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INDEX

TOMUS XXVII

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FIBRE BUNDLES OVER SUBSPACES OF PARTITION SPACE

By

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The collection of measurable partitions with finite entropy of a given probability space is a complete metric space using the Rokhlin metric. In this note a vector bundle is defined where the base of the bundle is the subspace of partitions into n atoms, n a fixed integer, the fibre is E^n, and the group is the symmetric group on n elements.

In the following n is a fixed positive integer greater than 1, Z denotes the collection of (mod 0) measurable partitions of a non-atomic probability space $(X, \mathcal{A}, m)$, and Z_n denotes the partitions in Z which contain n sets of positive measure. Elements of Z will be denoted by small Greek letters. If α and β are members of Z then $H(\alpha)$ and $H(\alpha/\beta)$ denote the entropy of α and the conditional entropy of α given β, respectively. These objects are defined by

$$H(\alpha) = -\sum_{A\in\alpha} m(A)\log m(A)$$

$$H(\alpha/\beta) = -\sum_{A\in\alpha}\sum_{B\in\beta} m(A\cap B)\log m(A\cap B)/m(B).$$

For properties of entropy one may consult BILLINGSLY [1], PARRY [2], or SMORODINSKY [3].

For $\alpha, \beta\in Z_n$, $\varrho(\alpha, \beta)=H(\alpha/\beta)+H(\beta/\alpha)$ and $m(\alpha)=\min\{m(A): A\in\alpha\}$. For $\alpha\in Z$, $|\alpha|$ denotes the number of sets in α with positive measure, so that $\alpha\in Z_n$ if and only if $|\alpha|=n$.

1. LEMMA. *If $\alpha\in Z_n$, $\beta\in Z$, and $H(\alpha/\beta)<\varepsilon$ where $0<\varepsilon\leqq\frac{1}{4}m(\alpha)^2$ then*

(a) $|\alpha|\leqq|\beta|$

(b) *for each $A\in\alpha$ there is a β-set B_A, i.e., a union of atoms from β, such that $m(A\,\Delta\,B_A)<2\sqrt{\varepsilon}$*

(c) *if A_1, A_2 are distinct atoms of α, $B_{A_1}\cap B_{A_2}=\emptyset$*

(d) $m(\bigcup\{B_A: A\in\alpha\})\geqq 1-2n\sqrt{\varepsilon}$.

PROOF. (a) Let $A\in\alpha$ be given and let β_A denote the collection of elements of β which intersect A in a set of positive measure. Since $m(A)>0$ and $m(A)=\sum_{\beta} m(A\cap B)$, β_A is not empty. Since $H(\alpha/\beta)<\varepsilon$,

$$\sum_{\beta_A} m(A\cap B)[1-m(A\cap B)/m(B)] < \varepsilon. \tag{1.1}$$

Let $\mathscr{B}=\{B\in\beta_A : 1-m(A\cap B)/m(B)<\sqrt{\varepsilon}\}$. This family is not empty for if it were

$$1-m(A\cap B)/m(B)\geqq\sqrt{\varepsilon},\quad B\in\beta_A$$

and (1.1) implies that

$$\sqrt{\varepsilon}m(A)\leqq\sum_{\beta_A}\sqrt{\varepsilon}m(A\cap B)<\varepsilon$$

from which it follows that $m(A)<\frac{1}{2}m(\alpha)$. Thus $|\alpha|\leqq|\beta|$.

(b) Let B_A denote the union of the sets in $\mathscr{B}$. Since $m(B-A)<\sqrt{\varepsilon}m(B)$ for all $B\in\mathscr{B}$,

$$m(B_A-A)=\sum_{\mathscr{B}}m(B-A)<\sqrt{\varepsilon}m(B_A)\leqq\sqrt{\varepsilon}.$$

Also by summing $m(A\cap B)[1-m(A\cap B)/m(B)]$ over all $B\in\beta_A$ which are not in $\mathscr{B}$ we get from (1.1) that $m(A-B_A)<\sqrt{\varepsilon}$. Hence $m(A\Delta B_A)<2\sqrt{\varepsilon}$.

(c) Suppose $B_{A_1}\cap B_{A_2}\neq\emptyset$. Then there exists a $B\in\beta$ such that

$$m(A_i\cap B)>(1-\sqrt{\varepsilon})m(B),\qquad i=1,2$$

and hence

$$m(B)\geqq m[(A_1\cup A_2)\cap B]>2(1-\sqrt{\varepsilon})m(B)$$

so that $\sqrt{\varepsilon}>\frac{1}{2}$, a contradiction.

(d) $2n\sqrt{\varepsilon}\geqq\sum_{\alpha}m(A\Delta B_A)\geqq m(\bigcup_{\alpha}(A\Delta B_A))\geqq m(\bigcup_{\alpha}A-\bigcup_{\alpha}B_A)=1-m(\bigcup_{\alpha}B_A)$.

2. Corollary. *For every $n\geqq 2$, $\bigcup_{j=1}^{n}Z_j$ is a closed subset of the space of finite partitions.*

Proof. Let α be a finite partition which is not contained in $\bigcup_{j=1}^{n}Z_j$. Then $|\alpha|>n$ and if $S_\varepsilon=\{\beta:\varrho(\alpha,\beta)<\varepsilon\}$, where $\varepsilon<\frac{1}{4}m^2(\alpha)$, then $S_\varepsilon\cap\bigcup_{j=1}^{n}Z_n=\emptyset$.

3. Corollary. *If α and β are finite partitions such that $\varrho(\xi,\eta)<\varepsilon$ where $\varepsilon<$ $<\min\{m^2(\alpha)/4, m^2(\beta)/4\}$ then*

(a) $|\alpha|=|\beta|$,

(b) *there exists a unique* 1—1 *map f of the atoms in α onto the atoms in β such that $m(A\Delta f(A))<2\sqrt{\varepsilon}$, $A\in\alpha$.*

Proof. (a) Since both $H(\alpha/\beta)<\varepsilon$ and $H(\beta/\alpha)<\varepsilon$, $|\alpha|=|B|$.

(b) For each $A\in\alpha$, let $f(A)=B_A$ as defined in Lemma 1.

4. Corollary. *If α and β are in Z_n and $\varrho(\alpha,\beta)<\frac{1}{4}m^2(\alpha)$ then there exists a unique* 1—1 *map $f_{\beta\alpha}:\alpha\to\beta$ such that $m(A\Delta f_{\beta\alpha}(A))<2\sqrt{\varrho(\alpha,\beta)}$.*

Proof. Take $\varepsilon=\varrho(\alpha,\beta)$. Since $\varepsilon<\frac{1}{4}m^2(\alpha)$, for each A in a, there is the set B_A of Lemma 1. Since α and β have the same number of atoms, B_A is formed from exactly one atom of β. Define $f_{\beta\alpha}(A)=B_A$.

In the following a coordinate bundle will be constructed over Z_n for fixed n. The coordinate neighbourhoods are denoted by $V(\alpha)$ for $\alpha \in Z_n$, where $V(\alpha)=\{\beta \in Z_n : \varrho(\alpha, \beta) < m^2(\alpha)/64\}$.

5. Lemma. *If $\beta, \beta' \in V(\alpha)$ then there exists a unique 1—1 onto map $f^{\alpha}_{\beta\beta'} : \beta' \to \beta$ such that*

(a) $f^{\alpha}_{\beta\beta'} f^{\alpha}_{\beta'\alpha} = f^{\alpha}_{\beta\alpha}$.

If $\beta, \beta' \in V(\alpha) \cap V(\alpha')$ then

(b) $f^{\alpha}_{\beta\beta'} = f^{\alpha'}_{\beta\beta'}$.

Proof. (a) Since $\beta \in V(\alpha)$, Corollary 4 implies that there exists $f_{\beta\alpha} : \alpha \to \beta$ such that

$$m(A \,\Delta\, f_{\beta\alpha}(A)) < 2\sqrt{\varrho(\alpha, \beta)} < \frac{1}{4} m(\alpha)$$

and there exists $f_{\beta'\alpha} : \alpha \to \beta'$ such that

$$m(A \,\Delta\, f_{\beta'\alpha}(A)) < \frac{1}{4} m(\alpha),$$

so that for each $A \in \alpha$

$$m(f_{\beta'\alpha}(A)) > m(A) - \frac{1}{4} m(\alpha) \geqq \frac{3}{4} m(\alpha), \quad m(f_{\beta\alpha}(A)) > \frac{3}{4} m(\alpha).$$

Since both $f_{\beta\alpha}$ and $f_{\beta'\alpha}$ are onto maps these inequalities imply that

$$m(\alpha) \leqq \frac{4}{3} \min \{m(\beta), m(\beta')\}. \tag{5.1}$$

Since β and β' are in $V(\alpha)$,

$$\varrho(\beta, \beta') \leqq \varrho(\beta, \alpha) + \varrho(\alpha, \beta') < m^2(\alpha)/32$$

and it follows from (5.1) that $\varrho(\beta, \beta') < \frac{1}{18} \min \{m^2(\beta), m^2(\beta')\}$. Corollary 4 now implies that there exist maps $f_{\beta\beta'}$ and $f_{\beta'\beta}$ such that for $B \in \beta$ and $B' \in \beta'$

$$m(B \,\Delta\, f_{\beta'\beta}(B)) < 2\sqrt{\varrho(\beta, \beta')} < m(\alpha)/2\sqrt{2},$$

$$m(B' \,\Delta\, f_{\beta\beta'}(B)) < 2\sqrt{\varrho(\beta, \beta')} < m(\alpha)/2\sqrt{2}.$$

Define $f^{\alpha}_{\beta'\beta}$ to be the map $f_{\beta'\beta}$.

Let $A \in \alpha$ so that $m(A \,\Delta\, f^{\alpha}_{\beta\alpha}(A)) < m(\alpha)/4$ and $m(f^{\alpha}_{\beta\alpha}(A) \,\Delta\, f^{\alpha}_{\beta'\beta}(f_{\beta\alpha}(A))) < m(\alpha)/2\sqrt{2}$. Thus

$$m(A \,\Delta\, f^{\alpha}_{\beta'\beta}(f_{\beta\alpha}(A))) < \frac{2+\sqrt{2}}{4\sqrt{2}} m(\alpha).$$

Also $m(A \,\Delta\, f^{\alpha}_{\beta'\alpha}(A)) < m(\alpha)/4$ so that

$$m(f^{\alpha}_{\beta'\alpha}(A) \,\Delta\, f^{\alpha}_{\beta'\beta}(f^{\alpha}_{\beta\alpha}(A))) < \frac{2+2\sqrt{2}}{4\sqrt{2}} m(\alpha) < 2m(\beta') \tag{5.2}$$

using (5.1). Both $f^{\alpha}_{\beta'\alpha}(A)$ and $f^{\alpha}_{\beta'\beta}(f^{\alpha}_{\beta\alpha}(A))$ are atoms of β' so they either coincide or are disjoint. If they were disjoint

$$m(f^{\alpha}_{\beta'\alpha}(A)\,\Delta\, f^{\alpha}_{\beta'\beta}(f^{\alpha}_{\beta\alpha}(A))) \geqq 2m(\beta')$$

a contradiction to (5.2). Thus equation (a) is proven.

(b) Suppose β and β' are in $V(\alpha)\cap V(\alpha')$. By the first part of the proof, for $B\in\beta$

$$m(B\,\Delta\, f^{\alpha}_{\beta'\beta}(B)) < m(\alpha)/2\sqrt{2} < 2m(\beta')/3\sqrt{2}$$

and

$$m(B\,\Delta\, f^{\alpha'}_{\beta'\beta}(B)) < m(\alpha')/2\sqrt{2} < 2m(\beta')/3\sqrt{2}$$

so that

$$m(f^{\alpha}_{\beta'\beta}(B)\,\Delta\, f^{\alpha'}_{\beta'\beta}(B)) < 4m(\beta')/3\sqrt{2} < m(\beta'). \tag{5.3}$$

Since both $f^{\alpha}_{\beta'\beta}(B)$ and $f^{\alpha'}_{\beta'\beta}(B)$ are in β' they either coincide or are disjoint. In the latter case a contradiction to (5.3) would arise so that (b) follows.

In the following, for α, β, γ given with β and γ in $V(\alpha)$ we shall write $f_{\beta\gamma}$ to mean $f^{\alpha}_{\beta\gamma}$ when no confusion may arise.

6. Lemma. *If α, β, γ are in Z_n and $\beta, \gamma\in V(\alpha)$ then $f_{\beta\gamma}f_{\gamma\beta}=\mathrm{id}_\beta$ and $f_{\gamma\beta}f_{\beta\gamma}=\mathrm{id}_\gamma$ where id_β is the identity map of β onto β.*

Proof. It follows from Lemma 5 that $f_{\gamma\beta}f_{\beta\alpha}=f_{\gamma\alpha}$ and $f_{\beta\gamma}f_{\gamma\alpha}=f_{\beta\alpha}$ so that $f_{\beta\gamma}f_{\gamma\beta}f_{\beta\alpha}=f_{\beta\gamma}f_{\gamma\alpha}=f_{\beta\alpha}$ and the result follows since $f_{\beta\alpha}$ is a 1—1 map of β onto α.

Let S_n denote the symmetric group on n elements with the discrete topology and let S_n act on R^n by

$$(g, (x_1, \dots, x_n)) \to (x_{g(1)}, \dots, x_{g(n)}).$$

A system of coordinate transformations in Z_n with values in S_n will now be defined. One may consult Steenrod [4] for the existence of fibre bundles given coordinate bundles.

For each $\xi\in Z_n$, let i_ξ denote a given one to one map of $N=\{1, 2, \dots, n\}$ onto ξ, i. e., i_ξ is an indexing (or ordering of ξ).

7. Definition. For $\xi, \xi'\in Z_n$ define $g_{\xi\xi'}$ on $V(\xi)\cap V(\xi')$ into S_n by

$$g_{\xi\xi'}(\alpha) = i_\xi^{-1} f_{\xi\alpha} f_{\alpha\xi'} i_{\xi'}.$$

8. Theorem. *The functions $g_{\xi\xi'}$ together with the covering $\{V(\alpha):\alpha\in Z_n\}$ form a system of coordinate transformations in Z_n with values in S_n.*

Proof. First it will be shown that $g_{\alpha\beta}$ is constant on $V(\alpha)\cap V(\beta)$ so that the functions $g_{\alpha\beta}$ are continuous on their domains. Let ξ and η be in $V(\alpha)\cap V(\beta)$. Then

$$g_{\alpha\beta}(\xi) = i_\alpha^{-1} f_{\alpha\xi} f_{\xi\beta} i_\beta = i_\alpha^{-1} f_{\alpha\xi} f_{\xi\eta} f_{\eta\xi} f_{\xi\beta} i_\beta = i_\alpha^{-1} f_{\alpha\eta} f_{\eta\beta} i_\beta = g_{\alpha\beta}(\eta).$$

Next for $\xi\in V(\alpha)\cap V(\beta)\cap V(\gamma)$,

$$g_{\alpha\beta}(\xi)\cdot g_{\beta\gamma}(\xi) = (i_\alpha^{-1} f_{\alpha\xi} f_{\xi\beta} i_\beta)\cdot(i_\beta^{-1} f_{\beta\xi} f_{\xi\gamma} i_\gamma) = i_\alpha^{-1} f_{\alpha\xi} f_{\xi\gamma} i_\gamma = g_{\alpha\gamma}(\xi).$$

As a non-trivial example of the bundle defined by the coordinate transformations take $(X, \mathscr{B}, m)$ to be the circle with normalized Lebesgue measure. We shall identify an induced bundle in the bundle defined by the coordinate transformations given in 7 for $n=2$ which is equivalent to an infinite Mobius band crossed with the reals.

To achieve this consider the mapping of $[0, 1)$ into Z_2, the collection of 2-element partitions of the space X, defined by $t \to \alpha_t$ where α_t denotes the partition consisting of the two arcs joining $\exp(\pi i t)$ and $\exp(\pi i t + \pi i)$. Considering $[0,1)$ as the circle by identifying 0 and 1, $t \to \alpha_t$ is uniformly continuous since

$$\varrho(\alpha_t, \alpha_s) = -2|t-s| \log |t-s| - 2\{1-|t-s|\} \log \{1-|t-s|\},$$

and one to one. It can also be seen that the inverse function is continuous so that the image T of $[0, 1)$ in Z_2 is homeomorphic to the circle. The bundle induced by T is an infinite Mobius band crossed with the reals. This may be seen by observing that S_2 acting on R^2 leaves the line $x=y$ invariant.

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IDEALS OF MATRIXRINGS OVER NONASSOCIATIVE RINGS

By
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1. Introduction

In this paper we introduce a radical class Λ of nonassociative (i.e. not necessarily associative) rings. It is interesting that the class Λ consists of all nonassociative rings R such that each ideal of the matrixring R_n, for some $n>1$, is of the form M_n for some ideal M of R. This explains the significance of Λ-rings.

We also introduce two radical classes Λ_l and Λ_r of nonassociative rings. The rings in the classes Λ_l and Λ_r are characterized by means of some property of the left ideals and right ideals respectively in matrixring over such rings.

I would like to thank Professor F. A. Szász for his careful reading of the manuscript and his helpful advice.

2. A useful characterization of radical classes

For the definition of a radical class we refer to Divinsky's book [5], likewise for all other notions taken from the theory of radicals in rings.

If σ denotes a subclass of a universal (i.e. homomorphically closed and hereditary) class ω then the following theorem, which is due to Amitsur [1], gives a useful criterion for checking the class σ to be a radical subclass or not.

Theorem 1. *A subclass σ of a universal class ω of rings is radical if and only if σ statisfies the following three conditions:*

(A) *σ is homomorphically closed;*

(X) *σ is closed under σ-extensions (i.e. if A is an ideal of the ω-ring R and both A and R/A are in σ, then R is in σ);*

(Y) *σ has the inductive property (i.e. if $A_1\subseteq A_2\subseteq\ldots\subseteq A_n\subseteq\ldots$ is an ascending chain of σ-ideals of an ω-ring R, then $A=\bigcup_n A_n$ is a σ-ideal of R).*

The following lemma can be considered as an easy consequence of Theorem 1.

Lemma 1. *Let ω and Ω be two universal classes of nonassociative rings satisfying $\omega\subseteq\Omega$. If Σ is a radical subclass of Ω, then $\sigma=\Sigma\cap\omega$ is a radical subclass of ω.*

3. The radical class Λ

First we recall some definitions and results of B. de la Rosa. In his thesis [8] he only considers associative rings. An ideal Q of such a ring R is defined to be *quasisemiprime* if and only if for each ideal A of R the following is true: $RAR\subseteq Q\Rightarrow$ $\Rightarrow A\subseteq Q$.

If all ideals of the ring R are quasisemiprime, the ring R is called a *λ-ring*. The class λ, consisting of all λ-rings, is proved to be a radical subclass of the class of all associative rings.

In the last section of his thesis the λ-rings are characterized in a nice way. We can summarize the main results of this section as follows:

THEOREM 2. *For each associative ring R and for each natural number $n(n>1)$ the following three statements are equivalent:*

(1) *R is a λ-ring;*

(2) *R_n (the ring of all (n, n)-matrices over R) is a λ-ring;*

(3) *the ideals of R_n are of the form M_n, where M is an ideal of R.*

Having summarized the work of de la Rosa, as far as we need it, we now first give a new characterization of λ-rings.

LEMMA 2. *The associative ring R is a λ-ring if and only if for each ideal A of R the following holds: $A=RA=AR$.*

PROOF. Let A be an ideal of the λ-ring R. Then RA is an ideal of R, and because R is a λ-ring, this ideal is quasisemiprime. Therefore $RAR \subseteqq RA$ implies $A \subseteqq RA$, which proves that $A=RA$. Similarly one shows that $A=AR$.

Conversely, if $A=RA=AR$ for each ideal of the associative ring R, let Q be an arbitrary ideal of R. If B is an ideal of R such that $RBR \subseteqq Q$, then $B=RB= =(RB)R=RBR \subseteqq Q$. Thus the ideal Q is quasisemiprime. Because Q is arbitrary, this proves that R is a λ-ring.

In what follows each ring is nonassociative, unless stated otherwise. If M is a subset of the ring R, then $\langle M \rangle_R$ denotes the ideal of R generated by M. If there is no ambiguity the we shall write $\langle M \rangle$ instead of $\langle M \rangle_R$.

DEFINITION 1. *A ring R is called a Λ-ring if and only if for each ideal A of R the following holds: $A=\langle (RA)R \rangle$.*

REMARKS. 1. For each ideal A of a ring R we trivially have $(RA)R \subseteqq A$, hence $\langle (RA)R \rangle \subseteqq A$. Generally $(RA)R$ need not to be an ideal of R.

2. If R is an associative ring, then $(RA)R=RAR$ is an ideal of R, for each ideal A of R. So R is a Λ-ring if and only if $A=RAR$ for each ideal A of R, or equivalently if and only if $A=RA=AR$ for each ideal A of R. Therefore an associative ring is a Λ-ring if and only if it is a λ-ring, by Lemma 2.

3. If R is an alternative ring, the the subsets RA an AR of R are ideals of R, for each ideal A of R.[1] Therefore we then also have that R is a Λ-ring if and only if $A=RA=AR$ for each ideal A of R.

4. Each Λ-ring is idempotent. This follows from the definition by taking $A=R$.

5. Each ring with a unity element is a Λ-ring, because for such a ring we have $A=RA=AR$ for each ideal A of R, hence $(RA)R=AR=A$.

6. The class Λ is not hereditary. To see this, consider the ideal $2Z$ in the ring Z of the rational integers, and use Remarks 4 and 5.

PROPOSITION 1. *The ring A is a Λ-ring if and only if A has the following property: If R is an arbitrary ring containing A as an ideal, then for each ideal B of R the following holds: $\langle (AB)A \rangle_A = A \cap B$.*

[1] Using methods of [6], it is easy to prove this.

PROOF. Suppose the Λ-ring A is an ideal of the ring R and B is an arbitrary ideal of R. Then $A\cap B$ is an ideal of A. Because A is a Λ-ring we may write:

$$A\cap B = \langle (A(A\cap B))A\rangle_A \subseteqq \langle (AB)A\rangle_A \subseteqq A\cap B.$$

This proves that $\langle (AB)A\rangle_A = A\cap B$.

Conversely, if the ring A has the stated property, then considering the ring A as an ideal of itself, we find for each ideal B of A: $\langle (AB)A\rangle = A\cap B = B$, which means that A is a Λ-ring.

COROLLARY. *The associative ring A is a λ-ring if and only if A has the following property: If R is an associative ring containing A as an ideal, then for each ideal B of R the following holds: $AB=BA=A\cap B$.*

For the proof of the next theorem we need the following lemma, which is a corollary of the wellknown modular law for subgroups of an additive group, which states that if A, B and C are subgroups of an additive group, then:

$$B\subseteqq C \Rightarrow B+(A\cap C) = (B+A)\cap C.$$

LEMMA 3. *If R is a ring, and A, B and C are subgroups of the additive group $(R, +)$ of R satisfying:*
(1) $A\cap B = A\cap C$,
(2) $A+B=A+C$, *and*
(3) $B\subseteqq C$,
then $B=C$.

THEOREM 3. *The class Λ of all Λ-rings is a radical subclass of the class of all rings.*

PROOF. Referring to Theorem 1 we shall prove that the class Λ has the properties (A), (X) and (Y), where ω stands for the universal class of all rings. Then the theorem will be proved.

Let R be a Λ-ring, and let $\varphi: R\to R'$ be a ring-epimorphism. Then it is easy to verify that for each subset M of R the following is true: $\varphi\langle M\rangle = \langle \varphi M\rangle$. Now let A' be an ideal of R'. Then $A=\varphi^{-1}A'$ is an ideal of R. Since R is in Λ we have $A=\langle (RA)R\rangle$. Therefore we may write: $A'=\varphi A=\varphi\langle (RA)R\rangle = \langle \varphi[(RA)R]\rangle = \langle (\varphi R\cdot \varphi A)\varphi R\rangle = \langle (R'A')R'\rangle$. This proves that R' is in Λ, so Λ has (A).

Now suppose that A is an ideal of the ring R, and both A and R/A are in Λ. We then have to show that R is in Λ. Let B be an ideal of R. Using Proposition 1 we may write then:

$$A\cap B = \langle (AB)A\rangle_A \subseteqq A\cap\langle (RB)R\rangle \subseteqq A\cap B.$$

Therefore we have $A\cap B = A\cap\langle (RB)R\rangle$. Because $(A+B)/A$ is an ideal of R/A, and R/A is in Λ we may write:

$$(A+B)/A = \langle ((R/A)((A+B)/A))(R/A)\rangle.$$

Thus follows $A+B=\langle (R(A+B))R+A\rangle = A+\langle (RB)R\rangle$. Since B is an ideal of R we have $\langle (RB)R\rangle\subseteqq B$, so Lemma 3 yields $B=\langle (RB)R\rangle$. This proves that Λ has (X).

Finally, let $A_1 \subseteqq A_2 \subseteqq \ldots \subseteqq A_n \subseteqq \ldots$ be an ascending chain of Λ-ideals of a ring R, and let A denote their union. If B is an ideal of A, then we have by Proposition 1: $A_n \cap B = \langle (A_n B) A_n \rangle_A$ for each n. Therefore we may write:

$$B = A \cap B = (\bigcup_n A_n) \cap B = \bigcup_n (A_n \cap B) = \bigcup_n \langle (A_n B) A_n \rangle_A = \langle (AB) A \rangle_A .$$

This shows that $B = \langle (AB) A \rangle_A$, because B is an ideal of A. Therefore A is in Λ, and thus Λ has (Y). Thus the proof is complete.

Referring to Lemma 1 and Remark 2 we refind a result of de la Rosa as a corollary of Theorem 3:

COROLLARY. *The class λ is a radical subclass of the class of all associative rings.*

This result was proved by de la Rosa by using elementwise methods, whereas the proof just given is elementfree.

We conclude this section by giving some more properties of λ-rings, i.e. of associative Λ-rings.

DE LA ROSA states [8, page 28] that the λ-radical $\lambda(R)$ of an associative ring R has the property that each ideal of the radical $\lambda(R)$ is an ideal of R too. It is not difficult to generalize this statement to the following

PROPOSITION 2. *If A is a quasisemiprime ideal of B, and B is an ideal of the associative ring R, then A is an ideal of R too. If moreover B is quasisemiprime in R, then A is quasisemiprime in R.*

PROOF. Let A, B and R be given as in the proposition. Let A_1 denote the ideal of R generated by A. Then we have $A_1 = A + RA + AR + RAR$. Therefore $BA_1B = B(A + RA + AR + RAR)B \subseteqq BAB \subseteqq A$. Because A_1 is an ideal of B and A is quasisemiprime in B thus follows $A_1 \subseteqq A$. This proves that $A_1 = A$, and thus A is an ideal of R.

If moreover B is quasisemiprime in R, let C be an ideal of R such that $RCR \subseteqq A$. Then $RCR \subseteqq B$ implies $C \subseteqq B$, because B is quasisemiprime in R. Now C is an ideal of B satisfying $BCB \subseteqq RCR \subseteqq A$, which implies $C \subseteqq A$, because A is quasisemiprime in B. This shows that A is quasisemiprime in R.

COROLLARY. *If A is a λ-ideal of the associative ring R, then each ideal of A is an ideal of R too.*

Another property of λ-rings can be found by noting that a nonzero λ-ring cannot be nilpotent, because each λ-ring is idempotent. We can prove even more.

PROPOSITION 3. *A nonzero λ-ring is not locally nilpotent.*

PROOF. Suppose the λ-ring R is locally nilpotent, and $x \in R$. Let X denote the ideal of R generated by the element x. Then $X = Zx + Rx + xR + RxR$. Hence $RXR = R(Zx + Rx + xR + RxR)R \subseteqq RxR$. Because the ring R is a λ-ring, we have $X = RXR$. So $RXR \subseteqq RxR$ implies $X \subseteqq RxR$, hence $x \in RxR$.[2] Therefore there must exist a finite

[2] We note that if conversely $x \in RxR$ for each element x of the associative ring R, then clearly R is a λ-ring. So we find that the associative ring R is a λ-ring if and only if $x \in RxR$, for each $x \in R$. This characterization of λ-rings was given already in [8].

number of elements r_i, s_i ($1 \leqq i \leqq n$) in R such that $x = \sum_i r_i x s_i$. Let S be the subring of R generated by the elements r_i, s_i ($1 \leqq i \leqq n$). Because R is locally nilpotent and S is a finitely generated subring of R, the ring S must be nilpotent. So $S^k = 0$ for some natural number k. Now $x \in SxS$ implies $SxS \subseteq S^2xS^2$, which implies $S^2xS^2 \subseteq S^3xS^3$, etc. Continuing in this way we find $x \in S^kxS^k$; because $S^k = 0$, this implies $x = 0$. Thus R is the zeroring. This proves the proposition.

The existence or non-existence of a nonzero λ-ring which is nil seems to be unsettled. The famous Koethe-problem asks for the existence of a nontrivial simple ring which is nil. Since a nontrivial simple ring is a λ-ring, the first problem can be considered as a generalization of the second. In fact we can state:

PROPOSITION 4. *There exists a nontrivial simple nil ring if and only if there exists a nonzero nil λ-ring containing a maximal ideal.*

4. The radical classes Λ_l and Λ_r.

DEFINITION 2. *A ring R is called a Λ_l-ring if and only if for each left ideal L of R the following holds: $L = RL$.*

REMARKS. 1. For each left ideal L of a ring R the subset RL is a left ideal and trivially $RL \subseteq L$.

2. Each Λ_l-ring is idempotent. This follows from the definition by taking $L = R$.
3. Each ring with a left unity element is a Λ_l-ring.
4. A commutative ring is a Λ_l-ring if and only if it is a Λ-ring.
5. The class Λ_l is not hereditary. See Remark 6, Section 3.
6. The class of all associative Λ_l-rings shall be denoted by λ_l. The associative ring R is a λ_l-ring if and only if $x \in Rx$ for each $x \in R$. For let R be a λ_l-ring and $x \in R$. The left ideal X of R generated by the element x equals $Zx + Rx$. R being a λ_l-ring we have $X = RX = R(Zx + Rx) \subseteq Rx$. Hence $x \in Rx$. The converse is trivial.

In general Λ_l-rings cannot be characterized in this way, as the following example shows.

EXAMPLE 1. Let R be the algebra, generated by the elements a and b, over the field of two elements, with the multiplication as given in the table.

	a	b
a	b	a
b	0	0

Then R, considered as a ring, has only one nontrivial left ideal, namely $L = \{0, a+b\}$. Clearly $L = RL$ and $R = RR$. Therefore R is a Λ_l-ring. We note that for example $a \notin Ra$.

Before going on we shall give two more examples, showing that the classes Λ and Λ_l are not contained in one another.

EXAMPLE 2. Let R be the algebra, generated by the elements a and b, over an arbitrary field F, with the multiplication as shown in the table.

	a	b
a	a	b
b	0	0

Then R, considered as a ring, is associative and a is a left unity element. Hence R is in λ_l. R has only one twosided nontrivial ideal, namely the subset $B=\{\alpha b|\alpha\in F\}$. Since $BR=0$, R is not λ, hence not in Λ.

EXAMPLE 3. Let R be the algebra, generated by the elements a and b, over an arbitary field F, with the multiplication as shown in the table.

	a	b
a	b	0
b	a	0

It is easy to see that the ring R is nonassociative, and that R does not have nontrivial ideals. Since $RR\neq 0$ this implies $R=RR$. Therefore R is a Λ-ring. The subset $L=\{\alpha b|\alpha\in F\}$ is a left ideal of R. Since $RL=0$, the ring R is not in Λ_l.

REMARK 7. The ring in Example 3 is not associative. Later on in this section we shall give an example of an associative ring which is in λ but not in λ_l.

PROPOSITION 5. *The ring A is a Λ_l-ring if and only if A has the following property: If R is an arbitrary ring containing A as an ideal, then $AL=A\cap L$ for each left ideal L of R.*

PROOF. Suppose the Λ_l-ring A is an ideal of the ring R, and L is a left ideal of R. Then $A\cap L$ is a left ideal of A. Therefore we may write $A\cap L=A(A\cap L)\subseteq AL\subseteq A\cap L$. Hence $AL=A\cap L$. If conversely A has the stated property we have in particular for each left ideal L of A: $AL=A\cap L=L$, which shows that A is a Λ_l-ring.

THEOREM 4. *The class Λ_l of all Λ_l-rings is a radical subclass of the class of all rings.*

PROOF. The proof is quite similar to that of Theorem 3. Let R be a Λ_l-ring, and let $\varphi: R\to R'$ be a ringepimorphism. If L' is a left ideal of R' then $L=\varphi^{-1}L'$ is a left ideal of R. Since R is in Λ_l we have $L'=\varphi L=\varphi(RL)=(\varphi R)(\varphi L)=R'L'$, which shows that R' is a Λ_l-ring. This proves that the class Λ_l is homomorphically closed. To prove that Λ_l has property (X), let A be an ideal in the nonassociative ring R such that both A and R/A are in Λ_l, and let L be a left ideal in R. Using Proposition 5 we may write: $A\cap L=AL\subseteq A\cap RL\subseteq A\cap L$, which shows that $A\cap L=A\cap RL$.

Because $(A+L)/A$ is a left ideal in R/A, and R/A is in Λ_l, we have $(A+L)/A=(R/A)((A+L)/A)$, which implies $A+L=R(A+L)+A$ or $A+L=A+RL$.

Because $RL\subseteq L$ we have $L=RL$, by Lemma 3, and so we have proved that R is in Λ_l.

To prove that A_l has (Y) let $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ be an ascending chain of A_l-ideals in a nonassociative ring R, and let A denote their union. If L is a left ideal of A, then $A_n L = A_n \cap L$ for each n by Proposition 5. Therefore we may write:

$$AL = (\bigcup_n A_n)L = \bigcup_n A_n L = \bigcup_n (A_n \cap L) = (\bigcup_n A_n) \cap L = A \cap L = L,$$

which proves that A is a A_l-ideal of R.

COROLLARY. *The class λ_l, consisting of all associative A_l-rings, is a radical subclass of the class of all associative rings.*

Concerning the rings in the class λ_l we can give two propositions which are in correspondence with the Propositions 2 and 3 in Section 4.

PROPOSITION 5. *If A is a λ_l-ideal of the associative ring R, then each left ideal of A is a left ideal of R too.*

PROOF. Let L be a left ideal of the λ_l-ideal A of the associative ring R. Then the left ideal L_1 of R generated by L equals $L + RL$. Now L_1 is contained in A, because A is an ideal of R containing L. Since A is in λ_l we therefore have: $L_1 = AL_1 = A(L + RL) \subseteq AL + (AR)L \subseteq AL \subseteq L$. Hence $L_1 = L$, and thus L itself is a left ideal of R.

PROPOSITION 6. *A nonzero λ_l-ring is not Jacobson-radical (hence not nil!).*

PROOF. Let R be λ_l-ring which is Jacobson-radical, and suppose $x \in R$. Then $x \in Rx$ by Remark 6. Therefore there must exist an element c in R such that $x = cx$. Because R is assumed to be Jacobson-radical we have $c + d - dc = 0$, for some element d of R. Hence $x = cx = (dc - d)x = d(cx) - x = dx - dx = 0$. This shows that R is the zero ring.

REMARK 8. E. SĄSIADA [9] has constructed a nontrivial simple associative ring which is Jacobson-radical. Here we have an example of an associative ring which is not in λ_l by Proposition 6, but it is in λ because each nontrivial simple associative ring is a λ-ring.

REMARK 9. The A_l-rings (λ_l-rings) have their left-right dualized notions as A_r-rings (λ_r-rings). For these, all the results obtained before have their analogues.

5. One-sided and two-sided ideals in matrixrings

We consider the ring R_n of all (n, n)-matrices over a nonassociative ring R, with $n > 1$.

Although R may not have a unity element, we still use the matrix units E_{ij} $(1 \leqq i, j \leqq n)$ in a formal way: if $x \in R$ then xE_{ij} is to be interpreted as the matrix with the element x in the position (i, j) and the zero element in all other positions. Then the following lemma can be proved by direct calculation.

LEMMA 4. *Let p, q, r and s be arbitrary elements of the set $\mathbf{n} = \{1, 2, \ldots, n\}$, and x and y arbitrary elements of the ring R. Then we have $xE_{pq} \cdot yE_{rs} = \delta_{qr} xyE_{ps}$.*

For each left ideal L of R, let $L(j)$ denote the set of all matrices in R_n with entries running through L, which are zero in all but the j-th column, i.e. for each $j \in \mathbf{n}$:

$$L(j) = \left\{ \sum_{p=1}^{n} a_p E_{pj} | a_p \in L \right\} = \sum_{p=1}^{n} LE_{pj}.$$

Then $L(j)$ is a left ideal of R_n, for each $j \in \mathbf{n}$. Each left ideal of R_n which equals the left ideal $L(j)$ for some left ideal L of R and for some $j \in \mathbf{n}$, will be called an *elementary left ideal* of R_n. It is clear that the ring R_n can be written as a direct sum of elementary left ideals:

$$R_n = R(1) \oplus R(2) \oplus \ldots \oplus R(n).$$

The canonical projection $R_n \to R(j)$ induced by this decomposition shall be denoted by $\pi(j)$, for each $j \in \mathbf{n}$.

Having a right ideal K of R we can define in a similar way the *elementary right ideals* $K[i]$ of R_n, for each $i \in \mathbf{n}$, by

$$K[i] = \left\{ \sum_{p=1}^{n} b_p E_{ip} | b_p \in K \right\} = \sum_{p=1}^{n} KE_{ip}.$$

Then R_n is a direct sum of elementary right ideals:

$$R_n = R[1] \oplus R[2] \oplus \ldots \oplus R[n],$$

and the induced canonical projection $R_n \to R[i]$ shall be denoted by $\pi[i]$, for each $i \in \mathbf{n}$. For each matrix X in R_n the element of R in the position (i, j) shall be denoted by x_{ij}, and if $\mathscr{M}$ is a subset of R_n, then the set $\{m_{ij} | M \in \mathscr{M}\}$ consisting of all elements of R arising in the position (i, j) in matrices belonging to $\mathscr{M}$, shall be denoted by M_{ij}. In fact $M_{ij} = \pi[i]\pi(j)\mathscr{M}$. It is easy to verify that M_{ij} is a left ideal (right ideal) of R for all $i, j \in \mathbf{n}$, if $\mathscr{M}$ is a left ideal (right ideal respectively) of R_n.

Using these conventions, we can state

THEOREM 5. (α) *If $\mathscr{L}$ is a left ideal of R_n, and $j \in \mathbf{n}$, then $RL_{qj}E_{pj} \subseteq \pi(j)\mathscr{L}$, for all $p, q \in \mathbf{n}$.*

(β) *If $\mathscr{K}$ is a right ideal of R_n, and $i \in \mathbf{n}$, then $K_{ir}RE_{is} \subseteq \pi[i]\mathscr{K}$, for all $r, s \in \mathbf{n}$.*

(λ) *If $\mathscr{A}$ is an ideal of R_n, then $(RA_{pq})RE_{rs} \subseteq \mathscr{A}$, and $R(A_{pq}R)E_{rs} \subseteq \mathscr{A}$, for all p, q, r and $s \in \mathbf{n}$.*

PROOF. Let $\mathscr{L}$ be a left ideal of R_n, and $j \in \mathbf{n}$. Take $q \in \mathbf{n}$ and $a \in L_{qj}$. By the definition of L_{qj} there must exist then a matrix X in $\mathscr{L}$ with $a = x_{qj}$. Because $\mathscr{L}$ is a left ideal of R_n, the matrices of $RE_{pq} \cdot X$ must be in $\mathscr{L}$ too, for all $p \in \mathbf{n}$. Using Lemma 4 we thus find:

$$RE_{pq} \cdot X = RE_{pq} \cdot \sum_{r,s} x_{rs} E_{rs} = \sum_{r,s} \delta_{qr} R x_{rs} E_{ps} = \sum_{s} R x_{qs} E_{ps} \subseteq \mathscr{L}.$$

Therefore $\pi(j) \sum_{s} R x_{qs} E_{ps} = R x_{qj} E_{pj} = RaE_{pj} \subseteq \pi(j)\mathscr{L}$. The latter relation holds for each element a of L_{qj}, and thus we have $RL_{qj}E_{pj} \subseteq \pi(j)\mathscr{L}$. This proves part ($\alpha$) of the theorem.

