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ETUDE ARITHMÉTIQUE DES NOMBRES DE YOUNG

Par
M. MIGNOTTE (Paris)

Introduction

Les coefficients considérés ici ne sont autres que les dimensions des représentations linéaires irréductibles du groupe symétrique. Ils ont été en particulier calculés par Young, nous les avons appelés nombres de Young par souci de brièveté. Ces nombres apparaissent encore en combinatoire quand on énumère les tableaux standards attachés à une partition (à ce sujet, consulter par exemple [1], p. 49—59).

A l'heure actuelle, la démonstration de la formule qui fournit ces coefficients est loin d'être satisfaisante; elle consiste en une vérification par récurrence et n'apporte aucune information.

Nous avons délibérément fait abstraction des interprétations combinatoires de ces coefficients. Partant des formules, nous démontrons, de manière purement arithmétique, que ces coefficients sont entiers-ce qui n'a rien d'évident a priori. Cette démonstration fournit des renseignements importants sur les propriétés de divisibilité des nombres de Young, propriétés qui — semble-t-il — n'avaient pas été remarquées antérieurement.

Après cette étude des coefficients généraux, dans deux cas particuliers, coefficients «rectangulaires» et coefficients attachés à des «triangles isocèles», nous exhibons une formule explicite qui fournit la décomposition en facteurs premiers de ces coefficients (la terminologie a pour origine la nature géométrique des tableaux standards associés à ces deux familles de coefficients). Dans la suite, notre étude porte uniquement sur la seconde de ces familles dont les nombres — notés Y_n — ne dépendent que d'un paramètre, nous en donnons quelques propriétés arithmétiques remarquables.

Dans la partie suivante, nous utilisons les Y_n pour obtenir quelques renseignements sur la répartition des nombres premiers. Avant d'entrer dans les détails de ce travail, rappelons que les premiers résultats importants concernant la répartition des nombres premiers — plus particulièrement l'ordre de grandeur de la fonction $\Pi(x)=\sum_{p\leqq x} 1$ — ont été obtenus en 1850 par Tchebytcheff grâce à l'étude arithmétique de certains coefficients binomiaux. A ce propos, il est important de noter que le coefficient binomial $\binom{m}{n}$ est un certain coefficient de Young, celui associé à la partition $\underbrace{1+\ldots+1}_{\langle m \text{ fois}\rangle}+(n-m)$ de l'entier n. Les coefficients de Young apparaissent donc comme une généralisation des coefficients binomiaux et peuvent, comme ces derniers, être utilisés pour une étude élémentaire de certaines fonctions arithmétiques liées aux nombres premiers. Nous nous sommes limités ici aux applications arithmétiques de l'étude des Y_n. Et, pour ceci, nous avons d'abord calculé un développement asymptotique précis des Y_n. La principale application concerne la fonction $S(x)=\sum_{p\leqq x}\frac{\mathrm{Log}\, p}{p}$. Vu l'esprit élémentaire de cette étude, nous avons systématiquement utilisé les estimations de Tchebytcheff des fonctions Π, θ et ψ. Il ne semble pas que les nombres Y_n soient utilisables pour obtenir directement des renseignements précis sur la fonction Π par exemple.

Ce travail se termine par une étude précise de certaines fonctions arithmétiques très intimement liées aux Y_n. La démarche est inverse de celle de la partie précédente, nous utilisons cette fois des résultats profonds et non élémentaires sur les fonctions θ et S.

Il apparaîtra clairement au lecteur qu'une étude analogue aurait pu être faite sur d'autres familles de nombres de Young. C'est bien entendu le cas pour les coefficients que nous avons appelés rectangulaires. A ce sujet, il est utile de noter que les coefficients $Y_{n,2}$ (voir plus loin pour la signification de cette notation) — qui ne sont autres que les nombres de Catalan — peuvent être employés à la place des coefficients binomiaux pour obtenir des estimations analogues à celles de Tchebytcheff.

Je tiens à exprimer ma profonde gratitude à Monsieur le Professeur M. P. Schützenberger que m'a suggéré cette étude, et sans les conseils et encouragements duquel ce travail n'aurait jamais vu le jour.

I. Préliminaires

Soit $\alpha=(\alpha_1, \dots, \alpha_m)$ une partition de l'entier n, $n=\alpha_1+\dots+\alpha_m$, $\alpha_1\geqq\dots\geqq\alpha_m\geqq 1$. On appelle partage conjugué de α la partition $\alpha^*=(\alpha_1^*, \alpha_2^*, \dots)$ où α_i^* désigne le nombre de termes de la suite α qui sont au moins égaux à i.

En termes de diagramme de Ferrers, le diagramme associé à α^* se déduit de celui de α par symétrie par rapport à la diagonale.

EXEMPLE.

$$\alpha = (5, 4, 1, 1) \qquad \alpha^* = (4, 2, 2, 2, 1)$$

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. . . .          . .
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Posons $h_j^i=1+(\alpha_i-j)+(\alpha_j^*-i)$ pour $1\leqq i\leqq m$ et $1\leqq j\leqq \alpha_i$. On définit le coefficient $Y(\alpha)$ par la formule

$$Y(\alpha)=\frac{n!}{\prod_{i,j} h_j^i}. \tag{1}$$

Le résultat suivant nous sera utile.

LEMME 1. *Pour tout couple* (i,j) *fixé, la suite*

$$(h_j^i, h_{j+1}^i, \dots, h_{\alpha_i}^i, h_j^i-h_j^{i+1}, \dots, h_j^i-h_j^{\alpha_j^*})$$

constitue une permutation des entiers $1, 2, \dots, h_j^i$.

DÉMONSTRATION. Remarquons d'abord que la sous suite $h^i_j, \ldots, h^i_{\alpha_i}$ est strictement décroissante, tandis que la sous suite $h^i_j - h^{i+1}_j, \ldots, h^i_j - h^{\alpha^*_j}_j$ est strictement croissante. En particulier, les termes de cette première sous-suite sont au plus égaux à h^i_j et il en est évidemment de même pour ceux de la seconde.

Il résulte de ceci qu'il suffit de montrer que les termes de la suite totale sont deux à deux distincts. D'après la première remarque, il suffit pour cela de prouver qu'aucun terme de la première sous suite ne peut coïncider avec un terme de la seconde, ce qui équivaut encore au fait qu'une relation du type

$$h^i_j = h^i_{j+t} + h^i_{j+s}, \quad s \geqq 1, \quad t \geqq 1,$$

est impossible.

La décroissance de la suite α_i implique que seuls sont possibles les deux cas suivants,

$$\alpha^*_{j+t} \leqq i+s \quad \text{et} \quad \alpha_{i+s} \leqq j+t$$

ou

$$\alpha^*_{j+t} > i+s \quad \text{et} \quad \alpha_{i+s} > j+t.$$

Il est facile de vérifier que l'égalité ci-dessus n'a lieu dans aucun de ces deux cas, ce qui achève la démonstration.

Ce lemme permet de démontrer la formule

$$Y(\alpha) = n! \frac{\prod\limits_{j.\geqq i.\geqq 1} (h^i_1 - h^j_1)}{\prod\limits_{i \geqq 1} (h^i_1)!}. \tag{2}$$

(Une variable sera suivie d'un point lorsqu'elle varie dans l'expression et qu'il y a ambiguité.)

Compte tenu de la relation évidente,

$$h^i_1 = \alpha_i + m - i, \tag{3}$$

la formule (2) peut se mettre sous la forme

$$Y(\alpha) = n! \frac{\prod\limits_{j.\geqq i.\geqq 1} (j-i)}{\prod\limits_{i \geqq 1} (\alpha_i + m - i)!}. \tag{4}$$

II. Démonstration directe du fait que les $Y(\alpha)$ sont entiers

Il nous faut introduire quelques notations. Soit a un nombre entier fixé $\geqq 2$. Pour un entier x, on pose

$$\chi_a(x) = \begin{cases} 1 & \text{si} \quad a \mid x \\ 0 & \text{si} \quad a \nmid x \end{cases}$$

(le signe | signifie divise et $\nmid$ ne divise pas). Pour toute suite finie $S = (u_1, \ldots, u_k)$, on note

$$\#_a S = \sum_{i=1}^{k} \chi_a(u_i).$$

Considérons une partition α d'un entier n et le coefficient $Y(\alpha)$ associé. Nous nous proposons de démontrer que $Y(\alpha)$ est entier. La démonstration fournira comme sous produits quelques propriétés arithmétiques générales des coefficients $Y(\alpha)$.

PROPOSITION 1. *Soit* $Y(\alpha)$ *le coefficient associé à la partition* $\alpha=(\alpha_1, \dots, \alpha_m)$ *d'un entier* m. *Si* $\mathcal{N}=(1, 2, \dots, n)$ *désigne la suite des termes du numérateur de* $Y(\alpha)$ (*voir la formule* (1)) *et*

$$\mathcal{D} = (h_j^i)_{\substack{1\leqq i\leqq m\\ 1\leqq j\leqq \alpha_i}}$$

celle des termes du dénominateur de $Y(\alpha)$, *alors, pour tout entier* $a\geqq 2$, *on a l'inégalité*

$$\#_a\mathcal{D} \leqq \#_a\mathcal{N}.$$

DÉMONSTRATION. Posons $x_i=h_1^i$, $i=1, \dots, m$. D'après (2), on a

$$\#_a\mathcal{D} = \sum_i \left[\frac{x_i}{a}\right] - \sum_{j.\geqq i.\geqq 1} \chi(x_j-x_i), \tag{5}$$

où on a posé $\chi=\chi_a$ pour simplifier et où $[x]$ désigne l'entier q tel que $q\leqq x<q+1$. A chaque x_i, associons l'entier y_i qui vérifie $y_i\equiv -x_i$ modulo a, et $0\leqq y_i<a$. De (4), on déduit

$$\#_a\mathcal{D} = \sum_i \frac{x_i-y_i}{a} - \sum_{j.\geqq i.\geqq 1} \chi(y_j-y_i). \tag{6}$$

La relation (3) permet de transformer (6) en

$$\#_a\mathcal{D} = \sum_i \frac{\alpha_i}{a} + R_a = \frac{n}{a} + R_a \tag{7}$$

où R_a désigne la quantité

$$R_a = \sum \frac{i-1}{a} - \sum \frac{y_i}{a} - \sum_{j.\geqq i.\geqq 1} \chi(y_j-y_i) = \tag{8}$$

$$= \frac{m(m-1)}{2a} - \sum \frac{y_i}{a} - \sum_{j.\geqq i.\geqq 1} \chi(y_j-y_i).$$

Si on compare alors la formule (7) à la formule évidente

$$\#_a\mathcal{N} = \left[\frac{n}{a}\right] \tag{9}$$

on voit que la proposition cherchée équivaut à l'inégalité $R_a\leqq 0$. Remarquons encore que $R_a\leqq 0$ équivaut à

$$\Delta_a \geqq m^2-m \tag{10}$$

où

$$\Delta_a = 2\sum y_i + a\sum_{i.\neq j.} \chi_a(y_j-y_i). \tag{11}$$

Quitte à modifier la numérotation, on peut supposer que les y_i sont croissants.

On définit alors des entiers λ_k, $k=1, 0, \dots, a-1$, par

(12) $$y_i = k \quad \text{si} \quad \lambda_{k-1} < i \leqq \lambda_k, \quad \text{avec} \quad \lambda_{-1} = 0.$$

En particulier,

$$\sum_{0 \leqq k \leqq a-1} \lambda_k = m.$$

Avec ces notations, on a

(13) $$\Delta_a = 2 \sum_{0 \leqq k \leqq a-1} k\lambda_k + a \sum_k \lambda_k(\lambda_k - 1) = 2\sum k\lambda_k + a\sum \lambda_k^2 - am = F(\lambda) - am,$$

si on pose

(14) $$F(\lambda) = 2 \sum k\lambda_k + a \sum \lambda_k^2.$$

La comparaison de (10), (13) et (14) montre que tout revient à prouver que la fonction F vérifie l'inégalité

(15) $$F(\lambda) \geqq m^2 + am - m.$$

Posons

(16) $$c = m + \frac{a-1}{2},$$

(17) $$\mu_k = \frac{c-k}{a}$$

et

$$\nu_k = \lambda_k - \mu_k.$$

Il est alors facile de vérifier l'identité

(18) $$F(\lambda) = (m^2 + am - m) + \left(a \sum \nu_k^2 - \frac{a^2-1}{12}\right).$$

Cette dernière formule montre que l'on a toujours la minoration

(19) $$F(\lambda) \geqq (m^2 + am - m) + \left(a \sum \|\mu_k\|^2 - \frac{a^2-1}{12}\right),$$

où $\| \ \|$ désigne la distance algébrique à l'entier le plus proche (ce qui signifie que $\|x\| = x - [x]$ si $x - [x] \leqq [x] + 1 - x$ et $\|x\| = x - [x] - 1$ sinon).

L'inégalité (15) sera donc démontrée si on vérifie que la relation suivante a lieu

(20) $$a \sum_{0 \leqq k \leqq a-1} \|\mu_k\|^2 = \frac{a^2-1}{12}.$$

Distinguons deux cas suivant la parité de a. Si a est impair, la formule (16) montre d'abord que c est entier. Puis, la formule (17) implique que les $\|\mu_k\|$ parcourent l'ensemble $\left\{\pm\frac{1}{a}, \pm\frac{2}{a}, \dots, \pm\frac{\frac{a-1}{2}}{a}\right\}$ lorsque k varie de 0 à $a-1$. D'où les égalités

$$a\sum\|\mu_k\|^2=2a\sum_{k=1}^{(a-1)/2}\frac{k^2}{a^2}=\frac{2}{a}\,\frac{\frac{a-1}{2}\,\frac{a+1}{2}\,a}{6}=\frac{a^2-1}{12}$$

qui, dans ce cas, prouvent la relation (20).

Soit maintenant a pair. Dans ce cas, la relation (16) montre d'abord que c appartient à $\frac{1}{2}+\mathbf{N}$. Les formules (16) et (17) conduisent aux relations

$$\mu_k=\frac{2m+a-(2k+1)}{2a} \tag{21}$$

qui montrent que les $\|\mu_k\|$ parcourent l'ensemble

$$\left\{\pm\frac{1}{2a},\ \pm\frac{3}{2a},\ \ldots,\ \pm\frac{a-1}{2a}\right\}.$$

D'où

$$a\sum\|\mu_k\|^2=2a\left(\frac{1}{4a^2}+\frac{3^2}{4a^2}+\ldots+\frac{(a-1)^2}{4a^2}\right)=\frac{1}{2a}\left(\sum_{k=0}^{a}k^2-\sum_{0\leqq k\leqq a,\,k\text{ pair}}k^2\right)=$$

$$=\frac{1}{2a}\left(\sum_{k=0}^{a}k^2-4\sum_{k=0}^{a/2}k^2\right)=\frac{1}{2a}\left(\frac{a(a+1)(2a+1)}{6}-\frac{4}{6}\,\frac{a}{2}\left(\frac{a}{2}+1\right)(a+1)\right)=\frac{a^2-1}{12}.$$

Ainsi, la relation (20) et, par conséquent, l'inégalité (15) ont lieu dans tous les cas. D'après les remarques ci-dessus, ceci achève la démonstration de la proposition.

COROLLAIRE 1. *Les coefficients* $Y(\alpha)$ *sont entiers.*

Soit en effet un nombre premier p quelconque. Il est facile de voir que la valuation p-adique $|Y(\alpha)|_p$ de $Y(\alpha)$ (c'est-à-dire le plus grand exposant h tel que p^h divise $Y(\alpha)$) est donnée par

$$|Y(\alpha)|_p=\sum_{h\geqq 1}\left(\#_{p^h}(\mathcal{N})-\#_{p^h}(\mathcal{D})\right). \tag{22}$$

D'après la proposition, chaque terme de la somme qui figure au membre droit de (22) est positif ou nul, et donc $|Y(\alpha)|_p\geqq 0$.

Ceci étant vrai pour tout p premier, le nombre $Y(\alpha)$ est bien entier.

De la formule (22) et de la proposition, on déduit aussi le résultat suivant.

COROLLAIRE 2. *Soit* p *un nombre premier. S'il existe des entiers* h *et* l *tels que* $\#_{p^h}(\mathcal{N})$ *et* $\#_{p^h}(\mathcal{D})$ *vérifient l'inégalité*

$$\#_{p^h}(\mathcal{N})\geqq\#_{p^h}(\mathcal{D})+l,$$

alors p^l *divise* $Y(\alpha)$.

De façon moins formelle mais un peu imprécise, on peut dire que le corollaire 2 exprime la propriété suivante: Si dans la formule (1) il existe un nombre a, puissance d'un nombre premier p, tel que le nombre de multiples de a qui figurent au numérateur soit au moins égal au nombre de multiples de a qui figurent au dénominateur augmenté de l, alors p^l divise $Y(\alpha)$.

COROLLAIRE 3. *Soit* $\alpha=(\alpha_1, \dots, \alpha_m)$ *une partition d'un entier* n, $n=\alpha_1+\dots+\alpha_m$, $\alpha_1 \geqq \alpha_2 \geqq \dots \geqq \alpha_m \geqq 1$. *Alors, si on pose*

$$r = \left[\frac{n}{\beta}\right], \quad \text{où} \quad \beta = \max(\alpha_i + m - i), \tag{23}$$

le nombre $Y(\alpha)$ *est divisible par* $r!$.

Soit p un nombre premier. La formule (22) montre que la valuation p-adique de $Y(\alpha)$ vérifie

$$|Y(\alpha)|_p \geqq \sum_{p^h > \beta} \#_{p^h}(\mathcal{N}), \tag{24}$$

en effet, pour $p^h > \beta$, on a clairement $\#_{p^h}(\mathcal{D})=0$ et, pour $p^h \leqq \beta$, la proposition dit que les termes $(\#_{p^h}(\mathcal{N}) - \#_{p^h}(\mathcal{D}))$ sont positifs ou nuls. En utilisant la formule (9), on obtient alors la minoration

$$|Y(\alpha)|_p \geqq \sum_{p^h > \beta} \left[\frac{n}{p^h}\right]. \tag{25}$$

D'où aussitôt, les inégalités

$$|Y(\alpha)|_p \geqq \sum_{k \geqq 1} \left[\frac{n}{p^k \beta}\right] \geqq \sum_{k \geqq 1} \left[\frac{[n/\beta]}{p^k}\right] = \sum_{k \geqq 1} \left[\frac{r}{p^k}\right]. \tag{26}$$

La formule bien connue

$$|r!|_p = \sum_{k \geqq 1} \left[\frac{r}{p^k}\right],$$

jointe à (26), prouve l'inégalité

$$|Y(\alpha)|_p \geqq |r!|_p. \tag{27}$$

Cette inégalité, valable pour tout nombre premier p, montre que $r!$ divise $Y(\alpha)$.

REMARQUE 1. Chaque fois qu'un nombre rationnel $x=\dfrac{N}{D}$, où $D=\prod_{v \in \mathcal{D}} v$ et $N=\prod_{u \in \mathcal{N}} u$, vérifie la conclusion de la proposition 1, alors les analogues des corollaires 1 à 3 sont vrais. Il est facile de voir que ceci s'applique en particulier pour les coefficients multinomiaux. On obtient ainsi le résultat suivant.

COROLLAIRE 3 BIS. *Soit* $n=\alpha_1+\dots+\alpha_m$, $\alpha_1 \geqq \dots \geqq \alpha_m \geqq 1$. *Alors le coefficient multinomial*

$$\binom{n}{\alpha_1, \dots, \alpha_m} = \frac{n!}{\alpha_1! \dots \alpha_m!}$$

est divisible par $r!$, *où* $r=\left[\dfrac{n}{\alpha_1+1}\right]$.

REMARQUE 2. La majoration triviale $|Y(\alpha)|_p \leqq |n!|_p$ est, dans le cas général, la meilleure possible. En effet, soit p un nombre premier fixé, et soit la partition $\alpha=(1, 1+(p-1), 1+2(p-1), \dots, 1+k(p-1))$ de l'entier $n=(k+1)\left(1+k\dfrac{p-1}{2}\right)$. On peut vérifier que l'on a l'égalité $|Y(\alpha)|_p=|n!|_p$.

IV. Etude de quelques cas particuliers

1. *Cas des coefficients « rectangulaires ».* A la partition $nk=n+\ldots+n$ (k fois) est associé le coefficient $Y(n, \ldots, n)$, qu'on notera $Y_{n,k}$ pour simplifier. Ces coefficients sont donnés par les formules

$$Y_{n,k} = \frac{(nk)!}{\dfrac{n!(n+1)!\ldots(n+k-1)!}{1!2!\ldots(k-1)!}} \tag{28}$$

ce qui se vérifie immédiatement sur l'expression (2).

Nous nous proposons de calculer explicitement la valuation p-adique de $Y_{n,k}$. L'énoncé suivant nous sera utile.

LEMME 2. *Soit la suite* $\mathcal{S}_m=(1, 1, 2, 1, 2, 3, \ldots, 1, 2, \ldots, m)$ *et soit* a *un entier* $\geqq 2$. *Alors*

$$\#_a\mathcal{S}_m = \frac{\left(m+1-\frac{a}{2}\right)^2-\left(\frac{a}{2}-1-r\right)^2}{2a}, \tag{29}$$

où r *désigne le reste de la division euclidienne de* m *par* a.

On a clairement

$$\#_a\mathcal{S}_m = \left[\frac{1}{a}\right]+\left[\frac{2}{a}\right]+\ldots+\left[\frac{m}{a}\right]. \tag{30}$$

Posons $m=aq+r$, $0\leqq r<a$. La formule (30) conduit à la relation

$$\#_a\mathcal{S}_m = a+2a+\ldots+(q-1)a+(r+1)q.$$

D'où

$$\#_a\mathcal{S}_m = \frac{q(q-1)}{2}a+(r+1)q = \frac{q}{2}(m+r+2-a) =$$

$$= \frac{(m-r)(m+r+2-a)}{2a} = \frac{\left(m+1-\frac{a}{2}\right)^2-\left(\frac{a}{2}-1-r\right)^2}{2a}.$$

Ce qui démontre (29).

La formule (29) permet de calculer $\#_a\mathcal{D}$, où $\mathcal{D}$ désigne, comme au paragraphe II, la suite des facteurs du dénominateur de $Y_{n,k}$. On trouve

$$\#_a\mathcal{D} = \frac{1}{2a}\left(\left(n+k-\frac{a}{2}\right)^2-\left(\frac{a}{2}-1-r_{n+k}\right)^2-\left(n-\frac{a}{2}\right)^2+\right.$$

$$\left.+\left(\frac{a}{2}-1-r_n\right)^2-\left(k-\frac{a}{2}\right)^2+\left(\frac{a}{2}-1-r_k\right)^2\right), \tag{31}$$

où r_i (pour $i=k, n, n+k$) désigne le reste de la division de $i-1$ par a. En remarquant que l'on a $r_{n+k}=r_n+r_k+1-\varepsilon a$, où

$$\varepsilon = \left[\frac{r_n+r_k+2}{a+1}\right] \quad (= 0 \text{ ou } 1),$$

la formule (31) se réduit à

$$\#_a\mathcal{D} = \frac{1}{2a}\big((n^2+k^2-1-r_{n+k}^2+2nk-an-ak+a+ar_{n+k}-2r_{n+k})-$$

$$-(n^2-1-r_n^2-an+a+ar_n-2r_n)-(k^2-1-r_k^2-ak+a+ar_k-2r_k)\big) =$$

$$= \frac{1}{2a}\big(2nk+(a-2)(r_{n+k}-r_n-r_k)-(r_{n+k}^2-r_n^2-r_k^2)-(a-1)\big) =$$

$$= \frac{1}{2a}\big(2nk+(a+2)(1-\varepsilon a)-(1+\varepsilon a^2+2r_n+2r_k-2\varepsilon a+a-1+2r_nr_k-2r_n\varepsilon a-2r_k a\varepsilon)\big) =$$

$$= \frac{1}{2a}\big(2nk-\varepsilon a(a-2+a-2-2r_n-2r_k)-2(1+r_n+r_k+r_nr_k)\big),$$

et enfin

$$(32) \qquad \#_a\mathcal{D} = \frac{1}{a}\big(nk-\varepsilon a(a-2-r_n-r_k)-(1+r_n+r_k+r_nr_k)\big).$$

D'autre part, il est clair que $\#_a\mathcal{N}$ a pour valeur

$$\#_a\mathcal{N} = \left[\frac{nk}{a}\right],$$

soit

$$(33) \qquad \#_a\mathcal{N} = \frac{1}{a}\left(kn-\left((r_n+1)(r_k+1)-a\left[\frac{(r_n+1)(r_k+1)}{a}\right]\right)\right)$$

puisque

$$n = an'+r_n+1, \quad k = ak'+r_k+1,$$

$$kn = a\big(n'(r_k+1)+k'(r_n+1)\big)+(r_n+1)(r_k+1).$$

Les formules (32) et (33) donnent

$$(34) \qquad \#_a\mathcal{N}-\#_a\mathcal{D} = \left[\frac{(r_n+1)(r_k+1)}{a}\right]+\left[\frac{r_n+r_k+2}{a+1}\right](a-2-r_n-r_k).$$

Si on désigne par $r(x, a)$ le reste de la division-euclidienne de l'entier x par a, la formule précédente s'écrit

$$(35) \qquad \#_a\mathcal{N}-\#_a\mathcal{D} = \left[\frac{1}{a}r(n, a)r(k, a)\right]+\left[\frac{r(n, a)+r(k, a)}{a+1}\right]\big(a-r(n, a)-r(k, a)\big)$$

au moins dans le cas où a ne divise ni n, ni k (en effet, on a alors $r(n, a)=r_n+1$ et $r(k, a)=r_k+1$). Dans le cas où a divise au moins l'un des deux entiers n et k, les deux membres de droite de (34) et (35) coïncident (tous deux s'annulent). L'égalité (35) a donc toujours lieu. (Dans ce cas particulier, il n'est pas très difficile de vérifier l'inégalité $\#_a\mathcal{N} \geqq \#_a\mathcal{D}$). La formule (35) conduit au résultat suivant.

PROPOSITION 2. *Soit p un nombre premier. La valuation p-adique de $Y_{n,k}$ est donnée par la formule*

$$|Y_{n,k}|_p = \sum_{h\geqq 1}\left(\left[\frac{1}{p^h}r(n,p^h)r(k,p^h)\right]-\left[\frac{r(n,p^h)+r(k,p^h)}{p^h+1}\right]\right)(r(n,p^h)r(k,p^h)-p^h)$$

où $r(x,a)$ désigne le reste de la division euclidienne de x par a.

2. *Cas des « triangles rectangles isocèles »*. Les notations différeront légèrement de celles de II. Soit la partition $N=1+2+\ldots+n$ de l'entier $N=\frac{n(n+1)}{2}$. On posera $Y_n=Y(n,n-1,\ldots,1)$. [1] Il est facile de voir directement que les Y_n sont donnés par la formule

(36) $$Y_n = \frac{N!}{1.(1.3)(1.3.5)\ldots(1.3.5.\ldots.(2n-1))}.$$

Posons

(37) $$U_n = N!$$

(38) $$V_n = \prod_{h=1}^{n-1} w_h, \quad \text{avec} \quad w_h = \prod_{i=0}^{h}(1+2i).$$

Avec ces notations, la formule (36) s'écrit

(39) $$Y_n = \frac{U_n}{V_n}.$$

Soit maintenant un nombre p premier fixé. On sait d'abord que

$$|U_n|_p = \sum_{k\geqq 1}\left[\frac{N}{p^k}\right],$$

soit

(40) $$|U_n|_p = \sum_{k\geqq 1}\left[\frac{n(n+1)}{2p^k}\right].$$

D'autre part, il est facile de vérifier que

(41) $$|w_h|_p = \sum_{j\geqq 1}\left[\frac{h+q_j+1}{p^j}\right], \quad \text{avec} \quad q_j = \frac{p^j-1}{2}.$$

En faisant la somme des formules (41) pour $h=1,2,\ldots,n-1$, et en utilisant (38), il vient

(42) $$|V_n|_p = \sum_{j\geqq 1}\left(\sum_{h=1}^{n-1}\left[\frac{h+q_j+1}{p^j}\right]\right).$$

Posons, pour simplifier,

(43) $$c_j = \sum_{h=1}^{n-1}\left[\frac{h+q_j+1}{p^j}\right].$$

[1] Ces Y_n correspondent aux caractères principaux des représentations irréductibles de $\mathfrak{S}_n$.

Nous nous proposons de trouver une expression plus simple des c_j. Posons encore $n+q_j=a_jp^j+r_j$, $0\leqq r_j<p^j$. Avec ces nouvelles notations, on a

$$c_j=\left[\frac{1+q_j}{p^j}\right]+\ldots+\left[\frac{n+q_j}{p^j}\right]=\sum_{1}^{2p^j-1}1+\sum_{2p^j}^{3p^j-1}2+\ldots+\sum_{(a_j-1)p^j}^{a_jp^j-1}(a_j-1)+\sum_{a_jp^j}^{n+q_j}a_j=$$

$$=p^j\frac{a_j(a_j-1)}{2}+a_j(r_j+1)=\frac{a_j}{2}(p^j(a_j-1)+2r_j+2)=\frac{a_j}{2}(n+q_j-r_j+2r_j-p^j+2)=$$

$$=\frac{a_j}{2}(n+r_j-q_j+1)=\frac{(n+q_j-r_j)(n+1-(q_j-r_j))}{2p^j}$$

et finalement

$$c_j=\frac{n(n+1)-(q_j-r_j)(q_j-r_j-1)}{2p^j}. \tag{44}$$

Si on pose alors

$$\frac{n(n+1)}{2p^j}=e_j+\theta_j,\quad e_j \text{ entier},\quad 0\leqq\theta_j<1, \tag{45}$$

et $d_j=q_j-r_j$, les formules (40), (42), (43), (44) et (45) conduisent à l'expression suivante de $|Y_n|_p$

$$|Y_n|_p=|U_n|_p-|V_n|_p=\sum_{j\geqq1}\left(\left[\frac{n(n+1)}{2p^j}\right]-\frac{n(n+1)-d_j(d_j-1)}{2p^j}\right)=$$

$$=\sum_{j\geqq1}\left(e_j-e_j-\theta_j+\frac{d_j(d_j-1)}{2p^j}\right)=\sum_{j\geqq1}\left(\frac{d_j(d_j-1)}{2p^j}-\theta_j\right).$$

Mais, chaque terme $\frac{d_j(d_j-1)}{2p^j}-\theta_j$ est un entier et les θ_j verifient $0\leqq\theta_j<1$, on obtient donc

$$|Y_n|_p=\sum_{j\geqq1}\left[\frac{d_j(d_j-1)}{2p^j}\right].$$

On peut noter que d_j peut s'interpréter simplement comme la distance algébrique de $-n$ à l'élément de $p^j\mathbf{Z}$ le plus proche (de façon précise, si p^jn' désigne le multiple de p^j le plus proche de $-n$, et le plus petit en cas d'ambiguïté, d_j est égal à $-n-p^jn'$). On obtient ainsi l'énoncé suivant.

PROPOSITION 3. *Soit* $Y_n=Y(n,n-1,\ldots,1)$. *Pour* p *premier, on désigne par* d_j *la distance algébrique de* $-n$ *à l'ensemble* $p^j\mathbf{Z}$ *(voir ci-dessus pour une définition précise). La valuation p-adique de* Y_n *est alors donnée par la formule*

$$|Y_n|_p=\sum_{j\geqq1}\left[\frac{d_j(d_j-1)}{2p^j}\right]. \tag{46}$$

De cette formule, nous allons déduire quelques propriétés arithmétiques des Y_n.

Remarquons d'abord que, avec les notations précédentes, on a toujours les inégalités $-q_j \leqq d_j \leqq q_j$ et que, de plus,

$$\max d_j(d_j-1) = q_j(q_j-1) = \frac{p^{2j}-1}{4}. \tag{47}$$

En outre, si j est tel que l'on ait l'inégalité $n+q_j<p^j$, ou encore, ce qui est équivalent, si $2n-1<p^j$, alors r_j est donné par la formule $r_j=n+q_j$ et, dans ce cas, $d_j=-n$ et

$$\frac{d_j(d_j-1)}{2} = \frac{n(n+1)}{2}.$$

Compte tenu de ces remarques, la formule (46) équivaut à

$$|Y_n|_p = \sum_{\substack{j\geqq 1\\ p^j\leqq 2n-1}} \left[\frac{d_j(d_j-1)}{2p^j}\right] + \sum_{p^j\geqq 2n} \left[\frac{n(n+1)}{2p^j}\right]. \tag{46 bis}$$

D'où aussitôt les minorations

$$|Y_n|_p \geqq \sum_{p^j\geqq 2n} \left[\frac{n(n+1)}{2p^j}\right] \geqq \sum_{j\geqq 1} \left[\frac{[(n+1)/4]}{p^j}\right]. \tag{48}$$

Il en résulte:

Corollaire 1. *Le coefficient* Y_n *est divisible par* $r!$, *où* $r=\left[\frac{n+1}{4}\right]$.

Les formules (46 bis) et (47) permettent aussi de majorer $|Y_n|_p$ comme suit:

$$|Y_n|_p \leqq \sum_{j\geqq 1,\, p^j<2n} \left[\frac{p^{2j}-1}{8p^j}\right] + \sum_{p^j\geqq 2n} \left[\frac{n(n+1)}{2p^j}\right] = \sum_{j\geqq 1,\, p^j<2n} \left[\frac{p^j}{8}\right] + \sum_{p^j\geqq 2n} \left[\frac{n(n+1)}{2p^j}\right].$$

Pour p impair, il vient

$$|Y_n|_p \leqq \frac{p^{k+1}-p}{8(p-1)} - \frac{k}{8} + \frac{n(n+1)}{2p^k(p-1)}, \tag{49}$$

où k désigne l'entier tel que $p^k<2n<p^{k+1}$ $\left(\text{i.e. } k=\left[\frac{\operatorname{Log} 2n}{\operatorname{Log} p}\right]\right)$. Il est facile d'en déduire la majoration

$$|Y_n|_p \leqq \frac{2n(p+1)+p}{8(p-1)} - \frac{1}{8}\left[\frac{\operatorname{Log} 2n}{\operatorname{Log} p}\right]. \tag{50}$$

Si, dans cette inégalité on fixe p et on fait tendre n vers l'infini, il vient

$$\varlimsup_{n\to\infty} \frac{|Y_n|_p}{n} \leqq \frac{p+1}{4(p-1)}. \tag{51}$$

En considérant les valeurs particulières $n=\frac{p^l-1}{2}$, $l=1, 2, \dots,$ on obtient l'égalité

$$\varlimsup_{n\to\infty} \frac{|Y_n|_p}{n} = \frac{p+1}{4(p-1)}. \tag{52}$$

La formule (46 bis) fournit la minoration

$$\varliminf_{n\to\infty} \frac{|Y_n|_p}{n} \geqq \frac{1}{4(p-1)}. \tag{53}$$

Les cas particuliers $n=p^l$ conduisent à l'inégalité

$$\varliminf \frac{|Y_n|_p}{n} \leqq \frac{1}{2(p-1)}.$$

Résumons les résultats obtenus.

COROLLAIRE 2. *Soit p un nombre premier impair fixé. On a toujours la majoration*

$$|Y_n|_p \leqq \frac{2n(p+1)+p}{8(p-1)} - \frac{1}{8}\left[\frac{\operatorname{Log} 2n}{\operatorname{Log} p}\right].$$

De plus,

$$\varlimsup_{n\to\infty} \frac{|Y_n|_p}{n} = \frac{p+1}{4(p-1)}$$

et

$$\frac{1}{4(p-1)} \leqq \varliminf \frac{|Y_n|_p}{n} \leqq \frac{1}{2(p-1)}.$$

V. Application à l'étude de certaines fonctions arithmétiques

Nous nous proposons de montrer que l'étude arithmétique des Y_n permet de donner en particulier des estimations de sommes du type $\sum_{p\leqq m} \frac{\operatorname{Log} p}{p}$. Pour ceci il nous faut d'abord évaluer Y_n, ce qui peut se faire grâce à la formule d'Euler Mac-Laurin.

1. *Formule d'Euler Mac-Laurin.* LEMME 3. *Soit f une fonction trois fois continûment dérivable sur l'intervalle* $[0, n]$. *On désigne par Φ la fonction de période* 1 *caractérisée par*

$$\Phi(x) = x^3 - \frac{3}{2}x^2 + \frac{x}{2}, \quad pour \quad 0 \leqq x \leqq 1.$$

Alors

$$f(0)+\ldots+f(n) = \int_0^n f(t)\,dt + \frac{1}{2}(f(n)+f(0)) + \frac{1}{12}(f'(n)-f'(0)) - \tag{55}$$

$$- \int_0^n \frac{\Phi(t)}{6} f'''(t)\,dt.$$

DÉMONSTRATION. Dans tous les livres de calcul infinitésimal.

Posons

$$R = \frac{1}{6}\int_0^n \Phi(t) f'''(t)\,dt.$$

On peut majorer R comme suit,

(56) $$|R| \leqq \frac{1}{6} \max_{0 \leqq x \leqq 1} |\Phi(x)| \int_0^n |f'''(t)|dt = \frac{1}{72\sqrt{3}} \int_0^n |f'''(t)|.$$

De plus, il est facile de voir que R est du signe de $-f''(t)$ lorsque cette fonction conserve un signe constant sur l'intervalle $[0, n]$.

2. *Evaluation de* Y_n. Reprenons les notations $Y_n = \frac{U_n}{V_n}$. D'après (38), on a

$$V_n = 1^n \cdot 3^{n-1} \cdot 5^{n-2} \cdot \ldots \cdot (2n-1),$$

D'où

(57) $$\operatorname{Log} V_n = (n-1) \operatorname{Log} 3 + (n-2) \operatorname{Log} 5 + \ldots + \operatorname{Log}(2n-1).$$

Si on applique la formule d'Euler Mac-Laurin aux fonctions

$$f(x) = \operatorname{Log}(1+2x) \quad \text{et} \quad g(x) = x \operatorname{Log}(1+2x),$$

on obtient

(58) $$\sum_{k=0}^{n} \operatorname{Log}(1+2k) = (n+1) \operatorname{Log}(1+2n) - n + c_1 + \frac{1}{6(2n+1)} + R_2,$$

où R_2 vérifie

(59) $$-\frac{1}{18\sqrt{3}(1+2n)^3} \leqq R_2 \leqq 0,$$

et

(60) $$\sum_{k=0}^{n} k \operatorname{Log}(1+2k) = \frac{1}{2}\left(n^2 - \frac{1}{4}\right) \operatorname{Log}(1+2n) - \frac{n^2}{4} + \frac{n}{2} \operatorname{Log}(1+2n) - $$
$$- \frac{1}{12(n+1)} + R_3$$

avec

(61) $$-\frac{1}{36\sqrt{3}}\left(\frac{1}{2n+1} + \frac{1}{(2n+1)^3}\right) \leqq R_3 \leqq 0,$$

où c_1 et c_2 sont des constantes.

Les formules (58) à (61) conduisent à une estimation du type

$$\operatorname{Log} V_n = \frac{n^2}{2} \operatorname{Log}(2n+1) - \frac{3}{4} n^2 + \frac{n}{2} \operatorname{Log}(2n+1) + c_1' n + c_2' + \frac{\theta_3}{27\sqrt{3}(2n+1)}$$

avec $0 \leqq \theta_3 \leqq 1$, et c_1', c_2' constantes. Pour calculer $\operatorname{Log} U_n$, on déduit du lemme 3 l'estimation

$$\log m! = m \log m - m + \frac{1}{2} \log m + c_3 + \frac{1}{12m} - \frac{\theta_4}{72\sqrt{3}(m+1)^2}, \quad 0 \leqq \theta_4 \leqq 1.$$

On obtient finalement l'existence de constantes c et c' telles que

(62) $$\operatorname{Log} Y_n = \frac{n(n+1)}{2} \operatorname{Log} \frac{n(n+1)}{2(2n+1)} + \frac{n^2}{4} + cn + c' + \frac{\theta}{12n}, \quad \text{avec} \quad |\theta| \leqq 1.$$

3. *Estimation de* $\sum_{p \leqq x} \frac{\log p}{p}$ (méthode élémentaire). D'après la formule (46 bis), on peut écrire

$$\operatorname{Log} Y_n = A_1 + A_2 + A_3 + A_4,$$

où

$$A_1 = |Y_n|_2 \operatorname{Log} 2, \quad A_2 = \sum \left[\frac{n(n+1)}{2p^j}\right] \operatorname{Log} p,$$

où la sommation est étendue aux entiers p^j, p premier, tels que $2 < p \leqq 2n$ et $p^j > 2n$,

$$A_3 = \sum_{p^j < 2n} \left[\frac{d_j(d_j-1)}{2p^j}\right] \operatorname{Log} p, \quad \text{et} \quad A_4 = \sum_{p \geqq 2n} \left[\frac{n(n+1)}{2p}\right] \operatorname{Log} p.$$

Pour estimer A_4, il nous faut d'abord étudier les quantités A_1, A_2 et A_3.

a) *Estimation de* A_1. Il est facile de montrer que l'on a

(63) $$A_1 = \frac{n(n+1)}{2} \operatorname{Log} 2 - \eta_1 (2 \operatorname{Log} n + \operatorname{Log} 2), \quad \text{avec} \quad 0 \leqq \eta_1 \leqq 1.$$

b) *Estimation de* A_2. Nous nous contenterons de la majoration

$$A_2 = \sum_{2 < p \leqq 2n,\, p^j > 2n} \left[\frac{n(n+1)}{2p^j}\right] \operatorname{Log} p \leqq \operatorname{Log} 2n \left(\sum_{p \leqq 2n,\, p^j > 2n} \frac{n(n+1)}{2p^j}\right).$$

Posons, comme plus haut, $N = \frac{n(n+1)}{2}$. Le nombre d'entiers premiers au plus égaux à x sera noté $\Pi(x)$. On a

$$A_2 \leqq \left(\sum_{N^{1/3} \leqq p \leqq 2n} \frac{n(n+1)}{2p^2} + \frac{n+1}{4} \sum_{p \leqq N^{1/3},\, j \geqq 0} p^{-j}\right) \operatorname{Log} 2n.$$

D'où

(64) $$A_2 \leqq \left(\left(\frac{n(n+1)}{2}\right)\right)^{1/3} \Pi(2n) + \frac{n+1}{2} \Pi(N^{1/3}) \operatorname{Log} 2n.$$

La majoration triviale $\Pi(x) < x$ donne alors

$$A_2 \leqq N^{1/3} \frac{5n+1}{2} \operatorname{Log} 2n,$$

et en particulier

(65) $$A_2 = O(n^{5/3} \operatorname{Log} n).$$

c) *Estimation de* A_3. De la majoration $d_j(d_j-1) \leqq \frac{p^{2j}}{4}$ (voir (47)), on déduit

$$A_3 \leqq \frac{1}{8} \sum_{p^j \leqq 2n} p^j \operatorname{Log} p.$$

D'où, aisément,

(66) $$A_3 \leqq \frac{1}{8} \sum_{p \leqq 2n} p \operatorname{Log} p + \frac{n}{4} \operatorname{Log} n \cdot \Pi(\sqrt{2n}).$$

En utilisant encore une fois $\Pi(x)<x$, il vient

$$A_3 \leqq \frac{1}{8} \sum_{p \leqq 2n} p \operatorname{Log} p + \frac{n\sqrt{2n}}{4} \operatorname{Log} n. \tag{67}$$

d) *Estimation de* A_4. Depuis Tchebytcheff, on sait montrer de façon très élémentaire qu'il existe une constante a' telle que $\theta(x) \leqq a'x$ $\left(\text{où } \theta(x) = \sum_{p \leqq x} \operatorname{Log} p\right)$. Il en résulte aussitôt l'inégalité

$$A_3 \leqq \frac{a'}{2} n^2 + \frac{n}{4} \sqrt{2n} \operatorname{Log} n \quad \text{(cf. (67))}.$$

D'où l'encadrement

$$A_4 \geqq \operatorname{Log} Y_n - \frac{a'}{2} n^2 - \frac{n(n+1)}{2} \operatorname{Log} 2 - \left(\frac{n(n+1)}{2}\right)^{1/3} \frac{5n+1}{2} \operatorname{Log} 2n - \frac{n}{4} \sqrt{2n} \operatorname{Log} n,$$

$$A_4 \leqq \operatorname{Log} Y_n - \frac{n(n+1)}{2} \operatorname{Log} 2 + 2 \operatorname{Log} n + \operatorname{Log} 2,$$

en utilisant les inégalités (63) à (67) et le fait que les quantités A_i sont positives ou nulles. En particulier, (62) implique

$$\overline{\lim} \left(\frac{2A_4}{n^2} - \operatorname{Log} n + 3 \operatorname{Log} 2 - \frac{1}{2} \right) \leqq a'. \tag{69}$$

Remarque. D'après Rosser et Schoenfeld, Th. 9 p. 71, [3], on peut prendre $a' = 1{,}017$.

e) *Estimation de* $S = \sum_{2n \leqq p \leqq \frac{n(n+1)}{2}} \frac{\operatorname{Log} p}{p}$. Reprenons l'expression de A_4,

$$A_4 = \sum_{2n \leqq p \leqq N} \left[\frac{N}{p}\right] \operatorname{Log} p.$$

On a clairement

$$A_4 = NS - \eta_2 \theta(N), \quad 0 \leqq \eta_2 \leqq 1, \tag{70}$$

où θ désigne la fonction de Tchebytcheff.

D'où l'encadrement

$$\frac{A_4}{N} \leqq S \leqq \frac{A_4}{N} + a',$$

et, grâce à (69), l'inégalité

$$\overline{\lim} \, |S - \operatorname{Log} n - 2 \operatorname{Log} 2| \leqq \operatorname{Log} 2 + a'. \tag{71}$$

Des estimations de Y_n et A_4, on déduit:

PROPOSITION 5. *Il existe deux constantes* c *et* c' *telles que la quantité* $S=\sum_{2n\leq p\leq N} \operatorname{Log} p/p$, *avec* $N=n(n+1)/2$, *vérifie les inégalités*

$$-a'-\frac{5n+1}{2N^{2/3}}\operatorname{Log} 2n-\frac{2n}{2(2n+1)}\operatorname{Log} n+\frac{4c+1-2a'}{2(n+1)}+\frac{c'}{N}-\frac{1}{12nN}\leqq$$

$$\leqq S-\operatorname{Log}\frac{n(n+1)}{2(2n+1)}-\frac{1}{2}-\operatorname{Log} 2\leqq\frac{4c+1}{2(n+1)}+N^{-1}(\operatorname{Log} 2n^2+c'+1/12n)$$

où $a'=\max(\theta(x)/x)$.

4. *Applications de la théorie des nombres premiers (méthode non élémentaire).* On montre ([2] p. 91) qu'il existe une constante E telle que

(72) $$\sum_{p\leq x}\operatorname{Log} p/p=\operatorname{Log} x+E+o(1),$$

d'où

(73) $$S=\operatorname{Log} n-2\operatorname{Log} 2+o(1).$$

Nous allons déduire de (73) une estimation de A_4.

a) *Retour sur l'estimation de* A_4. Remarquons d'abord l'égalité

(74) $$\frac{n(n+1)}{2}S-A_4=\sum_{2n\leq p\leq N}\left\{\frac{N}{p}\right\}\operatorname{Log} p=S'\quad\text{(disons)}.$$

Nous sommes amenés à étudier la somme S' et pour ce faire le résultat suivant sera nécessaire

LEMME 4. *On a, pour* m *tendant vers l'infini,*

(75) $$\sum_{p\leq m}\left\{\frac{m}{p}\right\}\operatorname{Log} p\sim(1-\gamma)m,$$

où $\gamma=0{,}57721566490153286060\ldots$ *désigne la constante d'Euler.*

Il est bien connu que

$$\operatorname{Log} m!=\sum_{p^h\leq m}\left[\frac{m}{p^h}\right]\operatorname{Log} p.$$

Ce qui permet encore d'écrire

$$\operatorname{Log} m!=\sum_{p\leq m}\frac{m}{p}\operatorname{Log} p-\sum\left\{\frac{m}{p}\right\}\operatorname{Log} p+\sum_{p^h\leq m,\,h\geq 2} mp^{-h}\operatorname{Log} p-$$

$$-\sum_{p^h\leq m,\,h\geq 2}\{mp^{-h}\}\operatorname{Log} p.$$

Remarquons en outre que la série

$$\sum_{j\geq 2}\sum_{p}p^{-j}\operatorname{Log} p=\sum_{p}\frac{\operatorname{Log} p}{p(p-1)}$$

est convergente, soit c' sa somme. Alors,

$$\sum_{p^h \leqq m,\, h \geqq 2} (\operatorname{Log} p) p^{-h} = c' + o(1). \tag{76}$$

D'autre part

$$\sum_{p^h \leqq n,\, h \geqq 2} \{mp^{-h}\} \operatorname{Log} p = \sum_{p \leqq \sqrt{m},\, h \geqq 2,\, p^h \leqq n} \{mp^{-h}\} \operatorname{Log} p \leqq \sum_{p \leqq \sqrt{m}} \operatorname{Log} p (\sum_{h \geqq 2} 1)$$

dans cette dernière sommation la variable h doit bien sûr vérifier $p^h \leqq n$, ce qui implique

$$\sum_{p^h \leqq m,\, h \geqq 2} \{mp^{-h}\} \operatorname{Log} p \leqq \sum_{p \leqq \sqrt{m}} \operatorname{Log} p \left(\frac{\operatorname{Log} m}{\operatorname{Log} p}\right) = \Pi(\sqrt{m}) \operatorname{Log} m.$$

D'où, en particulier, (puisque $\Pi(x) < x$)

$$\sum_{p^h \leqq n,\, h \geqq 2} \{mp^{-h}\} \operatorname{Log} p \leqq \sqrt{m} \operatorname{Log} m. \tag{77}$$

Grâce à (72), (76) et (77), on obtient

$$\operatorname{Log} m! = m \operatorname{Log} m + Em + C'm - \sum_{p \leqq m} \left\{\frac{m}{p}\right\} \operatorname{Log} p + o(m).$$

Si on compare cette formule à la formule de Stirling (démontrée plus haut)

$$\log m! = m \log m - m + o(m),$$

il vient

$$\sum_{p \leqq m} \left\{\frac{m}{p}\right\} \operatorname{Log} p \sim (E + C' + 1) m.$$

D'après INGHAM, [2] p. 91, on peut montrer que $E = -C' - \gamma$, d'où le lemme. En utilisant (73), (74) et (75), on obtient l'estimation suivante.

PROPOSITION 6. *On a*

$$\sum_{2n \leqq p \leqq n(n+1)/2} [n(n+1)/2p] \operatorname{Log} p = \frac{n^2}{2} \operatorname{Log} n - n^2 \left(\operatorname{Log} 2 + \frac{1-\gamma}{2}\right) + o(n^2). \tag{78}$$

b) *Nouvelle étude de* A_3. De (78), (62) et (65) on déduit aisément le résultat arithmétique suivant.

PROPOSITION 7. *Soit n un entier. Pour p premier et j entier $\geqq 1$, on pose d_j = distance algébrique de $-n$ à $p^j\mathbf{Z}$. On a alors*

$$\sum_{p^j \leqq 2n} \left[\frac{d_j(d_j-1)}{2p^j}\right] \operatorname{Log} p \sim \left(\frac{3}{4} - \frac{\gamma + \operatorname{Log} 2}{2}\right) n^2.$$

VI. Applications arithmétiques-suite

L'étude des quotients Y_{n+1}/Y_n conduit à d'autres résultats arithmétiques qui concernent particulièrement les nombres entiers $\left(\frac{(n+1)(n+2)}{2}\right)!\Big/\left(\frac{n(n+1)}{2}\right)!$. Il nous faut d'abord, comme plus haut, estimer quelques quantités.

1. *Estimation de* Y_{n+1}/Y_n. Posons $\Delta_n=\log(Y_{n+1}/Y_n)$. D'après la formule

$$Y_m = \frac{(m(m+1)/2)!}{1^n \cdot 3^{n-1} \cdot \ldots \cdot (2n-1)},$$

on a

$$\Delta_n = \mathrm{Log}\left(((n+1)(n+2)/2)!\right) - \mathrm{Log}\left((n(n+1)/2)!\right) + \mathrm{Log}\,(2^n n!) - \mathrm{Log}\left((2n)!\right).$$

Grâce à la formule de Stirling sous la forme

$$\mathrm{Log}\, m! = m\,\mathrm{Log}\, m - m + \frac{1}{2}\mathrm{Log}\, m + C + O\left(\frac{1}{m}\right),$$

on obtient — après quelques calculs —

$$\Delta_n = n\,\mathrm{Log}\, n - (2\,\mathrm{Log}\, 2 - 1)n + 2\log n + \left(2 - \frac{3}{2}\mathrm{Log}\, 2\right) + O\left(\frac{1}{n}\right). \tag{79}$$

2. *Applications* (méthode élémentaire). Posons

$$u = \left(\frac{n(n+1)}{2}+1\right)\left(\frac{n(n+1)}{2}+2\right)\ldots\frac{(n+1)(n+2)}{2} = \frac{\left(\frac{(n+1)(n+2)}{2}\right)!}{\left(\frac{n(n+1)}{2}\right)!},$$

$$v = 3\cdot 5\cdot \ldots \cdot(2n-1), \quad (\text{d'où } Y_{n+1}/Y_n = u/v),$$

$$d_j' = \text{distance algèbrique de } -(n+1) \text{ à } p^j\mathbf{Z},$$

$$d_j = \text{distance algèbrique de } -n \text{ à } p^j\mathbf{Z},$$

$$B_1 = \sum_{p^j<2(n+1)}\left(\left[\frac{d_j'(d_j'-1)}{2p^j}\right]-\left[\frac{d_j(d_j-1)}{2p^j}\right]\right)\mathrm{Log} p,$$

$$B_2 = \sum_{2(n+1)\leq p^j\leq\frac{n(n+1)}{2}}\left([(n+1)(n+2)/2p^j]-[n(n+1)/2p^j]\right)\mathrm{Log}\, p,$$

$$B_3 = \sum_{p^j>n(n+1)/2}\left([(n+1)(n+2)/2p^j]\right)\mathrm{Log}\, p.$$

De la formule (46), on déduit

$$\Delta_n = B_1+B_2+B_3. \tag{80}$$

Nous nous proposons d'étudier les quantités B_i.

a) *Estimation de* B_1. Remarquons que $d'_j = d_j - 1$ sauf si $d_i = q_j$, auquel cas $d'_j = -d_j$. Ces deux formules montrent que, dans tous les cas, on a

$$\left| \left[\frac{d'_j(d'_j-1)}{2p^j} \right] - \left[\frac{d_j(d_j-1)}{2p^j} \right] \right| \leqq 1.$$

D'où la majoration de B_1

(81) $$|B_1| \leqq \sum_{p^j \leqq 2n+2} \operatorname{Log} p = \psi(2n+2), \quad \text{(fonction } \psi \text{ de Tchebytcheff)}.$$

b) *Estimation de* B_3. Si, dans l'expression de B_3, on partage la somme pour $j=1$ et celle pour $j \geqq 2$, pour $n \geqq 3$ (cas auquel nous nous limiterons désormais), il vient

$$B_3 = \theta\big((n+1)(n+2)/2\big) - \theta\big(n(n+1)/2\big) + B'_3,$$

avec

$$B'_3 = \sum_{n(n+1)/2 \leqq p^j \leqq (n+1)(n+2)/2,\; j \geqq 2} [(n+1)(n+2)/2p^j] \operatorname{Log} p.$$

Dès que n est assez grand, l'intervalle $\left[\frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2} \right]$ contient au plus un carré et un cube, il existe donc n_0 tel que $n \geqq n_0$ implique

$$B'_3 \leqq 2 \operatorname{Log}\big((n+1)(n+2)/2\big) + \sum_{p^j \leqq \frac{(n+1)(n+2)}{2},\; j \geqq 5} \operatorname{Log} p \leqq$$

$$\leqq 2 \operatorname{Log}\big((n+1)(n+2)/2\big) + \left(\frac{(n+1)(n+2)}{2} \right)^{\frac{1}{5}} \operatorname{Log} \left(\frac{(n+1)(n+2)}{2} \right)^{\frac{1}{5}}$$

(on a utilisé $\Pi(x) \leqq x$). D'où

(82) $$B_3 = \theta\big((n+1)(n+2)/2\big) - \theta\big(n(n+1)/2\big) + O(n^{2/5} \operatorname{Log} n).$$

c) *Etude de* B_2 *et applications*. Si on utilise le fait, très simple à démontrer (il suffit de remarquer que $\underline{\lim} \frac{\Phi(n)}{n} = 0$, Φ indicatrice d'Euler), que la densité des nombres premiers tend vers zéro, on obtient

$$B_1 = o(n \operatorname{Log} n) \quad \text{et} \quad B_3 = o(n \operatorname{Log} n).$$

Grâce à (79) et (80), il vient alors

(83) $$B_2 \sim n \operatorname{Log} n.$$

Pour p premier $\geqq 2n+2$, posons

$$\lambda_p = [(n+1)(n+2)/2p] - [n(n+1)/2p].$$

Il est facile de vérifier que $\lambda_p = 0$ ou 1 et que, de plus, $\lambda_p = 1$ si, et seulement si, p divise u.

L'expression de B_2 peut encore s'écrire

(84) $$B_2 = \sum_{2n+2 \leqq p \leqq n(n+1)/2} \lambda_p \operatorname{Log} p + B'_2,$$

où

$$B_2' = \sum_{2n+2 \leqq p^j \leqq n(n+1)/2,\ j \geqq 2} ([(n+1)(n+2)/2p^j] - [n(n+1)/2p^j]) \operatorname{Log} p.$$

Majorons B_2'

$$B_2' \leqq \sum_{2n+2 \leqq p^j \leqq n(n+1)/2,\ j \geqq 2} \operatorname{Log} p \leqq \sum_{p \leqq (n(n+1)/2)^{1/2}} \operatorname{Log}\left(\frac{n(n+1)}{2}\right),$$

soit

$$B_2' \leqq \operatorname{Log}\left(\frac{n(n+1)}{2}\right) \Pi\left(\left(\frac{n(n+1)}{2}\right)^{1/2}\right) = o(n \operatorname{Log} n).$$

Ce résultat, joint à (83) et (84), donne

$$\sum_{2n \leqq p \leqq n(n+1)/2} \lambda_p \operatorname{Log} p \sim n \operatorname{Log} n.$$

D'après l'interprétation de λ_p, on obtient le résultat suivant.

PROPOSITION 8. *Soit* $u = \prod_{n(n+1)/2 < k \leqq (n+1)(n+2)/2} k$. *On a l'équivalence*

(85) $$\sum_{p|u} \operatorname{Log} p \sim n \operatorname{Log} n.$$

La relation (85) a plusieurs conséquences:

COROLLAIRE 1. *Soit* Q *le nombre de diviseurs premiers de* u. *Soit* $\varepsilon > 0$ *fixé. Alors, pour* n *assez grand, on a*

$$(1-\varepsilon)\frac{n}{2} \leqq Q \leqq (1+\varepsilon)n.$$

COROLLAIRE 2. *Il existe* $b > 0$ *tel que, pour tout* n, *le nombre* u *ait un diviseur premier* p *tel que* $p > bn \operatorname{Log} n$.

3. *Retour à* B_1 (méthode non élémentaire). Suivant la même démarche que dans la paragraphe précédent, nous allons utiliser des résultats arithmétiques précis pour estimer la quantité B_1 en particulier.

Posons $J = \sum_{p \geqq 2n,\ p|u} \operatorname{Log} p$. D'après (80) et (83) on a d'une part

(86) $$\Delta_n = B_1 + J + o(n).$$

D'autre part on peut évaluer J en étudiant le nombre

$$c_n = \binom{(n+1)(n+2)/2}{n+1}.$$

(87) $$\operatorname{Log} c_n = n \operatorname{Log} n + (1 - \operatorname{Log} 2)n + O(\operatorname{Log} n)$$

et

(88) $$\operatorname{Log} c_n = J + B_5$$

avec

$$B_5 = \sum_{p < 2n} (|u|_p - |(n+1)!|_p) \operatorname{Log} p.$$

Pour majorer B_5 remarquons que, pour tout nombre a, le nombre de facteurs de u divisibles par a est égal au nombre de facteurs de $(n+1)!$ divisibles par

a augmenté au plus de 1. Il en résulte facilement la majoration

$$B_5 \leqq \operatorname{Log} K\Pi(K^{1/3})+\theta(K^{1/2})+\theta(2n)$$

où $K=n(n+1)/2$.

D'où (utiliser le théorème des nombres premiers)

$$0 \leqq B_5 \leqq 2n+o(n). \tag{89}$$

La comparaison de (87), (88) et (89) donne

$$n \operatorname{Log} n-(1+\operatorname{Log} 2)n-o(n) \leqq J \leqq n \operatorname{Log} n+(1-\operatorname{Log} 2)n+o(n). \tag{90}$$

Si on reporte (90) en (86), il vient, grâce à (79),

$$-n \operatorname{Log} 2-o(n) \leqq B_1 \leqq (2-\operatorname{Log} 2)n+o(n). \tag{91}$$

De plus, posons

$$B_1' = \sum_{p^j \leqq 2(n+1)} \left| \left[\frac{d_j'(d_j'-1)}{2p^j}\right] - \left[\frac{d_j(d_j-1)}{2p^j}\right] \right| \operatorname{Log} p.$$

Le théorème des nombres premiers, joint à la démonstration de (81), conduit à la majoration

$$|B_1| \leqq B_1' \leqq 2n+o(n). \tag{92}$$

De (79), (86) et (92) on tire

$$n \operatorname{Log} n-(3-2\operatorname{Log} 2)n-o(n) \leqq J \leqq n \operatorname{Log} n+(2\operatorname{Log} 2+1)n+o(n). \tag{93}$$

La conjonction de (90) et (93) fournit l'encadrement

$$n \operatorname{Log} n-(3-2\operatorname{Log} 2)n-o(n) \leqq J \leqq n \operatorname{Log} n+(1-\operatorname{Log} 2)n+o(n). \tag{94}$$

D'où

$$-n \operatorname{Log} 2-o(n) \leqq B_1 \leqq 4(1-\operatorname{Log} 2)n+o(n). \tag{95}$$

QUESTIONS. Quelle est la meilleure constante c_0 telle que $|B_1| \leqq c_0 n+o(n)$? Existe-t-il α tel que $B_1=\alpha n+o(n)$?

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DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ LOUIS PASTEUR
67084 — STRASBOURG
FRANCE

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TWO ELEMENTARY COMMUTATIVITY THEOREMS FOR RINGS

By

A. HARMANCI (Ankara)

In a paper [5], JOHNSEN, OUTCALT and YAQUB have proved that a ring R having a unity and satisfying $(xy)^2=x^2y^2$, for all $x, y\in R$ is commutative. They produced an example for a non-commutative ring R with unity and satisfying $(xy)^k=x^ky^k$ for all $x, y\in R$ and for any integer $k>2$. In [6], LUH proved the following: Let R be ring with unity and J be its Jacobson radical and $n>1$ be an integer such that $(xy)^k=x^ky^k$, $k=n$, $n+1$, $n+2$, $x, y\in R$, holds in R and R/J is simple, then R is commutative. He also showed by an example that, for $k=3, 4$, there exists a non-commutative ring R with unity satisfying the identity $(xy)^k=x^ky^k$, for all $x, y\in R$. The question therefore naturally arises: Under what conditions is a ring R with unity, satisfying $(xy)^k=x^ky^k$ for just two consecutive integers, commutative?

In Section I of this note, we prove the following:

THEOREM A. *Let R be a ring with unity which satisfies the identities:*

$$(xy^n) = x^n y^n, \quad (xy)^{n+1} = x^{n+1} y^{n+1}, \ x, y\in R \quad \text{and} \quad n>1 \quad \text{fixed integer.}$$

If the characteristic of R does not divide $n(n!)^2$, then R is commutative.

BELL [1] proved that for a fixed integer $n>1$, a ring R generated by the n-th powers of its elements and satisfying the identity $[x^n, y]=[x, y^n]$ is commutative. In this direction we shall prove the following

THEOREM B. *If R is a ring with unity and satisfying the identities:*

$$[x^n, y] = [x, y^n], \quad [x^{n+1}, y] = [x, y^{n+1}] \quad \text{for all} \quad x, y\in R,$$

where $n>1$ is a fixed integer then R is commutative.

In what follows $[a, b]$ will denote the commutator $ab-ba$ and R an associative ring with unity.

Section I

LEMMA 1. *For any positive integer n,*

$$n! = \sum_{r=0}^{n} (-1)\binom{n}{r}(n-r)^n.$$

For the proof see [2], page 58.

LEMMA 2. *For any positive integer* m,

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^m=(-1)^m m!.$$

PROOF. Multiply the equality

$$m!=\sum_{r=0}^{m}(-1)^r\binom{m}{r}(m-r)^m$$

by $(-1)^m$ and use

$$\sum_{s=0}^{m}(-1)^{m-s}\binom{m}{m-s}(m-s)^m=\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^m,\quad \binom{m}{s}=\binom{m}{m-s}$$

and $(-1)^r=(-1)^{-r}$ to obtain

$$(-1)^m m!=\sum_{s=0}^{m}(-1)^m(-1)^s\binom{m}{s}(m-s)^m=\sum_{s=0}^{m}(-1)^m(-1)^{-s}\binom{m}{m-s}(m-s)^m=$$

$$=\sum_{s=0}^{m}(-1)^{m-s}\binom{m}{m-s}(m-s)^m=\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^m,$$

from which the lemma follows.

LEMMA 3. *Let* x *be a variable and* m, n *be integers satisfying* $n>1$, $0\leqq m\leqq n$. *Then the degree of the polynomial*

$$f_m(x)=\sum_{r=0}^{m}(-1)^r\binom{m}{r}(r+x)^n$$

is $n-m$.

PROOF. We first show that $f_m(x)=f_{m-1}(x)-f_{m-1}(1+x)$. In the identity

$$f_{m-1}(x)=\sum_{r=0}^{m-1}(-1)^r\binom{m-1}{r}(r+x)^n,$$

replace x by $1+x$ to get

$$f_{m-1}(1+x)=\sum_{r=0}^{m-1}(-1)^r\binom{m-1}{r}(r+1+x)^n.$$

On the other hand,

$$\sum_{r=0}^{m-1}(-1)^r\binom{m-1}{r}(r+x+1)^n=\sum_{r=1}^{m}(-1)^{r-1}\binom{m-1}{r-1}(r+x)^n=$$

$$=\sum_{r=1}^{m-1}(-1)^{r-1}\binom{m-1}{r-1}(r+x)^n+(-1)^{m-1}\binom{m-1}{m-1}(m+x)^n.$$

Hence

$$f_{m-1}(x)-f_{m-1}(1+x) = (-1)^0\binom{m-1}{0}(0+x)+\sum_{r=1}^{m-1}(-1)^r\binom{m-1}{r}(r+x)^n-$$

$$-\sum_{r=1}^{m-1}(-1)^{r-1}\binom{m-1}{r-1}(r+x)^n-(-1)^{m-1}\binom{m-1}{m-1}(m+x)^n = (-1)^0\binom{m-1}{0}(0+x)^n+$$

$$+\sum_{r=1}^{m-1}(-1)^r\left(\binom{m-1}{r}+\binom{m-1}{r-1}\right)(r+x)^n+(-1)^m\binom{m}{m}(m+x)^n =$$

$$= \sum_{r=0}^{m}(-1)^r\binom{m}{r}(r+x)^n=f_m(x).$$

To prove that the degree of $f_m(x)$ is $n-m$, we use induction on m. Since $f_0(x)=x^n$, and $f_1(x)=f_0(x)-f_0(1+x)=x^n-(1+x)^n$, the lemma is true for $m=0$ and $m=1$. Suppose the lemma is true for $m-1$. Since

$$f_m(x) = \sum_{r=0}^{m}(-1)^r\binom{m}{r}(r+x)^n = \sum_{r=0}^{m}\sum_{k=0}^{n}(-1)^r\binom{m}{r}\binom{m}{k}r^kx^{n-k},$$

the coefficient of x^{n-m} in $f_m(x)$ is $\binom{n}{m}(-1)^n m!$, by the induction hypothesis the degree of $f_{m-1}(x)$ and $f_{m-1}(1+x)$ is $n-m+1$, and since $f_m(x)=f_{m-1}(x)-$ $-f_{m-1}(1+x)$, the degree of $f_m(x)$ is just $n-m$.

As a generalization of Lemma 1, we prove

LEMMA 4. *Let $n>1$ be an integer. For any real number x,*

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}(x+r)^{n-k} = \begin{cases}(-1)^n n & \textit{if } k=0\\ 0 & \textit{if } 1\leqq k\leqq n.\end{cases}$$

PROOF. If $k=0$, then $\sum_{r=0}^{n}(-1)^r\binom{n}{r}(x+r)^n$ is just the polynomial $f_n(x)$ defined in Lemma 3 which has degree $n-n=0$. Hence $f_n(x)$ must be constant, and so

$$f_n(x) = \sum_{r=0}^{n}(-1)^r\binom{n}{r}(r+x)^n = \sum_{r=0}^{n}(-1)^r\binom{n}{r}r^n = (-1)^n n!.$$

Assume $1\leqq k\leqq n$. Since, by Lemma 3, the polynomial $f_n(x)$ is a constant, its k-th derivative

$$\frac{n!}{(n-k)!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}(r+x)^{n-k}$$

must be zero. This and $\frac{n!}{(n-k)!}\neq 0$ imply

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}(r+x)^{n-k}=0,$$

which completes the proof.

COROLLARY 1. *Let $n>1$ be an integer and R a ring with unity whose characteristic does not divide $n!$. Then, for each $x\in R$,*

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}(x+r)^{n-k}=\begin{cases}(-1)^n n! & \text{if } k=0\\ 0 & \text{if } k<m.\end{cases}$$

LEMMA 5. *Let m and n be positive integers such that $0\leqq m\leqq n$. Then the degree of the polynomial*

$$F_m(y)=\sum_{r=0}^{m}(-1)^r\binom{m}{r}[x,(r+y)^n](r+y)$$

in y, is $n-m+1$, where x and y are non-commuting variables.

PROOF. As in the proof of Lemma 3, it may be easily shown that $F_m(y)=$ $=F_{m-1}(y)-F_{m-1}(1+y)$. Since the commutator is linear,

$$F_m(y)=\sum_{r=0}^{m}\sum_{k=0}^{n}(-1)^r\binom{m}{r}\binom{n}{k}r^k[x,y^{n-k}](r+y)=\sum_{r=0}^{m}\sum_{k=0}^{n}(-1)^r\binom{m}{r}\binom{n}{k}r^{k+1}[x,y^{n-k}]+$$

$$+\sum_{r=0}^{m}\sum_{k=0}^{n}(-1)^r\binom{m}{r}\binom{n}{k}r^k[x,y^{n-k}]y.$$

The terms $[x,y^{n-m+1}]$ and $[x,y^{n-m}]y$, which occur in $F_m(y)$, give the only terms of degree $n-m+1$. The coefficients of $[x,y^{n-m+1}]$ and $[x,y^{n-m}]y$ are

$$\binom{n}{m-1}\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^m=(-1)^m\binom{n}{m-1}m!$$

and

$$\binom{n}{m}\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^m=(-1)^m\binom{n}{m}m!$$

resp. which are different from zero. Now we show that the coefficients of terms which have degree greater than $n-m+1$ are zero.

$$\binom{n}{k-1}\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^k \quad\text{and}\quad \binom{n}{k}\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^k$$

are the coefficients of the terms $[x,y^{n-k+1}]$ and $[x,y^{n-k}]y$ respectively. Since $n-k+1>n-m+1$ or $n-k>n-m$ which implies $k<m$ and $\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^k=0$. For $k<m$, $F_m(y)$ does not contain terms which have degrees greater than $n+1-m$. Hence $F_m(y)$ has degree $n+1-m$.

COROLLARY. $F_n(y)=(-1)^n\cdot n\cdot n!\cdot[x,y]$.

PROOF. The lemma implies that the degree of $F_n(y)$ is one. Hence $F_n(y)$ should include terms of the form $[x,y]$. Thus in

$$F_n(y)=\sum_{r=0}^{n}\sum_{k=0}^{n}(-1)^r\binom{n}{r}\binom{n}{k}r^{k+1}[x,y^{n-k}]+\sum_{r=0}^{n}\sum_{k=0}^{n}(-1)^r\binom{n}{r}\binom{n}{k}r^k[x,y^{n-k}]y,$$

$n-k$ must be taken 1. For this value of $n-k$,

$$F_n(y) = \sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{n}{n-1}r^n[x, y] = \binom{n}{n-1}\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^n[x, y] =$$

$$= \binom{n}{n-1}(-1)^n n![x, y] = (-1)^r \cdot n \cdot n!\,[x, y].$$

Proof of Theorem A. Multiply the identity $(xy)^n = x^n y^n$ by xy on the right and use $(xy)^{n+1} = x^{n+1}y^{n+1}$ to get $x^{n+1}y^{n+1} = x^n y^n xy$ or $x^n[x, y^n]y = 0$. Let

(1) $$G(x, y) = x^n[x, y^n]y.$$

By using (1) and cosidering $f_i(x)$ as in Lemma 3 and $F_i(y)$ as in Lemma 5, set

$$G_{10}(x, y) = G(x, y) - G(1+x, y) = f_1(x)[x, y^n]y$$

$$G_{11}(x, y) = G_{10}(x, y) - G_{10}(x, 1+y) = f_1(x)F_1(y)$$

$$G_{21}(x, y) = G_{11}(x, y) - G_{11}(1+x, y) = f_2(x)F_1(y)$$

$$G_{22}(x, y) = G_{21}(x, y) - G_{21}(x, 1+y) = f_2(x)F_2(y)$$

....

and so, for each m, $1 \leqq m \leqq n$, we find, for all $x, y \in R$,

(2) $$G_{mm}(x, y) = f_m(x)\,F_m(y).$$

For $m=n$, (2) becomes

$$0 = G_{nn}(x, y) = f_n(x)F_n(y) = (-1)^n n!(-1)^n nn![x, y] = n(n!)^2[x, y], \quad \text{for all} \quad x, y \in R.$$

The condition on the characteristic of R implies that $[x, y]=0$ for all $x, y \in R$ Hence R is commutative.

The condition on the unity in Theorem A is essential as it is shown by

Example 1. Let

$$R = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}: a, b \text{ integers}\right\}.$$

It is easy to check that R has no unity and $(xy)^k = x^k y^k$ for all $k \geqq 1$ and all $x, y \in R$. But R is not commutative.

Example 2. Let

$$R = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}$$

be a subring of 2×2 matrices over Z_2, the ring of integers mod 2. In R, $(xy)^k = x^k y^k$ for all x, y in R and all $k \geqq 1$. But R has characteristic 2 and it is not commutative.

Another example with respect to the characteristic of R, a ring with unity was essentially given in [6]:

EXAMPLE 3. Let S be a non-commutative ring of characteristic 3 such that $S^3=0$. Let R be the ring consisting of the pairs (a, n), $a\in S$, $n\in Z_3$, the ring of integers mod 3. The addition and multiplication in R are defined by

$$(a, n)+(b, m) = (a+b, n+m), \quad (a, n)\cdot(b, m) = (ab+nb+ma, nm).$$

The characteristic of R is 3, the identity of R is $(0, 1)$ and $(xy)^3=x^3y^3$, $(xy)^4=x^4y^4$ hold in R. Let $a, b\in S$ be such that $[a, b]\neq 0$, then $(a, 0)(b, 0)\neq(b, 0)(a, 0)$ shows that R is not commutative.

Section II

We begin this section with a theorem due to Herstein, which will be extensively used in the proof of our main theorem. For the sake completeness, we first prove

THEOREM (HERSTEIN, [3]). *If for every x and y in a ring R we can find a polynomial $p_{x,y}(t)$ with integer coefficients which depend on x and y such that $[x^2p_{x,y}(x)-x, y]=0$, then R is commutative.*

PROOF. Suppose R is not commutative. We can find $x, y\in R$ such that $[x, y]\neq 0$. Let T be the subring of R generated by x and y. Take $a\in T$. For a and x there exists a polynomial $p_{a,x}(t)$ with integer coefficients which depend on a and x such that $[a^2p_{a,x}(a)-a, x]=0$. For $b=a^2p_{a,x}(a)-a$ and y there exists a polynomial $p_{b,y}(t)$ with integer coefficients which depend on b and y such that $[b^2p_{b,y}(b)-b, y]=0$. Since b commutes with x, so does $c=b^2p_{b,y}(b)-b$. Thus c commutes with x and y, and so with every element in the subring T they generate. Hence c is in the centre of T. On the other hand, $c=b^2p_{b,y}(b)-b=(a^2p_{a,x}(a)-a)^2-(a^2p_{a,x}(a)-a)=-(a^2q(a)-a)$, where $q(t)$ is a polynomial with integer coefficients. So, for every $a\in T$, we can find a polynomial $q(t)$ with integer coefficients so that $a^2q(a)-a$ lies in the centre of T. By the main theorem of [4], T is commutative. Hence $[x, y]=0$ which is a contradiction to $[x, y]\neq 0$. This proves the theorem.

PROOF OF THEOREM B. Replace x by $1+x$ in $[x^n, y]=[x, y^n]$ to obtain $[(1+x)^n, y]=[1+x, y^n]=[x, y^n]=[x^n, y]$. On the other hand,

$$[(1+x)^n, y] = n[x, y]+\sum_{k=2}^{n-1}\binom{n}{k}[x^k, y]+[x^n, y],$$

by combining the last two results we get

$$n[x, y]+\sum_{k=1}^{n-1}\binom{n}{k}[x^k, y] = 0. \tag{3}$$

Again replace x by $1+x$ in $[x^{n+1}, y]=[x, y^{n+1}]$ to get $[(1+x)^{n+1}, y]=[1+x, y^{n+1}]=[x^{n+1}, y]$, and

$$[(1+x)^{n+1}, y] = (n+1)[x, y]+\sum_{k=2}^{n}\binom{n+1}{k}[x^k, y]+[x^{n+1}, y].$$

From these identities, it follows

$$(n+1)[x, y]+\sum_{j=2}^{n}\binom{n+1}{j}[x^j, y] = 0. \tag{4}$$

By subtracting (4) from (3) we find

$$0 = \left[\sum_{k=2}^{n-1}\binom{n}{k}x^k - \sum_{j=2}^{n}\binom{n+1}{j}x^j, y\right] - [x, y] = [x^2p(x)-x, y]$$

where $p(t)$ is a polynomial with integer coefficients. Hence, for all $x, y \in R$, there is a polynomial $p(t)$ with integer coefficients, which may depend on x and y, such that $x^2p(x)-x, y=0$. Hence by Herstein theorem, R is commutative.

The hypothesis of the existence of a unity in R is not superfluous in Theorem B as it is shown by the following two examples.

EXAMPLE 1. Let R be the subring generated by the matrices

$$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \quad \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \quad \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$$

in the ring of all 3×3 matrices over Z_2, the ring of integers mod 2. For each integer $k\geqq1$ and all $x, y \in R$ $[x^k, y]=[x, y^k]$ holds. However, R is not commutative.

EXAMPLE 2. Let R be the subring generated by the matrices

$$\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}, \quad \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \quad \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$$

in the ring of all 3×3 matrices over Z_2, the ring of integers mod 2. For each integer $k\geqq1$ and all $x, y \in R$ $[x^k, y]=[x, y^k]$ is satisfied in R, but R is not commutative.

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HACETTEPE UNIVERSITY
DEPARTMENT OF MATHEMATICS
ANKARA, TURKEY

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ON THE APPLICATION OF THE AVERAGING METHOD FOR SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS OF STANDARD TYPE WITH DISCONTINUOUS RIGHT-HAND SIDE

By

A. M. SAMOILENKO (Kiev), D. D. BAINOV and S. D. MILUSHEVA (Sofia)

The averaging method for systems of ordinary differential equations with discontinuous right-hand side has been justified by A. M. SAMOILENKO [1]—[6]. The present paper considers a version of the averaging method for systems of integro-differential equations of Volterra type

$$\frac{dx}{dt} = \varepsilon X\Big(t, x, \int_0^t \varphi\big(t, s, x(s)\big)\, ds\Big) \tag{1}$$

where $x, X \in R_n$, $\varphi \in R_m$, and $\varepsilon > 0$ is a small parameter.

Let the hypersurfaces

$$t = t_i(x), \quad t_i(x) < t_{i+1}(x), \quad i = 1, 2, \ldots \tag{2}$$

be given in the space (t, x).

Under the assumption that outside the hypersurface (2) the movement takes place according to equations (1), and, on every hypersurface $t_i(x)$, at the point x, the trajectory of the system (1) undergoes a momentary discontinuity, following the law

$$\Delta x|_{t=t_i(x)} = x_+ - x_- = \varepsilon I_i(x) \tag{3}$$

where x_- and x_+ are the points at which the trajectory meets and leaves the hypersurface $t = t_i(x)$, respectively, we put in correspondence with the system (1), the averaged system

$$\frac{d\bar{x}}{dt} = \varepsilon[X_0(\bar{x}) + I_0(\bar{x})] \tag{4}$$

where

$$\begin{cases} X_0(x) = \lim\limits_{T\to\infty} \dfrac{1}{T} \displaystyle\int_t^{t+T} X\Big(\theta, x, \int_0^\theta \varphi(\theta, s, x)\, ds\Big)\, d\theta, \\ I_0(x) = \lim\limits_{T\to\infty} \dfrac{1}{T} \displaystyle\sum_{t<t_i<t+T} I_i(x). \end{cases} \tag{5}$$

The following theorem for proximity of the solutions of the Cauchy problem of systems (1) and (4) holds:

THEOREM 1. *Let the following assumptions be fulfilled:*

1. *The function* $X(t, x, u)$ *is defined and continuous in the domain* $\{t \geqq 0, x \in D \subset R_n, u \in R_m\}$.

The function $\varphi(t, s, x)$ *is defined and continuous in the domain* $\{t \geqq 0, s \geqq 0, x \in D\}$.

2. *There exist positive constants* M, N, C, K, M^* *and the function* $\sigma(t, s)$, *such that*

(6)

$$\begin{cases} \left\| \dfrac{\partial t_i(x)}{\partial x} \right\| + \| X(t, x, u)\| + \|I_i(x)\| \leqq M, \quad \left\| \dfrac{\partial^2 t_i(x)}{\partial x^2} \right\| \leqq C, \\ \left\| \dfrac{\partial t_i(x)}{\partial x} - \dfrac{\partial t_i(x')}{\partial x} \right\| + \|I_i(x) - I_i(x')\| \leqq K\|x - x'\|, \\ \|X(t, x, u) - X(t, x', u')\| \leqq K[\|x - x'\| + \|u - u'\|], \\ \|\varphi(t, s, x) - \varphi(t, s, x')\| \leqq \sigma(t, s)\|x - x'\|, \quad \int\limits_0^t \sigma(t, s)\, ds \leqq M^*, \quad t \int\limits_0^t \sigma(t, s)\, ds \leqq N \end{cases}$$

for every $t \geqq 0, s \geqq 0, x, x' \in D, u, u' \in R_m, i = 1, 2, \ldots$.

3. *Uniformly with respect to* $t \geqq 0$ *and* $x \in D$ *there exist the finite limits* (5) *and*

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t < t_i < t + T} 1 = d_0, \quad d_0 = \text{const.}$$

4. *The system* (1) *has a unique solution* $x_t(x^*)$ *at every* $t^* > 0$ *and at any fixed* x^* *from the domain* D_0, $x_{t^*}(x^*) = x^*$.

5. *The averaged system* (4) *has a solution* $\bar{x} = \bar{x}(\varepsilon t, x_0)$, $\bar{x}(0, x_0) = x_0$, *which, when* $\varepsilon = 1$, *belongs to the domain* D *for every* $t \in [0, L]$, $0 < L = \text{const.}$ *together with some* ϱ*-neighbourhood* ($0 < \varrho = \text{const.}$), *and satisfies the inequalities*

$$\begin{cases} \dfrac{\partial t_1(x_0)}{\partial x} I_1(x_0) \leqq \beta < 0, \quad I_i(x_{kT}(x_0)) = I_i^k, \\ \dfrac{\partial t_i(\bar{x}(\varepsilon t, x_0))}{\partial x} I_i(\bar{x}(\varepsilon t, x_0)) \leqq \beta < 0, \quad t_i' < t < t_i'', \\ t_i = \inf\limits_{x \in D} t_i(x), \quad t_i'' = \sup\limits_{x \in D} t_i(x), \quad i = 2, 3, \ldots, d, \\ t_d < L\varepsilon^{-1} < t_{d+1} \end{cases} \tag{7}$$

or the condition $\dfrac{\partial t_i(x)}{\partial x} \equiv 0$, *when* $t = t_i = \text{const.}$ *is a hyperplane.*

Then, for any $\eta > 0$ *and any* $L > 0$ *there can be found an* $\varepsilon_0 > 0$ *such that, when* $\varepsilon < \varepsilon_0$, *the system of equations* (1) *has a solution* $x_t(x_0)$, $x_0(x_0) = x_0$, *defined for every* $t \in [0, L\varepsilon^{-1}]$, *and such that*

$$\|x_t(x_0) - \bar{x}(\varepsilon t, x_0)\| < \eta \quad \textit{when} \quad t \in [0, L\varepsilon^{-1}]. \tag{8}$$

PROOF. According to the assumptions of the theorem there exists a monotonically decreasing function $\alpha(t)$ $(\alpha(t)\to 0$ as $t\to\infty)$, such that

$$\text{(9)}\qquad \begin{cases} \left\| \int\limits_{t}^{t+T} \left[X\left(\theta, x, \int\limits_{0}^{\theta} \varphi(\theta, s, x)\, ds\right) - X_0(x) \right] d\theta \right\| \leqq \frac{1}{2}\alpha(T)\cdot T, \\ \left\| \sum\limits_{t<t_i<t+T} I_i(x) - TI_0(x) \right\| \leqq \frac{1}{2}\alpha(T)T. \end{cases}$$

Let T be a fixed and sufficiently large number, and let in the interval $(0, T)$ lie d_1 points

(10)

$$t_1(x_0) = t_1^0, \ldots, t_{d_1}(x_0) = t_{d_1}^0, \quad t_i^0 < t_{i+1}^0, \quad 0 < t_1^0, t_{d_1}^0 < T, \quad i = 1, 2, \ldots, (d_1 - 1).$$

Denote by $x(t, \tau, c)$, $x(\tau, \tau, c) = c$ the solution of the system

$$\text{(11)}\qquad x(t, \tau, c) = c + \varepsilon \int\limits_{\tau}^{t} X\left(\theta, x(\theta, \tau, c), \int\limits_{0}^{\theta} \varphi(\theta, s, x(s, \tau, c))ds\right) d\theta.$$

It is obvious that

$$x(t, \tau, c) = c + \varepsilon \int\limits_{\tau}^{t} X\left(\theta, c, \int\limits_{0}^{\theta} \varphi(\theta, s, c)\, ds\right) d\theta + R(t, \varepsilon, \tau),$$

where, when $0 < \tau \leqq t \leqq T$,

$$\|R(t, \varepsilon, \tau)\| = \varepsilon \left\| \int\limits_{\tau}^{t} \left[X\left(\theta, x(\theta, \tau, c), \int\limits_{\tau}^{\theta} \varphi(\theta, s, x(s, \tau, c))ds\right) - \right.\right.$$

$$\left.\left. - X\left(\theta, c, \int\limits_{0}^{\theta} \varphi(\theta, s, c)ds\right)\right] d\theta \right\| < \frac{1}{2}\varepsilon^2 KMT(T + 2N).$$

Consider now the solution $x_t(x_0)$ of the system (1) on the segment $[0, T]$. This solution consists of pieces determining (11). For this reason, to an accuracy of values of order ε^2 on the segment $0 < \tau \leqq t \leqq T$, the solution $x_t(x_0)$ may be defined by

$$\text{(12)}\qquad x_t(x_0) = c + \varepsilon \int\limits_{\tau}^{t} X\left(\theta, c, \int\limits_{0}^{\theta} \varphi(\theta, s, c)\, ds\right) d\theta + O(\varepsilon^2) = x_1(t, \tau, c) + O(\varepsilon^2).$$

Whence, to an accuracy of values of order ε^2, one can obtain $x_t(x_0) = x_1(t, 0, x_0)$, $0 \leqq t \leqq t_1^*$, where t_1^* is the root of the equation $t = t_1(x_1(t, 0, x_0))$, or

$$\text{(13)}\qquad t = t_1\left(x_0 + \varepsilon \int\limits_{0}^{t} X\left(\theta, x_0, \int\limits_{0}^{\theta} \varphi(\theta, s, x_0)ds\right) d\theta\right) =$$

$$= t_1^0 + \varepsilon \frac{\partial t_1(x_0)}{\partial x} \int\limits_{0}^{t} X\left(\theta, x_0, \int\limits_{0}^{\theta} \varphi(\theta, s, x_0)\, ds\right) d\theta + O(\varepsilon^2).$$

From (13), to an accuracy of ε^2, one can find

$$t_1^* = t_1^0 + \varepsilon \frac{\partial t_1(x_0)}{\partial x} \int_0^{t_1^0} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\, ds\Big)\, d\theta \equiv t_1^0 + \varepsilon \Theta_1. \tag{14}$$

Indeed,

$$t = t_1^0 + \varepsilon \frac{\partial t_1(x_0)}{\partial x} \int_0^{t_1^0} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0) ds\Big)\, d\theta +$$

$$+ \varepsilon \frac{\partial t_1(x_0)}{\partial x} \int_{t_1^0}^{t} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\, ds\Big)\, d\theta + O(\varepsilon^2) =$$

$$= t_1^0 + \varepsilon \frac{\partial t_1(x_0)}{\partial x} \int_0^{t_1^0} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0) ds\Big)\, d\theta +$$

$$+ \varepsilon \frac{\partial t_1(x_0)}{\partial x} (t - t_1^0) X\Big(\tilde{t}, x_0, \int_0^{\tilde{t}} \varphi(\tilde{t}, s, x_0) ds\Big) + O(\varepsilon^2),$$

where $\tilde{t} = t_1^0 + \mu(t - t_1^0)$, $0 \leqq \mu \leqq 1$. (For the different components of the vector X. the constant μ assumes different values from the interval $[0, 1]$.)

So,

$$x_t(x_0) = x_1(t, 0, x_0), \quad 0 < t < t_1^0 + \varepsilon \Theta_1 = t_1^*. \tag{15}$$

Further,

$$x_{t_1^*}^+(x_0) = x_1(t_1^*, 0, x_0) + \varepsilon I_1\big(x_1(t_1^*, 0, x_0)\big) \equiv$$

$$\equiv x_0 + \varepsilon \int_0^{t_1^*} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0) ds\Big)\, d\theta + \varepsilon I_1^0.$$

The solution of the system (1) between the hyperplanes $t = t_1(x)$ and $t = t_2(x)$ is described by (12), where $\tau = t_1^*$ and $c = x_{t_1^*}^+(x_0)$:

$$x_t(x_0) = x_{t_1^*}^+(x_0) + \varepsilon \int_{t_1^*}^{t} X\Big(\theta, x_{t_1^*}^+(x_0), \int_0^{\theta} \varphi(\theta, s, x_{t_1^*}^+(x_0)) ds\Big)\, d\theta + O(\varepsilon^2) = \tag{16}$$

$$= x_0 + \varepsilon \int_0^{t_1^*} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0) ds\Big)\, d\theta + \varepsilon I_1^0 +$$

$$+ \varepsilon \int_{t_1^*}^{t} X\Big(\theta, x_{t_1^*}^+(x_0), \int_0^{\theta} \varphi(\theta, s, x_{t_1^*}^+(x_0))\, ds\Big)\, d\theta + O(\varepsilon^2) =$$

$$= x_0 + \varepsilon \int_0^{t} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0) ds\Big)\, d\theta + \varepsilon I_1^0 +$$

$$+ \varepsilon \int_{t_1^*}^{t} \Big[X\Big(\theta, x_{t_1^*}^+(x_0), \int_0^{\theta} \varphi(\theta, s, x_{t_1^*}^+(x_0)) ds\Big) - X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0) ds\Big)\Big]\, d\theta + O(\varepsilon^2).$$

Since

$$\Big\|\varepsilon \int_{t_1^*}^{t} \Big[X\Big(\theta, x_{t_1^*}^{+}(x_0), \int_0^\theta \varphi(\theta, s, x_{t_1^*}^{+}(x_0))\,ds\Big) - X\Big(\theta, x_0, \int_0^\theta \varphi(\theta, s, x_0)ds\Big)\Big]\,d\theta\Big\| \leqq$$

$$\leqq \varepsilon K\Big|\int_{t_1^*}^{t} \Big[\|x_{t_1^*}^{+}(x_0)-x_0\| + \int_0^\theta \sigma(\theta, s)\|x_{t_1^*}^{+}(x_0)-x_0\|\,ds\Big]\,d\theta\Big| \leqq$$

$$\leqq \varepsilon K(1+M^*)(t-t_1^*)\|x_{t_1^*}^{+}(x_0)-x_0\| \leqq$$

$$\leqq \varepsilon^2 K(1+M^*)T\Big\|\int_0^{t_1^*} X\Big(\theta, x_0, \int_0^\theta \varphi(\theta, s, x_0)ds\Big)\,d\theta + I_1^0\Big\| \leqq$$

$$\leqq \varepsilon^2 KM(1+M^*)(1+T)T,$$

the last two summands in (16) may be ruled out and, to an accuracy of values of order ε^2, one can obtain

$$(17) \qquad x_t(x_0) = x_0+\varepsilon \int_0^t X\Big(\theta, x_0, \int_0^\theta \varphi(\theta, s, x_0)ds\Big)d\theta + \varepsilon I_1^0 = x_1(t, 0, x_0)+\varepsilon I_1^0.$$

The following question arises: will trajectory (17), after the moment t_1^*, meet the surface $t=t_1(x)$ or not, i.e. will there occur a beat or not?

In order to answer this question let us solve the system

$$x_t(x_0) = x_1(t, 0, x_0)+\varepsilon I_1^0, \quad t = t_1(x).$$

Ruling out x, one obtains

$$(18) \qquad t = t_1\big(x_1(t, 0, x_0)+\varepsilon I_1^0\big).$$

Let now $\bar{t}_1$ be the root of equation (18), i.e.

$$\bar{t}_1 = t_1\big(x_1(\bar{t}_1, 0, x_0)+\varepsilon I_1^0\big).$$

From (18) one can find

$$t = t_1\big(x_1(t, 0, x_0)+\varepsilon I_1^0\big) = t_1\Big(x_0+\varepsilon \int_0^t X\Big(\theta, x_0, \int_0^\theta \varphi(\theta, s, x_0)ds\Big)\,d\theta+\varepsilon I_1^0\Big) =$$

$$= t_1^0+\varepsilon \frac{\partial t_1(x_0)}{\partial x}\Big[I_1^0+\int_0^t X\Big(\theta, x_0, \int_0^\theta \varphi(\theta, s, x_0)\,ds\Big)\,d\theta\Big]+O(\varepsilon^2) =$$

$$= t_1^0+\varepsilon \frac{\partial t_1(x_0)}{\partial x}\Big[I_1^0+\int_0^{t_1^0} X\Big(\theta, x_0, \int_0^\theta \varphi(\theta, s, x_0)ds\Big)\,d\theta\Big]+$$

$$+\varepsilon \frac{\partial t_1(x_0)}{\partial x}\int_{t_1^0}^{t} X\Big(\theta, x_0, \int_0^\theta \varphi(\theta, s, x_0)ds\Big)\,d\theta+O(\varepsilon^2) =$$

$$= t_1^0+\varepsilon\frac{\partial t_1(x_0)}{\partial x}\int_0^{t_1^0} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\,ds\Big)\,d\theta+\varepsilon\frac{\partial t_1(x_0)}{\partial x} I_1^0+$$

$$+\varepsilon\frac{\partial t_1(x_0)}{\partial x}(t-t_1^0)X\Big(\tilde t, x_0, \int_0^{\tilde t} \varphi(\tilde t, s, x_0)\,ds\Big)+O(\varepsilon^2),$$

$$\tilde t = t_1^0+\mu(t-t_1^0), \quad 0 \leqq \mu \leqq 1.$$

Hence, it follows that

$$\bar t_1 = t_1^0+\varepsilon\frac{\partial t_1(x_0)}{\partial x}\int_0^{t_1^0} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\,ds\Big)\,d\theta+\varepsilon\frac{\partial t_1(x_0)}{\partial x} I_1^0+O(\varepsilon^2) =$$

$$= t_1^*+\varepsilon\frac{\partial t_1(x_0)}{\partial x} I_1^0+O(\varepsilon^2).$$

Because of the fact that the second summand is negative, we have $\bar t_1 < t_1^*$. In this way, when $t > t_1^*$ the solution $x_t(x_0)$ leaves the hypersurface $t=t_1(x)$, i.e. the point (t_1^0, x_0) is not a beat point of the solution onto the surface $t=t_1(x)$.

Let us find the moment when the solution (17) reaches the hypersurface $t=t_2(x)$. For this reason let us define the root t_2^* of the equation

$$t = t_2\big(x_1(t, 0, x_0)+\varepsilon I_1^0\big) = t_2\Big(x_0+\varepsilon\int_0^{t} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\,ds\Big)\,d\theta+\varepsilon I_1^0\Big) =$$

$$= t_2^0+\varepsilon\frac{\partial t_2(x_0)}{\partial x}\Big[I_1^0+\int_0^{t} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\,ds\Big)\,d\theta\Big]+O(\varepsilon^2) =$$

$$= t_2^0+\varepsilon\frac{\partial t_2(x_0)}{\partial x}\Big[I_1^0+\int_0^{t_2^0} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\,ds\Big)\,d\theta\Big]+$$

$$+\varepsilon\frac{\partial t_2(x_0)}{\partial x}(t-t_2^0)X\Big(\tilde t, x_0, \int_0^{\tilde t} \varphi(\tilde t, s, x_0)\,ds\Big)+O(\varepsilon^2),$$

$$\tilde t = t_2^0+\mu(t-t_2^0), \quad 0 \leqq \mu \leqq 1.$$

We have

$$t_2^* = t_2^0+\varepsilon\frac{\partial t_2(x_0)}{\partial x}\Big[I_1^0+\int_0^{t_2^0} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\,ds\Big)\,d\theta\Big] \equiv t_2^0+\varepsilon\Theta_2.$$

Since, by the definition $t_2^0 > t_1^0$, then (if ε is sufficiently small) $t_2^* > t_1^*$, and, therefore, the solution of the system (1) on the interval $t_1^* < t < t_2^*$ is expressed by the formula:

$$x_t(x_0) = x_1(t, 0, x_0)+\varepsilon I_1^0.$$

Further,

$$x_{t_2^*}^+(x_0) = x_1(t_2^*, 0, x_0)+\varepsilon I_1^0+\varepsilon I_2\big(x_1(t_2^*, 0, x_0)+\varepsilon I_1^0\big) \equiv$$

$$\equiv x_1(t_2^*, 0, x_0)+\varepsilon(I_1^0+I_2^0).$$

It is not difficult to show (as it can be seen from the previous calculations), that the system (1) under the initial condition $c=x^+_{t_2^*}(x_0)$ at the moment t_2^* has, to an accuracy of values of order ε^2 on the interval $t_2^*<t<t_3^*$, the solution

$$x_t(x_0) = x_1(t, 0, x_0)+\varepsilon(I_1^0+I_2^0),$$

where t_3^* is defined as a root of the equation

$$t = t_3\big(x_1(t, 0, x_0)+\varepsilon(I_1^0+I_2^0)\big)$$

under the condition that

$$\frac{\partial t_2(x_0)}{\partial x} I_2^0 \leqq \beta < 0 \quad \left(\text{or} \quad \frac{\partial t_2(x)}{\partial x} \equiv 0\right).$$

As before, we have

$$t_3^* = t_3^0+\varepsilon\frac{\partial t_3(x_0)}{\partial x}\left[I_1^0+I_2^0+\int_0^{t_3^0} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\,ds\Big)\,d\theta\right] \equiv t_3^0+\varepsilon\Theta_3.$$

It turns out as well that the trajectory $x_t(x_0)=x_1(t, 0, x_0)+\varepsilon(I_1^0+I_2^0)$ after the moment t_2^* does not meet anymore the surface $t=t_2(x)$.

It is easy to see, following the method of induction, that under the condition

$$\text{(19)} \qquad \frac{\partial t_i(x_0)}{\partial x} I_i^0 \leqq \beta < 0 \quad \left(\text{or} \quad \frac{\partial t_i(x)}{\partial x} \equiv 0\right), \quad i = 1, \ldots, d_1,$$

for $x_t(x_0)$ one can get the following expression:

$$\text{(20)} \qquad x_t(x_0) = x_1(t, 0, x_0)+\varepsilon \sum_{i=0}^{k} I_i^0, \quad t_k^0+\varepsilon\Theta_k < t < t_{k+1}^0+\varepsilon\Theta_{k+1},$$

where

$$\text{(21)} \qquad \begin{cases} \Theta_k = \dfrac{\partial t_k(x_0)}{\partial x}\left[\displaystyle\sum_{i=0}^{k-1} I_i^0+\int_0^{t_k^0} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\,ds\Big)\,d\theta\right], \\ t_0^0 = \Theta_0 = I_0^0 = \Theta_{d_1+1} = 0, \quad t_{d_1+1}^0 = T, \quad K = 1, \ldots, d_1. \end{cases}$$

Therefore, when condition (19) is fulfilled the solution $x_t(x_0)$ exists on the segment $[0, T]$ and is defined to an accuracy of values of order ε^2, according to (15) and (20).

Now, one can calculate

$$\text{(22)} \qquad x_T(x_0) = x_1(T, 0, x_0)+\varepsilon \sum_{i=1}^{d_1} I_i^0+O(\varepsilon^2) =$$

$$= x_0+\varepsilon \int_0^{T} X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\,ds\Big)\,d\theta+\varepsilon \sum_{i=1}^{d_1} I_i^0+O(\varepsilon^2) =$$

$$= x_0+\varepsilon[X_0(x_0)+I_0(x_0)]T+\varepsilon \int_0^{T} \Big[X\Big(\theta, x_0, \int_0^{\theta} \varphi(\theta, s, x_0)\,ds\Big)-$$

$$-X_0(x_0)\Big]\,d\theta+\varepsilon\left[\sum_{i=1}^{d_1} I_i^0-I_0(x_0)T\right]+O(\varepsilon^2).$$

Define the operator A_0, as follows:

$$A_0 x_0 = x_0 + \varepsilon T [X_0(x_0) + I_0(x_0)].$$

Then, from (22), one can find

$$\|x_T(x_0) - A_0 x_0\| \leqq \varepsilon \left\| \int_0^T \left[X\left(\theta, x_0, \int_0^\theta \varphi(\theta, s, x_0)\,ds\right) - X_0(x_0)\right] d\theta \right\| + \tag{23}$$

$$+ \varepsilon \left\| \sum_{i=1}^{d_1} I_i^0 - I_0(x_0) T \right\| + O(\varepsilon^2) \leqq$$

$$\leqq \frac{1}{2} \varepsilon \alpha(T) T + \varepsilon \left\| \sum_{i=1}^{d_1} I_i(x_0) - I_0(x_0) T \right\| + \varepsilon \left\| \sum_{i=1}^{d_1} (I_i^0 - I_i(x_0)) \right\| + O(\varepsilon^2) \leqq$$

$$\leqq \varepsilon \alpha(T) T + \varepsilon \sum_{i=1}^{d_1} \|I_i^0 - I_i(x_0)\| + O(\varepsilon^2) =$$

$$= \varepsilon \alpha(T) T + \varepsilon \sum_{i=1}^{d_1} \left\| I_i \left(x_1(t_i^*, 0, x_0) + \varepsilon \sum_{k=0}^{i-1} I_k^0 \right) - I_i(x_0) \right\| + O(\varepsilon^2) \leqq$$

$$\leqq \varepsilon \alpha(T) T + \varepsilon \sum_{i=1}^{d_1} K \left\| x_1(t_i^*, 0, x_0) + \varepsilon \sum_{k=0}^{i-1} I_k^0 - x_0 \right\| + O(\varepsilon^2) \leqq$$

$$\leqq \varepsilon \alpha(T) T + \varepsilon K \sum_{i=1}^{d_1} \|x_1(t_i^*, 0, x_0) - x_0\| + \varepsilon^2 K \sum_{i=1}^{d_1} \sum_{k=0}^{i-1} \|I_k^0\| + O(\varepsilon^2) \leqq$$

$$\leqq \varepsilon \alpha(T) T + \varepsilon^2 K \sum_{i=1}^{d_1} \left\| \int_0^{t_i^*} X\left(\theta, x_0, \int_0^\theta \varphi(\theta, s, x_0)\,ds\right) d\theta \right\| + \varepsilon^2 K M \frac{d_1(d_1-1)}{2} + O(\varepsilon^2) \leqq$$

$$\leqq \varepsilon \alpha(T) T + \varepsilon^2 K M \left(T + \frac{d_1 - 1}{2} \right) d_1 + O(\varepsilon^2) \leqq \varepsilon \alpha(T) T + \varepsilon^2 M_1,$$

where $M_1 = M_1(T, d_1)$ is constant.

Let $\bar{x} = \bar{x}(\varepsilon t, x_0)$ be a solution of the averaged system (6) with the initial condition $\bar{x}(0, x_0) = x_0$. According to the integralrepresentation

$$\bar{x}(\varepsilon t, x_0) = x_0 + \varepsilon \int_0^t \left[X_0(\bar{x}(\varepsilon\theta, x_0)) + I_0(\bar{x}(\varepsilon\theta, x_0))\right] d\theta$$

one can obtain that

$$\bar{A} x_0 = \bar{x}(\varepsilon T, x_0) = x_0 + \varepsilon \int_0^T \left[X_0(\bar{x}(\varepsilon\theta, x_0)) + I_0(\bar{x}(\varepsilon\theta, x_0)) d\theta\right].$$

When $t \geqq 0$ and $x \in D$, we have:

$$\|X_0(x)\| \leqq M, \quad \|I_0(x)\| \leqq Md_0, \quad \|\bar{x}(\varepsilon t, x_0)-x_0\| \leqq \varepsilon(1+d_0)MT,$$

$$\|X_0(\bar{x})-X_0(x_0)\| \leqq \lim_{T\to\infty} \frac{1}{T}\int_0^{t+T} \Big\|X\Big(\theta, \bar{x}(\varepsilon t, x_0), \int_0^\theta \varphi(\theta, s, \bar{x}(\varepsilon t, x_0))ds\Big)-$$

$$-X\Big(\theta, x_0, \int_0^\theta \varphi(\theta, s, x_0)ds\Big)\Big\|\, d\theta \leqq$$

$$\leqq K\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\Big[\|\bar{x}(\varepsilon t, x_0)-x_0\|+\int_0^\theta \sigma(\theta, s)\|\bar{x}(\varepsilon t, x_0)-x_0\|\, ds\Big]\, d\theta \leqq$$

$$\leqq \varepsilon(1+d_0)(1+M^*)KMT,$$

$$\|I_0(\bar{x})-I_0(x_0)\| \leqq \lim_{T\to\infty}\frac{1}{T}\sum_{t<t_i<t+T}\|I_i(\bar{x})-I_i(x_0)\| \leqq \varepsilon(1+d_0)KMTd_0.$$

Then,

$$\|\bar{A}x_0-A_0x_0\| \leqq \varepsilon\int_0^T [\|X_0(\bar{x}(\varepsilon\theta, x_0))-X_0(x_0)\|+\|I_0(\bar{x}(\varepsilon\theta, x_0))-I_0(x_0)\|]d\theta \leqq \tag{24}$$

$$\leqq \varepsilon^2(1+d_0)(1+\tilde{d}_0)KMT^2, \quad \tilde{d}_0 = d_0+M^*$$

and

$$\|x_T(x_0)-\bar{A}x_0\| \leqq \|x_T(x_0)-A_0x_0\|+\|\bar{A}x_0-A_0x_0\| \leqq \tag{25}$$

$$\leqq \varepsilon\alpha(T)T+\varepsilon^2[M_1+(1+d_0)(1+\tilde{d}_0)KMT^2].$$

Since $\bar{A}x_0$ belongs to D with a ϱ-neighbourhood, then, from (24) and (25) it follows that $x_T(x_0)$ and A_0x_0 belong to D with the neighbourhoods

$$\varrho_1 = \varrho-\varepsilon\{\alpha(T)T+\varepsilon[M_1+(1+d_0)(1+\tilde{d}_0)KMT^2]\},$$

$$\varrho_1' = \varrho-\varepsilon^2(1+d_0)(1+\tilde{d}_0)KMT^2,$$

respectively.

Now, assume that in the semi-interval $(T, 2T)$ there lie d_2 points:

$$T < t_{d_1+1}(\bar{A}x_0), \dots, t_{d_1+d_2}(\bar{A}x_0) < 2T, \tag{26}$$

but then, according to the estimation (25) and the continuity of the functions $t_i(x)$, it follows that on the interval $(T, 2T)$ there lie d_2 points:

$$T < t_{d_1+1}(x_T(x_0)) = t_1^{(1)}, \dots, t_{d_1+d_2}(x_T(x_0)) = t_{d_2}^{(1)} < 2T.$$

From condition (7) of the theorem it follows that

$$\frac{\partial t_{d_1+i}(x_T(x_0))}{\partial x} I_{d_1+i}(x_T(x_0)) = \frac{\partial t_{d_1+i}(x_T(x_0))}{\partial x} I_i^{(1)} \leqq \beta_1 < 0,$$

$$\beta_1 = \beta-\varepsilon_2, \quad i = 1, \dots, d_2.$$

Let us extend the solution $x_t(x_0)$ of the system (1), constructed for $0 \leqq t < T$, onto the segment $[T, 2T]$, denoting $x_T(x_0) = x_T$:

$$x_t(x_0) = x(t, T, x_T(x_0)) = x_T + \varepsilon \int_T^t X\Big(\theta, x_T, \int_0^\theta \varphi(\theta, s, x_T) ds\Big) d\theta +$$

$$+O(\varepsilon^2) = x_1(t, T, x_T) + O(\varepsilon^2), \quad T \leqq t < t^*_{d_1+1},$$

where $t^*_{d_1+1}$ is the solution of the equation

$$t = t_{d_1+1}\Big(x_T + \varepsilon \int_T^t X\Big(\theta, x_T, \int_0^\theta \varphi(\theta, s, x_T) ds\Big) d\theta + O(\varepsilon^2)\Big),$$

i.e.

$$t^*_{d_1+1} = t_1^{(1)} + \varepsilon \frac{\partial t_{d_1+1}(x_T)}{\partial x} \int_T^t X\Big(\theta, x_T, \int_0^\theta \varphi(\theta, s, x_T) ds\Big) d\theta + O(\varepsilon^2) =$$

$$= t_1^{(1)} + \varepsilon \frac{\partial t_{d_1+1}(x_T)}{\partial x} \int_T^{t_1^{(1)}} X\Big(\theta, x_T, \int_0^\theta \varphi(\theta, s, x_T) ds\Big) d\theta + O(\varepsilon^2) = t_1^{(1)} + \varepsilon \Theta_1^{(1)} + O(\varepsilon^2).$$

Therefore, to an accuracy of ε^2, one can obtain

(27) $$x_t(x_0) = x_1(t, T, x_T) \quad \text{when} \quad T \leqq t < t_1^{(1)} + \varepsilon \Theta_1^{(1)} = t^*_{d_1+1},$$

$$x^+_{t^*_{d_1+1}}(x_0) = x_1(t^*_{d_1+1}, T, x_T) + \varepsilon I_1^{(1)}(x_1(t^*_{d_1+1}, T, x_T)),$$

whence

$$x_t(x_0) = x^+_{t^*_{d_1+1}}(x_0) + \varepsilon \int_{t^*_{d_1+1}}^t X\Big(\theta, x^+_{t^*_{d_1+1}}(x_0), \int_0^\theta \varphi(\theta, s, x^+_{t^*_{d_1+1}}(x_0)) ds\Big) d\theta =$$

$$= x_1(t^*_{d_1+1}, T, x_T) + \varepsilon I_1^{(1)} + \varepsilon \int_{t^*_{d_1+1}}^t X\Big(\theta, x^+_{t^*_{d_1+1}}(x_0), \int_0^\theta \varphi(\theta, s, x^+_{t^*_{d_1+1}}(x_0)) ds\Big) d\theta =$$

$$= x_T + \varepsilon \int_T^{t^*_{d_1+1}} X\Big(\theta, x_T, \int_0^\theta \varphi(\theta, s, x_T) ds\Big) d\theta + \varepsilon I_1^{(1)} + \varepsilon \int_{t^*_{d_1+1}}^t X\Big(\theta, x^+_{t^*_{d_1+1}}(x_0),$$

$$\int_0^\theta \varphi(\theta, s, x^+_{t^*_{d_1+1}}(x_0)) ds\Big) d\theta = x_T + \varepsilon \int_T^t X\Big(\theta, x_T, \int_0^\theta \varphi(\theta, s, x_T) ds\Big) d\theta +$$

$$+ \varepsilon I_1^{(1)} + \varepsilon \int_{t^*_{d_1+1}}^t \Big[X\Big(\theta, x^+_{t^*_{d_1+1}}(x_0), \int_0^\theta \varphi(\theta, s, x^+_{t^*_{d_1+1}}(x_0)) ds\Big) -$$

$$- X\Big(\theta, x_T, \int_0^\theta \varphi(\theta, s, x_T) ds\Big)\Big] d\theta = x_1(t, T, x_T) + \varepsilon I_1^{(1)},$$

$$t^*_{d_1+1} \leqq t < t_2^{(1)} + \varepsilon \Theta_2^{(1)} = t^*_{d_1+2},$$

etc.

In the general case one has

$$(28)\qquad x_t(x_0) = x_1(t, T, x_T)+\varepsilon \sum_{i=0}^{k} I_i^{(1)}, \quad t_k^{(1)}+\varepsilon\Theta_k^{(1)} \leqq t < t_{k+1}^{(1)}+\varepsilon\Theta_{k+1}^{(1)},$$

where

$$\Theta_k^{(1)} = \frac{\partial t_{d_1+k}(x_T(x_0))}{\partial x}\Big[\int_T^{t_k^{(1)}} X\Big(\theta, x_T, \int_0^\theta \varphi(\theta, s, x_T)ds\Big)\,d\theta + \sum_{i=0}^{k} I_i^{(1)}\Big],$$

$$K = 1, \dots, d_2, \quad t_0^{(1)} = I_0^{(1)} = \Theta_0^{(1)} = \Theta_{d_2+1}^{(1)} = 0, \quad t_{d_2+1}^{(1)} = 2T.$$

So, $x_t(x_0)$ is defined on the segment $[T, 2T]$ to an accuracy of values of order ε^2, according to formulae (27) and (28).

Therefore,

$$x_{2T}(x_0) = x_1(2T, T, x_T)+\varepsilon \sum_{i=1}^{d_2} I_i^{(1)}+O(\varepsilon^2) =$$

$$= x_T+\varepsilon \int_T^{2T} X\Big(\theta, x_T, \int_0^\theta \varphi(\theta, s, x_T)\,ds\Big)\,d\theta+\varepsilon \sum_{i=1}^{d_2} I_i^{(1)}+O(\varepsilon^2) =$$

$$= x_T+\varepsilon[X_0(x_T)+I_0(x_T)]T+\varepsilon \int_T^{2T} \Big[X\Big(\theta, x_T, \int_0^\theta \varphi(\theta, s, x_T)\,ds\Big)-$$

$$-X_0(x_T)\Big]\,d\theta+\varepsilon\left[\sum_{i=1}^{d_2} I_i^{(1)}-I_0(x_T)T\right]+O(\varepsilon^2).$$

Whence, according to the definition of the operator A_0, one can find

$$(29)\qquad \|x_{2T}(x_0)-A_0x_T\| \leqq \varepsilon\alpha(T)T+\varepsilon^2M_2,$$

where $M_2=M_2(T, d_2)$ is constant.

Further, we have

$$(30)\qquad \bar{A}^2x_0 = \bar{x}(2\varepsilon T, x_0) = x_0+\varepsilon \int_O^{2T} \big[X_0(\bar{x}(\varepsilon\theta, x_0))+I_0(\bar{x}(\varepsilon\theta, x_0))\big]\,d\theta =$$

$$= \bar{A}x_0+\varepsilon \int_T^{2T} \big[X_0(\bar{x}(\varepsilon\theta, x_0))+I_0(\bar{x}(\varepsilon\theta, x_0))\big]\,d\theta,$$

$$\|A_0x_T-A_0(\bar{A}x_0)\| = \|x_T+\varepsilon T\,[X_0(x_T)+I_0(x_T)]-\bar{A}x_0-\varepsilon T\,[X_0(\bar{A}x_0)+I_0(\bar{A}x_0)]\| \leqq$$

$$\leqq \|x_T-\bar{A}x_0\|+\varepsilon T\,\{\|X_0(x_T)-X_0(\bar{A}x_0)\|+\|I_0(x_T)-I_0(\bar{A}x_0)\|\} \leqq$$

$$\leqq \|x_T-\bar{A}x_0\|+\varepsilon T\{K(1+M^*)\|x_T-\bar{A}x_0\|+Kd_0\|x_T-\bar{A}x_0\|\} \leqq$$

$$\leqq [1+\varepsilon(1+\tilde{d}_0)KT]\|x_T-\bar{A}x_0\| \leqq$$

$$\leqq [1+\varepsilon(1+\tilde{d}_0)KT]\{\varepsilon\alpha(T)T+\varepsilon^2[M_1+(1+d_0)(1+\tilde{d}_0)KMT^2]\},$$

$$\|\bar{x}(\varepsilon t, x_0)-\bar{A}x_0\| = \|\bar{x}(\varepsilon t, x_0)-\bar{x}(\varepsilon T, x_0)\| \leqq$$

$$\leqq \varepsilon \int_T^t \left[\|X_0(\bar{x}(\varepsilon\theta, x_0))\| + \|I_0(\bar{x}(\varepsilon\theta, x_0))\|\right] d\theta \leqq \varepsilon(1+d_0)MT \quad \text{when} \quad T < t < 2T,$$

$$\text{(31)} \qquad \|A_0(\bar{A}x_0) - \bar{A}(\bar{A}x_0)\| = \|\bar{A}x_0 + \varepsilon T[X_0(\bar{A}x_0) + I_0(\bar{A}x_0)] -$$

$$-\bar{A}x_0 - \varepsilon \int_T^{2T} \left[X_0(\bar{x}(\varepsilon\theta, x_0)) + I_0(\bar{x}(\varepsilon\theta, x_0))\right] d\theta\| \leqq$$

$$\leqq \varepsilon \int_T^{2T} \left[\|X_0(\bar{x}(\varepsilon\theta, x_0)) - X_0(\bar{A}x_0)\| + \|I_0(\bar{x}(\varepsilon\theta, x_0)) - I_0(\bar{A}x_0)\|\right] d\theta \leqq$$

$$\leqq \varepsilon K \int_T^{2T} \left\{ \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \left[\|\bar{x}(\varepsilon\theta, x_0) - \bar{A}x_0\| + \int_0^l \sigma(l, s)\|\bar{x}(\varepsilon\theta, x_0) - \bar{A}x_0\| ds\right] dl + \right.$$

$$\left. + \lim_{T\to\infty} \frac{1}{T} \sum_{t<t_i<t+T} \|\bar{x}(\varepsilon\theta, x_0) - \bar{A}x_0\| \right\} d\theta \leqq \varepsilon^2(1+d_0)(1+\tilde{d}_0)KMT^2.$$

From (29)—(31) one can obtain

$$\|x_{2T}(x_0) - \bar{A}^2x_0\| \leqq \|x_{2T}(x_0) - A_0x_T\| + \|A_0x_T - A_0(\bar{A}x_0)\| + \|A_0(\bar{A}x_0) - \bar{A}^2x_0\| \leqq$$

$$\leqq \varepsilon\alpha(T)T + \varepsilon^2M_2 + [1+\varepsilon(1+\tilde{d}_0)KT]\{\varepsilon\alpha(T)T + \varepsilon^2[M_1 + (1+d_0)(1+\tilde{d}_0)KMT^2]\} +$$

$$+\varepsilon^2(1+d_0)(1+\tilde{d}_0)KMT^2 = \varepsilon\{1+[1+\varepsilon(1+\tilde{d}_0)KT]\}[\alpha(T)T + \varepsilon\bar{M}],$$

$$\bar{M} = (1+d_0)(1+\tilde{d}_0)KMT^2 + \max(M_1, M_2).$$

Therefore, $x_{2T}(x_0)$ belongs to the domain D together with its ϱ_2-neighbourhood

$$\varrho_2 = \varrho - \varepsilon \sum_{i=0}^{1} [1+\varepsilon(1+\tilde{d}_0)KT]^i[\alpha(T)T + \varepsilon\bar{M}].$$

Now, along $x_{2T}(x_0)$ there can be constructed the solution $x_t(x_0)$ for $t \in [2T, 3T]$. Accomplishing the calculations, one can find

$$\|x_{3T}(x_0) - A_0x_{2T}\| \leqq \varepsilon\alpha(T)T + \varepsilon^2M_3(T, d_3),$$

where d_3 is the number of the points

$$2T < t_{d_1+d_2+1}(\bar{x}(2\varepsilon T, x_0)), \ldots, t_{d_1+d_2+d_3}(\bar{x}(2\varepsilon T, x_0)) < 3T$$

lying in the interval $(2T, 3T)$.

Further, as we did before, let us find

$$\|x_{3T}(x_0) - \bar{x}(3\varepsilon T, x_0)\| = \|x_{3T}(x_0) - \bar{A}^3x_0\| \leqq$$

$$\leqq \|x_{3T}(x_0) - A_0x_{2T}\| + \|A_0x_{2T} - A_0(\bar{A}^2x_0)\| + \|A_0(\bar{A}^2x_0) - \bar{A}(\bar{A}^2x_0)\| \leqq$$

$$\leqq \varepsilon\alpha(T)T + \varepsilon^2M_3 + \varepsilon[1+\varepsilon(1+\tilde{d}_0)KT]\{1+[1+\varepsilon(1+\tilde{d}_0)KT]\}[\alpha(T)T + \varepsilon\bar{M}] +$$

$$+\varepsilon^2(1+d_0)(1+\tilde{d}_0)KMT^2 \leqq$$

$$\leqq \varepsilon\{1+[1+\varepsilon(1+\tilde{d}_0)KT] + [1+\varepsilon(1+\tilde{d}_0)KT]^2\}[\alpha(T)T + \varepsilon\bar{M}],$$

$$\bar{M} = (1+d_0)(1+\tilde{d}_0)KMT^2 + \max_{i=1,2,3,} M_i.$$

Therefore, $x_{3T}(x_0)$ lies in the domain D together with the ϱ_3-neighbourhood

$$\varrho_3 = \varrho - \varepsilon \sum_{i=0}^{2} [1+\varepsilon(1+d_0)KT]^i[\alpha(T)T+\varepsilon\overline{M}].$$

Proceeding with the process being described, at the k-th step let us construct the solution $x_t(x_0)$ for $t\in[(k-1)T, kT]$, $kT\leqq L\varepsilon^{-1}$. As a result one can get (and it is not difficult to prove this by mathematical induction):

$$\|x_{kT}(x_0)-\bar{x}(k\varepsilon T, x_0)\| = \|x_{kT}(x_0)-\bar{A}^k x_0\| \leqq \varepsilon \sum_{i=0}^{k-1} [1+\varepsilon(1+\tilde{d}_0)KT]^i[\alpha(T)T+\varepsilon\overline{M}],$$

$$\overline{M} = (1+d_0)(1+\tilde{d}_0)KMT^2 + \max_{i=1,\dots,k} M_i.$$

Since $d_i \leqq c < \infty$,

$$\overline{M} = (1+d_0)(1+\tilde{d}_0)KMT^2+\max_i M_i(T, d_i) \leqq M_0(T) < \infty,$$

whence

$$\|x_{kT}(x_0)-\bar{x}(k\varepsilon T, x_0)\| \leqq \varepsilon \sum_{i=0}^{k-1} [1+\varepsilon(1+\tilde{d}_0)KT]^i[\alpha(T)T+\varepsilon M_0(T)] <$$

$$< [\alpha(T)T+\varepsilon M_0(T)]\frac{[1+\varepsilon(1+\tilde{d}_0)KT]^K}{(1+\tilde{d}_0)KT} \leqq$$

$$\leqq \frac{1}{(1+\tilde{d}_0)KT}[\alpha(T)T+\varepsilon M_0(T)][e^{(1+\tilde{d}_0)KL}+O(\varepsilon)].$$

Now, choose T and ε_0 such that, when $\varepsilon<\varepsilon_0$, the conditions

$$e^{(1+\tilde{d}_0)KL}\frac{\alpha(T)}{(1+\tilde{d}_0)K} < \frac{\eta}{4}$$

will be fulfilled, and

$$O(\varepsilon)\frac{\alpha(T)}{(1+\tilde{d}_0)K}+\varepsilon[e^{(1+\tilde{d}_0)KL}+O(\varepsilon)]\frac{M_0(T)}{(1+\tilde{d}_0)KT} < \frac{\eta}{4}.$$

Then,

$$\|x_{kT}(x_0)-\bar{x}(k\varepsilon T, x_0)\| < \frac{\eta}{2}, \quad k = 0, 1, \dots, \left[\frac{L}{\varepsilon T}\right]. \tag{32}$$

On the interval $[(k-1)T, kT]$ we have the estimations

$$\|\bar{x}(\varepsilon t, x_0)-\bar{x}((k-1)\varepsilon T, x_0)\| \leqq \varepsilon \int_{(k-1)T}^{t} [\|X_0(\bar{x}(\varepsilon\theta, x_0))\| + \tag{33}$$

$$+\|I_0(\bar{x}(\varepsilon\theta, x_0))\|]d\theta \leqq \varepsilon(1+d_0)MT,$$

$$\|x_t(x_0)-x_{(k-1)T}(x_0)\| = \|x_1(t, (k-1)T, x_{(k-1)T})+\varepsilon \sum_{i=0}^{d_k} I_i^{(k-1)} - x_{(k-1)T}(x_0)\| = \tag{34}$$

$$= \left\|\varepsilon \int_{(k-1)T}^{kT} X\Big(\theta, x_{(k-1)T}, \int_0^\theta \varphi(\theta, s, x_{(k-1)T})\, ds\Big)\, d\theta+\varepsilon \sum_{i=0}^{d_k} I_i^{(k-1)}\right\| \leqq \varepsilon M(T+c).$$

It can be seen from (32)—(34) that if T is sufficiently large, and ε is sufficiently small $\left(\varepsilon<\min\left(\varepsilon_0, \frac{\eta}{4M(T+c)}, \frac{\eta}{4(1+d_0)MT}\right)\right)$, then, on the interval $[(k-1)T, kT]$ the following estimation holds:

$$\|x_t(x_0)-\bar{x}(\varepsilon t, x_0)\| \leqq$$

$$\leqq \|x_t(x_0)-x_{(k-1)T}(x_0)\|+\|x_{(k-1)T}(x_0)-\bar{x}((k-1)\varepsilon T, x_0)\|+\|\bar{x}((k-1)\varepsilon T, x_0)-$$

$$-\bar{x}(\varepsilon t, x_0)\| \leqq \frac{\eta}{4}+\frac{\eta}{2}+\frac{\eta}{4} = \eta.$$

Therefore, on the whole interval $0\leqq t\leqq L\varepsilon^{-1}$ the inequality

$$\|x_t(x_0)-\bar{x}(\varepsilon t, x_0)\| < \eta$$

is fulfilled.

In this way Theorem 1 is proved.

REMARK 1. Condition 1 of Theorem 1 might be weakened but that leads to some complications in the proof.

Consider now the problem of the qualitative correspondence between the exact solutions of the system (1), (3) and its approximated $\bar{x}$-solutions of the system (4).

Assume that the averaged system (4) has an isolated state of equilibrium $\bar{x}=x^0$:

$$X_0(x^0)+I_0(x^0) = 0. \tag{35}$$

The following theorem holds:

THEOREM 2. *Let conditions* 1–4 *of Theorem* 1 *be fulfilled. Then, if the state of equilibrium* $\bar{x}=x^0$ *of the averaged system is asymptotically stable and*

$$\frac{\partial t_i(x)}{\partial x} I_i(x) \leqq \beta < 0 \quad \left(or \quad \frac{\partial t_i(x)}{\partial x} \equiv 0\right) \tag{36}$$

for every $i=1, 2, \ldots$ *and every* x *on a certain* ϱ_0*-neighbourhood of the point* x^0, *then, there exist a* ϱ*-neighbourhood* $D_\varrho(\varrho\leqq\varrho_0)$ *of the point* x^0 *and an* $\varepsilon^0>0$ *such that at every* $\varepsilon<\varepsilon^0$ *and every* $x\in D_\varrho$ *the solutions* $x_t(x)$, $x_t(x)|_{t=0}=x_0(x)=x$ *of system* (1) *are uniformly bounded when* $t\in(0, \infty)$.

PROOF. Let $\bar{x}(t, x)$, $\bar{x}(0, x)=x$ be the solution of the averaged system when $\varepsilon=1$. Since $\bar{x}(t, x^0)=x^0$ is an asymptotically stable solution of the averaged system, then, this fact, as well as the continuous dependence on the initial conditions, determine the existence of a ϱ' such that

$$\|\bar{x}(t, x)-x^0\| \leqq \varrho_0 \quad \text{when} \quad \|x-x^0\| \leqq \varrho' \quad \text{and} \quad t \geqq 0. \tag{37}$$

(37) and (36) show that for the solutions $\bar{x}(\varepsilon t, x)$ when $x\in T_{\varrho'}= =\{x, \|x-x^0\|\leqq\varrho'\}$ condition 5 of Theorem 1 holds for every $t>0$. But then, according to the assertion of Theorem 1, there may be found an $\varepsilon^0=\varepsilon^0(L, \varrho)$ such that, at every $\varepsilon<\varepsilon^0$, $t\in[0, L\varepsilon^{-1}]$ and some ϱ $(\varrho\leqq\min(\varrho_0, \varrho'))$ the inequality

$$\|x_{t+\tau}(x, \tau)-\bar{x}(\varepsilon t, x)\| \leqq \frac{\varrho}{2} \tag{38}$$

will be fulfilled, where $x_{t+\tau}(x, \tau)$ is the value at the point $t+\tau$ of the solution of system (1), passing, when $t=\tau$, through the point $x \in T_\varrho$.

Choose ϱ such that T_ϱ will belong to the domain of asymptotic stability of the solution $\bar{x}=x^0$, and L in such a way that

$$\|\bar{x}(t, x \in T_\varrho)-x^0\| \leqq \frac{\varrho}{2} \quad \text{when} \quad t \geqq L. \tag{39}$$

Inequalities (37)—(39) then lead to the estimations

$$\|x_{t+\tau}(x \in T_\varrho, \tau)-x^0\| \leqq \|x_{t+\tau}(x \in T_\varrho, \tau)-\bar{x}(\varepsilon t, x \in T_\varrho)\|+\|\bar{x}(\varepsilon t, x \in T_\varrho)-x^0\| \leqq \tag{40}$$

$$\leqq \frac{\varrho}{2}+\varrho_0 \quad \text{when} \quad t \in [0, L\varepsilon^{-1}),$$

$$\|x_{\frac{L}{\varepsilon}}(x \in T_\varrho, 0)-x^0\| \leqq \|x_{\frac{L}{\varepsilon}}(x \in T_\varrho, 0)-\bar{x}(L, x \in T_\varrho)\|+\|\bar{x}(L, x \in T_\varrho)-x^0\| \leqq \varrho,$$

the last of which means that

$$x_{\frac{L}{\varepsilon}}(x \in T_\varrho, 0) \in T_\varrho. \tag{41}$$

Taking into consideration (41), as well as the fact that

$$x_{t+\tau}(x, 0)=x_{t+\tau}(x_\tau(x, 0), \tau), \tag{42}$$

one can obtain

$$x_{t+\frac{L}{\varepsilon}}(x \in T_\varrho, 0)=x_{t+\frac{L}{\varepsilon}}\left(x_{\frac{L}{\varepsilon}}(x \in T_\varrho, 0), \frac{L}{\varepsilon}\right)=x_{t+\frac{L}{\varepsilon}}\left(x' \in T_\varrho, \frac{L}{\varepsilon}\right),$$

whence, taking into consideration (40), it follows that

$$\|x_{t+\frac{L}{\varepsilon}}(x \in T_\varrho, 0)-x^0\|=\left\|x_{t+\frac{L}{\varepsilon}}\left(x' \in T_\varrho, \frac{L}{\varepsilon}\right)-x^0\right\| \leqq \frac{\varrho}{2}+\varrho_0 \quad \text{when} \quad t \in\left[0, \frac{L}{\varepsilon}\right),$$

$$\|x_{2\frac{L}{\varepsilon}}(x \in T_\varrho, 0)-x^0\|=\left\|x_{2\frac{L}{\varepsilon}}\left(x' \in T_\varrho, \frac{L}{\varepsilon}\right)-x^0\right\| \leqq \tag{43}$$

$$\leqq\left\|x_{2\frac{L}{\varepsilon}}\left(x' \in T_\varrho, \frac{L}{\varepsilon}\right)-\bar{x}(L, x' \in T_\varrho)\right\|+\|\bar{x}(L, x' \in T_\varrho)-x^0\| \leqq \varrho.$$

The last of equalities (43) proves that

$$x_{2\frac{L}{\varepsilon}}(x \in T_\varrho, 0) \in T_\varrho$$

and leads to the estimations

$$\|x_{t+k\frac{L}{\varepsilon}}(x \in T_\varrho, 0)-x^0\| \leqq \frac{\varrho}{2}+\varrho_0 \quad \text{when} \quad t \in\left[0, \frac{L}{\varepsilon}\right),$$

$$\|x_{(k+1)\frac{L}{\varepsilon}}(x \in T_\varrho, 0)-x^0\| \leqq \varrho, \quad k=0,1,2, \ldots. \tag{44}$$

Inequalities (44) mean that

$$\|x_t(x\in T_\varrho)-x^0\| \leqq \frac{\varrho}{2}+\varrho_0 \tag{45}$$

for every $t\in[0, \infty)$.

In this way Theorem 2 is proved.

THEOREM 3. *Let the system* (1) *satisfy conditions* 1—4 *of Theorem* 1 *when* $t>0$, *as well as when* $t<0$. *Let*

$$I_0(x) = \lim_{T\to\infty}\frac{1}{T}\sum_{t<t_i<t+T} I_i(x) \quad as \quad t \geqq 0, \tag{46}$$

$$I^0(x) = \lim_{T\to\infty}\frac{1}{T}\sum_{t-T<t_i<t} I_i(x) \quad as \quad t \leqq 0.$$

Assume that the averaged system

$$\frac{d\bar{x}}{dt} = \varepsilon[X_0(\bar{x})+I_0(\bar{x})], \tag{47}$$

when $t\geqq 0$, *has an asymptotically stable state of equilibrium* $x=x^0$, *satisfying* (36) *for* x *out of some* ϱ_0*-neighbourhood. Let in the* ϱ*-neighbourhood* D_ϱ *of the solution* x^0, *pointed in Theorem* 2, *the averaged system*

$$\frac{d\bar{x}_1}{dt} = \varepsilon[X_0(\bar{x}_1)+I^0(\bar{x}_1)] \tag{48}$$

when $t\leqq 0$, *have a state of equilibrium* $\bar{x}_1=x_1^0$, *for which*

$$\frac{\partial t_i(x)}{\partial x} I_i(x) \leqq \beta < 0 \quad \left(or \quad \frac{\partial t_i(x)}{\partial x} \equiv 0\right) \tag{49}$$

for every $i=-1, -2, \ldots$ *and* x *out of some* ϱ_0'*-neighbourhood of the state* x_1^0. *Then*

1. *If the state of equilibrium* x_1^0 *of the system* (48) *is asymptotically stable when* $t<0$, *then, there can be found an* $\varepsilon^0>0$ *and a domain* D_{ϱ_1}, *including* x^0 *and* x_1^0 *such that, when* $\varepsilon<\varepsilon^0$, *all the solutions* $x_t(x)$ *of* (1), (3), *for which* $x\in D_{\varrho_1}$, *are uniformly bounded when* $t\in(-\infty, \infty)$.

2. *If the state of equilibrium* x_1^0 *of the system* (48) *is asymptotically stable, then, there can be found an* $\varepsilon_0>0$ *and* x^* *such that the solution* $x_t(x^*)$ *of the system* (1), (3) *is bounded when* $t\in(-\infty, \infty)$.

PROOF. Let the state of equilibrium x_1^0 of system (48) be asymptotically unstable. Applying Theorem 2 for the intervals $t\geqq 0$ and $t\leqq 0$, we are convinced that there exist ϱ- and $\bar{\varrho}$-neighbourhoods of the points x^0 and x_1^0, such that

$$\|x_t(x\in D_\varrho)\| \leqq c_1 \quad \text{as} \quad t\in(0, \infty)$$

and

$$\|x_t(x\in T_{\bar{\varrho}})\| \leqq c_2 \quad \text{as} \quad t\in(-\infty, 0).$$

Since $x_1^0 \in D_\varrho$, the set $D_\varrho \cap T_{\bar\varrho} = D_{\varrho_1}$ is not empty and therefore

$$\|x_t(x \in D_{\varrho 1})\| \leqq c = \max(c_1, c_2) \quad \text{as} \quad t \in (-\infty, \infty).$$

Now let x_1^0 be an asymptotically stable solution of the system (48). Applying Theorem 1 to (1), (3), we are convinced that for $x_t(x \in T_{\varrho' 1}, \tau)$ the estimations of type (40) hold, i.e.

$$\|x_{t+\tau}(x \in T_{\varrho_1'}, \tau) - x_1^0\| \leqq \frac{\varrho_1'}{2} + \varrho_1^0 \quad \text{when} \quad t \in \left[0, \frac{L}{\varepsilon}\right), \quad \tau \geqq -\frac{L}{\varepsilon}$$

and

(50) $$\left\| x_0 \left(x \in T_{\varrho_1'}, -\frac{L}{\varepsilon} \right) - x_1^0 \right\| \leqq \varrho_1',$$

where $T_{\varrho' 1}$ is a certain ϱ_1'-neighbourhood of the point x_1^0.

Inequalities (50) lead to the estimations

(51) $$\left\| x_{t - k\frac{L}{\varepsilon}} \left(x \in T_{\varrho_1'}, -k\frac{L}{\varepsilon} \right) - x_1^0 \right\| \leqq \frac{\varrho_1'}{2} + \varrho_1^0,$$

$$\left\| x_0 \left(x \in T_{\varrho_1'}, -k\frac{L}{\varepsilon} \right) - x_1^0 \right\| \leqq \varrho_1'$$

when $t \in \left[0, k\frac{L}{\varepsilon}\right)$, $k = 1, 2, \ldots$.

Choose from the set $D_{\varrho_2} = T_{\varrho_1'} \cap D_\varrho$ the convergent subsequence of the points

(52) $$y_k = x_0 \left(x_k \in D_{\varrho_2}, -k\frac{L}{\varepsilon} \right), \quad k = 1, 2, \ldots, \quad \lim_{k \to \infty} y_k = y_0$$

and consider the solution $x_t(y_0)$. It will be shown that $x_t(y_0)$ is defined when $t \in (-\infty, 0)$, and bounded as well. Indeed, the sequence of the solutions $x_t(y_k)$ is defined when $-k\frac{L}{\varepsilon} \leqq t \leqq 0$, and uniformly bounded when $k = 1, 2, \ldots$, since it coincides with the sequence $x_{\bar t - k\frac{L}{\varepsilon}}\left(x_k, -k\frac{L}{\varepsilon}\right)$ when $\bar t \in \left[0, k\frac{L}{\varepsilon}\right)$. Suppose that $\|x_t(y_0)\| > D$ at a certain $t = t_0 < 0$, $D > \frac{1}{2}\varrho_1' + \varrho_1^0 + \|x_1^0\|$.

Since $x_t(y_0)$ is a piecewise continuous function of t on any interval, then the inequality

$$\|x_t(y_0)\| > D > \frac{1}{2}\varrho_1' + \varrho_1^0 + \|x_1^0\|$$

holds for some interval (t_0, t_0'), the latter being an interval of continuity for $x_t(y_0)$. Since the points of discontinuity of the solution $x_t(x)$ depend continuously on x on the finite interval of time, there may be chosen a $\tau_1 \in (t_0, t_0')$ such that

(53) $$x_{\tau_1}(y_k) \to x_{\tau_1}(y_0) \quad \text{as} \quad k \to \infty.$$

Relation (53) leads to the inequality

$$x_{\tau_1}(y_k) > \frac{1}{2}\varrho_1' + \varrho_1^0 + \|x_1^0\|. \tag{54}$$

But (54) contradicts (51), since

$$x_{\tau_1}(y_k) = x_{\bar{\tau}-k\frac{L}{\varepsilon}}\left(x_k, -k\frac{L}{\varepsilon}\right).$$

The contradiction proves that $x_t(y_0)$ is bounded when $t\in(-\infty, 0)$. Since $y_0\in D_\varrho$, $x_t(y_0)$ is bounded when $t\geqq 0$ as well, and therefore

$$\|x_t(y_0)\| \leqq c < \infty,$$

and, in this way Theorem 3 is proved.

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RADICALS AND SEMIPRIME IDEALS IN SEMIGROUPS

By

H. L. CHOW (Hong Kong)

Let S be a compact topological semigroup and $I \subset S$ an ideal of S. We define the *radical* of I as the set $R(I)=\{x \in S: x^n \in I$ for some positive integer $n\}$. Evidently $R(I) \supset I$ and $R(R(I))=R(I)$. The ideal I is said to be *semiprime* if $R(I)=I$. If S is commutative, $R(I)$ is obviously an ideal and so a semiprime ideal. In this note, we will study radicals of open ideals and obtain results for open semiprime ideals.

Following NUMAKURA [4], we say that an element $x \in S$ is *nilpotent* with respect to an ideal I of S if, for any open set U containing I, there is an integer n_0 such that $x^n \in U$ for $n \geqq n_0$.

THEOREM 1. *If I is an open ideal of S, then $R(I)$ is the set of nilpotent elements with respect to I.*

PROOF. Let x be a nilpotent element with respect to I; then $x^n \in I$ for sufficiently large n, whence $x \in R(I)$. Conversely, take any $x \in R(I)$ i.e. $x^m \in I$ for some m. Let $\Gamma(x)$ denote the closed semigroup generated by x. It is well-known that the minimal ideal $K(x)$ of $\Gamma(x)$ is a group with identity e; moreover, $K(x)$ is the set of cluster points of the sequence $\{x^n\}$ [5, Theorem 3.1.1]. Since I is an ideal of S, we see that $x^m e \in I$ and so $K(x) \subset I$, i.e. I contains every cluster point of the sequence $\{x^n\}$. Now suppose x is not nilpotent with respect to I, i.e. $x^p \in S \setminus I$ for infinitely many p. Since $S \setminus I$ is compact, it follows that the sequence $\{x^n\}$ has a cluster point in $S \setminus I$. This contradiction proves the theorem.

The next corollary is now clear.

COROLLARY. *Let I be an open ideal of S. Then I is semiprime if and only if there are no nilpotent elements with respect to I lying outside of I.*

Next, let S be a compact affine semigroup i.e. the compact semigroup S is a convex subset of a topological vector space such that $x(ty+(1-t)z)=txy+(1-t)xz$ and $(ty+(1-t)z)x=tyx+(1-t)zx$ for $x, y, z \in S$, $0 \leqq t \leqq 1$.

THEOREM 2. *If I is an open ideal of S, then $R(I)$ is connected and dense in S. Further, $R(I)$ is convex if it is an ideal.*

PROOF. Let $N(I)$ be the largest ideal contained in $R(I)$. By a similar argument to that given in the proof of [4, Theorem 1 (1)], we can show that $N(I)$ is the intersection of open prime ideals containing I. In the proof of [1, Theorem 6], any open prime ideal P of S is so-called *ultra-convex*, i.e. P contains the set $\{ta+(1-t)b: a \in I, b \in S, 0<t \leqq 1\}$. Then it is easily seen that $N(I)$ is also ultra-convex; therefore, $N(I)$ is convex and dense in S. The result now follows easily.

COROLLARY. *Let I be an open semiprime ideal of S; then I is convex and dense in S.*

PROOF By the above proof, $N(I)$ is convex and dense in S. Since $R(I)\supset\supset N(I)\supset I$ and $R(I)=I$, we have $N(I)=I$, giving the result.

Finally, we consider the set $P(S)$ of probability measures on a compact semigroup S. It is known [3] that $P(S)$ is a compact semigroup under convolution and the weak* topology. Let the support of a measure $\mu\in P(S)$ be denoted by $\operatorname{supp}\mu$. For an ideal $\Omega\subset P(S)$, let $\mathscr{D}(\Omega)=\bigcup_{\mu\in\Omega}\operatorname{supp}\mu$, which is obviously an ideal of S.

THEOREM 3. *If Ω is an open ideal of $P(S)$, then $R(\mathscr{D}(\Omega))=\mathscr{D}(R(\Omega))=S$.*

PROOF. Since $P(S)$ is a compact affine semigroup, we see, from the proof of Theorem 2, that $N(\Omega)$ is the intersection of open prime ideals of $P(S)$ containing Ω. As shown in [2, Corollary 1], we have $\mathscr{D}(N(\Omega))=S$, whence $\mathscr{D}(R(\Omega))=S$. Now take any $x\in S$. Then $x\in\mathscr{D}(R(\Omega))$ implies $x\in\operatorname{supp}\mu$ for some $\mu\in R(\Omega)$. Since $\mu^n\in\Omega$ for some n, it follows that $x^n\in\operatorname{supp}\mu^n$, and so $x^n\in\mathscr{D}(\Omega)$. Thus $x\in R(\mathscr{D}(\Omega))$, concluding the proof.

COROLLARY. *Let Ω be an open semiprime ideal of $P(S)$. Then $\mathscr{D}(\Omega)=S$.*

PROOF. We observe that $R(\Omega)=\Omega$, and apply the preceding theorem to obtain the result.

Note that the openness of ideals I and Ω in the theorems above is essential, as the following example shows.

EXAMPLE. Let $S=[0, 1]$ with usual topology and usual multiplication. Let $I=\{0\}$. Then $R(I)=\{0\}$, but the set of nilpotent elements with respect to I is $[0, 1)$. Moreover, $R(I)$ is not dense in S, while S is affine. If δ_0 denotes the Dirac measure at 0, let $\Omega=\{\delta_0\}$; then $R(\mathscr{D}(\Omega))=\mathscr{D}(R(\Omega))=\{0\}\neq S$.

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MATHEMATICS DEPARTMENT
THE CHINESE UNIVERSITY OF HONG KONG
HONG KONG

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ON THE POTENCY OF SPACES WITH GENERALIZED DISPERSION POINTS UNDER MAPS AND FUNCTIONS

By

J. F. CHEW and P. H. DOYLE (East Lansing)

In [1] C. A. COPPIN proved the following:

THEOREM (Coppin). *Let X be a connected, locally connected Hausdorff space and Y a connected Hausdorff space with a dispersion point. Then any map from X to Y is necessarily a constant.*

The following represents a slight improvement of Coppin's result.

THEOREM 1. *Let X be a connected, locally connected space and Y a connected T_1-space with a dispersion point. If $f: X\to Y$ is a connected function from X to Y such that f has closed point inverses, then f is a constant.*

REMARK. Coppin assumed both X and Y are Hausdorff and that f is continuous. His proof, however, shows that it is enough to assume that Y is T_1 and that f takes connected sets onto connected sets and f has closed point inverses. Connected functions have the property that $f(\overline{C})\subset\overline{f(C)}$ for each connected set C in X whenever Y is T_1 ([2], Theorem 3). On the other hand, requiring f to have closed point inverses entitles us to conclude $f^{-1}(Y-\{p\})$ is an open set in X because $f^{-1}(Y-\{p\})=X-f^{-1}(p)$ is an open set in X. The proof of the theorem given in [1] now goes through unchanged for Theorem 1.

As an example of a non-continuous connected function with closed point inverses, consider the following. Let $X=$the Sorgenfrey line, $Y=$the real line. Define $f: X\to Y$ by

$$f(x)=\begin{cases} x & \text{if } x\leqq 4 \\ -x+7 & \text{if } x>4. \end{cases}$$

We now prove the following companion theorem to the theorem of Coppin.

THEOREM 2. *Let X be a space with a generalized dispersion point p and assume also that X is dense-in-itself (each point of X is a limit point of X). Let Y be a locally connected T_1-space. If $f: X\to Y$ is a map such that f^{-1} preserves connected sets, then f is a constant. Let X be dense-in-itself and p a point in X. If $X-\{p\}$ is totally disconnected then p is a generalized dispersion point of X.*

PROOF. Suppose $f\neq$constant. Then there exists $a\in X-\{p\}$ such that $f(a)\neq f(p)$. Let $b=f(a)$ and $q=f(p)$. Pick an open set V in Y containing b such that $q\notin V$. Since Y is locally connected, we may assume that V is also connected. Note that $U=f^{-1}(V)$ is an open set containing a. Hence there exists $a'\in U$ such that $a'\neq a$. Also $U\subset X-\{p\}$, hence, being connected, U must be a singleton.

This contradiction (that U is a singleton containing two distinct points a and a') shows that f must be a constant.

REMARK. Let us re-emphasize that X is *not* assumed to be connected in Theorem 2 so that the theorem is applicable, for example, if X is the Sorgenfrey line with $p \in X$ arbitrary.

By Lemma 1 of [2], we know that a non-degenerate T_1 connected space is dense-in-itself. Hence we have the following:

COROLLARY 1. *Let X be a non-degenerate connected T_1-space with a dispersion point and let Y be a locally connected T_1-space. If $f: X \to Y$ is a map such that f^{-1} preserves connected sets, then f is a constant.*

REMARK. Theorem 2 of [5] states that if $f: X \to Y$ is a closed function such that $f^{-1}(y)$ is connected for each $y \in Y$, then f^{-1} preserves connected sets. Hence we have a second corollary to Theorem 2.

