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Az Acta Mathematica angol, német, francia és orosz nyelven közöl értekezéseket a matematika köréből. Váltakozó terjedelmű füzetekben jelenik meg, több füzet alkot egy kötetet. A közlésre szánt kéziratok a szerkesztőség, minden más levelezés a kiadóhivatal címére küldendő.

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INDEX

TOMUS XXVIII

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ALMOST CONTACT HYPERBOLIC-(f, g, η, ξ) STRUCTURE

By

M. D. UPADHYAY and K. K. DUBE (Lucknow)

1. Introduction. Let us consider an n-dimensional $(n=m+1)$ real differentiable manifold V_n of class C^∞. Let there exist a C^∞ function F, a contravariant vector field η and a 1-form ξ which satisfy the following conditions:

$$f^2X = X + \xi(X)\eta \tag{1.1a}$$

for an arbitrary vector field X.

$$\overline{X} \overset{\text{def}}{=} F(X), \tag{1.1b}$$

$$\xi(\eta) = -1, \tag{1.2}$$

$$\bar{\eta} = 0 \tag{1.3}$$

and

$$\xi(\overline{X}) = 0. \tag{1.4}$$

Let g be the Riemannian metric tensor such that

$$g(\overline{X}, \overline{Y}) \overset{\text{def}}{=} -g(X, Y) - \xi(X)\xi(Y). \tag{1.5}$$

The manifold V_n satisfying all the conditions from (1.1)a to (1.5), yields a hyperbolic structure. We call it as a hyperbolic almost contact-(f, g, η, ξ) structure.

Let N be the Nijenhuis tensor [2] then,

$$N(X, Y) \overset{\text{def}}{=} [fX, fY] + f^2[X, Y] - f[X, fY] - f[fX, Y]. \tag{1.6}$$

By virtue of (1.1) and (1.6), we get

$$N(X, Y) = [fX, fY] + [X, Y] - f[X, fY] - f[fX, Y] + \xi([X, Y])\eta. \tag{1.7}$$

Let

$$F(X, Y) \overset{\text{def}}{=} g(fX, Y). \tag{1.8}$$

Then from (1.8), (1.1)a and (1.5) we have

$$F(X, fY) = g(fX, fY) = -g(X, Y) - \xi(X)\xi(Y) = -F(fX, Y). \tag{1.9}$$

From (1.1), (1.9) and (1.5) we have

$$F(fX,fY) = g(X,fY) = -g(fX,Y) = -F(X,Y). \tag{1.10}$$

Thus we have

THEOREM 1.1. *$F(X, Y)$ is hybrid in both the slots X and Y.*

By virtue of (1.1), (1.5) and (1.8) we can show that

$$F(X,Y) = -F(Y,X). \tag{1.11}$$

Thus we have

THEOREM 1.2. *The tensor F of type* (0,2) *is skew-symmetric in X and Y.*

Let $X=\eta$, then from (1.5) and (1.3) we get

$$g(\bar{\eta}, \bar{Y}) = -g(\eta, Y) - \xi(\eta)\xi(Y) \tag{1.12}$$

which yields

$$g(\eta, Y) = \xi(Y). \tag{1.13}$$

When $X=\eta$, from (1.8) we get

$$F(\eta, Y) = 0. \tag{1.14}$$

From (1.13) we have

$$g(\eta, \bar{Y}) = 0. \tag{1.15}$$

2. Some results. THEOREM 2.1. *Let*

$$P(X,Y) \stackrel{\text{def}}{=} [fX,fY] - f[X,fY] \tag{2.1}$$

then

$$fP(X,fY) = f[fX,Y] + f(\xi(Y)[fX,\eta]) - [X,Y] - \xi(Y)[X,\eta] - \xi([X,Y])\eta - \\ - \xi(\xi(Y)[X,\eta])\eta \tag{2.2}$$

and

$$P(X,Y) - fP(X,fY) = N(X,Y) - f(\xi(Y)[fX,\eta]) + \xi(Y)[X,\eta] + \\ + \xi(\xi(Y)[X,\eta])\eta. \tag{2.3}$$

PROOF. The proof of (2.2) and (2.3) follows easily by means of (2.1) and (1.1)a.

COROLLARY 2.1. *We have*

$$P(X,\eta) = N(X,\eta) + f[fX,\eta] - \xi([X,\eta])\eta - [X,\eta]. \tag{2.4}$$

PROOF. Putting $Y=\eta$ in (2.3) and by virtue of (1.2) we get (2.4) .

THEOREM 2.2. *Let*

$$Q(X,Y) \stackrel{\text{def}}{=} [fX,fY] - f[fX,Y] \tag{2.5}$$

then

$$fQ(fX, Y) = f[X, fY] + f(\xi(X)[\eta, fY]) - \\ - [X, Y] - \xi([X, Y])\eta - \xi(X)([\eta, Y]) - \xi(\xi(X)[\eta, Y])\eta \tag{2.6}$$

and

$$Q(X, Y) - fQ(fX, Y) = N(X, Y) - f(\xi(X)[\eta, fY]) + \xi(X)[\eta, Y] + \xi(\xi(X)[\eta, Y])\eta. \tag{2.7}$$

PROOF. Proof of (2.6) and (2.7) follows immediately in consequence of (1.1)a and (2.5).

THEOREM 2.3. *Let*

$$H(X, Y) \overset{\text{def}}{=} [fX, fY] + [X, Y] \tag{2.8}$$

then

$$fH(fX, Y) = f[X, fY] + f(\xi(X)[\eta, fY]) + f[fX, Y] \tag{2.9}$$

and

$$H(X, Y) - fH(fX, Y) = N(X, Y) - f(\xi(X)[\eta, fY]) - \xi([X, Y])\eta. \tag{2.10}$$

PROOF. (2.10) and (2.9) follows easily by using (2.8) and (1.1)a.

COROLLARY 2.2. *We have*

$$\begin{aligned} &\text{a) } H(\eta, Y) = N(\eta, Y) + f[\eta, fY] - \xi([\eta, Y])\eta. \\ &\text{b) } H(X, \eta) - fH(fX, \eta) = N(X, \eta) - \xi([X, \eta])\eta. \\ &\text{c) } Q(\eta, Y) = N(\eta, Y) + f[\eta, fY] - \xi([\eta, Y])\eta - [\eta, Y]. \end{aligned} \tag{2.11}$$

PROOF. The proof is obvious.

THEOREM 2.4. *We have*

$$fN(fX, Y) + N(X, Y) = f(\xi(X)[\eta, fY]) - \xi([fX, fY])\eta - \\ - \xi(X)[\eta, Y] - \xi(\xi(X)[\eta, Y])\eta \tag{2.12}$$

$$fN(fX, \eta) + N(X, \eta) = 0 \tag{2.13}$$

and

$$N(\eta, Y) = [\eta, Y] + \xi([\eta, Y])\eta - f[\eta, fY]. \tag{2.14}$$

PROOF. From (1.7) and (1.1)a we have

$$fN(fX, Y) = f[X, fY] - [fX, fY] - [X, Y] + f[fX, Y] - \\ - \xi([X, Y])\eta + f(\xi(X)[\eta, fY]) - \xi([fX, fY])\eta - \xi(X)[\eta, Y] - \xi(\xi(X)[\eta, Y])\eta + \\ + f(\xi([fX, Y]))\xi. \tag{2.15}$$

which with the help of (1.7) yields (2.12).

The proof of (2.13) and (2.14) follows at once by virtue of (1.2), (1.3) and (2.12).

3. Affine connection. Let D be an affine connection in V_n and let a vector valued bilinear function S be its torsion tensor, so that

$$S(X, Y) = D_X Y - D_Y X - [X, Y].$$

Here and in what follows square brackets [ ] stand for Lie brackets.

The affine connection D satisfies the following properties:

$$\text{(3.1)} \qquad \begin{aligned} &\text{a) } D_X \eta = fX, \\ &\text{b) } D_X fY = D_Y fX + f[X, Y] + X\xi(X) - Y\xi(X), \\ &\text{c) } (D_X \xi)(Y) + (D_Y \xi)(X) = 0. \end{aligned}$$

From (3.1)b we get the following

THEOREM 3.1. *We have*

$$\text{(3.2)} \qquad D_{fX} fY - D_Y X = f[fX, Y] + f(Y)\xi(X) + f(X)\xi(Y) + \eta Y(\xi(X)),$$

$$\text{(3.3)} \qquad D_X Y - D_{fY} fX = f[X, fY] - f(Y)\xi(X) - f(X)\xi(Y) + \eta X(\xi(Y)),$$

$$\text{(3.4)} \qquad fD_X fY - fD_Y fX = [X, Y] + \xi([X, Y])\eta + f(X)\xi(Y) - f(Y)\xi(X).$$

PROOF. The proof follows easily.

THEOREM 3.2. *We have*

$$N(X, Y) = \xi([X, Y])\eta + \eta Y(\xi(X)) + \eta X(\xi(Y)).$$

PROOF. The proof follows from (3.2), (3.3) and (1.7).

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DEPARTMENT OF MATHEMATICS AND ASTRONOMY,
LUCKNOW UNIVERSITY,
LUCKNOW, INDIA

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CORATIONAL EXTENSIONS AND PSEUDO-PROJECTIVE MODULES

By
L. BICAN (Praha)

1. Introduction

The restricted injective envelopes with respect to various types of preradicals have been studied by many authors (see e.g. [2], [6], [8], [10]). In the second paragraph we shall dualize this notion for a class of preradicals and we shall state its basic properties. In § 3 we restate some results of [5] with very simple proofs and we improve a theorem from [4]. In the final section we dualize the notion of pseudo-injective modules (see [3], [13]) and we give some alternative characteristic of such modules.

All the rings considered below will be associative with identity and all modules will be unitary left R-modules. The category of left R-modules is denoted by ${}_R\mathcal{M}$.

A preradical ϱ for ${}_R\mathcal{M}$ is any subfunctor of the identity, i.e. ϱ assigns to each module M a submodule $\varrho(M)$ in such a way that every homomorphism $M\to N$ induces $\varrho(M)\to\varrho(N)$ by restriction. A preradical ϱ is said to be idempotent if $\varrho^2=\varrho$ and is called a radical if $\varrho({}^M/_{\varrho(M)})=0$. For a preradical ϱ, a module M is called ϱ-torsion if $\varrho(M)=M$ and ϱ-torsion-free if $\varrho(M)=0$. For two preradicals ϱ, σ we shall write $\varrho\geqq\sigma$ if $\varrho(M)\supseteqq\sigma(M)$ for every $M\in{}_R\mathcal{M}$.

Recall that a submodule K of a module M is said to be small in M if $K+M'=$ $=M$, M' a submodule of M necessarily yields $M'=M$. A short exact sequence $0\to K\to$ $\to P\to M\to 0$ is called a projective presentation of M if P is projective and it is called a cover of M if K is small in P and it is called a projective cover of M if it is both projective presentation and a cover of M (see [1] for details). A module is said to be cocyclic if it has essential simple socle.

2. ϱ-projective modules and covers

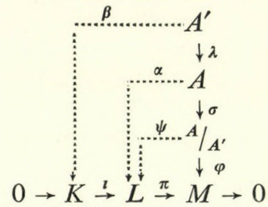
Definition 1. Let ϱ be a preradical for ${}_R\mathcal{M}$. A module A is said to be ϱ-projective if for every diagram

$$\begin{array}{ccccccccc} & & & & & & A & & \\ & & & & & & \downarrow\varphi & & \\ 0 & \to & K & \to & L & \xrightarrow{\pi} & M & \to & 0\end{array}$$

with exact row and K ϱ-torsion-free there exists $\psi: A\to L$ such that $\psi\pi=\varphi$.

Proposition 1. *Let ϱ be a preradical for ${}_R\mathcal{M}$ and A a ϱ-projective module. If A' is a ϱ-torsion submodule of A then ${}^A/_{A'}$ is ϱ-projective.*

PROOF. Let us consider the following diagram



with exact row, K ϱ-torsion-free and λ, σ the natural embedding and projection, respectively. By hypothesis, there exists $\alpha\colon A \to L$ with $\alpha\pi = \sigma\varphi$. α induces $\beta\colon A' \to K$ with $\lambda\alpha = \beta\iota$ and $\beta = 0$ since A' is ϱ-torsion and K is ϱ-torsion-free. Hence there exists $\psi\colon {}^{A}/_{A'} \to L$ with $\sigma\psi = \alpha$. Consequently, $\sigma\psi\pi = \alpha\pi = \sigma\varphi$ and $\psi\pi = \varphi$, σ being an epimorphism.

DEFINITION 2. A preradical ϱ for ${}_R\mathcal{M}$ is said to be cohereditary if the class of ϱ-torsion-free modules is closed under homomorphic images.

LEMMA 1. *If ϱ is a cohereditary radical for ${}_R\mathcal{M}$ then the class of ϱ-torsion modules is closed under covers.*

PROOF. Let $0 \to K \to L \to M \to 0$ be a cover of M, $\varrho(M) = M$ and let $\varrho(L) = L' \subseteqq L$. Then ${}^{L}/_{(L'+K)}$ is ϱ-torsion-free as a homomorphic image of ${}^{L}/_{L'}$ and it is ϱ-torsion as a homomorphic image of ${}^{L}/_{K} \cong M$. Hence $L' + K = L$, contradicting the fact K is small in L.

DEFINITION 3. Let ϱ be a preradical for ${}_R\mathcal{M}$. A short exact sequence $0 \to A' \to A \to M \to 0$ is said to be a ϱ-projective cover of M if

(1) A is a ϱ-projective module,
(2) A' is small in A,
(3) no factor ${}^{A}/_{A''}$, $0 \neq A'' \subseteqq A'$ is ϱ-projective.

THEOREM 1. *Let ϱ be an idempotent radical for ${}_R\mathcal{M}$. If a module A has a projective cover then it has a ϱ-projective cover which is unique up to isomorphism over the identity on A.*

PROOF. Let $0 \to N \to P \to A \to 0$ be a projective cover of A and let $X = \varrho(N)$. We claim that $0 \to {}^{N}/_{X} \to {}^{P}/_{X} \to A \to 0$ is a ϱ-projective cover of A. ${}^{P}/_{X}$ is ϱ-projective by Proposition 1 and ${}^{N}/_{X}$ is clearly small in ${}^{P}/_{X}$. Suppose that ${}^{P/X}/_{T/X} \cong {}^{P}/_{T}$ is ϱ-projective for some $0 \neq {}^{T}/_{X} \subseteqq {}^{N}/_{X}$. In this case the exact sequence $0 \to {}^{T}/_{X} \to {}^{P}/_{X} \to {}^{P}/_{T} \to 0$ must split since ${}^{T}/_{X} \subseteqq {}^{N}/_{X}$ is ϱ-torsion-free. This contradicts the fact ${}^{T}/_{X}$ is small in ${}^{P}/_{X}$. Now we proceed to the uniqueness. Suppose in the following diagram the exact rows are two ϱ-projective covers of M.

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & A & \xrightarrow{\pi} & M & \to & 0 \\ & & & & \Big\downarrow{\scriptstyle\varphi} & & \Big\| & & \\ 0 & \to & A_1' & \to & A_1 & \xrightarrow{\pi_1} & M & \to & 0 \end{array}$$

By the property (3) of Definition 3 and by Proposition 1, A_1' is ϱ-torsion-free. Therefore the homomorphism $\varphi: A \to A_1$ with $\varphi\pi_1 = \pi$ exists and it is in fact an epimorphism since A_1' is small in A_1. Now $\operatorname{Ker}\varphi \subseteqq A'$ as is easily seen and $A/_{\operatorname{Ker}\varphi} \cong A_1$ is ϱ-projective and $\operatorname{Ker}\varphi = 0$ by (3) of Definition 3.

THEOREM 2. *Let ϱ be an idempotent radical for ${}_R\mathcal{M}$, let $0 \to L \to N \to M \to 0$ be any cover of M with L ϱ-torsion-free, $0 \to K \to P \to M \to 0$ be a projective cover of M and $X = \varrho(K)$. Then*

(i) *there exists an epimorphism ${}^P/_X \to N$ over the identity on M,*

(ii) *any epimorphism $N \overset{\psi}{\to} {}^P/_X$ over the identity on M is necessarily an isomorphism.*

PROOF. (i) In the diagram

$$\begin{array}{ccccccccc} 0 & \to & {}^K/_X & \to & {}^P/_X & \overset{\pi}{\to} & M & \to & 0 \\ & & & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \to & L & \to & N & \overset{\sigma}{\to} & M & \to & 0 \end{array}$$

the homomorphism φ with $\varphi\sigma = \pi$ exists since ${}^P/_X$ is ϱ-projective by Proposition 1 and L is ϱ-torsion-free. Further, $\operatorname{Im}\varphi + L = N$ yields $\operatorname{Im}\varphi = N$, L being small in N.

(ii) Consider the following commutative diagram

$$\begin{array}{ccccccccc} & & & & U & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & L & \to & N & \to & M & \to & 0 \\ & & & & {\scriptstyle\psi}\downarrow\uparrow{\scriptstyle\varphi} & & \| & & \\ 0 & \to & {}^K/_X & \to & {}^P/_X & \to & M & \to & 0 \\ & & & & \uparrow & & & & \\ & & & & {}^H/_X & & & & \end{array}$$

where ψ is a given epimorphism, φ is the epimorphism constructed in the part (i), ${}^H/_X = \operatorname{Ker}\varphi$, $U = \operatorname{Ker}\psi$. Suppose $U \neq 0$. Then clearly $U \subseteqq L$, ${}^H/_X \subseteqq {}^K/_X$ and ${}^N/_U \cong {}^P/_X$, ${}^P/_H \cong {}^{P/_X}/_{H/_X} \cong N$. Under the last isomorphism U corresponds to ${}^G/_H$ and it is an easy exercise that $X \subseteqq H \subseteqq G \subseteqq K$. But then ${}^{P/_X}/_{G/_X} \cong {}^P/_G \cong {}^{P/_H}/_{G/_H} \cong {}^N/_U \cong {}^P/_X$ is ϱ-projective contradicting the definition of ϱ-projective cover (and the fact ${}^P/_X$ is a ϱ-projective cover of M).

3. Corational extensions

For a module M let us put $\varrho^M(N) = \sum_{f \in \operatorname{Hom}_R(M,N)} \operatorname{Im} f$. It is easily seen that ϱ^M is an idempotent preradical for ${}_R\mathcal{M}$. For a preradical ϱ we denote by $\bar{\varrho}$ the smallest radical containing ϱ (see. e.g. [10] for the construction and details). It can be easily checked that for a projective module P, ϱ^P is an idempotent cohereditary radical.

PROPOSITION 2. *Let A be a module having a projective cover $0 \to K \to P \to A \to 0$. Then a module M is ϱ^P-torsion-free iff its every factor is ϱ^A-torsion-free.*

PROOF. If M is ϱ^P-torsion-free then every factor of M is ϱ^A-torsion-free since ϱ^P is cohereditary and $\varrho^P \geqq \varrho^A$. Conversely, let every factor of M be ϱ^A-torsion-free and let $\varphi \in \mathrm{Hom}_R(P, M)$. Then φ induces $\bar{\varphi}: A \cong P/K \to \varphi(P)/\varphi(K) \subseteqq M/\varphi(K)$ and $\varphi(P) = \varphi(K)$, since $\bar{\varphi} = 0$ by hypothesis. Hence $\mathrm{Ker}\,\varphi + K = P$ and $\varphi = 0$, K being small in P.

Now we are in position to show that some of the results of § 2 in COURTER [5] are simple corollaries of the preceding theory. In our terminology (compare with Theorem 1.1 of [5]), the short exact sequence $0 \to L \to N \to M \to 0$ is a corational extension by M if every factor of L is ϱ^N-torsion-free, and it is a maximal corational extension by M if every corational extension $0 \to L' \to N' \to M \to 0$ satisfies the following conditions:

(i) There exists an epimorphism $\varphi: N \to N'$ over the identity on M,

(ii) every epimorphism $\varphi: N' \to N$ over the identity on M is necessarily an isomorphism.

LEMMA 2. *Let $0 \to K \to P \to M \to 0$ be a projective cover of M and $0 \to L \to N \to M \to 0$ a corational extension by M. Then*

(i) *L is small in N,*

(ii) $\varrho^P(L) = 0$.

PROOF. (i) (See [5], Theorem 2.3.) If $B \subsetneqq N$, $L + B = N$ then the composed homomorphism $N \to N/B \cong L/L \cap B$ is non-zero, contradicting the corationality.

(ii) Since $\varrho^N \geqq \varrho^M$, every factor of L is ϱ^M-torsion-free and it suffices to use Proposition 2.

THEOREM 3. (See [5], Theorem 2.12.) *Let $0 \to K \to P \to M \to 0$ be a projective cover of M and $X = \varrho^P(K)$. Then $0 \to K/X \to P/X \to M \to 0$ is a maximal corational extension by M.*

PROOF. Follows immediately from Proposition 2, Lemma 2, Theorem 2 and the simple fact that $0 \to X \to P \to P/X \to 0$ is a projective cover of P/X.

Now we are going to generalize Proposition (2F) from [4]. Recall that a module M is corationally complete (see [4]) if for every exact sequence $0 \to V \to N \to N/V \to 0$ such that every factor module of V is ϱ^M-torsion-free the induced homomorphic $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N/V)$ is onto.

THEOREM 4. *Let M be a module having a projective cover $0 \to K \to P \to M \to 0$. Then the following conditions are equivalent:*

(i) *M is corationally complete.*

(ii) *M has no proper corational extension.*

(iii) $\varrho^P(K) = K$.

(iv) *M is ϱ^P-projective.*

PROOF. (i)$\Rightarrow$(ii) Let M be corationally complete and let $0 \to L \to N \to M \to 0$ be a corational extension by M. By Lemma 2 $\varrho^P(L) = 0$ so that every factor of L is ϱ^M-torsion-free and hence $0 \to L \to N \to M \to 0$ splits. Therefore $L = 0$, L being small in N by Lemma 2.

(ii)$\Rightarrow$(i) $\varrho^P(K) = K$ since there is no proper corational extension by M and M is ϱ^P-projective by the proof of Theorem 1. Now it suffices to use Proposition 2.

(ii)$\Rightarrow$(iii) follows by Theorem 3, (iii)$\Rightarrow$(iv) by the proof of Theorem 1 and (iv)$\Rightarrow$(i) by Proposition 2.

THEOREM 5. *A direct sum of a finite number of corationally complete modules having projective covers is corationally complete.*

PROOF. It clearly suffices to consider two modules M_1, M_2 with projective covers $0 \to K_i \to P_i \to M_i \to 0$, $i=1, 2$. It is well-known (see [1]) that $0 \to K_1 \oplus K_2 \to P_1 \oplus P_2 \to M_1 \oplus M_2 \to 0$ is a projective cover of $M_1 \oplus M_2$. Further $\varrho^{M_1 \oplus M_2}(N) = \varrho^{M_1}(N) + \varrho^{M_2}(N)$ since $\mathrm{Hom}_R(M_1 \oplus M_2, N) \cong \mathrm{Hom}_R(M_1, N) \oplus \mathrm{Hom}_R(M_2, N)$. Now

$$\varrho^{P_1 \oplus P_2}(K_1 \oplus K_2) = \varrho^{P_1 \oplus P_2}(K_1) \oplus \varrho^{P_1 \oplus P_2}(K_2) = \big(\varrho^{P_1}(K_1) + \varrho^{P_2}(K_1)\big) \oplus$$
$$\oplus \big(\varrho^{P_1}(K_2) + \varrho^{P_2}(K_2)\big) = K_1 \oplus K_2$$

and it suffices to use Theorem 4.

REMARK. The converse does not hold in general, since for a projective generator P and every module M, $P \oplus M$ is corationally complete.

4. Pseudo-projective modules

PROPOSITION 3. *The following conditions for an idempotent preradical ϱ for ${}_R\mathcal{M}$ are equivalent:*

(i) *ϱ is a radical,*

(ii) $\varrho = \bar{\varrho}$,

(iii) *the class of ϱ-torsion modules is closed under extensions.*

PROOF. Obvious.

THEOREM 6. *Let A be a module having a projective cover $0 \to K \to P \to A \to 0$. Then the class of ϱ^A-torsion-free modules is closed under factors iff $\overline{\varrho^A} = \varrho^P$.*

PROOF. ϱ^A and $\overline{\varrho^A}$ clearly have the same classes of torsion-free modules so that it suffices to use Proposition 2.

DEFINITION 4. We shall call a module A pseudo-projective with respect to the epimorphism $\sigma\colon B \to C$ if for every $0 \neq f\colon A \to C$ there are $\varphi\colon A \to A$ and $\bar{f}\colon A \to B$ such that $0 \neq \varphi f = \bar{f}\sigma$. A is called pseudo-projective if it is pseudo-projective with respect to all epimorphisms.

THEOREM 7. *Let A be a module having a projective cover $0 \to K \to P \to A \to 0$. Then the following conditions are equivalent:*

(i) *A is pseudo-projective,*

(ii) *A is pseudo-projective with respect to all epimorphisms $P' \to C$ with C cocyclic and P' projective,*

(iii) *for every module B, every epimorphism $\sigma\colon B \to C$ with C cocyclic and every non-zero homomorphism $f\colon A \to C$ there exists $\bar{f}\colon A \to B$ such that $\bar{f}\sigma \neq 0$,*

(iv) $\varrho^A = \varrho^P$,

(v) *ϱ^A is a cohereditary radical,*

(vi) $\varrho^A(P) = P$,

(vii) *P is a homomorphic image of some direct power of A.*

PROOF. (i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(iii) Let $\sigma: B\to C$ be an epimorphism with C cocyclic and let $0\neq f: A\to C$ be arbitrary. Let $\pi: P'\to C$ be any projective presentation of C and consider the following diagram

$$\begin{array}{ccccc} & \overset{\varrho}{\dashleftarrow} & P' & \overset{\bar f}{\dashleftarrow} & A \\ \downarrow & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow{\sigma} & C & \xleftarrow{f} & A \end{array}$$

The homorphism $\varrho: P'\to B$ with $\varrho\sigma=\pi$ exists by the projectivity of P'. By hypothesis there exist $\varphi: A\to A$ and $\bar f: A\to P'$ such that $0\neq\varphi f=\bar f\pi$. Hence $\bar f\varrho\sigma=\bar f\pi\neq 0$ and we are through.

(iii)$\Rightarrow$(iv) Since $\varrho^A\leqq\varrho^P$, we can suppose $\varrho^A(M)=M_1\subsetneqq M_2=\varrho^P(M)$ for some module M. Then M/M_1 has a factor which is not ϱ^A-torsion-free, for otherwise it would be ϱ^P-torsion-free by Proposition 2, contradicting $\varrho^P(M_2/M_1)=M_2/M_1\neq 0$. Hence there is $M_1\subsetneqq M_3$ with $\varrho^A(M/M_3)=M_4/M_3\neq 0$. Taking $x\in M_4\dot{-}M_3$ arbitrarily we choose N maximal with respect to $x\notin N$, $M_3\subseteqq N$. Then we have an epimorphism $\sigma: M_4\to M_4/N$ with M_4/N cocyclic (as easily seen) and there is $0\neq f: A\to M_4/N$ since M_4/N is ϱ^P-torsion. By (iii) there is $\bar f: A\to M_4$, $\bar f\sigma\neq 0$, contradicting the fact $\varrho^A(M)=M_1$.

(iv)$\Rightarrow$(v) is trivial.

(v)$\Rightarrow$(vi) Suppose $\varrho^A(P)=P'\subsetneqq P$. The natural map $A\cong P/K\to P/(K+P')$ must be zero, since $\varrho^A(P/(K+P'))=0$. Hence $K+P'=P$ yields a contradiction with the fact K is small in P.

(vi)$\Rightarrow$(vii) By the well-known properties of direct sums there exists a (unique) homomorphism $\psi: \dot{\sum}_{f\in\mathrm{Hom}_R(A,P)} A_f\to P$ with $\iota_f\psi=f$ where $A_f\cong A$ and $\iota_f: A_f\to \dot{\sum}_{f\in\mathrm{Hom}_R(A,P)} A_f$ is the canonical embedding. ψ is an epimorphism since $\varrho^A(P)=P$.

(vii)$\Rightarrow$(i) Consider the diagram

$$\begin{array}{ccccccccccc} B & & & \xleftarrow{\quad\eta\quad} & & & & & & & \\ \downarrow{\scriptstyle\sigma} & & & & \uparrow & & & & & & \\ C & \xleftarrow{f} & A & \xleftarrow{\pi} & P & \xleftarrow{\psi} & \dot{\sum}_{\lambda\in\Lambda} A_\lambda & \xleftarrow{\iota_\lambda} & A_\lambda & \overset{\varrho}{\cong} & A \end{array}$$

where $\sigma: B\to C$ is an arbitrary epimorphism, $0\neq f: A\to C$ is an arbitrary homomorphism, $0\to K\to P\xrightarrow{\pi} A\to 0$ is the projective cover of A, ψ is an epimorphism which exist by (vii) and ι_λ is the canonical embedding. Since $f\neq 0$, there exists $\lambda\in\Lambda$ with $\iota_\lambda\psi\pi f\neq 0$. The homomorphism $\eta: P\to B$ with $\eta\sigma=\pi f$ exists by projectivity of P. Now the homomorphism $\varphi=\varrho\iota_\lambda\psi\pi: A\to A$ and $\bar f=\varrho\iota_\lambda\psi\eta: A\to B$ satisfies $\varphi f=\varrho\iota_\lambda\psi\pi f=\varrho\iota_\lambda\psi\eta\sigma=\bar f\sigma\neq 0$ and Theorem 7 is therefore proved.

REMARK. Note that the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) hold without the assumption that A has a projective cover.

Recall that a module A is a generator for ${}_R\mathcal{M}$ (in the categorical sense) if every module is ϱ^A-torsion.

PROPOSITION 4. *A pseudo-projective module A such that* $\mathrm{Hom}_R(A, C)\neq 0$ *for every cocyclic module C, is a generator for* ${}_R\mathcal{M}$.

PROOF. Suppose on the contrary $\varrho^A(M)=M_1 \subsetneqq M$ for some module M and for arbitrary $x \in M \dot{-} M_1$ choose $N \subseteqq M$ maximal with respect to $M_1 \subseteqq N$, $x \notin N$. Then M/N is cocyclic and our hypothesis together with the condition (iii) of the above theorem gives a homomorphism $\bar{f}\colon A \to M$, $\operatorname{Im} \bar{f} \nsubseteq N$, which contradicts $\varrho^A(M)= =M_1 \subsetneqq N$.

THEOREM 8. *Let A, B be modules having projective covers $0 \to K \to P \to A \to 0$, $0 \to K' \to P' \to B \to 0$, respectively and let A be pseudo-projective. If $\varrho^A(B)=B$, then $A \oplus B$ is pseudo-projective. Conversely, if $\varrho^A(P') \supseteqq \varrho^B(P')$ and $A \oplus B$ is pseudo-projective then $\varrho^A(B)=B$.*

PROOF. As was remarked in the proof of Theorem 5, $0 \to K \oplus K' \to P \oplus P' \to \to A \oplus B \to 0$ is a projective cover of $A \oplus B$ and $\varrho^{M_1 \oplus M_2}(N)=\varrho^{M_1}(N)+\varrho^{M_2}(N)$. Hence

$$\varrho^{A \oplus B}(P \oplus P') = \big(\varrho^A(P)+\varrho^B(P)\big) \oplus \big(\varrho^A(P')+\varrho^B(P')\big).$$

Now $\varrho^A(P) = P$ by Theorem 7 (vi) and for $\varrho^A(P') = P'' \subsetneqq P'$ we have $P'/(P''+K')$ ϱ^A-torsion as a factor-module of B and ϱ^A-torsion-free as a factor-module of P'/P''. This contradiction shows that $\varrho^A(P')=P'$ and the direct part follows by Theorem 7 (vi).

Conversely,

$$P \oplus P' = \varrho^{A \oplus B}(P \oplus P') = P \oplus \varrho^A(P'),$$

whence $\varrho^A(P')=P'$ and $\varrho^A(B)=B$ since B is an epimorphic image of P'.

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FACULTY OF MATHEMATICS AND PHYSICS
KARLOVY UNIVERSITY
18600 SOKOLOWSKA' 83
PRAHA 8—KARLÍN
ČSSR

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EINIGE GRUPPENTHEORETISCHE ÄQUIVALENTE ZUM AUSWAHLAXIOM

Von

U. FELGNER (Heidelberg)

E. Steinitz, O. Teichmüller und andere haben überzeugend gezeigt, daß das Auswahlaxiom der Mengenlehre notwendig beim Aufbau vieler Gebiete der Algebra ist. Im Laufe der Zeit gelang es nachzuweisen, daß einige Aussagen aus der Algebra nicht nur aus dem Auswahl-Axiom folgen, sondern auch umgekehrt dieses implizieren (vergl. H. RUBIN—J. E. RUBIN [11] p. 37—46). Das vielleicht schönste Beispiel dafür ist die Aussage „*Jede Gruppe besitzt maximale abelsche Untergruppen*". Die Äquivalenz dieser Aussage mit dem Auswahlaxiom wurde von G. KLIMOVSKY [7] in einem sehr umfangreichen Beweis nachgewiesen. Ich möchte hier einen neuen sehr viel kürzeren Äquivalenz-Beweis geben, der zugleich zwei allgemeinere Ergebnisse gestattet. Es sollen hier die folgenden beiden Sätze bewiesen werden:

SATZ 1. *Sei $k \geqq 1$ eine beliebige natürliche Zahl. Dann ist die Aussage* (MN_k) *„Jede Gruppe besitzt maximale nil-k-Untergruppen" mit dem Auswahlaxiom* (AC) *äquivalent.*

SATZ 2. *Sei $k \geqq 1$ eine beliebige natürliche Zahl. Dann ist die Aussage* (MA_k) *„Jede Gruppe besitzt maximale Untergruppen der Auflösbarkeits-Stufe $\leqq k$" mit dem Auswahlaxiom* (AC) *äquivalent.*

Sowohl aus Satz 1 als auch aus Satz 2 folgt unmittelbar das folgende Korollar, indem man $k=1$ setzt:

KOROLLAR 1. (G. KLIMOVSKY [7]). *Das Auswahlaxiom ist mit der Aussage „Jede Gruppe besitzt maximale abelsche Untergruppen" äquivalent.*

KOROLLAR 2. *Das Auswahlaxiom ist mit der Aussage „Jede Gruppe besitzt maximale metabelsche Untergruppen" äquivalent.*

Es sei betont, daß sich in den beiden Sätzen und in den beiden Korollaren die Maximalität auf die Inklusionsrelation $\subseteq$ bezieht; die Untergruppe H einer Gruppe G, welche maximal in Bezug auf die Eigenschaft $\mathfrak{G}$ ist, kann also gleich der ganzen Gruppe G sein. In einer partiell geordneten Menge $\langle M, \leqq \rangle$ wird ein Element $x \in M$ *maximal* genannt, wenn es kein $y \in M$ mit $x \leqq y$ und $x \neq y$ gibt. H ist eine maximale nil-k Untergruppe von G, wenn H ein maximales Element von $\langle \{A, A \leqq G \;\&\; A$ hat nil-$k\}, \subseteq \rangle$ ist. Dabei bedeutet $A \leqq G$, daß A eine Untergruppe von G ist, $A \subseteq G$.

§ 1. Gruppentheoretische Vorbemerkung

Bevor wir mit dem Beweis der beiden Sätze beginnen, erinnern wir an die folgenden Definitionen und Ergebnisse aus der Gruppentheorie.

(i) Wenn x und y Elemente der Gruppe G sind, dann sei $(x, y)=x^{-1}y^{-1}xy$ der Kommutator von x und y. Wir setzen $(x_1, x_2, \ldots, x_{n-1}, x_n)=((x_1, \ldots, x_{n-1}), x_n)$. Eine Gruppe G hat die Eigenschaft *nil-k*, wenn $(x_1, x_2, \ldots, x_k, x_{k+1})=1$ für alle $x_i \in G$ gilt $(1 \leqq i \leqq k+1)$. G hat also genau dann die Eigenschaft nil-k, wenn G nilpotent der Klasse m ist für $m \leqq k$ (vergl. M. HALL [5] p. 153). Eine Gruppe G hat offenbar genau dann die Eigenschaft nil-1, wenn G abelsch ist. Eine Gruppe G heißt *metabelsch*, wenn $((u, v), (x, y))=1$ für alle $u, v, x, y \in G$ gilt. Für eine Gruppe G sei G' die Kommutator-Untergruppe von G, $G''=(G')'$, und allgemein $G^{(n+1)}=(G^{(n)})'$. Für eine natürliche Zahl $k \geqq 1$ wird definiert: G hat die *Auflösbarkeits-Stufe k* genau dann wenn $G^{(k)}=\{1\}$ und $G^{(k-1)} \neq \{1\}$. Die metabelschen Gruppen sind also gerade die Gruppen der Auflösbarkeits-Stufe $\leqq 2$.

(ii) Für eine Menge $\{G_i, i \in I\}$ von Gruppen G_i wird die Menge D aller Funktionen g von I in $\bigcup \{G_i; i \in I\}$ so daß $g(i) \in G_i$ für alle $i \in I$ und $\{j \in I; g(j) \neq 1\}$ endlich ist, das *schwache direkte Produkt* der Gruppen G_i genannt („direct product“ in M. HALL [5] p. 33).

(iii) Wenn S eine beliebige Menge ist, dann sei $F=F_S$ die von S erzeugte freie Gruppe (cf. M. HALL [5] p. 91—92, W. MAGNUS—A. KARRAS—D. SOLITAR [9] p. 4—19). S ist ein freies Erzeugenden-System von F_S. Der Satz von Schreier, daß Untergruppen freier Gruppen frei sind, ist nur unter Zuhilfenahme des Auswahl-Axiomes beweisbar (cf. H. LÄUCHLI [8], JECH-SOCHOR [6]). Der schwächere Satz von Nielsen (1921), daß endlich-erzeugte Untergruppen freier Gruppen frei sind, ist hingegen ohne Zuhilfenahme des Auswahl-Axiomes beweisbar (cf. M. HALL [5] p. 106—109).

(iv) Sei $F=F_S$ eine freie Gruppe mit dem freien Erzeugenden-System S und $w=x_1^{\eta_1}x_2^{\eta_2} \ldots x_n^{\eta_n}$ $(x_i \in S, \eta_i=\pm 1)$ ein Wort über dem Alphabet S. Dann ist w in der Gruppe F einem *reduzierten Wort* („freely reduced word“ in [9] p. 33—34) gleich. Es gibt einen Algorithmus der w in endlich vielen Schritten in dieses eindeutig bestimmte reduzierte Wort, das wir mit $\varrho(w)$ bezeichnen, überführt, $w \approx \varrho(w)$ (siehe [5] p. 91, [9] p. 34—35).

(v) In einer freien Gruppe F_S sind zwei Elemente u und v dann und nur dann vertauschbar, wenn sie in einer zyklischen Untergruppe von F_S liegen. Dies ist ohne Auswahl-Axiom beweisbar (siehe MAGNUS—KARRAS—SOLITAR [9] p. 42, problem 6).

(vi) Den folgenden Satz hat G. BAUMSLAG in [2] p. 285—286 bewiesen: *Falls $n \geqq 2$ eine natürliche Zahl ist und falls a, b, c Elemente einer freien Gruppe $F=F_S$ sind, welche durch die Beziehung $(a, b)=c^n$ miteinander verbunden sind, dann gilt $a \cdot b=b \cdot a$ und folglich ist $c=1$ und a und b liegen in einer zyklischen Untergruppe von F.* Der Beweis dieses Satzes verwendet nicht das Auswahlaxiom.

(vii) Von A. I. MALZEW [10] stammt der folgende Satz: *Sei $\{a, b\}=S$ ein freies Erzeugenden-System der freien Gruppe $F=F_S$ und seien u und v Elemente von F, welche der Gleichung $u^{-1}v^{-1}uv=a^{-1}b^{-1}ab$ genügen. Dann existiert eine ganze Zahl m so, daß entweder $u=b^m a$ und $v=b$ oder $u=a$ und $a=a^m b$ gilt.* Der Beweis dieses Satzes verwendet nicht das Auswahl-Axiom.

(viii) Wir machen noch die folgende Bemerkung mengentheoretischer Natur. Mit ZF bezeichnen wir das Axiomensystem der Zermelo—Fraenkelschen Axiome der Mengenlehre. Mit (AC) bezeichnen wir das übliche Auswahlaxiom: Zu jeder Menge s von nicht-leeren Mengen existiert eine Funktion f so daß $f(x)\in x$ für alle $x\in s$. Das Auswahlaxiom (AC) kommt unter den ZF-Axiomen nicht vor. Wir betrachten noch das folgende multiple Auswahlaxiom:

(MC_2) *Zu jeder Menge s von nicht-leeren Mengen existiert eine Funktion f so, daß $\emptyset \neq f(x)\subseteq x$ und* Card $(f(x))\leqq 2$ *für alle $x\in s$.*

Dabei ist Card(y) die Kardinalität (oder Mächtigkeit) von y. A. Lévy hat gezeigt, daß (AC) und (MC_2) äquivalent sind (siehe H. Rubin—J. E. Rubin [11] p. 5—8). Das Fundierungsaxiom wird in dem Äquivalenzbeweis (AC)$\Leftrightarrow$(MC_2) nicht benötigt.

§ 2. Beweis von Satz 1

Es ist klar, daß aus dem Auswahlaxiom für jede natürliche Zahl $k\geqq 1$ die Existenz maximaler nil-k-Untergruppen in jeder Gruppe folgt. Wir müssen zeigen, daß umgekehrt für jede gegebene Zahl $k\geqq 1$, (MN_k)$\Rightarrow$(MC_2) gilt.

Sei also $k\geqq 1$ vorgegeben und sei S eine Menge von nicht-leeren Mengen. Für jedes Element $t\in S$ sei F_t die freie Gruppe, welche von der Menge t frei erzeugt wird, und sei G das schwache direkte Produkt der Gruppen F_t (für $t\in S$). Aus (MN_k) folgt, daß G eine maximale nil-k Untergruppe H besitzt. Sei H_t die Projektion von H auf die t-Koordinate:

$$H_t = \{g\in H;\quad \forall u\in S[t\neq u \Rightarrow g(u) = 1]\}.$$

Falls $H_t=\{1\}$ für ein $t\in S$, dann sei $z\in t$ (also $z\neq 1$) und $\langle z\rangle$ die von z erzeugte zyklische Untergruppe von F_t. Dann ist die direkte Summe $H\oplus\langle z\rangle$ eine nil-k Untergruppe von G im Widerspruch zur Maximalität von H. Also gilt $H_t\neq\{1\}$ für alle $t\in S$. Nach dem Satz von Nielsen (siehe § 1, (iii)) ist jede Gruppe H_t lokal-frei (d. h. endlich erzeugte Untergruppen von H_t sind frei). Wir wollen zeigen, daß H_t abelsch ist.

Angenommen es gäbe ein $t\in S$ so, daß H_t nicht abelsch ist. Dann gibt es $a\in H_t$ und $b\in H_t$ mit $ab\neq ba$. Sei A die von $\{a, b\}$ erzeugte Untergruppe von H_t. A ist eine freie Gruppe vom Rang 2. Setze $x_1=a$, $x_2=b$, $x_3=x_4=\ldots=x_{k+1}=a^2$. Weil H_t eine nil-k Gruppe ist, gilt

$$(a, b, a^2, \ldots, a^2) = (x_1, x_2, \ldots, x_{k+1}) = 1.$$

Die Elemente $(x_1, x_2, \ldots, x_k)$ und x_{k+1} sind also vertauschbar und nach § 1 (v) gibt es $w\in A$ und ganze Zahlen d_1 und d_2 mit $d_1\geqq 0$ so daß $(x_1, x_2, \ldots, x_k)=w^{d_1}$ und $x_{k+1}=w^{d_2}$. Nach dem Satz von Baumslag (§ 1, (vi)) gilt im Falle $d_1\geqq 2$:

$$(x_1, x_2, \ldots, x_k) = 1.$$

Im Falle $d_1=0$ folgt ebenso $(x_1, x_2, \ldots, x_k)=1$. Der Fall $d_1=1$ kann nicht auftreten, denn sonst wäre $(x_1, x_2, \ldots, x_k)^{d_2}=w^{d_2}=x_{k+1}=a^2$. Aber $(x_1, x_2, \ldots, x_k)^{d_2}$ liegt in der Kommutator-Untergruppe von A, während a^2 nicht in A' liegt (siehe W. Magnus—A. Karras—D. Solitar [9] p. 79). A hat also sogar die Eigenschaft nil-$(k-1)$.

Durch Induktion folgt $(a, b)=1$, ein Widerspruch. Es folgt, daß alle Gruppen H_t abelsch sind.

Nach § 1 (v) ist jede Gruppe H_t (für $t \in S$) lokal-zyklisch. Jedes Element $x \in H_t$ ist ein Wort über dem Alphabet t. Für $x \in H_t$ sei $\varphi(x)$ das erste und $\lambda(x)$ das letzte Symbol aus t, das in dem reduzierten Wort $\varrho(x)$ vorkommt. Für $\eta_1 = \pm 1$ und $\eta_2 = \pm 1$ und ein reduziertes Wort r, das nicht mit $\varphi(x)^{-\eta_1}$ anfängt und nicht auf $\lambda(x)^{-\eta_2}$ endet, gilt also $\varrho(x)=\varphi(x)^{\eta_1} r \lambda(x)^{\eta_2}$. Jedem Element $x \in H_t$ ordnen wir die Menge $C_t(x)=\{\varphi(x), \lambda(x)\}$ zu. Weil H_t lokal-zyklisch ist, gilt $C_t(x)=C_t(y)$ für alle $x \in H_t$ und alle $y \in H_t$ mit $x \neq 1$, $y \neq 1$. Damit haben wir jeder Menge $t \in S$ eindeutig eine nicht-leere höchstens zwei-Elementige Menge $C_t(x)$ (wobei $1 \neq x \in H_t$ beliebig ist) zugeordnet. Aus (MN_k) folgt also (MC_2) und nach A. Lévy folgt daraus das Auswahlaxiom, Q.E.D.

§ 3. Beweis von Satz 2

Wir verfahren wie im Beweis von Satz 1 und bilden zur vorgegebenen Menge S von nicht-leeren Mengen das schwache direkte Produkt G der freien Gruppen F_t für $t \in S$. Sei H eine Untergruppe von G, die maximal ist in Bezug auf die Eigenschaft, daß ihre Auflösbarkeits-Stufe $\leqq k$ ist. Sei H_t die Projektion von H auf die t-Koordinate, $t \in S$. Es folgt $H_t \neq \{1\}$ für alle $t \in S$. Wir müssen zeigen, daß die Gruppen H_t abelsch sind.

Angenommen es gibt ein $t \in S$ so, daß H_t nicht abelsch ist. Dann gibt es $a, b \in H_t$ mit $ab \neq ba$. Sei A die von $\{a, b\}$ erzeugte Untegruppe von H_t. Weil H_t lokal-frei ist, ist A eine freie Gruppe. mit dem freien Erzeugenden-System $\{a, b\}$ (siehe M. HALL [5] p. 109, Theorem 7.3.3, W. MAGNUS—A. KARRAS—D. SOLITAR [9] p. 110, Corollary 2.13.1).

Wir definieren wie üblich die folgenden höheren Kommutatoren: $s_1(x, y) = x^{-1}y^{-1}xy = (x, y)$, $s_2(x, y, u, v) = \big(s_1(x, y), s_1(u, v)\big)$, $s_{n+1}(\vec{x}, \vec{y}) = \big(s_n(\vec{x}), s_n(\vec{y})\big)$, wobei $\vec{x}$ und $\vec{y}$ Folgen von 2^n paarweise verschiedenen Variablen sind. G ist auflösbar der Stufe $\leqq k$ genau dann wenn $s_k(\vec{g})=1$ für alle 2^k-Tupel $\vec{g}$ von Elementen aus G gilt.

Mit H ist auch A eine k-stufig auflösbare Gruppe und dasselbe gilt von der Kommutatorgruppe A'. Aber A' ist eine freie Gruppe unendlichen Ranges (cf. [9] p. 112 (2), p. 114 (13)) — diese Behauptung folgt ohne Verwendung des Auswahlaxiomes, da A abzählbar, also wohlordenbar ist. Sei T ein freies Erzeugenden-System von A' und $\vec{g}$ eine Folge von 2^k vielen paarweise verschiedenen Elementen aus T. Dann kann aber $s_k(\vec{g})=1$ nicht gelten, denn $s_k(\vec{g})$ ist bereits ein reduziertes Wort, welches vom leeren Wort verschieden ist: $s_k(\vec{g})=\varrho\big(s_k(\vec{g})\big) \neq \emptyset$ (hier fassen wir $s_k(\vec{g})$ als Wort über dem Alphabet T auf). Dieser Widerspruch zeigt, daß A nicht den Rang 2 haben kann. Es gilt also $ab=ba$ für alle $a, b \in H_t$ und H_t ist lokal-zyklisch. Der Rest des Beweises folgt jetzt wörtlich dem Beweis von Satz 1, indem man jeder Menge $t \in S$ die zwei-elementige Menge $C_t(x)$ zuordnet, Q.E.D.

BEMERKUNG. Korollar 2 folgt sofort aus Satz 2, indem man $k=2$ setzt. Korollar 2 könnte man auch dadurch beweisen, indem man zunächst wie im Beweis von Satz 1 verfährt und dann die Kommutativität der Gruppen H_t unter Verwendung der Sätze von Malzew (§ 1, (vii)) und Baumslag (§ 1, (vi)) beweist (für zwei Elemente $a \in H_t$ und $b \in H_t$ mit $ab \neq ba$ betrachtet man $((a, b), (a^2, b^2))=1$). Wir bemerken schließlich noch, daß man im Beweis von Satz 1 die Kommutativität der Gruppen H_t auch ohne Verwendung des Satzes von Baumslag nachweisen kann, indem man wie im Beweis von Satz 2 die Tatsache ausnutzt, daß die Kommutator-Untergruppe A' von A unendlichen Rang hat und daß zwischen den freien Erzeugenden $g_1, g_2, \dots, g_n, \dots$ $(n \in \omega)$ von A' die Relation $(g_1, g_2, \dots, g_{k+1}) = 1$ nicht gelten kann, weil $(g_1, g_2, \dots$ $\dots, g_{k+1})$ bereits ein reduziertes Wort in den freien Erzeugenden von A' ist, welches nicht leer ist, also von 1 verschieden ist.

§ 4. Bemerkungen und Probleme

Es ist bekannt, daß die Äquivalenz mancher Aussagen der Mengenlehre mit dem Auswahlaxiom (AC) nur unter Verwendung des Fundierungsaxiomes $\forall x[x \neq \emptyset \Rightarrow \Rightarrow \exists y \in x[x \cap y = \emptyset]]$ beweisbar ist (siehe etwa H. Rubin—J. E. Rubin [11], Felgner [3] p. 61—67 und Felgner—Jech [4]). Sei ZF^0 das Axiomen-System bestehend aus dem Null-Mengenaxiom, Axiom der Bestimmtheit, Paarmengenaxiom, Summenaxiom, Unendlichkeitsaxiom, Potenzmengenaxiom und dem Schema der Ersetzungsaxiome (cf. [3] p. 10—11). Das System ZF aller Zermelo—Fraenkelschen Axiome entsteht aus ZF^0 durch Hinzunahme des Fundierungsaxiomes. Die Beweise der Sätze 1 und 2 zeigen, daß die Äquivalenzen (AC)$\Leftrightarrow$(MN_k) und (AC)$\Leftrightarrow$(MA_k) im System ZF^0 gültig sind.

Es ist bemerkenswert, daß es auch in der Algebra Aussagen gibt, die im System ZF^0 nicht mit (AC) äquivalent sind, im System ZF hingegen mit (AC) äquivalent sind. In Verallgemeinerung des Axiomes (MC_2) betrachten wir:

(MC) Multiples Auswahlaxiom: *Zu jeder Menge S von nicht-leeren Mengen existiert eine Funktion f so, daß für alle $x \in S$, $f(x)$ eine nicht-leere endliche Teilmenge von x ist.*

Es gilt in ZF: (AC)$\Leftrightarrow$(MC), aber (MC)$\Rightarrow$(AC) ist kein beweisbarer Satz von ZF^0 (siehe [4]). M. K. Armbrust [1] hat gezeigt, daß die Aussage:

(TDS) *Jede torsionsfreie teilbare Untergruppe H einer abelschen Gruppe G ist ein direkter Summand von G,*

mit dem multiplen Auswahlaxiom äquivalent ist: $ZF^0 \vdash$ (MC)$\Leftrightarrow$(TDS). Daher folgt: Im System ZF sind die Aussagen (TDS) und das Auswahlaxiom (AC) äquivalent, im System ZF^0 hingegen sind (AC) und (TDS) nicht äquivalent! Daraus folgt eine partielle Lösung des Problems von Armbrust [1]: Im System ZF sind die Aussagen (AC), (MC), (TDS) und (DS): *„Jede teilbare Untergruppe einer abelschen Gruppe G ist ein direkter Summand von G"* alle untereinander äquivalent. In ZF^0 gilt (AC)$\Rightarrow$ $\Rightarrow$(DS)$\Rightarrow$(TDS)$\Leftrightarrow$(MC), aber es ist unbekannt, welcher der Implikationspfeile im System ZF^0 umkehrbar ist.

Es gibt viele Sätze der Gruppentheorie, die nur unter Zuhilfenahme des Auswahlaxiomes beweisbar sind. Um die mengentheoretische Stärke dieser Sätze übersehen zu können, wäre es interessant zu wissen, welche von ihnen mit dem Auswahlaxiom äquivalent sind. Es ist bisher immer noch unbekannt, ob der Satz von Schreier, daß Untergruppen freier Gruppen frei sind, mit (AC) äquivalent ist. Ist die

Aussage, daß die Frattini-Untergruppe $\mathrm{Fr}(G)$ einer Gruppe G gerade die Menge der Nicht-Generatoren ist, mit (AC) äquivalent? Die Klärung von Fragen dieser Art gibt Aufschluß über die mengentheoretische Stärke gewisser gruppentheoretischer Konstruktionen.

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MATHEMATISCHES INSTITUT
DER UNIVERSITÄT
69 HEIDELBERG
BUNDESREPUBLIK DEUTSCHLAND

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SOME COMMUTATIVITY THEOREMS FOR RINGS AND NEAR RINGS

By

S. LIGH (Lafayette) and J. LUH (Raleigh)

1. Introduction

Many generalizations of the famous Wedderburn's Theorem which states that a finite division ring is commutative have appeared recently and in various directions. In [12] the first author obtained the following generalization of a theorem of OUTCALT and YAQUB [15]:

THEOREM A. *Let R be a ring with a left identity e and the set N of nilpotent elements is an ideal. If* (i) *R/N is finite and* (ii) $x\equiv y$ (mod N) *implied that $x^2=y^2$ or both x and y commute with all elements of N, then R is commutative.*

One of the purposes of this paper is to generalize Theorem A further and furnish an example to settle a conjecture in [12].

Several generalizations of some of the commutativity theorems for rings to near rings have appeared recently. The following result [10] is a generalization of a theorem of BELL [2].

THEOREM B. *A d.g. near ring R is commutative if and only if for each x, y in R, there is an integer $n=n(x, y)>1$, such that $(xy-yx)^n=xy-yx$.*

In this paper we shall extend Theorem B to a larger class of near rings and examine some other commutativity theorems for near rings.

For various definitions and elementary facts about near rings, see [2].

2. A commutativity theorem for rings

It was conjectured in [12] whether or not one can drop the assumption that R has a left identity in Theorem A. The following example settles the conjecture negatively.

EXAMPLE. Let F be a field of characteristic $\neq 2$, and $S=\{(a, b, c)|a, b, c\in F\}$. Define addition and multiplication in S by $(a, b, c)+(a', b', c')=(a+a', b+b', c+c')$ and $(a, b, c)\cdot(a', b', c')=(0, 0, ab'-ba')$ for all (a, b, c), (a', b', c') in S. Then clearly S is a noncommutative ring without one sided identities, $s^2=0$ for every $s\in S$, and $S^3=(0)$.

Now let D be an arbitrary finite field and $R=S\oplus D$, the direct sum of S and D. Then the set N of nilpotent elements in R is an ideal of R isomorphic to S and $R/N\cong D$ is a finite field. Also it is easy to see that condition (ii) in Theorem A holds in R. However, R is not commutative.

We now further generalize Theorem A as follows.

THEOREM 2.1. *Let R be a ring with a left identitiy e and N be the set of nilpotent elements in R. If*

(i) *for each* $x \in R$, *there exists an integer* $n(x) > 1$, *such that* $x^{n(x)} - x \in N$,
(ii) $x \equiv y \pmod N$ *implies that* $x^2 = y^2$ *or both* x *and* y *commute with all elements of* N, *and*
(iii) *for* $a \in N$ *and* $r \in R$, $ar \in N$,
then R *is a commutative ring.*

PROOF. We shall show first that N is an ideal of R. Let $a \in N$ and $r \in R$. By (iii), $(ar)^m = 0$ for some positive integer m, so $(ra)^{m+1} = r(ar)^m a = 0$. Hence $ra \in N$. Now assume $a, b \in N$. By (ii), since $a \equiv 0 \pmod N$ either $a^2 = 0$ or a commutes with every element of N. Likewise, since $b \equiv 0 \pmod N$ ekiher $b^2 = 0$ or b commutes with every element of N. Thus either $a^2 = b^2 = 0$ or a or b commutes with every element of N. In the latter case, since $ab = ba$, it follows immediately that $a - b \in N$. In the case that $a^2 = b^2 = 0$, since ab and ba both are in N by (iii), there exists a positive integer m, such that $(ab)^m = (ba)^m = 0$. Consequently, $(a-b)^{2m} = ((a-b)^2)^m = (-ab-ba)^m = (-ab)^m + (-ba)^m = 0$, so $a - b \in N$. Therefore, N is an ideal of R.

By (i) R/N is a commutative ring [5, Theorem 1, p. 217] and every finitely generated subring of R/N is isomorphic to a finite direct sum of finite fields [16, Theorem 3.4]. Using the similar proof of Theorem 2 in [12], we can show that e is a unique left identity of R, and hence e is a two-sided identity of R.

Next, we shall show that N is a commutative subring of R. Let $a, b \in N$. Suppose $ab \neq ba$. Since $a, b, a+b$ are elements in N which do not commute with every element in N, by (ii), $a^2 = b^2 = (a+b)^2 = 0$. This implies $ab + ba = 0$. On the other hand, since $(a+e) \equiv e \pmod N$, and $a+e$ does not commute with every element in N, we get $(a+e)^2 = e^2 = e$, so $2a = 0$. Thus $2ba = 0$ and $ab = -ba = ba$, a contradiction.

Finally, we shall show that N is in the centre of R and then applying a well known result of HERSTEIN [5, Theorem 2, p. 221] we obtain that R is commutative.

Suppose that N is not contained in the centre of R. Then there exist $a \in N$ and $b \in R$ such that $ab \neq ba$. Let S be the subring of R generated by e, b and N. Since S/N is a finitely generated subring of R/N, $S/N = S_1/N \oplus S_2/N \oplus \ldots \oplus S_j/N$ where S_i is an ideal of S and S_i/N is a finite field $GF(p_i^{k_i})$ of characteristic p_i. Let $b + N = (b_1 + N) + (b_2 + N) + \ldots + (b_j + N)$ where $b_i \in R_i$. Then $ab_i \neq b_i a$ for some i. But $(a + b_i) \equiv b_i \pmod N$, by (ii) $(a+b_i)^2 = b_i^2$. Consequently $0 = (a+b_i)(a+b_i)^2 - (a+b_i)^2(a+b_i) = (a+b_i)b_i^2 - b_i^2(a+b_i)$, and hence $ab_i^2 = b_i^2 a$. Likewise, $a(b_i+e)^2 = (b_i+e)^2 a$. Thus, $2(ab_i - b_i a) = 0$. We have the following two cases:

Case 1. $p_i \neq 2$. Since $p_i(ab_i - b_i a) = a(p_i b_i) - (p_i b_i)a = a \cdot 0 - 0 \cdot a = 0$ and since 2 and p_i are relatively prime, we obtain $ab_i - b_i a = 0$, a contradiction.

Case 2. $p_i = 2$. Since S_i/N is a field of order 2^{k_i}, $b_i^{2^{k_i}} - b_i \in N$. It follows that $a(b_i^{2^{k_i}} - b_i) = (b_i^{2^{k_i}} - b_i)a$. On the other hand, since $ab_i^2 = b_i^2 a$, $ab_i^{2^{k_i}} = b_i^{2^{k_i}} a$ and hence $ab_i = b_i a$, again a contradiction. This completes the proof of the theorem.

3. Commutativity theorems for near rings

The study of commutativity for near rings began in [6] and BELL proved the following two theorems in [2].

THEOREM C. *Let* R *be a d.g. near ring with* 1 *and for each* x, y *in* R, *there exists an integer* $n = n(x, y) > 1$, *such that* $(xy - yx)^n = xy - yx$. *Then* R *is a commutative ring.*

THEOREM D. *Let R be a d.g. near ring with* 1 *and for each x in R, there exists an integer* $n=n(x)>1$, *such that* x^n-x *is in the centre of R. Then R is a commutative ring.*

In [10] it was shown that the existence of an identitiy is not required for Theorem C. However, whether an identity is needed for Theorem D is an open problem. In this section we further generalize Theorem B to a larger class of near rings. To facilitate our discussion we begin with the following definition.

DEFINITION. A near ring R is called a D-near ring if every non-zero homomorphic image T of R satisfies the following two conditions:

(i) T has a non-zero right distributive element.

(ii) That $(T, +)$ is abelian implies that $(T, +, \cdot)$ is a ring.

Examples of D-near rings:

(1) All rings.

(2) All d.g. near rings.

(3) The example given in [3, 2.5, # 6].

REMARK. It was shown in [7] that if a d.g. near ring has a unique left identity, then it is also a right identity. The example (3) above has a unique left identity, yet it is not a right identity. Thus we see that the class of D-near rings is larger than the class of d.g. near rings.

The following results are extensions of previously known results.

THEOREM 3.1. *Let R be a finite D-near ring with no zero divisors. Then it is a field.*

THEOREM 3.2. *Let R be a finite D-near ring with no nilpotent elements. Then R is a commutative ring.*

The proofs of the above results are similar to the ones given in [13] and [9], hence omitted.

It was shown in [4] that a finite ring is commutative if and only if all the nilpotent elements are central. It is not known whether this particular result is valid for finite D-near rings or not. However, using Theorem D we obtain the following result.

THEOREM 3.3. *Let R be a finite D-near ring with an identitiy* 1. *If all the nilpotent elements in R are central, then R is a commutative ring.*

PROOF. Let N be the set of nilpotent elements in R. Then N is an ideal of R and R/N is a D-near ring with no nilpotent elements. Hence by Theorem 3.2, R/N is a finite commutative ring with no nilpotent elements. Thus for each x in R, there is an integer $n=n(x)$, such that x^n-x is in N. Now by Theorem D, R is a commutative ring.

Before we state and prove our main result, the following lemmas are needed.

LEMMA 3.4. *Let R be a near ring with no nilpotent elements. Then R contains a family of completely prime ideals with trivial intersection.*

LEMMA 3.5. *Let R be a D-near ring such that for each x, y in R, there exists* $n=n(x, y)>1$, *such that* $(xy-yx)^n=xy-yx$. *Then the set of nilpotents is an ideal of R.*

The proof of Lemma 3.4 is in [1] and Lemma 3.5 can be found in [2].

LEMMA 3.6. *Let R be a D-near ring with no zero divisors and for each x, y in R, there is an integer $n=n(x, y)>1$, such that $(xy-yx)^n=xy-yx$. Then R is a commutative ring.*

PROOF. For each x, y in R, $(xy-yx)^{n-1}=e$ is idempotent. Since $e(er-r)=0$ and R has no zero divisors, it follows that e is a left identity. Since a D-near ring has a nonzero right distributive element, we see that e is also a right identity, hence the identity of R. Let $x\neq 0$ be in R. If x is not in Z, the centre of R, then $xy-yx\neq 0$ for some y in R. Hence $0\neq x(xy-yx)=x(xy)-(xy)x$ implies that $e=(x(xy)-(xy)x)^{n-1}=(x(xy-yx))^{n-1}$ and thus x has a right inverse. If x is in Z and there is a $y\neq 0$ in R such that xy is not in Z, then from above x has a right inverse. Hence R is a near field. Since the additive group of a near field is abelian [11] and R is a D-near ring, it follows that R is a commutative ring.

Now suppose x is in Z and xr is in Z for each r in R. Let a, b be arbitrary elements of R. Then $x(ab)=(xa)b=b(xa)=xba$ implies that $ab=ba$, and R is a commutative near ring. But a commutative near ring with an identity [8, Corollary 3] is a commutative ring.

We are now ready to prove our main result which is a generalization of Theorems B and C.

THEOREM 3.7. *A D-near ring R is commutative if and only if for each x, y in R, there exists an integer $n=n(x, y)>1$, such that $(xy-yx)^n=xy-yx$.*

PROOF. If every element of R is nilpotent, then clearly R is commutative. If R has no nilpotent elements, then by Lemma 3.4, R contains a family of completely prime ideals P with trivial intersection. For each P, R/P is a D-near ring and has no zero divisors and $(xy-yx)^n=xy-yx$ for each x, y in R/P. Thus by Lemma 3.6, R/P is commutative. Hence for each a, b in R, $ab-ba$ is in P. Consequently $ab=ba$.

Now let N be the set of nilpotent elements. By Lemma 3.5, N is an ideal of R. Now R/N has no nilpotent elements. By the above argument R/N is commutative and hence $ab-ba$ is in N for each a, b in R. Thus $ab=ba$.

This completes the proof.

In view of the above result, we make the following conjecture.

CONJECTURE. Theorem D is valid for D-near rings.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHWESTERN LOUISIANA
LAFAYETTE, LOUISIANA
U.S.A.

DEPARTMENT OF MATHEMATICS
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NORTH CAROLINA
U.S.A.

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ON COGENERATORS IN ABELIAN GROUPS

By
S. M. YAHYA (Karachi)

1. Introduction

In [5] we introduced the concept of cogenerators in abelian groups and described the structure of cocyclic and finitely cogenerated abelian groups (see also [1]). We also proved that an abelian group G cogenerated by a set S of non-zero elements can be embedded as a subgroup in the direct product of cocyclic quotient groups of G corresponding to the elements of S. We recall that a subset S of an abelian group G is a *set of cogenerators* (or a *cogenerating set*) of G if every non-trivial subgroup of G contains a non-zero element of S. Clearly if S cogenerates G, then so does $S\setminus\{0\}$ if $0\in S$. A group cogenerated by a single non-zero element is called a *cocyclic group* and that cogenerated by a finite set of non-zero elements a *finitely cogenerated group*. A group is cocyclic iff it is isomorphic to $Z(p^k)$ for some prime p, where $k=1, 2, \dots,$ or ∞. A group is finitely cogenerated iff it is a direct product of a finite number of cocyclic groups. If S is a set of non-zero elements of an abelian group G, then by H_S we denote a subgroup of G (not necessarily unique) maximal with respect to missing S. We remark that $G/_{H_S}$ is cogenerated by the cosets corresponding to the elements of S. When there is no danger of confusion we shall simply say that $G/_{H_S}$ is cogenerated by S. Thus corresponding to every subset of non-zero elements of an abelian group G there exists a quotient group of G which it cogenerates. Conversely, a quotient group of G determines a subset of G which cogenerates the quotient group. The object of this paper is to develop further the theory of cogenerators.

In section 2 we state elementary results about cogenerators some of which are implicitly used in the subsequent sections, while in section 3 we systematically study the cogenerating sets of an abelian group. We introduce the concept of codependence and prove, among other results, that a cogenerating set of an abelian group contains a minimal cogenerating set iff it is a torsion group. We call the cardinality of a minimal cogenerating set, which is shown to be an invariant of the group, the *corank* of the group, and prove that two p-groups are of the same corank iff they have the same rank. We also show that an abelian group is divisible iff it is cogenerated by a set of elements of infinite height.

In section 4 we introduce the concept of copure subsets in an abelian group. A subset of an abelian group is called pure if it generates a pure subgroup. Dually, we call a set S of non-zero elements in an abelian group G a *copure subset* of G if it always cogenerates a pure quotient group of G, i.e., if every subgroup H_S maximal with respect to missing S is a pure subgroup of G. In our study of copure subsets of an abelian group we arrive at certain results which include, as a particular case, some theorems of IRWIN and WALKER (see [2], [3], and [4]). We observe that purity, in a way, is a self-dual notion, for an exact sequence $0\to A'\to A\to A''\to 0$ of abelian groups is pure exact if the heights of elements of the subgroup A' are preserved in A, or equivalently if the orders of elements of the quotient group A'' are preserved in A in the sense that every element of A'' has a pre-image in A of the same order. Indeed, heights and orders play dual roles in the theory of abelian groups. We say that A' is a pure subgroup and A'' a pure quotient group of A. We emphasize that sometimes it is more fruitful to examine the quotient group A'' to show the purity of the subgroup A' in A.

For the sake of brevity we have omitted the proofs which are either trivial or are otherwise evident.

We follow [1] for notation and terminology, and by a group in the rest of the paper we mean an abelian group. We may also mention here that p invariably denotes a prime.

2. Some elementary results

We state here some elementary results regarding cogenerators.

(i) A group G is cogenerated by a subset S iff every nonzero cyclic subgroup of G contains a non-zero element of S.

(ii) Let M be an independent set of elements in a group G, then G is cogenerated by $\langle M\rangle$ iff M is maximal.

(iii) If H is an essential subgroup of a group G, then H cogenerates G, and also any set of cogenerators of H cogenerates G.

(iv) A torsion group G is cogenerated by its socle $S(G)$ and also by any cogenerating set of $S(G)$. (Note that if G is a p-group, then $S(G)=G[p]$.)

(v) If a non-zero subgroup H of a group G cogenerates G, then $H\supseteq S(G)$.

(vi) An infinite cyclic subgroup of a group G cogenerates G iff G is isomorphic to a subgroup of the group of rationals.

(vii) A group of rank 1 is cogenerated by any of its non-zero subgroups.

(viii) Let x be a non-zero element of a group G, and H_x a subgroup of G maximal with respect to missing x. Then ${}^{G}/_{H_x}$ is a cocyclic group cogenerated by the coset $(x+H_x)$. Following are the immediate consequences of this observation:

(a) If the order of x is a power of p, where p is a given prime, then $px\in H_x$.

(b) If x is of finite order m, then $px\in H_x$ for some prime factor p of m.

(c) If x is of infinite order, then $px\in H_x$ for some prime p.

(ix) Let G be a p-group and let S be a set of m non-zero elements of G, where $0<m<p$. Then there exists an element $x\in S$ such that H_S is also maximal with respect to missing x.

(x) Let M be a subgroup of a group G, and let H be an M-high subgroup of G, i.e., H is maximal with respect to $H\cap M=0$. Then H is also maximal with respect to missing S, where S is a set of non-zero cogenerators of M.

3. Properties of cogenerating sets

3.1 Proposition. *Let S be a set of cogenerators of a group G, and let S' be a proper subset of S. If H is a non-zero subgroup of G containing S' and missing $S\setminus S'$, then H is cogenerated by S'.*

3.2 Proposition. *If a group G is cogenerated by a set of elements of finite order, then G is a torsion group.*

PROOF. Let $g(\neq 0)\in G$. Then $\langle g\rangle$ contains a non-zero element x of S. Hence $x=ng$ for some integer $n\neq 0$. If x is of order m, then $mng=0$, and so g is of finite order.

3.3 PROPOSITION. *A group cogenerated bv a set of elements of order a power of p, where p is a prime, is a p-group.*

3.4 PROPOSITION. *A group cogenerated by a set of elements of infinite order is torsion-free.*

3.5 PROPOSITION. *Let a group G be cogenerated by a set S of non-zero elements of G and let $S=S_1\cup S_2$, where S_1 consists of elements of finite order and S_2 those of infinite order. If H_i is a subgroup of G containing S_i and maximal with respect to missing S_j, where $i,j=1,2$, and $i\neq j$, then H_i is cogenerated by S_i and $H_1\cap H_2=0$. Moreover, H_1 is unique, being the torsion subgroup $T(G)$ of G.*

PROOF. S_i cogenerates H_i by Proposition 3.1. H_1 is a torsion group and H_2 a torsion-free one by Propositions 3.2 and 3.4, so $H_1\cap H_2=0$. Let $g\neq 0$ be an element of finite order in G. If $g\notin H_1$, then the maximality of H_1 implies that $H_1+\langle g\rangle$ contains an element of S_2, which is impossible as $H_1+\langle g\rangle$ is a torsion group.

3.6 PROPOSITION. Let $S=\bigcup_{p\in P} S_p$ be a cogenerating set of a group G, where P is a family of primes, and S_p consists of non-zero elements of order a power of p. Let G_p be a subgroup of G containing S_p and maximal with respect to missing $S\setminus S_p$, then G_p is cogenerated by S_p and it is unique, being the p-component of G. Moreover, $G=\bigoplus_{p\in P} G_p$.

3.7 PROPOSITION. *If S_p cogenerates the p-component G_p of a group G, then $\bigcup_p S_p$ cogenerates $T(G)$, where p runs over all primes.*

We note that the intersection of an infinite descending chain of cogenerating sets of a group need not be a cogenerating set of the group, e.g., $Z\supset 2Z\supset 4Z\supset\dots$. However, if the group is a torsion group, then the result holds.

3.8 DEFINITION. Let x, y be two non-zero elements of a group G. We say that y is *codependent* on x if x is a multiple of y, i.e., $my=x$ for some integer $m\neq 0$. A subset S_1 of G is *codependent* on a subset S_2 of G if each non-zero element of S_1 is *codependent* on some element of S_2. A set S is *coindependent* if it is not codependent on any of its proper subsets. A set S is *codependent* if it is not coindependent. An element of S is a *codependent element* of S if it codepends on some other element of S, otherwise it is a *coindependent element* of S.

Note that a set S is coindependent iff it has no codependent elements and $0\notin S$.

We observe that S is a cogenerating set of a group G iff G is codependent on S, and that if S cogenerates G, then we can exclude from S the codependent elements of S and the remaining set will still cogenerate G. This leads to a natural question: Does a cogenerating set always contain a minimal cogenerating set? This is equivalent to asking whether a cogenerating set of a group always contains a coindependent cogenerating set. It is clear that this is not true in torsion-free groups. In fact, we have the following theorem in this direction.

3.9 THEOREM. *A cogenerating set of a group G contains a minimal cogenerating set of G iff G is a torsion group.*

PROOF. Let S be a cogenerating set of a torsion group G. Let $C=\{S_i\}_{i\in I}$ be the collection of all subsets of S which cogenerate G. We partially order C by setting $S_i\leqq S_j$ if $S_j\subseteqq S_i$. Let $C'=\{S_j\}_{j\in J}$, $J\subseteqq I$, be a linearly ordered subset of C. It is easy to check that $\bigcap_{j\in J} S_j$ cogenerates G, and so the Zorn's condition is satisfied. Hence there exists a maximal element in C which is a minimal cogenerating set of G.

Conversely, let S be a minimal cogenerating set of a group G. If G is not a torsion group, then by Proposition 3.2 S contains at least one element, say x, of infinite order. Also $\langle 2x\rangle$ contains an element of S which is different from x. Hence x is a codependent element of S, which is a contradiction.

It is evident that a torsion group may have more than one minimal cogenerating set. However, the cardinality of a minimal cogenerating set is an invariant of the group. We shall prove this for a p-group, but the proof can be generalized for any torsion group in an obvious manner.

3.10 LEMMA. *Let G be a p-group and let ϱ be a relation on the set $L=G[p]\setminus\{0\}$, defined by setting $x\varrho y$ iff x is a multiple of y. Then ϱ is an equivalence relation. Moreover, S is a minimal cogenerating set of G iff S is a complete set of representatives of the equivalence classes determined by ϱ.*

PROOF. It is easy to check that ϱ is an equivalence relation. We note that G is cogenerated by L and that any set of cogenerators of G contains a subset of L as a cogenerating set of G. Since elements of one equivalence class are codependent on each other and no element of one class is codependent on any element of the other class the result follows.

Note that a cogenerating set of a p-group G is minimal iff it is a minimal cogenerating set of $G[p]$.

An immediate consequence of the above lemma is the following theorem.

3.11 THEOREM. *Any two minimal cogenerating sets of a p-group have the same cardinal number.*

3.12 DEFINITION. We call the cardinal number of a minimal cogenerating set of a torsion group G the *corank* of G (denoted by $\operatorname{cor}(G)$).

3.13 PROPOSITION. *Let G be a torsion group, then*

$$\operatorname{cor}(G)=\sum_p \operatorname{cor}(G_p),$$

where p runs over all primes.

3.14 THEOREM. *Two p-groups G and G' have the same corank iff they have the same rank.*

PROOF. If G and G' have the same rank, then $G[p]\cong G'[p]$, and so they have the same corank. Conversely, if G and G' are of the same corank, then it follows from Lemma 3.10 that $G[p]$ and $G'[p]$ are of the same order and so of the same rank.

3.15 COROLLARY. *Two divisible p-groups are isomorphic iff they have the same corank.*

We shall now prove that a group is divisible iff it is cogenerated by a set of elements of infinite height. It suffices to prove only the 'if' part. We do this in three steps.

3.16 THEOREM. *A p-group G is divisible if it is cogenerated by a set S of elements of infinite height.*

PROOF. Since S contains a minimal cogenerating set, it follows from Lemma 3.10 that all elements of order p in G have infinite height. Hence G is divisible.

3.17 COROLLARY. *A torsion group cogenerated by a set of elements of infinite height is divisible.*

3.18 THEOREM. *A torsion-free group G cogenerated by a set S of elements of infinite height is divisible.*

PROOF. Let $x(\neq 0)\in G$ and $n(\neq 0)\in Z$. There exists an element $y(\neq 0)\in S$ such that $y=mx$ for some $m(\neq 0)\in Z$. Since y is of infinite height there exists an element $z\in G$ such that $y=mnz$, which implies that $x=nz$.

3.19 THEOREM. *A group G is divisible if it is cogenerated by a set S of elements of infinite height.*

PROOF. Let $S=S_1\cup S_2$, where S_1 consists of elements of finite order and S_2 those of infinite order. Then by 3.5 $T(G)$ is cogenerated by S_1 and is maximal with respect to missing S_2. Since elements of S_1 are of infinite height in $T(G)$ also, $T(G)$ is divisible by 3.17. Now ${}^G/_{T(G)}$ is torsion-free and is cogenerated by the cosets corresponding to the elements of S_2 which are of infinite height in ${}^G/_{T(G)}$. Hence ${}^G/_{T(G)}$ is divisible by 3.18, and so G, being isomorphic to $T(G)\oplus {}^G/_{T(G)}$, is divisible.

4. Copure subsets

In this section we consider some copure subsets. For this we first prove a theorem which has also its own importance.

4.1 THEOREM. *Let H be a subgroup of a group G and let* $\varphi\colon G\to K={}^G/_H$ *be the natural homomorphism. Let* $K=\bigoplus_{i\in I} C_i$, *where each* C_i *is a cocyclic group. Then H is a pure subgroup (or K a pure quotient group) of G if a cogenerator of each* C_i *has a pre-image (with respect to* φ*) of the same order and the same height in G.*

PROOF. We prove the theorem by showing that each element of K has a pre-image (with respect to φ) of the same order in G. This is equivalent to showing that each element of C_i has a pre-image of the same order in G. Since a cogenerator of each C_i is of order p_i for some prime p_i and every other element of order p_i in C_i is a multiple of it, the result holds for all elements of order up to p_i in each C_i. We apply induction and suppose that the result is true for all elements of order p_i^m in each C_i. We shall show that it is true for all elements of order up to p_i^{m+1} in each C_i. Let c_i be a cogenerator of C_i, $i\in I$, and let g_i be a pre-image of c_i of the same order and the same height. Let x_i be any element of p_i^{m+1} in C_i. Then $p_i^m x_i$ is of order p_i, and so $p_i^m x_i=\lambda c_i$ for some integer λ with $0<\lambda<p_i$. Let $g_i=p_i^m g$

for some g of order p_i^{m+1} in G. Then $p_i^m \lambda g\varphi = \lambda g_i \varphi = \lambda c_i = p_i^m x_i$, which implies that $p_i^m(\lambda g\varphi - x_i) = 0$. Hence, by the induction hypothesis, there exists $g' \in G$ such that $p_i^m g' = 0$ and $g'\varphi = \lambda g\varphi - x_i$, whence it follows that $(\lambda g - g')\varphi = x_i$, where $\lambda g - g'$ is an element of order p_i^{m+1} in G.

4.2 Corollary. *Let x be an element of order p and of infinite height in a group G. Then $\{x\}$ is a copure subset of G, i.e., every subgroup H_x of G maximal with respect to missing x is a pure subgroup of G.*

4.3 Corollary. *Let x be an element of order p and p-height k in a group G. Then a subgroup H_x of G containing $p^{k+1}G$ and maximal with respect to missing x is a pure subgroup of G.*

4.4 Corollary. *Let G be a bounded p-group satisfying $p^{m+1}G = 0$, $p^m G \neq 0$. Then a non-zero element x of height m in G constitutes a copure subset, and H_x is a direct summand.*

4.5 Theorem. *Let G be a group and let $S = \bigcup_{i \in I} S_i$, where each S_i is a copure subset of G. If every quotient group K of G cogenerated by S is a direct sum of groups K_i cogenerated by S_i, then S is also a copure subset.*

Proof. Let $K = \bigoplus_{i \in I} K_i$ be a quotient group of G, cogenerated by S. Each K_i, being a direct summand of K, is a quotient group of K, and so a pure quotient group of G. Hence each element of K_i has a pre-image of the same order in G, for all $i \in I$. Consequently, each element of K has a pre-image of the same order in G, so K is a pure quotient group.

4.6 Corollary. *Let $S = \bigcup_{p \in P} S_p$, where P is a family of primes and S_p consists of elements of order a power of p. If each S_p is a copure subset of G, then S is also a copure subset of G.*

Proof. This follows from Proposition 3.6 and Theorem 4.5.

4.7 Lemma. *A p-group G cogenerated by a set of elements of order p and height m is a direct sum of cocyclic groups of height p^m, i.e., of cyclic groups of order p^{m+1}.*

Proof. It is clear that each element of order p in G is then of height m. Let $G[p] = \bigoplus_{i \in I} \langle x_i \rangle$. Then it is easy to check that $G = \bigoplus_{i \in I} C_i$, where each C_i is a cocyclic group of height p^m, cogenerated by x_i.

4.8 Theorem. *Let H be a subgroup of a group G and let $\varphi\colon G \to K = {}^G/_H$ be the natural homomorphism. If K is a torsion group such that its each primary component is cogenerated by a set, of which each element has a pre-image (with respect to φ) of the same order and height in G, then H is a pure subgroup of G.*

Proof. In the light of the proof of Theorem 4.1 if suffices to consider the case when K is a p-group. It follows immediately that each element of order p in K has a pre-image of the same order and height in G. Then the result follows by induction as in the proof of Theorem 4.1.

We can now at once deduce a number of important results, including some known ones.

4.9 Corollary. *Let S be a set of elements of order p and of infinite height in a group G. Then S is a copure subset of G, i.e., H_S is pure in G. Moreover, H_S contains a p-basic subgroup of G, where p is any prime.*

Proof. The purity of H_S follows from Theorem 4.8. Also, since G/H_S is divisible by 3.16, H_S contains a p-basic subgroup.

4.10 Corollary. *Let G be a bounded p-group satisfying $p^{m+1}G=0$, $p^mG\neq 0$, then any subset S of G consisting of non-zero elements of height m is a copure subset.*

4.11 Corollary. *Let G and S be as in* 4.10, *then H_S is a direct summand.*

Proof. This follows from Corollary 4.10 and Lemma 4.7.

4.12 Corollary. *A bounded p-group is a direct sum of cocyclic groups of finite height, i.e., a direct sum of cyclic groups.*

4.13 Corollary. *Let S be the set of all elements of order p which have heights $\geqq m \geqq 0$ in a p-group G. Then S is a copure subset. In fact, H_S is a direct summand of G.*

4.14 Corollary. *Let G be a p-group, then a p^mG-high subgroup is a direct summand of G.*

4.15 Corollary. *Let M be a subgroup of G with $M\subseteqq G'_p$. Then an M-high subgroup H is pure in G.*

Proof. Notice that H is H_S, where $S=M[p]\setminus\{0\}$.

4.16 Corollary. *Let M be a subgroup of G with $M\subseteqq T(G')$. Then an M-high subgroup H is pure in G.*

Proof. We note that H is maximal with respect to missing $\bigcup_p M[p]\setminus\{0\}$.

4.17 Lemma. *Let S be a subset of non-zero elements in the socle $S(G)$ of a group G, then there exists a subgroup $M\subseteqq\langle S\rangle$ such that H_S is maximal with respect to $H_S\cap M=0$.*

Proof. Apply Zorn's lemma.

4.18 Theorem. *Let G be a group, then any subset S of non-zero elements in $S(G')$ is a copure subset of G.*

Proof. By Lemma 4.17 there exists a subgroup $M\subseteqq\langle S\rangle\subseteqq T(G')$ such that H_S is maximal with respect to $H_S\cap M=0$. The result follows from Corollary 4.16.

4.19 Definition. We call a set S of a group G closed under multiplication by positive integers if $nS\subseteqq S$, where n is any positive integer.

4.20 Theorem. *Let S be a subset of a group G such that it is closed under multiplication by positive integers and consists of elements of infinite order. Then S is a copure subset of G.*

PROOF. S cogenerates a torsion-free quotient group (by 3.4) which is therefore a pure quotient group.

4.21 COROLLARY. *If M is a torsion-free subgroup of a group, then an M-high subgroup is pure in G.*

4.22 THEOREM. *Let G be any group. Let S_1 be a set of non-zero elements in $S(G')$, and let S_2 be a set of elements of infinite order and closed under multiplication by positive integers. Then $S=S_1\cup S_2$ is a copure subset of G.*

PROOF. Let H' be a subgroup of G containing H_S and maximal with respect to missing S_2. H' is pure in G by Theorem 4.20. We note that $H'\supseteqq T(G)\supseteqq S_1$, and that H_S is maximal with respect to missing S_1 in H'. Hence H_S is pure in H' (by Theorem 4.18), and so pure in G.

4.23 COROLLARY. *Let M be a subgroup of a group G with $M\subseteqq G'$. Then an M-high subgroup H is pure in G.*

PROOF. Note that H is maximal with respect to missing the non-zero elements of $S(M)\cup(M\setminus T(M))$.

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DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF LIBYA
TRIPOLI, LIBYA

DEPARTMENT OF MATHEMATICS,
KARACHI UNIVERSITY
KARACHI, PAKISTAN

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ON A COMMUTATIVITY THEOREM OF LUH

By
A. KAYA (Ankara)

In his paper [4], J. LUH have proved that a primary ring with unity element 1 and satisfying the identities

$$(xy)^k = x^k y^k, \quad k = n, n+1, n+2$$

for all $x, y \in R$, where n is a fixed integer >1, is commutative. In this paper we will prove that a ring satisfying the identities

$$(xy)^k = x^k y^k, \quad k = n(x, y),\ n(x, y)+1,\ n(x, y)+2$$

for all $x, y \in R$, where $n(x, y)$ is an integer >1 which depends on x and y, is commutative if it is a primary ring with unity element 1 or if it is semi-prime.

Let R be a ring having Jacobson radical J. R is called primary if and only if R has a unity element and R/J is a simple ring (not necessarily artinian). R is completely primary if and only if R has a unity element and R/J is a division ring.

We begin with a

THEOREM 1. *Let R be a ring in which the identities*

$$(1) \qquad (xy)^k = x^k y^k, \ k = n(x, y),\ n(x, y)+1$$

hold for every $x, y \in R$, where $n(x, y)$ is an integer >1 which depends on x and y, then

(i) *R is commutative if it is a division ring,*

(ii) *R is commutative if it is a primitive ring,*

(iii) *R is commutative if it is a semi-simple ring,*

(iv) *the commutator ideal $C(R)$ is contained in the Jacobson radical J of R,*

(v) *if R has the unity element 1, then for $x \in R$, $x^m \in J$ implies $x \in J$, where m is any integer >1.*

PROOF. (i) For any $x, y \in R$ we have

$$x^{n(x,y)+1} y^{n(x,y)+1} = (xy)^{n(x,y)+1} = (xy)(xy)^{n(x,y)} = xyx^{n(x,y)} y^{n(x,y)}$$

which gives

$$x^{n(x,y)} y = yx^{n(x,y)}.$$

Hence R is commutative by [1; Lemma 1].

(ii) The proof is based on the idea used by HERSTEIN in [3; p. 73]. Since R is a primitive ring, either $R \cong D$ for some division ring D, in this case R is commutative by (i), or for some $k>1$, D_k is a homomorphic image of a subring of R. In the latter case, D_k would inherit the property (1). But taking

$$a=\begin{pmatrix}0&1&0&\dots&0\\0&0&0&\dots&0\\ \cdot&\cdot&\cdot&\dots&\cdot\\0&0&0&\dots&0\end{pmatrix},\qquad b=\begin{pmatrix}0&0&\dots&0\\1&0&\dots&0\\ \cdot&\cdot&\dots&\cdot\\0&0&\dots&0\end{pmatrix}$$

we see that

$$ab=\begin{pmatrix}1&0&\dots&0\\0&0&\dots&0\\ \cdot&\cdot&\dots&\cdot\\0&0&\dots&0\end{pmatrix}\quad\text{and}\quad (ab)^2=ab,\quad a^2=0.$$

Hence

$$(ab)^{n(a,b)}=a^{n(a,b)}b^{n(a,b)}$$

yields $ab=0$, i.e., $1=0$ which is impossible. Therefore R is a division ring so must be commutative.

(iii) and (iv) are clear by (ii) and (iii), respectively.

(v) R/J is commutative by (iii). So for any $r\in R$ we have

$$\overline{(xr)^m}=(\bar{x}\bar{r})^m=\bar{x}^m\bar{r}^m=\bar{0},$$

which gives $(xr)^m\in J$ for any $r\in R$. It follows that

$$1-(xr)^m=(1-xr)[(xr)^{m-1}+\dots+xr+1]=[(xr)^{m-1}+\dots+xr+1](1-xr)$$

is invertible, so is 1-xr for every $r\in R$, that is to say, $x\in J$. This completes the proof of the theorem.

REMARK. Condition (ii) of Theorem 1 yields exactly those rings as condition (i), for a primitive ring is commutative if and only if it is a field.[1]

LEMMA 1. (See [4].) *Let R be a completely primary ring with unity element* 1 *and with the Jacobson radical J. If $x\in R$ and $x\notin J$, then there exists $u\in R$ such that $ux=xu=1$.*

LEMMA 2. *Let G be a multiplicative group. If the identities*

$$(ab)^k=a^kb^k,\quad k=n(a,b),\ n(a,b)+1,\ n(a,b)+2$$

hold for every $a,b\in G$, where $n(a,b)$ is an integer >1 which depends on a and b, then G is abelian.

PROOF. In the proof of Theorem 1 (i) we got

$$a^{n(a,b)}b=ba^{n(a,b)}.$$

Similarly we have

$$a^{n(a,b)+1}b=ba^{n(a,b)+1}$$

[1] The author thanks to the referee for his suggestions.

for every $a, b \in G$. Hence,

$$a^{n(a,b)+1}b = ba^{n(a,b)+1} = ba^{n(a,b)}a = a^{n(a,b)}ba$$

which implies that $ab=ba$ for every $a, b \in G$.

COROLLARY 1. *Let R be a ring with unity element 1. If the identities*

$$(xy)^k = x^k y^k, \quad k = n(x, y), n(x, y)+1, n(x, y)+2$$

hold for every $x, y \in R$, where $n(x, y)$ is an integer >1 which depends on x and y, then the Jacobson radical J of R is commutative.

PROOF. $S=\{1-x|x\in J\}$ is a multiplicative group.

LEMMA 3. *Let R be a completely primary ring with unity element 1. If the identities*

$$(2) \qquad (xy)^k = x^k y^k, \quad k = n(x, y), n(x, y)+1, n(x, y)+2, n(x, y) > 1,$$

hold for every $x, y \in R$, then

$$C(J) = \{x\in R|x\notin J\}$$

is commutative.

PROOF. For $x, y \notin J$, there exist $x', y' \in R$ such that

$$xx' = x'x = yy' = y'y = 1$$

by Lemma 1. Hence, $x', y' \notin J$. Furthermore,

$$1 = (y'x')(xy) = (xy)(y'x').$$

Therefore, $xy, y'x' \notin J$ and xy has $y'x'$ as a multiplicative inverse in $C(J)$. So, $C(J)$ is a multiplicative group which satisfies the condition (2). Consequently, it is an abelian group by Lemma 2.

THEOREM 2. *Let R be a ring with unity element 1 in which the identities*

$$(xy)^k = x^k y^k, \quad k = n(x, y), n(x, y)+1, n(x, y)+2$$

hold for every $x, y \in R$, where $n(x, y)$ is an integer >1 which depends on x and y, then R is commutative if one of the following conditions is satisfied:

(i) *R is primary,*

(ii) *R is semi-prime.*

PROOF. (i) By Theorem 1, R/J is commutative, so it is a subdirect sum of fields, hence R/J is a field, namely R is completely primary. Now, if $x, y\in J$ or if $x, y\notin J$, then $xy=yx$ by corollary to Lemma 2 and by Lemma 3, respectively. If, on the other hand, $x\in J$ and $y\notin J$, then $1-x\notin J$ so we have $y(1-x)=(1-x)y$ which gives us that $xy=yx$.

(ii) For every $x, y\in R$, $xy-yx\in J$ and J is commutative by Corollary 1, so $xy-yx\in Z$, the centre of R, by [2; Lemma 1.5]. Hence R is commutative by [5; Corollary 1].

COROLLARY 2 (LUH). *If R is a primary ring with unity element 1 in which the identities*

$$(xy)^k = x^k y^k, \quad k = n, n+1, n+2$$

hold for every $x, y\in R$, where n is an integer >1, then R is commutative.

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MIDDLE EAST TECHNICAL UNIVERSITY
DEPARTMENT OF MATHEMATICS
ANKARA, TURKEY

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ON EXTENSIONS OF ORDERS OF GROUPS AND RINGS

By
J. RACHŮNEK (Olomouc)

1.1. As it is well-known, any lattice can be considered as an algebraic system with one binary relation or as an algebraic system with two binary operations (i.e. as a universal algebra). Likewise an ordered group is usually defined as an algebraic system with one binary operation and with one binary relation while this paper presents an ordered group (directed group, lattice-ordered group) G expressed as a set with one binary operation and one generalized binary operation (generalized in the sense that to each ordered pair of elements in G a subset of G corresponds).

In this paper we prove a necessary and sufficient condition for the existence of a directed (lattice) order of a group G (Theorem 2.4), a necessary and sufficient condition of the possibility for the extension of the given order of a group G to a directed (lattice) one (Corollary 2.4.1) and a necessary and sufficient condition such that every order of a group G can be extended to a directed (lattice) one (Corollary 2.4.2). Analogous problems are solved for rings.

Throughout this paper we shall follow the terminology and results of [3].

2.1. Let S be a nonempty set. As it is known, any mapping $\varphi\colon S^n\to S$ is called n-ary algebraic operation on S. In this paper any mapping $\psi\colon S^n\to 2^S$ will be called a generalized n-ary algebraic operation on S. The generalized binary operations have been used e.g. in the theory of multigroups and multigroupoids (see [5], [6]) and in the theory of multilattices (see [1], [2], [4], [7]).

Let $G=[G,+]$ be a group, and $V(x,y)\colon G\times G\to 2^G$ a generalized binary operation on G. $V(x,x)$ will be denoted by $V(x)$. Now we consider the following conditions:

(D1) $V(x,y)\neq\emptyset$ for each $x,y\in G$.

(D2) $V(x)\subseteqq V(y)\Rightarrow V(u+x+v)\subseteqq V(u+y+v)$ for each $x,y,u,v\in G$.

(D3) $V(x)=V(y)\Rightarrow x=y$ for each $x,y\in G$.

(D4) $V(x,y)=V(x)\cap V(y)$ for each $x,y\in G$.

(D5) $y\in V(x)\Leftrightarrow V(y)\subseteqq V(x)$ for each $x,y\in G$.

2.2. Theorem. *The conditions* D1—D5 *are mutually independent.*

Proof. For the independence of D1 let us consider the group Z_2 of the numbers 0, 1 with the addition mod 2. Let $V(0)=\{0\}$, $V(1)=\{1\}$, $V(0,1)=V(1,0)=\emptyset$.

For D2 let us consider Z_2. Denote $V(0)=\{0\}$, $V(1)=\{0,1\}$, $V(0,1)=V(1,0)=\{0\}$.

For D3 let us consider Z_2. Denote $V(x,y)=\{0,1\}$ for each $x,y\in Z_2$.

For D4 let us consider Z_2. Denote $V(0)=\{0\}$, $V(1)=\{1\}$, $V(0,1)=V(1,0)=\{0,1\}$.

For D5 let us consider the additive group Z of the integer numbers. We denote $V(x,y)=\{z\in Z\colon z>\max(x,y)$ in the natural order$\}$.

2.3. THEOREM. *Let $G=[G, +]$ be a group, $V(x, y)$ a generalized binary operation on G. If V satisfies the conditions* D2, D3, D5, *then $V(0)$ is the positive cone of a certain order of the group G.*

PROOF. Let V satisfy D2, D3, D5.

a) By D5 $x \in V(x)$ for each $x \in G$. Therefore $0 \in V(0)$ that means $V(0) \neq \emptyset$. Let $a, b \in V(0)$. Then by D5 $V(a) \subseteq V(0)$, $V(b) \subseteq V(0)$. Thus by D2 $V(a+b)=V(a+b+0) \subseteq \subseteq V(a+0+0)=V(a)$. Now by D5 $a+b \in V(a)$, therefore also $a+b \in V(0)$ holds. This implies that $V(0)$ is a subsemigroup of G.

b) Let $a \in G$, $a \in V(0)$, $-a \in V(0)$. Then by D5 $V(a) \subseteq V(0)$, $V(-a) \subseteq V(0)$. Herefrom by D2 $V(0)=V(-a+a+0) \subseteq V(-a+0+0)=V(-a)$, therefore $V(0)= =V(-a)$. Thus by D3 $-a=0$, and so $a=0$.

c) Let $x \in G$, $a \in V(0)$. Then by D5 $V(a) \subseteq V(0)$, and so by D2 $V(-x+a+x) \subseteq \subseteq V(-x+0+x)=V(0)$, thus by D5 $-x+a+x \in V(0)$.

Let us state the following condition:

(D6) For each $x, y \in G$ there exists $z \in G$ such that $V(x, y)=V(z)$.

2.4. THEOREM. *The group $G=[G, +]$ admits a directed (lattice) order if and only if there exists a mapping $V(x, y)$: $G \times G \to 2^G$ satisfying* D1—D5 (D1—D6).

PROOF. a) Assume that there exists a mapping V with the properties D1—D5. By Theorem 2.3, $V(0)$ is the positive cone of a certain order which will be denoted by $\leqq$. Thus $a \leqq b$ iff $b-a \in V(0)$. Let us show that the ordered group $G=[G, +, \leqq]$ is directed. First, let us prove that the upper cone $U(x, y)=\{z \in G: z \geqq x, z \geqq y\}$ of the elements $x, y \in G$ is equal to $V(x, y)$. Thus let $z \geqq x, y$, i.e. $z \in U(x, y)$. Then $z-x \in V(0)$, $z-y \in V(0)$, and so by D5 $V(z-x) \subseteq V(0)$, $V(z-y) \subseteq V(0)$. In accordance with D2 $V(z)=V(0+(z-x)+x) \subseteq V(0+0+x)=V(x)$ holds. Herefrom and from D5 it follows that $z \in V(x)$. Similarly $z \in V(y)$. Then by D4 $z \in V(x, y)$, i.e. $U(x, y) \subseteq V(x, y)$. Conversely, let $z \in V(x, y)$. Then by D4 $z \in V(x)$, $z \in V(y)$. D5 implies $V(z) \subseteq V(x)$, $V(z) \subseteq V(y)$ and therefore by D2 $V(z-x)=V(0+z-x) \subseteq V(0+x-x)=V(0)$. Then by D5 $z-x \in V(0)$, i.e. $z \geqq x$. Similarly $z \geqq y$. That means $V(x, y) \subseteq U(x, y)$. By the assumption D1 $U(x, y)=V(x, y) \neq \emptyset$, thus G is directed. The condition D6 makes a lattice order possible.

b) The converse implication is evident for $V(x, y)=U(x, y)$.

LEMMA. *Let $\leqq_i$ be an order of a group G, $U_i(x, y)$ the upper cone of the elements $x, y \in G$ $(i=1, 2)$. Then the order $\leqq_2$ is an extension of the order $\leqq_1$ if and only if for each $x, y \in G$, $U_1(x, y) \subseteq U_2(x, y)$ holds.*

The proof is evident.

Corollaries 2.4.1 and 2.4.2 follow immediately from the preceding results.

2.4.1. COROLLARY. *Let $G=[G, +, \leqq]$ be an ordered group, $U(x, y)$ the upper cone of the elements $x, y \in G$. Then the order $\leqq$ admits the extension to a directed (lattice) one if and only if there exists a mapping $V(x, y)$: $G \times G \to 2^G$ such that*

1. $V(x, y)$ *satisfies* D1—D5;

(1′. $V(x, y)$ satisfies D1—D6;)

2. $V(x, y) \supseteq U(x, y)$ for each $x, y \in G$.

2.4.2. COROLLARY. *Let $G=[G, +]$ be a group. Then each order of G admits the extension to a directed (lattice) one if and only if for each mapping $U(x, y)$: $G\times G\to 2^G$ satisfying* D2, D3, D5 *there exists a mapping $V(x, y)$: $G\times G\to 2^G$ such that*

1. *$V(x, y)$ satisfies* D1—D5;

(1′. *$V(x, y)$ satisfies* D1—D6;)

2. *$V(x, y)\supseteqq U(x, y)$ for each $x, y\in G$.*

3.1. Let now $R=[R, +, \cdot]$ be a ring, $V(x, y)$: $R\times R\to 2^R$ be a generalized binary operation on R. We shall again denote $V(x, x)$ by $V(x)$. Let us state the following conditions:

(D′1) $V(x, y)\neq\emptyset$ for each $x, y\in R$.

(D′2) $V(x)\subseteqq V(y)\Rightarrow V(u+x)\subseteqq V(u+y)$ for each $x, y, u\in R$.

(D′3) $V(x)\subseteqq V(y)\Rightarrow V(ux)\subseteqq V(uy)$ for each $x, y\in R$, $u\in V(0)$.

(D′4) $V(x)=V(y)\Rightarrow x=y$ for each $x, y\in R$.

(D′5) $V(x, y)=V(x)\cap V(y)$ for each $x, y\in R$.

(D′6) $y\in V(x)\Leftrightarrow V(y)\subseteqq V(x)$ for each $x, y\in R$.

3.2. THEOREM. *The conditions* D′1—D′6 *are mutually independent.*

PROOF. For the conditions D′1, D′2, D′4—D′6 we can use the ring Z_2 of the numbers 0, 1 with the addition mod 2 and the multiplication mod 2 and the ring of the integer numbers with the same mappings $V(x, y)$ as in Theorem 2.2 for groups.

For D′3 we consider the field of the complex numbers C. It is well-known that C is, with respect to the order $\leqq$ given by

$$x_1+y_1 i \leqq x_2+y_2 i \quad \text{iff} \quad x_1 \leqq x_2, y_1 \leqq y_2,$$

a lattice-ordered (and also directed) group. We denote the upper cones of this order by $V(x, y)$. $V(2+2i)\subseteqq V(1+i)$, $i\in V(0)$ holds hereby $V(i(2+2i))=V(-2+2i)\nsubseteqq V(-1+i)=V(i(1+i))$.

3.3. THEOREM. *Let $R=[R, +, \cdot]$ be a ring, $V(x, y)$ a generalized binary operation on R. If V satisfies the conditions* D′2—D′4, D′6, *then $V(0)$ is the positive cone of a certain order of the ring R.*

PROOF. By Theorem 2.3 it suffices to show that $V(0)$ is closed with respect to the multiplication. Thus let $a, b\in V(0)$. Then by D′6 $V(b)\subseteqq V(0)$ and therefore by D′3 $V(ab)\subseteqq V(a\cdot 0)=V(0)$, and so by D′6 $ab\in V(0)$ holds.

Let us state the following condition:

(D′7) For each $x, y\in R$ there exists $z\in R$ such that $V(x, y)=V(z)$.

3.4. THEOREM. *The ring $R=[R, +, \cdot]$ admits a directed (lattice) order if and only if there exists a mapping $V(x, y)$: $R\times R\to 2^R$ satisfying* D′1—D′6 (D′1—D′7).

3.4.1. COROLLARY. *Let $R=[R, +, \cdot, \leqq]$ be an ordered ring, $U(x, y)$ the upper cone of the elements $x, y\in R$. Then the order $\leqq$ admits the extension to a directed (lattice) one if and only if there exists a mapping $V(x, y)$: $R\times R\to 2^R$ such that*

1. *$V(x, y)$ satisfies* D′1—D′6;

(1′. *$V(x, y)$ satisfies* D′1—D′7;)

2. *$V(x, y)\supseteqq U(x, y)$ for each $x, y\in R$.*

3.4.2. COROLLARY. *Let* $R=[R, +, \cdot]$ *be a ring. Then each order of R admits the extension to a directed (lattice) one if and only if for each mapping* $U(x, y)$: $R\times R\to 2^R$ *satisfying* D′2—D′4, D′6 *there exists a mapping* $V(x, y)$: $R\times R\to 2^R$ *such that*

1. $V(x, y)$ *satisfies* D′1—D′6;

(1′. $V(x, y)$ *satisfies* D′1—D′7;)

2. $V(x, y)\supseteq U(x, y)$ *for each* $x, y\in R$.

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DEPARTMENT OF MATHEMATICS
PALACKÝ UNIVERSITY
OLOMOUC, CZECHOSLOVAKIA

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NUMERICAL RELATIONSHIPS IN DIRECT PRODUCTS OF GROUPS

By

H. FINKELSTEIN (Atlanta)

Introduction

There are various summation and divisibility properties among the elements of finite groups. In a previous paper [1] some numerical relationships were obtained for cyclic groups. In this paper other divisibility properties will be investigated.

The first section deals with some counting formulas for certain subsets of the elements in a group. Some of these formulas are then applied to nilpotent groups to give a necessary condition similar to those obtained in [1] for cyclic groups. In the last section several explicit formulas are obtained for counting elements of an abelian group.

The notation is the same as in [1], but will be included for easier reference. Throughout this paper G will denote a finite group of order $|G|$. Let $o(g)$ denote the order of the element g belonging to group G. If A and B are groups, then $A\times B$ denotes the direct product of A and B. Let $H[s,k](G)\equiv\{x\in G\colon k|o(x)|sk\}$ where $a|b$ means a divides b, ($a\nmid b$ means a does not divide b), and let $h[s,k](G)\equiv|H[s,k](G)|$, the size of the set $H[s,k](G)$. If there is no ambiguity the notations will be simplified to $H[s,k]$ and $h[s,k]$. Unless otherwise stated, when the expression $h[s,k](G)$ appears it will be assumed that $sk\,||G|$. If $|G|=p^a t$ and $(p,t)=1$ we write $p^a\,\|\,|G|$. Let Φ denote Euler's phi-function. Also l.c.m. $\{x,y\}$ denotes the least common multiple of x and y.

I. Some counting formulas

LEMMA 1. *Let A and B be groups. Then*

$$h[1,t](A\times B)=\sum_{d|t}h[1,d](A)\sum h[1,k](B)$$

where the inner sum runs over those k for which l.c.m. $\{d,k\}=t$.

The proof of this lemma easily follows from the fact that $o((a,b))=$ l.c.m. $\{o(a),o(b)\}$ where $(a,b)\in A\times B$.

The following special case of this lemma will be useful later on.

COROLLARY 1.1. *Let A and B be groups, and let α be a positive integer. Then*

$$h[1,p^\alpha](A\times B)=h[1,p^\alpha](B)\sum_{i=0}^{\alpha}h[1,p^i](A)+h[1,p^\alpha](A)\sum_{i=0}^{\alpha-1}h[1,p^i](B).$$

PROOF. From Lemma 1, $h[1,p^\alpha](A\times B)=\sum_{d|p^\alpha} h[1,d](A)\sum h[1,k](B)$, where the inner sum runs over those k for which l.c.m. $\{d,k\}=p^\alpha$.

$$= h[1,1](A)h[1,p^\alpha](B)+\ldots+h[1,p^{\alpha-1}](A)h[1,p^\alpha](B)+$$
$$+h[1,p^\alpha](A)\left[h[1,1](B)+h[1,p](B)+\ldots+h[1,p^\alpha](B)\right]=$$
$$= h[1,p^\alpha](B)\sum_{i=0}^{\alpha} h[1,p^i](A)+h[1,p^\alpha](A)\sum_{i=0}^{\alpha-1} h[1,p^i](B).$$

COROLLARY 1.2. *Let A and B be groups. Then*

$$h[r,t](A\times B)=\sum_{s|r}\Big(\sum_{d|st} h[1,d](A)\sum h[1,k](B)\Big)$$

where the innermost sum runs over those k for which l.c.m. $\{d,k\}=st$.

In the evaluation of these sums one needs to know the number of elements of a given order in a group. In particular when C is a cyclic group, it is easily seen that

$$h[1,d](C)=\begin{cases}0 & \text{if } d\nmid |C|\\ \Phi(d) & \text{if } d\,|\,|C|.\end{cases}$$

From the definition of $H[s,k]$ let us write

$$h[r,t](G)=\sum_{d|r} h[1,dt](G).$$

An application of the standard Möbius inversion formula (see for example [4] par. 16.4) to this equation yields

LEMMA 2. *Let G be a group. Then*

$$h[1,rt](G)=\sum_{d|r} h[d,t](G)\,\mu(rd^{-1}),$$

where μ is the Möbius function.

A similar result can be obtained in which the number of elements of order n is calculated by summing over all the divisors of n.

LEMMA 3. *Let G be a group. Then*

$$h[1,n](G)=\sum_{d|n}\mu(d)\,h[p_\gamma bd^{-1},p_\gamma^{e_\gamma-1}](G)$$

where p_γ is any prime divisor of n, $p_\gamma^{e_\gamma}\|n$, and $p_\gamma^{e_\gamma}b=n$.

PROOF. Let $n=rt$. Then using Lemma 2 one obtains

$$h[1,n]=\sum_{d|nt^{-1}} h[d,t]\,\mu\big(n(td)^{-1}\big).$$

Let us express this as a sum over the divisors of n. Suppose the prime decomposition of n is $n=\prod_{i=1}^{m} p_i^{e_i}$ where $p_i<p_{i+1}$. Let $t=p_\gamma^{e_\gamma-1}$ where γ is arbitrarily chosen from

the set $\{1, 2, \ldots, m\}$. Then $nt^{-1}=p_\gamma \prod_{i=1}^{m}{}' p_i^{e_i}=p_\gamma b$ and $(p_\gamma, b)=1$. The symbol $\prod'$ means that the factor $p_\gamma^{e_\gamma}$ has been removed from the original product. Now if $d|nt^{-1}$ then $d=p_\gamma^j r$ where $j=0$ or 1 and $r|b$.

Hence

$$h[1,n]=\sum_{r|b}\sum_{j=0}^{1} h[p_\gamma^j r, p_\gamma^{e_\gamma-1}]\mu(p_\gamma b(p_\gamma^j r)^{-1})=\sum_{r|b}(h[r,p_\gamma^{e_\gamma-1}]\mu(p_\gamma br^{-1})+$$
$$+h[p_\gamma r, p_\gamma^{e_\gamma-1}]\mu(br^{-1}))=\sum_{r|b}(-h[r,p_\gamma^{e_\gamma-1}]\mu(br^{-1})+h[p_\gamma r,p_\gamma^{e_\gamma-1}]\mu(br^{-1}))=$$
$$=\sum_{r|b}\mu(br^{-1})(h[p_\gamma r,p_\gamma^{e_\gamma-1}]-h[r,p_\gamma^{e_\gamma-1}])=(h[p_\gamma b,p_\gamma^{e_\gamma-1}]-h[b,p_\gamma^{e_\gamma-1}])-$$
$$-\left(\sum_{i=1}^{m}{}' h[p_\gamma bp_i^{-1},p_\gamma^{e_\gamma-1}]-h[bp_i^{-1},p_\gamma^{e_\gamma-1}]\right)+$$
$$+\left(\sum_{\substack{i=1\\ i\neq j}}^{m}{}' h[p_\gamma b(p_ip_j)^{-1},p_\gamma^{e_\gamma-1}]-h[b(p_ip_j)^{-1},p_\gamma^{e_\gamma-1}]\right)-$$
$$-\ldots(+)(-1)^{m-1}\left(h\left[p_\gamma b\left(\prod_{i=1}^{m}{}' p_i\right)^{-1},p_\gamma^{e_\gamma-1}\right]-h\left[b\left(\prod_{i=1}^{m}p_i\right)^{-1},p_\gamma^{e_\gamma-1}\right]\right)$$

where $\sum'$ means that terms involving p_γ do not appear. Upon regrouping one obtains

$$=h[p_\gamma b,p_\gamma^{e_\gamma-1}]-\sum_{i=1}^{m}h[p_\gamma bp_i^{-1},p_\gamma^{e_\gamma-1}]+\sum_{1\leqq i<j\leqq m}h[p_\gamma b(p_ip_j)^{-1},p_\gamma^{e_\gamma-1}]-$$
$$-\ldots+(-1)^{m-1}\sum_{1\leqq i<j\ldots<l\leqq m}h[p_\gamma b(p_ip_j\ldots p_l)^{-1},p_\gamma^{e_\gamma-1}]+$$
$$+(-1)^m h[p_\gamma b(p_1p_2\ldots p_m)^{-1},p_\gamma^{e_\gamma-1}]=\sum_{d|n}\mu(d)b[p_\gamma bd^{-1},p_\gamma^{e_\gamma-1}].$$

REMARKS 1. It should be noted from the proof of this lemma, that the choice of t as $p_\gamma^{e_\gamma-1}$ was somewhat arbitrary. By letting $t=p_\gamma^\lambda$, $0\leqq\lambda<e_\gamma$ and proceeding as in the proof one would obtain an alternate formula, viz.

$$h[1,n](G)=\sum_{d|n}\mu(d)h[p_\gamma^{e_\gamma-\lambda}bd^{-1},p_\gamma^\lambda](G). \tag{1}$$

The case $t=p_\gamma^{e_\gamma}$ must be treated separately. In this case $nt^{-1}=b$ and one obtains directly

$$h[1,n](G)=\sum_{d|b}\mu(bd^{-1})h[d,p_\gamma^{e_\gamma}](G)=\sum_{d|b}\mu(d)h[bd^{-1},p_\gamma^{e_\gamma}](G).$$

2. If $\lambda=0$ in equation (1) then the summands are of the form $\mu(d)h[nd^{-1},1](G)$. Applying this formulation and Lemma 1 of [1] to a cyclic group of such an order n yields the well-known number-theoretic identity $\Phi(n)=n\sum_{d|n}\frac{\mu(d)}{d}$. Further information on these summands can also be obtained using a theorem of FROBENIUS [2].

3. It should be obvious from the above calculations that choosing t as a non-prime divisor would not yield as simple a formula as obtained above.

COROLLARY 3.1. *Let G be a group and p a prime. Then for $i \geqq 1$*

$$h[1, p^i](G) = \sum_{k=1}^{i} (-1)^{k+1} h[p, p^{i-k}](G) - (-1)^{i+1}.$$

PROOF. Let $n = p^i$ in the lemma and note that $h[1, p^i] = h[p, p^{i-1}] - h[1, p^{i-1}]$.

II. Application to nilpotent groups

By using some of the formulas of the previous section one can obtain an interesting result about nilpotent groups.

THEOREM 4. *Let S_p be a Sylow p-subgroup of G. Suppose $G = S_p \times B$ where $p^\alpha \| |G|$. Then*

$$h[r, p^\alpha](G) = 0 \quad or \quad \Phi(p^\alpha) h[r, 1](B).$$

PROOF. Use the formula in Corollary 1.2. Then

$$h[r, p^\alpha](G) = \sum_{s|r} \Big(\sum_{d|sp^\alpha} h[1, d](S_p) \sum h[1, k](B) \Big) \tag{2}$$

where the innermost sum runs over those k for which l.c.m. $\{d, k\} = sp^\alpha$. Let us first evaluate the inner sum for $s = 1$ i.e.

$$\sum_{d|p^\alpha} h[1, d](S_p) \sum h[1, k](B).$$

If $d \neq p^\alpha$ then $k = p^\alpha$ in the inner sum. But $p \nmid |B|$ and so $h[1, p^\alpha](B) = 0$. Thus the sum reduces to

$$h[1, p^\alpha](S_p) \sum_{i=0}^{\alpha} h[1, p^i](B).$$

Again $h[1, p^i](B) = 0$ for $i \geqq 1$. So the final result is

$$h[1, p^\alpha](S_p).$$

This value is zero unless S_p is cyclic, in which case its value is clearly $\Phi(p^\alpha)$.

Now suppose s is any divisor, say r_1, of r. The sum is then

$$\sum_{d|r_1 p^\alpha} h[1, d](S_p) \sum h[1, k](B).$$

It is easily seen that the only non-zero summands occur when $d = p^\alpha$ and in this case the sum simplifies to

$$h[1, p^\alpha](S_p) h[1, r_1](B)$$

and so if S_p is cyclic, equation (2) becomes

$$h[r, p^\alpha](G) = \sum_{s|r} h[1, p^\alpha](S_p) h[1, s](B) = \Phi(p^\alpha) \sum_{s|r} h[1, s](B) = \Phi(p^\alpha) h[r, 1](B).$$

In the special case that G is a cyclic group, one obtains the same result as in Lemma 1 of [1].

COROLLARY 4.1. *The same hypothesis as in the theorem yield*

$$h[|G|p^{-\alpha}, p^{\alpha}](G) = 0 \quad or \quad |G|p^{-\alpha}\Phi(p^{\alpha}).$$

PROOF. In this case $|B|=|G|p^{-\alpha}$ and $h[|G|, 1](G)=|G|$.

One can apply Corollary 4.1 to a nilpotent group G for which $p^{\alpha}\| |G|$. It should be recalled that the quantity $h[s, k]$ is a sum over a certain range of divisors in a group. For a given value of k, these sums are maximal when $s=|G|k^{-1}$. Choose k to be the highest power of a given prime which divides the order of the group. Then these quantities offer some measure of the Sylow structure of a group. For a cyclic group of order $|G|$ where $p^{\alpha}\| |G|$, it is known that $h[|G|p^{-\alpha}, p^{\alpha}]=|G|p^{-\alpha}\Phi(p^{\alpha})$ (see [1], Theorem 2). This value coincides with the corresponding value for all nilpotent groups of order $|G|$, unless that value is zero. This result about nilpotent groups does not extend to arbitrary groups. For example $h[24, 3]=8$ in the symmetric group of degree 4. Empirical data seems to suggest that divisibility conditions involving $h[np^{-\alpha}, p^{\alpha}]$ hold among all the groups of order n for every integer n.

The next result gives a necessary and sufficient condition for this last possibility.

LEMMA 5. *Let* $p^{\alpha}\| |G|$. *Then* $h[|G|p^{-\alpha}, p^{\alpha}](G)=0$ *if and only if* G *has no cyclic Sylow p-subgroups.*

PROOF. Suppose S_p is a cyclic Sylow p-subgroup of G. Then $|S_p|=p^{\alpha}$ and any generator of S_p belongs to $h[|G|p^{-\alpha}, p^{\alpha}](G)$. Conversely, suppose $x\in H[|G|p^{-\alpha}, p^{\alpha}](G)$. Then $o(x)=p^{\alpha}t$ $(p, t)=1$, and $o(x^t)=p^{\alpha}$. So G has a cyclic Sylow p-subgroup.

III. Application to abelian groups

The h-values of cyclic groups are easily evaluated by number-theoretic expressions (see Lemma 1 and Theorem 2 of [1]). Let us now obtain expressions for calculating the number of elements of a given order in any abelian group, knowing only the type of the abelian group. The rather complicated formulas will be derived for abelian p-groups. These results, together with the formulas of Section I, can be used to obtain the general formula. An explicit formula for calculating $h[n, 1](G)$ for any abelian group G is also obtained.

Let A be an abelian p-group of type $(m_1, m_2, \dots, m_k)$ where $m_j\geqq m_{j+1}$. Let $m_{k+1}=0$. Define the length of group A to be k, the number of integers in the type of A. So $A\cong\prod_{i=1}^{k} C(m_i)$ where $C(m_i)$ is a cyclic group of order p^{m_i}. Suppose also that $|A|=p^m$. Let us compute $h[1, p^{\alpha}](A)$ where $1\leqq\alpha\leqq m$. Of course if $m_1<\alpha\leqq m$, then $h[1, p^{\alpha}](A)=0$ and so let us restrict α to be a positive integer no larger than m_1.

It is easily seen that for a given α there exists a unique integer $i(\alpha)$, depending on α, such that $m_{i(\alpha)}\geqq\alpha>m_{i(\alpha)+1}$ where $1\leqq i(\alpha)\leqq k$.

LEMMA 6. *Let B be an abelian p-group of type* $(m_1, m_2, \dots, m_i)$. *If* $1\leqq\alpha\leqq m_i$ *then*

$$h[1, p^{\alpha}](B) = \Phi(p^{\alpha})\sigma(p^{i-1})p^{(i-1)(\alpha-1)}$$

where $\sigma(m)$ *represents the sum of all divisors of* m.

PROOF. Proceed by induction on the length of B. If $i=2$ then $B\cong C(m_1)\times C(m_2)$. Using Lemma 1

$$h[1,p^\alpha](B)=\sum_{d|p^\alpha} h[1,d](C(m_1))\sum h[1,k](C(m_2))$$

where the inner sum runs over those k for which l.c.m. $\{d,k\}=p^\alpha$. Using Corollary 1.1 this can be rewritten as

$$=h[1,p^\alpha](C(m_2))\sum_{i=0}^{\alpha} h[1,p^i](C(m_1))+h[1,p^\alpha](C(m_1))\sum_{j=0}^{\alpha-1} h[1,p^j](C(m_2)).$$

Since $m_1\geqq m_2\geqq\alpha$ this sum becomes

$$=\Phi(p^\alpha)\big(1+\Phi(p)+\ldots+\Phi(p^\alpha)\big)+\Phi(p^\alpha)\big(1+\Phi(p)+\ldots+\Phi(p^{\alpha-1})\big).$$

Now $\sum_{d|n}\Phi(d)=n$, hence

$$=\Phi(p^\alpha)p^\alpha+\Phi(p^\alpha)\,p^{\alpha-1}=\Phi(p^\alpha)\,(p^\alpha+p^{\alpha-1})=\Phi(p^\alpha)\,(p+1)p^{\alpha-1}=\Phi(p^\alpha)\sigma(p)p^{\alpha-1}.$$

Now assume the result is valid for groups whose length is $i-1$, for $i\geqq 3$. Suppose B is a group of type $(m_1,\ldots,m_i)$ and write B as $B\cong C(m_1)\times\big(C(m_2)\times\ldots\times C(m_i)\big)$. Corollary 1.1 yields

$$h[1,p^\alpha](B)=h[1,p^\alpha]\left(\prod_{r=2}^{i}C(m_r)\right)\sum_{t=0}^{\alpha}h[1,p^t](C(m_1))+$$

$$+h[1,p^\alpha](C(m_1))\left(1+\sum_{j=1}^{\alpha-1}h[1,p^j]\left(\prod_{r=2}^{i}C(m_r)\right)\right).$$

Since $\alpha\leqq m_i$ apply the induction step and get

$$=\Phi(p^\alpha)\sigma(p^{i-2})p^{(i-2)(\alpha-1)}\cdot p^\alpha+\Phi(p^\alpha)\left(1+\sum_{j=1}^{\alpha-1}\Phi(p^j)\sigma(p^{i-2})p^{(i-2)(j-1)}\right)=$$

$$=\Phi(p^\alpha)\sigma(p^{i-2})\left\{p^{(i-2)(\alpha-1)+\alpha}+\sum_{j=1}^{\alpha-1}\Phi(p^j)p^{(i-2)(j-1)}\right\}+\Phi(p^\alpha).$$

Using $\Phi(p^j)=p^{j-1}(p-1)$ the expression becomes

$$=\Phi(p^\alpha)\sigma(p^{i-2})\left\{p^{(i-2)(\alpha-1)+\alpha}+(p-1)-(p-1)\sum_{j=1}^{\alpha-2}p^{j(i-1)}\right\}+\Phi(p^\alpha).$$

Also $\sigma(p^j)=\dfrac{p^{j+1}-1}{p-1}$, and so

$$=\Phi(p^\alpha)\left\{\sigma(p^{i-2})\big(p^{(i-2)(\alpha-1)+\alpha}+(p-1)\big)+(p^{i-1}-1)\sum_{j=1}^{\alpha-2}p^{j(i-1)}\right\}+\Phi(p^\alpha)=$$

$$=\Phi(p^\alpha)\{\sigma(p^{i-2})\big(p^{(i-2)(\alpha-1)+\alpha}+(p-1)\big)+(p^{(\alpha-1)(i-1)}-p^{(i-1)})\}+\Phi(p^\alpha)=$$

$$=\frac{\Phi(p^\alpha)}{p-1}\{(p^{i-1}-1)\big(p^{(i-2)(\alpha-1)+\alpha}+(p-1)\big)+(p^i-p^{i-1})(p^{(\alpha-2)(i-1)}-1)\}+\Phi(p^\alpha).$$

By expanding and cancelling within the curved brackets this becomes

$$= \frac{\Phi(p^\alpha)}{p-1}\{p^{(i-1)(\alpha-1)+i} - p^{(i-1)(\alpha-1)} - p + 1\} + \Phi(p^\alpha) =$$

$$= \Phi(p^\alpha)\left[1 + \frac{1}{p-1}\{p^{(i-1)(\alpha-1)}(p^i-1) - (p-1)\}\right] =$$

$$= \Phi(p^\alpha)[1 + p^{(i-1)(\alpha-1)}\sigma(p^{i-1}) - 1] =$$

$$= \Phi(p^\alpha)\sigma(p^{i-1})p^{(i-1)(\alpha-1)}.$$

Let us next use Lemma 6 to obtain the desired formula for counting the number of elements of a given order in an abelian p-group.

THEOREM 7. *Let $A \cong \prod_{i=1}^{k} C(m_i)$ be an abelian p-group of type $(m_1, \ldots, m_k)$ where $m_j \geqq m_{j+1}$ and set $m_{k+1}=0$. Let $1 \leqq \alpha \leqq m_1$ and let $i(\alpha)$ be the unique integer satisfying $m_{i(\alpha)} \geqq \alpha > m_{i(\alpha)+1}$ where $1 \leqq i(\alpha) \leqq k$. Then $h[1, p^\alpha](A)=$*

$$\Phi(p^\alpha)\sigma(p^{i(\alpha)-1})p^{(i(\alpha)-1)(\alpha-1)}\left(1 + \sum_{\beta=1}^{\alpha-1} h[1, p^\beta]\left(\prod_{j=i(\alpha)+1}^{k+1} C(m_j)\right)\right). \tag{3}$$

The $\alpha-1$ summands are each computed in a similar manner: let $i(\beta)$ be the unique integer satisfying $m_{i(\beta)} \geqq \beta > m_{i(\beta)+1}$ where $i(\alpha)+1 \leqq i(\beta) \leqq k$. Then

$$h[1, p^\beta]\left(\prod_{j=i(\alpha)+1}^{k+1} C(m_j)\right) = \Phi(p^\beta)\sigma(p^{i(\beta)-(i(\alpha)+1)})p^{(i(\beta)-(i(\alpha)+1))(\beta-1)} \times$$

$$\times (1 + \left(\sum_{\gamma=1}^{\beta-1} h[1, p^\gamma]\left(\prod_{s=i(\beta)+1}^{k+1} C(m_s)\right)\right).$$

The $\beta-1$ summands are treated in a similar manner.

Before proceeding with the proof of this theorem let us note certain facts. First if $\alpha=1$, then $i(\alpha)=k$ and the group involved in (3) is non-existent. Hence the entire sum will be zero. The same comment holds when β, γ, etc. are equal to 1. Also note that $\alpha > \beta > \gamma > \ldots$ and that $i(\alpha) < i(\beta) < i(\gamma) \ldots$. Thus the process stops in a finite number of steps.

PROOF. First write

$$A \simeq \prod_{j=1}^{i(\alpha)} C(m_j) \times \prod_{j=i(\alpha)+1}^{k+1} C(m_i) \equiv B \times C.$$

Then Corollary 1.1 yields

$$h[1, p^\alpha](A) = h[1, p^\alpha](C)\sum_{i=0}^{\alpha} h[1, p^i](B) + h[1, p^\alpha](B)\sum_{\beta=0}^{\alpha-1} h[1, p^\beta](C).$$

Since $\alpha > m_{i(\alpha)+1} \geqq \dots \geqq m_k$ one obtains $h[1, p^\alpha](C)=0$. An application of Lemma 6 yields

$$= \Phi(p^\alpha)\sigma(p^{i(\alpha)-1})p^{(i(\alpha)-1)(\alpha-1)}\left(1+\sum_{\beta=1}^{\alpha-1} h[1,p^\beta](C)\right).$$

For a given β first determine where it "fits into" the type of C, which is $(m_{i(\alpha)+1}, \dots$ $\dots, m_k)$. Let $i(\beta)$ be the integer for which $m_{i(\beta)} \geqq \beta > m_{i(\beta)+1}$. Then write

$$C \cong \prod_{j=i(\alpha)+1}^{i(\beta)} C(m_j) \times \prod_{j=i(\beta)+1}^{k+1} C(m_j) \equiv D \times E.$$

Now compute $h[1, p^\beta](C)$ as above and get

$$h[1, p^\beta](C) = h[1, p^\beta](E) \sum_{i=0}^{\beta} h[1, p^i](D) + h[1, p^\beta](D) \sum_{\gamma=0}^{\beta-1} h[1, p^\gamma](E).$$

Simplifying again this yields

$$= \Phi(p^\beta)\sigma(p^{i(\beta)-(i(\alpha)+1)})p^{(i(\beta)-(i(\alpha)+1))(\beta-1)}\left(1+\sum_{\gamma=1}^{\beta-1} h[1, p^\gamma](E)\right).$$

The procedure given by this result is as follows: Given an abelian p-group A of type $(m_1, \dots, m_k)$ and a positive integer α. If $\alpha > m_1$ then $h[1, p^\alpha](A)=0$. If $\alpha=1$, then $i(\alpha)=k$ and $h[1, p](A)=\Phi(p)\sigma(p^{k-1})$. For any other value of α, one first finds $i(\alpha)$. Then for each value of β from $\beta=1$ to $\beta=\alpha-1$, one computes $h[1, p^\beta]\left(\prod_{j=i(\alpha)+1}^{k+1} C(m_j)\right)$, and repeats.

As a numerical illustration let A be an abelian group of order 32 and type (3, 1, 1). The number of elements of order 2 is found as follows: $\alpha=1$ and $i(\alpha)=3$ so $h[1, 2](A)=\Phi(2)\sigma(2^{3-1})=7$. To find $h[1, 4](A)$ set $\alpha=2$ and $i(\alpha)=1$. By the formula $h[1, 4](A)=\Phi(4)\sigma(1)(1+h[1, 2](C(1)\times C(1)))$. Since the four-group has three elements of order 2, the answer is clearly 8. However, continuing with the procedure, there is only one value of β viz $\beta=1$ and so $i(\beta)=3$. Thus

$$h[1, 2](C(1)\times C(1)) = \Phi(2)\sigma(2^{3-(1+1)})2^{(3-(1+1))(1-1)} = \sigma(2) = 3.$$

For another numerical illustration let A be an abelian 2-group of type (5, 2, 1). To find the number of elements of order 8 set $\alpha=3$ and $i(\alpha)=1$. So

$$h[1, 8](A) = \Phi(8)\sigma(1)\left(1+\sum_{\beta=1}^{2} h[1, 2^\beta]\left(\prod_{j=2}^{4} C(m_j)\right)\right).$$

For $\beta=1$ $i(\beta)=3$ and

$$h[1, 2](C(2)\times C(1)) = \Phi(2)\sigma(2^{3-(1+1)})2^{(3-(1+1))(1-1)} = 3.$$

For $\beta=2$ $i(\beta)=2$ and

$$h[1, 4](C(2)\times C(1)) = \Phi(4)\sigma(2^{2-(1+1)})2^{(2-(1+1))(2-1)}(1+h[1, 2](C(1))) = 4.$$

Thus $h[1, 8](A)=4(1+3+4)=32$.

It is an easy consequence of Theorem 7 to obtain an explicit formula for the number of cyclic subgroups of a given order p^α in an abelian p-group A. Since the number of cyclic subgroups of order m times $\Phi(m)$ is the same as the number of elements of order m, the desired formula is obtained by deleting the factor $\Phi(p^\alpha)$ from the expression for $h[1, p^\alpha](A)$ in Theorem 7.

As a final result let us use one of the corollaries from Section 1 to obtain an explicit formula for the expression $h[n, 1](G)$ when G is an abelian group. Suppose $|G| = \prod_{i=1}^{r} p_i^{f_i}$ and $G \cong \prod_{i=1}^{r} A(p_i)$ where $A(p_i)$ is the subgroup of G, all of whose elements have order a multiple of p_i. Let the type of $A(p_i)$ be $(m_{i,1}, m_{i,2}, \ldots, m_{i,k_i})$ and let $m_{i,k_i+1}=0$ for $1 \leqq i \leqq r$.

THEOREM 8. *Let G be the abelian group described above. Let $n = \prod_{i=1}^{r} p_i^{d_i}$ for $0 \leqq d_i \leqq f_i$ be a divisor of $|G|$. Then*

$$h[n, 1](G) = \prod_{i=1}^{r} p_i^{e_i}$$

where

$$e_i = \begin{cases} 0 & \text{if } d_i = 0 \\ \theta_i(d_i)d_i + \sum_{\alpha=\theta(i)+1}^{k_i+1} m_{i,\alpha} & \text{if } 1 \leqq d_i < m_{i,1} \\ f_i & \text{if } d_i \geqq m_{i,1} \end{cases}$$

where

$$\theta_i(t) = \min\left\{ j \Big/ \left[\frac{m_{i,j+1}}{t}\right] \leqq 1, \quad 1 \leqq j \leqq k_i \right\}$$

and $[x]$ denotes the greatest integer function.

PROOF. Proceed by induction on r, the number of distinct primes in $|G|$. If $r=1$, the result has already been proved. (See Lemma 5 of [1].) In order to simplify the notation the result will be shown for $r=2$. The procedure is easily extended to any value of r.

Let $G \cong A(p_1) \times A(p_2)$. Applying Corollary 1.2 yields

$$h[n, 1](G) = \sum_{s|n} \Big(\sum_{l|s} h[1, l](A(p_1)) \sum h[1, k](A(p_2)) \Big)$$

where the innermost sum runs over those k for which l.c.m. $\{l, k\} = s$. Since s is a divisor of n write $s = p_1^{\alpha_1} p_2^{\alpha_2}$, $0 \leqq \alpha_i \leqq d_i$. If $l \neq p_1^{\alpha_1} \cdot \gamma$ then the innermost sum is always zero. On the other hand the other summands are zero if $\gamma \neq 1$. Thus the formula can be written as

$$= \sum_{\alpha_2=0}^{d_2} \sum_{\alpha_1=0}^{d_1} h[1, p_1^{\alpha_1}](A(p_1)) h[1, p_2^{\alpha_2}](A(p_2)) = h[p_1^{d_1}, 1](A(p_1)) h[p_2^{\alpha_2}, 1](A(p_2)).$$

The final result now comes from applying Lemma 5 of [1] to each factor.

Since $\sum_{j=1}^{k_i} m_{i,j} = f_i$ and $\theta_i(d_i)d_i \leqq \sum_{j=1}^{\theta_i(d_i)} m_{i,j}$, one sees that $e_i \leqq f_i$ for $1 \leqq i \leqq r$

in all cases. Hence for an abelian group G one obtains as a corollary to Theorem 8 that $h[n, 1](G)$ divides $|G|$. This fact is a special case of a much deeper result of P. HALL [3] Theorem 1.6. The extension of this statement to non-abelian p-groups and the converse of this result do not hold. The two non-abelian groups of order 8 provide the necessary counterexamples. There also exist non-abelian non-p-groups G which satisfy the condition $h[n, 1](G)/|G|$. The direct product of a cyclic group of order 3 and the dicyclic group of order 8 is one such example.

Returning to Theorem 8 it is easily seen that $d_i \leqq e_i$ for $1 \leqq i \leqq r$. This yields $n/h[n, 1](G)$, for any abelian group G. This result is, of course, a special case of Frobenius' fundamental lemma [2].

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EMORY UNIVERSITY
DEPARTMENT OF MATHEMATICS
ATLANTA, GEORGIA
USA

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KOMPAKTE SUMMIERBARKEIT VON POTENZREIHEN IM EINHEITSKREIS

Von
W. LUH (Gießen)

1. Einleitung

Es sei $A=(\alpha_{nk})$ eine unendliche Matrix mit komplexen Koeffizienten. Eine Folge $\{s_n\}$ heißt A-limitierbar zum Wert s, wenn jede der Reihen $\sigma_n=\sum_{k=0}^{\infty}\alpha_{nk}s_k$ $(n=0, 1, \dots)$ konvergiert und außerdem $A\text{-}\lim_{n\to\infty} s_n=\lim_{n\to\infty}\sigma_n=s$ ist. Das durch A definierte Limitierungsverfahren heißt permanent, wenn für jede Folge $\{s_n\}$ mit $\lim_{n\to\infty} s_n=s$ auch $A\text{-}\lim_{n\to\infty} s_n=s$ gilt. Nach dem Satz von Toeplitz (siehe etwa [6], S. 11) ist A genau dann permanent, wenn folgende drei Eigenschaften erfüllt sind:

(1) $$\lim_{n\to\infty}\alpha_{nk}=0,\quad k=0,1,\dots;$$

(2) $$\lim_{n\to\infty}\sum_{k=0}^{\infty}\alpha_{nk}=1;$$

(3) $$\text{es gibt eine Konstante } C \text{ mit } \sum_{k=0}^{\infty}|\alpha_{nk}|\leqq C.$$

In mehreren neueren Arbeiten ist untersucht worden, unter welchen notwendigen und hinreichenden Bedingungen über A alle Folgen $\{a_k\}$, für welche die Reihe $\sum_{k=0}^{\infty}a_k z^k$ den Konvergenzradius 1 hat, A-limitierbar sind. Man beachte hierzu die Arbeiten von Haplanov [2], Tonne [7], [8] sowie Köthe—Toeplitz [4], Ganapathy Iyer [1]. Die Beweismethoden hierbei sind meist rein limitierungstheoretischer Art (Methode des gleitenden Buckels usw.).

Wir interessieren uns hier für ein weitergehendes Problem; unser Ziel ist die Gewinnung notwendiger und hinreichender Kriterien für A, damit für alle Potenzreihen $f(z)=\sum_{k=0}^{\infty}a_k z^k$ vom Konvergenzradius 1 die A-Transformierte

$$\sigma_n(z)=\sum_{k=0}^{\infty}\alpha_{nk}s_k(z)\quad\text{mit}\quad s_k(z)=\sum_{j=0}^{k}a_j z^j$$

im Einheitskreis kompakt gegen $f(z)$ konvergiert (eine Folge von Funktionen heißt kompakt konvergent in einem Gebiet, wenn sie in jedem kompakten Teil gleichmäßig konvergiert). Es ist zu erwarten, daß wir schwächere als die Toeplitzschen Bedingungen erhalten werden. Wir zeigen:

SATZ. *Die Folge* $\{\sigma_n(z)\}$ *konvergiert für alle Potenzreihen* $f(z)=\sum_{j=0}^{\infty} a_j z^j$ *vom Konvergenzradius* 1 *in* $|z|<1$ *kompakt gegen* $f(z)$ *genau dann, wenn A die Eigenschaften* (1), (2) *erfüllt und außerdem*

(4) *zu jedem* $r\in(0, 1)$ *gibt es eine Konstante* $C(r)$ *mit* $|\alpha_{nk}|r^k \leqq C(r)$, $k=0, 1, \ldots;$ $n=0, 1, \ldots$.

Es ist klar, daß die Bedingung (4) sicher dann erfüllt ist, wenn (3) gilt, aber nicht umgekehrt. Zur kompakten Summierung von Potenzreihen im Einheitskreis können also auch gewisse nicht permanente Verfahren herangezogen werden.

Unser Satz ist eine direkte Konsequenz aus zwei Hilfssätzen, welche mit rein funktionentheoretischen Mitteln bewiesen werden.

2. Beweis des Satzes

Wir betrachten zunächst die Folge $\{z^n\}$ und erörtern die Frage, unter welchen notwendigen und hinreichenden Bedingungen die A-Transformierte $\tau_n(z)=\sum_{k=0}^{\infty}\alpha_{nk} z^k$ im Einheitskreis kompakt gegen Null konvergiert.

HILFSSATZ 1. *Die Folge* $\{\tau_n(z)\}$ *konvergiert in* $|z|<1$ *kompakt gegen Null genau dann, wenn A die Eigenschaften* (1) *und* (4) *erfüllt.*

BEWEIS. 1) Wir zeigen, daß (1) und (4) hinreichend sind. Zu einem kompakten Teil B des Einheitskreises gibt es ein $q\in(0, 1)$ so, daß $|z|<q$ für alle $z\in B$. Wir wählen ein $r\in(q, 1)$ und erhalten:

$$|\alpha_{nk}|\cdot|z^k| \leqq |\alpha_{nk}|r^k\left(\frac{q}{r}\right)^k \leqq C(r)\cdot\left(\frac{q}{r}\right)^k,$$

so daß für jedes n die Reihe $\tau_n(z)$ in B absolut und gleichmäßig konvergiert.