The proof of part (β) is quite similar, we therefore omit it. To prove part (γ) let $\mathscr{A}$ be an ideal of R_n. Take $p, q \in \mathbf{n}$ and $a \in A_{pq}$. Then there must exist a matrix X in $\mathscr{A}$ with $a = x_{pq}$ by the definition of A_{pq}. Because $\mathscr{A}$ is an ideal of R_n the matrices belonging to $(RE_{rp} \cdot X) RE_{qs}$ must be in $\mathscr{A}$, for all $r, s \in \mathbf{n}$. Using Lemma 4 agaiu we therefore have:

$$(RE_{rp} \cdot X) RE_{qs} = (RE_{rp} \cdot \sum_{u,v} x_{uv} E_{uv}) RE_{qs} = (\sum_{u,v} \delta_{pu} R x_{uv} E_{rv}) RE_{qs} =$$

$$= (\sum_{v} R x_{pv} E_{rv}) RE_{qs} = \sum_{v} \delta_{vq} (R x_{pv}) RE_{rs} = (R x_{pq}) RE_{rs} = (Ra) RE_{rs} \subseteq \mathscr{A}.$$

Because the latter inclusion holds for each element a of A_{pq}, we have $(RA_{pq}) RE_{rs} \subseteq \mathscr{A}$. Similarly one shows $R(A_{pq}R) E_{rs} \subseteq \mathscr{A}$. This proves the theorem. Using the notation of Theorem 5 we state as a

COROLLARY. (α) *$RL_{qj} \subseteq L_{pj}$, for all $p, q, j \in \mathbf{n}$.*

(β) *$K_{ir} R \subseteq K_{is}$, for all $r, s, i \in \mathbf{n}$.*

(γ) *$(RA_{pq}) R \subseteq A_{rs}$ and $R(A_{pq}R) \subseteq A_{rs}$, for all $p, q, r, s \in \mathbf{n}$.*

THEOREM 6. (α) *If $\mathscr{L}$ is a left ideal of R_n such that $\mathscr{L} = R_n \mathscr{L}$, then the projections $\pi(j)\mathscr{L}$ ($j \in \mathbf{n}$) are all elementary left ideals.*

(β) *If $\mathscr{K}$ is a right ideal of R_n such that $\mathscr{K} = \mathscr{K} R_n$, then the projections $\pi[i]\ \mathscr{K}$ ($i \in \mathbf{n}$) are all elementary right ideals.*

(γ) *If $\mathscr{A}$ is an ideal of R_n such that $\mathscr{A} = \langle (R_n \mathscr{A}) R_n \rangle$, then $\mathscr{A}$ is of the form A_n for some ideal A of R.*

PROOF. Let $\mathscr{L}$ be a left ideal of R_n satisfying $\mathscr{L} = R_n \mathscr{L}$. Then $\pi(j)\mathscr{L} = \pi(j)(R_n \mathscr{L}) = R_n \pi(j) \mathscr{L}$, for each $j \in \mathbf{n}$. Using Lemma 4 we may write therefore:

$$\pi(j)\mathscr{L} = R_n \pi(j) \mathscr{L} \subseteq \sum_{p,q} RE_{pq} \cdot \sum_{r} L_{rj} E_{rj} = \sum_{p,q} RL_{qj} E_{pj}.$$

By Theorem 5 (α) we also have the inverse inclusion. Hence

$$\pi(j)\mathscr{L} = \sum_{p,q} RL_{qj} E_{pj} = \sum_{p} (\sum_{q} RL_{qj}) E_{pj}.$$

Putting $L = \sum_{q} RL_{qj}$ we find $\pi(j)\mathscr{L} = \sum_{p} LE_{pj} = L(j)$, which proves part ($\alpha$) of the theorem.

The proof of part (β) is similar.

To prove part (γ) we need the identity $\langle M_n \rangle = (\langle M \rangle)_n$, which holds for each subset M of R. The inclusion $M \subseteq \langle M \rangle$ implies $M_n \subseteq (\langle M \rangle)_n$, and because $(\langle M \rangle)_n$ is an ideal this implies $\langle M_n \rangle \subseteq (\langle M \rangle)_n$. Conversely we have for all $r, s \in \mathbf{n}: ME_{rs} \subseteq M_n$. Hence $ME_{rs} \subseteq \langle M_n \rangle$. This implies $\langle M \rangle E_{rs} \subseteq \langle M_n \rangle$ as is easy to see. Since the subsets $\langle M \rangle E_{rs}$ ($r, s \in \mathbf{n}$) generate the ideal $(\langle M \rangle)_n$ of R_n we find $(\langle M \rangle)_n \subseteq \langle M_n \rangle$. Now assume that $\mathscr{A}$ is an ideal of R_n satisfying $\mathscr{A} = \langle (R_n \mathscr{A}) R_n \rangle$. Using Lemma 4 again we may write then:

$$\mathscr{A} = \langle (R_n \mathscr{A}) R_n \rangle \subseteq \left\langle (\sum_{r,u} RE_{ru} \cdot \sum_{p,q} A_{pq} E_{pq}) \sum_{v,s} RE_{vs} \right\rangle = \left\langle \sum_{r,s} (\sum_{p,q} (RA_{pq}) R) E_{rs} \right\rangle.$$

By Theorem $5(\gamma)$ we also have the inverse inclusion. Therefore we find, putting $A=\sum_{p,q}(RA_{pq})R$,

$$\mathscr{A}=\Big\langle\sum_{r,s}AE_{rs}\Big\rangle=\langle A_n\rangle=(\langle A\rangle)_n,$$

which completes the proof of the theorem.

6. Matrices over Λ-rings

First we give a generalization of Theorem 2 to nonassociative rings.

THEOREM 7. *For each ring R and for each natural number n $(n>1)$, the following three statements are equivalent:*

(1) *R is a Λ-ring;*

(2) *R_n is a Λ-ring;*

(3) *each ideal of R_n is of the form A_n for some ideal A of R.*

PROOF. We shall prove the theorem by proving successively the implications (1)⇒(2), (2)⇒(3) and (3)⇒(1). Before doing so we remark that for each two subgroups M and N of the additive group $(R, +)$ of a ring R and for each natural number n $(n>1)$ the following identity holds: $(MN)_n=M_nN_n$.

(1)⇒(2): Suppose R is a Λ-ring and $\mathscr{A}$ is an ideal of R_n. Using the notation of Section 5 we may write then:

$$\mathscr{A}\subseteqq\sum_{r,s}A_{rs}E_{rs}=\sum_{r,s}\langle(RA_{rs})R\rangle E_{rs}\subseteqq\sum_{r,s}\big(\sum_{p,q}\langle(RA_{pq})R\rangle\big)E_{rs}.$$

By Theorem $5(\gamma)$ we also have the inverse inclusion. Therefore, putting $A=\sum_{p,q}\langle(RA_{pq})R\rangle$ we find $\mathscr{A}=\sum_{r,s}AE_{rs}=A_n$, where A is an ideal of R. Hence

$$\langle(R_n\mathscr{A})R_n\rangle=\langle(R_nA_n)R_n\rangle=\langle((RA)R)_n\rangle=(\langle(RA)R\rangle)_n=A_n=\mathscr{A},$$

which proves that R_n is a Λ-ring.

(2)⇒(3): If R_n is a Λ-ring then each ideal $\mathscr{A}$ of R_n satisfies $\mathscr{A}=\langle(R_n\mathscr{A})R_n\rangle$. Therefore the implication (2)⇒(3) is a direct consequence of Theorem $6(\gamma)$.

(3)⇒(1): Let R be a ring satisfying (3) for some $n>1$, and let A be an ideal of R. Then we have to show that $A=\langle(RA)R\rangle$. We shall do this by showing successively the equalities $A=\langle RA\rangle$ and $\langle RA\rangle=\langle(RA)R\rangle$.

To prove the first equality, consider the subset of R_n consisting of all matrices with entries running through the ideals A and $\langle RA\rangle$ of R, as indicated in the diagram.

$$\begin{pmatrix}\langle RA\rangle & \langle RA\rangle & \dots & \langle RA\rangle\\ A & A & \dots & A\\ \vdots & \vdots & & \vdots\\ A & A & \dots & A\end{pmatrix}$$

Because $R\cdot A\subseteqq\langle RA\rangle$ and $R\cdot\langle RA\rangle\subseteqq A$, this subset is an ideal of R_n. By our assumption this implies $A=\langle RA\rangle$. To get the second equality we use the following identity: $\langle RA\rangle=RA+\langle(RA)R\rangle$, which holds for each ideal A of an arbitrary ring R, as is easy to verify. This identity proves that $\langle (RA)R\rangle \subseteq \langle RA\rangle$, and $\langle RA\rangle \cdot R = [RA + \langle (RA)R\rangle]R \subseteq (RA)R + \langle (RA)R\rangle \subseteq \langle (RA)R\rangle$. Therefore the subset of R_n consisting of all matrices with entries running through the ideals $\langle RA\rangle$ and $\langle (RA)R\rangle$ of R as indicated in the diagram below, is an ideal of R_n.

$$\begin{pmatrix} \langle (RA)R\rangle & \langle RA\rangle \dots \langle RA\rangle \\ \langle (RA)R\rangle & \langle RA\rangle \dots \langle RA\rangle \\ \vdots & \vdots \quad\quad \vdots \\ \langle (RA)R\rangle & \langle RA\rangle \dots \langle RA\rangle \end{pmatrix}$$

By our assumption this implies that $\langle RA\rangle = \langle (RA)R\rangle$, which completes the proof.

REMARK 1. Theorem 7 makes clear that we can characterize Λ-rings as follows: A ring R is a Λ-ring if and only if the map $A \to A_n$ from the lattice of the ideals of R into the lattice of the ideals of R_n, is a lattice-isomorphism. In particular it follows that if R is a nontrivial simple ring, then so is R_n.

REMARK 2. Let us define a class Γ of rings such that R is a Γ-ring if and only if for each ideal A of R the following holds: $A = \langle R(AR)\rangle$. The class Γ is a radical class and moreover, Theorem 7 remains true if we replace the class Λ by the class Γ. Hence $\Lambda = \Gamma$. This yields the following equivalence for each ring R: $A = \langle (RA)R\rangle$ for each ideal A of R if and only if $A = \langle R(AR)\rangle$ for each ideal A of R. In the associative case this is trivial, but in the nonassociative case it seems to be nontrivial.

THEOREM 8. *For each ring R and for each natural number n* ($n > 1$) *the following holds:* $\Lambda(R_n) = (\Lambda R)_n$.

PROOF. Let B and $\mathscr{A}$ denote the Λ-radical of R and R_n respectively. Then we must prove that $\mathscr{A} = B_n$. Since B is in Λ, B_n is in Λ by Theorem 7. Therefore B_n is a Λ-ideal of R_n. Hence $B_n \subseteq \mathscr{A}$. So it remains to prove that $\mathscr{A} \subseteq B_n$. Since $\mathscr{A}$ is a Λ-ring, $\mathscr{A}$ is idempotent. Therefore we have

$$\mathscr{A} = (\mathscr{A}\mathscr{A})\mathscr{A} \subseteq (R_n\mathscr{A})R_n \subseteq \langle (R_n\mathscr{A})R_n\rangle.$$

The inverse inclusion is trivial. So we have $\mathscr{A} = \langle (R_n\mathscr{A})R_n\rangle$. By Theorem 6($\gamma$) this implies that $\mathscr{A} = A_n$ for some ideal A of R. Now A_n is in Λ, hence A is Λ by Theorem 7. Thus follows $A \subseteq B$, which implies $\mathscr{A} = A_n \subseteq B_n$. This proves the theorem. The above proof makes clear that in fact we can state

COROLLARY. *Let σ be a radical class satisfying the following two conditions:*
(1) *each σ-ring is idempotent;*
(2) *R is a σ-ring $\Leftrightarrow$ R_n is a σ-ring, for each $n > 1$.*
Then for each ring R and for each $n > 1$ the following holds: $\sigma(R_n) = (\sigma R)_n$.

7. Matrices over Λ_l- and Λ_r-rings

In this section we show that the class Λ_l is closed under taking matrixings and conversely if a matrixring over a ring R is in Λ_l, then R itself is in Λ_l.

Furthermore we characterize the rings in the class Λ_l by means of the structure of the left ideals in matrixrings over such rings.

Before doing so we need to remark that each left ideal $\mathscr{L}$ in the matrixring R_n $(n>1)$ over an arbitrary (not necessarily associative) ring R is a subdirect sum of the projections $\pi(j)\mathscr{L}(j\in\mathbf{n})$. In general the left ideals $\pi(j)\mathscr{L}$ are not elementary as the following example shows.

EXAMPLE 4. Let R be the ring of the even rational integers. Then the set $\mathscr{L}$ of all matrices of the form

$$\begin{pmatrix} 2x & 0 \\ y & 0 \end{pmatrix}, \quad \text{with} \quad x, y \in R,$$

is a left ideal of R_2. The projection $\pi(1)\mathscr{L}=\mathscr{L}$ is not elementary.

THEOREM 9. *For each ring R and for each natural number n $(n>1)$ the following three statements are equivalent:*

(1) *R is a Λ_l-ring;*

(2) *R_n is a Λ_l-ring;*

(3) *each left ideal of R_n is a subdirect sum of elementary left ideals.*

PROOF. We shall prove the theorem by proving successively the implications (1)⇒(2), (2)⇒(3) and (3)⇒(1). Before doing so we note that the identity $R_nL(j)=(RL)(j)$ holds for each left ideal L of an arbitrary ring R and for each $j\in\mathbf{n}$. The inclusion $R_nL(j)\subseteqq(RL)(j)$ is trivial. To get the inverse inclusion we note that the left ideal $(RL)(j)$ is generated by the subsets RLE_{kj} $(k\in\mathbf{n})$. Since $RLE_{kj}=RE_{kk}\cdot LE_{kj}$, we have $RLE_{kj}\subseteqq R_nL(j)$ for each $k\in\mathbf{n}$. Hence $(RL)(j)\subseteqq R_nL(j)$.

(1)⇒(2): Suppose R is a Λ_l-ring and $\mathscr{L}$ is a left ideal of R_n. Then we have to prove that $\mathscr{L}=R_n\mathscr{L}$. Because $\mathscr{L}$ is a left ideal it suffices to prove that $\mathscr{L}\subseteqq R_n\mathscr{L}$. This will be done by induction to the number of nonzero projections $\pi(j)\mathscr{L}$ of $\mathscr{L}$. Let us call this number $\delta(\mathscr{L})$. Thus

$$\delta(\mathscr{L}) = |\{j\in\mathbf{n}|\pi(j)\mathscr{L}\neq 0\}|.$$

If $\delta(\mathscr{L})=0$, then clearly $\mathscr{L}=0$, hence $\mathscr{L}=R_n\mathscr{L}$. If $\delta(\mathscr{L})=1$, then $\mathscr{L}=\pi(j)\mathscr{L}$ for some $j\in\mathbf{n}$. Using that R is in Λ_l we may write then:

$$\pi(j)\mathscr{L} = \mathscr{L} \subseteqq \sum_p L_{pj}E_{pj} = \sum_p RL_{pj}E_{pj} \subseteqq \sum_p (\sum_q RL_{qj})E_{pj}.$$

By Theorem 5 (α) we also have the inverse inclusion. Therefore, putting $L=\sum_q RL_{qj}$, we find $\mathscr{L}=\sum_p LE_{pj}=L(j)$. Hence $R_n\mathscr{L}=R_nL(j)=(RL)(j)=L(j)=\mathscr{L}$.

Let us assume now that $\delta(\mathscr{L})=k>0$, and for each left ideal $\mathscr{N}$ of R_n with $\delta(\mathscr{N})<k$ it is true that $\mathscr{N}=R_n\mathscr{N}$. Since $k>0$, there must exist a column-number $j\in\mathbf{n}$ such that $\pi(j)\mathscr{L}\neq 0$. The subset $\mathscr{L}_1=\{A\in\mathscr{L}|\pi(j)A=0\}$ of $\mathscr{L}$, consisting of all matrices belonging to $\mathscr{L}$ with zero j-th column, is a left ideal of R_n, and clearly $\delta(\mathscr{L}_1)<\delta(\mathscr{L})=k$. Applying our assumption on $\mathscr{L}_1$ we find $\mathscr{L}_1=R_n\mathscr{L}_1$, hence $\mathscr{L}_1\subseteqq R_n\mathscr{L}$. Now suppose that $A\in\mathscr{L}$. Then $\pi(j)A\in\pi(j)\mathscr{L}$. Since $\pi(j)\mathscr{L}\neq 0$, we have $\delta(\pi(j)\mathscr{L})=1$. Thus follows $\pi(j)\mathscr{L}=R_n\pi(j)\mathscr{L}$. Hence $\pi(j)A\in R_n\pi(j)\mathscr{L}$. So we may write:

$$\pi(j)A = P_1X_1+\ldots+P_sX_s,$$

for certain matrices P_r and X_r taken from R_n and $\pi(j)\mathscr{L}$ respectively. Since X_r $(1\leqq r\leqq s)$ is in $\pi(j)\mathscr{L}$, there must be a matrix Y_r in $\mathscr{L}$ such that $\pi(j)Y_r=X_r$, for each r. Now consider the matrix:

$$A_1 = A-(P_1Y_1+\ldots+P_sY_s).$$

Clearly $A_1 \in \mathscr{L}$. Moreover

$$\pi(j)A_1 = \pi(j)A-(P_1\pi(j)Y_1+\ldots+P_s\pi(j)Y_s) =$$

$$= \pi(j)A-(P_1X_1+\ldots+P_sX_s) = 0,$$

hence $A_1 \in \mathscr{L}_1$, so it follows that $A_1 \in R_n\mathscr{L}$. Now $A = A_1+(P_1Y_1+\ldots+P_sY_s)$ implies $A \in R_n\mathscr{L}$, because A_1 is in $R_n\mathscr{L}$ and P_rY_r is in $R_n\mathscr{L}$ for each r. Because the matrix A is arbitrary in $\mathscr{L}$ this proves that $\mathscr{L} \subseteqq R_n\mathscr{L}$. Hence R_n is in Λ_l.

(2)⇒(3): If R_n is in Λ_l and $\mathscr{L}$ is a left ideal of R_n, then $\mathscr{L} = R_n\mathscr{L}$. Therefore the implication (2)⇒(3) is an immediate consequence of Theorem 6(α).

(3)⇒(1): If each left ideal of R_n is a subdirect sum of elementary left ideals, let L be a left ideal of R. Because RL is a left ideal of R it is easy to verify that the subset $\mathscr{L}$ of R_n consisting of all matrices with entries running through L and RL as indicated in the diagram, is a left ideal of R_n.

$$\begin{pmatrix} RL & 0\ldots 0 \\ L & 0\ldots 0 \\ \vdots & \vdots \quad \vdots \\ L & 0\ldots 0 \end{pmatrix}$$

Clearly $\mathscr{L} = \pi(1)\mathscr{L}$. By our assumption this implies that the left ideal $\mathscr{L}$ is elementary. Hence $L = RL$, which proves that R is in Λ_l. This completes the proof.

THEOREM 10. *For each ring R and for each natural number n* ($n > 1$) *the following holds:* $\Lambda_l(R_n) = (\Lambda_l R)_n$.

PROOF. This is a consequence of Theorem 9 and the corollary of Theorem 8, for each Λ_l-ring is idempotent.

REMARK 1. It is clear that by considering matrices over Λ_r-rings, we can get left-right analogues of the last two theorems.

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ON THE GAUSS TYPE QUADRATURE WITH PARAMETRIC WEIGHT FUNCTION

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If the function $W(k, x)$ is nonnegative, measurable and not identically equal to zero for $x \in [a, b]$ and $k \in [k_1, k_2]$ whereby the moments $W_m(k) = \int_a^b x_m W(k, x)\,dx$ exist for arbitrary nonnegative integer m, then the quadrature formula of Gauss type with weight function $W(k, x)$ exists, i.e.

$$\int_a^b f(x) W(k, x)\,dx = \sum_{i=1}^{n} A_{in}(k) f(x_{in}(k)) + R_{n,k}(f) \tag{1}$$

holds, where $R_{n,k}(x^j) = 0$ for $j = 0, 1, \ldots, 2n-1$. The knots $\{x_{in}(k)\}$, the corresponding coefficients $\{A_{in}(k)\}$ and the remainder $R_{n,k}(f)$ in (1) in this case also depend on the parameter k. This paper makes an inquiry into this dependency.

It is known that with a fixed k, $\omega_n(k, x) = \prod_{i=1}^{n} (x - x_{in}(k))$ will be an orthogonal polynomial with weight $W(k, x)$ on $[a, b]$. Let us further have $x_{in}(k) < x_{i+1,n}(k)$ for $i = 1, 2, \ldots, n-1$. About $x_{in}(k)$ and $A_{in}(k)$ we have

THEOREM 1. *Let $W(k, x)$ be a positive and continuous function on $[a, b]$ for $k_1 < k < k_2$ and let on these intervals the partial derivative $\dfrac{\partial W(k, x)}{\partial k}$ exist and be continuous whereby the integrals $\int_a^b x^m \dfrac{\partial W(k, x)}{\partial k}\,dx$ for $m = 0, 1, \ldots, 2n-1$ converge uniformly on each interval $k' \leqq k \leqq k''$ lying inside (k_1, k_2). Then for the i-th (i fixed) knot $x_{in}(k)$ and the coefficients $A_{in}(k)$ of (1), we have*

a) $\operatorname{sign} x'_{in}(k) = \operatorname{sign} F_i^{(1)}(k)$,

b) $\operatorname{sign} A'_{in}(k) = \operatorname{sign} F_i^{(4)}(k)$,

where

$$F_i^{(1)}(k) = \int_a^b L_i^2(k, x)(x - x_{in}(k)) \frac{\partial W(k, x)}{\partial k}\,dx, \quad F_i^{(4)}(k) = 2F_i^{(1)}(k) F_i^{(3)}(k) + F_i^{(2)}(k),$$

$$F_i^{(2)}(k) = \int_a^b L_i^2(k, x) \frac{\partial W(k, x)}{\partial k}\,dx, \quad F_i^{(3)}(k) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{x_{jn}(k) - x_{in}(k)}$$

and

$$L_i(k, x) = \frac{\omega_n(k, x)}{(x - x_{in}(k))\omega_n'(k, x_{in}(k))},$$

i.e. on intervals where presumptions given by the following table are simultaneously satisfied, $x_{in}(k)$ and $A_{in}(k)$, respectively, are increasing or decreasing functions of the parameter k.

Assumptions:			Statements:	
sign $F_i^{(1)}(k)$	sign $F_i^{(2)}(k)$	sign $F_i^{(3)}(k)$	sign $x_{in}'(k)$	sign $A_{in}'(k)$
+	+	+	+	+
+	−	−	+	−
−	+	−	−	+
−	−	+	−	−

REMARK. About the knots $x_{in}(k)$, A. A. MARKOV proved in [1] the following statement: If the assumptions of Theorem 1 are fulfilled and if $F(k, x) = \frac{\partial W(k, x)}{\partial k} W^{-1}(k, x)$ is an increasing function of x on $[a, b]$, then $x_{in}(k)$ (with fixed i) is an increasing function of the parameter k. Theorem 1 generalizes this statement, paying attention also to the coefficients $A_{in}(k)$.

PROOF. It is proved in [1] that under the given conditions the coefficients of the polynomial $\omega_n(k, x)$, the knots $x_{in}(k)$ and coefficients $A_{in}(k)$ have continuous derivatives with respect to k in $[k_1, k_2]$ and that $W_m(k)$, $m=0, 1, \dots, 2n-1$ may also be differentiated with respect to k.

Let $f(x)$ in (1) be a polynomial of degree $2n-1$. After differentiating (1) with respect to k at a fixed n we obtain [1]

$$\text{(2)} \qquad \int_a^b f(x) \frac{\partial W(k, x)}{\partial k} dx = \sum_{i=1}^{n} \left[A_{in}'(k) f(x_{in}(k)) + A_{in}(k) \frac{\partial f(x_{in}(k))}{\partial x} x_{in}'(k) \right].$$

If $f(x) = L_i^2(k, x)\big(x - x_{in}(k)\big)$ then $f\big(x_{in}(k)\big) = 0$ for $i = 1, 2, \dots, n$

and

$$\frac{\partial f(x_{jn}(k))}{\partial x} = \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases}$$

and thus we get from (2)

$$\text{(3)} \qquad x_{in}'(k) = \frac{F_i^{(1)}(k)}{A_{in}(k)}$$

from which a) follows.

If we consider the quadrature (1) for $f(x) = L_i^2(k, x)$ then

$$f(x_{jn}(k)) = \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases}$$

and $\dfrac{\partial f(x_{in}(k))}{\partial x} = -2F_i^{(3)}(k)$. After substituting in (2) we have

(4) $$F_i^{(2)}(k) = A'_{in}(k) - 2A_{in}(k) x'_{in}(k) F_i^{(3)}(k).$$

If we substitute (3) in (4) we obtain

(5) $$A'_{in}(k) = 2F_i^{(1)}(k) F_i^{(3)}(k) + F_i^{(2)}(k)$$

from which b) follows.

REMARK. Our considerations might involve also the zero points of the functions $F_i^{(j)}(k)$, $j=1, 2, 3, 4$ whereby we would get information also about the extremal $x_{in}(k)$ and $A_{in}(k)$.

From Theorem 1 there follow several corollaries.

COROLLARY 1. *Let* $\dfrac{\partial W(k, x)}{\partial k}$ *be a nonnegative (nonpositive) function not identically equal to zero for* $x \in [a, b]$ *and* $k \in [k_1, k_2]$.

If $y_{in}(k) \notin [a, b]$ *where* $y_{in}(k) = x_{in}(k) - \dfrac{1}{2F_i^{(3)}(k)}$ *then* $A_{in}(k)$ *is an increasing (decreasing) function of the parameter* k *for* $k \in [k_1, k_2]$.

PROOF. Substituting the corresponding formulae in the right side of (5) we get

(6) $$A'_{in}(k) = \int_a^b L_i^2(k, x) p_{in}(k, x) \frac{\partial W(k, x)}{\partial k} \, dx$$

where $p_{in}(k, x) = 1 + 2\big(x - x_{in}(k)\big) F_i^{(3)}(k)$ is a linear function of x. If $p_{in}(k, x)$ does not change sign for $x \in [a, b]$ i.e. if $y_{in}(k) \notin [a, b]$ then on $[a, b]$ $p_{in}(k, x) > 0$, because $p_{in}\big(k, x_{in}(k)\big) = 1$. Our statement follows now from (6).

Let us further denote $\dfrac{\partial \omega_n(k, x)}{\partial x} = \omega'_n(k, x)_{(x)}$ and $\dfrac{\partial^2 \omega_n(k, x)}{\partial x^2} = \omega''_n(k, x)_{(x)}$.

REMARKS. 1) It can be shown that $F_i^{(3)}(k) = -\dfrac{\omega''_n(k, x_{in}(k))_{(x)}}{2\omega'_n(k, x_{in}(k))_{(x)}}$ holds. After substituting in (6) we get

(7) $$A'_{in}(k) = \int_a^b H_i(k, x) \, \frac{\partial W(k, x)}{\partial k} \, dx$$

where

$$H_i(k, x) = L_i^2(k, x) \left[1 - \big(x - x_{in}(k)\big) \frac{\omega''_n(k, x_{in}(k))_{(x)}}{\omega'_n(k, x_{in}(k))_{(x)}} \right]$$

is the fundamental polynomial of degree $2n-1$ of the Hermite's first order interpolation pertaining to knot $x_{in}(k)$. Equality (7) might be thus obtained directly from (2) by substituting for $f(x)=H_i(k,x)$. Along with the application of the Gauss—Jacobi quadrature this is done in [2].

2) The value $F_i^{(3)}(k)$ may be obtained also in another way. If the system of orthogonal polynomials $\{\omega_n(k,x)\}$ satisfies the homogeneous linear differential equation of the form

$$A_n(k,x)\omega_n''(k,x)_{(x)}+B_n(k,x)\omega_n'(k,x)_{(x)}+C_n(k,x)\omega_n(k,x)=0$$

then for $x=x_{in}(k)$ if $A_n(k,x_{in}(k))\neq 0$, we get

$$\frac{\omega_n''(k,x_{in}(k))_{(x)}}{\omega_n'(k,x_{in}(k))_{(x)}}=-\frac{B_n(k,x_{in}(k))}{A_n(k,x_{in}(k))}$$

and thus

$$F_i^{(3)}(k)=\frac{B_n(k,x_{in}(k))}{2A_n(k,x_{in}(k))}.$$

By integrating (3) and (5), we get

$$x_{in}(k)=x_{in}(k_0)+\int_{k_0}^{k}F_i^{(1)}(k)A_{in}^{-1}(k)\,dk,\quad A_{in}(k)=A_{in}(k_0)+\int_{k_0}^{k}F_i^{(4)}(k)\,dk$$

resp. Another relation for $A_{in}(k)$ is formulated by

COROLLARY 2. *If the functions* $F_i^{(2)}(k)$ *and*

$$F_i^{(5)}(k)=\frac{x_{in}'(k)\omega_n''(k,x_{in}(k))_{(x)}}{\omega_n'(k,x_{in}(k))_{(x)}}$$

are continuous on $[k_0,k]$ *then*

$$A_{in}(k)=\left|\omega_n'(k,x_{in}(k))_{(x)}\right|^{-1}\left[A_{in}(k_0)+\int_{k_0}^{k}\left|\omega_n'(k,x_{in}(k))_{(x)}\right|F_i^{(2)}(k)\,dk\right]. \tag{8}$$

PROOF. If (4) is considered as a linear differential equation

$$A_{in}'(k)+F_i^{(5)}(k)A_{in}(k)=F_i^{(2)}(k)$$

of the first order, then its solution is

$$A_{in}(k)=\exp\left(-\int F_i^{(5)}(k)\,dk\right)\left[\int\exp\left(\int F_i^{(5)}(k)\,dk\right)F_i^{(2)}(k)\,dk+c\right]$$

from which we get (8) under the initial conditions $A_{in}(k)_{k=k_0}=A_{in}(k_0)$.

COROLLARY 3. *If for* $k\in[k_1,k_2]$ $F_i^{(2)}(k)>0$ *holds then*

$$A_{in}(k_2)\left|\omega_n'(k_2,x_{in}(k_2))_{(x)}\right|>A_{in}(k_1)\left|\omega_n'(k_1,x_{in}(k_1))_{(x)}\right|. \tag{9}$$

PROOF. Because $F_i^{(2)}(k)>0$ on $[k_1,k_2]$ we obtain by integrating (4)

$$\int_{k_1}^{k_2}\frac{A_{in}'(k)}{A_{in}(k)}\,dk>-\int_{k_1}^{k_2}F_i^{(5)}(k)\,dk$$

and thus

$$[\ln A_{in}(k)]_{k_1}^{k_2} > \left[\ln \left|\omega_n'(k, x_{in}(k))_{(x)}\right|\right]_{k_2}^{k_1}$$

from which we get, after rearrangement (9).

It is obvious that if $F_i^{(2)}(k)<0$ on $[k_1, k_2]$, then (9) holds with the reverse sign of inequality.

Theorem 1 allows to compare the knots and coefficients of two Gauss type quadratures with various weight functions.

Let us have two Gauss type quadratures on $[a, b]$, with weights $W(x)$ and $w(x)$, knots $x_i^{(n)}$ and $\bar{x}_i^{(n)}$, coefficients $A_i^{(n)}$ and $\bar{A}_i^{(n)}$, resp. Let us now set up the weight function $W(k, x)$, $k_1 \leqq k \leqq k_2$, such that for $k_1<k_2$, $W(k_1, x)=w(x)$ and $W(k_2, x)=W(x)$ i.e. $\omega_n(k_1, x)$ and $\omega_n(k_2, x)$ will be orthogonal polynomials with weight functions $w(x)$ and $W(x)$ on $[a, b]$ with knots $x_{in}(k_1)=\bar{x}_i^{(n)}$ and $x_{in}(k_2)=x_i^{(n)}$, resp. Then

$$W(k, x) = \frac{1}{k_2-k_1}[(k-k_1)W(x)+(k_2-k)w(x)]. \tag{10}$$

In this case

$$F_i^{(1)}(k) = \frac{1}{k_2-k_1}\int_a^b L_i^2(k, x)(x-x_{in}(k))(W(x)-w(x))\,dx$$

and thus considering the orthogonality of the polynomials $\{\omega_n(k, x)\}$ with the respective weights to each polynomial of degree less than n we obtain

$$F_i^{(1)}(k_1) = \frac{1}{k_2-k_1}\int_a^b L_i^2(k_1, x)(x-\bar{x}_i^{(n)})W(x)\,dx$$

and

$$F_i^{(1)}(k_2) = \frac{1}{k_1-k_2}\int_a^b L_i^2(k_2, x)(x-x_i^{(n)})w(x)\,dx.$$

If we find k_1, k_2 such that for $k\in[k_1, k_2]$, $F_i^{(1)}(k)$ will be of constant sign, then $\bar{x}_i^{(n)} < x_i^{(n)}$ if $F_i^{(1)}(k_1) > 0$ and $\bar{x}_i^{(n)} > x_i^{(n)}$ if $F_i^{(1)}(k_1) < 0$.

REMARK. It follows from MARKOV's theorem [1] that if for the weight (10) in which $w(x)$ and $W(x)$ are from $C[a, b]$ we have for $x\in[a, b]$

$$\frac{\partial F(k, x)}{\partial x} = \left[\frac{W(x)-w(x)}{(k-k_1)W(x)+(k_2-k)w(x)}\right]' > 0$$

i.e. if $\left[\dfrac{W(x)}{w(x)}\right]' > 0$ then $x_{in}'(k) > 0$ i.e. $\bar{x}_i^{(n)} < x_i^{(n)}$, $i=1, 2, \dots n$.

To compare the quadrature coefficients we use (7) and (9), respectively. If considering again $W(k, x)$ from (10), then $A_{in}(k_1) = \bar{A}_i^{(n)}$ and $A_{in}(k_2) = A_i^{(n)}$. Find k_1, k_2 such that for $k\in[k_1, k_2]$ $F_i^{(4)}(k)$ will not change sign. If $F_i^{(4)}(k_1) > 0$ then $\bar{A}_i^{(n)} < A_i^{(n)}$, if $F_i^{(4)}(k_1) < 0$ then $\bar{A}_i^{(n)} > A_i^{(n)}$. Similarly if for $k\in[k_1, k_2]$ we have $F_i^{(2)}(k) > 0$ then we get from (9)

$$\bar{A}_i^{(n)} < A_i^{(n)}\left|\frac{\omega_n'(k_2, x_i^{(n)})_{(x)}}{\omega_n'(k_1, \bar{x}_i^{(n)})_{(x)}}\right|.$$

Because

$$F_i^{(2)}(k) = \frac{1}{k_2-k_1}\int_a^b L_i^2(k,x)\big(W(x)-w(x)\big)\,dx$$

this condition is satisfied e.g. if $W(x) \geqq w(x)$ for $x\in[a,b]$.

For $F_i^{(2)}(k)<0$ the reverse inequalities hold everywhere.

REMARK. Since the functions $F_i^{(j)}(k)$, $j=1,2,4$ are in fact linear combinations of the moments $\int_a^b x^m \frac{\partial W(k,x)}{\partial k}\,dx$, $m=0,1,\dots,2n-1$ their values may be obtained at a given k also by the interpolation quadrature with p knots having weight function $\frac{\partial W(k,x)}{\partial k}$. If this is a Newton—Cotes type quadrature, then $p\geqq 2n$, if it is of Gauss type, then $p\geqq n$.

Let us yet investigate the dependency of the integral $N_n(k)=\int_a^b \omega_n^2(k,x)W(k,x)\,dx$ from k at a fixed n. This integral normalizes the system $\{\omega_n(k,x)\}$, because for $Q_n(k,x)=N_n^{-\frac{1}{2}}(k)\,\omega_n(k,x)$, $\int_a^b Q_n^2(k,x)W(k,x)\,dx=1$ holds, and for $f(x)\in C^{2n}[a,b]$ it is also in the remainder in (1) because then $R_{n,k}(f)=\frac{f^{(2n)}(\xi)}{(2n)!}N_n(k)$ where $a\leqq\xi\leqq b$. We have

THEOREM 2. *If* $\frac{\partial W(k,x)}{\partial k}$ *is a nonnegative (nonpositive) function not identically zero for* $x\in[a,b]$ *and* $k\in[k_1,k_2]$, *then* $N_n(k)$ *is an increasing (decreasing) function of the parameter* k *for* $k\in[k_1,k_2]$.

PROOF. If $f(x)$ in (1) is a polynomial of degree $2n$ with main coefficient 1, then $R_{n,k}(f)=N_n(k)$. After differentiating (1) with respect to k we get

$$\int_a^b f(x)\frac{\partial W(k,x)}{\partial k}\,dx = \sum_{i=1}^{n}\left[A'_{in}(k)f(x_{in}(k))+A_{in}(k)\frac{\partial f(x_{in}(k))}{\partial x}x'_{in}(k)\right]+N'_n(k).$$

If $f(x)=\omega_n^2(k,x)$ then $f(x_{in}(k))=\frac{\partial f(x_{in}(k))}{\partial x}=0$, $i=1,2,\dots n$ and thus

$$N'_n(k)=\int_a^b \omega_n^2(k,x)\frac{\partial W(k,x)}{\partial k}\,dx$$

from which the statement of the theorem follows.

Theorem 2 allows to compare the integrals $\overline{N}_n$ with N_n, where $\overline{N}_n=N_n(k_1)= =\int_a^b \omega_n^2(k_1,x)w(x)\,dx$ and $N_n=N_n(k_2)=\int_a^b \omega_n^2(k_2,x)W(x)\,dx$. For the weight (10) we get

$$N_n'(k) = \frac{1}{k_2-k_1}\int_a^b \omega_n^2(k,x)\big(W(x)-w(x)\big)\,dx.$$

Hence if e.g. $W(x) \geqq w(x)$ $\big(W(x) \leqq w(x)\big)$ for $x \in [a, b]$ then $\bar{N}_n < N_n$ $(\bar{N}_n > N_n)$.

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A NOTE ON NOETHER LATTICES IN WHICH EACH ELEMENT IS PRIMARY

By

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§ 1. Introduction

A generalized primary ring is a commutative ring with identity which satisfies the condition that every ideal is primary. This concept was introduced recently by M. SATYANARAYANA in [6]. In his paper he states the following theorem ([6], Theorem 4.5):

(1.1) *Let R be a Noetherian domain with identity. Then R is generalized primary if and only if R is a one dimensional local ring.*

Since any field is a generalized primary ring with (Krull) dimension zero, the above theorem tacitly assumes that R is not a field.

In this paper we obtain a complete characterization of all generalized primary (commutative) Noetherian rings with identity as a corollary to a more general result (Corollary 2.4).

§ 2. The results

For terminology used in the remainder of this paper which is not defined here, the reader is directed to the references. Most of the basic concepts can be found in [1].

Let A be an arbitrary element of a Noether lattice L. Rad (A) is defined to be the join of all elements X of L for which there exists a natural number n such that $X^n \leqq A$. Clearly $A \leqq \mathrm{Rad}(A)$, and since L satisfies the ascending chain condition, for each element A of L there exists a natural number m such that $(\mathrm{Rad}(A))^m \leqq A$. By a maximal element of L we shall mean a maximal element different from the unit element of L.

The rank of a prime element P of a Noether lattice L is defined to be the supremum of all integers n for which there exists a prime chain $P_0 < P_1 < P_2 < \ldots < P_n = P$ in L. The altitude of L is defined to be the supremum of the ranks of the prime elements of L.

We will make use of the following result in the sequel.

THEOREM 2.1. *Let L be a Noether lattice. If every element of L is primary, then*
(2.1) *L is a local Noether lattice, and*
(2.2) *the prime elements of L are linearly ordered.*

PROOF. Assume every element of L is primary. Let A and B be arbitrary prime elements of L. Since $AB \leqq A \wedge B$, and $A \wedge B$ is primary by hypothesis, we have that either $A \leqq A \wedge B$ or that $B^n \leqq A \wedge B$, for some natural number n. Since A is prime, it follows that either $A \leqq B$ or $B \leqq A$, and thus the prime elements are linearly ordered which establishes (2.2). Since maximal elements in a Noether lattice are prime, it follows from (2.2) that L has only one maximal element, and thus L is local which establishes (2.1) and completes the proof.

We will require the following result in the proof of Theorem 2.3. The proof is straightforward and will be omitted.

LEMMA 2.2. *Let L be a Noether lattice and let A be an element of L. If* Rad (A) *is a maximal element of L, then A is primary.*

We can now establish the following theorem.

THEOREM 2.3. *Let L be a Noether lattice. Then the following two statements are equivalent:*
(2.3) *Every element of L is primary.*
(2.4) *Either* (a) *L is local with altitude zero, or* (b) *L is local with altitude one and the null element of L is prime.*

PROOF. Assume every element of L is primary. By Theorem 2.1, L is a local Noether lattice. Since every element of L is primary, L is a unique normal decomposition lattice. Thus, by Lemma 3 of [4], L is either a primary lattice or L is a one dimensional lattice in which 0 is prime. Hence, (2.3) implies (2.4).

Assume now that (2.4) holds and let M be the maximal element of L. Let A be an arbitrary element of L. If the altitude of L is zero, then Rad $(A)=M$, and thus A is primary by Lemma 2.2. Similarly, if the altitude of L is one, 0 is prime, and $A\neq 0$, then A is primary. If the altitude of L is one, 0 is prime, and $A=0$, then A is prime and hence A is primary. Thus (2.3) is established, and the proof is complete.

An application of the above theorem to the ring case yields the following result.

COROLLARY 2.4. *Let R be a (commutative) Noetherian ring with identity. Then the following two statements are equivalent:*
(2.5) *R is generalized primary,*
(2.6) *Either* (a) *R is (Zariski) primary, or* (b) *R is a one dimensional local domain.*

PROOF. The lattice of ideals $L(R)$ of R is a Noether lattice [1]. Since R is Noetherian it is easy to see that A is a primary element of $L(R)$ if and only if A is a primary ideal of R. The proof is now completed by applying Theorem 2.3 to the Noether lattice $L(R)$.

