de translations*

(ii) *il n'y a pas de translations dans Σ autres que celles de G*

(iii) $\mathfrak{S}' = \Sigma/G$ *est fini.*

On dit alors que $\mathfrak{S}'$ et G engendrent Σ. Dans la suite Σ est toujours un groupe cristallographique, $\mathfrak{S}'$ peut être considéré comme un sous-groupe fini de $O(n)$ et $\sigma \in \Sigma$ peut s'écrire de façon unique sous la forme $\sigma = \tau\bar{\sigma}$, τ étant une translation et $\bar{\sigma} \in \mathfrak{S}'$.

On dit que $\mathfrak{S} \subset \Sigma$ est un *système complet* représentant $\mathfrak{S}'$ si quel que soit $s' \in \mathfrak{S}'$ il existe *un et un seul* $s \in \mathfrak{S}$ tel que $\bar{s} = s'$. Dans la suite, $\mathfrak{S}$ représente toujours un tel système. Tout $\sigma \in \Sigma$ peut s'écrire de façon unique sous la forme $\sigma = t_g s$ avec $s \in \mathfrak{S}$, $g \in G$, t_g étant la translation associée à g. L'ordre f de $\mathfrak{S}'$ s'appelle aussi l'indice de Σ et on pose dens $\Sigma = f$ dens $G = f/d(G)$ ($d(G)$ déterminant positif de G, dens G, resp. dens Σ densité de G, resp. Σ). Pour les groupes cristallographiques on pourra consulter J. A. Wolf [24] dont l'étude s'appuie sur les résultats de L. Bieberbach [4] et de J. Burckhardt [5], [6]. Si $\mathfrak{S}'' = \{\sigma_1'', \ldots, \sigma_f''\}$ est un ensemble de f isométries distinctes telles que quels que soitent les éléments distincts τ, ν de $\mathfrak{S}''$ on ait

$$\tau\nu^{-1} \notin G \quad \text{(où } G \text{ est un réseau de } \mathbf{R}^n\text{)}.$$

$\Gamma = \{t_p\tau | g \in G \text{ et } \tau \in \mathfrak{S}''\}$ est appelé famille *semi-cristallographique* engendrée par $\mathfrak{S}''$ et G d'indice f. (Pour $\gamma \in \Gamma$, γ peut se mettre sous la forme $t_g\tau$ de façon unique). [Des exemples de tels Γ sont donnés par $\Gamma = \Sigma$ ou encore si $\mathfrak{S}''$ est un groupe fini d'isométries (car alors pour deux éléments distincts τ, ν de $\mathfrak{S}''\tau\nu^{-1}$ n'est jamais une translation); on peut prendre aussi pour $\mathfrak{S}''$ un ensemble de translations $\{t_1, ..., t_f\}$ telles que pour $i, j=1, ..., f$ on n'ait pas $t_i-t_j \in G$ si $t_i \neq t_j$, dans ce cas Γ est une grille multiple d'indice f qui est un réseau multiple ssi[1] cette grille contien tun élément de G (par exemple pour $1 \in \mathfrak{S}''$)]. Pour un tel Γ, on définit encore la densité de Γ par

$$\text{dens}\, \Gamma = f/d(G).$$

Pour une famille quelconque Γ d'isométries et pour $K \subset \mathbf{R}^n$, (K, Γ) désigne la famille $\{\gamma K | \gamma \in \Gamma\}$ appelée *étalement de K par Γ*. Dans le cas où Γ est semi-cristallographique et vol $K>0$ on pose dens $(K, \Gamma) = \text{vol}\, K$. dens Γ: c'est la densité de (K, Γ). [Notons que si τ est une similitude affine de rapport non nul, dens $(\tau K, \tau \Gamma) =$ $=$ dens (K, Γ).] On parle alors d'étalement semi-cristallographique.

On dit que (K, Γ) est un *recouvrement d'une partie A* de $\mathbf{R}^n$ quand $\bigcup_{\gamma \in \Gamma} \gamma K$ couvre A à un ensemble de mesure nulle près.

On dit que (K, Γ) est un *empilement* quand quels que soient γ_1, γ_2 distincts de Γ, $\gamma_1 K$ et $\gamma_2 K$ sont disjoints. (Si Γ est un groupe cristallographique Σ il faut et il suffit pour cela que quel que soit $\sigma \in \Sigma$ on ait pour $\sigma \neq 1$, $K \cap \sigma K = \emptyset$).

Quand (K, Γ) est à la fois un empilment et un recouvrement de $\mathbf{R}^n$ on dit que c'est un *pavement*.

Pour $v \in \mathbf{R}^n$, on pose $K_v = t_v K = K+v$, $\{K_g | g \in G\}$ est appelé *étalement régulier* de translatés de K par G. C'est un étalement cristallographique correspondant à $\mathfrak{S} = \mathfrak{S}' = \{1\}$. Dans la suite, on suppose que *K est borné et de mesure strictement positive.*

PROPOSITION 1.1. *Si (K, G) est un empilement, on a* dens $(K, G) \leqq 1$ *et si (K, G) est un recouvrement,* dens $(K, G) \geqq 1$.

PREUVE. C'est connu. On peut utiliser le

LEMME. *Si P est un parallélotope fondamental de G, alors quel que soit $u \in \mathbf{R}^n$*

$$\sum_{x \in G+u} \text{vol}\,(K_x \cap P) = \text{vol}\, K.$$

PROPOSITION 1.2. *Un empilement régulier de densité* 1 *est un pavement. De plus, si K est ouvert, un recouvrement régulier de densité* 1 *est un pavement.*

PREUVE. Si (K, G) est un empilement de densité 1,

$$\sum_{x \in G} \text{vol}\,(K_x \cap P)/\text{vol}\, P = \text{vol}\, K/d(G) = 1.$$

Par suite $\{K_x \cap P | x \in G\}$ recouvre P à un ensemble de mesure nulle près. Cela suffit.

D'autre part si (K, G) est un recouvrement et qu'on peut trouver, dans G, g_1 et g_2 distincts et $x_0 \in \mathring{K}_{g_1} \cap \mathring{K}_{g_2} \cap \mathring{P}$, une boule $B(x_0)$ de centre x_0 et de rayon convenable $\varrho > 0$ est contenue dans $K_{g_1} \cap K_{g_2} \cap P$. Soit alors $\varepsilon = \text{vol}\, B(x_0)/\text{vol}\, P$. On a

$$\sum_{x \in G} \text{vol}\,(K_x \cap P) - \varepsilon\, \text{vol}\, P \geqq \text{vol}\, P$$

ce qui, vu le lemme, donne

$$\text{dens}\,(K, G) \geqq (1+\varepsilon)\, \text{vol}\, P/\text{vol}\, P = 1+\varepsilon.$$

[1] ssi est une abreviation pour "si et seulement si".

C'est exclu pour dens $(K, G)=1$ de sorte que dans ce cas pour g_1 et g_2 distincts éléments de G on a

$$\mathring{K}_{g_1} \cap \mathring{K}_{g_2} \cap P = (\mathring{K}_{g_1} \cap P) \cap (\mathring{K}_{g_2} \cap P) = \emptyset.$$

Quand K est ouvert, pour x parcourant G, les $K_x \cap P$ sont disjoints. Cela suffit.

PROPOSITION 1.3. *Σ étant un groupe cristallographique et $\mathfrak{S} \subset \Sigma$ un système complet associé à Σ,*

(a) (K, Σ) *est un empilement ssi* $(K, \mathfrak{S})$ *est un empilement et* $\left(\bigcup_{s \in \mathfrak{S}} sK, G\right)$ *aussi;*

(b) (K, Σ) *est un recouvrement ssi* $\left(\bigcup_{s \in \mathfrak{S}} sK, G\right)$ *en est un.*

PREUVE. Pour les recouvrements c'est immédiat. Pour les empilements: notons $A \oplus B$ l'union des parties A et B de $\mathbf{R}^n$ lorsqu'elles sont disjointes et $s_1, \ldots, s_f$ les éléments distincts de $\mathfrak{S}$.

Si $(K, \mathfrak{S})$ est un empilement, $\bigcup_{s \in \mathfrak{S}} sK = \bigoplus_{i=1}^{f} s_i K$. Si de plus, pour $K' = \bigcup_{s \in \mathfrak{S}} sK$, (K', G) est un empilement c'est que pour $g \neq 1$ élément de G, $K' \cap t_g K' = \emptyset$. Comme

$$t_g \left(\bigoplus_{i=1}^{f} s_i K \right) = \bigoplus_{i=1}^{f} t_g s_i K,$$

(K, Σ) est bien un empilement. D'autre part la réciproque est immédiate.

PROPOSITION 1.4. (a) *Si* (K, Σ) *est un empilement,* dens $(K, \Sigma) \leqq 1$.

(b) *Si* (K, Σ) *est un recouvrement,* dens $(K, \Sigma) \geqq 1$.

(c) *Si* (K, Σ) *est un pavement,* dens $(K, \Sigma) = 1$.

PREUVE. On applique les propositions 1.1 et 1.3 en notant que dans le cas d'un empilement vol $K' = f$ vol K et que dans le cas d'un recouvrement vol $K' \leqq f$ vol K.

PROPOSITION 1.5. *Si* (K, Σ) *est un empilement de densité* 1 *ou pour K ouvert un recouvrement de densité* 1, *c'est un pavement.*

C'est là une conséquence immédiate des propositions 1.2 et 1.3.

REMARQUE. Les propositions 1.3, 1.4 et 1.5 s'étendent au cas où (K, Σ) est remplacé par (K, Γ) où Γ est une famille semi-cristallographique engendrée par $\mathfrak{S}''$ et G, si on remplace dans ces propositions $\mathfrak{S}$ par $\mathfrak{S}''$. Elles sont vraies en particulier quand Γ est un h-réseau.

2. Constantes d'empilements cristallographiques et Animaux strictement cristallographiques

On note $\eta_{C'}(K)$, resp. $\bar{\eta}_{C'_f}(K)$ la borne supérieure de dens (K, Γ) prise pour les (K, Γ) qui sont des empilements semi-cristallographiques quelconques, resp. d'indice f. On note $\eta_C(K)$, resp. $\bar{\eta}_{C_f}(K)$ la borne supérieure de dens (K, Σ) prise par les (K, Σ) qui sont des empilements cristallographiques quelconques, resp.

d'indice f. On pose

$$\eta_{C'_f}(K) = \sup_{1 \leqq g \leqq f} \bar{\eta}_{C'_g}(K) \quad \text{et} \quad \eta_{C_f}(K) = \sup_{1 \leqq g \leqq f} \bar{\eta}_{C_g}(K).$$

On a

$$\eta_{C'_f}(K) \leqq \eta_{C'}(K) \leqq 1 \quad \text{et} \quad \eta_{C_f}(K) \leqq \eta_C(K) \leqq 1.$$

Rappelons quelques notations concernant les empilements par des translatés de K. Pour $X \subset \mathbf{R}^n$, on pose $(K, X) = \{K_x | x \in X\}$ et $\text{dens}\,(K, X) = \text{vol}\, K \cdot \text{dens}\, X$ (cf. par ex. [2] ou [3]). $\eta_*(K)$, $\bar{\eta}_h(K)$ désigne la borne supérieure de dens (K, X) pour X quelconque et quand X est un h-réseau. On pose

$$\eta_h(K) = \sup_{1 \leqq k \leqq h} \bar{\eta}_h(K); \quad \text{on a} \quad \eta_h(K) \leqq \eta_*(K) \leqq 1.$$

Evidemment $\eta_1(K) = \eta_{C_1}(K) = \eta_{C'_1}(K)$. Cette quantité notée $\eta(K)$ est appelée densité régulière (d'empilement) de K. (K, X) est un empilement ssi $DX \cap DK = \{0\}$ (Pour $Y \subset \mathbf{R}^n$, $DY = Y - Y$ est l'ensemble « différence » de Y.) Comme $DK \subset D\left(\frac{1}{2} DK\right)$, $D\left(\frac{1}{2} DK\right) \cap DX = \{0\}$ entraîne $DK \cap DX = \{0\}$ donc si $\left(\frac{1}{2} DK, X\right)$ est un empilement (de translatés de $\frac{1}{2} DK$) (K, X) est aussi un empilement. On en déduit

$$\eta_*\left(\tfrac{1}{2} DK\right)/\text{vol}\left(\tfrac{1}{2} DK\right) \leqq \eta_*(K)/\text{vol}\, K$$

et

$$\eta_h\left(\tfrac{1}{2} DK\right)/\text{vol}\left(\tfrac{1}{2} DK\right) \leqq \eta_h(K)/\text{vol}\, K.$$

On peut écrire aussi

$$2^n \eta_*(DK)/\text{vol}\,(DK) \leqq \eta_*(K)/\text{vol}\, K \quad \text{et} \quad 2^n \eta_h(DK)/\text{vol}\, DK \leqq \eta_h(K)/\text{vol}\, K$$

en se servant de $\eta_*(DK) = \eta_*\left(\frac{1}{2} DK\right)$, $\eta_h(DK) = \eta_h\left(\frac{1}{2} DK\right)$ et $\text{vol}\,(DK) = 2^n \text{vol}\left(\frac{1}{2} DK\right)$.

Notons que *lorsque K est convexe,* il en est de même de DK et que $D\left(\frac{1}{2} DK\right) = 2\left(\frac{1}{2} DK\right) = DK$. Il en résulte qu'alors les inégalités précédentes deviennent des égalités. On sait que pour $n = 2$ lorsque *C est convexe symétrique* on a $\eta_*(C) = \eta(C)$, cette propriété ayant été trouvée indépendamment par C. A. ROGERS [21] et L. FEJES TÓTH [11] (cf. par exemple J. W. S. CASSELS [7] ou L. FEJES TÓTH [12]).

Il résulte alors des égalités précédentes que pour K convexe de $\mathbf{R}^2$ (non nécessairement symétrique) on a $\eta_*(K) = \eta(K)$ (cf. aussi C. G. LEKKERKEKER [19] p. 195).

Notons aussi que pour n quelconque, $K \subset \mathbf{R}^n$ on a $\eta(K) = \text{vol}\, K/\Delta(DK)$ où $\Delta(DK)$ est la constante critique de DK (cf. par ex. [2] ou [3]). Cela nous servira.

Soit P un polytope régulier ouvert de $\mathbf{R}^n$ d'adhérence $\bar{P}$. On dit que *Π_k est un k-animal fermé régulier construit sur P si l'on a*

$$\Pi_k = \overline{\bigcup_{i=1}^{k} P_i} = \bigcup_{i=1}^{k} \bar{P}_i$$

avec

(i) pour $i = 1, \ldots, k$ P_i est isométrique à P

(ii) pour $i = 1, \ldots, k-1$ $\bigcup_{j=1}^{i} \bar{P}_j \cap P_{i+1} = \emptyset$

(iii) pour $i = 1, \ldots, k-1$ $\bar{P}_i \cap \bar{P}_{i+1}$ est une face à $n-1$ dimensions de P_i (et de P_{i+1}).

On en déduit que pour $k > 1$ $\bigcup_{i=1}^{k-1} \bar{P}_i$ est un $(k-1)$ animal régulier construit sur P

et que $\operatorname{vol} \Pi_k = k \operatorname{vol} P$. On considère souvent les animaux réguliers modulo le groupe des similitudes affines parfois modulo le groupe des similitudes directes. Rappelons qu'une famille *infinie dénombrable* de polytopes de $\mathbf{R}^n$ *fermés convexes* $\{\Pi_i\}_{i \in \mathbf{N}}$ forme une *mosaïque* de $\mathbf{R}^n$ quand

(i) pour $i \neq j$ $\Pi_i \cap \Pi_j$ est une face à $n-1$ dimensions commune à Π_i et Π_j ou $\dim (\Pi_i \cap \Pi_j) < n-1$

(ii) $\mathbf{R}^n = \bigcup_{i \in \mathbf{N}} \Pi_i$. Dans ce cas $\{\mathring{\Pi}_i\}_{i \in \mathbf{N}}$ est un pavement de $\mathbf{R}^n$ qu'on appelle aussi mosaïque.

Rappelons aussi qu'on dit qu'un polytope convexe Π est un *parallélоèdre* lorsqu'il pave régulièrement $\mathbf{R}^n$.

On dit qu'un *animal K est cristallographique lorsqu'on peut trouver un groupe cristallographique Σ tel que $(\mathring{K}, \Sigma)$ soit un pavement.*

Lorsque K est construit à partir de P, d'un tel (K, Σ) on déduit naturellement un pavement de $\mathbf{R}^n$ à l'aide d'isométriques de P. S'il est possible de choisir Σ de façon que ce pavement soit une mosaïque on dit que l'animal est *strictement cristallographique.*

PROPOSITION 2.1. *Un animal régulier est strictement cristallographique seulement s'il provient*

pour $n=2$ d'un triangle équilatéral, d'un carré ou d'un hexagone;

pour $n=3$ et $n \geqq 5$ d'un (hyper) cube;

pour $n=4$ d'un hypercube ou d'un des deux autres polytopes réguliers permettant de former une mosaïque de $\mathbf{R}^4$.

PREUVE. Si K est un tel animal, $\bar{P}_1$ forme avec des isométriques une mosaïque de $\mathbf{R}^n$. La proposition résulte alors de l'étude de ces mosaïques (cf. par ex. l'ouvrage [8] de COXETER et spécialement sa table II).

REMARQUE. On peut considérer des animaux non réguliers.

En effet appelons *parallélоèdre mosaïcal* un polytope ouvert convexe P tel que l'on puisse trouver un réseau G avec

(i) $\mathbf{R}^n = \bigcup_{g \in G} \bar{P}_g$,

(ii) quels que soient g, g' éléments distincts de G, $\bar{P}_g \cap \bar{P}_{g'}$ est une face commune à $\bar{P}_g$ et $\bar{P}_{g'}$, ou a une dimension strictement inférieure à $n-1$.

Si dans la définition d'un animal régulier P est maintenant un parallélоèdre mosaïcal, on définit un animal dit mosaïcal. On peut se demander si un tel animal peut être strictement cristallographique: c'est le cas pour un 1-animal ou un 2-animal mosaïcal. On peut se demander aussi s'il existe des parallélоèdres mosaïcaux non réguliers. C'est la cas par exemple du parallélоèdre de AP SIMON [1]. Un tel parallélоèdre L est défini par la donnée de n nombres $c_i > 1$ comme l'ensemble des x de $\mathbf{R}^n$ dont les coordonnées vérifient $|x_r| < 1 + c_r$, $|x_r - x_s| < c_r + c_s - 2$ pour $r, s = 1, \ldots, n$. Si G est le réseau engendré par les points $a_1, \ldots, a_n$ où a_i est le point dont la i-ème coordonnée vaut $2(1+c_i)$ et les autres 4, $\bigcup_{g \in G} L_g$ constitue un pavement mosaïcal de $\mathbf{R}^n$ (cf. [1] et [2]) et il n'existe pas de pavements de $\mathbf{R}^n$ par des translatés de L contenant L distinct de $\bigcup_{g \in G} L_g$. Pour $n=2$, les k-animaux réguliers triangulaires sont appelés k-iamonds, les k-animaux carrés k-iominos, les k-animaux réguliers hexagonaux k-hexes (polyamonds, polyminos, polyhexes quand k n'est pas précisé).

La remarque précédente montre qu'on peut aussi considérer dans ce cas les k-animaux mosaïcaux construits sur des parallélogrammes ou des hexagones convexes centrés.

Dans la suite, on aura à déterminer DK pour $K\subset\mathbf{R}^n$. Notons que pour $x_0\in K$, $K'=K_{-x_0}$ on a

$$DK=DK'=\bigcup_{x\in K'}(-K')_x=\bigcup_{x\in -K'}K'_x.$$

C'est semble-t-il la seule formule pratique pour déterminer l'ensemble différence d'un animal triangulaire. Par contre, si l'on considère le k-animal hypercubique $K=\bigcup_{i=1}^{k} C_{a_i}$ où C est un hypercube fermé, on a aussi

$$(*)\qquad DK=\bigcup_{i=1}^{k}\bigcup_{j=1}^{k}(2C)_{a_i-a_j}.$$

[Comme pour $K=A\cup B$, $DK=DA\cup DB\cup(B-A)\cup(A-B)$ on a $D(C_{a_1}\cup C_{a_2})=$ $=DC\cup(DC)_{a_1-a_2}\cup(DC)_{a_2-a_1}$. Comme $DC=C+C=2C$ car C est convexe et symétrique $(*)$ est prouvé pour $k=2$ (et $k=1$); on conclut par récurrence.] De façon analogue si $K=\bigcup_{i=1}^{k}H_{a_i}$ où H est un hexagone régulier fermé on a

$$(**)\qquad DK=\bigcup_{i=1}^{k}\bigcup_{j=1}^{k}(2H)_{a_i-a_j}.$$

3. Animaux réguliers plans

On désigne par T, C, H respectivement un polyamond, un polymino, un polyhexe (T_k ... désignant un k-iamond ...). Il existe 17 groupes cristallographiques du plan distincts (c.à.d. non isomorphes) dont on connaît d'ailleurs la définition abstraite par générateurs et relations (cf. [9]). Pour fixer les idées on emploie la notation utilisée par L. FEJES TÓTH dans [13]. Ces 17 groupes sont

1. Les 4 groupes engendrés uniquement par des symétries $\mathfrak{W}_2^2$, $\mathfrak{W}_3^1$, $\mathfrak{W}_4^1$, $\mathfrak{W}_6^1$;
2. Les 4 groupes ne contenant que l'identité comme rotation $\mathfrak{W}_1$, $\mathfrak{W}_1^1$, $\mathfrak{W}_1^2$, $\mathfrak{W}_1^3$;
3. Les 9 autres groupes (figure entre parenthèses l'ordre de leurs rotations)
$\mathfrak{W}_2$, $\mathfrak{W}_2^1$, $\mathfrak{W}_2^3$, $\mathfrak{W}_2^4$ (ordre 2)
$\mathfrak{W}_3$, $\mathfrak{W}_3^2$ (ordre 3)
$\mathfrak{W}_4$, $\mathfrak{W}_4^2$ (ordre 4, 2)
$\mathfrak{W}_6$ (ordre 6, 3, 2).

De plus, seuls les 5 groupes $\mathfrak{W}_1$, $\mathfrak{W}_2$, $\mathfrak{W}_3$, $\mathfrak{W}_4$, $\mathfrak{W}_6$ ne contiennent que des isométries directes. Rappelons que dans le plan une isométrie directe est nécessairement une rotation ou une translation et une isométrie indirecte une symétrie glissante (produit commutatif d'une symétrie droite et d'une translation (qui peut être nulle) parallèle à l'axe de symétrie). Notons qu'un groupe cristallographique de la liste 1 n'admet pas de partie fondamentale d'intérieur connexe autre qu'un triangle pour $\mathfrak{W}_3^1$, $\mathfrak{W}_4^1$, $\mathfrak{W}_6^1$ et qu'un rectangle pour $\mathfrak{W}_2^2$ (cf. [9]). Dans la suite on appelle animal régulier un animal T, C, H distinct d'un rectangle ou d'un triangle.

Notons que dire qu'un animal régulier K est cristallographique revient à dire qu'on peut trouver Σ tel qu'à un ensemble de mesure nulle près de fr K, K est une partie fondamentale de Σ. Ce qui précède montre que pour un tel K, Σ ne peut être dans la liste 1.

D'autre part si $\Omega \in \Sigma$ est une rotation de centre ω pour $g \in G$, $t_g \Omega t_g^{-1} \in \Sigma$. Comme $t_g \Omega t_g^{-1}$ n'est autre que la rotation, déduite de Ω par la translation t_g, de centre $\omega_g = \omega + g$ on voit que si Σ contient une rotation, il contient aussi celles qui s'en dèduisent par les translations de G. De plus comme on ne peut avoir $\omega \in \widehat{\sigma' \mathring{K}}$ quel que soit $\sigma' \in \Sigma$ on peut trouver $\sigma \in \Sigma$ tel que $\omega \in \text{fr}\,(\sigma K)$. On a alors $\sigma^{-1}\omega \in \text{fr}\ K$. Or $\sigma^{-1}\Omega\sigma$ est une rotation de Σ de centre $\sigma^{-1}\omega$. Il en résulte qu'on a à étudier les angles formés par L en un point de fr L pour savoir si L peut être un animal régulier cristallographique. Par exemple si L est un polyhexe, l'angle de L en un point de fr L est $2\pi/3$, π ou $4\pi/3$. Si L est partie fondamentale pour Σ, Σ ne peut contenir de rotations d'ordre 6 ou d'ordre 4. Les groupes $\mathfrak{W}_6$, $\mathfrak{W}_4$ et $\mathfrak{W}_4^2$ sont donc à éliminer. De même pour un polymino on peut éliminer $\mathfrak{W}_3$, $\mathfrak{W}_3^2$ et $\mathfrak{W}_6$ et pour un polyamond $\mathfrak{W}_4$ et $\mathfrak{W}_4^2$ (et même si ce polyamond n'a pas d'angle égal à $2\pi/3$, $\mathfrak{W}_3$ et $\mathfrak{W}_3^2$). Pour un animal régulier K, on a défini $\bar{\eta}_{C_f}(K)$ ($f = 1, 2, 3, 4$ ou 6). Si Φ est l'un des 17 groupes cristallographiques, on peut considérer la borne supérieure de dens (K, Σ) prise sur les empilements cristallographiques de « type » $\Phi (\Sigma \approx \Phi)$. On la note $\bar{\eta}_\Phi(K)$. La considération des 17 $\bar{\eta}_\Phi(K)$ est évidemment plus fine que celle des 5 $\bar{\eta}_{C_f}(K)$. Ce qui précède montre que si Φ est dans la liste 1 on a $\bar{\eta}_\Phi(K) < 1$. Dans la suite on étudie seulement les 13 groupes des listes 2 et 3. (Evidemment les $\bar{\eta}_\Phi(K)$ sont définis aussi pour des K qui ne sont pas des animaux réguliers, et en particulier quand K est un rectangle ou un triangle mais on ne peut plus éliminer $\mathfrak{M}_2^2$ pour le rectangle et $\mathfrak{W}_3^1$ pour le triangle). On a joint au papier une table donnant la concordance entre la notation de Fejes Tóth et celle (des cristallographes) de Hermann et Mauguin (cf. [18]) reprise par Coxeter et Moser qui y ont permuté $p3m1$ et $p31m$. Considérons les figures 7 et 8; 7 donne l'exemple d'une partie fondamentale pour $\mathfrak{W}_1^1$ et 8 d'une partie

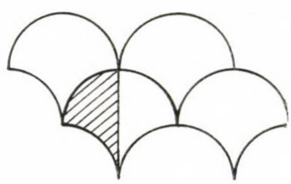


Fig. 7

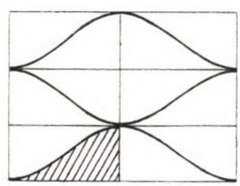


Fig. 8

fondamentale pour $\mathfrak{W}_2^1$ (Pour $\mathfrak{W}_1^1$ cf. la planche I de [13] ou la page 1 de [17]). Pour les 11 autres groupes on a donné aussi dans la table I le numéro de la planche de l'ouvrage de C. H. MACGILLAVRY [20] où l'on trouve une partie fondamentale d'intérieur connexe non convexe pour un de ces groupes. (C. H. Macgillavry s'est servi des illustrations du graveur hollandais M. C. Escher et ne considère ni $\mathfrak{W}_1^1$ ni $\mathfrak{W}_2^1$). Notons des critères qui permettent de savoir si un polygone simplement connexe K vérifie $\bar{\eta}_H(K) = 1$ pour un groupe cristallographique H d'isométries directes. On suppose le plan orienté et fr K orienté en conséquence. Si un polygone a ses côtés

Table I[2]

F.T	H.M	Planche de C.M
$\mathfrak{W}_1$	p1	1 et 14
$\mathfrak{W}_1^1$	cm	cf. figure 7
$\mathfrak{W}_1^2$	pm	19
$\mathfrak{W}_1^3$	pg	3, 17 et 18
$\mathfrak{W}_2$	p2	2 et 16
$\mathfrak{W}_2^1$	cmm	cf. figure 8
$\mathfrak{W}_2^2$	pmm	—
$\mathfrak{W}_2^3$	pmg	20
$\mathfrak{W}_2^4$	pgg	5
$\mathfrak{W}_3$	p3	7 et 38
$\mathfrak{W}_3^1$	p3m1	—
$\mathfrak{W}_3^2$	p31m	40
$\mathfrak{W}_4$	p4	24 et 37
$\mathfrak{W}_4^1$	p4m	—
$\mathfrak{W}_4^2$	p4g	6
$\mathfrak{W}_6$	p6	27 et 39
$\mathfrak{W}_6^1$	p6m	—

sur fr K on cite ses sommets dans l'ordre de l'orientation. Les critères sont valables pour un compact de Jordan K

(a) On a prouvé (cf. [2]) et c'était sans doute connu que $\eta(K)=1$ ssi on peut trouver un polygone $Q=\alpha\beta\gamma\delta$ ou $\alpha\beta\gamma\delta\varepsilon\zeta$ qui soit un parallélogramme ou un hexagone centré dont les sommets sont sur fr K tel que si à tout couple de côtés équipollents de Q on associe les arcs de fr K ayant les mêmes extrémités ces arcs sont congrus par translation.

(b) D'autre part J. H. Conway a remarqué (cf. [14]) qu'on a $\bar{\eta}_{\mathfrak{W}_2}(K)=1$ lorsqu'on peut trouver un polygone $Q'=\alpha\beta\gamma\delta\varepsilon\zeta$ (certains points peuvent être confondus) dont les sommets sont sur fr K tel que $\alpha\beta$ soit équipollent à $\varepsilon\delta$ et que si a, b, c, d, e, f désignent les arcs $\widehat{\alpha\beta}$, $\widehat{\beta\gamma}$, $\widehat{\gamma\delta}$, $\widehat{\varepsilon\delta}$, $\widehat{\varepsilon\zeta}$, $\widehat{\zeta\alpha}$, de fr K a et d se correspondent par translation et b, c, e, f admettent un centre de symétrie. En se servant du fait que dans (a) figure une condition nécessaire on voit que le critère de Conway est également nécessaire.

(c) On a $\bar{\eta}_{\mathfrak{W}_3}(K)=1$ lorsque avec les notations de (b) (certains points peuvent ici aussi être confondus) on peut trouver Q' tel que des rotations d'angle égal valant $\pm 2\pi/3$ de centres respectifs β, δ, ζ transforment $\widehat{\beta\alpha}$ en $\widehat{\beta\gamma}$, $\widehat{\delta\gamma}$ et $\widehat{\delta\varepsilon}$ et $\widehat{\zeta\varepsilon}$ en $\widehat{\zeta\alpha}$.

(d) On a $\bar{\eta}_{\mathfrak{W}_4}(K)=1$ quand on peut trouver un pentagone $\alpha\beta\gamma\delta\varepsilon$ dont les sommets sont sur fr K tel que les rotations d'angle égal valant $\pm\pi/2$ de centre β et ε respectivement transforment respectivement $\widehat{\beta\alpha}$ en $\widehat{\beta\gamma}$, $\widehat{\varepsilon\delta}$ en $\widehat{\varepsilon\alpha}$ et que l'arc $\widehat{\gamma\delta}$ admette un centre de symétrie (ou peut avoir ici $\alpha=\beta=\gamma$ ou $\gamma=\delta$).

(e) On a $\bar{\eta}_{\mathfrak{W}_6}(K)=1$ si

(1) On peut trouver un pentagone $\alpha\beta\gamma\delta\varepsilon$ dont les sommets sont sur fr K tel qu'une rotation de centre β d'angle $\varphi=\pm\pi/3$ transforme $\widehat{\beta\alpha}$ en $\widehat{\beta\gamma}$, qu'une rotation de centre δ d'angle 2φ transforme $\widehat{\delta\gamma}$ en $\widehat{\delta\varepsilon}$ et que l'arc $\widehat{\varepsilon\alpha}$ admette un centre de symétrie

[2] Notons que dans la table un — dans la 3ème colonne signale que le groupe correspondant est dans la liste 1.

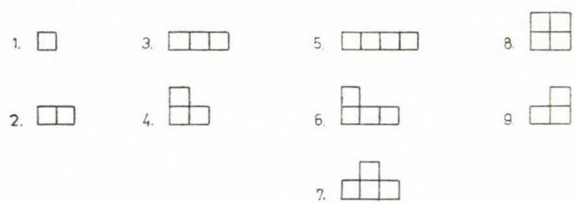
(on peut avoir $\alpha=\varepsilon$, dans ce cas les angles successifs du quadrilatère $\alpha\beta\gamma\delta$ sont $\pi/2, \pi/3, \pi/2, 2\pi/3$).

(2) On peut trouver un triangle équilatéral $\alpha\beta\gamma$ dont les sommets sont sur fr K tel qu'une rotation de centre β d'angle $\pm\pi/3$ transforme $\widehat{\beta\alpha}$ en $\widehat{\beta\gamma}$ et que $\widehat{\gamma\alpha}$ admette un centre symétrie (on peut considérer ce cas comme le cas particulier de (1) correspondant à $\gamma=\delta=\varepsilon$).