Zu gegebenem $\varepsilon>0$ wählen wir ein N so, daß $C(r)\cdot\sum_{k=N}^{\infty}\left(\frac{q}{r}\right)^k<\varepsilon$. Es ergibt sich dann:

$$\max_B|\tau_n(z)| \leqq \sum_{k=0}^{N-1}|\alpha_{nk}|q^k+\sum_{k=N}^{\infty}|\alpha_{nk}|r^k\left(\frac{q}{r}\right)^k < \sum_{k=0}^{N-1}|\alpha_{nk}|+\varepsilon.$$

Wegen (1) folgt die Behauptung.

2) Wir nehmen an, daß $\tau_n(z)$ für jedes n existiert und in $|z|<1$ kompakt gegen Null konvergiert. Da dann bei festem n die Potenzreihe $\tau_n(z)$ auf kompakten Teilen von $|z|<1$ gleichmäßig konvergiert, ergibt sich für die m-te Ableitung:

$$\tau_n^{(m)}(z) = m!\cdot\sum_{k=m}^{\infty}\alpha_{nk}\binom{k}{m}z^{k-m}, \quad m=0, 1, \ldots.$$

Für $n\to\infty$ konvergiert $\tau_n^{(m)}(z)$ im Einheitskreis kompakt gegen Null, und es folgt für $z=0$:

$$m!\cdot\lim_{n\to\infty}\alpha_{nm} = \lim_{n\to\infty}\tau_n^{(m)}(0) = 0.$$

Also ist die Bedingung (1) notwendig.

Für jedes $r \in (0, 1)$ und $k=0, 1, \dots$ erhalten wir:

$$|\alpha_{nk}| = \left| \frac{1}{2\pi i} \cdot \int\limits_{|z|=r} \frac{\tau_n(z)}{z^{k+1}} dz \right| \leqq \frac{1}{r^k} \cdot \max_{|z|=r} |\tau_n(z)|.$$

Daraus folgt (4).

Als nächstes untersuchen wir die A-Transformierten $\sigma_n(z)$ von Potenzreihen mit dem Konvergenzradius 1.

HILFSSATZ 2. *Die Folge* $\{\sigma_n(z)\}$ *konvergiert für alle Potenzreihen* $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ *vom Konvergenzradius* 1 *in* $|z|<1$ *kompakt gegen* $f(z)$ *genau dann, wenn* $\{\tau_n(z)\}$ *in* $|z|<1$ *kompakt gegen Null konvergiert und* A *die Eigenschaft* (2) *erfüllt.*

BEWEIS. 1) Wir zeigen, daß die Bedingungen hinreichend sind. Es sei wieder B ein kompakter Teil des Einheitskreises und $q \in (0, 1)$ so, daß $|z|<q$ für alle $z \in B$. Bei festem n ist die Reihe $\sum\limits_{k=0}^{\infty} |\alpha_{nk}| q^k$ offenbar konvergent. Für die Reihenreste $r_k(z) = s_k(z) - f(z)$ gilt mit einer geeigneten Konstanten M die Abschätzung $\max\limits_B |r_k(z)| \leqq M q^k$ für alle k. Hieraus folgt (bei festem n) die absolute und gleichmäßige Konvergenz der Reihe $\varrho_n(z) = \sum\limits_{k=0}^{\infty} \alpha_{nk} r_k(z)$ in B. Mit (2) ergibt sich dann

$$\varrho_n(z) + f(z) \sum_{k=0}^{\infty} \alpha_{nk} = \sigma_n(z), \tag{5}$$

so daß für festes n und $z \in B$ die Reihe $\sigma_n(z)$ konvergiert. Mit Hilfe der Cauchyschen Integralformel erhalten wir für alle $z \in B$:

$$\varrho_n(z) = -\frac{z}{2\pi i} \sum_{k=0}^{\infty} \alpha_{nk} \cdot \int\limits_{|\zeta|=q} \frac{f(\zeta)}{\zeta(\zeta - z)} \left(\frac{z}{\zeta}\right)^k d\zeta = -\frac{z}{2\pi i} \cdot \int\limits_{|\zeta|=q} \frac{f(\zeta)}{\zeta(\zeta - z)} \tau_n\left(\frac{z}{\zeta}\right) d\zeta.$$

Wegen $\left|\frac{z}{\zeta}\right| \leqq \frac{1}{q} \cdot \max\limits_B |z| < 1$ konvergiert $\{\varrho_n(z)\}$ auf B glechmäßig gegen Null. Aus (5) folgt wegen (2), daß $\{\sigma_n(z)\}$ auf B gleichmäßig gegen $f(z)$ konvergiert.

2) Wir nehmen an, daß $\{\sigma_n(z)\}$ für alle Potenzreihen $f(z)$ vom Konvergenzradius 1 existiert und in $|z|<1$ kompakt gegen $f(z)$ konvergiert. Für die geometrische Reihe $\sum\limits_{k=0}^{\infty} z^k$ haben wir dann

$$\sigma_n(z) = \sum_{k=0}^{\infty} \alpha_{nk} \left\{ \frac{1}{1-z} - \frac{z^{k+1}}{1-z} \right\}.$$

Aus $\lim\limits_{n \to \infty} \sigma_n(0) = 1$ folgt (2), und wir erhalten

$$\sigma_n(z) - \frac{1}{1-z} \sum_{k=0}^{\infty} \alpha_{nk} = -\frac{z}{1-z} \tau_n(z).$$

Hieraus ergibt sich die Existenz von $\tau_n(z)$ in $0<|z|<1$ und damit auch in $|z|<1$. Außerdem folgt die Kompakte Konvergenz von $\{\tau_n(z)\}$ in $|z|<1$ gegen Null. Damit ist Hilfssatz 2 bewiesen.

Aus Hilfssatz 1 und Hilfssatz 2 ergibt sich sofort unser Satz.

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D 63 GIEẞEN
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SPECIAL CLOSURE, M-RADICALS, AND RELATIVE COMPLEMENTS

By

W. G. LEAVITT* (Lincoln) and J. WATTERS (Leicester)

Dedicated to the memory of Andor Kertész

1. Introduction

It is well-known that given a suitably conditioned class T of associative rings (for example, a hereditary class), then

$$UT = \{R|R \text{ has no non-zero image in } T\}$$

is a radical class. Hence every non-zero UT-semi-simple ring has a non-zero image in T. On the other hand, given a radical class P it is not immediately obvious how to determine a class T such that $P=UT$, apart from the obvious candidate $T=SP$, the semi-simple class associated with P. In an attempt to deal with this problem in some generality we discuss in Section 4 the concept of an M-radical. For an arbitrary class M we define an M-radical to be a radical P such that every P-semi-simple ring has a non-zero image in $SP\cap M$. We then show, in Theorem 3, that $P=U(SP\cap M)$. Also in Section 4 we construct the smallest M-radical to contain a given radical. When M is hereditary this is just $U(SP\cap M)$. Special M-radicals are also discussed and sufficient conditions for $SP\cap M$ to be special are given in Theorem 6. Note that since for most of our constructions it is immaterial whether or not 0 is a member of a class, we will assume for any class M that $0\in M$ unless otherwise stated.

To facilitate the discussion of special radicals we begin in Section 3 with a construction of the smallest special class to contain a given class of prime rings. This class, called the special closure of the given class, is described in Theorem 1.

Finally in Section 5 some of the earlier results are used to establish generalizations of ANDRUNAKIEVITCH's results on complementary radicals [1, p. 181]. For radicals P and Q we will call G the P-relative complementary radical to Q (and designate it as $G=(P:Q)$) if G is the largest radical such that in all rings R, $G(R)\cap Q(R)\subseteqq P(R)$ where $P(R)$, $Q(R)$, $G(R)$ are the P, Q, and G-radicals of R respectively. The radical $(P:Q)$ exists when both P and Q are hereditary [7]. In Section 5 necessary and suficient conditions are derived so that $(P:Q)$ will be hypernilpotent. We also obtain suficient conditions for $(P:Q)$ to be a subidempotent or a hereditary M-radical, and this yields as a special case a condition for a special $(P:Q)$.

2. Terminology and notation

A class M of associative rings has been called *special* by V. A. ANDRUNAKIEVITCH [1, p. 192] if it is a hereditary class of prime rings with the property:

(1) If $I\lhd R$ with $I\in M$, then $R/I^*\in M$, where I^* is the annihilator of I in R.

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We remark that, since $I \in M$ is semi-prime, the ideal I^* can also be characterized as the largest of the ideals J of R for which $I \cap J = 0$. A hereditary class M of semi-prime rings satisfying (1) has been called *weakly special* by JU. M. RJABUHIN [2, p. 66]. In the following there will often be a discussion for semi-prime rings, parallel to that for prime rings, which we will enclose in brackets. Thus if we define

$$FM = \{R/I^* | I \lhd R \text{ with } I \in M\},$$

then a hereditary class M of prime [semi-prime] rings is special [weakly special] if and only if $FM = M$. Other operators related to F are given by:

$$F'M = \{R | I \lhd R \text{ with } I \in M \text{ and } I^* = 0\}$$

and

$$M^e = \{R | R \text{ is prime [semi-prime] and there is } 0 \neq I \lhd R \text{ with } I \in M\}.$$

If M is a class of prime [semi-prime] rings, then $M \subseteq F'M \subseteq FM$ and $M \subseteq M^e$. For a class M of prime rings $M^e \subseteq F'M$.

We will denote the class of all prime [semi-prime] rings by $W[W_1]$.

For an arbitrary class M of rings we define

$$UM = \{R | \text{ every } 0 \neq R/I \notin M\}$$

and

$$SM = \{R | \text{ if } 0 \neq I \lhd R, \text{ then } I \notin M\}.$$

A class of rings is called radical if it is radical in the Kurosh-Amitsur sense. See [3, p. 3]. It is known [4, Theorem 1] that UM is radical if and only if for every $0 \neq R \in M$ there is some $0 \neq R/I \in SUM$. This is true, in particular, if M is a special [weakly special] class, in which case UM is called a *special [weakly special]* radical.

A radical class P is called hypernilpotent if it is hereditary and contains all nilpotent rings. RJAHUBIN has shown [2, p. 67] that P is hypernilpotent if and only if it is weakly special, that is, if and only if $P = UM$ for some weakly special class M. As a corollary one obtains the result [1, p. 125]: Every special radical is hypernilpotent.

A radical class P is called *subidempotent* if it is hereditary and is contained in the class of idempotent rings.

3. Special and weakly special closure

Let M be a class of prime [semi-prime] rings. Since the intersection of any family of special [weakly special] classes is special [weakly special], the intersection $\overline{M}$ of all special [weakly special] classes containing M (of which the class of all prime [semi-prime] rings is one) will be the smallest special class to contain M. We will call $\overline{M}$ the *special [weakly special] closure* of M. Our intention now is to give an alternative (and more useful) characterization of $\overline{M}$. A subring A of a ring R is accessible in R if $A = I_0 \lhd I_1 \lhd \ldots \lhd I_n = R$ for some n. The hereditary closure of M is

$$IM = \{A | A \text{ is accessible in some } R \in M\}.$$

In Theorem 1 we show that $\overline{M} = FIM$.

LEMMA 1 (ANDRUNAKIEVITCH). *If* $M \subseteq W[W_1]$, *then* $FM \subseteq W[W_1]$.

PROOF. Let $I \lhd R$ with $I \in M$ and suppose A, B are ideals of R such that $AB \subseteq I^* [A^2 \subseteq I^*]$. Then $(AI)(BI)=0\ [(AI)^2=0]$ and since I is a prime [semi-prime] ring we have, say, $AI=0$. But then $IA=0$ and so $A \subseteq I^*$.

LEMMA 2. *If* $M \subseteq W[W_1]$, *then* $FM=F'M$.

PROOF. Clearly $F'M \subseteq FM$, so suppose $\bar{R}=R/I^* \in FM$ where $I \lhd R$ with $I \in M$. Then $\bar{I}=(I+I^*)/I^* \cong I$ since $I \cap I^*=0$. Thus $\bar{I} \lhd \bar{R}$ with $\bar{I} \in M$. If $\bar{I}^*=J/I^*$, then $J \cap I \subseteq I^* \cap I=0$ and so $J=I^*$. Hence $\bar{I}^*=0$ and $R \in F'M$.

REMARK 1. It follows from Lemmas 1 and 2 that for a class M of prime [semi-prime] rings $FM=F'M \subseteq M^e$ and so for a class M of prime rings $FM=F'M=M^e$. Thus a hereditary class of prime rings is special if and only if $M=M^e$ i.e., M has the *prime extension* property.

COROLLARY 1. *If* $M \subseteq W[W_1]$ *and* R *is simple, then* $R \in FM$ *if and only if* $R \in M$.

LEMMA 3. *If* M *is a hereditary class of prime [semi-prime] rings, then so is* FM.

PROOF. Let $0 \neq R \in F'M$. Then there is $I \lhd R$ with $I \in M$ and $I^*=0$. Hence if $0 \neq A \lhd R$, then $I \cap A \neq 0$. If J is the annihilator of $I \cap A$ in A and $\bar{J}$ is the ideal of R generated by J, then $\bar{J}^3 \subseteq J$. Hence $I \cap A \cap \bar{J}^3=0$ and so $A \cap \bar{J}^3=0$. But $\bar{J} \subseteq A$ so $\bar{J}^3=0$. Since R is semi-prime, by Lemma 1, we have $\bar{J}=J=0$. Thus M is hereditary.

LEMMA 4. *If* M *is a hereditary class of prime [semi-prime] rings, then* $FFM=FM$.

PROOF. By Lemma 2 we have $FM=F'M$. Suppose $0 \neq J \lhd I \lhd R$ where $J \in M$ and both $I^*=0$ and $J^{(*)}=0$ (where $J^{(*)}$ is the annihilator of J in I). Let $\bar{J}$ be the ideal of R generated by J. Put $A=\bar{J}^3 \subseteq J$. Then $A \lhd R$ with $A \in M$. Let $K \lhd R$ and write $B=K \cap I$. If $AB=0$, then $(\bar{J}B)^3=0$ and, since $\bar{J}B \lhd I \in FM$, it follows that $\bar{J}B=0$. But then $JB=0$ and so $B=0$. Thus if $AK=0$ we would have $K \cap I=0$ so that $K \subseteq I^*=0$. Therefore $A^*=0$ and so $R \in F'M=FM$.

It is known [1, Theorem 6, p. 198] that if T is a special class, then $UT=U(SUT \cap W)$. Similarly, if T is weakly special, then [2, Remark (3), p. 67] SUT is weakly special so $UT=U(SUT \cap W_1)$. Indeed if P is any hypernilpotent (and so weakly special) radical, then $U(SP \cap W)$ is the smallest special radical containing P.

Here and in the remainder of the paper we will let N denote the class of all zero rings (rings R for which $R^2=0$). Then, since the lower radical of a hereditary class is hereditary [5, Theorem 2, p. 64], the smallest hypernilpotent (hence weakly special) radical containing a given class M is $L(IM \cup N)$. It is easy to check that this is equal to $U(SLIM \cap W_1)$. Since the smallest hereditary radical to contain M is LIM and the smallest hypernilpotent radical to contain M is $L(IM \cup N)$, we have that the smallest special radical to contain M is $U(SLIM \cap W)=U(SL(IM \cup N) \cap W)$.

The example in [7, p. 210] shows that, in general, a radical need not have a largest special radical contained in it. The example given is a hypernilpotent, but not special, radical which is the join, in the lattice of all radicals, of two special radicals. However, in Theorem 1 we characterize the largest special radical contained in an upper radical determined by a hereditary class of prime rings.

THEOREM 1. *If* M *is a class of prime [semi-prime] rings, then*

(a) $FIM=\bar{M}$ *(the special [weakly special] closure of* M*), and*

(b) $U\bar{M}$ *is the largest special [weakly special] radical contained in* UIM.

PROOF. (a) By Lemma 4 we have $FFIM=FIM$ so FIM is special [weakly special]. But if T is a special [weakly special] class with $M\subseteqq T$ then $FIM\subseteqq FIT==FT=T$. Thus $FIM=\overline{M}$.

(b) Since $IM\subseteqq\overline{M}$, $U\overline{M}\subseteqq UIM$. If T is a special [weakly special] class such that $UT\subseteqq UIM$, then $IM\subseteqq SUIM\subseteqq SUT$ and so $IM\subseteqq SUT\cap W$. Thus $\overline{M}\subseteqq F(SUT\cap W)==SUT\cap W$ and hence $UT=U(SUT\cap W)\subseteqq U\overline{M}$ [for the weakly special proof substitute W_1 for W].

Within a given radical $P=UM$ we need not have, as the next example shows, $U\overline{M}$ equal to the largest special radical in P.

EXAMPLE 1. Let R be the ring of all linear transformations of a countably infinite dimensional vector space, with $J\lhd R$ the set of members of R of finite rank. We have J and R/J are non-isomorphic simple rings. Put $M=\{R, R/J\}$. Then $UM==U\{R/J\}$, so that UM is a special radical, since any class of simple rings with unit is a special class [3, p. 145]. However, $J\in IM\subseteqq FIM=\overline{M}$, so that $J\notin U\overline{M}$. On the other hand J is simple and $J\notin M$ so $J\in UM$. Thus, $U\overline{M}\subset UM=P$ which is itself special. (Note: by "simple" we will always mean simple idempotent.)

COROLLARY 2 (ANDRUNAKIEVITCH). *If S is any class of simple rings and M is the class of all subdirectly irreducible rings with hearts in S, then M is a special class.*

PROOF. This follows from the fact that S is a hereditary class of prime rings for which $M=F'S=\bar{S}$.

THEOREM 2. *Let $M\subseteqq W[W_1]$. If IM contains no simple rings, then the special [weakly special] radical $U\overline{M}$ has a semi-simple class $SU\overline{M}$ containing no subdirectly irreducible rings.*

PROOF. A subdirectly irreducible ring $R\in SU\overline{M}$ would have a simple heart which would be in $\overline{M}=FIM$. But then from Corollary 1, IM would contain a simple ring.

This theorem is of interest since it shows that there are special radicals, for example $U\overline{M}$ where $M=\{Z\}$, where $Z=$integers, whose semi-simple class contains no subdirectly irreducible rings.

It has already been noted that not every hypernilpontent radical contains a largest special radical. We now present some necessary and sufficient conditions for a radical to contain a largest weakly special radical. First of all we have

PROPOSITION 1. *Let M be an arbitrary class. Then the following are equivalent:*
(1) *UM is a radical containing a special radical;*
(2) *UM is a radical containing a weakly special radical;*
(3) (a) *Every $0\neq R\in M$ has $0\neq R/I\in SUM$, and*
(b) *$SUM\subseteqq W_1$.*

PROOF. From [4] we have that UM is a radical class if and only if M satisfies 3(a). Thus (1)⇒(2)⇒(3), and 3(a)⇒UM is a radical. But $SUM\subseteqq W_1$ implies that $UM\supseteqq N$ and so UM contains the lower Baer radical LN, which is the special radical UW. Hence (3)⇒(1).

It should be noted that the class M of Proposition 1 need not itself be a class of semi-prime rings. For example, if Z_n denotes the integers mod n, $M=\{Z_4, Z_2\}$ has $UM=U\{Z_2\}$ which is special.

PROPOSITION 2. *A radical P contains a largest weakly special radical if and only if* $P \supseteqq N$.

PROOF (F. SZÁSZ and R. WIEGANDT [9, Theorem 2, p. 237]). If P contains a weakly special radical, then clearly $P \supseteqq N$.

Conversely, since the hereditary radicals form a complete sublattice of the lattice of all radicals [7, p. 208], it follows that the join in that lattice of all weakly special radicals contained in P, of which LN is one, is the largest weakly special radical contained in P. This weakly special radical is just the largest hereditary radical in P (since $P \supseteqq LN$) which has been characterized [6, Theorem 3, p. 682] as the class $\{R | I\{R\} \subseteqq P\}$.

4. *M*-radicals

By Theorem 2 there are radicals (even special radicals) whose semi-simple classes contain no subdirectly irreducible rings. On the other hand, as we shall see, there are radicals P such that every $0 \neq R \in SP$ has a non-zero semi-simple image R/I which is subdirectly irreducible with simple heart. We shall call such a radical an *SI*-radical.

From this definition it follows that an *SI*-radical P contains N, the class of all zero rings. Thus if P is hereditary, then P is hypernilpotent. In fact, every hereditary *SI*-radical is special, as we now show.

PROPOSITION 3. *Let V be the class of all subdirectly irreducible rings with simple hearts. P is an SI-radical if and only if* $P=U(SP \cap V)$.

PROOF. The definition of an *SI*-radical is clearly equivalent to $SP \subseteqq SU(SP \cap V)$. But since $SP \cap V \subseteqq SP$ it follows that $SU(SP \cap V) \subseteqq SP$. Thus we have equality and since $U(SP \cap V)$ is radical, $U(SP \cap V)=P$.

COROLLARY 3. *The hereditary SI-radicals are the dual special radicals of Andrunakievitch.*

PROOF. Dual special radicals were described in [1, Theorem 10, p. 203] as upper radicals UQ where Q is the class of all subdirectly irreducible rings with simple hearts having a specified algebraic property. Thus from Proposition 3 every hereditary *SI*-radical P is a dual special radical; the algebraic property being P-semi-simple. Conversely, since Q is hereditary, $Q \subseteqq SUQ$ and so every $R \in SUQ$ has an image in $(SUQ) \cap V \supseteqq Q$. Hence UQ is an *SI*-radical; it is hereditary since Q is special.

COROLLARY 4. *If Q is any class of subdirectly irreducible rings with simple hearts then UQ is an SI-radical.*

PROOF. As in the proof of Corollary 3.

As examples of *SI*-radicals we have, in particular, any upper radical determined by a class of simple rings. Thus the Brown—McCoy radical is an *SI*-radical. Note that by Theorem 2 there are special radicals which are not SI-radicals. On the other hand there are SI-radicals which are not hereditary and hence not special (see Corollary 7).

From the example we mentioned earlier [7, p. 210] it also follows that the join of two SI-radicals need not be an SI-radical. However, the intersection of a family of SI-radicals is an SI-radical. This will be derived as a particular case of a more general result later.

We will now generalize the concept of an SI-radical to that of an M-radical which we define as follows:

Let M be any class of rings. A radical class P is called an M-radical if every $0 \neq R \in SP$ has a non-zero image in $SP \cap M$.

Thus the SI-radicals are M-radicals with $M = V$. We have seen for an SI-radical P that $P = U(SP \cap V)$, so here we define $P' = U(SP \cap M)$ and consider the generalization of Proposition 3 for M-radicals. In Proposition 3 we were assisted by the fact that $SP \cap V$ is hereditary; it may not happen that $SP \cap M$ is hereditary and so it would be possible for P' not to be radical.

The class M may satisfy;

$(*)$ Every non-zero ideal B of any $R \in M$ has an image $0 \neq B/J$ such that:

(a) $B/J \in M$, and

(b) if $0 \neq K/J \lhd B/J$, then there is some $0 \neq (K/J)/(H/J) \cong K/H \in I\{R\}$.

Remark that any hereditary class satisfies $(*)$.

LEMMA 5. *The class M satisfies $(*)$ if and only if $SP' \cap M = SP \cap M$ for every radical class P.*

PROOF. Since $SP \cap M \subseteqq SP$ we have in any case $P = USP \subseteqq U(SP \cap M) = P'$ and so $SP' \cap M \subseteqq SP \cap M$. Now suppose that M satisfies $(*)$ and let $R \in SP \cap M$. If $0 \neq B \lhd R$, then there is some non-zero $B/J \in M \cap SUI\{R\}$. But $R \in SP$ so $I\{R\} \subseteqq SP$ and thus $B/J \in M \cap SP$. Hence $B \notin P'$ and $R \in SP'$. Therefore $SP' \cap M = SP \cap M$.

Conversely, if there is $0 \neq B \lhd R \in M$ such that every $0 \neq B/J \notin M \cap SUI\{R\}$, then setting $P = UI\{R\}$ we have $I\{R\} \subseteqq SP$ and $R \in SP \cap M$. But B has no non-zero image in $M \cap SP$ so $B \in P'$ and $R \notin SP' \cap M$.

COROLLARY 5. *If M is a hereditary class of rings, then $SP' \cap M = SP \cap M$ for every radical class P.*

LEMMA 6. *Let P be a radical class and H a homomorphically closed class of rings. If $P \subseteqq H$ and $SP = SH$, then $P = H$.*

PROOF. Since $USH \supseteqq H$ we have $P = USP = USH \supseteqq H$ and so $P = H$.

THEOREM 3. *For an arbitrary class M of rings, the radical P is an M-radical if and only if $P = P'$.*

PROOF. As in the proof of Lemma 5 we have $P \subseteqq P'$ in any case. If $P = P'$, then $SP = SU(SP \cap M)$ and P is an M-radical. Conversely, if P is an M-radical, then the hereditariness of SP implies that $SP \subseteqq SU(SP \cap M) = SP'$. Hence $SP = SP'$ with P' homomorphically closed. By Lemma 6 $P = P'$.

We now describe two constructions which, for an arbitrary class M, yield the smallest M-radical to contain a given radical P.

PROPOSITION 4. *Let* $\{P_\lambda | \lambda \in \Lambda\}$ *be a family of M-radicals. Then* $P \cap P_\lambda$ *is an M-radical.*

PROOF. The class P is radical and so $P \subseteq U(SP \cap M)$. Further, $P = \cap P_\lambda = \cap U(SP_\lambda \cap M) \supseteq U(\cup (SP_\lambda \cap M))$. On the other hand, if $R \in P$ and $R/I \in SP_\lambda \cap M$ for some index λ, then $I = R$ since $R \in P_\lambda = U(SP_\lambda \cap M)$, by Theorem 3. Therefore $P = U(\cup (SP_\lambda \cap M)) = U(\cup (SP_\lambda) \cap M)$ and, since $\cup (SP_\lambda) \subseteq SP$, $P \supseteq U(SP \cap M)$. This establishes the equality $P = U(SP \cap M)$ and P is an M-radical by Theorem 3.

It follows from this proposition that the intersection of all (hereditary, special) M-radicals containing a given class A is the smallest (hereditary, special) M-radical to contain A. The collection of all M-radicals is then a complete lattice, although not in general a sublattice of the lattice of all radicals.

Our second construction of the smallest M-radical to contain P involves the construction of a transfinite chain of classes whose union is the required radical.

Define $P_0 = P$ and, for any ordinal $\beta > 0$,

$$P_\beta = \begin{cases} \bigcup\limits_{\alpha < \beta} P_\alpha & \text{if } \beta \text{ is a limit ordinal} \\ U(SP_{\beta-1} \cap M) & \text{if } \beta \text{ is a non-limit ordinal.} \end{cases}$$

Define $P_M = \bigcup\limits_\beta P_\beta$. Immediately from the construction we have

LEMMA 7. *Each class* P_β *is homomorphically closed.*

LEMMA 8. *For all* $\alpha \leqq \beta$, $P_\alpha \subseteq P_\beta$.

PROOF. It suffices to show that $P_\beta \subseteq P_{\beta+1}$ for all β. We have $USP_\beta \subseteq P_{\beta+1}$ from the definition of $P_{\beta+1}$. Hence, since P_β is homomorphically closed, $P_\beta \subseteq USP_\beta \subseteq P_{\beta+1}$.

LEMMA 9. *For each ordinal* γ, $\bigcap\limits_{\beta \geqq \gamma} SP_\beta = SP_M$.

PROOF. Clearly $SP_M \subseteq \bigcap\limits_{\beta \geqq \gamma} SP_\beta$.

Suppose $R \notin SP_M$. Then there is some $0 \neq I \lhd R$ with $I \in P_M$. But then $I \in P_\beta$, for some $\beta \geqq \gamma$, and so $R \notin SP_\beta$. Hence $SP_M = \bigcap\limits_{\beta \geqq \gamma} SP_\beta$.

THEOREM 4. *For an arbitrary class M and any radical class P,* P_M *is the smallest M-radical containing P.*

PROOF. From Lemma 7 P_M is homomorphically closed. We show now that if $0 \neq R \notin P_M$, then R has a non-zero image in SP_M. This will establish that P_M is radical.

If $R \notin P_M$, then, for any α, R has a non-zero image in $SP_\alpha \cap M$. Let $G_\alpha = \{K | K$ is an ideal of some image of R and $K \in P_\alpha\}$. Since $\{P_\alpha\}$ is a chain so also is $\{G_\alpha\}$ which, being a chain of subsets, must stabilize at some ordinal γ, so that $G_\gamma = G_{\gamma+1} = \dots$. Now there is a non-zero image $R/I \in SP_\gamma \cap M$, so $R/I \in SP_\beta \cap M$ for all $\beta \geqq \gamma$. Therefore $R/I \in \bigcap\limits_{\beta \geqq \gamma} (SP_\beta \cap M) = \left(\bigcap\limits_{\beta \geqq \gamma} SP_\beta\right) \cap M = SP_M \cap M$ by Lemma 9. Thus $0 \neq R/I \in SP_M$ and P_M is a radical class. Furthermore $U(SP_M \cap M) \subseteq P_M$, the opposite inclusion holds since P_M is radical, and so $P_M = U(SP_M \cap M)$. Then, by Theorem 3, P_M is an M-radical which clearly contains P.

Finally, if Q is any M-radical and $Q \supseteq P = P_0$, then an easy induction argument shows that $Q \supseteq P_\beta$ for every ordinal β. Thus $Q \supseteq P_M$ and the theorem is proved.

For an arbitrary class A the smallest M-radical containing A is thus $(LA)_M$ which by Proposition 4 is the intersection of all M-radicals containing A. In the particular case $M = V$, the radical $(LA)_V$ is the smallest SI-radical containing A. The smallest hereditary M-radical containing A is $(LIA)_M$.

COROLLARY 6. *If M is a class of rings with property $(*)$ then the smallest M-radical containing a radical P is $P' = U(SP \cap M)$.*

PROOF. Since $P_1 = P'$, by Lemma 5, we have $SP_1 \cap M = SP \cap M$. Hence $P_2 = U(SP_1 \cap M) = P_1$ and $P_M = P_1 = U(SP \cap M)$.

It is an easy induction argument to show that if $A \subseteq B$, then $P_B \subseteq P_A$. Alternatively, this inclusion follows immediately from our first construction. Also, by minimality, if Q is a radical with $P \subseteq Q \subseteq P_A$, then $Q_A = P_A$. Thus $(P_B)_A = P_A$.

If P is a hereditary radical, then it is well-known that $\tilde{P} = U(SP \cap W)$ is the smallest special radical to contain P. If M is a hereditary class, then $(\tilde{P})'$ is the smallest M-radical to contain $\tilde{P}$. But if the members of M are prime rings, then $P' = U(SP \cap M) \supseteq \tilde{P} \supseteq P$ and so $(\tilde{P})' = P'$.

For an arbitrary class M there exists a largest subclass M^* with property (*), namely the union of all subclasses of M with property (*) (non-empty since $0 \in M$). An M^*-radical containing P will be an M-radical but even the smallest M^*-radical containing P, $U(SP \cap M^*)$, will in general be larger than P_M.

We now determine necessary conditions for an M-radical to be special. A ring A with no proper ideals and $A^2 = 0$ is called a *prime order zero ring*.

THEOREM 5. *Let M and H be classes of rings such that H is homomorphically closed. If there is a ring $A \in M \setminus H$ and A is either a simple ring without unit or a prime order zero ring, then there is a non-hereditary (and so non-special) M-radical containing H.*

PROOF. Let R be the split extension of A by either Z_p if A has characteristic p or by the rationals Q if A has characteristic 0 (note that when A has characteristic 0 it is automatically a Q-algebra). In either case $A \lhd R$ and is the only proper ideal of R. Since $A \notin H \cup \{R\}$ it follows that $A \notin P = L(H \cup \{R\}) = P_0$. If $A \notin P_\beta$ for any ordinal β, then $A \in SP_\beta$ since A is simple, and so $A \in SP_\beta \cap M$. Hence $A \notin P_{\beta+1}$. Thus, by induction, $A \notin P_M$. However, $R \in P \subseteq P_M$ and so P_M is not hereditary.

COROLLARY 7. *If M contains a simple ring without unit, then there is a non-hereditary M-radical containing all nilpotent rings. Thus with $M = W$ we have a non-special radical P such that $P = U(SP \cap W)$. Also, letting $M = V$, there exist non-special SI-radicals.*

THEOREM 6. *Let M be a special [weakly special] class of rings. If P is a radical for which $P \cap M$ is hereditary then $SP \cap M$ is a special [weakly special] class (and so $P' = U(SP \cap M)$ is a special [weakly special] M-radical).*

PROOF. We know that $SP \cap M$ is a hereditary class of prime [semi-prime] rings. Let $R \in F'(SP \cap M)$ then R has an ideal $0 \neq I \in SP \cap M$ and since $SP \cap M \subseteq M$ it follows from Remark 1 that $R \in F'M = M$. If $R \notin SP$ then there is some $0 \neq J \lhd R$ with $J \in P$. But since $I^* = 0$ this would imply $0 \neq I \cap J$ would be contained in SP and also $P \cap M$ both of which are hereditary.

COROLLARY 8. *Let M be a special [weakly special] class of rings and let P be an M-radical. Then P is special if and only if $P \cap M$ is hereditary.*

Many upper radicals have the property that their semi-simple classes are subdirect closures of proper subclasses. Accordingly we say that an upper radical $P = UM$ has property (*Int*) relative to M if, for every $R \in SP$, we have $0 = \bigcap_{I \in T} I$ where $T = \{I \lhd R \mid R/I \in M\}$ or, equivalently, if every $R \in SP$ is a subdirect sum of members of M.

From [1, Remark 14, p. 195] and [2, Proposition 1, p. 67] the special [weakly special] radical $P' = U(SP \cap M)$ of Theorem 6 has property (Int) relative to $SP \cap M$.

It is not sufficient to have M a hereditary class of prime rings in Theorem 6 as we now show.

EXAMPLE 2. Let P be the lower Baer radical and M the class of all simple rings without unit. As in Example 1, let J be the ideal of all linear transformations of finite rank of the ring R of all linear transformations of a countably infinite dimensional vector space. In [8, Example 2] it was shown that R has a subring K with exactly one non-zero proper ideal I and $J \cong I \cong K/I$. Then $J \in SP \cap M$ so $K \in SP'$. Hence P' does not have property (Int) relative to $SP \cap M$ since K is subdirectly irreducible but not in $SP \cap M$. Also, R/J is a simple ring with unit so $R \in P'$, whereas $J \in SP'$. Thus P' is not hereditary. However, it is (vacuously) true that $P \cap M$ is hereditary.

It is also true that the hypothesis $P \cap M$ hereditary cannot in general be omitted from Theorem 6:

EXAMPLE 3. If K is a Q-algebra, where Q = rationals, we will write K_1 for the split extension of K by Q. Let M be a special class of rings containing two (non-isomorphic) simple rings without unit A, B. (The special closure $\overline{\{A, B\}}$, which is the class of all subdirectly irreducible rings with heart either A or B, would suffice.) We will show that for $P = L\{N \cup B_1\}$ the radical $P' = U(SP \cap M)$ does not have property (Int) and is not hereditary. From $A, B \in SP \cap M = SP' \cap M$ it follows that both $P \cap M$ and P' are non-hereditary. Now let $R = (A \oplus B)_1$ then $R \notin P'$ since R has an image $R/B \cong$ $\cong A_1 \in SP \cap M$. The same is true of A, B, and $A \oplus B$ so in fact $R \in SP'$. However, R/B is the only image of R in $SP \cap M$ so P' does not have property (Int) relative to $SP \cap M$.

From the following example it follows that the converse of Theorem 6 is not valid:

EXAMPLE 4. Let R be the ring of all linear transformations of a vector space of dimension $\aleph_1$ and $J \lhd I \lhd R$ where I is the set of all transformations of rank $\leqq \aleph_0$ and J those of finite rank. Let A be a simple ring without unit such that $A \not\cong J$, $A \not\cong I/J$. Let $M = F'\{J, A\}$, so M is a special class, and $P = L(N \cup \{J, R\})$. It is known that $I/J \not\cong J$ so $I \notin P$ whereas $R \in M$. Thus $P \cap M$ is not hereditary. On the other hand, if $K \in SP \cap M$ then clearly $A \lhd K$ with $A^* = 0$. Thus $SP \cap M$ is the class of all subdirectly irreducible rings with heart A and so is special.

Another question which naturally arises is whether every M-radical P, M special, with property (Int) relative to $SP \cap M$ is special. Thus we pose the following related problems.

PROBLEM 1. Does there exist a special class M and an M-radical P with property (Int) relative to $SP\cap M$, but $SP\cap M$ is not special?

PROBLEM 2 ([8]). Does there exist a non-special class A of simple rings such that UA has property (Int) relative to A?

If A were a non-special class of simple rings such that UA has property (Int) relative to A, then with $M=F'A$ and $P=UA$ so we would have a positive answer to Problem 1, since $SP\cap M=A$ in this case.

From Theorem 6 it follows that if M is special and the radical P is such that $P\cap M$ is hereditary, then P' has property (Int) relative to $SP\cap M$. It is not necessary for M to be special in order for this to hold as we now show by Example 5.

PROPOSITION 5. *Let M and H be classes of rings such that every $R\in H$ is a subdirect sum of members of M. Then $P=UM=U(M\cup H)$ and if P has property (Int) relative to $M\cup H$, then it also has property (Int) relative to M.*

PROOF. Since $M\subseteq M\cup H$ then $U(M\cup H)\subseteq UM$. But every $R\in H$ has an image in M so we have $SU(M\cup H)\subseteq SUM$ and so $UM=U(M\cup H)$. The second part follows immediately from the hypothesis.

EXAMPLE 5. Let $H=\{Z\}$ and $M=\{R\in W|R\not\cong Z\}$. Let P be any hereditary radical not containing any Z_p. By the previous proposition we have

$$P' = U(SP\cap M) = U((SP\cap M)\cup(SP\cap H)) = U(SP\cap W).$$

5. Complementary radicals

Complementary radicals were defined in [1] as follows. Given a radical Q, a radical G is called *complementary* to Q if G is the largest radical (assuming such a radical exists) such that $G(R)\cap Q(R)=0$ for every ring R. We generalize this concept and define the P-relative complementary radical to Q, where P is also a radical. A radical G is called *P-relative complementary* to Q if G is the largest radical (assuming such a radical exists) such that $G(R)\cap Q(R)\subseteq P(R)$ for every ring R. If we denote the class $\{0\}$, by 0, then the radical 0-relative complementary to Q is the complementary radical to Q as defined in [1] by ANDRUNAKIEVITCH. We will denote the P-relative complementary radical to Q by $(P:Q)$. It is clear that $(P:(P:Q))\supseteq Q$ when the P-relative complementary radicals exist.

The question of the existence of P-relative complementaries was partially settled, although not in this terminology, in [7]. There it was shown that, for hereditary radicals P and Q, the P-relative complementary radical to Q exists and is equal to $\{R|Q(R/I)\subseteq P(R/I)$ for every ideal I of $R\}$.

PROPOSITION 6. *Let P and Q be hereditary radical classes. Then*

$$(P:Q) = \{R|R/I\in SP \Rightarrow R/I\in SQ\}.$$

PROOF. We use the characterization of $(P:Q)$ derived by SNIDER in [7, p. 216] and quoted above. From that result it is clear that $(P:Q)\subseteq\{R|R/I\in SP\Rightarrow R/I\in SQ\}$. But if R is such that whenever $R/I\in SP$, then $R/I\in SQ$ and if $P(R/K)=J/K$, then $R/J\in SP$ and so is in SQ. Hence $Q(R/K)\subseteq J/K$ and we have the required equality.

COROLLARY 8. *When P and Q are hereditary $(P:Q)\supseteq P\supseteq(P:Q)\cap Q$ and $(P:Q)$ contains all strongly Q-semi-simple rings, where R is strongly Q-semi-simple if every $R/I\in SQ$.*

COROLLARY 9. *When P and Q are hereditary, $Q\cap SP\subseteq S(P:Q)$.*

PROOF. Let $R\in Q\cap SP$ and $I\lhd R$ with $I\in(P:Q)$. Then $I\in(P:Q)\cap Q$, so $I\in P$ by Corollary 8. But $R\in SP$ so $I=0$ and $R\in S(P:Q)$.

THEOREM 7. *Let P and Q be hereditary radical classes. The radical $(P:Q)$ is hypernilpotent if and only if $Q\cap SP\subseteq W_1$.*

PROOF. Suppose $Q\cap SP\subseteq W_1$ and let $R\in(P:Q)$, $K\lhd I\lhd R$. Put $P(I/K)=A/K$, $Q(I/K)=B/K$ so that A and B are ideals of I containing K. If $\bar{A}$ is the ideal of R generated by A, then $\bar{A}\subseteq I$ and $\bar{A}^3\subseteq A$. Hence $(A+B)/A\cap\bar{A}/A\in SP\cap Q\cap LN\subseteq W_1\cap LN=0$. Thus $(A+B)\cap\bar{A}=A$.

Consider the ring $R/\bar{A}$. Since $R\in(P:Q)$, $P(R/\bar{A})\supseteq Q(R/\bar{A})$. Also $(B+\bar{A})/\bar{A}\subseteq Q(R/\bar{A})$ and P is hereditary, so $(B+\bar{A})/\bar{A}\in P$. But

$$\frac{B+\bar{A}}{\bar{A}}=\frac{(B+A)+\bar{A}}{\bar{A}}\cong\frac{B+A}{(B+A)\cap\bar{A}}=\frac{B+A}{A}\lhd\frac{I}{A}\in SP$$

and so $B+\bar{A}=\bar{A}$. Hence $B\subseteq A$ and thus $I\in(P:Q)$. Therefore $(P:Q)$ is hereditary.

Further, if $R\in N$ and $A=P(R)$, $B=Q(R)$, then $(A+B)/A\in Q\cap SP\cap N=0$. Hence $A\supseteq B$ and so $R\in(P:Q)$. Thus $(P:Q)$ is a hypernilpotent radical class.

Conversely, we have seen in Corollary 9 that $Q\cap SP\subseteq S(P:Q)$ so if $(P:Q)$ is hypernilpotent $Q\cap SP\subseteq W_1$.

COROLLARY 10. *If P is a hypernilpotent radical class and Q is a hereditary radical class, then $(P:Q)$ is hypernilpotent.*

COROLLARY 11. *If P is a hereditary radical class and Q is a subidempotent radical class, then $(P:Q)$ is hypernilpotent.*

It follows from Corollary 11 that if Q is a subidempotent radical class, then its complementary radical class $(0:Q)$ is hypernilpotent. It is known that $(0:Q)$ is, in fact, special. This will be derived as a corollary of a later result.

THEOREM 8. *Let P and Q be hereditary radicals. If $Q\cap SP\supseteq N$, then the radical $(P:Q)$ is subidempotent.*

PROOF. Let $R\in(P:Q)$ and $I\lhd R$. Then $Q(R/I^2)\supseteq I/I^2$ and so $I/I^2\subseteq P(R/I^2)$. Hence $I/I^2\in P\cap N\subseteq P\cap SP=\{0\}$. Therefore $I=I^2$, every ideal in R is idempotent, and every accessible subring of R is an ideal of R. Let $J\lhd I\lhd R$ with $P(I/J)=A/J$ and $Q(I/J)=(B/J)$. Then $P(R/A)\supseteq Q(R/A)\supseteq(B+A)/A$. But P is hereditary, so $(B+A)/A\in P$. However, $(B+A)/A\lhd I/A\in SP$ and so $B\subseteq A$. Hence $I\in(P:Q)$ and $(P:Q)$ is subidempotent.

COROLLARY 12. *If P is a subidempotent radical class and Q is a hypernilpotent radical class, then $(P:Q)$ is subidempotent.*

COROLLARY 13 (ANDRUNAKIEVITCH). *If Q is a hypernilpotent radical class, then $(0:Q)$ is subidempotent.*

In the following theorems we include the hypothesis $Q \subsetneqq P$ to avoid the trivial case $(P:Q) = \{\text{all rings}\}$.

THEOREM 9. *Let P and Q be hereditary radicals with $Q \subsetneqq P$. A sufficient condition for $(P:Q) = U(M \setminus SQ)$ is that P have property (Int) relative to a class M for which $U(M \setminus SQ)$ is radical. This is true in particular if M is a hereditary class of rings.*

PROOF. If $0 \neq R \in S(P:Q)$ then by Proposition 6 every $0 \neq I \lhd R$ has an image $I/J \in SP$ but not in SQ. Hence by the intersection property I/J has a non-zero image in $(M \setminus SQ)$. Thus $I \notin U(M \setminus SQ)$ and $R \in SU(M \setminus SQ)$, so $S(P:Q) \subseteq SU(M \setminus SQ)$.

On the other hand if $R \in SU(M \setminus SQ)$ then every ideal of R has a non-zero image K in $(M \setminus SQ)$, which is not in $(P:Q)$ since $K \in (P:Q) \cap SP$ would imply $K \in SQ$. Thus $S(P:Q) \supseteq SU(M \setminus SQ)$, and since $U(M \setminus SQ)$ is radical we have $(P:Q) = U(M \setminus SQ)$.

Now if M is hereditary and $R \in M \setminus SQ$ then R has a non-zero Q-radical $I \in M \cap Q \subseteq SP \cap Q$. Let $J \lhd R$ be maximal relative to $I \cap J = 0$ then $I \cong \bar{I} = (I+J)/J$ is essential in $\bar{R} = R/J$. Thus if $0 \neq \bar{A} \lhd \bar{R}$ then $\bar{A} \in SP$ since otherwise its P-radical would meet $\bar{I} \in SP$, and also $\bar{A} \notin SQ$ since $0 \neq \bar{I} \cap \bar{A}$. By property (Int) $\bar{A}$ has an image in $M \setminus SQ$ so $\bar{R} \in SU(M \setminus SQ)$ and hence by [4, Theorem 1] $U(M \setminus SQ)$ is radical.

COROLLARY 14 (ANDRUNAKIEVITCH). *The radical class $(0:Q)$ complementary to $Q \neq 0$ is the upper radical determined by all subdirectly irreducible rings with a Q-radical heart. The radical $(0:Q)$ is special if Q is subidempotent.*

PROOF. We take $P=0$ which has property (Int) relative to M, the class of all subdirectly irreducible rings. Also $M \setminus SQ$ is the class of all subdirectly irreducible rings with a Q-radical heart. If Q is subidempotent, then $M \setminus SQ$ is a special class.

We look finally at M-radicals, where M is a class of rings with the property (*) of Section 4. Thus, by Theorem 3, P is an M-radical if and only if $P = U(SP \cap M)$.

THEOREM 10. *Let P and Q be hereditary radical classes with $Q \subsetneqq P$. If P is an M-radical with property (Int) relative to $SP \cap M$ and $SP \cap M \setminus SQ$ is a class of prime rings, then $(P:Q)$ is a hereditary M-radical.*

PROOF. Let $0 \neq I \lhd R \in SP \cap M \setminus SQ$. Since I meets the Q-radical of R then $I \notin SQ$. But $I \in SP$ so by property (Int) I has a non-zero image in $SP \cap M \setminus SQ$. Thus $U(SP \cap M \setminus SQ)$ is radical so by Theorem 9, $(P:Q) = U(SP \cap M \setminus SQ)$ and clearly, since $(P:Q)$ is radical, $(P:Q) \subseteq U(S(P:Q) \cap M)$. Now if $R \notin (P:Q)$, then there is $I \lhd R$ with $0 \neq R/I \in SP \cap M \setminus SQ$. Let $I \subseteq B \lhd R$ with $0 \neq B/I \in Q$. Suppose that $I \subseteq J \lhd R$ with $J/I \in (P:Q)$. Since $R/I \in SP$ we have $J/I \in (P:Q) \cap SP$ and so $J/I \in SQ$. Hence $(B/I)(J/I) \in Q \cap SQ$ since Q is hereditary. Therefore $BJ \subseteq I$. Now $SP \cap M \setminus SQ$ is a class of prime rings so either $B \subseteq I$ or $J \subseteq I$. We must have $J \subseteq I$ and so $R/I \in S(P:Q)$. In fact, $R/I \in S(P:Q) \cap M$ so that $R \notin U(S(P:Q) \cap M)$. Thus $(P:Q) = U(S(P:Q) \cap M)$ and $(P:Q)$ is an M-radical.

Let $R \in (P:Q)$ and $I \lhd R$ with $I \notin (P:Q)$. Then there is $J \lhd I$ with $0 \neq I/J \in SP \cap M \setminus SQ$. But $SP \cap M \setminus SQ$ is a class of prime rings, so $J \lhd R$. Let $A/J = P(R/J)$, the P-radical of R/J. Then $(I \cap A)/J \in P \cap SP = \{0\}$. Hence $I \cap A = J$. Also $R/A \in SP$, so by Proposition 6, $R/A \in SQ$. However, $(I+A)/A \cong I/(I \cap A) = I/J \notin SQ$ which contradicts the hereditariness of SQ. Therefore we must have $(P:Q)$ hereditary.

COROLLARY 14. *Let P and Q be hereditary radicals with $Q \subsetneqq P$. If P is special, then $(P:Q)$ is special.*

PROOF. Take $M=W$, then $(P:Q)=U(S(P:Q)\cap W)$ by Theorem 10. Hence $(P:Q)$ is special.

COROLLARY 15. *Let M be a special class. If P is a hereditary M-radical and Q is a hereditary radical, then $(P:Q)$ is a special M-radical.*

PROOF. By Theorem 6, since $(P:Q)$ is hereditary, $(P:Q)=U(S(P:Q)\cap M)$ is a special M-radical.

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UNIVERSITY OF NEBRASKA
LINCOLN, NEBRASKA, USA

UNIVERSITY OF LEICESTER
LEICESTER, ENGLAND

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NOTE ON A THEOREM OF PÓLYA AND CATHERINE RÉNYI

By
R. R. HALL (York)

Introduction

In this note I prove the following result:

THEOREM. *Let $f(x)$ be a polynomial of degree n over a field F of characteristic zero and for each $w \in F$ set*

$$n(w,f) = \text{card}\,\{r : f^{(r)}(w) = 0\}.$$

Then for any distinct w_1, w_2 we have

$$n(w_1,f)+n(w_2,f) \leqq n.$$

REMARKS. This was proved by PÓLYA [1] and CATHERINE RÉNYI [2] in the case $F=\mathbf{C}$, by a simple method depending on Rolle's theorem. It seems worthwile to give a purely algebraic proof; and this permits the generalization to fields of characteristic zero.

If $w_1, w_2, \ldots, w_t$ are distinct elements of F then for $t \geqq 2$ we may deduce from the above that

$$n(w_1,f)+n(w_2,f)+\ldots+n(w_t,f) \leqq \tfrac{1}{2}\,tn, \tag{1}$$

and the polynomial $f(x)=(1-x^2)^m$ shows that there can be equality in (1) if $t=2$ or 3. Notice that if $t \geqq 3$ and (1) holds with equality, it is necessary that

$$n(w_1,f) = n(w_2,f) = \ldots = n(w_t,f) = n/2 \tag{2}$$

(so that n must be even), and it seems possible, though I am unable to prove, that for $t \geqq 4$ (2) cannot hold.

PROOF OF THE THEOREM. This is based on the following result:

LEMMA. *Let $0 \leqq n_1 < n_2 < \ldots < n_k$; $0 \leqq r_1 < r_2 < \ldots < r_k$ be given sets of integers. Then*

$$\det\left[\binom{n_i}{r_j}\right] \neq 0 \quad \textit{if and only if} \quad r_i \leqq n_i \quad \textit{for} \quad 1 \leqq i \leqq k.$$

The binomial coefficient $\binom{n}{r}$ is taken to be zero when $r>n$, and the condition on the right is necessary, since if $r_h > n_h$ for some $h \leqq k$, the first h rows of the determinant are dependent.

To prove sufficiency, it will be enough to show that under the conditions stated,

$$F(n_1, n_2, \ldots, n_k; r_1, r_2, \ldots, r_k) = \det\left[\frac{1}{(n_i-r_j)!}\right] > 0,$$

where $(-m)!=0$ for positive integers m and $0!=1$. The proof is by induction on

$$S=\sum_{i=1}^{k}(n_i-r_i)\geqq 0.$$

First, if $S=0$ we infer that $r_i=n_i$ for $1\leqq i\leqq k$ and the determinant is triangular: its value is 1. Suppose then that $S\geqq 1$ and that the result holds upto $S-1$. For any permutation τ,

$$S\prod_{i=1}^{k}\frac{1}{(n_i-r_{\tau(i)})!}=\sum_{h=1}^{k}(n_h-r_{\tau(h)})\prod_{i=1}^{k}\frac{1}{(n_i-r_{\tau(i)})!}$$

and so

$$SF(n_1, n_2, \dots, n_k; r_1, r_2, \dots, r_k)=\sum_{h=1}^{k}F(\dots, n_h-1, \dots; \dots)$$

the other $2k-1$ variables being unaltered. Notice that on the right, $S-1$ replaces S in every term.

By the induction hypothesis each term on the right is positive provided the new variables satisfy the conditions of the lemma, and this is the case unless either

(i) $n_h-1=n_{h-1}$

or

(ii) $n_h=r_h$.

In either of these cases, $F(\dots, n_h-1, \dots; \dots)=0$ so it will be necessary and sufficient to show that there exists an $h\leqq k$ for which both (i) and (ii) are false. Suppose that

$$r_{l+1}=a+1 \quad \text{and} \quad n_{l+u}=a+u \quad \text{for} \quad 1\leqq u\leqq v.$$

Then the condition

$$r_{l+1}<r_{l+2}<\dots<r_{l+v}\leqq n_{l+v}=a+v$$

requires that

$$r_{l+u}=a+u \quad \text{for} \quad 1\leqq u\leqq v$$

and so

$$\sum_{u=1}^{v}(n_{l+u}-r_{l+u})=0. \tag{3}$$

Therefore, if we bunch together any consecutive n_i's, the condition (i) does not hold at the left-hand end-points of these bunches, moreover (ii) cannot hold at all of them, in view of (3) and the fact that $S>0$. This completes the proof of the lemma.

It may be that quite general results can be found for determinants of the form $|g(n_i-r_j)|$.

It remains to prove the theorem. We may assume that $w_1=0$ and we put $w_2=w\neq 0$, and write

$$f(x)=\sum_{i=1}^{k}c_i x^{n_i}, \quad n_k=n, c_i\neq 0,$$

so that $n(0,f)=n+1-k$. Suppose that $n(w,f)\geqq k$: we have to show that this leads to a contradiction. There would exist integers $0\leqq r_1<r_2<\ldots<r_k$ such that

$$w^r \frac{f^{(r)}(w)}{r!} = \sum_{i=1}^{k} c_i \binom{n_i}{r} w^{n_i} = 0 \tag{4}$$

when $r=r_j$, $1\leqq j\leqq k$. We distinguish two cases, whether or not $r_i\leqq n_i$ for every $i\leqq k$. In the first case, (4) gives a system of k simultaneous equations with non--vanishing determinant for the k unknowns $c_i w^{n_i}$ which by hypothesis are non-zero, and this is the required contradiction.

In case two, let $h\geqq 1$ be the least integer with the property that $r_i\leqq n_i$ for $i>h$. Since in any event $r_k<n=n_k$ such an h exists. It will be seen that if we begin by differentiating f r_h times, the resulting polynomial is in case one. This completes the proof.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF YORK
HESLINGTON, YORK Y01 5DD
ENGLAND

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ÜBER EINEN SATZ VON A. KERTÉSZ

Von

DINH VAN HUYNH (Hanoi/Halle)

Unter einem Ring verstehen wir in dieser Note stets einen *assoziativen* Ring. Was die Terminologie betrifft, halten wir an diejenige des Buches [2]. In [3] hat A. KERTÉSZ den folgenden Satz bewiesen:

Es sei die additive Gruppe $(A, +)$ *eines Unterringes* A *des Ringes* R *ein direkter Summand der Gruppe* $(R, +)$: $(R, +)=(A, +)\oplus(B, +)$. *Gibt es in* R *ein Linkseinselement modulo* A, *d. h. ein Element* e *mit* $x-ex\in A$ *für jedes* $x\in R$, *und gilt* $AB=BA=(0)$, *so ist* A *ein ringtheoretischer direkter Summand von* R.

Das Ziel dieser kleinen Note ist es, diesen Satz in allgemeinerer Form zu beweisen. Es gilt nämlich der folgende

SATZ. *Es sei die additive Gruppe* $(A, +)$ *eines Unterringes* A *des Ringes* R *ein direkter Summand der Gruppe* $(R, +)$:

$$(1)\qquad (R, +) = (A, +)\oplus(B, +) \quad \textit{mit} \quad AB = BA = (0).$$

Gibt es zu jedem Element $x(\in R)$ *ein* e_x $(\in R)$ *mit*

$$(2)\qquad x-e_x x\in A,$$

und gibt es zu je endlich vielen $e_{x_i}(x_i, e_{x_i}\in R, i=1, 2, \ldots, n)$ *mit* $x_i-e_{x_i}x_i\in A$ *ein* y $(\in R)$ *derart, daß*

$$(3)\qquad e_{x_1}B+\ldots+e_{x_n}B\subseteqq yB$$

gilt, so ist A *ein ringtheoretischer direkter Summand von* R.

BEWEIS. Es sei R ein Ring mit den Bedingungen des Satzes und sei x ein Element aus R. Dann gibt es ein $e_x\in R$ mit $x-e_x x\in A$. Wegen (1) gilt $x=a+b$ $(a\in A, b\in B)$ und $e_x=e_a+e_b$ $(e_a\in A, e_b\in B)$. Daher folgt

$$x-e_x x = (a+b)-(e_a+e_b)(a+b) = b-e_b b+a-e_a a\in A.$$

Da A ein Unterring ist, folgt

$$(4)\qquad b-e_b b\in A.$$

Es gibt also zu jedem $b\in B$ ein $e_b\in B$ mit der Relation (4). Die Menge $e_b B\overset{\text{def}}{=}\{e_b b'\mid\forall b'\in B\}$ mit festem $e_b\in B$ ist offenbar eine Untergruppe von $(R, +)$. Mit B^* bezeichnen wir die von allen $e_b B$ $(e_b\in B)$ erzeugte Untergruppe von $(R, +)$. Wir wollen zunächst zeigen, daß B^* ein Unterring von R ist. Es genügt, $(e_{b_1}B)(e_{b_2}B)\subseteqq$

$\subseteq e_{b_1}B$ für beliebige $b_1, b_2 \in B$ zu zeigen. Wegen (1) gilt $b' e_{b_2} b'' = a + b$ $(a \in A, b, b', b'' \in B)$. Daraus ergibt sich

$$(e_{b_1}b')(e_{b_2}b'') = e_{b_1}a + e_{b_1}b = e_{b_1}b \in e_{b_1}B.$$

Die Unterringe B^* und A annullieren einander: $B^*A = AB^* = (0)$. Daß $A \cap B^* = (0)$ ist, sieht man so ein: Ist $r^* \in A \cap B^*$, so gilt etwa

$$r^* = e_{b_1}b_1' + \ldots + e_{b_t}b_t' = a \in A \quad (b_i, b_i' \in B).$$

Wegen (3) gibt es ein $y = y_1 + y_2$ $(y_1 \in A, y_2 \in B)$, so daß $r^* \in yB = y_2B$ gilt. Es gibt also ein b' in B mit $r^* = y_2 b' = a \in A$. Es existiert ein e_{y_2} in B mit $y_2 - e_{y_2}y_2 \in A$, woraus $y_2 b' = e_{y_2} y_2 b'$ folgt. Daher ergibt sich

$$r^* = y_2 b' = e_{y_2}(y_2 b') = e_{y_2} a = 0.$$

Es sei $r \in R$. Wegen (1) gilt $r = a + b$ $(a \in A, b \in B)$. Es gibt in B ein e_b, so daß $b - e_b b = a' \in A$ gilt. Es ist also $b = a' + e_b b$. Setzen wir dieses in der Darstellung von r ein, so haben wir

$$r = (a + a') + e_b b \in A + B^*.$$

R ist also die Summe von A und B^*. Folglich ist die ringtheoretische direkte Zerlegung $R = A \boxplus B^*$ gültig.

Damit ist der Beweis des Satzes erbracht.

Wir formulieren jetzt eine Folgerung, die wir in [1] als Satz 7 angegeben haben:

FOLGERUNG ([1], Satz 7). *Es sei R ein MHR-Ring (d. h. ein Ring mit Minimalbedingung für Hauptrechtsideale). Ist auch das Jacobsonsche Radikal $J(R)$ von R ein MHR-Ring, so ist R die ringtheoretische direkte Summe seines maximalen Torsionsideales und eines radikalfreien torsionsfreien MHR-Ringes.*

BEWEIS. Aus [5] (Satz 3.1) ergibt sich die gruppentheoretische direkte Zerlegung $(R, +) = F \oplus T$ für jeden MHR-Ring R, wobei F torsionsfreie (teilbare) Untergruppe und T die maximale Torsionsuntergruppe von $(R, +)$ ist, für die $FT = TF = (0)$ gilt. Es sei $J(R)$ auch ein MHR-Ring. Dann gilt $J(R) \subseteq T$ (vgl. [5], Satz 3.3). Da jedes homomorphes Bild eines MHR-Ringes wieder ein MHR-Ring ist, ist $\bar{R} \overset{\text{def}}{=} R/T$ ein torsionsfreier radikalfreier MHR-Ring. Es sei r ein beliebiges Element aus R. Dann gibt es nach [4] (Satz 4) ein e_r in R derart, daß $r - e_r r \in T$ gilt.

Es seien $r_1, r_2, \ldots, r_n$ endlich viele Elemente aus R und seien $e_{r_1}, \ldots, e_{r_n}$ derartige Elemente aus R, die die Bedingung $r_i - e_{r_i} r_i \in T$ $(i = 1, 2, \ldots, n)$ erfüllen. Ferner sei R' das von $e_{r_1}, \ldots, e_{r_n}$ und T erzeugte Rechtsideal von R. Dann besitzt R' nach [4] (Satz 4) ein Linkseinselement e *modulo* T. Da T ein Ideal von R ist, können wir $e \in F \cap R'$ wählen. Für e_{r_i} gilt also

$$ee_{r_i} = e_{r_i} + x_i \quad (x_i \in T, \quad i = 1, 2, \ldots, n).$$

Daraus und aus $e_{r_1}F + \ldots + e_{r_n}F \subseteq R = F + T$ und $FT = TF = (0)$ folgt

$$e(e_{r_1}F + \ldots + e_{r_n}F) = e_{r_1}F + \ldots + e_{r_n}F \subseteq e(F + T) = eF.$$

Setzen wir $A = T$, $B = F$, so erfüllt R die Bedingungen des Satzes. Damit folgt die Behauptung aus obigem Satz.

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SEKTION MATHEMATIK DER UNIVERSITÄT
DDR-401 HALLE, UNIVERSITÄTSPLATZ 8/9

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MAXIMAL CIRCUITS OF GRAPHS. I

By
D. R. WOODALL (Nottingham)

1. Introduction. All graphs considered are finite, undirected, and without loops or multiple edges. Circuits and paths are 'elementary', i.e., have no repeated vertices. $V(G)$ denotes the set of vertices of G. $[x]$ denotes the greatest integer less than or equal to x.

In [2, Theorem (2.7)], ERDŐS and GALLAI proved that, if $d \geqq 2$ and G is a graph on n vertices with more than $\frac{1}{2}d(n-1)$ edges, then G contains a circuit of length at least $d+1$. They pointed out that this result is best possible when n is of the form $t(d-1)+1$, in view of the graph consisting of t copies of K_d all having exactly one vertex in common. Here I obtain the slight improvement:

THEOREM 1. *If $d \geqq 2$, and $n=t(d-1)+p+1$ where $t \geqq 0$ and $0 \leqq p < d-1$, and G is a graph on n vertices with more than $t\binom{d}{2}+\binom{p+1}{2}$ edges, then G contains a circuit of length at least $d+1$.*

This result is best possible for every value of n, in view of the graph consisting of t copies of K_d and one copy of K_{p+1}, all having exactly one vertex in common.

An exactly analogous situation holds for paths, where ERDŐS and GALLAI [2, Theorem (2.6)] proved that, if $d \geqq 0$ and G is a graph on n vertices with more than $\frac{1}{2}dn$ edges, then G contains a path of length at least $d+1$. This is best possible when n is of the form $t(d+1)$, in view of the graph consisting of t disjoint copies of K_{d+1}. The analogous improvement, best possible for all values of n, is given in Corollary 1.1. (This result was first proved by FAUDREE and SCHELP [3, Theorem 5], who also characterized the extremal graphs.)

COROLLARY 1.1. *If $d \geqq 0$, and $n=t(d+1)+p$ where $t \geqq 0$ and $0 \leqq p < d+1$, and G is a graph on n vertices with more than $t\binom{d+1}{2}+\binom{p}{2}$ edges, then G contains a path of length at least $d+1$.*

PROOF. Add a new vertex to G, joined to all the vertices of G by edges, to form a new graph G^* with $n+1=t(d+1)+p+1$ vertices and more than

$$t\binom{d+1}{2}+\binom{p}{2}+n = t\binom{d+2}{2}+\binom{p+1}{2}$$

edges. By Theorem 1, G^* contains a circuit of length at least $d+3$, and so G contains a path of length at least $d+1$. This completes the proof.

If $a, b, c \geqq 0$, let $K(a, b, c)$ denote the graph consisting of K_{a+b} and $K_{b,c}$ with b of the vertices of K_{a+b} identified with the 'first' b vertices of $K_{b,c}$ (so that $K(a, b, c)$ has $a+b+c$ vertices). If $d \geqq 2$, $j \leqq \frac{1}{2}d$ and $n \geqq d+1-j$, let

$$f(n,j,d) := \binom{d-j+1}{2} + j(j+n-d-1),*$$

the number of edges in the graph $K(d-2j+1, j, j+n-d-1)$, which has n vertices and in which the longest circuit has length d $\left(\text{if } j \leqq \frac{1}{2}d\right)$. The proof of Theorem 1 uses:

THEOREM 2. *If $d \geqq 2$ and $n \geqq \frac{3}{2}d-1$, and G is a 2-connected graph on n vertices with more than $f\left(n, \frac{1}{2}d, d\right)$ edges, then G contains a circuit of length at least $d+1$.*

Note that this bound agrees (in effect) with that of Theorem 1 if $p=\frac{1}{2}d-1$, $\frac{1}{2}d-\frac{1}{2}$ or $\frac{1}{2}d$; otherwise it is less than that of Theorem 1. If d is even, Theorem 2 is best possible except for the restriction on the value of n. The following conjecture would be best possible for all values of n, in view of the graphs $K(d-2j+1, j, j+n-d-1)$ $\left(k \leqq j \leqq \left[\frac{1}{2}d\right]\right)$.

CONJECTURE. *If $d \geqq 2$, $2 \leqq k \leqq \frac{1}{2}d$ and $n \geqq d+1$, and G is a 2-connected graph on n vertices with more than*

$$\max\left(f(n, k, d), f\left(n, \left[\tfrac{1}{2}d\right], d\right)\right)$$

edges in which each vertex has valency at least k, then G contains a circuit of length at least $d+1$. (If nothing is known about the valencies, replace k by 2.)

Note that $f(n, k, d) \leqq f\left(n, \left[\frac{1}{2}d\right], d\right)$ whenever k is greater than about $\frac{1}{6}(5d-4n)$, so that the bound in the conjecture is always equal to $f\left(n, \left[\frac{1}{2}d\right], d\right)$ if $n > \frac{5}{4}d$. The conjecture is true for any values of n and $d (=: n-r-1)$ for which the conjecture on page 747 of [4] is true. (I have recently noticed that the latter conjecture can be false if $n \geqq \frac{3}{2}d+2$, in view of graphs consisting of three or more copies of $K\left(0, \left[\frac{1}{4}d\right], \left[\frac{1}{4}(d+10)\right]\right)$, disjoint except for two vertices which appear among the $\left[\frac{1}{4}(d+10)\right]$ vertices in each copy.)

Theorem 2 has the following corollary, which was proved by ERDŐS and GALLAI [2, Theorem (3.4)] subject to the stronger restriction that $n > k^2-k+6$.

COROLLARY 2.1. *If $n \geqq 3k+2$ and $k \geqq 0$, and G is a connected graph on n vertices with more than $\binom{k}{2}+k(n-k)$ edges, then G contains a path of length at least $2k+1$.*

PROOF. Add a new vertex to G, joined to all the vertices of G, to form a 2-connected graph G^* with $n+1$ vertices and more than

$$\binom{k}{2}+k(n-k)+n = \binom{k+2}{2}+(k+1)((n+1)-k-2)$$

* Throughout the paper the symbol := or =: indicates that the equation in which it occurs acts as the definition of (some part of) the expression on the same side of the equality sign as the colon.

edges. By Theorem 2 with $d=2k+2$, G^* contains a circuit of length at least $2k+3$, and so G contains a path of length at least $2k+1$.

2. Proofs of the theorems. LEMMA. *If $d\geqq 2$ and $d+1\leqq n\leqq 2d-1$, and G is a graph on n vertices with more than $\binom{d}{2}+\binom{n-d+1}{2}$ edges, then G contains a circuit of length at least $d+1$.*

PROOF. Put $d+1=n-r$ in Corollary 11.1 of [4].

PROOF OF THEOREM 2 by induction on n. If $n=\frac{3}{2}d-1$, then $f\left(n,\frac{1}{2}d,d\right)=$ $=\binom{d}{2}+\binom{\frac{1}{2}d}{2}$ and the result follows by the Lemma. If $n=\frac{3}{2}d-\frac{1}{2}$, then $f\left(n,\frac{1}{2}d,d\right)=$ $=\binom{d}{2}+\binom{\frac{1}{2}(d+1)}{2}+\frac{1}{8}$, and the result follows similarly. So the induction starts.

If every vertex of G has valency at least $\frac{1}{2}(d+1)$, then the result follows by Theorem 4 of DIRAC [1]. If G contains a vertex v with valency $\leqq\frac{1}{2}d$, then $G\setminus\{v\}$ has more than $f\left(n-1,\frac{1}{2}d,d\right)$ edges, and the result follows by the induction hypothesis if $G\setminus\{v\}$ is 2-connected. So we may suppose that G contains at least one vertex with valency $\leqq\frac{1}{2}d$, and that, if v is any such vertex, then $G\setminus\{v\}$ is not 2-connected.

Let $\{a,b\}$ be a separating set of two vertices, and let L be a *lune* of G attached at a and b, i.e., a subgraph with $|V(L)|\geqq 3$ such that a and b are the only vertices of L joined to anything outside L, and $L\setminus\{a,b\}$ is connected; and choose a, b and L so that L is minimal (by inclusion). Suppose first that $L\setminus\{a,b\}$ contains a vertex v with valency $\leqq\frac{1}{2}d$, and consider the possibilities for a vertex w such that $\{v,w\}$ is a separating set. Certainly $w\notin L$, or there would be a smaller lune within L, attached at v and w. But if $w\notin L$, the only way in which we can avoid $\{v,a\}$ or $\{v,b\}$ being a separating set (giving a smaller lune) is to have $L=\{a,b,v,(a,v),(v,b)\}$, and now $\{(a,b)\}\cup G\setminus\{v,(a,v),(v,b)\}$ satisfies the hypotheses of the theorem and the result follows by induction. So we may suppose that every vertex of $L\setminus\{a,b\}$ has valency at least $\frac{1}{2}(d+1)$. By Lemma 12.4 of [5], a and b are connected by a path of length at least $\frac{1}{2}(d+1)$ in L.

Let L' be another minimal lune of G, attached at c and d. (Possibly $\{c,d\}=$ $=\{a,b\}$.) By the same argument, c and d are connected by a path of length at least $\frac{1}{2}(d+1)$ in L'. Since G is 2-connected, $\{a,b\}$ is connected to $\{c,d\}$ by two disjoint paths, which clearly do not contain any vertices of L or L' apart from a,b,c,d themselves. So G contains a circuit of length at least $d+1$. This completes the proof.

PROOF OF THEOREM 1 by induction on n. The result is (vacuously) true if $n\leqq d$, and it follows from the Lemma if $d+1\leqq n\leqq 2d-1$; so suppose that $n\geqq 2d$. If G is 2-connected, the result follows from Theorem 2; so we may suppose that $G=G_1\cup G_2$, where G_1 and G_2 either are disjoint or have exactly one vertex in common. Let G_i have n_i vertices, where

$$n_i=t_i(d-1)+p_i+1\quad (0\leqq p_i<d-1;\ i=1,2).$$

Then $n_1+n_2=n$ or $n+1$, and either

$$t_1+t_2 = t \quad \text{and} \quad p_1+p_2 = p-1 \quad \text{or} \quad p,$$

or

$$t_1+t_2 = t-1 \quad \text{and} \quad p_1+p_2 = d-1+p-1 \quad \text{or} \quad d-1+p.$$

Suppose that neither G_1 nor G_2 satisfies the hypotheses of the theorem. Then the number of edges in G is at most

$$t_1\binom{d}{2}+\binom{p_1+1}{2}+t_2\binom{d}{2}+\binom{p_2+1}{2} =: N.$$

If $t_1+t_2 = t$, then $(p_1+1)+(p_2+1)\leqq p+2$, and $p_1+1\leqq p+1$ and $p_2+1\leqq p+1$, and so

$$N \leqq t\binom{d}{2}+\binom{p+1}{2}+\binom{1}{2} = t\binom{d}{2}+\binom{p+1}{2},$$

contrary to hypothesis. If, on the other hand, $t_1+t_2=t-1$, then $(p_1+1)+(p_2+1)\leqq$ $\leqq d+p+1$, and $p_1+1<d$ and $p_2+1<d$, and so

$$N < (t-1)\binom{d}{2}+\binom{d}{2}+\binom{p+1}{2} = t\binom{d}{2}+\binom{p+1}{2},$$

again contrary to hypothesis. Thus one of G_1 and G_2 must in fact satisfy the hypotheses of the theorem, and the result follows by induction.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NOTTINGHAM
NOTTINGHAM, ENGLAND NG7 2RD

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ON INTEGRAL INEQUALITIES OF BIHARI TYPE

By

P. R. BEESACK (Ottawa)

In a recent paper, S. G. DEO and M. G. MURDESHWAR [3] presented two theorems, one of which can be regarded as a generalization of a basic inequality of I. BIHARI [1], and the other of which is false. In this note, we obtain several extensions of Bihari's result including an extended version of Theorem 1 of [3] and a replacement for the false Theorem 2.

For brevity we shall use the common notations: $f\in\uparrow$ or $f\in\downarrow$ to indicate that a function f is nondecreasing or nonincreasing on its domain. We also will, occasionally, use the logical symbols $\vee$, $\wedge$ to denote the logical "or", "and".

The following lemma, which is a simple extension of Bihari's lemma [1, § 3], is required.

LEMMA. *Let g be a continuous, monotonic function which is never zero on an interval I containing a point u_0. Let x and k be continuous functions on an interval $J=[\alpha, \beta]$ such that $x(J)\subset I$ and k does not change sign on J. Let $a\in I$, and*

$$(1)\qquad x(t) \leqq a+\int_{\alpha}^{t} k(s)g\big(x(s)\big)\,ds, \quad t\in J.$$

If either g is nondecreasing and $k\geqq 0$, or g is nonincreasing and $k\leqq 0$, then

$$(1')\qquad x(t) \leqq G^{-1}\Big\{G(a)+\int_{\alpha}^{t} k\,ds\Big\}, \quad \alpha \leqq t < \beta_1,$$

where $G(u)\equiv \int_{u_0}^{u} dy/g(y)$, $u\in I$, and $\beta_1=\min(u_1, u_2)$ with

$$u_1 = \sup\Big\{u\in J\colon a+\int_{\alpha}^{t} k(s)g\big(x(s)\big)\,ds\in I,\ \alpha \leqq t \leqq u\Big\},$$

$$u_2 = \sup\Big\{u\in J\colon G(a)+\int_{\alpha}^{t} k\,ds\in G(I),\ \alpha \leqq t \leqq u\Big\}.$$

The result is also valid if $\leqq$ is replaced by $\geqq$ in both (1) *and* (1′). *Finally, either result is valid if* $[\alpha, \beta)$, $[\alpha, \beta_1)$, $\int_{\alpha}^{t}$ *are replaced throughout by* $(\alpha, \beta]$, $(\alpha_1, \beta]$, $\int_{t}^{\beta}$, *where* *now* $\alpha_1 = \max(v_1, v_2)$ *with*

$$v_1 = \inf\left\{v \in J:\ a + \int_t^\beta k(s)g(x(s))\,ds \in I,\ v \leqq t \leqq \beta\right\},$$

$$v_2 = \inf\left\{v \in J:\ G(a) + \int_t^\beta k\,ds \in G(I),\ v \leqq t \leqq \beta\right\}.$$

PROOF. We suppose first that $g>0$. Let

$$U(t) \equiv a + \int_\alpha^t k(s)g(x(s))\,ds, \quad t \in J.$$

Then $U(t) \in I$ for $\alpha \leqq t < \beta_1$ and since $x(t) \leqq U(t)$ by (1),

$$\frac{d}{ds} G(U(s)) = \frac{g(x(s))}{g(U(s))} k(s) \leqq k(s)$$

whether $k \geqq 0$ and $g \in \uparrow$, or $k \leqq 0$ and $g \in \downarrow$. Integrating over $[\alpha, t]$ we obtain

$$G(U(t)) \leqq G(a) + \int_\alpha^t k\,ds$$

in both cases. Since $t<\beta_1$ implies $t<u_2$ and since G^{-1} (with G) is strictly increasing for $g>0$, it follows that in both cases

$$(1'') \qquad U(t) \leqq G^{-1}\left\{G(a) + \int_\alpha^t k\,ds\right\}, \quad \alpha \leqq t < \beta_1,$$

The inequality (1′) now follows from (1) and (1″).

If $x(t) \geqq U(t)$, all of the inequalities of the proof are reversed, including that in (1′). Now, suppose that $g<0$. If we define $g_1 = -g$, $k_1 = -k$, then the functions x, k_1, g_1 satisfy the hypotheses already dealt with for x, k, g. Moreover, if $G_1(u) \equiv \int_{u_0}^u dy/g_1(y)$, then $G_1(u) \equiv -G(u)$, whence $G_1^{-1}(v) \equiv G^{-1}(-v)$, so that the conclusion

$$x(t) \leqq G_1^{-1}\left\{G_1(a) + \int_\alpha^t k_1\,ds\right\} = G^{-1}\left\{G(a) + \int_\alpha^t k\,ds\right\}$$

follows.

To prove the final statement of the lemma, it suffices to make the change of variables $t=-u$, $s=-\sigma$, $k_1(u)=k(-u)$, $x_1(u)=x(-u)$ in

$$x(t) \leqq a + \int_t^\beta k(a)g(x(s))\,ds, \quad \alpha < t \leqq \beta,$$

to obtain

$$x_1(u) \leqq a + \int_{-\beta}^u k_1(\sigma)g(x_1(\sigma))\,d\sigma, \quad -\beta \leqq u < -\alpha.$$

The conclusion now follows on applying the parts already proved to the functions x_1, k_1 on $[-\beta, \alpha)$, and g on I.

Bihari's lemma is the special case having $I=J=[0, \infty)$ and $k \geqq 0$ with $g>0$. In this case, $u_1=\infty$ and $\beta_1=u_2$. As noted in [1], the estimate (1′) is independent of the choice of $u_0 \in I$.

The extended version of [3, Th. 1] is given by the following theorem. Before stating the result we note that the *proof* of Theorem 1 of [3] is defective, and in fact has the same error as that made in obtaining the (false) Theorem 2: one can *not* assume that $x(s) \geqq a(s)$ for $\alpha \leqq s \leqq t$ in order to prove (2′) — when $h>0$ — although one *can* assume $x(t) \geqq a(t)$.

THEOREM 1. *Let x, a, k be continuous functions such that k does not change sign on $J=[\alpha, \beta)$. Let g be continuous, monotonic, and never zero on an interval I_0 such that $x(J) \subset I_0$ and $a(J) \subset I_0$. Suppose also that the function h is continuous and monotonic on an interval I such that $0 \in I$, $h(I) \subset I_0$, and that any one of the conditions*

(i) $h \in \uparrow$, $g \in \uparrow$, *g is subadditive*, $k \geqq 0$, $g>0$,
(ii) $h \in \downarrow$, $g \in \uparrow$, *g is subadditive*, $k \leqq 0$, $g>0$,
(iii) $h \in \uparrow$, $g \in \downarrow$, *g is superadditive*, $k \leqq 0$, $g<0$,
(iv) $h \in \downarrow$, $g \in \downarrow$, *g is superadditive*, $k \geqq 0$, $g<0$,

is satisfied. Then

$$x(t) \leqq a(t)+h\Big(\int_\alpha^t k(s)g(x(s))\,ds\Big), \quad t \in J, \tag{2}$$

implies

$$x(t) \leqq a(t)+h\Big\{G^{-1}\Big[\int_\alpha^t k\,ds+G\Big(\int_\alpha^t k(s)g(a(s))\,ds\Big)\Big]\Big\} \tag{2'}$$

for $\alpha \leqq t < \beta_1$, *where* $G(u) \equiv \int_{u_0}^{u} dy/g(h(y))$ *for* $u \in I (u_0 \in I)$, *and* $\beta_1 = \min_{1 \leqq i \leqq 3} u_i$, *with*

$$u_1 = \sup\Big\{u \in J\colon a(t)+h\Big(\int_\alpha^t k(s)g(x(s))\,ds\Big) \in I_0,\ \alpha \leqq t \leqq u\Big\},$$

$$u_2 = \sup\Big\{u \in J\colon \int_\alpha^u k(s)\Big\{g(a(s))+g\circ h\Big(\int_\alpha^u k(r)g(x(r))\,dr\Big)\Big\}\,ds \in I\Big\},$$

$$u_3 = \sup\Big\{u \in J\colon \int_\alpha^t k\,ds+G\Big(\int_\alpha^T k(s)g(a(s))\,ds\Big) \in G(I),\ \alpha \leqq t \leqq T \leqq u\Big\}.$$

The result is valid if $\leqq$ is replaced by $\geqq$ in both (2) *and* (2′) *provided the conditions* (i)—(iv) *are replaced by*

(i′) $h \in \downarrow$, $g \in \downarrow$, *g is subadditive*, $k \geqq 0$, $g>0$,
(ii′) $h \in \uparrow$, $g \in \downarrow$, *g is subadditive*, $k \leqq 0$, $g>0$,
(iii′) $h \in \downarrow$, $g \in \uparrow$, *g is superadditive*, $k \leqq 0$, $g<0$,
(iv′) $h \in \uparrow$, $g \in \uparrow$, *g is superadditive*, $k \geqq 0$, $g<0$.

Finally, both results remain valid if $[\alpha,\beta)$, $[\alpha,\beta_1)$, $\int_\alpha^t$ *are replaced by* $(\alpha,\beta]$, $(\alpha_1,\beta]$, $\int_t^\beta$ *where now* $\alpha_1=\max_{1\le i\le 3} v_i$, *with*

$$v_1=\inf\Big\{v\in J\colon a(t)+h\Big(\int_t^\beta k(s)g(x(s))\,ds\Big)\in I_0,\ v\leqq t\leqq\beta\Big\},$$

$$v_2=\inf\Big\{v\in J\colon \int_v^\beta k(s)\Big\{g(a(s))+g\circ h\Big(\int_s^\beta k(r)g(a(r))\,dr\Big\}\,ds\in I\Big\},$$

$$v_3=\inf\Big\{v\in I\colon \int_t^\beta k\,ds+G\Big(\int_T^\beta k(s)g(a(s))\,ds\Big)\in G(I),\ v\leqq T\leqq t\leqq\beta\Big\}.$$

PROOF. Define $U(t)\equiv\int_\alpha^t k(s)g(x(s))ds$ and note that (2) implies that $U(J)\subset I$, and that

$$g(x(s))\lesseqqgtr g[a(s)+h(U(s))]\lesseqqgtr g(a(s))+g\circ h(U(s))$$

where $\leqq$ or $\geqq$ holds according as g is nondecreasing and subadditive, or nonincreasing and superadditive. Therefore

$$U'(s)\lesseqqgtr k(s)g(a(s))+k(s)g\circ h(U(s))$$

where $\leqq$ or $\geqq$ holds according as (a_1): ($g\in\uparrow$, g subadditive, $k\geqq 0$) $\vee$ ($g\in\downarrow$, g superadditive, $k\leqq 0$), or (b_1): ($g\in\uparrow$, g subadditive, $k\leqq 0$) $\vee$ ($g\in\downarrow$, g superadditive, $k\geqq 0$). Integrating from α to t, this reduces to

$$U(t)\lesseqqgtr\int_\alpha^t k(s)g(a(s))\,ds+\int_\alpha^t k(s)g\circ h(U(s))\,ds \tag{3}$$

where $\leqq$ or $\geqq$ holds according as (a_1) or (b_1) holds.

Now, fix $T\in(\alpha,\beta_1)$. Then by (3), if we set $A(t)\equiv\int_\alpha^t k(s)g(a(s))\,ds$, it follows that for $\alpha\leqq t\leqq T$,

$$U(t)\lesseqqgtr A(T)+\int_\alpha^t k(s)g\circ h(U(s))\,ds \tag{4}$$

where $\leqq$ or $\geqq$ holds according as (a_1) holds and k,g have the same sign, or (b_1) holds and k,g have the opposite sign. Since $U([\alpha,T])\subset I$, it follows from the lemma that

$$U(t)\lesseqqgtr G^{-1}\Big[\int_\alpha^t k\,ds+G(A(T))\Big]\quad\text{for}\quad \alpha\leqq t\leqq T. \tag{4'}$$

(Observe that $\alpha\leqq t\leqq T<\beta_1$ implies that $A(T)+\int_\alpha^t k(s)g\circ h(U(s))ds$ lies between 0 and $A(T)+\int_\alpha^T k(s)g\circ h(U(s))ds$ in all cases (i)–(iv), (i′)–(iv′), and hence

$A(T)+\int_\alpha^t k(s)g\circ h(U(s))ds\in I$ follows.) By the lemma, $\leqq$ or $\geqq$ holds in (4′) according as (a_2): $((a_1),\ g\circ h\in\uparrow,\ k\geqq 0,\ g\geqq 0)\vee((a_1),\ g\circ h\in\downarrow,\ k\leqq 0,\ g\leqq 0)$, or (b_2): $((b_1),\ g\circ h\in\uparrow,\ k\geqq 0,\ g\leqq 0)\vee((b_1),\ g\circ h\in\downarrow,\ k\leqq 0,\ g\geqq 0)$. On analysis, these conditions reduce to (a_2): (i) or (iii) hold, or (b_2): (ii) or (iv) hold. From (4′), with $t=T$ and a change of notation,

$$h(U(t))\leqq h\left\{G^{-1}\left[\int_\alpha^t k\,ds+G(A(t))\right]\right\} \tag{5}$$

holds for either (a_2) or (b_2). Hence, (2′) follows in all four cases (i)—(iv), as asserted.

If $\leqq$ is replaced by $\geqq$ in (2), the only change in the analysis preceding conditions (a_2), (b_2) is that the roles of "$g\in\uparrow$" and "$g\in\downarrow$" are interchanged. The new (a_2) reduces to conditions (i′) or (iii′), and the new (b_2) reduces to conditions (ii′) or (iv′). Hence we obtain (5) and (2′) with $\leqq$ replaced by $\geqq$ in all four cases (i′)—(iv′). The final part of the theorem follows precisely as in the proof of the lemma by the change of variables $t=-u$, $s=-\sigma$, $x_1(u)=x(-u)$, $a_1(u)=a(-u)$, $k_1(s)=k(-s)$.

Theorem 1 of [3] is the special case of the first part of the above theorem having $J=I=[0,\infty]$, and $x\geqq 0$, $a\geqq 0$, $k\geqq 0$, $h\geqq 0$, $h\in\uparrow$; moreover $g>0$, $g\in\uparrow$ and g is subadditive. The domain I_0 of g must *include* J. Note that in this case, $u_1=u_2=\infty$, so $\beta_1=u_3$ which, because k and G have the same sign, reduces to the result given in [3]. Again it is easy to verify that the estimate in (2′) is independent of the choice of $u_0\in I$.

To obtain a theorem involving an inequality of the form

$$x(t)\geqq a(t)-h\left(\int_\alpha^t k(s)g(x(s))\,ds\right),\quad \alpha\leqq t\leqq\beta, \tag{6}$$

one needs only replace h by $(-h)$ in the second part of Theorem 1. Thus, *with the preceding hypotheses, except that now* $(-h)(I)\subset I_0$, $G_1(u)\equiv\int_{u_0}^u dy/g(-h(y))$, *and any of the conditions*

(i′) $h\in\uparrow$, $g\in\downarrow$, *g subadditive*, $k\geqq 0$, $g>0$,
(ii′) $h\in\downarrow$, $g\in\downarrow$, *g subadditive*, $k\leqq 0$, $g>0$,
(iii′) $h\in\uparrow$, $g\in\uparrow$, *g superadditive*, $k\leqq 0$, $g<0$,
(iv′) $h\in\downarrow$, $g\in\uparrow$, *g superadditive*, $k\geqq 0$, $g<0$,

the inequality (6) *implies*

$$x(t)\geqq a(t)-h\left\{G_1^{-1}\left[\int_\alpha^t k\,ds+G_1\left(\int_\alpha^t k(s)g(a(s))\,ds\right)\right]\right\} \tag{6′}$$

for $\alpha\leqq t<\beta_1$. Here, β_1 is defined as before, but with G replaced by G_1 and h by $(-h)$.

In particular, contrary to the result stated in [3, Th. 2], no implication of the form (6)$\Rightarrow$(6′) is valid under the hypotheses: $h\in\uparrow$, $g\in\uparrow$, g subadditive, $k\geqq 0$, $g>0$. A simple counter-example is given by the inequality

$$x(t)\geqq 1-\int_0^t x(s)\,ds,\quad 0\leqq t\leqq T/2,$$

where $Te^T=1$, valid for $x(t)=\alpha(1-2T^{-1}t)$ with $\alpha\geqq 4T^{-1}$. The inequality (6′) — as well as the inequality in [3] — reduces to

$$x(t) \geqq 1-te^{-t}, \quad 0 \leqq t \leqq T/2,$$

and this is clearly false for t near $T/2$. In this context, we note that it was proved generally by C. E. LANGENHOP [5, Th. 2] that an implication

$$x(t) \geqq a-\int_\alpha^t k(s)g(x(s))\,ds \Rightarrow x(t) \geqq G^{-1}\Big\{G(a)-\int_\alpha^t k\,ds\Big\}$$

is *always* false when $k\geqq 0$, $g>0$, and g is nondecreasing.

We conclude this paper by noting that precisely the same method used in proving Theorem 1 also yields

THEOREM 2. *Let x, a, k, g, h satisfy the preliminary hypotheses of Theorem* 1, *and let b be a function which is continuous, does not change sign on $J=[\alpha,\beta)$, and satisfies $b(J)\subset I_0$. Let either one of the conditions*

(i) $h\in\uparrow$, $g\in\uparrow$, *g subadditive, submultiplicative,* $k\geqq 0$, $g>0$, $b\geqq 0$,
(ii) $h\in\downarrow$, $g\in\uparrow$, *g subadditive, submultiplicative,* $k\leqq 0$, $g>0$, $b\geqq 0$,

be satisfied. Then

$$x(t) \leqq a(t)+b(t)h\Big(\int_\alpha^t k(s)g(x(s))\,ds\Big), \quad t\in J, \tag{7}$$

implies that

$$x(t) \leqq a(t)+b(t)h\Big\{G^{-1}\Big[\int_\alpha^t k(s)g(b(s))\,ds+G\Big(\int_\alpha^t k(s)g(a(s))\,ds\Big)\Big]\Big\} \tag{7′}$$

for $\alpha\leqq t<\beta_1$, where G is defined as in Theorem 1, *and $\beta_1=\min_{1\leqq i\leqq 4} u_i$ with*

$$u_1 = \sup\{u\in J\colon\ a(t)+b(t)h(U(t))\in I_0,\ \alpha \leqq t \leqq u\},\quad U(t) \equiv \int_\alpha^t k(s)g(x(s))\,ds,$$

$$u_2 = \sup\{u\in J\colon\ b(t)h(U(t))\in I_0,\ \alpha \leqq t \leqq u\},$$

$$u_3 = \sup\Big\{u\in J\colon\ \int_\alpha^T k(s)g(a(s))\,ds+\int_\alpha^t k(s)g(b(s))g\circ h(U(s))\,ds\in I,\ \alpha \leqq t \leqq T \leqq u\Big\},$$

$$u_4 = \sup\Big\{u\in J\colon\ \int_\alpha^t k(s)g(b(s))\,ds+G\Big(\int_\alpha^T k(s)g(a(s))\,ds\Big)\in G(I),\ \alpha \leqq t \leqq T \leqq u\Big\}.$$

The result remains valid if $\leqq$ is replaced by $\geqq$ in both (7) *and* (7′), *provided the conditions* (i)—(ii) *are replaced by*

(i′) $h\in\downarrow$, $g\in\downarrow$, *g subadditive, submultiplicative,* $k\geqq 0$, $g>0$, $b\geqq 0$,
(ii′) $h\in\uparrow$, $g\in\downarrow$, *g subadditive, submultiplicative,* $k\leqq 0$, $g>0$, $b\geqq 0$.

Finally, both of these results remain valid if $[\alpha, \beta)$, $[\alpha, \beta_1)$, $\int_\alpha^t$ *are replaced throughout by* $(\alpha, \beta]$, $(\alpha_1, \beta]$, $\int_t^\beta$ *where now* $\alpha_1 = \max_{1 \leqq i \leqq 4} v_i$ *with the* v_i *defined in an obvious manner (cf. Theorem* 1).

The estimate in (7′) is again independent of the choice of $u_0 \in I$ used in defining G. As for the proof, we only note that the preceding proof applies down to (4) but with $g \circ h(U(s))$ replaced by $g[b(s)h(U(s))]$ throughout. Before applying the lemma, one uses the submultiplicative property of g; the lemma then applies to

$$U(t) \leqq A(T) + \int_\alpha^t K(s) g \circ h(U(s))\, ds,$$

with $K(s) \equiv k(s) g(b(s))$.

An inequality of the form (7) with $h = g^{-1}$ was considered by GOLLWITZER [4, Th. 1]. However, the results are disjoint (except for the case $g(u) \equiv u$) since Gollwitzer's hypotheses: g convex and submultiplicative on $[0, \infty)$ with $g(0) = 0$, imply [6, p. 23] that g is superadditive so that none of (i)—(iv) apply. However, in the case $g(u) \equiv u$, with $J = [\alpha, \beta)$, $I_0 = [0, \infty)$, $I = [0, \infty)$, and x, a, b, k all nonnegative, both estimates apply. Our estimate (7′) reduces to

$$x(t) \leqq a(t) + b(t) \exp\Big(\int_\alpha^t k(s) b(s)\, ds\Big) \cdot \int_\alpha^t k(s) a(s)\, ds, \tag{8}$$

while Gollwitzer's estimate reduces to

$$x(t) \leqq a(t) + b(t) \exp\Big(\int_\alpha^t k(s) b(s)\, ds\Big) \int_\alpha^t k(s) a(s) \exp\Big(-\int_\alpha^s k(r) b(r)\, dr\Big)\, ds, \tag{9}$$

both for $t \in J$. Gollwitzer's estimate is clearly better in this case, and indeed is best possible since if equality holds in

$$x(t) \leqq a(t) + b(t) \int_\alpha^t k(s) x(s)\, ds, \tag{10}$$

it also holds in (9) (cf. [7, Th. 0]).

More recently, S. G. DEO and U. D. DHONGADE [2, Lemma 2] also considered the inequality (7). Again their hypotheses require (among other things) that g be submultiplicative but superadditive, so that the two results are not in general comparable. Both apply, however, to the inequality (10) (provided $b(t) \geqq 1$, $a \in \uparrow$, $a > 0$, $x \geqq 0$, for [2], and $k \geqq 0$ for both), where now $I = I_0 = J = [0, \infty)$, and the estimate in [2] reduces to

$$x(t) \leqq a(t) b(t) (1 + u_0) \exp\Big(\int_0^t k(s) b(s)\, ds\Big), \qquad t > 0, \tag{11}$$

for arbitrary $u_0 > 0$. In (11) we may take $u_0 = 0$; taking $a(t)$, $b(t)$ constant ($a > 0$, $b > 1$), it is easy to see that (8) is better for small t while (11) is better for all t with $\int_0^t k\, ds \geqq 1$.

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CARLETON UNIVERSITY
OTTAWA, ONTARIO
CANADA

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ON THE ABSOLUTE CONVERGENCE OF LACUNARY ORTHONORMAL SERIES

By

W. LUH and G. SCHROETER (Giessen)

Let X be a measurable space with a positive measure μ. On a measurable set $E\subset X$ with finite measure we consider a system $\{f_n(x)\}$ of μ-integrable functions and the corresponding productsystem $\{g_n(x)\}$, which is defined by $g_0(x)\equiv 1$ and

$$g_n(x) = f_{\nu_1+1}(x)\cdot f_{\nu_2+1}(x)\dots f_{\nu_k+1}(x) \quad \text{for} \quad n = \sum_{j=1}^{k} 2^{\nu_j}$$

(especially we have $g_{2^k-1}(x)=f_k(x)$). The system $\{f_n(x)\}$ is called weakly multiplicative on E, if

$$\sum_{n=1}^{\infty}\Big|\int_E g_n(x)\,d\mu(x)\Big| < \infty$$

and it is called multiplicatively orthogonal on E, if

$$\int_E g_n(x)\,d\mu(x) = 0 \quad (n = 1, 2, \dots).$$

Recently G. ALEXITS, who has introduced these notions ([1] and [2]), proved the following theorem which states under certain hypotheses the absolute convergence of expansions in the functions $f_n(x)$ of a weakly multiplicative system.

THEOREM A (ALEXITS [1]). *Let $\{f_n(x)\}$ be a bounded, weakly multiplicative system on E which satisfies*

(1) $$\varliminf_{n\to\infty}\int_E |f_n(x)|\,d\mu(x) > 0.$$

Let $f(x)$ be a one-sided bounded, μ-integrable function on E and assume that the expansion-coefficients of f in the procudt functions $g_n(x)$:

$$(f, g_n) = \int_E f(x)\,g_n(x)\,d\mu(x) \quad (n = 1, 2, \dots)$$

vanish except perhaps the coefficients

$$(f, g_{2^n-1}) = (f, f_n) = \int_E f(x)\,f_n(x)\,d\mu(x).$$

Then the series $\sum_{n=1}^{\infty}|(f, f_n)|$ *is convergent.*

We shall prove that the hypotheses of Theorem A can be weakened somewhat and applied to the proof of classical results.

THEOREM 1. *Let $\{f_n(x)\}$ be a bounded weakly multiplicative system on E which satisfies* (1). *Suppose $\{f_n(x)\}$ to be splitted into p subsequences $\{f_n^{(j)}(x)\}$ $(j=1, \ldots, p)$ and design by $\{g_n^{(j)}(x)\}$ the product-system according to $\{f_n^{(j)}(x)\}$. Let $f(x)$ be a one-sided bounded, μ-integrable function on E and assume that for $j=1, \ldots, p$ the expansion-coefficients of f in the product-functions $g_n^{(j)}(x)$:*

$$(f, g_n^{(j)}) = \int_E f(x) g_n^{(j)}(x)\, d\mu(x) \quad (n = 1, 2, \ldots)$$

vanish except perhaps the coefficients

$$(f, g_{2^n-1}^{(j)}) = (f, f_n^{(j)}) = \int_E f(x) f_n^{(j)}(x)\, d\mu(x).$$

Then the series $\sum_{n=1}^{\infty} |(f, f_n)|$ is convergent.

PROOF. For $j=1, \ldots, p$ we have

$$\sum_{n=1}^{\infty} \left| \int_E g_n^{(j)}(x)\, d\mu(x) \right| \leqq \sum_{n=1}^{\infty} \left| \int_E g_n(x)\, d\mu(x) \right| < \infty.$$

Therefore $\{f_n^{(j)}(x)\}$ is weakly multiplicative on E and satisfies the hypotheses of Theorem A. Hence the series $\sum_{n=1}^{\infty} |(f, f_n^{(j)})|$ is convergent. It follows

$$\sum_{n=1}^{\infty} |(f, f_n)| = \sum_{j=1}^{p} \sum_{n=1}^{\infty} |(f, f_n^{(j)})| < \infty$$

which proves our assertion.

REMARK. A product function of $\{f_n(x)\}$ is not necessarily also a product function of any system $\{g_n^{(j)}(x)\}$.

We next consider systems $\{\varphi_n(x)\}$ of bounded, μ-integrable otrhonormal functions on E, which satisfy the relation

$$\varphi_k(x)\varphi_l(x) = \sum_{j=1}^{c(k+l)} \gamma_j^{(k,l)} \varphi_j(x) \tag{2}$$

for fixed $c \in \mathbf{N}$ with $\gamma_j^{(k+l)}$ real.

We remark that among the many systems with this recurrence-property, there are especially the trigonometric system, any μ-orthonormal polynomial system $\{\varphi_n(x)\}$ with $\varphi_n(x)$ exactly of degree n, the Walsh-system and many others.

Our purpose is to prove a theorem on the absolute convergence of orthonormal series with „Hadamard-gaps".

THEOREM 2. *Given any bounded system* $\{\varphi_n(x)\}$ *which satisfies* (1) *and* (2). *For a sequence* $\{n_k\}$ *of integers with*

$$\frac{n_{k+1}}{n_k} \geqq q > 1 \quad (k = 1, 2, \ldots)$$

denote by $\{\psi_n(x)\}$ *the product-system of* $\{\varphi_{n_k}(x)\}$. *Let* f *be one-sided bounded,* μ*-integrable on* E *and assume that the expansion-coefficients*

$$(f, \psi_n) = \int_E f(x)\psi_n(x)\,d\mu(x)$$

vanish except perhaps the coefficients

$$(f, \psi_{2^k-1}) = (f, \varphi_{n_k}) = \int_E f(x)\varphi_{n_k}(x)\,d\mu(x).$$

Then the series $\sum_{k=1}^{\infty} |(f, \varphi_{n_k})|$ *converges.*

PROOF. Put first $c=1$. We choose an integer p with $q^p>2$ and split the sequence $\{n_k^{(j)}\}$ into p subsequences $\{n_k^{(j)}\}$, $j=1, \ldots, p$ where $n_k^{(j)}=n_{(k-1)p+j}$. We obtain for every k:

$$\frac{n_{k+1}^{(j)}}{n_k^{(j)}} = \prod_{\nu=0}^{p-1} \frac{n_{kp+j-\nu}}{n_{kp+j-1-\nu}} \geqq q^p > 2.$$

Obviously the system $\{\varphi_{n_k}^{(j)}(x)\}$ satisfies (1). Furthermore it is multiplicatively orthogonal and all the more weakly multiplicative. To prove this, we proceed by induction. First we have

$$\int_E \varphi_{n_1^{(j)}}(x)\,d\mu(x) = 0.$$

Suppose now

(3) $$\int_E \prod_{\nu=1}^{r} \varphi_{n_{k_\nu}^{(j)}}(x)\,d\mu(x) = 0$$

for arbitrary indices $k_1, \ldots, k_r$ with $k_1<k_2<\ldots<k_r\leqq m$. We have to show that (3) implies

(4) $$\int_E \varphi_{n_{m+1}^{(j)}}(x) \prod_{\nu=1}^{r} \varphi_{n_{k_\nu}^{(j)}}(x)\,d\mu(x) = 0.$$

From (2) we obtain

$$\prod_{\nu=1}^{r} \varphi_{n_{k_\nu}^{(j)}}(x) = \sum_{\nu=1}^{N_m^{(j)}} \delta_\nu^{(m)} \varphi_\nu(x)$$

where $\delta_\nu^{(m)} \in \mathbf{R}$ and

$$N_m^{(j)} \leqq \sum_{k=1}^{m} n_k^{(j)} < n_m^{(j)} \sum_{k=0}^{\infty} \frac{1}{2^k} = 2n_m^{(j)} < n_{m+1}^{(j)}.$$

Since $\{\varphi_n(x)\}$ is an orthonormal system (4) follows. Hence the system $\{\varphi_{n_k}(x)\}$ satisfies the hypotheses of Theorem 2, consequently $\sum_{k=1}^{\infty} |(f, \varphi_{n_k})|$ converges. Similar arguments hold for $c>1$.

Our Theorem 2 contains implicitely the well known theorem of Sidon on the absolute convergence of lacunary Fourier series.

THEOREM 3 (SIDON [3]). *Suppose $f(x)$ is integrable and one-sided bounded in* $[0,2\pi]$. *Let*

$$\sum_{k=1}^{\infty}\{a_k \cos n_k x + b_k \sin n_k x\}$$

be the Fourier-series of f, where

$$\frac{n_{k+1}}{n_k} \geqq q > 1 \quad (k = 1, 2, \ldots).$$

Then the series $\sum_{k=1}^{\infty}\{|a_k|+|b_k|\}$ *converges.*

Tihs theorem is obviously true for $q \geqq 3$, because then the trigonometric gap-system is multiplicatively orthogonal. In the case $1<q<3$ one proceeds by splitting the sequence $\{n_k\}$ into $p$suitable subsequences $\{n_k^{(j)}\}$ $(j=1, \ldots, p)$ from which Sidon's theorem follows.

The authors are indebted to G. Alexits for having raised the problem and for his suggestions.

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MATHEMATISCHES INSTITUT
DER UNIVERSITÄT
D 63 GIESSEN
ARNDTSTRASSE 2
BUNDESREPUBLIK DEUTSCHLAND

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ГРУППЫ КОБОРДИЗМОВ *l*-ПОГРУЖЕНИЙ. II

А. СЮЧ (Сегед)

Г) ВЫЧИСЛЕНИЕ ГРУППЫ УШИДА

§ 1.

Ушида в своей работе [1] определил группы кобордизмов l-погружений n-мерных многообразий в $(n+k)$-мерные многообразия следующим образом. Приведем определение этих групп:

Определение. Рассмотрим всевозможные погружения $f\colon M\to N$, где M произвольное n-мерное многообразие, N произвольное $(n+k)$-мерное многообразие, а f некоторое l-погружение. Два таких l-погружения $f\colon M\to N$ и $f'\colon M'\to N'$ называются кобордантными, если существуют $(n+1)$-мерное многообразие W с краем $\partial W=M\cup M'$, $(n+k+1)$-мерное многообразие U с краем $\partial U=N\cup N'$ и l-погружение $F\colon W\to U$ такое, что $F|_M$ и $F|_{M'}$ совпадают с отображениями $i\circ f$ и $i'\circ f'$, где i и i' — это включения $N\subset\partial U$ и $N'\subset\partial U$.

Множество классов эквивалентности образует группу относительно сложения, определенного через несвязную сумму. Полученную группу Ушида обозначает через $C(n, k; l)$ (он показал, что естественно возникающее отображение $C(n, k; l)\to C(n, k; l+1)$ есть мономорфизм). Итак, группа Ушида — это группа кобордизмов l-погружений второго типа (см. Введение работы [5]).

Теорема 1. $C(n, k; l)\approx\mathfrak{N}_{n+k}(\Gamma_l MO(k))$.

Доказательство полностью аналогично доказательству изоморфизма $G^l(n, k)\approx\pi_{n+k}(\Gamma_l MO(k))$ из первой части.

Следствие 1. *Отображение* $C(n, k; l)\to C(n, k; l+1)$ *является мономорфизмом.*

Замечание. Это результат Ушида. Он доказал это утверждение геометрически. Мы его выведем из гомологических свойств пространств $\Gamma_l MO(k)$.

Доказательство следствия 1. По теореме Баррата—Эклеса [2] отображение $H_*(\Gamma_l MO(k); Z_2)\to H_*(\Gamma_{l+1} MO(k); Z_2)$, индуцированное отображением вложения $\Gamma_l MO(k)\subset\Gamma_{l+1} MO(k)$ есть мономорфизм. Отсюда следует, что и отображение $\mathfrak{N}_*(\Gamma_l MO(k))\to\mathfrak{N}_*(\Gamma_{l+1} MO(k))$ групп бордизмов также есть мономорфизм, что и влечет следствие 1.

Следствие 2. *Группа кобордизмов m-мерных многообразий с k-мерным векторным расслоением и l-листным накрытием (введенная Ушида как вспомогательный объект для изучения групп $C(n, k; l)$ и обозначенная через $B(m, k; l)$; см.* [1]) *имеет следующую гомотопическую интерпретацию:* $B(m, k; l)\approx\approx\pi_{m+k+N}(MO(N)\wedge MO^{(l)}(k))$, *где N достаточно большое число.*

Доказательство. Из диаграммы

$$\begin{array}{ccccccc} 0 \to \mathfrak{N}_{n+k}(\Gamma_l MO(k)) & \to & \mathfrak{N}_{n+k}(\Gamma_{l+1} MO(k)) & \to & \mathfrak{N}_{n+k}(\Gamma_{l+1} MO(k), \Gamma_l MO(k)) & \to & 0 \\ \| & & \| & & & & \\ 0 \to C(n, k; l) & \to & C(n, k; l+1) & \to & B(n-(l+1)k, k; l) & \to & 0 \end{array}$$

следует, что

$$B(n-(l+1)k, k; l) \approx \mathfrak{N}_{n+k}(\Gamma_{l+1} MO(k)/\Gamma_l MO(k)) = \mathfrak{N}_{n+k}(MO^{(l+1)}(k)).$$

Отсюда ввиду изоморфизма $\mathfrak{N}_n(X) \approx \pi_{n+N}(MO(N) \wedge X)$ следует доказываемое.

§ 2. Вычисление групп $C(n, k; l)$

Известно, что группы бордизмов любого пространства выражаются через группы гомологий с коэффициентами в Z_2 того же пространства

$$\mathfrak{N}_*(X) \approx H_*(X; Z_2) \otimes \mathfrak{N}_*.$$

Поэтому по теореме 1 из [5] проблема определения группы $C(n, k; l)$ сводится к проблеме вычисления групп $H_j(\Gamma_l MO(k); Z_2)$. Однако по теореме Баррата—Эклеса

$$H_*(\Gamma_l MO(k); Z_2) \approx \sum_{i=1}^{l} H_*(MO^{(i)}(k); Z_2),$$

где через $MO^{(i)}(k)$ обозначается факторпространство $\Gamma_i MO(k)/\Gamma_{i-1} MO(k)$. Это факторпространство, как видно из конструкции $\Gamma_l MO(k)$, является пространством Тома для универсального ik-мерного векторного расслоения с группой $O^{(i)}(k)$ (см. [5]).

Определим группу $O_2^{(i)}(k)$. Эта группа есть сплетение группы $O(k)$ с группой $S(i, 2)$, где $S(i, 2)$ есть силовская 2-подгруппа в $S(i)$.

По изоморфизму Тома $H_{n+lk}(MO^{(l)}(k); Z_2) \approx H_n(BO^{(l)}(k); Z_2)$. По теореме Серра о гомологиях накрытий $H_n(BO^{(l)}(k); Z_2) \approx H_n(BO_2^{(l)}(k); Z_2)$.

Известно, что силовская 2-подгруппа группы $S(l)$ разлагается в прямую сумму: $S(l, 2) \approx \sum_r S(2^{a_r}, 2)^{a_r}$, где $l = \sum a_r \cdot 2^r$ есть двоичная запись числа l. Этому разложению соответствует разложение группы $O_2^{(l)}(k)$, которому в свою очередь соответствует разложение пространства $BO_2^{(l)}(k)$ в произведение пространств. А это разложение пространства $BO_2^{(l)}(k)$ индуцирует разложение ряда Пуанкаре для $BO_2^{(l)}(k)$ в произведение рядов Пуанкаре для сомножителей.

Далее известно следующее:

1) $S(2^r, 2) = S(2^{r-1}, 2) \sim Z_2$ (где знак $\sim$ означает сплетение) (см. Накаока [3]).

2) Если ряд Пуанкаре над Z_2 для X есть $f(t)$, то ряд Пуанкаре для пространства $\tilde{X} = (X \times X) \underset{Z_2}{\times} WZ_2$ (где WZ_2 — ациклический свободный Z_2-комплекс,

а на $X\times X$ группа Z_2 действует, переставляя множители) имеет вид:

$$\varphi(t)=\frac{1}{2}[f(t)^2-f(t)]+\frac{f(t^2)}{1-t^2}$$

(см. Богаченко [4]).

Из сказанного следует, что ряд Пуанкаре пространства $BO^{(i)}(k)$ есть произведение

$$F_i(t)=\prod_r f_r^{a_r(i)}$$

где f_1 ряд Пуанкаре пространства $BO(k)$ и

$$f_{j+1}=\frac{1}{2}[f_j(t)^2-f_j(t^2)]+\frac{f_j(t^2)}{1-t^2},$$

а $i=\sum_r a_r(i)\cdot 2^r$ есть двоичная запись числа i. Тем самым вычислены группы $H_n(BO^{(i)}(k);Z_2)$ и значит проблема вычисления групп $C(n,k;l)$ решена.

Для ясности в нижеследующей теореме явно опишу аглорифм нахождения чисел $\dim_{Z_2} C(n,k;l)$.

Теорема 2. 1) *Пусть $f(t)$ — ряд Пуанкаре пространства $BO(k)$ (т.е. $f(t)=\sum_i t^i\pi_k(i)$, где $\pi_k(i)$ — число разбиений числа i на слагаемые, не превосходящие k).*

2) *Пусть $f_1(t)=f(t)$ и*

$$f_{i+1}(t)=\frac{1}{2}[f_i(t)^2-f_i(t^2)]+\frac{f_i(t^2)}{1-t^2},\quad i=1,2,\dots.$$

3) *Распишем каждое число i от 1 до l в двоичной записи*

$$i=\sum_r a_r(i)\cdot 2^r\quad где\quad 1\leqq i\leqq l,\quad a_r(i)=0,1.$$

4) *Пусть $F_i(t)=\prod_r f_r^{a_r(i)}$ для $i=1,2,\dots,l$.*

5) *Пусть $M(t)=\sum_{i=1}^{l} F_i(t)\cdot t^{ik}$. Обозначим j-ый коэффициент этого ряда через m_j, значит*

$$M(t)=\sum_{j=0}^{\infty} m_j t^j.$$

6) *Тогда*

$$\dim_{Z_2} C(n,k;l)=\sum_{\substack{(j,s)\\ j+s=n+k}} m_j\dim_{Z_2}\mathfrak{N}_s.$$

Суммирование производится по таким j и s, для которых $j+s=n+k$.

Д) ДОПОЛНЕНИЯ

Настоящая часть нашей работы содержит разные дополнения к теме l-погружений.

1) В ней мы обобщим точные последовательности: а) Уайтхеда, содержащую гомотопические группы сфер, б) Саломонсена, содержащую группы кобордизмов вложений и погружений, и в) Тодда;

2) докажем некоторый аналог теоремы Бурлета из [9];

3) применяя метод оснащенных многообразий Понтрягина для 2-погружений, приводим новое доказательство одной известной теоремы теории гомотопий;

4) в последнем параграфе исследуем образы естественных гомоморфизмов

$$G^l(n,k) \to \mathfrak{N}_n \quad \text{и} \quad G^l_\Omega(n,k) \to \Omega_n.$$

Полученные здесь результаты можно рассматривать как необходимые условия об l-погружаемости. Эти условия имеют такой вид:

Необходимым условием существования l-погружения $f\colon M^n \to R^{n+k}$ является принадлежность многообразия к определенным классам кобордизмов.

§ 1. Обобщение точной последовательности Уайтхеда

В работе [6] Уайтхед доказал точность следующей последовательности при $n \leqq 2k-2$

$$\pi_{n+k}(S^k) \xrightarrow{S} \pi_{n+k+1}(S^{k+1}) \xrightarrow{H} \pi_{n+k-1}(S^{2k-1}) \to \pi_{n+k-1}(S^k) \to .$$

Мы здесь докажем следующую теорему:

Теорема 3. *Для любого натурального числа l и для всех пар (n,k), для которых $n \leqq (l+1)k-2$ имеет место точная последовательность*

$$\bar{G}^l_{\mathrm{fr}}(n,k) \to \pi_{n+k+1}(S^{k+1}) \to \pi_{n+k-1}(S^{(l+1)k-1}) \to \bar{G}^l_{\mathrm{fr}}(n-1,k) \to .$$

Замечание. а) При $l=1$ наша последовательность совпадает с последовательностью Уайтхеда [6];

b) наше доказательство аналогично доказательству последовательности Уайтхеда из книги Спеньера [7].

Доказательство. Рассмотрим пару $(\bar{\Gamma}_{l+1}S^k, \bar{\Gamma}_l S^k)$, где S^k — сфера (а определение $\bar{\Gamma}_l$ см. в части I).

Напишем для нее точную гомотопическую последовательность

$$\to \pi_{n+k}(\bar{\Gamma}_{l+1}S^k) \to \pi_{n+k}(\bar{\Gamma}_{l+1}S^k, \bar{\Gamma}_l S^k) \to \pi_{n+k-1}(\bar{\Gamma}_l S^k) \to \pi_{n+k-1}(S^k) \to .$$

Пространство $\bar{\Gamma}_l S^k$ является $(k-1)$-связным, а пара $(\bar{\Gamma}_{l+1}S^k, \bar{\Gamma}_l S^k)$ является $(l+1)k-1$ связной. Пространство $\bar{\Gamma}_{l+1}S^k \diagdown \bar{\Gamma}_l S^k$ гомоморфно шару D^{lk}. Поэтому по теореме о гомотопическом вырезании для $n \leqq (l+1)k-2$ имеются изоморфизмы

$$\pi_{n+k}(\bar{\Gamma}_{l+1}S^k, \bar{\Gamma}_l S^k) \approx \pi_{n+k}(D^{(l+1)k}, S^{(l+)k-1}) \approx \pi_{n+k-1}(S^{(l+1)k-1}).$$

С другой стороны, для рассмотренных пар (n, k) имеет место изоморфизм $\pi_{n+k}(\bar{\Gamma}_\infty S^k) \approx \pi_{n+k}(\bar{\Gamma}_{l+1}S^k)$. А по теореме Джеймса $\pi_{n+k}(\bar{\Gamma}_\infty S^k) \approx \pi_{n+k+1}(S^{k+1})$. Значит $\pi_{n+k}(\bar{\Gamma}_{l+1}S^k) \approx \pi_{n+k+1}(S^{k+1})$.

Напишем еще раз точную последовательность пары используя полученные изоморфизмы:

$$\begin{array}{ccccccccc} \ldots \to \pi_{n+k}(\bar{\Gamma}_{l+1}S^k) & \to & \pi_{n+k}(\bar{\Gamma}_{l+1}S^k, \bar{\Gamma}_l S^k) & \to & \pi_{n+k-1}(\bar{\Gamma}_l S^k) & \to & \pi_{n+k-1}(\bar{\Gamma}_{l+1}S^k) \to \\ \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx \\ \to \pi_{n+k+1}(S^k) & \to & \pi_{n+k-1}(S^{(l+1)k-1}) & \to & \bar{G}^l_{\mathrm{fr}}(n-1, k) & \to & \pi_{n+k}(S^k) \to \end{array}$$

Вертикальные стрелки изоморфизмы, а стрелки нижней последовательности определены так, чтобы диаграмма была коммутативной. Тогда из точности верхней последовательности следует точность нижней, а эта последняя совпадает с последовательностью из формулировки теоремы.

§ 2. Обобщение теоремы Саломонсена

В работе [8] имеется точная последовательность, которую в наших обозначениях можно переписать так:

$$G^1(n, k) \xrightarrow{\varphi} G^\infty(n, k) \xrightarrow{\psi} H_2(n-k, k) \to G^1(n-k, k) \to .$$

Последовательность точна при $n \leqq 2k-2$. Определение групп $H_2(m, k)$ — и более общо группы $H_l(m, k)$, где l — натуральное число, — таково:

Определение. $H_l(m, k)$ — это группа кобордизмов вложенных в $(m+l\cdot k)$-мерную сферу S^{m+lk} m-мерных многообразий с дополнительным условием: нормальное расслоение вложения сведено к подгруппе $O^{(l)}(k)$ группы $O(l\cdot k)$. Это означает другими словами, что нормальное расслоение *локально* раскладывается в уитневскую сумму l-штук k-мерных расслоений.

Аналогичное сведение нормального расслоения к группе должно быть задано и для «пленок», осуществляющих кобордизм вложений.

Замечание 1. При $n \leqq 2k-2$ группу $G^\infty(n, k)$ можно заменить на группу $G^2(n, k)$ и получить точную последовательность

$$\to G^1(n, k) \xrightarrow{\varphi} G^2(n, k) \xrightarrow{\psi} H_2(n-k, k) \to G^1(n-1, k) \to$$

Замечание 2. Отображения φ и ψ описываются так: φ сопоставляет классу кобордизмов произвольного вложения класс кобордизмов 2-погружений того же вложения, рассмотренного как 2-погружение. ψ сопоставляет классу кобордизмов 2-погружения класс многообразия двойных точек 2-погружения из $H_2(n-k, k)$.

Замечение 3. Эта последовательность аналогична последовательности Уайтхеда; только здесь рассматриваются произвольные, не обязательно оснащенные вложения и погружения, и погружения, не обязательно разделяющиеся.

Теорема 4. *При $n \leqq lk-2$ имеет место точная последовательность*

$$\to G^{l-1}(n, k) \to G^l(n, k) \to H_l(n-(l-1)k, k) \to G^{l-1}(n-1, k) \to .$$

Замечание 4. При $l=2$ эта последовательность совпадает с последовательностью Саламонсена (в форме, описанной в замечании 1).

Замечание 5. Как будет видно из доказательства, отображения φ и ψ описываются аналогично тому, как они были описаны для $l=2$ в замечании 2.

Доказательство. $\pi_i(\Gamma_{l-1}MO(k))=0$ при $i \leqq k-1$ и

$$\pi_i(\Gamma_l MO(k), \Gamma_{l-1}MO(k))=0 \text{ при } i \leqq lk-1.$$

Поэтому для $n<lk-2$

$$\pi_{n+k}(\Gamma_l MO(k), \Gamma_{l-1}MO(k)) \approx \pi_{n+k}(\Gamma_l MO(k)/\Gamma_{l-1}MO(k)) \approx \pi_{n+k}(MO^{(l)}(k)).$$

Поэтому для $n<lk-2$ гомотопическую последовательность пары $(\Gamma_l MO(k), \Gamma_{l-1}MO(k))$ можно писать так:

$$\pi_{n+k}(\Gamma_{l-1}MO(k)) \to \pi_{n+k}(\Gamma_l MO(k)) \to \pi_{n+k}(MO^{(l)}(k)) \to \pi_{n+k-1}(\Gamma_{l-1}MO(k))$$

то есть

$$\to G^{l-1}(n, k) \xrightarrow{\varphi} G^l(n, k) \xrightarrow{\psi} H_l(n-(l-1)k, k)) \to G^{l-1}(n-1, k) \to .$$

А это есть последовательность, точность которой хотели доказать.

Замечание. Эти последовательности можно написать и для групп $G^l_{fr}(n, k)$ тоже. Например последовательность, написанная в замечании 1 для групп $G^l_{fr}(n, k)$ оснашенных кобордизмов перейдет в точную при $n \leqq lk-2$ последовательность

$$\to \pi_{n+k}(S^k) \to \pi^S_{n+k}(S^k) \to \pi^S_n(P_\infty R) \to \pi_{n+k-1}(S^{k-1}) \to$$

которая есть последовательность Тодда.

§ 3. Аналог теоремы Бурлета

Теорема Бурлета утверждает, что если погружение ориентированного многообразия в сферу имеет нормальное поле, то некоторое его кратное кобордантно вложению. Мы докажем аналогичное утверждение для произвольных (не обязательно ориентируемых) погружений, однако только для нечетной коразмерности.

Доказательство. Пусть $f: M^n \to S^{n+k}$ погружение с нормальным полем. По теореме Хирша существует погружение $\bar{f}: M^n \to S^{n+k-1}$, такое что композиция $\overline{M} \xrightarrow{\bar{f}} S^{n+k-1} \subset S^{n+k}$ регулярно гомотопно f. Теперь в виду четности числа $(k-1)$ некоторое кратное погружения f будет кобордантно разделяющемуся погружению $g: M^n \to S^{n+k-1}$, композиция $i \circ g$ дает искомое вложение.

§ 4.

Известно, что Л. С. Понтрягин и В. А. Рохлин вычислили первые стабильные гомотопические группы сфер с помощью группы кобордизмов вложенных в сферу оснащенных многообразий. После введения группы кобордизмов оснащенных l-погружений естественно возникает вопрос:

Нельзя ли распространить их методы, применяя уже не группу кобордизмов оснащенных вложений, а группу кобордизмов оснащенных l-погружений.

Таким геометрическим методом мы докажем следующее (известное) утверждение:

Утверждение 1. *При нечетном k отображение двойной надстройки $S^2\colon \pi_{2k}(S^k)\to\pi_{2k+2}(S^{k+2})$ является эпиморфизмом.*

Доказательство. Знаем, что при $n\leqq 2k-2$, $\bar{G}^2_{fr}(n,k)\approx\pi_{n+k+1}(S^{k+1})$ и $G^2_{fr}(n,k)\approx\pi^s_n$.

Итак, нужно доказать, что отображение $\bar{G}^2_{fr}(n,k)\to G^2_{fr}(n,k)$ есть эпиморфизм при $n=k+1$ и нечетном k.

Но в этих размерностях двойные точки образуют одномерное и значит ориентируемое многообразие. Тогда из-за нечетности любое 2-погружение будет разделяющимся.

Значит отображение $\bar{G}^2_{fr}(k+1,k)\to G^2_{fr}(k+1,k)$ эпиморфно. Доказательство закончено.

Замечание. Доказанное утверждение *не* следует из теоремы Фрейденталя.

§ 5. Каковы образы естественных гомоморфизмов $G^l_\Omega(n,k)\to\Omega_n$ и $G^l(n,k)\to\mathfrak{N}_n$?

Вопрос, сформулированный в заглавии параграфа, можно рассматривать как обобщение вопроса Брауна:

Какие n-мерные многообразия погружаются (вкладываются) с точностью до кобордизма в эвклидово пространство R^{n+k}, т.е. каковы образы отображений $G^1(n,k)\to\mathfrak{N}_n$ и $G^\infty(n,k)\to\mathfrak{N}_n$. Его главный результат заключается в том, что при $k\geqq n-\alpha(n)$ (соответственно $k\geqq n-\alpha(n)+1$) отображение $G^\infty(n,k)\to\mathfrak{N}_n$ (соответственно $G^1(n,k)\to\mathfrak{N}_n$) есть эпиморфизм.

Теорема 5. a) *Если нечетномерное ориентируемое n-мерное многообразие M^n погружается в R^{n+2}, то оно кобордантно нулю.*

b) Если четномерное ориентируемое многообразие M^n погружается в R^{n+2} без $\left(\frac{n}{2}\right)$-кратных точек, то оно кобордантно нулю.

Следствие. *Гомоморфизм $G^l_\Omega(n,2)\to\Omega_n$ имеет нулевой образ*

а) *при любом l, если n нечетно*

б) *при $l<\frac{n}{2}$, если n четно.*

При доказательстве этой и следующих теорем нам будет необходимо следствие следующей теоремы:

Теорема Смейла—Лашофа [10]. *Пусть* $f\colon M^n \to R^{n+k}$ *есть погружение с нормальным классом Эйлера* $\bar{\chi}(f)$ *и со старшим нормальным классом Штифеля—Уитни* $\bar{w}_k(f)$. *Тогда многообразие l-кратных точек реализует класс гомологий, двойственный классу когомологий* $\bar{w}_k^{l-1}$, *если коэффициенты из* Z_2, *и классу* $\bar{\chi}^{l-1}$, *если коэффициенты из* Z.

Следствие (теоремы Смейла—Лашофа). *Если* f *есть l-погружение, то* $\bar{w}_k^l(f)=0$ *и* $\bar{\chi}^l(f)=0$.

Доказательство теоремы 5. Вычислим числа Штифеля—Уитни рассматриваемого многообразия M^n. Легко видеть, что $w_{2k+1}=0$ и $w_{2k}=(\bar{w}_2)^k$ при $k=0, 1, 2, \dots$. Поэтому при нечетном n все числа Штифеля—Уитни — нули. При четном n все числа Штифеля—Ушини равны числу $\langle (\bar{w}_2)^{\frac{n}{2}}, [M^n]\rangle$. Однако по следствию теоремы Смейла—Лашофа $\bar{w}_2^{\frac{n}{2}}=0$. Значит все числа Штифеля—Уитни многообразия M^n равны нулю.

Аналогичное вычисление для чисел Понтрягина дает, что они также все нули. Доказательство закончено.

Доказательство нижеследующего утверждения полностью аналогично предыдущему доказательству, поэтому я его опущу.

Утверждение 2. *Обозначим через* $T(l; n, k)$ *размерность группы* $\operatorname{Tor}\operatorname{Im}\left(G_\Omega^l(n, k) \to \Omega_n\right)$ *над* Z_2. *Тогда*

1) $T(2; n, 1) = 0$

2) $T(2; n, 2) = 0$ при $n \geqq 4$

3) $T(2; n, 3) \leqq 1$, причем при $n \equiv 1 \pmod 4$ $\quad T(2; n, 3) = 0$.

Про свободную часть группы $\operatorname{Im}\left(G_\Omega^l(n, k) \to \Omega_n\right)$ докажем такую теорему:

Теорема 6. *Пусть n такое делящееся на 4 натуральное число, что сумма коэффициентов n-ого многочлена Хирцебруха не равна нулю. Тогда* $\operatorname{rang}\operatorname{Im}\left(G_\Omega^2(n, 4) \to \Omega_n\right) \leqq 1$.

При этом единственным целочисленным инвариантом является индекс многообразия.

Поясним последнюю фразу:

Пусть $f\colon M^n \to R^{n+4}$ 2-погружение ориентированного многообразия. Тогда если index $M^n=0$, то класс кобордизмов $[M^n]$ имеет конечный порядок в Ω_n.

Следствие (теорема о 2-погружаемости). *Если n-мерное ориентируемое многообразие имеет нулевой индекс, но его класс кобордизмов в* Ω_n *имеет бесконечный порядок, то его нельзя 2-погрузить в* R^{n+4}.

Замечание. Очевидно, что из предыдущих утверждений этого параграфа также можно получить теоремы о 2-погружаемости.

Доказательство теоремы 6. Аналогичное утверждение доказано для вложений в статье Томонага [11]. Его доказательство с помощью теоремы Смейла—Лашофа дословно переносится на случай 2-погружений. Чтобы это показать, я напомню формулировку и доказательство этой теоремы Томонага.

Теорема Томонага. *Пусть X_n есть n-мерное, ориентируемое многообразие, вложенное в R^{n+4} и имеющее нулевой индекс. Пусть n такое делящееся на 4 натуральное число, что сумма коэффициентов n-ого многочлена Хирцебруха отлична от нуля. Тогда $2X_n$ кобордантно нулю в Ω_n.*

Доказательство. Поскольку $X_n \subset R^{n+4}$, то все нормальные классы Понтрягина $\bar{p}_i$ этого вложения равны нулю при $i \geqq 2$. Отсюда следует, что касательные классы Понтрягина равны $p_i(X) = (\bar{p}_1)^i$. Поэтому все числа Понтрягина имеют вид $\langle (\bar{p}_1)^{\frac{n}{4}}, [X_{4n}] \rangle$. Формула Хирцебруха для индекса в данном случае имеет такой вид:

$$\tau(X_n) = \alpha_n \cdot \langle \bar{p}_1^{\frac{n}{4}}, [X_n] \rangle$$

где α_n есть ненулевая сумма коэффициентов n-ого многочлена Хирцебруха.

Поскольку $\tau(X_n) = 0$, то $\langle \bar{p}_1^{\frac{n}{4}}, [X_n] \rangle = 0$ и поэтому кобордантно нулю. Теорема Томонага доказана.

Пусть теперь мы имеем 2-погружение. Тогда по следствию теоремы Смейла—Лашофа $\bar{\chi}^2 = 0$. Но известно, что $\bar{p}_2 = \bar{\chi}^2$ и значит $\bar{p}_2 = 0$. Классы $\bar{p}_i$ при $i \geqq 3$ равны нулю уже по размерностным соображениям.

Дальше доказательство совпадает с доказательством теоремы Томонага.

В другой работе Томонага [12] имеются следующие утверждения.

1) Если M^{4k} $4k$-мерное ориентируемое многообразие, вложенное в эвклидово пространство с коразмерностью 3, то

а) при $k=2$ $\tau(M^8)$ четно и

$$[M^8] = \frac{\tau}{2}[3P_4C - 5P_2C^2]$$

б) при $k=3$ $\tau(M^{12})$ делится на 17 и

$$[M^{12}] = \frac{\tau}{17}[45P_6C - 168P_4C \times P_2C + 140(P_2C)^3].$$

Эти утверждения аналогично предыдущему обобщаются на случай 2-погружений; более того, при этом коразмерность 3 можно заменить на коразмерность 4.

Следствие. 1) $\operatorname{Im}(G^2(8,4) \to \Omega_8)$ *есть циклическая подгруппа* Ω_8, *порожденная элементом* $[3P_4C - 5P_2C^2]$.

2) $\operatorname{Im}(G^2(12,4) \to \Omega_{12})$ *есть циклическая подгруппа порожденная элементом* $[45P_6C - 168P_4C \times P_2C + 140(P_2C)^3]$.

Замечание. Аналогичная формула для $k=4$ тоже имеется в работе Томанага. Она тоже обобщается на наш случай. Мы ее опустили лишь из-за ее громоздкости.

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A. SZŰCS
JÓZSEF ATTILA TUDOMÁNYEGYETEM
BOLYAI INTÉZET
6720 SZEGED, ARADI VÉRTANÚK TERE 1.

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ÜBER DIE LEBESGUESCHEN FUNKTIONEN

Von

K. TANDORI (Szeged), Mitglied der Akademie

1. Es sei $\lambda=\{\lambda_n\}_1^\infty$ eine monoton nichtabnehmende Zahlenfolge mit $\lambda_1\geqq 1$. Für ein im Intervall (0, 1) orthonormiertes Funktionensystem $\varphi=\{\varphi_n(x)\}_1^\infty$ bilden wir die Lebesgueschen Funktionen

$$L_n(\varphi;x)=\int_0^1\left|\sum_{k=1}^n\varphi_k(x)\varphi_k(t)\right|dt\quad (x\in(0,1);\ n=1,2,\ldots).$$

Für eine meßbare Menge $E(\subseteqq(0,1))$ betrachten wir die folgenden Klassen der in (0, 1) orthonormierten Funktionensysteme φ:

$$\Omega_E=\left\{\varphi:\sup_n L_n(\varphi:x)/\lambda_n\in L^\infty(E)\right\},\quad \Omega_E^{**}=\left\{\sup_n L_n(\varphi:x)/\lambda_n<\infty \text{ f. ü. in } E\right\}.$$

Offensichtlich gilt $\Omega_E\subseteqq\Omega_E^{**}$.

Es sei M_0 die Klasse der Folgen $a=\{a_n\}_1^\infty$, die die folgende Eigenschaft besitzen: für jede meßbare Menge $E(\subseteqq(0,1))$ und für jedes $\varphi\in\Omega_E$ ist die Reihe

(1) $$\sum_{n=1}^\infty a_n\varphi_n(x)$$

in E fast überall konvergent. Weiterhin sei M_0^{**} die Klasse der Folgen a mit der Eigenschaft, daß für jede meßbare Menge $E(\subseteqq(0,1))$ und für jedes $\varphi\in\Omega_E^{**}$ die Reihe (1) in E fast überall konvergiert.[1]

In dieser Arbeit werden wir erstens den folgenden Satz beweisen.

SATZ. I. *Es gilt* $M_0=M_0^{**}$.

Da $M_0\supseteqq M_0^{**}$ evident is, haben wir nur

(2) $$M_0\subseteqq M_0^{**}$$

zu zeigen.

S. KACZMARZ [2] hat den folgenden Satz bewiesen. Sind $\lambda_n=O(1)$, $E(\subseteqq(0,1))$ eine meßbare Menge und $\varphi\in\Omega_E$, so ist die Reihe (1) im Falle $a\in l^2$ in E fast überall konvergent.

Diese Behauptung kann man folgenderweise verschärfen. Sind $\lambda_n=O(1)$, $E(\subseteqq(0,1))$ eine meßbare Menge un $\varphi\in\Omega_E^{**}$, so ist die Reihe (1) im Falle $a\in l^2$ in E fast überall konvergent.

[1] Diese und die weiterfolgenden Klassen und Normen hängen auch von der Folge λ ab. Da λ in dieser Arbeit immer dieselbe Folge ist, werden wir die Abhängigkeit davon nicht eigens bezeichnen.

Es sei nämlich

$$E_N = \{x\in(0,1): L_n(\varphi; x) \leqq N\} \quad (N=1,2,\dots).$$

Nach dem erwähnten Kaczmarzschen Satz ist (1) im Falle $a\in l^2$ in E_N fast überall konvergent ($N=1,2,\dots$). Da mes $\left(E\setminus \bigcup_{N=1}^{\infty} E_N\right)=0$ ist, ergibt sich daraus die letzte Behauptung. Nach dieser Behauptung ist $l^2\subseteqq M_0^{**}$.

Weiterhin haben wir in [3] bewiesen, daß im Falle $\lambda_n=O(1)$, $a\notin l^2$ ein System $\varphi\in M_0$ derart existiert, daß die Reihe (1) in (0, 1) fast überall divergiert, d.h. $l^2\supseteqq M_0$ und so im Falle $\lambda_n=O(1)$ $M_0=M_0^{**}=l^2$ gilt.

Zum Beweis von (2) können wir also $\lambda_n\nearrow\infty$ annehmen.

2. Zum Beweis der Relation (2) werden wir mehrere Hilfssätze vorauschicken. Für eine Zahl $K\geqq 1$ bezeichnen wir mit $M_0(K)$ die Klasse der Folgen a, die die folgende Eigenschaft besitzen: für jede meßbare Menge $E(\subseteqq(0,1))$ und für jedes in (0, 1) orthonormiertes System φ mit

$$\text{(3)}\qquad L_n(\varphi; x) \leqq K\lambda_n \quad (x\in E;\ n=1,2,\dots),$$

ist die Reihe (1) in E fast überall konvergent.

HILFSSATZ I. *Für jede Zahl* $1<K<\infty$ *ist* $M_0(K)=M_0(1)$.

Da im Falle $1<K_1<K_2<\infty$, $M_0(1)\supseteqq M_0(K_1)\supseteqq M_0(K_2)$ gilt, genügt es, $M_0(K)\supseteqq M_0(1)$ $(1<K<\infty)$ zu zeigen.

Im entgegengesetzten Falle gibt es eine Zahl $1<K$ und eine Folge $a\in M_0(1)$, mit $a\notin M_0(K)$. Dann existiert aber eine meßbare Menge $E(\subseteqq(0,1))$ und ein in (0, 1) orthonormiertes System φ mit (3) derart, daß die Reihe (1) auf einer meßbaren Teilmenge von E mit positiven Maß divergiert. Es sei $\eta=\frac{1}{4K}$, und bilden wir die Funktionen

$$\psi_n(x)=\begin{cases}\dfrac{1}{\sqrt{K}}\varphi_n(2x) & (x\in(0,1/2)),\\[2ex] \sqrt{\dfrac{1-1/2K}{\eta}}\,\varphi_n\left(\dfrac{x-1/2}{\eta}\right) & (x\in(1/2,1/2+\eta))\\[2ex] 0 & \text{sonst.}\end{cases}\qquad (n=1,2,\dots)$$

Das System $\psi=\{\psi_n(x)\}_1^\infty$ ist in (0, 1) orthonormiert. Es sei F die Bildmenge von E, die sich mit der Transformation $y=x/2$ ergibt. Durch einfache Rechnung bekommen wir

$$L_n(\psi; x) \leqq \lambda_n \quad (x\in F;\ n=1,2,\dots),$$

und nach der Definition von ψ ist die Reihe

$$\sum_{n=1}^{\infty} a_n\psi_n(x)$$

auf einer Teilmenge von F von positivem Maß divergent, was der Annahme $a \in M_0(1)$ widerspricht.

Aus dem Hilfssatz I folgt:

$$M_0(1) = \bigcap_{K \geqq 1} M_0(K) = M_0. \tag{4}$$

3. Für eine Zahl $1 \leqq K < \infty$ betrachten wir die Klasse der in (0, 1) orthonormierten Systeme φ:

$$\Omega_K^* = \left\{\varphi\colon \int_0^1 \sup_n \frac{L_n(\varphi\colon x)}{\lambda_n}\, dx \leqq K\right\}.$$

Es sei $M^*(K)$ die Klasse der Folgen a, die die folgende Eigenschaft besitzen: für jedes $\varphi \in \Omega_K^*$ ist die Reihe (1) in (0, 1) fast überall konvergent. Da im Falle $1 < K_1 < K_2 < \infty$, $\Omega_1^* \subseteqq \Omega_{K_1}^* \subseteqq \Omega_{K_2}^*$ gilt, besteht auch $M^*(K_2) \subseteqq M^*(K_1) \subseteqq M^*(1)$.

Es sei endlich für eine Folge a

$$\|a; K\|^* = \sup_{\varphi \in \Omega_K^*} \int_0^1 \sup_{1 \leqq i \leqq j} |a_i \varphi_i(x) + \ldots + a_j \varphi_j(x)|\, dx.$$

Es ist klar, daß im Falle $1 < K_1 < K_2 < \infty$

$$\|a; 1\|^* \leqq \|a; K_1\|^* \leqq \|a; K_2\|^*$$

für jede Folge a besteht.

HILFSSATZ II. *Es gilt* $M_0(1) \subseteqq M^*(1)$.

Es sei nämlich $a \in M_0(1)$ und $\varphi \in \Omega_1^*$. Wir setzen

$$E_N = \{x \in (0, 1)\colon\ L_n(\varphi; x) \leqq N\lambda_n;\ \ n = 1, 2, \ldots\} \quad (N = 1, 2, \ldots).$$

Aus dem Hilfssatz I folgt, daß die Reihe (1) in E_N fast überall konvergiert. Da $\operatorname{mes}\left((0, 1) \setminus \bigcup_{N=1}^{\infty} E_N\right) = 0$ ist, erhalten wir, daß die Reihe (1) in (0, 1) fast überall konvergiert, d.h. $a \in M^*(1)$ ist.

In der Arbeit [4] haben wir die folgende Behauptung bewiesen.