COROLLARY 2. *Let X be a non-degenerate connected T_1-space with dispersion point and let Y be a locally connected T_1-space. If $f: X \to Y$ is a closed map such that $f^{-1}(y)$ is connected for each $y \in Y$, then f is a constant.*

REMARK. Let X be the example of KNASTER and KURATOWSKI [3] of a connected space in the plane with a dispersion point. Then X is a completely regular Hausdorff space and so there exist non-constant continuous functions from X onto $[0, 1]$. Hence the hypothesis that f^{-1} preserves connected sets cannot be dropped. The proof of Theorem 2, however, shows that if Y is regular, it is enough to require that $f^{-1}(V)$ be a connected set in X for each connected *closed* set V in Y. Functions f which have the stronger property that f^{-1} preserves closed connected sets have been called semiconnected (see [5], for example).

Incidentally, Theorem 1 of [5] is a much weaker result than Theorem 9 of [2], so that it is wise to consult [2] before launching a program to investigate generalizations of or sufficient conditions for continuity.

Certainly the condition that f^{-1} preserves connected sets has some force even in the absence of continuity. This suggests a theorem without the continuity assumption.

THEOREM 3. *Let $f: X \to Y$ be a function from a space X with generalized dispersion point p to an arbitrary topological space Y. If f^{-1} preserves connected sets, then either f is constant or $f(X)$ has a generalized dispersions point.*

PROOF. Let $f(p)=q$. Then $B=f^{-1}(q)$ is connected and $f|X-B$ is a $1-1$ function onto $f(X)-\{f(p)\}$ since f^{-1} preserves connected sets, and $f(X)-$ $-\{f(p)\}$ is totally disconnected or void. Consequently either $f(X)=\{f(p)\}$ or $f(p)$ is a generalized dispersion point of $f(X)$.

Let E be the category of topological spaces whose morphisms are the functions of Theorem 3.

COROLLARY 3. *There exists a category $F \subset E$ that is a non-full subcategory with morphisms that are epimorphisms and whose objects are points or spaces with generalized dispersion points.*

We noted that in Theorem 3 there was no continuity assumption on f and wondered whether we might be able to get a result suggested by Theorem 2 without assuming f to be continuous and without assuming Y to be locally connected. For this purpose it will be convenient to isolate a property of topological spaces which has (apparently) hitherto gone unnoticed.

DEFINITION. A topological space Y is LW if given $y \in Y$ and given an open set U in Y such that $y \in U$ there exists a non-degenerate connected set V such that $y \in V \subset U$.

Note that the closed up sine curve is LW and that any discrete space is a locally connected space which fails to be LW.

THEOREM 4. *Let X be a space with a generalized dispersion point p and let Y be a LW T_1-space. Then there does NOT exist a function $f: X \twoheadrightarrow Y$ from X onto Y with the property that f^{-1} preserves connected sets.*

PROOF. Suppose such an f exists. Since Y is, *a priori*, non-degenerate and since f is surjective, it follows that f is non-constant. Choose $a \in X - \{p\}$ such that $f(a) \neq f(p) = q$. Select an open set U in Y such that $b = f(a) \in U$ while $q \notin U$. If W is a non-degenerate connected set such that $b \in W \subset U$, then $f^{-1}(W)$ is a connected set in X and $f^{-1}(W) \subset X - \{p\}$. Hence $f^{-1}(W)$ is a singleton. But the non-degeneracy of W implies that $f^{-1}(W)$ is more than a singleton since f is surjective. This contradiction proves that such an f cannot exist.

Recently (see [4], for example) there have been examples of *countable* connected Hausdorff spaces with dispersion points in the literature. If X is a countable connected space and $Y = [0, 1]$, then, of course, the only continuous functions from X to Y are constants. We close this paper with the following:

QUESTION. Let X be a countable connected Hausdorff space which has a dispersion point and let Y be a connected, locally connected Hausdorff space. Are constants the only continuous functions from X to Y?

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MICHIGAN STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS
EAST LANSING, MICHIGAN 48 823
USA

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ZUM GÜLTIGKEITSBEREICH DES ZENTRALEN GRENZWERTSATZES UND DES GESETZES DER GROßEN ZAHLEN

Von

W. KRATZ und R. TRAUTNER (Ulm)

0. In der vorliegenden Arbeit werden hinreichende Bedingungen für die Gültigkeit des Zentralen Grenzwertsatzes (ZG) diskutiert. Wir wollen annehmen, daß der zugrunde gelegte Wahrscheinlichkeitsraum $\Omega=[0, 1]$ mit Lebesgue-Maß ist. (Eine Übertragung auf beliebige Wahrscheinlichkeitsräume ist möglich, da nur die Verteilungsfunktionen der betreffenden Zufallsvariablen untersucht werden.)

Seien $\Phi_n(x)$ Zufallsvariable mit Erwartungswerten $\int\limits_0^1 \Phi_n(x)\,dx=0$ und Varianzen $\int\limits_0^1 \Phi_n^2(x)\,dx=1$, und sei (c_n) eine Folge reeller Zahlen mit $c_n>0$, $C_N^2=\sum\limits_{\nu=1}^{N} c_\nu^2\to\infty$, $c_N=o(C_N)$. Man sagt, daß der ZG für die Zufallsvariablen $\Phi_n(x)$ bezüglich der Folge (c_n) gilt, falls die Verteilung der Zufallsvariablen $\frac{1}{C_N}\sum\limits_{n=1}^{N} c_n\Phi_n(x)$ gegen die Normalverteilung $\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x} e^{-t^2/2}\,dt$ konvergiert. Beim klassischen ZG ist $c_n\equiv\sigma$, und die $\Phi_n(x)$ sind identischverteilt und unabhängig. Wir untersuchen hier, inwieweit man die Voraussetzung der Unabhängigkeit abschwächen kann.

Révész [5] und Móricz [4] gaben folgende Richtung an: Man ersetzt die Voraussetzung „Φ_n unabhängig" im wesentlichen durch „Φ_n multiplikativ orthogonal" und weitere Zusatzbedingungen. Dabei heißt nach Alexits [1] ein Funktionensystem multiplikativ orthogonal, wenn gilt:

$$b_{n_1\dots n_k}=\int\limits_0^1 \Phi_{n_1}(x)\dots\Phi_{n_k}(x)\,dx=0 \quad \text{für alle} \quad 1\leqq n_1<\dots<n_k,\ n_i\in\mathbf{N},\ k\in\mathbf{N}.$$

Der Beweis bei [4] zeigt, daß folgende Bedingungen hinreichend sind für die Gültigkeit des ZG:

$$\text{(1)}\qquad \frac{1}{C_N^2}\sum_{n=1}^{N} c_n^2\Phi_n^2(x)\to 1 \quad \text{dem Maß nach auf } [0, 1];$$

das heißt für die Folge $\Psi_n(x)=\Phi_n^2(x)-1$ gilt das Gesetz der Großen Zahlen bezüglich der Folge (c_n^2),

$$\text{(2)}\qquad \lim_{N\to\infty}\int\limits_0^1 \prod_{n=1}^{N}\left(1-it\frac{c_n}{C_N}\Phi_n(x)\right)dx=1 \quad \text{für} \quad t\in\mathbf{R}$$

und

$$(3)\qquad \int_0^1 \left| \prod_{n=1}^{N} \left(1 - it\frac{c_n}{C_N}\Phi_n(x)\right)\right| dx \leqq K(t) \quad \text{für} \quad N \in \mathbf{N},\ t \in \mathbf{R}.$$

(Siehe auch [3], Proposition.) Hinreichend für die Bedingung (3) ist

$$(4)\qquad \int_0^1 \Phi_{n_1}^2(x)\dots\Phi_{n_k}^2(x)dx \leqq C^k \quad \text{für alle} \quad 1 \leqq n_1 < \dots < n_k,\ n \in \mathbf{N}$$

(siehe dazu [3], (6)); denn aus (4) folgt:

$$\left(\int_0^1 \left| \prod_{n=1}^{N} \left(1 - it\frac{c_n}{C_N}\Phi_n(x)\right)\right| dx\right)^2 \leqq \int_0^1 \prod_{n=1}^{N} \left(1 + t^2\frac{c_n^2}{C_N^2}\Phi_n^2(x)\right) dx \leqq$$

$$\leqq 1 + \sum_{k=1}^{N} \frac{t^{2k}}{C_N^{2k}} \times \sum_{1 \leqq n_1 < \dots < n_k \leqq N} c_{n_1}^2 \dots c_{n_k}^2 \int_0^1 \Phi_{n_1}^2(x)\dots\Phi_{n_k}^2(x)dx \leqq$$

$$\leqq 1 + \sum_{k=1}^{N} \frac{t^{2k}}{C_N^{2k}} \times \sum_{1 \leqq n_1 < \dots < n_k \leqq N} c_{n_1}^2 \dots c_{n_k}^2 C^k \leqq \sum_{k=0}^{\infty} \frac{C^k t^{2k}}{k!} = e^{Ct^2}.$$

Die Bedingung (4) ist z. B. erfüllt, wenn die Folge $(\Phi_n(x))$ gleichmäßig beschränkt ist, d. h. $|\Phi_n(x)| \leqq K$ für $n \in \mathbf{N}$, $x \in [0, 1]$.

Wir geben schwächere Bedingungen für die Gültigkeit von (1) und (2) an. Hinreichend für (2) ist etwa „$\Phi_n(x)$ schwach multiplikativ", d. h.

$$\sum_{1 \leqq n_1 < \dots < n_k}^{\infty} |b_{n_1 \dots n_k}| < \infty.$$

Weitere, feinere Kriterien erhält man, wenn man die l_2-Normen

$$\|B_k\|_2 := \left(\sum_{1 \leqq n_1 < \dots < n_k}^{\infty} b_{n_1 \dots n_k}^2\right)^{1/2}$$

oder die Verteilung der $b_{n_1 \dots n_k}$ oder für monotone Folgen (c_n) Mittelwertsätze betrachtet.

Mit ähnlichen Methoden (Verteilung von $\int_0^1 \Phi_{n_1}(x)\Phi_{n_2}(x)\,dx$, Mittelwertsätze) erhält man Bedingungen für die Gültigkeit des Gesetzes der Großen Zahlen (GG), die auch unabhängig von den Anwendungen auf den ZG von Interesse sind.

1. Bezeichnungen und Hilfsmittel. Für eine Folge (c_n) reeller Zahlen und eine Folge $(\Phi_n(x))$ auf [0, 1] Lebesgue-meßbarer Funktionen sei

$$C_N^2 = \sum_{n=1}^{N} c_n^2, \quad S_N(x) = \sum_{n=1}^{N} c_n\Phi_n(x) \quad \text{für} \quad x \in [0, 1],$$

und $F_N(Y)$ sei die Verteilungsfunktion von $\frac{S_N(x)}{C_N}$; d. h.

$$F_N(y) = \mu\left(\left\{x \;\middle|\; \frac{S_N(x)}{C_N} < y\right\}\right), \quad y \in \mathbf{R}.$$

($\mu(M)$ bezeichnet dabei das Lebesgue-Maß der Menge M.)

$\varphi_N(t)$ sei die charakteristische Funktion von $F_N(Y)$, also

$$\varphi_N(t) = \int_{-\infty}^{\infty} e^{-ity}\, dF_N(y) = \int_0^1 e^{-itS_N(x)/C_N}\, dx$$

(nach [4], (7)). Über die beiden Folgen (c_n) und $(\Phi_n(x))$ werden wir im allgemeinen folgende Voraussetzungen machen:

(5) $\quad \Phi_n(x) \in L_2[0, 1]$ und $\left|\int_0^1 \Phi_n(x)\Phi_m(x)\,dx\right| \leqq K$ für $n \neq m, \quad n, m \in \mathbf{N}$.

(6) $\quad c_n > 0,\ C_N \to \infty,\ c_N = o(C_N)$ für $N \to \infty$.

Als Hilfsmittel in (2) und (3) benötigen wir:

LEMMA 1 (Mittelwertsatz). *Seien* $N, k \in \mathbf{N}$, $b_{\nu_1 \dots \nu_k} \in \mathbf{R}$ *für* $\{\nu_1, \dots, \nu_k\} \subset \subset \{1, \dots, N\}$,

$$B_k(N) = \operatorname{Max}\left(\left\{\left|\sum_{\nu_1 = n_1}^{m_1} \dots \sum_{\nu_k = n_k}^{m_k} b_{\nu_1 \dots \nu_k}\right| : \ 1 \leqq n_\nu \leqq m_\nu \leqq \mathbf{N},\ \nu = 1, 2, \dots, k\right\}\right),$$

und sei (c_n) *eine monotone, nichtnegative Zahlenfolge. Dann gilt:*

(7) $$\left|\sum_{\nu_1, \dots, \nu_k = 1}^{N} c_{\nu_1} \dots c_{\nu_k} b_{\nu_1 \dots \nu_k}\right| \leqq (2c_N)^k B_k(N), \text{ falls } c_n \uparrow$$

und

(8) $$\left|\sum_{\nu_1, \dots, \nu_k = 1}^{N} c_{\nu_1} \dots c_{\nu_k} b_{\nu_1 \dots \nu_k}\right| \leqq (2c_1)^k B_k(N), \text{ falls } c_n \downarrow.$$

BEWEIS (durch vollständige Induktion nach k): Sei zuerst $c_n \uparrow$. Für $k=1$ liefert der 2. Mittelwertsatz der Integralrechnung mit $g(x) = b_{[x]}$ für $1 \leqq x \leqq N+1$ und $0 \leqq f(x) = c_{[x]} \uparrow$ für $1 \leqq x < N+1)$, $f(N+1) = c_N$:

$$\sum_{\nu=1}^{N} c_\nu b_\nu = \int_1^{N+1} g(x) f(x)\, dx = f(N+1) \int_\xi^{N+1} g(x)\, dx =$$

$$= c_N \int_\xi^{N+1} g(x) dx = c_N \sum_{\nu = [\xi]}^{N} b_\nu - c_N(\xi - [\xi]) b_{[\xi]} \quad (1 \leqq \xi \leqq N+1).$$

Also gilt:

(9) $$\sum_{\nu=1}^{N} c_\nu b_\nu = c_N \left(\sum_{\nu = n_0}^{N} b_\nu + \theta b_{n_0}\right)$$

für ein $n_0 \in \{1, \ldots, N\}$ und ein $\theta \in [-1, 1]$ und somit

$$\left|\sum_{\nu=1}^{N} c_\nu b_\nu\right| \leqq c_N \left(\left|\sum_{\nu=n_0}^{N} b_\nu\right| + |b_{n_0}|\right) \leqq 2c_N B_1(N).$$

Die Behauptung gelte also für $k \in \mathbf{N}$. Dann folgt zuerst mit (9):

$$\sum_{\nu_1, \ldots, \nu_{k+1}=1}^{N} c_{\nu_1} \ldots c_{\nu_{k+1}} b_{\nu_1 \ldots \nu_{k+1}} = \sum_{\nu_{k+1}=1}^{N} c_{\nu_{k+1}} \left(\sum_{\nu_1, \ldots, \nu_k=1}^{N} c_{\nu_1} \ldots c_{\nu_k} b_{\nu_1 \ldots \nu_{k+1}}\right) =$$

$$= c_N \left(\sum_{\nu_1, \ldots, \nu_k=1}^{N} c_{\nu_1} \ldots c_{\nu_k} \left(\sum_{\nu_{k+1}=n_0}^{N} b_{\nu_1 \ldots \nu_k \nu_{k+1}}\right) + \theta \times \sum_{\nu_1, \ldots, \nu_k=1}^{N} c_{\nu_1} \ldots c_{\nu_k} b_{\nu_1 \ldots \nu_k n_0}\right)$$

für ein $n_0 \in \{1, \ldots, N\}$ und ein $\theta \in [-1, 1]$. Die Induktionsbehauptung liefert:

$$\left|\sum_{\nu_1, \ldots, \nu_{k+1}=1}^{N} c_{\nu_1} \ldots c_{\nu_{k+1}} b_{\nu_1 \ldots \nu_{k+1}}\right| \leqq$$

$$\leqq c_N \left((2c_N)^k \operatorname{Max}\left(\left\{\left|\sum_{\nu_1=n_1}^{m_1} \ldots \sum_{\nu_k=n_k}^{m_k} \sum_{\nu_{k+1}=n_0}^{N} b_{\nu_1 \ldots \nu_k \nu_{k+1}}\right| : 1 \leqq n_\nu \leqq m_\nu \leqq N, \nu = 1, \ldots, k\right\}\right) +\right.$$

$$\left. + (2c_N)^k \max\left(\left\{\left|\sum_{\nu_1=n_1}^{m_1} \ldots \sum_{\nu_k=n_k}^{m_k} b_{\nu_1 \ldots \nu_k n_0}\right| : 1 \leqq n_\nu \leqq m_\nu \leqq N, \nu = 1, \ldots, k\right\}\right)\right) \leqq$$

$$\leqq c_N\big((2c_N)^k B_{k+1}(N) + (2c_N)^k B_{k+1}(N)\big) = (2c_N)^{k+1} B_{k+1}(N).$$

Für $c_n \downarrow$ folgt völlig analog die Ungleichung (8).

Bemerkung. Analog zur Gleichung (9) für $k=1$ erhält man auch im allgemeinen Falle eine Identität, in der dann 2^k Summanden der Form

$$\sum_{\mu_1, \ldots, \mu_j=1}^{N} b_{n_1 \ldots \mu_1 \ldots \mu_j \ldots n_k}$$

auftreten; wir benötigen allerdings nur die Ungleichungen (7) und (8).

2. Zentraler Grenzwertsatz (ZG). Wir fragen in diesem Abschnitt nach der Gültigkeit des ZG; d. h. unter welchen Voraussetzungen über die Folge (c_n) und die Funktionen $\Phi_n(x)$

(10) $$\lim_{N \to \infty} F_N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt \quad \text{für alle } y \in \mathbf{R} \text{ gilt.}$$

Dies ist äquivalent mit [2]:

(11) $$\lim_{N \to \infty} \varphi_N(t) = \lim_{N \to \infty} \int_{-\infty}^{\infty} e^{-ity} dF_N(y) = e^{-t^2/2} \quad \text{für alle } t \in \mathbf{R}.$$

In diesem Zusammenhang werden wir noch stets Folgendes voraussetzen:

(12) $$\int_0^1 \Phi_n^2(x) dx = 1 \quad \text{und} \quad \int_0^1 \Phi_n(x) dx = 0 \quad \text{für alle } n \in \mathbf{N},$$

d. h. die Erwartungswerte der Zufallsvariablen sollen gleich 0 und die Varianzen gleich 1 sein.

LEMMA 2. *Für die Folgen* (c_n) *und* $(\Phi_n(x))$ *gelte* (4), (6) *und* (12). *Außerdem gelte:*

$$(1)\qquad \frac{1}{C_N^2}\sum_{n=1}^{N} c_n^2\Phi_n^2(x)\to 1 \quad \textit{dem Maß nach auf } [0,1]$$

und

$$(2)\qquad \lim_{N\to\infty}\int_0^1 \prod_{n=1}^{N}\left(1-it\frac{c_n\Phi_n(x)}{C_N}\right)dx=1 \quad \textit{für alle } t\in\mathbf{R}.$$

Dann gilt der ZG, d. h. $\lim_{N\to\infty}\varphi_N(t)=e^{-t^2/2}$ *für alle* $t\in\mathbf{R}$.

BEWEIS. Móricz führt in [4] (siehe auch [6]) den Beweis, indem er $e^z=(1+z)e^{z^2/2+o(1)}$ für $|z|\to 0$ und damit

$$\varphi_N(t)=\int_{-\infty}^{\infty} e^{-ity}dF_N(y)=\int_0^1 e^{-itS_N(x)/C_N}dx=$$

$$=\int_0^1\prod_{n=1}^{N}\left(1-\frac{itc_n\Phi_n(x)}{C_N}\right)\exp\left(-\left(\frac{t^2}{2}\sum_{n=1}^{N}\frac{c_n^2\Phi_n^2(x)}{C_N^2}+o(1)\right)\right)dx$$

benutzt.

In [4], Satz 2, zeigt MÓRICZ, indem er ein Ergebnis von RÉVÉSZ [5] verbessert, daß unter den Voraussetzungen (6), (12) und der gleichmäßigen Beschränktheit der Folge $(\Phi_n(x))$ folgende Bedingungen für die Gültigkeit des ZG hinreichend sind:

$$(13)\qquad \int_0^1 \Phi_n^2(x)\Phi_m^2(x)dx=\int_0^1\Phi_n^2(x)dx\int_0^1\Phi_m^2(x)dx=1 \quad \text{für} \quad n\neq m$$

und

$$(14)\qquad \int_0^1 \Phi_{n_1}(x)\dots\Phi_{n_k}(x)dx=0 \quad \text{für} \quad n_1<\dots<n_k;\ k\in\mathbf{N}.$$

Letzteres bedeutet: „$(\Phi_n(x))$ ist ein multiplikatives System" (ALEXITS [1]). Das Ergebnis von Móricz ergibt sich aus Lemma 2, da aus (13) die Bedingung (1) und aus (14) die Bedingung (2) folgt.

Wir werden zunächst die Bedingung (14) abschwächen, und zeigen aber vorher, daß im Satz von Móricz die Bedingung (13) nicht vollkommen wegfallen kann. Eine entsprechende Bemerkung steht in einer Arbeit von KOMLÓS [3].

LEMMA 3. *Es gibt Folgen* (c_n) *und gleichmäßig beschränkte Folgen* $(\Phi_n(x))$, *die die Voraussetzungen* (6), (12) *und* (14) *erfüllen (d. h. multiplikative Systeme), für die der ZG nicht gilt.*

BEWEIS. Wir gehen aus von Folgen (c_n) und gleichmäßig beschränkten Folgen $(\Psi_n(x))$, die die Voraussetzungen (6), (12), (13) und (14) erfüllen. Wir betrachten jetzt den Wahrscheinlichkeitsraum $\Omega^*=[0,2]$ mit dem Maß $d\mu(x)=dx/2$ und definieren: $\Phi_n(t)=\sqrt{2}\Psi_n(t)$ für $0\leq t\leq 1$ und $\Phi_n(t)=0$ für $1<t\leq 2$.

Dann ist

$$\int_0^2 \Phi_n^2(t)\,d\mu(t) = \int_0^1 2\Psi_n^2(t)\,dt/2 = 1$$

und

$$\int_0^2 \Phi_{n_1}(x)\dots\Phi_{n_k}(x)\,d\mu(x) = 2^{k/2-1}\times\int_0^1 \Psi_{n_1}(x)\dots\Psi_{n_k}(x)\,dx = 0$$

für $1\leqq n_1<\dots<n_k$; also erfüllen die Folgen (c_n) und $\big(\Phi_n(x)\big)$ die Voraussetzungen (6), (12) und (14), und die $\Phi_n(x)$ sind gleichmäßig beschränkt (entsprechend für das Intervall [0, 2]). Aber es gilt für $\big(\Psi_n(x)\big)$ bzgl. (c_n) der ZG. Es folgt:

$$\lim_{N\to\infty}\frac{1}{C_N}\sum_{n=1}^{N} c_n^2\Phi_n^2(x) = \begin{cases} 2 \text{ dem Maß nach auf } [0,1] \\ 0 \text{ dem Maß nach auf } [1,2], \end{cases}$$

und damit:

$$\lim_{N\to\infty}\int_0^2 \prod_{n=1}^{N}\left(1-itc_n\frac{\Phi_n(x)}{C_N}\right)\exp\left(-\frac{t^2}{2}\left(\sum_{n=1}^{N}\frac{c_n^2\Phi_n^2(x)}{C_N^2}\right)+o(1)\right)d\mu(x) =$$

$$= \frac{1}{2}e^{-t^2}+\frac{1}{2}\neq e^{-t^2/2};$$

d. h. für die Folgen (c_n) und $\big(\Phi_n(x)\big)$ gilt der ZG nicht.

Wir geben jetzt einige hinreichende Bedingungen für die Gültigkeit von (2) an. (In 3. geben wir noch hinreichende Bedingungen für (1) an, d. h. die Gültigkeit des GG.) Dazu definieren wir für $k\in\mathbf{N}$:

$$\|B_k\|_1 = \sum_{1\leqq n_1<\dots<n_k}^{\infty} |b_{n_1\dots n_k}| \quad\text{und}\quad \|B_k\|_2 = \left(\sum_{1\leqq n_1<\dots<n_k}^{\infty} b_{n_1\dots n_k}^2\right)^{1/2}.$$

SATZ 1. *Die Folgen (c_n) und $\big(\Phi_n(x)\big)$ erfüllen* (1), (4), (6) *und* (12). *Dann ist jede der folgenden, zusätzlichen Voraussetzungen hinreichend für die Gültigkeit des Zentralen Grenzwertsatzes:*

(ZG$_1$) $\sum_{k=2}^{\infty}\|B_k\|_1<\infty$ (*d. h.* $\big(\Phi_n(x)\big)$ *schwach multiplikativ*);

(ZG$_2$) $(\|B_k\|_2)^{1/k} = o(\sqrt{k})$ *für* $k\to\infty$;

(ZG$_3$) *Für alle* $\varepsilon>0$ *mit*

$$F_\varepsilon(k,N) = \left|\left\{(n_1,\dots,n_k)\colon\ |b_{n_1\dots n_k}|>\frac{\varepsilon}{N^{k/2}},\quad 1\leqq n_1<\dots<n_k\leqq N\right\}\right|^{1}$$

gelte:

$$\lim_{N\to\infty}\left(\max_{n\leqq N} c_n\,\frac{(F_\varepsilon(k,N))^{1/k}}{C_N}\right) = 0 \text{ gleichmäßig in } k=2,3,\dots;$$

[1] $|M|$ bedeutet die Anzahl der Elemente der endlichen Menge M.

(ZG$_4$) *Die Folge* (c_n) *sei monoton, und es gelte für*

$$B_k(N) = \mathrm{Max}\left(\left\{\left|\sum_{\nu_1=n_1}^{m_1} \dots \sum_{\nu_k=n_k}^{m_k} b_{\nu_1\dots\nu_k}\right| : 1 \leqq n_\nu \leqq m_\nu \leqq N,\ \nu = 1, \dots, k\right\}\right)$$

$$\lim_{N\to\infty} (B_k(N))^{1/k} \left(\frac{c_N+c_1}{C_N}\right) = 0 \text{ gleichmäßig in } k = 2, 3, \dots .$$

BEWEIS. Nach Lemma 2 genügt es jeweils (2) zu beweisen.

$$\int_0^1 \prod_{n=1}^N (1 - itc_n\Phi_n(x)/C_N)\,dx = 1 + \sum_{k=1}^N (-it)^k \sum_{1\leqq n_1<\dots<n_k}^N \frac{c_{n_1}\dots c_{n_k}}{C_N^k} \int_0^1 \Phi_{n_1}\dots\Phi_{n_k}(x)dx =$$

$$= 1 + \sum_{k=2}^N (-it)^k \sum_{1\leqq n_1<\dots<n_k}^N c_{n_1}\dots c_{n_k} b_{n_1\dots n_k} C_N^{-k}$$

wegen $\int_0^1 \Phi_n(x)dx = 0$. Mit

$$d_{Nk} = \sum_{1\leqq n_1<\dots<n_k}^N c_{n_1}\dots c_{n_k} b_{n_1\dots n_k} C_N^{-k}$$

ist (2) also äquivalent mit:

(15) $$\lim_{N\to\infty} \sum_{k=2}^N d_{Nk}(-it)^k = 0 \quad \text{für alle } t\in\mathbf{R}.$$

Hinreichend für (15) ist:

(16) $$\lim_{N\to\infty} \sqrt[k]{|d_{Nk}|} = 0 \text{ gleichmäßig in } k = 2, 3, \dots .$$

Dabei ist (16) sicher immer dann erfüllt, wenn gilt:

(17) $$\lim_{N\to\infty} d_{Nk} = 0 \text{ für } k = 2, 3, \dots$$

und

(18) $$|d_{Nk}| \leqq d_k \text{ für alle } N\in\mathbf{N} \text{ mit } \lim_{k\to\infty} \sqrt[k]{d_k} = 0.$$

Wir werden jeweils (16) oder (17) und (18) als hinreichende Bedingungen für (2) beweisen.

(a) Zunächst folgt (ZG$_2$) aus (ZG$_1$); denn aus $\sum_{k=2}^\infty \|B_k\|_1 < \infty$ folgt: $\sum_{k=2}^\infty \|B_k\|_2^2 < \infty$, also gilt $\lim_{k\to\infty} \|B_k\|_2 = 0$ und damit auch $\sqrt[k]{\|B_k\|_2} = O(\sqrt{k})$ für $k\to\infty$.

(b) Es gelte (ZG$_2$). Wir zeigen (17) und (18).

$$d_{Nk}^2 \leqq \left(\sum_{1\leqq n_1<\dots<n_k}^{N\,\prime} c_{n_1}\dots c_{n_k} |b_{n_1\dots n_k}| C_N^{-k}\right)^2 \leqq$$

$$\leqq \sum_{1\leqq n_1<\dots<n_k}^N b_{n_1\dots n_k}^2 \sum_{1\leqq n_1<\dots<n_k}^N c_{n_1}^2\dots c_{n_k}^2 C_N^{-2k} \leqq \|B_k\|_2^2/k! =: d_k;$$

denn

$$\sum_{1\leqq n_1<\ldots<n_k}^{N} c_{n_1}^2\ldots c_{n_k}^2=\frac{1}{k!}\sum_{\substack{n_1,\ldots,n_k=1\\ n_i\neq n_j,\, i\neq j}}^{N} c_{n_1}^2\ldots c_{n_k}^2\leqq\frac{1}{k!}\sum_{n_1,\ldots,n_k=1}^{N} c_{n_1}^2\ldots c_{n_k}^2=C_N^{2k}/k!.$$

Mit $\|B_k\|_2^{1/k}=o(\sqrt{k})$ folgt $\sqrt[k]{d_k}=o(1)$ für $k\to\infty$ und damit (18). Um (17) zu zeigen, beachten wir, daß zu jeder Folge (x_n) mit $\sum_{n=1}^{\infty}|x_n|<\infty$ eine Folge $y_n\uparrow\infty$ existiert mit $\sum_{n=1}^{\infty}|x_n y_n|<\infty$. Wenden wir dies auf

$$x_n=\sum_{n\leqq n_2<\ldots<n_k}^{\infty} b_{nn_2\ldots n_k}^2$$

an, so gilt $\sum_{n=1}^{\infty}|x_n|<\infty$, und wir erhalten eine Folge $y_n\uparrow\infty$ mit $\sum_{1\leqq n_1<\ldots<n_k}^{\infty} b_{n_1\ldots n_k}^2 y_{n_1}^2<\infty$. Dann folgt:

$$d_{Nk}^2=\left(\sum_{1\leqq n_1<\ldots<n_k}^{N}\frac{c_{n_1}\ldots c_{n_k}}{y_{n_1}C_N^k}b_{n_1\ldots n_k}y_{n_1}\right)^2\leqq$$

$$\leqq\left(\sum_{1\leqq n_1<\ldots<n_k}^{N} c_{n_1}^2\ldots c_{n_k}^2 y_{n_1}^{-2}C_N^{-2k}\right)^2\left(\sum_{1\leqq n_1<\ldots<n_k}^{N} b_{n_1\ldots n_k}^2 y_{n_1}^2\right)^2=$$

$$=O(1)\sum_{n=1}^{N}c_n^2y_n^{-2}\left(\sum_{n=1}^{N}c_n^2\right)^{k-1}C_N^{-2k}=O(1)\frac{1}{C_N^2}\times\sum_{n=1}^{N}c_n^2y_n^{-2}=o(1)\text{ für }N\to\infty.$$

(c) Es gelte jetzt (ZG_3). Sei $\varepsilon>0$ beliebig, und sei

$$d_{Nk}^{(1)}=\sum_{F_\varepsilon(k,N)}c_{n_1}\ldots c_{n_k}b_{n_1\ldots n_k}C_N^{-k}\quad\text{und}\quad d_{Nk}^{(2)}=d_{Nk}-d_{Nk}^{(1)}.$$

Wir zeigen zunächst, daß aus (ZG_3) (16) folgt für $d_{Nk}^{(1)}$

$$|d_{Nk}^{(1)}|\leqq F_\varepsilon(k,N)\cdot K^k(\max_{n\leqq N}c_nC_N^{-1})^k$$

(beachte $|b_{n_1\ldots n_k}|\leqq K^k$ wegen (4) mit $K=C^{1/2}$.).
Es folgt mit (ZG_3)

$$\lim_{N\to\infty}\sqrt[k]{|d_{Nk}^{(1)}|}\leqq\lim_{N\to\infty}K\cdot\max_{n\leqq N}c_n\cdot C_N^{-1}(F_\varepsilon(k,N))^{1/k}=0$$

gleichmäßig in $k=2,3,\ldots$. Also gilt

(19) $$\lim_{N\to\infty}\sum_{k=2}^{N}d_{Nk}^{(1)}(-it)^k=0\text{ für alle }t\in\mathbf{R},$$

$$\left|\sum_{k=2}^{N}d_{Nk}^{(2)}(-it)^k\right|\leqq\sum_{k=2}^{N}\varepsilon N^{-k/2}|t|^kC_N^{-k}\frac{1}{k!}\sum_{n_1,\ldots,n_k=1}^{N}c_{n_1}\ldots c_{n_k}\leqq\sum_{k=2}^{N}\varepsilon|t|^k/k!\leqq\varepsilon e^{|t|}$$

wegen

$$\sum_{n_1,\dots,n_k=1}^{N} c_{n_1}\dots c_{n_k} = \left(\sum_{n=1}^{N} c_n\right)^k \leqq \left(\left(\sum_{n=1}^{N} c_n^2\right)^{1/2} N^{1/2}\right)^k = C_N^k N^{k/2}.$$

Es folgt zusammen mit (19) Bedingung (15).

(d) Es gelte (ZG_4). Wir zeigen (16). Mit Lemma 1 folgt:

$$|d_{Nk}| = \left|\sum_{1\leqq n_1<\dots<n_k}^{N} c_{n_1}\dots c_{n_k} b_{n_1\dots n_k} C_N^{-k}\right| \leqq \left(2(c_1+c_N)\right)^k \cdot C_N^{-k}\cdot B_k(N),$$

und folglich gilt

$$\lim_{N\to\infty} \sqrt[k]{|d_{Nk}|} \leqq \lim_{N\to\infty} 2(c_1+c_N)C_N^{-1}(B_k(N))^{1/k} = 0$$

gleichmäßig in $k=2, 3, \dots$. Damit ist Satz 1 vollständig bewiesen.

3. Gesetz der Großen Zahlen (GG). Wir werden in diesem Abschnitt stets noch voraussetzen:

(20) (d_n) sei eine Folge positiver Zahlen, $D_N = \sum_{n=1}^{N} d_n$ mit

$$\lim_{N\to\infty} d_N/D_N = 0 \quad \text{und} \quad D_N \to \infty.$$

Die Folge $(\Psi_n(x))$ erfülle (5) und (d_n) erfülle (20). Wir sagen dann: $(\Psi_n(x))$ erfüllt bezüglich (d_n) das Gesetz der Großen Zahlen, wenn gilt:

(21) $$\lim_{N\to\infty} \frac{1}{D_N} \sum_{n=1}^{N} d_n \Psi_n(x) = 0 \text{ dem Maß nach auf } [0, 1].$$

Mit diesen Bezeichnungen ist die Bedingung (1) (Lemma 1) äquivalent mit:

Die Folge $(\Psi_n(x)=\Phi_n^2(x)-1)$ erfüllt bezüglich $(d_n=c_n^2)$ das Gesetz der Großen Zahlen, und für das so definierte $\Psi_n(x)$ ist die Bedingung (13) von Móricz äquivalent mit:

$$\int_0^1 \Psi_n(x)\Psi_m(x)dx = 0 \text{ für } m \neq n, \text{ d. h. } (\Psi_n(x)) \text{ bildet ein Orthogonalsystem.}$$

Wir werden im Folgenden diese Bedingung der Orthogonalität und die gleichmäßige Beschränktheit der $\Psi_n(x)$ abschwächen. Sei dazu:

$$B = (b_{ik}) \text{ mit } b_{ik} = \int_0^1 \Psi_i(x)\Psi_k(x)dx \text{ für } i, k = 1, 2, \dots.$$

SATZ 2. *Für* $(\Psi_n(x))$ *und* (d_n) *gelte* (5), (20) *sowie* $b_{nn}d_n/D_n\to 0$ *für* $n\to\infty$. *Dann sind die folgenden Bedingungen jeweils hinreichend für die Gültigkeit des Gesetzes der Großen Zahlen, d. h.* (21):

(GG_1) *Für alle* $\varepsilon>0$ *existiert* $M(\varepsilon)$ *mit* $|b_{ik}|<\varepsilon$ *für* $i\neq k$ *und* $i>M(\varepsilon)$, $k>M(\varepsilon)$;

(GG$_2$) *Es existiert eine Funktion* $\Phi(N)$ *mit* $\sum_{i\in I_N} d_i = o(D_N)$ *für jede beliebige Indexmenge mit* $I_N \subset \{1, \ldots, N\}$, $|I_N| \leqq \sqrt{\Phi(N)}$, *und für alle* $\varepsilon > 0$ *existiert* $M(\varepsilon)$, *so daß für*

$$F_\varepsilon(N) = \{(i, k): |b_{ik}| > \varepsilon,\ i \neq k,\ M(\varepsilon) < i \leqq N, M(\varepsilon) < k \leqq N\}$$

$|F_\varepsilon(N)| \leqq \Phi(N)$ *gilt;*

(GG$_3$) *Die Folge* (d_n) *sei monoton, und es gelte für*

$$B(N) := \operatorname{Max}\left(\left|\sum_{\nu_1=n_1}^{m_1} b_{\nu_1 \nu_2}\right|:\quad 1 \leqq n_1 \leqq m_1 \leqq N,\ 1 \leqq \nu_2 \leqq N\}\right)$$

$$\lim_{N\to\infty} (d_1 + d_N) B(N)/D_N = 0.$$

BEMERKUNG. Bildet die Folge $(\Psi_n(x))$ ein Orthogonalsystem, so sind alle Bedingungen (GG$_{1,2,3}$) erfüllt, soweit sie die Folge $(\Psi_n(x))$ betreffen; d. h. wir bekommen schwächere hinreichende Bedingungen für Gültigkeit des GG.

BEWEIS. Hinreichend für (21) ist

(22) $$\lim_{N\to\infty} \int_0^1 \left(\sum_{n=1}^{N} d_n \Psi_n(x)/D_N\right)^2 dx = \lim_{N\to\infty} \sum_{i,j=1}^{N} d_i d_j b_{ij}/D_N^2 = 0.$$

Wir zeigen jeweils (22).

(a) Zunächst folgt aus (GG$_1$) unmittelbar (GG$_2$). Man wähle $\Phi(N)=0$ in (GG$_2$).

(b) Es gelte (GG$_2$). Für $\varepsilon > 0$ sei:

$$A = \{(i, k):\ 1 \leqq i \leqq N,\ 1 \leqq k \leqq N\};$$

$$A_1 = \{(i, k) \in A:\ i \leqq M(\varepsilon) \quad \text{oder} \quad k \leqq M(\varepsilon), i \neq k\};$$

$$A_2 = \{(i, k) \in A:\ i > M(\varepsilon) \text{ und } k > M(\varepsilon),\ i \neq k,\ (i, k) \notin F_\varepsilon(N)\};$$

also

$$A = A_1 \cup A_2 \cup F_\varepsilon(N) \cup \{(i, i):\ 1 \leqq i \leqq N\}.$$

Es folgt:

$$\sum_A d_i d_k b_{ik}/D_N^2 = \sum_{i=1}^{N} d_i^2 b_{ii}/D_N^2 + \Big(\sum_{A_1} + \sum_{A_2} + \sum_{F_\varepsilon(N)}\Big) d_i d_k b_{ik}/D_N^2 = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

$$|\mathrm{I}| \leqq \frac{1}{D_N} \sum_{i=1}^{N} d_i d_i b_{ii}/D_i = o(1) \frac{1}{D_N} \sum_{i=1}^{N} d_i = o(1) \text{ für } N \to \infty;$$

denn $D_N \to \infty$ und $d_i b_{ii}/D_i \to 0$ für $N \to \infty$ bzw. $i \to \infty$.

$$|\mathrm{II}| \leqq 2K \cdot D_N^{-2} \sum_{i=1}^{N} \sum_{k=1}^{M(\varepsilon)} d_i d_k = 2K \cdot D_N^{-1} \sum_{i=1}^{M(\varepsilon)} d_i = o(1) \text{ für } N \to \infty$$

wegen $D_N \uparrow \infty$ (20) und wegen (5); d.h. $|b_{ik}| \leqq K$.

$$|\mathrm{III}| \leqq \varepsilon \sum_{i,k=1}^{N} d_i d_k / D_N^2 = \varepsilon.$$

Zur Abschätzung von IV können wir $d_n\downarrow$ für $1\leqq n\leqq N$ annehmen; denn anderenfalls ordne man die $d_1, \ldots, d_N$ geeignet um. Sei $\psi(N)=[\sqrt{\Phi(N)}+1]$ und

$$F_1 = \{(i,k)\in F_\varepsilon(N)\colon\ i < \Psi(N)\},\quad F_2 = \{(i,k)\in F_\varepsilon(N)\colon\ k < \Psi(N)\},$$
$$F_3 = \{(i,k)\in F_\varepsilon(N)\colon\ i \geqq \Psi(N),\ k \geqq \Psi(N)\}.$$

Dann gilt für alle (i,k) mit $i\leqq\Psi(N)$, $k\leqq\Psi(N)$ und alle (m,n) mit $m\geqq\Psi(N)$, $n\geqq\Psi(N)$ $d_i d_k\geqq d_m d_n$. Damit folgt:

$$|\mathrm{IV}| \leqq K\Big(\sum_{F_1}+\sum_{F_2}+\sum_{F_3}\Big)d_i d_k/D_N^2 \leqq 2K\sum_{i=1}^{N}\sum_{k=1}^{\Psi(N)-1} d_i d_k/D_N^2+K\sum_{F_3} d_i d_k/D_N^2$$

und weiter wegen $|F_\varepsilon(N)|\leqq\Psi^2(N)$

$$\leqq 2K\left(\sum_{i=1}^{\Psi(N)} d_i\right)\Big/D_N+K\sum_{i=1}^{\Psi(N)}\sum_{k=1}^{\Psi(N)} d_i d_k/D_N^2 = o(1)$$

für $N\to\infty$ wegen (GG_2) $(\psi(N)-1\leqq\sqrt{\Phi(N)})$ und (20).

(c) Es gelte (GG_3). Dann folgt mit Lemma 1:

$$\left|\sum_{i,k=1}^{N} d_i d_k b_{ik}/D_N^2\right| = \left|\sum_{i=1}^{N} d_i D_N^{-2}\left(\sum_{k=1}^{N} d_k b_{ik}\right)\right| =$$

$$= \left|\sum_{i=1}^{N} d_i D_N^{-2}\left(\sum_{k=n_1(i)}^{n_2(i)} b_{ik}+\theta_i b_{in_0(i)}\right)\right| \begin{matrix} d_N, & \text{falls } d_n\uparrow \\ d_1, & \text{falls } d_n\downarrow \end{matrix} \leqq$$

$$\leqq 2B(N)(d_1+d_N)\sum_{i=1}^{N} d_i/D_N^2 = 2B(N)(d_1+d_N)/D_N = o(1)$$

für $N\to\infty$ wegen (GG_3).

BEMERKUNGEN. (1) Die Bedingung (GG_1) besagt, daß in der Kovarianzmatrix $B=(b_{ik})$ die Nichtdiagonalelemente gegen Null gehen; dies wird in (GG_2) dahingehend abgeschwächt, daß lediglich der „größte Teil" gegen Null konvergiert.

(2) Ist $\Phi(N)=o(N^2)$ in (GG_2), so lauten die Bedingungen: $\sum_{n\in I_N} d_n=o(D_N)$ für $|I_N|=o(N)$ sowie

$$|\{(i,k)\colon\ |b_{ik}|>\varepsilon,\ i\neq k,\ i>M(\varepsilon),\ k>M(\varepsilon)\}| = o(N^2).$$

Dies ist in einem gewissen Sinne die schwächste Bedingung, die für die Verteilung $(|F_\varepsilon(N)|)$ der b_{ik} möglich ist. Obige Bedingungen für die Folge (d_n) sind etwa erfüllt für $d_n=n^\alpha$ für $\alpha>-1$, aber nicht erfüllt für $d_n=n^{-1}$ oder $d_n=e^{n^\alpha}$ mit $\alpha>0$.

(3) Die Autorem danken Herrn F. Móricz die Vereinfachung des ursprünglichen Beweises von Satz 2.

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UNIVERSITÄT ULM
ABTEILUNG MATHEMATIK
ULM
BUNDESREPUBLIK DEUTSCHLAND

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NOTE ON A PROBLEM OF CATHERINE RÉNYI ABOUT JULIA LINES

By
J. S. HWANG* (Hamilton)

Let $f(z)$ be an entire function. The ray $\arg z=\vartheta$ is called a Julia line if in every angle $|\arg z-\vartheta|<\varepsilon$ the function takes every finite value infinitely often with at most one exception. CATHERINE RÉNYI asked (see [1], Problem 2.4) if an entire function can have different exceptional values at different Julia lines. This is shown to be the case by TOPPILA [2] and later by a different method by BARTH and SCHNEIDER [3]. A notion connected with exceptional values is that of asymptotic values. A path tending to infinity is called asymptotic path with corresponding asymptotic value w if $f(z)\to w$ as $z\to\infty$ along the path. In this connection we prove the following

THEOREM. *Let E be an arbitrary nowhere dense closed set on the real line periodic with respect to 2π and $w(\vartheta)$ $(\vartheta\in E)$ an arbitrary function periodic with respect to 2π and belonging to the first Baire class on E. Then there is an entire function $f(z)$ which has all the rays $\arg z=\vartheta\in E$ both as Julia lines and asymptotic paths with corresponding asymptotic value $w(\vartheta)$.*

We remark that the assumption on $w(\vartheta)$, i.e. that it be the pointwise limit of a sequence of continuous functions is obviously also necessary.

PROOF. We can add to E a countable set E' such that $E\cap E'=\emptyset$, $E\subset\bar{E}'$, the closure of E' and $E''=E\cup E'$ also satisfies the hypothesis of the theorem. We can then extend $w(\vartheta)$ to be a periodic function on E'' such that for every $\varepsilon>0$ and $\vartheta_0\in E$ the set $\{w(\vartheta)\colon |\vartheta-\vartheta_0|<\varepsilon,\ \vartheta\in E'\}$ be e.g. everywhere dense in the complex plane. Since E' is countable the extended function will also belong to the first Baire class and so $w(\vartheta)=\lim_{n\to\infty} g_n(\vartheta)$ $(\vartheta\in E'')$ with $g_n(\vartheta)$ $(n=1, 2, \dots)$ continuous on E''. Let $g(ne^{i\vartheta})=g_n(\vartheta)$ and $g(xe^{i\vartheta})$ e.g. linear for $n\leqq x\leqq n+1$. We have thus defined a continuous function $g(z)$ on the union of rays $\arg z\in E''$, $|z|\geqq 1$, and so by a result of A. ROTH (see [4]), one can find and entire function $f(z)$ such that $f(z)-g(z)\to 0$ (even uniformly) as $z\to\infty$, $\arg z\in E''$. This obviously implies the statement concerning asymptotic values. It remains to show that each ray $\arg z=\vartheta_0\in E$ is a Julia line.

Assuming the contrary there is a sector $|\arg z-\vartheta_0|<\varepsilon(\leqq 1)$, $|z|\geqq R_0$ where $f(z)$ omits two finite values. So does then the family of functions

$$f_R(s)=f(Re^{i\vartheta_0}+sR\varepsilon/2)$$

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for $|s|<1$, $R \geqq 2R_0$. Now, a family with this property is a normal family (see e.g. [5]) and since $f_R(0) \to w(\vartheta_0)$, as $R \to \infty$, $f_R(s)$ is bounded for $|s| \leqq r < 1$, uniformly in R, i.e. $f(z)$ is bounded in $|\arg z - \vartheta_0| < \varepsilon/2$, a case ruled out by the construction of E' and $w(\vartheta)$ on E' which implies that the asymptotic values of $f(z)$ corresponding to asymptotic rays in $|\arg z - \vartheta_0| < \varepsilon/2$ are everywhere dense and the proof is completed.

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DEPARTMENT OF MATHEMATICS
MCMASTER UNIVERSITY
HAMILTON, ONTARIO, CANADA

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APPROXIMATION THEOREMS FOR COSINE OPERATOR FUNCTIONS

By

B. NAGY (Budapest)

1. Introduction. Let X denote a complex Banach space, $B(X)$ the space of bounded linear operators from X into X, R the real field. A cosine operator function $C(t)$ is a mapping of R into $B(X)$ such that $C(0)=I$ (the identity operator), for every $s, t\in R$

$$C(s+t)+C(s-t) = 2C(s)\,C(t), \tag{1}$$

and $t\to t_0$ implies $C(t)\to C(t_0)$ in the strong operator topology of $B(X)$. The generator operator A of $C(t)$ is defined as $Ax=\lim_{t\to 0}\frac{2}{t^2}[C(t)-I]x$ for exactly those $x\in X$ for which the above limit exists in the strong topology of X. The fundamental results on cosine operator functions can be found, e.g., in [3], [6] and [7].

In this paper we will study the behaviour of the operators

$$T_n(t) = C(t)-\sum_{k=0}^{n-1}\frac{t^{2k}}{(2k)!}A^k \quad (n \text{ positive integer})$$

in the neighbourhood of $t=0$. The starting point is Taylor's formula [6], stating that $x\in D(A^n)$, $t\in R$ imply

$$T_n(t)x = \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!}C(s)\,A^n x\,ds.$$

In Section 2 the saturation problem is solved, making use of the concept of relative completion (cf. [1]). In Section 3 some applications on function spaces are presented. Concerning similar problems for semigroups of linear operators see, e.g., [1] and [2].

We note that in the Banach spaces dealt with the norm will be denoted by $|\cdot|$ (with a subscript if necessary), and for the natural pairing of X^* and X we shall write x^*x or (x^*, x). Also, for an operator A, we shall denote $(A^*)^k$ by A^{*k} and $(A^k)^*$ by A^{k*}.

2. Saturation theorems. If $C(t)$ is a cosine operator function and A is its generator, then for $n=1, 2, \dots$ A^n is a closed operator with domain $D(A^n)$ dense in X. Introduce in $D(A^n)$ the n-norm

$$|x|_n = |x|+|Ax|+\dots+|A^n x|,$$

then $D_n=\{D(A^n), |\cdot|_n\}$ is a Banach space ($D_0=X$, by definition).

LEMMA 1. *If* $y \in D(A^n)$, *then*

$$|(2n)!\,t^{-2n}T_n(t)y - A^n y| = o(1) \quad (t \to 0).$$

Further, if $x \in D(A^{n-1})$, $x_k \in D(A^n)$ *for* $k = 1, 2, \dots$, $|x_k - x|_{n-1} \to 0$ *and* $\{A^n x_k\}_k$ *is a bounded sequence in* X, *then*

$$|T_n(t)x| = O(t^{2n}) \quad (t \to 0).$$

PROOF. Since $C(t)$ is an even function, we may and will suppose $t>0$. From Taylor's formula

$$|(2n)!\,t^{-2n}T_n(t)y - A^n y| =$$

$$= \left|(2n)!\,t^{-2n}\int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!}C(s)A^n y\,ds - 2nt^{-2n}\int_0^t (t-s)^{2n-1}A^n y\,ds\right| =$$

$$= 2nt^{-2n}\left|\int_0^t (t-s)^{2n-1}(C(s)-I)A^n y\,ds\right| \leqq \sup_{0 \leqq s \leqq t} |(C(s)-I)A^n y| \to 0$$

as $t \to 0$. Moreover, for every k

$$t^{-2n}|T_n(t)x_k| \leqq (2n)!^{-1} \sup_{0 \leqq s \leqq t} |C(s)||A^n x_k| \leqq M|A^n x_k|$$

for $0 < t \leqq 1$. By assumption,

$$t^{-2n}|T_n(t)x| = \lim_{k \to \infty} t^{-2n}|T_n(t)x_k| \leqq M \sup_k |A^n x_k| \leqq M_1,$$

and the second statement is also proved.

The following definition is suggested by Lemma 1 (cf. [1]).

DEFINITION 1. The completion of D_n relative to D_{n-1} is denoted by $\tilde{D}_n^{D_{n-1}}$ and defined as

$$\tilde{D}_n^{D_{n-1}} = \bigcup_{R>0} \overline{S_n(R)}^{n-1}, \quad \text{where} \quad S_n(R) = \{y \in D_n : |y|_n \leqq R\},$$

and $\overline{S_n(R)}^{n-1}$ denotes the strong closure of $S_n(R)$ in D_{n-1}.

In this terminology Lemma 1 states that $x \in \tilde{D}_n^{D_{n-1}}$ implies $|T_n(t)x| = O(t^{2n})$. In the converse direction we have

LEMMA 2. *If* $x \in D(A^{n-1})$ *and* $\varliminf_{t \to 0} t^{-2n}|T_n(t)x| < \infty$, *then* $x \in \tilde{D}_n^{D_{n-1}}$.

PROOF. For every $x \in D_{n-1}$ we define

$$x(t) = \frac{2n}{t^{2n}}\int_0^t (t-s)^{2n-1}C(s)x\,ds \quad (t \in R,\ t \neq 0).$$

We have for $k = 0, 1, \dots, n-1$

$$A^k x(t) = \frac{2n}{t^{n2}}\int_0^t (t-s)^{2n-1}C(s)A^k x\,ds,$$

for A^k is closed and $A^k C(s)x = C(s)A^k x$. Hence

$$|A^k x(t) - A^k x| \leqq \sup_{0 \leqq s \leqq |t|} |[C(s)-I]A^k x| \to 0 \quad \text{as} \quad t \to 0,$$

thus $|x(t)-x|_{n-1} \to 0$, as $t \to 0$. On the other hand, for every $y \in D_n$ we get, by Taylor's theorem,

$$A^n y(t) = \frac{(2n)!}{t^{2n}} \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} C(s) A^n y\, ds = \frac{(2n)!}{t^{2n}} T_n(t) y. \tag{2}$$

Since A also generates a semigroup of class (C_0), a suitable modification of I. GELFAND's proof (see, e.g., [2], p. 12) yields that $D(A^n)$ is $(n-1)$-dense in D_{n-1}, i.e. for $x \in D_{n-1}$ there exists a sequence $\{y_r\} \subset D_n$ with $|y_r - x|_{n-1} \to 0$ $(r \to \infty)$. But then for $r \to \infty$ in the norm topology of X we have

$$y_r(t) = \frac{2n}{t^{2n}} \int_0^t (t-s)^{2n-1} C(s) y_r\, ds \to x(t)$$

and, because of (2)

$$A^n y_r(t) = \frac{(2n)!}{t^{2n}} \left[C(t) - \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} A^k \right] y_r \to \frac{(2n)!}{t^{2n}} T_n(t) x.$$

Since A^n is closed, $A^n x(t) = \frac{(2n)!}{t^{2n}} T_n(t) x$. By assumption, there is a sequence $\{t_r\} \subset R$ such that $t_r^{-2n} |T_n(t_r)x| \leqq K < \infty$ for every r, and $t_r \to 0$. Then $x(t_r) \in D_n$, $|x(t_r) - x|_{n-1} \to 0$, while $|A^n x(t_r)| \leqq K(2n)!$, thus $|x(t_r)|_n = |x(t_r)| + |Ax(t_r)| + \ldots$ $\ldots + |A^n x(t_r)|$ remains bounded when $r \to \infty$. Hence $x \in \tilde{D}_n^{D_{n-1}}$, which was to be proved.

COROLLARY 1. *If* $x \in D(A^{n-1})$ *and for some sequence* $\{t_r\} \to 0$ *we have* $(2n)!\, t_r^{-2n} T_n(t_r) x \xrightarrow{w} y$, *then* $x \in D(A^n)$ *and* $A^n x = y$. ($\xrightarrow{w}$ and $\xrightarrow{s}$ will denote weak and strong convergences in X, respectively).

PROOF. From the proof of Lemma 2, $x(t_r) \in D(A^n)$, $x(t_r) \xrightarrow{s} x$ and $A^n x(t_r) =$ $= (2n)!\, t_r^{-2n} T_n(t_r) x \xrightarrow{w} y$. Then, by a result of MAZUR (see, e.g., [8], Theorem 2 on p. 120), some sequence of convex combinations of the elements $A^n x(t_r)$ converges strongly to y, and the sequence of the same convex combinations of the elements $x(t_r)$ converges strongly to x. Since A^n is closed, we obtain the assertion.

COROLLARY 2. $x \in D(A^{n-1})$ *and* $\varliminf_{t \to 0} t^{-2n} |T_n(t) x| = 0$ *imply* $T_n(t) x = 0$ *for every real* t.

PROOF. Corollary 1 yields that $A^n x = 0$, and Taylor's formula proves the statement.

Now we give the following definition of saturation (cf. [1], [2]).

DEFINITION 2. Let X be a Banach space, Y a linear subset of X, $T(t)$ $(t > 0)$ a family of linear operators from Y to X. Suppose there is a positive r such that for any $y \in Y$ the limit relation $|T(t)y| = o(t^r)$ $(t \to 0+)$ implies $T(t)y = 0$

for every $t>0$, while $F\{T(t);\ Y, X\}=\{y\in Y: |T(t)y|=O(t^r)\ (t\to 0+)\}$ contains at least one y with $T(t)y\neq 0$ for some $t>0$. Then $\{T(t); t>0\}$ is saturated in (Y, X) with order $O(t^r)$ and $F\{T(t);\ Y, X\}$ is its saturation (or Favard) class.

THEOREM 1. *Suppose $C(t)$ is a cosine operator function and*

$$T_n(t) = C(t) - \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} A^k$$

is not identically $0\in B(X)$ for every $t>0$. Then $\{T_n(t)\}$ is saturated in $(D(A^{n-1}), X)$ with order $O(t^{2n})$, and its Favard class is $\tilde{D}_n^{D_{n-1}}$.

PROOF. In view of Lemmas 1, 2 and Corollary 2 we need only to show that $\tilde{D}_n^{D_{n-1}}$ contains an element y with $T_n(t)y\neq 0$ for some $t>0$. Suppose that for every $y\in D_n$, $t>0$ we have $T_n(t)y=0$. Then $A^n y=C^{(2n)}(0)y=T_n^{(2n)}(0)y=0$, hence $A^n=0\in B(X)$, because A^n is closed and has dense domain in X. But then Taylor's theorem yields $T_n(t)=0\in B(X)$ for every $t>0$, a contradiction.

COROLLARY 3. *Under the conditions of Theorem 1 assume that X is a reflexive Banach space. Then $F\{T_n(t); D(A^{n-1}), X\}=D_n$.*

PROOF. $x\in\tilde{D}_n^{D_{n-1}}$ if and only if the function $t^{-2n}T_n(t)x$ is bounded in a neighbourhood of $t=0$, by Theorem 1. Hence, according to the Theorem on p. 141 in [8], for some sequence $\{t_r\}\to 0$ we have $t_r^{-2n}T_n(t_r)x\xrightarrow{w} y$. Thus $x\in D(A^n)$, by Corollary 1, and Theorem 1 yields the statement.

In what follows we will study the saturation behaviour on $D(A^{*n-1})$ (n positive integer) of the operators

$$S_n^*(t) = C(t)^* - \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} A^{*k} \quad (t\in R).$$

Note that the family of dual operators $\{C(t)^*: t\in R\}$ is not, as a rule, strongly continuous on X^*, and is *not* what we called the adjoint cosine operator function in [6]. We shall call it the conjugate function of $C(t)$.

LEMMA 3. *If A is the generator of a cosine operator function $C(t)$, then $A^{*k}=A^{k*}$ $(k=1, 2, \ldots)$.*

PROOF. For every k the set $D(A^k)$ is dense in X, and we clearly have $A^{*k}\subset A^{k*}$. According to [3], Lemma 5.6, $\sigma(A)$, the spectrum of A, is contained in $\{z;\ \mathrm{Re}\, z\leq w^2-(\mathrm{Im}\, z)^2/4w^2\}$ for some $w>0$, hence the resolvent set of A^k, $\varrho(A^k)$ is nonvoid. Indeed, it is easy to see that for $p>0$ large enough every $\sqrt{pi}$ belongs to $\varrho(A)$, thus the spectral mapping theorem (cf. [4], Theorem 5.12.3) yields that $pi\in\varrho(A^k)=\varrho(A^{k*})$. By the same reason, we obtain $pi\in\varrho(A^{*k})$, thus $pi-A^{*k}$ and $pi-A^{k*}$ are both 1—1 mappings onto X^*, hence $A^{*k}=A^{k*}$.

LEMMA 4. *If $x^*\in D(A^{*n-1})$, then*
a) *$x^*\in D(A^{n*})$ implies $|S_n^*(t)x^*|=O(t^{2n})$ $(t\to 0)$,*
b) *$\varliminf_{t\to 0} t^{-2n}|S_n^*(t)x^*|<\infty$ implies $x^*\in D(A^{n*})$,*
c) *$\varliminf_{t\to 0} t^{-2n}|S_n^*(t)x^*|=0$ implies $S_n^*(t)x^*=0^*$ for $t\in R$.*

PROOF. a) Suppose $x^* \in D(A^{n*})$, $x \in D(A^n)$, then

$$(S_n^*(t)x^*, x) = \left(\left[C(t)^* - \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} A^{k*}\right] x^*, x\right) =$$

$$= \left(x^*, \left[C(t) - \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} A^k\right] x\right) = \left(x^*, A^n \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} C(s)x\,ds\right) =$$

$$= \left(A^{n*}x^*, \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} C(s)x\,ds\right).$$

Since $D(A^n)$ is dense in X, we get for every $x \in X$

$$(S_n^*(t)x^*, x) = \left(A^{n*}x^*, \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} C(s)x\,ds\right).$$

Hence for $0 < |t| \leqq 1$ we obtain

$$t^{-2n}|S_n^*(t)x^*| \leqq |A^{n*}x^*| \cdot (2n)!^{-1} \sup_{0 \leqq |s| \leqq 1} |C(s)| < \infty,$$

and our statement follows.

b) Here we apply an idea of K. DE LEEUW [5]. Suppose $\{t_j\}$ is a sequence of real numbers converging to 0 and such that $t_j^{-2n}|S_n^*(t_j)x^*| \leqq K < \infty$ for every t_j. Define for $r = 1, 2, \ldots$ $H_r = \{(2n)!\, t_j^{-2n} S_n^*(t_j)x^* : j \geqq r\}$ and G_r to be the w*-closure of H_r in X^*. Then G_r is strongly bounded and, by Alaoglu's theorem, compact in the w*-topology of X^*. The sequence G_r is decreasing and has the finite intersection property, hence there exists a $y^* \in \bigcap_{r=1}^{\infty} G_r$.

Let $x \in D(A^n)$, then Lemma 1 yields

$$A^n x = \lim_{t \to \infty} \frac{(2n)!}{t^{2n}} \left[C(t) - \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} A^k\right] x. \tag{3}$$

On the other hand, $y^*x \in \bigcap_{r=1}^{\infty} G_r x \subset \bigcap_{r=1}^{\infty} \overline{H_r x}$, hence there is a subsequence $t_{j_m} = t_{j_m}(x) \to 0$ such that

$$\lim_{m \to \infty} ((2n)!\, t_{j_m}^{-2n} S_n^*(t_{j_m})x^*, x) = (y^*, x).$$

Because of $x^* \in D(A^{k*})$ $(k = 0, 1, \ldots, n-1)$ we obtain

$$(x^*, A^n x) = \lim_{m \to \infty} \left(x^*, (2n)!\, t_{j_m}^{-2n} \left[C(t_{j_m}) - \sum_{k=0}^{n-1} \frac{t_{jm}^{2k}}{(2k)!} A^k\right] x\right) =$$

$$= \lim_{m \to \infty} ((2n)!\, t_{j_m}^{-2n} S_n^*(t_{j_m})x^*, x) = (y^*, x),$$

hence $x^* \in D(A^{n*})$.

c) If the assumption holds, then b) and Lemma 3 imply that $x^* \in D(A^{*n})$. If $C_0^*(t)$ $(t \in R)$ denotes the cosine operator function adjoint to $C(t)$ (see [6]), and A_0^* its generator operator, then it is seen from [6], Lemma 3 that $x^* \in D(A_0^{*\,n-1})$ and

$$S_n^*(t)x^* = \left[C_0^*(t) - \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} A_0^{*k}\right] x^* \quad \text{for} \quad t \in R.$$

It then follows that Corollary 2 applies and the proof is complete.

THEOREM 2. *If $C(t)$ is a cosine operator function, then the following condition are equivalent:*

a) $T_n(t)x=0$ *for every* $x \in D(A^{n-1})$, $t \in R$,

b) $A \in B(X)$, $A^n=0$ *and* $C(t)=\sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} A^k$ *for every* $t \in R$,

c) $S_n^*(t)x^*=0^*$ *for every* $x^* \in D(A^{*\,n-1})$, $t \in R$.

Moreover, if none of these conditions hold, then $\{S_n^*(t)\}$ is saturated in $\left(D(A^{*n-1}), X^*\right)$ with order $O(t^{2n})$, and its Favard class is $D(A^{*n})$.

PROOF. If a) holds, then (3) yields that $A^n x=0$ for $x \in D(A^n)$. Since A^n is densely defined and closed, therefore $A^n=0 \in B(X)$ and b) follows. If b) is true, then $S_n^*(t)=T_n(t)^*=0^* \in B(X^*)$, thus c) holds. Finally, if c) is valid, then

$$\left(x^*, T_n(t)x\right) = \left(S_n^*(t)x^*, x\right) = 0 \quad \text{for} \quad x^* \in D(A^{*n-1}), x \in D(A^{n-1}), \quad t \in R.$$

On the other hand, Lemma 3 and [2], Proposition 1.4.2 yield that for every positive integer k the set $D(A^{*k})$ is w*-dense in X^*, hence a) is true.

Concerning the saturation problem, by Lemma 4, we have only to prove that there exists a $y^* \in D(A^{*n})$ with $S_n^*(t)y^* \neq 0^*$ for some $t>0$. If this were false, we would have

$$\left(y^*, T_n(t)x\right) = 0 \quad \text{for} \quad y^* \in D(A^{*n}), \quad x \in D(A^{n-1}), \quad t \in R.$$

By the above reasoning, then a) holds, which is a contradiction.

3. Applications. As an illustration, we apply the above results to the cosine function of symmetric translations on some of the function spaces listed below (cf. [2]).

$UCB(R)$: the space of all bounded uniformly continuous complex-valued functions $x(t)$ defined on R, with norm

$$|x| = \sup \{|x(t)|;\ t \in R\}.$$

$C_0(R)$: the subspace of $UCB(R)$ for whose elements $\{t \in R; |x(t)| \geqq \varepsilon\}$ is compact for every $\varepsilon>0$.

$AC(R)$, $AC_{\text{loc}}(R)$: the space of all absolutely continuous (locally a.c.) functions on R.

$L^p(R)$: the Lebesgue spaces on $R (1 \leqq p \leqq \infty)$.

$NBV(R)$: the space of all normalized complex-valued functions $x(t)$ of bounded variation on R.

It is well-known that if $X=UCB(R)$ or $X=L^p(R)$ $(1\leq p<\infty)$, then $G(t)$ $(t\in R)$ defined by

$$[G(t)x](s) = x(s+t)$$

is a strongly continuous group of operators for whose generator

$$D(B) = \{x\in X;\ x\in AC_{\text{loc}}(R) \text{ and } x'\in X\} \quad \text{and} \quad Bx = x'.$$

Then it can be shown that

$$[C(t)x](s) = \frac{1}{2}[x(s+t)+x(s-t)]$$

is a strongly continuous cosine operator function with generator $A=B^2$. We then obtain the following

THEOREM 3. *Suppose* $1<p<\infty$ *and* n *is a positive integer,* $x, x', \ldots, x^{(2n-3)}\in L^p(R)\cap AC_{\text{loc}}(R)$ *and* $x^{(2n-2)}\in L^p(R)$. *We then have* $x^{(2n-2)}, x^{(2n-1)}\in L^p(R)\cap AC_{\text{loc}}(R)$ *and* $x^{(2n)}\in L^p(R)$ *if and only if*

$$\varliminf_{t\to 0} t^{-2np} \int_{-\infty}^{\infty} \left|\frac{1}{2}[x(s+t)+x(s-t)] - \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} x^{(2k)}(s)\right|^p ds < \infty.$$

PROOF. Follows from Lemmas 1, 2 and Corollary to Theorem 1.

It follows from the results of [2], 1.4.2 that the operator function

$$[D(t)y](s) = \frac{1}{2}[y(s+t)+y(s-t)]$$

on $Y=NBV(R)$ or $Y=L^\infty(R)$ is the conjugate function of the cosine function $C(t)$ of symmetric translations on $C_0(R)$ and $L^1(R)$, respectively:

$$D(t) = C(t)^* = \frac{1}{2}[G(t)^*+G(-t)^*].$$

Hence $A^*=B^{2*}$ and, since B is the generator of a strongly continuous group of operators, it can be shown similarly as in Lemma 3 that $B^{2*}=B^{*2}$. Using [2], Theorem 1.4.9, we then obtain from Lemma 4

THEOREM 4. *Suppose* n *is a positive integer,* $y, y', \ldots, y^{(2n-3)}\in AC(R)$ *and* $y^{(2n-2)}\in NBV(R)$. *Then* $y^{(2n-2)}, y^{(2n-1)}\in AC(R)$ *and* $y^{(2n)}\in NBV(R)$ *if and only if*

$$\varliminf_{t\to 0} t^{-2n}\cdot \text{Var}_s\left\{\frac{1}{2}[y(s+t)+y(s-t)] - \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} y^{(2k)}(s)\right\} < \infty.$$

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DEPARTMENT OF MATHEMATICS
FACULTY OF CHEMICAL ENGINEERING
TECHNICAL UNIVERSITY
1521 BUDAPEST, STOCZEK U. 2—4.

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SELECTIVE DERIVATES

By

R. J. O'MALLEY (Milwaukee)[1]

In this paper we introduce a new form of differentiation for functions $f: [0, 1] \to R$. This new form, while natural for Baire 1, Darboux functions, exhibits some surprising contrasts to the properties of established derivatives.

Our derivates are defined by a process which we label selective. In this paper a process will be called selective if it has as its first step the selection of a fixed point from the interior of each closed non-degenerate subinterval of [0, 1]. Selective processes have been used to obtain characterizations of Baire 1, Darboux functions [10], derivatives [10], and, more recently, M_3 functions [11].

This paper consists of three sections. In the first, we give the necessary definitions and consider mainly the lower selective derivate. In this section we lay the foundations for most of the results of the second section. In the second section we show why selective differentiation is natural for Baire 1, Darboux functions and pass to a consideration of the finite selective derivative. At that time the similarities and contrasts between this derivative and the approximate derivative are shown. Finally, in the last section we look at the one-sided selective derivates and point out their fundamental pathology.

1. The lower selective derivate

To simplify the later computations we will use the notation $[a, b]$ to denote the closed interval having endpoints a and b irregardless of whether $a>b$ or $a<b$. Let $[a, b]$ be a fixed subinterval of [0, 1]. We select one point p from the interior of $[a, b]$ and label it $p_{[a,b]}$. The collection of p's thus obtained we call the selection S.

Definition. For a given selection S the lower selective derivate, ${}_s f'(x)$, of a function $f: [0, 1] \to R$ at a point x is

$$\liminf_{h \to 0} \frac{f(p_{[x,x+h]}) - f(x)}{p_{[x,x+h]} - x}.$$

It should be clear from the above definition how we would define the upper selective derivate ${}^s f'(x)$, selective derivative $\mathrm{s}f'(x)$, and one-sided selective derivates. It should also be emphasized that these derivates depend on the choice of $p_{[a,b]}$, different selections possibly leading to different derivates. This fact will be exploited several times to point out interesting aspects of the results. For example,

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the selection of $p_{[a,b]}=\frac{a+b}{2}$ gives as selective derivates the classical Dini derivates. Further, it should be noted that if f has a derivative at x it has a selective derivative at x for every selection, and the two are equal.

The concept of metric density and the accompanying concepts of approximate differentiation will play an important role in our development of the theory. We give several basic definitions here and refer the reader to [12] and [5] as important sources.

DEFINITION. Let E be a measurable subset of $[0, 1]$ and x be a fixed point. If

$$\lim_{h\to 0} \frac{m[E\cap[x, x+h]]}{m[x, x+h]}$$

exists and equals α we say that E has density α at x. In particular when $\alpha=1$ we say that x is a point of density of x.

DEFINITION. The lower approximate derivate ${}_{\mathrm{ap}}f'(x_0)$ of a measurable function f at the point x_0 is

$$\sup\left[y:\left\{x:\frac{f(x)-f(x_0)}{x-x_0}>y\right\}\text{ has density 1 at } x_0\right],$$

where the sup over the empty set is defined as $-\infty$.

DEFINITION. A measurable function f has an approximate derivative $\mathrm{ap}f'(x_0)$ at x_0 if and only if for every $\varepsilon>0$

$$\left\{x:\left|\frac{f(x)-f(x_0)}{x-x_0}-\mathrm{ap}f'(x_0)\right|<\varepsilon\right\}$$

has density 1 at x_0.

Standard changes should be made in the above definition to allow for the case of an infinite approximate derivative.

Finally, it should be noted that we will make use of the properties of functions of generalized bounded variation several times in this paper. For a good introduction to these functions again we single out [12] as a reference. We give here only the basic definition.

DEFINITION. A function f: $[0, 1]\to R$ is of generalized bounded variation **BVG** if there is a countable collection of sets E_n such that $\bigcup_{n=1}^{\infty} E_n=[0, 1]$ and f is of bounded variation on E_n for each n.

Our first two lemmas are very simple. However, these lemmas and variations of them will play an essential role in our work.

LEMMA 1. *Let* f:$[0, 1]\to R$ *and let* S *be a fixed selection. Let*

$$P_n=\left\{x:\frac{f(p_{[x,x+h]})-f(x)}{p_{[x,x+h]}-x}>0 \textit{ for all } h \textit{ with } |h|<\frac{1}{n}\right\}.$$

If $x<y$ *both belong to* P_n *and* $y-x<\frac{1}{n}$ *then* $f(x)<f(y)$. *Hence* f *is of bounded variation on* P_n.

PROOF. Let $x<y$ belong to P_n and $y-x<\frac{1}{n}$. Consider the interval $[x, y]$ and the point $p_{[x,y]}$. We have that

$$\frac{f(p_{[x,y]})-f(x)}{p_{[x,y]}-x}>0 \quad \text{and} \quad \frac{f(p_{[x,y]})-f(y)}{p_{[x,y]}-y}>0.$$

Therefore $f(x)<f(y)$. Then let $P_{ni}=P_n\cap\left[\frac{i-1}{2n}, \frac{i}{2n}\right]$ for $i=1, \dots, 2n$. On each P_{ni} f is increasing, from which it follows that f is VB on P_n.

LEMMA 2. *Let $f: [0, 1]\to R$ and let S be a fixed selection. Let P_n be defined as above, and let P_n^* be its closure. Let $x<y$ be any two points with*

i) *The distance between x and y is less than $\frac{1}{n}$.*

ii) *There is a decreasing sequence x_k of points of P_n converging to x.*

iii) *There is an increasing sequence y_j of points of P_n converging to y.*

iv) $\min[{}_sf'(x), {}_sf'(y)]>-\infty$.

Then $f(x)<f(y)$.

PROOF. We may assume that $x_k<\frac{y+x}{2}<y_j$ for all k and j. Therefore by Lemma 1 we have

$$f(x_{k+1})<f(x_k)<f(y_j)<f(y_{j+1})$$

for all k and j. We claim that $f(x)<f(x_k)$ for all k. Suppose instead that there is an integer K such that $f(x)\geqq f(x_K)$. For $k>K+1$, $f(x_k)<f(x_{K+1})<f(x_K)\leqq f(x)$. Let the point selected from $[x, x_k]$ be denoted by p_k. We have $f(p_k)<f(x_k)$, since x_k belongs to P_n and $\frac{1}{n}<x_k-x>0$. Thus for $k>K+1$

$$f(p_k)-f(x)<f(x_k)-f(x)<f(x_{K+1})-f(x)=\alpha<0.$$

The sequence x_k converges to x, so that

$$\liminf_{k\to\infty}\frac{f(p_k)-f(x)}{p_k-x}=-\infty,$$

contradicting (iv), ${}_sf'(x)>-\infty$. Thus $f(x)<f(x_k)$.

The same argument applies to $f(y)$, so that $f(y)>f(y_j)$ for all j. Thus $f(x)<f(x_1)<f(y_1)<f(y)$.

We now show that in some senses the lower selective derivate behaves like the classical lower Dini derivate.

THEOREM 1. *Let $f: [0, 1]\to R$ have ${}_sf'>0$ for all x for a fixed selection S. Then f is increasing.*

PROOF. Let E be the collection of all x for which there exists no open interval containing x on which f is increasing. It is easy to show that the complement of E is an open set U. Further f is increasing on the closure of each component interval of U. Thus U can contain no two intervals of the form (a, b) and (b, c), which implies that E is a perfect set. If E is shown to be empty, we will have that f is increasing.

Suppose E is a non-empty perfect set. Let P_n and P_n^* be defined as in Lemmas 1 and 2. We have that $\bigcup_{n=1}^{\infty} P_n \cap E = E$ because ${}_sf'(x) > 0$ for all x. Thus $\bigcup_{n=1}^{\infty} P_n^* \cap \cap E = E$, and the Baire category theorem guarantees the existence of an open interval (a, b) and an integer N such that

$$(a, b) \cap E \neq \emptyset \quad \text{and} \quad (a, b) \cap E \subset (a, b) \cap P_N^*.$$

If such an interval (a, b) exists, it does not affect the generality of the argument to assume that (a, b) has length less than $\frac{1}{N}$. However, we claim that in this case f is increasing on (a, b), which contradicts $E \cap (a, b) \neq \emptyset$. Let $x < y$ be any two points of (a, b). We wish to show that $f(x) < f(y)$. If x and y satisfy conditions i)—iv) of Lemma 2, we have $f(x) < f(y)$. It is clear that any x and y in (a, b) satisfy i) and iv). Thus we need only consider three cases:

1) The point x does not satisfy ii), but y satisfies iii).
2) The point y does not satisfy iii), but x satisfies ii).
3) The point x does not satisfy ii), and y does not satisfy iii).

We will give the proof only for the third case. The proof in cases 1) and 2) will then be clear.

If x does not satisfy ii), there is an open interval (x, x_1) with $(x, x_1) \cap P_N = \emptyset$. Since P_N is dense in P_N^* it follows that $(x, x_1) \cap P_N^* = \emptyset$ and hence $(x, x_1) \cap E = \emptyset$. Similarly if y does not satisfy iii), there is an open interval (y_1, y) with $(y_1, y) \cap \cap P_N^* = \emptyset$ and $(y_1, y) \cap E = \emptyset$.

We recall that the complement of E relative to (a, b) is an open set and that f is increasing on the closure of each component of this open set. Let (c_1, d_1) be the component containing (x, x_1) and (c_2, d_2) the component containing (y_1, y). If $(c_1, d_1) = (c_2, d_2)$, $f(x) < f(y)$. If $(c_1, d_1) \neq (c_2, d_2)$ then $(c_1, d_1) \cap \cap (c_2, d_2) = \emptyset$, and $d_1 < c_2$. These two points belong to E, and since E is perfect $E \cap (a, b)$ contains a decreasing sequence r_k converging to d_1 and an increasing sequence s_k converging to c_1. Since $E \cap (a, b) \subset P_N^*$ and P_n is dense in P_n^* it follows that d_1 and c_2 satisfy conditions i)—iv) of Lemma 2. Thus $f(d_1) < f(c_2)$, and finally $f(x) < f(d_1) < f(c_2) < f(y)$ which completes the proof.

It follows easily that if we require only ${}_sf'(x) \geqq 0$ for all x we would obtain that f is non-decreasing. In fact we have:

THEOREM 2. *Let f: [0, 1] $\to R$, and let S be a selection for which ${}_sf'(x) > -\infty$ for all x and ${}_sf'(x) \geqq 0$ for almost all x. Then f is non-decreasing.*

PROOF. The proof of this theorem is a standard argument which can be found, for example, on pages 267—268 of [9]. For this reason we delete the details.

THEOREM 3 (Dini Theorem). *Let f: [0, 1] $\to R$, and let S be a fixed selection. If*

$$\beta = \inf_{x \neq y} \left[\frac{f(x) - f(y)}{x - y} \right] \quad \textit{and} \quad \alpha = \inf_x [{}_sf'(x)]$$

then $\alpha = \beta \leqq +\infty$.

Again this is a standard argument which follows the same format as [12, p. 204].

The above theorem will be of use in the second part of this paper. A similar theorem holds for the upper selective derivate.

We note that while Theorems 1, 2 and 3 make no restrictions such as measurability on the function f we nevertheless can show the following.

THEOREM 4. *Let* $f\colon [0,1]\to R$, *and let* S *be a selection for which* ${}_s f'(x) > -\infty$ *for all* x *in* $[0,1]$. *Then*

a) *The function* f *is measurable and of generalized bounded variation.*

b) *The interval* $[0,1]$ *is the union of a countable collection of closed sets* A_n^*, *and for almost all* x *in* A_n^* f *has an approximate derivative* $\mathrm{ap} f'(x) \geqq -n$.

c) *There is a dense open set* V *on which* f *is differentiable for almost all* x *in* V.

PROOF. For each n we define

$$A_n = \left\{x\colon \frac{f(p_{[x,x+h]})-f(x)}{p_{[x,x+h]}-x} > -n \text{ for all } h \text{ with } |h| < \frac{1}{n}\right\}$$

and A_n^* as its closure. Since ${}_s f'(x) > -\infty$ for all x, the union of the sets A_n, $n=1,2,\dots$, is $[0,1]$. Let n be fixed, and consider the function $h(x)=f(x)+nx$. Then the set A_n^* will be precisely the set P_n^* of Lemma 2 for the function $h(x)$. We denote the set of bilateral limit points of P_n^* as Q_n. Then $A_n^*=P_n^*=Q_n\cup R_n$ where R_n is at most countable. If x and y are any two points of Q_n with $x<y<x+\frac{1}{n}$ then these two points also satisfy conditions ii), iii) and iv) of Lemma 2. Thus $h(x)<h(y)$. If we partition Q_n into subsets $Q_{ni}=Q_n\cap\left[\frac{i-1}{2n},\frac{i}{2n}\right]$, $i=1,\dots,2n$, then h is increasing on Q_{ni} for each i. Thus since Q_{ni} is also measurable, we have h is measurable and of bounded variation on Q_n. It is clear that any function is of generalized bounded variation and measurable on a countable set. We therefore have h measurable and of generalized bounded variation on $Q_n\cup R_n=A_n^*$. Since $f(x)=h(x)-nx$ the same is true of $f(x)$ on A_n^*. We have $\bigcup_{n=1}^{\infty} A_n^*=[0,1]$, so that a) of the conclusion is proven.

For b), Theorem 4.3 of page 222 of [12] states: A function which is measurable and of generalized bounded variation on a set is approximately derivable at almost all points of this set.

Therefore, f is approximately derivable at almost all points of Q_n for each n. Let x be a point of Q_n at which f is approximately derivable. If y belongs to Q_n and $|x-y|<\frac{1}{n}$ then $\frac{f(x)-f(y)}{x-y}>-n$. Hence at x, $\mathrm{ap} f'(x)\geqq -n$. Since Q_n has the same measure as A_n^*, b) of the conclusion is proven.

For c), let $[a,b]$ be a closed subinterval of $[0,1]$. Application of the Baire category theorem to the sequence of sets $A_n^*\cap[a,b]$ guarantees the existence of an open interval (c,d) and an integer N such that $(c,d)\subset A_N^*\cap[a,b]\subset A_N^*$. We may assume that $d-c<\frac{1}{N}$, which means that if x and y belong to (c,d)

then $\frac{f(x)-f(y)}{x-y}>-n$. The function f is thus differentiable at almost all x in (c, d). This establishes part c).

At this point we provide the first indication of some of the interesting differences between these derivates and the classical Dini derivates. It is clear that if ${}_sf'(x)>-m$ then f is differentiable for a.e. x. At any point of differentiability ${}_sf'(x)=f'(x)$, and therefore we are assured that ${}_sf'$ is measurable. However, the next example shows that even if f is Baire 1 and Darboux and ${}_sf'(x)>-\infty$ for all x we may still have ${}_sf'$ non-measurable. (In particular ${}_sf'(x)\neq \operatorname{ap} f'(x)$ for almost all x.) This non-measurability of ${}_sf'(x)$ is in sharp constrast to HÁJEK'S result [4] which states that for any function f (measurable or not) the lower derivate of f must be of Baire class 2.

EXAMPLE 1. Let C be a Cantor set of positive measure, and let $U=\bigcup_{i=1}^{\infty}(c_n, d_n)$ be its complement. We define $f(x)$ as

$$f(x)=\begin{cases}0, & \text{for } x \text{ in } C\\ \sin\left(\dfrac{1}{(x-c_n)(d_n-x)}\right), & c_n<x<d_n.\end{cases}$$

We decompose C into two disjoint non-measurable sets C_1 and C_2. It is clear from the definition of f that a selection S can be made as follows. Let $[a, b]$ be a fixed sub-interval with $a<b$. If a and b both belong to U let $p_{[a,b]}=\frac{a+b}{2}$. If b belongs to C let $p_{[a,b]}$ be chosen from (a, b) so that $f(p_{[a,b]})=0$. If a belongs to C_1 and b belongs to U choose $p_{[a,b]}$ so that

$$\frac{f(p_{[a,b]})-f(a)}{p_{[a,b]}-a}=-1.$$

If a belongs to C_2 and b belongs to U choose $p_{[a,b]}$ so that

$$\frac{f(p_{[a,b]})-f(a)}{p_{[a,b]}-a}=-2.$$

We leave to the reader the easy details of showing that for all x in C_1, ${}_sf'(x)=-1$, and for all x in C_2 ${}_sf'(x)=-2$. Meanwhile, for all x in U, f has a finite derivative f' and ${}_sf'=f'$. Therefore ${}_sf'(x)$ is non-measurable. This leads us to the following unsolved problem: What conditions, other than being bounded below, guarantee the measurability of ${}_sf'$.

In the next section we show a strong relation between the selective derivative and the approximate derivative. In preparation for this we show the following theorem.

THEOREM 5. *If* $f\colon [0, 1]\to R$ *and* S *is a selection for which* ${}_sf'(x)>-\infty$ *for all* x, *then* ${}_sf'(x)\leqq \operatorname{ap} f'(x)$ *for almost all* x.

PROOF. Let $B=\{x\colon {}_sf'(x)>\operatorname{ap} f'(x)\}$. We need to show that B has measure zero. The set B is the union of the sets $\{x\colon {}_sf'(x)>r>\operatorname{ap} f'(x)\}$ where r ranges over the rational numbers. We will only show $C=\{x: {}_sf'(x)>0>\mathrm{ap}f'(x)\}$ is of measure zero. This will be sufficient to prove the general case. Now $\{x: {}_sf'(x)>0\}$ is the union of the sets P_n of Lemma 1. As in Theorem 4, we have that $\mathrm{ap}f'(x)\geqq 0$ for almost all x in P_n^*. Therefore,

$$C=\{x: {}_sf'(x)>0\}\cap\{x:\mathrm{ap}f'(x)<0\}\subset\bigcup_{n=1}^{\infty}P_n^*\cap\{x:\mathrm{ap}f'(x)<0\}.$$

Since $P_n^*\cap\{x:\mathrm{ap}f'(x)<0\}$ has measure zero for all n, so does C.

With ${}_sf'(x)\leqq\mathrm{ap}f'(x)$ a.e. it is somewhat surprising that we have the following:

THEOREM 6. *If $f: [0, 1]\to R$ is measurable, then there is a selection S for which ${}_sf'\geqq{}_{\mathrm{ap}}f'$ for all x in* $[0, 1]$.

PROOF. We make the convention for this proof that if $c=+\infty$ then for $\varepsilon>0$, $c-\varepsilon=\frac{1}{\varepsilon}$. Let $[a, b]$ be a fixed subinterval of $[0, 1]$ with $a<b$. Let

$$a(y)=\sup\left\{y: m\left[\left\{x: \frac{f(x)-f(a)}{x-a}>y\right\}\cap[a, b]\right]>\frac{1}{2}m([a, b])\right\},$$

and

$$b(y)=\sup\left\{y: m\left[\left\{x: \frac{f(x)-f(b)}{x-b}>y\right\}\cap[a, b]\right]>\frac{1}{2}m([a, b])\right\},$$

where the sup over the empty set is defined to be equal to $-\infty$. If $a(y)$ and $b(y)$ both equal $-\infty$ we select $p_{[a,b]}$ to be the midpoint of (a, b). If $a(y)>-\infty$ and $b(y)=-\infty$ we select $p_{[a,b]}$ so that

$$\frac{f(p_{[a,b]})-f(a)}{p_{[a,b]}-a}>a(y)-(b-a).$$

If $b(y)>-\infty$ and $a(y)=-\infty$ we select $p_{[a,b]}$ so that

$$\frac{f(p_{[a,b]})-f(b)}{p_{[a,b]}-b}>b(y)-(b-a).$$

If $a(y)>-\infty$ and $b(y)>-\infty$, then

$$\left\{x: \frac{f(x)-f(a)}{x-a}>a(y)-(b-a)\right\}\quad\text{and}\quad\left\{x: \frac{f(x)-f(b)}{x-b}>b(y)-(b-a)\right\}$$

both have measure greater than $\frac{1}{2}m([a, b])$ in $[a, b]$. Therefore we may select as $p_{[a,b]}$ a point from $[a, b]$ which is in both sets. Now we prove that for this selection we have ${}_sf'(x)\geqq{}_{\mathrm{ap}}f'(x)$ for all x. Let x_0 be fixed. If ${}_{\mathrm{ap}}f'(x_0)=-\infty$ then there is nothing to show. If ${}_{\mathrm{ap}}f'(x_0)>-\infty$ let $\varepsilon>0$ be given. By the definition of ${}_{\mathrm{ap}}f'(x_0)$ there is a δ with $0<\delta<\frac{\varepsilon}{2}$ such that if $|h|<\delta$, then

$$m\left[\left\{x: \frac{f(x)-f(x_0)}{x-x_0}>{}_{\mathrm{ap}}f'(x_0)-\frac{\varepsilon}{2}\right\}\cap[x_0, x_0+h]\right]>\frac{1}{2}(b-a).$$

By the above selection formula it is clear that for $|h|<\delta$ we have chosen $p_{[x,x+h]}$ so that

$$\frac{f(p_{[x_0,x_0+h]})-f(x_0)}{p_{[x_0,x_0+h]}-x_0} > {}_{\mathrm{ap}}f'(x_0)-\frac{\varepsilon}{2}-|h| > {}_{\mathrm{ap}}f'(x_0)-\varepsilon.$$

This completes the proof of Theorem 6.

Combining Theorems 5 and 6 we get:

THEOREM 7. *If $f\colon[0,1]\to R$ is measurable and ${}_{\mathrm{ap}}f'(x)>-\infty$ for all x, then there is a selection S such that ${}_{s}f'(x)={}_{\mathrm{ap}}f'(x)$ for almost all x.*

Also, by combining Theorems 2 and 6 we have the following new theorem:

THEOREM 8. *If $f\colon[0,1]\to R$ is measurable and ${}_{\mathrm{ap}}f'(x)\geqq 0$ for almost all x and ${}_{\mathrm{ap}}f'(x)>-\infty$ for all x then f is non-decreasing.*

A study of the essential points of the proof of Theorem 6 shows that the set

$$\left\{x\colon \frac{f(x)-f(x_0)}{x-x_0} > {}_{\mathrm{ap}}f'(x_0)-\varepsilon\right\}$$

need not have density 1 at x_0. We need only

$$m\left(\left\{x\colon \frac{f(x)-f(x_0)}{x-x_0} > {}_{\mathrm{ap}}f'(x_0)-\varepsilon\right\}\cap I\right) > \frac{1}{2}m(I)$$

for all intervals sufficiently small containing x_0. This is precisely the case with the preponderant lower derivate. Therefore, Theorems 6 and 8 could just as well be stated for these derivates. Similarly, it requires little to see that only minor changes are needed to make Theorems 6 and 8 apply to the qualitative lower derivate. For a treatment of both the preponderant and qualitative derivates and further background information about them, we refer the reader to [8].

To end this section we present a result dealing with the comparison of two different selections. The above material has illustrated that different selections can lead to very different derivates. The next theorem establishes a result which can be considered an analogue of G. C. Young's result about one-sided Dini derivates [12, p. 261]. It also shows that there is some stability as we switch from selection to selection.

THEOREM 9. *Let $f\colon[0,1]\to R$, and let S and T be two selections. Then $\{x\colon {}_{s}f'(x)>{}^{t}f'(x)\}$ is at most countable.*

PROOF. It is sufficient to show that $\{x\colon {}_{s}f'(x)>0>{}^{t}f'(x)\}=B$ is countable. Let $[a,b]$ be a fixed subinterval. Let us denote the point $p_{[a,b]}$ from selection S as $s_{[a,b]}$ and that from selection T as $t_{[a,b]}$. Then

$$\{x\colon {}_{s}f'(x)>0\}=\bigcup_{n=1}^{\infty}\left\{x\colon \frac{f(s_{[x,x+h]})-f(x)}{s_{[x,x+h]}-x}>0,\ h<\frac{1}{n}\right\}=\bigcup_{n=1}^{\infty}S_n,$$

and

$$\{x\colon {}^{t}f'(x)<0\}=\bigcup_{n=1}^{\infty}\left\{x\colon \frac{f(t_{[x,x+h]})-f(x)}{t_{[x,x+h]}-x}<0,\ h<\frac{1}{n}\right\}=\bigcup_{n=1}^{\infty}T_n.$$

We are interested in $B=\bigcup_{n=1}^{\infty} S_n \cap \bigcup_{n=1}^{\infty} T_n$. Since S_n and T_n are increasing sequences of sets we have that $B=\bigcup_{n=1}^{\infty} (S_n \cap T_n)$. Therefore if we show that $T_n \cap S_n$ is countable for each n we are finished. Let n be fixed. If $T_n \cap S_n$ contains two points $x<y$ within $\frac{1}{n}$ units of each other we have by Lemma 1 that $f(x)<f(y)$, and also $f(x)>f(y)$. This contradiction means no two points of $T_n \cap S_n$ are within $\frac{1}{n}$ units of each other, and $T_n \cap S_n$ consists of at most n points.

2. The selective derivative

In the introduction to this paper it was mentioned that this form of differentation was a "natural" one for Baire 1, Darboux functions. However, throughout the first section the most restrictive condition placed on the function f was measurability. Before we proceed to study the selective derivative we will make clear the connections between selective derivates and Baire 1, Darboux functions.

In [10], NEUGEBAUER obtained a theorem which, when stated in the terminology of the present paper, becomes

THEOREM A. *A function* f: $[0, 1]\to R$ *is a Baire* 1, *Darboux function if and only if there is a selection* S *with the property that for all* x *in* $[0, 1]$ *and for every sequence of intervals* $[a_n, b_n]$ *satisfying* $a_n \leqq x \leqq b_n$ *and* $\lim_{n\to+\infty} a_n = x = \lim_{n\to+\infty} b_n$ *we have that* $\lim_{n\to+\infty} f(p_{[a_n, b_n]}) = f(x)$.

Basically, this theorem states that a function f is Baire 1 Darboux if and only if there is a selection relative to which f satisfies a continuity condition.

Now it can be shown that the above theorem can be changed to the following:

THEOREM B. *A function* f: $[0, 1]\to R$ *is Baire* 1, *Darboux if and only if there is a selection S such that for all x in* $[0, 1]$, $\lim_{h\to 0} f(p_{[x, x+h]}) = f(x)$. *(That is, we have placed the condition on the sequences* $[a_n, b_n]$ *that* x *is an endpoint of each interval.)*

PROOF. The only part of the proof of Theorem B that is not immediate is the following: if $\lim_{h\to 0} f(p_{[x,x+h]})=f(x)$, then f is Baire 1. One way this property can be shown is by looking at the function $\varphi(a, b)=f(p_{[a,b]})$ defined on the open upper half of the unit square. For each point $0<x<1$ and each point (x, x) on the diagonal we have $\lim_{h\to 0^+} \varphi(x, x+h) = \lim_{h\to 0^+} \varphi(x-h, x) = f(x)$. Therefore, the theorem of L. E. SNYDER [13, p. 422] can be applied, which yields that f is Baire 1.

The above discussion guarantees that if f is Baire 1, Darboux then there is a natural selection with which to define our selective derivates. Thus, the results of Section 1 can be interpreted as a demonstration that this natural selection will be useful in revealing the monotonicity and differentiability properties of the function f.

Further, the following is obvious:

COROLLARY TO THEOREM B. *If* $f\colon [0, 1]\to R$ *and* S *is a selection such that* $-\infty < {}_sf'(x) < {}^sf'(x) < +\infty$ *for all* x, *then* f *is Baire* 1, *Darboux.*

We will show that more can actually be said about the structure of f. To do this we will need the following lemma, which is similar to Lemma 2 but with a stronger conclusion.

LEMMA 3. *Let* $f\colon [0, 1]\to R$, *and let* S *be a selection such that for all* x, $\lim f(p_{[x, x+h]}) = f(x)$. *Let*

$$E_n = \left\{x : |f(p_{[x,x+h]}) - f(x)| \leqq n|p_{[x,x+h]} - x|,\ |h| < \frac{1}{n}\right\}.$$

Let E_n^* *be the closure of* E_n. *If* x *and* y *are any two points of* E_n^* *with* $x-y<\frac{1}{n}$, *then* $|f(x)-f(y)|\leqq n|x-y|$.

PROOF. If x and y belong to P_n and $|x-y|<\frac{1}{n}$ we have

$$|f(p_{[x,y]})-f(x)| \leqq n|p_{[x,y]}-x| \quad \text{and} \quad |f(p_{[x,y]})-f(y)| \leqq n|p_{[x,y]}-y|.$$

Also,

$$|p_{[x,y]}-x|+|p_{[x,y]}-y| = |x-y|.$$

Therefore,

$$|f(x)-f(y)| \leqq n|x-y|.$$

Now let x and y be any two points of P_n^* with $|x-y|<\frac{1}{n}$. There are two sequences x_k and y_k, possibly constant sequences, of points from P_n with $|x_k-y_k|<\frac{1}{n}$, $|x_k-x|<\frac{1}{n}$, $|y_k-y|<\frac{1}{n}$, $|x_k-x_{2k}|<\frac{1}{n}$, and $|y_k-y_{2k}|<\frac{1}{n}$ for all k. Then we have

$$|f(p_{[x_k,x]})-f(x_k)| \leqq n|p_{[x_k,x]}-x_k|$$

$$|f(x_k)-f(x_{2k})| \leqq n|x_k-x_{2k}|$$

$$|f(x_{2k})-f(y_{2k})| \leqq n|x_{2k}-y_{2k}|$$

$$|f(y_{2k})-f(y_k)| \leqq n|y_{2k}-y_k|$$

$$|f(y_k)-f(p_{[y_k,y]})| \leqq n|y_k-p_{[y_k,y]}|.$$

Therefore,

$$|f(p_{[x_k,x]})-f(p_{[y_k,y]})| \leqq n[|p_{[x_k,x]}-x_k|+|x_k-x_{2k}|+|x_{2k}-y_{2k}|+|y_{2k}-y_k|+|y_k-p_{[y_k,y]}|].$$

Letting $k\to\infty$, we have

$$|f(x)-f(y)| \leqq n|x-y|.$$

This completes the proof.

We can now improve the corollary to Theorem B. A well-known characterization of Baire 1 functions is that in every perfect set C there must be a point x such that f is continuous, relative to C, at x. Here, we have:

THEOREM 10. *Let* $f\colon [0,1]\to R$, *and let* S *be a selection such that* $-\infty<{}_s f'(x)<{}^s f'(x)<+\infty$. *Then* f *is Baire* 1, *Darboux. In fact,* f *has the additional property that for every closed set* C *there is an open interval* (a,b), *with* $(a,b)\cap C\neq\emptyset$, *on which* f *is continuous relative to* C.

PROOF. As was mentioned before, f is clearly Baire 1, Darboux. From the hypothesis we have $\bigcup_{n=1}^{\infty} E_n=[0,1]$, where the E_n are defined as in Lemma 3. Therefore $\bigcup_{n=1}^{\infty} E_n^*\cap C=C$, and once again the Baire category theorem yields an open interval (a,b) and an integer N with $(a,b)\cap C\neq\emptyset$ and $(a,b)\cap C\subset E_N^*\cap(a,b)$. Since $|f(x)-f(y)|\leqq N|x-y|$ for all x,y in E_N^* with $|x-y|<1/N$, we are finished.

We now proceed to the study of the selective derivative. We restrict our attention, in general, to selective derivatives which are finite for each x.

The main properties of functions f having a finite selective derivative, and also the main properties of this finite selective derivative, are stated in the next theorem.

THEOREM 11. *Let* $f\colon [0,1]\to R$, *and let* S *be a selection for which* f *has a finite selective derivative* $\mathrm{s}f'(x)$ *for all* x *in* $[0,1]$. *Then*

a) *There is a sequence of closed sets* E_n^* *whose union is* $[0,1]$ *such that* f *is continuous on each* E_n^*, *relative to* E_n^*.

b) *The function* f *has the Darboux property.*

c) *The function* f *has an approximate derivative* $\mathrm{ap}f'(x)$ *for almost all* x.

d) *There is a dense open set* U *on which* f *is continuous, and, further,* f *is differentiable for almost all* x *in* U.

e) *The selective derivative* $\mathrm{s}f'(x)=\mathrm{ap}f'(x)$ *for almost all* x.

f) *The selective derivative has the Darboux property.*

PROOF. Properties a), b), c) and d) are clear from Theorems 4 and 10. We give here the proof of e) because this property will be of interest in the sequel. By Theorem 5, we have $-\infty<\mathrm{s}f'(x)={}_s f'(x)<\mathrm{ap}\,f'(x)$ for almost all x. Further, $+\infty>\mathrm{s}f'(x)={}^s f'(x)\geqq\mathrm{ap}\,f'(x)$ for almost all x. This establishes e). Next we consider part f), the Darboux property. It is sufficient to assume that $\mathrm{s}f'(0)<0$ and $\mathrm{s}f'(1)>0$ and to show that there is a point x_0 where $\mathrm{s}f'(x)=0$. Let V be the set of all x for which there is a neighbourhood about x on which f is continuous. By d) we know that V contains the dense open set U. If f has a local extremum at any point of V it is clear that at that point $\mathrm{s}f'$ is equal to zero, and we are finished. If we assume that f has no such local extrema we must assume that f is monotone on each component of V. Hence by the finiteness of $\mathrm{s}f'$ we would have f monotone on the closure of each component of V. Since $\mathrm{s}f'(0)<0$ and $\mathrm{s}f'(1)>0$, we cannot have $V=(0,1)$. Therefore, the complement E of V is non-empty, and further we are assured that it is a perfect set. In this case, Theorem 10 asserts the existence of an interval (a,b) such that $(a,b)\cap E\neq\emptyset$ and f is continuous, relative to E, at every point of $(a,b)\cap E$. However, since f is continuous and monotone on the closure of each component of $(a,b)\setminus E$ it is not hard to see that f is actually continuous on (a,b). Thus $(a,b)\subset V$, which contradicts $(a,b)\cap E\neq\emptyset$. Therefore, we have that f must have a local extremum at some x_0 in V and $\mathrm{s}f'(x_0)=0$. We further note that this same property can be used to prove the Mean Value Theorem for $\mathrm{s}f'$.

Theorem 11 establishes the measurability of the finite selective derivative, improving the situation found to exist with regard to the lower selective derivate. It also shows that the finite selective derivative is closely related to the approximate derivative. Further, both these derivatives also have the Darboux property [3]. It is therefore somewhat unexpected that the relation does not extend to other properties possessed by the approximate derivative. There are some interesting contrasts which we will give now.

One property of the approximate derivative, and other generalizations of the derivative, is that a finite approximate derivative is Baire 1. This is not true for the selective derivative.

EXAMPLE. Let C, U, and f be defined as in Example 1 of Section 1. Let K be the collection of right end-points of the components of U. We will make a selection, differing from that of Example 1, to obtain a finite selective derivative which equals 1 at every point of K and 0 at every point of $C\setminus K$. Since both K and $C\setminus K$ are dense in C there can be no point of C at which $\mathrm{s}f'$ is continuous relative to C. Therefore $\mathrm{s}f'$ will not be Baire 1.

Again the oscillatory behaviour of f on U permits a selection as follows:

If a belongs to U and b belongs to U let $p_{[a,b]}=\dfrac{a+b}{2}$.

If a belongs to U and b belongs to $C\setminus K$ let $p_{[a,b]}$ be chosen so that $f(p_{[a,b]})=0$.

If a belongs to U and b belongs to K let $p_{[a,b]}$ be chosen so that $\dfrac{f(p_{[a,b]})-f(b)}{p_{[a,b]}-b}=1$.

If a belongs to $C\setminus K$ let $p_{[a,b]}$ be chosen so that $f(p_{[a,b]})=0$.

If a belongs to K and b belongs to U or K let $p_{[a,b]}$ be chosen so that $\dfrac{f(p_{[a,b]})-f(a)}{p_{[a,b]}-a}=1$.

If a belongs to K and b belongs to $C\setminus K$ let $p_{[a,b]}$ be chosen so that both

$$\frac{f(p_{[a,b]})-f(a)}{p_{[a,b]}-a}=1 \quad \text{and} \quad a-b<\frac{f(p_{[a,b]}-f(b))}{p_{[a,b]}-b}<0.$$

We delete the details of showing that f has a finite selective derivative everywhere and that this selective derivative has the desired properties on $C\setminus K$ and K.

This brings two problems to our attention: What, if any, is the Baire class of a finite selective derivative $\mathrm{s}f'$? Under what conditions does $\mathrm{s}f'$ become Baire class 1?

We can give partial answers to these questions. Under a certain additional condition we can prove that a selective derivative is Baire class 2. Namely,

THEOREM 12. *Let f: $[0, 1]\to R$ be given. Let S be a selection such that f has a finite selective derivative $\mathrm{s}f'(x)$ for all x. Suppose further that for all x*

$$\lim_{[a,b]\to x}\frac{f(p_{[a,b]})-f(x)}{p_{[a,b]}-x}=\mathrm{s}f'(x),$$

where $[a, b] \to x$ *means that* $a \leqq x \leqq b$ *and* a *and* b *converge to* x. (*If* $p_{[a,b]} = x$ *we set* $\frac{f(p_{[a,b]}) - f(x)}{p_{[a,b]} - x} = \mathrm{s}f'(x)$.) *Then* $\mathrm{s}f'(x)$ *is Baire* 2.

Proof. Let n be fixed. We define a function $f_n\colon [0, 1] \to R$ as follows:

$$f_n(x) = \begin{cases} \dfrac{f(p_{[\frac{i-1}{n}, \frac{i}{n}]}) - f(x)}{p_{[\frac{i-1}{n}, \frac{i}{n}]} - x} & \text{if } \dfrac{i-1}{n} \leqq x < \dfrac{i}{n} \text{ and } x \neq p_{[\frac{i-1}{n}, \frac{i}{n}]}, \\ \mathrm{s}f'(x) & \text{if } x = p_{[\frac{i-1}{n}, \frac{i}{n}]} \text{ or } 1. \end{cases}$$

For each n and for all x in $\left(\frac{i-1}{n}, p_{[\frac{i-1}{n}, \frac{i}{n}]}\right)$ or $\left(p_{[\frac{i-1}{n}, \frac{i}{n}]}, \frac{i}{n}\right)$, $f_n(x)$ is the quotient of a Baire 1 function and a continuous function. Hence $f_n(x)$ is Baire class 1 on $[0, 1]$ for each n. We claim $f_n(x)$ converges to $\mathrm{s}f'(x)$ pointwise on $[0, 1]$. This is clear if $x = 1$. Let $0 \leqq x < 1$, and let n be fixed with $\frac{i-1}{n} \leqq x < \frac{i}{n}$. If $p_{[\frac{i-1}{n}, \frac{i}{n}]} = x$ we have $f_n(x) = \mathrm{s}f'(x)$. If $p_{[\frac{i-1}{n}, \frac{i}{n}]} \neq x$, we have

$$f_n(x) = \frac{f(p_{[\frac{i-1}{n}, \frac{i}{n}]}) - f(x)}{p_{[\frac{i-1}{n}, \frac{i}{n}]} - x}.$$

Further, $\left[\frac{i-1}{n}, \frac{i}{n}\right] \to x$ as $n \to +\infty$. Hence by the additional condition, $f_n(x)$ converges to $\mathrm{s}f'(x)$ as $n \to +\infty$, and f is Baire class 2.

In connection with both the selective derivative and selective derivate a worthwhile line of investigation is the selection function $\varphi(a, b) = p_{[a,b]}$. The placing of conditions such as continuity on this function $\varphi(a, b)$ leads to illuminating results. For example:

Theorem 13. *Let* $f\colon [0, 1] \to R$, *and let* S *be a selection for which* f *has a finite selective derivative for all* x, *Let* $g_n(x) = p_{[x, x+\frac{1}{n}]}$. *If* $g_n(x)$ *is a continuous function of* x *for each* n *then* $\mathrm{s}f'(x)$ *is Baire class* 2. *If* f *is continuous,* $\mathrm{s}f'$ *is Baire* 1.

Proof. It is clear that the functions

$$h_n(x) = \frac{f(p_{[x, x+\frac{1}{n}]}) - f(x)}{p_{[x, x+\frac{1}{n}]} - x}$$

will be Baire 1 because f is Baire 1 and $p_{[x, x+\frac{1}{n}]}$ is continuous. If f is continuous, then $h_n(x)$ is continuous for each n. In either case, $h_n(x)$ converges to $\mathrm{s}f'(x)$ pointwise on $[0, 1]$, and this completes the proof.

It is still an unsolved problem whether the finite selective derivative is always Baire 2 without any additional conditions.

Another property of the approximate derivative is that for a monotone function any point of approximate derivability is actually a point of differentiability of f, [6, p. 240]. This is another property not shared by the selective derivative. It is easy to construct examples of functions and selections illustrating this fact, but we will not give any here. We point out, however, that it is this fact which keeps us from improving d) of Theorem 11 to: f is differentiable at every point of U.

We now consider the conditions under which a point of selective differentiability is a point of differentiability.

Theorem 3 for the lower selective derivative and the analogous result for the upper selective derivative combine to give:

THEOREM 14. *Let f: $[0, 1]\to R$, and let S be a selection for which f has a selective derivative (possibly infinite) at every point of* $[0, 1]$. *If*

$$\beta = \inf_{x\neq y}\left[\frac{f(x)-f(y)}{x-y}\right] \quad \textit{and} \quad \alpha = \inf_{x}[sf'(x)],$$

and

$$\gamma = \sup_{x\neq y}\left[\frac{f(x)-f(y)}{x-y}\right] \quad \textit{and} \quad \delta = \sup_{x}[sf'(x)],$$

then $\alpha=\beta$ and $\delta=\gamma$. Further, f is differentiable at any point of continuity of the selective derivative.

PROOF. As was mentioned in Theorem 3, the proof of the first part of the conclusion follows the same reasoning as in [12, p. 204]. The second part is also immediate, but since it is short we give the proof here:

Let x_0 be a point of continuity of $sf'(x)$. Let $\varepsilon>0$ be given. There is a $\delta>0$ such that if $|x-x_0|<\delta$ then

$$sf'(x_0)-\varepsilon < \sup_{|x-x_0|<\delta}[sf'(x)] - \inf_{|x-x_0|<\delta}[sf'(x)] < sf'(x_0)+\varepsilon.$$

Hence by the first part of the conclusion

$$sf'(x_0)-\varepsilon < \frac{f(x)-f(x_0)}{x-x_0} < sf'(x_0)+\varepsilon$$

for all x with $0\leqq x-x_0<\delta$. This completes the proof of Theorem 14.

For monotone functions it is possible to put additional conditions on the selection S under which a point of selective differentiation becomes a point of differentiability. We give one example of such a condition.

THEOREM 15. *Let f: $[0, 1]\to R$ be non-decreasing, and let S be a selection such that*

1) *f has a selective derivative at* 0, *and*
2) $\lim_{n\to\infty} n\cdot p_{\left[0, \frac{1}{n}\right]} = 1.$

Then f has a derivative at 0.

PROOF We assume that $f(0)=0$. Let h_k be a decreasing sequence converging to 0. For each k there is a largest integer n_k such that $p_{\left[0,\frac{1}{n_k}\right]} \geqq h_k$. Then $p_{\left[0,\frac{1}{n_k+1}\right]} < h_k$, and

$$f\left(p_{\left[0,\frac{1}{n_k}\right]}\right) \geqq f(h_k) \geqq f\left(p_{\left[0,\frac{1}{n_k+1}\right]}\right).$$

Let us relabel $p_{\left[0,\frac{1}{n_k}\right]}$ as p_k and $p_{\left[0,\frac{1}{n_k+1}\right]}$ as q_k. We have

$$\frac{f(p_k)}{q_k} \geqq \frac{f(h_k)}{h_k} \geqq \frac{f(q_k)}{p_k}$$

for all k. However,

$$\frac{f(p_k)}{q_k} = \left(\frac{f(p_k)}{p_k}\right)(n_k p_k)\left(\frac{n_k+1}{n_k}\right)\left(\frac{1}{(n_k+1)q_k}\right) = a_k b_k c_k d_k,$$

and $\lim\limits_{k\to+\infty} a_k = \mathrm{s}f'(0)$ and $\lim\limits_{k\to+\infty} (b_k c_k d_k)=1$. Hence $\lim\limits_{k\to+\infty} \frac{f(p_k)}{q_k} = \mathrm{s}f'(0)$. The same holds for $\lim\limits_{k\to+\infty} \frac{f(q_k)}{p_k}$. Therefore $\lim\limits_{h\to 0^+} \frac{f(h)}{h} = \mathrm{s}f'(0)$, which says that f is differentiable at 0.

The following analogoue of Theorem 6 further connects the selective derivative and the approximate derivative.

THEOREM 16. *If $f\colon [0, 1]\to R$ is measurable and has an approximate derivative $\mathrm{ap}f'(x)$ (possibly infinite) at all x in $[0, 1]$, then there is a selection S such that $\mathrm{s}f'(x)=\mathrm{ap}f'(x)$ for all x.*

PROOF. We make the convention for this proof that if $\lambda=+\infty$ then for $\varepsilon>0$, $\lambda-\varepsilon=\frac{1}{\varepsilon}$ and $\lambda+\varepsilon=+\infty$; and if $\lambda=-\infty$, then $\lambda-\varepsilon=-\infty$ and $\lambda+\varepsilon=-\frac{1}{\varepsilon}$. For any point c in $[0, 1]$ and for each $\varepsilon>0$ we define

$$C(\varepsilon) = \left\{x\colon \mathrm{ap}f'(c)-\varepsilon < \frac{f(x)-f(c)}{x-c} < \mathrm{ap}f'(c)+\varepsilon\right\}.$$

Let $[a, b]$ be a fixed subinterval of $[0, 1]$ with $a<b$. Let

$$\varepsilon_1 = \inf\left\{\varepsilon\colon m\big(A(\varepsilon)\cap[a, b]\big) > \frac{1}{2}(b-a)\right\},$$

and let

$$\varepsilon_2 = \inf\left\{\varepsilon\colon m\big(B(\varepsilon)\cap[a, b]\big) > \frac{1}{2}(b-a)\right\}.$$

Then the sets

$$A\big(\varepsilon_1+(b-a)\big)\cap[a, b] \quad\text{and}\quad B\big(\varepsilon_2+(b-a)\big)\cap[a, b]$$

both have measure greater than $\frac{1}{2}$ the length of $[a, b]$. Therefore $A\big(\varepsilon_1+(b-a)\big)\cap \cap B\big(\varepsilon_2+(b-a)\big)\cap(a, b)$ is not empty. We select $p_{[a,b]}$ from their intersection. We show that this selection gives a selective derivative $\mathrm{s}f'(x)$ which equals $\mathrm{ap}f'(x)$ at all x.

Let c be fixed and $\varepsilon>0$ be given. Let $\varepsilon^*=\frac{\varepsilon}{2}$. There is a number δ such that $\varepsilon^*>\delta>0$, and if $|h|<\delta$ then $m(C(\varepsilon^*)\cap[c, c+h])>\frac{1}{2}|h|$. Then $p_{[c,c+h]}$ has been selected so that

$$\operatorname{ap}f'(x)-\varepsilon<\operatorname{ap}f'(c)-\varepsilon^*-|h|<\frac{f(p_{[c,c+h]})-f(c)}{p_{[c,c+h]}-c}<$$

$$<\operatorname{ap}f'(c)+\varepsilon^*+|h|<\operatorname{ap}f'(c)+\varepsilon.$$

This completes the proof. We remark that the comments made after Theorem 6 can be carried over here to apply to the preponderant derivative and the qualitative derivative.

We have mentioned above that selective differentiation is natural to Baire 1, Darboux functions. Our next results further emphasize this fact.

Regarding the monotonicity of Baire 1, Darboux functions, perhaps the most encompassing result is that of BRUCKNER [1].

THEOREM C. *Let P be any function-theoretic property sufficiently strong to imply conditions* i) *and* ii) *below:*

i) *If f is continuous and of bounded variation on an interval I_0 and possesses property P on I_0, then f is non-decreasing on I_0.*

ii) *If f is a Darboux function in Baire class* 1 *and possesses property P on I_0, then f is of generalized bounded variation on I_0.*

Then any Darboux function in Baire class 1 *which possesses property P on I_0 is continuous and non-decreasing on that interval.*

Bruckner obtained this result while answering affirmatively a problem presented by ZAHORSKI in [15]. (This question was also answered independently by SWIATOWSKI [14].) Zahorski's problem was: Let f be a finite real-valued function defined on an interval I_0 and satisfying the following conditions:

1) f is a Darboux function,

2) f is in the first class of Baire,

3) f possesses an approximate derivative $\operatorname{ap}f'$, finite or infinite except perhaps on a denumerable set,

4) $\operatorname{ap}f'\geqq 0$ almost everywhere.

Is it necessarily true that f is continuous and non-decreasing on I_0?

Recently LEONARD in [7] used Bruckner's Theorem C to obtain monotonicity results in cases where the approximate derivative in Zahorski's problem above is replaced by other forms of derivative. Here we use Theorem C to show:

THEOREM 17. *Let f: $[0, 1]\to R$ be Baire* 1, *Darboux. Let S be a selection such that*

a) $sf'(x)$ *exists except on a denumerable set, and*

b) $sf'(x)\geqq 0$ *almost everywhere.*

Then f is non-decreasing and continuous on $[0, 1]$.

PROOF. We show first that any function having a selective derivative except in a denumerable set is of generalized bounded variation. This is again an application of Lemma 1. Let

$$H_n=\left\{x\colon \frac{f(p_{[x,x+h]})-f(x)}{p_{[x,x+h]}-x}>-2,\ |h|<\frac{1}{n}\right\},$$

and

$$J_n=\left\{x\colon \frac{f(p_{[x,x+h]})-f(x)}{p_{[x,x+h]}-x}<-1,\ |h|<\frac{1}{n}\right\}.$$

The same reasoning as in Lemma 1 gives that f is of bounded variation on H_n and J_n for all n. Since $[0,1]=\bigcup_{n=1}^{\infty} H_n\cup\bigcup_{n=1}^{\infty} J_n\cup K$, where K is the denumerable set where sf' does not exist, we have that f is of generalized bounded variation on $[0,1]$. Therefore, condition ii) of Theorem C is satisfied.

Now let f be any continuous function satisfying conditions a) and b) above. Let H_n^* denote the closure of H_n and J_n^* the closure of J_n. The continuity of f on $[0,1]$ easily gives that if x and y are any two points of H_n^*, (J_n^*), with $|x-y|<\frac{1}{n}$ then $\frac{f(y)-f(x)}{y-x}\geqq -2$, $\left(\frac{f(y)-f(x)}{y-x}\leqq -1\right)$. From this point the proof is very similar to that of Theorem 1. We define E to be the collection of all x for which there exists no interval containing x on which f is non-decreasing. We denote the complement of E as U. The set U is open, and the function f is non-decreasing on each component of U. Further, the continuity of f forces it to be non-decreasing on the closure of each component of U. Thus E contains no isolated points and is a perfect set. If E is shown to be empty we will have that f is non-decreasing.

Suppose E is a non-empty perfect set. We have $E\cap\left[\bigcup_{n=1}^{\infty} H_n^*\cup\bigcup_{n=1}^{\infty} J_n^*\cup C\right]=E$. Therefore the Baire category theorem once again yields the existence of an open interval (a,b) and an integer N with either $(a,b)\cap E\subset H_n^*$ and $E\cap(a,b)\neq\emptyset$ or $E\cap(a,b)\subset J_n^*$ and $E\cap(a,b)\neq\emptyset$. If such an interval (a,b) exists, then we may assume that the length of (a,b) is less than $\frac{1}{N}$. We will show that no such (a,b) can exist. Suppose $E\cap(a,b)\subset H_N^*$ and $E\cap(a,b)\neq\emptyset$. Then $(a,b)\setminus H_N^*$ is an open subset of U, so that f is non-decreasing on the closure of any component of $(a,b)\setminus H_N^*$. We claim that for any x_0 in (a,b) we have

$$\liminf_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\geqq -2.$$

This is clear if x_0 belongs to $(a,b)\setminus H_N^*$. Let x_0 belong to $(a,b)\cap H_N^*$. For any other point y of $H_N^*\cap(a,b)$ we have $\frac{f(y)-f(x_0)}{y-x_0}\geqq -2$. Therefore we need only consider points y which belong to $(a,b)\setminus H_N^*$. Let y_0 be a fixed point of $(a,b)\setminus H_N^*$. Let (c_0,d_0) be the component of $(a,b)\setminus H_N^*$ to which y_0 belongs. Then $f(c_0)\leqq f(y_0)\leqq f(d_0)$. If $x_0\leqq c_0<y_0<d_0$, then

$$f(y_0)\geqq f(c_0)\geqq f(x_0)-2(c_0-x_0)\geqq f(x_0)-2(y_0-x_0).$$

If $c_0<y_0<d_0\leqq x_0$ then

$$f(x_0)\geqq f(d)-2(x_0-d_0)\geqq f(y_0)-2(x_0-y_0).$$

This establishes that

$$\liminf_{h\to 0}\frac{f(x+h)-f(x)}{h}\geqq -2$$

for all x in (a,b). In particular we have ${}_sf'(x)\geqq -2$ for all x in (a,b), and by hypothesis also $_sf'(x)\geqq 0$ for almost all x in (a,b). Therefore, an application of Theorem 2 yields that f is non-decreasing on (a,b), contradicting $(a,b)\cap E\neq\emptyset$.

Suppose $(a,b)\cap E\subset J_N^*$ and $E\cap(a,b)\neq\emptyset$. We claim that $(a,b)\cap J_N^*$ is nowhere dense. For any two points x and y of $J_N\cap(a,b)$, $\dfrac{f(x)-f(y)}{x-y}\leqq -1$. If $J_N^*\cap(a,b)$ contains an interval (c,d) then since $_sf'$ exists except for a countable set, we would have $_sf'(x)\leqq -1$ for all x in (c,d) except a countable set. This would contradict $_sf'(x)\geqq 0$ for almost all x in $[0,1]$. Therefore, $J_N^*\cap(a,b)$ is nowhere dense. Hence, because E is perfect and $E\cap(a,b)\subset J_N^*$, we can find a component interval (r,s) of $(a,b)\setminus J_N^*$ with $a<r<s<b$. Then we have that r and s belong to J_N^* and $0<s-r<\dfrac{1}{N}$. Hence $f(s)-f(r)\leqq(-1)(s-r)<0$. However, on $[r,s]$ f is non-decreasing, so that $f(s)-f(r)\geqq 0$. This contradiction shows that no such (a,b) can exist, which in turn implies that E must be empty. Hence f is non-decreasing, and i) of Theorem C is satisfied. This completes the proof of Theorem 17.

We present one more simple theorem and question before we end this section. The theorem deals with an intriguing contrast between the selective derivative and the classical derivative. Up to this point all the contrasts have been of one type. Namely, the derivative or derivate has possessed properties not held by the corresponding selective concept. Now we show a property of selective differentiation not held by the derivative.

It is well-known that there exist continuous functions which possess no derivative at any point of $[0,1]$. In fact there are continuous functions which have no one-sided derivative, infinite or finite, at any point. That is, $D_+f<D^+f$ and $D_-f<D^-f$ for all x, where D_+, etc., denote the classical one-sided Dini derivates. See Jeffery [5, p. 172] for an example of such a function. This is not the case with selective differentiation. We have the following simple theorem:

Theorem 18. *Let $f\colon[0,1]\to R$ be continuous. Then there is a selection S and a point x_0 such that f has a selective derivative at x_0.*

Proof. Let x be fixed. A derived number of f at x is a number λ, possibly infinite, such that there exists a sequence of points x_n converging to x with

$$\lim_{n\to+\infty}\frac{f(x_n)-f(x)}{x_n-x}=\lambda.$$

For a continuous function the set of right derived numbers of f at x is a closed interval with $D_+f(x)$ and $D^+f(x)$ as endpoints. The same is true for the left derived numbers at x. Now Young's theorem [12, p. 261] is that the set of points x at which $D^+f(x)<D_-f(x)$ or $D^-f(x)<D_+f(x)$ is a countable set, which we label as Q. Let x_0 be any point of $(0, 1)\setminus Q$. Then at x_0 the intervals $[D_+f(x_0), D^+f(x_0)]$ and $[D_-f(x_0), D^-f(x_0)]$ have non-empty intersection. Let λ be any point in this intersection. There are two sequences of positive numbers, h_n, k_n, converging to x_0 such that

$$\lim_{n\to+\infty} \frac{f(x_0+h_n)-f(x_0)}{h_n} = \lambda = \lim_{n\to+\infty} \frac{f(x_0-k_n)-f(x_0)}{k_n}.$$

We make our selection in the most obvious way. Namely, let $[a, b]$ be any interval with $a<b$.

If $a\neq x_0$ and $b\neq x_0$, let $p_{[a,b]}=\dfrac{a+b}{2}$.

If $a=x_0$, let $p_{[a,b]}$ be any $x+h_n$ with $x_0<x+h_n<b$.

If $b=x_0$, let $p_{[a,b]}$ be any $x-k_n$ with $a<x-k_n<x_0$.

Relative to this selection, $\mathrm{s}f'(x_0)=\lambda$, and we are done.

This theorem can obviously be improved as to the number of points x at which f has a selective derivative. Our question is:

If f is a continuous function, for how large a set A does there have to exist a selection S with respect to which f has a selective derivative at every point of A?

3. One-sided selective derivates

The previous two sections have shown that the bilateral selective derivates and the selective derivative can be useful in the study of functions. Unfortunately, it does not appear that the one-sided derivates can play a similar role without additional restrictions on the functions f or the selection function.

We denote the one-sided selective derivates as ${}_{\mathrm{s}^+}f$, ${}^{\mathrm{s}^+}f$, ${}_{\mathrm{s}^-}f$, and ${}^{\mathrm{s}^-}f$. The one-sided selective derivative is denoted by s^+f' and s^-f'. We have

$$D_+f \leqq {}_{\mathrm{s}^+}f \leqq {}^{\mathrm{s}^+}f \leqq D^+f$$

with the same relation on the left. Therefore for continuous functions we could make obvious statements such as: If $f\colon [0, 1]\to R$ is continuous and S is a selection such that ${}^{\mathrm{s}^+}f(x)\geqq 0$ for all x then f is non-decreasing. Such statements, however, shed no new light on the study of continuous functions. For Baire 1, Darboux functions we have an even worse situation. For example, LEONARD in [8] used Theorem C of Bruckner to obtain:

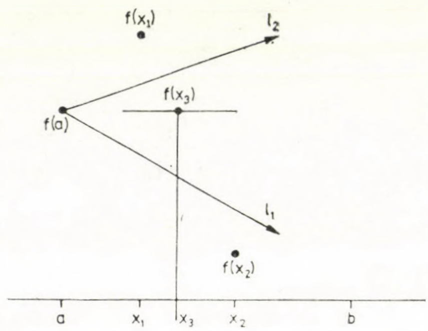
THEOREM D. *Let $f\colon [0, 1]\to R$ be Baire 1 and Darboux. Let the right-sided derivative of f, f'_+, exist except for a countable set, and, further, let $f'_+(x)\geqq 0$ for almost all x in $[0, 1]$. Then f is non-decreasing.*

Such a theorem is not possible for the right-sided selective derivative. To see this, we note that CROFT [2] constructed a Baire 1, Darboux function f with the property that $f(x)\geqq 0$ for all x in $[0, 1]$ and $f(x)=0$ for almost all x. We take $g(x)=-f(x)$. Then for each $[a, b]$ we select $p_{[a,b]}$ to be a point from (a, b) at which $g(p_{[a,b]})=0$. Then g will have a right selective derivative everywhere, and $s^+g'(x)\geqq 0$ for all x in $[0, 1]$. In fact, $s^+g'(x)=0$ for almost all x. However, clearly g is not non-decreasing.

Finally for Baire class 2 functions having the Darboux property we present a simple theorem which exposes a basic pathology for the one-sided selective derivative.

THEOREM 19. *Let $f: [0, 1]\to R$ be a Baire 2, Darboux function which takes on every value between $-\infty$ and $+\infty$ on every sub-interval of* $[0, 1]$ *(see* KURATOWSKI [8], p. 82). *Let $g: [0, 1]\to R$ be any fixed finite-valued function. Then there is a selection S for which $s^+f'(x)=g(x)$ for all x in* $[0, 1)$.

PROOF. Let $[a, b]$ be a fixed sub-interval of $[0, 1]$ with $a<b$. Let $g(a)=\lambda$. The function f takes on all values between $-\infty$ and $+\infty$ on the interval $[a, b]$. Consider the two lines l_1 and l_2 passing through the point $(a, f(a))$, having slopes $g(a)-(b-a)$ and $g(a)+(b-a)$ respectively. There is a point x_1 in (a, b) with $f(x_1)$ above both of these lines. Similarly, there is a point x_2 in (a, b) with $f(x_2)$ below both of these lines. The Darboux property guarantees that there is a point x_3 with $f(x_3)$ between the two lines. Diagram:



We select $p_{[a,b]}=x_3$. Then

$$g(a)-(b-a) < \frac{f(p_{[a,b]})-f(a)}{p_{[a,b]}-a} < g(a)+(b-a).$$

Clearly, with this selection S we have $s^+f'(x)=g(x)$ for all x in $[0, 1)$. This completes the paper.

Added in proof (February 22, 1977). It has come to the author's attention that M. Laczkovich has established that a finite selective derivative must be of Baire class 2. His proof and other interesting results can be found in this same issue of Acta Math. Hungar.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN-MILWAUKEE
MILWAUKEE, WISCONSIN 53 201
USA

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ON THE BAIRE CLASS OF SELECTIVE DERIVATIVES

By
M. LACZKOVICH (Budapest)

1. The notion of the selective derivative was introduced by R. J. O'MALLEY who presented some interesting theorems and problems on selective derivatives ([1]). In this paper we solve a problem of O'Malley showing that every selective derivative is of Baire class 2 and prove that every selective derivative is continuous on an everywhere dense subset of [0, 1]. Our method leads us to a more general notion of derivation which possesses the most important properties of selective derivatives.

By a selection we mean an interval function $p_{[a,b]}$ for which $a<p_{[a,b]}<b$ holds for every $0\leqq a<b\leqq 1$. We define the selective derivative $sf'(x)$ of the function $f(x)$ by

$$sf'(x)=\lim_{h\to 0}\frac{f(p_{[x,x+h]})-f(x)}{p_{[x,x+h]}-x}$$

if the limit exists and finite (for $h<0$ $[x, x+h]$ denotes the interval $[x+h, x]$).

Suppose that $f(x)$ has the selective derivative $sf'(x)$ for every $x\in[0,1]$ and define the interval functions $l(x,y)$ and $r(x,y)$ by

(1) $$l(x,y)=\frac{f(y)-f(p_{[x,y]})}{y-p_{[x,y]}}\quad (0\leqq x<y\leqq 1);$$

(2) $$r(x,y)=\frac{f(p_{[x,y]})-f(x)}{p_{[x,y]}-x}\quad (0\leqq x<y\leqq 1).$$

Then we have

(3) $$\lim_{y\to x-0} l(y,x)=sf'(x)\quad (0<x\leqq 1)$$

and

(4) $$\lim_{y\to x+0} r(x,y)=sf'(x)\quad (0\leqq x<1).$$

Since

$$\min\left(\frac{f(b)-f(a)}{b-a},\ \frac{f(c)-f(b)}{c-b}\right)\leqq\frac{f(c)-f(a)}{c-a}\leqq\max\left(\frac{f(b)-f(a)}{b-a},\ \frac{f(c)-f(b)}{c-b}\right)$$

holds for every $f(x)$ and $0\leqq a<b<c\leqq 1$ we have

(5)

$$\min\left(l(x,y),r(x,y)\right)\leqq\frac{f(y)-f(x)}{y-x}\leqq\max\left(l(x,y),r(x,y)\right)\quad (0\leqq x<y\leqq 1).$$

LEMMA 1. *Let the interval functions* $l(x, y)$ *and* $r(x, y)$ *be defined on the subintervals of* $[0, 1]$ *and let* $\varphi(x, y)$ *be a Baire* 1 *function defined on the set* $\{(x, y); 0 \leqq x < y \leqq 1\}$ *with*

$$\min\big(l(x, y), r(x, y)\big) \leqq \varphi(x, y) \leqq \max\big(l(x, y), r(x, y)\big) \quad (0 \leqq x < y \leqq 1). \tag{6}$$

If the limits

$$\lim_{y \to x-0} l(y, x) = \lim_{y \to x+0} r(x, y) = g(x) \tag{7}$$

exist for every $0 < x < 1$ *then the function* $g(x)$ *is of Baire class* 2 *on* $(0, 1)$.

PROOF. First we show that for every $a < b$ there exist disjoint $G_{\delta\sigma}$ sets $H_{a,b}$ and $K_{a,b}$ such that $E_a = \{x \in (0, 1); g(x) < a\} \subset H_{a,b}$ and $E^b = \{x \in (0, 1); g(x) > b\} \subset \subset K_{a,b}$.

Let $\varepsilon = \frac{1}{3}(b-a)$ and

$$A_n = \Big\{x \in (0, 1);\ |l(y, x) - g(x)| < \varepsilon \ \text{ if } \ 0 < x - y < \frac{1}{n} \ \text{ and}$$
$$|r(x, y) - g(x)| < \varepsilon \ \text{ if } \ 0 < y - x < \frac{1}{n}\Big\}.$$

It follows from (7) that

$$A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots; \quad \bigcup_{n=1}^{\infty} A_n = (0, 1). \tag{8}$$

We claim that the sets $A_n \cap E_a$ and $A_n \cap E^b$ cannot be everywhere dense in any non-empty set simultaneously (that is $X = \overline{X \cap (A_n \cap E_a)}$, $X = \overline{X \cap (A_n \cap E^b)}$ imply $X = \emptyset$). In fact, suppose indirectly

$$X = \overline{X \cap A_n \cap E_a}, \quad X = \overline{X \cap A_n \cap E^b} \quad \text{and} \quad X \neq \emptyset. \tag{9}$$

Then the set X cannot have isolated points hence there exist intervals (c_1, d_1) and (c_2, d_2) with $d_1 < c_2$, $X_1 = (c_1, d_1) \cap X \neq \emptyset$, $X_2 = (c_2, d_2) \cap X \neq \emptyset$ and $d_2 - c_1 < \frac{1}{n}$. Let

$$P = \overline{X_1 \times X_2} \subset \{(x, y);\ 0 \leqq x < y \leqq 1\},$$

P is obviously perfect.

If $x \in X_1 \cap A_n \cap E_a$, $y \in X_2 \cap A_n \cap E_a$ then $0 < y - x < \frac{1}{n}$ and by the definition of A_n and E_a we have $|l(x, y) - g(y)| \leqq \varepsilon$, $|r(x, y) - g(x)| \leqq \varepsilon$, $g(x) < a$ and $g(y) < a$ from which $l(x, y) < a + \frac{b-a}{3}$ and $r(x, y) < a + \frac{b-a}{3}$ follows. Thus $\varphi(x, y) < < a + \frac{b-a}{3}$ by (6). This and (9) imply that $\varphi(x, y) < a + \frac{b-a}{3}$ holds on a set everywhere dense in P. The same argument shows that the set $\Big\{(x, y) \in P; \varphi(x, y) > > b - \frac{b-a}{3}\Big\}$ is everywhere dense in P, as well. This contradicts the fact that the Baire 1 function $\varphi(x, y)$ has a point of continuity restricted to the perfect set P. This contradiction proves the impossibility of (9).

Now this implies that there exist F_σ and G_σ sets U_n and V_n such that

$$A_n \cap E_a \subset U_n, \quad A_n \cap E^b \subset V_n, \quad U_n \cap V_n = \emptyset .$$

(See [2] Chapter I, § 12, III. 1, p. 65). Let

$$H_{a,b} = \bigcup_{n=1}^{\infty} \left(U_n - \bigcup_{i=n}^{\infty} V_i \right) \quad \text{and} \quad K_{a,b} = \bigcup_{n=1}^{\infty} \left(V_n - \bigcup_{i=n}^{\infty} U_i \right).$$

Since U_n and V_n are F_σ and G_δ sets, $H_{a,b}$ and $K_{a,b}$ are $G_{\delta\sigma}$ sets. It is easy to see that $H_{a,b} \cap K_{a,b} = \emptyset$. In addition $E_a \subset H_{a,b}$ and $E^b \subset K_{a,b}$. In fact, if $x \in E_a$ then $x \in A_n \cap E_a$ for a suitable n and thus $x \in U_n$. On the other hand (8) and $U_i \cap V_i = \emptyset$ imply $x \notin V_i$ for $i \geqq n$ that is $x \in U_n - \bigcup_{i=n}^{\infty} V_i \subset H_{a,b}$. Similarly $E^b \subset K_{a,b}$.

The function $g(x)$ is proved to be Baire 2 if we show that the sets E_c and E^c are $G_{\delta\sigma}$ sets for every c. Since

$$E_{c-\frac{1}{n}} \subset H_{c-\frac{1}{n}, c-\frac{1}{n+1}} \quad \text{and} \quad H_{c-\frac{1}{n}, c-\frac{1}{n+1}} \cap E^{c-\frac{1}{n+1}} = \emptyset$$

hold for every n we have

$$E_c = \bigcup_{n=1}^{\infty} H_{c-\frac{1}{n}, c-\frac{1}{n+1}}$$

and similarly

$$E^c = \bigcup_{n=1}^{\infty} K_{c+\frac{1}{n+1}, c+\frac{1}{n}}.$$

Hence E_c and E^c are $G_{\delta\sigma}$ and the lemma is proved.

THEOREM 1. *If* $f(x)$ *has the selective derivative* $\mathrm{s}f'(x)$ *everywhere on* $[0, 1]$ *then* $\mathrm{s}f'(x)$ *is of Baire class* 2.

PROOF. By a theorem of O'MALLEY $f(x)$ is Baire 1 ([1], Theorem 10. A generalization of O'Malley's theorem will be proved in Theorem 6.) Thus the function $\varphi(x, y) = \dfrac{f(y)-f(x)}{y-x}$ is Baire 1 on the set $\{(x, y);\ 0 \leqq x < y \leqq 1\}$. Consequently (3), (4), (5) and Lemma 1 prove Theorem 1.

2. Now we define the new form of differentation mentioned in Section 1.

DEFINITION. Let $f(x)$ be an arbitrary function on $[0, 1]$. Suppose that the interval functions $l(x, y)$ and $r(x, y)$ are defined on the subintervals of $[0, 1]$ and satisfy (5). The upper derivate of $f(x)$ with respect to the interval functions $l(x, y)$ and $r(x, y)$ is defined as $\max\left(\overline{\lim}_{y \to x-0} l(y, x),\ \overline{\lim}_{y \to x+0} r(x, y)\right)$ and is denoted by ${}^r_l\overline{f}(x)$. (At the endpoints of $[0, 1]$ ${}^r_l\overline{f}(x)$ is defined by ${}^r_l\overline{f}(0) = \overline{\lim}_{y \to 1+0} r(0, y)$ and ${}^r_l\overline{f}(1) = \overline{\lim}_{y \to 1-0} l(y, 1)$.) The definition of the lower derivate ${}^r_l\underline{f}(x)$ is similar. $f(x)$ is said to be differentiable at the point x with respect to $l(x, y)$ and $r(x, y)$ if $-\infty < {}^r_l\underline{f}(x) = {}^r_l\overline{f}(x) < \infty$ holds and the derivative is denoted by ${}^r_l f'(x)$.

If $f(x)$ has the selective derivative for the selection $p_{[a,b]}$ then $f(x)$ is differentiable with respect to the interval functions defined by (1) and (2) and $sf'(x) = {}_{l}^{r}f'(x)$. That is the derivative with respect to $l(x, y)$ and $r(x, y)$ can be regarded as a generalization of the selective derivative. On the other hand we shall see that the main properties of the selective derivative can be extended to the $l-r$-derivative.

In the sequel we shall suppose that $f(x)$, $l(x, y)$ and $r(x, y)$ are given and satisfy (5).

At first sight it may seem that there is only a loose connection between the function $f(x)$ and its derivative ${}_{l}^{r}f'(x)$.

EXAMPLE 1. *Let* $H \subset [0, 1]$ *be countable and let* $g(x)$ *be an arbitrary function defined on* H. *Then there exist* $l_g(x, y)$ *and* $r_g(x, y)$ *satisfying* (5) *such that* ${}_{l_g}^{r_g}f'(x) = g(x)$ *holds for every* $x \in H$.

PROOF. Let $H = \{x_n\}_{n=1}^{\infty}$ and let

$$l_g(x, y) = r_g(x, y) = \frac{f(y)-f(x)}{y-x} \quad \text{if} \quad x, y \notin H;$$

$$l_g(y, x_n) = g(x_n) \quad \text{and} \quad r_g(y, x_n) = \frac{f(x_n)-f(y)}{x_n-y}$$

if $y < x_n$, $y \notin H$ or $y = x_k \in H$ and $k > n$;

$$r_g(x_n, y) = g(x_n) \quad \text{and} \quad l_g(x_n, y) = \frac{f(y)-f(x_n)}{y-x_n}$$

if $y > x_n$, $y \notin H$ or $y = x_k \in H$ and $k > n$.

It is easy to see that ${}_{l_g}^{r_g}f'(x_n) = g(x_n)$ holds for every n.

THEOREM 2. *If both* $l_1(x, y)$, $r_1(x, y)$ *and* $l_2(x, y)$, $r_2(x, y)$ *satisfy* (5) *then the set* $\{x \in [0, 1]; {}_{l_1}^{r_1}\overline{f}(x) < {}_{l_2}^{r_2}\underline{f}(x)\}$ *is countable.*

PROOF. It is enough to show that the set $H_s = \{x; {}_{l_1}^{r_1}\overline{f}(x) < s < {}_{l_2}^{r_2}\underline{f}(x)\}$ is countable for every rational s. Let $A_n = \Big\{x \in H_s;\ r_1(x, y) < s < r_2(x, y) \text{ if } 0 < y-x < \frac{1}{n}$ and $l_1(y, x) < s < l_2(y, x)$ if $0 < x-y < \frac{1}{n}\Big\}$.

Then $x, y \in A_n$ implies $|y-x| \geqq \frac{1}{n}$ because otherwise

$$\frac{f(y)-f(x)}{y-x} \leqq \max\big(l_1(x, y), r_1(x, y)\big) < s < \min\big(l_2(x, y), r_2(x, y)\big) \leqq \frac{f(y)-f(x)}{y-x}$$

would hold. Thus A_n is finite and $\bigcup_{n=1}^{\infty} A_n = H_s$ is countable.

The following theorem (which is a trivial consequence of Theorem 2) shows that the derivative with respect to $l(x, y)$ and $r(x, y)$ is essentially independent of $l(x, y)$ and $r(x, y)$.

THEOREM 3. *If* $f(x)$ *is differentiable with respect to both* $l_1(x, y)$, $r_1(x, y)$ *and* $l_2(x, y)$, $r_2(x, y)$ *then* ${}_{l_1}^{r_1}f'(x) = {}_{l_2}^{r_2}f'(x)$ *holds on* $[0, 1]$ *apart from a countable set.*

THEOREM 4. *The set of points at which one of the inequalities* $\underline{f}(x) \leqq {}^r_l\bar{f}(x)$ *and* ${}^r_l\underline{f}(x) \leqq \bar{f}(x)$ *does not hold is countable, where* $\underline{f}$ *and* $\bar{f}$ *denote the lower and upper derivate numbers of* f, *respectively.*

This easily follows from Theorem 2 again putting $l_2(x, y) = r_2(x, y) = \dfrac{f(y)-f(x)}{y-x}$.
Our further results are based on the following

LEMMA 2. *Suppose that* $f(x)$ *is differentiable with respect to* $l(x, y)$ *and* $r(x, y)$. *Let* $P \subset [0, 1]$ *be perfect and let the set* $H = \{x \in P;\ c < {}^r_l f'(x) < d\}$ *be of the second category in* P. *Then there exists an interval* (a, b) *such that* $(a, b) \cap P \neq \emptyset$ *and* $c \leqq \dfrac{f(y)-f(x)}{y-x} \leqq d$ *holds for every* $x<y$, $x, y \in (a, b) \cap P$.

PROOF. Let $H_n = \left\{x \in P;\ c < l(y, x) < d \text{ if } 0 < x-y < \frac{1}{n} \text{ and } c < r(x, y) < d \text{ if } 0 < y-x < \frac{1}{n}\right\}$.

Since $H \subset \bigcup_{n=1}^{\infty} H_n$ and H is of the second category in P, there exist an interval (a, b) and a natural number N such that $(a, b) \cap P \neq \emptyset$ and H_N is everywhere dense in $(a, b) \cap P$. We may assume $b-a < \frac{1}{N}$. Then for every $x<y$, $x, y \in (a, b) \cap \cap H_N$ we have $c < l(x, y) < d$ and $c < r(x, y) < d$ and thus $c < \dfrac{f(y)-f(x)}{y-x} < d$ by (5). In order to show $c \leqq \dfrac{f(y)-f(x)}{y-x} \leqq d$ for every $x<y$, $x, y \in (a, b) \cap P$ it is enough to prove $\lim\limits_{\substack{x \to x_0 \\ x \in H_N}} f(x) = f(x_0)$ for any $x_0 \in (a, b) \cap P$.

If $x_n \to x_0$, $x_n < x_0$ and $x_n \in H_N$ then $c < r(x_n, x_0) < d$ and $l(x_n, x_0) \to {}^r_l f'(x_0)$ nnd thus $l(x_n, x_0)$ is bounded. Hence by (5), $\dfrac{f(x_n)-f(x_0)}{x_n-x_0}$ is bounded, too, which implies $f(x_n) \to f(x_0)$. A similar argument shows that $f(x_n) \to f(x_0)$ holds if $x_n \to x_0$, $x_n > x_0$ and $x_n \in H_N$. This proves the lemma.

THEOREM 5. *If* $f(x)$ *is differentiable with respect to* $l(x, y)$ *and* $r(x, y)$ *and* ${}^r_l f'(x) > 0$ *for every* $x \in [0, 1]$ *then* $f(x)$ *is non-decreasing on a subinterval of* $[0, 1]$. *If in addition* $f(x)$ *is a Darboux function then* $f(x)$ *is non-decreasing on* $[0, 1]$.

PROOF. Since the set $\{x;\ 0 < {}^r_l f'(x) < n\}$ is of the second category in $[0, 1]$ for a suitably large n, Lemma 2 proves the first statement. The second statement can be shown in the same way as Theorem 1 in [1].

THEOREM 6. *Suppose that* $f(x)$ *is differentiable on* $[0, 1]$ *with respect to* $l(x, y)$ *and* $r(x, y)$. *Then* $f(x)$ *is Baire* 1 *and there exists an everywhere dense open set* U *such that* $f(x)$ *is continuous and almost everywhere differentiable on* U.

PROOF. Let $P \subset [0, 1]$ be an arbitrary perfect set and let $H_n = \{x \in P;\ -n < {}^r_l f'(x) < n\}$. Since $\bigcup_{n=1}^{\infty} H_n = P$, hence H_n is of the second category in P for a suitable n. Thus by Lemma 2 there exists an interval (a, b) such that $(a, b) \cap$

$\cap P \neq \emptyset$ and $|f(x)-f(y)| \leqq n|x-y|$ holds for every $x, y \in (a, b) \cap P$. Hence $f(x)|_P$ has some points of continuity for every non-empty perfect P which assures $f(x)$ to be Baire 1. In addition every $[a, b] \subset [0, 1]$ has an open subinterval on which $f(x)$ is Lipschitz 1. This trivially implies the existence of the open set U with the required properties.

THEOREM 7. *If $f(x)$ is differentiable on* $[0, 1]$ *with respect to $l(x, y)$ and $r(x, y)$ then the set of points of continuity of the derivate numbers $\underline{f}$ and $\bar{f}$ is everywhere dense in* $[0, 1]$.

PROOF. It is enough to prove that there exists a point x_0 at which $\underline{f}$ and $\bar{f}$ are continuous.