The following two concepts are closely related to the concepts of the weak union condition and the union condition on primes introduced in [2] and [3] by E. W. JOHNSON and J. P. LEDIAEV.

A Noether lattice L is said to satisfy the strong union condition on elements if, given a collection $\{P_\alpha : \alpha \in S\}$ of primes of L and an element A of L such that $A \not\leqq P_\alpha$, for each α in S, then there exists a principal element E of L such that $E \leqq A$ and $E \not\leqq P_\alpha$, for each α in S.

A Noether lattice L is said to satisfy the strong union condition on primes if, given a collection $\{P_\alpha : \alpha \in S\}$ of primes of L and a prime element P of L such that $P \not\leqq P_\alpha$, for each α in S, then there exists a principal element E of L such that $E \leqq P$ and $E \not\leqq P_\alpha$, for each α in S.

COROLLARY 2.5. *Let L be a Noether lattice. If every element of L is primary, then the following four statements hold in L.*
(2.7) *L satisfies the strong union condition on elements.*
(2.8) *L satisfies the strong union condition on primes.*
(2.9) *For each prime element P of L, there exists a principal element E of L such that* $P = \mathrm{Rad}\,(E)$.
(2.10) *For each element A of L, there exists a principal element E of L such that* $\mathrm{Rad}\,(A) = \mathrm{Rad}\,(E)$.

PROOF. It was shown in Theorem 3.1 of [5] that these four conditions are equivalent for any Noether lattice. Since (2.8) follows immediately from (2.4) of Theorem 2.3, the proof is complete.

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GROUPS WHOSE PROPER FACTORS ARE ABELIAN

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In [2] the author defines a Z-group as a group all of whose proper factors are abelian and he discusses Z-groups whose centre is nontrivial. In this paper, we discuss Z-groups whose centre is trivial and we give a structure theorem for solvable and supersolvable Z-groups.

DEFINITION. We will call a finite non abelian group a Z-group if all its proper factors are abelian; a $Z-p$-group if G is also a p-group.

We give a simple criterion for determining if a group is a Z-group. Let $Z(G)$ denote the centre of G, and G' the derived group of G.

THEOREM 1. *Let G be a finite non abelian group. G is a Z-group if and only if G' is the unique minimal normal subgroup of G.*

PROOF. Let G be a Z-group. Then G has a unique minimal normal subgroup. For if N_1 and N_2 are minimal normal in G, then by hypothesis G/N_1 and G/N_2 are abelian, hence $G/N_1 \cap N_2$ or G abelian. Since G is a finite non abelian group, G has a unique minimal normal subgroup N and since G/N is abelian this unique minimal normal subgroup of G is G'.

Conversely, suppese that G' is the unique minimal normal subgroup of G. We distinguish between the case $Z(G)$ is nontrivial and the case $Z(G)$ is trivial.

Let $Z(G)$ be nontrivial and suppose that G' is the unique minimal normal subgroup of G. Since every subgroup of $Z(G)$ is normal in G, and G' is the unique minimal normal subgroup of G, we must have that $Z(G)$ has a unique minimal subgroup. The structure theorem for finite abelian groups implies that $Z(G)$ is a cyclic p-group for some prime p and that G' has order p. By Theorem 2 of [2], we have that G is a $Z-p$-group.

Let $Z(G)$ be trivial and let G' be the unique minimal normal subgroup of G. If N is a nontrivial normal subgroup of G, then $[N, G]=\langle [n, g]/n\in N, g\in G\rangle$ is a normal subgroup of G. Since $Z(G)=\{1\}$, $[N, G]$ is nontrivial. The normality of N and minimality of G^1 imply $G^1\subseteqq [N, G]\subseteqq N$ or G/N is abelian and G is a Z-group.

We saw in Theorem 1 that in any Z-group G, the derived group G' is the unique minimal normal subgroup of G. We use this result to obtain the following:

LEMMA. *Let G be a Z-group with trivial centre. Then:*
(i) *G' is characteristically simple.*
(ii) *G' is a direct product of conjugates of a simple group H.*
(iii) *G is isomorphic to a primitive permutation group on the cosets of a non-normal maximal subgroup M in G.*

PROOF. If H is a nontrivial characteristic subgroup of G', then H is a characteristic subgroup of G. Hence H is surely normal in G. This contradicts the minimality of G' and G' is characteristically simple.

Let H be a minimal normal subgroup of G'. Let $T=\langle H^g | g\in G\rangle$. Then T is a nontrivial normal subgroup of G and $T\subseteqq G'$. Hence $T=G'$. Consequently, $G'=H_1\times \times H_2\times\ldots\times H_r$ with $H_i=H^{g_i}$ for some $g_i\in G$. Since every normal subgroup of H_i is a normal subgroup of G' and since the H_i are minimal normal in G', we must have that the H_i are simple.

Since G is not nilpotent, G has a non-normal maximal subgroup, say M. Now $G/\text{core } M$ is isomorphic to a primitive permutation group on the cosets of M in G. If core $M\neq\{1\}$, then $G'\subseteqq \text{core } M\subseteqq M$ implies that M is normal in G. Hence G is isomorphic to a primitive permutation group on the cosets of M in G.

If G' is a direct product of conjugates of a simple abelian group H, then H is a cyclic group of order p, p a prime, and G^1 is an elementary abelian p-group. Furthermore, G is solvable. The above Lemma allows us to give a structure theorem for solvable Z-group with trivial centre.

THEOREM 2. *Let G be a solvable Z-group with trivial centre, then G is a Frobenius group with the following properties:*

(i) *G' is the unique minimal normal subgroup of G and G^1 is an elementary abelian p-group of order p^d, p a prime.*

(ii) *G' is the Frobenius kernel of G.*

(iii) *G' is the fitting subgroup of G.*

(iv) *If M is a non-normal maximal subgroup of G, then M is a Frobenius complement.*

(v) *If M is a non-normal maximal subgroup of G, then $|M|$ divides $|G'|-1$.*

(vi) *All non-normal maximal subgroups are conjugate and cyclic.*

PROOF. Let G be a solvable Z-group with trivial centre. Since G is not nilpotent, G has a non-normal maximal subgroup, say M. By the Lemma, G is isomorphic to a solvable primitive permutation group on the cosets of M. By [1, p. 159], we have that $G=G'M$, $G'\cap M=\{1\}$ and $C_G(G')=G'$. As a minimal normal subgroup of a solvable group, G^1 is contained in the centre of the fitting subgroup of G. Since $C_G(G')=G'$, we must have that G' is the fitting subgroup of G.

To show that G is a Frobenius group, it suffices to show that for all $g\in G/M$, we have $M\cap M^g=\{1\}$.

Let $g\in G/M$ and suppose that $M\cap M^g\neq\{1\}$. Then there exists $m\in M$ such that $m^g\in M$. Consequently, $m^{-1}m^g\in M$. This implies that $m^{-1}m^g=[m, g]\in G^1\cap M= =(1)$ or $m^g=m$. Since M is abelian, we have that $C_G(\langle m\rangle)\supseteqq M$ and $C_G(\langle m\rangle)\supseteqq\langle g\rangle$. Consequently, $C_G(\langle m\rangle)\supseteqq M\langle g\rangle=G$ or $m\in Z(G)$. This is a contradiction. Hence $M\cap M^g=\{1\}$ for all $g\in G/M$ and G is a Frobenius group with Frobenius complement M.

From the structure of a Frobenius group (see [2]) we obtain (i), (ii), (iv) and (v).

Since the Frobenius complements are all conjugate and any maximal non-normal subgroup is a Frobenius complement, all maximal non-normal subgroups are conjugate. The Sylow subgroups of a Frobenius complement are cyclic for $p>2$ and cyclic or generalized quarternion for $p=2$. Since the Frobenius complements are abelian, we must have that they are cyclic.

In [2], the author shows that $Z-p$-groups can be characterized in terms of $Z-p$-groups of rank 2. For solvable groups with trivial centre, we obtain:

COROLLARY. *Let G be a solvable Z-group with trivial centre, then G is generated by 2 elements.*

PROOF. By Theorem 2, $G=G'M$ with G' an elementary abelian group and M a cyclic group. Let a be any element of G', $a\neq 1$. Let b be a generator of the cyclic group M. Then $\langle a, b\rangle = G$.

Solvable Z-groups with trivial centre need not be supersolvable, as the following example shows:

EXAMPLE. Let $G=A_4$, the alternating group on four letters. Then G' is the Klein four group and is the unique minimal normal subgroup of G. On the other hand, G is not supersolvable.

In case G is a supersolvable Z-group with trivial centre, we obtain:

THEOREM 3. *Let G be a finite group with trivial centre. Then the following are equivalent:*

(a) *G is a supersolvable Z-group.*

(b) *G' has prime order.*

(c) *All Sylow subgroups of G are cyclic and the Sylow subgroup corresponding to the largest prime has prime order and is self centralizing.*

(d) *G is generated by two elements a and b with defining relations $a^p=1$; $b^m=1$; $b^{-1}ab=a^r$; $r^m=1 \bmod p$, p a prime.*

PROOF. (a)⇒(b) if G is a supersolvable Z-group, a minimal normal subgroup has prime order. Since G' is a minimal subgroup in a Z-group, G^1 has prime order.

(b)⇒(c) Let G^1 have prime order, say p. Let M be a self normalizing maximal subgroup of G. Then $G=G'M$ and $G'\cap M=\{1\}$. Since $C_M(G^1)$ is centralized by M as well as G', we have $C_M(G')\subseteq Z(G)$. By hypothesis $Z(G)=\{1\}$, hence $C_M(G')= =\{1\}$. Therefore M is isomorphic to a subgroup of the automorphism group of a cyclic group of order p. Consequently, it is cyclic of order dividing $p-1$ and (c) holds.

(c)⇒(d) A group having all Sylow subgroup cyclic is a Sylow—Tower group. Consequently, the Sylow subgroup P corresponding to the largest prime is normal in G. Since P is selfcentralizing, we have that $G/C_G(P)=G/P$ is cyclic. Consequently, $G^1\leqq P$. Therefore $G^1=P$ has order p. Furthermore, if M is a self normalizing maximal subgroup, then $G=G^1M$ and $G'\cap M=\{1\}$. Let $a\in G'$; $a\neq 1$; and let b be a generator of M, then a and b satisfy the defining relations given in (d).

(d)⇒(a) A group C with the above defining relations has $G^1=\langle a\rangle$. If H is a normal subgroup of G and $H\cap G^1=\{1\}$, then $[H, G]=\{1\}$ and $H\subseteq Z(G)$, so $H=\{1\}$. Consequently, $H\cap G'\neq\{1\}$, and since $|G^1|=p$, we must have $G^1\subseteq H$. Therefore, G is a supersolvable Z-group.

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ON ANALYTIC FUNCTIONS WHIT GAPS. II

By

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1. Introduction

Let f be an entire function of finite exponential type τ andlet $\lambda>0$. The behaviour of zeros of the derivatives of f under the constraint $\tau<\lambda$ has been the subject of considerable investigation [1, 2]. Roughly speaking, a function f of bounded exponential type cannot have too many derivatives vanishing in a certain neighbourhood of the origin, unless f reduces to a constant. The supremum of numbers λ for which the conditions $\tau<\lambda$, $f^{(k)}(z_k)=0$ and $|z_k|\leqq 1$ $(k=0, 1, 2, \dots)$ imply $f\equiv 0$ is called the *Whittaker constant* [1]. Whittaker's constant is usually denoted by W, and satisfies $0.7259<W<0.7378$.

From results of M. DRAGILEV [4] and J. D. BUCKHOLTZ [3] it is known that the Whittaker constant appears in similar capacity in the corresponding problem for functions analytic in only a neighbourhood of 0. Dragilev proved that if f has radius of convergence greater than W^{-1} and if $f^{(k)}(z_k)=0$ for some $|z_k|\leqq 1/(k+1)$, $k=0, 1, 2, \dots$, then $f\equiv 0$. Buckholtz proved that W is the sharp constant.

In the present paper, we determine the analogous sharp constant for functions f analytic in a neighbourhood of 0 and satisfying the additional constraint that f have gaps of length p, p a positive integer, in its Taylor series: $f(z)=\sum_{k=0}^{\infty} a_{kp}z^{kp}$. Equivalently, f must satisfy $f^{(j)}(0)=0$, for $j\neq mp$, $m=0, 1, 2, \dots$.

H. S. WILF [7] defines the constant W_p to be the supremum of numbers $\lambda>0$ such that if $f(z)=\sum_{k=0}^{\infty} a_{kp}z^{kp}$ is entire with exponential type $\tau(f)$ less than λ and each of $f, f', f'', \dots$ has a zero in $|z|\leqq 1$, then $f\equiv 0$. Wilf proved that W_p satisfies the asymptotic formula $W_p\sim p/e$. In [5] these authors obtained an exact determination of W_p and improved the bounds in [7] to

$$(p!/2)^{1/p} \leqq W_p \leqq (p!)^{1/p}. \tag{1.1}$$

Our results here are contained in the following two theorems.

THEOREM A. *Suppose that $f(z)=\sum_{k=0}^{\infty} a_{kp}z^{kp}$ has radius of convergence $c(f)>1$ and suppose there exists a sequence $\{z_{kp}\}_{k=0}^{\infty}$ such that $|z_{kp}|\leqq W_p/(kp+1)$ and $f^{(kp)}(z_{kp})=0$, $k=0, 1, 2, \dots$. Then $f\equiv 0$.*

While this result is new, it is a fairly straightforward extension of DRAGILEV's theorem ([4]), with the same methods being applied. For notation and a treatment of the Dragilev theory, see §2 below and [6]. The details of the proof of Theorem A are left to the reader.

For a function f analytic in a neighbourhood of 0, let $r_n(f)$ denote the smallest modulus of a zero of $f^{(n)}(z)$. In case some $f^{(m)}$ does not vanish, we take $r_m(f)=\infty$.

THEOREM B. *Suppose that* $f(z)=\sum_{k=0}^{\infty} a_{kp}z^{kp}$ *has radius of convergence* $c(f)>1$ *and is not a polynomial. Then*

$$\limsup_{n\to\infty}(np+1)r_{np}(f)\geqq W_p. \tag{1.2}$$

Moreover, there exists a function $F(z)=\sum_{k=0}^{\infty} A_{kp}z^{kp}$, *with* $c(F)=1$, *such that equality holds in* (1.2).

The estimate (1.2) is an easy consequence of Theorem A.

If $\varepsilon>0$, the function F of Theorem B satisfies $(np+1)r_{np}(F)<W_p+\frac{\varepsilon}{2}$, for all but finitely many n. Choose a number $r<1$ so that $r(W_p+\varepsilon)>W_p+\frac{\varepsilon}{2}$. For some sufficiently large integer N the function $G(z)=F^{(Np)}(rz)$ satisfies $c(G)>1$ and $(kp+1)r_{kp}(G)<W_p+\varepsilon$, $k=0,1,2,\ldots$. Since $G\not\equiv 0$, it follows that the constant W_p is best possible in Theorem A.

2. Proof of Theorem B

Let p be a positive integer and $\{z_j\}_{j=0}^{\infty}$ a sequence of complex numbers such that $z_j=0$ whenever $j\neq kp$, $k=0,1,2,\ldots$. The Gončarov polynomial ([2]) of degree np, formed with respect to the sequence $\{z_j\}$, is found inductively by the formula

$$G_{np}(z;z_0,0,\ldots,0,z_p,0,\ldots,0,z_{(n-1)p},0,\ldots,0)=$$
$$=\frac{z^{np}}{(np)!}-\sum_{k=0}^{n-1}\frac{z_{kp}^{np-kp}}{(np-kp)!}G_{kp}(z;z_0,0,\ldots,0,z_p,0,\ldots,0,z_{(k-1)p},0,\ldots,0),$$

and, as usual, $G_0(z)=1$. Let

$$H_{np}=\max|G_{np}(0;\zeta_0,0,\ldots,0,\zeta_p,0,\ldots,0,\zeta_{(n-1)p},0,\ldots,0)|, \tag{2.2}$$

where $|\zeta_{kp}|\leqq 1$, $0\leqq k<n$. In [5] we proved that

$$W_p=\{\sup_{1\leqq n<\infty} H_{np}^{1/np}\}^{-1}=\{\lim_{n\to\infty} H_{np}^{1/np}\}^{-1} \tag{2.3}$$

and we characterized W_p as the sharp constant in the following expansion theorem: if $f(z)=\sum_{k=0}^{\infty}a_{kp}z^{kp}$ is entire with $\tau(f)<W_p$ and $\{z_{kp}\}_{k=0}^{\infty}$ is a sequence of points in $|z|\leqq 1$, then

$$f(z)=\sum_{k=0}^{\infty}f^{(kp)}(z_{kp})G_{kp}(z;z_0,0,\ldots,0,z_p,0,\ldots,0,z_{(k-1)p},0,\ldots,0) \tag{2.4}$$

for all z. Together with (2.4), we will need the following properties of Gončarov polynomials in the sequel (see [2]):

$$G_{np}(z_0;\, z_0, 0, \dots, z_p, 0, \dots, z_{(n-1)p}, 0, \dots) = 0, \tag{2.5}$$

$$\begin{aligned} &G_{np}(\alpha z;\, \alpha z_0, 0, \dots, \alpha z_p, 0, \dots, \alpha z_{(n-1)p}, 0, \dots) = \\ &= \alpha^{np} G_{np}(z;\, z_0, 0, \dots, z_p, 0, \dots, z_{(n-1)p}, 0, \dots), \end{aligned} \tag{2.6}$$

$$\begin{aligned} &G_{np}(z;\, z_0, 0, \dots, z_p, 0, \dots, z_{(n-1)p}, 0, \dots) = \\ &= \sum_{k=0}^{n} G_{np-kp}(0;\, z_{kp}, 0, \dots, z_{(k+1)p}, 0, \dots, z_{(n-1)p}, 0, \dots) \frac{z^{kp}}{(kp)!}, \end{aligned} \tag{2.7}$$

$$H_{np} \geqq H_{mp} H_{(n-m)p}, \quad 0 \leqq m \leqq n, \tag{2.8}$$

$$\begin{aligned} &G_{np}^{(kp)}(z;\, z_0, 0, \dots, z_p, 0, \dots, z_{(n-1)p}, 0, \dots) = G_{np-kp}(z;\, z_{kp}, 0, \dots, z_{(n-1)p}, 0, \dots), \\ &0 \leqq k \leqq n. \end{aligned} \tag{2.9}$$

LEMMA 2.1. *If* $f(z) = \sum_{k=0}^{\infty} a_{kp} z^{kp}$ *is entire and* $z_0, z_p, \dots, z_{(m-1)p}$ *are complex numbers, where* m *is a positive integer, then*

$$\begin{aligned} f(z) = &\sum_{k=0}^{m-1} f^{(kp)}(z_{kp}) G_{kp}(z;\, z_0, 0, \dots, z_p, 0, \dots, z_{(k-1)p}, 0, \dots) + \\ &+ \sum_{k=m}^{\infty} f^{(kp)}(0) G_{kp}(z;\, z_0, 0, \dots, z_p, 0, \dots, z_{(m-1)p}, 0, \dots, 0, \dots). \end{aligned} \tag{2.10}$$

PROOF. Replace $f^{(kp)}(z_{kp})$ by $\sum_{m=0}^{\infty} f^{(mp+kp)}(0) \frac{z_{kp}^{mp}}{(mp)!}$ in the first sum on the right, and interchange the order of summation. Using (2.1), the resulting expression reduces to (2.10).

For the remainder of this paper, matters will be greatly simplified if we take $p=2$. This we will do, leaving the case of longer gaps as an obvious generalization.

The following lemma yields information about the growth of the sequence of numbers defined by (2.2).

LEMMA 2.2. *There exists an infinite set* S *of positive integers such that* $n \in S$ *implies*

$$H_{2n}^{1/2n} \geqq H_{2k}^{1/2k}, \quad 0 \leqq k \leqq n.$$

PROOF. Recall that $W_2^{-1} = \sup_{1 \leqq m < \infty} H_{2m}^{1/2m}$. Suppose first that $W_2^{-1} = H_{2m}^{1/2m}$ for some m. If j is a positive integer, then (2.8) implies

$$H_{2jm}^{1/2jm} \geqq \{H_{2(j-1)m} H_{2m}\}^{1/2jm} \geqq \{H_{2(j-2)m} H_{2m}^2\}^{1/2jm} \geqq \dots \geqq \{H_{2m}^j\}^{1/2jm} = H_{2m}^{1/2m} = W_2^{-1}.$$

Therefore $H_{2jm}^{1/2jm} = W_2^{-1}$, $j = 1, 2, 3, \dots$, and we may take $S = \{jm\}_{j=1}^{\infty}$.

Suppose, on the other hand, that $H_{2n}^{1/2n} < W_2^{-1}$ for all n. Set $M_n = \max_{0 \leqq k < n} H_{2k}^{1/2k}$. Then $\{M_n\}$ is nondecreasing, converges to W_2^{-1} (by (2.3)) and every term is less than W_2^{-1}. Thus $M_{n+1} > M_n$ for infinitely many n. For such n, $\max_{0 \leqq k \leqq n} H_{2k}^{1/2k} > \max_{0 \leqq k < n} H_{2k}^{1/2k}$, so that $H_{2n}^{1/2n} > \max_{0 \leqq k < n} H_{2k}^{1/2k}$. This completes the proof of the lemma.

We are now ready to construct the function F of Theorem B. Consider the set S of Lemma 2.2; for each $n \in S$ choose a configuration $z_0, 0, z_2, 0, z_4, 0, \ldots, \ldots, z_{2(n-1)}, 0$ such that $|z_{2j}|=1$, $0 \leqq j \leqq n-1$, and

$$H_{2n} = |G_{2n}(0; z_0, 0, z_2, 0, \ldots, z_{2(n-1)}, 0)|.$$

Set $\alpha_n = H_{2n}^{1/2n} W_2$ (note $\alpha_n \geqq 1$) and

$$P_n(z) = \frac{G_{2n}(\alpha_n z; z_0, 0, z_2, 0, \ldots, z_{2(n-1)}, 0)}{G_{2n}(0; z_0, 0, z_2, 0, \ldots, z_{2(n-1)}, 0)}$$

for $n \in S$. We have $P_n(0) = 1$ and, by (2.7),

$$P_n(z) = \sum_{k=0}^{n} \frac{G_{2n-2k}(0; z_{2k}, 0, \ldots, z_{2(n-1)}, 0)}{G_{2n}(0; z_0, 0, \ldots, z_{2(n-1)}, 0)} \cdot \frac{\alpha_n^{2k} z^{2k}}{(2k)!}.$$

Thus the modulus of the coefficient of z^{2k} does not exceed $H_{2n-2k}\alpha_n^{2k}/(H_{2n}(2k)!)$. But since $n \in S$, $H_{2n}^{1/2n} \geqq H_{2n-2k}^{1/(2n-2k)}$, so

$$\frac{H_{2n-2k}}{H_{2n}} \leqq H_{2n}^{\frac{2n-2k}{2n}-1} = H_{2n}^{-k/n}.$$

Also, $\alpha_n^{2k} = [H_{2n}^{1/2n} W_2]^{2k} = H_{2n}^{k/n} W_2^{2k}$. Therefore, the modulus of the coefficient of z^{2k} does not exceed $W_2^{2k}/(2k)!$. Hence $P_n(z)$ is majorized by the function $\cosh(W_2 z)$. In particular, $\{P_n\}_{n \in S}$ is uniformly bounded on compact sets. Let $\{P_{n_j}\}$ be a subsequence uniformly convergent on compact subsets to a function $P(z)$. Then $P(0)=1$, $P^{(2n+1)}(0)=0$ and $|P^{(2n)}(0)| \leqq W_2^{2n}$, $n=0, 1, 2, \ldots$. For fixed k, (2.9) and (2.5) imply that each of $P_n^{(2k)}(z)$, $n > k$, has a zero on the circle $|z| = \alpha_n^{-1}$. Since $\lim_{n \to \infty} \alpha_n = 1$, uniform convergence of derivatives implies that each of $P^{(2k)}(z)$ has a zero on $|z|=1$. The conditions on the sequence $P^j(0)$, $0 \leqq j < \infty$, imply that $\tau(P) \leqq W_p$. The expansion theorem (2.4) implies that $\tau(P) = W_2$.

Now let $Q(z) = P(z/W_2)$. Then $\tau(Q)=1$, $Q^{(2n+1)}(0)=0$, $n=0, 1, 2, \ldots$, and each of $Q^{(2n)}(z)$ has a zero on the circle $|z| = W_2$. Let m be a positive integer and choose complex numbers $z_0, z_2, z_4, \ldots, z_{2(m-1)}$ so that $|z_{2j}| = W_2$ and $Q^{(2j)}(z_{2j}) = 0$, $0 \leqq j < m$. By (2.10)

$$1 = Q(0) = \sum_{k=m}^{\infty} Q^{(2k)}(0) G_{2k}(0; z_0, 0, z_2, 0, \ldots, z_{2(m-1)}, 0, \ldots, 0).$$

By (2.1), (2.6) and (2.3),

$$\begin{aligned} &|G_{2k}(0; z_0, 0, z_2, 0, \ldots, z_{2(m-1)}, 0, \ldots, 0)| = \\ &= \left| -\sum_{j=0}^{m-1} \frac{z_{2j}^{2k-2j}}{(2k-2j)!} G_{2j}(0; z_0, 0, z_2, 0, \ldots, z_{2(j-1)}, 0) \right| \leqq \\ &\leqq \sum_{j=0}^{m-1} \frac{W_2^{2k-2j}}{(2k-2j)!} = \frac{W_2^{2k-2m}}{(2k-2m)!} \sum_{j=0}^{m-1} \frac{W_2^{2m-2j}}{(2k-2j)!} (2k-2m)! \leqq \\ &\leqq \frac{W_2^{2k-2m}}{(2k-2m)!} \sum_{j=0}^{m-1} \frac{W_2^{2m-2j}}{(2m-2j)!} < \frac{W_2^{2k-2m}}{(2k-2m)!} \cosh(W_2). \end{aligned}$$

Hence

$$1 \leqq \cosh(W_2) \sum_{k=m}^{\infty} |Q^{(2k)}(0)| \frac{W_2^{2k-2m}}{(2k-2m)!} = \cosh(W_2) \sum_{k=0}^{\infty} |Q^{(2m+2k)}(0)| \frac{W_2^{2k}}{(2k)!}.$$

It follows that there exists a number β, $0<\beta<1$, and a positive integer N_0 such that

$$\max\{|Q^{(2k)}(0)|, |Q^{(2k+2)}(0)|, \ldots, |Q^{(2k+2N_0)}(0)|\} \geqq \beta \tag{2.11}$$

for all k.

Now let $F(z) = 1 + \sum_{k=0}^{\infty} Q^{(2k)}(0) z^{2k}$. Since $\tau(Q) = 1$, then $c(F) = 1$. We will show that $\limsup\,(2n+1) r_{2n}(F) \leqq W_2$. Define

$$f_n(z) = \frac{F^{(2n)}\left(\frac{z}{2n+1}\right)}{(2n)!} = \sum_{k=0}^{\infty} Q^{(2n+2k)}(0) \frac{(2n+2k)!}{(2n)!\,(2k)!} \left(\frac{z}{2n+1}\right)^{2k} \qquad (n = 1, 2, 3, \ldots).$$

If $|z| = r < 2$, then $|Q^{(2j)}(0)| \leqq 1$ $(j = 0, 1, 2, \ldots)$ implies

$$|f_n(z) - Q^{(2n)}(z)| = \left| \sum_{k=0}^{\infty} \frac{Q^{(2n+2k)}}{(2k)!}(0) \left\{ \frac{(2n+2k)!}{(2n)!\,(2n+1)^{2k}} - 1 \right\} z^{2k} \right| \leqq$$

$$\leqq \sum_{k=0}^{\infty} \frac{r^{2k}}{(2k)!} \left\{ \frac{(2n+2k)!}{(2n)!\,(2n+1)^{2k}} - 1 \right\} \leqq \sum_{k=0}^{\infty} \frac{r^k}{k!} \left\{ \frac{(2n+k)!}{(2n)!\,(2n+1)}k - 1 \right\} =$$

$$= \sum_{k=0}^{\infty} \binom{k+2n}{2n} \left(\frac{r}{2n+1}\right)^k - \sum_{k=0}^{\infty} \frac{r^k}{k!} = \left(1 - \frac{r}{2n+1}\right)^{-2n-1} - e^r \qquad (n = 1, 2, 3, \ldots).$$

Therefore

$$f_n(z) = Q^{(2n)}(z) + o(1) \tag{2.12}$$

on compact subsets of $|z|<2$ (recall $W_2<2$).

Suppose now that $\limsup\,(2n+1) r_{2n}(F) > W_2$. There exists an $\varepsilon>0$ and a sequence of integers $\{n_m\}_{m=1}^{\infty}$ such that

$$(2n_m+1) r_{2n_m}(F) > W_2 + \varepsilon \qquad (m = 1, 2, 3, \ldots) \tag{2.13}$$

and such that $\{Q^{(2n_m)}(z)\}_{m=1}^{\infty}$ converges uniformly on compact sets to an entire function g. Since each of $Q^{(2n_m)}(z)$ has a zero on $|z| = W_2$, uniform convergence implies that g has a zero on $|z| = W_2$. The condition (2.11) implies that $g \not\equiv 0$. By (2.12), we know that $f_{n_m} \to g$ and therefore by Hurwitz's Theorem each of f_{n_m} has a zero in $|z| \leqq W_2 + \frac{\varepsilon}{2}$, for m sufficiently large. But since

$$f_{n_m}(z) = \frac{F^{(2n_m)}\left(\frac{z}{2n_m+1}\right)}{(2n_m)!}$$

then (2.13) implies that f_{n_m} does not vanish in $|z| \leqq W_2 + \varepsilon$. This contradiction show that $\limsup\,(2n+1) r_{2n}(F) \leqq W_2$, and this completes the proof.

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ON DIRECT PRODUCTS OF SOME BURNSIDE GROUPS

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Given an integer $m\geqq 1$ we denote by $\mathbf{G}_m$ the class of groups $\mathfrak{G}=(G;\cdot,1)$ satisfying the identity $x^m=1$ called Burnside groups. If $\mathbf{K}_1$ and $\mathbf{K}_2$ are two equational classes of the same type, we denote by $\mathbf{K}_1\times\mathbf{K}_2$ the class of all products $\mathfrak{A}\times\mathfrak{B}$ where $\mathfrak{A}\in\mathbf{K}_1$, $\mathfrak{B}\in\mathbf{K}_2$. We shall prove that if q_1 and q_2 are two relatively prime integers then $\mathbf{G}=\mathbf{G}_{q_1}\times\mathbf{G}_{q_2}$ is an equational class of groups described by two identities:

(1) $$x^{q_1\cdot q_2}=1$$

(2) $$(xy)^{n_1q_1}=x^{n_1\cdot q_1}\cdot y^{n_1q_1}$$

where n_1 is the smallest integer such that $n_1q_1\equiv 1 \pmod{q_2}$.

§ 1

LEMMA. *If* $\mathfrak{G}=(G;\cdot,1)$ *is a group satisfying* (1) *and* (2) *then the following equations hold identically in* $\mathfrak{G}$:

(3) $$(x\cdot y)^{n_2q_2}=x^{n_2q_2}\cdot y^{n_2q_2}$$

(*here* n_2 *is the smallest integer such that* $n_2\cdot q_2\equiv 1 \pmod{q_1}$)

(4) $$x^{n_1q_1}y^{n_2q_2}=y^{n_2q_2}\cdot x^{n_1q_1}$$

(5) $$(x\cdot y)^{n_1q_1}(uv)^{n_2q_2}=x^{n_1q_1}u^{n_2q_2}y^{n_1q_1}v^{n_2q_2}.$$

PROOF. Observe first that:

(6) $$n_1q_1+n_2q_2\equiv 1 \pmod{q_1\cdot q_2}$$

In fact $n_1q_1-1=k\cdot q_2$, $n_2q_2-1=l\cdot q_1$ for some k and l. Hence $(n_1q_1-1)(n_2q_2-1)=$ $=k\cdot l\cdot q_1q_2$ and $n_1q_1+n_2q_2-1=-k\cdot l\cdot q_1q_2+n_1n_2q_1q_2$ thus (6) holds. Hence

$$(yx)^{n_1q_1+n_2q_2-1}=y^{n_1q_1+n_2q_2-1}=1,$$

and by (2) we have:

$$y^{n_1q_1}x^{n_1q_1}(yx)^{n_2q_2-1}=y^{n_1q_1+n_2q_2-1}.$$

Multiplying the last identity on the left by $y^{-n_1q_1}$ we get

$$x^{n_1q_1}(yx)^{n_2q_2-1}=y^{n_2q_2-1}.$$

Hence

$$x^{n_1 q_1}(yx)^{n_2 q_2 - 1} = x^{n_1 q_1 + n_2 q_2 - 1} y^{n_2 q_2 - 1}.$$

Multiplying this identity on the left by $x^{-n_1 q_1}$ we get

$$(yx)^{n_2 q_2 - 1} = x^{n_2 q_2 - 1} y^{n_2 q_2 - 1}.$$

Multiplying the last identity on the left by x and on the right by y we get (3).

To prove (4) observe that, by (6) $(yx)^{n_1 q_1 + n_2 q_2} = yx$. By (2) and (3) we have

$$y^{n_1 q_1} x^{n_1 q_1} y^{n_2 q_2} x^{n_2 q_2} = yx.$$

Multiplying the last equality on the left by $y^{-n_1 q_1}$ and on the right by $x^{-n_2 q_2}$ we get

$$x^{n_1 q_1} y^{n_2 q_2} = y^{-n_1 q_1 + 1} x^{-n_2 q_2 + 1}.$$

Hence in view of (6) we get (4). (5) follows from (2), (3) and (4).

THEOREM. *If q_1 and q_2 are two relatively prime integers then $\mathbf{G} = \mathbf{G}_{q_1} \times \mathbf{G}_{q_2}$ is an equational class of groups described by two identities: $x^{q_1 q_2} = 1$ and $(xy)^{n_1 q_1} = x^{n_1 q_1} y^{n_1 q_1}$ where n_1 is the smallest integer such that $n_1 q_1 \equiv 1 \pmod{q_2}$.*

PROOF. Obviously, any product $\mathfrak{G}_1 \times \mathfrak{G}_2$ where $\mathfrak{G}_1 \in \mathbf{G}_{q_1}$, $\mathfrak{G}_2 \in \mathbf{G}_{q_2}$ satisfies (1) and (2). Thus $\mathbf{G}_{q_1} \times \mathbf{G}_{q_2} \subseteq \mathbf{G}$.

Let $\mathfrak{G} = (G; \cdot, 1) \in \mathbf{G}$. Put $x \circ y = x^{n_1 q_1} y^{n_2 q_2}$. By (6) we have

$$x \circ x = x. \tag{7}$$

Observe that $n_1 q_1 n_2 q_2 \equiv 0 \pmod{q_1 \cdot q_2}$. Further, because $n_1 q_1 \equiv 1 \pmod{q_2}$, we get $n_1 q_1 - 1 = r q_2$ for some r. Multiplying the last formula by $n_1 q_1$ we obtain $n_1^2 \cdot q_1^2 - n_1 \cdot q_1 = n_1 \cdot r \cdot q_1 \cdot q_2$ what means $n_1^2 q_1^2 \equiv m_1 \cdot q_1 \pmod{q_1 \cdot q_2}$. Analogously $n_2^2 \cdot q_2^2 \equiv n_2 q_2 \pmod{q_1 q_2}$. Thus using (2) and (3) we have

$$x \circ (y \circ z) = x^{n_1 q_1} \cdot (y^{n_1 q_1} \cdot z^{n_2 q_2})^{n_2 q_2} = x^{n_1 q_1} \cdot y^{n_1 q_1 n_2 q_2} \cdot z^{n_2^2 q_2^2} = x^{n_1 q_1} \cdot z^{n_2 q_2} = x \circ z$$

and

$$(x \circ y) \circ z = (x^{n_1 q_1} \cdot y^{n_2 q_2})^{n_1 q_1} z^{n_2 q_2} = x^{n_1^2 q_1^2} y^{n_1 q_1 n_2 q_2} z^{n_2 q_2} = x^{n_1 q_1} z^{n_2 q_2} = x \circ z.$$

Hence $x \circ y$ satisfies

$$x \circ (y \circ z) = (x \circ y) \circ z = x \circ z. \tag{8}$$

Define two relations R_i $(i = 1, 2)$ in $\mathfrak{G}$ by the formula

$$a_1 R_i a_2 \Leftrightarrow a_1 \circ a_2 = a_i \qquad (i = 1, 2).$$

Reflexivity and transitivity of R_i follows from (7) and (8). If $a_1 \circ a_2 = a_1$ then multiplying this equality on the left by a_2 and using (7) and (8) we get $a_2 \circ a_1 = a_2$. Hence, R_1 is symmetrical and analogously R_2 is symmetrical. By (5) R_i are congruences in $\mathfrak{G}$. Obviously, $R_1 \cap R_2 = \iota$, where ι is the identity. Further for any a_1 and $a_2 \in G$ we have $a_1 R_1 a_2 \circ a_1$ and $a_2 \circ a_1 R_2 a_2$. Thus the product $R_1 \cdot R_2 = G \times G$. Hence $\mathfrak{G}$ is isomorphic to $(\mathfrak{G}/R_1 \times \mathfrak{G}/R_2)$. Since for any $x \in G$ we have $1 R_1 x^{q_1}$ and $x^{q_2} R_2 1$ we get $\mathfrak{G}/R_1 \in \mathbf{G}_{q_1}$, $\mathfrak{G}/R_2 \in \mathbf{G}_{q_2}$, q.e.d.

REMARK. If we want $\mathbf{G}_{q_1}$ to be abelian we must modify the Theorem by adding to (1) and (2) the identity:

$$(xy)^{n_1 q_1}(yx)^{n_2 q_2} = x \cdot y. \tag{9}$$

In fact, if $\mathbf{G}_{q_1}$ is abelian then the identity (9) holds in $\mathbf{G}_{q_1} \times \mathbf{G}_{q_2}$. Conversely, if we have the identity (9) in $\mathfrak{G}$ then $x \cdot y R_1 y \cdot x$ and $\mathfrak{G}/R_1$ is abelian.

PROBLEM. Does $\mathbf{G}_{q_1} \times \mathbf{G}_{q_2}$ form an equational class even if q_1 and q_2 are not relatively prime?

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MAXIMAL ASYMMETRY OF GRAPHS*

By

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§ 1. Introduction and statement of main results

NOTATION. If G is a graph, $V(G)$=vertex set, G=edge set. $S_x=\{y: \{x, y\}\in G\}$, defined for $x\in V(G)$. $A\,\Delta\,B=(A\cup B)-(A\cap B)$=symmetric difference of A, B. $|X|=$ =number of elements in X. If σ is a permutation, $q(\sigma)=|\{i: \sigma(i)\neq i\}|$.

A graph G is called *symmetric* if there is a permutation $\sigma\neq 1$ on $V(G)$ so that $\sigma(G)=G$ (identifying σ with the induced permutation on edges). Define

$$B_\sigma(G) = |G\,\Delta\,\sigma(G)| \tag{1}$$

$$B(G) = \min_{\sigma\neq 1} B_\sigma(G) \tag{2}$$

$$B(n) = \max_{|V(G)|=n} B(G). \tag{3}$$

$B(G)$ is called the *asymmetry* of G. A slightly different notion of asymmetry is considered in [1]. See §4 for a comparison of the two notions.

In this paper we shall bound $B(n)$. We show that to determine $B(n)$ we need look „essentially" only at those $\sigma=(ij)$. Note that it is not true for any particular G that $B(G)$ may be approximated by looking only at $\sigma=(ij)$; there are G with „global" symmetries.

DEFINITION.

$$v(G) = \min_{\substack{x\neq y\\ x,y\in V(G)}} |S_x\,\Delta\,S_y-\{x, y\}| \tag{4}$$

$$v(n) = \max_{|V(G)|=n} v(G). \tag{5}$$

In [1] it is shown that $v(n)\leqq n/2$. If $\sigma=(xy)$ then

$$G\,\Delta\,\sigma(G)=\big\{\{i, z\}: i=x, y;\ z\in S_x\,\Delta\,S_y-\{x, y\}\big\}.$$

Thus $B(G)\leqq 2v(G)$, so $B(n)\leqq 2v(n)$. Our main result is

THEOREM 1. $2v(n)-O(1)\leqq B(n)\leqq 2v(n)$.

We shall, instead, prove

THEOREM 2. *If n is sufficiently large and $v(n)\geqq 0.49n$ then $2v(n)-400\leqq B(n)$.*

We shall show in §3 that $v(n)\geqq 0.49n$ for n sufficiently large. The „sufficiently large" is absorbed in the $O(1)$ of Theorem 1. Thus Theorem 2 shall imply Theorem 1.