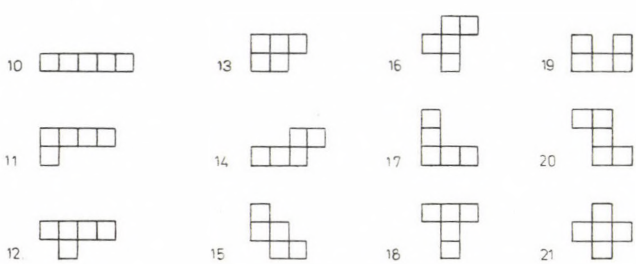
(3) On peut trouver un triangle $\alpha\delta\varepsilon$ d'angle $\pi/6, 2\pi/3, \pi/6$ tel qu'une rotation de centre δ d'angle $\pm 2\pi/3$ transforme $\widehat{\delta\alpha}$ en $\widehat{\delta\varepsilon}$ et que $\widehat{\varepsilon\alpha}$ admette un centre de symétrie (on peut considérer ce cas comme le cas particulier de (1) correspondant à $\alpha=\beta=\gamma$).

Il est possible de montrer qu'une des conditions (a)—(e) doit nécessairement être vérifiée par K compact de Jordan pour que K soit partie fondamentale d'un groupe H d'isométries directes. *Cela sera prouvé ailleurs.* Notons que dans le cas où K est un animal régulier cristallographique comme le pavement carré, triangulaire ou hexagonal qu'on en déduit est bien déterminé, les points éventuels $\alpha, \ldots, \zeta$ des critères (a) et (b) sont bien fixés sur fr K. Cela permet en se servant de la condition nécessaire de ces critères de vérifier que dans certains cas on a $\eta(K)<1$ ou $\bar{\eta}_{\mathfrak{W}_2}(K)<1$. Notons aussi qu'un polygone connexe vérifiant (a) a nécessairement un nombre pair de côtés parallèles à une direction donnée. Nous indiquons quelques résultats concernant les k-iominos, les k-iamonds et les k-hexes relatifs au pavage du plan par des isométriques d'un tel animal K. On pose $x=\eta(K)$.

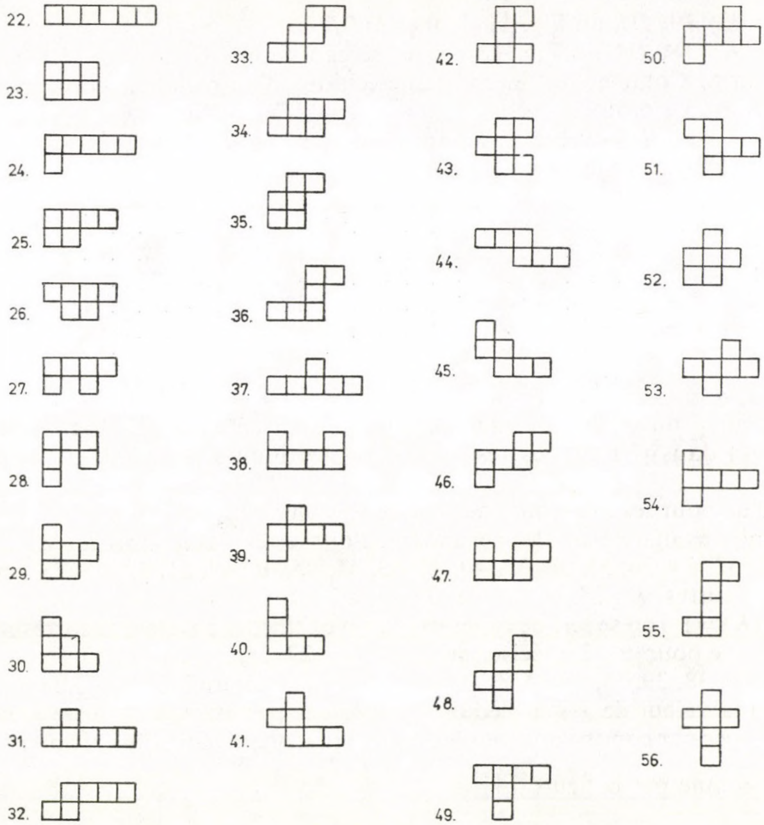
A. *Les k-iominos.* On emploie la numérotation de GOLOMB dans [16]. Le monomino, le domino, les deux triminos et les cinq tetraminos sont numérotés de 1 à 9, es 12 pentaminos de 10 à 21 et les 35 hexaminos de 22 à 56.



Les pentaminos sont



Les hexaminos sont



Il y a 108 heptaminos et 369 octaminos.

Considérons d'abord les k-iominos pour $k \leqq 5$. Pour $K=1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21$ on a $x=1$. Par contre pour les 3 pentaminos 16, 18, 19 on a $x<1$ (Dans ces cas et des cas analogues on peut montrer que quel que soit $k \in \mathbf{N}^*$ on a $\eta_k(K)<1$ en se servant de la généralisation aux h-réseaux de la proposition 1.5.)

On a vu dans l'introduction que $\eta(18) \geqq 5/6$.

Les figures 9 et 10 montrent que $\eta(16) \geqq 5/6$ et $\eta(19) \geqq 5/6$.

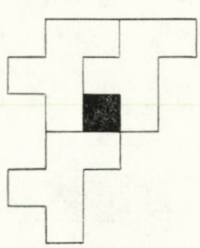


Fig. 9

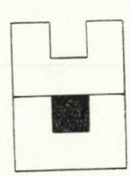


Fig. 10

Pour $K=19$ DK du Fig. 11 donne $\Delta(DK)=6$ et $x=5/6$.

Pour $K=18$ DK du Fig. 12 a pour réseau permis G de base (2,1) et (0,3) de déterminant 6. Comme 6 est la constante critique d'un rectangle contenu dans DK, $\Delta(DK)=6$ et $x=5/6$;

Pour $K=16$ DK du Fig. 13 a pour réseau permis G^t de base (2, 0) et $(t, 3)$ (t réel quelconque) de déterminant 6.

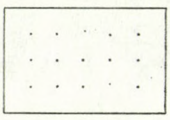


Fig. 11

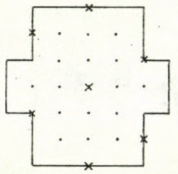


Fig. 12

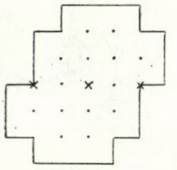


Fig. 13

On peut prouver (en utilisant par exemple la méthode de MORDELL employée par CASSELS dans [7]) que les réseaux G^t sont critiques pour DK et par suite que $x=5/6$.

De plus pour les 21 k-iominos avec $k\leqq 5$ on a $\bar{\eta}_{\mathfrak{W}_2}(K)=1$.

Examinons maintenant les hexaminos. Pour les 24 hexaminos $K=22$, 23, 24, 25, 26, 29, 31, 32, 33, 34, 35, 36, 37, 40, 42, 43, 44, 45, 46, 48, 51, 52, 53, 55 on a $x=1$; pour les 11 autres $K=27$, 28, 30, 38, 39, 41, 47, 49, 50, 54, 56 on a $x<1$. Pour tous on a $\bar{\eta}_{\mathfrak{W}_2}(K)=1$ (on se sert de (a) et (b) ou on exhibe les pavements correspondants); on peut même pour $x<1$ préciser que pour $K=27$, 28, 30, 41, 47, 50, 56 on a $x=6/7$ et pour $K=38$, 39, 54, $x=6/8$. Quand on peut déterminer facilement la valeur de $\Delta(DK)$ la valeur de x s'en déduit. C'est le cas par exemple pour $K=38$, 54. On peut raisonner par contre dans tous les cas de la façon suivante illustrée pour $K=39=$ $=\bigcup_{i=1}^{6}(i)$ donné par la figure 14

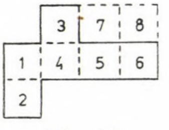


Fig. 14

On pose $K'=K\cup(7)$, $K''=K'\cup(8)$.

Comme K'' pave régulièrement $\mathbf{R}^2$, $\eta(39)\geqq 6/8$. On montre que $\eta(39)\leqq 6/8$ en prouvant que si G empile K, il empile K''. Tout d'abord un tel G empile K'. En effet soit y un point de l'intérieur de (7). Si pour $g\in G$ on avait $y\in K_g$ on pourrait

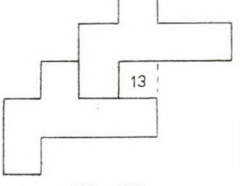


Fig. 15.

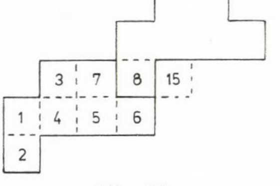


Fig. 16

trouver $x \in K$ avec $y - x \in G$. Pour $x \in (i)$ avec $i = 1, 3, 4, 5$ on a une contradiction immédiate et pour $i = 2$ la figure 15 conduit à une contradiction pour le carré 13. Reste alors à montrer que si G empile K' il empile K''. Le procédé ci-dessus montre qu'on a seulement à examiner le cas où pour y intérieur à (8) on a $x \in (2)$. La contradiction provient alors de l'examen du carré (15) dans la figure 16. Notons que pour le pentamino 16 le procédé précédent redonne assez simplement $\eta(16) = 5/6$. D'après [14], 101 des 108 heptaminos vérifient le critère de Conway et sont donc tels que $\bar{\eta}_{\mathfrak{W}_2}(K) = 1$. Les 7 autres sont donnés par la figure 17. L'heptamino 1 n'est pas

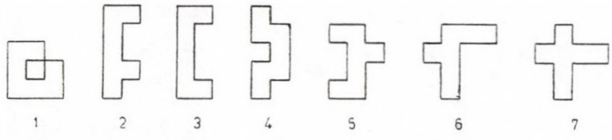


Fig. 17

simplement connexe et évidemment ne peut paver le plan à l'aide d'isométriques. On peut montrer qu'il en est de même pour les heptaminos 3, 6 et 7. La figure 18 montre qu'on a $\bar{\eta}_{\mathfrak{W}_4}(5) = \bar{\eta}_{\mathfrak{W}_2^1}(5) = 1$ ce qui donne l'exemple d'un K pour lequel $\bar{\eta}_{C_2}(K) = 1$ avec $\bar{\eta}_{\mathfrak{W}_2}(K) \neq 1$.

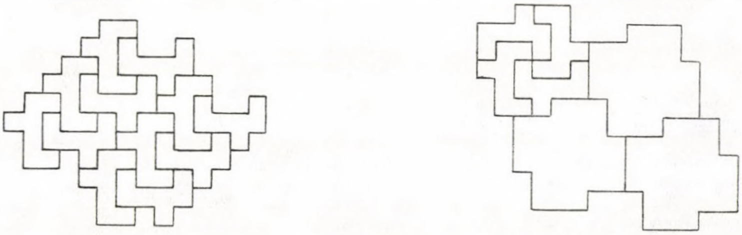


Fig. 18

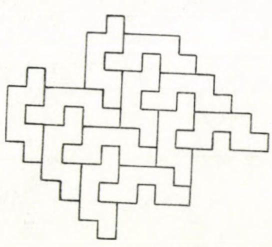


Fig. 19

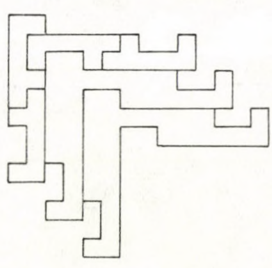
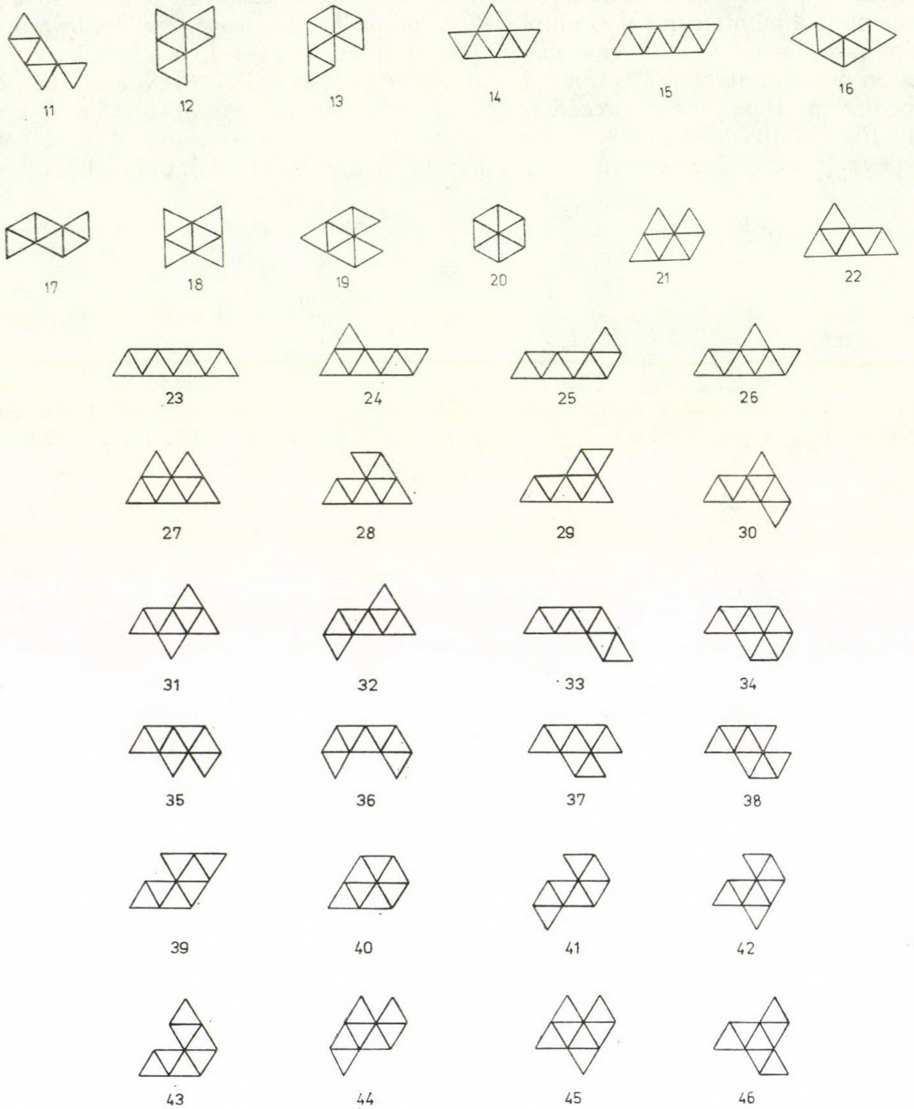


Fig. 20

Les figures 19 et 20 montrent qu'il existe des pavements semi-cristallographiques d'indice 2 pour l'heptamino 4 et d'indice 4 pour l'heptamino 2.

B. *Les k-iamonds* ($k \leqq 7$). On les numérote comme il suit:

(10 k-iamonds pour $k \leqq 5$). 12 hexiamonds (11-22), 24 heptiamonds (23-46).



On pose toujours $x = \eta(K)$.

Considérons d'abord les k-iamonds pour $k \leqq 5$.

Pour $K=2, 4, 5$, $x=1$; pour $K=7$, $x=10/11$; pour $K=3$, $x=6/7$; pour $K=8, 9, 10$, $x=5/6$ et pour $K=1, 6$, $x=2/3$. De plus pour tous ces polyamonds $\bar{\eta}_{\mathfrak{W}_2}(K)=1$. $x=1$ pour $K=2, 4, 5$ résulte du critère (a) et la valeur 1 de $\bar{\eta}_{\mathfrak{W}_2}(K)$ du critère (b).

Les valeurs de x pour $K=7$, 3, 1 proviennent des déterminations respectives de $\Delta(DK)$ qui valent respectivement $\frac{11}{2}|T|$, $\frac{7}{2}|T|$ et $\frac{3}{2}|T|$ (où $|T|$ est la surface de l'élément $T=1$) valeurs déterminées facilement car alors DK est un hexagone; en particulier pour $K=1$, DK est l'hexagone régulier H formé de six triangles équilatéraux isométriques à K. Pour $K=8$, DK donné par la Fig. 21 est tel que $\Delta(DK)=$ $=\Delta(2H)=6|T|$ d'où x. Pour $K=9=\bigcup_{i=1}^{5}(i)$, donné par la figure 22, $K'=K\cup(6)$

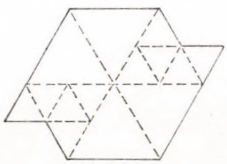


Fig. 21

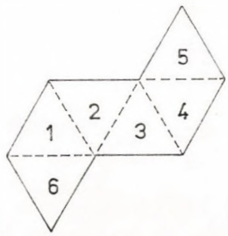


Fig. 22

où a $\eta(K')=1$ d'où $\eta(K)\geqq 5/6$; on conclut en montrant que si G empile K, il empile aussi K'. Considérons maintenant les hexiamonds. On a pour $K=11$, 14, 15, 16, 17, 18, 19, 20 $x=1$, pour $K=12$, 13, 21, 22, $6/8\leqq x<1$; on présume que $x=6/8$ pour $K=12$, 22. Montrons ici que $\eta(13)=\eta(21)=4/5$. Pour $K=13$ DK est un hexagone H' et pour $K=21$ un hexagone H'' (cf. figure 23) de constantes

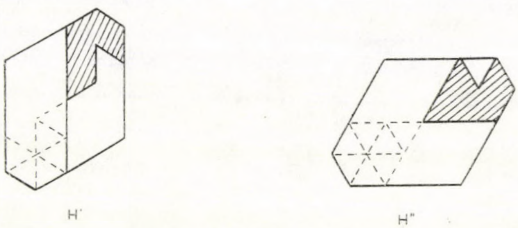


Fig. 23

critiques respectives $\frac{1}{4}|H'|$ et $\frac{1}{4}|H''|$ avec $|H'|=|H''|=30|T|$ d'où $\eta(13)=\eta(21)=$ $=6/(15/2)=4/5$ (H'' et H' sont isométriques). (Notons qu'on obtient aussi $\eta(40)=$ $=14/15$.)

Finissons par les 24 heptiamonds dont nous faisons une étude sommaire. Pour tous ceux-ci on a $x<1$; de plus pour 19 d'entre eux $K=23$, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 44, 45 on $\bar{\eta}_{\mathfrak{W}_2}(K)=1$. Par contre pour $K=36$, 41, 42, 46 on a $\bar{\eta}_{\mathfrak{W}_6}(K)=1$ et il semble que $\bar{\eta}_{\mathfrak{W}_2}(K)<1$. De plus on sait (cf. [15]) que pour $K=43$ il n'existe pas de pavements du plan par des isométriques de K. Notons que bien qu'on ait $\bar{\eta}_{\mathfrak{W}_2}(K)=1$ pour $K=30$, 32, 35, 40 on a aussi $\bar{\eta}_{\mathfrak{W}_6}(K)=1$. Que pour $K=30$, 32, 35, 36, 40, 41, 42, 46 on a $\bar{\eta}_{\mathfrak{W}_6}(K)=1$ est donné par la figure 24 (les rosaces formées pavant régulièrement le plan). Notons que le critère (e 2) est vérifié par les heptiamonds 30 et 46 et le critère (e 1) par les heptiamonds 32, 35, 36, 40, 41 et 42.

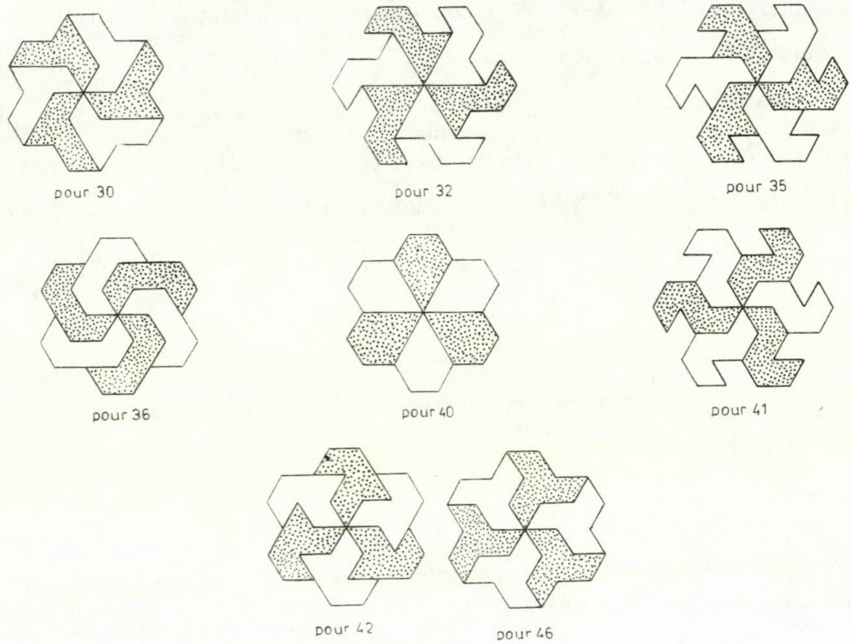
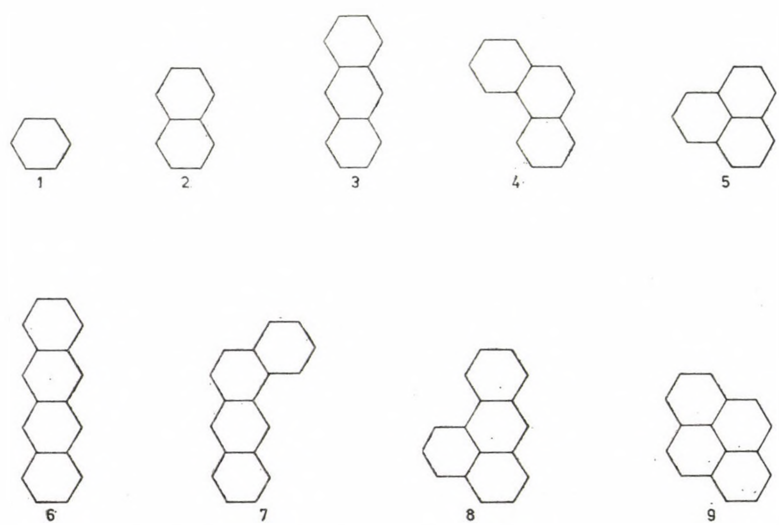
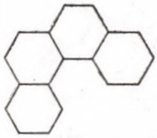


Fig. 24

C. *Les k-hexes* ($k \leqq 5$). On les numérote comme il suit (un monohexe et un bihexe, trois trihexes, sept quadrhexes et vingt-deux penthexes)

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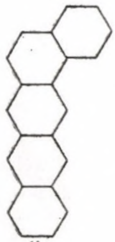
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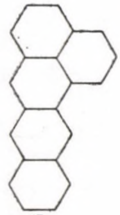
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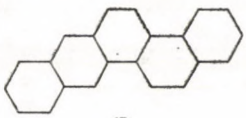
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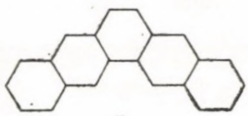
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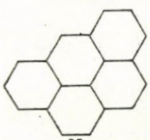
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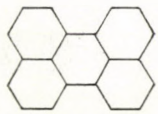
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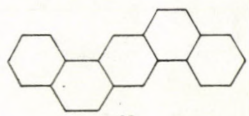
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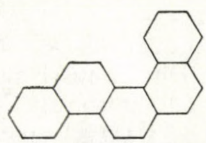
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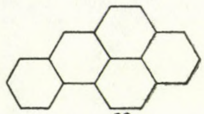

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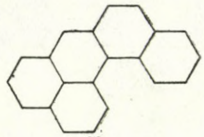

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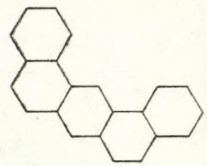

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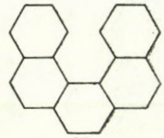

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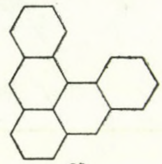

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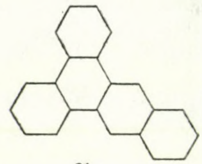

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Notons simplement en posant $x=\eta(K)$, $y=\bar{\eta}_{\mathfrak{W}_2}(K)$ qu'on a

(i) $x=1$ pour $K=1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 23, 24, 25, 26, 27, 29, 33$.

(ii) $x\neq 1$ sinon mais $y=1$ pour $K=10, 19, 20, 21, 22, 28, 30, 32, 34$.

(iii) De plus on a $\eta_{C_2'}(K)=1$ pour $K=31$ comme le montre la figure 25 l'union K'' de $K=31$ et de son isométrique inverse K' vérifiant le critère (a) (on passe de K à K' par $\sigma=s_\Delta\tau$).

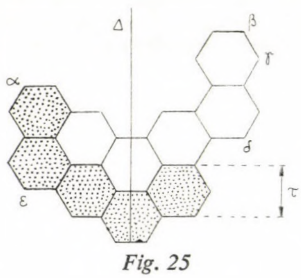

Fig. 25

De plus on peut montrer (par la méthode qui a permis de trouver $\eta(K)$ pour l'hexamino 39) qu'on a $x=4/5$ pour le quadrhexe 10, $x=5/6$ pour les penthexes 19, 20, 21, 22, 28, 30, $x=5/7$ pour les penthexes 31, 32 et $x=5/9$ pour le penthexe 34.

REMARQUES. Disons avec GARDNER (cf. [14]) qu'un pavement de $\mathbf{R}^2$ par des isométriques de $K\subset\mathbf{R}^2$ est *périodique* s'il est semi-cristallographique *apériodique* sinon. Disons aussi que K est *périodique* (resp. apériodique) s'il existe un pavement périodique (resp. apériodique) de $\mathbf{R}^2$ par des isométriques de K (K est périodique ssi $\eta_{C'}(K)=1$) et que K est *strictement apériodique* s'il est apériodique sans être périodique. On connaît des K apériodiques. Par exemple tout polymino « reptile » K

(c.à.d. tel que des isométriques de K pavent une réplique semblable de K, est apériodique comme l'a montré GOLOMB dans [16]. De même le nonagone de VODERBERG (cf. les originaux [22] et [23] ou [13]) pave apériodiquement le plan en forme de spirale.

Cependant on ne connaît pas de K strictement apériodique. Par exemple tous les polyominos reptiles connus pavent un rectangle et comme ce dernier pave régulièrement le plan, on en déduit que ces reptiles sont périodiques. D'autre part pour le nonagone V de Voderberg on a $\bar{\eta}_{\mathfrak{W}_2}(V)=1$.

Note ajoutée aux épreuves (28. décembre 1977). On a signalé au § 3 que les conditions (a)—(e) etaient pour un compact de Jordan K nécessaires et suffisantes pour que celui-ci soit partie fondamentale d'un groupe H d'isométries directes. Cela a été précisé dans [25] qui rappelle aussi quelques propriétés des animaux plans.

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ON MODULAR RIGHT IDEALS OF A RING

By
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A right ideal R of a ring A is called *modular*, if there exists a left identity mod R, i.e. there is an element a in A such that $x-ax\in R$ for all x in A. If we define

$$M(a, A)\overset{\text{def}}{=}\{x-ax : x\in A\},$$

then it is clear, that $M(a, A)$ is a right ideal in A, and R is modular if and only if it contains an $M(a, A)$ for a suitable a. A ring A is an *MD-ring* ($A\in MD$), if the intersection of two modular right ideals of A is modular. The notion of modularity is important in the theory of primitive rings.

The following statements are evident:

(i) A is an MD-ring iff for every $a, b\in A$ there is an element c in A, such that $M(c, A)\subseteq M(a, A)\cap M(b, A)$.

(ii) (0) is modular iff there exists a left identity in A.

A. KERTÉSZ [1] has shown, that if R_1 and R_2 are modular right ideals of A, and their sum is A, then their intersection is modular. (This result is an immediate consequence of our Theorem 3.) A. KERTÉSZ [1] asked the following:

PROBLEM 1. Do there exist non-MD-rings?

F. SZÁSZ [3] solved this problem by constructing rings without left identity for all cardinals K with K modular right ideals such that every two distinct ones of them intersect each other in (0). In his paper he gave some conditions for a ring to be an MD-ring:

Let $Q(a, A)\overset{\text{def}}{=}a+M(a, A)$. The ring A has the property Q, if for each $a, b\in A$ the intersection $Q(a, r)\cap Q(b, A)$ is non-empty. Then (cf. SZÁSZ [3]):

(iii) Every ring with property Q is an MD-ring.

(iv) Every commutative ring, and every ring with a right identity is a Q-ring, and hence, an MD-ring.

PROBLEM 2. (F. SZÁSZ [2], Problem 54, [3], Problem 1.) Do all MD-rings without left identity satisfy Q?

PROBLEM 3. (F. SZÁSZ [4], Problem 3.) Let A be a non-MD-ring. Is it true, that the exponent of its additive group is 2?

The aim of this paper is to give another example for non-MD-ring, and to solve Problems 2 and 3.

We shall write the operators on the right.

LEMMA 1. *If* $\varphi : A_1\to A_2$, *is an epimorphism, then*

$$M(a, A_1)\varphi = M(a\varphi, A_2).$$

PROOF. The statement is a consequence of the equation

$$(x-ax)\varphi = (x\varphi)-(a\varphi)(x\varphi).$$

LEMMA 2. *The epimorphic image of an MD-ring is an MD-ring.*

PROOF. Assume $\varphi: A_1 \to A_2$ is an epimorphism, and $a\varphi, b\varphi \in A_2$. Since A_1 is an *MD*-ring, there exists a c in A_1 such that $M(c, A_1) \subseteqq M(a, M_1) \cap M(b, A_1)$. Hence, by Lemma 1, $c\varphi$ satisfies the conditions in (i).

LEMMA 3. *Let R be the complete (discrete) direct sum of the rings A_i $(i \in I)$. Then A is an MD-ring iff all the rings A_i are MD-rings.*

PROOF. Let p_i denote the projection of A onto A_i. Since this is an epimorphism, we get from Lemma 2 the first part of the lemma. Now assume, that all the A_i-s are *MD*-rings. If $a, b \in A$, then let us choose elements c_i to ap_i and bp_i by (i), and a $c \in A$, with the property $cp_i = c_i$. This is possible, because for almost all i (in the case of the discrete direct sum) $ap_i = bp_i = 0$, hence, $c_i = 0$. By (i) it is enough to verify that $M(c, A) \subseteqq M(a, A)$. If x is arbitrary in A, then $(x-cx)p_i = (xp_i) - (cp_i)(xp_i) = y_i - ap_i y_i$ for suitable $y_i \in R_i$. It is clear, that in the case of the discrete direct sum we can choose $y_i = 0$ for almost all indices i. This implies the existence of a y, which satisfies the equation $y_i = yp_i$ for every i. Hence $x - cx = y - ay$, and the proof is complete.

It is easy to check that

(v) $M(a, A)$ is always the complete (discrete) direct sum of its images under the p_i-s.

Now, if there exists a non-*MD*-ring, then we can construct another one with exponent greater than 2, which is the solution of Problem 3.

Lemma 2 shows, that if we want to construct a non-*MD*-ring, we must investigate the free rings.

Let $X = \{x_i\}$ be an abstract set, and J an arbitrary integral domain. We define the following:

S_X is the free semigroup generated by X. S_X consists of the formal products made from the elements of X. The length of a "word" s is the number of the symbols in s counted with multiplicities. (By a symbol we mean an element in X; the length of s will be denoted by $l(s)$.) If we have a new element e, and S_X^l is the semigroup containing S_X, e, and the formal products se (here s is arbitrary in S_X) then we have got the free semigroup on X with a left identity. We get the product of two elements, if we delete e from the middle of their formal product. The notion of S_X^r, the free semigroup with a right identity is defined analogously. By the length of se we mean the length of s, and the length of e is 0.

Now, if S is a semigroup, then $J(S)$ denotes the semigroup-algebra over J corresponding to S. We say that an element s in S is a summand of an element a in $J(S)$, if it has a non-zero coefficient in a. (As it is known, $J(S)$ is the set of the formal linear combinations.) If S is free, then the length of a is the length of the longest summand of it; the length of $0 \in J(S)$ is zero. We shall assume in the following, that J has an identity, and identify S with the subset $S \cdot 1$ in $J(S)$. At last we mention, that if $J = Z$ (the ring of integers), then $Z(S_X)$ is the free ring generated by X.

S_X has the following evident property:

(vi) If a, b, c, d are in S_X, $ab=cd$ and $l(a)=l(c)$, then $a=c$, and $b=d$.

LEMMA 4. a) *$J(S_X)$ has no left (right) identity.*

b) *If $i \neq j$, then* $M(x_i, J(S_X)) \cap M(x_j, J(S_X)) = (0)$.

c) *If $i \neq j$, then* $Q(x_i, J(S_X^l)) \cap (x_j, J(S_X^l)) = \emptyset$.

PROOF. a) If e is a left identity, then $ex_i = x_i$ for every i. But $x_i \neq 0$ implies $e \neq 0$, and consequently the length of ex_i is at least two which is a contradiction. The other case is analogous.

b) If $a - x_i a = b - x_i b \neq 0$, then $a, b \neq 0$, and hence $l \overset{\text{def}}{=} \max(l(a), l(b))$ is greater than 0. If for example $l = l(s)$, and s is a summand of a, then $l(x_i a) = l+1$, so $x_i a$ is a summand of the left, and hence of the right side, too. But this is a contradiction, because $l(b) \leqq 1$, and each summand of $x_j b$ begins with $x_j \neq x_i$.

c) If $x_i + a - x_i a = x_j + b - x_j b$, then there are two cases. If the length of a equals 0, then the length of the left, and hence of the right side is one, so $a = \alpha e$ and $b = \beta e$, $\alpha, \beta \in J$. So we have $x_i + \alpha e - \alpha x_i e = x_j + \beta e - \beta x_j e$, and here x_i is not a summand on the right side. Hence, the length of a is positive, and the proof is almost the same as in b).

We mention, that we used the following evident consequence of (vi): In all rings considered the length of the product of two elements is the sum of their lengths. This fact implies that these rings have no divisors of zero.