Let I_0 be an interval on which $f(x)$ is Lipschitz 1 and let

$$\sup_{\substack{x, y \in I_0 \\ x \neq y}} \frac{f(y)-f(x)}{y-x} = \sup_{x \in I_0} \bar{f}(x) = M_0,$$

$$\inf_{\substack{x, y \in I_0 \\ x \neq y}} \frac{f(y)-f(x)}{y-x} = \inf_{x \in I_0} \underline{f}(x) = m_0, \quad d_0 = M_0 - m_0.$$

It follows from Theorem 4 that $I_0 \setminus \{x;\ m_0 \leqq {}^r_l f'(x) \leqq M_0\}$ is countable. Hence either

$$\left\{x \in I_0;\ m_0 - 1 < {}^r_l f'(x) < M_0 - \frac{d_0}{3}\right\} \quad \text{or} \quad \left\{x \in I_0;\ m_0 + \frac{d_0}{3} < {}^r_l f'(x) < M_0 + 1\right\}$$

is of the second category in I_0. Thus by Lemma 2 there is a closed interval $I_1 \subset \operatorname{int} I_0$ such that either

$$\sup_{\substack{x, y \in I_1 \\ x \neq y}} \frac{f(y)-f(x)}{y-x} = \sup_{x \in I_1} \bar{f}(x) \leqq M_0 - \frac{d_0}{3}$$

or

$$\inf_{\substack{x, y \in I_1 \\ x \neq y}} \frac{f(y)-f(x)}{y-x} \inf_{x \in I_1} \underline{f}(x) \geqq m_0 + \frac{d_0}{3}$$

holds. In both cases we have

$$\sup_{x \in I_1} \bar{f}(x) - \inf_{x \in I_1} \underline{f}(x) = d_1 \leqq \frac{2}{3} d_0.$$

Repeating this argument we get a sequence of intervals I_n such that $\bar{I}_n \subset \operatorname{Int} I_{n-1}$ and

$$\sup_{x \in I_n} \bar{f}(x) - \inf_{x \in I_n} \underline{f}(x) \leqq \left(\frac{2}{3}\right)^n d_0$$

hold. Then $x_0 \in \bigcap_{n=1}^{\infty} I_n$ is a point of continuity of both $\underline{f}$ and $\bar{f}$.

THEOREM 8. *If $f(x)$ has the selective derivative $\mathrm{s}f'(x)$ for a given selection $p_{[a, b]}$ then the set of points of continuity of $\mathrm{s}f'(x)$ is everywhere dense in* $[0, 1]$.

PROOF. By the preceding theorem it is enough to show that ${}_sf'(x)$ is continuous at the points of continuity of $\underline{f}$ and $\bar{f}$. For such a point x_0 $f(x)$ is differentiable at x_0 and thus the inequality $\underline{f}(x) \leqq {}_sf'(x) \leqq \bar{f}(x)$ proves the theorem.

It should be noted that Theorem 8 fails to hold for $l-r$-derivatives. For example consider Example 1 putting $f(x) \equiv 0$, $H = \{x \in [0, 1];\ x \text{ is rational}\}$ and $g(x) \equiv 1$. Then $f(x)$ is differentiable with respect to $l(x, y)$ and $r(x, y)$ and ${}_l^r f'(x) = 1$ or 0 according to x is rational or not.

THEOREM 9. *If $f(x)$ is differentiable on $[0, 1]$ with respect to $l(x, y)$ and $r(x, y)$ then there exists a function $g(x)$ such that $\{x;\ {}_l^r f'(x) \neq g(x)\}$ is countable and the set of points of continuity of $g(x)$ is everywhere dense in $[0, 1]$.*

PROOF. It easily follows from Theorems 4 and 7 that $g(x) = \max\left(\underline{f}(x), \min\left(\bar{f}(x), {}_l^r f'(x)\right)\right)$ is a suitable function.

Our last theorem shows the close relation between the $l-r$-derivative and the usual derivative of the function $f(x)$.

THEOREM 10. *If $f(x)$ is differentiable on $[0, 1]$ with respect to $l(x, y)$ and $r(x, y)$ then there is a set $H \subset [0, 1]$ such that $f(x)$ is differentiable at the points of H, ${}_l^r f'(x) = f'(x)$ holds for every $x \in H$ and $[0, 1] \setminus H$ is of the first category.*

PROOF. Let H_1 be the set of points of continuity of $\bar{f}(x)$. Then H_1 is a G_δ set and by Theorem 7 it is everywhere dense in $[0, 1]$. Thus $[0, 1] \setminus H_1$ is of the first category. Let U be an everywhere dense, open set on which $\underline{f}(x)$ is continuous. Then by Dini's theorem $f(x)$ is differentiable at the points of $H_1 \cap U$ (see [3], Chapter VI. § 7, p. 204). Let $K = \{x \in [0, 1];\ f'(x)$ exists and differs from ${}_l^r f'(x)\}$, then K is countable by Theorem 4. Thus for the set $H = (H_1 \cap U) \setminus K$, $[0, 1] \setminus H = ([0, 1] \setminus H_1) \cup ([0, 1] \setminus U) \cup K$ is of the first category and ${}_l^r f'(x) = f'(x)$ holds for every $x \in H$. This proves the theorem.

Finally we remark that if $f(x)$ is differentiable on $[0, 1]$ with respect to $l(x, y)$ and $r(x, y)$ then the function ${}_l^r f'(x)$ is Baire 2. In fact, this trivially follows from Lemma 1 and Theorem 6. This raises the following question: does the function ${}_l^r f'(x)$ belong to the family of the honorary functions of the second class? (A function $g(x)$ is an honorary function of the second class if there is a function $h(x)$ in the first Baire class such that $g(x) = h(x)$ except on a countable set.) Is this true for the selective derivatives? We note that O'Malley has constructed a selective derivative which is not Baire 1 but his function is an honorary function of the second class.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT I OF ANALYSIS
1088 BUDAPEST, MÚZEUM KRT. 6—8.

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A FINITE 2-DIMENSIONAL CW COMPLEX WHICH CANNOT BE TRIANGULATED

By
M. BOGNÁR (Budapest)

Introduction

The book of Lundell and Weingram presents a finite 3-dimensional CW complex which cannot be triangulated ([1] p. 81). In this example it is possible to choose the family of cells such that it consists of three cells: a 3-cell, a 2-cell and a 0-cell.

The purpose of this paper is to construct a 2-dimensional CW complex, which cannot be triangulated and contains also exactly three cells, a 2-dimensional, a 1-dimensional and a 0-dimensional one. This is, in a certain sense, the simplest non-triangulable CW complex. In particular — as we shall show — if the set of the dimensions of the cells contains at most two numbers, then the finite CW complex can always be triangulated.

1. Let $\psi: [0, 2\pi] \to [0, 2\pi]$ denote the function which is linear on the intervals $\left[\frac{2\pi}{n+1}, \frac{2\pi}{n}\right]$, $n \in \mathbf{N}$ ($\mathbf{N}$ is the set of all positive integers) and satisfies the conditions:

$$\psi(0) = 0, \ \psi(2\pi) = 2\pi, \ \psi\left(\frac{2\pi}{2r+1}\right) = \frac{2\pi}{2r-1} \quad (r \in \mathbf{N}),$$

$$\psi\left(\frac{2\pi}{2r}\right) = 0 \quad (r \in \mathbf{N}),$$

(See Figure 1.)

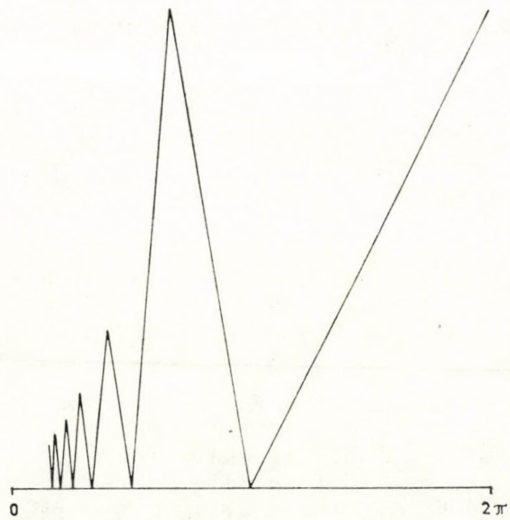

Fig. 1

2. ψ is continuous. Furthermore $\psi^{-1}\left(\left]\frac{2\pi}{2r+1}, \frac{2\pi}{2r-1}\right[\right)$ consists of $2r+1$ components: $]x_r^1, x_r'^1[, \ldots,]x_r^{2r+1}, x_r'^{2r+1}[,$ where $\psi(]x_r^j, x_r^{j\prime}[)=\left]\frac{2\pi}{2r+1}, \frac{2\pi}{2r-1}\right[$ for every $j\in[1, 2r+1]\cap\mathbf{N}$.

3. Let L be the unit disc and J the unit circle in the complex number space. Let J' be $J+3$. L and J' are disjoint sets. J' is obviously a CW complex with respect to a family of cells consisting of one 1-dimensional and one 0-dimensional cell.

4. Let ω denote the function $\omega\colon [0, 2\pi]\to J\colon x\mapsto e^{ix}$, and let $f\colon J\to J'$ be defined by $f(\omega(x))=\omega(\psi(x))+3$.

5. $X=J'\cup_f L$ ([1] p. 37) *is a CW complex consisting of one 2-dimensional, one 1-dimensional and one 0-dimensional cell.* (See [1] Proposition 2.2. p. 46.) ($J'\cup_f L$ is the quotient space of $J'\cup L$ under the equivalence relation which identifies $z\in J$ with $f(z)\in J'$.)

We shall prove that X *cannot be triangulated.* At first we are doing some preliminary steps.

6. PROPOSITION. *Let A be a simple arc in the plane $\mathbf{R}^2$ and G an open set of $\mathbf{R}^2$ intersecting A. Then there exists an open subset G' of G for which $G'\cap A\neq\emptyset$ and $G'\setminus A$ consists of two components.*

PROOF. Let s and t be the end points of A and let G_1 be an open circular disc lying in $G\setminus\{s, t\}$ and intersecting A. Let q_1, q_2, q_3 be three distinct points of G_1 satisfying the conditions:

1) $q_1\in G_1\setminus A$; $q_2, q_3\in A\cap G_1$; q_1, q_2, q_3 are non-collinear,

2) $\overline{q_1q_i}\cap A=\{q_i\}$ for $i=2, 3$ where $\overline{q_1q_i}$ is the linear segment with the end points q_1 and q_i.

3) The subarc B of A with the end points q_2 and q_3 is contained in G_1.

There exist obviously such three points.

The Jordan curve $J^*=B\cup\overline{q_1q_2}\cup\overline{q_1q_3}$ and hence the bounded component H_1 of $\mathbf{R}^2\setminus J^*$ is contained in G_1.

Futhermore let G_2 be an open circular disc lying in G, intersecting B and not intersecting $(A\setminus B)\cup\overline{q_1q_2}\cup\overline{q_1q_3}$. Let q_4, q_5, q_6 be three distinct points of G_2 satisfying the conditions:

4) $q_4\in G_2\setminus(A\cup H_1)$; $q_5, q_6\in A\cap G_2$; q_4, q_5, q_6 are non-collinear.

5) $\overline{q_4q_i}\cap A=\{q_i\}$ for $i=5, 6$.

6) The subarc C of A with the end points q_5, q_6 is contained in G_2.

There exist obviously such three points.

The Jordan curve $J^{**}=C\cup\overline{q_4q_5}\cup\overline{q_5q_6}$ and hence the bounded component H_2 of $\mathbf{R}^2\setminus J^{**}$ is contained in G_2.

Denoting C by v_1, $\overline{q_4q_5}\cup\overline{q_4q_6}$ by v_2 and $\overline{q_1q_2}\cup\overline{q_1q_3}\cup(B\setminus C)\cup\{q_5, q_6\}$ by v_3, $v_1\cup v_2\cup v_3$ is a Θ curve in $\mathbf{R}^2$ and the three components of $\mathbf{R}^2\setminus(v_1\cup v_2\cup v_3)$ are H_1, H_2 and the non-bounded component of $\mathbf{R}^2\setminus(v_2\cup v_3)$. (See Figure 2.) Let G' be the bounded component of $\mathbf{R}^2\setminus(v_2\cup v_3)$. Then $G'=H_1\cup\cup H_2\cup C\setminus\{q_5, q_6\}$ and therefore $G'\subset G$, $G'\cap A\neq\emptyset$. Finally H_1 and H_2 are the two components of $G'\setminus A$ and this proves the proposition.

7. Let $p\colon J'\cup L\to X=J'\cup_f L$ be the quotient map. p is injective on the subsets J' and $L\setminus J$, $p|_{J'}\colon J'\to p(J')$ and $p|_{L\setminus J}\colon L\setminus J\to p(L\setminus J)$ are homeomorphisms, moreover $p(L\setminus J)$ is an open and $p(J')$ a closed subset of X.

Now $f(J)=J'$ and thus $p(J)=p(J')$ and since $L\setminus J$ is dense in L, $p(L\setminus J)$ is dense in X.

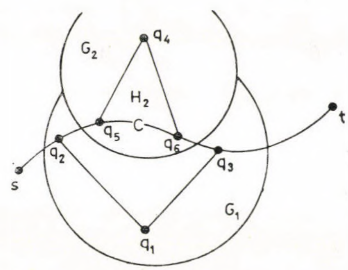

Fig. 2

8. Let

$$U_r^j=\{t\cdot\omega(x);\, t\in]0,1[,\, x\in]x_r^j, x_r'^j[\};\quad r\in\mathbf{N},\quad j\in[1,2r+1]\cap\mathbf{N}$$

(see Section 2). The open sectors $U_r^1,\dots,U_r^{2r+1}$ of $L\setminus J$ are disjoint. Let

$$V_r=p(U_r^1\cup\dots\cup U_r^{2r+1})\cup p\left(\omega\left(\left]\frac{2\pi}{2r+1},\frac{2\pi}{2r-1}\right[\right)+3\right);\quad r\in\mathbf{N}.$$

Since

$$f^{-1}\left(\omega\left(\left]\frac{2\pi}{2r+1},\frac{2\pi}{2r-1}\right[\right)+3\right)=\omega\left(\psi^{-1}\left(\left]\frac{2\pi}{2r+1},\frac{2\pi}{2r-1}\right[\right)\right)=$$
$$=\omega(]x_r^1, x_r'^1[\cup\dots\cup]x_r^{2r+1}, x_r'^{2r+1}[)$$

(see Sections 4 and 2) it holds

$$p^{-1}\left(p\left(\omega\left(\left]\frac{2\pi}{2r+1},\frac{2\pi}{2r-1}\right[\right)+3\right)\right)=\left(\omega\left(\left]\frac{2\pi}{2r+1},\frac{2\pi}{2r-1}\right[\right)+3\right)\cup$$
$$\cup f^{-1}\left(\omega\left(\left]\frac{2\pi}{2r+1},\frac{2\pi}{2r-1}\right[\right)+3\right)=\left(\omega\left(\left]\frac{2\pi}{2r+1},\frac{2\pi}{2r-1}\right[\right)+3\right)\cup$$
$$\cup\,\omega(]x_r^1, x_r'^1[)\cup\dots\cup\omega(]x_r^{2r+1}, x_r'^{2r+1}[),$$

and therefore

$$p^{-1}(V_r)=U_r^1\cup\dots\cup U_r^{2r+1}\cup p^{-1}\left(p\left(\omega\left(\left]\frac{2\pi}{2r+1},\frac{2\pi}{2r-1}\right[\right)+3\right)\right)$$

is open in $L\cup J'$, hence V_r is open in X.

Furthermore

$$V_r\setminus p(J')=V_r\setminus p\left(\omega\left(\left]\frac{2\pi}{2r+1},\frac{2\pi}{2r-1}\right[\right)+3\right)$$

consists of $2r+1$ disjoint open sets, i.e.: $p(U_r^1), \dots, p(U_r^{2r+1})$ and because

$$p(\omega(]x_r^j, x_r'^j[)) = p(f(\omega(]x_r^j, x_r'^j[))) =$$

$$= p(\omega(\psi(]x_r^j, x_r'^j[))+3) = p\left(\omega\left(\left]\frac{2\pi}{2r+1}, \frac{2\pi}{2r-1}\right[\right)+3\right)$$

(see Sections 4 and 2) we find that the boundary of each $p(U_r^j)$ contains the set

$$p\left(\omega\left(\left]\frac{2\pi}{2r+1}, \frac{2\pi}{2r-1}\right[\right)+3\right) = p(J')\cap V_r.$$

Consequently *for every open set* G *of* X *intersecting* $p(J')$ *and contained in* V_r

$$G\setminus p(J') = (G\cap p(U_r^1))\cup \dots \cup (G\cap p(U_r^{2r+1})),$$

where $\{G\cap p(U_r^1), \dots, G\cap p(U_r^{2r+1})\}$ is a disjoint family of nonempty open sets, and so *the number of the components of* $G\setminus p(J')$ *is at least* $2r+1$.

9. In contrast to our assertion let us assume that X can be triangulated, and let K be a simplicial complex such that the polyhedron $|K|$ of K should be homeomorphic to X. X is compact, hence K must be finite. So it may be supposed that K is a finite geometric simplicial complex in some Euclidean space $\mathbf{R}^n$. Let $\varphi\colon |K|\to X$ be a homeomorphism.

We first show that K is 2-dimensional.

In fact, considering an (open) simplex σ^t in K of largest dimension, σ^t is open in $|K|$, hence $\varphi(\sigma^t)$ is open in X. $p(L\setminus J)$ is, however, dense and open in X (see Section 7) and this implies that $\varphi(\sigma^t)\cap p(L\setminus J)$ is nonempty and open. $p(L\setminus J)$ is homeomorphic to $L\setminus J$ and thus every nonempty open subset of $p(L\setminus J)$ is 2-dimensional. Consequently $\varphi^{-1}(p(L\setminus J)\cap\varphi(\sigma^t))$ is a nonempty open 2-dimensional subset of σ^t and we have $t=2$ as required.

10. We show in this section the impossibility of $\varphi(\sigma^2)\cap p(J')\neq\emptyset$ for each $\sigma^2\in K$.

In fact, let us suppose in contrast to the statement that $\varphi(\sigma^2)\cap p(J')\neq\emptyset$ for some $\sigma^2\in K$. Since $\varphi(\sigma^2)$ is open in X and

$$\bigcup\left\{\omega\left(\left]\frac{2\pi}{2r+1}, \frac{2\pi}{2r-1}\right[\right)+3;\ r\in\mathbf{N}\right\}$$

is dense in J' there exists an $r\in\mathbf{N}$ and a point $q\in\sigma^2$ such that

$$\varphi(q)\in p\left(\omega\left(\left]\frac{2\pi}{2r+1}, \frac{2\pi}{2r-1}\right[\right)+3\right)\subset V_r.$$

Let s be a simple arc in $V_r\cap p(J')\cap\varphi(\sigma^2)$ containing $\varphi(q)$ as its cut point. Denoting the set of cut points of s by $\operatorname{int} s$ we get $\varphi(q)\in\operatorname{int} s$. $p(J')\setminus\operatorname{int} s$ is closed, and therefore $X\setminus(p(J')\setminus\operatorname{int} s)=p(L\setminus J)\cup\operatorname{int} s$ is open in X. Let us observe that $G\setminus s=G\setminus p(J')$ holds for each open subset G of $p(L\setminus J)\cup\operatorname{int} s$.

$\varphi(\sigma^2)$ is homeomorphic to $\mathbf{R}^2$, $G=V_r\cap\varphi(\sigma^2)\cap(p(L\setminus J)\cup\operatorname{int} s)$ is an open subset of $\varphi(\sigma^2)$ intersecting s, and hence according to Section 6 there exists an open subset G' of G such that $G'\cap s\neq\emptyset$ and $G'\setminus s=G'\setminus p(J')$ consists of two components. However G' intersects $p(J')$, it is contained in V_r, and so according to Section 8 the number of the components of $G'\setminus s=G'\setminus p(J')$ is at least three. The contradiction proves the statement, thus we have $\varphi(\sigma^2)\cap p(J')=\emptyset$ for each $\sigma^2\in K$, consequently $p(J')\subset\varphi(|K^1|)$, where K^1 is the 1-skeleton of K.

11. Choose an $r\in\mathbf{N}$ and take a simple arc s in $(p(J')\cap V_r)\setminus\varphi(|K^0|)$, where K^0 is the 0-skeleton of K. The finiteness of K^0 assures the existence of such an arc. According to Section 10 $\varphi^{-1}(s)$ is a connected subset of $|K^1|\setminus|K^0|$, therefore it is contained in some (open) 1-simplex σ^1 of K. $\varphi^{-1}(s)$ is a subsegment of the open segment σ^1 and $\varphi^{-1}(s)$ is contained in $\varphi^{-1}(V_r)$. Let q be a cut point of $\varphi^{-1}(s)$. Considering the (open) 2-simplexes $\sigma_1^2, \dots, \sigma_m^2$ of the star of σ^1 in K

$$H=\sigma_1^2\cup\dots\cup\sigma_m^2\cup\varphi^{-1}(\operatorname{int} s) \tag{1}$$

is open in $|K|$ (if $m=0$ then $H=\varphi^{-1}(\operatorname{int} s)$; $\operatorname{int} s$ is the set of cut points of s), and therefore there exists a spherical neighbourhood $S=S_\varepsilon(q)$ of q in $\mathbf{R}^n$ such that

$$(S\cap|K|)\subset\big(H\cap\varphi^{-1}(V_r)\big). \tag{2}$$

$\sigma_j^2\cap S$ is a nonempty convex set for $j\in[1,m]\cap\mathbf{N}$ and hence it is connected. It is also open in $|K|$. This implies that $(S\cap|K|)\setminus\varphi^{-1}(s)$ has exactly m components, namely: $\sigma_1^2\cap S, \dots, \sigma_m^2\cap S$.

Let us denote $\varphi(S\cap|K|)$ by G. $G\setminus s$ also has exactly m components. The open set G of X is contained in V_r, it intersects $p(J')$ and since by (2) and (1) $G\subset\varphi(H)=\varphi(\sigma_1^2\cup\dots\cup\sigma_m^2)\cup\operatorname{int} s$ and according to Section 10 $\varphi(\sigma_1^2\cup\dots$ $\dots\cup\sigma_m^2)\cap p(J')=\emptyset$ we have $G\setminus s=G\setminus p(J')$. From this it follows by Section 8 that the number of the components of $G\setminus s$ is at least $2r+1$. Hence $m\geqq 2r+1$.

That means, the number of the 2-simplexes of K is at least $2r+1$ and this assertion is true for every $r\in\mathbf{N}$. But this is impossible, since K is finite. The contradiction proves our statement about the non-existence of a polyhedron homeomorphic to X. The proof of the non-triangulability of X is complete.

12. Our aim in this section is to prove the assertion at the end of the introduction.

Let X be a finite CW complex with cells $\mathscr{S}$ such that the set M of the dimensions of the cells of $\mathscr{S}$ contains at most two numbers. Hence either $M=\{0\}$ or $M=\{0,t\}$ where $t>0$.

In the first case X can be evidently triangulated, and we shall prove this fact also in the second case.

Let σ^t be a closed t-cell of $\mathscr{S}$. Then

$$\dot\sigma^t=\sigma_{i_1}^0\cup\dots\cup\sigma_{i_l}^0, \tag{1}$$

where the $\sigma_{i_j}^0$-s are the proper faces of σ^t. Since $\dot\sigma^t\neq\emptyset$ we have $l\neq 0$. Choose the open neighbourhoods U_{i_j} of $\sigma_{i_j}^0$ such that $U_{i_1}, \dots, U_{i_l}$ should be mutually disjoint sets. Denoting $\sigma^t\setminus\bigcup_{j=1}^{l} U_{i_j}=(\sigma^t\setminus\dot\sigma^t)\setminus\bigcup_{j=1}^{l} U_{i_j}$ by H, H is closed in X and thus it is compact.

Let $\varphi\colon E^t\to\sigma^t$ be a characteristic map for the cell σ^t, where E^t is a t-disc.

Then $\varphi^{-1}(H)$ is compact, it is contained in $E^t \setminus \dot{E}^t$ and therefore there exists a t-disc E'^t concentric with E^t, containing $\varphi^{-1}(H)$ and contained in $E^t \setminus \dot{E}^t$.

Let us denote $\sigma^t \setminus (\dot{\sigma}^t \cup \varphi(E'^t))$ by G. $\dot{\sigma}^t$ lies on the boundary of G and so by (1) $U_{i_j} \cap G \neq \emptyset$, $j=1, \dots, l$. Furthermore $\{U_{i_1} \cap G, \dots, U_{i_l} \cap G\}$ is a covering of G, where $U_{i_j} \cap G$-s are mutually disjoint nonempty open sets.

Consequently:

a) In the case $t>1$, G is homeomorphic to the connected set $E^t \setminus (\dot{E}^t \cup E'^t)$, hence $l=1$ and σ^t is homeomorphic to the t-sphere $\mathbf{S}^t$.

b) In the case $t=1$, G is homeomorphic to $E^1 \setminus (\dot{E}^1 \cup E'^1)$ and thus it consists of two components. Hence either $l=1$ and then σ^1 is homeomorphic to S^1 or $l=2$ and then σ^1 is homeomorphic to E^1.

The polyhedron homeomorphic to X can be constructed now as follows:

a) In the case $t>1$:

Let $\sigma_1^0, \dots, \sigma_m^0$ be those 0-cells which are contained in some closed t-cell of $\mathcal{S}$ and $\sigma_{m+1}^0, \dots, \sigma_v^0$ the other 0-cells of $\mathcal{S}$. Let $\sigma_{i,1}^t, \dots, \sigma_{i,q_i}^t$ be the closed t-cells of $\mathcal{S}$ containing σ_i^0, $i=1, \dots, m$. Let us take mutually disjoint $(t+1)$-dimensional parallel strips $L_1, \dots, L_v$ in the $(t+1)$-dimensional Euclidean space $\mathbf{R}^{t+1}$. Choose the hyperplane $\mathbf{R}^t$ in $\mathbf{R}^{t+1}$ such that $\mathbf{R}^t$ intersects each of the strips.

Let p_i be a point in $L_i \setminus \mathbf{R}^t$, $i=1, \dots, v$. Let $s_{i,1}^t, \dots, s_{i,q_i}^t$ be mutually disjoint closed t-simplexes in $L_i \cap \mathbf{R}^t$ for $i=1, \dots, m$. Take the cones $s_{i,j}^{t+1}$ with base $s_{i,j}^t$ and vertex p_i $(j=1, \dots, q_i;\ i=1, \dots, m)$. The proper faces of the simplexes $s_{i,j}^{t+1}$ and the singletons $\{p_i\}$ are forming a simplicial complex K and the polyhedron $|K|$ is evidently homeomorphic to X.

b) In the case $t=1$:

Let $\sigma_1^0, \dots, \sigma_m^0$ be the 0-cells and $\sigma_1^1, \dots, \sigma_n^1$ the 1-cells of $\mathcal{S}$. Choose the points $p_1, \dots, p_{2n+m}$ in $\mathbf{R}^3$ such that they should be in general position. Let us take the segments $\overline{p_{2j-1}p_{2j}}$ for $j=1, \dots, n$. If σ_j^1 has the single 0-face $\sigma_{k_j}^0$ then take the segments $\overline{p_{2j-1}p_{2n+k_j}}$ and $\overline{p_{2j}p_{2n+k_j}}$. If σ_j^1 has two proper faces $\sigma_{k_j^1}^0$ and $\sigma_{k_j^2}^0$ with $k_j^1 < k_j^2$ then take the segments $\overline{p_{2j-1}p_{2n+k_j^1}}$ and $\overline{p_{2j}p_{2n+k_j^2}}$. These segments and the singletons $\{p_i\}$, $i=1, \dots, 2n+m$ constitute a simplicial complex K and the polyhedron $|K|$ is obviously homeomorphic to X.

So we have proved the following statement: *if the set of the dimensions of the cells contains at most two numbers then the finite CW complex can always be triangulated.*

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF GEOMETRY
1088 BUDAPEST, MÚZEUM KRT. 6—8.

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STANDARD IDEALS IN WILCOX LATTICES

By
S. MAEDA (Matsuyama)

1. Introduction. It is well known that any affine matroid lattice is an atomistic Wilcox lattice. In his paper [5], M. STERN proved that a non-modular affine matroid lattice satisfying Euclid's strong parallel axiom is simple, and in [6] he proved that the ideal $F(L)$ formed by all elements of an atomistic Wilcox lattice L is standard if and only if every imaginary element is finite in the modular extension of L.

Now we remark that any atomistic Wilcox lattice is section complemented. The main purpose of this paper is to investigate standard ideals in section complemented Wilcox lattices in order to obtain some results which imply the above Stern's theorems as special cases.

2. Standard ideals in section complemented lattices. Let L be a lattice with 0, and let $a, b \in L$. We say that a is *subperspective* to b when $a \leqq b \vee x$ and $a \wedge x = 0$ for some $x \in L$, and say that a is *subprojective* to b when there exist $a_0, a_1, \dots, a_n \in L$ such that $a_0 = a$, $a_n = b$ and a_{i-1} is subperspective to a_i $(i = 1, \dots, n)$ ([4], (6.1) and (6.6)).

An ideal J of L is called a *p-ideal* when any element subperspective to an element of J belongs to J ([4], (35.3)). An ideal J is called *standard* when, in the lattice of all ideals of L, it holds that $I_1 \vee (J \wedge I_2) = (I_1 \vee J) \wedge (I_1 \vee I_2)$ for all I_1, I_2, or equivalently, it holds that $I \vee J = \{x \vee y;\ x \in I, y \in J\}$ for all I ([1], Theorem 2). Any standard ideal is a homomorphism kernel (= the kernel of a congruence relation) by [1], Lemma 3, and any homomorphism kernel is evidently a p-ideal.

A lattice L with 0 is called *section complemented* when the interval $L[0, a] = \{x \in L;\ 0 \leqq x \leqq a\}$ is complemented for every $a > 0$ ([1], p. 27).

We shall generalize [3], Theorem 5.3 as follows:

THEOREM 1. *Let L be a section complemented lattice. For any subset M of L, the set*

$$\mathbf{I}(M) = \{a_1 \vee \dots \vee a_n;\ \textit{each } a_i \textit{ is subprojective to an element of } M\}$$

is a standard ideal. (This is the smallest standard ideal including M.)

PROOF. Let J be an arbitrary ideal of L. We shall prove that

(*) if $b \leqq x \vee y$ with $x \in \mathbf{I}(M)$ and $y \in J$ then $b = x_1 \vee y_1$ for some $x_1 \in \mathbf{I}(M)$ and $y_1 \in J$.

We put $x = a_1 \vee \dots \vee a_n$ where a_i is subprojective to $m_i \in M$ $(i = 1, \dots, n)$, and put

$$b_0 = y \wedge b, \quad b_i = (a_1 \vee \dots \vee a_i \vee y) \wedge b \quad (i = 1, \dots, n).$$

Since L is section complemented, there exist c_i $(i=1, \ldots, n)$ such that $c_i \vee b_{i-1} = b_i$ and $c_i \wedge b_{i-1} = 0$. Then, c_i is subperspective to a_i, since $c_i \leqq b_i \leqq a_1 \vee \ldots \vee a_i \vee y$ and $c_i \wedge (a_1 \vee \ldots \vee a_{i-1} \vee y) \leqq c_i \wedge b_{i-1} = 0$. Hence, c_i is subprojective to m_i, whence $c_1 \vee \ldots \vee c_n \in \mathbf{I}(M)$. It is evident that $b = c_1 \vee \ldots \vee c_n \vee b_0$ and $b_0 \in J$. Thus the statement (*) has been proved.

It is evident that $x_1, x_2 \in \mathbf{I}(M)$ implies $x_1 \vee x_2 \in \mathbf{I}(M)$. Putting $J = \{0\}$ in (*), it follows that $b \leqq x \in \mathbf{I}(M)$ implies $b \in \mathbf{I}(M)$. Hence, $\mathbf{I}(M)$ is an ideal. Moreover, $\mathbf{I}(M)$ is standard, since (*) implies $\mathbf{I}(M) \vee J = \{x \vee y;\ x \in \mathbf{I}(M),\ y \in J\}$.

COROLLARY 1. *Let J be an ideal of a section complemented lattice. The following four statements are equivalent.*

(α) *J is standard.*

(β) *J is a homomorphism kernel.*

(γ) *J is a p-ideal.*

(δ) *$J = \mathbf{I}(J)$.*

PROOF. The implications $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma) \Rightarrow (\delta)$ are evident. The implication $(\delta) \Rightarrow (\alpha)$ follows from the theorem.

REMARK 1. The equivalence of (α) and (β) is included in [1], Theorem 11. The equivalence of (α) and (γ) is the same as [2], Theorem 4.2 (i), because if a lattice L with 0 is section complemented then it is easy to verify that an ideal of L is 0-projective in the sense of [2] if and only if it is a p-ideal in our sense. (In [2], the term "p-ideal" is used in a different way.)

A lattice with 0 is called *atomistic* when every element is the join of a family of atoms (= points), and an element is called *finite* when it is zero or the join of a finite number of atoms ([4], (7.1) and (8.1)). If an atomistic lattice L has the *covering property* (i.e. L is an AC-lattice), then the set $F(L)$ of all finite elements of L forms an ideal ([4], (7.4) and (8.8)). Any upper continuous AC-lattice, which is called a *matroid lattice*, is relatively complemented ([4], (7.16) and (8.7)), whence it is section complemented.

COROLLARY 2. *Let L be an irreducible matroid lattice. The set $\mathbf{I}(F(L))$ is the smallest non-zero standard ideal in L, and hence L is simple if and only if $\mathbf{I}(F(L))$ contains* 1.

PROOF. If J is a non-zero standard ideal, then J contains an atom of L. It follows from [4], (13.6) that every two atoms of L are perspective. Hence, J contains all atoms and all finite elements, whence $\mathbf{I}(F(L)) \subset J$.

3. Standard ideals in Wilcox lattices. A Wilcox lattice L is defined as follows ([4], (3.11) and (3.12)). Let Λ be a complemented modular lattice where the join and the meet are denoted by $\cup$ and $\cap$ respectively, and let S be an ideal of Λ with 0 deleted. In the set $L \equiv \Lambda - S$, we define the same order as Λ. Then, in L, the join $a \vee b$ and the meet $a \wedge b$ exist for every $a, b \in L$. In fact,

$$a \vee b = a \cup b, \quad a \wedge b = \begin{cases} a \cap b & \text{if } a \cap b \in L \\ 0 & \text{if } a \cap b \in S. \end{cases}$$

The lattice Λ is called the *modular extension* of L and each element of S is called an *imaginary element*. If S has the greatest element i, then i is called an *imaginary unit*.

We remark that any p-ideal of Λ is standard since Λ is relatively complemented. Hereafter, we assume that S is not empty.

LEMMA 1. *Let* $L \equiv \Lambda - S$ *be a Wilcox lattice. If* Γ *is a standard ideal of* Λ *and* $S \subset \Gamma$, *then* $\Gamma - S$ *is a homomorphism kernel of* L.

PROOF. We consider, in Λ, the congruence relation with the kernel Γ (see [1], Theorem 2 (γ'')), and we restrict it to L. That is, for $x, y \in L$ we define $x \equiv y$ when there exists $\gamma \in \Gamma$ such that $(x \cap y) \cup \gamma = x \cup y$ $(= x \vee y)$. To prove that $x \equiv y$ is a congruence relation in L we need to verify the following four statements ([1], Lemma II): (a) $x \equiv x$, (b) $x \equiv y$ is equivalent to $x \vee y \equiv x \wedge y$, (c) $x \geqq y \geqq z$, $x \equiv y$, $y \equiv z$ imply $x \equiv z$, (d) $x \geqq y$, $x \equiv y$ imply $x \vee z \equiv y \vee z$ and $x \wedge z \equiv y \wedge z$ for all $z \in L$. The statement (a) is trivial. If $x \wedge y \neq 0$, then (b) is evident since $x \wedge y = x \cap y$. If $x \wedge y = 0$, then $x \cap y \in S \cup \{0\} \subset \Gamma$. Hence, $x \equiv y \Leftrightarrow x \cup y \in \Gamma \Leftrightarrow x \vee y \equiv 0 = x \wedge y$. Next, (c) and the first part of (d) are evident. If $x \geqq y$ and $x \equiv y$, then for any $z \in L$ there exists $\gamma \in \Gamma$ such that $(y \cap z) \cup \gamma = x \cap z$, since Γ is standard. If $y \wedge z \neq 0$, then $x \wedge z \neq 0$. Hence, $x \wedge z = x \cap z \equiv y \cap z = y \wedge z$. If $y \wedge z = 0$, then $y \cap z \in S \cup \{0\} \subset \Gamma$. Since $x \wedge z \leqq x \cap z = (y \cap z) \cup \gamma \in \Gamma$, we have $x \wedge z \in \Gamma$, whence $x \wedge z \equiv 0 = y \wedge z$. Therefore, $x \equiv y$ is a congruence relation in L, and evidently $x \equiv 0$ if and only if $x \in \Gamma - S$.

REMARK 2. In this lemma, if L is section complemented, then $\Gamma - S$ is a standard ideal of L (Corollary 1 of Theorem 1). But, otherwise, $\Gamma - S$ is not necessarily standard. Fig. 1 shows a Wilcox lattice where Λ is a Boolean lattice (with 8 elements) and $S = \{i\}$. The ideal $\Gamma = \{a, b, i, 0\}$ of Λ is standard, but the ideal $\Gamma - S = \{a, b, 0\}$ of L is not standard, since the set $\{x \vee y;\ x \in \Gamma - S, y \in \in L[0, c]\}$ is not an ideal.

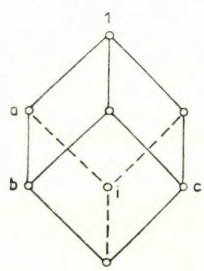

Fig. 1

A Wilcox lattice $L \equiv \Lambda - S$ is *left complemented* if and only if S satisfies the following condition (see [4], (3.13)):

(**) If $b \leqq a$ in Λ and if $a \notin S$ then there exists $c \notin S$ such that $a = b \cup c$ and $b \cap c = 0$.

LEMMA 2. *Let L be a Wilcox lattice. The following three statements are equivalent.*

(α) *L is left complemented.*

(β) *L is relatively complemented.*

(γ) *L is section complemented.*

PROOF. The implication $(\alpha)\Rightarrow(\beta)$ follows from [4], (3.9) and (2.10), and $(\beta)\Rightarrow(\gamma)$ is trivial. We shall prove that $(\gamma)\Rightarrow(^{**})$. Let $b\leqq a$ in Λ with $a\notin S$. If $b\in S$, then a complement c of b in the interval $\Lambda[0, a]$ of Λ does not belong to S and hence it has the desired property. If $b\in L$, then, by (γ), there exists $d\in L$ such that $b\vee d=a$ and $b\wedge d=0$. If $b\cap d=0$, then $c=d$ is the desired. If $b\cap d\neq 0$, then $b\cap d\in S$. Let c be a complement of $b\cap d$ in $\Lambda[0, d]$. Then $c\notin S$, and moreover

$$b\cup c=b\cup(b\cap d)\cup c=b\cup d=a \quad \text{and} \quad b\cap c=b\cap d\cap c=0.$$

Thus $(^{**})$ has been proven.

In the next lemma and theorem, we assume that a Wilcox lattice L satisfies the following condition:

(†) There exist $a, b\in L$ such that $0<a<b<1$.

By this assumption we exclude only the trivial case that the length of L is 2.

LEMMA 3. *Let $L\equiv\Lambda-S$ be a section complemented Wilcox lattice satisfying (†). For a non-empty subset M of $L-\{0\}$, we put*

$$M_S=\{x\cup u;\ 0\neq x\in M,\ u\in S\}.$$

Then, $\mathbf{I}(M_S)=\mathbf{I}(M)$.

PROOF. We have $M\subset\mathbf{I}(M_S)$, since $x\leqq x\cup u\in M_S$ for every $0\neq x\in M$. We shall prove $M_S\subset\mathbf{I}(M)$. Let $a=x\cup u$ with $0\neq x\in M$ and $u\in S$. If $a\neq 1$, then by the condition $(^{**})$ there exists $0\neq c\in L$ such that $a\cup c=1$ and $a\cap c=0$. We put $y=c\cup u$. Since Λ is modular, we have

$$a\cap y=(u\cup c)\cap a=u\cup(c\cap a)=u\in S,$$

whence $a\wedge y=0$. Since $a=x\cup u\leqq x\vee y$, a is subperspective to $x\in M$, and hence $a\in\mathbf{I}(M)$. If $a=1$, then we may assume $x\cap u=0$, for u can be replaced by a complement of $x\cap u$ in $\Lambda[0, u]$. If x is not an atom of L, then there is $x_1\in L$ with $0<x_1<x$, and by $(^{**})$ there is $0\neq x_2\in L$ such that $x_1\cup x_2=x$, $x_1\cap x_2=0$. Put $a_i=x_i\cup u$ $(i=1, 2)$. We have $a_1\neq 1$, since $a_1\cap x_2=(x_1\cup u)\cap x\cap x_2=\{x_1\cup(u\cap x)\}\cap x_2=x_1\cap x_2=0$, and hence $a_1\in\mathbf{I}(M)$ as above. Similarly $a_2\in\mathbf{I}(M)$, and hence $a=a_1\vee a_2\in\mathbf{I}(M)$. If x is an atom, then u is not an atom by (†). Hence, there exist $u_1, u_2\in S$ such that $u=u_1\cup u_2$, $u_1\cap u_2=0$. Putting $a_i=x\cup u_i$, we have $a_i\neq 1$, and hence $a=a_1\vee a_2\in\mathbf{I}(M)$. This completes the proof.

Let $L\equiv\Lambda-S$ be a Wilcox lattice, and let J be a non-zero ideal of L. We say that J *dominates* S when for every $u\in S$ there exists $x\in J$ such that $u<x$. It is easy to verify that the following three statements are equivalent: (α) J dominates S, (β) the union $J\cup S$ is an ideal of Λ, (γ) J includes the set $J_S=\{x\cup u;\ 0\neq x\in J,\ u\in S\}$.

THEOREM 2. *Let* $L \equiv \Lambda - S$ *be a section complemented Wilcox lattice satisfying* (†). *A non-zero ideal* J *of* L *is standard if and only if* J *dominates* S *and the ideal* $J \cup S$ *of* Λ *is standard.*

PROOF. The "if" part follows from Lemma 1. Next, let J be a non-zero standard ideal of L, and let $0 \neq x \in J$. It follows from Lemma 3 that $u < x \cup u \in \mathbf{I}(J) = J$ for every $u \in S$. Hence, J dominates S, and $J \cup S$ is an ideal of Λ. It suffices to prove that $J \cup S$ is a p-ideal. If $0 \neq a \in L$ is subperspective to $\lambda \in J \cup S$ in Λ, then there exists $\mu \in \Lambda$ such that $a \leqq \lambda \cup \mu$ and $a \cap \mu = 0$. If $\mu \in S$, then $\lambda \in J$, since otherwise $a \leqq \lambda \cup \mu \in S$, a contradiction. Since $x \cup \mu \in \mathbf{I}(J) = J$ and since $a \leqq \lambda \vee (x \cup \mu)$, we have $a \in J$. If $\mu \in L$ and $\lambda \in J$, then a is subperspective to λ in L and hence $a \in J$. If $\mu \in L$ and $\lambda \in S$, then a is subperspective to $x \cup \lambda \in J$ in L and hence $a \in J$. Therefore, $J \cup S$ is a p-ideal of Λ.

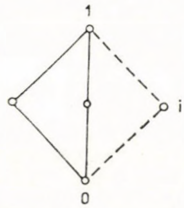

Fig. 2

Fig. 2 shows that if we omit the assumption (†), then L may have a non-zero standard ideal which does not dominate S.

COROLLARY. *The lattice* $L \equiv \Lambda - S$ *given in Theorem* 2 *is simple if and only if there is no standard ideal* Γ *of* Λ *such that* $S \cup \{0\} \subsetneqq \Gamma \neq \Lambda$.

4. Atomistic Wilcox lattices. If a Wilcox lattice $L \equiv \Lambda - S$ is atomistic, then it is an AC-lattice by [4], (20.1), and hence $F(L)$ is an ideal of L. If a covers b in L then a covers b in Λ also, and hence an element p of L is an atom if and only if p is an atom of Λ ([4], (20.2)). Moreover, it follows from [4], (20.3) that Λ is atomistic.

LEMMA 4. *If a Wilcox lattice* $L \equiv \Lambda - S$ *is atomistic then it is section complemented.*

PROOF. Let $0 < a < b$ in L. Then, there exists a non-zero element λ of Λ such that $a \cup \lambda = b$ and $a \cap \lambda = 0$. If $\lambda \in L$, then evidently λ is a complement of a in $L[0, b]$. If $\lambda \in S$, then, taking an atom t of Λ with $t \leqq \lambda$, a complement λ_1 of t in $\Lambda[0, \lambda]$ is covered by λ. Then, $a \cup \lambda_1$ belongs to L and it is covered by $a \cup \lambda = b$, since $\Lambda[0, \lambda]$ is isomorphic to $\Lambda[a, b]$ by the mapping: $\mu \to a \cup \mu$. Since L is atomistic, there exists an atom p of L such that $p \not\leqq a \cup \lambda_1$ and $p \vee (a \cup \lambda_1) = b$. If we put $c = p \cup \lambda_1$, then $c \in L$ and $c \vee a = b$. Moreover, since Λ is modular, we have

$$c \wedge a \leqq c \cap a = (\lambda_1 \cup p) \cap (a \cup \lambda_1) \cap a = [\{\lambda_1 \cup \{p \cap (a \cup \lambda_1)\}] \cap a = \lambda_1 \cap a \leqq \lambda \cap a = 0.$$

Hence, c is a complement of a in $L[0, b]$.

THEOREM 3. *Let* $L \equiv \Lambda - S$ *be an atomistic Wilcox lattice of infinite length. The following statements are equivalent.*

(α) $F(L)$ *is standard.*

(β) $F(L)$ *dominates* S.

(γ) $S \subset F(\Lambda)$.

PROOF. We remark that $F(L) = F(\Lambda) \cap L$ (see [4], (20.4)). The equivalence of (β) and (γ) is evident, and (γ) implies that $F(L) \cup S = F(\Lambda)$. Since Λ is modular, $F(\Lambda)$ is a p-ideal by [4], (11.9), and hence it is standard. Hence, the equivalence of (α) and (β) follows from Theorem 2 and Lemma 4.

The equivalence of (α) and (γ) and Theorem 4 (iii) below were proved by STERN [6] using a different method.

A matroid lattice of length $\geqq 4$ is called an *affine matroid lattice* when it is weakly modular and satisfies Euclid's weak parallel axiom ([4], (18.3)). It follows from [4], (19.14) and (20.15) that a non-modular affine matroid lattice L is an atomistic Wilcox lattice with imaginary unit i and that L satisfies Euclid's strong parallel axiom if and only if i is a dual-atom (= hyperplane) of the modular extension of L.

THEOREM 4. *Let* L *be a non-modular affine matroid lattice.*

(i) L *is simple if and only if there exist* $a_1, \ldots, a_n \in L$ *such that* $a_1 \vee \ldots \vee a_n = 1$ *and each* a_k *is subprojective to an element* $x_k \cup i$ *with* $0 \neq x_k \in F(L)$.

(ii) *If* i *is dual-finite in the modular extension* Λ (especially, if L satisfies Euclid's strong parallel axiom), *then* L *is simple.*

(iii) *The ideal* $F(L)$ *is standard if and only if* i *is finite in* Λ.

PROOF. (i) is a consequence of Corollary 2 of Theorem 1 and Lemma 3, and (ii) follows from (i). (iii) follows from Theorem 3.

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DEPARTMENT OF MATHEMATICS
EHIME UNIVERSITY
BUNKYO-CHO, MATSUYAMA-SHI
JAPAN

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UNIFORM RATIONAL APPROXIMATION OF THE CLASS V_r AND ITS APPLICATIONS

By

V. A. POPOV (Sofia)

We denote by V_r the class of functions f, defined in the interval $[a, b]$, which have an absolutely continuous $(r-1)$-th derivative $f^{(r-1)}$ in the interval $[a, b]$, which is an integral of a function $f^{(r)}$ of bounded variation in the interval $[a, b]$. There exist many works on the uniform rational approximation of this class of functions. First P. TURÁN and P. SZÜSZ [1] (see also P. TURÁN [2]), on the basis of the famous result of NEWMAN [3], have shown that the functions of the class V_1 (more exactly the convex functions of the class Lip 1) can be uniformly approximated by means of rational functions of degree n better than by means of algebraic polynomials of the same degree. They showed that

$$\sup_{V_b^a f' \leqq 1} R_n(f; [a, b]) = O(\ln^4 n/n^2)$$

where

$$R_n(f; [a, b]) = \inf_{q \in R_n} \|f-q\|_{C[a,b]}, \quad \|f-q\|_{C[a,b]} = \sup_{x \in [a,b]} |f(x)-q(x)|,$$

R_n denotes the set of all rational functions of degree at most n, i.e. $q \in R_n$ if $q(x)= =(a_k x^k + \dots + a_0)/(b_m x^m + \dots + b_0)$, $k, m \leqq n$, and $V_a^b \varphi$ denotes the variation of the function φ in the interval $[a, b]$.

In [4] G. FREUD obtained that

$$\sup_{V_0^1 f^{(r)} \leqq 1} R_n(f; [0, 1]) = O(\ln^2 n/n^{r+1}), \quad r \geqq 1.$$

In [5], using a modification of the method of Freud we obtained the following result: for every natural number $k>0$ the following estimate is valid:

$$R_n(f;[0, 1]) \leqq c_{k,r} \frac{(V_0^1 f^{(r)}) \overbrace{\ln \dots \ln}^{k} n}{n^{r+1}}$$

where $c_{k,r}$ is a constant, depending only on k and r.

Let us mention that the uniform rational approximation of some classes of functions, which are near the classes V_r, are considered in the papers of BULANOV [6], [7], ABDULGAPAROV [8], HATAMOV [9]. For the rational approximation of some other classes of functions see A. A. GONČAR [10], G. FREUD—J. SZABADOS [11], J. SZABADOS [12].

The main aim of this work is to prove

THEOREM 1. *The following estimate is valid:*

$$\sup_{V_a^b f^{(r)} \leqq M} R_n(f; [a, b]) \leqq c(r) \frac{(b-a)^r M}{n^{r+1}}, \quad r \geqq 1$$

where $c(r)$ *is a constant, depending only on* r.

One can easily see that

$$c'(r) \frac{(b-a)^r M}{n^{r+1}} \leqq \sup_{V_a^b f^{(r)} \leqq 1} R_n(f; [a, b]),$$

where $c'(r)$ is a constant, depending only on r (see also G. FREUD [13]). Therefore the order $O(n^{-r-1})$ in Theorem 1 is exact.

In § 1 we give some lemmas. In § 2 we give the proof of Theorem 1 and in § 3 we give some applications.

The results are announced in [14].

§ 1. Some lemmas

LEMMA 1 (see [5], pp. 61). *Let*

$$\Phi_n^r = \sup_{V_1^0 f^{(r)} \leqq 1} R_n(f; [0, 1]).$$

Then

$$\sup_{V_a^b f^{(r)} \leqq M} R_n(f; [a, b]) \leqq (b-a)^r M \Phi_n^r.$$

LEMMA 2. *Let*

$$\Phi_{n,A}^r = \sup_{\substack{V_0^1 f^{(r)} \leqq 1 \\ f^{(s)}(0)=0,\, s=0, \dots, r}} \inf_{\substack{q \in R_n \\ \|q\|_{C(-\infty, \infty)} \leqq A}} \|f-q\|_{C[0,1]}.$$

Then for every $\varepsilon > 0$ *we have:*

$$\sup_{\substack{V_a^b f^{(r)} \leqq M \\ \|f^{(s)}\|_{C[a,b]} \leqq 1,\, s=0, \dots, r}} \inf_{\substack{q \in R_{n+2r} \\ \|q\|_{C(-\infty,\infty)} \leqq M(b-a)^r A + \varepsilon^{-1/2}}} \|f-q\|_{C[a,b]} \leqq$$

$$\leqq (b-a)^r M \Phi_{n,A}^r + \varepsilon \left(\sum_{s=0}^r \frac{(b-a)^s}{s!} \right)^3 \quad \left(\|q\|_{C(-\infty,\infty)} = \sup_x |q(x)|\right).$$

PROOF. Let $V_a^b f^{(r)} \leqq M$, $\|f^{(s)}\|_{C[a,b]} \leqq 1$,[1] $s=0, \dots, r$. Then for

$$g(x) = \frac{1}{M(b-a)^r} f(a+(b-a)x) - \sum_{s=0}^r \frac{f^{(s)}(a) x^s}{s!\, M(b-a)^{r-s}}$$

[1] As usual, $C[a, b]$ denotes the set of all functions continuous in the interval $[a, b]$.

we have $V_0^1 g^{(r)} \leqq 1$, $g^{(s)}=0$, $s=0, \dots, r$, and therefore there exists a rational function $q \in R_n$ such that

$$\|g-q\|_{C[0,1]} \leqq \Phi_{n,A}^r; \quad \|q_{-\infty}\|_{C(,\infty)} \leqq A,$$

i.e.

$$\max_{x\in[0,1]}\left|\frac{1}{M(b-a)^r} f(a+(b-a)x) - \sum_{s=0}^{r} \frac{f^{(s)}(a)(b-a)^s}{s!} x^s\right) - q(x)\right| =$$

$$= \frac{1}{M(b-a)^r} \max_{x\in[a,b]}\left|f(x) - \sum_{s=0}^{r} \frac{f^{(s)}(a)(x-a)^s}{s!} - M(b-a)^r q(x)\right| \leqq \Phi_{n,A}^r.$$

Let us set:

$$p(x) = \sum_{s=0}^{r} \frac{f^{(s)}(a)(x-a)^s}{s!}, \quad \theta(x) = p(x)/(1+\varepsilon p^2(x)).$$

Obviously we have: $\theta \in R_{2r}$, $\|\theta\|_{C(-\infty,\infty)} \leqq \varepsilon^{-1/2}$,

$$\|p(x)\|_{C[a,b]} \leqq \sum_{s=0}^{r} \frac{(b-a)^s}{s!}; \quad \|\theta - p\|_{C[a,b]} \leqq \varepsilon \left(\sum_{s=0}^{r} \frac{(b-a)^s}{s!}\right)^3;$$

$$\theta + M(b-a)^r q \in R_{n+2r},$$

therefore

$$\|f-(\theta+M(b-a)^r q)\|_{C[a,b]} \leqq M(b-a)^r \Phi_{n,A}^r + \varepsilon\left(\sum_{s=0}^{s} \frac{(b-a)^s}{s!}\right)^3,$$

$$\|\theta + M(b-a)^r q\|_{C(-\infty,\infty)} \leqq M(b-a)^r A + \varepsilon^{-1/2},$$

which proves the lemma.

LEMMA 3. *Let* $V_0^1 \varphi \leqq 1$, $\varphi \in C[0,1]$, *and let* $m \geqq 1$ *and* $r \geqq 0$ *be integers. Then there exist* $m+1$ *points* x_i, $i=0, \dots, m$, $0=x_0<x_1<\dots<x_m=1$, *such that*

$$(V_{x_{i-1}}^{x_i} \varphi)(x_i - x_{i-1})^r \leqq 1/m^{r+1}, \quad i=1, \dots, m.$$

PROOF. Let us set $\varphi(x)=\varphi(1)$ for $x \geqq 1$ and construct (as far as it is possible) the points x_i, $i=0, \dots, s$, $x_0=0$, such that $(V_{x_{i-1}}^{x_i} \varphi)(x_i-x_{i-1})^r = m^{-r-1}$. Denote:

$$a_i = V_{x_{i-1}}^{x_i} \varphi, \quad b_i = x_i - x_{i-1}, \quad i=1, \dots, s.$$

Then $a_i b_i^r = m^{-r-1}$, $i=1, \dots, s$, $\sum_{i=1}^{s} a_i \leqq V_0^1 \varphi \leqq 1$. If $s<m$, then the lemma is proved. Let $s \geqq m$ and let us estimate x_m. We have:

$$\sum_{i=1}^{m} b_i = x_m, \quad b_i = (a_i^{-1/r} m^{-1-1/r}), \quad i=1, \dots, s; \quad \sum_{i=1}^{m} a_i = A \leqq 1.$$

Let us find $\min_{a_i} \sum_{i=1}^{m} (a_i^{-1/r} m^{-1-1/r})$ if $\sum_{i=1}^{m} a_i = A \leqq 1$. We have $a_i = \text{const} = A/m$,

$i=1, \dots, m$ and

$$x_m \geqq \min_{a_i,\ \sum_{i=1}^m a_i = A \leqq 1} \sum_{i=1}^m (a_i^{-1/r} m^{-1-1/r}) = A^{-1/r} \geqq 1,$$

since $A \leqq 1$. The lemma is proved.

LEMMA 4. *There exists a constant $d(r)$, depending only on r, such that for every natural number $n>1$ there exists a rational function $\sigma_n \in R_N$, $N \leqq d(r) \ln^2 n$, for which*

$$|\sigma_n(x)| \leqq n^{-2r-4} \quad \textit{for} \quad -1 \leqq x \leqq -n^{-r-2};$$

$0 \leqq \sigma_n(x) \leqq 1$ *for* $|x| \leqq 1$; $\sigma_n(x)$ *is a monotone increasing function in the interval* $|x| \leqq n^{-r-2}$;

$$|1-\sigma_n(x)| \leqq n^{-2r-4} \quad \textit{for} \quad n^{-r-2} \leqq x \leqq 1;$$

$$\|\sigma_n\|_{C(-\infty,\infty)} \leqq \sqrt{2}\, n^{r+2}.$$

PROOF. From the lemma of GONČAR [10] it follows that there exists a constant $d'(r)$ (depending only on r) such that for every integer $n>1$, there exists a rational function $\bar{\sigma} \in R_{\bar{N}}, \bar{N} \leqq d'(r) \ln^2 n$, such that

$$|\bar{\sigma}_n(x)| \leqq \frac{1}{2} n^{-2r-4} \quad \text{for} \quad -1 \leqq x \leqq -n^{-r-2};$$

$0 \leqq \bar{\sigma}_n(x) \leqq 1$ for $|x| \leqq 1$, $\bar{\sigma}_n(x)$ is a monotone increasing function in the interval $|x| \leqq n^{-r-2}$,

$$|1-\bar{\sigma}_n(x)| \leqq 1/(2n^{2r+4}) \quad \text{for} \quad n^{-r-2} \leqq x \leqq 1.$$

Then for $\sigma_n(x) = \bar{\sigma}_n(x)/(1+n^{-2r-4}\bar{\sigma}_n^2(x)/2)$ we have:

$$\sigma_n \in R_N, \quad N \leqq 2d'(r) \ln^2 n, \quad 0 \leqq \sigma_n(x) \leqq 1 \quad \text{for} \quad |x| \leqq 1,$$

$|\sigma_n(x)| \leqq n^{-2r-4}$ for $-1 \leqq x \leqq -n^{-r-2}$ and $|1-\sigma_n(x)| \leqq n^{-2r-4}$ for $n^{-r-2} \leqq x \leqq 1$.

On the other hand

$$\sigma_n'(x) = \bar{\sigma}_n'(x) \left(1 - \frac{1}{2} n^{-2r-4} \bar{\sigma}_n^2(x)\right) \Big/ \left(1 + \frac{1}{2} n^{-2r-4} \bar{\sigma}_n^2(x)\right)^2$$

therefore $\sigma_n(x)$ is monotone increasing in $|x| \leqq n^{-r-2}$. Finally $\|\sigma_n\|_{C(-\infty,\infty)} \leqq \leqq \sqrt{2}\, n^{r+2}$. The lemma is proved.

LEMMA 5 (the Main Lemma). *There exists a natural number $n_0(r)$, depending only on r, such that for $n \geqq n_0(r)$ we have*

$$\Phi^r_{n, n^{2r+4}} \leqq \varphi(k) n^{-r-1} (1+3/\ln n)^{r+1}, \quad r \geqq 1,$$

if $\Phi^r_{k, k^{2r+4}} \leqq \varphi(k) k^{-r-1}$, $\varphi(k) \geqq 1$, *for* $k = [d(r) \ln^3 n]$, *where* $d(r)$ *is the constant from Lemma* 3.

(Here and in what follows we use the notations of Lemmas 1—3).

PROOF. Let $V_0^1 f^{(r)} \leqq 1$, $f^{(s)}(0)=0$, $s=0, \dots, r$. We may assume that $f^{(r)} \in \in C[0, 1]$ since the set $A = \{f\colon V_0^1 f^{(r)} \leqq 1,\ f^{(s)}=0,\ s=0, \dots, r,\ f^{(r)} \in C[0, 1]\}$ is a dense subset (in $C[0,1]$) in the set $B=\{f\colon V_0^1 f^{(r)}\leqq 1,\ f^{(s)}(0)=0,\ s=0,\dots,r\}$.[2] Applying Lemma 3 it follows that there exist $m+1$ points x_i, $i=0,\dots,m$, $0=x_0<x_1<\dots<x_m=1$, $m\geqq 1$, such that

$$(V_{x_{i-1}}^{x_i} f^{(r)})(x_i-x_{i-1})^r \leqq m^{-r-1}, \quad i=1,\dots,m. \tag{1}$$

Let us set $f(x)=f(0)=0$ for $x\leqq 0$ and $f(x)=f(1)$ for $x\geqq 1$. Denote $\Delta_i=[x_{i-1}-n^{-r-2}, x_i+n^{-r-2}]$ and let T_i be the linear transformation of the interval Δ_i into the interval $[x_{i-1}, x_i]$. Then

$$|T_i x - x| \leqq n^{-r-2} \quad \text{for} \quad x\in\Delta_i. \tag{2}$$

Let r_i, $i=1,\dots,m$, be rational functions of degree $\leqq k+2r$ such that

$$\begin{cases} \|f-r_i\|_{C[x_{i-1},x_i]} \leqq m^{-r-1}\Phi^r_{k,k^{2r+4}}+n^{-r-2}\left(\sum\limits_{s=0}^{r}\dfrac{1}{s!}\right)^3, \\ \|r_i\|_{C(-\infty,\infty)} \leqq k^{2r+4}m^{-r-1}+n^{1+r/2}, \quad i=1,\dots,m. \end{cases} \tag{3}$$

Such rational functions exist by (1) and Lemma 2. From (2) and (3) we obtain

$$\|f(x)-r_i(T_i x)\|_{C(\Delta_i)} \leqq \max_{x\in\Delta_i}|f(x)-f(T_i x)|+\max_{x\in\Delta_i}|f(T_i x)-r_i(T_i x)| \leqq \tag{4}$$

$$\leqq n^{-r-2}+\|f-r_i\|_{C[x_{i-1},x_i]} \leqq m^{-r-1}\Phi^r_{k,k^{2r+4}}+\left(1+\left(\sum_{s=0}^{r}\frac{1}{s!}\right)^3\right)n^{-r-2},$$

(since $f^{(s)}(0)=0$, $s=0,\dots,r$, $V_0^1 f^{(r)}\leqq 1$, we have $|f(x)-f(y)|\leqq|x-y|$), where

$$\|r_i\|_{C(\infty,\infty)} \leqq k^{2r+4}m^{-r-1}+n^{1+r/2}, \quad i=1,\dots,m. \tag{5}$$

Let us consider the rational function $q\in R_N$, where $N\leqq m(k+2r+d(r)\ln^2 n)$:

$$q(x) = r_1(T_1 x)+\sum_{i=1}^{m-1}\sigma_n(x-x_i)\big(r_{i+1}(T_{i+1}x)-r_i(T_i x)\big)$$

where σ_n is the function from Lemma 4.

Let $x\in[0,1]$. Denote $i_0=\max\{i\colon x_i<x-n^{-r-2}\}$, $i_N=\max\{i\colon x_i<x+n^{-r-2}\}$, $d_{i_0+1}=1-\sigma_n(x-x_{i_0+1})$, $d_i=\sigma_n(x-x_{i-1})-\sigma_n(x-x_i)$, $i=i_0+2,\dots,i_N$, $d_{i_N+1}=\sigma_n(x-x_{i_N})$. Then $x\in\Delta_i$, $i=i_0+1,\dots,i_N+1$ and from Lemma 4 (the monotonicity

[2] If $f\in B$ we can take for example

$$f_h^{[r]}(x)=\frac{1}{h}\int_0^h f^{(r)}(x-t)\,dt \quad (f^{(r)}(x)=f^{(r)}(0)=0 \quad \text{for} \quad x\leqq 0).$$

Then $V_0^1 f_h^{[r]}\leqq 1$, $f_h^{[r]}\in C[0,1]$, moreover for

$$f_h(x)=\int_0^x\int_0^{t_1}\dots\int_0^{t_{r-1}} f_h^{[r]}(t_r)\,dt_r\dots dt_1$$

we have $f_h^{(s)}(0)=0$, $s=0,\dots,r$ and

$$\|f_h-f\|_{C[0,1]}\leqq\|f_h^{[r]}-f^{(r)}\|_{L[0,1]}\to 0 \quad \text{as} \quad h\to 0.$$

of the function σ_n in $|x|\leqq n^{-r-2}$) it follows that $d_i\geqq 0$, $\sum_{i=i_0+1}^{i_N+1} d_i=1$. Using (4), (5) and Lemma 4 we obtain:

(6)

$$|f(x)-q(x)| \leqq |r_1(T_1x)+\sum_{i=1}^{i_0}\sigma_n(x-x_i)\big(r_{i+1}(T_{i+1}x)-r_i(T_ix)\big)-r_{i_0+1}(T_{i_0+1}x)|+$$

$$+\left|r_{i_0+1}(T_{i_0+1}x)+\sum_{i=i_0+1}^{i_N}\sigma_n(x-x_i)\big(r_{i+1}(T_{i+1}x)-r_i(T_ix)\big)-f(x)\right|+$$

$$+\left|\sum_{i=i_N+1}^{m-1}\sigma_n(x-x_i)\big(r_{i+1}(T_{i+1}x)-r_i(T_ix)\big)\right| \leqq$$

$$\leqq 2mn^{-2r-4}\max_{1\leqq i\leqq m}\|r_i\|_{C(-\infty,\infty)}+\sum_{i=i_0+1}^{i_N+1}\alpha_i\|f(x)-r_i(T_ix)\|_{C(\Delta_i)}\leqq$$

$$\leqq 2mn^{-2r-4}\left(\frac{k^{2r+4}}{m^{r+1}}+n^{1+r/2}\right)+m^{-r-1}\Phi^r_{k,k^{2r+4}}+n^{-r-2}\left(1+\left(\sum_{s=0}^{r}\frac{1}{s!}\right)^3\right),$$

$$\|q\|_{C(-\infty,\infty)}\leqq 2^{3/2}mn^{r+2}(k^{2r+4}m^{-r-1}+n^{1+r/2}).$$

Let us set: $k=[d(r)\ln^3 n]$, $m=\big[n/\big(d(r)\,(\ln^3 n+\ln^2 n)+2r\big)\big]$. Then $q\in R_n$ and

$$mk\geqq n\left(\frac{1}{1+1/\ln n+2r/\big(d(r)\ln^3 n\big)}-\frac{d(r)\ln^3 n}{n}-\frac{1}{d(r)(\ln^3 n+\ln n)+2r}\right).$$

Now let $n_0(r)$ be such that for $n\geqq n_0(r)$ we have:

1) $[d(r)\ln^3 n]\geqq 1$;

2) $\big[n/\big(d(r)(\ln^3 n+\ln^2 n)+2r\big)\big]\geqq 1$;

3) $2\sqrt{2}mn^{r+2}(k^{2r+4}m^{-r-1}+n^{1+r/2})\leqq n^{2r+4}$;

4) $2mn^{-2r-4}(k^{2r+4}m^{-r-1}+n^{1+r/2})+(1+e^3)n^{-r-2}\leqq 30n^{-r-2}$;

5) $\left(\frac{1}{1+1/\ln n+2r/\big(d(r)\ln^3 n\big)}-\frac{d(r)\ln^3 n}{n}-\frac{1}{d(r)(\ln^3 n+\ln^2 n)+2r}\right)\geqq\left(1+\frac{2}{\ln n}\right)^{-1}$;

6) $\big(30n^{-1}+(1+2/\ln n)^{r+1}\big)\leqq(1+3/\ln n)^{r+1}$.

Then if $\Phi^r_{k,k^{2r+4}}\leqq\varphi(k)k^{-r-1}$, $\varphi(k)\geqq 1$, we have

$$\|f-q\|_{C[0,1]}\leqq\varphi(k)n^{-r-1}(1+3/\ln n)^{r+1},\quad \|q\|_{C(-\infty,\infty)}\leqq n^{2r+4},\quad q\in R_n,$$

which proves the lemma.

REMARK. The method proving Lemma 5 is an improvement of the methods from [5] and [15].

§ 2. Proof of Theorem 1

In what follows we shall denote the constants, which may depend only on r, with $c_1(r), c_2(r), \ldots$.

In view of Lemma 1 to prove Theorem 1 it is sufficient to prove that $\Phi^r_{n,n^{2r+4}} \leqq c_1(r) n^{-r-1}$, since

$$\Phi^r_n \leqq \Phi_{n-r,\,(n-r)^{2r+4}} \leqq \frac{c_1(r)}{(n-r)^{r+1}} \leqq \frac{c_2(r)}{n^{r+1}} \quad (n > r).$$

Let $N_0(r)$ be such that a) $N_0(r) \geqq n_0(r)$, where $n_0(r)$ is from Lemma 5, b) for $n \geqq N_0(r)$ we have

$$2 \leqq [d(r) \ln^3 n]^2 \leqq n,$$

where $d(r)$ is the constant from Lemma 4. Obviously there exists a constant $c_3(r) \geqq 1$ such that for $n \leqq N_0(r)$ we have

$$\Phi^r_{n,n^{2r+4}} \leqq \frac{c_3(r)}{n^{r+1}}.$$

Denote $y(x) = [d(r) \ln^3 x]$, $y^s(x) = y(y^{s-1}(x))$. Let $n > N_0(r)$. Then there exists s_0 depending on n such that $y^{s_0}(n) \leqq N_0(r)$ and $y^{s_0-1}(n) > N_0(r)$. Using Lemma 5, we obtain successively:

$$\Phi^r_{y^{s_0-1}(n),\,(y^{s_0-1}(n))^{2r+4}} \leqq \frac{c_3(r)}{(y^{s_0-1}(n))^{r+1}} \left(1 + \frac{3}{\ln y^{s_0-1}(n)}\right)^{r+1},$$

$$\Phi^r_{y^{s_0-2}(n),\,(y^{s_0-2}(n))^{2r+4}} \leqq \frac{c_3(r)}{(y^{s_0-2}(n))^{r+1}} \left(1 + \frac{3}{\ln y^{s_0-1}(n)}\right)^{r+1} \left(1 + \frac{3}{\ln y^{s_0-2}(n)}\right)^{r+1},$$

$$\Phi^r_{n,n^{2r+4}} \leqq \frac{c_3(r)}{n^{r+1}} \prod_{i=0}^{s_0-1} \left(1 + \frac{3}{\ln y^i(n)}\right)^{r+1}; \quad y^0(n) = n.$$

Since $y^{s_0-1}(n) > N_0(r)$, we have $(y^i(n))^2 \leqq y^{i-1}(n)$, i.e. $\ln y^i(n)/\ln y^{i-1}(n) \leqq 1/2$. Therefore there exists a constant $c_4(r)$ (independent of n and s_0) such that

$$\prod_{i=0}^{s_0-1} (1 + 3/\ln y^i(n))^{r+1} \leqq c_4(r)$$

which proves Theorem 1.

§ 3. Some applications

THEOREM 2. *Let the function f have an absolutely continuous $(r-1)$-th derivative $f^{(r-1)}$ in the interval* $[0, 1]$. *Then*

$$R_n(f; [0, 1]) = O(\omega_{r+1}(f; m_n^{-1}) + m_n^{-r}),$$

where $n^{-1} m_n \to \infty$ *as* $n \to \infty$, $\omega_k(f; \delta) = \sup_{x, |h| \leqq \delta} |\Delta^k_h f(x)|$,

$$\Delta^k_h f(x) = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k+m} f(x+mh).$$

PROOF. We shall use the method from [16] (see also [17]). In [16], pp. 166—169 it is proved that for every function f with absolutely continuous $(r-1)$-th derivative $f^{(r-1)}$ in the interval $[0, 1]$ (i.e. $f^{(r)}\in L[0, 1]$) and for every $h>0$ there exists a function $\hat{f}_h\in C^r[0, 1]$ such that

$$\|f-\hat{f}_h\|_{C[0,1]} \leqq c_5(r)\omega_{r+1}(f; h),$$

$$V_0^1 \hat{f}_h^{(r)} \leqq c_6(r)\{h^{-1}\omega_1(f^{(r)}; h)_L + h^{-r}\omega_{r+1}(f; h)\}$$

where

$$\omega_1(g; h)_L = \sup_{0<\delta\leqq h} \int_0^{1-\delta} |g(x+\delta)-g(x)|\,dx$$

is the integral modulus of continuity of the function g.

Using Theorem 1 we obtain:

$$R_n(f; [0, 1]) \leqq \|f-\hat{f}_h\|_{C[0,1]} + R_n(\hat{f}_h; [0, 1]) \leqq \tag{7}$$

$$\leqq c_7(r)\{\omega_{r+1}(f; h) + n^{-r-1}[h^{-1}\omega_1(f^{(r)}; h)_L + h^{-r}\omega_{r+1}(f; h)]\}.$$

Since $f^{(r)}\in L[0, 1]$, $\omega_1(f^{(r)}; h)\to 0$ as $h\to 0$. Therefore there exists a sequence $\{m_n\}_1^\infty$, $n^{-1}m_n\to\infty$ $(n\to\infty)$, such that

$$R_n(f; [0, 1]) = O\big(\omega_{r+1}(f; m_n^{-1}) + m_n^{-r}\big)$$

(we can set for example $h=h(n)=\sqrt[r+1]{\omega_1(f^{(r)}; 1/n)_L}/n$ in (7)).

The theorem is proved.

COROLLARY 1. *If the function f has an absolutely continuous $(r-1)$-th derivative $f^{(r-1)}$ in the interval $[0, 1]$, which satisfies the Zygmund condition $\omega_2(f^{(r-1)}; \delta) = O(\delta)$ in the interval $[0, 1]$, then*

$$R_n(f; [0, 1]) = o(n^{-r}).$$

COROLLARY 2. (The Newman's conjecture). *If* $f\in \mathrm{Lip}\,1$, *i.e.* $\omega_1(f; \delta)=O(\delta)$, *then* $R_n(f; [0, 1])=o(n^{-1})$.

REMARK 1. G. Freud remarked that from the validity of Theorem 1 follows the Newman's conjecture.

REMARK 2. From the results of J. SZABADOS [18] it follows, that Theorem 2 (and Corollaries 1 and 2) cannot be improved.

We shall give also some applications of Theorem 1 to the approximation of functions with respect to the Hausdorff distance between functions (for the definition and main properties of the Hausdorff distance between functions see [19], [20]) and to the local approximation of functions.

We shall denote by $r(f, g)$ the Hausdorff distance between the bounded functions f and g on the interval $[0, 1]$. This may be considered as a generalization of the uniform distance (see [20]) and is useful for the approximation of discontinuous functions [19].

The best Hausdorff approximation of the function f bounded on the interval $[0, 1]$ by means of rational functions of degree n is given by

$$R_n(f)_r = \inf_{q \in R_n} r(f, q).$$

In [21] we proved that if the function f is of bounded variation on the interval $[0, 1]$ then

$$R_n(f)_r = O(\ln \ln n/n).$$

Using Theorem 1 we shall prove that in this case we have $R_n(f)_r = O(n^{-1})$.

Denote by $V(m)$ the class of all step functions φ with $V_0^1 \varphi \leqq 1$, which have jumps only at the points $x_i = i/m$, $i = 1, \ldots, m-1$.

LEMMA 6. *Let* $V_0^1 f \leqq 1$. *Then there exists* $g \in V(2m)$ *such that* $r(f, g) \leqq 1/m$.

PROOF. We may suppose that $f \in C[0, 1]$ since the class of these functions is dense in the class $V_0^1 f \leqq 1$ with respect to the Hausdorff distance. Denote

$$y_i = i/m, \quad i = 0, \ldots, m, \quad x_i = i/2m, \quad i = 0, \ldots, 2m,$$

$$M_i = \max_{x \in [y_{i-1}, y_i]} f(x), \quad N_i = \min_{x \in [y_{i-1}, y_i]} f(x), \quad i = 1, \ldots, m.$$

If there exist $x' \in [y_{i-1}, y_i]$, $x'' \in [y_{i-1}, y_i]$, such that $f(x') = M_i$, $f(x'') = N_i$, $x' < x''$, we set

$$g(x) = M_i \quad \text{for} \quad x \in [x_{2i-2}, x_{2i-1}), \quad g(x) = N_i \quad \text{for} \quad x \in [x_{2i-1}, x_{2i})$$

(if $i = m$, then $g(x) = N_m$ for $x \in [x_{2m-1}, x_{2m}]$), in the converse case the roles of M_i and N_i should be interchanged. Obviously $V_0^1 g \leqq 1$ and from the definition of the Hausdorff distance it follows that $r(f, g) \leqq 1/m$.

THEOREM 3. *We have*

$$\sup_{V_0^1 f \leqq 1} R_n(f)_r \leqq c/n$$

where c *is an absolute constant.*

PROOF. Let $V_0^1 f \leqq 1$. Applying Lemma 6, there exists $g \in V(2n)$ such that $r(f, g) \leqq 1/n$. Let us set $g(x) = g(1)$ for $x \geqq 1$, $h = 1/4n$. Denote

$$g_h(x) = \frac{1}{h} \int_0^h g(x+t)\,dt.$$

It is easy to see that

$$V_0^1 g_h' = \frac{1}{h} V_0^1 \big(g(x+h) - g(x)\big) \leqq \frac{2}{h}, \quad r(g, g_h) \leqq h = 1/4n.$$

Using Theorem 1 we obtain:

$$R_n(f)_r \leqq r(f, g) + r(g, g_h) + R_n(g_h) \leqq \frac{5}{4n} + \frac{c(1)2}{hn^2} = c/n.$$

The theorem is proved.

The problem of local approximation of functions was set in [22] as follows: Let $\{\Phi_n\}_0^\infty$, be a given series of functions and let A be a given class of functions.

What should be the maximal speed of decreasing of the function $\varphi(n)$ so that for every function $f\in A$ there exists a function $\Psi_n\in\Phi_n$ such that for every $x\in[0, 1]$ we have

$$|f(x)-\psi_n(x)| \leqq \omega(f, x; \varphi(n))+O(n^{-1})$$

where $\omega(f, x; \delta)$ is the local modulus of continuity of the function f in the point $x\in[0, 1]$:

$$\omega(f,x; \delta) = \sup_{|y-x|\leqq\delta} |f(x)-f(y)|.$$

In [22] we proved that if Φ_n is the set of all algebraic polynomials of degree n then the order of the function $\varphi(n)$ is $\ln n/n$ if A is the set of all continuous functions in $[a, b]$, or the set of all monotone functions with a variation $\leqq 1$. In [21] we proved that if $\Phi_n=R_n$, then the order of the function $\varphi(n)$ is $\leqq \ln\ln n/n$ for the class of functions with bounded variation. Using Theorem 3 it is easy to see that in this case the order of the function $\varphi(n)$ is $O(n^{-1})$.

LEMMA 7 (see [20], [22]). *Let f be a continuous function on the interval* $[0, 1]$. *Then for every $x\in[0, 1]$ and $g\in C[0, 1]$ we have*

$$|f(x)-g(x)| \leqq r(f, g)+\omega(f, x; r(f, g)).$$

Using Lemma 7 and Theorem 3 we obtain immidiately

THEOREM 4. *Let $f\in C[0, 1]$ and $V_0^1 f\leqq V$. Then for every natural number $n>0$ there exists $q_n\in R_n$ such that*

$$|f(x)-q_n(x)| \leqq \omega\left(f, x; \frac{c(V)}{n}\right)+\frac{c(V)}{n}, \quad x\in[0, 1],$$

where $c(V)$ is a constant, depending only on V.

It is easy to see that the order $O(n^{-1})$ in Theorem 3 is exact. We do not know if the order $O(n^{-1})$ in Theorem 4 is exact.

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INSTITUTE OF MATHEMATICS AND MECHANICS
OF BULGARIAN ACADEMY OF SCIENCES
SOFIA, BULGARIA

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ON A DOWKER-TYPE THEOREM OF EGGLESTON

By
G. FEJES TÓTH (Budapest)

§1. Introduction

Let us recall the fundamental theorems of DOWKER [2]:

If $i(n)$ is the area of an n-gon of maximal area inscribed in an arbitrarily given convex domain then

$$i(n-1)+i(n+1) \leqq 2i(n) \qquad (n = 4, 5, \ldots).$$

If $c(n)$ is the area of an n-gon of minimal area circumscribed about an arbitrarily given convex domain then

$$c(n-1)+c(n+1) \geqq 2c(n) \qquad (n = 4, 5, \ldots).$$

By the proof of these theorems Dowker was led to two further theorems:

Among the $2m$-gons of maximal area inscribed in a centro-symmetric convex domain there is one which has central symmetry $(m=2, 3, \ldots)$.

Among the $2m$-gons of minimal area circumscribed about a centro-symmetric convex domain there is one which has central symmetry $(m=2, 3, \ldots)$.

The intrinsic beauty of these theorems, as well as their central part in the theory of packing and covering [1, 6, 7, 9], started further investigations [3, 5, 8, 10, 11]. We emphasize the following two theorems:

Theorem of Eggleston: If $m(n)$ is the minimum of the area-deviation of a convex n-gon from a given convex domain then

$$m(n-1)+m(n+1) \geqq 2m(n) \qquad (n = 4, 5, \ldots).$$

The area-deviation of two domains is defined as the area of the set of those points which belong to exactly one of the domains.

As to the following theorem see [5].

If k and m are positive integers such that $km>2$ then among the km-gons of maximal (minimal) area inscribed in (circumscribed about) a convex domain of k-fold rotatory symmetry there is one which has k-fold rotatory symmetry.

In the present paper we shall give certain generalizations of these theorems which have important applications in the theory of packing and covering [4].

We shall denote the area of a domain X by $|X|$.

Let p and q be positive, finite or infinite numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Let A and B be two measurable domains. We define the weighted area deviation, in short the *deviation of B from A* by

$$a(A, B) = p|A-(A\cap B)|+q|B-(B\cap A)|.$$

In this formula a product of the form $\infty \cdot 0$ can occur. Let the value of such a product be 0 by definition. Observe that apart from the case $p=q$ the deviation $a(A, B)$ is not symmetric in A and B.

We phrase our results in the following two theorems:

THEOREM 1. *For fixed weights p and q $\left(1 \leqq p, q \leqq \infty, \frac{1}{p}+\frac{1}{q}=1\right)$ let $a(n)$ be the minimum of the deviation of a convex n-gon from a strictly convex domain C. Then*

$$a(n-1)+a(n+1) > 2a(n) \qquad (n = 4, 5, \ldots).$$

THEOREM 2. *Let C be a strictly convex domain with k-fold rotatory symmetry. Then for arbitrary weights p and q $\left(1 \leqq p, q \leqq \infty, \frac{1}{p}+\frac{1}{q}=1\right)$ any convex polygon with $km>3$ sides having minimal deviation from C has k-fold rotatory symmetry with the same centre of symmetry as C.*

Since any convex domain can be approximated by strictly convex ones and the deviation is a continuous function of the domains, Theorems 1 and 2 immediately imply the following theorems:

THEOREM 1*. *For fixed weights p and q $\left(1 \leqq p, q \leqq \infty, \frac{1}{p}+\frac{1}{q}=1\right)$ let $a(n)$ be the minimum of the deviation of a convex n-gon from a convex domain C. Then*

$$a(n-1)+a(n+1) \geqq 2a(n) \qquad (n = 4, 5, \ldots).$$

THEOREM 2*. *Let C be a convex domain with k-fold rotatory symmetry. Then for arbitrary weights p and q $\left(1 < p, q < \infty, \frac{1}{p}+\frac{1}{q}=1\right)$ there is a convex km-gon ($km>3$) with minimal deviation from C having k-fold rotatory symmetry with the same centre of symmetry as C.*

The existence of a convex n-gon having minimal deviation from C can be reduced to a wellknown theorem of Weierstrass. A similar reduction in an even more complicated case will be presented in § 3.

It is easily seen that Theorems 1 and 2 contain the above theorems of Dowker concerning inscribed ($p=1, q=\infty$) and circumscribed ($p=\infty, q=1$) polygons. For $p=q=2$ Theorem 1 is equivalent to the above mentioned theorem of Eggleston.

§2. Sketch of the proof of Theorems 1 and 2

We shall use a slight modification of Eggleston's method.

First of all we extend the notion of deviation to point-sets in which each point occurs with a certain non negative integer multiplicity. Such a *multiple set* A is defined by its characteristic function $\chi_A(P)$ which gives the multiplicity of the point P. The set-operations for multiple sets are defined by the following relations:

$$\chi_{A\cup B}(P) = \max\{\chi_A(P), \chi_B(P)\},$$

$$\chi_{A\cap B}(P) = \min\{\chi_A(P), \chi_B(P)\},$$

$$\chi_{A-B}(P) = \begin{cases} \chi_A(P)-\chi_B(P), & \text{if } \chi_A(P) \geqq \chi_B(P) \\ 0, & \text{if } \chi_A(P) < \chi_B(P). \end{cases}$$

We define two further operations, the *sum of two multiple sets* and the *n-fold of a multiple set* as follows:

$$\chi_{A+B}(P) = \chi_A(P)+\chi_B(P), \quad \chi_{nA}(P) = n\chi_A(P).$$

If $\chi_A(P)$ is L-measurable then the area of A is defined by

$$|A| = \sum_{k=1}^{\infty} k|\{P, \chi_A(P) = k\}|.$$

Now the deviation of multiple sets can be defined by the above formula concerning simple sets.

Let us assign in the plane a positive direction of rotation in the clockwise sense. Let T be a finite set of closed oriented polygonal lines. Let U be the point-set union of the polygons of T. Let P be a point not belonging to U. Consider the angle described by the halfline PX when X moves along a side of a polygon of T in accordance with its orientation. We assume that the sum of all these angles is, for any point P not belonging to U, nonnegative. Then T forms the boundary of a *multiple polygonal region* A defined as follows: For $P\notin U$ let $\chi_A(P)$ be the above sum of angles divided by 2π. (Obviously $\chi_A(P)$ is an integer). For $P\in U$ let $\chi_A(P)$ be equal to $\liminf_{Q\to P, Q\notin U} \chi_A(Q)$.

A is uniquely defined by T but not vice versa. For instance the multiple polygonal region defined by a pentagram may be generated also by a convex pentagon and a non-convex simple decagon. To avoid this ambiguity it is convenient to define a multiple polygonal region, instead of its characteristic function, by the polygons forming its boundary. We shall call the vertices and the oriented sides of the polygons also vertices and sides of the multiple polygonal region.

We are going to define a special family of multiple polygonal regions which plays a distinguished part in the proof. Let $H(XY)$ denote the open halfplane lying on the right hand side of the oriented line XY. We define the *core* $M(A)$ of a multiple polygonal region A as the intersection of all halfplanes $H(XY)$ belonging to the sides XY of A. We call A a *double-polygon* if all of its angles are convex, $M(A)\neq\emptyset$ and $\chi_A(P)=2$ for all $P\in M(A)$. Let us mention some simple properties of a double-polygon.

(1) Let the segment PQ intersect the side XY of the double-polygon A. It is easily seen that if PQ does not intersect any other side of A and $P\in H(XY)$, $Q\in H(YX)$ then $\chi_A(P)>\chi_A(Q)$.

Let XY be a side of A. If $P\in M(A)$ then the oriented angle $\sphericalangle XPY$ is positive, otherwise we would have $P\in H(YX)$, i.e. $P\notin M(A)$. Since $\chi_A(P)=2$ therefore, by the definition of $\chi_A(P)$, any halfline issuing from P intersects the boundary of

A in two points dividing the halfline into three parts. According to (1) the points of these parts of the halfline belong to A with multiplicity 2, 1 and 0. It follows immediately that

(2) the set of points Q with $\chi_A(Q)>0$ is a star domain with respect to any point $P\in M(A)$.

The above considerations also show that

(3) $M(A)=\{P, \chi_A(P)=2\}$

and that

(4) the boundary of a double-polygon consists of one or two closed polygonal lines (Fig. 1).

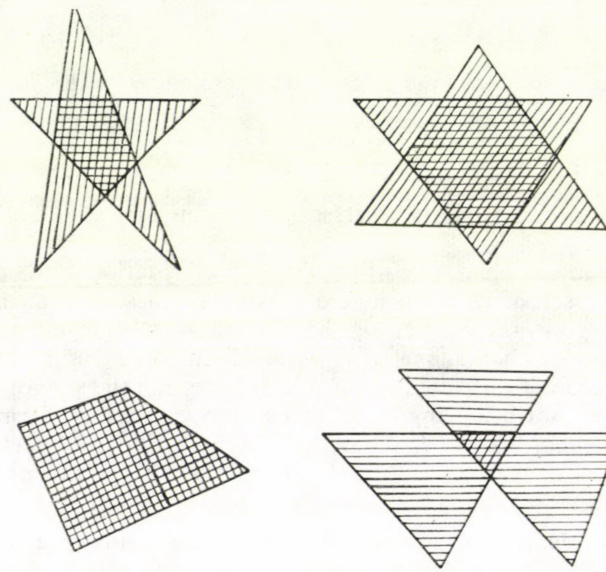

Fig. 1

Let X, P and Y be three non-collinear points. We define the set $S(XPY)$ as follows: $S(XPY)$ consists of the points of the open angular region $\sphericalangle XPY$, of the points of the closed segment PX and of the points of the halfline PY other than the points of the closed segment PY (Fig. 2). Let $\mathcal{D}(n)$ be the family of

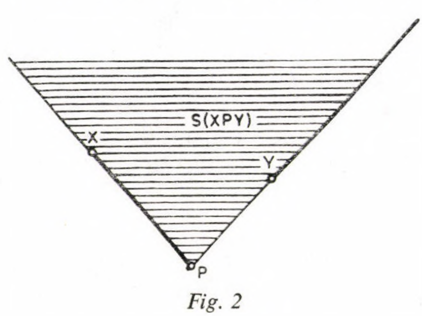

Fig. 2

the double-polygons with at most n sides. For a fixed convex domain C we write

$$d(n) = \inf a(2C, D), \quad D \in \mathcal{D}(n).$$

Then we have the following

LEMMA 1. *Let $n \geqq 8$ be an integer and C a strictly convex domain. Then there is a double-polygon $D \in \mathcal{D}(n)$ such that $a(2C, D) = d(n)$. Any $D \in \mathcal{D}(n)$ with $a(2C, D) = d(n)$ has the following properties: $C \cap M(D) \neq \emptyset$, D has exactly n sides, finally if XY is a side of D and $P \in C \cap M(D)$ then the region $S(XPY)$ contains exactly one vertex of D besides X.*

The proof of Lemma 1 follows in § 3. Here we shall show how Theorems 1 and 2 can be proved with the help of this lemma.

We start to show that for $n \geqq 4$ we have

$$d(2n) = 2a(n).$$

According to Lemma 1 there is a double-polygon $D \in \mathcal{D}(n)$ such that $a(2C, D) = d(2n)$ $(n \geqq 4)$ and this D has exactly $2n$ sides. Suppose that the boundary of D consists of a single polygonal line $X_1 \dots X_{2n}$. Consider a point $P \in C \cap M(D)$ and a sequence of vertices $X_1, \dots, X_k$ of D such that $\sphericalangle X_1PX_k < 2\pi$ but $\sphericalangle X_1PX_{k+1} \geqq 2\pi$. Then by Lemma 1 $X_{k+1} \in S(X_1PX_2), \dots, X_{2n} \in S(X_{k-1}PX_k)$ showing that the number $2n$ of vertices of D is $2k-1$. This contradiction shows that the boundary of D consists of two polygonal lines. Again, by Lemma 1 each of these polygons must be an n-gon. Thus D is the sum of two convex polygons D_1 and D_2 and we have obviously $a(C, D_1) = a(C, D_2) = a(n)$, consequently

$$d(2n) = a(2C, D) = a(C, D_1) + a(C, D_2) = 2a(n)$$

as claimed.

Denote by B_m a convex m-gon whose deviation from C is equal to $a(m)$ $(m = 3, 4, \dots)$. We claim that for $n + n' \geqq 8$, $n, n' > 2$ we have $B_n + B_{n'} \in \mathcal{D}(n+n')$. To see this we have only to show that $B_n \cap B_{n'} \neq \emptyset$. For $p = \infty$, $q = 1$ this is obvious because now both poligons contain C. If $p < \infty$ then the supposition $B_n \cap B_{n'} = \emptyset$ would imply that $a(n) + a(n') = a(C, B_n) + a(C, B_{n'}) > p|C|$. On the other hand, by a theorem of E. SAS [12], C contains a convex n-gon of area $s(n) = \dfrac{n|C|}{2\pi} \sin \dfrac{2\pi}{n}$. Hence $a(n) + a(n') \leqq p(|C| - s(n)) + p(|C| - s(n'))$. Since $s(3), s(4), \dots$ is an increasing concave sequence, we have $a(n) + a(n') \leqq p(2|C| - s(3) - s(5)) < p|C|$. This contradiction proves the assertion.

Now, Theorem 1 is an immediate consequence of the relations $d(2n) = 2a(n)$ and $B_{n-1} + B_{n+1} \in \mathcal{D}(2n)$, $n \geqq 4$. For we have

$$a(n-1) + a(n+1) = a(C, B_{n-1}) + a(C, B_{n+1}) = a(2C, B_{n-1} + B_{n+1}) \geqq d(2n) = 2a(n).$$

Since by Lemma 1 $B_{n-1} + B_{n+1}$ cannot have minimal deviation from $2C$ among the double-polygons with at most $2n$ sides, we see that for a strictly convex domain C the sequence $a(3), a(4), \dots$ is strictly convex.

We continue to prove Theorem 2. Let C be a strictly convex domain having k-fold rotatory symmetry $(k \geqq 2)$. Let F_0 be a convex polygon with $km > 3$ sides such that $a(C, F_0) = a(km)$. Let P be the centre of symmetry of C. The rotations about P through the angles $2\pi/k, \ldots, (k-1)2\pi/k$ carry C into itself and F_0 into $F_1, \ldots, F_{k-1}$. According to the above considerations we have $F_i+F_j \in \mathcal{D}(2km)$ and $a(2C, F_i+F_j)=d(2km)$, $0 \leqq i, j \leqq k-1$. Let X_0 be an arbitrary vertex of F_0 and $X_0, X_1, \ldots, X_{n-1}, X_n=X_0$ consecutive vertices of F_0. We have obviously $P \in C \cap M(F_0+F_j)$ $(j=1, \ldots, k-1)$. Therefore, by Lemma 1, any region $S(X_iPX_{i+1})$ $(i=0, \ldots, n-1)$ contains exactly one vertex of the polygon F_j $(j=1, \ldots, k-1)$.

Let G be the pointset consisting of the vertices of the polygons $F_0, \ldots, F_{k-1}$. Here coincident vertices of different polygons are considered to be different. Since G has k-fold rotatory symmetry, there is a point $Z \in G$ such that $\sphericalangle X_0PZ=2\pi/k$ and $X_0P=ZP$. Obviously $S(X_0PZ)$ contains exactly km points of G. Since, on the other hand, any region $S(X_iPX_{i+1})$ $(i=0, \ldots, n-1)$ contains exactly k points of G therefore $S(X_0PX_m)$ contains also km points of G. But as we have $Z, X_m \in G$, $S(X_0PZ)$ and $S(X_0PX_m)$ can contain the same number of points of G only if X_m coincides with Z. This means that the rotation about P through the angle $2\pi/k$ carries X_0 into X_m, i.e. F_0 has, indeed, k-fold rotatory symmetry.

§3. Proof of Lemma 1

Since in the cases when $p=1$, $q=\infty$ or $p=\infty$, $q=1$ the lemma is known we restrict ourselves, for the sake of simplicity, to the case when $1<p, q<\infty$.

Observe that for an arbitrary double-polygon D we have

$$a(2C, D) \geqq p\big(|C|-|C \cap M(D)|\big).$$

On the other hand, we have by the theorem of Sas for $n \geqq 8$

$$d(n) \leqq 2a(4) \leqq 2p\big(|C|-s(4)\big) = p|C|\frac{2(\pi-2)}{\pi} = p|C|0.726\ldots < p|C|.$$

It follows that for any double-polygon $D \in \mathcal{D}(n)$ with $a(2C, D)=d(n)$ $(n \geqq 8)$ we have

$$|C \cap M(D)| \geqq p|C|\left(1-\frac{2(\pi-2)}{\pi}\right) > 0.$$

Thus for a $D \in \mathcal{D}(n)$ having the least deviation from $2C$ we have, indeed, $C \cap M(D) \neq \emptyset$.

In order to see that for $n \geqq 8$ there is a $D \in \mathcal{D}(n)$ for which $a(2C, D)=d(n)$ we consider a sequence $D_1, D_2, \ldots$ of double-polygons with at most n sides such that $\lim_{i\to\infty} a(2C, D_i)=d(n)$. Obviously we may suppose without loss of generality that $a(2C, D_i) \leqq 2a(4)$ $(i=1, 2, \ldots)$. The above considerations show that then $a(2C, D_i)<p|C|$ and, writing $t_0=\left(1-\dfrac{2(\pi-2)}{\pi}\right)p|C|$, we have $|C \cap M(D_i)| \geqq t_0$ $(i=1, 2, \ldots)$. We claim that under these conditions the D_i's are uniformly bounded.

Let w_i and d_i be the width and the diameter of $C \cap M(D_i)$, respectively. If d is the diameter of C then $w_id \geqq w_id_i \geqq t_0/2$, i.e. $w_i \geqq t_0/2d$. Let X be an arbitrary point of D_i outside of C. Let P be the point of the boundary of C nearest to X. The set $C \cap M(D_i)$ contains a segment of any direction of length

w_i, thus it contains a segment EF of length $t_0/2d$ perpendicular to the line PX. Since $EF \subset M(D_i)$ and the set of those points Q for which $\chi_{D_i}(Q) > 0$ is a star-domain with respect to any point of $M(D_i)$, therefore any point of the triangle EFX belongs to D_i with multiplicity at least one. Let the line through P perpendicular to XP intersect the lines XE and XF in E' and F' (Fig. 3), respectively.

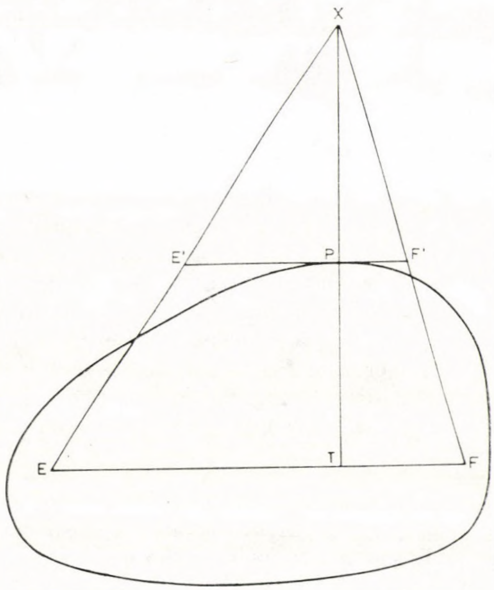

Fig. 3

Let the lines PX and EF intersect in T. Since the points of the triangle $E'F'X$ do not belong to C but they belong to D_i with multiplicity at least one, we have

$$a(2C, D_i) \geqq q|E'F'X| = q\left(\frac{XP}{XT}\right)^2 \frac{1}{2} EF \cdot XT \geqq \frac{qt_0 XP^2}{4d(XP+d)}.$$

This yields, along with the inequality $p|C| > a(2C, D_i)$, an upper bound for XP depending besides p and q only on $|C|$ and d.

We associate with D_i $(i=1, 2, \ldots)$ a $(2n+4)$-dimensional vector $\mathbf{z}_i$ as follows: The boundary of D_i consists either of one closed polygonal line $X_1 \ldots X_m X_1$ or of two closed convex polygonal lines $X_1 \ldots X_l X_1$, $X_{l+1} \ldots X_m X_{l+1}$ $(m \leqq n)$. We obtain the coordinates of $\mathbf{z}_i$ by writing down successively the coordinates of the vertices in the above order and writing at the end as many zeros as necessary. Though the mapping $D_i \to \mathbf{z}_i$ is not unique, the vector $\mathbf{z}_i$ determines D_i uniquely. Since the sequence $\{\mathbf{z}_i\}_{i=1}^{\infty}$ is bounded we can select a subsequence from it which tends to a vector $\mathbf{z}$. This vector represents a double-polygon $D \in \mathcal{D}(n)$ for which $a(2C, D) = d(n)$.

In the rest of the proof D will always denote a double-polygon belonging to $\mathcal{D}(n)$ such that $a(2C, D) = d(n)$. By $\ldots, X_{i-1}X_i, X_iX_{i+1}, \ldots$ we shall mean successive sides of D belonging to the same polygonal line. In accordance with our previous notations let $H(XY)$ be the open halfplane lying on the right hand side of the oriented line XY.

Translate the line X_iX_{i+1} through d into $H(X_iX_{i+1})$ or into $H(X_{i+1}X_i)$ according as d is positive or negative. Let the new line intersect the lines $X_{i-1}X_i$ and $X_{i+1}X_{i+2}$ in X_i' and X_{i+1}', respectively. Replace the sides $X_{i-1}X_i$, X_iX_{i+1} and $X_{i+1}X_{i+2}$ by $X_{i-1}X_i'$, $X_i'X_{i+1}'$ and $X_{i+1}'X_{i+2}$, respectively. For sufficiently small values of d we obtain a new double-polygon $D'\in\mathscr{D}(n)$. If u is the length of the segment $X_iX_{i+1}\cap C$ and v is the total length of the parts of the segment X_iX_{i+1} lying outside of C then

$$a(2C, D') = a(2C, D)+(pu-qv)d+O(d^2).$$

Since the devation of D from $2C$ is minimal, therefore $pu=qv$. It follows that the side X_iX_{i+1} intersects C. Let the endpoints of the segment $X_iX_{i+1}\cap C$ be $\overline{X}_i$ and $\underline{X}_{i+1}$ choosing the notations so that the order of the points on the line X_iX_{i+1} should be X_i, $\overline{X}_i$, $\underline{X}_{i+1}$, X_{i+1}. Let M_i be the midpoint of the segment $\overline{X}_i\underline{X}_{i+1}$. Scrutinizing now the variation of the deviation effected by small rotations of the side X_iX_{i+1} about M_i we see that $X_i\overline{X}_i=\underline{X}_{i+1}X_{i+1}$. Thus every vertex of D lies outside of C. It follows that D has exactly n vertices. For if D had less than n vertices then we could cut off from D a triangle lying completely outside of C obtaining a double-polygon $D'\in\mathscr{D}(n)$ such that $a(2C, D')<a(2C, D)$. We summarize the above properties of D in the following

LEMMA 2. *D has exactly n vertices all lying outside of C. Each side X_iX_{i+1} of D intersects the boundary of C in two points $\overline{X}_i$ and $\underline{X}_{i+1}$ lying on X_iX_{i+1} in the order X_i, $\overline{X}_i$, $\underline{X}_{i+1}$, X_{i+1} and satisfying the equalities*

$$qX_i\overline{X}_i = q\underline{X}_{i+1}X_{i+1} = p\overline{X}_i\underline{X}_{i+1}.$$

We continue to show that if two vertices of D coincide then D is the double of a convex polygon. To see this it is obviously enough to prove the following

LEMMA 3. *From a point X exterior to the strictly convex domain C one can draw at most two halflines intersecting the boundary of C in two points so that the quotient of the distances of these points from X assumes a prescribed value.*

Suppose that there are three halflines intersecting the boundary of C in the points Y_i and Z_i $(i=1, 2, 3)$ in the order X, Y_i, Z_i so that

$$\frac{XZ_1}{XY_1} = \frac{XZ_2}{XY_2} = \frac{XZ_3}{XY_3}.$$

Let the halfline XY_2Z_2 lie in the convex angular region Z_1XZ_3. Let Y_2' and Z_2' be the points of intersection of XZ_2 and Y_1Y_3 and XZ_2 and Z_1Z_3, respectively (Fig. 4). Then

$$\frac{XZ_2}{XY_2} > \frac{XZ_2'}{XY_2'} = \frac{XZ_1}{XY_1} = \frac{XZ_3}{XY_3}.$$

This contradiction proves the Lemma.

Let X_1 be a vertex of D. Let $C(X_i)$ be the open "triangle" bounded by the segments $X_i\underline{X}_i$ and $X_i\overline{X}_i$ and the arc $\underline{X}_i\overline{X}_i$ of the boundary of C lying in the convex angular region $\underline{X}_iX_i\overline{X}_i$.

LEMMA 4. *If X_i is a vertex of D then there is no vertex of D in $C(X_i)$.*

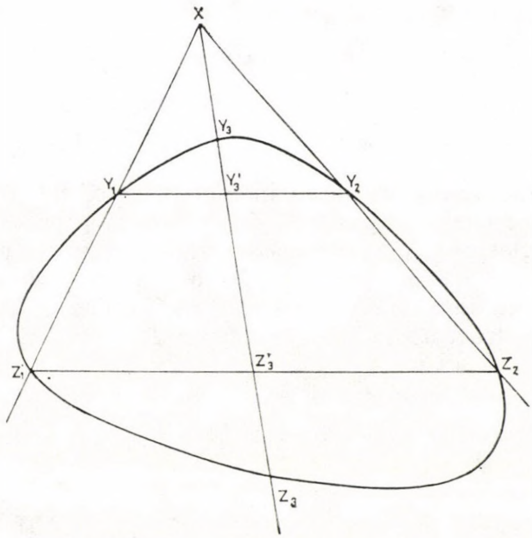

Fig. 4

Assume that the vertex X_j of D lies in $C(X_i)$. Referring to Lemma 2, the considerations of the proof of Lemma 3 show that neither of the halflines X_jX_{j-1} and X_jX_{j+1} can lie in the convex angular region $X_{i-1}X_iX_{i+1}$. Neither can it occur that both of these halflines intersect the same of the two halflines X_iX_{i-1} and X_iX_{i+1}. For suppose that both of the halflines X_jX_{j-1} and X_jX_{j+1} intersect X_iX_{i-1}. We consider the two halflines emanating from X_j the first being parallel to X_iX_{i+1} and intersecting C, the second being tangent to C and intersecting X_iX_{i+1}. For the different halflines emanating from X_j and intersecting the boundary of C at the points Y and Z, in this order, consider the quotient X_jY/X_jZ. For the first line this quotient is less than $\frac{2}{p+2q}$ and for the second it is equal to 1, i.e. greater than $\frac{p}{p+2q}$. Therefore there is a halfline "between" these halflines with a quotient equal to $\frac{p}{p+2q}$. Since by Lemma 2 the halflines X_jX_{j-1} and X_jX_{j+1} have the same quotient we get in contradiction with Lemma 3. Finally, it is impossible that the halfline X_jX_{j-1} intersects X_iX_{i+1} and the halfline X_jX_{j+1} intersects X_iX_{i-1}. Otherwise the angle of D at X_j would not be convex. Thus we have a situation as shown in Fig. 5.

The halflines $X_{j-1}X_j$ and $X_{j+1}X_j$ intersect the segments $X_i\overline{X}_i$ and $X_i\underline{X}_i$ in the points, say, K and L, respectively. We construct a new double-polygon $D'\in\mathcal{D}(n)$ by replacing the sides $X_{i-1}X_i$, X_jX_{j+1}, $X_{j-1}X_j$ and X_iX_{i+1} by $X_{i-1}L$, LX_{j+1}, $X_{j-1}K$ and KX_{i+1}, respectively (Fig. 5). The points of the quadrangle X_iKX_jL are exterior to C, they belong to D with multiplicity one and to D' with multiplicity zero. Any other point of the plane belongs to D and D' with the same multiplicity. Therefore

$$a(2C,D') = a(2C,D) - q|X_iKX_jL|,$$

in contradiction with the hypothesis that D has minimal deviation from $2C$.

LEMMA 5. *No vertex of D can lie in the interior of a side of D.*

Suppose that the vertex X_j is an interior point of the side X_iX_{i+1}. Since X_j cannot lie on the closed segment $\overline{X}_i\underline{X}_{i+1}$, we may suppose that X_j belongs, say, to the open segment $X_i\overline{X}_i$. The considerations used in the proof of Lemmas 3 and 4 show that the halfline X_jX_{j+1} lies outside the convex angular region $X_{i-1}X_iX_{i+1}$. Thus the line X_jX_{j+1} intersects the halfline X_iX_{i-1} in some point K (Fig. 6). We construct a new double-polygon $D'\in\mathcal{D}(n)$ by replacing the sides

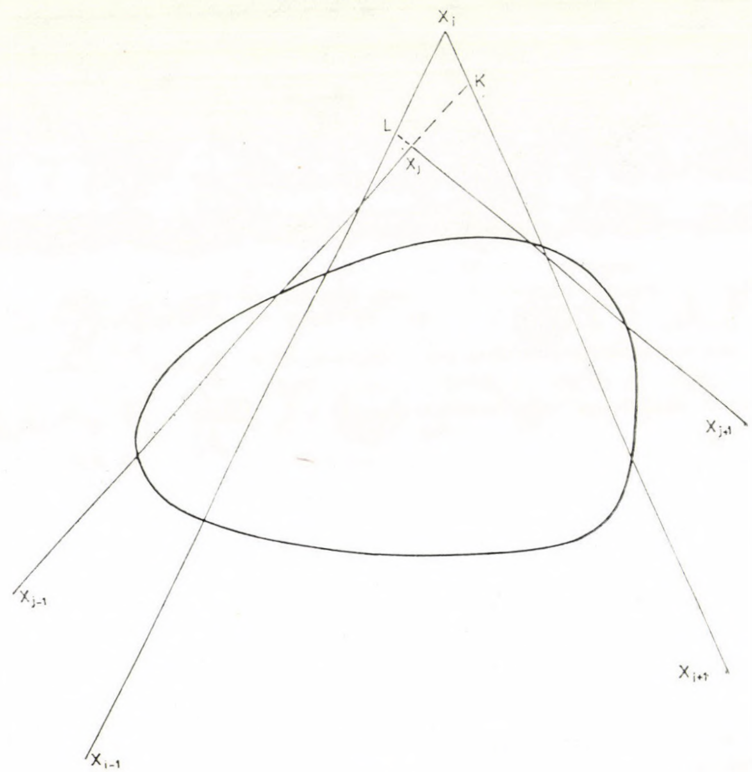

Fig. 5

$X_{i-1}X_i$, X_iX_{i+1} and X_jX_{j+1} by $X_{i-1}K$, X_jX_{i+1} and KX_{j+1}, respectively. Now we have $a(2C, D')=a(2C, D)-q|KX_iX_j|$, which is impossible.

LEMMA 6. *If* X_iX_{i+1} *is a side of* D *then the halfplane* $H(X_{i+1}X_i)$ *contains at most one vertex of* D.

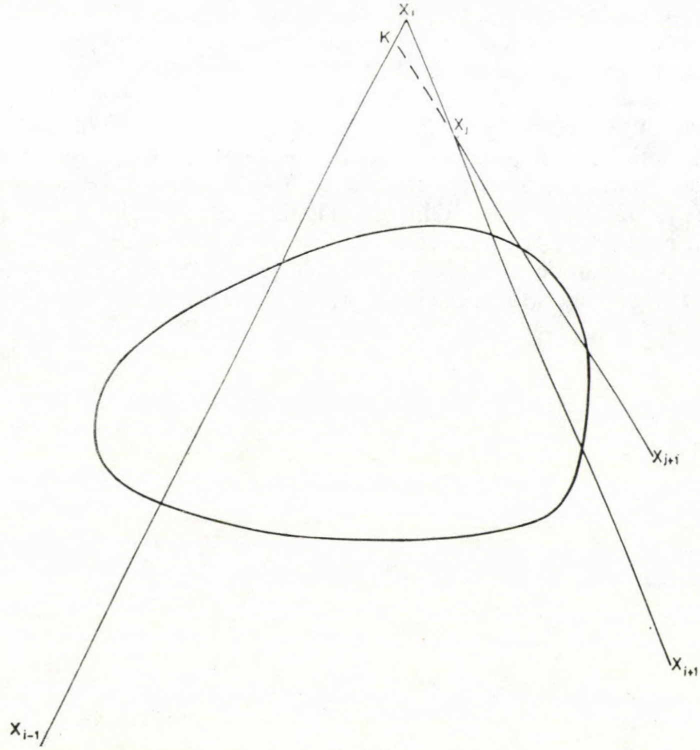

Fig. 6

Suppose that $H(X_{i+1}X_i)$ contains two vertices of D, say, X_k and X_l. We claim that then $H(X_{i+1}X_i)$ contains a side of D. Suppose that the boundary of D consists of one closed polygonal line $X_k\dots X_iX_{i+1}\dots X_l\dots X_k$. Let P be a point in $M(D)$. Consider the oriented angle swept over by the halfline PQ while Q runs over the boundary of D. If $H(X_{i+1}X_i)$ does not contain any side of D then X_{k+1} and X_{l+1} lie either in $H(X_iX_{i+1})$ or on the line X_iX_{i+1}. Using the fact that all angles of D are convex, it easily follows that the angle under consideration is at least 6π, which is impossible. The case when the boundary of D consists of two closed polygonal lines can be settled in a similar way. Thus, in order to prove Lemma 6 it suffices to prove the following

LEMMA 7. *If* X_iX_{i+1} *is a side of* D *then there is no side of* D *which lies in* $H(X_{l+1}X_i)$.

Suppose that the side X_jX_{j+1} of D is contained in $H(X_{i+1}X_i)$. Then $H(X_jX_{j+1})$ contains at least one of the vertices X_i and X_{i+1}. To see this we consider an arbitrary point P of $C\cap M(D)$. If non of X_i and X_{i+1} lies in $H(X_jX_{j+1})$ then the triangle $P\overline{X}_i\overline{X}_j$ contains either X_i and X_{j+1} or X_{i+1} and X_j. But this is impossible because the triangle $P\overline{X}_i\overline{X}_j$ is contained in C. Thus we had a vertex of D in C which contradicts Lemma 2. Therefore it suffices to scrutinize the following two cases:

(i) From among X_i and X_{i+1} exactly one lies in $H(X_jX_{j+1})$,

(ii) Both X_i and X_{i+1} lie in $H(X_jX_{j+1})$.

Case (i). Assume that X_i lies either in $H(X_{j+1}X_j)$ or on the line X_jX_{j+1}. Then the line X_jX_{j+1} intersects the side X_iX_{i+1} in a certain point K which may coincide with X_i. K cannot lie on the segment $\overline{X}_iX_{i+1}$. For it we had $K\in\overline{X}_iX_{i+1}$ then choosing an arbitrary point $P\in C\cap M(D)$, X_i would lie in the triangle $P\overline{X}_i\overline{X}_j$ and consequently in C. The line X_jX_{j+1} intersects also the side $X_{i-1}X_i$ in a point, say, L because otherwise either the side $X_{i-1}X_i$ would not intersect C or the angle of D at X_i would be concave. Similarly as above we can see that L must lie on the segment $X_i\underline{X}_i$.

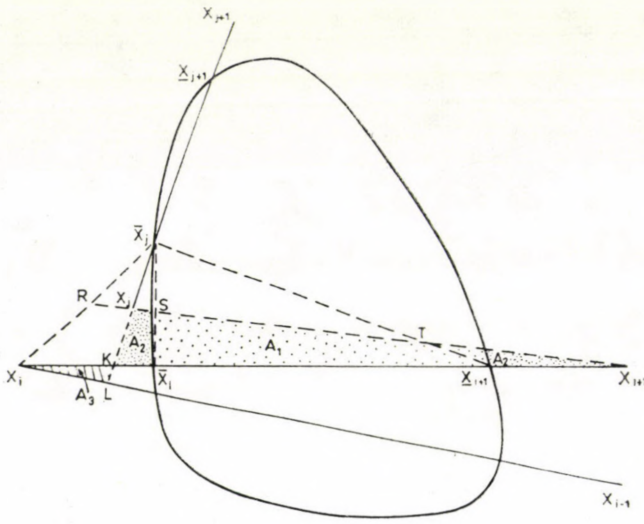

Fig. 7

From D we obtain a new double-polygon $D'\in\mathcal{D}(n)$ by replacing the sides $X_{i-1}X_i$, X_jX_{j+1} and X_iX_{i+1} by $X_{i-1}L$, LX_{j+1} and X_jX_{i+1} (Fig. 7). Letting $A_1=C\cap X_jX_{i+1}K$, $A_2=X_jX_{i+1}K-A_1$ and $A_i=LX_iK$ we have

$$a(2C,D')=a(2C,D)-p|A_1|+q|A_2|-q|A_3|.$$

We claim that $p|A_1|>q|A_2|$. Let the line X_jX_{i+1} intersect the lines $\overline{X}_jX_i$, $\overline{X}_j\overline{X}_i$ and $\overline{X}_j\underline{X}_{i+1}$ in R, S and T, respectively. By a convenient area preserving affinity we can attain that $RX_i=RX_{i+1}$ and $\sphericalangle X_{i+1}RX_i=90°$. Denote the distance of $\overline{X}_j$ from the line X_jX_{i+1} by m. Then we have by Lemma 2

$$\begin{aligned} p|A_1|-q|A_2| &> p|\overline{X}_i\underline{X}_{i+1}TS|-q(|\underline{X}_{i+1}X_{i+1}T|+|X_i\overline{X}_iSR|) = \\ &= p|\overline{X}_j\overline{X}_i\underline{X}_{i+1}|-q(|\overline{X}_j\underline{X}_{i+1}X_{i+1}|+|\overline{X}_jX_i\overline{X}_i|)- \\ &-m(pST-q(RS+TX_{i+1})) = m(q(RS+TX_{i+1})-pST). \end{aligned}$$

Therefore it suffices to show that

$$pST < q(RS+TX_{i+1}).$$

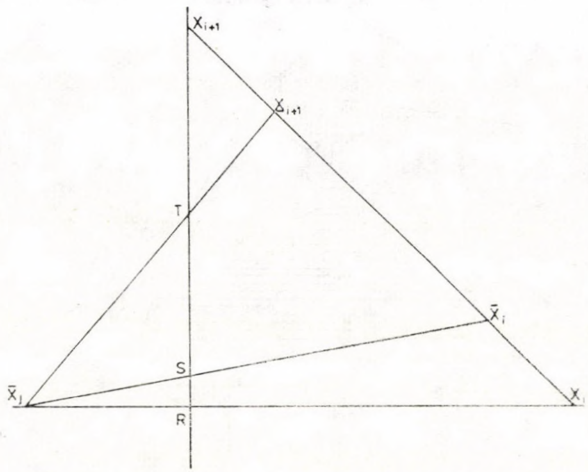

Fig. 8

Let R be the origin of a Cartesian coordinate system in which $X_i=(2pq, 0)$, $X_{i+1}=(0, 2pq)$ and $\overline{X}_j=(-\xi, 0)$ (Fig. 8). Then, taking into account the relation $pq=p+q$ and Lemma 2, we obtain by a simple computation that

$$\overline{X}_i=(pq+q, p), \quad \underline{X}_{i+1}=(p, pq+q),$$

$$S=\left(0, \frac{p\xi}{pq+q+\xi}\right), \quad T=\left(0, \frac{(pq+q)\xi}{p+\xi}\right),$$

$$pST=2pq\frac{\xi^2+2pq\xi}{\xi^2+2pq\xi+p(pq+q)}<2pq=RS+ST+TX_{i+1},$$

$$pST<\frac{p}{p-1}(RS+TX_{i+1})=q(RS+TX_{i+1}).$$

Thus we have indeed $p|A_1|>q|A_2|$ and consequently $a(2C, D')<a(2C, D)$ which is impossible. The case when X_i lies in $H(X_jX_{j+1})$ and X_{i+1} lies either in $H(X_{j+1}X_j)$ or on the line X_jX_{j+1} can be settled in the same way.

Case (ii). Assume that both X_i and X_{i+1} lie in $H(X_jX_{j+1})$. Then the segments X_iX_{j+1} and X_jX_{i+1} intersect in a point, say O. Replacing the sides X_iX_{i+1} and X_jX_{j+1} by X_iX_{j+1} and X_jX_{i+1} we obtain a new double-polygon $D'\in\mathcal{D}(n)$ such that

$$a(2C, D') = a(2C, D)-p(|A_1|-|A_2|)-q(|A_3|-|A_4|),$$

where $A_1=C\cap OX_iX_{i+1}$, $A_2=C\cap OX_jX_{j+1}$, $A_3=OX_jX_{j+1}-A_2$ and $A_4= =OX_iX_{i+1}-A_1$ (Fig. 9).

We shall show that

$$p(|A_1|-|A_2|)+q(|A_3|-|A_4|) > 0.$$

Let E be the point of intersection of the boundary of C and the segment X_jX_{i+1}

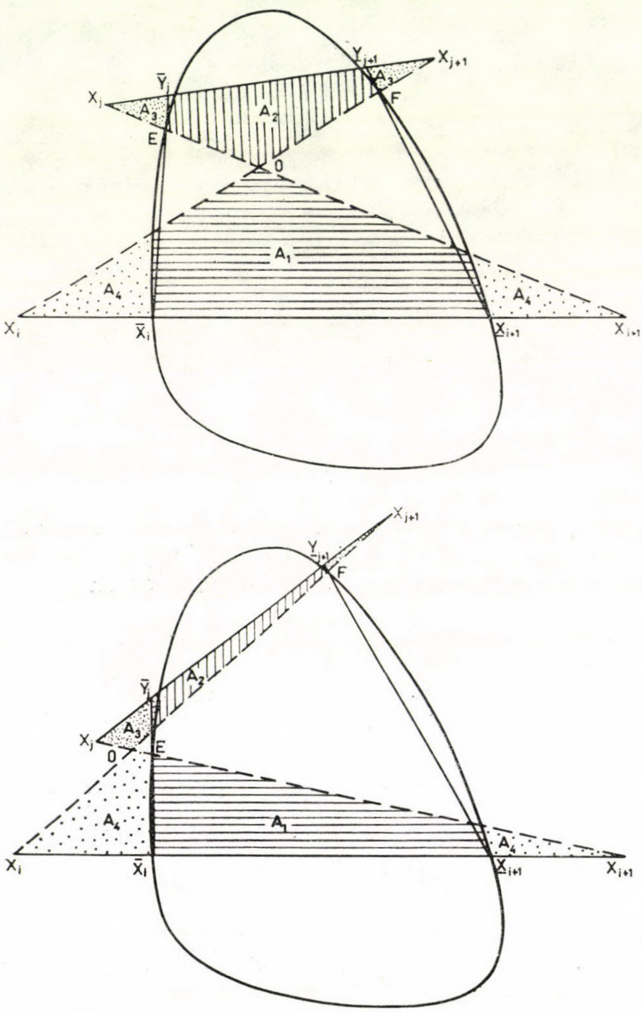

Fig. 9

nearer to X_j. Let F be the point of intersection of the boundary of C and the segment $X_{j+1}X_i$ nearer to X_{j+1}. Let the line X_jX_{j+1} intersect the lines $\overline{X}_iE$ and $\underline{X}_{i+1}F$ in $\overline{Y}_j$ and $\underline{Y}_{j+1}$, respectively. Instead of $A_1, \ldots, A_4$ we introduce new sets defined as follows:

$$A'_1 = C' \cap OX_iX_{i+1}, \qquad A'_2 = C' \cap OX_jX_{j+1},$$

$$A'_3 = OX_jX_{j+1} - A'_2, \qquad A'_4 = OX_iX_{i+1} - A'_1,$$

where C' denotes the quadrangle $\overline{Y}_j\underline{Y}_{j+1}\underline{X}_{i+1}\overline{X}_i$. Since $A_1 \supset A'_1$, $A_2 \subset A'_2$, $A_3 \supset A'_3$ and $A_4 \subset A'_4$ it is enough to show that

$$p(|A'_1| - |A'_2|) + q(|A'_3| - |A'_4|) > 0.$$

By Lemma 2, $C \cap H(X_{i+1}X_i)$ is contained in the parallel strip bounded by the lines $\underline{X}_i\overline{X}_i$ and $\overline{X}_{i-1}\underline{X}_{i+1}$. This implies, along with the strict convexity of C, that the lines $\overline{X}_i\overline{X}_j$ and $\underline{X}_{i+1}\underline{X}_{j+1}$ intersect one another in $H(X_{i+1}X_i)$. Since $X_i\overline{X}_i = \underline{X}_{i+1}X_{i+1} = \frac{1}{2p}X_iX_{i+1}$ and $X_j\overline{X}_j = \underline{X}_{j+1}X_{j+1} = \frac{1}{2p}X_jX_{j+1}$ the lines X_iX_j and $X_{j+1}X_{j+1}$ also intersect one another in $H(X_{i+1}X_i)$. It follows immediately that $|OX_iX_{i+1}| > |OX_jX_{j+1}|$.

Consider an area preserving affinity carrying the segments OX_i and OX_{i+1} into equal segments perpendicular to one another. In the transformed figure we preserve the notations of the original one. Choose a Cartesian coordinate-system such that $O=(0, 0)$, $X_i=(2pq, 0)$, $X_{i+1}=(0, 2pq)$, $X_j=(0, -2pq\eta)$ and $X_{j+1}= =(-2pq\xi, 0)$. Then $\overline{X}_i=(pq+q, p)$ and $\underline{X}_{i+1}=(p, pq+q)$ (Fig. 10).

Let the lines $\underline{X}_{i+1}\underline{X}_{j+1}$ and $\overline{X}_i\overline{Y}_j$ intersect the axes y and x in G and H, respectively. In accordance with our previous notations let these lines intersect the axes x and y in F and E, respectively. Fixing the points $O, X_i, X_{i+1}, \overline{X}_i$,

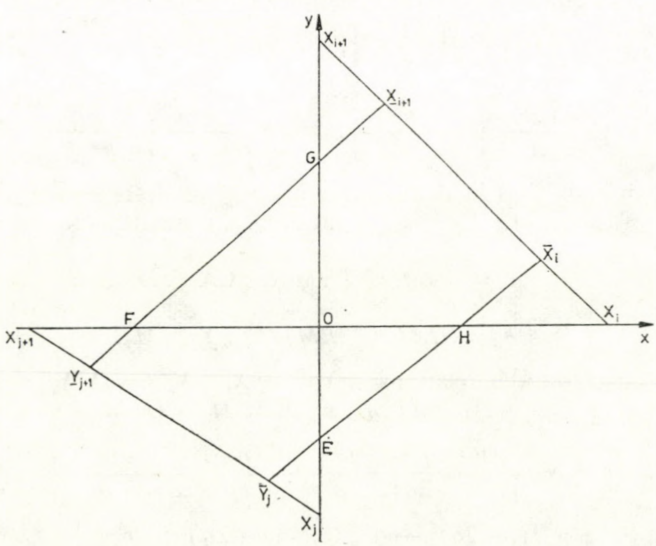

Fig. 10

$\underline{X}_{i+1}$, E, F, G and H, let the line $X_j\overline{Y}_j\underline{Y}_{j+1}X_{j+1}$ vary under the following conditions:

$$(*)\qquad \begin{cases} \xi \geqq 0, \eta \geqq 0, \xi\eta \leqq 1, \\ X_j\overline{Y}_j \leqq \dfrac{1}{2p} X_jX_{j+1}, \\ X_{j+1}\underline{Y}_{j+1} \leqq \dfrac{1}{2p} X_jX_{j+1}. \end{cases}$$

Then the quantity $f=p(|A_1'|-|A_2'|)+q(|A_3'|-|A_4'|)$ obviously attains its minimum. Observe that f cannot attain its minimum but in one of the following cases: a) $\xi\eta=0$, b) $\xi\eta=1$, c) $0<\xi\eta<1$, $X_j\overline{Y}_j=X_{j+1}\underline{Y}_{j+1}=\dfrac{1}{2p}X_jX_{j+1}$. For suppose that $0<\xi\eta<1$ and let, say, $X_j\overline{Y}_j<\dfrac{1}{2p}X_jX_{j+1}$. Choose a point $X_{j+1}'\in OX_{j+1}$. Let the line $X_{j+1}'\overline{Y}_j$ intersect the lines OX_j, $O\overline{Y}_j$ and $O\underline{Y}_{j+1}$ in X_j', $\overline{Y}_j'$ $(=\overline{Y}_j)$ and $\underline{Y}_{j+1}'$, respectively. If X_{j+1}' is sufficiently near to X_{j+1} then the points X_j', $\overline{Y}_j'$, $\underline{Y}_{j+1}$ and X_{j+1}' continue to satisfy the conditions $(*)$, but for the new points the value of f is smaller than originally.

If $\xi=\eta=0$, then we have, obviously, $f=0$. The considerations of case (i) show that if exactly one of ξ and η is equal to 0 then $f>0$. If $\xi\eta=1$ then the lines X_iX_j and $\underline{X}_{i+1}X_{j+1}$ are parallel and the conditions $(*)$ cannot be satisfied but if the lines $\overline{X}_i\overline{Y}_j$ and $\underline{X}_{i+1}\underline{Y}_{j+1}$ are parallel too. But then $f=0$.

Let us now turn to the case $0<\xi\eta<1$, $X_j\overline{Y}_j=X_{j+1}\underline{Y}_{j+1}=\dfrac{1}{2p}X_jX_{j+1}$. Now we have $\overline{Y}_j=(-p\xi,\ -(p+2q)\eta)$ and $\underline{Y}_{j+1}=(-(p+2q)\xi, p\eta)$. Hence we obtain by a simple computation that

$$E=\left(0,\ \frac{p^2\xi-(p+2q)^2\eta}{p\xi+p+2q}\right),\quad F=\left(\frac{p^2\eta-(p+2q)^2\xi}{p\eta+p+2q},\ 0\right),$$

$$G=\left(0,\ \frac{(p+2q)^2\xi-p^2\eta}{(p+2q)\xi+p}\right),\quad H=\left(\frac{(p+2q)^2\eta-p^2\xi}{(p+2q)\eta+p},\ 0\right).$$

First suppose that O lies in the interior of the quadrangle $\overline{Y}_j\underline{Y}_{j+1}\underline{X}_{i+1}\overline{X}_i$. Let the midpoints of the segments X_iX_{i+1} and X_jX_{j+1} be M_i and M_j, respectively. Then

$$A_1'=OGM_iH\cup HM_i\overline{X}_i\cup G\underline{X}_{i+1}M_i$$

and

$$A_2'=OEM_jF\cup EM_j\overline{Y}_j\cup F\underline{Y}_{j+1}M_j.$$

Thus, making use of the relations $pX_i\overline{X}_i=p\underline{X}_{i+1}X_{i+1}=q\overline{X}_i\underline{X}_{i+1}$ and $pX_j\overline{Y}_j=$ $=p\underline{Y}_{j+1}X_{j+1}=q\overline{Y}_j\underline{Y}_{j+1}$, we obtain $f=p(|OGM_iH|-|OEM_jF|)$, i.e.

$$f=p^2q\left[\frac{(p+2q)^2\xi-p^2\eta}{(p+2q)\xi+p}+\frac{(p+2q)^2\eta-p^2\xi}{(p+2q)\eta+p}-\right.$$
$$\left.-\xi\frac{(p+2q)^2\eta-p^2\xi}{p+2q+p\xi}-\eta\frac{(p+2q)^2\xi-p^2\eta}{p+2q+p\eta}\right].$$

It is easily seen that this formula continues to hold if O does not lie in the quadrangle $\overline{Y}_j\underline{Y}_{j+1}\underline{X}_{i+1}\overline{X}_i$.

A simple transformation shows that

$$\frac{(p+2q)^2\xi-p^2\eta}{(p+2q)\xi+p}=p+2q-\frac{p}{\xi}+(1-\xi\eta)\frac{p^2}{\xi[(p+2q)\xi+p]},$$

$$\frac{(p+2q)^2\eta-p^2\xi}{(p+2q)\eta+p}=p+2q-\frac{p}{\eta}+(1-\xi\eta)\frac{p^2}{\eta[(p+2q)\eta+p]},$$

$$-\xi\frac{(p+2q)^2\eta-p^2\xi}{p+2q+p\xi}=p+2q-p\xi+(1-\xi\eta)\frac{(p+2q)^2}{p+2q+p\xi},$$

$$-\eta\frac{(p+2q)^2\xi-p^2\eta}{p+2q+p\eta}=p+2q-p\eta+(1-\xi\eta)\frac{(p+2q)^2}{p+2q+p\eta}.$$

Using these relations we obtain

$$f=p^2q(1-\xi\eta)\left[\frac{p^2}{\xi[(p+2q)\xi+p]}+\frac{(p+2q)^2}{p+2q+p\xi}-\frac{p}{\xi}+\frac{p^2}{\eta[(p+2q)\eta+p]}+\right.$$
$$\left.+\frac{(p+2q)^2}{p+2q+p\eta}-\frac{p}{\eta}\right]=4p^3q^3(p+2q)(1-\xi\eta)\left[\frac{\xi}{[(p+2q)\xi+p](p+2q+p\xi)}+\right.$$
$$\left.+\frac{\eta}{[(p+2q)\eta+p](p+2q+p\eta)}\right]>0.$$

This completes the proof of Lemma 7.

Now we are in a position to prove the last statement of Lemma 1. Consider a point $P\in C\cap M(D)$ and a side X_iX_{i+1} of D. We may suppose that X_i is not a double-vertex of D because otherwise D is the double of a convex polygon, and the statement of the lemma is obviously fulfiled. Then Lemmas 4, 5 and 6 imply that $S(X_iPX_{i+1})$ contains, besides X_i, at most one vertex of D. Suppose that $S(X_iPX_{i+1})$ contains no vertex of D other than X_i. Then D has a side X_jX_{j+1} which intersects the halflines PX_i and PX_{i+1}, say in the points L_i and L_{i+1}, respectively. The vertex X_i cannot lie on the segment PL_i, because then X_i would lie in the set $C(X_j)$ which contradicts Lemma 4. In the same way we see that X_{i+1} cannot lie on the segment PL_{i+1}. It follows that both X_i and X_{i+1} lie in the halfplane $H(X_{j+1}X_j)$ which is impossible by Lemma 6. Therefore $S(X_iPX_{i+1})$ contains besides X_i exactly one further vertex of D.

This completes the proof of Lemma 1 and simultaneously the proof of Theorems 1 and 2.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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ON THE INTERSECTION OF A CONVEX DISC AND A POLYGON

By

G. FEJES TÓTH (Budapest)

Distribute in the plane translates of a centro-symmetric convex disc C with a given density d. Let d be too big to be able to arrange the discs without overlapping but too small to cover the plane completely with them. How should the discs be arranged so as to cover the greatest possible part of the plane, more precisely so as to maximize the area-density of the part of the plane covered by the discs? In a previous paper [1] I showed that the solution of this problem depends on the behaviour of the function $f(x)$ defined as the maximum of the area of the intersection of C and a convex hexagon of area x. If $\bar{f}(x)$ is the least concave majorant of $f(x)$ and $|C|$ denotes the area of C then the density of the part of the plane covered by the discs cannot exceed $d\bar{f}(|C|/d)$. This bound is exact. If $f(|C|/d)=\bar{f}(|C|/d)$ then we obtain a best distribution by arranging the centres of the discs so as to form a conveniently chosen point-lattice. If $f(|C|/d)<\bar{f}(|C|/d)$ then a best distribution can be obtained by arranging the discs in a certain angular region in a suitably chosen lattice and in the complementary angular region in another lattice. However the existence of a disc with a non-concave $f(x)$ remained open. Here a centro-symmetric convex disc will be constructed for which $f(x)$ is not concave.

More generally, let $f_n(x)$ be the maximum of the area of the part of a convex disc C which can be covered by a convex n-gon of area x. For any $n\geqq 4$ we shall show the existence of a convex disc with n-fold rotatory symmetry for which $f_n(x)$ is not concave.

Let C be a convex disc having n-fold rotatory symmetry. Let $g_n(x)$ be the maximum of the area of the intersection of C and a regular n-gon of area x concentric with C. Probably we have $f_n(x)=g_n(x)$ for any convex disc with n-fold rotatory symmetry. Unfortunately I was not able to prove this conjecture. But the following weaker result will be enough for our purpose:

LEMMA 1. *Let n be an integer greater than 3 and C a strictly convex disc having n-fold rotatory symmetry. Suppose that for a given value x_0 we have $f_n(x_0)=\bar{f}_n(x_0)$. Then we have $f_n(x_0)=g_n(x_0)$.*

In the proof of Lemma 1 we shall make use of the notion of the weighted area deviation. Let p and q be positive numbers and A and B two convex domains. We define the weighted area deviation, in short the *deviation of B from A* by

$$a(A,B) = p|A-(A\cap B)|+q|B-(B\cap A)|.$$

Lemma 1 is a simple consequence of a rather deep theorem (Theorem 2 in [2]) which we phrase here as

LEMMA 2. *Let n be an integer greater than 3 and C a strictly convex disc having n-fold rotatory symmetry. If Z_0 is a convex n-gon having minimal deviation from C for given weights p and q then Z_0 is a regular n-gon concentric with C.*

In [2] the condition $\frac{1}{p}+\frac{1}{q}=1$ is made on the weights p and q but it is easily seen that all the results of [2] hold without this condition on the weights.

Consider a strictly convex disc C having n-fold rotatory symmetry $(n>3)$ and suppose that for the value x_0 we have $f_n(x_0)=\bar{f}_n(x_0)$. Then there is a line containing the point $(x_0, f_n(x_0))$ which lies completely above the graph of $f_n(x)$. Let the slope of this line be q. Then we have

$$f_n(x) \leqq f_n(x_0)+q(x-x_0)$$

for any x. Let Z_0 be a convex n-gon of area x_0 such that $|C\cap Z_0|=f_n(x_0)$. It is obvious from the definition of the function $f_n(x)$ that $0\leqq q\leqq 1$. We write $p=1-q$ and consider the deviation of an arbitrary convex n-gon Z from C with the weights p and q. We have

$$\begin{aligned} a(C,Z) = p|C|+q|Z|-|C\cap Z| &\geqq p|C|+q|Z|-f_n(|Z|) \geqq \\ &\geqq p|C|+q|Z|-\big(f_n(x_0)+q(|Z|-x_0)\big) = \\ &= p|C|+qx_0-f_n(x_0) = p|C|+q|Z_0|-|C\cap Z_0| = a(C,Z_0). \end{aligned}$$

Thus we have $a(C,Z)\geqq a(C,Z_0)$ showing that Z_0 has a least possible deviation from C from among all convex n-gons with the above choice of the weights p and q. By Lemma 2, Z_0 is a regular n-gon concentric with C, i.e. we have $f_n(x_0)=g_n(x_0)$.

Now we turn to the construction of a convex disc C with n-fold rotatory symmetry $(n>3)$ for which $f_n(x)$ is not concave. By Lemma 1 it suffices to construct a strictly convex disc C with n-fold rotatory symmetry for which the function $g_n(x)$ is not concave.

Let K be a unit circle with centre O and $Y=Y_1\dots Y_n$ a regular n-gon inscribed in K. Let K' be a circle concentric with K whose radius r is less than that of K and greater than the inradius of Y. Let the points of intersection of the side Y_iY_{i+1} $(Y_{n+1}=Y_1)$ with the boundary of K' be $\bar{R}_i$ and $\underline{R}_{i+1}$ choosing the notations so that the order of the points should be Y_i, $\bar{R}_i$, $\underline{R}_{i+1}$, Y_{i+1} (see the figure). Choose the points A_i and B_i on the boundary of K in the cyclic order A_i, Y_i, B_i so that Y_iA_i and Y_iB_i are tangent to K' $(i=1, 2, \dots, n)$. Obviously, we have

$$\lim_{r\to 1} \sphericalangle\, A_iOB_i = \lim_{r\to 1} \sphericalangle\, \underline{R}_iO\bar{R}_i = 0.$$

Furthermore, a simple computation shows that

$$\lim_{r\to 1} \frac{\sphericalangle\, \underline{R}_iO\bar{R}_i}{\sphericalangle\, A_iOB_i} = 0.$$

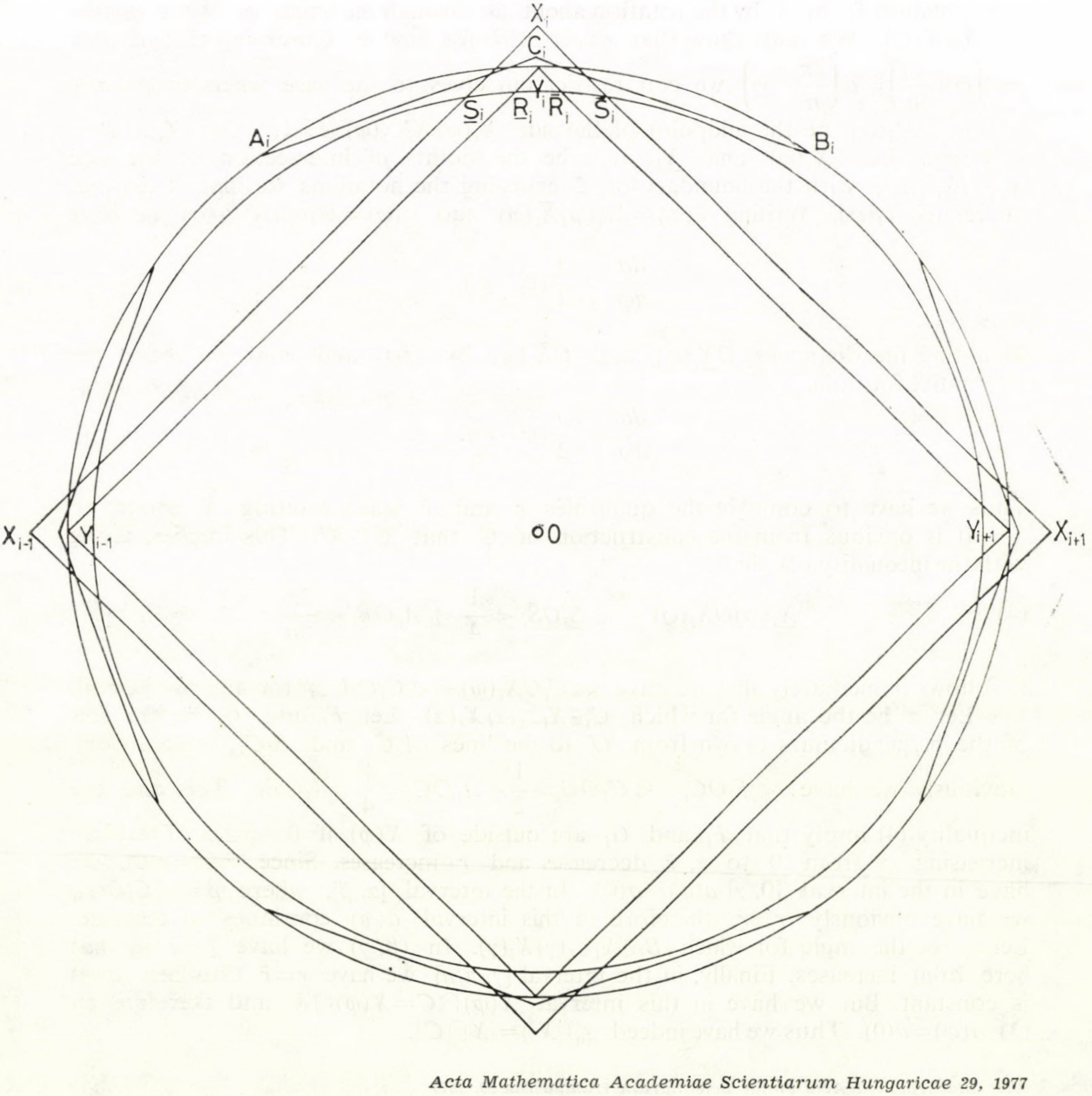
Therefore, choosing r sufficiently near to 1 we have

$$(1) \qquad 4\sphericalangle\, \underline{R}_iO\bar{R}_i < \sphericalangle\, A_iOB_i < \frac{8\pi}{5n}.$$

Enlarge Y by a similitude with respect to O so as to obtain a regular polygon $X=X_1\dots X_n$ with an inradius less than r. Let $\underline{S}_i$ be the point of intersection of the boundary of K' and the side $X_{i-1}X_i$ nearer to X_i. Let $\bar{S}_i$ be the point of intersection of the boundary of K' and the side X_iX_{i+1} nearer to X_i. Because of the inequality (1) we can choose X so that

$$4 \sphericalangle \underline{S}_iO\bar{S}_i < \sphericalangle A_iOB_i < \frac{8\pi}{5n}. \tag{2}$$

We choose a point C_i on each of the segments Y_iX_i at equal distances from O so that the segments A_iC_i intersect the boundary of K in the interior of X.



Replacing the segments of K cut off by the lines A_iB_i by the isosceles triangles $A_iB_iC_i$ we obtain a convex disc C with n-fold rotatory symmetry. If C_i coincides with Y_i then we have $C\cap X\subset K\cap X$. On the other hand, in the other limiting case when the point of intersection of A_iC_i and the boundary of K lies on the side $X_{i-1}X_i$ then $C\cap X\supset K\cap X$. Therefore we can choose the points C_i so that

$$|C\cap X| = |K\cap X|. \tag{3}$$

We claim that for the disc C constructed in this way the function $g_n(x)$ is not concave.

First we show that $g_n(|X|)=|C\cap X|$. Let $X(\varphi)=X_1(\varphi)\dots X_n(\varphi)$ be the n-gon obtained from X by the rotation about O through the angle φ. Write $a(\varphi)=|X(\varphi)\cap C|$. We must show that $a(\varphi)\leqq a(0)$ for any φ. Observing that $a(\varphi)=a\left(\varphi+\frac{2\pi}{n}\right)=a\left(\frac{2\pi}{n}-\varphi\right)$ we can restrict ourselves to the case when $0\leqq\varphi\leqq\pi/n$.

Let $M_i(\varphi)$ be the midpoint of the side $X_i(\varphi)X_{i+1}(\varphi)$ $(i=1,\dots,n;\ X_{n+1}(\varphi)=X_1(\varphi))$. Let $\overline{X}_i(\varphi)$ and $\underline{X}_{i+1}(\varphi)$ be the points of intersection of the side $X_i(\varphi)X_{i+1}(\varphi)$ with the boundary of C choosing the notations so that $\overline{X}_i(\varphi)$ lies nearer to $X_i(\varphi)$. Writing $\bar{v}(\varphi)=M_i(\varphi)\overline{X}_i(\varphi)$ and $\underline{v}(\varphi)=M_i(\varphi)\underline{X}_{i+1}(\varphi)$ we have

$$\frac{da}{d\varphi}=\frac{n}{2}(\bar{v}^2-\underline{v}^2).$$

Denoting the distances $O\underline{X}_i(\varphi)$ and $O\overline{X}_i(\varphi)$ by $\underline{r}(\varphi)$ and $\bar{r}(\varphi)$ we obtain the alternative formula

$$\frac{da}{d\varphi}=\frac{n}{2}(\bar{r}^2-\underline{r}^2).$$

Thus we have to compare the quantities $\underline{r}$ and $\bar{r}$ when rotating X about O.

It is obvious from the construction of C that $C\supset K'$. This implies, along with the inequality (2), that

$$\sphericalangle\ \underline{X}_i(\varphi)O\overline{X}_i(\varphi)<\sphericalangle\ \underline{S}_iO\overline{S}_i<\frac{1}{4}\sphericalangle\ A_iOB_i<\frac{2\pi}{5n}. \tag{4}$$

It follows immediately that we have $\sphericalangle C_iO\overline{X}_i(\varphi)<\sphericalangle C_iOA_{i+1}$ for any $\varphi\in[0,\pi/n]$.

Let α be the angle for which $C_i\in X_{i-1}(\alpha)X_i(\alpha)$. Let F_i and G_i be the feet of the perpendiculars drawn from O to the lines A_iC_i and B_iC_i, respectively. Obviously we have $\sphericalangle F_iOC_i=\sphericalangle C_iOG_i>\frac{1}{2}\sphericalangle A_iOC_i=\frac{1}{4}\sphericalangle A_iOB_i$. This and the inequality (4) imply that F_i and G_i are outside of $X(\varphi)$ if $0\leqq\varphi\leqq\alpha$. Therefore increasing φ from 0 to α, $\bar{r}$ decreases and $\underline{r}$ increases. Since $\bar{r}(0)=\underline{r}(0)$, we have in the interval $[0,\alpha]$ $a(\varphi)\leqq a(0)$. In the interval $[\alpha,\beta]$, where $\beta=\sphericalangle C_iOG_i$, we have obviously $\underline{r}>\bar{r}$, therefore in this interval $a(\varphi)$ continues to decrease. Let γ be the angle for which $B_i\in X_{i-1}(\gamma)X_i(\gamma)$. In (β,γ) we have $\underline{r}<\bar{r}$ so that here $a(\varphi)$ increases. Finally, in the interval $[\gamma,\pi/n]$ we have $\underline{r}=\bar{r}$ thus here $a(\varphi)$ is constant. But we have in this interval $X(\varphi)\cap C=X(\varphi)\cap K$ and therefore by (3) $a(\varphi)=a(0)$. Thus we have indeed $g_n(|X|)=|X\cap C|$.

Let $X'(0)$ be a regular n-gon concentric and homothetic with $X=X(0)$ of area $|X|-h$. Let $X'(\pi/n)$ be a regular n-gon concentric and homothetic with $X(\pi/n)$ of area $|X|+h$. Writing

$$w(\varphi)=\frac{\overline{X}_i(\varphi)\underline{X}_{i+1}(\varphi)}{X_i(\varphi)X_{i+1}(\varphi)}$$

we have

$$|C\cap X'(0)|=|C\cap X(0)|-w(0)h+O(h^2)$$

and

$$\left|C\cap X'\left(\frac{\pi}{n}\right)\right|=\left|C\cap X\left(\frac{\pi}{n}\right)\right|-w\left(\frac{\pi}{n}\right)h+O(h^2).$$

Since we have $w(\pi/n)>w(0)$ and $|C\cap X(0)|=\left|C\cap X\left(\frac{\pi}{n}\right)\right|=g_n(|X|)$ this shows that the right hand derivative of $g_n(x)$ at the point $|X|$ is greater than the left hand derivative. Thus the function $g_n(x)$ cannot be concave.

The existence of a strictly convex disc with n-fold rotatory symmetry for which $g_n(x)$ is not concave can be seen immediately if we approximate C by strictly convex discs with n-fold rotatory symmetry.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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ON A GENERALIZATION OF THE CONCEPT OF DERIVATIVE

By

J. PÁL and P. SIMON (Budapest)

Introduction

In this paper we are concerned with a generalization of the concept of derivative introduced by J. E. GIBBS, P. L. BUTZER and H. J. WAGNER ([2], [1], [7]). This generalization is due to C. W. ONNEWEER [3].

We introduce the inverse operator of the derivative, namely the integral operator and prove the strong differentiability of the integral function. This is a generalization of a theorem proved by P. L. BUTZER and H. J. WAGNER in [1]. Applying these ideas to the $(C, 1)$ summability of Fourier series with respect to the bounded Vilenkin systems, we prove the a.e. convergence of $(C, 1)$ means for functions in L^1.