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§ 2. Proof of Theorem 2

Fix G, $v(G)=v(n)\geqq 0.49n$. We shall see that $B_\sigma(G)\geqq 2v(n)$ if $q(\sigma)$ is small but we need worry about „global" σ. To this end, we shall alter G by a small random graph, throwing off the global near symmetries σ. Let $\mathscr{F}$ be the set of 1-factors on $V(G)$. (A 1-factor is a set of $\left[\frac{n-1}{2}\right]$ disjoint edges on $V(G)$, not necessarily in G.) Let $\mathbf{F}_1, \ldots, \mathbf{F}_{100}$ be independently randomly chosen from $\mathscr{F}$. Let $\mathbf{F}=\mathbf{F}_1\cup\ldots\cup\mathbf{F}_{100}$. Set $\mathbf{H}=G\,\Delta\,\mathbf{F}$. We shall show

(6) $$\operatorname{Prob}[B(\mathbf{H})<2v(n)-400]<1.$$

This will imply there exists a specific H, $B(H)\geqq 2v(n)-400$, proving Theorem 2. Define

(7) $$P_\sigma=\operatorname{Prob}[B_\sigma(\mathbf{H})<2v(n)-400].$$

Note

(8) $$B_\sigma(\mathbf{H})=\left|(G\,\Delta\,\sigma(G))\,\Delta\,(\mathbf{F}\,\Delta\,\sigma(\mathbf{F}))\right|.$$

Since $\operatorname{Prob}[B(\mathbf{H})<2v(n)-400]\leqq\sum_{\sigma\neq 1}P_\sigma$ it suffices to prove

(9) $$\sum_{\sigma\neq 1}P_\sigma<1$$

to imply (6) and thus our Theorems. We split the summation of (9) into cases depending on $q=q(\sigma)$.

Case I: $q=2$. This is the only case where the -400 of (7) is needed. We know $G\,\Delta\,\sigma(G)$ consists of at least $2v(n)$ edges of the form $\{x,z\}$ and $\{y,z\}$. For any value F of $\mathbf{F}$ there are at most 400 edges of that form in $F\,\Delta\,\sigma(F)$, so

$$B_\sigma(H)\geqq|(G\,\Delta\,\sigma(G))-(F\,\Delta\,\sigma(F))|\geqq 2v(n)-400$$

for *all* H; hence $P_\sigma=0$.

Case II: $3\leqq q\leqq 0.48n$. We shall again show $P_\sigma=0$. Let $A=\{i\colon\sigma(i)\neq i\}$. For $x\in A$

(10) $$\begin{aligned}|(S_x\,\Delta\,S_{\sigma(x)})\cap A^c|&=|S_x\,\Delta\,S_{\sigma(x)}|+|A^c|-|(S_x\,\Delta\,S_{\sigma(x)})\cup A^c|\geqq\\&\geqq v(n)-2+n-q-n\geqq v(n)-(q+2).\end{aligned}$$

If $x\in A$ and $z\in(S_x\,\Delta\,S_{\sigma(x)})\cap A^c$ then $\sigma(\{x,z\})=\{\sigma(x),z\}$ so $\{\sigma(x),z\}\in G\,\Delta\,\sigma(G)$. For each $x\in A$ there are at least $v(n)-(q+2)$ such edges so $|G\,\Delta\,\sigma(G)|\geqq q[v(n)-(q+2)]$, all such edges containing at least one point from A. For any value F of $\mathbf{F}$ there are at most $200q$ edges in $F\,\Delta\,\sigma(F)$ containing a point of A so

$$B\sigma(H)\geqq|(G\,\Delta\,\sigma(G))-(F\,\Delta\,\sigma(F))|\geqq q[v(n)-(q+2)-200]\geqq 2v(n)$$

for all H; hence $P_\sigma=0$.

Case III: $0.48n<q$. Here we may not show $P_\sigma=0$, as G may have „global” symmetries, we show instead that

$$(11)\qquad P_\sigma < n!^{-1}.$$

$$(12)\qquad P_\sigma = \text{Prob}\,[B_\sigma(\mathbf{H}) < 2v(n)-400] = \sum_S \text{Prob}\,[\mathbf{F}\,\Delta\,\sigma(\mathbf{F}) = S]$$

where the summation runs over all S,

$$(13)\qquad |G\,\Delta\,\sigma(G)\,\Delta\,S| < 2v(n)-400$$

(using (8)). There are

$$\sum_{i=0}^{2v(n)-400} \binom{\binom{n}{2}}{i} \leqq n^{2n}$$

such S (using (6)). The above approximation, and the ones we shall use later, may appear exceedingly coarse, but they shall suffice for our purposes. We have

$$(14)\qquad P_\sigma \leqq n^{2n} \max_S \text{Prob}\,[\mathbf{F}\,\Delta\,\sigma(\mathbf{F}) = S].$$

We now fix S and give an upper bound to Prob $[\mathbf{F}\,\Delta\,\sigma(\mathbf{F})=S]$. We are guided by the following intuitive argument. Think of $\mathbf{F}$ as a random set of $50n$ edges and $\mathbf{F}\,\Delta\,\sigma(\mathbf{F})$ as a random set of $100n$ edges. Here it is critical that $q(\sigma)$ be large as otherwise $\mathbf{F}$ and $\sigma(\mathbf{F})$ would „cancel”. If all possible sets of $100n$ edges were equally possible as $\mathbf{F}\,\Delta\,\sigma(\mathbf{F})$ then for any particular S, $|S|=100n$,

$$\text{Prob}\,[\mathbf{F}\,\Delta\,\sigma(\mathbf{F}) = S] = \binom{\binom{n}{2}}{100n}^{-1} = n^{-(200+o(1))n}.$$

It would suffice to show Prob$[\mathbf{F}\,\Delta\,\sigma(\mathbf{F})=S]\leqq n^{-cn}$ where $c>3$ so there is plenty of „room” when we give our (unfortunately lengthy) precise argument. We note

$$(15)\qquad \text{Prob}\,[\mathbf{F}\,\Delta\,\sigma(\mathbf{F}) = S] = \frac{|\{(F_1, \dots, F_{100}) : F\,\Delta\,\sigma(F) = S\}|}{|\{(F_1, \dots, F_{100})\}|}$$

where $F_1, \dots, F_{100} \in \mathscr{F}$, $F = F_1 \cup \dots \cup F_{100}$. It is easy to see that

$$|\mathscr{F}| = (n-1)(n-3)(n-5)\dots = n^{n(\frac{1}{2}+o(1))}.$$

(It shall turn out that $n^{o(n)}$ factors can be ignored so that exceedingly coarse estimates will suffice.) Thus

$$(16)\qquad |\{(F_1, \dots, F_{100})\}| = n^{n(50+o(1))}.$$

We assume $|S|\leqq 100n$, as otherwise $F\,\Delta\,\sigma(F)$ could not equal S. We set $N_S=|\{(F_1, \dots, \dots, F_{100}: F\,\Delta\,\sigma(F)=S\}|$ for convenience. Set $W_x=\{y\colon \{x, y\}\in S\}$. Then $\sum |W_x|= =2|S|\leqq 200n$. Thus $|\{x\colon |W_x|\geqq 10^4\}|\leqq 2\cdot 10^{-2}n$. Let $A_1=A\cap\{x\colon |W_x|\leqq 10^4\}$. Then $|A_1|\geqq q-2\cdot 10^{-2}n\geqq 0.46n$. (Here we have deleted the potentially „bad” points of A.) Set $A_2=A_1-\sigma^{-1}[A-A_1]$. Then

$$A_2 \subseteqq A_1, \quad \sigma(A_2) \subseteqq \sigma(A_1)-[A-A_1] \subseteqq A_1, \quad |A_2|\geqq|A_1|-|A-A_1| \geqq 0.44n.$$

Define an undirected graph σ^* on A_2 by the edges $\{x, \sigma(x)\}$, $x, \sigma(x) \in A_2$. Each point of A_2 will have degree at most 2 so σ^* decomposes into isolated points, paths, and circuits. We find an independent set $A_3 \subseteqq A_2$, $|A_3| \geqq \frac{1}{3} |A_2|$. (The worst case being if σ^* is the union of triangles.) So A_3 has the properties

$$(17) \quad |A_3| \geqq \frac{1}{3} |A_2| \geqq 0.14n, \quad A_3 \cap \sigma(A_3) = \emptyset, \quad A_3 \cup \sigma(A_3) \subseteqq A_2 \cup \sigma(A_2) \subseteqq A_1.$$

Set

$$(18) \quad A_3 = \{c_1, \dots, c_k\}, \quad k \geqq 0.14n, \qquad \sigma(A_3) = \{a_1, \dots, a_k\}, \quad a_i = \sigma(c_i).$$

We assume, at this point, that n is even. This is only a convenience, the case n odd being left to the reader. We wish to bound N_S from above. Assume $F \Delta \sigma(F) = S$. Let $f_j(x)$ be that y such that $\{x, y\} \in F_j$. For $1 \leqq i \leqq k$, $1 \leqq j \leqq 100$ either $f_j(a_i) \in W_{a_i}$ or $\{a_i, f_j(a_i)\} \in F \cap S^c$ so must be in $\sigma(F)$, so $\sigma^{-1}(\{a_i, f_j(a_i)\}) = \{c_i, \sigma^{-1}(f_j(a_i))\} \in F_t$ for some $1 \leqq t \leqq 100$. Intuitively, in the first case a factor of $n/|W_{a_i}|$ is lost in determining $f_j(a_i)$ and in the second case one of the $f_t(c_i)$ is determined, again losing a factor of n in the number of $(F_1, \dots, F_{100})$. As this occurs $100k \geqq 14n$ times, a factor of n^{14n} will be lost.

To return to rigour, consider all matrices $X = [x_{ij}]$, $1 \leqq i \leqq k$, $1 \leqq j \leqq 100$ where either

$$x_{ij} \in W_{a_i} \cup \{c_i\} \overset{\text{def}}{=} V_i$$

or

$$(19) \qquad 1 \leqq x_{ij} \leqq 100.$$

(The $c_i \in V_i$ is a technicality.) Let $\mathscr{A}_X$ be the set of $(F_1, \dots, F_{100})$ so that for $1 \leqq i \leqq k$, $1 \leqq j \leqq 100$

(a) If $x_{ij} \in V_i$, $f_j(a_i) = x_{ij}$

(b) If $1 \leqq x_{ij} \leqq 100$, $\{c_i, \sigma^{-1}(f_j(a_i))\} \in F_{x_{ij}}$

and also (again the technicality) $f_j(a_i) \neq c_i$.

By our previous remarks, $F \Delta \sigma(F) = S$ implies $(F_1, \dots, F_{100}) \in \mathscr{A}_X$ for some X. Letting $\alpha_X = |\mathscr{A}_X|$

$$(20) \qquad N_s \leqq \sum_X \alpha_X.$$

There are at most $10^4 + 1 + 10^2$ choices for each x_{ij} so at most $(10101)^{100k} \leqq n^{o(n)}$ choices of X. Thus

$$(21) \qquad N_S \leqq n^{o(n)} \alpha_X$$

for some X, which we fix. We now bound α_X. (Note how the seemingly enormous number of X evaporates as we are dealing with functions of an even higher order of magnitude.)

All $(F_1, \dots, F_{100}) \in \mathscr{A}_X$ may be determined by the following procedure. For $x_{ij} \in V(G)$, place $\{a_i, x_{ij}\} \in F_j$. Put the ordered pairs (a, j), $a \in V(G)$, $1 \leqq j \leqq 100$ in some order. Now run through this list deciding $f_j(a)$ whenever it has not already been decided. If $1 \leqq x_{ij} \leqq 100$ then whenever $f_j(a_i)$ is determined as, say, y this determines $f_{x_{ij}}(c_i)$ as $\sigma^{-1}(y)$. Conversely $f_{x_{ij}}(c_i)=z$ forces $f_j(a_i)=\sigma(z)$. (If we were counting all $(F_1, \ldots, F_{100})$ the number of decision would equal the number of edges $= 50n$, each decision may be made in $\leqq n$ ways, so the number of $(F_1, \ldots, F_{100})$ would be $\leqq n^{50n}$.) Look at the s edges involving $a_1, \ldots, a_k$ in all F_j and c_i in $F_{x_{ij}}$ where $1 \leqq x_{ij} \leqq 100$. The value of s depends on the decisions on the edges (e.g.: some a's could be joined to other a's) but $s \geqq \frac{k}{2}(100) \geqq 7n$ in any case. At most $s/2$ decisions are made in determining these edges since if $x_{ij} \in V(G)$, $f_j(a_i)$ is determined and the other edges are paired. (The technicality $f_j(a_i) \neq c_i$ is required as *if* $x_{ij}=j$ and $\sigma(a_i)=c_i$, $\sigma(c_i)=a_i$ then if $f_j(a_i)$ is determined as c_i, placing $\{a_i, c_i\} \in F_j$ the edge forced would be $\{c_i, \sigma^{-1}(c_i)\} = \{c_i, a_i\} \in F_{x_{ij}} = F_j$, the same edge. However, aside from this technicality, each edge $\{a_i, y\} \in F_j$ is paired with a different $\{c_i, z\} \in F_t$.) Now the remaining $50n-s$ edges require at most $50n-s$ decisions, yielding at most $50n-s/2 \leqq 46.5n$ decisions. Since each decision is made in $\leqq n$ ways

$$\alpha_X \leqq n^{46.5n}. \tag{22}$$

So by (21)

$$N_S \leqq n^{(46.5+o(1))n} \tag{23}$$

so that, using (15),

$$\text{Prob}\,[\mathbf{F} \,\Delta\, \sigma(\mathbf{F}) = S] \leqq \frac{n^{(46.5+o(1))n}}{n^{(50+o(1))n}} = n^{-(3.5+o(1))n} \tag{24}$$

so, by (14),

$$P_\sigma \leqq n^{-(1.5+o(1))n} \tag{25}$$

implying (11) and thus completing Case III.

We now note

$$\sum_{\sigma \neq 1} P_\sigma = \sum_{\mathrm{I}} P_\sigma + \sum_{\mathrm{II}} P_\sigma + \sum_{\mathrm{III}} P_\sigma \leqq 0+0+n!\, n^{-(1.5+o(1))n} < 1 \tag{26}$$

yielding (9) and thus our Theorems 1 and 2.

§ 3. The function $v(n)$

Our first object is to show that $v(n) \geqq 0.49n$ for n sufficiently large. We show the following stronger result.

THEOREM 3. $\frac{n}{2} - (1+o(1))\sqrt{n \log n} \leqq v(n) \leqq \left[\frac{n-1}{2}\right]$.

PROOF. The upper bound is given in [1]. Let $\mathbf{G}$ be a random graph on n points where $\text{Prob}\,[\{i,j\} \in \mathbf{G}] = \frac{1}{2}$ and these probabilities are independent. Set $\mathbf{T}_{ij} = |S_i \,\Delta\, S_j - \{i,j\}|$ in $\mathbf{G}$. Then $\mathbf{T}_{ij}$ has binomial distribution $B(n-2, \frac{1}{2})$ so, by using the appropriate normal approximation

$$\text{Prob}\,[\mathbf{T}_{ij} \leqq c] < n^{-2} \tag{27}$$

if $c=\frac{n}{2}-(1+\varepsilon)\sqrt{n\log n}$

$$\text{(28)}\qquad \text{Prob}\,[v(\mathbf{G})\leqq c]\leqq\sum_{i,j}\text{Prob}\,[\mathbf{T}_{ij}\leqq c]\leqq\binom{n}{2}n^{-2}<1$$

so there exists G, $v(G)\leqq c$, proving Theorem 3.

It shall be convenience to consider a function

$$\text{(29)}\qquad w(n)=\frac{n-1}{2}-v(n)$$

and attempt to bound $w(n)$ from above. In [1], G is called a Δ-graph if $v(G)=\left[\frac{n-1}{2}\right]$. It is shown that such G exists for $n\equiv 1\,(\text{mod }4)$, $n=$ prime power.

THEOREM 4. *If* $s<n$, $v(n-s)\geqq v(n)-s$.

PROOF. Let G be a graph, $|V(G)|=n$, $v(G)=v(n)$. Delete any s points from G yielding H. Then $v(H)\geqq v(G)-s$ so $v(n-s)\geqq v(H)\geqq v(G)-s$.

In terms of w

$$\text{(30)}\qquad w(n-s)\leqq w(n)+s+\frac{n-s-1}{2}-\frac{n-1}{2}\leqq w(n)+\frac{s}{2}$$

so

$$\text{(31)}\qquad w(m)\leqq\alpha(m)/2$$

where $\alpha(m)=\min p-m$: $p\geqq m$, p prime power, $p\equiv 1\,(\text{mod }4)$.

CONJECTURE. $w(m)=O(1)$.

§ 4. Another notion of asymmetry

ERDŐS and RÉNYI [1] defined the asymmetry $A(G)$ of a graph G to be the minimal number of edges that need be added and/or deleted to make the graph symmetric. Formally

$$\text{(32)}\qquad A(G)=\min|D|:\ \exists\sigma\neq 1\quad \sigma(G\,\Delta\,D)=G\,\Delta\,D$$

Since $\sigma(G\,\Delta\,D)=\sigma(G)\,\Delta\,\sigma(D)$, if (32) holds, $G\,\Delta\,\sigma(G)=D\,\Delta\,\sigma(D)$ so $|D|\geqq\frac{1}{2}|G\,\Delta\,\sigma(G)|$. Hence

$$\text{(33)}\qquad A(G)\geqq\frac{1}{2}B(G).$$

Set

$$\text{(34)}\qquad A(n)=\min_{|V(G)|=n}A(G).$$

Then

$$\text{(35)}\qquad A(n)\geqq\frac{1}{2}B(n)\geqq\frac{1}{2}[2v(n)-O(1)]\geqq\frac{n}{2}-\big(1+o(1)\big)\sqrt{n\log n}$$

as was shown in [1]. In [1] it is shown that

$$A(n) \leqq \left[\frac{n-1}{2}\right]. \tag{36}$$

A proof of the conjecture in §3 would give $A(n)$ within a constant additive factor.

Reference

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ON ASSOCIATIVE CURVES. I

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1. Introduction. With this paper we are initiating a new series of papers dealing with study of some new curves on the surface of reference S: $x^i = x^i(u^\alpha)$ $(i=1, 2, 3;$ $\alpha=1, 2)$ of a rectilinear congruence, a line of which through a point $P(x^i)$ is represented by a vector λ^i which may be expressed as

$$\lambda^i = p^\alpha x^i_{,\alpha} + qX^i, \tag{1.1}$$

where p^α and q are parameters, X^i the components of unit normal to S at P and $x^i_{,\alpha}$ $(\equiv \partial x^i/\partial u^\alpha)$ may be considered to be the covariant derivative of x^i with respect to u^α based on the first fundamental tensor $g_{\alpha\beta}(\equiv x^i_{,\alpha} \cdot x^i_{,\beta})$ of S.

Let C: $x^i = x^i(s)$ be a curve in S through P. Let V^i be the components of a unit vector depending linearly on λ^i and dx^i/ds and orthogonal to dx^i/ds. Then, if $\lambda^i \cdot dx^i/ds = \cos\theta$, we have

$$V^i = e \operatorname{cosec}\theta(\lambda^i - \cos\theta\, dx^i/ds), \tag{1.2}$$

where e is $+1$ or -1 according as $\theta(\neq 0)$ is acute or obtuse.

Throughout this paper we assume the following:

(1) Latin and Greek indices take values 1, 2, 3 and 1, 2 respectively and repetition of indices indicates the sum of terms over the range of repeated indices.

(2) The curves under consideration are not straight lines, i.e., they can not be geodesics and asymptotic lines simultaneously.

(3) All the functions are differentiable up to the required order.

(4) We follow the usual notations of EISENHART [1] except that we use $e_{\alpha\beta}$, instead of $\varepsilon_{\alpha\beta}$ for $[x^i_{,\alpha}\, x^i_{,\beta}\, X^i]$ and $x^i_{,\alpha} \times x^i_{,\beta} \cdot X^i$ instead of $\varepsilon^{123}_{ijk} x^i_{,\alpha} x^j_{,\beta} X^k$.

The following equations will find ample use in the text hence being given:

$$x^i_{,\alpha\beta} = d_{\alpha\beta} X^i, \tag{1.3}$$

$d_{\alpha\beta}$ being the second fundamental tensor of S,

$$X^i_{,\alpha} = -d_{\alpha\beta}\, g^{\beta\gamma} x^i_{,\gamma}; \tag{1.4}$$

$$d^2x^i/ds^2 = \varrho^\alpha x^i_{,\alpha} + k_n X^i; \tag{1.5}$$

$k_n = d_{\alpha\beta} u'^\alpha u'^\beta$ is the normal curvature of S along C, dashes having been used for differentiation, with respect to s; and

$$d\lambda^i/ds = (\mu^\alpha_\beta x^i_{,\alpha} + \nu_\beta X^i) u'^\beta \tag{1.6}$$

where

(1.7)a $$\mu^{\alpha}_{\beta} \equiv p^{\alpha}_{,\beta} - q d_{\beta\gamma} g^{\gamma\alpha}$$

and

(1.7)b $$\nu_{\beta} \equiv p^{\alpha} d_{\alpha\beta} + q_{,\beta}.$$

2. Definitions. DEFINITION 2.1. *(TA)-vector.* Vector with contravariant components V^i as given by (1.2) will be called "tangent associate vector" or TA-vector.

DEFINITION 2.2. *(TA)-plane.* The plane formed by TA-vectors at some point P of the curve C and at a consecutive point will be known as TA-plane.

DEFINITION 2.3. *(TTA)-curve.* We shall call the curve C, (TTA)-curve if its tangent at P lies in the (TA)-plane.

DEFINITION 2.4. *(P_nTA)-curve.* The curve C will be called (P_nTA)-curve if at any point of the curve its principal normal lies in the (TA)-plane at that point.

DEFINITION 2.5. *(B_nTA)-curve.* If the binormal to C lies in the (TA)-plane, C will be known as (B_nTA)-curve.

3. (TTA)-curve. By Definition 2.3 for this curve dx^i/ds, V^i and $V^i + dV^i/ds$ are coplanar. So we have

$$[dx^i/ds \quad V^i \quad dV^i/ds] = 0,$$

which on solving with the use of (1.1), (1.2), (1.5) and (1.6) takes the form

(3.1) $$e_{\alpha\beta} u'^{\alpha} \{p^{\beta}(\nu_{\delta} u'^{\delta} - k_n \cos\theta) + q(\varrho^{\beta}\cos\theta - \mu^{\beta}_{\delta} u'^{\delta})\} = 0.$$

This is a differential equation of second degree. Hence there are ∞^2 such curves on S.

Denoting the term in curly bracket by T^{β} we can write (3.1) as

$$e_{\alpha\beta} u'^{\alpha} T^{\beta} = 0.$$

We shall call the vector with contravariant components T^{β}, T-curvature vector of C and its magnitude, say k_T will be called T-curvature of C. Thus:

THEOREM (3.1). *A curve C is a (TTA)-curve if and only if its T-curvature vanishes, i.e., T-curvature vector is a null vector at each point of C.*

Since dx^i/ds is perpendicular to V^i and V^i is orthogonal to dV^i/ds, we have

(3.2) $$dx^i/ds = e/L \cdot dV^i/ds,$$

where L is the magnitude of dV^i/ds and e is 1 or -1 according as dx^i/ds and dV^i/ds are in the same or opposite directions.

Multiplying (3.2) by dx^i/ds we get

(3.3) $$L = \operatorname{cosec}\theta(\mu_{\alpha\beta} u'^{\alpha} u'^{\beta} + \theta' \cdot \sin\theta),$$

where $\mu_{\alpha\beta} \equiv \mu^{\gamma}_{\beta} g_{\alpha\gamma}$. Again multiplying (3.2) by d^2x^i/ds^2 and solving with the help of (1.1), (1.2), (1.5) and (1.6) we get

(3.4) $$k^2 = \sec\theta\,[(\mu_{\alpha\beta}\varrho^{\alpha} + \nu_{\beta} k_n) u'^{\beta} - \theta' \cot\theta \cdot k_C],$$

where k_C is the curvature of the congruence section of S in the direction of C as defined by MISHRA [2]. However, for brevity, we shall call it hyperasymptotic curvature, because its vanishing means that the curve C is a hyperasymptotic curve. Equation (3.4) gives the value of the curvature of (TTA)-curves.

For a normal congruence, $p^\alpha=0$, $q=1$ and $\theta=\pi/2$, therefore in view of (1.7), equation (3.4) reduces to $d_{\alpha\beta}\varrho^\alpha u'^\beta=0$, which implies:

THEOREM (3.2). *For a normal congruence, the (TTA)-curve is such that its first curvature vector is conjugate to the tangent vector.*

REMARK. One may easily obtain some results by multiplying equation (3.2) by $dx^i/ds\times d^2x^i/ds^2$, X^i and λ^i etc.

4. ($\mathbf{P}_n$TA)-curve. In view of Definition (2.4), we have

$$[d^2x^i/ds^2 \quad V^i \quad dV^i/ds] = 0,$$

which when solved with the help of (1.1), (1.2), (1.5) and (1.6) takes the form

$$\begin{aligned}(4.1)\qquad & e_{\alpha\beta}[\varrho^\alpha\{v_\delta u'^\delta(p^\beta-\cos\theta\cdot u'^\beta)-q(\mu^\beta_\delta u'^\delta+\theta'\sin\theta u'^\beta)\}+\\ &+k_n(p^\alpha\mu^\beta_\delta u'^\delta+\theta'\sin\theta p^\alpha u'^\beta+\cos\theta\mu^\alpha_\delta u'^\beta u'^\delta)]=0.\end{aligned}$$

Equation (4.1) is the differential equation of (P_nTA)-curves.

For a normal congruence by virtue of equation (1.7), the equation (4.1) reduces to $e_{\alpha\beta}\varrho^\alpha d_{\delta\gamma}g^{\gamma\beta}u'^\delta=0$, which becomes an identity when either $\varrho^\alpha=0$, i.e., C is a geodesic or $d_{\delta\gamma}=0$, i.e., S is totally geodesic. Thus:

THEOREM (4.1). *For a normal congruence,* (P_nTA)*-curves on S are indeterminate when either they are geodesics or S is a totally geodesic surface.*

Next, let the congruence be formed of tangents to a one parameter family of curves, then $q=0$ and in view of (1.7), the equation (4.1) reduces to

$$\begin{aligned}(4.2)\qquad & \{e_{\alpha\beta}\varrho^\alpha(p^\beta-u'^\beta\cos\theta)\}d_{\gamma\delta}p^\gamma u'^\delta+k_n\{e_{\alpha\beta}p^\alpha p^\beta_{,\delta}u'^\delta+\\ &+\theta'\sin\theta e_{\alpha\beta}p^\alpha u'^\beta+\cos\theta\, e_{\alpha\beta}p^\alpha_{,\delta}u'^\delta u'^\beta\}=0,\end{aligned}$$

If, in addition, we assume that C is an asymptotic line, (4.2) becomes

$$e_{\alpha\beta}\varrho^\alpha(p^\beta-u'^\beta\cos\theta)d_{\gamma\delta}p^\gamma u'^\delta=0,$$

which leads to

THEOREM (4.2). *If the congruence is formed of tangents to a one parameter family of curves in S and if (P_nTA)-curve C is an asymptotic line then either of the following holds:*

(ii) *the (TA)-vector is parallel to the first curvature vector of C,*

(i) *the lines of congruence are conjugate to the tangents to C at the points of intersection of the curve C and the lines.*

REMARK. We have not considered the case of the curves, being simultaneously asymptotic lines and geodesics in conformity to the second assumption of Section 1.

If C is geodesic, (4.2) reduces to

$$e_{\alpha\beta}\, p^{\alpha} p^{\beta}_{,\delta} u'^{\delta} + \theta' \sin\theta e_{\alpha\beta}\, p^{\alpha} u'^{\beta} + \cos\theta e_{\alpha\beta}\, p^{\alpha}_{,\delta} u'^{\delta} u'^{\beta} = 0,$$

which implies:

THEOREM (4.3). *For the congruence being formed of tangents to a one parameter family F of curves and the (P_nTA)-curve C being geodesic, if any two of the following statements are true, the third is also true:*

(i) *the congruence is geodesic relative to C,*

(ii) *either C is being met by the lines of congruence at a constant angle or C is a member of F,*

(iii) *either C is being intersected orthogonally by the lines of congruence or the curvature vector of the congruence relative to C is parallel to the tangent to C.*

Since for C to be (P_nTA)-curve, d^2x^i/ds^2, V^i and dV^i/ds are coplanar, therefore we have

$$d^2x^i/ds^2 = aV^i + b\,dV^i/ds,$$

where a and b are to be determined. Taking help of (1.1), (1.2) and (1.5) we can write the last relation as

$$d^2x^i/ds^2 = e \operatorname{cosec}\theta \left[k_C V^i + \frac{1}{L}\sqrt{k^2\sin^2\theta - k_C^2}\, dV^i/ds\right]. \tag{4.3}$$

Multiplying (4.3) by dx^i/ds and solving with the help of (1.2) and (1.6) we get

$$\sqrt{k^2\sin^2\theta - k_C^2}\,(\theta'\sin\theta + \mu_{\alpha\beta} u'^{\alpha} u'^{\beta}) = 0,$$

which yields either

$$k_C/k = \sin\theta, \tag{4.4)a}$$

or

$$\cos\theta = \int (\mu_{\alpha\beta} u'^{\alpha} u'^{\beta})\, ds + \text{constant of integration}.$$

This constant of integration is determined by the fact that when the congruence is normal $\theta = \pi/2$, $\mu_{\alpha\beta} = -d_{\alpha\beta}$. Thus the last equation becomes

$$\cos\theta = \int (\mu_{\alpha\beta} - d_{\alpha\beta}) u'^{\alpha} u'^{\beta}\, ds. \tag{4.4)b}$$

From equations (4.4) we obtain:

THEOREM (4.4). *For a (P_nTA)-curve C either of the following holds:*

(i) *the ratio of its hyperasymptotic curvature and curvature is numerically equal to the sine of the angle between tangent to C at the point of intersection and the intersecting line of the congruence,*

(ii) $$\cos\theta = \int (\mu_{\alpha\beta} - d_{\alpha\beta}) u'^{\alpha} u'^{\beta}\, ds.$$

For a normal congruence (4.4)a yields $k_C = k_n = k$, i.e., $k_g = 0$ and (4.4)b reduces to $k_n = 0$. Hence we have:

THEOREM (4.5). *For a normal congruence, (P_nTA)-curves are either geodesics or asymptotic lines.*

From Theorems (4.1) and (4.5) we have:

THEOREM (4.6). *For a normal congruence the only determinate (P_nTA)-curves are asymptotic lines.*

REMARK. Similar results can be obtained when the congruences are formed of tangents to a one parameter family of curves.
Multiplication of (4.3) by λ^i gives

$$k_C\sqrt{k^2\sin^2\theta-k_C^2}\cot\theta=0,$$

which yields:

THEOREM (4.7). *A (P_nTA)-curve must satisfy one of the following:*
(i) *the ratio of the hyperasymptotic curvature and the curvature of C is numerically equal to the angle at which the lines of congruence cut the curve,*
(ii) *the lines of congruence intersect the curve orthogonally,*
(iii) *the curve is a hyperasymptotic curve.*

Taking product of (4.3) by X^i we obtain

$$k_n=\frac{\{qk_C+(v_\alpha u'^\alpha-\theta' q\cot\theta)\sqrt{k^2\sin^2\theta-k_C^2}\}\operatorname{cosec}^2\theta}{\left(1+\dfrac{\operatorname{cosec}^2\theta}{L}\cot\theta\sqrt{k^2\sin^2\theta-k_C^2}\right)},$$

which gives an expression for the normal curvature of S in the direction of C.

REMARK. Multiplication of (4.3) by $dx^i/ds\times d^2x^i/ds^2$, $\varrho^\beta x^i_{,\beta}$ etc. will give some other results.

5. (B_nTA)-curve. From Definition (2.5) of Section 2, we have

$$[dx^i/ds\times d^2x^i/ds^2 \quad V^i \quad dV^i/ds]=0,$$

i.e.,

$$(dV^i/ds\cdot dx^i/ds)(V^i\cdot d^2x^i/ds^2)=0,$$

which on solving with the use of (1.1), (1.2), (1.5) and (1.6) becomes

$$k_C(\mu_{\alpha\beta}u'^\alpha u'^\beta+\theta'\sin\theta)=0. \tag{5.1}$$

From (5.1) we observe:

THEOREM (5.1). *(B_nTA)-curves are either hyperasymptotic curves or they are given by the equation $\mu_{\alpha\beta}u'^\alpha u'^\beta+\theta'\sin\theta=0$.*

If the congruence is normal, (5.1) reduces to $k_n=0$. Thus:

THEOREM (5.2). *For a normal congruence (B_nTA)-curves are asymptotic lines.*

For the congruence formed of tangents to a one parameter family of curves in S, (5.1) reduces to

$$(p_\alpha\varrho^\alpha)(p_{\beta,\gamma}u'^\beta u'^\gamma+\theta'\sin\theta)=0,$$

which implies either

(5.2)a $$p_\alpha \varrho^\alpha = 0$$

or

(5.2)b $$\cos\theta = \int (p_{\alpha,\beta} u'^\alpha u'^\beta)\,ds + \text{constant}.$$

When $\theta = 0$, the term within integration sign vanishes, as it then becomes tendency of the tangent to the curve in its own direction. Hence constant becomes equal to unity and (5.2)b becomes

(5.2)c $$\cos\theta = 1 + \int (p_{\alpha,\beta} u'^\alpha u'^\beta)\,ds.$$

Thus from (5.2)a, c we obtain:

THEOREM (5.3). *For a congruence formed of tangents to a one parameter family of curves, $(B_n TA)$-curve satisfies either of the following:*

(i) *the lines of congruence are orthogonal to the first curvature vector of C,*
(ii) *θ, the angle between the line of congruence and tangent to C is given by*

$$\cos\theta = 1 + \int (p_{\alpha,\beta} u'^\alpha u'^\beta)\,ds.$$

Since $dx^i/ds \times d^2x^i/ds^2$, V^i and dV^i/ds lie in the same plane therefore

(5.3) $$dx^i/ds \times d^2x^i/ds^2 = AV^i + B\,dV^i/ds,$$

where A and B are to be determined.

Multiplication of (5.3) by V^i, on solving in view of (1.1), (1.2) and (1.5) yields:

(5.4) $$A = e \operatorname{cosec}\theta \cdot q k_u,$$

where k_u is the union curvature [3].

Squaring (5.3) and making use of (5.4) we get

(5.5) $$B = \frac{e \operatorname{cosec}\theta}{L} \sqrt{k^2 \sin^2\theta - q^2 k_u^2}.$$

Use of (5.4) and (5.5) in (5.3) give

(5.6) $$dx^i/ds \times d^2x^i/ds^2 = e \operatorname{cosec}\theta \left[q k_u V^i + \frac{1}{L} \sqrt{k^2 \sin^2\theta - q^2 k_u^2}\, dV^i/ds \right].$$

Multiplying equation (5.6) by dx^i/ds we obtain

$$\sqrt{k^2 \sin^2\theta - q^2 k_u^2}\,(\theta' \sin\theta + \mu_{\alpha\beta} u'^\alpha u'^\beta) = 0,$$

which gives either

(5.7)a $$k = q k_u \operatorname{cosec}\theta,$$

or

(5.7)b $$\cos\theta = \int (\mu_{\alpha\beta} - d_{\alpha\beta}) u'^\alpha u'^\beta\, ds.$$

Thus we have:

THEOREM (5.4). *(B_nTA)-curves satisfy either of the equations* (5.7).

Multiplying (5.6) by d^2x^i/ds^2 and solving with the help of (1.1), (1.2) and (1.6) we get

(5.8) $$\{\mu_{\alpha\beta}\varrho^\alpha u'^\beta + k_n \gamma_\delta u'^\delta - k^2\cos^2\theta - \theta' k_C \cot\theta\}\sqrt{k^2\sin^2\theta - q^2 k_u^2}/L + qk_u k_C = 0,$$

which is a relation between curvature, hyperasymptotic curvature and union curvature of C.

REMARK. A number of results can be obtained by multiplying equation (5.6) by λ^i, X^i etc. and considering the congruences to be formed of tangents to a one parameter family of curves.

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ON ASSOCIATIVE CURVES. II

By

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1. Preliminaries. The unit vector $\hat{\lambda}$ in the direction of a line of a rectilinear congruence may be expressed as

$$\hat{\lambda} = p^\alpha x_\alpha + q\hat{X} \tag{1.1}$$

where $x_\alpha = \partial \mathbf{x}/\partial u^\alpha$ and other terms have their usual meaning.

Differentiating (1.1) with respect to s and making use of Gauss and Weingarten equations

(1.2) a $\quad x_{\alpha\beta} = d_{\alpha\beta}\hat{X}\quad$ and $\quad$ (1.2) b $\quad \hat{X}_\alpha = -d_{\alpha\beta}\, g^{\beta\gamma} x_\gamma,$

where $g_{\alpha\beta}\,(=x_\alpha \cdot x_\beta)$ and $d_{\alpha\beta}$ are first and second fundamental tensors of S respectively, we obtain

$$\hat{\lambda}' = (\mu^\alpha_\beta x_\alpha + \nu_\beta \hat{X})\, u'^\beta, \tag{1.3}$$

where dashes indicate differentiation with respect to s and

(1.4) a $$\mu^\alpha_\beta \equiv p^\alpha_{,\beta} - q d_{\beta\gamma}\, g^{\gamma\alpha}$$

and

(1.4) b $$\nu_\beta \equiv p^\alpha d_{\alpha\beta} + q_\beta .$$

Comma followed by a Greek index and q_β having been used for covariant derivatives based on $g_{\alpha\beta}$ and $\partial q/\partial u^\beta$ respectively.

If k and τ stand for the curvature and torsion of C, we have the following well known Frenet equations:

$$\hat{t} = k\hat{n} = \varrho^\alpha x_\alpha + k_n \hat{X}, \quad \hat{n}' = -k\hat{t} + \tau\hat{b}, \quad \hat{b}' = -\tau\hat{n}, \tag{1.5}$$

where ϱ^α are the components of the geodesic curvature vector of C and k_n is the normal curvature.

Throughout this paper Greek indices will take values 1, 2. The functions will be assumed to be differentiable to the required order and $k \neq 0$.

2. Definitions. DEFINITION (2.1). *(P_nA)-vector.* A unit vector $\hat{U}$ associated with any point P of a curve C on the surface of reference of a rectilinear congruence whose line passing through P is represented by a unit vector $\hat{\lambda}$, will be called (P_nA)-vector if it linearly depends on the principal normal $\hat{n}$ to C at P and $\hat{\lambda}$ and is orthogonal to $\hat{n}$.

If we put $\cos \Phi=\hat{\lambda}\cdot\hat{n}$, we can express $\hat{U}$ as

$$\hat{U} = e \operatorname{cosec} \Phi (\hat{\lambda} - \cos \Phi \hat{n}), \tag{2.1}$$

e being equal to 1 or -1 according as Φ is acute or obtuse.

DEFINITION (2.2). *(P_nA)-plane.* We define (P_nA)-plane at a point P of C as a plane which is generated by the (P_nA)-vectors at P and at a consecutive point.

DEFINITION (2.3). *(TP_nA)-curve.* A curve C on S will be called (TP_nA)-curve if its tangent at P lies in the (P_nA)-plane at P.

DEFINITION (2.4). *(P_nP_nA)-curve.* We shall call C to be (P_nP_nA)-curve if the principal normal to it at any point P lies in the (P_nA)-plane at P.

DEFINITION (2.5). *(B_nP_nA)-curve.* We define C to be (B_nP_nA)-curve if the binormal to it at any point P lies in the (P_nA)-plane.

3. (TP_nA)-curve. According to Definition (2.3) we have

$$[\hat{t} \quad \hat{U} \quad \hat{U}+\hat{U}'] = 0, \quad \text{i.e.} \quad [\hat{t} \quad \hat{U} \quad \hat{U}'] = 0,$$

which with the use of (2.1) becomes

$$[\hat{t} \quad \hat{\lambda} \quad \hat{\lambda}'] + \sin \Phi \Phi' [\hat{t} \quad \hat{\lambda} \quad \hat{n}] - \cos \Phi [\hat{t} \quad \hat{n} \quad \hat{\lambda}'] = 0. \tag{3.1}$$

In this equation the first term vanishes if C is a α-curve [2], vanishing of second term means that either Φ=constant or C is a union curve [3] and equality to zero of the third term implies either $\Phi=\pi/2$, which means that C is a hyperasymptotic curve [1] or C is a 0^*-curve [4]. From this discussion we conclude the following:

THEOREM (3.1). *If the (TP_nA)-curve C satisfies any two of the following properties it staisfies the third also:*

(i) *C is a α-curve,*

(ii) *either C is a union curve or the lines of the congruence intersect C such that they are inclined at a constant angle to the principal normal to C at the point of intersection,*

(iii) *C is either a hyperasymptotic curve or a 0^*-curve.*

Solving (3.1) with the help of (1.1), (1.3) and (1.5)a we get

$$e_{\alpha\beta} u'^{\alpha} L^{\beta} = 0, \tag{3.2}$$

where $e_{\alpha\beta} \equiv [x_\alpha \quad x_\beta \quad \hat{X}]$ and

$$L^{\beta} \equiv k(v_\delta p^\beta - q\mu_\delta^\beta)u'^\delta + \sin\Phi\Phi'(k_n p^\beta - q\varrho^\beta) - \cos\varphi(v_\delta \varrho^\beta - k_n \mu_\delta^\beta)u'^\delta.$$

We shall call the vector with contravariant components L^β (TP_nA)-vector and its magnitude (TP_nA)-curvature.