HILFSSATZ III. *Es sei* $1 \leqq K < \infty$. *Ist* $\|a; K\|^* < \infty$, *so gehört a zu* $M^*(K)$. *Ist aber* $\|a; K\|^* = \infty$, *so gibt es ein* $\varphi \in \Omega_K^*$ *derart, daß die Reihe* (1) *in* (0, 1) *fast überall divergiert, d.h.* $a \notin M^*(K)$ *ist.*

Weiterhin ist in der Note [5] folgendes bewiesen.

HILFSSATZ IV. *Für jede Zahl* $1 \leqq K < \infty$ *und für jede Folge a sind die Größen* $\|a; 1\|^*$ *und* $\|a; K\|^*$ *beide enclich oder beide unendlich.*

Aus diesen letzten Hilfssätzen ergibt sich der

HILFSSATZ V. *Für jede Zahl* $1 < K < \infty$ *gilt* $M^*(1) = M^*(K)$.

Es sei M^* die Klasse der Folgen a, die die folgende Eigenschaft besitzen: für jedes in (0, 1) orthonormierte System φ mit $\sup_n L_n(\varphi; x)/\lambda_n \in L(0, 1)$ ist die Reihe (1) in (0, 1) fast überall konvergent. Nach dem Hilfssatz V gilt

$$M^*(1) = \bigcap_{K \geqq 1} M^*(K) = M^*. \tag{5}$$

4. Es sei $E(\subseteqq(0,1))$ eine meßbare Menge mit positivem Maß, und $\Omega_{1,E}^{*}$ die Klasse der in $(0, 1)$ orthonormierten Funktionensysteme φ, für die

$$\int_E \sup_n \frac{L_n(\varphi;x)}{\lambda_n}\,dx \leqq 1$$

besteht. Weiterhin sei für eine Folge

$$\|a;1\|_E^* = \sup_{\varphi\in\Omega_{1,E}^*}\int_E \sup_{1\leqq i\leqq j}|a_i\varphi_i(x)+\ldots+a_j\varphi_j(x)|\,dx.$$

HILFSSATZ VI. *Für jede Folge a gilt* $\|a;1\|_E^*\leqq\|a;2\|^*$.

Für eine beliebige Folge a sei

$$a(1,N)=\{a_1,\ldots,a_N,0,\ldots\}\quad (N=1,2\ldots).$$

Nach den Definitionen gelten

$$\|a(1,N);\,2\|^*\nearrow\|a;2\|^*,\quad \|a(1,N);\,1\|_E^*\nearrow\|a;1\|_E^*\quad (N\nearrow\infty),$$

so können wir uns auf endliche Folgen $a(1, N)$ beschränken. Man kann auch $\mathrm{mes}\,(E)<1$ voraussetzen.

Ohne Beschränkung der Allgemeinheit können wir auch annehmen, daß E ein Intervall ist: $E=(0,\alpha)$ $(0<\alpha<1)$. Es seien nämlich $E(\subseteqq(0,1))$ eine meßbare Menge mit $\alpha=\mathrm{mes}\,E$, $0<\alpha<1$ und φ ein beliebiges in $(0, 1)$ orthonormiertes Funktionensystem. Wir betrachen die Funktionen

$$f(x)=\mathrm{mes}\,((0,x)\cap E),\quad g(x)=\mathrm{mes}\,((0,x)\cap((0,1)\setminus E))+\alpha.$$

$f(x)$ und $g(x)$ sind Integralfunktionen charakteristischer Funktionen der Menge E bzw. der Menge $(0,1)\setminus E$, und so ist $f'(x)=1$ fast überall in E, und $g'(x)=1$ fast überall in $(0,1)\setminus E$. Weiterhin sind $f(x)\colon E\to(0,\alpha)$, $g(x)\colon(0,1)\setminus E\to(\alpha,1)$. Es seien $f^{-1}(x)$, $g^{-1}(x)$ die Inversen dieser Funktionen, und wir betrachten die Funktionen

$$\psi_n(x)=\begin{cases}\varphi_n(f^{-1}(x)) & (x\in(0,\alpha)),\\ \varphi_n(g^{-1}(x)) & (x\in(\alpha,1))\end{cases}\quad (n=1,2,\ldots).$$

Eine einfache Rechnung ergibt

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx=\int_0^\alpha \varphi_n(f^{-1}(x))\varphi_m(f^{-1}(x))dx+\int_\alpha^1 \varphi_n(g^{-1}(x))\varphi_m(g^{-1}(x))\,dx=$$

$$=\int_E \varphi_n(x)\varphi_m(x)\,dx+\int_{(0,1)\setminus E}\varphi_n(x)\varphi_m(x)\,dx=\int_0^1 \varphi_n(x)\varphi_m(x)\,dx,$$

d.h. das System ψ ist in $(0, 1)$ orthonormiert.

Weiterhin gelten nach dem Obigen

$$\int_0^1 \sup_{1\le i\le j} |a_i\varphi_i(x)+\ldots+a_j\varphi_j(x)|\,dx = \int_0^1 \sup_{1\le i\le j} |a_i\psi_i(x)+\ldots+a_j\psi_j(x)|\,dx,$$

$$\int_E \sup_{1\le i\le j} |a_i\varphi_i(x)+\ldots+a_j\varphi_j(x)|\,dx = \int_0^\alpha \sup_{1\le i\le j} |a_i\psi_i(x)+\ldots+a_j\psi_j(x)|\,dx$$

für jede Folge a, und für jedes n ist für $x\in(0,\alpha)$

$$L_n(\psi;x) = \int_0^1 \left|\sum_{k=1}^n \psi_k(x)\psi_k(t)\right| dt =$$

$$= \int_0^\alpha \left|\sum_{k=1}^n \varphi_k(f^{-1}(x))\varphi_k(f^{-1}(t))\right| dt + \int_\alpha^1 \left|\sum_{k=1}^n \varphi_k(f^{-1}(x))\varphi_k(g^{-1}(t))\right| dt =$$

$$= \int_E \left|\sum_{k=1}^n \varphi_k(f^{-1}(x))\varphi_k(t)\right| dt + \int_{(0,1)\setminus E} \left|\sum_{k=1}^n \varphi_k(f^{-1}(x))\varphi_k(t)\right| dt =$$

$$= \int_0^1 \left|\sum_{k=1}^n \varphi_k(f^{-1}(x))\varphi_k(t)\right| dt = L_n(\varphi; f^{-1}(x)) \quad (f^{-1}(x)\in E),$$

und ähnlicherweise für $x\in(\alpha,1)$

$$L_n(\psi;x) = L_n(\varphi; g^{-1}(x)) \quad (g^{-1}(x)\in(0,1)\setminus E).$$

Daher hat das System ψ dieselbe Eigenschaften wie das System φ; im Falle ψ ist aber die entsprechende Menge E ein Intervall.

Es sei $\varepsilon>0$ beliebig. Dann gibt es ein $\varphi\in\Omega^*_{1,E}$ derart, daß

$$(6)\qquad \int_E \sup_{1\le i\le j\le N} |a_i\varphi_i(x)+\ldots+a_j\varphi_j(x)|\,dx \ge \|a(1,N);1\|_E^* - \varepsilon$$

ist. Es seien $\eta>0$, $\alpha+\eta<1$, und betrachten wir die folgenden Funktionen:

$$\psi_n(x) = \begin{cases} \varphi_n(x) & (x\in(0,\alpha)),\\ \dfrac{1}{\sqrt{\eta}}\varphi_n\left(\dfrac{1-\alpha}{\eta}x+1-\dfrac{1-\alpha}{\eta}(\alpha+\eta)\right) & (x\in(\alpha,\alpha+\eta)) \end{cases} \quad (n=1,2,\ldots,N),$$

$$\psi_{N+1+k}(x) = \begin{cases} \dfrac{1}{\sqrt{1-(\alpha+\eta)}}\chi_k\left(\dfrac{x-(\alpha+\eta)}{1-(\alpha+\eta)}\right) & (x\in(\alpha+\eta,1), \quad k=0,1,\ldots,\\ 0 & (x\in(0,\alpha+\eta)), \end{cases}$$

wobei $\chi_n(x)$ die n-te Haarsche Funktion bezeichnet. Für das Haarsche System $\chi = \{\chi_n(x)\}_1^\infty$ gilt

$$(7)\qquad L_n(\chi;x) \le 1 \quad (x\in(0,1);\ n=0,1,\ldots)$$

(s. z. B. [1], S. 49).

Das System $\psi=\{\psi_n(x)\}_1^\infty$ ist orthonormiert in (0, 1). Nach der Definition von ψ gilt

$$(8)\qquad \int\limits_E \sup_{1\le i\le j\le N} |a_i\varphi_i(x)+\ldots+a_j\varphi_j(x)|\,dx = \int\limits_E \sup_{1\le i\le j\le N} |a_i\psi_i(x)+\ldots+a_j\psi_j(x)|\,dx,$$

und auf Grund von (7) erhalten wir durch einfache Rechnung, daß für genügend kleine η auch $\psi\in\Omega_2^*$ besteht. Daraus und aus (6) und (8) ergibt sich

$$\|a(1,N);1\|_E^*-\varepsilon \le \int\limits_E \sup_{1\le i\le j\le N} |a_i\psi_i(x)+\ldots+a_j\psi_j(x)|\,dx \le$$

$$\le \int\limits_0^1 \sup_{1\le i\le j\le N} |a_i\psi_i(x)+\ldots+a_j\psi_j(x)|\,dx \le \|a;2\|^*.$$

Da $\varepsilon>0$ beliebig war, erhalten wir die Behauptung des Hilfssatzes VI.

HILFSSATZ VII. *Es gilt* $M^*(1)=M_0(1)$.

Da nach dem Hilfssatz II $M_0(1)\subseteq M^*(1)$ ist, braucht nur mehr $M^*(1)\subseteq \subseteq M_0(1)$ bewiesen werden. Es sei nun $a\in M^*(1)$, $E(\subseteq(0,1))$ eine meßbare Menge und φ ein in (0, 1) orthonormiertes System, für welches

$$L_n(\varphi;x)\le\lambda_n \quad (x\in E;\quad n=1,2,\ldots)$$

erfüllt ist. Dann gilt auch

$$(9)\qquad \varphi\in\Omega_{1,E}^*.$$

Weiterhin ist wegen $a\in M^*(1)$, auf Grund des Hilfssatzes III $\|a;1\|^*<\infty$, und so folgt $\|a;2\|^*<\infty$ aus dem Hilfssatz IV.

Für eine Folge a und für natürliche Zahlen $N_1\le N_2$ setzen wir $a(N_1,N_2)= =\{\underbrace{0,\ldots,0,}_{N_1-1}a_{N_1},\ldots,a_{N_2},0,\ldots\}$, weiterhin sei $a(N,\infty)=\{\underbrace{0,\ldots,0,}_{N-1}a_N,a_{N+1},\ldots\}$ ($N= =1,2,\ldots$). In der Arbeit [4] haben wir gezeigt, daß im Falle $\|a;2\|^*<\infty$ $\lim_{N\to\infty}\|a(N,\infty);2\|^*=0$ ist. Daraus folgt, daß eine Indexfolge $(0=)n_0<\ldots<n_k<\ldots$ derart existiert, daß

$$\sum_{k=0}^\infty \|a(n_k+1,n_{k+1});2\|^*<\infty$$

ist. Auf Grund des Hifssatzes VI besteht also

$$(10)\qquad \sum_{k=0}^\infty \|a(n_k+1,n_{k+1});1\|_E^*<\infty.$$

Die n-te Partialsumme der Reihe (1) (mit dem System (9)) bezeichnen wir mit $s_n(x)$. Aus (10) folgt

$$\sum_{k=0}^\infty \int\limits_E |s_{n_{k+1}}(x)-s_{n_k}(x)|\,dx<\infty,$$

woraus folgt, daß $\lim_{k\to\infty} s_{n_k}(x)$ in E fast überall existiert. Es sei

$$\delta_k(x) = \sup_{n_k < i < n_{k+1}} |s_i(x) - s_{n_k}(x)| \quad (k = 0, 1, \dots).$$

Aus (10) erhalten wir

$$\sum_{k=0}^{\infty} \int_E \delta_k(x)\, dx < \infty,$$

woraus sich $\lim_{k\to\infty} \delta_k(x) = 0$ in E fast überall ergibt. Damit haben wir gezeigt, daß im Falle (9) die Reihe (1) in E fast überall konvergiert. Da $E(\subseteqq (0, 1))$ eine beliebige meßbare Menge ist, gilt $a \in M_0(1.)$

Damit haben wir Hilfssatz VII bewiesen.

Aus (4), (5) erhalten wir auf Grund des Hilfssatzes VII

$$M_0 = M^*. \tag{11}$$

5. Es sei Ω^{**} die Klasse der in (0, 1) orthonormierten Systeme φ, für welche in (0, 1) fast überall

$$\sup_n \frac{L_n(\varphi; x)}{\lambda_n} < \infty$$

gilt. Weiterhin sei M^{**} die Klasse der Folgen a, für die die Reihe (1) bei jedem $\varphi \in \Omega^{**}$ in (0, 1) fast überall konvergiert. Da $\Omega_1^* \subseteqq \Omega^{**}$ ist, besteht $M^{**} \subseteqq M^*(1)$. Es seien $\varphi \in \Omega^{**}$ und

$$E_N = \left\{x \in (0, 1) \colon \sup_n \frac{L_n(\varphi; x)}{\lambda_n} \leqq N\right\} \quad (N = 1, 2, \dots).$$

Ist $a \in M^*(1)$, so folgt $a \in M_0(N)$ $(N = 1, 2, \dots)$ aus den Hilfssätzen I und VII. Daher konvergiert die Reihe (1) in jeder Menge E_N fast überall. Da $\operatorname{mes}\left((0, 1) \setminus \bigcup_{N=1}^{\infty} E_N\right) = 0$ ist, konvergiert die Reihe (1) in (0, 1) fast überall, d.h. $a \in M^{**}$. Damit haben wir folgendes gezeigt:

HILFSSATZ VIII. *Es gilt* $M^{**} = M^*(1)$.

Aus (4) und (5) folgt also

$$M_0 = M^* = M^{**}. \tag{12}$$

6. Offensichtlich gilt $M^{**} \supseteqq M_0^{**}$. Zum Beweis des Satzes I werden wir noch den folgenden Hilfssatz heranziehen.

HILFSSATZ IX. *Es gilt* $M^{**} \subseteqq M_0^{**}$.

Es seien nämlich $a \in M^{**}$, $E(\subseteqq (0, 1))$ eine meßbare Menge, und $\varphi \in \Omega_E^{**}$. Wir betrachten die Mengen

$$E_N = \left\{x \in E \colon \sup_n \frac{L_n(\varphi; x)}{\lambda_n} \leqq N\right\} \quad (N = 1, 2, \dots).$$

Aus dem Hilfssatz I und aus (12) ergibt sich $a\in M_0(N)$ $(N=1, 2, \ldots)$, woraus folgt, daß die Reihe (1) in jeder Menge E_N fast überall konvergiert. Da mes $\left(E\setminus\bigcup_{N=1}^{\infty} E_N\right)=0$ ist, konvergiert die Reihe (1) auch in E fast überall. So ist $a\in M_0^{**}$.

Aus (12) und aus dem Hilfssatz IX erhalten wir

$$M_0 = M^* = M^{**} = M_0^{**}. \tag{13}$$

Aus den Hilfssätzen III, IV und aus (13) folgt:

SATZ II. *Die Elemente der Klassen M_0, M^*, M^{**}, M_0^{**} sind jene und nur jene Folgen, für die $\|a; 1\|^*<\infty$ besteht.*

7. Für eine Zahl $1\leqq K<\infty$ bezeichnen wir mit $\Omega(K)$ die Klasse der in (0, 1) orthonormierten Funktionensysteme φ, für die

$$\sup_n \frac{L_n(\varphi; x)}{\lambda_n} \leqq K \quad (x\in(0, 1))$$

gilt, und es sei $M(K)$ die Klasse der Folgen a, für die die Reihe (1) bei jedem $\varphi\in\Omega(K)$ in (0, 1) fast überall konvergiert.

Offensichtlich gelten $M(K)\supseteqq M^*(K)$ $(1\leqq K<\infty)$ und $M(1)\supseteqq M(K_1)\supseteqq M(K_2)$ $(1\leqq K_1<K_2<\infty)$. Es sei weiterhin

$$M = \bigcap_{K\geqq 1} M(K).$$

Es besteht nach (5) auch $M\supseteqq M^*$.

Wir erwähnen zwei Probleme. 1. Gilt $M(1)=M(K)$ für jedes $1\leqq K<\infty$? 2. Gilt $M=M^*$?

Die Klasse M kann man aber charakterisieren. Es sei $1\leqq K<\infty$. Für eine Folge a setzen wir

$$\|a; K\| = \sup_{\varphi\in\Omega(K)} \int_0^1 \sup_{1\leqq i\leqq j} |a_i\varphi_i(x)+\ldots+a_j\varphi_j(x)|\, dx.$$

Offensichtlich gilt für jede Folge a

$$\|a; 1\| \leqq \|a; K_1\| \leqq \|a; K_2\| \quad (1\leqq K_1<K_2<\infty).$$

Wir werden noch den folgende Satz beweisen.

SATZ III. *$a\in M$ gilt dann und nur dann, wenn $\|a; K\|<\infty$ für jedes $1\leqq K<\infty$ gilt.*

Der Satz III folgt unmittelbar aus dem folgenden Hilfssatz.

HILFSSATZ X. *Ist für ein K mit $1\leqq K<\infty$ $a\notin M(K)$ dann gilt $\|a; 24K\|=\infty$, umgekehrt ist $\|a; K\|=\infty$, dann gilt $a\notin M(24K)$.*

Den Hilfssatz X kann man mit einer in der Arbeit [6] angewandten Methode beweisen. Zum Beweis müssen wir aber mehrere weitere Hilfssätze vorausschicken.

8. HILFSSATZ XI. *Es seien* $1 \leqq K < \infty$ *und* $a = \{a_1, \ldots, a_N, 0, \ldots\}$ *eine endliche Folge mit* $\|a; K\| > 4\sqrt{2}$. *Dann gibt es ein in* $(0, 1)$ *orthonormiertes System* ψ *von Treppenfunktionen* $\psi_n(x)$ $(n = 1, \ldots, N)$ *derart, daß*

$$\sup_{n \leqq N} \frac{L_n(\psi; x)}{\lambda_n} \leqq 4K \quad (x \in (0, 1))$$

und

$$\sup_{1 \leqq i \leqq j \leqq N} |a_i \psi_i(x) + \ldots + a_j \psi_j(x)| \geqq 1$$

in einer einfachen Menge $E(\subseteqq (0, 1))$ *mit* $\operatorname{mes}(E) \geqq 1/4$ *besteht. (Die Menge E wird einfach genannt, wenn sie die Vereiningung endlich vieler Intervalle ist.)*

Auf Grund der Definition von $\|a; K\|$ gibt es ein in $(0, 1)$ orthonormiertes System $h = \{h_n(x)\}_1^N$ mit

$$\int_0^1 \sup_{1 \leqq i \leqq j \leqq N} |a_i h_i(x) + \ldots + a_j h_j(x)| \, dx > 4\sqrt{2} \tag{14}$$

und

$$\sup_{n \leqq N} \frac{L_n(h; x)}{\lambda_n} \leqq K \quad (x \in (0, 1)). \tag{15}$$

Für ein $\eta > 0$ wählen wir die Treppenfunktionen $\tilde{h}_n(x)$ $(n = 1, \ldots, N)$ mit

$$\int_0^1 |h_n(x) - \tilde{h}_n(x)| \, dx \leqq \eta \quad (n = 1, \ldots, N),$$

und es sei

$$\tilde{\alpha}_{i,j} = \int_0^1 \tilde{h}_i(x) \tilde{h}_j(x) \, dx \quad (i, j = 1, \ldots, N).$$

Ist η genügend klein $(\eta < \eta_1)$, so folgt aus (14)

$$\int_0^1 \sup_{1 \leqq i \leqq j \leqq N} |a_i \tilde{h}_i(x) + \ldots + a_j \tilde{h}_j(x) \, dx| > 4\sqrt{2}. \tag{i}$$

Wegen der Normiertheit von $h_n(x)$ gilt

$$|h_n(x)| \leqq \Big(\int_0^1 |h_n(t)| \, dt\Big)^{-1} |L_n(h; x) - L_{n-1}(h; x)| \leqq \max_{n \leqq N} \Big(\int_0^1 |h_n(t)| \, dt\Big)^{-1} 2\lambda_N^K = A < \infty$$

für $n = 1, \ldots, N$ und $x \in (0, 1)$. So können wir auch $|\tilde{h}_n(x)| \leqq A$ $(x \in (0, 1); n = 1, \ldots, N)$ voraussetzen. Dann ist

$$|\tilde{\alpha}_{i,j}| \leqq \Big|\int_0^1 h_i(x) h_j(x) \, dx\Big| + \Big|\int_0^1 h_i(x)\big(h_j(x) - \tilde{h}_j(x)\big) \, dx\Big| + \Big|\int_0^1 \tilde{h}_j(x)\big(h_i(x) - \tilde{h}_i(x)\big) \, dx\Big| \leqq$$

$$\leqq \Big|\int_0^1 h_i(x) h_j(x) \, dx\Big| + A \int_0^1 |h_j(x) - \tilde{h}_j(x)| \, dx + A \int_0^1 |h_i(x) - \tilde{h}_i(x)| \, dx,$$

woraus

(ii) $|\tilde{\alpha}_{i,j}| \leqq 2A\eta$ $(i, j=1, \dots, N;\ i\neq j)$, $1-2A\eta \leqq \tilde{\alpha}_{i,i} \leqq 1+2A\eta$ $(i = 1, \dots, N)$

folgt.

Es sei weiterhin

$$E(\eta) = \{x\in(0, 1)\colon |h_n(x)-\tilde{h}_n(x)| \leqq \sqrt{\eta};\ n = 1, \dots, N\}.$$

Durch einfache Rechnung erhalten wir $\operatorname{mes}(E(\eta))\geqq 1-N\sqrt{\eta}$. Aus (15) folgt

$$L_n(\tilde{h}; x) = \int_0^1 \left|\sum_{k=1}^n h_k(x)h_k(t)\right| dt + \int_0^1 \left|\sum_{k=1}^n (h_k(x)-\tilde{h}_k(x))h_k(t)\right| dt +$$

$$+\int_0^1 \left|\sum_{k=1}^N \tilde{h}_k(x)(h_k(t)-\tilde{h}_k(t))\right| dt \leqq K\lambda_n + \sqrt{\eta}\sum_{k=1}^N \int_0^1 |h_k(t)|\, dt + AN\eta$$

$(x\in E(\eta);\ n=1, \dots, N)$. Ist η genügend klein $(\eta<\eta_2\leqq\eta_1)$, so gilt

$$\text{(iii)} \qquad \sup_{n\leqq N} \frac{L_n(\tilde{h}; x)}{\lambda_n} \leqq \frac{5}{4}K \quad (x\in E(\eta)).$$

Es sei

$$H(\eta) = \left\{x\in(0, 1)\colon \sup_{n\leqq N} \frac{L_n(\tilde{h}; x)}{\lambda_n} > \frac{5}{4}K\right\}.$$

Im Falle $\eta<\eta_2$ ist $H(\eta)\subseteqq(0, 1)\setminus E(\eta)$ nach (iii), und so ist $\operatorname{mes}(H(\eta))<N\sqrt{\eta}$. Es sei

$$\bar{h}_n(x) = \begin{cases}\tilde{h}_n(x) & (x\in(0, 1)\setminus H(\eta)),\\ 0 & \text{sonst}\end{cases} \qquad (n = 1, \dots, N).$$

Da $H(\eta)$ eine eifache Menge ist, sind $\bar{h}_n(x)$ Treppenfunktionen. Offensichtlich gilt

$$\int_0^1 |\tilde{h}_n(x)-\bar{h}_n(x)|\, dx \leqq A \operatorname{mes}(H(\eta)) \leqq AN\sqrt{\eta} \quad (n = 1, \dots, N).$$

Ist η genügend klein $(\eta<\eta_3\leqq\eta_2)$, so folgt aus (i):

$$\text{(16)} \qquad \int_0^1 \sup_{1\leqq i\leqq j\leqq N} |a_i\bar{h}_i(x)+\dots+a_j\bar{h}_j(x)|\, dx > 4\sqrt{2}.$$

Wegen (iii) und der Definition von $\bar{h}_n(x)$ gilt

$$\text{(17)} \qquad \sup_{n\leqq N} \frac{L_n(\bar{h}; x)}{\lambda_n} \leqq \frac{5}{4}K \quad (x\in(0, 1)),$$

wenn $\eta<\eta_2$. Mit der obigen Rechnung bekommen wir

$$|\bar{\alpha}_{ij}| = \left|\int_0^1 \bar{h}_i(x)\bar{h}_j(x)\, dx\right| \leqq |\tilde{\alpha}_{i,j}|+NA^2\sqrt{\eta} \quad (i, j = 1, \dots, N),$$

und so folgt aus (ii)

$$|\bar{\alpha}_{i,j}| \leqq 2A\eta + NA^2\sqrt{\eta} \quad (i, j = 1, \dots, N;\ i \neq j),$$

$$1 - 2A\eta + NA^2\sqrt{\eta} \leqq \bar{\alpha}_{i,i} \leqq 1 + 2A\eta + NA^2\sqrt{\eta} \quad (i = 1, \dots, N).$$

Ist endlich η genügend klein $\left(\eta < \eta_3 \leqq \min\left(\eta_2, \tfrac{1}{2}\right)\right)$, so besteht auch

$$(18) \qquad |\bar{\alpha}_{i,j}| < \varepsilon \ (i, j = 1, \dots, N;\ i \neq j), \quad 1 - \varepsilon \leqq \bar{\alpha}_{i,i} \leqq 1 + \varepsilon \ (i = 1, \dots, N)$$

mit einem beliebigen $\varepsilon > 0$.

Die Funktionen $\bar{\varphi}_n(x)$ definieren wir folgenderweise. Wir teilen das Intervall $(1, 2)$ in $N(N-1)$ gleiche Teilintervalle $I_{i,j}$ $(i, j = 1, \dots, N;\ i \neq j)$ ein. Wir setzen

$$\bar{\varphi}_n(x) = \begin{cases} \bar{h}_n(x) & (x \in (0, 1)), \\ \sqrt{2^{-1}N(N-1)|\bar{\alpha}_{n,i}|} & (x \in I_{n,i};\ i = 1, \dots, N;\ i \neq n), \\ -\sqrt{2^{-1}N(N-1)|\bar{\alpha}_{n,i}|}\,\operatorname{sign}\bar{\alpha}_{n,i} & (x \in I_{i,n};\ i = 1, \dots, N;\ i \neq n), \\ 0 & \text{sonst.} \end{cases} \quad (n = 1, \dots, N)$$

Diese Treppenfunktionen $\bar{\varphi}_n(x)$ bilden in (0,2) ein orthogonales System, und nach (18) gilt

$$(19) \qquad 1 - \varepsilon \leqq \bar{\alpha}_{n,n} + \sum_{\substack{1 \leqq j \leqq N \\ i \neq n}} |\bar{\alpha}_{n,i}| = \int_0^2 \bar{\varphi}_n^2(x)\,dx \leqq 1 + N\varepsilon \quad (n = 1, \dots, N).$$

Nach (17) und (18) ist im Falle $x \in (0, 1)$, $1 \leqq n \leqq N$

$$(20) \qquad \frac{L_n(\bar{\varphi}; x)}{\lambda_n} = \frac{1}{\lambda_n}\int_0^2 \left|\sum_{k=1}^n \bar{\varphi}_k(x)\bar{\varphi}_k(t)\right| dt = \frac{1}{\lambda_n}\int_0^1 \left|\sum_{k=1}^n \bar{h}_k(x)\bar{h}_k(t)\right| dt +$$

$$+ \frac{1}{\lambda_n}\int_1^2 \left|\sum_{k=1}^n \bar{h}_k(x)\bar{\varphi}_k(t)\right| dt \leqq \frac{5}{4}K + \frac{1}{\lambda_n}\sup_{k \leqq N}\int_1^2 |\bar{\varphi}_k(t)|\,dt \sum_{k=1}^N |\bar{h}_k(x)| \leqq$$

$$\leqq \frac{5}{4}K + \sqrt{2N(N-1)} \max_{\substack{1 \leqq i, j \leqq N \\ i \neq j}} \sqrt{|\bar{\alpha}_{i,j}|} \sum_{k=1}^N |\bar{h}_k(x)| \leqq \frac{3}{2}K,$$

wenn ε genügend klein ist. Im Falle $x \in I_{i,j}$ $(i \neq j)$, $1 \leqq n \leqq N$ folgt aus (18) und (19) weiterhin

$$(21) \qquad \frac{L_n(\bar{\varphi}; x)}{\lambda_n} = \frac{1}{\lambda_n}\int_0^1 \left|\sum_{k=1}^n \bar{\varphi}_k(x)\bar{h}_k(t)\right| dt + \frac{1}{\lambda_n}\int_1^2 \left|\sum_{k=1}^n \bar{\varphi}_k(x)\bar{\varphi}_k(t)\right| dt \leqq$$

$$\leqq \sqrt{2N(N-1) \sup_{\substack{1 \leqq i, j \leqq N \\ i \neq j}} |\bar{\alpha}_{i,j}|} \left(\sum_{k=1}^N \int_0^1 |\bar{h}_k(t)|\,dt + \sum_{k=1}^N \int_1^2 |\bar{\varphi}_k(t)|\,dt\right) \leqq \frac{3}{2} \leqq \frac{3}{2}K.$$

Aus (20) und (21) ergibt sich

$$(22) \qquad \sup_{n \leqq N} \frac{L_n(\bar{\varphi}; x)}{\lambda_n} \leqq \frac{3}{2}K \quad (x \in (0, 2)).$$

Wir setzen

$$\bar{\psi}_n(x) = \sqrt{2}\,\bar{\varphi}_n(2x)\Big(\int_0^2 \bar{\varphi}_u^2(x)\,dx\Big)^{-1/2} \quad (n = 1, \dots, N).$$

Ist $\varepsilon(>0)$ genügend klein, so erhalten wir aus (16), (17) und (22)

$$\int_0^1 \sup_{1\le i\le j\le N} |a_i\bar{\psi}_i(x)+\dots+a_j\bar{\psi}_j(x)|\,dx \geqq 4$$

und

$$\text{(23)} \qquad \sup_{n\le N} \frac{L_n(\bar{\psi}:x)}{\lambda_n} \leqq 2K \quad (x\in(0,1)).$$

Offensichtlich ist das System $\bar{\psi}=\{\bar{\psi}_n(x)\}_1^N$ in (0, 1) orthonormiert. Ohne Beschränkung der Allgemeinheit können wir

$$\text{(24)} \qquad \int_0^1 \sup_{1\le i\le j\le N} \left|\frac{a_i}{2}\bar{\psi}_i(x)+\dots+\frac{a_j}{2}\bar{\psi}_j(x)\right| dx = 2$$

voraussetzen.

Es sei $I_r=(\bar{a}_r, \bar{b}_r)$ $(r=1, \dots, \varrho)$ eine Zerlegung von (0, 1) in paarweise disjunkte Intervalle derart, daß jede Funktion $\bar{\psi}_n(x)$ in jedem I_r konstant ist. Es sei

$$F(x) = \sup_{1\le i\le j\le N} \left|\frac{a_i}{2}\bar{\psi}_i(x)+\dots+\frac{a_j}{2}\bar{\psi}_j(x)\right|;$$

die Werte von $F(x)$ in den Intervallen I_r bezeichnen wir der Reihe nach mit w_r. Nach (24) ist

$$\text{(25)} \qquad \sum_{r=1}^{\varrho} w_r \operatorname{mes}(I_r) = 2.$$

Es seien $1\leqq r_1<\dots<r_s\leqq\varrho$ diejenigen Indizes r, für die $w_r\geqq 1$ ist. Nach (25) gilt

$$\text{(26)} \qquad (2\geqq)\sum_{i=1}^{s} w_{r_i}\operatorname{mes}(I_{r_i}) \geqq 1.$$

Es seien $J_r=(\alpha_r, \beta_r)$ $(r=1, \dots, \varrho)$ nacheinander folgende Intervalle in (0, 3) mit $\operatorname{mes}(J_r)=\operatorname{mes}(I_r)$ $(r\ne r_i;\ i=1, \dots, s)$ bzw. mit $\operatorname{mes}(J_{r_i})=w_{r_i}\operatorname{mes}(I_{r_i})$ $(i=1, \dots, s)$, und $\bar{J}_{r_i}=(\bar{\alpha}_{r_i}, \bar{\beta}_{r_i})$ $(i=1, \dots, s)$ nacheinander folgende Intervalle in (3, 4) mit $\operatorname{mes}(\bar{J}_{r_i})=$ $=\operatorname{mes}(I_{r_i})$ $(i=1, \dots, s)$. Wir setzen

$$\tilde{\psi}_n(x) = \begin{cases} \bar{\psi}_n(x-\alpha_r+\bar{a}_r) & (x\in J_r;\ r\ne r_i;\ i=1,\dots,s),\\ \dfrac{1}{w_{r_i}}\bar{\psi}_n\left(\dfrac{x-\alpha_{r_i}}{w_{r_i}}+\bar{a}_{r_i}\right) & (x\in J_{r_i};\ i=1,\dots,s),\\ \left(1-\dfrac{1}{w_{r_i}}\right)^{1/2}\bar{\psi}_n(x-\bar{\alpha}_{r_i}+\bar{a}_{r_i}) & (x\in\bar{J}_{r_i};\ i=1,\dots,s),\\ 0 & \text{sonst.}\end{cases} \qquad (n=1,\dots,N)$$

Auf Grund von (26) ist diese Definition sinnvoll.

Die Treppenfunktionen $\tilde{\psi}_n(x)$ bilden in (0, 4) ein orthonormiertes System. Es sei

$$\tilde{E} = \bigcup_{i=1}^{s} J_{r_i}.$$

$\tilde{E}$ ist einfach, und es gilt

$$\text{mes}(\tilde{E}) \geqq 1 \tag{27}$$

nach (26). Ist $x \in \tilde{E}$, so gilt offensichtlich

$$\sup_{1 \leqq i \leqq j \leqq N} \left| \frac{a_i}{2} \tilde{\psi}_i(x) + \dots + \frac{a_j}{2} \tilde{\psi}_j(x) \right| \geqq 1. \tag{28}$$

Auf Grund der Definition von $\tilde{\psi}_n(x)$ folgt

$$\begin{aligned} L_n(\tilde{\psi}; x) = & \sum_{\substack{r=1 \\ r \neq r_i;\, i=1,\dots,s}}^{\varrho} \int_{J_r} \left| \sum_{k=1}^{n} \tilde{\psi}_k(x) \overline{\psi}_k(t - \alpha_r + \bar{a}_r) \right| dt + \\ & + \sum_{i=1}^{s} \frac{1}{w_{r_i}} \int_{J_{r_i}} \left| \sum_{k=1}^{n} \tilde{\psi}_k(x) \overline{\psi}_k \left(\frac{t - \alpha_{r_i}}{w_{r_i}} + \bar{a}_{r_i} \right) \right| dt + \\ & + \sum_{i=1}^{s} \left(1 - \frac{1}{w_{r_i}} \right)^{1/2} \int_{\bar{J}_{r_i}} \left| \sum_{k=1}^{n} \tilde{\psi}_k(x) \overline{\psi}_k (t - \bar{\alpha}_{r_i} + \bar{a}_{r_i}) \right| dt = \\ = & \sum_{\substack{r=1 \\ r \neq r_i;\, i=1,\dots,s}}^{\varrho} \int_{I_r} \left| \sum_{k=1}^{n} \tilde{\psi}_k(x) \overline{\psi}_k(t) \right| dt + \sum_{i=1}^{s} \int_{I_{r_i}} \left| \sum_{k=1}^{n} \tilde{\psi}_k(x) \overline{\psi}_k(t) \right| dt + \\ & + \sum_{i=1}^{s} \left(1 - \frac{1}{w_{r_i}} \right)^{1/2} \int_{I_r} \left| \sum_{k=1}^{n} \tilde{\psi}_k(x) \overline{\psi}_k(t) \right| dt. \end{aligned} \tag{29}$$

Daraus ergibt sich

$$L_n(\tilde{\psi}; x) \leqq \begin{cases} 2L_n(\overline{\psi}; x - \alpha_{r_0} + \bar{a}_{r_0}) & (x \in J_r;\, r \neq r_i \ (i = 1, \dots, s)), \\ \dfrac{2}{w_{r_{i_0}}} L_n \left(\overline{\psi}; \dfrac{x - \alpha_{r_{i_0}}}{w_{r_{i_0}}} + \bar{a}_{r_{i_0}} \right) & (x \in J_{r_{i_0}};\, 1 \leqq i_0 \leqq s), \\ 2L_n(\overline{\psi}; x - \bar{\alpha}_{r_{i_0}} + \bar{a}_{r_{i_0}}) & (x \in \bar{J}_{r_{i_0}};\, 1 \leqq i_0 \leqq s), \end{cases} \tag{30}$$

wovon man

$$\sup_{n \leqq N} \frac{L_n(\tilde{\psi}; x)}{\lambda_n} \leqq 4K \quad (x \in (0, 4)) \tag{31}$$

ableitet. Endlich setzen wir

$$\psi_n(x) = 2\tilde{\psi}_n(4x) \quad (x \in (0, 1);\ n = 1, \dots, N). \tag{32}$$

Weiterhin bezeichnet E die Bildmenge von $\tilde{E}$ bei der Transformation $y = x/4$. Auf Grund von (27), (28) und (31) ist es offensichtlich, daß die Funktionen $\psi_n(x)$ und die Menge E allen Forderungen des Hilfssatzes XI genügen.

9. Wir beweisen noch einen Hilfssatz, woraus Hilfssatz X unmittelbar folgt.

HILFSSATZ XII. *Es sei* $1 \leqq K < \infty$. *Gilt für eine Folge* $\lim_{N\to\infty} \|a(N, \infty); K\| = 0$, *so ist* $a \in M(K)$. *Gilt aber* $\lim_{N\to\infty} \|a(N, \infty); K\| \neq 0$, *so gibt es ein in* (0, 1) *orthonormiertes System* $\varphi \in \Omega(24K)$ *derart, daß*

$$\overline{\lim_{i,j\to\infty}} \left| \sum_{n=i}^{j} a_n \varphi_n(x) \right| = \infty$$

in (0, 1) *fast überall besteht.*

BEWEIS DES HILFSATZES XII. Gilt $\lim_{N\to\infty} \|a(N, \infty); K\| = 0$, so gibt es eine Indexfolge $(0=) n_0 < \ldots < n_k < \ldots$ mit folgender Eigenschaft:

$$\sum_{k=0}^{\infty} \|a(n_k+1, n_{k+1}); K\| < \infty. \tag{33}$$

Es sei $\varphi \in \Omega(K)$, und die n-te Partialsumme der Reihe (1) bezeichnen wir mit $s_n(x)$. Nach (33) ist

$$\sum_{k=0}^{\infty} \int_0^1 |s_{n_{k+1}}(x) - s_{n_k}(x)|\, dx < \infty,$$

woraus folgt, daß $\lim_{k\to\infty} s_{n_k}(x)$ in (0, 1) fast überall existiert. Es sei

$$\delta_k(x) = \sup_{n_k < n < n_{k+1}} |s_n(x) - s_{n_k}(x)| \quad (k = 0, 1, \ldots).$$

Nach (33) gilt

$$\sum_{k=0}^{\infty} \int_0^1 \delta_k(x)\, dx < \infty,$$

woraus sich $\lim_{k\to\infty} \delta_k(x) = 0$ in (0, 1) fast überall ergibt. Daraus folgt endlich, daß im Falle $\varphi \in \Omega(K)$ die Reihe (1) in (0, 1) fast überall konvergiert, d.h. $a \in M(K)$ ist.

Endlich beweisen wir die zweite Behauptung des Hilfssatzes XII. Ist $\lim_{N\to\infty} \|a(N, \infty); K\| > 0$, so gibt es eine Indexfolge $(0=) n_0 < \ldots < n_k < \ldots$ mit $\|a(n_k+1, n_{k+1}); K\| > \varrho (>0)$ $(k=0, 1, \ldots)$. Ohne Beschränkung der Allgemeinheit können wir

$$\|a(n_k+1, n_{k+1}); K\| > 4\sqrt{2} \quad (k = 0, 1, \ldots) \tag{34}$$

und

$$\lambda_{n_1} + \ldots + \lambda_{n_k} \leqq 2\lambda_{n_k} \quad (k = 1, 2, \ldots) \tag{35}$$

voraussetzen.

Durch Induktion werden wir ein in (0, 1) orthonormiertes System φ von Treppenfunktionen $\varphi_n(x)$ und eine Folge einfacher Mengen $E_k (\subseteqq (0, 1))$ $(k=1, 2, \ldots)$ derart definieren, daß die folgenden Bedingungen erfüllt seien:

für jedes $k (=1, 2, \ldots)$ gilt

$$\sup_{n_{k-1} < N \leqq n_k} \frac{1}{\lambda_N} \int_0^1 \left| \sum_{n=n_{k-1}+1}^{N} \varphi_k(x) \varphi_k(t) \right| dt \leqq 8K \quad (x \in (0, 1)); \tag{36}$$

die Mengen E_k sind stochastisch unabhängig, und für jedes $k(=1, 2, ...)$ gilt

(37) $$\text{mes}\,(E_k) \geqq 1/4k;$$

weiterhin besteht für jedes $k(=1, 2, ...)$

(38) $$\sup_{n_{k-1}<i\leqq j\leqq n_k} |a_i\varphi_i(x)+...+a_j\varphi_j(x)| \geqq k \quad (x\in E_k).$$

Wir wenden den Hilfssatz XI auf die Folge $\{a_1, ..., a_{n_1}, 0, ...\}$ an. Die entsprechenden Treppenfunktionen bzw. die entsprechende einfache Menge bezeichnen wir mit $\varphi_1(x), ..., \varphi_{n_1}(x)$ bzw. mit E_1. Nach dem Hilfssatz XI sind (36), (37) und (38) für $k=1$ erfüllt.

Es sei k_0 eine natürliche Zahl, und nehmen wir an, daß die Treppenfunktionen $\varphi_n(x)$ $(n=1, ..., n_{k_0})$ und die einfachen Mengen $E_1, ..., E_{k_0}(\subseteqq(0, 1))$ schon derart definiert sind, daß das System $\{\varphi_n(x)\}_1^{n_{k_0}}$ in $(0, 1)$ orthonormiert ist, die Mengen $E_1, ..., E_{k_0}$ stochastisch unabhängig sind, und (36), (37), (38) im Falle $k=1, ..., k_0$ erfüllt sind.

Wir teilen das Intervall $(0, 1)$ in paarweise disjunkte Intervalle I_r $(r=1, ..., \varrho)$ derart ein, daß jede Funktion $\varphi_n(x)$ $(n=1, ..., n_{k_0})$ in jedem I_r konstant ist, und jede Menge E_k $(k=1, ..., k_0)$ als die Vereinigung gewisser I_r entsteht. Die zwei Hälften des Intervalls I_r bezeichnen wir mit I_r' bzw. mit I_r''.

Wir wenden den Hilfssatz XI auf die Folge $\{\underbrace{0, ..., 0}_{n_{k_0}}, a_{n_{k_0}+1}, ..., a_{n_{k_0+1}}, 0, ...\}$ an. Die entsprechende Funktionen und die entsprechende Menge bezeichnen wir mit $\tilde{\varphi}_n(x)$ $(n=1, ..., n_{k_0+1})$ bzw. mit $\tilde{E}_{k_0+1}$. Wir setzen

$$\bar{\varphi}_n(x) = \begin{cases} \sqrt{k_0+1}\,\tilde{\varphi}_n((k_0+1)x) & (x\in(0, 1/(k_0+1)) \\ 0 & \text{sonst} \end{cases} \quad (n = n_{k_0}+1, ..., n_{k_0+1}),$$

und es sei $\bar{E}_{k_0+1}$ die Bildmenge von $\tilde{E}_{k_0+1}$, die mit der Transformation $y=x/(k_0+1)$ aus $\tilde{E}_{k_0+1}$ entsteht. Auf Grund des Hilfssatzes XI ergibt sich:

(39) $$\int_0^1 \left| \sum_{n=n_{k_0}+1}^{n} \bar{\varphi}_n(x)\bar{\varphi}_n(t) \right| dt \leqq 8K\lambda_n \quad (x\in(0,1);\ n = n_{k_0}+1, ..., n_{k_0+1}),$$

(40) $$\text{mes}\,(\bar{E}_{k_0+1}) \geqq 1/4(k_0+1)$$

und

(41) $$\sup_{n_{k_0}<i\leqq j\leqq n_{k_0+1}} |a_i\bar{\varphi}_i(x)+...+a_j\bar{\varphi}_j(x)| \geqq \sqrt{k_0+1} \quad (x\in\bar{E}_{k_0+1}).$$

Wir setzen endlich

$$\varphi_n(x) = \sum_{r=1}^{\varrho} \bar{\varphi}_n(I_r'; x) - \sum_{r=1}^{\varrho} \bar{\varphi}_n(I_r''; x) \quad (n = n_{k_0+1}+1, ..., n_{k_0+1}),$$

$$E_{k_0+1} = \bigcup_{r=1}^{\varrho} (\bar{E}_{k_0+1}(I_r') \cup \bar{E}_{k_0+1}(I_r'')),$$

wobei für eine in (0, 1) definierte Funktion $f(x)$ und für ein Intervall $I=(a, b)(\subseteqq(0, 1))$

$$f(I; x) = \begin{cases} f\left(\frac{x-a}{b-a}\right) & (x\in(a, b)), \\ 0 & \text{sonst} \end{cases}$$

ist, und für eine Menge $H(\subseteqq(0, 1))$ bezeichnet $H(I)$ die Bildmenge von H, die aus H mit der Transformation $y=(b-a)x+a$ entsteht.

Offensichtlich bilden die Treppenfunktionen $\varphi_n(x)$ $(n=1, \ldots, n_{k_0+1})$ in (0, 1) ein orthonormiertes System, die einfachen Mengen $E_1, \ldots, E_{k_0+1}$ sind stochastisch unabhängig, wegen (40) und (41) gelten (37) im Falle $k=k_0+1$; weiterhin folgt durch einfache Rechnung, daß auch (36) im Falle $k=k_0+1$ besteht. Damit können wir für die Definition der Funktionen $\varphi_n(x)$ und der Mengen E_k Induktion anwenden.

Es sei $E=\overline{\lim\limits_{k\to\infty}} E_k$. Ist $x\in E$, dann gilt

$$\overline{\lim_{i,j\to\infty}}\left|\sum_{n=i}^{j} a_n\varphi_n(x)\right| = \infty \tag{42}$$

wegen (38). Da die Mengen E_k stochastisch unabhängig sind und wegen (37)

$$\sum_{k=1}^{\infty} \operatorname{mes}(E_k) = \infty$$

gilt, erhalten wir durch Anwendung des Borel—Cantellischen Lemmas, daß $\operatorname{mes}(E)=1$ ist, und so besteht (42) in (0, 1) fast überall. Es sei N eine beliebige natürliche Zahl, $n_{k_0}<N\leqq n_{k_0+1}$. Dann ergibt sich aus (35) und (36):

$$L_N(\varphi; x) = \sum_{k=0}^{k_0-1}\int_0^1\left|\sum_{n=n_k+1}^{n_{k+1}}\varphi_n(x)\varphi_n(t)\right|dt + \int_0^1\left|\sum_{n=n_{k_0}+1}^{N}\varphi_n(x)\varphi_n(t)\right|dt \leqq$$

$$\leqq 8K(\lambda_1+\ldots+\lambda_{n_{k_0}}+\lambda_N) \leqq 8K(2\lambda_{n_{k_0}}+\lambda_N) \leqq 24K\lambda_N$$

überall in (0, 1).

Damit haben wir Hilfssatz II bewiesen.

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JÓZSEF ATTILA UNIVERSITÄT
BOLYAI INSTITUT
SZEGED, ARADI VÉRTANÚK TERE 1

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WEITERE BEMERKUNGEN ÜBER DIE KONVERGENZ UND SUMMIERBARKEIT DER FUNKTIONENREIHFN

Von

K. TANDORI (Szeged), Mitglied der Akademie

1. In dieser Note werden wir einige für Orthogonalreihen bekannte Sätze auf den nicht orthogonalen Fall übertragen.

Für eine Folge $f=\{f_n(x)\}_0^\infty$ der reellen Funktionen $f_n(x)\in L(0,1)$ bilden wir die Lebesgueschen Funktionen

$$L_n(f;x)=\int_0^1\left|\sum_{k=0}^n f_k(x)f_k(t)\right|dt,\qquad L_n^1(f;x)=\int_0^1\left|\sum_{k=1}^n\left(1-\frac{k}{n+1}\right)f_k(x)f_k(t)\right|dt.$$

Es sei $\lambda=\{\lambda_n\}_0^\infty$ eine nicht-abnehmende Folge von positiven Zahlen mit $\lim_{n\to\infty}\lambda_n=\infty$. Die Klasse der Folgen $a=\{a_n\}_0^\infty$ mit

$$\sum_{n=0}^\infty a_n^2\lambda_n<\infty$$

bezeichnen wir mit $l^2(\lambda)$.

Es sei A eine Klasse der Folgen a. Wir sagen, daß das System f ein Konvergenzsystem für A in $(0, 1)$, bzw. ein Konvergenzsystem für A in $(0, 1)$ dem Maß nach ist, wenn die Reihe

$$\sum_{n=0}^\infty a_nf_n(x) \tag{1}$$

im Falle $a\in A$ in $(0, 1)$ fast überall konvergiert, bzw. in $(0, 1)$ dem Maß nach konvergiert. Weiterhin sagen wir, daß das System f ein $(C, \alpha>0)$-Konvergenzsystem für A in $(0, 1)$ ist, wenn die Reihe (1) im Falle $a\in A$ in $(0, 1)$ fast überall $(C, \alpha>0)$-summierbar ist.

Wir werden erstens die folgenden Sätze beweisen.

SATZ I. *Gilt*

$$\sup_n\frac{L_n(f;x)}{\lambda_n}<\infty$$

fast überall in $(0, 1)$ *und ist* f *ein Konvergenzsystem für* l^2 *dem Maß nach, dann ist* f *ein Konvergenzsystem für* $l^2(\lambda)$ *in* $(0, 1)$.

SATZ II. *Es sei* λ *eine von unten konkave Folge. Gilt*

$$\sup_n\frac{L_n^1(f;x)}{\lambda_n}<\infty$$

in (0, 1) *fast überall und ist f ein Konvergenzsystem für* l^2 *dem Maß nach, dann ist f ein* $(C, \alpha>0)$*-Konvergenzsystem für* $l^2(\lambda)$ *in* (0, 1).

SATZ III. *Es sei* $\{\nu_n\}_0^\infty$ *eine Indexfolge mit* $1<q\leqq \nu_{n+1}/\nu_n<s(<\infty)$ $(n=0, 1, \ldots)$. *Gilt*

$$\sup_n \frac{L_{\nu_n}(f; x)}{\lambda_{\nu_n}} < \infty$$

in (0, 1) *fast überall und ist f ein Konvergenzsystem für* l^2 *dem Maß nach, dann ist f ein* $(C, \alpha>0)$*-Konvergenzsystem für* $l^2(\lambda)$ *in* (0, 1).

Für orthonormierte Funktionensysteme f sind diese Behauptungen wohlbekannt. (S. z. B. G. ALEXITS [1].) Die Sätze I—II sind gewisse Verschärfungen der Sätze von G. ALEXITS und A. SHARMA [2].

2. BEWEIS DES SATZES I. Wir brauchen den folgenden

HILFSSATZ I. *Es sei* $\{\nu_n\}_0^\infty$ *eine streng wachsende Indexfolge, und* $E(\subseteqq(0, 1))$ *eine meßbare Menge. Sind die Funktionen* $L_{\nu_n}(f; x)/\lambda_{\nu_n}$ *in E gleichmäßig beschränkt, dann gilt*

$$|s_{\nu_n}(x)|/\sqrt{\lambda_{\nu_n}} < \infty$$

im Falle $a\in l^2$ *fast überall in E, wobei* $s_n(x)$ *die n-te Partialsumme der Reihe* (1) *bezeichnet.*

Diese Behauptung stammt von G. ALEXITS und A. SHARMA [2].

Ist das System f ein Konvergenzsystem für l^2 in (0, 1), dem Maß nach, so ist f nach einem Satz von E. M. NIKISCHIN [4] fast orthonormiert; d.h. zu jeder positiven Zahl ε gibt es eine meßbare Menge $E_\varepsilon(\subseteqq(0, 1))$, eine positive Zahl M_ε und ein in (0, 1) orthonormiertes System $\psi=\{\psi_n(\varepsilon; x)\}_0^\infty$ derart, daß mes $(E_\varepsilon)\geqq 1-\varepsilon$ und

$$f_n(x) = M_\varepsilon \psi_n(\varepsilon; x) \quad (x\in E\ ;\ n = 0, 1, \ldots)$$

bestehen.

Es sei

$$E_N = \left\{x\in(0, 1)\colon \frac{L_{\nu_n}(f; x)}{\lambda_{\nu_n}} \leqq N\right\} \quad (N = 1, 2, \ldots).$$

Auf Grund des Hilfssatzes I gilt $|s_{\nu_n}(x)|/\sqrt{\lambda_{\nu_n}}<\infty$ im Falle $a\in l^2$ in jeder Menge E_N, fast überall. Da mes $\left((0, 1)\setminus \bigcup_{N=1}^{\infty} E_N\right)=0$ ist, gilt (2) im Falle $a\in l^2$ fast überall in (0, 1).

Es sei nun $a\in l^2(\lambda)$. Dann gibt es eine nichtabnehmende Folge der positiven Zahlen $\mu=\{\mu_n\}_0^\infty$ mit $\lim_{n\to\infty} \mu_n=\infty$ und $a\sqrt{\mu}\in l^2(\lambda)$. Die n-te Partialsumme der Reihe

(2) $$\sum_{n=0}^{\infty} a_n \sqrt{\lambda_n \mu_n} f_n(x)$$

bezeichnen wir mit $s_n^*(x)$. Dann ist

(3) $$s_n(x)=\sum_{k=0}^{n} a_k f_k(x)=\sum_{k=0}^{n}\frac{1}{\sqrt{\lambda_k\mu_k}}a_k\sqrt{\lambda_k\mu_k}f_k(x)=\sum_{k=0}^{n-1}\Delta\left(\frac{1}{\sqrt{\lambda_k\mu_k}}\right)s_k^*(x)+\frac{1}{\sqrt{\lambda_n\mu_n}}s_n^*(x).$$

Nach obigem gilt $\lim_{n\to\infty} s_n^*(x)/\sqrt{\lambda_n\mu_n}=0$ fast überall in $(0, 1)$. Weiterhin ist

$$\sum_{n=0}^{\infty}\Delta\left(\frac{1}{\sqrt{\lambda_n\mu_n}}\right)\int_{E_\varepsilon}|s_n^*(x)|\,dx = M_\varepsilon\sum_{n=0}^{\infty}\Delta\left(\frac{1}{\sqrt{\lambda_n\mu_n}}\right)\int_{E_\varepsilon}\left|\sum_{k=0}^{n}\sqrt{\lambda_k\mu_k}a_k\psi_k(\varepsilon;x)\right|dx \leqq$$

$$\leqq M_\varepsilon\sum_{n=0}^{\infty}\Delta\left(\frac{1}{\sqrt{\lambda_n\mu_n}}\right)\sqrt{\int_0^1\left(\sum_{k=0}^{n}\sqrt{\lambda_k\mu_k}a_k\psi_k(\varepsilon;x)\right)^2dx} \leqq M_\varepsilon\frac{1}{\sqrt{\lambda_0\mu_0}}\left(\sum_{k=0}^{\infty}\lambda_k\mu_k a_k^2\right)^{1/2},$$

woraus folgt, daß die Reihe

(4) $$\sum_{n=0}^{\infty}\Delta\left(\frac{1}{\sqrt{\lambda_n\mu_n}}\right)s_n^*(x)$$

in E_ε fast überall konvergiert. Da $\sup_{\varepsilon>0}\operatorname{mes}(E_\varepsilon)=1$ ist, konvergiert (4) in $(0, 1)$ fast überall. Auf Grund von (3) ergibt sich also, daß $\lim_{n\to\infty} s_n(x)$ fast überall in $(0, 1)$ existiert.

3. Beweis des Satzes II. Der Beweis gründet sich auf den folgenden Hilfssatz.

Hilfssatz II. *Es sei λ eine von unten konkave Folge, und $E(\subseteqq(0, 1))$ eine meßbare Menge. Sind die Funktionen $L_n^1(f;x)/\lambda_n$ in E gleichmäßig beschränkt, dann gilt*

(5) $$|\sigma_n(x)|/\sqrt{\lambda_n}<\infty$$

im Falle $a\in l^2$ fast überall in E, wobei $\sigma_n(x)$ das n-te $(C, 1)$-Mittel der Reihe (1) *bezeichnet.*

Diese Behauptung stammt von G. Alexits und A. Sharma [2].

Es sei

$$E_N=\left\{x\in(0, 1): \frac{L_n^1(f;x)}{\lambda_n}\leqq N\right\}\quad (N=1, 2, \ldots).$$

Auf Grund des Hilfssatzes II gilt (5) im Falle $a\in l^2$ in jeder Menge E_N fast überall, daher gilt (4) im Falle $a\in l^2$ wegen $\operatorname{mes}\left((0, 1)\setminus\bigcup_{N=1}^{\infty}E_N\right)=0$ fast überall in $(0, 1)$.

Es sei nun $a\in l^2(\lambda)$, und $\mu=\{\mu_n\}_0^\infty$ eine nichtabnehmende von unten konkave Folge der positiven Zahlen mit $\lim_{n\to\infty}\mu_n=\infty$, und $a\sqrt{\mu}\in l^2(\lambda)$. Die n-te $(C, 1)$-Mittel der Reihe (2) bezeichnen wir mit $\sigma_n^*(x)$. Durch Abelsche Unformung ergibt sich

(6) $$\sigma_n(x)=\sum_{k=0}^{n-1}\left(1-\frac{k}{n+1}\right)\Delta^2\left(\frac{1}{\sqrt{\lambda_k\mu_k}}\right)(k+1)\sigma_k^*(x)+$$
$$+\frac{2}{n+1}\sum_{k=0}^{n-1}\Delta\left(\frac{1}{\sqrt{\lambda_k\mu_k}}\right)(k+1)\sigma_k^*(x)+\frac{1}{\sqrt{\lambda_n\mu_n}}\sigma_n^*(x).$$

Nach obigem gilt $\lim_{n\to\infty} \sigma_n^*(x)/\sqrt{\lambda_n \mu_n}=0$ fast überall in (0, 1).

Es gilt weiterhin

$$\sum_{k=0}^{\infty} \Delta^2\left(\frac{1}{\sqrt{\lambda_k \mu_k}}\right)(k+1)\int_{E_\varepsilon}|\sigma_k^*(x)|\,dx =$$

$$= M_\varepsilon \sum_{k=0}^{\infty} \Delta^2\left(\frac{1}{\sqrt{\lambda_k \mu_k}}\right)(k+1)\int_{E_\varepsilon}\left|\sum_{l=1}^{k}\left(1-\frac{l}{k+1}\right)\sqrt{\lambda_l \mu_l}\,a_l \psi_l(\varepsilon; x)\right| dx \leqq$$

$$\leqq M_\varepsilon \sum_{k=0}^{\infty} \Delta^2\left(\frac{1}{\sqrt{\lambda_k \mu_k}}\right)(k+1)\sqrt{\int_0^1\left(\sum_{l=0}^{k}\left(1-\frac{l}{k+1}\right)\sqrt{\lambda_l \mu_l}\,a_l \psi_l(\varepsilon; x)\right)^2 dx} \leqq$$

$$\leqq M_\varepsilon \sqrt{\sum_{l=0}^{\infty} a_l^2 \lambda_l \mu_l}\sum_{k=0}^{\infty} \Delta^2\left(\frac{1}{\sqrt{\lambda_k \mu_k}}\right)(k+1),$$

woraus folgt, daß die Reihe

$$\sum_{k=0}^{\infty} \Delta^2\left(\frac{1}{\sqrt{\lambda_k \mu_k}}\right)(k+1)\sigma_k^*(x)$$

in E_ε fast überall konvergiert, und so erhalten wir, daß

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\left(1-\frac{1}{n+1}\right)\Delta^2\left(\frac{1}{\sqrt{\lambda_k \mu_k}}\right)(k+1)\sigma_k^*(x)$$

in E_ε fast überall existiert. Ähnlicherweise gilt

$$\sum_{k=0}^{\infty} \Delta\left(\frac{1}{\sqrt{\lambda_k \mu_k}}\right)\int_{E_\varepsilon}|\sigma_k^*(x)|\,dx = M_\varepsilon \sum_{k=0}^{\infty} \Delta\left(\frac{1}{\sqrt{\lambda_k \mu_k}}\right)\int_{E_\varepsilon}\left|\sum_{l=0}^{k}\left(1-\frac{l}{k+1}\right)\sqrt{\lambda_l \mu_l}\,a_l \psi_l(\varepsilon; x)\right| dx \leqq$$

$$\leqq M_\varepsilon \sum_{k=0}^{\infty} \Delta\left(\frac{1}{\sqrt{\lambda_k \mu_k}}\right)\sqrt{\int_0^1\left(\sum_{l=0}^{k}\left(1-\frac{l}{k+1}\right)\sqrt{\lambda_l \mu_l}\,a_l \psi_l(\varepsilon; x)\right)^2 dx} \leqq$$

$$\leqq M_\varepsilon \sqrt{\sum_{l=0}^{\infty} a_l^2 \lambda_l \mu_l}\cdot\frac{1}{\sqrt{\lambda_0 \mu_0}} < \infty,$$

woraus folgt, daß

$$\sum_{k=0}^{\infty} \Delta\left(\frac{1}{\sqrt{\lambda_k \mu_k}}\right)|\sigma_k^*(x)| < \infty$$

in einer Menge $\tilde{E}_\varepsilon(\subseteqq E_\varepsilon)$ mit mes $(\tilde{E}_\varepsilon)=$mes (E_ε) besteht. Es sei $\varepsilon>0$ beliebig, und $x\in\tilde{E}_\varepsilon$. Dann gibt es einen Index N mit

$$\sum_{k=N+1}^{\infty} \Delta\left(\frac{1}{\sqrt{\lambda_k \mu_k}}\right)|\sigma_k^*(x)| < \frac{\varepsilon}{4}.$$

Dann gilt aber

$$\left|\frac{2}{n+1}\sum_{k=0}^{n-1}\Delta\left(\frac{1}{\sqrt{\lambda_k\mu_k}}\right)(k+1)\sigma_k^*(x)\right| \leqq$$

$$\leqq \frac{2}{n+1}\sum_{k=0}^{N}\Delta\left(\frac{1}{\sqrt{\lambda_k\mu_k}}\right)(k+1)|\sigma_k^*(x)|+2\sum_{k=N+1}^{\infty}\Delta\left(\frac{1}{\sqrt{\lambda_k\mu_k}}\right)|\sigma_k^*(x)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

für genügend großes n. D. h. ist

$$\lim_{n\to\infty}\frac{2}{n+1}\sum_{k=0}^{n-1}\Delta\left(\frac{1}{\sqrt{\lambda_k\mu_k}}\right)(k+1)\sigma_k^*(x) = 0$$

fast überall in E_ε. Auf Grund von (6) erhalten wir somit, daß $\lim_{n\to\infty}\sigma_n(x)$ in E_ε fast überall existiert. Das bedeutet, daß die Orthogonalreihe

$$\sum_{n=0}^{\infty} a_n\psi_n(\varepsilon; x) \tag{7}$$

in E_ε fast überall $(C, 1)$-summierbar ist, woraus erhalten wir durch Anwendung eines Satzes von A. ZYGMUND [6], daß die Reihe (7) und mithin auch die Reihe (1) im Falle $a\in l^2(\lambda)$ in E_ε fast überall $(C, \alpha>0)$-summierbar ist. Da $\sup_{\varepsilon>0}\text{mes}(E_\varepsilon)=1$ ist, ergibt sich die Behauptung.

4. BEWEIS DES SATZES III. Es sei

$$E_N = \left\{x\in(0, 1)\colon \frac{L_{\nu_n}(f; x)}{\lambda_{\nu_n}} \leqq N\right\} \quad (N = 1, 2, \ldots).$$

Auf Grund des Hilfssatzes I folgt, daß im Falle $a\in l^2$

$$|s_{\nu_n}(x)|/\sqrt{\lambda_{\nu_n}} < \infty$$

in jeder Menge E_N fast überall besteht; und so gilt diese Abschätzung fast überall in $(0, 1)$. Es sei $a\in l^2(\lambda)$ und $\mu=\{\mu_n\}_0^\infty$ eine nichtabnehmende Folge von positiven Zahlen mit $\lim_{n\to\infty}\mu_n=\infty$, und $a\sqrt{\mu}\in l^2(\lambda)$. Die n-te Partialsumme der Reihe (2) bezeichnen wir mit $s_n^*(x)$. Dann ist

$$s_{\nu_n}(x) = \sum_{k=0}^{\nu_n-1}\Delta\left(\frac{1}{\sqrt{\lambda_k\mu_k}}\right)s_k^*(x)+\frac{1}{\sqrt{\lambda_{\nu_n}}}s_{\nu_n}^*(x),$$

also erhalten wir wie in dem Beweis des Satzes I, daß $\lim_{n\to\infty}s_{\nu_n}(x)$ im Falle $a\in l^2(\lambda)$ in E_ε fast überall existiert. Das bedeutet, daß die Folge der ν_n-ten Partialsummen der Orthogonalreihe (7) im E_ε fast überall konvergiert. Nach einem Satz von A. ZYGMUND [6] folgt daraus, daß die Reihe (7) in E_ε fast überall $(C, \alpha>0)$-summierbar ist, woraus sich wegen $\sup_{\varepsilon>0}\text{mes}(E_\varepsilon)=1$, die Behauptung des Satzes III ergibt.

5. Wir sagen, daß die Reihe (1) in (0, 1) fast überall H_2-summierbar ist, wenn

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N}(s_n(x)-f(x))^2=0$$

mit einer fast überall endlichen Funktion $f(x)$ in (0, 1) fast überall gilt.

Man kann die Sätze II und III auch auf die H_2-Summation erweitern, da auch die folgenden Behauptungen gelten.

SATZ IV. *Unter den Bedingungen des Satzes* II *ist die Reihe* (1) *im Falle* $a\in l^2(\lambda)$ *in* (0, 1) *fast überall* H_2*-summierbar.*

SATZ V. *Unter den Bedingungen des Satzes* III *ist die Reihe* (1) *im Falle* $a\in l^2(\lambda)$ *in* (0, 1) *fast überall* H_2*-summierbar.*

Unter den Bedingungen des Satzes II oder des Satzes III ist nämlich die Orthogonalreihe (7) im Falle $a\in l^2(\lambda)$ in E_ε fast überall $(C, 1)$-summierbar. Daraus folgt durch Anwendung eines Satzes von A. ZYGMUND [6], daß die Reihe (7) und somit auch die Reihe (1) im Falle $a\in l^2(\lambda)$ in E_ε fast überall H_2-summierbar ist, woraus sich die Sätze IV, V ergeben.

7. Wir bemerken, daß in den vorigen Sätzen die Bedingung, daß das System f in (0, 1), für l^2 ein Konvergenzsystem dem Maß nach sei, wesentlich ist. Wir werden es mit einem Gegenbeispiel zeigen.

Es sei $\lambda=\{\lambda_n\}_0^\infty$ eine nichtabnehmende Folge von positiven Zahlen mit $\lambda_{2^{2^n}}=2^n$ $(n=0, 1, \ldots)$, und

$$f_{2^{2^n}}(x)=\sqrt{2^n}\quad (x\in(0,1);\, n=0,1,\ldots),\quad f_k(x)=0\quad (x\in(0,1);\, k\neq 2^{2^n};\, n=0,1,\ldots).$$

Dann ist im Falle $2^{2^{n-1}}\leqq k<2^{2^n}\,(n=1, 2, \ldots)$

$$L_k(f;x)=L_{2^{2^{n-1}}}(f;x)\leqq 2\lambda_{2^{2^{n-1}}}\leqq 2\lambda_k\quad (x\in(0,1)).$$

Weiterhin sei

$$a_{2^{2^n}}=\frac{1}{\sqrt{2^n(n+1)\log^2(n+2)}}\quad (n=0,1,\ldots),\quad a_k=0\quad (k\neq 2^{2^n};\, n=0,1,\ldots).$$

Dann ist

$$\sum_{k=0}^{\infty}a_k^2\lambda_k=\sum_{n=0}^{\infty}\frac{1}{(n+1)\log^2(n+2)}<\infty,$$

und doch gilt

$$\sum_{k=0}^{\infty}a_kf_k(x)\equiv\sum_{n=0}^{\infty}\frac{1}{\sqrt{(n+1)\log^2(n+2)}}=\infty\quad (x\in(0,1)).$$

8. Endlich bemerken wir, daß in diesen Betrachtungen ein modifizierter Begriff der Lebesgueschen Funktionen brauchbarer ist.

Es sei $f=\{f_n(x)\}_0^\infty$ eine Folge der reellen Funktionen $f_n(x)\in L(0, 1)$, und $\lambda=\{\lambda_n\}_0^\infty$ eine nichtabnehmende Folge der positiven Zahlen mit $\lim_{n\to\infty} \lambda_n=\infty$. Wir setzen

$$L_n\left(\frac{f}{\sqrt{\lambda}}; x\right) = \int_0^1 \left|\sum_{k=0}^n \frac{f_k(x)f_k(t)}{\lambda_k}\right| dt, \quad L_n^1\left(\frac{f}{\sqrt{\lambda}}; x\right) = \int_0^1 \left|\sum_{k=0}^n \left(1-\frac{k}{n+1}\right)\frac{f_k(x)f_k(t)}{\lambda_k}\right| dt.$$

Durch einfache Rechnung ergibt sich, daß im Falle

$$\sup_n L_n\left(\frac{f}{\sqrt{\lambda}}; x\right) < \infty \quad \text{(fast überall in (0, 1))}$$

$$L_n(f; x) \leqq \sum_{k=0}^{n-1} |\Delta(\lambda_k)| L_k\left(\frac{f}{\sqrt{\lambda}}; x\right) + \lambda_n L_n\left(\frac{f}{\sqrt{\lambda}}; x\right) \leqq C_1(x)\lambda_n \quad (x\in(0, 1))$$

mit einer in (0, 1) fast überall endlichen positiven Funktion $C_1(x)$ erfüllt ist.

Es sei λ von unten konkav. Durch einfache Rechnung ergibt sich, daß im Falle

$$\sup_n L_n^1\left(\frac{f}{\sqrt{\lambda}}; x\right) < \infty \quad \text{(fast überall in (0, 1))}$$

$$L_n^1(f; x) \leqq \sum_{k=0}^{n-1}\left(1-\frac{k}{n+1}\right)|\Delta^2(\lambda_k)|(k+1)L_k\left(\frac{f}{\sqrt{\lambda}}; x\right) +$$

$$+\frac{2}{n+1}\sum_{k=0}^{n-1}|\Delta(\lambda_k)|(k+1)L_k\left(\frac{f}{\sqrt{\lambda}}; x\right) + \lambda_n L_n\left(\frac{f}{\sqrt{\lambda}}; x\right) \leqq C_2(x)\lambda_n \quad (x\in(0, 1))$$

mit einer in (0, 1) fast überall endlichen positiven Funktion $C_2(x)$ erfüllt wird. Dann gelten die folgenden Behauptungen.

SATZ VI. *Gilt*

$$\sup_n L_n\left(\frac{f}{\sqrt{\lambda}}; x\right) < \infty$$

fast überall in (0, 1), *dann ist f ein Konvergenzsystem für* $l^2(\lambda)$ *in* (0, 1).

SATZ VII. *Es sei* λ *von unten konkav. Gilt*

$$\sup_n L_n^1\left(\frac{f}{\sqrt{\lambda}}; x\right) < \infty$$

fast überall in (0, 1), *dann ist f ein* $(C, \alpha>0)$*-Konvergenzsystem für* $l^2(\lambda)$ *in* (0, 1).

SATZ VIII. *Es sei* $\{v_n\}_0^\infty$ *eine Indexfolge mit* $1<q\leqq v_{n+1}/v_n<s(<\infty)$ $(n=0, 1, \ldots)$. *Gilt*

$$\sup_n L_{v_n}\left(\frac{f}{\sqrt{\lambda}}; x\right) < \infty$$

fast überall in (0, 1), *dann ist f ein* $(C, \alpha>0)$*-Konvergenzsystem für* $l^2(\lambda)$ *in* (0, 1).

Das Analogon der Sätze IV und V kann man auch beweisen; die ausführliche Fassung der entsprechenden Behauptungen überlassen wir dem Leser.

9. BEWEIS DES SATZES VI. Wir wenden den folgenden Hilfssatz an.

HILFSSATZ III. *Es sei* $\{\nu_n\}_0^\infty$ *eine streng wachsende Indexfolge. Gilt*

$$\sup_n L_{\nu_n}(f; x) < \infty$$

fast überall in (0, 1), *dann konvergiert die Folge der* ν_n*-ten Partialsummen der Reihe* (1) *im Falle* $a \in l^2$ *in* (0, 1) *fast überall.*

Diese Behauptung ist eine gewisse Verschärfung eines Satzes von G. ALEXITS [3]. Betreffs des Beweises s. K. TANDORI [5].

Unter der Bedingung des Satzes VI ist das System $f/\sqrt{\lambda}$ nach dem Hilfssatz III ein Konvergenzsystem für l^2 in (0, 1), woraus Satz VI unmittelbar folgt.

10. BEWEIS DES SATZES VII. Wir benutzen den folgenden

HILFSSATZ IV. *Gilt*

$$\sup_n L_n^1(f; x) < \infty$$

fast überall in (0, 1), *dann ist* f *ein* $(C, 1)$*-Konvergenzsystem für* l^2 *in* (0, 1).

Es sei

$$E_N = \left\{x \in (0, 1): \sup_n L_n^1(f; x) \leqq N\right\} \quad (N = 1, 2, \ldots).$$

Für jeden Index N definieren das Funktionensystem $f_N = \{f_{N,n}(x)\}_0^\infty$ folgenderweise: es sei

$$f_{N,n}(x) = \begin{cases} f_n(x) & (x \in E_N), \\ 0 & \text{sonst} \end{cases} \quad (n = 0, 1, \ldots).$$

Offensichtlich gilt

$$\sup_n L_n(f_N; x) \leqq N \quad \big(x \in (0, 1);\ N = 1, 2, \ldots\big).$$

Nach einem Satz von G. ALEXITS und A. SHARMA [2] folgt, daß f_N für jedes N ein $(C, 1)$-Konvergenzsystem für l^2 in (0, 1) ist. Da $\operatorname{mes}\left((0, 1) \setminus \bigcup_{N=1}^{\infty} E_N\right) = 0$, und $f_{N,n}(x) = f_n(x)$ $(x \in E_N;\ n = 0, 1, \ldots)$ für jedes N gelten, ergibt sich, daß f ein $(C, 1)$-Konvergenzsystem für l^2 in (0, 1) ist.

Damit haben wir den Hilfssatz IV bewiesen.

Durch Anwendung des Hilfssatzes IV ergibt sich, daß unter der Bedingung des Satzes VII das System $f/\sqrt{\lambda}$ ein $(C, 1)$-Konvergenzsystem für l^2 in (0, 1) ist. Nach einem bekannten Satz (s. z. B. K. TANDORI [5]) ist also das System $f/\sqrt{\lambda}$ fast orthonormiert, und der Satz VII folgt ähnlich wie im Beweis des Satzes I.

11. BEWEIS DES SATZES VIII. Wir definieren die Matrix $T = \|t_{n,k}\|_{n,k=0}^\infty$ folgenderweise:

$$t_{n,k} = 1 \quad (k = 0, \ldots, \nu_n), \quad t_{n,k} = 0 \quad (k = \nu_n + 1, \nu_n + 2, \ldots) \quad (n = 0, 1, \ldots).$$

Durch Anwendung des Hilfssatzes III ergibt sich, daß unter der Bedingung des Satzes VIII $f/\sqrt{\lambda}$ ein T-Konvergenzsystem für l^2 in (0, 1) ist. (D.h. im Falle $a \in l^2$ konvergieren die Summen

$$t_n(a; x) = \sum_{k=0}^{\nu_n} t_{n,k} a_k f_k(x)/\sqrt{\lambda_k} \quad (n = 0, 1, \ldots)$$

in (0, 1) fast überall.) Nach einem bekannten Satz (s. z. B. K. TANDORI [5]) folgt daraus, daß das System $f/\sqrt{\lambda}$ fast orthonormiert ist. Es sei $\varepsilon > 0$ beliebig angegeben, und seien E_ε, M_ε, $\psi = \{\psi_n(\varepsilon; x)\}_0^\infty$ wie in der Definition der Fastorthonormalität.

So folgt, daß die Folge der ν_n-ten Partialsummen der Orthogonalreihe (7) im Falle $a \in l^2(\lambda)$ in E fast überall konvergiert. Daraus erhalten wir durch Anwendung eines Satzes von A. ZYGMUND [6], daß im Falle $a \in l^2(\lambda)$ die Reihe (7) und somit auch die Reihe (1) in E_ε fast überall $(C, \alpha > 0)$-summierbar ist. Wegen $\sup_{\varepsilon > 0} \operatorname{mes}(E_\varepsilon) = 1$ folgt die Behauptung des Satzes VIII.

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JÓZSEF ATTILA UNIVERSITÄT
BOLYAI INSTITUT
SZEGED, ARADI VÉRTANÚK TERE 1.

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ON SOME CONNECTIVITY PROPERTIES OF EULERIAN GRAPHS

By
L. LOVÁSZ (Budapest)

There are several facts known in graph theory that suggest that connectivity properties of Eulerian graphs are much nicer than those of general graphs. KOTZIG [2, 3] gave a construction which presents all $(2k)$-regular $(2k)$-edge-connected graphs (he also gave an analogous construction for $(2k-1)$-regular $(2k-1)$-edge-connected graphs; this, however, does not concern us in this paper). Similar results have been obtained by G. J. SIMMONS [5]. It is known (ROTSCHILD and WHINSTON [4]) that the so-called 2-commodity-flow theorem holds in a stronger (integral) form for Eulerian graphs.

In the present paper we are going to describe a method which is essentially that a given point x of an Eulerian graph can be split so that connectivity relations between the rest of the points remain unchanged. This generalises the above-mentioned result of Kotzig. As another application we discuss a conjecture of Gallai. This concerns the maximum number of edge-disjoint paths connecting two points of a specified set X. We show that for Eulerian graphs this number can be expressed in a very simple form, thus verifying the conjecture for this class of graphs (even in a slightly sharpened form).

It is should be remarked that the point-splitting method applies to non-Eulerian graphs as well in many cases. We are going to discuss this in a forthcoming paper.

1. Let G be a graph. (We allow multiple edges.) For $X \subseteqq V(G)$, denote[1] by $\delta_G(X)$ the number of edges connecting X to $V(G)-X$.

We denote by $c_G(A, B)$ the maximum number of edge-disjoint (A, B)-paths ($A, B \subseteqq V(G)$ are disjoint). By the max-flow-min-cut theorem (or Menger's theorem)

$$c_G(A, B) = \min_{A \subseteqq X \subseteqq V(G)-B} \delta_G(X).$$

The following observation will be crucial throughout the paper:

LEMMA 1. $\delta(X \cap Y) + \delta(X \cup Y) \leqq \delta(X) + \delta(Y)$. *If equality holds here then no edge connects* $X-Y$ *to* $Y-X$.

This lemma follows by simple counting.

Given $x \in V(G)$ and two edges (x, u) and (x, v) adjacent to it, the operation of removing these two edges and creating a new edge which connects u to v, is called *splitting off* the two edges (x, u) and (x, v) from x. If x has even degree and we remove x and create $\frac{1}{2}\delta(x)$ new edges which connect neighbours of x such that in the arising

[1] We remove the subscript in the case of the graph denoted by G.

graph G' $\delta_{G'}(y)=\delta(y)$ for each $y\neq x$, the operation is called *splitting* of x. It is clear that splitting x can be carried out by repeatedly splitting off two edges adjacent to x and, finally, removing the isolate x. It is evident that splitting off two edges from a point or splitting a point of an Eulerian graph results in an Eulerian graph.

LEMMA 2. *Let us split off two edges from a point x of G and denote by G' the resulting graph. Then*

$$c_{G'}(A, B) \leqq c(A, B)$$

for any two disjoint sets $A, B \subseteqq V(G)$.

LEMMA 3. *If G is an Eulerian graph then (X) and $c(A, B)$ are even for any $X, A, B \subseteqq \subseteqq V(G)$.*

The proofs are again trivial.

THEOREM 1. *Let G be an Eulerian graph, $x \in V(G)$ and (x, u) an edge adjacent to x. Then there exists another edge (x, v) adjacent to x such that splitting off the edges (x, u) and (x, v) from x, the resulting graph G' satisfies*

$$c_{G'}(a, b) = c(a, b)$$

for any two points $a, b \in V(G)-\{x\}$.

Before proving this theorem we formulate two other assertions of similar nature and then prove them simultaneously.

THEOREM 2. *Let A, B, C, D be sets of points in an Eulerian graph G, $A\cap B=\emptyset$, $C\cap D=\emptyset$. Let $x\in V(G)-A-B-C-D$ and u a neighbour of x. Then there exists another edge (x, v) such that splitting off (x, u) and (x, v) from x the resulting graph G' satisfies*

$$c_{G'}(A, B) = c(A, B), \quad c_{G'}(C, D) = c(C, D).$$

THEOREM 3. *Let G be an Eulerian graph, $A=\{x_1, \dots, x_k\}\subseteqq V(G)$, $x\in V(G)-A$ and (x, u) an edge adjacent to x. Then there exists another edge (x, v) adjacent to x such that splitting off (x, u) and (x, v) from x, the arising graph G' satisfies*

$$c_{G'}(x_i, A-\{x_i\}) = c(x_i, A-\{x_i\}) \qquad (i = 1, \dots, k).$$

PROOF. All of Theorems 1—3 have the following form:

$(*)$ Let $A_i, B_i \subseteqq V(G)$, $A_i\cap B_j=\emptyset$ $(i=1, \dots, k)$. Let, furthermore, $x\in V(G)$, $x\notin A_i, B_i$, and (x, u) an edge adjacent to x. Then there exists another edge (x, v) adjacent to x such that if we split off edges (x, u) and (x, v) from x, the resulting graph G' satisfies

$$c_{G'}(A_i, B_i) = c(A_i, B_i) \quad (i = 1, \dots, k).$$

This assertion, however, does not hold in general, and to formulate a condition on the collection $A_1, \dots, A_k, B_1, \dots, B_k$, though possible, would be complicated. So we try to prove $(*)$ in general and then observe that in the special cases formulated in Theorems 1—3 the proof can be completed.

Let M be the set of sets S such that

(a) $x\in S$, $u\notin S$.

(b) For some $1\leqq i\leqq k$, $A_i\subseteqq S\subseteqq V(G)-B_i$ or $B_i\subseteqq S\subseteqq V(G)-A_i$ and $\delta(S)=c(A_i, B_i)$.

(c) S is minimal with respect to properties (a)—(b).

If $S\in M$ and $1\leqq i\leqq k$ is an index such that (b) holds for this i we call S a *set of index* i (the same set may have several indices).

CLAIM 1. *For each i, there exists at most one set of index i.*

PROOF. Assume S, S' are distinct sets of index i. There are three cases:

Case 1. $A_i\subseteqq S$, $A_i\subseteqq S'$. Then consider $S_1=S\cap S'$ and $S_2=S\cup S'$. By the minimality of S and S', $\delta(S_1)>c(A_i, B_i)$, and clearly $\delta(S_2)\geqq c(A_i, B_i)$. This is a contradiction since by Lemma 1,

$$\delta(S_1)+\delta(S_2) \leqq \delta(S)+\delta(S') = 2c_G(A_i, B_i).$$

Case 2. $B_i\subseteqq S$, $B_i\subseteqq S'$. Here we get contradiction analogously.

Case 3. $A_i\subseteqq S$, $B_i\subseteqq S'$. Set $S''=V(G)-S'$. Since (u, x) connects $S-S''$ to $S''-S$, we have by Lemma 1

$$\delta(S\cap S'')+\delta(S''\cup S) < \delta(S)+\delta(S'') = 2c(A_i, B_i).$$

This contradicts the obvious facts that

$$\delta(S\cap S'') \geqq c(A_i, B_i), \quad \delta(S''\cup S) \geqq c(A_i, B_i).$$

This completes the proof of Claim 1. We remark that by Claim 1, we may assume that the notation is chosen so that, for each set S of index i, $A_i\subseteqq S$.

CLAIM 2. *Let S be a set of index i and T a set of index $j(i\neq j)$. Then $A_i\cup B_i\cup A_j\cup B_j$ must intersect each of the four sets $S\cap T$, $S-T$, $T-S$, $V(G)-S-T$ and at least one of A_i, B_i, A_j, B_j must intersect at least two of $S\cap T$, $S-T$, $T-S$, $V(G)-S-T$.*

PROOF. I. Suppose $A_i\cup B_i\cup A_j\cup B_j$ does not intersect one of $S\cap T$, $S-T$, $T-S$, $V(G)-S-T$. There are essentially three cases to consider.

Case 1. $(A_i\cup B_i\cup A_j\cup B_j)\cap S\cap T=\emptyset$. Then $A_i\subseteqq S-T$ and $A_j\subseteqq T-S$. Thus

$$\delta(S-T) \geqq c(A_i, B_i), \quad \delta(T-S) \geqq c(A_j, B_j);$$

on the other hand, by Lemma 1 (with $X=S$, $Y=V(G)-T$),

$$\delta(S-T)+\delta(T-S) < \delta(S)+\delta(T) = c(A_i, B_i)+c(A_j, B_j),$$

a contradiction.

Case 2. $(A_i\cup B_i\cup A_j\cup B_j)\cap(V(G)-S-T)=\emptyset$. We get a contradiction as above.

Case 3. $(A_i \cup B_i \cup A_j \cup B_j) \cap (S-T) = \emptyset$ (say). Then $A_i \subseteq S \cap T$, $B_j \subseteq V(G)-T-S$. By the minimality property of S and T, $\delta(S \cap T) > c(A_i, B_i)$ and clearly $\delta(S \cup T) \geqq c(A_j, B_j)$. This contradicts Lemma 1 since

$$\delta(S \cup T) + \delta(S \cap T) \leqq \delta(S) + \delta(T) = c(A_i, B_i) + c(A_j, B_j).$$

II. Assume now each of A_i, B_i, A_j, B_j is contained in one of the sets $S \cap T$, $S-T$, $T-S$, $V(G)-S-T$. By Part I we have $A_i \subseteq S \cap T$, $B_i \subseteq V(G)-S-T$, $A_j \subseteq T-S$, $B_j \subseteq S-T$ for a suitable choice of indices i, j. Then

$$\delta(S) \geqq c(A_j, B_j) = \delta(T) \geqq c(A_i, B_i) = \delta(S),$$

whence

$$\delta(S) = \delta(T) = c(A_i, B_i) = c(A_j, B_j).$$

Clearly

$$\delta(S-T) \geqq c(A_j, B_j), \quad \delta(T-S) \geqq c(A_j, B_j),$$

which is impossible since by Lemma 1,

$$\delta(S-T) + \delta(T-S) < \delta(S) + \delta(T) = 2c(A_j, B_j).$$

The proof of Claim 2 is complete.

CLAIM 3. *If* $|M| \leqq 2$ *then* $(*)$ *is true.*

PROOF. Let e.g. $M = \{S, T\}$;if $|M| \leqq 1$ the proof is essentially the same but simpler. We claim x has a neighbour v in $S \cap T$. In fact, let d_1, d_2, d_3, d_4 be the numbers of neighbours of x in $S \cap T$, $S-T$, $T-S$, $V(G)-S-T$, respectively. We have

$$\delta(S - \{x\}) \geqq c(A_i, B_i) = \delta(S)$$

(where i is an index of S) and hence

$$(1) \qquad d_1 + d_2 - d_3 - d_4 = \delta(S - \{x\}) - \delta(S) \geqq 0.$$

Similarly,

$$(2) \qquad d_1 + d_3 - d_2 - d_4 \geqq 0.$$

Adding (1) and (2) we get $2d_1 - 2d_4 \geqq 0$, $d_1 \geqq d_4 > 0$.

Let v be a neighbour of x in $S \cap T$. We claim that splitting off (x, u) and (x, v) the resulting graph G' satisfies

$$c_{G'}(A_i, B_i) = c(A_i, B_i) \quad (i = 1, \dots, k)$$

Assume indirectly $c_{G'}(A_i, B_i) < c(A_i, B_i)$ for some $1 \leqq i \leqq k$. G, G' being Euleraian, these numbers are even, and hence

$$c_{G'}(A_i, B_i) \leqq c(A_i, B_i) - 2.$$

Let $X \subseteq V(G)$ be such that $x \in X$ and either $A_i \subseteq X \subseteq V(G) - B_i$ or $B_i \subseteq X \subseteq V(G) - A_i$; moreover $\delta_{G'}(X) = c_{G'}(A_i, B_i)$. Then

$$\delta_{G'}(X) \leqq c(A_i, B_i) - 2 \leqq \delta(X) - 2.$$

This is possible only if $\delta(X)=c(A_i, B_i)$ and $u, v \notin X$. Thus, X satisfies (a), (b) in the definition of M. Therefore, $S \subseteq X$ or $T \subseteq X$, so $S \cap T \subseteq X$. But this is not the case because $v \notin X$. This contradiction proves Claim 3.

Now we can easily prove Theorems 1—3.

PROOF OF THEOREM 1. We, in fact, show $|M| \leqq 1$. Assume $S, S' \in M$. Let S be of index i, S' be of index j. Then, since in this case A_i, B_i, A_j, B_j are singletons, the second assertion of Claim 2 is not satisfied, a contradiction.

PROOF OF THEOREM 2. $|M| \leqq 2$ by Claim 1.

PROOF OF THEOREM 3. We again show $|M| \leqq 1$. Assume $S, S' \in M$, S, S' having indices i, j, resp. Then $A_i \cup B_i \cup A_j \cup B_j = A_i \cup B_i$ and since one of A_i, B_i is a singleton the first assertion of Claim 2 cannot be satisfied.

We now deduce certain corollaries of Theorems 1—3.

COROLLARY 2. *Given an Eulerian graph G and four points a, b, a', b', the maximum number of edge-disjoint paths each of which connects a to a' or b to b' equals to*

$$c = \min \{c(\{a, b\}, \{a', b'\}), c(\{a, b'\}, \{a', b\})\}.$$

PROOF. It is trivial that there cannot exist more than c paths with the given properties. We show by induction on $|E(G)|$ that c paths do exist. If there is no point other than a, b, a', b' the assertion is easily proved. Assume $x \in V(G) - \{a, a', b, b'\}$. Then by Theorem 2 it is possible to split off two edges from x so that the resulting graph G' satisfies

$$c_{G'}(\{a, b\}, \{a', b'\}) = c(\{a, b\}, \{a', b'\}),$$

$$c_{G'}(\{a, b'\}, \{a', b\}) = c(\{a, b'\}, \{a', b\}).$$

By induction hypothesis, G' contains c edge-disjoint paths each connecting a to a' or b to b'. These clearly yield c such paths in G.

Let, now, $A \subseteq V(G)$ be any fixed subset of G. By an A-path we mean a path whose endpoints belong to A but which has no other point common with A. In particular, an edge connecting two points of A is an A-path. The following conjecture is a version of a conjecture of GALLAI [1]:

CONJECTURE 1. *Let ν be the maximum number of edge-disjoint A-paths in an arbitrary graph G. Then there are $\leqq 2\nu$ edges which represent all A-paths, i.e. which completely separate the points of A.*

Note that for $|A|=m=2$, 2ν can be replaced by ν by Menger's theorem.

We will prove that for Eulerian graphs this conjecture holds in a slightly more general form:

THEOREM 4. *Let G be an Eulerian graph and $A \subseteq V(G)$, $|A|=m$. Denote by ν the maximum number of edge-disjoint A-paths in G. Then there are $2\nu\left(1-\frac{1}{m}\right)$ edges which represent all A-paths, i.e. such that removing them the points of A will be in distinct components.*

Also, we have the following expression for v:

$$v = \frac{1}{2} \sum_{a \in A} c(a, A - \{a\}). \tag{4}$$

REMARKS. 1. Note that the "edge-disjoint path"-version of Menger's theorem, for Eulerian graphs, is a special case for $m=2$.

2. It would be enough to assume that the points of $V(G)-A$ have even degrees. This can be seen by pairing up the odd-valenced points in A by new edges.

PROOF. I. First we prove formula (4). Let F_a be a minimum set of edges separating a from $A-\{a\}$, then $|F_a|=c(a, A-a)$. Let $P_1, \dots, P_v$ be a maximum set of edge-disjoint A-paths. Then each P_i meets at least two F_a, hence $2v \leqq \sum_{a \in A} |F_a|$ which proves $\leqq$ in (4).

On the other hand, we prove by induction on $|E(G)|$ that there are

$$\frac{1}{2} \sum_{a \in A} c(a, A - \{a\})$$

edge-disjoint A-paths. If $A=V(G)$, the assertion is trivial: there are $|E(G)|$ edge-disjoint A-paths. Let $x \in V(G)-A$, and let us split off two edges from x so that the resulting graph G' satisfies

$$c_{G'}(a, A-\{a\}) = c(a, A-\{a\})$$

for every $a \in A$. By the induction hypothesis, there are

$$\frac{1}{2} \sum_{a \in A} c_{G'}(a, A-\{a\}) = \frac{1}{2} \sum_{a \in A} c(a, A-\{a\})$$

edge-disjoint A-paths in G', which yield the same number of edge-disjoint A-paths in G. Thus (4) is proved.

II. Observe that

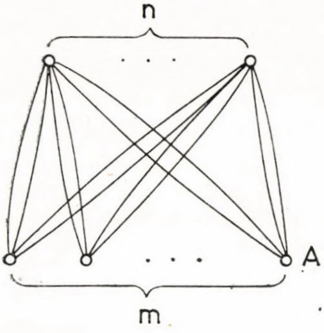
$$\bigcup_{\substack{a \in A \\ a \neq a_0}} F_a$$

covers all A-paths for any $a_0 \in A$. Let $a_0 \in A$ with $|F_{a_0}|$ maximal, then

$$\Big|\bigcup_{\substack{a \in A \\ a \neq a_0}} F_a\Big| \leqq \sum_{\substack{a \in A \\ a \neq a_0}} |F_a| \leqq \left(1-\frac{1}{m}\right) \sum_{a \in A} |F_a| = \left(1-\frac{1}{m}\right) 2v$$

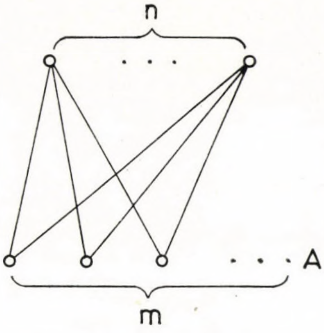
as stated.

We remark that the bound is sharp, as shown by the graph on Fig. 1.a. Fig. 1.b shows that for non-Eulerian graphs, the bound in Conjecture 1 cannot be improved for $m \geqq 3$.



a) $\nu = mn$

Fig. 1)a



b) $\nu = n$

Fig. 1)b

Appendix

We add some theorems concerning the above problems for arbitrary graphs, partly as applications of the preceding Theorem 4.

Let G be an arbitrary graph and $A \subseteqq V(G)$. A 2-*packing* is a collection of A-paths such that each edge belongs to at most 2 of them (the same A-path may occur in the collection twice). A 2-*cover* is a collection of edges such that each A-path contains at least 2 of them; again, the same edge may be taken more than once. Let ν_2 and τ_2 be the maximum size of a 2-packing and the minimum size of a 2-cover, respectively. We still use ν for the maximum number of edge-disjoint A-paths.

THEOREM 5. *For an arbitrary graph G and any $A \subseteqq V(G)$, $|A| \geqq 2$,*

$$\nu_2 = \tau_2 = \sum_{a \in A} c(a, A - \{a\}).$$

PROOF. Let us double each edge of G and let G^* be the resulting Eulerian graph. Clearly

$$c_{G^*}(a, A - \{a\}) = 2c(a, A - \{a\})$$

and the maximum number of edge-disjoint A-paths in G is ν_2. Hence

$$\nu_2 = \sum_{a \in A} c(a, A - \{a\})$$

follows by Theorem 4.

On the other hand, let F_a be a minimum set of edges separating a from $A - \{a\}$. Let F be the collection $\sum F_{a'}$ i.e. let an edge a occur in F l times iff it is contained in l sets F_a. Then F is a 2-cover and hence,

$$\tau_2 \leqq |F| = \sum_{a \in A} c(a, A - \{a\}).$$

Since $\tau_2 \geqq \nu_2$ follows by a trivial counting argument, Theorem 5 is proved.

THEOREM 6. $\tau_2 \leqq \left(4 - \dfrac{2}{\left[\dfrac{m+1}{2}\right]}\right) \nu.$

PROOF. Set

$$\mu = \max_{X \subseteq A} c(X, A-X).$$

It is evident that $\nu \geqq \mu$. We prove

$$\tau_2 \leqq \left(4 - \frac{2}{\left[\frac{m+1}{2}\right]}\right) \mu.$$

This could be obtained from Theorem 5 using certain inequalities similar to Lemma 1, but here we choose another way.

Let, for each $X \subseteq A$, $|X| = \left[\frac{m}{2}\right]$, F_X be a minimum set of edges separating X from $A-X$, and let $P = \{P_i \colon 1 \leqq i \leqq \nu_2\}$ be a maximum 2-packing of paths. Let us count the incidences between paths in P and edges in F_X. Each path P_i meets F_X for each X separating its endpoints; there are $2\binom{m-2}{\left[\frac{m}{2}\right]-1}$ such sets X, so we count at least $2\nu_2 \binom{m-2}{\left[\frac{m}{2}\right]-1}$ incidences. A given edge of F_X occurs in at most two paths P_i, so there are at most

$$2 \sum_X F_X \leqq 2 \binom{m}{\left[\frac{m}{2}\right]} \mu$$

incidences. Hence

$$\tau_2 = \nu_2 \leqq \mu \binom{m}{\left[\frac{m}{2}\right]} \bigg/ \binom{m-2}{\left[\frac{m}{2}\right]-1} = \left(4 - \frac{2}{\left[\frac{m+1}{2}\right]}\right) \mu$$

which proves the theorem.

It follows as in Theorem 3:

COROLLARY. *All A-paths can be covered by at most*

$$4\left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{2\left[\frac{m+1}{2}\right]}\right)$$

edges.

REMARK. We have the following numbers here in consideration:

$$\mu = \max_{X \subseteq A} c(X, A-X);$$

ν, the maximum number of edge-disjoint A-paths; ν_2, the maximum size of a 2-packing; τ_2, the minimum size of a 2-cover; τ, the minimum number of edges totally separating the points of A. We know

$$\mu \leqq \nu \leqq \frac{\nu_2}{2} = \frac{\tau_2}{2} \leqq \tau$$

moreover

$$\tau \leqq \left(1-\frac{1}{m}\right)\tau_2, \quad \tau_2 \leqq \left(4-\frac{2}{\left[\frac{m+1}{2}\right]}\right)\mu,$$

and that, for Eulerian graphs, $\nu=\frac{\nu_2}{2}$. I conjectured $\tau \leqq 2\mu$ but recently P. D. Seymour kindly showed me a counterexample (Fig. 2).

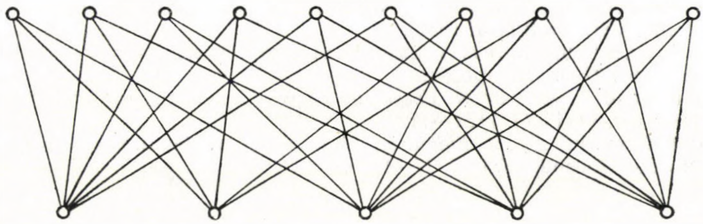


Fig. 2

THEOREM 7. *Let G be a graph and let us associate a capacity $c(e)$ with each edge e of G. Suppose $\sum_{e \ni y} c(e)$ is even for each point y. Let $x \in V(G)$. Then the maximum number k of circuits containing x such that each edge e is contained in at most $c(e)$ of them equals to*

$$\min_F \frac{1}{2} \sum_{e \in F} c(e)$$

where F ranges through all collections of edges representing every circuit through x twice.

PROOF. Let us construct a graph G' from G as follows. Let $a_1, \dots, a_d$ be the neighbours of x. Let us replace x by d new points $x_1, \dots, x_d$, where x_i is connected to a_i by $c\big((x, a_i)\big)$ edges (and to no other point) and also replace each edge e of $G-x$ by $c(e)$ parallel edges with the same endpoints. The resulting graph G' has even degrees except, possibly, in the points $x_1, \dots, x_d$. Let $A=\{x_1, \dots, x_d\}$. Observe now that k is, trivially, the same as the maximum number of edge-disjoint A-paths in G'. Moreover, a collection F of edges of G represents all circuits through x twice iff the corresponding collection F' of edges of G, of cardinality

$$\sum_{e \in F} c(e),$$

represents all A-paths in G' twice; and if a collection F_1 of edges of G' represents all A-paths twice and is minimal, then trivially, it consists of the

$$\sum_{e \in F_2} c(e)$$

edges corresponding to a collection F_2 of edges of G, which represents all circuits through x twice. Hence Theorem 5 implies Theorem 7.

Added in proof (October 1, 1976). P. D. Seymour informed me (private correspondence) that he improved Theorem 6 by showing that $\tau_2 \leqq 3\nu$.

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JÓZSEF ATTILA UNIVERSITY
BOLYAI INSTITUTE
SZEGED, HUNGARY

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ON A PROBLEM OF J. SZABADOS

By
P. VÉRTESI (Budapest)

1. Introduction and preliminary results. One of the most fundamental theorems of the approximation theory is the Jackson estimation which states that if $f(x)\in C^{(r)}[-1, 1](=f^{(r)}$ is continuous on $[-1, 1])$ then there exist polynomials $P_n(f; x)$ of degree $\leqq n$ such that

$$(1.1)\qquad \|f(x)-P_n(f;x)\| = O(n^{-r})\omega\left(f^{(r)};\frac{1}{n}\right)\quad (n>r\geqq 0)$$

where $\|\cdot\|$ is the maximum norm on $[-1, 1]$. As it is well known there exists a better pointwise estimation due to Timan, namely

$$(1.2)\quad |f(x)-P_n(f;x)| = O(1)\left(\frac{\sqrt{1-x^2}}{n}+\frac{|x|}{n^2}\right)^r\omega\left(f^{(r)};\frac{\sqrt{1-x^2}}{n}+\frac{|x|}{n^2}\right)\quad (n>r\geqq 0).$$

(Here $\omega_k(f; t)$ is the k-th modulus of smoothness of $f(x)$ on $[-1, 1]$, $\omega(f; t)=$ $=\omega_1(f; t)$.)

For $r=0$ many authors have proved these estimations using interpolatory polynomials

$$Q_n(f;x)\overset{\text{def}}{=}\sum_{k=1}^{n}f(x_{k,n})q_{k,n}(x),$$

i.e. for which

$$(1.3)\qquad q_{k,n}(x_{i,n}) = \delta_{k,i}\qquad (i, k=1, 2, \ldots, n)$$

and the degree of $q_{k,n}(x)$ is not greater than $4n$. Here

$$(1.4)\qquad -1\leqq x_{n,n}<x_{n-1,n}<\ldots<x_{1,n}\leqq 1\qquad (n=1, 2, 3, \ldots)$$

is a suitable point-system (see e.g. G. FREUD [1], M. SALLAY [2], R. B. SAXENA [3], P. VÉRTESI [4], G. FREUD and P. VÉRTESI [5], O. KIS and P. VÉRTESI [6]).

In his paper [7] R. B. SAXENA proved that for the interpolatory polynomials $Q_n(f; x)$ defined in [3] we can get the Telyakovski's estimation

$$|f(x)-Q_n(f;x)| = O(1)\omega\left(f;\frac{\sqrt{1-x^2}}{n}\right)$$

for any $f\in C[-1, 1]$.

In the papers [1]—[7] the node-systems are of "Chebyshev type", i.e., they are "very far" from the equidistant nodes which generally show bad behaviour in interpolatory questions. So it was interesting when J. SZABADOS, answering a question of DeVore, gave polynomials $p_{j,k,r,n}(x)$ of degree $\leqq n$ such that for the (not interpolating) operator

$$S_{n,r}(f;x) \stackrel{\text{def}}{=} \sum_{k=0}^{n}{}' \sum_{j=0}^{r} f^{(j)}\left(\frac{k}{n}\right) p_{j,k,r,n}(x) \tag{1.5}$$

we have (1.1) for any $f(x)\in C^{(r)}[-1,1]$ (see [8]).

In the same paper J. Szabados raised the problem as follows.

"Is it possible to improve the Jackson order $n^{-r}\omega(f^{(r)}; n^{-1})$ to the Timan—Telyakowski pointwise estimate $(\sqrt{1-x^2}/n)^r\omega(f^{(r)}; \sqrt{1-x^2}/n)$ by operators like in (1.5)?"

2. New results. We give a negative answer for the above mentioned problem. Namely we have

Theorem 2.1. *If $r\geqq 0$ is a fixed integer, $l_{j,k,r,n}(x)$ are arbitrary continuous functions and*

$$L_{n,r}(f;x) = \sum_{k=0}^{n}\sum_{j=0}^{r} f^{(j)}\left(-1+\frac{2k}{n}\right) l_{j,k,r,n}(x)$$

then there exist a function $f_1\in C^{(r)}[-1,1]$ and a sequence $\{y_n\}\subset[-1,1]$ such that

$$|f_1(y_n)-L_{n,r}(f_1;y_n)| \neq O(1)\left(\frac{\sqrt{1-y_n^2}}{n}+\frac{|y_n|}{n^2}\right)^r \omega\left(f_1^{(r)};\frac{\sqrt{1-y_n^2}}{n}+\frac{|y_n|}{n^2}\right). \tag{2.1}$$

2.1. Proof. If there exists a sequence $\{s_{n_i}\}_{i=1}^{\infty}$ such that $L_{n_i,r}(1;s_{n_i})\neq 1$ then we have (2.1) with $f_1(x)=1$ and $\{y_{n_i}\}=\{s_{n_i}\}$. So we can suppose that

$$L_{n,r}(1;x) = \sum_{k=0}^{n} l_{0,k,r,n}(x) \equiv 1 \quad (n \geqq n_0). \tag{2.2}$$

Let further

$$x_{k,n} = -1+\frac{2k}{n} \quad (k=0,1,\dots,n); \qquad x_{n+1,n} \stackrel{\text{def}}{=} y_n = 1-\frac{1}{n} \quad (n=1,2,3,\dots). \tag{2.3}$$

2.2. Now we prove the following

Lemma 2.1. *If for each $f\in C^{(r)}[-1,1]$*

$$|f(x)-L_{n,r}(f;x)| = O(1)\left(\frac{\sqrt{1-x^2}}{n}+\frac{|x|}{n^2}\right)^r \omega\left(f^{(r)};\frac{\sqrt{1-x^2}}{n}+\frac{|x|}{n^2}\right), \tag{2.4}$$

then for any $\eta_n\nearrow\infty$ we can choose a subsequence $M=\{m_i\}_{i=1}^{\infty}$ such that

$$\sum_{k=0}^{n} |l_{j,k,r,n}(y_n)| \leqq \frac{c\cdot\eta_n}{n^{\frac{r+1}{2}}} \quad (j=0,1,\dots,r;\quad n=m_1,m_2,m_3,\dots; c,\eta_n>0). \tag{2.5}$$

In proving this lemma we use the ideas of [9]. Here we give some correspondences: $T_n=L_{n,r}$, $U_n=E$, $m=r+1$, $\omega_m=t^{r+1}$, $\lambda_n(x)=\sum_{k=0}^{n}|l_{j,k,r,n}(x)|$, $z_n=y_n$, $e_n=\eta_n^{-1}$, $\delta_n=\frac{1}{n}$, $\tilde f=f_2$. We shall only sketch the proof.

For a fixed j we define $g_{3^l}(l=1, 2, 3, \ldots)$ as follows:

$$(2.6)\qquad \begin{cases} g_3^{(s)}(x_{k,3}) = 0 \quad (0 \leqq s \leqq r,\ s \neq j,\ k = 0, 1, 2, 3), \\ g_3^{(j)}(x_{k,3}) = \operatorname{sg}\ l_{j,k,r,3}(y_3) \quad (k = 0, 1, 2, 3), \\ g_3^{(s)}(y_3) = 0 \quad (s = 0, 1, \ldots, r). \end{cases}$$

In the disjoint open intervals defined by $\{x_{k,3}\}_{k=0}^{4}$ let $g_3(x)$ be the corresponding Hermite polynomial of degree $\leqq 2r+1$. If we have determined the functions $g_3(x), g_9(x), \ldots, g_{3^{p-1}}(x)$, we define $g_{3^p}(x)$ at the nodes $\{x_{k,3^p}\}_{k=0}^{3^p+1}$ as in (2.6) completing with the conditions

$$(2.7)\qquad g_{3^p}^{(s)}(y_{3^i}) = 0 \qquad (s=0, 1, \ldots, r;\ i=1, 2, \ldots, p-1).$$

We define $g_{3^p}(x)$ in the open intervals determined by the points $\{x_{k,3^p}\}_{k=0}^{3^p} \cup \{y_{3^i}\}_{i=1}^{p}$ as above. It is easy to see that

$$(2.8)\qquad \{x_{k,3^{p-1}}\}_{k=1}^{3^{p-1}} \subset \{x_{i,3^p}\}_{i=0}^{3^p}, \qquad \left(\bigcup_{i=1}^{\infty} y_{3^i}\right) \cap \left(\bigcup_{p=1}^{\infty} \{x_{k,3^p}\}_{k=0}^{3^p}\right) = \emptyset,$$

$$(2.9)\qquad \min_{\substack{0 \leqq k \leqq 3^p \\ 1 \leqq i \leqq p}} |x_{k,3^p} - y_{i,3^i}| = \frac{1}{3^p}.$$

Let

$$(2.10)\qquad f_2(x) = Q \sum_{i=1}^{\infty} \frac{e_{n_i}}{n_i^{r+1}} g_{n_i}(t)$$

where $e_n = \eta_n^{-1}$, $T \stackrel{\text{def}}{=} \{n_i\}_{i=1}^{\infty} \subset \{3^i\}_{l=1}^{\infty}$, is a lacunary subsequence (see later).

Using the notation

$$(2.11)\qquad \omega_{r+1}(t) = t^{r+1}$$

we can see that $g_n(x) \in C(\omega_{r+1})$ $\big(f \in C(\omega_{r+1})$ if $\omega_{r+1}(f; t) \leqq a(f)\omega_{r+1}(t)\big)$, similarly $f_2 \in C(\omega_{r+1})$ for suitable T (see ([9], 2.4 and 3.3).

So by (2.4) we have for each fixed $n \in T$

$$(2.12)\qquad |L_{N,r}(g_n; y_N) - g_n(y_N)| \leqq c_n \left(\frac{1}{N^{3/2}}\right)^{r+1}.$$

Using the definition of $g_n(x)$ we get

$$(2.13)\qquad |L_{n,r}(g_N; y_n)| \leqq \sum_{k=1}^{r} |l_{j,k,r,n}(y_n)| \quad (N > n;\ n, N \in T),$$

$$(2.14)\qquad g_N^{(s)}(y_n) = 0 \qquad (s=0, 1, \ldots, r;\ N \geqq n;\ n, N \in T).$$

Finally by (2.6)

$$(2.15)\qquad L_{n,r}(g_n; y_n) = \sum_{k=1}^{n} |l_{j,k,r,n}(y_n)| \quad (n \in T).$$

By (2.10)—(2.13) we get

$$L_{n,r}(f_2; y_n) - f_2(y_n) > c_7 e_n n^{-r-1} \sum_{k=1}^{n} |l_{j,k,r,n}(y_n)| \quad (n \in T)$$

(see [9], 2.6). On the other hand by (2.4)

$$|L_{n,r}(f_2; y_n) - f_2(y_n)| = O(n^{-\frac{3}{2}(r+1)}) \quad (n = 1, 2, 3, \ldots)$$

from where we get (2.5) for a fixed j. I.e., we can get a set of indices M_0 for which (2.5) is true if $j=0$. From M_0, using the proof of the lemma, we can choose M_1 such that (2.5) will be true for $j=1$ if $n\in M_1$. Continuing this process we get M_r, which can be the desired M.

2.3. Now we can easily prove our theorem. By (2.2) $\sum_{k=0}^{n} |l_{0,k,r,n}(y_n)| \geqq 1$ which contradicts (2.5).

3. Remarks. 3.1. With the same method we can prove the following theorem. If $r\geqq 0$ is a fixed integer, further for the fixed $\omega_{r+1}(t)\neq c_1 t^{r+1}$ we have $\omega_{r+1}(n^{-3/2})=o(1)\omega_{r+1}(n^{-1})$, then there exists a function $f_3\in C(\omega_{r+1})$ for which (2.1) is true. (This statement is interesting when $r=0$ because $\omega_{r+1}(n^{-3/2})=o(1)\omega_{r+1}(n^{-1})$ if $r\geqq 1$.)

3.2. As in the lemma we can prove the relation

$$\sum_{k=0}^{n} |l_{j,k,r,n}(x_0)| \leqq c_2 \quad (j=0,1,\dots,r;\ n\in M)$$

where $x_0\in(-1,1)$, $\omega_{r+1}(t)\neq c_1 t^{r+1}$ supposing only Jackson order of convergence.

3.3. We have seen that if the nodes are "far" from each other at the end-points then we generally cannot obtain a Timan-type estimation (see Theorem 2.1). On the other hand if for the minimal distance of the nodes

$$d_n \overset{\text{def}}{=} \min_{1\leqq i<k<n} |x_{i,n}-x_{k,n}| \sim n^{-2} \tag{3.1}$$

then we can get the Timan-estimate (see [1]—[7]). A natural question is whether there exists a point system such that $\overline{\lim_{n\to\infty}}\, d_n\cdot n^2=\infty$ for which we can prove a Timan type estimation.

The following example gives a negative answer for this question.

Let $\{\varphi_n\}_{n=1}^{\infty}$ be a subsequence of the set of positive integers, $\varphi_i\leqq\varphi_{i+1}$ ($i=1,2,3,\dots$), $\varphi_n=O(\sqrt{n})$. Let further

$$\begin{cases} h_{k,n} = \cos\dfrac{2k-1}{2n}\pi \quad (k=1,2,\dots,n), \\[2ex] v_{k,n} = \cos\dfrac{2k-1}{2n}\varphi_n\pi \quad (k=1,2,\dots,n), \\[2ex] H \overset{\text{def}}{=} \left\{h_{k,n};\ |h_{k,n}|\leqq \dfrac{1}{2}\right\}, \\[2ex] V \overset{\text{def}}{=} \left\{v_{k,n};\ |v_{k,n}|\geqq \dfrac{1}{2}+\dfrac{1}{n}\right\}, \\[2ex] \{x_{i,n}\} = H\cup V,\quad y_n = \dfrac{x_{1,n}+x_{2,n}}{2}. \end{cases} \tag{3.2}$$

It is easy to see that $d_n \sim \varphi_n^2 n^{-2}$, $n^{-1}\sqrt{1-y_n^2}+|y_n|n^{-2} \sim \varphi_n n^2$.

By the arguments used above we can prove the following

THEOREM 3.1. *If $r \geqq 0$ is a fixed integer, $l_{j,k,r,n}(x)$ are arbitrary continuous functions and*

$$L_{n,r}(f;x) = \sum_{k=1}^{n}\sum_{j=0}^{r} f^{(j)}(x_{k,n})\, l_{j,k,r,n}(x)$$

then, supposing that $\varphi_n \nearrow \infty$, there exist a function $h(x) \in C^{(r)}[-1,1]$ and a sequence $\{v_n\} \subset [-1,1]$ such that

$$|h(x)-L_{n,r}(h;v_n)| \neq O(1)\left(\frac{\sqrt{1-v_n^2}}{n}+\frac{|v_n|}{n^2}\right)^r \omega\left(h^{(r)};\frac{\sqrt{1-v_n^2}}{n}+\frac{|v_n|}{n^2}\right).$$

We can prove the analogon of 3.1, too.

On the other hand if $\varphi_n=1$ then the points $x_{k,n}$ are the Chebyshev nodes. So using the result of [5] and standard arguments (see e.g. [8], Corollary) we get an operator of the form

$$L_{n,r}(f;x) = \sum_{k=1}^{n}\sum_{j=0}^{r} f^{(j)}(x_{k,n})\, q_{j,k,r,n}(x)$$

for which we have the Timan type convergence for $f \in C^{(r)}[-1,1]$. ($q_{j,k,r,n}(x)$ are polynomials of degree $\leqq c_3 rn$.)

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST
REÁLTANODA U, 13—15. HUNGARY

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ON THE DYADIC DERIVATIVE

By

F. SCHIPP (Budapest)

1. Introduction

In [2], P. L. BUTZER and H. J. WAGNER introduced the following concept of derivate for a function f given on the interval $[0, 1)$.

DEFINITION. For $f\colon [0, 1)\to \mathbf{R}$ let

$$(1)\qquad (d_n f)(x) = \sum_{i=0}^{n-1} 2^{i-1}(f(x)-f(x\dotplus 2^{-i-1}))\quad (x\in[0, 1),\, n\in\mathbf{P}),$$

where $\dotplus$ denotes the addition introduced by N. J. FINE [4], and $\mathbf{P}=\{1, 2, 3, \dots\}$.

(i) If for the function f there exists

$$\lim_{n\to\infty} (d_n f)(x) = c$$

at some point $x\in[0, 1)$, then c is called the pointwise dyadic derivative of f at x denoted by $f^{[1]}(x)=c$.

(ii) If for $f\in L^p(0, 1)$ $(1\leqq p\leqq\infty)$ there exists $g\in L^p(0, 1)$ such that

$$\lim_{n\to\infty} \|d_n f-g\|_p = 0,^1$$

then g is called the strong dyadic derivative of f in $L^p(0, 1)$ denoted by $Df=g$.

This is a modified form of GIBBS' definition, introduced in [6], [7], [8].

The strong dyadic derivative turned out to be a linear, closed operator, its inverse operator or "integral" was introduced, and the fundamental theorem of the calculus was also found to hold for these two concepts [2], [3], [12].

The most important property of the dyadic derivative is the fact that the Walsh—Paley functions ψ_k $(k\in\mathbf{N}=\mathbf{P}\cup\{0\})$ are arbitrarily often differentiable (in both sense) and one has

$$\psi_k^{[1]} = D\psi_k = k\psi_k\quad (k\in\mathbf{N}).$$

The inverse operator to the differential operator D, namely the "integral" operator I, is defined by the convolution

$$(2)\qquad (If)(x) = (f*W)(x) = \int_0^1 f(u)W(x\dotplus u)\,du,$$

[1] $\|\cdot\|_p$ denotes the $L^p(0,1)$-norm $(1\leqq p\leqq\infty)$.

where the function $W \in L^1(0, 1)$ is given by its Walsh—Fourier coefficients

$$\hat{W}(0) = 1, \qquad \hat{W}(k) = k^{-1} \quad (k \in \mathbf{P}). \tag{3}$$

(See [2], [3].)

In [10], the following analogon of the fundamental theorem of the differential and integral calculus is proved for the pointwise dyadic derivative:

THEOREM 1. *For $f \in L^1(0, 1)$ and for almost every $x \in (0, 1)$ we have*

$$(If)^{[1]}(x) = f(x). \tag{4}$$

In this paper we give the following generalization of the above theorem.

THEOREM 2. *If F is a function of bounded variation on* $[0, 1]$, *then for almost every* $x \in (0, 1)$

$$(W * dF)^{[1]}(x) = F'(x), \tag{5}$$

where $(W * dF)(x) = \int_0^1 W(x \dotplus t)\, dF(t)$ and the integral is taken in the Lebesgue—Stieltjes sense.

It is obvious, that Theorem 1 follows from Theorem 2, if we apply it to the absolutely continuous function $F(x) = \int_0^x f(t)\, dt$ $(x \in [0, 1])$. To the proof of Theorem 2 we apply the Calderón—Zygmund decomposition lemma in a form concerning Borel measures, and use stopping time technics. (For these concepts see e.g. [13].)

An immediate consequence of Theorem 2 is

COROLLARY 1. *If*

$$\sum_{n=0}^{\infty} n a_n \psi_n \tag{6}$$

is the Walsh—Fourier—Stieltjes series of a function F of bounded variation, then the series

$$\sum_{n=0}^{\infty} a_n \psi_n \tag{7}$$

converges a.e. to a function $f \in L^2(0, 1)$, moreover f is a.e. dyadic differentiable and for almost every $x \in (0, 1)$

$$f^{[1]}(x) = F'(x) = \lim_{n \to \infty} \left(\sum_{k=0}^{2^n - 1} k a_k \psi_k(x) \right). \tag{8}$$

Indeed, $n a_n = \int_0^1 \psi_n\, dF = O(1)$ implies that (7) is the Walsh—Fourier series of a function $f \in L^2(0, 1)$ and by a result of P. BILLARD [1] the series (7) converges a.e. to f. On the other hand by $W \in L^1(0, 1)$ (see [2]) $(W * dF) \in L^1(0, 1)$ and $(W * dF)^{\wedge}(n) = a_n$ we have $W * dF = f$ a.e., thus from Theorem 2 follows $f^{[1]} = F'$ a.e.

For every $n\in\mathbf{N}$ and $x\in(0, 1)$ we define the pair of dyadic rationals $\alpha_n(x)$, $\beta_n(x)$ by the inequalities

$$\alpha_n(x) = k2^{-n} \leqq x < (k+1)2^{-n} = \beta_n(x) \qquad (k<2^n,\ k, n\in\mathbf{N}).$$

Then the well-known relation

$$D_{2^n}(x \dotplus y) = \sum_{k=0}^{2^n-1} \psi_k(x \dotplus y) = \begin{cases} 2^n & (y\in[\alpha_n(x), \beta_n(x))), \\ 0 & (y\notin[\alpha_n(x), \beta_n(x))) \end{cases} \tag{9}$$

gives

$$\sum_{k=0}^{2^n-1} ka_k\psi_k(x) = \int_0^1 D_{2^n}(x \dotplus y)\, dF(y) = 2^n\big(F(\beta_n(x)) - F(\alpha_n(x))\big) \to F'(x) \tag{10}$$

a.e. if $n\to\infty$. Thus Corollary 1 is proved.

A standard argument shows (see e.g. [5], [9]), that if

$$\left\|\sum_{k=0}^{2^n-1} b_k\psi_k\right\|_1 = O(1),$$

and for every dyadic national point α we have

$$\left(\sum_{k=0}^{2^n-1} b_k\psi_k\right)(\alpha-0) = O_\alpha(2^n),$$

then $\sum_{k=0}^{\infty} b_k\psi_k$ is a Walsh—Fourier—Stieltjes series. This implies

COROLLARY 2. *If*

$$\left\|\sum_{k=0}^{2^n-1} ka_k\psi_k\right\|_1 = O(1), \qquad \sum_{k=0}^{2^n-1} ka_k\psi_k(\alpha-0) = O_\alpha(2^n)$$

(α dyadic rational), then (7) *converges a.e. to a function $f\in L^2(0, 1)$ which is a.e. dyadic differentiable and for almost every $x\in(0, 1)$ we have*

$$f^{[1]}(x) = \lim_{n\to\infty}\left(\sum_{k=0}^{2^n-1} ka_k\psi_k(x)\right).$$

Let

$$V_n^0 = D_{2^n}, \qquad V_n^1(x) = \sum_{i=0}^{n-1} 2^i \sum_{k=n}^{\infty} 2^{-k} D_{2^k}(x \dotplus e_i),$$

$$V_n^2(x) = \sum_{i=0}^{n-1} 2^i \sum_{k=i}^{n-1} 2^{-n} D_{2^k}(x \dotplus e_i) \quad (x\in[0, 1],\ n\in\mathbf{P}), \tag{11}$$

where $e_i=2^{-i-1}$ $(i\in\mathbf{N})$ and for an arbitrary Borel measure μ we set

$$(T^i\mu)(x) = \sup_n |V_n^i * \mu)(x)| \quad (i = 0, 1, 2). \tag{12}$$

Here $V*\mu$ denotes the dyadic convolution of the measure μ with the function $V\in L(0, 1)$, i.e.

$$(V*\mu)(x) = \int_0^1 V(x \dotplus y)\, d\mu(y) \quad (x\in[0, 1]).$$

We shall prove, that Theorem 2 is an immediate consequence of the following

THEOREM 3. *Let μ be a Borel measure and $y>0$. Then*

(13) $$\lambda\{x: (T^i\mu)(x) > y\} \leqq 16\|\mu\|/y,$$

where λ denotes the Lebesgue measure and $\|\mu\| = \sup_B |\mu(B)|$ (the sup *is taken over all Borel sets $B\subset(0, 1)$).*

Theorem 3 is a generalization of part b) of the theorem in [10].

By means of the method, used in the proof of Theorem 3, we can also show the following theorem of N. J. FINE (see [5], Theorem 8).

THEOREM 4. *Let F be a function of bounded variation and denote $\sigma_n(dF)$ the n-th $(C, 1)$ mean of the Walsh—Fourier—Stieltjes series of F. Then for almost every $x\in(0, 1)$ we have*

(14) $$\lim_{n\to\infty} \sigma_n(dF)(x) = F'(x).$$

2. Proofs

To prove Theorem 3 we shall use the following decomposition lemma of Calderón—Zygmund type.

LEMMA. *Let μ be a positive Borel measure and $y>\mu([0, 1))=\|\mu\|$. Then there exist a sequence of dyadic intervals $(I_m, m\in\mathbf{P})$ and Borel-measures μ_m $(m\in\mathbf{N})$ such that*

(15) (i) $I_n \cap I_m = \emptyset \quad (n \neq m)$ *and* $\sum_{n=1}^{\infty} \lambda(I_n) \leqq \|\mu\|/y$,

(ii) $\mu = \sum_{n=0}^{\infty} \mu_n$,

(iii) $\mu_n(H) = 0$, *if* $H \supseteqq I_n$ $(n\in\mathbf{P})$, $\operatorname{supp} \mu_n \subseteqq I_n$ *and* $\|\mu_n\| \leqq 2\mu(I_n)$,

(iv) $\|\mu_0 * D_{2^n}\|_\infty \leqq 3y \quad (n\in\mathbf{N})$.

PROOF. Denote $\mathcal{A}_n$ $(n\in\mathbf{N})$ the σ-algebra, generated by the dyadic intervals of length 2^{-n}, and set

$$\tau(x) = \inf\{n\in\mathbf{N}: (\mu * D_{2^n})(x) > y\},$$

where $\inf \emptyset = \infty$. Then τ is a stopping time with respect to the sequence $(\mathcal{A}_n, n\in\mathbf{N})$ and the set

$$X = \bigcup_{\tau(x)<\infty} [\alpha_{\tau(x)}(x), \beta_{\tau(x)}(x)) = \{x: \tau(x) < \infty\}$$

is the union of mutually disjoint dyadic intervals: $X=\bigcup_{m=1} I_m$, and by the definition of τ and I_m we have

$$\sum_{m=1}^{\infty} \lambda(I_m) \leqq \sum_{m=1}^{\infty} \mu(I_m)/y \leqq \|\mu\|/y.$$

For an arbitrary Borel set $B\subseteq[0, 1)$ let

$$\mu_m(B) = \mu(B\cap I_m) - \frac{\mu(I_m)}{\lambda(I_m)} \lambda(B\cap I_m) \quad (m\in\mathbf{P}),$$

$$\mu_0(B) = \mu(X'\cap B) + \sum_{m=1}^{\infty} \frac{\mu(I_m)}{\lambda(I_m)} \lambda(B\cap I_m),$$

where $X'=[0, 1)\setminus X$. Then $\mu=\sum_{m=0} \mu_m$ and $B\supseteqq I_m$, $m\in\mathbf{P}$ implies $\mu_m(B)=0$ and obviously $\|\mu_m\|\leqq 2\mu(I_m)$ and $\operatorname{supp}\mu_m\subseteqq I_m$ $(m\in\mathbf{P})$. By the definition of τ we have $\tau\geqq 1$ and $(\mu * D_{2^n})(x)\leqq y$, if $n<\tau(x)$, thus

$$(\mu * D_{2^{\tau(x)}})(x) \leqq 2(\mu * D_{2^{\tau(x)-1}})(x) \leqq 2y \quad (x\in[0, 1]).$$

From this we get

$$\frac{\mu(I_m)}{\lambda(I_m)} \leqq 2y, \quad (\mu_0 * D_{2^m})(x) \leqq 3y \quad (m\in\mathbf{N}, x\in[0, 1])$$

and the lemma is proved.

Proof of Theorem 3. To prove (13) we need only to consider a positive measure. We shall apply the Lemma and show, that

(16) (i) $\|T^i\mu_0\|_\infty \leqq 6y,$

(ii) $\|\chi_{X'} T^i \mu_m\|_1 \leqq 2\mu(I_m) \quad (m\in\mathbf{P},\ i = 0, 1, 2),$

where $\chi_{X'}$ denotes the characteristic function of the set X'. It is easy to see that (16) implies (13). Indeed, from (16) (ii) for the measure $\mu'=\sum_{m=1}^{\infty} \mu_m$ by (15) (i) and (ii) we have

$$\|\chi_{X'} T^i\mu\|_1 \leqq 2\|\mu\| \quad (i=0, 1, 2),$$

thus

$$\lambda\{x\colon (T^i\mu)(x) > 12y\} \leqq \lambda\{x\colon (T^i\mu_0)(x) > 6y\}+\lambda\{x\in X'\colon (T^i\mu')(x) > 6y\}+$$
$$+\lambda\{x\in X\colon (T^i\mu')(x) > 6y\} \leqq (1/3+1)\,\|\mu\|/y.$$

Hence (13) follows.

To see (16), we use (15) (iv). Hence $\|\mu_0 * V_n^0\|_\infty\leqq 3y$,

$$\|\mu_0 * V_n^1\|_\infty \leqq \sum_{i=0}^{n-1} 2^i \sum_{k=n}^{\infty} 2^{-k}\|\mu_0 * D_{2^k}\|_\infty \leqq 3y2\sum_{i=0}^{n-1} 2^{i-n} \leqq 6y,$$

and

$$\|\mu_0 * V_n^2\|_\infty \leqq \sum_{i=0}^{n-1} 2^i \sum_{k=i}^{n-1} 2^{-n}\|\mu_0 * D_{2^k}\|_\infty \leqq 3y\sum_{i=0}^{n-1} 2^{i-n}(n-i) \leqq 6y,$$

which gives (16) (i).

To prove (16) (ii) we note that $e_i \leqq \lambda(I_m)$ and $x \in I_m' = [0, 1) \setminus I_m$ implies $x \dotplus e_i \notin I_m$, thus by (15) (iii) we have

$$(\mu_m * D_{2^k})(x \dotplus e_i) = 0$$

if $2^{-k} \geqq \lambda(I_m)$, or $2^{-k} < \lambda(I_m)$ and $e_i \leqq \lambda(I_m)$. This gives for $x \notin I_m$ $(\mu_m * V_n^0)(x) = 0$ $(n \in \mathbf{N})$ and

$$|(\mu_m * V_n^i)(x)| \leqq \sum_{e_i \in \lambda(I_m)} 2^i \sum_{2^{-k} \leqq \lambda(I_m)} 2^{-k} (\mu_m * D_{2^k})(x \dotplus e_i) \quad (n \in \mathbf{N},\ i = 1, 2).$$

Hence by $\|\mu * D_{2^m}\|_1 \leqq \|\mu\| \, \|D_{2^m}\|_1 = \|\mu\|$ $(m \in \mathbf{N})$, we get

$$\|\chi_{X'} T^i \mu_m\|_1 \leqq \|\mu\| \sum_{e_i \leqq \lambda(I_m)} 2^i 2\lambda(I_m) \leqq 2\|\mu_m\| \quad (m \in \mathbf{P}).$$

This completes the proof of Theorem 3.

Proof of Theorems 2 and 4. To prove Theorems 2 and 4, by Theorem 1 we need only to consider an increasing singular function F. (See also [11], Korollar 2).

We shall use some facts from [10] (see Hilfssatz 2, 3). For $d_n W$ $(n \in \mathbf{P})$ the following estimates holds:

$$(17) \qquad |d_n W| \leqq A_1 D_{2^n} + A_2 \left(2^{-n} \sum_{i=1}^{2^n} |K_i| + |K_{2^n - 1}| \right) + \sum_{k=0}^{\infty} 2^{-k} D_{2^{n+k}} + V_n^1,$$

where

$$D_n = \sum_{k=0}^{n-1} \psi_k, \quad K_n = (D_1 + \ldots + D_n)/n \quad (n \in \mathbf{P})|$$

and A_1, A_2 are constants. Further for $2^{n-1} \leqq m < 2^n$

$$(18) \qquad |K_m| \leqq \sum_{i=0}^{n-1} 2^{i-n} \sum_{j=i}^{n-1} D_{2^j} + V_n^2$$

holds.

We shall prove that for almost every $x \in (0, 1)$

$$(19) \qquad \lim_{n \to \infty} (V_n^i * dF)(x) = 0 \quad (i = 1, 2).$$

From this using Toeplitz summation theorem by

$$\lim_{n \to \infty} (D_{2^n} * dF)(x) = \lim_{n \to \infty} 2^n \big(F(\beta_n(x)) - F(\alpha_n(x)) \big) = 0 \quad \text{a.e.},$$

and by (17) and (18) we get

$$\lim_{n \to \infty} (d_n W * dF)(x) = \lim_{n \to \infty} \sigma_n(dF)(x) = 0 \quad \text{a.e.}$$

Let

$$H_0 = \left\{ x \in [0, 1) : \lim_{n \to \infty} (D_{2^n} * dF)(x) \neq 0 \right\}, \quad H = H_0 \bigcup_{i=0}^{\infty} (H_0 \dotplus e_i).$$

Then $\lambda(H_0)=\lambda(H)=0$ and using Toeplitz summation theorem we get

$$\text{(20)}\quad \text{(i)}\ \lim_{n\to\infty}\left(\sum_{k=n}^{\infty} 2^{-k+n}(D_{2^k}*dF)(x\dot{+}a)\right) = 0 \quad (x\in H,\ a\in\{0, e_i\colon i\in\mathbf{N}\}$$

$$\text{(ii)}\ \lim_{n\to\infty}\left(\sum_{k=0}^{n-1} 2^{-n}(D_{2^k}*dF)(x\dot{+}e_i)\right) = 0 \quad (x\in H,\ i\in\mathbf{P}),$$

$$\text{(iii)}\ \lim_{n\to\infty} \sum_{i=0}^{n-1} 2^i \sum_{k=i}^{n-1} 2^{-n}(D_{2^k}*dF)(x)\Bigg) = 0 \quad (x\in H_0).$$

Let $I\subset[0, 1)$ be a dyadic interval and introduce the Borel-measure $\mu_I(B)=\int\limits_{B\cap I} dF$. Since $x\notin I$, $e_i\leqq\lambda(I)$ and $2^{-k}\leqq\lambda(I)$ implies $(\mu_I*D_{2^k})(x\dot{+}e_i)=0$, from (20) follows, that for all $x\notin I\cap H$ we have

$$\text{(21)}\qquad \lim_{n\to\infty}(\mu_I*V_n^i)(x) = 0 \quad (i = 1, 2,\ x\notin I\cap H).$$

To prove (19) it is enough to show that for every $\varepsilon>0$

$$\text{(22)}\qquad \lambda\left\{x\colon \limsup_{n\to\infty}(V_n^i*dF)(x) > 2\sqrt{\varepsilon}\right\} \leqq 3\varepsilon + 16\sqrt{\varepsilon}.$$

Since the Borel measure $\mu(B)=\int\limits_B dF$ is singular with respect to the Lebesgue measure λ, there exist mutually disjoint sets A and B such that $\lambda(A)=\mu(B)=0$ and $A\cup B=[0, 1]$. It is obvious, that we can cover A and B with open intervals I_n and J_n, respectively ($n\in\mathbf{P}$) such that

$$\text{(23)}\qquad \sum_{i=1}^{\infty}\lambda(I_i) < \varepsilon, \qquad \sum_{j=1}^{\infty}\mu(J_j) < \varepsilon$$

and thus by Borel theorem there exist finitely many intervals $I_{m_1}, \dots, I_{m_s}, J_{n_1}, \dots, J_{n_r}$ with the property

$$\left(\bigcup_{k=1}^{r} J_{n_k}\right) \cup \left(\bigcup_{i=1}^{s} I_{m_i}\right) \supseteqq [0, 1].$$

Let $\tilde{\mu}(H)=\mu(H\cap Y)$, where $Y=\bigcup_{k=1}^{r} J_{n_k}$. Then by (23) $\tilde{\mu}(Y)<\varepsilon$. It is easy to see that for every I_{m_i} there exist at most three dyadic intervals I_i^j ($j=1, 2, 3$) such that

$$\sum_{j=1}^{3}\lambda(I_i^j) \leqq 3\lambda(I_{m_i}) \quad \text{and} \quad \bigcup_{j=1}^{3} I_j^i \supseteqq I_{m_i}\cap[0, 1].$$

Since

$$\mu \leqq \tilde{\mu} + \sum_{i=1}^{s}\sum_{j=1}\mu_{I_i^j} = \tilde{\mu} + \tilde{\tilde{\mu}},$$

and by (21)

$$\lim_{n\to\infty}(\mu_{I_i^j}*V_n^k)(x) = 0, \quad \text{if} \quad x\notin I_i^j \quad (k = 1, 2),$$

we have

$$\lambda\left\{x: \limsup_{n\to\infty} (V_n^k * \tilde{\tilde{\mu}})(x) > \sqrt{\varepsilon}\right\} \leqq 3\varepsilon.$$

Finally by (23) and Theorem 2

$$\lambda\left\{x: \limsup_{n\to\infty} (V_n^k * \tilde{\mu})(x) > \sqrt{\varepsilon}\right\} \leqq 16\,\|\tilde{\mu}\|/\sqrt{\varepsilon} \leqq 16\sqrt{\varepsilon}$$

and this yields (22).

Theorems 2 and 4 are proved.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT II OF ANALYSIS
1088 BUDAPEST, MÚZEUM KRT. 6—8.

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APPROXIMATIVE REPRESENTATION OF FOURIER TRANSFORM

By

CATHERINE BALÁZS (Budapest)

1. The Fourier transform of a function $f(t)$ defined on $[0, \infty)$ is

$$F(f, x) = \int_0^\infty f(t) e^{-ixt}\, dt. \tag{1}$$

The physicists are often interested in approximative expression of $F(f, x)$ when $f(t)$ is only experimentally known, i.e. when the values of $f(t)$ are given in a finite number of points on $[0, \infty)$.

We shall give a method for solving this problem. Let $f(t)$ be a continuous function defined on $[0, \infty)$, $\varphi(t) \overset{\text{def}}{=} f(t)e^{ct}$ where $c>0$ is an arbitrary constant, and suppose that

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} f(t)e^{ct} \tag{2}$$

exists, moreover

$$F(f, x) \overset{\text{def}}{\sim} I_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{a_n}\right) A_k^{(n)}(x) = \sum_{k=0}^{n} f\left(\frac{k}{a_n}\right) \frac{e^{c\frac{k}{a_n}} a_n^k}{(c+a_n+ix)^{k+1}}, \tag{3}$$

where c is the same as in (2) and $\{a_n\}$ is a sequence of positive numbers. We shall prove the following

THEOREM. *If condition* (2) *is fulfilled and* $a_n = \dfrac{c}{2}\dfrac{n}{\log n}$ *then*

$$|F(f, x) - I_n(f, x)| = O\left(\omega_\varphi\left(\sqrt{\frac{\log n}{n}}\right)\right) + O\left(\frac{1}{n^2}\right) \quad (n\to\infty, 0 \leqq x < \infty) \tag{4}$$

where $\omega_\varphi(\delta)$ *is the modulus of continuity of* $\varphi(t) = f(t)e^{ct}$ *on* $[0, \infty)$, *c is the same as in* (2), *O depends only on c and* $\max\limits_{0\leqq t\leqq\infty} |\varphi(t)|$.

J. BALÁZS and P. TURÁN [1] investigated the same problem and gave another approximative expression for $F(f, x)$. The method published here has the following advantages: 1) we can choose equidistant nodes $\left(x_k^{(n)} = \dfrac{k}{a_n}\right)$, 2) our fundamental functions $A_k^{(n)}(x)$ are easy calculable, 3) our theorem gives an estimate for the rate of convergence. However, now $F(f, x) \neq I_n(f, x)$ if $f(t) = P_n(t)e^{-ct}$ where $P_n(t)$ is a polynomial of degree $\leqq n$, while in the Balázs—Turán case the equality holds.

2. To prove the theorem we need the following

LEMMA. *For* $y>0$, $c>0$ *we have*

(5)
$$\int_0^\infty \left(\sum_{k=n+1}^\infty \frac{(yt)^k}{k!}\right) e^{-(c+y)t}\,dt \leqq \frac{1}{c} e^{-c\frac{n+1}{y+c}} \quad (n = 0, 1, 2, \ldots).$$

PROOF. Integrating by parts on the left hand side of (5) $n+1$ times we have

$$y^{n+1} \sum_{k=n+1}^\infty \int_0^\infty y^{k-n-1} \frac{t^k}{k!} e^{-(c+y)t}\,dt = \left(\frac{y}{y+c}\right)^{n+1} \int_0^\infty \left(\sum_{k=0}^\infty \frac{(yt)^k}{k!}\right) e^{-(c+y)t}\,dt =$$

$$= \left[\left(1-\frac{c}{y+c}\right)^{y+c}\right]^{\frac{n+1}{y+c}} \int_0^\infty e^{-ct}\,dt < \frac{1}{c} e^{-c\frac{n+1}{y+c}}.$$

Here we used the inequality $\left(1-\frac{c}{y+c}\right)^{y+c} < e^{-c}$ which follows from the relations:

$$\lim_{y\to\infty} \left(1-\frac{c}{y+c}\right)^{y+c} = e^{-c}, \quad \frac{d}{dy}\left(1-\frac{c}{y+c}\right)^{y+c} > 0 \quad (y > 0, c > 0).$$

So the lemma is proved.