§. 1

In this section we introduce some notations and definitions. Let

(1) $$m = (m_0, m_1, \dots, m_k, \dots) \qquad (2 \leqq m_k,\ m_k \in \mathbf{N},\ k \in \mathbf{N} := \{0, 1, 2, \dots\})!$$

be a sequence of natural numbers and denote by $\mathbf{Z}_{m_k}$ the m_kth discrete cyclic group, i.e.

(2) $$\mathbf{Z}_{m_k} := \{0, 1, \dots, m_k - 1\} \quad (k \in \mathbf{N}).$$

Furthermore, if we define the group G_m as the direct product of the groups $\mathbf{Z}_{m_k}$, then G_m is a compact Abelian group. Thus the elements of G_m are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $0 \leqq x_k < m_k$ $(k \in \mathbf{N})$ and for x, y in G_m their sum $x \dotplus y$ is obtained by adding the n^{th} coordinates of x and y modulo m_k $(k \in \mathbf{N})$. We define the subgroups $I_n(x)$ of G_m as follows:

(3) $$I_n(x) := \{y \in G_m\colon\ y = (x_0, \dots, x_{n-1}, y_n, \dots)\} \quad (x \in G_m,\ n \in \mathbf{N}).$$

Then the $I_n(0)$'s $(n \in \mathbf{N})$ form a basis for the neighbourhoods of $0 \in G_m$ in G_m and these subgroups completely determine the topology of G_m.

Next, let $\Gamma(m) = \{\psi_n\colon n \in \mathbf{N}\}$ denote the character group of G_m. We enumerate the elements of $\Gamma(m)$ as follows. For $k \in \mathbf{N}$ and $x \in G_m$ let r_k be the function defined by

(4) $$r_k(x) := \exp \frac{2\pi i x_k}{m_k} \quad (x \in G_m,\ i := \sqrt{-1},\ k \in \mathbf{N}).$$

If we define the sequence $(M_k,\ k \in \mathbf{N})$ by $M_0 := 1$ and $M_k := m_0 m_1 \dots m_{k-1}$

($k \in \mathbf{P} := \{1, 2, \ldots\}$), then each $n \in \mathbf{N}$ has a unique representation of the form

$$n = \sum_{k=0}^{\infty} n_k M_k,$$

where $0 \leqq n_k < m_k$ ($k \in \mathbf{N}$). For such $n \in \mathbf{N}$ we define the function ψ_n by

$$\psi_n := \prod_{k=0}^{\infty} (r_k)^{n_k}. \tag{5}$$

We remark that $\Gamma(m)$ is a complete orthonormal system with respect to the normalized Haar measure dx on G_m ([6]). Furthermore, if $m_n = 2$ for all $n \in \mathbf{N}$, then G_m is the so-called dyadic group and the elements of the character group $\Gamma(m)$ are the Walsh—Paley functions.

For $f \in L^1(G_m)$ we define its partial sums and $(C, 1)$ means by

$$\begin{cases} \hat{f}(n) := \int\limits_{G_m} f(t)\overline{\psi_n(t)}\,dt, \quad S_n f := \sum\limits_{k=0}^{n-1} \hat{f}(k)\psi_k \quad (n \in \mathbf{P}, S_0 f := 0), \\ \sigma_n f := \dfrac{1}{n} \sum\limits_{k=0}^{n-1} S_k f \quad (n \in \mathbf{P}). \end{cases} \tag{6}$$

Then we have the formulae

$$\begin{cases} (S_n f)(x) = \int\limits_{G_m} f(t) D_n(x \dot{-} t)\,dt = (f * D_n)(x) \\ (\sigma_n f)(x) = \int\limits_{G_m} f(t) K_n(x \dot{-} t)\,dt = (f * K_n)(x) \end{cases} \quad (x \in G_m,\ n \in \mathbf{P}), \tag{7}$$

where D_n, K_n are the so-called Dirichlet and Fejér kernels, resp. In [5] it was proved the following formula:

$$D_n = \psi_n \sum_{k=0}^{\infty} \left(\sum_{j=m_k-n_k}^{m_k-1} (r_k)^j \right) D_{M_k} \quad (n \in \mathbf{P}). \tag{8}$$

C. W. ONNEWEER has given the following generalization of the concept of derivative due to P. L. BUTZER and H. J. WAGNER:

DEFINITION ([3]). The function $f \in L^1(G_m)$ has a (strong) derivative $df \in L^1(G_m)$ if

$$\lim_n \|df - d_n f\|_1 = 0,$$

where

$$(d_n f)(x) := \sum_{j=0}^{n-1} M_j \sum_{k=0}^{m_j-1} k \cdot m_j^{-1} \sum_{l=0}^{m_j-1} \overline{r_j(le_j)^k} f(x \dotplus le_j) \tag{9}$$

$$(x \in G_m,\ n \in \mathbf{P},\ e_j := (0, 0, \ldots, 0, \underset{j}{1}, 0, \ldots) \in G_m,\ le_j := \underset{1}{e_j} \dotplus \underset{2}{e_j} \dotplus \ldots \dotplus \underset{l}{e_j}).$$

The following relation is easily verified:

$$d\psi_n = n\psi_n \quad (n \in \mathbf{N}). \tag{10}$$

Further we suppose that the sequence (1) is bounded.

§. 2

Let us denote by $w\in L^2(G_m)$ the function for which

$$\hat{w}(k) := \begin{cases} 1 & (k=0) \\ \dfrac{1}{k} & (k\in\mathbf{P}) \end{cases}$$

holds and let

$$If := w*f \quad (f\in L^1(G_m)).$$

The system $\Gamma(m)$ is complete, therefore the function w is uniquely determined. Then we have

THEOREM 1. *Let $f\in L^1(G_m)$ be a function and $\hat{f}(0)=0$. Then*
i) *if f has a (strong) derivative, then*

$$I(df) = f,$$

ii) *the function $If\in L^1(G_m)$ has a (strong) derivative and*

$$d(If) = f.$$

On the basis of this theorem the operator I is called the integral operator.
In the following theorem we give three equivalent conditions for the existence of the (strong) derivative.

THEOREM 2. *The following conditions are equivalent:*
i) *the function $f\in L^1(G_m)$ has a (strong) derivative,*
ii) *there exists a function $g\in L^1(G_m)$ for which*

$$\hat{g}(k) = k\hat{f}(k) \quad (k\in\mathbf{N})$$

holds,
iii) *there exists a function $g\in L^1(G_m)$ for which*

$$f = Ig+\hat{f}(0)$$

holds.

The following theorems show applicability of the concept of derivative in the theory of Fourier series with respect to the Vilenkin systems. From (10) follows easily that

$$K_n = D_n - \frac{1}{n}\, dD_n \quad (n\in\mathbf{P}). \tag{11}$$

Using this we prove

THEOREM 3. $\|K_n\|_1 = O(1)$ $(n\to\infty)$.

Let σ^* denote the maximal operator of the operator sequence $\sigma_n\colon L^1(G_m)\to\to L^1(G_m)$ $(n\in\mathbf{N})$, i.e.

$$(\sigma^* f)(x) := \sup_n |(\sigma_n f)(x)| \quad (f\in L^1(G_m),\ x\in G_m).$$

Then the following maximal theorem holds:

THEOREM 4. *The operator* $\sigma^*: L^1(G_m)\to L^1(G_m)$ *has weak type* $(1, 1)$ *and for all* $p, 1<p\leqq\infty$ *has type* (p, p), *i.e. the following inequalities hold:*

i) $\text{mes}\,\{x: (\sigma^* f)(x)>y\}\leqq C_1\|f\|_1/y \quad (f\in L^1(G_m), y>0)$,

ii) $\|\sigma^* f\|_p\leqq C_p\|f\|_p \quad (f\in L^p(G_m))$,

where the constants $C_p\,(1\leqq p\leqq\infty)$ *depend only on* p.

An immediate consequence of this theorem is the following

COROLLARY. *Let* f *be an arbitrary function in* $L^1(G_m)$. *Then*

$$\lim_n (\sigma_n f)(x)=f(x) \quad (\text{a.e. } x\in G_m).$$

Here we should like to pose two problems. Are the previous theorems valid also in the case $\overline{\lim_n}\, m_n=+\infty$? In [4], F. SCHIPP proved the following statement: if the sequence in (1) is $m_n=2$ for all $n\in\mathbf{N}$, then

$$\lim_n d_n(If)(x)=f(x) \quad (\text{a.e. } x\in G_m,\ f\in L^1(G_m)).$$

Is this theorem true for an arbitrary sequence (1)?

§. 3. Proof of the theorems

PROOF OF THEOREM 3. Let $n\in\mathbf{P}$ be an arbitrary fixed number and $s\in\mathbf{P}$ be the natural number for which $M_{s-1}\leqq n<M_s$. Then from (11) we get

$$K_n=D_n-\frac{1}{n}dD_n=D_n-\frac{1}{n}d_sD_n,$$

i.e.

$$nK_n=nD_n-d_sD_n.$$

Hence applying (8) and (9) the following estimate is valid:

$$nK_n(x)=\psi_n(x)\left(\sum_{v=0}^{s-1}n_vM_v\right)\sum_{i=0}^{s-1}\sum_{j=m_i-n_i}^{m_i-1}(r_i(x))^jD_{M_i}(x)-$$

$$-\sum_{v=0}^{s-1}M_v\sum_{k=0}^{m_v-1}km_v^{-1}\sum_{l=0}^{m_v-1}\overline{r_v(le_v)^k}D_n(x\dotplus le_v)=$$

$$=\psi_n(x)\sum_{v=0}^{s-1}M_v\sum_{i=0}^{s-1}\sum_{j=m_i-n_i}^{m_i-1}c_{ij}^{(v)}(x) \quad (n\in\mathbf{P}, x\in G_m),$$

where

$$c_{ij}^{(v)}(x)=[r_i(x)]^j\left\{n_vD_{M_i}(x)-\sum_{k=0}^{m_v-1}k\cdot m_v^{-1}\sum_{l=0}^{m_v-1}\overline{r_v(le_v)^k}\cdot[r_i(le_v)]^jD_{M_i}(x\dotplus le_v)\right\}\psi_n(le_v).$$

Since $\text{supp}\, D_{M_k}=I_k(0)\ (k\in\mathbf{N})$, we have in the case $0\leqq i<v$

$$D_{M_i}(x\dotplus le_v)=D_{M_i}(x) \quad (x\in G_m),\quad r_i(le_v)=1,$$

thus also

$$c_{ij}^{(v)}(x) = 0 \quad (x \in G_m,\ j = m_i - n_i, \dots, m_i - 1).$$

From this it follows that

$$n|K_n(x)| = \left| \psi_n(x) \sum_{v=0}^{s-1} M_v \sum_{i=v}^{s-1} \sum_{j=m_i-n_i}^{m_i-1} c_{ij}^{(v)}(x) \right| =$$

$$= O(1) \sum_{v=0}^{s-1} M_v \sum_{i=v}^{s-1} \left[D_{M_i}(x) + \sum_{l=0}^{m_v-1} D_{M_i}(x \dotplus le_v) \right] \quad (x \in G_m).$$

Since $\|D_{M_i}\|_1 = 1$ $(i \in \mathbf{N})$ ([6]), the last estimation implies

$$\|K_n\|_1 = O(1) \frac{1}{n} \sum_{v=0}^{s-1} (s-v) M_v = O(1) \frac{1}{M_{s-1}} \sum_{v=0}^{s-1} (s-v) M_v =$$

$$= O(1) \sum_{v=0}^{s-1} \frac{s-v}{m_v \dots m_{s-2}} = O(1) \sum_{v=1}^{s} \frac{v}{2^v} = O(1) \quad (s \to \infty).$$

Theorem 3 is proved.

In order to prove Theorem 1, we shall need the following generalizations of some lemmas of BUTZER and WAGNER [1], [8].

LEMMA 1. *Let*

$$w_k := w * D_{M_k}$$

for an arbitrary $k \in \mathbf{N}$. *Then*

$$\lim_k \|w - w_k\|_1 = 0.$$

PROOF. Let $n, k \in \mathbf{N}$, be fixed numbers and suppose that $n > k$. Then applying Abel transformation we obtain that

$$w_n - w_k = \sum_{l=M_k}^{M_n-2} \frac{1}{l(l+1)} D_{l+1} - \frac{1}{M_k} D_{M_k} + \frac{1}{M_n - 1} D_{M_n}.$$

By $\|D_k\|_1 = O(\log k)$ $(k \to \infty)$ ([6]), we have

$$\|w_n - w_k\|_1 = O(1) \left(\sum_{l=M_k}^{M_n-1} \frac{\log l}{l^2} + \frac{1}{M_k} + \frac{1}{M_n - 1} \right).$$

From this immediately follows that

$$\lim_{k,n} \|w_n - w_k\|_1 = 0,$$

and also there exists a function $g \in L^1(G_m)$ for which

$$\lim_k \|g - w_k\|_1 = 0$$

holds. Because the system $\Gamma(m)$ is complete, so $g = w$. This proves Lemma 1.

LEMMA 2. *For an arbitrary but fixed* $k\in\mathbf{N}$ *let* $w^{(k)}\in L^2(G_m)$ *be the function for which*

$$(w^{(k)})\hat{}\,(l) := \begin{cases} 0 & (0 \leqq l < M_k) \\ \dfrac{1}{l} & (M_k \leqq l). \end{cases}$$

Then

i) $\|w^{(k)}\|_1 = O(1/M_k) \quad (k\to\infty)$,

ii) $\|w(\cdot)-w(\cdot \dotplus h)\|_1 = O(1/M_k) \quad (h\in I_k(0),\ k\to\infty)$.

PROOF. i) Let $n, k\in\mathbf{N}$ be fixed numbers and $n>k$. Then, as in the proof of Lemma 1, we obtain that

$$\sum_{l=M_k}^{M_n-1}\frac{1}{l}\psi_l = \sum_{l=M_k}^{M_n-3}\left(\frac{1}{l}-\frac{1}{l+2}\right)K_{l+1}-\frac{1}{M_k+1}K_{M_k}+$$

$$+\frac{1}{M_n-2}K_{M_n-1}-\frac{1}{M_k}D_{M_k}+\frac{1}{M_n-1}D_{M_n}.$$

Since from Theorem 3, $\|K_l\|_1=O(1)$ $(l\to\infty)$ is valid, we have the following estimation:

$$\left\|\sum_{l=M_k}^{M_n-1}\frac{1}{l}\psi_l\right\|_1 = O(1)\left[\left(\sum_{l=M_k}^{M_n-3}\frac{1}{l^2}\right)+\frac{1}{M_k+1}+\frac{1}{M_n-2}+\frac{1}{M_k}+\frac{1}{M_n-1}\right] = O(1/M_k).$$

From this we obtain the statement i) by $n\to\infty$.

ii) Since

$$w_k(\cdot) = w_k(\cdot\dotplus h) \quad (h\in I_k(0),\ k\in\mathbf{N}),$$

we have

$$\|w(\cdot)-w(\cdot\dotplus h)\|_1 \leqq \|w-w_k\|_1+\|w_k(\cdot\dotplus h)-w(\cdot\dotplus h)\|_1 =$$

$$= 2\|w-w_k\|_1 = 2\|w^{(k)}\|_1.$$

From this ii) follows directly on the basis of the proof of i). Lemma 2 is proved.

LEMMA 3. i) *If the function* $f\in L^1(G_m)$ *has a (strong) derivative, then*

$$(df)\hat{}\,(s) = s\hat{f}(s) \quad s\in\mathbf{N}).$$

ii) *If there exists a function* $g\in L^1(G_m)$ *for which*

$$\hat{g}(s) = s\hat{f}(s) \quad (s\in\mathbf{N})$$

holds, then

$$f = Ig+\hat{f}(0).$$

PROOF. i) Since

$$\lim_n \|d_n f-df\|_1 = 0,$$

we have

$$\lim_n (d_n f)\hat{}\,(s) = (df)\hat{}\,(s) \quad (s\in\mathbf{N}).$$

On the other hand by (9) we have

$$(d_n f)^\wedge(s) = \sum_{j=0}^{n-1} M_j \sum_{k=0}^{m_j-1} k\cdot m_j^{-1} \sum_{l=0}^{m_j-1} \overline{r_j(le_j)^k} \int_{G_m} f(x \dotplus le_j)\overline{\psi_s(x)}\,dx =$$

$$= \sum_{j=0}^{n-1} M_j \sum_{k=0}^{m_j-1} k\cdot m_j^{-1} \sum_{l=0}^{m_j-1} \overline{r_j(le_j)^k}\psi_s(le_j) \int_{G_m} f(x)\overline{\psi_s(x)}\,dx =$$

$$= \hat{f}(s)\left(\sum_{j=0}^{n-1} s_j M_j\right).$$

Hence we obtain

$$\lim_n (d_n f)^\wedge(s) = s\hat{f}(s) \quad (s\in\mathbf{N}).$$

ii) Since

$$(Ig)^\wedge(s) = \hat{w}(s)\hat{g}(s) = \begin{cases} 0 & (s=0) \\ \dfrac{1}{s}\hat{g}(s) & (s\in\mathbf{P}), \end{cases}$$

we have by the assumption

$$\hat{f}(s) = (Ig)^\wedge(s) \quad (s\in\mathbf{P}).$$

This completes the proof of Lemma 3.

Proof of Theorems 1 and 2. Part i) of Theorem 1 is a direct consequence of Lemma 3.

ii) Let $n\in\mathbf{N}$ and $F_n := d_n w^{(n)}$. Then by means of the method used in the proof of Lemma 3, we have

$$\hat{F}_n(k) = \begin{cases} 0 & (0\leqq k < M_n), \\ \dfrac{1}{k}\sum_{j=0}^{n-1} k_j M_j & (M_n \leqq k). \end{cases}$$

By comparison of the Fourier coefficients it is easily verified that

$$d_n(If) - f = S_{M_n}f - f + (f * F_n) \quad (n\in\mathbf{P}).$$

Since (cf. [6])

$$\lim_n \|S_{M_n}f - f\|_1 = 0,$$

it suffices to prove that

$$\lim_n \|f * F_n\|_1 = 0.$$

For the proof of this relation we remark that for a "polynomial" of the form $P = \sum_{i=0}^{l} a_i\psi_i$ $(l < M_n,\ a_i\in\mathbf{C})$ we have $F_n * P = 0$. On the other hand by Lemma 2 we get

$$\|F_n\|_1 \leqq \sum_{j=0}^{n-1} M_j \sum_{k=0}^{m_j-1} k\|w^{(n)}\|_1 = O(1)\frac{1}{M_n}\sum_{j=0}^{n-1} M_j = O(1) \quad (n\to\infty).$$

Applying the inequality $\|f * F_n\|_1 \leqq \|F_n\|_1\|f\|_1$ and the theorem of Banach and Steinhaus, Theorem 1 is proved.

The implications i)⇒ii) and ii)⇒iii) of Theorem 2 follow from Lemma 3 and the implication iii)⇒i) from Theorem 1.

To prove Theorem 4, we shall use the following decomposition lemma of Calderon—Zygmund type.

LEMMA 4 ([5], Lemma 1). *Let* $f\in L^1(G_m)$ *and* $y>\|f\|_1$. *Then there exist decompositions*

$$G_m = F\cup \bar{F} \quad (\bar{F} := G_m \setminus F), \quad f = f_0 + \sum_{i=1}^{\infty} f_i$$

such that

i) $F = \bigcup_{i=1}^{\infty} J_i \quad (J_i \cap J_k = \emptyset,\ i\neq k)$,

where $J_i = I_{k_i}(x^{(i)})(k_i\in \mathbf{N},\ x^{(i)}\in G_m,\ i\in\mathbf{P})$,

ii) $\|f_0\|_\infty \leqq 2y$,

iii) $\operatorname{supp} f_k \subset J_k, \quad \int_{J_k} f_k(x)\,dx = 0,$

$$\frac{1}{\operatorname{mes} J_k}\int_{J_k} |f_k(x)|\,dx \leqq 4y \quad (k\in\mathbf{P}),$$

iv) $\operatorname{mes} F \leqq \|f\|_1/y$.

PROOF OF THEOREM 4. Let

$$\Phi_n(x) := \frac{1}{M_n}\sum_{\nu=0}^{n-1} M_\nu \sum_{i=\nu}^{n-1}\left[D_{M_i}(x) + \sum_{l=0}^{m_\nu-1} D_{M_i}(x \dotplus l e_\nu)\right] \quad (x\in G_m,\ n\in\mathbf{P}).$$

By the proof of Theorem 3, $|K_n(x)| = O(1)|\varphi_n(x)|$ $(n\to\infty,\ x\in G_m)$. Thus it suffices to prove that the maximal operator

$$(Tf)(x) := \sup_n |(f*\Phi_n)(x)| \quad (f\in L^1(G_m),\ x\in G_m)$$

has weak type (1, 1) and type (p, p) $(1<p\leqq\infty)$.

Let us first consider the case $p=\infty$. Then we have

$$\|Tf\|_\infty = O(1)\big(\sup_n \|\Phi_n\|_1\big)\cdot \|f\|_\infty .$$

Since $\|\varphi_n\|_1 = O(1)$ $(n\to\infty)$ (see the proof of Theorem 3), the operator T has type (∞, ∞).

Next we prove that T has weak type (1, 1). Let $f\in L^1(G_m)$, $y>0$ be arbitrarily given. We can suppose that $y>\|f\|_1$. Applying Lemma 4 with notation used there, we get

$$f*\Phi_n = f_0 * \Phi_n + \sum_{i=1}^{\infty} f_i * \Phi_n .$$

According to the Lemma 4 ii) and $\|\Phi_n\|_1 = O(1)$ $(n\to\infty)$, there exists an absolute constant $C>0$ for which

$$\|f_0 * \Phi_n\|_\infty \leqq Cy.$$

Let $i\in\mathbf{P}$ be fixed and $x\in\mathbf{P}$. Then we have

$$(f_i*\Phi_n)(x)=\frac{1}{M_n}\sum_{v=0}^{n-1}M_v\sum_{j=v}^{n-1}\left[2(f_i*D_{M_j})(x)+\sum_{l=1}^{m_v-1}(f_i*D_{M_j})(x\dotplus le_v)\right].$$

Since in the case $0\leqq j<k_i$ we have $J_i\cap I_j(x)=\emptyset$ or J_i and by $x\in\bar{F}$ in the case $k_i\leqq j$ $J_i\cap I_j(x)=\emptyset$ holds, thus according to Lemma 4 iii) we obtain

$$(f_i*D_{M_j})(x)=M_j\int\limits_{J_i\cap I_j(x)}f_i(t)\,dt=0\quad(j\in\mathbf{N}).$$

It is proved similarly that in the case $k_i>j$ we get

$$(f_i*D_{M_j})(x\dotplus le_v)=0\quad(l=1,2,\ldots,m_v-1;\ v\in\mathbf{N}),$$

thus

$$(f_i*\Phi_n)(x)=\frac{1}{M_n}\sum_{v=0}^{n-1}M_v\sum_{j=k_i}^{n-1}\sum_{l=1}^{m_v-1}(f_i*D_{M_j})(x\dotplus le_v)\quad(x\in\bar{F}).$$

We remark that if $n-1<k_i$, then $(f_i*\Phi_n)(x)=0$. Thus we can suppose that $n\geqq k_i+1$ and on the other hand in the case $x\in\bar{F}$ for $v>k_i-1$ we have $x\dotplus le_v\notin J_i$, thus

$$(f_i*D_{M_j})(x\dotplus le_v)=0\quad(l=1,2,\ldots,m_v-1).$$

From the above it follows

$$|(f_i*\Phi_n)(x)|\leqq\sum_{v=0}^{k_i-1}\sum_{j=k_i}^{n-1}\frac{M_v}{M_n}\sum_{l=1}^{m_v-1}(|f_i|*D_{M_j})(x\dotplus le_v)\leqq$$

$$\leqq\sum_{v=0}^{k_i-1}M_v\sum_{j=k_i}^{\infty}\frac{1}{M_j}\sum_{l=1}^{m_v-1}(|f_i|*D_{M_j})(x\dotplus le_v)\quad(x\in\bar{F}).$$

Let us denote by χ_A the characteristic function of the set $A\subset G_m$. Then we have

$$\|\chi_F(Tf_i)\|_1=O(1)\left(\sum_{v=0}^{k_i-1}\sum_{j=k_i}^{\infty}\frac{M_v}{M_j}\right)\|f_i\|_1\leqq C_1\|f_i\|_1,$$

where $C_1>0$ is an absolute constant. Since T is sublinear, we have

$$Tf\leqq Tf_0+\sum_{i=1}^{\infty}Tf_i\leqq Cy+\sum_{i=1}^{\infty}Tf_i,$$

so

$$\text{mes}\,\{x\colon (Tf)(x)>2Cy\}\leqq\text{mes}\,\{x\colon (Tf_0)(x)>Cy\}+$$

$$+\,\text{mes}\left\{x\colon x\in F,\left(\sum_{i=1}^{\infty}Tf_i\right)(x)>Cy\right\}+\text{mes}\left\{x\colon x\in\bar{F},\left(\sum_{i=1}^{\infty}Tf_i\right)(x)>Cy\right\}\leqq$$

$$\leqq\text{mes}\,F+\frac{1}{Cy}\left\|\left(\sum_{i=1}^{\infty}Tf_i\right)\chi_F\right\|_1\leqq\text{mes}\,F+\frac{1}{Cy}\sum_{i=1}^{\infty}\|(Tf_i)\chi_F\|_1\leqq\text{mes}\,F+\frac{1}{Cy}C_1\sum_{i=1}^{\infty}\|f_i\|_1.$$

Now, applying Lemma 4 we have the following estimation:

$$\text{mes}\,\{x\colon (Tf)(x) > 2Cy\} \leqq \frac{\|f\|_1}{y} + 4\frac{C_1}{C}\frac{\|f\|_1}{y} \leqq C_2\frac{\|f\|_1}{y},$$

where $C_2 > 0$ is an absolute constant. So we have proved that the operator T has weak type $(1, 1)$. Since T has type (∞, ∞) by the interpolation theorem of MARCINKIEWICZ [8], T has type (p, p) $(1 < p \leqq \infty)$. This proves Theorem 4.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT II. FOR ANALYSIS
1088. BUDAPEST, MÚZEUM KRT. 6—8.

EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT FOR COMPUTER MATHEMATICS
1088. BUDAPEST, MÚZEUM KRT. 6—8.

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CONTRIBUTION TO THE THEORY OF INTERPOLATION

By
P. VÉRTESI (Budapest)

1. Introduction

On a joint American-Hungarian seminar held in 1975 in Budapest, P. Turán raised the following two problems on the connection of the Hermite—Fejér and Lagrange interpolations (see further [1, Problems XXII and XXIII]).

a) For a fixed $0<\alpha<1$ let us give a matrix of nodes $X\subset[-1,1]$ such that $L_n(f;X,x)$ uniformly tends to $f(x)$ in $[-1,1]$ for each $f\in\operatorname{Lip}\alpha$, but for a suitable $f_1(x)\in\operatorname{Lip}\alpha$ we have $\overline{\lim\limits_{n\to\infty}}\,\|H_n(f_1;X,x)\|=\infty$.

b) If for a matrix X, $L_n(f;X,x)$ uniformly tends to $f(x)$ in $[-1,1]$ for each $f\in\operatorname{Lip}\alpha$ $(0<\alpha<1)$, then can we choose a positive integer $r=r(\alpha)$ such that $H_n(g;X,x)$ tends uniformly to $g(x)$ (in $[-1,1]$) supposing that $g^{(r)}(x)$ is continuous on $[-1,1]$? (Here, as usual, $\operatorname{Lip}\alpha=\{f(x);\,|f(x)-f(y)|\leqq c(f)\,|x-y|^{\alpha},$ $x,y\in[-1,1]\}$; $\|g\|=\max\limits_{-1\leqq x\leqq 1}|g(x)|$; $L_n(f;X,x)$ and $H_n(f;X,x)$ are defined by (2.4) and (2.9)).

The positive solutions of these problems would mean that the Hermite—Fejér interpolation can be worse (Problem a)) but not much worse than the Lagrange one (Problem b)).

I proved some theorems concerning a) in [7].

In the present paper we give a negative answer for b), moreover, we gain some new results for a), too. A third aim is to contribute to the fine and rough theory of Hermite—Fejér interpolation.

2. Notations

Let us consider an arbitrary system of nodes

$$-1\leqq x_{n,n}<x_{n-1,n}<\ldots<x_{2,n}<x_{1,n}\leqq 1\quad(n=1,2,\ldots)\tag{2.1}$$

in $[-1,1]$, further denote

$$\Omega_n(X,x)\overset{\text{def}}{=}c(x-x_{1,n})(x-x_{2,n})\ldots(x-x_{n,n}),\quad c\neq 0,\tag{2.2}$$

$$l_{k,n}(X,x)\overset{\text{def}}{=}\frac{\Omega_n(X,x)}{\Omega_n'(X,x_{k,n})(x-x_{k,n})}\qquad(k=1,2,\ldots,n),\tag{2.3}$$

$$L_n(f;X,x)\overset{\text{def}}{=}\sum_{k=1}^{n}f(x_{k,n})\,l_{k,n}(X,x)\qquad(n=1,2,\ldots),\tag{2.4}$$

$$(2.5)\qquad v_{k,n}(X,x) \overset{\text{def}}{=} 1-2l'_{k,n}(X,x_{k,n})(x-x_{k,n}) \qquad (k=1,2,\dots,n),$$

$$(2.6)\qquad h_{k,n}(X,x) \overset{\text{def}}{=} v_{k,n}(X,x)\,l^2_{k,n}(X,x) \qquad (k=1,2,\dots,n),$$

$$(2.7)\qquad \mathfrak{H}_{k,n}(X,x) \overset{\text{def}}{=} (x-x_{k,n})\,l^2_{k,n}(X,x) \qquad (k=1,2,\dots,n),$$

$$(2.8)\qquad H_n(f;X,x) \overset{\text{def}}{=} \sum_{k=1}^{n} f(x_{k,n})h_{k,n}(X,x) \qquad (n=1,2,\dots),$$

$$(2.9)\qquad H_n^*(f;X,x) \overset{\text{def}}{=} H_n(f;X,x)+\sum_{k=1}^{n} f'(x_{k,n})\mathfrak{H}(X,x) \qquad (n=1,2,\dots).$$

Here X is the matrix $\{x_{k,n}\}_{k=1}^{n}$ $(n=1,2,\dots)$, $f\in C$ (i.e., $f(x)$ is continuous on $[-1,1]$), moreover in (2.9) we suppose that $f'\in C$, too. As it is wellknown, L_n is the Lagrange— and H_n (or H_n^*) is the Hermite—Fejér interpolation.

3. Results

I.

3.1. If X is a given matrix of nodes let

$$(3.1)\qquad \lambda_n(L,X) \overset{\text{def}}{=} \max_{-1\leqq x\leqq 1} \sum_{k=1}^{n} |l_{k,n}(X,x)|,$$

$$(3.2)\qquad \lambda_n(H,X) \overset{\text{def}}{=} \max_{-1\leqq x\leqq 1} \sum_{k=1}^{n} |h_{k,n}(X,x)|,$$

$$(3.3)\qquad \Gamma(\lambda_n) \overset{\text{def}}{=} \{X;\ \lambda_n(L,X)\sim\lambda_n\}.$$

Let us suppose that for a certain sequence $\{\lambda_n\}$ and function class $F\subseteqq C$ we have $E_n(f)=o(\lambda_n^{-1})$ if $f\in F$.[1] Then, as it is wellknown, for each $X\in\Gamma(\lambda_n)$, $L_n(f;X,x)$ uniformly converges to $f(x)$ if $f\in F$. Can we define a suitable subclass $F_1\subset F$ such that $H_n(f;X,x)$ tends to $f(x)$ uniformly for $f\in F_1$, whenever $x\in\Gamma(\lambda_n)$? If we suppose that F_1 contains the very good function $f_1(x)=x$ then we can give a negative answer. Our assertion follows from the following

THEOREM 3.1. *If $\lambda_n\geqq\ln n$ $(n\geqq 2)$, then for each sequence $\{\lambda_n\}$ there exists a matrix $Y=Y_{\{\lambda_n\}}$ such that $Y\in\Gamma(\lambda_n)$, moreover, for each fixed $x\neq 0$, $-1\leqq x\leqq 1$*

$$(3.4)\qquad \overline{\lim_{n\to\infty}} \frac{|H_n(f_1;Y,x)|}{\lambda_n^2} > 0 \quad (x\neq 0)$$

where $f_1(x)=x$.

3.2. Definitions. A) We say that F is an L-good class of functions $f(x)$ for the matrix X if $L_n(f;X,x)$ tends uniformly to $f(x)$ in $[-1,1]$ whenever $f\in F$.

[1] $E_n(f)$ is the deviation of the best approximating polynomial of degree $\leqq n$ from $f(x)$ on $[-1,1]$ in C.

Similar definitions hold for the procecces H_n and H_n^*.

B) F is L-good for $\Gamma(\lambda_n)$ if F is L-good for any $X\in\Gamma(\lambda_n)$. We say that F is L-bad for $\Gamma(\lambda_n)$ if there does not exist an L-good $X\in\Gamma(\lambda_n)$. One can analogously define F which is H (or H^*)-good (or -bad) for $\Gamma(\lambda_n)$ (see [2, 4] and 3.8).

C) G is a non-trivial function-class if $x\in G$.

By these definitions we can assert the following consequence of Theorem 3.1 (see b):

THEOREM 3.2. *If F is a non-trivial function class and $X\in\Gamma(\lambda_n)$ for certain $\{\lambda_n\}$ $(\lambda_n\geqq\ln n)$ then we can choose $Y\in\Gamma(\lambda_n)$ such that there does not exist a non-trivial subset F_1 of F which would be H-good for Y.*

3.3. Now let us see another consequence of (3.4). Let

$$C(\omega_m)=\{f(x);\ f^{(m-1)}\in C \quad\text{and}\quad \omega(f^{(m-1)};t)\leqq a_m(f)\omega(t)\}.$$

So, if $\omega_m(f;t)$ is the m-th modulus of smoothness of $f(x)$, then

$$\omega_m(t)\leqq a_m(f)t^{m-1}\omega(t)\overset{\text{def}}{=}a_m(f)\omega_m(t).$$

(Here $\omega(t)$ is a modulus of smoothness, $m\geqq 1$.) Then we have another answer to a).

THEOREM 3.3. *If we suppose that for a certain $\omega(t)$ and $\{\lambda_n\}$ $(\lambda_n\geqq\ln n)$ we have*

$$\lim_{n\to\infty}\omega\left(\frac{1}{n}\right)\lambda_n=0, \tag{3.5}$$

then with a suitable matrix $Y=Y_{\{\lambda_n\}}\in\Gamma(\lambda_n)$

$$\lim_{n\to\infty}\|f(x)-L_n(f;Y,x)\|=0 \quad \textit{for each} \quad f\in C(\omega), \tag{3.6}$$

but for a certain $f_2\in C(\omega)$

$$\overline{\lim_{n\to\infty}}\,|H_n(f_2;Y,x)-f_2(x)|=\infty \quad \textit{for any fixed} \quad x\neq 0,\ x\in[-1,1]. \tag{3.7}$$

We remark that $f_2'(x)$ does not exist. (Compare this theorem with [7, Theorem 3.1]).

II.

3.4. Considering the above results we may conclude that knowing only $\{\lambda_n\}$, we cannot determine a non-trivial F so that it would be H-good for $\Gamma(\lambda_n)$. The following natural step is the investigation of H_n^* from this point of view. We shall see that now the situation is more favourable.

3.5. At first we prove an estimation (as (3.8), see 5.4).

THEOREM 3.4. *We have for any matrix X*

$$\lambda_n(H,X)=O(1)n^2\lambda_n^3(L,X) \qquad (n=1,2,\dots). \tag{3.8}$$

3.6. Using (3.8) we shall prove the following statement on the rapidity of the convergence of H_n^*.

THEOREM 3.5. *If* $m \geqq 2$ *then for each* $f \in C(\omega_n)$

$$\|H_n^*(f; X, x) - f(x)\| = O(1)\omega_m\left(\frac{1}{n}\right)[\lambda_n(H, X) + n\lambda_n^2(L, X)]. \tag{3.9}$$

3.7. By (3.8) and (3.9) we immediately have the following convergence-theorem:

THEOREM 3.6. *If we suppose that for a certain* $\omega_m(t)$ *and* $\{\lambda_n\}$ $(\lambda_n \geqq \ln n)$ *we have*

$$\lim_{n\to\infty} \omega_m\left(\frac{1}{n}\right) n^2 \lambda_n^3 = 0, \tag{3.10}$$

then for each $f \in C(\omega_m)$ $f' \in C$ *and*

$$\lim_{n\to\infty} \|H_n^*(f; X, x) - f(x)\| = 0 \quad \textit{whenever} \quad X \in \Gamma(\lambda_n). \tag{3.11}$$

3.7.1. The above mentioned theorem can be formulated as follows: If (3.10) is valid then $C(\omega_m)$ is H^*-good for $\Gamma(\lambda_n)$.

3.7.2. Now we prove that the order of (3.8) is the best possible.

THEOREM 3.7. *For each* $\{\lambda_n\}$ $(\lambda_n \geqq \ln n)$ *there exists an* $Y_{\{\lambda_n\}} = Y \in \Gamma(\lambda_n)$ *for which*

$$\lambda_n(H, Y) \geqq cn^2\lambda_n^3(L, Y) \qquad (n = 1, 2, \ldots). \tag{3.12}$$

III.

3.8. In their paper [2], P. ERDŐS and P. TURÁN introduced the notion of the rough and fine theory of Lagrange interpolation: If, knowing only the Lebesgue constants $\{\lambda_n\}$, we can determine whether a certain F is L-good or L-bad for $\Gamma(\lambda_n)$, then the finer structure of $X \in \Gamma(\lambda_n)$, cannot essentially influence the convergence-divergence behaviour of F. This case was called rough theory. E.g., let $F = \mathrm{Lip}\,\alpha$ $(0 < \alpha \leqq 1)$ and $\lambda_n \sim n^\beta$ $(0 < \beta < 1)$. Erdős and Turán proved the following:

If $\alpha < \dfrac{\beta}{\beta+2}$ then $\mathrm{Lip}\,\alpha$ is L-bad for $\Gamma(n^\beta)$;

if $\alpha > \beta$ then $\mathrm{Lip}\,\alpha$ is L-good for $\Gamma(n^\beta)$;

but if $\dfrac{\beta}{\beta+2} < \alpha < \beta$ then $\mathrm{Lip}\,\alpha$ is neither L-good nor L-bad for $\Gamma(n^\beta)$.

This means that in the third case the convergence-divergence behaviour is not determined by the Lebesgue constants alone but it depends on the finer structure of $X \in \Gamma(\lambda_n)$, too. This case belongs to a finer theory.

3.9. Now we wish to extend the above mentioned fine and rough theory for the Hermite—Fejér interpolation. This program also was suggested in [2] and [1, XIX]. At first we quote a theorem which can be included in the fine theory but it was essentially proved by G. GRÜNWALD ([3]).

THEOREM 3.8. *If* $\lambda_n \sim \ln n$ *or* $\lambda_n \sim n^\beta$ $(0<\beta<0.5)$ *then there exists a* $Z_{\{\lambda_n\}}= =Z\in\Gamma(\lambda_n)$ *so that the function-class* C *is* H*-good for* Z.

3.9.1. A consequence of Theorems 3.1 and 3.8 is the following. Let F be a non-trivial function-class with $F\subseteqq C$. If $0<\beta<0.5$ then F is neither H-good nor H-bad for $\Gamma(n^\beta)$.

3.10. It seems that $\lambda_n(L, X)$ is not suitable for investigating the convergence-divergence behaviour of the Hermite—Fejér interpolation. This may be true for H but as to H^*, considering Theorem 3.6, we can expect a better behaviour. First we prove the following theorem. Using the same matrix as in Theorem 3.1 we get

THEOREM 3.9. *For each* $\{\lambda_n\}$ $(\lambda_n \geqq \ln n)$ *there exists an* $Y\in\Gamma(\lambda_n)$ *so that for any fixed* $u\neq 0$, $u\in[-1, 1]$ *we can define* $f_3(x)\in C(\omega_m)$ $(m\geqq 2)$ *for which*

$$\overline{\lim_{n\to\infty}} \frac{H_n^*(f_3; Y, u)-f_3(u)}{\omega_m\left(\dfrac{1}{n\lambda_n}\right) n\lambda_n^3} > 1, \tag{3.13}$$

supposing $\lim\limits_{n\to\infty} \omega_m\left(\dfrac{1}{n\lambda_n}\right) n\lambda_n^3=\infty$.

3.10.1. By (3.10) and (3.13) we can assert the following statement which can be included in the fine and rough theory of Hermite—Fejér interpolation.

Let $f\in W^{(2)}(\alpha)\big(=\{f(x);\ f^{(2)}(x)\in \mathrm{Lip}\,\alpha\}\big)$ and $\lambda_n\sim n^\beta$ $(\beta>0)$.

If $\alpha>3\beta$ then $W^{(2)}(\alpha)$ is H^*-good for $\Gamma(n^\beta)$; on the other hand if $0<\alpha<1$ and $\alpha<\dfrac{\beta-1}{\beta+1}$ then $W^{(2)}(\alpha)$ is not H^*-good for $\Gamma(n^\beta)$.

3.10.2. The above mentioned theorems on fine and rough theory of the Hermite—Fejér interpolation mean only the initial steps. We wish to do a much more detailed investigation in another paper.

3.11. By Theorem 3.9 we can state

THEOREM 3.10. *If for a certain* $\omega_m(t)$ $(m\geqq 2)$ *and* $\{\lambda_n\}$ $(\lambda_n\geqq \ln n)$ *we have*

$$\lim_{n\to\infty} \omega_m\left(\frac{1}{n}\right)\lambda_n = 0 \quad \textit{and} \quad \lim_{n\to\infty} \omega_m\left(\frac{1}{n\lambda_n}\right) n\lambda_n^3 = \infty \tag{3.14}$$

then with a suitable matrix $Y\in Y_{\{\lambda_n\}}\in\Gamma(\lambda_n)$ $\lim\limits_{n\to\infty} \|f(x)-L_n(f; Y, x)\|=0$ *holds for each* $f\in C(\omega_m)$ *but for a certain* $f_3(x)\in C(\omega_m)$ *and any fix* $u\neq 0$ $(u\in[-1, 1])$

$$\overline{\lim_{n\to\infty}}\, [H_n^*(f_3; Y, u)-f_3(u)] = \infty.$$

E.g., (3.14) is valid if $C(\omega_m)=W^{(2)}(\alpha)$, $\lambda_n\sim n^\beta$ and $0<\alpha<1$, $\alpha>\beta-2$, $\alpha< <(\beta-1)(\beta+1)^{-1}$ (see 1. b) and [7]).

4. Proofs

4.1. PROOF OF THEOREM 3.1. Let $T_n(x)=\cos n\vartheta$, $x=\cos\vartheta$. If T is the Chebyshev matrix, i.e.

$$x_{k,n}=\cos\frac{2k-1}{2n}\pi,\quad \vartheta_{k,n}=\frac{2k-1}{2n}\pi\quad (k=1,2,\dots,n), \tag{4.1}$$

then we define $Y=Y_{\{\lambda_n\}}$ for $n=2s-1$ $(s\geqq 2)$ as

$$\begin{cases} y_{k,n}=x_{k,n} & (k=1,2,\dots,n),\\ y_{0,n}=\cos\left(\frac{\pi}{2}-\varrho_n\right) & \text{with}\quad 0<\varrho_n=O\left(\frac{1}{n\log n}\right),\quad n=2s-1.\end{cases} \tag{4.2}$$ [2]

It is easy to verify the following relations, sometimes omitting the superfluous indices.

$$l_{k,n}(Y,x)=l_{k,n}(T,x)\frac{x-y_0}{y_k-y_0}\quad (k=1,2,\dots,n), \tag{4.3}$$

$$l_{0,n}(Y,x)=\frac{T_n(x)}{T_n(y_0)}, \tag{4.4}$$

$$l'_{k,n}(Y,y_k)=\frac{y_k}{2(1-y_k^2)}+\frac{1}{y_k-y_0}\quad (k=1,2,\dots,n), \tag{4.5}$$

$$l'_{0,n}(Y,y_0)=\frac{T'_n(y_0)}{T_n(y_0)}, \tag{4.6}$$

$$|T_n(y_0)|=|\sin n\varrho_n|\sim n\varrho_n,\quad |T'_n(y_0)|=|n\cos n\varrho_n|\sim n. \tag{4.7}$$

4.11. Now we prove

LEMMA 4.1. *We have uniformly in* x

$$\sum_{\substack{k=1\\k\neq s}}^{n}|l_{k,n}(Y,x)|=O(\ln n)\quad (-1\leqq x\leqq 1). \tag{4.8}$$

Indeed, using $l_{k,n}(T,x)=(-1)^{k-1}T_n(x)\sin\vartheta_k n^{-1}(x-x_k)^{-1}$ and (4.3),

$$\sum_{\substack{k=1\\k\neq s}}^{n}|l_{k,n}(Y,x)|=\sum_{\substack{k=1\\k\neq s}}^{n}|l_{k,n}(T,x)|\frac{|y_0-y_k+y_k-x|}{|y_0-y_k|}=$$

$$=O(1)\left[\sum_{\substack{k=1\\k\neq s}}^{n}|l_{k,n}(T,x)|+\frac{1}{n}\sum_{\substack{k=1\\k\neq s}}^{n}\frac{1}{|y_0-y_k|}\right]==O(\ln n).$$

Here we supposed $x\neq y_i$ $(i=0,1,\dots,n)$ because of $l_k(Y,y_i)=\delta_{k,i}$, and we used $\sum_{k=1}^{n}|l_k(T,x)|=O(\ln n)$.

[2] Now the n-th row of Y contains $n+1$ nodes; the degree of $\Omega_n(Y,x)$ equals $n+1$, etc. (compare (2.1)—(2.9)).

4.12. With similar estimations we prove

LEMMA 4.2. *We have for each fixed* $x\in[-1, 1]$

$$\sum_{\substack{k=1\\k\neq s}}^{n} |y_{k,n}|\,|v_{k,n}(Y, x)|\,l_{k,n}^2(Y, x) = O_x(1). \tag{4.9}$$

If $x=y_{k,n}$ $(k=0, 1, \dots, n)$ then $\sum\dots<1$. So we suppose $x\neq y_{k,n}$. Let further

$$\min_{\substack{k=1,2,\dots,n\\k\neq s}} |x-y_{k,n}| = |x-y_{j,n}|.$$

Then using (4.5), (4.3), (2.3) and (4.2) we have for $n\geqq n_0(x)$

$$\sum_{\substack{k=1\\k\neq s}}^{n} |y_{k,n}v_{k,n}(Y, x)|\,l_{k,n}^2(Y, x) = O(1)\sum_{\substack{k=1\\k\neq s}}^{n} |y_k|\left[\left(\frac{|y_k|}{\sin^2\vartheta_k}+\frac{1}{|y_k-y_0|}\right)|x-y_k|+1\right]\cdot$$

$$\cdot\frac{T_n^2(x)\cdot\sin^2\vartheta_k}{n^2(x-y_k)^2}\frac{(x-y_0)^2}{(y_k-y_0)^2} = O(1)\sum_{\substack{k=1\\k\neq s}}^{n}\left[\frac{T_n^2(x)}{n^2|x-y_k|}+\frac{T_n^2(x)\sin^2\vartheta_k}{n^2|x-y_k|y_k^2}+\frac{T_n^2(x)\sin^2\vartheta_k}{n^2(x-y_k)^2}\frac{1}{|y_k|}\right].$$

Here

$$\sum_{\substack{k=1\\k\neq s}}^{n}\frac{T_n^2(x)}{n^2|x-y_k|} = \begin{cases} O(n^{-2})\left[\sum\limits_{|y_k|\geqq\frac12}\sin^{-2}\dfrac{\vartheta_k}{2}+\sum\limits_{|y_k|<\frac12}1\right] = O(1) & \text{if } x=\pm1,\\[2ex] O_x(n^{-2})\left[\sum\limits_{|x-y_k|\leqq c}\dfrac{n}{|j-k|+1}+\sum\limits_{\substack{|x-y_k|>c\\k\neq s}}1\right] = O_x\left(\dfrac{\ln n}{n}\right) & \text{if } x\neq\pm1,\end{cases} \tag{4.10}$$

$$\sum_{\substack{k=1\\k\neq s}}^{n}\frac{T_n^2(x)\sin^2\vartheta_k}{n^2|x-y_k|y_k^2} = \begin{cases} O(n^{-2})\sum\limits_{\substack{k=1\\k\neq s}}^{n} y_k^{-2} = O(1) & \text{if } x=\pm1,\\[2ex] O_x(n^{-2})\left[\sum\limits_{|x-y_k|\leqq c}\dfrac{n}{|j-k|+1}+\sum\limits_{\substack{|x-y_k|>c\\k\neq s}} y_k^{-2}\right] = O_x(1) & \text{if } x\neq\pm1,\end{cases} \tag{4.11}$$

$$\sum_{\substack{k=1\\k\neq s}}^{n}\frac{T_n^2(x)\sin^2\vartheta_k}{n^2(x-y_k)^2|y_k|} = \begin{cases} O(n^{-2})\left[\sum\limits_{|y_k|\geqq\frac12}\sin^{-2}\dfrac{\vartheta_k}{2}+\sum\limits_{\substack{|y_k|<\frac12\\k\neq s}}\dfrac{n}{|k-s|}\right] = O(1) & \text{if } x=\pm1,\\[2ex] O_x(n^{-2})\left[\sum\limits_{|x-y_k|\leqq c}\dfrac{n^2}{(j-k)^2+1}+\sum\limits_{\substack{|x-y_k|>c\\k\neq s}}|y_k|^{-1}\right] = O_x(1) & \text{if } x\neq\pm1.\end{cases} \tag{4.12}$$

Here $x\neq0$ and $2c=2c(x)=\min(1+x, 1-x)$.

4.13. LEMMA 4.3. *For each* $x\in[-1, 1]$

$$|l_{s,n}(Y, x)| = O\left(\frac{1}{n\varrho_n}\right), \tag{4.13}$$

$$|l_{0,n}(Y, x)| = O\left(\frac{1}{n\varrho_n}\right). \tag{4.14}$$

As for (4.13), we can suppose $x\neq 0$. But then $|l_{s,n}(Y, x)| = |T_n(x)n^{-1}x^{-1}(x-y_0)y_0^{-1}|$, from where we get (4.13). (For $x<0$, $|x|<2y_0$ we use $|T_n(x)|\sim|nx|$.) To prove (4.14) we use (4.4) and (4.7).

4.14. Using (4.8), (4.13), (4.14) and $|l_0(Y, 1)|\sim(n\varrho_n)^{-1}$ we have

$$\lambda_n(L, Y)\sim\frac{1}{n\varrho_n}. \tag{4.15}$$

4.15. Later we shall use the important

LEMMA 4.4. *For each fixed* $x\neq 0$ *there exist* $N_x\subset P\overset{\text{def}}{=}\{2k-1\}_{k=1}^{\infty}$ *such that*

$$|l_{0,n}(Y, x)|\sim\frac{1}{n\varrho_n}\quad(n\in N_x). \tag{4.16}$$

Indeed, if $x\neq 0$ and fixed, then for a suitable N_x, $|T_n(x)|\geqq c_1(x)>0$ $(n\in N_x)$ (see, e.g., the ideas of [9, 14.4 (2)]).

4.16. LEMMA 4.5. *For each fixed* $x\neq 0$

$$\frac{|v_{0,n}(Y, x)|\,l_{0,n}^2(Y, x)}{|x|}\sim\frac{1}{n^2\varrho_n^3}\quad \textit{if}\quad n\in N_x. \tag{4.17}$$

We get (4.17) if we use (2.5), (4.6), (4.7) and (4.16).

4.17. Finally, considering the relations (2.5), (4.17), $y_0\sim\varrho_n$, $y_s=0$ and (4.9) we get

$$|H_n(f_1; Y, x)| = \left|\sum_{k=0}^{n} y_k h_k(Y, x)\right| \geqq |y_0||v_0(x)|l_0^2(x) - \sum_{k=1}^{n}|y_k||h_k(x)| \geqq \frac{c_2(x)}{n^2\varrho_n^2}$$

$$(n\in N_x;\ x\neq 0,\ f_1\equiv x).$$

With $\varrho_n\sim(n\lambda_n)^{-1}$ we obtain (3.4).

4.2. PROOF OF THEOREM 3.2. F_1 is non-trivial so $x\in F_1$. But then by (3.4) we get our statement with Y defined by (4.2), because F_1 is not H-good for Y.

4.3. PROOF OF THEOREM 3.3. The relation (3.6) immediately follows from (3.5). To prove (3.7) let

$$f_2(x) = \begin{cases} x & \text{if } -1\leqq x\leqq 0.5,\\[4pt] 0.5-\dfrac{\omega(x-0.5)}{2\omega(0.5)} & \text{if } 0.5\leqq x\leqq 1.\end{cases} \tag{4.18}$$

It is easy to see that $\omega(f_2; t)=\omega(t)$ $(0\leqq t\leqq 0.5)$ further $|f_2(x)|\leqq|x|$ $(-1\leqq x\leqq 1)$. So $f_2(x)\in C(\omega)$ and by (4.9)

$$\sum_{k=1}^{n} |f_2(y_k)|\,|h_k(x)| = O_x(1)$$

with the same matrix $Y_{\{\lambda_n\}}$. The remaining part is the same as that of Theorem 3.1. For completing our statement we remark that by (4.18), $f_2'(0.5)$ does not exist.

4.3.1. If we had wanted to prove only Theorem 3.1, we could have argued as follows. At first we prove

$$\sum_{\substack{k=1\\k\neq s}}^{n} l_{k,n}^2(Y, x) = O_x(1).$$

Then using

$$x \equiv H_n(f_1; Y, x)+\sum_{k=1}^{n} \mathfrak{H}_{k,n}(Y, x)$$

we get

$$H_n(f_1; Y, x) = (y_0-x)\,l_0^2(x)+(y_s-x)\,l_s^2(x)+O(1).$$

By (4.16) we get (3.4) if $x\neq 0$. For $x=0$ let $n=2s$, $y_k=x_k$ $(k=1, 2, \ldots, n)$ and $y_0=\cos(\vartheta_s-\varrho_n)$. We get as above

$$H_n(f_1; Y, 0) = y_0\,l_0^2(0)+y_s\,l_s^2(0)+O(1)\sim n^{-1}\lambda_n^2 \quad (s = 1, 2, \ldots)$$

from where we obtain a divergence-theorem for $x=0$ if $n=O(\lambda_n^2)$. But if we wish to prove (3.7) we must have

$$\sum_{\substack{k=1\\k\neq s}}^{n} |f_2(y_k)|\,|h_k(x)| = O_x(1) \tag{4.19}$$

which came from (4.9). We omit the details.

4.4. PROOF OF THEOREM 3.4. For an arbitrary polynomial $P_n(x)$ of degree $\leqq n$ we have the Markov-inequality

$$|P_n'(x)| \leqq n^2\|P_n(x)\| \qquad (x\in[-1, 1]) \tag{4.20}$$

So we obviously have

$$|l_k'(x_k)| \leqq n^2\lambda_n(L, X) \quad (k = 1, 2, \ldots, n). \tag{4.21}$$

So using (2.5) and (4.21) we get

$$|v_{k,n}(X, x)| = O(1)n^2\lambda_n(L, X) \qquad (n \geqq 1,\ k = 1, 2, \ldots, n). \tag{4.22}$$

By (2.6), (4.22) and (3.1)

$$\lambda_n(H, X) = \left\|\sum_{k=1}^{n} |v_k(x)|\,l_k^2(x)\right\| = O(1)n^2\lambda_n(L, X)\left\|\sum_{k=1}^{n} l_k^2(x)\right\| = O(1)n^2\lambda_n^3(L, X),$$

as we stated.

4.5. PROOF OF THEOREM 3.5. Let $P_n(f; x)$ be a polynomial of degree $\leqq n$ for which

$$\|f(x)-P_n(f; x)\| = O(1)\,\omega_m\left(\frac{1}{n}\right), \tag{4.23}$$

$$\|f'(x)-P_n'(f; x)\| = O(n)\,\omega_m\left(\frac{1}{n}\right) \tag{4.24}$$

if $f\in C(\omega_m)$ $(m\geqq 2)$. (As for the existence of $P_n(f; x)$ see e.g. [9, Theorem 1.3.3].) Then

$$|H_n^*(f; X, x)-f(x)| = |H_n^*(f; x)-P_n(f; x)+P_n(f; x)-f(x)| \leqq$$

$$\leqq \sum_{k=1}^{n} |f(x_k)-P_n(x)||h_k(x)| + \sum_{k=1}^{n} |f'(x_k)-P_n'(x_k)||x-x_k| l_k^2(x) +$$

$$+O(1)\,\omega_m\left(\frac{1}{n}\right) = O(1)\,\omega_m\left(\frac{1}{n}\right)[\lambda_n(H, X)+n\lambda_n^2(L, X)+1]$$

from where we get (3.9).

4.6. PROOF OF THEOREM 3.7. Let

$$\begin{cases} y_{k,n} = \cos\dfrac{2k-1}{2n}\pi \\ y_{0,n} = \cos\left(\dfrac{\pi}{2n}-\varrho_n\right), \quad 0<\varrho_n = O\left(\dfrac{1}{n\ln n}\right) \quad (n=1, 2, \dots). \end{cases} \tag{4.25}$$

We can verify the following relations

$$|l_{1,n}(Y, x)| = O(n^{-2}d_n^{-1}), \quad |l_{0,n}(Y, x)| = O(n^{-2}d_n^{-1}),$$

$$|l_{0,n}'(Y, y_0)| \sim \frac{1}{d_n}, \quad \sum_{k=2}^{n} |l_{k,n}(Y, x)| = O(\ln n),$$

where $d_n = d_n(Y) = \min\limits_{0\leqq k\leqq n-1}(y_k-y_{k+1}) = y_0-y_1 \sim \varrho_n n^{-1}$ (see (4.3)—(4.7)). But $|l_0(Y, -1)| \sim n^{-2}d_n^{-1}$ so $\lambda_n(L, Y) \sim |l_0(Y, -1)| \sim n^{-2}d_n^{-1}$. Now we get

$$\lambda_n(H, Y) \geqq |v_0(Y, -1)| l_0^2(Y, -1) \sim d_n^{-1}\lambda_n^2(L, Y) \sim n^2\lambda_n^3(L, Y),$$

as we stated.

4.7. PROOF OF THEOREM 3.8. Let us denote by $z_{k,n}^{(\gamma,\gamma)}$ the roots of the Jacobi polynomials $P_n^{(\gamma,\gamma)}(x)$ ([9, § 2.4.1]). If $-1<\gamma<0$ then these roots form a strictly normal system ([4, Part III, § 6]), and hence $H_n(f; Z; x)$ uniformly tends to $f(x)$ for each $f\in C$ ([3, Theorem 3]). The only relation we have to refer to is $\lambda_n(L, Z) \sim \ln n\,(\gamma=-0.5)$ or $\lambda_n(L, Z) \sim n^{\gamma+0.5}$ $(-0.5<\gamma<0)$ ([9, 14.4]).

4.8. Proof of Theorem 3.9. We use the matrix defined by (4.2) and [5, Theorem 2.1]. We only sketch the proof. Let

$$g_n(y_k)=\begin{cases}\operatorname{sign} h_0(Y,u) & \text{if} \quad k=0,\\ 0 \quad \text{for} \quad k=1,2,\dots,n+1 & \text{where}\\ y_{n+1,n}=0.5y_{0,n};\ n\in N_u.\end{cases} \tag{4.26}$$

In the intervals defined by y_i $(i=0,1,\dots,n+1)$ let $g_n(x)$ be the Hermite interpolatory polynomials of degree $2m-1$ which is equal to 0 or 1 according to (4.16). Let further at these end-points $g_n'(y_i)=g_n''(y_i)=\dots=g_n^{(m-1)}(y_i)=0$ $(k=0,1,2,\dots,n+1)$. For $x\in[-1,y_n]\cup[y_0,1]$ let $g_n(x)=0$.

Let us notice that we can choose a set $N_1\subset N_u$ such that

$$g_n(x)\cdot g_N(x)=0 \quad (x\in[-1,1],\ n,\ N\in N_1,\ N>n). \tag{4.27}$$

Using the correspondences $z_n=u$, $\Lambda_n(z_n)=|h_0(Y,u)|\sim|u|n\lambda_n^3$, $\delta_n=\varrho_n(\sim n^{-1}\lambda_n^{-1})$, $T_n=H_n$, $U_n=E$ (unit-operator), $e_n=1$ and $C_p^{\{a,b\}}(\omega_m)=C(\omega_m)$ we have that for a certain

$$h(x)=c_2\sum_{n\in N_2}\omega_m(\varrho_n)g_n(x) \quad (c_2>0,\ N_2\subset N_1)$$

$h(x)\in C(\omega_m)$, moreover,

$$H_n(h;Y,u)-h(u)>\omega_m(\varrho_n)|u|n\lambda_n^3 \quad (n\in N_2).$$

(To obtain the conditions A, B and C of [5, Theorem 2.1] see (4.16), (4.17) and (4.27) (for A) (4.27) and [6, Lemma 2.1] (for B) and [5, C1] (for C)).

If we remark that

$$|h'(y_{k,r})|=c_2\Big|\sum_{n\in N_2}\omega_m(\varrho_n)g_n'(y_{k,r})\Big|=\Big|c_2\sum_{\substack{n\in N_2\\ n<r}}\omega_m(\varrho_n)g_n'(y_{k,r})\Big|=$$

$$=O(1)\sum_{\substack{n\in N_2\\ n<r}}\varrho_n\omega(\varrho_n)\varrho_n^{-1}=O(1) \quad (\text{if } N_2 \text{ is lacunary enough}),$$

we have our statement if $y_{n+1,n}>y_{s-1,N}$ $(n<N,\ n,\ N\in N_2)$. Indeed, by the above estimations and (4.3)

$$\Big|\sum_{k=0}^{n}h'(y_{k,n})(x-y_{k,n})l_{k,n}^2(Y,x)\Big|=O(1)\sum_{k\le s-1}\frac{\sin^2\vartheta_k T_n^2(x)(x-y_0)^2}{n^2|x-x_k|(x_k-y_0)^2}=$$

$$=O(1)\Big[\sum_{|x|\le 2x_k}+\sum_{|x|>2x_k}\Big]=O(1)$$

from where we get (3.14).

5. Remarks

5.1. In his paper [8], D. L. Berman proved that for the equidistant nodes $x_{k,n}=-1+2k(n-1)^{-1}$ $(k=0,1,\dots,n-1)$ in $[-1,1]$, the sequence $\{H_n(f_1;X,x)\}$ diverges unboundedly for $n\to\infty$ at each point of $[-1,1]$ except the point $x=0$ (compare with Theorem 3.1).

5.2. We can prove theorems for the trigonometric case, too.

5.3. It is easy to see that the order of divergence in (3.4) is the best possible.

5.4. The estimation (3.8) was proved by M. Sallay (oral communication). I was able to prove only the weaker estimation

$$\lambda_n(H, X) = O(1)n^2 \ln n \lambda_n^3(L, X) \quad (n \geqq 2).$$

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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ÜBER DIE LEBESGUESCHEN FUNKTIONEN. II

Von

K. TANDORI (Szeged), Mitglied der Akademie

1. Für ein orthonormiertes System $\varphi=\{\varphi_n(x)\}_1^\infty$ im Intervall $(0, 1)$ betrachten wir die Lebesgueschen Funktionen

$$L_n(\varphi; x) = \int_0^1 \left|\sum_{k=1}^n \varphi_k(x)\varphi_k(t)\right| dt \quad \big(x\in(0, 1);\; n = 1, 2, \ldots\big).$$

Es sei $\lambda=\{\lambda_n\}_1^\infty$ eine nichtabnehmende Folge von positiven Zahlen mit $\lambda_1\geqq 1$.
Mit Ω_λ, bzw. Ω_λ^* bezeichnen wir die Klasse der orthonormierten Funktionensysteme φ, für die

$$L_n(\varphi; x) = O(\lambda_n) \quad \big(x\in(0, 1)\big) \tag{1}$$

bzw.

$$L_n(\varphi/\sqrt{\lambda}; x) = O(1) \quad \big(x\in(0, 1)\big) \tag{2}$$

erfüllt ist.

Durch einfache Rechnung folgt, daß sich aus (2) die Abschätzung (1) ergibt, also gilt

$$\Omega_\lambda^* \subseteqq \Omega_\lambda. \tag{3}$$

Weiterhin kann man leicht einsehen, daß im Falle $\lambda_n=O(1)$ aus (1) auch (2) folgt, d.h. im Falle $\lambda_n=O(1)$ besteht

$$\Omega_\lambda^* = \Omega_\lambda.$$

Es sei ferner $M(\lambda)$, bzw. $M^*(\lambda)$ die Klasse der Koeffizientenfolgen $a=\{a_n\}_1^\infty$, für die die Reihe

$$\sum_{n=1}^\infty a_n\varphi_n(x) \tag{4}$$

bei jedem System φ aus Ω_λ, bzw. aus Ω_λ^* im Intervall $(0, 1)$ fas überall konvergiert.

Aus (3) erhalten wir:

$$M(\lambda) \subseteqq M^*(\lambda). \tag{5}$$

Nach den obigen besteht

$$M(\lambda) = M^*(\lambda) \quad \big(\lambda_n = O(1)\big). \tag{6}$$

Diese Gleichung ist im Falle $\lambda_n\nearrow\infty$ nicht bewiesen.

Mit $l^2(\lambda)$ bezeichnen wir die Klasse der Folgen a mit

$$\sum_{n=1}^{\infty} a_n^2 \lambda_n < \infty.$$

Es sei weiterhin M die Klasse der Folgen $a=\{a_n\}_1^\infty$, mit $|a_n| \geqq |a_{n+1}|$ $(n=1, 2, \ldots)$. In dieser Note werden wir die folgende Behauptung beweisen:

SATZ I. *Es sei* λ *von unten konkav mit* $\lambda_n = O(\log^2 n)$. *Dann gilt*

$$M(\lambda) \cap M = M^*(\lambda) \cap M \quad (= l^2(\lambda) \cap M).$$

D.h. im Falle der Reihen mit dem absoluten Betrage nach monoton nichtwachsenden Koeffizienten sind die Bedingungen (1) und (2) im allgemeinen gleichbedeutend.

Nach (6) braucht man den Satz I nur im Falle $\lambda_n \nearrow \infty$ zeigen.

Nach einem bekannten Sazt von S. KACZMARZ [2] und (5) gilt

$$l^2(\lambda) \subseteqq M(\lambda) \subseteqq M^*(\lambda)$$

für monotone Folgen λ. Weiterhin hat der Verf. gezeigt, daß unter den Bedingungen des Satzes I im Falle

$$a \notin l^2(\lambda), \quad a \in M$$

es ein orthonormiertes System φ mit (1) gibt derart, daß die Reihe (4) im Intervall (0, 1) fast überall divergiert [4]. Somit folgt Satz I aus dem folgenden

SATZ II. *Es sei* λ *von unten konkav mit* $\lambda_n \nearrow \infty$, $\lambda_n = O(\log^2 n)$. *Ist* $a \in M$ *und* $a \notin l^2(\lambda)$, *dann gibt es ein orthonormiertes System* φ *mit* (2) *derart, daß die Reihe* (4) *im Intervall* (0, 1) *fast überall divergiert.*

2. VORBEREITUNGEN. Zum Beweis des Satzes II brauchen wir zwei bekannte Hilfssätze.

HILFSSATZ I ([1], S. 46—50). *Für das Haarsche System* $\chi = \{\chi_m(x)\}_1^\infty$ *gilt*

$$L_n(\chi, x) \leqq 1 \quad (x \in (0, 1);\ n = 1, 2, \ldots).$$

HILFSSATZ II ([3], Hilfssatz XIV). *Es seien* $p(\geqq 2)$, q *natürliche Zahlen. Dann gibt es ein in* (0, 1) *orthonormiertes System von Treppenfunktionen* $g_l(p, q; x)$ $(l=1, \ldots, 2pq)$ *mit den folgenden Eigenschaften: Es gilt*

$$\int_0^1 \left| \sum_{l=1}^{n} g_l(p, q; x) g_l(p, q; t) \right| dt \leqq C_1 \log^2 p \quad (x \in (0, 1);\ n = 1, \ldots, 2pq;\ C_1 \geqq 1),$$

und es gibt eine in einfache Menge $E(\subseteqq (0, 1))$ *(d.h. ist* E *die Vereinigung endlichvieler Intervalle) mit* $m(E) = \dfrac{1}{5}$ *derart, daß für jedes* $x \in E$ *ein Index* $m(x)(<2pq)$ *existiert mit* $g_l(p, q; x) \geqq 0$ $(l=1, \ldots, 2pq)$ *und*

$$\sum_{l=1}^{m(x)} g_l(p, q; x) \geqq C_2 \sqrt{2pq} \log p.$$

($C_1, C_2, \ldots$ bezeichnen positive Konstanten. In dieser Note bezeichnet $\log \alpha$ den Logarithmus mit der Basis 2.)

3. Beweis des Satzes II. Es sei $a=\{a_n\}_1^\infty$ eine Folge aus M mit $a \notin l^2(\lambda)$. Dann gibt es eine nichtabnehmende, nach Unendlich strebende, von unten konkave Folge $\mu=\{\mu(n)\}_1^\infty$ mit $\mu(1)\geqq 1$ und

$$\frac{\mu(n)}{\lambda_n} \searrow 0 \quad (n\to\infty), \tag{7}$$

für die

$$\sum_{n=1}^{\infty} a_n^2 \mu(n) = \infty \tag{8}$$

besteht.

Nach der Voraussetzung für λ und nach (7) gibt es eine positive Konstante $C_3(\geqq 1)$ mit

$$\mu(n) \leqq C_3 \log^2 n \quad (n = 2, 3, \ldots). \tag{9}$$

Wir werden eine Indexfolge $\{m_k\}$ folgenderweise definieren: Es sei $m_1=1$ und m_{k+1} die kleinste natürliche Zahl mit $\mu(m_{k+1})>2\mu(m_k+1)$ $(k=1, 2, \ldots)$. Wegen der Konkavität gilt

$$\frac{\mu(2m_k)-\mu(m_k+1)}{m_k-1} \leqq \frac{\mu(m_k+1)-\mu(m_1)}{m_k-m_1+1},$$

woraus

$$\mu(2m_k)-\mu(m_k+1) \leqq \frac{m_k-1}{m_k}\mu(m_k+1) \leqq \mu(m_k+1)$$

folgt. Nach der Definition von m_{k+1} gilt also $m_{k+1}>2m_k$ $(k\geqq 2)$. Daraus erhalten wir nach (9)

$$C_3 \log^2 (m_{k+1}-m_k) > C_3 \log^2 m_k \geqq \mu(m_k) \quad (k = 2, 3, \ldots).$$

Ist k genügend groß $(k>k_0)$, dan gelten die Ungleichungen

$$\mu(m_k) \leqq \mu(m_k-1)+1, \quad \mu(m_k)/C_1C_3 \geqq 8,$$

und es gibt eine natürliche Zahl $\bar{q}_k$ mit $m_{k+1}-m_k>2\bar{q}_k$ und

$$4C_1C_3 \leqq \frac{\mu(m_k)}{2} \leqq C_1C_3 \log^2 \left[\frac{m_{k+1}-m_k}{2\bar{q}_k}\right] \leqq \mu(m_k).$$

($[\alpha]$ bezeichnet den ganzen Teil von α; die Konstanten C_1, C_3 sind in dem Hilfssatz II, bzw. in (9) definiert.) Es seien $n_0=1$, $n_k=m_{k+k_0}$, $q_k=\bar{q}_{k+k_0}$ $(k=1, 2, \ldots)$. Dann gelten die Beziehungen

$$n_{k+1} > 2n_k \quad (k = 1, 2, \ldots), \tag{10}$$

$$\mu(n_{k+1}) \leqq 4\mu(n_k-1) \quad (k = 1, 2, \ldots), \tag{11}$$

$$4C_1C_3 \leqq \frac{\mu(n_k)}{2} \leqq C_1C_3 \log \left[\frac{n_{k+1}-n_k}{2q_k}\right] \leqq \mu(n_k) \quad (k = 1, 2, \ldots). \tag{12}$$

Ohne Beschränkung der Allgemeinheit können wir $a_n \geqq 0$ $(n=1, 2, \ldots)$ voraussetzen.

Wir werden erstens für jedes k $(k \geqq 2)$ ein in $(0, 1)$ orthonormiertes System von Treppenfunktionen $\varphi_l(k; x)$ $(l=n_k, \ldots, n_{k+1}-1)$ definieren derart, daß für jedes k gilt:

$$\int_0^1 \left| \sum_{l=n_k}^{n} \frac{\varphi_l(k; x)\varphi_l(k; t)}{\lambda_l} \right| dt \leqq C_4 \frac{\mu(n_k)}{\lambda_{n_k}} \quad (x \in (0, 1);\ n = n_k, \ldots, n_{k+1}-1), \tag{13}$$

$$\max_{n_k \leqq n \leqq m < n_{k+1}} |a_n \varphi_n(k; x) + \ldots + a_m \varphi_m(k; x)| \geqq C_5 A_k \quad (x \in E_k), \tag{14}$$

wobei

$$A_k = \left\{ \sum_{i=1}^{C(k)} (n_k - n_{k-1}) a^2_{n_k + i(n_k - n_{k-1})} \mu(n_k) \right\}^{1/2}, \quad c(k) = \left[\frac{n_{k+1} - n_k}{n_k - n_{k-1}} \right]$$

bedeutet, und für die einfache Menge $E_k(\subseteqq (0, 1))$ besteht

$$m(E_k) = \frac{1}{10}. \tag{15}$$

Wir werden zu diesem Zweck den Hilfssatz II im Falle

$$p = \left[\frac{n_k - n_{k-1}}{2q_{k-1}} \right], \quad q = q_{k-1}$$

an. Die entsprechenden Funktionen bezeichnen wir mit $g_s(x)$ $(s=1, \ldots, 2pq)$. Dann gelten auf Grund des Hilfssatzes II und der Ungleichung (12):

$$\int_0^1 \left| \sum_{s=1}^{\sigma} g_s(x) g_s(t) \right| dt \leqq \mu(n_k) \quad (x \in (0, 1);\ \sigma = 1, \ldots, 2pq), \tag{16}$$

$$\max_{1 \leqq m < n < 2pq} |a_{n_k + (i-1)(n_k - n_{k-1}) + m} g_m(x) + \ldots + a_{n_k + (i-1)(n_k - n_{k-1}) + n} g_n(x)| \geqq \tag{17}$$

$$\geqq C_6 \sqrt{n_k - n_{k-1}}\, a_{n_k + i(n_k - n_{k-1})} \sqrt{\mu(n_k)} \quad (x \in E,\ i = 1, \ldots, c(k)).$$

Aus (16) folgt auf Grund von (7);

$$\int_0^1 \left| \sum_{s=1}^{\sigma} \frac{g_s(x) g_s(t)}{\lambda_{n_k+s-1}} \right| dt \leqq \sum_{s=1}^{\sigma-1} \left(\frac{1}{\lambda_{n_k+s-1}} - \frac{1}{\lambda_{n_k+s}} \right) \int_0^1 \left| \sum_{l=1}^{s} g_l(x) g_l(t) \right| dt + \tag{18}$$

$$+ \frac{1}{\lambda_{n_k+\sigma}} \int_0^1 \left| \sum_{l=1}^{\sigma} g_l(x) g_l(t) \right| dt \leqq 2 \frac{\mu(n_k)}{\lambda_{n_k}} \quad (\sigma = 1, \ldots, 2pq).$$

Es seien $s_0 = 0$,

$$s_i = \sum_{j=1}^{i} a^2_{n_k + j(n_k - n_{k-1})} \Big/ 2 \sum_{j=1}^{c(k)} a^2_{n_k + j(n_k - n_{k-1})} \quad (i = 1, \ldots, c(k)),$$

$s_{c(k)+1}=1$ und $I_i=(s_{i-1},s_i)$ $(i=1,\dots,c(k)+1)$. Wit setzen

$$\varphi_n(k;x)=\begin{cases}\dfrac{1}{\sqrt{m(I_i)}}\,g_{n-(n_k+(i-1)(n_k-n_{k-1}))+1}\left(\dfrac{x-s_{i-1}}{m(I_i)}\right), & x\in I_i,\\ 0 \quad \text{sonst}\end{cases}$$

für $n_k+(i-1)(n_k-n_{k-1})\leqq n<n_k+(i-1)(n_k-n_{k-1})+2pq$ und $i=1,\dots,c(k)$.

Weiterhin sei $\bar{E}_i$ die Menge die aus E mit der linearen Transformation $y=(s_i-s_{i-1})x+s_{i-1}$ ensteht, und

$$E_k=\bigcup_{i=1}^{c(k)}\bar{E}_i.$$

Aus der Definition der Menge E_k und aus dem Hilfssatz II folgt (15). Endlich seien die Funktionen $\varphi_n(k;x)$ für $n_k+(i-1)(n_k-n_{k-1})+2pq\leqq n<n_k+i(n_k-n_{k-1})$; $i=1,\dots,c(k)$, und $n_k+c(k)(n_k-n_{k-1})+2pq<n<n_{k+1}$ der Reihe nach gleich den Funktionen

$$f_n(x)=\begin{cases}\sqrt{2}\,\chi_n(2(x-1/2)), & x\in I_{c(k)+1},\\ 0 & \text{sonst}\end{cases}\qquad (n=1,2,\dots).$$

Aus dem Hilfssatz I und aus (18) erhalten wir durch einfache Rechnung

$$\int_0^1\left|\sum_{l=n_k}^{n}\frac{\varphi_l(k;x)\varphi_l(k;t)}{\lambda_l}\right|dt\leqq\begin{cases}2\dfrac{\mu(n_k)}{\lambda_{n_k}}, & x\in\bigcup_{i=1}^{c(k)}I_i,\\[2mm] 2\dfrac{1}{\lambda_{n_k}}, & x\in I_{c(k)+1}\end{cases}\qquad (n_k\leqq n<n_{k+1}).$$

Daraus ergibt sich (13) wegen $\mu(n)\geqq 1$ $(n\geqq 1)$. Endlich folgt auf Grund der Definition der Funktionen $\varphi_l(k;x)$ und aus (17) auch die Ungleichung (14).

Aus (8), (10), (11), aus der Monotonie der Folge a und aus der Definition der $c(k)$ folgt

$$\text{(19)}\qquad \sum_{k=2}^{\infty}A_k^2=\sum_{k=2}^{\infty}\sum_{i=1}^{c(k)}(n_k-n_{k-1})a^2_{n_k+i(n_k-n_{k-1})}\mu(n_k)\geqq C_7\sum_{n=n_2}^{\infty}a_n^2\mu(n)=\infty.$$

Wir definieren eine Indexfolge $\{k_r\}_0^\infty$ folgenderweise: Es sei $k_0=2$; wenn k_r $(r\geqq 0)$ schon definiert ist, dann sei k_{r+1} die kleinste natürliche Zahl (größer als k_r), für die

$$\text{(20)}\qquad \sqrt{\sum_{k=k_r+1}^{k_{r+1}}A_k^2}\geqq r+1$$

und

$$\text{(21)}\qquad \frac{\mu(n)}{\lambda_n}\leqq\frac{1}{2^r}\quad (k_{r+1}\leqq n)$$

erfüllt sind. Wegen (7) und (19) existiert eine solche Indexfolge.

Für jeden Index $r(\geqq 1)$ definieren wir ein orthonormiertes System von Treppenfunktionen $\psi_n(r;x)$ $(n=n_{k_r+1},\dots,n_{k_{r+1}+1}-1)$ mit folgenden Eigenschaften:

Es gilt

$$\int_0^1 \left| \sum_{l=n_{k_r}+1}^{n} \frac{\psi_l(r;x)\psi_l(r;t)}{\lambda_l} \right| dt \leqq C_4 \frac{1}{2^r} \quad (x\in(0,1);\ n_{k_r+1} \leqq n < n_{k_{r+1}+1}) \tag{22}$$

und

$$\max_{n_{k_r+1}\leqq n\leqq m<n_{k_{r+1}+1}} |a_n\psi_n(r;x)+\ldots+a_m\psi_m(r;x)| \geqq C_5(r+1) \quad (x\in H_r), \tag{23}$$

wobei für die einfache Menge $H_r(\subseteqq(0,1))$ besteht

$$m(H_r) = \frac{1}{10}. \tag{24}$$

Es sei nähmlich $\bar{s}_0=0$ und

$$\bar{s}_i = \sum_{j=1}^{i} A^2_{k_r+j} \Big/ \sum_{j=k_r+1}^{k_{r+1}} A_j^2 \quad (i=1,\ldots,k_{r+1}-k_r),$$

und $\bar{I}_i=(\bar{s}_{i-1},\bar{s}_i)$ $(i=1,\ldots,k_{r+1}-k_r)$. Wir setzen

$$\psi_n(r;x) = \begin{cases} \dfrac{1}{\sqrt{m(\bar{I}_i)}}\,\varphi_n\left(k;\dfrac{x-\bar{s}_{i-1}}{m(\bar{I}_i)}\right), & x\in\bar{I}_i, \\ 0 \quad \text{sonst} \end{cases}$$

$$(n_{k_r+i} \leqq n < n_{k_r+i+1};\ i=1,\ldots,k_{r+1}-k_r).$$

Weiter sei $\bar{H}_i$ die Menge, die aus der Menge E_{k_r+i} mit der Transformation $y=(\bar{s}_i-\bar{s}_{i-1})x+\bar{s}_{i-1}$ entsteht. Wir setzen

$$H_r = \bigcup_{i=1}^{k_{r+1}-k_r} \bar{H}_i.$$

Aus der Definition der Menge $\bar{H}_i$ und aus (15) ergibt sich (24). Aus der Definition der Funktionen $\psi_n(r;x)$, und (14), (20) erhalten wir (23). Endlich aus der Definition der Funktionen $\psi_n(r;x)$ und aus (7), (13), (21) bekommen wir auch (22).

Endlich definieren wir durch Induktion ein orthonormiertes System von Treppenfunktionen $\varphi_n(x)$ $(n=1,2,\ldots)$ im Intervall $(0,1)$, und eine Folge von einfachen Mengen $G_r(\subseteqq(0,1))$ $(r=1,2,\ldots)$, für welche die folgenden Bedingungen erfüllt sind: Es gilt

$$\int_0^1 \left| \sum_{l=1}^{n} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt \leqq C_8 \quad (x\in(0,1);\ n=1,\ldots,n_{k_1+1}-1); \tag{25}$$

die Mengen G_r $(r=1,2,\ldots)$ sind stochastisch unabhängig und für jede natürliche Zahl r gilt

$$m(G_r) = \frac{1}{10}. \tag{26}$$

Ferner bestehen für jede natürliche Zahl r die Abschätzungen

$$\int_0^1 \left| \sum_{l=n_{k_r+1}}^{n} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt \leqq C_4 \frac{1}{2^r} \quad (x\in(0,1);\ n_{k_r+1} \leqq n < n_{k_{r+1}+1}), \tag{27}$$

$$\max_{n_{k_r+1}\leqq n\leqq m<n_{k_{r+1}+1}} |a_n\varphi_n(x)+\ldots+a_m\varphi_m(x)| \geqq C_5(r+1) \quad (x\in G_r). \tag{28}$$

Es sei $\varphi_n(x)=\chi_n(x)$ $(n=1, \ldots, n_{k_1+1}-1)$. Auf Grund des Hilfssatzes I erhalten wir dann durch einfache Rechnung, daß auch (25) besteht.

Es sei r_0 eine natürliche Zahl. Nehmen wir an, daß die Treppenfunktionen $\varphi_n(x)$ $(n=1, \ldots, n_{k_{r_0}+1}-1)$ und die einfachen Mengen $G_1, \ldots, G_{r_0-1}$ $(\subseteqq(0, 1))$ derart definiert sind, daß diese Funktionen in $(0, 1)$ ein orthonormiertes System bilden, diese Mengen stochastisch unabhängig sind, ferner für $r=1, \ldots, r_0-1$ (26), (27) und (28) bestehen.

Dann gibt es eine Einteilung des Intervalls $(0, 1)$ in endlichviele disjunkte Intervalle $J_s(s=1, \ldots, \sigma)$ derart, daß jede Funktion $\varphi_n(x)$ $(n=1, \ldots, n_{k_{r_0}+1}-1)$ in jedem Teilintervall J_s konstant und jede Menge G_r $(r=1, \ldots, r_0-1)$ die Vereinigung gewisser J_s ist. Die zwei Hälften des Intervalls J_s bezeichnen wir mit J_s' bzw. J_s''.

Wir setzen

$$\varphi_n(x) = \sum_{s=1}^{\sigma} \psi_n(r_0;\, J_s';\, x) - \sum_{s=1}^{\sigma} \psi_n(r_0;\, J_s'';\, x) \quad (n = n_{k_{r_0}+1}, \ldots, n_{k_{r_0+1}+1}-1),$$

wobei gilt:

$$\psi_n(r_0;\, J_s';\, x) = \begin{cases} \psi_n\left(r_0; \dfrac{x-a_s}{b_s-a_s}\right), & x\in J_s' = (a_s, b_s), \\ 0 & \text{sonst}, \end{cases}$$

$$\psi_n(r_0;\, J_s'';\, x) = \begin{cases} \psi_n\left(r_0; \dfrac{x-A_s}{B_s-A_s}\right), & x\in J_s'' = (A_s, B_s), \\ 0 & \text{sonst} \end{cases}$$

für $n=n_{k_{r_0}+1}, \ldots, n_{k_{r_0+1}+1}-1$. Weiter sei

$$G_{r_0} = \bigcup_{s=1}^{\sigma} \left(H_{r_0}(J_s') \cup H_{r_0}(J_s'')\right),$$

wobei $H_{r_0}(J_s')$, bzw. $H_{r_0}(J_s'')$ jene Menge bezeichnet die aus der Menge H_{r_0} durch die lineare Transformation entsteht, welche das Intervall $(0, 1)$ auf das Intervall J_s', bzw. auf das Intervall J_s'' abbildet.

Offensichtlich bilden die Treppenfunktionen $\varphi_1(x), \ldots, \varphi_{n_{k_{r_0+1}+1}-1}(x)$ ein orthonormiertes System in $(0, 1)$. Die einfachen Mengen $G_1, \ldots, G_{r_0}$ sind offensichtlich stochastisch unabhängig, ferner besteht (26) auch für $r=r_0$ auf Grund von (24). Aus (23) erhalten wir (28) für $r=r_0$ offensichtlich. Endlich, aus (22) bekommen wir (27) für $r=r_0$ durch einfache Rechnung.

Das ganze Funktionensystem $\{\varphi_n(x)\}$ und die Mengenfolge $\{G_r\}$ mit den geforderten Eigenschaften erhalten wir also dann durch Induktion.

Aus (26) folgt nämlich

$$\sum_{r=1}^{\infty} m(G_r) = \infty.$$

Da die Mengen G_r stochastisch unabhängig sind, erhalten wir durch Anwendung des zweiten Borel—Cantellischen Lemmas, daß

$$m(\overline{\lim_{r\to\infty}}\, G_r) = 1.$$

Daraus und aus (29) ergibt sich, daß die Reihe (4) im Intervall (0, 1) fast überall divergiert derart, daß

$$\overline{\lim_{N\to\infty}} \left| \sum_{n=1}^{N} a_n \varphi_n(x) \right| = \infty$$

in (0, 1) fast überall gilt.

Endlich sei n eine natürliche Zahl mit $n_{k_{r_0}+1} \leqq n < n_{k_{r_0+1}+1}$ $(r_0 \geqq 1)$. Dann gilt die Abschätzung

$$\int_0^1 \left| \sum_{l=1}^{n} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt \leqq \int_0^1 \left| \sum_{l=1}^{n_{k_1+1}-1} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt +$$

$$+ \sum_{r=1}^{r_0-1} \int_0^1 \left| \sum_{l=n_{k_r+1}}^{n_{k_{r+1}+1}-1} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt + \int_0^1 \left| \sum_{l=n_{k_{r_0}+1}}^{n} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt \leqq$$

$$\leqq C_8 + C_4 \left(\frac{1}{2} + \ldots + \frac{1}{2^{r_0}} \right) \leqq \max(C_8, C_4) = C_9$$

für jedes $x \in (0, 1)$ auf Grund von (25) und (27).

Daraus, und aus (25) ergibt sich endlich auch

$$\int_0^1 \left| \sum_{l=1}^{n} \frac{\varphi_l(x)\varphi_l(t)}{\lambda_l} \right| dt \leqq C_9 \quad (x \in (0, 1);\ n = 1, 2, \ldots).$$

Damit haben wir den Satz II vollständig bewiesen.

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JÓZSEF ATTILA UNIVERSITÄT
BOLYAI INSTITUT
6720 SZEGED, ARADI VÉRTANÚK TERE 1.
UNGARN

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MÜNTZ—JACKSON TYPE THEOREMS VIA INTERPOLATION

By

L. MÁRKI, G. SOMORJAI and J. SZABADOS (Budapest)

Let $\Lambda=\{\lambda_1, \lambda_2, \ldots\}$ be an infinite sequence of positive exponents. For $0\leqq a<1$ and $f(x)\in C[a, 1]$ put

$$E_{n,a}(f, \Lambda) = \min_{c_0, c_1, \ldots, c_n}{}^{*} \max_{a\leqq x\leqq 1} \left| f(x) - c_0 - \sum_{k=1}^{n} c_k x^{\lambda_k} \right|$$

where $*$ indicates that $c_0=0$ has to be put if $a>0$. By the Müntz theorem, $\lim_{n\to\infty} E_{n,0}(f, \Lambda)=0$ is true for each $f(x)\in C[0, 1]$ if and only if

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = +\infty. \tag{1}$$

Clarkson and Erdős [1] generalized this theorem for any $0<a<1$. Suppose (1) is valid. The Müntz—Jackson type theorems (see e.g. D. Newman [4]) are two-sided estimates concerning the rate of convergence of the sequences $E_{n,0}(f, \Lambda)$.

A possible way of obtaining an upper bound for $E_{n,a}(f, \Lambda)$ consists of two steps: first we choose an integer m and take the best approximating polynomial to $f(x)$ in $[a, 1]$ $P_m(x)=\sum_{j=0}^{m} b_j x^j$, i.e. for which

$$\max_{a\leqq x\leqq 1} |f(x)-P_m(x)| = E_{m,a}(f) = \text{minimum}, \tag{2}$$

and then we approximate $P_m(x)$ term by term by linear combinations of the powers x^{λ_k} $(1\leqq k\leqq n)$.

For performing this second step we introduce a new, interpolation theoretical method which gives error estimates directly in the supremum norm.

Lemma 1. *Let* $\lambda\geqq 0$, $m\geqq 0$, *and* $0<x\leqq 1$ *be real numbers, then*

$$\varepsilon = \left| x^{\lambda} - \sum_{k=1}^{n} x^{\lambda_k} \frac{2(\lambda+m)}{\lambda_k+\lambda+2m} \prod_{j=1}^{n} \frac{\lambda_k+\lambda_j+2m}{\lambda+\lambda_j+2m} \cdot \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{\lambda-\lambda_j}{\lambda_k-\lambda_j} \right| \leqq \frac{1}{x^m} \prod_{k=1}^{n} \frac{|\lambda-\lambda_k|}{\lambda+\lambda_k+2m}. \tag{3}$$

Proof. For x fixed consider the function

$$\varphi_x(z) = \frac{x^z}{z+\lambda+2m} \prod_{j=1}^{n} (z+\lambda_j+2m)$$

of the complex variable z. $\varphi_x(z)$ is analytic in the half plane $\{\mathrm{Re}\, z\geqq -m\}$. Let

$p_x(z)$ denote the Lagrange-polynomial which interpolates $\varphi_x(z)$ at the nodes $\lambda_1, \lambda_2, \ldots, \lambda_n$. Explicity,

$$p_x(z) = \sum_{k=1}^{n} \frac{x^{\lambda_k}}{\lambda_k+\lambda+2m} \prod_{j=1}^{n} (\lambda_k+\lambda_j+2m) \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{z-\lambda_j}{\lambda_k-\lambda_j}.$$

By a known formula (see e.g. WALSH [5]) the error of the interpolation is

$$\varphi_x(z)-p_x(z) = \frac{1}{2\pi i} \oint_C \frac{\varphi_x(z)}{\zeta-z} \prod_{k=1}^{n} \frac{z-\lambda_k}{\zeta-\lambda_k}\, d\zeta, \tag{4}$$

where C is an arbitrary rectifiable closed Jordan-curve lying in $\{\mathrm{Re}\, z \geqq -m\}$ to which $z, \lambda_1, \ldots, \lambda_n$ are interior. We put $C=C_R$, where

$$C_R = \{\zeta:\ \mathrm{Re}\,\zeta = -m,\ |\mathrm{Im}\,\zeta| \leqq R\} \cup \{\zeta:\ \mathrm{Re}\,\zeta > -m,\ |\zeta+m| = R\} = C_R' \cup C_R''$$

and let $R\to\infty$. On the half-circle C_R'' we have $\varphi_x(\zeta)=O(R^{n-1})$ while the kernel of integration is $O(R^{-n-1})$ $(R\to\infty)$. Hence it follows that $\lim\limits_{R\to\infty} \int\limits_{C_R''} = 0$, and by (4)

$$\varphi_x(z)-p_x(z) = \frac{1}{2\pi i} \int_{-m-i\infty}^{-m+i\infty} \frac{\varphi_x(\zeta)}{\zeta-z} \prod_{k=1}^{n} \frac{z-\lambda_k}{\zeta-\lambda_k}\, d\zeta \tag{5}$$

holds for any z in $\{\mathrm{Re}\, z > -m\}$. It follows from the definitions of $\varphi_x(z)$ and $p_x(z)$ that

$$\varepsilon = \frac{2(\lambda+m)}{\prod\limits_{j=1}^{n}(\lambda+\lambda_j+2m)} |\varphi_x(\lambda)-p_x(\lambda)|,$$

therefore (5) implies (with $z=\lambda$)

$$\varepsilon = \frac{2(\lambda+m)}{\prod\limits_{j=1}^{n}(\lambda+\lambda_j+2m)} \left| \frac{1}{2\pi} \int_{-m-i\infty}^{-m+i\infty} \frac{\varphi_x(\zeta)}{\zeta-\lambda} \prod_{k=1}^{n} \frac{\lambda-\lambda_k}{\zeta-\lambda_k}\, d\zeta \right| \leqq \tag{6}$$

$$\leqq \frac{\lambda+m}{\pi} \prod_{k=1}^{n} \frac{|\lambda-\lambda_k|}{\lambda+\lambda_k+2m} \int_{-m-i\infty}^{-m+i\infty} \left| \frac{x^\zeta}{(\zeta-\lambda)(\zeta+\lambda+2m)} \prod_{k=1}^{n} \frac{\zeta+\lambda_k+2m}{\zeta-\lambda_k} \right| |d\zeta|.$$

Now we notice that for $\mathrm{Re}\,\zeta = -m$ we have

$$|x^\zeta| = x^{-m},\quad |\zeta-\lambda| = |\zeta+\lambda+2m|,\quad |\zeta+\lambda_k+2m| = |\zeta-\lambda_k| \qquad (1 \leqq k \leqq n)$$

further

$$\int_{-m-i\infty}^{-m+i\infty} \frac{|d\zeta|}{|\zeta-\lambda|^2} = \int_{-\infty}^{\infty} \frac{dy}{y^2+(\lambda+m)^2} = \frac{\pi}{\lambda+m}. \tag{7}$$

(6) and the facts above prove (3).

COROLLARY.

$$E_{n,a}(x^\lambda, \Lambda) \leqq \frac{1}{a^m} \prod_{k=1}^{n} \frac{|\lambda-\lambda_k|}{\lambda+\lambda_k+2m}. \tag{8}$$

This estimate is due to M. GOLITSCHEK [3], who obtained it via L_2 norm. Our proof is entirely different from that of Golitschek.

G. Halász conjectured that for $a>0$ fixed and any function $f(x)$ analytic in $[a, 1]$ $E_{n,a}(f, \Lambda) \ll E_{n,0}(f, \Lambda)$ $(n\to\infty)$, more exactly,

$$\log E_{n,a}(f, \Lambda) < -ce^{\sum_{k=1}^{n} \frac{1}{\lambda_k}} \tag{9}$$

with some $c=c(a,f)>0$ for n large enough. The conjecture is motivated by Bernstein's theorem: (9) is true if $\lambda_k=k$ $(k=1, 2, \ldots)$. (9) will be a consequence of our more general Theorem 1.

LEMMA 2.

$$\sum_{j=0}^{m} |b_j| \leqq \left(\frac{12}{1-a}\right)^m \|f\| \tag{10}$$

where the b_j's are the coefficients of the polynomial $P_m(x)$ in (2), $\|f\| = \max_{a\leqq x\leqq 1} |f(x)|$.

PROOF. Consider the factorized form of $P_m(x)$:

$$P_m(x) = b_m \prod_{j=1}^{r} (x-x_j) \prod_{j=r+1}^{n} (x-x_j) = b_m \pi_1(x)\pi_2(x), \tag{11}$$

where $|x_j|<2$ if and only if $1\leqq j\leqq r$. Applying Chebyshev's theorem we obtain

$$\|\pi_1(x)\| = |\pi_1(x_0)| \geqq \frac{1}{2^{r-1}}\left(\frac{1-a}{2}\right)^r. \tag{12}$$

(2) implies $|P_m(x_0)|\leqq 2\|f\|$, thus (11) and (12) yield

$$\prod_{j=r+1}^{n} (|x_j|-1) \leqq |\pi_2(x_0)| \leqq \left(\frac{4}{1-a}\right)^r \frac{\|f\|}{|b_m|}. \tag{13}$$

Evidently

$$\sum_{j=0}^{n} |b_j| \leqq |b_m| \prod_{j=1}^{n} (|x_j|+1) \leqq |b_m| 3^r \prod_{j=r+1}^{n} (|x_j|+1), \tag{14}$$

$$\prod_{j=r+1}^{n} (|x_j|+1) = \prod_{j=r+1}^{n} \frac{|x_j|+1}{|x_j|-1} \prod_{j=r+1}^{n} (|x_j|-1) \leqq 3^{n-r} \prod_{j=r+1}^{n} (|x_j|-1). \tag{15}$$

(13), (14), and (15) prove (10).

THEOREM 1. *Assume there is a $K>0$ such that*

$$\sum_{\lambda_k<m} \frac{1}{\lambda_k} \leqq K \log m \quad (m = 1, 2, \ldots). \tag{16}$$

Then for any $0<a<1$, $f\in C[a, 1]$ and integer n the inequality

$$E_{n,a}(f, \Lambda) \leqq E_{m,a}(f) + \|f\| e^{-\frac{2}{3}Km} \tag{17}$$

is valid, where

$$m = m(n) = \left[e^{\frac{1}{K} \sum_{k=1}^{n} \frac{1}{\lambda_k} - 1} \cdot \left(\frac{a(1-a)}{12} \right)^{\frac{3}{2K}} \right]. \tag{18}$$

PROOF. (2) and Lemma 2 yield the estimate

$$E_{n,a}(f, \Lambda) \leqq E_{m,a}(f) + \sum_{j=0}^{m} |b_j| \cdot E_{n,a}(x^j, \Lambda) \leqq \tag{19}$$

$$\leqq E_{m,a}(f) + \left(\frac{12}{1-a} \right)^m \|f\| \max_{1 \leqq j \leqq m} E_{n,a}(x^j, \Lambda).$$

Now we apply Lemma 1 with $m=m(n)$, $\lambda=j\leqq m$, $a\leqq x\leqq 1$. Omitting on the right-hand side of (3) the factors belonging to the λ_k's smaller than m, we obtain

$$E_{n,a}(x^j, \Lambda) \leqq \frac{1}{a^m} \prod_{\substack{1 \leqq k \leqq n \\ \lambda_k \geqq m}} \frac{\lambda_k - j}{\lambda_k + j + 2m} \quad (0 < j \leqq m). \tag{20}$$

Here

$$\frac{\lambda_k - j}{\lambda_k + j + 2m} \leqq \frac{\lambda_k}{\lambda_k + 2m} = 1 - \frac{2m}{\lambda_k + 2m} \leqq 1 - \frac{2m}{3\lambda_k} \leqq e^{-\frac{2m}{3\lambda_k}}. \tag{21}$$

It follows from (16) and (18) that

$$\sum_{\substack{1 \leqq k \leqq n \\ \lambda_k \geqq m}} \frac{1}{\lambda_k} \geqq \sum_{k=1}^{n} \frac{1}{\lambda_k} - K \log m \geqq K + \frac{3}{2} \log \frac{12}{a(1-a)}. \tag{22}$$

(20), (21) and (22) imply

$$\max_{0 \leqq j \leqq m} E_{n,a}(x^j, \Lambda) \leqq e^{-\frac{2}{3} Km} \left(\frac{1-a}{12} \right)^m. \tag{23}$$

(19) and (23) prove (17).

Remarks and corollaries

1. Our Theorem 1 can be considered as a generalization of some results of M. VON GOLITSCHEK [3, Sätze 2,3].

2. In case $\lambda_1, \lambda_2, \ldots$ are integers >1, (16) is satisfied with $K=1$.

3. a) Applying Jackson's theorems and Theorem 1 we obtain:

$$E_{n,a}(f, \Lambda) = O\left(e^{-\frac{r}{K} \sum_{k=1}^{n} \frac{1}{\lambda_k}}\right) \omega\left(f^{(r)}, e^{-\frac{1}{K} \sum_{k=1}^{n} \frac{1}{\lambda_k}}\right) \quad (n \to \infty)$$

if $f^{(r)}(x) \in C[a, 1]$ and $f^{(r)}(x)$ is not identically constant $(r=0, 1, \ldots)$.

b) Bernstein's theorem states that for each f analytic in $[a, 1]$ there is a $q=q(a, f)$ $(0<q<1)$ such that

$$E_{m,a}(f) = O(q^m) \quad (m \to \infty). \tag{24}$$

(17) and (24) yield a Müntz—Bernstein-type theorem:

$$\log E_{n,a}(f, \Lambda) \leqq m \cdot \max\left(\log q, -\frac{2}{3}K\right) + O(1) \tag{25}$$

$$(m = m(n) \quad \text{as in (18);} \quad n \to \infty).$$

As for lower estimates of $E_{n,a}(f, \Lambda)$ we prove

4. STATEMENT 1. *If* $\lambda_k = k+1$ $(k=1, 2, \ldots)$ *then for any* $a>0$ *we have*

$$E_{n,a}(x, \Lambda) \geqq 2a\left(\frac{1-a}{4}\right)^n \quad (n = 1, 2, \ldots). \tag{26}$$

PROOF. Consider the polynomial $P(x) = x - \sum_{k=1}^{n} c_k x^{k+1}$ with $E_{n,a}(x, \Lambda) = \|P(x)\|$. Applying Chebyshev's theorem to the polynomial $P^*(x) = x^{n+1} P\left(\frac{1}{x}\right)$ in the interval $\left[1, \frac{1}{a}\right]$ we obtain in fact

$$\|P(x)\| \geqq \frac{a^{n+1}}{2^{n-1}} \left(\frac{\frac{1}{a} - 1}{2}\right)^n = 2a\left(\frac{1-a}{4}\right)^n.$$

Thus, in case of the above example the order of the estimate (25) is exact for $f(x) = x$.

However, this fails to hold in general; moreover, there is no general lower estimate for $E_{n,a}(f, \Lambda)$ depending only on $\sum_{k=1}^{n} \frac{1}{\lambda_k}$ and $f(x)$. This will be shown in

STATEMENT 2. *Let* $F(u)$ *be a positive-valued decreasing function for* $0 \leqq u < \infty$. *Then there is a sequence of integers* $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ *such that* $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty$ *and*

$$E_{n,a}(f, \Lambda) = O\left(\omega\left(f, F\left(\sum_{k=1}^{n} \frac{1}{\lambda_k}\right)\right)\right) \quad (n \to \infty) \tag{27}$$

holds for each fixed $0<a<1$ *and* $f(x) \in C[a, 1]$.

PROOF. First define the sequence $\{\mu_r\}_{r=1}^{\infty}$ as follows: $\mu_1 = 1$, further

$$\mu_{r+1} = \left[e\mu_r + \frac{1}{F(2r)}\right] \quad (r = 1, 2 \ldots). \tag{28}$$

Let $\lambda_1 < \lambda_2 < \ldots$ be all the integers in the set $\bigcup_{r=1}^{\infty} [\mu_r + 1, e\mu_r]$. Define

$$t = t(a) = \left[\frac{3}{2} \log \frac{12}{a(1-a)}\right] + 1. \tag{29}$$

We need the following consequence of (19), (20) and (21):

$$E_{n,a}(f,\Lambda)=O\left(\omega\left(f,\frac{1}{m}\right)\right)+\left(\frac{12}{a(1-a)}\right)^m\|f\|e^{-\frac{2}{3}m\sum\limits_{\substack{1\le k\le n\\ \lambda_k\ge m}}\frac{1}{\lambda_k}}. \tag{30}$$

Now use (30) with $m=\mu_{r-t}$, where $r=r(n)$ is defined by $\mu_r<n<e\mu_r$ (a and f are fixed, $n\to\infty$). It follows from the definitions that

$$\sum_{\substack{1\le k\le n\\ \lambda_k>\mu_{r-t}}}\frac{1}{\lambda_k}>t-\frac{t}{\mu_{r-t}}\to t\quad (r=r(n),\ n\to\infty). \tag{31}$$

(29) and (31) imply that the second term of the right side of (30) is less than q^{μ_r}- ($0<q<1$ is a constant, n is large enough). Thus the term $O\left(\omega\left(f,\frac{1}{m}\right)\right)$ dominates. Therefore the inequalities (for sufficiently large n's)

$$\frac{1}{\mu_{r-t}}\le F\big(2(r-t-1)\big)\le F(r)\quad\text{and}\quad \sum_{k=1}^{n}\frac{1}{\lambda_k}\le r$$

prove (27). (We used the monotonicity of F.)

Finally we deal with bounded sequences $\Lambda=\{\lambda_k\}_{k=1}^{\infty}$.

LEMMA 3. *Suppose that* $0<\lambda_k\le L$ $(k=1,2,\dots)$, $0<a<1$. *Then for any* $\lambda\ge 0$

$$E_{n,a}(x^\lambda,\Lambda)\le\left(\frac{e\max(L,\lambda)\cdot\log\frac{1}{a}}{n}\right)^n\quad (n=1,2,\dots). \tag{32}$$

PROOF. By Lemma 1 we have

$$E_{n,a}(x^\lambda,\Lambda)\le\frac{1}{a^m}\prod_{k=1}^{n}\frac{|\lambda_k-\lambda|}{\lambda_k+\lambda+2m}\le\left(\frac{\max(L,\lambda)}{2m}\right)^n\frac{1}{a^m}. \tag{33}$$

Choosing $m=\dfrac{n}{\log\frac{1}{a}}$, (33) yields (32).

By (19) and Lemma 3 we obtain now

THEOREM 2. *For any* $f(x)\in C[a,1]$, $m>L$ *we have*

$$E_{n,a}(f,\Lambda)\le E_{m,a}(f)+\|f\|\left(\frac{12}{1-a}\right)^m\left(\frac{em\log\frac{1}{a}}{2n}\right)^n\quad (n=1,2,\dots). \tag{34}$$

COROLLARIES. 1.

$$E_{n,a}(f,\Lambda)\le E_{[cn],a}(f)+\|f\|e^{-n},$$

where

$$c=c(a)=\min\left(1,\ \frac{1-a}{6e^2\log\frac{1}{a}}\right).$$

By the Jackson theorem this improves the results of GOLITSCHEK [3, Sätze 5, 6] corresponding to intervals $[a, b]$, $0<a<b$ as indicated in his remark on p. 105.

2. $\lim_{n\to\infty} E_{n,a}(f, \Lambda)^{\frac{1}{n}}=0$ *if* $f(x)$ *is an entire function.*

This follows from (34) by taking $m=\left[\frac{n}{K}\right]$, where $K>0$ is an arbitrary fixed number, and by using the relation $\lim_{m\to\infty} E_{m,a}(f)^{\frac{1}{m}}=0$.

3. A direct consequence of Lemma 3 is that $E_{n,a}(f, \Lambda)^{\frac{1}{n}}$ *tends to zero at the rate of* $\frac{1}{n}$ $(n\to\infty)$ *if* $f(x)=\sum_{j=1}^{r} c_j x^{\mu_j}$, *where* $\mu_j>0$, c_j *are real numbers.*

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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ASYMMETRIC TREES WITH TWO PRESCRIBED DEGREES

By
L. BABAI (Budapest)

1. Introduction. In view of their use in constructing graphs and other structures with prescribed automorphism groups and other properties, asymmetric graphs (having no nonidentity automorphism) have been studied by a number of authors [1, 2, 4, 5, 6, 13, 14 and others].

In the 60's, due to the work of category theorists in Prague, construction of rigid graphs (with no nonidentity endomorphism) became more important [see e.g. 3, 7, 8, 9, 10, 11, 12, 15].

The rigid graph constructions are generally much harder than the analogous problems for asymmetric graphs. One of the aims of this paper is to present a (hopefully nontrivial) family of asymmetric graphs.

While investigating rigid graphs in several papers, trees have been neglected for the simple reason that a tree with at least 2 vertices is never rigid (Cf. Problem 1.4). This explains that very little seems to be known about asymmetric trees. It is an easy exercise to prove that, given an infinite cardinal $\aleph$, there are $2^{\aleph}$ nonisomorphic asymmetric trees of power $\aleph$ — however, to the author's knowledge, this has not been published so far.

Constructing an asymmetric graph means, intuitively, to code different information around every vertex.

At the first look one would expect that almost the single reason for which a *tree* may be asymmetric at all, is that the valencies distinguish the points. In fact, a tree, all vertices of which have the same degree, cannot be asymmetric. It is quite natural to ask then, what is the minimum cardinal of an asymmetric tree with infinite degrees. By the above, it is not countable. I hope it is a little surprising that the answer is ω_1: if we allow the 2 possibilities that a vertex have degree ω or ω_1, this suffices to find an appropriate asymmetric tree. More generally, we prove the following

MAIN THEOREM 1.1. *Let* $1 \leqq \beta < \alpha$ *be cardinals. Then there is an asymmetric tree having degrees* α *and* β *only. For* $\alpha \geqq 3$ *there are exactly* $\max(2^{\alpha}, 2^{\omega})$ *nonisomorphic trees of this kind.*

From this theorem one can easily derive (we omit the proof):

COROLLARY 1.2. *For* α *a limit cardinal and* H *a set of cardinals less than* α *such that* $\sup H = \alpha$, *there are* 2^{α} *asymmetric trees of power* α *with all degrees belonging to* H.

The principal tool of the proof is the construction of asymmetric colourings of trees. A vertex-colouring of a graph is asymmetric if no non-identity automorphism preserves it. We prove that, if all vertices of a tree have the same degree $\alpha \geqq 2$ then its vertices possess an asymmetric 2-colouring (Theorem 3.2). We close this section with two related problems:

PROBLEM 1.3. *Which trees possess an asymmetric colouring by a finite number of colours?*

One can also ask for *rigid* colourings of trees. (Endomorphisms are colour- and edge-preserving mappings of the vertex set into itself.)

PROBLEM 1.4. *Given a cardinal* α, *does there exist a tree of power* α *which has a rigid colouring by a finite number of colours?*

For α less than the first uncountable inaccessible cardinal the answer is affirmative. — We remark that a positive answer to Problem 1.4 implies that there are rigid graphs of power α all finite subgraphs of which are *planar*.

2. Basic definitions. We consider graphs without loops and multiple edges. An *automorphism* of a graph is its isomorphism onto itself. A graph is *asymmetric* if it has no non-identity automorphism.

The set of vertices of a graph X is denoted by $V(X)$, the set of edges by $E(X)$.

A *rooted* graph is a graph with a distinguished vertex called the *root.*

We call a *rooted* graph α*-valent* if all vertices *with the possible exception of the root* have degree α, and the degree of the root does not exceed α.

Note that the power of an α-valent rooted tree is $\max(\alpha, \omega)$ $(\alpha \geqq 2)$.

By a *colouring* of a graph X we simply mean a mapping $V(X) \to C$ of the vertex set into some set C of colours. (Not necessarily a "good" colouring in the usual sense).

The *isomorphisms* of rooted and/or coloured graphs should preserve the root and the colours, resp. A coloured and/or rooted graph is *asymmetric* if it has no nonidentity automorphisms in the above sense.

An *isomorphism of two colourings* of a (rooted) graph is an isomorphism of the resulting coloured (rooted) graphs. A colouring is *asymmetric* if it has no non-identity automorphisms.

The edges of the *rooted trees* can be *directed* in a natural way *from* the root. Thus the root has *out-neighbours* only, and the single *in-neighbour* of any other vertex v lies on the path connecting v to the root.

3. Asymmetric α-valent coloured trees. The main result of this section is the

MAIN LEMMA 3.1. *Let* $\alpha \geqq 3$. *Given an* α*-valent rooted tree* T *and* 2 *colours (yellow and green, say), there are* $\max(2^{\alpha}, 2^{\omega})$ *non-isomorphic asymmetric colourings of* T *such that*

(*i*) *all neighbours of the root be yellow and all other vertices have at least one green out-neighbour.*

(*ii*) *For* $\alpha \geqq 4$ *one can additionally require that all vertices have at least one yellow out-neighbour.*

The Lemma is rather technically formulated. We mention that it has a root-less corollary:

THEOREM 3.2. *Let* $\alpha \geqq 2$. *Given* 2 *colours there are* $\max(2^\alpha, 2^\omega)$ *non-isomorphic asymmetric* α*-valent coloured trees* (with these colours).

Namely, for $\alpha \geqq 3$, the colourings of the lemma are asymmetric even in the root-less sense, the root being distinguished by the property that its neighbours are yellow. The case $\alpha = 2$ is obvious.

The proof of the main lemma will be done through a series of lemmas.

LEMMA 3.3. *Given an integer* $m \geqq 2$, *a cardinal* λ, *and an* α*-valent* ($\alpha \geqq 3$) *rooted tree* T, *assume that* T *has* λ *non-isomorphic asymmetric colourings with given* $\leqq 2^m - 1$ *colours. Then it has* λ *asymmetric colourings with* $\leqq m$ *colours.*

NOTATION. Let $d(x, y)$ denote the distance of the vertices x and y. For T a rooted tree, rooted at x_0, $h \geqq 0$ an integer, and x a vertex of T, set

$$N^h(x) = \{y \in V(T)\colon d(x, y) = h,\ d(x_0, y) = d(x_0, x) + h\}.$$

Thus $N^0(x) = \{x\}$, $N^1(x)$ is the set of out-neighbours of x, and

$$N^{h+k}(x) = \bigcup \{N^k(y)\colon y \in N^h(x)\}.$$

PROOF OF LEMMA 3.3. $m < (\alpha - 1)^n$ for some integer $n \geqq 1$.

Thus, for any vertex x of T different from the root x_0 we have

$$|N^n(x)| = (\alpha - 1)^n \geqq m.$$

Let C denote the set of colours ($|C| \leqq 2^m - 1$) and D a set of power m (the new colours). Let f denote a one-to-one mapping of C into the set of non-empty subsets of D. Let c denote one of the original colourings. Now, change the colouration of T such that the members of $N^n(x)$ $(x \neq x_0)$ obtain colours from the set $f(c(x))$ only and every member of $f(c(x))$ occurs as the colour of some vertex in $N^n(x)$. Such a colouration exists since

$$1 \leqq |f(c(x))| \leqq m \leqq |N^n(x)|$$

and for $x \neq y$, $N^n(x)$ and $N^n(y)$ are disjoint. We colour the set $\bigcup_{i=0}^{n} N^i(x_0)$ arbitrarily (taking colours from D).

If f and n are fixed, c can be uniquely recovered from the obtained new colouring. This observation shows that every isomorphism of the new colourings induces an isomorphism of the original ones, thus proves the lemma.

LEMMA 3.4. *If an* α*-valent* ($\alpha \geqq 3$) *rooted tree* T *has an asymmetric colouring by a finite number of colours then it has* $\max(2^\alpha, 2^\omega)$ *non-isomorphic asymmetric colourings by* 2 *colours.*

PROOF. Repeated application of Lemma 3.3 shows that T has an asymmetric 2-colouring c. Let $V(T) = A \cup B$ be the partition of the vertex set induced by this colouring. Clearly, at least one (actually both) of A and B, say A, has power $\max(\alpha, \omega)$, hence A can be partitioned $\max(2^\alpha, 2^\omega)$-ways as $A_1 \cup A_2$. The partition (A_1, A_2, B) induces a 3-colouring of T which is clearly asymmetric, and different partitions correspond to non-isomorphic colourings (since any such isomorphism would be a non-trivial automorphism of the original colouring).

Hence we have $\max(2^\alpha, 2^\omega)$ non-isomorphic asymmetric 3-colourings. Now set $m=2$ and apply Lemma 3.3.

LEMMA 3.5. *Every α-valent $(\alpha \geqq 2)$ rooted tree T has an asymmetric colouring by a finite number of colours.*

PROOF. We proceed by transfinite induction on α. For α finite one can colour the tree by α colours such that all neighbours of any vertex have different colours, an obviously sufficient condition on a coloured rooted tree to be asymmetric.

Assume now that $\alpha \geqq \omega$ and 3.5 holds for all cardinals less than α. Using Lemma 3.4 we obtain a family

$$\{T_\beta:\ \beta < \alpha\} \quad (\beta \text{ ordinal})$$

of pairwise non-isomorphic asymmetric 2-coloured (yellow and green, say) rooted trees of degrees less than α. Let t_β denote the root of T_β; set $D=\{t_\beta: \beta<\alpha\}$. Let the root of our tree have degree $\varrho\,(\varrho \leqq \alpha)$.

Set

$$V = \{x_0\} \cup \bigcup \{V(F_\beta):\ \beta < \alpha\}.$$

(x_0 is a new point, and the sets here are assumed to be disjoint.) As $|V|=|D|=\alpha \geqq \omega$, there is a partition

$$D = \bigcup \{D_x:\ x \in V\}$$

such that

(iii) $|D_{x_0}|=\varrho,\quad |D_x|=\alpha \quad (x \in V,\ x \neq x_0)$

(iv) $x \in V(T),\quad t_\gamma \in D_x$ implies $\gamma > \beta$.

We define the graph X by

$$V(X) = V;$$

$$E(X) = \bigcup \{E(T_\beta):\ \beta < \alpha\} \cup \{[x, y]:\ y \in D_x,\ x \in V\}.$$

We consider x_0 as the root of X. Let us colour X such as to preserve the yellow-green colouring of $T_\beta - \{t_\beta\}\,(\beta<\alpha)$; the roots t_β and x_0 get red colour.

By (iii), the degree of x_0 is ϱ; all other vertices of X have degree α. By (iv), X is a tree. Hence the rooted tree X is isomorphic to T. We prove that the yellow-green-red colouring of X is asymmetric.

Omitting x_0 and those edges of X whose end (the one being more distant from x_0) is red, the rest is the disjoint union of the trees T_β with the original yellow-green colouring except for t_β which is red now. This distinguishes the original roots, hence the obtained root-less yellow-green-red trees are likewise asymmetric and pairwise non-isomorphic. This in turn implies the asymmetry of the colouring of X, proving Lemma 3.5.

Now we can prove the Main Lemma 3.1. Let T be the α-valent rooted tree of 3.1 $(\alpha \geqq 3)$. By Lemmas 3.4 and 3.5, T has $\max(2^\alpha, 2^\omega)$ non-isomorphic asymmetric red-blue colourings. We may assume x_0 is always red. Let c denote one of these colourings. Clearly, there is a yellow-green colouring c' of T satisfying (i), (ii) (if $\alpha \geqq 4$) and (v):

(v) for any vertex x of T, $c(x)$ is red if and only if there are at least as many yellow out-neighbours of x as green ones (with respect to c').

By (v), c can be uniquely recovered from c'. Hence, every isomorphism of the new colourings induces an isomorphism of the original ones, thus it is the identity. This completes the proof of 3.1.

4. Proof of the main theorem. I. Assume $3 \leqq \beta < \alpha$.

Let T be a tree rooted at x_0. Let c be a 4-colouring (yellow-green-red-blue) of T. We associate a pair (T', c') of the same kind with (T, c). The root of T' will be the same x_0, and $V(T')$ will contain $V(T)$.

Start from (T, c) and consider an $x \neq x_0$ with $c(x)$ yellow. Subdivide the edge joining x to its (single) in-neighbour by a new vertex x'. Identify x' with the root of some tree T_x consisting of new vertices. T_x should be α-valent and the degree of its root $\beta-2$. (Note that $\beta \geqq 3$ is assumed now. For β infinite, $\beta-2=\beta$.) For different x, y take disjoint trees. Let T' be the union of these trees T_x for all x with $c(x)$ yellow, plus the original tree subdivided by the x'-s. Hence

$$V(T') = V(T) \cup \bigcup \{V(T_x)\colon\ x \in V(T),\ c(x) \text{ yellow}\}.$$

Let c' be defined on T as follows:

$$c'(x) = \begin{cases} c(x), & \text{if } c(x) \text{ is not yellow} \\ \text{red}, & \text{if } c(x) \text{ is yellow.} \end{cases}$$

The new vertices should be coloured as follows:

$c'(x')$ is blue (for x' corresponding to x, where $c(x)$ is yellow); moreover, let T_x be given a yellow-green asymmetric colouring (with blue root x') such as to satisfy (i) and (ii) of Lemma 3.1. (Now we have $\alpha \geqq 4$, as $\beta \geqq 3$. The colour of the root does not incfluence the asymmetry of a rooted graph.)

Observe the following properties of (T', c'):

(1) The vertices of T have the same degree in T as in T'. Their colour is also the same except if $c(x)$ is red.

(2) If $c'(v)$ is yellow then neither v nor any of its neighbours belong to T.

(3) If x, y are vertices of T and neither $c(x)$ nor $c(y)$ is yellow then x and y are neighbours in T iff they are neighbours in T'. In any case, the distance in T' of two neighbours in T is at most 2; if z is between them then $c'(z)$ is blue (cf. (2)).

(4) The degrees of the vertices of T' not in T are α and β. Those of degree β are blue, the others are yellow and green.

(5) If the degrees of T are α and β, and $c(x)$ $(x \in V(T))$ is blue iff the degree of x is β, then the same holds for (T', c') (Cf. (1) and (4)).

(6) If no out-neighbour of some vertex v of T' is yellow (with respect to c') then v belongs to T. (This follows from (ii) in Lemma 3.1.)

(7) If $c(x)$ is yellow for some x in T then the in-neighbour of x in T' does not belong to T; it has degree β and c'-colour blue.

(8) If v is vertex of T', not in T, and $c'(v)$ is blue then all neighbours of v outside T are yellow (with respect to c').

Let now X_1 denote an α-valent rooted tree, whose root x_0 has degree α, too. Let c_1 be a yellow-green asymmetric colouring of X_1 such that x_0 and its neighbours are green, all other vertices have out-neighbours of both colours. (Such a colouring exists by Lemma 3.1, interchanging the roles of "yellow" and "green".)

Let

$$(X_{n+1}, c_{n+1}) = (X'_n, c'_n).$$

Define the direct limit X of the sequence X_n to be a graph defined by

$$V(X) = \bigcup_{n<\omega} V(X_n) \quad \text{(join of an ascending chain)};$$

$E(X)$=the set of those pairs which are edges in all but a finite number of X_n-s.

As every X_n is a tree, X is a tree as well by (1), (2) and (3).

By (1), (2), (3) and (5), the degrees in X are α and β. We consider X to be root-less. We prove that X is asymmetric.

Let Y_1 be a coloured rooted tree having the properties required from X_1 and starting from it, obtain a tree Y. We shall prove simultaneously that any isomorphism of X and Y induces an isomorphism of X_1 and Y_1, thus proving (in view of 3.1) that there are $\max(2^\alpha, 2^\omega)$ non-isomorphic trees satisfying the main theorem. (Y may coincide with X; this allows us to argue simultaneously.)

x_0 is the unique vertex of X_n all of whose neighbours are green with respect to c_n. This follows by induction on n, using (6), (1) and (2).

It follows that x_0 is the unique vertex of X of degree α all of whose neighbours have degree α. (Namely, if $x \in V(X_n) - V(X_{n-1})$ then $c_n(x)$ is not red (4), hence it is either blue and thus of degree β (4), or x has a yellow out-neighbour (6) hence it has a neighbour of degree β in X_{n+1} (3), (7). They remain neighbours in X_{n+2}, etc. (3).)

We conclude that any isomorphism φ of X onto Y takes x_0 to the root y_0 of Y_1.

We define a colouring c of X (and, similarly, of Y) by green, red and blue colours. By the above, we may refer to x_0 as to the root.

Let us colour the vertices of degree β blue, those of degree α whose in-neighbour has degree β red, the rest (including x_0) green.

Clearly, the isomorphism φ keeps the colours just introduced. By (5), (7) and (1), (2), (3) we have

$$c(x) = c_n(x)$$

for any x in X and sufficiently large n (depending on x). Using (7), and (8) we obtain

(9) For any $x \in V(X_n) - V(X_{n-1})$, if $c(x)$ is blue then one of the out-neighbours of x is red and belongs to X_{n-1}, all other out-neighbours of x are blue and do not belong to X_{n-1}.

From this we easily deduce that X_1 is characterized by the following:

A vertex x of X belongs to X_1 iff it is either red or green (with respect to c), and the path $x_0 - x$ does not contain two neighbouring blue vertices.

Hence, φ maps $V(X_1)$ onto $V(Y_1)$.

We prove by induction on n that φ maps $V(X_n)$ onto $V(Y_n)$. The induction step consists of the following observations:

If $c(x)$ is not blue then x belongs to $V(X_n) - V(X_{n-1})$ iff the $x_0 - x$ path contains exactly $n-1$ edges both ends of which are blue.

If $c(x)$ is blue then x belongs to $V(X_n) - V(X_{n-1})$ iff it does not belong to X_{n-1} but some neighbour of it does.

Now we easily recover (X_n, c_n). All we have to do is to replace each c-blue vertex x of $V(X_{n+1})$ by an edge joining those 2 neighbours of x belonging to $V(X_n)$, and to change the colour of one of them, which is an out-neighbour (the other one is the in-neighbour of x) from red to yellow.

Consequently, φ preserves c_n-colours and X_n-edges, thus defines an isomorphism of X_n onto Y_n. This is a contradiction if $X_1 \not\cong Y_1$. If $X=Y$ we obtain that φ defines anautomorphism of X_n thus, by induction, φ is the identity. This completes the proof of the theorem for $\beta \geqq 3$.

II. The rest is quite straightforward.

First, let $\beta=2$.

Take an α-valent rooted tree T whose root x_0 has degree α, too. Colour it yellow and green according to Lemma 3.1.

Subdivide every edge of T joining a green vertex to its in-neighbour by a new vertex of degree 2. The obtained tree X will be considered root-less. Its degrees are 2 and α.

By 3.1 (i), x_0 is characterized in X as the unique vertex having no neighbour of degree 2. Thus, again, the root is recognized.

Now one can easily recover T and its colouring, proving as above, both the asymmetry of X and that there are max $(2^\alpha, 2^\omega)$ non-isomorphic trees X satisfying the requirements of the theorem.

III. Let now $\beta=1, 3 \leqq \alpha < \omega$. Let T be an asymmetric tree with degrees α and $\alpha-1$ (root-less, uncoloured: it exists by the above). Join new vertices of degree 1 to each vertex of degree $\alpha-1$. The obtained tree obviously complies with the requirements.

IV. Assume now $\beta=1, \alpha \geqq \omega$. Let T be a (root-less) yellow-green-coloured asymmetric tree, all vertices of which have degree α. (See Theorem 3.2.) Join new vertices of degree 1 to each green vertex. T and its colouring can be trivially recognized from the obtained (uncoloured) tree. The conclusion follows as in the other cases.

V. Finally, the one-way infinite path, as a unique example, proves the theorem for $\beta=1, \alpha=2$.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ALGEBRA AND NUMBER THEORY
1088 BUDAPEST, MÚZEUM KRT. 6—8.

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ON SUMMABILITY OF ORTHOGONAL SERIES

By

G. ALEXITS (Budapest), member of the Academy, I. JOÓ
and K. TANDORI (Szeged), member of the Academy

1. It is an old question, how to generalize the Kolmogoroff—Seliverstoff—Plessner—Kaczmarz theorem on the convergence of orthogonal series to summation ([2], [4], [7]). First it was generalized to Cesàro summation ([2], [3], [9]), then to Riesz summation ([5], [8]), and recently F. MÓRICZ [6] proved an appropriate theorem for a general summation process generated by a matrix from which the previous results follow as a special case.

In this paper, continuing F. Móricz's considerations, we shall prove a further theorem concerning this problem. We assume for the Lebesgue functions of the summation more than required by F. Móricz, but we shall not need further assumptions on the convergence properties of the orthonormal system. In our note we use an idea of F. Móricz.

In this paper we confine ourselves to the orthogonal case, but using the results of F. MÓRICZ's paper [6] one can extend our theorem to arbitrary, not necessarily orthogonal integrable function systems.

2. In this paper let $T=\|t_{i,k}\|_{i,k=0}^{\infty}$ be a matrix, such that

$$\sum_{k=0}^{\infty} t_{i,k}^2 < \infty \quad (i = 0, 1 \ldots), \tag{1}$$

$$\lim_{i\to\infty} t_{i,k} = t_k \quad (k = 0, 1, \ldots) \tag{2}$$

where $t_{i,k}$-s and t_k-s are real numbers.

Let $\varphi=\{\varphi_k\}_0^{\infty}$ be an orthonormal system on the measure space $(\Omega, \mathscr{A}, \mu)$. If $c=\{c_k\}_0^{\infty}\in l_2$ then according to (1), the sums

$$t_i(c, x) = \sum_{k=0}^{\infty} t_{i,k} c_k \varphi_k(x) \quad (i = 0, 1, \ldots)$$

converge in the metric of the space $L^2(\Omega, \mathscr{A}, \mu)$, so the functions $t_i(c, x)$ are finite almost everywhere on Ω.

The series

$$\sum_{k=0}^{\infty} c_k \varphi_k(x) \tag{3}$$

is said to be T summable on the set $E(\in\mathscr{A})$ almost everywhere to the function $t(c, x)$ if

$$\lim_{i\to\infty} t_i(c, x) = t(c, x)$$

on E almost everywhere. We remark that in the case of $c_N=\{c_k\}_0^N$ ($N=0, 1, \ldots$), the sum $\sum_{k=0}^{N} c_k \varphi_k(x)$ is T summable to the function $\sum_{k=0}^{N} t_k c_k \varphi_k(x)$ almost everywhere on Ω.

According to

$$\sum_{k=1}^{\infty} t_{i,k}^2 \int_{\Omega} \varphi_k^2(x)\, d\mu(x) = \sum_{k=0}^{\infty} t_{i,k}^2 < \infty \quad (i = 0, 1, \ldots)$$

the series

$$K_i(T, \varphi; x, t) = \sum_{k=0}^{\infty} t_{i,k} \varphi_k(x) \varphi_k(t)$$

converges for almost every $x \in \Omega$, in the metric of $L^2(\Omega, \mathcal{A}, \mu)$ and

$$K_i(T, \varphi; x, t) \in L^2(\Omega, \mathcal{A}, \mu) \quad (i = 0, 1, \ldots).$$

Define the i-th Lebesgue function of the system φ, belonging to the summation T by

$$L_i(T, \varphi; x) = \int_{\Omega} |K_i(T, \varphi; x, t)|\, d\mu(t) \quad (i = 0, 1, \ldots).$$

Let $\lambda = \{\lambda_k\}_0^\infty$ be a nondecreasing sequence of positive numbers. Denote by $l^2(\lambda)$ the set of sequences $c = \{c_k\}_0^\infty$ for which

$$\sum_{k=0}^{\infty} c_k^2 \lambda_k < \infty.$$

Obviously $l^2(\lambda)$ is a Banach space if it is endowed with the norm

$$\|c\| = \left\{\sum_{k=0}^{\infty} c_k^2 \lambda_k\right\}^{1/2}$$

and the elements of the form $c_N = \{c_k\}_0^N$ ($N=0, 1, \ldots$) are dense in $l^2(\lambda)$.

The system φ is said to be a T-convergence system for $l^2(\lambda)$ if $c \in l^2(\lambda)$ implies the T summability of the series (3).

At last denote $\varphi/\sqrt{\lambda}$ and $\varphi/\lambda^{1/4}$ the system $\left\{\frac{\varphi_k}{\sqrt{\lambda_k}}\right\}_0^\infty$ and $\left\{\frac{\varphi_k}{\lambda_k^{1/4}}\right\}_0^\infty$, respectively.

We shall prove the following

THEOREM. *Let $E \in \mathcal{A}$. Suppose the system φ satisfies the conditions*

$$L_i(T, \varphi/\sqrt{\lambda}; x) = O_x(1) \quad (x \in E), \tag{4}$$

$$\left| \int_{\Omega} K_i(T, \varphi/\lambda^{1/4}; x, t) K_j(T, \varphi/\lambda^{1/4}; y, t)\, d\mu(t) \right| \leqq \sum_{k=0}^{\infty} \beta_k |K_k(T, \varphi/\sqrt{\lambda}; x, y)| \tag{5}$$
$$(x, y \in E;\ i, j = 0, 1, \ldots),$$

where $0 \leqq \beta_k = \beta_k(i, j)$ ($k=0, 1, \ldots$) and $\sum_{k=0}^{\infty} \beta_k = O(1)$. Then φ is a T-convergence system for $l^2(\lambda)$ on E.

We remark, that condition (5) is fulfilled for the $(C, 1)$ summation and in the case of lower concave λ the condition (4) is stronger than

$$L_i(T, \varphi; x) = O_x(\sqrt{\lambda_i}) \quad (x \in E)$$

(see for instance [6], [10]).

PROOF. First suppose that instead of (4) the following stronger condition is fulfilled:

$$L_i(T, \varphi/\sqrt{\lambda}; x) = O(1) \quad (x \in E). \tag{6}$$

Let I be a nonnegative integer and $c \in l^2(\lambda)$, further

$$t_I^*(x) = \sup_{i \leqq I} t_i(c, x) = t_{I(x)}(c, x).$$

Obviously, $\{t_I^*(x)\}_{I=0}^\infty$ is monotone nondecreasing. Using the orthonormality of the system φ, the Buniakowski—Schwarz inequality gives for arbitrary I:

$$\int_E t_I^*(x)\,d\mu(x) = \int_E \left[\int_\Omega \left(\sum_{k=0}^\infty c_k \sqrt{\lambda_k}\varphi_k(t)\right) K_{I(x)}(T, \varphi/\lambda^{1/4}; x, t)\,d\mu(t)\right] d\mu(x) \leqq \tag{7}$$

$$\leqq \sqrt{\sum_{k=1}^\infty c_k^2 \lambda_k} \left\{\int_\Omega \left[\int_E K_{I(x)}(T, \varphi/\lambda^{1/4}; x, t)\,d\mu(x)\right]^2 d\mu(t)\right\}^{1/2} = A \cdot J_I,$$

where

$$A = \sqrt{\sum_{k=0}^\infty c_k^2 \lambda_k} \quad \text{and} \quad J_I^2 = \int_\Omega \left[\int_E K_{I(x)}(T, \varphi/\lambda^{1/4}; x, t)\,d\mu(x)\right]^2 d\mu(t).$$

From (5) we obtain by easy computation:

$$J_I^2 \leqq \int_E \int_E \left|\int_\Omega K_{I(x)}(T, \varphi/\lambda^{1/4}; x, t) K_{I(y)}(T, \varphi/\lambda^{1/4}; y, t)\,d\mu(t)\right| d\mu(x)\,d\mu(y) \leqq$$

$$\leqq \sum_{k=0}^\infty \beta_k \int_E \int_E \left|K_k(T, \varphi/\sqrt{\lambda}; x, y)\right| d\mu(x)\,d\mu(y) \leqq$$

$$\leqq \sum_{k=0}^\infty \beta_k \int_E \left[\int_\Omega |K_k(T, \varphi/\sqrt{\lambda}; x, y)|\,d\mu(x)\right] d\mu(y) = O(1),$$

which shows by (7) that

$$\int_E t_I^*(x)\,d\mu(x) = O(1) \quad (I = 0, 1, \ldots).$$

The sequence $\{t_I^*(x)\}_{I=0}^\infty$ being monotone nondecreasing, we obtain $\sup_I t_I^*(x) < +\infty$ almost everywhere on E. The same argument applied to $\{-c_k\}_0^\infty$ gives $\inf_I t_I^*(x) > -\infty$ almost everywhere on E; that is under condition (6) we have $|t_I^*(x)| = O_x(1)$ almost everywhere on E.

The strong condition (6) one can replace by that of the weaker (4), namely let

$$E_N = \{x \in E \colon \sup_i L_i(T, \varphi/\sqrt{\lambda}; x) \leqq N\},$$

then obviously $E=\bigcup_{N=1}^{\infty} E_N$. Because of $|t_I^*(x)|=O_x(1)$ on every E_N, we obtain

(8) $$|t_I^*(x)| = O_x(1) \quad (c\in l^2(\lambda);\, x\in E).$$

Denote $S(E, \mathscr{A}, \mu)=S$ the set of $\mathscr{A}$-measurable, μ-almost everywhere finite functions on E, endowed with the complete metric of convergence in measure. It is easy to see that for fixed i and N the linear operators

$$L_N^i(c) = \sum_{k=0}^{N} t_{i,k} c_k \varphi_k; \quad l^2(\lambda) \to S$$

are continuous, and for every fixed i the sequence $\{L_N^i(c)\}$ converges in the metric of S as N tends to infinity, for every $c\in l^2(\lambda)$; further

$$\lim_{N\to\infty} L_N^i(c)(x) = t_i(c, x) \quad (i = 0, 1, \ldots).$$

By a theorem in Dunford—Schwarz's book ([11] p. 52) it follows that the linear operators

$$L^i(c); \quad l^2(\lambda) \to S, \quad L^i(c)(x) = t_i(c, x) \qquad (i = 0, 1, \ldots)$$

are also linear and continuous. From (8) it follows that

(9) $$\sup_i |L^i(c)(x)| < \infty$$

almost everywhere on E, further we recall the fact that for $c_N\in l^2(\lambda)$ $(c_N=\{c_k\}_0^N)$ the limit

(10) $$\lim_{i\to\infty} L^i(c_N)(x) = \lim_{i\to\infty} t_i(c_N, x)$$

exists almost everywhere on E. From (9) and (10) it follows according to a theorem of Banach ([1], Theorem II) that for every $c\in l^2(\lambda)$ the limit

$$\lim_{i\to\infty} L^i(c)(x) = \lim_{i\to\infty} t_i(c, x)$$

exists almost everywhere on E. So our theorem is proved.

Remark. We have mentioned in the introduction that our theorem is valid also for arbitrary (not necessarily orthogonal) integrable systems φ. But for an orthogonal system φ the condition (4) can be weakened to

(11) $$L_i(T, \varphi; x) = O_x(\sqrt{\lambda_i}) \quad (x\in E)$$

if we restrict T on the Cesàro or Riesz summation ([3], [5], [8], [9]). On the contrary, we want now to show that, *for general systems* φ *the condition* (11) *is no more sufficient for any permanent method* T. We prove this by elementary construction of a counter-example.

Let $E=(0, 1)$, $\lambda_n=\log n$, $\varphi_n(x)\equiv 1/\sqrt{n}$ and consider the common Lebesgue functions:

$$L_i(\varphi; x) = \int_0^1 \left|\sum_{k=1}^{i} \varphi_k(x)\varphi_k(y)\right| dy = \sum_{k=1}^{i} \frac{1}{k} = O(\log i) = O(\lambda_i).$$

So the condition (11) is satisfied. (One can easily see that also (5) is satisfied trivially when we choose $\beta_k = k^{-1}\log^{-1} k\,(\log\log k)^{-2}$, because in this case the right hand side of (5) becomes infinite.) Put

$$c_n = \frac{1}{\sqrt{n}\log n\cdot\log\log n},$$

then $c\in l^2(\lambda)$, but

$$\sum_{n=3}^{\infty} c_n\varphi_n(x) = \sum_{n=3}^{\infty}\frac{1}{n\log n\cdot\log\log n} = \infty.$$

All the terms of the series $\sum c_n\varphi_n(x)$ being positive, from the divergence to infinity it follows that $\sum c_n\varphi_n(x)$ is nowhere summable in E by any permanent method T.

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ASYMPTOTIC DISTRIBUTION AND ASYMPTOTIC INDEPENDENCE OF SEQUENCES OF INTEGERS

By

M. B. NATHANSON (Princeton)

0. Introduction. The idea of asymptotic distribution and asymptotic independence modulo m of sequences of integers was introduced by KUIPERS, NIEDERREITER, and SHIUE [1, 2] in their study of the uniform distribution of sequences of integers. The following question arose in their work: If A is a sequence with asymptotic distribution $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ modulo m, and if $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ is a distribution, then is there a sequence B with asymptotic distribution $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ modulo m such that A and B are independent? The existence of such a sequence is proved constructively in this paper. NIEDERREITER [4, 5] has obtained some non-constructive generalizations of these results.

1. Asymptotic distribution modulo m. Let $A=\{a_i\}_{i=1}^{\infty}$ be a sequence of integers. For integers $m \geqq 2$ and r, let $A(N, r, m)$ denote the number of a_i with $i \leqq N$ such that $a_i \equiv r \pmod m$. If the limits

$$\lim_{N\to\infty} \frac{A(N, r, m)}{N} = \alpha_r$$

exist for $r=0, 1, \ldots, m-1$, then the sequence A has asymptotic distribution $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ modulo m. Clearly, $0 \leqq \alpha_r \leqq 1$ and $\alpha_0+\alpha_1+\ldots+\alpha_{m-1}=1$.

For example, the sequence $\mathbf{N}$ of the natural numbers in their natural order has asymptotic distribution $(1/m, 1/m, \ldots, 1/m)$ modulo m for all $m \geqq 2$. The sequence of squares $\{i^2\}_{i=1}^{\infty}$ has asymptotic distribution (1/3, 2/3, 0) modulo 3 and (1/2, 1/2, 0, 0) modulo 4. Let $[x]$ denote the integral part of the real number x. NIVEN [6] has proved that if θ is irrational, then the sequence $\{[i\theta]\}_{i=1}^{\infty}$ has asymptotic distribution $(1/m, 1/m, \ldots, 1/m)$ modulo m for all $m \geqq 2$. If $n=md$, then $A(N, r, m)= \sum_{k=0}^{d-1} A(N, r+km, n)$. It follows that if the sequence A has asymptotic distribution $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ modulo n, then A has asymptotic distribution $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ modulo m, where $\beta_r = \sum_{k=0}^{d-1} \alpha_{r+km}$.

An m-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ of nonnegative real numbers whose sum is 1 is called an asymptotic distribution modulo m. In this section we show how to construct a sequence of integers that satisfies a given family of asymptotic distributions.

If $X=\{x_j\}_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers, let $X(N)$ denote the number of x_j such that $x_j \leqq N$. The sequence X has asymptotic density δ if

$$\lim_{N\to\infty} \frac{X(N)}{N} = \delta.$$

Clearly, $0 \leqq X(N) \leqq N$, and so $0 \leqq \delta \leqq 1$. If X has asymptotic density δ, then $\mathbf{N} \setminus X = \{i \in \mathbf{N} \mid i \notin X\}$ has asymptotic density $1-\delta$. If $A=\{a_i\}_{i=1}^{\infty}$ and $B=\{b_i\}_{i=1}^{\infty}$ are two sequences of integers such that $\{i \mid a_i \not\equiv b_i (\mathrm{mod}\, m)\}$ has density 0, then A and B have the same asymptotic distribution modulo m.

LEMMA 1. *Let $0 \leqq \delta \leqq 1$. Then there exists a sequence $X=\{x_j\}_{j=1}^{\infty}$ with asymptotic density δ.*

PROOF. If $\delta=0$, let X be a finite sequence or the sequence $\{2^j\}_{j=1}^{\infty}$. If $0<\delta\leqq 1$, let $x_1=1$ if and only if $\delta=1$. Suppose that $X(N)=k$ and that $x_1<x_2<\ldots<x_k$ have been determined so that

$$\frac{X(N)}{N} = \frac{k}{N} \leqq \delta < \frac{k+1}{N} = \frac{X(N)+1}{N}.$$

Let $X_{k+1}=N+1$ if and only if $\delta \geqq (k+1)/(N+1)$. Since

$$\frac{k}{N+1} \leqq \frac{k}{N} \leqq \frac{k+1}{N+1} \leqq \frac{k+1}{N} \leqq \frac{k+2}{N+1},$$

it follows that

$$\frac{X(N+1)}{N} \leqq \delta < \frac{X(N+1)+1}{N+1}.$$

The sequence X constructed inductively in this way satisfies $0 \leqq \delta - X(N)/N < 1/N$ for all N, and so has asymptotic density δ.

LEMMA. 2. *Let $0 \leqq \gamma \leqq \delta \leqq 1$. If the sequence $X=\{x_j\}_{j=1}^{\infty}$ has asymptotic density δ, then there is a subsequence Z of X with asymptotic density γ.*

PROOF. If $\gamma=\delta=0$, let Z be any subsequence of X. If $\delta>0$, then $0 \leqq \gamma/\delta \leqq 1$. By Lemma 1, there exists a sequence $Y=\{y_j\}_{j=1}^{\infty}$ with asymptotic density γ/δ. Let $Z=\{x_{y_j}\}_{j=1}^{\infty}$. Then $Z(N)=Y(X(N))$, and so

$$\lim_{N\to\infty} \frac{Z(N)}{N} = \lim_{N\to\infty} \frac{Y(X(N))}{X(N)} \frac{X(N)}{N} = \frac{\gamma}{\delta} \cdot \delta = \gamma.$$

Let $[a, b]$ denote the set of integers n such that $a \leqq n \leqq b$.

THEOREM 1. *Let $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ be an asymptotic distribution modulo m. Then there exists a sequence of integers with asymptotic distribution $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ modulo m.*

PROOF. It suffices to partition $\mathbf{N}$ into m disjoint sequences $X_0, X_1, \ldots, X_{m-1}$ such that X_r has asymptotic density α_r for all $r \in [0, m-1]$. Then any sequence of integers $A=\{a_i\}_{i=1}^{\infty}$ with $a_i \equiv r \pmod m$ for all $i \in X_r$ has asymptotic distribution $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ modulo m.

By Lemma 1, there is a sequence X_0 of positive integers with asymptotic density α_0. Then $\mathbf{N}\setminus X_0$ has asymptotic density $\alpha_1+\alpha_2+\dots+\alpha_{m-1}\geqq\alpha_1$. By Lemma 2, there is a subsequence X_1 of $\mathbf{N}\setminus X_0$ with asymptotic density α_1. Then $\mathbf{N}\setminus(X_0\cup X_1)$ has asymptotic density $1-\alpha_0-\alpha_1=\alpha_2+\dots+\alpha_{m-1}\geqq\alpha_2$. Continuing in this way, we obtain the desired partition of $\mathbf{N}$. Clearly, whenever $\alpha_r=0$ we can make X_r either a finite or an infinite sequence.

THEOREM 2. *Let $A=\{a_i\}_{i=1}^{\infty}$ be a sequence of integers. Then some rearrangement of A has asymptotic distribution $(\alpha_0, \alpha_1, \dots, \alpha_{m-1})$ modulo m if and only if the congruence $a_i\equiv r \pmod m$ holds for infinitely many i whenever $\alpha_r>0$.*

PROOF. If, for some $r\in[0, m-1]$, there are only finitely many i such that $a_i\equiv r \pmod m$, then for any rearrangement A' of A we have $\lim_{N\to\infty} A'(N, r, m)/N=$ $=0$, and so the stated congruence conditions are necessary.

Conversely, let $A_r=\{a_{j,r}\}_{j=1}^{\infty}$ be the subsequence of A consisting of all $a_i\equiv$ $\equiv r \pmod m$, and suppose that A_r is an infinite sequence whenever $\alpha_r>0$. Partition $\mathbf{N}$ into m disjoint sequences $X_0, X_1, \dots, X_{m-1}$ such that X_r has asymptotic density α_r and $\operatorname{card}(X_r)=\operatorname{card}(A_r)$ whenever $\alpha_r=0$. Let $X_r=\{x_{j,r}\}_{j=1}^{\infty}$. For any $i\in\mathbf{N}$, there exist unique $j\in\mathbf{N}$ and $r\in[0, m-1]$ such that $i=x_{j,r}$. Let $a_i'=a_{j,r}$. Then $A'=\{a_i'\}_{i=1}^{\infty}$ is a rearrangement of A with asymptotic distribution $(\alpha_0, \alpha_1, \dots, \alpha_{m-1})$ modulo m.

THEOREM 3. *Let $\{m_k\}$ be a finite or infinite family of moduli, and let $(\alpha_{0,k}, \alpha_{1,k}, \dots$ $\dots, \alpha_{m_k-1,k})$ be an asymptotic distribution modulo m_k, such that*

(i) $m_k|m_{k+1}$ *for* $k=1, 2, 3, \dots$; *and*

(ii) $$\alpha_{r,k}=\sum_{d=0}^{\frac{m_{k+1}}{m_k}-1}\alpha_{r+dm_k, k+1} \quad \textit{for} \quad r\in[0, m_k-1].$$

Then there exists a sequence of integers $A=\{a_i\}_{i=1}^{\infty}$ such that A has asymptotic distribution $(\alpha_{0,k}, \alpha_{1,k}, \dots, \alpha_{m_k-1,k})$ modulo m_k for all k.

PROOF. If the family of moduli is finite, then it suffices to construct a sequence with the desired asymptotic distribution with respect to the largest modulus in the family. Then conditions (i) and (ii) imply that this sequence has the right asymptotic distribution with respect to all other moduli in the family.