THEOREM 1. *If $|X| \geqq 2$, then $J(S_X)$ is not an MD-ring. ($|X|$ means the power of X.)*

PROOF. The statement is a consequence of (i), (ii), and Lemma 4.

It is interesting (because of Problem 3), that if J has characteristic p (0 or a prime), then so does $J(S_X)$.

Now we solve Problem 2.

THEOREM 2. *If A' is an MD-ring without a left identity, and X has at least two elements, then $A \overset{\text{def}}{=} J(S_X^l) \boxplus A'$ is an MD-ring without the property Q, and without a left identity.*

PROOF. (ii) and Lemma 3 show that A is an MD-ring. It does not have a left identity, because A' does not have. Finally if $i \neq j$ and

$$(x_i, 0) + (a, a') - (x_i, 0) \cdot (a, a') = (x_j, 0) + (b, b') - (x_j, 0) \cdot (b, b'),$$

then the same holds for the first components which contradicts Lemma 4. So A does not have the property Q.

We are going to give a necessary and sufficient condition for a ring to be an MD-ring. It is similar to Q.

If R is a right ideal in a ring A, then we write $R^- \overset{\text{def}}{=} \{r \in A : rA \subseteq R\}$. This is a right ideal in A containing R.

THEOREM 3. *If R_1 and R_2 are modular right ideals in A, and a, b are left identities* mod R_1 *and R_2, resp., then $R_1 \cap R_2$ is modular iff $a - b \in R_1^- + R_2^-$.*

PROOF. We show: The set of left identities mod R_1 is $a + R_1^-$. Indeed, $x - ax \in R_1$, and hence $a - a' \in R_1^-$ iff $x - a'x$ is in R_1 for any x in A. Now $R_1 \cap R_2$ is modular if and only if there exists a common left identity for R_1 and R_2, i.e. $a+R_1^-\cap b+$ $+R_2^-\neq\emptyset$, and this condition is the same as in the theorem.

It is of interest, that in all cases we have $a-b\in(R_1+R_2)^-$, because for every x in A, $x-ax-(x-bx)\in R_1+R_2$ holds.

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LOCALIZING THE LOZINSKI—HARSHILADZE THEOREM ON PROJECTIONS INTO THE SPACE OF TRIGONOMETRIC POLYNOMIALS

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§ 1. We think of periodic continuous functions as functions of the complex variable $z=e^{ix}$ on the unit circle with Fourier series

$$f(z) \sim \sum_{k=-\infty}^{\infty} a_k z^k$$

and norm

$$\|f\| = \max_{|z|=1} |f(z)|.$$

By a projection T_n we mean a linear operator that makes correspond to every such function a trigonometric polynomial of order at most n (i.e. an f with vanishing Fourier coefficients for $|k|>n$) but leaves each trigonometric polynomial of order $\leqq n$ invariant. $f \to \sum_{k=-n}^{n} a_k z^k$ is such an operator and the theorem referred to in the title (see e.g. [1]) states that this has minimal norm among all projections T_n. The norm is defined as

$$|||T_n||| = \sup_{\|f\| \leqq 1} \|T_n f\|$$

and as a consequence

$$|||T_n||| > c \log n. \tag{1}$$

(Here and in what follows $c, c_1, \ldots$ are positive universal constants.) This implies that there exists no sequence $\{T_n\}_{n=0}^{\infty}$ of projections with $T_n f \to f$ uniformly as $n \to \infty$ for each f. It has been asked (see [1]) whether one can have point-wise convergence. Denoting by $T_n(z) f$ the value of the trigonometric polynomial $T_n f$ at z this latter is equivalent to

$$\|T_n(z)\| = \sup_{\|f\| \leqq 1} |T_n(z) f|$$

being (ultimately) bounded in n for each fixed z. We first show in an even stronger form that this is not possible, either.

The space of trigonometric polynomials being separable, in taking the supremum in the last definition we can restrict ourselves to a denumerable set showing that $\|T_n(z)\|$ is a measurable function of z with respect to linear Lebesgue measure on $|z|=1$.

THEOREM 1. *For any sequence of projections*

$$\limsup_n \|T_n(z)\| = +\infty$$

on a set of positive measure on $|z|=1$.

However, the measure can be made as small as we please. Our example is, in a sense, best possible (see the remark after its proof).

THEOREM 2. *For any closed subarc Γ of $|z|=1$ one can construct a sequence $\{T_n\}_{n=1}^{\infty}$ with $\|T_n(z)\|$ uniformly bounded on Γ.*

As the price of this nice behaviour on Γ such a sequence must, as shown by the proof of Theorem 1, behave extremely badly outside it. If we want nice behaviour on the whole circle and we cannot have uniform boundedness, not even everywhere or almost everywhere boundedness, then the most we can expect is boundedness in integral norm.

THEOREM 3. (i) *For any T_n,*

$$\int_{|z|=1} \log \|T_n(z)\|\, |dz| > 2\pi \log\log n - c_1.$$

(ii) *One can construct a sequence $\{T_n\}_{n=0}^{\infty}$ such that*

$$\int_{|z|=1} \log \|T_n(z)\|/[\log\log(\|T_n(z)\|+3)]^3\, |dz| < c_2.$$

If one gives up boundedness, one can construct sequences of arbitrarily slow point-wise growth, an easy consequence of Theorem 2, implying point-wise convergence for functions restricted by a modulus of continuity.

§ 2. The case of ordinary polynomials and continuous functions on $-1 \leqq t \leqq 1$ is, by the substitution $t=\cos x$, equivalent to the case of cosine polynomials and even periodic functions of x. This seemingly minor difference makes difficulties and our results are far from satisfactory.[1] Therefore, we only outline them, indicating the changes needed after the proof of Theorem 1.

(1) is known to hold in this case as well but we cannot generalize it further than

$$\int_{|z|=1} \|T_n(z)\|\, |dz| > c_3 \log n,$$

nor can we get anything better than Theorem 3, (ii). $\|T_n(z)\|$ cannot be bounded everywhere, the set where $\limsup_n \|T_n(z)\| = +\infty$ is either of positive measure or everywhere dense but whether the first alternative must always hold remains open. In the special case of Lagrange interpolation this is even known to take place almost everywhere for arbitrary node systems (ERDŐS [2], see this for further references to earlier work on this special case) but not in general: Theorem 2 extends mutatis mutandis.

§ 3. We first prove the negative results.

PROOF OF THEOREM 3, (i). The method is essentially due to FABER [3] who used it for Lagrange interpolation.

With

$$f(z) = \sum_{k=-3n}^{n} a_k z^k$$

[1] See the remark added in proof at the end of the paper.

to be specified later we take $f_\varrho = f(\varrho z)$ $(|\varrho|=1)$ as test functions:

$$|T_n(z)f_\varrho| = \left|\sum_{k=-n}^{n} a_k \varrho^k z^k + \sum_{k=-3n}^{-n-1} a_k \varrho^k p_k(z)\right| \leqq \|T_n(z)\| \cdot \|f\| \quad (|\varrho| = 1).$$

Here $p_k(z)$ are trigonometric polynomials of order $\leqq n$. In particular, for $\varrho = 1/z$ our quantity becomes

$$g(z) \stackrel{\text{def}}{=} \sum_{k=-n}^{n} a_k + \sum_{k=-3n}^{-n-1} a_k z^{-k} p_k(z).$$

The second sum when expanded into powers of z contains only positive exponents and $g(z)$ is in fact regular in z with the first sum as $g(0)$. Hence, by Jensen's inequality

$$\log \left|\sum_{k=-n}^{n} a_k\right| \leqq (1/2\pi) \int_{|z|=1} \log (\|T_n(z)\| \cdot \|f\|)\, |dz|.$$

Fejér's classical example

$$a_k = \begin{cases} \dfrac{1}{k+n} & \text{for} \quad 0 < |k+n| \leqq 2n \\ 0 & \text{otherwise} \end{cases}$$

for which

$$|f(z)| = \left|\sum_{\substack{k=-3n \\ k \neq -n}}^{n} z^k/(k+n)\right| = 2\left|\sum_{k=1}^{2n} \sin kx/k\right| < c_4$$

and

$$\sum_{k=-n}^{n} a_k = \sum_{k=1}^{2n} 1/k > c_5 \log n,$$

shows that

$$(1/2\pi) \int_{|z|=1} \log \|T_n(z)\|\, |dz| > \log\log n + \log (c_5/c_4).$$

PROOF OF THEOREM 1. Now we take

$$a_k = \begin{cases} \dfrac{1}{k+n} & \text{for} \quad \delta n \leqq |k+n| \leqq 2n \\ 0 & \text{otherwise.} \end{cases}$$

(δ and δ_1, ... in the sequel are small positive constants that will only be fixed some time after their first appearance.) The advantage is the large gap in the expansion

$$g(z) = b_0 + \sum_{\delta n \leqq j \leqq 4n} b_j z^j,$$

even though we have the weaker

$$b_0 = \sum_{k=-n}^{n} a_k = \sum_{\delta n \leqq k \leqq 2n} 1/k > c_6 \log (1/\delta).$$

As before,

$$|f(z)| = 2\left|\sum_{\delta n \leqq k \leqq 2n} \sin kx/k\right| < 2c_4$$

implies

$$|g(z)| < 2c_4\|T_n(z)\|. \tag{2}$$

Suppose that $\|T_n(z)\|<K$ on a fixed set E of positive measure that we can then also assume to be closed. If we choose δ as

$$b_0 > c_6 \log(1/\delta) = 8c_4K, \quad \text{i.e.} \quad \delta = \exp(-c_7K),$$

then

$$|g(z)| = \Big|b_0 + \sum_{\delta n \leqq j \leqq 4n} b_j z^j\Big| < b_0/4 \quad (z\in E). \tag{3}$$

The idea of the proof is to show that unless $g(z)$ is large on $|z|=1$ this inequality can be extended into some part of $|z|<1$ which is impossible since b_0 is constant whereas the rest sharply decreases.

Generally, let $p(z)$ be any polynomial. With the notation $\overset{+}{\log}\, a = \max(\log a, 0)$ $\overset{+}{\log}\,|p(z)|$ is subharmonic and by the Poisson—Jensen inequality

$$\overset{+}{\log}\,|p(z)| \leqq (1/2\pi) \int\limits_{|\xi|=1} \frac{1-|z|^2}{|\xi-z|^2} \overset{+}{\log}\,|p(\xi)|\,|d\xi| \quad (|z|<1).$$

Putting $z=r\varrho$ $(r<1, |\varrho|=1)$ and integrating

$$\int\limits_E \overset{+}{\log}\,|p(r\varrho)|\,|d\varrho| \leqq (4(1-r^2)/2\pi) \int\limits_{|\xi|=1} \overset{+}{\log}\,|p(\xi)| \int\limits_E \frac{|d\varrho|}{|\xi-\varrho|^2}\,|d\xi|, \tag{4}$$

as $|\xi-r\varrho| \geqq |\xi-\varrho|/2$. If also $|p(\xi)|\leqq 1$ on E, then

$$\overset{+}{\log}\,|p(\xi)| \leqq mG(\xi), \tag{5}$$

where m is (a bound for) the degree of $p(z)$ and $G(z)$ is the Green function of E with pole at infinity, i.e. the harmonic function on the plane outside E, vanishing on E and behaving like $\log|z|+$ harmonic at infinity; our inequality follows from the maximum principle: it holds on E and the difference is subharmonic including at infinity.

In order to estimate $G(z)$ we introduce

$$\omega(z) = (1/2\pi) \int\limits_{\bar{E}} \frac{1-|z|^2}{|\varrho-z|^2}\,|d\varrho| \quad (|z|<1),$$

the harmonic measure in $|z|<1$ of $\bar{E}$, the complement of E with respect to $|z|=1$, i.e. the harmonic function assuming 1 on $\bar{E}$ and 0 on E and apply, as H. SELBERG [4] did, Green's formula

$$\int\limits_{|\xi|=1} G(\xi) \frac{\partial\omega(\xi)}{\partial n}\,|d\xi| = \int\limits_{|\xi|=1} \omega(\xi) \frac{\partial G(\xi)}{\partial n}\,|d\xi|,$$

where $\partial/\partial n$ denotes the derivative in the normal, i.e. radial direction. Both $G(\xi)$ and $\omega(\xi)$ vanish on E. On $\bar{E}$ $\omega(\xi)=1$ and

$$1-\omega(z) = (1/2\pi) \int\limits_E \frac{1-|z|^2}{|\varrho-z|^2}\,|d\varrho| \quad (|z|<1),$$

hence

$$\frac{\partial \omega(\xi)}{\partial n} = (1/\pi) \int_E \frac{|d\varrho|}{|\varrho-\xi|^2} \quad (\xi \in \bar{E}).$$

As to $\partial G/\partial n$, the function $G(1/\bar{z})+\log|z|$ is harmonic in $|z|<1$ including $z=0$ with the same boundary value as $G(z)$ implying

$$G(z) = G(1/\bar{z})+\log|z| \quad (|z|<1) \tag{6}$$

and since $\partial G(\xi)/\partial n = -\partial G(1/\xi)/\partial n$ on $\bar{E}$ and $\partial \log|\xi|/\partial n = 1$ $(|\xi|=1)$, we have $\partial G(\xi)/\partial n = 1/2$ on $\bar{E}$. The Green formula becomes

$$\int_{\bar{E}} G(\xi) \int_E \frac{|d\varrho|}{|\varrho-\xi|^2} |d\xi| = \pi|\bar{E}|/2,$$

where $|\ |$ stands also for linear measure.

Our formal arguments can be made precise if E consists of a finite number of closed arcs, because in this case the derivatives of G and ω behave at the worst like $|z-z_0|^{-1/2}$ and $|z-z_0|^{-1}$, respectively, in the neighbourhood of an endpoint z_0 of an arc and continuously elsewhere. In the general case we first cover E with a decreasing sequence of closed sets with E as limit, each consisting of a finite number of arcs. Since the corresponding Green functions increase and the inner integrals converge to a limit >0 (owing to $|E|>0$), from Fatou's lemma we infer that the double integral is finite that is all we actually need. (This shows, incidentally, the existence of $G(z)<+\infty$.) Thus, for any $\delta_1>0$ there is a $\mu(\delta_1)$ $(=\mu(\delta_1, E))>0$ such that

$$\int_A G(\xi) \int_E \frac{|d\varrho|}{|\xi-\varrho|^2} |d\xi| < \delta_1 \quad (A \subset \bar{E}),$$

whenever $|A|<\mu(\delta_1)$. Let us fix a closed set $B \subset \bar{E}$ depending only on δ_1 (and E) such that $|\bar{E} \setminus B| < \mu(\delta_1)/2$. For any set $A \subset B$ subject only to $|A|<\mu(\delta_1)/2$. we then have

$$\int_{A\cup(\bar{E}\setminus B)} G(\xi) \int_E \frac{|d\varrho|}{|\xi-\varrho|^2} |d\xi| < \delta_1.$$

Recalling (5), this can be directly applied to estimate the contribution of $\xi \in A \cup \cup(\bar{E} \setminus B)$ in (4) where we define

$$A = \{\xi : \xi \in B, |p(\xi)| > \exp(\delta_2 m)\}.$$

In $B \setminus A$ $\overset{+}{\log}|p(\xi)| \leqq \delta_2 m$ and denoting by $d(\delta_1)$ the positive distance of B and E, $|\xi-\varrho| \geqq d(\delta_1)$ $(\xi \in B \setminus A \subset B,\ \varrho \in E)$, while in the remaining E $\overset{+}{\log}|p(\xi)|=0$, giving

$$\int_E \overset{+}{\log}|p(r\varrho)||d\varrho| < (2(1-r^2)/\pi)(m\delta_1+\delta_2 m(2\pi)^2/d^2(\delta_1)) < c_8(1-r)\, m\delta_1,$$

where we have chosen $\delta_2 = \delta_1 d^2(\delta_1)$, provided that $|A|<\mu(\delta_1)/2$.

We use this inequality with $p(z)=4g(z)/b_0$ of (3), $m=4n$:

$$\int_E \overset{+}{\log} |4g(r\varrho)/b_0||d\varrho| < 4c_8(1-r)n\delta_1$$

and with $p(z)=4(g(z)-b_0)/5b_0 z^{[\delta n]}$, $m=4n$:

$$\int_E \overset{+}{\log} |4(g(r\varrho)-b_0)/5b_0 r^{[\delta n]}||d\varrho| < 4c_8(1-r)n\delta_1,$$

both satisfying the condition $|p(\xi)|\leqq 1$ on E. Let $5r^{[\delta n]}=1$, i.e. $r=\exp(-\log 5/[\delta n])$. Observing $\overset{+}{\log}(a+b)\leqq \overset{+}{\log} a+\overset{+}{\log} b+\log 2$, it follows

$$\int_E \log 4|d\varrho| < 8c_8(1-r)\,n\delta_1+\int_E \log 2|d\varrho|,$$

$$|E|\log 2 < 8c_8[1-\exp(-\log 5/[\delta n])]n\delta_1 < 8c_8\delta_1 n\log 5/[\delta n] < 30c_8\delta_1/\delta,$$

a contradiction, if $\delta_1 \overset{\text{def}}{=} \delta|E|\log 2/30c_8$. This contradiction shows that with our choices of the δ's either of the two $p(z)$ above does not fulfil $|A|<\mu(\delta_1)/2$. Recalling the definitions and (2) we see that

$$\|T_n(z)\| > c_9\exp(4\delta_2 n)$$

on a set of measure at least $\mu(\delta_1)/2$. This set may depend on n but it follows that the inequality holds on a fixed set of measure $\mu(\delta_1)/2$ for infinitely many n and the proof is complete.

In the situation of § 2 f_ϱ has to be defined as $f(\varrho z)+f(\varrho^{-1}z)$.[1] $g(z)$ will also contain negative exponents and Jensen's inequality has to be replaced by

$$b_0 \leqq (1/2\pi)\int_{|z|=1} |g(z)|\,|dz|.$$

As to Theorem 1, $g(z)$ will have the form

$$b_0+\sum_{\delta n\leqq|j|\leqq 2n} b_j z^j.$$

If $\|T_n(z)\|$ is bounded point-wise on an arc, then by an elementary category argument E, too, can be taken as an arc and one can use $\int_E \overset{+}{\log}|p(r\varrho)|\,|d\varrho|$ as norm to show that the contribution of the positive j's sharply increases with r or one can multiply $g(z)$ with an appropriate trigonometric polynomial of order $<\delta n$ and integrate over $|z|=1$ to show that $g(z)$ must be exponentially large but we have not succeeded to carry over either of these methods to general E.

Proof of Theorem 2. Without loss of generality let

$$\Gamma=\{z:|z|=1,\ \alpha\leqq\arg z\leqq 2\pi-\alpha\}\quad(0<\alpha<1/2).$$

In the estimations below we shall also take into account the dependence on α; this and some of the estimates will only be made use of in the next proof.

The de la Vallée-Poussin operator defined for

$$f(z) \sim \sum_{k=-\infty}^{\infty} a_k z^k$$

by

$$K_{n,m} f = \sum_{k=-n}^{n} a_k z^k + \sum_{n<|k|<n+m} a_k \left(1 - \frac{|k|-n}{m}\right) z^k$$

can be written as

$$\frac{n+m}{m} K_{0,n+m} - \frac{n}{m} K_{0,n},$$

where $K_{0,k} f$ are the Fejér means for which, as is well-known, $|||K_{0,k}|||=1$ and we have the rough estimate

$$|||K_{n,m}||| \leqq (2n+m)/m, \tag{7}$$

showing that in case m is proportional to n this operator fulfils all the conditions imposed on T_n except that it contains terms of order higher than n.

For $n<k<n+m$ we therefore approximate z^k by an ordinary polynomial $p_k(z)$ of degree $<n$ on Γ, conveniently by interpolating at the zeros of a polynomial $\pi(z)$ of degree n to be determined later. We represent it by Nörlund's formula (see [5]) that can easily be proved by the residue theorem,

$$z^k - p_k(z) = (\pi(z)/2\pi i) \int_{|\xi|=2} \frac{\xi^k}{\pi(\xi)(\xi - z)} d\xi \quad (|z| < 2).$$

All we need is that $p_k(z)$ is a polynomial of degree $<n$ (actually the Lagrange or in case of multiple zeros the Hermite interpolating polynomial interpolating z^k at zeros lying inside and 0 at zeros outside the path of integration). Trivially

$$|z^k - p_k(z)| \leqq 2|\pi(z)| 2^k / \min_{|\xi|=2} |\pi(\xi)| \quad (|z| = 1). \tag{8}$$

It is the Tchebycheff polynomial of Γ or what is almost the same but easier to handle its Faber polynomial that makes the right hand side as small as possible for $z \in \Gamma$. Let $w=\varphi(z)$ map the complement of Γ conformally onto $|w|>1$, $\varphi(\infty)=\infty$. The polynomial part of

$$\varphi^n(z) = d_n z^n + \ldots + d_0 + d_{-1}/z + \ldots \quad (|z| > 1),$$

$$\pi(z) = d_n z^n + \ldots + d_0$$

is called the nth Faber polynomial. By a well-known formula, also proved easily using the residue theorem,

$$\pi(z) - \varphi^n(z) = (1/2\pi i) \int_{\Gamma} \frac{\varphi^n(\xi)}{\xi - z} d\xi \quad (z \notin \Gamma)$$

integrated on both sides of Γ implying

$$|\pi(z) - \varphi^n(z)| \leqq 2/d(z, \Gamma), \tag{9}$$

where $d(z, \Gamma)$ stands for the distance of z from Γ.

Now, $\varphi(z)$ can be obtained as the composition of the following mappings.

$$w_1 = \frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\, e^{i\alpha}$$

maps onto the plane slit along the positive axis, $z=\infty$ into $e^{i\alpha}$;

$$w_2 = \sqrt{w_1}$$

maps onto the upper half-plane, $z=\infty$ into $e^{i\alpha/2}$ and

$$|w| = |\varphi(z)| = \left|\frac{w_2-e^{-i\alpha/2}}{w_2-e^{i\alpha/2}}\right|.$$

The following inequalities can, one after the other, be readily checked.

For $|z|=1/2$

$$1/3<|w_1|<3, \quad 2\pi-c_{10}\alpha<\arg w_1<2\pi-c_{11}\alpha,$$
$$1/2<|w_2|<2, \quad \pi-c_{10}\alpha/2<\arg w_2<\pi-c_{11}\alpha/2,$$
$$|\varphi(z)|>\exp(c_{12}\alpha^2) \quad (|z|=1/2).$$

Since $|\varphi(z)/z|=|\varphi(1/\bar{z})|$ (this is the same as (6)),

$$|\varphi(z)|>2\exp(c_{12}\alpha^2) \quad (|z|=2)$$

implying in (9)

$$(10) \qquad |\pi(z)|>2^n\exp(c_{12}\alpha^2 n)-2>2^{n-1}\exp(c_{12}\alpha^2 n) \quad (|z|=2).$$

For $d(z,\Gamma)\leqq\alpha^3/n^2$ and $\operatorname{Im} z\leqq 0$

$$|w_1|<1, \quad d(w_1,\Gamma_1)<c_{13}\alpha^2/n^2$$

(Γ_1 and Γ_2 denote the positive and real axis, respectively),

$$d(w_2,\Gamma_2)<c_{14}\alpha/n, \quad |\varphi(z)|<\exp(c_{15}/n)$$

(we always think $n>c_{16}$) which by symmetry also holds for $\operatorname{Im} z>0$, implying in (9) first for $d(z,\Gamma)=\alpha^3/n^2$ but by the maximum principle also for $d(z,\Gamma)<\alpha^3/n^2$

$$(11) \qquad |\pi(z)|<\exp c_{15}+2n^2/\alpha^3<3n^2/\alpha^3 \quad (d(z,\Gamma)\leqq\alpha^3/n^2).$$

Finally, for $|z|=1$, $z\notin\Gamma$

$$w_1<0, \quad iw_2<0, \quad |\varphi(z)|<\exp(c_{17}\alpha)$$

implying in (9), first for $d(z,\Gamma)>\alpha^3/n^2$

$$(12) \qquad |\pi(z)|<\exp(c_{17}\alpha n)+3n^2/\alpha^3 \quad (|z|=1)$$

but by (11) also for $d(z,\Gamma)\leqq\alpha^3/n^2$.

Putting (10) and (11) into (8)

$$(13) \qquad |z^k-p_k(z)|<2(3n^2/\alpha^3)2^{k-n+1}\exp(-c_{12}\alpha^2 n)<1/n^2 \quad (z\in\Gamma),$$

provided that $0<k-n<c_{12}\alpha^2 n$ and $\alpha>c_{18}\sqrt{\log n/n}$. With (12) in place of (11)

$$|z^k-p_k(z)|<\exp(c_{17}\alpha n) \quad (|z|=1). \tag{14}$$

If $p_k(z)=p_{-k}(1/z)$ for negative k, then these are valid for all $0<|k|-n\leqq c_{12}\alpha^2 n$.

We can now correct the only defect of $K_{n,m}$ by the modification

$$T_n f = T_{n,m} f = \sum_{k=-n}^{n} a_k z^k + \sum_{n<|k|<n+m} a_k\left(1-\frac{|k|-n}{m}\right) p_k(z).$$

Owing to our good approximation in (13) with $m=[c_{12}\alpha^2 n]$, $c_{18}\sqrt{\log n/n}<\alpha<1/2$ the error made is (noting $|a_k|\leqq\|f\|$)

$$\|T_{n,m}(z)-K_{n,m}(z)\|\leqq 2m/n^2<1/n \quad (z\in\Gamma) \tag{15}$$

and recalling (7)

$$\|T_{n,m}(z)\|<c_{19}n/m<c_{20}/\alpha^2 \quad (z\in\Gamma).$$

Q.e.d.

More careful calculation than (7) shows that $|||K_{n,m}|||<c_{21}\log(n/m)$ $(m<n/2)$ implying here a bound $c_{22}\log(1/\alpha)$, best possible as could be shown by the method of Theorem 1: For any T_n and any set A with $|A|>1/n$

$$\sup_{z\notin A}\|T_n(z)\|>c_{23}\log(1/|A|).$$

By (14) we also have

$$\|T_{n,m}(z)-K_{n,m}(z)\|<2m\exp(c_{17}\alpha n) \quad (|z|=1). \tag{16}$$

Proof of Theorem 3, (ii). To contain the growth of our approximating polynomials in (14) we use (13) and (14) with smaller h and l in place of k and n, respectively, and substitute z by z^s and multiply by z^d. In order to regain z^k and a polynomial of degree $\leqq n$ we need

$$hs+d=k, \quad ls+d\leqq n, \quad d\geqq 0$$

and define accordingly, keeping $h<k/2$ as a parameter for the time being,

$$s=[k/h], \quad d=k-hs, \quad l=[(n-d)/s].$$

To satisfy the conditions for (13) and (14) set $\alpha=\sqrt{(h-l)/c_{12}l}$ and require $c_{18}\sqrt{\log l/l}<\alpha<1/2$, i.e.

$$c_{24}\log l<h-l<c_{25}l.$$

Now,

$$h-l=h-(n-d)/s+O(1)=(k-n)/[k/h]+O(1)$$

and this will in fact be satisfied with the minimal choice

$$h\overset{\text{def}}{=}\left[c_{26}\frac{n}{k-n}\log\frac{n}{k-n}\right]$$

if

$$c_{27}\log n<k-n<c_{28}n,$$

where $3c_{26}<c_{27}$ are sufficiently large, c_{28} sufficiently small. (13) gives with $q_k(z)= =z^d p_h(z^s)$

$$|z^k-q_k(z)|<1/l^2<(k-n)/n \tag{17}$$

no longer outside a small arc but outside a small set of the same measure

$$2\alpha=2\sqrt{(h-l)/c_{12}l}<c_{29}\sqrt{(k-n)/n}$$

with our choices. (14) gives

$$|z^k-q_k(z)|<\exp(c_{17}\alpha l)<\exp\left(c_{30}\sqrt{n/(k-n)}\log\frac{n}{k-n}\right)\quad(|z|=1). \tag{18}$$

With m and u to be specified later let $\Delta=2^u m$. In the decomposition

$$K_{n-m+\Delta,\Delta}=\sum_{\nu=1}^{u}(K_{n-m+\Delta 2^{-\nu+1},\Delta 2^{-\nu+1}}-K_{n-m+\Delta 2^{-\nu},\Delta 2^{-\nu}})+K_{n,m}\overset{\text{def}}{=}\sum_{\nu=1}^{u+1}L_\nu,$$
$$L_\nu=L'_\nu+L''_\nu\quad(1\leqq\nu\leqq u),$$

where the operators L'_ν and L''_ν act on $f(z)\sim\sum_{j=-\infty}^{\infty}a_j z^j$ by multiplying the jth term with an $e(j)$ of support $(n-m+\Delta 2^{-\nu}, n-m+\Delta 2^{-\nu+2})$ and its reflection on the negative axis, respectively. Let, therefore, $k=k_\nu=n-m+\Delta 2^{-\nu+2}$ and

$$M'_\nu=z^{-k}q_k(z)L'_\nu\quad(1\leqq\nu\leqq u)$$

meaning pointwise multiplication. We see that M'_ν maps into trigonometric polynomials of order $\leqq n$ if $(3/2)\Delta\leqq n$ and carries each such polynomial into 0, since so does L'_ν. We also have $|||L'_\nu|||\leqq 3/2$, for its $e(j)$ is, as can easily be checked, the sum of two triangular functions of equal sides height 1 and 1/2, respectively, and such functions correspond to translates of Fejér means having norm 1. Hence, if we define u by $c_{28}n/2\leqq 2\Delta=2^{u+1}m<c_{28}n$, (17) gives

$$\|M'_\nu(z)-L'_\nu(z)\|\leqq|1-z^{-k}q_k(z)|\cdot|||L'_\nu|||<(3/2)(k-n)/n<(3/2)\Delta 2^{-\nu+2}/n<c_{31}2^{-\nu}$$

outside a set A_ν of measure

$$|A_\nu|<c_{29}\sqrt{(k-n)/n}<c_{32}2^{-\nu/2}$$

and (18) gives

$$\|M'_\nu(z)-L'_\nu(z)\|<(3/2)\exp(c_{33}2^{\nu/2}\nu)\quad(|z|=1)$$

everywhere. We define mutatis mutandis M''_ν for L''_ν and $M_\nu=M'_\nu+M''_\nu$ for L_ν satisfying the same inequalities. The only condition left is

$$k_\nu-n=-m+\Delta 2^{-\nu+2}>c_{27}\log n\quad(1\leqq\nu\leqq u),$$

i.e.

$$3m>c_{27}\log n.$$

With $m=[c_{34}\log n]$, c_{34} sufficiently large, we can also approximate $L_{u+1}=K_{n,m}$ by $M_{u+1}=T_{n,m}$ of (15) and (16) (mapping into trigonometric polynomials of order $\leqq n$ and leaving each such polynomial invariant) if we put there $\alpha=\sqrt{m/c_{12}n}$ so that our estimates remain valid for $\nu=u+1$ as well. Hence, letting

$$B_0 = \{z\colon |z| = 1\},\quad B_i = \bigcup_{\nu=i}^{u+1} A_\nu \quad (1 \leqq i \leqq u+1),\quad B_{u+2} = \emptyset,$$

where

$$|B_i| < c_{32} \sum_{\nu=i}^{\infty} 2^{-\nu/2} < c_{35} 2^{-i/2} \quad (0 \leqq i \leqq u+1),$$

we have for $z \notin B_i$

$$D(z) \overset{\text{def}}{=} \sum_{\nu=1}^{u+1} \|M_\nu(z)-L_\nu(z)\| < 3 \sum_{\nu=1}^{i-1} \exp(c_{33}2^{\nu/2}\nu)+$$

$$+2c_{31}\sum_{\nu=i}^{u+1} 2^{-\nu} < c_{36}\exp(c_{33}2^{i/2}i) \quad (1 \leqq i \leqq u+2)$$

implying

$$\int_{B_{i-1}\setminus B_i} \log D(z)/[\log\log(D(z)+3)]^3\,|dz| < |B_{i-1}|\,c_{37}2^{i/2}i/i^3 < c_{38}/i^2 \quad (1 \leqq i \leqq u+2),$$

$$\int_{|z|=1} = \sum_{i=1}^{u+2} \int_{B_{i-1}\setminus B_i} < 2c_{38}.$$

But $\sum_{\nu=1}^{u+1} L_\nu = K_{n-m+\Delta,\Delta}$ is bounded by (7) as Δ is proportional to n and to complete the proof we set $T_n = \sum_{\nu=1}^{u+1} M_\nu$.

I am indebted to Mr. G. Somorjai for useful information.

Added in proof (December 28, 1977). Concerning the case of power polynomials of § 2, Mr. G. Somorjai has, after reading the manuscript, pointed out that instead of the clumsy ones propesed after the proof of Theorem 1, one shoult take $f_\varrho=f(\varrho z)+f(\varrho z^{-1})$ with $f(z)=\sum_{\delta n<|k+n|<n/2} z^k/(k+n)$ as test functions giving rise to a $g(z)$ with the same properties as required in the proof of Theorem 1; so that contrary what has been said in the paper, the same method applies giving (mutatis mutandis) the same results Theorem 1, 2 and 3 for power polynomials as well.