The well-known relation $\omega_\varphi(\lambda\delta) \leqq (\lambda+1)\omega_\varphi(\delta)$ gives

(6)
$$\left|\varphi\left(\frac{k}{a_n}\right) - \varphi(t)\right| \leqq \omega_\varphi\left(\left|\frac{k}{a_n} - t\right|\right) \leqq \omega_\varphi\left(\frac{1}{\sqrt{a_n}}\right)\left\{\sqrt{a_n}\left|\frac{k}{a_n} - t\right| + 1\right\}.$$

It is easy to see that

(7)
$$I_n(f, x) = \sum_{k=0}^n f\left(\frac{k}{a_n}\right) e^{c\frac{k}{a_n}} \int_0^\infty \frac{(a_n t)^k}{k!} e^{-(c+a_n+ix)t}\,dt = \int_0^\infty e^{-a_n t} \sum_{k=0}^n \varphi\left(\frac{k}{a_n}\right) \frac{(a_n t)^k}{k!} e^{-(c+ix)t}\,dt.$$

We have from (3), (6) and (7)

(8)
$$|F(f, x) - I_n(f, x)| = \left|\int_0^\infty \left\{\varphi(t) - e^{-a_n t} \sum_{k=0}^n \varphi\left(\frac{k}{a_n}\right) \frac{(a_n t)^k}{k!}\right\} e^{-(c+ix)t}\,dt\right| =$$

$$= \left|\int_0^\infty e^{-a_n t} \sum_{k=0}^n \left(\varphi(t) - \varphi\left(\frac{k}{a_n}\right)\right) \frac{(a_n t)^k}{k!} + \varphi(t) e^{-a_n t} \sum_{k=n+1}^\infty \frac{(a_n t)^k}{k!}\right\} e^{-(c+ix)t}\,dt\right| \leqq$$

$$\leqq \int_0^\infty \omega_\varphi\left(\frac{1}{\sqrt{a_n}}\right)\left\{\sqrt{a_n} \sum_{k=0}^n \left|\frac{k}{a_n} - t\right| \frac{(a_n t)^k}{k!} + \sum_{k=0}^n \frac{(a_n t)^k}{k!}\right\} e^{-(c+a_n)t}\,dt +$$

$$+ \left(\max_{0\leqq t<\infty} |\varphi(t)|\right) \int_0^\infty \left(\sum_{k=n+1}^\infty \frac{(a_n t)^k}{k!}\right) e^{-(c+a_n)t}\,dt = A + B.$$

Using Schwarz inequality and the relation

$$\sum_{k=0}^{n}(a_n t-k)^2\frac{(a_n t)^k}{k!}\leqq a_n t e^{a_n t}$$

which follows from the trivial identity

$$\sum_{k=0}^{\infty}(a_n t-k)^2\frac{(a_n t)^k}{k!}=a_n t e^{a_n t}\quad (t\geqq 0),$$

we obtain

$$A\leqq \omega_\varphi\left(\frac{1}{\sqrt{a_n}}\right)\int_0^\infty e^{-ct}dt+ \tag{9}$$

$$+\omega_\varphi\left(\frac{1}{\sqrt{a_n}}\right)\frac{1}{\sqrt{a_n}}\int_0^\infty\left\{\left(\sum_{k=0}^{n}(a_n t-k)^2\frac{(a_n t)^k}{k!}\right)\left(\sum_{k=0}^{n}\frac{(a_n t)^k}{k!}\right)\right\}^{1/2}e^{-(c+a_n)t}\,dt\leqq$$

$$\leqq\omega_\varphi\left(\frac{1}{\sqrt{a_n}}\right)\left\{\frac{1}{c}+\int_0^\infty\sqrt{t}\,e^{-ct}\,dt\right\}\leqq\omega_\varphi\left(\frac{1}{\sqrt{a_n}}\right)\left(\frac{2}{c}+\frac{e^{-c}}{c^2}\right).$$

From (5), (8) and (9) we have (4) with $a_n=\dfrac{c}{2}\dfrac{n}{\log n}$, $y=a_n$, so the theorem is proved.

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KARL MARX UNIVERSITY
DEPARTMENT FOR COMPUTER SCIENCE
1093 BUDAPEST, DIMITROV TÉR 8.
HUNGARY

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ИССЛЕДОВАНИЕ ОДНОГО ИНТЕРПОЛЯЦИОННОГО ПРОЦЕССА. III

О. КИШ (Будапешт) и ХО ТХО КАУ (Ханой)

Введем следующие обозначения: C множество вещественных непрерывных функций, определенных на отрезке $[-1, 1]$; если не оговорено противное, то x произвольное число из $[-1, 1]$, f любой элемент из C, $\omega(h)$ модуль непрерывности функции f, i произвольное целое число, k любое неотрицательное целое число, n произвольное положительное целое число и $P_k(x)$ тот многочлен не выше чем $n-1$-ой степени, который удовлетворяет условию

$$P_k\left(\cos\frac{2i-1}{2n}\pi\right) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} f\left(\cos\frac{2i+2j-k-1}{2n}\pi\right) \quad (i = 1, 2, \ldots, n). \tag{1}$$

Заметим, что $P_0(x)$ обычный интерполяционный многочлен Лагранжа, $P_1(x)$ некоторое видоизменение многочленов, которые ввел С. Н. Бернштейн [3] и Г. Грюнвалд [6], $P_2(x)$ определил С. Н. Бернштейн [3]; в общем случае $P_k(x)$ некоторое видоизменение многочленов, которые введ Д. Л. Берман [1].

Теорема 1. *Выполняются следующие неравенства:*

$$|P_0(x)-f(x)| \leqq \left(\frac{1}{\pi}\ln n+1\right)\omega\left(\frac{\pi}{n}\right); \tag{2}$$

$$|P_1(x)-f(x)| \leqq \frac{1}{2}\omega\left(\frac{\pi}{n}\right)+\left(\frac{1}{2}+\frac{1}{\pi}\right)\omega\left(\frac{\pi}{2n}\right) \quad (n = 1, 3, 5, \ldots); \tag{3}$$

$$|P_2(x)-f(x)| \leqq \frac{5}{4}\omega\left(\frac{\pi}{n}\right); \tag{4}$$

$$|P_3(x)-f(x)| \leqq \frac{3}{4}\omega\left(\frac{\pi}{n}\right)+\left(\frac{1}{2}+\frac{2}{3\pi}\right)\omega\left(\frac{\pi}{2n}\right) \quad (n = 2, 4, 6, \ldots); \tag{5}$$

$$|P_4(x)-f(x)| \leqq \frac{23}{16}\omega\left(\frac{\pi}{n}\right). \tag{6}$$

Соотношение (2) следует из двух теорем, которые опубликовали Х. Брасс и Р. Гюнттнер в статьях [5] и [7]; неравенства (3), (4), (6) получены в работе [11]; соотношение (5) будет доказано ниже.

Заметим, что многочлен $P(x)$ не выше чем $n-1$-ой степени, для которого

$$|P(x)-f(x)| = O\left[\omega\left(\frac{1}{n}\right)\right],$$

впервые построил Д. Джексон [8]; для введенных им, уже упомянутых интерполяционных многочленов при $k=4$ такое соотношение впервые доказал Д. Л. Берман [2]. Доказывая неравенства (3)—(6) и нижеследующие соотношения (7)—(12) и (22)—(25), мы стремились получить как можно более точные коэффициенты при модуле непрерывности.

Ниже доказывается

Теорема 2. *Выполняются следующие соотношения:*

$$(7)\quad |P_0(x)-f(x)| \leqq \left(\frac{1}{\pi}\ln n+1\right)\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+\frac{2}{\pi}\sum_{i=1}^{2n-1}\frac{1}{i}\,\omega\left(\frac{i\pi^2}{2n^2}|x|\right)+\omega\left(\frac{\pi^2|x|}{8n^2}\right);$$

$$(8)\quad |P_1(x)-f(x)| \leqq \frac{1}{2}\,\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+\left(\frac{1}{2}+\frac{1}{\pi}\right)\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)+$$
$$+\frac{2}{\pi}\sum_{i=2}^{2n-1}\frac{1}{i^2-1}\,\omega\left(\frac{i\pi^2}{2n^2}|x|\right)+\left(\frac{35}{24}+\frac{\pi}{4}-\frac{11}{12\pi}\right)\omega\left(\frac{\pi^2|x|}{2n^2}\right)\quad (n=1,3,5,\ldots);$$

$$(9)\quad |P_1(x)-f(x)| \leqq \frac{1}{2}\,\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+\left(\frac{1}{2}+\frac{1}{\pi}\right)\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)+$$
$$+\left(\frac{1}{2\pi}\ln n+\frac{1}{2}\right)\omega\left(\frac{\pi^2}{n^2}|x|\right)+\left(\frac{1}{2}+\frac{1}{\pi}\right)\omega\left(\frac{\pi^2|x|}{8n^2}\right)\quad (n=1,3,5,\ldots);$$

$$(10)\quad |P_2(x)-f(x)| \leqq \frac{5}{4}\,\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+\frac{7}{4}\,\omega\left(\frac{\pi^2|x|}{2n^2}\right);$$

$$(11)\quad |P_3(x)-f(x)| \leqq \frac{3}{4}\,\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+\left(\frac{1}{2}+\frac{2}{3\pi}\right)\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)+$$
$$+\frac{7}{8}\,\omega\left(\frac{\pi^2|x|}{2n^2}\right)+\left(\frac{1}{2}+\frac{2}{3\pi}\right)\omega\left(\frac{\pi^2|x|}{8n^2}\right)\quad (n\leqq 2,4,6,\ldots);$$

$$(12)\quad |P_4(x)-f(x)| \leqq \frac{23}{16}\,\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+\frac{39}{16}\,\omega\left(\frac{\pi^2|x|}{2n^2}\right).$$

Заметим, что похожее на (7), но менее точное неравенство было доказано в статье [9]; многочлен $P(x)$ не выше чем $n-1$-ой степени, для которого

$$|P(x)-f(x)| = O\left[\omega\left(\frac{1}{n}\sqrt{1-x^2}\right)+\omega\left(\frac{|x|}{n^2}\right)\right],$$

впервые построил А. Ф. Тиман [13]; в работе [12] Т. М. Миллс и А. К. Варма опубликовали такое соотношение для уже упомянутых многочленов, которые ввел Г. Грюнвалд [6]; (12) и похожее на (11), но менее точное неравенство приведено без доказательства в статье [10].

Порядок оценок (7)—(8) и (10)—(12) не хуже, чем порядок оценок (2)—(6); для (7) и (8) это следует из неравенства

$$\omega\left(\frac{i\pi}{2n^2}|x|\right) \leqq \omega\left(\frac{\pi^2}{n}|x|\right)\quad (i=1,2,\ldots 2n).$$

Для „плохих" функций, например, если

$$\omega(h) = -\frac{1}{\ln h} \quad \left(0 < h \leqq \frac{1}{e}\right), \quad \omega(h) = 1 \quad \left(\frac{1}{e} < h \leqq 2\right),$$

(7)—(8) и (10)—(12) не лучше, чем (2)—(6); для (7) и (8) в этом можно убедиться, используя неравенство

$$\omega\left(\frac{i\pi^2}{2n^2}|x|\right) \geqq \omega\left(\frac{\pi^2|x|}{2n^2}\right) \quad (i = 1, 2, 3, \ldots).$$

Для „хороших" функций оценки (7)—(8) и (10)—(12) лучше, чем (2)—(6) в концах отрезка $[-1, 1]$; если, например, выполняется условие Липшица

(13) $$|f(x+h)-f(x)| = Mh^\alpha \quad (0<h\leqq 2,\ -1\leqq x\leqq 1-h,\ M>0,\ 0<\alpha\leqq 1),$$

то из (7)—(9) получаем:

(14) $$|P_0(x)-f(x)| \leqq M\left[\left(\frac{1}{\pi}\ln n+1\right)\left(\frac{\pi}{n}\sqrt{1-x^2}\right)^\alpha + \frac{2}{\pi\alpha}\left(\frac{\pi^2|x|}{n}\right)^\alpha + \left(\frac{\pi^2|x|}{8n^2}\right)^\alpha\right];$$

(15) $$|P_1(x)-f(x)| \leqq M\left\{\left[\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{\pi}\right)\frac{1}{2^\alpha}\right]\left(\frac{\pi}{n}\sqrt{1-x^2}\right)^\alpha + \right.$$

$$\left. + \left(\frac{2}{\pi}\frac{2-\alpha}{1-\alpha}+\frac{35}{24}+\frac{\pi}{4}-\frac{11}{12\pi}\right)\left(\frac{\pi^2|x|}{2n^2}\right)^\alpha\right\} \quad (\alpha < 1,\ n = 1, 3, 5, \ldots);$$

(16) $$|P_1(x)-f(x)| \leqq M\left[\left(\frac{3}{4}\pi+\frac{1}{2}\right)\frac{1}{n}\sqrt{1-x^2} + \left(\frac{\pi}{2}\ln n+\frac{9}{16}\pi^2+\frac{\pi}{8}\right)\frac{|x|}{n^2}\right]$$

$$(\alpha = 1,\ n = 1, 3, 5, \ldots).$$

Можно показать, что оценки (2)—(8), (10)—(12) и (14)—(16) нельзя существенно улучшить. Существуют, например, зависящие от n функции f, для которых выполняется условие Липшица

$$|f(x+h)-f(x)| \leqq h \quad (0 < h \leqq 2,\ -1 \leqq x \leqq 1-h)$$

и одно из следующих неравенств:

(17) $$|P_0(0)-f(0)| \geqq \frac{\ln n}{n} - \frac{\pi^2}{8n} \quad (n = 2, 4, 6, \ldots);$$

(18) $$|P_0(1)-f(1)| \geqq \frac{\pi}{2n} - \frac{\pi}{2n^2};$$

(19) $$|P_1(1)-f(1)| \geqq \frac{\pi}{2}\frac{\ln n}{n^2}.$$

Мы определим такие функции в конце настоящей статьи.

Введем следующие обозначения: $C_{2\pi}$ множество вещественных, 2π-периодических, непрерывных функций, определенных на множестве вещественных чисел; если не оговорено противное, то ϑ произвольное вещественное число, g любая функция из $C_{2\pi}$, а $\Omega(h)$ ее модуль непрерывности; $a_0, a_1, \ldots, a_{n-1}$ и $b_1, b_2, \ldots, b_n$ вещественные числа; $S_k(\vartheta)$ тот тригонометрический многочлен вида

$$\sum_{i=0}^{n-1} a_i \cos i\vartheta + \sum_{i=1}^{n} b_i \sin i\vartheta,$$

который удовлетворяет условиям

$$(20) \qquad S_k\left(\frac{2i-1}{2n}\pi\right) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} g\left(\frac{2i+2j-k-1}{2n}\pi\right) \quad (i = 0, \pm 1, \pm 2, \ldots).$$

Заметим, что $S_0(\vartheta)$ обычный тригонометрический интерполяционный многочлен; $S_1(\vartheta)$ и $S_2(\vartheta)$ определил С. Н. Бернштейн в статьях [3] и [4].

Теорема 3. *Выполняются следующие неравенства:*

$$(21) \qquad |S_0(\vartheta) - g(\vartheta)| \leqq \left(\frac{1}{\pi}\ln n + 1\right)\Omega\left(\frac{\pi}{n}\right);$$

$$(22) \qquad |S_1(\vartheta) - g(\vartheta)| \leqq \frac{1}{2}\Omega\left(\frac{\pi}{n}\right) + \left(\frac{1}{2} + \frac{1}{\pi}\right)\Omega\left(\frac{\pi}{2n}\right) \quad (n = 1, 3, 5, \ldots);$$

$$(23) \qquad |S_2(\vartheta) - g(\vartheta)| \leqq \frac{5}{4}\Omega\left(\frac{\pi}{n}\right);$$

$$(24) \qquad |S_3(\vartheta) - g(\vartheta)| \leqq \frac{3}{4}\Omega\left(\frac{\pi}{n}\right) + \left(\frac{1}{2} + \frac{2}{3\pi}\right)\Omega\left(\frac{\pi}{2n}\right) \quad (n = 2, 4, 6, \ldots);$$

$$(25) \qquad |S_4(\vartheta) - g(\vartheta)| \leqq \frac{23}{16}\Omega\left(\frac{\pi}{n}\right).$$

Соотношение (21) следует из двух теорем, которые Х. Брасс и Р. Гюнттнер доказали в работах [5] и [7]; (22), (23) и (25) доказано в статье [11], а неравенство (24) мы докажем в настоящей работе.

Введем следующие обозначения:

$$(26) \qquad d(\vartheta) = \frac{1}{2n} + \frac{1}{n}\sum_{i=1}^{n-1}\cos i\vartheta + \frac{1}{2n}\cos n\vartheta;$$

если k четное число, то

$$(27) \qquad s_i(\vartheta) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} d\left(\vartheta - \frac{2i+2j-k-1}{2n}\pi\right),$$

а если k нечетное число, то

$$(28) \qquad s_i(\vartheta) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} d\left(\vartheta - \frac{2i+2j-k}{2n}\pi\right);$$

m то целое число, для которого

$$(29)\qquad \frac{2m-1}{2n}\pi < \arccos x \leqq \frac{2m+1}{2n}\pi, \quad \text{если} \quad k = 0, 2, 4, \ldots;$$

$$\frac{m\pi}{n} < \arccos x \leqq \frac{m+1}{n}\pi, \quad \text{если} \quad k = 1, 3, 5, \ldots;$$

$$(30)\qquad t = \arccos x - \frac{m\pi}{n};$$

$$(31)\qquad \sigma_i(t) = \sum_{j=1-n}^{i} s_j(t) \quad (i = 0, -1, -2, \ldots, 1-n);$$

$$(32)\qquad \sigma_i(t) = \sum_{j=i}^{n} s_j(t) \quad (i = 1, 2, \ldots, n).$$

Доказательство теоремы 2 опирается на следующее вспомогательное предположение:

Лемма. *Если* $k = 0, 2, 4, \ldots$ *и* $-\frac{\pi}{2n} < t \leqq 0$, *то*

$$(33)\qquad |P_k(x)-f(x)| \leqq \omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)\sum_{i=1-n}^{n}|\sigma_i(t)| + \sum_{i=1}^{n-1}\omega\left(\frac{i\pi^2}{n^2}|x|\right)|\sigma_{-i}(t)| +$$
$$+\omega\left(\frac{\pi^2|x|}{8n^2}\right)|\sigma_0(t)| + \sum_{i=0}^{n-1}\omega\left(\frac{2i+1}{2n^2}\pi^2|x|\right)|\sigma_{i+1}(t)|;$$

если $k=0, 2, 4, \ldots$ *и* $0 \leqq t \leqq \frac{\pi}{2n}$, *то*

$$(34)\qquad |P_k(x)-f(x)| \leqq \omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)\sum_{i=1-n}^{n}|\sigma_i(t)| + \sum_{i=0}^{n-1}\omega\left(\frac{2i+1}{2n^2}\pi^2|x|\right)|\sigma_{-i}(t)| +$$
$$+\omega\left(\frac{\pi^2|x|}{8n^2}\right)|\sigma_1(t)| + \sum_{i=1}^{n}\omega\left(\frac{i\pi^2}{n^2}|x|\right)|\sigma_{i+1}(t)|;$$

если $k=1, 3, 5, \ldots$ *и* $0 < t \leqq \frac{\pi}{2n}$, *то*

$$(35)\qquad |P_k(x)-f(x)| \leqq \omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)\sum_{\substack{i=1-n\\ i\neq 0}}^{n}|\sigma_i(t)| + \omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)|\sigma_0(t)| +$$
$$+\sum_{i=1}^{n-1}\omega\left(\frac{i\pi^2}{n^2}|x|\right)|\sigma_{-i}(t)| + \omega\left(\frac{\pi^2|x|}{8n^2}\right)|\sigma_0(t)| + \sum_{i=1}^{n}\omega\left(\frac{2i-1}{2n^2}|x|\right)|\sigma_i(t)|;$$

если $k=1, 3, 5, \ldots$ *и* $\frac{\pi}{2n}<t\leqq\frac{\pi}{n}$, *то*

$$(36)\qquad |P_k(x)-f(x)|\leqq\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)\sum_{\substack{i=1-n\\ i\neq 1}}^{n}|\sigma_i(t)|+\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)|\sigma_1(t)|+$$

$$+\sum_{i=0}^{n-1}\omega\left(\frac{2i+1}{2n^2}|x|\right)|\sigma_{-i}(t)|+\omega\left(\frac{\pi^2|x|}{8n^2}\right)|\sigma_1(t)|+\sum_{i=1}^{n-1}\omega\left(\frac{i\pi^2}{n^2}|x|\right)|\sigma_{i+1}(t)|.$$

Доказательство леммы. Если

$$(37)\qquad g(\vartheta)=f(\cos\vartheta),$$

то

$$(38)\qquad P_k(x)=S_k(\arccos x).$$

Эта формула фигурирует на странице 364 работы [10]. В этой же статье получена (при $m=n$) следующая формула (5):

$$(39)\qquad S_k(\vartheta)=\sum_{i=m+1-n}^{m+n}g\left(\frac{2i-1}{2n}\pi\right)s_i(\vartheta)\quad(k=0, 2, 4, \ldots).$$

Запишем ее в виде

$$S_k(\vartheta)=\sum_{i=1-n}^{n}g\left(\frac{2i+2m-1}{2n}\pi\right)s_i\left(\vartheta-\frac{m\pi}{n}\right)\quad(k=0, 2, 4, \ldots).$$

Полагая здесь $\vartheta=\arccos x$ и используя (38), (37) и (30) получаем:

$$P_k(x)=\sum_{i=1-n}^{n}f\left(\cos\frac{2i+2m-1}{2n}\pi\right)s_i(t)\quad(k=0, 2, 4, \ldots).$$

Отсюда с помощью преобразования Абеля и равенства (37) получаем следующий аналог формулы (25) из [11]:

$$(40)\quad P_k(x)-f(x)=\sum_{i=1-n}^{-1}\left[f\left(\cos\frac{2i+2m-1}{2n}\pi\right)-f\left(\cos\frac{2i+2m+1}{2n}\pi\right)\right]\sigma_i(t)+$$

$$+\left\{f\left(\cos\frac{2m-1}{2n}\pi\right)-f\left[\cos\left(\frac{m\pi}{n}+t\right)\right]\right\}\sigma_0(t)+\left\{f\left(\cos\frac{2m+1}{2n}\pi\right)-\right.$$

$$\left.-f\left[\cos\left(\frac{m\pi}{n}+t\right)\right]\right\}\sigma_1(t)+\sum_{i=1}^{n-1}\left[f\left(\cos\frac{2i+2m+1}{2n}\pi\right)-f\left(\cos\frac{2i+2m-1}{2n}\pi\right)\right]\sigma_{i+1}(t)$$

$$(k=0, 2, 4, \ldots).$$

Принимая во внимание (30), получаем:

$$\cos\frac{2i+2m-1}{2n}\pi-\cos\frac{2i+2m+1}{2n}\pi = 2\sin\frac{\pi}{2n}\sin\frac{i+m}{n}\pi =$$

$$= 2\sin\frac{\pi}{2n}\left[\sqrt{1-x^2}\cos\left(\frac{i\pi}{n}-t\right)+x\sin\left(\frac{i\pi}{n}-t\right)\right].$$

Ввиду (29) и (30) $-\frac{\pi}{2n}<t\leqq\frac{\pi}{2n}$. Предположим, что $-\frac{\pi}{2n}<t\leqq 0$; случай $0<t\leqq\frac{\pi}{2n}$ рассматривается аналогичным образом. Тогда

(41)

$$\left|\cos\frac{2i+2m-1}{2n}\pi-\cos\frac{2i+2m+1}{2n}\pi\right|\leqq\frac{\pi}{n}\sqrt{1-x^2}+\frac{2i+1}{2n^2}\pi^2|x| \quad (i=1,2,\dots,n-1),$$

(42)

$$\left|\cos\frac{2i+2m-1}{2n}\pi-\cos\frac{2i+2m+1}{2n}\pi\right|\leqq\frac{\pi}{n}\sqrt{1-x^2}+\frac{|i|}{n^2}\pi^2|x| \quad (i=-1,-2,\dots,1-n).$$

Так как

$$\cos\frac{2m-1}{2n}\pi-\cos\left(\frac{m\pi}{n}+t\right) = \cos\left(\frac{m\pi}{n}+t\right)\left[\cos\left(\frac{\pi}{2n}+t\right)-1\right]+$$

$$+\sin\left(\frac{m\pi}{n}+t\right)\sin\left(\frac{\pi}{2n}+t\right) = \sqrt{1-x^2}\sin\left(\frac{\pi}{2n}+t\right)-2x\sin^2\left(\frac{\pi}{4n}+\frac{t}{2}\right),$$

то

$$\left|\cos\frac{2m-1}{2n}\pi-\cos\left(\frac{m\pi}{n}+t\right)\right| \leqq \frac{\pi}{n}\sqrt{1-x^2}+\frac{\pi^2|x|}{8n^2}. \tag{43}$$

Аналогичным образом получаем:

$$\left|\cos\left(\frac{m\pi}{n}+t\right)-\cos\frac{2m+1}{2n}\pi\right| \leqq \frac{\pi}{n}\sqrt{1-x^2}+\frac{\pi^2|x|}{2n^2}. \tag{44}$$

Из (40)—(44) получаем (33). Аналогичным образом доказывается (34).

Чтобы доказать (35) и (36), вместо (39) следует воспользоваться доказанным при $m=n$ тождеством (6) из [10]:

$$S_k(\vartheta) = \sum_{i=m+1-n}^{m+n} g\left(\frac{i\pi}{n}\right)s_i(\vartheta) \quad (k = 1, 3, 5, \dots). \tag{45}$$

Из него получается следующий аналог формулы (40):

$$(46)\qquad P_k(x)-f(x) = \sum_{i=1-n}^{-1}\left[f\left(\cos\frac{i+m}{n}\pi\right)-f\left(\cos\frac{i+m+1}{n}\pi\right)\right]\sigma_i(t)+$$

$$+\left\{f\left(\cos\left(\frac{m\pi}{n}\right)-f\left[\cos\left(\frac{m\pi}{n}+t\right)\right]\right\}\sigma_0(t)+\right.$$

$$+\left\{f\left(\cos\frac{m+1}{n}\pi\right)-f\left[\cos\left(\frac{m\pi}{n}+t\right)\right]\right\}\sigma_1(t)+$$

$$+\sum_{i=1}^{n-1}\left[f\left(\cos\frac{i+m+1}{n}\pi\right)-f\left(\cos\frac{i+m}{n}\pi\right)\right]\sigma_{i+1}(t)\qquad (k=1,3,5,\dots).$$

Отсюда получаем (35) и (36) также, как (33) из (40).

Доказательство неравенства (7). Предположим, что $k=0$. Замечание 1 на странице 179 из [11] можно записать в виде

$$(47)\qquad \sum_{i=1-n}^{n}|\sigma_i(t)| = \frac{1}{2}\sum_{i=1-n}^{n}\left|d\left(t-\frac{2i-1}{2n}\pi\right)\right|+\frac{1}{2}.$$

Р. Гюнттнэр [7] опубликовал неравенство

$$(48)\qquad \sum_{i=1-n}^{n}\left|d\left(\vartheta-\frac{2i-1}{2n}\pi\right)\right| \leqq \frac{2}{\pi}\ln n+1.$$

Поэтому

$$(49)\qquad \sum_{i=1-n}^{n}|\sigma_i(t)| \leqq \frac{1}{\pi}\ln n+1.$$

Из определения (27) функций $s_i(\vartheta)$ видно, что

$$(50)\qquad s_i(\vartheta) = d\left(\vartheta-\frac{2i-1}{2n}\pi\right).$$

Из определения (26) функции $d(\vartheta)$ получаем:

$$(51)\qquad d(\vartheta) = \frac{1}{2n}\sin n\vartheta\operatorname{ctg}\frac{\vartheta}{2}.$$

Поэтому

$$(52)\qquad d\left(\vartheta-\frac{2i-1}{2n}\pi\right) = (-1)^i\frac{1}{2n}\cos n\vartheta\operatorname{ctg}\left(\frac{\vartheta}{2}-\frac{2i-1}{4n}\pi\right).$$

Отсюда следует, что

$$(53)\qquad (-1)^{i+1}d\left(t-\frac{2i-1}{2n}\pi\right)\geqq 0,\ (-1)^{i+1}d\left(t+\frac{2i-1}{2n}\pi\right)\geqq 0\quad (i=1,2,\dots,n)$$

и эти последовательности убывают если i возрастает. Поэтому, ввиду (50) и определения (31)—(32) функций $\sigma_i(t)$

(54) $$|\sigma_i(t)| \leqq |s_i(t)| \quad \big(i = 0, \pm 1, \pm 2, \dots, \pm(n-1), n\big).$$

Предположим, что $-\frac{\pi}{2n} < t \leqq 0$; случай $0 < t \leqq \frac{\pi}{2n}$ рассматриваетсяя аналогичным образом. Из (50) и (52) получаем:

(55) $$|s_{-i}(t)| \leqq \frac{1}{2n} \operatorname{ctg} \frac{i\pi}{2n} < \frac{1}{\pi i} \quad (i = 1, 2, \dots, n-1);$$

(56) $$|s_{i+1}(t)| \leqq \frac{1}{2n} \operatorname{ctg} \frac{2i+1}{4n} \pi < \frac{2}{\pi} \frac{1}{2i+1} \quad (i = 0, 1, \dots, n-1).$$

Из определения (26) функции $d(\vartheta)$ видно, что

(57) $$|d(\vartheta)| \leqq 1.$$

Отсюда и из (50) следует:

(58) $$|s_0(t)| \leqq 1.$$

Из (33), (49), (54)—(56) и (58) получаем доказываемое неравенство (7).

Доказательство соотношения (8). Предположим, что $k=1$ и $0 < t \leqq \frac{\pi}{2n}$; случай $\frac{\pi}{2n} < t \leqq \frac{\pi}{n}$ рассматривается аналогичным образом. В [11] доказано следующее тождество (62):

(59) $$\sum_{\substack{i=1-n \\ i \neq 0}}^{n} |\sigma_i(t)| = \frac{1}{2}.$$

Используя из [11] формулы (17), (64), (65) и замечание к теореме 3, получаем:

(60) $$|\sigma_0(t)| = \sigma_0(t) \leqq \sigma_0(0) = \frac{1}{2} + \frac{1}{2n} \operatorname{ctg} \frac{\pi}{2n} < \frac{1}{2} + \frac{1}{\pi} \quad (n = 1, 3, 5, \dots);$$

(61)
$$|\sigma_0(t)| = \sigma_0(t) \leqq \sigma_0(0) = \frac{1}{2} + \frac{1}{2n} \operatorname{cosec} \frac{\pi}{2n} < \frac{1}{2} + \frac{1}{\pi} + \left(\sqrt{2} - \frac{4}{\pi}\right) \frac{1}{n^2} \quad (n=2,4,6,\dots).$$

Из определения (28) функций $s_i(\vartheta)$ и из (52) следует:

(62) $$s_i(\vartheta) = \frac{1}{2} d\left(\vartheta - \frac{2i-1}{2n} \pi\right) + \frac{1}{2} d\left(\vartheta - \frac{2i+1}{2n} \pi\right);$$

(63) $$s_i(\vartheta) = \frac{1}{4n} \sin \frac{\pi}{2n} (-1)^i \cos n\vartheta \operatorname{cosec}\left(\frac{\vartheta}{2} - \frac{2i+1}{4n} \pi\right) \operatorname{cosec}\left(\frac{\vartheta}{2} - \frac{2i-1}{4n} \pi\right).$$

Отсюда видно, что последовательности $s_{-1}(t), s_{-2}(t), \ldots, s_{1-n}(t)$ и $s_1(t), s_2(t), \ldots$ $\ldots, s_n(t)$ знакопеременны и их абсолютные значения убывают. Поэтому и ввиду определения (31)—(32) функций $\sigma_i(t)$

$$(64)\qquad |\sigma_i(t)| \leqq |s_i(t)| \quad (i = \pm 1, \pm 2, \ldots, \pm(n-1), n).$$

Известно, что

$$(65)\qquad \operatorname{cosec} \vartheta = \frac{1}{\vartheta} + \sum_{i=0}^{\infty} a_i \vartheta^{2i+1} \quad (0 < \vartheta < \pi,\ a_i > 0).$$

Поэтому

$$(66)\qquad \operatorname{cosec} \vartheta \leqq \frac{1}{\vartheta} + \vartheta \sum_{i=0}^{\infty} a_i \left(\frac{\pi}{2}\right)^{2i} = \frac{1}{\vartheta} + \frac{2}{\pi}\left(1 - \frac{2}{\pi}\right)\vartheta \quad \left(0 < \vartheta \leqq \frac{\pi}{2}\right).$$

Отсюда и из (63) получаем:

$$(67)\qquad |s_i(t)| \leqq \frac{1}{4n} \sin\frac{\pi}{2n} \operatorname{cosec}\frac{i\pi}{2n} \operatorname{cosec}\frac{i-1}{2n}\pi \leqq$$

$$\leqq \frac{\pi}{8n^2}\left[\frac{2n}{\pi i} + \left(1 - \frac{2}{\pi}\right)\frac{i}{n}\right]\left[\frac{2}{\pi}\frac{n}{i-1} + \left(1 - \frac{2}{\pi}\right)\frac{i-1}{n}\right] =$$

$$= \frac{1}{2\pi}\frac{1}{i(i-1)} + \frac{1}{2}\left(1 - \frac{2}{\pi}\right)\left[1 + \frac{1}{2i(i-1)}\right]\frac{1}{n^2} + \frac{\pi}{8}\left(1 - \frac{2}{\pi}\right)^2 \frac{i(i-1)}{n^4} \leqq$$

$$\leqq \frac{1}{2\pi}\frac{1}{i(i-1)} + \left(\frac{1}{8} + \frac{\pi}{8} - \frac{3}{4\pi}\right)\frac{1}{n^2} \quad (i = 2, 3, \ldots, n).$$

Аналогичным образом

$$(68)\qquad |s_{-i}(t)| \leqq \frac{2}{\pi}\frac{1}{4i^2-1} + \left(\frac{1}{3} + \frac{\pi}{8} - \frac{7}{6\pi}\right)\frac{1}{n^2} \quad (i = 1, 2, \ldots, n-1).$$

Из (62) видно, что $s_1(t) \geqq 0$. Ввиду (52) $d\left(t - \frac{3\pi}{2n}\right) \leqq 0$. Отсюда, из (62) и (57) получаем:

$$(69)\qquad |s_1(t)| \leqq \frac{1}{2}.$$

Из (35), (59)—(61), (64) и (67)—(69) следует:

$$|P_1(x) - f(x)| \leqq \frac{1}{2}\,\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right) + \sigma_0(0)\,\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right) +$$

$$+ \sum_{i=1}^{n-1} \omega\left(\frac{i\pi^2}{n^2}|x|\right)\left[\frac{2}{\pi}\frac{1}{4i^2-1} + \left(\frac{1}{3} + \frac{\pi}{8} - \frac{7}{6\pi}\right)\frac{1}{n^2}\right] + \sigma_0(0)\,\omega\left(\frac{\pi^2|x|}{8n^2}\right) +$$

$$+ \frac{1}{2}\,\omega\left(\frac{\pi^2|x|}{2n^2}\right) + \sum_{i=2}^{n} \omega\left(\frac{2i-1}{2n^2}\pi^2|x|\right)\left[\frac{1}{2\pi}\frac{1}{i(i-1)} + \left(\frac{1}{8} + \frac{\pi}{8} - \frac{3}{4\pi}\right)\frac{1}{n^2}\right].$$

Так как

$$\sum_{i=1}^{n-1} i = \frac{n^2}{2} - \frac{n}{2}, \qquad \sum_{i=2}^{n} (2i-1) = n^2 - 1,$$

то отсюда и из (60)—(61) получаем доказываемое неравенство (8) и аналогичное соотношение

(70) $$|P_1(x) - f(x)| \leqq \frac{1}{2}\,\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right) + \left[\frac{1}{2} + \frac{1}{\pi} + \left(\sqrt{2} - \frac{4}{\pi}\right)\frac{1}{n^2}\right]\cdot$$

$$\cdot\,\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right) + \frac{2}{\pi}\sum_{i=2}^{2n-1}\frac{1}{i^2-1}\,\omega\left(\frac{i\pi^2}{2n^2}|x|\right) +$$

$$+\left[\frac{35}{24} + \frac{\pi}{4} - \frac{11}{12\pi} + \left(\sqrt{2} - \frac{4}{\pi}\right)\frac{1}{n^2}\right]\omega\left(\frac{\pi^2|x|}{2n^2}\right) \quad (n = 2, 4, 6, \ldots).$$

Доказательство формулы (9). Пусть $0 < t \leqq \frac{\pi}{2n}$; случай $\frac{\pi}{2n} < t \leqq \frac{\pi}{n}$ рассматривается аналогичным образом. Из (35) получаем:

(71) $$|P_k(x) - f(x)| \leqq \omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)\sum_{\substack{i=1-n\\ i\neq 0}}^{n} |\sigma_i(t)| + \omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)|\sigma_0(t)| +$$

$$+\,\omega\left(\frac{\pi^2|x|}{n^2}\right)\sum_{i=1-n}^{n}|i\sigma_i(t)| + \omega\left(\frac{\pi^2|x|}{8n^2}\right)|\sigma_0(t)| \quad (k = 1, 3, 5, \ldots).$$

Предположим, что $k=1$ и воспользуемся леммой 3 из [11]:

(72) $$\sigma_0(t) \geqq 0,\ |\sigma_i(t)| = (-1)^{i+1}\sigma_i(t) \quad (i = \pm 1, \pm 2, \ldots, \pm(n-1), n).$$

Отсюда следует:

$$\sum_{i=1-n}^{n} |i\sigma_i(t)| = \sum_{i=1-n}^{n} (-1)^{i+1}|i|\sigma_i(t).$$

Принимая во внимание (31)—(32) и (62), видим, что

$$\sum_{i=1-n}^{n}|i\sigma_i(t)| = \frac{1}{4}\sum_{i=1}^{n}[1+(-1)^{i+1}]\left[d\left(t - \frac{2i-1}{2n}\pi\right) + d\left(t + \frac{2i-1}{2n}\pi\right)\right].$$

Отсюда и из известного тождества

(73) $$\sum_{i=1-n}^{n} d\left(\vartheta - \frac{2i-1}{2n}\pi\right) = 1,$$

а также из соотношений (53) и (48) получаем:

(74) $$\sum_{i=1-n}^{n}|i\sigma_i(t)| \leqq \frac{1}{\pi}\ln n + 1.$$

Из (71), (59)—(61) и (74) следует доказываемое неравенство (9) и аналогичное соотношение

$$(75)\qquad |P_1(x)-f(x)| \leqq \frac{1}{2}\,\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+\left[\frac{1}{2}+\frac{1}{\pi}+\left(\sqrt{2}-\frac{4}{n}\right)\frac{1}{n^2}\right]\cdot$$

$$\cdot\,\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)+\left(\frac{1}{\pi}\ln n+1\right)\omega\left(\frac{\pi^2|x|}{n^2}\right)+$$

$$+\left[\frac{1}{2}+\frac{1}{\pi}+\left(\sqrt{2}-\frac{4}{\pi}\right)\frac{1}{n^2}\right]\omega\left(\frac{\pi^2|x|}{8n^2}\right)\quad (n=2,4,6,\dots).$$

Доказательство неравенства (10). Из (33) и (34) получаем:

$$(76)\qquad |P_k(x)-f(x)| \leqq \omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)\sum_{i=1-n}^{n}|\sigma_i(t)|+$$

$$+\,\omega\left(\frac{\pi^2|x|}{2n^2}\right)\sum_{i=1}^{n}(2i-1)[|\sigma_i(t)|+|\sigma_{1-i}(t)|]\quad (k=2,4,6,\dots).$$

Пусть $k=2$. Лемму 5 и теорему 1 из [11] можно записать в виде

$$(77)\qquad \sum_{i=1-n}^{n}|\sigma_i(t)| \leqq \frac{5}{4}\,.$$

Из леммы 3 статьи [11] получаем:

$$(78)\qquad \sigma_0(t)\geqq 0,\ \sigma_1(t)\geqq 0,\ |\sigma_i(t)|=(-1)^i\sigma_i(t),\ |\sigma_{1-i}(t)|=(-1)^i\sigma_{1-i}(t)$$
$$(i=2,3,\dots,n).$$

Формулу (24) из [11] можно записать в виде

$$(79)\qquad \sigma_0(t)+\sigma_1(t)=1\quad (k=0,1,2,\dots).$$

Из (78), (79) и (31), (32) следует:

$$\sum_{i=1}^{n}(2i-1)[|\sigma_i(t)|+|\sigma_{1-i}(t)|=2+\sum_{i=1}^{n}(-1)^i i[s_i(t)+s_{1-i}(t)].$$

Из (27) получаем:

$$(80)\qquad s_i(\vartheta)=\frac{1}{4}\left[d\left(\vartheta-\frac{2i-3}{2n}\pi\right)+2d\left(\vartheta-\frac{2i-1}{2n}\pi\right)+d\left(\vartheta-\frac{2i+1}{2n}\pi\right)\right]$$

Поэтому

$$\sum_{i=1}^{n}(2i-1)[|\sigma_i(t)|+|\sigma_{1-i}(t)|]=2-\frac{1}{4}\left[d\left(t+\frac{\pi}{2n}\right)+d\left(t-\frac{\pi}{2n}\right)\right]+$$

$$+\frac{2n+1}{4}(-1)^n\left[d\left(t+\frac{2n-1}{2n}\pi\right)+d\left(t-\frac{2n-1}{2n}\pi\right)\right].$$

Обозначим через $l_i(x)$ фундаментальные многочлены Лагранжева интерполирования по узлам $\cos\frac{2i-1}{2n}\pi$:

$$(81)\qquad l_i(\cos\vartheta)=\frac{1}{n}(-1)^{i+1}\sin\frac{2i-1}{2n}\pi\frac{\cos n\vartheta}{\cos\vartheta-\cos\frac{2i-1}{2n}\pi}\quad (i=1,2,\dots,n).$$

Известно, что

$$(82)\qquad l_i(\cos\vartheta)=d\left(\vartheta+\frac{2i-1}{2n}\pi\right)+d\left(\vartheta-\frac{2i-1}{2n}\pi\right)\quad (i=1,2,\dots,n).$$

Поэтому

$$\sum_{i=1-n}^{n}(2i-1)|\sigma_i(t)|=2-\frac{1}{4}l_1(\cos t)+\frac{2n+1}{4}(-1)^n l_n(\cos t).$$

Из (81) видно, что

$$(83)\qquad l_1(\cos t)\geqq 1,$$

$$(84)\qquad (-1)^n l_n(\cos t)\leqq 0.$$

Поэтому

$$(85)\qquad \sum_{i=1}^{n}(2i-1)[|\sigma_i(t)|+|\sigma_{1-i}(t)|]\leqq\frac{7}{4}.$$

Из (76), (77) и (85) получаем доказываемое неравенство (10).

Доказательство соотношения (24). Предположим, что $k=3$. Из формулы (58) статьи [11] получаем:

$$\sum_{\substack{i=1-n\\ i\neq 0}}^{n}|\sigma_i(t)|=\frac{3}{2}-2s_0(t).$$

Поэтому и ввиду (28), (26)

$$\sum_{\substack{i=1-n\\ i\neq 0}}^{n}|\sigma_i(t)|=\frac{3}{2}-\frac{1}{4}d\left(t+\frac{3\pi}{2n}\right)-\frac{3}{4}d\left(t+\frac{\pi}{2n}\right)-\frac{3}{4}d\left(t-\frac{\pi}{2n}\right)-\frac{1}{4}d\left(t-\frac{3\pi}{2n}\right)=$$

$$=\frac{3}{2}-\frac{1}{n}-\frac{1}{2n}\sum_{i=1}^{n-1}\cos it\left(3\cos\frac{i\pi}{2n}+\cos\frac{3i\pi}{2n}\right)=\frac{3}{2}-\frac{1}{n}-\frac{2}{n}\sum_{i=1}^{n-1}\cos it\left(\cos\frac{i\pi}{2n}\right)^3.$$

Следовательно на отрезке $-\frac{\pi}{2n}\leqq t\leqq\frac{\pi}{2n}$ функция $\sum_{\substack{i=1-n\\ i\neq 0}}^{n}|\sigma_i(t)|$ принимает свое наибольшее значение в точке $\frac{\pi}{2n}$, которое можно вычислить с помощью тождества (51)

$$(86)\qquad \sum_{\substack{i=1-n\\ i\neq 0}}^{n}|\sigma_i(t)|\leqq\sigma_0\left(\frac{\pi}{2n}\right)=\frac{3}{2}-\frac{1}{4}d\left(\frac{2\pi}{n}\right)-\frac{3}{4}d\left(\frac{\pi}{n}\right)-\frac{3}{4}d(0)-\frac{1}{4}d\left(-\frac{\pi}{n}\right)=\frac{3}{4}.$$

В силу (32), (28) и (73)

$$(87)\qquad \sigma_0(t) = \sum_{i=1-n}^{0} s_i(t) = \frac{1}{2} d\left(t+\frac{2n-1}{2n}\pi\right)+\frac{7}{8} d\left(t+\frac{2n-3}{2n}\pi\right)+$$

$$+\sum_{i=2}^{n-2} d\left(t+\frac{2i-1}{2n}\pi\right)+\frac{7}{8} d\left(t+\frac{\pi}{2n}\right)+\frac{1}{2} d\left(t-\frac{\pi}{2n}\right)+\frac{1}{8} d\left(t-\frac{3\pi}{2n}\right)+$$

$$+\frac{1}{8} d\left(t-\frac{2n-1}{2n}\pi\right) = \frac{1}{2}+\frac{3}{8} d\left(t+\frac{2n-3}{2n}\pi\right)+\frac{1}{2}\sum_{i=2}^{n-2} d\left(t+\frac{2i-1}{2n}\pi\right)+$$

$$+\frac{3}{8} d\left(t+\frac{\pi}{2n}\right)-\frac{3}{8} d\left(t-\frac{3\pi}{2n}\right)-\frac{1}{2}\sum_{i=2}^{n-2} d\left(t-\frac{2i+1}{2n}\pi\right)-\frac{3}{8} d\left(t-\frac{2n-1}{2n}\pi\right).$$

Из (26) получаем:

$$d\left(t+\frac{2i-1}{2n}\pi\right)-d\left(t-\frac{2i+1}{2n}\pi\right) = \frac{2}{n}\sum_{j=1}^{n-1} \sin j\left(\frac{\pi}{2n}-t\right)\sin\frac{ij\pi}{n}.$$

Очевидно

$$\sum_{i=1}^{n-1} \sin\frac{ij\pi}{n} = \begin{cases} 0 & (j = 2, 4, 6, \ldots), \\ \operatorname{ctg}\dfrac{i\pi}{2n} & (j = 1, 3, 5, \ldots); \end{cases}$$

$$\sin\frac{j\pi}{n}+\sin\frac{n-1}{n} j\pi = \begin{cases} 0 & (j = 2, 4, 6, \ldots), \\ 2\sin\dfrac{j\pi}{n} & (j = 1, 3, 5, \ldots); \end{cases}$$

$$\operatorname{ctg}\frac{j\pi}{2n}-\frac{1}{2}\sin\frac{j\pi}{n} = \left(\cos\frac{j\pi}{2n}\right)^3 \operatorname{cosec}\frac{j\pi}{2n}.$$

Поэтому на отрезке $0 \leqq t \leqq \dfrac{\pi}{2n}$

$$(88)\qquad \sigma_0(t) = \frac{1}{2}+\frac{1}{n}\sum_{j=1}^{\left[\frac{n}{2}\right]} \sin(2j-1)\left(\frac{\pi}{2n}-t\right)\left(\cos\frac{2j-1}{2n}\pi\right)^3 \operatorname{cosec}\frac{2j-1}{2n}\pi \leqq \sigma_0(0).$$

Из (87) и (51) получаем:

$$\sigma_0(0) = \frac{1}{2}+\frac{1}{8} d\left(\frac{3\pi}{2n}\right)+\frac{3}{8} d\left(\frac{\pi}{2n}\right)-\frac{1}{8} d\left(\frac{2n-3}{2n}\pi\right)-\frac{3}{8} d\left(\frac{2n-1}{2n}\pi\right) =$$

$$= \frac{1}{2}+\frac{3}{16n}\operatorname{ctg}\frac{\pi}{4n}-\frac{1}{16n}\operatorname{ctg}\frac{3\pi}{4n}+\frac{1}{16n}(-1)^{n+1}\operatorname{tg}\frac{3\pi}{4n}+\frac{3}{16n}(-1)^n\operatorname{tg}\frac{\pi}{4n}.$$

В силу (65)

$$3\operatorname{cosec}\vartheta-\operatorname{cosec}3\vartheta = \frac{8}{3\vartheta}-\sum_{i=1}^{n} 3a_i(9^i-1)\vartheta^{2i+1} < \frac{8}{3\vartheta}\qquad\left(0<\vartheta<\frac{\pi}{3}\right).$$

Поэтому

$$(89)\qquad \sigma_0(0)=\frac{1}{2}+\frac{3}{8n}\operatorname{cosec}\frac{\pi}{2n}-\frac{1}{8n}\operatorname{cosec}\frac{3\pi}{2n}<\frac{1}{2}+\frac{2}{3\pi}\quad (n=2,4,6,\ldots).$$

Так как

$$(90)\qquad \operatorname{ctg}\vartheta=\frac{1}{\vartheta}-\sum_{i=1}^{\infty}b_i\vartheta^{2i+1}\quad (0<\vartheta<\pi,\ b_i>0),$$

то

$$3\operatorname{ctg}\vartheta-\operatorname{ctg}3\vartheta=\frac{8}{3\vartheta}+\vartheta^3\sum_{i=1}^{\infty}3b_i(9^i-1)\vartheta^{2i-2}\leqq$$

$$\leqq\frac{8}{3\vartheta}+\vartheta^3\sum_{i=1}^{\infty}3b_i(9^i-1)\left(\frac{\pi}{6}\right)^{2n-2}=$$

$$=\frac{8}{3\vartheta}+\left(3\sqrt{3}-\frac{16}{\pi}\right)\left(\frac{6}{\pi}\right)^3\vartheta^3\quad\left(0<\vartheta\leqq\frac{\pi}{6}\right).$$

Поэтому

(91)

$$\sigma_0(0)=\frac{1}{2}+\frac{3}{8n}\operatorname{ctg}\frac{\pi}{2n}-\frac{1}{8n}\operatorname{ctg}\frac{3\pi}{2n}\leqq\frac{1}{2}+\frac{2}{3\pi}+\left(3\sqrt{3}-\frac{16}{\pi}\right)\frac{27}{8n^4}\quad (n=3,5,7,\ldots).$$

Очевидно

$$(92)\qquad \sigma_0(t)=\frac{1}{2}\quad (n=1).$$

Неравенство (54) из [11] можно записать в виде

$$|S_3(t)-g(t)|\leqq\Omega\left(\frac{\pi}{n}\right)\sum_{\substack{i=1-n\\ i\neq 0}}^{n}|\sigma_i(t)|+\Omega\left(\frac{\pi}{2n}\right)|\sigma_0(t)|\quad\left(0\leqq t\leqq\frac{\pi}{2n}\right).$$

Отсюда, из (86), (89), (91) и (92) получаем при $0<t\leqq\frac{\pi}{2n}$ доказываемое неравенство (24) и аналогичное соотношение

$$(93)\quad |S_3(t)-g(t)|\leqq\frac{3}{4}\Omega\left(\frac{\pi}{n}\right)+\left[\frac{1}{2}+\frac{2}{3\pi}+\left(3\sqrt{3}-\frac{16}{\pi}\right)\frac{27}{8n^4}\right]\Omega\left(\frac{\pi}{2n}\right)\quad (n=1,3,5,\ldots).$$

Случай $\frac{\pi}{2n}<t\leqq\frac{\pi}{n}$ рассматривается аналогичным образом.

Доказательство формулы (5). Неравенство (5) и аналогичное неравенство

$$(94)\quad |P_3(x)-f(x)|\leqq\frac{3}{4}\omega\left(\frac{\pi}{n}\right)+\left[\frac{1}{2}+\frac{2}{3\pi}+\left(3\sqrt{3}-\frac{16}{\pi}\right)\frac{27}{8n^4}\right]\omega\left(\frac{\pi}{2n}\right)\quad (n=1,3,5,\ldots)$$

следует из (37), (38), (24), (93) и соотношения $\Omega(h)\leqq\omega(h)$.

Доказательство неравенства (11). Из леммы 3 статьи [11] получаем:

(95) $\sigma_1(t)\geqq 0, \quad \sigma_{-1}(t)\geqq 0, \quad |\sigma_i(t)|=(-1)^i\sigma_i(t) \quad (i=0, \pm 2, \pm 3, \ldots, \pm(n-1), n).$

Отсюда, из (79), (31), (32), (28) и (82) следует:

$$\sum_{i=1-n}^{n}|i\sigma_i(t)| = 2\sigma_1(t)+2\sigma_{-1}(t)+\sum_{i=1-n}^{n}(-1)^i|i|\sigma_i(t) = \tag{96}$$

$$=\frac{7}{4}-2s_0(t)+\frac{1}{4}\sum_{i=1-n}^{n}(-1)^i(2|i|+1)s_i(t)=\frac{7}{4}-\frac{7}{8}\left[d\left(t+\frac{\pi}{2n}\right)+d\left(t-\frac{\pi}{2n}\right)\right]-$$

$$-\frac{1}{4}\left[d\left(t+\frac{3\pi}{2n}\right)+d\left(t-\frac{3\pi}{2n}\right)\right]+\frac{1}{8}(-1)^n\left[d\left(t+\frac{2n-1}{2n}\pi\right)+d\left(t-\frac{2n-1}{2n}\pi\right)\right]=$$

$$=\frac{7}{4}-\frac{7}{8}l_1(\cos t)-\frac{1}{4}l_2(\cos t)+\frac{1}{8}(-1)^n l_n(\cos t).$$

Вопользуемся леммой 8 из [11]:

$$17l_1(\cos t)+9l_2(\cos t)\geqq 17 \quad \left(n=3,4,5,\ldots;\ -\frac{\pi}{2n}\leqq t\leqq\frac{\pi}{2n}\right). \tag{97}$$

Отсюда, из (83), (84) и (96) получаем:

$$\sum_{i=1-n}^{n}|i\sigma_i(t)|\leqq\frac{7}{8}. \tag{98}$$

Из (71), (86), (88), (89), (91), (92) и (98) получаем доказываемое неравенство (11) и аналогичное соотношение

$$|P_3(x)-f(x)|\leqq\frac{3}{4}\,\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+\left[\frac{1}{2}+\frac{2}{3\pi}+\left(3\sqrt{3}-\frac{16}{\pi}\right)\frac{27}{8n^4}\right]\cdot \tag{99}$$

$$\cdot\,\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)+\frac{7}{8}\,\omega\left(\frac{\pi^2|x|}{2n^2}\right)+\left[\frac{1}{2}+\frac{2}{3\pi}+\left(3\sqrt{3}-\frac{16}{\pi}\right)\frac{27}{n^2}\right]\omega\left(\frac{\pi^2|x|}{8n^2}\right)$$

$$(n=1,3,5,\ldots)$$

при $0<t\leqq\frac{\pi}{2n}$. Случай $\frac{\pi}{2n}<t\leqq\frac{\pi}{n}$ рассматривается аналогичным образом.

Доказательство соотношения (12). Предположим, что $k=4$. Из леммы 5 и теоремы 2 статьи [11] получаем:

$$\sum_{i=1-n}^{n}|\sigma_i(t)|\leqq\frac{23}{16}. \tag{100}$$

Из леммы 3 статьи [11] следует:

$$\sigma_{-1}(t)\geqq 0, \quad \sigma_2(t)\geqq 0, \tag{101}$$

(102) $|\sigma_i(t)|=(-1)^{i+1}\sigma_i(t), \quad |\sigma_{1-i}(t)|=(-1)^{i+1}\sigma_{1-i}(t) \quad (i=1,3,4,\ldots,n).$

Отсюда, из (31), (32), (79), (27) и (82) получаем при $n \geqq 3$:

$$(103) \quad \sum_{i=1}^{n} (2i-1)[|\sigma_i(t)|+|\sigma_{1-i}(t)|] = 6\sigma_{-1}(t)+6\sigma_2(t)+$$

$$+\sum_{i=1}^{n}(-1)^{i+1}(2i-1)[\sigma_i(t)+\sigma_{1-i}(t)] = 6-6s_0(t)-6s_1(t)+$$

$$+\sum_{i=1}^{n}(-1)^{i+1}i[s_i(t)+s_{1-i}(t)] = 6-\frac{1}{16}\{57l_1(\cos t)+$$

$$+29l_2(\cos t)+6l_3(\cos t)+(2n+1)(-1)^n[l_{n-1}(\cos t)+3l_n(\cos t)]\}.$$

Из (81) следует:

$$(104) \quad l_3(\cos t) \geqq 0.$$

На странице 183 статьи [11] фигурирует неравенство

$$(105) \quad (-1)^n[l_{n-1}(\cos t)+3l_n(\cos t)] \geqq 0.$$

Из (103), (97), (83), (104) и (105) получаем при $n \geqq 3$

$$(106) \quad \sum_{i=1}^{n}(2i-1)[|\sigma_i(t)|+|\sigma_{1-i}(t)|] \leqq \frac{39}{16}.$$

Аналогичным образом можно получить даже более точную оценку при $n=1$ и $n=2$. Из (76), (100) и (106) следует доказываемое неравенство (12).

Пример 1. Обозначим через n четное число и определим четную функцию $f \in C$ следующим образом:

$$(107) \quad f(0) = 0,$$

$$(108) \quad f\left(\cos\frac{2i-1}{2n}\pi\right) = (-1)^{i+\frac{n}{2}}\operatorname{tg}\frac{\pi}{2n}\sin\frac{2i-1}{2n}\pi \quad \left(i = 1, 2, \ldots, \frac{n}{2}\right),$$

f линейна на отрезках

$$\left[0, \cos\frac{n-1}{2n}\pi\right], \quad \left[\cos\frac{2i+1}{2n}\pi, \cos\frac{2i-1}{2n}\pi\right] \quad \left(i = 1, 2, \ldots, \frac{n}{2}-1\right)$$

и постоянна на отрезке $\left[\cos\frac{\pi}{2n}, 1\right]$. Очевидно

$$f\left(\cos\frac{n-1}{2n}\pi\right)-f(0) = \cos\frac{n-1}{n}\pi,$$

$$f\left(\cos\frac{2i-1}{2n}\pi\right)-f\left(\cos\frac{2i+1}{2n}\pi\right) = (-1)^{i+\frac{n}{2}}\left(\cos\frac{2i-1}{2n}\pi-\cos\frac{2i+1}{2n}\pi\right)$$

и поэтому функция f удовлетворяет условию Липшица

$$(109)\qquad |f(x+h)-f(x)| \leqq h \quad (0 < h \leqq 2,\ -1 \leqq x \leqq 1-h).$$

Покажем, что для нее выполняется неравенство (17).

Из определения (81) многочленов $l_i(x)$ получаем:

$$(110)\qquad l_i(0) = \frac{1}{n}(-1)^{i+\frac{n}{2}} \operatorname{tg} \frac{2i-1}{2n}\pi \quad (i = 1, 2, \ldots, n).$$

Принимая во внимание четность функции f, тождество

$$(111)\qquad P_k(x) = \sum_{i=1}^{n} P_k\left(\cos\frac{2i-1}{2n}\pi\right) l_i(x) \quad (k = 0, 1, 2, \ldots)$$

и формулы (108) и (1), получаем:

$$(112)\qquad P_0(0) = \frac{2}{n}\operatorname{tg}\frac{\pi}{2n}\sum_{i=1}^{n/2}\left(\sin\frac{2i-1}{2n}\pi\right)^2 \sec\frac{2i-1}{2n}\pi =$$

$$= \frac{2}{n}\operatorname{tg}\frac{\pi}{2n}\sum_{i=1}^{n/2}\left(\sec\frac{2i-1}{2n}\pi - \cos\frac{2i-1}{2n}\pi\right) =$$

$$= \frac{2}{n}\operatorname{tg}\frac{\pi}{2n}\sum_{i=1}^{n/2}\left(\operatorname{cosec}\frac{2i-1}{2n}\pi - \sin\frac{2i-1}{2n}\pi\right) \geqq$$

$$\geqq \frac{\pi}{n^2}\sum_{i=1}^{n/2}\left(\frac{2}{\pi}\frac{n}{2i-1} - \frac{\pi}{2}\frac{2i-1}{2n}\right) \geqq \frac{\ln n}{n} - \frac{\pi^2}{8n}.$$

Отсюда и из (107) получаем доказываемое неравенство (17).

Пример 2. Пусть $n \geqq 2$. Определим зависяшую от n функцию $f \in C$ следующим образом:

$$(113)\qquad f(1) = \operatorname{tg}\frac{\pi}{2n}\operatorname{tg}\frac{\pi}{4n};$$

$$(114)\qquad f\left(\cos\frac{2i-1}{2n}\pi\right) = (-1)^{i+1}\operatorname{tg}\frac{\pi}{2n}\sin\frac{2i-1}{2n}\pi \quad (i = 1, 2, \ldots, n);$$

f постоянна на отрезке $\left[-1, -\cos\frac{\pi}{n}\right]$ и линейна на отрезках

$$\left[\cos\frac{2i+1}{2n}\pi, \cos\frac{2i-1}{2n}\pi\right] \quad (i = 1, 2, \ldots, n-1), \quad \left[\cos\frac{\pi}{n}, 1\right].$$

Условие (109) снова выполняется. Докажем неравенство (18).

Из определения (81) многочленов $l_i(x)$ получаем:

$$(115)\qquad l_i(x) = \frac{1}{n}(-1)^{i+1}\operatorname{ctg}\frac{2i-1}{4n}\pi \quad (i = 1, 2, \dots, n).$$

Из (111), (1), (114) и (115) следует:

$$(116)\quad P_0(1) = \frac{2}{n}\operatorname{tg}\frac{\pi}{2n}\sum_{i=1}^{n}\left(\cos\frac{2i-1}{4n}\pi\right)^2 = \frac{1}{n}\operatorname{tg}\frac{\pi}{2n}\sum_{i=1}^{n}\left(1+\cos\frac{2i-1}{2n}\pi\right) = \operatorname{tg}\frac{\pi}{2n}.$$

Отсюда, из (113) и из неравенства

$$(117)\qquad \vartheta \leqq \operatorname{tg}\vartheta \leqq \frac{4}{\pi}\vartheta \quad \left(0 \leqq \vartheta \leqq \frac{\pi}{4}\right),$$

получаем:

$$P_0(1)-f(1) = \operatorname{tg}\frac{\pi}{2n}\left(1-\operatorname{tg}\frac{\pi}{4n}\right) \geqq \frac{\pi}{2n}\left(1-\frac{1}{n}\right),$$

что и требовалось доказать.

Пример 3. Пусть $n \geqq 2$. Определим зависящую от n функцию $f \in C$ следующим образом:

$$(118)\qquad f\left(\cos\frac{i\pi}{n}\right) = (-1)^{i+1}\operatorname{tg}\frac{\pi}{2n}\sin\frac{i\pi}{n} \quad (i = 0, 1, \dots, n);$$

f линейна на отрезках

$$\cos\frac{i\pi}{n} \leqq x \leqq \cos\frac{i-1}{n}\pi \quad (i = 1, 2, \dots, n).$$

Так как

$$f\left(\cos\frac{i-1}{n}\pi\right) - f\left(\cos\frac{i\pi}{n}\right) = (-1)^i\left(\cos\frac{i-1}{n}\pi - \cos\frac{i\pi}{n}\right) \quad (i = 1, 2, \dots, n),$$

то условие (109) выполняется. Докажем неравенство (19).

Очевидно

$$f\left(\cos\frac{i-1}{n}\pi\right) + f\left(\cos\frac{i\pi}{n}\right) = 2(-1)^{i+1}\operatorname{tg}\frac{\pi}{2n}\sin\frac{\pi}{2n}\cos\frac{2i-1}{2n}\pi \quad (i = 1, 2, \dots, n).$$

Отсюда, из (115), (1) и (111) получаем:

$$P_1(1) = \frac{1}{n}\operatorname{tg}\frac{\pi}{2n}\sin\frac{\pi}{2n}\sum_{i=1}^{n}\left(\operatorname{ctg}\frac{2i-1}{4n}\pi - \sin\frac{2i-1}{2n}\pi\right).$$

Р. Гюнттнэр [7] опубликовал неравенство

$$\frac{1}{n}\sum_{i=1}^{n}\operatorname{ctg}\frac{2i-1}{2n}\pi \geqq \frac{2}{\pi}\ln n + 0{,}962.$$

Так как

$$\sum_{i=1}^{n} \sin(2n-1)\vartheta = \sin^2 n\vartheta \operatorname{cosec} \vartheta,$$

то

$$\sum_{i=1}^{n} \sin \frac{2i-1}{2n}\pi = \operatorname{cosec} \frac{\pi}{2n}.$$

Следовательно

$$P_1(1) \geqq \operatorname{tg}\frac{\pi}{2n} \sin\frac{\pi}{2n}\left(\frac{2}{\pi}\ln n + 0{,}962\right) - \frac{1}{n}\operatorname{tg}\frac{\pi}{2n}.$$

Очевидно

$$\operatorname{tg}\vartheta \sin\vartheta = \sec\vartheta - \cos\vartheta \geqq 1 + \frac{\vartheta^2}{2} + 5\frac{\vartheta^2}{24} - \left(1 - \frac{\vartheta^2}{2} + \frac{\vartheta^4}{24}\right) \geqq \vartheta^2 \quad \left(|\vartheta| < \frac{\pi}{2}\right).$$

Ввиду (118) $f(1)=0$. Поэтому и в силу (117) действительно имеет место доказываемое неравенство (19):

$$P_1(1) - f(1) \geqq \frac{\pi^2}{4n^2}\left(\frac{2}{\pi}\ln n + 0{,}962\right) - \frac{2}{n^2} \geqq \frac{\pi}{2}\frac{\ln n}{n^2}.$$

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SOME REMARKS TO A PAPER BY E. CSÁKI AND G. TUSNÁDY ON THE BALLOT THEOREM

By

M. FOLLEDO (Bahia Blanca) and I. VINCZE (Budapest)

Introduction

In their paper [1] the authors considered the ballot lemma of TAKÁCS [2] giving its generalizations and extensions in different ways; the results contain as special cases some well known relations in the theory of order statistics. Our aim is now twofold. In § 1 we give a proof for the ballot lemma of TAKÁCS [2] as elementary and simple as the statement of the lemma itself is; the proof concerns a "sweeping" (balayage) procedure which transforms the original numbers to a 0, 1 sequence for which the statement is trivial. In § 2 we improve the proof of a lemma occuring in § 2 of the quoted paper; the same incorrect proof occurs in the paper by I. VINCZE [3] as well. This lemma is a geometrical analogon of the ballot lemma of Takács.

1. Proof of the ballot lemma

BALLOT LEMMA (TAKÁCS). *Let $k_1, k_2, \dots k_n$ be non-negative integers with $\sum_{i=1}^{n} k_i = k < n$; let further $k_{i+n} = k_i$, $i = 1, 2, \dots n$. Then the number of indices h for which the system of relations*

$$\sum_{j=1}^{i} k_{h+j-1} < i, \quad i = 1, 2, \dots n \tag{2.1}$$

holds, equals to $n-k$.

The case $k=n$ being trivial can be left out of considerations.

PROOF. An index h ($1 \leqq h \leqq n$) will be called a *good starting index,* when for it the relations in (2.1) hold.

Let us consider the case when the set of k_j's contains zeros and ones only; in this case the terms $k_j = 0$ and only these will be the good starting terms. So the theorem is trivial for such a configuration.

We shall consider now the sequence of the k_j's for the general case in cyclic order, fixed them around a circle. Using the notation $\varphi(u)$ for the function

$$\varphi(u) = \begin{cases} 0, & \text{if} \quad u = 0 \quad \text{or} \quad 1 \\ u-1, & \text{if} \quad u > 1 \end{cases}$$

defined for nonnegative integers, the following transformation will be applied on the k_j's in the cyclic sense:

$$k_j \to k'_j = k_j - \varphi(k_j) + \varphi(k_{j+1}). \tag{2.2}$$

Now the following simple statements lead to the proof of the lemma:

a) There exists a good starting index: the last index for which the quantity $k_1+k_2+\ldots+k_h-h$ $(h=1, 2, \ldots, n)$ attains its maximum is a good starting index.

b) The transformation (2.2) does not alter the set $\{h_1^*, h_2^*, \ldots, h_l^*\}$ of the good starting indices; for any good starting index h from the relation

$$k_h+k_{h+1}+\ldots+k_{h+j-1}+k_{h+j}<j+1 \tag{2.3}$$

if $k_{h+j}>1$, the relation

$$k'_h+k'_{h+1}+\ldots+k'_{h+j-1}+1<j+1 \tag{2.4}$$

follows which means that the jth relation of (2.1) remains valid after using the transformation (2.2) for $j=1, 2, \ldots, n-1$; if $k_{h+j}=0$ or 1 this statement trivially holds.

As from a positive term $(k_j>0)$ a transformed k'_j with value 0 cannot occur, the number l of good starting indices will not increase.

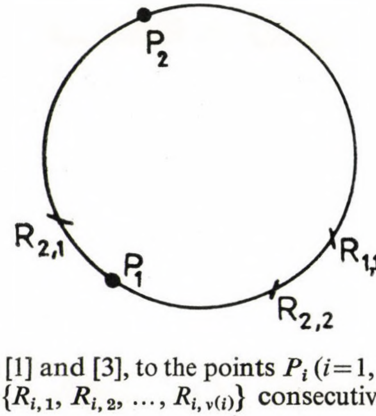
c) Using the transformation (2.2) repeatedly until a term $k_j^{(r)}$ (after the r'st step) larger than one exist, the transformation will lead to a new configuration; the procedure cannot have a period as the "values flow in negative direction" but the good starting terms k_{h^*} remain zeros.

d) As the number of all possible configurations of nonnegative numbers with the sum $k<n$ is finite, the procedure will stop, i.e. it will lead after a finite step to a sequence containing zeros and ones only. This completes the proof.

2. A geometrical ballot lemma

We turn now to the discussion and proof of the following

LEMMA (CSÁKI and TUSNÁDY). *Let us given n points $P_1, P_2, \ldots, P_n$ on a directed circle having circumference* 1 *and let q be a number such that* $0<nq<1$. *If Q is an arbitrary point on the circle, put the points $Q_1, Q_2, \ldots, Q_n$ on the circle in positive direction from the point Q such that the length of arc QQ_k is kq. We say that Q is a point of first category if the arc QQ_k contains less than k of the points $P_1, P_2, \ldots, P_n$, for $k=1, 2, \ldots, n$. Then the measure of the set of points first category is* $1-nq$.



In the proof given in [1] and [3], to the points P_i $(i=1, 2, \ldots, n)$ a chain is ordered i.e. a set of points $C_i=\{R_{i,1}, R_{i,2}, \ldots, R_{i,\nu(i)}\}$ consecutive on the circle but in the negative direction such that the length of arc $P_i R_{i,j}$ is jq and the integer is chosen in such a way that the arc $P_i R_{i,j}$ contains at least j of the points $P_1, P_2, \dots, P_n$ (including the point P_i) for $j=1, 2, \dots, \nu(i)$, while less than j for $j=\nu(i)+1$. Now it is stated in the quoted papers that (i) if a chain C_i covers the point P_j then it covers the chain C_j. As the simple example with $n=2$, $0<q<\widehat{P_1P_2}<\frac{1}{2}$ shows, this statement (even the statement in (iii)) is not true. Namely, in this case $\nu(2)=2$ and $\nu(1)=1$, but the chain C_2, although it covers the point P_1, it does not cover the chain C_1. (If the value $\nu(i)>n$ is allowed, then the chain C_2 will cover the chain C_1, but in this case the statement in (ii) — according to which a "maximal chain covers exactly $\nu(i)$ of the points" — will be violated.)

A slight modification of the definition of the chain leads to the

PROOF OF THE LEMMA. To any point P_i $(i=1, 2, \dots, n)$ a chain $C_i=\{R_{i,1}, R_{i,2}, \dots$ $\dots, R_{i,\nu(i)}\}$ will be ordered with the following properties:

a) $\widehat{P_i R_{i,j}}=jq$, b) the arc $P_i R_{i,j}$ contains at least $j+1$ of the points $P_1, P_2, \dots, P_n$ (including P_i itself) for $j=1, 2, \dots, \nu(i)-1$ and exactly j points for $j=\nu(i)$. In this way if the arc $P_i R_{i,1}$ does not contain any further point then $\nu(i)=1$. Now the following statements are true:

(i) If a chain C_i contains the point P_k then it contains the chain C_k as well. This follows from the fact that the last interval $(R_{i,\nu(i)-1}, R_{i,\nu(i)})$ is free from the P_j's, the last two $(R_{i,\nu(i)-2}, R_{i,\nu(i)})$ contain less than two of the P_j's etc. Consequently, to any P_k occurring in the jth interval the length of the chain C_k is at most $\nu(i)-j$, i.e. it ends at most in the last interval.

(ii) If we omit the chains covered by any other one, then the remaining chains are disjoints and have the total length nq; this is an immediate consequence of (i).

(iii) A point Q of the circle is of the first category if and only if it is not covered by any chain. Consequently the measure of all points of first category is $1-nq$, as stated.

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UNIVERSIDAD NACIONAL DEL SUR
BAHIA BLANCA
ARGENTINA

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.
HUNGARY

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ON RELATIVE UNIVERSAL EMBEDDING SPACES

By
L. ÚRY (Budapest)

Introduction

Tietze's well known theorem can be generalized in many different ways. One of the possible conceptions is the following. Let F be a subspace of a topological space. When can a continuous map of F into any other space be extended to a continuous map of the whole space? In this generalized case up to day no exact answer has been given.

Another generalization is the following: when does the extension of the map inherit not only the continuity but other properties too. The following theorem proved by HAUSDORFF [1] and independently by BING [2] is of this type (cf. also [3]).

THEOREM. *Let X be a metrizable topological space, and let $F \subset X$ be a closed set. Then for every metric ϱ on F which induces the relative topology on F, there is a metric $\tilde{\varrho}$ on X which is an extension of ϱ and is compatible with the topology of X.*

First of all we need two definitions.

DEFINITION. Let Σ be a metric space with weight ω, where ω can be any cardinal. Σ is called a universal embedding space, if for every metric space X with weight ω there is a topological embedding of X into Σ.

DEFINITION. Let Σ be a metric space with weight ω, and let F be a metric space. Σ is called a universal embedding space modulo F, if there is an isometric embedding i of F into Σ such that for every metric space X with weight ω containing F as a closed subspace there is a topological embedding j of X into Σ such that the following diagramm is commutative:

$$\begin{array}{ccc} F & \xrightarrow{i} & \Sigma \\ \cap & \nearrow_{j} & \\ X & & \end{array}$$

(e.g. if $a \in F$ then $i(a) = j(a)$).

Denote by $S(\omega)$ the hedgehog with weight ω. It is well known that if $\omega \geqq \aleph_0$, then $\prod_{i=1}^{\infty} S(\omega)$ is a universal embedding space. The aim of this paper is to construct the universal embedding space, if F is a "good" topological space. Evidently if we construct a universal embedding space modulo F, then we prove again the theorem of Hausdorff—Bing for this F.

Basic definitions and constructions

J is the closed interval $[0, L]$, where L will be chosen later. In this section due to technical reasons a cone of a topological space H is denoted by $\varkappa(H)$ is obtained from the set $H\times J$ by identifying the set $H\times\{L\}$ to a single point $\varkappa$. Let us denote by p the projection of $H\times J$ into $\varkappa(H)$. The *usual topology* on $\varkappa(H)$ is the topology induced by p. Let us denote by $i_\varkappa$ the natural embedding of H into $\varkappa(H)$. H and $i_\varkappa(H)$ can be identified and H is called the basic surface of the cone. $(i_\varkappa(x)=(x, 0).)$

THEOREM 1. *Let $(H, \tilde{\varrho})$ be a bounded metric space. Then there is a bounded metric $\varrho_\varkappa$ on $\varkappa(H)$ which is identical with $\tilde{\varrho}$ on H. The topology, induced by $\varrho_\varkappa$ is weaker than the usual topology on $\varkappa(H)$. These two topologies coincide if and only if H is compact.*

PROOF. Let K be a bound of $\tilde{\varrho}$ on H, and $L=K/2$. Let f be a function from J onto the closed unit interval $[0, 1]$ given by $f(t)=1-2t/K$. It is evident, that f is non-negative, strictly monotone and $f(0)=1$, $f(L)=0$.

The sequence $a_1, a_2, \dots, a_n$ of points of $\varkappa(H)$ is called a *road* if the first or the second coordinates of a_i and a_{i+1} are equal. Let $a_i=(x_i, t_i)$ and $a_{i+1}=(x_{i+1}, t_{i+1})$. The pair a_i, a_{i+1} is called a *step up* or a *step down* if $x_i=x_{i+1}$ and $t_i<t_{i+1}$ or $x_i=x_{i+1}$ and $t_i>t_{i+1}$ respectively. If $t_i=t_{i+1}$ then the pair a_i, a_{i+1} is called a *step side*. Therefore each road consists of some step sides, step downs and step ups. Let us define the length of steps by

$$d\big((x, t), (x', t)\big) = f(t)\cdot\tilde{\varrho}(x, x'), \quad d\big((x, t), (x, t^1)\big) = |t'-t|.$$

The *length of the road* is the sum of the length of the steps, the road consists of. Let a, b be two points of $\varkappa(H)$ and define $\varrho_\varkappa$ by

$$\varrho_\varkappa(a, b) = \inf\{\text{ length of the road between } a \text{ and } b\}.$$

We prove that

$$(*)\qquad \varrho_\varkappa\big((x, t), (x', t')\big) = f\big(\max(t', t)\big)\cdot\tilde{\varrho}(x, x')+|t'-t|.$$

From this immediately follows, that $\varrho_\varkappa$ is a bounded metric on $\varkappa(H)$. It is easy to see, that the topology induced by $\varrho_\varkappa$ is weaker than the usual topology on $\varkappa(H)$.

To prove $(*)$ we show that a sequence of steps of a certain type can be changed for a not longer one. By means of these changes — through shorter and shorter roads — finally we get the road which has only one step up or/and one step side. This proves $(*)$.

(a) A sequence of step sides can be changed into one step side, since for fixed t the distances are the same as on H (apart from a constant).

(b) The sequence of one step down, one step side and one step up can be changed into one step side (see Fig. 1, 2 and 3). The length of $\rightarrow$ is $2(t_1-t_2)+f(t_2)\cdot\tilde{\varrho}(x_1, x_2)$ and the length of $\dashrightarrow$ is $f(t_1)\cdot\tilde{\varrho}(x_1, x_2)$ but $f(t_1)<f(t_2)$.

(c) The sequence of one step up one step side and one step down can be changed into one step side. The length of $\rightarrow$ is $2(t_1-t_2)+f(t_1)\cdot\tilde{\varrho}(x_1, x_2)$ and the length of $\dashrightarrow$ is $f(t_2)\tilde{\varrho}(x_1, x_2)$, but $2(t_1-t_2)+f(t_1)\cdot\tilde{\varrho}(x_1, x_2)\geqq f(t_2)\tilde{\varrho}(x_1, x_2)$ iff $\tilde{\varrho}(x_1, x_2)\leqq \leqq 2(t_1-t_2)/\big(f(t_2)-f(t_1)\big)=K$, which is always true, as K is a bound of $\tilde{\varrho}$.

(d) The sequence of one step side and one step up can be changed into a sequence of one step up and one step side. The length of $\rightarrow$ is $(t_1-t_2)+f(t_2)\cdot\tilde{\varrho}(x_1, x_2)$ and the length of $\dashrightarrow$ is $(t_1-t_2)+f(t_1)\cdot\tilde{\varrho}(x_1, x_2)$ but $f(t_1)<f(t_2)$.

We omitted the details of the proof of $(*)$.

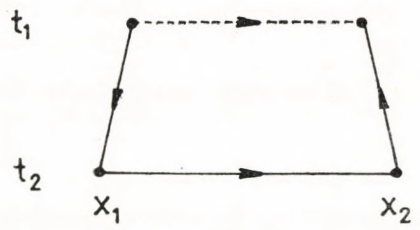

Fig. 1

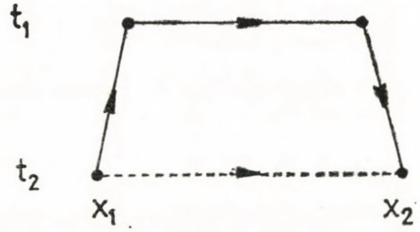

Fig. 2

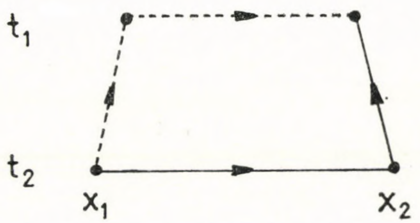

Fig. 3

Now let H be a compact space and U an open neighbourhood of $\varkappa$. Hence H is compact, thus there is a T, such that $0<T<L$ and $p((T, L]\times H)\subset U$ and so the topology induced by $\varrho_\varkappa$ is identical with the usual topology on $\varkappa(H)$. If H is not compact, then there is a discrete set of H, $\{x_i\}_{i=1}^{\infty}$. Let $\{U_i\}_{i=1}^{\infty}$ be a discrete family consisting of open sets, and $x_i\in U_i$. Let J_i be the interval $(L-1/i, L]$. Let

$$V=\left(\bigcup_{i=1}^{\infty} J_i\times U_i\right)\cup\left(J\times(H\setminus\{x_i\}_{i=1}^{\infty})\right)$$

and let $U=p(V)$. Then U is open in the usual topology but not open in the topology induced by $\varrho_\varkappa$. Q.e.d.

Let S be a set so that $|S|=\omega$, where ω is any cardinal. Let $\equiv$ be an equivalence relation defined on the set $I\times S$, where I is the closed unit interval $[0, 1]$, and $(x_1, S_1)\equiv\equiv(x_2, S_2)$ iff $S_1=S_2$ and $x_1=x_2$ or $x_1=x_2=0$. $[x_1, S_1]$ is the coset containing the point (x_1, S_1). Let

$$\varrho_1([x_1, S_1], [x_2, S_2])=\begin{cases}|x_1-x_2| & \text{if } S_1=S_2,\\ x_1+x_2 & \text{otherwise.}\end{cases}$$

It is well known, that ϱ_1 is a metric on the quotient set $S\times I/\equiv$. The topological space induced by ϱ_1 is called a hedgehog with weight ω and is denoted by $S(\omega)$. If $\omega\geqq\aleph_0$, the weight of the space $S(\omega)$ is indeed ω, otherwise it is $\aleph_0$. Denote by S_C the point $[0, s]$, where s is any point of S.

Let $(H, \tilde{\varrho})$ be a bounded metric space, and let $(\varkappa(H), \varrho_\varkappa)$ be the metric space constructed in Theorem 1. Let $\sim$ be an equivalence relation on the set $S(\omega) \vee \varkappa(H)$, where $\vee$ denotes the disjoint union. $a \sim b$ iff $a = S_C$ and $b = \varkappa$ or $b = S_C$ and $a = \varkappa$ or $a = b$. Denote by $S(\omega, H, \tilde{\varrho})$ the set $S(\omega) \vee \varkappa(H)/\sim$. Let

$$\varrho_S([a],[b]) = \begin{cases} \varrho_\varkappa(a, b) & \text{if } a, b \in \varkappa(H), \\ \varrho_1(a, b) & \text{if } a, b \in S(\omega), \\ \varrho_\varkappa(a, \varkappa) + \varrho_1(S_C, b) & \text{if } a \in \varkappa(H), b \in S(\omega), \\ \varrho_\varkappa(b, \varkappa) + \varrho_1(S_C, a) & \text{if } a \in S(\omega), b \in \varkappa(H). \end{cases}$$

Obviously ϱ_S is a metric on $S(\omega, H, \tilde{\varrho})$ and induces just the quotient topology. It is evident, that $S(\omega, H, \tilde{\varrho})$ has a system of coordinates with axis $S \vee H$ and I. We shall denote by (t, x) the elements of $S(\omega, H, \tilde{\varrho})$, where $x \in S \vee H$ and $t \in I$.

Let i_s be a map from H into $S(\omega, H, \tilde{\varrho})$ given by $i_S(x) = (1, x)$. i_S is an isometrical embedding. Denote by $j_\varkappa \colon \varkappa(H) \to S(\omega, H, \tilde{\varrho})$ the embedding. We shall prove, if H is a "good" topological space, then $S(\omega, H, \tilde{\varrho})^{\mathbf{N}}$ with $i_S^{\mathbf{N}}$ is a universal ω-embedding space modulo H. (See Fig. 4.)

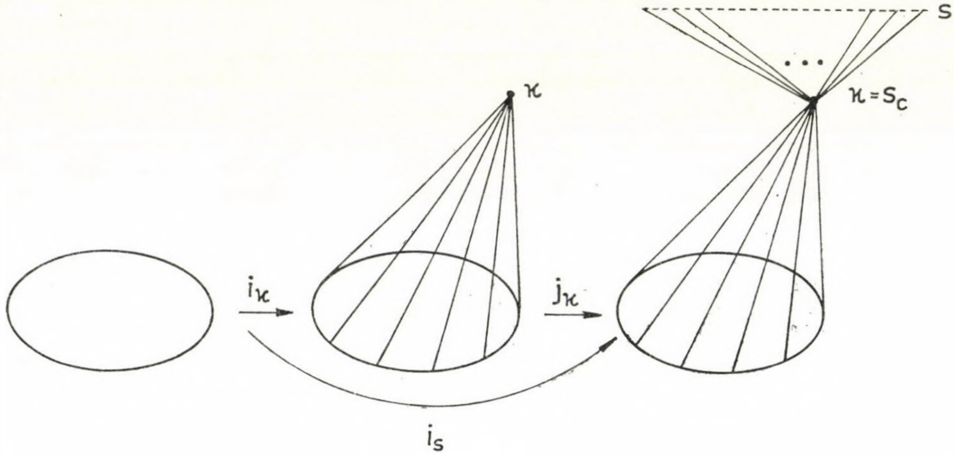

Fig. 4

Upraising systems

Definition. Let F be a closed subspace of any topological space X. A set $\{(V_n, \varphi_n) \colon n \in \mathbf{N}\}$ or simply $(V_n, \varphi_n)_{n=1}^{\infty}$ is an *upraising system of F for X* iff:

(1) V_n is an open neighbourhood of F,
(2) if $n > m$ then $V_n \subset V_m$,
(3) if $x \in X \setminus F$ then there is an $n \in \mathbf{N}$, such that $x \notin \overline{V}_n$,
(4) $\varphi_n \colon X \to \varkappa(F)$ is a continuous map,
(5) $\varphi_n[X \setminus V_n] \subset \{\varkappa\}$,
(6) $\varphi_n(x) = i(x)$ if $x \in F$,
(7) if $A \subset X$ is a closed set and $A \cap F = \emptyset$ then for every $n \in \mathbf{N}$ $\overline{\varphi_n(A)} \cap \overline{\varphi_n(F)} = \emptyset$.

LEMMA 1. *If F is a closed, discrete subspace of a metrizable topological space X, then there is an upraising system of F for X.*

PROOF. Let d be any metric on X, inducing the topology of X, and let $F=\{x_i: i\in\Gamma\}$, where $x_i\neq x_j$ if $i\neq j$. Hence X is a metrizable space, thus X is multinormal and M_1. So there is a system

$$\{U_i, U_i^j, \varepsilon_{ij}: i\in\Gamma, j\in\mathbf{N}\}$$

with the following properties:

(i) $\{U_i\}_{i\in\Gamma}$ is a discrete, open family in X,

(ii) U_i^j is an open d-ball with radius ε_{ij} and centre x_i and $U_i^j\subset U_i$,

(iii) if $j>k$ then $\varepsilon_{ij}<\varepsilon_{ik}$,

(iv) $\{U_i^j\}_{j\in\mathbf{N}}$ is the open neighbourhood base of x_i.

Let $V_n=\bigcup_{i\in\Gamma} U_i^n$, $n=1, 2, \ldots$ and let $\varphi_n: X\to\varkappa(H)$ be a map given by

$$\varphi_n(x)=\begin{cases}[x_i, \varepsilon_{in}^{-1}\cdot d(x, x_i)] & \text{if } x\in U_i^n\\ \varkappa & \text{otherwise}\end{cases}$$

It is evident, that $(V_n, \varphi_n)_{n=1}^{\infty}$ satisfies properties (1)—(6). To prove (7), let A be a closed set in X and let $A\cap F=\emptyset$. Let

$$V'=\bigcup_{i\in\Gamma}\{[x_i, t\}: t<\min(2^{-1}, \varepsilon_{in}^{-1}\cdot d_i)\},$$

where $d_i=2^{-1}\cdot d(A, x_i)>0$. Since $\overline{\varphi_n(A)}\cap\overline{V}'=\emptyset$ and $V'\supset F$ thus $\overline{\varphi_n(A)}\cap\overline{\varphi_n(F)}=\emptyset$. Q.e.d.

LEMMA 2. *If F is a compact neighbourhood retract of a metrizable space X, then there is an upraising system of F for X.*

PROOF. Let $r: V\to F$ be a retraction, where V is an open neighbourhood of F and let d be any metric on X inducing the topology of X. Since F is compact, thus $d(F, X\setminus V)>1/N$, where $N\in\mathbf{N}$. Let

$$V_n=\{x: d(x, F)<1/n\}$$

if $n>N$. Let $\varphi_n: X\to\varkappa(F)$ be a continuous map if $n>N$ given by

$$\varphi_n(x)=\begin{cases}[r(x), n\cdot d(x, F)] & \text{if } x\in V_n\\ \varkappa & \text{otherwise.}\end{cases}$$

$(N_{n+N}, \varphi_{n+N})_{n=1}^{\infty}$ satisfies properties (1)—(6). Let A be a closed set in X, and suppose that $A\cap F=\emptyset$. Thus $d(A, F)>2/m$ for a suitable constant $m\in\mathbf{N}$. Fix $n>N$ and let

$$U=\left\{[x, t]: x\in F, t<\min\left(\frac{1}{2}, \frac{n}{m}\right)\right\}.$$

Thus $\overline{\varphi_n(F)}\subset\overline{U}$ and $\overline{U}\cap\overline{\varphi_n(A)}=\emptyset$. Q.e.d.

LEMMA 3. *Let X be a regular T_0 space, and let F be a closed subspace of X. Let $(V_n, \varphi_n)_{n=1}^{\infty}$ be an upraising system of F for X. Let $B=\bigcup_{i=1}^{\infty} B_i$ be a σ-discrete open base for X, and let the B_i's be discrete. Let $\tilde{\varrho}$ be a bounded metric on F inducing the relative topology on F. Let ω be any cardinal greater than or equal to* $\max(|B|, \aleph_0)$.

Then there is a family $\{f_n\}_{n=1}^{\infty}$ with the following properties:

(a) *$f_n\colon x \to S(\omega, F, \tilde{\varrho})$ is a continuous map for every $n \in \mathbf{N}$,*

(b) *$\{f_n\}_{n\in\mathbf{N}}$ separates the points and the closed sets,*

(c) *if $n\in\mathbf{N}$, $x\in F$ then $f_n(x)=i_S(x)$.*

PROOF. Since $|B|\leqq|S|$ there is an injection σ from B into S. Fix n, k and let $\hat{B}_{n,k}=\{U\in B_n\colon \overline{U}\cap\overline{V}_k=\varnothing\}$. Fix m and let

$$V_U = \bigcup\{Z\colon Z\in B_m,\ \overline{Z}\subset U\}.$$

Since B_m is discrete thus $\overline{V}_U=\bigcup\{\overline{Z}\colon Z\in B_m, \overline{Z}\subset U\}$ and $\overline{V}_U\subset U$. It is well known, that if a regular T_0 space has a σ-discrete base, then the space is normal. Let f_U separate $\overline{V}_U$ and $X\setminus U$ so that $f_U(\overline{V}_U)\subset\{1\}$. Let $C=\bigcup_{U\in\hat{B}_{nk}}\overline{U}$ and let $D=X\setminus\bigcup_{U\in\hat{B}_{nk}} U$. Let α, β, γ be continuous maps from C, $D\setminus V_k$ and $\overline{V}_k$ respectively into $S(\omega, F, \tilde{\varrho})$ given by

$$\alpha(x) = \big(f_U(x), \sigma(U)\big) \quad\text{if}\quad x\in\overline{U},\ \beta(x) = [\varkappa] = [S_C],\ \gamma(x) = j_\varkappa(\varphi_k(x)).$$

Since α, β and γ are compatible in X, there is a continuous map δ from X into $S(\omega, F, \tilde{\varrho})$ such that $\delta|C=\alpha$, $\delta|D\setminus V_k=\beta$ and $\delta|\overline{V}_k=\gamma$. Let $h_{nmk}=\delta$.

If $x\in F$ then $h_{nmk}(x)=i_S(x)$ and thus (c) is true. In order to prove (b) let $G\subset X$ be a closed set, and $x\notin G$. First let $x\in F$. Since φ_n is continuous and $\varkappa(F)$ is a metric space, there is an open neighbourhood V of $F\cap G$ contained in V_k and $\overline{\varphi_k(V)}\not\ni i_\varkappa(x)=\varphi_k(x)$. Applying the property (7) of upraising system for $A=G\setminus V$, $\overline{\varphi_k(x)}\cap\overline{\varphi_k(A)}=\varnothing$. Thus:

$$\{h_{nmk}(x)\}\cap\overline{h_{nmk}(G)} \subset \big(\overline{h_{nmk}(A)}\cup\overline{h_{nmk}(V)}\big)\cap\{h_{nmk}(x)\} = \varnothing.$$

Now let $x\in X\setminus F$. By (3) there is a K such that $\overline{V}_k\ni x$. Since $\overline{V}_k\cup G$ is closed in X, thus there is a $U\in B$ such that $x\in U$, $\overline{U}\cap(\overline{V}_k\cup G)=\varnothing$. Suppose $U\in B_N$. Then there is an L such that $x\in\overline{Z}\subset U$ for a $Z\in B_L$. Thus h_{NLK} separates the sets $\{x\}$ and G. Q.e.d.

The main theorem

It is well known that if the family $\{f_n\colon x\to Y_n\}_{n=1}^{\infty}$ consisting of continuous maps separates the points and the closed sets then the map $f\colon x\to\prod_{i=1}^{\infty} Y_n$ given by $f(x)_n=f_n(x)$ is an embedding.

It is well known too, that if $(x_n, \varrho_n)_{n=1}^{\infty}$ is a sequence of uniformly bounded metric spaces then

$$\varrho(x, y) = \sum_{n=1}^{\infty} \frac{\varrho_n(x_n, y_n)}{2^n}$$

is a metric on $\prod_{i=1}^{\infty} X_n$ inducing the product topology of x_n's.

THEOREM 2. *Let $(F, \tilde{\varrho})$ be a bounded metric space with weight ω, where ω is any cardinal greater than or equal to $\aleph_0$. Let F be* (a) *a discrete or* (b) *a compact ANR. Then $\sum = \prod_{i=1}^{\infty} S(\omega, F, \tilde{\varrho})$ is a universal ω-embedding space modulo F.*

PROOF. Evidently $\varrho_\Sigma(x, y) = \sum_{n=1}^{\infty} 2^{-n} \cdot \varrho_S(x_n, y_n)$ exists, since if F is bounded then so is $S(\omega, F, \tilde{\varrho})$.

Let the isometric embedding $i\colon F \to \Sigma$ be given by $i(x)_n = i_S(x)$. It is evident, that the weight of Σ is ω. We prove that (Σ, ϱ_Σ) with map i is a universal ω-embedding space modulo F.

In order to prove this, let X be a metric space with ω-weight containing F as a closed subspace. In the case of (a) by Lemma 1, and in the case of (b) by Lemma 2 there is an upraising system of F for X. Since X is metrizable and $\omega \geqq \aleph_0$, there is a σ-discrete open base with cardinal ω. Applying Lemma 3 and the first remark in this section there is an $f\colon x \to \Sigma$ embedding. If $x \in F$ then evidently $f(x) = i(x)$. Q.e.d.

A consequence

Let F be a not closed subspace of a metrizable space X, then there is a metric on F, which induces the relative topology on F, and this metric cannot be extended to a metric on the whole space inducing the topology on X.

PROOF. Let (x_n) be a sequence convergent to a point of $X \setminus F$ and $(x_n) \subset F$. Let $H = \{Y_n\colon n \in \mathbf{N}\}$ and let d be the following metric on H:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise.} \end{cases}$$

Since (H, d) is bounded and H is discrete, closed in F, thus there is a universal embedding space modulo H. The embedding induces an extension of d to the whole space F. Now the new metric $\tilde{\varrho}$ cannot be extended to a metric of X, inducing the topology of X, since $x_n \to x$ but $\tilde{\varrho}(x_n, x_m)$ does not satisfy the Cauchy's property. Q.e.d.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF MATHEMATICS
1088 BUDAPEST, MÚZEUM KRT. 6—8.

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[1] G. Szegő, *Orthogonal polynomials*, AMS Coll. Publ. Vol. XXIII (Providence, 1939).
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РЕЗУЛЬТАНТНАЯ МАТРИЦА И ЕЕ ОБОБЩЕНИЯ

II. КОНТИНУАЛЬНЫЙ АНАЛОГ РЕЗУЛЬТАНТНОГО ОПЕРАТОРА

И. Ц. ГОХБЕРГ и Г. ХАЙНИГ (Карл-Маркс-Штадт)

Пусть $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$ $(\lambda \in \mathbf{C}^1)$ — целые функции вида

$$\mathcal{A}(\lambda) = a_0 + \int_0^\tau a(t)e^{i\lambda t}dt, \qquad \mathcal{B}(\lambda) = b_0 + \int_{-\tau}^0 b(t)e^{i\lambda t}dt \tag{0.1}$$

где $a_0, b_0 \in \mathbf{C}^1$; $a(t) \in L_1(0, \tau)$, $b(t) \in L_1(-\tau, 0)$ и τ — некоторое положительное число.

Этой паре функций поставим в соответствие линейный ограниченный оператор $R_0(\mathcal{A}, \mathcal{B})$, действующий в простанстве $L_1(-\tau, \tau)$ по формуле

$$(R_0(\mathcal{A}, \mathcal{B})f)(t) = \begin{cases} f(t) + \int_{-\tau}^{\tau} a(t-s)f(s)ds & (0 \leqq t \leqq \tau) \\ f(t) + \int_{-\tau}^{\tau} b(t-s)f(s)ds & (-\tau \leqq t < 0) \end{cases}$$

где полагается, что $a(t) = 0$ при $t \notin [0, \tau]$ и $b(t) = 0$ при $t \notin [-\tau, 0]$. Этот оператор естественно считать континуальным аналогом результантного оператора для двух полиномов (см. [1]). Оператор $R_0(\mathcal{A}, \mathcal{B})$ будем называть результантным оператором функций $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$.

Имеет место следующая теорема.

Теорема 0.1. *Пусть $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$ — целые функции вида* (0.1), $\lambda_1, \lambda_2, \dots, \lambda_l$ *— полный набор различных общих нулей функций $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$ и k_j — кратность общего нуля λ_j. Тогда система функций.*

$$\psi_{jk}(t) = t^k e^{-i\lambda_j t} \qquad (k = 0, 1, \dots, k_j - 1; \quad j = 1, 2, \dots, l)$$

образует базис подпространства Ker $R_0(\mathcal{A}, \mathcal{B})$. *В частности,*

$$\sum_{j=1}^{l} k_j = \dim \operatorname{Ker} R_0(\mathcal{A}, \mathcal{B}).$$

Эта теорема является континуальным аналогом теоремы 0.1 из [1].

Настоящая статья является продолжением статьи авторов [1], в ней устанавливаются континуальные аналоги и других теорем из [1].

Отметим, что в континуальном матричном случае определение результантного оператора усложняется. По аналогии с дискретным случаем континуальный результантный оператор для матрицы-функции действует из одного пространства вектор-функций в другое с более широким носителем вектор-функций.

Между дискретным и континуальным результантными операторами для матриц-функций имеется существенное различие. Оно состоит в том, что выбор пространств, в которых действует результантный оператор в континуальном случае, не зависит от самих матриц и их порядков.

Статья состоит из шести параграфов; в первом формулируется основная теорема. Она доказывается в третьем параграфе. В четвертом параграфе доказывается теорема 0.1. Второй параграф носит вспомогательный характер. В последних двух параграфах приводятся некоторые приложения основной теоремы.

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§1. Формулировка основной теоремы

1. Пусть α и $\beta\,(-\infty < \alpha < \beta < \infty)$ — пара вещественных чисел и d — натуральное число. Через $L_1^d(\alpha, \beta)$ обозначается банахово пространство вектор-функций $f(t) = (f_j(t))_{j=1}^d$ с компонентами из $L_1(\alpha, \beta)$. Аналогично $L_1^{d\times d}(\alpha, \beta)$ будет обозначать пространство d-мерных матриц-функций $a(t) = \| a_{jk}(t) \|_{j,k=1}^d$ с элементами из $L_1(\alpha, \beta)$.

Условимся через $F^{d\times d}(\alpha, \beta)$ обозначать пространство всех матриц-функций вида

$$\mathcal{A}(\lambda) = a_0 + \int_\alpha^\beta a(t)e^{i\lambda t}dt \tag{1.1}$$

где $a_0 \in L(\mathbf{C}^d)$[1] и $a(t) \in L_1^{d\times d}(\alpha, \beta)$, а через $F_0^{d\times d}(\alpha, \beta)$ — подмножество $F^{d\times d}(\alpha, \beta)$ с обратимыми первыми слагаемыми.

Пространство $F^{d\times d}(\alpha, \beta)$ состоит из целых матриц-функций.

Пусть τ — некоторое положительное число. Каждой паре матриц-функций вида

$$\mathcal{A}(\lambda) = a_0 + \int_0^\tau a(t)e^{i\lambda t}dt \in F^{d\times d}(0, \tau)$$

и

$$\mathcal{B}(\lambda) = a_0 + \int_{-\tau}^0 b(t)e^{i\lambda t}dt \in F^{d\times d}(-\tau, 0)$$

и каждому числу $\varepsilon > 0$ поставим в соответствие оператор $R_\varepsilon(\mathcal{A}, \mathcal{B})$, действующий из пространства $L_1^d(-\tau, \tau+\varepsilon)$ в пространство $L_1^d(-\tau-\varepsilon, \tau+\varepsilon)$ по правилу

$$(R_\varepsilon(\mathcal{A}, \mathcal{B})\phi)(t) = \begin{cases} \phi(t) + \int_{-\tau}^{\tau+\varepsilon} a(t-s)\phi(s)ds & (0 \leqq t \leqq \tau+\varepsilon)^2 \\ \phi(t+\varepsilon) + \int_{-\tau}^{\tau+\varepsilon} b(t+\varepsilon-s)\phi(s)\,ds & (-\tau-\varepsilon \leqq t < 0). \end{cases}$$

[1] Через $L(\mathbf{C}^d)$ обозначается пространство квадратных матриц d-го порядка.

[2] Здесь и в дальнейшем полагается $a(t) = 0$ при $t \notin [0, \tau]$ и $b(t) = 0$ при $t \notin [-\tau, 0]$.

По аналогии с дискретным случаем оператор $R_\varepsilon(\mathcal{A}, \mathcal{B})$ будем называть результантным оператором матриц-функций $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$. Оператор $R_0(\mathcal{A}, \mathcal{B})$, действующий в пространстве $L_1^d(-\tau, \tau)$, будем называть классическим результантным оператором.

Легко видеть, что результантный оператор $R_\varepsilon(\mathcal{A}, \mathcal{B})$ тесно связан с оператором $\tilde{R}_\varepsilon(\mathcal{A}, \mathcal{B})$, действующим из пространства $L_1^d(-\tau, \tau+\varepsilon)$ в пространство $L_1^d(-\tau, \varepsilon) \dot{+} L_1^d(0, \tau+\varepsilon)$ по формуле

$$(\tilde{R}_\varepsilon(\mathcal{A}, \mathcal{B})\phi)(t) = \begin{cases} \phi(t) + \int\limits_{-\tau}^{\tau+\varepsilon} a(t-s)\phi(s)ds & (0 \leqq t \leqq \tau+\varepsilon) \\ \phi(t) + \int\limits_{-\tau}^{\tau+\varepsilon} b(t-s)\phi(s)ds & (-\tau \leqq t \leqq \varepsilon). \end{cases}$$

Действительно, если $\tilde{R}_\varepsilon(\mathcal{A}, \mathcal{B})\phi = (f_1, f_2)$, где $f_1 \in L_1(-\tau, \varepsilon)$ и $f_2 \in L_1^d(0, \tau+\varepsilon)$, то

$$(R_\varepsilon(\mathcal{A}, \mathcal{B})\phi)(t) = \begin{cases} f_1(t+\varepsilon) & (-\tau-\varepsilon \leqq t < 0) \\ f_2(t) & (0 \leqq t \leqq \tau+\varepsilon). \end{cases}$$

Очевидно, имеет место равенство

$$\operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B}) = \operatorname{Ker} \tilde{R}_\varepsilon(\mathcal{A}, \mathcal{B}).$$

2. Напомним некоторые определения и обозначения из [1].[3] Пусть $\mathcal{A}(\lambda)$ — целая матрица-функция. Число $\lambda_0 \in \mathbf{C}^1$ называется характеристическим числом матрицы-функции $\mathcal{A}(\lambda)$, если $\det \mathcal{A}(\lambda_0) = 0$. Вектор $\phi_0 \in \mathbf{C}^d$ называется собственным вектором, отвечающим характеристическому числу λ_0, если $\mathcal{A}(\lambda_0)\phi_0 = 0$. Последовательность векторов $\phi_0, \phi_1, \ldots, \phi_r$ называется цепочкой из собственного и присоединенных векторов длины $r+1$, если имеют место равенства

$$\mathcal{A}(\lambda_0)\phi_k + \frac{1}{1!}\left(\frac{d}{d\lambda}\mathcal{A}\right)(\lambda_0)\phi_{k-1} + \ldots + \frac{1}{k!}\left(\frac{d^k}{d\lambda^k}\mathcal{A}\right)(\lambda_0)\phi_0 = 0$$

при $k = 0, 1, \ldots, r$.

Пусть λ_0 — характеристическое число матрицы-функции $\mathcal{A}(\lambda)$. Без труда доказывается, что в ядре оператора $\mathcal{A}(\lambda_0)$ можно построить базис $\phi_{10}, \phi_{20}, \ldots, \phi_{r0}$ со следующим свойством: для каждого вектора существует цепочка из собственного и присоединенных векторов $\phi_{j0}, \phi_{j1}, \ldots, \phi_{j,k_j-1}$, где $k_1 \geqq k_2 \geqq \ldots \geqq k_r$ и $\sum_j k_j$ равно кратности нуля функции $\det \mathcal{A}(\lambda)$ в точке λ_0. Числа k_j $(j = 1, 2, \ldots, r)$ называются частными кратностями характеристического числа λ_0, а система $\phi_{j0}, \phi_{j1}, \ldots, \phi_{j,k_j-1}$ $(j = 1, 2, \ldots, r)$ — канонической системой собственных и присоединенных векторов матрицы-функции $\mathcal{A}(\lambda)$, отвечающих характеристическому числу λ_0.

Рассмотрим две целых матрицы-функции $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$. Пусть λ_0 — общее характеристическое число матриц-функций $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$ и

$$\mathfrak{R} = \operatorname{Ker} \mathcal{A}(\lambda_0) \cap \operatorname{Ker} \mathcal{B}(\lambda_0).$$

[3] См. также [2] и [3].

Пусть $\phi_0, \phi_1, \ldots, \phi_r$ — цепочка собственного и присоединенных векторов одновременно для пучков $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$, соответствующая характеристическому числу λ_0. Число $r+1$ называется длиной этой цепочки. Наибольшая длина такой цепочки, начинающейся вектором ϕ_0, назовем рангом общего собственного вектора ϕ_0 и обозначается через $\operatorname{rang}_{\lambda_0}\phi_0$.

В подпространстве $\mathfrak{N}$ выберем базис $\phi_{10}, \phi_{20}, \ldots, \phi_{l0}$, ранги k_j векторов которого обладают следующими свойствами: k_1 является максимальным из чисел $\operatorname{rang}_{\lambda_0}\phi$ $(\phi \in \mathfrak{N})$, а k_j $(j = 2, 3, \ldots, l)$ является максимальным из чисел $\operatorname{rang}_{\lambda_0}\phi$ для всех векторов ϕ прямого дополнения к $\operatorname{lin}\{\phi_{10}, \phi_{20}, \ldots, \phi_{j-1,0})\}$ в $\mathfrak{N}$, содержащего ϕ_{j0}.

Легко видеть, что число $\operatorname{rang}_{\lambda_0}\phi_0$ для любого вектора $\phi_0 \in \mathfrak{N}$ равно одному из чисел k_j $(j = 1, 2, \ldots, l)$. Следовательно, числа k_j $(j = 1, 2, \ldots, l)$ определяются однозначно пучками $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$. Обозначим через $\phi_{j1}, \phi_{j2}, \ldots, \phi_{j,k_j-1}$ соответствующую общую для $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$ цепочку присоединенных векторов к собственному вектору ϕ_{j0} $(j = 1, 2, \ldots, l)$.

Систему

$$\phi_{j0}, \phi_{j1}, \ldots, \phi_{j,k_j-1} \qquad (j = 1, 2, \ldots, l)$$

назовем канонической системой общих собственных и присоединенных векторов матриц-функций $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$, отвечающих характеристическому числу λ_0, а число

$$\nu(\mathcal{A}, \mathcal{B}, \lambda_0) \overset{\text{def}}{=\!=} \sum_{j=1}^{l} k_j$$

назовем общей кратностью характеристического числа λ_0 матриц-функций $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$.

Условимся еще в следующем обозначении

$$\nu(\mathcal{A}, \mathcal{B}) \overset{\text{def}}{=\!=} \sum_{l} \nu(\mathcal{A}, \mathcal{B}\ \lambda_l)$$

где λ_l пробегает все общие характеристические числа матриц-функций $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$.

Отметим, что для любой пары матриц-функций $\mathcal{A}(\lambda) \in F_0^{d\times d}(0, \tau)$ и $\mathcal{B}(\lambda) \in$ $\in F_0^{d\times d}(-\tau, 0)$ число $\nu(\mathcal{A}, \mathcal{B})$ конечно. В самом деле, легко видеть, что матрица-функция $\mathcal{A}(\lambda)$ ограничена в верхней полуплоскости и $\lim\limits_{\operatorname{Im}\lambda \geqq 0,\, \lambda\to\infty} \mathcal{A}(\lambda) = a_0$, а матица-функция $\mathcal{B}(\lambda)$ ограничена в нижней полуплоскости и $\lim\limits_{\operatorname{Im}\lambda \leqq 0,\, \lambda\to\infty} \mathcal{B}(\lambda) = b_0$. Отсюда вытекает, что функция $\det \mathcal{A}(\lambda)$ имеет не больше конечного числа нулей в верхней полуплоскости, а функция $\det \mathcal{B}(\lambda)$ имеет не больше конечного числа нулей в нижней полуплоскости.

3. Основной в этой статье является следующая

Теорема 1.1. *Пусть $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$ — две матрицы-функции из $F_0^{d\times d}(0, \tau)$ и $F_0^{d\times d}(-\tau, 0)$ соответственно, $\Lambda = \{\lambda_1\, \lambda_2, \ldots, \lambda_l\}$ — полный набор различных общих характериситческих чисел матриц-функций $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$ и*

$$\mathfrak{F}_l = \{\phi_{jk,l} : k = 0, 1, \ldots, k_{jl} - 1; \quad j = 1, 2, \ldots, j_l\} \qquad (l = 1, 2, \ldots, l_0)$$

каноническая система общих собственных и присоединенных векторов, отвечающих характеристическому числу λ_l.

Тогда для любого $\varepsilon > 0$ *система функций*

$$(1.4)\qquad \phi_{jk,l}(t) = e^{-i\lambda_l t}\left(\frac{(it)^k}{k!}\phi_{j0,l} + \frac{(it)^{k-1}}{(k-1)!}\phi_{j1,l} + \ldots + \phi_{jk,l}\right) \quad (-\tau \leqq t \leqq \tau+\varepsilon)$$

где $k = 0, 1, \ldots, k_{jl}-1$; $j = 1, 2, \ldots, j_l$; $l = 1, 2, \ldots, l_0$, *образует базис результантного оператора* $R_\varepsilon(\mathcal{A}, \mathcal{B})$. *В частности, имеет место равенство*

$$(1.5)\qquad \nu(\mathcal{A}, \mathcal{B}) = \dim \operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B}).$$

4. Оказывается, что отыскание ядра результантного оператора можно свести к отысканию ядер двух операторов, действующих (в отличие от оператора $R_\varepsilon(\mathcal{A}, \mathcal{B})$) в одном пространстве. Введем операторы $R'_\varepsilon(\mathcal{A}, \mathcal{B})$ и $R''_\varepsilon(\mathcal{A}, \mathcal{B})$, действующие в пространстве $L_1^d(-\tau, \tau+\varepsilon)$ по формулам

$$(1.6)\qquad (R'_\varepsilon(\mathcal{A}, \mathcal{B})\phi)(t) = \begin{cases} a_0\phi(t) + \int\limits_{-\tau}^{\tau+\varepsilon} a(t-s)\phi(s)ds & (0 \leqq t \leqq \tau+\varepsilon) \\ b_0\phi(t) + \int\limits_{-\tau}^{\tau+\varepsilon} b(t-s)\phi(s)ds & (-\tau \leqq t < 0) \end{cases}$$

и

$$(R''_\varepsilon(\mathcal{A}, \mathcal{B})\phi)(t) = \begin{cases} a_0\phi(t) + \int\limits_{-\tau}^{\tau+\varepsilon} a(t-s)\phi(s)ds & (\varepsilon \leqq t \leqq \tau+\varepsilon) \\ b_0\phi(t) + \int\limits_{-\tau}^{\tau+\varepsilon} b(t-s)\phi(s)ds & (-\tau \leqq t < \varepsilon). \end{cases}$$

Без труда можно убедиться в том, что имеет место равенство

$$(1.7)\qquad \operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B}) = \operatorname{Ker} R'_\varepsilon(\mathcal{A}, \mathcal{B}) \cap \operatorname{Ker} R''_\varepsilon(\mathcal{A}, \mathcal{B}).$$

Отметим еще, что роль результантного оператора может играть также оператор $\hat{R}_\varepsilon(\mathcal{A}, \mathcal{B})$, действующий из пространства $L_1^d(-\tau-\varepsilon, \tau)$ в пространство $L_1^d(-\tau-\varepsilon, \tau+\varepsilon)$ по правилу

$$(\hat{R}_\varepsilon(\mathcal{A}, \mathcal{B})\phi)\,(t) = \begin{cases} a_0\phi(t-\varepsilon) + \int\limits_{-\tau-\varepsilon}^{\tau} a(t-\varepsilon-s)\phi(s)ds & (0 \leqq t \leqq \tau+\varepsilon) \\ b_0\phi(t) + \int\limits_{-\tau-\varepsilon}^{\tau} b(t-s)\phi(s)ds & (-\tau-\varepsilon \leqq t < 0). \end{cases}$$

По существу, оператор $\hat{R}_\varepsilon(\mathcal{A}, \mathcal{B})$ совпадает с оператором $R_\varepsilon(\mathcal{A}, \mathcal{B})$.

5. Приведем простой пример, показывающий, что теорема 1.1 в матричном случае перестает быть верной при $\varepsilon = 0$, т.е. теорема 0.1 не допускает непосредственное матричное обобщение.

Пусть

$$a(t) = \begin{Vmatrix} 1 & 1 \\ 1 & -1 \end{Vmatrix} \quad (0 \leqq t \leqq 1); \qquad a(t) = 0 \qquad (t \notin [0, 1])$$

$$b(t) = \begin{Vmatrix} 1 & 1 \\ -1 & -1 \end{Vmatrix} \quad (-1 \leqq t \leqq 0); \qquad b(t) = 0 \qquad (t \notin [-1, 0])$$

и $a_0 = b_0 = I$, где I — единичная матрица.

Тогда

$$\mathcal{A}(\lambda) = a_0 + \int_0^1 a(t) e^{i\lambda t} dt = \frac{1}{i\lambda} \begin{Vmatrix} i\lambda + e^{i\lambda} - 1 & e^{i\lambda} - 1 \\ e^{i\lambda} - 1 & i\lambda - e^{i\lambda} + 1 \end{Vmatrix}$$

и

$$\mathcal{B}(\lambda) = b_0 + \int_{-1}^0 b(t) e^{i\lambda t} dt = \frac{1}{i\lambda} \begin{Vmatrix} i\lambda + 1 - e^{-i\lambda} & 1 - e^{-i\lambda} \\ e^{i\lambda} - 1 & i\lambda - 1 + e^{-i\lambda} \end{Vmatrix}$$

при $\lambda \neq 0$ и

$$\mathcal{A}(0) = \begin{Vmatrix} 2 & 1 \\ 1 & 0 \end{Vmatrix}, \qquad \mathcal{B}(0) = \begin{Vmatrix} 2 & 1 \\ -1 & 0 \end{Vmatrix}.$$

Имеем

$$\det \mathcal{A}(\lambda) = 1 - 2 \left(\frac{e^{i\lambda} - 1}{i\lambda} \right)^2 \qquad (\lambda \neq 0); \qquad \det \mathcal{A}(0) = -1.$$

и

$$\det \mathcal{B}(\lambda) \equiv 1.$$

Таким образом, $\nu(\mathcal{A}, \mathcal{B}) = 0$.

Оператор $R_0(\mathcal{A}, \mathcal{B})$ определяется равенством

$$(R_0(\mathcal{A}, \mathcal{B})\phi)(t) = \phi(t) + \begin{cases} \begin{Vmatrix} 1 & 1 \\ 1 & -1 \end{Vmatrix} \int_0^1 \phi(t - s) ds & (0 \leqq t \leqq 1) \\ \begin{Vmatrix} 1 & 1 \\ -1 & -1 \end{Vmatrix} \int_{-1}^0 \phi(t - s) ds & (-1 \leqq t < 0). \end{cases}$$

Без труда проверяется, что для вектор-функции

$$\phi(t) = \begin{Vmatrix} 1 + t - \theta(t) \\ -1 - t \end{Vmatrix}$$

где

$$\theta(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t \geqq 0) \end{cases}$$

выполнено равенство $R_0(\mathcal{A}, \mathcal{B})\phi = 0$. Следовательно, $1 \leqq \dim \operatorname{Ker} R_0(\mathcal{A}, \mathcal{B})$ и $\nu(\mathcal{A}, \mathcal{B}) \neq \dim \operatorname{Ker} R_0(\mathcal{A}, \mathcal{B})$.

§2. Одна лемма

Предложение, устанавливаемое в этом параграфе, используется в § 3 при доказательстве основной теоремы.

Л е м м а 2.1. *Пусть $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$ — две целых матрицы-функции из $F_0^{d\times d}(0,\tau)$ и $F_0^{d\times d}(-\tau, 0)$ соответственно. Тогда для любого $\varepsilon > 0$ ядро результантного оператора $R_\varepsilon(\mathcal{A}, \mathcal{B})$ состоит лишь из абсолютно непрерывных вектор-функций.*

Д о к а з а т е л ь с т в о. Докажем сначала, что ядро оператора $R'_\varepsilon(\mathcal{A}, \mathcal{B})$ состоит лишь из вектор-функций, которые абсолютно непрерывны на интервалах $[-\tau, 0)$ и $[0, \tau+\varepsilon]$. Рассмотрим оператор K'_ε, действующий в пространстве $L_1^d(-\tau, \tau+\varepsilon)$ по правилу

$$(K'_\varepsilon f)(t) = \begin{cases} \int\limits_{-\tau}^{\tau+\varepsilon} b(t-s)\phi(s)ds & (-\tau \leqq t < 0) \\ \int\limits_{-\tau}^{\tau+\varepsilon} a(t-s)\phi(s)ds & (0 \leqq t \leqq \tau+\varepsilon). \end{cases}$$

Очевидно, имеет место равенство

$$R'_\varepsilon(\mathcal{A}, \mathcal{B})f)(t) = (K'_\varepsilon f)(t) + \begin{cases} a_0 f(t) & (0 \leqq t \leqq \tau+\varepsilon) \\ b_0 f(t) & (-\tau \leqq t < 0). \end{cases} \tag{2.1}$$

Легко видеть, что оператор K'_ε является линейным ограниченным оператором в пространстве $L_1^d(-\tau, \tau+\varepsilon)$ и имеет место оценка

$$\| K'_\varepsilon \|_{L_1^d(-\tau,\tau)} \leqq \| a(t) \|_{L_1^d(0,\tau)} + \| b(t) \|_{L_1^d(-\tau,0)}. \tag{2.2}$$

Докажем, что оператор K'_ε является вполне непрерывным в пространстве $L_1^d(-\tau, \tau+\varepsilon)$. В самом деле, функции $a(t)$ и $b(t)$ можно аппроксимировать в норме пространства $L_1^d(-\tau, \tau+\varepsilon)$ с любой степенью точности матрицами-функциями вида

$$\sum_{j=-m}^{m} e^{2\pi ijt/(2\tau+\varepsilon)} a_j. \tag{2.3}$$

Пусть $\tilde{a}(t)$ и $\tilde{b}(t)$ — функции вида (2.3) такие, что $\| \tilde{a}(t) - a(t) \|_{L_1^d(-\tau,\tau)} < \delta$ и $\| \tilde{b}(t) - b(t) \|_{L_1^d(-\tau,\tau)} < \delta$, где $\delta > 0$ — заданное число.

Оператор $\tilde{K}$, определенный равенством

$$(\tilde{K}f)(t) = \begin{cases} \int\limits_{-\tau}^{\tau+\varepsilon} \tilde{a}(t-s)\,\phi(s)ds & (0 \leqq t \leqq \tau+\varepsilon) \\ \int\limits_{-\tau}^{\tau+\varepsilon} \tilde{b}(t-s)\,\phi(s)ds & (-\tau \leqq t < 0) \end{cases}$$

очевидно, конечномерен. Из оценки (2.2) вытекает

$$\| K'_\varepsilon - \tilde{K} \|_{L_1^d(-\tau,\tau+\varepsilon)} \leqq \| \tilde{a}(t) - a(t) \|_{L_1^d(0,\tau)} + \| \tilde{b}(t) - b(t) \|_{L_1^d(-\tau,0)} < 2\delta.$$

Таким образом, оператор K можно аппроксимировать (по норме) с любой степенью точности конечномерными операторами. Следовательно, оператор

K'_ε вполне непрерывен. Из полной непрерывности оператора K'_ε вытекает в силу (2.1) соотношение

$$\text{(2.4)} \qquad \dim \operatorname{Ker} R'_\varepsilon(\mathcal{A}, \mathcal{B}) = \dim \operatorname{Coker} R'_\varepsilon(\mathcal{A}, \mathcal{B}) < \infty.$$

Обозначим через $\hat{W}_0^d$ банахово пространство вектор-функций $\phi(t) \in \in L_1^d(-\tau, \tau+\varepsilon)$, которые абсолютно непрерывны на интервалах $[-\tau, 0)$ и $[0, \tau+\varepsilon]$ и имеют предел $\phi(-0)$ $(= \lim\limits_{t\to -0} \phi(t)) \in \mathbf{C}^1$. Очевидно, пространство $\hat{W}_0^d$ является прямой суммой пространства абсолютно непрерывных на $[-\tau, \tau]$ вектор-функций и пространства вектор-функций вида $c\Theta(t)$, где $c \in \mathbf{C}^d$ и

$$\theta(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t \geqq 0). \end{cases}$$

Норма в пространстве $\hat{W}_0^d$ определяется равенством

$$\| \phi(t) \|_{\hat{W}_0^d} = \| \phi(t) \|_{L_1^d(-\tau, \tau+\varepsilon)} + \| (D\phi)(t) \|_{L_1^d(-\tau, \tau+\varepsilon)},$$

где здесь и в дальнейшем полагается $(D\phi)(t) = \left(\dfrac{d}{dt}\phi\right)(t)$ при $t \neq 0$ (почти всюду) и $(D\phi)(0) = \lim\limits_{t\to+0} (D\phi)(t)$ (если этот предел существует).

Пусть $\hat{K}'_\varepsilon$ — сужение оператора K'_ε на пространство $\hat{W}_0^d$. Оператор $\hat{K}'_\varepsilon$ является вполне непревывным оператором в $\hat{W}_0^d$. В самом деле, для вектор-функции $f(t) \in \hat{W}_0^d$ имеет место

$$(K'_\varepsilon f)(t) = \begin{cases} f(t) + \int\limits_0^t a(r) f(t-r)\,dr + \int\limits_t^\tau a(r) f(t-r)\,dr & (0 \leqq t \leqq \tau + \varepsilon) \\ f(t) + \int\limits_{-\tau}^t b(r) f(t-r)\,dr + \int\limits_t^0 b(r) f(t-r)\,dr & (-\tau \leqq t < 0). \end{cases}$$

Очевидно, правая сторона последнего равенства почти всюду дифференцируема и

$$\left(\frac{d}{dt}\hat{K}'_\varepsilon f\right)(t) = \begin{cases} (Df)(t) + \int\limits_0^t a(r) Df(t-r)\,dr + \int\limits_t^\tau a(r) Df(t-r) + \\ \qquad + a(t)(f(0) - f(-0)) & (0 \leqq t \leqq \tau) \\ (Df)(t) + \int\limits_{-\tau}^t b(r) Df(t-r)\,dr + \int\limits_t^0 b(r) Df(t-r) + \\ \qquad + b(t)(f(0) - f(-0)) & (-\tau \leqq t < 0). \end{cases}$$

Так как $Df(t) \in L_1^d(-\tau, \tau+\varepsilon)$ и оператор K'_ε переводит пространство $L_1^d(-\tau, \tau+\varepsilon)$ в себя, то из последнего равенства вытекает, что $K'_\varepsilon f \in \hat{W}_0^d$ и имеет место оценка

$$\| \hat{K}'_\varepsilon f \|_{\hat{W}_0^d} \leqq \rho(\| a(t) \|_{L_1^d} + \| b(t) \|_{L_1^d}) \| f \|_{\hat{W}_0^d},$$

где ρ — некотороя константа, не зависящая от $f(t)$, $a(t)$ и $b(t)$. С помощью этой оценки полная непрерывность оператора $\hat{K}'_\varepsilon$ в пространстве $\hat{W}_0^d$ доказывается так же, как полная непрерывность оператора K'_ε в $L_1^d(-\tau, \tau+\varepsilon)$.

В силу полной непрерывности оператора $\hat{K}_\varepsilon$ и равенства (2.1) получаем

$$\dim \operatorname{Ker} \hat{R}'_\varepsilon = \dim \operatorname{Coker} \hat{R}'_\varepsilon. \tag{2.5}$$

где R'_ε — сужение оператора $R'_\varepsilon(\mathcal{A}, \mathcal{B})$ на пространство $\hat{W}^d_0$. Так как

$$\dim \operatorname{Ker} R'_\varepsilon \leqq \dim \operatorname{Ker} R'_\varepsilon(\mathcal{A}, \mathcal{B}) \text{ и } \dim \operatorname{Coker} \hat{R}'_\varepsilon \geqq \dim \operatorname{Coker} R'_\varepsilon(\mathcal{A}, \mathcal{B}),$$

то из (2.4) и (2.5) вытекает, что

$$\dim \operatorname{Ker} \hat{R}'_\varepsilon = \dim \operatorname{Ker} R'_\varepsilon(\mathcal{A}, \mathcal{B}) < \infty.$$

Следовательно, $\operatorname{Ker} R'_\varepsilon = \operatorname{Ker} R'_\varepsilon(\mathcal{A}, \mathcal{B})$ и

$$\operatorname{Ker} R'_\varepsilon(\mathcal{A}, \mathcal{B}) \subset \hat{W}^d_0. \tag{2.6}$$

Рассмотрим теперь оператор K''_ε, действующий в пространстве $L^d_1(-\tau, \tau+\varepsilon)$ по правилу

$$(K''_\varepsilon\phi)(t) = \begin{cases} b_0\phi(t) + \int\limits_{-\tau}^{0} b(s)\phi(t-s)ds & (-\tau \leqq t < \varepsilon) \\ a_0\phi(t) + \int\limits_{0}^{\tau} a(s)\phi(t-s)ds & (\varepsilon \leqq t \leqq \tau + \varepsilon). \end{cases}$$

Тогда имеет место равенство

$$(R''_\varepsilon\phi)(t) = (K''_\varepsilon\phi)(t) + \begin{cases} a_0\phi(t) & (\varepsilon \leqq t \leqq \tau + \varepsilon) \\ b_0\phi(t) & (-\tau \leqq t \leqq \varepsilon). \end{cases}$$

Обозначим через $\hat{W}^d_\varepsilon$ пространство вектор-функций $\phi(t) \in L^d_1(-\tau, \tau + \varepsilon)$, которые абсолютно непрерывны на интервалах $[-\tau, \varepsilon)$ и $[\varepsilon, \tau + \varepsilon]$ и имеют предел $\phi(\varepsilon - 0)$ $(= \lim\limits_{h\to\varepsilon-0} \phi(h)) \in \mathbf{C}^1$. С помощью предыдущих рассуждений можно доказать, что оператор K''_ε отображает пространство $\hat{W}^d_\varepsilon$ в себя и сужение $\hat{K}''_\varepsilon$ оператора K''_ε на пространство $\hat{W}^d_\varepsilon$ является вполне непрерывным оператором в $\hat{W}^d_\varepsilon$ и, следовательно, имеет место включение

$$\operatorname{Ker} R''_\varepsilon(\mathcal{A}, \mathcal{B}) \subset \hat{W}^d_\varepsilon.$$

Отсюда и из соотношения (2.6) согласно равенству (1.7) вытекает

$$\operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B}) \subset \hat{W}^d_\varepsilon \cap \hat{W}^d_0.$$

Так как $\hat{W}^d_\varepsilon \cap \hat{W}^d_0$ равно пространству абсолютно непрерывных на интервале $[-\tau, \tau + \varepsilon]$ вектор-функций, то отсюда непосредственно следует утверждение.

Лемма доказана.

§3. Доказательство основной теоремы

Введем в рассмотрение операторы $R_\varepsilon(\mathcal{A})$ и $R_\varepsilon,(\mathcal{B})$, полагая

$$(R_\varepsilon(\mathcal{A})\phi)(t) = \phi(t) + \int_{-\tau}^{\tau+\varepsilon} a(t-s)\phi(s)ds \qquad (0 \leqq t \leqq \tau + \varepsilon)$$

и

$$(R_\varepsilon(\mathcal{B})\phi)(t) = \phi(t) + \int_{-\tau}^{\tau+\varepsilon} b(t-s)\phi(s)ds \qquad (-\tau \leqq t < \varepsilon).$$

Оператор $R_\varepsilon(\mathcal{A})$ действует из пространства $L_1^d(-\tau, \tau+\varepsilon)$ в пространство $L_1^d(0, \tau+\varepsilon)$, а оператор $R_\varepsilon(\mathcal{B})$ — из $L_1^d(-\tau, \tau+\varepsilon)$ в $L_1^d(-\tau, \varepsilon)$. Очевидно, имеет место равенство

$$\text{Ker}\, R_\varepsilon(\mathcal{A}, \mathcal{B}) = \text{Ker}\, R_\varepsilon(\mathcal{A}) \cap \text{Ker}\, R_\varepsilon(\mathcal{B}). \tag{3.1}$$

Пусть ϕ_{0k} $(k = 0, 1, \dots, k_0 - 1)$ — общая цепочка из собственного и присоединенных векторов, отвечающих характеристическому числу λ_0 матриц-функции $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$. Докажем, что тогда все функции

$$\phi_{0k}(t) = e^{-i\lambda_0 t}\left(\frac{(it)^k}{k!}\phi_{00} + \dots + \frac{it}{1!}\phi_{0,k-1} + \phi_{0k}\right) \tag{3.2}$$

$$(-\tau \leqq t \leqq \tau + \varepsilon; \quad k = 0, 1, \dots, k_0 - 1)$$

принадлежат $\text{Ker}\, R_\varepsilon(\mathcal{A}, \mathcal{B})$. В самом деле, при $k = 0, 1, \dots, k_0 - 1$ и $0 \leqq t \leqq \tau + \varepsilon$

$$(R_\varepsilon(\mathcal{A})\phi_{0k})(t) = e^{-i\lambda_0 t}\sum_{r=0}^{k}\left(\frac{(it)^r}{r!} + \int_0^\tau a(s)e^{i\lambda_0 s}\frac{(t-s)^r}{r!}ds\right)\phi_{0,k-r} =$$

$$= e^{-i\lambda_0 t}\sum_{r=0}^{k}\sum_{p=0}^{r}\frac{i^r t^p}{r!}\left(\delta_{rp} + \int_0^\tau \binom{r}{p} a(s)e^{i\lambda_0 s}(-s)^{r-p}ds\right)\phi_{0,k-r}.$$

Так как

$$\left(\frac{d^p}{d\lambda^p}\mathcal{A}\right)(\lambda) = a_0\delta_{0p} + (-i)^p\int_0^\tau (-s)^p a(s)^p e^{i\lambda s}ds,$$

то

$$R_\varepsilon(\mathcal{A})\phi_{0k}(t) = -e^{-i\lambda_0 t}\sum_{r=0}^{k}\sum_{p=0}^{r}\frac{(-i)^p}{p!(r-p)!}t^p\left(\frac{d^{r-p}}{d\lambda^{r-p}}\mathcal{A}\right)(\lambda_0)\phi_{0,k-r} = \tag{3.3}$$

$$= -e^{-i\lambda_0 t}\sum_{p=0}^{k}\frac{(it)^p}{p!}\sum_{r=p}^{k}\frac{1}{(r-p)!}\left(\frac{d^{r-p}}{d\lambda^{r-p}}\mathcal{A}\right)(\lambda_0)\phi_{0,k-r}.$$

Отсюда в силу определения цепочки собственного и присоединенных векторов вытекает, что $R_\varepsilon(\mathcal{A})\phi_{0k}(t) = 0$. Аналогично доказывается равенство $R_\varepsilon(\mathcal{B})\phi_{0k}(t) =$

$= 0$. Следовательно, $\phi_{0k}(t) \in \operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B})$. Из доказанного, в частности, вытекает соотношение $\dim \operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B}) \geqq \nu(\mathcal{A}, \mathcal{B})$.

Предположим теперь, что вектор-функция $\phi(t)$ принадлежит ядру оператора $R_\varepsilon(\mathcal{A}, \mathcal{B})$. Тогда

$$\phi(t) + \int_0^\tau a(s)\phi(t-s)ds = 0 \qquad (0 \leqq t \leqq \tau + \varepsilon) \tag{3.4}$$

и

$$\phi(t) + \int_{-\tau}^0 b(s)\phi(t-s)ds = 0 \qquad (-\tau \leqq t < \varepsilon). \tag{3.4}$$

В силу леммы 2.1 вектор-функция $\phi(t)$ абсолютно непрерывна. Отсюда и из равенств (3.4) вытекает, что для любого $r = 0, 1, \ldots$ функция $\frac{d^r}{dt^r}\phi(t)$ принадлежит $\operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B})$. Так как $\dim \operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B}) < \infty$, то отсюда следует, что существуют числа $\alpha_j (j = 1, 2, \ldots, m_0)$ такие, что

$$\sum_{j=0}^{m_0} \alpha_j \frac{d^j}{dt^j}\phi(t) = 0 \qquad (-\tau \leqq t \leqq \tau + \varepsilon).$$

Следовательно, вектор-функция $\phi(t)$ имеет вид

$$\phi(t) = \sum_{j=1}^{q} p_j(t) e^{-i\lambda_j t},$$

где $p_j(t)$ — полиномы с векторными коэффициентами.

Покажем, что все слагаемые $p_j(t)e^{-i\lambda_j t} (j = 1, 2, \ldots, l)$ также принадлежат ядру оператора $R_\varepsilon(\mathcal{A}, \mathcal{B})$. Если $\phi(t) \in \operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B})$, то, в частности,

$$0 = \phi(t) + \int_0^\tau a(s)\phi(t-s)ds,$$

при $0 \leqq t \leqq \tau + \varepsilon$. Согласно (3.5) получим

$$0 = \sum_{j=1}^{l} \left(p_j(t)e^{-i\lambda_j t} + \int_0^\tau a(s)p_j(t-s)e^{-i\lambda_j(t-s)}ds\right) = \tag{3.6}$$

$$= \sum_{j=1}^{l} e^{-i\lambda_j t}\left(p_j(t) + \int_0^\tau a(s)p_j(t-s)e^{-i\lambda_j s}ds\right).$$

Легко убедиться в том, что вектор-функция

$$q_j(t) = p_j(t) + \int_0^\tau a(s)p_j(t-s)e^{i\lambda_j s}ds$$

представляет собой полином с векторными коэффициентами.

Как известно, система скалярных функций вида $e^{\mu_j t} r_j(t)$ $(j = 1, 2, \ldots, l)$ где $r_j(t)$ — полиномы и μ_j — различные между собой комплексные числа, линейно независима. Тем более система вектор-функций $q_j(t)e^{-i\lambda_j t}$ линейно независима. Отсюда и из (3.5) вытекает, что

$$0 = e^{-i\lambda_j t} q_j(t) = e^{-i\lambda_j t} p_j(t) + \int_0^\tau a(s) p_j(t-s) e^{-i\lambda_j (t-s)} ds.$$

Последнее равенство означает, что $e^{-i\lambda_j t} p_j(t) \in \operatorname{Ker} R_\varepsilon(\mathcal{A})$. Аналогично доказывается, что $e^{-i\lambda_j t} p_j(t) \in \operatorname{Ker} R_\varepsilon(\mathcal{B})$. Согласно равенству (3.1) отсюда вытекает, что

$$e^{-i\lambda_j t} \phi_j(t) \in \operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B}) \qquad (j = 1, 2, \ldots, l).$$

Пусть

$$p_j(t) = \frac{(it)^{k_j}}{k_j!} \phi_{j0} + \frac{(it)^{k_j - 1}}{(k_j - 1)!} \phi_{j1} + \ldots + \phi_{jk_j} \qquad (j = 1, 2, \ldots, l).$$

Тогда

$$0 = R_\varepsilon(\mathcal{A}) \phi_j(t) = e^{-i\lambda_j t} \sum_{k=0}^{k_j} \left(\frac{(it)^k}{k!} + \int_0^\tau a(s) e^{i\lambda_j s} \frac{(t-s)^k}{k!} i^k ds \right) \phi_{j, k_j - k}.$$

Учитывая, что

$$\left(\frac{d^k}{d\lambda^k} \mathcal{A} \right)(\lambda) = a_0 \delta_{k0} + \int_0^\tau (is)^k a(s) e^{i\lambda s} ds$$

получаем

$$0 = e^{-i\lambda_j t} \sum_{p=0}^{k_j} \frac{(-it)^p}{p!} \sum_{k=p}^{k_j} \frac{1}{(k-p)!} \left(\frac{d^{k-p}}{d\lambda^{k-p}} \mathcal{A} \right)(\lambda_j) \phi_{j, k_j - k}.$$

Таким образом,

$$\sum_{k=p}^{k_j} \frac{1}{(k-p)!} \left(\frac{d^{k-p}}{d\lambda^{k-p}} \mathcal{A} \right)(\lambda_j) \phi_{j, k_j - k} = 0 \qquad (0 \leqq t \leqq \tau + \varepsilon)$$

при $p = 0, 1, \ldots, k_j$ и $j = 1, 2, \ldots, l$. Последнее означает, что для каждого $j = 1, 2, \ldots, l$ векторы $\phi_{j0}, \phi_{j2}, \ldots, \phi_{jk}$ $(k = 0, 1, \ldots, k_j)$ образуют цепочку из собственного и присоединенных векторов, отвечающих характеристическому числу λ_j матрицы-функции $\mathcal{A}(\lambda)$. Аналогично доказывается, что $\phi_{j0}, \phi_{j1}, \ldots, \phi_{jk}$ также представляет собой цепочку из собственного и присоединенных векторов для матрицы-функции $\mathcal{B}(\lambda)$.

Следовательно, доказано, что каждая вектор-функция из $\operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B})$ является линейной комбинацией вектор-функций вида (1.4). В частности, отсюда вытекает равенство (1.5). Теорема доказана.

Из теоремы 1.1. и равенства (1.7) вытекает

Следствие 3.1. *Для того чтобы две целые матрицы-функции* $\mathcal{A}(\lambda) \in F_0^{d\times d}(0, \tau)$ *и* $\mathcal{B}(\lambda) \in F_0^{d\times d}(-\tau, 0)$ *не имели ни одного общего собственного вектора, отвечающего одному и тому же характеристическому числу, необходимо и достаточно, чтобы для некоторого* $\varepsilon > 0$ *оператор* $R'_\varepsilon(\mathcal{A}, \mathcal{B})$ *или оператор* $R''_\varepsilon(\mathcal{A}, \mathcal{B})$ *был обратим в пространстве* $L_1^d(-\tau, \tau+\varepsilon)$.

Следствие 3.2. *При условиях теоремы* 1.1 *имеет место соотношение*

$$\nu(\mathcal{A}, \mathcal{B}) \leqq \dim \operatorname{Ker} R_0(\mathcal{A}, \mathcal{B}).$$

Действительно, это вытекает из очевидного соотношения

$$\operatorname{Ker} R_\varepsilon(\mathcal{A}, \mathcal{B}) \subseteqq \operatorname{Ker} R_0(\mathcal{A}, \mathcal{B}).$$

§4. Скалярный случай

В этом параграфе доказывается теорема 0.1.

1. Нам понадобится следующая лемма.