We now consider two special cases.

Case I. If the congruence is normal, (3.2) reduces to

$$k_g[k_g e_{\alpha\beta} d_{\delta\gamma} g^{\gamma\beta} u'^\delta u'^\alpha - k \sin\Phi\Phi'] = 0, \tag{3.3}$$

which yields:

THEOREM (3.2). *For a normal congruence, the* (TP_nA)*-curves are either geodesics or they are given by the equation*

$$k_g e_{\alpha\beta} d_{\delta\gamma} g^{\gamma\beta} u'^{\delta} u'^{\alpha} - k \sin\varphi\varphi' = 0.$$

If C is a line of curvature, i.e., $e_{\alpha\beta} d_{\delta\gamma} g^{\gamma\beta} u'^{\alpha} u'^{\delta} = 0$, then equation (3.3) leads to

THEOREM (3.3). *For a normal congruence, a* (TP_nA)*-curve C is a line of curvature if and only if the lines of the congruence intersect C such that they are inclined at a constant angle to the principal normal to C, provided C is not a geodesic.*

Since $\cos\Phi = \hat{\lambda}\cdot\hat{n} = k_c/k$, $k_c = \varrho^{\alpha} p_{\alpha} + q k_n$ being the curvature of the congruence section of S along C (hereafter we shall call k_c the hyperasymptotic curvature), therefore for a normal congruence $\cos\Phi = k_n/k$ which vanishes for C to be asymptotic line. Thus as $k_g \neq 0$, when $k_n = 0$, from (3.3) we obtain

THEOREM (3.4). *For a normal congruence, if* (TP_nA)*-curve is an asymptotic line, it is a line of curvature.*

Case II. Let the congruence be formed of tangents to a one parameter family of curves on S (hereafter we shall call such a congruence the tangential congruence). Then (3.2) reduces to

$$d_{\gamma\delta} p^{\gamma} u'^{\delta} (e_{\alpha\beta} u'^{\alpha} p^{\beta} - \cos\Phi\, k_g) + k_n e_{\alpha\beta} u'^{\alpha} (p^{\beta} \cos\Phi)' = 0. \tag{3.4}$$

From this equation we derive the following:

THEOREM (3.5). *If the congruence is tangential, the necessary and sufficient condition for either the direction of* (TP_nA)*-curve C to be conjugate to the line of congruence at the point P of intersection or cosine of the angle between the line of congruence and the principal normal to C at P to be equal to the ratio of* $e_{\alpha\beta} u'^{\alpha} p^{\beta}$ *and* k_g *is that either C be an asymptotic line or the derived vector along C of the component of the unit vector in the direction of the line of the congruence through P along the principal normal to C at P be null or parallel to the tangent to C at P.*

If in addition $\cos\Phi = 0$, i. e., C is a hyperasymptotic curve, (3.4) reduces to

$$(d_{\gamma\delta} p^{\gamma} u'^{\delta})(e_{\alpha\beta} u'^{\alpha} p^{\beta}) = 0,$$

which implies:

THEOREM (3.6). *If the congruence is tangential and a curve is both a* (TP_nA)*-curve and a hyperasymptotic curve, then the directions of the lines of congruence are either conjugate or parallel to the directions of C at the points of intersection.*

Since $\hat{t}$, $\hat{U}$, $\hat{U}'$ are coplanar therefore $\hat{t}$ can be expressed as a linear combination of $\hat{U}$ and $\hat{U}'$. With the use of (1.1), (1.3), (1.5) and (2.1) we obtain

$$\hat{t} = e \operatorname{cosec}\Phi \Big[(p_{\alpha} u'^{\alpha})\hat{U} + \frac{1}{M^2}(\mu_{\alpha\beta} u'^{\alpha} u'^{\beta} - \cot\Phi' p_{\alpha} u'^{\alpha} + k\cos\Phi\Phi)\hat{U}'\Big], \tag{3.5}$$

where $M \equiv |\hat{U}'|$ (which we shall consider to be non zero).

Multiplication of (3.5) by $\hat{n}$ with the use of (1.1), (1.3), (1.5) and (2.1) yields

$$(\mu_{\alpha\beta}u'^{\alpha}u'^{\beta}-\cot\Phi\Phi' p_{\alpha}u'^{\alpha}+k\cos\Phi)(\mu_{\alpha\beta}\varrho^{\alpha}u'^{\beta}+k_n\nu_{\beta}u'^{\beta}+\sin\Phi\Phi' k)=0,$$

which gives:

THEOREM (3.7). *The (TP_nA)-curves satisfy either of the following:*

(i) $\mu_{\alpha\beta}u'^{\alpha}u'^{\beta}-\cot\Phi\Phi' p_{\alpha}u'^{\alpha}+k\cos\Phi=0$,

(ii) $\mu_{\alpha\beta}\varrho^{\alpha}u'^{\beta}-k_n\nu_{\beta}u'^{\beta}+\sin\Phi\Phi' k=0$.

Multiplication of (3.5) by $\hat{\lambda}$ results in

$$\cos\Phi(\mu_{\alpha\beta}u'^{\alpha}u'^{\beta}-\cot\Phi\Phi' p_{\alpha}u'^{\alpha}+k\cos\Phi)(k^2p_{\alpha}u'^{\alpha}\tau qk_u)=0. \tag{3.6}$$

Hence:

THEOREM (3.8). *The (TP_nA)-curve satisfies one of the following:*
(i) *it is a hyperasymptotic curve,*

(ii) $\mu_{\alpha\beta}u'^{\alpha}u'^{\beta}-\cot\Phi\Phi' p_{\alpha}u'^{\alpha}+k\cos\Phi=0$,

(iii) $k^2p_{\alpha}u'^{\alpha}-\tau qk_u=0$.

If C are hypernormal curves, i.e., if $p_{\alpha}u'^{\alpha}=0$, (3.6) reduces to

$$\tau qk_u\cos\Phi(\mu_{\alpha\beta}u'^{\alpha}u'^{\beta}+k\cos\Phi)=0$$

which leads to

THEOREM (3.9). *If a* (TP_nA)-*curve C is a hypernormal curve, it satisfies one of the following:*
(i) *C is a plane curve,*
(ii) *C is a union curve,*
(iii) *C is a hyperasymptotic curve,*
(iv) *C is given by the equation* $\mu_{\alpha\beta}u'^{\alpha}u'^{\beta}+k\cos\Phi=0$.

REMARK. A number of different forms of differential equations of these curves may be obtained by multiplying equation (3.5) by $\hat{b}$, $\hat{X}$ etc. and also a good number of results may be deduced out of them.

4. (P_nP_nA)-curves. By Definition (2.4) we have $[\hat{n}\quad\hat{U}\quad\hat{U}']=0$, which in view of (1.5) and (2.1) becomes

$$[\hat{n}\quad\hat{\lambda}\quad\hat{\lambda}']+\cos\varphi(k[\hat{n}\quad\hat{\lambda}\quad\hat{t}]-\tau[\hat{n}\quad\hat{\lambda}\quad\hat{b}])=0. \tag{4.1}$$

We know that $[n\quad\hat{\lambda}\quad\hat{\lambda}']=0$ for C to be β-curve [2], $\cos\Phi=0$ if C is a hyperasymptotic curve, $[\hat{n}\quad\hat{\lambda}\quad\hat{t}]=qk_u$, $[\hat{n}\quad\hat{\lambda}\quad\hat{b}]=-p_{\alpha}u'^{\alpha}$. Thus from (4.1) we have:

THEOREM (4.1). *The necessary and sufficient condition for a* (P_nP_nA)-*curve C to be a* β-*curve is that it satisfies either of the following:*
(i) *C is a hyperasymptotic curve,*
(ii) $qk_u+p_{\alpha}u'^{\alpha}=0$.

Solving (4.1) in view of (1.1), (1.3) and (1.5)a we get

$$e_{\alpha\beta}(\varrho^\alpha p^\beta v_\delta - q\varrho^\alpha \mu_\delta^\beta + k_n p^\alpha \mu_\delta^\beta)u'^\delta + k_c(qk_u + \tau p_\alpha u'^\alpha) = 0, \tag{4.2}$$

where $k_c \equiv k \cos \Phi$.

Now we consider some special cases:

Case I. For a normal congruence (4.2) reduces to

$$k_n k_g = e_{\alpha\beta} d_{\delta\gamma} g^{\gamma\alpha} \varrho^\beta u'^\delta,$$

which gives the product of geodesic and normal curvatures. If C is a geodesic this equation becomes identity and if C is an asymptotic line $e_{\alpha\beta} d_{\delta\gamma} g^{\gamma\alpha} \varrho^\beta u'^\delta = 0$. Thus:

THEOREM (4.2). *For a normal congruence (P_nP_nA)-curves are*
(i) *indeterminate if they are also geodesics,*
(ii) *and satisfy $e_{\alpha\beta} d_{\delta\gamma} g^{\gamma\alpha} \varrho^\beta u'^\delta = 0$, if they are also asymptotic lines.*

Case II. For the tangential congruence (4.2) takes the form

$$e_{\alpha\beta}[\varrho^\alpha p^\beta (d_{\delta\gamma} p^\gamma u'^\delta) + k_n p^\alpha (p^\beta_{,\delta} u'^\delta) + k_n p_\delta \varrho^\delta u'^\alpha p^\beta] + \tau p_\beta \varrho^\beta p_\alpha u'^\alpha = 0. \tag{4.3}$$

If in addition C is geodesic, (4.3) reduces to

$$e_{\alpha\beta} p^\alpha (p^\beta_{,\delta} u'^\delta) = 0$$

which implies $p^\beta_{,\delta} u'^\delta = 0$, because p^β is a unit vector. Hence:

THEOREM (4.3). *If the congruence is tangential and the (P_nP_nA)-curve C is a geodesic, the relative curvature vector of the congruence relative to C is a null vector.*

Next if C is an asymptotic line, (4.3) gives

$$(e_{\alpha\beta} \varrho^\alpha p^\beta)(d_{\delta\gamma} p^\gamma u'^\delta) + \tau (p_\beta \varrho^\beta)(p_\alpha u'^\alpha) = 0,$$

which yields:

THEOREM (4.4). *For the tangential congruence and the (P_nP_nA)-curve C being an asymptotic line, the necessary and sufficient condition for the line of the congruence to be either parallel to the first curvature vector of C or conjugate to the tangent to C is given by one of the following:*
(i) *C is a plane curve,*
(ii) *C is a hyperasymptotic curve,*
(iii) *C is a hypernormal curve.*

Since for C to be a (P_nP_nA)-curve $\hat{n}$, $\hat{U}$ and $\hat{U}'$ are coplanar and $\hat{n}$ is orthogonal to $\hat{U}$, therefore $\hat{n}$ must be parallel to $\hat{U}'$. Hence we have:

$$\hat{n} = \frac{e}{M} \hat{U}', \tag{4.4}$$

e being 1 or -1 according as $\hat{n}$ and $\hat{U}$ have the same or opposite directions.

Multiplication of (4.4) by $\hat{t}$ by virtue of (1.1), (1.3), (1.5) and (2.1) gives

$$\mu_{\alpha\beta} u'^\alpha u'^\beta + k \cos \Phi - \cot \Phi \Phi' p_\alpha u'^\alpha = 0, \tag{4.5}$$

which is an alternate form of the differential equation of these curves. It also gives an expression for the value of the curvature of the curve C.

From equation (4.5) we deduce the following:

THEOREM (4.5). *If a (P_nP_nA)-curve is a hyperasymptotic curve its differential equation is given by* $\mu_{\alpha\beta}u'^{\alpha}u'^{\beta}=0$.

THEOREM (4.6). *If a (P_nP_nA)-curve C is a hypernormal curve then its differential equation is given by* $\mu_{\alpha\beta}u'^{\alpha}u'^{\beta}+k\cos\Phi=0$. *Conversely, the above holds if and only if the lines of the congruence are not inclined to the normals to C at a constant angle.*

THEOREM (4.7). *If the congruence is tangential and the (P_nP_nA)-curve is also a hyperasymptotic curve, then the tendency of the congruence along C is zero.*

5. (B_nP_nA)-curve. From Definition (2.5) for these curves we have $[\hat{b}\quad \hat{U}\quad \hat{U}']=0$, which with the help of (2.1) becomes

$$[\hat{b}\quad \hat{\lambda}\quad \hat{\lambda}']-\cos\Phi[\hat{b}\quad \hat{\lambda}\quad \hat{\lambda}']+\sin\Phi\Phi'[\hat{b}\quad \hat{\lambda}\quad \hat{n}]=0. \tag{5.1}$$

It is known that $[\hat{b}\quad \hat{\lambda}\quad \hat{\lambda}']=0$, if C is a γ-curve [2], $\cos\Phi\cdot[\hat{b}\quad \hat{n}\quad \hat{\lambda}']=0$, if C is either a hyperasymptotic curve or a N^*-curve [5] and $\sin\Phi\Phi'[\hat{b}\quad \hat{\lambda}\quad \hat{n}]=0$, if either $\Phi=$constant or C is a hypernormal curve. Thus we have:

THEOREM (5.1). *If a (B_nP_nA)-curve C satisfies two of the following properties it satisfies the third also:*

(i) *it is a γ-curve,*

(ii) *it is either a hyperasymptotic curve or a N^*-curve,*

(iii) *either the line of the congruence is inclined at a constant angle with the principal normal to C or C is a hypernormal curve.*

Solving (5.1) with the help of (1.1), (1.3) and (1.5) we get

$$(p_\alpha u'^{\alpha})\{(\mu_{\alpha\beta}\varrho^{\alpha}+k_n\nu_\beta)u'^{\beta}+k\sin\Phi\Phi'\}=0, \tag{5.2}$$

which implies:

THEOREM (5.2). *A (B_nP_nA)-curve C is either a hypernormal curve or it is given by the differential equation*

$$(\mu_{\alpha\beta}\varrho^{\alpha}+k_n\nu_\beta)u'^{\beta}+k\sin\varphi\varphi'=0.$$

For a normal congruence (5.2) reduces to an identity except when C is not a hypernormal curve and in that case C satisfies

$$\cos\Phi=-\int\left(\frac{1}{k}d_{\alpha\beta}\varrho^{\alpha}u'^{\beta}\right)ds+\text{constant}. \tag{5.3}$$

Hence we have:

THEOREM (5.3). *For a normal congruence (B_nP_nA)-curves are indeterminate except when they are not hypernormal curves, in that case they satisfy* (5.3).

Next let the congruence be tangential, then (5.2) takes the form

$$(p_\alpha u'^{\alpha})\{(p_{\alpha,\beta}\varrho^{\alpha}+k_nd_{\beta\alpha}p^{\alpha})u'^{\beta}+k(p_\alpha\varrho^{\alpha})'\}=0. \tag{5.4}$$

In addition if C is a geodesic the last equation reduces to

$$(p_\alpha u'^\alpha) d_{\beta\alpha} p^\alpha u'^\beta = 0,$$

which yields:

THEOREM (5.4). *If the congruence is tangential and a (B_nP_nA)-curve C is a geodesic then either it is a hypernormal curve or the directions of the lines of congruence and tangents to C at the points of intersection are conjugate.*

Since for C to be a (B_nP_nA)-curve, $\hat{b}$ may be expressed as a linear combination of $\hat{U}$ and $\hat{U}'$. With the use of (1.1), (1.3), (1.5) and (2.1) we obtain

$$\hat{b} = \frac{e \operatorname{cosec} \Phi}{k}\left[qk_u \hat{U} - \left(\frac{1}{M^2}\right)(\cot \Phi \Phi' qk_u - k_g v_\delta u'^\delta + k_n e_{\alpha\beta} u'^\alpha \mu_\delta^\beta u'^\delta + k\tau \cos \Phi)\hat{U}'\right]. \tag{5.5}$$

Multiplying (5.5) by $\hat{t}$ we get

$$\begin{aligned} qk_u(p_\alpha u'^\alpha) + \frac{1}{M^2}(\cot \Phi\Phi' qk_u - k_g v_\delta u'^\delta + k_n e_{\alpha\beta} u'^\alpha \mu_\delta^\beta u'^\delta + k\tau \cos \Phi) \times \\ \times (\cot \Phi\Phi' p_\alpha u'^\alpha - \mu_{\alpha\beta} u'^\alpha u'^\beta - k \cos \Phi) = 0. \end{aligned} \tag{5.6}$$

For a normal congruence (5.6) reduces to an identity supporting Theorem 5.3.

If C is a union curve, (5.6) reduces to

$$(k_n e_{\alpha\beta} u'^\alpha \mu_\delta^\beta u'^\delta + k\tau \cos \Phi - k_g v_\delta u'^\delta)(\cot \Phi\Phi' p_\alpha u'^\alpha - \mu_{\alpha\beta} u'^\alpha u'^\beta k \cos \Phi) = 0,$$

which implies:

THEOREM (5.5). *If a (B_nP_nA)-curve C is a union curve it satisfies either of the following:*

(i) $k_n e_{\alpha\beta} u'^\alpha \mu_\delta^\beta u'^\delta + k\tau \cos \Phi - k_g v_\delta u'^\delta = 0,$

(ii) $\cot \Phi\Phi' p_\alpha u'^\alpha - \mu_{\alpha\beta} u'^\alpha u'^\beta - k \cos \Phi = 0.$

Multiplication of (5.5) by $\hat{\lambda}$ gives

$$(\cot \Phi\Phi' qk_u - k_g v_\delta u'^\delta - k_n e_{\alpha\beta} u'^\alpha \mu_\delta^\beta u'^\delta + k\tau \cos \Phi)(k^2 p_\alpha u'^\alpha \tau qk_u) = 0,$$

which yields:

THEOREM (5.6). *A (B_nP_nA)-curve satisfies either of the following:*

(i) $\cot \Phi\Phi' qk_u - k_g v_\delta u'^\delta + k_n e_{\alpha\beta} u'^\alpha \mu_\delta^\beta u'^\delta + k\tau \cos \Phi = 0,$

(ii) $k^2 p_\alpha u'^\alpha - \tau qk_u = 0.$

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DETACHABLE p-GROUPS AND QUASI-INJECTIVITY

By

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If E is a ring, and G is an E-module, then G is *quasi-injective* if every homomorphism from a submodule of G to G can be extended to an endomorphism of G. Poole and Reid [2] raise the question as to which abelian groups G are quasi-injective as modules over their endomorphism rings E. They prove that direct sums of cyclic p-groups are quasi-injective. This result is extended here to a large class of p-groups, including all p-groups that have no elements of infinite height, and all totally projective p-groups — in particular, all countable p-groups.

The central idea is a property shared by totally projective p-groups and p-groups with no elements of infinite height.

Definition. If G is a p-group and $x \in G[p]$, then x is said to be *detachable* if G can be written as $G_1 \oplus G_2$ with $\langle x \rangle = p^\alpha G_1[p]$ for some ordinal α. If every element of $G[p]$ is detachable, we say that G is *detachable.*

We write $p^\alpha G_1[p]$ instead of $p^\alpha G_1$ to allow G_1 to be divisible. Note that every element in $G[p]$ of finite height is detachable, so every p-group with no elements of infinite height is detachable. Hill's extension of Ulm's theorem to totally projective p-groups (see [3]) implies that these groups are also detachable.

Theorem 1. *If G is a detachable p-group, and $x, y \in G[p]$, then there is an endomorphism f of G such that $f(x) = y$ or $f(y) = x$.*

Proof. We may suppose that $\mathrm{ht}(x) \leqq \mathrm{ht}(y)$. Let $G = G_1 \oplus G_2$, where $p^\alpha G_1[p] = \langle x \rangle$. Let Π_{G_1} denote the projection of G on G_1. If $\Pi_{G_1}(y) \neq 0$, then $x = n\Pi_{G_1}(y)$ for some integer n, since $\mathrm{ht}(x) \leqq \mathrm{ht}(y) \leqq \mathrm{ht}(\Pi_{G_1}(y))$. Thus $f = n\Pi_{G_1}$ is the desired map. If $y \in G_2$ let $G = M \oplus L$ with $p^\beta M[p] = \langle x+y \rangle$. Now $\Pi_M(x) + \Pi_M(y) = \Pi_M(x+y) = x+y$ and, since $\mathrm{ht}(x+y) = \mathrm{ht}(x) \leqq \mathrm{ht}(y)$, then $\Pi_M(x)$ and $\Pi_M(y)$ are in $\langle x+y \rangle$. Hence $\Pi_M\langle x \rangle = \langle x+y \rangle$ or $\Pi_M\langle y \rangle = \langle x+y \rangle$. But $\Pi_{G_1}(x+y) = x$ and $\Pi_{G_2}(x+y) = y$, so either $\Pi_{G_1}\Pi_M$ or $\Pi_{G_2}\Pi_M$ is the desired map (up to a unit).

Theorem 1 cannot be strengthened to assert that if x and y are in $G[p]$ and $\mathrm{ht}(x) \leqq \mathrm{ht}(y)$ then there is an endomorphism f of G such that $f(x) = y$. This can be seen by applying the following partial converse of Theorem 1 to an example of Megibben [1].

Theorem 2. *Let G be a direct sum of detachable p-groups G_i. Then G is detachable if and only if for every pair $x \in G_i[p]$ and $y \in G_j[p]$, either there is a homomorphism $f\colon G_i \to G_j$ such that $f(x) = y$, or there is a homomorphism $g\colon G_j \to G_i$ such that $f(y) = x$.*

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PROOF. The "only if" comes from Theorem 1. For the "if" it suffices to prove the theorem for finite sums, since G is detachable if every finite sum of the G_i is detachable. Suppose $G=G_1\oplus\ldots\oplus G_n\oplus G_{n+1}$ and $K=G_1\oplus\ldots\oplus G_n$ is detachable. Let $x=x_1+\ldots+x_n$ be an element of K, and y be an element of G_{n+1}. Then either, for some $1\leqq i\leqq n$, there is a map $f\colon G_i\to G_{n+1}$ such that $f(x_i)=y$, and hence a map $f'\colon K\to G_{n+1}$ such that $f(x)=y$, or there are maps $g_i\colon G_{n+1}\to G_i$ such that $g_i(y)=x_i$, and hence a map $g\colon G_{n+1}\to K$ such that $g(y)=x$. Suppose we have the map f' (the argument with the map g is identical). Then $1+f'$ maps K isomorphically onto a complementary summand of G_{n+1} taking x to $x+y$. Since x is detachable from K, then $x+y$ is detachable from $(1+f')K$, and hence from G.

Note that the homomorphism f in Theorem 2 certainly exists if either x or y has finite height.

EXAMPLE 1 (Megibben). Let K and L be p-groups such that $K^1=\langle x\rangle$, $L^1=\langle y\rangle$, K/K^1 is torsion complete, L/L^1 is a direct sum of cyclic groups, and x and y have order p. Let $G=K\oplus L$. Since there is a homomorphism $f\colon L\to K$ such that $f(y)=x$, we may apply Theorem 2 to show that G is detachable. On the other hand, any endomorphism of G taking x to y induces a homomorphism $g\colon K\to L$ such that $g(x)=y$. But g then induces a map from K/K^1 to L/L^1 which must be small since K/K^1 is torsion complete and L/L^1 is a direct sum of cyclic groups. But this implies that $g(x)=0$, so there is no endomorphism of G taking x to y.

If G is a p-group with endomorphism ring E, then multiplication by a p-adic integer induces an E-map of G. Conversely, any E-endomorphism of G is induced by multiplication by a p-adic integer. To show that G is quasi-injective as an E-module, it is therefore necessary and sufficient to show that any E-map from a fully invariant subgroup of G to G is induced by multiplication by a p-adic integer.

EXAMPLE 2. If we take y of order p^2 in Example 1, we get an example of a p-group G that is not quasi-injective. Indeed, $\langle x\rangle$ and $\langle py\rangle$ are disjoint fully invariant subgroups, so projection of $\langle x\rangle\oplus\langle py\rangle$ on $\langle x\rangle$ is an E-map that cannot be extended to G. This group also shows that a direct sum of two detachable groups need not be quasi-injective, and hence need not be detachable.

THEOREM 3. *Let G be a detachable p-group with endomorphism ring E, and H a fully invariant subgroup of G. If $f\colon H\to G$ is an E-map, then there is a p-adic integer θ so that $f(x)=\theta x$ for all x in H.*

PROOF. We shall show, for $n=0,1,2,\ldots$, that if $f(x)=mx$ for all x in $H[p^n]$, then $f(x)=(m+sp^n)x$ for all x in $H[p^{n+1}]$. Suppose $x\in H[p^{n+1}]$. Then $p^nx\in H[p]$, so $G=G_1\oplus G_2$, with $p^\alpha G_1[p]=\langle p^nx\rangle$. Write $x=g_1+g_2$ with $g_1\in G_1$ and $g_2\in G_2$. Note that g_1 and g_2 are in H since H is fully invariant. Then $p^ng_2=0$ and $p^ng_1=p^nx$, so $f(g_2)=$ $=(m+sp^n)\,g_2$ for *any* integer s. Thus we may assume that $x=g_1$, that is, that $x\in G_1$.

Since $px\in H[p^n]$ w ehave $f(px)=mpx$. Thus $p\big(f(x)-mx\big)=0$, so we can write $G=K_1\oplus K_2$ with $p^\beta K_1[p]=\langle f(x)-mx\rangle$. If $f(x)=mx$, choose $s=0$ and we are done. Since f is an E-map, and x is in G_1, we have $f(x)-mx\in G_1$. Thus, if $f(x)\neq mx$, then $\mathrm{ht}\big(f(x)-mx\big)\leqq\mathrm{ht}(p^nx)$. If $\mathrm{ht}\big(f(x)-mx\big)=\mathrm{ht}(p^nx)$ then $f(x)-mx=sp^nx$ for some s, as desired. If $\mathrm{ht}\big(f(x)-mx\big)<\mathrm{ht}(p^nx)$ write $x=k_1+k_2$ with $k_1\in K_1$ and $k_2\in K_2$. Then $p^nk_1=0$ so $f(k_1)=mk_1$. Hence $f(x)-mx=f(k_2)-mk_2$ is in $K_1\cap K_2$ since K_2 is invariant under the E-map f. So $f(x)=mx$, a contradiction.

It remains to show that s is independent of x, modulo p. Suppose $f(x)=mx+$ $+s_1p^nx$ and $f(y)=my+s_2p^ny$ where $o(x)=o(y)=p^{n+1}$. Choose $\lambda\in E$ so that $\lambda(p^nx)=$ $=p^ny$ (or vice-versa). Then $p^n(\lambda x-y)=0$ so $f(\lambda x)-f(y)=f(\lambda x-y)=m(\lambda x-y)$. Therefore $m\lambda x+s_1p^ny=\lambda f(x)=f(\lambda x)=m\lambda x+f(y)-my=m\lambda x+s_2p^ny$. Since $o(y)=$ $=p^{n+1}$, we have $s_1\equiv s_2 \pmod p$.

COROLLARY. *If G is a detachable p-group with endomorphism ring E, then G is quasi-injective as an E-module.*

THEOREM 4. *Let K be a p-group that is quasi-injective as a module over its endomorphism ring. Let L be a p-group that is isomorphic to a subgroup of a product of copies of K. Then the group $G=K\oplus L$ is quasi-injective as a module over its endomorphism ring E.*

PROOF. Suppose $H\subseteqq G$ is fully invariant and $f\colon H\to G$ is an E-map. Then f induces an endomorphism of $H\cap K$ which, since K is quasi-injective over its endomorphism ring, is given by a p-adic integer θ. We shall show that $f(x)=\theta x$ for all x in H. Suppose $f(x)\neq\theta x$ for some x in H. Then, since L is isomorphic to a subgroup of a product of copies of K, there is a λ in E such that $\lambda(G)\subseteqq K$ and $\lambda(f(x)-$ $-\theta x)\neq 0$. But $\lambda(f(x)-\theta x)=\lambda f(x)-\lambda\theta x=f(\lambda x)-\theta\lambda x=\theta\lambda x-\theta\lambda x=0$.

COROLLARY 1. *If G is a p-group with endomorphism ring E, such that G has a nonzero divisible subgroup, then G is quasi-injective as an E-module.*

PROOF. Let K be a subgroup of G isomorphic to $Z(p^\infty)$.

COROLLARY 2. *There exists a reduced p-group that is quasi-injective as a module over its endomorphism ring, but is not detachable. This group has a summand that is not quasi-injective over its endomorphism ring.*

PROOF. Let $G=K\oplus L$ be as in Example 2. Let $M=(K\oplus L)/\langle x-py\rangle$. Let z denote the coset $x+\langle x-py\rangle$ in M. Then $M/\langle z\rangle\cong K/\langle x\rangle\oplus L/\langle py\rangle$ has length $\omega+1$ while z has height at least $\omega+2$ since it is an image of py. Hence $\langle z\rangle=M^1[p]$, so M is detachable and hence quasi-injective as a module over its endomorphism ring. Moreover, $K\oplus L$ is isomorphic to a subgroup of $M\oplus M$. Theorem 4 then tells us that $K\oplus L\oplus M$ is quasi-injective as a module over its endomorphism ring. But $K\oplus L\oplus M$ is not detachable because there is no endomorphism taking x to py or taking py to x, contrary to Theorem 1. Moreover, $K\oplus L$ is Example 2, and hence is not quasi-injective.

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A REMARK ON SEMIGROUPS WITH IDENTITY ELEMENT

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Recently F. SZÁSZ [1] has shown the following characterization of semigroups with an identity:

The following assertions concerning a semigroup S are equivalent:

(I) S is a semigroup with an identity;

(II) S has the following properties:

(α) there exists at least one left cancellable element f in S such that $Sf \subseteqq fS$;

(β) there exists at least one right cancellable element e in S such that $eS \subseteqq Se$;

(γ) no homomorphic image of S^0 has any non-zero left annihilator element;

(δ) no homomorphic image of S^0 has any non-zero right annihilator element;

where S^0 is defined in [2] or in [3].

In this connection I. KISS [2, p. 250] has proved the following characterization of semigroups with an identity:

The following assertions concerning a semigroup S are equivalent:

(I) S is a semigroup with an identity;

(II) S has properties $(\beta)'$ and (δ), where

$(\beta)'$ there exists at least one right cancellable element e in S such that $eS = Se$.

I. Kiss has also pointed out that the dual of the above theorem can be proved.

The object of this note is to remark that in addition to the theorem and its dual of Kiss, some other theorems involving the characterization of semigroups with identity element can be proved. The following is our characterization:

THEOREM. *The following assertions for a semigroup S are equivalent:*

(I) *S is a semigroup with an identity;*

(II) *S has properties $(\alpha)'$ and (δ), where*

$(\alpha)'$ there exists at least one left cancellable element f in S such that $Sf = fS$;

(III) *S has properties $(\beta)'$ and (γ), where*

$(\beta)'$ there exists at least one right cancellable element e in S such that $eS = Se$.

PROOF. It is clear that a semigroup S with an identity has properties $(\alpha)'$ and (δ) or $(\beta)'$ and (γ).

First let S be a semigroup having properties $(\alpha)'$ and (δ). By $(\alpha)'$ there exists at least one left cancellable element $f \in S$ such that $Sf = fS$. Thus S is a semigroup having property (δ) and f is an element of S such that $Sf \subseteqq fS$. Consequently by the lemma of KISS [2, p. 250], $f \in fS$. So $f = ff'$ for an element $f' \in S$. Now for every $x \in S$, we have $ff'x = fx \Rightarrow f'x = x$. Thus f' is a left identity element of S.

On the other hand, we have $Sf = fS$. Thus for every $x \in S$ there exists an $y \in S$ such that $fx = yf$. So $fxf' = yff' = yf = fx$, i.e. $xf' = x$. Thus f' is a right identity element of S. So f' is the identity of S. Thus we have proved (II).

Since (III) is the dual of (II), it can be easily proved.

Lastly we remark that our theorem, when combined with that of Kiss, gives the following characterization:

The following assertions for a semigroup S are equivalent:

(I) S is a semigroup with an identity;
(II) S has properties $(\beta)'$ and (δ);
(III) S has properties $(\alpha)'$ and (γ);
(IV) S has properties $(\alpha)'$ and (δ);
(V) S has properties $(\beta)'$ and (γ).

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A NOTE ON A THEOREM OF ALEXITS AND KRÁLIK

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Let $f(x)$ be a 2π periodic function integrable in the sense of Lebesgue in $[0, 2\pi]$ and let

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. We denote the n-th de la Vallée Poussin sum by

$$V_n(f) = \frac{1}{n} \sum_{k=n}^{2n-1} S_k(f) \tag{2}$$

where $S_n(f)$ is the n-th partial sum of (1). It is known (cf. ZYGMUND [4] Vol. I. p. 115) that $|V_n(f)-f| \leqq 4E_n(f)$ where $E_n(f)$ represents the best approximation of $f(x)$ by trigonometric polynomials of order n. We denote by Z the class of all continuous 2π periodic functions satisfying the inequality $\sup |f(x+t)-2f(x)+f(x-t)| \leqq \leqq K|t|$ for all x and t where K is an absolute positive constant. Earlier, G. ALEXITS and D. KRÁLIK [2] gave the necessary and sufficient condition for r-th derivative $f^{(r)}(x)$ of a function $f(x)$ to belong to the class Z in terms of the de la Vallée Poussin sum in strong sense:

$$\frac{1}{n} \sum_{k=n}^{2n-1} |S_k(f)-f| = O(n^{-r-1}).$$

The object of this note is to strengthen slightly this statement by proving the following

THEOREM. *Denote by $\{v_n\}$ an arbitrary sequence of integers $\geqq 1$. The derivative function $f^{(r)}(x)$ belongs to the class Z if and only if*

$$\left\{\frac{1}{n} \sum_{k=v_n}^{n+v_n-1} |S_k(f)-f|^p\right\}^{1/p} = O(v_n^{-r-1})$$

for any fixed $p \geqq 1$.

For $v_n = n$ and $p=1$ our theorem reduces to that of ALEXITS—KRÁLIK. Because of a general theorem of ALEXITS [3] we have only to prove that, if $|f(x)| \leqq M$, then

$$\left\{\frac{1}{n} \sum_{k=v_n}^{n+v_n-1} |S_k(f)|^p\right\}^{1/p} \leqq KM \tag{3}$$

for every $p \geqq 1$ where K is an absolute constant. Moreover, it is enough to prove (3) only for $p \geqq 2$, because the left hand side of (3) is a monotonically increasing function of $p > 1$. Hence suppose $p \geqq 2$ and consider

$$|S_k(f)-f|^p = \left| \frac{1}{n} \int_0^\pi \Phi_x(t) \frac{\sin(k+1/2)t}{2\sin t/2}\, dt \right|^p$$

where $\Phi_x(t) = f(x+t)+f(x-t)-2f(x)$. Then

$$|S_k(f)-f|^p \leqq 2^{p-1} \left\{ \left| \frac{1}{\pi} \int_0^{1/n} \Phi_x(t) \frac{\sin(k+1/2)t}{2\sin t/2}\, dt \right|^p + \left| \frac{1}{\pi} \int_{1/n}^{\pi} \Phi_x(t) \frac{\sin(k+1/2)t}{2\sin t/2}\, dt \right|^p \right\} =$$

$$= 2^{p-1}\{I_1+I_2\}.$$

We get for the first integral

$$I_1 \leqq \left(\frac{1}{\pi} \int_0^{1/n} \left| \Phi_x(t) \frac{\sin(k+1/2)t}{2\sin t/2} \right| dt \right)^p < \left(\frac{8M}{\pi} \right)^p,$$

but for the second, by applying a classical method (cf. [4], Vol. II, p. 182), we obtain easily

$$I_2 \leqq 2^{p-1} \left\{ \left| \frac{1}{\pi} \int_{1/n}^{\pi} \Phi_x(t) \frac{\sin kt}{2\,\mathrm{tg}\, t/2}\, dt \right|^p + \left| \frac{1}{2\pi} \int_{1/n}^{\pi} \Phi_x(t) \cos kt\, dt \right|^p \right\} \leqq 2^{p-1}\{|b_k|^p + (2M)^p\}$$

where b_k denotes the k-th sine Fourier coefficient of the function

$$g(t) = \begin{cases} \dfrac{\Phi_x(t)}{2\,\mathrm{tg}\, t/2} & \left(\dfrac{1}{n} \leqq t \leqq \pi \right) \\ 0 & \left(0 \leqq t < \dfrac{1}{n} \right). \end{cases}$$

Hence collecting the terms of I_1 and I_2 we obtain

$$\frac{1}{n} \sum_{k=\nu_n}^{n+\nu_n-1} |S_k(f)-f|^p \leqq K_1 M^p + \frac{K_2}{n} \sum_{k=\nu_n}^{n+\nu_n-1} |b_k|^p$$

where K_1, K_2, K_3, K_4 and K_5 denote absolute constants.

Choose now q such that $\frac{1}{p}+\frac{1}{q}=1$. Since $p \geqq 2$, we have $1 < q \leqq 2$. Hence applying the Hausdorff—Young theorem (cf. [4] Vol. II, p. 101), we obtain

$$\sum_{k=\nu_n}^{n+\nu_n-1} |b_k|^p \leqq \left(\frac{1}{\pi} \int_{1/n}^{\pi} |g(t)|^q\, dt \right)^{p/q} = \left(\frac{1}{\pi} \int_{1/n}^{\pi} \left| \frac{\Phi_x(t)}{2\,\mathrm{tg}\, t/2} \right|^q dt \right)^{p/q} \leqq$$

$$\leqq \left(\frac{(4M\pi)^q}{\pi} \cdot \frac{n^{q-1}}{q-1} \right)^{p/q} = K_3 M^p n^{p(q-1)/q}.$$

Consequently

(4) $$\frac{1}{n} \sum_{k=\nu_n}^{n+\nu_n-1} |b_k|^p \leqq K_3 M^p n^{\frac{p(q-1)-q}{q}}$$

which reduces to the constant due to hypothesis $p(q-1)-q=0$. Therefore,

$$\frac{1}{n}\sum_{k=\nu_n}^{n+\nu_n-1} |S_k(f)-f|^p \leqq K_4 M^p$$

and by

$$\left\{\frac{1}{n}\sum_{k=\nu_n}^{n+\nu_n-1} |S_k(f)|^p\right\}^{1/p} \leqq \left\{\frac{1}{n}\sum_{k=\nu_n}^{n+\nu_n-1} |S_k(f)-f|^p\right\}^{1/p} + K_5 M$$

we deduce indeed

$$\left\{\frac{1}{n}\sum_{k=\nu_n}^{n+\nu_n-1} |S_k(f)|^p\right\}^{1/p} \leqq KM$$

where K is an absolute positive constant and this completes the proof of our theorem.

Finally I like to express my sincere thanks to Professor G. Alexits for his suggestions and improvements.

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BEHANDLUNG EINES WHITTAKERSCHEN MEHR-PUNKTE-PROBLEMS MIT PERRONSCHEN SUMMENGLEICHUNGEN

Von

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Herrn Prof. Dr. Wolfgang R. Wasow zum 65. Geburtstag am 25. Juli 1974 gewidmet

1. Einleitung

Wir behandeln hier die folgende Aufgabe für ganze Funktionen.

Es seien in der komplexen Zahlenebene paarweise verschiedene Punkte A, $B_1, \dots, B_k$ gegeben. Ferner seien drei Folgen komplexer Zahlen $\{a_n\}_{n=0,1,2,\dots}$, $\{b_n\}_{n=0,1,2,\dots}$ und $\{c_n\}_{n=1,2,\dots,k}$ gegeben. Gesucht ist eine ganze Funktion f, welche den Bedingungen

$$\text{(1)} \qquad \sum_{\varkappa=1}^{k} c_\varkappa f^{(2n)}(B_\varkappa) = b_n, \quad f^{(2n)}(A) = a_n \quad (n = 0, 1, 2, \dots)$$

genügt.

Die Aufgabe stellt eine Verallgemeinerung einer von WHITTAKER behandelten Interpolationsaufgabe ($k=1$) dar (vgl. [13] Theorem 1, S. 455 und [14]). In Anlehnung an seine Terminologie bezeichen wir obige Aufgabe als „Mehr-Punkte-Problem". Whittakers ursprüngliche Aufgabe wurde später von PÓLYA [7], PORITSKY [8], SCHOENBERG [9], VERMES [10, 11, 12] und WHITTAKER [15] selbst wieder aufgegriffen und in verschiedener Weise verallgemeinert.

Ein seher spezielles Mehr-Punkte-Problem, welches in etwa unserer Aufgabe entspricht, findet man bei PORITSKY (vgl. [8] Theorem 11a, S. 308—309). Während die genannten Aufgaben üblicherweise mit funktionentheoretischen Methoden gelöst werden, zieht VERMES [10, 11] funktionalanalytische Mittel heran. Er geht von den Taylorentwicklungen von $f^{(2n)}(B_1)$ um den Punkt A aus und wird dabei auf ein unendliches lineares Gleichungssystem für die Ableitungen $f^{(2n+1)}(A)$ geführt. Dieses löst er unter Verwendung von Sätzen DIENES [2] über zeilenfinite Matrizen.