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DIFFERENTIAL PROPERTIES OF THE OPERATOR OF BEST APPROXIMATION

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1. Introduction and results

Let $C[0, 1]$ be the space of real valued functions continuous on $[0, 1]$, let the set of functions $\{\varphi_i\}_{i=0}^n$ be a Cebysev system on $[0, 1]$. Further let U_n be the set of polynomials of the system $\{\varphi_i\}_{i=0}^n$, so that for any $f \in C[0, 1]$ there exists the unique best approximaton $P_n(f) \in U_n$. In what follows under $E_n(f)$ we shall mean the measure of best approximation. Furthermore, it is known that there exists a system of points $0 \leqq x_0 < x_1 < \ldots < x_{n+1} \leqq 1$ such that

$$f(x_i) - P_n(f, x_i) = \gamma(-1)^i E_n(f) \quad (i = 0, 1, \ldots, n+1), \tag{1}$$

where $\gamma = \pm 1$. A system of points which has property (1) will be called $A(f)$ system. (It is obvious that, in general, for a function $f \in C[0, 1]$ there exist more than one $A(f)$ systems.)

Let us now discuss the differential properties of the operator P_n, which maps the function $f \in C[0, 1]$ into its best approximation $P_n(f)$:

G. FREUD [1] proved that for any $f \in C[0, 1]$ the operator of best approximation satisfies a Lipschitz condition at f i.e. for any $g \in C[0, 1]$

$$\|P_n(f) - P_n(g)\| \leqq C \cdot \|f - g\|$$

holds, where the constant C depends only on f and U_n.

S. B. Stečkin raised a problem of differentiation of the operator P_n by direction.

Let us give some definitions. We shall say that the operator P_n has left derivative at the point $f \in C[0, 1]$ if for any $g \in C[0, 1]$ the limit (in supremum norm)

$$\lim_{t \to -0} \frac{P_n(f+tg) - P_n(f)}{t} = D_f^- P_n(g)$$

exists. The operator $D_f^- P_n(g)$ of the variable $g \in C[0, 1]$ will be called left derivative of P_n at f. Analogously, we define the operator $D_f^+ P_n(g)$ as the right derivative of P_n at f. If for any $g \in C[0, 1]$

$$D_f^- P_n(g) = D_f^+ P_n(g) = D_f P_n(g),$$

we shall call $D_f P_n(g)$ the derivative of P_n at f.

Let $0 \leqq x_0 < x_1 < \ldots < x_{n+1} \leqq 1$ be a system of points in [0, 1] and define the following determinants:

$$U_i(x) = U_i(x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x) = \begin{vmatrix} \varphi_0(x_0) \ldots \varphi_0(x_{i-1})\ \varphi_0(x_{i+1}) \ldots \varphi_0(x) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varphi_n(x_0) \ldots \varphi_n(x_{i-1})\ \varphi_n(x_{i+1}) \ldots \varphi_n(x) \end{vmatrix}$$

$$(i = 0, 1, \ldots, n);$$

$$U_i = U_i(x_{n+1}) = U_i(x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}) > 0 \quad (i = 0, 1, \ldots, n+1).$$

By the points $\{x_i\}_{i=0}^n$, for any $g \in C[0, 1]$ define a polynomial Q_n, which interpolates g at these points:

$$Q_n(g \mid \{x_i\}_{i=0}^n) = \sum_{i=0}^n \frac{g(x_i)(-1)^{n-i} \cdot U_i(x)}{U_{n+1}},$$

and by the points $\{x_i\}_{i=0}^{n+1}$ introduce the linear operator

$$\bar{Q}(g \mid \{x_i\}_{i=0}^{n+1}) = \sum_{i=0}^{n+1} (-1)^i g(x_i) \cdot U_i$$

defined for any $g \in C[0, 1]$.

Take any $f \in C[0, 1]$ and let $R_n(f)$ be the number of those points $x \in [0, 1]$ at which $|f(x) - P_n(f, x)| = E_n(f)$. It is evident that for any $f \in C[0, 1]$, $n+2 \leqq R_n(f) \leqq \infty$. If for a given $f \in C[0, 1]$ $R_n(f) = n+2$ holds then there is a unique $A(f)$ system. In this case with the points of the $A(f)$ system we may introduce the linear operators $Q_n(g \mid \{x_i\}_{i=0}^n)$ and $\bar{Q}(g \mid \{x_i\}_{i=0}^{n+1})$ for $g \in C[0, 1]$ and use following notation:

$$Q_n(g \mid \{x_i\}_{i=0}^n) = Q_n(g \mid f), \quad \bar{Q}(g \mid \{x_i\}_{i=0}^{n+1}) = \bar{Q}(g \mid f).$$

THEOREM. *Let* $f \in C[0, 1] \setminus U_n$. *Then*

(i) *the operator* P_n *has left and right derivatives* $D_f^- P_n(g)$ *and* $D_f^+ P_n(g)$ *at the point* f;

(ii) *the operator* P_n *has derivative* $D_f P_n(g)$ *at the point* f *if and only if* $R_n(f) = n+2$, *and in this case* $D_f P_n(g)$ *is a linear operator. Moreover*

$$D_f P_n(g) = Q_n(g \mid f) - \frac{Q_n(f - P_n(f) \mid f)}{\bar{Q}(f \mid f)} \cdot \bar{Q}(g \mid f).$$

2. Some lemmas

Now we shall prove some lemmas.

LEMMA 1. *Let* $f \in C[0, 1] \setminus U_n$, *and let* $\{x_i\}_{i=0}^{n+1}$ *be an* $A(f)$ *system. Then* $\bar{Q}(f \mid \{x_i\}_{i=0}^{n+1}) \neq 0$.

PROOF. Using that the system of points $\{x_i\}_{i=0}^{n+1}$ has property (1), and $U_i>0$ for all $i=0, 1, ..., n+1$, we obtain

$$\bar{Q}(f|\{x_i\}_{i=0}^{n+1}) = \sum_{i=0}^{n+1} (-1)^i \cdot f(x_i) \cdot U_i =$$

$$= \sum_{i=0}^{n+1} (-1)^i P_n(f, x_i) \cdot U_i + \gamma \cdot E_n(f) \cdot \sum_{i=0}^{n+1} U_i = \gamma \cdot E_n(f) \cdot \sum_{i=0}^{n+1} U_i \neq 0.$$

Q.E.D.

The second lemma is a modification of results of D. NEWMAN and H. SHAPIRO [2].

LEMMA 2. *Let* $q_n \in U_n$, $\mu>0$, $|\gamma|=1$ *and*

$$\gamma \cdot (-1)^{i+1} q_n(x_i) \leqq \mu \quad (i = 0, 1, ..., n+1) \tag{2}$$

for some points $0 \leqq x_0 < x_1 < ... < x_{n+1} \leqq 1$. *Then*

$$\|q_n\| \leqq \frac{C_1 \cdot \mu}{U_{i0} \cdot U_{n+1}}$$

where $U_{i0} = \min_{i=0,1,...,n+1} U_i$ *and the constant* C_1 *depends only on* U_n.

PROOF. Take the following system of equations for $p_i \in \mathbf{R}$ $(i=0, 1, ..., n+1)$:

$$\left.\begin{aligned} &\sum_{i=0}^{n+1} p_i = 1 && (3)\\ &\sum_{i=0}^{n+1} (-1)^{i+1} \cdot p_i \cdot \varphi_k(x_i) = 0 \quad (k = 0, 1, ..., n). && (4) \end{aligned}\right\}$$

Then from (4) we obtain the following relations:

$$p_i = \frac{(-1)^{n+3} \cdot p_{n+1} \cdot (-1)^{n-i} \cdot U_i \cdot \prod_{\substack{j=0\\ j\neq i}}^{n} (-1)^{j+1}}{U_{n+1} \cdot \prod_{j=0}^{n} (-1)^{j+1}} = \frac{p_{n+1} \cdot U_i}{U_{n+1}} \quad (i = 0, 1, ..., n),$$

and by (3)

$$p_{n+1} = \frac{U_{n+1}}{\sum_{i=0}^{n+1} U_i}.$$

Hence

$$p_i = \frac{U_i}{\sum_{i=0}^{n+1} U_i} \quad (i = 0, 1, ..., n+1)$$

and $0 < p_i < 1$ $(i=0, 1, ..., n+1)$. For $q_n \in U_n$ we have from (4)

$$\gamma \cdot \sum_{i=0}^{n+1} (-1)^{i+1} \cdot p_i \cdot q_n(x_i) = 0$$

hence using (2) we obtain

$$(-1)^{j+1}\cdot p_j\cdot\gamma\cdot q_n(x_j) = -\sum_{\substack{i=0\\ i\neq j}}^{n+1} p_i(-1)^{i+1}\cdot\gamma\cdot q_n(x_i) > -\mu$$

for any $0\leqq j\leqq n+1$, i.e.

$$\gamma\cdot(-1)^{j+1}\cdot q_n(x_j) \geqq -\frac{\mu}{p_j} \geqq -\frac{\mu}{\min\limits_{i=0,1,\dots,n+1} p_i} = -\frac{\mu\cdot\sum\limits_{i=0}^{n+1} U_i}{U_{i0}}.$$

Combining this with (2) we have

(5) $$|q_n(x_i)| \leqq \mu\cdot\frac{\sum\limits_{i=0}^{n+1} U_i}{U_{i0}} \leqq C_2\cdot\frac{\mu}{U_{i0}} \quad (i=0,1,\dots,n+1).$$

Using the Lagrange interpolation formula

$$q_n(x) = \sum_{i=0}^{n} \frac{q_n(x_i)(-1)^{n-i}U_i(x)}{U_{n+1}}$$

and inequalities (5), we obtain the desired inequality.

COROLLARY. *Let $f\in C[0,1]$ and $\{x_i\}_{i=0}^{n+1}$ be an $A(f)$ system. Then*
a) *for any $q_n\in U_n$*

(6) $$\|P_n(f)-q_n\| \leqq C_3\cdot\frac{\{\|f-q_n\|-E_n(f)\}}{U_{i0}\cdot U_{n+1}},$$

where the constant C_3 depends only on U_n;

b) *for any $g\in C[0,1]$*

(7) $$\|P_n(f)-P_n(g)\| \leqq 2\cdot C_3\cdot\frac{\|f-g\|}{U_{i0}\cdot U_{n+1}}.$$

PROOF. Let us prove inequality (6). Take $\bar f=f-P_n(f)$ and $\bar q_n=q_n-P_n(f)$ for any $q_n\in U_n$. Then $P_n(\bar f)\equiv 0$ and $\{x_i\}_{i=0}^{n+1}$ is an $A(\bar f)$ system, i.e.

(8) $$\bar f(x_i) = \gamma(-1)^i\|\bar f\| \quad (i=0,1,\dots,n+1)$$

where $|\gamma|=1$. On the other hand

(9) $$\|\bar f-\bar q_n\| = \|\bar f\|+\mu$$

where

$$\mu = \|\bar f-\bar q_n\|-\|\bar f\| = \|f-q_n\|-E_n(f).$$

Combining (8) and (9) we obtain

$$\gamma(-1)^{i+1}\cdot\bar q_n(x_i) \leqq \mu \quad (i=0,1,\dots,n+1),$$

where $\mu>0$ when $q_n\neq P_n(f)$, so using Lemma 2 we get (6). Now

$$\|f-P_n(g)\| \leqq 2\|f-g\|+E_n(f)$$

and (6) imply (7). Q.E.D.

LEMMA 3. *Let* $f, g \in C[0, 1]$, $P_n(f) \equiv 0$. *Then for any* λ $\|P_n(\lambda f+g)\| \leqq M$ *where the constant* M *depends only on* f *and* g.

PROOF. Remark, that if $\{x_i\}_{i=0}^{n+1}$ is an $A(f)$ system then for any λ, $\{x_i\}_{i=0}^{n+1}$ is also an $A(\lambda f)$ system. Thus by (7) we have

$$\|P_n(\lambda f+g)\| = \|P_n(\lambda f+g)-P_n(\lambda f)\| \leqq 2\cdot C_3\cdot \frac{\|g\|}{U_{i0}\cdot U_{n+1}} = M.$$

Q.E.D.

LEMMA 4. *Let* $f, g \in C[0, 1]$, $P_n(f) \equiv 0$ *and* $R_n(f)=n+2$. *Then for any* $q_n \in U_n$ *and* $\lambda > \lambda_0 = \lambda_0(f, g) > 0$ *we have*

$$\|P_n(\lambda f+g)-q_n\| \leqq C_4\{\|\lambda f+g-q_n\|-E_n(\lambda f+g)\}$$

where the constant C_4 *depends only on* f.

PROOF. It is evident that we may assume $\|f\|=1$. For $\lambda>0$ let $\{x_{i,\lambda}\}_{i=0}^{n+1}$ be an $A(\lambda f+g)$ sytem, and let

$$U_{i,\lambda} = U_i(x_{0,\lambda}, \ldots, x_{i-1,\lambda}, x_{i+1,\lambda}, \ldots, x_{n+1,\lambda}) \quad (i = 0, 1, \ldots, n+1).$$

Then (6) implies

$$(10) \qquad \|P_n(\lambda f+g)-q_n\| \leqq C_3\cdot \frac{\{\|\lambda f+g-q_n\|-E_n(\lambda f+g)\}}{U_{i0,\lambda}\cdot U_{n+1,\lambda}}$$

where $U_{i0,\lambda} = \min_{i=0,1,\ldots,n+1} U_{i,\lambda}$. By Lemma 3 for any λ

$$(11) \qquad \|P_n(\lambda f+g)\| \leqq M$$

hence

$$(12) \qquad E_n(\lambda f+g) = \|\lambda f+g-P_n(\lambda f+g)\| \geqq \lambda-\|g\|-M \geqq \lambda-2M_1$$

where $M_1 = \max\{\|g\|, M\}$.

Let now $\{x_i\}_{i=0}^{n+1}$ be the $A(f)$ system, then

$$(13) \qquad f(x_i) = \gamma(-1)^i \quad (|\gamma| = 1,\ i = 0, 1, \ldots, n+1).$$

Further let

$$d = \min_{i=0,1,\ldots,n} (x_{i+1}-x_i)/3$$

and for any $i=0, 1, \ldots, n+1$ consider the intervals $U_d^1(x_i) = \{x\in[0, 1]: |x-x_i|<d\}$.

Moreover, using (13), we can introduce the intervals $U^2(x_i)$ $(i=0, 1, \ldots, n+1)$ such that for any $x \in U^2(x_i)$

$$(14) \qquad \gamma(-1)^i f(x) > \frac{1}{2} \quad (i = 0, 1, \ldots, n+1).$$

Setting $U_d(x_i) = U_d^1(x_i) \cap U^2(x_i)$ and using that $R_n(f)=n+2$ we have

$$(15) \qquad \sup_{x\in[0,1]\setminus \bigcup_{i=0}^{n+1} U_d(x_i)} |f(x)| = 1-\delta$$

where $0<\delta\leqq 1$.

Let us prove now that if $\lambda>\lambda_0=\frac{4M_1}{\delta}$ then $x_{i,\lambda}\in\bigcup_{l=0}^{n+1}U_d(x_l)$ for any $i=0, 1, \dots$ $\dots, n+1$. Indeed, if there exists a $j, 0\leqq j\leqq n+1$ such that $x_{j,\lambda}\notin\bigcup_{l=0}^{n+1}U_d(x_l)$ then $x_{j,\lambda}\in[0, 1]\setminus\bigcup_{l=0}^{n+1}U_d(x_l)$ and using (11), (12) and (15) we obtain

$$\lambda-2M_1\leqq E_n(\lambda f+g)=|\lambda f(x_{j,\lambda})+g(x_{j,\lambda})-P_n(\lambda f+g, x_{j,\lambda})|\leqq$$
$$\leqq\lambda(1-\delta)+\|g\|+M\leqq\lambda(1-\delta)+2M_1$$

or

$$\lambda\leqq\frac{4M_1}{\delta},$$

a contradiction. This means that $\{x_{i,\lambda}\}_{i=0}^{n+1}\in\bigcup_{l=0}^{n+1}U_d(x_l)$ when $\lambda>\lambda_0$. Let us prove now that two points $x_{i,\lambda}$ and $x_{j,\lambda}$ $(i<j)$ can not belong to the same interval $U_d(x_k)$ when $\lambda>\lambda_0$. This is evident because if $x_{i,\lambda}, x_{j,\lambda}\in U_d(x_k)$ then $x_{i,\lambda}, x_{i+1,\lambda}\in U_d(x_k)$ and

$$\lambda f(x_{i,\lambda})+g(x_{i,\lambda})-P_n(\lambda f+g, x_{i,\lambda})=\gamma_{i,\lambda}E_n(\lambda f+g),$$
$$\lambda f(x_{i+1,\lambda})+g(x_{i+1,\lambda})-P_n(\lambda f+g, x_{i+1,\lambda})=\gamma_{i+1,\lambda}E_n(\lambda f+g)$$

where $|\gamma_{i,\lambda}|, |\gamma_{i+1,\lambda}|=1$ and $\gamma_{i,\lambda}=-\gamma_{i+1,\lambda}$. It follows that

$$\lambda|f(x_{i,\lambda})+f(x_{i+1,\lambda})|=|g(x_{i,\lambda})+g(x_{i+1,\lambda})-P_n(\lambda f+g, x_{i,\lambda})-P_n(\lambda f+g, x_{i+1,\lambda})|$$

hence and from (14)

$$\lambda<\lambda|f(x_{i,\lambda})+f(x_{i+1,\lambda})|\leqq 4M_1,$$

again a contradiction. So we can conclude that $x_{i,\lambda}\in U_d(x_i)$ $(i=0, 1, \dots, n+1)$ when $\lambda>\lambda_0$ and this implies

$$x_{i+1,\lambda}-x_{i,\lambda}\geqq d\quad(i=0, 1, \dots, n).$$

Hence

$$U_{i,\lambda}\geqq C_5\quad(i=0, 1, \dots, n+1)$$

where $\lambda>\lambda_0$ and the constant C_5 depends only on f and U_n. Combining this with (10) we obtain the desired inequality.

Now let $f, g\in C[0, 1]$, $E_n(f)\neq 0$, and let $\{x_i\}_{i=0}^{n+1}$ be an $A(f)$ system. Then by Lemma 1, $\bar{Q}(f|\{x_i\}_{i=0}^{n+1})\neq 0$ so we may set

$$C=\frac{\bar{Q}(g|\{x_i\}_{i=0}^{n+1})}{\bar{Q}(f|\{x_i\}_{i=0}^{n+1})}=\frac{\sum_{i=0}^{n+1}(-1)^i\cdot g(x_i)\cdot U_i}{\sum_{i=0}^{n+1}(-1)^i\cdot f(x_i)\cdot U_i}. \tag{16}$$

Then the polynomial $\psi_n=Q_n(g|\{x_i\}_{i=0}^{n})-C\cdot Q_n(f|\{x_i\}_{i=0}^{n})$ interpolates the function $g-C\cdot f$ at the points $\{x_i\}_{i=0}^{n}$ and what is more we have the following

LEMMA 5. *Let* $f, g \in C[0, 1]$, $E_n(f) \neq 0$ *and let* $\{x_i\}_{i=0}^{n+1}$ *be an* $A(f)$ *system. Then the polynomial*

$$\psi_n = Q_n(g|\{x_i\}_{i=0}^n) - C \cdot Q_n(f|\{x_i\}_{i=0}^n)$$

interpolates the function $g - C \cdot f$ *at the points* $\{x_i\}_{i=0}^{n+1}$.

PROOF. By the definition of the polynomial ψ_n we have

$$\psi_n(x) = \sum_{i=0}^{n} \frac{\{g(x_i) - C \cdot f(x_i)\} \cdot (-1)^{n-i} \cdot U_i(x)}{U_{n+1}}. \tag{17}$$

In order to prove the lemma, it is enough to show that $\psi_n(x_{n+1}) = g(x_{n+1}) - C \cdot f(x_{n+1})$. (17) implies

$$\psi_n(x_{n+1}) = \sum_{i=0}^{n} \frac{\{g(x_i) - C \cdot f(x_i)\} \cdot (-1)^{n-i} \cdot U_i(x_{n+1})}{U_{n+1}} =$$

$$= \frac{(-1)^n}{U_{n+1}} \left\{ \sum_{i=0}^{n} (-1)^i g(x_i) \cdot U_i + C \cdot \sum_{i=0}^{n+1} (-1)^{i+1} f(x_i) \cdot U_i + C \cdot (-1)^{n+1} f(x_{n+1}) \cdot U_{n+1} \right\} =$$

$$= \frac{(-1)^n}{U_{n+1}} \left\{ \sum_{i=0}^{n} (-1)^i g(x_i) \cdot U_i - \sum_{i=0}^{n+1} (-1)^i g(x_i) \cdot U_i + C \cdot (-1)^{n+1} \cdot f(x_{n+1}) \cdot U_{n+1} \right\} =$$

$$= \frac{(-1)^n}{U_{n+1}} \{(-1)^n g(x_{n+1}) \cdot U_{n+1} + C \cdot (-1)^{n+1} \cdot f(x_{n+1}) \cdot U_{n+1}\} = g(x_{n+1}) - C \cdot f(x_{n+1}).$$

Q.E.D.

LEMMA 6. *Let* $f, g \in C[0, 1]$, $P_n(f) \equiv 0$, $R_n(f) = n+2$ *and let* $\{x_i\}_{i=0}^{n+1}$ *be the* $A(f)$ *system. Then*

$$P_n(\lambda f + g) \rightrightarrows \psi_n = Q_n(g|f) - C \cdot Q_n(f|f) \quad (\lambda \to +\infty)$$

where $C = \dfrac{\bar{Q}(g|f)}{\bar{Q}(f|f)}$.

PROOF. Remark at first that by $R_n(f) = n+2$, $E_n(f) \neq 0$, so by Lemma 1, $\bar{Q}(f|f) \neq 0$. Evidently we may assume that $\|f\| = 1$ and $f(x_i) = (-1)^i$ $(i = 0, 1, \dots$ $\dots, n+1)$. By Lemma 5

$$\psi_n(x_i) = g(x_i) - C \cdot (-1)^i \quad (i = 0, 1, \dots, n+1). \tag{18}$$

For any $\lambda > 0$ take the function $\lambda f + g$, then

$$(\lambda f + g - \psi_n)(x_i) = \lambda(-1)^i + g(x_i) - g(x_i) + C(-1)^i = (-1)^i(\lambda + C). \tag{19}$$

Let $\lambda > \max\{0, -C\}$, and introduce the sets

$$A_\lambda = \{x \in [0, 1] : |\lambda f + g - \psi_n| \leqq \lambda + C\};$$

$$B_\lambda^+ = \{x \in [0, 1] : \lambda f + g - \psi_n > \lambda + C\};$$

$$B_\lambda^- = \{x \in [0, 1] : \lambda f + g - \psi_n < -\lambda - C\}.$$

Thus $A_\lambda \cup B_\lambda^+ \cup B_\lambda^- = [0, 1]$, and (19) implies that ψ_n is the polynomial of best approximation to $\lambda f + g$ on A_λ. Observe, that because of $R_n(f) = n+2$, $-1 < f(x) < 1$ when $x \in B_\lambda^+ \cup B_\lambda^-$. Further for $x \in B_\lambda^+$

$$\lambda(f(x)-1) + (g(x) - C - \psi_n(x)) > 0, \tag{20}$$

and analogously for $x \in B_\lambda^-$

$$\lambda(f(x)+1) + (g(x) + C - \psi_n(x)) < 0. \tag{21}$$

Define

$$F_+(x) = g(x) - C - \psi_n(x), \quad F_-(x) = g(x) + C - \psi_n(x),$$

and let $\max\{\|F_+\|, \|F_-\|\} < M$. We may assume that $\lambda > \lambda_1 = \max\{0, -C, M\}$. By $f(x) < 1$ $(x \in B_\lambda^+)$ and (20) we obtain for $x \in B_\lambda^+$

$$1 - \frac{M}{\lambda} < f(x) < 1, \tag{22}$$

and analogously for $x \in B_\lambda^-$

$$-1 < f(x) < -1 + \frac{M}{\lambda}. \tag{23}$$

(18) implies that $F_+(x_i) = 0$ when $i = 0, 2, \dots, 2\left[\frac{n}{2}\right]$ and $F_-(x_i) = 0$ when $i = 1, 3, \dots, 2\left[\frac{n+1}{2}\right] - 1$. So for any $\varepsilon > 0$ there exists $h = h(\varepsilon) > 0$, $h < \min\limits_{i=0,1,\dots,n} \frac{(x_{i+1} - x_i)}{3}$, such that

$$|F_+(x)| < \varepsilon, \tag{24}$$

when $x \in [-h + x_i, x_i + h]$, $i = 0, 2, \dots, 2\left[\frac{n}{2}\right]$, and

$$|F_-(x)| < \varepsilon, \tag{25}$$

when $x \in [-h + x_i, x_i + h]$, $i = 1, 3, \dots, 2\left[\frac{n+1}{2}\right] - 1$. Moreover

$$\omega(g, 2h) < \varepsilon, \tag{26}$$

$$\omega(\psi_n, 2h) < \varepsilon. \tag{27}$$

Let

$$H_+ = \bigcup_{i=0}^{\left[\frac{n}{2}\right]} [-h + x_{2i}, x_{2i} + h], \quad H_- = \bigcup_{i=1}^{\left[\frac{n+1}{2}\right]} [-h + x_{2i-1}, x_{2i-1} + h].$$

Then using $R_n(f) = n+2$ we obtain

$$\sup_{x \in [0,1] \setminus H_+} f(x) = 1 - \delta_1, \quad \delta_1 > 0, \qquad \inf_{x \in [0,1] \setminus H_-} f(x) = -1 + \delta_2, \quad \delta_2 > 0.$$

Let now $\lambda>\lambda_2=\max\left\{\lambda_1, \frac{M}{\delta_1}, \frac{M}{\delta_2}\right\}$. Then it is easy to see (using (22) and (23)) that $B_\lambda^+\subseteqq H_+$, $B_\lambda^-\subseteqq H_-$. Thus (20) and (24) imply

$$-\varepsilon<\lambda\big(f(x)-1\big)<0 \tag{28}$$

when $x\in B_\lambda^+$, and

$$0<\lambda\big(f(x)+1\big)<\varepsilon \tag{29}$$

when $x\in B_\lambda^-$.

By Lemma 4, setting $q_n=\psi_n$ we have

$$\|\psi_n-P_n(\lambda f+g)\| \leqq C_4\{\|\lambda f+g-\psi_n\|-E_n(\lambda f+g)\} \tag{30}$$

when $\lambda>\lambda_0$, and we may assume that $\lambda>\max\{\lambda_0, \lambda_2\}$. Let

$$\|\lambda f+g-\psi_n\| = |(\lambda f+g-\psi_n)(\bar{x})|,$$

where $\bar{x}\in[0, 1]$.

If $\bar{x}\in A_\lambda$, then $\|\lambda f+g-\psi_n\|=\|\lambda f+g-\psi_n\|_{A_\lambda}=\lambda+C$ and (19) implies that $P_n(\lambda f+g)\equiv\psi_n$.

If $\bar{x}\notin A_\lambda^+$ then $\bar{x}\in B_\lambda^+\cup B_\lambda^-$. Let for example $\bar{x}\in B_\lambda^+$. Then $\bar{x}\in H_+$ and consequently $\bar{x}\in[-h+x_k, h+x_k]$ for some even k, $0\leqq k\leqq n+1$. Let $B_{\lambda,k}^+=B_\lambda^+\cap\cap[-h+x_k, x_k+h]$. It is evident that $B_{\lambda,k}^+$ is a non-empty open set, thus there exists $x^*\in[-h+x_k, h+x_k]$ such that $x^*\notin B_{\lambda,k}^+$ and x^* is a point of accumulation of $B_{\lambda,k}^+$. Then from (28) we obtain

$$-\varepsilon<\lambda\big(f(\bar{x})-1\big)<0, \tag{31}$$

$$-\varepsilon\leqq\lambda\big(f(x^*)-1\big)\leqq 0. \tag{32}$$

Furthermore $x^*\in[-h+x_k, h+x_k]$, $x^*\notin B_{\lambda,k}^+$ therefore $x^*\notin B_\lambda^+\cup B_\lambda^-$, i.e. $x^*\in A_\lambda$. Combining (31) and (32) we obtain

$$\lambda|f(\bar{x})-f(x^*)|\leqq\varepsilon. \tag{33}$$

Hence using (33), (26) and (27) we have

$$\|\lambda f+g-\psi_n\| = |\lambda f(\bar{x})+g(\bar{x})-\psi_n(\bar{x})| \leqq \lambda|f(\bar{x})-f(x^*)|+$$
$$+|g(\bar{x})-g(x^*)|+|\psi_n(\bar{x})-\psi_n(x^*)|+|\lambda f(x^*)+g(x^*)-\psi_n(x^*)| \leqq$$
$$\leqq 3\varepsilon+\|\lambda f+g-\psi_n\|_{A_\lambda}.$$

But ψ_n is the polynomial of best approximation to $\lambda f+g$ on A_λ, hence

$$\|\lambda f+g-\psi_n\| \leqq 3\varepsilon+\|\lambda f+g-\psi_n\|_{A_\lambda}\leqq$$
$$\leqq 3\varepsilon+\|\lambda f+g-P_n(\lambda f+g)\|_{A_\lambda}\leqq 3\varepsilon+\|\lambda f+g-P_n(\lambda f+g)\| = 3\varepsilon+E_n(\lambda f+g).$$

Combining this inequality with (30) we obtain

$$\|\psi_n-P_n(\lambda f+g)\| \leqq 3\cdot C_4\cdot\varepsilon \tag{34}$$

when $\lambda>\max\{\lambda_0, \lambda_2\}$. The same inequality can be proved in the case when $\bar{x}\in B_\lambda^-$ and recalling that $\psi_n\equiv P_n(\lambda f+g)$ if $\bar{x}\in A_\lambda$ we can conlude that (34) is true in any case when $\lambda>\max\{\lambda_0, \lambda_2\}$. This means that $P_n(\lambda f+g)\rightrightarrows\psi_n$ when $\lambda\to+\infty$. Q.E.D.

3. Proof of the theorem

Let us prove at first statement (i). It is enough to prove the existence of

$$\lim_{t\to+0}\frac{P_n(f+tg)-P_n(f)}{t}$$

for any $f, g\in C[0, 1]$, $E_n(f)\neq 0$. Put $\bar f=f-P_n(f)$, so $P_n(\bar f)\equiv 0$ and prove the existence of $\lim\limits_{\lambda\to+\infty} P_n(\lambda\bar f+g)$. Without loss of generality we may assume that $\|\bar f\|=1$. Take any sequence of positive numbers $\{\lambda^k\}_{k=1}^{\infty}$, $\lambda^k\to\infty$. By Lemma 3

$$\sup_{\lambda}\|P_n(\lambda\bar f+g)\|\leqq M,$$

so we may assume that

$$P_n(\lambda^k\bar f+g)\rightrightarrows \bar P_n\in U_n \quad (k\to\infty),$$

or setting $\bar g=g-\bar P_n$

$$P_n(\lambda^k\bar f+\bar g)\rightrightarrows 0 \quad (k\to\infty).$$

Take any $0<\varepsilon<1$, then there exists λ_0 such that

$$\|P_n(\lambda^k\bar f+\bar g)\|\leqq\varepsilon \tag{35}$$

when $\lambda^k\geqq\lambda_0$. Let $M_1=\|\bar g\|+M$ and take any $\lambda_1, \lambda_2>0$ such that $\lambda_2\geqq\lambda_1+2M_1$. Then

$$\begin{aligned}E_n(\lambda_2\bar f+\bar g) &= \|\lambda_2\bar f+\bar g-P_n(\lambda_2\bar f+\bar g)\|\geqq\lambda_2-M_1\geqq\\ &\geqq\lambda_1+M_1\geqq\|\lambda_1\bar f+\bar g-P_n(\lambda_1\bar f+\bar g)\| = E_n(\lambda_1\bar f+\bar g).\end{aligned}\tag{36}$$

Take now $h>0$ such that

$$\omega(\bar g, h)\leqq\varepsilon, \tag{37}$$

$$\omega(\bar f, 2h)\leqq 1. \tag{38}$$

Consider

$$M_+ = \{x\in[0, 1]:\bar f(x) = 1\};\quad M_- = \{x\in[0, 1]:\bar f(x) = -1\};$$

$$M_+^h = \{x\in[0, 1]:\ \text{there exists } x'\in M_+ \text{ such that } |x-x'|<h\};$$

$$M_-^h = \{x\in[0, 1]:\ \text{there exists } x'\in M_- \text{ such that } |x-x'|<h\}.$$

Let $\{x_{\lambda^k,i}\}_{i=1}^{n+1}$ be an $A(\lambda^k\bar f+\bar g)$ system, i.e.

$$\lambda^k\bar f(x_{\lambda^k,i})+\bar g(x_{\lambda^k,i})-P_n(\lambda^k\bar f+\bar g, x_{\lambda^k,i}) = \gamma_k(-1)^iE_n(\lambda^k\bar f+\bar g),$$

$$(|\gamma_k| = 1,\quad i = 0, 1, \dots, n+1).$$

For some $\delta_1, \delta_2>0$ depending on h

$$\sup_{x\in[0,1]\setminus M_+^h}\bar f(x) = 1-\delta_1,\qquad \inf_{x\in[0,1]\setminus M_-^h}\bar f(x) = -1+\delta_2.$$

It is easy to see that if $\lambda^k > \lambda_1 = \max\left\{\frac{2M_1}{\delta_1}, \frac{2M_1}{\delta_2}\right\}$, then

$$x_{\lambda^k, i} \in M_+^h, \quad \text{when} \quad \gamma_k(-1)^i > 0 \quad (i = 0, 1, \ldots, n+1), \tag{39}$$

$$x_{\lambda^k, i} \in M_-^h, \quad \text{when} \quad \gamma_k(-1)^i < 0 \quad (i = 0, 1, \ldots, n+1). \tag{40}$$

Take $\lambda^{\bar{k}} > \max\{\lambda_0, \lambda_1\}$, consider the function $\lambda^{\bar{k}}\bar{f}+\bar{g}$ and let $\{x_{\lambda^{\bar{k}}, i}\}_{i=0}^{n+1}$ be one of the $A(\lambda^{\bar{k}}\bar{f}+g)$ systems, i.e.

$$\begin{aligned} \lambda^{\bar{k}}\bar{f}(x_{\lambda^{\bar{k}}, i}) + \bar{g}(x_{\lambda^{\bar{k}}, i}) - P_n(\lambda^{\bar{k}}\bar{f}+\bar{g}, x_{\lambda^{\bar{k}}, i}) = \\ = \gamma_{\bar{k}}(-1)^i E_n(\lambda^{\bar{k}}\bar{f}+\bar{g}) \quad (|\gamma_{\bar{k}}| = 1, \ i = 0, 1, \ldots, n+1). \end{aligned} \tag{41}$$

Then from (39) and (40) we obtain that $x_{\lambda^{\bar{k}}, i} \in M_+^h$ when $\gamma_{\bar{k}}(-1)^i > 0$, $x_{\lambda^{\bar{k}}, i} \in M_-^h$ when $\gamma_{\bar{k}}(-1)^i < 0$.