Лемма 4.1. *Пусть* $a(t) \in L_1(0, \tau)$, $b(t) \in L_1(-\tau, 0)$, $\mathcal{A}(\lambda) = 1 + \int_0^\tau a(s)e^{i\lambda s}ds$ *и* $\mathcal{B}(\lambda) = 1 + \int_{-\tau}^0 b(s)e^{i\lambda s}ds$. *Если система уравнений*

$$(4.1)\qquad \begin{aligned} a(t) - \int_0^\tau a(s)\omega(t-s)ds &= \omega(t) \qquad (0 \leqq t \leqq \tau) \\ b(t) - \int_{-\tau}^0 b(s)\omega(t-s)ds &= \omega(t) \qquad (-\tau \leqq t \leqq 0) \end{aligned}$$

имеет решение $\omega(t) \in L_1(-\tau, \tau)$, *то классический результантный оператор* $R_0(\mathcal{A}, \mathcal{B})$ *обратим.*

Эта лемма установлена в работе [4] (см. также [5], § 3, 5°).

С помощью леммы 4.1 доказывается

Лемма 4.2. *Пусть* $\mathcal{A}(\lambda) \in F_0^{1\times 1}(0, \tau)$ *и* $\mathcal{B}(\lambda) \in F_0^{1\times 1}(-\tau, 0)$. *Тогда ядро классического результантного оператора* $R_0(\mathcal{A}, \mathcal{B})$ *состоит лишь из абсолютно непрерывных функций.*

Доказательство. С помощью рассуждений из доказательства леммы 2.1 можно показать, что каждую функцию $\phi(t) \in \operatorname{Ker} R_0(\mathcal{A}, \mathcal{B})$ можно представаить в виде $\phi(t) = \phi_0(t) - c\Theta(t)$, где $\phi_0(t)$ — абсолютно непрерывная функция, $c \in \mathbf{C}^1$.

Пусть $\mathcal{A}(\lambda) = a_0 + \int_0^\tau a(t)e^{i\lambda t}dt$ и $\mathcal{B}(\lambda) = b_0 + \int_{-\tau}^0 b(t)e^{i\lambda t}dt$. Предположим, что

$$(4.2)\qquad \phi(t) = \phi_0(t) - c\theta(t) \in \operatorname{Ker} R_0(\mathcal{A}, \mathcal{B})$$

и $c \neq 0$. Так как

$$R_0(\mathcal{A}, \mathcal{B})\theta(t) = \begin{cases} a_0 + \int\limits_0^t a(s)ds & (0 \leqq t \leqq \tau) \\ \int\limits_{-\tau}^t b(s)ds & (-\tau \leqq t < 0), \end{cases}$$

то

$$\phi_0(t) + \int\limits_0^\tau a(s)\phi(t-s)ds = (a_0 + \int\limits_0^t a(s)ds)c \qquad (0 \leqq t \leqq \tau)$$

$$\phi_0(t) + \int\limits_{-\tau}^0 b(s)\phi(t-s)ds = (\int\limits_{-\tau}^t b(s)ds)c \qquad (-\tau \leqq t < 0).$$

Дифференцируя последние равенства и полагая $\omega(t) = \frac{1}{c}\frac{d}{dt}\phi(t)$ $(t \neq 0)$ получим

$$a(t) - \int\limits_0^\tau a(s)\omega(t-s)ds = \omega(t) \qquad (0 \leqq t \leqq \tau)$$

$$b(t) - \int\limits_{-\tau}^0 b(s)\omega(t-s)ds = \omega(t) \qquad (-\tau \leqq t < 0).$$

В силу леммы 4.1 отсюда вытекает, что оператор $R_0(\mathcal{A}, \mathcal{B})$ обратим. Это противоречит предположению $\phi(t) \in \operatorname{Ker} R_0(\mathcal{A}, \mathcal{B})$ $(\phi \neq 0)$. Следовательно, в представлении (4.2) будем иметь $c = 0$. Последнее означает, что функция $\phi(t)$ абсолютно непрерывна.

Лемма доказана.

Отметим, что лемма 2.3 перестает быть верной для матричных функций. В этом можно убедиться на примере, приведенном в первом параграфе.

2. Доказательство теоремы 0.1. Для доказатеьлства теоремы 1.2 повторим доказательство теоремы 1.1, причем вместо леммы 2.1 будем пользоваться леммой 2.3. Получим, что ядро оператора $\operatorname{Ker} R_0(\mathcal{A}, \mathcal{B})$ состоит из линейной оболочки функций вида

$$\phi_{jk}(t) = e^{-i\lambda_j t}\left(\frac{(it)^k}{k!}\phi_{j0} + \ldots + \frac{it}{1!}\phi_{j.k-1} + \phi_{jk}\right)$$

$$(k = 0, 1, \ldots, k_j; \quad j = 1, 2, \ldots, l)$$

где ϕ_{jk} — комплексные числа и $\phi_{j0} \neq 0$.

Имеет место равенство

$$\phi_{jk}(t) = \sum_{r=0}^{k} \frac{i^r}{r!}\phi_{jr}\psi_{jr}(t)$$

с другой стороны,

$$\phi_{jk}(t) - \phi_{j,k-1}(t) = \frac{i^k}{k!}\phi_{j0}t^k.$$

Из последних двух равенств вытекает, что линейная оболочка функций $\phi_{jk}(t)$ совпадает с линейной оболочкой функций $\psi_{jk}(t)$.

Теорема доказана.

Следствие 4.1. *Две целых функции $\mathcal{A}(\lambda) \in F_0^{1\times 1}(0, \tau)$ и $\mathcal{B}(\lambda) \in F_0^{1\times 1}(-\tau, 0)$ не имеют ни одного общего нуля в том и только том случае, когда оператор $R_0(\mathcal{A}, \mathcal{B})$ обратим.*

§5. Приложения

Приведем пример приложения теорем о континуальном аналоге результантного оператора к задаче об исключении неизвестного из систем двух (вообще говоря, трансцендентных) уравенениий с двумя неизвестными.

1. Рассмотрим сначала скалярный случай. Пусть $\mathcal{A}(\lambda, \mu)$ и $\mathcal{B}(\lambda, \mu)$ — целые функции от λ и μ вида

$$\mathcal{A}(\lambda, \mu) = a_0 + \int_0^\tau \int_0^\tau a(t, s) e^{i(\lambda t + \mu s)} ds dt$$

(5.1)

$$\mathcal{B}(\lambda, \mu) = b_0 + \int_{-\tau}^0 \int_{-\tau}^0 b(t, s) e^{i(\lambda t + \mu s)} ds dt,$$

где $a_0, b_0 \in \mathbf{C}^1$; $a_0, b_0 \neq 0$; $0 < \tau < \infty$; $a(t, s) \in L^1([0, \tau] \times [0, \tau])$ и $b(t, s) \in \in L_1([-\tau, 0] \times [-\tau, 0])$. Рассмотрим систему уравнений

$$\text{(5.2)} \qquad \begin{cases} \mathcal{A}(\lambda, \mu) = 0 \\ \mathcal{B}(\lambda, \mu) = 0 \end{cases}$$

с неизвестными числами λ и $\mu \in \mathbf{C}^1$. Рассмотрим функции

$$a_\mu(t) = \int_0^\tau a(t, s) e^{i\mu s} ds \qquad (\mu \in \mathbf{C}^1;\ t \in [0, \tau])$$

и

$$b_\mu(t) = \int_{-\tau}^0 b(t, s) e^{i\mu s} ds \qquad (\mu \in \mathbf{C}^1;\ t \in [-\tau, 0]).$$

Очевидно, для любого фиксированного μ функции $a_\mu(t)$ $(b_\mu(t))$, принадлежат пространству $L_1(0, \tau)$ $(L_1(-\tau, 0))$. Следовательно, функции $a_\mu(t)$ и $b_\mu(t)$ можно рассматривать как вектор-функции от μ со значениями в пространствах $L_1(0, \tau)$

и $L_1(-\tau, 0)$ соответственно. Эти вектор-функции являются целыми. В самом деле, пусть

$$a'_\mu(t) = \int_0^\tau a(t, s)(is)e^{i\mu s}ds .$$

Тогда для $h \in \mathbf{C}^1$ будем иметь

$$\| a'_\mu - \frac{1}{h}(a_{\mu+h} - a_\mu) \|_{L_1(0,\tau)} \leqq \int_0^\tau \int_0^\tau | a(t, s) || \frac{1}{h}(e^{ihs} - 1) - is || e^{i\mu s} | dsdt \leqq$$

$$\leqq e^{|\mu|\tau} \int_0^\tau \int_0^\tau | a(t, s) | \left| \frac{1}{h}(e^{ihs} - 1) - is \right| dsdt .$$

Следовательно,

$$\lim \left\| a'_\mu - \frac{1}{h}(a_{\mu+h} - a_\mu) \right\|_{L_1(0,\tau)} = 0 .$$

Аналогично доказывается, что b_μ также является целой вектор-функцией.

Для каждого $\mu \in \mathbf{C}^1$ рассмотрим классический результантный оператор $R_0(\mu)$, действующий в пространстве $L_1(-\tau, \tau)$ по формуле

$$(R_0(\mu)\phi)(t) = \begin{cases} a_0\phi(t) + \int_{-\tau}^{\tau} a_\mu(t - s)\phi(s)ds \\ b_0\phi(t) + \int_{-\tau}^{\tau} b_\mu(t - s)\phi(s)ds . \end{cases} \tag{5.3}$$

Оператор-функция $R_0(\mu)$ является целой. В самом деле, положим

$$(R'_0(\mu)\phi)(t) = \begin{cases} \int_{-\tau}^{\tau} a'_\mu(t - s)\phi(s)ds & (0 \leqq t \leqq \tau) \\ \int_{-\tau}^{\tau} b'_\mu(t - s)\phi(s)ds & (-\tau \leqq t < 0). \end{cases}$$

Тогда в силу оценки (2.2) для любого $h \in \mathbf{C}^1$ получим

$$\left\| \frac{1}{h}\big(R_0(\mu + h) - R_0(\mu)\big) - R'_0(\mu) \right\|_{L_1(-\tau,\tau)} \leqq \left\| \frac{1}{h}(a_{\mu+h} - a_\mu) - a'_\mu \right\|_{L_1(0,\tau)} +$$

$$+ \left\| \frac{1}{h}(b_{\mu+h} - b_\mu) - b'_\mu \right\|_{L_1(-\tau,0)} .$$

Следовательно,

$$\lim_{h\to 0} \left\| \frac{1}{h}\big(R_0(\mu + h) - R_0(\mu)\big) - R'_0(\mu) \right\|_{L_1(-\tau,\tau)} = 0 \qquad (\mu \in \mathbf{C}^1).$$

Применяя теорему 0.1 к целым функциям $\mathcal{A}(\lambda, \mu)$ и $\mathcal{B}(\lambda, \mu)$ для фиксированного μ, получим, что множество характеристических чисел оператор-функции $R_0(\mu)$ совпадает с множеством точек μ' для которых функции $\mathcal{A}(\lambda, \mu')$ и $\mathcal{B}(\lambda, \mu')$ имеют общие нули. Без труда устанавливается, что для оператор-функции $R_0(\mu)$ существуют точки μ_0, для которых оператор $R_0(\mu_0)$ обратим. Следовательно, множество характеристических чисел оператор-функции $R_0(\mu)$ дискретно.

Таким образом, система (5.2) сводится к семейству систем уравнений с одной неизвестной

$$\begin{cases} \mathcal{A}(\lambda, \mu_j) = 0 \\ \mathcal{B}(\lambda, \mu_j) = 0 \end{cases}$$

где μ_j пробегает множество характеристических чисел оператор-функции$R_0(\mu)$. Отыскание характеристических чисел оператор-функции $R_0(\mu)$ в некоторых случаях можно свести к отысканию нулей некоторой целой функции.

В самом деле, если $a_\mu(t) \in L_2(0, \tau)$ и $b_\mu(t) \in L_2(-\tau, 0)$, то операторы

$$(K(\mu)\phi)(t) = -\phi(t) + \begin{cases} a_0^{-1} R_0(\mu)\phi(t) : & 0 \leqq t \leqq \tau \\ b_0^{-1} R_0(\mu)\phi(t) : & -\tau \leqq t \leqq 0. \end{cases}$$

принадлежат классу Гильберта-Шмидта. Следовательно, множество характеристических чисел оператор-функции $R_0(\mu)$ совпадает с множеством нулей целой функции $\widetilde{\det}(I + K(\mu))$, где $\widetilde{\det}(I + K(\mu))$ означает регуляризованный определитель оператора $I + K(\mu)$ (см. [2] гл. IV, § 2).

Поменяем теперь ролями переменные λ и μ и повторим описанный выше процесс относительно результантного оператора

$$(5.4) \qquad \big(R_0(\lambda)\phi\big)(t) \stackrel{\text{def}}{=} \begin{cases} a_0\phi(t) + \int\limits_{-\tau}^{\tau} a_\lambda(t-s)\phi(s)ds & (0 \leqq t \leqq \tau) \\ b_0\phi(t) + \int\limits_{-\tau}^{\tau} b_\lambda(t-s)\phi(s)ds & (0 \leqq t \leqq \tau) \end{cases}$$

где

$$a_\lambda(s) = \int\limits_0^\tau a(t, s)e^{i\lambda t}dt \quad \text{и} \quad b_\lambda(s) = \int\limits_{-\tau}^{\tau} b(t, s)e^{i\lambda t}dt.$$

Получим, что система уравнений (5.2) может удовлетворяться только характеристическими числами λ оператора $R_0(\lambda)$. Обозначим через $\{\lambda_j\}$ множество этих характеристических чисел. Наконец, можно сделать вывод, что все решения системы (5.2) содержатся среди пар чисел (λ_j, μ_k).

2. Все сказанное выше естественным образом обобщается на случай матриц-функций, т.е. на случай решения системы уравнений

$$(5.5) \qquad \begin{cases} \mathcal{A}(\lambda, \mu)\phi = 0 \\ \mathcal{B}(\lambda, \mu)\phi = 0 \end{cases}$$

с неизвестными числами λ и μ и неизвестным вектором $\phi \in \mathbf{C}^d$ в предположении что

$$(5.6) \qquad \mathcal{A}(\lambda, \mu) = a_0 + \int_0^\tau \int_0^\tau a(t, s) e^{i(\lambda t + \mu s)} dt ds$$

и

$$(5.6) \qquad \mathcal{B}(\lambda, \mu) = b_0 + \int_{-\tau}^0 \int_{-\tau}^0 b(t, s) e^{i(\lambda t + \mu s)} dt ds$$

где $a_0, b_0 \in L(\mathbf{C}^d)$; $a(t, s)$, $b(-t, -s) \in L_1^{d\times d}([0, \tau] \times [0, \tau])$.

В силу следствия 3.2 из конца § 3 здесь можно ограничиться применением классического результантного оператора и свести задачу (5.5) к решению систем уравнений

$$\begin{cases} \mathcal{A}(\lambda_j, \mu_k)\phi = 0 \\ \mathcal{B}(\lambda_j, \mu_k)\phi = 0 \end{cases}$$

где μ_k пробегает все характеристичекие числа оператора $R_0(\mu)$, определенного равенством (5.3), а λ_j — все характеристические числа оператора $R_0(\lambda)$, определенного равенством (5.4).

3. Отметим, что количество примеров приложений основных теорем можно значительно увеличить, варьируя классы рассматриваемых функций и матриц-функций.

§6. Континуальный аналог безутианты

Пусть

$$\mathcal{A}(\lambda) = 1 + \int_0^\tau a(t) e^{i\lambda t} dt \quad \text{и} \quad \mathcal{B}(\lambda) = 1 + \int_{-\tau}^0 b(t) e^{i\lambda t} dt$$

где $a(t) \in L_1(0, \tau)$ и $b(t) \in L_1(-\tau, 0)$.

Рассмотрим функцию

$$\mathcal{G}(\lambda, \mu) \stackrel{\text{def}}{=} \frac{i}{\lambda + \mu} (\mathcal{A}(\lambda)\mathcal{B}(-\mu) - e^{i\tau(\lambda+\mu)} \mathcal{A}(-\mu)\mathcal{B}(\lambda) \qquad (\lambda, \mu \in \mathbf{C}^1).$$

Легко проверяется (см. [5]), что

$$\mathcal{G}(\lambda, \mu) = \int_0^\tau \int_0^\tau \gamma(t, s) e^{i(\lambda t + \mu s)} dt ds$$

где

$$\gamma(t, s) \stackrel{\text{def}}{=} a(t-s) + b(t-s) + \int\limits_0^{\min(t,s)} (a(t-r)b(r-s) - b(t-r-\tau)a(r-s+\tau))dr$$

$$(0 \leqq t, s \leqq \tau).$$

Отметим, что $\gamma(t, s) \in L_1([0, \tau] \times [0, \tau])$.

Оператор $I + \Gamma(\mathcal{A}, \mathcal{B})$, где

$$(\Gamma(\mathcal{A}, \mathcal{B})\phi)(t) = \int\limits_0^\tau \gamma(t, s)\phi(s)ds$$

действующий в пространстве $L_1(0, \tau)$, будем называть безутиантным оператором функций $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$. Он является естественным континуальным аналогом безутиантного оператора в дискретном случае (см. [1, 6, 7]).

Отметим, что $\gamma(t, 0) = a(t)$ и $\gamma(0, s) = b(-s)$.

Континуальным аналогом теоремы 5.1 из [1] является

Теорема 6.1. *Пусть* $\mathcal{A}(\lambda) = 1 + \int\limits_0^\tau a(t)e^{i\lambda t}dt$ и $\mathcal{B}(\lambda) = 1 + \int\limits_{-\tau}^0 b(t)e^{i\lambda t}dt$ *где* $a(t) \in L_1(0, \tau)$ *и* $b(t) \in L_1(-\tau, 0)$. *Тогда ядро безутиантного оператора* $I + \Gamma(\mathcal{A}, \mathcal{B})$ *функций* $\mathcal{A}(\lambda)$ *и* $\mathcal{B}(\lambda)$ *состоит из линейной оболочки системы функций*

$$\phi_{jk}(t) = t^k e^{-i\lambda_j t} \qquad (k = 0, 1, \ldots, k_j - 1;\ j = 1, 2, \ldots, l)$$

где λ_j $(j = 1, 2, \ldots, l)$ *— все различные общие нули функций* $\mathcal{A}(\lambda)$ *и* $\mathcal{B}(\lambda)$, *а* k_j *— кратность общего нуля* λ_j.

В частности,

$$\nu(\mathcal{A}, \mathcal{B}) = \dim \operatorname{Ker}(I + \Gamma(\mathcal{A}, \mathcal{B})). \tag{6.1}$$

Доказательство. Введем в рассмотрение следующие операторы, действующие в пространстве $L_1(0, \tau)$:

$$(A\phi)(t) = \int\limits_0^\tau a(t-s)\phi(s)ds, \qquad (B\phi)(t) = \int\limits_0^\tau b(t-s)\phi(s)ds$$

$$(0 \leqq t \leqq \tau)$$

$$(\tilde{A}\phi)(t) = \int\limits_0^\tau a(t-s+\tau)\phi(s)ds, \qquad (\tilde{B}\phi)(t) = \int\limits_0^\tau b(t-s-\tau)\phi(s)ds.$$

С помощью этих операторов безутиантный оператор можно записать в виде

$$I + \Gamma(\mathcal{A}, \mathcal{B}) = (I + A)(I + B) - \tilde{B}\tilde{A}.$$

Введем еще операторы

$$(H_+\phi)(t) = \phi(t-\tau) \qquad (0 \leqq t \leqq \tau)$$

$$(H_-\phi)(t) = \phi(t-\tau) \qquad (-\tau \leqq t \leqq 0).$$

Оператор H_+ отображает $L_1(-\tau, 0)$ на $L_1(0, \tau)$, а оператор H_- является обратным к оператору H_+. Если отождествить пространство $L_1(-\tau, \tau)$ с прямой суммой подпространств $L_1(-\tau, 0)$ и $L_1(0, \tau)$, то результантный оператор $R_0(\mathcal{A}, \mathcal{B})$ функций $\mathcal{A}(\lambda)$ и $\mathcal{B}(\lambda)$ примет следующий блочный вид:

$$R_0(\mathcal{A}, \mathcal{B}) = \left\| \begin{matrix} I + H_-BH_+ & H_-\tilde{B} \\ \tilde{A}H_+ & I + A \end{matrix} \right\|.$$

Непосредственной проверкой убеждаемся в справедливости равенства

$$(6.2)\quad R_0(\mathcal{A}, \mathcal{B}) = \left\| \begin{matrix} I & H_-\tilde{B}(I+A)^{-1} \\ 0 & I \end{matrix} \right\| \left\| \begin{matrix} H_-CH_+ & 0 \\ 0 & I + A \end{matrix} \right\| \left\| \begin{matrix} I & 0 \\ (I+A)^{-1}\tilde{A}H_+ & I \end{matrix} \right\|$$

где $C = I + B - \tilde{B}(I + A)^{-1}\tilde{A}$. Так как операторы $\tilde{B}$ и A, очевидно, перестановочны, то

$$(6.3)\qquad C = (I + A)^{-1}((I + A)(I + B) - \tilde{B}\tilde{A}) = (I + A)^{-1}(I + \Gamma(\mathcal{A}, \mathcal{B})).$$

Пусть $f(t) \in \operatorname{Ker} R_0(\mathcal{A}, \mathcal{B})$, тогда в силу (6.2) и (6.3) $(I + \Gamma(\mathcal{A}, \mathcal{B}))\, f_1 = 0$, где $f_1(t) = f(t - \tau)\ (0 \leqq t \leqq \tau)$. Наоборот, из равенства $(I + \Gamma(A, B)) f_1(t) = 0$ вытекает, что $f(t) \in \operatorname{Ker} R_0(\mathcal{A}, \mathcal{B})$, где

$$f(t) = \begin{cases} f_1(t + \tau) & (-\tau \leqq t \leqq 0) \\ -(I + A)^{-1}\tilde{A}f_1(t) & (0 \leqq t \leqq \tau). \end{cases}$$

Для завершения доказательства осталось воспользоваться теоремой 3.1.

Отметим еще, что в случае, когда безутиантный оператор обратим, обратный к нему является интегральным оператором с ядром, зависящим только от разности аргументов (см. [4, 5]).

Для случая $\mathcal{A}(\lambda) = \mathcal{B}(-\lambda)$ теорема, содержащая континуальное обобщение теоремы Эрмита [6] (в ней, в частности, имеется равенство (6.1)), установлено М. Г. Крейном. Этот результат опубликован им лишь в дискретном случае (см. [8]).

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TECHNISCHE HOCHSCHULE
90 KARL-MARX-STADT
REICHENHAINER STR. 39
DDR

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A NOTE ON THE VALIRON METHOD OF SUMMABILITY

By

V. SWAMINATHAN (Kariavattom)

1. Introduction

The summability of series of Legendre polynomials by various methods has been studied by COWLING and KING [2], JAKIMOVSKI [6], [7], KING [8], POWELL [9], [10] and PRACHAR [11]. In Section 2 of this note, we study the summability of series of Legendre polynomials by regular (V, α) method (cf. HYSLOP [3]).

In Section 3 of this note, it is observed that when the (V, α) transform is regular, (i) it preserves the Gibbs phenomenon for Fourier series, and (ii) the Lebesgue constants are unbounded. The corresponding results for the case when $\alpha = \frac{1}{2}$ have been given by IKENO [4], [5]. The Valiron-transformation referred to by Ikeno corresponds to the case $\alpha = \frac{1}{2}$ of the (V, α) method studied in this note.

After Hyslop, we may define the (V, α) summability method as follows.* The sequence $\{s_n\}$ $(n = 0, 1, 2, \ldots)$ is said to be (V, α) summable to the sum s if

$$\lim_{\mu\to\infty} \mu^{-\alpha} \sqrt{\frac{a}{\pi}} \sum_{n=-\mu}^{\infty} e^{-an^2\mu^{-2\alpha}} s_{n+\mu} = s,$$

where a is a positive constant; or alternatively,

$$\lim_{\mu\to\infty} \sum_{k=0}^{\infty} c_{\mu k} s_k = s,$$

where

$$c_{\mu k} = \mu^{-\alpha} \sqrt{\frac{a}{\pi}}\, e^{-a(k-\mu^2)\mu^{-2\alpha}} \qquad (k = 0, 1, 2, \ldots). \tag{1.1}$$

Hyslop has shown that the (V, α) method is regular if and only if $0 < \alpha < 1$.

LEMMA. *For the summability matrix* $(c_{\mu k})$ *given by* (1.1),

$$\sum_{k=0}^{\infty} c_{\mu k} = 1 + O(\mu^{-\alpha}) \qquad (\mu \to \infty).$$

This can be obtained by an easy calculation.

* Hyslop chooses the constant $a = \frac{1}{2}$.

2. Summability of series of Legendre polynomials by the (V, α) method

Let $P_n(z)$ and $Q_n(w)$ denote the Legendre polynomials of the first and second kind of degree n, respectively. The Laplace integral representations of $P_n(z)$ and $Q_n(w)$ are given by ([12], Chapter XV)

$$P_n = \frac{1}{\pi}\int_0^{\pi} \zeta^n d\phi \tag{2.1}$$

and

$$Q_n = \int_0^{\infty} \tau^{-n-1} d\beta, \tag{2.2}$$

where

$$\zeta = \zeta(\phi) = z + (z^2-1)^{1/2}\cos\phi$$

and

$$\tau = \tau(\beta) = w + (w^2-1)^{1/2}\cosh\beta.$$

The branch of $(z^2-1)^{1/2}$ is chosen so that $z+(z^2-1)^{1/2}$ lies in the exterior of the unit circle. Write

$$s_k = s_k(z, w) = \sum_{n=0}^{k} (2n+1)P_n(z)Q_n(w) \tag{2.3}$$

and

$$d_n = d_n(z, w) = P_{n+1}(z)Q_n(w) - P_n(z)Q_{n+1}(w). \tag{2.4}$$

We have by the Christoffel formula [12],

$$\frac{1}{w-z} = s_n + (n+1)\frac{1}{w-z}d_n. \tag{2.5}$$

By Heine's theorem (cf. [12], p. 321), $\{s_n(z, w)\}$ converges to $(w-z)^{-1}$ in the interior of the ellipse with foci ± 1 and passing through w. The following theorem asserts that, under suitable conditions, the sequence $\{s_n(z, w)\}$ is summable by the (V, α) method to $(w-z)^{-1}$ in a wider region.

THEOREM. *Let $\zeta/\tau = re^{i\theta}$. Then the sequence $\{s_k\}$ of partial sums of the series of Legendre polynomials*

$$\sum_{n=0}^{\infty} (2n+1)P_n(z)Q_n(w)$$

is (V, α) summable to $(w-z)^{-1}$ in the region given by

$$\operatorname*{Max}_{\phi,\beta}\left|\frac{\zeta}{\tau}\right| < \begin{cases} 1 & \text{when } \alpha < 1/2, \\ e^{\sqrt{4a^2+\theta^2}-2a} & \text{when } \alpha = 1/2,\ -\pi \leqq \theta \leqq +\pi \\ e^{\theta} & \text{when } \alpha > 1/2,\ 0 \leq \theta \leqq \pi \\ 1 & \text{when } \alpha > 1/2,\ -\pi \leqq \theta \leqq 0. \end{cases}$$

PROOF. Let

$$T_\mu = \sum_{k=0}^{\infty} c_{\mu k} s_k ,$$

where $c_{\mu k}$ and s_k are given by (1.1) and (2.3), respectively. Then, using (2.5), we get

$$T_\mu = \sum_{k=0}^{\infty} c_{\mu k}\left[\frac{1}{w-z} - (k+1)\frac{1}{w-z}\, d_k\right] =$$

$$= \frac{1}{w-z}\sum_{k=0}^{\infty} c_{\mu k} - \frac{1}{w-z}\sum_{k=0}^{\infty}(k+1)c_{\mu k} d_k .$$

Using the above lemma, we obtain

$$T_\mu = \frac{1}{w-z}\,[1+o(1)] - \frac{1}{w-z}\sum_{k=0}^{\infty}(k+1)c_{\mu k} d_k \qquad (\mu\to\infty).$$

Hence

$$\lim_{\mu\to\infty} T_\mu = (w-z)^{-1}$$

if and only if

$$\sum_{k=0}^{\infty}(k+1)c_{\mu k} d_k = o(1) \qquad (\mu\to\infty). \tag{2.6}$$

We now investigate the region, where equation (2.6) is satisfied. Using the results (2.1), (2.2) and (2.4) in (2.6), we have

$$\sum_{k=0}^{\infty}(k+1)c_{\mu k} d_k = \frac{1}{\pi}\sum_{k=0}^{\infty} c_{\mu k}(k+1)\int_0^{\infty}\int_0^{\pi}\left(\frac{\zeta}{\tau}\right)^k\left(\frac{\zeta}{\tau} - \frac{1}{\tau^2}\right) d\phi\, d\beta. \tag{2.7}$$

Change of order of integration and summation in (2.7) is permissible provided

$$\left|\frac{\zeta}{\tau}\right| \leqq M < +\infty . \tag{2.8}$$

From (2.7), we obtain

$$\sum_{k=0}^{\infty}(k+1)c_{\mu k} d_k = \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi}\left(\frac{\zeta}{\tau} - \frac{1}{\tau^2}\right)\sum_{k=0}^{\infty}(k+1)c_{\mu k}\left(\frac{\zeta}{\tau}\right)^k d\phi\, d\beta = o(1) \qquad (\mu\to\infty),$$

provided

$$\sum_{k=0}^{\infty}(k+1)c_{\mu k}(\zeta/\tau)^k = o(1) \qquad (\mu\to\infty).$$

Let $\zeta/\tau = e^{\xi+i\eta}$. For convenience, we consider the cases $\operatorname{Max}_{\phi,\beta} |\zeta/\tau| < 1$ and $\operatorname{Max}_{\phi,\beta} |\zeta/\tau| \geqq 1$ separately; these correspond to $\operatorname{Max}_{\phi,\beta} \xi < 0$ and $\operatorname{Max}_{\phi,\beta} \xi \geqq 0$ respectively.

Case I. $\operatorname{Max}_{\phi,\beta} \xi < 0$.

Given $\varepsilon > 0$, choose M such that for $k > M$,

$$(k+1)\,|\,\zeta/\tau\,|^k < \varepsilon.$$

Then

$$\left|\sum_{k=0}^{\infty}(k+1)c_{\mu k}(\zeta/\tau)^k\right| \leqq \left|\sum_{k=0}^{M}\right| + \left|\sum_{k=M+1}^{\infty}\right| <$$

$$< \frac{C}{\mu^\alpha}(M+1)^2 + \varepsilon \sum_{k=M+1}^{\infty} c_{\mu k} = o(1) \qquad (\mu \to \infty),$$

C being an absolute constant. Thus, when $\operatorname*{Max}_{\phi,\beta}\,|\,\zeta/\tau\,| < 1$, the sequence $\{s_k\}$ given by (2.3), is (V, α) summable to $(w-z)^{-1}$ for $0 < \alpha < 1$.

Case II. $\operatorname*{Max}_{\phi,\beta} \xi \geqq 0$.

The required sum in the present case is given by

$$\sum_{k=0}^{\infty}(k+1)c_{\mu k}(\zeta/\tau)^k = \sum_{k=0}^{\infty}(k+1)\mu^{-\alpha}\sqrt{\frac{a}{\pi}}\,e^{-a(k-\mu)^2\mu^{-2\alpha}}(\zeta/\tau)^k =$$

$$= \mu^{-\alpha}\sqrt{\frac{a}{\pi}}\sum_{n=-\mu}^{\infty}(\mu+n+1)e^{-an^2\mu^{-2\alpha}}(\zeta/\tau)^{\mu+n} =$$

$$= \mu^{-\alpha}\sqrt{\frac{a}{\pi}}\sum_{n=-\mu}^{\infty}(\mu+n+1)e^{-an^2\mu^{-2\alpha}+(\mu+n)(\xi+i\eta)} =$$

$$= \mu^{-\alpha}\sqrt{\frac{a}{\pi}}\left(\sum_{n=-\infty}^{+\infty} - \sum_{n=-\infty}^{-\mu-1}\right)(\mu+n+1)e^{-an^2\mu^{-2\alpha}+(\mu+n)(\xi+i\eta)} = \Sigma_1 - \Sigma_2,$$

say. Clearly

$$|\,\Sigma_2\,| \leqq \mu^{-\alpha}\sqrt{\frac{a}{\pi}}\left|\sum_{-\infty}^{-\mu-1}(\mu+n+1)\,e^{-an^2\mu^{-2\alpha}+(\mu+n)(\xi+i\eta)}\right| \leqq$$

$$\leqq \mu^{-\alpha}\sqrt{\frac{a}{\pi}}\sum_{-\infty}^{-\mu-1}|\,\mu+n+1\,|\,e^{-an^2\mu^{-2\alpha}} <$$

$$< \mu^{-\alpha}\sqrt{\frac{a}{\pi}}\int_{\mu}^{\infty}e^{-ax^2\mu^{-2\alpha}}(x-\overline{\mu+1})\,dx = o(1) \qquad (\mu \to \infty).$$

Now

$$\Sigma_1 = \mu^{-\alpha}\sqrt{\frac{a}{\pi}}\,e^{\mu(\xi+i\eta)}\sum_{-\infty}^{+\infty}(\mu+n+1)e^{-an^2\mu^{-2\alpha}+n(\xi+i\eta)} =$$

$$= \mu^{-\alpha}\sqrt{\frac{a}{\pi}}\,e^{\mu(\xi+i\eta)+\frac{\xi^2}{4a}\mu^{2\alpha}}\sum_{-\infty}^{+\infty}(\mu+n+1)e^{-a\mu^{-2\alpha}\left(n-\mu^{2\alpha}\frac{\xi}{2a}\right)^2+ni\eta} =$$

$$= \mu^{-\alpha}\sqrt{\frac{a}{\pi}}\,e^{\mu(\xi+i\eta)+\frac{\xi^2}{4a}\mu^{2\alpha}+\frac{\mu^{2\alpha}}{2a}i\xi\eta}\sum_{m=-\infty}^{+\infty}\left(\mu+\mu^{2\alpha}\frac{\xi}{2a}+m+1\right)e^{-a\mu^{-2\alpha}m^2+im\eta} =$$

$$= F\cdot\sum_{-\infty}^{+\infty}\left(\mu+\mu^{2\alpha}\frac{\xi}{2a}+m+1\right)e^{-a\mu^{-2\alpha}m^2+im\eta},$$

say. Thus

$$\Sigma_1 = \Sigma_{1,1}+\Sigma_{1,2},$$

where

$$\Sigma_{1,1} = F(\mu+\mu^{2\alpha}\xi/2a+1)\sum_{-\infty}^{+\infty}e^{-a\mu^{-2\alpha}m^2+mi\eta}$$

and

$$\Sigma_{1,2} = F\sum_{-\infty}^{+\infty}e^{-a\mu^{-2\alpha}m^2+im\eta}\,m.$$

Here

$$|\Sigma_{1,1}| \leqq |F|\cdot|\mu+\mu^{2\alpha}\xi/2a+1|\cdot\left|\int_{-\infty}^{+\infty}e^{-ax^2\mu^{-2\alpha}+ix\eta}dx\right|. \tag{2.9}$$

By properties of theta functions ([1], p. 9), we have

$$\int_{-\infty}^{+\infty}e^{-tx^2+i2xy}dx = \sqrt{\frac{\pi}{t}}\,e^{-y^2/t}, \tag{2.10}$$

where $t>0$. Hence

$$|\Sigma_{1,1}| \leqq |F|\cdot|\mu+\mu^{2\alpha}\xi/2a+1|\cdot\sqrt{\frac{\pi}{a}}\,\mu^{\alpha}e^{-\eta^2\mu^{2\alpha}/4a}. \tag{2.11}$$

Also

$$|\Sigma_{1,2}| \leqq |F|\cdot\left|\int_{-\infty}^{+\infty}e^{-a\mu^{-2\alpha}x^2+ix\eta}x\,dx\right|. \tag{2.12}$$

Differentiating (2.10) with respect to y under the integral sign (which is justified on account of the uniform convergence of the integral), we obtain

$$\int_{-\infty}^{+\infty}e^{-tx^2+i2xy}x\,dx = \sqrt{\pi}\,ie^{-y^2/t}y/t^{3/2}. \tag{2.13}$$

Using (2.13) in (2.12), we get

$$|\Sigma_{1,2}| < |F| \sqrt{\pi}\mu^{3\alpha} e^{-\eta^2\mu^{2\alpha}/4a} |\eta/2|/a^{3/2}. \tag{2.14}$$

Combining (2.11) and (2.14), it is seen that

$$|\Sigma_1| \leqq \Sigma_{1,1}| + |\Sigma_{1,2}| <$$

$$< \{\mu + 1 + \mu^{2\alpha}/2a(|\xi| + |\eta|)\} e^{\mu\xi + \mu^{2\alpha}/4a(\xi^2 - \eta^2)} = o(1) \qquad (\mu \to \infty),$$

provided

$$\mu\xi + \frac{\mu^{2\alpha}}{4a}(\xi^2 - \eta^2) \to -\infty \tag{2.15}$$

as $\mu \to \infty$.

Now consider separately the cases when $\alpha = 1/2, \alpha < 1/2$ and $\alpha > 1/2$. For $\alpha = 1/2$, we have

$$\xi + (\xi^2 - \eta^2)/4a < 0$$

or

$$\xi < \sqrt{4a^2 + \eta^2} - 2a.$$

The domain of summability of the series of Legendre polynomials in this case is therefore given by

$$\operatorname*{Max}_{\phi, \beta} |\zeta/\tau| < e^{\sqrt{4a^2 + \theta^2} - 2a} \qquad (-\pi \leqq \theta \leqq +\pi).$$

For $\alpha < 1/2$, the region of convergence is easily seen from (2.15) to be $\operatorname*{Max}_{\phi, \beta} |\zeta/\tau| < 1$.

For $\alpha > 1/2$, we have again from (2.15) that $\xi^2 < \eta^2$. Since the case $\xi < 0$ has already been discussed, we need only consider the case when $\xi < \eta$. The region of summability in this case is given by

$$\operatorname*{Max}_{\phi, \beta} |\zeta/\tau| < \begin{cases} e^{\theta}, & 0 \leqq \theta \leqq \pi, \\ 1, & -\pi \leqq \theta \leqq 0. \end{cases}$$

3. Gibbs phenomenon and Lebesgue constants for the (V, α) summability transform

THEOREM 3.1. *For $0 < \alpha < 1$, the (V, α) transform completely preserves the Gibbs phenomenon for Fourier series.*

THEOREM 3.2. *The Lebesgue constants $L_V(\alpha, \mu)$ for the regular (V, α) method are given by*

$$L_V(\alpha, \mu) = \frac{2}{\pi^2} \log(4a\mu^{2-2\alpha}) + A + O(\mu^{\alpha-1}) \qquad (\mu \to \infty),$$

where

$$A = -\frac{C}{\pi^2} + \frac{2}{\pi}\int_0^1 \frac{\sin u}{u}\,du - \frac{2}{\pi}\int_1^\infty \left(\frac{2}{\pi} - |\sin u|\right)\frac{du}{u}$$

and

$$C = \int_0^1 \frac{1-e^{-u}}{u}\,du - \int_1^\infty \frac{1}{ue^u}\,du,$$

which is the Euler–Mascheroni's constant.

These theorems follow exactly on the same lines of argument adopted by Ikeno for the $F(a, q)$ family of summability methods. The particular case of the (V, α) method, when $\alpha = 1/2$, has been discussed by IKENO [4], [5].

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KERALA
KARIAVATTOM
TRIVANDRUM, INDIA

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DIFFERENCE OPERATORS AND PERIODIC SEQUENCES OVER FINITE MODULES

By

MELVYN. B. NATHANSON (Carbondale)

1. Introduction Let R be a ring with 1, and let M be a unitary R-module. A *sequence over* M is a function from the nonnegative integers N into the module M. Let M^N denote the set of all sequences over M. Then M^N is a R-module under coordinatewise addition and scalar multiplication: If $x, y \in M^N$ and $r \in R$, then $(rx+y)(i) = rx(i) + y(i)$ for all $i \in N$. Let F be a function from M^N to M^N, and let $x \in M^N$. The *F-orbit of* x is the sequence of sequences $x, F(x), F^2(x), \ldots$. In this paper we study periodic sequences over finite modules M, and their orbits under difference operators and the inverses of difference operators. We prove that for a difference operator D and $x \in M^N$, the D-orbit of x is eventually periodic if and only if the sequence x is eventually periodic. For a certain right inverse I of the difference operator, we show that the I-orbit of x is eventually periodic if and only if x is the constant sequence of zeros. These results generalize theorems obtained in the special case of the "derivative operator" $D(x)(i) = x(i) + x(i+1)$ acting on the space of binary sequences [1, 2].

2. Row-finite matrices. DEFINITION 1. Let $d_0, d_1, \ldots, d_{n-1}$ be in R, and let d_n be a unit in R. Define the operator D on M^N by

$$D(x)(i) = \sum_{t=0}^{n} d_t x(i+t)$$

for all $x \in M^N$ and $i \in N$. Then *D is a difference operator of degree n.*

Note that D is a homomorphism of the additive group M^N. If the ring R is commutative, then D is also an R-module homomorphism.

LEMMA 1. *Let D be a difference operator of degree n on M^N, let $(x_0, x_1, \ldots, x_{n-1})$ be an n-tuple of elements of M, and let $y \in M^N$. Then there exists a unique sequence x in M^N such that $D(x) = y$ and $x(i) = x_i$ for $i = 0, 1, \ldots, n-1$.*

PROOF. We construct the sequence x inductively. Let $x(i) = x_i$ for $i = 0, 1, \ldots, n-1$. If $x(0), x(1), \ldots, x(i+n-1)$ have been computed, then let

$$x(i+n) = d_n^{-1}\left[y(i) - \sum_{t=0}^{n-1} d_t x(i+t)\right].$$

Clearly, $D(x) = y$, and the sequence x is determined uniquely.

COROLLARY. Let D be a difference operator of degree n on M^N. Then D is surjective, and the kernel of D has cardinality $|M|^n$.

DEFINITION 2. An *infinite matrix over R* is a function (a_{ij}) from $N\times N$ into the ring R. A *row-finite matrix over R* is an infinite matrix (a_{ij}) over R such that for each i there exist only finitely many j with $a_{ij}\neq 0$.

The row-finite matrices over a ring R form a ring with the usual matrix addition and multiplication. Each row-finite matrix $A=(a_{ij})$ defines a homomorphism A of sequences x over the R-module M by

$$A(x)(i)=\sum_{j=0}^{\infty}a_{ij}x(j).$$

This sum is finite because the matrix A is row-finite. Every difference operator has a representation as a row-finite matrix. To the operator $D(x)(i)=\sum_{t=0}^{n}d_t\,x(i+t)$ corresponds the matrix

$$d_{ij}=\begin{cases}d_{j-i} & \text{if } j=i,i+1,\dots,i+n\\ 0 & \text{otherwise.}\end{cases}$$

The row-finite matrices do not represent all operators on M^N, since these matrices are completely determined by their action on the proper submodule of M^N consisting of all sequences over M with only finitely many non-zero coordinates.

LEMMA 2. *Let D be a difference operator of degree $n\geqq 1$. Then D has infinitely many right inverses that can be represented by row-finite matrices.*

PROOF. Let $D=(d_{ij})$ be the row-finite matrix corresponding to the difference operator $D(x)(i)=\sum_{t=0}^{n}d_tx(i+t)$. We construct a row-finite matrix $A=(a_{ij})$ such that $DA=(\delta_{ij})$, where $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$. Choose arbitrarily the first n row-finite rows of A. The ring R is a unitary R-module, and so D acts as a difference operator on sequences over R. By Lemma 1, for each j there is a sequence x_j over R such that $D(x_j)(i)=\delta_{ij}$ for all i and $x_j(i)=a_{ij}$ for $i=0,1,\dots,n-1$. Define $A=(a_{ij})$ by $a_{ij}=x_j(i)$ for all i,j. Then A is an infinite matrix over R which is a right inverse of D.

Clearly, the matrix A is uniquely determined by its first n rows, and there are infinitely many choices of n row-finite rows of elements of R.

Finally, we must show that A is a row-finite matrix. By construction, the first n rows of A have only finitely many nonzero entries. Suppose that for some $i_0\geqq n$ there exists an integer j_0 such that $a_{ij}=0$ if $i<i_0$ and $j>j_0$. Then for $j>\max(j_0,i_0-n)$,

$$0=\delta_{i_0-n,j}=\sum_{t=0}^{\infty}d_{i_0-n,t}a_{tj}=\sum_{t=i_0-n}^{i_0}d_{i_0-n,t}a_{tj}=d_{i_0-n,i_0}a_{i_0,j}=d_na_{i_0,j}.$$

Since d_n is a unit of the ring, $a_{i_0j}=0$ for $j>\max(j_0,i_0-n)$. Therefore, the matrix A is row-finite.

DEFINITION 3. Let D be a difference operator of degree n. The *integral of D* is the unique row-finite right inverse matrix of D whose first n rows are zero.

LEMMA 3. *Let M be a finite, unitary R-module, and let $x \in M^N$. Then the sequence x is eventually periodic if and only if there exist $r_0, r_1, \dots, r_{n-1}$ in R and r_n a unit in R such that*

$$(*) \qquad \sum_{t=0}^{n} r_t x(i+t) = 0$$

for all sufficiently large i.

PROOF. Suppose that (*) is satisfied for all $i \geqq i_0$. Then for $i \geqq i_0$,

$$x(i+n) = -r_n^{-1} \sum_{t=0}^{n-1} r_t x(i+t)$$

and so each n-tuple $(x(i), x(i+1), \dots, x(i+n-1))$ determines $x(j)$ for all $j \geqq i$. But there are only $|M|^n$ n-tuples of elements of M, and so $(x(i), x(i+1), \dots \dots, x(i+n-1)) = (x(k), x(k+1), \dots, x(k+n-1))$ for some $i_0 \leqq i < k \leqq \leqq i_0 + |M|^n$. But then the sequence x has the period $k-i$.

Conversely, if $x(i+P) + x(i)$ for some $P > 0$ and all $i \geqq i_0$, let $r_t = -1$ for $t = 0, 1, \dots, P-1$ and let $r_t = 1$ for $t = P, P+1, \dots, 2P-1$. Then $\sum_{t=0}^{2P-1} r_t x(i+t) = 0$ for $i \geqq i_0$.

LEMMA 4. *Let R be a finite ring with 1, let D be a difference operator of order n, and let I be the integral of D. Let A be any row-finite right inverse of D, and let $B = A - I$. Then B is a row-finite matrix, and the rows of B form an eventually periodic sequence of sequences over the ring R.*

PROOF. It suffices to show, first, that the matrix $B = (b_{ij})$ is row-bounded, that is, there exists an integer j_0 such that $b_{ij} = 0$ for $j \geqq j_0$ and all i, and, second, that each column of B is an eventually periodic sequence over R.

Since B is the difference of two row-finite matrices, B is row-finite, and so there exists an integer j_0 such that $b_{ij} = 0$ if $j \geqq j_0$ and $i = 0, 1, \dots, n-1$. Fix $j \geqq j_0$. Since $DB = D(A - I) = 0$, the j-th column of B is a sequence x_j over R such that $D(x_j) = 0$ and $x_j(i) = b_{ij} = 0$ for $i = 0, 1, \dots, n-1$. But the constant sequence of zeros also satisfies these conditions, and so, by the uniqueness statement of Lemma 1, it follows that $b_{ij} = x_j(i) = 0$ for all $j \geqq j_0$ and all i. Therefore, B is a row-bounded matrix.

Moreover, since $DB = 0$, we have

$$0 = \sum_{k=0}^{\infty} d_{ik} b_{kj} = \sum_{k=i}^{i+n} d_{ik} b_{kj} = \sum_{k=i}^{i+n} d_{k-i} b_{kj} = \sum_{k=0}^{n} d_k b_{k+i,j}.$$

By Lemma 3, this linear recurrence implies that each column of the matrix B is an eventually periodic sequence over R.

3. Periodicity and operators. THEOREM 1. *Let D be a difference operator of degree $n \geqq 1$, and let x be a sequence over a finite module M. Then x is eventually periodic if and only if the D-orbit of x is eventually periodic.*

PROOF. Let $D(x)(i) = \sum_{t=0}^{n} d_t x(i+t)$. If $x(i+P) = x(i)$ for some $P > 0$ and all $i \geqq i_0$, then

$$D(x)(i+P) = \sum_{t=0}^{n} d_t x(i+P+t) = \sum_{t=0}^{n} d_t x(i+t) = D(x)(i),$$

and so $D^k(x)(i+P) = D^k(x)(i)$ for all $k \geqq 0$ and $i \geqq i_0$. Therefore, each sequence $D^k(x)$ is completely determined by its first $P + i_0$ terms. Since there are only $|M|^{P+i_0}$ distinct $(P+i_0)$-tuples of elements of M, it follows that $D^k(x) = D^{k'}(x)$ for some $k \neq k'$, and so the D-orbit of x is eventually periodic.

Conversely, suppose that $D^{k+Q}(x) = D^k(x)$ for some $Q > 0$ and all $k \geqq k_0$. Since D^{k_0+Q} and D^{k_0} are difference operators on sequences over M of degrees $(k_0+Q)n$ and $k_0 n$, respectively, it follows that $D^{k_0+Q}(x)(i) - D^{k_0}(x)(i) = 0$ for all i. But this defines a linear recurrence on the sequence x, and, by Lemma 3, the sequence is eventually periodic.

COROLLARY. *Let A be any row-finite right inverse of the difference operator D, and let $x \in M^N$. If the A-orbit of x is eventually periodic, then x is eventually periodic.*

PROOF. If $A^{k+Q}(x) = A^k(x)$ for some $Q > 0$, then

$$x = D^{k+Q}A^{k+Q}(x) = D^{k+Q}A^k(x) = D^Q(x)$$

and so the D-orbit of x is eventually periodic. By Theorem 1, the sequence x is eventually periodic.

The difference operators are not the only functions on M^N with the periodicity property described in Theorem 1. For let σ be the shift operator: $\sigma(x)(i) = x(i+1)$. Define the operator F on M^N by $F(x) = 0$ if x is eventually periodic, and $F(x) = \sigma(x)$ if x is not eventually periodic. Then the F-orbit of a sequence x is eventually periodic if and only if x is eventually periodic, but F is not a difference operator.

THEOREM 2. *Let I be the integral of a difference operator D of degree $n \geqq 1$, and let $x \in M^N$. Then the I-orbit of x is eventually periodic if and only if x is the constant sequence of zeros.*

PROOF. Let D be the difference operator $D(x)(i) = \sum_{t=0}^{n} d_t x(i+t)$, and let $I = (z_{ij})$ be its integral. We show $z_{ij} = 0$ if $i < j + n$, and that $z_{j+n,j}$ is a unit in the ring R. Since $DI = (\delta_{ij})$, we have

$$\delta_{ij} = \sum_{k=0}^{\infty} d_{ik} z_{kj} = \sum_{k=i}^{i+n} d_{k-i} z_{kj} = \sum_{k=0}^{n} d_k z_{k+i,j}.$$

Fix j. Since I is the integral of D, then $z_{ij} = 0$ for $i = 0, 1, \ldots, n-1$. But $1 = \sum_{k=0}^{n} d_k z_{k+j,j}$ implies that $z_{ij} \neq 0$ for some $i \geqq n$. Let $i_0 + n$ be the least integer such that $z_{i_0+n,j} \neq 0$. Then

$$\delta_{i_0,j} = \sum_{k=0}^{n} d_k z_{k+i_0,j} = d_n z_{i_0+n,j} \neq 0$$

and so $i_0 = j$ and $d_n z_{j+n,j} = 1$. Therefore, $z_{j+n,j}$ is a unit in R.

Let $I^m = (z_{ij}^m)$. We shall show that $z_{ij}^m = 0$ for $i < j + n + m - 1$. This has just been proved for $m = 1$. Assume that $z_{ij}^m = 0$ for all $m \leqq M$ and $i < j + n + m - 1$. Choose $i < j + n + M$. Then

$$z_{ij}^{M+1} = \sum_{k=0}^{\infty} z_{ik} z_{kj}^M = \sum_{k=j+n+M-1}^{\infty} z_{ik} z_{kj}^M = \sum_{k=j+n+M-1}^{i-n} z_{ik} z_{kj}^M .$$

But $i - n < i \leqq j + n + M - 1$, and so the last sum is empty. Therefore, $z_{ij}^{M+1} = 0$ for $i < j + n + M$. It follows by induction on m that $z_{ij}^m = 0$ for $i < j + n + m - 1$. In particular, the first $n + m - 1$ rows of I^m are zero. Therefore, $I^m(x)(i) = 0$ for $i < n + m - 1$ and all $x \in M^N$.

Now let the I-orbit of x be eventually periodic. Then $I^{k+Q}(x) = I^k(x)$ for some $Q > 0$ and all $k \geqq k_0$. Let $i \in N$. Choose $u \in N$ such that $i < k_0 + uQ + n - 1$. Then

$$I^{k_0}(x)(i) = I^{k_0+uQ}(x)(i) = 0,$$

and so $I^{k_0}(x)$ is the constant sequence of zeros.

Since I is a right inverse of D, it follows that $x = D^{k_0} I^{k_0}(x) = D^{k_0}(0) = 0$.

Conversely, if $x = 0$, then $I^m(x) = 0$ for all m, and so the I-orbit of x is eventually periodic.

Let A be any row-finite right inverse matrix of the difference operator D, and let x be a nonzero, eventually periodic sequence over M. It is not known whether the A-orbit of x can be eventually periodic.

4. Doubly-infinite sequences and nonabelian groups. Theorem 1, that a sequence over a finite R-module is eventually periodic if and only if its orbit under a difference operator is eventually periodic, can be extended, first, to doubly-infinite sequences, and, second, to sequences over finite, nonabelian groups.

Let x be a doubly-infinite sequence over M, that is, a function from the integers Z into the finite module M. Define the difference operator D on a doubly-infinite sequence x by

$$D(x)(i) = \sum_{t=m}^{n} d_t x(i + t),$$

where m and n are integers not both zero, $m \leqq n$, and $d_m, d_{m+1}, \ldots, d_n$ are elements of R such that d_m or d_n is a unit of R, and if $m = 0$ (resp. $n = 0$), then d_m (resp. d_n) is a unit of R. Then x is periodic *(not* eventually periodic) if and only if its orbit under the difference operator D is eventually periodic. This was proved in [3] for doubly-infinite sequences over a finite field.

The operator D can also be represented by a doubly-infinite row-finite matrix (d_{ij}), where

$$d_{ij} = \begin{cases} d_{j-i} & \text{if } \; j = i + m, i + m + 1, \ldots, i + n \\ 0 & \text{otherwise.} \end{cases}$$

If both d_m and d_n are units in R, then the operator D is surjective on the set of all doubly-infinite sequences over M, and the matrix of D has infinitely many doubly-infinite row-finite right inverse matrices. Problem: Define an integral I for the operator D such that the I-orbit of a doubly-infinite sequence x is eventually periodic if and only if x is zero.

Let G be a finite, nonabelian group. Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be n-tuples of integers and nonnegative integers, respectively, such that

(i) $\max(b_1, b_2, \ldots, b_n) = b_{i_0} > 0$ for some unique i_0;

(ii) $a_{i_0} = \pm 1$.

Define a difference operator D on sequences x over G by

$$(**) \qquad D(x)(i) = x(i+b_1)^{a_1} x(i+b_2)^{a_2} \ldots x(i+b_n)^{a_n}.$$

Then a sequence x over G is eventually periodic if and only if its orbit under D is eventually periodic.

Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be n-tuples of integers such that

(i) $\max(b_1, b_2, \ldots, b_n) = b_{i_1} > 0$ for some unique i_1;

(ii) $\min(b_1, b_2, \ldots, b_n) = b_{i_2} < 0$ for some unique i_2;

(iii) $a_{i_1} = \pm 1$ and $a_{i_2} = \pm 1$.

Define the difference operator D on doubly-infinite sequences x over G by formula (**). Then x is periodic if and only if the D-orbit of x is eventually periodic.

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SCHOOL OF MATHEMATICS
THE INSTITUTE OF ADVANCED STUDY
PRINCETON, NEW JERSEY 08540
USA

AND

DEPARTMENT OF MATHEMATICS
SOUTHERN ILLINOIS UNIVERSITY
CARBONDALE, ILLINOIS 62901
USA

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NOTE ON A THEOREM OF BOSANQUET

By

B. P. MISHRA and D. SINGH (Gorakhpur)

1. Introduction. Let $\{a_k\}_{k=0}^{\infty}$ be a given sequence of real numbers and suppose that

$$H_n^r = \frac{H_0^{r-1} + H_1^{r-1} + \ldots + H_n^{r-1}}{n+1}$$

for $r = 1, 2, 3, \ldots$ where $H_n^0 \equiv s_n = \sum_{k=0}^{n} a_k$. We say that the sequence $\{s_n\}$ is (H, K) summable to the sum s provided $H_n^k \to s$, as $n \to \infty$ and we write $s_n \to s(H, K)$. Throughout the paper, a sum without limit is understood to run from zero to infinity.

Further suppose that $\lambda > -1$ and that

$$E_n^\lambda = \binom{n+\lambda}{n}, \qquad \Phi_\lambda(x) = (1-x)^{\lambda+1} \Sigma E_n^\lambda s_n x^n, \qquad \Phi_0(x) = \Sigma a_n x^n.$$

With BORWEIN [1] we say that the sequence $\{s_n\}$ is A_λ-summable to s, and write $s_n \to s(A_\lambda)$, if the series defining $\Phi_\lambda(x)$ is convergent for all x in the open interval $(0, 1)$ and tends to a finite limit s as $x \to 1$ in $(0, 1)$. The (A_0) method is the ordinary Abel method.

It is known (see HARDY [3, Theorems 49, 54, 55]) that $H_n^r \to s$ $(r \geqq 2$, s finite) implies $\Phi_0(x) \to s$, as $x \to 1-0$. BOSANQUET [2] has proved that the above result is false for s infinite in the following

THEOREM A. *There is a series Σa_k for which*

(i) $\lim\limits_{n\to\infty} H_n^2 = +\infty$,

(ii) $\Sigma a_n x^n$ *converges for* $0 < x < 1$,

(iii) $\varliminf\limits_{x\to 1-0} a_n x^n = -\infty$.

In this paper we obtain a result analogous to Theorem A for A_λ-summability which we state and prove in § 2.

2. THEOREM B. *There is a series Σa_k for which*

(i) $\lim\limits_{n\to\infty} H_n^2 = +\infty$,

(ii) $(1-x)^{\lambda+1}\Sigma E_n^\lambda s_n x^n$ *converges for* $0 < x < 1$,

(iii) $\varliminf\limits_{x\to 1-0} (1-x)^{\lambda+1}\Sigma E_n^\lambda s_n x^n = -\infty$,

where $-1 < \lambda < \infty$.

The case $\lambda = 0$ of Theorem B is due to BOSANQUET [2].

To prove Theorem B, we require the following

LEMMA (see [1]). *If* $s_n \to s(A_\lambda)$, *then* $s_n \to s(A_\mu)$ *whenever* $\lambda > \mu > -1$.

PROOF OF THEOREM B. We consider the case $\lambda > 0$ while the result for $-1 < \lambda < 0$ follows from the above lemma and Theorem A.

$$\Phi_\lambda(x) = (1-x)^{\lambda+1}\Sigma E_n^\lambda s_n x^n = (1-x)^{\lambda+1}\Sigma E_n^\lambda H_n^1\{(n+1)x^n - (\lambda+n+1)x^{n+1}\} =$$
$$= (1-x)^{\lambda+1}\Sigma E_n^\lambda H_n^2[(n+1)\{(n+1)-(\lambda+n+1)x\} -$$
$$-(\lambda+n+1)x\{(n+2)-(\lambda+n+2)x\}]x^n = (1-x)^{\lambda+1}\Sigma E_n^\lambda H_n^2 K(x,n)x^n,$$

where

$$K(x,n) = (n+1)\{(n+1)-(\lambda+n+1)x\} -$$
$$-(\lambda+n+1)x\{(n+2)-(\lambda+n+2)x\}.$$

It is easy to show that

$$\text{(2.1)}\qquad K(x,n)\begin{cases} \leqq -\lambda x(1-x), & \text{if } -2+\dfrac{(2\lambda-1-\sqrt{4\lambda+1})x+2}{2(1-x)} \leqq \\ & \leqq n \leqq -2+\dfrac{(2\lambda-1+\sqrt{4\lambda+1})x+2}{2(1-x)}, \\ > -\lambda x(1-x), & \text{for the remaining } n. \end{cases}$$

We write

$$N(x) = -2 + \left[\frac{(2\lambda-1+\sqrt{4\lambda+1})x+2}{2(1-x)}\right]$$

and

$$N_1(x) = -2 + \left[\frac{(2\lambda-1-\sqrt{4\lambda+1})x+2}{2(1-x)}\right].$$

Now consider the numbers $x_1, x_2, \dots x_{v-1}$ and integers $N_1, N_2, \dots N_{v-1}$ such that

$$\frac{2}{2\lambda+3+\sqrt{4\lambda+1}} \leqq x_1 < x_2 < \dots < x_{v-1}$$

and

$$0 < N_1 < N_2 < \dots < N_{v-1}.$$

We define H_n^2 for $n \leqq N_{v-1}$ so that

$$\text{(2.2)}\qquad H_n^2\begin{cases} > 0, & \text{if } n > N_1(x), \\ = 0, & \text{if } n \leqq N_1(x). \end{cases}$$

Then, if $N(x) > N_{v-1}$,

$$\text{(2.3)}\qquad (1-x)^{\lambda+1}\sum_{n=0}^{N_{v-1}} E_n^\lambda H_n^2 x^n K(x,n) \leqq 0.$$

We may choose $x_v < 1$, since

$$(1-x)^{\lambda+1}\Sigma E_n^\lambda x^n K(x,n) = (1-x)^{\lambda+1}\Sigma E_n^\lambda x^n = 1 \qquad (0<x<1)$$

and

$$(1-x)^{\lambda+1} E_n^\lambda x^n K(x,n) \to 0, \quad \text{as} \quad x \to 1-0 \quad \text{for each } n,$$

so that $N(x_v) > N_{v-1}$ and

$$(1-x_v)^{\lambda+1} \sum_{N_{v-1}+1}^{\infty} E_n^\lambda x_v^n K(x_v,n) < \lambda+2. \tag{2.4}$$

Then

$$(1-x_v)^{\lambda+1} \sum_{N_{v-1}+1}^{N} E_n^\lambda x_v^n K(x_v,n) < \lambda+2, \tag{2.5}$$

for $N > N(x_v)$. Choose $N_v > N(x_v)$ so that

$$(1-x_v)^{\lambda+1} \sum_{N_v+1}^{\infty} (\lambda+n+3)^{2\lambda+3} x_v^n K(x_v,n) < \tag{2.6}$$

$$< \frac{\Gamma(\lambda+1)}{1+\Gamma(\lambda+1)\cdot \frac{(\lambda+3)}{\lambda}\left(\frac{2\lambda+3+\sqrt{4\lambda+1}}{2}\right)^{\frac{2\lambda+3+\sqrt{4\lambda+1}}{2}}}.$$

We now define H_n^2 for $H_{v-1} < n \leqq N_v$ such that

$$H_n^2 = \begin{cases} v^{\lambda+1}, & \text{if} \quad N_{v-1} < n \leqq N_v \quad \text{and} \quad n \neq N(x_v), \\ v^{\lambda+1} + \dfrac{(\lambda+3)\Gamma(\lambda+2)}{\lambda\cdot(\lambda+1)}(N(x_v)+\lambda+1)(N(x_v)+3)^{\lambda+2}\cdot x^{-N(x_v)-1}, & \text{if} \quad n = N(x_v). \end{cases} \tag{2.7}$$

Similarly we can define H_n^2 for all n. Since $H_n^2 \geqq v^{\lambda+1}$ for $n > N_{v-1}$, we have

$$\lim_{n\to\infty} H_n^2 = +\infty. \tag{2.8}$$

If $\dfrac{2}{2\lambda+3+\sqrt{4\lambda+1}} \leqq x < 1$, we have

$$x^{-N(x)-1} < x^{-\left[\frac{(2\lambda-1+\sqrt{4\lambda+1})x+2}{2(1-x)}\right]} < x^{-\frac{2\lambda+1+\sqrt{4\lambda+1}}{2(1-x)}} = \left(1-\frac{a}{y}\right)^{-ay} <$$

$$< \left\{1-\left(\frac{2\lambda+1+\sqrt{4\lambda+1}}{2\lambda+3+\sqrt{4\lambda+1}}\right)\right\}^{-\frac{2\lambda+3+\sqrt{4\lambda+1}}{2}} =$$

$$= \left(\frac{2\lambda+3+\sqrt{4\lambda+1}}{2}\right)^{\frac{2\lambda+3+\sqrt{4\lambda+1}}{2}}.$$

We have, since $N(x_v) > N_{v-1} \geqq v - 1$,

$$0 \leqq H_n^2 < \left\{1 + \Gamma(\lambda + 1)\frac{(\lambda + 3)}{\lambda} \cdot \right. \tag{2.9}$$

$$\left. \cdot \left(\frac{2\lambda + 3 + \sqrt{4\lambda + 1}}{2}\right)^{\frac{2\lambda+3+\sqrt{4\lambda+1}}{2}}\right\}(\lambda + n + 3)^{\lambda+3}$$

which shows that $s_n = O(n^{\lambda+5})$. Hence $(1 - x)^{\lambda+1}\Sigma E_n^\lambda s_n x^n$ converges for $0 < x < 1$. Now

$$\Phi_\lambda(x_v) = (1 - x_v)^{\lambda+1}\left\{\sum_0^{N_v} E_n^\lambda H_n^2 x_v^n K(x_v, n) + \sum_{N_{v-1}+1}^{N_v} v^{\lambda+1} E_n^\lambda x_v^\lambda K(x_v, n) + \right.$$

$$\left. + \sum_{N_v+1}^{\infty} E_n^\lambda H_n^2 x_v^n K(x_v, n)\right\} + I_\lambda(v),$$

where

$$I_\lambda(v) \sim (1 - x_v)^{\lambda+1}\frac{\lambda + 1}{N(x_v) + \lambda + 1} \cdot \frac{N^{\lambda+1}(x_v)}{\Gamma(\lambda + 2)} \cdot \tag{2.10}$$

$$\cdot \frac{(\lambda + 3)\Gamma(\lambda + 2)}{\lambda(\lambda + 1)} \cdot (N(x_v) + \lambda + 1) \cdot$$

$$\cdot (N(x_v) + 3)^{\lambda+2} \cdot x_v^{-N(x_v)-1} \cdot x_v^{N(x_v)} K(x_v, N(x_v)) \leqq$$

$$\leqq -(1 - x_v)^{\lambda+2} \cdot N^{\lambda+1}(x_v) \cdot (N(x_v) + 3)^{\lambda+2} \cdot (\lambda + 3) < -(\lambda + 3)v^{\lambda+1},$$

since

$$1 - x_v > \left\{\left[\frac{(2\lambda - 1 + \sqrt{4\lambda + 1})x_v + 2}{2(1 - x_v)}\right] + 1\right\}^{-1} = (N(x_v) + 3)^{-1}$$

and

$$N(x_v) \geqq v.$$

Thus, we have by (2.6),

$$(1 - x_v)^{\lambda+1}\sum_{N_v+1}^{\infty} E_n^\lambda H_n^2 x_v^n K(x_v, n) < 1. \tag{2.11}$$

Here we have supposed that $K(x_v, n)$ is positive. The inequality is still true for negative $K(x_v, n)$. Thus, by (2.3), (2.5), (2.11) and (2.10), we see that

$$\Phi_\lambda(x_v) < 0 + (\lambda + 2)v^{\lambda+1} + 1 - (\lambda + 3)v^{\lambda+1} = 1 - v^{\lambda+1}.$$

Hence

$$\lim_{x \to 1-0} (x) = -\infty.$$

Thus Theorem **B** is completely established.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GORAKHPUR
GORAKHPUR (U. P.) INDIA

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ON THE „DEL" RELATION IN CERTAIN ATOMISTIC LATTICES

By

M. F. JANOWITZ (Amherst)

In a lattice with 0 the symbol $a \bigtriangledown b$ denotes the fact that $(a \vee x) \wedge b = x \wedge b$ for all $x \in L$. It is shown in [5], pp. 20—25 as well as in [6], pp. 22—24 that the $\bigtriangledown$-relation has an intimate connection with direct sum decompositions of a lattice. As a step toward understanding the structure of atomistic lattices with the covering property, this paper contains an investigation of the $\bigtriangledown$-relation in such a lattice. Specifically, it provides a characterization of $\bigtriangledown$-symmetry and $\bigtriangledown$-continuity for a class of lattices that includes all finite-statisch lattices with the covering property, and relates both $\bigtriangledown$-symmetry and $\bigtriangledown$-continuity to direct sum decompositions of hese lattices.

1. Preliminaries. Though our terminology will follow that of [6] we begin by introducing some of the more basic items. A lattice L with 0 is called $\bigtriangledown$*-symmetric* if $a \bigtriangledown b \Rightarrow b \bigtriangledown a$; it is called $\bigtriangledown$*-continuous* if $a_\alpha \bigtriangledown b$ for every index α and the existence of $a = \vee_\alpha a_\alpha$ together imply $a \bigtriangledown b$. (Note that we are *not* assuming L to be complete.) By [5], Anmerkung 2.2, p. 20 every modular lattice with 0 is $\bigtriangledown$-symmetric. By [2], Corollary 4.4, p. 69, every dual section semicomplemented lattice with 0 (hence every bounded relatively complemented lattice) is both $\bigtriangledown$-symmetric and $\bigtriangledown$-continuous. It is also worth mentioning that ([5], Hilfssatz 2.5, p. 23) every upper continuous lattice is $\bigtriangledown$-continuous. The relation of $\bigtriangledown$-continuity to direct sum decompositions can be seen by examining [6], Lemma 5.6, p. 23. Here it is shown that if a complete $\bigtriangledown$-continuous lattice is the direct sum of a family (S_α) of ideals, then each ideal S_α is a principal ideal generated by a central element z_α, and each $a \in L$ is uniquely expressed as

$$a = \vee_\alpha (z_\alpha \wedge a) \qquad z_\alpha \wedge a \in S_\alpha .$$

An element b of an atomistic lattice is called *finite* if it is 0 or the join of a finite number of atoms; otherwise, it is called *infinite*. An AC-*lattice* is an atomistic lattice. with the covering property:

p an atom, $p \not\leq a$ implies $p \vee a$ covers a.

An FAC-*lattice* is simply an atomistic lattice in which the above covering property holds for p an atom and a finite.

For complete lattices, the notion of a *statisch* atomistic lattice was introduced by WILLE in [7] and later generalized by the author [1] to the concept of a *finite-statisch* atomistic lattice. Without assuming completeness, let us agree to call a lattice *statisch* if p an atom, $p \leqq a \vee b$ implies $p \leqq a_1 \vee b_1$ with a_1, b_1 finite, $a_1 \leqq a$ and $b_1 \leqq b$; it will be called *finite-statisch* if $p \leqq q \vee a$ with p, q atoms implies $p \leqq \leqq q \vee a_1$ for some finite element $a_1 \leqq a$. It should be noted that for a complete lattice these definitions reduce to those earlier given ([7], Satz 3.8, p. 21 and [1], Theorem 3.2, p. 340). Finally, two elements a, b of a lattice with 0 are called *perspective* (denoted $a \sim b$) in case there is an element x such that $a \vee x = b \vee x$ and $a \wedge x = b \wedge x = 0$.

2. Lattices of subspaces. The basic tool we shall use throughout the paper is the embedding described by S. MAEDA ([6], pp. 61–67) of an atomistic lattice into its lattice of subspaces. For the readers convenience the highlights of the construction are reproduced here.

Let L be an atomistic lattice and $\Omega(L)$ the set of atoms of L. A subset ω of $\Omega(L)$ is called a *subspace* when it satisfies the following condition:

$$p \in \Omega(L), \ q_i \in \omega \ (i = 1, 2, \ldots, m) \text{ and } p \leqq q_1 \vee \ldots \vee q_m$$

imply that $p \in \omega$.

By [6], Theorem 15.5, p. 62, for any atomistic lattice L, the set $L(\Omega(L))$ of all subspaces of $\Omega(L)$ is a complete compactly atomistic lattice when ordered by set inclusion. For any $a \in L$, the set $\omega(a) = \{p \in \Omega(L); p \leqq a\}$ is a subspace of $\Omega(L)$. The mapping $a \to \omega(a)$ is a one-one isotone mapping of L into $L(\Omega(L))$. Furthermore, letting $\sqcup$ and $\sqcap$ denote the join and meet operations in $L(\Omega(L))$, one has:

(1) $\omega(0) = 0$ and $\omega(1) = 1$ (if L has a largest element).

(2) $\omega(\wedge_\alpha a_\alpha) = \sqcap_\alpha \omega(a_\alpha)$ (if $\wedge_\alpha a_\alpha$ exists in L).

(3) $\omega(a \vee b) \leqq \omega(a) \sqcup \omega(b)$ for every a and b,
and equality holds if both a and b are finite.

(4) If $F(L)$ denotes the join semilattice formed by the finite elements of L, then $F(L)$ and $F(L(\Omega(L)))$ are isomorphic by the mapping $a \to \omega(a)$.

(5) $a \to \omega(a)$ is an isomorphism of L onto $L(\Omega(L))$ if and only if L is compactly atomistic.

(6) $L(\Omega(L))$ is isomorphic to the lattice of ideals of $F(L)$. (This is stated for AC-lattices in [6] as Theorem 15.7. But the proof given there applies almost verbatim to establish this slight generalization.)

It should now be clear that

(7) L is statisch if and only if for all $a, b \in L$, $\omega(a \vee b) = \omega(a) \sqcup \omega(b)$.

(8) L is finite-statisch if and only if for $a \in F(L)$ and $b \in L$, $\omega(a \vee b) = \omega(a) \sqcup \omega(b)$.

Finally, by [6], Theorem 15.7, p. 63 we have

(9) If L is an FAC-lattice, then $L(\Omega(L))$ is a matroid lattice.

Suppose now that L is an FAC-lattice. For each $a \in L$, let $e(\omega(a))$ denote the meet of all central elements of $L(\Omega(L))$ that dominate $\omega(a)$. Since $L(\Omega(L))$ is a matroid lattice, it follows that $e(\omega(a))$ is the smallest central element of $L(\Omega(L))$ that dominates $\omega(a)$. For future reference we now provide a description of $e(\omega(a))$.

LEMMA 1. *Let L be an* FAC-*lattice, and let* $a \in L$. *Then* $e(\omega(a))$ *is the set of atoms that are perspective in* $F(L)$ *to some atom under* a.

PROOF. Let $p, q \in \Omega(L)$. If $p \sim q$ in $F(L)$ and $q \leqq a$ then for some $x \in F(L)$, $p \vee x = q \vee x$, $p \wedge x = q \wedge x = 0$. Then $\omega(p) \sqcup \omega(x) = \omega(p \vee x) = \omega(q \vee x) = \omega(q) \sqcup \omega(x)$ and $\omega(p) \sqcap \omega(x) = \omega(q) \sqcap \omega(x) = 0$ shows $\omega(p) \sim \omega(q)$ in $L(\Omega(L))$.

Since $\omega(p) \sim \omega(q) \leqq \omega(a) \leqq e(\omega(a))$ with $e(\omega(a))$ central, we must have $\omega(p) \leqq \leqq e(\omega(a))$, whence $p \in e(\omega(a))$.

Conversely, if $p \in e(\omega(a))$ then by [6] Theorem 13.5, p. 56 $\omega(p) \sim \omega(q)$ in $L(\Omega(L))$ for some $q \in \omega(a)$. Thus there is a subspace ω such that $\omega(p) \sqcup \omega = \omega(q) \sqcup \omega$, $\omega(p) \sqcap \omega = \omega(q) \sqcap \omega = 0$. Then $p \in \omega(q) \sqcup \omega$ so there exist $r_1, r_2, \ldots, r_k \in \omega$ such that $p \leqq q \vee r_1 \vee \ldots \vee r_k$. If $x = r_1 \vee r_2 \vee \ldots \vee r_k$ then $x \in F(L)$, $p \leqq q \vee x$ and $p \wedge x = q \wedge x = 0$. Since L is an FAC-lattice, this forces $p \sim q$ in $F(L)$, so p is perspective in $F(L)$ to some atom under a.

3. ***p*-compatible atomistic lattices.** Our basic plan of attack will be to use the fact that the lattice of subspaces of an FAC-lattice is a matroid lattice in order to obtain information about the ∇-relation in the original lattice. For there to be any hope of success in such an endeavor, there must be some connection between the ∇-relation in the two lattices. Our first result along these lines follows.

LEMMA 2. *Let L be an atomistic lattice, and let $a, b \in L$. Then:*

(1) *$a \nabla b$ in L implies $\omega(a) \nabla \omega(b)$ in $L(\Omega(L))$.*

(2) *If L is statisch, then $\omega(a) \nabla \omega(b)$ in $L(\Omega(L))$ implies $a \nabla b$ in L.*

(3) *If L is finite-statisch and if a is finite, then $\omega(a) \nabla \omega(b)$ in $L(\Omega(L))$ implies $a \nabla b$ in L.*

PROOF. (1) Let $\omega \in L(\Omega(L))$. Then for $p \in [\omega(a) \sqcup \omega] \sqcap \omega(b)$, $p \leqq a \vee q_1 \vee q_2 \vee \vee \ldots \vee q_k$ $(q_i \in \omega)$, so $p \leqq (a \vee q_1 \vee q_2 \vee \ldots \vee q_k) \wedge b = (q_1 \vee q_2 \vee \ldots \vee q_k) \wedge b$ puts p in $\omega \sqcap \omega(b)$. Thus

$$(\omega(a) \sqcup \omega) \sqcap \omega(b) \leqq \omega \sqcap \omega(b)$$

and this forces $\omega(a) \nabla \omega(b)$ in $L(\Omega(L))$.

(2) and (3). These both follow from the fact that for each $x \in L$, $\omega(a \vee x) = = \omega(a) \sqcup \omega(x)$.

Though our aim is to investigate the ∇-relation in a finite-statisch atomistic lattice, it turns out that it will cost nothing to present the results in a more general setting. Accordingly, we call an atomistic lattice L *p-compatible* if $p \sim q$ in L implies $p \sim q$ in $F(L)$. If all atoms of $L(\Omega(L))$ are perspective in $F(L)$, then L is clearly p-compatible, and the next lemma shows how the notion of p-compatibility generalizes that of a finite-statisch FAC-lattice.

LEMMA 3. *Every finite-statisch* FAC*-lattice L is a p-compatible* AC*-lattice.*

PROOF. Let $p \in \Omega(L)$, $x \in L$ with $p \not\leqq x$. Then if $q \in \Omega(L)$, $q \leqq p \vee x$ and $q \not\leqq x$, we have $q \leqq p \vee x_1$ with $x_1 \leqq x$ and x_1 finite. Since L is an FAC-lattice $q \vee x_1 = = p \vee x_1$ and so $q \vee x = p \vee x$. This not only shows that L is an AC-lattice, but also shows that it is p-compatible.

Before doing anything else, we present a pair of examples.

EXAMPLE 1. *A p-compatible* AC*-lattice that is not finite-statisch.* Let M be the lattice of subspaces of an infinite dimensional vector space V. Choose an infinite dimensional subspace A having an infinite dimensional complement. Then let $\mathfrak{L} = = \{S;\ S$ is finite dimensional, $S = A$ or $S = V\}$. When ordered by set inclusion, $\mathfrak{L}$ becomes a complete atomistic lattice. To see that it is not finite-statisch, we let S_1, S_2 be distinct one-dimensional subspaces disjoint from A. Then $S_1 \vee A =$

$= S_2 \vee A = V$, but $S_2 \not\leqq S_1 \vee A_1$ for any finite element A_1 under A. The fact that $\mathfrak{L}$ is p-compatible follows from the fact that its atoms are all perspective in $F(\mathfrak{L})$.

EXAMPLE 2. *An atomistic lattice that is not p-compatible.* Let X be an infinite set. Let A, B, C be pairwise disjoint infinite subsets of X whose union is X. Then let $\mathfrak{L} = \{S;\ S \text{ is finite}\} \cup \{A, B, C, X\}$. When ordered by set inclusion, $\mathfrak{L}$ forms a complete atomistic lattice. If $x, y \notin A$, $\{x\} \vee A = \{y\} \vee A = X$ so $\{x\} \sim \{y\}$ in L, but they are not perspective in $F(L)$. This shows that L is not p-compatible. We shall refer back to this example at the conclusion of the paper.

The next result shows how p-compatibility ties in with the ∇-relation.

THEOREM 4. *For an* FAC-*lattice L, the following are equivalent:*

(1) *L is p-compatible.*

(2) *For $p, q \in \Omega(L)$, $p \sim q \Leftrightarrow \omega(p) \sim \omega(q)$.*

(3) *For $a \in F(L)$, $b \in L$, $\omega(a) \nabla \omega(b) \Rightarrow a \nabla b$.*

(4) *For $a \in F(L)$, $x \in L$,*

$$\omega(a \vee x) \leqq e(\omega(a)) \sqcup \omega(x).$$

(5) *For $a \in F(L)$, $x \in L$*

$$\omega(a \vee x) \sqcup e(\omega(a)) = \omega(x) \sqcup e(\omega(a)).$$

PROOF. (1) $\Rightarrow$ (2). If $\omega(p) \sim \omega(q)$ there is a subspace ω such that $\omega(p) \sqcup \omega = \omega(q) \sqcup \omega$, $\omega(p) \sqcap \omega = \omega(q) \sqcap \omega = 0$. Then $p \leqq q \vee r_1 \vee r_2 \vee \dots \vee r_k$ with $r_i \in \omega$. Letting $x = r_1 \vee r_2 \vee \dots \vee r_k$, we have $p \leqq q \vee x$, $p \wedge x = q \wedge x = 0$, so $p \sim q$ in L. If, on the other hand, $p \sim q$ in L, then $p \sim q$ in $F(L)$, so for some finite element x, $p \vee x = q \vee x$, $p \wedge x = q \wedge x = 0$. Then $\omega(p) \sqcup \omega(x) = \omega(q) \sqcup \omega(x)$, $\omega(p) \sqcap \omega(x) = \omega(q) \sqcap \omega(x) = 0$ shows $\omega(p) \sim \omega(q)$.

(2) $\Rightarrow$ (3). Let $a \in F(L)$ and $\omega(a) \nabla \omega(b)$ in $L(\Omega(L))$. Then if $p \in \omega(a)$, $q \in \omega(b)$, $\omega(p) \nabla \omega(q)$ in $L(\Omega(L))$ and by (2), p is not perspective to q in L. By [6], Lemma 11.1, p. 48, $p \nabla q$. Since $p \nabla q$ for all atoms q under b, we may apply [6], Lemma 10.2, p. 44 to see that $p \nabla b$. Since a is finite, $a \nabla b$ now follows.

(3) $\Rightarrow$ (4). Assume $a \in F(L)$, $x \in L$ and $p \in \omega(a \vee x)$. If $p \leqq x$ there is nothing to prove, so assume $p \not\leqq x$. We cannot have $a \nabla p$, so $\omega(a) \nabla \omega(p)$, $\omega(p) \nabla \omega(a)$ and $p \nabla a$ must all fail. It follows that p must be perspective to some atom under a. By Lemma 1, this puts p in $e(\omega(a))$.

(4) $\Rightarrow$ (5). For $a \in F(L)$, $\omega(x) \sqcup e(\omega(a)) \leqq \omega(a \vee x) \sqcup e(\omega(a)) \leqq \omega(x) \sqcup e(\omega(a))$.

(5) $\Rightarrow$ (1). Let $p, q \in \Omega(L)$ with $p \sim q$ in L. Then $p \vee x = q \vee x$, $p \wedge x = q \wedge x = 0$ for some $x \in L$. But

$$\omega(p \vee x) \sqcup e(\omega(p)) = \omega(x) \sqcup e(\omega(p))$$

shows $q \in \omega(x) \sqcup e(\omega(p))$. Since $q \notin \omega(x)$ it follows that $q \in e(\omega(p))$, so $q \sim p$ in $F(L)$.

By [6], Theorem 11.4, p. 49, we now have

COROLLARY 5. *Let L be a p-compatible* FAC-*lattice. If p, q, r are atoms of L, then $p \sim q$, $q \sim r \Rightarrow p \sim r$.*

4. ∇-symmetry. This section is devoted to the investigation of ∇-symmetry in a p-compatible FAC-lattice. Our first result disposes of the statisch case.

THEOREM 6. *Every statisch* AC-*lattice is ∇-symmetric.*

PROOF. For if L is any such lattice, then by Lemma 2, $a \wedge b$ in L is equivalent to $\omega(a) \nabla \omega(b)$ in $L(\Omega(L))$. But since $L(\Omega(L))$ is a matroid lattice, it is relatively complemented, hence ∇-symmetric.

We now turn officially to the general case, breaking our proof up into a number of lemmas.

LEMMA 7. *Let L be a p-compatible* FAC*-lattice. If L is ∇-symmetric then for all $a, x \in L$,*

$$\omega(a \vee x) \leqq e(\omega(a)) \sqcup \omega(x).$$

PROOF. Suppose $\omega(a \vee x) \not\leqq e(\omega(a)) \sqcup \omega(x)$. There must then exist $p \in \omega(a \vee x)$ such that $p \notin e(\omega(a)) \sqcup \omega(x)$. Then $p \notin e(\omega(a))$, so $\omega(p) \nabla \omega(a)$ in $L(\Omega(L))$. By Theorem 4, $p \nabla a$ in L. But $a \nabla p$ fails since $p \leqq a \vee x$, $p \not\leqq a$ and $p \not\leqq x$. This contradiction establishes the lemma.

LEMMA 8. *Let L be an atomistic lattice having the property that for all $a, x \in L$*

$$\omega(x) \sqcup e(\omega(a)) = \omega(a \vee x) \sqcup e(\omega(a)).$$

Then $\omega(a) \nabla \omega(b)$ in $L(\Omega(L))$ implies $a \nabla b$ in L.

PROOF. Let $\omega(a) \nabla \omega(b)$ and p an atom under $(a \vee x) \wedge b$. Then $p \in \omega(a \vee x) \leqq \leqq \omega(a \vee x) \sqcup e(\omega(a)) = \omega(x) \sqcup e(\omega(a))$ so $p \in [e(\omega(a)) \sqcup \omega(x)] \sqcap \omega(b) = \omega(x) \sqcap \omega(b) = = \omega(x \wedge b)$. Thus $(a \vee x) \wedge b = x \wedge b$ and $a \nabla b$.

THEOREM 9. *For a p-compatible* FAC*-lattice L, the following conditions are equivalent:*

(1) *L is ∇-symmetric.*

(2) *$\omega(a) \nabla \omega(b) \Rightarrow a \nabla b$.*

(3) *$\omega(a \vee x) \leqq e(\omega(a)) \sqcup \omega(x)$ for all $a, x \in L$.*

(4) *$\omega(x) \sqcup e(\omega(a)) = \omega(a \vee x) \sqcup e(\omega(a))$ for every $a, x \in L$.*

(5) *$a \nabla b \Rightarrow x = (x \vee a) \wedge (x \vee b)$ for all $x \in L$.*

PROOF. (1) $\Rightarrow$ (3) by Lemma 7.

(3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (2) by Lemma 8.

(2) $\Rightarrow$ (1). Use Lemma 2 together with the fact that $L(\Omega(L))$ is ∇-*symmetric*.

Having established the equivalence of the first four conditions we now observe that (5) $\Rightarrow$ (1) is trivial. The proof will be completed if we can show that (3) $\Rightarrow$ (5). Suppose that $a \nabla b$. Then $\omega(a) \nabla \omega(b)$, so $e(\omega(a)) \sqcap e(\omega(b)) = 0$. For arbitrary $x \in L$,

$$\omega[(x \vee a) \wedge (x \vee b)] = \omega(x \vee a) \sqcap \omega(x \vee b) \leqq$$

$$\leqq [\omega(x) \sqcup e(\omega(a))] \sqcap [\omega(x) \sqcup e(\omega(b))] = \omega(x)$$

shows that $x = (x \vee a) \wedge (x \vee b)$, as desired.

REMARK 10. Lemma 1 may be used to provide another characterization of ∇-symmetry that does not involve the lattice of subspaces. For by Lemma 1 and Theorem 9 a p-compatible FAC-lattice is ∇-symmetric if and only if it satisfies the following condition:

(*) *p an atom, $p \leqq a \vee x$ implies the existence of a finite element $x_1 \leqq x$ and finitely many atoms $q_1, q_2, \ldots, q_k$ such that $p \leqq q_1 \vee \ldots \vee q_k \vee x_1$ and each q_i is perspective to some atom under a.*

In the same spirit Lemma 1 may be combined with Theorem 4 to deduce that an FAC-lattice is p-compatible if and only if condition (*) is satisfied for all *finite* elements a.

This also leads immediately to the following result on homomorphism kernels of such a lattice.

THEOREM 11. *Let L be a ∇-symmetric p-compatible* FAC-*lattice, and let $a \in L$. The following are then equivalent:*

(1) *a is a standard element of L.*

(2) *$\omega(a)$ is central in $L(\Omega(L))$.*

(3) *$[0, a]$ is the kernel of a homomorphism of L.*

(4) *$[0, a]$ is a p-ideal in the sense that $b \leqq a$, $b \sim c$ implies $c \leqq a$.*

(5) *If p, q are atoms, then $p \sim q \leqq a$ implies $p \leqq a$.*

PROOF. (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (2). By Lemma 1, $e(\omega(a))$ is the set of atoms perspective to some atom under a. But condition 5 says that this is just $\omega(a)$.

(2) $\Rightarrow$ (1). Let $x, b \in L$. Then $(a \vee x) \wedge b \geqq (a \wedge b) \vee (x \wedge b)$. If $p \in \omega[(a \vee x) \wedge \wedge b] = \omega(a \vee x) \sqcap \omega(b)$, then since $\omega(a) = e(\omega(a))$ we may apply Theorem 9 to see that

$$p \in [\omega(a) \sqcup \omega(x)] \sqcap \omega(b) = [\omega(a) \sqcap \omega(b)] \sqcup [\omega(x) \sqcap \omega(b)]$$

which shows that $p \in \omega[(a \wedge b) \vee (x \wedge b)]$. Hence $(a \vee x) \wedge b = (a \wedge b) \vee (x \wedge b)$ and a is a standard element of L.

The above theorem should be compared with [3], Theorem 6.7, p. 298.

5. An ideal-theoretic characterization of ∇-symmetry. An ideal I of a lattice L is called *projective* if $a \in I$, $a \vee x \geqq b \vee x$, $a \wedge x \geqq b \wedge x \Rightarrow b \in I$. Our characterization will be in terms of certain subsets of L being projective ideals. It should be pointed out that the kernel of any homomorphism is a projective ideal, and in a section complemented lattice ([3], Theorem 4.2, p. 293) the converse is true.

Assume L is a p-compatible FAC-lattice. For each central element ω of $L(\Omega(L))$, let $I_\omega = \{a \in L;\ \omega(a) \leqq \omega\}$. Suppose that L is ∇-symmetric. Then by Theorem 9, $a, b \in I_\omega$ imply that

$$\omega(a \vee b) \leqq e(\omega(a)) \sqcup \omega(b) \leqq e(\omega(a)) \sqcup e(\omega(b)) \leqq \omega$$

so $a \vee b \in I_\omega$. Since $a \in I_\omega$ clearly implies $c \in I_\omega$ for all $c \leqq a$ we see that I_ω is an ideal of L. To show that I_ω is projective, we assume $a \in I_\omega$, $a \vee x \geqq b \vee x$, $a \wedge x \geqq b \wedge x$, and must show $b \in I_\omega$. But $b \leqq a \vee x \Rightarrow \omega(b) \leqq \omega(a \vee x) \leqq \leqq e(\omega(a)) \sqcup \omega(x)$, so

$$\omega(b) = [\omega(b) \sqcap e(\omega(a))] \sqcup [\omega(b) \sqcap \omega(x)] \leqq$$

$$\leqq e(\omega(a)) \sqcup [\omega(a) \sqcap \omega(x)] \leqq e(\omega(a)) \sqcup \omega(a) \leqq \omega$$

and this does show $b \in I_\omega$.

Suppose on the other hand that for each central element ω of $L(\Omega(L))$, I_ω is a projective ideal of L. We wish to show that L is ∇-symmetric. To do this, we assume that $b \nabla a$ but $a \nabla b$ fails, and arrive at a contradiction. There must exist an $x \in L$ such that $(a \vee x) \wedge b > x \wedge b$. Hence there is an atom $p \leqq (a \vee x) \wedge b$ with $p \not\leqq x \wedge b$. Then $p \leqq a \vee x$ and $p \wedge x = 0$, so $p \vee x \leqq a \vee x$, $p \wedge x \leqq a \wedge x$. Using the fact that $I_{e(\omega(a))}$ is projective, it follows that $\omega(p) \leqq e(\omega(a))$. But $p \leqq b$ also says that $\omega(p) \leqq \omega(b) \leqq e(\omega(b))$, contrary to the fact that $e(\omega(a)) \sqcap e(\omega(b)) = 0$. We deduce from this that L is after all ∇-symmetric, and have thereby established the following theorem.

THEOREM 12. *Let L be a p-compatible* FAC*-lattice. A necessary and sufficient condition for L to be ∇-symmetric is that for each central element ω of $L(\Omega(L))$ I_ω be a projective ideal of L.*

In order to provide a version of Theorem 12 that does not mention the lattice of subspaces of L we need an internal characterization of the sets I_ω. To see how this goes we let P be a set of atoms of L, and set $E(P) = \bigsqcup\{e(\omega(p));\, p \in P\}$, observing that $E(P)$ is the smallest central element of $L(\Omega(L))$ containing all atoms in P. We also let ${}^\nabla P = \{a \in L;\, p \nabla a \text{ for all } p \in P\}$. Our claim is that

(1) ${}^\nabla P = I_{E'(P)}$

where $E'(P)$ is the complement of $E(P)$ in $L(\Omega(L))$. For if $p \nabla a$ then $\omega(p) \nabla \omega(a)$ for all $p \in P$ forces $\omega(a) \leqq e'(\omega(p))$, the complement of $e(\omega(p))$. Thus $a \in {}^\nabla P$ implies

$$\omega(a) \leqq \bigsqcap_{p \in P} e'(\omega(p)) = E'(P).$$

On the other hand, if $\omega(a) \leqq E'(P)$ then for all $p \in P$, $\omega(a) \sqcap e(\omega(p)) = 0$. By Theorem 4, $p \nabla a$. This completes the proof of (1) and establishes that every set of the form ${}^\nabla P$ is also of the form I_ω for suitable ω.

Now let $\omega \in L(\Omega(L))$ be central, and set $P = \{p \in \Omega(L);\, \omega(p) \not\leqq \omega\}$. We wish to prove that

(2) $I_\omega = {}^\nabla P$.

If $\omega(a) \leqq \omega$ we may use Theorem 4 to establish that $p \nabla a$ for all $p \in P$, so $a \in {}^\nabla P$. If on the other hand $p \nabla a$ for all $p \in P$, then $e(\omega(p)) \sqcap e(\omega(a)) = 0$ implies $\omega(a) \leqq$ $\leqq \bigsqcap_{p \in P} e'(\omega(p)) = \omega$, and (2) has been proven. Combining (1) and (2), we have

(3) *Let $A \subseteqq L$. Then $A = {}^\nabla P$ for some set P of atoms of $L \Leftrightarrow A = I_\omega$ for some central element ω of $L(\Omega(L))$.*

Using this we may phrase Theorem 12 in the following form.

THEOREM 12*. *Let L be a p-compatible* FAC*-lattice. A necessary and sufficient condition for L to be ∇-symmetric is that for each set P of atoms of L, ${}^\nabla P$ be a projective ideal of L.*

6. ∇-continuity. As we pointed out in the introduction ∇-continuity seems to have something to do with expressing a lattice as a direct sum of principal ideals generated by central elements. In the case of complete p-compatible FAC-lattices the precise nature of this relationship will be clarified by the results of this section. Before proceeding we shall need the following little generalization of [4], Lemma 5.5.

LEMMA 13. *Every ∇-continuous p-compatible* FAC*-lattice is ∇-symmetric.*

PROOF. If $a \bigtriangledown b$, then $a \bigtriangledown q$ for all $q \in \omega(b)$, $\omega(a) \bigtriangledown \omega(q)$, and by Theorem 4, $q \bigtriangledown a$. Since $q \bigtriangledown a$ for all atoms q under b, we may use $\bigtriangledown$-continuity to deduce that $b \bigtriangledown a$.

In connection with the next theorem it will prove convenient to call an ideal of a lattice *subcomplete* if it is closed under the formation of existing joins.

THEOREM 14. *For a p-compatible* FAC-*lattice, the following are equivalent:*

(1) *L is $\bigtriangledown$-continuous.*

(2) *Whenever $a = \vee_\alpha a_\alpha$ exists in L, then for arbitrary $x \in L$,*

$$\omega(a \vee x) \leqq \sqcup_\alpha e(\omega(a_\alpha)) \sqcup \omega(x).$$

(3) *If $a = \vee_\alpha a_\alpha$ exists in L, then p an atom, $p \leqq a \vee x$ implies the existence of a finite element $x_1 \leqq x$ and finitely many atoms $q_1, q_2, \ldots, q_k$ such that $p \leqq q_1 \vee \ldots \vee q_k \vee x_1$ and each q_i is perspective to an atom under a_{α_i} for some index α_i.*

(4) *For each central element ω of $L(\Omega(L))$, I_ω is a subcomplete projective ideal of L.*

PROOF. (1) $\Rightarrow$ (2). Let L be $\bigtriangledown$-continuous. By Lemma 13, it is $\bigtriangledown$-symmetric. Assume next that $a = \vee_\alpha a_\alpha$ exists in L. Evidently $a_\alpha \leqq a$ implies that $e(\omega(a_\alpha)) \leqq e(\omega(a))$, so that $\sqcup_\alpha e(\omega(a_\alpha)) \leqq e(\omega(a))$. Now if $p \notin \sqcup_\alpha e(\omega(a_\alpha))$, then $\omega(p) \sqcap e(\omega(a_\alpha)) = 0$, and since L is $\bigtriangledown$-symmetric, $a_\alpha \bigtriangledown p$ so that by $\bigtriangledown$-continuity, $a \bigtriangledown p$. This says that $e(\omega(a)) \sqcap e(\omega(p)) = 0$, and $p \notin e(\omega(a))$. It follows that $e(\omega(a)) = \sqcup_\alpha e(\omega(a_\alpha))$. The proof is completed by observing that by $\bigtriangledown$-symmetry, $\omega(a \vee x) \leqq e(\omega(a)) \sqcup \omega(x) = \sqcup_\alpha e(\omega(a_\alpha)) \sqcup \omega(x)$.

(2) $\Rightarrow$ (3). This is clear from Lemma 1.

(3) $\Rightarrow$ (4). By Remark 10, L is $\bigtriangledown$-symmetric. Let ω be a central element of $L(\Omega(L))$. By Theorem 12, I_ω is a projective ideal of L. If $a_\alpha \in I_\omega$ for all α and if $a = \vee_\alpha a_\alpha$ exists in L, then $p \in \omega(a)$ implies $p \leqq q_1 \vee \ldots \vee q_k$ with each $q_i \in e(\omega(a_{\alpha_i})) \leqq \omega$, whence $\omega(a) \leqq \omega$ and $a \in I_\omega$.

(4) $\Rightarrow$ (1). If $a_\alpha \bigtriangledown b$ for all α, then $\omega(a_\alpha) \leqq e'(\omega(b))$ for all α so that $a_\alpha \in I_{e'(\omega(b))}$. If $a = \vee_\alpha a_\alpha$ exists in L, condition (4) assures us that $a \in I_{e'(b))}$, so $\omega(a) \sqcap e(\omega(b)) = 0$, and $\omega(a) \bigtriangledown \omega(b)$. By Theorem 12, L is $\bigtriangledown$-symmetric, so we may apply Theorem 9 to deduce that $a \bigtriangledown b$, whence L is $\bigtriangledown$-continuous.

In the presence of completeness, the situation is even nicer, as indicated by the next theorem.

THEOREM 15. *Let L be a complete p-compatible* FAC-*lattice. The following are equivalent:*

(1) *L is $\bigtriangledown$-continuous.*

(2) *For each ω central in $L(\Omega(L))$ there is a central element $z \in L$ such that $\omega(z) = \omega$.*

PROOF. (1) $\Rightarrow$ (2). Let ω be central in $L(\Omega(L))$ and $z = \vee\{p;\, p \in \omega\}$. Clearly $\omega \leqq \omega(z)$ and by Theorem 14, $z \in I_\omega$ so that $\omega(z) \leqq \omega$; hence $\omega(z) = \omega$. By Theorem 11, z is a standard element of L. By the same token, if $\omega(z') = \omega'$, then z' is standard. Evidently $z \wedge z' = 0$ and $z \vee z' = 1$, so by [3], Theorem 7.3, p. 300 z is central.

(2) $\Rightarrow$ (1). Evidently each I_ω is a subcomplete projective ideal of L, so by Theorem 14, L is $\bigtriangledown$-continuous.

COROLLARY 16. *In a complete $\bigtriangledown$-continuous p-compatible* FAC-*lattice the mapping $z \rightarrow \omega(z)$ is an isomorphism of the centre of L onto the centre of $L(\Omega(L))$*

Several times we have mentioned a relationship between ∇-continuity and certain direct sum decompositions of a lattice. It is time to explore this relationship. In connection with this we agree to call a lattice with 0 *∇-irreducible* in case $a \nabla b$ implies $a = 0$ or $b = 0$. Evidently, a p-compatible FAC-lattice is ∇-irreducible if and only if all of its atoms are perspective. Now by [6], Remark 10.13, p. 47, every complete ∇-continuous atomistic lattice L is the direct product of a family $\{[0, z_\alpha]\}_{\alpha \in A}$ where $\{z_\alpha; \alpha \in A\}$ is the set of atoms of the centre of L. The trouble is that there is no reason to expect that the individual factors should be ∇-irreducible. For the case of a p-compatible FAC-lattice the situation is more gratifying, as is shown by the next theorem.

THEOREM 17. *A complete p-compatible* FAC-*lattice L is ∇-continuous if and only if it is isomorphic to a direct product of ∇-irreducible lattices.*

PROOF. If L is ∇-continuous, let $\{z_\alpha; \alpha \in A\}$ denote the set of atoms of the centre of L. By Theorem 9, each interval sublattice $[0, z_\alpha]$ is ∇-irreducible, and by [6], Remark 10.13, p. 47 L is isomorphic to the direct product of the family $\{[0, z_\alpha]\}_{\alpha \in A}$. If conversely L is isomorphic to a direct product of ∇-irreducible lattices, the proof is completed by observing that any ∇-irreducible lattice is ∇-continuous, and ∇-continuity is preserved by the formation of direct products.

7. Concluding remarks. Let us see how all of this applies to the theory of direct sum decompositions of a p-compatible FAC-lattice L. Suppose L is ∇-symmetric. By Theorem 9, $a \nabla b$ implies $\omega(a \vee b) = \omega(a) \sqcup \omega(b)$, so if $c \leqq a \vee b$ we have

$$\omega(c) = \omega(c) \sqcap \omega(a \vee b) = \omega(c) \sqcap [\omega(a) \sqcup \omega(b)] =$$
$$= [\omega(c) \sqcap \omega(a)] \sqcup [\omega(c) \sqcap \omega(b)] = \omega(c \wedge a) \sqcup \omega(c \wedge b) \leqq$$
$$\leqq \omega[(c \wedge a) \vee (c \wedge b)] \leqq \omega(c),$$

and $c = (c \wedge a) \vee (c \wedge b)$. It follows that a is central in $[0, a \vee b]$ with b as its unique complement. The converse of this also follows rather quickly from Theorem 9. Recall ([6], p. 16) that L is the *direct sum* of the ideals $S_1, S_2, \ldots, S_n$ when

(1) every element $a \in L$ can be expressed in the form

$$a = a_1 \vee \ldots \vee a_n \text{ with } a_i \in S_i \ (i = 1, \ldots, n)$$

and

(2) $a_i \nabla a_j$ if $a_i \in S_i$, $a_j \in S_j$ $(i \neq j)$.

In view of the above remarks, condition (1) may be replaced with

(1′) $S_1 \vee \ldots \vee S_n = L$ in the lattice of ideals of L.

Assume now that L is ∇-continuous, and has the property that if $a = \vee\{a_\alpha; \alpha \in A\}$ exists and if $a_\alpha \nabla a_\beta$ $(\alpha \neq \beta)$, then for each $\alpha \in A$, $\vee\{a_\beta; \beta \neq \alpha\}$ exists. If $p \in \omega(a)$ we may apply Theorem 14 to see that $p \in \sqcup_\alpha e(\omega(a_\alpha))$. It follows that for some index α, $p \nabla a_\alpha$ fails, and $p \nabla a_\beta$ for all $\beta \neq \alpha$. If $a'_\alpha = \vee\{a_\beta; \beta \neq \alpha\}$ we have $a'_\alpha \nabla p$, and $p = (a'_\alpha \vee a_\alpha) \wedge p = a_\alpha \wedge p$. The point to all of this is that $\omega(a) = \sqcup_\alpha \omega(a_\alpha)$. As in our discussion of the ∇-symmetric case, $c \leqq a$ implies $\omega(c) = \sqcup_\alpha \omega(c \wedge a_\alpha)$ from which it follows that $c = \vee_\alpha (c \wedge a_\alpha)$. If one drops the assumption of completeness from [6], Definition 5.2, p. 22, it is now an easy matter to show that L is the direct sum of the family S_α $(\alpha \in A)$ of ideals if and only if the following conditions all hold:

(1) Each ideal S_α is subcomplete in the sense that it is closed under the formation of any existing suprema.

(2) If $a \in S_\alpha$, $b \in S_\beta$ $(\alpha \neq \beta)$, then $a \bigtriangledown b$.

(3) L is the ideal generated by the set of existing suprema of the form $\vee_\alpha a_\alpha$ $(a_\alpha \in S_\alpha)$.

In closing we ask the reader to consider Example 2 of Section 3. This lattice is $\bigtriangledown$-irreducible, hence both $\bigtriangledown$-continuous and $\bigtriangledown$-symmetric. Since $F(\mathfrak{L})$ is distributive, it is easy to show that $L(\Omega(\mathfrak{L}))$ is a Boolean algebra. Yet Theorem 15 fails since, for example, $\omega(A) \cup \omega(B)$ is not of the form $\omega(S)$ for any $S \in \mathfrak{L}$. Even Theorem 9 fails since if $x \notin C$, then $\omega(\{x\}) \bigtriangledown \omega(C)$ without having $\{x\} \bigtriangledown C$ in $\mathfrak{L}$. This example shows that something like p-compatibility is needed before one can do very much with the $\bigtriangledown$-relation.

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UNIVERSITY OF MASSACHUSETTS
AMHERST, MASSACHUSETTS 01002
USA

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ÜBER EINE KLASSE VON L-SPLINE-FUNKTIONEN

Von

U. TIPPENHAUER (Kaiserslautern)

1. Einleitung

Auf der Suche nach einfachen, ökonomischen und flexiblen Approximations- und Interpolationsverfahren ließ man sich weitgehend von der Elastizitätstheorie im Sinne von Euler—Bernoulli leiten und gelangte zu dem Begriff der Spline-Funktion, wie etwa bei BIRKHOFF—DE BOOR [3] dargelegt wird. In den Arbeiten von ATTEIA [2], DELVOS—SCHEMPP [5] und TIPPENHAUER [14] werden die charakteristischen Minimaleigenschaften der Spline-Funktionen mit Hilfe von Projektionsoperatoren formuliert. Zur konkreten Darstellung dieser Spline-Funktionen müssen Aussagen über ihre Struktur gemacht werden, welche grundlegend von der Struktur der klassischen Polynom-Splines abweichen kann. Dieser Aspekt ist bisher in der Literatur kaum beachtet worden.

Wir werden daher auf eine Klasse von Spline-Funktionen in den Grundräumen H_2^k eingehen, welche — neben den natürlichen Polynom-Spline-Funktionen — die L-Splines von AHLBERG—NILSON—WALSH [1], DE BOOR—LYNCH [4] und JEROME—SCHUMAKER [6] umfaßt. Bezüglich der Grundräume H_2^k legen wir die Resultate zugrunde, welche SCHOENBERG [10] im Zusammenhang mit Fragen der optimalen Approximation im Sinne von SARD [7], SCHEMPP [8] im Rahmen der Schwartzschen Distributionentheorie und TIPPENHAUER [15] sowie SCHEMPP—TIPPENHAUER [9] unter dem Aspekt der von SCHWARTZ [11, 12, 13] entwickelten Theorie über Unterhilberträume hausdorffscher lokalkonvexer topologischer Vektorräume und deren assoziierte Kerne erzielt haben. In diesen Grundräumen lassen sich die zugehörigen reproduzierenden Kerne — kurz Reprokerne genannt — bestimmen und mit deren Hilfe Spline Interpolationsprojektoren zu sehr allgemeinen Interpolationsfunktionalen explizit angeben, was bisher nur in speziellen Fällen möglich war.

2. Die kanonische Isomorphie $H_2^{k'} \to H_2^k$

Es sei $\Omega = (a, b)$ ein offenes Intervall von $\mathbf{R}$ und $W_2^k(\Omega)$ der Sobolev-Raum mit dem üblichen Skalarprodukt

$$(1) \qquad (f\,|\,g)_W = \sum_{j=0}^{k} (D^j f\,|\,D^j g)_{L^2(\Omega)}.$$

Die Differentialoperatoren

$$(2) \qquad L_k = \sum_{j=0}^{k} a_j D^j$$

mit $a_k = 1$ und $a_i \in C(I)$, $0 \leqq i \leqq k-1$, $I = [a, b]$, sind Epimorphismen von $W_2^k(\Omega)$ auf $L^2(I)$. Im (starken) topologischen Dual $W_2^k(\Omega)'$ sind die Linearformen

$$(3) \qquad \mathfrak{L} : W_2^k(\Omega) \ni f \to \langle f, \mathfrak{L}\rangle_{W,W'} = \sum_{j=0}^{k-1} \int_I D^j f(t) d\mu_j(t) \in \mathbf{R}$$

enthalten, wobei μ_j, $0 \leqq j \leqq k-1$, reelle Radon-Maße auf I sind. Es seien nun L_k ein Differentialoperator (2) und $\mathfrak{L}_1, \ldots, \mathfrak{L}_k$ Linearformen (3), welche linear unabhängig auf $\operatorname{Ker} L_k$ sind. Durch.

$$(4) \qquad U : W_2^k(\Omega) \ni f \to Uf := (\langle f, \mathfrak{L}\rangle_{W,W'})_{1 \leqq i \leqq k} \in \mathbf{R}^k$$

wird ein Isomorphismus von $\operatorname{Ker} L_k$ auf $\mathbf{R}^k$ und durch

$$(5) \qquad (f, g) \to (f \mid g)_H := (Uf \mid Ug)_{\mathbf{R}^k} + (L_k f \mid L_k g)_{L^2(I)}$$

auf $W_2^k(\Omega)$ ein Skalarprodukt definiert; mit dem Sobolevschen Einbettungssatz läßt sich nachweisen, daß die von (1) und (5) induzierten Normen äquivalent sind. *Wird $W_2^k(\Omega)$ mit der Hilbertschen Struktur (5) versehen, so bezeichnen wir $W_2^k(\Omega)$ mit $H_2^k(I)$.* Es ist nun möglich, die kanonische Isomorphie

$$\phi : H_2^k(I)' \to H_2^k(I)$$

und damit den Darstellungssatz von Riesz für stetige Linearformen auf $H_2^k(I)$ zu realisieren: Ist A der eindeutig bestimmte Reprokern zu $H_2^k(I) \in \operatorname{Hilb}(\mathbf{C}^I)$, so gilt für alle Linearformen $\mathfrak{L} \in H_2^k(I)'$:

$$\phi(\mathfrak{L}) = \langle A, \mathfrak{L}\rangle_{H,H'}.$$

Die Bestimmung des Reprokernes A erfolgt mit

SATZ 1. *Es sei e_i die zu $\mathfrak{L}_i$ duale Basis in* $\operatorname{Ker} L_k$, $1 \leqq i \leqq k$, *und G die Greensche Funktion zu dem Anfangswertproblem*

$$\begin{cases} L_k h = 1, & 1 \in L^2(I) \\ Uh = 0 \end{cases}$$

in $H_2^k(I)$. Dann ist

$$A(x, y) = \sum_{i=1}^{k} e_i(x) \cdot e_i(y) + (G(x, t) \mid G(y, t))_{L^2(I)}.$$

BEWEIS. $f \in H_2^k(I) = \operatorname{Ker} L_k + (\operatorname{Ker} L_k)^\perp$ besitzt eine eindeutige Darstellung $f = f_1 + f_2$, wobei $f_1 \in \operatorname{Ker} L_k$ durch $Uf = Uf_1$ und $f_2 \in (\operatorname{Ker} L_k)^\perp$ durch $L_k f = L_k f_2$ bestimmt ist. Also gilt

$$f(y) = (A(x, y) \mid f(x))_H;$$

vgl. [9 ,15].

Somit gilt für alle $f \in H_2^k(I)$ und alle $\mathfrak{L} \in H_2^k(I)'$:

$$\langle f, \mathfrak{L}\rangle_{H,H'} = (f \mid \langle A, \mathfrak{L}\rangle_{H,H'})_H.$$

3. L-Spline-Interpolation

Es sei L_k ein Differentialoperator vom Typ (2); $\mathfrak{L}_1, \ldots, \mathfrak{L}_k, \ldots, \mathfrak{L}_n$ seien $n \geqq k$ Linearformen vom Typ (3). Ohne Beschränkung der Allgemeinheit seien die "ersten" k Linearformen linear unabhängig auf $\operatorname{Ker} L_k$, welche zur Definition des Skalarproduktes (5) auf $W_2^k(\Omega)$ herangezogen werden. In dem so erhaltenen Hilbertraum $H_2^k(I)$ wird das Problem der L-Spline-Interpolation folgendermaßen formuliert: *Ist $f \in H_2^k(I)$ gegeben, so heißt die Lösung des Minimalisierungsproblems*

$$\| L_k s \|_{L^2(I)} = \min_{h \in I(f)} \| L_k h \|_{L^2(I)}$$

L_k-Spline-Funktion, wobei

$$I(f) := \{h \in H_2^k(I) \mid \langle f, \mathfrak{L}_i \rangle_{H,H'} = \langle h, \mathfrak{L}_i \rangle_{H,H'},\ 1 \leqq i \leqq n\}$$

ist. Unter den obigen Voraussetzungen läßt sich nachweisen, daß stets eine eindeutig bestimmte L_k-Spline-Funktion $s \in I(f)$ existiert. Ist nämlich p die orthogonale Projektion von $H_2^k(I)$ auf den Unterraum S, welcher von den Repräsentanten $\phi(\mathfrak{L}_i) \in H_2^k(I)$ der zugrundeliegenden n stetigen Linearformen aufgespannt wird, so gilt $s = pf$. Für alle Funktionen $g \in S \subset H_2^k(I)$ gilt

$$\| L_k(f-g) \|_{L^2(I)} \geqq \| L_k(f-s) \|_{L^2(I)},$$

mit Gleichheit genau dann, wenn $g = s + e$ und $e \in \operatorname{Ker} L_k$ ist. Zu $\mathfrak{L} \in H_2^k(I)'$ erhält man die eindeutig beste Approximation durch eine Linearkombination $\sum_{i=1}^{n} c_i \mathfrak{L}_i$ der vorgegebenen Linearformen bezüglich der Norm

$$\left\| \mathfrak{L} - \sum_{i=1}^{n} c_i \mathfrak{L}_i \right\|_{H'} = \sup_{\|f\|_H \leqq 1} \left| \left\langle f, \mathfrak{L} - \sum_{i=1}^{n} c_i \mathfrak{L}_i \right\rangle_{H,H'} \right|$$

mit $p \circ \phi(\mathfrak{L}) = \sum_{i=1}^{n} c_i \phi(\mathfrak{L}_i)$; diese Approximation wird also durch

$$\left\langle f, \sum_{i=1}^{n} c_i \mathfrak{L}_i \right\rangle_{H,H'} = \langle s, \mathfrak{L} \rangle_{H,H'}, \qquad c_i \in \mathbf{R}$$

charakterisiert.

SATZ 2. *Es gilt* $\operatorname{Ker} L_k \subset S$.

BEWEIS. Die zu den Linearformen $\mathfrak{L}_1, \ldots, \mathfrak{L}_k$ duale Basis $e_1, \ldots, e_k$ in $\operatorname{Ker} L_k$ ist bezüglich (5) orthonormiert. Mit dem Reprokern A gemäß Satz 1 ist $\langle A(\cdot, x), \mathfrak{L}_j \rangle_{H,H'} = e_j$, $1 \leqq j \leqq k$, also $\operatorname{Ker} L_k \subset S$.

Werden also die Repräsentanten der "restlichen" Linearformen $\mathfrak{L}_{k+1}, \ldots, \mathfrak{L}_n$ derart orthonormiert, daß sie orthonormal zu der Basis $\{e_1, \ldots, e_k\}$ von $\operatorname{Ker} L_k$ sind, so ist das Problem der Spline-Interpolation gelöst. Die orthogonale Projektion

p von $H_2^k(I)$ auf S hat die Darstellung

$$pf = \sum_{i=1}^{n} (f \mid e_i)_H e_i,$$

wobei $\{e_1, \ldots, e_k, \ldots, e_n\}$ die erwähnte Orthonormalbasis von $S = \langle \phi(\mathfrak{L}_1), \ldots, \phi(\mathfrak{L}_n) \rangle$ ist.

4. Beispiele

BEISPIEL 1. Wir wählen $\Omega = (0, 1)$, $L_k = L_2 = D^2$ und die auf $W_2^2(\Omega)$ stetigen Linearformen vom Typ (3)

$$(6) \qquad \langle f, \mathfrak{L}_1 \rangle_{W,W'} = f(0), \qquad \langle f, \mathfrak{L}_2 \rangle_{W,W'} = f'(0), \qquad \langle f, \mathfrak{L}_3 \rangle_{W,W'} = f(1).$$

Das Skalarprodukt auf $H_2^2(I)$ ist somit

$$(f \mid g)_H = f(0) \cdot g(0) + f'(0) \cdot g'(0) + \int_0^1 D^2 f(t) D^2 g(t) dt.$$

Für den Reprokern A zu $H_2^2(I)$ ergibt sich

$$A(x, y) = 1 + x \cdot y + \frac{1}{2} \cdot x^2 \cdot y - \frac{1}{6} \cdot x^3 + \frac{1}{6} \cdot (x - y)_+^3,$$

wobei

$$z_+^k := \begin{cases} z, & z \leqq 0 \\ 0, & z < 0 \end{cases}$$

definiert ist. Zu $f \in H_2^2(I)$ ist

$$s(x) = pf(x) = f(0) + f'(0) \cdot x + \frac{1}{2} \cdot \{f(1) - f(0) - f'(0)\} \cdot (3x^2 - x^3)$$

die eindeutig bestimmte bezüglich (6) interpolierende Spline-Funktion. Der Repräsentant der stetigen Linearform

$$\mathfrak{L} : H_2^2(I) \ni f \to \langle f, \mathfrak{L} \rangle_{H,H'} = \int_0^1 f(t) dt \in \mathbf{R}$$

ist $h(x) = \dfrac{23}{24} + \dfrac{2}{3} \cdot x + \dfrac{1}{24} \cdot (1 - x)_+^4$. Die eindeutig beste Approximation von $\mathfrak{L}$ durch eine Linearkombination $\sum_{i=1}^{3} c_i \mathfrak{L}_i$ ergibt sich mit $c_1 = \dfrac{23}{24}$, $c_2 = \dfrac{1}{2}$ und $c_3 = \dfrac{\sqrt{3}}{8}$.

BEISPIEL 2. Unter den gleichen Voraussetzungen wie in Beispiel 1 wird der Differentialoperator $L_k = L_2 = D^2 \pm id$ gewählt. Das Skalarprodukt auf $H_2^2(I)$ ist somit

$$(f \mid g)_H = f(0) \cdot g(0) + f'(0) \cdot g'(0) + \int_0^1 [D^2 f(t) \pm f(t)][D^2 g(t) \pm g(t)]dt.$$

Im Falle $L_2 = D^2 + id$ ergibt sich für den Reprokern

$$A(x, y) = \cos(x) \cdot \cos(y) + \sin(x) \cdot \sin(y) +$$
$$+ \frac{1}{2} \cdot x \cdot \cos(x - y) + \frac{1}{4} \cdot \{\sin(y - x) - \sin(x + y)\} \cdot (y - x)_+^0 +$$
$$+ \frac{1}{2} \cdot y \cdot \cos(y - x) + \frac{1}{4} \cdot \{\sin(x - y) - \sin(y + x)\} \cdot (x - y)_+^0$$

und somit für die Spline-Funktion

$$s(x) = pf(x) = f(0) \cdot \cos(x) + f'(0) \cdot \sin(x) +$$
$$+ \{f(1) - f(0) \cdot \cos(1) - f'(0) \cdot \sin(1)\} \cdot \frac{x \cdot \cos(x - 1) - \cos(1) \cdot \sin(x)}{1 - \cos(1) \cdot \sin(1)}.$$

Ist $L_2 = D^2 - id$, so ergibt sich

$$A(x, y) = \cosh(x + y) +$$
$$+ \frac{1}{4} \cdot \{\sinh(x + y) + \sinh(x - y) - 2x \cdot \cosh(x - y)\} \cdot (y - x)_+^0 +$$
$$+ \frac{1}{4} \cdot \{\sinh(x + y) + \sinh(y - x) - 2y \cdot \cosh(x - y)\} \cdot (x - y)_+^0$$

und

$$s(x) = pf(x) = f(0) \cdot \cosh(x) + f'(0) \cdot \sinh(x) +$$
$$+ \{f(1) - f(0) \cdot \cosh(1) - f'(0) \cdot \sinh(1)\} \cdot \frac{x \cdot \sinh(x)}{\sinh(1)}.$$

5. Schlußbemerkungen

Mit ähnlichen Methoden lassen sich Spline-Funktionen in mehreren Veränderlichen bestimmen. Dabei ist es nicht notwendig, sich auf Tensorprodukte von Spline-Grundräumen des Types H_2^k zu beschränken. Für geeigneten $\Omega \subset \mathbf{R}^n$ lassen sich die Reprokerne zu $W_2^k(\Omega) \in \mathrm{Hilb}(C^\Omega)$, $k > \frac{n}{2}$, bestimmen und mit denen die entsprechenden Spline-Interpolationsprojektoren.

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FACHBEREICH MATHEMATIK
UNIVERSITÄT TRIER-KAISERSLAUTERN
6750 KAISERSLAUTERN
PFAFFENBERGSTRASSE 95
BUNDESREPUBLIK DEUTSCHLAND

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A NOTE ON RENORMING OF BANACH SPACES DECOMPOSABLE INTO CERTAIN OPERATOR RANGES

By

K. JOHN and V. ZIZLER (Karlin)

The purpose of this note is to present a remark to the renorming techniques of Day, Kadec and Trojanski. The paper is intended to be selfcontained.

The norm $\|\cdot\|$ of a Banach space X is rotund (R) if whenever $\|x\| = \|y\| = \frac{1}{2}\|x+y\|$, then $x = y$. The norm $\|\cdot\|$ is locally uniformly rotund (LUR) if whenever $\|x_n\| = \|x\| = 1$, $\lim \|x_n + x\| = 2$, then $\lim \|x_n - x\| = 0$.

THEOREM 1. *Let X be a Banach space and assume that there is an equicontinuous family $T_\alpha : X \to X$, $\alpha \in \Gamma$ of linear operators with the properties:*

(i) *For any $x \in X$ and $\varepsilon > 0$, the set*

$$\Gamma(x, \varepsilon) = \{\alpha \in \Gamma;\ |T_\alpha x| \geqq \varepsilon\}$$

is finite.

(ii) *For each $x \in X$ and $\varepsilon > 0$, there is a finite $K \subset \Gamma$ such that $|x - \sum_{\alpha \in K'} T_\alpha x| < \varepsilon$ for any finite $K' \supset K$.*

(iii) *For any $\alpha \in \Gamma$, $T_\alpha X$ has an equivalent* LUR *norm $\|\cdot\|_\alpha$. Then X has an equivalent* LUR *norm.*

THEOREM 2. *Let X be a Banach space and assume that there is an equicontinuous family $T_\alpha : X \to X$, $\alpha \in \Gamma$ of linear operators satisfying conditiin* (i) *of Theorem* 1 *as well as the following ones:*

(ii′) *For any $x \in X$, we have $x \in \overline{\text{sp}} \sum_{\alpha \in \Gamma} T_\alpha x$.*

(iii′) *For any $\alpha \in \Gamma$, $T_\alpha X$ has an equivalent R-norm $\|\cdot\|_\alpha$. Then X has an equivalent rotund norm.*

COROLLARY (Day, Lovaglia). *Let X_α, $\alpha \in \Gamma$ be a family of Banach spaces which all have equivalent R* (LUR) *norms. Then their $c_0(\Gamma)$ or $l_p(\Gamma)$, $p \in \langle 1, \infty)$-product has an equivalent R* (LUR) *norm.*

For the proof of Theorems 1, 2 we need the following

LEMMA 1. *Let $(X, |\cdot|)$ be a Banach space which has an equivalent R* (LUR) *norm $\|\cdot\|$. Then for any $\varepsilon > 0$, there is an R* (LUR) *norm $\|\cdot\|_\varepsilon$ such that $|\cdot| \leqq \|\cdot\|_\varepsilon \leqq (1+\varepsilon)|\cdot|$.*

PROOF. Suppose $|\cdot| \leqq \|\cdot\| \leqq K \cdot |\cdot|$. Given an $\varepsilon > 0$, define on X: $\|x\|_\varepsilon = (|x|^2 + \varepsilon K^{-1}\|x\|^2)^{1/2}$. Then, easily, $\|\cdot\|_\varepsilon$ is R (LUR) and $|\cdot| \leqq \|\cdot\|_\varepsilon \leqq (1+\varepsilon)|\cdot|$.

We also apply the following elementary observation used in [2].

LEMMA 2. *Let* $\{s_i\}$ *and* $\{t_i\}$ *be two sequences of nonnegative numbers with* $s_{i+1} \leqq s_i$, $t_{i+1} \leqq t_i$ *and let* β *be a permutation of positive integers. Then formally*

$$\sum_{k=1}^{\infty} s_k t_k - \sum_{k=1}^{\infty} s_k t_{\beta(k)} = \sum_{k=1}^{\infty} (s_k - s_{k+1}) \left(\sum_{i=1}^{k} t_i - \sum_{i=1}^{k} t_{\beta(i)} \right)$$

and thus:

a) $\sum_{k=1}^{\infty} s_k t_k \geqq \sum_{k=1}^{\infty} s_k t_{\beta(k)}$,

b) *for each integer* m,

$$\sum_{k=1}^{\infty} s_k t_k - \sum_{k=1}^{\infty} s_k t_{\beta(k)} \geqq (s_m - s_{m+1})(t_m - t_{m+1})$$

or β *permutes* $1, 2, \ldots, m$ *onto itself.*

The proof of the following lemma is a slight modification of some arguments of [2].

LEMMA 3. *Let* $\{p_\alpha\}$, $\alpha \in \Gamma$ *be a family of seminorms on a linear space* X. *Let the operator* Q *be defined by* $Qx = \{p_\alpha(x)\}$ *map* X *into* $c_0(\Gamma)$. *Let* J *be the Day's norm on* $c_0(\Gamma)$, *i.e.*

$$Jc = \sup \left\{ \left(\sum_{i=1}^{k} 2^{-i} c^2(\alpha_i) \right)^{1/2};\ (\alpha_i)_{i=1}^{k} \text{ finite ordered subsets of } \Gamma \right\}.$$

Then $q = JQ$ *is a seminorm on* X. *Moreover, if*

$$\lim_n \{2q^2(x) + 2q^2(x_n) - q^2(x + x_n)\} = 0, \tag{1}$$

then for each $\alpha \in \Gamma$,

$$\lim_n \{2p_\alpha^2(x) + 2p_\alpha^2(x_n) - p_\alpha^2(x + x_n)\} = 0. \tag{2}$$

REMARK. If for some $\alpha \in \Gamma$, $p_\alpha(x) = |f_\alpha(x)|$, f_α a linear functional on X, then from (2) if follows that $\lim f_\alpha(x_n) - f_\alpha(x) = 0$.

PROOF. The subadditivity of q follows easily from Lemma 2. Now let $\{\alpha_k\}_{k=1}^{\infty}$ be the set of all indices $\alpha \in \Gamma$ satisfying $p_\alpha(x) \neq 0$, ordered so that $p_{\alpha_{k+1}}(x) \leqq p_{\alpha_k}(x)$. Similarly $\{\alpha_k^n\}_k$, $\{\beta_k^n\}_k$ stand for the supports of $\{p_\alpha(x_n)\}_k$, $\{p_\alpha(x+x_n)\}_\alpha$, respectively. We have

$$2q^2(x) + 2q^2(x_n) - q^2(x + x_n) = \sum_k 2^{-k}(2p_{\alpha_k}^2(x) + 2p_{\alpha_k^n}^2(x_n) - \tag{3}$$

$$- p_{\beta_k^n}^2(x + x_n)) \geqq \sum_k 2^{-k}(2p_{\beta_k^n}^2(x) + 2p_{\beta_k^n}^2(x_n) - p_{\beta_k^n}^2(x + x_n)).$$

Suppose that for some α_{k_0},

$$2p_{\alpha_{k_0}}^2(x) + 2p_{\alpha_{k_0}}^2(x_n) - p_{\alpha_{k_0}}^2(x + x_n) \geqq \varepsilon > 0 \tag{4}$$

for some subsequence of $\{x_n\}$ which is again denoted by $\{x_n\}$. Let K be an integer such that $K \geqq k_0$ and $p_{\alpha_K}(x) > p_{\alpha_{K+1}}(x)$. Let us put

$$\delta = 2(2^{-K} - 2^{-K-1})\,\{p^2_{\alpha_K}(x) - p^2_{\alpha_{K+1}}(x)\}.$$

Then using (1) and (3) we have for $n \geqq n_0$

$$\sum_k 2^{-k}\{2p^2_{\alpha_k}(x) + 2p^2_{\alpha_k^n}(x_n) - p^2_{\beta_k^n}(x+x_n)\} - \sum_k 2^{-k}\{2p^2_{\beta_k^n}(x) + 2p^2_{\beta_k^n}(x_n) - p^2_{\beta_k^n}(x+x_n)\} < \delta .$$

Hence

$$2\sum_k 2^{-k}p^2_{\alpha_k}(x) - 2\sum_k 2^{-2}p^2_{\beta_k^n}(x) < 2(2^{-K} - 2^{-K-1})\{p^2_{\alpha_K}(x) - p^2_{\alpha_{K+1}}(x)\}.$$

So, for $n \geqq n_0$, $\{\beta_k^n\}_{k=1}^K$ is a permutation of $\alpha_1, \ldots, \alpha_K$. Consequently, there is a subsequence of $\{x_n\}$ denoted again by $\{x_n\}$ and a fixed permutation $\{\beta_k\}_{k=1}^K$ of $\{\alpha_1, \ldots, \alpha_K\}$ such that $\beta_k^n = \beta_k$, $k = 1, 2, \ldots, K$, $n \geqq n_1$. Thus by (3), (1) we have

$$\lim_n \sum_{k=1}^K 2^{-k}\{2p^2_{\beta_k}(x) + 2p^2_{\beta_k}(x_n) - 2p^2_{\beta_k}(x+x_n)\} = 0,$$

so that for $k = 1, 2, \ldots, K$,

$$\lim_n \{2p^2_{\beta_k}(x) + 2p^2_{\beta_k}(x_n) - p^2_{\beta_k}(x+x_n)\} = 0,$$

which contradicts (4).

So, (2) follows for any $\alpha \in \Gamma$ such that $p_\alpha(x) \neq 0$. If $p_\alpha(x) = 0$, then to prove (2) it suffices to show (by the subadditivity of the p_α's) that $\lim_n p_\alpha(x_n) = 0$.

For this purpose, given an $\varepsilon > 0$, let $K > 0$ be such that $p^2_{\alpha_k}(x) \leqq \varepsilon/4$ for $k \geqq K$. Then if we consider the sequence $\{\alpha_1, \ldots, \alpha_K, \alpha, 0, \ldots\}$ we see by the definition of q that

$$\sum_1^K 2^{-k}p^2_{\alpha_k}(x_n) + 2^{-K-1}p^2_\alpha(x_n) \leqq q^2(x_n),$$

and so

(5) $$2^{-K-1}p^2_\alpha(x_n) \leqq (q^2(x_n) - q^2(x)) +$$
$$+\left(\sum_1^K 2^{-k}p^2_{\alpha_k}(x) - \sum_1^K 2^{-k}p^2_{\alpha_k}(x_n)\right) + \sum_{k=K+1}^\infty 2^{-k}p^2_{\alpha_k}(x).$$

Since

$$2q^2(x_n) + 2q^2(x) - q^2(x+x_n) \geqq (q(x) - q(x_n))^2$$

by the subadditivity of q, we see from (5) and the first part of this proof that for $n \geqq n_0$,

$$2^{-K-1}p^2_\alpha(x_n) \leqq 2\cdot 2^{-K-1}\,\varepsilon/4 + 2\cdot 2^{-K-1}\,\varepsilon/4 .$$

Therefore $\lim p_\alpha(x_n) = 0$.

PROOF OF THEOREM 1. (A modification of the proof of Proposition 1 of [3].) First let for $\alpha \in \Gamma$, $|\cdot|_\alpha$ be an LUR norm on $T_\alpha X$ such that $|\cdot| \leqq |\cdot|_\alpha \leqq 2|\cdot|$ (see Lemma 1). Further assume $|T_\alpha| \leqq 1$. Denote by $\mathfrak{A}_n$ the system of all subsets $K \subset \Gamma$ such that card $K \leqq n$. Define

$$E_K^n(x) = \left| x - \sum_{\alpha \in K} T_\alpha(x) \right|, \qquad K \in \mathfrak{A}_n,$$

$$t_\alpha(x) = |T_\alpha x_\alpha|, \qquad \alpha \in \Gamma,$$

$$F_K(x) = \sum_{\alpha \in K} t_\alpha(x), \qquad K \in \mathfrak{A}_n,$$

$$G_1(x) = |x| \text{ (the original norm on } X),$$

$$G_{k+1}(x) = \sup_{K \in \mathfrak{A}_k} \{E_K^k(x) + k^2 F_K(x)\}; \qquad k = 0, 1, 2, \ldots .$$

Define $\Delta = \{-1, -2, \ldots\} \cup \Gamma$ (assume the last two sets disjoint) and put

$$Qx(\delta) = \begin{cases} 2^\delta G_{-\delta} & \text{for } \delta = -1, -2, \ldots, \\ t_\delta(x) & \text{for } \delta \in \Gamma. \end{cases}$$

Take $|||x||| = JQx$, where J is the Day's norm on $c_0(\Delta)$. Then $|||\cdot|||$ is an equivalent norm to $|\cdot|$. Furthermore if

$$\lim (2 |||x|||^2 + 2 |||x_n|||^2 - |||x + x_n|||^2) = 0,$$

then by Lemma 3,

$$\lim_n (2 |T_\alpha x_n|_\alpha^2 + 2 |T_\alpha x_n|_\alpha^2 - |T_\alpha(x + x_n)|_\alpha^2) = 0,$$

and thus by the LUR property of $|\cdot|_\alpha$, $\lim_n T_\alpha(x_n - x) = 0$ for $\alpha \in \Gamma$.

By (ii) it suffices to show that $\{x_k\}$ is totally bounded. So let $\varepsilon > 0$ be given. First we find by (ii) a $B \in \mathfrak{A}_m$, $B \subset \{\alpha \in \Gamma;\ T_\alpha(x) \neq 0\} = \Gamma(x)$ such that

$$E_B^m(x) < \varepsilon/3 .$$

Let $\Gamma_B(x) = \{\alpha \in \Gamma;\ t_\alpha(x) < \min (t_\beta(x);\ \beta \in B)\}$. Let

$$j = \text{card } \{\alpha \in \Gamma;\ \alpha \in \Gamma(x) \setminus \Gamma_B(x)\} \quad \text{and} \quad b = \min_{\substack{\alpha \in \Gamma_B(x) \\ \beta \in B}} (t_\beta(x) - t_\alpha(x)) .$$

Let

(6) $$n > \max (m, j, [|x| + \{|x|^2 + 8b (|x| + \varepsilon/3)\}^{1/2}] (2b)^{-1}).$$

We find $A \in \mathfrak{A}_n$ such that

(7) $$G_n(x) - \{E_A^n(x) + n^2 F(x)\} \leqq \varepsilon/3 .$$

Next we prove that $B \subset A$ (and thus $E_A^n(x) \leqq E_B^m(x)$, see (ii)). If $B \not\subset A$, then, as in [3], we can take a set $D \in \mathfrak{A}_n \setminus \mathfrak{A}_{n-1}$ such that $t_\alpha(x) \leqq t_\beta(x)$ for $\alpha \notin D$, $\beta \in D$; thus using $n > j$ we have $B \subset D$ and if $B \not\subset A$, then by (6),

$$G_n(x) - \{E_A^n(x) + n^2 F_A(x)\} \geqq E_D^n(x) + n^2 F_D(x) - \{E_A^n(x) + n^2 F_A(x)\} \leqq$$

$$\leqq n^2\{F_D(x) - F_A(x)\} - \{E_D^n(x) + E_A^n(x)\} \leqq n^2 b - (2 + n) \mid x \mid \geq 2\varepsilon/3,$$

contradicting (7).

So, as in [3], there is an integer k_0 such that for $k \geqq k_0$, $E_A^n(x_k) \leqq 2\varepsilon/3$, i.e. $\mid x_k - \sum_{\alpha \in A} T_\alpha x_k \mid \leqq 2\varepsilon/3$, and since we already know that $\lim_k \sum_{\alpha \in A} T_\alpha x_k = \sum_{\alpha \in A} T_\alpha x$, we can choose a finite ε-net for $\{x_k\}$.

Proof of Theorem 2. Choose again for $\alpha \in \Gamma$ an R-norm on $T_\alpha X$ so that $\mid \cdot \mid \leqq \mid \cdot \mid_\alpha \leqq 2 \mid \cdot \mid$ (see Lemma 1). Set $\triangle = \{1\} \cup \Gamma$ (supposing again the last two sets disjoint). Then define

$$Qx(\delta) = \begin{cases} \mid x \mid & \text{for } \delta = 1, \\ \mid T_\delta x \mid_\delta & \text{for } \delta \in \Gamma. \end{cases}$$

Put $||| x ||| = JQx$. Then if $2 ||| x |||^2 + 2 ||| y |||^2 - ||| x + y |||^2 = 0$, then by Lemma 3, $2 \mid T_\alpha x \mid_\alpha^2 + 2 \mid T_\alpha y \mid_\alpha^2 - \mid T_\alpha(x + y) \mid_\alpha^2 = 0$, so by the R-property of $\mid \cdot \mid_\alpha$, $T_\alpha(x - y) = 0$ and from the assumption (ii′) of Theorem 2, $x = y$.

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MATHEMATICAL INSTITUTE
CZECHOSLOVAK ACADEMY OF SCIENCES
AND
CHARLES UNIVERSITY, PRAGUE
CZECHOSLOVAKIA

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HAUSDORFF COMPACTIFICATIONS AND WALLMAN SPACES

By

A. SULTAN (Brooklyn)

Introduction. There are many known methods of constructing all T_2 compactifications of a completely regular T_1 space (see e.g. [1], [2], [5]). Of the methods known, all require the use of some very powerful theorems of analysis or are based on for the most part, indirect methods. It would seem desirable therefore, to devise a method of constructing all T_2 compactifications which is direct. In this paper we give such a method. It is particularly interesting in that it emphasizes in a marked way, the relationships between the structural properties of the compactification and the space we are compactifying. It uses nothing more than the most elementary facts about the Wallman compact space associated with a distributive lattice (see [6]) and point set topology. Furthermore, one gets as immediate corollaries, some important theorems concerning T_2 compactifications. As a further application one gets as a corollary of this general procedure a compactification devised by GORDON [4].

Notations and conventions. We will suppose that Y is any T_2 compactification of a completely regular T_1 space X. $C^*(Y, R)$ will represent the collection of all bounded real valued continuous functions defined on Y. $S(X, R)$ will denote the restrictions of these functions to X. We note that $S(X, R)$ is uniformly closed, contains constants, and separates points of X. If $\mathfrak{z}$ represents the collection of zero sets of functions in $S(X, R)$, and $\hat{\mathfrak{z}}$ represents the collection of zero sets of functions in $C^*(Y, R)$, then as is well known (see [3]), $\hat{\mathfrak{z}}$ forms a base for the closed sets of Y. Furthermore, given any neighbourhood N of $y \in Y$, N contains a zero set neighbourhood of y in Y. Since $S(X, R)$ contains constants, $\emptyset$ and X are elements of $\mathfrak{z}$. Also since the collection $\mathfrak{z}$ forms a distributive lattice, we can form the Wallman compact space $W(\mathfrak{z})$ associated with $\mathfrak{z}$ (see [6]). We say that a z-ultrafilter $F \in W(\mathfrak{z})$ converges to a point $y \in Y$ iff every neighbourhood in Y of y contains a Z in F where $Z \in \mathfrak{z}$. In view of the fact that every neighbourhood in Y contains a zero set neighbourhood, we see that $F \in W(\mathfrak{z})$ converges to $y \in Y$ iff every zero set neighbourhood of y in Y contains a $Z \in F$. We will use the notation $F \to y$ to express the fact that F converges to y, or equivalently that y is the limit of F (written $y = \lim F$).

Construction. It is clear that if $Z^*(f)$ is the zero set of a function f in $C^*(Y, R)$, then $Z^*(f|_X) = Z^*(f) \cap X \in \mathfrak{z}$, where $f|_X$ stands for the restriction of f to X. It is also clear from the compactness of Y that each $F \in W(\mathfrak{z})$ has an adherence point in Y. In fact:

LEMMA 1. *Each z-ultrafilter $F \in W(\mathfrak{z})$ converges to each of its adherence points.*

PROOF. If $y \in Y$ is an adherence point of F, then $y \in \bigcap \bar{Z}_a$, where the Z_a run through F. If Z^* is any zero set neighbourhood of y in Y, then $Z_a \cap Z^* \neq \emptyset$ for any $Z_a \in F$. Thus $Z \doteq Z^* \cap X$, intersects every Z_a in F and since F is maximal $Z \in F$. Since $Z \subset Z^*$, we see that $F \to y$.