Wir gehen im folgenden ähnlich wie Vermes vor, nützen aber aus, daß für unsere Aufgabe das auftretende Gleichungssystem seinem Typ nach zu den sog. Summengleichungen mit schwach variierenden Koeffizienten gehört. Über deren Lösungen gibt aber die Theorie von PERRON [5] und PAASCHE [4] Auskunft. Dieses einfache und wenig aufwendige Vorgehen liefert dann neben hinreichenden Bedingungen für die Existenz von Lösungen auch Aussagen über ihre Anzahl sowie über ihre Ordnung und ihren Typ als ganze Funktionen. Darüber hinaus gestattet es noch, einige Verallgemeinerungen der gestellten Aufgabe zu lösen.

Eine andersartige Aufgabe, welche sich mit der gleichen Methode lösen läßt, findet man in [6].

Einen Hinweis auf unser Vorgehen kann mann schließlich KAZMIN [3] Korollar 5, S. 224 entnehmen.

2. Eine hinreichende Lösbarkeitsbedingung

Die Anwendung der Theorie der Perronschen Summengleichungen auf unser Problem erfordert die Betrachtung der Funktion

$$F(z) = \frac{1}{\sqrt{z}} \cdot \sum_{\varkappa=1}^{k} c_\varkappa \sinh\left[(B_\varkappa - A)\sqrt{z}\right]. \tag{2}$$

Man überlegt sich leicht, daß F entweder identisch verschwindet, was im folgenden stets ausgeschlossen sein soll, oder eine ganze Funktion von z der Ordnung $\varrho=\frac{1}{2}$ ist. Als solche besitzt sie unendlich viele Nullstellen (vgl. etwa BIEBERBACH [1], S. 242). Wir werden die Anzahl der Nullstellen von F, mehrfache gemäß ihrer Vielfachheit gezählt, in der Kreisscheibe $|z| \leqq d$ mit $\alpha(d)$ bezeichnen. Abgesehen von kleineren Werten von d ist dann stets $\alpha(d) > 0$. Weiter setzen wir zur Abkürzung

$$A_n = b_n - \sum_{\nu=0}^{\infty}\left[\sum_{\varkappa=1}^{k} c_\varkappa \frac{(B_\varkappa - A)^{2\nu}}{(2\nu)!}\right] a_{n+\nu} \qquad (n = 0, 1, 2, \ldots) \tag{3}$$

und beweisen mit diesen Bezeichnungen

SATZ 1. *Es seien die folgenden Bedingungen erfüllt:*

$$\lim_{n\to\infty} \sqrt[2n]{\frac{|\alpha_n|}{(2n)!}} = 0 \quad (\alpha_n = a_n \text{ bzw. } \alpha_n = b_n), \tag{4}$$

$$\sum_{\varkappa=1}^{k} c_\varkappa (B_\varkappa - A) \neq 0, \tag{5}$$

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n|} = d < \infty. \tag{6}$$

Ist dann $\alpha(d) \geqq 0$, so gibt es stets eine ganze Funktion f, welche den Bedingungen (1) *genügt. Dabei können die Werte $f^{(2n+1)}(A)$ für $n = N+1, \ldots, N+\alpha(d)$ für genügend großes N beliebig vorgeschrieben werden und es gilt*

$$\overline{\lim_{n\to\infty}} \sqrt[2n+1]{|f^{(2n+1)}(A)|} = \sqrt{d}. \tag{7}$$

BEWEIS. Wir zeigen die Existenz von f in mehreren Schritten.

a) Zunächst folgt aus der Hadamardschen Formel für den Konvergenzradius einer holomorphen Funktion, daß die Bedingung (4) notwendig ist.

b) Zieht man die Taylorentwicklung

$$f^{(2n)}(B_\varkappa) = \sum_{\nu=0}^{\infty} \frac{(B_\varkappa - A)^\nu}{\nu!} f^{(2n+\nu)}(A) \qquad (n = 0, 1, 2, \ldots)$$

heran, so erhält man nach etwas Rechnung das mit (1) äquivalente Gleichungssystem

$$\sum_{\nu=0}^{\infty}\left[\sum_{\varkappa=1}^{k} c_\varkappa \frac{(B_\varkappa - A)^{2\nu+1}}{(2\nu+1)!}\right] f^{(2n+2\nu+1)}(A) = A_n \qquad (n = 0, 1, 2, \ldots). \tag{8}$$

Dabei sind die A_n gemäß (3) definiert. Man kann sich nun leicht überzeugen, (vgl. PERRON [5] bzw. PAASCHE [4]) daß die Gleichungen (8) sog. Summengleichungen mit schwach variierenden Koeffizienten für die Unbekannten $f^{(2m+1)}(A)$ $(m=0, 1, 2, \dots)$ darstellen.

c) Um Aussagen über Ordnung und Typ von f zu erhalten, stützen wir uns zur Lösung von (8) auf Satz 10 von PAASCHE [4] S. 28—29. Für die Anwendung dieses Satzes ist neben (5) notwending, daß

$$d^{-1} > \overline{\lim_{\nu\to\infty}} \left[\frac{1}{(2\nu+1)!}\left|\sum_{\varkappa=1}^{k} c_\varkappa (B_\varkappa - A)^{2\nu+1}\right|\right]^{\frac{1}{\nu}} = 0,$$

also $d<\infty$ ist.

Bildet man dann die in (2) definierte Funktion

$$F(z) = \sum_{\nu=0}^{\infty}\left[\sum_{\varkappa=1}^{k} c_\varkappa \frac{(B_\varkappa - A)^{2\nu+1}}{(2\nu+1)!}\right] z^\nu$$

und ist $\alpha(d)>0$, so erhält man eine Lösung $\{f^{(2m+1)}(A)\}_{m=0,1,2,\dots}$ von (8), welche die mit (7) äquivalente Grenzbeziehung

$$\overline{\lim_{m\to\infty}} \sqrt[m]{|f^{(2m+1)}(A)|} = d$$

sowie die restlichen im Satz genannten Bedingungen erfüllt.

Ist dagegen $\alpha(d)=0$, so liefert Satz 10 von PAASCHE [4] S. 28—29 eine eindeutig bestimmte Lösung, welche ebenfalls allen Bedingungen des Satzes genügt.

d) Man rechnet jetzt leicht nach, daß einmal alle Umformungen unter b) zulässig sind und daß man zum anderen so tatsächlich eine Lösung der gestellten Aufgabe erhält.

Unser Satz erlaubt uns auch, Aussagen über Ordnung ϱ und Typ σ der Lösungsfunktion f zu machen, wenn wir die bekannten Formeln (vgl. etwa BIEBERBACH [1] S. 238)

$$\varrho = \overline{\lim_{k\to\infty}} \frac{\ln k!}{\ln k! - k \ln \sqrt[k]{|f^{(k)}(A)|}}$$

und

$$(\sigma\varrho)^{\frac{1}{\varrho}} = \overline{\lim_{k\to\infty}} \left(\frac{k}{e}\right)^{\frac{1}{\varrho}-1} \sqrt[k]{|f^{(k)}(A)|}$$

heranziehen. Als unmittelbare Folge unseres Satzes beweisen wir

KOROLLAR 2. *Für jede Lösungsfunktion f in Satz 1 gilt im Falle $d>0$*

$$\varrho \geqq 1 \quad \textit{und} \quad \sigma \geqq \begin{cases} \sqrt{d} & \textit{für} \quad \varrho = 1 \\ 0 & \textit{für} \quad \varrho > 1. \end{cases}$$

BEWEIS. Verwenden wir obige Formeln lediglich für ungeradzahligen Index k, so erhalten wir unter Verwendung von (7) untere Schranken für ϱ und σ. Man überzeugt sich nun leicht, daß $z=0$ keine Nullstelle der in (2) definierten Funktion

F ist. Damit folgt die Behauptung sofort aus den beiden Beziehungen

$$\overline{\lim_{k\to\infty}} \frac{\ln(2k+1)!}{\ln(2k+1)!-(2k+1)\ln \sqrt[2k+1]{|f^{(2k+1)}(A)|}} = 1$$

und

$$\overline{\lim_{k\to\infty}} \left(\frac{2k+1}{e}\right)^{\frac{1}{\varrho}-1} \sqrt[2k+1]{|f^{(2k+1)}(A)|} \leqq \sqrt{d}\cdot \overline{\lim_{k\to\infty}} \left(\frac{2k+1}{e}\right)^{\frac{1}{\varrho}-1}.$$

Genauere Aussagen über ϱ und σ erhält man natürlich, wenn man über (4) hinausgehende, geeignete Bedingungen an die Folge der Werte a_n stellt.

Eine weitere Frage, die sich in natürlicher Weise stellt, ist die nach einer Charakterisierung solcher Folgen von Werten a_n und b_n, welche den Bedingungen (4) und (6) in Satz 1 genügen. Wir geben hier eine einfache Bedingung an, die durch Satz 11 bei PAASCHE [4] S. 33 nahegelegt wird.

KOROLLAR 3. *Es sei*

$$(10) \qquad |a_n| \leqq Mt^{2n}, \quad |b_n| \leqq Mt^{2n}$$

mit $0<M$, $t<\infty$ *für* $n=0, 1, 2, \ldots$. *Dann sind auch die Beziehungen* (4) *und* (6) *mit* $d\leqq t^2$ *erfüllt.*

BEWEIS. Zunächst rechnet man leicht nach, daß (4) eine unmittelbare Folge von (10) ist. Weiter erhält man nach etwas Rechnung

$$|A_n| \leqq M\left[1+\sum_{\varkappa=1}^{k} |c_\varkappa| \cos h(|B_\varkappa - A|t)\right] t^{2n}$$

und daraus folgt sofort (6) mit $d\leqq t^2$.

3. Eine allgemeinere Aufgabe

In ähnlicher Weise wie wir Aufgabe (1) gelöst haben, können wir auch das etwas allgemeinere Problem behandeln, eine ganze Funktion f zu finden, welche die Gleichungen

$$(11) \qquad \sum_{\varkappa=1}^{k} c_\varkappa f^{(q_n)}(B_\varkappa) = b_n, \quad f^{(p_n)}(A) = a_n \quad (n = 0, 1, 2, \ldots)$$

erfüllt. Dabei sollen die p_n und q_n ganze Zahlen sein, welche den Bedingungen

$$(12) \qquad 0 \leqq p_0<p_n<p_{n+1}; \quad 0 \leqq q_0<q_n<q_{n+1} \quad \text{für} \quad n = 0, 1, 2, \ldots$$

genügen.

Auch diese Aufgabe wurde für den Fall, daß $k=1$ ist, von den bereits erwähnten Autoren für gewisse Klassen von Folgen $\{p_n\}_{n=0,1,2,\ldots}$ und $\{q_n\}_{n=0,1,2,\ldots}$ untersucht.

Wollen wir (11) ähnlich wie (1) mit der Methode der Perronschen Summengleichungen behandeln, so sind wir gezwungen, die Folgen $\{p_n\}_{n=0,1,2,\ldots}$ und

$\{q_n\}_{n=0,1,2,\dots}$ so zu wählen, daß die (8) entsprechenden Gleichungen ebenfalls ihrem Typ nach Perronsche Summengleichungen werden. Sind die Folgen dementsprechend gewählt, so werden wir von einem Paar „zulässiger" Folgen sprechen.

Zulässige Folgen lassen sich in großer Zahl angeben. Wir erwähnen hier das folgende leicht zu beweisende Kriterium, auf das uns H. M. Möller aufmerksam machte.

LEMMA 4. *Die (12) genügenden Folgen* $\{p_n\}_{n=0,1,2,\dots}$ *und* $\{q_n\}_{n=0,1,2,\dots}$ *bilden ein zulässiges Paar, wenn die folgenden Bedingungen erfüllt sind.*

Es gibt eine Folge ganzer Zahlen r_m $(m=0, 1, 2, \dots)$ *mit den Eigenschaften:*

a) *zu jedem* q_n *gibt es genau ein* r_n *mit* $q_n \leqq r_n < q_{n+1}$

b) $\{p_n | n \geqq 0\} \cap \{r_m | m \geqq 0\} = \emptyset$

c) $\{p_n | n \geqq 0\} \cup \{r_m | m \geqq 0\} = \{k | k \geqq q_0\}$.

Es sei noch darauf hingewiesen, daß die r_m gerade die Ordnungen der Ableitungen von f in den (8) entsprechenden Gleichungen darstellen. Zulässig sind beispielsweise die Folgen mit

$$p_n = 2n, \quad q_n = 2n; \quad p_n = 2n+1, \quad q_n = 2n+1;$$

$$p_n = 2n+1, \quad q_n = 2n; \quad q_n = 3n; \quad r_m = 3m+1;$$

$$q_n = n\text{-te Primzahl mit } q_0 = 3, \quad r_m = 1+q_m$$

für $n=0, 1, 2, \dots$.

Mit diesen Vorbereitungen ist es nun leicht, in vielen Fällen für die mit einem Paar zulässiger Folgen gebildete Aufgabe (11) ähnliche Ergebnisse wie die für Aufgabe (1) abgeleiteten zu beweisen. Zusätzliche Überlegungen erfordert lediglich die Ableitung des Grenzwertes (7). Er hängt nämlich wesentlich vom Verhalten von $\frac{m}{r_m}$ für $m \to \infty$ ab, bleibt aber stets endlich wegen $\frac{m}{r_m} \leqq 1$.

Freilich kann man auch leicht nicht zulässige Folgen angeben, wie etwa

$$p_n = 3n, \quad q_n = 2n; \quad p_n = 3n, \quad q_n = 3n$$

oder

$$p_n = (k+1) \cdot n, \quad q_n = (k+1) \cdot n$$

für $n=0, 1, 2, \dots$. Dabei soll $k \geqq 2$ der Parameter aus (11) sein.

Der zuletzt genannte Fall läßt sich allerdings noch — wie eine Reihe anderer, ähnlicher Fälle — mit der Methode von PÓLYA [7] lösen. Für den Hinweis auf diese Möglichkeit möchten wir dem Referenten unseren Dank aussprechen.

Zur Anwendung dieser Methode verlangen wir, daß $c_1 \neq 0$ ist in (11), was ja ohne Einschränkungen möglich ist. Dann betrachten wir anstatt (11) die folgende Aufgabe:

$$f^{(kn+n)}(A) = a_n, \quad f^{(kn+n)}(B_1) = b_n \cdot c_1^{-1}, \quad f^{(kn+n)}(B_\varkappa) = 0$$

für $\varkappa=2, 3, \dots, k$ und $n=0, 1, 2, \dots$.

Man prüft nun leicht nach, daß sämtliche Bedingungen des Satzes von PÓLYA [7] S. 130 erfüllt sind, falls man noch

$$\lim_{n\to\infty} \sqrt[(k+1)n]{\frac{|a_n|}{(kn+n)!}} = 0, \quad \lim_{n\to\infty} \sqrt[(k+1)n]{\frac{|b_n|}{(kn+n)!}} = 0$$

verlangt. Damit gibt es dann aber eine ganze Funktion f, welche die obige Aufgabe wie auch die ursprüngliche Aufgabe (11) löst. Jedoch erhalten wir auf diesem Wege keine Aussagen über Ordnung und Typ von f.

4. Ein Spezialfall

Wie bereits erwähnt, wurde der Spezialfall $k=1$ unserer Aufgabe (11), nämlich die Lösung der Gleichungen

$$f^{(p_n)}(A) = a_n, \quad f^{(q_n)}(B) = b_n \quad (n = 0, 1, 2, \dots) \tag{13}$$

mittels einer ganzen Funktion f verschiedentlich in der Literatur untersucht. So hat WHITTAKER (vgl. [13], Theorem 1, S. 455) für $A=0$, $B=1$, $p_n=q_n=2n$ $(n=0, 1, 2, \dots)$ gezeigt, daß die Konvergenz der Reihen

$$\sum_{n=0}^{\infty} (-1)^n \frac{a_n}{\pi^{2n}} \quad \text{und} \quad \sum_{n=0}^{\infty} (-1)^n \frac{b_n}{\pi^{2n}} \tag{14}$$

die Existenz einer Lösung von (13) sichert und daß sich diese mit sog. Lidstoneschen Reihen darstellen läßt.

Seine Bedingung erfaßt also nur Folgen des Typs (10) mit einem Wert $t \leqq \pi$. Später hat dann PÓLYA (vgl. [7] S. 130) die wesentlich schwächere Bedingung

$$\lim_{n\to\infty} \sqrt[2n]{\frac{|a_n|}{(2n)!}} = 0, \quad \lim_{n\to\infty} \sqrt[2n]{\frac{|b_n|}{(2n)!}} = 0 \tag{15}$$

als hinreichend für die Existenz einer Lösung nachgewiesen.

VERMES (vgl. [10] Theorem 1, S. 113 und [11] Theorem 3, S. 77, Theorem 5, S. 79) hat für sehr allgemeine Folgen von Werten p_n und q_n unsere Bedingung (10) mit $t \leqq 1$ angegeben.

Wenden wir jetzt Satz 1 auf (13) mit $p_n=q_n=2n$ $(n=0, 1, 2, \dots)$ an, so werden wir auf die Funktion

$$F(z) = \frac{1}{\sqrt{z}} \sinh\left[(B-A)\sqrt{z}\right]$$

mit den Nullstellen

$$\alpha_j = -\left(\frac{\pi \cdot j}{B-A}\right)^2 \qquad (j = 1, 2, 3, \dots)$$

geführt. Somit erhalten wir für Werte d mit $|\alpha_1| \leqq d < \infty$ Lösungen f, in denen noch $\alpha(d)$ Parameter frei verfügbar sind, während wir für Werte d mit $0 \leqq d < |\alpha_1|$ eine eindeutig bestimmte Lösung f erhalten. Damit liefert unsere Methode in den von (14) erfaßten Fällen gerade die von Whittaker angegebene Lösung.

Ziehen wir schließlich noch Satz 7 von PAASCHE [4] S. 25—26 heran, so können wir auch Aussagen über den homogenen Fall (d. h. $a_n=b_n=0$ für $n=0, 1, 2, \ldots$) von Aufgabe (11) machen. Dieser wurde allerdings bereits von SCHOENBERG [9] für $p_n=q_n=2n$ ($n=0, 1, 2, \ldots$) eingehend untersucht. Er zeigte, daß jede Lösung f vom Exponentialtyp ein ungerades trigonometrisches Polynom ist.

5. Bemerkungen

Es sei darauf hingewiesen, daß die von uns in Satz 1 erhaltenen Lösungen f in gewissem Sinne „minimal" sind. Aus Satz 8 bei PAASCHE [4] S. 27—28 folgt nämlich, daß es keine Lösung f gibt, für die der Grenzwert (7) einen kleineren Wert als $\sqrt{d}$ hat.

Schließlich zeigt sich, daß wir gemäß Lemma 4 auch sehr unregelmäßig gebaute zulässige Folgen von Werten p_n und q_n konstruieren können; neben „vollständigen" im Sinne von WHITTAKER (vgl. [13] S. 452 bzw. [10] S. 109) auch „unvollständige" mit der Eigenschaft, daß sich die zugehörigen Lösungen f von (11) nicht als Reihen von Standardpolynomen darstellen lassen.

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ЧИСЛО ДЕРЕВЬЕВ ГРАФА, СОДЕРЖАЩИХ ЗАДАННЫЙ ЛЕС

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1. Задача определения числа деревьев графа, обладающих теми или иными свойствами (или более общая задача определения суммы весов всех деревьев, если вес дерева равен произведению весов его ребер) возникает при исследовании электрических цепей, при изучении надежности информационных сетей и в других случаях.

В настоящей статье речь идет о числе деревьев графа, содержащих заданный лес.[1] Эта задача просто сводится к задаче об определении числа всех деревьев некоторого другого графа. Поэтому способ определения числа деревьев графа, восходящий к Кирхгоффу и связанный с вычислением некоторого определителя (см., напр., [1]), пригоден и в этом случае. Наоборот, если для любого леса известно число деревьев полного графа, содержащих этот лес, то метод включения и исключения дает принципиальную возможность определить число деревьев любого графа [2, 3]. Однако некоторые классы графов допускают более простые решения. Так, в целом ряде работ приводятся формулы числа деревьев для различных серий графов, и более того, даются алгоритмы выписывания таких формул для определенных классов графов [3—9 и многие другие].

Что же касается задачи о числе деревьев графа, содержащих заданный лес, то пока выделено немного классов графов, для которых выписываются соответствующие формулы. А именно, было показано [2, 3], что число деревьев $T(K_n, F^k)$ полного графа K_n, содержащих лес F^k с k ребрами, равно

(1) $$T(K_n, F^k) = \gamma(F^k)\cdot n^{n-2-k} = \gamma(F^k)\cdot n^{-k}\cdot T(K_n),$$

где $\gamma(F^k)$ есть произведение чисел вершин всех $(n-k)$ компонент леса F^k, $T(K_n)=n^{n-2}$ — число деревьев полного графа K_n.

В [10] доказано, что если граф L получается из полного графа K_t удалением всех ребер, соединяющих вершины множества A с вершинами множества B ($A\cap B=\varnothing$; $A, B\subset K_t$), то число деревьев $T(L, H)$ графа L, содержащих дерево H на вершинах A, равно

(2) $$T(L, H) = a\cdot t^{t-a-b-1}(t-a)^{b-1}\cdot(t-b-a)$$

где $a=|A|$, $b=|B|$.

В настоящей работе показано, как для графов G некоторого класса получать формулы для доли $\dfrac{T(G,F)}{T(G)}$ деревьев, содержащих заданный лес F. Это

[1] Настоящая работа была доложена на Семинаре проф. А. А. Зыкова по теории графов и гиперграфов в сентябре 1972 года.

позволяет к тому же получать формулы числа деревьев различных серий графов.

Кроме того, указана точная нижняя оценка числа деревьев графа, содержащих заданный лес с одной нетривиальной компонентной. Из этой оценки вытекает экстремальность графов некоторого класса по числу деревьев.

2. Будем рассматривать неориентированные графы без петель, но быть может, с кратными ребрами [1].

Припишем каждой вершине x графа G натуральное число $p(x)$ — вес вершины x. Получившееся образование будем называть взвешенным графом и обозначать через G^p. Пусть при этом $\Sigma_G = \sum_{x \in G} p(x)$ — вес графа G; $|G|$ — число вершин G; $\Pi_G = \prod_{x \in G} p(x)$. Граф, состоящий всего из одной вершины веса s, будем обозначать через g^s.

Пусть $\Gamma^p L^p$ есть граф, который содержит все элементы графов Γ и L, и в котором каждые две вершины $x \neq y$, $x \in \Gamma$, $y \in L$ соединены дополнительно $p(x) \cdot p(y)$ ребрами. Если Φ — граф без ребер, то $\Phi^p \cdot \Phi^p = P$ есть граф, в котором каждые две вершины $x \neq y$ соединены $p(x) \cdot p(y)$ кратными ребрами.

Очевидно, $\Gamma^p L^p = L^p \Gamma^p$ и $(\Gamma^p L^p) R^p = \Gamma^p (L^p R^p)$. Поэтому вместо $(\Gamma^p L^p) R^p$ будем писать $\Gamma^p L^p R^p$. Так например, $P = \Phi^p \cdot \Phi^p = g_1^{p_1} \cdot g_2^{p_2} \cdot \ldots \cdot g_n^{p_n}$, где $n = |P|$ — число вершин P.

В дальнейшем указанная операция будет применяться к непересекащимся графом.

Пусть $Y_1, Y_2, \ldots, Y_k$ — попарно не пересекающиеся подмножества вершин (или подграфы) графа G. Обозначим через $G^p_{Y_1, Y_2, \ldots, Y_k}$ взвешенный граф, который получается из G^p, если каждое из множеств Y_i, $i = 1, 2, \ldots, k$, отождествить в одну вершину z_i (удалив получившиеся при этом петли) и приписать этой вершине вес $p(z_i) = \sum_{x \in Y_i} p(x)$. Соответствующий граф без весов будем обозначать $G_{Y_1, Y_2, \ldots, Y_k,}$. Заметим, что графы G^p и $G^p_{Y_1, Y_2, \ldots, Y_k}$ имеют один и тот же вес Σ_G. Очевидно,

$$\Gamma^p_Y L^p = (\Gamma^p L^p)_Y. \tag{3}$$

Два дерева графа будем считать различными, если различны множества их ребер. Пусть, как и выше, $T(G)$ есть число различных деревьев графа G; $T(G, F)$ — число деревьев G, содержащих лес F. При этом рассматриваются лишь такие деревья графа, которые содержат все его вершины.

Пусть $B(G) = \|b_{ij}\|$ — матрица, у которой b_{ii} равно числу ребер, инцидентных вершине x_i, а $-b_{ij}$, $i \neq j$, равно числу кратных ребер, соединяющих вершины x_i и x_j в G. Очевидно, $\det B(G) = 0$.

3. Если лес F графа G имеет компоненты $V_1, V_2, \ldots, V_k$, то, очевидно, $T(G, F) = T(G_{V_1, V_2, \ldots, V_k})$. Поэтому нам предстоит изучение числа деревьев графа при различных отождествлениях подмножеств его вершин.

Пусть граф L состоит из двух вершин u и v, соединенных $c(u, v)$ кратными ребрами; $G = R^p L^p$; z — вершина в $G_{(u, v)}$, получающаяся в результате отождествления вершин u и v в G; $\Sigma_R = \sum_{x \in R} p(x)$.

Лемма 1.

$$\frac{T(G)}{T(G_{(u,v)})} = \frac{p(u)\cdot p(v)\cdot \Sigma_R}{p(u)+p(v)} + c(u, v).$$

Доказательство. Пусть матрица $B_{x,y,\dots}(\Gamma)$ получается из $B(\Gamma)$ удалением строк и столбцов, соответствующих вершинам $x, y, \dots$ графа Γ. Так как $\det B_x(\Gamma) = T(\Gamma)$ [1], то

$$\det B_u(G) = T(G); \quad \det B_{u,v}(G) = \det B_z(G_{(u,v)}) = T(G_{(u,v)}). \tag{4}$$

Образуем матрицу $B'_u(G)$ из $B_u(G)$ умножением строки и затем столбца, соответствующих вершине v, на $\frac{p(u)+p(v)}{p(v)}$. Тогда

$$\det B'_u(G) = \left(\frac{p(u)+p(v)}{p(v)}\right)^2 \det B_u(G) = \tag{5}$$

$$= \left\{\left(\frac{p(u)+p(v)}{p(v)}\right)^2 \big(p(v)\cdot \Sigma_R + c(u, v)\big) - \big(p(u)+p(v)\big)\Sigma_R\right\} \det B_{u,v}(G) + \det B(G_{(u,v)}).$$

Так как $\det B(G_{(u,v)}) = 0$, то из (4) и (5) получаем требуемое утверждение.

Пусть $\{y_1, y_2, \dots, y_n\}$ — множество вершин графа $P = \Phi^p \cdot \Phi^p$, $p(y_i) = p_i$; $|P| = n$.

Лемма 2.

$$T([\Gamma^p P^p]_P) = \frac{\Sigma_P}{\Pi_P}[\Sigma_P + \Sigma_\Gamma]^{1-|P|}\cdot T(\Gamma^p P^p). \tag{6}$$

Доказательство. Пусть G_i и R_i — графы, получающиеся из $\Gamma^p P^p$ соответственно отождествлением и удалением (вместе с инцидентными ребрами) первых i вершин из P. Очевидно, $\Gamma^p P^p = G_0$; $T(\Gamma^p P^p) = T(G_0)$; $T([\Gamma^p P^p]_P) = T(G_n)$. Обозначим $\Sigma_i = \sum_{k=1}^{i} p_k$, $\Sigma^i = \sum_{k=i}^{n} p_k$, так что $\Sigma_i + \Sigma^{i+1} = \Sigma_P$. Так как $G_{i+1} = R^p_{i+1}\cdot g^{\Sigma_{i+1}}$ и $G_i = R^p_{i+1}\cdot g_1^{\Sigma_i}\cdot g_2^{p_{i+1}}$, то положив в лемме 1 $p(u) = \Sigma_i$, $p(v) = p_{i+1}$, $c(u, v) = \Sigma_i p_{i+1}$, $\Sigma_R = \Sigma_\Gamma + \Sigma^{i+2}$, получим:

$$\frac{T(G_{i+1})}{T(G_i)} = \frac{\Sigma_{i+1}}{p_{i+1}\Sigma_i[\Sigma_\Gamma + \Sigma_P]} \qquad (i = 1, 2, \dots, n-1).$$

Поэтому

$$\frac{T(G_n)}{T(G_1)} = \prod_{i=1}^{n-1}\frac{T(G_{i+1})}{T(G_i)} = \frac{\Sigma_P}{\Pi_P}(\Sigma_P + \Sigma_\Gamma)^{1-|P|},$$

что и требовалось доказать.

Заметим, что формула (2) может быть получена из (6). Действительно, граф L представим в виде: $L = \Gamma^p P^p$, где $\Gamma = K_{t-a}$, $P = K_a$, Γ и P не пересекаются, и $p(x) = 0$, если $x \in B \subset K_{t-a}$; $p(x) = 1$, если $x \in K_{t-a} \setminus B$; $p(y) = 1$ для $y \in P$. Поэтому согласно (6)

$$T(L, H) = T([\Gamma^p P^p]_P) = a(t-b)^{1-a}\cdot T(L).$$

Пользуясь методом из [6] (или следствием 2.4, когда G есть дерево с тремя вершинами) получим

$$T(L) = t^{t-a-b-1}(t-b)^{a-1}(t-a)^{b-1}(t-a-b).$$

Из последних двух соотношений вытекает (2).

Пусть F^s — лес с s ребрами в графе P, $\gamma(F^s, P)$ — произведение весов компонент леса F^s в P.

Теорема 1.

$$T(\Gamma^p P^p, F^s) = \frac{\gamma(F^s, P)}{\Pi_P}(\Sigma_{\Gamma \cup P})^{-s} \cdot T(\Gamma^p P^p). \tag{7}$$

Доказательство. Пусть лес F^s имеет компоненты с множествами вершин $V_1, V_2, \ldots, V_k$ и с числами вершин $v_1 = |V_1|, v_2 = |V_2|, \ldots, v_k = |V_k|$, где $k = |P| - s$, так что $\sum_{i=1}^{k} v_i = |P| = n$. Пусть, кроме того, $G_0 = \Gamma^p P^p$; $G_i = (\Gamma^p P^p)_{V_1, \ldots, V_i}$; $R_i = G_0 \setminus V_i$; $\mathscr{V}_i$ — подграф G на вершинах V_i. Тогда $G_0 = \mathscr{V}_i{}^p R_i^p$; $T(G_0, F^{n-k}) = T(G_k)$. Согласно (3) $G_i = (\mathscr{V}_{i+1}^p R_{i+1}^p)_{V_1, \ldots, V_i} = \mathscr{V}_{i+1}^p (R_{i+1})^p_{V_1, \ldots, V_i}$. Так как $\Sigma_{\mathscr{V}_{i+1}} + \Sigma_{R_{i+1}} = \Sigma_{G_0}$ и так как графы $(R_{i+1})^p_{V_1, \ldots, V_i}$ и R_{i+1}^p имеют один и тот же вес, то по лемме 2

$$\frac{T(G_{i+1})}{T(G_i)} = \frac{\Sigma_{\mathscr{V}_{i+1}}}{\Pi_{\mathscr{V}_{i+1}}}(\Sigma_{G_0})^{1-V_{i+1}} \qquad (i = 0, 1, \ldots, k-1).$$

Поэтому

$$\frac{T(G_k)}{T(G_0)} = \prod_{i=0}^{k-1} \frac{T(G_{i+1})}{T(G_i)} = \frac{\prod_{i=1}^{k} \Sigma_{\mathscr{V}_i}}{\Pi_P}(\Sigma_{G_0})^{-s},$$

что и требовалось доказать.

Заметим, что формула (1) получается из теоремы 1, когда Γ — полный граф и веса $p(x)$ всех вершин равны 1, т. к. $T(K_n) = n^{n-2}$.

Рассмотрим граф G без кратных ребер с множеством вершин $Z = \{z_1, \ldots, z_n\}$. Пусть $D_k = \{i | (z_k, z_i)$ — ребро в G или $z_i = z_k\}$; $P_k = \Phi^{p_k} \Phi^{p_k}$ — граф, в котором любые две вершины $x \neq y$ соединены $p_k(x) \cdot p_k(y)$ кратными ребрами, $p_k(x)$ — положительные целые числа.

Построим по графу G и графам $P_1, P_2, \ldots, P_n$ «расширенный» граф $\Gamma = \Gamma(G, P_1, P_2, \ldots, P_n)$ следующим образом:

а) каждая вершина z_k в G заменяется на P_k,

б) каждое ребро (z_i, z_j) в G заменяется на $P_i^{p_i} \cdot P_j^{p_j}$, т. е. любая вершина x из P_i соединяется с любой вершиной y из P_j $p_i(x) \cdot p_j(y)$ кратными ребрами. Это означает, что граф $\Gamma_{P_1, P_2, \ldots, P_n}$ получается из G заменой ребра (z_i, z_j) на $\Sigma_{P_i} \cdot \Sigma_{P_j}$ кратных ребер.

Теорема 2.

$$\frac{T(\Gamma_{P_1, P_2, \ldots, P_n})}{T(\Gamma)} = \prod_{k=1}^{n} \frac{\Sigma_{P_k}}{\Pi_{P_k}} \Big(\sum_{i \in D_k} \Sigma_{P_i}\Big)^{1-|P_k|}.$$

Доказательство. Пусть $\Gamma_k = \Gamma \setminus P_k$. Тогда для любого $k = 1, 2, \dots, n$ $\Gamma = \Gamma_k^{s_k} P_k^{p_k}$, где для $x \in P_i \subset \Gamma_k$

$$s_k(x) = \begin{cases} p_i(x), & \text{если } i \in D_k; \\ 0, & \text{если } i \notin D_k. \end{cases}$$

Поэтому, как и в доказательстве теоремы 1, последовательное применение леммы 2 и соотношения (1) дает требуемое утверждение.

При некоторых дополнительных ограничениях можно установить связь между числом деревьев графа $\Gamma = \Gamma(G, P_1, P_2, \dots, P_n)$ и числом деревьев графа G.

Следствие 2.1. *Пусть G — граф с n вершинами, $\Gamma = \Gamma(G, P_1, \dots, P_n)$ и $\Sigma_{P_i} = \tau$, $i = 1, 2, \dots, n$. Тогда*

$$T(\Gamma) = \left\{ \tau^{|\Gamma|-2} \prod_{k=1}^{n} \Pi_{P_k}(d_k+1)^{|P_k|-1} \right\} \cdot T(G),$$

где d_k — степень вершины z_k в G.

Доказательство. Если $\Sigma_{P_i} = \tau$ для $i = 1, 2, \dots, n$, то $\Gamma_{P_1, P_2, \dots, P_n}$ получается из G заменой каждого ребра на τ^2 кратных ребер. Поэтому $T(\Gamma_{P_1, P_2, \dots, P_n}) = \tau^{2(n-1)} T(G)$, и положив в теореме 2 $\Sigma_{P_i} = \tau$, получим требуемое утверждение.

Пусть $G \times K_\tau$ есть декартово произведение графа G и полного графа K_τ, т. е. граф, вершины которого соответствуют парам (x, y), $x \in G$, $y \in K_\tau$, и две вершины $(x_1, y_1) \neq (x_2, y_2)$ смежны, если x_1 и x_2 смежны или равны в G, а также y_1 и y_2 смежны или равны в K_τ. Тогда $G \times K_\tau = \Gamma(G, P_1, P_2, \dots, P_n)$, где P_i, $i = 1, 2, \dots, n$, есть полный граф с τ вершинами. Поэтому, если в следствии 2.1 положить $p_k(x) = 1$, $x \in P_k$, $k = 1, 2, \dots, n$, то получим

Следствие 2.2.

$$T(G \times K_\tau) = \left\{ \tau^{\tau n - 2} \prod_{k=1}^{n} (d_k+1)^{\tau-1} \right\} \cdot T(G),$$

где, как и выше $|G| = n$.

Пусть R — двудольный граф, т. е. граф, все вершины которого можно разбить на два класса X_1 и X_2 так, что вершины из X_i $(i = 1, 2)$ попарно не смежны; $|X_i| = n_i$; $|R| = n$.

Следствие 2.3. *Пусть $\Gamma = \Gamma(R, P_1, P_2, \dots, P_n)$; $\Sigma_{P_z} = \tau_1$, если $z \in X_1$ и $\Sigma_{P_z} = \tau_2$, если $z \in X_2$. Тогда*

$$T(\Gamma) = \tau_1^{n_2-1} \cdot \tau_2^{n_1-1} \left\{ \prod_{z \in X_1} \Pi_{P_z}(\tau_1 + \tau_2 d_z)^{|P_z|-1} \right\} \times \left\{ \prod_{z \in X_2} \Pi_{P_z}(\tau_2 + \tau_1 d_z)^{|P_z|-1} \right\} \cdot T(R).$$

Получается из теоремы 2, если заметить, что

$$T(\Gamma_{P_1, P_2, \dots, P_n}) = (\tau_1 \tau_2)^{n-1} \cdot T(R).$$

Таким образом, приведеные выше результаты позволяют не только получать в ряде случаев формулы для доли деревьев графа, содержащих заданный лес. Они к тому же дают возможность выписывать формулы числа деревьев для различных серий графов.

Вот еше один пример. Пусть каждый блок графа G есть либо цикл, либо ребро, и пусть $C_1, C_2, \dots, C_k$ — все блоки-циклы а $b_1, b_2, \dots, b_\tau$ — все блокиребра графа G. Если ребро u цикла C заменить s_u кратными ребрами, то число деревьев получившегося графа $C(s)$ равно

$$T(C(s)) = (\prod_{u\in C} s_u) \sum_{u\in C} \frac{1}{s_u}.$$

Поэтому для графа $\Gamma = \Gamma(G, P_1, \ldots, P_n)$ получим из теоремы 2

Следствие 2.4.

$$T(\Gamma) = \prod_{z\in G} \frac{\Pi_{P_z}}{\Sigma_{P_z}} (\sum_{y\in D_z} \Sigma_{P_y})^{|P_z|-1} \left\{\prod_{i=1}^{k} (\prod_{u\in C_i} s_u) \left(\sum_{u\in C_i} \frac{1}{s_u}\right)\right\} \cdot \prod_{j=1}^{\tau} s_{b_j},$$

где для ребра $u=(x, y)$ *из* G $s(u) = \Sigma_{P_x}\cdot\Sigma_{P_y}$.

4. Выше были выделены некоторые условия, при которых доля деревьев графа содержащих заданный лес, определяется достаточно просто. В общем случае картина, конечно, сложнее. Поэтому возникает вопрос, в каких пределах заключена доля деревьев с указанными свойствами для произвольного графа.

Рассмотрим случай, когда лес F имеет единственную, компоненту V с числом вершин $|V| \geqq 2$. Очевидно, $\frac{T(G, F)}{T(G)} \leqq 1$, и если каждое ребро F есть перешеек графа G, то $\frac{T(G, F)}{T(G)} = 1$.

Пользуясь изложенными выше результатами и результатами работы [12], можно получить следующую нижнюю оценку для $\frac{T(G, F)}{T(G)}$.

Пусть $t = t(G, F)$ — число ребер графа G, оба конца которых принадлежат V; $l = l(G, F)$ — число ребер G, ровно один конец которых принадлежит V; $\vartheta = \vartheta(F) = |V| \geqq 2$.

Теорема 3.

$$\frac{T(G, F)}{T(G)} \geqq \frac{\vartheta}{\left(\frac{2t}{\vartheta-1} + \frac{e}{\vartheta}\right)^{\vartheta-1}},$$

причем оценка достигается тогда и только тогда, когда G *представим в виде* $R^p P^p$, *где множество вершин* P *есть* V, $p(x) = p(y)$, *для* $x, y \in V$ (см. лемму 2).

Если F имеет лишь одно ребро $u = (x, y)$, то получаем:

Следствие 3.1. *Пусть* d_x — *степень вершины* x, *а* $c(x, y)$ — *число ребер, соединяющих вершины* x, y *в* G. *Тогда*

$$T(G_{(x,y)}) \geqq \frac{4}{d_x + d_y + 2c(x, y)} \cdot T(G)$$

причем оценка достигается тогда и только тогда, когда граф G *представим в виде* $G = \Gamma^p(x, y)^p$ *и* $p(x) = p(y)$ (см. лемму 1).

Пусть граф $\overline{N}_n^k$ получается из K_n удалением k попарно не смежных ребер, $0 \leqq k \leqq \frac{n}{2}$.

Следствие 3.2. *Для любого графа G с n вершинами и m ребрами*

$$T(G) \leqq n^{n-2}\left(1-\frac{2}{n}\right)^{\overline{m}}$$

где $\overline{m}=\binom{n}{2}-m$. *Равенство имеет место тогда и только тогда, когда* $G=\overline{N}_n^k$.

Это утверждение было ранее доказано в [11].

Следствие 3.3. *Для любого графа G с n вершинами*

$$\sum_{(x,y)\in G}\frac{c(x,y)}{d_x+d_y+2c(x,y)} \leqq \frac{n-1}{4},$$

где суммирование ведется по всем $\binom{n}{2}$ *неупорядоченным парам вершин из G. Равенство имеет место тогда и только тогда, когда в G любые две вершины x, y соединены одним и тем же числом ребер* $c(x,y)=c>0$.