Suppose that the family of moduli is infinite. We construct inductively a family of sequences $\{A_k\}$, where $A_k=\{a_{i,k}\}_{i=1}^{\infty}$, such that A_k has asymptotic distribution $(\alpha_{0,k}, \alpha_{1,k}, \dots, \alpha_{m_k-1,k})$ modulo m_k, and $a_{i,k+1}\equiv a_{i,k} \pmod{m_k}$ for all i and k. Let the sequence A_k be constructed by partitioning $\mathbf{N}$ into m_k disjoint sequences $X_0, X_1, \dots, X_{m_k-1}$ with asymptotic densities $\alpha_{0,k}, \alpha_{1,k}, \dots, \alpha_{m_k-1,k}$, respectively, and letting $a_{i,k}\equiv r \pmod{m_k}$ for all $i\in X_r$. For each $r\in[0, m_k-1]$, partition X into m_{k+1}/m_k subsequences Y_{r+dm_k} for $d\in[0, m_{k+1}/m_k-1]$ such that Y_{r+dm_k} has asymptotic density $\alpha_{r+dm_k, k+1}$. The sets Y_{r+dm_k} for $r\in[0, m_k-1]$ and $d\in[0, m_{k+1}/m_k-1]$ partition the natural numbers $\mathbf{N}$. Let $a_{i,k+1}\equiv r+dm_k \pmod{m_{k+1}}$ if $i\in Y_{r+dm_k}$. Then $A_{k+1}=\{a_{i,k+1}\}_{i=1}^{\infty}$ has asymptotic distribution $(\alpha_{0,k+1}, \alpha_{1,k+1}, \dots, \alpha_{m_{k+1}-1,k+1})$ modulo m_{k+1}, and $a_{i,k+1}\equiv a_{i,k} \pmod{m_k}$ for all i.

Now let $A=\{a_i\}_{i=1}^{\infty}$ be the "diagonal" sequence defined by $a_i=a_{i,i}$ for all i. Then $a_i\equiv a_{i,k} \pmod{m_k}$ for all $i\geqq k$, and so A has asymptotic distribution $(\alpha_{0,k}, \alpha_{1,k}, \ldots, \alpha_{m_k-1,k})$ modulo m_k, for all k.

THEOREM 4. *Let $\{m_k\}$ be a finite or infinite family of pairwise relatively prime moduli, and, for each k, let $(\alpha_{0,k}, \alpha_{1,k}, \ldots, \alpha_{m_k-1,k})$ be an asymptotic distribution modulo m_k. Then there exists a sequence $A=\{a_i\}_{i=1}^{\infty}$ with asymptotic distribution $(\alpha_{0,k}, \alpha_{1,k}, \ldots, \alpha_{m_k-1,k})$ modulo m_k for all k.*

PROOF. Let $M_k=m_1m_2\ldots m_k$. If $r\in[0, M_k-1]$, then there exist unique integers $r_j\in[0, m_j-1]$ such that $r\equiv r_j \pmod{m_j}$ for all $j\in[1, k]$. Let $\gamma_{r,k}=\prod_{j=1}^{k}\alpha_{r_j,j}$. Then $0\leqq\gamma_{r,k}\leqq 1$, and

$$\sum_{r=0}^{M_k-1}\gamma_{r,k}=\sum_{r=0}^{M_k-1}\prod_{j=1}^{k}\alpha_{r_j,j}=\prod_{j=1}^{k}\sum_{s=0}^{m_j-1}\alpha_{s,j}=1,$$

and so $(\gamma_{0,k}, \gamma_{1,k}, \ldots, \gamma_{M_k-1,k})$ is an asymptotic distribution modulo M_k. Clearly, $M_k|M_{k+1}$ for all k. Moreover, for $r\in[0, M_k-1]$ and $d\in[0, M_{k+1}/M_k-1]$,

$$\gamma_{r+dM_k,k+1}=\prod_{j=1}^{k+1}\alpha_{(r+dM_k)_j,j}=\prod_{j=1}^{k}\alpha_{(r+dM_k)_j,j}\cdot\alpha_{(r+dM_k)_{k+1},k+1}=$$

$$=\prod_{j=1}^{k}\alpha_{r_j,j}\,\alpha_{(r+dM_k)_{k+1},k+1}=\gamma_{r,k}\alpha_{(r+dM_k)_{k+1},k+1},$$

and so

$$\sum_{d=0}^{\frac{M_{k+1}}{M_k}-1}\gamma_{r+dM_k,k+1}=\sum_{d=0}^{m_{k+1}-1}\gamma_{r,k}\alpha_{(r+dM_k)_{k+1},k+1}=\gamma_{r,k}\sum_{d=0}^{m_{k+1}-1}\alpha_{d,k+1}=\gamma_{r,k}.$$

Since conditions (i) and (ii) of Theorem 3 are satisfied, there exists a sequence A with asymptotic distribution $(\gamma_{0,k}, \gamma_{1,k}, \ldots, \gamma_{M_k-1,k})$ modulo M_k for all k.

Let $r\in[0, m_k-1]$. Then

$$\sum_{d=0}^{\frac{M_k}{m_k}-1}\gamma_{r+dm_k,k}=\sum_{d=0}^{M_{k-1}-1}\prod_{j=1}^{k}\alpha_{(r+dm_k)_j,j}=\alpha_{r,k}\sum_{d=0}^{M_{k-1}-1}\prod_{j=1}^{k-1}\alpha_{(r+dm_k)_j,j}=$$

$$=\alpha_{r,k}\sum_{d=0}^{M_{k-1}-1}\prod_{j=1}^{k-1}\alpha_{d_j,j}=\alpha_{r,k}\sum_{d=0}^{M_{k-1}-1}\gamma_{d,k-1}=\alpha_{r,k},$$

and so the sequence A has the desired asymptotic distribution $(\alpha_{0,k}, \alpha_{1,k}, \ldots, \alpha_{m_k-1,k})$ modulo m_k for all k.

REMARK. Suppose that a sequence A has asymptotic distribution $(\alpha_{0,k}, \alpha_{1,k}, \ldots, \alpha_{m_k-1,k})$ modulo m_k for all moduli in a family $\{m_k\}$. If $d|m_1$ and $d|m_2$ for moduli m_1 and m_2 in this family, then A has asymptotic distribution $(\beta_0, \beta_1, \ldots, \beta_{d-1})$ modulo d, where for $r\in[0, d-1]$

$$(*)\qquad \beta_r=\sum_{i=0}^{\frac{m_1}{d}-1}\alpha_{r+id,1}=\sum_{j=0}^{\frac{m_2}{d}-1}\alpha_{r+jd,2}.$$

Conversely, is it true that if a family of asymptotic distributions satisfies the "consistency" conditions $(*)$, then there exists a sequence of integers with these asymptotic distributions?

2. Asymptotic independence modulo m. Let $A=\{a_i\}_{i=1}^{\infty}$ and $B=\{b_i\}_{i=1}^{\infty}$ be sequences of integers with asymptotic distributions $(\alpha_0, \alpha_1, \dots, \alpha_{m-1})$ and $(\beta_0, \beta_1, \dots, \beta_{m-1})$ modulo m, respectively. If $r, s \in [0, m-1]$, let $A\times B(N, r, s, m)$ denote the number of pairs (a_i, b_i) such that $a_i \equiv r \pmod m$ and $b_i \equiv s \pmod m$ and $i \leqq N$. If

$$\lim_{N\to\infty} \frac{A\times B(N, r, s, m)}{N} = \alpha_r \beta_s$$

for all $r, s \in [0, m-1]$, then the sequences A and B are *asymptotically independent modulo* m. In general, let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a finite family of sequences of integers, and let the sequence $A_\lambda = \{a_i^\lambda\}_{i=1}^{\infty}$ have asymptotic distribution $(\alpha_0^\lambda, \alpha_1^\lambda, \dots, \alpha_{m-1}^\lambda)$ modulo m. Let $\mathscr{R} = \prod_{\lambda\in\Lambda} [0, m-1]$. If $R = (r_\lambda)_{\lambda\in\Lambda} \in \mathscr{R}$, let $\prod_{\lambda\in\Lambda} A_\lambda(N, R, m)$ denote the number of Λ-tuples $(a_i^\lambda)_{\lambda\in\Lambda}$ such that $i \leqq N$ and $a_i^\lambda \equiv r_\lambda \pmod m$ for all $\lambda\in\Lambda$. The family of sequences $\{A_\lambda\}_{\lambda\in\Lambda}$ is asymptotically independent modulo m if

$$\lim_{N\to\infty} \frac{\prod_{\lambda\in\Lambda} A_\lambda(N, R, m)}{N} = \prod_{\lambda\in\Lambda} \alpha_{r_\lambda}^\lambda$$

for all $R\in\mathscr{R}$. An infinite family of sequences is asymptotically independent modulo m if every finite subfamily is.

For example, let A be the periodic sequence $0, 1, 0, 1, 0, 1, \dots,$ and let B be the periodic sequence $0, 0, 1, 1, 0, 0, 1, 1, \dots$. Both A and B have asymptotic distribution $(1/2, 1/2)$ modulo 2. For every $R \in \mathscr{R} = \{(0, 0), (0, 1), (1,0), (1,1)\}$, we have

$$\lim_{N\to\infty} \frac{A\times B(N, R, 2)}{N} = \frac{1}{4} = \frac{1}{2}\cdot\frac{1}{2},$$

and so A and B are asymptotically independent modulo 2. NIEDERREITER [3] has proved that if $1, \theta_1, \theta_2, \dots, \theta_n$ are linearly independent over the rational numbers, then the sequences $A_1 = \{[i\theta_1]\}_{i=1}^{\infty}, \dots, A_n = \{[i\theta_n]\}_{i=1}^{\infty}$ are asymptotically independent modulo m for all $m \geqq 2$.

Clearly, if a finite or infinite family of sequences is asymptotically independent modulo m, then every subfamily is also independent. The converse is false. For example, the three periodic sequences determined by the quadruples $(0, 0, 1, 1)$, $(0, 1, 0, 1)$, and $(0, 1, 1, 0)$ are pairwise asymptotically independent modulo 2, but are not independent.

In this section we show how to construct a sequence of integers that satisfies a given family of asymptotic distributions and that is asymptotically independent of a given family of sequences of integers.

LEMMA 3. *Let* $\{\beta_R\}_{R\in\mathscr{R}}$ *and* $\{\chi_R\}_{R\in\mathscr{R}}$ *be two families of sequences of real numbers such that*

(i) $\lim_{N\to\infty} \beta_R(N) = \beta$ *uniformly in* R;

(ii) *for each* $N=1, 2, \ldots,$ *we have* $\chi_R(N)\neq 0$ *for only finitely many* $R\in\mathcal{R}$;

(iii) $\lim_{N\to\infty} \sum_{R\in\mathcal{R}} \chi_R(N) = \chi$; *and*

(iv) $\limsup_{N\to\infty} \sum_{R\in\mathcal{R}} |\chi_R(N)| < \infty$.

Then

$$\lim_{N\to\infty} \sum_{R\in\mathcal{R}} \beta_R(N)\chi_R(N) = \beta\chi.$$

PROOF. By (iv), there is a constant C such that $\sum_{R\in\mathcal{R}} |\chi_R(N)|<C$ for all N. Fix $\varepsilon>0$. There exists an integer $N(\varepsilon)$ such that

$$\Big|\sum_{R\in\mathcal{R}} \chi_R(N)-\chi\Big| < \frac{\varepsilon}{C+|\beta|} \quad \text{and} \quad |\beta_R(N)-\beta| < \frac{\varepsilon}{C+|\beta|}$$

for all $R\in\mathcal{R}$ and $N>N(\varepsilon)$. Then for $N>N(\varepsilon)$,

$$\Big|\sum_{R\in\mathcal{R}} \beta_R(N)\chi_R(N)-\beta\chi\Big| \leqq \Big|\sum_{R\in\mathcal{R}} \beta_R(N)\chi_R(N)-\beta \sum_{R\in\mathcal{R}} \chi_R(N)\Big|+\Big|\beta \sum_{R\in\mathcal{R}} \chi_R(N)-\beta\chi\Big| \leqq$$

$$\leqq \sum_{R\in\mathcal{R}} |\beta_R(N)-\beta||\chi_R(N)|+|\beta|\Big|\sum_{R\in\mathcal{R}} \chi_R(N)-\chi\Big| < \frac{\varepsilon C}{C+|\beta|}+\frac{\varepsilon|\beta|}{C+|\beta|} = \varepsilon.$$

Note that condition (ii) and the uniformity condition in (i) of Lemma 3 are automatically satisfied when $\mathcal{R}$ is a finite set.

THEOREM 5. *Let* $\{A_\lambda\}_{\lambda\in\Lambda}$ *be a finite family of sequences of integers, and let the sequence* $A_\lambda=\{a_i^\lambda\}_{i=1}^\infty$ *have asymptotic distribution* $(\alpha_0^\lambda, \alpha_1^\lambda, \ldots, \alpha_{m-1}^\lambda)$ *modulo* m. *Let* $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ *be an asymptotic distribution modulo* m. *Then there exists a sequence* $B=\{b_i\}_{i=1}^\infty$ *such that* B *has asymptotic distribution* $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ *modulo* m, *and the sequences* A_λ *and* B *are asymptotically independent modulo* m *for all* $\lambda\in\Lambda$. *If the family* $\{A_\lambda\}_{\lambda\in\Lambda}$ *is asymptotically independent modulo* m, *then the sequence* B *can be chosen so that the family* $\{A_\lambda\}_{\lambda\in\Lambda}\cup\{B\}$ *is asymptotically independent modulo* m.

PROOF. Let $\mathcal{R}=\prod_{\lambda\in\Lambda}[0, m-1]$. For $R=(r_\lambda)_{\lambda\in\Lambda}\in\mathcal{R}$, define

$$X_R = \{i|a_i^\lambda \equiv r_\lambda \pmod m \quad \text{for all} \quad \lambda\in\Lambda\} = \{x_{j,R}\}_{j=1}^\infty,$$

where the positive integers $x_{j,R}$ are arranged in order of increasing magnitude: $x_{1,R}<x_{2,R}<\ldots$. Let $\mathcal{R}_0=\{R\in\mathcal{R}|0<\operatorname{card}(X_R)<\infty\}$ and let $\mathcal{R}_1=\{R\in\mathcal{R}|\operatorname{card}(X_R)=\infty\}$. Then

$$\sum_{R\in\mathcal{R}} X_R(N) = \sum_{R\in\mathcal{R}_0} X_R(N)+\sum_{R\in\mathcal{R}_1} X_R(N) = N$$

for all N, and

(1) $$\lim_{N\to\infty} \sum_{R\in\mathcal{R}_1} \frac{X_R(N)}{N} = 1.$$

By Theorem 1, there exists a sequence $B'=\{b'_j\}_{j=1}^{\infty}$ with asymptotic distribution $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ modulo m. Since the sets $\{X_R\}_{R\in\mathscr{R}}$ partition the positive integers, for any i there exist unique j and R such that $i=x_{j,R}$. Let $b_i=b'_j$, and let $B=\{b_i\}_{i=1}^{\infty}$. Then for $r\in[0, m-1]$,

$$B(N, r, m) = \sum_{R\in\mathscr{R}} B'\big(X_R(N), r, m\big)$$

and so

$$\frac{B(N, r, m)}{N} = \sum_{R\in\mathscr{R}_0} \frac{B'(X_R(N), r, m)}{N} + \sum_{R\in\mathscr{R}_1} \frac{B'(X_R(N), r, m)}{X_R(N)} \frac{X_R(N)}{N}.$$

But

$$\lim_{N\to\infty} \sum_{R\in\mathscr{R}_0} \frac{B'\big(X_R(N), r, m\big)}{N} = 0. \tag{2}$$

Also, $X_R(N)\to\infty$ as $N\to\infty$ for every $R\in\mathscr{R}_1$, and so

$$\lim_{N\to\infty} \frac{B'\big(X_R(N), r, m\big)}{X_R(N)} = \beta_r. \tag{3}$$

It follows from (1), (2), (3), and Lemma 3 that

$$\lim_{N\to\infty} \frac{B(N, r, m)}{N} =$$

$$= \lim_{N\to\infty} \sum_{R\in\mathscr{R}_0} \frac{B'\big(X_R(N), r, m\big)}{N} + \lim_{N\to\infty} \sum_{R\in\mathscr{R}_1} \frac{B'\big(X_R(N), r, m\big)}{X_R(N)} \frac{X_R(N)}{N} = \beta_r.$$

Therefore, the sequence B has asymptotic distribution $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ modulo m.

Choose $\lambda\in\Lambda$ and $r, s\in[0, m-1]$. Then

$$A_\lambda\times B(N, r, s, m) = \sum_{\substack{R\in\mathscr{R}\\ r_\lambda=r}} B'\big(X_R(N), s, m\big)$$

and so

$$\frac{A_\lambda\times B(N, r, s, m)}{N} = \sum_{\substack{R\in\mathscr{R}_0\\ r_\lambda=r}} \frac{B'\big(X_R(N), s, m\big)}{N} + \sum_{\substack{R\in\mathscr{R}_1\\ r_\lambda=r}} \frac{B'\big(X_R(N), s, m\big)}{X_R(N)} \frac{X_R(N)}{N}.$$

Clearly,

$$\sum_{\substack{R\in\mathscr{R}\\ r_\lambda=r}} X_R(N) = \sum_{\substack{R\in\mathscr{R}_0\\ r_\lambda=r}} X_R(N) + \sum_{\substack{R\in\mathscr{R}_1\\ r_\lambda=r}} X_R(N) = A_\lambda(N, r, m),$$

and so

$$\lim_{N\to\infty} \sum_{\substack{R\in\mathscr{R}_1\\ r_\lambda=r}} \frac{X_R(N)}{N} = \lim_{N\to\infty} \frac{A_\lambda(N, r, m)}{N} = \alpha_r^\lambda. \tag{4}$$

Also,

$$\lim_{N\to\infty} \frac{B'\big(X_R(N), s, m\big)}{X_R(N)} = \beta_s \tag{5}$$

for all $R\in\mathscr{R}_1$, and

$$\lim_{N\to\infty}\sum_{\substack{R\in\mathscr{R}_0\\ r_\lambda=r}}\frac{B'(X_R(N),s,m)}{N}=0. \tag{6}$$

It follows from (4), (5), (6), and Lemma 3 that

$$\lim_{N\to\infty}\frac{A_\lambda\times B(N,r,s,m)}{N}=$$

$$=\lim_{N\to\infty}\sum_{\substack{R\in\mathscr{R}_0\\ r_\lambda=r}}\frac{B'(X_R(N),s,m)}{N}+\lim_{N\to\infty}\sum_{\substack{R\in\mathscr{R}_1\\ r_\lambda=r}}\frac{B'(X_R(N),s,m)}{X_R(N)}\,\frac{X_R(N)}{N}=\alpha_r^\lambda\beta_s.$$

Therefore, the sequences A_λ and B are asymptotically independent modulo m for all $\lambda\in\Lambda$.

If $R\in\mathscr{R}$, then $X_R(N)=\prod_{\lambda\in\Lambda}A_\lambda(N,R,m)$. It follows that if the family of sequences $\{A_\lambda\}_{\lambda\in\Lambda}$ is asymptotically independent modulo m, then

$$\lim_{N\to\infty}\frac{X_R(N)}{N}=\lim_{N\to\infty}\frac{\prod_{\lambda\in\Lambda}A_\lambda(N,R,m)}{N}=\prod_{\lambda\in\Lambda}\alpha_{r_\lambda}^\lambda.$$

Let $(R,s)\in\mathscr{R}\times[0,m-1]$. Then

$$\Big(\prod_{\lambda\in\Lambda}A_\lambda\times B\Big)\big(N,(R,s),m\big)=B'\big(X_R(N),s,m\big).$$

Therefore,

$$\lim_{N\to\infty}\frac{\Big(\prod_{\lambda\in\Lambda}A_\lambda\times B\Big)\big(N,(R,s),m\big)}{N}=\lim_{N\to\infty}\frac{B'(X_R(N),s,m)}{N}=$$

$$=\lim_{N\to\infty}\frac{B'(X_R(N),s,m)}{X_R(N)}\,\frac{X_R(N)}{N}=\beta_s\prod_{\lambda\in\Lambda}\alpha_{r_\lambda}^\lambda,$$

and so the family of sequences $\{A_\lambda\}_{\lambda\in\Lambda}\cup\{B\}$ is asymptotically independent modulo m.

THEOREM 6. *Let* $\{(\alpha_{0,n},\alpha_{1,n},\dots,\alpha_{m-1,n})\}_{n=1}^\infty$ *be a countable family of asymptotic distributions modulo* m. *Then there exists an asymptotically independent family of sequences* $\{A_n\}_{n=1}^\infty$ *such that* A_n *has asymptotic distribution* $(\alpha_{0,n},\alpha_{1,n},\dots,\alpha_{m-1,n})$ *modulo* m.

PROOF. Suppose that an asymptotically independent family of $n-1$ sequences $\{A_k\}_{k=1}^{n-1}$ has been constructed such that A_k has asymptotic distribution $(\alpha_{0,k},\alpha_{1,k},\dots,\alpha_{m-1,n})$ modulo m. By Theorem 5, there exists a sequence A_n with asymptotic distribution $(\alpha_{0,n},\alpha_{1,n},\dots,\alpha_{m-1,n})$ modulo m such that the family $\{A_k\}_{k=1}^n$ is asymptotically independent modulo m. The theorem follows by induction on n.

THEOREM 7. *Let* $A=\{a_i\}_{i=1}^{\infty}$ *be a sequence of integers with asymptotic distribution* $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ *modulo* m. *The following are equivalent:*

(i) *The sequences* A *and* B *are asymptotically independent modulo* m *for every sequence* B *with asymptotic distribution modulo* m.

(ii) *The sequence* A *is asymptotically independent of itself modulo* m.

(iii) $\alpha_r=1$ *for some* $r\in[0, m-1]$ *and* $\alpha_s=0$ *for all* $s\in[0, m-1]$ *with* $s\neq r$.

PROOF. Clearly, (i) implies (ii).

If the sequences A and A are asymptotically independent, then $A\times A(N, r, r, m)=A(N, r, m)$ for all $r\in[0, m-1]$, and so

$$\alpha_r^2 = \lim_{N\to\infty} \frac{A\times A(N, r, r, m)}{N} = \lim_{N\to\infty} \frac{A(N, r, m)}{N} = \alpha_r.$$

Therefore, $\alpha_r=0$ or 1. Since $\alpha_0+\alpha_1+\ldots+\alpha_{m-1}=1$, it follows that $\alpha_r=1$ for some unique $r\in[0, m-1]$, and $\alpha_s=0$ for $s\neq r$. Thus, (ii) implies (iii).

Suppose that (iii) holds, and that B is a sequence with asymptotic distribution $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ modulo m. If $s, t\in[0, m-1]$ and $s\neq r$, then $0\leqq A\times B(N, s, t, m)\leqq A(N, s, m)$, and so

$$0 \leqq \lim_{N\to\infty} \frac{A\times B(N, s, t, m)}{N} \leqq \lim_{N\to\infty} \frac{A(N, s, m)}{N} = \alpha_s = 0.$$

Therefore,

$$\lim_{N\to\infty} \frac{A\times B(N, s, t, m)}{N} = \alpha_s\beta_t = 0$$

for $s\neq r$. Similarly,

$$B(N, t, m) = A\times B(N, r, t, m)+\sum_{\substack{s=0\\ s\neq r}}^{m-1} A\times B(N, s, t, m),$$

and so

$$\lim_{N\to\infty} \frac{A\times B(N, r, t, m)}{N} = \lim_{N\to\infty} \frac{B(N, t, m)}{N} - \lim_{N\to\infty} \sum_{\substack{s=0\\ s\neq r}}^{m-1} \frac{A\times B(N, s, t, m)}{N} = \\ = \beta_t = \alpha_r\beta_t.$$

Thus the sequences A and B are asymptotically independent modulo m, and so (iii) implies (i).

REMARKS. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an infinite family of sequences of integers with asymptotic distributions modulo m, and let $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ be an asymptotic distribution modulo m. Define the sets X_R for $R\in\mathscr{R}=\prod_{\lambda\in\Lambda}[0, m-1]$ exactly as in the proof of Theorem 5. If $\mathscr{R}_0=\{R\in\mathscr{R}|0<\text{card}\,(X_R)<\infty\}$ is a finite set, and if there exists a sequence B' with asymptotic distribution $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ modulo m such that

$$\lim_{N\to\infty} \frac{B'(X_R(N), r, m)}{X_R(N)} = \beta_r$$

uniformly for $R\in\mathscr{R}_1=\{R\in\mathscr{R}|\text{card}\,(X_R)=\infty\}$, then the construction in the proof of Theorem 5 yields a sequence B such that B has asymptotic distribution $(\beta_0, \beta_1, \ldots$

$\ldots, \beta_{m-1})$ modulo m, and the sequences A_λ and B are asymptotically independent modulo m for all $\lambda \in \Lambda$.

However, if $\mathscr{R}_0$ is an infinite set, this proof may fail. For example, define the countable family of sequences $A_n = \{a_{i,n}\}_{i=1}^\infty$ by $a_{i,n} = i$ if $i \neq n$ and $a_{n,n} = n+1$. Then card $(X_R) = 0$ or 1 for all $R \in \mathscr{R} = \prod_{n=1}^\infty [0, 1]$. If B' is any sequence with asymptotic distribution (1/2, 1/2) modulo 2, then the sequence B constructed by the method of Theorem 5 will be constant, and not have asymptotic distribution (1/2, 1/2) modulo 2. Of course, in this case there does exist a sequence with asymptotic distribution (1/2, 1/2) and asymptotically independent of all A_n; for example, the periodic sequence 0, 0, 1, 1, 0, 0, 1, 1, ...

But there do exist infinite families of sequences of integers such that no sequence with given asymptotic distribution is asymptotically independent of every sequence in the family. For example, take the "maximally saturated" family of all sequences with asymptotic distribution (1/2, 1/2) modulo 2. Clearly, no sequence with asymptotic distribution (1/2, 1/2) modulo 2 can be independent of all sequences in this family.

Theorems 5 and 6 imply that every finite family of sequences asymptotically independent modulo m is a subfamily of a countably infinite asymptotically independent family of sequences. Does there exist an uncountable asymptotically independent family of sequences? What is the cardinality of a maximal asymptotically independent family of sequences?

It follows from Theorem 7 that a sequence that is constant except perhaps for a subsequence of density zero is asymptotically independent of every sequence with respect to every modulus. Not only these constant sequences have this property; for example, so does the sequence $\{i!\}_{i=1}^\infty$.

Let S be any finite set. We can consider sequences whose elements lie in S, and define the asymptotic distribution and asymptotic independence of these sequences. Using the methods of Theorems 1 and 5, we can construct a sequence over S with given asymptotic distribution and asymptotically independent of any finite family of sequences over S.

3. Strong asymptotic independence. Let $\{m_\mu\}_{\mu \in M}$ be a family of moduli, and let $A = \{a_i\}_{i=1}^\infty$ and $B = \{b_i\}_{i=1}^\infty$ be sequences of integers with asymptotic distributions $(\alpha_{0,\mu}, \alpha_{1,\mu}, \ldots, \alpha_{m_\mu-1,\mu})$ and $(\beta_{0,\mu}, \beta_{1,\mu}, \ldots, \beta_{m_\mu-1,\mu})$ modulo m_μ, respectively, for all $\mu \in M$. If $\mu, \nu \in M$ and $r \in [0, m_\mu - 1]$ and $s \in [0, m_\nu - 1]$, let $A \times B(N, r, m_\mu, s, m_\nu)$ denote the number of (a_i, b_i) with $i \leq N$ such that $a_i \equiv r \pmod{m_\mu}$ and $b_i \equiv s \pmod{m_\nu}$. The sequences A and B are *strongly asymptotically independent modulo* $\{m_\mu\}_{\mu \in M}$ if

$$\lim_{N \to \infty} \frac{A \times B(N, r, m_\mu, s, m_\nu)}{N} = \alpha_{r,\mu} \beta_{s,\nu}$$

for all $\mu, \nu \in M$ and $r \in [0, m_\mu - 1]$ and $s \in [0, m_\nu - 1]$.

THEOREM 8. *Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a finite family of sequences of integers. For each $\lambda \in \Lambda$, let $\{m_\mu\}_{\mu \in M_\lambda}$ be a finite family of moduli such that the sequence $A_\lambda = \{a_i^\lambda\}_{i=1}^\infty$ has asymptotic distribution $(\alpha_{0,\mu}^\lambda, \alpha_{1,\mu}^\lambda, \ldots, \alpha_{m_\mu-1,\mu}^\lambda)$ modulo m_μ, for all $\mu \in M_\lambda$. Let $M = \bigcup_{\lambda \in \Lambda} M_\lambda$, and let $B' = \{b_j'\}_{j=1}^\infty$ be a sequence of integers such that B' has asymptotic distribution $(\beta_{0,\mu}, \beta_{1,\mu}, \ldots, \beta_{m_\mu-1,\mu})$ modulo m_μ for all $\mu \in M$. Then there* *exists a sequence* $B=\{b_i\}_{i=1}^{\infty}$ *such that* B *has asymptotic distribution* $(\beta_{0,\mu}, \beta_{1,\mu}, \dots, \beta_{m_\mu-1,\mu})$ *modulo* m_μ *for* $\mu\in M$, *and the sequences* A_λ *and* B *are strongly asymptotically independent modulo* $\{m_\mu\}_{\mu\in M_\lambda}$ *for all* $\lambda\in\Lambda$.

PROOF. Let $\mathscr{R}_\lambda=\prod_{\mu\in M_\lambda}[0, m_\mu-1]$, and let $\mathscr{R}=\prod_{\lambda\in\Lambda}\mathscr{R}_\lambda$. If $R_\lambda=(r_\mu^\lambda)_{\mu\in M_\lambda}\in\mathscr{R}_\lambda$ and if $R=(R_\lambda)_{\lambda\in\Lambda}\in\mathscr{R}$, define the sets X_{R_λ} and X_R by

$$X_{R_\lambda}=\{i|a_i^\lambda\equiv r_\mu^\lambda \pmod{m_\mu} \quad \text{for all} \quad \mu\in M_\lambda\},$$

$$X_R=\bigcap_{\Lambda\in\lambda} X_{R_\lambda}=\{i|a_i^\lambda\equiv r_\mu^\lambda \pmod{m_\mu} \quad \text{for all} \quad \lambda\in\Lambda \quad \text{and} \quad \mu\in M_\lambda\}=\{x_{j,R}\}_{j=1}^{\infty},$$

where the positive integers $x_{j,R}$ are arranged in order of increasing magnitude. The sets $\{X_R\}_{R\in\mathscr{R}}$ partition the positive integers, and so, for any i, there exist a unique integer j and a unique $R\in\mathscr{R}$ such that $i=x_{j,R}$. Let $b_i=b'_j$, and let $B=\{b_i\}_{i=1}^{\infty}$.

Let $\mu\in M$ and $r\in[0, m_\mu-1]$. Then

$$B(N, r, m_\mu)=\sum_{R\in\mathscr{R}} B'\big(X_R(N), r, m_\mu\big).$$

Let $\mathscr{R}_0=\{R\in\mathscr{R}|0<\operatorname{card}(X_R)<\infty\}$, and let $\mathscr{R}_1=\{R\in\mathscr{R}|\operatorname{card}(X_R)=\infty\}$. Then

$$\frac{B(N, r, m_\mu)}{N}=\sum_{R\in\mathscr{R}_0}\frac{B'\big(X_R(N), r, m_\mu\big)}{N}+\sum_{R\in\mathscr{R}_1}\frac{B'\big(X_R(N), r, m_\mu\big)}{X_R(N)}\,\frac{X_R(N)}{N}.$$

Since

$$\lim_{N\to\infty}\sum_{R\in\mathscr{R}_0}\frac{B'\big(X_R(N), r, m_\mu\big)}{N}=0, \tag{7}$$

and

$$\lim_{N\to\infty}\sum_{R\in\mathscr{R}_1}\frac{X_R(N)}{N}=1, \tag{8}$$

and

$$\lim_{N\to\infty}\frac{B'\big(X_R(N), r, m_\mu\big)}{X_R(N)}=\beta_{r,\mu} \tag{9}$$

for all $R\in\mathscr{R}_1$, it follows from (7), (8), (9), and Lemma 3 that

$$\lim_{N\to\infty}\frac{B(N, r, m_\mu)}{N}=\beta_{r,\mu}.$$

Therefore, the sequence B has asymptotic distribution $(\beta_{0,\mu}, \beta_{1,\mu}, \dots, \beta_{m_\mu-1,\mu})$ modulo m_μ for all $\mu\in M$.

Let $\mu, \nu\in M_\lambda$ and $r\in[0, m_{\mu-1}]$ and $s\in[0, m_\nu-1]$. Then

$$A_\lambda\times B(N, r, m_\mu, s, m_\nu)=\sum_{\substack{R\in\mathscr{R}\\ r_\mu^\lambda=r}} B'\big(X_R(N), s, m_\nu\big),$$

and so

$$\frac{A_\lambda\times B(n, r, m_\mu, s, m_\nu)}{N}=\sum_{\substack{R=\mathscr{R}_0\\ r_\mu^\lambda=r}}\frac{B'\big(X_R(N), s, m_\nu\big)}{N}+\sum_{\substack{R\in\mathscr{R}_1\\ r_\mu^\lambda=r}}\frac{B'\big(X_R(N), s, m_\nu\big)}{X_R(N)}\,\frac{X_R(N)}{N}.$$

Since

$$\lim_{N\to\infty} \sum_{\substack{R\in\mathscr{R}_0 \\ r_\mu^\lambda=r}} \frac{B'(X_R(N), s, m_\nu)}{N} = 0, \tag{10}$$

and

$$\lim_{N\to\infty} \sum_{\substack{R\in\mathscr{R}_1 \\ r_\mu^\lambda=r}} \frac{X_R(N)}{N} = \lim_{N\to\infty} \frac{A_\lambda(N, r, m_\mu)}{N} = \alpha_{r,\mu}^\lambda . \tag{11}$$

and

$$\lim_{N\to\infty} \frac{B'\big(X_R(N), s, m_\nu\big)}{X_R(N)} = \beta_{s,\nu} \tag{12}$$

for all $R\in\mathscr{R}_1$, it follows from (10), (11), (12), and Lemma 3 that

$$\lim_{N\to\infty} \frac{A_\lambda\times B(N, r, m_\mu, s, m_\nu)}{N} = \alpha_{r,\mu}^\lambda \beta_{s,\nu} .$$

Therefore, the sequences A_λ and B are strongly asymptotically independent modulo $\{m_\mu\}_{\mu\in M_\lambda}$, for all $\lambda\in\Lambda$.

REMARKS. The proof of Theorem 8 breaks down if the set of moduli is infinite. For example, let $A=\{a_i\}_{i=1}^\infty$ be a sequence of distinct integers with asymptotic distribution $(\alpha_{0,\mu}, \alpha_{1,\mu}, \dots, \alpha_{m_\mu-1,\mu})$ modulo m_μ, for all μ in an infinite set M. If $R\in\mathscr{R}=\prod_{\mu\in M}[0, m_\mu-1]$ and X_R is the set of integers constructed in Theorem 8, then $i,j\in X_R$ implies that $a_i\equiv a_j \pmod{m_\mu}$ for all $\mu\in M$. But M is infinite, and so $a_i=a_j$. Since A is a sequence of distinct integers, it follows that $i=j$, and so $\operatorname{card}(X_R)=0$ or 1 for all $R\in\mathscr{R}$. Then the sequence B constructed in the proof of Theorem 8 is constant.

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SCHOOL OF MATHEMATICS
THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY 08540
USA

AND

DEPARTMENT OF MATHEMATICS
SOUTHERN ILLINOIS UNIVERSITY
CARBONDALE, ILLINOIS 62901
USA

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RETRACTABLE GROUPS

By

R. D. BYRD, J. T. LLOYD, R. A. MENA and J. R. TELLER (Houston)

1. Introduction. One of the long standing unanswered questions in the theory of lattice-ordered groups, which was posed by G. BIRKHOFF [1, Problem 7], is to find necessary and sufficient conditions that an abstract group be isomorphic with the multiplicative group of a lattice-ordered group. Several authors have given such conditions for the analogous question for totally ordered groups (see [3, Chapter III] for an excellent account of some of these). The main result of this paper (Theorem 3.2) gives a partial answer to the question for lattice-ordered groups. In our investigation we have considered a class of groups that do not appear to have been extensively studied in the literature, viz., the class of retractable groups. The class of lattice-ordered groups is a subclass of the class of retractable groups and in Section 6 (Example 6.2) we give an example of a retractable group that does not admit a lattice-ordering.

Throughout this paper, G will denote a group, written multiplicatively and with identity i, and $F(G)$ will denote the collection of all finite, nonempty subsets of G. Then $F(G)$ is a join monoid, that is, $F(G)$ is a join semilattice in which $A\vee B=A\cup B$ for all $A, B\in F(G)$, $F(G)$ is a monoid in which $AB=\{ab|a\in A$ and $b\in B\}$ for all $A, B\in F(G)$, and if $A, B, C\in F(G)$ with $A\subseteqq B$, then $AC\subseteqq BC$ and $CA\subseteqq CB$. Moreover, if $A, B, C\in F(G)$, then $A(B\vee C)=AB\vee AC$ and $(A\vee B)C=$ $=AC\vee BC$, and the collection of all singleton subsets of G is the group of units of $F(G)$. A homomorphism σ of the monoid $F(G)$ into G such that $\{g\}\sigma=g$ for every $g\in G$ will be called a *retraction* of G. We will denote by Ret G the collection of retractions of G. If Ret G is nonempty, then G is said to be a *retractable* group.

For a retractable group, we give in Theorem 3.2, six conditions each of which is equivalent to the group admitting a lattice-ordering. This changes the above problem to determining which abstract groups are isomorphic to a retractable group, where it seems more likely that necessary and sufficient conditions can be found (we give one such in Corollary 2.11). Again, for retractable groups, in Corollary 3.4 we give three conditions each of which is equivalent to the group admitting a total ordering.

Most of the remainder of this paper is devoted to developing the fundamental properties which are satisfied by retractable groups. In Section 2 (Theorem 2.2) we show that a retractable group is torsion-free. Also, in this section, we establish several properties which are satisfied by retractable groups and which could be viewed as generalizations of corresponding properties that are satisfied by lattice-ordered groups. In Corollary 2.11 we show that a group G is retractable if and only if $F(G)$ possesses a subsemigroup that satisfies certain conditions. Also we show (Theorem 2.12) that the lattice of subgroups of G is lattice isomorphic to a sublattice of the lattice of all equivalence relations in $F(G)$. This isomorphism appears very useful in determining the structure of a retractable group.

In Section 4 we introduce the concept of a convex σ-subgroup and c-σ-subgroup, both of which could be viewed as a generalization of a convex l-subgroup in a lattice-ordered group. In Theorem 4.2 we show that the collection of c-σ-subgroups is a complete sublattice of the lattice of all subgroups of a retractable group. In Theorem 4.3 we show that the normal c-σ-subgroups of a retractable group are the subgroups that give rise, in a natural way, to retractable quotient groups. For these subgroups, the isomorphism theorems are valid.

In Section 5 we give some methods for constructing retractions from a given retraction of a group. In Section 6 (also in Sections 2 and 5) we give examples that illustrate the scope and limitations of our theory.

The reader is referred to [2] for properties of lattice-ordered groups. If $X \subseteqq G$, then $C(X)$ denotes the centralizer of X in G and $[X]$ denotes the subgroup of G generated by X. The natural numbers will be denoted by N, the integers by Z, and the rationals by Q. The cardinal number of a set Y is denoted by $|Y|$.

2. Properties of retractable groups. It is immediate from the definition of retraction, that if $\sigma \in \operatorname{Ret} G$, $\{a_1, \ldots, a_n\} \in F(G)$, and $x, y \in G$, then $x(\{a_1, \ldots, a_n\}\sigma)y = \{xa_1y, \ldots, xa_ny\}\sigma$. In this section, we establish some properties that are satisfied by every retractable group. Many of these are generalizations of corresponding properties that are satisfied by lattice-ordered groups (see [2, Chapter 0], and [3, Chapter V]). This is no casual relationship and for completeness we state the following well known result for lattice-ordered groups.

THEOREM 2.1. *If G is a lattice-ordered group and σ is given by $A\sigma = \vee A$, for each $A \in F(G)$, then $\sigma \in \operatorname{Ret} G$ and σ is a join homomorphism of the join semilattice $F(G)$ onto G. Thus, the class of lattice-ordered groups is a subclass of the class of retractable groups.*

In the above, we will call σ the retraction of G induced by the lattice-ordering of G.

THEOREM 2.2. *Let G be a retractable group and $\sigma \in \operatorname{Ret} G$.*

(i) *G is torsion-free.*

(ii) *If $g, h \in G$ such that $\{g, h\}\sigma \in C(g, h)$, then $gh = hg$. In particular, if $\{g, h\}\sigma = i$, then g and h commute.*

PROOF. (i) Let H be a finite subgroup of G and $h \in H$. Then $(H\sigma)h = (Hh)\sigma = (H\sigma)i$. Thus $h = i$ and so G is torsion-free.

(ii) Suppose that $\{g, h\}\sigma = a \in C(g, h)$. Then $\{ga^{-1}, ha^{-1}\}\sigma = i$, and since $a \in C(g, h)$, it suffices to show that if $\{g, h\}\sigma = i$, then $gh = hg$. Now $\{g, h\}\sigma = i$ implies that $g^{-1} = \{i, g^{-1}h\}\sigma = \{i, hg^{-1}\}\sigma$ and $h^{-1} = \{gh^{-1}, i\}\sigma = \{h^{-1}g, i\}\sigma$. Therefore,

$$g^{-1}h^{-1} = (\{i, hg^{-1}\}\sigma)(\{i, gh^{-1}\}\sigma) = \{i, hg^{-1}, gh^{-1}\}\sigma =$$

$$= (\{g, h\}\sigma)(\{g^{-1}, h^{-1}\}\sigma) = (\{g^{-1}, h^{-1}\}\sigma)(\{g, h\}\sigma) = \{i, h^{-1}g, g^{-1}h\}\sigma =$$

$$= (\{i, h^{-1}g\}\sigma)(\{i, g^{-1}h\}\sigma) = h^{-1}g^{-1}.$$

In Examples 2.7 and 2.8 we show that the class of retractable groups is a proper subclass of the class of all torsion-free groups. The proof of the next theorem is immediate from the definition of retraction and will be omitted.

THEOREM 2.3. *Let G be a retractable group and $\sigma\in\operatorname{Ret} G$.*

(i) *If $g\in G$, then $g=(\{i,g\}\sigma)(\{i,g^{-1}\}\sigma)^{-1}=(\{i,g^{-1}\}\sigma)^{-1}(\{i,g\}\sigma)$ and $\{(\{i,g\}\sigma)^{-1}, (\{i,g^{-1}\}\sigma)^{-1}\}\sigma=i$.*

(ii) *If $g,h\in G$, then $\{g,h\}\sigma=g(\{g^{-1},h^{-1}\}\sigma)h$. If $gh=hg$, then $gh=(\{g,h\}\sigma)(\{g^{-1},h^{-1}\}\sigma)^{-1}$.*

If G is a lattice-ordered group and σ is the retraction of G induced by the lattice-ordering of G, then (i) and (ii) of Theorem 2.3 correspond to the identities $g=(g\vee i)(g^{-1}\vee i)^{-1}$, where $(g\vee i)\wedge(g^{-1}\vee i)=i$, and $g\vee h=g(g\wedge h)^{-1}h$, respectively.

THEOREM 2.4. *Let G be a retractable group and $\sigma\in\operatorname{Ret} G$.*

(i) *If $A=\{g_1,\dots,g_n\}\in F(G)$ such that $g_ig_j=g_jg_i$ for every $g_i,g_j\in A$, then $A\sigma\in C(A)$. In particular, $\{i,g\}\sigma\in C(g)$ for every $g\in G$.*

(ii) *If $g\in G$, $\{i,g\}\sigma=a$, and $k_1,\dots,k_m\in Z$ with $k_1<\dots<k_m$, then $\{g^{k_1},\dots,g^{k_m}\}\sigma=a^{k_m-k_1}g^{k_1}$.*

PROOF. (i) If $A\sigma=a$, then $ag_1^{-1}=\{i,g_2g_1^{-1},\dots,g_ng_1^{-1}\}\sigma=\{i,g_1^{-1}g_2,\dots,g_1^{-1}g_n\}\sigma=g_1^{-1}a$. Thus $a\in C(g_1)$. Similarly, $a\in C(g_i)$ for every $g_i\in A$ and hence $a\in C(A)$.

(ii) First we show, by induction, that $\{i,g,\dots,g^n\}\sigma=a^n$ for each $n\in N$. Suppose that $\{i,g,\dots,g^n\}\sigma=a^n$. Then $a^{n+1}=a^na=(\{i,g,\dots,g^n\}\sigma)(\{i,g\}\sigma)=\{i,g,\dots,g^{n+1}\}\sigma$. It follows that if $k\in Z$ and $n\in N$, then $\{g^k,g^{k+1},\dots,g^{k+n}\}\sigma=(\{i,g,\dots,g^n\}\sigma)g^k=a^ng^k$. Finally, let $\{g^{k_1},\dots,g^{k_m}\}\sigma=b$. Then $a^{k_m-k_1}b=(\{i,g,\dots,g^{k_m-k_1}\}\{g^{k_1},\dots,g^{k_m}\})\sigma=\{g^{k_1},g^{k_1+1},\dots,g^{k_1+(2k_m-2k_1)}\}\sigma=a^{2k_m-2k_1}g^{k_1}$. Therefore, $b=a^{k_m-k_1}g^{k_1}$.

In Section 5 (Examples 5.6 and 5.7) we shall use (ii) of this theorem to classify $\operatorname{Ret} Z$ and $\operatorname{Ret} Q$.

Let $\sigma\in\operatorname{Ret} G$ and H be a subgroup of G. We say that H is a *σ-subgroup* of G if $A\sigma\in H$ for each $A\in F(H)$. Then the restriction of σ to $F(H)$ is a retraction of H. Trivially, a conjugate of a σ-subgroup of G is a σ-subgroup and the intersection of any collection of σ-subgroups is a σ-subgroup. Note, that if G is a lattice-ordered group and σ is the retraction of G induced by the lattice-ordering of G, then the collection of σ-subgroups of G is precisely the collection of l-subgroups of G.

The following corollary is immediate from the theorem and again shows a similarity of retractable groups and lattice-ordered groups.

COROLLARY 2.5. *Let G be a retractable group and $\sigma\in\operatorname{Ret} G$.*

(i) *If $g\in G$ and $\{i,g\}\sigma=a$, then $\{i,g^n\}\sigma=a^n$ for each $n\in N$.*

(ii) *If $g\in G$ and $\{i,g\}\sigma\in[g]$, then $[g]$ is a σ-subgroup of G.*

(iii) *If $g\in G$ and $\{i,g^m\}\sigma=g^m$, for some $m\in N$, then $\{i,g\}\sigma=g$.*

(iv) *If $g,h\in G$ such that $gh=hg$, and $\{g,h\}\sigma=a$, then $\{g^n,h^n\}\sigma=a^n$ for each $n\in N$.*

In the next theorem we give two necessary (and rather curious) conditions for a group to be retractable.

THEOREM 2.6. *Let* G *be a retractable group and* $\sigma \in \mathrm{Ret}\, G$.
(i) *If* $g, h \in G$ *with* $ghg^{-1} = h^{-1}$, *then* h *is a commutator.*
(ii) *If* $g, h \in G$ *with* $g^2 = h^2$, *then* g *and* h *are conjugate.*

PROOF. (i) If $\{i, h^{-1}\}\sigma = a$, then $\{i, h\}\sigma = ha$. Also, $g^{-1}ag = \{i, g^{-1}h^{-1}g\}\sigma = \{i, h\}\sigma$. Thus, $g^{-1}aga^{-1} = h$.
(ii) If $\{g, h\}\sigma = a$, then $ag = \{g^2, hg\} = \{h^2, hg\}\sigma = ha$. Thus $h = aga^{-1}$.

EXAMPLE 2.7 [2, Example 0.5, p. 0.10]. Let $H = Z \times Z \times Z$ and define

$$(a, b, c) + (x, y, z) = \begin{cases} (a+x, b+y, c+z) & \text{if } z \text{ is even} \\ (b+x, a+y, c+z) & \text{if } z \text{ is odd.} \end{cases}$$

Then H is a group and a splitting extension of $Z \times Z$ by Z. Let $g = (0, 0, 1)$, $h = (1, -1, 0)$, and K be the subgroup of H generated by g and h. Then $g + h - g = -h$ and it is easily verified that h is not a commutator in K. Thus, by (i) of Theorem 2.6, K is not retractable. Note that if L denotes the subgroup of K generated by h, then L and K/L are both infinite cyclic and hence are retractable (as they can be linearly ordered). It is well known that H can be lattice-ordered and hence K can be right ordered. Hence, the class of retractable groups is not identical with the class of right ordered groups. We do not know whether every retractable group can be right ordered.

EXAMPLE 2.8. Consider the group G given by the presentation $\langle x, y | x^2 = y^2 \rangle$. Then, G, being the free product with amalgamation of torsion-free groups, is torsion-free. By considering the homomorphism from G to C_2 (where C_2 is the cyclic group of order 2) which sends x to 0 and y to 1, it follows that x and y are not conjugates in G. Thus, G is not retractable.

A submonoid S of $F(G)$ will be called *normal* if $gSg^{-1} = S$ for every $g \in G$. A normal submonoid S of $F(G)$ is called a *G-complement* if

(1) $Cg \in S$, where $C \in S$ and $g \in G$, implies that $g = i$;
(2) if $A \in F(G)$, then there exists $C \in S$ and there exists $g \in G$ such that $A = Cg$.

We will denote by Com G the collection of G-complements of $F(G)$ and if $\sigma \in \mathrm{Ret}\, G$, then $\mathrm{Ker}\, \sigma = \{C \in F(G) | C\sigma = i\}$. The proof of the next theorem is straightforward and will be omitted.

THEOREM 2.9. *Let* G *be a retractable group.*
(i) *If* $\sigma \in \mathrm{Ret}\, G$, *then* $\mathrm{Ker}\, \sigma \in \mathrm{Com}\, G$.
(ii) *If* $\sigma, \tau \in \mathrm{Ret}\, G$ *with* $\mathrm{Ker}\, \sigma \subseteq \mathrm{Ker}\, \tau$, *then* $\sigma = \tau$.

THEOREM 2.10. *If* $S \in \mathrm{Com}\, G$, *then* G *is retractable and there exists* $\sigma \in \mathrm{Ret}\, G$ *such that* $S = \mathrm{Ker}\, \sigma$.

PROOF. If $A \in F(G)$, then $A = Cg$ for some $C \in S$ and some $g \in G$. Define $A\sigma = g$. Now σ is well defined, for if $D \in S$ and $h \in G$ with $A = Dh$, then $D = Cgh^{-1}$ and since $S \in \mathrm{Com}\, G$, $gh^{-1} = i$. The normality of S insures that σ is a homomorphism and since $\{i\} \in S$, $\{g\}\sigma = (\{i\}g)\sigma = g$ for each $g \in G$. Thus, $\sigma \in \mathrm{Ret}\, G$. The verification that $\mathrm{Ker}\, \sigma = S$ is routine.

Combining Theorems 2.9 and 2.10 we have

COROLLARY 2.11. *A necessary and sufficient condition for a group* G *to be retractable is that* $\mathrm{Com}\, G$ *is nonempty. Moreover, the mapping* $\sigma \to \mathrm{Ker}\, \sigma$ *is a one-to-one mapping of* $\mathrm{Ret}\, G$ *onto* $\mathrm{Com}\, G$.

Next let $\mathscr{S}(G)$ denote the lattice of subgroups of G and $\mathscr{E}(F(G))$ denote the lattice of equivalence relations on $F(G)$. For $H \in \mathscr{S}(G)$ and $\sigma \in \mathrm{Ret}\, G$, let

$$\varrho_{H,\sigma} = \{(A, B) | A, B \in F(G) \quad \text{and} \quad H(A\sigma) = H(B\sigma)\}.$$

THEOREM 2.12. *If* $\sigma \in \mathrm{Ret}\, G$, *then the mapping* Ψ *given by* $H\Psi = \varrho_{H,\sigma}$ *is a complete lattice isomorphism of* $\mathscr{S}(G)$ *into* $\mathscr{E}(F(G))$. *Moreover,*

(i) *if* $H \in \mathscr{S}(G)$, $(A, B) \in \varrho_{H,\sigma}$, *and* $C \in F(G)$, *then* $(AC, BC) \in \varrho_{H,\sigma}$;

(ii) *if* $H \in \mathscr{S}(G)$ *and* $x \in G$, *then* $\varrho_{x^{-1}Hx,\sigma} = \{(x^{-1}Ax, x^{-1}Bx) | (A, B) \in \varrho_{H,\sigma}\}$;

(iii) *if* $H \in \mathscr{S}(G)$, $A \in F(G)$, *and* $[A]\varrho_{H,\sigma}$ *denotes the equivalence class of* A *under* $\varrho_{H,\sigma}$, *then* $\{\{h\} | h \in H\} \subseteqq [\{i\}]\varrho_{H,\sigma}$ *and the mapping* $[A]\varrho_{H,\sigma} \to H(A\sigma)$ *is a one-to-one mapping of the colection of equivalence classes of* $F(G)$ *under* $\varrho_{H,\sigma}$ *onto the collection of right cosets of* H *in* G. *If* $A, B \in F(G)$, *then* $([A]\varrho_{H,\sigma})x = [B]\varrho_{H,\sigma}$ *for some* $x \in G$.

PROOF. It is easily verified that Ψ is a one-to-one inclusion preserving mapping of $\mathscr{S}(G)$ into $\mathscr{E}(F(G))$ that satisfies $(\bigcap H_\lambda)\Psi = \bigcap(H_\lambda \Psi)$ for every $\{H_\lambda | \lambda \in \Lambda\} \subseteqq \mathscr{S}(G)$. We show only that $(\bigvee H_\lambda)\Psi = \bigvee(H_\lambda \Psi)$. If $(A, B) \in (\bigvee H_\lambda)\Psi$, then $(\bigvee H_\lambda)(A\sigma) = (\bigvee H_\lambda)(B\sigma)$, and so $(A\sigma)(B\sigma)^{-1} \in \bigvee H_\lambda$. Therefore, $(A\sigma)(B\sigma)^{-1} \in H_{\lambda_1} \vee \ldots \vee H_{\lambda_n}$ for some $n \in N$ and some $\lambda_1, \ldots, \lambda_n \in \Lambda$. Consequently, there exist $h_1, \ldots, h_m \in \bigcup H_{\lambda_i}$ $(1 \leqq i \leqq n)$ such that $(A\sigma)(B\sigma)^{-1} = h_1 \ldots h_m$. For $1 \leqq k \leqq m$, let $h_k \in H_{\lambda_{i_k}}$. Then $H_{\lambda_{i_1}}(A\sigma) = H_{\lambda_{i_1}}(h_2 \ldots h_m B\sigma)$ implies that $(A, h_2 \ldots h_m B) \in (H_{\lambda_{i_1}})\Psi$; $H_{\lambda_{i_2}}(h_2 \ldots h_m B\sigma) = H_{\lambda_{i_2}}(h_3 \ldots h_m B\sigma)$ implies that $(h_2 \ldots h_m B, h_3 \ldots h_m B) \in (H_{\lambda_{i_2}})\Psi$; $\ldots$; $H_{\lambda_{i_m}}(h_m B\sigma) = H_{\lambda_{i_m}}(B\sigma)$ implies that $(h_m B, B) \in (H_{\lambda_{i_m}})\Psi$. It follows that $(A, B) \in \bigvee(H_\lambda \Psi)$. Since Ψ preserves inclusion, the reverse containment is immediate.

COROLLARY 2.13. *Let* $\sigma \in \mathrm{Ret}\, G$ *and* $H \in \mathscr{S}(G)$. *Then* H *is a* σ*-subgroup if and only if* $F(H) \subseteqq [\{i\}]\varrho_{H,\sigma}$.

If $H \in \mathscr{S}(G)$ and $X \subseteqq G$, let $C_H(X)$ denote the centralizer of X in H. We close this section with

THEOREM 2.14. *If* $\sigma \in \mathrm{Ret}\, G$, $X \subseteqq G$, *and* H *is a* σ*-subgroup of* G, *then* $C_H(X)$ *is a* σ*-subgroup of* G.

PROOF. Let $A \in F(C_H(X))$, $A\sigma = a$, and $x \in X$. Since H is a σ-subgroup and $A \in F(H)$, $a \in H$. Thus $xa = (xA)\sigma = (Ax)\sigma = ax$, and so $a \in C_H(X)$.

3. *l*-retractions. Throughout this section, let G be a retractable group. A retraction σ of G is said to be *convex* if $\mathrm{Ker}\, \sigma$ is a convex subset of $F(G)$ and is said to be an *l-retraction* if $\mathrm{Ker}\, \sigma$ is a convex subsemilattice of $F(G)$.

LEMMA 3.1. *Let* σ *be a convex retraction of* G *and for* $g, h \in G$ *define* $g \leqq h$ *if and only if* $\{g, h\}\sigma = h$. *Then* $(G, \leqq)$ *is a semiclosed partially ordered group.*

PROOF. Clearly $\leqq$ is reflexive and antisymmetric. Let $f, g, h \in G$ with $f \leqq g$ and $g \leqq h$. Then $\{fg^{-1}, i\}\sigma = i$ and $\{gh^{-1}, i\}\sigma = i$. Thus $i = (\{fg^{-1}, i\}\{gh^{-1}, i\})\sigma =$

$=\{fh^{-1}, fg^{-1}, gh^{-1}, i\}\sigma$. Since $\{i\}\subseteqq\{fh^{-1}, i\}\subseteqq\{fh^{-1}, fg^{-1}, gh^{-1}, i\}$ and σ is convex, $\{fh^{-1}, i\}\in\operatorname{Ker}\sigma$. Thus $\{f, h\}\sigma=h$ and so $\leqq$ is transitive. Since $x(\{g, h\}\sigma)y=$ $=\{xgy, xhy\}\sigma$ for all $g, h, x, y\in G$, $(G, \leqq)$ is a partially ordered group. It follows from Corollary 2.5 (iii), that the partial order is semiclosed.

THEOREM 3.2. *If $\sigma\in\operatorname{Ret} G$, then the following are equivalent:*

(i) *σ is an l-retraction;*

(ii) *for all $A, B, C, D\in F(G)$, if $A\subseteqq B\subseteqq C$ and $A\sigma=C\sigma$, then $A\sigma=B\sigma$ and $D\sigma=(D\cup\{D\sigma\})\sigma$;*

(iii) *if $\{a_1, \ldots, a_n\}\in F(G)$ with $n>1$, then $\{a_1, \ldots, a_n\}\sigma=\{\{a_1, \ldots, a_{n-1}\}\sigma, a_n\}\sigma$;*

(iv) *for all $A, B, C\in F(G)$, if $A\sigma=B\sigma$, then $(A\cup C)\sigma=(B\cup C)\sigma$;*

(v) *there is a lattice-ordering of G such that G is a lattice-ordered group in which $A\sigma=\bigvee A$, for all $A\in F(G)$;*

(vi) *there is a lattice-ordering of G such that G is a lattice-ordered group and σ is a join homomorphism of $F(G)$ onto G;*

(vii) *$\varrho_{\{i\},\sigma}$ is a join congruence.*

PROOF. (i) implies (ii). If $A\sigma=C\sigma=a$, then $Aa^{-1}, Ca^{-1}\in\operatorname{Ker}\sigma$ and $Aa^{-1}\subseteqq$ $\subseteqq Ba^{-1}\subseteqq Ca^{-1}$. Since $\operatorname{Ker}\sigma$ is convex, $Ba^{-1}\in\operatorname{Ker}\sigma$ and so $B\sigma=a=A\sigma$. Let $D\sigma=b$. Then $Db^{-1}\in\operatorname{Ker}\sigma$ and $\{i\}\in\operatorname{Ker}\sigma$. Since $\operatorname{Ker}\sigma$ is a subsemilattice, $Db^{-1}\cup\{i\}\in\operatorname{Ker}\sigma$. Therefore, $(D\cup\{D\sigma\})\sigma=(D\cup\{b\})\sigma=b$.

(ii) implies (iii). The proof is divided into steps.

(1) If $A, C\in F(G)$, $A\sigma=a$, and $a\in A$, then $(A\cup C)\sigma=(\{a\}\cup C)\sigma$.

Since $A\sigma=a$ and $a\in A$, $(Aa^{-1})\sigma=i$ and $i\in Aa^{-1}$. Let $b=(\{i\}\cup Ca^{-1})\sigma$. Then $b=ib=((Aa^{-1})(\{i\}\cup Ca^{-1}))\sigma=((Aa^{-1})\cup(Aa^{-1}Ca^{-1}))\sigma$. Now $\{i\}\cup Ca^{-1}\subseteqq$ $\subseteqq(Aa^{-1})\cup(Ca^{-1})\subseteqq(Aa^{-1})\cup(Aa^{-1}Ca^{-1})$. Hence, by (ii), $((Aa^{-1})\cup(Ca^{-1}))\sigma=b$. Then $ba=(\{a\}\cup C)\sigma=(A\cup C)\sigma$.

(2) If $A\sigma=a$ and $g\in G$, then $(A\cup\{g\})\sigma=\{a, g\}\sigma$.

Let $b=\{i, ga^{-1}\}\sigma$. Since $i=(Aa^{-1})\sigma$, $b=((Aa^{-1})(\{i, ga^{-1}\}))\sigma=((Aa^{-1})\cup$ $\cup(Aa^{-1}ga^{-1}))\sigma$. By (ii), $(Aa^{-1}\cup\{i\})\sigma=i$. Thus, $b=ib=((Aa^{-1}\cup\{i\})(\{i, ga^{-1}\}))\sigma=$ $=(\{i\}\cup Aa^{-1}\cup\{ga^{-1}\}\cup Aa^{-1}ga^{-1})\sigma$. Since $(Aa^{-1})\cup(Aa^{-1}ga^{-1})\subseteqq(Aa^{-1})\cup\{ga^{-1}\}\cup$ $\cup(Aa^{-1}ga^{-1})\subseteqq\{i\}\cup(Aa^{-1})\cup\{ga^{-1}\}\cup(Aa^{-1}ga^{-1})$, we have by (ii) that $((Aa^{-1})\cup$ $\cup\{ga^{-1}\}\cup(Aa^{-1}ga^{-1}))\sigma=b$. Also, $ga^{-1}=iga^{-1}=((Aa^{-1}\cup\{i\})ga^{-1})\sigma=((Aa^{-1}ga^{-1})\cup$ $\cup\{ga^{-1}\})\sigma$. By (1), $((Aa^{-1}ga^{-1})\cup\{ga^{-1}\}\cup(Aa^{-1}))\sigma=(\{ga^{-1}\}\cup(Aa^{-1}))\sigma$. Thus, $b=(\{ga^{-1}\}\cup(Aa^{-1}))\sigma$ and so $ba=(\{g\}\cup A)\sigma=\{a, g\}\sigma$.

We now show that (iii) is true. If $\{a_1, \ldots, a_{n-1}\}\sigma=a$, then by (2), $(\{a_1, \ldots, a_{n-1}\}\cup$ $\cup\{a_n\})\sigma=\{a, a_n\}\sigma$.

(iii) implies (iv). To prove that (iv) is true, we induct on $|C|$. If $C=\{c\}$, then by (iii), $(A\cup C)\sigma=(A\sigma\cup\{c\})\sigma=(B\sigma\cup\{c\})\sigma=(B\cup C)\sigma$. Next suppose that $C=$ $=\{c_1, \ldots, c_n\}$, where $n>1$, and that if $D\in F(G)$ with $|D|<|C|$, then $(A\cup D)\sigma=$ $=(B\cup D)\sigma$. Then $(A\cup C)\sigma=(A\cup\{c_1, \ldots, c_{n-1}\}\cup\{c_n\})\sigma=\{(A\cup\{c_1, \ldots, c_{n-1}\})\sigma, c_n\}\sigma=$ $=\{(B\cup\{c_1, \ldots, c_{n-1}\})\sigma, c_n\}\sigma=(B\cup C)\sigma$. Thus, if $A, B, C\in F(G)$ with $A\sigma=B\sigma$, then $(A\cup C)\sigma=(B\cup C)\sigma$.

(iv) implies (v). Let $A, B, C\in F(G)$ with $A\subseteqq B\subseteqq C$, and $A, C\in\operatorname{Ker}\sigma$. Then by (iv), $B\sigma=(A\cup B)\sigma=(C\cup B)\sigma=C\sigma=i$ and hence $B\in\operatorname{Ker}\sigma$. Thus, by Lemma 3.1, G is a partially ordered group, where $g\leqq h$ if and only if $\{g, h\}\sigma=h$, for all $g, h\in G$.

Let $g, h\in G$ and $x=\{g, h\}\sigma$. Since $\{x\}\sigma=x$, we have by (iv) that $x=(\{x\}\cup$ $\cup\{x\})\sigma=(\{g, h\}\cup\{x\})\sigma$. Again using (iv), $x=\{g, h, x\}\sigma=(\{g, h, x\}\cup\{g\})\sigma=$

$=(\{x\}\cup\{g\})\sigma$. Therefore, $g\leqq x$ and similarly, $h\leqq x$. Next suppose that $y\in G$, and that $g\leqq y$ and $h\leqq y$. Then $\{g,y\}\sigma=\{h,y\}\sigma=y$, and so by (iv), $\{g,y\}\sigma=$ $=(\{g,y\}\cup\{g\})\sigma=(\{h,y\}\cup\{g\})\sigma=\{g,h,y\}\sigma$. Since $\{x\}\sigma=\{g,h\}\sigma$, $\{x,y\}\sigma=(\{x\}\cup$ $\cup\{y\})\sigma=(\{g,h\}\cup\{y\})\sigma=\{g,h,y\}\sigma$. Hence, $x\leqq y$. Therefore, $x=g\vee h$ and it follows that G is a lattice-ordered group.

Next we show that $A\sigma=\vee A$ for every $A\in F(G)$, by induction on $|A|$. Trivially, if $|A|=1$, then $A\sigma=\vee A$. Let $A=\{a_1,\dots,a_n\}$, where $n>1$, and let $B=\{a_1,\dots$ $\dots,a_{n-1}\}$. If $B\sigma=b$, then by (iv) and the above, $b\vee a_n=\{b,a_n\}\sigma=(B\cup\{a_n\})\sigma=A\sigma$. Clearly, $b\vee a_n=\vee A$.

(v) implies (vi) by Theorem 2.1.

(vi) implies (vii). By Theorem 2.12, $\varrho_{\{i\},\sigma}$ is an equivalence relation on $F(G)$. Let $(A,B)\in\varrho_{\{i\},\sigma}$, $C\in F(G)$, and $H=\{i\}$. Then $H((A\cup C)\sigma)=H(A\sigma\vee C\sigma)=$ $=H(B\sigma\vee C\sigma)=H((B\cup C)\sigma)$. Therefore, $\varrho_{H,\sigma}$ is a join congruence on $F(G)$.

(vii) implies (i). The congruence class containing $\{i\}$ under $\varrho_{\{i\},\sigma}$ is precisely the kernel of σ. Since $\varrho_{\{i\},\sigma}$ is a join congruence, it is well known that each congruence class, and, in particular, $\operatorname{Ker}\sigma$, is a convex subsemilattice of $F(G)$.

We will call the lattice-ordering of G given in (v) above (or in (vi), since they are identical), the lattice-ordering of G induced by the l-retraction σ. From Theorems 2.1 and 3.2, we now have

COROLLARY 3.3. *Let G be a group. Then there is a one-to-one correspondence between the lattice-orderings of G and the l-retractions of G.*

COROLLARY 3.4. *Let σ be an l-retraction of G and let $\leqq$ denote the lattice-ordering of G induced by σ. Then the following are equivalent:*

(i) *$\leqq$ is a total ordering of G;*

(ii) *if $A\in\operatorname{Ker}\sigma$, then $i\in A$;*

(iii) *$\operatorname{Ker}\sigma$ is closed under intersections;*

(iv) *$\operatorname{Ker}\sigma$ is a lattice, where $A\wedge B=A\cap B$ for all $A,B\in\operatorname{Ker}\sigma$.*

PROOF. (i) implies (ii). If $A\in\operatorname{Ker}\sigma$, then $i=A\sigma=\vee A$. Since $\leqq$ is a total ordering of G and A is finite, $i\in A$.

(ii) implies (iii) and (iii) implies (iv) are immediate.

(iv) implies (i). Suppose that $\leqq$ is not a total ordering of G. Then there exists $g,h\in G$ with $g\neq i\neq h$ and $g\vee h=i$. Then $\{i\}$, $\{g,h\}\in\operatorname{Ker}\sigma$, but $\{i\}\cap\{g,h\}\notin\operatorname{Ker}\sigma$.

COROLLARY 3.5. *Let σ be an l-retraction of G, $\leqq$ be the lattice-ordering of G induced by σ, and H be a subgroup of G. Then the following are equivalent:*

(i) *H is a convex l-subgroup of G;*

(ii) *$\varrho_{H,\sigma}$ is a join congruence of $F(G)$;*

(iii) *$F(G)/\varrho_{H,\sigma}$ is a join semilattice in which $[A]\varrho_{H,\sigma}\vee[B]\varrho_{H,\sigma}=[A\cup B]\varrho_{H,\sigma}$, for every $A,B\in F(G)$.*

If H is a convex l-subgroup of G, then the mapping $[A]\varrho_{H,\sigma}\to H(A\sigma)$, given in Theorem 2.12, is a join isomorphism of $F(G)/\varrho_{H,\sigma}$ onto the collection of right cosets of H in G.

PROOF. The equivalence of (ii) and (iii) is well-known for any join semilattice.

(i) implies (ii). By Theorem 2.12, $\varrho_{H,\sigma}$ is an equivalence relation on $F(G)$. Let $(A,B)\in\varrho_{H,\sigma}$ and $C\in F(G)$. By the theorem, $(A\cup C)\sigma=A\sigma\vee C\sigma$. Since H is a convex l-subgroup of G, $H((A\cup C)\sigma)=H(A\sigma\vee C\sigma)=H(A\sigma)\vee H(C\sigma)=H(B\sigma)\vee\vee H(C\sigma)=H(B\sigma\vee C\sigma)=H((B\cup C)\sigma)$. Therefore, $\varrho_{H,\sigma}$ is a join congruence.

(ii) implies (i). Clearly, $\{h\}\in[\{i\}]\varrho_{H,\sigma}$ for each $h\in H$. Suppose that $\{a_1, \ldots, a_n\}\in F(H)$, where $n>1$, and $\{a_1, \ldots, a_{n-1}\}\in[\{i\}]\varrho_{H,\sigma}$. Since $(\{a_n\}, \{i\})\in\varrho_{H,\sigma}$ and $\varrho_{H,\sigma}$ is a join congruence, $(\{a_1, \ldots, a_n\}, \{i\})\in\varrho_{H,\sigma}$. Therefore, $F(H)\subseteq[\{i\}]\varrho_{H,\sigma}$ and hence, H is an l-subgroup of G. If $x\in G$ and $h\in H$ such that $i\leqq x\leqq h$, then $(\{i, x\}, \{h, x\})\in\varrho_{H,\sigma}$ and so $Hx=H(\{i, x\}\sigma)=H(\{x, h\}\sigma)=Hh=H$. Hence, $x\in H$ and so H is a convex l-subgroup of G.

If H is a convex l-subgroup of G, then the collection of right cosets of H in G is a lattice in which $Hx\vee Hy=H(x\vee y)$ for all $x, y\in G$. Thus, for $A, B\in F(G)$, $[A]\varrho_{H,\sigma}\vee[B]\varrho_{H,\sigma}=[A\cup B]\varrho_{H,\sigma}$ and $H(A\sigma)\vee H(B\sigma)=H((A\cup B)\sigma)$.

A convex l-subgroup H of a lattice-ordered group G is said to be *regular* if H is maximal with respect to not containing some g in G and is said to be *prime* if the collection of right cosets of H in G is totally ordered. The next two corollaries are immediate from Corollary 3.5 and their proofs will be omitted.

COROLLARY 3.6. *Let σ be an l-retraction of G, $\leqq$ be the lattice-ordering of G induced by σ, and H be a convex l-subgroup of G. Then the following are equivalent:*

(i) *H is prime in G;*

(ii) *$F(G)/\varrho_{H,\sigma}$ is totally ordered, in which $[A]\varrho_{H,\sigma}\vee[B]\varrho_{H,\sigma}=[A\cup B]\varrho_{H,\sigma}$, for all $A, B\in F(G)$;*

(iii) *$\varrho_{H,\sigma}$ is a join congruence of $F(G)$ and $[\{i\}]\varrho_{H,\sigma}$ is finitely join irreducible in $F(G)/\varrho_{H,\sigma}$.*

COROLLARY 3.7. *Let σ be an l-retraction of G, $\leqq$ be the lattice-ordering of G induced by σ, and H be a subgroup of G. Then the following are equivalent:*

(i) *H is a regular subgroup of G and the lattice of right cosets of H in G is discrete;*

(ii) *$\varrho_{H,\sigma}$ is a join congruence of $F(G)$ in which $[A]\varrho_{H,\sigma}\vee[B]\varrho_{H,\sigma}=[A\cup B]\varrho_{H,\sigma}$, for all $A, B\in F(G)$, $|F(G)/\varrho_{H,\sigma}|\geqq 2$, and $[\{i\}]\varrho_{H,\sigma}$ is infinitely join irreducible in $F(G)/\varrho_{H,\sigma}$.*

For $X\subseteq G$, let $X^{-1}=\{x^{-1}|x\in X\}$. To conclude this section, we state the following theorem, the proof of which is straightforward.

THEOREM 3.8. *Let $\sigma\in\operatorname{Ret} G$ and $H_\sigma=\{g|g\in G$ and $g\in A\cap B^{-1}$ for some $A, B\in\in\operatorname{Ker}\sigma\}$.*

(i) *H_σ is a normal subgroup of G.*

(ii) *If $g, h\in G$, define $g\leqq h$ if and only if $gh^{-1}\in A$, for some $A\in\operatorname{Ker}\sigma$. Then $\leqq$ is a quasi-ordering of G that is preserved by both right and left multiplication. Moreover, $\leqq$ is a partial ordering of G if and only if $H_\sigma=\{i\}$.*

4. c-σ-subgroups. Let σ be a retraction of G and H be a subgroup of G. In view of Corollary 3.5, we say that H is a *convex σ-subgroup* of G *if* $\varrho_{H,\sigma}$ is a join congruence of $F(G)$ (or equivalently, if (iii) of Corollary 3.5 is satisfied). It is well-known that the collection of join congruences of a join semilattice L is a complete sublattice of the lattice of all equivalence relations on L. Thus from Theorem 2.12 and Corollary 2.13 we have

THEOREM 4.1. *Let $\sigma\in\operatorname{Ret} G$.*

(i) *If H is a convex σ-subgroup of G and $x\in G$, then so is $x^{-1}Hx$.*

(ii) *If* $\mathscr{C}=\{H|H$ *is a convex* σ*-subgroup of* $G\}$, *then* $\mathscr{C}$ *is a sublattice of* $\mathscr{S}(G)$ *that contains* G; *the joins and meets of nonempty subcollections of* $\mathscr{C}$ *agree with those in* $\mathscr{S}(G)$. *Moreover,* $\{\varrho_{H,\sigma}|H\in\mathscr{C}\}$ *is a sublattice of* $\mathscr{E}(F(G))$ *that is lattice isomorphic to* $\mathscr{C}$.

(iii) *If* H *is a convex* σ*-subgroup of* G, *then* H *is a* σ*-subgroup of* G.

Thus, by (ii) of this theorem there is a smallest convex σ-subgroup of G, which is clearly normal, and by Theorem 3.2 (vii), it is the identity subgroup of G if and only if σ is an l-retraction. In Section 5 we give an example of a group G and $\sigma\in\text{Ret}\, G$ such that G is the only convex σ-subgroup of itself. This indicates that there is a need for a larger collection of distinguished subgroups of a retractable group if one is to determine the structure of the group from its subgroups.

Let $\sigma\in\text{Ret}\ G$. A subgroup H of G is said to be a *c-*σ*-subgroup* of G if $\{a_1, \ldots, a_n\}\in F(G)$ and $h_1, \ldots, h_n\in H$, then $(\{a_1, \ldots, a_n\}, \{h_1a_1, \ldots, h_na_n\})\in\varrho_{H,\sigma}$.

THEOREM 4.2. Let $\sigma\in\text{Ret}\, G$.

(i) *If* H *is a c-*σ*-subgroup of* G *and* $x\in G$, *then so is* $x^{-1}Hx$.

(ii) *If* H *is a convex* σ*-subgroup of* G, *then* H *is a c-*σ*-subgroup of* G.

(iii) *If* H *is a c-*σ*-subgroup of* G, *then* H *is a* σ*-subgroup of* G.

(iv) *If* $\{H_\lambda|\lambda\in\Lambda\}$ *is a collection of c-*σ*-subgroups of* G, *then* $\bigcap\{H_\lambda|\lambda\in\Lambda\}$ *and* $[\bigcup\{H_\lambda|\lambda\in\Lambda\}]$ *are c-*σ*-subgroups of* G. *Thus, the collection of c-*σ*-subgroups of* G *is a complete sublattice of the lattice* $\mathscr{S}(G)$ *of all subgroups of* G.

PROOF. (i) Let $\{a_1, \ldots, a_n\}\in F(G)$ and $h_1, \ldots, h_n\in x^{-1}Hx$. Then $\{xa_1x^{-1}, \ldots, xa_nx^{-1}\}\in F(G)$ and $xh_1x^{-1}, \ldots, xh_nx^{-1}\in H$. Since H is a c-σ-subgroup, $(\{xa_1x^{-1}, \ldots, xa_nx^{-1}\}, \{xh_1a_1x^{-1}, \ldots, xh_na_nx^{-1}\})\in\varrho_{H,\sigma}$. By Theorem 2.12 (ii), $(\{a_1, \ldots, a_n\}, \{h_1a_1, \ldots, h_na_n\})\in\varrho_{x^{-1}Hx,\sigma}$.

(ii) Let $\{a_1, \ldots, a_n\}\in F(G)$ and $h_1, \ldots, h_n\in H$. For $1\leqq i\leqq n$, $(\{a_i\}, \{h_ia_i\})\in\varrho_{H,\sigma}$. Since $\varrho_{H,\sigma}$ is a join congruence, $(\{a_1, \ldots, a_n\}, \{h_1a_1, \ldots, h_na_n\})\in\varrho_{H,\sigma}$. Thus, H is a c-σ-subgroup of G.

(iii) If $\{h_1, \ldots, h_n\}\in F(H)$, then $h_1^{-1}, \ldots, h_n^{-1}\in H$. Hence $(\{h_1, \ldots, h_n\}, \{h_1^{-1}h_1, \ldots, h_n^{-1}h_n\})\in\varrho_{H,\sigma}$. Therefore, $F(H)\subseteqq[\{i\}]\varrho_{H,\sigma}$ and so by Corollary 2.13, H is a σ-subgroup.

(iv) Clearly $\{i\}$ and G are c-σ-subgroups of G. Let $\{H_\lambda|\lambda\in\Lambda\}$ be a nonempty collection of c-σ-subgroups of G, and $\{a_1, \ldots, a_n\}\in F(G)$. If $h_1, \ldots, h_n\in\bigcap H_\lambda$ and $\gamma\in\Lambda$, then $(\{a_1, \ldots, a_n\}, \{h_1a_1, \ldots, h_na_n\})\in\varrho_{H_\gamma,\sigma}$. Hence, by Theorem 2.12, $(\{a_1, \ldots, a_n\}, \{h_1a_1, \ldots, h_na_n\})\in\varrho_{\cap H_\gamma,\sigma}$.

Next let $x_1, \ldots, x_n\in[\bigcup H_\lambda]$. The there exist $m, p\in N$, there exist $\lambda_1, \ldots, \lambda_m\in\Lambda$, and for $1\leqq i\leqq n$ and $1\leqq j\leqq m$ there exists $h_{i,km+j}\in H_{\lambda_j}$, where $0\leqq k<p$, such that $x_i=h_{i,1}\ldots h_{i,pm}$. Now $h_{1,1}^{-1}, \ldots, h_{1,n}^{-1}\in H_{\lambda_1}$ and since H_{λ_1} is a c-σ-subgroup, $(\{x_1a_1, \ldots, x_na_n\}, \{h_{1,2}\ldots h_{1,pm}a_1, \ldots, h_{n,2}\ldots h_{n,pm}a_n\})\in\varrho_{H_{\lambda_1},\sigma}$; since $h_{1,2}^{-1}, \ldots, h_{n,2}^{-1}\in H_{\lambda_2}$ and H_{λ_2} is a c-σ-subgroup, $(\{h_{1,2}\ldots h_{1,pm}a_1, \ldots, h_{n,2}\ldots h_{n,pm}a_n\}, \{h_{1,3}\ldots h_{1,pm}a_1, \ldots, h_{n,3}\ldots h_{n,pm}a_n\})\in\varrho_{H_{\lambda_2},\sigma}$; ...; since $h_{1,pm}^{-1}, \ldots, h_{n,pm}^{-1}\in H_{\lambda_m}$ and H_{λ_m} is a c-σ-subgroup, $(\{h_{1,mp}a_1, \ldots, h_{n,pm}a_n\}, \{a_1, \ldots, a_n\})\in\varrho_{H_{\lambda_m},\sigma}$. It follows that $(\{a_1, \ldots, a_n\}, \{x_1a_1, \ldots, x_na_n\})\in\bigvee\varrho_{H_\lambda,\sigma}$, and by Theorem 2.12, $\bigvee\varrho_{H_\lambda,\sigma}=\varrho_{[\cup H_\lambda],\sigma}$. Therefore, $[\bigcup H_\lambda]$ is a c-σ-subgroup of G.

THEOREM 4.3. *Let* $\sigma\in\text{Ret}\, G$ *and* J *be a normal subgroup of* G.

(i) *If* J *is a c-*σ*-subgroup of* G *and if* $X\sigma^*=J(\{a_1, \ldots, a_n\}\sigma)$, *for every* $X=$

$=\{Ja_1, \dots, Ja_n\}\in F(G/J)$, *then* σ^* *is a retraction of* G/J. *Moreover, if* H *is a subgroup of* G *that contains* J, *then* $\varrho_{H/J,\sigma^*}=\{(X, Y)|X=\{Ja_1, \dots, Ja_m\}, Y=\{Jb_1, \dots, Jb_n\}\in F(G/J)$ *and* $(\{a_1, \dots, a_m\}, \{b_1, \dots, b_n\})\in\varrho_{H,\sigma}\}$. *If, in addition, J is a convex* σ*-subgroup of* G, *then* σ^* *is an l-retraction of* G/J.

(ii) *If* τ *is a retraction of* G/J, *where* $\{Ja_1, \dots, Ja_n\}\tau=J(\{a_1, \dots, a_n\}\sigma)$, *for each* $\{Ja_1, \dots, Ja_n\}\in F(G/J)$, *then* J *is a c-*σ*-subgroup of* G. *If, in addition,* τ *is an l-retraction of* G/J, *then* J *is a convex* σ*-subgroup of* G.

PROOF. (i) Suppose that J is a c-σ-subgroup of G and that for $1\le i\le n$, $Ja_i=Jb_i$. Then $b_i=h_ia_i$ for some $h_i\in J$, and so $(\{a_1, \dots, a_n\}, \{h_1a_1, \dots, h_na_n\})\in\varrho_{J,\sigma}$. Therefore, σ^* is well defined, and clearly $\{Ja\}\sigma^*=Ja$ for each $Ja\in G/J$. Since σ is a homomorphism, it follows that σ^* is a homomorphism. Therefore, $\sigma^*\in\operatorname{Ret} G/J$.

The verification that $\varrho_{H/J,\sigma^*}=\{(X, Y)|X=\{Ja_1, \dots, Ja_m\}, Y=\{Jb_1, \dots, Jb_n\}$, and $(\{a_1, \dots, a_m\}, \{b_1, \dots, b_n\})\in\varrho_{H,\sigma}\}$ is straightforward and will be omitted.

Next suppose that J is a convex σ-subgroup of G. Then J is a c-σ-subgroup. Let $(X, Y)\in\varrho_{\{J\},\sigma^*}$, where $X=\{Ja_1, \dots, Ja_m\}$ and $Y=\{Jb_1, \dots, Jb_n\}$, and let $Z=\{Jc_1, \dots, Jc_p\}\in F(G/J)$. By the preceding $(\{a_1, \dots, a_m\}, \{b_1, \dots, b_n\})\in\varrho_{J,\sigma}$ and since $\varrho_{J,\sigma}$ is a join congruence, $(\{a_1, \dots, a_m, c_1, \dots, c_p\}, \{b_1, \dots, b_n, c_1, \dots, c_p\})\in\varrho_{J,\sigma}$. Therefore, $(X\cup Z, Y\cup Z)\in\varrho_{\{J\},\sigma^*}$ and so $\varrho_{\{J\},\sigma^*}$ is a join congruence. Hence, $\{J\}$ is a convex σ^*-subgroup of G/J. Thus, by Theorem 3.2, σ^* is an l-retraction of G/J.

(ii) Suppose that τ is a retraction of G/J. If $\{a_1, \dots, a_n\}\in F(G)$ and $h_1, \dots, h_n\in J$, then $\{Ja_1, \dots, Ja_n\}=\{Jh_1a_1, \dots, Jh_na_n\}$ and so $J(\{a_1, \dots, a_n\}\sigma)=\{Ja_1, \dots, Ja_n\}\tau=$ $=\{Jh_1a_1, \dots, Jh_na_n\}\tau=J(\{h_1a_1, \dots, h_na_n\}\sigma)$. Therefore, $(\{a_1, \dots, a_n\}, \{h_1a_1, \dots, h_na_n\})\in$ $\in\varrho_{H,\sigma}$ and J is a c-σ-subgroup of G. If σ^* is given by (i), then, clearly, $\tau=\sigma^*$.

Next assume that τ $(=\sigma^*)$ is an l-retraction of G/J. Then J is a c-σ-subgroup and by (i) $\varrho_{\{J\},\sigma^*}=\{(\{Ja_1, \dots, Ja_m\}, \{Jb_1, \dots, Jb_n\})|(\{a_1, \dots, a_m\}, \{b_1, \dots, b_n\})\in\varrho_{J,\sigma}\}$. Since τ is an l-retraction of G/J, G/J is a lattice-ordered group and $\{J\}$ is a convex l-subgroup of G/J. Hence, $\varrho_{\{J\},\sigma^*}$ is a join congruence and it follows that $\varrho_{J,\sigma}$ is a join congruence. Therefore, J is a convex σ-subgroup of G.

We will call σ^*, given in (i) of Theorem 4.3, the retraction of G/J induced by σ.

THEOREM 4.4. *Let* $\sigma\in\operatorname{Ret} G$, J *be a normal c-*σ*-subgroup of* G, H *be a subgroup of* G *that contains* J, *and* σ^* *be the retraction of* G/J *induced by* σ. *Then* H *is a c-*σ*-subgroup (resp. convex* σ*-subgroup) of* G *if and only if* H/J *is a c-*σ^**-subgroup (resp. convex* σ^**-subgroup) of* G/J.

Theorem 4.4 is easily proven using the description of $\varrho_{H/J,\sigma^*}$ given in Theorem 4.3.

COROLLARY 4.5. *Let* $\sigma\in\operatorname{Ret} G$, J *be a normal c-*σ*-subgroup of* G, *and* σ^* *be the retraction of* G/J *induced by* σ. *Then there is a lattice isomorphism between the c-*σ*-subgroups of* G *that contain* J *and the c-*σ^**-subgroups of* G/J. *Moreover, under this isomorphism the convex* σ*-subgroups of* G *that contain* J *map onto the convex* σ^**-subgroups of* G/J.

In the remarks after Theorem 4.1, we observed that if $\sigma\in\operatorname{Ret} G$, then there was a smallest convex σ-subgroup of G. This subgroup will be called the *convex* σ*-kernel* of G.

COROLLARY 4.6. *Let $\sigma\in\operatorname{Ret} G$ and J be the convex σ-kernel of G. Then there is a lattice isomorphism of the convex σ-subgroups of G onto the convex l-subgroups of G/J. Thus, the lattice of convex σ-subgroups of G is a Brouwerian lattice.*

PROOF. By Theorem 4.3, if σ^* is the retraction of G/J induced by σ, then σ^* is an l-retraction. Thus, G/J is a lattice-ordered group. By Corollary 4.5, the convex σ^*-subgroups of G/J are isomorphic to the convex σ-subgroups of G, and by Corollary 3.5, the convex σ^*-subgroups of G/J are precisely the convex l-subgroups of G/J. It is well-known that the lattice of convex l-subgroups of a lattice-ordered group is a Brouwerian lattice.

THEOREM 4.7. *If σ is an l-retraction of G, $\leqq$ is the lattice-ordering of G induced by σ, and H is a c-σ-subgroup of G, then H is a convex l-subgroup of G.*

PROOF. By Theorem 4.1, H is a σ-subgroup of G and hence an l-subgroup. Let $h\in H$ and $g\in G$ such that $i\leqq g\leqq h$. Then $\{i, h^{-1}g\}\in F(G)$, $i, h\in H$, and since H is a c-σ-subgroup, $(\{i, h^{-1}g\}, \{i, g\})\in\varrho_{H,\sigma}$. Thus $H=Hi=H(i\vee h^{-1}g)=$ $=H(\{i, h^{-1}g\}\sigma)=H(\{i, g\}\sigma)=H(i\vee g)=Hg$. Therefore, $g\in H$ and so H is a convex l-subgroup of G.

COROLLARY 4.8. *If $\sigma\in\operatorname{Ret} G$ and H is a c-σ-subgroup of G that contains the convex σ-kernel of G, then H is a convex σ-subgroup of G. Thus, the collection of convex σ-subgroups of G is a dual ideal in the lattice of c-σ-subgroups of G.*

PROOF. Let J be the convex σ-kernel of G. Since H is a c-σ-subgroup of G that contains J, H/J is a c-σ^*-subgroup of G/J by Theorem 4.4. By Theorem 4.7, H/J is a convex l-subgroup of G/J. Since the collection of convex l-subgroups of G/J is identical with the collection of convex σ-subgroups of G/J, we have by Theorem 4.4, that H is a convex σ-subgroup of G.

THEOREM 4.9. *Let $\sigma\in\operatorname{Ret} G$ and H be a c-σ-subgroup of G.*

(i) *If K is a σ-subgroup of G, then $H\cap K$ is a c-σ-subgroup of K.*

(ii) *If $n(H)$ denotes the normalizer of H in G, then $n(H)$ is a σ-subgroup of G.*

(iii) *Let $g\in n(H)$ and $\{i, g\}\sigma=a$. Then $H[g]$ is a σ-subgroup of G if and only if $a\in H[g]$, where $H[g]$ is the subgroup generated by H and g.*

PROOF. (i) Let $\{a_1,\dots,a_n\}\in F(K)$ and $h_1,\dots,h_n\in H\cap K$. Then $\{h_1a_1,\dots,h_na_n\}\in$ $\in F(K)$, $\{a_1, \dots, a_n\}\sigma\in K$, $\{h_1a_1, \dots, h_na_n\}\sigma\in K$, and $(\{a_1, \dots, a_n\}, \{h_1a_1, \dots, h_na_n\})\in$ $\in\varrho_{H,\sigma}$. Therefore, $(\{a_1, \dots, a_n\}\sigma)(\{h_1a_1, \dots, h_na_n\}\sigma)^{-1}=h$ for some $h\in H\cap K$, and so $H\cap K$ is a c-σ-subgroup of K.

(ii) Let $\{a_1, \dots, a_n\}\in F(n(H))$, $\{a_1, \dots, a_n\}\sigma=a$, $h\in H$ and for $1\leqq i\leqq n$, $a_iha_i^{-1}=$ $=h_i$. Then $h_i\in H$ and $Hah=H(\{a_1, \dots, a_n\}\sigma h)=H(\{a_1h, \dots, a_nh\}\sigma)=H(\{h_1a_1, \dots$ $\dots, h_na_n\}\sigma)=H(\{a_1, \dots, a_n\}\sigma)=Ha$. Therefore, $aha^{-1}\in H$ and so $a\in n(H)$. Consequently, $n(H)$ is a σ-subgroup of G.

(iii) Suppose that $a\in H[g]$ and let $\{h_1g^{m_1}, \dots, h_ng^{m_n}\}\in F(H[g])$, where $h_1, \dots$ $\dots, h_n\in H$ and $m_1<\dots<m_n$. Since H is a c-σ-subgroup, $H(\{h_1g^{m_1}, \dots, h_ng^{m_n}\}\sigma)=$ $=H(\{g^{m_1}, \dots, g^{m_n}\}\sigma)$. By Theorem 2.4, $\{g^{m_1}, \dots, g^{m_n}\}\sigma=a^{m_n-m_1}g^{m_1}$. Therefore, $\{h_1g^{m_1}, \dots, h_ng^{m_n}\}\sigma\in H[g]$ and so $H[g]$ is a σ-subgroup of G. The converse is trivial.

COROLLARY 4.10. *Let $\sigma\in\operatorname{Ret} G$ and H be a c-σ-subgroup of G. If $g\in G$ and $g^n\in H$ for some $n\in N$, then $g\in H$. In particular, H is a pure subgroup of G.*

PROOF. Let $K=\bigcap\{J|J$ is a c-σ-subgroup of G and $g^n\in J\}$. By Theorem 4.2, K is a c-σ-subgroup of G. If J is a c-σ-subgroup of G such that $g^n\in J$, then $g^n=g^{-1}g^ng\in g^{-1}Jg$. It follows that $g\in n(K)$. By the theorem $n(K)$ is a σ-subgroup of G, and hence $n(K)/K$ is retractable. By Theorem 2.2, $n(K)/K$ is torsion-free. Therefore, $g\in K\subseteqq H$.

Let G and L be retractable groups, $\sigma\in\operatorname{Ret} G$, and $\tau\in\operatorname{Ret} L$. A homomorphism φ of G into L is said to be a *σ-τ-homomorphism* provided that $\varphi\tau=\sigma\varphi$ (here, and in Section 5, we will not distinguish in notation between the image of an element under a function and the image of a subset under the function), i.e., if the following diagram is commutative:

$$\begin{array}{ccc} F(G) & \xrightarrow{\varphi} & F(L) \\ \sigma\downarrow & & \downarrow\tau \\ G & \xrightarrow{\varphi} & L. \end{array}$$

It both σ and τ are l-retractions, then this definition makes φ an l-homomorphism.

THEOREM 4.11. *Let G and L be retractable groups, $\sigma\in\operatorname{Ret} G$, and $\tau\in\operatorname{Ret} L$. If φ is a σ-τ-homomorphism of G into L, then the kernel of φ is a c-σ-subgroup of G.*

PROOF. Let J denote the kernel of σ, $\{a_1, \dots, a_n\}\in F(G)$, and $h_1, \dots, h_n\in J$. Then $\{a_1, \dots, a_n\}\sigma\varphi=\{a_1, \dots, a_n\}\varphi\tau=\{a_1\varphi, \dots, a_n\varphi\}\tau=\{(h_1a_1)\varphi, \dots, (h_na_n)\varphi\}\tau= =\{h_1a_1, \dots, h_na_n\}\varphi\tau=\{h_1a_1, \dots, h_na_n\}\sigma\varphi$. Therefore, $(\{a_1, \dots, a_n\}\sigma)(\{h_1a_1, \dots \dots, h_na_n\}\sigma)^{-1}\in J$ and hence, J is a c-σ-subgroup of G.

The next three theorems assert that the isomorphism theorems are valid for c-σ-subgroups. The proofs are straightforward and are omitted.

THEOREM 4.12. *Let G and L be retractable groups, $\sigma\in\operatorname{Ret} G$, $\tau\in\operatorname{Ret} L$, φ be a σ-τ-homomorphism of G onto L, J be the kernel of φ, and σ^* the be retraction of G/J induced by σ. Then the induced isomorphism φ^* of G/J onto L is a σ^*-τ-isomorphism.*

In diagram form:

$$\begin{array}{ccc} F(G/J) & \xrightarrow{\varphi^*} & F(L) \\ \sigma^*\downarrow & & \downarrow\tau \\ G/J & \xrightarrow{\varphi^*} & L. \end{array}$$

THEOREM 4.13. *Let $\sigma\in\operatorname{Ret} G$, J and K be normal c-σ-subgroups of G such that $J\subseteqq K$, σ_1 be the retraction of G/J induced by σ, σ_2 be the retraction of G/K induced by σ, and σ_3 be the retraction of $G/J/K/J$ induced by σ_1. Then the induced isomorphism of $G/J/K/J$ onto G/K is a σ_3-σ_2-isomorphism.*

In diagram form:

$$\begin{array}{ccccccc} & & & \eta^* & & & \\ F(G/K) & \xleftarrow{\eta_2} & F(G) & \xrightarrow{\eta_1} & F(G/J) & \xrightarrow{\eta_3} & F(G/J/K/J) \\ \sigma_2\downarrow & & \downarrow\sigma & & \sigma_1\downarrow & & \downarrow\sigma_3 \\ G/K & \xleftarrow{\eta_2} & G & \xrightarrow{\eta_1} & G/J & \xrightarrow{\eta_3} & G/J/K/J. \\ & & & \eta^* & & & \end{array}$$

THEOREM 4.14. *Let $\sigma \in \operatorname{Ret} G$, H and J be normal c-σ-subgroups of G, σ_1 be the retraction of $J/J\cap H$ induced by the restriction of σ to $F(J)$, and σ_2 be the retraction of JH/H induced by the restriction of σ to $F(JH)$. Then the induced isomorphism of $J/H\cap J$ onto JH/H is a σ_1-σ_2-isomorphism.*

5. Retractions. It appears to be a difficult problem to determine if a group admits a retraction. Even if a group is retractable, it is difficult to construct a retraction for the group. In this section we present (without proofs) some methods of constructing "new" retractions from a given retraction of a group.

THEOREM 5.1. *Let $\sigma \in \operatorname{Ret} G$ and let φ be an automorphism or an anti-automorphism of G. Then $\varphi\sigma\varphi^{-1} \in \operatorname{Ret} G$. If φ is an inner automorphism of G, then $\varphi\sigma\varphi^{-1}=\sigma$.*

Of some special interest in Theorem 5.1 is the retraction $\sigma'=\varphi\sigma\varphi^{-1}$, where φ is the anti-automorphism of G given by $g\varphi=g^{-1}$. If $\sigma=\sigma'$, then we say that σ is *self dual*. In Example 5.7 below, we give an example of a self dual retraction. If $S\subseteq F(G)$, let $S^{-1}=\{A^{-1}|A\in S\}$.

COROLLARY 5.2. *Let $\sigma \in \operatorname{Ret} G$.*

(i) $\operatorname{Ker} \sigma'=(\operatorname{Ker} \sigma)^{-1}$. *Thus, if S is a G-complement of $F(G)$, then so is S^{-1}.*

(ii) *If σ is an l-retraction of G, then so is σ' and the lattice-ordering of G induced by σ' is the dual of the lattice-ordering of G induced by σ.*

The next corollary shows that groups that admit a self dual retraction must be two divisible.

COROLLARY 5.3. *If $\sigma \in \operatorname{Ret} G$ and $\sigma=\sigma'$, then $G^2=G$.*

PROOF. First let $g, h\in G$ with $\{g, h\}\sigma=i$. Then, by the preceding corollary, $\{g^{-1}, h^{-1}\}\sigma=i$. Now $\{g, h\}\sigma=i$ implies that $g^{-1}h^{-1}=\{h^{-1}, g^{-1}\}\sigma=i$. Therefore, $g=h^{-1}$. Next let $g\in G$ and $\{i, g\}\sigma=a$. Then $\{a^{-1}, ga^{-1}\}\sigma=i$ and by what we have just proven $a^{-1}=(ga^{-1})^{-1}$. Hence, $g=a^2$.

THEOREM 5.4. *For each $\lambda\in\Lambda$, let G_λ be a group, let $\sigma_\lambda\in \operatorname{Ret} G_\lambda$, let $G=\Pi G_\lambda$, the unrestricted direct product of the G_λ, or let $G=\Sigma G_\lambda$, the restricted direct product of the G_λ, let π_λ be the projection of G onto G_λ, and let $A\in F(G)$. If $A\sigma=$ $=(\dots, A\pi_\lambda\sigma_\lambda, \dots)$, then $\sigma\in \operatorname{Ret} G$. Moreover,*

(i) *σ is an l-retraction of G if and only if σ_λ is an l-retraction of G_λ for each $\lambda\in\Lambda$;*

(ii) *σ is self dual if and only if σ_λ is self dual for each $\lambda\in\Lambda$.*

THEOREM 5.5. *Let G be an abelian group, $\sigma\in \operatorname{Ret} G$, and φ be an endomorphism of G. If $A\in F(G)$ and $A\tau=\big((AA^{-1})\sigma\varphi\big)(A\sigma)$, then $\tau\in \operatorname{Ret} G$.*

EXAMPLE 5.6. Let σ be the retraction of Z induced by the natural ordering of the integers. For $k\in Z$, let φ_k be the endomorphism of Z given by $x\varphi_k=kx$ and let σ_k be the retraction of Z given in Theorem 5.5 using σ and φ_k. Then if $A\in F(Z)$, $A\sigma_k=(k+1)(\max A-\min A)+\min A$.

Let $\tau\in \operatorname{Ret} Z$ and $\{0,1\}\tau=k$. If $A=\{a_1, \dots, a_n\}\in F(Z)$, with $a_1<\dots<a_n$, then by Theorem 2.4, $A\tau=(a_n-a_1)k+a_1=k(\max A-\min A)+\min A=A\sigma_{k-1}$. Therefore, $\operatorname{Ret} Z=\{\sigma_k|k\in Z\}$. If $k\neq-1, 0$, then Z is the only convex σ_k-subgroup of itself.

EXAMPLE 5.7. As above, if $r\in Q$, $A\in F(Q)$, and $A\sigma_r=(r+1)(\max A-\min A)+$ $+\min A$, then $\sigma_r\in \operatorname{Ret} Q$. Let $\tau\in\operatorname{Ret} Q$, $\{0,1\}\tau=k$, and $A=\left\{\frac{r_1}{s},\dots,\frac{r_n}{s}\right\}\in F(Q)$, where $r_1<\dots<r_n$, $r_1,\dots,r_n\in Z$, $s\in N$. By Corollary 2.5 (i), $k=\{0,1\}\tau=s(\{0,1/s\}\tau)$. By Theorem 2.4 (ii),

$$A\tau=(r_n-r_1)\frac{k}{s}+\frac{r_1}{s}=k\left(\frac{r_n}{s}-\frac{r_1}{s}\right)+\frac{r_1}{s}=A\sigma_{k-1}.$$

Therefore, $\operatorname{Ret} Q=\{\sigma_r|r\in Q\}$.

Now $A\sigma_{-1/2}=1/2(\max A-\min A)+\min A=1/2(\max A+\min A)=-((-A)\sigma_{-1/2})=A(\sigma_{-1/2})'$. Hence, $\sigma_{-1/2}$ is self dual.

For $r=-1$ or $r=0$, the kernel of σ_r is a convex subsemilattice of $F(Q)$. If $r>0$ or $r<-1$, the kernel of σ_r is a convex subset of $F(Q)$, but not a subsemilattice. If $-1<r<0$, the kernel of σ_r is a subsemlattice of $F(Q)$, but not convex.

6. Examples. In this section we present two examples. In the first example we exhibit a retractable group G and $\sigma,\tau\in\operatorname{Ret} G$ with the property that σ and τ agree on two element subsets of G, but $\sigma\neq\tau$. Clearly, σ and τ cannot be l-retractions.

In the second example we exhibit a retractable group that cannot be lattice-ordered, thus showing that the class of lattice-ordered groups is a proper subclass of the class of retractable groups.

EXAMPLE 6.1. Let $G_1=G_2=Q$, $\tau_1=\tau_2=\sigma_{-1/2}$, where $\sigma_{-1/2}$ is given in Example 5.7, and let $\sigma\in\operatorname{Ret}(G_1\times G_2)$ be given from τ_1 and τ_2 as in Theorem 5.4. Let φ be the automorphism of $G_1\times G_2$ given by $(x,y)\varphi=(x+y,y)$ and as in Theorem 5.1, let $\tau=\varphi\sigma\varphi^{-1}$. If $A=\{(a,b),(x,y)\}\in F(G_1\times G_2)$, then $A\sigma=\left(\frac{a+x}{2},\frac{b+y}{2}\right)=A\tau$. Thus, σ and τ agree on two element subsets of $G_1\times G_2$. However, if $B=\{(6,8),(4,2),(8,4)\}$, then $B\sigma=(6,5)$ and $B\tau=(15,5)$.

EXAMPLE 6.2. Let $H=Q\times Q\times Z$ and define the binary operation $+$ on H as was done in Example 2.7. For $(a,b,c)\in H$, define $(a,b,c)\geqq(0,0,0)$ if $c>0$, or if $c=0$, $a\geqq 0$, and $b\geqq 0$. Then it is well-known that H is a lattice-ordered group. It is easily verified that $\{(a,-a,0)|a\in Q\}$ is the commutator subgroup of H (in fact each element is a commutator), and $K=\{(a,-a,x)|\ a\in Q$, and $x\in Z\}$ is a subgroup of H.

CLAIM 1. *If* $\sigma\in\operatorname{Ret} K$, *then* $\{(0,0,0),(1,-1,0)\}\sigma=(1/2,-1/2,0)$.

Let $(a,-a,x)=\{(0,0,0),(1,-1,0)\}\sigma$. Then by Theorem 2.4, $(a,-a,x)\in$ $\in C((1,-1,0))$ and so x must be even. Now $(a-1,-(a-1),x)=\{(0,0,0),$ $(1,-1,0)\}\sigma+(-1,1,0)=\{(-1,1,0),(0,0,0)\}\sigma$. On the other hand, $(-a,a,x)=$ $=-(0,0,1)+\{(0,0,0),(1,-1,0)\}\sigma+(0,0,1)=\{(0,0,0),(-1,1,0)\}\sigma$. Therefore, $(-a,a,x)=(a-1,-(a-1),x)$ and so $a=1/2$. For $n\in N$, $\left(\frac{1}{2^n},-\frac{1}{2^n},0\right)\in K$. If $(b,-b,y)=\left\{(0,0,0),\left(\frac{1}{2^n},-\frac{1}{2^n},0\right)\right\}\sigma$, then by Corollary 2.5, $2^n(b,-b,y)=$

$=\left\{(0,0,0),\ 2^n\left(\frac{1}{2^n},\ -\frac{1}{2^n},\ 0\right)\right\}\sigma=(a,\ -a,\ x)$. Thus, $x=2^n y$ and since n was arbitrary, $x=0$.

CLAIM 2. *K does not admit a lattice-order.*

Suppose (by way of contradiction) that K admits a lattice-order and let σ be the l-retraction induced by the lattice-ordering. By Claim 1, $(1/2,\ -1/2,\ 0)=\{(0,0,0),\ (1,\ -1,\ 0)\}\sigma=(0,0,0)\vee(1,\ -1,\ 0)\geqq(0,0,0)$. But then $(-1/2,\ 1/2,\ 0)=-(0,0,1)+ +(1/2,\ -1/2,\ 0)+(0,0,1)\geqq(0,0,0)$, which is impossible as no nonzero element and its inverse can be positive.

CLAIM 3. *K is retractable.*

Let $A=\{(a_1,\ -a_1,\ x_1),\ \dots,\ (a_m,\ -a_m,\ x_m)\}\in F(K)$, where $x_1\leqq\dots\leqq x_m$. Define

$$A\sigma=\left(\frac{\bigvee_{i=p}^{m} a_i+\bigwedge_{i=p}^{m} a_i}{2},\ -\frac{\bigvee_{i=p}^{m} a_i+\bigwedge_{i=p}^{m} a_i}{2},\ x_m\right)$$

where $x_{p-1}<x_p=\dots=x_m$. A straightforward calculation shows that $\sigma\in\mathrm{Ret}(K)$.

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UNIVERSITY OF HOUSTON
UNIVERSITY OF WYOMING
GEORGETOWN UNIVERSITY

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KOMMUTATIONSGLEICHUNGEN IN SEMIFREIEN GRUPPEN

Von
A. BAUDISCH (Berlin)

1. Einleitung

Eine semifreie Gruppe sei eine Gruppe, deren definierende Relationen Kommutationsgleichungen zwischen den Erzeugenden sind.

Es wird gezeigt, daß zwei Elemente $\bar{p}$ und $\bar{q}$ einer semifreien Gruppe genau dann kommutieren, wenn es Elemente $\bar{w}, \bar{u}_i$ und ganze Zahlen α_i, β_i $(1 \leqq i \leqq n)$ gibt, so daß die Erzeugenden, die in $\bar{u}_i$ vorkommen, mit allen Erzeugenden, die in $\bar{u}_j$ für $j \neq i$ vorkommen, kommutieren, und

$$\bar{p} = \bar{w} \prod_{i=1}^{n} \bar{u}_i^{\alpha_i} \bar{w}^{-1} \quad \text{und} \quad \bar{q} = \bar{w} \prod_{i=1}^{n} \bar{u}_i^{\beta_i} \bar{w}^{-1}$$

gilt. Aus dem obigen Resultat folgt die Universalität und somit die rekursive Unentscheidbarkeit der elementaren Theorie der semifreien Gruppen, wie in [1] gezeigt wurde. Wenn **K** eine Klasse von Strukturen einer abzählbaren Signatur σ ist, so heißt Universalität der elementaren Theorie der semifreien Gruppen im Sinne von [1], daß durch elementare Definitionen der Zeichen aus σ, des Gleichheitszeichens und des Grundbereichs in der Sprache der Gruppentheorie sämtliche Elemente aus **K** is semifreien Gruppen verschlüsselt werden können.

Weiterhin kann gezeigt werden, daß jede abelsche Untergruppe einer semifreien Gruppe frei-abelsch ist. Es bleibt die Frage offen, ob es außer den semifreien Gruppen noch weitere Gruppen gibt, deren sämtliche abelsche Untergruppen frei-abelsch sind.

An dieser Stelle möchte ich Herrn Dr. K. Hauschild für die Anregung zu dieser Arbeit und Unterstützung bei ihrer Anfertigung danken.

2. Ausführliche Definition der semifreien Gruppen

$\mathfrak{A}$ sei ein irreflexiver symmetrischer Graph $\langle A, R\rangle$. Der Menge A sei durch eine eineindeutige Abbildung $\varphi(a)=a^{-1}$ eine zu A disjunkte Menge $A^{-1}=\{a^{-1}: a\in A\}$ zugeordnet. $\mathfrak{W}(A)$ sei die Menge aller endlichen Folgen $w=\langle c_1, c_2, \dots, c_n\rangle$ mit $c_i \in A\cup A^{-1}$ (einschließlich der leeren Folge ♀). Die Elemente von $\mathfrak{W}(A)$ werden Worte genannt. Über $\mathfrak{W}(A)$ wird definiert:

$p \approx_{\mathfrak{A}} q \overset{\text{Df}}{=} p$ kann durch endlich viele Anwendungen der folgenden Regeln in q umgewandelt werden:

1. Einsetzen von aa^{-1} oder $a^{-1}a$ für $a\in A$ an beliebiger Stelle.
2. Streichen von aa^{-1} oder $a^{-1}a$ für $a\in A$.

3. Kommutieren von zwei benachbarten Elementen $ab, a^{-1}b, ab^{-1}$ oder $a^{-1}b^{-1}$, falls $\neg R(a, b)$ in $\mathfrak{A}$ gilt.

Abkürzend wird statt „$\approx_{\mathfrak{A}}$" oft „$\approx$" geschrieben, „$\approx$" ist bezüglich der Verkettung $p \circ q$ von Worten p, q eine Kongruenzrelation. In $\mathfrak{W}(A)/\approx$ kann zwischen Äquivalenzklassen $\bar{p}$ und $\bar{q}$ eine Gruppenoperation $\bar{p} \circ \bar{q} = \overline{p \circ q}$ definiert werden. $\Gamma(\mathfrak{A})$ sei die Gruppe $\langle \mathfrak{W}(A)/\approx, \circ \rangle$. Eine Gruppe ist nun genau dann semifrei, wenn sie zu einer Gruppe der Form $\Gamma(\mathfrak{A})$ isomorph ist.

Wenn $\mathfrak{B} \subseteq \mathfrak{A}$, so sei $\Gamma(\mathfrak{A}) \restriction_B$ die von den Äquivalenzklassen der Elemente aus B in $\Gamma(\mathfrak{A})$ erzeugte Untergruppe.

Lemma 1. *Wenn* $\mathfrak{B} \subseteq \mathfrak{A}$, *so gilt* $\Gamma(\mathfrak{B}) \cong \Gamma(\mathfrak{A}) \restriction_B$.

Beweis. Für $p, q \in \mathfrak{W}(B)$ ist zu zeigen, daß $p \approx_{\mathfrak{B}} q$ genau dann gilt, wenn $p \approx_{\mathfrak{A}} q$ gilt.

$p \approx_{\mathfrak{B}} q$ impliziert nach Definition $p \approx_{\mathfrak{A}} q$.

Aus $p \approx_{\mathfrak{A}} q$ folgt $p \approx_{\mathfrak{B}} q$, indem man bei der Umrechnung von p in q alle Regeln fallen läßt, die Elemente aus $(A \cup A^{-1}) \setminus (B \cup B^{-1})$ enthalten. Dabei bleibt nämlich die Folge der Elemente aus $B \cup B^{-1}$ in jedem Schritt dieselbe wie in der Umrechnung $p \approx_{\mathfrak{A}} q$. q.e.d.

Mit Hilfe von Lemma 1 ergibt sich leicht folgendes

Lemma 2. *Wenn* $\mathfrak{A} = \bigcup_{i<\mu} \mathfrak{A}_i$ *eine Zerlegung des Graphen* $\mathfrak{A}$ *in seine Zusammenhangskomponenten ist, so gilt* $\Gamma(\mathfrak{A}) = \bigoplus_{i<\mu} \Gamma(\mathfrak{A}_i)$, *wobei* $\oplus$ *die direkte Summe bezeichnet.*

Beweis. Nach Lemma 1 genügt es, $\Gamma(\mathfrak{A}) = \bigoplus_{i<\mu} \Gamma(\mathfrak{A}) \restriction_{A_i}$ zu zeigen. Jedes $\Gamma(\mathfrak{A}) \restriction_{A_i}$ ist Normalteiler in $\Gamma(\mathfrak{A})$, Weiterhin ergibt sich aus den Regeln 1. bis 3. für „$\approx$" $\Gamma(\mathfrak{A}) \restriction_{A_i} \cap \sum_{j \neq i} \Gamma(\mathfrak{A}) \restriction_{A_j} = \bar{\varnothing}$. q.e.d.

3. Die Kongruenzrelation „$\approx$"

Jedem Wort p aus $\mathfrak{W}(A)$ ordnen wir eindeutig eine Potenzschreibweise $p = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}$ mit $a_i \in A$, α_i ganze Zahl und $\alpha_i \neq 0$ zu, indem maximale Teilworte der Form $aa \dots a$ zu a^{α} und maximale Teilworte der Form $a^{-1}a^{-1} \dots a^{-1}$ zu $a^{-\alpha}$ zusammengefaßt werden, wenn a in $aa \dots a$ bzw. a^{-1} in $a^{-1}a^{-1} \dots a^{-1}$ α-mal vorkommt. Es kann definiert werden: Ein Wort p ist eine Minimaldarstellung von $\bar{p}$, wenn in der Potenzschreibweise $p = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}$ n minimal ist. n wird als die Länge $\lambda(\bar{p})$ von $\bar{p}$ bezeichnet.

Die Elemente von A werden im weiteren Buchstaben genannt. Die a-Potenz a^{α} sei das Wort $aa \dots a$ für $\alpha > 0$, bzw. $a^{-1}a^{-1} \dots a^{-1}$ für $\alpha < 0$, wobei a bzw. a^{-1} $|\alpha|$-mal vorkommt. Alle Minimaldarstellungen lassen sich nur durch Anwendung von Regel 3. in einander überführen, wie noch in Lemma 5 gezeigt wird. Deshalb ist die Folge der a-Potenzen eines Buchstaben a in jeder Minimaldarstellung dieselbe. Es ist sinnvoll von der Folge der a-Potenzen in $\bar{p}$ zu sprechen.

Die Potenzfolgen zweier Buchstaben, die nicht miteinander kommutieren, sind nach Lemma 5 in eindeutiger Weise ineinander angeordnet.

Man kann definieren: a ist ein Buchstabe von $\bar{p}$ genau dann, wenn a oder a^{-1} in einer (und damit nach Lemma 5 in jeder) Minimaldarstellung von $\bar{p}$ vorkommt.

Weiterhin ist es sinnvoll zu sagen: a^{α_i} ist in $\bar{p}$ die i-te a-Potenz. a^{α_i} steht links (bzw. rechts) von b^{β_j}, falls a und b nicht kommutieren. Ein Buchstabe ist erster Buchstabe von $\bar{p}$ zur Potenz α, wenn es eine Minimaldarstellung $a^{\alpha}p'$ von $\bar{p}$ mit $\lambda(\bar{p}')<\lambda(\bar{p})$ gibt. Analog wird der Begriff letzter Buchstabe von $\bar{p}$ zur Potenz α definiert.

Zwei Elemente $\bar{p}, \bar{q}$ aus $\Gamma(\mathfrak{A})$ sind zusammenhängend, wenn der durch ihre Buchstaben erzeugte Teilgraph von $\mathfrak{A}$ zusammenhängend ist. Ein Element $\bar{p}$ ist zusammenhängend, wenn $\bar{p}$ und $\bar{\text{♀}}$ zusammenhängend sind.

Es sollen jetzt einige Lemmata zur Relation „$\approx$" angegeben werden:

LEMMA 3. *Es gelte $p \approx q$. Dann gibt es ein Umrechnungsverfahren, bei dem zuerst nur die Regeln* 1. *und* 3. *und dann nur die Regeln* 2. *und* 3. *verwandt werden.*

BEWEIS. Es sei ein Umrechnungsverfahren von p in q vorgegeben. Im i-ten Schritt soll aa^{-1} gestrichen werden, obwohl in einem späteren Schritt noch Regel 1. angewandt wird. Dieser i-te Schritt sei die letzte Anwendung von Regel 2., nach der noch eine Anwendung von Regel 1. kommt. Zwischen dem i-ten Schritt und der letzten Anwendung von Regel 1. erfolgt keine weitere Kürzung. Zum Beweis des Lemmas genügt es zu zeigen, daß man den i-ten Schritt bis nach der Anwendung von Regel 1. hinausschieben kann. Das durch Hinauszögern des i-ten Schrittes stehengebliebene Paar aa^{-1} oder $a^{-1}a$ wird gekennzeichnet. Bei Ausführung der Regel 1. stören stehengebliebene Paare der Form tt^{-1} $(t \in \mathfrak{W}(A))$ nicht.

Soll Regel 3. durchgeführt werden, und es liegt folgende Situation vor: $\dots btt^{-1}c\dots$, c und b sollen kommutiert werden, und tt^{-1} müßte eigentlich schon gestrichen sein, so wird noch einmal Regel 1. eingeschoben: $\dots bcc^{-1}tt^{-1}c\dots$, und dann Regel 3. mit dem Resultat $\dots cbc^{-1}tt^{-1}c\dots$ angewandt.

Die Kennzeichnung von tt^{-1} wird auf $c^{-1}tt^{-1}c$ ausgedehnt. Ist Regel 1. dann das letzte Mal in dem Ausgangsverfahren durchgeführt, so werden alle gekennzeichneten Paare tt^{-1} nach Regel 2. gekürzt. Danach wird das alte Verfahren zum Abschluß gebracht. q.e.d.

LEMMA 4. *Es gelte $p \approx q$, und q sei eine Minimaldarstellung. Dann gibt es ein Umrechnungsverfahren von p in q, das nur die Regeln* 2. *und* 3. *benutzt.*

BEWEIS. Es genügt für alle Verfahren zwischen p und q zu zeigen, daß man die letzte Anwendung von Regel 1. umgehen kann. Das Paar aa^{-1} werde o.B.d.A. in dieser letzten Anwendung von Regel 1. eingeschoben. Die beiden eingesetzten Elemente werden durch Unterstreichen gekennzeichnet. Da im weiteren Verlauf der Umrechnung nur noch gekürzt und kommutiert wird, können $\underline{a}$ und $\underline{a}^{-1}$ nach jedem Schritt durch Kommutieren wieder nebeneinander gebracht werden, falls noch keiner von beiden Buchstaben gekürzt wurde. Um zu einer Minimaldarstellung zu gelangen, muß mindestens einer von beiden weggekürzt werden, da man sie sonst nach Abschluß der Rechnung zusammenführen und kürzen könnte. Wird $\underline{a}\underline{a}^{-1}$ gekürzt, so braucht man $\underline{a}\underline{a}^{-1}$ gar nicht erst einzuführen.

Wird o.B.d.A. $a^{-1}\underline{a}$ im i-ten Schritt gekürzt, so braucht $\underline{a}\underline{a}^{-1}$ auch nicht eingeführt zu werden, und man muß nur a^{-1} nach dem $(i-1)$-ten Schritt durch

Kommutieren auf den Platz bringen, den sonst $\underline{a}^{-1}$ einnehmen würde. Dies ist, wie oben schon bemerkt, möglich. q.e.d.

Aus Lemma 4 ergibt sich sofort:

LEMMA 5.5.1. *Wenn* $p \approx q$ *gilt, und* p *und* q *Minimaldarstellungen sind, so kann* p *in* q *allein durch Anwendung von Regel* 3. *überführt werden.*

5.2. *Die Folge der Potenzen eines Buchstaben* a *ist in allen Minimaldarstellungen eines Elements* $\bar{p}$ *diesselbe. Das Gleiche gilt für die Folge der Potenzen zweier Buchstaben, die nicht miteinander kommutieren.*

LEMMA 6.6.1. *Falls kein Buchstabe zugleich erster Buchstabe von* $\bar{q}$ *und letzter Buchstabe von* $\bar{p}$ *ist, so erhält man für jeden Buchstaben* a *die Folge der* a*-Potenzen in* $\bar{p}\bar{q}$ *durch Hintereinanderschreiben der entsprechenden Folgen von* $\bar{p}$ *und* $\bar{q}$.

6.2. *Falls* a *erster Buchstabe von* $\bar{q}$ *aber nicht letzter Buchstabe von* $\bar{p}$ *oder letzter Buchstabe von* $\bar{p}$ *aber nicht erster Buchstabe von* $\bar{q}$ *ist, so erhält man die Folge der* a*-Potenzen in* $\bar{p}\bar{q}$ *durch Hintereinanderschreiben der entsprechenden Folgen aus* $\bar{p}$ *und* $\bar{q}$.

6.3. *Wenn* $\langle a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_n}\rangle$ *die Folge der* a*-Potenzen von* $\bar{p}$ *und* $\langle a^{\beta_1}, a^{\beta_2}, \ldots, a^{\beta_m}\rangle$ *die Folge der* a*-Potenzen von* $\bar{q}$ *ist, wobei* $\alpha_n+\beta_1 \neq 0$, *so hat die Folge der* a*-Potenzen von* $\bar{p}\bar{q}$ *das Aussehen* $\langle a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_n}, a^{\beta_1}, a^{\beta_2}, \ldots, a^{\beta_m}\rangle$ *oder* $\langle a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_n+\beta_1}, a^{\beta_2}, \ldots, a^{\beta_m}\rangle$.

BEWEIS. 6.1 ergibt sich sofort aus Lemma 4. Um Lemma 6.2 zu beweisen, wird o.B.d.A. angenommen, daß a erster Buchstabe von $\bar{q}$ zur Potenz β ist. Dann existiert ein Buchstabe b in $\bar{p}$, der in jeder Minimaldarstellung von $\bar{p}$ rechts von allen a-Potenzen vorkommt und nicht mit a kommutiert. p und q seien Minimaldarstellungen von $\bar{p}$ bzw. $\bar{q}$. Nach Lemma 4 wird jede Minimaldarstellung von $\bar{p}\bar{q}$ durch Kürzen und Kommutieren aus pq erhalten. Wie man sieht, können aber nur Buchstaben gekürzt werden, die mit a kommutieren, also nicht der Buchstabe b. Hieraus ergibt sich die Behauptung von 6.2.

6.3 beweist man induktiv über $\lambda(\bar{p})+\lambda(\bar{q})$. Für den Induktionsanfang $\lambda(\bar{p})+\lambda(\bar{q}) \leqq 1$ ist die Behauptung klar. Nun sei für $\lambda(\bar{p}')+\lambda(\bar{q}') \leqq n$ alles bewiesen. Es sei $\lambda(\bar{p})+\lambda(\bar{q})=n+1$.

Falls kein Buchstabe existiert, der zugleich letzter Buchstabe von $\bar{p}$ und erster Buchstabe von $\bar{q}$ ist, so folgt die Behauptung aus 6.1.

Ansonsten existieren ein Buchstabe c und ganze Zahlen α, β, so daß $\bar{p}=\bar{p}'\bar{c}^{\alpha}$ und $\bar{q}=\bar{c}^{\beta}\bar{q}'$ mit $\lambda(\bar{p}')<\lambda(\bar{p})$ und $\lambda(\bar{q}')<\lambda(\bar{q})$ gilt.

Im Fall $\alpha=-\beta$ folgt $\bar{p}\bar{q}=\bar{p}'\bar{q}'$, und die Behauptung ergibt sich aus der Induktionsvoraussetzung.

Im Fall $\alpha \neq -\beta$ werden $\bar{p}'\bar{c}^{\alpha+\beta}$ und $\bar{q}'$ betrachtet. Es gilt $\bar{p}\bar{q}=\bar{p}'\bar{c}^{\alpha+\beta}\bar{q}'$ mit $\lambda(\bar{p}'\bar{c}^{\alpha+\beta}) \leqq \lambda(\bar{p})$ und $\lambda(\bar{q}')<\lambda(\bar{q})$. Aus der Induktionsvoraussetzung erhält man die Behauptung für alle Buchstaben ungleich c. Für c ergibt sich die Behauptung aus 6.2. q.e.d.

Ein Element $\bar{p}$ einer semifreien Gruppe heiße zyklisch reduziert, wenn es keine Minimaldarstellung der Form $a^{\alpha}p'a^{\beta}$ besitzt, wobei α und β ungleich Null sind und verschiedene Vorzeichen haben, und $\lambda(\bar{p}')+1<\lambda(\bar{p})$ gilt. Ein Wort p, das durch Anwendung von Regel 3. in eine Minimaldarstellung von $\bar{p}$ umgewandelt werden kann, heiße eine Quasiminimaldarstellung von $\bar{p}$.

LEMMA 7. *Für jedes* $\bar{p}$ *existieren eindeutig bestimmte Elemente* $\bar{w}$ *und* $\bar{p}'$ *mit* $\bar{p}=\bar{w}\bar{p}'\bar{w}^{-1}$, *so daß gilt:*

i) *Wenn* w *und* p' *Minimaldarstellungen von* $\bar{w}$ *bzw.* $\bar{p}'$ *sind, so ist* $wp'w^{-1}$ *eine Quasiminimaldarstellung.*

ii) $\bar{p}'$ *ist zyklisch reduziert.*

BEWEIS. Falls $\bar{p}$ zyklisch reduziert ist, so ist nur zu zeigen, daß keine Elemente $\bar{w}\neq\bar{\varphi}$ und $\bar{p}'$ existieren, so daß i) und ii) erfüllt sind. Solche Elemente kann es nicht geben, da man sonst aus i) sofort einen Widerspruch dazu erhielte, daß $\bar{p}$ zyklisch reduziert ist.

Der Beweis wird nun induktiv über $\lambda(\bar{p})$ geführt. Für $\lambda(\bar{p})\leqq 2$ ist $\bar{p}$ zyklisch reduziert, und somit die Behauptung schon bewiesen.

Für $\lambda(\bar{q})<n$ sei alles gezeigt. Es sei $\lambda(\bar{p})=n$. Die Behauptung ist nur noch für den Fall zu zeigen, daß $\bar{p}$ nicht zyklisch reduziert ist. Deshalb wird angenommen, daß $\bar{p}$ eine Minimaldarstellung der Form $a^{\alpha}p'a^{\beta}$ besitzt, wobei α und β ungleich Null sind und verschiedene Vorzeichen haben. a sei nicht erster oder letzter Buchstabe von p', und o.B.d.A. $|\alpha|\leqq|\beta|$. Es gilt $\lambda(\bar{p}'\bar{a}^{\alpha+\beta})<\lambda(\bar{p})$. Es seien $\bar{v}$ und $\bar{r}$ mit $\bar{p}'\bar{a}^{\alpha+\beta}=\bar{v}\bar{r}\bar{v}^{-1}$ die nach Induktionsvoraussetzung eindeutig existierenden Elemente, die i) und ii) erfüllen. $\bar{a}^{\alpha}\bar{v}$ und $\bar{r}$ erfüllen dann i) und ii). Hierzu ist nur zu zeigen, daß wrw^{-1} eine Quasiminimaldarstellung von $\bar{p}$ ist, wenn w eine Minimaldarstellung von $\bar{a}^{\alpha}\bar{v}$ und r eine Minimaldarstellung von $\bar{r}$ ist. Nach Lemma 4 ist a nicht erster Buchstabe von $\bar{p}'\bar{a}^{\alpha+\beta}$. Da vrv^{-1} eine Quasiminimaldarstellung von $\bar{p}'\bar{a}^{\alpha+\beta}$ ist, wenn v eine Minimaldarstellung von $\bar{v}$ ist, folgt hieraus, daß a auch nicht erster Buchstabe von $\bar{v}$ ist. Nach Lemma 4 ist $a^{\alpha}v$ dann eine Minimaldarstellung von $\bar{a}^{\alpha}\bar{v}$. Nach Lemma 5 gelangt man nur durch Kommutieren von $a^{\alpha}v$ zu w. Wäre nun wrw^{-1} keine Quasiminimaldarstellung von $\bar{p}$, so auch $a^{\alpha}vr(a^{\alpha}v)^{-1}$ nicht. Da vrv^{-1} eine Quasiminimaldarstellung ist, müßte man nach Lemma 4 nach einigen Kommutationsschritten a in $a^{\alpha}vr(a^{\alpha}v)^{-1}$ kürzen können. Wegen Lemma 6 ist dies nicht möglich, da a nicht erster Buchstabe von $\bar{v}\bar{r}\bar{v}^{-1}$ und nicht letzter Buchstabe von $\bar{v}\bar{r}\bar{v}^{-1}$ oder letzter Buchstabe von $\bar{v}\bar{r}\bar{v}^{-1}$ zur Potenz $\alpha+\beta$ ist, wobei $\alpha+\beta$ das gleiche Vorzeichen wie $-\alpha$ hat.

Um den Beweis des Lemmas abzuschließen, bleibt zu zeigen, daß für alle Elemente $\bar{w}, \bar{p}'$ mit $\bar{p}=\bar{w}\bar{p}'\bar{w}^{-1}$, die i) und ii) erfüllen, $\bar{w}=\bar{a}^{\alpha}\bar{v}$ und $\bar{p}'=\bar{r}$ gilt.

w und p' seien Minimaldarstellungen von $\bar{w}$ bzw. $\bar{p}'$. Dann sind $wp'w^{-1}$ und $a^{\alpha}vr(a^{\alpha}v)^{-1}$ Quasiminimaldarstellungen von $\bar{p}$, d.h. sie lassen sich nach Lemma 5 durch Kommutieren ineinander überführen. Hieraus ergibt sich, daß a erster Buchstabe von w zur Potenz α ist.

Wäre dies nicht der Fall, so müßte in w ein Buchstabe b existieren, der nicht mit a kommutiert. Sonst wäre $\bar{p}'$ nicht zyklisch reduziert. Aus der Existenz von b in $\bar{w}$ würde dann ein Widerspruch dazu folgen, daß a erster Buchstabe von $\bar{p}$ zur Potenz α ist. Wenn $\bar{w}=\bar{a}^{\alpha}\bar{w}'$ mit $\lambda(\bar{w}')<\lambda(\bar{w})$, so folgt nach Induktionsvoraussetzung $\bar{w}'=\bar{v}$ und $\bar{p}'=\bar{r}$. q.e.d.

Mit Hilfe von Lemma 7 kann man zeigen:

LEMMA 8. *Wenn* ε *eine ganze Zahl ungleich Null ist, so gilt:*

i) $\bar{p}^{\varepsilon}$ *ist genau dann zyklisch reduziert, wenn* $\bar{p}$ *zyklisch reduziert ist.*

ii) $\bar{p}^{\varepsilon}$ *enthält dieselben Buchstaben wie* $\bar{p}$.

iii) *a ist genau dann erster (letzter) Buchstable von* $\bar{p}$ *zur Potenz* α, *wenn a erster (letzter) Buchstabe von* $\bar{p}^\varepsilon$ *zur Potenz* α *für* $\varepsilon > 0$ *ist, bzw. wenn a letzter (erster) Buchstabe von* $\bar{p}^\varepsilon$ *zur Potenz* $-\alpha$ *für* $\varepsilon < 0$ *ist.*

BEWEIS. Es sei $\bar{p} = \bar{w}\bar{r}\bar{w}^{-1}$ die nach Lemma 7 eindeutig existierende Darstellung von $\bar{p}$, so daß die Bedingungen i) und ii) von Lemma 7 erfüllt sind. Es genügt zu zeigen, daß $\bar{p}^\varepsilon = \bar{w}\bar{r}^\varepsilon\bar{w}^{-1}$ die entsprechende Darstellung von $\bar{p}^\varepsilon$ ist. Dies folgt mit Hilfe von Lemma 6.3 und Lemma 4. q.e.d.

LEMMA 9. *Es sei* ε *eine ganze Zahl ungleich Null. Weiterhin gelte* $\bar{p}\bar{q}^\varepsilon = \bar{q}^\varepsilon\bar{p}$. *Wenn a dann erster oder letzter Buchstabe von* $\bar{p}$ $(\bar{q})$ *ist und nicht in* $\bar{q}$ $(\bar{p})$ *vorkommt, so kommutiert a mit allen Buchstaben von* $\bar{q}$ $(\bar{p})$.

BEWEIS. Wegen Lemma 8 kann man o.B.d.A. $\varepsilon = 1$ annehmen. Weiter sei a o.B.d.A. erster Buchstabe von $\bar{p}$. Nach Lemma 6.2 ist die Folge der a-Potenzen von $\bar{q}\bar{p} = \bar{p}\bar{q}$ die Folge der a-Potenzen von $\bar{p}$. a muß dann erster Buchstabe von $\bar{p}\bar{q} = \bar{q}\bar{p}$ sein, wie aus Lemma 4 folgt. Nach Lemma 4 kommt man durch Kürzen und Kommutieren zu einer Minimaldarstellung von $\bar{q}\bar{p}$, bei der a erster Buchstabe ist. Alle Buchstaben, die dabei gekürzt werden, sind ungleich a und kommutieren mit a. Folglich müssen alle Buchstaben von $\bar{q}$ mit a kommutieren. q.e.d.

4. Das Haupttheorem

THEOREM 1. $\bar{p}$ *und* $\bar{q}$ *seien Elemente einer semifreien Gruppe* $\Gamma(\mathfrak{A})$. $\bar{p}$ *oder* $\bar{q}$ *sei zyklisch reduziert. Wenn* $\bar{p}$ *und* $\bar{q}$ *zusammenhängend sind, so gilt* $\bar{p}\bar{q}^\varepsilon = \bar{q}^\varepsilon\bar{p}$ *in* $\Gamma(\mathfrak{A})$ *genau dann, wenn in* $\Gamma(\mathfrak{A})$ *ein* $\bar{u}$ *und ganze Zahlen* α, β *mit* $\bar{p} = \bar{u}^\alpha$ und $\bar{q} = \bar{u}^\beta$ *existieren.*

BEWEIS. Es wird die nichttriviale Richtung gezeigt. Der Beweis wird induktiv über $\lambda(\bar{p}) + \lambda(\bar{q})$ geführt. Für $\lambda(\bar{p}) + \lambda(\bar{q}) \leqq 2$ ist die Behauptung klar. Nun sei für den Fall $\lambda(\bar{p}') + \lambda(\bar{q}') \leqq n$ alles bewiesen. $\bar{p}$ und $\bar{q}$ seien Elemente von $\Gamma(\mathfrak{A})$ mit $\lambda(\bar{p}) + \lambda(\bar{q}) = n + 1 > 2$. p und q seien Minimaldarstellungen von $\bar{p}$ bzw. $\bar{q}$. Die trivialen Fälle $\bar{p} = \bar{\varphi}, \bar{q} = \bar{\varphi}$ und $\varepsilon = 0$ seien ausgeschlossen. Zuerst wird gezeigt:

(1) Wenn a erster oder letzter Buchstabe von $\bar{p}$ $(\bar{q})$ ist, so kommt a in $\bar{q}$ $(\bar{p})$ vor.

Zum Beweis von (1) kann man wegen Lemma 8 $\varepsilon = 1$ annehmen. O.B.d.A. sei a erster Buchstabe von $\bar{p}$ zur Potenz α. Es wird angenommen, daß a nicht in $\bar{q}$ vorkommt. Nach Lemma 9 kommutiert a mit allen Buchstaben von $\bar{q}$. Wenn $\bar{p} = \bar{a}^\alpha\bar{p}'$ mit $\lambda(\bar{p}') < \lambda(\bar{p})$, so folgt $\bar{p}'\bar{q} = \bar{q}\bar{p}'$.

Zuerst wird angenommen, daß $\bar{q}$ oder $\bar{p}'$ zyklisch reduziert ist. $\bar{q} = \prod_{i=1}^{n} \bar{q}_i$ und $\bar{p}' = \prod_{i=1}^{n} \bar{r}_i$ seien Darstellungen von $\bar{q}$ und $\bar{p}'$, so daß $\bar{q}_i$ und $\bar{r}_i$ zusammenhängend sind, und alle Buchstaben von $\bar{q}_i$ und $\bar{r}_i$ mit allen Buchstaben von $\bar{q}_j$ und $\bar{r}_j$ für $j \neq i$ kommutieren. Man erhält diese Darstellungen, indem man den von den Buchstaben von $\bar{q}$ und $\bar{p}'$ erzeugten Teilgraphen von $\mathfrak{A}$ in seine Zusammenhängskomponenten zerlegt. Falls $\bar{q}$ zyklisch reduziert war, so sind nach Lemma 4 die $\bar{q}_i$ zyklisch reduziert, und falls $\bar{p}'$ zyklisch reduziert war, so sind die $\bar{r}_i$ zyklisch reduziert. Nach Lemma 1 und 2 gilt $\bar{q}_i\bar{r}_i=\bar{r}_i\bar{q}_i$. Da $\lambda(\bar{q}_i)+\lambda(\bar{r}_i)\leqq\lambda(\bar{q})+$ $+\lambda(\bar{p}')<\lambda(\bar{q})+\lambda(\bar{p})$, folgt nach Induktionsvoraussetzung die Existenz von Elementen $\bar{u}_i$ und ganzen Zahlen α_i, β_i $(1\leqq i\leqq n)$, so daß $\bar{r}_i=\bar{u}_i^{\alpha_i}$ und $\bar{q}_i=\bar{u}_i^{\beta_i}$. Hieraus ergibt sich ein Widerspruch, da $\bar{p}=\bar{a}^{\alpha}\prod_{i=1}^{n}\bar{u}^{\alpha_i}$ und $\bar{q}=\prod_{i=1}^{n}\bar{u}_i^{\beta_i}$ dann nicht zusammenhängend sein können. Wenn weder $\bar{p}'$ noch $\bar{q}$ zyklisch reduziert ist, dann ist nach Voraussetzung $\bar{p}$ zyklisch reduziert. Wenn $a_1^{\alpha_1}p_1a_1^{-\alpha_1}$ eine Minimaldarstellung von $\bar{p}'$ mit $\lambda(\bar{p}_1)<\lambda(\bar{p}')$ ist, so kommutiert a nicht mit a_1. Damit ist a_1 nicht aus $\bar{q}$. Nach Lemma 9 folgt dann aus $\bar{p}'\bar{q}=\bar{q}\bar{p}'$, daß a_1 mit allen Buchstaben von $\bar{q}$ kommutiert, und damit $\bar{p}_1\bar{q}=\bar{q}\bar{p}_1$ gilt. Induktiv über j kann die Existenz von Elementen $\bar{p}_j$, Buchstaben a_j und ganzen Zahlen α_j $(1\leqq j\leqq m)$ gezeigt werden, so daß gilt:

i) $\lambda(\bar{p}_{j-1})>\lambda(\bar{p}_j)$.

ii) $\bar{p}_{j-1}=\bar{a}_j^{\alpha_j}\bar{p}_j\bar{a}_j^{-\alpha_j}$.

iii) a_j kommutiert mit allen Buchstaben von $\bar{q}$ und kommt selbst nicht in $\bar{q}$ vor.

iv) $\bar{p}_m$ ist zyklisch reduziert, während die $\bar{p}_j$ mit $j<m$ nicht zyklisch reduziert sind.

Für $j<k$ seien $\bar{p}_j, a_j, \alpha_j$ konstruiert, und $\bar{p}_{k-1}$ sei nicht zyklisch reduziert. Aus den Bedingungen ii) und iii) folgt $\bar{q}\bar{p}_{k-1}=\bar{p}_{k-1}\bar{q}$. $a_k^{\alpha_k}p_ka_k^{-\alpha_k}$ mit $\lambda(\bar{p}_k)<\lambda(\bar{p}_{k-1})$ sei eine Minimaldarstellung von $\bar{p}_{k-1}$. Da $\bar{p}$ zyklisch reduziert ist, kann a_k nicht mit allen a_j für $j<k$ und a kommutieren. Damit ist a_k nach iii) kein Buchstabe aus $\bar{q}$. Nach Lemma 9 kommutiert a_k mit allen Buchstaben aus $\bar{q}$. i), ii) und iii) sind somit für $j=k$ erfüllt. Falls $\bar{p}_k$ zyklisch reduziert ist, sei $m=k$.

Aus ii) und iii) ergibt sich $\bar{q}\bar{p}_m=\bar{p}_m\bar{q}$. Wegen $\lambda(\bar{q})+\lambda(\bar{p}_m)<\lambda(\bar{q})+\lambda(\bar{p})$ kann man dann analog wie oben mit $\bar{q}$ und $\bar{p}'$ unter Ausnutzung der Induktionsvoraussetzung einen Widerspruch dazu konstruieren, daß $\bar{p}$ und $\bar{q}$ zusammenhängend sind.

Mit Hilfe von (1) ergibt sich:

(2) Wenn a erster (letzter) Buchstabe von $\bar{p}$ zur Potenz α ist, so ist a erster (letzter) Buchstabe von $\bar{q}$ oder $\bar{q}^{-1}$ zur Potenz α. Wenn a erster (letzter) Buchstabe von $\bar{q}$ zur Potenz α ist, so ist a erster (letzter) Buchstabe von $\bar{p}$ oder $\bar{p}^{-1}$ zur Potenz α.

Zum Beweis von (2) kann wegen Lemma 8 o.B.d.A. $\varepsilon=1$ angenommen werden. Weiterhin sei a o.B.d.A. erster Buchstabe von $\bar{p}$ zur Potenz α.

Falls a erster Buchstabe von $\bar{q}^{-1}$ zur Potenz α ist, so ist nichts zu zeigen. Gilt dies nicht, so erhält man die Folge der a-Potenzen von $\bar{q}\bar{p}=\bar{p}\bar{q}$ entsprechend Lemma 6.3. $\langle a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_n}\rangle$ sei die Folge der a-Potenzen von $\bar{p}$. Es gilt $n\geqq 1$ und $\alpha_1=\alpha$.

$\langle a^{\beta_1}, a^{\beta_2}, \ldots, a^{\beta_m}\rangle$ sei die Folge der a-Potenzen von $\bar{q}$. Wegen (1) gilt $m\geqq 1$.

1. *Fall.* Ist $\langle a^{\beta_1}, a^{\beta_2}, \ldots, a^{\beta_m}, a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_n}\rangle$ die Folge der a-Potenzen von $\bar{q}\bar{p}$, so folgt nach Lemma 4, daß $\langle a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_n}, a^{\beta_1}, a^{\beta_2}, \ldots, a^{\beta_m}\rangle$ die Folge der a-Potenzen von $\bar{p}\bar{q}=\bar{q}\bar{p}$ ist. Es ergibt sich $\alpha_1=\beta_1$. Würde es in $\bar{q}$ links von a^{β_1} noch einen Buschstaben b geben, der nicht mit a kommutiert, so würde sich bei Betrachtung der Folge der a- und b-Potenzen von $\bar{p}\bar{q}=\bar{q}\bar{p}$ unter Berücksichtigung von Lemma 4 und 5 ein Widerspruch ergeben.

2. *Fall.* Nach Lemma 6.3 kann noch der Fall eintreten, daß $\langle a^{\beta_1}, a^{\beta_2}, \ldots, a^{\beta_m+\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_n}\rangle$ die Folge der a-Potenzen von $\bar{q}\bar{p}$ ist. Die Folge von $\bar{p}\bar{q}=\bar{q}\bar{p}$ muß nach Lemma 4 die Gestalt $\langle a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_n+\beta_1}, a^{\beta_2}, \ldots, a^{\beta_m}\rangle$ haben.

Wenn nun $n=1$ oder $m=1$ gilt, so folgt, da sich $m\geqq 1$ aus (1) ergab, $n=m=1$. Für $n\geqq 2$ und $m\geqq 2$ beweist man (2) wie im 1. Fall.

Es wird gezeigt, daß der Fall $n=m=1$ nicht eintritt. Nach Lemma 4 kann $\bar{q}\bar{p}$ durch Kürzen und Kommutieren in eine Minimaldarstellung von $\bar{q}\bar{p}$ umgerechnet werden. Da a erster Buchstabe von $\bar{p}$ ist, kürzen sich hierbei nur Buchstaben, die mit a kommutieren. Um die angegebene Folge der a-Potenzen $\langle a^{\alpha_1+\beta_1}\rangle$ zu erhalten, muß a folglich letzter Buchstabe von $\bar{q}$ zur Potenz β_1 sein.

Nach Lemma 4 ergibt sich nun, daß $a^{\alpha_1+\beta_1}$ sowohl erster als auch letzter Buchstabe von $\bar{p}\bar{q}=\bar{q}\bar{p}$ ist. Da sich bei der Berechnung einer Minimaldarstellung von $\bar{p}\bar{q}=\bar{q}\bar{p}$ nur Buchstaben kürzen können, die mit a kommutieren, muß a mit allen Buchstaben von $\bar{p}$ und $\bar{q}$ kommutieren. Dies ergibt einen Widerspruch dazu, daß $\bar{p}$ und $\bar{q}$ zusammenhängend sind, und $\lambda(\bar{p})+\lambda(\bar{q})>2$ gilt.

$a^{\alpha}p'a^{\beta}$ mit $\lambda(\bar{p}')+1<\lambda(\bar{p})$ sei eine Minimaldarstellung von $\bar{p}$. Nach (2) ist a dann erster Buchstabe von $\bar{q}$ oder $\bar{q}^{-1}$ zur Potenz α. O.B.d.A. kann man annehmen daß $a^{\alpha}q'$ mit $\lambda(\bar{q}')<\lambda(\bar{q})$ eine Minimaldarstellung von $\bar{q}$ ist, da $\bar{q}^{\varepsilon}\bar{p}=\bar{p}\bar{q}^{\varepsilon}$ zu $\bar{q}^{-\varepsilon}\bar{p}=\bar{p}\bar{q}^{-\varepsilon}$ äquivalent ist. Aus

$$(\bar{a}^{\alpha}\bar{p}'\bar{a}^{\beta})(\bar{a}^{\alpha}\bar{q}')^{\varepsilon} = (\bar{a}^{\alpha}\bar{q}')^{\varepsilon}(\bar{a}^{\alpha}\bar{p}'\bar{a}^{\beta})$$

ergibt sich dann

$$(\bar{p}'\bar{a}^{\alpha+\beta})(\bar{q}'\bar{a}^{\alpha})^{\varepsilon} = (\bar{q}'\bar{a}^{\alpha})^{\varepsilon}(\bar{p}'\bar{a}^{\alpha+\beta}).$$

Nach Lemma 4 gilt $\lambda(\bar{p}'\bar{a}^{\alpha+\beta})<\lambda(\bar{p})$ und $\lambda(\bar{q}'a^{\alpha})\leqq\lambda(\bar{q})$. Falls $\bar{p}$ zyklisch reduziert ist, haben α und β gleiches Vorzeichen. Es gilt $a^{\alpha+\beta}\neq \text{ϙ}$. $\bar{p}'\bar{a}^{\alpha+\beta}$ und $\bar{q}'\bar{a}^{\alpha}$ sind dann zusammenhängend. Aus $\alpha+\beta\neq 0$ folgt weiterhin, daß $\bar{p}'\bar{p}^{\alpha+\beta}$ zyklisch reduziert ist. Falls $\bar{q}$ zyklisch reduziert ist, so ist a nicht letzter Buchstabe von $\bar{q}$ und $\bar{q}'$ zur Potenz $-\alpha$, und somit letzter Buchstabe von $\bar{q}'a^{\alpha}$. Damit ist dann $\bar{q}'\bar{a}^{\alpha}$ zyklisch reduziert, und $\bar{p}'\bar{a}^{\alpha+\beta}$ und $\bar{q}'\bar{a}^{\alpha}$ sind zusammenhängend.

Aus der Induktionsvoraussetzung folgt die Existenz eines $\bar{u}$ und ganzer Zahlen γ, δ, so daß

$$\bar{p}'\bar{a}^{\alpha+\beta} = \bar{u}^{\gamma} \quad \text{und} \quad \bar{q}'\bar{a}^{\alpha} = \bar{u}^{\delta}$$

gilt. Dies impliziert

$$\bar{p} = (\bar{a}^{\alpha}\bar{u}\bar{a}^{-\alpha})^{\gamma} \quad \text{und} \quad \bar{q} = (\bar{a}^{\alpha}\bar{u}\bar{a}^{-\alpha})^{\delta},$$

was zu beweisen war.