Suppose now that $F \in W(\mathfrak{z})$ has two adherence points y_1 and y_2 in Y. Then if Z_1^* and Z_2^* are disjoint zero set neighbourhoods in Y of y_1 and y_2 respectively, by the proof of the previous lemma it follows that the traces of these neighbourhoods are disjoint elements of F which is impossible. We therefore have:

LEMMA 2. *Each $F \in W(\mathfrak{z})$ has a unique adherence point in Y.*

LEMMA 3. *Let $y \in Y$. Then there is an $F \in W(\mathfrak{z})$ such that $F \to y$.*

PROOF. Consider the filter base of zero set neighbourhoods of y in Y. Let T be its trace on X. Since X is dense in Y, $T \cup \{X\}$ has the finite intersection property, and is extendable by Zorn's lemma to a z-ultrafilter $F \in W(\mathfrak{z})$. This is clearly the required z-ultrafilter.

Define for F_1 and $F_2 \in W(\mathfrak{z})$ $F_1 \sim F_2$ iff $\lim F_1 = \lim F_2$. $\sim$ is an equivalence relation. Let $W(\mathfrak{z})/\sim$ represent the resulting quotient space, and h the canonical map from $W(\mathfrak{z})$ to $W(\mathfrak{z})/\sim$. Let H be the map from $W(\mathfrak{z})/\sim$ to Y defined by $H([F]) = \lim F$.

LEMMA 4. *If $Z \in \mathfrak{z}$ then $h^{-1}(H^{-1}(\bar{Z}))$ is closed in $W(\mathfrak{z})$.*

PROOF. Let $A = W(\mathfrak{z}) - h^{-1}(H^{-1}(Z))$. Suppose that $F_0 \in A$. Then $\lim F_0 = y_0 \notin \bar{Z}$. It follows that y_0 has a closed neighbourhood V not intersecting $\bar{Z}$ and that this neighbourhood in turn contains a neighbourhood W which is a cozero set. The set $0 = \{F \in W(\mathfrak{z})$: there exists a $Z_F \in F$ with $Z_F \subset W\}$ is a basic open set in $W(\mathfrak{z})$, contains F_0, and is disjoint from $h^{-1}(H^{-1}(\bar{Z}))$ since if $F \in 0$, $\lim F \in \bar{Z}_F \subset V$. Thus A is open and $h^{-1}(H^{-1}(Z))$ is closed.

THEOREM. *The map H: $W(\mathfrak{z})/\sim \to Y$ where $H([F]) = \lim F$ is a homeomorphism.*

PROOF. H is clearly $1-1$ and is onto by lemma 3. To show that H is continuous, we note that $\{\bar{Z} : Z \in \mathfrak{z}\}$ forms a base for the closed sets of Y. It follows from the previous lemma and the definition of quotient topology, that if $\bar{Z}$ is any basic closed set in Y where $Z \in \mathfrak{z}$, then $H^{-1}(\bar{Z})$ is closed in $W(\mathfrak{z})/\sim$ and thus H is continuous. Since H is a $1-1$ continuous map from a compact space onto a Hausdorff space it is a homeomorphism.

COROLLARY 1. *Every Hausdorff compactification of a completely regular T_1 space is the quotient of a Wallman compact space.*

COROLLARY 2. *If the trace $\mathfrak{z}$ of the zero sets of Y on X has the property that whenever $Z_1 \cap Z_2 = \emptyset$, where $Z_1, Z_2 \in \mathfrak{z}$, then $\bar{Z}_1 \cap \bar{Z}_2 = \emptyset$, then Y is a Wallman compactification.*

PROOF. In this case there can only be one $F \in W(\mathfrak{z})$ converging to any $y \in Y$. For suppose that F_1 and F_2 were both in $W(\mathfrak{z})$ converging to $y \in Y$ and that $F_1 \neq F_2$. Then there would be a $Z_1 \in F_1$ and a $Z_2 \in F_2$ such that $Z_1 \cap Z_2 = \emptyset$. But then by hypothesis $\bar{Z}_1 \cap \bar{Z}_2 = \emptyset$ which is impossible since $y \in \bar{Z}_1 \cap \bar{Z}_2$.

COROLLARY 3. *If each zero set Z^* in $\hat{\mathfrak{z}}$ has the property that $\overline{Z^* \cap X} = Z^*$, then Y is homeomorphic to $W(\hat{\mathfrak{z}})$.*

PROOF. Suppose that Z_1 and $Z_2 \in \mathfrak{z}$. Then there exists Z_1^* and Z_2^* in $\hat{\mathfrak{z}}$ such that $Z_1 = Z_1^* \cap X$ and $Z_2 = Z_2^* \cap X$. By our hypothesis we have the following chain of equalities:

$$\bar{Z}_1 \cap \bar{Z}_2 = \overline{Z_1^* \cap X} \cap \overline{Z_2^* \cap X} = Z_1^* \cap Z_2^* = \overline{(Z_1^* \cap Z_2^*) \cap X} = \overline{(Z_1^* \cap X) \cap (Z_2^* \cap X)} = \overline{Z_1 \cap Z_2}.$$

Thus if $Z_1 \cap Z_2 = \emptyset$ where $Z_1, Z_2 \in \mathfrak{z}$ then $\overline{Z_1 \cap Z_2} = \emptyset$. The result now follows from corollary 2.

In [4] it is assumed that a Banach algebra B of real valued bounded functions defined on a set X is given and that B contains constants, and separates points of X. It is assumed that in X there is a collection of subsets S which is closed under finite intersections and has the property that for each $f \in B$ those neighbourhoods N of zero such that $f^{-1}(N) \in S$ form a fundamental system of neighbourhoods. $\tilde{X}$, the collection of S-ultrafilters is topologized with the topology having as a base for its closed sets, sets of the form $\tilde{f}^{-1}(0)$, where $\tilde{f}\colon \tilde{X} \to R$ and $\tilde{f}(F) = \lim f(F)$ for each $F \in \tilde{X}$ and $f \in B$. The resulting compact space is factored via the equivalenc relation $\sim$ where $F_1 \sim F_2$, $F_1, F_2 \in \tilde{X}$ iff $\lim f(F_1) = \lim f(F_2)$ for all $f \in B$ and is thus turned into a T_2 compactification $\hat{X}$ of X. Each $f \in B$ can be uniquely extended to a continuous function $\hat{f}$ defined on $\hat{X}$, and $\{\hat{f}: f \in B\} = C^*(X, R)$. We see, by applying the results of this paper that $\hat{X}$ is homeomorphic to $W(\mathfrak{z})/\sim$ where $\mathfrak{z} = \{Z(f): f \in B\}$.

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2222 EAST 1-ST STREET
BROOKLYN, NEW YORK 11223
USA

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A NOTE ON PRE-ARITHMETICAL RINGS

By

M. B. BOISEN, Jr. and P. B. SHELDON (Blacksburg)

Jensen has shown that if R is an integral domain for which every proper homomorphic image is a Bezout ring, then R is a Prüfer domain [4, p. 92]. (For convenience, we will adopt the terminology that a ring is a *pre-Bezout ring* in case all of its proper homomorphic images are Bezout rings.) In this note we consider some possible extensions of Jensen's result to the case where R is a commutative ring with unity. In this case, we have a choice of generalizations of the notion of a Prüfer domain to rings with divisors of zero. Those that we will consider are the Prüfer rings and the arithmetical rings, both of which are defined below. We show that every pre-Bezout ring is in the larger of these classes of rings, namely the Prüfer rings, and we show by an example that a pre-Bezout ring is not necessarily in the smaller class, that of the arithmetical rings. Moreover, we broaden the scope of Jensen's result to include *pre-arithmetical rings* — that is, rings for which all proper homomorphic images are arithmetical rings — and conclude that every pre-arithmetical ring is a Prüfer ring. It has been shown elsewhere that this result cannot be further broadened to conclude that a "pre-Prüfer ring" is necessarily a Prüfer ring, even in the domain case [1].

All rings considered will be assumed to be commutative with unity. We will adopt the terminology of Griffin and define a *Prüfer ring* to be a ring in which every finitely generated regular ideal is invertible [3]. An *arithmetical ring* is a ring in which the lattice of ideals is distributive, that is for any three ideals A, B and C, $A \cap (B+C) = (A \cap B) + (A \cap C)$. A *Bezout ring* is a ring in which every finitely generated ideal is principal. The class of all arithmetical rings is contained in the class of all Prüfer rings [3, Theorem 19, p. 65]. Furthermore, the class of all Bezout rings is contained in the class of all arithmetical rings [5, p. 119], and thus the class of all pre-Bezout rings is contained in the class of all pre-arithmetical rings.

THEOREM. *If R is a pre-arithmetical ring, then R is a Prüfer ring.*

PROOF. Suppose that R is a pre-arithmetical ring and that I is a finitely generated regular ideal of R. Then I is invertible in R if and only if IR_M is an invertible ideal of R_M for all maximal ideals M of R [2, Theorem 7.3 p. 71]. We note that since I is a regular ideal of R, IR_M is a regular ideal of R_M [2, apply Lemma 4.1, p. 34]. Also we note that R_M is a pre-arithmetical ring since every proper homomorphic image of R_M is the localization of a proper homomorphic image of R and the localization of an arithmetical ring is again an arithmetical ring [5, Lemma 2, p. 91]. Hence it is sufficient to show that I is invertible in the case where R is a local pre-arithmetical ring. (Note: our use of the word "local" does not imply that the ring is Noetherian.) Let x be a regular element in I. In the proper homomorphic image, $R/(x)$, the image of I is principal since in a local arithmetical ring the ideals are totally ordered [5, Theorem 1, p. 115]. Let a be a preimage of the generator of $I/(x)$. Then $I=(a, x)$. Since x is a regular element, (ax) is a nonzero ideal of R and the ring $R/(ax)$ is a local arithmetical ring. Hence the images in $R/(ax)$ of the ideals (x) and (a) are comparable; consequently, since both (x) and (a) contain (ax), (x) and (a) are comparable ideals of R. Thus either $I=(a)$ or $I=(x)$, and hence I is an invertible ideal.

The converse to this theorem has been shown to be false by an example [1, Example 3.3] of a Prüfer ring which has a homomorphic image which is not even a Prüfer ring, hence not an arithmetical ring. However, the converse is true in the domain case, and so we have the following corollary. The proof of the corollary is an immediate consequence of the facts that every homomorphic image of an arithmetical ring is again an arithmetical ring, and that for an integral domain, the conditions of being a Prüfer ring and being an arithmetical ring are equivalent.

COROLLARY. *An integral domain is a Prüfer domain if and only if it is pre-arithmetical.*

In view of an earlier remark, this corollary states that a pre-arithmetical domain is arithmetical. This result cannot be extended to the ring case as the following example shows. In fact, this example is a non-arithmetical pre-Bezout ring. Thus we have answered in the negative the question whether Jensen's result that a pre-Bezout domain is arithmetical can be extended to the ring case.

EXAMPLE. *Let K be a field and let D be the integral domain consisting of all polynomials in $K[X]$ with zero coefficient on the X term. Let $I=X^4K[X]\cap D$ and let $R=D/I$. Then R has the following two properties:*

(1) *Every proper homomorphic image of R is a Bezout ring.*

(2) *R is not an arithmetical ring.*

PROOF. From our definition of R it is clear that each element of R can be represented uniquely as a polynomial in D of degree less than four. If an element has a nonzero constant term it is a unit since $(\alpha_0+\alpha_2X^2+\alpha_3X^3)(\alpha_0^{-1}-\alpha_2\alpha_0^{-2}X^2-$ $-\alpha_3\alpha_0^{-2}X^3=1$. Hence the set of all polynomials in R with zero constant term is the unique maximal ideal of R. Let A be a proper ideal of R and let $a=\alpha_2X^2+\alpha_3X^3$ be a nonzero element of A. Then either $\alpha_2\neq 0$ or $\alpha_3\neq 0$. Now consider $b=\beta_2X^2+$ $+\beta_3X^3$ and $c=\gamma_2X^2+\gamma_3X^3$. If $\alpha_2\neq 0$, then $b+A=\beta_3'X^3+A$ and $c+A=$ $=\gamma_3'X^3+A$ for some β_3' and γ_3' in K. If either $\beta_3'\neq 0$ or $\gamma_3'\neq 0$, then $b+A$ and $c+A$ generate the ideal $((X^3)+A)/A$ in R/A. Otherwise they generate the zero ideal in R/A. In either case they generate a principal ideal.

Similar reasoning in the case where $\alpha_3\neq 0$ again leads to the conclusion that $b+A$ and $c+A$ generate a principal ideal in R/A.

Hence R/A is a Bezout ring.

Finally we note that (X^2) and (X^3) are incomparable ideals in the local ring R and hence R is not arithmetical.

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VIRGINIA POLYTECHNIC INSTITUTE
AND STATE UNIVERSITY
BLACKSBURG, VIRGINIA 24061
USA

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ON THE RATE OF CONVERGENCE TO NORMALITY FOR SUMS OF DEPENDENT RANDOM VARIABLES

By

A. K. BASU[1] (Sudbury)

1. Introduction. In this paper we attempt to get uniform bounds in the central limit theorem for a class of dependent random variables considered by DVORETZKY [2].

Throughout the sequel we shall consider a double array $\{X_{n,j}, j \leqq k_n\}$ of random variables. Relationships of equality or inequality stated between random variables are to be understood to hold only almost surely. Let $S_{n,k} = \sum_{j=1}^{k} X_{n,j}$ for $k = 0, 1, \ldots, k_n$ and $F_{n,k} = \mathcal{B}(S_{n,k})$ be the σ-field generated by $S_{n,k}$. Assume $\mu_{n,k} = \mathbf{E}(X_{n,k} \mid F_{n,k-1}) = 0$ and the conditional variances $\sigma^2_{n,k} = \mathbf{E}(X^2_{n,k} \mid F_{n,k-1})$ exists almost surely.

Furthermore $\sum_{k=1}^{k_n} \sigma^2_{n,k} = 1$ a.s. and the Lindberg condition

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbf{E}[X^2_{n,k} I(|X_{nk}| > \varepsilon)] = 0$$

for $\varepsilon > 0$ will be assumed to hold. ($I(\cdot)$ is the indicator function of the set within the bracket.)

In section three we shall find a rate of convergence under the additional condition

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbf{E}[|X_{nk}|^{2-\delta}] \to 0$$

for some $\delta > 0$. Of course this Liapaunov condition implies Lindberg condition.

Let $S_n = \sum_{j=1}^{k_n} X_{n,j}$. In section two we have derived estimate of the form $\mathbf{P}(S_n \geqq a_n) = \exp[-(a_n^2/2)(1+o(1))]$ as $n \to \infty$ where $a_n = (1 \pm \varepsilon)(2 \log\log n)^{1/2}$, $\varepsilon > 0$ by examining the convergence rate in Trotter's method of operators. The above large deviation estimate can be used to prove ordinary law of the iterated logarithm for dependent random variables. This motivates our results for section two. But a careful analysis shows that Trotter's method yields a slower rate of convergence to zero than is actually known to be the case for i.i.d. random variables. This point was noted in the introduction to PINSKY's note [8] and was pointed out more generally in FELLER [4], Chapter IX.

J. CHOVER [1] proved the functional law of the iterated logarithm for i.i.d. random variables. One of the key tools was a Berry–Esseen Theorem. Chover's idea

[1] This work was partly done when the author was at 1973 Carleton Summer Research Institute of Canadian Mathematical Congress. Appreciation is extended to N. R. C. of Canada and the Canadian Mathematical Congress for financial support of this work.

was more exploited by IOSIFESCU [6] and BERKES for dependent random variables. It is this result of Chover's that provided the stimulus for section three. Though we have not been successful to get Berry—Esseen bound as sharp as in the case of independent random variables, but yet better than many known bounds for dependent random variables (c.f. EBRAGIMOV [3] and GRAMS [5]). The rate of convergence obtained in section three will be used to prove the functional law of the iterated logarithm for dependent random variables, in a subsequent paper. After writing this manuscript we came to know that Theorem 1 and its corrollary is proved by PINSKY [8] when random variables are independent.

2. We shall state a lemma due to DVORETZKY [2] and it will permit us replacing the conditioning by $\mathcal{B}(S_{n,k-1})$ by a finer conditioning relative to an increasing sequence of σ-fields.

LEMMA (DVORETZKY [2]). *Let $S_1, S_2, \ldots, S_n$ be random variables defined on some probability space (Ω, F, P). Then there exists a probability space $(\bar{\Omega}, \bar{F}, \bar{P})$ and random variables $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_n$ defined on it such that*

$$\bar{\mathbf{P}}(\bar{S}_k \leqq u, \bar{S}_{k-1} \leqq v) = \mathbf{P}(S_k \leqq u, S_{k-1} \leqq v), \qquad k = 2, 3, \ldots, n$$

for all real u, v and

$$\bar{\mathbf{P}}(\bar{S}_k \mid \bar{F}_{k-1}) = \bar{\mathbf{P}}(\bar{S}_k \mid G_{k-1}), \qquad k = 2, 3, \ldots, n,$$

where $\bar{F}_k = \mathcal{B}(\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_k)$ and $G_k = \mathcal{B}(\bar{S}_k)$.

Now we may assume the σ-fields $F_{n,k}$ to be nondecreasing i.e.

$$F_{n,1} \subset F_{n,2} \subset \ldots \subset F_{n,k-1} \subset F_{n,k} \subset \ldots \subset F_{n,k_n}.$$

We denote by $F_{n,0}$ the trivial σ-field. ($X_{n,k}$ is assumed to be $F_{n,k}$-measurable but $\mathcal{B}(X_{n,1}, \ldots, X_{n,k})$ may be a proper subfield of $F_{n,k}$.)

THEOREM 1. *If f is a real-valued function on the real line with three bounded derivatives, then for $1 \geqq \delta > 0$*

$$\left| \mathbf{E}[f(S_n)] - \int_{-\infty}^{\infty} f(x)(2\pi)^{-\frac{1}{2}} \exp(-x^2/2)dx \right| \leqq$$

$$\leqq K \| f \| \left[\sum_{k=1}^{k_n} \mathbf{E}[| X_{n,k} |^{2+\delta}] + C_\delta \sum_{k=1}^{k_n} \mathbf{E} \, | \sigma_{n,k} |^2 \right] + \delta$$

where $\| f \| = \sup_x [| f''(x) | + | f'''(x) |]$, C_δ is a function of δ only, and K is a constant.

PROOF. Let $Y_{n,k}$ be completely independent standard normal random variables and independent of the $X_{n,k}$. Denote by $Z_{n,k} = X_{n,1} + \ldots + X_{n,k-1} + \sigma_{n,k+1} Y_{n,+1} + \ldots + \sigma_{n,k_n} Y_{n,k_n}$ for $k = 1, 2, \ldots, k_n$. Then

$$\left| \mathbf{E}[f(S_n)] - \int_{-\infty}^{\infty} f(x)(2\pi)^{-\frac{1}{2}} \exp(-x^2/2)dx \right| = \left| \mathbf{E}[f(S_n)] - \mathbf{E}\left[f\left(\sum_{k=1}^{k_n} \sigma_{n,k} Y_{n,k} \right) \right] \right| =$$

$$= \sum_{k=1}^{k_n} \{\mathbf{E}[f(Z_{n,k} + X_{n,k})] - \mathbf{E}[f(Z_{n,k} + \sigma_{n,k}Y_{n,k})]\} =$$

$$= \sum_{k=1}^{k_n} \mathbf{E}\{\mathbf{E}[f(Z_{n,k} + X_{n,k}) \mid Z_{n,k}] - \mathbf{E}[f(Z_{n,k} + \sigma_{n,k}Y_{n.k}) \mid Z_{n.k}]\} \leqq$$

$$\leqq \sum_{k=1}^{k_n} \mathbf{E} \mid \mathbf{E}[f(Z_{n,k} + X_{n,k}) \mid F_{n,k-1}] - \mathbf{E}[f(Z_{n,k} + \sigma_{n,k}Y_{n,k}) \mid F_{n,k-1}] \mid$$

(making use of the modified Taylor estimate $| f(x + h) - f(x) - hf'(x) - h^2/2f''(x) | \leqq | h |^{2+\delta} \| f \|$),

$$\leqq K \| f \| \left[\sum_{k=1}^{k_n} [\mathbf{E} \mid X_{n,k} \mid^{2+\delta}] + C_\delta \sum_{k=1}^{k_n} \mathbf{E}[\mid \sigma_{n,k} \mid^{2+\delta}] \right].$$

COROLLARY. *Let* $r_n = \sum_{k=1}^{k_n} \mathbf{E}[\mid X_{n,k} \mid^{2+\delta}]$ *for* $0 < \delta \leqq 1$ *and* $r_n \to 0$. *If* $\{a_n\}$ *is a sequence of real numbers such that* $a_n^2/\log(1/r_n) \to 0$ *as* $n \to \infty$, *then for each* $\varepsilon > 0$

$$\exp\{-(a_n^2/2)(1 + \varepsilon) \leqq \mathbf{P}(S_n \geqq a_n) \leqq \exp\{(-a_n^2/2)(1 - \varepsilon)\} \quad \textit{for} \quad n \geqq N_0(\varepsilon).$$

PROOF. Let $f_n^{\pm}(x) = f_0(x - a_n \mp 1/2)$ where f_0 is a fixed C_3-function vanishing for $x \leqq -1/2$, equal to one for $x \geqq 1/2$ with $0 \leqq f_0 \leqq 1$. Then $\mathbf{E}[f_n^-(S_n)] \leqq \mathbf{P}(S_n \geqq a_n) \leqq \mathbf{E}[f_n^+(S_n)]$. Applying Theorem 1,

$$1 - \Phi(a_n + 1) - \bar{K}r_n \leqq \mathbf{P}(S_n \geqq a_n) \leqq 1 - \Phi(a_n + 1) + \bar{K}r_n$$

where $\bar{K}$ is independent of n. Using a well-known estimate for the tail of normal distribution the result follows.

3. Let $T_n = \sum_{k=1}^{k_n} \sigma_{n,k}Y_{n,k}$, f_n be the characteristic function of S_n and g_n be the characteristic function of T_n. $f_{n,k}$ and $g_{n,k}$ are the characteristic functions of $X_{n,k}$ and $\sigma_{n,k}Y_{n,k}$ respectively. Stars will denote conditioning by $Z_{n,k}$; for example,

$$f_{n,k}^* = \mathbf{E}(\exp(iuX_{n,k}) \mid Z_{n,k}), \qquad g_{n,k}^* = \mathbf{E}(\exp(iu\sigma_{n.k}Y_{n,k}) \mid Z_{n,k}).$$

Let

$$\Phi(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{u} \exp(-x^2/2)dx.$$

THEOREM 2.

$$\sup_u \mid \mathbf{P}(S_n \leqq u) - \Phi(u) \mid = O\left(\sqrt{\sum_{k=1}^{k_n} \mathbf{E}[\mid X_{n,k} \mid^{2+\delta}] + \sum_{k=1}^{k_n} \mathbf{E}[\mid \sigma_{n,k} \mid^{2+\delta}]}\right) +$$

$$+ O\left[\sum_{k=1}^{k_n} \mathbf{E}[\mid X_{n,k} \mid^{2+\delta}] + \sum_{k=1}^{k_n} \mathbf{E}[\mid \sigma_{n,k} \mid^{2+\delta}]\right]^{\frac{1}{4+2\delta}}$$

for $0 < \delta \leqq 1$.

PROOF.

$$e^{iuS_n} - e^{iuT_n} = \sum_{k=1}^{k_n} (e^{iuX_{n,k}} - e^{iu\sigma_{n,k}+Y_{n,k}})e^{iuZ_{n,k}}.$$

Therefore,

$$|f_n - g_n| \leqq \sum_{k=1}^{k_n} \mathbf{E}|f^*_{n,k} - g^*_{n,k}|.$$

We know

$$|e^{iu} - 1 - iu - (iu)^2/2| \leqq C_\delta |u|^{2+\delta} \qquad (0 < \delta \leqq 1).$$

Therefore,

$$|f^*_{n,k}(u) - g^*_{n,k}(u)| \leqq |u| |\mathbf{E}^*(X_{n,k}) - \mathbf{E}^*(\sigma_{n,k}Y_{n,k})| +$$
$$+ u^2/2 |\mathbf{E}^*(X^2_{n,k}) - \mathbf{E}^*(\sigma^2_{n,k}Y^2_{n,k})| + C_\delta |u|^{2+\delta}\{\mathbf{E}^*|X_{n,k}|^{2+\delta} + \mathbf{E}^*|\sigma_{n,k}Y_{n,k}|^{2+\delta}\}.$$

Since the $Y_{n.k}$'s are completely independent standard normal variables $N(0, 1)$ and independent of the $X_{n,k}$'s, and $\sum_{k=1}^{k_n} \sigma^2_{n,k} = 1$ a.s.,

$$\mathbf{E}^*(X_{n,k}) = \mathbf{E}(X_{n,k} | Z_{n,k}) = \mathbf{E}(X_{n.k} | F_{n.k-1}) = 0,$$
$$\mathbf{E}^*(\sigma_{n,k}Y_{n,k}) = \mathbf{E}(\sigma_{n,k}Y_{n,k} | Z_{n.k}) = \sigma_{n.k}\mathbf{E}(Y_{n,k} | F_{n,k-1}) = \sigma_{n,k}\mathbf{E}(Y_{n,k}) = 0,$$
$$\mathbf{E}^*(X^2_{n,k}) - \mathbf{E}^*(\sigma^2_{n,k}Y_{n,k}) = \mathbf{E}(X^2_{n,k} | Z_{n,k}) - \mathbf{E}(\sigma^2_{n,k}Y^2_{n,k} | Z_{n,k}) =$$
$$= \mathbf{E}(X^2_{n.k} | F_{n,k-1}) - \sigma^2_{n,k}\mathbf{E}(Y^2_{n,k} | F_{n,k-1}) = \sigma^2_{n.k} - \sigma^2_{n,k}\mathbf{E}(Y^2_{n,k}) = 0.$$

Therefore,

$$|f_n(u) - g_n(u)| = |f_n(u) - e^{-u^2/2}| \leqq C_\delta |u|^{2+\delta} \sum_{k=1}^{k_n} \{\mathbf{E}|X_{n,k}|^{2+\delta} + \mathbf{E}|\sigma_{n,k}|^{2+\delta}\}.$$

Now take

$$U = C\left[\sum_{k=1}^{k_n} \{\mathbf{E}|X_{n,k}|^{2+\delta} + \mathbf{E}|\sigma_{n,k}|^{2+\delta}\right]^{-\frac{1}{4+2\delta}}$$

(C is a suitable constant) in the basic lemma of LOEVE [7, page 289]. Then,

$$\sup |\mathbf{P}(S_n \leqq u - \Phi(u)| \leqq 2\pi^{-1}C_\delta \int_0^U u^{1+\delta} \sum_{k=1}^{k_n} \{\mathbf{E}[|X_{n,k}|^{2+\delta}] + \mathbf{E}[|\sigma_{n,k}|^{2+\delta}]\}du +$$
$$+ 24(\pi U)^{-1}(2\pi)^{-1/2} = O\left(\sqrt{\sum_{k=1}^{k_n} (\mathbf{E}|X_{n.k}|^{2+\delta} + \sum_{k=1}^{k_n} \mathbf{E}|\sigma_{n,k}|^{2+\delta}}\right) +$$
$$+ O\left[\sum_{k=1}^{k_n} \mathbf{E}|X_{n,k}|^{2+\delta} + \sum_{k=1}^{k_n} \mathbf{E}|\sigma_{n,k}|^{2+\delta}\right]^{\frac{1}{4+2\delta}}. \text{ Q.e.d.}$$

Note that this bound is of $O(n^{-1/4})$ if random variables are independent, uniformly bounded and $\delta = 1$, and is not as good as $O(n^{-1/2})$ obtained by Berry and Esseen.

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LAURENTIAN UNIVERSITY
DEPARTMENT OF MATHEMATICS
SUDBURY, ONTARIO, CANADA

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Ext $(.,Z)$-REPRODUCED ABELIAN GROUPS ARE FINITE

By

F. R. BEYL (Heidelberg) and A. HANNA (Beirut)

1. Introduction and notations. The purpose of this note is to show that an isomorphism Ext $(A, Z) \cong A$ holds for an abelian group A if and only if A is finite. This result is derived from the formula Ext $(Z^p, Z) \cong Z(p^\infty) \oplus Q^{\aleph}$ (notation given below). Although this note relates to the extensive investigations of GRÄBE—VILJOEN [1], it does not seem that they observed the present main result. An immediate consequence is that Hom $(A, Q/Z) \cong A$ if and only if A is finite, a result proved by GROSSE [2].

Notation.

Z	additive group of integers
Q	additive group of rational numbers
R	additive group of real numbers
Z^p	additive group of p-adic integers
$Z(p^\infty)$	quasi-cyclic group
$C(p)$	cyclic group of order p
$\|I\|$	cardinality of a set I
$\aleph_0$	cardinality of Z
$\aleph$	cardinality of R
$A^{(I)}$	direct sum of $\|I\|$ copies of a group A
A^I	direct product of $\|I\|$ copies of a group A
tA	torsion part of a group A.

All groups considered are additively written abelian groups.

2. THEOREM. *A group A is such that* Ext $(A, Z) \cong A$ *if and only if A is finite.*

The proof is preceded by two lemmas.

LEMMA 1. Ext $(Z^p, Z) \cong Z(p^\infty) \oplus Q^{\aleph}$.

PROOF. Since Z^p is torsion free, Ext (Z^p, Z) is divisible, and since it is indecomposable, Hom $(Z^p, Z) = 0$. Therefore, the exact sequence

$$0 \to Z^p \xrightarrow{u} Z^p \to C(p) \to 0,$$

where u is multiplication by p, induces the exact sequence

$$0 \to C(p) \to \text{Ext}\,(Z^p, Z) \xrightarrow{u_*} \text{Ext}\,(Z^p, Z) \to 0,$$

where u_* is also multiplication by p. Hence Ext (Z^p, Z) contains exactly one copy of $Z(p^\infty)$. To determine the torsion free part of Ext (Z^p, Z), consider the following two subgroups of the p-adic field: Z as the subgroup generated by 1, and P the subgroup consisting of all rational numbers with denominators prime to p. Then Z^p/Z is divisible and $t(Z^p/Z) \cong P/Z \cong \bigoplus_{q \neq p} Z(q^\infty)$. Thus there is an exact sequence

$$0 \to Z \to Z^p \to Q^{\aleph} \bigoplus_{q \neq p} Z(q^\infty) \to 0.$$

This yields the exact sequence

$$0 \to Z \to R^{\aleph} \oplus \prod_{q \neq p} Z^q \to \operatorname{Ext}(Z^p, Z) \to 0,$$

which implies that $|\operatorname{Ext}(Z^p, Z)| = 2^{\aleph}$. Hence the torsion free part of $\operatorname{Ext}(Z^p, Z)$ is isomorphic to $Q^{\aleph}$. As Z^p is uniquely divisible by any prime $q \neq p$, so is $\operatorname{Ext}(Z^p, Z)$. Therefore $\operatorname{Ext}(Z^p, Z) \cong Z(p^\infty) \oplus Q^{\aleph}$.

LEMMA 2. *If* $\operatorname{Ext}(A, Z) \cong A$, *then* A *is reduced.*

PROOF. Suppose A is not reduced. Then the torsion free divisible part of A is $\neq 0$. For if $Z(p^\infty)$ is a direct summand of A, the previous lemma implies that $\operatorname{Ext}(\operatorname{Ext}(Z(p^\infty), Z), Z) \cong \operatorname{Ext}(Z^p, Z) \cong Z(p^\infty) \oplus Q^{\aleph}$ is also a direct summand of A. Let $Q^{(\alpha)}$, $\alpha \neq 0$, be the torsion free divisible part of A, then $Q^{(\alpha)}$ contains $\operatorname{Ext}(Q^{(\alpha)}, Z) \cong R^{\alpha}$. Since this contradicts $\aleph_0 \cdot \alpha < \aleph^{\alpha}$, A is reduced.

PROOF OF THE THEOREM. If A is finite, then $\operatorname{Ext}(A, Z) \cong \operatorname{Hom}(A, Q/Z) \cong A$.

Conversely, suppose $\operatorname{Ext}(A, Z) = A$. Consider the exact sequences

$$0 \to tA \to A \to A/tA \to 0$$

and

$$0 \to \operatorname{Ext}(A/tA, Z) \to \operatorname{Ext}(A, Z) \to \operatorname{Ext}(tA, Z) \to 0.$$

By Lemma 2, the divisible group $\operatorname{Ext}(A/tA, Z)$ must be zero. Consequently $A \cong \operatorname{Ext}(A, Z) \cong \operatorname{Ext}(tA, Z) \cong \prod \operatorname{Ext}(T_p, Z)$, where $\oplus T_p$ is a direct sum decomposition of tA into its primary components. Every primary component T_p must be a direct sum $\bigoplus_{i \in I_p} C_i$ of cyclic groups. For let B be the basic subgroup of T_p, and assume $B \neq T_p$. Then the non-trivial divisible group T_p/B possesses $Z(p^\infty)$ as a direct summand. Consequently there exists an epimorphism $tA \to T_p \to Z(p^\infty)$, and hence $Z^p = \operatorname{Ext}(Z(p^\infty), Z)$ is isomorphic to a subgroup of $\operatorname{Ext}(tA, Z) \cong A$. This implies that Z^p is isomorphic to a subgroup of A/tA which contradicts $\operatorname{Ext}(A/tA, Z) = 0$ and $\operatorname{Ext}(Z^p, Z) \neq 0$.

Next we prove that every primary component $T_p = \bigoplus_{i \in I_p} C_i$ of tA is finite. Equivalently, we prove that I_p is a finite set. Suppose it is infinite. If the orders of the p-groups C_i did not have a common bound, the direct summand $\prod_{i \in I_p} C_i = \operatorname{Ext}(\bigoplus_{i \in I_p} C_i, Z)$ of A would contain a subgroup isomorphic to Z^p. This would lead again to a contradiction. Due to the common bound, $\prod_{i \in I_p} C_i$ is a p-torsion group, and hence it is a subgroup of $\bigoplus_{i \in I_p} C_i$. But

$$\Big|\prod_{i \in I_p} C_i\Big| = 2^{|I_p|} > |I_p| = \Big|\bigoplus_{i \in I_p} C_i\Big|,$$

a contradiction. Therefore I_p must be finite.

Finally, we show that $T_p \neq 0$ for a finite number of prime numbers. Since every T_p is finite, $A \cong \prod \operatorname{Ext}(T_p, Z) \cong \prod T_p$. If the primary components $\neq 0$ were infinite, then $A/tA = \prod T_p/\oplus T_p$ would be a rational vector space, which contradicts $\operatorname{Ext}(A/tA, Z) = 0$.

COROLLARY. *Hom $(A, Q/Z) \cong A$ if and only if A is finite.*

PROOF. An argument very similar to that in the proof of Lemma 2 shows that $\operatorname{Hom}(A, Q/Z) \cong A$ implies that A is reduced. It follows that $\operatorname{Hom}(A, Z) \cong 0$, for otherwise Z and consequently $\operatorname{Hom}(Z, Q/Z) = Q/Z$ would be a direct summand of A. Thus the exact sequence

$$0 \to Z \to Q \to Q/Z \to 0$$

induces the exact sequence

$$0 \to \operatorname{Hom}(A, Q) \to \operatorname{Hom}(A, Q/Z) \to \operatorname{Ext}(A, Z) \to 0.$$

The latter sequence exhibits the rational vector space $\operatorname{Hom}(A, Q)$ as a subgroup of A. Hence $\operatorname{Hom}(A, Q) = 0$, and consequently $\operatorname{Hom}(A, Q/Z) \cong \operatorname{Ext}(A, Z)$. Thus $\operatorname{Hom}(A, Q/Z) \cong A$ implies A is finite.

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MATHEMATISCHES INSTITUT DER UNIVERSITÄT
69 HEIDELBERG 1
IM NEUENHEIMER FELD 288
GERMANY

DEPARTMENT OF MATHEMATICS
AMERICAN UNIVERSITY OF BEIRUT
BEIRUT, LEBANON

Acta Mathematica Academiae Scientiarum Hungaricae
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SOME INEQUALITIES FOR CERTAIN POWER SUMS

By

C. J. SMYTH (Turku)

1. Let $g(k)=\sum_{j=1}^{n} b_j z_j^k$, where $b_j>0$ $(j=1,\dots,n)$, and the z_j are complex numbers with $\max_{1\leqq j\leqq n}|z_j|=|z_1|=1$. We shall prove that for each λ: $0<\lambda\leqq 1$ and $k\in\mathbf{Z}$

$$\max_{1\leqq k\leqq K}\Re g(k)>\frac{b_1}{4}(1-\lambda) \tag{1}$$

for $K=[\lambda^{-1}(4\beta+3)]+1$. Here $\beta=(b_2+\dots+b_n)/b_1$. In particular $(\lambda=1)$

$$\max_{1\leqq k\leqq K}\Re g(k)>0 \tag{2}$$

for $K=[4\beta+4]$. This improves a result of CASSELS [2, § 4] who proved (2) for $K=2[\beta^2+6\beta]+7$. We shall give an example to show that $K=[4\beta+4]$ is best possible in (2) in the special case $b_1=\dots=b_n=1$.

Next, we let $|z_1|=\dots=|z_n|=1$, and derive the known result (due to Cassels and Turán) that for each $\varepsilon>0$

$$\max_{1\leqq k\leqq(1+\varepsilon)n}|g(k)|\geqq\frac{c(\varepsilon)}{\sqrt{n}}|g(0)|, \tag{3}$$

where we can take $c(\varepsilon)=\{\varepsilon/(1+\varepsilon)\}^{\frac{1}{2}}$.

Finally, we apply an extension of (1) to give (Lemma 2) a simple proof of Lemma 3 in BLANKSBY and MONTGOMERY [1]. (This lemma forms part of the proof of their important result [1, Theorem 1] concerning algebraic integers near the unit circle.) The proofs of all the results are based on Lemma 1, due essentially to BLANKSBY and MONTGOMERY [1].

I would like to thank Professors Cassels and Turán for their helpful comments.

2. For $\mathbf{x}=(x_1,\dots,x_n)\in\mathbf{R}^n$, $\mathbf{m}=(m_1,\dots,m_n)\in\mathbf{Z}^n$, we put $\mathbf{m}\cdot\mathbf{x}=m_1x_1+\dots+m_nx_n$. Let $e(z)=\exp(2\pi iz)$.

LEMMA 1 ([1, Lemma 1]). *Let, for each non-negative integer k, and $\mathbf{x}\in\mathbf{R}^n$,*

$$f(k,\mathbf{x})=\sum_{\mathbf{m}^n\in\mathbf{Z}}a(\mathbf{m})r(\mathbf{m})^k e(k(\mathbf{m}\cdot\mathbf{x})), \tag{4}$$

where $r(\mathbf{0}) = 1$, $\sum_{\mathbf{m}\in \mathbf{Z}^n} a(\mathbf{m})r(\mathbf{m})^k$ *converges, and, for each* $\mathbf{m} \in \mathbf{Z}^n$, $a(\mathbf{m}) = a(-\mathbf{m}) \geqq 0$, $r(\mathbf{m}) = r(-\mathbf{m})$ *and* $0 \leqq r(\mathbf{m}) \leqq 1$. *Then*

$$\sum_{k=1}^{K} \left(1 - \frac{k}{K+1}\right) f(k, \mathbf{x}) \geqq \frac{K+1}{2} a(\mathbf{0}) - \frac{1}{2} f(0, \mathbf{0}) \tag{5}$$

and hence

$$\max_{1\leqq k \leqq K} f(k, \mathbf{x}) \geqq \frac{K+1}{K} a(\mathbf{0}) - \frac{1}{K} f(0, \mathbf{0}). \tag{6}$$

The statement of equation (5) in [1] is only for the case when $r(\mathbf{m}) = 1$ for each $\mathbf{m} \in \mathbf{Z}^n$. However, the proof is very similar in the general case. It uses the non-negativity of Fejér's kernel:

$$\frac{1}{2} + \sum_{k=1}^{K} \left(1 - \frac{k}{K+1}\right) r^k \cos 2\pi k x \geqq 0,$$

for r, x real and $0 \leqq r \leqq 1$. We omit the details. Equation (6) follows immediately from (5).

3. We now prove (1). Put $z_j = r_j e(x_j)$ $(j = 1, \ldots, n)$, where $r_1 = 1$, and $0 \leqq r_j \leqq 1$ $(j = 2, \ldots, n)$ and define

$$f(k, \mathbf{x}) = 2(1 + \cos 2\pi x_1)\Re g(k) = \tag{7}$$

$$= \left(1 + \frac{1}{2} e(x_1) + \frac{1}{2} e(-x_1)\right) (b_1 r_1^k (e(x_1) + e(-x_1)) + \ldots + b_n r_n^k (e(x_n) + e(-x_n))).$$

This function satisfies the conditions of Lemma 1, with $f(0, \mathbf{0}) = 4g(0) = 4b_1(1 + \beta)$, and $a(\mathbf{0}) = b_1$. Hence from (6)

$$\max_{1\leqq k \leqq K} f(k, \mathbf{x}) \geqq b_1 \left(1 - \frac{4\beta + 3}{K}\right). \tag{8}$$

Taking $K = [\lambda^{-1}(4\beta + 3)] + 1$, where $0 < \lambda \leqq 1$, we have $K > \lambda^{-1}(4\beta + 3)$, and so

$$\max_{1\leqq k \leqq K} f(k, \mathbf{x}) > b_1(1 - \lambda) \geqq 0. \tag{9}$$

Now $f(k, \mathbf{x}) > 0 \Rightarrow \Re g(k) \geqq \frac{1}{4} f(k, \mathbf{x})$ from (7), and hence, using (9), we have (1).

When $b_1 = b_2 = \ldots = b_n = 1$, $[4\beta + 4] = 4n$. Let $z_j = \omega^{2j-1}$ $(j = 1, \ldots, n)$, where $\omega = e(1/4n)$. Then $\Re g(k) = 0$ for $k = 1, \ldots, 4n - 1$, $k \neq 2n$, and $g(-2n) = -n$. This shows that $K = [4\beta + 4]$ is best possible in (2) when $b_1 = \ldots = b_n$.

4. We now prove (3), for $|z_1| = \ldots = |z_n| = 1$. Put $z_j = e(x_j)$ $(j = 1, \ldots, n)$. We apply (6) to $f(k, \mathbf{x}) = g(k)\overline{g(k)}$. We have $a(\mathbf{0}) = \sum_{j=1}^{n} b_j^2 \geqq \frac{1}{n}(g(0))^2$, and $f(0, \mathbf{0}) = g(0)^2$, so that for $K = [(1+\varepsilon)n]$

$$\max_{1 \leqq k \leqq K} |g(k)|^2 \geqq g(0)^2 \frac{K+1-n}{nK} \geqq g(0)^2 \frac{n\varepsilon}{n^2(1+\varepsilon)}. \tag{10}$$

This gives (3).

Finally, we prove the following

LEMMA 2 ([1, Lemma 3]). *Let* $0 < \rho_j < \rho < 1$ $(j = 1, \ldots, n)$, $R = (1-\rho)^{-1}$, *and* $f(\mathbf{x}) = -\sum_{j=1}^{N} \log(\rho_j e(x_j) - 1)$. *Then for* $B \geqq 4/\rho_1$

$$\max_{1 \leqq k \leqq K} \Re f(kx) > \frac{\rho_1}{4} - B^{-1}, \tag{11}$$

where $K = [BN \log R]$.

PROOF. We first remark that (1) clearly still holds when $g(k) = \sum_{j=1}^{\infty} b_j z_j^k$, where the b_j are non-negative, $\Sigma\, b_j$ converges, and $\beta = \left(\sum_{j=2}^{\infty} b_j\right)\Big/ b_1$. (We could have stated (1) in this extended form.) Since $f(\mathbf{x}) = \sum_{j=1}^{N} \sum_{m=1}^{\infty} m^{-1} \rho_j^m e(mx_j)$, we can apply (1) to $g(k) = f(k\mathbf{x})$. Taking $z_1 = e(x_1)$, $b_1 = \rho_1$, and noting that $\sum_{j=1}^{\infty} b_j \leqq N \log R$, we have for $B \geqq 4/\rho_1$

$$K = [BN \log R] > B\rho_1(1+\beta) - 1 \geqq \frac{B\rho_1}{4}(4\beta + 3).$$

Hence, putting $\lambda = 4/B\rho_1$, $K \geqq [\lambda^{-1}(4\beta+3)] + 1$, so that (1) gives (11).

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UNIVERSITY OF TURKU
FINLAND

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AUFSTEIGENDE KETTENBEDINGUNG FÜR ZYKLISCHE MODULN UND PERFEKTE ENDOMORPHISMENRINGE

Von

G. HAUGER (München)

Ziel dieser Note ist es, folgenden Satz zu beweisen:

THEOREM. *Für einen endlich erzeugten selbstprojektiven Modul M_R mit Endomorphismenring S sind folgende Aussagen äquivalent:*

a) *S ist rechtsperfekt.*

b) *Der Radikalfaktormodul M/*Ra *M von M ist halbeinfach und jeder M-erzeugte Modul hat ein kleines Radikal.*

c) *Es gilt die aufsteigende Kettenbedingung für M-zyklische Moduln.*

d) *Jeder M-erzeugte Modul hat eine $\mathfrak{E}M$-projektive Hülle in $\mathfrak{E}M$.*

e) *Jeder M-erzeugte Modul ist komplementiert.*

Dabei heißt ein Modul *M-zyklisch (M-erzeugt)*, falls er epimorphes Bild von M (einer direkten Summe von Kopien von M) ist. Wir sagen, es gilt die *aufsteigende Kettenbedingung für M-zyklische Moduln*, falls für jeden Modul jede aufsteigende Folge M-zyklischer Untermoduln stationär wird (vergleiche [1]). Die Klasse der M-erzeugten Moduln bezeichnen wir mit $\mathfrak{E}M$. Ein wesentlicher Epimorphismus $P \to X$ heißt *$\mathfrak{E}M$-projektive Hülle in $\mathfrak{E}M$*, falls P $\mathfrak{E}M$-projektiv und M-erzeugt ist. Ein Modul X heißt *komplementiert* [2], wenn zu jedem Untermodul $U \subset X$ in der Menge der Untermoduln $V \subset X$ mit $X = U + V$ ein minimaler existiert; jeder solche minimale Untermodul heißt ein *Komplement* von U. Als Folgerung des Theorems erhalten wir die Charakterisierung perfekter Ringe ([1], Main Theorem): *Ein Ring ist genau dann rechtsperfekt, wenn die aufsteigende Kettenbedingung für zyklische Rechtsmoduln gilt.* Der hier geführte Beweis liefert, auf den Ring spezialisiert, einen neuen und einfacheren Beweis des Theorems von Jonah. Weiter gilt:

FOLGERUNG 1. a) *Der Endomorphismenring eines endlich erzeugten selbstprojektiven Rechtsmodul über einem rechtsperfekten Ring ist rechtsperfekt.*

b) *Sei $R \to T$ ein Ringhomomorphismus, R rechtsperfekt und T_R endlich erzeugt. Dann ist T rechtsperfekt.*

c) (MARES [3], WARE [7]) *Ein endlich erzeugter projektiver Modul ist genau dann perfekt, wenn dessen Endomorphismenring perfekt ist.*

BEWEIS von b). Mit R_R erfüllt auch R^n für $n \in \mathrm{N}$ die äquivalenten Eigenschaften des Theorems und damit auch T_R, also auch T_T.

Zum Beweis des Theorems benötigen wir eine Reihe von Sätzen:

SATZ 2. *Für einen Modul M sind folgende Aussagen äquivalent:*

(i) *M ist radikalfrei und es gilt die aufsteigende Kettenbedingung für M-zyklische Moduln.*

(ii) *M ist halbeinfach und endlich erzeugt.*

BEWEIS. Die aufsteigende Kettenbedingung gilt für M-zyklische Moduln, falls M endlicher Länge ist, woraus die Implikation (ii) $\Rightarrow$ (i) folgt. Sei nun M radikalfrei und es gelte die aufsteigende Kettenbedingung für M-zyklische Moduln. Wir nehmen an, M sei nicht endlicher Länge, und geben rekursiv eine absteigende Folge von Untermoduln M_i an, so daß M/M_i halbeinfach der Länge i ist. Wir setzen $M_0 = M$. Sei M/M_i halbeinfach der Länge i. Nach Voraussetzung gibt es einen maximalen Untermodul N von M, so daß $N \cap M_i =: M_{i+1}$ echt in M_i enthalten ist. Es gibt eine exakte Folge

$$0 \to M/M_{i+1} \to M/M_i \oplus M/N \to M/M_i + N \to 0.$$

Aus $M_i + N = M$ folgt $\text{Lä}\,(M/M_{i+1}) = \text{Lä}\,(M/M_i \oplus M/N) = i + 1$. Der natürliche Epimorphismus $M/M_{i+1} \to M/M_i$ zerfällt. Es gibt also einen Monomorphismus $f_i\colon M/M_i \to M/M_{i+1}$. Die Familie $(M/M_i, f_i)$ erzeugt ein induktives System mit induktivem Limes (X, g_i). Wir erhalten eine echt aufsteigende Folge $(\text{Bi}\, g_i)$ von M-zyklischen Untermoduln von X im Widerspruch zur Voraussetzung. M ist also halbeinfach und endlich erzeugt.

FOLGERUNG 3. *Ist M endlich erzeugt, $M/\text{Ra}\, M$ halbeinfach und hat jeder M-erzeugte Modul ein kleines Radikal, so gilt die aufsteigende Kettenbedingung für M-zyklische Moduln.*

BEWEIS. Sei (X_i) eine aufsteigende Folge von M-zyklischen Untermoduln von X. Ohne Einschränkung nehmen wir $X = \bigcup (X_i \mid i \in \mathbf{N})$ an. Mit $n\colon X \to X/\text{Ra}\, X$ ist (nX_i) eine aufsteigende Folge $M/\text{Ra}\, M$-zyklischer Untermoduln von $X/\text{Ra}\, X$. Diese wird stationär nach Satz 1. Aus $X = \bigcup X_i$ folgt $X/\text{Ra}\, X = \bigcup nX_i$. Es gibt also eine natürliche Zahl i mit $nX_i = X/\text{Ra}\, X$, d. h. $X_i + \text{Ra}\, X = X$, woraus $X_i = X$ folgt, da nach Voraussetzung der M-erzeugte Modul X ein kleines Radikal hat.

Mit Folgerung 3 haben wir b) $\Rightarrow$ c) bewiesen. Zum weiteren Beweis des Satzes benötigen wir einen verbandstheoretischen Satz:

SATZ 4 *(vergleiche* [6], *Theorem* 2.2*). Sei M selbstprojektiv und endlich erzeugt mit Endomorphismenring S, X ein beliebiger R-Modul. Für $Y \subset X$ sei $Y^0 = \text{Hom}(M, Y) \subset \text{Hom}\,(M, X)$ und für einen S-Rechtsuntermodul Z von $\text{Hom}\,(M, X)$ sei $Z^0 = \Sigma\,(\text{Bi}\, f \mid f \in Z)$. Dann gilt für $Y, Y' \subset X$ und $Z, Z' \subset \text{Hom}\,(M, Y)$:*

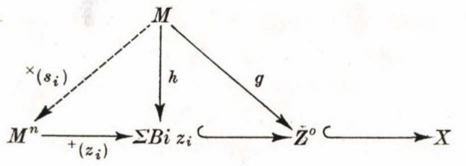
(i) $Y^{00} = \Sigma\,(\text{Bi}\, f \mid f \in \text{Hom}\,(M, Y)) \subset Y$

(ii) $Z = Z^{00}$

(iii) $(Y + Y')^0 = Y^0 + Y'^0$

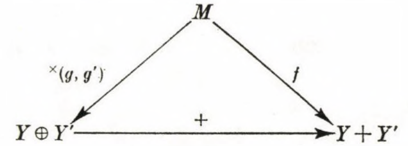
(iv) $(Z + Z')^0 = Z^0 + Z'^0$.

BEWEIS. (i), (iv) und $Z \subset Z^{00}$ sind trivial. Sei $g \in Z^{00}$, d. h. $\text{Bi}\, g \subset Z^0 \subset X$. Da $\text{Bi}\, g$ endlich erzeugt und $Z^0 = \Sigma\,(\text{Bi}\, z \mid z \in Z)$ ist, gibt es $z_1, \ldots, z_n$ mit $\text{Bi}\, g \subset \Sigma\, \text{Bi}\, z_i$ und wir erhalten folgendes kommutative Diagramm:



Nun gilt $g = {}^{+}(z_i)^{\times}(s_i) = \Sigma\, z_i s_i \in Z$.

(iii) Es genügt $(Y + Y')^0 \subset Y^0 + Y'^0$ zu zeigen. Sei f: $M \to Y + Y'$. Nach [5] ist M $Y \oplus Y'$-projektiv, also gibt es g, g', so daß folgendes Diagramm (+ ist die Additionsabbildung) kommutativ ist:



Es ist also $f = g + g' \in Y^0 + Y'^0$.

FOLGERUNG 5. *Mit den Voraussetzungen und Bezeichnungen von Satz* 4 *gilt:*

(i) *Mit X ist auch* Hom (M, X) *artinsch (noethersch).*

(ii) *(*[4], *Theorem* 1*) Mit X ist auch* Hom (M, X) *halbartinsch.*

(iii) *Ein M-erzeugter Modul X ist genau dann komplementiert, wenn* Hom (M, X) *es ist.*

(iv) *Für einen M-erzeugten Modul X und $Y \subset X$ ist Y genau dann klein in X, wenn Y^0 klein in X^0 ist.*

(v) *Es ist* Ra_S Hom $(M, X) =$ Hom $(M, \mathrm{Ra}_R\, X)$ *für jeden M-erzeugten Modul X.*

BEWEIS. (i) ist trivial.

(ii) Für $Y \subset$ Hom (M, X) ist die Existenz von Y' zu zeigen mit $Y \subset Y' \subset$ $\subset$ Hom (M, X) und Y'/Y einfach. Die Menge $\mathfrak{M} = \{N \subset X \mid N^0 = Y\}$ ist induktiv geordnet und nicht leer. Also hat $\mathfrak{M}$ ein maximales Element N. Da X halbartinsch ist, existiert $N' \subset X$ mit N'/N einfach. Dann ist $Y = N^0$ maximaler Untermodul von N'^0 wegen Satz 4 (ii) und der Maximalität von N.

(iii) Sei Hom (M, X) komplementiert und $Y \subset X$. Es existiert $Z \subset \mathrm{Hom}(M,X) =$ $= X^0$ minimal mit $Y^0 + Z = X^0$. Dann ist $X = X^{00} = (Y^0 + Z)^0 = Y^{00} + Z^0 =$ $= Y + Z^0$ wegen (i) und (iv) in Satz 4. Sei nun $X = Y + Y'$ und $Y' \subset Z^0$. Es gilt $X^0 = Y^0 + Y'^0$ und $Y'^0 \subset Z^{00} = Z$, woraus $Y'^0 = Z$, also $Y'^{00} = Z^0 \subset Y'$ und somit $Y' = Z^0$ folgt. Etwas einfacher zeigt man die andere Implikation in (iii), ebenso (iv).

(v) Da M endlich erzeugt ist, gilt $f \in$ Hom $(M, \mathrm{Ra}\, X)$ genau dann, wenn $\mathrm{Bi}\, f$ klein in X ist. Nach (iv) ist dies äquivalent damit, daß $(\mathrm{Bi}\, f)^0 = fS$ klein in X^0 ist. Das bedeutet, daß f Element von Ra_S Hom (M, X) ist.

BEWEIS von c) $\Rightarrow$ a). Nach 5 (v) ist $\mathrm{Ra}\, S =$ Hom $(M, \mathrm{Ra}\, M)$. Nach Satz 1 ist $M/\mathrm{Ra}\, M$ halbeinfach und artinsch, somit auch der Endomorphismenring von $M/\mathrm{Ra} M$, der isomorph zu $S/\mathrm{Ra}\, S$ ist. Wir zeigen nun, daß $\mathrm{Ra}\, S$ rechts-T-nilpotent ist. Sei (s_i) eine Folge von Elementen aus $\mathrm{Ra}\, S$. Mit $M_i = M$ erzeugt (M_i, s_i) ein induktives System mit induktivem Limes (L, u_i). $\mathrm{Bi}\, u_i$ ist M-zyklisch und $L = \bigcup \mathrm{Bi}\, u_i$. Es existiert also eine natürliche Zahl i mit $L = \mathrm{Bi}\, u_i$, woraus, wegen $u_{i+1}s_i = u_i$, $\mathrm{Bi}\, s_i + \mathrm{Ke}\, u_{i+1} = M$ folgt. $\mathrm{Bi}\, s_i$ ist klein, also $u_{i+1} = 0$ und somit $L = 0$. Sei $M = \Sigma\,(Rm_j \mid j = 1, \ldots, r)$. Zu m_j existiert $n_j \in \mathbf{N}$ mit $s_{n_j}s_{n_j-1} \ldots s_1 m_j = 0$. Mit $n = \max\{n_1, \ldots, n_r\}$ ist dann $s_n s_{n-1} \ldots s_1 = 0$.

a) $\Rightarrow$ b). Wegen Folgerung 3 gilt die aufsteigende Kettenbedingung für S-zyklische Rechtsmoduln. Wir zeigen als nächstes, daß dann auch die aufsteigende Kettenbedingung für M-zyklische Moduln gilt. Seien (X_i) eine aufsteigende Folge von Untermoduln von X und $f_i\colon M \to X_i$ Epimorphismen. Da M selbstprojektiv ist, gilt $f_iS = \mathrm{Hom}\,(M, X_i) \subset \mathrm{Hom}\,(M, X)$. Die aufsteigende Folge (f_iS) wird stationär. Es existiert i_0, so daß für $i \geqq i_0$ gilt $f_iS = f_{i+1}S$. Es gibt t_i mit $f_{i+1} = f_i t_i$, woraus $X_{i+1} = \mathrm{Bi}\, f_{i+1} \subset \mathrm{Bi}\, f_i = X_i$, also $X_i = X_{i+1}$ für $i \geqq i_0$ folgt. Nach Satz 1 ist nun $M/\mathrm{Ra}\, M$ halbeinfach. Nach Voraussetzung ist $\mathrm{Ra}_S\, \mathrm{Hom}\,(M, X)$ klein in $\mathrm{Hom}\,(M, X)$ für jeden M-erzeugten Modul X. Wegen 5 (v) und (iv) ist dann $\mathrm{Ra}\, X$ klein in X.

Die Äquivalenz von a) und e) wurde in Folgerung 5 (iii) gezeigt.

Den folgenden Satz zitieren wir, ohne ihn zu beweisen.

SATZ 6 (KASCH und MARES [2], Bemerkung). *Ist P selbstprojektiv und komplementiert, so ist jedes Komplement direkter Summand.*

BEWEIS von e) $\Rightarrow$ d). Sei X M-erzeugt, d. h., es gibt einen Epimorphismus $f\colon M^{(I)} \to X$. Da M selbstprojektiv und endlich erzeugt ist, ist $M^{(I)}$ $\mathfrak{G}M$-projektiv und nach Voraussetzung komplementiert. Sei $\mathrm{Ke}\, f + U = M^{(I)}$, U minimal mit dieser Eigenschaft. Dann ist $U \to M^{(I)} \to X$ wesentlicher Epimorphismus und U nach Satz 6 direkter Summand von $M^{(I)}$, also M-erzeugt und $\mathfrak{G}M$-projektiv.

d) $\Rightarrow$ e). Sei X M-erzeugt, $U \subset X$, $m\colon X \to X/U$ und $P \xrightarrow{n} X/U$ $\mathfrak{G}M$-projektive Hülle in $\mathfrak{G}M$. Es existiert $f\colon P \to X$ mit $mf = n$. Dann ist $X = \mathrm{Bi}\, f + \mathrm{Ke}\, m = \mathrm{Bi}\, f + U$ und $f(\mathrm{Ke}\, n) = \mathrm{Bi}\, f \cap U$ klein in $\mathrm{Bi}\, f$, d. h., $\mathrm{Bi}\, f$ ist minimal in $\{U' \mid U' + U = X\}$.

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MATHEMATISCHES INSTITUT
DER LUDWIG-MAXIMILIAN-UNIVERSITÄT
D 8000 MÜNCHEN 2
THERESIENSTR. 39
BUNDESREPUBLIK DEUTSCHLAND

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SOME COMMUTATIVITY RESULTS FOR PERIODIC RINGS

By

H. E. BELL (St. Catharines)

A ring R is called *periodic* if for each $a \in R$, there is a pair n, m of distinct positive integers for which $a^n = a^m$. There are numerous theorems in the literature, beginning with Wedderburn's theorem on finite division rings, which suggest that periodic rings are more likely to be commutative or nearly so than arbitrary rings [4, 5, 6]; on the other hand, periodic rings may be badly non-commutative, as in the case of the ring of $n \times n$ matrices over $GF(p)$ [1]. This note contains several results of the former kind, each dealing with periodic rings with constraints on the set of nilpotent elements, zero divisors, or commutators.

Section 1 deals with results which are easily established for alternative rings, while Section 2 contains results which I am able to prove only for the associative case. Throughout the paper, the term "zero divisor" refers to one-sided divisors of zero; it should be mentioned that even though one cannot prove that one-sided annihilators in alternative rings are left or right ideals, nonetheless if a is a left zero divisor, so is ra for every $r \in R$. The term *regular* element will be used to denote an element which is neither a left nor right zero divisor.

In both Sections 1 and 2, the following elementary proposition will be used.

PROPOSITION 0. *A periodic alternative ring has each of the following properties:*

(a) *For each $a \in R$ there exists some positive integer $k(a)$ such that $a^{k(a)}$ is idempotent.*

(b) *For each $a \in R$, there exists a positive integer $n(a) > 1$ for which $a - a^{n(a)}$ is nilpotent.*

(c) *If I is an ideal of R and $x + I$ is a non-zero nilpotent element of R/I, then R contains a nilpotent element u such that $x \equiv u \pmod{I}$.*

PROOF. (a) If $a^n = a^m$ for $n > m$, then $a^{j+k(n-m)} = a^j$ for each positive integer k and each $j \geqq m$. Thus, $a^{m(n-m)}$ is idempotent.

(b) Let $a^n = a^m, n > m > 1$. Then $a^{m-1}(a - a^{n-m+1}) = 0 = a^{m-2}a(a - a^{n-m+1}) = a^{m-2}a^{n-m+1}(a - a^{n-m+1})$; therefore, $a^{m-2}(a - a^{n-m+1})^2 = 0$ and the result follows by the obvious induction.

(c) If $x + I$ is a non-zero nilpotent element of R/I, then there is a positive integer k such that $x^q \in I$ for each $q \geqq k$. By the proofs of (a) and (b), R contains a nilpotent element $u = x - x^q$ with $q \geqq k$; obviously, $u \equiv x \pmod{I}$.

1. Periodic rings with commuting nilpotent elements or zero divisors

THEOREM 1. *Let R be a periodic alternative ring, the nilpotent elements of which commute with each other. Then the commutator ideal $C(R)$ is nil, and the nilpotent elements form an ideal.*

PROOF. We shall use without explicit mention Artin's result that any two elements of an alternative ring generate an associative subring.

We let N denote the set of nilpotent elements, and use the standard argument for the commutative case to show that $u_1 - u_2 \in N$ whenever $u_1, u_2 \in N$; we then prove by induction on k that if $u^k = 0$ and $r \in R$, it is true that $(ur)^k = (ru)^k = 0$.

Consider first the case $k = 2$. Let u be any element of R such that $u^2 = 0$ and j a positive integer for which $(ur)^j = e$ is idempotent (possibly zero). Then $re - ere$ is nilpotent and hence commutes with u — that is,

$$ur(ur)^j - u(ur)^j r(ur)^j = r(ur)^j u - (ur)^j r(ur)^j u; \tag{1.1}$$

and multiplying on the right by u yields $(ur)(ur)^j u = 0$, so that $(ur)^{j+2} = (ru)^{j+2} = 0$. It follows that u commutes with both ur and ru so that $(ru)^2 = (ur)^2 = 0$.

Now suppose the result holds for all y with $y^m = 0, m < k$; and suppose $u^k = 0$, $k \geqq 3$. Determining j as above and multiplying (1.1) by r on the left and u on the right, we get

$$(ru)^{j+2} = ru^2 s + tu^2, \tag{1.2}$$

where s and t are elements of the subring generated by r and u. Since $(u^2)^{k-1} = 0$, the inductive hypothesis implies both $ru^2 s$ and tu^2 are nilpotent; therefore (1.2) shows that ru and ur are nilpotent. Again, invoking the fact that u must commute with ur and ru, we see that $(ru)^k = (ur)^k = 0$.

Having finished the demonstration that N is an ideal, we apply (b) of Proposition 0 to show that R/N has the "$a^n = a$ property". By Jacobson's well-known theorem for the associative case [6] and Artin's theorem, R/N is commutative; and the proof of Theorem 1 is complete.

COROLLARY 1. *Let R be a periodic alternative ring with identity, and suppose that the invertible elements of R commute with the nilpotent elements. Then $C(R)$ is nil and the nilpotent elements form an ideal.*

PROOF. Let n_1, n_2 be nilpotent. Since $1 + n_1$ is invertible, it commutes with n_2, hence n_1 and n_2 commute.

COROLLARY 2. *Let R be a periodic alternative ring having at least one regular element. If the regular elements of R commute with the nilpotent elements, then $C(R)$ is nil and the nilpotent elements form an ideal.*

PROOF. If a is a regular element of R, $a^n = a$ for some $n > 1$ and a^{n-1} is a regular idempotent e. For all $y \in R$, we have $e(ye - eye) = (ey - eye)e = 0$, hence $ye - eye = ey - eye = 0$ and therefore $ey = ye$. This fact, together with substitution of e for x in the standard alternative-ring identity $(x^2, y, z) = x(x, y, z) + (x, y, z)x$

[3, p. 379], yields the result that

$$(e, y, z) = 0 \quad \text{for all} \quad y, z \in R; \tag{1.3}$$

and it follows easily that eR is a subring of R with identity e.

If ex is invertible in eR, there is some k such that $(ex)^k = ex^k = x^k e = e$; and the assumption that x is a zero divisor, together with (1.3), yields the contradiction that e is a zero divisor. Thus ex is invertible in eR only if x is regular in R; and since it is clear that ex is nilpotent in eR if and only if x is nilpotent in R, eR satisfies the hypotheses of Corollary 1. Thus nilpotent elements of eR commute, and cancellation of e shows that nilpotent elements of R do also.

Theorem 1 is a generalization of the corollary in [5]. We note that its hypotheses do not imply commutativity even for finite rings – we need only consider R having as its additive group the Klein 4-group with elements 0, a, b, c and multiplicaton given by $0x = cx = 0$ and $ax = bx = x$ for all $x \in R$.

THEOREM 2. *Let R be a periodic alternative ring whose zero divisors commute with each other. Then either* (i) *R is commutative or* (ii) *R has an identity and R/N is a field.*

PROOF. Note first that $ab = 0$ implies $ba = 0$, so we need make no distinction between left and right annihilators. We denote the annihilator of the element x by $A(x)$.

Now if R is nil, it is commutative; hence we may assume at the outset that R has non-zero idempotents e. If there exists such an idempotent which is not an identity element, at least one of the direct-sum decompositions $R^+ = Re \oplus A(e)$, $R^+ = eR \oplus A(e)$ of the additive group R^+ must be non-trivial; and e must be a zero divisor. Therefore $e(ex - exe) = (ex - exe)e$ and $e(xe - exe) = (xe - exe)e$; and it follows that e commutes with all $x \in R$. By computations like those in the proof of Corollary 2, R is a ring direct sum of the commutative rings Re and $A(e)$ and is therefore commutative.

The only remaining case to consider is that where R has an identity 1 and no other non-zero idempotents. The periodicity of R then implies that each element is either nilpotent or invertible; and invoking Theorem 1 to show that N is an ideal, we get an alternative division ring R/N, which is commutative by Jacobson's "$a^n = a$ theorem". To complete the proof, we need only observe that a commutative alternative ring without non-zero nilpotent elements is associative [10, Lemma 3].

COROLLARY 3. *If R is a periodic alternative ring whose zero divisors commute and whose invertible elements commute, then R is commutative.*

PROOF. Consider case (ii) of Theorem 2. Let a be invertible and $u \in N$. Then a and $1 + u$ commute, hence so do a and u. Since all elements of R are either nilpotent or invertible, nilpotent elements commute with all elements of R, which is therefore commutative by the corollary of [5] (applied to two-generator subrings of R).

It is to be noted that the hypotheses of Theorem 2 do not imply commutativity – see [2] for examples of finite associative rings whose zero divisors annihilate each other. It is also the case that the assumption that nilpotent elements commute will not yield the conclusion of Theorem 2; to see this consider the set $Z_6 \times Z_6$, where Z_6 denotes the integers mod 6, with componentwise addition and multiplication given by $(i, j)(p, q) = (ip, iq)$.

2. Periodic rings with restricted commutators

The generalized commutator $[x_1, x_2, \ldots, x_n]$ is defined inductively as follows: $[x_1, x_2] = x_1x_2 - x_2x_1$, and $[x_1, \ldots, x_k] = [[x_1, \ldots, x_{k-1}], x_k]$, $k \geqq 3$. We shall use the abbreviation $[x, y]_n$ to denote $[x, y, \ldots, y]$ with $n-1$ entries of y.

THEOREM 3. *Let R be an associative ring such that*

(1) *R is periodic;*

(2) *there exists a positive integer n such that $[u_1, u_2, u_3, \ldots, u_n] = 0$ for all nilpotent elements $u_1, \ldots, u_n$ of R;*

(3) *R is 2-torsion-free.*

Then $C(R)$ is nil and the nilpotent elements of R form an ideal.

PROOF. Let N denote the Jacobson radical of R. In view of (a) of Proposition 0, N must be nil, for the Jacobson radical can contain no non-zero idempotents; moreover, as is easily shown, $R/N = \bar{R}$ inherits the hypotheses of Theorem 3. We need only establish that $\bar{R}$ is commutative without non-zero nilpotent elements; accordingly, we shall show that primitive homomorphic images of $\bar{R}$ are fields.

As a first step, we note that if a is any non-nilpotent element of $\bar{R}$, then there exist m, n with $n > m$ such that $a^n = a^m$ and $(2a)^n = (2a)^m$, so that $(2^n - 2^m)a^m = 2^m(2^{n-m} - 1)a^m = 0$; and the absence of 2-torsion shows that some power of a is a q-torsion element with $(q, 2) = 1$.

Now let $\tilde{R}$ be a primitive homomorphic image of $\bar{R}$. If $\tilde{R}$ is not a division ring, then there exists a subring $\tilde{S}$ of $\tilde{R}$ and a homomorphism of $\tilde{S}$ onto the ring Δ_2 of 2×2 matrices over some division ring Δ. If e is the identity element of Δ_2, then e must be the image of a non-nilpotent element of $\tilde{S}$, which in turn has a non-nilpotent pre-image in $\bar{R}$; thus $qe = 0$ for some q relatively prime to 2 and Δ cannot have characteristic 2.

In view of (c) of Proposition 0, Δ_2 will inherit properties (1) and (2). However, for any division ring of characteristic different from 2 and any positive integer k, the nilpotent matrices $n_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $n_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ have the property that $[n_1, n_2, n_1, n_2, \ldots, n_1, n_2] \neq 0$, this generalized commutator having k entries of n_1 and k of n_2. Thus, we must have the result that $\tilde{R}$ is a division ring, which is commutative by the "$a^n = a$ theorem". This completes the proof of Theorem 3.

It is to be noted that the restriction on 2-torsion cannot be omitted from the hypotheses of Theorem 3, for the ring of 2×2 matrices over $GF(2)$ has the property that $[u_1, u_2, u_3] = 0$ for all nilpotent elements u_1, u_2, u_3.

The proof of Theorem 3 yields

COROLLARY 4. *If R is periodic and 2-torsion-free and if for each pair of nilpotent elements x, y some sufficiently long generalized commutator of form $[x, y, x, y, \ldots, x, y]$ vanishes, then $C(R)$ is nil and the nilpotent elements form an ideal.*

There has been extensive discussion in the literature of rings R with the property that for each $x, y \in R$, $[x, y]_n = 0$ [8, 9]. It is known that $C(R)$ is nil if n can be chosen independent of x and y; but the problem for variable n is still unresolved. Another unsolved problem is whether the nilpotence of commutators in the ring R implies that the nilpotent elements form an ideal; in this case, if there is a fixed n such that

$[x, y]^n = 0$ for all $x, y \in R$, the answer is affirmative [7, p. 29]. For periodic rings, the variable-n question is in both cases manageable; and our final two theorems state the results. The details of proof are omitted, since the strategy is the same as in the proof of Theorem 3 — it is only necessary to note in the end that the ring of 2×2 matrices over a division ring (of arbitrary characteristic in this case) fails to satisfy the hypothesis and thus the primitive rings are fields.

THEOREM 4. *If R is a periodic associative ring such that for each $x, y \in R$, $[x, y]_n = 0$ for some n depending on x and y, then $C(R)$ is nil and the nilpotent elements form an ideal.*

THEOREM 5. *Let R be a periodic associative ring in which all commutators of nilpotent elements are nilpotent. Then $C(R)$ is nil and the nilpotent elements of R form an ideal.*

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DEPARTMENT OF MATHEMATICS
BROCK UNIVERSITY
ST. CATHARINES, ONTARIO
CANADA

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CAUCHY'SCHE INTEGRALFORMELN ANALYTISCHER FUNKTIONEN AUF ALGEBREN

Von

J. EDENHOFER (München)

Die klassische Funktionentheorie kann als Funktionentheorie in einer kommutativen Algebra über $\mathbf{R}$ vom Rang 2 mit einer Nullteilerhypergeraden, nämlich dem Punkt $z=0$, aufgefaßt werden (aus Bequemlichkeitsgründe nennen wir auch das Nullelement Nullteiler). Unter gewissen Voraussetzungen sind die Cauchy'schen Integralformeln gültig. Wir zeigen hier die sehr schöne Analogie zur klassischen Funktionentheorie auf, daß es solche Formeln in allen kommutativen und assoziativen Algebren endlichen Ranges $n \geqq 2$ über $\mathbf{R}$ mit Einselement gibt, welche mindestens eine Nullteilerhypergerade besitzen. Daß diesen verallgemeinerten Cauchy'schen Integralformeln eine ähnliche Bedeutung (Laurentreihenentwicklungen hyperkomplexer Funktionen, Anwendungen auf partielle Differentialgleichungen) beizumessen ist, wie den Cauchy'schen Integralformeln der klassischen Funktionentheorie, braucht wohl nicht besonders hervorgehoben zu werden. Abschließend werden noch für eine nach E. Lammel konstruierte Algebra vom Range 3 mit einer Nullteilergeraden die verallgemeinerten Cauchy'schen Integralformeln angegeben.

Wir bezeichnen mit $\mathfrak{A}_{\mathbf{R}}^{(n)}$ die Klasse aller kommutativen und assoziativen Algebren von endlichem Rang n über den reellen Zahlen $\mathbf{R}$ mit Einselement e. Die Elemente von $A \in \mathfrak{A}_{\mathbf{R}}^{(n)}$ heißen hyperkomplexe Zahlen, eine Basis von A sei $e_0=e, e_1, \dots, e_{n-1}$. Durch

$$e_i \cdot e_j = \sum_{k=0}^{n-1} \gamma_{ijk} e_k; \qquad i,j = 0, 1, \dots, n-1;$$

sind die "Multiplikationskonstanten" γ_{ijk} in Bezug auf die gegebene Basis eindeutig festgelegt. Man prüft leicht nach, daß die Abbildung

$$a = \sum_{i=0}^{n-1} \alpha_i e_i \to \Gamma(a) := \left(\sum_{i=0}^{n-1} \alpha_i \gamma_{ikl} \right)^1 \qquad (a \in \mathbf{R} \text{ Komponenten von } a)$$

von A auf eine Unteralgebra der vollständigen Matrixalgebra vom Rang n^2 über $\mathbf{R}$ ein Isomorphismus bzgl. aller in A vorkommenden Verknüpfungen ist. Ist $a \in A$ Nullteiler von A (dies ist genau dann der Fall, wenn det $\Gamma(a)=0$), so auch $\rho \cdot a$ für jedes $\rho \in \mathbf{R}$. Wir bezeichnen deshalb die Menge aller Nullteiler von A als Nullteilerhyperkegel von A. Der Nullteilerhyperkegel von A zerfällt in Hyperebenen und Hypergeraden (wegen der Kommutativität von A, näheres siehe [2]), die wir als Nullteilerhyperebenen und Nullteilerhypergeraden bezeichnen.

[1] (β_{kl}); $\beta_{kl} \in \mathbf{R}$; n-reihige quadratische Matrix mit k als Zeilenindex und l als Spaltenindex.

Im folgenden nehmen wir an, daß $A \in \mathfrak{A}_{\mathbf{R}}^{(n)}$, als Vektorraum betrachtet, normiert ist. Wir verwenden speziell die Spektralnorm von $\Gamma(a)$ als Norm für $a \in A$:

$$(1) \qquad \| a \| := +\sqrt{\text{max. Eigenwert von } (\Gamma(a))' \cdot \Gamma(a)}.$$

Für $a, b \in A$ gilt

$$(2) \qquad \| a \cdot b \| \leqq \sigma \cdot \| a \| \cdot \| b \|,$$

wo $\sigma \in \mathbf{R}$ nur von der verwendeten Algebra $A \in \mathfrak{A}_{\mathbf{R}}^{(n)}$, nicht aber von a und b abhängt.

Unter einer hyperkomplexen Funktion f verstehen wir eine Abbildung von $D \subset A$ in A.

DEFINITION 1. Die hyperkomplexe Funktion f sei in einer Umgebung $U(z)$ der Stelle $z \in D$ definiert. f heißt an der Stelle z analytisch, wenn es eine von h unabhängige hyperkomplexe Zahl $f'(z) \in A$ gibt, so daß

$$(3) \qquad f(z+h) - f(z) = f'(z) \cdot h + \omega_f(h, z)$$

für $z + h \in U(z)$ und $\dfrac{\| \omega_f(h, z) \|}{\| h \|} \to 0$ für $h \to 0$. f heißt auf $M \subset D$ analytisch, wenn f dies in jedem Punkt von M ist.

Wie die höheren Ableitungen $f^{(k)}(z)$; $k = 2, 3, \ldots$; von $f(z)$ zu definieren sind, ist klar.

Seien ξ_i; $i = 0, 1, \ldots, n-1$; die Komponenten von z bzgl. der Basis $e_0 = e$, $e_1, \ldots, e_{n-1}$. Wie in der klassischen Funktionentheorie läßt sich zeigen: f ist in $z \in D$ genau dann analytisch, wenn die Komponenten von f in z linear approximierbar sind und den verallgemeinerten Cauchy–Riemann'schen Differentialgleichungen

$$(4) \qquad \frac{\partial f(z)}{\partial \xi_i} = e_i \cdot \frac{\partial f(z)}{\partial \xi_0}; \qquad i = 1, \ldots, n-1;$$

genügen. Bekanntlich bleibt auch der *Cauchy'sche Integralsatz* gültig:[2] Sei f eine auf dem einfach zusammenhängenden Gebiete G des $\mathbf{R}^n$ analytische hyperkomplexe Funktion. Dann gilt

$$(5) \qquad \int_C f(z)dz = 0$$

für jeden einfach geschlossenen Weg C, der ganz in G verläuft.

Wir versuchen jetzt ein Analogon der *Cauchy'schen Integralformeln* der gewöhnlichen Funktionentheorie für analytische Funktionen aufzustellen. Die gewöhnliche Funktionentheorie ist in einer Algebra aus $\mathfrak{A}_{\mathbf{R}}^{(2)}$ definiert, welche eine Nullteilerhypergerade, nämlich den Punkt $z = 0$, besitzt. In Verallgemeinerung davon gelten die im folgenden abgeleiteten Formeln für kommutative Algebren $A \in \mathfrak{A}_{\mathbf{R}}^{(n)}$, welche mindestens eine Nullteilerhypergerade $\mathfrak{G}$ besitzen.

[2] Siehe [4], [5].

$\mathfrak{S}$ ist ein $(n-2)$-dimensionaler linearer Unterraum von A. Zu $\mathfrak{S}$ gibt es daher zwei hyperkomplexe Zahlen $a, b \in A$, so daß

$$A = \{d \mid d = \lambda a + \mu b + c;\ \lambda, \mu \in R,\ c \in \mathfrak{S}\}.$$

Die Kurve ${}_0C_{2\pi}$ des $\mathbf{R}^n$

(6) $z := c + r(a \cos \phi + b \sin \phi);\quad 0 \leqq \phi \leqq 2\pi;\quad c \in \mathfrak{S};\quad r \in \mathbf{R}$ mit $r > 0$;

ist einfach, geschlossen und hat mit $\mathfrak{S}$ keinen Punkt gemeinsam. c sei so gewählt, daß c in keiner anderen Nullteilerhyperebene bzw. -gerade von A liegt. Dann kann man $r > 0$ so klein wählen, daß auf der Kurve ${}_0C_{2\pi}$ kein Nullteiler liegt. Denn nach Voraussetzung ist der Euklidische Abstand von c und den von $\mathfrak{S}$ verschiedenen Nullteilerhyperebenen und -geraden H positiv, also wegen der Stetigkeit des Abstandes auch der von $z = c + r(a \cos \phi + b \sin \phi)$ und H, wenn man nur $r > 0$ hinreichend klein wählt. Daher ist z für $0 \leqq \phi \leqq 2\pi$ invertierbar. Das Inverse z^{-1} ist eine stetige Funktion von ϕ, also existiert

(7) $$\oint_{{}_0C_{2\pi}} \frac{1}{z}\, dz,$$

wo der Integrationsweg die Nullteilerhypergerade "umschließt", d. h. man kann ${}_0C_{2\pi}$ nicht stetig auf einen Punkt zusammenziehen, der selbst nicht Nullteiler ist, ohne Nullteiler zu überstreichen.

Sei nun $f(z)$ an der Stelle 0 analytisch und in einer Umgebung $U(0)$ von 0 stetig. Wir betrachten die Integrationswege

(8) $$C_s : z_s = s[c + r(a \cos \phi + b \sin \phi)];\qquad 0 < s \leqq 1;$$

wo C_1 der schon beschriebene Weg ${}_0C_{2\pi}$ ist. Es gibt ein $\bar{s}$, so daß für s mit $0 < s \leqq \bar{s}$ alle C_s in $U(0)$ liegen. Aus (3) folgt

$$\frac{f(z_s) - f(0)}{z_s} = f'(0) + \frac{\omega_f(z_s, 0)}{z_s},$$

also

(9) $$I_s := \oint_{C_s} \frac{f(z_s) - f(0)}{z_s}\, dz_s = \oint_{C_s} \frac{\omega_f(z_s, 0)}{z_s}\, dz_s;\qquad 0 < s \leqq \bar{s}.$$

Die Abschätzung von I_s mittels (2) liefert:

(10) $$\| I_s \| \leqq k \cdot L_s \operatorname*{Max}_{z_s \in C_s} \left\| \frac{\omega_f(z_s, 0)}{z_s} \right\|,$$

wo L_s die Länge des Integrationsweges C_s und $k \in \mathbf{R}$ eine nur von $A \in \mathfrak{A}_{\mathbf{R}}^{(n)}$ abhängige Konstante ist.

Wegen (2) folgt weiter

$$(11) \qquad \left\| \frac{\omega_f(z_s, 0)}{z_s} \right\| \leqq \sigma \, \| \omega_f(z_s, 0) \| \cdot \| z_s^{-1} \| =$$

$$= \sigma \frac{\| \omega_f(z_s, 0) \|}{\| z_s \|} \cdot \sqrt{\frac{\text{max. Eigenwert von } \Gamma(z_s)' \, \Gamma(z_s)}{\text{min. Eigenwert von } \Gamma(z_s)' \, \Gamma(z_s)}}\,.[3]$$

Die Wurzel ist von s unabhängig, denn für die Eigenwerte λ_s von $\Gamma(z_s)' \, \Gamma(z_s)$ ist

$$\det\left(\Gamma(z_s)'\Gamma(z_s) - \lambda_s E\right) = \det\left(s^2 \Gamma(z_1)' \Gamma(z_1) - \lambda_s E\right) = s^{2n} \det\left(\Gamma(z_1)'\Gamma(z_1) = \frac{\lambda_s}{s^2} E\right) = 0.$$

Die Eigenwerte von $\Gamma(z_1)' \, \Gamma(z_1)$ sind also $\lambda_1 := \frac{\lambda_s}{s^2}$ oder $\lambda_s = s^2 \lambda_1$. Der Quotient zweier Eigenwerte von $\Gamma(z_s)' \, \Gamma(z_s)$ ist daher von s unabhängig. Daraus folgt

$$(12) \qquad \left\| \frac{\omega_f}{z_s} \right\| \leqq \sigma \frac{\| \omega_f \|}{\| z_s \|} \cdot \sqrt{\frac{\text{max. Eigenwert von } \Gamma(z_1)'\Gamma(z_1)}{\text{min. Eigenwert von } \Gamma(z_1)'\Gamma(z_1)}}\,.$$

Die Wurzel in (12) ist eine stetige Funktion von ϕ. Ihr Maximum auf $[0, 2\pi]$ sei M, also

$$(13) \qquad \left\| \frac{\omega_f}{z_s} \right\| \leqq \sigma M \frac{\| \omega_f \|}{\| z_s \|}\,.$$

Da f in 0 analytisch ist, folgt

$$(14) \qquad \lim_{s \to 0} \left\| \frac{\omega_f}{z_s} \right\| = 0$$

also

$$(15) \qquad \lim_{s \to 0} I_s = 0.$$

Ist nun $f(z)$ ind $U(0)$ nicht nur stetig, sondern auch analytisch, so hat I_s für s mit $0 < s \leqq \tilde{s}$ den gleichen Wert, der wegen (15) gleich Null ist.

SATZ. *Sei $A \in \mathfrak{A}_{\mathbf{R}}^{(n)}$ eine Algebra vom Rang $n \geqq 2$, welche eine Nullteilerhypergerade $\mathfrak{G}$ besitzt und $f(z)$ in einem einfach zusammenhängenden Gebiet G des $\mathbf{R}^n$, das den Punkt* 0 *enthält, analytisch. Dann ist*

$$(16) \qquad \oint_C \frac{f(z)}{z}\, dz = f(0) \oint_C \frac{1}{z}\, dz\,,$$

wo C ein Integrationsweg ist, der $\mathfrak{E}$ umschließt, ganz in G verläuft und auf dem keine Nullteiler liegen.

$$(17) \qquad \oint_C \frac{f(z)}{z - x}\, dz = f(x) \oint_C \frac{1}{z - x}\, dz$$

[3] Siehe [1], S. 148.

Translation liefert

$$i := \oint_C \frac{1}{z-x}\, dz \text{ ist von } x \text{ unabhängig.}$$

Differentiation von

$$(18) \qquad \oint_C \frac{f(z)}{z-x}\, dz = i \cdot f(x); \qquad i \in A;$$

nach x liefert die verallgemeinerten Cauchy'schen Integralformeln

$$(19) \qquad (i \cdot f(x))^{(k)} = k' \cdot \oint_C \frac{f(z)}{(z-x)^{k+1}}\, dz; \qquad k = 0, 1, \ldots .$$

Hieraus folgt i. a. nicht, daß $f(z)$ beliebig oft differenzierbar ist. $f(z)$ ist sicher dann beliebig oft differenzierbar, wenn i kein Nullteiler ist. Für welche Algebren dies der Fall ist, bleibt zu untersuchen. (18) kann man als Ausgangspunkt weiterer funktiontheoretischer Überlegungen, z. B. Laurentreihenentwicklungen hyperkomplexer Funktionen, benützen.

Auch ist (18) naturgemäß ein vorzügliches Instrument zur Lösung partieller Differentialgleichungen. Welche Gleichungen man damit behandeln kann, soll kurz erläutert werden.

Die gewöhnliche Funktionentheorie liefert uns die Hilfsmittel, Randwertaufgaben für die Cauchy–Riemann'schen Differentialgleichungen und die Potentialgleichung $u_{xx} + u_{yy} = 0$ zu lösen. Wie bekannt, genügen die Komponenten jeder analytischen Funktion dieser Gleichung. Um diesen Sachverhalt auf hyperkomplexe Funktionen zu verallgemeinern, geben wir die

DEFINITION 2.[4] Jede partielle Differentialgleichung, der in einem Gebiet G des $\mathbf{R}^n$ jede Komponente der analytischen Funktionen genügt, heißt eine den v. $CR - DGLn$ (4) in G zugeordnete Differentialgleichung.

Um den v. $CR - DGLn$ (4) der analytischen Funktionen zugeordnete Differentialgleichungen zu ermitteln, wenden wir den homogenen linearen Differentialoperator m-ter Ordnung $P_m\left(\frac{\partial}{\partial \xi_0}, \ldots, \frac{\partial}{\partial \xi_{n-1}}\right)$ mit konstanten Koeffizienten aus $\mathbf{R}$ auf eine in $G \subset A \in \mathfrak{A}_{\mathbf{R}}^{(n)}$ analytische Funktion f and, welche analytische Ableitungen bis mindestens zur Ordnung $m-1$ besitzt.

Es ergibt sich wegen (4):

$$(20) \qquad P_m(e_0, \ldots, e_{n-1}) \cdot f^{(m)}(x) = 0.$$

Hinreichend dafür, daß die partielle Differentialgleichung

$$(21) \qquad P_m\left(\frac{\partial}{\partial \xi_0}, \ldots, \frac{\partial}{\partial \xi_{n-1}}\right) u = 0$$

[4] Nach [3], S. 76.

den v. $CR - DGLn$ zugeordnet ist, ist also

$$(22) \qquad P_m(e_0, \ldots, e_{n-1}) = 0.^5$$

(22) ist auch notwendig, denn aus (20) folgt für $f(z) = z^m$ (22).

Im folgenden werden wir für eine nach E. Lammel konstruierte Algebra[6] vom Range 3 mit einer Nullteilergeraden die verallgemeinerten Cauchy–Riemann'schen Differentialgleichungen, eine ihnen zugeordnete Differentialgleichung und die verallgemeinerten Cauchy'schen Integralformeln angeben.

A sei die Menge aller Tripel reeller Zahlen $(\alpha_0, \alpha_1, \alpha_2)$. Addition 2-er Tripel und Multiplikation eines Tripels mit einer reellen Zahl sollen wie bei den Vektoren des $\mathbf{R}^3$ definiert sein. Die Multiplikation 2-er Tripel $(\alpha_0, \alpha_1, \alpha_2)$ und $(\beta_0, \beta_1, \beta_2)$ legen wir durch

$$(23) \qquad \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \alpha_0 & -\alpha_2 & -\alpha_1 \\ \alpha_1 & \alpha_0 & -\alpha_2 \\ \alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \Gamma(a) \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

fest. Damit wird A zu einer Algebra aus $\mathfrak{A}^{(3)}_{\mathbf{R}}$.

Die Determinante von $\Gamma(a)$ ist

$$(24) \qquad \det \Gamma(a) = (\alpha_0 - \alpha_1 + \alpha_2)\left[\left(\alpha_0 + \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2\right)^2 + \frac{3}{4}(\alpha_1 + \alpha_2)^2\right] =$$
$$= 3\alpha_0\alpha_1\alpha_2 + \alpha_0^3 - \alpha_1^3 + \alpha_2^3.$$

A ist also eine Algebra mit einer Nullteilerebene

$$(25) \qquad \alpha_0 - \alpha_1 + \alpha_2 = 0$$

und einer Nullteilergeraden

$$(26) \qquad \alpha_0 + \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 = 0, \qquad \alpha_1 + \alpha_2 = 0.$$

Die v. $CR - DGLn$ sind

$$(27) \qquad (u_y, v_y, w_y) = (-w_x, u_x, v_x), \qquad (u_z, v_z, w_z) = (-v_x, -w_x, u_x).$$

Jede Komponente einer analytischen Funktion, welche 3-mal stetig differenzierbar ist, genügt wegen $u_{yyy} = -w_{xyy}$; $w_{yyx} = v_{xyx}$; $v_{yxx} = u_{xxx}$ der partiellen Differentialgleichung

$$(28) \qquad u_{yyy} + u_{xxx} = 0,$$

welche den Gleichungen (27) zugeordnet ist.

[5] Der entsprechende Satz in [3] ist unter den dort angegebenen Voraussetzungen nicht zutreffend. Gegenbeispiel:

$$\Gamma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial x} \text{ in } C.$$

[6] Siehe [5].

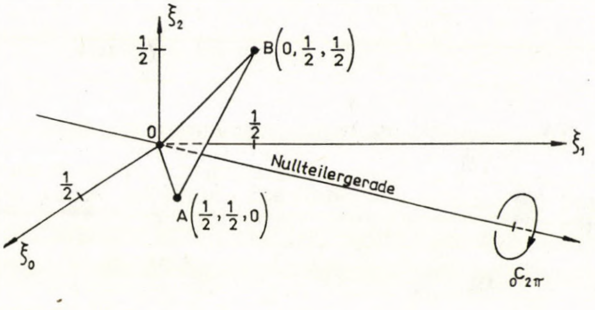
Da A eine Algebra mit einer Nullteilergeraden ist, können wir die verallgemeinerten Cauchy'schen Integralformeln (19) aufstellen. Dazu ist die Berechnung des Inversen a^{-1} von $a=(\alpha_0, \alpha_1, \alpha_2)$ nötig. Aus (23) folgt

$$(29)\qquad a^{-1}=\begin{pmatrix}\alpha_0 & -\alpha_2 & -\alpha_1\\ \alpha_1 & \alpha_0 & -\alpha_2\\ \alpha_2 & \alpha_1 & \alpha_0\end{pmatrix}^{-1}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=\frac{1}{\det\Gamma(a)}\begin{pmatrix}\alpha_0^2+\alpha_1\alpha_2\\ -\alpha_0\alpha_1-\alpha_2^2\\ \alpha_1^2-\alpha_0\alpha_2\end{pmatrix}.$$

Die Nullteilergerade hat die Parameterdarstellung $\lambda\cdot(-1, 1, -1)$. Als Integrationsweg ${}_0C_{2\pi}$ können wir daher

$$(30)\qquad {}_0C_{2\pi}: z=\begin{pmatrix}\xi_0\\ \xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}-1\\ 1\\ -1\end{pmatrix}+r\sin\phi\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}+r\cos\phi\begin{pmatrix}0\\ 1\\ 1\end{pmatrix},\qquad 0\leqq\phi\leqq 2\pi;$$

wählen, wo $r>0$ so klein zu nehmen ist, daß der Integrationsweg mit der Nullteilerebene (25) keinen Punkt gemeinsam hat:



ΔOAB liegt in der Nullteilerebene.

Es ist

$$\oint_{{}_0C_{2\pi}}\frac{1}{z}\,dz=\int_0^{2\pi}\frac{1}{z}\frac{dz}{d\phi}\,d\phi.$$

Wegen $\xi_0=-1+r\sin\phi$; $\xi_1=1+r\cos\phi$; $\xi_2=-1-r\sin\phi+r\cos\phi$ folgt aus (24) und (29)

$$(31)\qquad z^{-1}=\frac{1}{-9r^2\left(1-\frac{1}{2}\sin 2\phi\right)}\begin{pmatrix}r^2-3r\sin\phi-\frac{r^2}{2}\sin 2\phi\\ \frac{r^2}{2}\sin 2\phi-r^2+3r\cos\phi-3r\sin\phi\\ r^2+3r\cos\phi-\frac{r^2}{2}\sin 2\phi\end{pmatrix}.$$

Weiter ist

$$\frac{dz}{d\phi} = \begin{pmatrix} r\cos\phi \\ -r\sin\phi \\ -r\cos\phi - r\sin\phi \end{pmatrix}. \tag{32}$$

Jetzt ist z^{-1} mit $\frac{dz}{d\phi}$ nach (23) zu multiplizieren. Die Integration scheint schwierig, doch nach dem Cauchy'schen Integralsatz und einem Satz der reellen Analysis ist

$$\int_0^{2\pi} \frac{1}{z}\frac{dz}{d\phi}\,d\phi = \lim_{r\to 0}\int_0^{2\pi}\frac{1}{z}\frac{dz}{d\phi}\,d\phi = \int_0^{2\pi}\lim_{r\to 0}\left(\frac{1}{z}\frac{dz}{d\phi}\right)d\phi. \tag{33}$$

Das Integral ganz rechts ist

$$\int_0^{2\pi}\begin{pmatrix}\cos 2\phi \\ 1+\cos^2\phi \\ 1+\sin^2\phi\end{pmatrix}\frac{1}{-3\left(1-\frac{1}{2}\sin 2\phi\right)}\,d\phi = -\frac{2\pi}{\sqrt{3}}\begin{pmatrix}0\\1\\1\end{pmatrix}. \tag{34}$$

Sei nun $x \in A$ beliebig und $f(z)$ in einer Umgebung $U(x)$ von x holomorph. Dann gilt die Cauchy'sche Integralformel

$$\oint_C \frac{f(z)}{z-x}\,dz = -\frac{2\pi}{\sqrt{3}}\begin{pmatrix}0\\1\\1\end{pmatrix}\cdot f(x), \tag{35}$$

wo C ganz in $U(x)$ verläuft, die zur Nullteilergeraden parallele Gerade durch x umschließt und mit der Ebene durch x, welche zur Nullteilerebene parallel ist, keine Punkte gemeinsam hat. Zu beachten ist, daß C wie in der Figur orientiert ist (Rechtsschraube!).

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$$\left(a_0\frac{\partial^n}{\partial x^n} + a_1\frac{\partial^n}{\partial x^{n-1}\partial y} + \ldots + \frac{\partial^n}{\partial y^n}\right)u(x,y) = 0$$

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LEHRSTUHL FÜR MATHEMATIK
TECHNISCHE UNIVERSITÄT
8000 MÜNCHEN 2, ARCISSTRASSE 21
BUNDESREPUBLIK DEUTSCHLAND

[7] Der vorstehende Artikel setzt sich aus Auszügen von [2] zusammen.

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REFLECTED RADICAL CLASSES

By

B. J. GARDNER (Hobart) and P. N. STEWART (Halifax)

1. Introduction. Our categorical terminology is selected from the books of FREYD [5] and MITCHELL [11]. We shall not distinguish notationally between a subobject and a representative morphism.

On the basis of ŠUL'GEIFER's work in [13] we say that a category is *suitable for radical theory* if it has a zero object, kernels and conormal epimorphic images, if unions of ascending chains of normal subobjects exist and are normal and if images under conormal epimorphisms of normal subobjects are normal subobjects.

Throughout this paper "category" will mean a category suitable for radical theory in which the class of normal subobjects of each object is a set.

Let $\mathcal{R}$ be a non-empty, isomorphism closed class of objects in a category $\mathcal{K}$. A subobjects $S \to A$ is a *$\mathcal{R}$-subobject* if $S \in \mathcal{R}$. The class $\mathcal{R}$ is a *radical class* if (see [13, Theorem 3.1])

R1. *If $A \to B$ is a conormal epimorphism and $A \in \mathcal{R}$, then $B \in \mathcal{R}$.*

R2. *An object A is in $\mathcal{R}$ if, for every non-zero conormal epimorphism $A \to B$, B has a non-zero normal $\mathcal{R}$-subobject.*

If $\mathcal{R}$ is a radical class then each object A has a unique largest $\mathcal{R}$-subobject $\mathcal{R}(A) \to A$, and A is called $\mathcal{R}$ *semi-simple* if $\mathcal{R}(A) = 0$.

Suppose that $\mathcal{K}$ also satisfies: for each normal subobject $N \to B$ and conormal epimorphism $A \to B$ the pullback

$$\begin{array}{ccc} P & \to & A \\ \downarrow & & \downarrow \\ N & \to & B \end{array}$$

exists, $P \to A$ is a normal subobject of A, and $P \to N$ is a conormal epimorphism It is then routine to check that R2, in the presence of R1, is equivalent to:

R3. *If $A' \to A \to A'' \to 0$ is exact and $A', A'' \in \mathcal{R}$, then $A \in \mathcal{R}$,*

and

R4. *The union of any ascending chain of normal $\mathcal{R}$-subobjects of an object A is a normal $\mathcal{R}$-subobject of A.*

2. Reflections. For the remainder of this paper $\mathcal{K}_1$ and $\mathcal{K}_2$ will denote categories and $\Phi : \mathcal{K}_1 \to \mathcal{K}_2$ an object function.

The object function Φ is said to *reflect radical classes* if, for each radical class $\mathcal{R}$ in $\mathcal{K}_2$, the class

$$\mathcal{R}^* = \{A \mid \Phi(A) \in \mathcal{R}\}$$

is a radical class in $\mathcal{K}_1$. Throughout this section we assume that Φ is a functor.

THEOREM 1. *If $\mathcal{K}_1$ is a category in which radical classes are characterized by* R1, R3 *and* R4, *and if Φ is an exact functor which preserves unions of ascending chains of normal subobjects, then Φ reflects radical classes.*

The proof is immediate: conditions R1 and R3 for $\mathcal{R}^*$ follow from the exactness of Φ and R4 follows because Φ preserves unions of chains of normal subobjects.

Since a functor with a right adjoint preserves colimits, and hence unions, we have the following corollary.

COROLLARY 2. *If $\mathcal{K}_1$ satisfies the condition of the theorem and Φ is an exact functor with a right adjoint, then Φ reflects radical classes.*

We now give some examples of functors satisfying the conditions of Theorem 1. In each case the functor acts on morphisms in the obvious way.

Let $\mathcal{K}_1 = \mathcal{K}_2 = \mathbf{R}$, the category of associative rings, and let S be a fixed semigroup. For each $A \in \mathbf{R}$ define $\Phi(A)$ to be the semigroup ring $A[S]$. The case when S is the free semigroup on one generator with an identity adjoined (that is, $A[S]$ is the polynomial ring $A[x]$) has been considered in detail by the first author in [6]. A similar example can be obtained by defining $\Phi(A)$ to be the ring of $n \times n$ matrices with entries from A.

AMITSUR [2] considers some examples of exact functors which reflect radical classes and in a remark on page 53 asserts that all exact functors $\Phi : \mathbf{R} \to \mathbf{R}$ reflect radical classes. It can be seen that this assertion is incorrect by considering the exact functor which assigns to each associative ring A the power series ring $A[[x]]$: for each integer $n \geqq 2$ let A_n be a commutative ring satisfying $A_n^{n-1} \neq (0) = A_n^n$ and let $A = \bigoplus_n A_n$, then $A_n[[x]] \in \mathcal{N}$ (the nil radical class) for each $n \geqq 2$, but $A[[x]] \notin \mathcal{N}$.

Forgetful functors also provide examples. For instance, take $\mathcal{K}_1 =$ the category of associative rings with a set of operators and $\mathcal{K}_2 = \mathbf{R}$ (see also DIVINSKY and SULINSKI [4]), or take $\mathcal{K}_1 = \mathbf{R}$ and $\mathcal{K}_2 = \mathbf{AB} =$ the category of abelian groups (the reflected radical classes obtained in this case are called *A-radical classes* and have been studied by the first author in [7]).

A further class of examples can be obtained as follows. Let $\mathcal{K}_1 = \mathbf{R}$, $\mathcal{K}_2 =$ the category of Jordan rings, and for each $A \in \mathbf{R}$ define $\Phi(A)$ to be the Jordan ring obtained by defining the multiplication $x*y = xy + yx$ for all $x, y \in A$ (there is, of course, a similar example for Lie rings). For another example of this kind let $\mathcal{K}_1 = \mathbf{J} =$ the category of Jacobson radical rings, $\mathcal{K}_2 = \mathbf{GR} =$ the category of groups, and for each $A \in \mathbf{J}$ define $\Phi(A)$ to be the group obtained by defining the operation $x \circ y = x + y - xy$ for all $x, y \in A$. If $\mathcal{R}$ is a radical class in $\mathbf{GR}$ its reflection by the functor Φ in this example is not only a radical class in $\mathbf{J}$, but also in $\mathbf{R}$. This follows from the next proposition which generalizes a result of ARNAUTOV and VODINČAR [3] concerning topological rings.

PROPOSITION 3. *Let $\mathcal{K}$ be a category, $\mathcal{C}$ a hereditary (that is, closed under taking normal subobjects) radical class in $\mathcal{K}$, and $\mathcal{C}^\#$ the full subcategory of $\mathcal{K}$ generated* *by $\mathcal{C}$. Then $\mathcal{C}^{\#}$ is a category suitable for radical theory and every radical class in $\mathcal{C}^{\#}$ is a radical class in $\mathcal{K}$.*

PROOF. The suitability of $\mathcal{C}^{\#}$ for radical theory is easily verified.

Let $\mathcal{R}$ be a radical class in $\mathcal{C}^{\#}$. If $A \to B$ is a conormal epimorphism in $\mathcal{K}$ and $A \in \mathcal{R}$, then $A \in \mathcal{C}$ and hence $B \in \mathcal{C}$. It follows that $A \to B$ is a conormal epimorphism in $\mathcal{C}^{\#}$, whence $B \in \mathcal{R}$. This establishes R1. Suppose an object A of $\mathcal{K}$ is such that for every non-zero conormal epimorphism $A \to B$, B has a non-zero normal subobject $I \to B$ with $I \in \mathcal{R}$. Then $I \in \mathcal{C}$, so by R2 we have $A \in \mathcal{C}$. Finally, if $A \to C$ is a non-zero conormal epimorphism in $\mathcal{C}^{\#}$ it has the same property in $\mathcal{K}$ and so, by assumption, C has a non-zero normal $\mathcal{R}$-subobject $J \to C$. Since $\mathcal{R}$ satisfies R2 in $\mathcal{C}^{\#}$ it follows that $A \in \mathcal{R}$. Thus $\mathcal{R}$ satisfies R2 in $\mathcal{K}$ and is thus a radical class in $\mathcal{K}$, as asserted.

A radical class $\mathcal{R}$ is *strict* (KUROS [9]) if for each object A, $\mathcal{R}(A) \to A$ contains all $\mathcal{R}$-subobjects of A. In the next theorem we shall require the following characterization of strict radical classes.

PROPOSITION 4. *Let $\mathcal{K}$ be a category, $\mathcal{R}$ a radical class in $\mathcal{K}$, and $\mathcal{S}$ the class of $\mathcal{R}$ semi-simple objects. The radical class $\mathcal{R}$ is strict if and only if*

$$\mathcal{R} = \{A \mid [A, S]_{\mathcal{K}} = \{0\} \text{ for all } S \in \mathcal{S}\}.$$

PROOF. Assume that $\mathcal{R}$ is strict. For any morphism $A \to S$, where $A \in \mathcal{R}$ and $S \in \mathcal{S}$, $\mathrm{Im}\,(A \to S)$ is an $\mathcal{R}$-subobject of S and thus is contained in $0 = \mathcal{R}(S) \to S$. Hence $[A, S]_{\mathcal{K}} = \{0\}$ for $A \in \mathcal{R}$, $S \in \mathcal{S}$. On the other hand, if A is such that $[A, S]_{\mathcal{K}} = \{0\}$ for each $S \in \mathcal{S}$, it follows from the existence of an exact sequence

$$0 \to \mathcal{R}(A) \to A \to S_0 \to 0$$

with $S_0 \in \mathcal{S}$, that $A = \mathcal{R}(A) \in \mathcal{R}$.

Conversely, assume that

$$\mathcal{R} = \{A \mid [A, S]_{\mathcal{K}} = \{0\} \text{ for each } S \in \mathcal{S}\}.$$

For an arbitrary object A there exists an exact sequence

$$0 \to \mathcal{R}(A) \to A \to S \to 0$$

with $S \in \mathcal{S}$. For any $\mathcal{R}$-subobject $I \to A$ of A, we have $I \to A \to S \in [I, S]_{\mathcal{K}} = \{0\}$ so that $I \to A \subseteq \mathrm{Ker}\,(A \to S) = \mathcal{R}(A) \to A$, and $\mathcal{R}$ is therefore strict. This completes the proof.

We have seen that an exact functor with a right adjoint reflects radical classes. Where strict radical classes are concerned, the exactness requirement can be dropped.

THEOREM 5. *If Φ has a right adjoint, then Φ reflects strict radical classes; that is, if $\mathcal{R}$ is a strict radical class in $\mathcal{K}_2$, then $\mathcal{R}^* = \{A \mid \Phi(A) \in \mathcal{R}\}$ is a radical class in $\mathcal{K}_1$.*

PROOF. Let $\mathcal{R}$ be a strict radical class in $\mathcal{K}_2$, $\mathcal{S}$ the class of $\mathcal{R}$ semi-simple objects and $\Psi : \mathcal{K}_2 \to \mathcal{K}_1$ a right adjoint of Φ. Using Proposition 4 we see that

$$A \in \mathcal{R}^* \Leftrightarrow \Phi(A) \in \mathcal{R} \Leftrightarrow [\Phi(A), S]_{\mathcal{K}_2} = \{0\} \quad \text{for all} \quad S \in \mathcal{S}$$

$$\Leftrightarrow [A, \Psi(S)]_{\mathcal{K}_1} = \{0\} \quad \text{for all} \quad S \in \mathcal{S}.$$

Clearly $\mathcal{R}^*$ satisfies R1. Let A be an object of $\mathcal{K}_1$ such that for every non-zero conormal epimorphism $A \to B$, B contains a non-zero normal $\mathcal{R}^*$-subobject. If there exists a non-zero morphism $A \to \Psi(S)$ for some $S \in \mathcal{S}$, let $I \to \Psi(S) = \mathrm{Im}(A \to \Psi(S))$. By assumption, I has a non-zero normal $\mathcal{R}^*$-subobject, and so $\Psi(S)$ has a non-zero $\mathcal{R}^*$-subobject $C \to \Psi(S)$. But $[C, \Psi(S)]_{\mathcal{K}_1} = \{0\}$, so $C = 0$ which is a contradiction. Hence $[A, \Psi(S)]_{\mathcal{K}_1} = \{0\}$ for each $S \in \mathcal{S}$, and so $\mathcal{R}^*$ satisfies R2 and the proof is complete.

Since in an abelian category every radical class is strict, we have the following corollary.

COROLLARY 6. *If $\mathcal{K}_2$ is abelian and Φ has a right adjoint, then Φ reflects radical classes.*

We conclude this section with some examples of functors satisfying the conditions of Theorem 5. In each case the functors act on morphisms in the obvious way.

Let $\mathcal{K}_1$ be a category of universal algebras (which is suitable for radical theory), Σ a set of identities, and for each algebra $A \in \mathcal{K}_1$ let A_Σ denote the quotient algebra of A induced by Σ. Denote the full subcategory of $\mathcal{K}_1$ generated by the variety $\{A \mid A_\Sigma = A\}$ by v_Σ. The correspondence $A \to A_\Sigma$ defines, in the obvious way, a functor $\Phi : \mathcal{K}_1 \to v_\Sigma$ which has inclusion as a right adjoint. Consequently, $\{A \mid A_\Sigma \in \mathcal{R}\}$ is a radical class in $\mathcal{K}_1$ for every strict radical class $\mathcal{R}$ in v_Σ. In particular, when $\mathcal{K}_1 = \mathbf{GR}$ and $\mathcal{R} = \{0\}$ we obtain the verbal radical classes of ŠČUKIN [12]. We mention two other examples of functors of this kind. First, the functor from **GR** to **AB** which maps each group G to $G/[G, G]$ where $[G, G]$ is the commutator subgroup of G. Second, the functor from **R** to the full subcategory of rings with trivial multiplication which maps each associative ring A to A/A^2. (Since the full subcategory of associative rings with trivial multiplication is naturally isomorphic to the category **AB**, this is another example of a functor from **R** to **AB** which reflects radical classes.)

For another example let R_1 and R_2 be associative rings with identity and $\mathbf{MOD}(R_i)$ be the category of right unital R_i-modules for $i = 1, 2$. Let M be a left unital R_1, right unital R_2-module and define $\Phi : \mathbf{MOD}(R_1) \to \mathbf{MOD}(R_2)$ by $\Phi(N) = N \otimes_{R_1} M$. Since Φ has a right adjoint and $\mathbf{MOD}(R_2)$ is abelian, Φ reflects radical classes. In some cases the right adjoint Ψ of Φ also reflects radical classes. For instance, if there is an (identity preserving) ring homomorphism $f : R_1 \to R_2$ and we set $M = R_2$, then Ψ associates with each R_2-module K the R_1-module obtained by defining $kr = kf(r)$ for $k \in K$, $r \in R_1$. The functor Ψ reflects radical classes since it satisfies the conditions of Theorem 1 (alternatively, Ψ has a right adjoint of its own).

3. Invariant classes. In this section we shall assume that $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$ and drop the requirement that the object function $\Phi : \mathcal{K} \to \mathcal{K}$ be a functor. A class $\mathcal{C}$ of objects of $\mathcal{K}$ is Φ *invariant* if $\Phi[\mathcal{C}] \subseteq \mathcal{C}$.

PROPOSITION 7. *If Φ reflects radical classes and $\mathcal{C}$ is Φ invariant, then $L(\mathcal{C})$, the lower radical class determined by $\mathcal{C}$, is also Φ invariant.*

PROOF. If $\Phi[\mathcal{C}] \subseteq \mathcal{C}$, then for any $A \in \mathcal{C}$ we have $\Phi(A) \in \mathcal{C} \subseteq L(\mathcal{C})$, and so $A \in L(\mathcal{C})^*$. Thus $\mathcal{C} \subseteq L(\mathcal{C})^*$, so $L(\mathcal{C}) \subseteq L(\mathcal{C})^*$ as required.

For any class $\mathcal{C}$ of objects in $\mathcal{K}$, $\mathcal{C}^\uparrow = \{\Phi^n(A) \mid A \in \mathcal{C}, n = 0, 1, 2, \ldots\}$ is the smallest Φ invariant class of objects containing $\mathcal{C}$. If Φ reflects radical classes the above proposition implies that $L(C^{\uparrow})$ is the smallest Φ invariant radical class containing $\mathcal{C}$.

For any class $\mathcal{C}$ of objects in $\mathcal{K}$, $C^{\downarrow} = \{A \mid \Phi^n(A) \in \mathcal{C}$ for all $n = 0, 1, 2, \dots\}$ is the largest Φ invariant subclass in $\mathcal{C}$. If $\mathcal{C}$ is a radical class and Φ reflects radical classes, then

$$\mathcal{C}^{\downarrow} = \mathcal{C} \cap \mathcal{C}^* \cap (\mathcal{C}^*)^* \cap \dots$$

is a radical class and consequently the largest Φ invariant radical subclass of $\mathcal{C}$. However, an arbitrary class $\mathcal{C}$ may not contain a largest Φ invariant radical subclass; in fact, when Φ is the identity all classes are Φ invariant and it is known [10, Example 2] that an arbitrary class $\mathcal{C}$ need not contain a largest radical subclass.

These two constructions were first considered in connection with the polynomial ring functor $\Phi(A) = A[x]$ by the first author in [6]. We note that in this case the chain

$$\mathcal{R} \supseteqq \mathcal{R}^* \supseteqq (\mathcal{R}^*)^* \supseteqq \dots$$

need not terminate, thus answering a question raised in [6]. For let $p_1, p_2, \dots$ be an enumeration of the primes, F_n the field of p_n elements, $A_n = F_n[x_1, \dots, x_n]$ and $\mathcal{R} = L(\{A_n \mid n = 1, 2, \dots\})$. Arguing as in [6, Corollary 9] we see that F_n is in the n^{th} class of the chain but not in the $(n+1)^{\text{st}}$ class.

For the polynomial ring functor we have

$$L(\mathfrak{L}^{\uparrow}) = \mathfrak{L}^{\uparrow} = \mathfrak{L} = \mathfrak{L}^{\downarrow} \subseteqq \mathcal{N}^{\downarrow} \subseteqq \mathfrak{J}^{\downarrow} \subseteqq \mathcal{N} \begin{matrix} \subseteqq L(\mathcal{N}^{\uparrow}) \subseteqq \\ \subseteqq \quad \mathfrak{J} \quad \subseteqq \end{matrix} L(\mathfrak{J}^{\uparrow})$$

where $\mathfrak{J}$ is the Jacobson, $\mathcal{N}$ is the nil and $\mathfrak{L}$ is the Levitzki (locally nilpotent) radical class. All of the inclusions are straightforward except $\mathfrak{J}^{\downarrow} \subseteqq \mathcal{N}$ which follows from the result that for any ring A, $\mathfrak{J}(A[x]) \subseteqq \mathcal{N}(A)[x]$ (see [1]). The rings in $\mathcal{N}^{\downarrow}$ have been studied by Krempa in [8] where they are called *absolutely nil*.

We now consider the above constructions in the case of the matrix ring functor $\Phi(A) = A_N$ (N is a fixed positive integer). For any radical class $\mathcal{R}$,

$$\mathcal{R}^{\downarrow} = \{A \mid A_{N_k} \in \mathcal{R} \quad \text{for} \quad k = 0, 1, 2, \dots\} \supseteqq \{A \mid A_n \in \mathcal{R} \quad \text{for} \quad n = 0, 1, 2, \dots\}$$

with equality if $\mathcal{R}$ is closed under taking subrings. Since Krempa [8] has shown that $\{A \mid A_n \in \mathcal{N}$ for $n = 0, 1, 2, \dots\} = \{A \mid A[x] \in \mathfrak{J}\}$ we conclude that $\mathcal{N}^{\downarrow} = \{A \mid A[x] \in \mathfrak{J}\}$. It is now straightforward to check that

$$L(\mathfrak{L}^{\uparrow}) = \mathfrak{L}^{\uparrow} = \mathfrak{L} = \mathfrak{L}^{\downarrow} \subseteqq \mathcal{N}^{\downarrow} = \{A \mid A[x] \in \mathfrak{J}\} \subseteqq \mathcal{N} \subseteqq L(\mathcal{N}^{\uparrow}) \subseteqq L(\mathfrak{J}^{\uparrow}) = \mathfrak{J}^{\uparrow} = \mathfrak{J} = \mathfrak{J}^{\downarrow}$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TASMANIA
BOX 252-C, G. P. O.
HOBART, TASMANIA 7001
AUSTRALIA

DEPARTMENT OF MATHEMATICS
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA
CANADA

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ÜBER EIN PROBLEM VON J. M. ASH, P. ERDŐS UND L. A. RUBEL

Von

E. HEPPNER (Freiburg)

Wir wollen hier zu festem $k \in \mathbf{N} \cup \{0\}$ eine Funktion $f: \mathbf{R} \to \mathbf{R}$ konstruieren mit folgenden Eigenschaften:

(i) *Für jedes* $\alpha \in [0, 1]$ *gilt*

$$\lim_{x\to\infty} x \left(\prod_{i=1}^{k} \log_i x\right) (f(x+\alpha) - f(x)) = 0,$$

wobei $\log_i x$ *den i-fach iterierten Logarithmus von x bezeichnet, aber*

(ii) $\sup_{\alpha\in[0,1]} |f(x+\alpha) - f(x)| = \infty$ *für alle* $x \in \mathbf{R}$.

Für $k = 0$ beantwortet dies eine Frage von J. M. ASH, P. ERDŐS und L. A. RUBEL [1].

Wir betrachten $\mathbf{R}$ als Vektorraum über $\mathbf{Q}$. Aus dem Zornschen Lemma folgt die Existenz einer Basis B von $\mathbf{R}$ über $\mathbf{Q}$. Wir wählen eine abzählbare Teilmenge $\{x_i\}_{i\in\mathbf{N}}$ der Menge B und bezeichnen mit $V_i (i \in \mathbf{N} \cup \{0\})$ den $\mathbf{Q}$-Vektorraum, der aufgespannt wird von $\{x \mid x \in B, x \neq x_j \text{ für alle } j > i\}$. Dann gilt offenbar $V_i \subset \subset V_{i+1}$, $V_i \neq V_{i+1}$ und $\bigcup_{i=1}^{\infty} V_i = \mathbf{R}$.

Für $n \leqq |x| < n+1$ mit $n \in \mathbf{N}$ setzen wir nun mit $c_0 = 0$, $c_{i+1} = \exp(c_i)$,

$$f(x) = \begin{cases} 0 & \text{für } x \in V_n, \\ \log_{k+2}(m + c_{k+2}) - \log_{k+2}(n + c_{k+2}) & \text{für } x \notin V_n,\ x \in V_m,\ x \notin V_{m-1}. \end{cases}$$

Um die Eigenschaft (i) zu zeigen geben wir uns ein $\alpha \in [0, 1]$ vor. Es sei $\alpha \in V_l$. Für ein x mit $l < n \leqq x < n+1$ unterscheiden wir nun die Fälle $x \in V_n$ und $x \notin V_n$. Im ersten Fall ist $f(x) = f(x+\alpha) = 0$. Für $x \notin V_n$ gilt aber $x \in V_m$, $x \notin V_{m-1}$ genau für $x + \alpha \in V_m$, $x + \alpha \notin V_{m-1}$. Hieraus folgt nun leicht

$$f(x) - f(x+\alpha) = \begin{cases} 0 & \text{für } x + \alpha < n+1 \\ \log_{k+2}(n+1+c_{k+2}) - \log_{k+2}(n + c_{k+2}) & \text{für } x + \alpha \geqq n+1. \end{cases}$$

Wegen

$$\log_{k+2}(n+1+c_{k+2}) - \log_{k+2}(n+c_{k+2}) = O\left(\left(x \prod_{i=1}^{k+1} \log_i (x + c_{k+2})\right)^{-1}\right)$$

ist damit (i) gezeigt.

Für festes x wird aber $f(x+\alpha)$ beliebig groß, wenn wir $\alpha = r\,|x_m|$ mit m beliebig groß und $0 < r < |x_m|^{-1}$, $r \in \mathbf{Q}$, wählen, womit auch (ii) nachgewiesen ist.

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MATHEMATISCHES INSTITUT DER UNIVERSITÄT
78 FREIBURG
HEBELSTR. 40
BUNDESREPUBLIK DEUTSCHLAND

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REMARK ON A PAPER OF C. T. NG

By

I. FENYŐ (Budapest)

In a recent paper C. T. NG [1] considered the following functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F_{ij}(p_i q_j) = \sum_{i=1}^{n} G_i(p_i) + \sum_{j=1}^{m} H_j(q_j)$$

where p_i and q_j are probability distributions, i.e. $\sum_{j=1}^{m} p_i = 1$, $p_i \geqq 0$ $(i = 1, 2, \dots, n)$, $\sum_{j=1}^{m} q_j = 1$, $q_j \geqq 0$ $(j = 1, 2, \dots, m)$. Ng has given the general measurable solution of this equation for $n = 2$ and $m = 3$ and made the remark, that the general case can be reduced to the case considered by him. His result is based on the fact, that he succeeded to find the general measurable solution of the following functional equation

$$(1)\quad F_1(q_1) - F_2(pq_1) - F_3(q_1 - pq_1) + F_4(q_2) - F_5(pq_2) - F_6(q_2 - pq_2) = A(p, q_1 + q_2),$$

where p, q_1, q_2 are independent variables, $p \in [0, 1]$, $q_1, q_2, q_1 + q_2 \in [0, 1]$; F_i, A are the unknown functions $(i = 1, 2, \dots, 6)$.

Ng needed four lemmas to solve the functional equation (1). With a very little more assumption on A we can get the solution of Ng in a much shorter way by using the technique which we elaborated in some earlier papers [2], [3], [4]. This method has not only the advantage that it gives the asked solution in a much shorter way, but provides the general solution not only in the domain of measurable functions but also in the more general domain of distributions.

Let us denote by $I\,(= I^1)$ the interval $(0, 1)$, $I \times I = I^2$, $I^2 \times I = I^3$. $D(I^N)$ is the linear space of Schwartz testing functions with support in I^N, $D'(I^N)$ the space of distributions over I^N $(N = 1, 2, 3)$.

Now we consider in (1) F_i as unknown distributions from the class $D'(I)$ $(i = 1, \dots, 6)$ and A a distribution of $D'(I^2)$. Under $F_i(pq)$ we mean PF_i, where the operator P is defined in [3] ((3.4) p. 59), the sum standing on the left hand side of (1) (where distributions with different argument are) is understood in the sense of the operation $\oplus$, see [3] p. 55.

We assume (this assumption is not made in [1]) that $A(p, q) = U(p)q + V(p)$, where $U, V \in D'(I)$ are unknown, and $U(p)q$ means $U \otimes q$ (here the function q is also considered as a distribution of $D'(I)$) and $\otimes$ is the tensorial product of two distributions and of course under $+$ we mean again the operation $\oplus$. Under F_i', F_i'', U', ... we mean the distribution-derivatives of the first, second, ... order.

We state the following

THEOREM. *Let $A(p, q) = U(p) \otimes q + V(p)$, where U and V are unknown distributions of the class $D'(I)$. The most general solution of the functional equation* (1) *in the domain of distributions of the class $D'(I)$ are distributions generated by the following functions:*

$$F_i(t) = a_i t \log t + b_i t + c_i \qquad (i = 1, 2, \dots, 6)$$

$$U(p) = a(p \log p + (1 - p) \log (1 - p) + bp + c)$$

$$V(p) = d,$$

where a_i, b_i, c_i, a, b, c, d are certain constants.

This theorem is slightly more general than that of Ng, because it gives the most general solutions in a class of distributions and not only in a class of measurable functions as by Ng. But at the same time it is also a small restriction, because it makes an assumption on the form of the unknown $A(p, q)$. Our aim is using the technique of papers [2], [3], [4] to give a proof for the theorem above which is much more simpler and shorter than that of Ng.

PROOF. Using the notations, definitions above and rules (5.4), (5.1), (4.4) in [3] we form the second derivative of (1) with respect to q_1:

$$F_1''(q_1) - p^2 F_2''(pq_1) - F_3''(q_1 - pq_1)(1 - p^2) = 0. \tag{2}$$

If we take now the derivative of (2) with respect to p and multiply by q_1, we get

$$-2pq_1 F_2''(pq_1) - p^2 q_1^2 F_2'''(pq_1) - 2(1 - p)q_1 F_3''(q_1 - pq_1) - (1 - p)^2 q_1 F_3'''(q_1 - pq_1) = 0.$$

Let us introduce the new variables $t = pq_1$ and $q = q_1$ instead of p and q_1 (this can be made after the definition (2.1) in [3] p. 47), so we are led to

$$-2tF_2''(t) - t^2 F_2'''(t) = 2(q - t)F_3'''(q - t) + (q - t)^2 F_3'''(q - t). \tag{3}$$

If we differentiate (3) with respect to q, we see that

$$2tF_2''(t) + t^2 F_2'''(t) = C, \tag{4}$$

where C is a constant (i.e. the constant distribution). (4) is an ordinary differential equation with respect to an unknown distribution considered in the interval I, that means we have to apply to (4) testing functions of one variable which vanish (together with all their derivatives) for $t = 0$. Then after a well known theorem of L. SCHWARTZ [5] the differential equation (4) has no other solutions as the classical ones. Hence, if in (1) F_2 is not identically zero, then it can have only the form

$$F_2(t) = a_2 t \log t + b_2 t + c_2$$

(a_2, b_2, c_2 are constants).

Substituting a new variable into (3) for $q - t$ by means of the quoted rule, we get in the same manner

$$F_3(t) = a_3 t \log t + b_3 t + c_3.$$

Just in this way (also by reason of symmetry)

$$F_5(t) = a_5 t \log t + b_5 t + c_5, \qquad F_6(t) = a_6 t \log t + b_6 t + c_6.$$

If we know that F_3 has the form above, we get from (2) $F_1''(q_1) = 1/q_1$ and therefore

$$F_1(t) = a_1 t \log t + b_1 t + c_1$$

and in the same way

$$F_4(t) = a_4 t \log t + b_4 t + c_4 .$$

Substituting these functions into (1) we get

$$A(p, q) = a(p \log p + (1-p) \log (1-p) + bp + c)q + d.$$

If we substitute all these functions into (1) we get also the necessary relations between the constants as it is in [1].

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TECHNICAL UNIVERSITY
FACULTY OF ELECTRICAL ENGINEERING
DEPARTMENT OF MATHEMATICS
1111 BUDAPEST, STOCZEK U. 2—4

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AUTOMORPHISMEN GEWISSER FUNKTIONENALGEBREN. II

Von

K.-H. INDLEKOFER (Paderborn)

1. Einleitung. Es sei $A(D)$ die Algebra der im Einheitskreis $D=\{z\in\mathbf{C}:|z|<1\}$ holomorphen und auf der Hülle $\bar{D}$ stetigen Funktionen. Mit

$$A_a(D):=\left\{f\in A(D):f(z)=\sum_{n=0}^{\infty}a_nz^n,\ \sum_{n=0}^{\infty}|a_n|<\infty\right\} \tag{1.1}$$

bezeichnen wir die Algebra der Funktionen f aus $A(D)$, deren Maclaurinreihe für $|z|=1$ absolut konvergiert. In [7] untersuchten wir Teilalgebren von $A_a(D)$, die wir durch Vorgabe des Stetigkeitsmoduls bestimmten, und charakterisierten ihre Automorphismen. Genauer:

Sei $w:[0,\pi]\to\mathbf{R}$ ein *Stetigkeitsmodul*, d. h. eine Funktion mit den folgenden drei Eigenschaften

(i) *$w(h)\to 0$ für $h\to 0$, $w(0)=0$.*

(ii) *$w(h)$ ist positiv und monoton wachsend.*

(iii) *w ist subadditiv:*

$$w(h_1+h_2)\leqq w(h_1)+w(h_2).$$

Für stetige Funktionen $f:\bar{D}\to\mathbf{C}$ sei

$$\omega(f;h):=\sup_{|\vartheta|\leqq\pi}\ \sup_{|t|\leqq h}|f(e^{i(\vartheta+t)})-f(e^{i\vartheta})|$$

der *Stetigkeitsmodul von f.* Für jeden Stetigkeitsmodul $w:[0,\pi]\to\mathbf{R}$ setzten wir

$$A_{aw}:=\left\{f\in A_a(D):\sup_{h\in(0,\pi]}\frac{\omega(f;h)}{w(h)}<\infty\right\} \tag{1.2}$$

und bewiesen für diese Teilalgebren von $A_a(D)$

(I) *Sei*

$$\sum_{i=0}^{\infty}2^{i/2}w(2^{-i})=\infty. \tag{1.3}$$

Dann hat jeder Automorphismus τ der Algebra A_{aw} die Gestalt $(f\in A_{aw},\ z\in\bar{D})$

$$(\tau f)(z)=f(e^{i\alpha}z),\qquad \alpha=\alpha(\tau)\in\mathbf{R}.$$

Die von (I) nicht erfaßten Teilalgebren behandelte

(II) *Sei*

$$\sum_{i=0}^{\infty} 2^{i/2} w(2^{-i}) < \infty . \tag{1.4}$$

Dann existiert zu jedem Automorphismus τ der Algebra A_{aw} eine Möbiustransformation Φ von D, so daß für alle $f \in A_{aw}$

$$(\tau f)(z) = f(\Phi(z)), \qquad z \in \bar{D}, \tag{1.5}$$

ist. Umgekehrt definiert (1.5) *für jede solche konforme Abbildung Φ einen Automorphismus von A_{aw}.*

Bekanntlich hat jede Möbiustransformation $\Phi : D \to D$ die Gestalt ($z \in D$)

$$\Phi(z) = e^{i\alpha} \frac{z - \zeta_0}{1 - \bar{\zeta}_0 z}, \qquad \zeta_0 \in D. \tag{1.6}$$

In (I) sind nur die "Drehungen" ($\zeta_0 = 0$), in (II) auch die Transformationen Φ mit $\zeta_0 \neq 0$ zugelassen. Dies beruht darauf, daß, falls (1.3) gilt, zu jedem ζ_0, $0 < |\zeta_0| < 1$, Funktionen $f \in A_{aw}$ mit der Eigenschaft existieren, daß $f \circ \Phi \in A(D) \setminus A_a(D)$ ist, d. h. für $f(\Phi(z)) = \sum_{n=0}^{\infty} b_n z^n$ ist $\sum_{n=0}^{\infty} |b_n| = \infty$. Die Existenz solcher Funktionen $f \in A_a$ – also ohne die Nebenbedingung (1.3) – wurde zuerst von L. ALPÁR [1], [2] bewiesen. Das bedeutet, daß in diesen Fällen die Menge

$$M_{aw}(\zeta_0) := \left\{ f \in A_{aw} : f \circ \Phi \notin A_a(D), \quad \Phi(z) = e^{i\alpha} \frac{z - \zeta_0}{1 - \bar{\zeta}_0 z} \right\} \qquad (\zeta_0 \neq 0) \tag{1.7}$$

nicht leer ist. In dieser Arbeit wollen wir die Menge $M_{aw}(\zeta_0)$ genauer untersuchen und zeigen, daß der Durchschnitt

$$M_{aw} := \bigcap_{0 < |\zeta_0| < 1} M_{aw}(\zeta_0) \tag{1.8}$$

nicht leer ist und die Menge M_{aw} sogar dicht[1] in $A_a(D)$ liegt. Dabei wird u. a. ein von G. HALÁSZ [4] gestelltes Problem – teils unter Benützung einiger Gedanken von Halász – gelöst. Wir behandeln der Einfachheit halber nicht alle Stetigkeitsmoduln w, die (1.3) erfüllen. Da nämlich für

$$w(h) = \frac{h^{\frac{1}{2}}}{\log \frac{2\pi}{h}}$$

[1] Dabei legen wir die durch die Norm ($f \in A_a(D)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$)

$$\|f\| = \sum_{n=0}^{\infty} |a_n|$$

erzeugte Topologie zugrunde.

die Reihe

$$\sum_{i=0}^{\infty} 2^{i/2} w(2^{-i}) = \sum_{i=0}^{\infty} \frac{1}{i \log 2 + \log 2\pi}$$

divergiert, betrachten wir hier nur Stetigkeitsmoduln w der Gestalt

$$w(h) = \frac{h^{\frac{1}{2}}}{\log \frac{2\pi}{h}} \cdot w_1\left(\left(\log \frac{2\pi}{h}\right)^{-1}\right), \tag{1.9}$$

wobei wir voraussetzen, daß sich $w_1(u)$ für ein gewisses Intervall $[0, u_0]$ wie ein Stetigkeitsmodul verhält (d. h. (i) – (iii) sind erfüllt). Dann zeigen wir

SATZ 1. *Sei* $w: [0, \pi]$ *ein Stetigkeitsmodul der Gestalt* (1.9), *und sei für* $l_0 \geqq \frac{1}{u_0}$

$$\sum_{l \geqq l_0} l^{-1} w_1(l^{-1}) = \infty .$$

Dann ist die Menge M_{aw} *nicht leer.*

KOROLLAR 1. *Es existieren Funktionen* $f \in A_a(D)$ *mit folgenden Eigenschaften*

(a) $$\omega(f; h) = O\left(\frac{h^{\frac{1}{2}}}{\log \frac{1}{h} \cdot \log\log \frac{1}{h}}\right) \quad \text{für} \quad h \to 0^+.$$

(b) *Für alle Möbiustransformationen* Φ *von* D, $\Phi(z) \neq e^{i\alpha} z$, *ist* $\sum_{n=0}^{\infty} |b_n| = \infty$, *wobei* $f(\Phi(z)) = \sum_{n=0}^{\infty} b_n z^n$ *ist.*

$A_a(D)$ ist ein Banachraum mit der Norm ($f(z) = \Sigma a_n z^n$)

$$\|f\| = \sum_{n=0}^{\infty} |a_n| .$$

Wir zeigen

SATZ 2. *Die Menge* M_{aw} *aus Satz* 1 *liegt dicht im Banachraum* $A_a(D)$.

2. Beweis von Satz 1. Setzt man die Taylorreihe von $\Phi(z)$ in die von $f(w) = \Sigma a_\nu w^\nu$ ein, so wird der Zusammenhang zwischen $\{a_\nu\}$ und $\{b_n\}$ durch eine lineare Transformation

$$b_n = \sum_{\nu=0}^{\infty} \gamma_{n\nu} a_\nu \qquad (n = 0, 1, 2, \ldots) \tag{2.1}$$

gegeben. Grundlegend ist nun folgendes

LEMMA. *Sei* $r = |\zeta_0| \neq 0$ *und* $c_l > 1$ $(l = 1, 2, \ldots)$ *eine Folge positiver Zahlen mit* $\lim_{l \to \infty} c_l = 1$ *und*

$$\frac{1-r}{1+r} < \frac{1}{c_l} < c_l < \frac{1+r}{1-r} \qquad (l \leqq l_1) . \tag{2.2}$$

Dann gilt für $l \geqq l_0 \geqq l_1$ und jedes $\nu = 1, 2, \ldots$

$$(2.3) \qquad \sum_{c_l^{-1}\nu < n \leqq c_l \nu} |\gamma_{n\nu}| > K_1 \nu^{\frac{1}{2}}(c_l - c_l^{-1}) - K_2,$$

wobei die positiven Konstanten K_1 und K_2 von ν und der Folge $\{c_l\}$ unabhängig sind.

BEWEIS. In [5], II, § 5 werden Zahlen $c_0' < 1 < c_0$ mit

$$\frac{1-r}{1+r} < c_0' < c_0 < \frac{1+r}{1-r}$$

angegeben und danach das Intervall $[c_0', c_0]$ in const $\cdot\, \nu$ Intervalle der Länge $> 4\nu^{-1}$ eingeteilt. In jedem dieser Intervalle gibt es mindestens eine Stelle n/ν, so daß $|\gamma_{n\nu}| > K_3 \nu^{-\frac{1}{2}}$ ist (K_3 hängt nicht von ν ab). Zählt man die Intervalle ab, die ganz in $[c_l^{-1}, c_l]$ liegen, erhält man die Behauptung des Lemmas.

Sei nun

$$\sum_{l=l_0}^{\infty} l^{-1} w_1(l^{-1}) = \infty .$$

Dann existiert eine Folge $\{\delta(l)\}$ mit den Eigenschaften

$$0 \leqq \delta(l) \leqq \delta(l-1) \quad (l = 1, 2, \ldots), \qquad \delta(l) \to 0 \quad \text{für} \quad l \to \infty,$$

so daß auch

$$(2.4) \qquad \sum_{l=l_0}^{\infty} l^{-1} w_1(l^{-1}) \delta^2(l) = \infty$$

gilt. O. B. d. A. sei

$$\frac{2}{3}\delta(l-1) \leqq \delta(l) \leqq \delta(l-1) \leqq \frac{1}{3} \qquad (l = 2, 3, \ldots).$$

Wir bestimmen nun eine Folge $\{q_l\}$ ganzer Zahlen, so daß

$$(2.5) \qquad \delta(l) \leqq \frac{1}{\sqrt{q_l + 1}} \leqq \frac{1}{\log (q_l + 1)}$$

ist, wobei die q_l maximal[2] gewählt werden. Offenbar gilt dann

$$8 \leqq q_1 \leqq q_2 \leqq \ldots \to \infty .$$

Setzen wir nun

$$(2.6) \qquad c_l = 1 + \frac{1}{q_l}, \qquad s_l = (q_l + 1)^2, \qquad \vartheta_l = q_l + 1$$

[2] Maximal bedeutet, daß (2.5) und

$$\delta(l) > \frac{1}{\sqrt{q_l + 2}}$$

gilt.

und wählen (vgl. [8], Theorem 8) einen konkaven Stetigkeitsmodul w_1^* mit

(2.7) $$w_1(u) \leqq w_1^*(u) \leqq 2w_1(u), \qquad u \in [0, u_0].$$

Wir definieren

(2.8) $$\alpha_l := w_1^*(l^{-1}) - \frac{1}{2} w_1^*((l-1)^{-1}).$$

Dann erfüllt

(2.9) $$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu = \sum_{l=l_2}^{\infty} \alpha_l \frac{\delta(l)}{l} s_l^{-l/2} z^{\vartheta_l s_l^l}$$

die Behauptung des Satzes 1.

Wir zeigen zunächst, daß

(i) $$\alpha_l \geqq 0 \quad \text{für} \quad l \geqq l_0$$

(ii) $$\sum_{l=l_2}^{\infty} \alpha_l \frac{\delta(l)}{l} s_l^{-l/2} < \infty$$

(iii) $$\sum_{l=l_2}^{\infty} \alpha_l \frac{\delta(l)}{l} = \infty$$

und

(iv) $$\omega(f; h) = O(w(h))$$

ist. Wegen der Konkavität von w_1^* haben wir für $l \geqq l_2$

(2.10) $$w_1^*(l^{-1}) \geqq \frac{l-1}{l} w_1^*((l-1)^{-1}) > \frac{2}{3} w_1^*((l-1)^{-1}).$$

Damit ist (i) gezeigt. (ii) ist offensichtlich erfüllt. (iii) gilt wegen (2.4), (2.7), (2.10) und der Wahl von α_l und $\delta(l)$. Um den Stetigkeitsmodul von f aus (2.9) zu berechnen, beachten wir, daß für $|t_1 - t_2| \leqq h$ stets

$$|e^{it_1\vartheta_l s_l^l} - e^{it_2\vartheta_l s_l^l}| \leqq \min(2; h\vartheta_l s_l^l)$$

ist. Damit erhalten wir ($L \geqq l_2$)

$$|f(e^{it_1}) - f(e^{it_2})| \geqq h \sum_{l_2 \leqq l \leqq L} \alpha_l \frac{\delta(l)}{l} \vartheta_l s_l^{l/2} + 2 \sum_{l > L} \alpha_l \frac{\delta(l)}{l} s_l^{-l/2}.$$

Nun ist wiederum wegen der Konkavität von w_1^*

$$w_1^*\left(\frac{1}{l}\right) \geqq \frac{l-1}{2l} w_1^*\left(\frac{1}{l-1}\right) + \frac{l+1}{2l} w_1^*\left(\frac{1}{l+1}\right)$$

bzw.

$$w_1^*\left(\frac{1}{l+1}\right) \leqq \left(2 - \frac{2}{l+1}\right) w_1^*\left(\frac{1}{l}\right) - \left(1 - \frac{2}{l+1}\right) w_1^*\left(\frac{1}{l-1}\right).$$

Damit erhalten wir für $l \geqq l_2$

$$\alpha_l - \alpha_{l+1} = \frac{3}{2} w_1^* \left(\frac{1}{l}\right) - \frac{1}{2} w_1^* \left(\frac{1}{l-1}\right) - w_1^* \left(\frac{1}{l+1}\right) \geqq$$

$$\geqq \left(w_1^* \left(\frac{1}{l-1}\right) - w_1^* \left(\frac{1}{l}\right)\right)\left(\frac{1}{2} - \frac{2}{l+1}\right) \geqq 0,$$

d. h. die α_l fallen monoton. Andererseits ist

$$2^l \alpha_l = 2^l w_1^* \left(\frac{1}{l}\right) - 2^{l-1} w_1^* \left(\frac{1}{l-1}\right)$$

und

$$(3/2)^l \delta(l) \leqq (3/2)^{l+1} \delta(l+1).$$

Wir erhalten somit wegen $s_l \geqq 80$

$$|f(e^{it_1}) - f(e^{it_2})| \leqq h \sum_{l_2 \leqq l \leqq L} \frac{\vartheta_l (s_l/9)^{l/2} (3/2)^l \delta(l) 2^l \alpha_l}{l} + 2 \frac{\delta(L)}{L} \alpha_L \sum_{l \leqq L} \frac{1}{s_l^{l/2}} \ll$$

$$\ll h \frac{s_L^{L/2+\frac{1}{2}}}{L} w_1^* \left(\frac{1}{L}\right) \delta(L) + \frac{s_L^{-(L/2+\frac{1}{2})}}{L} \cdot w_1^* \left(\frac{1}{L}\right) \delta(L).$$

Wählt man L so, daß

$$s_L^{L/2+\frac{1}{2}} \leqq h^{-\frac{1}{2}} < s_{L+1}^{L/2+1}$$

ist, so wird wegen der Wahl von $\{\delta(l)\}$ und $\{s_l\}$

$$\omega(f; h) \ll \frac{h^{\frac{1}{2}}}{\log \frac{2\pi}{h}} w_1^* \left(\left(\log \frac{2\pi}{h}\right)^{-1}\right) \ll w(h),$$

und (iv) ist bewiesen.