Hence for any $x_{\lambda^{\bar{k}}, i}$ we can find a point $\bar{x}_i \in [0, 1]$ such that

$$|x_{\lambda^{\bar{k}}, i} - \bar{x}_i| < h \tag{42}$$

and $\bar{f}(\bar{x}_i) = \gamma_{\bar{k}}(-1)^i$.

Furthermore (38) implies that $0 \leqq \bar{x}_0 < \bar{x}_1 < \ldots < \bar{x}_{n+1} \leqq 1$, so $\{\bar{x}_i\}_{i=0}^{n+1}$ is an $A(\bar{f})$ system. By (37) and (42) we obtain

$$|\bar{g}(x_{\lambda^{\bar{k}}, i}) - \bar{g}(\bar{x}_i)| \leqq \varepsilon \quad (i = 0, 1, \ldots, n+1).$$

Denote $\bar{g}(x_{\lambda^{\bar{k}}, i}) - \bar{g}(\bar{x}_i) = a_i$, $P_n(\lambda^{\bar{k}}\bar{f}+\bar{g}, x_{\lambda^{\bar{k}}, i}) = b_i$ $(i = 0, 1, \ldots, n+1)$, where $|a_i|$, $|b_i| \leqq \varepsilon$. Then from (41) we have

$$\lambda^{\bar{k}}\bar{f}(x_{\lambda^{\bar{k}}, i}) + \bar{g}(\bar{x}_i) + a_i - b_i = \gamma_{\bar{k}}(-1)^i E_n(\lambda^{\bar{k}}\bar{f}+\bar{g}) \quad (i = 0, 1, \ldots, n+1). \tag{43}$$

Take any $\lambda \geqq \lambda^{\bar{k}} + 2M_1$, then by (36)

$$E_n(\lambda\bar{f}+\bar{g}) \geqq E_n(\lambda^{\bar{k}}\bar{f}+\bar{g}). \tag{44}$$

Now we prove that if $\gamma_{\bar{k}}(-1)^i > 0$ then

$$E_n(\lambda\bar{f}+\bar{g}) \leqq \lambda + \bar{g}(\bar{x}_i) + 3\varepsilon. \tag{45}$$

Indeed, if $E_n(\lambda\bar{f}+\bar{g}) > \lambda + \bar{g}(\bar{x}_i) + 3\varepsilon$ then (43) and (44) imply

$$\lambda + \bar{g}(\bar{x}_i) + 3\varepsilon - \lambda^{\bar{k}}\bar{f}(x_{\lambda^{\bar{k}}, i}) - \bar{g}(\bar{x}_i) - a_i + b_i < E_n(\lambda\bar{f}+\bar{g}) - E_n(\lambda^{\bar{k}}\bar{f}+\bar{g}) < \lambda - \lambda^{\bar{k}}$$

or $\lambda - \lambda^{\bar{k}} + \varepsilon < \lambda - \lambda^{\bar{k}}$.

This contradiction shows that (45) holds.

Consider now $P_n(\lambda\bar{f}+\bar{g}, \bar{x}_i)$, where $\gamma_{\bar{k}}(-1)^i > 0$. Then using (45) we obtain

$$\lambda + \bar{g}(\bar{x}_i) - P_n(\lambda\bar{f}+\bar{g}, \bar{x}_i) \leqq E_n(\lambda\bar{f}+\bar{g}) \leqq \lambda + \bar{g}(\bar{x}_i) + 3\varepsilon$$

or

$$P_n(\lambda\bar{f}+\bar{g}, \bar{x}_i) \geqq -3 \cdot \varepsilon \tag{46}$$

when $\lambda \geqq \lambda^{\bar{k}} + 2M_1$ and $\gamma_{\bar{k}}(-1)^i > 0$. Analogously we can prove that if $\lambda \geqq \lambda^{\bar{k}} + 2M_1$ and $\gamma_{\bar{k}}(-1)^i < 0$ then

(47) $$P_n(\lambda\bar{f} + \bar{g}, \bar{x}_i) \leqq 3 \cdot \varepsilon.$$

Combining (46) and (47) we obtain

$$\gamma_{\bar{k}}(-1)^{i+1} P_n(\lambda\bar{f} + \bar{g}, \bar{x}_i) \leqq 3 \cdot \varepsilon \quad (i = 0, 1, \ldots, n+1)$$

when $\lambda \geqq \lambda^{\bar{k}} + 2M_1$. Then by Lemma 2

(48) $$\|P_n(\lambda\bar{f} + \bar{g})\| \leqq \frac{C \cdot \varepsilon}{U_{i0} \cdot U_{n+1}},$$

where $U_i = U_i(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_{i+1}, \ldots, \bar{x}_n, \bar{x}_{n+1})$ $(i = 0, 1, \ldots, n+1)$, $U_{i0} = \min_{i=0,1,\ldots,n+1} U_i$ and the constant C depends only on U_n. But $\{\bar{x}_i\}_{i=0}^{n+1}$ is an $A(\bar{f})$ system hence

$$2 = |\bar{f}(\bar{x}_{i+1}) - \bar{f}(\bar{x}_i)| \leqq \omega(\bar{f}, \bar{x}_{i+1} - \bar{x}_i) \leqq \omega(\bar{f}, \bar{x}_{i+1} - \bar{x}_i) + \bar{x}_{i+1} - \bar{x}_i = \omega^*(\bar{x}_{i+1} - \bar{x}_i)$$

where ω^* is a convex modulus of continuity. Therefore $\bar{x}_{i+1} - \bar{x}_i \geqq (\omega^*)^{-1}(2)$ and $U_i \geqq C_1$ $(i = 0, 1, \ldots, n+1)$ where the constant C_1 depends only on $\bar{f}$ and U_n. Combining these inequalities with (48) we obtain $\|P_n(\lambda\bar{f} + \bar{g})\| \leqq C_2 \cdot \varepsilon$ when $\lambda \geqq \lambda^{\bar{k}} + 2M_1$. This means that $P_n(\lambda\bar{f} + \bar{g}) \rightrightarrows 0$ $(\lambda \to +\infty)$ or $P_n(\lambda\bar{f} + g) \rightrightarrows \bar{P}_n$ $(\lambda \to +\infty)$ so

$$\lim_{t \to +0} \frac{P_n(f + tg) + P_n(f)}{t} = \bar{P}_n$$

and statement (i) is proved.

Let us prove (ii). Let $f \in C[0, 1]$, $R_n(f) = n+2$ and let $\{x_i\}_{i=0}^{n+1}$ be the $A(f)$ system. Then the function $\bar{f} = f - P_n(f)$ has the same A-system and $P_n(\bar{f}) \equiv 0$, $R_n(\bar{f}) = n+2$. By Lemma 6, for any $g \in C[0, 1]$

$$P_n(\lambda\bar{f} + g) \rightrightarrows Q_n(g|\bar{f}) - \frac{\bar{Q}(g|\bar{f})}{\bar{Q}(\bar{f}|\bar{f})} \cdot Q_n(\bar{f}|\bar{f}) \quad (\lambda \to +\infty)$$

and this evidently implies that

$$\lim_{t \to +0} \frac{P_n(f + tg) - P_n(f)}{t} = Q_n(g|f) - \frac{Q_n(f - P_n(f)|f)}{\bar{Q}(f|f)} \cdot \bar{Q}(g|f).$$

Being the right derivative a linear operator of g, it is evident that the left derivative will be equal to the right, so the first part of statement (ii) is proved.

Let now $f \in C[0, 1] \setminus U_n$ and $R_n(f) > n+2$. We may assume again that $P_n(f) \equiv 0$ and $\|f\| = 1$. Let $\{x_i\}_{i=0}^{n+1}$ be an $A(f)$ system. Because of $R_n(f) > n+2$, there exists $x^* \in [0, 1]$, $x^* \neq x_i$, $i = 0, 1, \ldots, n+1$ such that $f(x^*) = \gamma$, where $|\gamma| = 1$. Assume for example $\gamma = 1$. Then there exists an interval $(x_1^*, x_2^*) \subset [0, 1]$ such that $x^* \in (x_1^*, x_2^*)$, $f(x) > \frac{1}{2}$ when $x \in (x_1^*, x_2^*)$ and $x_i \notin (x_1^*, x_2^*)$ $(i = 0, 1, \ldots, n+1)$. (Here we assume that $x^* \neq 0, 1$, otherwise the proof is practically the same.) Define now $g \in C[0, 1]$ by

$$g = \begin{cases} 0 \quad \text{if} \quad x \geqq x_2^*,\ x \leqq x_1^*; \\ \dfrac{x - x_1^*}{x^* - x_1^*}, \quad \text{if} \quad x_1^* < x \leqq x^*; \\ \dfrac{x_2^* - x}{x_2^* - x^*}, \quad \text{if} \quad x^* < x \leqq x_2^*. \end{cases}$$

Then by statement (i) of the theorem, there exist the left and right derivatives $D_f^- P_n(g)$ and $D_f^+ P_n(g)$ of P_n at the point f by the direction g. It is easy to see that

$$D_f^+ P_n(g) = \lim_{\lambda \to +\infty} P_n(\lambda f + g).$$

Let us prove now that if $\lambda > 2\|g\| + 1$ then

$$\|P_n(\lambda f + g)\| \geqq \frac{1}{2}. \tag{49}$$

Assume the contrary, i.e. for some $\bar{\lambda} > 2\|g\| + 1$

$$\|P_n(\bar{\lambda} f + g)\| < \frac{1}{2}. \tag{50}$$

Then because of $(\bar{\lambda} f + g)(x^*) = \bar{\lambda} + 1$ we obtain that $E_n(\bar{\lambda} f + g) > \bar{\lambda} + \frac{1}{2}$. In $[0, 1] \setminus (x_1^*, x_2^*)$, $|(\bar{\lambda} f + g)(x)| \leqq \bar{\lambda}$, thus there is no point of the $A(\bar{\lambda} f + g)$ system in $[0, 1] \setminus (x_1^*, x_2^*)$. Then there must be at least two points $\bar{x}_1$ and $\bar{x}_2$ of the $A(\bar{\lambda} f + g)$ system in (x_1^*, x_2^*), and we may assume that

$$\bar{\lambda} f(\bar{x}_1) + g(\bar{x}_1) - P_n(\bar{\lambda} f + g, \bar{x}_1) = \gamma E_n(\bar{\lambda} f + g),$$

$$\bar{\lambda} f(\bar{x}_2) + g(\bar{x}_2) - P_n(\bar{\lambda} f + g, \bar{x}_2) = -\gamma E_n(\bar{\lambda} f + g)$$

where $|\gamma| = 1$. Hence by (50)

$$\bar{\lambda} < \bar{\lambda}\big(f(\bar{x}_1) + f(\bar{x}_2)\big) < 2\|g\| + 1.$$

This contradiction shows that (49) is true and therefore $\|D_f^+ P_n(g)\| \geqq \frac{1}{2}$. On the other hand,

$$D_f^- P_n(g) = \lim_{\lambda \to -\infty} P_n(\lambda f + g)$$

and it is evident that $P_n(\lambda f + g) \equiv 0$ for $\lambda \leqq -1$, so $D_f^- P_n(g) = 0$. Q.E.D.

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ON THE ABSOLUTE CONVERGENCE OF CERTAIN FUNCTION SERIES

By

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1. Fifty years ago S. SIDON [2] proved the following

THEOREM A. *If the series*

$$\sum_{k=1}^{\infty} \varrho_k \cos(n_k x + \alpha_k) \quad \left(\frac{n_{k+1}}{n_k} \geqq q > 1,\ \varrho_k \geqq 0\right) \tag{1}$$

is the Fourier expansion of a one sided bounded function $f(x)$, then $\sum \varrho_k < \infty$.

Recently, as a generalization of the properties of lacunary series, ALEXITS [1] introduced the notion of weakly multiplicative systems and proved for them an analogon of Theorem A. The system $\Phi = \{\varphi_n\}_{n=0}^{\infty}$ defined in a probability space $(X, \mathcal{A}, \mu)$ is called weakly multiplicative, if the functions ψ_n of the product system Ψ of Φ have the property

$$\sum_{n=0}^{\infty} \left| \int_X \psi_n \, d\mu \right| < \infty$$

where the product system Ψ is defined by $\psi_0 \equiv 1$, and for $n \geqq 1$

$$\psi_n = \prod_{j=0}^{\infty} \varphi_j^{\delta_j} \quad \left(n = \sum_{j=0}^{\infty} \delta_j 2^j,\ \delta_j = 0,1\right).$$

Alexits' generalization is the following

THEOREM B. *Let Φ be a bounded weakly multiplicative system and for a function $f \in L(X, \mathcal{A}, \mu)$ put*

$$c_n(f) = \int_X f \psi_n \, d\mu.$$

If f is one sided bounded in X and $c_n(f) = 0$ for $\psi_n \notin \Phi$, then $\sum |c_n(f)| < \infty$.

Theorem A follows readily from B. In the proof of both A and B the one sided boundedness in the whole space plays an essential role, hence one might ask whether an analogous result can be obtained, if we localise the boundedness of f only to a subinterval I of $[-\pi, \pi]$? In this direction a result was obtained [3] by supposing the one sided boundedness of the series (1) itself instead of the one sided boundedness of the developed function f:

THEOREM C. *Denoting by* $s_n(x)$ *the* n^{th} *partial sum of the series* (1) *suppose that one of the relations*

$$\limsup_{n\to\infty} s_n(x) < \infty \quad \textit{or} \quad \liminf_{n\to\infty} s_n(x) > -\infty \tag{2}$$

is satisfied on an interval $I\subset[-\pi, \pi]$. *Then* $\sum \varrho_n < \infty$.

We shall show that a similar local theorem does not hold for general bounded weakly multiplicative systems, neither a localised form of A nor of C. But in the following we shall consider some other function systems which allow us to generalize both Theorems A and C in a proper form. Moreover, we shall see that the whole statement of A remains true even if we suppose that f is bounded from one side only in a subinterval of $[-\pi, \pi]$.

2. Beside B which generalises A for weakly multiplicative systems, we wish to show now that also C has an appropriate analogon for these systems, if we suppose that (2) is everywhere satisfied, but neither A nor C has a localised analogon for general bounded multiplicative systems.

THEOREM 1. (i) *Let* Φ *be a weakly multiplicative system with* $|\varphi_n(x)|\leqq 1$ $(x\in X,$ $n=0, 1, 2, \ldots)$ *and* Ψ *its product system. If the partial sums* $s_n(x)=\sum_{k=1}^{n} a_k\,\varphi_k(x)$ *are uniformly bounded from one side in* X *and*

$$\int_X \varphi_n \psi_m \, d\mu = 0 \quad (m \neq 2^n), \tag{3}$$

then $\sum |a_n| < \infty$.

(ii) *There exist a bounded strongly multiplicative system* Φ^*, *an interval* $I\subset[0, 1]$ *and a function* f *bounded on* I *such that the partial sums*

$$s_n(x) = \sum_{k=1}^{n} c_{2^k} \varphi_k^*(x)$$

are bounded on I, *nevertheless* $\sum |c_{2^k}(f)| = \infty$.

PROOF OF (i). Let M be an upper bound of all partial sums s_n and denote by r_n the nth Rademacher function $r_n(t)=\text{sign}\sin 2^n\pi t$. By $\{w_n\}$ we mean the product system of $\{r_n\}$, i.e. w_n is the nth Walsh function. By (3) and the definition of Φ one has

$$\sum_{k=0}^{n-1} a_k r_k(t) = \int_X s_n(x) \sum_{k=0}^{2^n-1} w_k(t)\psi_k(x)\, d\mu(x) =$$

$$= \int_X s_n(x) \prod_{k=0}^{n-1} \big(1 + r_k(t)\,\varphi_k(x)\big)\, d\mu(x).$$

The product in the last integral being $\geqq 0$ and $s_n(x) \leqq M$, we get

$$\sum_{k=0}^{n-1} a_k r_k(t) \leqq M \sum_{k=0}^{2^n-1} |w_k(t)| \left| \int_X \psi_k \, d\mu \right| \leqq MC, \tag{4}$$

where

$$C = \sum_{k=0}^{\infty} \left| \int_X \psi_k d\mu \right| < \infty.$$

Put here $t=t_0$ where t_0 is that point of [0, 1] for which $r_k(t_0)=\text{sign}\, a_k$ $(k<n)$, then (4) leads to $\sum |a_k| < \infty$ and (i) is proved.

PROOF OF (ii). Since the Rademacher series $\sum n^{-1} r_n(x)$ is convergent almost everywhere, there exists a set $E \subset [0, 1]$ of positive measure such that the partial sums are uniformly convergent on E, therefore the sum $f(x)$ of the series is bounded on E, too. It can be readily seen that one can choose an almost everywhere one-to-one transformation $T:[0, 1] \to [0, 1]$ such that T is measure preserving and transforms an interval $I \subset [0, 1]$ of length mes E in E. Put

$$\varphi_n^*(x) = r_n(T(x)) \quad (x \in [0, 1],\ n = 0, 1, 2, \ldots).$$

The transformation T being measure preserving, the system $\{\varphi_n^*\}$ remains strongly multiplicative like $\{r_n\}$, the sum of $\sum n^{-1} \varphi_n^*(x)$ is bounded and convergent on I to the bounded function $f(T(x))$. Hence all the conditions of Theorems B and C are satisfied. Nevertheless $\sum n^{-1} = \infty$.

3. Denote by φ an arbitrary finite function defined in the interval [0, 1] and having the property that there exist two subintervals $[\alpha_0, \beta_0]$, $[\alpha_1, \beta_1]$ each of length $l>0$ and a number $c>0$ such that

$$\varphi(x) \geqq c \quad \text{for} \quad x \in [\alpha_0, \beta_0] \quad \text{and} \quad \varphi(x) \leqq -c \quad \text{for} \quad x \in [\alpha_1, \beta_1].$$

Continuing φ periodically on the whole real line, we form the system $\{\varphi(nx)\}$. Further, a given positive finite summation matrix $\{\gamma_{nk}\}$ submitted to the only condition

$$\lim_{n\to\infty} \gamma_{nk} = 1 \quad (k = 0, 1, 2, \ldots)$$

defines the nth Toeplitz mean

$$t_n(x) = \sum_{k=0}^{m_n} \gamma_{nk} a_k \varphi(\mu_k x)$$

of the series $\sum a_k \varphi(\mu_k x)$, where $\{\mu_k\}$ denotes a sequence of positive numbers (not necessarily integers). As an analogon of Theorem C we want to prove the following

THEOREM 2. *If one has*

$$\frac{\mu_{k+1}}{\mu_k} \geqq \frac{2}{l}$$

and if for every point of a subinterval I of [0, 1] *at least one of the conditions*

$$\limsup_{n\to\infty} t_n(x) < \infty \quad or \quad \liminf_{n\to\infty} t_n(x) > -\infty$$

is satisfied, then the series $\sum |a_n|$ is convergent.

PROOF. Consider first the case $\limsup t_n(x)<\infty$. Let $\delta_n=0$ or 1, if $a_n\geqq 0$ or $a_n<0$, respectively, and put

$$I_{n,k_n}^{\delta_n}=\left[\frac{1}{\mu_n}(\alpha_{\delta_n}+k_n),\ \frac{1}{\mu_n}(\alpha_{\delta_n}+k_n+l)\right].$$

One can easily see that for sufficiently large $n\geqq n_0$ and proper choice of k_n

$$I_{n+1,k_{n+1}}^{\delta_{n+1}}\subset I_{n,k_n}^{\delta_n}\subset I\subset[0,1].$$

Since x is a point of $I_{n,k_n}^{\delta_n}$ if and only if

$$\alpha_{\delta_n}+k_n\leqq\mu_n x\leqq\alpha_{\delta_n}+k_n+l,$$

hence we have in every point $x\in I_{n,k_n}^{\delta_n}$

$$(-1)^{\delta_n}\varphi(\mu_n x)\geqq c>0.$$

It follows that in a common point

$$\xi\in\bigcap_{n=n_0}^{\infty}I_{n,k_n}^{\delta_n}$$

one gets

$$(-1)^{\delta_n}\varphi(\mu_n\xi)\geqq c\quad(n\geqq n_0).$$

Consequently, there exists a fixed constant K_ξ such that

$$\sum_{k=n_0}^{n}\gamma_{mk}|a_k|\leqq\frac{1}{c}\sum_{k=n_0}^{n}\gamma_{mk}|a_k|(-1)^{\delta_k}\varphi(\mu_k\xi)=\frac{1}{c}\sum_{k=n_0}^{n}\gamma_{mk}a_k\varphi(\mu_k\xi)\leqq\frac{K_\xi}{c}$$

for every $n\geqq n_0$. With respect to $\gamma_{mk}\geqq 1/2$, for every $k=n_0,\dots,n$ and sufficiently large $m\geqq n_0$ we get

$$\sum_{k=n_0}^{n}|a_k|\leqq\frac{2K_\xi}{c}<\infty,$$

thus $\sum_{k=0}^{\infty}|a_k|<\infty$, as we have stated. We proceed similarly in the case $\liminf t_n(x)>$ $>-\infty$.

4. Let $F=\{\varphi_n(x)\}_{n=0}^{\infty}$ be a complete orthonormal system and denote by $\Phi=$ $=\{\varphi_{\nu_n}(x)\}_{n=1}^{\infty}$ a subsystem of F. Denote by

$$c_n(f)=\int_0^1 f(x)\varphi_n(x)\,dx\quad(n=0,1,2\dots),$$

the Fourier coefficients of the function $f\in L^2(0,1)$ with respect to the system F. Generalizing some properties of the function systems used in the previous theorems we introduce the following concepts.

The system $\Phi\subset L^2(0,1)$ is said to be a local S system (shortly: LS-system) if for any interval $I\subset[0,1]$

(LS) $f\in L^2(0,1)$, $|f(x)|\leqq M(<\infty)$ for $x\in I$ and $c_n(f)=0$ $(\varphi_n\notin F)$

implies $\sum|c_n|<\infty$.

We say that the system Φ is a local Z system (shortly: LZ-system) with respect to the row finite matrix $\beta=\{\beta_{nk}\}$, if for an interval $I\subset[0, 1]$ the conditions

$$\text{(LZ)}\qquad \Big|\sum_{k=0}^{m_n} \beta_{n\nu_k} a_{\nu_k} \varphi_{\nu_k}(x)\Big| \leqq M \quad (x\in I,\ n = 1, 2, \ldots) \quad \text{and} \quad \sum_{k=1}^{\infty} a_{\nu_k}^2 < \infty$$

imply $\sum |a_{\nu_k}| < \infty$.

Replacing the interval I by $[0, 1]$ in (LS) and (LZ) we get the definition of the (global) S-system and Z-system, respectively. If we replace in (LS) and (LZ) the conditions of boundedness by one sided boundedness, we obtain the definition of LS*-, LZ*-, S*- and Z*-systems. By Theorems A and C the lacunary trigonometric system is a S*- and a LZ*-system. Theorem B and Theorem 1 show that every bounded weakly multiplicative system is a S- and a Z-system.

We now prove the following

THEOREM 3. *Let $\beta=\{\beta_{nk}\}$ be a positive permanent row finite matrix transforming series in sequences. Then every LS-system is a LZ-system with respect to β.*

PROOF. Let Φ be a LS-system and $I\subset[0, 1]$ an interval. Put

$$\|f\| = \Big(\int_0^1 [f(t)]^2\, dt\Big)^{1/2} + \sup_{x\in I} |f(x)|$$

and denote by X_I the set of $f\in L^2(0, 1)$ for which $\|f\|<\infty$ and $c_n(f)=0$ if $\varphi_n\notin\Phi$. X_I is a Banach space with this norm and

$$F_n(f) = \sum_{k=1}^{n} |c_{\nu_k}|$$

are continuous sublinear functionals defined on X_I. Since Φ is a LS-system, the sequence of the functionals F_n converges on X_I. Thus by the Banach—Steinhaus theorem there exists a constant $K_I>0$ such that

$$(5)\qquad F_n(f) \leqq K_I \|f\| \quad (n = 1, 2, \ldots)$$

for every $f\in X_I$.

We now suppose that for the sequence $\{a_{\nu_n}\}$ the condition (LZ) holds. Applying (5) to the functions

$$t_n = \sum_{k=1}^{m_n} \beta_{n\nu_k} a_{\nu_k} \varphi_{\nu_k}$$

we get

$$\sum_{k=1}^{m_n} \beta_{n\nu_k} |a_{\nu_k}| \leqq K_I \left(M + \left(\sum_{k=1}^{m_n} \beta_{n\nu_k}^2 a_{\nu_k}^2\right)^{1/2}\right) \leqq K_I \left(M + K^* \left(\sum_{k=1}^{\infty} a_{\nu_k}^2\right)^{1/2}\right) \quad (n = 1, 2, \ldots)$$

where $K^* = \sup_{n,k} \beta_{n\nu_k} < \infty$. Hence, it follows that $\sum_{k=1}^{\infty} |a_{\nu_k}| < \infty$, thus Φ is a LZ-system.

5. Now we shall consider the sums

$$\tau_n(f)(x) = \sum_{k=0}^{m_n} \beta_{nk} c_k(f) \varphi_k(x)$$

where $\{\beta_{nk}\}$ is a row finite permanent matrix transform series in sequences. Consider the nth kernel of this summation method:

$$K_n(x, y) = \sum_{k=0}^{m_n} \beta_{nk}\varphi_k(x)\varphi_k(y).$$

Set $\varphi_0=1$ and assume the following properties of the kernel K_n:

(6) $$K_n(x, y) \leqq \Psi(\delta) \quad (|x-y| \geqq \delta,\ n = 0, 1, 2, \ldots)$$

where $\Psi(\delta)$ is finite for every $\delta>0$, further

(7) $$\int_0^1 |K_n(x, y)|\, dy \leqq M^* < \infty \quad (x\in[0, 1],\ n = 0, 1, 2, \ldots)$$

or

(7′) $$K_n(x, y) \geqq 0 \quad (x, y\in[0, 1],\ n = 0, 1, 2, \ldots).$$

(We remark that (7′) implies (7) if $\varphi_0=1$.)

THEOREM 4. (i) *Suppose that the conditions* (6) *and* (7) *are satisfied. Then* Φ *is a LS-system if and only if it is a LZ-system.*

(ii) *If* K_n *satisfies* (6) *and* (7′), *then* Φ *is a* LS^**-system if and only if it is a* LZ^**-system.*

PROOF OF (i). Let first Φ be a LS-system and denote by $\{a_{\nu_k}\}$ a number sequence satisfying (LZ). Further let $f\in L^2$ the function for which $c_{\nu_k}(f)=a_{\nu_k}$ $(k=1, 2, \ldots)$ and $c_n(f)=0$ for $\varphi_n\notin\Phi$. Then

$$\sum_{k=1}^{m_n} \beta_{n\nu_k} a_{\nu_k}\varphi_{\nu_k} = \tau_n(f).$$

Since $\{\beta_{nk}\}$ is permanent and F is complete, $\{\tau_n(f)\}$ converges in L^2-norm to f. Then there exists a subsequence $\{\tau_{m_k}(f)\}$ of $\{\tau_n(f)\}$ such that $\lim_{k\to\infty}\tau_{n_k}(f)(x)=f(x)$ a.e., thus by the condition (LZ) one has $|f(x)|\leqq M$ $(x\in I)$. This means that for f the conditions (LS) are satisfied. The condition that Φ is a LS-system implies $\sum|a_{\nu_k}|=$ $=\sum|c_{\nu_k}(f)|<\infty$, i.e. Φ is a LZ-system.

Let now Φ be a LZ-system and denote by f a function for which (LS) holds, further put $I=[a, b]$. Then

$$\tau_n(f)(x) = \sum_{k=1}^{m_n} \beta_{n\nu_k} c_{\nu_k}\varphi_{\nu_k}(x) = \int_0^1 f(y)K_n(x, y)\, dy$$

and by (6) and (7) we have for $x\in[a+\delta, b-\delta]$

$$|\tau_n(f)(x)| \leqq \int_0^a |f(y)||K_n(x, y)|\, dy + \int_a^b |f(y)||K_n(x, y)|\, dy +$$

$$+\int_b^1 |f(y)||K_n(x, y)|\, dy \leqq \Psi(\delta)\int_0^1 |f(y)|\, dy + MM^*.$$

This means that for the sequence $\{c_{\nu_k}(f)\}$ the conditions (LZ) are satisfied, hence $\sum |c_{\nu_k}(f)| < \infty$.

(ii) The proof of (ii) is similar.

Let e.g. $\{\beta_{nk}\}$ the matrix of the $(C, 1)$-summation method and F the trigonometric system. Then K_n is the nth Fejér kernel and conditions (6) and (7′) are satisfied.

COROLLARY. *Let* $\frac{n_{k+1}}{n_k} \geqq q > 1$, *then the lacunary trigonometric system* $\{\cos n_k x, \sin n_k x\}$ *is a* LS^* *system.*

Remarking that in (2) the partial sums s_n can be replaced by the Fejér means σ_n of (1) (ZYGMUND [3]), our corollary follows immediately from Theorem B and Theorem 4 (ii).

A similar result is valid — mutatis mutandis — also for the sufficiently lacunary Walsh system.

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ON CERTAIN TYPES OF CONDITIONS CHARACTERIZING ADDITIVE ARITHMETICAL FUNCTIONS

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1. Throughout this paper f denotes an additive arithmetical function, A, B and C are subsequences of the natural numbers, composed of the elements $a_1 < a_2 < < a_3 < \dots$, $b_1 < b_2 < \dots$, and $c_1 < c_2 < \dots$ resp.

The general program of characterizing additive functions may be described as follows:

We are interested in sets A and conditions on the function values $f(a_k)$, so that if they are satisfied, then f belongs to a given class of additive functions.

We shall call A the *set of characterization* (from now on: *CH-set*), the conditions required for the $f(a_k)$ will be called the *conditions of characterization (CH-conditions)*, and the given class of the additive functions will be called the *class of functions to be characterized (CH-class)*.

The first results in this direction were obtained by P. ERDŐS [1], and later very many papers dealt with such types of problems (see e.g. [2]—[7]).

In most cases the CH-class either consisted of the single function $f=0$, or was the class of the functions $c \cdot \log n$. The CH-condition characterizing the function $f=0$ was generally $f(a_k)=0$, $k=1, 2, \dots$. In this case A is called a *U-set (set of uniqueness)*.

In [8], [9] and [10] we examined some further CH-conditions for the function $f=0$. We dealt also with the problem of characterizing the class of bounded fonctions.

In this paper we investigate some other problems in connection with the characterization of these two CH-classes.

2. In [9] we have proved that to an arbitrary function $g(n)$ there exists an A which is a U-set, and

$$\frac{a_{k+1}}{a_k} > g(k) \tag{1}$$

holds, but we cannot achieve

$$a_{k+1} > g(a_k) \tag{2}$$

if $g(n)$ increases too rapidly. We also determined the exact order of the functions $g(n)$ for which (2) can still be satisfied by a suitable U-set, i.e. we checked the maximal possible "rate of growth" of the elements of a U-set (this turned out to be roughly $a_{k+1} \sim a_k^2$).

Now we show that by suitable additional conditions on f, to an arbitrary $g(n)$ we can find a CH-set A, for which (2) holds, and fairly weak CH-conditions still guarantee $f=0$.

We assume first that f is completely additive.

THEOREM 1. *To any* $g(n)$ *we can construct an* A *for which* (2) *holds and* A *has the following characterizing property:*

If f *is completely additive and*

$$\frac{f(a_{k+1})-f(a_k)}{\log a_{k+1}} \to c, \tag{3}$$

then $f=c\cdot\log n$.

PROOF. Let $t_1, t_2, \ldots$ be a sequence of natural numbers, where $t_i>1$ and every $n>1$ appears infinitely many times among the t_i.

We put

$$a_{2i-1} = t_i^{m_i}, \quad a_{2i} = t_i^{r_i}, \quad i = 1, 2, \ldots,$$

where the m_i and r_i are specified in the following way:

Assuming that m_{i-1} and r_{i-1} have already been determined, we take m_i large enough to satisfy $a_{2i-1}>g(a_{2i-2})$, and then we select r_i so that $a_{2i}>g(a_{2i-1})$ and $\frac{r_i}{m_i}>i$ should hold.

Clearly, (2) is valid.

If f is completely additive and satisfies (3), then

$$\frac{f(a_{2i})-f(a_{2i-1})}{\log a_{2i}} = \frac{(r_i-m_i)\cdot f(t_i)}{r_i\cdot\log t_i} = \left(1-\frac{m_i}{r_i}\right)\cdot\frac{f(t_i)}{\log t_i} \to c,$$

and thus

$$\frac{f(t_i)}{\log t_i} \to c.$$

Since the sequence t_i contains every natural number greater than 1 infinitely many times, we obtain $\frac{f(n)}{\log n} \equiv c$.