Analog kommt man zur Behauptung des Theorems, wenn $\bar{q}$ eine Minimaldarstellung der Gestalt $a^{\alpha}q'a^{\beta}$ mit $\lambda(\bar{q}')+1<\lambda(\bar{q})$ hat. Folglich kann für den weiteren Beweis vorausgesetzt werden, daß weder $\bar{p}$ noch $\bar{q}$ eine Minimaldarstellung der Form $a^{\alpha}ra^{\beta}$ mit $\lambda(\bar{r})+1<\lambda(\bar{p})$ bzw. $\lambda(\bar{r})+1<\lambda(\bar{q})$ besitzt. Unter dieser Voraussetzung kommutiert ein Buchstabe, der erster und letzter Buchstabe von $\bar{p}$ oder $\bar{q}$ ist, mit allen Buchstaben von $\bar{p}$ bzw. $\bar{q}$.

O.B.d.A. sei $\bar{p}=\bar{a}^{\alpha}\bar{p}'$ mit $\lambda(\bar{p}')<\lambda(\bar{p})$, und a kommutiert mit allen Buchstaben von $\bar{p}'$.

Nach (2) ist a o.B.d.A. erster Buchstabe von $\bar{q}$ zur Potenz α. Da $\bar{p}$ und $\bar{q}$ zusammenhängend sind, und $\lambda(\bar{p})+\lambda(\bar{q})>2$ gilt, kommutiert, a nicht mit allen Buchstaben von $\bar{q}$.

Nach obiger Voraussetzung ist a dann nicht letzter Buchstabe von $\bar{q}$. Wenn $\langle a^{\alpha}, a^{\beta_1}, \ldots, a^{\beta_n}\rangle$ die Folge der a-Potenzen von q^{ε} ist, so ist nach Lemma 6.3

$\langle a^{2\alpha}, a^{\beta_1}, \ldots, a^{\beta_n}\rangle$ die Folge der a-Potenzen von $\bar{p}\bar{q}^{\varepsilon}$, und nach Lemma 6.2 $\langle a^{\alpha}, a^{\beta_1}, \ldots, a^{\beta_n}, a^{\alpha}\rangle$ die Folge der a-Potenzen von $\bar{q}^{\varepsilon}\bar{p}$. Dies widerspricht Lemma 5. Folglich kann die obige Voraussetzung zu folgendem verschärft werden:

(3) Ein Buchstabe ist nicht zugleich erster und letzter Buchstabe von $\bar{p}$ oder $\bar{q}$.

Dann sind $\bar{p}$ und $\bar{q}$ und nach Lemma 8 auch $\bar{q}^{\varepsilon}$ zyklisch reduziert. Da q als Minimaldarstellung von $\bar{q}$ vorausgesetzt war, gilt:

(4) q^{ε} ist Minimaldarstellung von $\bar{q}^{\varepsilon}$.

Es sei a erster Buchstabe von $\bar{p}$ zur Potenz α. Nach (2) ist a dann erster Buchstabe von $\bar{q}$ oder $\bar{q}^{-1}$. Da $\bar{p}\bar{q}^{\varepsilon}=\bar{q}^{\varepsilon}\bar{p}$ zu $\bar{p}\bar{q}^{-\varepsilon}=\bar{q}^{-\varepsilon}\bar{p}$ äquivalent ist, kann für den weiteren Beweis angenommen werden, daß a erster Buchstabe von $\bar{q}$ zur Potenz α ist, und $\varepsilon>0$ gilt. Nach (3) ist a weder letzter Buchstabe von $\bar{p}$ noch von $\bar{q}$. $\{c_i : 1\leqq i\leqq f\}$ sei eine Numerierung der Buchstaben von $\bar{p}$ und $\bar{q}$, so daß für jedes $i>1$ ein $j<i$ mit $c_i c_j \neq c_j c_i$ existiert, und $c_1=a$ gilt. Diese existiert, da $\bar{p}$ und $\bar{q}$ zusammenhängend sind. Induktiv über $i\leqq f$ wird gezeigt:

(5) Die Folge der c_i-Potenzen in $\bar{p}\bar{q}^{\varepsilon}=\bar{q}^{\varepsilon}\bar{p}$ ergibt sich durch Hintereinanderschreiben der entsprechenden c_i-Potenzen von $\bar{p}$ und q^{ε}. Jedes c_i kommt in $\bar{p}$ und $\bar{q}$ vor.

Für $c_1=a$ folgt (5) aus Lemma 6.2. Nun sei (5) für $1\leqq j<i$ bewiesen. (5) ist für c_i zu zeigen. Falls in $\bar{p}$ oder $\bar{q}$ links oder rechts von allen c_i-Potenzen ein Buchstabe c_j mit $j<i$ vorkommt, der nicht mit c_i kommutiert, so ergibt sich die Behauptung schon aus der Induktionsvoraussetzung, (4) und Lemma 4.

Dies sei nun nicht der Fall. c_i komme in $\bar{p}$ vor. Dann existiert in $\bar{p}$ eine Folge $\langle b_h^{\beta_h}, b_{h-1}^{\beta_{h-1}}, \ldots, b_1^{\beta_1}\rangle$, so daß $b_1=c_i$, $b_1^{\beta_1}$ steht links von allen c_j-Potenzen mit $j<i$ und $c_j c_i \neq c_i c_j$, $b_k b_{k+1} \neq b_{k+1} b_k$, $b_{k-1}^{\beta_{k+1}}$ steht links von allen $b_k^{\beta_k}$ und b_h ist erster Buchstabe von $\bar{p}$. Nach Induktionsvoraussetzung und Lemma 4 steht $b_1^{\beta_1}$ dann auch in $\bar{p}\bar{q}^{\varepsilon}=\bar{q}^{\varepsilon}\bar{p}$, in $\bar{q}^{\varepsilon}$ und nach (4) in $\bar{q}$ links von allen c_j mit $j<i$ und $c_i c_j \neq c_j c_i$. Induktiv kann man in $\bar{p}\bar{q}^{\varepsilon}=\bar{q}^{\varepsilon}\bar{p}$ und nach (4) damit in $\bar{q}$ die Existenz der Folge $\langle b_h^{\beta_h}, b_{h-1}^{\beta_{h-1}}, \ldots, b_1^{\beta_1}\rangle$ links von allen c_j mit $j<i$ und $c_j c_i \neq c_i c_j$ nachweisen. Weiterhin ergibt sich, daß b_h erster Buchstabe von $\bar{p}\bar{q}^{\varepsilon}=\bar{q}^{\varepsilon}\bar{p}$ und $\bar{q}$ ist. Nach (3) ist b_h nicht letzter Buchstabe von $\bar{p}$ oder $\bar{q}$, und somit folgt (5) für b_h nach Lemma 6.2.

Wie im ersten Fall kann man nun zeigen, daß (5) für b_k gilt, wenn (5) für b_{k+1} gilt $(1\leqq k\leqq h)$. Damit ergibt sich (5) für c_i. Falls c_i in $\bar{q}$ vorkommt, geht man analog unter Ausnutzung von (4) vor.

Da p und q Minimaldarstellungen von $\bar{p}$ bzw. $\bar{q}$ sind, folgt aus Lemma 4, (4) und (5), daß man jede Minimaldarstellung von $\bar{p}\bar{q}^{\varepsilon}$ und $\bar{q}^{\varepsilon}\bar{p}$ durch Hintereinanderschreiben von p und q und Kommutieren erhält. Insbesondere gilt:

(6) pq^{ε} und $q^{\varepsilon}p$ sind Minimaldarstellungen von $\bar{p}\bar{q}^{\varepsilon}=\bar{q}^{\varepsilon}\bar{p}$. Aus (6) folgt die Behauptung von Theorem 1 sehr einfach:

1. *Fall.* Für jeden letzten Buchstaben a von $\bar{p}$ oder $\bar{q}$ sind die Folgen der a-Potenzen in $\bar{p}$ und $\bar{q}$ gleich lang. Nach (2), (3) und (5) ist ein Buchstabe genau dann letzter Buchstabe von $\bar{p}$, wenn er letzter Buchstabe von $\bar{q}$ ist. Da die Potenzen eines Buchstaben in $\bar{p}$ und in $\bar{q}$ links von einem letzten Buchstaben liegen, ergibt sich aus (6), daß $p\approx q$ gilt. Damit ist das Theorem in diesem Falle bewiesen.

2. *Fall.* Es existiert ein letzter Buchstabe a von $\bar{p}$, so dad ß ie Folgeder a-Potenzen in $\bar{p}$ länger ist als die Folge der a-Potenzen in $\bar{q}$. Dies werde durch $\lambda_a(\bar{p}) > \lambda_a(\bar{q})$ abgekürzt.

$\{c_i : 1 \leqq i \leqq f\}$ sei wieder eine Aufzählung der Buchstaben aus $\bar{p}$ und $\bar{q}$, so daß für jedes $i > 1$ ein j mit $j < i$ und $c_j c_i \neq c_i c_j$ existiert, und $c_1 = a$ gilt. Da $\bar{p}$ und $\bar{q}$ zusammenhängend sind, ist es möglich, eine solche Aufzählung zu erhalten. Induktiv über i wird

$$(7) \qquad \lambda_{c_i}(\bar{p}) > \lambda_{c_i}(\bar{q})$$

gezeigt. Es wird angenommen, daß (7) für $j \leqq i$ nachgewiesen ist. Für $i=1$ ist (7) vorausgesetzt, c_j mit $j \leqq i$ sei ein Buchstabe, so daß $c_j c_{i+1} \neq c_{i+1} c_j$ gilt.

Liegen in $\bar{q}$ alle c_{i+1}-Potenzen links von der k-ten c_j-Potenz, so folgt aus der Induktionsvoraussetzung und (6), daß in $\bar{p}$ links von der k-ten c_j-Potenz dieselbe Folge von c_{i+1}-Potenzen liegt. Da $\lambda_{c_j}(\bar{p}) > \lambda_{c_j}(\bar{q})$, muß in $\bar{p}$ aber auch rechts von dieser k-ten c_j-Potenz noch eine c_{i+1}-Potenz vorkommen. Dies folgt, wenn man die Enden der Minimaldarstellungen $pq^\varepsilon \approx q^\varepsilon p$ vergleicht. Nun sei in $\bar{q}$ die k-te c_j-Potenz die letzte c_j-Potenz, und c_{i+1} komme noch rechts von dieser vor. Aus (6) ergibt sich, daß in $\bar{p}$ links von der nach Induktionsvoraussetzung existierenden $(k+1)$-ten c_j-Potenz sämtliche c_{i+1}-Potenzen stehen, die auch in $\bar{q}$ vorkommen. Beachtet man wiederum die Enden von $q^\varepsilon p \approx pq^\varepsilon$, so sieht man wegen (6), daß in $\bar{p}$ rechts von der $(k+1)$-ten c_j-Potenz weitere c_{i+1}-Potenzen vorkommen. Hiermit ist (7) für c_{i+1} bewiesen. Aus (7) und (6) ergibt sich die Existenz eines Wortes w, so daß qw eine Minimaldarstellung von $\bar{p}$ ist, und $\lambda(\bar{w}) < < \lambda(\bar{p})$ gilt. Aus $\bar{p}\bar{q}^\varepsilon = \bar{q}^\varepsilon \bar{p}$ folgt dann $\bar{q}\bar{w}\bar{q}^\varepsilon = \bar{q}^\varepsilon \bar{q}\bar{w}$ und somit $\bar{w}\bar{q}^\varepsilon = \bar{q}^\varepsilon \bar{w}$ mit $\lambda(\bar{w}) + \lambda(\bar{q}) < \lambda(\bar{p}) + \lambda(\bar{q})$. Nach Induktionsvoraussetzung existieren ein Wort u und ganze Zahlen γ, δ, so daß $\bar{w} = \bar{u}^\gamma$ und $\bar{q} = \bar{u}^\delta$ und damit $\bar{p} = \bar{u}^{\gamma+\delta}$, womit der Beweis in diesem Fall abgeschlossen ist.

3. *Fall.* Es existiert ein letzter Buchstabe a von $\bar{q}$, so daß die Folge der a- Potenzen in $\bar{q}$ länger als die in $\bar{p}$ ist. Analog wie im 2. Fall kann

(8) $\lambda_b(\bar{q}) > \lambda_b(\bar{p})$ für alle Buchstaben b aus $\bar{p}$ und $\bar{q}$

gezeigt werden. Aus (8) und (6) folgt die Existenz eines Wortes w, so da*ß* pw eine Minimaldarstellung von $\bar{q}$ ist, und $\lambda(\bar{w}) < \lambda(\bar{q})$ gilt. Aus $\bar{p}\bar{q}^\varepsilon = \bar{q}^\varepsilon \bar{p}$ folgt dann $\bar{p}(\bar{p}\bar{w})^\varepsilon = (\bar{p}\bar{w})^\varepsilon \bar{p}$, $(\bar{p}\bar{w})^\varepsilon = (\bar{w}\bar{p})^\varepsilon$ und hieraus mit Hilfe von Lemma 6 und 7 $\bar{p}\bar{w} = \bar{w}\bar{p}$. Sind nämlich $\bar{p}\bar{w} = \bar{v}\bar{r}\bar{v}^{-1}$ und $\bar{w}\bar{p} = \bar{t}\bar{s}\bar{t}^{-1}$ die nach Lemma 7 eindeutig existierenden Darstellungen von $\bar{p}\bar{w}$ und $\bar{w}\bar{p}$, die den Bedingungen 7 i) und 7 ii) genügen, so sind $\bar{v}\bar{r}^\varepsilon \bar{v}^{-1}$ und $\bar{t}\bar{s}^\varepsilon \bar{t}^{-1}$ die entsprechenden eindeutig bestimmten Darstellungen von $(\bar{p}\bar{w})^\varepsilon$ bzw. $(\bar{w}\bar{p})^\varepsilon$. Aus $(\bar{p}\bar{w})^\varepsilon = (\bar{w}\bar{p})^\varepsilon$ ergibt sich $t = \bar{v}$ und $\bar{r}^\varepsilon = \bar{s}^\varepsilon$ und nach Lemma 6.3 $\bar{r} = \bar{s}$, da $\bar{r}$ und $\bar{s}$ zyklisch reduziert sind. Analog wie im 2. Fall folgt die Behauptung aus der Induktionsvoraussetzung. q.e.d.

5. Schlußfolgerungen

THEOREM 2. *$\bar{p}\bar{q} = \bar{q}\bar{p}$ gilt in einer semifreien Gruppe genau dann, wenn es Elemente $\bar{w}$ und $\bar{u}_i$ sowie ganze Zahlen α_i, β_i für $1 \leqq i \leqq n$ gibt, so daß gilt:*

1) *Für $i \neq j$ kommutiert jeder Buchstabe aus $\bar{u}_i$ mit jedem Buchstaben aus $\bar{u}_j$.*

2) *Jedes* $\bar{u}_i$ *ist zusammenhängend.*

3) $\bar{p} = \bar{w} \prod_{i=1}^{n} \bar{u}_i^{\alpha_i} \bar{w}^{-1}$ *und* $\bar{q} = \bar{w} \prod_{i=1}^{n} \bar{u}_i^{\beta_i} \bar{w}^{-1}$.

BEWEIS. $\bar{p}=\bar{w}\bar{p}'\bar{w}^{-1}$ sei die in Lemma 7 angegebene eindeutig existierende Darstellung von $\bar{p}$. $\bar{p}'$ ist dann zyklisch reduziert. Aus $(\bar{w}\bar{p}'\bar{w}^{-1})\bar{q}=\bar{q}(\bar{w}\bar{p}'\bar{w}^{-1})$ ergibt sich $\bar{p}'(\bar{w}^{-1}\bar{q}\bar{w})=(\bar{w}^{-1}\bar{q}\bar{w})\bar{p}'$. Weiterhin sei $\bar{p}'=\prod_{i=1}^{n}\bar{p}_i$ und $\bar{w}^{-1}\bar{q}\bar{w}=\prod_{i=1}^{n}\bar{q}_i$, wobei $\bar{p}_i$ und $\bar{q}_i$ zusammenhängend sind, und die Buchstaben von $\bar{p}_i$ und $\bar{q}_i$ mit allen Buchstaben von $\bar{p}_j$ und $\bar{q}_j$ für $j\neq i$ kommutieren. Da $\bar{p}'$ zyklisch reduziert ist, sind auch die $\bar{p}_i$ zyklisch reduziert. Aus Lemma 2 folgt $\bar{p}_i\bar{q}_i=\bar{q}_i\bar{p}_i$. Nach Theorem 1 gibt es Elemente $\bar{u}_i$ und ganze Zahlen α_i, β_i, so daß $\bar{p}_i=\bar{u}_i^{\alpha_i}$ und $\bar{q}_i=\bar{u}_i^{\beta_i}$ gilt. Hieraus erhält man die Behauptung von Theorem 2. q.e.d.

KOROLLAR 1. *Für jedes zusammenhängende zyklisch reduzierte* $\bar{p}$ *existieren eindeutig bestimmt ein* $\bar{u}$ *und eine ganze Zahl* $\alpha\geqq 0$, *so daß* $\bar{p}=\bar{u}^{\alpha}$ *gilt, und es für jedes* $\bar{v}$ *und jedes* β *mit* $\bar{v}^{\beta}=\bar{p}$ *ein* γ *mit* $\bar{u}^{\gamma}=\bar{v}$ *gibt.*

BEWEIS. Es sei $\bar{v}^{\delta}=\bar{w}^{\varepsilon}=\bar{p}$. Da $\bar{p}$ zusammenhängend und zyklisch reduziert ist, sind $\bar{v}$ und $\bar{w}$ zyklisch reduziert und zusammenhängend. Es ergibt sich $\bar{w}\bar{v}^{\delta}= =\bar{w}^{\varepsilon+1}=\bar{v}^{\delta}\bar{w}$. Dann folgt aus Theorem 1 die Existenz eines Wortes $\bar{r}$ und ganzer Zahlen η, μ, so daß $\bar{w}=\bar{r}^{\eta}$ und $\bar{v}=\bar{r}^{\mu}$ gilt. Da ein Element einer semifreien Gruppe nur endlich viele Wurzeln besitzt, wie man mit Hilfe von Lemma 7 leicht zeigt, kann man durch Anwendung der obigen Überlegung in endlich vielen Schritten ein $\bar{u}$ finden, das Wurzel aller Wurzeln von $\bar{p}$ ist.

Die Eindeutigkeit von $\bar{u}$ und $\alpha\geqq 0$ ergibt sich aus Lemma 6. q.e.d.

KOROLLAR 2. *Aus* $\bar{p}^{\alpha}\bar{q}^{\beta}=\bar{q}^{\beta}\bar{p}^{\alpha}$ *folgt* $\bar{p}\bar{q}=\bar{q}\bar{p}$, *wenn* α *und* β *ungleich Null sind.*

BEWEIS. Es wird angenommen, daß $\bar{p}$ zyklisch reduziert ist. Ist dies nicht der Fall, so betrachtet man entsprechende automorphe Bilder von $\bar{p}$ und $\bar{q}$. Lemma 2 rechtfertigt es weiterhin anzunehmen, daß $\bar{p}$ und $\bar{q}$ zusammenhängend sind. Nach Theorem 1 existiert dann ein Wort $\bar{v}$ und ganze Zahlen γ, δ, so daß $\bar{p}^{\alpha}=\bar{v}^{\delta}$ und $\bar{q}=\bar{v}^{\delta}$. Auf $\bar{v}\bar{p}^{\alpha}=\bar{p}^{\alpha}\bar{v}$ wird nun noch einmal Theorem 1 angewandt. Es folgt die Existenz eines $\bar{u}$ und ganzer Zahlen η, μ, so daß $\bar{p}=\bar{u}^{\eta}$ und $\bar{v}=\bar{u}^{\mu}$. Hieraus ergibt sich die Behauptung. q.e.d.

Eine Darstellung $\bar{p}=\bar{w}\prod_{i=1}^{n}\bar{u}_i^{\alpha_i}\bar{w}^{-1}$ ist eine Standarddarstellung von $\bar{p}$, wenn gilt:

i) Wenn w eine Minimaldarstellung von $\bar{w}$ und u_i eine Minimaldarstellung von $\bar{u}_i$ $(1\leqq i\leqq n)$ ist, so ist $w\prod_{i=1}^{n}u_i^{\alpha_i}w^{-1}$ eine Quasiminimaldarstellung von $\bar{p}$.

ii) Für jedes i $(1\leqq i\leqq n)$ ist $\bar{u}_i$ zyklisch reduziert und zusammenhängend. Jeder Buchstabe von $\bar{u}_i$ kommutiert mit allen Buchstaben von $\bar{u}_j$ für $j\neq i$.

iii) $\alpha_i\geqq 0$, und für jedes $\bar{v}$ und jede ganze Zahl α mit $\bar{u}_i=\bar{v}^{\alpha}$ gilt $\bar{u}_i=\bar{v}$ oder $\bar{u}_i=\bar{v}^{-1}$ und $\alpha=1$ bzw. $\alpha=-1$.

LEMMA 10. *Jedes* $\bar{p}$ *besitzt eine eindeutig bestimmte Standarddarstellung.*

BEWEIS. $\bar{p}=\bar{w}\bar{p}'\bar{w}^{-1}$ sei die in Lemma 7 angegebene eindeutig bestimmte Darstellung von $\bar{p}$. Dann existiert eine eindeutig bestimmte Zerlegung $\bar{p}'=\prod_{i=1}^{n}\bar{v}_i$, wobei jedes $\bar{v}_i$ zyklisch reduziert und zusammenhängend ist, und jeder Buchstabe von $\bar{v}_i$ mit jedem Buchstaben von $\bar{v}_j$ für $j\neq i$ kommutiert. Wenn man dann $\bar{u}_i$ und α_i mit $\bar{v}_i=\bar{u}_i^{\alpha_i}$ gemäß Korollar 1 wählt, folgt die Behauptung. q.e.d.

Aus Theorem 1 ergibt sich:

KOROLLAR 3. *Wenn* $\bar{p}=\bar{w}\prod_{i=1}^{n}\bar{u}_i^{\alpha_i}\bar{w}^{-1}$ *die Standarddarstellung von* $\bar{p}$ *ist, und* $\bar{q}$ *mit* $\bar{p}$ *kommutiert, so existieren ganze Zahlen* β_i *und ein* $\bar{z}$, *dessen Buchstaben mit den Buchstaben der* $\bar{u}_i$ $(1\leqq i\leqq n)$ *kommutieren und in keinem* $\bar{u}_i$ *vorkommen, daß* $\bar{q}=\bar{w}\prod_{i=1}^{n}\bar{u}_i^{\beta_i}\bar{z}\bar{w}^{-1}$ *gilt.*

Zwei Standarddarstellungen $\bar{p}=\bar{w}\prod_{i=1}^{n}\bar{u}_i^{\alpha_i}\bar{w}^{-1}$ und $\bar{q}=\bar{v}\prod_{i=1}^{m}\bar{r}_i^{\beta_i}\bar{v}^{-1}$ sind verträglich, wenn gilt:

i) Es existieren $\bar{t}$, $\bar{v}_1$ und $\bar{w}_1$, so daß $\bar{w}=\bar{t}\bar{w}_1$ und $\bar{v}=\bar{t}\bar{v}_1$ gilt, alle Buchstaben von $\bar{w}_1$ mit allen Buchstaben der $\bar{r}_j$ $(1\leqq j\leqq m)$ und $\bar{v}_1$ kommutieren, und alle Buchstaben von $\bar{v}_1$ mit allen Buchstaben der $\bar{u}_i$ $(1\leqq i\leqq n)$ und $\bar{w}_1$ kommutieren.

ii) Für jedes i $(1\leqq i\leqq n)$ kommutieren die Buchstaben von $\bar{u}_i$ mit allen Buchstaben der $\bar{r}_j$ $(1\leqq j\leqq m)$ und sind von diesen verschieden, oder es gibt ein j mit $1\leqq j\leqq m$, so daß $\bar{u}_i=\bar{r}_j$ oder $\bar{u}_i=\bar{r}_j^{-1}$ gilt.

KOROLLAR 4. *Zwei Elemente* $\bar{p}$ *und* $\bar{q}$ *kommutieren genau dann, wenn ihre Standarddarstellungen verträglich sind.*

BEWEIS. Daß zwei Elemente mit verträglichen Standarddarstellungen kommutieren, ist klar.

Nun seien zwei kommutierende Elemente $\bar{p}, \bar{q}$ mit den Standarddarstellungen $\bar{p}=\bar{w}\prod_{i=1}^{n}\bar{u}_i^{\alpha_i}\bar{w}^{-1}$ und $\bar{q}=\bar{v}\prod_{i=1}^{m}\bar{r}_i^{\beta_i}\bar{v}^{-1}$ gegeben. Nach Korollar 3 existieren ganze Zahlen γ_i $(1\leqq i\leqq m)$ und ein $\bar{z}$, dessen Buchstaben mit den Buchstaben der $\bar{r}_i$ $(1\leqq i\leqq m)$ kommutieren und von diesen verschieden sind, so daß

$$\bar{p} = \bar{v}\prod_{i=1}^{m}\bar{r}_i^{\gamma_i}\bar{z}\bar{v}^{-1}$$

gilt. $\bar{z}=\bar{w}_1\prod_{i=1}^{k}\bar{s}_i^{\delta_i}\bar{w}_1^{-1}$ sei die Standarddarstellung von $\bar{z}$. Es ergibt sich $\bar{p}= =(\bar{v}\bar{w}_1)\prod_{i=1}^{m}\bar{r}_i^{\gamma_i}\prod_{i=1}^{k}\bar{s}_i^{\delta_i}(\bar{v}\bar{w}_1)^{-1}$. Es sei tv_1 eine Minimaldarstellung von $\bar{v}$, so daß alle Buchstaben von $\bar{v}_1$ mit allen Buchstaben von $\bar{w}_1$, $\bar{r}_i^{\gamma_i}$ $(1\leqq i\leqq m)$ und $\bar{s}_i$ $(1\leqq i\leqq k)$ kommutieren, und es zu jedem letzten Buchstaben von $\bar{t}$ einen Buchstaben von $\bar{w}_1$, einem $\bar{r}_i^{\gamma_i}$ $(1\leqq i\leqq m)$ oder einem $\bar{s}_i$ $(1\leqq i\leqq k)$ gibt, der nicht mit ihm kommutiert. Um die Verträglichkeit der Standarddarstellungen von $\bar{p}$ und $\bar{q}$ zu beweisen, genügt es jetzt zu zeigen:

(1) $\bar{p}=(\bar{t}\bar{w}_1)\prod_{i=1}^{m}\bar{r}_i'^{\gamma_i'}\prod_{i=1}^{k}\bar{s}_i^{\delta_i}(\bar{t}\bar{w}_1)^{-1}$ ist die Standarddarstellung von $\bar{p}$, wobei $\gamma_i'\geqq 0$, und $\bar{r}_i'=\bar{r}_i$ und $\gamma_i'=\gamma_i$ oder $\bar{r}_i'=\bar{r}_i^{-1}$ und $\gamma_i'=-\gamma_i$ gilt.

Wenn w_1 eine Minimaldarstellung von $\bar{w}_1$ ist, so ist tw_1 eine Minimaldarstellung von $\bar{t}\bar{w}_1$. Dies ergibt sich sofort aus Lemma 4, da ein letzter Buchstabe von $\bar{t}$ nicht aus $\bar{w}_1$ sein kann. Wäre ein letzter Buchstabe von $\bar{t}$ aus $\bar{w}_1$, so würde er mit jedem Buchstaben von $\bar{r}_i$ $(1\leqq i\leqq m)$ und von $\bar{v}_1$ kommutieren, und $\bar{t}\bar{v}_1\prod_{i=1}^{m}\bar{r}_i^{\beta_i}(\bar{t}\bar{v}_1)^{-1}$ wäre keine Standarddarstellung von $\bar{q}$. Die Gültigkeit von (1) ergibt sich dann sofort aus:

(2) $tw_1\prod_{i=1}^{m}r_i'^{\delta_i'}\prod_{i=1}^{k}s_i^{\delta_i}(tw_1)^{-1}$ ist eine Quasiminimaldarstellung von $\bar{p}$, wenn r_i' und s_i Minimaldarstellungen von $\bar{r}_i'$ bzw. $\bar{s}_i$ sind.

Da $\prod_{i=1}^{m}r_i'^{\delta_i'}\prod_{i=1}^{k}s_i^{\delta_i}$ nach Konstruktion eine Quasiminimaldarstellung ist, genügt es zu zeigen:

(3) Ein letzter Buchstabe von $\bar{t}\bar{w}_1$ kommutiert nicht mit allen Buchstaben von $\prod_{i=1}^{m}r_i'^{\delta_i'}\prod_{i=1}^{k}s_i^{\delta_i}$.

(4) Wenn a letzter Buchstabe von $\bar{t}\bar{w}_1$ zur Potenz α und erster Buchstabe von $\bar{r}_i'^{\gamma_i'}$ oder $\bar{s}_i^{\delta_i}$ zur Potenz β ist, so haben α und β gleiches Vorzeichen. Wenn a letzter Buchstabe von $\bar{t}\bar{w}_1$ zur Potenz α und letzter Buchstabe von $\bar{r}_i'^{\gamma_i'}$ oder $\bar{s}_i^{\delta_i}$ zur Potenz γ ist, so haben α und γ verschiedene Vorzeichen.

Um (3) und (4) nachzuweisen, wird angenommen, daß a letzter Buchstabe von $\bar{t}\bar{w}_1$ ist. Da tw_1 eine Minimaldarstellung von $\bar{t}\bar{w}_1$ ist, ist a dann entweder letzter Buchstabe von $\bar{w}_1$, oder a kommutiert mit allen Buchstaben aus $\bar{w}_1$ und ist letzter Buchstabe von $\bar{t}$.

Falls a letzter Buchstabe aus $\bar{w}_1$ ist, so folgen (3) und (4) daraus, daß a dann mit allen Buchstaben der $\bar{r}_i$ $(1\leqq i\leqq m)$ kommutiert und von diesen verschieden ist, und $\bar{w}_1\prod_{i=1}^{k}\bar{s}_i^{\delta_i}\bar{w}_1^{-1}$ eine Standarddarstellung ist. Ist a letzter Buchstabe von $\bar{t}$, so ist a nicht aus $\bar{z}$, da a sonst mit jedem Buchstaben von $\bar{v}_1$ und von $\bar{r}_i$ $(1\leqq i\leqq m)$ kommutieren würde, und $tv_1\prod_{i=1}^{m}r_i^{\beta_i}(tv_1)^{-1}$ keine Quasiminimaldarstellung von $\bar{q}$ wäre.

Es folgt, daß man sich beim Nachweis von (4) auf die $\bar{r}_i'^{\gamma_i'}$ $(1\leqq i\leqq m)$ beschränken kann. Ist a erster oder letzter Buchstabe eines $\bar{r}_i'^{\gamma_i'}$, so kommutiert a mit allen Buchstaben von $\bar{v}_1$, und (4) ergibt sich mit Hilfe von Lemma 8 daraus, daß $tv_1\prod_{i=1}^{m}r_i^{\beta_i}(tv_1)^{-1}$ eine Quasiminimaldarstellung ist. (3) ist in diesem Fall durch die Wahl von t und v_1 automatisch erfüllt. q.e.d.

LEMMA 11. *Sind die Standarddarstellungen* $\bar{p}=\bar{w}\prod_{i=1}^{n}\bar{u}_i^{\alpha_i}\bar{w}^{-1}$ *und* $\bar{q}=\bar{v}\prod_{i=1}^{m}\bar{r}_i^{\beta_i}\bar{v}^{-1}$ *verträglich, und sind* $\bar{t}, \bar{w}_1$ *und* $\bar{v}_1$ *so gewählt, daß* $\bar{t}\bar{w}_1=\bar{w}$ *und* $\bar{t}\bar{v}_1=\bar{v}$ *gilt, und jeder Buchstabe von* $\bar{w}_1$ *mit jedem Buchstaben von* $\bar{r}_i$ $(1\leqq i\leqq m)$ *und von* $\bar{v}_1$ *und jeder Buchstabe von* $\bar{v}_1$ *mit jedem Buchstaben von* $\bar{u}_i$ $(1\leqq i\leqq n)$ *und von* $\bar{w}_1$ *kommutiert, so kann* $\bar{p}\bar{q}=(\bar{t}\bar{w}_1\bar{v}_1)\prod_{i=1}^{n}\bar{u}_i^{\alpha_i}\prod_{i=1}^{m}\bar{r}_i^{\beta_i}(\bar{t}\bar{w}_1\bar{v}_1)^{-1}$ *in die Standarddarstellung von* $\bar{p}\bar{q}$ *umgeformt werden, wenn man in Fällen* $\bar{u}_i=\bar{r}_j$ *oder* $\bar{u}_i=\bar{r}_j^{-1}$ *die entsprechenden Potenzen geeignet zusammenfaßt.*

Lemma 11 wird ähnlich wie Korollar 4 bewiesen.

THEOREM 3. *Jede abelsche Untergruppe* $\mathfrak{B}$ *einer semifreien Gruppe ist frei-abelsch.*

BEWEIS. V sei die Menge aller $\bar{w}\bar{u}\bar{w}^{-1}$, wenn $\bar{w}\prod_{i=1}^{n}\bar{r}_i^{\beta_i}\bar{u}^{\alpha}\bar{w}^{-1}$ die Standarddarstellung eines Elements $\bar{p}$ von $\mathfrak{B}$ ist. Wenn $\bar{w}\bar{u}\bar{w}^{-1}$ und $\bar{v}\bar{r}\bar{v}^{-1}$ Elemente aus V sind, so folgt aus Korollar 4:

(1) Wenn $\bar{u}$ und $\bar{r}$ einen gemeinsamen Buchstaben besitzen, so gilt $\bar{u}=\bar{r}$ und $\bar{w}\bar{u}\bar{w}^{-1}=\bar{v}\bar{r}\bar{v}^{-1}$ oder $\bar{u}=\bar{r}^{-1}$ und $\bar{w}\bar{u}\bar{w}^{-1}=(\bar{v}\bar{r}\bar{v}^{-1})^{-1}$. Haben $\bar{u}$ und $\bar{r}$ keinen gemeinsamen Buchstaben, so kommutieren alle Buchstaben von $\bar{u}$ mit allen Buchstaben von $\bar{r}$.

U sei eine Teilmenge von V, die von jedem Paar $\bar{w}\bar{u}\bar{w}^{-1}$ und $(\bar{w}\bar{u}\bar{w}^{-1})^{-1}$ aus V genau ein Element enthält.

$\mathfrak{B}$ ist dann Untergruppe der von U erzeugten Gruppe $\mathfrak{C}$. Es wird gezeigt, daß $\mathfrak{C}$ die direkte Summe der freien zyklischen Gruppen ist, die durch die Elemente von U erzeugt werden. Als Untergruppe der frei-abelschen Gruppe $\mathfrak{C}$ ist dann auch $\mathfrak{B}$ frei-abelsch.

Zuerst wird nachgewiesen, daß die Elemente von U kommutieren. Wenn $\bar{w}\bar{u}\bar{w}^{-1}$ und $\bar{v}\bar{r}\bar{v}^{-1}$ Elemente aus U sind, so gibt es nach Korollar 4 Elemente $\bar{w}_1$ und $\bar{v}_1$, so daß $\bar{w}\bar{v}_1=\bar{v}\bar{w}_1$ gilt, und alle Buchstaben von $\bar{w}_1$ mit allen Buchstaben von $\bar{r}$ und alle Buchstaben von $\bar{v}_1$ mit allen Buchstaben von $\bar{u}$ kommutieren. Hieraus folgt

$$\bar{w}\bar{u}\bar{w}^{-1}\bar{v}\bar{r}\bar{v}^{-1}=\bar{w}\bar{v}_1\bar{u}\bar{v}_1^{-1}\bar{w}^{-1}\bar{v}\bar{w}_1\bar{r}\bar{w}_1^{-1}\bar{v}^{-1}=\bar{v}\bar{w}_1\bar{u}\bar{r}\bar{v}_1^{-1}\bar{w}^{-1}$$

und wegen (1)

$$=\bar{v}\bar{w}_1\bar{r}\bar{u}\bar{v}_1^{-1}\bar{w}^{-1}=\bar{v}\bar{w}_1\bar{r}\bar{w}_1^{-1}\bar{v}^{-1}\bar{w}\bar{v}_1\bar{u}\bar{v}_1^{-1}\bar{w}^{-1}=\bar{v}\bar{r}\bar{v}^{-1}\bar{w}\bar{u}\bar{w}^{-1}.$$

Es bleibt zu zeigen, daß die Elemente von U linear unabhängig sind.

Es gelte $\prod_{i=1}^{n}(\bar{w}_i\bar{u}_i\bar{w}_i^{-1})^{\alpha_i}=\bar{\varphi}$, wobei die $\bar{w}_i\bar{u}_i\bar{w}_i^{-1}$ verschiedene Elemente aus U sind. Wegen Lemma 11 und (1) findet man ein $\bar{w}$, so daß $\bar{w}\bar{u}_i\bar{w}^{-1}=\bar{w}_i\bar{u}_i\bar{w}_i^{-1}$ gilt. Dann ergibt sich $\bar{w}\left(\prod_{i=1}^{n}\bar{u}_i^{\alpha_i}\right)\bar{w}^{-1}=\bar{\varphi}$. Nach Anwendung des entsprechenden Automorphismus folgt $\prod_{i=1}^{n}\bar{u}_i^{\alpha_i}=\bar{\varphi}$. Aus (1) und Lemma 2 folgt dann $\alpha_i=0$. q.e.d.

THEOREM 4. *Die elementare Theorie der semifreien Gruppen ist im Sinne von* [1] *universell und damit erblich unentscheidbar.*

Einen Beweis von Theorem 4 findet man in [1], S. 149—150. Das hier bewiesene Theorem 2 liefert dabei den noch ausstehenden Beweis von Lemma 1 S. 149 in [1].

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HUMBOLDT-UNIVERSITÄT
SEKTION MATHEMATIK, BEREICH LOGIK
1068 BERLIN, UNTER DEN LINDEN 6
DDR

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A NOTE ON HILBERT CLASS FIELDS OF ALGEBRAIC NUMBER FIELDS

By
K. GYŐRY (Debrecen) and W. LEAHEY (El Paso)

Let L be an algebraic number field with the following properties:

(1) L is totally imaginary,

(2) L contains a totally real subfield L_0 such that $[L:L_0]=2$ and,

(3) L contains a subfield K such that L is the Hilbert class field of K.

Then K also satisfies (1) and (2). A great number of such pairs of number fields L and K are known (see e.g. [16], [1], [7], [8], [17] and the references given in these works). In this note we show (Theorem1) that the number of number fields L of bounded degree which satisfy (1),(2), and (3) are always finite. We also point out that aside from two exceptional cases these fields L and their subfields K, over which they are Hilbert class fields, can be effectively determined. In Theorem 2 we list all the imaginary bicyclic biquadratic fields which have the above properties.

In the following h_M and D_M denote the class number and the absolute value of the discriminant of an algebraic number field M, respectively.

THEOREM 1. *For any positive integer N there are only finitely many number fields L of degree less than or equal to N which satisfy conditions* (1), (2), *and* (3). *Moreover, for given N all such pairs of fields L and K with $[L:Q]\leqq N$ can be effectively determined except possibly*

(i) *one pair L, K in which K is imaginary quadratic field with $h_K \geqq 3$, and*

(ii) *at most* $30N^2 \log^2 (13N/6)$ *pairs L, K in which K are imaginary bicyclic biquadratic fields with $h_K \geqq 2$.*

PROOF. Let L be an algebraic number field of degree $\leqq N$ with properties (1), (2), and (3). Since L is the Hilbert class field of K it must be unramified at the infinite primes of K. Hence K also must be totally imaginary. Let $[K:Q]=2n$ and denote by K_0 the maximal real subfield of K. Then K_0 is totally real and $[K:K_0]=2$. (See e.g.[3].) Since $D_L = D_K^{h_K}$ and $h_K=[L:K]\leqq N/2n$, it is sufficient to show that for given N there are only finitely many totally imaginary number fields K of degree $2n\leqq N$ with $h_K \leqq N/2n$, which contain totally real subfields K_0 such that $[K:K_0]=2$. In the following K denotes such a number field.

By a theorem of H. M. STARK [11], for $0<\varepsilon\leqq 1/2$ there is an effectively computable function $c(\varepsilon)>0$ such that

$$h_K > [n(n!)]^{-1} c(\varepsilon)^n D_{K_0}^{\frac{1}{2}-\frac{1}{n}-\varepsilon} f^{\frac{1}{2}-\frac{1}{2n}}$$

where $D_K = D_{K_0}^2 f$. This implies

$$D_K^{\frac{1}{4}-\frac{1}{2n}-\frac{\varepsilon}{2}} < \frac{1}{2} n!\, c(\varepsilon)^{-n} N.$$

By taking ε to be some fixed positive number less than $1/6$ it follows from this last inequality that for given N there are only finitely many number fields K of degree $2n \geqq 6$ which have the required property. All these fields K can be effectively determined.

Suppose now that $2n=[K:Q]=2$. Such fields K with $h_K \leqq 2$ are explicitly listed by H. M. STARK [10], [12], and H. L. MONTGOMERY and P. J. WEINBERGER [6]. In the general case (when $h_K \leqq N/2$), by a theorem of T. TATUZAWA [13], we have

$$D_K \leqq 525 N^2 \log^2 (13N/2)$$

for any K, with one possible exception.

Suppose now $2n=[K:Q]=4$. Let $h_1 = h_K/h_{K_0}$ be the relative class number of K. Since $h_K \leqq N/4$ we also have $h_1 \leqq N/4$. Then by a result of K. UCHIDA ([15], p. 352, Corollary) if K is non-abelian, D_K is bounded above by some effectively determined number which depends only on N. If K is cyclic then by another result of K. UCHIDA ([14],Theorem 1′) there is an effectively computable upper bound for the conductor of K, i.e. for the smallest positive integer k for which K is contained in the field of k^{th} roots of unity. Again this upper bound depends only on N. Further, $D_K < k^{\varphi(k)}$. In both cases the number of possible fields K is finite and these fields can be effectively determined.

There are only finitely many imaginary bicyclic biquadratic number fields of class number 1 and all of these are known (see E. BROWN and C. J. PARRY [2]).

There remains the case when K is imaginary bicyclic biquadratic and $h_K \geqq 2$. Denote by $K_1 = Q(\sqrt{-D_1})$ and $K_2 = Q\sqrt{-D_2})$ the two imaginary quadratic subfields of K with discriminants $-D_1$ and $-D_2$ respectively. Then $D_2^2 \leqq D_K \leqq$ $\leqq (D_1 D_2)^2$ and, by a well-known theorem (see, for example, H. HASSE [5], pp. 72—78), $h_{K_1} h_{K_2} | 2h_K$, whence $h_{K_1} h_{K_2} \leqq N/2$. If neither K_1 nor K_2 is exceptional in the above sense, then we can give an explicitly calculated upper bound for D_K, which depends only on N. Therefore, these fields K can be effectively determined.

Let us now assume that K_2 is exceptional in the above sense. Then $h_{K_2} \geqq 3$ and $h_{K_1} \leqq N/6$. Thus, by Tatuzawa's theorem quoted above, we have

$$|D_1| \leqq 2100 (N/6)^2 \log^2 (13N/6).$$

Further, $-D_1 \equiv 0$ or $1 \pmod 4$. Thus, as is stated in (ii), the number of the exceptional imaginary bicyclic biquadratic fields K is less than $30N^2 \log^2 (13N/6)$. This completes the proof of the first part of Theorem 1.

Finally, consider those K above which can be effectively determined. We can now effectively determine, for any of these K, those possible extensions L of K for which $D_L = D_K^{h_K}$, $[L:Q] = h_K [K:Q]$, and (1), (2) hold. Moreover, for any such extension L of a given K we can always decide whether L is the Hilbert class field of K. Consequently, apart from the possible exceptions stated in (i) and (ii), we can determine all the pairs L, K with the required properties.

REMARK 1. An algorithm for determining all imaginary quadratic fields with class number h for any given h would provide an algorithm for finding all L and K which fall into cases (i) and (ii) of the theorem.

REMARK 2. A large collection of class fields over totally imaginary quadratic extensions of totally real fields can be obtained by complex multiplication of abelian varieties. (See [16], [4], [1], [7], [8], [9] and the references given there.)

By Theorem 1 there are only finitely many number fields L of degree $\leqq 4$ with properties (1), (2) and (3) and these fields L can be effectively determined. We now consider the special case where the L are imaginary bicyclic biquadratic fields. For brevity we write (a, b) for $Q(\sqrt{a}, \sqrt{b})$ and (a) for $Q(\sqrt{a})$. We have then the following

THEOREM 2. *There are exactly* 50 *imaginary bicyclic biquadratic number fields which are Hilbert class fields over one of their subfields. They are*

(i) $(-1, -2)$, $(-1, -3)$, $(-1, -7)$, $(-1, -11)$, $(-1, -19)$, $(-1, -43)$, $(-1, -67)$, $(-1, -163)$, $(-2, -3)$, $(-2, -7)$, $(-2, -11)$, $(-2, -19)$, $(-2, -43)$, $(-2, -67)$, $(-3, -7)$, $(-3, -11)$, $(-3, -19)$, $(-3, -43)$, $(-3, -67)$, $(-3, -163)$, $(-7, -11)$, $(-7, -19)$, $(-7, -43)$, $(-7, -163)$, $(-11, -19)$, $(-11, -67)$, $(-11, -163)$, $(-19, -67)$, $(-19, -163)$, $(-43, -67)$, $(-43, -163)$, $(-67, -163)$.

These fields are Hilbert class fields only over themselves.

(ii) $(-1, 5)$, $(-1, 13)$, $(-1, 37)$, $(-2, 5)$, $(-2, 29)$, $(-3, 2)$, $(-3, 5)$, $(-3, 17)$, $(-3, 41)$, $(-3, 89)$, $(-7, 5)$, $(-7, 13)$, $(-7, 61)$, $(-11, 2)$, $(-11, 17)$. *These fields are Hilbert class fields both over themselves and over the quadratic subfields* (-5), (-13), (-37), (-10), (-58), (-6), (-15), (-51), (-123), (-267), (-35), (-91), (-427), (-22), (-187) *respectively.*

(iii) $(-23, 5)$, $(-31, 13)$, $(-47, 5)$. *These fields are Hilbert class fields over the quadratic subfields* (-115), (-403), (-235) *respectively and only over these subfields.*

REMARK 3. It is easy to see that the fields listed in (ii) and (iii) are exactly those biquadratic fields which have properties (1) and (2) and which are Hilbert class fields over one of their proper subfields.

PROOF of THEOREM 2. By the result of E. BROWN and C. J. PARRY [2] the fields listed in (i) and (ii) are exactly the imaginary bicyclic biquadratic number fields of class number 1, that is exactly those imaginary bicyclic biquadratic number fields which are Hilbert class fields over themselves.

Suppose now that L is an imaginary bicyclic biquadratic field such that it is the Hilbert class field of one, say K, of its proper subfields. Let $K = Q(\sqrt{-D})$ where $-D$ is the discriminant of K. Then since $h_K = 2$, $-D$ must be one of the numbers -15, -20, -24, -35, -40, -51, -52, -88, -91, -115, -123, -148, -187, -232, -235, -267, -403, -427. (See H. M. STARK [12] and H. L. MONTGOMERY and P. J. WEINBERGER [6]). Each $-D$ can be written uniquely in the form $-D = p_1^* p_2^*$ where p_1^* and p_2^* are distinct prime discriminants. By a well-known theorem the genus fields Γ of K over Q is $Q(\sqrt{p_1^*}, \sqrt{p_2^*})$. Since Γ is contained in L and $[\Gamma : Q] = 4$, $\Gamma = L$. Therefore we can easily list all the imaginary bicyclic biquadratic number fields L which are Hilbert class fields over imaginary quadratic fields. The imaginary quadratic fields of class number 2 listed above are the only proper subfields of the fields L occurring in (ii) and (iii) whose Hilbert class fields are these fields L.

COROLLARY. *There are exactly* 11 *Dirichlet biquadratic number fields* $Q(\sqrt{d}, \sqrt{-d}) = Q(\sqrt{-1}, \sqrt{-d})$ *(d square free integer) which are Hilbert class fields. They are*

(i) $(-1, -2)$, $(-1, -3)$, $(-1, -7)$, $(-1, -11)$, $(-1, -19)$, $(-1, -43)$, $(-1, -67)$, $(-1, -163)$. *These fields are Hilbert class fields only over themselves.*

(ii) $(-1, -5)$, $(-1, -13)$, $(-1, -37)$. *These fields are Hilbert class fields over themselves and over the quadratic subfields* $(-5,)$ (-13), (-37) *respectively.*

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DEBRECEN
4010 DEBRECEN
HUNGARY

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TEXAS AT EL PASO
EL PASO, TEXAS 79968
USA

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ON CRITICALLY HAMILTONIAN GRAPHS

By
LINDA LESNIAK-FOSTER (Baton Rouge)

A *hamiltonian cycle (hamiltonian path)* of a graph G is a cycle (path) containing all the vertices of G. A graph possessing a hamiltonian cycle is called a *hamiltonian graph*. If v is a vertex of a graph G, then $G-v$ denotes the subgraph of G with vertex set $V(G)-\{v\}$ and whose edges are all those of G not incident with v. A graph G is called *critically hamiltonian* if G is hamiltonian and for each vertex v of G, the graph $G-v$ is not hamiltonian. Thus, for example, every cycle is a critically hamiltonian graph. It is known (see [1]) that if a graph G of order $p\geqq 4$ has at least $(p^2-3p+8)/2$ edges, then G is hamiltonian and $G-v$ is hamiltonian for each vertex v of G. Therefore the size (number of edges) of a critically hamiltonian graph of order $p\geqq 4$ is clearly less than $(p^2-3p+8)/2$. The purpose of this paper is to establish several results concerning the size of critically hamiltonian graphs of given order. In this regard, the following lemma will be useful.

LEMMA. *If G is a non-hamiltonian graph of order $p\geqq 3$ which contains a hamiltonian path $v_1, v_2, \ldots, v_p$, then $\deg_G v_1+\deg_G v_p\leqq p-1$. Moreover, if $\deg_G v_1+\deg_G v_p=p-1$, then the following must hold:*

(i) *if $v_1v_k\notin E(G)$, $2\leqq k\leqq p$, then $v_{k-1}v_p\in E(G)$,*

(ii) *if p is even and $\deg_G v_1\geqq\deg_G v_p$, then v_1v_l and v_1v_{l+1} are edges of G for some l satisfying $2\leqq l\leqq p-2$.*

PROOF. That $\deg_G v_1+\deg_G v_p\leqq p-1$ follows from a result in [2, p. 55]. Suppose $\deg_G v_1+\deg_G v_p=p-1$. Define the sets V_1 and V_2, where

$$V_1=\{v_i \mid 1\leqq i\leqq p-1 \text{ and } v_1v_{i+1}\in E(G)\}$$

and

$$V_2=\{v_i \mid 1\leqq i\leqq p-1 \text{ and } v_iv_{i+1}\notin E(G)\}.$$

Then $V_1\cup V_2=\{v_1, v_2, \ldots, v_{p-1}\}$ and $V_1\cap V_2=\emptyset$, implying that $|V_1|+|V_2|=p-1$. Moreover, $|V_1|=\deg_G v_1$. Since $\deg_G v_1+\deg_G v_p=p-1$, we conclude that $\deg_G v_p=|V_2|$. Now, v_p is adjacent to no vertex of V_1; for otherwise, there is an integer i $(1\leqq i\leqq p-1)$ such that $v_1v_{i+1}, v_pv_i\in E(G)$. But then G contains the hamiltonian cycle $v_1, v_2, \ldots, v_i, v_p, v_{p-1}, \ldots, v_{i+1}, v_1$ which is a contradiction. Therefore v_p is adjacent only to vertices of V_2. Since $\deg_G v_p=|V_2|$, the vertex v_p is adjacent to every vertex of V_2. Thus if $v_1v_k\notin E(G)$, $2\leqq k\leqq p$, we have that $v_{k-1}\in V_2$ so that $v_{k-1}v_p\in E(G)$.

In order to prove part (ii) we note that the result holds for $p=4$. Thus we assume that p is even and $p\geqq 6$. Since $\deg_G v_1\geqq\deg_G v_p$ and $\deg_G v_1+\deg_G v_p=p-1$, we have that $\deg_G v_1\geqq(p-1)/2$, which implies that $\deg_G v_1\geqq p/2$. Therefore v_1

is adjacent to at least $p/2$ vertices in the set $\{v_2, v_3, ..., v_{p-1}\}$ since $v_1v_p \notin E(G)$. But $|\{v_2, v_3, ..., v_{p-1}\}|=p-2$, so that v_1 is adjacent to both v_l and v_{l+1} for some l satisfying $2 \leqq l \leqq p-2$.

We now present an upper bound on the size of a critically hamiltonian graph in terms of the order of the graph. The corollary which follows is an immediate consequence of the method of proof employed in the theorem.

THEOREM 1. *If G is a critically hamiltonian graph of order $p \geqq 4$, then $|E(G)| \leqq p^2/4$.*

PROOF. Let $C: u_1, u_2, ..., u_p, u_1$ be a hamiltonian cycle of G. For each i, with $1 \leqq i \leqq p$, we wish to consider the sum $\deg_G u_i + \deg_G u_{i+2}$ (all subscripts expressed modulo p). Since G is hamiltonian, $G-u_{i+1}$ is not hamiltonian. However, $G-u_{i+1}$ contains a hamiltonian u_i-u_{i+2} path. Using the lemma we obtain the inequality

$$\deg_{G-u_{i+1}} u_i + \deg_{G-u_{i+1}} u_{i+2} \leqq (p-1)-1 = p-2.$$

Therefore, $\deg_G u_i + \deg_G u_{i+2} \leqq p$, and so

$$\sum_{i=1}^{p} (\deg_G u_i + \deg_G u_{i+2}) \leqq p^2.$$

Since

$$\sum_{i=1}^{p} (\deg_G u_i + \deg_G u_{i+2}) = 2 \sum_{i=1}^{p} \deg_G u_i = 4|E(G)|,$$

the proof is complete.

COROLLARY. *If G is a critically hamiltonian graph of order $p \geqq 4$ and size $p^2/4$ and $C: u_1, u_2, ..., u_p, u_1$ is a hamiltonian cycle of G, then $\deg_G u_i + \deg_G u_{i+2} = p$ (all subscripts expressed modulo p) for $1 \leqq i \leqq p$.*

Since, for even $p \geqq 4$, the complete bipartite graph $K(p/2, p/2)$ is a critically hamiltonian graph of order p and size $p^2/4$, the bound given in Theorem 1 is sharp for even values of p. We observe that if G is a critically hamiltonian graph of odd order p, then $|E(G)| \leqq [p^2/4] < p^2/4$. We improve this bound in Theorem 2.

THEOREM 2. *If G is a critically hamiltonian graph of odd order $p \geqq 5$, then $|E(G)| \leqq [p(p-1)/4]$.*

PROOF. Since the only critically hamiltonian graph of order five is the 5-cycle C_5 and $|E(C_5)| = 5 = [5(5-1)/4]$, we will assume $p \geqq 7$. Let $C: z_1, z_2, ..., z_p, z_1$ be a hamiltonian cycle of G and let i be an arbitrary integer satisfying $1 \leqq i \leqq p$. We consider the sum $\deg_G z_i + \deg_G z_{i+2}$ (all subscripts expressed modulo p). Without loss of generality, we may assume that $\deg_G z_{i+2} \geqq \deg_G z_i$. Relabel the vertices of G in the following fashion:

$$u_1 = z_{i+2}, u_2 = z_{i+3}, u_3 = z_{i+4}, \dots, u_{p-1} = z_i, u_p = z_{i+1}.$$

Then $u_1, u_2, ..., u_p, u_1$ is a hamiltonian cycle of G and $\deg_G u_1 \geqq \deg_G u_{p-1}$. We wish to show that $\deg_G u_1 + \deg_G u_{p-1} \leqq p-1$.

Since G is critically hamiltonian, $G-u_p$ is not hamiltonian. However, $G-u_p$ contains a hamiltonian u_1-u_{p-1} path. Using the lemma, we obtain the inequality $\deg_{G-u_p}u_1+\deg_{G-u_p}u_{p-1}\leqq(p-1)-1=p-2$. Thus $\deg_G u_1+\deg_G u_{p-1}\leqq p$. Suppose $\deg_G u_1+\deg_G u_{p-1}=p$. Then $\deg_{G-u_p}u_1+\deg_{G-u_p}u_{p-1}=(p-2)=(p-1)-1$ and $\deg_{G-u_p}u_1\geqq\deg_{G-u_p}u_{p-1}$. Since $|V(G-u_p)|$ is even, we may apply parts (i) and (ii) of the lemma. By part (ii), we have that $u_1u_l, u_1u_{l+1}\in E(G-u_p)$ for some l satisfying $2\leqq l\leqq(p-1)-2$, so that $u_1u_l, u_1u_{l+1}\in E(G)$. Since G is critically hamiltonian, $u_1u_3\notin E(G)$. Thus $u_1u_l, u_1u_{l+1}\in E(G-u_p)$ for some l satisfying $4\leqq l\leqq p-3$. Now, since $u_1u_3\notin E(G)$, u_1 and u_3 are not adjacent in $G-u_p$. Therefore, by part (i), u_2 and u_{p-1} are adjacent in $G-u_p$. But then $u_2, u_3, \dots, u_l, u_1, u_{l+1}, \dots, u_{p-1}, u_2$ is a hamiltonian cycle of $G-u_p$, which is a contradiction. Therefore, we must have $\deg_G u_1+\deg_G u_{p-1}\leqq p-1$.

In terms of the original labeling of the vertices of G, we have the inequality $\deg_G z_{i+2}+\deg_G z_i\leqq p-1$. Since i was arbitrary,

$$\sum_{i=1}^{p}(\deg_G z_{2i-1}+\deg_G z_{2i+1})\leqq p(p-1).$$

Equivalently, $2\sum_{i=1}^{p}\deg_G z_i\leqq p(p-1)$. Since $2\sum_{i=1}^{p}\deg_G z_i=4|E(G)|$, the proof is complete.

The bound given in Theorem 2 is best possible for $p=5$ and $p=7$. However, for larger values of odd p, this does not seem to be the case.

By Theorem 1, if G is a critically hamiltonian graph of order $p\geqq4$, then $|E(G)|\leqq p^2/4$. Our final theorem characterizes critically hamiltonian graphs which attain this bound.

THEOREM 3. *A graph G of order $p\geqq4$ and size $p^2/4$ is critically hamiltonian if and only if G is isomorphic to $K(p/2, p/2)$.*

PROOF. We first observe that if a graph has order p and size $p^2/4$, then p is even. As noted previously, the graph $K(p/2, p/2)$ is a critically hamiltonian graph of order p and size $p^2/4$, for even $p\geqq4$.

Conversely, let G be a critically hamiltonian graph of order $p\geqq4$ and size $p^2/4$. If $p=4$, then G is isomorphic to $K(2, 2)$. So we may assume that $p\geqq6$.

Let $C: z_1, z_2, \dots, z_p, z_1$ be a hamiltonian cycle of G and let i be an arbitrary integer satisfying $1\leqq i\leqq p$. We will show that

$$\{w\in V(G)|z_iw\in E(G)\}=\{w\in V(G)|z_{i+2}w\in E(G)\}=\{z_{i+3}, z_{i+5}, \dots, z_{i-1}, z_{i+1}\},$$

where all subscripts are expressed modulo p. Relabel the vertices of G in the following fashion:

$$u_1=z_{i+2}, u_2=z_{i+3}, u_3=z_{i+4}, \dots, u_{p-1}=z_i, u_p=z_{i+1}.$$

We note that $u_1u_p, u_{p-1}u_p\in E(G)$.

Since G is critically hamiltonian, $G-u_p$ is not hamiltonian. However, $G-u_p$ contains a hamiltonian u_1-u_{p-1} path. Applying the lemma, we obtain the inequality $\deg_{G-u_p}u_1+\deg_{G-u_p}u_{p-1}\leqq(p-1)-1$. However, we must have $\deg_{G-u_p}u_1+\deg_{G-u_p}u_{p-1}=(p-1)-1$; for otherwise, $\deg_G u_1+\deg_G u_{p-1}<p$, which contradicts the corollary to Theorem 1. Since G is critically hamiltonian, $u_1u_3, u_{p-3}u_{p-1}\notin E(G)$.

Therefore $u_1u_3, u_{p-3}u_{p-1} \notin E(G-u_p)$. By two applications of part (i) of the lemma, we have that $u_2u_{p-1}, u_1u_{p-2} \in E(G-u_p)$. Thus $u_2u_{p-1}, u_1u_{p-2} \in E(G)$.

We now observe that if $u_1u_l \in E(G)$ for some l satisfying $2 \leqq l \leqq p-2$, then $u_1u_{l+1} \notin E(G)$; for otherwise, $u_2, u_3, ..., u_l, u_1, u_{l+1}, ..., u_{p-1}, u_2$ is a hamiltonian cycle of $G-u_p$, which contradicts the fact that G is critically hamiltonian. Similarly, if $u_{p-1}u_l \in E(G)$ for some l satisfying $2 \leqq l \leqq p-2$, then $u_{p-1}u_{l-1} \notin E(G)$; for otherwise, $u_{p-2}, u_{p-3}, ..., u_l, u_{p-1}, u_{l-1}, ..., u_1, u_{p-2}$ is a hamiltonian cycle of $G-u_p$.

Since G is critically hamiltonian, $u_1u_{p-1} \notin E(G)$. Thus u_1 is adjacent to exactly $\deg_G u_1 - 1$ vertices in the set $\{u_2, u_3, ..., u_{p-2}\}$ and u_{p-1} is adjacent to exactly $\deg_G u_{p-1} - 1$ vertices in the set $\{u_2, u_3, ..., u_{p-2}\}$. Using the observations made above together with the fact that $\deg_G u_1 + \deg_G u_{p-1} = p$, we conclude that $\deg_G u_1 = \deg_G u_{p-1} = p/2$ and

$$\{w \in V(G) | u_{p-1}w \in E(G)\} = \{w \in V(G) | u_1 w \in E(G)\} = \{u_2, u_4, u_6, ..., u_p\}.$$

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LOUISIANA STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS
BATON ROUGE, LOUISIANA 70803
USA

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SPLINE FUNCTIONS AND THE CAUCHY PROBLEMS. II

APPROXIMATE SOLUTION OF THE DIFFERENTIAL EQUATION $y''=f(x, y, y')$ WITH SPLINE FUNCTIONS

By

T. FAWZY (Cairo)

1. Introduction and description of the method

The problem of approximating the solution of the non-linear second order differential equations was always of special interest. The problem was solved by K. D. SHARMA and R. G. GUPTA [4] by a one-step method depending upon the Lobatto four-point quadrature formula in which the function f is necessary to be sufficiently differentiable. The same problem has been also solved by GH. MICULA [2], [3] using spline functions, but under the restrictions that the first derivative is absent i.e. $y''=f(x, y)$ and that $f\in C^2$ at least. He has constructed a spline function of degree $m\geqq 3$ which approximates the solution of the Cauchy problem $y''=f(x, y)$, $y(x_0)=y_0$, $y'(x_0)=y_0'$ and his convergence theorem is as follows.

THEOREM 1.1 (Micula). *Let $s:[a, b]\to R$ be the constructed spline function. If $f\in C^{m-1}$ where $m\geqq 3$, then there is an $h_0>0$ such that for all $H\leqq h_0$ and for all $x\in[a, b]$, there exists the constants K_1, K_2, K_3 such that*

$$|s(x)-y(x)|< K_1H^m, \quad |s'(x)-y'(x)|< K_2H^{m-1}, \quad |s''(x)-y''(x)|< K_3H^{m-1}$$

where $y(x)$ is the exact solution.

More details on this theorem may be found in [3].

In this paper, following the same method presented in [5], we are going to approximate the solution of the prolem avoiding the restrictions in the above theorem and proving theorems of high rate of convergence and of best approximation.

For this purpose, consider the Cauchy problem in the non-linear ordinary differential equation

$$y''=f[x, y(x), y'(x)], \quad y(x_0)=y_0, \quad y'(x_0)=y_0' \tag{1.1}$$

where $f\in C^r([0, b]\times R\times R)$ and r is a finite positive integer, or zero. (The case $r=0$ is solved in [5].)

If $S(x)$ is the spline function approximating the solution of (1, 1), it satisfies

$$S(x)\in C^{r+2}[0, b] \tag{1.2}$$

$$S(x)\in \Pi_m \text{ in each subinterval } [x_k, x_{k+1}],\ k=0, 1, \dots, n-1, \tag{1.3}$$

where we define the knots by

$$0=x_0<x_1<\dots<x_n=b \tag{1.4}$$

and in our case we shall deal with equal subintervals and denote

$$x_{k+1}-x_k=h, \quad k=0, 1, \dots, n-1. \tag{1.5}$$

Also in what follows $c_0, c_1, c_2, \dots$ will denote constans independent of h and consequently independent of n.

Here Π_m denotes the set of all polynomials of degree $\leqq m$, and $m=2r+5$.

We assume that (1.1) represents a single scalar equation, but nearly all of the numerical and theoretical considerations in this paper carry over to systems of second order equations where (1.1) could be treated in vector form. Moreover we shall use the Lipschitz condition on f to guarantee the existence of a unique analytical solution of (1.1).

As in [5], our method of approximating the solution of (1.1) will be divided into two main approximation processes the first of which is to obtain, numerically, the approximate values $\bar{y}_k, \bar{y}'_k, \dots, \bar{y}_k^{(r+2)}$ where $k=0, 1, \dots, n$. The second approximation process contains the construction of the spline function approximating the solution and the convergence of this function to the exact solution.

2. The first approximation process

This section contains some assumptions concerning the function f and a method for obtaining the approximate values $\bar{y}_k, \bar{y}'_k, \dots, \bar{y}_k^{(r+2)}$ where $k=1, 2, \dots, n$ and also we discuss the convergence of these approximate values to the exact values.

2.1. Assumptions and procedures of the method. In this paper we assume that $f[x, y(x), y'(x)]$ satisfies the conditions

$$f[x, y(x), y'(x)] \in C^r([0, b] \times R \times R) \tag{2.1.1}$$

together with the Lispchitz conditions

$$|f^{(q)}(x, y, y'_1) - f^{(q)}(x, y_2, y'_2)| \leqq K(|y_1 - y_2| + |y'_1 - y'_2|) \tag{2.1.2}$$

for all $x \in [0, b]$ and all y_1, y'_1, y_2, y'_2 in R, where K is some Lipschitz constant and $q=0, 1, 2, \dots, r$. We assume also that $y^{(r+2)}(x) \equiv f^{(r)}$ has a modulus of continuity $\omega_r(f^{(r)}, h) = \omega_r(h)$.

Let $y(x)$ be the exact solution of (1.1) with the initial conditions $y(0)=y_0$ and $y'(0)=y'_0$. Then by integrating (1.1) from x_k to x where $x_k \leqq x \leqq x_{k+1}$, $k=0, 1, 2, \dots, n-1$, we get

$$y'(x) = y'_k + \int_{x_k}^{x} f\{t, y(t), y'(t)\}\, dt = \tag{2.1.3}$$

$$= y'_k + \int_{x_k}^{x} f\{t, y(t), y'_k + \int_{x_k}^{t} f[u, y(u), y'(u)]\, du\}\, dt$$

and

$$y(x) = y_k + y'_k(x - x_k) + \int_{x_k}^{x} \int_{x_k}^{t} f\{u, y(u), y'(u)\}\, du\, dt = \tag{2.1.4}$$

$$= y_k + y'_k(x - x_k) + \int_{x_k}^{x} \int_{x_k}^{t} f\{u, y(u), y'_k + \int_{x_k}^{u} f[v, y(v), y'(v)]\, dv\}\, du\, dt$$

and $y(x), y'(x)$ have the following Taylor expansion

$$(2.1.5)\quad y(x)=\sum_{j=0}^{r+1}\frac{y_k^{(j)}}{j!}(x-x_k)^j+\frac{y^{(r+2)}(\xi_k)}{(r+2)!}(x-x_k)^{r+2},\quad x_k<\xi_k<x_{k+1}$$

and

$$(2.1.6)\quad y'(x)=\sum_{j=0}^{r}\frac{y_k^{(j+1)}}{j!}(x-x_k)^j+\frac{y^{(r+2)}(\eta_k)}{(r+1)!}(x-x_k)^{r+1},\quad x_k<\eta_k<x_{k+1}$$

where $x_k\leqq x\leqq x_{k+1}$ and $k=0, 1, 2, \ldots, n-1$, and these expansions may be approximated by using the approximate values $\bar{y}_k, \bar{y}'_k, \ldots, \bar{y}_k^{(r+2)}$ to get

$$(2.1.7)\quad y_k^*(x)=\sum_{j=0}^{r+2}\frac{\bar{y}_k^{(j)}}{j!}(x-x_k)^j,\quad x_k\leqq x\leqq x_{k+1}$$

and

$$(2.1.8)\quad y_k^{*\prime}(x)=\sum_{j=0}^{r+1}\frac{\bar{y}_k^{(j+1)}}{j!}(x-x_k)^j,\quad x_k\leqq x\leqq x_{k+1}.$$

Setting $x=x_{k+1}$ in (2.1.4) and (2.1.3), we get

$$(2.1.9)\quad y(x_{k+1})=y_{k+1}=y_k+y'_kh+\int_{x_k}^{x_{k+1}}\int_{x_k}^{t} f\{u, y(u), y'(u)\}\,du\,dt=$$

$$=y_k+y'_kh+\int_{x_k}^{x_{k+1}}\int_{x_k}^{t} f\{u, y(u), y'_k+\int_{x_k}^{u} f[v, y(v), y'(v)]\,dv\}\,du\,dt$$

where $x_k\leqq v\leqq u\leqq t\leqq x_{k+1}$ and

$$(2.1.10)\quad y'(x_{k+1})=y'_{k+1}=y'_k+\int_{x_k}^{x_{k+1}} f\{t, y(t), y'(t)\}\,dt=$$

$$=y'_k+\int_{x_k}^{x_{k+1}} f\{t, y(t), y'_k+\int_{x_k}^{t} f[u, y(u), y'(u)]\,du\}\,dt$$

where $x_k\leqq u\leqq t\leqq x_{k+1}$.

Define the approximate values $\bar{y}_{k+1}, \bar{y}'_{k+1}$ by

$$(2.1.11)\quad \bar{y}_{k+1}=\bar{y}_k+\bar{y}'_kh+\int_{x_k}^{x_{k+1}}\int_{x_k}^{t} f\{u, y_k^*(u), y_k^{**\prime}(u)\}\,du\,dt$$

and

$$(2.1.12)\quad \bar{y}'_{k+1}=\bar{y}'_k+\int_{x_k}^{x_{k+1}} f\{ty_k^*(t), y_k^{**\prime}(t)\}\,dt$$

where

$$y_k^{**\prime}(x)=\bar{y}'_k+\int_{x_k}^{x}[f(t), y_k^*(t), y_k^{*\prime}(t)]\,dt.$$

Finally, by knowing that $y^{(q+2)}=f^{(q)}(x, y, y')$ which is expressed as a function of x, y and y' only where $q=0, 1, \ldots, r$, we define

$$\bar{y}_{k+1}^{(q+2)} = f^{(q)}(x_{k+1}, \bar{y}_{k+1}, \bar{y}'_{k+1}), \quad q = 0, 1, \ldots, r. \tag{2.1.13}$$

In the relations (2.1.9)—(2.1.13), $k=0, 1, 2, \ldots, n-1$, and we can start our calculations by using the substitutions $\bar{y}_0=y_0$, $\bar{y}'_0=y'_0$ and $\bar{y}_0^{(q+2)}=y_0^{(q+2)}$, $q=0, 1, \ldots, r$.

2.2. General convergence processes. In this paragraph we prove theorems dealing with the convergence of the approximate values $\bar{y}_{k+1}, \bar{y}'_{k+1}, \ldots, \bar{y}_{k+1}^{(r+2)}$ to the exact values $y(x_{k+1}), y'(x_{k+1}), \ldots, y^{(r+2)}(x_{k+1})$ where k in general may take the values $0, 1, 2, \ldots, n-1$. Before proving these theorems we are in need to prove some lemmas.

LEMMA 2.2.1. *The inequality*

$$|y'_{k+1}-\bar{y}'_{k+1}| \leqq |y'_k-\bar{y}'_k|(1+c_0 h)+Kh(1+c_1 h)|y_k-\bar{y}_k|+c_2\omega_r(h)h^{r+3}$$

is true for all $k=0, 1, 2, \ldots, n-1$.

PROOF. Using equations (2.1.10) and (2.1.12) together with the Lipschitz condition (2.1.2) we get

$$|y'_{k+1}-\bar{y}'_{k+1}| \leqq |y'_k-\bar{y}'_k|+K\int_{x_k}^{x_{k+1}}|y(t)-y_k^*(t)|\,dt+$$

$$+K\int_{x_k}^{x_{k+1}}\left|y'_k+\int_{x_k}^{t} f[u, y(u), y'(u)]\,du-\bar{y}'_k-\int_{x_k}^{t} f[u, y_k^*(u), y_k^{*\prime}(u)]\,du\right|dt.$$

Applying the Lipschitz condition once more, this will be

$$\leqq |y'_k-\bar{y}'_k|+K\int_{x_k}^{x_{k+1}}|y(t)-y_k^*(t)|\,dt+Kh|y'_k-\bar{y}'_k|+K^2\int_{x_k}^{x_{k+1}}\int_{x_k}^{t}|y(u)-y_k^*(u)|\,du\,dt+$$

$$+K^2\int_{x_k}^{x_{k+1}}\int_{x_k}^{t}|y'(u)-y_k^{*\prime}(u)|\,du\,dt.$$

Using equations (2.1.5), (2.1.7), (2.1.6), and (2.1.8) this becomes

$$\leqq |y'_k-\bar{y}'_k|+K\sum_{j=0}^{r+2}\frac{|y_k^{(j)}-\bar{y}_k^{(j)}|}{(j+1)!}h^{j+1}+K\omega_r(h)\frac{h^{r+3}}{(r+3)!}+$$

$$+Kh|y'_k-\bar{y}'_k|+K^2\sum_{j=0}^{r+2}\frac{|y_k^{(j)}-\bar{y}_k^{(j)}|}{(j+2)!}h^{j+2}+K^2\omega_r(h)\frac{h^{r+4}}{(r+4)!}+$$

$$+K^2\sum_{j=0}^{r+1}\frac{|y_k^{(j+1)}-\bar{y}^{(j+1)}|}{(j+2)!}h^{j+2}+K^2\omega_r(h)\frac{h^{r+3}}{(r+3)!}.$$

Using the Lipschitz condition (2.1.2) once more, knowing that $q=j-2$, i.e. for $j=2, 3, 4, ..., r+2$ we have

$$|y_k^{(j)}-\bar{y}_k^{(j)}| \leqq K(|y_k-\bar{y}_k|+|y_k'-\bar{y}_k'|),$$

i.e.

$$|y_{k+1}'-\bar{y}_{k+1}'| \leqq$$

$$\leqq |y_k-\bar{y}_k|\left\{Kh+K^2\sum_{j=2}^{r+2}\frac{h^{j+1}}{(j+1)!}+K^2\frac{h^2}{2}+K^3\sum_{j=2}^{r+2}\frac{h^{j+2}}{(j+2)!}+K^3\sum_{j=2}^{r+2}\frac{h^{j+1}}{(j+1)!}\right\}+$$

$$+|y_k'-\bar{y}_k'|\left\{1+Kh+K\frac{h^2}{2}+K^2\sum_{j=2}^{r+2}\frac{h^{j+1}}{(j+1)!}+K^2\frac{h^3}{6}+K^3\sum_{j=2}^{r+2}\frac{h^{j+2}}{(j+2)!}+\right.$$

$$\left.+K^3\sum_{j=2}^{r+2}\frac{h^{j+1}}{(j+1)!}\right\}+K\left(1+K+\frac{Kh}{r+4}\right)\omega_r(h)\frac{h^{r+3}}{(r+3)!}\leqq$$

$$\leqq |y_k'-\bar{y}_k'|(1+c_0h)+Kh(1+c_1h)|y_k-\bar{y}_k|+c_2\omega_r(h)h^{r+3}.$$

Thus the proof of the lemma is complete.

Definition 2.2.1. We shall denote the estimating errors of y and y' at any point $x_k\in[0, b]$, $k=0, 1, ..., n$, by

$$e_k=|y_k-\bar{y}_k| \quad\text{and}\quad e_k'=|y_k'-\bar{y}_k'|.$$

Lemma 2.2.2. *The inequality*

$$e_{k+1}'\leqq c_3e_{r_0}+c_4\omega_r(h)h^{r+2}$$

is true for all $k=0, 1, ..., n-1$, *where* $e_{r_0}=\max\{e_0, e_1, ..., e_k\}$.

Proof. By using Definition 2.2.1 and Lemma 2.2.1, the principle of successive substitution implies

$$e_{k+1}'\leqq e_k'(1+c_0h)+Kh(1+c_1h)e_k+c_2\omega_r(h)h^{r+3}$$

$$e_k'(1+c_0h)\leqq e_{k-1}'(1+c_0h)^2+Kh(1+c_1h)e_{k-1}(1+c_0h)+c_2\omega_r(h)h^{r+3}(1+c_0h)$$

$$e_k'(1+c_0h)^2\leqq e_{k-2}'(1+c_0h)^3+Kh(1+c_1h)e_{k-2}(1+c_0h)^2+c_2\omega_r(h)h^{r+3}(1+c_0h)^2$$

$$\dots$$

$$e_1'(1+c_0h)^k\leqq e_0'(1+c_0h)^{k+1}+Kh(1+c_1h)e_0(1+c_0h)^k+c_2\omega_r(h)h^{r+3}(1+c_0h)^k$$

and we obtain easily

$$e_{k+1}'\leqq e_0'(1+c_0h)^{k+1}+Kh(1+c_1h)\sum_{j=0}^{k}e_j(1+c_0h)^{k-j}+c_2\omega_r(h)h^{r+3}\sum_{j=0}^{k}(1+c_0h)^j.$$

Let $e_{r_0}=\max\{e_0, e_1, ..., e_k\}$, $0\leqq r_0\leqq k$, and substitute e_0' by zero to obtain

$$e_{k+1}'\leqq Kh(1+c_1h)e_{r_0}\sum_{j=0}^{k}(1+c_0h)^j+c_2\omega_r(h)h^{r+3}\sum_{j=0}^{k}(1+c_0h)^j=$$

$$=Kh(1+c_1h)e_{r_0}\frac{\{(1+c_0h)^{k+1}-1\}}{c_0h}+c_2\omega_r(h)h^{r+3}\frac{\{(1+c_0h)^{k+1}-1\}}{c_0h}$$

and

$$(1+c_0h)^{k+1} = \left(1+\frac{bc_0}{n}\right)^{k+1} \leqq \left(1+\frac{bc_0}{n}\right)^n \leqq e^{bc_0} = \text{const.}$$

This implies

$$e'_{k+1} \leqq c_3 e_{r_0} + c_4 \omega_r(h) h^{r+2}$$

which completes the proof.

LEMMA 2.2.3. *The inequality*

$$e_{k+1} \leqq e_k(1+c_5h^2) + c_6 h e'_k + c_7 \omega_r(h) h^{r+4}$$

is true for all $k=0, 1, \dots, n-1$.

PROOF. By the same way as in Lemma 2.2.1 and by using the equations (2.1.9), (2.1.11), (2.1.5), (2.1.6), (2.1.7), (2.1.8) and the Lipschitz condition (2.1.2) we can get the required result.

LEMMA 2.2.4. *The inequality*

$$e_{k+1} \leqq e_{r_0}(1+c_8 h) + c_9 \omega_r(h) h^{r+3}$$

is true for all $k=0, 1, \dots, n-1$ *where* $e_{r_0} = \max\{e_0, e_1, \dots, e_k\}$ *and* $0 \leqq r_0 \leqq k$.

PROOF. From Lemma 2.2.2 we get

$$e'_k \leqq c_3 e_{r_0^*} + c_4 \omega_r(h) h^{r+2}$$

where $e_{r_0^*} = \max\{e_0, e_1, \dots, e_{k-1}\}$ and by recalling $e_{r_0} = \max\{e_0, e_1, \dots, e_{k-1}, e_k\}$ then obviously $e_{r_0^*} \leqq e_{r_0}$ from which we get

$$e'_k \leqq c_3 e_{r_0} + c_4 \omega_r(h) h^{r+2}.$$

Using this result in Lemma 2.2.3 we get

$$e_{k+1} \leqq e_{r_0}(1+c_5h^2) + c_6 h\{c_3 e_{r_0} + c_4 \omega_r(h) h^{r+2}\} + c_7 \omega_r(h) h^{r+4}$$

i.e.

$$e_{k+1} \leqq e_{r_0}(1+c_8 h) + c_9 \omega_r(h) h^{r+3}$$

and thus the proof is complete.

LEMMA 2.2.5. *The inequality*

$$e'_{r_0} \leqq c_{10}(1+c_{11}h) e_{r_1} + c_{12} \omega_r(h) h^{r+2}$$

is true where r_0 *is the subscript of the maximum error* e_{r_0} *where* $e_{r_0} = \max\{e_0, e_1, \dots, e_k\}$ *and* $e_{r_1} = \max\{e_0, e_1, \dots, e_{r_0-1}\}$ *with some* r_1, $0 \leqq r_1 \leqq r_0 - 1$.

PROOF. By using Lemma 2.2.1 with $[x_k, x_{k+1}]$ replaced by $[x_{r_0-1}, x_{r_0}]$ and by similar procedures as shown in Lemma 2.2.2 by Definition 2.2.1 it is easy to obtain the required result.

LEMMA 2.2.6. *The inequality*

$$e_{r_0} \leqq e_{r_1}(1+c_{13}h) + c_{14} \omega_r(h) h^{r+3}$$

is true, where $e_{r_0}=\max\{e_0, e_1, \ldots, e_k\}$ *with some* r_0, $0 \leqq r_0 \leqq k$, *and* $e_{r_1}=\max\{e_0, e_1, \ldots, e_{r_0-1}\}$ *with some* r_1, $0 \leqq r_1 \leqq r_0-1$.

PROOF. From Lemma 2.2.3 by replacing $[x_k, x_{k+1}]$ by $[x_{r_0-1}, x_{r_0}]$ we obviously can get

$$e_{r_0} \leqq e_{r_0-1}(1+c_{15}h^2)+c_{16}he'_{r_0-1}+c_{17}\omega_r(h)h^{r+4}$$

and $e_{r_1}=\max\{e_0, e_1, \ldots, e_{r_0-1}\}$, as it was defined in Lemma 2.2.5, implies $e_{r_0}-1 \leqq e_{r_1}$ from which we get

$$e_{r_0} \leqq e_{r_1}(1+c_{15}h^2)+c_{16}he'_{r_0-1}+c_{17}\omega_r(h)h^{r+4}.$$

Lemma 2.2.5 implies

$$e'_{r_0-1} \leqq c_{10}(1+c_{11}h)e_{r_1^*}+c_{12}\omega_r(h)h^{r+2}$$

where $e_{r_1^*}=\max\{e_0, e_1, \ldots, e_{r_0-2}\}$ for some r_1^*, $0 \leqq r_1^* \leqq r_0-2$, and it is easy to use the fact that

$$e_{r_1^*} = \max\{e_0, e_1, \ldots, e_{r_0-2}\} \leqq \max\{e_0, e_1, \ldots, e_{r_0-2}, e_{r_0-1}\} = e_{r_1}$$

from which we get

$$e'_{r_0-1} \leqq c_{10}(1+c_{11}h)e_{r_1}+c_{12}\omega_r(h)h^{r+2}.$$

Returning to e_{r_0} and using the last inequality we get

$$e_{r_0} \leqq e_{r_1}(1+c_{15}h^2)+c_{16}h\{c_{10}(1+c_{11}h)e_{r_1}+c_{12}\omega_r(h)h^{r+2}\}+c_{17}\omega_r(h)h^{r+4} \leqq$$
$$\leqq e_{r_1}(1+c_{13}h)+c_{14}\omega_r(h)h^{r+3}$$

which completes the proof.

THEOREM 2.2.1. *The speed of convergence of the approximate value* $\bar{y}_{k+1}$ *given by the formula* (2.1.11) *to the exact value of the solution of* (1.1) *at* x_{k+1} *is estimated by the inequality*

$$e_{k+1} = |y_{k+1}-\bar{y}_{k+1}| \leqq c_{18}\omega_r(h)h^{r+2}$$

which holds for all $k=0, 1, \ldots, n-1$.

PROOF. From Lemma 2.2.4 we have

$$e_{k+1} \leqq e_{r_0}(1+c_8h)+c_9\omega_r(h)h^{r+3}$$

where $e_{r_0}=\max\{e_0, e_1, \ldots, e_k\}$ with some r_0, $0 \leqq r_0 \leqq k$, and $k=0, 1, \ldots, n-1$.

From Lemma 2.2.6 we know that

$$e_{r_0} \leqq e_{r_1}(1+c_{13}h)+c_{14}\omega_r(h)h^{r+3}$$

where $e_{r_1}=\max\{e_0, e_1, \ldots, e_{r_0-1}\}$ with some r_1, $0 \leqq r_1 \leqq r_0-1$.

Continuing by the same way as it was shown in Lemmas 2.2.4 and 2.2.6 we can obtain the inequalities

$$e_{r_1} \leqq e_{r_2}(1+c_1^*h)+c_2^{**}\omega_r(h)h^{r+3}$$

where $e_{r_2}=\max\{e_0, e_1, \ldots, e_{r_1-1}\}$ with some r_2, $0 \leqq r_2 \leqq r_1-1$, and

$$e_{r_2} \leqq e_{r_3}(1+c_2^*h)+c_2^{**}\omega_r(h)h^{r+3}$$

where $e_{r_3}=\max\{e_0, e_1, \dots, e_{r_2-1}\}$ with some r_3, $0\leqq r_3\leqq r_2-1$, and at the end we can get the inequality

$$e_{r_s} \leqq e_{r_{s+1}}(1+c_s^* h)+c_s^{**}\omega_r(h)\,h^{r+3}$$

where $e_{r_s}=\max\{e_0, e_1\}$ with some r_s, $0\leqq r_s\leqq 1$, and $e_{r_{s+1}}=\max\{e_0\}=e_0$ for some r_{s+1}, $0\leqq r_{s+1}\leqq 0$, i.e. $r_{s+1}=0$.

Now, taking $c_{19}=\max\{c_8, c_{13}, c_1^*, c_2^*, \dots, c_s^*\}$ and $c_{20}=\max\{c_9, c_{14}, c_1^{**}, c_2^{**}, \dots$ $\dots, c_s^{**}\}$ and by the rearrangement of the above inequalities we get

$$e_{k+1} \leqq e_{r_0}(1+c_{19}h)+c_{20}\omega_r(h)\,h^{r+3}$$

$$e_{r_0}(1+c_{19}h) \leqq e_{r_1}(1+c_{19}h)^2+c_{20}\omega_r(h)\,h^{r+3}(1+c_{19}h)$$

$$e_{r_1}(1+c_{19}h)^2 \leqq e_{r_2}(1+c_{19}h)^3+c_{20}\omega_r(h)\,h^{r+3}(1+c_{19}h)^2$$

$$\dots$$

$$e_{r_s}(1+c_{19}h)^{s+1} \leqq e_{r_{s+1}}(1+c_{19}h)^{s+2}+c_{20}\omega_r(h)\,h^{r+3}(1+c_{19}h)^{s+1}$$

from which we get

$$e_{k+1} \leqq e_{r_{s+1}}(1+c_{19}h)^{s+2}+c_{20}\omega_r(h)h^{r+3}\sum_{j=0}^{s+1}(1+c_{19}h)^j$$

and using the fact that $e_{r_{s+1}}=e_0=0$ this will be

$$\leqq c_{20}\omega_r(h)\,h^{r+3}\sum_{j=0}^{s+1}(1+c_{19}h)^j = c_{20}\omega_r(h)\,h^{r+3}\frac{\{(1+c_{19}h)^{s+2}-1\}}{c_{19}h} \leqq c_{18}\,\omega_r(h)\,h^{r+2}$$

which completes the proof.

THEOREM 2.2.2. *The convergence of the approximate value* $\bar{y}'_{k+1}$ *to* y'_{k+1} *given by the formula* (2.1.12) *is estimated by the inequality*

$$e'_{k+1} = |y'_{k+1}-\bar{y}'_{k+1}| \leqq c_{21}\omega_r(h)\,h^{r+2}$$

where $k=0, 1, \dots, n-1$.

PROOF. Lemma 2.2.2 tells us that

$$e'_{k+1} \leqq c_3 e_{r_0}+c_4\,\omega_r(h)\,h^{r+2}$$

where $e_{r_0}=\max\{e_0, e_1, \dots, e_k\}$ and from Theorem 2.2.1 we can obtain that

$$e_{r_0} \leqq c_{18}\,\omega_r(h)\,h^{r+2}$$

and thus we obviously get

$$e'_{k+1} \leqq c_3 c_{18}\,\omega_r(h)\,h^{r+2}+c_4\,\omega_r(h)\,h^{r+2} \leqq c_{21}\,\omega_r(h)h^{r+2}$$

which completes the proof.

THEOREM 2.2.3. *The error of the approximate values* $y_{k+1}^{(q+2)}$ *are estimated by the inequalities*

$$e_{k+1}^{(q+2)} = |y_{k+1}^{(q+2)}-\bar{y}_{k+1}^{(q+2)}| \leqq c_{22}\omega_r(h)\,h^{r+2}$$

where $k=0, 1, \dots, n-1$ *and* $q=0, 1, \dots, r$.

PROOF. Using equations (1.1) and (2.1.13) we get

$$|y_{k+1}^{(q+2)}-\bar{y}_{k+1}^{(q+2)}| \leqq |f^{(q)}(x_{k+1}, y_{k+1}, y'_{k+1})-f^{(q)}(x_{k+1}, \bar{y}_{k+1}, \bar{y}'_{k+1})|.$$

Applying the Lipschitz condition this will be

$$\leqq K(|y_{k+1}-\bar{y}_{k+1}|+|y'_{k+1}-\bar{y}'_{k+1}|).$$

Using Theorems 2.2.1 and 2.2.2 this redusec to

$$\leqq c_{18}\omega_r(h)h^{r+2}+Kc_{21}\omega_r(h)h^{r+2} \leqq c_{22}\omega_r(h)h^{r+2}$$

which completes the proof.

3. The second approximation process

We have obtained, as we have seen before, the sets of approximate values

$$\bar{Y}^{(q)}: \bar{y}_0^{(q)}, \bar{y}_1^{(q)}, \ldots, \bar{y}_n^{(q)}, \quad q = 0, 1, \ldots, r+2$$

which are approximating the values

$$Y^{(q)}: \; y_0^{(q)}, y_1^{(q)}, \ldots, y_n^{(q)}, \quad q = 0, 1, \ldots, r+2,$$

respectively. In this section and on the base of these sets of approximate values, we are going to construct a spline function $S_\Delta(x)$ which interpolates to the set $\bar{Y}$ on the mesh Δ and approximates the solution $y(x)$ of (1.1). Further we shall discuss the convergence of this function to $y(x)$.

3.1. Construction of the spline function. In this paragraph we introduce the spline function approximating the solution of our differential equation.

THEOREM 3.1. *For a given mesh of points*

$$\Delta: 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = b, \quad x_{k+1}-x_k = h$$

and for given sets of values

$$\bar{Y}^{(q)}: \; \bar{y}_0^{(q)}, \bar{y}_1^{(q)}, \ldots, \bar{y}_n^{(q)}, \quad q = 0, 1, \ldots, r+2$$

there is a unique spline function $S_\Delta(x)$ interpolating on the mesh Δ to the set $\bar{Y}$ and satisfying the conditions

(3.1.1) $$S_\Delta(\bar{Y}, x) = S_\Delta(x) \in C^{r+2}[0, b]$$

(3.1.2) $$S_k^{(q)}(x_k) = \bar{y}_k^{(q)}, \quad q = 0, 1, \ldots, r+2 \quad \textit{and} \quad k = 0, 1, \ldots, n$$

(3.1.3) *For* $x_k \leqq x \leqq x_{k+1}$ *and* $k = 0, 1, \ldots, n-1$

$$S_\Delta(x) = S_k(x) = \sum_{j=0}^{r+2} \frac{\bar{y}_k^{(j)}}{j!}(x-x_k)^j + \sum_{p=1}^{r+3} a_p^{(k)}(x-x_k)^{p+r+2}.$$

PROOF. From the continuity condition (3.1.1) and by using (3.1.2) for $x=x_{k+1}$ we get

$$S_k^{(t)}(x_{k+1}) = S_{k+1}^{(t)}(x_{k+1}) = \bar{y}_{k+1}^{(t)} \tag{3.1.4}$$

where $k=0, 1, \ldots, n-1$ and $t=0, 1, \ldots, r+2$. Substituting from (3.1.4) in (3.1.3) we get the system of equations

$$\sum_{p=1}^{r+1} t!\binom{p+r+2}{t} a_p^{(k)} h^{p-1} = h^{t-r-3}\left(\bar{y}_{k+1}^{(t)} - \sum_{j=0}^{r+2-t} \frac{\bar{y}_K^{(j+t)}}{j!} h^j\right) = F_t^{(k)} \tag{3.1.5}$$

where $t=0, 1, \ldots, r+2$. This system of eauations has a unique solution for the unknowns $a_p^{(k)}$ $(p=1, 2, \ldots, r+3)$ since its determinant is

$$D_r =$$

$$= \begin{vmatrix} 1 & h & \ldots & h^{p-1} & \ldots & h^{r+2} \\ \binom{r+3}{1} 1! & \binom{r+4}{1} 1!h & \ldots & \binom{r+2+p}{1} 1!h^{p-1} & \ldots & \binom{2r+5}{1} 1!h^{r+2} \\ \binom{r+3}{2} 2! & \binom{r+4}{2} 2!h & \ldots & \binom{r+2+p}{2} 2!h^{p-1} & \ldots & \binom{2r+5}{2} 2!h^{r+2} \\ \cdot & \cdot & \ldots & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot & \ldots & \cdot \\ \binom{r+3}{r+2} (r+2)! & \binom{r+4}{r+2} (r+2)!h & \ldots & \binom{r+2+p}{r+2} (r+2)!h^{p-1} & \ldots & \binom{2r+5}{r+2} (r+2)!h^{r+2} \end{vmatrix} =$$

$$= h^{\frac{1}{2}(r+2)(r+3)} \prod_{t=0}^{r+2} t!$$

which is different from zero for $h>0$. If we replace the p^{th} column in D_r by the column $(F_0^{(k)}, F_1^{(k)}, \ldots, F_{r+2}^{(k)})^T$ and denote the resulting determinant by D_r^p, the solution of the system (3.1.5) will be

$$a_p^{(k)} = \frac{D_r^p}{D_r}, \quad p = 1, 2, \ldots, r+3 \tag{3.1.6}$$

and after factorizing D_r^p in terms of $F_0^{(k)}, F_1^{(k)}, \ldots, F_{r+2}^{(k)}$ the solution (3.1.6) will take the form

$$a_p^{(k)} = \frac{1}{h^{p-1}} \sum_{i=0}^{r+2} c_{pi} F_i^{(k)}, \quad p = 1, 2, \ldots, r+3 \tag{3.1.7}$$

and, as we have said before, this solution is unique. This guarantees the uniqueness of the spline function $S_\Delta(x)$, consequently the existence of such a function and thus the theorem is proved.

3.2. Convergence of the spline function to the solution. In this paragraph we prove a theorem concerned with the convergence of our spline function constructed in Theorem 3.1 to the exact solution of (1.1). Moreover, we prove that this function satisfies this differential equation as $n\to\infty$.

THEOREM 3.2.1. *Let* $y(x)$ *be the solution of* (1.1) *and let* $f\in C^r([0,b]\times R\times R)$. *If* $S_\Delta(x)$ *is the spline function constructed in Theorem* 3.1, *then there exists a constant* E *independent of* h *such that*

$$|y^{(q)}(x)-S_\Delta^{(q)}(x)|\leqq E\omega_r(h)h^{r+2-q}$$

for all $x\in[0,b]$ *and* $q=0,1,\dots,r+2$.

For the proof we need the following

LEMMA 3.2.1. *We have*

$$|a_p^{(k)}|\leqq\frac{A_p}{h^p}\omega_r(h),\quad p=1,2,\dots,r+3$$

where A_p *are constants independent of* h.

PROOF OF THE LEMMA. At first we deduce some inequalities concerning the absolute values of $F_t^{(k)}$ $(t=0,1,\dots,r+2)$. They are calculated as follows.

From (3.1.5) we have

$$|F_t^{(k)}|=h^{t-r-3}\left|\bar{y}_{k+1}^{(t)}-\sum_{j=0}^{r+2-t}\frac{\bar{y}_k^{(j+t)}}{j!}h^j\right|$$

and if we define the Taylor expansion of $y^{(t)}(x)$ for $x_k\leqq x\leqq x_{k+1}$ to be

(3.2.1)
$$y^{(t)}(x)=\sum_{j=0}^{r+1-t}\frac{y_k^{(j+t)}}{j!}(x-x_k)^j+\frac{y_k^{(r+2)}(\xi_{kt})}{(r+2-t)!}(x-x_k^{r+2-t}),\quad x_k<\xi_{kt}<x_{k+1}.$$

So, we get for $x=x_{k+1}$

$$y_{k+1}^{(t)}=\sum_{j=0}^{r+1-t}\frac{y_k^{(j+t)}}{j!}h^j+\frac{y_k^{(r+2)}(\xi_{kt})}{(r+2-t)!}h^{r+2-t}\tag{3.2.2}$$

where $t=0,1,2,\dots,r+2$.

Using the last identity (3.2.2) together with (3.1.5) we get for $t=0,1,\dots,r+2$

$$|F_t^{(k)}|\leqq h^{t-r-3}\left\{|y_{k+1}^{(t)}-\bar{y}_{k+1}^{(t)}|+\sum_{j=0}^{r+2-t}\frac{|y_k^{(j+t)}-\bar{y}_k^{(j+t)}|}{j!}h^j+\frac{|y^{(r+2)}(\xi_{kt})-y_k^{(r+2)}|}{(r+2-t)!}h^{r+2-t}\right\}.$$

Using Theorems 2.2.1, 2.2.2 and 2.2.3 together with the definition of the modulus of continuity this will be

$$\leqq h^{t-r-3}\{C_t^*\omega_r(h)h^{r+2-t}\}$$

where C_t^* $(t=0,1,\dots,r+2)$ are constants independent of h and so we have the result that

$$|F_t^{(k)}|\leqq C_t^*\frac{\omega_r(h)}{h},\quad t=0,1,\dots,r+2.\tag{3.2.3}$$

Now, after obtaining the last inequality, we go on to prove Lemma 3.2.1, and for this purpose we combine equations (3.2.3) with (3.1.7) to get

$$|a_p^{(k)}| \leqq \frac{1}{h^{p-1}} \sum_{t=0}^{r+2} C_{pi} C_t^* \frac{\omega_r(h)}{h} \leqq A_p \frac{\omega_r(h)}{h^p}$$

where A_p is a constant independent of h, and thus the proof is complete.

PROOF OF THEOREM 3.2.1. By using equations (3.2.1) and (3.1.3) we get:

$$|y^{(q)}(x) - S_\Delta^{(q)}(x)| = \Bigg| \sum_{j=0}^{r+1-q} \frac{y_k^{(j+q)}}{j!} (x-x_k)^j + \frac{y^{(r+2)}(\xi_{kq})}{(r+2-q)!} (x-x_k)^{r+2-q} -$$

$$- \sum_{j=0}^{r+1-q} \frac{\bar{y}_k^{(j+q)}}{j!} (x-x_k)^j - \frac{\bar{y}_k^{(r+2)}}{(r+2-q)!} (x-x_k)^{r+2-q} -$$

$$- \sum_{p=1}^{r+3} q! \binom{p+r+2}{q} a_p^{(k)} (x-x_k)^{p+r+2-q} \Bigg| \leqq$$

$$\leqq \sum_{j=0}^{r+2-q} \frac{|y_k^{(j+q)} - \bar{y}_k^{(j+q)}|}{j!} h^j + \frac{|y^{(r+2)}(\xi_{kq}) - y_k^{(r+2)}|}{(r+2-q)!} h^{r+2-q} +$$

$$+ \sum_{p=1}^{r+3} q! \binom{p+r+2}{q} a_p^{(k)} h^{p+r+2-q}.$$

Using Theorems 2.2.1, 2.2.2 and 2.2.3 together with the definition of the modulus of continuity $\omega_r(h)$ and Lemma 3.2.1, this will be

$$\leqq C_q^{**} \omega_r(h)\, h^{r+2-q}.$$

Taking $E = \max C_q^{**}$ where $q = 0, 1, \ldots, r+2$ we get

$$|y^{(q)}(x) - S_\Delta^{(q)}(x)| \leqq E\omega_r(h)\, h^{r+2-q}$$

and thus the proof of the theorem is complete.

At last we are going to prove that our spline function $S_\Delta(x)$ satisfies the differential equation (1.1) as $n \to \infty$.

THEOREM 3.2.2. *If* $\bar{S}_\Delta''(x)$ *denotes the function*

$$\bar{S}_\Delta''(x) = f[x, S_\Delta(x), S_\Delta'(x)]$$

and $S_\Delta(x)$ *is the spline function given in Theorem* 3.1, *then for any* $x \in [0, b]$ *we have*

$$|\bar{S}_\Delta''(x) - S_\Delta''(x)| \leqq M\omega_r(h)\, h^r$$

where M *is some constant independent of* h. *Otherwise*

$$S_\Delta''(x) \cong \bar{S}_\Delta''(x) \quad \text{as} \quad n \to \infty \quad \text{or as} \quad h \to 0.$$

PROOF. We have

$$|\bar{S}_\Delta''(x) - S_\Delta''(x)| \leqq |\bar{S}''(x) - y''(x)| + |y''(x) - S_\Delta''(x)| =$$

$$= |f[x, S_\Delta(x), S_\Delta'(x)] - f[x, y(x), y'(x)]| + |y''(x) - S_\Delta''(x)|.$$

Applying the Lipschitz condition on f this will be

$$\leqq K|S_\Delta(x)-y(x)|+K|S'_\Delta(x)-y'(x)|+|y''(x)-S''_\Delta(x)|.$$

Applying Theorem 3.2.1 this becomes

$$\leqq KE\omega_r(h)h^{r+2}+KE\omega_r(h)h^{r+1}+E\omega_r(h)h^r = (E+KEh+KEh^2)\omega_r(h)h^r \leqq$$
$$\leqq M\omega_r(h)h^r$$

where M is some constant independent of h, and thus the proof is complete.

REMARK. In the case $f\in C^\infty[0, b]$ we can choose r to be finite in such a way that the error will be in the allowable range, because as we have seen in the convergence theorems, the error is $O(h^{r+2})$. Also in practical applications if $f\in C^r[0, b]$ where r is a large finite number, it is enough to choose a suitably small r in the sense that the error will be in the allowable range.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE
1088 BUDAPEST, MÚZEUM KRT. 6—8.

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ON CRITICAL 3-CHROMATIC HYPERGRAPHS

By
V. MÜLLER, V. RÖDL and D. TURZÍK (Prague)

Introduction. It is a well-known fact, that 3-critical graphs are just odd cycles. This means that the valence of x is equal to 2 for all 3-critical graphs H and for every vertex x of H. On the other hand DIRAC [1] proved that for $k \geqq 6$ and $h \in \mathbf{N}$ there exists a k-critical graph H with val $(x, H) \geqq h$ for every vertex $x \in V(H)$. The cases $k=4, 5$ were investigated by GALLAI and completely solved by TOFT [4] and SIMONOVITS [3] independently. The situation for hypergraphs is quite different. There is no simple characterization of 3-critical hypergraphs. We can easily find a 3-critical hypergraph H with val $(x, H) \geqq h$. Toft asked how the situation changes when we replace the valence of x by the quasivalence. (The quasivalence of x is the maximal number of edges containing x such that intersection of any pair of them is the singleton point x.) In [5], page 1456, problem 3, TOFT raised the following problem: Let r, h be natural numbers. Does there exist a 3-critical r-uniform hypergraph with all quasivalences $\geqq h$? ERDŐS and SPENCER ([2], page 21, problem 4) formulated this question in a stronger form. They asked if there exists a 3-critical uniform hypergraph H such that

1) $e_1, e_2 \in E(H),\ e_1 \neq e_2 \Rightarrow |e_1 \cap e_2| \leqq 1$

2) val $(x, H) \geqq h$ for every $x \in V(H)$.

In this paper a construction of a hypergraph satisfying the conditions of Toft's question is given (Theorem 4). Theorem 3 gives an answer to the Erdős—Spencer's modification.

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I. Preliminaries

A hypergraph H is a finite set $V(H)$ and a set $E(H)$ of subsets of $V(H)$, such that each element of $E(H)$ contains at least two elements of $V(H)$.

A hypergraph H is a k-uniform, $k \in N$, if $|e|=k$ for every $e \in E(H)$.

The valence of a vertex x of a hypergraph H, denoted by val (x, H) is the number of elements of $E(H)$ containing x. The quasivalence of x, denoted by qval (x, H), is the maximal number of elements in a subset $E' \subset E(H)$ satisfying:

1. For all $e \in E' : x \in e$
2. For all pairs $e_1, e_2 \in E'$ with $e_1 \neq e_2$: $e_1 \cap e_2 = \{x\}$.

A k-colouring of a hypergraph H is a decomposition of $V(H)$ into k classes (called colour classes) $A_1, \dots, A_k$ such that $1 \leqq j \leqq k, e \in E(H)$ implies $e \not\subset A_j$ (i.e. $e \cap \bigcup_{i \neq j} A_i \neq \emptyset$).

The chromatic number $\chi(H)$ of a hypergraph H is the minimal integer k, such that there exists a k-colouring $A_1, \dots, A_k$ of H.

A hypergraph H is called k-critical if $\chi(H)=k$ and $\chi(H-e)<k$ for every edge $e \in E(H)$ (where $(H-e)$ means the hypergraph $\langle V(H), E(H)-\{e\}\rangle$).

If Y is a set then $[Y]^2$ denotes the set of all pairs of elements of Y, e.g. $[Y]^2 = \{X; X \subset Y \text{ and } |X|=2\}$.

II. Construction 1

THEOREM 1. *Let* $n \in \mathbf{N}$. *Then there exists a* 3-*critical hypergraph* $G=\langle V(G), E(G)\rangle$ *such that* $\operatorname{qval}(x, G) \geqq n$ *for every* $x \in V(G)$.

PROOF. We can assume that n is a prime number, $n \geqq 5$. Put

$$V(G) = \{a_{i,j};\ 0 \leqq i \leqq n-1,\ 0 \leqq j \leqq n-1\} \cup \{a, a_0, \dots, a_{n-1}\},$$

$$E(G) = E_1 \cup E_2 \cup E_3 \cup E_4, \text{ where}$$

$$E_1 = \big\{\{a, a_i\};\ i = 0, 1, \dots, n-1\big\}$$

$$E_2 = \big\{\{a, a_{i,0}, a_{i,1}, \dots, a_{i,n-1}\};\ i = 0, 1, \dots, n-1\big\}$$

$$E_3 = \big\{\{a_i, a_{0,j}, a_{1,j+i}, a_{2,j+2i}, \dots, a_{n-1,j+(n-1)i}\}; 0 \leqq i \leqq n-1,\ 0 \leqq j \leqq n-1\big\}$$

(the addition is taken $\operatorname{mod} n$)

$$E_4 = \big\{\{a_0, \dots, a_{n-1}, a_{0,j_0}, a_{1,j_1}, \dots, a_{n-1,j_{n-1}}\};\ 0 \leqq j_i \leqq n-1,\ 0 \leqq i \leqq n-1$$

$$\text{and } \{a_i, a_{0,j_0}, \dots, a_{n-1,j_{n-1}}\} \notin E_3 \quad \text{for every } i = 0, 1, \dots, n-1\big\}.$$

REMARK. E_1 is the set of ordinary edges of G, i.e. $e \in E_1$ with $|e|=2$. The sets E_2, E_3 and E_4 are sets of hyperedges of G. It holds $e \in E_2 \cup E_3 \Rightarrow |e|=n+1$, $e \in E_4 \Rightarrow |e|=2n$.

The hypergraph G can be viewed upon as follows: We take the affine plane of order n on the set $\{a_{i,j}; 0 \leqq i \leqq n-1, 0 \leqq j \leqq n-1\}$. There are exactly $n+1$ directions of lines in that plane. We associate each of the directions to one of the points $a, a_0, \dots, a_{n-1}$. Thus we obtain the edges in E_2 and E_3. An edge $e \in E_4$ contains always the points $a_0, a_1, \dots, a_{n-1}$ and points $a_{0,j_0}, a_{1,j_1}, \dots, a_{n-1,j_{n-1}}$ provided that the later ones do not form a line.

It is easy to see that $\operatorname{qval}(x, G) \geqq n$ for every $x \in V(G)$. So *we are to prove* only that G is 3-critical:

1. $\chi(G) \geqq 3$. Suppose that in the contrary, there exists a 2-colouring M, N of G. Let $a \in M$. Then necessarily $a_0 \in N, a_1 \in N, \dots, a_{n-1} \in N$ and there exist $j_i \in \{0, \dots \dots, n-1\}, i=0, 1, \dots, n-1$ such that $a_{i,j_i} \in N$ for every $i=0, 1, \dots, n-1$. Then

either $\{a_0, \ldots, a_{n-1}, a_{0,j_0}, \ldots, a_{n-1,j_{n-1}}\}\in E_4$ or there exists an $i\in\{0, \ldots, n-1\}$ with $\{a_i, a_{0,j_0}, \ldots, a_{n-1,j_{n-1}}\}\in E_3$. So we have an edge lying in one colour class N, a contradiction.

2. $\chi(G-e)=2$ for every $e\in E(G)$. a) Let $e=\{a, a_i\}, 0\leq i\leq n-1$. We construct the 2-colouring M, N of $G-e$. Put $M=\{a_j, j\neq i\}\cup\{a_{0,0}, a_{1,1+i}, a_{2,2+2i}, \ldots$ $\ldots, a_{n-3,n-3+(n-3)i}, a_{n-2,n-1+(n-2)i}, a_{n-1,n-2+(n-1)i}\}, N=V(G)-M$.

b) Let $e=\{a, a_{i,0}, \ldots, a_{i,n-1}\}, 0\leq i\leq n-1$. Put $M=e, N=V(G)-M$.

c) Let $e=\{a_i, a_{0,j}, a_{1,j+i}, \ldots, a_{n-1,j+(n-1)i}\}$. Put $M=e\cup\{a_0, a_1, \ldots, a_{n-1}\}$. $N=V(G)-M$.

d) Let $e=\{a_0, \ldots, a_{n-1}, a_{0,j_0}, \ldots, a_{n-1,j_{n-1}}\}, e\in E_4$. Put $M=e, N=V(G)-M$.

We obtain a 2-colouring of $G-e$ in every case. This proves that G is 3-critical and also Theorem 1.

REMARK. It is possible to get a $2n$-uniform hypergraph with properties of Theorem 1 by a simple modification of Construction 1.

III. Construction 2

In this section we give the second example of a 3-critical hypergraph with large quasivalences.

THEOREM 2. *Let* $h\in\mathbf{N}$. *Then there exists a 3-critical uniform hypergraph* $\tilde{H}$ *with* $\operatorname{qval}(z, \tilde{H})\geq h$ *for every* $z\in V(\tilde{H})$.

PROOF. Since $R(k, k)\geq 2^{k/2}$, [2], we can find $k\in\mathbf{N}$ satisfying $\frac{R(k, k)-2}{2(k-2)}\geq h$. Fix k with this property. Set $n=R(k, k)$ and $K=\{1, 2, \ldots, n\}$. Define a $\binom{k}{2}$-uniform hypergraph H by:

$$\begin{cases} V(H) = [K]^2 \\ E(H) = \{[Y]^2;\ Y\subset K \quad \text{and} \quad |Y| = k\}. \end{cases}$$

Then $\chi(H)\geq 3$ according Ramsey Theorem.

Denote $\tilde{H}=\langle V(H), E(\tilde{H})\rangle$ an arbitrary 3-critical subhypergraph of H. We prove that $\tilde{H}$ has all required properties. It is enough to prove $\operatorname{qval}(\{x, y\}, \tilde{H})\geq$ $\geq\frac{n-2}{2(k-2)}$. Suppose in the contrary, that there exists $\{x, y\}\in V(\tilde{H})$ with $\operatorname{qval}(\{x, y\}, \tilde{H})=p<\frac{n-2}{2(k-2)}$. Then there exist edges $[Y_1]^2, \ldots, [Y_p]^2$ of $\tilde{H}$ such that $\{x, y\}\subset Y_i$ for $i=1, \ldots, p$ and $Y_i\cap Y_j=\{x, y\}$ for $i\neq j$. Put $B=\bigcup_{i=1}^{p} Y_i-\{x, y\}$, i.e. $|B|=p(k-2)<\frac{n-2}{2}$. As $\operatorname{qval}(\{x, y\}, \tilde{H})=p$ it holds $[Y]^2\in E(\tilde{H})$ and $\{x, y\}\subset Y$ implies $Y\cap B\neq\emptyset$. Now we can easily find a 2-colouring M, N with the property $\{x, y\}\in N$ and $\{x, z\}\in M\,\{y, z\}\in M$ for all $z\in B$.

IV. Construction 3

In this part we prove the existence of a 3-uniform 3-critical hypergraph X with the properties

$$e_1, e_2 \in E(X);\ e_1 \neq e_2 \Rightarrow |e_1 \cap e_2| \leqq 1 \tag{1}$$

$$\operatorname{val} x \geqq k \quad \text{for every} \quad x \in V(X). \tag{2}$$

The proof consists of two parts. In the first one (Lemma 1) we construct a 3-critical hypergraph with the properties (1) and (2) which is "almost" uniform. With the help of this we construct the required hypergraph.

Construction (i). Let $k\in\mathbf{N}, k\geqq 2$ be given. Define a hypergraph $G=\langle V(G), E(G)\rangle$ by $V(G)=A\cup B\cup C$ where A, B, C are pairwise disjoint with $|A|=|B|=|C|=k$, $A=\{a_0, a_1, \dots, a_{k-1}\}$, $B=\{b_0, b_1, \dots, b_{k-1}\}$, $C=\{c_0, c_1, \dots, c_{k-1}\}$.

$E(G)=\{A\}\cup\{B\}\cup\big\{\{a_i, b_j, c_{i+j}\};\ 0\leqq i\leqq k-1,\ 0\leqq j\leqq k-1\big\}$ (the addition is taken mod k).

LEMMA 1. *The hypergraph defined above is* 3-*critical.*

PROOF. a) Given $e\in E(G)$, we prove $\chi(G-e)=2$. If $e=A$ put $M=A\cup\{b_1\}$, $N=C\cup B-\{b_1\}$. Then M, N constitute a 2-colouring of $G-\{A\}$. The proof for $e=B$ is analogous. Let $e=\{a_i, b_j, c_{i+j}\}$. Put $M=\{a_i\}\cup\{b_j\}\cup C$, $N=A\cup B-\{a_i, b_j\}$. Then M, N constitute a 2-colouring of $G-e$.

b) $\chi(G)>2$. Assume that in the contrary, there exists a 2-colouring M, N of G. If $a_i\in M\cap A$, $b_j\in M\cap B$ then necessarily $c_{i+j}\in C\cap N$. Fix i so that $a_i\in M\cap A$ $(\neq\emptyset)$. Then clearly $|C\cap N|\geqq|B\cap M|$ since $j\neq j' \pmod k$ implies $i+j\neq i+j' \pmod k$. In the same way we obtain $|C\cap N|\geqq|A\cap M|$, $|C\cap M|\geqq|A\cap N|$, $|C\cap M|\geqq|B\cap N|$. Since $|A|=|B|=|C|$, necessarily $|A\cap M|=|B\cap M|=|C\cap N|$, $|A\cap N|=|B\cap N|=|C\cap M|$. Then for every $c_r\in C\cap N$, $b_j\in B\cap M$ there follows $a_{r-j}\in A\cap M$.

Suppose that e.g. $|B\cap M|\geqq k/2$. Then there exists j, $0\leqq j\leqq k-1$ such that $b_j\in B\cap M$ and, at the same time $b_{j+1}\in B\cap M$. Let $c_r\in C\cap N$ arbitrary. Then $a_{r-j}\in M$ and since $b_{j+1}\in B\cap M$ we get $c_{r-j+j+1}=c_{r+1}\in N$. Thus $c_r\in N$ implies $c_{r+1}\in N$ and $C\subset N$. Thus $A\subset M$, a contradiction.

Therefore there remain only two posibilities: $B\cap M=\{b_0, b_2, b_4, \dots, b_{k-2}\}$ or $B\cap M=\{b_1, b_3, \dots, b_{k-1}\}$, k even. Analogous possibilities remain for $A\cap M$. By testing each of the possible cases we see that they all lead to contradiction.

Thus $\chi(G)=3$ and so G is 3-critical.

Construction (ii). Let $k\in\mathbf{N}$ be given. Define a hypergraph $H=\langle V(H), E(H)\rangle$ by $V(H)=\bigcup\limits_{i=0}^{k}\bigcup\limits_{j=0}^{k^i-1} A_{i,j}$, where $A_{i,j}$ are pairwise disjont sets, $|A_{i,j}|=3\cdot 2^i$ for every i,j. Put $A_{i,j}=\{a^1_{i,j}, \dots, a^{3\cdot 2^i}_{i,j}\}$. For $0\leqq j\leqq k^k-1$ denote $f^i(j)=\left[\frac{j}{k^i}\right]$ (integer part). To have less indices, we put $f^1(j)=f(j)$.

Let us define $E(H)$:

$$E(H)=\{A_{0,0}\}\cup\{\{a_{i,j}^{2r-1},\,a_{i,j}^{2r},\,a_{i-1,f(j)}^{r}\},\quad i=1,2,\ldots,k,\ j=0,1,\ldots,k^{i}-1,\ r=1,2,\ldots,3\cdot 2^{i-1}\}.$$

Clearly, H is a 3-uniform hypergraph. For $e_1\in E(H)$, $e_2\in E(H)$ we have $|e_1\cap e_2|\leqq\leqq 1$. If $x\in A_{i,j}$, $i\leqq k-1$ then $\operatorname{val} x=k+1$.

LEMMA 2. *Let* $0\leqq j\leqq k^k-1$. *Then there exists no 2-colouring* M, N *of the above hypergraph such that* $A_{k,j}\subset M$.

PROOF. Assume that M, N is a 2-colouring such that $A_{k,j}\subset M$. It is clear from the construction that $A_{k-1,f(j)}\subset N$. In a succession we get $A_{k-2,f^2(j)}\subset M$, $A_{k-3,f^3(j)}\subset N,\ldots, A_{0,0}\subset M$ or $A_{0,0}\subset N$ (depending on the parity of k). This is a contradiction to $A_{0,0}\in E(H)$.

LEMMA 3. *Choose* $e\in E(H)$. *There exists* j_0, $0\leqq j_0\leqq k^k-1$ *and a 2-colouring* M, N *of the hypergraph* H-e *such that* $A_{k,j_0}\subset M$.

PROOF. If $e=A_{0,0}$, the proposition is clear. Put simply

$$M=\bigcup_{k-i\text{ even}}A_{i,j},\quad N=\bigcup_{k-i\text{ odd}}A_{i,j}.$$

Let $e=\{a_{\bar i,\bar j}^{2\bar r-1},\,a_{\bar i,\bar j}^{2\bar r},\,a_{\bar i-1,f(\bar j)}^{\bar r}\}$. Put $j_0=k^{k-\bar i}\cdot\bar j$ and let us construct a 2-colouring M, N of the hypergraph H:

1) $i\geqq\bar i,\ j\neq\bar j\cdot k^{i-\bar i}$ $\quad r$ even $\Rightarrow a_{i,j}^r\in M$

r odd $\Rightarrow a_{i,j}^r\in N$

2) $i\geqq\bar i,\ j=\bar j\cdot k^{i-\bar i}$ $\quad k-i$ even $\Rightarrow A_{i,j}\subset M$

$k-i$ odd $\Rightarrow A_{i,j}\subset N$

3) $i<\bar i,\ j\neq f^{\bar i-i}(\bar j)$ $\quad r$ even $\Rightarrow a_{i,j}^r\in M$

r odd $\Rightarrow a_{i,j}^r\in N$

4) $i<\bar i,\ j=f^{\bar i-i}(\bar j)$ $\quad k-i$ odd, $\quad r\neq\left[\dfrac{2\bar r+1}{2^{\bar i-i}}\right]\Rightarrow a_{i,j}^r\in N$

$$r=\left[\frac{2\bar r+1}{2^{\bar i-i}}\right]\Rightarrow a_{i,j}^r\in M$$

$k-i$ even, $\quad r\neq\left[\dfrac{2\bar r+1}{2^{\bar i-i}}\right]\Rightarrow a_{i,j}^r\in M$

$$r=\left[\frac{2\bar r+1}{2^{\bar i-i}}\right]\Rightarrow a_{i,j}^r\in N.$$

We can easily see that M, N is a 2-colouring of $H-e$ with the required properties.

LEMMA 4. *Let* $0 \leqq j_0 \leqq k^k - 1$, $0 \leqq \bar{j}_0 \leqq k^k - 1$, $\emptyset \neq V \subsetneqq \{1, 2, \ldots, 3.2^k\}$. *Then there exists a 2-colouring* M, N *of* H *such that* $a^r_{k, j_0} \in M \Leftrightarrow r \in V$, $a^r_{k, \bar{j}_0} \in M \Leftrightarrow r \in V$ *and for* $j \neq j_0, \bar{j}_0$, $a^r_{k,j} \in M \Leftrightarrow r$ *odd*, $a^r_{k,j} \in M \Leftrightarrow r$ *even.*

Sketch of the proof. Suppose we have constructed a 2-colouring M, N of $H/\bigcup_{i \geqq i_0} \bigcup_{j=0}^{k^i-1} A_{i,j}$ $(1 \leqq i_0 \leqq k)$ such that there exists $j_{i_0}, \bar{j}_{i_0}$ (it is possible that $j_{i_0} = \bar{j}_{i_0}$) and it holds for $j \neq j_{i_0}, j \neq \bar{j}_{i_0}$ $a^r_{i_0,j} \in M \Leftrightarrow r$ odd, $a^r_{i_0,j} \in N \Leftrightarrow r$ even, $M \cap A_{i_0, j_{i_0}} \neq \emptyset$, $N \cap A_{i_0, j_{i_0}} \neq \emptyset$ and $a^r_{i_0, \bar{j}_{i_0}} \in M \Leftrightarrow a^r_{i_0, j_{i_0}} \in M$.

We shall extend the 2-colouring M, N with the same property on $\bigcup_{i \geqq i_0 - 1} \bigcup_{j=0}^{k^i-1} A_{i,j}$. We distinguish two cases: 1) $f(j_{i_0}) \neq f(\bar{j}_{i_0})$ and 2) $f(j_{i_0}) = f(\bar{j}_{i_0})$. The proof of the induction step is easy to see in both cases.

Construction (iii). Let k be a positive integer and let $R = \langle V(R), E(R) \rangle$ be a 3-critical hypergraph with the following properties:

a) $V(R) = T \cup U \cup V$ where T, U, V are pairwise disjoint

b) $e \in E(R)$ implies $|e \cap T| \leqq 1$ or $e \subset T$

c) $e_1, e_2 \in E(R)$ implies $|e_1 \cap e_2| \leqq 1$

d) $x \in V(R)$ implies val $x \geqq k$

e) $V \in E(R)$, $|V| = 3 \cdot 2^k$

f) for every edge $e \in E(R) - \{V\}$, $|e \cap T| \leqq 1$ there exist two 2-colourings M_1, N_1 and M_2, N_2 of $R - \{V\}$ and a 2-colouring M_3, N_3 of $R - e$ such that $M_1 \cap T = M_2 \cap T = M_3 \cap T$ and $V \subset M_1$, $V \subset N_2$ (i.e. all three colourings coincide on T and the first two differs on V).

To every 3-critical hypergraph having properties a)$-$f) we construet a 3-critical hypergraph $\tilde{B}$. This operation will be defined in 3 steps. (Then we will use this operation iterated twice on G of lemma 1 to get Theorem 3.)

(1) First we choose $K \geqq k^{k+1}$ (K odd) copies of $R - \{V\}$ which we "merge" in the set T. More preciesly: We shall construct a hypergraph $R_1 = \langle V(R_1), E(R_1) \rangle$ as follows:

$$V(R_1) = T \cup (V(R) - T) \times \{0, 1, 2, \ldots, K-1\} \quad \text{(a disjoint union)}$$

$$B = \{\langle b_1, i_1 \rangle, \ldots, \langle b_s, i_s \rangle, t_1, t_2, \ldots, t_{\bar{s}}\} \quad \text{where} \quad t_1, t_2, \ldots, t_{\bar{s}} \in T,$$

$$b_1, b_2, \ldots, b_s \in V(R) - T, \quad 0 \leqq i_1, \ldots, i_s \leqq K-1$$

$$B \in E(R_1) \Leftrightarrow i_1 = i_2 = \ldots = i_s \quad \text{and} \quad \{b_1, b_2, \ldots, b_s, t_1, \ldots, t_{\bar{s}}\} \in E(R) - \{V\}.$$

(2) LEMMA 5. *Let* $K \geqq k^{k+1}$ *(K odd). Denote* $M = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle, \ldots, \langle K-2, K-1 \rangle, \langle K-1, 0 \rangle\}$. *Then there exists a mapping* $g\colon M \to \{0, 1, \ldots, k^k - 1\}$ *such that for all* $i = 0, 1, \ldots, K-1$, $g(\langle i-1, i \rangle) \neq g(\langle i, i+1 \rangle)$ *(addition is taken mod K) and* $|g^{-1}(i)| \geqq k$.

PROOF. Clear.

In what follows, we put $g(i, i+1)$ instead of $g(\langle i, i+1 \rangle)$.

Put $V = \{v_r\colon 1 \leqq r \leqq 3 \cdot 2^k\}$.

Now we shall construct a hypergraph $R_2=\langle V(R_2), E(R_2)\rangle$ as follows:

$$V(R_2)=V(R_1)\cup\bigcup_{j=0}^{k^k-1}A_{k,j},\quad |A_{k,j}|=3\cdot 2^k,\quad A_{k,j}=\{a_j^1,\dots,a_j^{3\cdot 2^k}\},$$

$$E(R_2)=E(R_1)\cup\big\{\{\langle v_r,i\rangle,\langle v_r,i+1\rangle,a^r_{g(i,i+1)}\};\ 0\leqq i\leqq K-1,\ 1\leqq r\leqq 3\cdot 2^k\big\}.$$

Here $i+1$ is taken $\bmod K$ and g is a function from the foregoing Lemma.

(3) To the hypergraph R_2 we shall adjoin a hypergraph H by identifying the sets $A_{k,j}$ in both constructions. Hence we shall construct a hypergraph $\tilde{R}=\langle V(\tilde{R}), E(\tilde{R})\rangle$ where $V(\tilde{R})=V(R_2)\cup V(H)$ (suppose $V(R_2)\cap V(H)=\bigcup_{j=0}^{k^k-1}A_{k,j}$), $E(\tilde{R})=E(R_2)\cup E(H)$.

Lemma 6. *The above hypergraph $\tilde{R}$ has the following properties:*

1) $x\in V(\tilde{R})\Rightarrow \operatorname{val} x\geqq k$

2) $e_1,e_2\in E(\tilde{R}),\ e_1\neq e_2\Rightarrow |e_1\cap e_2|\leqq 1$

3) *$\tilde{R}$ is a 3-critical hypergraph.*

Proof. Properties 1) and 2) are obvious. Let us prove property 3).

a) $\chi(G)\geqq 3$. On the contrary, let M, N be a 2-colouring of the hypergraph $\tilde{R}$. Since R is 3-critical the set $V\times\{i\}$ is always one-coloured for $i=0,1,\dots,K-1$. As K is an odd number there exists $i\in\{0,1,\dots,K-1\}$ such that the sets $V\times\{i\}$, $V\times\{i+1\}$ are coloured by the same colour ($i+1$ is taken mod K). Let e.g. $V\times\{i\}\subset M$, $V\times\{i+1\}\subset M$. From the construction of R_2 there follows $A_{k,j}\subset N$ where $j=g(i,i+1)$.

Thus $M\cap V(H), N\cap V(H)$ is a 2-colouring of the hypergraph H such that the set $A_{k,j}$ is one-coloured. This contradicts Lemma 2.

b) Let $e\in E(\tilde{R})$. We shall prove $\chi(\tilde{R}-e)=2$.

b1) Let $e\in E(H)$. According to Lemma 3 there exists $j\in\{0,1,\dots,k^k-1\}$ and a 2-colouring M, N of the hypergraph H such that $A_{k,j}\subset M$. Let us choose $i\in\{0,1,\dots,K-1\}$ such that $g(i,i+1)=j$. We shall extend the colouring M, N of the hypergraph H to $\tilde{R}$ as follows: $V\times\{i\}\subset N$, $V\times\{i+1\}\subset N$, $V\times\{i+i'\}\subset M$ for i' even, $2\leqq i'\leqq K-1$, $V\times\{i+i'\}\subset N$ for i' odd, $1\leqq i'\leqq K-2$. (The addition is taken mod K.) Due to property f) of R, we can extend this colouring of sets $V\times\{i\}$, $i=0,1,\dots,K-1$ to the whole of R_1.

b2) Let $e\in E(R_2)-E(R_1)$ (i.e. $e=\{\langle v_{\bar r},\bar i\rangle,\langle v_{\bar r},\bar i+1\rangle,a^{\bar r}_{g(\bar i,\bar i+1)}\}$). Let us define a colouring M, N of $\tilde{R}-e$:

$$V\times\{\bar i\}\subset M,\quad V\times\{\bar i+1\}\subset M,\quad V\times\{\bar i+i'\}\subset M\ \text{ if } i' \text{ is odd},\ 1\leqq i'\leqq K-2,$$

$$V\times\{\bar i+i'\}\subset N\ \text{ if } i' \text{ is even},\ 2\leqq i'\leqq K-1.$$

Analagously to b1), we can extend the colouring to $V(R_1)$. Put further $j_0=g(\bar i,\bar i+1)$, $a^{\bar r}_{k,j_0}\in M$, $a^r_{k,j_0}\in N$ for $r\neq\bar r$. Let $j\neq j_0$ then $a^r_{k,j}\in M$ iff r is odd, $a^r_{k,j}\in N$ iff r is even.

Via Lemma 4 we can extend this colouring to the whole of $V(H)$.

b3) Let $e \in E(R_1), e \subset T \cup V(R) \times \{i_0\}$ and $|e \cap T| \leqq 1$. Via f) there exists a 2-colouring M, N of $R_1 - \{e\}$ such that

$$V \times \{i_0 + i'\} \subset N \quad \text{if} \quad i' \quad \text{is odd}, \quad 1 \leqq i' \leqq K-2,$$

$$V \times \{i_0 + i'\} \subset M \quad \text{if} \quad i' \quad \text{is even}, \quad 2 \leqq i' \leqq K-1,$$

$$V \times \{i_0\} \cap M \neq \emptyset \neq V \times \{i_0\} \cap N.$$

We extend this colouring to $V(R_2)$. For $j \neq g(i_0, i_0+1)$, $j \neq g(i_0-1, i_0)$ put $a^r_{k,j} \in M$ iff r is odd, $a^r_{k,j} \in N$ iff r is even. Let $j = g(i_0, i_0+1)$ or $j = g(i_0-1, i_0)$. Put $a^r_{k,j} \in N$ iff $\langle v_r, i_0 \rangle \in M$, $a^r_{k,j} \in M$ iff $\langle v_r, i_0 \rangle \in N$. According to Lemma 4 this 2-colouring M, N can be extended to the hypergraph $\tilde{R}$.

4) Let $e \subset T$. The proof in this case is only a matter of routine.

THEOREM 3. *Let $k \in N$. Then there exists a 3-uniform 3-critical hypergraph $X = \langle V(X), E(X) \rangle$ such that:*

1) $e_1, e_2 \in E(X)$, $e_1 \neq e_2 \Rightarrow |e_1 \cap e_2| \leqq 1$,

2) $x \in V(X) \Rightarrow \operatorname{val} x \geqq k$.

PROOF. The hypergraph G from Lemma 1 fulfils almost all conditions of our theorem; only two hyperedges in G have cardinality bigger than 3. We shall remove this deficiency in two steps.

1) There exists a hypergraph which fulfils 1), 2) and contains only one hyperedge with cardinality bigger than 3. Let namely $G = \langle V(G), E(G) \rangle$ be the hypergraph of Lemma 1 with $3 \cdot 2^k$ instead of k. We are using the notation of Construction (iii). Put $T = B$, $U = C$, $V = A$. It is easy to check that G fulfils all of the conditions in Construction (iii), hence we can construct $\tilde{G}$ in the described way.

By the above Proposition, $\tilde{G}$ is 3-critical and fulfils:

$$e_1, e_2 \in E(\tilde{G}), \quad e_1 \neq e_2 \Rightarrow |e_1 \cap e_2| \leqq 1, \quad x \in V(\tilde{G}) \Rightarrow \operatorname{val} x \geqq k.$$

Moreover, $\tilde{G}$ contains just one hyperedge with more than three points.

2) We apply the above construction to $\tilde{G}$ once more (put $T = \emptyset$, $U = V(\tilde{G}) - B$, ($V = B$ to obtain a hypergraph $\tilde{\tilde{G}} = X$ which fulfils all of the conditions in our theorem.

THEOREM 4. *Let k, r, h be natural numbers, $k \geqq 3, r \geqq 3$. Then there exists a k-critical r-uniform hypergraph X with* $\operatorname{qval}(x) \geqq h$ *for every vertex $x \in V(X)$.*

Outline of the proof. For $k \geqq 4$ the theorem is proved in [5], Theorem 8. The case $k=3, r=3$ is proved in Theorem 3. The rest $(k=3, r \geqq 4)$ can be proved by induciton on r. Let $G = \langle V(G), E(G) \rangle$ be a 3-critical r-uniform hypergraph with all quasivalences $\geqq h$ which contains at least h disjoint edges (the hypergraph from Theorem 3 satisfies this condition with $r=3$). We shall construct such a hypergraph for $r+1$. First in the induction step we prove the folloving

LEMMA 7. *Let r, h be natural numbers, $r \geqq 3$. Then there exists a 3-critical hypergraph $H = \langle V(H), E(H) \rangle$ and $x \in V(H)$ such that the following conditions hold:*

1) $e \in E(H),\ x \in e \Rightarrow |e| = 2$

2) $e \in E(H),\ x \notin e \Rightarrow |e| = r+1$

3) $\operatorname{val}(x) \geqq h$

4) $\operatorname{qval}(y) \geqq h$ *for every* $y \in V(H)$ *such that* $\{x, y\} \notin E(H)$.

Outline of the proof of Lemma 7. Let G be a 3-critical r-uniform hypergraph with $\operatorname{qval}(y) \geqq h$ for every $y \in V(G)$. Let $G_1, G_2, \ldots, G_{r+1}$ be $r+1$ disjoint copies of G. Define the hypergraph H by

$$V(H) = \{x\} \cup \bigcup_{i=1}^{r+1} V(G_i) \cup \bigcup_{i=1}^{r+1} \big(E(G_i) \times \{1, 2\}\big)$$

$$E(H) = E_1 \cup E_2 \cup E_3 \cup E_4, \quad \text{where} \quad E_1 = \Big\{\{x, y\},\ y \in \bigcup_{i=1}^{r+1} E(G_i) \times \{1\}\Big\}$$

$$E_2 = \Big\{e \cup \{y\},\ e \in \bigcup_{i=1}^{r+1} E(G_i) \quad \text{and} \quad y = \langle e, 1\rangle\Big\}$$

$$E_3 = \Big\{e \cup \{y\},\ e \in \bigcup_{i=1}^{r+1} E(G_i) \quad \text{and} \quad y = \langle e, 2\rangle\Big\}$$

$$E_4 = \Big\{\{y_1, y_2, \ldots, y_{r+1}\}, \quad \text{where} \quad y_i \in E(G_i) \times \{2\}, \quad i = 1, 2, \ldots, r+1\Big\}.$$

It is only matter of routine to prove that H satisfies all the conditions of Lemma 7.

Now we can prove Theorem 4 using the Toft's method (see [5], Theorem 8). We apply this method to the hypergraph H just constructed and to the hypergraph G from the induction assumption.

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MATHEMATICAL INSTITUTE ČSAV
110 00 PRAHA, ŽITNÁ 28
CZECHOSLOVAKIA

FACULTY OF NUCLEAR ENGINEERING ČVUT
DEPARTMENT OF MATHEMATICS
110 00 PRAHA 1, HUSOVA 5

VŠCHT DEPARTMENT OF MATHEMATICS
160 00 PRAHA 6, STUDENSKÁ 6
CZECHOSLOVAKIA

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EXISTENCE AND BOUNDEDNESS OF RANDOM SOLUTIONS TO STOCHASTIC FUNCTIONAL INTEGRAL EQUATIONS

By
A. N. V. RAO and C. P. TSOKOS (Tampa)*

1. Introduction

The object of the present paper is to study stochastic functional integral equations of the type

$$x(t;\omega) = h(t;\omega)+\int_0^t k(t,\tau;\omega)f(\tau, x_\tau(\omega))\,d\beta(\tau) \tag{1.1}$$

and

$$x(t;\omega) = h(t;\omega)+\int_0^\infty k(t-\tau;\omega)f(\tau, x_\tau(\omega))\,d\beta(\tau) \tag{1.2}$$

where $t\in\mathbf{R}_+$, $\beta(t)$ a process, $\omega\in\Omega$ the supporting set of a complete probability measure space (Ω, A, μ); $x(t;\omega)$ is the unknown random function defined for $t\in\mathbf{R}_+$ and $\omega\in\Omega$; $h(t;\omega)$ is the *stochastic free term* defined for $t\in\mathbf{R}_+$ and $\omega\in\Omega$; $k(t,\tau;\omega)$ is the *stochastic kernel* defined for $0\leqq t\leqq\tau<\infty$ and $\omega\in\Omega$; $x_t(\omega)$ is the restriction of the function $x(\tau)$ to the interval $[0, t]$, $\tau\leqq t<\infty$, $L_2=L_2(\Omega, A_t, \mu)$.

We shall be concerned with the existence of a random solution, a second order stochastic process and conditions will be given so that the second moments of the random solution are bounded.

Stochastic functional integral equations are important in engineering sciences, especially in feed-back control systems with hysterisis, [2], [3]. That is, functional integral equations where the kernel is random and the integration is with respect to a process, a special case of which is Brownian motion.

2. Basic concepts

We shall assume for the process $\beta(t)$ that for every $t\in\mathbf{R}_+$, a minimal σ-algebra $A_t\subset A$ is defined so that the random variable $\beta(t;\omega)$ is measurable. In addition we shall assume that the process $\{\beta(t;\omega), A_t, 0\leqq t<\infty\}$ is a martingale and that there is a continuous, monotone increasing function $F(t)$, $t\in\mathbf{R}_+$, such that, for $s<t$,

$$E\{|\beta(t;\omega)-\beta(s;\omega)|^2\} = E\{|\beta(t;\omega)-\beta(s;\omega)|^2A_s\} = F(t)-F(s), \quad \mu\text{-a.e.}$$

For details of the above statements see Doob [1]. We should also remark that if $F(t)=ct$, c a constant, and if the martingale is real and almost all sample functions are continuous, the process $\beta(t;\omega)$ is a Brownian process.

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We shall denote by $C_c=C_c[\mathbf{R}_+, L_2(\Omega, A_t, \mu)]$ a linear space of all continuous maps $x(t;\omega)$, from $\mathbf{R}_+$ into L_2. We shall define the topology in this space by means of the following family of semi-norms:

$$\|x(t;\omega)\|_n = \sup_{0\leqq t\leqq n}\Big\{\int_\Omega |x(t;\omega)|^2\,d\mu(\omega)\Big\}^{1/2}.$$

This topology is metrizable and the space C_c is complete, [4]. For convenience we shall dentoe

$$\|x(t;\omega)\|_{L_2} = \Big\{\int_\Omega |x(t;\omega)|^2\,d\mu(\omega)\Big\}^{1/2}.$$

Further, we shall assume that $k(t,\tau;\omega)$ is A_τ measurable, μ-essentially bounded and are continuous as maps from

$$\Delta = \{(t,\tau);\ 0\leqq\tau\leqq t<\infty\}\quad\text{into}\quad L_\infty = L_\infty(\Omega, A, \mu),$$

and we shall denote by

$$|||k(t,\tau;\omega)||| = \mu\text{-}\operatorname*{ess\,sup}_{\omega\in\Omega}|k(t,\tau;\omega)|.$$

We shall call $x(t;\omega)$ a *random solution* of the stochastic functional integral equation (1.1) or (1.2) if $x(t;\omega)\in C$ for each $t\in\mathbf{R}_+$ and satisfies equation (1.1) or (1.2) μ-a.e.

Finally, we shall employ the following well known fixed point theorem.

THEOREM 2.1 (Schauder and Tychonoff). *Let G be a closed convex set in a Banach space and let T be a completely continuous operator on B, such that, $T(G)\subset G$. Then T has at least one fixed point in G. That is, there is at least one $x^*\in G$, such that, $T(x^*)=x^*$.*

3. Existence theorems

THEOREM 3.1. *Consider the stochastic functional integral equation* (1.1) *under the following conditions:*

(i) *the mapping $x(t;\omega)\to f(t, x_t(\omega))$ is a completely continuous map from $C_c[\mathbf{R}_+, L_2(\Omega, A_t, \mu)]$ into itself;*

(ii) *there exist two continuous non-negative real functions $g(t)$ and $l(t), t\in\mathbf{R}_+$, such that,*

a) $\|x(t;\omega)\|_{L_2}\leqq g(t)$ *implies* $\|f(t,x_t(\omega))\|_{L_2}\leqq l(t)$;

b) $\|h(t;\omega)\|_{L_2}+\Big\{\int_0^t |||k(t,\tau;\omega)|||^2 l^2(\tau)\,d\tau\Big\}^{1/2}\leqq g(t)$.

Then there is at least one random solution, $x(t;\omega)$, to equation (1.1) *in C_c, such that,*

$$\|x(t;\omega)\|_{L_2}\leqq g(t),\quad t\in\mathbf{R}_+.$$

PROOF. In the space C_c we shall define the set

$$\Lambda = \{x(t;\omega)\colon\ \|x(t;\omega)\|_{L_2}\leqq g(t),\quad t\in\mathbf{R}_+\}.$$

It is clear that the set Λ is a closed convex set. For each $x(t;\omega)\in\Lambda$, let us define the operator U, by

$$(Ux)(t;\omega)=h(t;\omega)+\int_0^t k(t,\tau;\omega)f(\tau,x_\tau(\omega))\,d\beta(\tau).$$

From the assumptions of continuity on h, k and f, it is clear that the operator U maps Λ into C_c. We shall show that $U\Lambda\subset\Lambda$. Let $x(t;\omega)\in\Lambda$. Then

$$\|(Ux)(t;\omega)\|_{L_2}\leqq\|h(t;\omega)\|_{L_2}+\Big\|\int_0^t k(t,\tau;\omega)f(\tau,x_\tau(\omega))\,d\beta(\tau)\Big\|_{L_2}=$$

$$=\|h(t;\omega)\|_{L_2}+\Big\{\int_0^t\|k(t,\tau;\omega)f(\tau,x_\tau(\omega))\|_{L_2}^2\,d\tau\Big\}^{1/2}\leqq$$

$$\leqq\|h(t;\omega)\|_{L_2}+\Big\{\int_0^t|||k(t,\tau;\omega)|||^2\|f(\tau,x_\tau(\omega))\|_{L_2}^2\,d\tau\Big\}^{1/2}\leqq$$

$$\leqq\|h(t;\omega)\|_{L_2}+\Big\{\int_0^t|||k(t,\tau;\omega)|||^2 l^2(\tau)\,d\tau\Big\}^{1/2}\leqq g(t).$$

This shows that $U\Lambda\subset\Lambda$. Furthermore, U is a continuous map from Λ into itself, as shown by the following chain of inequalities. On each compact interval $[0,T]$ and for $x(t;\omega), y(t;\omega)\in\Lambda$, we have

$$\sup_{0\leqq t\leqq T}\|(Ux)(t;\omega)-(Uy)(t;\omega)\|_{L_2}=$$

$$=\sup_{0\leqq t\leqq T}\Big\|\int_0^t k(t,\tau;\omega)\{f(\tau,x_\tau(\omega))-f(\tau,y_\tau(\omega))\}\,d\beta(\tau)\Big\|_{L_2}=$$

$$=\sup_{0\leqq t\leqq T}\Big\{\int_0^t\|k(t,\tau;\omega)\{f(\tau,x_\tau(\omega))-f(\tau,y_\tau(\omega))\}\|_{L_2}^2\,dF(\tau)\Big\}^{1/2}\leqq$$

$$\leqq\sup_{0\leqq t\leqq T}\Big\{\int_0^t|||k(t,\tau;\omega)|||^2\|\{f(\tau,x_\tau(\omega))-f(\tau,y_\tau(\omega))\}\|_{L_2}^2\,dF(\tau)\Big\}^{1/2}\leqq$$

$$\leqq\sup_{0\leqq t\leqq T}\|f(\tau,x_\tau(\omega))-f(\tau,y_\tau(\omega))\|_{L_2}\sup_{0\leqq t\leqq T}\Big\{\int_0^t|||k(t,\tau;\omega)|||^2\,dF(\tau)\Big\}^{1/2}.$$

The continuity assumptions on f and k and the last inequality proves the continuity of U. Since f is completely continuous by assumption, it follows that $U\Lambda$ is relatively compact. This allows us to apply the Schauder—Tychonoff fixed point theorem to the pair (Λ, U), which proves the existence of a random solution to equation (1.1) in the set Λ.

THEOREM 3.2. *Assume that the stochastic functional integral equation* (1.2) *satisfies the following conditions:*

(i) $\sup_{0 \leqq t} \|h(t; \omega)\|_{L_2} < M, \quad M \in \mathbf{R}_+;$

(ii) $\int_0^\infty |||k(t; \omega)|||^2 \, dt \leqq K, \quad K \in \mathbf{R}_+;$

(iii) *the mapping* $x(t; \omega) \to f(t, x_t(\omega))$ *is a completely continuous map from* C_c *into itself, such that,*

$$\|x(t; \omega)\|_{L_2} \leqq M + \sqrt{k}\Phi(M)$$

implies

$$\|f(t, x_t(\omega))\|_{L_2} \leqq \Phi(M), \quad t \in \mathbf{R}_+,$$

where $\Phi(M)$ *is positive real valued function defined for sufficiently large* M.

Then there exists at least one solution to equation (3.2) *in* C_c *such that* $\|x(t; \omega)\|_{L_2}$ *is bounded.*

PROOF. We shall give only a sketch of the proof since most of the details are similar to that of Theorem 3.1.

Define the set Λ,

$$\Lambda = \{x(t; \omega)\colon x(t; \omega) \in C_c[\mathbf{R}_+, L_2(\Omega, A_t, \mu)] \quad \text{and} \quad \|x(t; \omega)\|_{L_2} \leqq M + k\Phi(M)\}$$

and operator U,

$$(Ux)(t; \omega) = h(t; \omega) + \int_0^\infty k(t-\tau; \omega) f(\tau, x_\tau(\omega)) \, d\tau.$$

Using the assumption placed on the stochastic kernel and the continuity of f, it can be shown that U is a continuous map from Λ into C_c. Also, if $x(t; \omega) \in \Lambda$, we have

$$\|(Ux)(t; \omega)\|_{L_2} \leqq \|h(t; \omega)\|_{L_2} + \left\| \int_0^\infty k(t-\tau; \omega) f(\tau, x_\tau(\omega)) \, d\beta(\tau) \right\|_{L_2} \leqq$$

$$\leqq \|h(t; \omega)\|_{L_2} + \left\{ \int_0^\infty |||k(t-\tau; \omega)|||^2 \|f(\tau, x(\tau; \omega))\|_{L_2}^2 \, dF(\tau) \right\}^{1/2} \leqq$$

$$\leqq \|h(t; \omega)\|_{L_2} + \sqrt{K}\Phi(M) \leqq M + \sqrt{K}\Phi(M)$$

which implies that $U\Lambda \subset \Lambda$.

The rest of the proof follows that of Theorem 3.1.

4. Boundedness of random solutions

In this section we shall give conditions to insure that the second moments of the random solution are bounded.

COROLLARY 4.1. *Consider the stochastic functional integral equation* (1.1) *subject to the following conditions:*

(i) *the mapping* $x(t;\omega)\to f(t, x_t(\omega))$ *is a completely continuous map from* C_c *into itself;*

(ii) $\|h(t;\omega)\|_{L_2}$ *is bounded on* $t\in\mathbf{R}_+$;

(iii) $\|f(t, x_t(\omega))\|_{L_2}\leqq\lambda(t)|x_t(\omega)|$, $\lambda(t)$ *is positive continuous function on* $\mathbf{R}_+$ *and* $|x_t(\omega)|=\sup\limits_{0\leqq s\leqq t}\|x(s;\omega)\|_{L_2}$;

(iv) *there exists a real number* $m<1$, *such that,*

$$\int_0^t |||k(t,\tau;\omega)|||^2\lambda^2(s)\,ds\leqq m,\quad t\in\mathbf{R}_+.$$

Then, there exists at least one random solution of equation (1.1) *in* C_c *such that,* $\|x(t;\omega)\|_{L_2}\leqq M$, $M>0$.

PROOF. It is clear that we only need to show that condition (ii) of Theorem 3.1 is satisfied. We shall choose $g(t)$ to be equal to m and let $l(t)=M\lambda(t)$, where M is some sufficiently large positive real number. Then condition (ii) a) of Theorem 3.1 is satisfied. Further,

$$\|h(t;\omega)\|_{L_2}+\Big\{\int_0^t |||k(t,\tau;\omega)|||^2 l^2(\tau)\,dF(\tau)\Big\}^{1/2}\leqq\sup_{0\leqq t}\|h(t;\omega)\|_{L_2}+M\sqrt{m}\leqq M,$$

if $M\geqq U(-1\sqrt{m})^{-1}\{\sup\limits_{0\leqq t}\|h(t;\omega)\|_{L_2}\}$. Thus, the corollary follows from Theorem 3.1.

The following corollary is concerned with the exponential decay of the L_2-norm of the random solutions to equation (1.1).