В заключение заметим, что полученные выше результаты верны также для графов со взвешенными ребрами, когда вес дерева есть произведение весов его ребер, а $T(G)$ есть сумма весов деревьев. При этом веса вершин, рассматриваемые выше, следует считать произвольными числами, а не обязательно целыми.

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ON THE GROWTH RATE OF WEIGHTED AVERAGES OF EXPONENTIAL RANDOM VARIABLES

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Throughout this note let, for each $n \geqq 1$, $Y_{n1}, Y_{n2}, \ldots, Y_{nn}$ denote a sequence of independent identically distributed (iid) random variables (rv) each with the exponential distribution function

$$F(x) = \begin{cases} 1-e^{-x} & x > 0 \\ 0 & x \leqq 0. \end{cases}$$

Define $S_n = \sum_{k=1}^{n} Y_{nk}/k$. This note investigates certain aspects of the limiting behaviour of S_n.

Let $X_1, X_2, \ldots$ be iid rv with common distribution function $F(x)$ as above, and let $M_n = \max(X_1, X_2, \ldots, X_n)$. Rényi [3] has shown that, for each $n \geqq 1$, iid exponential rv $\delta_{n1}, \delta_{n2}, \ldots, \delta_{nn}$ exist satisfying $M_n = \sum_{k=1}^{n} \delta_{nk}/k$. Much is known about the behaviour of M_n (e.g. see [1] and [5]). In view of Rényi's result, one is moved to compare the limiting behaviour of S_n with that of M_n.

The simplest form for S_n occurs when $Y_1, Y_2, \ldots$ are iid exponential rv and one defines $Y_{nk} = Y_k$ for all $n \geqq 1$ and $k \leqq n$. In this case the sum corresponding to S_n is $T_n \equiv \sum_{k=1}^{n} Y_k/k$. Now $\lim_{n\to\infty} \operatorname{Var}(T_n) = \sum_{k=1}^{\infty} k^{-2} < \infty$ so $\lim_{n\to\infty} (T_n - ET_n)$ exists and is finite a.s. by the Kolmogorov Convergence Theorem ([2], p. 236). It is well-known that $ET_n - \log n \downarrow \gamma > 0$ (where γ is Euler's constant). From these facts if follows that $\lim_{n\to\infty} T_n/\log n = 1$ a.s. and

$$\lim_{n\to\infty} \frac{T_n - \sum_{m=1}^{N} \log_m n}{\log_{N+1} n} = \begin{cases} 0 & \text{if } N = 1 \\ -\infty & \text{if } N > 1, \end{cases}$$

where $\log_1 x \equiv \log x$ and $\log_{m+1} x \equiv \log(\log_m x)$ for $m \geqq 1$.

However, Resnick [4] has observed that it is evident from the fact that $M_n \geqq b_n$ infinitely often (i.o.) if and only if $X_n \geqq b_n$ i.o. for any real sequence $b_n \to \infty$ and from the Borel Zero—One Law that for each integer $N \geqq 1$,

$$\limsup_{n\to\infty} \frac{M_n - \sum_{m=1}^{N} \log_m n}{\log_{N+1} n} = 1 \quad \text{a.s.,}$$

since

$$\sum_{n=1}^{\infty} (n \log n \ldots \log_N n)^{-1} (\log_{N+1} n)^{-x} < \infty$$

if and only if $x > 1$. This rules out the possibility that the converse to Rényi's theorem is valid; i.e. given S_n as defined above, it is not always possible to find an iid exponential sequence $\{Z_n\}$ such that $S_n = \max(Z_1, \ldots, Z_n)$ for all n.

Nevertheless, we can establish some evidence as to the behaviour of S_n with the following theorem.

THEOREM. *For each* $n \geqq 1$, *let* $Y_{n1}, Y_{n2}, \ldots, Y_{nn}$ *be iid exponential rv and write* $S_n = \sum_{k=1}^{n} Y_{nk}/k$. *Then* $\liminf_{n\to\infty} \frac{S_n}{\log n} = 1$ *a.s. and* $1 \leqq \limsup_{n\to\infty} \frac{S_n}{\log n} \leqq 2$ *a.s.*

Moreover, this result cannot be improved without stregthening the hypotheses.

PROOF. Using either Rényi's result mentioned above or a momentgenerating function argument, one discovers that, for all $x > 0$, $\mathbf{P}[S_n \leqq x] = (1-e^{-x})^n$; i.e. S_n and M_n (as defined earlier) are identically distributed. For any number $\sigma > 0$,

$$\mathbf{P}[|S_n/\log n - 1| \leqq \sigma] = \mathbf{P}[(1-\sigma)\log n \leqq S_n \leqq (1+\sigma)\log n] =$$

$$= (1-n^{-1-\sigma})^n - (1-n^{-1+\sigma})^n \to 1 - 0 = 1.$$

So $S_n/\log n \to 1$ in probability. Hence a sequence $\{n_j\}$ of positive integers exists for which $S_{n_j}/\log n_j \to 1$ a.s., so that $\liminf S_n/\log n \leqq 1$ a.s. and $\limsup S_n/\log n \geqq 1$ a.s.

For $\varepsilon > 0$ define the events $A_n = [S_n < (1-\varepsilon)\log n]$ and $B_n = [S_n > (1+\varepsilon)\log n]$. By the Borel—Cantelli lemma the theorem will be established if $\sum_{n=1}^{\infty} \mathbf{P}[A_n] < \infty$ for all $\varepsilon > 0$ and $\sum_{n=1}^{\infty} \mathbf{P}[B_n] < \infty$ for all $\varepsilon > 1$.

It is well-known that $1 - t < e^{-t}$ if $0 < t < 1$. Therefore,

$$\sum_{n=1}^{\infty} \mathbf{P}[A_n] = \sum_{n=1}^{\infty} (1-n^{-1+\varepsilon})^n < \sum_{n=1}^{\infty} \exp\{-n^{\varepsilon}\} < \infty.$$

For the other series, note that $(1-n^{-1-\varepsilon})^n \to 1$ for all $\varepsilon > 0$ and use the fact that $-\log(1-x) \sim x$ as $x \to 0$ twice to get $\mathbf{P}[B_n] = 1-(1-n^{-1-\varepsilon})^n \sim -n\log(1-n^{-1-\varepsilon}) \sim \sim n \cdot n^{-1-\varepsilon} = n^{-\varepsilon}$. Clearly, then $\sum \mathbf{P}[B_n] < \infty$ if $\varepsilon > 1$.

To establish the last statement of the theorem, let $X_1, X_2, \ldots$ be iid exponential rv. Let $V_n = n(n+1)/2$ for all $n \geqq 0$. Put $Y_{nk} = X_{v_{n-1}+k}$ where $n \geqq 1$ and $k \leqq n$. Defining B_n as above and again using $\mathbf{P}[B_n] \sim n^{-\varepsilon}$ we discover that $\sum \mathbf{P}[B_n] = \infty$ if $0 < \varepsilon < 1$. But the events $\{B_n\}$ are independent, so the Borel Zero—One Law applies to yield $\mathbf{P}[B_n \text{ i.o.}] = 1$. Hence $\limsup S_n/\log n = 2$ in this case.

REMARKS. From the proof of the theorem it is evident that the conclusions of the theorem apply to any sequence $\{S_n\}$ of rv satisfying $\mathbf{P}[S_n \leqq x] = (1-e^{-x})^n$.

The theorem easily generalizes to include exponential distributions with parameters other than 1.

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ON WEIGHTED POLYNOMIAL APPROXIMATION WITH A WEIGHT $(1-x)^{\alpha/2}(1+x)^{\beta/2}$ IN L_2-SPACE

By

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1. Introduction

In the present paper we are giving an extension of a previous paper (NGUYEN XUAN KY [4]) concerning weighted polynomial approximation.

Let L_2 be, as usual, the space of real square-integrable functions in $[-1, 1]$ with norm $\|\cdot\|_2$. Let $\mathscr{P}_n$ be a set of algebraic polynomials of degree not greater than n $(n=0, 1, \dots)$. Let

$$W_{(\alpha,\beta)}(x) = (1-x)^{\alpha/2}(1+x)^{\beta/2}, \quad x\in[-1, 1], \quad \alpha, \beta \geqq -1/2. \tag{1}$$

For $W_{(\alpha,\beta)}f\in L_2$, we define

$$E_n(W_{(\alpha,\beta)};f) = \inf \|(f-\Pi)\, W_{(\alpha,\beta)}\|_2, \qquad n = 0, 1, \dots. \tag{2}$$

We denote by

$$\tilde{P}_n^{(\alpha,\beta)}(x) = \gamma_n(\alpha, \beta)x^n+\dots, \quad \gamma_n(\alpha, \beta) > 0 \tag{3}$$

the n-th orthonormal polynomial with respect to the weight $W^2_{(\alpha,\beta)}(x)$. That is $\tilde{P}_n^{(\alpha,\beta)}(x)$ is the known orthonormal Jacobi-polynomial. For $W_{(\alpha,\beta)}f\in L_2$, we denote by $S(\alpha, \beta; f, x)$ the orthonormal expansion of f with respect to the system $\{\tilde{P}_n^{(\alpha,\beta)}(x)\}$ that is

$$\begin{cases} f(x) \sim S(\alpha, \beta; f, x) = \sum\limits_{k=0}^{\infty} c_k(\alpha, \beta; f)\, \tilde{P}_k^{(\alpha,\beta)}(x) \quad \text{where} \\ c_k(\alpha, \beta; f) = \int\limits_{-1}^{1} f(x)\tilde{P}_k^{(\alpha,\beta)}(x) W^2_{(\alpha,\beta)}(x)\, dx \qquad k = 0, 1, \dots \end{cases} \tag{4}$$

It is known that

$$E_n(W_{(\alpha,\beta)}; f) = \left\{\sum_{k=n+1}^{\infty} c_k^2(\alpha, \beta; f)\right\}^{1/2}, \qquad n = 0, 1, \dots. \tag{5}$$

For $W_{(\alpha,\beta)}f\in L_2$, we define the following continuity modulus:

$$\begin{aligned} \omega(W_{(\alpha,\beta)}; f; \delta) = \sup_{0\leqq t\leqq \delta} \left\{\int\limits_0^{5\pi/8} |f^*(\theta+t)-f^*(\theta)|^2 W^{*2}_{(\alpha,\beta)}(\theta)\sin\theta\, d\theta\right\}^{1/2} + \\ + \sup_{0\leqq t\leqq \delta} \left\{\int\limits_{3\pi/8}^{\pi} |f^*(\theta-t)-f^*(\theta)|^2 W^{*2}_{(\alpha,\beta)}(\theta)\sin\theta\, d\theta\right\}^{1/2}, \qquad 0\leqq \delta \leqq \pi/3 \end{aligned} \tag{6}$$

where the function $f^*(\theta)$ is defined by

(7) $$f^*(\theta) = f(\cos\theta), \quad 0 \leqq \theta \leqq \pi.$$

The modulus $\omega(W_{(\alpha,\beta)};f;\delta)$ exists for every $W_{(\alpha,\beta)}f \in L_2$ and $0 \leqq \delta \leqq \pi/3$, and we have

$$\omega(W_{(\alpha,\beta)};f;\delta) \to 0 \quad (\delta \to 0).$$

In what follows $c_i(\alpha, \beta, k, \dots)$ denote constants, which depend only on $\alpha, \beta, k, \dots$. Let $S_k^{(\alpha,\beta)}$ $(k=1, 2, \dots)$ be the set of f satisfying

(i) f is a k-times iterated integral function of $f^{(k)}$ in $(-1, 1)$
(ii) $f^{(l)}W_{(\alpha+l,\beta+l)} \in L_2 \qquad (l = 0, 1, \dots, k) \quad (f^{(0)} \overset{\text{def}}{=} f)$.

Furthermore, we denote by $S_0^{(\alpha,\beta)}$ the set of $f(x)$ satisfying $W_{(\alpha,\beta)}f \in L_2$.

We give here the analogue of Jackson- and Bernstein-type approximation theorems, as follows:

Direct theorem. *Let k be an arbitrary non-negative integer. For each $f \in S_k^{(\alpha,\beta)}$ we have*

(8) $$E_{n+k}(W_{(\alpha,\beta)}; f) \leqq \frac{c_1(\alpha, \beta, k)}{n^k}\,\omega(W_{(\alpha+k,\beta+k)}; f^{(k)}; 1/n), \qquad n = 1, 2, \dots.$$

Inverse theorem 1. *For each $f \in S_0^{(\alpha,\beta)}$, we have*

(9) $$\omega(W_{(\alpha,\beta)}; f; \delta) \leqq c_1(\alpha, \beta)\delta \sum_{0 \leqq n \leqq \delta^{-1}} E_n(W_{(\alpha,\beta)}; f).$$

Inverse theorem 2. *Let k be a positive integer, $f \in S_0^{(\alpha,\beta)}$. If*

(10) $$\sum_{\nu=0}^{\infty} (\nu+1)^{k-1} E_\nu(W_{(\alpha,\beta)}; f) < \infty$$

then $f \in S_k^{(\alpha,\beta)}$; also

(11) $$E_n(W_{(\alpha+k,\beta+k)}; f^{(k)}) \leqq$$
$$\leqq c_2(\alpha, \beta, k)\Big[(n+1)^{k-1} E_n(W_{(\alpha,\beta)}; f) + \sum_{\nu=n+1}^{\infty} (\nu+1)^{k-1} E_\nu(W_{(\alpha,\beta)}; f)\Big];$$

furthermore

(12) $$\omega(W_{(\alpha+k,\beta+k)}; f^{(k)}; 1/n) \leqq$$
$$\leqq c_3(\alpha, \beta, k) n^{-1} \sum_{\nu=0}^{n} \Big[(\nu+1)^{k-1} E_\nu(W_{(\alpha,\beta)}; f) + \sum_{s=\nu+1}^{\infty} (s+1)^{k-1} E_s(W_{(\alpha,\beta)}; f)\Big].$$

Remark. It follows from the above theorems that $E_n(W_{(\alpha,\beta)};f) = O(n^{-(k+\alpha)})$ $(k=0, 1, \dots, 0<\alpha<1)$ if and only if $f \in S_k^{(\alpha,\beta)}$ and $\omega(W_{(\alpha+k,\beta+k)};f^{(k)};\delta) = O(\delta^\alpha)$.

2. Lemmata

The proof of our theorems is based on some properties of orthonormal Jacobi polynomials.

We have

$$[\tilde{P}_n^{(\alpha,\beta)}(x)]' = \frac{n\gamma_n(\alpha,\beta)}{\gamma_{n-1}(\alpha+1,\beta+1)}\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{13}$$

$$\frac{-\gamma_n(\alpha,\beta)}{(n+\alpha+\beta+1)\gamma_{n-1}(\alpha+1,\beta+1)}[W^2_{(\alpha+1,\beta+1)}(x)\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x)]' = W^2_{(\alpha,\beta)}(x)\tilde{P}_n^{(\alpha,\beta)}(x). \tag{14}$$

$$\gamma_n(\alpha,\beta) = \frac{1}{2^n}\binom{2n+\alpha+\beta}{n}\left\{\frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}}\cdot\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right\}^{1/2} \tag{15}$$

(see (4.3.1); (4.3.4); (4.5.5); (4.2.1) and (4.21.7) in SZEGŐ [5]). From (15) it follows that

$$\frac{k}{(k+\alpha+\beta+1)}\left[\frac{\gamma_k(\alpha,\beta)}{\gamma_{k-1}(\alpha+1,\beta+1)}\right]^2 = 1 \qquad (k=1,2,\dots). \tag{16}$$

We begin with the proof of the following lemmata.

LEMMA 1. *If $f\in S_1^{(\alpha,\beta)}$, then*

$$S'(\alpha,\beta;f,x) = S(\alpha+1,\beta+1;f',x). \tag{17}$$

PROOF.. Since $f\in S_1^{(\alpha,\beta)}$ we have

$$W^2_{(\alpha+1,\beta+1)}(x)f(x) = o(1) \qquad (|x|\to 1). \tag{18}$$

Indeed, (18) is trivial if $\alpha,\beta<0$. If $\alpha,\beta\geqq 0$, then from the monotonicity of the function $W^2_{(\alpha+1,\beta+1)}(x)$ in the neighbourhood of 1 and -1 we get

$$|W^2_{(\alpha+1,\beta+1)}(x)f(x)| \leqq |f(0)W^2_{(\alpha+1,\beta+1)}(x)| + \left|W_{(\alpha+1,\beta+1)}(x)\int_0^x W_{(\alpha+1,\beta+1)}(x)f'(t)\,dt\right| \leqq$$

$$\leqq o(1) + c_2(\alpha,\beta)W_{(\alpha+1,\beta+1)}(x)\left|\int_0^x |W_{(\alpha+1,\beta+1)}(t)f'(t)|\,dt\right| \leqq$$

$$\leqq o(1) + 2c_2(\alpha,\beta)W_{(\alpha+1,\beta+1)}(x)\,\|W_{(\alpha+1,\beta+1)}f'\|_2 = o(1) \qquad (|x|\to 1).$$

Now, from (14) and (18) we obtain by integration by parts

$$c_k(\alpha,\beta;f) = \int_{-1}^{1} f(t)\tilde{P}_k^{(\alpha,\beta)}(t)W^2_{(\alpha,\beta)}(t)\,dt =$$

$$= \left[\frac{-\gamma_k(\alpha,\beta)}{(k+\alpha+\beta+1)\gamma_{k-1}(\alpha+1,\beta+1)}f(t)W^2_{(\alpha+1,\beta+1)}(t)\tilde{P}_{k-1}^{(\alpha+1,\beta+1)}(t)\right]_{-1}^{1} +$$

$$+\frac{\gamma_k(\alpha,\beta)}{(k+\alpha+\beta+1)\gamma_{k-1}(\alpha+1,\beta+1)}\int_{-1}^{1} f'(t)\tilde{P}_k^{(\alpha+1,\beta+1)}(t)W^2_{(\alpha+1,\beta+1)}(t)\,dt = \tag{19}$$

$$= \frac{\gamma_k(\alpha,\beta)}{(k+\alpha+\beta+1)\gamma_{k-1}(\alpha+1,\beta+1)}c_{k-1}(\alpha+1,\beta+1;f').$$

(17) is indeed a consequence of (13), (19) and (16). Q.e.d.

LEMMA 2. *If* $f \in S_1^{(\alpha,\beta)}$ *then*

$$E_{n+1}(W_{(\alpha,\beta)}; f) \leqq \frac{c_4(\alpha,\beta)}{n+1} E_n(W_{(\alpha+1,\beta+1)}; f') \leqq \tag{20}$$

$$\leqq \frac{c_4(\alpha,\beta)}{(n+1)} \|W_{(\alpha+1,\beta+1)} f'\|_2, \qquad n = 0, 1, \dots.$$

PROOF. (20) is a consequence of (5), (19) and the asymptotic formula (see (15))

$$\frac{\gamma_k(\alpha,\beta)}{(k+\alpha+\beta+1)\gamma_{k-1}(\alpha+1,\beta+1)} \simeq \frac{1}{k} \qquad (k = 1, 2, \dots). \tag{21}$$

(The relation $A_n \simeq B_n$ means that $A < \frac{A_n}{B_n} < B$, where A and B are independent of n.)

LEMMA 3. *We have for every* $\Pi_n \in \mathscr{P}_n$ $(n = 0, 1, \dots)$

$$\|W_{(\alpha+1,\beta+1)} \Pi_n'\|_2 \leqq c_5(\alpha,\beta) n \|W_{(\alpha,\beta)} \Pi_n\|_2. \tag{22}$$

$$\Big\{\int_0^{5\pi/8} |\Pi_n^*(\theta+h) - \Pi_n^*(\theta)|^2 W_{(\alpha,\beta)}^{*2}(\theta) \sin\theta \, d\theta\Big\}^{1/2} \leqq \tag{23}$$

$$\leqq c_6(\alpha,\beta) hn \Big\{\int_0^{\pi} |\Pi_n^*(\theta) W_{(\alpha,\beta)}^*(\theta)|^2 \sin\theta \, d\theta\Big\}^{1/2}.$$

$$\Big\{\int_{3\pi/8}^{\pi} |\Pi_n^*(\theta-h) - \Pi_n^*(\theta)|^2 W_{(\alpha,\beta)}^{*2}(\theta) \sin\theta \, d\theta\Big\}^{1/2} \leqq \tag{24}$$

$$\leqq c_7(\alpha,\beta) hn \Big\{\int_0^{\pi} |\Pi_n^*(\theta) W_{(\alpha,\beta)}^*(\theta)|^2 \sin\theta \, d\theta\Big\}^{1/2}.$$

PROOF. Since

$$S_n(\alpha,\beta; \Pi_n) \equiv \Pi_n, \quad \Pi_n \in \mathscr{P}_n$$

we have by Parseval's formula and (19), (21)

$$\|W_{(\alpha+1,\beta+1)} \Pi_n'\|_2 = \|W_{(\alpha+1,\beta+1)} S_{n-1}(\alpha+1, \beta+1; \Pi_n')\|_2 =$$

$$= \Big\{\sum_{k=0}^{n-1} c_k^2(\alpha+1, \beta+1, \Pi_n')\Big\}^{1/2} =$$

$$= \Big\{\sum_{k=0}^{n-1} \Big[\frac{(k+\alpha+\beta+2)\gamma_k(\alpha+1,\beta+1)}{\gamma_{k+1}(\alpha,\beta)} c_{k+1}(\alpha,\beta; \Pi_n)\Big]^2\Big\}^{1/2} \leqq$$

$$\leqq \max_{0 \leqq k \leqq n-1} \Big|\frac{(k+\alpha+\beta+2)\gamma_k(\alpha+1,\beta+1)}{\gamma_{k+1}(\alpha,\beta)}\Big| \Big\{\sum_{k=0}^{n-1} c_{k+1}^2(\alpha,\beta; \Pi_n)\Big\}^{1/2} \leqq$$

$$\leqq c_8(\alpha,\beta) n \|S_n(\alpha,\beta; \Pi_n) W_{(\alpha,\beta)}\|_2 = c_8(\alpha,\beta) n \|W_{(\alpha,\beta)} \Pi_n\|_2.$$

Finally, (23) and (24) are consequences of (22) and the following inequalities valid for $0 \leqq h \leqq \pi/3$:

$$\text{(25)} \qquad \Big\{\int_0^{5\pi/8} |\Pi_n^*(\theta+h)-\Pi_n^*(\theta)|^2 W_{(\alpha,\beta)}^{*2}(\theta)\sin\theta\, d\theta\Big\}^{1/2} \leqq$$

$$\leqq c_9(\alpha,\beta)h\Big\{\int_0^{\pi} |\Pi_n^{*\prime}(\theta)W_{(\alpha,\beta)}^{*}(\theta)|^2 \sin\theta\, d\theta\Big\}^{1/2}.$$

$$\text{(26)} \qquad \Big\{\int_{3\pi/8}^{\pi} |\Pi_n^*(\theta-h)-\Pi_n^*(\theta)|^2 W_{(\alpha,\beta)}^{*2}(\theta)\sin\theta\, d\theta\Big\}^{1/2} \leqq$$

$$\leqq c_{10}(\alpha,\beta)h\Big\{\int_0^{\pi} |\Pi_n^{*\prime}(\theta)W_{(\alpha,\beta)}^{*}(\theta)|^2 \sin\theta\, d\theta\Big\}^{1/2}.$$

To prove (25) and (26) we notice only that

$$\text{(27)} \qquad W_{(\alpha,\beta)}^{*2}(\theta)\sin\theta \leqq c_{11}(\alpha,\beta)W_{(\alpha,\beta)}^{*2}(\theta+t)\sin(\theta+t), \qquad 0 \leqq \theta \leqq 5\pi/8,$$

$$0 \leqq t \leqq \pi/3 \quad \text{and}$$

$$\text{(28)} \qquad W_{(\alpha,\beta)}^{*2}(\theta)\sin\theta \leqq c_{12}(\alpha,\beta)W_{(\alpha,\beta)}^{*2}(\theta-t)\sin(\theta-t), \qquad 3\pi/8 \leqq \theta \leqq \pi,$$

$$0 \leqq t \leqq \pi/3.$$

Thus we completely proved our lemma.

3. Proof of the main theorems

PROOF OF THE DIRECT THEOREM. Since, for $n=1, 2, \dots$

$$\text{(29)} \qquad \Big\{\int_{3\pi/8}^{5\pi/8} \Big(2n\int_{n^{-1}/2}^{n^{-1}} [f^*(\theta+t)-f^*(\theta-t)]\,dt\Big)^2 d\theta\Big\}^{1/2} \leqq$$

$$\leqq c_{13}(\alpha,\beta)W_{(\alpha+1,\beta+1)}^{*-1}(3\pi/8)\int_{n^{-1}/2}^{n^{-1}}\Big\{\int_{3\pi/8}^{5\pi/8} |f^*(\theta+t)-f^*(\theta-t)|^2 W_{(\alpha,\beta)}^{*2}(\theta)\sin\theta\, d\theta\Big\}^{1/2} dt \leqq$$

$$\leqq c_{13}(\alpha,\beta)W_{(\alpha+1,\beta+1)}^{*-1}(3\pi/8)\,\omega(W_{(\alpha,\beta)}; f; 1/n),$$

there exists a θ_n, $3\pi/8 \leqq \theta_n \leqq 5\pi/8$ so that

$$\text{(30)} \qquad \Big|2n\int_{n^{-1}/2}^{n^{-1}} [f^*(\theta_n+t)-f^*(\theta_n-t)]\,dt\Big| \leqq$$

$$\leqq c_{13}(\alpha,\beta)W_{(\alpha+1,\beta+1)}^{*-1}(3\pi/8)\,\omega(W_{(\alpha,\beta)}; f; 1/n)\cdot(4/\pi)^{1/2}.$$

Let

$$\text{(31)} \qquad \varrho_n = 2n\int_{n^{-1}/2}^{n^{-1}} [f^*(\theta_n+t)-f^*(\theta_n-t)]\,dt.$$

We introduce the following function:

(32) $$\varphi_n^*(\theta) = \begin{cases} 2n \int\limits_{n^{-1}/2}^{n^{-1}} f^*(\theta+t)\,dt & 0 \leqq \theta \leqq \theta_n \\ 2n \int\limits_{n^{-1}/2}^{n^{-1}} f^*(\theta-t)\,dt + \varrho_n, & \theta_n < \theta \leqq \pi, \end{cases}$$

and let $\varphi_n(x) = \varphi_n^*(\arccos x)$. We obtain from (32) and (30)

(33) $$\|(\varphi_n - f)W_{(\alpha,\beta)}\|_2 = \Big\{\int\limits_0^\pi |\varphi_n^*(\theta) - f^*(\theta)|^2 W^{*2}_{(\alpha,\beta)}(\theta)\sin\theta\,d\theta\Big\}^{1/2} \leqq$$

$$\leqq 2n \int\limits_{n^{-1}/2}^{n^{-1}} \Big\{\int\limits_0^{\theta_n} |f^*(\theta+t) - f^*(\theta)|^2 W^{*2}_{(\alpha,\beta)}(\theta)\sin\theta\,d\theta\Big\}^{1/2} dt +$$

$$+ 2n \int\limits_{n^{-1}/2}^{n^{-1}} \Big\{\int\limits_{\theta_n}^{\pi} |f^*(\theta-t) - f^*(\theta)|^2 W^{*2}_{(\alpha,\beta)}(\theta)\sin\theta\,d\theta\Big\}^{1/2} dt +$$

$$+ c_{13}(\alpha,\beta) W^{*-1}_{(\alpha+1,\beta+1)}(3\pi/8)\cdot\omega(W_{(\alpha,\beta)}; f; 1/n)\Big\{\int\limits_{\theta_n}^{\pi} W^{*2}_{(\alpha,\beta)}(\theta)\sin\theta\,d\theta\Big\}^{1/2} \leqq$$

$$\leqq [1 + c_{14}(\alpha,\beta) W^{*-1}_{(\alpha+1,\beta+1)}(3\pi/8)]\,\omega(W_{(\alpha,\beta)}; f; 1/n).$$

Again, from (32), we get

(34) $$\|W_{(\alpha+1,\beta+1)}(x)\varphi_n'(x)\|_2 = \Big\{\int\limits_0^\pi |\varphi_n^{*\prime}(\theta) W^*_{(\alpha,\beta)}(\theta)|^2 \sin\theta\,d\theta\Big\}^{1/2} \leqq$$

$$\leqq 2n\Big\{\int\limits_0^{\theta_n} |f^*(\theta+n^{-1}) - f^*(\theta+n^{-1}/2)|^2 W^{*2}_{(\alpha,\beta)}(\theta)\sin\theta\,d\theta\Big\}^{1/2} +$$

$$+ 2n\Big\{\int\limits_{\theta_n}^{\pi} |f^*(\theta-n^{-1}) - f^*(\theta-n^{-1}/2)|^2 W^{*2}_{(\alpha,\beta)}(\theta)\sin\theta\,d\theta\Big\}^{1/2} \leqq 4n\omega(W_{(\alpha,\beta)}; f; 1/n).$$

From (31), (32), (33), (34) it follows that $\varphi_n(x) \in S_1^{(\alpha,\beta)}$, and thus by Lemma 2, we get

(35) $$E_n(W_{(\alpha,\beta)}; f) \leqq \|(\varphi_n - f)W_{(\alpha,\beta)}\|_2 + E_n(W_{(\alpha,\beta)}; \varphi_n) \leqq$$

$$\leqq \|(\varphi_n - f)W_{(\alpha,\beta)}\|_2 + \frac{c_{14}(\alpha,\beta)}{n+1}\|W_{(\alpha+1,\beta+1)}\varphi_n'\|_2 \leqq$$

$$\leqq [1 + c_{14}(\alpha,\beta) W^{*-1}_{(\alpha+1,\beta+1)}(3\pi/8) + c_{15}(\alpha,\beta)]\,\omega(W_{(\alpha,\beta)}; f; 1/n).$$

Thus the direct theorem is valid for $k=0$. The case $k>0$ is a consequence of the case $k=0$, and the relation

(36) $$E_{n+k}(W_{(\alpha,\beta)}; f) \leqq \frac{c_3(\alpha,\beta,k)}{n^k} E_n(W_{(\alpha+k,\beta+k)}; f^{(k)}), \quad n = 1, 2, \dots, \quad k = 0, 1, \dots,$$

which follows from (20). Thus we completely proved the direct theorem.

The proof of the inverse Theorem 1 is similar to the proof of Theorem 2 in NGUYEN XUAN KY [4]. We apply only inequalities (23) and (24).

PROOF OF THE INVERSE THEOREM 2. Let $q_n(x)$ be a polynomial of best approximation of order n to $f(x)$ with weight $W_{(\alpha,\beta)}(x)$ in L_2. Let $1 \leqq l \leqq k$. Using inequality (22) and the fact that the sequence $E_n(W_{(\alpha,\beta)};f)$ does not increase, we obtain

$$\|(q^{(l)}_{2^\nu n} - q^{(l)}_{2^{\nu+1}n})W_{(\alpha+l,\beta+l)}\|_2 \leqq c_4(\alpha,\beta,l)(2^{\nu+1}n)^l \|(q_{2^\nu n} - q_{2^{\nu+1}n})W_{(\alpha,\beta)}\|_2 \leqq$$

$$\leqq c_5(\alpha,\beta,l)(2^{\nu+1}_n)^l E_{2^\nu n}(W_{(\alpha,\beta)};f) \leqq c_6(\alpha,\beta,l) \sum_{j=2^\nu n-1}^{2^{\nu+1}n} (j+1)^{l-1} E_j(W_{(\alpha,\beta)};f).$$

Therefore, the series

$$\sum_{\nu=0}^{\infty} (q^{(l)}_{2^\nu n} - q^{(l)}_{2^{\nu+1}n})W_{(\alpha+l,\beta+l)} \tag{37}$$

converges according to the norm of the complete space to some element $q_n^{(l)} - f_{(l)}$ $(l=1,2,\ldots,k)$ and

$$\|(f_{(l)} - q_n^{(l)})W_{(\alpha+l,\beta+l)}\|_2 \leqq \sum_{\nu=0}^{\infty} \|q^{(l)}_{2^\nu n} - q^{(l)}_{2^{\nu+1}n})W_{(\alpha+l,\beta+l)}\|_2 \quad (l=1,2,\ldots,k), \tag{38}$$

from which (11) follows at once. From (38), we integrate k-times, and we take the convergence of (37) in consideration and obtain that f is a k-times iterated integral function of $f_{(k)}$, i.e. $f \in S_k^{(\alpha,\beta)}$.

The inequality (12) is the consequence of (11) and the inverse Theorem 1 for $W_{(\alpha+k,\beta+k)}$. Q.e.d.

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ESTIMATIONS FOR SOME INTERPOLATORY PROCESSES

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1. Introduction and preliminary results

1.1. In his paper [1] O. KIS proved some theorems for the Lagrange interpolation process based on the Chebysheff nodes

$$x_{kn} = \cos\frac{2k-1}{2n}\pi \qquad (k = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots). \tag{1.1}$$

E.g., he proved the following statements. Let $f(x)$ be a continuous function on $[-1, 1]$ with the modulus of continuity $\omega(f;\ t)$ such that $\omega(f; t) = O(\omega(t))$ for a certain modulus of continuity $\omega(t)$. If

$$\begin{cases} L_n(f;\ x) = \sum_{k=1}^{n} f(x_{kn}) l_{kn}(x), \quad l_{kn}(x) = \dfrac{\Omega(x)}{\Omega'(x_{kn})(x - x_{kn})}, \\ \Omega(x) = c \prod_{k=1}^{n} (x - x_{kn}),\ ^1 \end{cases} \tag{1.2}$$

then we have

THEOREM 1.1 (O. KIS).

$$|f(x) - L_n(f;\ x)| \leqq c_{12}\left[\omega\left(\frac{\sqrt{1-x^2}}{n}\right) + \sum_{k=1}^{n} \omega\left(\frac{\sqrt{1-x_{kn}^2}}{n}\right)|l_{kn}(x)|\right] \tag{1.3}$$

or

$$|f(x) - L_n(f;\ x)| \leqq c_2\left[\omega\left(\frac{\sqrt{1-x^2}}{n}\right)\ln n + \sum_{i=1}^{n} \frac{1}{i}\,\omega\left(\frac{i}{n^2}\right)\right] \tag{1.4}$$

$$(-1 \leqq x \leqq 1;\ n = 2, 3, 4, \ldots).$$

(See [1], (1.2), (4.1).).

He also gave some lower estimations but for proving these he had to settle the case $x = \pm 1$ separately.

The main aim of this paper is to give such kind of lower estimation which can be applied for *any* point of the interval $[-1, 1]$. For the sake of generality we prove our theorems for the Jacobi roots. In the remaining part we apply this method for another interpolatory process.

1.2. We need the following tools.

Let us denote by $\omega_m(t)$ a function for which

[1] All constants c and c_i are positive.

(i) $\omega_m(t)>0$ for $t>0$, $\omega_m(0)=0$, $\omega_m(T)\geqq\omega_m(t)$ for $T\geqq t$, $\omega_m(t)$ is a continuous function for $t\geqq 0$;

(ii) $\dfrac{t^m}{\omega_m(t)}$ is a monotone increasing function for $t\geqq 0$ ($m\geqq 1$ is a fixed integer).

Let $C(\omega_m)$ be the class of functions continuous on $[-1, 1]$ such that

$$\omega_m(f; t) \leqq a_m(f)\omega_m(t) \tag{1.5}$$

where $\omega_m(f; t)$ is the m-th modulus of smoothness of $f(x)$. For $m=1$ we use the notations $\omega(t)$ and $C(\omega)$.

Let $\{T_n\}_{n=1}^{\infty}$ denote linear operators on $C(\omega_m)$.

Let us suppose the following restrictions.

(A*) There exist certain functions $g_n(x)$ $(n=1, 2, 3, \dots)$ such that

$$g_n(x)\in C(\omega_m), \tag{a1}$$

$$T_n(g_n; z_n) \geqq c_4\lambda_n(z_n) \quad \text{for certain } \{z_n\}\subset[-1, 1] \tag{a2}$$

(definition of $\lambda_n(z_n)$).

Considering Theorem 1.2 we may suppose that

$$|T_n(g_N; z_n)-g_N(z_n)| \leqq \eta^N e_n\lambda_n(z_n) \quad (n > M(N), 0 < \eta < 1), \tag{1.6}$$

where $0<e_n\leqq 1$, $e_{n+1}\leqq e_n$.

(B*) Suppose $\tilde{f}(x)\in C(\omega_m)$ where

$$\tilde{f}(x) \overset{\text{def}}{=} Q \sum_{i=1}^{\infty} e_{n_i} g_{n_i}(x), \quad n_{i+1} > M(n_i), \quad Q > 0. \tag{b1}$$

(C*) We suppose that for certain $\bar{c}_4<c_4$ $(0<\bar{c}_4)$

$$\bar{c}_4\lambda_{n_k}(z_{n_k}) > \sum_{i=k+1}^{\infty} e_{n_i}|T_{n_k}(g_{n_i}; z_{n_k})| + \sum_{i=k}^{\infty} e_{n_i}|g_{n_i}(z_{n_k})|. \tag{c1}$$

Then, applying Theorem 3.1 from [3] for $B_{0,P}^{[-1,1]}$ where $P=\{f(x)\in C(\omega_m)\}$ (i.e. now $B_{0,P}^{[-1,1]}\equiv C(\omega_m)$) we have

THEOREM 1.2 ([3]). *If the conditions A^*, B^* and C^* are satisfied then there exist an $f(x)\in C(\omega_m)$ and $\{n_i\}_{i=1}^{\infty}$ such that*

$$T_{n_k}(f; z_{n_k})-f(z_{n_k}) > e_{n_k}\lambda_{n_k}(z_{n_k}) \qquad (k = 1, 2, 3, \dots). \tag{1.7}$$

2. New results

2.1. Let us consider the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of degree n defined by

$$(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}] \qquad (\alpha,\beta > -1). \tag{2.1}$$

If we consider the Lagrange interpolatory process $L^{(\alpha,\beta)}$ for the roots

$$-1 < x_{nn}^{(\alpha,\beta)} < x_{n-1,n}^{(\alpha,\beta)} < \dots < x_{2n}^{(\alpha,\beta)} < x_{n}^{(\alpha,\beta)} < 1 \tag{2.2}$$

of $P_n^{(\alpha,\beta)}(x)$ (see (1.2)), then we have

THEOREM 2.1. *For any $f(x)\in C(\omega)$*

$$|L_n^{(\alpha,\beta)}(f;\,x)-f(x)| \leqq c_1 \sum_{k=1}^{n} \omega\left(\frac{\sqrt{1-x_{kn}^{2(\alpha,\beta)}}}{n}\right)|l_{kn}^{(\alpha,\beta)}(x)| \tag{2.3}$$

where $c_1=c_1(\alpha,\beta)>0$.

This theorem is the best possible in the following sense. Let us suppose that $m\geqq 1$ is an arbitrary fixed integer. Then we have

THEOREM 2.2. If $\lim_{t=+0}(t^m/\omega_m(t))=0$ *then for any fixed* $x^*\in[-1,1]$ *there exists an* $f_1(x)\in C(\omega_m)$ *and a subsequence* $\{n_i\}_{i=1}^{\infty}$ *such that*

$$L_n^{(\alpha,\beta)}(f_1;\,x^*)-f_1(x^*) > \sum_{k=1}^{n} \omega_m\left(\frac{\sqrt{1-x_{kn}^{2(\alpha,\beta)}}}{n}\right)|l_{k,n}^{(\alpha,\beta)}(x^*)|, \quad n=n_1,n_2,n_3,\ldots. \tag{2.4}$$

Moreover if $\omega_m(t)=t^m$, *then for any* $\{\varepsilon_n\}$ $(\varepsilon_n\searrow 0)$ *and fixed* $x^*\in[-1,1]$ *there exists an* $f_2(x)\in C(t^m)$ *and* $\{p_i\}_{i=1}^{\infty}$ *such that*

$$L_n^{(\alpha,\beta)}(f_2;\,x^*)-f_2(x^*) > \varepsilon_n \sum_{k=1}^{n}\left(\frac{\sqrt{1-x_{kn}^{2(\alpha,\beta)}}}{n}\right)^m |l_{kn}^{(\alpha,\beta)}(x^*)|, \quad n=p_1,p_2,p_3,\ldots. \tag{2.5}$$

2.2. Remarks. 2.21. The expression of type $\omega(((1-x_{kn}^2)n^{-1})^{1/2})|l_{kn}(x)|$ was used e.g. by O. KIS [1] and P. O. RUNCK [2].

2.22. If

$$x_{j(n)+1,n}^{(\alpha,\beta)} \leqq x \leqq x_{j(n),n}^{(\alpha,\beta)} \qquad (j=0,1,\ldots,n;\; x_0\equiv 1,\; x_{n+1}\equiv -1) \tag{2.6}$$

then with $x_{kn}^{(\alpha,\beta)}=\cos\vartheta_{kn}^{(\alpha,\beta)}$, $x=\cos\vartheta$

$$\begin{cases} \vartheta_{k+1,n}^{(\alpha,\beta)}-\vartheta_{kn}^{(\alpha,\beta)} \sim \dfrac{1}{n} & (k=0,1,\ldots,n), \\[2ex] |x-x_{kn}^{(\alpha,\beta)}| \sim \dfrac{j^2-k^2}{n^2} & (k\neq j,\; j+1) \end{cases} \tag{2.7}$$

($f_n\sim g_n$ means that $f_n=O(g_n)$ and $g_n=O(f_n)$; see [10]).