REMARKS. 1. Condition (3) cannot be replaced by an even weaker one.

2. Using the same ideas we can also prove

THEOREM 2. *To any* $g(n)$ *and* $h(n)$ *we can construct an* A, *for which* (2) *holds, and if for a completely additive* f *we have*

$$f(a_{k+1})-f(a_k) = O\big(h(a_k)\big), \tag{4}$$

then $f=0$.

In fact, Theorem 2 is a corollary of Theorem 1.

Now we assume that f is bounded.

THEOREM 3. *To any* $g(n)$ *we can construct an* A, *for which* (2) *holds, and if for a bounded* f

$$f(a_{k+1})-f(a_k) \quad \textit{is convergent,} \tag{5}$$

then $f=0$.

PROOF. Let $t_1, t_2, \ldots$ be the same sequence as in the proof of Theorem 1.

We define now the sequence A:

$$u_1, t_1 \cdot u_2, u_3, t_2 \cdot u_4, \ldots, u_{2i-1}, t_i \cdot u_{2i}, \ldots,$$

where the u_k are different primes and $(t_i, u_{2i})=1$.

Clearly, (2) can be guaranteed.

Let f be bounded, $|f(n)|<L$, and $f(a_{k+1})-f(a_k)\to c$.

$f(1)=0$ by the additivity.

Let $m>1$ be arbitrary. We consider all the values of $k=2i-1$, where $t_i=m$. Then for these i we have

$$f(m)+f(u_{2i})-f(u_{2i-1}) \to c.$$

Let $\varepsilon>0$ be arbitrary, and i so large that

$$|f(m)+f(u_{2i})-f(u_{2i-1})-c| < \varepsilon$$

should hold. Taking N such values of i, we obtain

$$|N\cdot f(m)+f(u_{2i_1}\cdot \ldots \cdot u_{2i_N})-f(u_{2i_1-1}\cdot \ldots \cdot u_{2i_N-1})-N\cdot c| < N\varepsilon.$$

Dividing by N:

$$|f(m)-c| < \varepsilon+\frac{2L}{N}$$

and if $N\to\infty$, we have $|f(m)-c|\leqq\varepsilon$. Hence $f(m)=c$, but then by the additivity only $f=0$ is possible.

Now we assume that f is real-valued, and $\liminf f(n)\geqq 0$.

THEOREM 4. *To any $g(n)$ we can construct an A, for which* (2) *holds, and if*

$$\liminf f(n) \geqq 0, \tag{6}$$

then (5) *implies $f=0$.*

PROOF. Let $t_1, t_2, \ldots$ be the usual sequence.

A will be the union of successive blocks, where each block consists of three elements.

Let $m>1$ be a natural number.

The sequences $\{u_{mj}\}_{j=1}^{\infty}$, $\{v_{mj}\}_{j=1}^{\infty}$ and $\{w_{mj}\}_{j=1}^{\infty}$ should be composed of different primes not dividing m.

We take now those i, for which $t_i=m$; let these be $i_1, i_2, \ldots$.

The i_j-th block is defined in the following way:

$$u_{mj}, \quad u_{mj}\cdot v_{mj}, \quad m\cdot w_{mj}\cdot u_{mj}\cdot v_{mj}\cdot v_{m,j-1}\cdot v_{m,j-2} \quad (v_{m0}=v_{m,-1}=1).$$

Since each element contains a "new" factor (u_{mj}, v_{mj} and w_{mj} resp.), condition (2) can be satisfied.

We take now an f satisfying (6) and $f(a_{k+1})-f(a_k)\to c$. Considering the blocks with $t_i=m$ we have

$$f(u_{mj}\cdot v_{mj})-f(u_{mj}) \to c, \quad \text{i.e.} \quad f(v_{mj}) \to c,$$

further

$$f(m \cdot w_{mj} \cdot u_{mj} \cdot v_{mj} \cdot v_{m,j-1} \cdot v_{m,j-2}) - f(u_{mj} \cdot v_{mj}) \to c,$$

i.e.

$$f(n_j) = f(m \cdot w_{mj} \cdot v_{m,j-1} \cdot v_{m,j-2}) \to c \quad (j \to \infty).$$

Clearly,

$$f(n_j) = f(m) + f(w_{mj}) + f(v_{m,j-1}) + f(v_{m,j-2}),$$

and here the left hand side tends to c, while the sum of the last two terms on the right hand side tends to $2c$. Using (6) we obtain $c=0$, and $f(m)=0$.

3. We consider now some other CH-conditions. Put

$$F(n) = f(1) + f(2) + \ldots + f(n).$$

THEOREM 5. I. *There exists an* $f \neq 0$, *for which* $F(n)$ *is bounded.*

II. *To any* $g(n)$ *we can construct an* A, *for which*

$$a_k > g(k) \tag{7}$$

holds, and if

$$F(a_k) \textit{ is convergent,} \tag{8}$$

then $f=0$.

REMARK. For the CH-condition

$$\text{“}f(a_1) + f(a_2) + \ldots + f(a_k) \quad \text{is convergent“},$$

see Theorem 2 in [9].

THEOREM 6. I. *Let* $h(n)$ *tend to infinity (arbitrarily slowly). There exists an unbounded* f *satisfying* $|F(n)| \leqq h(n)$.

II. *To any* $g(n)$ *we can construct an* A *for which* (7) *holds, and if* $F(a_k)$ *is bounded, then so is* f.

REMARK. For the CH-condition

$$\text{“}f(a_1) + f(a_2) + \ldots + f(a_k) \quad \text{is bounded“},$$

see Remark 4 after the proof of Theorem 2/II in [9].

THEOREM 7. *If* A *has upper density* 1, *and* $F(a_k)$ *is convergent or bounded, then* $f=0$ *or* f *is bounded, resp.*

THEOREM 8. I. *There exists a completely additive function* $f \neq 0$, *satisfying* $F(n) = O(\log n)$.

II. *To any* $g(n)$ *we can construct an* A *for which* (7) *holds, and if* f *is completely additive and*

$$\frac{F(a_k)}{\log a_k} \quad \textit{is convergent,} \tag{9}$$

then $f=0$.

III. *To any* $g(n)$ *we can construct an* A *for which* (2) *holds, and if* f *is completely additive and*

$$\frac{f(a_1)+f(a_2)+\ldots+f(a_k)}{\log a_k} \to c, \tag{10}$$

then $f=c\cdot\log n$.

THEOREM 9. I. *There exists a completely additive, non-negative* $f\neq 0$ *satisfying* $F(n)/n\to 1$.

II. *If* f *satisfies* (6) *and*

$$\liminf \frac{F(n)}{n} = 0, \tag{11}$$

then $f=0$.

PROOFS. We may assume that $g(n)$ is an increasing function.

PROOF OF THEOREM 5. I. E.g.: $f(2)=1$, $f(2^r)=-1$ for $r\geqq 2$, and $f(p^m)=0$ if p is an odd prime.

II. In [8] we have proved (Theorem 1) that there exists a B, for which $b_k > g(2k)+1$, and if $f(b_k)$ is convergent, then $f=0$.

Put

$$a_n = \begin{cases} b_k - 1 & \text{for} \quad n = 2k-1 \\ b_k & \text{for} \quad n=2k. \end{cases}$$

Then (7) holds.

Further, (8) implies $f(b_k)=F(a_{2k})-F(a_{2k-1})\to 0$, and so by the result quoted above we have $f=0$.

PROOF OF THEOREM 6. I. We select n_i so that for $n\geqq n_i$, $h(n)\geqq i$ should hold. Let p_i be different primes, $p_i\geqq n_i$. We define f by $f(p_i)=\dfrac{1}{p_i-1}$, $f(p_i^r)=-1$ for $r\geqq 2$, and $f(p^m)=0$ if $p\neq p_i$, $i=1, 2, \ldots, p$ is a prime.

Then f is unbounded, since

$$f(p_1^2\cdot\ldots\cdot p_j^2) = -j.$$

On the other hand

$$F(n) = \sum_{p_i\leqq n}\Bigg\{\sum_{\substack{p_i\| k\\ k\leqq n}}\frac{1}{p_i-1} - \sum_{\substack{p_i^2| k\\ k\leqq n}} 1\Bigg\} = \sum_{p_i\leqq n} S_i.$$

Here $0\leqq S_i\leqq 1$, and thus for $n_j\leqq n<n_{j+1}$

$$0 \leqq F(n) \leqq j \leqq h(n).$$

(We observe that by a slight modification of the construction we can produce an f which is unbounded already on the set of the prime-squares, and still $|F(n)|\leqq h(n)$.)

II. We use Theorem 2 from [8], and argue in the same way as we did in the proof of Theorem 5/II above.

PROOF OF THEOREM 7. Consider the sequence B of those numbers n, for which both n and $n-1$ belong to A.

First we show that also B has upper density 1. Indirectly, assume that for a $d>0$ and large enough m, at least $d\cdot m$ numbers i in the interval $[1, m]$ have the property that either i or $i-1$, or both are not elements of A. Then at least for $\frac{d}{2}$ numbers $j\in[1, m]$, $j\notin A$ (such a "missing" j may be counted both as an "i" and as an "$i-1$"), but this is a contradiction, since A has upper density 1.

Consider now only B. Then $f(b_k)=F(b_k)-F(b_k-1)=F(b_k)-F(b_{k-1})$ tends to 0 or is bounded, and thus $f=0$ or is bounded by Theorems 4 and 5 in [8], resp.

PROOF OF THEOREM 8. I. We take $f(2)=1$, $f(3)=-2$ and $f(p)=0$ for the other primes.

By the complete additivity

$$F(n) = \sum_{\substack{p^s\leqq n\\ s\geqq 1}} f(p)\cdot\left[\frac{n}{p^s}\right] = \sum_{2^s\leqq n}\left[\frac{n}{2^s}\right]-2\cdot\sum_{3^s\leqq n}\left[\frac{n}{3^s}\right] = U-2V.$$

Here

$$U\leqq n\cdot\sum_{2^s\leqq n}2^{-s} = n\cdot(1-2^{-t})\leqq n\cdot\left(1-\frac{1}{n}\right) = n-1,$$

where $t=[\log_2 n]$. On the other hand

$$U>\Big\{n\cdot\sum_{2^s\leqq n}2^{-s}\Big\}-\log_2 n = n\cdot(1-2^{-t})-\log_2 n >$$

$$> n\cdot\left(1-\frac{1}{\frac{n}{2}}\right)-\log_2 n = n-2-\log_2 n.$$

Similarly

$$\frac{n-3}{2}-\log_3 n < V\leqq\frac{n-1}{2}.$$

Hence

$$-1-\log_2 n < U-2V < 2+\log_3 n,$$

i.e.

$$|F(n)|\leqq\log_2 n+2 = O(\log n).$$

II. We take a sequence guaranteed by Theorem 1, and denote it by B.

Put $a_{2i-1}=b_i-1$ and $a_{2i}=b_i$.

Let f be completely additive, for which (9) holds; $\frac{F(a_k)}{\log a_k}\to c$.

To any $\varepsilon>0$ we can find an N, such that for $i\geqq N$

$$(c-\varepsilon)\cdot\log(b_i-1) < F(b_i-1) < (c+\varepsilon)\cdot\log(b_i-1)$$

and

$$(c-\varepsilon)\cdot\log b_i < F(b_i) < (c+\varepsilon)\cdot\log b_i.$$

After subtraction, and division by $\log b_i$ we obtain

$$\left|\frac{f(b_i)}{\log b_i}\right| < 3\varepsilon$$

taking i large enough. Thus $\frac{f(b_i)}{\log b_i} \to 0$, and from the proof of Theorem 1 we see that in this case $f=0$.

III. We take the same A as in Theorem 1.

Similarly to the calculations in II, we obtain

$$(c-\varepsilon)-(c+\varepsilon)\cdot\frac{\log a_{i-1}}{\log a_i} < \frac{f(a_i)}{\log a_i} < (c+\varepsilon)-(c-\varepsilon)\cdot\frac{\log a_{i-1}}{\log a_i}.$$

We may assume that $\frac{\log a_{i-1}}{\log a_i} \to 0$, and thus $\frac{f(a_i)}{\log a_i} \to c$. Then from the proof of Theorem 1 we have obviously $f=c\cdot\log n$.

Proof of Theorem 9. I. E.g. the function defined by $f(2)=1$, $f(p)=0$ for the other primes, has this property, as we can see it from the proof of Theorem 8/I.

II. We assume indirectly that for some $k, f(k)=d\neq 0$.

Case (i): $d>0$. Let $\varepsilon>0$ be fixed. We can find an N such that for $n \geqq N$, $f(n) > -\varepsilon$. Further, if $(t,k)=1$ and $t \geqq N$, then $f(k\cdot t)=f(k)+f(t)>d-\varepsilon$. Thus

$$F(n) \geqq \left[\frac{n}{k}\right]\cdot\frac{\varphi(k)}{k}\cdot(d-\varepsilon)-\varepsilon\cdot n-T,$$

where T is a constant depending only on N.

If ε was chosen small enough, then for $n\to\infty$ the right hand side is greater than $c\cdot n$, which contradicts (11).

Case (ii): $d<0$. Then

$$\liminf_{\substack{t\to\infty \\ (t,k)=1}} f(t\cdot k) = \liminf_{\substack{t\to\infty \\ (t,k)=1}} \{f(t)+f(k)\} \geqq 0,$$

thus we can find a t with $f(t)>0$, and we can apply Case (i).

4. Finally we examine certain relations between CH-conditions for the function $f=0$ and those for the class of bounded functions. Several results of [8], [9] and [10], and also Theorem 7 suggest that there are strong connections between these two CH-classes.

Now — by very sharp counterexamples — we point out some great differences between the two types of characterization.

Theorem 10. *Let* $2=p_1<p_2<\ldots$ *denote the sequence of primes.*

I. *We can construct an* A, *for which* (5) *implies* $f=0$, *but even*

(12) *the boundedness of the sums* $\sum_{k=1}^{r} f(a_k)$ *does not imply that* f *is bounded, moreover, there exists a completely additive* f, *for which*

(13) $$|f(p_i)| \to \infty,$$

though (12) *holds.*

II. *We can construct an* A *which is a U-set, but even*

(14) $$f(a_k)=0, \quad k=2,3,\ldots,$$

does not imply the boundedness of f, moreover, there exists a completely additive f, for which both (13) *and* (14) *hold.*

III. *We can construct an A, for which*

(15) *the boundedness of* $f(a_{k+1})-f(a_k)$ *implies that f is bounded,*

but A is not a U-set, moreover we can construct an f with $f(a_k)=0$ *for all k, but* $f(q)\neq 0$ *for infinitely many prime powers q. (By the other condition on A this f must be bounded, and so e.g. cannot be completely additive.)*

PROOF. I. Defining $p_0=1$ we take

$$H_1 = \{p_k \cdot p_{k+1},\quad k \geqq 0\},$$
$$H_2 = \{p_k^s \cdot p_{ks+1},\ p_k^s \cdot p_{ks-1},\ k \geqq 2,\ s \geqq 2,\ k \text{ is even}\},$$
$$H_3 = \{p_k^s \cdot p_{ks+1},\ p_k^s \cdot p_{ks-1},\ k \geqq 1,\ s \geqq 3,\ \text{both } k \text{ and } s \text{ are odd}\},$$
$$H_4 = \{p_k^s \cdot p_{ks},\ k \geqq 1,\ s \geqq 2,\ k \text{ is odd, } s \text{ is even}\}$$

and $B=H_1\cup H_2\cup H_3\cup H_4$.

We form now the numbers t_i so that every b_j should appear infinitely often among the t_i, further,

$$\text{for } H_1,\ t_i = p_{2k}\cdot p_{2k+1} \text{ if and only if } t_{i+1} = p_{2k+1}\cdot p_{2k+2};$$
$$\text{for } H_2,\ t_i = p_k^s\cdot p_{ks+1} \text{ if and only if } t_{i+1} = p_k^s\cdot p_{ks-1};\ \text{ and}$$
$$\text{for } H_3,\ t_i = p_k^s\cdot p_{ks-1} \text{ if and only if } t_{i+1} = p_k^s\cdot p_{ks+1}.$$

Now we define u_i so that $u_i > u_{i-1}\cdot t_{i-1}$, $(u_i, t_i)=1$ and $u_i=p_{2r}\cdot p_{2r+1}\cdot p_{2r+3}\cdot p_{2r+4}$ for some suitable r. Put $c_{2i-1}=u_i$, $c_{2i}=u_i\cdot t_i$. Then we chose v_i so that $v_i > v_{i-1}\cdot c_{i-1}$, $(v_i, c_i)=1$, and $v_i=p_{2m}\cdot p_{2m+1}\cdot p_{2m+3}\cdot p_{2m+4}$ for some suitable m. Finally we put $a_{2i-1}=v_i$, $a_{2i}=c_i\cdot v_i$.

Consider now an f satisfying (5). Then $f(c_i)=f(a_{2i})-f(a_{2i-1})$ is convergent, and so $f(t_k)=f(c_{2k})-f(c_{2k-1})\to 0$. Since every b_j occurs infinitely many times among the t_k, we obtain $f(b_j)=0$ for all j.

Successively we find that $f(p_i)=0$: $f(p_1)=0$, $f(p_1\cdot p_2)=f(p_1)+f(p_2)=0$, i.e. $f(p_2)=0$, etc.

Now for $s\geqq 2$, $p_k^s\cdot p_{ks}\in B$ or $p_k^s\cdot p_{ks+1}\in B$, i.e. $f(p_k^s)+f(p_{ks})=0$ or $f(p_k^s)+$ $+f(p_{ks+1})=0$, which means that $f(p_k^s)=0$, and thus $f=0$.

To show that A is an „anti-CH-set" for the bounded functions, we define f by

$$f(p_k) = (-1)^k\cdot k.$$

Clearly, (13) holds. On the other hand, we easily see that $f(t_i)=0$ if $t_i\in H_4$, $f(t_i)=1$ or -1 if $t_i\in H_1$, H_2 or H_3, and in this latter case $f(t_i)=-1$ if and only if $f(t_{i+1})=$ $=1$. Further $f(u_i)=f(v_i)=0$ for all i.

Thus $\sum_{k=1}^{r} f(a_k)=0$, 1 or -1 for all r, and so (12) holds as well.

II. Put

$$A = \{2,\ 2^s\cdot 3^s,\ 3^s\cdot 5^s,\ 2^s\cdot p_{2k-1}^s\cdot p_{2k}^s,\ p_{2k}^s\cdot p_{2k+1}^s,\ s\geqq 1,\ k\geqq 2\}\cup\{2^s\cdot p_{2s},\ s\geqq 2\}.$$

First we assume $f(a_i)=0$ for all i. Then $f(2)=f(2\cdot 3)=f(3\cdot 5)=f(2\cdot 5\cdot 7)=f(7\cdot 11)=$ $=\ldots=0$, and so $f(p_k)=0$ for all k.

Now, for $s \geqq 2$ $f(2^s \cdot p_{2s})=0$ implies $f(2^s)=0$, and so using $f(2^s \cdot 3^s)=f(3^s \cdot 5^s)= =f(2^s \cdot 5^s \cdot 7^s)=\ldots=0$ we obtain $f(p_k^s)=0$ for all k, i.e. $f=0$. This means that A is a U-set.

On the other hand, we define f by

$$f(2)=-1, \quad f(p_{2k})=k, \quad f(p_{2k+1})=-k \quad (k \geqq 1).$$

Then f obviously satisfies (13), and by a simple calculation we find that (14) holds as well.

III. Consider $B=\{t;\ (t,6)=1\} \cup \{2^s,\ s \geqq 2\} \cup \{2 \cdot 3^s,\ s \geqq 1\}$.

Let $u_i > 3$ be arbitrary primes, $u_i > u_{i-1} \cdot b_{i-1}$ and $(u_i, b_i)=1$. Put $a_{2i-1}=u_i$, $a_{2i}=u_i \cdot b_i$.

Then (15) implies that $f(b_i)$ is bounded, $|f(b_i)|<L$.

Now for any n

$$f(n)=f(2^k)+f(3^m)+f(t) \quad ((t,6)=1),$$

where

$$|f(2^k)| \leqq \max\{|f(2)|, L\}, \quad |f(3^m)| \leqq |f(2 \cdot 3^m)|+|f(2)| < L+|f(2)|, \quad |f(t)| < L,$$

and so f is bounded.

On the other hand, put $f(2)=1$, $f(3^m)=-1$ and $f(q)=0$ for all other prime powers q, then $f(a_k)=0$ for all k.

REMARK. By refining the proofs of parts I and II we can replace (13) even by stronger conditions, namely $\frac{2^k}{f(p_k)}=o(1)$ or — with an arbitrary $g(n)$ — $\limsup \frac{f(p_k)}{g(k)}=\infty$ (the latter one is clearly equivalent to $\limsup \frac{f(p_k)}{g(p_k)}=\infty$).

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TORSION THEORIES IN AFFINE CATEGORIES

By

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1. Introduction. The aim of this paper is to investigate the torsion theories in certain so called affine categories, which generalize the well-known properties of the category of all affine modules over a ring with identity. Thus our result, in particular, answers a question, raised by R. Wiegandt during a personal discussion, asking for a characterization of torsion theories in the category of affine modules. Affine modules are closely related to modules; however, they have a property in common with graphs and topological spaces, namely that under a homomorphism the inverse image of any element is a substructure. On the other hand, since any abelian category turns out to be an affine category, our result yields a characterization for the torsion theories in abelian categories.

In §2 the concept of an A-category is introduced such that it includes additive categories as well as the category of affine modules over a ring with identity. Several examples of A-categories are also given.

In §§3 and 4 we discuss the basic properties of A-categories emphasizing the close connection and the differences between additive categories and A-categories (in particular, between abelian and affine categories).

Finally, §5 is devoted to the proof of the characterization theorem of torsion theories in affine categories. Because of the great similarity between abelian and affine categories, it is not surprising that our result is very similar to that of DICKSON [2]. Dickson's torsion theory and some generalizations of several related results on modules summarized by LAMBEK [6] are treated also in [3]. Furthermore, our result is analogous also to the characterization of semisimple classes of rings obtained by SANDS [11] and by LEEUWEN, ROOS and WIEGANDT [8], to the description of torsion theories of rings given by LEAVITT and WIEGANDT [7], and to the characterization of connectednesses and disconnectednesses in topological spaces and in graphs found by ARHANGEL'SKIĬ and WIEGANDT [1] and by FRIED and WIEGANDT [5], respectively.

We use the terminology of [9] and for the basic results concerning additive and abelian categories the reader is referred to [4] and [9]. Throughout the paper every category is assumed to be well-powered and to have small hom-sets.

Let R be a ring with identity, written 1. An affine right R-module A is an algebraic system for which there exists a unitary right R-module M such that their universes coincide and the operations of A are the linear forms $\sum_{i=1}^{n} x_i \cdot r_i$ on M with $r_i \in R$, $n \geqq 1$, $\sum_{i=1}^{n} r_i = 1$. The affine modules can also be directly defined by identities (see

[10]). For the ring Z of integers an affine Z-module will be called the full idempotent reduct of an abelian group. It is easy to see that the full idempotent reduct of an abelian group can be defined by a single ternary operation, namely by $\sum_{i=1}^{3} x_i \cdot (-1)^{i-1}$ (i.e., any other linear form $\sum_{i=1}^{n} x_i \cdot r_i$ with $r_i \in Z$, $\sum_{i=1}^{n} r_i = 1$ arises from this fundamental operation by superposition and collapsing variables). ($\sum_{i=1}^{n} \sqcup \cdot (-1)^{i-1}$ is considered as an operational symbol, we shall "forget" that it has been built up from the Z-module operations and shall keep only the identities concerning it.)

2. A-categories. It is well known that in the category of all affine right R-modules every homomorphism can be uniquely represented as a composition of a module homomorphism and a translation. A translation on an affine right R-module A is defined to be an automorphism assigning to every element a in A the element $a \cdot 1 + a_1 \cdot 1 + a_0 \cdot (-1)$ where a_1 and a_0 are fixed elements of A. Furthermore, it is easy to show that every hom-set forms the full idempotent reduct of an abelian group under the componentwise operations, moreover, the composition of homomorphisms is distributive with respect to these operations. On the other hand, the one-element affine module is clearly a terminal object in the category, and any constant mapping is a homomorphism.

This leads to the following general

DEFINITION 1. A category $\mathbf{C}$ will be called an *A-category* if and only if the following two conditions are satisfied:

(i) There exists a terminal object 0 in $\mathbf{C}$ such that for every object A in $\mathbf{C}$ the set $\hom_{\mathbf{C}}(0, A)$ is nonvoid.

(ii) Every hom-set in $\mathbf{C}$ is the full idempotent reduct of an abelian group and for any objects A, B, C, D and arrows $f_1, f_2, f_3 \in \hom_{\mathbf{C}}(A, B)$, $g \in \hom_{\mathbf{C}}(B, C)$ and $h \in \hom_{\mathbf{C}}(D, A)$ we have

$$g(f_1 \cdot 1 + f_2 \cdot 1 + f_3 \cdot (-1)) = gf_1 \cdot 1 + gf_2 \cdot 1 + gf_3 \cdot (-1),$$
$$(f_1 \cdot 1 + f_2 \cdot 1 + f_3 \cdot (-1))h = f_1 h \cdot 1 + f_2 h \cdot 1 + f_3 h \cdot (-1).$$

The arrows factorizable through the terminal object 0 will be called *zero arrows*. The single element of $\hom_{\mathbf{C}}(A, 0)$ will be denoted by 0^A. By analogy, the arrows of the form $1_A \cdot 1 + z0^A \cdot 1 + z'0^A \cdot (-1)$ with $A \in \mathrm{Obj}\,\mathbf{C}$ and $z, z' \in \hom_{\mathbf{C}}(0, A)$ will be called *translations* on A. For any arrow $f \in \hom_{\mathbf{C}}(A, B)$ the set of all arrows arising from f by composing it with a translation on B will be denoted by $\mathrm{tr}\, f$; i.e. using distributivity

$$\mathrm{tr}\, f = \{f \cdot 1 + z0^A \cdot 1 + z'0^A \cdot (-1) \mid z, z' \in \hom_{\mathbf{C}}(0, B)\}.$$

Obviously, for any object A the set $\mathrm{tr}\, 1_A$ of all translations on A constitutes a group under composition.

The following theorem states an important connection between A-categories and additive categories.

THEOREM 1. *Let* $\mathbf{C}$ *be a small A-category with finite products. Then there exists an additive category* $\mathbf{D}$ *such that the following conditions are satisfied:*

(a) Obj $\mathbf{C}$=Obj $\mathbf{D}$.

(b) *For any objects* A, B *we have* $\hom_{\mathbf{D}}(A, B) \subseteqq \hom_{\mathbf{C}}(A, B)$. *Furthermore,* $\hom_{\mathbf{D}}(0, 0) = \hom_{\mathbf{C}}(0, 0)$.

(c) *For any object* A, $\hom_{\mathbf{C}}(0, A)$ *is the full idempotent reduct of an abelian group such that* $f \in \hom_{\mathbf{D}}(A, B)$ *implies that* $fz \in \hom_{\mathbf{C}}(0, B)$ *provided* $z \in \hom_{\mathbf{C}}(0, A)$.

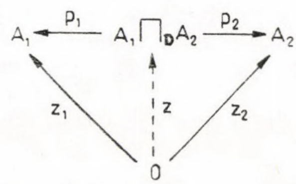
(d) *For any objects* A, B, C *and arrows* $z_i \in \hom_{\mathbf{C}}(0, A)$, $f_i \in \hom_{\mathbf{D}}(A, B)$ $(i=1, 2, 3)$ *and* $g_1 \in \hom_{\mathbf{D}}(B, C)$ *we have the following equalities:*

$$f_1(z_1 \cdot 1 + z_2 \cdot 1 + z_3 \cdot (-1)) = f_1 z_1 \cdot 1 + f_1 z_2 \cdot 1 + f_1 z_3 \cdot (-1),$$

$$(g_1 f_1) z_1 = g_1 (f_1 z_1), \quad (f_1 + f_2 - f_3) z_1 = f_1 z_1 \cdot 1 + f_2 z_1 \cdot 1 + f_3 z_1 \cdot (-1).$$

(e) *If* $A_1 \xleftarrow{p_1} A_1 \sqcap_{\mathbf{D}} A_2 \xrightarrow{p_2} A_2$ *is a product diagram in* $\mathbf{D}$ *then for any arrows* $z_1 \in \hom_{\mathbf{C}}(0, A_1)$ *and* $z_2 \in \hom_{\mathbf{C}}(0, A_2)$ *there exists an arrow* $z \in \hom_{\mathbf{C}}(0, A_1 \sqcap_{\mathbf{D}} A_2)$ *such that the following diagram (in* $\mathbf{C}$*) is commutative:*

(1)



(f) *In particular, if* $z_1 \in \hom_{\mathbf{D}}(0, A_1)$ *and* $z_2 \in \hom_{\mathbf{D}}(0, A_2)$ *then* $z \in \hom_{\mathbf{D}}(0, A_1 \sqcap_{\mathbf{D}} A_2)$ *is the unique arrow in* $\hom_{\mathbf{C}}(0, A_1 \sqcap_{\mathbf{D}} A_2)$ *making the diagram* (1) *commutative.*

(g) *Every element in* $\hom_{\mathbf{C}}(A, B)$ *has the form* $f \cdot 1 + z 0^A \cdot 1 + z' 0^A \cdot (-1)$ *for some* $z, z' \in \hom_{\mathbf{C}}(0, B)$ *and for an* $f \in \hom_{\mathbf{D}}(A, B)$ *where* $\{0^A\} = \hom_{\mathbf{D}}(A, 0)$. *Moreover, two such arrows* $f_1 \cdot 1 + z_1 0^A \cdot 1 + z_1' 0^A \cdot (-1)$ *and* $f_2 \cdot 1 + z_2 0^A \cdot 1 + z_2' 0^A \cdot (-1)$ *coincide if and only if* $f_1 = f_2$ *and* $z_1 = z_1' \cdot 1 + z_2 \cdot 1 + z_2' \cdot (-1)$.

(h) *For any objects* A, B, C *and arrows* $f_i \in \hom_{\mathbf{D}}(A, B)$, $z_i, z_i' \in \hom_{\mathbf{C}}(0, B)$ $(i=1, 2, 3)$, $f_4 \in \hom_{\mathbf{D}}(B, C)$, $z_4, z_4' \in \hom_{\mathbf{C}}(0, C)$ *we have*

(2)
$$\sum_{i=1}^{3} (f_i \cdot 1 + z_i 0^A \cdot 1 + z_i' 0^A \cdot (-1)) \cdot (-1)^{i-1} =$$
$$= (f_1 - f_2 + f_3) \cdot 1 + \left(\sum_{i=1}^{3} z_i \cdot (-1)^{i-1} \right) 0^A \cdot 1 + \left(\sum_{i=1}^{3} z_i' \cdot (-1)^{i-1} \right) 0^A \cdot (-1)$$

and

(3)
$$(f_4 \cdot 1 + z_4 0^B \cdot 1 + z_4' 0^B \cdot (-1))(f_1 \cdot 1 + z_1 0^A \cdot 1 + z_1' 0^A \cdot (-1)) =$$
$$= f_4 f_1 \cdot 1 + (f_4 z_1 \cdot 1 + f_4 z_1' \cdot (-1) + z_4 \cdot 1) 0^A \cdot 1 + z_4' 0^A \cdot (-1).$$

Conversely, if we are given a small additive category $\mathbf{D}$ *and a system of objects* Obj $\mathbf{C}$ *together with hom-sets* $\hom_{\mathbf{C}}(A, B)$ $(A, B \in \text{Obj}\ \mathbf{C})$ *defined by* (a) — (c) *and* (g) *in such a way that they also satisfy conditions* (d) — (f) *then* $\mathbf{C}$ *becomes an A-category with finite products provided we define the fundamental ternary operation on hom-sets by* (2) *and the composition of arrows by* (3).

PROOF. Suppose first that $\mathbf{C}$ is an A-category and consider a choice function Φ picking out an arrow from each hom-set $\hom_{\mathbf{C}}(0, A)$. (In the sequel we will say briefly that Φ is a choice function on $\mathbf{C}$.) If A is an object of $\mathbf{C}$ then the arrow selected from $\hom_{\mathbf{C}}(0, A)$ will be denoted by 0_A^Φ or briefly by 0_A if there is no danger of confusion. Define $\mathbf{C}^\Phi$ to be the subcategory of $\mathbf{C}$ with $\mathrm{Obj}\,\mathbf{C}^\Phi = \mathrm{Obj}\,\mathbf{C}$ and with hom-sets

$$\hom_{\mathbf{C}^\Phi}(A, B) = \{f \mid f \in \hom_{\mathbf{C}}(A, B),\ f0_A^\Phi = 0_B^\Phi\}$$

for any $A, B \in \mathrm{Obj}\,\mathbf{C}^\Phi$.