COROLLARY 4.2. *Consider the stochastic functional integral equation* (1.1) *subject to the following conditions:*

(i) *the mapping* $x(t;\omega)\to f(t, x_t(\omega))$ *is a completely continuous map from* C_c *into itself;*

(ii) $\|h(t;\omega)\|_{L_2}\leqq h_0e^{-\alpha t},\quad t\in\mathbf{R}_+,\ h_0>0;$

(iii) $\|f(t, x_t(\omega))\|_{L_2}\leqq Le^{-\beta t}|x_t(\omega)|,\ \alpha<\beta,\ L>0;$

(iv) $|||k(t,\tau;\omega)|||\leqq Ke^{-\alpha(t-\tau)},\ 0\leqq\tau\leqq t<\infty.$

Then, there exists at least one random solution to equation (1.1), *such that,*

$$\|x(t;\omega)\|_{L_2}\leqq\gamma e^{-\alpha t},\quad t\in\mathbf{R}_+,$$

provided $h_0+KL\gamma\{2(\beta-\alpha)\}^{-1/2}\leqq\gamma$.

For the proof, choose $g(t)=\gamma e^{-\alpha t}$ and $l(t)=\gamma L e^{-\beta t}$. With these choices, it is not difficult to show that the conditions (ii) a) and b) of Theorem 3.1 are satisfied. Thus, the corollary follows from Theorem 3.1.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH FLORIDA
TAMPA, FLORIDA 33620
USA

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INTERVALLES A RESTES MAJORÉS POUR LA SUITE $\{n\alpha\}$

Par
Y. DUPAIN (Talence)

§. 0. Introduction

Soient $u=(u_n)_{n\in\mathbf{N}}$ une suite de points de $[0, 1[$ et I un intervalle de $[0, 1]$. Nous appellerons $N^{\text{ième}}$ reste de I pour la suite u la quantité $\varphi(I, N)=$ $=\text{card}\,\{n;\, n\in\mathbf{N},\, n<N,\, u_n\in I\}-N\mu(I)$ (μ désignant la mesure de Lebesgue de $[0, 1[$). La suite u étant fixée, nous dirons que I est à restes bornés (respectivement majorés, minorés) si la suite des restes $N\to\varphi(I, N)$ est bornée (respectivement majorée, minorée). Dans cet article nous considérerons uniquement la suite $n\to\{n\alpha\}$, où α est un irrationnel fixé. Dans ce cadre, H. KESTEN démontre en 1966 [2] qu un intervalle I est à restes bornés si et seulement si $\mu(I)\equiv z\alpha \pmod 1$ (où $z\in\mathbf{Z}$), en étudiant des intervalles ayant 0 pour origine et en utilisant le théorème de BOHL [1] « pour un intervalle la propriété d'être à restes bornés ou non ne dépend que de la longueur de l'intervalle ». Nous nous intéressons au problème suivant: un intervalle à restes non bornés peut-il être à restes majorés? Notons que VERA T. SÓS dans [5] énonce le résultat: « Il existe des irrationnels α et des intervalles de longueur β ($\beta\not\equiv z\alpha$) à restes majorés pour la suite $\{n\alpha\}$ ». Nous précisons ce résultat, et remarquons que, contrairement au cas précédent, la propriété pour un intervalle d'être à restes majorés dépend de la longueur de l'intervalle mais aussi de son origine (§. 6).

Nous obtiendrons, entre autres, les résultats suivants:

1) Si la constante de Markov de α $M(\alpha)$[1] est infinie il existe des intervalles à restes majorés et non minorés (§. 3).

2) Si $\alpha=\dfrac{1+\sqrt{5}}{2}$, tout intervalle I ayant 0 pour extrémité à restes non bornés $\big(\mu(I)\not\equiv z\alpha\,[\text{mod } 1]\big)$ est à restes non majorés et non minorés (§. 4).

3) Nous répondrons négativement à une question posée par M. Keane aux journées ergodiques de Rennes (juin 1974): « Pour tout irrationnel α, tout intervalle de longueur 1/2 est-il à restes non majorés »? (§. 5).

§. 1. Définitions. Notations. Rappels

Nous nous servirons des notations et résultats dus à J. LESCA ([3] et [4]). Dans tout ce qui suit α est un irrationnel fixé.

Pour $\beta\in[0, 1[$ et $u\in\mathbf{N}^*$ posons

$$\varphi^-(\beta, u) = \text{card}\,\big\{n;\ n\in\mathbf{N},\ 0\leqq n<u,\ \{n\alpha\}\in[0, \beta[\big\}-u\mu([0, \beta[)$$

$$\varphi^+(\beta, u) = \text{card}\,\big\{n;\ n\in\mathbf{N},\ 0<n\leqq u,\ \{n\alpha\}\in]0, \beta]\big\}-u\mu(]0, \beta]).$$

[1] $M(\alpha)=\overline{\lim}\,\dfrac{1}{q\,\|q\alpha\|}$, où $\|q\alpha\|$ désigne la distance de $q\alpha$ à l'entier le plus proche.

Relation de réciprocité (J. LESCA [3] et [4]).

1.0. — Pour tout $(u, v) \in \mathbf{N}^* \times \mathbf{N}^*$

$$\varphi^+(\{u\alpha\}, v) = \varphi^-(\{v\alpha\}, u).$$

Soient (p_n/q_n) la suite des convergents du développement de α en fraction continue et (a_n) la suite des quotients partiels $(q_{n+1} = a_n q_n + q_{n-1})$

1.1. — Pour tout $\beta \in [0, 1[$ et tout indice $k \geqq 1$,

$$|\varphi^-(\beta, q_k)| < 1.$$

Posons $\theta_n = q_n \alpha - p_n$ et $\lambda_n = |\theta_n|$. Alors:

1.2. $$\theta_n = (-1)^{n+1} \lambda_n$$

1.3. $$\lambda_n q_{n+1} + \lambda_{n+1} q_n = 1$$

1.4. $$\lambda_n q_{n+1} > 1/2$$

1.5. $$\lambda_n = a_{n+1} \lambda_{n+1} + \lambda_{n+2} \quad \text{et} \quad \sum_{k=0}^{\infty} a_{2k+1} \lambda_{2k+1} = 1.$$

Développement de β par rapport à α (J. LESCA [3]).

Soit $\beta \in [0, 1[$ développer β par rapport à α consiste à déterminer une suite d'entiers $(b_n)_{n \in \mathbf{N}^*}$ vérifiant les conditions suivantes:

1.6. $$\begin{cases} 0 \leqq b_1 \leqq a_1 - 1 \\ 0 \leqq b_n \leqq a_n \quad (n = 2, 3, \ldots) \\ b_n = a_n \Rightarrow b_{n-1} = 0 \quad (n = 2, 3, \ldots) \\ \text{Il n'existe pas d'indice } u \text{ impair tel que:} \\ b_n = a_n,\ b_{n+1} = 0 \quad (n = u, u+2, u+4, \ldots) \\ \beta \equiv \sum_{n=1}^{\infty} b_n \theta_n \equiv \lim \{r_n \alpha\} \quad \text{où} \quad r_n = \sum_{k=1}^{n} b_k q_k. \end{cases}$$

Un tel développement existe et est unique. (On peut remarquer que ce développement est différent de celui utilisé par H. KESTEN [2].)

$\beta \equiv \sum_1^{\infty} b_n \theta_n \equiv u\alpha$ $(u \in \mathbf{Z})$ est caractérisé par:

1.7. $$\begin{cases} u \geqq 0 & \text{tous les } b_n \text{ sont nuls à partir d'un certain rang} \\ u < 0 & \text{tous les } b_n \text{ sont alternativement égaux à } a_n \ (n \text{ pair}) \text{ et à } 0 \\ & (n \text{ impair}) \text{ à partir d'un certain rang.} \end{cases}$$

Remarquons que

1.8. $\quad \forall n \in \mathbf{N}^* - \{1\} \; \forall p \in \mathbf{N}^* \left| \sum_{k=n}^{n+p} b_n \theta_n \right| < \lambda_{n-1}$

et

1.9. $\quad r_n < q_{n+1}$.

Soit $\beta \in [0, 1[$. Développons β par rapport à α; $\beta \equiv \lim \{r_n \alpha\}$,

1.10. $\quad$ Pour $x < r_n \quad |\varphi^-(\beta, x) - \varphi^-(\{r_n \alpha\}, x)| \leqq 1$.

Relations linéaires (J. LESCA [3]).

Soient u, v et n des entiers tels que $n > 0, u < q_n, v < q_{n+1}$ $\big((q_n)$ suite des dénominateurs des convergents de $\alpha\big)$. Il existe deux entiers b et $v', v' < q_n$ tels que $v = bq_n + v'$, alors

1.11. $\quad \begin{cases} \text{si } n \text{ est pair} & \varphi^+(\{u\alpha\}, v) = \varphi^+(\{u\alpha\}, v') + b\varphi^+(\{u\alpha\}, q_n) \\ \text{si } n \text{ est impair} & \varphi^-(\{u\alpha\}, v) = \varphi^-(\{u\alpha\}, v') + b\varphi^-(\{u\alpha\}, q_n). \end{cases}$

Nous appellerons S_n l'ensemble des points de $[0, 1[$ $\big\{0, \{\alpha\}, \{2\alpha\}, \ldots, \{(n-1)\alpha\}\big\}$ et T_n l'ensemble $\big\{\{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}\big\}$. Soit δ_n la longueur du plus petit intervalle découpé sur $[0, 1[$ par S_n $\big(\delta_n = \inf(\|\beta - \gamma\|, \beta \neq \gamma, \beta$ et $\gamma \in S_n)\big)$

1.12. $\quad$ alors $n < q_k \Rightarrow \delta_n \geqq \lambda_{k-1}$.

Développement d'un entier.

(q_n) désigne toujours la suite des dénominateurs des convergents de α. Alors:

1.13. $\quad$ tout entier $x < q_{n+1}$ se décompose de façon unique sous la forme

$$x = \sum_{i=1}^{n} d_i q_i \quad \text{où} \quad 0 \leqq d_i \leqq a_i \quad \text{et} \quad \forall j = 1, 2, \ldots, n, \quad \sum_{i=1}^{j} d_i q_i < q_{j+1}.$$

Extension de la notion d'intervalle.

Pour $\beta, \gamma \in [0, 1[$ $\beta < \gamma$, (γ, β) représentera la réunion des deux intervalles, $(\gamma, 1) \cup (0, \beta)$ et sera considéré comme un intervalle.

§. 2. Lemmes préliminaires

Soit n un entier. Tout entier $x < q_{n+1}$ se décompose de façon unique sous la forme $x = y + dq_n$ où $y < q_n$ et $d \leqq a_n$ (d'aprés 1.13). Nous nous proposons d'exprimer $\varphi^-(\{r_n \alpha\}, x)$ en fonction de $\varphi^-(\{r_{n-1}\alpha\}, y)$.

LEMME 2.1. — *Soient n un entier impair et x un entier tel que $x < q_{n+1}$, $x = y + dq_n$ $(y < q_n$ et $d \leqq a_n)$. Alors:*

$$\varphi^-(\{r_n \alpha\}, x) = \varphi^-(\{r_{n-1}\alpha\}, y) - \lambda_n(dr_{n-1} + b_n x) + A_n(x)$$

où

$$A_n(x) = \begin{cases} \text{Min}\,(d+1, b_n) & si \quad y > r_{n-1} \\ \text{Min}\,(d, b_n) & si \quad y \leqq r_{n-1}. \end{cases}$$

LEMME 2.2. — *Soient* n *un entier pair et* x *un entier tel que* $x<q_{n+1}$, $x=y+$ $+dq_n$ $(y<q_n,\ d\leqq a_n)$. *Alors:*

$$\varphi^-(\{r_n\alpha\},x)=\varphi^-(\{r_{n-1}\alpha\},y)+\lambda_n(dr_{n-1}+b_nx)-B_n(x)$$

où:

$$B_n(x)=\begin{cases}\mathrm{Min}\,(d,b_n) & si \quad y>r_{n-1}\\ \mathrm{Min}\,(d,b_n+1) & si \quad y\leqq r_{n-1} \quad et \quad y\neq 0\\ \mathrm{Min}\,(\mathrm{Max}\,(d-1,0),b_n) & si \quad y=0.\end{cases}$$

DÉMONSTRATION DU LEMME 2.1. Comparons tout d'abord $\varphi^-(\{r_n\alpha\},x)$ et $\varphi^-(\{r_{n-1}\alpha\},x)$.

Si $b_n=0$, $r_n=r_{n-1}$ et ces deux quantités sont égales.

Si $b_n\neq 0$, l'intervalle $[0,\{r_n\alpha\}[$ contient $[0,\{r_{n-1}\alpha\}[$ (car d'après 1.2 $\theta_n>0$) et la différence est union disjointe des b_n intervalles de longueur λ_n, $[\{r_{n-1}\alpha\}+$ $+(i-1)\{q_n\alpha\},\{r_{n-1}\alpha\}+i\{q_n\alpha\}[$ $(i=1,2,\dots,b_n)$. Evaluons le nombre de points de S_x appartenant à $[\{r_{n-1}\alpha\},\{r_n\alpha\}[$ x étant inférieur à q_{n+1}, d'après 1.12, nous obtenons l'équivalence

$$\left.\begin{matrix}\{t\alpha\}\in[\{r_{n-1}\alpha\},\{r_n\alpha\}[\\ \\ 0\leqq t<x\end{matrix}\right\}\Rightarrow\begin{cases}t=r_{n-1}+cq_n\\ 0\leqq c<b_n\\ t<x=y+dq_n\end{cases}$$

soit: $\mathrm{card}\,\{t,\ 0\leqq t<x,\ t\alpha\in[0,\{r_n\alpha\}[\}-\mathrm{card}\,\{t,\ 0\leqq t<x,\ t\alpha\in[0,\{r_{n-1}\alpha\}[=A_n(x)$ où:

$$A_n(x)=\begin{cases}\mathrm{Min}\,(d,b_n) & \text{si} \quad y\leqq r_{n-1}\\ \mathrm{Min}\,(d+1,b_n) & \text{si} \quad y>r_{n-1}\end{cases}$$

ce qui entraîne:

$$\varphi^-(\{r_n\alpha\},x)=\varphi^-(\{r_{n-1}\alpha\},x)+A_n(x)-b_n\lambda_nx.$$

Remarquons que cette formule reste valable dans le cas $b_n=0$.

Exprimons maintenant $\varphi^-(\{r_{n-1}\alpha\},x)$ en fonction de $\varphi^-(\{r_{n-1}\alpha\},y)$. Comme r_{n-1} est inférieur à q_n (d'aprés 1.9), nous pouvons appliquer la relation linéaire 1.11 dans le cas n impair:

$$\varphi^-(\{r_{n-1}\alpha\},x)=\varphi^-(\{r_{n-1}\alpha\},y)+d\varphi^-(\{r_{n-1}\alpha\},q_n).$$

Or d'aprês 1.0.:

$$\varphi^-(\{r_{n-1}\alpha\},q_n)=\varphi^+(\{q_n\alpha\},r_{n-1}).$$

r_{n-1} étant inférieur à q_n, $]0,\{q_n\alpha\}]$ ne contient pas de points de $T_{r_{n-1}}$, d'où $\varphi^+(\{q_n\alpha\},r_{n-1})=-r_{n-1}\lambda_n$, ce qui entraîne:

$$\varphi^-(\{r_n\alpha\},x)=\varphi^-(\{r_{n-1}\alpha\},y)-\lambda_n(b_nx+dr_{n-1})+A_n(x).$$

DÉMONSTRATION DU LEMME 2.2. En appliquant le même raisonnement que précédemment nous obtenons:

$$\varphi^-(\{r_n\alpha\}, x) = \varphi^-(\{r_{n-1}\alpha\}, x)+b_n\lambda_n x-c_n(x)$$

où:

$$c_n(x) = \begin{cases} \mathrm{Min}\,(d, b_n) & \text{si} \quad y > r_{n-1} \\ \mathrm{Min}\,(\mathrm{Sup}\,(d-1, 0), b_n) & \text{si} \quad y \leqq r_{n-1}. \end{cases}$$

Mais n étant pair, nous ne pouvons plus appliquer la relation linéaire 1.11.

Appliquons la relation de réciprocité 1.0.:

$$\varphi^-(\{r_{n-1}\alpha\}, x) = \varphi^+(\{x\alpha\}, r_{n-1}).$$

Par un raisonnement analogue nous obtenons:

$$\varphi^+(\{x\alpha\}, r_{n-1}) = \varphi^+(\{y\alpha\}, r_{n-1})+d\lambda_n r_{n-1}+D_n(x)$$

où:

$$D_n(x) = \begin{cases} -1 & \text{si} \quad d \neq 0 \quad \text{et} \quad 0 < y \leqq r_{n-1} \\ \ \ 0 & \text{sinon} \end{cases}$$

et en appliquant 1.0.:

$$\varphi^-(\{r_n\alpha\}, x) = \varphi^-(\{r_{n-1}\alpha\}, y)+\lambda_n(b_n x+dr_{n-1})-B_n(x).$$

§. 3. Intervalles à restes majorés et non minorés

THÉORÈME 3.1. — *Si la constante de Markov de* α *est infinie, il existe des intervalles à restes majorés et non minorés (ces intervalles peuvent être choisis avec une extrèmité en* 0 *ou en* 1*).*

L'hypothèse faite sur α est équivalente au fait que la suite $(a_i)_{i\in\mathbf{N}^*}$ des quotients partiels de α est non majorée.

Supposons tout d'abord la sous-suite $(a_{2k})_{k\in\mathbf{N}^*}$ non majorée. Considérons l'intervalle $(0, \beta)$ où β est déterminé par son développement par rapport à α, c'est-à-dire par la suite $(b_i)_{i\in\mathbf{N}^*}$ définie ci-dessous.

Soit $K=\{2k_1, 2k_2, \ldots, 2k_n, \ldots\}$ l'ensemble d'indices défini par récurrence de la façon suivante:

$k_1 = 1$

k_i est le plus petit entier supérieur à k_{i-1} tel que $a_{2k_i} \geqq 2^{i-1}$ $(i=2, 3, \ldots)$.

Cette construction est possible car la suite $(a_{2k})_{k\in\mathbf{N}^*}$ est non majorée.

Définissons b_i par $b_i=0$ pour $i\notin K$ et $b_i=1$ pour $i\in K$. La suite $(b_i)_{i\in\mathbf{N}^*}$ ainsi construite vérifie les relations 1.6. et $\beta \equiv \sum_{n=1}^{\infty} b_n\theta_n \not\equiv z\alpha$ $(z\in\mathbf{Z})$ d'après 1.7. Le théorème de Kesten prouve que l'intervalle $(0, \beta)$ est à restes non bornés. Il suffit donc de montrer que $(0, \beta)$ est à restes majorés.

Soit x un entier, $x<q_{n+1}$; d'aprés 1.13. $x=y+dq_n$ où $y<q_n$ et $0\leqq d\leqq a_n$. Evaluons $\varphi^-(\{r_n\alpha\}, x)$ en fonction de $\varphi^-(\{r_{n-1}\alpha\}, y)$ $\left(r_p=\sum_{k=1}^{p} b_k q_k\right)$.

a) $n=2k+1$:

$$\varphi^-(\{r_{2k+1}\alpha\}, x) = \varphi^-(\{r_{2k}\alpha\}, y)-\lambda_{2k+1}(dr_{2k}+b_{2k+1}x)+A_{2k+1}(x)$$

(d'après 2.1). Or

$$b_{2k+1} = 0 \Rightarrow A_{2k+1}(x) = 0, \quad \text{d'où} \quad \varphi^-(\{r_{2k+1}\alpha\}, x) \leqq \varphi^-(\{r_{2k}\alpha\}, y).$$

b) $n=2k$ et $b_{2k}=1$:

$$\varphi^-(\{r_{2k}\alpha\}, x) = \varphi^-(\{r_{2k-1}\alpha\}, y)+\lambda_{2k}(dr_{2k-1}+b_{2k}x)-B_{2k}(x)$$

(d'aprés 2.2). Si $d=0$:

$$\varphi^-(\{r_{2k}\alpha\}, x) = \varphi^-(\{r_{2k-1}\alpha\}, y)+\lambda_{2k}x,$$

or: $x=y+dq_{2k}=y<q_{2k}$

$$\varphi^-(\{r_{2k}\alpha\}, x) \leqq \varphi^-(\{r_{2k-1}\alpha\}, y)+\lambda_{2k}q_{2k} \leqq \varphi^-(\{r_{2k-1}\alpha\}, y)+\frac{1}{a_{2k}}$$

en effet, $a_{2k}q_{2k}<q_{2k+1}$ et $\lambda_{2k}q_{2k+1}<1$ d'après 1.3.

Si $d\neq 0$ et $y\neq 0$:

$$\varphi^-(\{r_{2k}\alpha\}, x) \leqq \varphi^-(\{r_{2k-1}\alpha\}, y)+\lambda_{2k}(dr_{2k-1}+x)-1$$

$\left(\text{car } y\neq 0\Rightarrow B_n(x)\geqq 1\right)$

$$\varphi^-(\{r_{2k}\alpha\}, x) \leqq \varphi^-(\{r_{2k-1}\alpha\}, y)+a_{2k}\lambda_{2k}r_{2k-1} \leqq \varphi^-(\{r_{2k-1}\alpha\}, y)+\lambda_{2k-1}r_{2k-1}$$

(en utilisant $d\leqq a_{2k}$, $x<q_{2k+1}$, 1.3. et 1.5.).

Si $d\neq 0$ et $y=0$:

$$\varphi^-(\{r_{2k}\alpha\}, x) \leqq \varphi^-(\{r_{2k-1}\alpha\}, y)+\lambda_{2k-1}r_{2k-1}+1.$$

c) $n=2k$ et $b_{2k}=0$:

$$\begin{aligned}\varphi^-(\{r_{2k}\alpha\}, x) &= \varphi^-(\{r_{2k-1}\alpha\}, y)+\lambda_{2k}(dr_{2k-1}+b_{2k}x)-B_{2k}(x)\\ &\leqq \varphi^-(\{r_{2k-1}\alpha\}, y)+\lambda_{2k}dr_{2k-1} \quad (\text{car } b_{2k}=0)\\ &\leqq \varphi^-(\{r_{2k-1}\alpha\}, y)+r_{2k-1}\lambda_{2k-1} \quad (\text{d'aprés 1.5}).\end{aligned}$$

Soient $2k$ et $2k+2p$ deux éléments consécutifs de K. Soit x un entier. $0<x<q_{2k+2p+1}$. D'après 1.13, x se décompose de façon unique sous la forme:

$$x = \sum_{j=1}^{2k+2p} e_j q_j$$

où $\sum_{j=1}^{m} e_j q_j<q_{m+1}$ pour tout $m=1, 2, \ldots, 2k+2p$.

Posons $x=y+\sum_{i=1}^{2p} d_i q_{2k+i}$ et majorons la différence

$$\varphi^-(\{r_{2k+2p}\alpha\}, x)-\varphi^-(\{r_{2k}\alpha\}, y) =$$

$$= \sum_{m=1}^{2p} \varphi^-\left(\{r_{2k+m}\alpha\}, y+\sum_{i=1}^{m} d_i q_{2k+i}\right)-\varphi^-\left(\{r_{2k+m-1}\alpha\}, y+\sum_{i=1}^{m-1} d_i q_{2k+i}\right).$$

Comme $\sum_{j=1}^{m} e_j q_j < q_{m+1}$, nous pouvons appliquer les formules a), b), c) pour majorer cette expression.

1. Si $y\neq 0$, pour tout m, $y+\sum_{i=1}^{m-1} d_i q_{2k+i}\neq 0$, alors: si m impair,

$$\varphi^-\left(\{r_{2k+m}\alpha\}, y+\sum_{i=1}^{m} d_i q_{2k+i}\right)-\varphi^-\left(\{r_{2k+m-1}\alpha\}, y+\sum_{i=1}^{m-1} d_i q_{2k+i}\right) \leqq 0$$

(d'après a), si m pair, $m\neq 2p$,

$$\varphi^-\left(\{r_{2k+m}\alpha\}, y+\sum_{i=1}^{m} d_i q_{2k+i}\right)-\varphi^-\left(\{r_{2k+m-1}\alpha\}, y+\sum_{i=1}^{m-1} d_i q_{2k+i}\right) \leqq \lambda_{2k+m-1} r_{2k+m-1}$$

(d'après c), si m pair, $m=2p$,

$$\varphi^-\left(\{r_{2k+m}\alpha\}, y+\sum_{i=1}^{m} d_i q_{2k+i}\right)-\varphi^-\left(\{r_{2k+m-1}\alpha\}, y+\sum_{i=1}^{m-1} d_i q_{2k+i}\right) \leqq$$

$$\leqq \frac{1}{a_{2k+2p}}+\lambda_{2k+2p-1} r_{2k+2p-1}$$

soit:

$$\varphi^-(\{r_{2k+2p}\alpha\}, x)-\varphi^-(\{r_{2k}\alpha\}, y) \leqq \frac{1}{a_{2k+2p}}+\sum_{j=1}^{p} \lambda_{2k+2j-1} r_{2k+2j-1}.$$

Or pour $j=1, 2, \ldots, p$; $r_{2k+2j-1}=r_{2k}$ (car $b_{2k+1}=b_{2k+2}=\ldots=b_{2k+2p-1}=0$)

$$\varphi^-(\{r_{2k+2p}\alpha\}, x)-\varphi^-(\{r_{2k}\alpha\}, y) \leqq \frac{1}{a_{2k+2p}}+r_{2k}\sum_{j=1}^{p} \lambda_{2k+2j-1}.$$

Or pour tout entier n, $\lambda_{n+2}\leqq 1/2\lambda_n$ (d'après 1.5) et $\sum_{j=1}^{p} \lambda_{2k+2j-1}<2\lambda_{2k+1}$ d'où

$$\varphi^-(\{r_{2k+2p}\alpha\}, x)-\varphi^-(\{r_{2k}\alpha\}, y) \leqq \frac{1}{a_{2k+2p}}+2r_{2k}\lambda_{2k+1}.$$

Or, $r_{2k}=q_{2k}+r_{2k-1}<2q_{2k}$ (car $b_{2k}=1$ et $r_{2k-1}<q_{2k}$) et $q_{2k}<\frac{q_{2k+1}}{a_{2k}}$.

Alors

$$\varphi^-(\{r_{2k+2p}\alpha\}, x)-\varphi^-(\{r_{2k}\alpha\}, y) \leqq \frac{1}{a_{2k+2p}}+\frac{4}{a_{2k}}. \tag{3.1}$$

2°) Si $y=0$, soit m_0 le plus petit indice tel que $d_{m_0}\neq 0$. Alors

$$\varphi^-(\{r_{2k+2p}\alpha\}, x)-\varphi^-(\{r_{2k}\alpha\}, y) =$$

$$= \sum_{m=m_0}^{2p} \varphi^-\Big(\{r_{2k+m}\alpha\}, \sum_{i=m_0}^{m} d_i q_{2k+i}\Big)-\varphi^-\Big(\{r_{2k+m-1}\alpha\}, \sum_{i=m_0}^{m-1} d_i q_{2k+i}\Big).$$

Les majorations précédentes s'appliquent ici, sauf, peut être pour $m=m_0$ cas dans lequel il faut éventuellement ajouter 1 (cf c) et

$$\varphi^-(\{r_{2k+2p}\alpha\}, x)-\varphi^-(\{r_{2k}\alpha\}, y) \leqq \frac{1}{a_{2k+2p}}+\frac{4}{a_{2k}}+1. \tag{3.2}$$

Soit x un entier, $x<q_{m+1}$, alors $x=\sum_{j=1}^{m} d_j q_j$. Soit $2k_i\in K$ tel que $x<q_{2k_i}$. Majorons $\varphi^-(\{r_{2k_i}\alpha\}, x)$:

$$\varphi^-(\{r_{2k_i}\alpha\}, x) = \sum_{n=2}^{i}\Big(\varphi^-\Big(\{r_{2k_n}\alpha\}, \sum_{j=1}^{\inf(m,2k_n)} d_j q_j\Big)-\varphi^-\Big(\{r_{2k_{n-1}}\alpha\}, \sum_{j=1}^{\inf(m,2k_{n-1})} d_j q_j\Big)\Big)+$$

$$+\varphi^-\Big(\{r_2\alpha\}, \sum_{j=1}^{\inf(m,2)} d_j q_j\Big).$$

Il existe au plus un rang n tel que $\sum_{j=1}^{\inf(m,2k_n)} d_j q_j\neq 0$ et $\sum_{j=1}^{\inf(m,2k_{n-1})} d_j q_j=0$, nous pouvons appliquer les majorations (3.1) et (3.2), la majoration (3.2) s'appliquant au plus une fois

$$\varphi^-(\{r_{2k_i}\alpha\}, x) \leqq \varphi^-\Big(\{r_2\alpha\}, \sum_{j=1}^{\inf(m,2)} d_j q_j\Big)+1+\sum_{n=1}^{i-1}\frac{1}{a_{2k_n+1}}+\frac{4}{a_{2k_n}}.$$

Or, par construction: $a_{2k_n}\geqq 2^{n-1}$ de plus $\sum_{j=1}^{\inf(m,2)} d_j q_j<q_3$, d'où:

$$\varphi^-(\{r_{2k_i}\alpha\}, x) \leqq q_3+11.$$

Il est alors clair, d'après 1.10, que l'intervalle $(0, \beta)$ est à restes majorés.

Supposons maintenant que la sous-suite $(a_{2k+1})_{k\in\mathbf{N}}$ soit non majorée. Considérons (a_n') suite des quotients partiels de $(1-\alpha)$. La suite des $(a_{2k}')_{k\in\mathbf{N}^*}$ est (à un terme près) celle des $(a_{2k+1})_{k\in\mathbf{N}}$ donc non majorée. Il existe un intervalle $(0, \beta)$ à restes majorés et non minorés pour la suite $\{-n\alpha\}$, l'intervalle $(1-\beta, 1)$ est donc à restes majorés et non minorés pour la suite $\{n\alpha\}$, ce qui achève la démonstration.

§. 4. Etude du cas où la constante de Markov de α est finie

a) *Une condition nécessaire pour que l'intervalle* $(0, \beta)$ *soit à restes majorés.*

LEMME 4.1. — *Soient* n *un entier impair,* x *et* s *deux entiers tels que* $x<r_n$ *et* $s\geqq n$. *Soit* m_0 *le plus petit entier impair (s'il existe) tel que* $m_0>n$ *et* $b_{m_0}\neq 0$. *Alors*

$$\varphi^-(\{r_s\alpha\}, x) \geqq \varphi^-(\{r_n\alpha\}, x)-x\lambda_{m_0-1}.$$

Ce lemme est une conséquence simple des lemmes 2.1, 2.2 et 1.8.

REMARQUE. Si tout entier impair $m>n$ est tel que $b_m=0$, $\varphi^-(\{r_s\alpha\}, x)\geqq \geqq\varphi^-(\{r_n\alpha\}, x)$, pour tout entier $s\geqq n$.

THÉORÈME 4.1. — *Soit $\alpha\in[0, 1[$ dont la constante de Markov est finie. Soit $\beta\in[0, 1[$, non congru modulo 1 à $z\alpha$ $(z\in\mathbf{Z})$ de développement $(b_i)_{i\in\mathbf{N}^*}$ par rapport à α. Une condition nécessaire pour que l'intervalle $(0, \beta)$ soit à restes majorés est qu'il n'existe qu'un nombre fini d'indices impairs, n, tels que b_n soit non nul.*

Nous raisonnerons par l'absurde et montrerons que la négation de cette propriété entraîne l'existence d'une constante strictement positive c telle que:

$$\begin{cases} \text{si } n_0 \text{ et } x_0 \text{ sont tels que pour tout } n\geqq n_0\ \varphi^-(\{r_n\alpha\}, x_0)\geqq M \\ \text{il existe } n_1 \text{ et } x_1 \text{ tels que pour tout } n\geqq n_1\ \varphi^-(\{r_n\alpha\}, x_1)\geqq M+c \end{cases}$$

ce qui est en contradiction (d'après 1.10) avec le fait que $(0, \beta)$ soit à restes majorés.

Supposons donc qu'il existe une infinité d'indices impairs n tels que b_n soit non nul. La constante de Markov de α étant finie, les quotients partiels $(a_i)_{i\in\mathbf{N}^*}$ de α sont majorés. Soit $A=\mathrm{Sup}\,\{a_i, i\in\mathbf{N}^*\}$. Posons $K=A+1$.

1°) Supposons qu'il existe une infinité d'indices impairs i tels que $0<b_i\leqq a_i-1$; n_0 et x_0 étant deux entiers tels que, pour $n\geqq n_0$, $\varphi^-(\{r_n\alpha\}, x_0)\geqq M$, il existe un entier impair $n_1>n_0$ tel que:

$$x_0<r_{n_1-1}, \quad 0<b_{n_1}\leqq a_{n_1}-1, \quad x_0\lambda_{n_1-1}<\frac{1}{8K^2}.$$

Prenons $x_1=x_0+b_{n_1}q_{n_1}$. Alors, d'après le lemme 2.1.:

$$\varphi^-(\{r_{n_1}\alpha\}, x_1) = \varphi^-(\{r_{n_1-1}\alpha\}, x_0)-\lambda_{n_1}(b_{n_1}r_{n_1-1}+b_{n_1}(x_0+b_{n_1}q_{n_1}))+b_{n_1}\geqq$$
$$\geqq \varphi^-(\{r_{n_1-1}\alpha\}, x_0)+b_{n_1}[1-\lambda_{n_1}[q_{n_1}+(a_{n_1}-1)q_{n_1}]]-x_0b_{n_1}\lambda_{n_1}$$

car $r_{n_1-1}<q_{n_1}$ (d'après 1.9) et $b_{n_1}\leqq a_{n_1}-1$.

De plus, $x_0<r_{n_1-1}$ entraîne $x_1<r_{n_1}$ et nous pouvons appliquer le lemme 4.1. (avec $m_0\geqq n_1+2$).

Pour tout entier $n\geqq n_1$ nous avons:

$$\varphi^-(\{r_n\alpha\}, x_1)\geqq\varphi^-(\{r_{n_1-1}\alpha\}, x_0)+b_{n_1}[1-\lambda_{n_1}(a_{n_1}q_{n_1})]-x_0b_{x_1}\lambda_{n_1}-(x_0+b_{n_1}q_{n_1})\lambda_{n_1+1}.$$

Or $a_{n_1}q_{n_1}=q_{n_1+1}-q_{n_1-1}$ et $\lambda_{n_1+1}q_{n_1}+\lambda_{n_1}q_{n_1+1}=1$ (1.3) d'où:

$$\varphi^-(\{r_n\alpha\}, x_1)\geqq\varphi^-(\{r_{n_1-1}\alpha\}, x_0)+b_{n_1}[\lambda_{n_1}q_{n_1-1}]-x_0[b_{n_1}\lambda_{n_1}+\lambda_{n_1+1}].$$

Or $b_{n_1}\lambda_{n_1}<a_{n_1}\lambda_{n_1}<\lambda_{n_1-1}$ (d'après 1.5) et $\lambda_{n_1+1}<\lambda_{n_1-1}$, d'où:

$$\varphi^-(\{r_n\alpha\}, x_1)\geqq\varphi^-(\{r_{n_1-1}\alpha\}, x_0)+\lambda_{n_1}q_{n_1-1}-2x_0\lambda_{n_1-1}.$$

Or

$$q_{n_1-1}>\frac{q_{n_1}}{a_{n_1-1}+1}>\frac{q_{n_1+1}}{(a_{n_1}+1)(a_{n_1-1}+1)}\geqq\frac{q_{n_1+1}}{K^2}$$

(car $q_{n+1}=a_nq_n+q_{n-1}$ et tous les quotients partiels a_i de α sont majorés par $K-1$) et $q_{n_1+1}\lambda_{n_1}>1/2$ d'après 1.4; d'autre part $n_1-1\geqq n$ et:

$$\varphi^-(\{r_{n_1-1}\alpha\}, x_0) \geqq M \quad \text{et} \quad x_0\lambda_{n_1-1} < \frac{1}{8K^2}$$

par construction. Nous obtenons finalement, pour tout entier $n\geqq n_1$,

$$\varphi^-(\{r_n\alpha\}, x_1) \geqq M+\frac{1}{2K^2}-\frac{1}{4K^2} = M+\frac{1}{2K^2}$$

ce qui achève la démonstration.

2°) S'il n'existe pas une infinité d'indices impairs n tels que $0<b_n\leqq a_n-1$, à partir d'un certain rang, pour tout n impair $b_n=0$ ou $b_n=a_n$. Comme β n'est pas congru à $z\alpha$ ($z\in\mathbf{Z}$), il existe une infinité d'indices impairs n tels que $b_n=a_n$ et $b_{n+2}=0$ (d'aprés 1.7).

De même que précédemment, x_0 et n_0 étant deux entiers tels que pour $n\geqq n_0$, $\varphi^-(\{r_n\alpha\}, x_0)\geqq M$, il existe un entier impair $n_1>n_0$ vérifiant:

$$x_0 < r_{n_1-1}, \quad b_{n_1} = a_{n_1}, \quad b_{n_1+2} = 0, \quad \lambda_{n_1-1}x_0 < \frac{1}{8K^3}.$$

Prenons $x_1=x_0+b_{n_1}q_{n_1}$, alors d'après le lemme 2.1.

$$\varphi^-(\{r_{n_1}\alpha\}, x_1) = \varphi^-(\{r_{n_1-1}\alpha\}, x_0)+a_{n_1}-\lambda_{n_1}(a_{n_1}r_{n_1-1}+(x_0+a_{n_1}q_{n_1})a_{n_1})$$

$x_0<r_{n_1-1}$ entraîne $x_1<r_{n_1}$; appliquons le lemme 4.1. en remarquant que $b_{n_1+2}=0$. Pour tout entier $n\geqq n_1$, nous avons

$$\varphi^-(\{r_n\alpha\}, x_1) \geqq \varphi^-(\{r_{n_1-1}\alpha\}, x_0)+a_{n_1}(1-\lambda_{n_1}(r_{n_1-1}+a_{n_1}q_{n_1}))-$$
$$-a_{n_1}\lambda_{n_1}x_0-\lambda_{n_1+3}(x_0+a_{n_1}q_{n_1}).$$

Or $b_{n_1}=a_{n_1}$ entraîne $b_{n_1-1}=0$ (d'après 1.6) c'est-à-dire $r_{n_1-1}<q_{n_1-1}$, et $r_{n_1-1}+$ $+a_{n_1}q_{n_1}<q_{n_1+1}$

$$\varphi^-(\{r_n\alpha\}, x) \geqq \varphi^-(\{r_{n_1-1}\alpha\}, x_0)+a_{n_1}((\lambda_{n_1+1}-\lambda_{n_1+3})q_{n_1})-$$
$$-x_0(a_{n_1}\lambda_{n_1}+\lambda_{n_1+3}) \geqq M+\lambda_{n_1+2}q_{n_1}-2x_0\lambda_{n_1-1}$$

car $n_1-1\geqq n_0$, $\lambda_{n_1+1}-\lambda_{n_1+3}=a_{n_1+2}\lambda_{n_1+2}$ et $a_{n_1}\lambda_{n_1}<\lambda_{n_1-1}$ (d'après 1.5). De même que précédemment $\lambda_{n_1+2}q_{n_1}>\frac{1}{2K^3}$ (d'après 1.4 et 1.5) et $2x_0\lambda_{n_1-1}<\frac{1}{4K^3}$ ce qui entraîne: pour tout entier $n\geqq n_1$, $\varphi^-(\{r_n\alpha\}, x_1)\geqq M+\frac{1}{2K^3}$, ce qui achève la démonstration.

De même une condition nécessaire pour que l'intervalle $(0, \beta)$ soit à restes minorés est qu'il n'existe qu'un nombre fini d'indices pairs n tels que $b_n\neq 0$ ($(b_i)_{i\in\mathbf{N}^*}$ développement de β par rapport à α).

b) *Cas particulier* $\alpha=\frac{\sqrt{5}-1}{2}$.

THÉORÈME 4.2. — *Soit* $\alpha=\frac{\sqrt{5}-1}{2}$. *Si* $\beta\in[0, 1[$ *est non congru à* $z\alpha$ $(z\in\mathbf{Z})$, *l'intervalle* $(0, \beta)$ *n'est ni à restes majorés, ni à restes minorés.*

Les quotients partiels de α sont tous égaux à 1. Supposons que l'intervalle $(0, \beta)$ soit à restes majorés. D'après le théorème 4.1., il existe un entier N_0 tel que pour tout entier impair $n\geqq N_0$, $b_n=0$. Or b_n étant inférieur à a_n ne peut prendre que les valeurs 0 et 1, et β n'étant pas congru à $z\alpha$ $(z\in\mathbf{Z})$, d'après 1.7, il existe une infinité d'indices pairs n tels que $b_n=1$ et $b_{n-1}=b_{n-2}=b_{n-3}=0$. Nous allons raisonner comme dans la partie a).

Soint n_0 et x_0 deux entiers tels que pour $n\geqq n_0$. $\varphi^-(\{r_n\alpha\}, x_0)>M$.

Il existe un entier pair $n_1>n_0+1$ tel que

$$x_0<q_{n_1-1},\quad b_{n_1}=1,\quad b_{n_1-1}=b_{n_1-2}=b_{n_1-3}=0,\quad n_1>N_0.$$

Prenons $x_1=x_0+q_{n_1-1}$

$$\varphi^-(\{r_{n_1-1}\alpha\}, x_1)=\varphi^-(\{r_{n_1-2}\alpha\}, x_0)-\lambda_{n_1-1}r_{n_1-2}$$

(d'après le lemme 2.1)

$$\varphi^-(\{r_{n_1}\alpha\}, x_1)=\varphi^-(\{r_{n_1-1}a\}, x_1)+\lambda_{n_1}(x_0+q_{n_1-1})$$

$$\varphi^-(\{r_{n_1}\alpha\}, x_1)=\varphi^-(\{r_{n_1-2}\alpha\}, x_0)-\lambda_{n_1-1}r_{n_1-2}+\lambda_{n_1}(x_0+q_{n_1-1}).$$

Or $b_{n_1-2}=b_{n_1-3}=0$ entraîne $r_{n_1-2}<q_{n_1-3}$ et $n_1-2\geqq n_0$ entraîne $\varphi^-(\{r_{n_1-2}\alpha\}, x_0)>$ $<M$. D'après le lemme 4.1, pour tout entier $n>n_1$ nous avons

$$\varphi^-(\{r_n\alpha\}, x_1)>M-\lambda_{n_1-1}q_{n_1-3}+\lambda_{n_1}q_{n_1-1}=M+\lambda_{n_1}\left[q_{n_1-1}-\frac{1+\sqrt{5}}{2}q_{n_1-3}\right]$$

en effet, $\lambda_n=\frac{1+\sqrt{5}}{2}\lambda_{n+1}$ (obtenu facilement en iterrant 1.5), de plus $q_{n_1-1}>2q_{n_1-3}$ et $\lambda_{n_1}q_{n_1-3}>\lambda_{n_1}q_{n_1+1}\frac{1}{2^4}>\frac{1}{2^5}$ (d'après 1.4). Nous obtenons donc, pour tout entier $n\geqq n_1$:

$$\varphi^-(\{r_n\alpha\}, x_1)>M+\frac{1}{2^7},$$

ce qui prouve, comme dans la partie a, que $(0, \beta)$ n'est pas à restes majorés.

De même l'intervalle $(0, \beta)$ ne peut être à restes minorés. Il suffit de remarquer que la suite des quotients partiels de $1-\alpha$ est $(2, 1, 1, \ldots, 1, \ldots)$ et que la même démonstration s'applique pour la suite $\{-n\alpha\}$; l'intervalle $(0, 1-\beta)$ est à restes non majorés pour la suite $\{-n\alpha\}$, $(1-\beta, 1)$ est à restes non minorés pour la suite $\{-n\alpha\}$ et $(0, \beta)$ est à restes non minorés pour la suite $\{n\alpha\}$ ce qui achève la démonstration.

§. 5. Construction d'un intervalle de longeur 1/2 à restes majorés

THÉORÈME 5.1. — *Soit* $\alpha\in[0, 1[$ *un irrationnel dont la suite des quotients partiels* $(a_i)_{i\in\mathbf{N}^*}$ *vérifie:*

$$\forall k\in\mathbf{N} \quad a_{2k+1}=2, \quad \sum_{k\in\mathbf{N}^*}\frac{1}{a_{2k}}<\infty.$$

Alors l'intervalle (1/2, 1) *est à restes majorés.*

Nous allons montrer la propriété équivalente, l'intervalle (0, 1/2) est à restes minorés.

Déterminons le développement de 1/2 par rapport à α; d'après (1.5)

$$1=\sum_{k=0}^{\infty}a_{2k+1}\theta_{2k+1}=\sum_{k=0}^{\infty}2\theta_{2k+1}$$

d'où:

$$\frac{1}{2}=\sum_{k=0}^{\infty}\theta_{2k+1}=\sum_{n=1}^{\infty}b_n\theta_n \quad \text{où} \quad \begin{cases} b_{2k}=0 & k\in\mathbf{N}^* \\ b_{2k+1}=1 & k\in\mathbf{N}. \end{cases}$$

La suite $(b_n)_{n\in\mathbf{N}^*}$ ainsi définie, vérifie les relations 1.6. C'est le développement de 1/2 par rapport à α.

Soient k un entier au moins égal à 2 et x un entier inférieur à q_{2k+2}. D'après 1.13, x s'écrit $x=y+d_1q_{2k}+d_2q_{2k+1}$. Exprimons $\varphi^-(\{r_{2k+1}\alpha\}, x)$ en fonction de $\varphi^-(\{r_{2k-1}\alpha\}, y)$:

$$\varphi^-(\{r_{2k}\alpha\}, y+d_1q_{2k})=\varphi^-(\{r_{2k-1}\alpha\}, y)+\lambda_{2k}\cdot d_1r_{2k-1}-B_{2k}(y+d_1q_{2k})$$

(d'après le lemme 2.2)

$$\varphi^-(\{r_{2k+1}\alpha\}, x)=\varphi^-(\{r_{2k}\alpha\}, y+d_1q_{2k})-\lambda_{2k+1}(d_2r_{2k}+x)+A_{2k+1}(x)$$

(d'après le lemme 2.1)

$$=\varphi^-(\{r_{2k-1}\alpha\}, y)+d_1(r_{2k-1}\lambda_{2k}-q_{2k}\lambda_{2k+1})-$$
$$-\lambda_{2k+1}(d_2r_{2k}+y)+A_{2k+1}(x)-B_{2k}(y+d_1q_{2k})-d_2q_{2k+1}\lambda_{2k+1}.$$

Minorons $d_1(r_{2k-1}\lambda_{2k}-q_{2k}\lambda_{2k+1})$

$$\lambda_{2k}=2\lambda_{2k+1}+\lambda_{2k+2} \quad \text{(d'après 1.5) et} \quad q_{2k}=2q_{2k-1}+q_{2k-2}$$

d'autre part: $r_{2k-1}\geqq q_{2k-1}$ et $d_1\leqq a_{2k}$ d'où:

$$d_1(r_{2k-1}\lambda_{2k}-q_{2k}\lambda_{2k+1})>-a_{2k}\lambda_{2k+1}q_{2k-2}>-\frac{\lambda_{2k+1}q_{2k+2}}{4a_{2k-2}}>-\frac{1}{4a_{2k-2}}$$

en effet: $q_n<\dfrac{q_{n+1}}{a_n}$ et $\lambda_nq_{n+1}<1$ (d'après 1.3). Minorons $-\lambda_{2k+1}(d_2r_{2k}+y)$ $b_{2k=0}$ entraîne $r_{2k}=r_{2k-1}<q_{2k}$ (d'après 1.9); $y<q_{2k}$ (d'après 1.13), $d_2\leqq a_{2k+1}=2$, d'où

$$-\lambda_{2k+1}(d_2r_{2k}+y)>-3\lambda_{2k+1}q_{2k}>-\frac{3}{2a_{2k}}.$$

Calculons enfin l'expression $A_{2k+1}(x)-B_{2k}(y+d_1q_{2k})-d_2q_{2k+1}\lambda_{2k+1}$ que nous désignons par $\Delta_k(x)$

1) $y>r_{2k-1}$, $B_{2k}(y+d_1q_{2k})=0$ (car $b_{2k}=0$) et $y+d_1q_{2k}>r_{2k-1}=r_{2k}$ entraîne $A_{2k+1}(x)=1$, $\Delta_k(x)=1-d_2q_{2k+1}\lambda_{2k+1}$.

2) $y\leqq r_{2k-1}$

a) $d_1=d_2=0$ $\quad A_{2k+1}(x)=B_{2k}(y+d_1q_{2k})=0$ et $\Delta_k(x)=0$

b) $d_1=0$, $d_2\neq 0$ $\quad A_{2k+1}(x)=1$; $B_{2k}(y+d_1q_{2k})=0$ et $\Delta_k(x)=1-d_2q_{2k+1}\lambda_{2k+1}$

c) $d_1\neq 0$, $d_2=0$ $\quad A_{2k+1}(x)=1$ car $y+d_1q_{2k}>r_{2k}=r_{2k-1}$ et $\Delta_k(x)\geqq 0$

d) $d_1\neq 0$, $d_2\neq 0$ $\quad A_{2k+1}(x)=B_{2k}(y+d_1q_{2k})=1$, $\Delta_k(x)=-d_2q_{2k+1}\lambda_{2k+1}$.

Soit x un entier. D'après 1.5, x s'écrit

$$x=y_2+\sum_{j=2}^{k_0}d_1^jq_{2j}+d_2^jq_{2j+1}.$$

Posons

$$y_k=y_2+\sum_{j=2}^{k-1}d_1^jq_{2j}+d_2^jq_{2j+1}\quad (k=2,3,\dots,k_0)$$

$$\varphi^-(\{r_{2k_0+1}\alpha\},x)=\sum_{k=2}^{k_0}(\{\varphi^-(\{r_{2k+1}\alpha\},y_k+d_1^kq_{2k}+d_2^kq_{2k+1})-\varphi^-(\{r_{2k-1}\alpha\},y_k))+$$

$$+\varphi^-(\{r_3\alpha\},y_2)\geqq -q_4-\frac{1}{4}\sum_{k=2}^{k_0}\frac{1}{a_{2k-2}}-\frac{3}{2}\sum_{k=2}^{k_0}\frac{1}{a_{2k}}+\sum_{k=2}^{k_0}\Delta_k(y_k+d_1^kq_{2k}+d_2^kq_{2k+1}).$$

Montrons que $\sum_{k=2}^{k_0}\Delta_k(y_k+d_1^kq_{2k}+d_2^kq_{2k+1})$ est minoré.

Considérons les quatre cas possibles

α) $y_k>r_{2k-1}$ et $d_2^k\neq 0$;

$$\Delta_k(y_k+d_1^kq_{2k}+d_2^kq_{2k+1})=1-d_2^kq_{2k+1}\lambda_{2k+1}\quad\text{(d'après 1)}$$
$$\geqq 1-a_{2k+1}q_{2k+1}\lambda_{2k+1}\geqq 0\quad\text{(d'après 1.3).}$$

β) $y_k>r_{2k-1}$ et $d_2^k=0$:

$\Delta_k(y_k+d_1^kq_{2k}+d_2^kq_{2k+1})=1$ (d'après 1)).

γ) $y_k\leqq r_{2k-1}$ et d_1^k ou $d_2^k=0$:

$\Delta_k(y_k+d_1^kq_{2k}+d_2^kq_{2k+1})\geqq 0$ (d'après 2) a, b, c).

δ) $y_k \leqq r_{2k-1}$, $d_1^k \neq 0$ et $d_2^k \neq 0$:

$$\Delta_k(y_k+d_1^k q_{2k}+d_2^k q_{2k+1}) = -d_2^k q_{2k+1}\lambda_{2k} \quad \text{(d'après 2), d)}$$
$$\geqq -1 \quad \text{(d'après 1.3).}$$

Remarquons enfin que si $y_k > r_{2k-1}$ et $d_2^k \neq 0$, alors:

$$y_{k+1} > r_{2k-1}+q_{2k+1} = r_{2(k+1)-1}.$$

Soient A, B, C, D les ensembles des indices k $(2 \leqq k \leqq k_0)$ correspondant à chacun des cas précédents (α, β, γ, δ respectivement). Δ_k ne peut prendre une valeur négative que si $k \in D$. Supposons qu'il existe $k_1 \in D$, alors $d_1^{k_1} \neq 0$ et $d_2^{k_1} \neq 0$ ce qui entraîne $y_{k_1+1} > r_{2(k_1+1)-1}$. S'il existe $k_2 > k_1$, $k_2 \in D$, d'après la remarque précédente, il existe $k \in B$ $(k_1 < k < k_2)$. Par conséquent, card $(D) \leqq$ card $(B)+1$, ce qui entraîne

$$\sum_{k=2}^{k_0} \Delta_k(y_k+d_1^k q_{2k}+d_2^k q_{2k+1}) \geqq -1$$

$\varphi^-(\{r_{2k_0+1}\alpha\}, x) \geqq -q_4-1-2 \sum_{k \in \mathbf{N}^*} \frac{1}{a_{2k}}$, ce qui prouve (d'après 1.10) que l'intervalle $(0, 1/2)$ est à restes minorés.

§. 6. Translations d'intervalles

Nous savons que pour un intervalle I, le fait d'avoir des restes bornés ne dépend que de la longueur de l'intervalle [1]. Par contre, le fait d'avoir des restes majorés ou minorés dépend aussi de l'origine de cet intervalle.

Si $I=(a, b)$ est un intervalle de $[0, 1[$, pour $\beta \in \mathbf{R}$, nous désignerons par $\beta+I$ l'intervalle $(\{a+\beta\}, \{b+\beta\})$. Soit I un intervalle de $[0, 1[$ à restes majorés. Nous nous proposons de caractériser les $\beta \in [0, 1[$ tels que l'intervalle $\beta+I$ soit encore à restes majorés. Pour cela nous utiliserons le développement de $(1-\beta)$ par rapport à α: $1-\beta=\lim \{r_n\alpha\}$ (voir 1.6).

Pour $\gamma \in [0, 1[$, introduisons la notation:

$$\varphi_\gamma(I, n) = \text{card}\{k, k < n, \{k\alpha\} \in 1-\gamma+I\} - n\mu(I) = \varphi(1-\gamma+I, n)$$

(nous avons alors $\varphi(I, n) = \varphi_0(I, n)$).

LEMME (P. BOHL [1]). — *Si δ et γ sont deux éléments de $[0, 1[$ tels que $\|\delta-\gamma\| < \lambda_k$, pour tout $n < q_{k+1}$*

$$|\varphi_\delta(I, n) - \varphi_\gamma(I, n)| \leqq 1.$$

PROPOSITION. — *I étant un intervalle de $[0, 1[$ à restes majorés, l'intervalle $\beta+I$ est à restes majorés si et seulement si l'ensemble $\{\varphi(I, r_n), n \in \mathbf{N}^*\}$ est minoré* $(1-\beta=\lim \{r_n\alpha\})$.

Remarquons tout d'abord que:

$$\varphi_{\{r_k\alpha\}}(I, n) = \varphi(I, r_k+n) - \varphi(I, r_k).$$

1°) Supposons l'ensemble $\{\varphi(I, r_n)_{n \in \mathbf{N}^*}\}$ minoré.

Soit n un entier, soit k un indice tel que $n<q_{k+1}$. D'après le lemme, $\varphi_{1-\beta}(I,n)\leqq \leqq\varphi_{\{r_k\alpha\}}(I,n)+1$, car $\|1-\beta-\{r_k\alpha\}\|<\lambda_k$ c'est-à-dire $\varphi_{1-\beta}(I,n)=\varphi(\beta+I,n)\leqq \leqq\varphi(I,r_k+n)-\varphi(I,r_k)+1$, quantité majorée indépendamment de n.

2°) Supposons l'ensemble $\{\varphi(I,r_n)_{n\in\mathbf{N}^*}\}$ non minoré.

Soit k un indice suffisamment grand pour que $r_k\neq 0$. Considérons $n=q_{k+1}-r_k$: $0<n<q_{k+1}$

$$\varphi(\beta+I,n)=\varphi_{1-\beta}(I,n)\geqq\varphi(I,r_k+n)-\varphi(I,r_k)-1=\varphi(I,q_{k+1})-1-\varphi(I,r_k).$$

Or $|\varphi(I,q_{k+1})|\leqq 3$ d'après (1.1) ce qui achève la démonstration.

§. 7. Remarques

a) Dans un article en préparation, nous montrons que, pour tout α irrationnel, il existe des intervalles à restes majorés et non minorés.

b) Nous avons montré l'existence de nombres α, tels qu'il existe des intervalles de longueur $\frac{1}{2}$ à restes majorés mais existe-t-il des nombres α tels que tout intervalle de longueur $\frac{1}{2}$ soit à restes non majorés?

c) Les mêmes prolèmes se posent pour des intervalles de longueur quelconque. Ils sont associès à la connaissance du développement par rapport à α de la longueur de l'intervalle.

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LABORATOIRE DE MATHÉMATIQUES ET D'INFORMATIQUE
DÉPENDANT DE L'UNIVERSITÉ DE BORDEAUX I ASSOCIÉ AU C. N. R. S.
351, COURS DE LA LIBÉRATION
33405 TALENCE
FRANCE

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КОБОРДИЗМЫ ГЛАДКИХ ОТОБРАЖЕНИЙ С ОСОБЕННОСТЬЮ

А. СЮЧ (Сегед)

0. Введение

В работе [2] построено классифицирующее пространство для l-погружений, т. е. такое пространство $\Gamma_l MO(k)$, для которого $\pi_{n+k}(\Gamma_l MO(k))\approx G^l(n, k)$, где $G^l(n, k)$ обозначает группу кобордизмов l-погружений n-мерных многообразий в $(n+k)$-мерную сферу (см. [2]). Однако, далеко не любое непрерывное отображение $M^n \to S^{n+k}$ может быть аппроксимировано погружением (и тем более l-погружением при $l<\infty$). То есть l-погружения не являются общими (т.е. плотными в пространстве всех отображений). В связи с этим возникает вопрос — нельзя ли построить классифицирующее пространство для общих отображений?

Конечно, тип общих отображений зависит от соотношения между размерностью n и коразмерностью k.

Напомним, что когда $n<2k$, то Σ^1-отображения (опр. см. ниже) являются общими (см., например, Хефлигер [1]).

Определение. Отображение $f\colon M^n \to N^{n+k}$ называется Σ^1-отображением, если

1) $\operatorname{rank} df(x) \geqq n-1$ и

2) если $f(x_1)=f(x_2)$, то $\operatorname{Im} df(x_1)$ и $\operatorname{Im} df(x_2)$ порождают все касательное производство в точке $f(x_1)=f(x_2)$.

Данная работа содержит описание классифицирующего пространства для Σ^1-отображений. При этом описании мы будем пользоваться обозначениями работы [2].

Введем некоторые новые обозначения. Группа кобордизмов Σ^1-отображений n-мерных многообразий в $(n+k)$-мерную сферу определяется стандартно и поэтому мы ее определение опустим. Обозначение этой группы будет $\Sigma^1(n, k)$. Искомое пространство, которое мы обозначим через $S(k)$, должно обладать свойством:

$$\pi_{n+\cdot}\,(S(k)) \approx \Sigma^1(n, k).$$

1. Конструкция пространства $S(k)$

Напомним, что для Σ^1-отображений двойные точки образуют внутренность некоторого многообразия с краем, а край этого многообразия ∂V состоит из точек, в которых ранг дифференциала df не максимален (т. е. $\{f(x) \mid \operatorname{rank} df(x)=n-1\}$).

Вне некоторой окрестности этого края отображение есть просто 2-погружение. Поэтому естественно исходить при построении искомого классифицирующего пространства для Σ^1-отображений из классифицирующего пространства для 2-погружений, которое мы обозначали через $\Gamma_2 MO(k)$.

Классифицирующие пространства всегда строились, исходя из нормального расслоения многообразий. Поэтому, чтобы понять, что нужно приклеить к пространству $\Gamma_2 MO(k)$, чтобы получить $S(k)$, посмотрим, каково нормальное расслоение многообразия ∂V.

Нетрудно видеть, что оно допускает группу $O^{(2)}(k)\oplus 1\subset O(2k+1)$. (Здесь $O^{(2)}(k)$ есть сплетение $O(k)\sim Z_2$, см. [2]). Универсальное расслоение с такой группой имеет вид:

$$EO^{(2)}(k)\times I \to BO^{(2)}(k).$$

Слои $EO^{(2)}(k)$ мы представляем себе, как $2k$-мерные шары, разложенные в прямое произведение $D^k\times D^k$. Поэтому слои пространства $EO^{(k)}(k)\times I$ имеют вид $I\times D^k\times D^k$. Это пространство мы и приклеим вдоль своего края к пространству $\Gamma_2 MO(k)$. Мы сейчас опишем приклеивающее отображение: Край пространства $EO^{(2)}(k)\times I$ есть объединение краев слоев. Край каждого слоя есть:

$$\partial(D^k\times D^k\times I) = \underbrace{D^k\times D^k\times\partial I}_{\{x_1\}}\cup \underbrace{\partial D^k\times D^k\times I}_{\{x_2\}}\cup \underbrace{D^k\times\partial D^k\times I}_{\{x_3\}}.$$

Точки первого множества этого объединения имеют вид

$$x_1 = x+\varepsilon \quad \text{где} \quad \varepsilon = 0\vee 1 \quad \text{и} \quad x\in EO^{(2)}(k).$$

Их мы приклеим к точке $x\in EO^{(2)}(k)\subset\Gamma_2 MO(k)$. Точки второго множества имеют вид:

$$x_2 = \{y, x, t\} \quad \text{где} \quad t\in I, \quad \text{а} \quad (y, x)\in\partial EO^{(2)}(k) \quad \text{причем} \quad \|y\| = 1.$$

Эти точки мы приклеим к точке $x\in MO(k)\subset\Gamma_2 MO(k)$.

Аналогично точки $x_3=\{x, y, t\}$ приклеим также к точке $x\in MO(k)\subset$ $\subset\Gamma_2 MO(k)$.

Полученное после приклеивания пространство и будет $S(k)$. Значит:

$$S(k) = \Gamma_2 MO(k) \bigcup_{\partial(EO^{(2)}(k)\times I)} EO^{(2)}(k)\times I.$$

Утверждение 1. $\pi_{n+k}(S(k)) \approx \Sigma^1(n, k)$.

Доказательство аналогично доказательству Теоремы 1 из [2].

2. Обобщение

Можно определить, конечно, кобордизмы Σ^1-отображений n-мерных многообразий в произвольное, но фиксированное $n+k$-мерное многообразие N^{n+k}. (Пленка натягивается в цилиндре $N^{n+k}\times I$). Множество таких кобордизмов обозначим через $\Sigma^1(n, N)$.

Нетрудно видеть следующий аналог утверждения 1:

Утверждение 2. $\Sigma^1(n, N)=[N, S(k)]$ *(равенство множеств).*

Замечание. В случае $n<2k-2$ любое непрерывное отображение можно аппроксимировать Σ^1-отображением, поэтому

$$\Omega_n(N) = \Sigma^1(n, N) = [N, S(k)].$$

3. Некоторые следствия

Напишем точную гомотопическую последовательность пары $(S(k), \Gamma_2 MO(k))$

$$(*) \quad \to \pi_{n+k+1}(S(k)) \to \pi_{n+k+1}(S(k), \Gamma_2 MO(k)) \to \pi_{n+k}(\Gamma_2 MO(k)) \to \pi_{n+k}(S(k)) \to.$$

Обозначим относительную группу $\pi_{n+k}(S(k), \Gamma_2 MO(k))$ через $X(n, k)$. Предположим $n<2k-2$. Тогда $(*)$ перепишется так:

$$\begin{array}{ccccccc} \Omega_{n+1}(S^{n+k+1}) & \to & X(n+1, k) & \to & G^2(n, k) & \to & \Omega_n(S^{n+k}). \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

То есть $G^2(n, k) \approx X(n+1, k)$. Но пространство $\Gamma_2 MO(k)$ k-связно, а пара $(S(k), \Gamma_2 MO(k))$ $2k+1$-связна, поэтому при $n<2k-2$

$$X(n+1, k) \approx \pi_{n+k+1}(S(k)/\Gamma_2 MO(k)) = \pi_{n+k+1}(SMO^{(2)}(k)) = G^1(n-k, 1 \oplus O^{(2)}(k)).$$

Здесь $G^1(m, H)$, где $H \subset O(j)$ означает группу кобордизмов вложений m-мерных многообразий в $m+j$-мерную сферу, норм. расслоение которых допускает группу H.

В нашем случае H будет группа $1 \oplus O^{(2)}(k)$, где 1 означает тривиальную группу $SO(1)$. Значит доказано:

Утверждение 3. *Если $n<2k-2$, то $G^2(n, k) \approx G^1(n-k, 1 \oplus O(k) \sim Z_2)$ которое сводит 2-погружения к вложениям.*

Замечание. Не вызывает сомнения, что аналогично можно построить группы кобордизмов и классифицирующие пространства для любых особенностей Σ^I. Интересно было бы выяснить, нельзя ли применить эти группы для получения теорем о несуществовании некоторых неособых отображений.

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ANDRÁS SZŰCS
JÓZSEF ATTILA TUDOMÁNYEGYETEM
BOLYAI INTÉZET
H-6720 SZEGED, ARADI VÉRTANÚK TERE 1.

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ON CONDITIONAL MEDIANS AND A LAW OF ITERATED LOGARITHM FOR STRONGLY MULTIPLICATIVE SYSTEMS[1]

By

M. CSÖRGŐ (Ottawa) and D. L. MCLEISH (Edmonton)

1. Introduction. In a recent paper TOMKINS [4] studies certain properties of conditional medians. In an earlier paper CSÖRGŐ [2] employs the notion of a conditional median and proves a law of iterated logarithm for strongly multiplicative systems of r. v. The aim of this note is to list two further properties of conditional medians and to study again, with their help, the earlier result of CSÖRGŐ [2] for strongly multiplicative systems.

2. The conditional Chebyshev inequality and bounds on the conditional median. Here we prove two simple propositions. TOMKINS' [4] definition of a conditional median is used throughout in the sequel.

PROPOSITION 1. *Suppose X is a r. v., $\mathscr{F}$ a sigma-field and ε is a non-negative $\mathscr{F}$-measurable r. v.. Then*

$$\mathbf{P}\{|X| > \varepsilon|\mathscr{F}\} \leqq \varepsilon^{-\alpha}\mathbf{E}(|X|^{\alpha}|\mathscr{F}) \quad a.s.\ for \quad \alpha > 0.$$

PROOF. Put $Y=\varepsilon^{-1}|X|$. Then

$$\mathbf{E}(|X|^{\alpha}|\mathscr{F})\varepsilon^{-1} = \mathbf{E}(|Y|^{\alpha}|\mathscr{F}) \geqq \mathbf{E}(Y^{\alpha}I_{[Y>1]}|\mathscr{F}) \geqq \mathbf{E}(I_{[Y>1]}|\mathscr{F}) = \mathbf{P}\{|X| > \varepsilon|\mathscr{F}\}.$$

PROPOSITION 2. *If $\mu(X|\mathscr{F})$ denotes the conditional median of X given $\mathscr{F}$, then for any $\alpha>0$*

$$|\mu(X|\mathscr{F})| \leqq (2\mathbf{E}(|X|^{\alpha}|\mathscr{F}))^{1/\alpha} \quad a.s.$$

PROOF. Clearly for any $\delta>0$

$$\mathbf{P}\{|X| > ((2+\delta)\mathbf{E}(|X|^{\alpha}|\mathscr{F}))^{1/\alpha}|\mathscr{F}\} \leqq \frac{\mathbf{E}(|X|^{\alpha}|\mathscr{F})}{(2+\delta)\mathbf{E}(|X|^{\alpha}|\mathscr{F})} = \frac{1}{2+\delta} < 1/2 \quad \text{a.s.}$$

by Proposition 1. The existence of $\mu(X|\mathscr{F})$ for every r.v. is guaranteed by Theorem 1 of TOMKINS [4].

3. A law of iterated logarithm for strongly multiplicative systems. In this section we work with r.v. satisfying the conditions of the following

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DEFINITION. A sequence $\{X_n\}$ of r.v. will be called a normed strongly multiplicative system (NSMS) if

$$(3.1)\qquad \begin{cases} \mathbf{E}(X_i)=0, \quad \mathbf{E}(X_i^2)=\delta_i^2, \quad i=1,2,\ldots, \\ \mathbf{E}(X_{i_1}^{r_1}X_{i_2}^{r_2}\ldots X_{i_k}^{r_k})=\mathbf{E}(X_{i_1}^{r_1})\mathbf{E}(X_{i_2}^{r_2})\ldots\mathbf{E}(X_{i_k}^{r_k}),\end{cases}$$

where $i_1<i_2<\ldots<i_k$, $k=1,2,\ldots$, and $r_1,r_2,\ldots,r_k$, $k=1,2,\ldots$, can be equal to 1 or 2.

We note that a martingale difference sequence of r.v. $\{X_n\}$ (that is $\mathbf{E}(X_1)=0$ and $\mathbf{E}(X_n|X_1,\ldots,X_{n-1})=0$, a.s. $n\geqq 2$) is a NSMS if it also satisfies the conditions

$$\mathbf{E}(X_1^2)=\delta_1^2, \quad \mathbf{E}(X_n^2|X_1,\ldots,X_{n-1})=\delta_n^2, \quad \text{a.s.} \quad n\geqq 2,$$

where $\delta_1,\delta_2,\ldots$ are non-negative constants.

Studies of strongly multiplicative systems of r.v. usually require the sequence $\{X_n\}$ to be uniformly bounded and equinormed (i.e., δ_i^2 are assumed to be equal), and the a.s. behaviour of such sequences is essentially known (cf. ALEXITS [1], RÉVÉSZ [3]). Not all strongly multiplicative systems can, however, be made equinormed, and the uniform boundedness condition is also quite restrictive. It should, therefore, be of interest to study sequences of r.v. satisfying our NSMS definition. In this regard the following is known.

THEOREM A (CSÖRGŐ [2]). *Let* $\{X_n\}$ *be a NSMS, put* $S_k=X_1+\ldots+X_k$ *and* $\hat{\delta}_n^2=\delta_1^2+\ldots+\delta_n^2$. *Assume that the conditional median* $\mu(S_k-S_n|S_1,\ldots,S_k)$ *of the r.v.* S_k-S_n *satisfies the condition*

$$(3.2)\qquad |\mu(S_k-S_n|S_i,\ldots,S_k)|\leqq(2\hat{\delta}_n^2)^{1/2}.$$

If $\hat{\delta}_n^2\to+\infty$ *and* $|X_n|/\hat{\delta}_n=o(\log\log^{-1/2}\hat{\delta}_n^2)$, $t_n=(2\log\log\hat{\delta}_n^2)^{1/2}$, *then*

$$(3.3)\qquad \mathbf{P}\left\{\limsup_{n\to\infty}\frac{|S_n|}{\hat{\delta}_n t_n}\leqq 1\right\}=1.$$

In the light of Proposition 2, we can drop condition (3.2), and make Theorem A somewhat more plausible and also more comparable to STOUT's [5] martingale analogue of Kolmogorov's law of iterated logarithm. We have

THEOREM. *Let* $\{X_n\}$ *be a NSMS and let* S_k, $\hat{\delta}_n^2$, t_n *and* $|X_n|/\hat{\delta}_n$ *be as in Theorem* A. *Put* $\sum_k^n=\mathbf{E}((S_k-S_n)^2|S_1,\ldots,S_k)$. *Assume, for every* $k\geqq 1$, *that*

$$(3.4)\qquad \textstyle\sum_k^n/\hat{\delta}_n^2\xrightarrow{\text{a.s.}}1 \quad (n\to\infty).$$

Then, for every $k\geqq 1$, *we have*

$$(3.5)\qquad \mathbf{P}\left\{\limsup_{n\to\infty}\frac{|S_n|}{(2\sum_k^n\log\log\sum_k^n)^{1/2}}\leqq 1\right\}=1.$$

PROOF. Proposition 2 implies that, almost surely

$$|\mu(S_k-S_n|S_1,\ldots,S_k)|\leqq(2\mathbf{E}((S_k-S_n)^2|\leqq(2\mathbf{E}((S_k-S_n)^2|S_1,\ldots,S_k))^{1/2}.$$

Using now the assumption (3.4) instead of (3.2), one first proves (exactly as in [2]) that (3.3) holds. Then, applying again (3.4), we get (3.5).

The constant 1 of Theorem is likely the best possible one.

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DEPARTMENT OF MATHEMATICS
CARLETON UNIVERSITY
COLONEL BY DRIVE
OTTAWA KLS 5B6
CANADA

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA T6G 2G1
CANADA

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DIMENSION TRANSITIV ORIENTIERBARER GRAPHEN[1]

Von
R. GYSIN (Dietikon)

1. Einleitung

Ist $\leqq_0$ eine partielle Ordnung der Menge P und sind a, b zwei bezüglich $\leqq_0$ unvergleichbare Elemente von P, so gibt es nach E. SZPILRAJN [5] eine totale Ordnung $\leqq$, welche $\leqq_0$ erweitert und für welche gilt: $a \leqq b$. Hieraus ergibt sich unmittelbar: Ist I die Menge aller totalen Erweiterungen von $\leqq_0$, so ist $\leqq_0$ der Durchschnitt der Ordnung aus I:

$$a \leqq_0 b \leftrightarrow (\forall \leqq_1)(\leqq_1 \in I \rightarrow a \leqq b).$$

Diese Äquivalenz gilt im allgemeinen offenbar nicht nur für die Menge I aller totalen Erweiterungen, sondern auch schon für gewisse ihrer Teilmengen I'. Nach DUSHNIK und MILLER [1] heißt die kleinste Mächtigkeit m, zu der eine solche Teilmenge I' mit der Mächtigkeit m existiert, die Dimension von $\leqq_0$. Für endliche partielle Ordnungen (was im folgenden stets vorausgesetzt wird) ist diese Dimension endlich.

Die Dimension von $\leqq_0$ ist offenbar genau dann 0, wenn P die leere Menge ist; sie ist 1, genau wenn P mindestens ein Element besitzt und $\leqq_0$ selbst eine totale Ordnung ist.

Um auch für den Fall der Dimension 2 eine notwendige und hinreichende Bedingung angeben zu können, wird einer partiellen Ordnung $(P, \leqq)$ nach folgender Vorschrift ein Graph $G = G(P, \leqq)$ zugeordnet:

Eckpunkte von G sind die Elemente von P; zwei verschiedene Elemente x, y von P sind genau dann verbunden, wenn sie vergleichbar sind (d. h. $x \leqq y$ oder $y \leqq x$).

Zu einem Graphen G wird ferner der komplementäre Graph $\bar{G}$ folgendermaßen definiert: $\bar{G}$ besitzt die selben Eckpunkte wie G und zwei verschiedene Elemente x, y sind in $\bar{G}$ genau dann verbunden, wenn sie in G nicht verbunden sind.

Es gilt dann der Satz (1): *Eine partielle Ordnung $(P, \leqq_0)$ besitzt genau dann eine Dimension $\leqq 2$, falls der zu $G(P, \leqq_0)$ komplementäre $\bar{G}$ transitiv orientierbar ist, d. h. falls es eine partielle Ordnung $(P, \leqq_1)$ gibt mit $G(P, \leqq_1) = \bar{G}$.*

Aus diesen Kriterien ergibt sich für partielle Ordnungen der Dimension $\leqq 2$ der folgende

[1] Diplomarbeit bei Prof. Dr. E. SPECKER an der E.T.H. Zürich.

SATZ. *Sind* $(P_1, \leqq_1)$ *und* $(P_2, \leqq_2)$ *partielle Ordnungen, deren zugeordnete Graphen* $G(P_1, \leqq_1)$ *und* $G(P_2, \leqq_2)$ *isomorph sind, so ist*

$$\dim (P_1, \leqq_1) = \dim (P_2, \leqq_2).$$

Das Ziel der vorliegende Arbeit ist der Beweis dieses Satzes für beliebige endliche partielle Ordnungen.

BEMERKUNG. Für endliche Dimensionen überträgt sich der Satz leicht auch auf unendliche partielle Ordnungen. Die Frage ob es auch für unendliche Dimensionen gilt, muß offen bleiben.

Ein wesentliches Hilfsmittel für den Beweis ist die folgende Äquivalenzrelation Δ auf der Kantenmenge eines Graphen G, welche von P. C. GILMORE und A. J. HOFFMAN in [3] eingeführt worden ist. Die Kanten (a, b) und (c, d) stehen genau dann in der Relation Δ, wenn es eine Folge von Kanten (x_i, x_{i+1}) $(i=1, \ldots, n, 2\leqq n)$ mit $x_1=a, x_2=b, x_n=c, x_{n+1}=d$ und $(x_i, x_{i+2})\notin G$ gibt.

Für transitiv orientierbare Graphen G ist die Relation Δ zu folgender Beziehung äquivalent:

Für alle transitive Orientierungen O_1, O_2 von G stimmen O_1, O_2 genau dann auf der Kante (a, b) überein, wenn sie auf der Kante (c, d) übereinstimmen.

2. Hilfssätze

SATZ 1 (T. HIRAGUTI, Satz 6.1 in [4]). *Sei ein System* $\{(A_s, \leqq_s) | s\in S\}$ *von punktfremden partiellen Ordnungen und eine partielle Ordnungsrelation* $\leqq_q$ *auf* S *gegeben, ferner soll ein* $\sigma\in S$ *existieren mit* $\dim (A_s, \leqq_s) \leqq \dim (A_\sigma, \leqq_\sigma)$ *für alle* $s\in S$. *Dann gilt:*

$$\dim \sum_{(S, \leqq_q)} (A_s, \leqq_s) = \max\big(\dim (A_\sigma, \leqq_\sigma), \dim (S, \leqq_q)\big).$$

Dabei bezeichnet $\sum_{(S, \leqq_q)} (A_s, \leqq_s)$ die Ordinalsumme der partiellen Ordnungen $(A_s, \leqq_s)$ $(s\in S)$ über der partiellen Ordnung $(S, \leqq_q)$.

SATZ 2 (T. GALLAI, Satz 1.2, Pkt. 2, Pkt. 3a, 3b in [2]). *Sei* G *ein endlicher Graph, ohne Schlingen und ohne Doppelkanten.*

(1) *Ist der komplementäre Graph* $\bar{G}$ *von* G *nicht zusammenhängend und sind* $P_1, \ldots, P_n$ *die Punktmengen der Komponenten von* $\bar{G}$, *so gilt, daß für alle Indexpaare* i, j $(1\leqq i, j\leqq n)$ *jeder Punkt aus* P_i *mit jedem Punkt aus* P_j *durch eine Kante von* G *verbunden ist, und daß die Kanten, die* P_i *mit* P_j *verbinden, eine* Δ-*Klasse* E_{ij} *bilden.*

(2) *Sind* G *und sein Komplement* $\bar{G}$ *beide zusammenhängend und mehrpunktig, so existiert eine echte Zerlegung* $\{P_1, \ldots, P_n\}$ *der Punktmenge von* G *mit den folgenden Eigenschaften:*

(a) *Für alle Indexpaare* i, j $(1\leqq i<j\leqq n)$ *entweder ist kein Punkt aus* P_i *mit einem Punkt aus* P_j *durch eine Kante von* G *verbunden oder jeder Punkt aus* P_i *mit jedem Punkt aus* P_j.

(b) *Die Kanten* (x, y) *von* G *für welche es kein* i $(1\leqq i\leqq n)$ *gibt mit* $x\in P_i$, $y\in P_i$ *bilden eine einzige* Δ-*Klasse* E *von* G.

Dabei heißt $\{P_1, \ldots, P_n\}$ eine echte Zerlegung der Menge P, wenn gilt: $2 \leqq n$, $P_i \cap P_j = \emptyset$ $(i \neq j, i, j = 1, \ldots, n)$ und $\bigcup_{i=1}^{n} P_i = P$.

SATZ 3 (T. GALLAI, Satz 1.9, Pkt. 1 in [2]). *Sei G wie in Satz 2. Man betrachte die Zerlegung $\{P_1, \ldots, P_n\}$ mit den Eigenschaften von Satz 2, und P_i, P_j seien verschiedene Mengen aus $\{P_1, \ldots, P_n\}$, so daß die Punkte von P_i mit jenen von P_j durch Kanten verbunden sind. Ist G transitiv orientierbar, so erhalten bei einer jeden transitiven Orientierung von G sämtliche Kanten aus G die P_i und P_j verbinden bezüglich der Mengen P_i und P_j die gleiche Richtung. (D. h. entweder sind alle von P_i nach P_j orientiert, oder sind alle umgekehrt orientiert.)*

LEMMA. *$P_1, \ldots, P_n$ seien die Punktmengen der Komponenten des Komplementären Graphen des Graphen der partiellen Ordnung $(P, \leqq)$. Dann gilt:*

$$\dim (P, \leqq) = \max_{1 \leqq i \leqq n} \dim (P_i, \leqq).$$

BEWEIS.[2] Wir setzen $P_i \leqq_s P_j$ genau dann, wenn $x \in P_i$ und $y \in P_j$ mit $x \leqq y$ existieren. Nach Satz 2, Pkt. (1) und Satz 3 folgt, daß $\leqq_s$ eine totale Ordnungsrelation auf der Menge $S = \{P_1, \ldots, P_n\}$ ist, und daß $\sum_{(S, \leqq_s)} (P_i, \leqq) = (P, \leqq)$ gilt. Nach Satz 1 folgt die Richtigkeit des Lemmas.

3. Hauptsatz

SATZ. *Sind $(P_1, \leqq_1)$ und $(P_2, \leqq_2)$ partielle Ordnungen deren zugeordnete Graphen $G(P_1, \leqq_1)$ und $G(P_2, \leqq_2)$ isomorph sind, so ist*

$$\dim (P_1, \leqq_1) = \dim (P_2, \leqq_2).$$

BEWEIS. Wir nehmen $P_1 = P_2 = P$ und $G(P_1, \leqq_1) = G(P_2, \leqq_2) = G$ an und führen den Beweis durch Induktion nach der Anzahl der Elemente von P. Für card $(P) = 1$ gilt: $\dim (P, \leqq_1) = \dim (P, \leqq_2) = 1$.

Sei nun m $(2 \leqq m)$ gegeben. Unter der Annahme, daß der Satz für alle partielle Ordnungen mit card $(P) < m$ gilt, werden wir den Satz auch für partielle Ordnungen mit card $(P) = m$ beweisen. Es bestehen folgende Möglichkeiten:

1. Der Graph G der beiden partiellen Ordnungen ist nicht zusammenhängend. Seien $P_1, \ldots, P_n$ die Punktmengen der Komponenten von G, dann nach dem Satz 1 folgt:

$$\dim (P, \leqq_1) = \max \big(\max_{1 \leqq i \leqq n} \dim (P_i, \leqq_1), 2 \big) \quad \text{und}$$

$$\dim (P, \leqq_2) = \max \big(\max_{1 \leqq i \leqq n} \dim (P_i, \leqq_2), 2 \big).$$

Aus der Induktionsannahme und weil card $(P_i) < m$ $(i = 1, \ldots, n)$ ist, folgt:

$$\dim (P_i, \leqq_1) = \dim (P_i, \leqq_2) \quad (i = 1, \ldots, n)$$

also:

$$\dim (P, \leqq_1) = \dim (P, \leqq_2).$$

[2] Für den vereinfachten Beweis des Lemmas möchte ich dem Referenten danken.

2. Das Komplement $\bar{G}$ von G ist nicht zusammenhängend. Seien $P_1, \ldots, P_n$ die Punktmengen der Komponenten von $\bar{G}$. Nach dem Lemma folgt:

$$\dim(P, \leqq_1) = \max_{1 \leqq i \leqq n} \dim(P_i, \leqq_1) \quad \text{und} \quad \dim(P, \leqq_2) = \max_{1 \leqq i \leqq n} \dim(P_i, \leqq_2).$$

Ferner wegen der Induktionsannahme und weil $\operatorname{card}(P_i) < m$ ist, gilt auch:

$$\dim(P_i, \leqq_1) = \dim(P_i, \leqq_2) \quad (i = 1, \ldots, n)$$

also:

$$\dim(P, \leqq_1) = \dim(P, \leqq_2).$$

3. G und $\bar{G}$ sind zusammenhängend. In diesem Fall betrachten wir die echte Zerlegung $Z = \{P_1, \ldots, P_n\}$ von P aus dem Satz 2 und für $i, j = 1, \ldots, n$ setzen wir $P_i R_1 P_j$ $(P_i R_2 P_j)$ genau dann, wenn $x \in P_i$ und $y \in P_j$ mit $x \leqq_1 y$ $(x \leqq_2 y)$ existieren. Nach Satz 2, Pkt. 2(a) und Satz 3 folgt, daß R_1 und R_2 partielle Ordungsrelationen auf Z sind, und daß

$$\sum_{(Z, R_1)} (P_i, \leqq_1) = (P, \leqq_1) \quad \text{sowie} \quad \sum_{(Z, R_2)} (P_i, \leqq_2) = (P, \leqq_2)$$

gilt. Nach dem Satz 1 gilt auch:

$$\dim \sum_{(Z, R_1)} (P_i, \leqq_1) = \max\big(\dim(Z, R_1), \max_{1 \leqq i \leqq n} \dim(P, \leqq_1)\big)$$

und

$$\dim \sum_{(Z, R_2)} (P_i, \leqq_2) = \max\big(\dim(Z, R_2), \max_{1 \leqq i \leqq n} \dim(P_i, \leqq_2)\big).$$

Ferner, da sämtliche Kanten von G, die verschiedene P_i Mengen verbinden eine einzige Δ-Klasse (Satz 2 Pkt. 2 (b)) bilden, ist R_1 gleich R_2 oder ihrer Inverse; insbesoderes gilt:

$$\dim(Z, R_1) = \dim(Z, R_2).$$

Aus der Induktionsannahme und weil $\operatorname{card}(P_i) < m$ ist, ergibt sich $\dim(P_i, \leqq_1) = \dim(P_i, \leqq_2)$. Also ist auch in diesem Fall

$$\dim(P, \leqq_1) = \dim(P, \leqq_2).$$

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VORSTADTSTR. 26
SCHWEIZ

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ON THE SUM OF DISTANCES BETWEEN n POINTS ON A SPHERE. II

By

R. ALEXANDER (Urbana)

§ 1. Introduction

By *sphere* we will always mean those points (x, y, z) in E^3 for which $x^2+y^2+z^2=1$. If p is a point on a sphere and $0<\varphi\leqq\pi/2$, those points on the sphere which with p form a central angle $\theta\leqq\varphi$ will be called a *spherical cap*. All distances, unless otherwise stated, will be Euclidean.

If $p_1, p_2, \ldots, p_n$ are variable points on the sphere, we let $S(n)$ denote the maximum value of the function $\sum_{i<j} |p_i-p_j|$. In our article [1] we proved that

$$\frac{2}{3}n^2-c_1 n^{\frac{1}{2}} < S(n) < \frac{2}{3}n^2-c_2, \tag{1}$$

and neither of these inequalities has been improved. However, K. B. STOLARSKY [6] by using a nice improvement of our method combined with results of W. M. Schmidt has given a much better version of the right inequality in (1) for the distance sum in higher dimensions.

STOLARSKY [7] has also studied the function $d(p_i)=\min_{i\neq j} |p_i-p_j|$ under the assumption that the distance sum is maximal, and shown that $d(p_i)>c_3/n$ for each i. The obvious conjecture that $d(p_i)$ is of order $n^{-1/2}$ seems beyond any present technique. Later, using a method of integral geometry, we will show that $d(p_i)<c_4/n^{3/16}$ when the p_i are in an extremal configuration. However, we first prove a general result relating distance sums and uniform distribution on the sphere.

§ 2. Uniformly distributed sequences of sets on the sphere

For $k=1, 2, \ldots$, let A_k be a set of k points on the sphere. We say that the sequence A_k is *uniformly distributed* if for each spherical cap C,

$$\lim_{k\to\infty} |A_k\cap C|k^{-1} = (\text{Area } C)/4\pi, \tag{2}$$

where $|X|$ denotes the cardinality of X. For convenience we define $s(A_k)=k^{-2}\sum_{i<j} |p_i-p_j|$ where p_i, p_j are members of A_k.

THEOREM 1. *For each positive integer k, let A_k be a set of k points on the sphere. Then the sequence of sets $A_1, A_2, \ldots$ is uniformly distributed if and only if*

$$\lim_{k\to\infty} s(A_k) = \frac{2}{3}. \tag{3}$$

PROOF. Let us suppose that the limit in (3) is indeed 2/3, but that the sequence $A_1, A_2, \dots$ is not uniformly distributed. Choose a subsequence, indexed by j, such that $\lim_{j\to\infty} |A_j \cap C| j^{-1}$ exists but does not equal (Area C)/4π for some spherical cap C.

Let μ_j be the measure on the sphere defined by placing mass j^{-1} at each point in A_j. By the Helly compactness theorem there will be a subsequence, indexed by m, of measures which converge weakly to a Borel measure μ on the sphere. By well-known properties of weak convergence it must be true that

$$\lim_{m\to\infty} s(A_m) = \frac{1}{2}\iint |p-q|\, d\mu(p)\, d\mu(q),$$

and this limit is 2/3 by assumption. However, BJŐRCK [4], has shown that $\iint |p-q|\, d\mu(p)\, d\mu(q) = 4/3$, an absolute maximum, only for the uniform measure among positive Borel measures of total mass 1 on the sphere. This contradicts the fact that μ is not uniform.

Because it is straightforward, we omit the proof in the other direction.

The following Corollary answers a question raised by Professor Erdős on his recent visit.

COROLLARY 1. *If* A_k *is chosen so that* $s(A_k) = k^{-2} S(k)$, *then the sequence* $A_1, A_2, \dots$ *is uniformly distributed.*

PROOF. The corollary follows at once from (1) and Theorem 1.

Another consequence of Theorem 1 is that

$$\lim_{k\to\infty} \left[\max\left(d(p_i):\ p_i \in A_k\right)\right] = 0$$

for the A_k in Corollary 1. We will strengthen this result in the next section.

§ 3. Inequalities for certain generalized distance sums

In this section μ will always be a *signed* Borel measure of total mass 1 concentrated on the sphere. We let $I(\mu)$ denote the integral $\iint |p-q|\, d\mu(p)\, d\mu(q)$.

THEOREM 2. *Let* C *be a spherical cap of spherical radius* φ *and let* μ *be such that* $|\mu| C = 0$. *Then*

$$\frac{1}{2} I(\mu) < \frac{2}{3} - c_5 \varphi^8. \tag{4}$$

The proof of Theorem 2 depends on integration with respect to the usual measure τ on the planes of E^3. This measure is invariant under Euclidean motions, and the measure of the set of planes which cut a line segment is the length of the segment. In general, if T is a Borel set of planes and $\bar{u}$ is a unit vector, let $T(\bar{u})$ be those planes in T which are orthogonal to $\bar{u}$, and let $h(\bar{u})$ be the Lebesgue measure of $T(\bar{u}) \cap L$ where L is any line parallel to $\bar{u}$. Then $\tau(T) = \frac{1}{2\pi} \int h(\bar{u})\, d\sigma(\bar{u})$ where σ is the usual surface measure of the sphere.

Let t be any plane which cuts the sphere, and let A_t and B_t be the portoins of the sphere which lie in the two open halfspaces determined by the plane t. In our article [2] we established the following formulas:

$$(5)\qquad I(\mu) = 2\int \mu(A_t)\mu(B_t)\,d\tau(t)$$

$$(6)\qquad I\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{1}{2}[I(\mu_1)+I(\mu_2)] = \frac{1}{2}\int[\mu_1(A_t)-\mu_2(A_t)]^2\,d\tau(t).$$

We shall apply the identity (6) for the case where μ_1 is the uniform measure on the sphere and μ_2 is a measure for which $|\mu_2|C=0$. Here $I\left(\frac{\mu_1+\mu_2}{2}\right)<\frac{4}{3}$ and $\frac{1}{2}[I(\mu_1)+I(\mu_2)]>I(\mu_2)$ by Bjőrck's theorem, which itself follows easily from (6) as we shall later show. Therefore if μ_1 is uniform,

$$(7)\qquad \frac{4}{3}-I(\mu_2) > \frac{1}{2}\int[\mu_1(A_t)-\mu_2(A_t)]^2\,d\tau(t).$$

In order to get a lower bound for the right side of (7), we shall integrate only over those special planes which cut the sphere in a small circle which lies entirely in the spherical cap C. Because of the general nature of μ_2, it would be difficult to consider the effect of other planes. However, for these special planes, $[\mu_1(A_t)-\mu_2(A_t)]^2=a^2(t)$ where $a(t)$ is the spherical area of the spherical cap determined by t and which lies in the cap C.

Suppose the centre of the cap C is given by the unit vector $\bar{u}_0$; let $\bar{u}$ be a unit vector which makes an angle $\theta<\varphi$ with $\bar{u}_0$. For the special set of planes we are considering, $h(\bar{u})=1-\cos(\varphi-\theta)$, where $h(\bar{u})$ is as above. Also it is easily seen that the average value of $a^2(t)$ for t in $T(\bar{u})$ exceeds $c_6\sin^4(\varphi-\theta)$. Thus the incremental contribution of the planes in $T(\bar{u})$ will exceed $c_6[1-\cos(\varphi-\theta)]\cdot\sin^4(\varphi-\theta)\,d\sigma(\bar{u})$. A Riemann integral which sums these contributions over all vectors $\bar{u}$ which lie in C is given by

$$(8)\qquad c_7\int_0^{\varphi}[1-\cos(\varphi-\theta)]\sin^4(\varphi-\theta)[2\pi\sin\theta\,d\theta].$$

Since $1-\cos(\varphi-\theta)>c_8(\varphi-\theta)^2$ and $\sin\theta>c_9\theta$, a simple computation shows that the right side of (7) exceeds $c_5\varphi^8$, and this proves Theorem 2.

COROLLARY 2. *Let the points* $p_1, p_2, \ldots, p_n$ *be placed on a sphere so that* $\sum_{i<j}|p_i-p_j|=S(n)$. *Then for each* i, $d(p_i)<c_4n^{-3/16}$.

PROOF. From (1) we know that $S(n)>\frac{2}{3}n^2-c_1n^{1/2}$. Let μ_2 be the measure formed by placing weight n^{-1} at each p_i and let $d(p_i)$ equal $2\varphi_i$.

By Theorem 2, $\frac{1}{2}I(\mu_2)<\frac{2}{3}-c_5\varphi_i^8$. Thus it must be true that $\frac{2}{3}-c_5\varphi_i^8>>\frac{2}{3}-c_1n^{-3/2}$ and $\varphi_i<\left(\frac{c_1}{c_5}\right)n^{-3/16}$. Since spherical distance exceeds Euclidean distance, $c_4=\frac{c_1}{2c_5}$ works.

In the case of the circle, L. FEJES TÓTH [5] showed that $d(p_i)=2\pi n^{-1}$ by identifying the vertices of the regular n-gon as being the extremal configuration. Our method, applied to the circle, would give $d(p_i)<c_{10}n^{-2/5}$. Since we are working with a local distortion technique, it is not surprising that our result is far from optimal. By considering an inequality $\frac{1}{2}I(\mu_2)<\frac{2}{3}-c_5\sum_i \varphi_i^8$, thus bringing in the local influence of all points, one might hope to obtain (for the sphere) $d(p_i)<c_{11}n^{-1/4}$ or perhaps a slightly better inequality.

Also, if μ_1 is the uniform measure on the sphere and μ_2 is arbitrary, $\int\int|p-q|d\mu_1(p)d\mu_2(q)=\frac{4}{3}$. From this it follows that $I\left(\frac{\mu_1+\mu_2}{2}\right)=\frac{1}{4}I(\mu_2)+1$. Substitution into (6) gives the identity

$$\frac{2}{3}-\frac{1}{2}I(\mu_2)=\int[\mu_1(A_t)-\mu_2(A)]^2\,d\tau(t), \tag{9}$$

which is perhaps more pleasing than the inequality (7). An analogue of (9) for positive measures, using Haar integration, was developed by STOLARSKY [6], and he should be credited with discovering this very pretty class of identities for $I(\mu)$ on Euclidean spheres. Also, we note that (9) immediately shows that $I(\mu_2)\leqq\frac{4}{3}$, and hence Bjőrck's result holds for signed measures which satisfy $\int d\mu=1$.

However, from the viewpoint of integral geometry (9) is a special case of (6), which holds for a broad class of metrics and arbitrary compact sets in Euclidean space as discussed in [2]. Thus the problems studied in [3] and [5] continue to be a source of interesting results.

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UNIVERSITY OF ILLINOIS
DEPARTMENT OF MATHEMATICS
URBANA, ILLINOIS 61801
USA

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ON ALMOST EVERYWHERE T-CONVERGENCE SYSTEMS

By
I. JOÓ (Szeged)

1. Let (Ω, A, μ) be a measure space with finite positive measure μ, let $S = S(\Omega, A, \mu)$ be the class of almost everywhere finite measurable functions on Ω with the usual metric of convergence in measure. Let T be a summation process generated by a matrix $\|t_{i,j}\|$ (not necessarily with finite rows) having the following properties:

$$|t_{i,j}| \leqq K \qquad (i, j = 1, 2, \dots) \tag{1}$$

and

$$\lim_{i\to\infty} t_{i,j} = 1 \quad (j = 1, 2, \dots), \tag{2}$$

where K is an absolute constant. Furthermore, let B be the Banach space of sequences[1] such that

$$\begin{cases} a = (a_1, a_2, \dots)\in B \quad \text{implies} \\ a(N_1, N_2) = (0, \dots, 0, a_{N_1}, a_{N_1+1}, \dots, a_{N_2}, 0, 0, \dots)\in B, \\ \|a(N_1, N_2)\|_B \leqq \|a\|_B, \quad \lim\limits_{N\to\infty} \|a(0, N) - a\|_B = 0, \\ \text{and} \quad |a_i| \leqq \|a\|_B.^2 \end{cases} \tag{3}$$

For example l_p $(1 \leqq p < \infty)$ is such a space. A sequence $\{f_n\} \subset S$ is called *T-convergence system for* B if for every $a \in B$ the limit

$$\hat{T}(a) = \lim_{i\to\infty} \lim_{N\to\infty} \tau_i^N(a) \quad \left(\tau_i^N(a) = \sum_{j=1}^{N} a_j t_{i,j} f_j(t)\right) \tag{4}$$

exists for almost every $t \in \Omega$.

The aim of this paper is to give a necessary and sufficient structural condition for $\{f_n\}$; that is, which does not use the elements of B under which it is a T-convergence system for B. We think these results (when $B = l_2$) may be useful in understanding the reason of the following fact, proved by MENSOV [2]: There exist a uniformly bounded orthonormal system $\{\varphi_n\}$ on $(0, 1)$ and a regular positive summation method T that is not equivalent, in the class of $L^2(0, 1)$ with respect to this system, to the a.e. convergence of any fixed subsequence of partial sums.

Our results will generalize those of NIKIŠIN [3] to summation processes; however, his proof cannot work in all parts of our proof. In the general case we need some results of Banach.

[1] Algebraic operations are defined in the natural way (by coordinates).
[2] The condition $|a_i| \leqq \|a\|_B$ is tacitly assumed in NIKISIN's work [3].

2. First we prove the following

THEOREM I. *Under the assumptions* (1), (2), *and* (3) *the sequence* $\{f_n\}\subset S$ *is a T-convergence system for* B *if and only if for all* $\varepsilon>0$ *and* $0<\delta<1$ *there exist a measurable set* $E_{\varepsilon,\delta}\subset\Omega$ *with* $\mu E_{\varepsilon,\delta}<\varepsilon$ *and a constant* $C_{\varepsilon,\delta}$ *depending only on* ε *and* δ *such that*

$$\int_{\Omega-E_{\varepsilon,\delta}} \sup_{i\geqq 1}\left(\limsup_{N\to\infty}\left|\sum_{j=1}^{N} a_j t_{i,j} f_j(t)\right|^{1-\delta}\right) d\mu(t) \leqq C_{\varepsilon,\delta}\|a\|_B^{1-\delta} \tag{5}$$

or all $a\in B$.

We need three lemmas. To this effect, let E be an arbitrary Banach space. A mapping G of E into S is called an *operator with convex absolute value* (briefly *convex*) if for every $x, y\in E$ and $\lambda\in R$ (real) the relations

$$|G(x+y)| \leqq |G(x)|+|G(y)| \tag{6}$$

and

$$|G(\lambda x)| = |\lambda|\cdot|G(x)| \tag{7}$$

hold true a.e.

G is called *bounded* if the image of the unit ball of E is bounded in measure in S, that is, for every $x\in E$ with $\|x\|_E\leqq 1$ and for every $\varepsilon>0$ there exists a $K>0$ such that

$$\mu\{t\in\Omega:\ |G(x)(t)|>K\}<\varepsilon. \tag{8}$$

According to (7) a convex operator is bounded if and only if it is continuous in the origin of E. From (6) it follows that

$$\big||G(x)|-|G(y)|\big| \leqq |G(x-y)|.$$

This shows that a convex operator is continuous if and only if it is continuous in the origin of E.

LEMMA 1(NIKIŠIN [3], Theorem 1). *Let* G *be a convex bounded mapping of* E *into* S. *Then for all* $\varepsilon>0$ *and* $0<\delta<1$ *there exist a measurable set* $E_{\varepsilon,\delta}\subset\Omega$ *with* $\mu E_{\varepsilon,\delta}<\varepsilon$ *and a constant* $C_{\varepsilon,\delta}$ *depending only on* ε *and* δ *such that*

$$\int_{\Omega-E_{\varepsilon,\delta}} |G(x)|^{1-\delta}\, d\mu \leqq C_{\varepsilon,\delta}\|x\|_E^{1-\delta} \tag{9}$$

for all $x\in E$.

LEMMA 2 (BANACH [1], Theorem I). *Let* $\{L_n\}$ *be a sequence of bounded linear mappings of* E *into* S *such that for every* $x\in E$ *the function*

$$G(x)(t) = \sup_{n\geqq 1}|L_n(x)(t)|$$

is finite for almost every $t\in\Omega$. *Then* G *is a bounded convex mapping of* E *into* S.

LEMMA 3 (BANACH [1], Theorem III). *Let* $\{L_n\}$ *be a sequence of bounded linear mappings of* E *into* S *such that for every* $x\in E$, $\limsup_{n\to\infty} |L_n(x)(t)|$ *is finite for almost every* $t\in\Omega$. *Suppose that for all* x *from a dense subset of* E *the limit* $\lim_{n\to\infty} L_n(x)(t)$ *exists for almost every* $t\in\Omega$. *Then* $\lim_{n\to\infty} L_n(x)(t)$ *exists for almost every* $t\in\Omega$ *for all* $x\in E$.

PROOF of THEOREM I. *Necessity*. Suppose that $\{f_n\}\subset S$ is a T-convergence system for B. Because the f_n's are finite for almost every $t\in\Omega$, further (1), $|a_i|\leqq \|a\|_B$, and $\|a(N_1, N_2)\|_B\leqq\|a\|_B$ are fulfilled (see (3)), one can easily see that the operators τ_i^N in (4) are bounded and linear, for arbitrary i and N. Apply Lemma 2, with a fixed i, to the sequence $\{\tau_i^N\}$ of operators and to $E=B$. Thus we obtain that the limit operator (denote it by τ_i) is also bounded and linear. We must only notice that for every $a\in B$

$$\left|\lim_{N\to\infty} \tau_i^N(a)(t)\right| \leqq \sup_{N\geqq 1} |\tau_i^N(a)(t)|$$

for almost every $t\in\Omega$. Now applying Lemma 2 to the sequence τ_i, we obtain that the operator

$$G(a)(t) = \sup_{i\geqq 1}\left(\limsup_{N\to\infty}\left|\sum_{j=1}^{N} a_j t_{i,j} f_j(t)\right|\right)$$

is a convex bounded mapping of B into S, and hence Lemma 1 gives (5).

Sufficiency. Conversely, suppose that (5) is fulfilled. Then

$$\int_{\Omega-E_{\varepsilon,\delta}} \limsup_{N\to\infty} |\tau_i^N(a)(t)|^{1-\delta}\, d\mu(t) \leqq C_{\varepsilon,\delta}\|a\|_B^{1-\delta} \quad (i = 1, 2, \ldots)$$

for all $a\in B$, which implies that for every $a\in B$, $\limsup_{N\to\infty} \tau_i^N(a)(t)$ is finite for almost every $t\in\Omega-E_{\varepsilon,\delta}$. As we noticed above, the operators τ_i^N $(i, N=1, 2, \ldots)$ are bounded and linear. It is clear that the set $A\overset{\text{def}}{=}\{a(0, M): a\in B, M=1, 2, \ldots\}\subset B$ is dense in B. This follows from the part $\lim_{N\to\infty}\|a(0, N)-a\|_B=0$ in (3). For any fixed i and $a(0, M)\in A$, the limit $\lim_{N\to\infty}\tau_i^N(a(0, M))(t)$ exists for almost every $t\in\Omega$. Thus Lemma 3 gives that for every $a\in B$, $\lim_{N\to\infty}\tau_i^N(a)(t)$ exists for almost every $t\in\Omega-E_{\varepsilon,\delta}$. Since $\varepsilon>0$ is arbitrary, for every $a\in B$ the limit $\lim_{N\to\infty}\tau_i^N(a)(t)$ exists for almost every $t\in\Omega$.

Denoting by τ_i the limit operator, Lemma 2 implies that the τ_i's are bounded and linear. In fact, repeat the above argument to the sequence τ_i. By (2), for every $a(0, M)\in A$ the limit $\lim_{i\to\infty}\tau_i\,(a(0, M))(t)$ exists for almost every $t\in\Omega$, further (5) gives

$$\int_{\Omega-E_{\varepsilon,\delta}} \sup_{i\geqq 1}|\tau_i(a)(t)|^{1-\delta}\, d\mu(t) \leqq C_{\varepsilon,\delta}\|a\|_B^{1-\delta},$$

which shows that for every $a\in B$, $\sup_{i\geqq 1}|\tau_i(a)(t)|$ is finite for almost every $t\in\Omega-E_{\varepsilon,\delta}$. Applying Lemma 3 we obtain that for all $a\in B$, $\lim_{i\to\infty}\tau_i(a)(t)$ exists for almost every $t\in\Omega-E_{\varepsilon,\delta}$. Since $\varepsilon>0$ is arbitrary, Theorem I is proved.

REMARK. The proof of Theorem I gives also the following statement. Under assumption (1), (2) and (3) the sequence $\{f_n\}\subset S$ is a T convergence system for B if and only if

$$\sup_{i\geq 1} \limsup_{N\to\infty} \left| \sum_{j=1}^{N} t_{i,j} a_j f_j \right| < \infty$$

in almost every point of Ω, for all $a\in B$.

3. We shall give a sharper form of Theorem I in case $B=l_p$ $(1\leq p<\infty)$ as follows.

THEOREM II. *Under the assumptions* (1), (2), *and* (3) *the sequence* $\{f_n\}\subset S$ *is a T-convergence system for* l_p $(1\leq p<\infty)$ *if and only if for all* $\varepsilon>0$ *and* $0<q<\min(p,2)$ *there exist a measurable set* $E_{\varepsilon,q}\subset\Omega$ *with* $\mu E_{\varepsilon,q}<\varepsilon$ *and a constant* $C_{\varepsilon,q}$ *depending only on* ε *and* q *such that*

$$\left\{ \int_{\Omega-E_{\varepsilon,q}} \sup_{i\geq 1}\left(\limsup_{N\to\infty} \left| \sum_{j=1}^{N} a_j t_{i,j} f_j(t) \right|^q \right) d\mu(t) \right\}^{1/q} \leq C_{\varepsilon,q} \|a\|_p \tag{10}$$

for all $a\in l_p$.

The proof runs along the same lines as that of Theorem I, but instead of Lemma 1 we use another result of Nikisin. For this we recall a definition of NIKIŠIN [3]. A mapping G of E into S is called sublinear if for all $x\in E$ there exists a linear mapping T_x of E into S depending on x such that

$$T(x)(t) = G(x)(t) \tag{11}$$

and for every $y\in E$

$$|T_x(y)(t)| \leq |G(y)(t)| \tag{12}$$

for almost every $t\in\Omega$. It is easy to prove that every sublinear operator is convex[3] (see also [3]). It is obvious that every linear operator is sublinear.

LEMMA 4 (NIKIŠIN [3], Theorem 2). *Let* G *be a bounded sublinear mapping of* l_p $(1\leq p<\infty)$ *into* S. *Then for all* $\varepsilon>0$ *and* $0<q<\min(p,2)$ *there exist a measurable set* $E_{\varepsilon,q}\subset\Omega$ *with* $\mu E_{\varepsilon,q}<\varepsilon$ *and a constant* $C_{\varepsilon,q}$ *depending only on* ε *and* q *such that*

$$\left\{ \int_{\Omega-E_{\varepsilon,q}} |G(a)(t)|^q d\mu(t) \right\}^{1/q} \leq C_{\varepsilon,q} \|a\|_{l_p} \tag{13}$$

for all $a\in l_p$.

PROOF OF THEOREM II. First we prove that if $\{f_n\}\subset S$ is a T-convergence system for l_p $(1\leq p<\infty)$, then the operator

$$G(a) \stackrel{\text{def}}{=} \sup_{i\geq 1}\left(\limsup_{N\to\infty} \left| \sum_{j=1}^{N} a_j t_{i,j} f_j(t) \right| \right) \quad (a\in l_p)$$

[3] In fact, every convex operator is also sublinear. This follows from the proof of the Hahn—Banach theorem, using the well-known fact that S is an ordered complete lattice.

is sublinear from l_p into S. To this effect, fix an $a\in l_p$ and define a function $i_a(t)$ by the equality:

$$G(a)(t)=\sup_{i\ge 1}\left(\lim_{N\to\infty}\left|\sum_{j=1}^{N}a_j t_{i,j}f_j(t)\right|\right)=\left|\sum_{j=1}^{\infty}a_j t_{i_a(t),j}f_j(t)\right|,$$

and set

$$T_a(b)(t)=\left[\operatorname{sign}\left(\sum_{j=1}^{\infty}a_j t_{i_a(t),j}f_j(t)\right)\right]\left[\sum_{j=1}^{\infty}b_j t_{i_a(t),j}f_j(t)\right].$$

It is easy to see that (11) and (12) are fulfilled. Now we can apply Lemma 4 and the proof of Theorem II is similar to that of Theorem I, thus we omit it.

4. In this section we consider the very important special case $B=l_2$. We shall prove the following

THEOREM III. *Under the assumptions* (1), (2), *and* (3) *the sequence* $\{f_n\}\subset S$ *is a T-convergence system for* l_2 *if and only if for all* $\varepsilon>0$ *there exist a measurable set* $E_\varepsilon\subset\Omega$ *with* $\mu E_\varepsilon<\varepsilon$ *and a constant* C_3 *depending only on* ε *such that*

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}(t_{k,j}-t_{k-1,j})\int_{(\Omega-E_\varepsilon)\cap E_k}f_j(t)\,d\mu(t)\right)^2\le C_\varepsilon<\infty \tag{14}$$

holds for an arbitrary decreasing sequence $\{E_k\}$ *of measurable subsets of* Ω $(t_{0,j}=0$ *for* $j=1,2,\ldots)$.

This is a structural condition for $\{f_n\}$ in order to be a T-convergence system for l_2, because it does not contain the elements of l_2. In the special case, when $t_{i,j}=0$ if $i<j$ and $=1$ otherwise, (14) reduces to

$$\sum_{j=1}^{\infty}\left(\int_{(\Omega-E_\varepsilon)\cap E_j}f_j(t)\,d\mu(t)\right)^2\le C_\varepsilon<\infty,$$

and this is a result of NIKIŠIN [3], p. 159.

PROOF. By Theorem II one can obtain the following assertion: $\{f_n\}\subset S$ is a T-convergence system for l_2 if and only if for all $\varepsilon>0$ there exist a measurable set $E_3\subset\Omega$ with $\mu E_\varepsilon<\varepsilon$ and a constant C_ε depending only on ε such that

$$\int_{\Omega-E_\varepsilon}\sup_{i\ge 1}\left(\limsup_{N\to\infty}\left|\sum_{j=1}^{N}a_j t_{i,j}f_j(t)\right|\right)d\mu(t)\le C_\varepsilon\|a\|_{l_2} \tag{15}$$

for all $a\in l_2$. In virtue of this, it is enough to prove the equivalence of (14) and (15).

First suppose that (15) is fulfilled, and let E_k be an arbitrary decreasing sequence of measurable subsets of Ω. Define a function $i(t)$ by setting $i(t)=k$ if $t\in E_k-E_{k+1}$

Then we have

$$(16)\qquad \int\limits_{\Omega-E_\varepsilon}\sum_{j=1}^{\infty} a_j t_{i(t),j} f_j(t)\,d\mu(t) = \int\limits_{\Omega-E_\varepsilon}\sum_{k=1}^{i(t)}\left(\sum_{j=1}^{\infty} a_j(t_{k,j}-t_{k-1,j})f_j(t)\right)d\mu(t) =$$

$$= \sum_{k=1}^{\infty}\int\limits_{(\Omega-E_\varepsilon)\cap E_k}\sum_{j=1}^{\infty} a_j(t_{k,j}-t_{k-1,j})f_j(t)\,d\mu(t) =$$

$$= \sum_{j=1}^{\infty} a_j \sum_{k=1}^{\infty}(t_{k,j}-t_{k-1,j})\int\limits_{\Omega-E_\varepsilon)\cap E_k} f_j(t)\,d\mu(t).$$

The change of the sums in the last step is permitted, because for almost every $t\in\Omega$ the sum with the running index k contains only a finite number of terms different from zero. Taking the upper bound of the right-hand side of (16) on the set $\{a\in l_2: \|a\|_{l_2}=1\}$, further estimating its left-hand side by (15), we obtain (14).

Conversely, suppose that (14) is fulfilled, but (15) not. Then there exists an $a\in l_2$ such that

$$(17)\qquad G(a) = \int\limits_{\Omega-E_\varepsilon}\sup_{i\geqq 1}\left(\limsup_{N\to\infty}\left|\sum_{j=1}^{N} a_j t_{i,j} f_j(t)\right|\right)d\mu(t) = \infty.$$

In fact, if $G(a)$ is finite for all $a\in l_2$ then the method of proving Theorem I gives that $\{f_n\}$ is a T-convergence system for l_2 and hence Theorem II implies that (15) is fulfilled, which is a contradiction.

By (17) it follows the existence of a measurable function $i(t)$ on Ω such that

$$(18)\qquad \int\limits_{\Omega-E_\varepsilon}\sum_{j=1}^{\infty} a_j t_{i(t),j} f_j(t)\,d\mu(t) = +\infty \quad\text{or}\quad -\infty.$$

Put $E_k=\{t\in\Omega: i(t)\geqq k\}$. From (16) it follows that

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}(t_{k,j}-t_{k-1,j})\int\limits_{(\Omega-E_\varepsilon)\cap E_k} f_j(t)\,d\mu(t)\right)^2 = \infty,$$

and this is a contradiction to our assumption. Thus Theorem III is proved.

In the special case

$$t_{i,j} = \begin{cases} 1 & \text{if } j\leqq n_i,\\ 0 & \text{if } j>n_i,\end{cases}$$

the T-summability means the convergence of n_k^{th} partial sums. Then (14) is of the special form

$$(19)\qquad \sum_{k=1}^{\infty}\sum_{j=n_{k-1}+1}^{n_k}\left(\int\limits_{(\Omega-E_\varepsilon)\cap E_k} f_j(t)\,d\mu(t)\right)^2 \leqq C_\varepsilon < \infty.$$

Comparing conditions (14) and (19), it is obvious that one can make the right-hand side of (14) less than that of (19). We think that this is the turning point of MENSOV's result [2]. However, we are unable to give a new proof of Mensov's result on the basis of the difference in conditions (14) and (19).

REMARK. The core of BANACH's paper [1] is our Lemma 2 and the other results of [1] follow from this easily. But Banach's proof for this lemma gives slightly more than this, namely it gives the following more general statement:

THEOREM IV. *If* $\{G_n\}_1^\infty$ *is a sequence of bounded convex mappings of the Banach space* E *into* S *(*S *is the same as before throughout in our paper) such that for every* $x \in E$ *the function*

$$G(x)(t) = \sup_{n \geq 1} |G_n(x)(t)|$$

is finite for almost every $t \in \Omega$, *then* G *is also a convex and bounded mapping of* E *into* S.

In the special case when Ω consists of only one point, further its measure is positive and finite (say, equals 1) then $S=R$ (real line) and Theorem IV reduces to a well known lemma of Gelfand, which is very useful many times in the theory of orthogonal series (see e.g. [3] or С. Качмаж, Г. Штейнгауз: *Теория ортогональных рядов*, Государственное Издательство Физ.-Мат. Литературы (Москва, 1958)). Gelfand's proof given in the just mentioned book does not seem to work for the general case of Theorem IV above.

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JÓZSEF ATTILA UNIVERSITY
BOLYAI INSTITUTE
6720 SZEGED, ARADI VÉRTANÚK TERE 1
HUNGARY

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ISOMORPHISM PROBLEM FOR A CLASS OF POINT-SYMMETRIC STRUCTURES

By

L. BABAI (Budapest)

1. Introduction. For a group G, the right regular representation G_R of G is a permutation group acting on G, which consists of the right multiplications by members of G. A graph X is a Cayley graph[1] of G if $V(X)=G$ and the automorphism group $\operatorname{aut} X$ of X contains G_R. It is easily seen, that the image X^α of X under an automorphism α of the *group* G, is again a Cayley graph. It is natural to ask the converse:

1.1. PROBLEM. *Given a group G and a Cayley graph X of G, are those of the form X^α ($\alpha \in \operatorname{Aut} G$) the only Cayley graphs of G isomorphic to X?*

The graph X (the group G, resp.) has the *Cayley isomorphism property,* if the answer to 1.1. is positive (for all Cayley grahps X of G). A. ÁDÁM conjectured [1] that the cyclic groups have this property. ELSPAS and TURNER [5] have found that this is false for Z_{16}, but they and DJOKOVIĆ [4] have proved that Z_p *(p prime) has the Cayley isomorphism property.* Their elegant proof uses the spectra of the adjacency matrices of these Cayley graphs. Quite surprisingly, this result extends far beyond the possible range of applications of adjacency matrices. One can extend the definition of Cayley graphs to any kind of (algebraic, combinatorial, topological, etc.) structures (objects of a concrete category). If $G=Z_p$, problem 1.1 has a positive answer even for *Cayley objects of any concrete category,* as can be shown by elementary group theoretic arguments (Theorem 2.3). Refining the method we obtain that Z_{2p} and D_p have the Cayley isomorphism property for ternary but not for quaternary relational structures (Theorems 4.4, 4.6). The corollary that D_p has the Cayley isomorphism property for graphs, and a formula, expressing the spectrum of a Cayley graph of G in terms of irreducible characters of G, are used in [2] to construct large families of cospectral Cayley graphs of D_p, thus showing that the spectrum cannot be used to prove that D_p has the Cayley isomorphism property for graphs.

In a forthcoming paper we shall prove that $Z_p \times Z_p$ has but Z_{p^2} has not the Cayley isomorphism property for graphs [3].

2. Definitions, preliminaries. Throughout this paper, G denotes a group, 1 its unit and S_G the symmetric group acting on the set G. Thus permutations from S_G operate on the members of G. We write the operation to the right. The right (left) regular representation $G_R(G_L)$ of G is defined by

$$G_R = \{g_R\colon\ g \in G\}, \quad G_L = \{g_L\colon\ g \in G\}$$

[1] Contrary to the usual definition, we do not require in this paper the connectedness of a Cayley graph. For basic facts concerning Cayley graphs see [6, 7, 8].

where $g_R, g_L \in S_G$ act on G according to

$$xg_R = xg, \quad xg_L = gx \quad (x \in G).$$

A *colour-graph* is a binary relational structure (containing any number of binary relations). A *graph* is a colour-graph with one colour (relation) only, which is symmetric and irreflexive.

Although we are interested in isomorphisms only, it will be convenient to use category language to formulate the problem in its most general form.

A *concrete category* is a pair $\mathcal{C}=(C, F)$ where C is a category and F, the *forgetful functor,* is a faithful set-functor $F: C \to \text{Sets}$ (associating an underlying set with each object and an underlying mapping with each morphism of C; for α, $\beta \in \text{Hom}_C(a, b)$, $\alpha \neq \beta$ implies $F(\alpha) \neq F(\beta)$).

We shall write

$$\text{iso}\,(X, Y) = \{F(f): f \text{ is an isomorphism of } X \text{ to } Y\};$$

$$\text{aut}\, X = \text{iso}\,(X, X) \quad (X, Y \in \text{0b}\, C)$$

(the *concrete* isomorphisms and automorphisms, resp.). Clearly, iso (X, Y) consists of bijections; aut (X) is a permutation group.

We shall require the *uniqueness* property: if $\text{id}_{F(X)} \in \text{iso}\,(X, Y)$ then $X=Y$. Moreover, we require that $\mathcal{C}$ be *closed under isomorphisms,* thus

given an object X and a bijection defined on $F(X)$ there is an object Y such that $\varphi \in \text{iso}\,(X, Y)$.

In view of the uniqueness property, Y is unique and will be denoted by X^φ.

Henceforth we always assume that $\mathcal{C}$ has the above properties.

We remark, that, from our point view, all morphisms except the isomorphisms can be omitted, hence the "category of graphs" means the category of graphs and graph-isomorphisms, etc.

An object X of $\mathcal{C}$ is a *Cayley-object* of the group G if

(i) $F(X)=G$;
(ii) aut $X \cong G_R$.

PROPOSITION 2.1. *If α is an automorphism of the group G and X is a Cayley-object of G in $\mathcal{C}$ then so is X^α.*

PROOF. $\alpha^{-1}g_R\alpha = (g^\alpha)_R$, since $x(\alpha^{-1}g_R\alpha) = (x^{\alpha^{-1}}g)^\alpha = xg^\alpha$. Hence

$$\text{aut}\,(X^\alpha) = \alpha^{-1}(\text{aut}\, X)\alpha \cong \alpha^{-1}G_R\alpha = G_R.$$

Let X be a Cayley-object of G in the concrete category $\mathcal{C}$. We shall say that X has the *Cayley-isomorphism property* (shortly X is a *CI*-object) for G in $\mathcal{C}$, if, given any Cayley-object Y of G in $\mathcal{C}$, if $X \cong Y$ then some automorphism of G belongs to iso (X, Y).

The group G is a *CI-group* with respect to $\mathcal{C}$ (shortly, a $\mathcal{C}$-*CI*-group), if every Cayley-object of G in $\mathcal{C}$ has the *CI* property.

In this terminology, the result of Djoković, Elspas and Turner, mentioned in the introduction, asserts that

2.2. Z_p *is a CI-group with respect to the category of graphs.*

We shall prove

THEOREM 2.3. Z_p *is a CI-group with respect to any concrete category.*

3. Characterization and application to p-groups. The following lemma provides a characterization of the Cayley objects.

3.1. LEMMA. *For a Cayley-object X of G, the following are equivalent:*

(i) X *is a CI-object;*

(ii) *given a permutation* $\varphi\in S_G$ *such that* $\varphi G_R\varphi^{-1}\leqq \operatorname{aut} X$, G_R *and* $\varphi G_R\varphi^{-1}$ *are conjugate in* aut X.

PROOF. (ii) implies (i). Let Y be another Cayley-object and φ an isomorphism: $Y=X^\varphi$. (So, $\varphi\in S_G$.) Clearly,

$$\operatorname{aut} X=\varphi(\operatorname{aut} Y)\varphi^{-1}\geqq\varphi G_R\varphi^{-1},$$

hence by (ii), $\varphi G_R\varphi^{-1}=\beta^{-1}G_R\beta$ for some $\beta\in\operatorname{aut} X$. Let $\beta\varphi=\gamma$, thus $\gamma^{-1}G_R\gamma==G_R$. If $1\gamma^{-1}=g$, set $\delta=g_R\gamma$. So, $1\delta=1$ and $\delta^{-1}G_R\delta=G_R$. Hence δ induces an automorphism α on G_R:

$$\delta^{-1}g_R\delta=(g^\alpha)_R=\alpha^{-1}g_R\alpha$$

for some $\alpha\in\operatorname{Aut} G$. Consequently, $\delta\alpha^{-1}$ belongs to the centralizer of G_R in S_G. As well known, this centralizer is G_L, the left regular representation of G, whence (as $\delta\alpha^{-1}$ fixes a letter)

$$\delta\alpha^{-1}=\mathrm{id}_G,\quad g_R\beta_\gamma=g_R\gamma=\delta=\alpha,\quad \varphi=\beta^{-1}g_R^{-1}\alpha,$$

$$Y=X^\varphi=X^{\beta^{-1}g_R^{-1}\alpha}=X^\alpha\quad(\text{as } \beta, g_R\in\operatorname{aut} X),$$

proving that X is a CI-object.

(i) implies (ii). Suppose $\varphi\in S_G$ and $\varphi G_R\varphi^{-1}\leqq\operatorname{aut} X$. Let $Y=X^\varphi$. Y is a Cayley-object of G since aut $Y=\varphi^{-1}(\operatorname{aut} X)\varphi\geqq G_R$. If X is a CI-object we have $Y=X^\alpha$ for some $\alpha\in\operatorname{Aut} G$. Hence $X^\alpha=X^\varphi$, $X^{\varphi\alpha^{-1}}=X$, consequently $\beta=\varphi\alpha^{-1}\in\operatorname{aut} X$. We conclude that

$$\varphi G_R\varphi^{-1}=\beta\alpha G_R\alpha^{-1}\beta^{-1}=\beta G_R\beta^{-1},$$

as stated.

3.2. COROLLARY. *For a group G the following are equivalent:*

(i) G *is a CI-group with respect to any concrete category.*

(ii) *Given any permutation* $\varphi\in S_G$, G_R *and* $\varphi G_R\varphi^{-1}$ *are conjugate in the subgroup* $\langle G_R,\varphi G_R\varphi^{-1}\rangle$ *of* S_G.

PROOF. The implication (ii)⇒(i) directly follows from 3.1, (ii)⇒(i). To derive (i)⇒(ii) from the corresponding implication of 3.1 we only have to observe that for $|G|=n$ there is an object X of the category $\mathcal{R}_n$ of n-ary relational structures with automorphism group aut $X=\langle G_R,\varphi G_R\varphi^{-1}\rangle$. (Any permutation group of degree n trivially coincides with the automorphism group of some n-ary relational structure.) X is a Cayley-object of $\mathcal{R}_n$, hence a CI-object by 3.2 (i). Thus 3.1 (ii) proves 3.2 (ii).

Now Theorem 2.3 easily follows: the group $G=Z_p$ (p prime) satisfies (ii) since now both G_R and $\varphi G_R \varphi^{-1}$ are p-Sylow subgroups of $\langle G_R, \varphi G_R \varphi^{-1}\rangle$.

One can easily see that Z_4 and $Z_2 \times Z_2$ satisfy 3.2 (ii), too, hence

3.3. Corollary. *Z_4 and $Z_2 \times Z_2$ are CI-groups.*

3.4. Problem. *Are there infinitely many CI-groups other than the cyclic groups of prime order?*

Similarly to our proof of Theorem 2.3 we obtain

3.5. Lemma. *For p a prime and G a finite p-group assume that G_R is a p-Sylow subgroup of* aut X *for some Cayley-object X of G. Then X is a CI-object.*

Proof. $|\varphi G_R \varphi^{-1}|=|G_R|$ hence an application of the Sylow theorem proves that X satisfies 3.1 (ii).

This lemma leads to another generalization of 2.2.

3.6. Theorem. *Let G be a finite p-group and X a connected Cayley graph of G. If the degree of the vertices of X is less than p then X is a CI-graph (for G).*

Proof. If G_R is a p-Sylow subgroup of aut X we are done by 3.5. Otherwise there is an automorphism $\pi \in \text{aut}\, X$ of order p^k fixing at least one letter. In view of the connectedness of X there is an edge $[x, y]$ in X such that $x\pi=x$, $y\pi \neq y$. The order of the orbit of y under $\langle \pi \rangle$ divides p^k hence $y, y\pi, \ldots, y\pi^{p-1}$ are different neighbours of x, a contradiction, proving the theorem.

4. Groups of order $2p$. 4.1. Lemma. *Let G be a finite group, Q a subgroup of S_G containing G_R, Q_p a p-Sylow subgroup of Q and $\varphi \in S_G$. Assume that both Q_p and G_R are contained in $\varphi^{-1} Q \varphi$. Then there is a $\psi \in S_G$ such that*

(i) $\psi \in N_{S_G}(Q_p)$;

(ii) $\psi \varphi^{-1} \in Q$;

(iii) *if, moreover,* $G_R \leqq N_{S_G}(Q_p)$, *then* $1\psi=1$.

Proof. Both Q_p and $\varphi Q_p \varphi^{-1}$ are p-Sylow subgroups of Q, hence

$$\varphi Q_p \varphi^{-1} = \beta^{-1} Q_p \beta$$

for some $\beta \in Q$. Set $g=1(\beta\varphi)^{-1}$ if $G_R \leqq N_{S_G}(Q_p)$ and $g=1$ otherwise. Let $\psi = g_R \beta \varphi$. (ii) and (iii) hold obviously. As $g_R \in N_{S_G}(Q_p)$, (i) holds, too.

4.2. Lemma. *Let G be a finite group, $\varphi \in S_G$, and Q a subgroup of S_G containing G_R. Assume that G_R contains a p-Sylow subgroup P of Q, and $G_R \leqq \varphi^{-1} Q \varphi$. Assume further that all automorphisms of P extend to automorphisms of G_R. Then there is an $\alpha \in$ Aut G and $\gamma \in C_{S_G}(P)$ such that*

(i) $\gamma\alpha\varphi^{-1} \in Q$;

(ii) *if, moreover,* $P \lhd G_R$, *then* $1\gamma=1$.

Proof. The conditions of 4.1 are satisfied now (in case (ii) even the assumption of 4.1 (iii) holds). Let ψ satisfy 4.1 (i), (ii) and (iii). By 4.1 (i), conjugation by ψ

induces an automorphism of P. Let this automorphism be the restriction of $\alpha\in\text{Aut } G$ to P. As $(g^\alpha)_R=\alpha^{-1}g_R\alpha$ $(g\in G)$, we have

$$\psi^{-1}g_R\psi = \alpha^{-1}g_R\alpha \quad (g_R\in P),$$

hence

$$\gamma = \psi\alpha^{-1}\in C_{S_G}(P).$$

Now (i) follows from 4.1 (ii) and (ii) from 4.1 (iii).

4.3. THEOREM. *Let* $|G|=2p, p$ *prime* $\geqq 3$. *If* X *is a Cayley-object for* G *and* $p^2\nmid |\text{aut } X|$ *then* X *is a CI-object.*

PROOF. Let $D_p=\langle\varrho, \sigma\colon \varrho^p=\sigma^2=1, \sigma\varrho\sigma=\varrho^{-1}\rangle$, $Z_{2p}=\langle\varrho, \sigma\colon \varrho^p=\sigma^2=1, \sigma\varrho=\varrho\sigma\rangle$. G is either D_p or Z_{2p}. Let $H=\{1, \varrho, \ldots, \varrho^{p-1}\}=\langle\varrho\rangle$. We thus have $G=H\cup\sigma H$. Let π_1 denote the cyclic permutation $(1, \varrho, \ldots, \varrho^{p-1})$ and π_2 the cyclic permutation $(\sigma, \sigma\varrho, \ldots, \sigma\varrho^{p-1})$, both considered as members of S_G.

Clearly, $\varrho_R=\pi_1\pi_2$ generates the p-Sylow subgroup P of G_R, being now a p-Sylow subgroup of $Q=\text{aut } X$, too. Let τ denote the involution $\tau= =(1, \sigma)(\varrho, \sigma\varrho)\ldots(\varrho^{p-1}, \sigma\varrho^{p-1})$. Actually, $\tau=\sigma_L$, hence for $G=Z_{2p}$, $\tau=\sigma_R$.

Let $\varphi\in S_G$ such that $G_R\leqq\varphi^{-1}Q\varphi$.

Now, all assumptions of 4.2 (including that of 4.2 (ii)) are satisfied. Hence, by 4.2 there exist $\alpha\in\text{Aut } G, \beta\in Q$ and $\gamma\in C_{S_G}(P)$ such that

$$\varphi = \beta\gamma\alpha$$

and $1\gamma=1$.

The centralizer of P in S_G is the group of order $2p^2$ generated by π_1, π_2 and τ, hence the stabilizer of 1 in $C_{S_G}(P)$ is $\langle\pi_2\rangle$. Consequently, $\gamma=\pi_2^k$ for some integer k.

Now we split the proof according to the two possibilities for G.

I. If $G=D_p$, observe that $\pi_2\in\text{Aut } G$, hence $\gamma\in\text{Aut } G, \gamma\alpha\in\text{Aut } G$, and we conclude that

$$\varphi G_R\varphi^{-1} = \beta(\gamma\alpha)G_R(\gamma\alpha)^{-1}\beta^{-1} = \beta G_R\beta^{-1}$$

thus G_R and $\varphi G_R\varphi^{-1}$ are conjugate in $Q=\text{aut } X$ proving the theorem by 3.1.

II. If $G=Z_{2p}$ we have $\tau\pi_2\tau=\pi_1$, hence $\tau\gamma\tau=\pi_1^k$ and $\gamma\tau\gamma^{-1}\tau=\pi_1^k\pi_2^{-k}$. But now we have $\tau\in G_R$,

$$\gamma\tau\gamma^{-1}\in\gamma G_R\gamma^{-1} = \beta^{-1}\varphi\alpha^{-1}G_R\alpha\varphi^{-1}\beta = \beta^{-1}\varphi G_R\varphi^{-1}\beta \leqq \beta^{-1}Q\beta = Q$$

hence $\gamma\tau\gamma^{-1}\tau\in Q$. Consequently,

$$\pi_2^{2k} = (\pi_1\pi_2)^k\pi_2^k\pi_1^{-k} = (\varrho_R)^k(\gamma\tau\gamma^{-1}\tau)\in Q.$$

If $k\not\equiv 0 \mod p$, $\langle\pi_2^{2k}\rangle=\langle\pi_2\rangle$ hence Q contains the subgroup $\langle\pi_1, \pi_2\rangle$ of order p^2, a contradiction. Thus $k\equiv 0 \mod p$, γ is the identity and

$$\varphi = \beta\alpha, \quad \varphi G_R\varphi^{-1} = \beta\alpha G_R\alpha^{-1}\beta^{-1} = \beta G_R\beta^{-1}$$

at once proving the theorem by 3.1.

4.4. THEOREM. *For p a prime and $|G|=2p$, G is a CI-group with respect to the category $\mathcal{R}_3$ of ternary relational structures (with any number of ternary relations defined on them).*

PROOF. For $p=2$ we are done by 3.3. Let $p\geqq 3$.

Let X be a Cayley object for G in $\mathcal{R}_3$. If p^2 does not divide the order of $Q=\text{aut}\,X$, X is a *CI*-object by 4.3. Henceforth we assume that $p^2||Q|$. We shall use the notation introduced in the proof of 4.3.

$P=\langle\pi_1\pi_2\rangle$ is the p-Sylow subgroup of G_R. P is contained in a unique p-Sylow subgroup $Q_p=\langle\pi_1,\pi_2\rangle$ of S_G, hence Q_p is a p-Sylow subgroup of Q. Let $\varphi\in S_G$ such that $\varphi G_R\varphi^{-1}\leqq Q$. The mentioned uniqueness of Q_p implies $\varphi Q_p\varphi^{-1}\leqq Q$. Now all assumptions of 4.1 (including that of 4.1 (iii)) are satisfied. Thus we deduce that 4.1 (i), (ii), (iii) hold for some $\psi\in S_G$. By 4.1 (i) and (iii),

$$\psi^{-1}\pi_1\psi=\pi_1^k$$

for some k ($\not\equiv 0 \bmod p$). Some automorphism α of G produces the same:

$$\psi^{-1}\pi_1\psi=\alpha^{-1}\pi_1\alpha.$$

(α is defined by $\varrho^\alpha=\varrho^k$, $\sigma^\alpha=\sigma$.) Let

$$\lambda=\psi\alpha^{-1};$$

thus λ fixes each member of $H=\langle\varrho\rangle$ and

$$\lambda^{-1}\pi_2\lambda=\pi_2^l$$

for some l ($\not\equiv 0 \bmod p$). By 4.1 (ii), $\varphi=\beta\psi$ for some $\beta\in Q$, hence (as $\alpha\in\text{Aut}\,G\leqq$ $\leqq N_{S_G}(G_R)$)

$$\varphi G_R\varphi^{-1}=\beta\psi G_R\psi^{-1}\beta^{-1}=\beta\lambda G_R\lambda^{-1}\beta^{-1}.$$

By 3.1 we only have to prove that G_R and $\lambda G_R\lambda^{-1}$ are conjugate in Q. ($\lambda^{-1}G_R\lambda\leqq Q$ since $\varphi G_R\varphi^{-1}\leqq Q$.) Actually, we assert that $\lambda\in Q$.

Let $\nu=\lambda\sigma_R\lambda^{-1}\sigma_R$. As $\lambda G_R\lambda^{-1}\leqq Q$ we have $\nu\in Q$. Note that the restriction of ν to σH coincides with that of λ, as $\sigma_R\lambda^{-1}\sigma_R$ fixes each member of σH.

In order to prove $\lambda\in Q$, let $x,y,z\in G$, and assume $(x,y,z)\in T$ for some ternary relation T belonging to X. We have to show that $(x\lambda,y\lambda,z\lambda)\in T$. This is clear if x,y,z each belong to H: then $x\lambda=x, y\lambda=y, z\lambda=z$. If $x,y\in H$, $z\in\sigma H$ then $z\lambda\in\sigma H$ hence $z\lambda=z\pi_2^t$ for some t, thus $(x\lambda,y\lambda,z\lambda)=(x\pi_2^t,y\pi_2^t,z\pi_2^t)\in T$ (since $\pi_2\in Q_p\leqq Q=\text{aut}\,X$). We proceed similarly if $x\in H$ and exactly one of y and z belongs to σH. Assume now $x\in H$ and $y,z\in\sigma H$. As $x\nu\in H$ there is a t such that $x\nu=x\pi_1^t$. Hence by a remark above.

$$(x\lambda,y\lambda,z\lambda)=(x,y\nu,z\nu)=(x\nu\pi_1^{-t},y\nu\pi_1^{-t},z\nu\pi_1^{-t})\in T$$

since both ν and π_1 belong to Q. We argue similarly if x and exactly one of y and z are in σH. Finally, if x,y,z each belong to σH we have

$$(x\lambda,y\lambda,z\lambda)=(x\nu,y\nu,z\nu)\in T.$$

The proof is complete.

4.5. COROLLARY. *For p a prime and $|G|=2p$, G is a CI-group with respect to the category of colour-graphs.*

PROOF. The rule $(x, y)\mapsto(x, x, y)$ associates a ternary relational structure with each colour-graph. Trivially, the isomorphisms remain the same, hence 4.4 in turn implies 4.5.

In contrast with theorem 4.4, we prove

4.6. THEOREM. *For p a prime, $p\geqq 3$ and $|G|=2p$, G is not a CI-group with respect to the category of quaternary relational structures.*

PROOF. Let ϑ_2 act on G as

$$\varrho^j\vartheta_2=\varrho^j;\quad (\varrho\sigma^j)\vartheta_2=\sigma\varrho^{-j}.$$

Let $\vartheta_1=\sigma_R\vartheta_2\sigma_R$ and

$$Q=\langle\vartheta_1\vartheta_2,\pi_1,\pi_2,\sigma_R\rangle.$$

Clearly, $\langle\vartheta_1\vartheta_2,\pi_1,\pi_2\rangle$ is a subgroup of index 2 in Q, $\langle\vartheta_1,\pi_1\rangle\cong\langle\vartheta_2,\pi_2\rangle\cong D_p$.

We assert that G_R and $\vartheta_2 G_R\vartheta_2^{-1}$ are not conjugate in Q. ($\vartheta_2 G_R\vartheta_2^{-1}\leqq Q$, since $\vartheta_1^2=\vartheta_2^2=id$, $\vartheta_2\pi_2\vartheta_2=\pi_2^{-1}$, $\vartheta_2\pi_1\vartheta_2=\pi_1$, $\vartheta_2\sigma_R\vartheta_2=\vartheta_1\vartheta_2\sigma_R$.) Consider $\pi_1\pi_2\in G_R$. $\vartheta_2\pi_1\pi_2\vartheta_2=\pi_1\pi_2^{-1}\in\vartheta_2 G_R\vartheta_2$. Assume that $\pi_1\pi_2^{-1}$ is a conjugate of some member η of G_R in Q. As $o(\pi_1\pi_2^{-1})=p$, we have $\eta=(\pi_1\pi_2)^m$ for some $m\not\equiv 0 \bmod p$. However, $Q\leqq N_{S_G}(\pi_1\pi_2)$, hence $\beta^{-1}\eta\beta$ is again a power of $\pi_1\pi_2$ for any $\beta\in Q$. Thus, there is an s such that

$$(\pi_1\pi_2)^S=\pi_1\pi_2^{-1}$$

hence $s\equiv 1 \pmod p$ and $s\equiv -1 \pmod p$, a contradiction, proving the assertion.

In view of 3.1 it remains to prove that one can define a quaternary relational structure X on G such that $\operatorname{aut} X=Q$. Let us define the quaternary relation T as follows: let $\pi=\pi_1\pi_2$,

$$T=\{(x,x,y,y)\colon x=y\pi \quad\text{or}\quad y=x\pi,\ x,y\in G\}\cup$$
$$\cup\{(x,x\pi,y,y\pi)\colon x\in H,\ y\in\sigma H \quad\text{or}\quad x\in\sigma H,\ y\in H\}\cup$$
$$\cup\{(x\pi,x,y\pi,y)\colon x\in H,\ y\in\sigma H \quad\text{or}\quad x\in\sigma H,\ y\in H\}.$$

Let $X=(G;T)$. Clearly, $\operatorname{aut} X\geqq Q$. As Q acts transitively on G, it suffices to prove that the stabilizers $(\operatorname{aut} X)_1$ and Q_1 coincide. The relations (x,x,y,y) guarantee that $(\operatorname{aut} X)_1\leqq\langle\vartheta_1,\vartheta_2,\pi_2\rangle$. We have to show that $(\operatorname{aut} X)_1\leqq\langle\vartheta_1\vartheta_2,\pi_2\rangle$, thus if some $\beta\in\operatorname{aut} X$ fixes all members of H then $\beta=\pi_2^m$ for some m. Otherwise β would be $\vartheta_2\pi_2^m$; here $\pi_2^m\in\operatorname{aut} X$ but $\vartheta_2\notin\operatorname{aut} X$: for $x\in H$, $y\in\sigma H$, ϑ_2 takes $(x,x\pi,y,y\pi)\in T$ to $(x,x\pi,y\vartheta_2,y\vartheta_2\pi^{-1})\notin T$, a contradiction. This completes the proof.

Added in proof (August 17, 1977). Several authors have obtained further partial results on Ádám's problem and its generalizations given in the present note. For p, q different primes, Z_{pq} has been proved to be a *CI*-group with respect to the category of graphs by M. H. Klin (Soviet Union) and R. Pöschel (GDR) already in 1975, and by C. D. Godsil (Australia), as well as py T. D. Parsons (U. S.) and B. Alspach (Canada) independently in 1977. (Recent personal commutations.)

Babai and Frankl have obtained strong constraints on the structure of *CI*-groups with respect to the category of graphs [3]. Problem 3.4 has been answered to the positive by the following remarkable result of P. P. Pálfy:

THEOREM (P. P. Pálfy). *Let* $n=p_1...p_k$ *where the* p_i*'s are prime numbers,* $p_{i+1}>p_1...p_i$ $(1\leqq i\leqq k-1)$, *and* g.c.d. $(n, \varphi(n))=1$. *Then* Z_n *is a CI-group with respect to any concrete category.* (φ denotes Euler's function.)

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ALGEBRA AND NUMBER THEORY
H-1088 BUDAPEST, MÚZEUM KRT. 6—8.

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ON SOME PROBLEMS OF P. TURÁN

By
P. VÉRTESI (Budapest)

1. Introduction. We investigate certain orthogonal systems and by these we prove that there does not exist rough theory for the Hermite—Fejér step parabolas. (We use the expressions $\left\|\sum_{k=1}^{n} v_{kn}(x)\, l_{kn}^2(x)\right\|$ as Lebesque-constants; see (2.13).)

2. Notations and preliminary results. 2.1. Let us consider an arbitary system of nodes

$$(2.1)\qquad -1 \leqq x_{n,n} < x_{n-1,n} < \ldots < x_{2,n} < x_{1,n} \leqq 1 \quad (n = 1, 2, 3, \ldots)$$

in $[-1, 1]$, further denote

$$(2.2)\qquad \Omega_n(X, x) \overset{\text{def}}{=} c(x-x_{1,n})(x-x_{2,n})\ldots(x-x_{n,n}) \quad (c \neq 0),$$

$$(2.3)\qquad l_{k,n}(X, x) \overset{\text{def}}{=} \frac{\Omega_n(X, x)}{\Omega_n'(X, x_{k,n})(x-x_{k,n})} \quad (k = 1, 2, \ldots, n),$$

$$(2.4)\qquad L_n(f;\, X, x) \overset{\text{def}}{=} \sum_{k=1}^{n} f(x_{k,n})\, l_{k,n}(X, x) \quad (n = 1, 2, 3, \ldots),$$

$$(2.5)\qquad v_{k,n}(X, x) \overset{\text{def}}{=} 1 - 2l_{k,n}'(X, x_{k,n})(x-x_{k,n}) \quad (k = 1, 2, \ldots, n),$$

$$(2.6)\qquad h_{k,n}(X, x) \overset{\text{def}}{=} v_{k,n}(X, x)\, l_{k,n}^2(X, x) \quad (k = 1, 2, \ldots, n),$$

$$(2.7)\qquad \mathfrak{h}_{k,n}(X, x) \overset{\text{def}}{=} (x-x_{k,n})\, l_{k,n}^2(X, x) \quad (k = 1, 2, \ldots, n),$$

$$(2.8)\qquad H_n(f;\, X, x) \overset{\text{def}}{=} \sum_{k=1}^{n} f(x_{k,n})\, h_{k,n}(X, x) \quad (n = 1, 2, 3, \ldots),$$

$$(2.9)\qquad H_n^*(f;\, X, x) \overset{\text{def}}{=} H_n(f;\, X, x) + \sum_{k=1}^{n} f'(x_{k,n})\, \mathfrak{h}_{k,n}(X, x) \quad (n = 1, 2, 3, \ldots),$$

where X is the matrix $\{x_{k,n}\}_{k=1}^{n}$ $(n=1, 2, 3, \ldots)$, $f\in C(=f(x)$ is continuous on $[-1, 1])$, moreover in (2.9) we suppose that $f'\in C$, too.

2.2. As FABER [1] proved, for *any* fixed system of nodes there exists a continuous function for which the Lagrange parabolas L_n do not converge uniformly to the function considered. But studying the Lebesgue constants

$$(2.10)\qquad \lambda_n(X) \overset{\text{def}}{=} \left\|\sum_{k=1}^{n} |l_{k,n}(X, x)|\right\| \quad (n = 1, 2, 3, \ldots),$$

in many cases we can decide the convergence-divergence behaviour of $L_n(f; X, x)$ for a well-defined class of function ($\|g\|$ stands for $\max_{-1\leq x\leq 1} |g(x)|$ for $g\in C$).

As in their paper [2] P. ERDŐS and P. TURÁN proved, if $\lambda_n(X)\sim n^\delta$ $(0<\delta<1)$ then

a) If $\gamma<\frac{\delta}{\delta+2}$ then there is an $f_1\in \mathrm{Lip}\,\gamma$ $(=\{f(x);\,|f(x)-f(y)|\leq c(f)\,|x-y|^\gamma,$ $x, y\in[-1, 1], 0<\gamma\leq 1\})$ such that the sequence $\|L_n(f_1; X, x)\|$ is unbounded;

b) If $\gamma>\delta$ then $\lim \|L_n(f; X, x)-f(x)\|=0$ whenever $f\in \mathrm{Lip}\,\gamma$;

c) If $\frac{\delta}{\delta+2}<\gamma<\delta$ then there is a matrix Z with $\lambda_n(Z)\sim n^\delta$ such that $\lim_{n\to\infty} \|L_n(f; Z, x)-f(x)\|=0$ whenever $f\in$ Lip γ, but we can define another matrix Y with $\lambda_n(Y)\sim n^\delta$ and $f_2\in$ Lip γ such that the sequence $\|L_n(f_2; Y, x)\|$ is unbounded.

As one can see, in a) and b) only the order of $\lambda_n(X)$ decides the convergence-divergence behaviour for Lip γ ("rough" theory); in c) we have to investigate the finer structure of our matrix, too ("fine" theory).

2.3. Now we turn to the Hermite—Fejér step-parabolas H_n. Here, contrary to the Faber's theorem, as L. FEJÉR [3] proved for the Chebyscheff-matrix $T=\left\{\cos\frac{2k-1}{2n}\pi\right\}$, $H_n(f; T, x)$ uniformly converges to $f(x)$ in $[-1, 1]$, supposing $f\in C$. More generally, as later G. GRÜNWALD [4] showed, if X is a so-called "strongly normal point-system", which means

$$(2.11)\qquad v_{k,n}(X, x) \geq a > 0 \quad (k = 1, 2, ..., n;\ n = 1, 2, 3, ...;\ x\in[-1, 1])$$

then also

$$(2.12)\qquad \lim_{n\to\infty} \|H_n(f;\ X, x)-f(x)\| = 0 \quad \text{for} \quad f\in C.$$

Denoting by $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta>-1)$ the Jacobi polynomials of degree n (which are orthogonal in $[-1, 1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$), (2.11) holds if $\Omega_n(X, x)=P_n^{(\alpha,\beta)}(x)$ and $-1<\alpha, \beta<0$. We notice that $\Omega_n(T, x)=P_n^{(-1/2,\,-1/2)}(x)$.