Sei jetzt

$$\Phi(z) = e^{i\alpha} \frac{z - \zeta_0}{1 - \bar{\zeta}_0 z}$$

mit $0 < r = |\zeta_0| < 1$. Dann ist für $l \geqq l_3(r)$

$$q_l > \frac{1+r}{1-r} > \frac{1-r}{1+r} > \frac{1}{q_l}$$

und

$$c_0' < \frac{1}{c} < 1 < c_l < c_0.$$

Wir wollen zeigen, daß

$$\sum_{n=0}^{\infty} |b_n| = \infty$$

ist, und betrachten dazu die n aus den disjunkten[3] Teilintervallen

$$I_l := \left(s_l^l q_l, \frac{s_l^{l+1}}{q_l} \right].$$

Die Intervalle sind nicht leer, da

$$q_l^2 < (q_l + 1)^2$$

ist. Für jedes $l \geqq l^3$ setzen wir (vgl. [6], S. 175)

$$a_{\nu l}^* = \begin{cases} a_\nu, & \text{falls } s_l^l < \nu \leqq s_l^{l+1} \\ 0 & \text{sonst,} \end{cases}$$

$$f_l^*(w) = \sum_{\nu=0}^{\infty} a_{\nu l}^* w^\nu = \sum_{s_l^l < \nu < s_l^{l+1}} a_\nu w^\nu \tag{2.11}$$

$$b_{nl}^* = \sum_{\nu=0}^{\infty} \gamma_{n\nu} a_{\nu l}^*.$$

Dann gilt (vgl. [3], II, S. 25) wegen $| a_\nu | \leqq c_1$

$$| b_n - b_{nl}^* | = \left| \sum_{\nu=0}^{\infty} \gamma_{n\nu}(a_\nu - a_{\nu l}^*) \right| \leqq$$

$$\leqq c_1 \left\{ \sum_{\nu \leqq s_l^l} | \gamma_{n\nu} | + \sum_{\nu > s_l^{l+1}} | \gamma_{n\nu} | \right\} \leqq c_2 e^{-c_3 n} \leqq c_2 e^{-c_3 s_l^l q_l}.$$

Dies bedeutet, daß es für den Nachweis der Divergenz von $\sum_{n=0}^{\infty} | b_n |$ ausreicht,

$$\sum_{l \geqq l_3} \sum_{n \in I_l} | b_{nl}^* | = \infty$$

zu zeigen.

Sei $l \geqq l_3 \geqq l_2$ und $m := \vartheta_l s_l^l$. Aus (2.11), (2.9) und mit dem Lemma folgt sofort

$$\sum_{n \in I_l} | b_{nl}^* | = \alpha_l \frac{\delta(l)}{l} s_l^{-l/2} \sum_{s_l^l q_l < n < \frac{s_l^{l+1}}{q_l}} | \gamma_{nm} | = \alpha_l \frac{\delta(l)}{l} s_l^{-l/2} \sum_{\frac{1}{c_l} m \quad n < mc_l} | \gamma_{nm} | >$$

$$> K_1 \alpha_l \frac{\delta(l)}{l} s_l^{-l/2} (\vartheta_l s_l^l)^{1/2} \left(c_l - \frac{1}{c_l} \right) - K_2 \alpha_l \frac{\delta(l)}{l} s_l^{-l/2}.$$

Daraus erhalten wir

$$\sum_{n=0}^{\infty} | b_n | \gg \sum_{l \geqq l_3}^{\infty} \alpha_l \frac{\delta(l)}{l} \vartheta_l^{1/2} \left(c_l - \frac{1}{c_l} \right).$$

[3] Die Intervalle I_l sind disjunkt, da

$$\frac{s_l^{l+1}}{s_{l+1}^{l+1}} < q_{l+1} q_l$$

ist.

Nun ist

$$\vartheta_l^{1/2}\left(c_l - \frac{1}{c_l}\right) = (q_l + 1)^{1/2}\frac{2 + q_l^{-1}}{q_l + 1} > (q_l + 1)^{-1/2},$$

und wegen der Wahl von $\delta(l)$ ist

$$\sum_{n=0}^{\infty} | b_n | \gg \sum_{l>l_3} \alpha_l \frac{\delta^2(l)}{l} = \infty .$$

Damit ist Satz 1 bewiesen. Korollar 1 folgt sofort, da $w_1(h) = \left(\log \frac{1}{h}\right)^{-1}$ ein Stetigkeitsmodul ist.

3. Beweis von Satz 2. Sei $M_{aw} \neq \emptyset$ wie in Satz 1 und $[M_{aw}]$ der von M_{aw} erzeugte Vektorraum. Wir zeigen zuerst, daß M_{aw} dicht in $[M_{aw}]$ liegt, und danach, daß der Abschluß von $[M_{aw}]$ gleich $A_a(D)$ ist.

Für jedes $f \in M_{aw}$ liegt auch $\lambda f\, (\lambda \neq 0 \in \mathbf{C})$ in M_{aw}. Seien jetzt f_1, f_2 Funktionen aus M_{aw} und $\varepsilon > 0$ beliebig. Dann liegt $f_1 + f_2$ entweder in M_{aw}, oder für mindestens ein $\Phi\, (\zeta_0 \neq 0)$ gilt

$$f_1 \circ \Phi + f_2 \circ \Phi \in A_a(D) .$$

Im zweiten Fall sei $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ und n_0 derart, daß $f_1^*(z) = \sum_{n \leqq n_0} a_n z^n$ und

$$\| f_1 - f_1^* \| < \varepsilon$$

ist. Da f_1^* ein Polynom ist, gilt

$$f_1^* \circ \Phi \in A_a(D) \quad \text{für alle } \Phi,$$

und somit ist

$$f_1^* + f_2 \in M_{aw} .$$

Wir erhalten

$$\| (f_1 + f_2) - (f_1^* + f_2) \| = \| f_1 - f_1^* \| < \varepsilon,$$

und M_{aw} liegt dicht in $[M_{aw}]$.

Für den zweiten Teil des Beweises beachten wir, daß $A_a(D)$ bzw. $[M_{aw}]$ isomorph zum Folgenraum l_1 bzw. zu einem (nichtleeren) linearen Teilraum M von l_1 ist. Dann müssen wir nach dem Satz von Hahn–Banach nur zeigen, daß jedes beschränkte lineare Funktional ϕ auf l_1, das

(3.1) $$\phi(x) = 0 \quad \text{für alle} \quad x \in M$$

erfüllt, identisch 0 ist. Da jedes beschränkte lineare Funktional ϕ auf l_1 die Gestalt

(3.2) $$\phi(x) = \sum_{\nu=0}^{\infty} \alpha_\nu a_\nu , \qquad x = (a_1, \ldots) \in l_1$$

mit $\sup_{\nu} |\alpha_\nu| < \infty$ hat (vgl. [10], S. 194), müssen wir aus (3.1)

$$\alpha_\nu = 0 \qquad (\nu = 0, 1, 2, \ldots) \tag{3.3}$$

folgern. Nun ist aber für jedes $f \in M_{aw}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, auch $\bar{f}$ mit $\bar{f}(z) = f(z) - z^N$ aus M_{aw}. Daraus folgt wegen $f - \bar{f} \in [M_{aw}]$

$$\alpha_N = 0$$

für jedes N, und Satz 2 ist bewiesen.

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GESAMTHOCHSCHULE PADERBORN
FACHBEREICH MATHEMATIK-INFORMATIK
D-479 PADERBORN
BUNDESREPUBLIK DEUTSCHLAND

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ON THE RATIONAL APPROXIMATION OF FUNCTIONS WITH CONVEX r-TH DERIVATIVE

By

P. P. PETRUSHEV (Sofia)

Let us consider the class of all functions, the r-th derivative of which is of bounded variation in the interval [0, 1]. For the functions of this class G. FREUD [1] estimated the best uniform approximation $R_n(f)$ by means of rational functions of degree n. He has shown that

$$\sup_{V_0^1(f^{(r)})\leqq 1} R_n(f) = O\left(\frac{\ln^2 n}{n^{r+1}}\right), \tag{1}$$

where $V_0^1(g)$ denotes the variation of the function g on the interval [0, 1], $r \geqq 1$. This result is a generalization and an improvement of the result of P. TURÁN and P. SZÜSZ [2].

In [3] V. A. POPOV improved the result of G. Freud as follows: for every integer $K \geqq 1$

$$\sup_{V_0^1(f^{(r)})\leqq 1} R_n(f) \leqq C_{k,r}\frac{\ln^{(k)} n}{n^{r+1}}, \tag{2}$$

where $C_{k,r}$ is a constant, depending only on k and r; $\ln^{(k)} n = \overbrace{\ln \ln \ldots \ln}^{k} n$.

For the functions, with convex r-th derivative in [0, 1] ($r \geqq 1$), A. A. ABDUGAPPAROV [4] has shown that

$$R_n(f) \leqq C_r M_r \frac{\ln^3 n}{n^{r+2}}, \tag{3}$$

where C_r is a constant, depending on r; $M_r = \max_{0\leqq x\leqq 1} |f^{(r)}(x)|$.

We shall obtain in this note an improvement of (3): for every integer $k \geqq 1$

$$R_n(f) \leqq C_{K,r} M_r \frac{\ln^{(k)} n}{n^{r+2}} \qquad (r \geqq 1), \tag{4}$$

where $C_{k,r}$ is a constant, depending only on k and r.

We shall use the method of V. A. POPOV [3].

Let R_n be the set of all rational functions of degree n and $R_n(f; \Delta)$ the best uniform approximation of the function f by means of the elements of R_n in the interval $\Delta = [a, b]$:

$$R_n(f; \Delta) = \inf_{q\in R_n} \| f - q \|_{C[a,b]}.$$

Let us denote by $K_r(M; [a, b])$ the class of all functions f, r-times differentiable on $[a, b]$, $f^{(r)}$ is convex or concave and $\max_{a \leqq x \leqq b} |f^{(r)}(x)| \leqq M$.

We denote by $\hat{K}_r(M; [a, b])$ the set of all functions $f \in K_r(M; [a, b])$, such that $f^{(r)}$ is concave and increasing in the interval $[a, b]$, $f^{(r)}(a) = 0$ and $f \in C^{r+1}([a, b])$.

It is obvious that $\hat{K}_r(M; [a, b]) \subset K_r(M; [a, b])$ and

$$\sup_{f \in \hat{K}_r(M; [a,b])} R_n(f; [a, b]) \leqq \sup_{f \in K_r(M; [a,b])} R_n(f; [a, b]).$$

For later applications we need the following two lemmas.

LEMMA 1. *We have*

$$\sup_{f \in K_r(M; [a,b])} R_n(f; [a, b]) = (b-a)^r M \sup_{f \in K_r(1; [0,1])} R_n(f; [0, 1]), \tag{5}$$

$$\sup_{f \in \hat{K}_r(M; [a,b])} R_n(f; [a, b]) = (b-a)^r M \sup_{f \in \hat{K}_r(1; [0,1])} R_n(f; [0, 1]), \tag{6}$$

$$\sup_{V_a^b(f^{(r)}) \leqq M} R_n(f; [a, b]) = (b-a)^r M \sup_{V_0^1(f^{(r)}) \leqq 1} R_n(f; [0, 1]) \tag{7}$$

(*see* [3]).

LEMMA 2. *For every integer $r \geqq 0$ there exists a constant C_r, depending only on r, such that*

$$\sup_{f \in K_r(M; [a,b])} R_n(f; [a, b]) \leqq C_r \sup_{f \in \hat{K}_r(M; [a; b])} R_n(f; [a, b]),$$

(*see* [4] *and* [6]).

THEOREM 1. *Let $r \geqq 1$ be an integer and for every $n \geqq N_0$*

$$\sup_{f \in \hat{K}_r(1; [0,1])} R_n(f; [0, 1]) \leqq A \frac{\ln^\alpha n \ln^{(k)} n}{n^{r+2}},$$

where $0 < \alpha \leqq 3$; $k \geqq 1$, and A is a constant. Then there exist constants D, C and N_1 ($D \geqq 6$), depending only on k and r, such that for $n \geqq \max\{N_0, N_1\}$

$$\sup_{f \in \hat{K}_r(1; [0,1])} R_n(f; [0, 1]) \leqq \left(\frac{3A}{D^{r+3}} + C\right) D^{r+2} \frac{\ln^{\frac{\alpha}{2}} n \ln^{(k)} n}{n^{r+2}}.$$

PROOF. Let $f \in \hat{K}_r(1; [0, 1])$. From the result in [5] it follows that there exist rational functions $\phi_n(x)$ of degree $[B_r \ln^2 n]$, where B_r is a constant, depending only on r, such that

$$\begin{cases} |\phi_n(x)| \leqq n^{-2(r+2)} & \text{for } -1 \leqq x \leqq -\dfrac{1}{4(B_r+6)^4 \ln^3 n} \\ |\phi_n(x)| \leqq 2 & \text{for } |x| \leqq 1 \\ |1-\phi_n(x)| \leqq n^{-2(r+2)} & \text{for } \dfrac{1}{4(B_r+6)^4 \ln^3 n} \leqq x \leqq 1. \end{cases} \tag{8}$$

Denote $D = B_r + 6$ and

$$N_1 = \min\left\{n : n \geqq D^2 \text{ and } \ln^{(k)}\left(\left[\frac{n}{[D]}\right] + 1\right) \geqq 1\right\}.$$

Let $n \geqq \max\{N_0, N_1\}$ and let us denote $m = \left[\frac{n}{[D]}\right] + 1$. Consider the points

$$z_1 = \frac{1}{D^4 \ln^{\frac{\alpha}{2}} m} \quad \text{and} \quad z_2 = \frac{z_1}{2} = \frac{1}{2D^4 \ln^{\frac{\alpha}{2}} m} \qquad (z_2 < z_1).$$

Applying Lemma 1, (6) for the interval $\delta_1 = [0, z_1]$ we obtain the estimate

$$R_m(f; \delta_1) \leqq \left(\frac{1}{D^4 \ln^{\frac{\alpha}{2}} m}\right)^r A \frac{\ln^\alpha m \ln^{(k)} m}{m^{r+2}}.$$

Consequently, there exists a function $q_1(x) \in R_m$ such that

$$\| f(x) - q_1(x) \|_{C[\delta_1]} \leqq \frac{A}{D^{r+3}} \cdot \frac{\ln^{\frac{\alpha}{2}} m \ln^{(k)} m}{m^{r+2}}. \tag{9}$$

From the result of V. A. Popov [2] and Lemma 1, (7) we obtain for the interval $\delta_2 = [z_2, 1]$ the estimate

$$R_m(f; \delta_2) \leqq C_{k,r} 2D^4 \ln^{\frac{\alpha}{2}} m \frac{\ln^{(k)} m}{m^{r+2}}.$$

Hence, there exists a function $q_2(x) \in R_m$ such that

$$\| f(x) - q_2(x) \|_{C[\delta_2]} \leqq 2C_{k,r} D^4 \frac{\ln^{\frac{\alpha}{2}} m \ln^{(k)} m}{m^{r+2}}. \tag{10}$$

Denote (see J. Szabados [7])

$$Q_i(x) = \frac{q_i(x)}{1 + m^{-r-2} q_i^2(x)}, \qquad i = 1, 2.$$

Since $f \in \hat{K}_r(1; [0, 1])$, we can assume that $\| f \|_{C[0,1]} \leqq 1$. Consequently $\| q_i(x) \|_{C[\delta_i]} \leqq 2$, and we have

$$\| Q_i(x) - q_i(x) \|_{C[\delta_i]} = \left\| \frac{1}{m^{r+2}} \cdot \frac{q_i^3(x)}{1 + m^{-r-2} q_i^2(x)} \right\|_{C[\delta_i]} \leqq \frac{8}{m^{r+2}}. \tag{11}$$

On the other hand

$$\| Q_i(x) \|_{C[0,1]} \leqq m^{r+2}. \tag{12}$$

Now consider the rational function

$$R(x)=Q_1(x)+(Q_2(x)-Q_1(x))\phi_m\left(x-\frac{z_1+z_2}{2}\right)$$

of degree $\leqq 3m+[B_r\ln^2 m]\leqq(3+B_r)m\leqq([D]-1)m$.

First of all we remark that

$$z_1-z_2=\frac{1}{2D^4\ln^{\frac{\alpha}{2}}m}\geqq\frac{1}{2(B_r+6)^4\ln^3 m}\quad\text{and}\quad \ln^{\frac{\alpha}{2}}m\ln^{(k)}m\geqq 1\,.$$

For $x\in[0,z_2]$ we obtain (using (8), (9), (11) and (12))

$$(13)\qquad |f(x)-R(x)|\leqq|f(x)-q_1(x)|+|q_1(x)-Q_1(x)|+$$

$$+(|Q_1(x)|+|Q_2(x)|)\left|\phi_m\left(x-\frac{z_1+z_2}{2}\right)\right|\leqq\left(\frac{A}{D^{r+3}}+10\right)\frac{\ln^{\frac{\alpha}{2}}m\ln^{(k)}m}{m^{r+2}}.$$

For $x\in[z_2,z_1]$ we obtain (using (8)–(12))

$$(14)\quad |f(x)-R(x)|\leqq|f(x)-q_1(x)|+|q_1(x)-Q_1(x)|+(|Q_2(x)-q_2(x)|+$$

$$+|q_2(x)-f(x)|+|f(x)-q_1(x)|+|q_1(x)-Q_1(x)|)\left|\phi_m\left(x-\frac{z_1+z_2}{2}\right)\right|\leqq$$

$$\leqq\left(\frac{3A}{D^{r+2}}+4C_{k,r}D^4+40\right)\frac{\ln^{\frac{\alpha}{2}}m\ln^{(k)}m}{m^{r+2}}.$$

Denote $C=4C_{k,2}D^4+40$. For $x\in[z_1,1]$ we obtain from (8), (10), (11) and (12)

$$(15)\qquad |f(x)-R(x)|\leqq|Q_1(x)|\left|1-\phi_m\left(x-\frac{z_1+z_2}{2}\right)\right|+$$

$$+(|Q_2(x)-q_2(x)|+|q_2(x)-f(x)|)\left|\phi_m\left(x-\frac{z_1+z_2}{2}\right)\right|+$$

$$+|f(x)|\left|1-\phi_m\left(x-\frac{z_1+z_2}{2}\right)\right|\leqq(4C_{k,r}D^4+18)\frac{\ln^{\frac{\alpha}{2}}m\ln^{(k)}m}{m^{r+2}}.$$

By (13)–(15) it follows that

$$\|f(x)-R(x)\|_{C[0,1]}\leqq\left(\frac{3A}{D^{r+3}}+C\right)\frac{\ln^{\frac{\alpha}{2}}m\ln^{(k)}m}{m^{r+2}},$$

where $R(x)\in R([D]-1)m$.

Since $m = \left[\frac{n}{[D]}\right] + 1$ and $n \geqq N_1$, then $([D]-1)m \leqq n$, $m \leqq n$ and $\frac{1}{m} \leqq \frac{D}{n}$.

Consequently $R(x) \in R_n$ and

$$\|f(x) - R(x)\|_{C[0,1]} \leqq \left(\frac{3A}{D^{r+3}} + C\right) D^{r+2} \frac{\ln^{\frac{\alpha}{2}} n \ln^{(k)} n}{n^{r+2}}.$$

The proof is complete.

THEOREM 2. *For every integers $k \geqq 1$, $r \geqq 1$, there exist constants $\hat{C}_{k,r}$ and $\hat{N}_{k,r}$, depending only on k and r such that for $n \geqq \hat{N}_{k,r}$*

$$\hat{\Phi}_n^r = \sup_{f \in \hat{K}_r(1;[0,1])} R_n(f;[0,1]) \leqq \hat{C}_{k,r} \frac{\ln^{(k)} n}{n^{r+2}}.$$

PROOF. Let $\hat{N}_{k,r} = N_1$ (N_1 is a constant from Theorem 1, N_1 depends only on k and r).

Let $n \geqq \hat{N}_{k,r}$. By (3) we obtain

$$\hat{\Phi}_n^r \leqq A_0 \frac{\ln^3 n}{n^{r+2}} \leqq A_0 \frac{\ln^3 n \ln^{(k)} n}{n^{r+2}},$$

where A_0 is a constant, depending on r.

From Theorem 1 and the fact that $\hat{N}_{k,r} = N_0 = N_1$ we obtain

$$\hat{\Phi}_n^r \leqq \left(\frac{3A_0}{D^{r+3}} + C\right) D^{r+2} \frac{\ln^{\frac{3}{2}} n \ln^{(k)} n}{n^{r+2}} = A_1 \frac{\ln^{\frac{3}{2}} n \ln^{(k)} n}{n^{r+2}}.$$

A second application of Theorem 1 gives

$$\hat{\Phi}_n^r \leqq \left(\frac{3A_1}{D^{r+2}} + C\right) D^{r+2} \frac{\ln^{\frac{3}{2^2}} n \ln^{(k)} n}{n^{r+2}} = A_2 \frac{\ln^{\frac{3}{2}} n \ln^{(k)} n}{n^{r+2}}.$$

With i iterations of the above process we get

$$\hat{\Phi}_n^r \leqq \left(\frac{3A_{i-1}}{D^{r+3}} + C\right) D^{r+2} \frac{\ln^{\frac{3}{2^i}} n \ln^{(k)} n}{n^{r+2}} = A_i \frac{\ln^{\frac{3}{2^i}} n \ln^{(k)} n}{n^{r+2}}.$$

Since $D \geqq 6$,

$$A_1 \leqq (A_0 + 2C) D^{r+2},$$

$$A_2 = \left(\frac{3A_1}{D^{r+3}} + C\right) D^{r+2} \leqq \left(\frac{3(A_0+2C)D^{r+2}}{D^{r+3}} + C\right) D^{r+2} \leqq (A_0 + 2C) D^{r+2},$$

$$\vdots$$

$$A_i \leqq (A_0 + 2C) D^{r+2}.$$

Take i such that $\ln^{\frac{3}{2^i}} n \leqq 2$ (i depends on n). Then we obtain the estimate

$$\hat{\Phi}_n^r \leqq 2(A_0 + 2C)D^{r+2}\frac{\ln^{(k)}n}{n^{r+2}}.$$

Theorem 2 is proved.

From Theorem 2 and Lemma 2 immediately follows

THEOREM 3. *For every integers $k \geqq 1$, $r \geqq 1$, there exists a constant $C_{k,r}$, depending only on k and r such that*

$$\sup_{f \in K_r(1;\,[0,1])} R_n(f;\,[0,1]) \leqq C_{k,r}\frac{\ln^{(k)}n}{n^{r+2}}.$$

COROLLARY. *For every integers $k \geqq 1$, $r \geqq 1$ there exists a constant $C_{k,r}$, depending only on k and r such that for every function f, defined in the interval $[a, b]$ and such that $f^{(r)}$ is convex we have*

$$R_n(f;\,[a,b]) \leqq C_{k,r}M_r(b-a)^r\frac{\ln^{(k)}n}{n^{r+2}},$$

where $M_r = \max_{a \leqq x \leqq b} |f^{(r)}(x)|$.

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MATHEMATICAL INSTITUTE
OF THE BULGARIAN ACADEMY OF SCIENCES
SOFIA, BULGARIA

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ÜBER ALLGEMEINE ÜBERTRAGUNGSTHEORIEN IN METRISCHEN LINIENELEMENTRÄUMEN

Von

A. MOÓR (Sopron)

§ 1. Einleitung

Heute kann die Cartansche Theorie der Finslerräume (vgl. [1])[1] schon als eine klassische Übertragungstheorie der metrischen Linienelementräume betrachtet werden, in der alle Größen von einer — die Metrik bestimmenden — Grundfunktion $F(x, \dot{x})$ abgeleitet sind. Die Übertragung der Vektoren $\xi^i(x, \dot{x})$ ist in dieser Übertragungstheorie durch ein invariantes Differential von der Form:

$$D\xi^i = d\xi^i + C^i_{jk}(x, \dot{x})\xi^j d\dot{x}^k + \Gamma^i_{jk}(x, \dot{x})\xi^j dx^k \tag{1.1}$$

definiert, wo die Übertragungsparameter C^i_{jk} und Γ^i_{jk} von dem metrischen Grundtensor

$$g_{ij} \overset{\text{def}}{=} \frac{1}{2}\, \partial^2_{\dot{x}^i \dot{x}^j} F^2(x, \dot{x}) \tag{1.2}$$

abgeleitet werden. Da die Grundfunktion in den $\dot{x}^i$ immer positiv homogen von erster Ordnung vorausgesetzt ist, gilt nach der Formel (1.2):

$$F(x, \dot{x}) \equiv \sqrt{g_{ij}(x, \dot{x})\dot{x}^i \dot{x}^j}, \tag{1.2*}$$

und außerdem genügt sie gewissen wohlbekannte Regularitätsbedingungen (vgl. [5], Kap. I. § 1).

In unseren Arbeiten [2]—[4] haben wir verschiedene Verallgemeinerungen der Cartanschen Theorie gegeben. In [2] und [3] wurde eine solche Geometrie entwickelt, in der der metrische Grundtensor nicht die Form (1.2) hatte, ferner: die Übertragungsparameter L^{*i}_{jk} waren auch nicht symmetrisch; die Grundfunktion ist aber selbstverständlich durch (1.2*) auch jetzt definiert. Die kovarianten Ableitungen von $g_{ij}(x, \dot{x})$ waren hingegen in beiden Arbeiten Null, d. h. die Übertragung war metrisch. In [4] wurde eine solche Geometrie begründet, in der die Übertragung zwar nichtmetrisch ist, die Länge der Vektoren veränderte sich aber bei einer Parallelübertragung zu sich selbst proportional, ferner, die Übertragungsparameter waren symmetrisch. Dies war also die Verallgemeinerung der wohlbekannten Weylschen Räume zu Linienelementmannigfaltigkeiten.

Im folgenden wollen wir eine solche Übertragungstheorie der Linienelementräume entwickeln, in der die Metrik durch einen metrischen Grundtensor $g_{ij}(x, \dot{x})$ definiert ist, der aber nicht notwendigerweise die Form (1.2) haben muß, das inva-

[1] Die Zahlen in eckigen Klammern sind Hinweise auf das Literaturverzeichnis am Ende unserer Arbeit.

riante Differential die Form (1.1) hat, die Übertragungsparameter aber durch (2.16) und (2.20) definiert (vgl. die nachfolgenden Paragraphen) sind. Diese Übertragungsparameter sind aus der Forderung abgeleitet, daß die kovarianten Ableitungen von g_{ij} den vorgegebenen Tensoren f_{ijk} und f^*_{ijk} gleich seien. Somit entsteht eine neue Übertragung, die die vorigen in sich enthält, außerdem aber auch neue Type bestimmt. *Außer g_{ij} sind jetzt auch f_{ijk} und f^*_{ijk} Grundgrößen des Raumes; im nichtsymmetrischen Fall rechnet man auch noch σ_{ijk} und μ_{ijk} hinzu.*

§ 2. Grundformeln und invariantes Differential

Zu Grunde gelegt seien eine Mannigfaltigkeit $\mathfrak{L}_n$ der Linienelemente: $(x^i, \dot{x}^i)$ $(i = 1, 2, \ldots, n)$, die wir kurz mit $(x, \dot{x})$ bezeichnen werden, und ein in (i, j) symmetrischer Tensor $g_{ij}(x, \dot{x})$, womit die Länge ξ eines Vektorfeldes $\xi^i(x, \dot{x})$ durch die Formel

$$\xi \overset{\text{def}}{=} \sqrt{g_{ij}(x, \dot{x})\xi^i(x, \dot{x})\xi^j(x, \dot{x})}$$

festgelegt ist. Die quadratische Form $g_{ij}\xi^i\xi^j$ sei immer positiv definit. Auf Grund dieser Bedingung können die Indizes der Tensoren in der gewöhnlichen Weise herauf- bzw. heruntergezogen werden.

Bemerkung. Die positiv-definite Eigenschaft ist bei gewissen Untersuchungen in den physikalischen Feldtheorien nicht erfüllt; beschränkt man sich aber auf solche Teilgebiete von $\mathfrak{L}_n$, in denen $g_{ij}\xi^i\xi^j > 0$ gilt, so werden unsere Resultate in diesen Teilgebieten bestehen.

Das invariante Differential von ξ^i habe die Form:

$$D\xi^i = d\xi^i + M_j{}^i{}_k\xi^j d\dot{x}^k + L_j{}^i{}_k\xi^j dx^k. \tag{2.1}$$

Mit Hilfe des Einheitsvektors $l^i = \dot{x}^i F^{-1}$ kann man (2.1) in die Form

$$D\xi^i = d\xi^i + M_j^{*i}{}_k\xi^j Dl^k + L_j^{*i}{}_k\xi^j dx^k \tag{2.2}$$

transformieren, wenn noch die – wegen der Übereinstimmung der durch $\dot{x}^i$ und $\rho\dot{x}^i$ $(\rho > 0)$ definierten Richtungen – notwendige Relation

$$M_j{}^i{}_0 \overset{\text{def}}{=} M_j{}^i{}_k l^k \equiv 0 \tag{2.3}$$

bedingt wird. Diese Transformation findet sich mit vollständiger Ausführlichkeit in unserem Aufsatz [3], § 2, auf den wir uns öfters berufen werden. Auch wollen wir die dort gebrauchten Bezeichnungen benützen. Für ein kovariantes Vektorfeld $\eta_i(x, \dot{x})$ gilt statt (2.2):

$$D\eta_i = d\eta_i - M_i^{*j}{}_k\eta_j Dl^k - L_i^{*j}{}_k\eta_j dx^k. \tag{2.4}$$

Die zum invarianten Differential (2.2) bzw. (2.4) gehörigen kovarianten Ableitungen erhält man durch die Darstellung von $d\xi^i$ bzw. $d\eta^i$ mittels dx^k und Dl^k (vgl.

[3], § 3). Es werden

$$(2.5a)\qquad D\xi^i = \overset{*}{\nabla}_k\xi^i Dl^k + \nabla_k\xi^i dx^k,$$

$$(2.5b)\qquad D\eta_i = \overset{*}{\nabla}_k\eta_i Dl^k + \nabla_k\eta_i dx^k,$$

wo die Formeln

$$(2.6a)\qquad \overset{*}{\nabla}_k\xi^i \overset{\text{def}}{=} (\xi^i\,||_t + \overline{M}_j{}^i{}_t\xi^j)J_k^{*t}, \qquad ||_t \equiv F\partial_{\dot{x}^t},$$

$$(2.6b)\qquad \overset{*}{\nabla}_k\eta_i \overset{\text{def}}{=} (\eta_i\,||_t - \overline{M}_i{}^j{}_t\eta_j)J_k^{*t}, \qquad \overline{M}_i{}^j{}_t \overset{\text{def}}{=} FM_i{}^j{}_t,$$

$$(2.7a)\qquad \nabla_k\xi^i \overset{\text{def}}{=} \partial_{x^k}\xi^i - \xi^i\,||_t L_0^{*t}{}_k + L_j^{*i}{}_k\xi^j,$$

$$(2.7b)\qquad \nabla_k\eta_i \overset{\text{def}}{=} \partial_{x^k}\eta_i - \eta_i\,||_t L_0^{*t}{}_k - L_i^{*j}{}_k\eta_j$$

die kovarianten Ableitungen definieren und J_k^{*t} durch

$$(2.7c)\qquad (\delta_k^i + \overline{M}_0{}^i{}_k)J_j^{*k} = \delta_j^i$$

festgelegt ist.

Die durch (2.1) bzw. (2.2) bestimmte Übertragung ist in der durch (2.2)–(2.7) angegebenen Form im wesentlichen eine affine Übertragung, da die Übertragungsparameter nicht von einem Fundamentaltensor abgeleitet sind. Diese affine Übertragung wollen wir jetzt mit dem metrischen Fundamentaltensor in Zusammenhang bringen. Es seien außer g_{ij} noch zwei in (i, j) symmetrische rein kovariante Tensoren dritter Stufe f_{ijk} und f_{ijk}^* angegeben. Wir stellen die Forderungen:

$$(2.8a)\qquad \nabla_k g_{ij} \equiv f_{ijk},$$

$$(2.8b)\qquad \overset{*}{\nabla}_k g_{ij} \equiv f_{ijk}^*,$$

wo die kovarianten Ableitungen durch (2.6b) bzw. (2.7b) festgelgt sind. Die Formeln (2.8a) bzw. (2.8b) definieren je ein Gleichungssystem für die Übertragungsparameter $L_j^{*i}{}_k$ bzw. $\overline{M}_j{}^i{}_k$. In ausführlicher Form lauten die Formeln (2.8):

$$(2.9a)\qquad \partial_{x^k}g_{ij} - 2A_{ijt}L_0^{*t}{}_k - L_{jik}^* - L_{ijk}^* \equiv f_{ijk}, \qquad L_{ijk}^* \overset{\text{def}}{=} g_{jt}L_i^{*t}{}_k,$$

$$(2.9b)\qquad (2A_{ijt} - \overline{M}_{ijt} - \overline{M}_{jit})J_k^{*t} \equiv f_{ijk}^*, \qquad A_{ijt} \overset{\text{def}}{=} \frac{1}{2}g_{ij}\,||_t.$$

Die Übertragungsparameter sind durch diese Gleichungen noch nicht bestimmt, da $L_j^{*i}{}_k$ bzw. $\overline{M}_{ijk}$ je n^3 Komponenten hat, wogegen (2.9a) bzw. (2.9b) wegen der Symmetrie in (i, j) nur aus $\frac{1}{2}n^2(n + 1)$ Komponenten besteht.

Wir bestimmen zuerst die Übertragungsparameter $L_i^{*t}{}_k$. Wir setzen

$$(2.10)\qquad L_j^{*i}{}_k = \Gamma_j^{*i}{}_k + \Lambda_j{}^i{}_k,$$

wo $L_j^{*i}{}_k$ die aus g_{ij} gebildeten in (j, k) symmetrischen Übertragungsparameter bedeuten, d. h. es ist

$$(2.11)\qquad g_{tj}\Gamma_j^{*t}{}_k \equiv \Gamma_{jik}^* = \left(\frac{1}{2}\,\partial_{x^k} g_{ji} - A_{jit}\Gamma_0^{*t}{}_k\right) + \{jik\}$$

wo $\{jik\}$ die zyklische Permutation des vorigen Ausdrucks bedeutet, wo aber im letzten Glied noch eine Vorzeichenänderung durchgeführt werden soll (vgl. [2], (2.18)–(2.21), oder [3], S. 150).

Durch Einsetzen von (2.10) in (2.9a) erhält man[2]

$$(2.12)\qquad \Lambda_{(ijk)k} = -A_{ijr}\Lambda_0{}^r{}_k - \frac{1}{2}f_{ijk}.$$

Bestimmen wir nun Λ_{ijk} in der Form

$$(2.13)\qquad \Lambda_{ijk} \equiv \Lambda_{(ij)k} + \sigma_{ijk}, \qquad \sigma_{ijk} \overset{\text{def}}{=} \Lambda_{[ij]k},$$

so wird nach (2.12) und (2.13)

$$(2.14)\qquad \Lambda_{ijk} = -A_{ijr}\Lambda_0{}^r{}_k - \frac{1}{2}f_{ijk} + \sigma_{ijk}.$$

Nehmen wir an, daß Det $(\delta_r^j + A_0{}^j{}_r) \neq 0$. Es ist somit der Tensor J_j^m durch

$$(2.15)\qquad (\delta_r^j + A_0{}^j{}_r)J_r^m = \delta_r^m$$

eindeutig festgelgt. Überschieben wir nun (2.14) mit l^i, so wird

$$\Lambda_0{}^r{}_k(\delta_r^j + A_0{}^j{}_r) = -\frac{1}{2}f_0{}^j{}_k + \sigma_0{}^j{}_k,$$

woraus nach einer neuen Überschiebung mit J_j^m, auf Grund von (2.15)

$$\Lambda_0{}^m{}_k = -J_t^m\left(\frac{1}{2}f_0{}^t{}_k - \sigma_0{}^t{}_k\right)$$

folgt. Substituieren wir das in (2.14), und dann Λ_{ijk} in (2.10), so wird

$$(2.16)\quad L_j^{*i}{}_k = \Gamma_j^{*i}{}_k + A_j{}^i{}_r J_t^r\left(\frac{1}{2}f_0{}^t{}_k - \sigma_0{}^t{}_k\right) - \frac{1}{2}f_j{}^i{}_k + \sigma_j{}^i{}_k, \qquad \sigma_{jik} \equiv -\sigma_{ijk}.$$

Die Formel (2.16) ist die allgemeinste von $L_j^{*i}{}_k$, die der Gleichung (2.8a) bzw. (2.9a) genügt, und in der $\sigma_{ijk} = -\sigma_{jik}$ noch frei wählbar ist. Dieser Tensor kann durch weitere Bedingungen, z. B. durch die Symmetrieforderung von L_{jik}^* in (j, k), festgelegt werden.

Jetzt gehen wir zur Bestimmung der allgemeinsten Form von $\overline{M}_{ijk}$ in (2.9b) über. Auf Grund von

$$(\delta_m^k + \overline{M}_0{}^k{}_m)J_k^{*t} = \delta_m^t$$

[2] Wir benützen im folgenden die Schoutensche Symbolik der Tensorrechnung.

(vgl. [3], (2.5)) geht (2.9b) in

$$\overline{M}_{(ij)m} = A_{ijm} - \frac{1}{2} f^*_{ijt}(\delta^t_m + \overline{M}_0{}^t{}_m)$$

über. Diese Formel bestimmt also den in (i,j) symmetrischen Teil von $\overline{M}_{ijm}$. Da

$$\overline{M}_{ijm} \equiv \overline{M}_{(ij)m} + \mu_{ijm}, \qquad \mu_{ijm} \overset{\text{def}}{=} \overline{M}_{[ij]m}$$

besteht, hat man

$$\overline{M}_{ijm} = A_{ijm} - \frac{1}{2} f^*_{ijt}(\delta^t_m + \overline{M}_0{}^t{}_m) + \mu_{ijm}, \tag{2.17}$$

wo wegen (2.3) auch $\mu_{ij0} \equiv 0$ ist.

Nehmen wir jetzt an, daß Ψ^k_j durch

$$\left(\delta^j_t + \frac{1}{2} f_0^*{}^j{}_t\right) \Psi^k_j = \delta^k_t \tag{2.18}$$

eindeutig festgelegt ist; ziehen wir dann in (2.17) den Index „j" herauf, so wird nach einer Kontraktion mit $l^i\Psi^k_j$:

$$\overline{M}_0{}^k{}_m = \left(A_0{}^t{}_m + \mu_0{}^t{}_m - \frac{1}{2} f_0^*{}^t{}_m\right) \Psi^k_t. \tag{2.19}$$

Substituieren wir das in (2.17), so erhalten wir

$$\overline{M}_{ijm} = A_{ijm} - \frac{1}{2} f^*_{ijt}\left[\delta^t_m + \left(A_0{}^s{}_m + \mu_0{}^s{}_m - \frac{1}{2} f_0^*{}^s{}_m\right) \Psi^t_s\right] + \mu_{ijm}, \quad \mu_{ijm} \equiv -\mu_{jim}, \tag{2.20}$$

was die allgemeinste Form von $\overline{M}_{ijm}$ ist. Der Tensor μ_{ijm} ist in (2.20) ein noch frei wählbarer in (i,j) schiefsymmetrischer Tensor, für den aber $\mu_{ij0} \equiv 0$ bestehen muß.

§ 3. Die symmetrische Übertragung

Die Forderung, daß $\overline{M}_{jkl}$ bzw. $L^*_j{}^i{}_k$ in den Indizes (j,k) symmetrisch sei, führt auf die symmetrischen Übertragungen. Wir beweisen den folgenden

SATZ 1. *Die symmetrischen Übertragungen sind bei einer Finslerschen Metrik durch die Forderungen* (2.8a) *und* (2.8b) *eindeutig festgelegt.*

BEWEIS. Aus (2.8a) bzw. (2.8b) folgt nach dem vorigen, die Formel (2.16) bzw. (2.20). Da f^*_{ijk} in (i,j) nach (2.8b) symmetrisch ist, bekommt man aus (2.20) die symmetrischen Übertragungsparameter $\overline{M}_{ijm}$ dadurch, daß man $\mu_{ijm} = 0$ setzt. Auf Grund von $M^*_{ijm} = \overline{M}_{ijt}J^{*t}_m$ (vgl. [3], (2.7a)) folgt dann, daß auch M^*_{ijm} in (i,j) symmetrisch wird.

Wir bestimmen nun die in (j,k) symmetrischen Übertragungsparameter L^*_{jik} Da der in (j,k) schiefsymmetrische Teil von L^*_{jik} verschwinden muß, ferner, die

Metrik eine Finslersche Metrik ist – d. h. die Relation (1.2) besteht – gilt nach (2.15):

$$(3.1)\qquad J_j^m = \delta_j^m, \qquad A_{0ik} = A_{k0i} = A_{ki0} = 0,$$

und auf Grund von (2.16) wird nach Herunterziehen des Indexes „i":

$$(3.2)\qquad 2A_{ir[j}\left(\frac{1}{2}f_{|0|}{}^r{}_{k]} - \sigma_{|0|}{}^r{}_{k]}\right) - f_{i[jk]} = 2\sigma_{i[jk]},$$

wo wir die Totalsymmetrie von A_{ijk}, die Symmetrie von f_{ijk} in (i, j) und die schiefe Symmetrie von σ_{ijk} in (i, j) beachtet haben. (A_{ijk} ist übrigens nur im Finslerschen Fall totalsymmetrisch). Überschieben wir unsere letzte Gleichung mit l^k und dann die erhaltene Gleichung mit l^i, so bekommen wir wegen $\sigma_{00j} = 0$ die Gleichungen:

$$(3.3)\qquad \sigma_{0ij} = \sigma_{ji0} - \frac{1}{2}(f_{ij0} - f_{0ji}) + A_{irj}\left(\frac{1}{2}f_0{}^r{}_0 - \sigma_0{}^r{}_0\right),$$

$$(3.4)\qquad \sigma_{0j0} = \frac{1}{2}(f_{00j} - f_{j00}), \qquad \sigma_0{}^r{}_0 = \frac{1}{2}(f_{00}{}^r - f^r{}_{00}).$$

Substituieren wir nun $\sigma_0{}^r{}_0$ aus (3.4) in (3.3), so wird unter Beachtung der Symmetrie von f_{ijk} in (i, j):

$$(3.5)\qquad \sigma_{0ij} = \sigma_{ji0} + \phi_{ji},$$

$$(3.5\text{a})\qquad \phi_{ji} \stackrel{\text{def}}{=} A_{irj}\left(f_0{}^r{}_0 - \frac{1}{2}f_{00}{}^r\right) - \frac{1}{2}(f_{ij0} - f_{0ij}).$$

Bilden wir jetzt den in (i, j) symmetrischen Teil von (3.5), so wird:

$$(3.6\text{a})\qquad \sigma_{0(ij)} = \phi_{(ij)}.$$

Um $\sigma_{0[ij]}$ zu bestimmen, überschieben wir (3.2) mit l^i, so erhält man nach entsprechenden Vertauschungen der Indizes:

$$(3.6\text{b})\qquad \sigma_{0[ij]} = \frac{1}{4}(f_{j0i} - f_{i0j}).$$

Addieren wir (3.6a) und (3.6b), so erhält man in Hinsicht auf (3.5a):

$$\sigma_{0ij} = A_{itj}\left(f_0{}^t{}_0 - \frac{1}{2}f_{00}{}^t\right) - \frac{1}{2}(f_{ij0} - f_{j0i}).$$

Substituiert man diesen Tensor in (3.2), so wird:

$$(3.7)\qquad \sigma_{ijk} - \sigma_{ikj} = \Phi_{ijk},$$

$$(3.7\text{a})\qquad \Phi_{ijk} \stackrel{\text{def}}{=} f_{i[kj]} - 2A_{ir[j}A_{k]}{}^r{}_t\left(f_0{}^t{}_0 - \frac{1}{2}f_{00}{}^t\right) + A_{ir[j}(f_{|0|}{}^r{}_{k]} + f_{k]}{}^r{}_0 - f_{k]0}{}^r).$$

Da σ_{ijk} in (i, j) schiefsymmetrisch ist, folgt aus (3.7):

$$\sigma_{ijk} = \frac{1}{2}(\Phi_{ijk} + \Phi_{jki} - \Phi_{kij}), \tag{3.8}$$

wie das aus (3.7) in Hinsicht auf die schiefe Symmetrie von σ_{ijk} in (i, j) unmittelbar berechnet werden kann. Der Tensor σ_{ijk} ist also durch (3.8) und (3.7a) eindeutig bestimmt, und da alle Schritte eindeutig waren, ist nach (3.8) und (2.16) der symmetrische $L^{*i}_{j\ k}$ für eine Finslersche Metrik eindeutig bestimmt, wie das im Satz behauptet wurde.

Die explizite Form des Tensors σ_{ijk} ist also im Fall der symmetrischen Übertragung die folgende:

$$\sigma_{ijk} = f_{k[ij]} + A_{kr[j}(f_{|0|}{}^{r}{}_{i]} + f_{i]}{}^{r}{}_{0} - f_{i]0}{}^{r}) + 2A_{kr[i}A_{j]}{}^{r}{}_{t}\left(f_{0}{}^{t}{}_{0} - \frac{1}{2}f_{00}{}^{t}\right). \tag{3.9}$$

Die in (j, k) symmetrischen Übertragungsparameter sind also – falls (1.2) gilt – durch (2.16) und (3.9) festgelgt (es ist jetzt $J^s_r = \delta^s_r$).

Bestimmt der Tensor $g_{ij}(x, \dot{x})$ nicht eine Finslersche Metrik, so ist die Bestimmung von σ_{ijk} sehr kompliziert, da dann die Relationen (3.1) nicht bestehen werden. Die Zahl der Gleichungen $L^{*\ i}_{[j\ k]} = 0$ stimmt aber wegen der schiefen Symmetrie auch in diesem Falle mit der Komponentenzahl von σ_{ijk} überein.

§ 4. Spezialfälle

In diesem Paragraphen wollen wir die wichtigsten Spezialfälle der im vorigen entwickelten Übertragungstheorie angeben.

a) *Der Fall der Punkträume.* Dieser Fall ist durch $A_{ijk} = 0, f^*_{ijk} = 0$ und $\mu_{ijk} = 0$ charakterisiert. Selbstverständlich sind die übrigen Größen immer nur vom Orte x^i abhängig. Ist $f_{ijk} = 0$, $\sigma_{ijk} = 0$, so geht der Raum in den Riemannschen Raum über; wenn $f_{ijk} = 0$, $\sigma_{ijk} \neq 0$ besteht, so erhält man nach (2.8a) eine metrische, aber nicht symmetrische Übertragung. $\sigma_{ijk} = 0, f_{ijk} = \kappa_k g_{ij}$ charakterisiert die Weylsche Geometrie. Mit Hilfe von (3.9) und (2.16) können verschiedenartige symmetrische Übertragungen gekennzeichnet werden, die in verschiedenen physikalischen einheitlichen Feldtheorien benützt wurden.

b) *Finslerräume mit Cartanscher Übertragungstheorie.* Dieser Typus ist durch (1.2), ferner durch

$$f_{ijk} = 0, \quad f^*_{ijk} = 0, \quad \sigma_{ijk} = 0, \quad \mu_{ijk} = 0$$

charakterisiert (vgl. [1]). Die Formeln (2.16) und (2.20) reduzieren sich auf:

$$L^{*i}_{j\ k} = \Gamma^{*i}_{j\ k}, \quad \overline{M}_{ijm} = A_{ijm}.$$

c) *Allgemeine metrische Übertragungen.* Diese Räume sind durch

$$f_{ijk} = 0, \quad f^*_{ijk} = 0$$

gekennzeichnet; wir verweisen darauf, daß (1.2) nicht bestehen muß (vgl. [2] und [3]).

d) *Weylsche Linienelementräume.* Diese Räume sind durch

$$f_{ijk} = \gamma_k g_{ij}, \qquad f^*_{ijk} = \gamma^*_k g_{ij} \tag{4.1}$$

gekennzeichnet. Sind die Übertragungsparameter noch symmetrisch, so entsteht unsere in [4] entwickelte Übertragungstheorie. Besteht (1.2), so gehen z. B. die durch (2.16) und (3.9) bestimmten Übertragungsparameter eben in die Formel (6.5) von [4] über, wie das durch eine einfachen Rechnung leicht bewiesen werden kann. Wird aber die Symmetrie nicht gefordert, d. h. gelten (4.1), und sind ferner σ_{ijk} und μ_{ijk} als weitere, in (i, j) schiefsymmetrische Grundgrößen zu betrachten, so erhält man einen nichtsymmetrische Weylsche Linienelementraum, der – nach dem Wissen des Verfassers – noch nicht eingehend untersucht wurde.

e) *Weitere spezielle Type.* Man erhält weitere spezielle Type, die aber den klassischen Cartanschen Fall verallgemeinern, falls man

$$f_{ijk} = \gamma_k f_{ij}, \qquad f^*_{ijk} = \gamma^*_k f^*_{ij}, \tag{4.2}$$

oder, z. B.

$$f_{ijk} = \gamma_k f_i f_j, \qquad f^*_{ijk} = \gamma^*_k f_i f_j, \tag{4.3}$$

oder aber

$$f_{ijk} = \gamma_k h_{(i} f_{j)}, \qquad f^*_{ijk} = \gamma^*_k h_{(i} f_{j)} \tag{4.4}$$

setzt.

f) *Verallgemeinerung der Minkowskischen Räume.* Sind die Grundgrößen vom Orte x^i unabhängig, so erhält man die Verallgemeinerungen der Minkowskischen Räume (vgl. [7]). In diesem Falle muß

$$f_{ijk} = 0, \qquad \sigma_{ijk} = 0$$

gesetzt werden und die übrigen Größen sind nur von der Richtung $\dot{x}^i$ abhängig.

§ 5. Torsion und Krümmung des Raumes

Die Torsion und die Krümmung des Raumes kann in der gewöhnlichen Weise nach der Cartanschen Methode bestimmt werden (vgl. [1], oder [5] Kap. IV. § 1). Diese Theorie unterscheidet sich nicht wesentlich von der, die in unserer Arbeit [2] § 7 entwickelt wurde; nur an jenen Stellen, wo die Differentiale von g_{ik} vorkommen, also z. B. bei den Symmetrieeigenschaften der Krümmungstensoren kommen Unterschiede vor.

Die Torsion ist nach (2.2) durch die in (d, δ) alternierende Form:

$$\Omega^i \overset{\text{def}}{=} (\Delta D - D\Delta)x^i = M_j^{*i}{}_k dx^j \wedge Dl^k + L^{*}_{[j}{}^{i}{}_{k]} dx^j \wedge dx^k \tag{5.1}$$

festgelegt,[3] wo „d" und „δ" die zu den invarianten Differentialen „D" und „Δ" gehörigen vertauschbaren Differentiale bedeuten. Nach (5.1) sind also die Torsionstensoren des Raumes: $M_j^{*i}{}_k$ und $L^{*}_{[j}{}^{i}{}_{k]}$.

Berechnen wir der Cartanschen Theorie entsprechend die alternierende Differentialform Ω_j^i durch die Formel

$$(5.2) \qquad \Omega_j^i(d, \delta)\xi^j \equiv (\Delta D - D\Delta)\xi^i,$$

so gibt Ω_j^i eben die Krümmungstensoren des Raumes an. Bei der Berechnung von Ω_j^i müssen wir die folgenden Relationen beachten. Da die kennzeichnenden Funktionen in den metrischen Linienelementräumen in den $\dot{x}^i$ homogen von nullter Dimension sind, gelten für diese die Eulerschen Relationen, ferner sind $M_j^{*i}{}_0 \equiv 0$, $f_j^{*i}{}_0 \equiv 0$. Es ist hiernach für eine Funktion $Q(x, \dot{x})$ (vgl. [3], (3.1)):

$$(5.3) \qquad Q_{\dot{x}^i}(x, \dot{x})d\dot{x}^i \equiv (J_t^{*i}Dl^t - L_0^{*i}{}_t dx^t)Q\,||_i\,.$$

Bei der expliziten Berechnung von (5.2) kommt auch das Glied $(\delta Dl^k - d\Delta l^k)$ vor. Bei der Berechnung dieses Gliedes müssen wir (vgl. [2], (2.26) und (2.27)) die Formeln

$$(5.4) \qquad \partial_{x^r} l^j = -\, l^j(\Gamma^*_{00r} + A_{00t}\Gamma^{*t}_{0}{}_{r}), \qquad l^j||_t = \delta_t^j - l^j(l_t + A_{00t})$$

beachten. Benützen wir in den Rechnungen die Übertragungsparameter in der Form (2.10), so müssen wir noch die aus (2.14) nach Kontraktion mit $l^i l^j$ folgende Relation

$$(5.5) \qquad A_0{}^r{}_k(l_r + A_{00r}) = -\frac{1}{2} f_{00k}$$

in Betracht ziehen, womit in Hinsicht auf (5.4) auch die Formeln

$$(5.6a) \qquad \nabla_k l^i = -\frac{1}{2}\, l^i f_{00k}\,,$$

$$(5.6b) \qquad \overset{*}{\nabla}_k l^i = \delta_k^i - l^i(l_t + A_{00t})J_k^{*t}$$

abgeleitet werden können.

Wir benötigen noch die wichtige Relation

$$(5.7) \qquad (l_t + A_{00t})J_k^{*t}Dl^k = -\frac{1}{2} f_{00k}dx^k,$$

die auf Grund der Formel (2.5a) des invarianten Differentials für $\xi^i = l^i$ und mittels der Formeln (5.6a) und (5.6b) unmittelbar bewiesen werden kann.

Aus der Formel (5.2) erhält man auf Grund der Formeln (5.3)–(5.7) durch explizite Berechnung von $(\Delta D - D\Delta)\xi^i$:

$$(5.8) \qquad \Omega_j^i(d, \delta) \overset{\text{def}}{=} \frac{1}{2} R_j{}^i{}_{kl}dx^k \wedge dx^l + P_j{}^i{}_{kl}dx^k \wedge Dl^l + \frac{1}{2} S_j{}^i{}_{kl}Dl^k \wedge Dl^l,$$

[3] Es ist $dx^j \wedge dx^k \overset{\text{def}}{=} dx^j\delta x^k - \delta x^j dx^k$. Entsprechendes gilt für „d" und „δ" in den übrigen Symbolen von „$\wedge$".

wo die Krümmungstensoren durch die Definitionsformeln

$$R_j{}^i{}_{kl} \overset{\text{def}}{=} \hat{R}_j{}^i{}_{kl} + M_j^{*i}{}_t(\delta_r^t + \overline{M}_0{}^t{}_r)\hat{R}_0{}^r{}_{kl}, \tag{5.9}$$

$$\frac{1}{2}\hat{R}_j{}^i{}_{kl} \overset{\text{def}}{=} \partial_{x[l}L^*_{|j|}{}^i{}_{k]} - L_j^{*i}{}_{[k}\|_{|t|}L^*_{|0|}{}^t{}_{l]} + L_j^{*t}{}_{[k}L^*_{|t|}{}^i{}_{l]}, \tag{5.9a}$$

$$P_j{}^i{}_{kl} \overset{\text{def}}{=} (L_j^{*i}{}_k\|_t - \nabla_k\overline{M}_j{}^i{}_t + \overline{M}_j{}^i{}_r L_p^{*r}{}_k\|_t l^p)J_l^{*t}, \tag{5.10}$$

$$\frac{1}{2}S_j{}^i{}_{kl} \overset{\text{def}}{=} J_{[l}^{*s}M^*_{|j|}{}^i{}_{k]}\|_s + M_j^{*t}{}_{[k}M^*_{|t|}{}^i{}_{l]} + M_j^{*i}{}_t(\overline{M}_0{}^t{}_m\|_s)J_{[k}^{*m}J_{l]}^{*s} \tag{5.11}$$

angegeben sind. Wir verweisen darauf, daß bei der Herleitung von (5.10) auch die Formeln (2.5) und (2.7a) von [3] benützt wurden.

Die Formeln (5.1), (5.2), (5.8)–(5.11) beweisen in trivialer Weise den

SATZ 2. *Die Torsionstensoren $M_j^{*i}{}_k$ und $L^*_{[j}{}^i{}_{k]}$ und die Krümmungstensoren* (5.9)–(5.10) *verallgemeinern die entsprechenden Tensoren der in den Arbeiten* [1]–[6] *entwickelten Theorien.*

BEMERKUNG 1. Der ausführliche Beweis dieses Satzes kann offensichtlich ganz leicht durchgeführt werden, nur bei der Formel (7.6) von [2] verweisen wir auf die Tatsache: gilt statt der Definitionsgleichung (2.7c) von J_k^{*t} die Relation $\overline{M}_0{}^j{}_t\overline{M}_0{}^t{}_k = 0$, wie es in unserer Arbeit [2] der Fall ist, so ist auf Grund von (2.6a) von [2] auch $\overline{M}_0{}^i{}_k = M_0^{*i}{}_k$. Somit geht unsere Formel (5.9) unmittelbar in die Formel (7.6) von [2] über.

BEMERKUNG 2. Die Formeln (5.9)–(5.11) sind von f_{ijk} und f^*_{ijk} formal unabhängig, da die Formel (2.2) des invarianten Differentials im wesentlichen eine affine Übertragung bestimmt. Selbstverständlich kommen aber diese Tensoren in den Formeln der Übertragungsparameter und folglich in den im folgenden zu diskutierenden Symmetrieeigenschaften der Krümmungstensoren vor.

BEMERKUNG 3. Auf Grund von (5.1) existieren in den allgemeinen Übertragungen zwei Torsionstensoren, die voneinander vollständig unabhängig sind.

Die Formel (5.11) des dritten Krümmungstensors kann auch allein durch $\overline{M}_j{}^i{}_k$ und J_k^{*i} ausgedrückt werden. Auf Grund von (2.7c) wird wegen $\delta_m^t\|_s = 0$:

$$\overline{M}_0{}^t{}_m\|_s J_k^{*m} = -J_k^{*m}\|_s(\delta_m^t + \overline{M}_0{}^t{}_m). \tag{5.12}$$

Da wegen (2.7c) auch

$$J_t^{*r}(\delta_m^t + \overline{M}_0{}^t{}_m) = \delta_m^r \tag{5.13}$$

besteht, bekommt man aus (5.12) auf Grund der Relation

$$M_j^{*i}{}_t \overset{\text{def}}{=} \overline{M}_j{}^i{}_r J_t^{*r} \tag{5.14}$$

die Formel

$$M_j^{*i}{}_t(\overline{M}_0{}^t{}_m\|_s)J_k^{*m} = -\overline{M}_j{}^i{}_r(J_k^{*r}\|_s). \tag{5.15}$$

Aus (5.11), (5.14) und (5.15) folgt:

$$\frac{1}{2}S_j{}^i{}_{kl} = J_{[k}^{*m}J_{l]}^{*s}(\overline{M}_j{}^i{}_m\|_s + \overline{M}_j{}^t{}_m\overline{M}_t{}^i{}_s). \tag{5.16}$$

§ 6. Schiefsymmetrische Eigenschaften der Krümmungstensoren

Aus den Definitionsformeln (5.9), (5.9a) und (5.11) folgt unmittelbar die folgende Behauptung:

Die Krümmungstensoren $\hat{R}_j{}^i{}_{kl}$, $R_j{}^i{}_{kl}$ *und* $S_j{}^i{}_{kl}$ *sind in den beiden letzten Indizes schiefsymmetrisch.*

Ziehen wir nun den Index „i" in (5.9), (5.10) und (5.11) herunter, so sind bekanntlich diese Krümmungstensoren *im metrischen Fall* in den ersten beiden Indizes schiefsymmetrisch (vgl. [2] § 7 bzw. [5], Kap. IV. § 2.).

Im allgemeinen Fall gilt der

SATZ 3. *Der* (i, j) *symmetrische Teil der Krümmungstensoren ist der folgende:*

$$R_{(ij)kl} = \nabla_{[k} f_{|ij|l]} - f_{ijt} L^{*}{}_{[k}{}^{t}{}_{l]} - \frac{1}{2} f^{*}_{ijt} R_0{}^t{}_{kl}, \tag{6.1}$$

$$2P_{(ij)kl} = \nabla_k f^{*}_{ijl} - \overset{*}{\nabla}_l f_{ijk} - f_{ijt} M^{*}_{k}{}^{t}{}_{l} - f^{*}_{ijt} P_0{}^t{}_{kl}, \tag{6.2}$$

$$S_{(ij)kl} = \overset{*}{\nabla}_{[k} f^{*}_{|ij|l]} - \frac{1}{2} f^{*}_{ijt} S_0{}^t{}_{kl}. \tag{6.3}$$

BEWEIS. Wir berechnen die Form $(\Delta D - D\Delta) g_{ij}$ mittels zwei verschiedener Methoden. In der ersten benützen wir die Formel

$$Dg_{ij} = \partial_{x^k} g_{ij} - 2A_{ijt} L^{*t}_0{}_k - 2L^{*}_{(ij)k},$$

und somit erhält man nach der Bezeichnung unter (5.8):

$$(\Delta D - D\Delta) g_{ij} = -\Omega_{ij} - \Omega_{ji}. \tag{6.4}$$

In der zweiten benützen wir die Formel

$$Dg_{ij} = \nabla_k g_{ij} dx^k + \overset{*}{\nabla}_k g_{ij} Dl^k,$$

woraus auf Grund der Forderungen (2.8a) und (2.8b) und in Hinsicht auf die Formel

$$(\Delta D - D\Delta) l^i = \Omega_0{}^i(d, \delta)$$

die folgende Formel ableitbar wird:

$$\begin{aligned}(\Delta D - D\Delta) g_{ij} = {} & (\nabla_{[l} f_{|ij|k]} + f_{ijt} L^{*}{}_{[k}{}^{t}{}_{l]} + \\ & + \frac{1}{2} f^{*}_{ijt} R_0{}^t{}_{kl}) dx^k \wedge dx^l + (\overset{*}{\nabla}_l f_{ijk} - \nabla_k f^{*}_{ijl} + f_{ijt} M^{*}_{k}{}^{t}{}_{l} + \\ & + f^{*}_{ijt} P_0{}^t{}_{kl}) dx^k \wedge Dl^l + (\overset{*}{\nabla}_{[l} f^{*}_{|ij|k]} + \frac{1}{2} f^{*}_{ijt} S_0{}^t{}_{kl}) Dl^k \wedge Dl^l.\end{aligned} \tag{6.5}$$

Auf Grund der Formeln (6.4) und (6.5) erhält man im Hinblick auf (5.8) eben die zu beweisenden Formeln (6.1)–(6.3).

Der Satz 3 zeigt, daß in gewissen Fällen die Krümmungstensoren auch in den ersten beiden Indizes schiefsymmetrisch sein können, ohne daß die Relation $Dg_{ij} = 0$ bestehe. Z. B. folgt aus (6.1) das

KOROLLAR A. *Gelten die Relationen:*

$$f_{ijk} = 0, \qquad R_0{}^t{}_{kl} = 0, \tag{6.6a}$$

oder

$$\nabla_{[k} f_{|ij|l]} = 0, \qquad L^{*\,t}_{[k\,l]} = 0, \qquad R_0{}^t{}_{kl} = 0 \tag{6.6b}$$

so ist R_{ijkl} *in* (i, j) *schiefsymmetrisch.*

Da nach (2.16) $R_0{}^t{}_{kl}$ von g_{ij}, σ_{ijk} und von f_{ijk} abhängig ist, bestimmen die Gleichungen (6.6) partielle Differentialgleichungssysteme für g_{ij}, σ_{ijk} und f_{ijk}.

Auf Grund von (6.3) folgt das

KOROLLAR B. *Notwendig und hinreichend für die schiefe Symmetrie von* S_{ijkl} *in* (i, j) *ist:*

$$\overset{*}{\nabla}_{[k} f^*_{|ij|l]} - \frac{1}{2} f^*_{ijt} S_0{}^t{}_{kl} = 0. \tag{6.7}$$

(6.7) ist nach (2.20) ein partielles Differentialgleichungs-system bezüglich g_{ij}, f^*_{ijk} und μ_{ijk}.

Zum Abschluß dieses Paragraphen wollen wir einen interessanten Spezialfall untersuchen. Der metrische Grundtensor sei der eines Finslerraumes, d. h. es soll (1.2) bestehen, woraus auch $A_0{}^i{}_k = 0$ folgt. Die Tensoren σ_{ijk} und μ_{ijk} in den Formeln der Übertragungsparameter (2.16) und (2.20) seien Null, ferner nehmen wir an, daß auch f^*_{0ik} in der Formel (2.20) verschwindet. Die Übertragungsparameter sind somit, nach den Formeln (2.16) und (2.20),

$$L^{*i}_{j\ k} = \Gamma^{*i}_{j\ k} + \frac{1}{2}(A_j{}^i{}_r f_0{}^r{}_k - f_j{}^i{}_k), \qquad \overline{M}_j{}^i{}_m = A_j{}^i{}_m - \frac{1}{2} f_j{}^{*i}{}_m. \tag{6.8}$$

Offensichtlich folgt aus unseren Annahmen, daß die Relationen

$$M^{*i}_{0\ m} = \overline{M}_0{}^t{}_m = 0, \qquad J^{*i}_k = \delta^i_k$$

bestehen. Bezeichnen wir mit $\Lambda_j{}^i{}_k$ den Ausdruck (vgl. (2.10) und (2.16)):

$$\Lambda_j{}^i{}_k = \frac{1}{2}(A_j{}^i{}_r f_0{}^r{}_k - f_j{}^i{}_k), \tag{6.9}$$

so wird nach (5.9a)

$$\hat{R}_j{}^i{}_{kl} = K_j{}^i{}_{kl} + \Phi_j{}^i{}_{kl}, \tag{6.10}$$

wo $K_j{}^i{}_{kl}$ den aus $\Gamma^{*i}_{j\ k}$ gebildete Krümmungstensor bedeutet (vgl. [5] Kap. IV. Formel (1.7))[4] der – in der Cartanschen Theorie – unserem Tensor (5.9a) entspricht; ferner bedeutet das zweite Glied unter (6.10) den Tensor:

$$\Phi_j{}^i{}_{kl} \overset{\text{def}}{=} 2\{\Lambda_j{}^i{}_{[k\|l]} - \Gamma^{*i}_{j\ [k}\|_{|t|}\Lambda_{|0|}{}^t{}_{l]} - \Lambda_j{}^i{}_{[k}\|_{|t|}\Lambda_{|0|}{}^t{}_{l]} + \Lambda_j{}^t{}_{[k}\Lambda_{|t|}{}^i{}_{l]}\}, \tag{6.10a}$$

wobei „$\|l$" die Cartansche kovariante Ableitung definiert.

Aus den Formeln (6.9)–(6.10a) und (5.9) folgt der

[4] In [5] ist $\partial_{\dot{x}^r} G^i \equiv \Gamma^{*i}_{l\ r}\dot{x}^l \equiv F^{-1}\Gamma^{*i}_{0\ r}$ (vgl. [5], III. (1.27')).

SATZ 4. *Ist* $f_j{}^i{}_k = c l_j l^i l_k$ $(c = \text{Konst.})$ *und bestehen die Formeln* (1.2) *und* (6.8) *mit* $f_0{}^{*i}{}_m = 0$, *so stimmen die Krümmungstensoren* $\hat{R}_j{}^i{}_{kl}$ *und* $K_j{}^i{}_{kl}$ *überein, und es gilt auch:*

$$R_0{}^i{}_{kl} = \hat{R}_0{}^i{}_{kl} = K_0{}^i{}_{kl}. \tag{6.11}$$

BEWEIS. Die Cartansche kovariante Ableitung von l^i ist bekanntlich identisch Null, woraus nach (6.10a), im Hinblick auf die Homogenität nullter Dimension in den $\dot{x}^i$ der in (6.10a) vorkommenden Größen, die Relation $\Phi_j{}^i{}_{kl} = 0$ folgt. Das beweist schon die erste Hälfte des Satzes; wegen $M_0^{*i}{}_k = 0$ erhält man aus (5.9) unmittelbar auch (6.11), womit der Satz 4 vollständig bewiesen ist.

Wir bemerken noch, daß der Krümmungstensor (5.9) nicht mit dem entsprechenden Krümmungstensor der Cartanschen Theorie zusammenfällt, da nach (6.8) $\overline{M}_j{}^i{}_m \neq A_j{}^i{}_m$ ist. Es folgt aber aus (6.11) das

KOROLLAR C. *Bestehen die Bedingungen des Satzes* 4, *existiert ferner ein absoluter Parallelismus der Linienelemente bezüglich der Cartanschen Übertragung, so ist*

$$R_j{}^i{}_{kl} = \hat{R}_j{}^i{}_{kl} = K_j{}^i{}_{kl}.$$

BEWEIS. Die notwendigen und hinreichenden Bedingungen der Existenz des absoluten Parallelismus der Linienelemente ist bekanntlich $K_0{}^i{}_{kl} = 0$. Aus (6.11) folgt somit auf Grund von (5.9) das Korollar.

§ 7. Charakteristische Linien

Während im Riemannschen Raum nur die geodätischen Linien — die gleichzeitig auch autoparallele Kurven sind — als charakteristische Kurven des Raumes betrachtet werden können, existieren im Linienelementraum $\mathfrak{L}_n$ mit allgemeiner Übertragung mehrere charakteristische Kurven.

Nehmen wir wieder an, daß der Linienelementraum $\mathfrak{L}_n$ eine Finslersche Metrik hat. Nehmen wir überdies noch an, daß eine Kurve $x^i(s)$ angegeben ist, wo der Parameter „s" die Bogenlänge bezüglich des Richtungsfeldes $\dot{x}^i(s)$ bedeutet, und ferner

$$\dot{x}^i(s) = \frac{dx^i}{ds} = l^i(s)$$

besteht. Die Kurve ist also die Mannigfaltigkeit der tangenten Linienelemente. In diesem Falle ist

$$\frac{d^2x^i}{ds^2} + \Gamma_0^{*i}{}_0 = 0, \qquad l^i = \frac{dx^i}{ds} \tag{7.1}$$

das charakteristische Differentialgleichungssystem der geodätischen Linien. Das Differentialgleichungssystem

$$\frac{d^2x^i}{ds^2} + L_0^{*i}{}_0 = 0, \qquad l^i = \frac{dx^i}{ds} \tag{7.2}$$

charakterisiert die autoparallelen Kurven. Diese beiden Kurven sind identisch, falls $L_0^{*i}{}_0 = \Gamma_0^{*i}{}_0$ besteht. Aus (2.16) folgt somit der

SATZ 5. *Bei einer Finslerschen Metrik stimmen die autoparallelen Kurven und die geodätischen Linien mit den Übertragungsparametern* (2.16) *dann und nur dann überein, wenn* $f_{0i0} = 2\sigma_{0i0}$ *gilt.*

Zum Schluß bemerken wir, daß man weitere charakteristische Kurven des Raumes erhält, wenn man die Bedingung $l^i = dx^i/ds$ fallen läßt. In diesem Fall kann $l^i(s)$ neben $x^i(s)$ von $\dot{x}^i(s)$ unabhängig vorgegeben werden oder durch die Gleichungen $Dl^i = 0$ bestimmt sein. Dadurch entstehen die von A. RAPCSÁK untersuchten quasiautoparallelen Kurven. (Vgl. [6], §§ 1, 2.).

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ERDÉSZETI ÉS FAIPARI EGYETEM
9401 SOPRON, PF. 132
UNGARN

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FINITE ORDER EXTENSIONS OF A PRIMARY GROUP BY A TORSION-FREE GROUP

By

A. MADER (Honolulu)

1. Introduction. In 1958 R. BAER published his fundamental paper [1] on the structure of the groups Ext (K, T) with T a torsion group and K torsion-free. It is easy to see that Ext (K, T) is divisible so that the structure of Ext (K, T) is determined by a set of ranks. In some cases Ext (K, T) is torsion-free but it may happen that Ext (K, T) $[p] \neq 0$ for some prime p. The problem of determining when Ext (K, T) is torsion-free derives added interest from results of C. WALKER [8] and A. MADER [4].

The present paper is concerned with this problem. Section 2 contains some reduction theorems. In Section 3 a necessary condition on K for Ext (K, T) to be torsion-free is established (Theorem 3.4). We arrive at the class of „*barred*" torsion-free groups (Definitions 3.18 and 3.24) which may deserve further study. Barredness is defined with reference to a p-basis (= maximal p-independent set) but turns out to be independent of the choice of a p-basis (Theorem 3.19). In Section 4 we utilize barred groups in order to answer negatively a question posed in [4] (Theorem 4.5). It becomes clear that groups K with Ext (K, T) $[p] \neq 0$ for some T are more common than it appeared previously.

Our standard reference for notions and notations is FUCHS [2]. In the whole paper p is a fixed prime number. The *p-rank* of a torsion-free group K is the vector space dimension of K/pK. A *p-basis* of K is a maximal p-independent subset of K. We let $N = \{1, 2, \dots\}$. We frequently deal with non-decreasing functions $f: N \to N$ and will simply call them *monotone*.

2. Reductions. Let K be torsion-free and let T be a p-group. For the purpose of studying Ext (K, T) we first of all may and will assume that T is reduced. It is routine to show that Ext (K, T) $[p] \cong$ Ext $(K/p^\omega K, T)$ $[p]$. Thus for the purpose of studying Ext (K, T) $[p]$ we may and will assume that K is p-reduced, i.e. $p^\omega K = 0$.

A useful reduction is due to NUNKE [6]. He showed that the structure of Ext (K, T) depends on a single known invariant of T, the so-called „critical number" which is the smallest among the cardinals dim $(p^n B)$ $[p]$, $n = 1, 2, \dots$, where B is any basic subgroup of T. Thus T may without loss of generality be assumed to be a direct sum of cyclic groups.

In [5] another reduction was pointed out. The group K may be replaced by a module K^P over the ring P of p-adic integers. The module K^P is the so-called p-adic hull of K and has the following characteristic properties: K^P contains K and is a reduced P-module, (K^P/K) $[p] = 0$ and $K^P = PK$ i.e. K generates K^P as a P-module. If A is a p-basic submodule of K^P, $A = \oplus Pa_i$, then every $x \in K^P$ has a unique representation $x = \Sigma x_i a_i$ where $x_i \in P$, and for every natural number n, $p^n \mid x_i$ for almost all i. The last statement implies that $x_i = 0$ for all but countably many i. If $\{a_i\}$ is a p-basis of K then $A = \oplus Pa_i$ is a p-basic submodule of K^P ([5], p. 219, 2.5 (j)). Hence every $x \in K$ has a unique representation $x = \Sigma x_i a_i$, $x_i \in P$, as described above. It is this last fact which we will use consistently; otherwise the p-adic hull will not enter the picture.

The following proposition is included as a matter of curiosity, it will not be used in this paper. We use concepts and facts from [5].

PROPOSITION. *Let K be a torsion-free p-reduced group and T a reduced p-group. Then* $\mathrm{Ext}\,(K, T)\,[p] \cong \mathrm{Ext}_P(K^P, T)\,[p]$. *If* $E : T \overset{\alpha}{\rightarrowtail} M \overset{\beta}{\twoheadrightarrow} K \in\in \mathrm{Ext}\,(K, T)\,[p]$ *the isomorphism is given by* $E \to E^P$ *where* $E^P : T \overset{\alpha^P}{\rightarrowtail} M^P \overset{\beta^P}{\twoheadrightarrow} K^P \in\in \mathrm{Ext}_P(K^P, T)\,[p]$.

PROOF. The main difficulty is the exactness of E^P. E^P is not exact for every E. For suppose K^P is a free P-module and E is not splitting. If E^P were exact, then E^P would split and so would E; contradicting the choice of E. It is therefore essential that E has finite order.

(a) Since $pE \equiv 0$ we have a commutative diagram

$$\begin{array}{ccccc} & T & \overset{\bar\alpha}{\rightarrowtail} & N & \overset{\bar\beta}{\twoheadrightarrow} & K \\ & \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle p} \\ E: & T & \overset{\alpha}{\rightarrowtail} & M & \overset{\beta}{\twoheadrightarrow} & K \end{array}$$

with top row splitting exact. Again by [5], p. 221, 2.6 and p. 219, 2.5 (e) we obtain the commutative diagram

$$\begin{array}{ccccc} & T & \overset{\bar\alpha^P}{\rightarrowtail} & N^P & \overset{\bar\beta^P}{\twoheadrightarrow} & K^P \\ & \| & & \downarrow{\scriptstyle\sigma^P} & & \downarrow{\scriptstyle p} \\ E^P: & T & \overset{\alpha^P}{\rightarrowtail} & M^P & \overset{\beta^P}{\twoheadrightarrow} & K^P \end{array}$$

with top row splitting exact. Again by [5], p. 221, 2.6 $\alpha^P\beta^P = (\alpha\beta)^P = 0$. α^P is injective and β^P is surjective. It remains to show that $\mathrm{Ker}\,\beta^P \subset \mathrm{Im}\,\alpha^P$. Given $x \in M$, there is $y \in N$ such that $x\beta = y\bar\beta$, and $(y\sigma - px)\beta = y\sigma\beta - px\beta = y\bar\beta p - px\beta = 0$. Hence $y\sigma - px \in T\alpha$ and $px \in T\alpha + N\sigma = T\bar\alpha\sigma + N\sigma = N\sigma$. So $pM \subset N\sigma$ and $pM^P = p(PM) = P(pM) \subset P(N\sigma) = (PN)\sigma^P = N^P\sigma^P$. Now suppose that $x \in M^P$ and $x\beta^P = 0$. Then $px = y\sigma^P$, $y \in N^P$, and $y\bar\beta^P p = y\sigma^P\beta^P = px\beta^P = 0$. So $y\bar\beta^P = 0$, $y \in T\bar\alpha^P$, and $px = y\sigma^P \in T\bar\alpha^P\sigma^P = T\alpha^P = T\alpha \subset M$. Since $(M^P/M)\,[p] = 0$, $x \in M$. Since $(M/T\alpha)\,[p] \cong K[p] = 0$, we have $x \in T\alpha = T\alpha^P$. So $\mathrm{Ker}\,\beta^P \subset \mathrm{Im}\,\alpha^P$ as claimed.

(b) From the diagram it is clear that $E^P \in\in \mathrm{Ext}_P(K^P, T)\,[p]$. It is easy to check that $E \mapsto E^P$ is homomorphic: simply attach superscripts P to each group and each map in the diagram defining $E_1 + E_2 = \nabla(E_1 \oplus E_2)\Delta$ and observe that the rows are exact by (a). The map is injective since T is a direct summand of M if T is a direct summand of M^P. To show that the map is surjective, let $T \overset{\alpha}{\rightarrowtail} N \overset{\beta}{\twoheadrightarrow} K^P \in\in \mathrm{Ext}_P(K^P, T)$ be given. Let $M = \{x \in N \mid x\beta \in K\}$. Then $T \rightarrowtail M \overset{\beta|M}{\twoheadrightarrow} K \in\in \mathrm{Ext}\,(K, T)$. Since $N/M \cong (N/T)/(M/T) \cong K^P/K$ we have $(N/M)\,[p] = 0$. For $x \in N$, $x\beta = \sum_i \lambda_i x_i$ with $x_i \in K$ and $x_i = m_i\beta$ with $m_i \in M$. Hence $(x - \Sigma\lambda_i m_i)\beta = 0$, and $x = \Sigma_i \lambda_i m_i + t$ with $t \in T\alpha \subset M$. So $N = M^P$ and $E \mapsto E^P$ is surjective. This proves the proposition.

The proposition reduces the problem to the case of P-modules. This simplifies certain matters. In particular we have the following corollary.

PROPOSITION. *If K^P is a free P-module then,* Ext $(K, T)[p] = 0$ *for all p-groups T. In particular, if K is countable or K/pK finite then* Ext $(K, T)[p] = 0$.

PROOF. For K^P free, Ext $(K, T)[p] \cong \text{Ext}_p(K^P, T)[p] = 0$. If K is countable, then K^P is countably generated hence free. If K/pK is finite, then $K^P/pK^P \cong K/pK$ is finite hence K^P is free.

It is perhaps unlikely that Ext $(K, T)[p] = 0$ for all T if and only if K^P is free but this is an open and fascinating question.

3. The class of groups K with Ext $(K, T)[p] = 0$. Let $\mathcal{K}$ be the class of torsion-free groups K such that Ext $(K, T)[p] = 0$ for every p-group T. In this paper we will use the following characterization of $\mathcal{K}$ which was established in [3].

3.1 PROPOSITION. *Let A be any p-basic subgroup of K. Then $K \in \mathcal{K}$ if and only if for every reduced p-group T,* $\text{Hom}(A, T) = p\,\text{Hom}(A, T) + \text{Hom}(K, T)$ *where* Hom (K, T) *is identified with the group of maps of* Hom (A, T) *which extend to K.*

PROOF. [3], p. 130, 4.3.

We remark that BAER [1], p. 226, 3.2, established the following different characterization.

3.2 PROPOSITION. *$K \in \mathcal{K}$ if and only if for every p-group T,* $\text{Hom}(K, T/pT) = \text{Hom}(K, T)\phi$ *where $\phi : T \to T/pT$ is the natural map.*

It is easy to derive 3.2 from 3.1 and conversely. The following closure property easily follows from 3.1.

3.3 PROPOSITION. *$\mathcal{K}$ is closed under taking p-pure subgroups.*

PROOF. [3], p. 130, 4.5.

On the other hand it is clear that $\mathcal{K}$ is not closed under taking subgroups since $\mathcal{K}$ contains the divisible hull of any torsion-free group but not all torsion-free groups. As was pointed out in Section 2 $\mathcal{K}$ contains all countable torsion-free groups, all torsion-free groups of finite p-rank, and more generally all torsion-free groups K for which the p-adic hull $(K/p^\omega K)^P$ is a free P-module.

The following theorem establishes a necessary condition for belonging to $\mathcal{K}$.

3.4 THEOREM. *Let K be a p-reduced torsion-free group of countable p-rank. If $K \in \mathcal{K}$ and $\mathcal{A} = \{a_1, a_2, \dots\}$ is any p-basis of K, then there is a function $f : N \to N$ such that for each $x = \sum_{i=1}^{\infty} x_i a_i \in K$ $(x_i \in P)$ there is $\bar{x} \in N$ such that for all $m \in N$, $p^{\bar{x}} x_i \equiv 0\ (p^m)$ whenever $i \geqq f(m)$.*

PROOF. By [6], p. 598, 1.3, Ext $(K, T) \cong$ Ext (K, B) for a basic subgroup B of T. Hence we assume without loss of generality that T is itself a direct sum of cyclic groups. Write $T = (\bigoplus_{i=1}^{\infty} \langle b_i \rangle) \oplus T'$ with $e_i =$ exponent of $b_i \geqq i$. Let $B = \bigoplus_{i=1}^{\infty} \langle b_i \rangle$ and let $\pi : T \to B$ be the projection corresponding to the decomposition $T = B \oplus T'$. Let $A = \langle \mathcal{A} \rangle$. By 3.1

$$\text{(3.5)} \qquad \text{Hom}(A, T) = p\,\text{Hom}(A, T) + \text{Hom}(K, T).$$

Define $\xi \in \text{Hom}(A, T)$ by $a_i \xi = b_i$. By (3.5) $\xi = p\xi' + \eta$ with $\xi' \in \text{Hom}(A, T)$ and

$\eta \in \mathrm{Hom}(K, T)$. Further $\xi\pi = p(\xi'\pi) + \eta\pi$ and hence

$$a_i(\eta\pi) = b_i + pc_i \tag{3.6}$$

with $\eta\pi \in \mathrm{Hom}(K, B)$ and $c_i = -a_i\xi'\pi \in B$. Each c_i has a unique representation

$$c_i = \sum_{j=1}^{\infty} n_{ij}b_j, \quad n_{ij} \in Z, \quad n_{ij} = 0 \quad \text{for} \quad j \geqq N(i). \tag{3.7}$$

Given $x = \sum_{i=1}^{\infty} x_i a_i \in K$, $x_i \in P$. Then

$$x\eta\pi = \sum_{i=1}^{\infty} x_i(a_i\eta\pi) = \sum_{i=1}^{\infty} x_i(b_i + pc_i) =$$

$$= \sum_{i=1}^{\infty} x_i \left(\sum_{j=1}^{\infty} \delta_{ij}b_j + \sum_{j=1}^{\infty} pn_{ij}b_j \right) = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} x_i(\delta_{ij} + pn_{ij}) \right) b_j .$$

Since $x\eta\pi \in B$ there is an integer $\bar{x} \geqq 1$ such that

$$\sum_{i=1}^{\infty} x_i(\delta_{ij} + pn_{ij}) \equiv 0\,(p^{e_j}) \quad \text{for all} \quad j \geqq \bar{x} . \tag{3.8}$$

Without loss of generality assume that $N(i+1) \leqq N(i) + 1$ and $N(1) \leqq 1$. We shall prove by induction on m that

$$x_j \equiv 0(p^m) \quad \text{for all} \quad j \geqq N^{m-1}(\bar{x}) . \tag{3.9}$$

Simple induction yields for any integer $k \geqq 1$

$$N^m(k) \geqq k + m; \text{ for } i \geqq j,\ N^j(k) \geqq N^i(k) \text{ and } N(i) \geqq N(j) + (i-j) \geqq N(j) . \tag{3.10}$$

If $m = 1$, then (3.9) is immediate from (3.8). Now suppose (3.9) holds and let $j \geqq N^m(\bar{x})$. If $i \leqq N^{m-1}(\bar{x})$, then $j \geqq N^m(\bar{x}) \geqq N(i)$ hence $n_{ij} = 0$. Thus (3.8) implies

$$\sum_{i > N^{m-1}(\bar{x})} x_i(\delta_{ij} + pn_{ij}) \equiv 0\,(p^{e_j}) \quad \text{for all} \quad j \geqq N^m(\bar{x}) . \tag{3.11}$$

By induction hypothesis $x_i \equiv 0\ (p^m)$ for all x_i occurring in (3.11). Also $e_j \geqq j \geqq N^m(\bar{x}) \geqq \bar{x} + m \geqq m + 1$. Thus (3.11) yields

$$x_j \equiv 0\,(p^{m+1}) \quad \text{for all} \quad j \geqq N^m(\bar{x}) . \tag{3.12}$$

This is just (3.9) with m replaced by $m+1$ and the induction proof is complete.

Now let $f(m) = N^{2(m-1)}(1)$. Then

$$f(m) \geqq N^{m-1}(k) \quad \text{for} \quad m \geqq k . \tag{3.13}$$

Proof. $f(m) = N^{2(m-1)}(1) = N^{m-1}(N^{m-1}(1)) \geqq N^{m-1}(m) \geqq N^{m-1}(k)$. We claim that

$$p^{\bar{x}}x_j \equiv 0\,(p^m) \quad \text{for all} \quad j \geqq f(m) . \tag{3.14}$$

Proof. If $m \geqq \bar{x}$, then $f(m) \geqq N^{m-1}(\bar{x})$ and for $j \geqq f(m)$ we have the stronger statement (3.9). If $m < \bar{x}$, then (3.14) is trivially true.

The condition of Theorem 3.4 can be recast as follows.

3.15 LEMMA. *Let K be a torsion-free p-reduced group of countable p-rank. Let $\mathcal{A} = \{a_1, a_2, \dots\}$ be a p-basis of K. Then the following are equivalent:*

(3.16) *There exists a function $f: N \to N$ such that for all $x = \sum_{i=1}^{\infty} x_i a_i \in K$ there is $\bar{x} \in N$ such that for all m $p^{\bar{x}} x_i \equiv 0\ (p^m)$ whenever $i \geqq f(m)$.*

(3.17) *There exists a monotone function $g: N \to N$ such that for all $x = \sum_{i=1}^{\infty} x_i a_i \in K$ there is $\bar{x} \in N$ such that $p^{\bar{x}} x_i \equiv 0\ (p^{g(i)})$ for all i.*

PROOF. Suppose (3.16) holds. Without loss of generality we assume that $f(m+1) > f(m)$. Define $g(i) = m$ if $f(m) \leqq i < f(m+1)$ with the convention $f(0) = 0$. For $f(m) \leqq i < f(m+1)$, $m \geqq g(i)$ hence $p^{\bar{x}} x_i \equiv 0\ (p^{g(i)})$. Now suppose (3.17) holds. Define $f(m) = \max \{j+1 \mid g(j) < m\}$ and $f(m) = 1$ if $\{j \mid g(j) < m\} = \emptyset$. Given m and $i \geqq f(m)$. Then $g(i) \geqq m$ and hence $p^{\bar{x}} x_i \equiv 0\ (p^m)$.

To simplify statements we introduce the following terminology.

3.18 DEFINITION. Let K be a p-reduced torsion-free group of countable p-rank. Let $\mathcal{A} = \{a_1, a_2, \dots\}$ be an (ordered) p-basis of K. We call *K barred by f with respect to $\mathcal{A}$* if f is a monotone unbounded function on N to N such that for every $x = \Sigma x_i a_i \in K$ there is $\bar{x} \in N$ such that $p^{\bar{x}} x_i \equiv 0\ (p^{f(i)})$ for all i.

In this language Theorem 3.4 reads: If $K \in \mathcal{K}$ and $\mathcal{A} = \{a_1, a_2, \dots\}$ is a p-basis of K, then K is barred with respect to $\mathcal{A}$. It is easy to construct torsion-free groups which are barred with respect to a given p-basis. We shall show that such groups are automatically barred with respect to any other p-basis also.

3.19 THEOREM. *Let K be a p-reduced torsion-free group. If K is barred with respect to some p-basis, then K is barred with respect to any p-basis.*

PROOF. (a) It is well-known and easy to check that $\{a_i\}$ is a p-basis of the torsion-free group K if and only if $\{a_i + pK\}$ is a vector space basis of K/pK. Thus if $\{a_i\}$ and $\{b_i\}$ are any two p-bases of K, then there are elements $c_i \in K$ and an integral row-finite matrix $[t_{ij}]$ such that $a_i = \sum_{j=1}^{\infty} t_{ij} b_j + pc_i$. Writing $c_i = \sum_{j=1}^{\infty} u_{ij} a_j$, and letting $[a_i]$, $[b_i]$ denote column vectors we obtain in matrix form

$$(I - p[u_{ij}])\,[a_j] = [t_{ij}]\,[b_j]$$

Noting that $(I - p[u_{ij}])^{-1} = \sum_{k=0}^{\infty} p^k [u_{ij}]^k$ makes sense in an obvious way we obtain

$$[a_j] = (I - p[u_{ij}])^{-1}\,[t_{ij}]\,[b_j]\,.$$

Now assume that K is barred with respect to $\{a_i\}$. In order to show that K is barred with respect to $\{b_i\}$ it suffices to consider the two cases $[a_i] = [t_{ij}]\,[b_j]$ and $[a_i] = (I - p[u_{ij}])^{-1}[b_j]$.

(b) Let $[a_i] = [t_{ij}]\,[b_j]$. Suppose that K is barred by f with respect to $\{a_i\}$. If $x = \sum_{i=1}^{\infty} x_i a_i \in K$ and $x = \sum_{i=1}^{\infty} y_i b_i$, then a simple computation yields $y_i = \sum_{j=1}^{\infty} x_j t_{ji}$. Since $[t_{ij}]$ is row finite there is a function $n: N \to N$ such that $t_{ji} = 0$ whenever $i \geqq n(j)$. Without loss of generality assume that n is monotone and not bounded. Define $g(i) = f(\min_j \{j \mid n(j) < i\})$. Then g is well-defined, monotone and not bounded.

Finally $p^{\bar{x}}y_i = \sum_{j=1}^{\infty} p^{\bar{x}}x_j t_{ji} = \sum_{n(j)>i} p^{\bar{x}}x_j t_{ji} \equiv 0\,(p^{g(i)})$ since $p^{\bar{x}}x_j \equiv 0\,(p^{f(j)})$ and $n(j) > i$ implies $g(i) \leqq f(j)$. Thus K is barred by g with respect to $\{b_i\}$.

(c) Now let $[a_i] = (I - p[u_{ij}])^{-1}[b_i]$. We assume condition (3.16) with monotone f. Hence there is a function $n : N \to N$ such that

$$p^{n(i)}u_{ij} \equiv 0\ (p^m) \text{ whenever } j \geqq f(m)\,. \tag{3.20}$$

Without loss of generality we assume that n is monotone. Let $[v_{ij}] = (I - p[u_{ij}])^{-1} = \sum_{k=0}^{\infty} p^k[u_{ij}]^k$, i.e.

$$v_{ij} = \delta_{ij} + pu_{ij} + p^2 \sum_{i_1} u_{ii_1}u_{i_1 j} + \ldots + p^{k+1} \sum_{i_1, i_2, \ldots, i_k} u_{ii_1}u_{i_1 i_2} \ldots u_{i_k j} + \ldots\,. \tag{3.21}$$

We claim

(3.22) *There is a function $g : N \times N \to N$ such that g is monotone in each variable and $v_{ij} \equiv 0\ (p^k)$ whenever $j > g(k, i)$.*

Proof. We first define inductively functions h_m by $h_0(r, s) = s$, $h_{m+1}(r, s) = f(r + nh_m(r, s))$. Obviously h_m is monotone in each variable. We next prove by induction on $r \geqq 0$ that for any $k \geqq r + 2$

$$u_{ii_1}u_{i_1 i_2} \ldots u_{i_r j} \equiv 0\ (p^{k-r-1}) \quad \text{if} \quad j \geqq h_{r+1}(k - r - 1, i)\,. \tag{3.23}$$

For $r = 0$ the claim is $u_{ij} \equiv 0\ (p^{k-1})$ if $j \geqq h_1(k-1, i) = f(k - 1 + n(i))$. This is immediate from (3.20). Now suppose $1 \leqq r \leqq k - 2$. By induction hypothesis with k replaced by $k - 1$ we have $u_{ii_1}u_{i_1 i_2} \ldots u_{i_{r-1} i_r} \equiv 0\,(p^{(k-1)-(r-1)-1})$ if $i_r \geqq h_r(k - r - 1, i)$, and in this case the claim follows upon multiplying this congruence by $u_{i_r j}$. On the other hand if $i_r < h_r(k - r - 1, i)$ then $f(k - r - 1 + n(i_r)) \leqq f(k - r - 1 + nh_r(k - r - 1, i)) = h_{r+1}(k - r - 1, i) \leqq j$. So $u_{i_r j} \equiv 0\,(p^{k-r-1})$ by (3.20) and the claim follows upon multiplying this congruence by $u_{ii_1} \ldots u_{i_{r-1} i_r}$. This concludes the proof of (3.23).

Define $g(k, i) = \max\,\{h_0(k, i), h_1(k - 1, i), \ldots, h_{k-1}(1, i)\}$. Clearly g is monotone in each variable. Assume $j > g(k, i)$ and consider (3.21). Since $j > h_0(k, i) = i$, $\delta_{ij} = 0$. Since $j \geqq h_{r+1}(k - r - 1, i)$ for $0 \leqq r \leqq k - 2$ it follows from (3.23) that $p^{r+1}u_{ii_1}u_{i_1 i_2} \ldots u_{i_r j} \equiv 0\ (p^k)$. Hence $v_{ij} \equiv 0\ (p^k)$. This proves (3.22).

If $x = \sum_{i=1}^{\infty} x_i a_i \in K$ and $x = \sum_{j=1}^{\infty} y_j b_j$, then an easy computation yields $y_j = \sum_{i=1}^{\infty} x_i v_{ij}$. Let $h(k) = g(k, f(k))$. Then $j > h(k)$ implies $j > g(k, i)$ for $1 \leqq i \leqq f(k)$, and by (3.22) $v_{ij} \equiv 0\ (p^k)$ for $1 \leqq i \leqq f(k)$ while $p^{\bar{x}}x_i \equiv 0\,(p^k)$ for $i \geqq f(k)$. Hence $p^{\bar{x}}y_j \equiv 0\ (p^k)$. This proves that K is barred with respect to $\{b_i\}$.

Proposition 3.19 permits a further simplification of the language.

3.24 Definition. The torsion-free p-reduced group K is *barred* if it is barred with respect to some (and hence every) p-basis.

For later use we now prove that a barred group possesses an unbounded reduced p-primary homomorphic image. A consequence is that $\mathrm{Ext}\,(K, T)\,[p] \neq 0$ for some p-group T if K has countable p-rank and every reduced p-primary homomorphic image of K is bounded. However, this fact is more easily obtained in other ways and the restriction on p-ranks is not necessary (see 4.3 and 4.4 below).

3.25 PROPOSITION. *If K is barred then K has an unbounded reduced p-primary homomorphic image.*

PROOF. Suppose K is barred by f with respect to $\{a_i\}$. Let $\langle b_i\rangle$ be cyclic of order $p^{f(i)}$. Let $B=\prod_{i=1}^{\infty}\langle b_i\rangle$. Define $\alpha: K\to B: \left(\sum_{i=1}^{\infty} x_i a_i\right)\alpha=(\dots, x_i b_i, \dots)$. Then $a_i\alpha = b_i$ and $K\alpha$ is unbounded. Clearly B is reduced and hence so is $K\alpha$. Since $p^{\bar{x}}x_i\equiv 0$ $(p^{f(i)})$, $p^{\bar{x}}(x\alpha)=(\dots, p^{\bar{x}}x_i b_i, \dots)=0$ so $K\alpha$ is p-primary and the proof is complete.

4. Groups K with Ext (K, T) $[p]\neq 0$. The following proposition, due to BAER [1], is an immediate corollary of 3.4.

4.1 PROPOSITION. *Let K_0 be the group of sequences of integers such that, for each m, almost all terms of the sequence are divisible by p^m. Then* Ext (K_0, T) $[p]\neq 0$ *for any unbounded reduced p-group T.*

Baer proved this proposition by first showing that every reduced p-primary homomorphic image of K_0 is bounded. In [3] it was shown that for any group K with this property the rank of Ext (K, T) $[p]$ can be determined and if K has infinite p-rank, then Ext (K, T) $[p]\neq 0$ for every p-group T with unbounded basic subgroup. The question arises whether Baer's detour was necessary, or more generally whether Ext (K, T) $[p]\neq 0$ implies that K contains a p-pure subgroup L of infinite p-rank such that every reduced p-primary homomorphic image of L is bounded. We shall show that the answer is no.

4.2 THEOREM. *Let T be an unbounded reduced p-group. Let K be torsion-free of infinite p-rank, and suppose that K contains a subgroup L such that K/L is p-primary and* Hom (L, T) *is a torsion group. Then* Ext (K, T) $[p]\neq 0$.

PROOF. Suppose Ext (K, T) $[p]=0$. If B is a basic subgroup of T, then Ext (K, B) $[p]=0$ by [6], p. 158, 1.3, and Hom (L, B) is torsion. Hence without loss of generality we assume that T is a direct sum of cyclic groups. Let $\{a_1, a_2, \dots\}$ be a p-independent subset of K and let $n_i=$ exponent of a_i+L. Write $T=B\oplus T'$, $B=\oplus\langle b_i\rangle$ such that $e_i=$ exponent of $b_i\geqq n_i+i$ for $i=1,2,\dots$. Let $\pi: T\to B$ be the projection corresponding to $T=B\oplus T'$. Extend $\{a_1, a_2, \dots\}$ to a maximal p-independent subset of K and let A be the p-basic subgroup it generates. Let $\xi: A\to T$ be such that $a_i\xi=b_i$. Proceed as in the proof of 3.4 to obtain

$$(4.3)\qquad a_i(\eta\pi)=\sum_{j=1}^{\infty}(\delta_{ij}+pn_{ij})b_j,\ \eta\pi\in \text{Hom}\,(K, B).$$

$\eta\pi\mid L\in$ Hom (L, B) hence $p^n(\eta\pi\mid L)=0$ for some n. Hence $0=(p^{n_i}a_i)(p^n\eta\pi)=$ $=\sum_{j=1}^{\infty}p^{n+n_i}(\delta_{ij}+pn_{ij})b_j$. Thus $p^{n+n_i}(\delta_{ij}+pn_{ij})b_j=0$ for all j, in particular $p^{n+n_i}(1+pn_{ii})b_i=0$. So $n+n_i\geqq e_i\geqq n_i+i$, resulting in the contradiction $n\geqq i$ for each i.

4.4 COROLLARY. *Let T be an unbounded reduced p-group, and K torsion-free of infinite p-rank such that* Hom (K, T) *is torsion. Then* Ext (K, T) $[p]\neq 0$.

PROOF. $L=K$ in 4.2.

4.5 THEOREM. *There exist p-reduced torsion-free groups K such that every p-pure subgroup of K of infinite p-rank has reduced unbounded p-primary homomorphic images and* Ext (K, T) $[p] \neq 0$ *for any p-group T with unbounded basic subgroup.*

We remark that 4.3 answers negatively a question stated in [4]: If Ext (K, T) $[p] \neq 0$ then this does not necessarily happen because K contains a p-pure subgroup of infinite p-rank for which every reduced p-primary homomorphic image is bounded.

PROOF OF 4.5. (a) Let $P^{\aleph_0}$ be the group of all sequences of p-adic integers, A the subgroup of sequences of integers with finitely many non-zero terms. Clearly A is free and p-pure in $P^{\aleph_0}$. Let $f: N \to N$ be monotone and unbounded. Let $L = \{(\ldots, p^{f(i)}x_i, \ldots) \mid x_i \in P\}$. Clearly L is an endomorphic image of $P^{\aleph_0}$ and $L \cong P^{\aleph_0}$. Let K be the p-purification of L in $P^{\aleph_0}$.

(b) It is easily seen that A is a p-basic subgroup of K and by construction K is barred by f with respect to the obvious basis of A. Let M be any p-pure subgroup of K of infinite p-rank. Extend some p-basis $\mathfrak{M}$ of M to a p-basis of K. By 3.19 K is barred with respect to this p-basis and therefore M is barred with respect to $\mathfrak{M}$. By 3.25 M possesses an unbounded reduced p-primary homomorphic image, and it follows easily (using for example Szele's Theorem, [7], by which a basic subgroup of a p-group is an endomorphic image) that Hom (M, T) is not torsion for any unbounded p-group T.

(c) By definition of K, K/L is p-primary. Since $L \cong P^{\aleph_0}$ and $P^{\aleph_0}$ is algebraically compact, so is L and Hom (L, T) is torsion for any given reduced p-group T. By 4.2 Ext (K, T) $[p] \neq 0$ for T a reduced unbounded p-group. This completes the proof.

In the proof of 4.5 we constructed a special type of barred groups which seem interesting and not too hard to investigate. The example also shows that the condition of Theorem 3.4 is not necessary, i.e. a barred group K need not belong to $\mathfrak{K}$.

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UNIVERSITY OF HAWAII
DEPARTMENT OF MATHEMATICS
HONOLULU, HAWAII 96822
USA

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THE RAPIDITY OF CONVERGENCE OF QUASI-HERMITE—FEJÉR INTERPOLATION POLYNOMIALS

By

R. B. SAXENA* and K. K. MATHUR (Lucknow)

1. The quasi-Hermite—Fejér interpolation polynomial $Q_n(f)$ of degree $\leqq 2n+1$ introduced by P. SZÁSZ [3] is given by

$$(1)\qquad Q_n(f,x)=\left[\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)\right]\frac{U_n^2(x)}{(n+1)^2}+$$
$$+\sum_{k=1}^{n}f(x_{kn})(1-x^2)(1-xx_{kn})\left[\frac{U_n(x)}{(n+1)(x-x_{kn})}\right]^2,$$

where $x_{kn}=\cos\dfrac{k\pi}{n+1}$, $k=1,2,\ldots,n$ are the zeros of Cebyshev polynomial of the second kind $U_n(x)=\dfrac{\sin(n+1)\theta}{\sin\theta}$, $x=\cos\theta$. P. SZÁSZ [3] has proved that

$$\lim_{n\to\infty}Q_n(f)=f \quad \text{uniformly on} \quad [-1,1].$$

As to the rapidity of convergence the first of us [4] gave the estimate

$$(2)\qquad \|Q_n(f)-f\|\leqq 5\omega_f\left(\frac{\log n}{n}\right)$$

where $\|f\|=\sup\{|f|,\ -1\leqq x\leqq 1\}$ and ω_f is the modulus of continuity of $f(x)$. In this paper, we obtain more precise estimates for the sequence $\|Q_n(f)-f\|$.

Let us denote $C_\omega[-1,1]$ the class of all functions $f(x)$ defined on $[-1,1]$ and satisfy the inequality

$$(3)\qquad \omega_f(h)\leqq c\omega(h)$$

where $\omega(h)$ is a certain modulus of continuity. We shall prove the following

THEOREM 1. *If $f(x)\in C_\omega[-1,1]$, then for all $x\in[-1,1]$, we have*

$$(4)\ |Q_n(f;x)-f(x)|\leqq c_1\left[\sum_{i=1}^{n}\frac{1}{i^2}\omega\left(\frac{i+1}{n+1}\pi\sqrt{1-x^2}\right)+\sum_{i=1}^{n}\frac{1}{i^2}\omega\left(\left(\frac{i+1}{n+1}\pi\right)^2\right)\right],$$

where c_1 (later on $c_2, c_3, \ldots$) is an absolute constant.

The estimate of this type for Hermite—Fejér interpolation has recently been given by P. O. H. VÉRTESI [6].

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Owing to the inequality (3), we get from (4)

$$(5)\quad | Q_n(f, x) - f(x) | \leqq c_2\left[\sum_{i=1}^{n} \frac{1}{i^2}\omega_f\left(\frac{(i+1)\sqrt{1-x^2}}{n+1}\pi\right) + \sum_{i=1}^{n} \frac{1}{i^2}\omega_f\left(\left(\frac{i+1}{n+1}\pi\right)^2\right)\right].$$

The inequality (5) is more precise than (2) in the sense that (2) follows from (5). For functions belonging to Lipschitz class of order α $(0<\alpha<1)$ i.e. $\omega_f(h) \leqq h^\alpha$, we get from (5)

$$(6)\qquad | Q_n(f; x) - f(x) | \leqq c_3\left[\left(\frac{\sqrt{1-x^2}}{n}\right)^\alpha + \varepsilon_n\right], \qquad -1 \leqq x \leqq 1$$

where

$$\varepsilon_n = \begin{cases} \dfrac{1}{n^{2\alpha}}, & 0<\alpha<\dfrac{1}{2} \\[2mm] \dfrac{\log n}{n}, & \alpha = \dfrac{1}{2} \\[2mm] \dfrac{1}{n}, & \dfrac{1}{2} < \alpha < 1; \end{cases}$$

and

$$(7)\qquad | Q_n(f, x) - f(x) | \leqq c_4\left[\frac{\sqrt{1-x^2}}{n}\log n + \frac{1}{n}\right], \qquad \alpha = 1; \qquad -1 \leqq x \leqq 1.$$

Evidently our estimation is best possible for $f(x) \in \operatorname{Lip}\alpha$ $(0<\alpha<1)$ when $-1<$ $<x<1$. For $f(x) \in \operatorname{Lip} 1$, the estimation is also precise when $-1<x<1$ as shown by the following

THEOREM 2. *There exists a function $f(x) \in \operatorname{Lip} 1$ and a constant c_5 such that*

$$(8)\qquad | Q_n(f, 0) - f(0) | \geqq c_5 \frac{\log n}{n}, \qquad n = 2, 4, 6, \dots .$$

Further on using the inequalities [5]

$$(9)\qquad \frac{\pi}{n+1}\int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega(t\sqrt{1-x^2})}{t^2}\,dt \leqq \sum_{i=1}^{n}\frac{1}{i^2}\omega\left(\frac{i+1}{n+1}\pi\sqrt{1-x^2}\right) \leqq$$

$$\leqq \frac{8\pi}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\omega(t\sqrt{1-x^2})}{t^2}\,dt$$

and

$$(10)\qquad \frac{\pi}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\omega(t^2)}{t^2}\,dt \leqq \sum_{i=1}^{n}\frac{1}{i^2}\omega\left(\frac{(i+1)^2\pi^2}{(n+1)^2}\right) \leqq \frac{8\pi}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\omega(t^2)}{t^2}\,dt,$$

the estimates in theorem 1 can be given in terms of the arithmetic means of the sequences $\left\{\omega\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right\}$. Namely the following theorem holds.

THEOREM 3. *If $f(x)\in C_\omega[-1,1]$, then for all $x\in[-1,1]$, we have*

$$|Q_n(f;x)-f(x)| \leqq \frac{c_6}{n}\sum_{k=1}^{n}\omega\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right). \tag{11}$$

This form of estimates for Hermite—Fejér interpolation has recently been given by BOJANIC [1] and SAXENA [5] and the proof follows on the same pattern as in [5].

2. We now give the proof of the above theorems. First we require the following lemma, the proof of which can be obtained by some modification in the proof of Lemma 1 given by KIS [2].

LEMMA. *If*

$$\frac{j-1}{n+1}\pi \leqq \theta \leqq \frac{j}{n+1}\pi, \qquad j=1,2,\ldots,n+1;$$

$x=\cos\theta$, *then*

$$|f(x)-f(x_{kn})| \leqq \begin{cases} c_7\left[\omega\left(\frac{\pi\sin\theta}{n+1}\right)+\omega\left(\frac{\pi^2}{(n+1)^2}\right)\right] \\ \quad \text{if } k=j-1,\ 1\leqq j\leqq n-1 \text{ and } k=j,\ 2\leqq j\leqq n \\ c_8\left[\omega\left(\frac{\pi(i+1)\sin\theta}{n+1}\right)+\omega\left(\left(\frac{i+1}{n+1}\pi\right)^2\right)\right] \\ \quad \text{if } 1\leqq j<k=j+i\leqq n \text{ and } 2\leqq k=j-i\leqq n+1. \end{cases} \tag{12}$$

PROOF OF THEOREM 1. With the usual convention

$$f(x)-Q_{2n+1}(f;x)= \tag{13}$$

$$=\left[\frac{1-x}{2}\{f(x)-f(-1)\}+\frac{1+x}{2}\{f(x)-f(1)\}\right]\frac{U_n^2(x)}{(n+1)^2}+$$

$$+\sum_{k=1}^{n}[f(x)-f(x_{kn})]\frac{(1-x^2)(1-xx_{kn})}{(n+1)^2}\left[\frac{U_n(x)}{x-x_{kn}}\right]^2 \equiv S_1+S_2.$$

We shall estimate S_1 and S_2 separately. Setting

$$x=\cos\theta,\quad 0\leqq\theta\leqq\pi,\quad x_{kn}=\cos\theta_{kn}=\cos\frac{k\pi}{n+1},\quad k=1,2,\ldots,n,$$

we have

$$S_2=\frac{1}{(n+1)^2}\sum_{k=1}^{n}[f(\cos\theta)-f(\cos\theta_{kn})](1-\cos\theta\cos\theta_{kn})\left[\frac{\sin(n+1)\theta}{\cos\theta-\cos\theta_{kn}}\right]^2. \tag{14}$$

Now

$$(15)\qquad (1-\cos\theta\;\cos\theta_{kn})\left[\frac{\sin(n+1)\theta}{\cos\theta-\cos\theta_{kn}}\right]^2\leqq$$

$$\leqq(1-\cos\theta\cos\theta_{kn}+\sin\theta\sin\theta_{kn})\left[\frac{\sin(n+1)\theta-\sin(n+1)\theta_{kn}}{\cos\theta-\cos\theta_{kn}}\right]^2\leqq$$

$$\leqq 2\left[\frac{\sin\dfrac{n+1}{2}(\theta-\theta_{kn})}{\sin\dfrac{1}{2}(\theta-\theta_{kn})}\right]^2.$$

Using the definition of j and θ_{kn} it follows that

$$(16)\qquad \frac{1}{\sin\dfrac{\theta-\theta_{kn}}{2}}\leqq\frac{n+1}{i\pi},\qquad k\neq j,\,j-1.$$

Hence from (14) on account of (15), (16) and the lemma, we get

$$(17)\qquad |S_2|\leqq c_9\left\{\omega\left(\frac{\pi\sin\theta}{n+1}\right)+\omega\left(\frac{\pi^2}{(n+1)^2}\right)\right\}+$$

$$+\frac{2c_8}{\pi}\sum_{k\neq j,j-1}\frac{1}{i^2}\left\{\omega\left(\frac{(i+1)\sin\theta}{n+1}\pi\right)+\omega\left(\left(\frac{i+1}{n+1}\pi\right)^2\right)\right\}.$$

For

$$S_1=\left[\frac{1-x}{2}\{f(x)-f(-1)\}+\frac{1+x}{2}\{f(x)-f(1)\}\right]\frac{U_n^2(x)}{(n+1)^2},$$

we have

$$|S_1|\leqq\frac{1}{2}\left[(1-x)\,\omega(1+x)+(1+x)\,\omega(1-x)\right]\frac{U_n^2(x)}{(n+1)^2}.$$

Let $x\geqq 0$ then

$$\frac{\omega(1+x)}{1+x}\leqq 2\,\frac{\omega(1-x)}{1-x}$$

and therefore

$$|S_1|\leqq\frac{3}{2}(1+x)\omega(1-x)\frac{U_n^2(x)}{(n+1)^2}\leqq 3\,\omega(1-x^2)\frac{U_n^2(x)}{(n+1)^2}\leqq$$

$$\leqq 3\left[\frac{n+1}{\pi}\sqrt{1-x^2}+1\right]\frac{U_n^2(x)}{(n+1)^2}\,\omega\left(\frac{\pi\sqrt{1-x^2}}{n+1}\right)\leqq 6\omega\left(\frac{\pi\sqrt{1-x^2}}{n+1}\right)$$

owing to the inequalities

$$\sqrt{1-x^2}\,|\,U_n(x)\,|\leqq 1, \qquad |\,U_n(x)\,|\leqq n+1, \qquad (-1\leqq x\leqq 1).$$

Thus from (13), (17) and (18) we have the theorem.

PROOF OF THEOREM 2. Let $f(x)=|\,x\,|$ and $n=2p$, then from (13), we have

$$Q_{2n+1}(f;0)-f(0)=\frac{U_n^2(0)}{(n+1)^2}+2\sum_{k=1}^{p}x_k\frac{U_n^2(0)}{(n+1)^2x_k^2}=$$

$$=\frac{1}{(n+1)^2}\left[1+2\sum_{k=1}^{p}\frac{1}{x_k}\right]\geqq\frac{2}{(n+1)^2}\sum_{k=1}^{p}\frac{1}{x_k}$$

and the theorem follows by similar argument as in [6].

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UNIVERSITY OF ALBERTA
EDMONTON, CANADA

AND

LUCKNOW UNIVERSITY
LUCKNOW, INDIA

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COMPARISON OF LAGRANGE- AND HERMITE—FEJÉR INTERPOLATIONS

By

P. VÉRTESI (Budapest)

1. Introduction. In many cases the Hermite—Fejér interpolation is better than the Lagrange one. The aim of this paper is to give a positive answer for a problem raised by P. TURÁN on a joint American—Hungarian seminar held in 1975 in Budapest: Do there exist a system of nodes and function class for which the Lagrange process is better than the Hermite—Fejér one?

2. Notations and preliminary results. 2.1. Let us consider an arbitrary system of nodes

$$-1 \leqq x_{n,n} < x_{n-1,n} < \dots < x_{2,n} < x_{1,n} \leqq 1 \qquad (n \in \mathbf{N}) \tag{2.1}$$

in $[-1; 1]$, further denote

$$\omega_n(X, x) \overset{\text{def}}{=} c(x - x_{1,n})(x - x_{2,n}) \dots (x - x_{n,n}), \qquad c \neq 0, \tag{2.2}$$

$$l_{k,n}(X, x) \overset{\text{def}}{=} \frac{\omega_n(X, x)}{\omega_n'(X, x_{k,n})(x - x_{k,n})} \qquad (k = 1, \dots, n), \tag{2.3}$$

$$L_n(f; X, x) \overset{\text{def}}{=} \sum_{k=1}^{n} f(x_{k,n}) l_{k,n}(X, x) \qquad (n \in \mathbf{N}),^1 \tag{2.4}$$

$$v_{k,n}(X, x) \overset{\text{def}}{=} 1 - \frac{\omega_n''(X, x_{k,n})}{\omega_n'(X, x_{k,n})}(x - x_{k,n}) \qquad (k = 1, \dots, n), \tag{2.5}$$

$$h_{k,n}(X, x) \overset{\text{def}}{=} v_{k,n}(X, x) l_{k,n}^2(X, x) \qquad (k = 1, \dots, n), \tag{2.6}$$

$$\mathfrak{H}_{k,n}(X, x) \overset{\text{def}}{=} (x - x_{k,n}) l_n^2(X, x) \qquad (k = 1, \dots, n), \tag{2.7}$$

$$H_n(f; X, x) \overset{\text{def}}{=} \sum_{k=1}^{n} f(x_{k,n}) h_{k,n}(X, x) \qquad (n \in \mathbf{N}), \tag{2.8}$$

$$H_n^*(f; X, x) \overset{\text{def}}{=} H_n(f; X, x) + \sum_{k=1}^{n} f'(x_{k,n}) \mathfrak{H}_{k,n}(X, x) \quad (n \in \mathbf{N}).^2 \tag{2.9}$$

Here $\mathbf{N} = \{1, 2, 3, \dots\}$, X stands for the matrix $\{x_{k,n}\}_{k=1,\dots,n,\, n \in \mathbf{N}}$, $f \in C$ (i.e. $f(x)$ is continuous on $[-1, 1]$), moreover, in (2.9) we suppose that $f' \in C$, too.

[1] L_n is the wellknown Lagrange interpolation.
[2] H_n (or H_n^*) is the wellknown Hermite—Fejér interpolation.

G. FABER proved the surprising fact that there exists no matrix X for which $L_n(f; X, x)$ uniformly converges in $[-1, 1]$ for every $f \in C$ (see e.g. [1], Part III, Chapter 2). On the other hand, if we consider, say, the Chebyshev matrix T, i.e., when

$$x_{k,n} = \cos \frac{2k-1}{2n} \pi \qquad (k = 1, \dots, n;\ n \in \mathbf{N}) \tag{2.10}$$

then, as FEJÉR proved, $H_n(f; T, x)$ (or $H_n^*(f; T, x)$ if $f' \in C$) uniformly tends to $f(x)$ in $[-1, 1]$ for every $f \in C$ (see [1], Part III, Chapter 3).

So it would be interesting — in accordance with a problem raised by P. TURÁN — to construct a matrix X such that $L_n(f; X, x)$ uniformly tends to $f(x)$ in $[-1, 1]$ if $f \in F$ but for a suitable $f_1 \in F$ and $x_0 \in [-1, 1]$ $H_n(f;; X, x_0)$ (or $H_n^*(f_1; X, x)$ if $f_1' \in C$) does not converge to $f_1(x_0)$ ($F \subseteq C$).

In his paper [2] D. L. BERMAN proved that for the equdistant nodes $x_{k,n} = -1 + 2k(n-1)^{-1}$ ($k = 0, \dots, n-1$) in $[-1, 1]$ with $f_1(x) = x$ the process $H_n(f_1; X, x)$ diverges unboundedly for $n \to \infty$ at each point of $[-1, 1]$ except the point $x = 0$. But, of course, $H_n^*(f_1; X, x) \equiv x$ if $n \geqq 1$.

3. New results. The aim of this paper is to construct such a system of nodes for which the „most reasonable“ Hermite–Fejér process (i.e., the process H_n if f' does not exist, or the process H_n^* if f' exists) behaves worse for a suitable function class than the corresponding Lagrange interpolation.

3.1. Using the usual notation $\| g(x) \|_{[a,b]} = \sup_{a \leqq x \leqq b} | g(x) |$ and $\operatorname{Lip} \alpha = \{f(x);\ |f(x_1) - f(x_2)| \leqq M_f |x_1 - x_2|^\alpha;\ x_1, x_2 \in [-1, 1]\}$, we have

THEOREM 3.1. *For any fixed* $0 < \alpha \leqq 1$ *there exists a matrix* $Y = Y_\alpha$ *such that*

$$\lim_{n\to\infty} \| L_n(f; Y, x) - f(x) \|_{[-1,1]} = 0 \tag{3.1}$$

for any $f \in \operatorname{Lip} \alpha$ *and*

$$\overline{\lim_{n\to\infty}} \, [H_n(f_1; Y, z_n) - f_1(z_n)] = \infty \tag{3.2}$$

for a suitable $f_1 \in \operatorname{Lip} \alpha$ *and a sequence* $\{z_n\}$ *where* $\lim_{n\to\infty} z_n = 0$.

For the completeness we remark that $f_1'(x)$ does not exist on $E \subset [-1, 1]$ where E is an infinite denumerable set.

3.2. If $f(x)$ is k-times differentiable ($k \geqq 1$) then we have

THEOREM 3.2. *For any fixed integer* $k \geqq 1$ *and* $0 < \alpha \leqq 1$ *we have a matrix* Y *such that*

$$\lim_{n\to\infty} \| L_n(f; Y, x) - f(x) \|_{[-1,1]} = 0 \tag{3.3}$$

if $f^{(k)}(x) \in \operatorname{Lip} \alpha$, *further, there exists a function* $f_2(x)$ *with* $f^{(k)}(x) \in \operatorname{Lip} \alpha$ *such that*

$$H_n^*(f_2; Y, x) = H_n(f_2; Y, x) + O\left(\frac{\ln n}{n}\right) \qquad (x \in [-1, 1]), \tag{3.4}$$

and for a suitable $\{u_n\}$

$$\text{(3.5)}\qquad \overline{\lim_{n\to\infty}}\,[H_n(f_2;\,Y,u_n)-f_2(u_n)]=\overline{\lim_{n\to\infty}}\,[H_n^*(f_2;\,Y,u_n)-f_2(u_n)]=\infty\,.$$

Here, as above, $\lim u_n=0$ *and* $Y=Y_{\alpha,k}$.

4. Proofs. Starting from the Chebyshev matrix T (see (2.10)) we construct the matrix $Y=\{y_{k,n}\}$ as follows.[3]

4.1. Let $r\geqq 0$ and $s>0$ be integers where r is fixed and

$$\text{(4.1)}\qquad s=s(n)=\left[\frac{n+3}{2}\right].$$

Sometimes omitting the superfluous notations, let

$$\text{(4.2)}\qquad y_{l,n}=\cos\frac{2l-1}{2n}\pi\qquad (l\neq s,\dots,s+r,\ n\geqq 4r+1),$$

$$\text{(4.3)}\qquad \begin{cases} y_{s+i,n}=\cos\left[\dfrac{2s+2r+1}{2n}-(r+1-i)\rho_n\right]\pi, \\ \text{where}\quad 0<\rho_n=O\left(\dfrac{1}{n\ln n}\right), \qquad i=0,\dots,r,\ n\geqq 4r+1\,. \end{cases}$$

It is easy to see that $y_{s-1,2l+1}=0$. By an easy computation we can verify the relations

$$\text{(4.4)}\qquad l_{k,n}(Y,x)=l_{k,n}(T,x)\prod_{t=0}^{r}\frac{x-y_{s+t}}{x-x_{s+t}}\prod_{t=0}^{r}\frac{x_k-x_{s+t}}{x_k-y_{s+t}}\qquad (k\neq s,\dots,s+r),$$

$$\text{(4.5)}\qquad l_{s+i,n}(Y,x)=\frac{T_n(x)}{T_n(y_{s+i})}\prod_{\substack{t=0\\ t\neq i}}^{r}\frac{x-y_{s+t}}{y_{s+i}-y_{s+t}}\prod_{t=0}^{r}\frac{y_{s+i}-x_{s+t}}{x-x_{s+t}}\qquad (i=0,\dots,r),$$

where, as usual, for $-1\leqq x\leqq 1$

$$\text{(4.6)}\qquad T_n(x)=\cos n\vartheta,\qquad \cos\vartheta=x\,.$$

4.2. Now we prove some properties of Y.

4.21. By (4.2) and (4.3) we have

$$\text{(4.7)}\qquad y_{s+i+1}-y_{s+i}\sim\rho_n\qquad (i=0,\dots,r),$$

$$\text{(4.8)}\qquad y_{k+1}-y_k\sim\frac{1}{n}\qquad \left(\left[\frac{n}{4}\right]\leqq k\leqq\left[\frac{3n}{4}\right],\ k\neq s,\dots,s+r\right).$$

4.22. We prove

[3] The proofs run together for any $Y_{\alpha,k}$ ($k\geqq 0$; $Y_{\alpha,0}\equiv Y_\alpha$).

LEMMA 4.1. *We have the relation*

$$\max_{-1\le x\le 1} \sum_{\substack{k=1\\ k\ne s,\dots,s+r+1}}^{n} |l_{k,n}(Y,x)| = O(\ln n). \tag{4.9}$$

For getting (4.9) at first let us remark that now

$$\prod_{t=0}^{r} \frac{|x_k - x_{s+t}|}{|x_k - y_{s+t}|} = O(1) \qquad (k \ne s,\dots,s+r+1) \tag{4.10}$$

To go further we have to consider different cases.

α. Let $-1 \leqq x \leqq y_s$ or $x_{s-1} \leqq x \leqq 1$. Now $\prod_{t=0}^{r} \frac{|x-y_{s+t}|}{|x-x_{s+t}|} = O(1)$, so using (4.10) and the wellknown relation

$$\max_{-1\le x\le 1} \sum_{k=1}^{n} |l_{k,n}(T,x)| = O(\ln n), \tag{4.11}$$

we get

$$\max_{-1\le x\le 1} \sum_{\substack{k=0\\ k\ne s,\dots,s+r+1}}^{n} |l_{k,n}(Y,x)| = O(1) \max_{-1\le x\le 1} \sum_{k=1}^{n} |l_{k,n}(T;x)| = O(\ln n).$$

β. Let $y_s < x < x_{s-1}$. If $x - y_s < x_{s+r} - x$ then we can use the previous estimation. On the other hand, let

$$x - y_s > |x_{s+r} - x| \quad \text{and} \quad \min_{i=-1,\dots,r} |x - x_{s+i}| = |x - x_j|.$$

At first we suppose that $j \geqq s$. Using the formula

$$l_{k,n}(T;x) = \frac{(-1)^{k-1}T_n(x)}{x-x_k} \cdot \frac{\sqrt{1-x_k^2}}{n},$$

we have by (4.4)

$$l_{k,n}(Y,x) = \frac{(-1)^{k-1}T_n(x)\sqrt{1-x_k^2}}{(x-x_k)n} \prod_{t=0}^{r} \frac{x-y_{s+t}}{x-x_{s+t}} \prod_{t=0}^{r} \frac{x_k - x_{s+t}}{x_k - y_{s+t}} =$$

$$= \frac{(-1)^{k-1}T_k(x)\sqrt{1-x_k^2}(x-y_j)}{(x-x_j)(x-x_k)n} \prod_{\substack{t=0\\ t\ne j-s}}^{r} \frac{x-y_{s+t}}{x-x_{s+t}} \prod_{t=0}^{r} \frac{x_k-x_{s+t}}{x_k-y_{s+t}}.$$

We use the relations

(4.12)

$$\frac{|T(x)|}{|x-x_j|} \leqq |T_n'(\xi)| = O(n) \ (y_s < \xi < x_s), \qquad \sqrt{1-x_k^2} = \sin\frac{2k-1}{2n}\pi \sim \frac{k}{n}.$$

Having the notation $x = \cos\vartheta$, (4.2) and (4.3), it is easy to see that

(4.13)
$$|x - x_i| \sim \frac{|j-i|}{n} \quad (i \neq j), \qquad |x - y_{s+t}| = O\left(\frac{1}{n}\right) \qquad (t = 0, \dots, r).$$

By these estimations

$$|l_{k,n}(Y, x)| = O(1) n \frac{n}{|j-k|} \frac{k}{n^2} \frac{1}{n^{r+1}} n^r = O\left(\frac{1}{|j-k|}\right).$$

So

$$\max_{-1 \leqq x \leqq 1} \sum_{k \neq s, \dots, s+r-1} |l_{k,n}(Y, x)| = O(1) \sum_{j \neq k} \frac{1}{|k-j|} = O(\ln n).$$

Finally, let $j = s-1$. Now $(x - y_{s+i})(x - x_{s+i})^{-1} = O(1)$ $(i = 0, \dots, r)$, so we get (4.9) as in the case α.

4.23. To prove another important property of Y let $4\varphi_n = x_{s+r+1} - x_{s+r+2}$,

(4.14) $$J_{1,n} = [x_{s+r+2} + \varphi_n, x_{s+r+1} - \varphi_n], \; J_{2,n} = [x_{s+r+1} + \varphi_n, x_{s+r} - \varphi_n],$$

(4.15) $$I_n = J_{1,n} \cup J_{2,n}.$$

Now we prove

LEMMA 4.2. *By the previous notations*

(4.16) $$|l_{s+i,n}(Y, x)| \sim \frac{1}{(n\rho_n)^{r+1}} \quad \textit{for} \quad x \in I_n \quad \textit{and} \quad 0 \leqq i \leqq r+1.$$

At first we suppose $0 \leqq i \leqq r$. For these indices

$$|T_n(y_{s+i})| = \left|\cos\frac{2s+2r+1}{2}\pi \cos n(r+1-i)\rho_n\pi + \right.$$

$$\left. + \sin\frac{2s+2r+1}{2}\pi \sin n(r+1-i)\rho_n\pi\right| = |\sin n(r+1-i)\rho_n\pi| \sim n\rho_n$$

$$(0 \leqq i \leqq r).$$

Further, it is easy to see that for $x \in I_n$ we have $|T_n(x)| \geqq c_1 > 0$, $|x - y_l| \sim |x - x_l|$ $(1 \leqq l \leqq n)$, $|y_{s+i} - y_{s+t}| \sim \rho_n$ $(0 \leqq i, t \leqq r, \; i \neq t)$, $|y_{s+i} - x_{s+t}| \sim \frac{1}{n}$ $(0 \leqq i, t \leqq r)$ and $|x - x_{s+i}| \sim \frac{1}{n} (0 \leqq i \leqq r)$.

By these estimations

$$|l_{s+i,n}(Y, x)| \sim \frac{n}{n\rho_n} \frac{1}{\rho_n^r} \frac{1}{n^{r+1}} \sim \frac{1}{(n\rho_n)^{r+1}} \qquad (x \in I_n, \; 0 \leqq i \leqq r).$$

If $i = r+1$ we can use the relations

$$|l_{s+r+1,n}(T; x)| \sim 1 \quad (x \in I_n), \quad |x_{s+r+1} - x_{s+t}| \sim \frac{1}{n} \quad (t = 0, \dots, r)$$

and

$$|x_{s+r+1} - y_{s+t}| \sim \rho_n \ (0 \leqq t \leqq r).$$

So

$$|l_{s+r+1,n}(Y, x)| \sim 1 \cdot 1 \cdot \frac{1}{n^{r+1}\rho_n^{r+1}} \quad (x \in I_n),$$

as we stated.

4.24. Using the above arguments and the „O" where it is necessary we can obtain

LEMMA 4.3. *For an arbitrary point of* $[-1; 1]$

$$|l_{s+i}(Y, x)| = O\left(\frac{1}{n^{r+1}\rho_n^{r+1}}\right) \quad (x \in [-1, 1], 0 \leqq i \leqq r+1). \tag{4.17}$$

4.3. Introducing the notations

$$\lambda_n(L, X) \overset{\text{def}}{=} \max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_{k,n}(X, x)|,$$

$$\lambda_n(H, X) \overset{\text{def}}{=} \max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |h_{k,n}(X, x)|$$

we have

LEMMA 4.4. *For our special matrix Y*

$$\lambda_n(L, Y) \sim \frac{1}{(n\rho_n)^{r+1}} \quad (n \geqq 4r+1), \tag{4.18}$$

$$\lambda_n(H, Y) \geqq \frac{c_2}{(n\rho_n)^{2r+2}} \quad (n \geqq 4r+1;\ c_2 > 0). \tag{4.19}$$

Indeed, (4.18) immediately follows from Lemmas 4.1–4.3. For (4.19) we remark sign $(w_n - x_{s+r+1}) = -\text{sign}\,(w_n^* - x_{s+r+1})$ if $w_n \in J_{1,n}$ and $w_n^* \in J_{2,n}$. So for suitable $\eta_n \in I_n$ $|v_{s+r+1,n}(Y, \eta_n)| \geqq 1$, i.e.

$$\lambda_n(H, Y) \geqq \max_{-1 \leqq x \leqq 1} |h_{r+s+1}(Y, x)| = \max_{-1 \leqq x \leqq 1} \left| \left[1 - \frac{\omega''(x_{s+r+1})}{\omega'(x_{s+r+1})}(x - x_{s+r+1})\right] \cdot \right.$$

$$\left. \cdot\, l_{s+r+1}^2(Y, x) \right| \geqq l_{s+r+1}^2(Y, \eta_n) \sim \frac{1}{(n\rho_n)^{2r+2}}.$$

4.4. Now we have the main tools to prove our theorems. To prove (3.2) and (3.5) we shall apply the methods and theorems of [3] and [4].

4.41. First we construct the functions $g_n(x)$ as follows

$$g_n(y_i) = \begin{cases} \operatorname{sign} v_{s+r+1}(Y, \eta_n) & \text{if } \ i = s+r+1, \\ 0 & \text{otherwise } \ (n \geqq 4r+1). \end{cases} \tag{4.20}$$

In the interval (y_{i+1}, y_i) we define $g_n(x)$ as the Hermite interpolatory polynomials of degree $2k+1$ $(k \geqq 0)$ which is equal to $g_n(y_{i+1})$ or $g_n(y_i)$ at the end-points, respectively, and at these end-points $g_n'(y_i) = g_n'(y_{i+1}) = \ldots = g_n^{(k)}(y_i) = g_n^{(k)}(y_{i+1}) = 0$. If $x \in [-1, y_n] \cup [y_1, 1]$ let $g_n(x) = 0$.

Let us notice that we can choose a set $\mathbf{N}_1 \subseteqq \mathbf{N}$ such that

$$g_n(x) \cdot g_m(x) = 0 \qquad (x \in [-1, 1],\ n, m \in \mathbf{N}_1,\ n \neq m) \tag{4.21}$$

(see (4.1) and the definition of $g_n(x)$).

4.42. So using the correspondences $\lambda_n = (n\rho_n)^{-2r-2}$, $e_n = 1$, $\delta_n = \rho_n$, $T_n = H_n$, $U_n = E$ (unit operator), $z_n = \eta_n$ and $C_p^{\{a,b\}}(\omega_m) = C(\omega_{k+1}) (= \{f(x);\ \omega(f^{(k)}; t) \leqq \leqq a(f)\omega(t)$ where $\omega(t)$ is a modulus of continuity; $k \geqq 0\})$[4] then by [3, Theorem 2.1] we have that for a certain

$$h(x) \overset{\text{def}}{=} c_3 \sum_{n \in \mathbf{N}_2} \omega_{k+1}(\rho_n) g_n(x) \qquad (c_3 > 0,\ \mathbf{N}_2 \subseteq \mathbf{N}_1) \tag{4.22}$$

$h(x) \in C(\omega_{k+1})$ and

$$H_n(h; Y, \eta_n) - h(\eta_n) > \frac{\omega_{k+1}(\rho_n)}{(n\rho_n)^{2r+2}} \qquad (n \in \mathbf{N}_2), \tag{4.23}$$

where we shall suppose that if $n \in \mathbf{N}_2$ then n is odd, further for $n, m \in \mathbf{N}_2$, $n < m$ we have

$$y_{s,n} < y_{s+r+2,m} < y_{s-1,n} \qquad (= y_{s-1,m} = 0). \tag{4.25}$$

(To obtain the conditions A, B and C of [3, Theorem 2.1], see (4.1), (4.24) and [5] (for A); (4.21) and [4, Lemma 2.1] (for B); finally (4.19) and [3, C2] (for C)).

4.43. On the other hand, for any $f(x) \in C(\omega_{k+1})$ we have by usual argument and (4.18)

$$\| L_n(f; Y, x) - f(x) \|_{[-1,1]} = O\left(\frac{1}{(n\rho_n)^{r+1}}\right) \omega_{k+1}\left(f; \frac{1}{n}\right) (n \in \mathbf{N}). \tag{4.24}$$

4.44. To prove the estimations (3.1), (3.2), (3.3) and (3.5) we shall choose the suitable r and ρ_n (depending on α and k) as follows.

[4] We define $\omega_{k+1}(t) = t^k \omega(t)$.

k	α	ρ_n	r	$\|\| L_n(f; Y, x) - f(x) \|\|_{[-1,1]}$	$H_n(h; Y, \eta_n) - h(\eta_n)$
0	$0 < \alpha < 1$	$\frac{\ln n}{n^{1+\alpha}}$	0	$O\left(\frac{1}{\ln n}\right)$	$\frac{n^{\alpha(1-\alpha)}}{(\ln n)^{2-\alpha}}$
0	1	$\frac{1}{n^{1.4}}$	1	$O\left(\frac{1}{n^{0.2}}\right)$	$n^{0.2}$
1	$0 < \alpha < 1$	$\frac{1}{n^{1+\varepsilon}}$ $0 < \varepsilon = \frac{1}{q} < \alpha$	q	$O\left(\frac{1}{n^{\alpha-\varepsilon}}\right)$	$n^{(1-\alpha)(1+\varepsilon)}$
1	1	$\frac{1}{n^{1.6}}$	2	$O\left(\frac{1}{n^{0.2}}\right)$	$n^{0.4}$
$\geqq 2$	$0 < \alpha \leqq 1$	$\frac{1}{n^{1+\varepsilon}}$ $0 < \varepsilon = \frac{1}{q} <$ $< \min\left(\alpha, \frac{1}{2k}\right)$	$q \cdot k$	$O\left(\frac{1}{n^{\alpha-\varepsilon}}\right)$	$n^{k-\alpha+\varepsilon(2-\alpha-k)} >$ $> n^{k-\alpha-k\varepsilon} >$ $> n^{0.5}$

Table 1

The first two rows prove (3.1) and (3.2); the remaining ones give (3.3) and (3.5). At the last two columns we used (4.23) and (4.25) and the restrictions of the quoted formulas.

4.45. By (4.1) and (4.15) obviously $\lim_{x \to \infty} \eta_n = 0$.

4.46. To get the set E mentioned in Theorem 3.1 we remark that for $k = 0$ $g_n'(x)$ does not exist on $E_n = \{y_{s+i,n};\, i = r, \dots, r+2\}$; so by (4.22) and (4.24) $E = \bigcup_{n \in \mathbf{N}_2} E_n$.

4.47. Finally, let us prove (3.4). Denoting by Δ_n the interval where $g_n(x) \neq 0$ we get for a fixed $m \in \mathbf{N}_2$ (by (4.22), (4.24) and the definition of $g_n(x)$)

$$h'(y_{i,m}) = c_3 \sum_{n \in \mathbf{N}_2} \omega_{k+1}(\rho_n) g_n'(y_{i,m}) = \tag{4.26}$$

$$= c_3 \sum_{\substack{n<m \\ n \in \mathbf{N}_2}} \omega_{k+1}(\rho_n) g_n'(y_{i,m}) = O(1) \sum_{\substack{n<m \\ n \in \mathbf{N}_2}} \omega(\rho_n) \rho_n \rho_n^{-1} = O(1)$$

if $\mathbf{N}_2$ is lacunary enough. Further by (4.24)

$$h'(y_{i,m}) = 0 \quad \text{if} \quad i \leqq s(m) + r + 2.$$

So as above

$$\left|\sum_{i=1}^{m} h'(y_{i,m})(x - y_{i,m})l_{i,m}^2(Y, x)\right| =$$

$$= O(1) \sum_{i \geq s(m)+r+3} |x - y_{i,m}|\, l_{i,m}^2(Y, x) = O\left(\frac{\ln n}{n}\right).$$

Here we used that

$$\sum_{i \geq s+r+3} h_{i,m}(Y, x) = O(1) \sum_{i \geq s+r+3} h_{i,m}(T, x) = O\left(\frac{\ln n}{n}\right)$$

(see (4.4)). Now by (2.9) we have (3.4).

5. Remarks. 5.1. We can prove our theorems using more general functions $\omega_{k+1}(t)$. E.g., one additional row to Table 1 could be

$$0 \quad (\omega(t)) = \frac{1}{|\ln t|^{4/3}} \quad \frac{1}{n \ln n} \quad 0 \quad \frac{1}{(\ln n)^{1/3}} \quad (\ln n)^{2/3}$$

5.2. By similar arguments we can prove our theorems not only for the point 0, but for arbitrary fixed $x_0 \in [-1, 1]$.

5.3. We can obtain similar results for the trigonometric case also.

5.4. Our procedure applied for the Chebyshev matrix T to gain Y is similar to the method used by ERDŐS and TURÁN (see [6, 10—12]).