From now on keep the choice function Φ fixed and set $\mathbf{D} = \mathbf{C}^\Phi$. We prove that $\mathbf{D}$ is an additive category satisfying conditions (a) — (h). Clearly, $\mathbf{D}$ is in fact a category, furthermore, it is easy to verify that for any objects A and B the set $\hom_{\mathbf{D}}(A, B)$ forms an abelian group if we define addition by the following rule:

$$f_1 + f_2 = f_1 \cdot 1 + f_2 \cdot 1 + 0_B 0^A \cdot (-1).$$

Note that $f_1 + f_2 - f_3 = f_1 \cdot 1 + f_2 \cdot 1 + f_3 \cdot (-1)$ holds for any $f_i \in \hom_{\mathbf{D}}(A, B)$ $(i = 1, 2, 3)$. Moreover, for any objects A, B, C, D and arrows $f_1, f_2 \in \hom_{\mathbf{D}}(A, B)$, $g \in \hom_{\mathbf{D}}(B, C)$, $h \in \hom_{\mathbf{D}}(D, A)$ we have

$$g(f_1 + f_2) = gf_1 + gf_2 \quad \text{and} \quad (f_1 + f_2)h = f_1 h + f_2 h.$$

The proof of all these facts is straightforward and is therefore omitted.

Next we show that (g) holds for $\mathbf{D}$. Indeed, take an arbitrary arrow f in $\hom_{\mathbf{C}}(A, B)$. Then, clearly,

$$f = (f \cdot 1 + 0_B 0^A \cdot 1 + f 0_A 0^A \cdot (-1)) \cdot 1 + f 0_A 0^A \cdot 1 + 0_B 0^A \cdot (-1)$$

and here $(f \cdot 1 + 0_B 0^A \cdot 1 + f 0_A 0^A \cdot (-1)) 0_A = 0_B$. To verify the second statement of (g) assume that $f_1 0_A = f_2 0_A = 0_B$ and $z_1, z_2, z_1', z_2' \in \hom_{\mathbf{C}}(0, B)$ such that

$$f_1 \cdot 1 + z_1 0^A \cdot 1 + z_1' 0^A \cdot (-1) = f_2 \cdot 1 + z_2 0^A \cdot 1 + z_2' 0^A \cdot (-1).$$

Then multiplying on the right by 0_A we obtain that

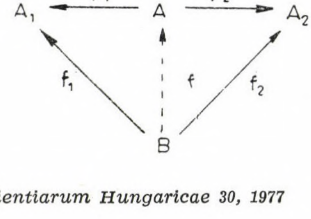
$$0_B \cdot 1 + z_1 \cdot 1 + z_1' \cdot (-1) = 0_B \cdot 1 + z_2 \cdot 1 + z_2' \cdot (-1),$$

i.e. $z_1 = z_1' \cdot 1 + z_2 \cdot 1 + z_2' \cdot (-1)$, which implies also that $f_1 = f_2$.

Since conditions (a)—(d) and (h) obviously hold for $\mathbf{D}$, it remains to prove that if $A_1 \xleftarrow{p_1'} A \xrightarrow{p_2'} A_2$ is a product diagram in $\mathbf{C}$ such that $p_i' = p_i \cdot 1 + z_i 0^A \cdot 1 + z_i' 0^A \cdot (-1)$ with $p_i \in \hom_{\mathbf{D}}(A, A_i)$ then

$$A_1 \xleftarrow{p_1} A \xrightarrow{p_2} A_2 \tag{4}$$

is a product diagram in $\mathbf{C}$ and in $\mathbf{D}$ as well. In fact, since $p_i' = t_i p_i$ where $t_i = 1_{A_i} \cdot 1 + z_i 0^{A_i} \cdot 1 + z_i' 0^{A_i} \cdot (-1)$ is an isomorphism for $i = 1, 2$, (4) is a product diagram in $\mathbf{C}$. To show that (4) is a product diagram in $\mathbf{D}$ it suffices to note that if $f_i \in \hom_{\mathbf{D}}(B, A_i)$ then for the unique arrow f in $\hom_{\mathbf{C}}(B, A)$ making the diagram



commutative we have

$$p_i(f0_B) = f_i 0_B = 0_{A_i} \quad (i = 1, 2).$$

On the other hand,

$$p_i 0_A = 0_{A_i} \quad (i = 1, 2),$$

implying that $f0_B = 0_A$, i.e. $f \in \mathrm{hom}_{\mathbf{D}}(B, A)$.

In particular, we obtain that **D** is a category with finite products and conditions (e) — (f) are fulfilled by **D**. The proof of the first statement of the theorem is complete.

Conversely, suppose that **D** is an additive category and we define the hom-sets $\mathrm{hom}_{\mathbf{C}}(A, B)$ with $A, B \in \mathrm{Obj}\,\mathbf{D}$ by the rules (b)—(c) and (g) such that they satisfy conditions (d) — (f). (Note that we identify $f \in \mathrm{hom}_{\mathbf{D}}(A, B)$ with $f \cdot 1 + z0^A \cdot 1 + z0^A \cdot (-1)$ where $z \in \mathrm{hom}_{\mathbf{C}}(0, B)$).

Applying (d) one can verify by a straightforward computation that the composition of two arrows is well defined by (3) and, moreover, this composition makes the system of objects Obj **D** together with the hom-sets $\mathrm{hom}_{\mathbf{C}}(A, B)$ a category **C**. We show that **C** is an A-category.

To prove that (i) of Definition 1 holds for **C** observe that, on the one hand, if 0 is the zero object in **D** then by (c) $\mathrm{hom}_{\mathbf{C}}(0, A)$ is never void, and, on the other hand, 0 is a terminal object in **C**. Indeed, any arrow f' in $\mathrm{hom}_{\mathbf{C}}(A, 0)$ can be represented by (g) in the form $f \cdot 1 + z0^A \cdot 1 + z'0^A \cdot (-1)$ where $f, 0^A \in \mathrm{hom}_{\mathbf{D}}(A, 0)$ and $z, z' \in \mathrm{hom}_{\mathbf{C}}(0, 0)$. However, (b) implies that $z = z'$, so that $f' = f = 0^A$, i.e. $\mathrm{hom}_{\mathbf{C}}(A, 0)$ is a one-element set, as required.

Turn to prove (ii) of Definition 1. First of all, again an easy computation shows that the fundamental operation is well-defined by (2) on any hom-set $\mathrm{hom}_{\mathbf{C}}(A, B)$. Furthermore, (2) also shows that for any n-ary operation P arising from the fundamental ternary operation by superposition and by collapsing variables and for any arrows $f_i \cdot 1 + z_i 0^A \cdot 1 + z_i' 0^A \cdot (-1)$ $(i = 1, \ldots, n)$ we have

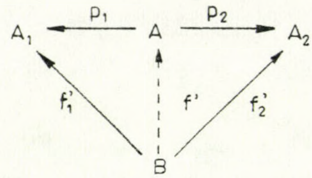
$$P(f_1 \cdot 1 + z_1 0^A \cdot 1 + z_1' 0^A \cdot (-1), \ldots, f_n \cdot 1 + z_n 0^A \cdot 1 + z_n' 0^A \cdot (-1)) =$$
$$= P(f_1, \ldots, f_n) \cdot 1 + P(z_1, \ldots, z_n) 0^A \cdot 1 + P(z_1', \ldots, z_n') 0^A \cdot (-1),$$

implying that any identity holding in the full idempotent reducts of abelian groups holds also in $\mathrm{hom}_{\mathbf{C}}(A, B)$, i.e. $\mathrm{hom}_{\mathbf{C}}(A, B)$ is itself the full idempotent reduct of an abelian group. The "distributivity" rules required in (ii) are also satisfied by **C**. The proof which makes use of conditions (c)—(d) and (h) only is again straightforward and is therefore left to the reader.

Finally, we show that any product diagram $A_1 \xleftarrow{p_1} A \xrightarrow{p_2} A_2$ in **D** is a product diagram in **C**, too. Let us consider first zero arrows $z_i \in \mathrm{hom}_{\mathbf{C}}(0, A_i)$ $(i = 1, 2)$. Then condition (e) implies the existence of an arrow $z \in \mathrm{hom}_{\mathbf{C}}(0, A)$ such that $p_i z = z_i$ $(i = 1,2)$. Assume that $p_i z' = z_i$ $(i = 1,2)$ also holds for another $z' \in \mathrm{hom}_{\mathbf{C}}(0, A)$. Then, by (d),

$$p_i(z \cdot 1 + z' \cdot (-1) + 0_A \cdot 1) = p_i 0_A = 0_{A_i}$$

where 0_A and 0_{A_i} stand for the single arrows in $\mathrm{hom}_{\mathbf{D}}(0, A)$ and $\mathrm{hom}_{\mathbf{D}}(0, A_i)$, resp. Applying condition (f) we obtain that $z \cdot 1 + z' \cdot (-1) + 0_A \cdot 1 = 0_A$, i.e. $z = z'$. Let us consider now the diagram

(5) 

in **C**. Condition (g) implies that f_i' can be uniquely written in the form $f_i'=f_i\cdot 1+ +z_i 0^B\cdot 1+0_{A_i}0^B\cdot(-1)$ where $f_i\in\hom_{\mathbf{D}}(B, A_i)$ and $z_i\in\hom_{\mathbf{C}}(0, A_i)$ $(i=1, 2)$. Suppose that the arrow $f'=f\cdot 1+z0^B\cdot 1+0_A 0^B\cdot(-1)$ with $f\in\hom_{\mathbf{D}}(B, A)$ and $z\in\hom_{\mathbf{C}}(0, A)$ makes the diagram (5) commutative. Then, by (h),

$$f_i\cdot 1+z_i 0^B\cdot 1+0_{A_i}0^B\cdot(-1)=f_i'=p_i f'=p_i f\cdot 1+p_i z0^B\cdot 1+p_i 0_A 0^B\cdot(-1)$$

where $p_i 0_A=0_{A_i}$ $(i=1, 2)$, whence, by (g),

$$f_i=p_i f \quad\text{and}\quad z_i=p_i z \quad (i=1, 2). \tag{6}$$

This implies that f is uniquely determined. On the other hand, if we choose f and z according to (6) then, obviously, $f'=f\cdot 1+z0^B\cdot 1+0_A 0^B\cdot(-1)$ makes the diagram (5) commutative.

This completes the proof of the theorem.

It is clear that translations play an important role in A-categories, namely condition (g) in the previous theorem says that, just as in the categories of affine modules, every arrow in $\hom_{\mathbf{C}}(A, B)$ is the composition of an arrow in $\hom_{\mathbf{D}}(A, B)$ and a translation on B.

The following statement is obvious.

PROPOSITION 2. *Let* **C** *be a small A-category with finite products. Then for any choice functions* Φ *and* Ψ *on* **C** *the additive categories* $\mathbf{C}^{\Phi}$ *and* $\mathbf{C}^{\Psi}$ *are isomorphic. The functor F defined on objects identically and on arrows by*

$$F(f)=f\cdot 1+0^{\Psi}_{\operatorname{cod} f}0^{\operatorname{dom} f}\cdot 1+f0^{\Psi}_{\operatorname{cod} f}0^{\operatorname{dom} f}\cdot(-1)$$

is clearly an isomorphism.

We conclude this section by presenting several examples of A-categories with finite products.

EXAMPLE 0. Obviously, any additive category is an A-category.

EXAMPLE 1. The most natural nontrivial example is, of course, the category **Aff-***R* of all right affine R-modules where R is an arbitrary ring with identity. Similarly, *R*-**Aff** stands for the category of all left affine R-modules. Note that **Aff-***R* is just the category arising from the category **Mod-***R* of all unitary right R-modules by adding the constant mappings $0\to A$ $(0\mapsto a)$ for every $A\in \mathrm{Obj}\,\mathbf{Aff}\text{-}R$ and $a\in A$, and following the construction described in Theorem 1.

EXAMPLE 2. Let **Abp** be the category in which the objects are the abelian groups and the hom-sets are defined for any abelian groups A and B to be the collection of all maps $f+c_b^{A,B}$ where f is an abelian group homomorphism of A into B and $c_b^{A,B}$ is the constant mapping assigning the periodic element b in B to every element in A.

One can prove easily that **Abp** is an A-category. However, this follows also from Theorem 1 if we observe that **Abp** is just the category constructed from the category **Ab** of abelian groups by adding the constant mappings $c_a^{0,A}$ with $a \in A$ periodic. In particular, this implies that **Abp** can be considered as a subcategory of **Aff**-Z where Z stands for the ring of integers.

EXAMPLE 3. Let n be any natural number. $\mathbf{Abp}_n$ will stand for the subcategory of **Abp** with Obj $\mathbf{Abp}_n$=Obj **Abp** and with arrows of the form $f+c_b^{A,B}$ where b is an element of the abelian group B with $nb=0$. It is clear, that $\mathbf{Abp}_n$ is an A-category.

EXAMPLE 4. By an analogous construction we get the subcategory $\mathbf{Abd}_n$ of **Aff**-Z as follows: Obj $\mathbf{Abd}_n$ is the class of all abelian groups and the arrows are the maps of the form $f+c_b^{A,B}$ where $f: A \to B$ is a group homomorphism and b is an element in B such that there exists a $b' \in B$ with $nb'=b$.

3. Products and coproducts. From now on in this section and in the next one, **C** will stand for a fixed A-category. For convenience, **C** will be supposed to be small.

We have seen in Theorem 1 that finite products in **C** and $\mathbf{C}^{\Phi}$ essentially coincide provided **C** is with finite products. Moreover, a similar argument shows that this holds also for all products existing in **C** (even if **C** is not with finite products). However, this is not the case with coproducts. In particular, it is easy to see that $0 \cong 0 \sqcup 0$ holds in **C** if and only if each $\hom_{\mathbf{C}}(0, A)$ is a one-element set, i.e. **C** is a preadditive category.

The following propositions generalize the inner characterization of product diagrams and coproduct diagrams in preadditive categories, respectively.

PROPOSITION 3. *Let* $n(\geqq 1)$ *be a natural number. Then*

$$A \xrightarrow{p_i} A_i \quad (i = 1, \dots, n)$$

is a product diagram in **C** *if and only if for any choice of arrows* $z_i \in \hom_{\mathbf{C}}(0, A_i)$ $(i=1, \dots, n)$ *there exist arrows* $u_i \in \hom_{\mathbf{C}}(A_i, A)$ *and* $z \in \hom_{\mathbf{C}}(0, A)$ *such that*

(i) $p_j u_i = z_j 0^{A_i}$ *if* $i \neq j$ *and* $p_i u_i = 1_{A_i}$ $(i, j = 1, \dots, n)$,

(ii) $p_i z = z_i$ $(i = 1, \dots, n)$,

(iii) $\sum_{k=1}^{n} u_k p_k \cdot 1 + z 0^A \cdot (-n+1) = 1_A$.

PROOF. One has to repeat the standard reasoning used in preadditive categories.

PROPOSITION 4. *Let* $n(\geqq 1)$ *be a natural number. Then*

$$A_i \xrightarrow{v_i} A \quad (i = 1, \dots, n)$$

is a coproduct diagram in **C** *if and only if whenever we fix arrows* $z_i \in \hom_{\mathbf{C}}(0, A_i)$, *the following conditions are satisfied:*

(i) *There exist arrows* $q_i \in \hom_{\mathbf{C}}(A, A_i)$ *such that*

$$q_j v_i = z_j 0^{A_i} \quad \text{if} \quad i \neq j \quad \text{and} \quad q_i v_i = 1_{A_i} \quad (i, j = 1, \dots, n).$$

(ii) *For any object* B *and any arrows* $z_i' \in \hom_{\mathbf{C}}(0, B)$ *there exists a unique arrow* $f \in \hom_{\mathbf{C}}(A, B)$ *such that*

$$f v_i = z_i' 0^{A_i} \quad (i = 1, \dots, n).$$

Denoting by g the arrow corresponding to the system $z_i' = v_i z_i \in \hom_{\mathbf{C}}(0, A)$ *we have*

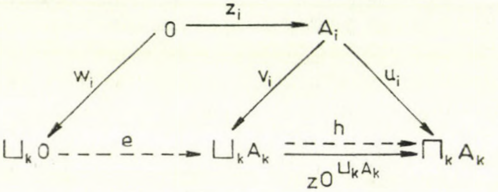
(iii) $\sum_{k=1}^{n} v_k q_k \cdot 1 + \sum_{k=1}^{n} v_k z_k 0^A \cdot (-1) + g \cdot 1 = 1_A$.

The proof is again standard and therefore omitted.

We remark that both of the propositions remain valid if we require only that the conditions be satisfied for a particular choice of the arrows z_i $(i=1, \dots, n)$. This follows immediately from Proposition 2.

It is well known that in preadditive categories the product and the coproduct of any finite family of objects are isomorphic whenever they exist. The following theorem is a generalization of this fact, since in preadditive categories $0 \sqcup 0 \cong 0$.

THEOREM 5. *Let n be a natural number and suppose that* $\sqcap_k A_k$, $\sqcup_k A_k$ *and* $\sqcup_k 0$ $(1 \leq k \leq n)$ *exist in* **C**. *Let us consider the following diagram:*



$$(i = 1, \dots, n)$$

where $z_i \in \hom_{\mathbf{C}}(0, A_i)$ *are arbitrary arrows,* w_i *and* v_i *are the injections corresponding to the coproducts and, finally,* u_i *and* z *are the arrows constructed in Proposition* 3. *Then there exists a unique arrow* $h \in \hom_{\mathbf{C}}(\sqcup_k A_k, \sqcap_k A_k)$ *which makes the upper triangle on the right commutative, and this arrow has a right inverse and is therefore epi. Moreover, the unique arrow e making the parallelogram on the left commutative is the equalizer of h and* $z 0^{\sqcup_k A_k}$.

PROOF. We claim that for any j $(j=1, \dots, n)$

$$f_j = \sum_{k=1}^{n} v_k p_k \cdot 1 + \sum_{\substack{k=1 \\ k \neq j}}^{n} v_k z_k 0^{\sqcap_k A_k} \cdot (-1)$$

is a right inverse of h. Indeed, since

$$p_i h v_k = \begin{cases} 1_{A_i}, & \text{if } \ i = k \\ z_i 0^{A_k}, & \text{otherwise,} \end{cases}$$

therefore $p_i h f_j = p_i$ for every $i=1, \dots, n$. Hence by uniqueness it follows that $h f_j = 1_{\sqcap_k A_k}$, as required.

Before we turn to prove that e is an equalizer we show that for any $i=1, \dots, n$ we have $u_i z_i = z$. To this end it suffices to note that, on the one hand, by (i) of Proposition 3 we have for any $j=1, \dots, n$ that $p_j(u_i z_i) = z_j$, on the other hand, (ii) of Proposition 3 requires that $p_j z = z_j$ $(j=1, \dots, n)$.

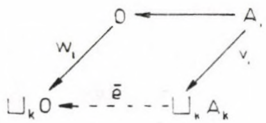
Since

$$(he) w_i = h v_i z_i = u_i z_i = z \quad \text{and} \quad (z 0^{\sqcup_k 0}) w_i = z \quad (i = 1, \dots, n),$$

uniqueness immediately implies that

$$he = z0^{\sqcup_k 0} = (z0^{\sqcup_k A_k})e,$$

as desired. Consider now an object X and an arrow $f \in \hom_{\mathbf{C}}(X, \sqcup_k A_k)$ such that $hf = (z0^{\sqcup_k A_k})f$, i.e. $hf = z0^X$. Let $\bar{e}$ denote the unique arrow making the diagram below commutative.



$$(i = 1, \ldots, n).$$

Observe on the one hand, that $(\bar{e}e)w_i = \bar{e}v_i z_i = w_i 0^{A_i} z_i = w_i$ $(i=1, \ldots, n)$ implying that $\bar{e}e = 1_{\sqcup_k 0}$. On the other hand, $(e\bar{e})v_i = ew_i 0^{A_i} = (v_i z_i)0^{A_i}$ $(i=1, \ldots, n)$ whence $e\bar{e} = g$, the arrow defined in Proposition 4. Making use of (iii) in Proposition 4 we obtain that

$$hf = h1_{\sqcup_k A_k} f = \sum_{k=1}^{n} u_k q_k f \cdot 1 + \sum_{k=1}^{n} u_k z_k 0^X \cdot (-1) + he\bar{e}f \cdot 1$$

where $u_k z_k = z$ $(k=1, \ldots, n)$ and $he = z0^{\sqcup_k 0}$. Thus

$$z0^X = hf = \sum_{k=1}^{n} u_k q_k f \cdot 1 + z0^X \cdot (-n+1).$$

Multiplying both sides by p_i $(i=1, \ldots, n)$ on the left and applying again (i)—(ii) of Proposition 3 we have

$$z_i 0^X = q_i f \quad (i = 1, \ldots, n).$$

Hence

$$f = \left(\sum_{k=1}^{n} v_k q_k \cdot 1 + \sum_{k=1}^{n} v_k z_k 0^{\sqcup_k A_k} \cdot (-1) + e\bar{e} \cdot 1 \right) f = e(\bar{e}f),$$

i.e. f factors through e. This factorization is unique for $\bar{e}$ is a left inverse of e.

The proof is complete.

4. Kernels and cokernels. Let us have a look at the category **Aff-R** from the point of view of kernels. It is clear that whenever we consider a homomorphism and the natural congruence induced by it then any congruence class is an affine R-module, furthermore, these submodules are isomorphic, more precisely, there exist translations that map a congruence class onto another. Thus we have a kernel system consisting of monomorphisms instead of a kernel. Generally, a kernel system can be defined in an A-category **C** as follows:

Definition 2. Let $f \in \hom_{\mathbf{C}}(A, B)$ be an arbitrary arrow. We say that the object K is a *kernel object* of f and $\ker f \subseteq \hom_{\mathbf{C}}(K, A)$ is a *kernel system* of f if the following conditions are satisfied:

(i) For any k in $\ker f$ the composition fk is a zero arrow.

(ii) For any $k, k' \in \ker f$ there exist arrows $z, z' \in \hom_{\mathbf{C}}(0, A)$ such that $k' = k \cdot 1 + z0^K \cdot 1 + z'0^K \cdot (-1)$ (i.e. $k' \in \operatorname{tr} k$).

(iii) Whenever $c \in \hom_{\mathbf{C}}(C, A)$ is an arrow such that fc is a zero arrow then there exists a unique arrow $k \in \ker f$ such that c factors through k, moreover this factorization is unique.

Obviously, if $\ker f$ exists then every arrow in it is monic.

In what follows we always assume that $\mathbf{C}$ is a small A-category with finite products.

THEOREM 6. *Let $f \in \hom_{\mathbf{C}}(A, B)$ and $z \in \hom_{\mathbf{C}}(0, A)$. Let us consider a choice function Φ on $\mathbf{C}$ such that $0_A^\Phi = z$ and $0_B^\Phi = fz$. Then $\ker f$ exists in $\mathbf{C}$ if and only if f has a kernel in $\mathbf{C}^\Phi$. Furthermore, if $k_0 \in \hom_{\mathbf{C}^\Phi}(K, A)$ is a kernel of f in $\mathbf{C}^\Phi$ then the relation $\sim$ defined on $\operatorname{tr} k_0$ by $k_1 \sim k_2$ if and only if there exist arrows $z_1, z_2 \in \hom_{\mathbf{C}}(0, K)$ such that $k_1 z_1 = k_2 z_2$, is an equivalence relation and every representation system for this equivalence relation is a kernel system of f.*

The following lemma will be useful in the proof.

LEMMA 7. *Given $k_1, k_2 \in \operatorname{tr} k_0$ we have $k_1 \sim k_2$ if and only if there exists a translation $t \in \operatorname{tr} 1_K$ such that $k_2 = k_1 t$.*

PROOF. Sufficiency follows immediately if we multiply the equality $k_2 = k_1 t$ by a zero arrow $z \in \hom_{\mathbf{C}}(0, K)$ on the right. Conversely, suppose that $k_1 z_1 = k_2 z_2$ where $z_1, z_2 \in \hom_{\mathbf{C}}(0, K)$. Since $k_1, k_2 \in \operatorname{tr} k_0$ implying that $k_2 \in \operatorname{tr} k_1$, we have

$$k_2 = k_1 \cdot 1 + k_2 z_2 0^K \cdot 1 + k_1 z_2 0^K \cdot (-1).$$

Consequently,

$$k_2 = k_1 (1_K \cdot 1 + z_1 0^K \cdot 1 + z_2 0^K \cdot (-1)),$$

which was to be proved.

PROOF OF THEOREM 6. First we prove necessity. Choose by (iii) of Definition 2 k to be the unique arrow such that 0_A^Φ factors through k and let $0_A^\Phi = kz'$ $(z' \in \hom_{\mathbf{C}}(0, K))$. We are going to show that $k_0 = k \cdot 1 + 0_A^\Phi 0^K \cdot 1 + k 0_K^\Phi 0^K \cdot (-1)$ is a kernel of f in $\mathbf{C}^\Phi$. It is clear that k_0 is an arrow in $\mathbf{C}^\Phi$ and $fk_0 = 0_B^\Phi 0^K$.

Now let $c \in \hom_{\mathbf{C}^\Phi}(C, A)$ be an arrow such that $fc = 0_B^\Phi 0^C$. Then by (iii) of Definition 2 there exists an arrow k' in $\ker f$ such that c factors through k', i.e. $c = k'x$ $(x \in \hom_{\mathbf{C}}(C, K))$. However,

$$k' x 0_C^\Phi = c 0_C^\Phi = 0_A^\Phi = kz'$$

whence by Lemma 7 k' factors through k implying that $k' = k$. Applying again (iii) of Definition 2 we obtain that c factors uniquely through k_0, since $kz' = 0_A^\Phi = k_0 0_K^\Phi$ and thus by Lemma 7 there exists a translation $t \in \operatorname{tr} 1_K$ with $k_0 = kt$. Moreover, if $c = k_0 x$, then $k_0(x 0_C^\Phi) = c 0_C^\Phi = 0_A^\Phi = k_0 0_K^\Phi$ and therefore $x 0_C^\Phi = 0_K^\Phi$, i.e. $x \in \hom_{\mathbf{C}^\Phi}(C, K)$. This completes the proof of necessity.

Sufficiency will be proved by showing that the system $\ker f$ of arrows described in the theorem is in fact a kernel system of f. Lemma 7 together with the fact that $\operatorname{tr} 1_K$ is a group immediately implies that $\sim$ is an equivalence relation. All that needs proof is that (iii) of Definition 2 is satisfied. Let $c \in \hom_{\mathbf{C}}(C, A)$ such that fc is a zero arrow and suppose that there exist arrows $x_1, x_2 \in \hom_{\mathbf{C}}(C, K)$ such that $c = k_1 x = k_2 x_2$ where $k_1, k_2 \in \ker f$. Then $k_1(x_1 0_C^\Phi) = k_2(x_2 0_C^\Phi)$ so that $k_1 \sim k_2$, implying $k_1 = k_2$.

Let $c_0 = c \cdot 1 + 0_A^\Phi 0^C \cdot 1 + c 0_C^\Phi 0^C \cdot (-1)$. Clearly, c_0 is an arrow in $\mathbf{C}^\Phi$ and $fc_0 = 0_B^\Phi$, thus c_0 has a unique factorization through k_0. Hence c has a unique factorization through $k = k_0 \cdot 1 + c 0_C^\Phi 0^K \cdot 1 + 0_A^\Phi 0^K \cdot (-1)$ and therefore through the element of $\ker f$ contained in the same $\sim$-class as k.

The proof of the theorem is complete.

The definition of the kernel system immediately implies

PROPOSITION 8. *Let $f \in \hom_{\mathbf{C}}(A, B)$. The kernel object of f and the kernel system of f is uniquely determined up to isomorphism in the sense that if $\ker f \subseteq \hom_{\mathbf{C}}(K, A)$ and $\ker' f \subseteq \hom_{\mathbf{C}}(K', A)$ are kernel systems then there exists an isomorphism $i \in \hom_{\mathbf{C}}(K, K')$ and a one-to-one correspondence $\varphi : \ker' f \to \ker f$ such that for any $k \in \ker' f$ we have $k\varphi \sim ki$.*

In defining the cokernel of an arrow we have no difficulties.

DEFINITION 3. Let $f \in \hom_{\mathbf{C}}(A, B)$. The *cokernel* of f is defined to be the coequalizer of f and $fz0^A$ where $z \in \hom_{\mathbf{C}}(0, A)$ and is denoted by $\operatorname{coker} f$.

It is easy to see that $\operatorname{coker} f$ is independent of the choice of z. In fact, more generally, for any $z_1, z_2 \in \hom_{\mathbf{C}}(0, A)$ and $u \in \hom_{\mathbf{C}}(B, C)$ $uf = u(fz_1 0^A)$ holds if and only if $uf = u(fz_2 0^A)$. Indeed, a short computation shows that the first equality implies that

$$ufz_2 0^A = ufz_1 0^A z_2 0^A = ufz_1 0^A = uf.$$

Clearly, $\operatorname{coker} f$ is epi and it is uniquely determined up to isomorphism.

THEOREM 9. *Let $f \in \hom_{\mathbf{C}}(A, B)$ and $z \in \hom_{\mathbf{C}}(0, A)$. Let us consider a choice function Φ on $\mathbf{C}$ such that $0_A^\Phi = z$ and $0_B^\Phi = fz$. Then $\operatorname{coker} f$ exists in $\mathbf{C}$ if and only if f has a cokernel in $\mathbf{C}^\Phi$.*

PROOF. The basic idea of the proof is similar to that of Theorem 6. However, the argument is much simpler and is therefore omitted.

One can naturally ask whether we could have defined the kernel system of an arrow $f \in \hom_{\mathbf{C}}(A, B)$, analogously to Definition 3, to be the set of equalizers of the couples $f, fz0^A$ where z runs over the zero arrows in $\hom_{\mathbf{C}}(0, A)$. Clearly, this depends only on whether these equalizers exists at all or not. Observe that the answer is positive in **Aff**-R and in Examples 2 and 3. However, Example 4 shows that the answer is, in general, negative. All these statements follow immediately by applying the following result.

THEOREM 10. *Given $f \in \hom_{\mathbf{C}}(A, B)$ and $w \in \hom_{\mathbf{C}}(0, B)$ the equalizer of f and $w0^A$ exists if and only if the following conditions hold:*

(i) *There exists an arrow $z \in \hom_{\mathbf{C}}(0, A)$ with $w = fz$.*

(ii) *f has a kernel system (with kernel object K).*

(iii) *There exists an arrow $k \in \ker f$ with $fk = w0^K$ such that for any $z' \in \hom_{\mathbf{C}}(0, A)$ with $fz' = w$ there exists an arrow $v \in \hom_{\mathbf{C}}(0, K)$ with $kv = z'$.*

The arrow k is uniquely determined by (iii) *and is an equalizer of f and $w0^A$.*

PROOF. Let $e \in \hom_{\mathbf{C}}(E, A)$ be an equalizer of f and $w0^A$. Then for a $u \in \hom_{\mathbf{C}}(0, E)$ we have $feu = w0^A u = w$, hence (i) holds. It is clear that e is the kernel of f in $\mathbf{C}^\Phi$ provided $0_A^\Phi = z$ and $0_B^\Phi = w$. Thus (ii) follows by applying Theorem 6. To prove (iii), observe that fe is a zero arrow and therefore there exists a unique

$k \in \ker f$ such that e factors through k, say $kx=e$. Since fk is a zero arrow, $fk=fkxu0^K=feu0^K=w0^K$. Take now an arbitrary arrow z' in $\hom_{\mathbf{C}}(0, A)$ with $fz'=w$. Since $fz'=w=(w0^A)z'$ and e is an equalizer, z' factors through e, i.e. $z'=ey$ for a $y \in \hom_{\mathbf{C}}(0, E)$. Therefore $z'=k(xy)$, as desired.

Conversely, suppose that conditions (i)—(iii) are satisfied. Observe first that if $k' \in \ker f$ such that $fk'=w0^K$, then $k' \sim k$, i.e. $k'=k$. Indeed, let $v' \in \hom_{\mathbf{C}}(0, K)$. Then $k'v' \in \hom_{\mathbf{C}}(0, A)$ and $f(k'v')=w$. Thus by (iii) $k'v'$ factors through k whence $k' \sim k$, as required.

Now it is easy to show that k is an equalizer of f and $w0^A$. Let $c \in \hom_{\mathbf{C}}(C, A)$ be an arrow with $fc=(w0^A)c$ $(=w0^C)$. Since k is monic it suffices to show that c factors through k. The fact that fc is a zero arrow implies the existence of an arrow $k' \in \ker f$ such that c factors through k', asy $c=k'd$. On the other hand, fk' is a zero arrow, hence $fk'=fk'du'0^K=fcu'0^K=w0^K$ where $u' \in \hom_{\mathbf{C}}(0, C)$ is arbitrary. The observation made above implies $k'=k$, which completes the proof of the theorem.

Theorems 6 and 9 show that there is a close connection between kernels [resp. cokernels] in $\mathbf{C}$ and in $\mathbf{C}^{\Phi}$ and suggest also that all concepts and basic theorems well-known in additive or abelian categories can be carried over with minor changes to A-categories. We mention here only those ones which will be used in the next section.

We suppose in the sequel that $\mathbf{C}$ is a special A-category, a so called affine category meaning that $\mathbf{C}^{\Phi}$ is abelian. Equivalently,

Definition 4. An A-category $\mathbf{C}$ with finite products is called an *affine category* if the following conditions hold:

(i) Every arrow in $\mathbf{C}$ has a kernel system and a cokernel.

(ii) Every epi is a cokernel of an arrow in $\mathbf{C}$ and every monic is contained in a kernel system of an arrow in $\mathbf{C}$.

It is easy to see that Examples 1—4 are all affine categories.

Naturally, for any arrow f in $\mathbf{C}$ $\operatorname{im} f$ is defined to be the kernel system of $\operatorname{coker} f$ while $\operatorname{coim} f$ means the cokernel of k where $k \in \ker f$. Observe that $\operatorname{coim} f$ does not depend on the choice of k in $\ker f$. Indeed, if $k_1, k_2 \in \hom_{\mathbf{C}}(K, A)$ and $k_2=tk_1$ with $t \in \operatorname{tr} 1_A$ then it is easy to verify that $\operatorname{coker} k_1=\operatorname{coker} k_2$. The proof is left to the reader.