Let us introduce

$$(2.13)\qquad \mu_n(X) \stackrel{\text{def}}{=} \left\|\sum_{k=1}^{n} |v_{k,n}(X, x)|\, l_{k,n}^2(X, x)\right\|.$$

Knowing that for any X, $H_n^*(f; X, x)\equiv f(x)$, supposing f is a polynomial of degree $\leq n$, we have

$$(2.14)\qquad \sum_{k=1}^{n} v_{k,n}(X, x)\, l_{k,n}^2(X, x) \equiv 1.$$

So if $v_{k,n}(X, x)\geq 0$ for any k and n then $\mu_n(X)\equiv 1$. This is true, e.g., if (2.11) is valid. Another well-known example is the case $\Omega_n(X, x)=P_n^{(0,0)}(x)$ (Legendre polynomials; see e.g. [4]).

Following P. Turán, one may ask whether — using $\mu_n(X)$ — there exists the fine and rough theory for the H_n process and if it does, let us determine those γ's for which $H_n(f; X, x)$ uniformly converges to $f(x)$ in $[-1+\varepsilon, 1-\varepsilon]$ whenever $f\in$ Lip γ and $\mu_n(X)=O(n^\delta)$ $(0<\gamma, \delta<1)$ (see [5], Problems XIX and XX).

2.4. Let us remark that here we have to restrict ouselves to closed subintervals of $(-1, 1)$, because for the Legendre abscissas if

$$\lim_{n\to\infty} |H_n(f;\ P^{(0,0)}, x)-f(x)| = 0 \quad (|x| = 1) \tag{2.15}$$

then

$$\frac{1}{2}\int_{-1}^{1} f(x)\,dx = f(-1) = f(1) \quad (f\in C)$$

(see [6]). On the other hand, $H_n(f;\, P^{(\alpha,\alpha)}, x)$ uniformly tends to $f(x)$ if $|x|\leqq 1-\varepsilon$, $f\in C$ and $\mu_n(P^{(\alpha,\alpha)})\sim n^{2\alpha}$ for $\alpha\geqq 0$ (see [7], 14.6).

3. **Results. 3.1.** First we deal with the second problem raised in 2.3 and prove that there do not exist γ and δ satisfying the requirements.

For this aim let

$$R_{2n}^{(\alpha,\beta)}(x) \overset{\text{def}}{=} P_n^{(\alpha,\beta)}(1-2x^2) \quad (n = 1, 2, 3, \ldots;\ \alpha, \beta > -1). \tag{3.1}$$

Denoting by $R^{(\alpha,\beta)}$ the matrix formed by the roots of the polynomials $R_{2n}^{(\alpha,\beta)}(x)$ of degree $2n$ we state

THEOREM 3.1. *If* $\alpha\geqq 0$, $\alpha\geqq\beta>-1$ *then we have*

$$\mu_{2n}(R^{(\alpha,\beta)})\sim n^{2\alpha} \quad (\alpha \geqq 0, \alpha \geqq \beta > -1) \tag{3.2}$$

and for the function $f_1(x)=x^2$

$$H_{2n}(f_1;\ R^{(\alpha,\beta)}, 0) \sim n^{2\alpha} \quad (\alpha \geqq 0, \alpha \geqq \beta > -1). \tag{3.3}$$

It seems to be worth to formulate the following special case.

COROLLARY 3.1. *There exists such matrix* X *for which* $\mu_{2n}(X)=O(1)$ *but for* $f_1(x)=x^2$

$$\varliminf_{n\to\infty} |H_{2n}(f_1;\ X, 0)-f_1(0)| > 0.$$

If we investigate X for which $n\mu_n^{-1}(X)=o(1)$ we can use the following

THEOREM 3.2. *Let* $n\mu_n^{-1}=o(1)$. *Then for each such a sequence* $\{\mu_n\}_{n=1}^{\infty}$ *there exists a matrix* $Y=Y_{\{\mu_n\}}$ *such that* $\mu_n(Y)\sim\mu_n$ $(n=1, 3, 5, \ldots)$ *further for the function* $f_2(x)=x$ *and for arbitrary but fixed* $x\neq 0$, $|x|\leqq 1$

$$\varlimsup_{n\to\infty} |H_n(f_2;\ Y, x)-f_2(x)|\left(\frac{n}{\mu_n}\right)^{2/3} > 0 \quad (|x| \leqq 1, x \neq 0, \mathit{fix}). \tag{3.4}$$

3.2. To complete our assertion we prove a convergence-theorem. Let $\omega(t)$ be a modulus of continuity on $[-1, 1]$, $\omega(f;t)$ the modulus of continuity of $f\in C$, further $C(\omega)=\{f(x);\ \omega(f;t)\leqq a(f)\omega(t)\}$. We prove the following

THEOREM 3.3. *Let* $1\leqq\mu_n=o(n)$. *Then for each sequence* $\{\mu_n\}_{n=1}^{\infty}$ *there exist a matrix* $Z=Z(\{\mu_n\})$ *and a function-class* $C(\omega)$ *depending on the sequence* $\{\mu_n\}$ *such that* $\mu_n(Z)\sim\mu_n$, *moreover*

$$\lim_{n\to\infty} \|H_n(f;\ Z, x)-f(x)\| = 0 \quad \textit{if} \quad f\in C(\omega). \tag{3.5}$$

By 3.1, 3.2 and (4.46)—(4.48) we obtain the following

COROLLARY 3.2. *Let* δ *be fixed with* $0<\delta<1$.
a) *If* $\gamma>\delta$, *then there is a matrix* Z *with* $\mu_n(Z)\sim n^\delta$ *such that*

$$\lim_{n\to\infty} \|H_n(f;\, Z, x)-f(x)\| = 0 \quad \textit{for} \quad f\in \mathrm{Lip}\,\gamma.$$

b) *There exist a matrix* Y *with* $\mu_{2n}(Y)\sim n^\delta$ *and* $f_1\in\mathrm{Lip}\,1$ *such that*

$$\lim_{n\to\infty} |H_{2n}(f_1;\, Y, 0)-f_1(0)| = \infty,$$

i.e., using $\mu_n(X)$ *and* Lip γ, *there does not exist rough theory for the Hermite—Fejér step-parabolas, either on the whole* $[-1, 1]$ *or on a closed subinterval* (see [5], Problem XIX).

3.3. In this part we state further properties of the polynomials defined by (3.1). If

$$R_{2n+1}^{(\alpha,\beta)}(x) \overset{\text{def}}{=} xP_n^{(\alpha+1,\beta)}(1-2x^2) \quad (n=0,1,2,\dots;\ \alpha,\beta>-1) \tag{3.6}$$

then using similar ideas as in [7], Theorem 4.1, we can verify that *the polynomials* $R_n^{(\alpha,\beta)}(x)$ $(\alpha,\beta>-1)$ *of degree* n *defined by* (3.1) *and* (3.6) *form an orthogonal system in* $[-1, 1]$ *with respect to the wight function*

$$p(\alpha,\beta;\, x) = |x|^{2\alpha+1}(1-x^2)^\beta \quad (\alpha,\beta>-1). \tag{3.7}$$

By (3.7) we obtain

COROLLARY 3.3. *Supposing* $\alpha,\beta>-1$, *we have*

$$R_n^{(-1/2,\beta)}(x) = c(n,\beta)\cdot P_n^{(\beta,\beta)}(x),$$

where $c(n,\beta)$ *does not depend on* x.

Now we state

THEOREM 3.4. *If* $\alpha\geqq\beta>-1$ *then we have*

$$\lambda_{2n}(R^{(\alpha,\beta)}) \sim \begin{cases} \ln n & (-1<\alpha\leqq -0.5),\\ n^{\alpha+1/2} & (-0.5<\alpha)\end{cases} \tag{3.8}$$

and

$$\mu_{2n}(R^{(\alpha,\beta)}) \sim \begin{cases} O(1) & (\alpha\leqq 0),\\ n^{2\alpha} & (\alpha\geqq 0).\end{cases} \tag{3.9}$$

3.4. Remarks. 3.4.1. The following problem is also due to P. TURÁN ([5], Problem XII).

For a fixed $0<\gamma<1$ let us give a matrix of nodes X such that $\lim_{n\to\infty}\|L_n(f;\, X, x)-f(x)\|=0$ if $f\in\mathrm{Lip}\,\gamma$, but for a suitable $f_1(x)\in\mathrm{Lip}\,\gamma$ we have $\overline{\lim_{n\to\infty}}\|H_n(f_1;\, X, x)\|=\infty$.

In [8] and [9] we gave some positive solutions for this problem. Now we provide another particular answer. Indeed, using (3.8), we get by standard argument

$$\|L_{2n}(f;\, R^{(\alpha,\beta)}, x)-f(x)\| = O(1)\,\omega\left(f;\, \frac{1}{n}\right) n^{\alpha+1/2} \quad (-0.5<\alpha,\ \alpha\geqq\beta), \tag{3.10}$$

from where

$$\lim_{n\to\infty} \|L_{2n}(f;\ R^{(\alpha,\beta)}, x)-f(x)\| = 0 \quad \text{if} \quad f\in \operatorname{Lip}\gamma,$$

moreover, applying (3.3) we get

$$\overline{\lim_{n\to\infty}} \|H_{2n}(f_1;\ R^{(\alpha,\beta)}, x)\| = \infty \quad \text{if} \quad f_1 = x^2 (\in \operatorname{Lip}\gamma),$$

supposing $\alpha>0, \alpha\geqq\beta$ and $\gamma>\alpha+0.5$.

3.4.2. In [8] and [9] we handled those cases when, in order to characterize the Hermite—Fejér interpolation, one uses $\lambda_n(X)$ instead of $\mu_n(X)$.

4. Proofs. 4.1. PROOF OF THEOREM 3.1. **4.1.1.** Using [7], (4.3.1) we have

$$(4.1)\quad (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\} \quad (\alpha,\beta>-1).$$

As we know the roots of $P_n^{(\alpha,\beta)}(x)$ satisfy

$$(4.2)\qquad -1 < x_{n,n}^{(\alpha,\beta)} < x_{n-1,n}^{(\alpha,\beta)} < \ldots < x_{2,n}^{(\alpha,\beta)} < x_{1,n}^{(\alpha,\beta)} < 1.$$

With $x=\cos\vartheta$ and $x_{k,n}^{(\alpha,\beta)}=\cos\vartheta_{k,n}^{(\alpha,\beta)}$ $(0\leqq\vartheta, \vartheta_{k,n}\leqq\pi)$ we have the following relations (sometimes omitting the superfluous notations)

$$(4.3)\qquad \frac{c_1}{n} \leqq \vartheta_{k+1}-\vartheta_k \leqq \frac{c_2}{n} \quad (k=0,1,\ldots,n)$$

with $0<c_1=c_1(\alpha,\beta)$; $c_2=c_2(\alpha,\beta)$, $x_0=\cos\vartheta_0=1$ and $x_{n+1}=\cos\vartheta_{n+1}=-1$;

$$(4.4)\quad |x-x_k| \sim n^{-2}|j^2-k^2| \quad \text{if} \quad x\in[x_{j+1},x_j],\quad k=0,1,\ldots,n+1,\ k\neq j, j+1$$

(see [10], Lemma 1 and Lemma 2). Further

$$(4.5)\qquad P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x),\ P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{\alpha} \sim n^{\alpha},$$

$$(4.6)\qquad |P_n^{(\alpha,\beta)}(\cos\vartheta)| = \begin{cases} O(\vartheta^{-\alpha-1/2}n^{-1/2}) & \text{if} \quad \dfrac{c_3}{n}\leqq\vartheta\leqq\pi-\varepsilon, \\ O(n^{\alpha}) & \text{if} \quad 0\leqq\vartheta\leqq\dfrac{c_3}{n}, \end{cases}$$

$$(4.7)\qquad [P_n^{(\alpha,\beta)}(x_k)]' \sim \begin{cases} k^{-\alpha-3/2}n^{\alpha+2} & (0<\vartheta_k\leqq\pi-\varepsilon), \\ (n-k+1)^{-\beta-3/2}n^{\beta+2} & (\varepsilon\leqq\vartheta_k<\pi), \end{cases}$$

$$(4.8)\qquad [P_n^{(\alpha,\beta)}(x)]' = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

$$(4.9)\qquad \frac{[P_n^{(\alpha,\beta)}(x_k)]''}{[P_n^{(\alpha,\beta)}(x_k)]'} = \frac{\alpha-\beta+(\alpha+\beta+2)x_k}{1-x_k^2}$$

(see [7], (4.1.3), (4.1.1), (7.32.5), (8.9.2), (8.9.9), (4.21.7) and (14.5.1)).

By (3.1), denoting the roots of $R_{2n}^{(\alpha,\beta)}(x)$ by $y_{k,n}^{(\alpha,\beta)}$, we have

$$(4.10)\qquad y_{k,n}^{(\alpha,\beta)} = -y_{-k,n}^{(\alpha,\beta)} = \sqrt{\frac{1-x_{k,n}^{(\alpha,\beta)}}{2}} = \sin\frac{\vartheta_{k,n}}{2} \quad (k=1,2,\dots,n),$$

$$(4.11)\qquad R_{2n}'(y_k) = -4y_k P_n'(x_{|k|}) \quad (k=\pm1,\pm2,\dots,\pm n),$$

$$(4.12)\qquad R_{2n}''(y_k) = 16y_k^2 P_n''(x_{|k|}) - 4P_n'(x_{|k|}) \quad (k=\pm1,\pm2,\dots,\pm n).$$

So for the fundamental functions, by (2.3) and (3.1)

$$(4.13)\qquad l_{k,n}(x) = -\frac{P_n(1-2x^2)}{4y_k P_n'(x_{|k|})(x-y_k)} \quad (k=\pm1,\pm2,\dots,\pm n),$$

further by (2.5), (4.11), (4.12) and (4.9)

$$(4.14)\qquad v_{k,n}(x) = 1+\left[4\frac{\alpha-\beta+(\alpha+\beta+2)x_{|k|}}{1-x_{|k|}^2}y_k - \frac{1}{y_k}\right](x-y_k).$$

4.1.2. Let us prove now (3.3). By (4.13), (4.5), (4.7) and (4.10)

$$(4.15)\qquad \sum_{-1+\delta<x_{|k|}<-\delta} l_k^2(0) \sim \sum_{-1+\delta<x_k<-\delta} \frac{n^{2\alpha}}{n} \sim n^{2\alpha} \quad (\alpha>-1,\ 0<\delta<1).$$

By (4.13), (4.10), (4.6) and (4.7), using that for $f_1 = x^2$

$$(4.16)\qquad x^2 \equiv H_{2n}(f_1;\, x) + \sum_{|k|=1}^{n} 2y_k(x-y_k)\, l_k^2(x) \quad (n \geqq 2)$$

$\left(\text{where } \sum\limits_{|k|=1}^{n} \text{ stands for } \sum\limits_{\substack{k=-n\\ k\neq 0}}^{n}\right)$ we obtain

$$H_{2n}(f_1;\, 0) = 2\sum_{|k|=1}^{n} y_k^2 l_k^2(0) \sim n^{2\alpha}\left\{\sum_{1\leqq k\leqq \frac{n}{2}} \frac{k^{2\alpha+3}}{n^{2\alpha+4}}\frac{n^2}{k^2} + \sum_{\frac{n}{2}<k\leqq n} \frac{(n-k+1)^{2\beta+3}}{n^{2\beta+4}}\right\} \sim n^{2\alpha}.$$

Let us see now μ_{2n}. By (4.14) we have

$$(4.17)\ v_k(0) = 2 - \frac{4[(\alpha-\beta)+(\alpha+\beta+2)x_{|k|}]}{1-x_{|k|}^2}\, y_k^2 \geqq 2 \quad \text{for} \quad \alpha\geqq\beta, \quad x_{|k|} \leqq \frac{\beta-\alpha}{\alpha+\beta+2},$$

so by (4.15)

$$(4.18)\qquad \mu_{2n}(R^{(\alpha,\beta)}) \geqq c(\alpha)\, n^{2\alpha} \quad (\alpha \geqq \beta > -1).$$

But, of course, we need an upper estimation, too.

4.1.3. LEMMA 4.1. *We have*

$$\mu_{2n}(R^{(\alpha,\beta)}) \sim n^{2\alpha} \quad (\alpha\geqq 0,\ \alpha\geqq\beta>-1)$$

(*see* (3.2)).

A. We can suppose $x \neq y_k$ $(k=\pm 1, \pm 2, \dots, \pm n)$. Let, e.g., $0 \leqq x \equiv \cos \vartheta \leqq 1$, so $1-2x^2=1-2\cos^2\vartheta = -\cos 2\vartheta = \cos(\pi-2\vartheta)$. Let $\min\limits_{1\leqq k\leqq n} |x-y_k|=|x-y_j|$ which means $|\pi-2\vartheta-\vartheta_j|=O(n^{-1})$.

B. At first let $y_1<x<z_0=y_{[n/2]}$. Then

$$c_4\frac{\pi}{4}<\vartheta<\frac{\pi}{2}-\frac{c_5}{n} \quad \text{where} \quad 0<c_4=c_4(\alpha,\beta)<2 \quad \text{and} \quad 0<c_5=c_5(\alpha,\beta)$$

(see [7], 6.21). To prove (3.2) we write, using (4.10) and (4.14),

$$(4.19) \qquad v_k(x)=1+\frac{1}{y_k}\left[2\frac{\alpha-\beta+(\alpha+\beta+2)x_{|k|}}{1+x_{|k|}}-1\right](x-y_k)=$$

$$=2\frac{1-x_{|k|}}{1+x_{|k|}}+\frac{x}{y_k}\,\frac{3x_{|k|}-1}{1+x_{|k|}}+\frac{2(\alpha-\beta)+2(\alpha+\beta)x_{|k|}}{y_k(1+x_{|k|})}(x-y_k).$$

We shall break up $\sum\limits_{|k|=1}^{n}|v_k(x)|\,l_k^2(x)$ according to (4.19). We write

$$(4.20) \qquad \sum_{|k|=1}^{n}|v_k(x)|\,l_k^2(x)=$$

$$=O(1)\left[\sum_{k=1}^{n}\frac{1-x_k}{1+x_k}\,l_k^2(x)+\sum_{k=1}^{n}\frac{x}{y_k(1+x_k)}\,l_k^2(x)+\sum_{k=1}^{n}\frac{|x-y_k|}{y_k(1+x_k)}\,l_k^2(x)\right].$$

We use the notations

$$\sum_{\substack{1\leqq k\leqq \frac{n}{4}\\ k\neq j}}=\sum\nolimits^{(1)}, \qquad \sum_{\substack{\frac{n}{4}<k\leqq n\\ k\neq j}}=\sum\nolimits^{(2)}.$$

Then, by (4.13), (4.4), (4.5), (4.6), (4.7) and (4.10), we have

$$\sum\nolimits^{(1)}\frac{1-x_k}{1+x_k}\,l_k^2(x)=O(1)\sum\nolimits^{(1)}\frac{k^2}{n^2}\,\frac{[\sin(\pi-2\vartheta)]^{-2\alpha-1}}{n\left[\sin\left(\frac{\pi}{2}-\vartheta\right)-y_k\right]^2 y_k^2}\cdot\frac{k^{2\alpha+3}}{n^{2\alpha+4}}=$$

$$=\frac{O(1)}{\sin^{2\alpha+1}\vartheta_j}\sum\nolimits^{(1)}\frac{k^{2\alpha+5}}{n^{2\alpha+7}\sin^2\left(\frac{\vartheta_j}{4}-\frac{\vartheta_k}{4}\right)\cos^2\left(\frac{\vartheta_j}{4}+\frac{\vartheta_k}{4}\right)}\cdot\frac{n^2}{k^2}=$$

$$=O(1)\frac{n^{-2\alpha-5}}{\left(\frac{j}{n}\right)^{2\alpha+1}}\sum\nolimits^{(1)}\frac{k^{2\alpha+3}}{\frac{(j-k)^2}{n^2}}=\frac{O(1)}{j^{2\alpha+1}n^2}\sum\nolimits^{(1)}\frac{k^{2\alpha+3}}{(j-k)^2}.$$

For the $\sum^{(2)}$ with similar argument, moreover, using that now $y_k \sim 1$, we get

$$\sum^{(2)} \frac{1-x_k}{1+x_k} l_k^2(x) = \frac{O(n^{2\alpha-2\beta})}{j^{2\alpha+1}} \sum^{(2)} \frac{(n-k+1)^{2\beta+1}}{(j-k)^2}.$$

If $j \geqq \frac{n}{9}$ then $\sum^{(1)} = O(1)$ and $\sum^{(2)} = O(1)$. So let $2 \leqq j < \frac{n}{9}$. Then

$$\frac{1}{j^{2\alpha+1} n^2} \sum^{(1)} \frac{k^{2\alpha+3}}{(j-k)^2} = \frac{1}{j^{2\alpha+1} n^2} \left[\sum_{\substack{k=1 \\ k \neq j}}^{2j} \frac{k^{2\alpha+3}}{(j-k)^2} + \sum_{2j < k \leqq \frac{n}{4}} \frac{k^{2\alpha+3}}{(j-k)^2} \right] =$$

$$= \frac{O(1)}{j^{2\alpha+1} n^2} \left[\sum_{\substack{k=1 \\ k \neq j}}^{2j} \frac{j^{2\alpha+3}}{(j-k)^2} + \sum_{2j < k \leqq \frac{n}{4}} \frac{k^{2\alpha+3}}{k^2} \right] = O(n^{2\alpha}),$$

further

$$\frac{n^{2\alpha-2\beta}}{j^{2\alpha+1}} \sum^{(2)} \frac{(n-k+1)^{2\beta+1}}{(j-k)^2} = \frac{O(n^{2\alpha-2\beta})}{j^{2\alpha+1}} \sum^{(2)} \frac{(n-k+1)^{2\beta+1}}{n^2} = O(n^{2\alpha}).$$

So we have

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1-x_k}{1+x_k} l_k^2(x) = O(n^{2\alpha}), \quad y_1 < x < z_0.$$

C. If $0 \leqq x < y_1$ we get by similar estimations

$$\sum^{(1)} \frac{1-x_k}{1+x_k} l_k^2(x) = \frac{O(1)}{n^2} \sum^{(1)} \frac{k^{2\alpha+3}}{(j-k)^2} = \frac{O(1)}{n^2} \sum^{(1)} k^{2\alpha+1} = O(n^{2\alpha})$$

and

$$\sum^{(2)} \frac{1-x_k}{1+x_k} l_k^2(x) = O(n^{2\alpha-2\beta}) \sum^{(2)} \frac{(n-k+1)^{2\beta+1}}{(j-k)^2} =$$

$$= O(n^{2\alpha-2\beta}) \sum^{(2)} \frac{(n-k+1)^{2\beta+1}}{n^2} = O(n^{2\alpha}).$$

D. Let us see now the second part of (4.20). Using that $|x-y_j| \leqq \frac{c}{n}$ we have for $y_1 < x < z_0$

$$\sum^{(1)} \frac{x}{y_k(1+x_k)} l_k^2(x) = \frac{O(1)}{j^{2\alpha}} \sum^{(1)} \frac{k^{2\alpha}}{(j-k)^2} = O(n^{2\alpha}),$$

$$\sum^{(2)} \frac{x}{y_k(1+x_k)} l_k^2(x) = O(1) \sum^{(2)} \frac{1-x_k}{1+x_k} l_k^2(x) = O(n^{2\alpha}).$$

For the interval $0 \leqq x \leqq y_1$ by similar estimations we get $\sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{x}{y_k(1+x_k)} l_k^2(x) = O(n^{2\alpha})$.

E. Finally, let us see the third part. If $\alpha=\beta=0$ then this term is 0 (see (4.19)). So we can suppose $\alpha>0$. Let $y_1<x<z_0$. We have as above

$$\sum^{(1)} \frac{|x-y_k|}{y_k(1+x_k)} l_k^2(x) = \frac{O(1)}{j^{2\alpha+1}} \sum^{(1)} \frac{k^{2\alpha}}{|j-k|} = O(n^{2\alpha}) \quad (\alpha>0)$$

and

$$\sum^{(2)} \frac{|x-y_k|}{y_k(1+x_k)} l_k^2(x) = \frac{O(n^{2\alpha-2\beta-1})}{j^{2\alpha+1}} \sum^{(2)} \frac{(n-k+1)^{2\beta+1}}{|j-k|} = O(n^{2\alpha}).$$

If $0\leqq x<y_1$, we have

$$\sum^{(1)} \frac{|x-y_k|}{y_k(1+x_k)} l_k^2(x) = O(1) \sum^{(1)} \frac{k^{2\alpha}}{|j-k|} = O(n^{2\alpha}) \quad (\alpha>0)$$

and

$$\sum^{(2)} \frac{|x-y_k|}{y_k(1+x_k)} l_k^2(x) = O(n^{2\alpha-2\beta-1}) \sum^{(2)} \frac{(n-k+1)^{2\beta+1}}{|j-k|} = O(n^{2\alpha}).$$

So we proved (see B, C, D and E)

$$\sum_{\substack{|k|=1\\k\neq j}}^{n} |v_k(x)|\, l_k^2(x) = O(n^{2\alpha}) \quad \text{if} \quad 0\leqq x<z_0. \tag{4.21}$$

F. Let us see now the j-th term. By (4.19) we have for any $0\leqq x\leqq 1$

$$v_j(x) = O(1)\left[1+\frac{|x-y_j|}{y_j(1+x_j)}\right] l_j^2(x). \tag{4.22}$$

If $1\leqq j\leqq\frac{n}{2}$ then $|x-y_j|\,[y_j(1+x_j)]^{-1}=O(j^{-1})=O(1)$, on the other hand, if $\frac{n}{2}<j\leqq n$ then by (4.10)

$$\frac{|x-y_j|}{y_j(1+x_j)} = O(1)\frac{(1-y_j)(1+y_j)}{y_j(1+x_j)} = \frac{O(1)}{y_j}\,\frac{1-\sin^2\frac{\vartheta_j}{2}}{2\cos^2\frac{\vartheta_j}{2}} = O(1).$$

So using (4.22) we have to prove that $l_j^2(x)=O(n^{2\alpha})$ $(0\leqq x\leqq 1)$. By (4.8)

$$l_j^2(x) = \left[\frac{P_n(1-2x^2)}{4y_jP_n'(x_j)(x-y_j)}\right]^2 = \left[\frac{4zP_n'(1-2z^2)}{4y_jP_n'(x_j)}\right]^2 =$$

$$= O(n^2)\left[\frac{z}{y_j}\cdot\frac{P_{n-1}^{(\alpha+1,\beta+1)}(1-2z^2)}{P_n'^{(\alpha,\beta)}(x_j)}\right]^2$$

where, e.g., $y_{j-1}<x<z<y_j$, so by $z=\cos\xi$ we have $|\pi-2\xi-\vartheta_j|=O(n^{-1})$. Then, by (4.6) and (4.7)

$$l_j^2(x)=O(n^2)\left[\frac{\max\limits_{x_j\le x\le x_{j-1}}|P_{n-1}^{(\alpha+1,\beta+1)}(x)|}{P_n'^{(\alpha,\beta)}(x_j)}\right]^2=O(1)\quad(0\le x\le 1),$$

from where using (4.21) we get

$$\sum_{|k|=1}^{n}|v_k(x)|\,l_k^2(x)=O(n^{2\alpha})\quad(0\le x<z_0). \tag{4.23}$$

G. Now we investigate the case $z_0<x<y_n$, i.e., $\frac{c_6}{n}<\vartheta=c_4\frac{\pi}{4}$ $(0<c_6=$ $=c_6(\alpha,\beta))$. We use another form of $v_k(x)$. By (4.19)

$$v_k(x)=\frac{(1-x_{|k|})\,(2y_k-1)+2x_{|k|}}{y_k(1+x_k)}+\frac{(x-1)(3x_{|k|}-1)}{y_k(1+x_k)}+ \tag{4.24}$$
$$+\frac{2(\alpha-\beta)+2(\alpha+\beta)x_{|k|}}{y_k(1+x_k)}(x-y_k).$$

As at B we shall break up $\sum_{|k|=1}^{n}|v_k(x)|\,l_k^2(x)$ according to (4.24). First of all we prove

(4.25)
$$|(1-x_{|k|})(2y_k-1)+2x_{|k|}|=O[(\pi-\vartheta_k)^4]=O\left[\frac{(n-k+1)^4}{n^4}\right]\quad(k=1,2,\dots,n).$$

Indeed, we have

$$y_k=\sin\frac{\vartheta_k}{2}=1-\frac{1}{2}\left(\frac{\pi-\vartheta_k}{2}\right)^2+\frac{1}{24}\sin\xi\left(\frac{\pi-\vartheta_k}{2}\right)^4,$$

$$x_{|k|}=\cos\vartheta_k=-1+\frac{1}{2}(\pi-\vartheta_k)^2+\frac{1}{24}\cos\eta\,(\pi-\vartheta_k)^4,$$

from where using (4.3) and (4.10) we get (4.25). Further we shall use that now $\frac{c}{n}<$ $<\pi-\vartheta_j\le\pi-\varepsilon$, i.e.,

$$\sin(\pi-2\vartheta)\sim\sin\vartheta_j=\sin(\pi-\vartheta_j)\sim\frac{n-j+1}{n}. \tag{4.26}$$

At $\sum^{(1)}$ we use that now $x-y_k\ge c$. For $\sum^{(2)}$ we shall apply

$$\cos^2\left(\frac{\pi}{4}-\frac{\vartheta}{2}+\frac{\vartheta_k}{4}\right)=\cos^2\left(\frac{\pi}{2}-\frac{\vartheta}{2}+\frac{\pi-\vartheta_k}{4}\right)=\sin^2\left(\frac{\vartheta}{2}+\frac{\pi-\vartheta_k}{4}\right)\ge \tag{4.27}$$
$$\ge c\cdot\sin^2\frac{\pi-\vartheta_k}{4}\sim\frac{(n-k+1)^2}{n^2}.$$

H. We have by (4.25) and (4.26) for $z_0<x<y_n$

$$\sum^{(1)} \frac{|(1-x_k)(2y_k-1)+2x_k|}{y_k(1+x_k)} l_k^2(x) = \frac{O(n^{2\beta})}{(n-j+1)^{2\beta+1}} \sum^{(1)} \frac{n^3}{k^3} \frac{k^{2\alpha+3}}{n^{2\alpha+4}} =$$

$$= \frac{O(n^{2\beta-2\alpha-1})}{(n-j+1)^{2\beta+1}} \sum^{(1)} k^{2\alpha} = O(n^{2\beta})$$

further as in B, by (4.25)—(4.27)

$$\sum^{(2)} \frac{|(1-x_k)(2y_k-1)+2x_k|}{y_k(1+x_k)} l_k^2(x) = \frac{O(n^{2\beta})}{(n-j+1)^{2\beta+1}} \sum^{(2)} \frac{(n-k+1)^4}{n^4} \frac{n^2}{(n-k+1)^2} \cdot$$

$$\cdot \frac{n^2}{(j-k)^2} \frac{n^2}{(n-k+1)^2} \frac{(n-k+1)^{2\beta+3}}{n^{2\beta+4}} = \frac{O(1)}{(n-j+1)^{2\beta+1} n^2} \sum^{(2)} \frac{(n-k+1)^{2\beta+3}}{(j-k)^2} =$$

$$= \frac{O(1)}{u^{2\beta+1} n^2} \sum_{\substack{1 \leqq i \leqq \frac{3n}{4} \\ i \neq u}} \frac{i^{2\beta+3}}{(u-i)^2} = O(n^{2\beta}) \quad \left(1 \leqq n-j+1 = u \leqq \frac{n}{2}+1\right).$$

I. At the second part of (4.24) we shall use that

(4.28)

$$1-x \sim 1-\sin\frac{\vartheta_j}{2} = 2\cos\left(\frac{\pi}{4}+\frac{\vartheta_j}{4}\right)\sin\left(\frac{\pi}{4}-\frac{\vartheta_j}{4}\right) = 2\cos\left(\frac{\pi}{2}+\frac{\vartheta_j-\pi}{4}\right)\sin\frac{\pi-\vartheta_j}{4} \sim$$

$$\sim \frac{(n-j+1)^2}{n^2} \quad \text{if} \quad k \neq j.$$

So we have by (4.26)—(4.28) for $z_0<x<y_n$

$$\sum^{(1)} \frac{1-x}{y_k(1+x_k)} l_k^2(x) = \frac{O(n^{2\beta})}{(n-j+1)^{2\beta+1}} \frac{(n-j+1)^2}{n^2} \sum^{(1)} \frac{n^3}{k^3} \frac{k^{2\alpha+3}}{n^{2\alpha+4}} = O(n^{2\beta})$$

and

$$\sum^{(2)} \frac{1-x}{y_k(1+x_k)} l_k^2(x) =$$

$$= \frac{O(n^{2\beta})}{(n-j+1)^{2\beta+1}} \frac{(n-j+1)^2}{n^2} \sum^{(2)} \frac{n^2}{(n-k+1)^2} \frac{n^2}{(j-k)^2} \frac{n^2}{(n-k+1)^2} \frac{(n-k+1)^{2\beta+3}}{n^{2\beta+4}} =$$

$$= \frac{O(1)}{(n-j+1)^{2\beta-1}} \sum^{(2)} \frac{(n-k+1)^{2\beta-1}}{(j-k)^2} = \frac{O(1)}{u^{2\beta-1}} \sum_{\substack{1 \leqq i \leqq \frac{3n}{4} \\ i \neq u}} \frac{i^{2\beta-1}}{(u-i)^2} =$$

$$= \frac{O(1)}{u^{2\beta-1}} \sum_{1 \leqq i \leqq \frac{u}{2}} \frac{i^{2\beta-1}}{(u-i)^2} + \frac{O(1)}{u^{2\beta-1}} \sum_{\substack{\frac{u}{2} \leqq i \leqq \frac{3n}{4} \\ i \neq u}} \frac{i^{2\beta-1}}{(i-u)^2} = O(1) \quad (2\beta-1<0).$$

For $2\beta-1 \geqq 0$ we have the estimation $O(n^{2\beta-1})$.

J. At the third part we have, supposing $\beta>0$ (see E), for $z_0<x<y_n$

$$\sum{}^{(1)} \frac{|x-y_k|}{y_k(1+x_k)} l_k^2(x) = \frac{O(n^{2\beta-2\alpha-1})}{(n-j+1)^{2\beta+1}} \sum{}^{(1)} k^{2\alpha} = O(n^{2\beta})$$

and

$$\sum{}^{(2)} \frac{|x-y_k|}{y_k(1+x_k)} l_k^2(x) = \frac{O(1)}{(n-j+1)^{2\beta+1}} \sum{}^{(2)} \frac{(n-k+1)^{2\beta}}{|j-k|} = O(n^{2\beta}).$$

K. With similar argument we have

$$\sum_{\substack{|k|=1\\k\neq j}}^{n} |v_k(x)|\, l_k^2(x) = O(n^{2\beta}) \quad (y_n < x \leqq 1).$$

So we completely proved the relation (3.2).

4.2. Proof of Theorem 3.2. **4.2.1.** Let us consider the matrix Y defined in [8], 4.1, i.e. let for odd n's

$$\text{(4.29)} \qquad \begin{cases} y_{k,n} = \cos\dfrac{2k-1}{2n}\pi & (k=1,2,\ldots,n;\ n=2s-1) \\ y_{0,n} = \cos\left(\dfrac{\pi}{2}-\varrho_n\right), & \text{where} \quad 0<\varrho_n \leqq \dfrac{\pi}{2n}. \end{cases}$$

Further, by (4.29), the n-th row of Y contains $n+1$ nodes, $\Omega_n(Y,x)$ is a polynomial of degree $n+1$, etc.

At first we prove

$$\mu_n(Y) \sim n^{-2}\varrho_n^{-3}.$$

By (4.29), with $T_n(x)=\cos n\vartheta$ we have

$$\text{(4.30)} \qquad \begin{cases} l_{k,n}(Y,x) = l_{k,n}(T,x)\dfrac{x-y_0}{y_k-y_0} & (k=1,2,\ldots,n), \\ l_{0,n}(Y,x) = \dfrac{T_n(x)}{T_n(y_0)}, \end{cases}$$

$$\text{(4.31)} \qquad \begin{cases} l'_{k,n}(Y,y_k) = \dfrac{y_k}{2(1-y_k^2)} + \dfrac{1}{y_k-y_0} & (k=1,2,\ldots,n), \\ l'_{0,n}(Y,y_0) = \dfrac{T'_n(y_0)}{T_n(y_0)}, \end{cases}$$

$$\text{(4.32)} \qquad |T_n(y_0)| = |\sin n\varrho_n| \sim n\varrho_n, \quad |T'_n(y_0)| = |n\cos n\varrho_n| \sim n.$$

So we have, using $l_{k,n}(T,x)=(-1)^{k-1}T_n(x)\sin\vartheta_k n^{-1}(x-y_k)^{-1}$

$$\sum_{\substack{k=1\\k\neq s}}^{n} |v_{k,n}(Y,x)|\, l_{k,n}^2(Y,x) = O(1) \sum_{\substack{k=1\\k\neq s}}^{n} \left[\left(\frac{|y_k|}{\sin^2\vartheta_k}+\frac{1}{|y_k|}\right)|x-y_k|+1\right] \frac{T_n^2(x)\sin^2\vartheta_k(x-y_0)^2}{n^2(x-y_k)^2y_k^2}.$$

4.2.2. We sketch the estimations as follows.

$$\sum_{k=1}^{n}{}' \frac{T_n^2(x)(x-y_0)^2}{n^2|x-y_k|y_k} = \begin{cases} \sum'_{|y_k|\le\frac{1}{4}} + \sum_{|y_k|>\frac{1}{4}} = O(n) & \text{if } |x| \geqq \frac{1}{2}, \\ \sum'_{|y_k|\le\frac{3}{4}} + \sum_{|y_k|>\frac{3}{4}} = O(\ln n) & \text{if } |x| < \frac{1}{2}, \end{cases}$$

where $\sum'$ means that we omit the s-th term. Further

$$\sum_{k=1}^{n}{}' \frac{T_n^2(x)(x-y_0)^2 \sin^2 \vartheta_k}{n^2|x-y_k|\, y_2^3} = \begin{cases} \sum'_{|y_k|\le\frac{1}{4}} + \sum_{|y_k|>\frac{1}{4}} = O(n) & \text{if } |x| \geqq \frac{1}{2}, \\ \sum'_{|y_k|\le\frac{3}{4}} + \sum_{|y_k|>\frac{3}{4}} = O(1) & \text{if } |x| \leqq \frac{c}{n}, \\ \sum_{\substack{|x-y_k|\le\frac{x}{2}\\ |y_k|\le\frac{3}{4}}} + \sum'_{\substack{|x-y_k|>\frac{x}{2}\\ |y_k|\le\frac{3}{4}}} + \sum_{|y_k|>\frac{3}{4}} = O(n) & \text{if } \frac{c}{n} < x < \frac{1}{2}. \end{cases} \tag{4.33}$$

For $\sum_{k=1}^{n}{}' \frac{T_n^2(x)(x-y_0)^2 \sin^2 \vartheta_k}{n^2(x-y_k)^2 y_k^2}$ we use the ideas of (4.33).
So we get

$$\sum_{\substack{k=1\\k\ne s}}^{n} |v_{k,n}(Y,x)|\, l_{k,n}^2(Y,x) = O(n). \tag{4.34}$$

4.2.3. We know that

$$\begin{cases} l_{s,n}(Y,x) = O\left(\frac{1}{n\varrho_n}\right), \\ l_{0,n}(Y,x) = O\left(\frac{1}{n\varrho_n}\right) \end{cases} \tag{4.35}$$

(see [8], 4.13). So by (4.31)

$$\begin{cases} |v_{s,n}(Y,x)|\, l_{s,n}^2(Y,x) = O\left(\frac{1}{n^2\varrho_n^3}\right), \\ |v_{0,n}(Y,x)|\, l_{0,n}^2(Y,x) = O\left(\frac{1}{n^2\varrho_n^3}\right). \end{cases} \tag{4.36}$$

Finally, using (4.30) and (4.31) we get $|v_{0,n}(Y,1)|\, l_{0,n}^2(Y,1) \sim n^{-2}\varrho_n^{-3}$, form where by (4.34) and (4.36) we have

$$\mu_n(Y) \sim \frac{1}{n^2\varrho_n^3}. \tag{4.37}$$

4.24. In [8], 4.3.1 we have shown that

$$|H_n(f_2;\, Y, x)| \sim \frac{1}{n^2\varrho_n^2} \quad (x \neq 0,\ |x| \leqq 1,\ \text{fix}). \tag{4.38}$$

By $\mu_n = n^{-2}\varrho_n^{-3}$ we get our theorem.

4.3. PROOF OF THEOREM 3.3. Using $\mu_n(T) \equiv 1$ we can suppose $\mu_n \neq O(1)$. We shall use the matrix Z as follows. Let $n=2s$ or $n=2s-1$ and

$$\begin{cases} z_{k,n} = x_{k,n} = \cos \dfrac{2k-1}{2n}\pi & (k=1,2,\dots,n,\ k \neq s) \\ z_{s,n} = \cos\left(\dfrac{2s+1}{2n} - \varrho_n\right)\pi & \text{where} \quad 0 < \varrho_n = o\left(\dfrac{1}{n}\right). \end{cases} \tag{4.39}$$

Then, as in [9], 4.1, we have

$$\begin{cases} l_k(Z, x) = l_k(T, x)\dfrac{x-z_s}{x-x_s}\dfrac{z_k-x_s}{z_k-z_s} & \left(k \neq s,\ x_s = \cos\dfrac{2s-1}{2n}\pi\right), \\ l_s(Z, x) = \dfrac{T_n(x)}{x-x_s}\dfrac{z_s-x_s}{T_n(z_s)}, \end{cases} \tag{4.40}$$

$$\begin{cases} l_k'(Z, z_k) = l_k'(T, z_k) \quad (k \neq s), \\ l_s'(Z, z_s) = \dfrac{T_n'(z_s)}{T_n(z_s)} + \dfrac{1}{x_s - z_s}, \end{cases} \tag{4.41}$$

$$|T_n(z_s)| \sim n\varrho_n, \quad |T_n'(z_s)| \sim n, \tag{4.42}$$

finally, if $4\varphi_n = z_{s+1} - z_{s+2}$ and $I_n = [z_{s+2} + \varphi_n;\ z_{s+1} - \varphi_n]$, then

$$\begin{cases} \|l_s(Z, x)\| = O\left(\dfrac{1}{n\varrho_n}\right), \quad \|l_{s+1}(Z, x)\| = O\left(\dfrac{1}{n\varrho_n}\right), \\ |l_s(Z, x)| \sim \dfrac{1}{n\varrho_n}, \quad |l_{s+1}(Z, x)| \sim \dfrac{1}{n\varrho_n} \quad \text{for} \quad x \in I_n. \end{cases} \tag{4.43}$$

Let us prove the relation

$$\mu_n(Z) \sim \frac{1}{(n\varrho_n)^3}. \tag{4.44}$$

Indeed, by (2.6), (4.40), (4.41) and (4.42)

$$\sum_{k \neq s, s+1} |v_k(Z, x)\, l_k^2(Z, x)| = \tag{4.45}$$

$$= O(1) \sum_{k \neq s, s+1} v_k(T, x)\, l_k^2(T, x) \frac{(x - x_s + x_s - z_s)^2}{(x-x_s)^2} =$$

$$= O(1) \sum_{k \neq s, s+1} (1 - xz_k) \frac{T_n^2(x)}{n^2(x-z_k)^2} \left[1 + \frac{1}{n^2(x-x_s)^2} + \frac{2}{n|x-x_s|}\right] = O(1),$$

further

$$|h_s(Z,x)| = O(1)\left(1+\frac{|x-z_s|}{\varrho_n}\right)\frac{T_n^2(x)}{(x-x_s)^2}\frac{1}{n^4\varrho_n^2} = O(1)\left(1+\frac{|x-x_s+x_s-z_s|}{\varrho_n}\right)\cdot$$

$$\cdot\frac{T_n^2(x)}{(x-x_s)^2}\frac{1}{n^4\varrho_n^2} = O(1)\left(\frac{1}{n^2\varrho_n^2}+\frac{1}{n^3\varrho_n^3}+\frac{1}{n^3\varrho_n^3}\right) = O\left(\frac{1}{n^3\varrho_n^3}\right),$$

finally

$$|h_{s+1}(Z,x)| = (1-xz_{s+1})\frac{T_n^2(x)}{n^2(x-z_{s+1})^2}\frac{(x-z_s)^2}{(x-x_s)^2}\frac{(z_{s+1}-x_s)^2}{(z_{s+1}-z_s)^2} = O\left(\frac{1}{n^2\varrho_n^2}\right).$$

We obtain (4.44) if we remark

$$|h_s(Z,x)| \sim \frac{1}{n^3\varrho_n^3} \quad \text{for} \quad x\in I_n.$$

Now we have by usual argument

$$|H_n(f;Z,x)-f(x)| = O(1)\Big[\sum_{\substack{k=1\\ k\neq s,s+1}}^{n} |f(z_k)-f(x)||h_k(Z,x)|+$$

$$+|f(z_s)-f(x)||h_s(Z,x)|+|f(z_{s+1})-f(x)||h_{s+1}(Z,x)|\Big] \quad (f\in C(\omega)).$$

Let us estimate the parts figuring here. Using the argument applied at (4.45) we can get

(4.46) $$\sum_{k\neq s,s+1} |f(x_k)-f(x)||h_k(Z,x)| = o(1).$$

For the remaining part we estimate as follows. If $z_{s+2}<x<z_{s-1}$ we have

(4.47) $$\sum_{k=s,s+1} |f(z_k)-f(x)||h_k(Z,x)| =$$

$$= O(1)\,\omega\left(\frac{1}{n}\right)(n\varrho_n)^{-3} = O(1)\,\omega\left(\frac{1}{n}\right)\mu_n(Z).$$

On the other hand, let $x\in[-1,z_{s+2}]\cup[z_{s-1},1]$. Then

(4.48) $$\sum_{k=s,s+1} |f(z_k)-f(x)||h_k(Z,x)| =$$

$$= O(1)\left[\omega(|x-z_s|)\frac{|x-z_s|}{|x-x_s|^2}\frac{1}{n^4\varrho_n^3}+\omega(|x-z_{s+1}|)\frac{1}{(x-z_{s+1})^2}\frac{1}{n^4\varrho_n^2}\right] =$$

$$= O(1)\,\omega\left(\frac{1}{n}\right)(n\varrho_n)^{-3} = O(1)\,\omega\left(\frac{1}{n}\right)\mu_n(Z).$$

By (4.46)—(4.48), $\varrho_n=\dfrac{1}{n^3\sqrt{\mu_n}}$ and $\omega\left(\dfrac{1}{n}\right)=o(\mu_n^{-1})$ we have got our assertion.

4.4. PROOF OF THEOREM 3.4. By the method used in 4.1 we can prove more. Namely, supposing $n=2, 4, 6, \dots$ we have

$$\sum_{|k|=1}^{n} |l_{k,n}(R^{(\alpha,\beta)}, x)| = \begin{cases} O(\ln n) & \text{for } -1<\alpha\leqq -0.5,\ |x|\leqq 1-\varepsilon, \\ O(n^{\alpha+1/2}) & \text{for } -0.5<\alpha,\ |x|\leqq 1-\varepsilon, \end{cases}$$

$$\sum_{|k|=1}^{n} |l_{k,n}(R^{(\alpha,\beta)}, y)| \sim \begin{cases} \ln n & \text{if } y \text{ is fix, } |y|<1,\ n\in N_y,\ -1<\alpha\leqq -0.5, \\ n^{\alpha+1/2} & \text{if } y=0,\ \alpha>-0.5,\ n=1,3,5,\dots, \end{cases}$$

$$\sum_{|k|=1}^{n} |l_{k,n}(R^{(\alpha,\beta)}, x)| = \begin{cases} O(\ln n) & \text{for } -1<\beta\leqq -0.5,\ \varepsilon\leqq |x|\leqq 1, \\ O(n^{\beta+1/2}) & \text{for } -0.5<\beta,\ \varepsilon\leqq |x|\leqq 1, \end{cases}$$

$$\sum_{|k|=1}^{n} |l_{k,n}(R^{(\alpha,\beta)}, y)| \sim \begin{cases} \ln n, & \text{if } y \text{ is fix, } 0<|y|\leqq 1,\ n\in N_y,\ -1<\beta\leqq -0.5, \\ n^{\beta+1/2} & \text{if } y=1,\ \beta>-0.5,\ n=1,3,5,\dots. \end{cases}$$

By these relations we gain (3.8).

For (3.9) we can prove as in 4.1 the following:

$$\sum_{|k|=1}^{n} |h_{k,n}(R^{(\alpha,\beta)}, x)| \begin{cases} = O(1) & \text{for } -1<\alpha\leqq 0,\ |x|\leqq 1-\varepsilon, \\ = O(n^{2\alpha}) & \text{for } \alpha\geqq 0,\ |x|\leqq 1-\varepsilon, \\ \sim n^{2\alpha} & \text{for } \alpha\geqq 0,\ x=0, \end{cases}$$

$$\sum_{|k|=1}^{n} |h_{k,n}(R^{(\alpha,\beta)}, x)| \begin{cases} = O(1) & \text{if } \begin{cases} -1<\beta\leqq -0.5, & \varepsilon\leqq |x|\leqq 1 \text{ or} \\ -0.5<\beta\leqq\alpha\leqq 0, & \varepsilon\leqq |x|\leqq 1 \text{ or} \\ -1<\beta\leqq 0, \ \alpha\geqq -0.5, & \varepsilon\leqq |x|\leqq 1, \end{cases} \\ = O(n^{2\beta}) & \text{if } 0\leqq\beta,\ \alpha\geqq -0.5,\ \varepsilon\leqq |x|\leqq 1, \\ \sim n^{2\beta} & \text{if } 0\leqq\beta,\ \alpha\geqq -0.5,\ x=1. \end{cases}$$

By these and (4.18) we obtain (3.9). (The crucial part of these estimations is $\sum^{(1)}$ at H and J.) We remark that by (4.24), e.g., $\sum_{\substack{k \\ x_k\geqq 0.9}} |h_{k,n}(R^{(-0.9,0)}, 1)| \sim n^{0.8}$, from where, e.g.,

$$\overline{\lim_{n\to\infty}}\ \mu_{2n}(R^{(-0.9,0)})\, n^{-0.8} > 0.$$

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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ILLUMINATION OF CONVEX DISCS

By

L. FEJES TÓTH (Budapest), member of the Academy

We say that a point-set $\{A\}$ is illuminated by the point-set $\{B\}$ if to any boundary point A of $\{A\}$ there is a point B of $\{B\}$ such that no point of $\{A\}$ lies on the open segment AB. (According to this definition a parallelogram can be illuminated by two points, while according to a more restrictive definition used in the literature [1] a parallelogram cannot be illuminated but by four points.)

THEOREM 1. *The point-set-union of $n>1$ circular discs can always be illuminated by $2n$ points.*

The example of successively touching circles whose centres lie on a line shows that $2n$ lighting points are actually needed (Fig. 1). We shall see that in any other case at most $\max(2n-2, 3)$ points will suffice.

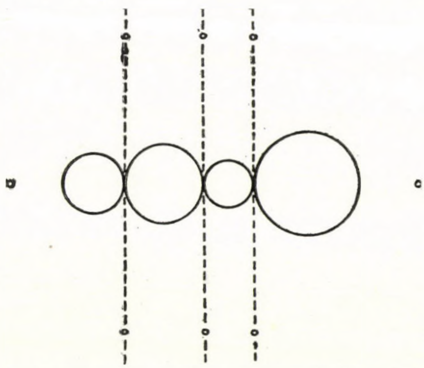

Fig. 1

The case when the circles are centred on a line but some of them overlap or successive circles are disjoint can easily be settled. Thus we may suppose that the centres are not collinear. Obviously we also may assume that no circle is covered by the other ones.

Consider the generalized Dirichlet cell [2] of a circle c defined as the set of those points whose power with respect to c is smaller than with respect to any other circle of the set. These cells are convex polygons or unbounded convex polygonal regions which fill the plane without overlapping and without interstices, in short, they form a tessellation. The vertices of a bounded cell will illuminate the arcs of the corresponding circle lying in the cell. (About the arcs lying outside the cell we need not care because they are contained in other circles of the set.) In order to illuminate the corresponding arcs of a circle whose cell is unbounded we must add to the vertices of the cell two further lighting points one on each infinite side of the cell sufficiently far from the circle (Fig. 2). Thus, if there are k infinite edges, then the union of the circles can be illuminated by the vertices of the tessellation along with k further points.

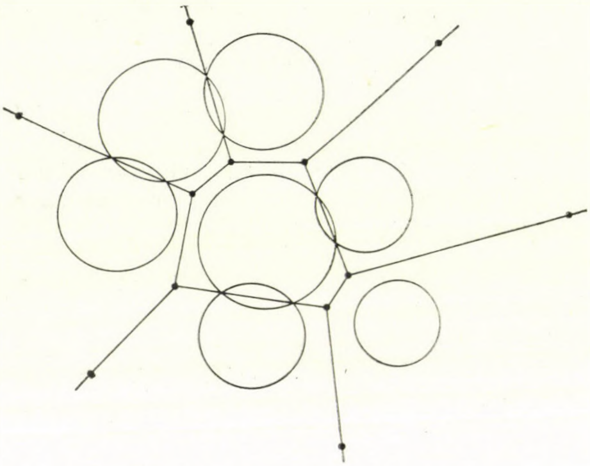

Fig. 2

In order to give an upper bound for the total number p of these points it is convenient to suppose that the infinite edges meet in an ideal vertex. Thus we have $p=v-1+k$, where v is the total number of the vertices of the tessellation. If e is the total number of the edges then

(1) $$n+v=e+2.$$

On the other hand, if v_i is the number of ordinary vertices of order i then

(2) $$1+v_3+v_4+\dots=v$$

and

(3) $$k+3v_3+4v_4+\dots=2e.$$

Hence

$$2e \geqq 3(v-1)+k.$$

Combining this with (1) we have

$$v \leqq 2n-k-1,$$

whence $p \leqq 2n-2$.

THEOREM 2. *The point-set-union of* $n>1$ *non-overlapping open convex discs can always be illuminated by* $\max(2n, 4n-7)$ *points.*

We shall see that the number max $(2n, 4n-7)$ cannot be replaced by a smaller one. For closed discs the theorem does not hold any more, as can be seen by the example of a square and a circle touching a side of the square. If these two discs are closed their union cannot be illuminated by a finite set of points.

The proof rests on a construction which turned out to be useful in the theory of packing [2, 3, 4]. Suppose that each disc tends to grow unboundedly in all directions, say, through the continuous set of outer parallel domains, but the growth is limited by certain "walls". These walls consist of the supporting lines which separate a disc, either in its original or increased state, from those other discs which have a boundary point in common with it. In short, whenever two discs collide, a wall comes into being, preventing them of growing into one another (but not hindering the growth of the rest of the discs). In this way each disc will expand into a convex region bounded by a finite number of straight lines. With the possible exception of a finite number of bounded or unbounded gaps, these regions fill the plane.

It may occur that a region abuts along a whole side on a gap, so that the region can further be enlarged without loosing the convexity and without overlapping other regions. This situation is caused by a "wrong" supporting line between two regions (Fig. 3). Replacing the wrong supporting lines by "good" ones and continuing to expand the regions according to the above principle, such situations can be eliminated. Thus we may assume that to each side of a gap there is a region leaning partly or entirely against this side and simultaneously partly against the elongation of this side in one direction. Let us elongate the sides of a gap in accordance with adjacent regions. Since from two sides meeting at a vertex only one is elongated beyond this vertex, the sides of a gap are cyclically unidirected (Fig. 4). It follows immediately that the gaps are convex polygons and each side of a gap leans against exactly one region. Thus the set of the vertices of the gaps coincides with the set of the vertices of the regions.

In the following discussion we first suppose that among the regions under consideration there is neither a half-plane nor a parallel strip.

Consider the "tessellation" consisting of the regions. We call the points at which more than two regions meet along with the gaps vertices of the tessellation. As a tentative definition we call the set of points at which exactly two particular regions meet an edge of the tessellation. An edge may consist of a point, namely of coincident vertices of two gaps (Fig. 5). Extending the above definition, we call a point in which a vertex of the tessellation and a vertex of a gap coincide also an edge. Similarly as in the proof of Theorem 1, we consider the unbounded regions to meet in an ideal vertex. Preserving the notations of the proof of Theorem 1, the relations (1), (2) and (3) continue to hold for our tessellation.

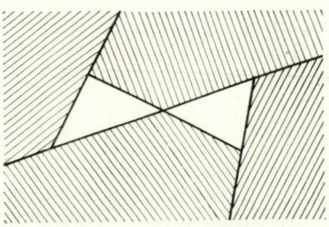

Fig. 3

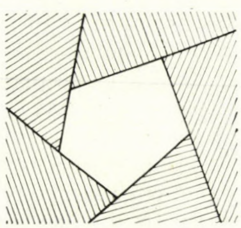

Fig. 4

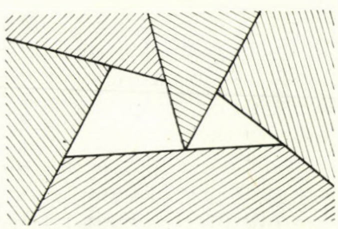

Fig. 5

Now we choose the lighting points as follows: 1. the ordinary vertices of the tessellation consisting of one point, 2. two arbitrarily chosen vertices of each triangular gap, 3. the vertices of all non-triangular gaps and finally 4. one lighting point on each infinite edge of the tessellation so that they span a convex polygon P containing all the discs. We claim that these points jointly illuminate the discs.

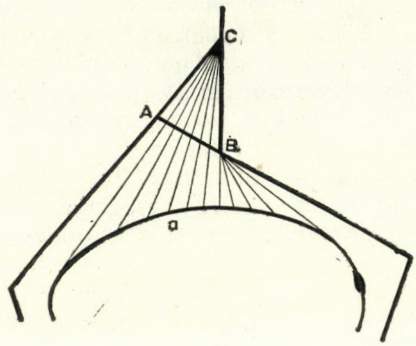

Fig. 6

Let d be one of the discs and D the intersection of P and the region pertaining to d. Translate the sides of D inwards so as to obtain a polygon circumscribed about d, and decompose the boundary of d into arcs whose extremities lie on adjacent sides of this polygon. Let a be one of these arcs and A the corresponding vertex of D. Since a is obviously illuminated by A, we have to consider the case when A is not among our lighting points. This occurs only if there is a triangular gap ABC in which B and C have been chosen as lighting points, and we have only to observe that, without being impeded by other discs, B and C together also illuminate a (Fig. 6).

Denoting the total number of the lighting points with t and using (1), (2) and (3), we have

$$t \leqq k+2v_3+4v_4+5v_5+\ldots \leqq k+2(v_3+2v_4+3v_5+\ldots) =$$

$$= k+2(2e-2v-k+2) = 4n-k-4 \leqq 4n-7.$$

We continue to consider the case when some of the regions are half-planes or parallel strips. If all of the regions are either half-planes or parallel strips then the discs can be illuminated by $2n$ points. Therefore we exclude this case. On the other hand, we first assume that there is at least one parallel strip. Let this parallel strip be in a vertical position. Let l be a horizontal line "far below" the discs. Let l intersect the vertical edges in m points $L_1, \ldots, L_m$. Cut off the parts of these edges lying below a horizontal line below l and bend the truncated edges so as to meet at a point V. Adding a new infinite edge issuing from V and pointing vertically downwards, we obtain a tessellation with n faces with a topological type considered previously (Fig. 7).

Now we distribute the lighting points essentially as described above: we put two lighting points in each trigonal vertex, i lighting points in each i-gonal vertex with $i>3$ and one lighting point on each infinite edge. The total number of these points is at most $4n-7$. Among these points there are some which must be rearranged. These are the coincident points at V and one point on the new edge. Since the order of V is $m+1$, the number of these points is $m+1$ or $m+2$ according as $m=2$ or $m>2$. On the other hand, besides the points which are already at their proper places, we need at most $m+2$ further lighting points, namely the points $L_1, \ldots, L_m$ and one additional point in each half-plane. But since in the case when $m=2$ we have at most one half-plane, the number of the necessary lighting points is again at most $4n-7$.

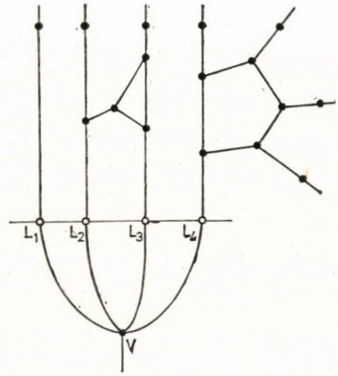

Fig. 7

We still have to consider the case when we have one or two half-planes but no parallel strip. Now there is at least one vertex on each line bounding a half-plane. But such a vertex necessarily consists of one point. Thus, choosing the lighting points as above, with each half-plane we gain at least one point. This suffices to complete the illumination of the corresponding extreme disc.

This completes the proof of Theorem 2.

For each $n>1$ it is easy to construct a set of discs which cannot be illuminated by less than $\max(2n, 4n-7)$ points. The case $n<4$ being trivial, we assume that $n>3$. Consider three mutually touching smooth discs which include a "general sharp-cornered triangular gap". The attribute "general" means that the boundary of the gap cannot be illuminated from inside but by two ponits. To illuminate these discs $5=4\cdot 3-7$ points are needed. Inscribe in the gap a new smooth disc so as to obtain three smaller gaps of the same type as the original one. Since the number of the gaps has increased by two, we need $5+4=4\cdot 4-7$ points to illuminate these four discs. Inscribing in any of the gaps a new disc and continuing this process we see that the number of points needed to illuminate the discs increases with each new disc by four.

A practicable characterisation of all cases when $4n-7$ lighting points are claimed is not known.

Let us still mention that by a slight generalisation Theorem 2 can be brought into a closer analogy with Theorem 1. We say that two discs cross each other if removing their intersection causes both discs to fall into disjoint pieces. Theorem 2 continues to hold if we replace then term "non-overlapping" by "non-crossing". This follows from the fact that in a finite set of non-crossing convex discs the discs can be contracted into non-overlapping discs which cover the same part of the plane as the original discs [2, 5].

What happens if we drop the condition that the discs do not cross each other? The fact that the plane can be decomposed by n straight lines into $\frac{1}{2}(n^2+n+2)$ regions from which $\frac{1}{2}(n^2+n+2)-2n$ are bounded implies that the order of magnitude of the minimal number of points by which the point-set-union of any set of n convex discs can be illuminated is at least $n^2/2$.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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ON SOME PROBLEMS OF THE STATISTICAL THEORY OF PARTITIONS WITH APPLICATION TO CHARACTERS OF THE SYMMETRIC GROUP. I

By

M. SZALAY and P. TURÁN, member of the Academy (Budapest)

To the 65th birthday of Theodor Schneider

1. In one of the papers of the second named author with P. Erdős on statistical group theory (see [1]) the following theorem is proved. If C runs over the conjugacy classes of S_n, the symmetric group of n letters, and $|C|$ stands for the cardinality of C then for an arbitrarily small $\varepsilon>0$ the inequality

$$(1.1)\qquad n!\exp\left\{-(1+\varepsilon)\frac{\sqrt{6}}{4\pi}\sqrt{n}\log^2 n\right\}\leqq |C|\leqq n!\exp\left\{-(1-\varepsilon)\frac{\sqrt{6}}{4\pi}\sqrt{n}\log^2 n\right\}$$

holds for almost all classes C, i.e. with the exception of $o(p(n))$ classes at most. Here $p(n)$ stands, as usual, for the number of partitions of n, i.e. for the number of solutions of

$$(1.2)\qquad n = 1x_1+2x_2+\dots,\quad x_j\geqq 0,\quad \text{integers.}$$

We remind the reader that the total number of the C's is $p(n)$ and the o-sign (and later also the O-sign) refers to $n\to\infty$.

The following corollary of (1.1) was observed much later by the second of us via the second orthogonality relation of the group-characters. Denoting the characters belonging to irreducible representations of S_n by

$$\chi_\nu(C)\quad (\nu=1,2,\dots,p(n))$$

we have for arbitrarily small $\varepsilon>0$ and almost all C's the inequality

$$(1.3)\qquad |\chi_\nu(C)|\leqq\exp\left\{(1+\varepsilon)\frac{\sqrt{6}}{8\pi}\sqrt{n}\log^2 n\right\}$$

for all ν's and this is no more true for all ν's replacing $1+\varepsilon$ by $1-\varepsilon$. Among the many important natural questions in connection with this we mention only four.

I. What analogous "best" inequality can be given which holds for almost all χ_ν's and all C's?

II. What analogous "best" inequality can be given which holds for almost all χ_ν's and almost all C_ν's?

III. Does there exist a "best" inequality which holds in two-dimensional sense for almost all pairs (χ_ν, C_μ)?

IV. Does there exist in this last case perhaps a value-distribution theorem?

Mentioning these problems in connection with a lecture in Rome in 1973 (see [2]), Professor M. P. Schützenberger called afterwards our attention to the interest of the further question what can be said on the distribution of the numbers $\chi_\nu(E)$ (E is the unit class). The most direct plausible attack on this problem can be based on the classical explicit formula for the numbers $\chi_\nu(E)$ due to Frobenius and I. Schur (see [11] p. 119). Introducing for the generic partition Π of n the notation

$$\Pi: \begin{cases} n = \lambda_1+\lambda_2+\ldots+\lambda_m \\ \lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_m \geqq 1 \end{cases} \tag{1.4}$$

and using the fact that the conjugacy classes are in one to one correspondence with the Π's it is better to use $\chi_\Pi(E)$ instead of $\chi_\nu(E)$; then the formula of Frobenius—Schur asserts that

$$\chi_\Pi(E) = n!\,\frac{\prod\limits_{1\leqq\mu<\nu\leqq m}(\lambda_\mu-\lambda_\nu+\nu-\mu)}{\prod\limits_{\mu=1}^{m}(\lambda_\mu+m-\mu)!}. \tag{1.5}$$

The statistical treatment of these weird expressions seemed impossible without a statistical investigation of the distribution of summands and the first such theorem for the number of summands $\leqq n^\alpha$ $\left(0<\alpha<\frac{1}{2}\text{ and fixed}\right)$ for almost all Π's was announced among others in a lecture in May 1974 (see [3]). Still the first result for the problem of Schützenberger was attained without such investigations in a lecture in October 1974 (see [4]) which asserted that for almost all Π's the inequality

$$\left(\log\sqrt{n!}\geqq\right)\log\chi_\Pi(E)\geqq\log\sqrt{n!}-n\log\log n-n \tag{1.6}$$

holds.

Owing to the relations of orthogonality, this result gave the following answer to the question I. For almost all Π's the estimate

$$\max_C|\chi_\Pi(C)| = \exp\left\{\frac{1}{2}n\log n+O(n\log\log n)\right\}$$

holds. (Here the factor $\log\log n$ can be omitted in view of Theorem VI of the next paragraph.)

2. The main aim of the first three instalments of this series of papers is to prove the following

THEOREM VI. *If A stands for a well-defined constant with the value*

$$A > \frac{6}{\pi^2}\cdot 0.02 \tag{2.1}$$

and c is an explicitly calculable positive constant then for an arbitrarily small $\varepsilon>0$ and $n>n_0(\varepsilon)$ the number of Π's satisfying the inequality

$$|\log\chi_\Pi(E)-\log\left(\sqrt{n!}\right)+An| < cn^{7/8}\log^4 n \tag{2.2}$$

is greater than $(1-\varepsilon)p(n)$.

Shortly the inequality (2.2) holds for almost all irreducible characters on S_n. It would be very risky conjecture that the relation

$$\chi_\Pi(E) = \sqrt{n!}\exp\{-An + A_1\sqrt{n}\log^2 n + A_2\sqrt{n}\log n + O(\sqrt{n}\log\log n)\} \tag{2.3}$$

holds with suitable constants A_1 and A_2 for almost all irreducible characters of S_n. Since the numbers $\chi_\Pi(E)$ are equal to the dimensions of the matrices of the corresponding irreducible representations, (2.2) asserts that these dimensions are "relatively equal in a weak sense". The truth of (2.3) would imply the same in a much stronger sense.

3. The proof of theorem VI will be given in the third paper of this series. The first two papers will deal with the statistical theory of partitions, which are partly needed for theorems I and II, but all are, in our opinion, of independent interest, some even for statistical physics. The fact that the theory of partitions can be used at all in statistical physics is well-known. (See e.g. BOHR and KALCKAR [5], AULUCK and KOTHARI [6], TEMPERLEY [7] and others. They use essentially the partition formula

$$p(n) = (1+o(1))\frac{1}{4n\sqrt{3}}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right) \tag{3.1}$$

of HARDY and RAMANUJAN [8]).

4. The first theorem in the statistical theory of partitions is due to ERDŐS and LEHNER from 1941 (see [9]). They proved that if m in (1.4) is denoted by $l(\Pi)$, the "length of the partition Π" then for almost all Π's the inequality

$$\left|l(\Pi) - \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n\right| < \sqrt{n}\,\omega(n) \tag{4.1}$$

holds if $\omega(n)\nearrow\infty$ arbitrarily slowly. Though this theorem played an important role as background, it will not be used in the sequel; we shall show in the paper II how theorem I of this paper and theorem III in the second paper give another proof for it which in addition will give an explicit upper bound for the cardinality of the exceptional set of Π's. This would be rather difficult to deduce from the rather concisely written original paper of Erdős—Lehner (which contains no upper bound for the exceptional set apart from $o(p(n))$). The same applies for the formula of SZEKERES [10]. This slightly generalized form of Erdős—Lehner's result will be theorem IV.

5. Based on the representation (1.4) let $S_1(n, \Pi, \Lambda)$ stand for the number of λ_j's satisfying the inequality

$$\lambda_j \geqq \Lambda. \tag{5.1}$$

Confining Λ first to the interval

$$11\log n \leqq \Lambda \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - \sqrt{n}\log\log n \tag{5.2}$$

we assert the following two theorems.

THEOREM I. *If* Λ *is restricted by* (5.2) *then the inequality*

$$\left| S_1(n, \Pi, \Lambda) - \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi\Lambda}{\sqrt{6n}}\right)}\right| < c\sqrt{\frac{n\log n}{\Lambda}} \tag{5.3}$$

holds for all but $cp(n)n^{-7/4}$ Π*'s at most.*

Requiring a bit stronger than (5.2)

$$11\log n \leqq \Lambda \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 3\sqrt{n}\log\log n \tag{5.4}$$

we assert

THEOREM II. *If* Λ *is restricted by* (5.4) *then the relation*

$$S_1(n, \Pi, \Lambda) = \left(1+O\left(\frac{1}{\log n}\right)\right)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi\Lambda}{\sqrt{6n}}\right)} \tag{5.5}$$

holds for all but

$$cp(n)n^{-7/4} \tag{5.6}$$

Π*'s at most.*

Throughout this paper c's stand for explicitly calculable positive constants not necessarily the same in different occurrences. As we shall see it is important to have a good command of the size of the exceptional set.

6. To see a bit closer the content of theorems I and II say let first

$$\Lambda = \Lambda_1 \overset{\text{def}}{=} \frac{\sqrt{6}}{\pi}n^{\alpha} \quad \left(0 < \alpha < \frac{1}{2} \text{ and fixed constant}\right). \tag{6.1}$$

Then for almost all Π's the number of summands $\geqq \Lambda_1$ is, owing to theorem I,

$$\frac{\sqrt{6}}{\pi}\left(\frac{1}{2}-\alpha\right)\sqrt{n}\log n + O\left(n^{(1-\alpha)/2}\cdot\sqrt{\log n}\right) + O(n^{\alpha}) \tag{6.2}$$

which is better than the formula given by theorem II. Next let

$$\Lambda = \Lambda_2 \overset{\text{def}}{=} \frac{\sqrt{6}}{\pi}\beta\sqrt{n}\log n \quad \left(0 < \beta < \frac{1}{2} \text{ and fixed constant}\right). \tag{6.3}$$

Then for almost all Π's the number of summands $\geqq \Lambda_2$ is, owing to theorem I,

$$\frac{\sqrt{6}}{\pi}n^{1/2-\beta} + O(n^{1/4}) + O(n^{1/2-2\beta}) \tag{6.4}$$

which makes sense only for $\beta<\frac{1}{4}$ whereas theorem II gives

$$\frac{\sqrt{6}}{\pi}n^{1/2-\beta}\cdot\left(1+O\left(\frac{1}{\log n}\right)\right) \tag{6.5}$$

for the whole range $0<\beta<\frac{1}{2}$. In the "border line case"

$$\Lambda=\Lambda_3\overset{\text{def}}{=}\frac{\sqrt{6}}{\pi}\gamma\sqrt{n}\quad(\gamma>0 \text{ and fixed constant}) \tag{6.6}$$

again theorem I is the stronger; then for almost all Π's the number of summands $\geqq\Lambda_3$ is

$$\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-\gamma)}+O\left(n^{1/4}\sqrt{\log n}\right). \tag{6.7}$$

In the most delicate case

$$\Lambda=\Lambda_4\overset{\text{def}}{=}\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n-3\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n \tag{6.8}$$

theorem I gives nothing whereas theorem II gives for almost all Π's the number of summands $\geqq\Lambda_4$ is

$$\frac{\sqrt{6}}{\pi}\log^3 n+O(\log^2 n). \tag{6.9}$$

For even bigger Λ's we shall return later.

The previous comparison motivates that we actually prove theorem I with the more complicated error term

$$O\left(\sqrt{\frac{\sqrt{n}\log n}{\exp\left(\frac{\pi\Lambda}{\sqrt{6n}}\right)-1}}\right)\quad\text{instead of}\quad O\left(\sqrt{\frac{n\log n}{\Lambda}}\right).$$

7. In the second paper we shall investigate — based again on the representation (1.4) — the quantity $S_2(n,\Pi,\Lambda)$ which stands for the number of λ_j's satisfying the inequality

$$\lambda_j\leqq\Lambda=\text{integer}. \tag{7.1}$$

Here the "fine" case is when Λ is "small", let

$$\Lambda\leqq 13\log n. \tag{7.2}$$

Let $A(n)$ be such that

$$A(n)\nearrow\infty\quad\text{arbitrarily slowly for}\quad n\to\infty \tag{7.3}$$

further

$$A(n) \leqq \frac{1}{1000} \Lambda \tag{7.4}$$

$\left(\text{and a fortiori } A(n) \leqq \frac{1}{1000} \log n\right)$. Then we assert the

THEOREM III. *If* Λ *is restricted by*

$$(1000\, A(n) \leqq)\ \Lambda \leqq 13 \log n,$$

$A(n)$ *by* (7.3)—(7.4) *then the inequality*

$$\frac{\sqrt{6}}{\pi}\sqrt{n}\log \Lambda - \frac{\sqrt{6}}{\pi}\sqrt{n}\log A(n) \leqq S_2(n, \Pi, \Lambda) \leqq \frac{\sqrt{6}}{\pi}\sqrt{n}\log \Lambda + \frac{\sqrt{6}}{\pi}\sqrt{n}\, A(n)$$

holds with the exception of

$$8p(n)\exp(-A(n))$$

Π*'s at most.*

Theorems I and II have important consequences. The exceptional sets in these theorems can vary with Λ but only at integer values of Λ. Using the restrictions for Λ we have the

COROLLARY OF THEOREM I. *The inequality* (5.3) *holds uniformly for the* Λ*'s in* (5.2) *apart from* $cp(n)\frac{\log n}{n^{5/4}}$ *exceptional* Π*'s at most.*

COROLLARY OF THEOREM II. *The inequality* (5.5) *holds uniformly for the* Λ*'s in* (5.4) *apart from* $cp(n)\frac{\log n}{n^{5/4}}$ *exceptional* Π*'s at most.*

8. In order to give an interpretation of theorems II—III in statistical mechanics let us consider the assembly consisting of K identical linear harmonic oscillators, with the same eigenfrequency ν_0, without interaction, and totalenergy E. As well known the possible values of the energy of each oscillator are given by $h\nu_0\left(n+\frac{1}{2}\right)$ (h is the Planck constant). Let us suppose the applicability of the Bose—Einstein statistics. If α_n stands for the number of oscillators with the energy $h\nu_0\left(n+\frac{1}{2}\right)$ and with

$$\sum_{n \geqq 1} n\alpha_n = N \tag{8.1}$$

we have obviously

$$E = h\nu_0\left(N+\frac{K}{2}\right). \tag{8.2}$$

Supposing

$$E \leqq \frac{3}{2} K h\nu_0 \tag{8.3}$$

(which can be checked from the initial data) we get from (8.2)—(8.3)

$$N \leqq K. \tag{8.4}$$

Conversely, if (8.3) is satisfied and N (the "normalized total nonzero-point-energy") satisfies (8.4) then the partitions of N give all possible energy distributions. Then theorem II (or III) gives for all sufficiently large N's a bit unprecisely that for $\log N < \Lambda < \frac{\sqrt{N}}{\log N}$ the total number of oscillators with the energy $\geqq \left(\Lambda+\frac{1}{2}\right) h\nu_0$ is

$$\frac{\sqrt{6}}{\pi}\sqrt{N}\log\frac{\sqrt{6N}}{\pi\Lambda}$$

with a probability nearly 1.

Theorems I and II settle the behaviour of $S_1(n, \Pi, \Lambda)$ for almost all Π's in the range (5.2) more or less satisfactorily. The case of the complementary ranges will be settled in the last few pages of the second paper by theorem V.

The continuation of this sequence of papers will refer — we hope — to problems II, III and IV mentioned in section 1.

9. Before turning to the proof of theorems I and II we shall need some simple lemmas. Let for $x>0, y>0$

$$D(x, y) = \sum_{l=1}^{\infty} \frac{1}{\exp(lx+y)-1}. \tag{9.1}$$

Then we shall need

LEMMA I. *We have the inequalities*

$$\frac{1}{x}\log\frac{1}{1-\exp(-x-y)} \leqq D(x, y) \leqq \frac{1}{x}\log\frac{1}{1-\exp\left\{-\frac{x}{2}-y\right\}}.$$

Namely we have

$$D(x, y) = \sum_{l=1}^{\infty} \frac{\exp\{-(lx+y)\}}{1-\exp\{-(lx+y)\}} =$$

$$= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \exp\{-m(lx+y)\} = \sum_{m=1}^{\infty} \exp(-my) \sum_{l=1}^{\infty} \exp\{-l(mx)\}$$

which gives for $D(x, y)$ the alternative representation

$$D(x, y) = \sum_{m=1}^{\infty} \frac{\exp\{-m(x+y)\}}{1-\exp(-mx)}. \tag{9.2}$$

Since for $t>0$, $k\geqq 1$ we have

$$\frac{t^k}{(k+1)!} \geqq \frac{1}{k!}\left(\frac{t}{2}\right)^k$$

we get the inequality

$$t\exp t > \exp t - 1 = t\left(1+\frac{t}{2!}+\frac{t^2}{3!}+\dots\right) \geqq$$

$$\geqq t\left\{1+\frac{1}{1!}\frac{t}{2}+\frac{1}{2!}\left(\frac{t}{2}\right)^2+\dots\right\} = t\exp\frac{t}{2}$$

for $t>0$, i.e.

(9.3) $$\frac{1}{t} < \frac{1}{1-\exp(-t)} < \frac{1}{t}\exp\frac{t}{2}.$$

Hence this and (9.2) give

(9.4) $$D(x, y) > \sum_{m=1}^{\infty}\frac{1}{mx}\exp\{-m(x+y)\} = \frac{1}{x}\log\frac{1}{1-\exp\{-(x+y)\}}$$

and

(9.5) $$D(x, y) < \sum_{m=1}^{\infty}\frac{1}{mx}\exp\left\{-m\left(\frac{x}{2}+y\right)\right\} = \frac{1}{x}\log\frac{1}{1-\exp\left\{-\frac{x}{2}-y\right\}}.$$

Q.e.d.

10. We shall need two simple lemmas on the sum

(10.1) $$\sum_{\substack{l\geqq \Lambda \\ (l \text{ integer})}}\frac{1}{\exp\{lx+y\}-1}.$$

LEMMA II. *For* $x>0, y>0, \Lambda>1$ *we have*

$$\sum_{l\geqq\Lambda}\frac{1}{\exp\{lx+y\}-1} \geqq \frac{1}{x}\log\frac{1}{1-\exp\{-(x+y+\Lambda x)\}}.$$

Namely, owing to

$$\Lambda \leqq -[-\Lambda] < \Lambda+1, \quad \Lambda-1 \leqq -[-\Lambda]-1 < \Lambda,$$

we have

$$\sum_{l\geqq\Lambda}\frac{1}{\exp(lx+y)-1} = D\big(x, y+(-[-\Lambda]-1)x\big) \geqq$$

$$\geqq D(x, y+\Lambda x) \geqq \frac{1}{x}\log\frac{1}{1-\exp(-x-y-\Lambda x)}$$

owing to (9.4), indeed.

The next lemma refers to (10.1) but with negative y.

LEMMA III. *For* $\Lambda>1, x>0$ *and* $|y|<(\Lambda-1)x$ *we have the inequality*

$$\sum_{l\geqq\Lambda}\frac{1}{\exp\{lx-|y|\}-1} \leqq \frac{1}{x}\log\frac{1}{1-\exp\left(-\Lambda x+|y|+\frac{x}{2}\right)}.$$

Namely, the left side expression is

$$= \sum_{l=1}^{\infty} \frac{1}{\exp\{lx-|y|+(-[-\Lambda]-1)x\}-1} \leqq$$
$$\leqq D(x, \{(\Lambda-1)x-|y|\})$$

and applying (9.5) the lemma is proved.

We shall also need the classical formula

$$f(x) \overset{\text{def}}{=} \prod_{\nu=1}^{\infty} \frac{1}{1-\exp(-\nu x)} = (1+o(1))\sqrt{\frac{x}{2\pi}}\exp\left(\frac{\pi^2}{6x}\right) \tag{10.2}$$

for $x \to +0$.

11. Next we assert the more difficult

LEMMA IV. *For $\Lambda > 1$ and $n > c$ the inequality*

$$S_1(n, \Pi, \Lambda) \geqq \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi\Lambda}{\sqrt{6n}}\right)} - 3\left\{\frac{\sqrt{n}\log n}{\exp\left(\frac{\pi\Lambda}{\sqrt{6n}}\right)-1}\right\}^{1/2}$$

holds except perhaps

$$cp(n)n^{-7/4}$$

Π's at most.

For the proof of this lemma let $g(n, k)$ be the number of Π's with *exactly* k summands $\geqq \Lambda$; here by definition

$$g(0, 0) = 1, \quad g(0, k) = 0 \quad \text{for} \quad k \geqq 1. \tag{11.1}$$

Then we have for $x>0, y>0$ (with notation (10.2))

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g(n, k)\exp(-nx-ky) = \tag{11.2}$$
$$= \prod_{\nu<\Lambda}\frac{1}{1-\exp(-\nu x)}\prod_{l\geqq\Lambda}\frac{1}{1-\exp(-lx-y)} =$$
$$= f(x)\prod_{l\geqq\Lambda}\frac{1-\exp(-lx)}{1-\exp(-lx-y)}.$$

Let $G(n, k)$ stand for the number of Π's with *at most* k summands $\geqq \Lambda$; here by definition

$$G(0, k) = 1. \tag{11.3}$$

Then we have

$$L(x, y) \overset{\text{def}}{=} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} G(n, k)\exp(-nx-ky) = \tag{11.4}$$
$$= \frac{f(x)}{1-\exp(-y)}\prod_{l\geqq\Lambda}\left\{1-\frac{\exp y-1}{\exp(lx+y)-1}\right\}.$$

Next we shall apply the simplest Tauberian argument. Let $n=N, k=K$ be fixed and observe that $G(n, k)$ is nonnegative and nondecreasing in n as well as in k. If $x_0>0, y_0>0$ will be chosen appropriately we get from (11.4)

$$L(x_0, y_0) \geqq G(N, K) \sum_{n=N}^{\infty} \sum_{k=K}^{\infty} \exp(-nx_0-ky_0) =$$

$$= G(N, K) \frac{\exp(-Nx_0-Ky_0)}{\{1-\exp(-x_0)\}\{1-\exp(-y_0)\}} \geqq G(N, K) \frac{\exp(-Nx_0-Ky_0)}{x_0\{1-\exp(-y_0)\}}.$$

Hence from this and (11.4) we get

$$G(N, K) \leqq L(x_0, y_0) x_0 \exp(Nx_0+Ky_0) \cdot \{1-\exp(-y_0)\}$$

i.e.

$$\text{(11.5)} \qquad G(N, K) \leqq f(x_0) \exp(Nx_0+Ky_0) x_0 \prod_{l \geqq \Lambda} \left\{1-\frac{\exp y_0 -1}{\exp(lx_0+y_0)-1}\right\} <$$

$$< f(x_0) x_0 \exp\left\{Nx_0+Ky_0-y_0 \sum_{l \geqq \Lambda} \frac{1}{\exp(lx_0+y_0)-1}\right\}.$$

Using (10.2) in the form that for sufficiently small x-values

$$f(x) < c\sqrt{x} \exp \frac{\pi^2}{6x}$$

we get

$$\text{(11.6)} \qquad G(N, K) < cx_0^{3/2} \exp\left\{\frac{\pi^2}{6x_0}+Nx_0+Ky_0-y_0 \sum_{l \geqq \Lambda} \frac{1}{\exp(lx_0+y_0)-1}\right\}.$$

Choosing

$$\text{(11.7)} \qquad x_0 = \frac{\pi}{\sqrt{6N}}, \quad N > c$$

and taking into account also (3.1) we have further

$$x_0^{3/2} \exp\left(\frac{\pi^2}{6x_0}+Nx_0\right) < \frac{c}{N^{3/4}} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{N}\right) < cp(N) N^{1/4}.$$

(11.6) takes the form

$$G(N, K) < cp(N) N^{1/4} \exp\left\{y_0\left(K-\sum_{l \geqq \Lambda} \frac{1}{\exp(lx_0+y_0)-1}\right)\right\}.$$

Hence if we choose K and y_0 so that

$$\text{(11.8)} \qquad K \leqq \sum_{l \geqq \Lambda} \frac{1}{\exp\left(\frac{l\pi}{\sqrt{6N}}+y_0\right)-1} - 2\frac{\log N}{y_0},$$

say, then we have

$$\text{(11.9)} \qquad G(N, K) < cp(N) N^{-7/4}.$$

12. Applying lemma II we get

$$\sum_{l \geqq \Lambda} \frac{1}{\exp(lx_0+y_0)-1} \geqq \frac{1}{x_0} \log \frac{1}{1-\exp(-x_0-y_0-\Lambda x_0)} =$$

$$= \frac{1}{x_0} \log \frac{1}{1-\exp(-\Lambda x_0)} - \frac{1}{x_0} \log \left(1+\frac{1-\exp(-x_0-y_0)}{\exp(\Lambda x_0)-1}\right) \geqq$$

$$\geqq \frac{1}{x_0} \log \frac{1}{1-\exp(-\Lambda x_0)} - \frac{1}{x_0} \frac{1-\exp(-x_0-y_0)}{\exp(\Lambda x_0)-1} \geqq$$

$$\geqq \frac{1}{x_0} \log \frac{1}{1-\exp(-\Lambda x_0)} - \frac{x_0+y_0}{x_0} \cdot \frac{1}{\exp(\Lambda x_0)-1}.$$

Hence (11.8) is certainly fulfilled if

$$\text{(12.1)} \quad K \leqq \frac{1}{x_0} \log \frac{1}{1-\exp(-\Lambda x_0)} - \frac{1}{\exp(\Lambda x_0)-1} - \left(\frac{y_0}{x_0} \cdot \frac{1}{\exp(\Lambda x_0)-1} + 2\frac{\log N}{y_0}\right).$$

We choose finally y_0 to minimize the expression in bracket as

$$\text{(12.2)} \qquad y_0 = \sqrt{2x_0 \log N\{\exp(\Lambda x_0)-1\}}$$

i.e. the requirement for K is from (12.1)

$$\text{(12.3)} \qquad K \leqq \frac{\sqrt{6N}}{\pi} \log \frac{1}{1-\exp(-\Lambda x_0)} - \frac{1}{\exp(\Lambda x_0)-1} - 2\sqrt{\frac{\frac{2\sqrt{6}}{\pi}\sqrt{N}\log N}{\exp(\Lambda x_0)-1}}.$$

We have obviously

$$\frac{1}{\exp(\Lambda x_0)-1} = \frac{1}{\sqrt{\exp(\Lambda x_0)-1}} \cdot \frac{1}{\sqrt{\exp\frac{\Lambda\pi}{\sqrt{6N}}-1}} \leqq$$

$$\leqq c\sqrt{\frac{N^{1/2}}{\exp(\Lambda x_0)-1}}.$$

Hence for $N > c$ (11.9) certainly holds if we choose

$$\text{(12.4)} \qquad K = \left[\frac{\sqrt{6}}{\pi}\sqrt{N} \log \frac{1}{1-\exp\left(-\frac{\pi\Lambda}{\sqrt{6N}}\right)} - 3\sqrt{\frac{\sqrt{N}\log N}{\exp\frac{\pi\Lambda}{\sqrt{6N}}-1}}\right].$$

In other words this means that the inequality

$$\text{(12.5)} \qquad S_1(N, \Pi, \Lambda) > \frac{\sqrt{6}}{\pi}\sqrt{N} \log \frac{1}{1-\exp\left(-\frac{\pi\Lambda}{\sqrt{6N}}\right)} - 3\sqrt{\frac{\sqrt{N}\log N}{\exp\frac{\pi\Lambda}{\sqrt{6N}}-1}}$$

holds with the exception of $cp(N)N^{-7/4}$ Π's. Writing back n instead of N, lemma IV is proved.

13. Next we prove the

LEMMA V. *Let* $n>c$ *and*

$$\text{(13.1)} \qquad 11\log n \leqq \Lambda \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - \sqrt{n}\log\log n.$$

Then the inequality

$$S_1(n, \Pi, \Lambda) \leqq \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi\Lambda}{\sqrt{6n}}\right)} + 100\sqrt{\frac{\sqrt{n}\log n}{\exp\frac{\pi\Lambda}{\sqrt{6n}}-1}}$$

holds for all but $cp(n)n^{-7/4}$ *exceptional* Π*'s at most.*

For the proof of this lemma we start from the identity of (11.2)

$$\text{(13.2)} \qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g(n,k)\exp(-nx-ky) = f(x)\prod_{l\geqq\Lambda}\left(1-\frac{\exp y-1}{\exp(lx+y)-1}\right),$$

valid for $x>0, y>0$. It is easy to see that the identity (13.2) holds also for the domain

$$x>0, \quad y\leqq 0, \quad |y|\leqq(\Lambda-1)x.$$

Hence

$$\text{(13.3)} \qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g(n,k)\exp(-nx+k|y|) = f(x)\prod_{l\geqq\Lambda}\left(1+\frac{1-\exp(-|y|)}{\exp(lx-|y|)-1}\right).$$

Since $g(n,k)$ is nonnegative and $g(n,k)$ is nondecreasing in n we get, fixing $n=N$ and $k=K$,

$$\text{(13.4)} \qquad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g(n,k)\exp(-nx+k|y|) \geqq \sum_{n=N}^{\infty}\sum_{k=K}^{\infty} g(N,k)\exp(-nx+k|y|) \geqq$$

$$\geqq \exp(K|y|)\sum_{n=N}^{\infty}\exp(-nx)\sum_{k=K}^{\infty} g(N,k) = H(N,K)\frac{\exp(-Nx+K|y|)}{1-\exp(-x)}$$

where

$$\text{(13.5)} \qquad H(N,K) = \sum_{k=K}^{\infty} g(N,k).$$

Using (9.3) we get from (13.4) and (13.3) for all $x>0$ and $-(\Lambda-1)x<y\leqq 0$

$$H(N,K) \leqq x\exp(Nx-K|y|)f(x)\prod_{l\geqq\Lambda}\left(1+\frac{1-\exp(-|y|)}{\exp(lx-|y|)-1}\right).$$

Choosing here again

$$x = x_0 = \frac{\pi}{\sqrt{6N}}, \quad N>c$$

we get as in section 11

$$H(N, K) < cp(N)N^{1/4}\exp(-K|y|)\prod_{l\geqq\Lambda}\left(1+\frac{1-\exp(-|y|)}{\exp(lx_0-|y|)-1}\right). \tag{13.6}$$

Since

$$\prod_{l\geqq\Lambda}\left(1+\frac{1-\exp(-|y|)}{\exp(lx_0-|y|)-1}\right) \leqq \exp\left\{|y|\sum_{l\geqq\Lambda}\frac{1}{\exp(lx_0-|y|)-1}\right\}$$

we get from (13.6)

$$H(N, K) < cp(N)N^{1/4}\exp\left\{|y|\left(\sum_{l\geqq\Lambda}\frac{1}{\exp(lx_0-|y|)-1}-K\right)\right\}. \tag{13.7}$$

Hence if

$$K \geqq \sum_{l\geqq\Lambda}\frac{1}{\exp(lx_0-|y|)-1}+2\frac{\log N}{|y|} \tag{13.8}$$

we get from (13.7)

$$H(N, K) < cp(N)N^{-7/4}. \tag{13.9}$$

14. Applying lemma III we get

$$\sum_{l\geqq\Lambda}\frac{1}{\exp(lx_0-|y|)-1} \leqq \frac{1}{x_0}\log\frac{1}{1-\exp\left(-\Lambda x_0+|y|+\frac{x_0}{2}\right)} =$$

$$= \frac{1}{x_0}\log\frac{1}{1-\exp(-\Lambda x_0)}+\frac{1}{x_0}\log\left(1+\frac{1-\exp\left(-|y|-\frac{x_0}{2}\right)}{\exp\left(\Lambda x_0-|y|-\frac{x_0}{2}\right)-1}\right) \leqq$$

$$\leqq \frac{1}{x_0}\log\frac{1}{1-\exp(-\Lambda x_0)}+\frac{1}{x_0}\cdot\frac{1-\exp\left(-|y|-\frac{x_0}{2}\right)}{\exp(\Lambda x_0)-1}\cdot\frac{\exp(\Lambda x_0)-1}{\exp\left(\Lambda x_0-|y|-\frac{x_0}{2}\right)-1} \leqq$$

$$\leqq \frac{1}{x_0}\log\frac{1}{1-\exp(-\Lambda x_0)}+\frac{1}{x_0}\cdot\frac{|y|+\frac{x_0}{2}}{\exp(\Lambda x_0)-1}\cdot\frac{\exp(\Lambda x_0)-1}{\exp\left(\Lambda x_0-|y|-\frac{x_0}{2}\right)-1}.$$

Hence (13.8) is certainly satisfied if

$$K \geqq \frac{1}{x_0}\log\frac{1}{1-\exp(-\Lambda x_0)}+ \tag{14.1}$$

$$+\left(\frac{1}{x_0}\cdot\frac{|y|+\frac{x_0}{2}}{\exp(\Lambda x_0)-1}\cdot\frac{\exp(\Lambda x_0)-1}{\exp\left(\Lambda x_0-|y|-\frac{x_0}{2}\right)-1}+\frac{2\log N}{|y|}\right).$$

Here the minimizing value for $|y|$ cannot be found directly; we shall perform a "partial minimization" for

$$\frac{1}{x_0}\cdot\frac{|y|}{\exp(\Lambda x_0)-1}+\frac{2\log N}{|y|}$$

and show afterwards that owing to the restriction (13.1) we did not lose much. Hence we choose

(14.2) $$y=y_1\overset{\text{def}}{=}-\sqrt{2x_0\{\exp(\Lambda x_0)-1\}\log N}\quad(<0);$$

we have to verify by this choice that

(14.3) $$|y_1|<(\Lambda-1)x_0$$

is satisfied and also that for $N>c$

(14.4) $$\frac{\exp(\Lambda x_0)-1}{\exp\left(\Lambda x_0-|y_1|-\frac{x_0}{2}\right)-1}<30.$$

Supposing (14.3)—(14.4), writing n instead of N, the inequality (14.1) is certainly fulfilled if

(14.5) $$K\geqq\frac{1}{x_0}\log\frac{1}{1-\exp(-\Lambda x_0)}+15\frac{1}{\exp(\Lambda x_0)-1}+60\sqrt{\frac{2\log n}{x_0\{\exp(\Lambda x_0)-1\}}}.$$

But owing to $\Lambda\geqq 11\log n$ we have for $n>c$

$$\exp(\Lambda x_0)-1\geqq\exp\left(\frac{11\pi}{\sqrt{6n}}\log n\right)-1>\frac{\log n}{\sqrt{n}}>\frac{x_0}{\log n}$$

i.e.

$$\frac{1}{\exp(\Lambda x_0)-1}<\sqrt{\frac{\log n}{x_0\{\exp(\Lambda x_0)-1\}}}$$

and thus (14.5) is certainly fulfilled if

(14.6) $$K\geqq\frac{1}{x_0}\log\frac{1}{1-\exp(-\Lambda x_0)}+100\sqrt{\frac{\sqrt{n}\log n}{\exp(\Lambda x_0)-1}}.$$

But this means that choosing

(14.7) $$K=K_0\overset{\text{def}}{=}1+\left[\frac{1}{x_0}\log\frac{1}{1-\exp(-\Lambda x_0)}+100\sqrt{\frac{\sqrt{n}\log n}{\exp(\Lambda x_0)-1}}\right]$$

the inequality

(14.8) $$H(n,K_0)<cp(n)n^{-7/4}$$

holds. Hence under supposition (14.3)—(14.4) lemma V is proved.

15. In order to verify (14.3) we write it in the form

$$\sqrt{\frac{2}{x_0}\log n\{\exp(\Lambda x_0)-1\}} < \Lambda - 1$$

which will be certainly true if we can show

$$2\log n\frac{\exp(\Lambda x_0)-1}{\Lambda x_0} < 0.99\,\Lambda. \tag{15.1}$$

Let first $\Lambda \leqq \frac{1}{x_0}$. Then (15.1) is certainly true if

$$2(e-1)\log n < 0{,}99\,\Lambda.$$

But this is true for $n>c$ owing to (13.1). Next let $\Lambda > \frac{1}{x_0}$. Then (15.1) is certainly satisfied if

$$2\log n\frac{\exp(\Lambda x_0)-1}{(\Lambda x_0)^2} < 0.99\frac{1}{x_0}.$$

Putting $\Lambda x_0 = t$ we have to show that for

$$1 < t \leqq \frac{1}{2}\log n - \frac{\pi}{\sqrt{6}}\log\log n \tag{15.2}$$

the inequality

$$\frac{e^t-1}{t^2} \leqq 0.49\frac{\sqrt{6}}{\pi}\frac{\sqrt{n}}{\log n}$$

holds. But $\frac{e^t-1}{t}$ being monotonically increasing we have in the range (15.2) for $n>c$

$$\frac{e^t-1}{t^2} < \frac{e^t-1}{t} \leqq \frac{\sqrt{n}\log^{-\pi/\sqrt{6}} n}{\frac{1}{2}\log n - \frac{\pi}{\sqrt{6}}\log\log n} < 0.49\frac{\sqrt{6}}{\pi}\cdot\frac{\sqrt{n}}{\log n}$$

indeed. Hence (14.3) is completely verified.

16. In order to verify (14.4) let first

$$(11\log n \leqq)\,\Lambda \leqq \frac{2}{x_0}. \tag{16.1}$$

If we succeed in showing that in this case

$$|y_1| + \frac{x_0}{2} < \frac{4}{5}\Lambda x_0 \tag{16.2}$$

holds then this is settled since

$$\frac{\exp(\Lambda x_0)-1}{\exp\left(\Lambda x_0 - |y_1| - \frac{x_0}{2}\right)-1} < \frac{\exp(\Lambda x_0)-1}{\exp\left(\frac{1}{5}\Lambda x_0\right)-1} < \frac{e^2-1}{e^{2/5}-1} < 30. \tag{16.3}$$

To show (16.2) in the case (16.1) we get using (14.2) the equivalent form

$$2\log n\,\frac{\exp(\Lambda x_0)-1}{\Lambda x_0}<\left(\frac{4}{5}-\frac{1}{2\Lambda}\right)^2\Lambda$$

which is certainly satisfied for $n>c$ if

$$2\log n\,\frac{\exp(\Lambda x_0)-1}{\Lambda x_0}<0.79^2\,\Lambda$$

which in turn would follow from

$$\frac{e^2-1}{0.79^2}\log n<\Lambda$$

owing to (16.1). But owing to

$$\frac{e^2-1}{0.79^2}<11$$

this follows from (13.1). Hence (14.4) is proved in the case (16.1).

17. Finally we have to verify (14.4) in the case

$$\frac{2}{x_0}<\Lambda\leqq\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n-\sqrt{n}\log\log n. \tag{17.1}$$

In this verification we shall need two simple observations. Firstly

$$\frac{\exp(t)-1}{t^2}\quad\text{is monotonically increasing for } t\geqq 2. \tag{17.2}$$

Secondly, if $\frac{1}{2}\leqq a\leqq 1$ and $t\geqq 2$ then

$$\frac{1-\exp(-t)}{1-\exp(-at)}\leqq\frac{1-\exp(-t)}{1-\exp\left(-\frac{1}{2}t\right)}<2. \tag{17.3}$$

For the verification we shall need first that

$$|y_1|<\frac{1}{2\sqrt{\log n}}\Lambda x_0. \tag{17.4}$$

Owing to (14.2) this is equivalent to

$$8\log^2 n\,\frac{\exp(\Lambda x_0)-1}{(\Lambda x_0)^2}<\frac{1}{x_0}=\frac{\sqrt{6}}{\pi}\sqrt{n}. \tag{17.5}$$

But for $t=\Lambda x_0$ we have owing to (17.1)

$$2<t\leqq\frac{1}{2}\log n-\frac{\pi}{\sqrt{6}}\log\log n$$

and hence using (17.2) the left side expression in (17.5) is indeed

$$< 8\log^2 n \frac{\sqrt{n}\exp\left(-\frac{\pi}{\sqrt{6}}\log\log n\right)}{\frac{1}{5}\log^2 n} < \frac{\sqrt{6}}{\pi}\sqrt{n}$$

for $n>c$. Owing to $\Lambda \geqq 11\log n$ it follows for $n>c$ at once

$$|y_1|+\frac{x_0}{2} < \frac{1}{\sqrt{\log n}}\Lambda x_0. \tag{17.6}$$

Now we can quickly finish the verification of (14.4)—(17.1). We have, using (17.6) and (17.3)

$$\frac{\exp(\Lambda x_0)-1}{\exp\left(\Lambda x_0-|y_1|-\frac{x_0}{2}\right)-1} = \exp\left(|y_1|+\frac{x_0}{2}\right)\frac{1-\exp(-\Lambda x_0)}{1-\exp\left(-\Lambda x_0+|y_1|+\frac{x_0}{2}\right)} \leqq$$

$$\leqq \exp\left(|y_1|+\frac{x_0}{2}\right)\frac{1-\exp(-\Lambda x_0)}{1-\exp\left\{-\left(1-\frac{1}{\sqrt{\log n}}\right)\Lambda x_0\right\}} \leqq 2\exp\left(|y_1|+\frac{x_0}{2}\right) < 3\exp|y_1|$$

for $n>c$. But using (14.2) and (17.1) we have for $n>c$

$$3\exp\sqrt{\frac{2\pi}{\sqrt{6n}}\log n\{\exp(\Lambda x_0)-1\}} <$$

$$< 3\exp\left\{\sqrt{\frac{10\log n}{\sqrt{n}}\cdot\exp\left(\frac{1}{2}\log n-\frac{\pi}{\sqrt{6}}\log\log n\right)}\right\} < 4.$$

Hence the verification of (14.3)—(14.4) and also the proof of lemma V are complete.

18. From lemmas IV and V we can complete easily the proof of theorem I. These give namely for the Λ's in (5.2)

$$\left|S_1(n,\Pi,\Lambda)-\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi\Lambda}{\sqrt{6n}}\right)}\right| < 100\sqrt{\frac{\sqrt{n}\log n}{\exp\left(\frac{\pi\Lambda}{\sqrt{6n}}\right)-1}} < c\sqrt{\frac{n\log n}{\Lambda}}$$

for all but $cp(n)n^{-7/4}$ Π's at most indeed.

Theorem II follows also easily from Lemmas IV and V. For this sake we investigate the quotient

$$\frac{\sqrt{\dfrac{\sqrt{n}\log n}{\exp\dfrac{\pi\Lambda}{\sqrt{6n}}-1}}}{\sqrt{n}\log\dfrac{1}{1-\exp\left(-\dfrac{\pi\Lambda}{\sqrt{6n}}\right)}} \stackrel{\text{def}}{=} M. \tag{18.1}$$

First let

$$\frac{\sqrt{6}}{\pi e^2}\sqrt{n} \leqq \Lambda \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 3\sqrt{n}\log\log n. \tag{18.2}$$

Then

$$M \leqq \sqrt{\frac{\log n}{\sqrt{n}\left\{\exp\dfrac{\pi\Lambda}{\sqrt{6n}}-1\right\}}}\cdot\exp\frac{\pi\Lambda}{\sqrt{6n}} < \tag{18.3}$$

$$< c\sqrt{\frac{\log n}{\sqrt{n}}\exp\frac{\pi\Lambda}{\sqrt{6n}}} < c\log^{1/2-\pi/\sqrt{6}\cdot 3/2} n < \frac{c}{\log n}.$$

Finally let

$$11\log n \leqq \Lambda < \frac{\sqrt{6}}{\pi e^2}\sqrt{n}. \tag{18.4}$$

Then we have

$$M < c\frac{\sqrt{\dfrac{\log n}{\Lambda}}}{\log\dfrac{\sqrt{6n}}{\pi\Lambda}} \leqq c\sqrt{\log n}\,\frac{1}{\min\sqrt{x}\log\dfrac{\sqrt{6n}}{\pi x}}$$

where the minimum refers to $11\log n \leqq x \leqq \frac{\sqrt{6}}{\pi e^2}\sqrt{n}$. But this minimum is taken at $x = 11\log n$ and hence also in this case

$$M < \frac{c}{\log n}.$$

This completes the proof of theorem II.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ALGEBRA AND NUMBER THEORY
1088 BUDAPEST, MÚZEUM KRT. 6—8.

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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ON SOME PROBLEMS OF THE STATISTICAL THEORY OF PARTITIONS WITH APPLICATION TO CHARACTERS OF THE SYMMETRIC GROUP. II

By

M. SZALAY and P. TURÁN, member of the Academy (Budapest)

To the 65th birthday of George Szekeres

1. As in Part I of this paper, Π stands for a generic partition of n

$$\Pi: \begin{array}{l} n = \lambda_1+\lambda_2+\ldots+\lambda_m \\ \lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_m \geqq 1 \end{array} \tag{1.1}$$

and $S_2(n, \Pi, \Lambda)$ gives the number of λ_j's in (1.1) satisfying

$$\lambda_j \leqq \Lambda = \text{integer}. \tag{1.2}$$

Let further

$$A(n) \nearrow \infty \quad \text{arbitrarily slowly,} \tag{1.3}$$

$$A(n) \leqq \frac{1}{1000}\Lambda \tag{1.4}$$

and

$$A(n) \leqq \frac{1}{5}\log n, \tag{1.5}$$

further

$$(1000\,A(n) \leqq)\,\Lambda \leqq 13\log n. \tag{1.6}$$

Then as indicated in Part I, we assert

THEOREM III. *Under the above restrictions the inequality*

$$\frac{\sqrt{6}}{\pi}\sqrt{n}\log\Lambda - \frac{\sqrt{6}}{\pi}\sqrt{n}\log A(n) \leqq S_2(n, \Pi, \Lambda) \leqq \frac{\sqrt{6}}{\pi}\sqrt{n}\log\Lambda + \frac{\sqrt{6}}{\pi}\sqrt{n}\,A(n)$$

holds with the exception of

$$8p(n)\exp(-A(n))$$

Π's at most.

2. For the proof we shall need a number of observations. Let for $1 \leqq l \leqq k$

$$f_{k,l}(x) = \frac{(1-x^k)(1-x^{k-1})\ldots(1-x^{k-l+1})}{(1-x)(1-x^2)\ldots(1-x^l)} \tag{2.1}$$

and

$$f_{k,0}(x) \equiv 1. \tag{2.2}$$

These are polynomials of degree $l(k-l)$.

Based on the recursion formula

$$f_{k+1,l+1}(x) = f_{k,l}(x) + x^{l+1} f_{k,l+1}(x) \quad (\text{where} \quad f_{k,k+1}(x) \equiv 0) \tag{2.3}$$

we get at once for the coefficients $d_\nu^{(k,l)}$ defined by

$$f_{k,l}(x) = \sum_{\nu=0}^{l(k-l)} d_\nu^{(k,l)} x^\nu \tag{2.4}$$

that

$$d_\nu^{(k,l)} \geqq 0, \quad \nu = 0, 1, \dots, l(k-l) \tag{2.5}$$

and also

$$\sum_{\nu=0}^{l(k-l)} d_\nu^{(k,l)} = \binom{k}{l}. \tag{2.6}$$

We shall need further the easily provable formula

$$\frac{1}{1-z} \prod_{l=1}^{K} \frac{1-x^l}{1-x^l z} = \frac{1}{1-z} + \sum_{l=1}^{K} \frac{(-1)^l x^{\binom{l+1}{2}} f_{K,l}(x)}{1-x^l z}; \tag{2.7}$$

further the easy fact that if $\varphi(z)$ is regular for $|z| < \varrho$ with $\varrho > 1$ and

$$\varphi(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu, \quad r_\nu = \sum_{j=\nu}^{\infty} c_j \tag{2.8}$$

then for $|z| \leqq 1$ and

$$h(z) = \sum_{\nu=0}^{\infty} r_\nu z^\nu \tag{2.9}$$

we have

$$h(z) = \frac{\varphi(1) - z\varphi(z)}{1-z}. \tag{2.10}$$

3. Now we turn to the proof of theorem III. Let first $a(n, m, K)$ be defined as the number of the partitions of n where the number of summands not exceeding K is exactly m. If for $|x| < 1$, $|z| \leqq 1$, $F_0(x, z)$ is defined by

$$F_0(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a(n, m, K) x^n z^m = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a(n, m, K) x^n z^m \tag{3.1}$$

then we have

$$F_0(x, z) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots x^{1k_1 + 2k_2 + \dots} z^{k_1 + \dots + k_K} = \prod_{\nu=1}^{K} \frac{1}{1-x^\nu z} \cdot \prod_{\nu=K+1}^{\infty} \frac{1}{1-x^\nu}. \tag{3.2}$$

Defining $F(x)$ for $|x| < 1$ by

$$F(x) = 1 + \sum_{\nu=1}^{\infty} p(\nu) x^\nu = \prod_{\nu=1}^{\infty} \frac{1}{1-x^\nu} \tag{3.3}$$

we get from (3.2)

$$F_0(x, z) = F(x) \prod_{\nu=1}^{K} \frac{1-x^\nu}{1-x^\nu z}. \tag{3.4}$$

Let now $b(n, m, K)$ stand for the number of partitions of n where the number of summands not exceeding K is

$$\geqq m. \tag{3.5}$$

Obviously we have

$$b(n, m, K) = a(n, m, K) + a(n, m+1, K) + \dots;$$

hence, defining $F_1(x, z)$ by

$$F_1(x, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b(n, m, K) x^n z^m \tag{3.6}$$

we get, using (2.10)—(2.9) and (3.4)

$$F_1(x, z) = \frac{F_0(x, 1) - zF_0(x, z)}{1-z} = \frac{F(x) - zF_0(x, z)}{1-z} = \tag{3.7}$$

$$= F(x) \frac{1}{1-z} \left\{1 - z \prod_{\nu=1}^{K} \frac{1-x^\nu}{1-x^\nu z}\right\}.$$

Hence, using (3.3) we have for $m \geqq 1$

$$b(n, m, K) = \text{coeffs. } x^n z^m \text{ in } F_1(x, z) =$$

$$= p(n) - \text{coeffs. } x^n z^{m-1} \text{ in } F(x) \left\{\frac{1}{1-z} \prod_{\nu=1}^{K} \frac{1-x^\nu}{1-x^\nu z}\right\}.$$

Applying (2.7) with the abbreviation

$$(-1)^l x^{\binom{l+1}{2}} f_{K,l}(x) = D_l \tag{3.8}$$

we get for $m \geqq 1$

$$b(n, m, K) = -\text{coeffs. } x^n z^{m-1} \text{ in } \left\{F(x) \sum_{l=1}^{K} \frac{D_l}{1-x^l z}\right\} =$$

$$= -\text{ coeffs. } x^n \text{ in } \left\{F(x) \sum_{l=1}^{K} D_l x^{l(m-1)}\right\},$$

or

$$b(n, m, K) = \sum_{l=1}^{K} (-1)^{l+1} \text{ coeffs. } x^n \text{ in } \{F(x) x^{lm+\binom{l}{2}} f_{K,l}(x)\}. \tag{3.9}$$

Using finally (3.3) and (2.4) we get the sieve-type representation

$$b(n, m, K) = \sum_{l=1}^{K} (-1)^{l+1} \sum_{\nu=0}^{l(K-l)} d_\nu^{(K,l)}\, p\left(n - lm - \binom{l}{2} - \nu\right). \tag{3.10}$$

4. We shall use the classical formula of Hardy and Ramanujan on $p(n)$ in the form

$$\left|p(n)-\frac{1}{4n\sqrt{3}}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right)\right| < cn^{-3/2}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right). \tag{4.1}$$

We shall need also the following simple inequality. If a, b, c, d are positive numbers so that

$$\frac{|a-c|}{a} \leqq \frac{1}{2}, \quad \frac{|b-d|}{b} \leqq \frac{1}{2} \tag{4.2}$$

then we have

$$\left|\frac{a}{b}-\frac{c}{d}\right| \leqq 2\frac{c}{d}\left(\frac{|a-c|}{c}+\frac{|b-d|}{d}\right). \tag{4.3}$$

For the reader's convenience we remark that

$$\frac{a}{b}-\frac{c}{d} = \frac{c}{d}\left(\frac{a-c}{c}+\frac{d-b}{d}\right)\frac{d}{b}$$

and from (4.2)

$$\frac{d}{b} \leqq \frac{b+|d-b|}{b} \leqq \frac{3}{2}$$

which proves (4.3). We apply (4.2)—(4.3) with

$$\begin{cases} a = p(n-t), \quad b = p(n), \\ c = \dfrac{1}{4(n-t)\sqrt{3}}\exp\left(\dfrac{2\pi}{\sqrt{6}}\sqrt{n-t}\right), \quad d = \dfrac{1}{4n\sqrt{3}}\exp\left(\dfrac{2\pi}{\sqrt{6}}\sqrt{n}\right) \end{cases} \tag{4.4}$$

where

$$m \leqq t \leqq K(m+K) \tag{4.5}$$

where we require on m and K provisorily only

$$10^{-100}\sqrt{n} \leqq m \leqq 2\sqrt{n}\log n, \quad K \leqq 100\log n. \tag{4.6}$$

This implies for $n>c$

$$\frac{1}{10^{100}}\sqrt{n} \leqq t \leqq 300\sqrt{n}\log^2 n. \tag{4.7}$$

Then, owing to (4.1), (4.2) is satisfied for $n>c$ and hence from (4.3) we get for the t's in (4.5)

$$\left|\frac{p(n-t)}{p(n)}-\frac{n}{n-t}\exp\left\{\frac{2\pi}{\sqrt{6}}(\sqrt{n-t}-\sqrt{n})\right\}\right| <$$

$$< c\frac{n}{(n-t)^{3/2}}\exp\left\{\frac{2\pi}{\sqrt{6}}(\sqrt{n-t}-\sqrt{n})\right\} < \frac{c}{\sqrt{n}}\exp\left\{-\frac{2\pi}{\sqrt{6}}\frac{t}{\sqrt{n-t}+\sqrt{n}}\right\},$$

or after easy estimations

$$\left|\frac{p(n-t)}{p(n)}-\exp\left(-\frac{\pi}{\sqrt{6}}\frac{t}{\sqrt{n}}\right)\right| < c\left(\frac{t}{n}+\frac{t^2}{n^{3/2}}\right)\exp\left(-\frac{\pi}{\sqrt{6}}\frac{t}{\sqrt{n}}\right).$$

Using the lower bound in (4.7) we get finally the useful inequality

$$\left|\frac{p(n-t)}{p(n)}-\exp\left(-\frac{\pi t}{\sqrt{6n}}\right)\right| < c\frac{t^2}{n^{3/2}}\exp\left(-\frac{\pi t}{\sqrt{6n}}\right) \tag{4.8}$$

if (4.7) holds.

5. We write now the sieve formula (3.10) in the form

$$\frac{b(n,m,K)}{p(n)} = \sum_{l=1}^{K}(-1)^{l+1}\sum_{\nu=0}^{l(K-l)}d_\nu^{(K,l)}\frac{p\left(n-lm-\binom{l}{2}-\nu\right)}{p(n)}. \tag{5.1}$$

For the quantities

$$lm+\binom{l}{2}+\nu$$

we get the lower bound m and the upper bound

$$Km+\frac{l(l-1)}{2}+l(K-l)\leqq K(m+l)\leqq K(m+K)$$

and hence under restrictions of (4.6) the requirement (4.7) is fulfilled and (4.8) is applicable to each term in (5.1). Let us estimate first the contribution of the error term. This is absolutely

$$\leqq\frac{c}{n^{3/2}}\sum_{l=1}^{K}\sum_{\nu=0}^{l(K-l)}d_\nu^{(K,l)}\left\{lm+\binom{l}{2}+\nu\right\}^2\exp\left\{-\frac{\pi}{\sqrt{6n}}\left(lm+\binom{l}{2}+\nu\right)\right\}. \tag{5.2}$$

Using (4.6) and (2.6) this is

$$< c\frac{m^2}{n^{3/2}}\sum_{l=1}^{K}l^2\exp\left\{-\frac{\pi}{\sqrt{6n}}\left(lm+\binom{l}{2}\right)\right\}\sum_{\nu=0}^{l(K-l)}\left(d_\nu^{(K,l)}\exp\left(-\frac{\pi\nu}{\sqrt{6n}}\right)\right) <$$

$$< c\frac{m^2}{n^{3/2}}\sum_{l=1}^{K}l^2\binom{K}{l}\exp\left\{-\frac{\pi}{\sqrt{6n}}\left(lm+\binom{l}{2}\right)\right\}$$

and using (4.6) further

$$< c\frac{\log^4 n}{\sqrt{n}}\sum_{l=1}^{K}\binom{K}{l}\exp\left(-\frac{\pi ml}{\sqrt{6n}}\right) < c\frac{\log^4 n}{\sqrt{n}}\left\{1+\exp\left(-\frac{\pi m}{\sqrt{6n}}\right)\right\}^K. \tag{5.3}$$

So far the restrictions for m and K in (4.6) were enough. In order to make the right side of (5.3) small we must be more careful. We shall replace (4.6) by

$$X\sqrt{n}\leqq m\leqq 2\sqrt{n}\log n,\quad K\leqq Y\log n \tag{5.4}$$

where X and Y are appropriate constants (however not violating (4.6)). Then the right side of (5.3) is

$$< c\frac{\log^4 n}{\sqrt{n}} n^U$$

with

$$U = Y\log\left\{1+\exp\left(-\frac{\pi X}{\sqrt{6}}\right)\right\}.$$

Hence requiring

$$Y\log\left\{1+\exp\left(-\frac{\pi X}{\sqrt{6}}\right)\right\} \leqq \frac{1}{4}, \tag{5.5}$$

say, (5.1) and (4.8) give

(5.6)

$$\left|\frac{b(n, m, K)}{p(n)} - \sum_{l=1}^{K}(-1)^{l+1}\exp\left\{-\frac{\pi}{\sqrt{6n}}\left(lm+\binom{l}{2}\right)\right\}\sum_{\nu=0}^{l(K-l)} d_\nu^{(K,l)}\exp\left(-\frac{\pi\nu}{\sqrt{6n}}\right)\right| < \\ < \frac{c\log^4 n}{n^{1/4}}.$$

Next we want to replace the factor $\exp\left\{-\frac{\pi}{\sqrt{6n}}\binom{l}{2}\right\}$ by 1. The error made is absolutely

$$\leqq \frac{c}{\sqrt{n}}\sum_{l=1}^{K} l^2\binom{K}{l}\exp\left(-\frac{\pi ml}{\sqrt{6n}}\right) < \frac{c\log^2 n}{\sqrt{n}}\left\{1+\exp\left(-\frac{\pi m}{\sqrt{6n}}\right)\right\}^K \leqq c\frac{\log^2 n}{n^{1/4}}.$$

Hence from (5.6) we get — under the restrictions (5.4)—(5.5) — that

$$\left|\frac{b(n, m, K)}{p(n)} - \sum_{l=1}^{K}(-1)^{l+1}\exp\left\{-\frac{\pi lm}{\sqrt{6n}}\right\} f_{K,l}\left(\exp\left(-\frac{\pi}{\sqrt{6n}}\right)\right)\right| < c\frac{\log^4 n}{n^{1/4}}. \tag{5.7}$$

6. To deal with the last sum we write it as

$$1-\sum_{l=0}^{K}(-1)^l f_{K,l}(1)\exp\left\{-\frac{\pi lm}{\sqrt{6n}}\right\}- \tag{6.1}$$

$$-\sum_{l=1}^{K}(-1)^{l+1}\left\{f_{K,l}(1)-f_{K,l}\left(\exp\left(-\frac{\pi}{\sqrt{6n}}\right)\right)\right\}\exp\left(-\frac{\pi lm}{\sqrt{6n}}\right) =$$

$$= 1-\left\{1-\exp\left(-\frac{\pi m}{\sqrt{6n}}\right)\right\}^K - Z.$$

Since

$$f_{K,l}(1)-f_{K,l}\left(\exp\left(-\frac{\pi}{\sqrt{6n}}\right)\right) = \sum_{\nu=0}^{l(K-l)} d_\nu^{(K,l)}\left\{1-\exp\left(-\frac{\nu\pi}{\sqrt{6n}}\right)\right\} <$$

$$< c\sum_{\nu=0}^{l(K-l)} d_\nu^{(K,l)}\frac{\nu}{\sqrt{n}} < c\frac{K^2}{\sqrt{n}}\binom{K}{l}$$

we get using (5.4) and (5.5)

$$(6.2)\qquad |Z| < c\frac{K^2}{\sqrt{n}}\sum_{l=1}^{K}\binom{K}{l}\exp\left(-\frac{\pi l m}{\sqrt{6n}}\right) < c\frac{K^2}{\sqrt{n}}\left\{1+\exp\left(-\frac{\pi m}{\sqrt{6n}}\right)\right\}^K < c\frac{\log^2 n}{n^{1/4}}$$

and hence finally

$$(6.3)\qquad \frac{b(n,m,K)}{p(n)} = 1-\left\{1-\exp\left(-\frac{\pi m}{\sqrt{6n}}\right)\right\}^K + O\left(\frac{\log^4 n}{n^{1/4}}\right).$$

7. In order to get upper and lower bounds for the left side in (6.3) we shall apply it twice (taking care of (5.4)—(5.5) and (4.6)). Let $Y=13$ and $X=X_0$ so large that

$$(7.1)\qquad \log\left(1+\exp\left(-\frac{\pi X_0}{\sqrt{6}}\right)\right) = \frac{1}{52}.$$

We choose

$$(7.2)\qquad m = m_1 \stackrel{\text{def}}{=} \left[\frac{\sqrt{6}}{\pi}\sqrt{n}\left(\log \Lambda + A(n)\right)\right],\quad K = K_1 = \Lambda.$$

Then (5.5) is fulfilled; owing to $A(n)\nearrow\infty$, (1.5) and (1.6) also (5.4) (and (4.6)) for $n>c$. Then we have on the right side of (6.3) with a $-1\leqq\vartheta\leqq+1$

$$(7.3)\qquad 1-\left\{1-\exp\left(-\frac{\pi m_1}{\sqrt{6n}}\right)\right\}^{K_1} = 1-\left\{1-\exp\left(-\log\Lambda - A(n)+\frac{\vartheta\pi}{\sqrt{6n}}\right)\right\}^{\Lambda} =$$

$$= 1-\left\{1-\frac{\exp\left(-A(n)+\dfrac{\vartheta\pi}{\sqrt{6n}}\right)}{\Lambda}\right\}^{\Lambda}.$$

Using the inequality

$$\left(1-\frac{x}{d}\right)^d \geqq 1-2x \qquad \left(0\leqq x\leqq \frac{1}{2},\ d \text{ positive integer}\right)$$

and for $n>c$

$$\exp\left(-A(n)+\frac{\pi}{\sqrt{6n}}\right) \leqq \frac{1}{2}$$

we get from (7.3) and (1.5)

$$\frac{b(n,m_1,K_1)}{p(n)} \leqq 2\exp\left(-A(n)+\frac{\pi}{\sqrt{6n}}\right)+c\frac{\log^4 n}{n^{1/4}} <$$

$$< 3\exp\left(-A(n)\right)+c\frac{\log^4 n}{n^{1/4}} < 4\exp\left(-A(n)\right)$$

for $n>c$. Hence the number of Π's with the property that the number of summands not exceeding Λ is

$$> \frac{\sqrt{6}}{\pi}\sqrt{n}\left(\log\Lambda + A(n)\right),$$

is $4p(n)\exp(-A(n))$ at most. In other words the inequality

$$S_2(n, \Pi, \Lambda) \leqq \frac{\sqrt{6}}{\pi}\sqrt{n}(\log \Lambda + A(n)) \tag{7.4}$$

holds with the exception of $4p(n)\exp(-A(n))$ Π's at most.

8. In order to get a lower bound for $S_2(n, \Pi, \Lambda)$ let again $Y=13$, $X=X_0$ by (7.1), further

$$m = m_2 \overset{\text{def}}{=} \left[\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{\Lambda}{A(n)}\right]+1, \quad K=K_2=\Lambda. \tag{8.1}$$

Using also (1.4) one can verify (5.4)—(5.5)—(4.6) as before. Then we have on the right side of (6.3)

$$\left\{1-\exp\left(-\frac{\pi m_2}{\sqrt{6n}}\right)\right\}^{K_2} \leqq \left\{1-\exp\left(-\log\frac{\Lambda}{A(n)}-\frac{\pi}{\sqrt{6n}}\right)\right\}^{\Lambda} =$$

$$= \left(1-e^{-\frac{\pi}{\sqrt{6n}}}\frac{A(n)}{\Lambda}\right)^{\Lambda} < 3\exp(-A(n))$$

and

$$O\left(\frac{\log^4 n}{n^{1/4}}\right) > -c\frac{\log^4 n}{n^{1/4}} > -\exp(-A(n)),$$

hence the number of Π's with the property that the number of summands not exceeding Λ is

$$\geqq \frac{\sqrt{6}}{\pi}\sqrt{n}(\log\Lambda - \log A(n)),$$

is greater than $(1-4\exp(-A(n)))\,p(n)$. In other words the inequality

$$S_2(n, \Pi, \Lambda) \geqq \frac{\sqrt{6}}{\pi}\sqrt{n}(\log\Lambda - \log A(n)) \tag{8.2}$$

holds, with the exception of $4p(n)\exp(-A(n))$ Π's at most. (7.4) and (8.2) prove theorem III.

Next we shall show how one can deduce the strong form of Erdős—Lehner's theorem mentioned in Part I. This will be

THEOREM IV. *If* $\omega(n)\nearrow\infty$ *arbitrarily slowly and satisfies*

$$\omega(n) = o(\log n) \tag{8.3}$$

and $l(\Pi)$ *stands for the number of summands in* Π *(with multiplicity) then the inequality*

$$\left|l(\Pi) - \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n\right| < c\sqrt{n}\,\omega(n)$$

holds with the exception of $cp(n)\exp(-\omega(n))$ Π*'s at most.*

For the proof let

$$(8.4)\qquad \Lambda_0 = [12 \log n]$$

and we start from the relation

$$(8.5)\qquad l(\Pi) = S_1(n, \Pi, \Lambda_0+1)+S_2(n, \Pi, \Lambda_0).$$

We apply theorems I and III with $\Lambda=\Lambda_0+1$, resp. $\Lambda=\Lambda_0$ (for which both are applicable) and in theorem III with $A(n)=\omega(n)$. These give on one hand

$$(8.6)\qquad l(\Pi) \leqq \frac{\sqrt{6}}{\pi}\sqrt{n}\log \frac{1}{1-\exp\left(-\frac{\pi(\Lambda_0+1)}{\sqrt{6n}}\right)}+c\sqrt{n}+$$

$$+\frac{\sqrt{6}}{\pi}\sqrt{n}\log \Lambda_0+c\sqrt{n}\,\omega(n) < \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n+c\sqrt{n}\,\omega(n)$$

apart from $cp(n)\exp\left(-\omega(n)\right)$ Π's at most. On the other hand we have with an exceptional set of the same measure

$$(8.7)\qquad l(\Pi) \geqq \frac{\sqrt{6}}{\pi}\sqrt{n}\log \frac{1}{1-\exp\left(-\frac{\pi(\Lambda_0+1)}{\sqrt{6n}}\right)}-c\sqrt{n}+$$

$$+\frac{\sqrt{6}}{\pi}\sqrt{n}\log \Lambda_0-c\sqrt{n}\log \omega(n) > \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n-c\sqrt{n}\,\omega(n)$$

for $n>c$. (8.6)—(8.7) prove theorem IV.

9. As mentioned in Section 8 of Part I, here we shall investigate $S_1(n, \Pi, \Lambda)$ for the ranges

$$(9.1)\qquad B(n) \leqq \Lambda \leqq 13 \log n$$

where $B(n)\nearrow\infty$ arbitrarily slowly and

$$(9.2)\qquad \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n-\sqrt{n}\log\log n \leqq \Lambda \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n-\sqrt{n}\,\omega(n)$$

where $\omega(n)\nearrow\infty$. The range (9.1) can be settled easily. Applying theorem III with $A(n)=\sqrt{\log\Lambda}$ we get for this range

$$(9.3)\qquad S_2(n, \Pi, [\Lambda]) = \frac{\sqrt{6}}{\pi}\sqrt{n}\log \Lambda+O\left(\sqrt{n\log\Lambda}\right)$$

with $O(p(n))\exp(-\sqrt{\log \Lambda})$ exceptional Π's at most. Then applying theorem IV with $\omega(n)=\sqrt{\log \Lambda}$ we obtain for the Λ's in (9.1)

$$S_1(n,\Pi,\Lambda)=\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n+O(\sqrt{n\log \Lambda})-S_2(n,\Pi,-[-\Lambda]-1)= \tag{9.4}$$

$$=\frac{\sqrt{6}}{2\pi}\sqrt{n}\,\log\frac{n}{\Lambda^2}+O(\sqrt{n\log \Lambda})$$

with the exception of $O(p(n))\exp(-\sqrt{\log \Lambda})$ exceptional Π's.

10. The treatment of the range (9.2) is a bit more difficult and the result we can prove is of weaker character but this seems to be inherent in the matter. We remind the reader to the combinatorial proof of the theorem (due to Euler) according to which the number of Π's with $l(\Pi)=k(\leqq n)$ is the same as the number of Π's with the maximal summand k. The same reasoning gives the following

LEMMA. *If*

$$1\leqq r,\, l\leqq n \tag{10.1}$$

then the number of Π's with

$$S_1(n,\Pi,r)\geqq l \tag{10.2}$$

is equal to the number of the Π's with

$$S_1(n,\Pi,l)\geqq r. \tag{10.3}$$

Hence if (10.2) holds for almost all Π's with $U(n)$ exceptions at most then the same holds for the Π's satisfying (10.3).

Let now l_0 be such that

$$B(n)\leqq l_0\leqq \log^2 n. \tag{10.4}$$

Then (9.4) and theorem I give that

$$\begin{cases} S_1(n,\Pi,l_0)<\dfrac{\sqrt{6}}{2\pi}\sqrt{n}\log\dfrac{n}{l_0^2}+c\sqrt{n\log l_0}\overset{\text{def}}{=}r_1 \\ S_1(n,\Pi,l_0)\geqq\dfrac{\sqrt{6}}{2\pi}\sqrt{n}\log\dfrac{n}{l_0^2}-c\sqrt{n\log l_0}\overset{\text{def}}{=}r_2 \end{cases} \tag{10.5}$$

with the exception of $O(p(n))\exp(-\sqrt{\log l_0})$ Π's at most. Applying now the lemma with $r=r_1$ resp. $r=r_2$ and $l=l_0$ we get

$$S_1(n,\Pi,r_1)<l_0,\quad S_1(n,\Pi,r_2)\geqq l_0 \tag{10.6}$$

with the exception of $O(p(n))\exp(-\sqrt{\log l_0})$ Π's at most. If l_0 varies in the interval (10.4) then each r_j $(j=1,2)$ fills an interval of the form

$$(10.7)\qquad \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n-2\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n+O(\sqrt{n\log\log n})\leqq$$

$$\leqq r_j\leqq\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n-\frac{\sqrt{6}}{\pi}\sqrt{n}\log B(n)+O(\sqrt{n\log B(n)})$$

and l_0 has the form

$$(10.8)\qquad l_0=\exp\left\{\frac{1}{2}\log n-\frac{r_j\pi}{\sqrt{6n}}+O(1)\sqrt{\frac{1}{2}\log n-\frac{r_j\pi}{\sqrt{6n}}}\right\}.$$

Replacing r_j by Λ and

$$\frac{\sqrt{6}}{\pi}\log B(n)+O(1)\sqrt{\log B(n)}=\omega(n)$$

we obtained

THEOREM V. *If* $\omega(n)\nearrow\infty$ *arbitrarily slowly and*

$$\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n-\sqrt{n}\log\log n\leqq\Lambda\leqq\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n-\sqrt{n}\,\omega(n)$$

then we have

$$S_1(n,\Pi,\Lambda)=\exp\left\{\left(\frac{1}{2}\log n-\frac{\Lambda\pi}{\sqrt{6n}}\right)+O(1)\sqrt{\frac{1}{2}\log n-\frac{\Lambda\pi}{\sqrt{6n}}}\right\}$$

with the exception of

$$O(p(n))\exp\left\{-\sqrt{\frac{1}{2}\log n-\frac{\Lambda\pi}{\sqrt{6n}}-c\sqrt{\frac{1}{2}\log n-\frac{\Lambda\pi}{\sqrt{6n}}}}\right\}$$

Π's at most. For the range

$$B(n)\leqq\Lambda\leqq 13\log n,\quad B(n)\nearrow\infty$$

we have

$$S_1(n,\Pi,\Lambda)=\frac{\sqrt{6}}{2\pi}\sqrt{n}\log\frac{n}{\Lambda^2}+O(\sqrt{n\log\Lambda})$$

with the exception of $O(p(n))\exp(-\sqrt{\log\Lambda})$ *Π's at most.*

11. Analogous reasoning can be applied to $S_2(n,\Pi,\Lambda)$; we shall not perform it. In this case the bounded Λ's present novelty, which is easy but we mention it for the sake of completeness. Let us investigate the case $\Lambda=1$ i.e. the statistical situation on the number of occurrences of 1 as summand in Π. The number of Π's containing 1 as summand k-times at most is (with notation (3.3))

$$\text{coeffs. } x^n \text{ in } (1+x^{1\cdot1}+x^{1\cdot2}+\ldots+x^{1\cdot k})\prod_{\nu=2}^{\infty}\frac{1}{1-x^\nu}=(1-x^{k+1})F(x)$$

i.e. the number of Π's with

$$S_2(n, \Pi, 1) \leqq k \tag{11.1}$$

is

$$p(n) - p(n-k-1) = p(n)\left(1 - \frac{p(n-k-1)}{p(n)}\right). \tag{11.2}$$

Using the weaker form of (4.1)

$$p(n) = (1+o(1))\frac{1}{4n\sqrt{3}}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right)$$

and supposing $k=o(n)$ a priori we get for the cardinality of Π's satisfying (11.1)

$$p(n)\left\{1-(1-o(1))\exp\left(-\frac{2\pi}{\sqrt{6}}\cdot\frac{k+1}{\sqrt{n}+\sqrt{n-k-1}}\right)\right\} =$$

$$= \begin{cases} o(p(n)) & \text{if} \quad k = o(\sqrt{n}) \\ (1-o(1))p(n) & \text{if} \quad \frac{1}{k} = o\left(\frac{1}{\sqrt{n}}\right) \quad \text{and} \quad k = o(n). \end{cases}$$

Hence if $\omega(n)\nearrow\infty$ arbitrarily slowly then for almost all Π's

$$\frac{\sqrt{n}}{\omega(n)} \leqq S_2(n, \Pi, 1) \leqq \sqrt{n}\,\omega(n)$$

and this is no more true replacing $\frac{\sqrt{n}}{\omega(n)}$ or $\sqrt{n}\,\omega(n)$ by $b\sqrt{n}$ (with a constant $b>0$).

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ALGEBRA AND NUMBER THEORY
1088 BUDAPEST, MÚZEUM KRT. 6—8.

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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APPROXIMATION OF UNBOUNDED FUNCTIONS ON UNBOUNDED INTERVAL

By
T. HERMANN (Budapest)

1. Let $S_n[f; x]$ be the well-known Szász operator, that is

$$S_n[f;x] = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_{n,k}(x) \tag{1}$$

where

$$p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

It is known [5] that if

$$f(x) = O(e^{\alpha x}) \quad (\alpha > 0,\ x \to \infty) \tag{2}$$

and f is continuous in $[0, \infty)$ then for all $A>0$

$$|f(x)-S_n[f;\ x]| \leqq O(\omega_{2A}(f;\ n^{-1/2})), \quad x\in[0, A] \tag{3}$$

where

$$\omega_A(f;\delta) = \sup\{|f(x+t)-f(x)|: |t| \leqq \delta, \quad x\in[0, A]\}.$$

We investigate two questions. The first question is: For which similar operators does the convergence hold under the restriction (2)? The second question is: May (2) be changed to a stronger condition or not?

2. In the following all functions will mean continuous functions. For the first problem we need the next

LEMMA. *Let $\{L_n\}_{n=1}^{\infty}$ be a sequence of positive linear operators so that for all bounded functions f*

$$L_n[f;x] \to f(x), \quad x\in I \tag{4}$$

holds, where I is a bounded or unbounded interval. If for any function $F(x)>0$

$$L_n[F;x] \to F(x), \quad x\in D \tag{5}$$

where $D\subset I$ and D is an open set, then for all $f(x)$ with $f(x)=O(F(x))$

$$L_n[f;x] \to f(x), \quad x\in D.$$

PROOF. Let x be any fixed element of D. D is an open set so there exists $\delta>0$ such that $[x-2\delta, x+2\delta]\subset D$. Let

$$\chi(t) = \chi_{x,\delta}(t) = \begin{cases} 0 & \text{if } |t-x| \geqq 2\delta \\ 1 & \text{if } |t-x| \leqq \delta \\ \text{linear otherwise.} \end{cases}$$

Let $\bar{\chi}(t)=1-\chi(t)$. From the linearity of L_n it follows

$$L_n[f; x] = L_n[\chi f; x]+L_n[\bar{\chi} f; x].$$

χf is a bounded function, so according to (4)

$$L_n[\chi f; x] \to \chi(x)f(x) = f(x).$$

If we prove that $L_n[\bar{\chi} f; x]=o(1)$ then we are ready. It is obvious that $\bar{\chi} f=O(\bar{\chi} F)$ so from the positivity of L_n it follows:

$$L_n[\bar{\chi} f; x] = O(L_n[\bar{\chi} F; x]).$$

Because χF is bounded, by (4)

$$L_n[\chi F; x] \to F(x)\chi(x) = F(x)$$

and since $x\in D$ so according to (5)

$$L_n[F; x] \to F(x).$$

Consequently

$$L_n[\bar{\chi} F; x] = L_n[F; x]-L_n[\bar{\chi} F; x] \to 0.$$

This completes the proof.

Let $X_1, X_2, \ldots, X_n, \ldots$ be independent random variables of the same distribution with expectation x. Let $Y_n=\frac{1}{n}\sum_{i=1}^{n} X_i$, let the distribution of the Y_n's be $F_{n,x}(t)$ and

$$L_n[f; x] = \int_{-\infty}^{\infty} f(t)\, dF_{n,x}(t). \tag{6}$$

It is well-known [3, p. 218] that if f is bounded then (4) is true for this operator. The problem is: if we replace the boundedness by (2), when will (4) be true? According to the Lemma, it is enough to test $e^{\alpha x}$ only.

We get with easy computation (similarly as for the characteristic function):

$$L_n[e^{\alpha t}; x] = (L_1[e^{\frac{\alpha}{n}t}; x])^n.$$

From this it is obviosusly enough to prove:

$$\lim_{n\to\infty} n \log L_1[e^{\frac{\alpha}{n}t}; x] = \alpha x.$$

The left part is a derivative:

$$\lim_{s\to 0}\frac{\log L_1[e^{\alpha ts}; x]}{s} = \left(\frac{d}{ds}\log L_1[e^{\alpha ts}; x]\right)_{s=0} =$$

$$= \left(\frac{\int\limits_{-\infty}^{\infty} \alpha t e^{\alpha ts}\, dF_{1,x}(t)}{\int\limits_{-\infty}^{\infty} e^{\alpha ts}\, dF_{1,x}(t)}\right)_{s=0} = \alpha \int\limits_{-\infty}^{\infty} t\, dF_{1,x}(t) = \alpha x.$$

So we have

THEOREM 1. *Let* $\{L_n\}_{n=1}^{\infty}$ *be a sequence of positive linear operators of type* (6), $f(x)=O(e^{\alpha x})$. *Then* (5) *is fulfilled if and only if* $L_1[e^{\alpha t}; x]$ *exists.*

3. In many cases $L_n[e^{\alpha t}; x]$ is easily computable.

EXAMPLES. a) If $F_{n,x}$ is the Poisson distribution (that is L_n is the Szász operator [3, p. 219], [7]), then

$$S_n[e^{\alpha t}; x] = \left(e^{-x}\sum_{k=0}^{\infty}\frac{(xe^{\alpha/n})^k}{k!}\right)^n = e^{nx(e^{\alpha/n}-1)}.$$

b) If $F_{n,x}$ is a gamma distribution, we get the gamma operator of FELLER [3, p. 219]:

$$G_n[f; x] = \int\limits_0^{\infty} f(l)\frac{n^n l^{n-1} e^{-nl/x}}{(n-1)!}\frac{n}{x}\, dl.$$

Then

$$G_n[e^{\alpha t}; x] = \left(\frac{1}{x}\int\limits_0^{\infty} e^{\left(\frac{\alpha}{n}-\frac{1}{x}\right)l}\, dl\right)^n = \left(1-\frac{\alpha x}{n}\right)^{-n}.$$

c) If $F_{n,x}$ is the negative binomial distribution, then L_n is the BASKAKOV operator [1], [7]:

$$L_n[f; x] = (1+x)^{-n}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k.$$

Then

$$L_n[e^{\alpha t}; x] = \left(\frac{1}{1+x}\sum_{k=0}^{\infty}\left(\frac{xe^{\alpha/n}}{1+x}\right)^k\right)^n = (1-x(e^{\alpha/n}-1))^{-n}.$$

REMARK. It is obvious that the Baskakov operator is equivalent to the Z_n operators of Meyer-König and Zeller with the substitution $y=\frac{x}{1+x}$, $g(y)=f(x)$. Then $g\in C[0, 1]$.

$$Z_n[g; y] = (1-y)^{n+1}\sum_{k=0}^{\infty} g\left(\frac{k}{n+k}\right)\binom{n+k-1}{k} y^k.$$

Here the equivalent of the condition (2) is

$$g(y) = O\left(e^{\alpha\frac{y}{1-y}}\right).$$

For this condition the convergence was proved by MEYER-KŐNIG and ZELLER in [6].

A generalisation of the Z_n operators due to CHENEY and SHARMA [2] is the following:

$$C_n[g;\, y] = (1-y)^{n+1} e^{t\frac{y}{1-y}} \sum_{k=0}^{\infty} L_k^{(n)}(t)\, y^k \cdot g\left(\frac{k}{n+k}\right)$$

where $\{L_k^{(n)}\}$ are the Laguerre polynomials. Cheney and Sharma proved that for all bounded functions the convergence is valid if $t=o(n)$ $(n\to\infty)$.

Let $f_0(y)=\exp \alpha \left(\dfrac{y}{1-y}\right)$. Then

$$C_n[f_0;\, y] = (1+y-ye^{\alpha/n})^{-(n+1)} \exp\left(ty\left(1-\frac{e^{\alpha/n}}{1+y-ye^{\alpha/n}}\right)\right).$$

Since in the case $t=0$

$$C_n[f_0;\, y] = Z_n[f_0;\, y] \to e^{\alpha\frac{y}{1-y}}$$

for the convergence it is enough to prove that the limit of the second part of the product equals 1. But

$$ty\left(1-\frac{e^{\alpha/n}}{1+y-ye^{\alpha/n}}\right) = O_y\left(\frac{t}{n}\right) = o(1).$$

According to our Lemma $C_n[f;\, y]\to f(y)$ for all $f(y)\in O(f_0)$.

4. Our second problem: is it possible to weaken (2) in the case of the Szász operator? (This problem was proposed by J. GRÓF [4].)

THEOREM 2. *If* $f(x)=O(x^{\alpha x})$ $(\alpha>0)$ *then* (3) *is fulfilled. If* $f(x)\geqq x^{\Phi(x)x}$ *where* $\Phi(x)$ *is any monotonically increasing function so that* $\lim\limits_{x\to\infty}\Phi(x)=\infty$ *then* $S_n[f;\, x]$ *does not exist.*

PROOF. First we prove the second part of the theorem.

$$S_n[t^{\Phi(t)t};\, x] = \sum_{k=0}^{\infty} a_k$$

where

$$a_k = \left(\frac{k}{n}\right)^{\Phi\left(\frac{k}{n}\right)\frac{k}{n}} p_{n,k}(x).$$

We use the Cauchy criterion:

$$\frac{a_{k+1}}{a_k} = \frac{nx}{k+1} e^{\left\{\Phi\left(\frac{k+1}{n}\right)\frac{k+1}{n}\log\frac{k+1}{n} - \Phi\left(\frac{k}{n}\right)\frac{k}{n}\log\frac{k}{n}\right\}},$$

$$\Phi\left(\frac{k+1}{n}\right)\frac{k+1}{n}\log\frac{k+1}{n} - \Phi\left(\frac{k}{n}\right)\frac{k}{n}\log\frac{k}{n} =$$

$$= \left(\Phi\left(\frac{k+1}{n}\right)-\Phi\left(\frac{k}{n}\right)\right)\frac{k+1}{n}\log\frac{k+1}{n} + \Phi\left(\frac{k}{n}\right)\frac{1}{n}\log\frac{k+1}{n} + \Phi\left(\frac{k}{n}\right)\frac{1}{n}\log\left(1+\frac{1}{k}\right)^k \geqq$$

$$\geqq \Phi\left(\frac{k}{n}\right)\frac{1}{n}\log\frac{k+1}{n} \quad (k\geqq n).$$

So

$$\frac{a_{k+1}}{a_k} \geqq \frac{nx}{k+1}\left(\frac{k+1}{n}\right)^{\Phi\left(\frac{k}{n}\right)\frac{1}{n}} = x\left(\frac{k+1}{n}\right)^{\frac{1}{n}\Phi\left(\frac{k}{n}\right)-1}.$$

If k is large enough then $\frac{a_{k+1}}{a_k}>1$; hence the sum is divergent.

Now we prove the first part of the theorem. Let $I=[0, 2A]$, $x\in[0, A]$. Then

$$|S_n[f; x]-f(x)| = |S_n[f\chi_I+f\bar{\chi}_I; x]-f(x)| \leqq$$
$$\leqq |S_n[f\chi_I; x]-(f\chi_I)(x)|+|S_n[\bar{\chi}_I f; x]| = R_1+R_2.$$

For $\chi_I f$ (3) is fulfilled because $\chi_I f$ is bounded. Obviously $\omega_{2A}(f\chi_I; \delta)=\omega_{2A}(f; \delta)$ hence $R_1=O(\omega_{2A}(f; \delta))$.

Now it is enough to prove that $R_2=O\left(\frac{1}{n}\right)$.

$$R_2 = \sum_{k\geqq 2An}^{\infty} e^{\alpha\frac{k}{n}\log\frac{k}{n}} p_{n,k}(x) = \sum_{k\geqq 2An}^{\infty} b_k,$$

$$\frac{b_{k+1}}{b_k} = \frac{nx}{k+1}\frac{\left(\frac{k+1}{n}\right)^{\alpha(k+1)/n}}{\left(\frac{k}{n}\right)^{\alpha k/n}} = x\frac{\left(1+\frac{1}{k}\right)^{k\alpha/n}}{\left(\frac{k+1}{n}\right)^{1-\alpha/n}} \leqq A\frac{e^{\alpha/n}}{(2A)^{1-\alpha/n}} \leqq \frac{2}{3}$$

if n is large enough. Consequently

$$R_2 \leqq b_{2An}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = O(p_{n,2An}(x)), \quad p_{n,2An}(x) = e^{-nx}\frac{(nx)^{2An}}{(2An)!}.$$

With the Stirling formula we get:

$$p_{n,2An}(x) \simeq e^{-nx}\frac{(nx)^{2An}}{\left(\frac{2An}{e}\right)^{2An}}\cdot\frac{1}{\sqrt{4\pi An}} = \left(e^{1-\frac{x}{2A}}\frac{x}{2A}\right)^{2An}\frac{1}{\sqrt{4\pi An}}.$$

The right hand side equals $O(e^{-cn})$ for any $c>0$ if $0\leqq x\leqq A$ which proves the Theorem.

REMARK. For operators of type (4) which satisfy Theorem 1, (2) generally cannot be strengthened. For example if L_n is the Baskakov operator, then $L_n[e^{\Phi(t)t}; x]$ diverges where $\Phi(t)\to\infty$ $(t\to\infty)$ and Φ is monotone.

$$L_n[e^{t\Phi(t)}; x] = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}e^{\frac{k}{n}\Phi\left(\frac{k}{n}\right)} = \sum_{k=0}^{\infty} c_k.$$

We apply again the Cauchy criterion:

$$\frac{c_{k+1}}{c_k} = \frac{x}{1+x}\cdot\frac{n+k}{k+1}e^{\left(\frac{k+1}{n}\Phi\left(\frac{k+1}{n}\right)-\frac{k}{n}\Phi\left(\frac{k}{n}\right)\right)} \geqq \frac{x}{1+x}e^{\frac{1}{n}\Phi\left(\frac{k+1}{n}\right)}.$$

The right part tends to the infinity so that the sum is divergent.

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[1] G. Szegő, *Orthogonal polynomials*, AMS Coll. Publ. Vol. XXIII (Providence, 1939).

[2] A. Zygmund, Smooth functions, *Duke Math. J.*, **12** (1945), 47—76.

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