5.5. It would be interesting to have similar theorems for analytic functions. The main difficulty is that we do not have a statement like Theorem 2.1 from [3].

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15
HUNGARY

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EINE BEMERKUNG ZUM SATZ VON FAVARD ÜBER ORTHOGONALE POLYNOMSYSTEME

Von

K. ENDL (Gießen)

§ 1. Einleitung

Ein orthogonales Polynomsystem $\{P_n(x)\}_0^\infty$ mit der Normierung

$$P_n(x) = x^n + \sum_{\nu=0}^{n-1} p_\nu^{(n)} x^\nu \qquad (p_\nu^{(n)} \in \mathbf{R}; n \in \mathbf{N}_0) \tag{1.1}$$

genügt bekanntlich einer dreigliedrigen Rekursionsformel ([1], S. 30—31)

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x) \tag{1.2}$$

$$(\alpha_n, \beta_n \in \mathbf{R}, \beta_n > 0; n \in \mathbf{N}_0; P_0(x) \equiv 1, P_{-1}(x) \equiv 0).$$

Umgekehrt gilt ([2]; für einen ausführlichen Beweis [3], S. 64 ff) der

SATZ VON FAVARD. *Genügt ein Polynomsystem* $\{P_n(x)\}_0^\infty$ *der Form* (1.1) *einer dreigliedrigen Rekursionsformel* (1.2) *so ist es bezüglich einer Riemann—Stieltjes Belegung* $d\mu(x)$ *orthogonal.*

G. Alexits warf das Problem auf, Funktionensysteme $\{f_n(x)\}_0^\infty$ zu untersuchen, welche mit einer beliebigen festen Funktion $\psi(x)$ einer Rekursionsformel

$$f_{n+1}(x) = (\psi(x)-\alpha_n)f_n(x) - \beta_n f_{n-1}(x) \tag{1.3}$$

$$(\alpha_n, \beta_n \in \mathbf{R}, \ \beta_n > 0; \ n \in \mathbf{N}_0; \ f_0(x) \equiv 1, \ f_{-1}(x) \equiv 0)$$

genügen.

Man kann sofort Beispiele von solchen Systemen angeben: Ist ein beliebiges Polynomsystem (1.1) gegeben, das einer Rekursionsformel (1.2) genügt, so gewinnt man mit $P_n(\Psi(x)) := f_n(x)$ $(n \in \mathbf{N}_0)$ sofort ein System, das (1.3) genügt. Es wird sich ohne Mühe herausstellen, daß jedes System $\{f_n\}_0^\infty$, das (1.3) genügt, auch so dargestellt werden kann (§2, Satz 2.2).

Wir behandeln hier das Problem, ob jedes Funktionensystem $\{f_n(x)\}_0^\infty$, das einer Rekursionsformel (1.3) genügt, in irgendeinem Sinn orthogonal ist. Beispiele hierzu lassen sich sofort durch Substitution mit passenden Funktionen $\Psi(x)$ aus orthogonalen Polynomsystemen gewinnen. Doch ist das Problem im Fall einer komplizierten, etwa nicht Lebesgue-meßbaren Funktion $\Psi(x)$ nicht auf diese Art zu lösen. Wir werden zeigen, daß im allgemeinen Fall Orthogonalität vorliegt, falls man diesen Begriff genügend allgemein faßt. Es wird sich herausstellen, daß — unabhängig von Funktionensystemen oder gar Polynomsystemen — der Satz von Favard eine Äquivalenzaussage zwischen einer dreigliedrigen Rekursionsformel und der Existenz eines linearen Funktionals in einer Algebra über dem Körper der reellen Zahlen beinhaltet. Es ist vielleicht nicht ohne Interesse, diesen abstrakten Zusammenhang klar herauszustellen, der in einer Richtung ([3], S. 64 ff.) schon im klassischen Fall vorgezeichnet ist.

Es sei X eine kommutative, nullteilerfreie Algebra mit Einselement über $\mathbf{R}$. Das Einselement von X werde mit e bezeichnet. Wir betrachten im folgenden Systeme (Folgen) $\{f_n\}_0^\infty$ aus X, wobei wir ein System als trivial bezeichnen, falls $f_n = 0$ für $n \in \mathbf{N}_0$.

In Analogie zur Definition von G. Alexits geben wir

DEFINITION 1.1. Ein System $\{f_n\}_0^\infty$ aus X heißt polynomartig, falls gilt:

$$f_k f_l = \sum_{\nu=0}^{k+l} a_\nu^{k,l} f_\nu \qquad (a^{k,l} \in \mathbf{R};\ k, l \in \mathbf{N}_0). \tag{1.4}$$

Es heißt echt polynomartig, falls $a_{k+l}^{k,l} \neq 0$ $(k, l \in \mathbf{N}_0)$, und normiert, falls $a_{k+l}^{k,l} = 1$ $(k, l \in \mathbf{N}_0)$.

Das Hauptresultat dieser Arbeit ist im folgenden Satz zusammengefaßt

SATZ 1.1. *Es sei $\{f_n\}_0^\infty$ ein nichttriviales, polynomartiges, normiertes System aus X. Dann sind die folgenden Aussagen äquivalent:*

(1) *Das System $\{f_n\}_0^\infty$ genügt einer dreigliedrigen Rekursionsformel:*

$$f_{n+1} = (f_1 - \alpha_n) f_n - \beta_n f_{n-1} \ (\alpha_n, \beta_n \in \mathbf{R},\ \beta_n > 0;\ n \in \mathbf{N}_0;\ f_0 = e, f_{-1} = 0)$$

(2) *Das System $\{f_n\}_0^\infty$ ist „orthogonal", d. h. es existiert auf X ein eindeutig bestimmtes lineares Funktional μ mit*

$$\mu(f_k f_l) = 0 \qquad k \neq l \qquad (k, l \in \mathbf{N}_0)$$

$$\mu(f_k^2) > 0 \qquad (k \in \mathbf{N}_0).$$

§ 2. Polynomartige Systeme, dreigliedrige Rekursionsformeln

Ein echt polynomartiges System ist genau dann trivial, wenn $f_0 = 0$ ist. Denn sicher ist diese Bedingung notwendig. Sie ist aber auch hinreichend. Denn betrachten wir die Gleichungen

$$f_k f_0 = \sum_{\nu=0}^{k} a_\nu^{k,0} f_\nu \qquad (k \in \mathbf{N}_0)$$

so folgt sofort durch Induktion wegen $a_k^{k,0} \neq 0$ $f_n = 0$ auch für $n = 1, 2 \ldots$.

Nach dieser Vorbemerkung beweisen wir:

SATZ 2.1. *Ein nichttriviales System $\{f_n\}_0^\infty$ aus X ist genau dann echt polynomartig, falls gilt:*

$$f_n = \sum_{\nu=0}^{n} c_\nu^{(n)} (f_1)^\nu \qquad (c_\nu^{(n)} \in \mathbf{R},\ c_n^{(n)} \neq 0;\ n \in \mathbf{N}_0). \tag{2.1}$$

Es ist darüber hinaus genau dann normiert falls $c_n^{(n)} = 1$ $(n \in \mathbf{N}_0)$.

BEWEIS. Wir nehmen an, $\{f_n\}_0^\infty$ sei echt polynomartig. Betrachten wir zuerst (1.4) für $k = l = 0$:

$$f_0 f_0 = a_0^{0,0} f_0 \qquad \text{oder} \qquad f_0(f_0 - a_0^{0,0} e) = 0$$

so folgt wegen $f_0 \neq 0$ und der Nullteilerfreiheit in X:

$$f_0 = a_0^{0,0} e := c_0^{(0)} \cdot f_1^0 \qquad \text{mit} \qquad c_0^{(0)} = a_0^{0,0} \neq 0 .$$

Als nächstes betrachten wir (1.4) für $k = 1$ und $l = 0$:

$$f_1 f_0 = a_0^{1,0} f_0 + a_1^{1,0} f_1. \tag{2.2}$$

Durch Einsetzen von f_0 ergibt sich:

$$f_1 = a_1^{1,0} f_1^0 + \frac{a_1^{1,0}}{a_0^{0,0}} f_1 := \sum_{\nu=0}^{1} c_\nu^{(1)} f_1^0 \qquad \text{mit} \qquad c_1^{(1)} = \frac{a_1^{1,0}}{a_0^{0,0}} \neq 0 .$$

Wir nehmen nun an, (2.1) sei bewiesen für $0, 1, \ldots, n-1$. Betrachten wir (1.4) mit $k = 1$, $l = n - 1$:

$$f_1 f_{n-1} = \sum_{\nu=0}^{n} a_\nu^{1,n-1} f_\nu ,$$

so folgt hieraus:

$$f_n = \frac{1}{a_n^{1,n-1}} f_1 f_{n-1} - \sum_{\nu=0}^{n-1} \frac{a_\nu^{1,n-1}}{a_n^{1,n-1}} f_\nu = \frac{1}{a_n^{1,n-1}} f_1 \sum_{\nu=0}^{n-1} c_\nu^{(n-1)} f_1^\nu - \sum_{\nu=0}^{n-1} \frac{a_\nu^{1,n-1}}{a_n^{1,n-1}} \sum_{k=0}^{\nu} c_k^{(\nu)} f_1^\nu =$$

$$= \sum_0^n c_\nu^{(n)} f_1^\nu \qquad \text{mit} \qquad c_n^{(n)} = \frac{c_{n-1}^{(n-1)}}{a_n^{1,n-1}} \neq 0 .$$

Ist $\{f_n\}_0^\infty$ normiert, so ist $c_0^{(0)} = a_0^{0,0} = 1$, und ebenfalls $c_1^{(1)} = \dfrac{a_1^{1,0}}{a_0^{0,0}} = 1$. Nehmen wir in unserer Induktion $c_\nu^{(\nu)} = 1$ für $0, 1, \ldots, n-1$ an, so folgt $c_n^{(n)} = \dfrac{c_{n-1}^{(n-1)}}{a_n^{1,n-1}} = 1$. Ist umgekehrt $\{f_n\}_0^\infty$ von der Form (2.1), so bilden wir

$$f_k f_l = \left(\sum_{\kappa=0}^{k} c_\kappa^{(k)} f_1^\kappa \right) \cdot \left(\sum_{\lambda=0}^{l} c_\lambda^{(l)} f_1^\lambda \right) := \sum_{m=0}^{k+l} d_m^{k,l} f_1^m$$

wobei $d_{k+l}^{k,l} = c_k^{(k)} \cdot c_l^{(l)} \neq 0$ ist. Da die Matrix $(c_\nu^{(n)})$ invertierbar ist, folgt mit

$$f_1^m = \sum_{\nu=0}^{m} c_\nu^{-1(m)} f_\nu , \qquad f_k f_l = \sum_{m=0}^{k+l} d_m^{k,l} \sum_{\nu=0}^{m} c_\nu^{-1(m)} f_\nu := \sum_{\nu=0}^{k+l} a_\nu^{k,l} f_\nu$$

wobei $a_{k+l}^{k,l} = d_{k+l}^{k,l} \cdot c_{k+l}^{-1(k+l)} \neq 0$ ist. Gilt $c_n^{(n)} = 1$ und damit $c_n^{-1(n)} = 1$ $(n \in \mathbf{N}_0)$, so ist $d_{k+l}^{k,l} = c_k^{(k)} c_l^{(l)} = 1$ $(k, l \in \mathbf{N}_0)$, also auch $a_{k+l}^{k,l} = d_{k+l}^{k,l} c_{k+l}^{-1(k+l)} = 1$ $(k, l \in \mathbf{N}_0)$.

BEMERKUNG 1. Im Fall eines echt polynomartigen Systems werden also zur Herleitung von (2.1) nur die Gleichungen $k = l = 0$ und $k = 1$, $l = 0, 1, 2, \ldots$ benötigt.

BEMERKUNG 2. Für f_1 gilt nach (2.2):

$$f_1(a_0^{0,0} - a_1^{1,0}) = a_0^{1,0} \cdot a_0^{0,0} \cdot e.$$

Ist also $a_0^{0,0} \neq a_1^{1,0}$, so folgt

$$f_1 = \frac{a_0^{1,0} a_0^{0,0}}{a_0^{0,0} - a_1^{1,0}} \cdot e$$

und $\{f_n\}_0^\infty$ besteht aus Vielfachen von e, falls noch $a_0^{1,0} \neq 0$. Ist $a_0^{0,0} = a_1^{1,0}$, so folgt $a_0^{1,0} = 0$ und f_1 ist beliebig.

Wir untersuchen noch Systeme $\{f_n\}_0^\infty$ die einer dreigliedrigen Rekursionsformel genügen.

SATZ 2.2. *Es sei $\{f_n\}_0^\infty$ ein System aus X das mit einem festen Element $\psi \in X$ einer Rekursionsformel*

$$f_{n+1} = (\psi - \alpha_n)f_n - \beta_n f_{n-1} \quad (\alpha_n, \beta_n \in \mathbf{R}, \beta_n > 0;\ n \in \mathbf{N}_0;\ \psi \neq 0;\ f_0 = e,\ f_{-1} = 0)$$

genügt. Dann ist $\{f_n\}_0^\infty$ polynomartig und normiert, und es gilt

$$f_n = \psi^n + \sum_0^{n-1} d_\nu^{(n)} \psi^\nu \qquad (n \in \mathbf{N}_0).$$

Zum Beweis schreiben wir die Rekursionsformel in der Form

$$\psi f_n = \beta_n f_{n-1} + \alpha_n f_n + f_{n+1}.$$

Wegen $f_1 = \psi - \alpha_0 e$ folgt:

$$f_1 f_n = \beta_n f_{n-1} + (\alpha_n - \alpha_0) f_n + f_{n+1}.$$

Da auch $f_0 f_0 = 1 f_0$ gilt, folgt nach dem Beweis von Satz 2.1 mit Bemerkung 1:

$$f_n = \sum_{\nu=0}^{n} c_\nu^{(n)} f_1^\nu = \sum_{\nu=0}^{n} c_\nu^{(n)} (\psi - \alpha_0 e)^\nu = \psi^n + \sum_{\nu=0}^{n-1} d_\nu^{(n)} \psi^\nu,$$

da $c_n^{(n)} = 1$ $(n \in \mathbf{N}_0)$ ist. Hieraus folgt nach Satz 2.1, daß das System $\{f_n\}_0^\infty$ polynomartig und normiert ist.

§ 3. Beweis von Satz 1.1

Wir nehmen zuerst an, es gelte (1). Zum Existenznachweis des linearen Funktionals definieren wir die Momente von f_1^n. Wir setzen $\mu(f_0) = \mu(e) = \mu(f_1^0) = 1$. Nehmen wir an, daß $\mu(f_1^0), \ldots, \mu(f_1^n)$ schon bestimmt sind, so ist wegen der gewünschten Linearität für ein beliebiges Polynom in f_1:

$$\mu\left(\sum_{\nu=0}^{n} a_\nu f_1^\nu\right) = \sum_{\nu=0}^{n} a_\nu \mu(f_1^\nu).$$

Das nächste Moment $\mu(f_1^{n+1})$ ist jetzt schon eindeutig bestimmt. Wegen der Forderung $\mu(f_{n+1}) = \mu(f_0 f_{n+1}) = 0$ und

$$f_{n+1} = \sum_{\nu=0}^{n} c_\nu^{(n+1)} f_1^\nu + f_1^{n+1}$$

folgt nämlich

$$\mu(f_1^{n+1}) = \mu(f_{n+1} + (f_1^{n+1} - f_{n+1})) = \mu(f_{n+1}) + \mu(f_1^{n+1} - f_{n+1}) = -\sum_{\nu=0}^{n} c_\nu^{(n+1)} \mu(f_1^\nu).$$

Hierdurch ist μ auf der durch $\{f_n\}_0^\infty$ bzw. durch die Potenzen $\{f_1^n\}_0^\infty$ erzeugten Unteralgebra und damit durch Fortsetzung auf X definiert.

Wir haben nur noch die zwei Bedingungen in (2) zu verifizieren. Es genügt zu zeigen, daß

$$\mu(f_k f_l) = 0 \qquad \text{für} \qquad 0 \leqq k < l = 1, 2, \ldots .$$

Dies ist wegen $f_k = \sum_{\nu=0}^{k} c_\nu^{(k)} f_1^\nu$ und $c_k^{(k)} = 1$ äquivalent zu

$$\mu(f_1^k f_l) = 0 \qquad \text{für} \qquad 0 \leqq k < l = 1, 2, \ldots .$$

Wir führen den Beweis durch Induktion nach k. Für $k = 0$ ist die Behauptung richtig wegen $f_1^0 = f_0$. Es gelte nun für alle κ mit $0 \leqq \kappa \leqq k$ und $l > k$

$$\mu(f_1^\kappa f_l) = 0 .$$

Nun folgt aus der Rekursionsformel für $l \in \mathbf{N}_0$:

$$f_{l+1} = f_1 f_l - \alpha_l f_l - \beta_l f_{l-1}$$

nach Multiplikation mit f_1^k:

$$f_1^{k+1} f_l = f_1^k f_{l+1} + \alpha_l f_1^k f_l + \beta_l f_1^k f_{l-1}$$

und hieraus

$$\mu(f_1^{k+1} f_l) = \mu(f_1^k f_{l+1}) + \alpha_l \mu(f_1^k f_l) + \beta_l \mu(f_1^k f_{l-1}). \tag{3.1}$$

Für $l > k+1$ sind alle drei Terme auf der rechten Seite Null, d. h. es gilt für $l >$ $> k+1$

$$\mu(f_1^{k+1} f_l) = 0.$$

Es bleibt noch zu zeigen, daß

$$\mu(f_n^2) > 0 \qquad (n \in \mathbf{N}_0).$$

Für $n = 0$ ist $\mu(f_0^2) = \mu(f_0) = 1$. Für $n \geqq 1$ folgt aus (3.1) mit $k = n-1$, $l = n$:

$$\mu(f_1^n f_n) = \beta_n \cdot \mu(f_1^{n-1} f_{n-1}) = \beta_n \beta_{n-1} \ldots \beta_1 \cdot \mu(f_1^0 f_0) = \prod_{\nu=1}^{n} \beta_\nu$$

und hieraus wegen $\mu((f_1^n - f_n)f_n) = 0$:

$$\mu(f_n^2) = \mu((-f_1^n + f_n + f_1^n)f_n) = -\mu((f_1^n - f_n)f_n) + \mu(f_1^n f_n) = \prod_{\nu=1}^{n} \beta_\nu > 0.$$

Es sei nun (2) erfüllt. Wir betrachten die Gleichungen

$$f_1 f_n = \sum_{\nu=0}^{n+1} a_\nu^{1,n} f_\nu \qquad (n \in \mathbf{N}). \tag{3.2}$$

Für $n = 1$ gilt

$$f_1 f_1 = a_0^{1,1} f_0 + a_1^{1,1} f_1 + f_2,$$

und es folgt für $n = 1$ die gesuchte Formel

$$f_2 = (f_1 - a_1^{1,1}) f_1 - a_0^{1,1} f_0 := (f_1 - \alpha_1) f_1 - \beta_1 f_0.$$

Es sei nun $n \geqq 2$. Wir multiplizieren (3.2) mit f_k $(0 \leqq k \leqq n-2)$

$$f_1 f_k f_n = \sum_{\nu=0}^{1+n} a_\nu^{1,n} f_\nu f_k$$

und erhalten durch Anwendung des linearen Funktionals μ

$$\mu(f_1 f_k f_n) = \sum_{\nu=0}^{1+n} a_\nu^{1,n} \mu(f_\nu f_k) = a_k^{1,n} \mu(f_k^2). \tag{3.3}$$

Nun ist

$$f_1 f_k = \sum_{\nu=0}^{1+k} a_\nu^{1,k} f_\nu$$

d. h.

$$f_1 f_k f_n = \sum_{\nu=0}^{1+k} a_\nu^{1,k} f_\nu f_n.$$

Es folgt, da $\nu \leqq 1 + k \leqq n - 1$

$$\mu(f_1 f_k f_n) = \sum_{\nu=0}^{1+k} a_\nu^{1,k} \mu(f_\nu f_n) = 0. \tag{3.4}$$

Also ist mit (3.3) und (3.4):

$$a_k^{1,n} = 0 \qquad (0 \leqq k \leqq n-2).$$

Es gilt also

$$f_1 f_n = a_{n-1}^{1,n} f_{n-1} + a_n^{1,n} f_n + f_{n+1}$$

oder

$$f_{n+1} = (f_1 - a_n^{1,n}) f_n - a_{n-1}^{1,n} f_{n-1} := (f_1 - \alpha_n) f_n - \beta_n f_{n-1}.$$

Multiplizieren wir die letzte Gleichung für $n \geqq 1$ mit f_{n-1}, so erhalten wir

$$\mu(f_{n-1}f_{n+1}) = \mu(f_1 f_{n-1} f_n) - \alpha_n \mu(f_{n-1}f_n) - \beta_n \mu(f_{n-1}^2)$$

und hieraus wegen der Orthogonalität

$$\beta_n \mu(f_{n-1}^2) = \mu(f_1 f_{n-1} f_n) = \mu(f_n^2)$$

da

$$\mu(f_1 f_{n-1} f_n) = \mu\left(\sum_{\nu=0}^{n} a_\nu^{1,n-1} f_\nu \cdot f_n\right) = \mu(f_n^2).$$

Hieraus folgt aber $\beta_n > 0$.

Die vorangehenden Definitionen und Sätze gelten unverändert auch für den Fall, daß X eine (i. a. nicht nullteilerfreie) Funktionen-Algebra ist:

$$X = X(D) = \{f \mid f: D \to \mathbf{R};\ D \in \mathbf{R}\}.$$

Hierbei definieren wir eine Folge $\{f_n(x)\}_0^\infty$ als nichttrivial, wenn $f_0(x) \neq 0$ auf D. (Dies ist keine Einschränkung, da für die uns interessierenden echt polynomähnlichen Systeme an Punkten $\bar{x}$ mit $f_0(\bar{x}) = 0$ schon $f_n(\bar{x}) = 0$ ($n \in \mathbf{N}$) folgt). Die Sätze 2.1–2 folgen jetzt durch punktweise Anwendung dieser Sätze, wobei **R** die Rolle der nullteilerfreien Algebra spielt.

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MATHEMATISCHES INSTITUT
DER JUSTUS LIEBIG-UNIVERSITÄT
6300 GIESSEN, IHERINGSTR. 6
BUNDESREPUBLIK DEUTSCHLAND

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ОБ ОДНОМ МЕТОДЕ ПРИБЛИЖЕНИЯ НЕПРЕРЫВНЫХ ПЕРИОДИЧЕСКИХ ФУНКЦИЙ ТРИГОНОМЕТРИЧЕСКИМИ МНОГОЧЛЕНАМИ

О. КИШ (Будапешт) и ХО ТХО КАУ (Ханой)

Введем следующие обозначения: $g(t)$ вещественная, 2π—периодическая и непрерывная функция вещественной переменной; m и r положительные целые числа; n целая часть числа $\frac{m}{2}$;

$$d_m(t) = 1 + 2\sum_{i=1}^{n} \cos it \qquad (m = 1, 3, 5, \dots); \tag{1}$$

$$d_m(t) = 1 + 2\sum_{i=1}^{n-1} \cos it + \cos nt \qquad (m = 2, 4, 6, \dots); \tag{2}$$

τ вещественное, k неотрицательное целое число;

$$S_{0,r}(t, \tau) = \frac{1}{mr} \sum_{i=0}^{mr-1} g\left(\tau + \frac{2i\pi}{mr}\right) d_m\left(t - \tau - \frac{2i\pi}{mr}\right); \tag{3}$$

$$S_{k,r}(t, \tau) = \frac{1}{2^k} \sum_{i=0}^{k} \binom{k}{i} S_{0,r}\left(t + \frac{2i-k}{m}\pi, \tau\right). \tag{4}$$

Заметим, что функция $S_{k,r}(t, \tau)$ очевидно есть тригонометрический многочлен не выше чем n—ого порядка. При $k = 0$ и нечетном m он получается методом наименьших квадратов. Выражение $S_{0,1}(t, \tau)$ интерполирует функцию $g(t)$ в узлах $\tau + \frac{2i\pi}{m}$ $(i = 0, \pm 1, \pm 2, \dots)$. Тригонометрический многочлен $S_{1,r}(t, \tau)$ ввел С. Н. Бернштейн в статье [1].

В настоящей работе мы оценим погрешность $|S_{k,r}(t, \tau) - g(t)|$ при $k = 0, 1$ и 2.

Обозначим через $\omega(g, t)$ модуль непрерывности функции $g(t)$ и положим

$$\lambda_{m,r} = \frac{1}{mr} \max_{t \in (\infty, \infty)} \sum_{i=0}^{mr-1} \left| d_m\left(t - \frac{2i\pi}{mr}\right)\right|. \tag{5}$$

Для случая $k = 0$ будет доказана

Теорема 1. *Выполняется тождество*

$$\max_{\substack{t \in (\infty, \infty) \\ g(t) \neq \text{const}}} \frac{|S_{0,r}(t, \tau) - g(t)|}{\omega\left(g, \frac{2\pi}{m}\right)} = \frac{1}{2} + \frac{1}{2}\lambda_{m,r}. \tag{6}$$

При $r = 1$ этот результат впервые доказали А. А. Лигун, Х. Брасс и Р. Гюнттнер в статьях [5] и [2].

Теорема 2. *Если m четное число, то*

$$\lambda_{m,r} = \frac{1}{nr} \sum_{i=1}^{r} \sin \frac{2i-1}{2r} \pi \sum_{j=0}^{n-1} \operatorname{ctg} \frac{2jr+2i-1}{2mr} \pi . \tag{7}$$

Для $r = 1$ формулу (7) можно записать в виде

$$\lambda_{m,1} = \frac{1}{n} \sum_{j=1}^{n} \operatorname{ctg} \frac{2j-1}{2m} \pi . \tag{8}$$

Это тождество впервые доказали Х. Эхлих и К. Зеллер в работе [3]. В статье [4] Р. Гюнттнер опубликовал неравенство

$$0{,}962 + \frac{2}{\pi} \ln n < \lambda_{m,1} \leqq 1 + \frac{2}{\pi} \ln n \qquad (m = 2, 4, 6, \ldots). \tag{9}$$

При $r = 2$ из (7)—(9) получаем:

$$\lambda_{m,2} = \frac{1}{\sqrt{2}m} \sum_{j=1}^{m} \operatorname{ctg} \frac{2j-1}{4m} \pi = \frac{1}{\sqrt{2}} \lambda_{2m,1} \leqq \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{\pi} \ln m .$$

Эта оценка примерно в $\sqrt{2}$ раз лучше чем (9).

Теорема 3. *Если m нечетное число, то*

(10)

$$\lambda_{m,r} = \frac{1}{mr} \sum_{i=1}^{r} \sin \frac{2i-1}{2r} \pi \left(2 \sum_{j=1}^{n-1} \operatorname{cosec} \frac{2jr+2i-1}{2mr} \pi + \operatorname{cosec} \frac{2nr+2i-1}{2mr} \pi \right) .$$

Для $r = 1$ из (10) получается доказанное в [3] соотношение

$$\lambda_{m,1} = \frac{1}{m} + \frac{2}{m} \sum_{j=1}^{n} \operatorname{cosec} \frac{2j-1}{2m} \pi .$$

В [4] опубликовано неравенство

$$0{,}962 + \frac{2}{\pi} \ln m < \lambda_{m,1} \leqq 1 + \frac{2}{\pi} \ln m \qquad (m = 1, 3, 5, \ldots).$$

При $r = 2$ из (10) следует:

$$\lambda_{m,2} = \frac{1}{\sqrt{2}m} \sum_{j=1}^{m} \operatorname{cosec} \frac{2j-1}{4m} \pi .$$

Это число примерно и $\sqrt{2}$ раз меньше чем $\lambda_{m,1}$.

Из нижеследующих формул (31)—(33) и (36) следуют соотношения

$$\lambda_{m,r} = \frac{1}{m} + \frac{2}{mr}\sum_{i=1}^{n} \operatorname{tg}\frac{i\pi}{m}\operatorname{cosec}\frac{i\pi}{mr} \qquad (m = 1, 3, 5, \dots);$$

$$\lambda_{m,r} = \frac{1}{n} + \frac{2}{nr}\sum_{i=1}^{\frac{n-1}{2}} \operatorname{tg}\frac{i\pi}{n}\operatorname{cosec}\frac{i\pi}{nr} \qquad (m = 2, 6, 10, \dots);$$

$$\lambda_{m,r} = \frac{2}{nr}\sum_{i=1}^{n/2} \operatorname{tg}\frac{2i-1}{m}\pi\operatorname{cosec}\frac{2i-1}{mr}\pi \qquad (m = 4, 8, 12, \dots).$$

Отсюда видно, что при фиксированном m последовательность $\lambda_{m,r}$ $(r = 1, 2, 3, \dots)$ убывает.

Для случая $k = 1$ будет доказана

Теорема 4. *Если m четное число, то*

$$(11) \qquad |S_{1,r}(t, \tau) - g(t)| \leqq \frac{1}{2}\omega\left(g, \frac{\pi}{n}\right) + a_{m,r}\omega\left(g, \frac{\pi}{m}\right),$$

где

$$(12) \qquad a_{m,r} = \frac{1}{2} + \frac{1}{nr}\sum_{1\leqq i\leqq n/2} \operatorname{ctg}\frac{2i-1}{m}\pi\operatorname{cosec}\frac{2i-1}{mr}\pi\left(1 - b_r\cos\frac{2i-1}{m}\pi\right),$$

$$(13) \qquad b_r = 1 \qquad (r = 2, 4, 6, \dots),$$

$$(14) \qquad b_r = \cos\frac{2i-1}{mr}\pi \qquad (r = 1, 3, 5, \dots).$$

При $r = 1$ из (12) и (14) получаем:

$$(15) \qquad a_{m,1} = \frac{1}{2} + \frac{1}{n}\sum_{1\leqq i\leqq n/2} \cos\frac{2i-1}{m}\pi.$$

Для случая $r = 1$ неравенство (11) было доказано в работе [6]. Там же отмечено, что из (15) следует:

$$a_{m,1} = \frac{1}{2} + \frac{1}{m}\operatorname{ctg}\frac{\pi}{m} \leqq \frac{1}{2} + \frac{1}{\pi} \qquad (m = 2, 6, 10, \dots).$$

$$a_{m,1} = \frac{1}{2} + \frac{1}{m}\operatorname{cosec}\frac{\pi}{m} < \frac{1}{2} + \frac{1}{\pi} + \left(\frac{1}{2\sqrt{2}} - \frac{1}{\pi}\right)\frac{1}{m^2} \qquad (m = 4, 8, 12, \dots).$$

Если $r = 2$, то формулу (12) можно записать в виде

$$a_{m,2} = \frac{1}{2} + \frac{1}{n}\sum_{1\leqq i\leqq n/2}\left(\cos\frac{2i-1}{2m}\pi - \frac{1}{2}\sec\frac{2i-1}{2m}\pi\right).$$

Очевидно

$$\lim_{m\to\infty} a_{m,2} = \frac{1}{2} + \frac{2}{\pi}\int\limits_0^{\pi/4}\left(\cos t - \frac{1}{2}\sec t\right)dt = \frac{1}{2} + \frac{\sqrt{2}}{\pi} - \frac{1}{\pi}\ln\operatorname{ctg}\frac{\pi}{8} =$$

$$= \frac{1}{2} + \frac{\sqrt{2}}{\pi} - \frac{1}{\pi}\ln(\sqrt{2}+1) = 0{,}67\ldots.$$

Пусть

$$\text{(16)}\qquad c_{m,r} = \frac{3}{2} - \frac{1}{4r} - \begin{cases} \dfrac{1}{2mr}\displaystyle\sum_{i=1}^{r-1}\sin\frac{i\pi}{r}\operatorname{cosec}\frac{i\pi}{mr} & (m = 1, 3, 5, \ldots), \\ \dfrac{1}{mr}\displaystyle\sum_{i=1}^{r-1}\sin\frac{i\pi}{r}\operatorname{cosec}\frac{2i\pi}{mr} & (m = 2, 4, 6, \ldots). \end{cases}$$

Для случая $k = 2$ будет доказана

Т е о р е м а 5. *Выполняется тождество*

$$\text{(17)}\qquad \sup_{\substack{t\in(-\infty,\infty)\\ g(t)\neq\text{const}}} \frac{|S_{2,r}(t,\tau) - g(t)|}{\omega\left(g, \dfrac{2\pi}{m}\right)} = c_{m,r}.$$

Полагая $r = 1$ и $r = 2$, получаем из (16):

$$c_{m,1} = \frac{5}{4}, \quad c_{m,2} < \frac{11}{8} - \frac{1}{2\pi} = 1{,}21\ldots.$$

При $r = 1$ теорема 5 была доказана в [6] и [7].

Из доказательства нижеследующей леммы 5 видно, что

$$c_{m,r} = \frac{3}{2} - \frac{1}{2m} - \frac{1}{2mr}\sum_{i=1}^{n}\sin\frac{2i\pi}{m}\operatorname{ctg}\frac{i\pi}{mr} \qquad (m = 1, 3, 5, \ldots).$$

Аналогичным образом можно показать, что

$$c_{m,r} = \frac{3}{2} - \frac{1}{m} - \frac{1}{mr}\sum_{i=1}^{\frac{n-1}{2}}\sin\frac{2i\pi}{n}\operatorname{ctg}\frac{i\pi}{nr} \qquad (m = 2, 6, 10, \ldots);$$

$$c_{m,r} = \frac{3}{2} - \frac{1}{mr}\sum_{i=1}^{n/2}\sin\frac{2i-1}{n}\pi\operatorname{ctg}\frac{2i-1}{mr}\pi \qquad (m = 4, 8, 12, \ldots).$$

Отсюда видно, что при фиксированном m последовательность $c_{m,r}$ ($r = 1, 2, 3, \ldots$) убывает.

Введем следующее обозначение:

$$\text{(18)}\qquad S_{k,\infty}(t) = \lim_{r\to\infty} S_{k,r}(t,\tau).$$

Если m нечетно, то $S_{0,\infty}(t)$ частичная сумма n-того порядка ряда Фурье функции $g(t)$, а если m четно, то $S_{0,\infty}(t)$ соответствующая модифицированная сумма. $S_{1,\infty}(t)$ сумма Рогозинского. Пусть

$$\lambda_{m,\infty} = \lim_{r\to\infty} \lambda_{m,r} = \frac{1}{2\pi}\int_0^{2\pi} |\, d_m(t)\,|\, dt, \tag{19}$$

где ввиду (1) и (2)

$$d_m(t) = \sin\frac{mt}{2}\operatorname{cosec}\frac{t}{2} \qquad (m = 1, 3, 5, \ldots); \tag{20}$$

$$d_m(t) = \sin nt \operatorname{ctg}\frac{t}{2} \qquad (m = 2, 4, 6, \ldots).$$

Из (6), (18) и (19) при $r \to \infty$ следует доказанное в [7] тождество

$$\sup_{g(t)\neq\text{const}} \frac{|\, S_{o,\infty}(t) - g(t)\,|}{\omega\left(g, \dfrac{2\pi}{m}\right)} = \frac{1}{2} + \frac{1}{2}\lambda_{m,\infty}.$$

Из последнего замечания к теоремам 2 и 3 следуют формула Фейера

$$\lambda_{m,\infty} = \frac{1}{m} + \frac{2}{\pi}\sum_{i=1}^{n}\frac{1}{i}\operatorname{tg}\frac{i\pi}{m} \qquad (m = 1, 3, 5, \ldots)$$

и аналогичные соотношения

$$\lambda_{m,\infty} = \frac{1}{n} + \frac{2}{\pi}\sum_{i=1}^{\frac{n-1}{2}}\frac{1}{i}\operatorname{tg}\frac{i\pi}{n} \qquad (m = 2, 6, 10, \ldots);$$

$$\lambda_{m,\infty} = \frac{4}{\pi}\sum_{i=1}^{n/2}\frac{1}{2i-1}\operatorname{tg}\frac{2i-1}{m}\pi \qquad (m = 4, 8, 12, \ldots).$$

Из теоремы 4 следует неравенство

$$|\, S_{1,\infty}(t) - g(t)\,| \leqq \frac{1}{2}\omega\left(g, \frac{\pi}{n}\right) + a_{m,\infty}\,\omega\left(g, \frac{\pi}{m}\right) \qquad (m = 2, 4, 6, \ldots.)$$

где

$$a_{m,\infty} = \frac{1}{2} + \frac{2}{\pi}\sum_{1\leqq i\leqq n/2}\frac{1}{2i-1}\operatorname{ctg}\frac{2i-1}{m}\pi\left(1 - \cos\frac{2i-1}{m}\pi\right).$$

Пусть

$$c_{m,\infty} = \lim_{r\to\infty} c_{m,r}.$$

Очевидно

$$c_{m,\infty} = \frac{3}{2} - \frac{1}{2m\pi}\int_0^\pi \sin t \operatorname{cosec}\frac{t}{m}\,dt \qquad (m = 1, 3, 5, \dots);$$

$$c_{m,\infty} = \frac{3}{2} - \frac{1}{m\pi}\int_0^\pi \sin nt \operatorname{cosec}\frac{2t}{m}\,dt \qquad (m = 2, 4, 6, \dots);$$

$$\sup_{m\geqq 1} c_{m,\infty} = \frac{3}{2} - \frac{1}{2\pi}\int_0^\pi \frac{\sin t}{t}\,dt = 1{,}20\dots.$$

$$c_{m,\infty} = \frac{3}{2} - \frac{1}{2m} - \frac{1}{2\pi}\sum_{i=1}^{n}\frac{1}{i}\sin\frac{2i\pi}{m} \qquad (m = 1, 3, 5, \dots);$$

$$c_{m,\infty} = \frac{3}{2} - \frac{1}{m} - \frac{1}{2\pi}\sum_{i=1}^{\frac{n-1}{2}}\frac{1}{i}\sin\frac{2i\pi}{n} \qquad (m = 2, 6, 10, \dots);$$

$$c_{m,\infty} = \frac{3}{2} - \frac{1}{\pi}\sum_{i=1}^{n/2}\frac{1}{2i-1}\sin\frac{2i-1}{n}\pi \qquad (m = 4, 8, 12, \dots).$$

Из теоремы 5 следует доказанное в [7] соотношение

$$\sup_{g(t)\neq \text{const}} \frac{|S(t)-g(t)|}{\omega\left(g, \frac{2\pi}{m}\right)} = c_{m,\infty}.$$

Введем следующие обозначения: $f(x)$ определенная на отрезке $-1 \leqq x \leqq 1$, вещественная и непрерывная функция; $\tau = 0$ или $\tau = \frac{\pi}{mr}$;

$$g(t) = f(\cos t), \qquad P_{k,r}(x, \tau) = S_{k,r}(\arccos x, \tau).$$

Так как тригонометрический многочлен $S_{k,r}(t, \tau)$ четный, если $\tau = 0$ или $\tau = \frac{\pi}{mr}$ и $g(t)$ четная функция, то $P_{k,r}(x, \tau)$ степенной многочлен не выше чем n-той степени. Если $r = 1$ и $k = 0$, то он очевидно интерполирует функцию $f(x)$ в узлах

$$\cos\left(\tau + \frac{2i\pi}{m}\right) \quad (i = 0, \pm 1, \pm 2, \dots).$$

Из полученных выше оценок (6), (15), и (17) следуют неравенства

$$(21) \qquad |P_{0,r}(x, \tau) - f(x)| \leqq \left(\frac{1}{2} + \frac{1}{2}\lambda_{m,r}\right)\omega\left(f, \frac{2\pi}{m}\right);$$

$$(22) \qquad |P_{1,r}(x, \tau) - f(x)| \leqq \frac{1}{2}\omega\left(f, \frac{\pi}{n}\right) + a_{m,r}\,\omega\left(f, \frac{\pi}{m}\right) \qquad (m = 2, 4, 6, \dots);$$

$$(23) \qquad |P_{2,r}(x, \tau) - f(x)| \leqq c_{m,r}\,\omega\left(f, \frac{2\pi}{m}\right),$$

где числа $\lambda_{m,r}$, $a_{m,r}$ и $c_{m,r}$ определяются формулами (7), (10), (12)—(14) и (16). Для случая $r=1$ оценки (21)—(23) были получены в [6].

В дальнейшем нам потребуется

Лемма 1. *На отрезке*

$$-\frac{2\pi}{m} \leqq t \leqq 0 \tag{24}$$

выполняются тождества

$$\sum_{i=0}^{m-1}\left|d_m\left(t-\frac{i\pi}{n}\right)\right| = 4\sum_{i=1}^{n/2}\sec\frac{2i-1}{m}\pi\cos(2i-1)\left(t+\frac{\pi}{m}\right) \tag{25}$$

$$(m=4, 8, 12, \dots);$$

$$\sum_{i=0}^{m-1}\left|d_m\left(t-\frac{i\pi}{n}\right)\right| = 2+4\sum_{i=1}^{\frac{n-1}{2}}\sec\frac{i\pi}{n}\cos 2i\left(t+\frac{\pi}{m}\right) \tag{26}$$

$$(m=2, 6, 10, \dots.);$$

$$\sum_{i=0}^{m-1}\left|d_m\left(t-\frac{2i\pi}{m}\right)\right| = 1+2\sum_{i=1}^{n}\sec\frac{i\pi}{m}\cos i\left(t+\frac{\pi}{m}\right) \tag{27}$$

$$(m=1, 3, 5, \dots).$$

Доказательство леммы 1. Формулу (2) очевидно можно записать в виде

$$d_m(t) = \sin nt\ \operatorname{ctg}\frac{t}{2} \qquad (m=2, 4, 6, \dots). \tag{28}$$

Поэтому на отрезке (24) при четном m

$$\left|d_m\left(t-\frac{i\pi}{n}\right)\right| = (-1)^i d_m\left(t-\frac{i\pi}{n}\right) \qquad (i=0, 1, \dots, n-1);$$

$$\left|d_m\left(t+\frac{i\pi}{n}\right)\right| = (-1)^{i+1} d_m\left(t+\frac{i\pi}{n}\right) \qquad (i=1, 2, \dots, n).$$

Отсюда и из (2) получаем:

$$\sum_{i=0}^{m-1}\left|d_m\left(t-\frac{i\pi}{n}\right)\right| = \sum_{i=-n}^{n-1}\left|d_m\left(t-\frac{i\pi}{n}\right)\right| = \tag{29}$$

$$= \sum_{i=0}^{n-1}(-1)^i\left[d_m\left(t-\frac{i\pi}{n}\right)+d_m\left(t+\frac{i+1}{n}\pi\right)\right] =$$

$$= 2\sum_{i=0}^{n-1}(-1)^i\sum_{j=1}^{n-1}\left[\cos j\left(t-\frac{i\pi}{n}\right)+\cos j\left(t+\frac{i+1}{2}\pi\right)\right] =$$

$$= 4\sum_{j=1}^{n-1}\cos j\left(t+\frac{\pi}{m}\right)\sum_{i=0}^{n-1}(-1)^i\cos j\frac{2i+1}{m}\pi \qquad (m=4, 8, 12, \dots).$$

Известно, что

$$\sum_{i=0}^{2l-1}(-1)^i\cos(2i+1)x=\sin^2 2lx\sec x\,.$$

Положим здесь $l=\dfrac{n}{2}$ и $x=\dfrac{j\pi}{m}$:

$$\sum_{i=0}^{n-1}(-1)^i\cos j\frac{2i+1}{m}\pi=\sin^2\frac{j\pi}{2}\sec\frac{j\pi}{m}=\begin{cases}0 & (j=2,4,6,\dots);\\ \sec\dfrac{j\pi}{m} & (j=1,3,5,\dots).\end{cases}$$

Отсюда и из (29) следует докаываемое тождество (25). Равенства (26) и (27) доказываются аналогичным образом.

Л е м м а 2. *На отрезке*

$$-\frac{2\pi}{mr}\leqq t\leqq 0 \tag{30}$$

выполняются тождества

$$\sum_{i=0}^{mr-1}\left|d_m\left(t-\frac{i\pi}{nr}\right)\right|=4\sum_{i=1}^{n/2}\operatorname{tg}\frac{2i-1}{m}\pi\operatorname{cosec}\frac{2i-1}{mr}\pi\cos(2i-1)\left(t+\frac{\pi}{mr}\right) \tag{31}$$

$$(m=4,8,12,\dots);$$

$$\sum_{i=0}^{mr-1}\left|d_m\left(t-\frac{i\pi}{nr}\right)\right|=2r+4\sum_{i=1}^{\frac{n-1}{2}}\operatorname{tg}\frac{i\pi}{n}\operatorname{cosec}\frac{i\pi}{nr}\cos 2i\left(t+\frac{\pi}{mr}\right) \tag{32}$$

$$(m=2,6,10,\dots)$$

$$\sum_{i=0}^{mr-1}\left|d_m\left(t-\frac{2i\pi}{mr}\right)\right|=r+2\sum_{i=1}^{n}\operatorname{tg}\frac{i\pi}{m}\operatorname{cosec}\frac{i\pi}{mr}\cos i\left(t+\frac{\pi}{mr}\right) \tag{33}$$

$$(m=1,3,5,\dots).$$

Д о к а з а т е л ь с т в о л е м м ы 2. Ввиду (25) на отрезке (30)

$$\sum_{i=0}^{mr-1}\left|d_m\left(t-\frac{i\pi}{nr}\right)\right|=\sum_{i=0}^{m-1}\sum_{j=0}^{r-1}\left|d_m\left(t-\frac{i\pi}{n}-\frac{j\pi}{nr}\right)\right|= \tag{34}$$

$$=4\sum_{i=1}^{n/2}\sec\frac{2i-1}{m}\pi\sum_{j=0}^{r-1}\cos(2i-1)\left(t+\frac{\pi}{m}-\frac{j\pi}{nr}\right)\qquad(m=4,8,12,\dots).$$

Известно, что

$$\sum_{j=0}^{r-1}\cos(x+jy)=\cos\left(x+\frac{r-1}{2}y\right)\sin\frac{ry}{2}\operatorname{cosec}\frac{y}{2}. \tag{35}$$

Полагая здесь $x=-(2i-1)\left(t+\dfrac{\pi}{m}\right)$ и $y=\dfrac{2i-1}{nr}\pi$, получаем:

$$\sum_{j=0}^{r-1}\cos(2i-1)\left(\frac{j\pi}{nr}-t-\frac{\pi}{m}\right)=\cos(2i-1)\left(t+\frac{\pi}{mr}\right)\sin\frac{2i-1}{mr}\pi\operatorname{cosec}\frac{2i-1}{mr}\pi.$$

Отсюда и из (34) следует (31). Тождества (32) и (33) доказываются аналогичным образом.

Доказательство теорем 2 и 3. Так как тригонометрический многочлен $d_m(t)$ 2π-периодичен, то функция

$$\sum_{i=0}^{mr-1}\left|d_m\left(t-\frac{2i\pi}{mr}\right)\right|$$

имеет период $\frac{2\pi}{mr}$. Поэтому и ввиду (5)

$$\lambda_{m,r}=\frac{1}{mr}\max_{-\frac{2\pi}{mr}\le t\le 0}\sum_{i=0}^{mr-1}\left|d_m\left(t-\frac{2i\pi}{mr}\right)\right|.$$

Отсюда, из (31), (32) и (33) следует:

$$\lambda_{m,r}=\frac{1}{mr}\sum_{i=0}^{mr-1}\left|d_m\left(\frac{2i+1}{mr}\pi\right)\right|. \tag{36}$$

Из (36) и (28) получаем:

$$\lambda_{m,r}=\frac{1}{mr}\sum_{i=0}^{r}\sin\frac{2i-1}{r}\pi\sum_{j=0}^{m-1}\left|\operatorname{ctg}\frac{2jr+2i-1}{2mr}\pi\right|.$$

Отсюда следует доказываемое тождество (7). Аналогичным образом из (36) и (20) следует (10).

Доказательство теорем 1, 4, и 5 опирается на следующее вспомогательное предложение:

Лемма 3. *Выполняется тождество*

$$S_{k,r}(t,\tau)=\frac{1}{r}\sum_{i=0}^{r-1}S_{k,1}\left(t,\tau+\frac{2i\pi}{mr}\right). \tag{37}$$

Доказательство леммы 3. Если $k=0$, то (37) следует из (3):

$$S_{0,r}(t,\tau)=\frac{1}{mr}\sum_{i=0}^{mr-1}g\left(\tau+\frac{2i\pi}{mr}\right)d_m\left(t-\tau-\frac{2i\pi}{mr}\right)= \tag{38}$$

$$=\frac{1}{r}\sum_{j=0}^{r-1}\frac{1}{m}\sum_{j=0}^{m-1}g\left(\tau+\frac{2i\pi}{mr}+\frac{2j\pi}{m}\right)d_m\left(t-\tau-\frac{2i\pi}{mr}-\frac{2j\pi}{m}\right)=$$

$$=\frac{1}{r}\sum_{i=0}^{r-1}S_{0,1}\left(t,\tau+\frac{2i\pi}{mr}\right).$$

В общем случае (37) следует из (4) и (38):

$$S_{k,r}(t,\tau)=\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}S_{0,r}\left(t+\frac{2j-k}{m}\pi,\tau\right)=$$

$$=\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{r}\sum_{i=0}^{r-1}S_{0,1}\left(t+\frac{2j-k}{m}\pi,\tau+\frac{2i\pi}{mr}\right)=\frac{1}{r}\sum_{i=0}^{r-1}S_{k,1}\left(t,\tau+\frac{2i\pi}{mr}\right).$$

Лемма 4. *Выполняется неравенство*

$$(39)\qquad | S_{0,r}(t, \tau) - g(t) | \leqq \frac{1}{2}(1 + \lambda_{m,r})\omega\left(g, \frac{2\pi}{m}\right).$$

Доказательство леммы 4. В [2] доказано соотношение:

$$| S_{0,1}(t, \tau) - g(t) | \leqq \left[\frac{1}{2} + \frac{1}{2m}\sum_{j=0}^{m-1}\left| d_m\left(t - \tau - \frac{2j\pi}{m}\right)\right|\right]\omega\left(g, \frac{2\pi}{m}\right).$$

Отсюда и из (37) получаем:

$$| S_{0,r}(t, \tau) - g(t) | \leqq \left[\frac{1}{2} + \frac{1}{2mr}\sum_{i=0}^{r-1}\sum_{j=0}^{m-1}\left| d_m\left(t - \tau - \frac{2i\pi}{mr} - \frac{2j\pi}{m}\right)\right|\right]\omega\left(g, \frac{2\pi}{m}\right) =$$

$$= \left[\frac{1}{2} + \frac{1}{2mr}\sum_{i=0}^{mr-1}\left| d_m\left(t - \tau - \frac{2i\pi}{mr}\right)\right|\right]\omega\left(g, \frac{2\pi}{m}\right).$$

Отсюда и из (5) следует доказываемое неравенство (39).

Доказательство теоремы 1. Определим непрерывную и 2π-периодическую функцию $g(t)$ следующим образом:

$$(40)\qquad g\left(\tau + \frac{2i\pi}{mr}\right) = \operatorname{sign} d_m\left(\frac{2i+1}{mr}\pi\right) \qquad (i = 0, 1, \dots, mr - 1),$$

$$(41)\qquad g\left(\tau - \frac{\pi}{mr}\right) = g\left(\tau - \frac{\pi}{mr} + 2\pi\right) = -1,$$

функция $g(t)$ линейна на отрезках

$$\tau - \frac{\pi}{mr} \leqq t \leqq \tau, \qquad \tau + \frac{2i\pi}{mr} \leqq t \leqq \tau + 2\,\frac{i+1}{mr}\pi \quad (0 \leqq i \leqq mr - 2),$$

$$\tau + 2\pi - \frac{2\pi}{mr} \leqq t \leqq \tau + 2\pi - \frac{\pi}{mr}.$$

Очевидно

$$(42)\qquad \omega\left(g, \frac{2\pi}{m}\right) = 2.$$

Из (3), (40), (41), (36) и (42) получаем:

$$S_{0,r}\left(\tau - \frac{\pi}{mr}, \tau\right) - g\left(\tau - \frac{\pi}{mr}\right) = \frac{1}{mr}\sum_{i=0}^{mr-1}\left| d_m\left(\frac{2i+1}{mr}\pi\right)\right| + 1 =$$

$$= \lambda_{m,r} + 1 = \frac{1}{2}(1 + \lambda_{m,r})\omega\left(g, \frac{2\pi}{m}\right).$$

Отсюда и из (39) следует доказываемое тождество (6).

Доказательство теоремы 4. В [6] доказано, что при четном m на отрезке $|t-\tau|\leqq\frac{\pi}{m}$ выполняется неравенство

$$|S_{1,1}(t,\tau)-g(t)|\leqq\frac{1}{2}\omega\left(g,\frac{\pi}{n}\right)+$$

$$+\left[\frac{1}{2}+\frac{1}{n}\sum_{1\leqq j\leqq n/2}\operatorname{ctg}\frac{2j-1}{m}\pi\sin(2j-1)\left(\frac{\pi}{m}-|\tau-t|\right)\right]\omega\left(g,\frac{\pi}{m}\right).$$

Отсюда и из (37) получаем: если

(43) $$\left|t-\tau-\frac{\pi}{m}+\frac{\pi}{mr}\right|\leqq\frac{\pi}{mr},$$

то

(44) $$|S_{1,r}(t,\tau)-g(t)|\leqq\frac{1}{2}\omega\left(g,\frac{\pi}{n}\right)+$$

$$+\left[\frac{1}{2}+\frac{1}{nr}\sum_{1\leqq j\leqq\frac{n}{2}}\operatorname{ctg}\frac{2j-1}{m}\pi\sum_{i=0}^{r-1}\sin(2j-1)\left(\frac{\pi}{m}-\left|\tau+\frac{i\pi}{nr}-t\right|\right)\right]\omega\left(g,\frac{\pi}{m}\right).$$

Здесь

(45)

$$\sin(2j-1)\left(\frac{\pi}{m}-\left|\tau+\frac{i\pi}{nr}-t\right|\right)+\sin(2j-1)\left(\frac{\pi}{m}-\left|\tau+\frac{r-i-1}{nr}\pi-t\right|\right)=$$

$$=2\sin(2j-1)\frac{2i+1}{mr}\pi\cos(2j-1)\left(t-\tau-\frac{\pi}{m}+\frac{\pi}{mr}\right)\qquad\left(0\leqq i\leqq\frac{r-1}{2}\right).$$

Известно, что

$$2\sum_{i=1}^{k}\sin(2i-1)x=(1-\cos 2kx)\operatorname{cosec}x.$$

Поэтому

$$2\sum_{i=0}^{\frac{r}{2}-1}\sin(2i+1)\frac{2j-1}{mr}\pi=\left(1-\cos\frac{2j-1}{m}\pi\right)\operatorname{cosec}\frac{2j-1}{mr}\pi$$

$$(r=2,4,6,\dots).$$

Отсюда, из (44), (45), (12) и (13) получаем доказываемое неравенство (11) для четных r при условии (43). Чтобы получить (11) на отрезках

$$\left|t-\tau-\frac{\pi}{m}+\frac{\pi}{mr}+\frac{i\pi}{nr}\right|\leqq\frac{\pi}{mr}\qquad(i=\pm1,\pm2,\pm3,\dots),$$

достаточно воспользоваться неравенством (11) для случая (43) и функции $g\left(t+\frac{i\pi}{nr}\right)$. При нечетных r теорема 4 доказывается аналогичным образом.

Лемма 5. *Выполняется неравенство*

$$|S_{2,r}(t,\tau)-g(t)|\leqq c_{m,r}\,\omega\left(g,\frac{2\pi}{m}\right). \tag{46}$$

Доказательство леммы 5. В [7] доказано, что на отрезке $\tau-\frac{2\pi}{m}\leqq t\leqq\tau$ выполняется неравенство

$$|S_{2,1}(t,\tau)-g(t)|\leqq\left[\frac{3}{2}-\frac{1}{4m}d_m\left(t-\tau+\frac{2\pi}{m}\right)-\frac{1}{4m}d_m(t-\tau)\right]\omega\left(g,\frac{2\pi}{m}\right)$$
$$(m=1,3,5.,\dots).$$

Отсюда и из (37) получаем: если

$$\tau-\frac{2\pi}{mr}\leqq t\leqq\tau, \tag{47}$$

то

$$|S_{2,r}(t,\tau)-g(t)|\leqq\left[\frac{3}{2}-\frac{1}{4mr}\sum_{i=1-r}^{r}d_m\left(t-\tau+\frac{2i\pi}{mr}\right)\right]\omega\left(g,\frac{2\pi}{m}\right) \tag{48}$$
$$(m=1,3,5,\dots).$$

Ввиду (1) и (35)

$$\sum_{i=1-r}^{r}d_m\left(t-\tau+\frac{2i\pi}{mr}\right)=$$
$$=\sum_{i=1}^{r}\left[d_m\left(t-\tau+\frac{\pi}{mr}+\frac{2i-1}{mr}\pi\right)+d_m\left(t-\tau+\frac{\pi}{mr}-\frac{2i-1}{mr}\pi\right)\right]=$$
$$=2r+4\sum_{j=1}^{n}\cos j\left(t-\tau+\frac{\pi}{mr}\right)\sum_{i=1}^{r}\cos j\frac{2i-1}{mr}\pi=$$
$$=2r+2\sum_{j=1}^{n}\sin\frac{2j\pi}{m}\operatorname{cosec}\frac{j\pi}{mr}\cos j\left(t-\tau+\frac{\pi}{mr}\right)\qquad(m=1,3,5,\dots).$$

Поэтому и в силу (20) на отрезке (47)

$$\sum_{i=1-r}^{r}d_m\left(t-\tau+\frac{2i\pi}{mr}\right)\geqq\sum_{i=1-r}^{r}d_m\left(\frac{2i\pi}{mr}\right)=m+2\sum_{i=1}^{r-1}\sin\frac{i\pi}{r}\operatorname{cosec}\frac{i\pi}{mr} \tag{49}$$
$$(m=1,3,5,\dots).$$

Отсюда, из (48) и (16) получаем доказываемое неравенство (46) для нечетных

m при условии (47). Чтобы доказать (46) для нечетных m на отрезке

$$\tau + 2\frac{i-1}{mr}\pi \leqq t \leqq \tau + \frac{2i\pi}{mr} \qquad (i = 1, \pm 2, \pm 3, \dots),$$

достаточно воспользоваться неравенством (46) при условии (47) для функции $g\left(t + \dfrac{2i\pi}{mr}\right)$. Для четных m лемма доказывается аналогичным образом.

Доказательство теоремы 5. Будем считать m нечетным числом, для четного m теорема доказывается аналогичным образом. Пусть

$$0 < \varepsilon < \frac{2\pi}{m}.$$

Определим зависящую от ε и m функцию $g(t)$ на отрезке

$$\tau - \varepsilon \leqq t \leqq \tau + 2\pi - \varepsilon \tag{50}$$

следующим образом:

$$g(\tau - \varepsilon) = g(\tau + 2\pi - \varepsilon) = -1, \tag{51}$$

$$g(t) = 0 \quad \Bigg(\tau \leqq t \leqq \tau + \frac{2\pi}{m} - \varepsilon,\ \tau + \frac{4i\pi}{m} \leqq t \leqq \tau + 2\frac{2i+1}{m}\pi - \tag{52}$$

$$- \frac{2\pi}{mr}\ (1 \leqq i < n), \quad \tau + 2\pi - \frac{2\pi}{m} - \varepsilon \leqq t \leqq \tau + 2\pi - \frac{2\pi}{mr}\Bigg);$$

$$g(t) = 1\ \left(\tau + 2\frac{2i-1}{m}\pi \leqq t \leqq \tau + \frac{4i\pi}{m} - \frac{2\pi}{mr}, \qquad 1 \leqq i \leqq n\right); \tag{53}$$

$g(t)$ линейна на отрезках

$$\tau - \varepsilon \leqq t \leqq \tau, \quad \tau + \frac{2\pi}{m} - \varepsilon \leqq t \leqq \tau + \frac{2\pi}{m},$$

$$\tau + \frac{2i\pi}{m} - \frac{2\pi}{mr} \leqq t \leqq \tau + \frac{2i\pi}{m} \qquad (2 \leqq i \leqq m-2),$$

$$\tau + 2\pi - \frac{2\pi}{m} - \frac{2\pi}{mr} \leqq t \leqq \tau + 2\pi - \frac{2\pi}{m} - \varepsilon,\ \tau + 2\pi - \frac{2\pi}{mr} \leqq t \leqq \tau + 2\pi - \varepsilon.$$

Вне отрезка (50) определим $g(t)$ так, чтобы она оказалась 2π-периодичной и непрерывной. Очевидно

$$\omega\left(g, \frac{2\pi}{m}\right) = 1.$$

В силу (3), (52) и (53)

$$S_{0,1}\left(t, \tau+\frac{2j\pi}{mr}\right)=\frac{1}{m}\sum_{i=0}^{m-1} g\left(\tau+\frac{2i\pi}{m}+\frac{2j\pi}{mr}\right) d_m\left(t-\tau-\frac{2i\pi}{m}-\frac{2j\pi}{mr}\right)=$$

$$=\frac{1}{m}\sum_{i=1}^{n} d_m\left(t-\tau-2\,\frac{2i-1}{m}\pi-\frac{2j\pi}{mr}\right) \qquad (0 \leqq j < r).$$

Поэтому и ввиду (4)

$$S_{2,1}\left(t, \tau+\frac{2j\pi}{mr}\right)=\frac{1}{4m}\sum_{i=1}^{n}\left[d_m\left(t-\tau-\frac{4i\pi}{m}-\frac{2j\pi}{mr}\right)+\right.$$

$$\left.+2d_m\left(t-\tau-2\,\frac{2i-1}{m}\pi-\frac{2j\pi}{mr}\right)+d_m\left(t-\tau-4\,\frac{i-1}{m}\pi-\frac{2j\pi}{mr}\right)\right]=$$

$$=\frac{1}{2m}\sum_{i=0}^{m-1} d_m\left(t-\tau-\frac{2i\pi}{m}-\frac{2j\pi}{mr}\right)-\frac{1}{4m}d_m\left(t-\tau-\frac{2j\pi}{mr}\right)-$$

$$-\frac{1}{4m}\,d_m\left(t-\tau-2\pi+\frac{2\pi}{m}-\frac{2j\pi}{mr}\right)=\frac{1}{2}-\frac{1}{4m}\,d_m\left(t-\tau-\frac{2j\pi}{mr}\right)-$$

$$-\frac{1}{4m}\,d_m\left(t-\tau+\frac{2\pi}{m}-\frac{2j\pi}{mr}\right) \qquad (0 \leqq j < r).$$

Отсюда и из (37) получаем:

$$S_{2,r}(t, \tau)=\frac{1}{2}-\frac{1}{4mr}\sum_{i=1-r}^{r} d_m\left(t-\tau+\frac{2i\pi}{mr}\right).$$

Поэтому и ввиду (51)

$$S_{2,r}(\tau-\varepsilon, \tau)-g(\tau-\varepsilon)=\frac{3}{2}-\frac{1}{4mr}\sum_{i=1-r}^{r} d_m\left(\frac{2i\pi}{mr}-\varepsilon\right).$$

Отсюда, из (49) и (16) получаем:

$$\text{(54)} \qquad \lim_{\varepsilon\to 0}[S_{2,r}(\tau-\varepsilon,\tau)-g(\tau-\varepsilon)]=\frac{3}{2}-\frac{1}{4mr}\sum_{i=1-r}^{r} d_m\left(\frac{2i\pi}{mr}\right)=c_{m,r}\,.$$

Из (46) и (54) следует доказываемое соотношение (17).

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KIS OTTÓ, HO THO CAO
BUDAPESTI MŰSZAKI EGYETEM
VILLAMOSKARI MATEMATIKA TANSZÉK
1111 BUDAPEST, STOCZEK U. 2—4

Acta Mathematica Academiae Scientiarum Hungaricae
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A TOPOLOGICAL CRITERION FOR PRIMARY DECOMPOSITION

By

T. S. SHORES (Lincoln)

There are available in the literature many criteria for a ring to have the property that its torsion modules (in the sense of Dickson) decompose into direct sums of primary submodules. The study of rings with this property was inauguarated by S. E. DICKSON in [4] and [5], where such rings were termed T-rings, and has been continued by numerous authors ([1], [2], [9], [11], [16], et al.). In this note we confine our attention to commutative rings with identity and develop a criterion for such a ring to be a T-ring in terms of the prime spectrum. The prime spectrum has been used as a tool for investigating torsion of commutative regular rings by C. NĂSTĂSESCU in [10]. What motivated us to find our criterion were several queries by Tom Cheatham (private communication), namely: What commutative regular rings are T-rings? In particular, is an arbitrary product of fields a T-ring? The criteria in the above mentioned papers do not seem to be readily applicable to these questions. However, Theorem 1 below yields a fairly straightforward answer to the first question and an affirmative answer to the second one. In fact, our results show that self-injective regular rings and rings of continuous functions on completely regular spaces are T-rings. In this connection I would like to thank Roger Wiegand for showing me how to prove the latter assertion for compact spaces. The general result needed only a modification of his proof.

Before stating our criterion, we require some notation. Rings are assumed to be commutative with identity and modules are left unital modules. A module is *torsion (in the sense of Dickson)* if every nonzero homomorphic image contains a simple submodule. The basic properties of such modules are detailed in [5]. A torsion module is said to *admit a primary decomposition* if it is a direct sum of submodules, each of which has every nonzero homomorphic image containing a simple submodule isomorphic to some *fixed* simple. If every torsion R-module admits a primary decomposition, then the ring R is said to be a *T-ring*. Also Spec (R) denotes the topological space of all prime ideals of R endowed with the Zariski (hull-kernel) topology. Specifically, the closed sets of Spec (R) are all sets of the form $V(I)=\{P\in \text{Spec}\,(R)\mid I\subseteq P\}$ and the open sets are the set differences $D(I)=\text{Spec}\,(R)-V(I)$, where I runs over the ideals of R. We assume a number of standard facts about Spec and refer the reader to [3] for details. Recall that for a radical ideal I and prime ideal P or R, $V(I)$ is *clopen* (closed and open) if and only if I is a ring summand of R; also, the closure of P in Spec (R) is $V(P)$. It follows that for a regular ring R there is a one to one correspondence between simple submodules of R and isolated points of Spec (R). (This correspondence has been explored by C. NĂSTĂSESCU in [10].) Also, Spec (R) is Hausdorff if and only if every prime ideal is maximal; moreover, R is regular if and only if R is reduced and Spec (R) is Hausdorff. Throughout, N^* denotes the one point compactification of the space of natural numbers with the discrete topology. The main result of this note follows. Its proof involves ideas from [1], [10] and [15], but the proof presented here is essentially self contained.

THEOREM 1. *Let* $X=$ Spec (R). *The following are equivalent conditions on* R: (1) *R is a T-ring;* (2) *X contains no closed subspaces homeomorphic to* N^*; (3) *X contains no countably infinite closed Hausdorff spaces.*

PROOF. It is easily seen that a torsion module admits a primary decomposition if and only if every cyclic submodule admits a decomposition. Hence, R is a T-ring if and only if every torsion ring (i.e. ring which is a torsion module over itself) homomorphic image of R admits a primary decomposition. Since a module decomposition of a cyclic R-module has only finitely many nonzero summands, an equivalent formulation is that every torsion ring image R/I of R is a finite direct product of local rings. But torsion domains are easily seen to be fields, so the prime ideals of a torsion ring are maximal and consequently the intersection of all maximal ideals consists of nilelements. It follows from the standard theorem on lifting idempotents mod nil ideals that a torsion ring is a finite product of local rings if and only if the ring has finitely many prime ideals. To summarize, R is a T-ring if and only if, for every ideal I of R such that R/I is torsion, $V(I)$ is finite.

To see that (1) implies (2), suppose that I is an ideal of R such that $V(I)$ is homeomorphic to N^*. We can assume that $I=\bigcap V(I)$, so that R/I is a reduced ring whose spectrum is homeomorphic to N^*; hence R/I is regular. Let $Z=\{M_i \mid i==1,2,\dots\}$ be the isolated points of Spec (R/I) and M_0 the limit point of Z. The ideals M_i, $i>0$, are summands of R/I since $V(M_i)$ is clopen in Spec (R/I), say $M_i==e_i(R/I)$ for suitable idempotents e_i. Let $S=\Sigma\{(1-e_i)(R/I) \mid i=1,2,\dots\}$ so that S is torsion and clearly $V(S)=\{M_0\}$. Thus $S=M_0$ and R/I is a torsion module, since R/I is regular. The implication now follows from the first paragraph of the proof. That (2) implies (3) is clear.

Finally, suppose by way of contradiction that R satisfies (3) but not (1). By the first paragraph, there is an ideal I of R such that $I=\bigcap V(I)$, R/I is torsion and $V(I)$ is infinite. Thus R/I is a regular ring whose socle (sum of all simple submodules) is S, say. Then S is not finitely generated, else S is a direct summand of R/I. It would then follow, from the facts that every nonzero homomorphic image of R/I has nonzero socle and $V(I)$ is infinite, that $S=R/I$, a contradiction. Pick a submodule $\oplus\{e_i(R/I) \mid i=1,2,\dots\}$ of S with each $e_i(R/I)$ simple and e_i idempotent. Then set $M_i==(1-e_i)(R/I)$; the M_i are maximal ideals of R/I and isolated points in $Y==$ Spec $(R/I)=V(I)$. Set $Y_0=\{M_1,M_2,\dots\}$ and $Y_1=Y-Y_0$. Then Y_1 is closed in Spec (R/I) and by the above remarks Y_1 contains an isolated point M_0. Since Spec (R/I) has a basis of clopen sets, there is a closed neighbourhood (in Y) N of M_0 such that $N\cap Y_1=\{M_0\}$. But $Y=Y_0\cup Y_1$, so that N is a countable closed subset of Y; hence Spec (R/I) has closed countably infinite subsets, which violates condition (3) and completes the proof.

COROLLARY 1. *A regular ring R is a T-ring if and only if every convergent sequence in* Spec (R) *is eventually constant.*

PROOF. The space $X=$ Spec (R) is a compact Hausdorff space. If $\{x_n \mid n==1,2,\dots\}$ were a sequence in X with limit x_0 in X and not eventually constant, then x_0 would be the only limit point of the set of x_n's, $n > 0$ (since X is Hausdorff). It follows that the subspace $\{x_n \mid 0_n = 0, 1, \ldots\}$ of X is homeomorphic to N^* and closed in X. Conversely, if X has a subspace homeomorphic to N^*, then the limit point of N^* is the limit of the remaining points of N^* in any sequential ordering. The Corollary now follows from Theorem 1.

THEOREM 2. *Each of the following conditions on the ring R implies the next:* (1) *R is a direct product of fields;* (2) *R is a self-injective regular ring;* (3) Spec (R) *is an extremally disconnected Hausdorff space;* (4) *R is a T-ring.*

PROOF. It is easily seen that (1) implies (2). That (2) implies (3) follows from Proposition 24.1 of [13]. The proof that (3) implies (4) is by way of contradiction. Suppose that R satisfies (3) but not (4). In light of Theorem 1, Spec (R) has a subspace Y homeomorphic to N^*, say $Y = \{y_0, y_1, \ldots\}$ and $\{y_1, y_2, \ldots\}$ is a discrete subspace of Y. We may select disjoint open neighbourhoods of y_1 and y_0. Moreover, any neighbourhood of y_0 contains all but a finite number of the y_i. So we can find disjoint open sets (in Spec (R)) M_1 and N_1 such that $y_1 \in M_1$ and $Y - \{y_1\} \subseteq N_1$. Now replace Spec (R) by N_1 and repeat the argument. In this way we obtain a sequence of pairwise disjoint open sets $M_1, M_2, \ldots$ such that $M_i \cap Y = \{y_i\}$ for $i > 0$. Let $S_i = \bigcup \{M_{2n} \mid n > 0\}$ and $S_2 = \bigcup \{M_{2n-1} \mid n > 0\}$. Then y_0 belongs to the closures of S_1 and S_2. Since the S_i are open and y_0 is a limit point of both S_i, the fact that disjoint open sets of Spec (R) have disjoint closures is contradicted. The result follows.

REMARK 1. It is clear that for commutative regular rings which are torsion over themselves (i e , semiartinian rings [12] or Loewy rings [14]), the property of being a T-ring is equivalent to semisimplicity. Therefore, examples of regular rings which are not T-rings abound in the literature (see e.g. [6], [11] or [14]) Infinite products of fields provide us with interesting examples of T-rings, as do the rings of continuous functions in Theorem 3 below. A certain amount of noncommutativity can be allowed: the preceding results remain valid for left modules and left duo rings, i.e. rings whose left ideals are two-sided. These include the strongly regular rings of [8]. In particular, an infinite product of division rings is a T-ring.

We might also note that another way to show that products of division rings are T-rings is to observe that the spectrum of $\pi\{D_\alpha \mid \alpha \in I\}$, D_α division rings, is just βI, the Stone–Cech compactification of I. (Here the index set I has the discrete topology.) One can verify that the map $P \to \{\{\alpha \in I \mid x_\alpha \neq 0\} \mid x_\alpha \notin P\}$ sends the spectrum of πD_α homeomorphically onto βI, which in this case has the set of all ultrafilters on I as its underlying set. By using the remark in [7, p. 134] one can see that for discrete I, βI has no countably infinite closed subsets. Thus Theorem 1 applies to yield the desired result.

THEOREM 3. *The ring of all continuous real valued functions on a completely regular space is a T-ring.*

PROOF. We refer the reader to [7] for notations and definitions. For an arbitrary ring R, we let Maxspec (R) denote the subspace of Spec (R) consisting of all maximal ideals of R. By the Gelfand–Kolmogoroff Theorem, the mapping given by $M \to Z(M)$, where $Z(M)$ is the set of zero sets of elements of the maximal ideal M, defines a homeomorphism from Maxspec $(C(X))$ onto Maxspec $(C(\beta X))$. The inverse mapping is given by $p \to M^P$, where for $p \in \beta X$, $M^p = \{f \in C(X) \mid p \in \mathrm{cl}_{\beta X} Z_X(f)\}$.

We also have the standard ring isomorphism $\beta : C^*(X) \to C(\beta X)$, $C^*(X)$ being the ring of bounded real valued continuous functions on X. Suppose by way of contradiction that $C(X)$ is not a T-ring, so that Spec $(C(X))$ contains a closed subspace $Y = \{M_i \mid i = 0, 1, \dots\}$ homeomorphic to N^* with, say, M_0 as the limit point. Each M_i is a maximal ideal of $C(X)$, since otherwise $\{M_i\} \neq V(M_i) \subseteqq Y$. Under the Gelfand–Kolmogoroff correspondence there are points $p_i \in \beta X$ such that $M_i = M^{p_i}$, $i = 0, 1, \dots$ and the (closed) subspace $\{p_i \mid i = 0, 1, \dots\}$ of X is homeomorphic to Y. Define the (continuous) function F on $\{p_i \mid i = 0, 1, \dots\}$ by setting $F(p_0) = 0$ and $F(p_n) = 1/n$, $n > 0$. Now use the Tietze Extension Theorem to extend the domain of F to all of βX. Let f be the restriction of F to X, so that $f \in C^*(X)$ and $f^\beta = F$. Also let $S = \{f^n g \mid g \in C^*(X), g^\beta(p_0) \neq 0 \text{ and } n \geqq 0\}$. Then S is a multiplicatively closed subset of $C(X)$. Set $I = \bigcap \{M^{p_n} \mid n = 1, 2, \dots\}$ and note that $S \cap I = \emptyset$; for if $f^n g \in S \cap I$, then $(f^n g)^\beta(p_n) = (f^n(p_n))^n g^\beta(p_n) = 0$ for all $n > 0$. Hence, $g^\beta(p_n) = 0$ for all $n > 0$ and $g^\beta(p_0) = 0$ by continuity, a contradiction. Now expand I to an ideal P (necessarily prime) maximal with respect to avoiding S. Then P belongs to the closure of Y (which is just $V(I)$). Since Y is closed in Spec$(C(X))$, $P \in Y$. But $f \in M^{p_0} - P$ and $P \subseteqq M^{p_0}$, so that P is strictly contained in M^{p_0}, a contradiction.

REMARK 2. Theorem 2 shows that the property of being a T-ring is not entirely determined by Maxspec; for $C(N^*)$ has Maxspec homeomorphic to N^* and $C(N^*)$ is a T-ring, while the ring R of all continuous functions from N^* into the integers mod 2 provides an example of a regular ring whose Maxspec (= Spec) is homeomorphic to N^*, whence R is not a T-ring.

Finally ,we would like to mention that Tom Cheatham and Jimmie Smith have pointed out the following fact to us: If R is a T-ring such that $M = M^2$ for all maximal ideals M of R, then every torsion R-module is semisimple, i.e. a direct sum of simples. (To see this, first decompose a given torsion module T into primary components T_M, M maximal, and note that if S_1 is the socle of T_M, S_2/S_1 the socle of T_M/S_1 and $x \in S_2 - S_1$, then $Mx \neq 0$ and $M^2x = 0$; hence $M \neq M^2$.) This result obviously applies to regular rings. Moreover, the result applies to $C(X)$; for if $f \in C(X)$, then $f = ((f^+)^{\frac{1}{2}})^2 - ((f^-)^{\frac{1}{2}})^2$, where f^+ and f^- are the positive and negative parts of f. Since maximal ideals are z-ideals (see [7, p. 27]), it is easily verified that if $f \in M$, a maximal ideal, then $(f^+)^{\frac{1}{2}}$ and $(f^-)^{\frac{1}{2}}$ belong to M. Therefore $M = M^2$, as desired. Theorem 3 and the preceding discussion also apply to the ring $C^*(X)$ via the standard isomorphism between $C^*(X)$ and $C(\beta X)$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NEBRASKA
LINCOLN, NEBRASKA 68588
USA

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REMARKS ON THE REMAINDER IN BIRKHOFF'S ERGODIC THEOREM

By

G. HALÁSZ (Budapest)

By a dynamic system we shall mean in this paper a probability space $(\Omega, \mathcal{A}, \mu)$ together with an ergodic measure preserving transformation T of Ω into itself. (For the basic notions and results of ergodic theory to be used see e.g. [1].) Birkhoff's ergodic theorem states that for any measurable set A with characteristic function $\chi_A(\omega)$ $(\omega \in \Omega)$

$$\frac{1}{n}\sum_{i=0}^{n-1} \chi_A(T^i\omega) \to \mu(A)$$

almost everywhere or more generally for any integrable $f(\omega)$

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i\omega) \to \int_\Omega f(\omega)d\mu \qquad \text{a.e.}$$

In this paper we make some remarks on the rate of convergence some of which, as far as we know, has not been observed.

The problem is trivial if T is a permutation of Ω consisting of a finite number of points with equal measures and also for systems, called atomic and easily seen to be equivalent in this respect to a permutation, which contain no sets of sufficiently small positive measure. In no other case can, however, Birkhoff's theorem be improved as shown in a somewhat stronger form by

THEOREM 1. *If the dynamic system is non-atomic, then for any sequence of integers $m(n) = o(n)$ there exists a measurable set A such that almost everywhere*

$$\frac{1}{m(n)} \sum_{i=n-m(n)+1}^{n} \chi_A(T^i\omega) \nrightarrow \mu(A).$$

At the other extreme, one can ask how small the remainder can be.

THEOREM 2. *If for a measurable set A*

$$\sum_{i=0}^{n-1} \chi_A(T^i\omega) - n\mu(A)$$

is bounded below on a set of ω of positive measure, then $e^{2\pi i\mu} = e^{2\pi i\mu(A)}$ is an eigenvalue of T, i.e. there exists a function $g(\omega)$ not identically 0 a.e. such that $g(T\omega) = e^{2\pi i\mu}g(\omega)$.

Conversely, if $e^{2\pi i\mu}$ $(0 \leqq \mu \leqq 1)$ is an eigenvalue, then one can find a set A of measure μ with

$$\left|\sum_{i=0}^{n-1} \chi_A(T^i\omega) - n\mu\right| \leqq 1 \qquad \textit{a.e.}$$

The first result of this kind is due to HECKE [2] (sufficiency) and KESTEN [3] (necessity) in the special case when Ω is the real line (mod 1) and $T\omega = \omega + \alpha$ with α irrational: If $A = \{\omega\colon 0 \leqq \omega \leqq \beta \pmod 1\}$ $(0 < \beta < 1)$, then for any fixed ω

$$\sum_{i=0}^{n-1} \chi_A(i\alpha + \omega) - n\beta = \sum_{i=0}^{n-1} \chi_A(T^i\omega) - n\beta$$

is bounded if and only if β is congruent (mod 1) to an integral multiple of α, i.e. an eigenvalue of T. We have learned that essentially the same generalization as our Theorem 2 has been done by FURSTENBERG, KEYNES and L. SHAPIRO [4] for a single ω in the framework of topological dynamics except for one-sided boundedness in the first part. In this respect between „single" and „of positive measure" there is indeed an essential difference as shown by a discovery of VERA T. SÓS [5]: In the above special case one can construct an α and a β not an integral multiple (mod 1) of α, yet $\sum_{i=0}^{n-1} \chi_A(i\alpha + \omega) - n\beta$ (though unbounded by Kesten's result) is bounded below for $\omega = 0$. Our result shows that this can only happen for ω belonging to a set of measure 0. We do not know how far this can be improved.

Theorem 2 settles the case of bounded remainders for functions f assuming two values. On the basis of eigenvalues one can rule out the same way some other discrete cases, too, while it can be shown that in a non-atomic system all symmetric distributions with values $+1$, -1 and 0, $+1$ and -1 taken with equal measures and 0 with positive measure always occur, but already this problem of three values, let alone the general problem of characterizing the distribution functions $\mu(f < x)$ of functions f with bounded remainder remain open.

We have seen that the remainder can only be bounded in some special cases. However,

THEOREM 3. *If the dynamic system is non-atomic, then for any non-decreasing $\varphi(n)$ tending to infinity however slowly with $\varphi(0) \geqq 2$ one can find a set A of measure $\mu = 1/2$ such that*

$$\left| \sum_{i=0}^{n-1} \chi_A(T^i\omega) - n\mu \right| \leqq \varphi(n) \qquad \text{a.e.}$$

for all $n = 1, 2, \ldots$.

In fact, this holds for any $0 \leqq \mu \leqq 1$ (see the remark after the proof).

Finally, for general f we observe

THEOREM 4. *For any integrable $f(\omega)$*

$$\sum_{i=0}^{n-1} f(T^i\omega) - n \int_\Omega f(\omega)d\mu$$

almost everywhere changes sign infinitely often in the weaker sense that it cannot be ultimately positive or negative.

PROOF OF THEOREM 4. We may assume without loss of generality that $\int_\Omega f(\omega)d\mu = 0$ and have to prove that for almost each ω

$$\sum_{i=0}^{n-1} f(T^i\omega)$$

cannot be ultimately positive, the negative case being similar. Suppose that this quantity is bounded below on a set of positive measure, otherwise there is nothing to prove. This set is easily seen to be invariant under T, hence it is in fact of measure 1 and thus we can define

$$h(\omega) = \inf_{n=1,2,\dots} \sum_{i=0}^{n-1} f(T^i\omega)$$

for almost every ω as a finite measurable function. We have

$$h(T\omega) = \inf_{n=1,2,\dots} \sum_{i=0}^{n-1} f(T^{i+1}\omega) = \inf_{n=2,3,\dots} \sum_{i=0}^{n-1} f(T^i\omega) - f(\omega) \geqq$$

$$\geqq \inf_{n=1,2,\dots} \sum_{i=0}^{n-1} f(T^i\omega) - f(\omega) = h(\omega) - f(\omega) \qquad \text{a.e.}$$

Hence

$$\kappa(\omega) \overset{\text{def.}}{=\!=} f(\omega) + h(T\omega) - h(\omega) \geqq 0 \qquad \text{a.e.}$$

and by the ergodic theorem

$$\frac{1}{n} \sum_{i=0}^{n-1} \kappa(T^i\omega) \to \int_\Omega \kappa(\omega)d\mu \qquad \text{a.e.}$$

(even if $\int_\Omega \kappa(\omega)d\mu = +\infty$ as follows easily from the finite case). The left hand side is

$$\frac{1}{n} \sum_{i=0}^{n-1} f(T^i\omega) + \frac{1}{n}\left(h(T^n\omega) - h(\omega)\right) \qquad \text{a.e.}$$

The first term tends to $\int_\Omega f(\omega)\, d\mu = 0$ a.e. by the ergodic theorem. Since $h(\omega)$ is finite a.e., there is a level set $\{\omega : |\, h(\omega)\, | \leqq M\}$ with positive measure and by the recurrence theorem (a weak version of the ergodic theorem) for almost every ω $T^n\omega$ falls into this set for infinitely many n. For such n $h(T^n\omega) - h(\omega)$ is bounded and so the second term also tends to 0. We conclude $\int_\Omega \kappa(\omega)d\mu = 0$ and $\kappa(\omega)$ being non-negative,

$$\kappa(\omega) = 0, \text{ i.e. } f(\omega) = h(\omega) - h(T\omega) \qquad \text{a.e.}$$

Thus we can write

$$\sum_{i=0}^{n-1} f(T^i\omega) = h(\omega) - h(T^n\omega) \qquad \text{a.e.}$$

Using the recurrence theorem similarly as before,

$$\limsup_{n\to\infty} h(T^n\omega) = \text{essential supremum of } h(\omega) \qquad \text{a.e.}$$

If $h(\omega) < h$ we get that $h(\omega) - h(T^n\omega) < 0$ infinitely often. If $h(\omega) = h$ and the set of these ω's has positive measure, then also $T^n\omega$ belongs to this set for infinitely many n and $h(\omega) - h(T^n\omega) = 0$ infinitely often. We see in any case that it cannot be ultimately positive and the proof is completed.

PROOF OF THEOREM 2. If

$$\sum_{i=0}^{n-1} (\chi_A(T^i\omega) - \mu(A))$$

is bounded below on a set of positive measure, then, as we proved before,

$$\chi_A(\omega) - \mu(A) = h(\omega) - h(T\omega) \qquad \text{a.e.}$$

and thus for $g(\omega) = e^{2\pi i h(\omega)}$

$$g(T\omega) = g(\omega)e^{2\pi i\mu(A) - 2\pi i \chi_A(\omega)} = g(\omega)e^{2\pi i \mu(A)} \qquad \text{a.e.}$$

since $\chi_A(\omega)$ takes only integral vaues. This means that $e^{2\pi i\mu(A)}$ is an eigenvalue with corresponding eigenfunction $g(\omega)$.

Conversely, if $g(\omega)$ is an eigenfunction with eigenvalue $e^{2\pi i\mu}$, then by ergodicity $g(\omega) \neq 0$ a.e., $\{\omega : g(\omega) \neq 0\}$ being invariant, and we can define

$$h(\omega) = \frac{1}{2\pi} \arg g(\omega)$$

with $0 \leqq \arg g(\omega) < 2\pi$, i.e. $0 \leqq h(\omega) < 1$. $g(T\omega) = g(\omega)e^{2\pi i\mu}$ implies

$$h(T\omega) = h(\omega) + \mu \pmod 1,$$

but because of $0 \leqq h(\omega) < 1$ this is only possible if $h(T\omega) - h(\omega) = \mu$ or $\mu - 1$ and so $\chi(\omega) = h(\omega) - h(T\omega) + \mu$ is either 0 or 1, i.e. the characteristic function of a set of measure

$$\int_\Omega \chi(\omega) d\mu = \int_\Omega h(\omega) d\mu - \int_\Omega h(T\omega) d\mu + \mu = \mu$$

such that

$$\left| \sum_{i=0}^{n-1} \chi(T^i\omega) - n\mu \right| = | h(\omega) - h(T^n\omega) | \leqq 1. \qquad \text{Q. e. d.}$$

PROOF OF THEOREM 1. It can be deduced easily from the following result due to ROHLIN ([1], Theorem 8.1):

In a non-atomic dynamic system for every $\varepsilon > 0$ and integer d one can find a measurable set D_0 with the property that $D_l = T^{-l}D_0$ $(l = 0, \ldots, d)$ are disjoint and $1 \geqq \mu\left(\bigcup_{l=0}^{d} D_l\right) \geqq 1 - \varepsilon = 1 - \frac{1}{d}$ where we have chosen $\varepsilon = \frac{1}{d}$.

Consider

$$A_d = \bigcup_{l=0}^{m(d)-1} D_l, \qquad B_d = \bigcup_{l=m(d)}^{d} D_l.$$

Here

$$0 < \mu(A_d) = m(d)\mu(D_0) \leqq \frac{m(d)}{d+1} \to 0$$

and

$$\mu(B_d) = \mu\left(\bigcup_{l=0}^{d} D_l \setminus A_d\right) \geqq 1 - \frac{1}{d} - \frac{m(d)}{d+1} \to 1 .$$

If $\omega \in B_d$, e.g. $\omega \in D_n$ $(m(d) \leqq n \leqq d)$, then we have for each i in $n - m(d) < i \leqq n$ and consequently (since we may assume $m(n)$ increasing) in $n - m(n) < i \leqq n$

$$T^i\omega \in A_d .$$

Choosing now a subsequence d_j such that

$$\sum_{j=1}^{\infty} \frac{m(d)}{d_j + 1} \leqq 1/2 ,$$

and defining

$$A = \bigcup_{j=1}^{\infty} A_{d_j}$$

we have $\mu(A) \leqq 1/2$, while $\mu(B_d) \to 1$ implies that almost every ω belongs to infinitely many B_{d_j} and if j runs through these indices, $T^i\omega \in A_{d_j} \subset A$ for all i in $n - m(n) < i \leqq n$ with a suitable n in $m(d_j) \leqq n \leqq d$. For this n which, as $m(d)$ can be assumed to tend to ∞, also tends to ∞ with j we have

$$\frac{1}{m(n)} \sum_{i=n-m(n)+1}^{n} \chi_A(T^i\omega) = 1 \nrightarrow \mu(A) \qquad (< 1). \qquad \text{Q. e. d.}$$

PROOF OF THEOREM 3. Using again Rohlin's lemma, for every d_1 we can find a set D_0 such that $D_l = T^{-l}D_0$ $(l = 0, \ldots, d_1)$ are disjoint. Let $n(\omega)$ be the smallest index $n \geqq 0$ with $T^n\omega \in D_{d_1}$, ($n(\omega)$ is finite a.e. by the recurrence theorem), and put

$$\kappa_1(\omega) = \begin{cases} +1, \text{ if } n(\omega) \text{ is even and } > 0 \text{ or } n(\omega) = 0 \text{ and } n(T\omega) \text{ is odd} \\ -1, \text{ if } n(\omega) \text{ is odd} \\ 0 \text{ otherwise} \end{cases}$$

and

$$R_1 = \{\omega : \kappa_1(\omega) = 0\}.$$

Here, since $R_1 \subset D_{d_1}$, we have

$$\mu(R_1) \leqq \frac{\mu(\Omega)}{d_1 + 1} = \frac{1}{d_1 + 1}$$

and also

$$\left| \sum_{i=0}^{n-1} \kappa_1(T^i\omega) \right| \leqq 1$$

where the latter can be seen by analyzing the sequence $\kappa_1(T^i\omega)$ $(i = 0, 1, \ldots)$: A $+1$ must always be followed by a -1; a -1 is followed either by a $+1$ or by a 0; finally, since $\omega \in R_1 \subset D_{d_1}$ implies $T^i\omega \in D_{d_1 - i}$ and so $\notin R_1 \subset D_{d_1}$ for $i = 1$, a 0 is seen to be a.e. followed by a $+1$. The latter implication holds for $i = 1, 2, \ldots, d_1$, too, and we also get the important fact that whenever $T^{i_1}\omega,\ T^{i_2}\omega \in R_1$ $(i_1 \neq i_2)$, necessarily $|\, i_1 - i_2 \,| \geqq d_1$ a.e.

Let us now consider the transformation $T_1\omega = T^{m(\omega)}\omega$ of R_1 into itself called the derivative or T with respect to R_1 where $m(\omega)$ is defined as the smallest index $m \geqq 1$ for which $T^m\omega \in R_1$. This also preserves μ and is ergodic (see KAKUTANI [6]) giving rise to a new dynamic system with the only defect that its measure is not normalized, a property that will not be used during this proof. We apply the same construction to the new system with d_2 in place of d_1 and get a $\kappa_2(\omega)$ which we define to be zero outside R_1. We have

$$\left| \sum_{i=0}^{n-1} \kappa_2(T^i\omega) \right| \leqq 1 \qquad (\omega \in \Omega)$$

since this sum arises from a sum like $\sum_{i=0}^{m} \kappa_2(T_1^i\omega_1)$ $(\omega_1 \in R_1)$ by inserting zeros at certain places. For the same reason $T^{i_1}\omega,\ T^{i_2}\omega \in R_2\,(i_1 \neq i_2)$ implies $|\, i_1 - i_2 \,| \geqq d_2$ where $R_2 \subset R_1$ is the set $\{\omega\colon \kappa_2(\omega) = 0,\ \omega \in R_1\}$;

$$\mu(R_2) \leqq \frac{\mu(R_1)}{d_2+1} \leqq \frac{1}{(d_1+1)(d_2+1)}\,.$$

Next we consider the derivative of T with respect to R_2 and so on. In this way we obtain sets $R_1 \supset R_2 \supset \ldots$ with

$$\mu\left(\bigcap_{j=1}^{\infty} R_j\right) = \lim_{j\to\infty} \mu(R_j) \leqq \lim_{j\to\infty} \frac{1}{(d_1+1)\ldots(d_j+1)} = 0$$

and such that $T^{i_1}\omega,\ T^{i_2}\omega \in R_j$ $(i_1 \neq i_2)$ implies $|\, i_1 - i_2 \,| \geqq d_j$ a.e. and also functions $\kappa_j(\omega)$ with disjoint supports $R_{j-1} \setminus R_j$ $(R_0 = \Omega)$, assuming $+1$ and -1 there and satisfying

$$\left| \sum_{i=0}^{n-1} \kappa_j(T^i\omega) \right| \leqq 1\,.$$

Defining

$$\kappa(\omega) = \sum_{j=1}^{\infty} \kappa_j(\omega)\,,$$

these properties imply in particular that $\kappa(\omega)$ takes only $+1$ and -1 a.e. Let $l = l(\omega, n)$ be the largest index for which there exist i_1 and i_2 $(i_1 \neq i_2)$ both in $[0, n)$ such that $T^{i_1}\omega,\ T^{i_2}\omega \in R_l$. If there is no such index, we put $l = 0$; if there are infinitely many such indices, we need not define l as this can only happen on a set of measure zero. We note that $n \geqq |i_2 - i_1| \geqq d_l$ (putting $d_0 = 0$). According to this definition there is at most one index $i_3 \in [0, n)$ with $T^{i_3}\omega \in R_{l+1}$ and since $\kappa_j(\omega) = 0$ if $\omega \notin R_{l+1}$ and $j > l+1$, we get

$$\kappa(T^i\omega) = \sum_{j=1}^{l+1} \kappa_j(T^i\omega) \qquad (i = 0, 1, \ldots, n-1)$$

with at most one exception $i = i_3$ where the left is $+1$ or -1 and the right vanishes.

Hence

$$\left|\sum_{i=0}^{n-1} \kappa(T^i\omega)\right| \leqq \left|\sum_{i=0}^{n-1} \sum_{j=1}^{l+1} \kappa_j(T^i\omega)\right| + 1 \leqq \sum_{j=1}^{l+1} \left|\sum_{i=0}^{n-1} \kappa_j(T^i\omega)\right| + 1 \leqq l + 2.$$

In view of $n \geqq d_l$ let us choose the sequence d_j in such a way that $\varphi(d_j) \geqq j + 2$, yielding

$$\left|\sum_{i=0}^{n-1} \kappa(T^i\omega)\right| \leqq \varphi(n) \qquad \text{a.e.}$$

and $\frac{1+\kappa(\omega)}{2}$ being the characteristic function of a set, obviously of measure 1/2, the proof is completed.

The case of rational μ is similar except that R_1 and $\kappa_1(\omega)$ (with values μ, $\mu - 1$ and 0) have to be defined by congruence relations with the denominator of μ as modulus in place of 2. The distance of two consecutive recurrences in A can even be made $> \frac{1}{3\mu}$. In the general case one constructs increasing sets A_k and B_k, $A_k \cap B_k = \emptyset$, of rational measures by induction as follows:

For the derivative of T with respect to $C_k \overset{\text{def.}}{=} \Omega \setminus (A_k \cup B_k)$ one chooses a $D_k \subset C_k$ with small rational $\mu(C_k \setminus D_k)$ and small remainder and then an E_k for the derivative with respect to D_k with rational measure and small remainder and puts $A_{k+1} = A_k \cup E_k$, $B_{k+1} = B_k \cup (D_k \setminus E_k)$.

If one makes the recurrences in $C_k \setminus D_k$ scarce (which, as has been remarked, is also possible) and $\mu(A_k)$ and $\mu(B_k)$ rapidly converge to μ and $1 - \mu$ respectively then $A = \bigcup A_k$ will also have small remainder.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13–15

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ON THE ELEMENTARY SYMMETRIC POLYNOMIALS OF INDEPENDENT RANDOM VARIABLES

By

G. HALÁSZ and G. J. SZÉKELY (Budapest)

Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of independent identically distributed random variables. In the classical theorems of probability theory the asymptotic behaviour of the partial sums $S_n = \sum_{j=1}^{n} \xi_j$ of $\{\xi_j\}_{j=1}^{\infty}$ is investigated. In this paper we discuss a natural generalization of this problem: we deal with the asymptotic properties of the elementary symmetric polynomials of $\xi_1, \xi_2, \ldots, \xi_n$:

$$S_n^{(k)} = \sum_{1 \leqq j_1 < j_2 < \ldots < j_k \leqq n} \xi_{j_1}\xi_{j_2} \ldots \xi_{j_k}.$$

THEOREM. *If* $\lim_{n\to\infty} \frac{k}{n} = c$ $(0 \leqq c \leqq 1)$, $\xi_j > 0$ *and*

(i) $E|\log \xi_j| < +\infty$ *when* $c = 1$,

(ii) $E\log(1+\xi_j) < +\infty$ *when* $0 < c < 1$,

(iii) $E\xi_j < +\infty$ *when* $c = 0$,

then $\lim_{n\to\infty} \sqrt[k]{S_n^{(k)} \Big/ \binom{n}{k}} = S^{(c)}$ *exists with probability one,* $S^{(0)} = E\xi_j$ *and*

$$S^{(c)} = c(1-c)^{\frac{1-c}{c}} \exp\left\{\frac{1}{c}\left[E\log(r_c+\xi_j) + (c-1)\log r_c\right]\right\}, \tag{1}$$

if $0 < c \leqq 1$, *where* r_c *is the unique nonnegative root of the equation*

$$E\frac{r}{r+\xi_j} = 1 - c \tag{2}$$

($0 \log 0 = 0$ *by definition*).

PROOF. Let us denote by $\rho = \rho(n, k, \xi_1, \xi_2, \ldots, \xi_n)$ the unique positive root of the equation

$$\frac{1}{n}\sum_{j=1}^{n} \frac{\rho}{\rho+\xi_j} = 1 - \frac{k}{n} \qquad (1 < k \leqq n). \tag{3}$$

First we prove that if c is strictly between 0 and 1 then $S_n^{(k)}$ asymptotically equals

$$Cn^{-1/2}\rho^{k-n}\prod_{j=1}^{n}(\rho+\xi_j) \tag{4}$$

with probability one as $n \to \infty$, where C is a positive finite constant

$$\left(C = \left[2\pi E \frac{r_c \xi_j}{(r_c + \xi_j)^2} \right]^{-1/2} \right).$$

It is useful to observe that $\lim_{n\to\infty} \rho = r_c$ with probability one, when $\lim_{n\to\infty} \frac{k}{n} = c$ because if e.g. $\rho > r_c + \varepsilon$ occurs infinitely often (where $\varepsilon > 0$) then for some sequence $\{n_s\}_{s=1}^{\infty}$ of integers

$$\frac{1}{n_s} \sum_{j=1}^{n_s} \frac{\rho}{\rho + \xi_j} \geqq \frac{1}{n_s} \sum_{j=1}^{n_s} \frac{r_c + \varepsilon}{r_c + \varepsilon + \xi_j} \xrightarrow{\text{a.s.}} E \frac{r_c + \varepsilon}{r_c + \varepsilon + \xi_j} > 1 - c$$

which is a contradiction.

Evidently

$$\sum_{k=0}^{n} S_n^{(k)} z^{n-k} = \prod_{j=1}^{n} (z + \xi_j)$$

where z is an arbitrary complex number, thus by Cauchy's coefficient formula

$$S_n^{(k)} = \frac{1}{2\pi i} \int_{|z|=\rho} \prod_{j=1}^{n} (z + \xi_j) z^{k-n-1} dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{j=1}^{n} (\rho e^{i\vartheta} + \xi_j) \rho^{k-n} e^{(k-n)i\vartheta} d\vartheta =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\{u(\vartheta)\} d\vartheta,$$

where

$$u(\vartheta) = \sum_{j=1}^{n} \log(\rho e^{i\vartheta} + \xi_j) + (k-n)i\vartheta + (k-n)\log \rho.$$

The function $|\exp\{u(\mu)\}|$ is a decreasing function of $|\vartheta|$ $(0 \leqq |\vartheta| \leqq \pi)$ since $|\rho e^{i\vartheta} + \xi_j|$ has this property, hence for any $0 < \delta \leqq \frac{\pi}{2}$

$$\left| \frac{1}{2\pi} \int_{\delta \leqq |\vartheta| \leqq \pi} \exp\{u(\vartheta)\} \right| \leqq |\exp\{\delta\}|.$$

Calculating the derivatives of $u(\vartheta)$,

$$u'(0) = i \left[\sum_{j=1}^{n} \frac{\rho}{\rho + \xi_j} + k - n \right] = 0$$

by (2),

$$u''(0) = -\sum_{j=1}^{n} \frac{\xi_j \rho}{(\rho + \xi_j)^2} \sim -nE \frac{\xi_j r_c}{(r_c + \xi_j)^2}$$

with probability 1 by $\rho \to r_c$ and the law of large numbers, finally

$$|u'''(\vartheta)| = \left| \sum_{j=1}^{n} \frac{\xi_j \rho (\rho e^{i\vartheta} - \xi_j)}{(\rho e^{i\vartheta} + \xi_j)^3} \right| < n$$

for $|\vartheta| \leqq \pi/4$. If δ is a sufficiently small fixed number, by Taylor's formula these imply on the one hand

$$\frac{1}{2\pi} \int\limits_{\delta<|\vartheta|\leqq\pi} \exp\{u(\vartheta)\}d\vartheta \leqq \exp\{u(0) - c_1\delta^2 n\} \qquad (c_1 > 0)$$

and on the other,

$$\frac{1}{2\pi} \int\limits_{|\vartheta|\leqq\delta} \exp\{u(\vartheta)\}d\vartheta = \frac{1}{2\pi} \exp\{u(0)\} \left\{ \int\limits_{-\delta}^{\delta} \exp\{u''(0)\vartheta^2/2\}d\vartheta + \right.$$

$$\left. + O(n) \int\limits_{-\infty}^{\infty} |\vartheta^3| \exp\{-c_2 n\vartheta^2\}\, d\vartheta \right\} \sim \frac{1}{2\pi} \exp\{u(0)\} \int\limits_{-\infty}^{\infty} \exp\{u''(0)\vartheta^2/2\}d\vartheta =$$

$$= \sqrt{\frac{1}{2\pi(-u''(0))}} \exp\{u(0)\},$$

as $n \to \infty$. Thus we have proved (4.)

The Theorem is a simple consequence of (4) if $0 < c < 1$. For the proof of the cases $c = 0$ and $c = 1$ it is enough to mention that $\sqrt[k]{S_n^{(k)} \Big/ \binom{n}{k}}$ is a decreasing function of k ([1], p. 51), therefore $S^{(c)}$ is also a decreasing function of c. (E.g. if $c = 0$, then

$$E\xi_j = \lim_{n\to\infty} S_n/n \geqq \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \sqrt[k]{S_n^{(k)} \Big/ \binom{n}{k}} \geqq \lim_{c\to+0} S^{(c)}$$

but this last limit equals $E\xi_j$ because from (2)

$$\lim_{c\to+0} cr_c = \lim_{r\to\infty} E\xi_j \frac{r}{r+\xi_j} = E\xi_j,$$

hence

$$\lim_{c\to+0} S^{(c)} = e^{-1}E\xi_j \lim_{c\to+0} \exp\left\{E \log\left(1 + \frac{\xi_j}{r_c}\right)^{\frac{1}{c}}\right\} = E\xi_j.$$

REMARKS 1. A special case of the Theorem is proved in [2] by another method.
2. The fact that $S_n^{(k)}$ asymptotically equals (4) implies that the limit distribution of

$$\left(\sqrt[k]{S_n^{(k)} \Big/ \binom{n}{k}} \Big/ S^{\left(\frac{k}{n}\right)}\right)^{\sqrt{n}}$$

is lognormal when $\lim_{n\to\infty} \frac{k}{n} = c > 0$. This is not true when $c = 0$. In connection with this case HOEFFDING [3] proved that if k is a fixed number and $E|\xi_j|^2 < +\infty$, then the distribution of

$$\sqrt{n}\left(S_n^{(k)}\Big/\binom{n}{k} - (E\xi_j)^k\right)$$

tends to a normal distribution with 0 expectation as $n \to \infty$.

3. In the exponent of (1)

$$s(c) = E\log(r_c + \xi_j) + (c-1)\log r_c = \inf_{r \geqq 0}\{E\log(r+\xi_j) + (c-1)\log r\}$$

is a Chernoff type function.

$$E\log(r+\xi_j) = \sup_{0 \leqq c \leqq 1}\{s(c) + (1-c)\log r\},$$

$$\frac{d^m E\log(r+\xi_j)}{d^m r^m} = E(r+\xi_j)^{-m}(-1)^{m+1},$$

thus $s(c)$ (and $S^{(c)}$ also) determines uniquely the distribution of ξ_j because the distribution of the bounded random variable $(r+\xi_j)^{-1}$ is uniquely determined by its moments.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15

DEPARTMENT OF PROBABILITY THEORY AND STATISTIC
EÖTVÖS LORÁND UNIVERSITY
1088 BUDAPEST, MÚZEUM KRT. 6—8

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ОБ ОДНОМ МЕТОДЕ ПРИБЛИЖЕНИЯ НЕПРЕРЫВНЫХ ФУНКЦИЙ МНОГОЧЛЕНАМИ

О. КИШ (Будапешт) и ХО ТХО КАУ (Ханой)

Введем следующие обозначения: $f(x)$ определенная на отрезке $-1 \leqq x \leqq 1$, вещественная, непрерывная функция; k неотрицательное, n и r положительное целое число:

$$D(\vartheta) = \frac{1}{2} + \sum_{i=1}^{n-1} \cos i\vartheta + \frac{1}{2}\cos n\vartheta = \frac{1}{2}\sin n\vartheta \operatorname{ctg}\frac{\vartheta}{2} \tag{1}$$

модифицированное ядро Дирихле:

$$P_{k,r}(x) = 2^{-k}\sum_{q=0}^{k}\binom{k}{q}\frac{1}{nr}\sum_{p=1}^{2nr} f\left(\cos\frac{p\pi}{nr}\right) D\left(\arccos x - \frac{p\pi}{nr} + \frac{2q-k}{2n}\pi\right). \tag{2}$$

Легко видеть, что $P_{k,r}(x)$ многочлен, степень которого не выше чем n. В статье [2] была получена оценка погрешности $P_{k,r}(x) - f(x)$ при $k = 0, 1$ и 2. Цель настоящей работы получить оценку, более точную в концах отрезка $-1 \leqq x \leqq 1$.

Формулу (21) из [2] можно записать в виде

$$| P_{0,r}(x) - f(x) | \leqq a_{n,r}\,\omega\left(\frac{\pi}{n}\right), \tag{3}$$

где $\omega(h)$ модуль непрерывности функции $f(x)$ и

$$\begin{aligned} a_{n,r} &= \frac{1}{2} + \frac{1}{2nr}\max_{\vartheta\in(-\infty,\infty)}\sum_{p=1}^{2nr}\left| D\left(\vartheta - \frac{p\pi}{nr}\right)\right| = \\ &= \frac{1}{2} + \frac{1}{2nr}\sum_{p=1}^{2nr}\left| D\left(\frac{2p-1}{2nr}\pi\right)\right| = \\ &= \frac{1}{2} + \frac{1}{2nr}\sum_{j=1}^{r}\sin\frac{2j-1}{2r}\pi\sum_{i=0}^{n-1}\operatorname{ctg}\frac{2ir+2j-1}{4nr}\pi = \\ &= \frac{1}{2} + [1-(-1)^n]\frac{1}{4n} + \frac{1}{2nr}\sum_{i=1}^{n-1}[1-(-1)^{i+n}]\operatorname{tg}\frac{i\pi}{2n}\operatorname{cosec}\frac{i\pi}{2nr}. \end{aligned} \tag{4}$$

Ниже будет получена

Т е о р е м а 1. *На отрезке* $-1 \leqq x \leqq 1$

$$\begin{aligned} | P_{0,r}(x) - f(x) | &\leqq a_{n,r}\,\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right) + \\ &+ \frac{2}{r\pi}\operatorname{cosec}\frac{\pi}{2r}\sum_{i=1}^{2n-1}\frac{1}{i}\,\omega\left(\frac{i\pi^2|x|}{2n^2}\right) + \omega\left(\frac{\pi^2|x|}{8n^2}\right). \end{aligned} \tag{5}$$

В [2] опубликовано неравенство

$$(6)\qquad |P_{1,r}(x)-f(x)|\leqq \frac{1}{2}\omega\left(\frac{\pi}{n}\right)+b_{n,r}\omega\left(\frac{\pi}{2n}\right),$$

здесь

$$b_{n,r}=\frac{1}{2}+\frac{1}{nr}\sum_{1\leqq i\leqq \frac{n}{2}}\operatorname{ctg}\frac{2i-1}{2n}\pi\operatorname{cosec}\frac{2i-1}{2nr}\pi\left(1-\cos\frac{2i-1}{2n}\pi\right)$$

$$(r=2,4,6,\dots);$$

$$b_{n,r}=\frac{1}{2}+\frac{1}{nr}\sum_{1\leqq i\leqq n/2}\operatorname{ctg}\frac{2i-1}{2n}\pi\operatorname{cosec}\frac{2i-1}{2nr}\pi\left(1-\cos\frac{2i-1}{2n}\pi\cos\frac{2i-1}{2nr}\pi\right)$$

$$(r=1,3,5,\dots).$$

В настоящей статье будет доказана

Теорема 2. *Имеют место неравенства*

$$(7)\qquad |P_{1,r}(x)-f(x)|\leqq\frac{1}{2}\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+b_{n,r}\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)+$$

$$+\left[\frac{2}{\pi}+\left(1-\frac{2}{\pi}\right)\frac{1}{n^2}\right]\frac{1}{r}\operatorname{cosec}\frac{\pi}{2r}\sum_{i=1}^{2n-1}\frac{1}{i^2-1}\omega\left(\frac{i\pi^2|x|}{2n^2}\right)+$$

$$+\left[\frac{1}{2}+\left(\frac{1}{2}+\frac{\pi}{8}-\frac{3}{2\pi}\right)\frac{1}{r}\operatorname{cosec}\frac{\pi}{2r}\right]\omega\left(\frac{\pi^2|x|}{2n^2}\right)+b_{n,r}\omega\left(\frac{\pi^2|x|}{8n^2}\right);$$

$$(8)\qquad |P_{1,r}(x)-f(x)|\leqq\frac{1}{2}\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+b_{n,r}\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)+$$

$$+\frac{1}{2}a_{n,r}\omega\left(\frac{\pi^2|x|}{n^2}\right)+b_{n,r}\omega\left(\frac{(\pi^2|x|}{8n^2}\right).$$

Неравенство (8) проще чем (7), но менее точно. В [2] получено также соотношение

$$(9)\qquad |P_{2,r}(x)-f(x)|\leqq c_{n,r}\omega\left(\frac{\pi}{n}\right),$$

где

$$(10)\qquad c_{n,r}=\max_{0\leqq t\leqq\frac{\pi}{nr}}\left[\frac{3}{2}-\frac{1}{4nr}\sum_{j=-r}^{r-1}D\left(t+\frac{j\pi}{nr}\right)+\right.$$

$$\left.+\frac{1}{4nr}(-1)^n\sum_{j=-r}^{r-1}D\left(t+\frac{j\pi}{nr}+\pi\right)\right]=\frac{3}{2}-\frac{1}{4r}-\frac{1}{2nr}\sum_{j=1}^{r-1}\sin\frac{j\pi}{r}\operatorname{cosec}\frac{j\pi}{nr}=$$

$$=\frac{3}{2}+[(-1)^n-1]\frac{1}{4n}+\frac{1}{4nr}\sum_{i=1}^{n-1}[(-1)^{i+n}-1]\sin\frac{i\pi}{n}\operatorname{ctg}\frac{i\pi}{nr}.$$

Ниже будет доказана

Теорема 3. *Выполняется неравенство*

$$|P_{2,r}(x)-f(x)\leqq c_{n,r}\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+\left(\frac{1}{2}+c_{n,r}\right)\omega\left(\frac{\pi^2|x|}{2n^2}\right). \tag{11}$$

Замечания. Из доказательства теорем 1—3 видно, что они остаются в силе, если многочлен (2) заменяется на

$$\frac{1}{2^k nr}\sum_{q=0}^{k}\binom{k}{q}\sum_{p=1}^{2nr} f\left(\cos\frac{2p-1}{2nr}\pi\right) D\left(\arc\cos x-\frac{2p-1}{2nr}\pi+\frac{2q-k}{2n}\pi\right).$$

Аналогичные теоремам 1—3 неравенства можно получить и при $k\geqq 3$.

Для случая $r=1$ теоремы 1—3 были получены в работе [1]. Величина чисел $a_{n,r}$, $b_{n,r}$, $c_{n,r}$ оценена в статье [2].

Для хороших функций в концах отрезка $-1\leqq x\leqq 1$ оценки (5), (7), (8) и (11) очевидно лучше чем неравенства (3), (6) и (9). Если, например, выполняется условие Липшица

$$|f(x+h)-f(x)|\leqq Mh^\alpha\,(0\leqq h\leqq 2,\ -1\leqq x\leqq 2-h,\ 0<M,\ 0<\alpha\leqq 1),$$

то в формулах (5) и (7) очевидно

$$\sum_{i=1}^{2n-1}\frac{1}{i}\,\omega\left(\frac{i\pi^2|x|}{2n^2}\right)\leqq\frac{M}{\alpha}\left(\frac{\pi^2|x|}{n}\right)^\alpha,$$

$$\sum_{i=1}^{2n-1}\frac{1}{i^2-1}\,\omega\left(\frac{i\pi^2|x|}{2n^2}\right)\leqq M\left(\frac{1}{3}+\frac{1}{1-\alpha}\right)\left(\frac{\pi^2|x|}{2n^2}\right)^\alpha\qquad(\alpha<1).$$

Легко видеть, что теоремы 1—3 остаются в силе, если величины

$$P_{k,r}(x),\quad a_{n,r},\ \frac{1}{r}\operatorname{cosec}\frac{\pi}{2r},\quad b_{n,r},\quad c_{n,r}$$

заменяются на их предел при $r\to\infty$.

В дальнейшем все доказательства проводятся для отрезков

$$\cos\left(\frac{l\pi}{n}+\frac{2m+1}{2nr}\pi\right)\leqq x\leqq\cos\left(\frac{l\pi}{n}+\frac{m\pi}{nr}\right) \tag{12}$$

$$(l=0,1,\dots,n-1;\qquad m=0,1,\dots,r-1).$$

Аналогичным образом можно рассматривать случай

$$\cos\left(\frac{l\pi}{n}+\frac{m+1}{nr}\pi\right)\leqq x\leqq\cos\left(\frac{l\pi}{n}+\frac{2m+1}{2nr}\pi\right).$$

Введем следующие обозначения:

$$t=\arc\cos x-\frac{l\pi}{n}-\frac{m\pi}{nr}; \tag{13}$$

$$x_{i,j}=\cos\left(\frac{i+l}{n}\pi+\frac{m-j}{nr}\pi\right)\qquad(-n<i\leqq n,\quad 0\leqq j<r). \tag{14}$$

Лемма 1. *Если* $-n < i < 0$ *и* $0 \leqq j < \frac{r}{2}$ *или* $0 < i < n$ *и* $\frac{r}{2} \leqq j \leqq r$, *то*

$$|x_{i,j} - x_{i+1,j}| < \frac{\pi}{n}\sqrt{1-x^2} + \frac{\pi^2|ix|}{n^2};$$

а если $-n < i < 0$ *и* $\frac{r}{2} \leqq j < r$ *или* $0 < i < n$ *и* $0 \leqq j < \frac{r}{2}$, *то*

$$|x_{i,j} - x_{i+1,j}| < \frac{\pi}{n}\sqrt{1-x^2} + \frac{2|i|+1}{2n^2}\pi^2|x|.$$

Доказательство. Из (13) и (14) получаем:

$$x_{i,j} - x_{i+1,j} = 2\sin\frac{\pi}{2n}\sin\left(\frac{i+l}{n}\pi + \frac{\pi}{2n} + \frac{m-j}{nr}\pi\right) = \tag{15}$$

$$= 2\sin\frac{\pi}{2n}\sin\left(\arccos x + \frac{2i+1}{2n}\pi - t - \frac{j\pi}{nr}\right) =$$

$$= 2\sin\frac{\pi}{2n}\left[\sqrt{1-x^2}\cos\left(\frac{2i+1}{2n}\pi - t - \frac{j\pi}{nr}\right) - x\sin\left(\frac{2i+1}{2n}\pi - t - \frac{j\pi}{nr}\right)\right].$$

Из (12) и (13) следует:

$$0 \leqq t \leqq \frac{\pi}{2nr}. \tag{16}$$

Отсюда и из (15) получаем лемму.

Лемма 2. *Если* $i = 0$ *и* $0 \leqq j < \frac{r}{2}$ *или* $i = 1$ *и* $\frac{r}{2} \leqq j < r$, *то*

$$|x_{i,j} - x| < \frac{\pi}{2n}\sqrt{1-x^2} + \frac{\pi^2|x|}{8n^2};$$

а если $i = 0$ *и* $\frac{r}{2} \leqq j < r$ *или* $i = 1$ *и* $0 \leqq j < \frac{r}{2}$, *то*

$$|x_{i,j} - x| < \frac{\pi}{n}\sqrt{1-x^2} + \frac{\pi^2|x|}{2n^2}.$$

Доказательство. Из (13) и (14) получаем:

$$x - x_{i,j} = x - \cos\left(\arccos x + \frac{ir-j}{nr}\pi - t\right) =$$

$$= \sqrt{1-x^2}\sin\left(\frac{ir-j}{nr}\pi - t\right) + 2x\sin^2\left(\frac{ir-j}{2nr}\pi - \frac{t}{2}\right).$$

Отсюда и из (16) следует лемма 2.

Введем следующие обозначения:

$$s_i(\vartheta) = \frac{1}{2^k n} \sum_{q=0}^{k} \binom{k}{q} D\left(\vartheta + \frac{2q-2i-k}{2n}\pi\right) \qquad (-n < i \leqq n). \tag{17}$$

$$\sigma_i(\vartheta) = \sum_{j=1-n}^{i} s_j(\vartheta) \qquad (-n < i \leqq 0), \tag{18}$$

$$\sigma_i(\vartheta) = \sum_{j=i}^{n} s_j(\vartheta) \qquad (0 < i \leqq n). \tag{19}$$

Лемма 3. *На отрезках* (12)

$$\begin{aligned} r\,|P_{k,r}(x) - f(x)| \leqq{}& \omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)\Bigg[\sum_{0\leqq j<\frac{r}{2}} \sum_{\substack{i=1-n\\ i\neq 0}}^{n} \left|\sigma_i\left(t+\frac{j\pi}{nr}\right)\right| + \\ &+ \sum_{\frac{r}{2}\leqq j<r} \sum_{\substack{i=1-n\\ i\neq 1}}^{n} \left|\sigma_i\left(t+\frac{j\pi}{nr}\right)\right|\Bigg] + \omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)\Bigg[\sum_{0\leqq j<\frac{r}{2}} \left|\sigma_0\left(t+\frac{j\pi}{nr}\right)\right| + \\ &+ \sum_{\frac{r}{2}\leqq j<r} \left|\sigma_1\left(t+\frac{j\pi}{nr}\right)\right|\Bigg] + \sum_{i=1}^{n-1} \omega\left(\frac{i\pi^2|x|}{n^2}\right)\Bigg[\sum_{0\leqq j<\frac{r}{2}} \left|\sigma_{-i}\left(t+\frac{j\pi}{nr}\right)\right| + \\ &+ \sum_{\frac{r}{2}\leqq j<r} \left|\sigma_{i+1}\left(t+\frac{j\pi}{nr}\right)\right|\Bigg] + \\ &+ \sum_{i=0}^{n-1} \omega\left(\frac{2i+1}{2n^2}\pi^2|x|\right)\Bigg[\sum_{\frac{r}{2}\leqq j<r} \left|\sigma_{-i}\left(t+\frac{j\pi}{nr}\right)\right| + \sum_{0\leqq j<\frac{r}{2}} \left|\sigma_{i+1}\left(t+\frac{j\pi}{nr}\right)\right|\Bigg] + \\ &+ \omega\left(\frac{\pi^2|x|}{8n^2}\right)\Bigg[\sum_{0\leqq j<\frac{r}{2}} \left|\sigma_0\left(t+\frac{j\pi}{nr}\right)\right| + \sum_{\frac{r}{2}\leqq j<r} \left|\sigma_1\left(t+\frac{j\pi}{nr}\right)\right|\Bigg]. \end{aligned}$$

Доказательство. Так как $f(\cos\vartheta)$ и $D(\vartheta)$ 2π-периодические функции, то в формуле (2) условие $0 < p \leqq 2nr$ можно заменить на соотношение $lr + m - nr < p \leqq lr + m + nr$. Полагая $p = lr + m + ir - j$ $(-n < i \leqq n,$ $0 \leqq j < r)$ и используя обозначения (13), (14) и (17), можно переписать равенство (2) в виде

$$P_{k,r}(x) = \frac{1}{r}\sum_{j=0}^{r-1} \sum_{i=1-n}^{n} f(x_{i,j})s_i\left(t+\frac{j\pi}{nr}\right). \tag{20}$$

Из формул (1) и (17) легко следует тождество

$$\sum_{i=1-n}^{n} s_i(\vartheta) = 1. \tag{21}$$

С помощью преобразования Абеля и равенств (18)–(21) получаем:

$$P_{k,r}(x)-f(x)=\frac{1}{r}\sum_{j=0}^{r-1}\left\{\sum_{i=1-n}^{-1}[f(x_{i,j})-f(x_{i+1,j})]\sigma_i\left(t+\frac{j\pi}{nr}\right)+\right.$$

$$+[f(x_{0,j})-f(x)]\sigma_0\left(t+\frac{j\pi}{nr}\right)+[f(x_{1,j})-f(x)]\sigma_1\left(t+\frac{j\pi}{nr}\right)+$$

$$\left.+\sum_{i=2}^{n}[f(x_{i,j})-f(x_{i-1,j})]\sigma_i\left(t+\frac{j\pi}{nr}\right)\right\}.$$

Отсюда и из лемм 1—2 получаем лемму 3.

Лемма 4. *Если* $k=0$, *то*

$$\text{(22)}\qquad \sum_{0\leqq j<\frac{r}{2}}\left|\sigma_{-i}\left(t+\frac{j\pi}{nr}\right)\right|+\sum_{\frac{r}{2}\leqq j<r}\sigma_{i+1}\left(t+\frac{j\pi}{nr}\right)\Bigg|<\frac{1}{i\pi}\operatorname{cosec}\frac{\pi}{2r}\qquad(0<i<n),$$

$$\text{(23)}\qquad \sum_{\frac{r}{2}\leqq j<r}\left|\sigma_{-i}\left(t+\frac{j\pi}{nr}\right)\right|+\sum_{0\leqq j<r/2}\left|\sigma_{i+1}\left(t+\frac{j\pi}{nr}\right)\right|<\frac{2}{\pi}\frac{1}{2i+1}\operatorname{cosec}\frac{\pi}{2r}$$

$$(0\leqq i<n).$$

Доказательство. Пусть $k=0$. Формулу (54) из [1] можно записать в виде

$$|\sigma_i(\vartheta)|\leqq\frac{1}{n}\left|D\left(\vartheta-\frac{i\pi}{n}\right)\right|\qquad\left(-n<i\leqq n,\ 0\leqq\vartheta\leqq\frac{\pi}{n}\right).$$

Отзюда и из (1) получаем:

$$\text{(24)}\qquad |\sigma_i(\vartheta)|\leqq\frac{1}{2n}\sin n\vartheta\operatorname{ctg}\left|\frac{\vartheta}{2}-\frac{i\pi}{2n}\right|.$$

Поэтому и ввиду (16) при $0<i<n$ и $0\leqq j<r$

$$\left|\sigma_{-i}\left(t+\frac{j\pi}{nr}\right)\right|\leqq\frac{1}{2n}\sin\left(nt+\frac{j\pi}{r}\right)\operatorname{ctg}\frac{i\pi}{2n}<\frac{1}{i\pi}\sin\left(nt+\frac{j\pi}{r}\right),$$

$$\left|\sigma_{i+1}\left(t+\frac{j\pi}{nr}\right)\right|\leqq\frac{1}{2n}\sin\left(nt+\frac{j\pi}{r}\right)\operatorname{ctg}\frac{i\pi}{2n}<\frac{1}{i\pi}\sin\left(nt+\frac{j\pi}{r}\right).$$

Следовательно

$$\text{(25)}\qquad \sum_{0\leqq j<\frac{r}{2}}\sigma_{-i}\left(t+\frac{j\pi}{nr}\right)\Bigg|+\sum_{\frac{r}{2}\leqq j<r}\left|\sigma_{i+1}\left(t+\frac{j\pi}{r}\right)\right|<\frac{1}{i\pi}\sum_{j=0}^{r-1}\sin\left(nt+\frac{j\pi}{r}\right).$$

Так как

$$\text{(26)}\qquad \sum_{j=0}^{r-1}\sin(\vartheta+j\varphi)=\sin\left(\vartheta+\frac{r-1}{2}\varphi\right)\sin\frac{r\varphi}{2}\operatorname{cosec}\frac{\varphi}{2},$$

то здесь

$$\sum_{j=0}^{r-1} \sin\left(nt + \frac{j\pi}{r}\right) = \sin\left(nt + \frac{r-1}{2r}\pi\right) \operatorname{cosec}\frac{\pi}{2r} \leqq \operatorname{cosec}\frac{\pi}{2r}.$$

Отсюда и из (25) следует доказываемое неравенство (22). Соотношение (23) доказывается аналогичным образом.

Доказательство теоремы 1. В [2] доказано неравенство

$$\sum_{j=0}^{r-1} \sum_{i=1-n}^{n} \left| \sigma_i\left(t + \frac{j\pi}{nr}\right) \right| \leqq r a_{n,r}. \tag{27}$$

Из (24) следует:

$$| \sigma_i(\vartheta) | \leqq 1 \qquad \left(-n \leqq i \leqq n, \quad 0 \leqq \vartheta \leqq \frac{\pi}{n}\right). \tag{28}$$

Из лемм 3—4 и из формул (27)—(28) получаем теорему 1.

Лемма 5. *Если* $k = 1$ *и* $0 < i < n$, *то*

$$\sum_{0 \leqq j < r/2} \left| \sigma_{-i}\left(t + \frac{j\pi}{nr}\right) \right| + \sum_{r/2 \leqq j < r} \left| \sigma_{i+1}\left(t + \frac{j\pi}{nr}\right) \right| < \tag{29}$$

$$< \operatorname{cosec}\frac{\pi}{2r} \left\{ \left[\frac{2}{\pi} + \left(1 - \frac{2}{\pi}\frac{1}{n^2}\right]\frac{1}{4i^2 - 1} + \left(\frac{1}{2} - \frac{1}{\pi}\right)\frac{1}{n^2} + \right.\right.$$

$$\left. + \left(\frac{\pi}{32} - \frac{1}{8} + \frac{1}{8\pi}\right) \frac{4i^2 - 1}{n^4} \right\},$$

$$\sum_{r/2 \leqq j < r} \left| \sigma_{-i}\left(t + \frac{j\pi}{nr}\right) \right| + \sum_{0 \leqq j < r/2} \left| \sigma_{i+1}\left(t + \frac{j\pi}{nr}\right) \right| < \tag{30}$$

$$< \operatorname{cosec}\frac{\pi}{2r} \left\{ \left[\frac{2}{\pi} + \left(1 - \frac{2}{\pi}\right)\frac{1}{n^2}\right]\frac{1}{4i^2 + 4i} + \left(\frac{1}{2} - \frac{1}{\pi}\right)\frac{1}{n^2} + \right.$$

$$\left. + \left(\frac{\pi}{32} - \frac{1}{8} + \frac{1}{8\pi}\right) \frac{4i^2 + 4i}{n^4} \right\}.$$

Доказательство. Пусть $k = 1$. В [1] доказана следующая формула (64):

$$| \sigma_i(\vartheta) | \leqq | s_i(\vartheta) | \qquad \left(-n < i \leqq n, \quad i \neq 0, \quad 0 \leqq \vartheta \leqq \frac{\pi}{2n}\right). \tag{31}$$

Аналогичным образом доказывается неравенство

$$| \sigma_i(\vartheta) | \leqq | s_i(\vartheta) | \qquad \left(-n < i \leqq n, \quad i \neq 1, \quad \frac{\pi}{2n} \leqq \vartheta \leqq \frac{\pi}{n}\right). \tag{32}$$

Из (1) и (17) получаем:

$$s_i(\vartheta) = (-1)^i \frac{1}{4n} \sin\frac{\pi}{2n} \cos n\vartheta \operatorname{cosec}\left(\frac{\vartheta}{2} - \frac{2i+1}{4n}\pi\right) \operatorname{cosec}\left(\frac{\vartheta}{2} - \frac{2i-1}{4n}\pi\right). \tag{33}$$

В силу (31)—(33)

$$|\sigma_{-i}(\vartheta)| < \frac{\pi}{8n^2}\cos n\vartheta \operatorname{cosec}\frac{2i-1}{4n}\pi \operatorname{cosec}\frac{2i+1}{4n}\pi$$

$$\left(0 < i < n,\ 0 \leqq \vartheta \leqq \frac{\pi}{2n}\right),$$

$$|\sigma_{i+1}(\vartheta)| < -\frac{\pi}{8n^2}\cos n\vartheta \operatorname{cosec}\frac{2i-1}{4n}\pi \operatorname{cosec}\frac{2i+1}{4n}\pi$$

$$\left(0 < i < n,\ \frac{\pi}{2n} \leqq \vartheta \leqq \frac{\pi}{n}\right).$$

Поэтому и ввиду (16)

$$\text{(34)}\qquad \sum_{0\leqq j<r/2}\left|\sigma_{-i}\left(t+\frac{j\pi}{nr}\right)\right| + \sum_{r/2\leqq j<r}\left|\sigma_{i+1}\left(t+\frac{j\pi}{nr}\right)\right| <$$

$$< \frac{\pi}{8n^2}\operatorname{cosec}\frac{2i-1}{4n}\pi \operatorname{cosec}\frac{2i+1}{4n}\pi \times$$

$$\times\left[\sum_{0\leqq j<r/2}\cos\left(nt+\frac{j\pi}{r}\right) - \sum_{r/2\leqq j<r}\cos\left(nt+\frac{j\pi}{r}\right)\right].$$

Если r четное число, то здесь ввиду (26)

$$\text{(35)}\qquad \sum_{j=0}^{\frac{r}{2}-1}\cos\left(nt+\frac{j\pi}{r}\right) - \sum_{j=\frac{r}{2}}^{r-1}\cos\left(nt+\frac{j\pi}{r}\right) =$$

$$= 2\sum_{j=0}^{\frac{r}{2}-1}\sin\frac{r-1-2j}{2r}\pi \sin\left(nt+\frac{r-1}{2r}\pi\right) \leqq 2\sum_{j=0}^{\frac{r}{2}-1}\sin\frac{r-1-2j}{2r}\pi =$$

$$= 2\sum_{j=0}^{\frac{r}{2}-1}\sin\frac{2j+1}{2r}\pi = \operatorname{cosec}\frac{\pi}{2r}.$$

Из (34) и (35) следует:

$$\text{(36)}\qquad \sum_{j=0}^{\frac{r}{2}-1}\left|\sigma_{-i}\left(t+\frac{j\pi}{nr}\right)\right| + \sum_{j=\frac{r}{2}}^{r-1}\left|\sigma_{i+1}\left(t+\frac{j\pi}{nr}\right)\right| <$$

$$< \frac{\pi}{8n^2}\operatorname{cosec}\frac{\pi}{2r}\operatorname{cosec}\frac{2i-1}{4n}\pi \operatorname{cosec}\frac{2i+1}{4n}\pi.$$

В [1] доказано, что

$$\operatorname{cosec}\vartheta \leqq \frac{1}{\vartheta} + \frac{2}{\pi}\left(1-\frac{2}{\pi}\right)\vartheta \qquad \left(0 < \vartheta \leqq \frac{\pi}{2}\right).$$

Отсюда и из (36) получаем доказываемое неравенство (29) для четных r. Соотношение (29) для нечетных r и формула (30) доказываются аналогичным образом.

Доказательство теоремы 2. Доказывая теорему 4 из [2], мы получили:

$$\sum_{0\leqq j<r/2}\sum_{\substack{i=1-n\\ i\neq 0}}^{n}\left|\sigma_i\left(t+\frac{j\pi}{nr}\right)\right|+\sum_{r/2\leqq j<r}\sum_{\substack{j=1-n\\ i\neq 1}}^{n}\left|\sigma_i\left(t+\frac{j\pi}{nr}\right)\right|=\frac{r}{2}, \tag{37}$$

$$\sum_{0\leqq j<r/2}\left|\sigma_0\left(t+\frac{j\pi}{nr}\right)\right|+\sum_{r/2\leqq j<r}\left|\sigma_1\left(t+\frac{j\pi}{nr}\right)\right|\leqq rb_{n,r}. \tag{38}$$

Из (31)—(33) следует:

$$|\sigma_0(\vartheta)|\leqq\frac{1}{2}\qquad\left(\frac{\pi}{2n}\leqq\vartheta\leqq\frac{\pi}{n}\right),$$

$$|\sigma_1(\vartheta)|\leqq\frac{1}{2}\qquad\left(0\leqq\vartheta\leqq\frac{\pi}{2n}\right).$$

Поэтому

$$\sum_{\frac{r}{2}\leqq j<r}\left|\sigma_0\left(t+\frac{j\pi}{nr}\right)\right|+\sum_{0\leqq j<\frac{r}{2}}\left|\sigma_1\left(t+\frac{j\pi}{nr}\right)\right|\leqq\frac{r}{2}. \tag{39}$$

Из лемм 3 и 5 и из формул (37)—(39) получаем:

$$\begin{aligned} |P_{1,r}(x)-f(x)|\leqq{}&\frac{1}{2}\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+b_{n,r}\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)+\\ &+\frac{1}{r}\operatorname{cosec}\frac{\pi}{2r}\sum_{i=2}^{2n-1}\omega\left(\frac{i\pi^2|x|}{2n^2}\right)\left\{\left[\frac{2}{\pi}+\left(1-\frac{2}{\pi}\right)\frac{1}{n^2}\right]\frac{1}{i^2-1}+\right.\\ &\left.+\left(\frac{1}{2}-\frac{1}{\pi}\right)\frac{1}{n^2}+\left(\frac{\pi}{32}-\frac{1}{8}+\frac{1}{8\pi}\right)\frac{i^2-1}{n^4}\right\}+\frac{1}{2}\omega\left(\frac{\pi^2|x|}{2n^2}\right)+b_{n,r}\omega\left(\frac{\pi^2|x|}{8n^2}\right). \end{aligned} \tag{40}$$

Здесь

$$\sum_{i=2}^{2n-1}\omega\left(\frac{i\pi^2|x|}{2n^2}\right)\leqq\omega\left(\frac{\pi^2|x|}{2n^2}\right)\sum_{i=2}^{2n-1}i=(2n^2-n-1)\,\omega\left(\frac{\pi^2|x|}{2n^2}\right)\leqq 2n^2\omega\left(\frac{\pi^2|x|}{2n^2}\right), \tag{41}$$

$$\sum_{i=2}^{2n-1}i^2\omega\left(\frac{i\pi^2|x|}{2n^2}\right)\leqq(4n^4-4n^3+n^2-1)\,\omega\left(\frac{\pi^2|x|}{2n^2}\right)\leqq 4n^4\omega\left(\frac{\pi^2|x|}{2n^2}\right). \tag{42}$$

Из (40)—(42) получаем доказываемое неравенство (7).

Из леммы 3 и из формул (37)—(38) получаем:

$$(43)\qquad |P_{1,r}(x)-f(x)|\leqq \frac{1}{2}\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)+b_{n,r}\,\omega\left(\frac{\pi}{2n}\sqrt{1-x^2}\right)+$$

$$+\frac{1}{r}\omega\left(\frac{\pi^2|x|}{n^2}\right)\sum_{i=1-n}^{n}\left[\sum_{0\leqq j<\frac{r}{2}}\left|i\sigma_i\left(t+\frac{j\pi}{nr}\right)\right|+\right.$$

$$\left.+\sum_{r/2\leqq j<r}|i-1|\cdot\left|\sigma_i\left(t+\frac{j\pi}{nr}\right)\right|\right]+b_{n,r}\,\omega\left(\frac{\pi^2|x|}{8n^2}\right).$$

Доказывая формулу (9) из [1], мы получили:

$$\sum_{i=1-n}^{n}|i\sigma_i(\vartheta)|=\frac{1}{4}+\frac{1}{4n}\sum_{i=1-n}^{n}\left|D\left(\vartheta-\frac{2i-1}{2n}\pi\right)\right|\qquad\left(0\leqq\vartheta\leqq\frac{\pi}{2n}\right).$$

Аналогичным образом можно доказать тождество

$$\sum_{i=1-n}^{n}|i-1|\cdot|\sigma_i(\vartheta)|=\frac{1}{4}+\frac{1}{4n}\sum_{i=1-n}^{n}\left|D\left(\vartheta-\frac{2i-1}{2n}\pi\right)\right|\qquad\left(\frac{\pi}{2n}\leqq\vartheta\leqq\frac{\pi}{n}\right).$$

Поэтому и в силу (4)

$$\frac{1}{r}\sum_{i=1-n}^{n}\left[\sum_{0\leqq j<\frac{r}{2}}\left|\sigma_i\left(t+\frac{j\pi}{nr}\right)\right|+\sum_{\frac{r}{2}\leqq j<r}|i-1|\cdot\left|\sigma_i\left(t+\frac{j\pi}{nr}\right)\right|\right]<$$

$$<\frac{1}{4}+\frac{1}{4nr}\sum_{p=1-nr}^{nr}\left|D\left(t+\frac{\pi}{2n}+\frac{p\pi}{nr}\right)\right|\leqq a_{n,r}.$$

Отсюда и из (43) следует доказываемое соотношение (8).

Доказательство теоремы 3. Из леммы 3 получаем:

$$(44)\qquad |P_{2,r}(x)-f(x)|\leqq\frac{1}{r}\omega\left(\frac{\pi}{n}\sqrt{1-x^2}\right)\sum_{j=0}^{r-1}\sum_{i=1-n}^{n}\left|\sigma_i\left(t+\frac{j\pi}{nr}\right)\right|+$$

$$+\frac{1}{r}\omega\left(\frac{\pi^2|x|}{2n^2}\right)\sum_{j=0}^{r-1}\sum_{i=0}^{n-1}(2i+1)\left[\left|\sigma_{-i}\left(t+\frac{j\pi}{nr}\right)\right|+\left|\sigma_{i+1}\left(t+\frac{j\pi}{nr}\right)\right|\right].$$

Также, как была доказана лемма 5 из [2], можно показать, что

$$(45)\qquad \frac{1}{r}\sum_{j=0}^{r-1}\sum_{i=1-n}^{n}\left|\sigma_i\left(t+\frac{j\pi}{nr}\right)\right|=\frac{3}{2}-\frac{1}{4nr}\sum_{j=-r}^{r-1}D\left(t+\frac{j\pi}{nr}\right)+$$

$$+(-1)^n\frac{1}{4nr}\sum_{=-r}^{r-1}D\left(t+\frac{j\pi}{nr}+\pi\right).$$

Доказывая формулу (10) из [1], мы получили:

$$\sum_{i=0}^{n-1}(2i+1)\left[|\sigma_{-i}(\vartheta)|+|\sigma_{i+1}(\vartheta)|\right]=2-\frac{1}{4n}\left[D(\vartheta)+D\left(\vartheta-\frac{\pi}{n}\right)\right]+$$

$$+(-1)^n\frac{2n+1}{4n}\left[D(\vartheta+\pi)+D\left(\vartheta-\frac{\pi}{n}+\pi\right)\right]\qquad\left(0\leqq\vartheta\leqq\frac{\pi}{n}\right).$$

Поэтому

$$\frac{1}{r}\sum_{j=0}^{r-1}\sum_{i=0}^{n-1}(2i+1)\left[\left|\sigma_{-i}\left(t+\frac{j\pi}{nr}\right)\right|+\left|\sigma_{i+1}\left(t+\frac{j\pi}{nr}\right)\right|\right]= \tag{46}$$

$$=2-\frac{1}{4nr}\sum_{j=-r}^{r-1}D\left(t+\frac{j\pi}{nr}\right)+(-1)^n\frac{2n+1}{4nr}\sum_{j=-r}^{r-1}D\left(t+\frac{j\pi}{nr}+\pi\right).$$

Ввиду (1)

$$(-1)^n D(\vartheta+\pi)=-\frac{1}{2}\sin n\vartheta \operatorname{tg}\frac{\vartheta}{2}\leqq 0 \quad \left(|\vartheta|\leqq\frac{\pi}{n}\right).$$

Поэтому и в силу (16)

$$\frac{1}{2r}(-1)^n\sum_{j=-r}^{r-1}D\left(t+\frac{j\pi}{nr}+\pi\right)\leqq 0. \tag{47}$$

Из (10) и (44)—(47) получаем теорему 3.

Цитированная литература

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KIS OTTÓ, HO THO CAO
BUDAPESTI MÜSZAKI EGYETEM
VILLAMOKARI MATEMATIKA TANSZÉK
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