The canonical factorization theorem reads as follows: Let f be an arbitrary arrow in $\mathbf{C}$. Then the image and coimage objects of f are isomorphic, one can suppose without loss of generality that they coincide, and there exists a unique arrow $k \in \operatorname{im} f$ such that f factors through k and $f=k(\operatorname{coim} f)$. This arrow will be denoted by $\operatorname{im}_0 f$.

A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbf{C}$ is called exact at B if $\operatorname{im} f=\ker g$ or, equivalently, $\operatorname{coker} f=\operatorname{coim} g$. Note that this holds if and only if the same sequence is exact in $\mathbf{C}^{\Phi}$ where $0_B^{\Phi}=f0_A^{\Phi}$ and $0_C^{\Phi}=gf0_A^{\Phi}$. This implies immediately that the well-known theorems concerning the exactness of certain commutative diagrams, in particular the 3×3 Lemma, hold in affine categories (even in large ones).

5. Torsion theories. In this section we characterize the torsion and the torsion free classes in Dickson's sense in certain affine categories, or in Wiegandt's terminology, the radical classes and the semisimple classes in these categories.

Let $\mathbf{C}$ be an affine category. The definition of a torsion theory in an abelian category (see [2]) can be literally carried over to $\mathbf{C}$. Namely, a pair $(\mathbf{T}, \mathbf{F})$ of classes of objects of $\mathbf{C}$ is said to be a *torsion theory* if the following conditions hold.

(I) $\mathbf{T} \cap \mathbf{F} = \{0\}$.
(II) If $T \to A \to 0$ is exact with $T \in \mathbf{T}$ then $A \in \mathbf{T}$.
(III) If $0 \to A \to F$ is exact with $F \in \mathbf{F}$ then $A \in \mathbf{F}$.
(IV) For each object X of $\mathbf{C}$ there is an exact sequence

$$0 \to T \to X \to F \to 0$$

with $T \in \mathbf{T}$ and $F \in \mathbf{F}$.

$\mathbf{T}$ will be called a *torsion class* and $\mathbf{F}$ a *torsion free class*. It is easy to verify that assuming the validity of (IV), conditions (I)—(III) are equivalent to

(V) $\mathbf{T}$ and $\mathbf{F}$ are closed under isomorphisms and for any $T \in \mathbf{T}$ and $F \in \mathbf{F}$ $\hom_{\mathbf{C}}(T, F)$ contains zero arrows only.

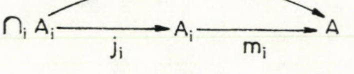
Let us introduce the following relations between the objects of $\mathbf{C}$: $A \prec B$ will mean that $\hom_{\mathbf{C}}(A, B)$ contains a nonzero monic while $A \mapsto B$ will mean that $\hom_{\mathbf{C}}(A, B)$ contains a non-zero epi. A class $\mathbf{R}$ of objects of $\mathbf{C}$ will be called a *radical class* if $A \in \mathbf{R}$ holds if and only if for any $B \in \mathbf{C}$ with $A \mapsto B$ there exists a $C \in \mathbf{R}$ such that $C \prec B$. Dually, a class $\mathbf{S}$ of objects of $\mathbf{C}$ is called a *semisimple class* if $A \in \mathbf{S}$ holds if and only if for any $B \in \mathbf{C}$ with $B \prec A$ there exists a $C \in \mathbf{S}$ such that $B \mapsto C$.

The definition of a radical class and a semisimple class in general was introduced in [5].

Observe that by the definitions it follows easily that a class $\mathbf{K}$ of objects of an affine category $\mathbf{C}$ is a torsion class [torsion free class, radical class, semisimple class] in $\mathbf{C}$ if and only if $\mathbf{K}$ is a torsion class [torsion free class, radical class, semisimple class, respectively] in $\mathbf{C}^{\Phi}$ where Φ is an arbitrary choice function on $\mathbf{C}$.

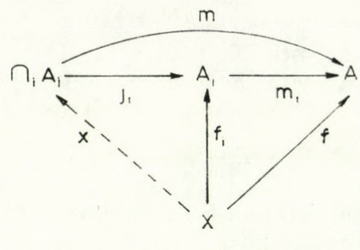
We are going to give an inner characterization, using closure properties only, of torsion classes, torsion free classes, radical classes and semisimple classes in certain affine categories. Taking into consideration the observation made above it is not surprising that our characterization is very similar to that of Dickson [2]. However, we impose much weaker conditions on the category $\mathbf{C}$ than he did.

In order to be able to describe these conditions we need some concepts and notations. The subobjects of an object A in $\mathbf{C}$ will be represented by couples of the form $\langle A', m\rangle$ where m is a monic in $\hom_{\mathbf{C}}(A', A)$. Further, if $\langle A', m\rangle$ is a subobject of A then the cokernel object of m will be denoted by A/A'. Let $\{\langle A_i, m_i\rangle | i \in I\}$ be a set of subobjects of A. We say that they have an element in common, in notation: $\bigcap_i A_i \neq \emptyset$ if there exist arrows $z_i \in \hom_{\mathbf{C}}(0, A_i)$ and $z \in \hom_{\mathbf{C}}(0, A)$ such that $m_i z_i = z$ $(i \in I)$. The intersection of the subobjects $\langle A_i, m_i\rangle$ of A is defined to be the subobject $\langle \bigcap_i A_i, m\rangle$ if, on the one hand, there exist monics $j_i \in \hom_{\mathbf{C}}(\bigcap_i A_i, A_i)$ $(i \in I)$ making the diagram



(7) $(i \in I)$

commutative and, on the other hand, whenever $\langle X, f_i\rangle$ is a subobject of A_i with $m_i f_i = f$ for all $i \in I$ in such a way that there are arrows $u \in \hom_{\mathbf{C}}(0, \bigcap_i A_i)$ and $v \in \hom_{\mathbf{C}}(0, X)$ with $mu = fv$ then there exists a monic x making the diagram

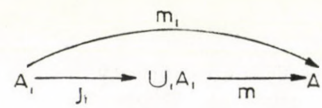


$(i \in I)$

commutative. (x is clearly unique.) Obviously, if $\bigcap_i A_i$ exists in $\mathbf{C}$ then $\bigcap_i A_i \neq \emptyset$.

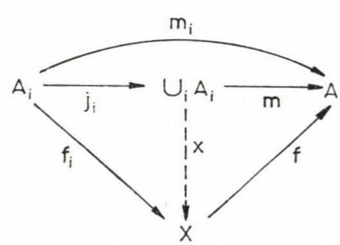
Note that the subobject $\langle \bigcap_i A_i, m \rangle$ of A is the intersection of the subobjects $\langle A_i, m_i \rangle$ $(i \in I)$ of A in $\mathbf{C}$ if and only if $\langle \bigcap_i A_i, m \rangle$ is the greatest lower bound (the intersection in the usual sense) of the subobjects $\langle A_i, m_i \rangle$ $(i \in I)$ of A in $\mathbf{C}^{\Phi}$ for a choice function (and therefore by Proposition 2 for every choice function) Φ with $0^{\Phi}_{\bigcap_i A_i} = u \in \hom_{\mathbf{C}}(0, \bigcap_i A_i)$, $0^{\Phi}_{A_i} = j_i u$ and $0^{\Phi}_A = mu$.

As usual, the union of the subobjects $\langle A_i, m_i \rangle$ of A is their least upper bound, i.e. it is defined to be the subobject $\langle \bigcup_i A_i, m \rangle$ of A if, on the one hand, there exist monics $j_i \in \hom_{\mathbf{C}}(A_i, \bigcup_i A_i)$ making the diagram



(8) $(i \in I)$

commutative and, on the other hand, whenever $\langle X, f \rangle$ is a subobject of A such that for each $i \in I$ $\langle A_i, f_i \rangle$ is a subobject of X with $ff_i = m_i$, then there exists a monic $x \in \hom_{\mathbf{C}}(\bigcup_i A_i, X)$ such that $fx = m$. This implies in particular that x is uniquely determined and the following diagram is commutative:



$(i \in I)$

It is clear that both $\bigcap_i A_i$ and $\bigcup_i A_i$ are uniquely determined up to isomorphism.

An affine category $\mathbf{C}$ will be called *chain-$\bigcup$-subcomplete* if for any object A every chain $\{\langle A_i, m_i \rangle | i \in I\}$ (i.e. a set ordered under the natural partial order of subobjects) of subobjects of A possesses a union. Similarly, $\mathbf{C}$ is called *chain-$\bigcap$-subcomplete* if any chain $\{\langle A_i, m_i \rangle | i \in I\}$ of subobjects of an arbitrary object A in $\mathbf{C}$

with $\bigcap_i A_i \neq \emptyset$ has an intersection in **C**. Examples 1—4 are obviously both chain-$\bigcup$- and chain-$\bigcap$-subcomplete.

Furthermore, according to the terminology of Dickson, **C** is called *subcomplete* if for any set $\{\langle A_i, m_i\rangle | i \in I\}$ of subobjects of an arbitrary object A the coproduct $\sqcup_i A_i$ and the product $\sqcap_i (A/A_i)$ exist in **C**. If the existence of the latter is required only then **C** is called *semisubcomplete*. It is clear that subcompleteness implies both chain-$\bigcup$- and chain-$\bigcap$-subcompleteness. In particular, semisubcompleteness implies the latter. Note that semisubcompleteness is a rather strong condition on affine categories, since, for instance, Example 2 fails to be semisubcomplete.

THEOREM 11. *Given a chain-$\bigcap$-subcomplete affine category* **C** *and a subclass* **F** *of objects of* **C** *the following conditions are equivalent.*

(i) **F** *is a torsion free class in* **C**.

(ii) **F** *is a semisimple class in* **C**.

(iii) **F** *is closed under subobjects and extensions and, furthermore, whenever* $\{\langle A_i, m_i\rangle | i \in I\}$ *is a chain of subobjects of* A ($\in \mathrm{Obj}\,\mathbf{C}$) *such that* $\bigcap_i A_i \neq \emptyset$ *and* $A/A_i \in \mathbf{F}$ $(i \in I)$ *then* $A/\bigcap_i A_i \in \mathbf{F}$.

If, moreover, **C** *is semisubcomplete then* (i)—(iii) *are equivalent to*

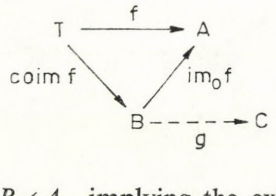
(iv) **F** *is closed under subobjects, extensions and infinite products.*

PROOF. First we show that (i) implies (ii). Let **T** be the torsion class corresponding to **F**, i.e. (**T**, **F**) is a torsion theory in **C**. Observe that **F** is complete with respect to (V). Indeed, let $X \in \mathbf{C}$ be an object such that for any $T \in \mathbf{T}$ $\hom_{\mathbf{C}}(T, X)$ contains zero arrows only. Since by (IV) there exists an exact sequence

$$0 \to T \to X \to F \to 0$$

and $T \to X$ is a zero arrow, therefore X is isomorphic to F, hence $X \in \mathbf{F}$, as was to be proved. A similar argument shows that **T** is also complete with respect to (V).

Suppose now that $A \in \mathbf{F}$. If $A \cong 0$ we have nothing to prove. Let $A \in \mathbf{F} \setminus \{0\}$. If $B \prec A$ then by (III) $B \in \mathbf{F} \setminus \{0\}$ and thus $B \mapsto B$. Conversely, suppose that $A \in \mathbf{C} \setminus \{0\}$ and for any $B \in \mathbf{C}$ with $B \prec A$ there exists a $C \in \mathbf{F}$ such that $B \mapsto C$. Consider an arbitrary arrow $f \in \hom_{\mathbf{C}}(T, A)$, $T \in \mathbf{T}$. Let B be the image object of f and take the canonical factorization of f:

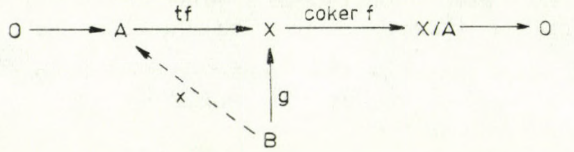


Then $B \cong 0$, for, otherwise $B \prec A$, implying the existence of a nonzero epi $g \in \hom_{\mathbf{C}}(B, C)$ such that $C \in \mathbf{F}$ and thus $g(\mathrm{coim}\, f)$ is a zero arrow, which is impossible. Thus f is necessarily a zero arrow. Therefore, by the completeness of **F**, $A \in \mathbf{F}$, as desired.

We turn to prove that (ii) implies (iii). The fact that **F** is closed under subobjects is trivial. Let us consider an exact sequence

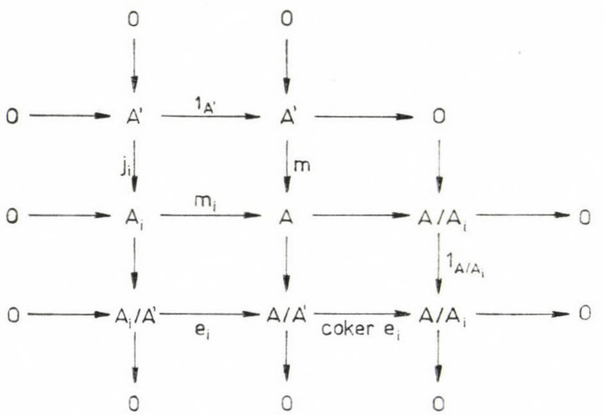
$$0 \longrightarrow A \xrightarrow{f} X \xrightarrow{\mathrm{coker} f} X/A \longrightarrow 0$$

in **C** such that $A, X/A \in \mathbf{F}$. We are going to prove that $X \in \mathbf{F}$. If any one of A and X/A is isomorphic to 0 then this is obvious. Suppose therefore that $A, X/A \in \mathbf{F} \setminus \{0\}$. Consider an object $B \in \mathbf{C}$ with $B \prec X$, i.e. $\langle B, g\rangle$ is a subobject of X and $B \not\cong 0$. If $(\operatorname{coker} f)g$ is not a zero arrow then the coimage object C of it is not isomorphic to 0, consequently $B \mapsto C$, where C being a subobject of X/A is contained in **F**. If in turn $(\operatorname{coker} f)g$ is a zero arrow then g factors through tf where t is a suitable translation in tr 1_X:



x is obviously monic, so that $B \mapsto B$ with $B \in \mathbf{F}$. Hence by the definition of a semi-simple class $X \in \mathbf{F}$, which is what we had to prove.

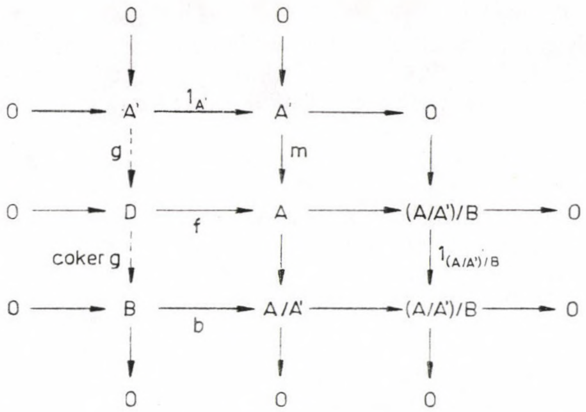
Instead of the third closure property we prove a stronger statement, namely we show that whenever $\{\langle A_i, m_i\rangle | i \in I\}$ is a set of subobjects of A such that $\bigcap_i A_i$ exists and $A/A_i \in \mathbf{F}$ $(i \in I)$, then $A/\bigcap_i A_i \in \mathbf{F}$. We use the notations of diagram (7). For brevity we write A' for $\bigcap_i A_i$. Let us take a choice function Φ on **C** such that $m0^{\Phi}_{A'} = 0^{\Phi}_A$ and for all $i \in I$ $j_i 0^{\Phi}_{A'} = 0^{\Phi}_{A_i}$. (If **C** is a large category then in order to avoid confusion we should consider a small full chain-$\bigcap$-subcomplete affine subcategory of **C** containing A, A' and all A_i $(i \in I)$.) Clearly, we have an exact commutative diagram in $\mathbf{C}^{\Phi}$:

(9)  $(i \in I)$.

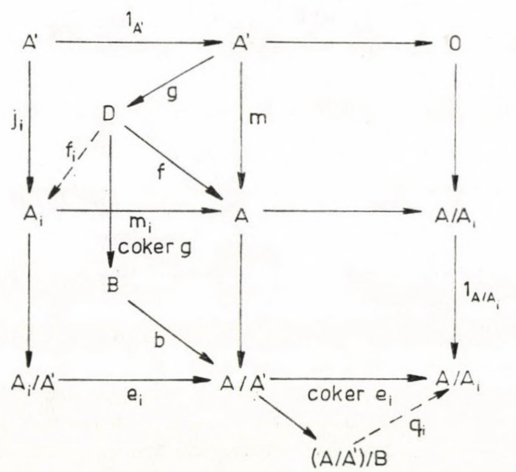
Let us consider a subobject $\langle B, b\rangle$ of A/A' with $B \not\cong 0$, i.e. $B \prec A/A'$. We have to show the existence of an object $C \in \mathbf{F}$ with $B \mapsto C$. This is clear if there exists an $i \in I$ such that $(\operatorname{coker} e_i)b$ is not a zero arrow. In fact, in this case the image object of $(\operatorname{coker} e_i)b$ being a nonzero subobject of $A/A_i \in \mathbf{F}$ is suitable.

It remains to show that if every arrow $(\operatorname{coker} e_i)b$ is a zero arrow then necessarily $B \cong 0$. Obviously, we can suppose without loss of generality that b is an arrow in $\mathbf{C}^{\Phi}$. It is easy to see that there are uniquely determined arrows g and coker g making the following diagram in $\mathbf{C}^{\Phi}$ commutative and exact:

(10)



Here f is the kernel of (coker b) (coker m) in $\mathbf{C}^{\Phi}$. Let $i\in I$. Put diagrams (9) and (10) together to yield the commutative diagram below.



$(i\in I)$

Since, by assumption, $(\operatorname{coker} e_i)b=0^{\Phi}_{A/A_i}0^B$, there exists a unique arrow q_i with $q_i(\operatorname{coker} b)=\operatorname{coker} e_i$. Hence $(\operatorname{coker} m_i)f=(\operatorname{coker} e_i)(\operatorname{coker} m)f=q_i(\operatorname{coker} b)\cdot$ $\cdot(\operatorname{coker} m)f=q_i(\operatorname{coker} f)f=0^{\Phi}_{A/A_i}0^D$ implying the existence of a uniquely determined arrow f_i with $m_i f_i=f$. Clearly, f_i is monic. On the other hand, by the definition of A' there exists a monic $x\in\hom_{\mathbf{C}}(D, A')$ such that $mx=f$. This together with the equality $fg=m$ implies that g is an isomorphism and thus $B\cong 0$, completing the proof.

To show that (iii) implies (i) we have to repeat the reasoning of Dickson with appropriate modifications. Suppose that $\mathbf{F}$ satisfies the closure properties described in (iii) and define $\mathbf{T}$ to be the class of all those objects T in $\mathbf{C}$ for which $\hom_{\mathbf{C}}(T, F)$ contains zero arrows only for every $F\in\mathbf{F}$. All that needs proof is that (IV) holds. Let us consider an object A in $\mathbf{C}$ and take $\mathbf{A}$ to be the set of all subobjects $\langle A_j, m_j\rangle$

of A with $A/A_j \in \mathbf{F}$. We can suppose without loss of generality that every m_j is an arrow in $\mathbf{C}^\Phi$. This can be reached by multiplying each m_j with a suitable translation on A. Then, in particular, $\bigcap \mathbf{A} \neq \emptyset$ and thus by (iii) and by Zorn's lemma $\mathbf{A}$ has a minimal element $\langle A', m' \rangle$. The proof of the fact that $A' \in \mathbf{T}$ can be carried out just as in [2].

Finally, the equivalence of (iv) and (i) follows from the characterization theorem of DICKSON [2] if we take into consideration that $\mathbf{C}$ is semisubcomplete if and only if $\mathbf{C}^\Phi$ is semisubcomplete and conditions analogous to (e) and (f) of Theorem 1 are required also for all existing infinite products. Moreover, if this is the case, then kernels, cokernels and products coincide in $\mathbf{C}$ and in $\mathbf{C}^\Phi$.

The proof of the theorem is complete.

THEOREM 12. *Given a chain-$\bigcup$-subcomplete affine category* $\mathbf{C}$ *and a subclass* $\mathbf{T}$ *of objects of* $\mathbf{C}$ *the following conditions are equivalent.*

(i)′ $\mathbf{T}$ *is a torsion class in* $\mathbf{C}$.

(ii)′ $\mathbf{T}$ *is a radical class in* $\mathbf{C}$.

(iii)′ $\mathbf{T}$ *is closed under images, extensions and forming the union of any chain of such subobjects of a fixed object which belong to* $\mathbf{T}$.

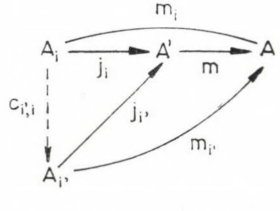
REMARK. A characterization dual to (iv) of Theorem 11 does not hold in general, even if $\mathbf{C}$ is supposed to be subcomplete. In fact, such characterization holds for every torsion class of a subcomplete affine category $\mathbf{C}$ if and only if $\mathbf{C}$ is abelian. This follows easily from the observation that for a coproduct diagram

$$0 \xrightarrow{j_1} 0 \sqcup 0 \xleftarrow{j_2} 0$$

$0 \sqcup 0$ is the union of the subobjects $\langle 0, j_i \rangle$ $(i=1,2)$ and $0 \sqcup 0$ is contained in the trivial torsion class only if $0 \cong 0 \sqcup 0$, i.e. if $\mathbf{C}$ is abelian.

PROOF. All steps of the proof of this theorem are essentially dual to those of Theorem 11, except the one showing that a radical class is closed under forming the union of any chain of subobjects of a fixed object. We discuss this in more detail.

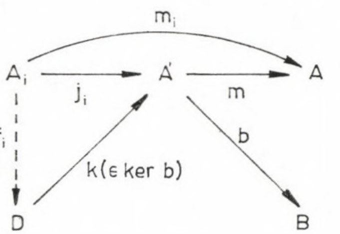
Let $\mathbf{T}$ be a radical class in $\mathbf{C}$ and consider a chain $\{\langle A_i, m_i \rangle | i \in I\}$ of subobjects of A. We use the notations of diagram (8) and write for brevity A' for $\bigcup_i A_i$. Furthermore we introduce the notation $i \prec i'$ $(i, i' \in I)$ to mean that $\langle A_i, m_i \rangle$ is a subobject of $\langle A_{i'}, m_{i'} \rangle$, i.e. there exists a uniquely determined monic $c_{i',i} \in \hom_{\mathbf{C}}(A_i, A_{i'})$ with $m_{i'} c_{i',i} = m_i$. In particular, if $i \prec i' \prec i''$ then $c_{i'',i'} c_{i',i} = c_{i'',i}$. Since m is monic, the following diagram is commutative:



$i \prec i' \, (i \in I)$.

Suppose now that for every $i \in I$, $A_i \in \mathbf{T}$. We are going to show that $A' \in \mathbf{T}$. Consider an object B with $A' \mapsto B$, i.e. $b \in \hom_{\mathbf{C}}(A', B)$ is a nonzero epi in $\mathbf{C}$. If there exists an $i \in I$ such that $b j_i$ is a nonzero arrow then the image object C of this arrow belongs to $\mathbf{T}$ as $A_i \rightarrowtail C$ and, clearly, $C \prec B$. By the definition of a radical class this implies that $A' \in \mathbf{T}$.

It remains to prove that if bj_i is a zero arrow for all $i \in I$ then, necessarily, $B \cong 0$. Observe first that every j_i $(i \in I)$ factors through the same arrow k in $\ker b$.



Indeed, if $i, i' \in I$ and, say, $i \prec i'$, then $kf_{i'} = j_{i'}$ implies $k(f_{i'} c_{i',i}) = j_i$. On the other hand, since A' is the least upper bound, there exists a monic $x \in \hom_{\mathbf{C}}(A', D)$ with $(mk)x = m$. Thus k is an isomorphism implying that $b = \operatorname{coker} k$ is a zero arrow and thus $B \cong 0$. This completes the proof of the theorem.

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AN INTERSECTION PROBLEM FOR FINITE SETS

By

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1. Introduction. Let X be a finite set of cardinality n and t a positive integer. Define the following families of sets:

a) for $n+t=2s$ let $\mathcal{F}(n,t)=\{F\subseteqq X \mid |F|\geqq s\}$,

b) for $n+t=2s+1$ let $\mathcal{F}(n,t)=\{F\subseteqq X \mid |F\cap(X-x)|\geqq s\}$, where x is some particular element of X.

KATONA [2] proved the following:

THEOREM 1. *Let $\mathcal{F}$ be a family of subsets of X and suppose that for $F_1, F_2\in\mathcal{F}$ we always have $|F_1\cap F_2|\geqq t$. Then $|\mathcal{F}|\leqq|\mathcal{F}(n,t)|$, and if $t\geqq 2$ then equality can hold only in the case $\mathcal{F}=\mathcal{F}(n,t)$.*

In [1], ERDŐS raised the problem to determine the maximum of $|\mathcal{F}|$ if we assume only $|F_1\cap F_2|\neq t-1$. As he remarks the case $t=1$ coincides with that case of Theorem 1.

The aim of this paper to solve this problem for $t=2$. We prove the following:

THEOREM 2. *Let $\mathcal{F}$ be a family of subsets of X. Assume that for any $F_1, F_2\in\mathcal{F}$ either $|F_1\cap F_2|\geqq t$ or $F_1\cap F_2=\varnothing$ holds, $t\geqq 2$. Then $|\mathcal{F}|\leqq|\mathcal{F}(n,t)|+1$ with equality holding if and only if $\mathcal{F}=\mathcal{F}(n,t)\cup\{\varnothing\}$.*

2. The proof of the result. Katona's proof of Theorem 1 relied upon the following.

THEOREM 3 (KATONA [2]). *If $1\leqq g<h$ and $g+t\geqq h$ (g, h, t are integers, and $\mathcal{A}$ is a family of h-subsets of X such that any two members of $\mathcal{A}$ intersect in at least t points. Then*

$$|\mathcal{A}^g|\geqq|\mathcal{A}|\frac{\binom{2h-t}{g}}{\binom{2h-t}{h}}, \tag{1}$$

where $\mathcal{A}^g=\{B \mid |B|=g,\ \exists A\in\mathcal{A},\ B\subset A\}$.

Moreover, equality holds in (1) only if for some $(2h-t)$-subset Y of X we have $\mathcal{A}=\{A\subset Y \mid |A|=h\}$.

We need the following generalization of Theorem 3:

THEOREM 4. *Let g, h, t be integers satisfying the assumptions of Theorem 3. Let $\mathcal{A}$ be a family of h-subsets of X such that for any $A, A'\in\mathcal{A}$ either $|A\cap A'|\geqq t$ or $A\cap A'=\varnothing$ holds. Then* (1) *holds. Moreover, equality is possible if and only if for* *some r there are r pairwise disjoint* $(2h-t)$*-subsets,* $Y_1, \ldots, Y_r \subseteq X$ *such that* $\mathcal{A} = \{A \mid |A| = h, \exists i, 1 \leqq i \leqq r$ *such that* $A \subset Y_i\}$.

PROOF OF THEOREM 4. Let $\mathcal{A}_1$ be a maximal collection of members of $\mathcal{A}$ such that the members of $\mathcal{A}_1$ have pairwise non-empty intersection. If $\mathcal{A} = \mathcal{A}_1$ then the assertion follows from Theorem 3. If $\mathcal{A} - \mathcal{A}_1$ is a non-empty family of subsets of X then let $\mathcal{A}_2$ be a maximal collection of members of it such that any two members of $\mathcal{A}_2$ have pairwise non-empty intersection, and so on. After a finite number, say q, of steps we arrive at a partition $\mathcal{A} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_q$ such that for any $1 \leqq i \leqq q$, and $A, A' \in \mathcal{A}_i$ we have $A \cap A' \neq \emptyset$ whence by the assumptions of the theorem $|A \cap A'| \geqq t$. Thus the assumptions of Theorem 3 are satisfied for $\mathcal{A}_i$ for any $1 \leqq i \leqq q$. We assert that for $1 \leqq i < j \leqq q$

$$\mathcal{A}_i^g \cap \mathcal{A}_j^g = \emptyset. \tag{2}$$

If (2) is not true then we can find a g-set B and two sets A_1, A_2 such that $A_1 \in \mathcal{A}_i$, $A_2 \in \mathcal{A}_j$, $A_1 \supset B$, $A_2 \supset B$. By the maximal choice of $\mathcal{A}_i$ there exists a member A_3 of $\mathcal{A}_i$ such that $A_2 \cap A_3 = \emptyset$. This implies $B \cap A_3 = \emptyset$ whence it follows $|A_1 \cap A_3| \leqq |A_1 - B| < t$, a contradiction.

Using (1) for the $\mathcal{A}_i$'s it follows from (2):

$$|\mathcal{A}^g| = \left|\bigcup_{i=1}^{q} \mathcal{A}_i^g\right| = \sum_{i=1}^{q} |\mathcal{A}_i^g| \geqq \sum_{i=1}^{q} \frac{\binom{2h-t}{g}}{\binom{2h-t}{h}} |\mathcal{A}_i| = \frac{\binom{2h-t}{g}}{\binom{2h-t}{h}} \sum_{i=1}^{q} |\mathcal{A}_i| = \frac{\binom{2h-t}{g}}{\binom{2h-t}{h}} |\mathcal{A}|,$$

proving (1). If we have equality then for each i $\exists Y_i \subset X$, $|Y_i| = 2h-t$, $\mathcal{A}_i = \{A \subset Y_i \mid |A| = h\}$. Now $Y_i \cap Y_j = \emptyset$ for $1 \leqq i < j \leqq q$ follows from the assumptions. Q.e.d.

Now we turn to the proof of Theorem 2. Let $\mathcal{F}_i$ denote the family of i-subsets in $\mathcal{F}$.

We assert that for $t \leqq i \leqq (n+t-1)/2$

$$\{X - G \mid G \in \mathcal{F}_i^{i-t+1}\} \cap \mathcal{F}_{n-i+t-1} = \emptyset.$$

The contrary would mean that there exist two sets F, F' in $\mathcal{F}$ such that $|F| = i$, $|F'| = n-i+t-1$ and F has an $(i-t+1)$-element subset $G = X - F'$, but then $|F \cap F'| = |F - G| = t-1$, a contradiction. Hence

$$|\mathcal{F}_i^{i-t+1}| + |\mathcal{F}_{n-i+t-1}| \leqq \binom{n}{n-i+t-1}.$$

Using Theorem 4 we obtain

$$\frac{i}{i-t+1} |\mathcal{F}_i| + |\mathcal{F}_{n-i+t-1}| \leqq \binom{n}{n-i+t-1} \quad \left(t \leqq i \leqq \frac{n+t-1}{2}\right),$$

yielding

$$|\mathcal{F}_i| + |\mathcal{F}_{n-i+t-1}| \leqq \binom{n}{n-i+t-1} \tag{3}$$

with equality holding if and only if $\mathcal{F}_i=\emptyset$, $\mathcal{F}_{n-i+t-1}=\{F\subset X \,\|\, |F|=n-i+t-1\}$, unless $i=(n+t-1)/2$ in which case (3) yields

$$(4)\qquad |\mathcal{F}_{(n+t-1)/2}| \leqq \frac{\binom{n}{\frac{n+t-1}{2}}}{\frac{n}{\frac{n-t+1}{2}}} = \binom{n-1}{\frac{n+t-1}{2}},$$

with equality holding if and only if for some $x\in X$ we have

$$\mathcal{F}_{(n+t-1)/2} = \left\{F \subset (X-x) \,\Big\|\, |F| = \frac{n+t-1}{2}\right\}.$$

We also have the following trivial inequalities:

$$(5)\qquad |\mathcal{F}_0| \leqq 1, \quad |\mathcal{F}_n| \leqq 1, \quad |\mathcal{F}_i| \leqq 0 \quad \text{for} \quad 1 \leqq i \leqq t-1.$$

Suppose now that $n+t=2s$. Then summing up the inequalities (5) along with the inequalities (3) we obtain

$$|\mathcal{F}| \leqq 1+\sum_{j=s}^{n}\binom{n}{j},$$

with equality holding if and only if $\mathcal{F}=\mathcal{F}(n,t)\cup\{\emptyset\}$. In the case $n+t=2s+1$ we replace the inequality (3) by the inequality (4) in the case $i=s$, then the summation yields:

$$|\mathcal{F}| \leqq 1+\binom{n-1}{s}+\sum_{j=s+1}^{n}\binom{n}{j},$$

with equality holding if and only if $\mathcal{F}=\mathcal{F}(n,t)\cup\{\emptyset\}$, q.e.d.

Theorem 2 suggests the following

CONJECTURE. Let t and t' be non-negative integers, $t'<t$. Let $\mathcal{F}$ be a family of subsets of X such that for any $F, G\in\mathcal{F}$ either $|F\cap G|<t'$ or $|F\cap G|\geqq t$ holds. Then for $n>n_0(t)$ we have

$$|\mathcal{F}| \leqq |\mathcal{F}(n,t)\cup\{F\subset X \,\|\, |F|<t'\}|.$$

Theorem 1 corresponds to the case $t'=0$, and Theorem 2 to the case $t'=1$ of the conjecture.

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