So for suitably chosen $\{n_i\}$ (sometimes omitting the superfluous indices)

$$\sum_{k=1}^{n}\omega_m\left(\frac{\sqrt{1-x_k^2}}{n}\right)|l_k(0)| \sim \sum_{k=\left[\frac{n}{4}\right]}^{\left[\frac{3n}{4}\right]}\omega_m\left(\frac{1}{n}\right)|l_k(0)| \sim \omega_m\left(\frac{1}{n}\right)\log n$$

(see [14.4.7][2]). That means if $\lim_{t=+0}(\omega(t)/t)=0$ we have for a suitable $f_3\in C(\omega)$

$$L_n^{(\alpha,\beta)}(f_3;\,0)-f_3(0) \sim \omega\left(\frac{1}{n}\right)\log n \qquad (n=q_1,q_2,q_3,\ldots). \tag{2.8}$$

[2] [14.4.7] means the formula (14.4.7) from [8].

If $x=1$ we have by (2.7), [4.1.1] and [8.9.2]

$$\sum_{k=1}^{n} \omega_m\left(\frac{\sqrt{1-x_k^2}}{n}\right)|l_k(1)| \sim \sum_{k=1}^{n} \omega_m\left(\frac{k}{n^2}\right) k^{\alpha-1/2}$$

from where e.g.

$$L_n^{(\alpha,\beta)}(f_4;\, 1)-f_4(1) \sim \omega\left(\frac{1}{n}\right) n^{\alpha+1/2} \qquad \left(\alpha > -\frac{1}{2},\ n = p_1, p_2, p_3, \ldots\right),$$

$$L_n(f_5;\, 1)-f_5(1) > \omega_m\left(\frac{1}{n}\right) \qquad \left(\alpha = \beta = -\frac{1}{2},\ n = r_1, r_2, r_3, \ldots\right),$$

$$L_n^{(\alpha,\beta)}(f_6;\, 1)-f_6(1) > \omega_m\left(\frac{1}{n^2}\right) \qquad \left(\alpha, \beta < -\frac{1}{2},\ n = s_1, s_2, s_3, \ldots\right)$$

for suitable f_4, f_5 and f_6.

2.23. In (2.4) and (2.5) we can substitute x^* by $x^{(n)}$ if we suppose that

$$|x_{j,n}-x^{(n)}| \sim |x_{j+1,n}-x^{(n)}| \sim \frac{j}{n^2},$$

instead of (3.8).

2.3. Let us consider the uniquely determined Hermite—Fejér interpolating polynomials of degree $\leqq 2n-1$ for the roots (1.1), i.e. let

$$H_n^*(f;\, x) = \sum_{k=1}^{n} f(x_{kn}) v_{kn}(x) l_{kn}^2(x) + \sum_{k=1}^{n} f'(x_{kn})(x-x_{kn}) l_{kn}^2(x) \tag{2.9}$$

where

$$v_{kn}(x) = \frac{1-xx_{kn}}{1-x_{kn}^2}, \quad v_{kn}(x) > \frac{1}{2}, \tag{2.10}$$

and $f'(x)$ is continuous. As it is well-known

$$H_n^*(f;\, x_{kn}) = f(x_{kn}), \quad H_n'^*(f;\, x_{kn}) = f'(x_{kn}). \tag{2.11}$$

We can prove a convergence theorem (see [11]). To give the order of the convergence we prove the following

THEOREM 2.3. *If $f'(x) \in C(\omega)$, then we have*

$$|H_n^*(f;\, x)-f(x)| \leqq c_{10} \sum_{k=1}^{n} \omega\left(\frac{\sqrt{1-x_{kn}^2}}{n}\right)|x-x_{kn}|\, l_{kn}^2(x). \tag{2.12}$$

On the other hand, for any fixed integer $m \geqq 1$ we have

THEOREM 2.4. *If* $\lim_{t=+0} (t^m/\omega_m(t))=0$ *then for any fixed* $x^* \in [-1, 1]$ *there exist an* $f_1'(x) \in C(\omega_m)$ *and a subsequence* $\{n_i\}$ *such that*

$$H_n^*(f_1;\, x^*)-f_1(x^*) > \sum_{k=1}^{n} \omega_m\left(\frac{\sqrt{1-x_{kn}^2}}{n}\right)|x^*-x_{kn}|\, l_{kn}^2(x^*), \qquad n = n_1, n_2, n_3, \ldots. \tag{2.13}$$

Moreover, if $\omega_m(t)=t^m$, then for any $\{\varepsilon_n\}$ $(\varepsilon_n\searrow 0)$ and fixed $x^*\in[-1,1]$ there exist an $f_2'(x)\in C(t_m)$ and $\{p_i\}$ such that

$$H_n^*(f_2;x^*)-f_2(x^*) > \varepsilon_n \sum_{k=1}^{n}\left(\frac{\sqrt{1-x_{kn}^2}}{n}\right)^m |x^*-x_{kn}|\, l_{kn}^2(x^*), \qquad n=p_1,p_2,p_3,\ldots. \tag{2.14}$$

2.4. Remarks. 2.41. We have, as in 2.22 the following formulae. If $\lim_{t=+0}(\omega(t)/t)=0$, then for suitable subsequences and functions

$$H_n^*(f_3;0)-f_3(0)\sim\omega\left(\frac{1}{n}\right)\frac{\log n}{n} \qquad (n=q_1,q_2,q_3,\ldots;\ f_3'\in C(\omega)),$$

$$H_n^*(f_4;1)-f_4(1)\sim\frac{1}{n^2}\sum_{k=1}^{n}\omega\left(\frac{k}{n^2}\right)\sim\frac{1}{n^{\alpha+1}}$$

$$\left(n=s_1,s_2,s_3,\ldots;\ \omega(t)=t^\alpha,\ 0<\alpha<1\right).$$

2.42. We can apply a sequence $\{x^{(n)}\}$ instead of x^* if we ensure (3.8).

2.43. We shall return to the case of Jacobi roots and the lacunary interpolation in another paper.

3. Proofs

3.1. Proof of Theorem 2.1. We need the following important

3.11. Lemma 3.1. *For any $x\in[-1,1]$ we have the relation*

$$\omega_m\left(\frac{\sqrt{1-x^2}}{n}\right)=O(1)\sum_{k=1}^{n}\omega_m\left(\frac{\sqrt{1-x_{kn}^{2(\alpha,\beta)}}}{n}\right)|l_{kn}^{(\alpha,\beta)}(x)| \tag{3.1}$$

where the O sign does not depend on x.

3.12. Using the notations $x=\cos\vartheta$, $x_k=x_{kn}^{(\alpha,\beta)}=\cos\vartheta_{kn}^{(\alpha,\beta)}$ etc., we shall apply the relations

$$P_n^{(\alpha,\beta)}(\cos\vartheta)=n^{-1/2}k(\vartheta)\{\cos(N\vartheta+\gamma)+(n\sin\vartheta)^{-1}O(1)\} \tag{8.21.18}$$

$$(cn^{-1}\leqq\vartheta\leqq\pi-cn^{-1})$$

where

$$k(\vartheta)=\pi^{-1/2}\left(\sin\frac{\vartheta}{2}\right)^{-\alpha-1/2}\left(\cos\frac{\vartheta}{2}\right)^{-\beta-1/2},\quad N=n+\frac{\alpha+\beta+1}{2},\quad \gamma=-\left(\alpha+\frac{1}{2}\right)\frac{\pi}{2};$$

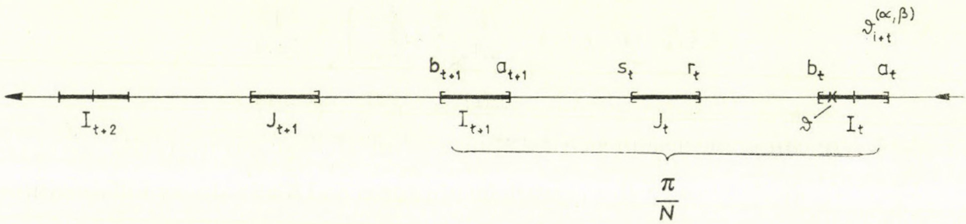
$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=\frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{4.21.7}$$

$$P_n^{(\beta,\alpha)}(x)=(-1)^nP_n^{(\alpha,\beta)}(-x), \tag{4.1.3}$$

$$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{\alpha}\sim n^\alpha, \tag{4.1.1}$$

$$|P_n'^{(\alpha,\beta)}(x_k)|\sim k^{-\alpha-3/2}n^{\alpha+2} \qquad \left(0<\vartheta_k\leqq\frac{\pi}{2}\right). \tag{8.9.2}$$

We may suppose that $x \geqq 0$ and $x \neq x_{j,n}^{(\alpha,\beta)}$, $x_{j+1,n}^{(\alpha,\beta)}$. Using the ideas of [8], Theorem 8.9.1, we can choose the intervals $I_\nu = [a_\nu, b_\nu]$ and $J_\nu = [r_\nu, s_\nu]$ on the unit circle $\left(\nu = t, t+1, \dots, \left[\frac{3n}{4}\right]\right)$, where t is fixed for $n \geqq n_0(\alpha, \beta)$ and $1 \leqq t \leqq c_3(\alpha, \beta)$, such that $\mu(I_\nu) = \mu(J_\nu) = \frac{\pi}{20N}$ (where μ stands for the Lebesgue-measure), $a_\nu - a_{\nu+1} = b_\nu - b_{\nu+1} = \frac{\pi}{N}$ and $\vartheta_{i+\nu,n}^{(\alpha,\beta)}$, $\eta_{\nu,n-1}^{(\alpha+1,\beta+1)} \in I_\nu$; $\vartheta_{f+\nu}^{(\alpha+1,\beta+1)}$, $\eta_{\nu,n}^{(\alpha,\beta)} \in J_\nu$ for suitably chosen fixed i and f. Here the points $\cos \eta_s^{(\gamma,\delta)}$ are the corresponding local maxima of $P_n^{(\gamma,\delta)}(x)$ on I_ν (or J_ν). (See this scheme.)



First we suppose that $\vartheta > \vartheta_{i+t}^{(\alpha,\beta)}$, e.g. $\frac{b_t + r_t}{2} \leqq \vartheta \leqq \frac{s_t + a_{t+1}}{2}$. Then by [8.9.6] $|P_n^{(\alpha,\beta)}(x)| \sim n^{-1/2} \vartheta^{-\alpha-1/2}$. Further using that now $j \sim n\vartheta$, $|\cos\vartheta - \cos\vartheta_j| = 2 \sin \frac{\vartheta + \vartheta_j}{2} \sin \frac{|\vartheta - \vartheta_j|}{2} \sim \vartheta n^{-1}$, we have

$$|l_j(x)| \sim \frac{n^{-1/2} \vartheta^{-\alpha-1/2}}{n^{-\alpha-3/2} \vartheta^{-\alpha-3/2} n^{\alpha+2} \vartheta n^{-1}} \sim 1. \tag{3.2}$$

Considering that $\sqrt{1-x^2} \sim \sqrt{1-x_j^2}$, we have

$$\omega_m \left(\frac{\sqrt{1-x^2}}{n} \right) \sim \omega_m \left(\frac{\sqrt{1-x_j^2}}{n} \right) |l_j(x)| \qquad (n \geqq n_0). \tag{3.3}$$

If $a_t \leqq \vartheta \leqq \frac{b_t + r_t}{2}$ then $|P_{n-1}^{(\alpha+1,\beta+1)}(x)| \sim n^{-1/2} \vartheta^{-\alpha-3/2}$ (see [8.9.6.]). So by [4.21.7]

$$|l_j(x)| = \left| \frac{P_n^{(\alpha,\beta)}(x)}{P_n'^{(\alpha,\beta)}(x_j)(x-x_j)} \right| \sim \frac{n |P_{n-1}^{(\alpha+1,\beta+1)}(x)|}{|P_n'^{(\alpha,\beta)}(x_j)|} \sim \frac{n n^{-1/2} \vartheta^{-\alpha-3/2}}{n^{-\alpha-3/2} \vartheta^{-\alpha-3/2} n^{\alpha+2}} \sim 1. \tag{3.5}$$

If $\frac{s_t + a_{t+1}}{2} \leqq \vartheta < b_t$ we get that $|l_{j+1}(x)| \sim 1$. Using (3.3), we get (3.1).

Supposing that $\vartheta < \vartheta_{i+t}$ we have $\vartheta \leqq c_3 n^{-1}$. So

$$\sum_{k=1}^{n} \omega_m \left(\frac{\sqrt{1-x_k^2}}{n} \right) |l_k(x)| \geqq c \omega_m \left(\frac{1}{n^2} \right) \sum_{k=1}^{n} |l_k(x)| \geqq c \omega_m \left(\frac{1}{n^2} \right) \geqq \omega_m \left(\frac{\sqrt{1-x^2}}{n} \right).$$

That means we completely proved (3.1).

3.13. The remaining part is simple. We can choose a polynomial $p_{n-1}(x)$ of degree $\leqq n-1$ such that

$$(3.6)\qquad |f(x)-p_{n-1}(x)| \leqq c_6\omega\left(\frac{\sqrt{1-x^2}}{n}\right) \qquad (-1 \leqq x \leqq 1;\ n = 2, 3, 4, \dots)$$

(see [1], [9]). So using the relation

$$L_n(f;\,x)-f(x) = p_{n-1}(x)-f(x)+\sum_{k=1}^{n}[f(x_k)-p_{n-1}(x_k)]\,l_k(x),$$

(3.6) and (3.1) for $m=1$, we get (2.3).

3.2. Proof of Theorem 2.2. We wish to apply Theorem 1.2. Let

$$H_n = \bigcup_{i=1}^{n} x_i \cup \{x^*\} = \bigcup_{i=1}^{n+1} y_i \quad \text{and} \quad y_i > y_{i+1},$$

$$(3.7)\qquad \begin{cases} g_n(x_k) = \operatorname{sign} l_k(x^*)\,\omega_m\left(\dfrac{\sqrt{1-x_k^2}}{n}\right) & k = 1, 2, \dots, n, \\ g_n(x^*) = 0, & \\ g_n(x) = \begin{cases} g_n(y_1) & \text{for}\quad x_1 \leqq x \leqq 1, \\ g_n(y_{n+1}) & \text{for}\quad -1 \leqq x \leqq x_n. \end{cases} \end{cases}$$

Let us define $g_n(x)$ for $x\in(y_{k+1}, y_k)$. In this interval let $g_n(x)$ be the Hermite interpolatory polynomial of degree $\leqq 2n-1$ which is equal to $g_n(y_k)$ or $g_n(y_{k+1})$ at the endpoints, respectively, and at these endpoints $g_n'(y_s)=g_n''(y_s)=\dots=g_n^{(m-1)}(y_s)=0$ $(s=k, k+1)$ (see [7]).

Using e.g. [8.21.18] and the ideas of [8], Theorem 8.9.1 (or (2.7) for $x^*\neq\pm 1$) we can choose the subsequence $\{r_i\}$ such that

$$(3.8)\qquad |x_{j,n}-x^*| \sim |x_{j+1,n}-x^*| \sim \frac{j}{n^2} \qquad (n = r_1, r_2, r_3, \dots).$$

If $x^*=\pm 1$ we have only one $|\dots|$. Then, supposing from now on that $n=r_1, r_2, r_3, \dots$, we have

$$(3.9)\qquad |y_i-y_{i+1}| \sim \frac{i}{n^2}.$$

Using that $\omega_m(g_n;t)\leqq \max_{1\leqq i\leqq n} \omega_m(g_n;t;[y_{i+1}, y_i])$ [3] we get that for a certain s

$$(3.10)\qquad \omega_m(g_n;\,t) \leqq B_m t^m \left(\frac{n^2}{s}\right)^m \omega_m\left(\frac{s}{n^2}\right) \qquad \left(0 < t \leqq c_8\frac{s}{n^2}\right).$$

Here we used (3.9), $\sqrt{1-x_s^2}\sim\dfrac{s}{n}$ (see 2.7), further the ideas of [7] obtaining (31) (from [7]). From (3.10) by (ii)

$$(3.11)\qquad \omega_m(g_n;\,t) \leqq B_m t^m n^{2m}\omega_m\left(\frac{1}{n^2}\right), \qquad 0 < t \leqq \frac{c_8}{n^2}.$$

[3] $\omega_m(f;\,t;\,[a, b])$ is the $\omega_m(f;\,t)$ restricted for $[a, b]$.

So

$$\omega_m(g_n;\, t) \leqq 2^{2m} B_m t^m n^{2m}\left(\frac{1}{n^{2m}t^m}+1\right)\omega_m(t) \leqq 2^{2m} B_m(1+c_8^m)\omega_m(t),$$

i.e. $g_n(x)\in C(\omega_m)$, so we obtain (a1).

By (3.7) we have

$$L_n(g_n;\, x^*) = \sum_{k=1}^{n} |l_k(x^*)|\omega_m\left(\frac{\sqrt{1-x_k^2}}{n}\right), \tag{3.12}$$

i.e. we have (a2) with $T_n=L_n$, $z_n=x^*$, $\lambda_n(x^*)=\sum_{k=1}^{n}|l_k(x^*)|\omega_m\left(\frac{\sqrt{1-x_k^2}}{n}\right)$ and $c_4=1$.

By (3.7) we get for any fixed $0<q<1$

$$L_n(g_N;\, x^*) \leqq \sum_{k=1}^{n} |l_{kn}(x^*)|\omega_m\left(\frac{1}{N}\right) \leqq q\sum_{k=1}^{n} |l_{kn}(x^*)|\omega_m\left(\frac{1}{n^2}\right) \quad \text{if} \quad N > S_q(n). \tag{3.13}$$

Let

$$\tilde{f}(x) = Q\sum_{i=1}^{\infty} g_{n_i}(x), \quad \{n_i\}\subset\{r_j\}, \quad n_{i+1} > M(n_i), \quad n_{i+1} > S(n_i). \tag{3.14}$$

Let $n_{j+1}^{-1} < t \leqq n_j^{-1}$. Then

$$\omega_m(\tilde{f};\, t) \leqq Q\sum_{i=1}^{\infty}\omega_m(g_{n_i};\, t) = Q\left[\sum_{i=1}^{j}+\sum_{i=j+1}^{\infty}\right].$$

But by (3.11)

$$\sum_{i=1}^{j}\omega_m(g_{n_i};\, t) \leqq B_m t^m \sum_{i=1}^{j} n_i^{2m}\omega_m(n_i^{-2}) \leqq 2B_m t^m n_j^m \omega_m(n_j^{-1}) \leqq 2B_m\omega_m(t)$$

if $\lim_{t=+0}\omega_m(t)t^{-m}=\infty$, further $\{n_i\}$ is so lacunary that

$$\sum_{i=1}^{j-1} n_i^{2m}\omega_m\left(\frac{1}{n_i^2}\right) \leqq n_j^m\omega_m\left(\frac{1}{n_j}\right). \tag{3.15}$$

Further by (3.7)

$$\sum_{i=j+1}^{\infty}\omega_m(g_{n_i};\, t) \leqq 2^m\sum_{i=j+1}^{\infty}\omega_m\left(\frac{1}{n_i}\right) \leqq 2^m\omega_m\left(\frac{1}{n_{j+1}}\right)\sum_{i=0}^{\infty}q^i \leqq \frac{2^m}{1-q}\omega_m(t)$$

because of $\{n_i\}$ is so lacunary that $\omega_m(n_1^{-1})\leqq q$, $\omega_m(n_{i+1}^{-1})\leqq q\omega_m(n_i^{-2})$. So we get (B*) because of

$$\omega_m(\tilde{f};\, t) \leqq Q\left(2B_m+\frac{2^m}{1-q}\right)\omega_m(t).$$

To prove (C*) we remark that $g_{n_i}(x^*)=0$, further by (3.13)

$$\sum_{i=k+1}^{\infty}|L_{n_k}(g_{n_i};\, x^*)| \leqq \frac{q}{1-q}\sum_{j=1}^{n_k}|l_{j,n_k}(x^*)|\omega_m\left(\frac{1}{n_k^2}\right) < \frac{1}{2}\lambda_{n_k}(x^*).$$

So we perfectly proved (2.4).

In proving (2.5) we have to choose $\{n_i\}$ such that $\sum_{i=1}^{\infty} \varepsilon_{n_i} < \infty$, $\varepsilon_{n_i} \searrow 0$, $0 < \varepsilon_{n_i} \leqq 1$. Then instead of (3.15) we get

$$\sum_{i=1}^{j} \varepsilon_{n_i} \omega_m(g_{n_i};\, t) \leqq B_m t^m \sum_{i=1}^{j} \varepsilon_{n_i} n_i^{2m} n_i^{-2m} \leqq c_{10} B_m t^m.$$

The remaining parts are the same as above. So we get (2.5) with $e_n = \varepsilon_n$.

3.3. Proof of Theorem 2.3. As we know there exist polynomials $Q_n(x)$ such that

$$|f^{(k)}(x) - Q_n^{(k)}(x)| \leqq c_{11} \left(\frac{\sqrt{1-x^2}}{n}\right)^{1-k} \omega\left(f';\, \frac{\sqrt{1-x^2}}{n}\right) \qquad (x \in [-1, 1],\ k = 0, 1)$$

(see [9]). So we have

$$|H_n^*(f;\, x) - f(x)| = \left|\sum_{k=1}^{n} [f(x_k) - Q_n(x_k)] v_k(x) l_k^2(x) + \sum_{k=1}^{n} [f'(x_k) - Q'(x_k)](x - x_k) l_k^2(x) + \right.$$

$$\left. + Q_n(f;\, x) - f(x)\right| \leqq c_{11} \left[\left(\frac{\sqrt{1-x^2}}{n}\right) \omega\left(f';\, \frac{\sqrt{1-x^2}}{n}\right) + \right.$$

$$\left. + \sum_{k=1}^{n} \frac{\sqrt{1-x_k^2}}{n} \omega\left(f';\, \frac{\sqrt{1-x_k^2}}{n}\right) v_k(x) l_k^2(x) + \sum_{k=1}^{n} \omega\left(f';\, \frac{\sqrt{1-x_k^2}}{n}\right) |x - x_k| l_k^2(x)\right].$$

Using that for $k \neq j,\ j+1$

$$\frac{\sqrt{1-x_k^2}}{n} v_k(x) = \frac{1 - x x_k}{n\sqrt{1-x_k^2}} \leqq \frac{1 - \cos\vartheta \cos\vartheta_k + \sin\vartheta \sin\vartheta_k}{n \sin\vartheta_k} = \frac{2}{n} \sin^2 \frac{\vartheta + \vartheta_k}{2} \leqq$$

$$\leqq 2 \sin\frac{\vartheta + \vartheta_k}{2} \sin\frac{|\vartheta - \vartheta_k|}{2} = |x - x_k|$$

and for $k = j$ or $k = j+1$

$$\frac{\sqrt{1-x_k^2}}{n} v_k(x) = \frac{1 - x_k^2 + x_k(x_k - x)}{n\sqrt{1-x_k^2}} = \frac{\sqrt{1-x_k^2}}{n} + \frac{x_k(x_k - x)}{n\sqrt{1-x_k^2}} \leqq c_{12} \frac{\sqrt{1-x_k^2}}{n},$$

further

$$|x - x_{j+1}| \sim \frac{j}{n^2} \sim \frac{\sqrt{1-x_j^2}}{n}, \quad \text{from where} \quad \sum_{k=1}^{n} \frac{\sqrt{1-x_k^2}}{n} v_k(x) = O\left(\sum_{k=1}^{n} |x - x_k|\right),$$

we get (2.12) if we apply the relation

$$\frac{\sqrt{1-x^2}}{n} \omega\left(f';\, \frac{\sqrt{1-x^2}}{n}\right) = O(1) \sum_{k=j}^{j+1} \omega\left(f';\, \frac{\sqrt{1-x_k^2}}{n}\right) \frac{\sqrt{1-x_k^2}}{n} v_k(x) l_k^2(x)$$

which can be proved by the method of Lemma 3.1.

3.4. Proof of Theorem 2.4. The ideas are similar to [5], Theorems 4.1 and 3.2. So I sometimes omit the details.

We can suppose that we have for $\{r_i\}$ (3.8) and

$$\bigcup_{k=1}^{r_i} x_{k,r_i} \subset \bigcup_{s=1}^{r_{i+1}} x_{s,r_{i+1}} \qquad (i=1,2,3,\dots). \tag{3.16}$$

(We obtain (3.8) and (3.16) e.g. by making the sequence $\{3^i\}_{i=1}^{\infty}$ suitable lacunary (see [6]).)

Now let

$$H_n = \left(\bigcup_{i=0}^{n+1} x_i\right)\left(\bigcup_{\substack{i=0\\ i\neq j}}^{n} \frac{x_{i+1}+x_i}{2} \stackrel{\text{def}}{\equiv} \tilde{x}_i\right)\cup\{x^*\} \stackrel{\text{def}}{\equiv} \cup y_i, \quad y_i > y_{i+1}.$$

We define $h_n(x)$ and $g_n(x) = h_n'(x)$ as follows:

$$\begin{cases} h_n(y_k) = 0 \quad \text{for any} \quad y_i, \\ g_n(x_k) = \operatorname{sign}(x-x_k)\,\omega_m\left(\dfrac{\sqrt{1-x_k^2}}{n}\right) & k=0,1,\dots,n+1, \\ g_n(\tilde{x}_k) = \operatorname{sign}(\tilde{x}_k - x)\,\omega_m\left(\dfrac{\sqrt{1-x_k^2}}{n}\right) & k=0,1,\dots,n,\ k\neq j, \\ g_n(x^*) = 0. \end{cases} \tag{3.17}$$

For $x\in(y_{k+1}, y_k)$ we apply the definition of 3.2.

If $T_n(f;x) \stackrel{\text{def}}{=} \sum_{k=1}^{n} f(x_k)(x-x_k)l_k^2(x)$ then we have

$$T_n(g_n; x^*) = \sum_{k=1}^{n} |x^*-x_k|\,l_k^2(x^*)\,\omega_m\left(\frac{\sqrt{1-x_k^2}}{n}\right) \stackrel{\text{def}}{=} \lambda_n(x^*). \tag{3.18}$$

Because of $g_n(x)\in C(\omega_m)$ (see 3.2) by (3.18) we get (A*). For (C*) we remark that $g_n(x^*)=0$ further

$$\sum_{i=k+1}^{\infty} |T_{n_k}(g_{n_i}; x^*)| \leqq \sum_{i=k+1}^{\infty} \sum_{j=1}^{n_k} |x^*-x_{j,n_k}|\,l_{j,n_k}^2(x^*)\,\omega_m\left(\frac{1}{n_i}\right) \leqq \frac{1}{2}\lambda_{n_k}(x^*)$$

if $\{n_i\}$ is lacunary enough. Similarly we have that for suitably lacunary $\{n_i\}$

$$t_1^*(x) \stackrel{\text{def}}{=} \varrho \sum_{k=1}^{\infty} g_{n_i}(x) \in C(\omega_m).$$

But then according to (1.7)

$$T_{n_j}(t_1^*; x^*) - t_1^*(x^*) > \sum_{k=1}^{n_j} |x^*-x_k|\,l_k^2(x^*)\,\omega_m\left(\frac{\sqrt{1-x_k^2}}{n_j}\right) \qquad (j=1,2,3,\dots). \tag{3.19}$$

Noticing that for $t_1(x) = \sum_{k=1}^{\infty} h_{n_i}(x)$, $t_1(x^*) = t_1^*(x^*) = 0$, $t_1'(x) = t_1^*(x)$, we get

$$H_{n_j}^*(t_1; x^*) - t_1(x^*) = \sum_{k=1}^{n_j} t_1(x_{k,n_j})\,v_{k,n_j}(x^*)\,l_{k,n_j}^2(x^*) + T_{n_j}(t_1'; x^*)\,t_1'(x^*). \tag{3.20}$$

Here

$$(3.21)\qquad \sum_{k=1}^{n_j} t_1(x_{k,n_j})v_{k,n_j}(x^*)l_{k,n_j}^2(x^*) = \sum_{i=1}^{\infty}\sum_{k=1}^{n_j} h_{n_i}(x_{k,n_j})v_{k,n_j}(x^*)l_{k,n_j}^2(x^*).$$

By (3.16) and (3.17) $h_{n_i}(x_{k,n_j})=0$ if $i\leqq j$. Further by (3.17) $|h_n(x)|\leqq \leqq c_{13}\omega_m(n^{-1})n^{-1}$. So we get by (2.10)

$$(3.22)\qquad \left|\sum_{k=1}^{n_j} h_{n_i}(x_{k,n_j})v_{k,n_j}(x^*)l_{k,n_j}^2(x^*)\right| \leqq c_{14}\sum_{k=1}^{n_j}\omega_m\left(\frac{1}{n_i}\right)\frac{n_j^2}{n_i}l_{k,n_j}^2(x^*)\leqq$$

$$\leqq q\sum_{k=1}^{n_j}\omega_m\left(\frac{1}{n_{i-1}^2}\right)n_{i-1}^{-2}l_{k,n_j}^2(x^*)\leqq q^{i-j}\sum_{k=1}^{n_j}\omega_m\left(\frac{1}{n_j^2}\right)n_j^{-2}l_{k,n_j}(x^*)\leqq q^{i-j}T_{n_j}(g_{n_j};\, x^*)$$

for $i>j$ if $\{n_i\}$ is lacunary enough. By (3.19)—(3.22) we get (2.13) with $f_1(x)= =2t_1(x)$. To obtain (2.14), see 3.2.

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ON THE DEFINITION OF SATURATION

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1.0. In this paper I should like to compare two definitions of saturation and give examples showing which of them is more effective for different positive operator sequences.

1.1. If $C^*=C^*(-\pi,\pi)$ denotes the space of 2π-periodic continuous functions on the line with the norm $\|f\|=\max_{-\infty<x<\infty}|f(x)|$, then we have the following usual definition:

(D1) Let $\{L_n\}_{n=1}^{\infty}$ be a sequence of operators from C^* into C^* and $\{\Phi_n\}_{n=1}^{\infty}$ a sequence of positive real numbers which converges to 0. We say that $\{L_n\}$ is saturated with order $\{\Phi_n\}$ if the following two conditions are satisfied.

(a1) If $f\in C^*$ then $\lim_{n\to\infty}\dfrac{\|f-L_n(f)\|}{\Phi_n}=0$ if and only if f is a constant;

(b) There is a non-constant function $f_0\in C^*$ for which $\|f_0-L_n(f_0)\|=O(\Phi_n)$.

In [1] DeVore used a slightly different definition (D2) for the saturation. The only difference is as follows:

(a2) If $f\in C^*$ then $\varliminf_{n\to\infty}\dfrac{\|f-L_n(f)\|}{\Phi_n}=0$

if and only if f is a constant (i.e. the using of $\varliminf$ instead of lim ([1], 3.1)).

He shows that under (D2) the saturation order $\{\Phi_n\}$ of $\{L_n\}$ (if it exists) is not unique, but for any two saturation orders $\{\Phi_n\}$ and $\{\Phi_n'\}$ we have $\Phi_n\sim\Phi_n'$ [1] ([1], 3.1). He also gives examples showing that under (D1) it is possible to have two saturation orders $\{\Phi_n\}$ and $\{\Phi_n'\}$ for which we do not have the relation $\Phi_n\sim\Phi_n'$ and even the saturation classes for these two orders may be different ([1], 3.10. and 3.13.5).

1.2. Let $\{L_n\}$ be a sequence of positive convolution operators; that is for $f\in C^*$

$$L_n(f;x)=(f*d\mu_n)(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)\,d\mu_n(t) \tag{1.1}$$

where $d\mu_n$ is a non-negative, even Borel measure on $(-\pi,\pi)$ such that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}d\mu_n(t)=1. \tag{1.2}$$

[1] $a_n\sim b_n$ means that there are constants $0<c_1\leqq c_2<\infty$ such that $c_1\leqq a_nb_n^{-1}\leqq c_2$, $n=1,2,3,\ldots$.

A typical example is the Fejér operator

$$\sigma_n(f;\, x) = (f * F_n)(x) = \frac{1}{2\pi(n+1)} \int_{-\pi}^{\pi} f(x+t) \left[\frac{\sin\,(n+1)\dfrac{t}{2}}{\sin\dfrac{t}{2}} \right]^2 dt \tag{1.3}$$

which is saturated with order n^{-1} using either (D1) or (D2) (see [1], 3.6).

However, we can give a sequence of positive linear operators which is saturated neither under (D2) nor D1) (see [1], 3.4 and [2], Theorems 3.1 and 3.2).

1.3. Our first aim is to give a sequence of positive convolution operators which is (D1)-saturated but not (D2)-saturated.

If $d\delta_x$ denotes the Dirac measure at x, i.e. for $f \in C^*$ we have

$$\int_{-\pi}^{\pi} f(t)\, d\delta_x(t) = f(x),$$

then let us define the following infinite matrix:

$$\begin{matrix} d\alpha^{(1,3)} & d\alpha^{(2,3)} & d\alpha^{(3,3)} & d\alpha^{(4,3)} \dots \\ & d\alpha^{(2,4)} & d\alpha^{(3,4)} & d\alpha^{(4,4)} \dots \\ & & d\alpha^{(3,5)} & d\alpha^{(4,5)} \dots \\ & & & d\alpha^{(4,6)} \dots \\ & & \dots & \end{matrix} \qquad \text{or} \qquad \begin{matrix} d\alpha_1 & d\alpha_2 & d\alpha_4 & d\alpha_7 \dots \\ & d\alpha_3 & d\alpha_5 & d\alpha_8 \dots \\ & & d\alpha_6 & d\alpha_9 \dots \\ & & & d\alpha_{10} \dots \\ & & \dots & \end{matrix}$$

where

$$d\alpha^{(r,s)} = \frac{\pi}{2}\left(1-\frac{1}{r^2}\right)(d\delta_{-\frac{1}{r^2}} + d\delta_{\frac{1}{r^2}}) + \frac{\pi}{2r^2}(d\delta_{-\frac{2\pi}{s}} + d\delta_{\frac{2\pi}{s}}) \tag{1.5}$$
$$(s = 3, 4, 5, \dots,\ r = 1, 2, 3, \dots).$$

We remark that

$$\text{if} \quad d\alpha^{(r,s)} \equiv d\alpha_n \quad \text{then} \quad r^2 \sim n. \tag{1.6}$$

Let

$$D_n(f;\, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, d\alpha_n(t) \quad \text{or} \quad D^{(r,s)}(f;\, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, d\alpha^{(r,s)}(t). \tag{1.7}$$

It is easy to see that D_n is a positive convolution operator. Nevertheless, we have the following

THEOREM 1.1. *Using* (D2) *the sequence* $\{D_n\}$ *is not saturated.*

PROOF. At first let us consider another usual definition

$$\varrho_{kn} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos kt\, d\alpha_n(t) \quad \text{or} \quad \varrho_k^{(r,s)} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos kt\, d\alpha^{(r,s)}(t) \qquad (k = 0, 1, 2, \dots). \tag{1.8}$$

By (1.4)—(1.8) we have

$$\|D_n(\cos t;\,x)-\cos x\| = \|(1-\varrho_{1n})\cos x\| = 1-\varrho_{1n} = 1-\varrho_1^{(r,s)} = \tag{1.9}$$
$$= 2\left(1-\frac{1}{r^2}\right)\sin^2\frac{1}{2r^2}+\frac{2}{r^2}\sin^2\frac{\pi}{s} = O\left(\frac{1}{r^2}\right) = O\left(\frac{1}{n}\right).$$

(1.9) will be used later. Using (1.5) and (1.6) we have for a fixed integer $p>0$

$$\|D^{(i,s)}(\cos pt;\,x)-\cos px\| = 1-\varrho_p^{(i,s)} = 2\left(1-\frac{1}{i^2}\right)\sin^2\frac{p}{2i^2}+\frac{2}{i^2}\sin^2\frac{\pi p}{s} \tag{1.10}$$
$$(i=1,2,3,\ldots;\ s=3,4,\ldots,i+2).$$

Now we can prove our theorem. According to Theorem 3.1 in [1] the sequence $\{D_n\}$ is (D2)-saturated if and only if for a certain positive integer m

$$\lim_{n\to\infty}\frac{1-\varrho_{kn}}{1-\varrho_{mn}} = \psi_k > 0 \quad \text{for every } k. \tag{1.11}$$

(If (1.11) holds then $1-\varrho_{mn}$ is a saturation order.) Let us suppose that we have (1.11) for a certain $m>0$. If we choose the subsequence $\{n_i\}$ such that

$$d\alpha^{(i,4m)} = d\alpha_{n_i} \tag{1.12}$$

we have, as in (1.10), for $k=4m$

$$\frac{1-\varrho_{4m,n_i}}{1-\varrho_{m,n_i}} = \frac{1-\varrho_{4m}^{(i,4m)}}{1-\varrho_m^{(i,4m)}} = \frac{2\left(1-\frac{1}{i^2}\right)\sin^2\frac{2m}{i^2}+\frac{2}{i^2}\sin^2\pi}{2\left(1-\frac{1}{i^2}\right)\sin^2\frac{m}{2i^2}+\frac{2}{i^2}\sin^2\frac{\pi}{4}} \sim \frac{\frac{1}{i^4}}{\frac{1}{i^2}} \sim \frac{1}{i^2} \sim \frac{1}{n_i} \quad (i>1). \tag{1.13}$$

So $\lim\limits_{i\to\infty}\dfrac{1-\varrho_{4m,n_i}}{1-\varrho_{m,n_i}} = 0$, i.e. (1.11) does not hold.

1.4. Now we prove the following

THEOREM 1.2. *If we use the definition* (D1) *then* $\{D_n\}$ *is saturated with order* $\{n^{-1}\}$.

PROOF. We use the transform technique. Let

$$\hat{f}(x) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt \qquad (k=0,\pm1,\pm2,\ldots), \tag{1.14}$$

$$\check{\mu}(k) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\,d\mu(t) \qquad (k=0,\pm1,\pm2,\ldots) \tag{1.15}$$

be the complex Fourier coefficients of $f\in C^*$ and the Borel measure $d\mu$, respectively.

Then as it is well known

$$\widehat{(f*d\mu)}(k) = 2\check{\mu}(k)\hat{f}(k) \tag{1.16}$$

(see e.g. [1], 1.4). So by (1.7) and (1.14)—(1.16) we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)-(f*d\alpha_n)(x)]e^{-ikx}\,dx = \hat{f}(k)(1-\varrho_{|k|,n}). \tag{1.17}$$

If for a function $f\in C^*$ $\|f-D_n(f)\|=o(n^{-1})$, then by (1.17) $|\hat{f}(m)(1-\varrho_{|m|,n})|= =o(n^{-1})$ $(m=0, \pm 1, \pm 2, \ldots)$. Then for any fixed $m\neq 0$ choosing the subsequence $\{n_i\}$ according to (1.12), we get that $1-\varrho_{|m|,n_i}\sim i^{-2}\sim n_i^{-1}$, so $\hat{f}(m)=0$, i.e. $f(x)$ is constant. On the other hand, by (1.9) we have (b).

1.5. Here we prove a statement which means that in many cases (D2) is more restrictive than (D1).

THEOREM 1.3. *If a sequence $\{L_n\}$ of arbitrary constant-preserving operators (from C^* into C^*) is* (D2)*-saturated, then $\{L_n\}$ is* (D1)*-saturated, too. (We do not require that L_n should be linear.)*

PROOF. Let $\{L_n\}$ be (D2)-saturated with order $\{\Phi_n\}$. It is easy to see that $\{L_n\}$ is (D1)-saturated with the same order. Here we construct another, different saturation order.

Let $\psi_n=\sup_{k\geqq n}\Phi_k$. Then $\psi_n\searrow 0$, $\psi_n\geqq\Phi_n$,

$$\frac{\|f-L_n(f)\|}{\psi_n}\leqq\frac{\|f-L_n(f)\|}{\Phi_n}$$

for any $f\in C^*$ but for a suitably chosen $\{n_i\}$ we have

$$\frac{\|f-L_n(f)\|}{\psi_n}=\frac{\|f-L_n(f)\|}{\Phi_n}\qquad (n=n_1, n_2, n_3, \ldots). \tag{1.18}$$

We prove that $\{L_n\}$ is saturated with $\{\psi_n\}$ under (D1). Indeed, if $\|f-L_n(f)\|=o(\psi_n)$ then $\|f-L_{n_i}(f)\|=o(\psi_{n_i})$ $(i=1, 2, 3, \ldots)$. But using (1.18) we get

$$\|f-L_{n_i}(f)\| = o(\Phi_{n_i}),\quad \text{i.e.}\quad \underline{\lim}_{n\to\infty}\frac{\|f-L_n(f)\|}{\Phi_n}=0.$$

So by (D2) $f\equiv$constant. On the other hand, if $f\equiv$constant, then $\|f-L_n(f)\|\equiv 0= =o\ (\psi_n)$.

To prove (b) we have to consider the function f_0 for which $\|f_0-L_n(f_0)\|=O(\Phi_n)$, $f_0\neq$constant. By $\psi_n\geqq\Phi_n$ we get $\|f_0-L_n(f_0)\|=O(\psi_n)$.

1.6. Now, if we drop the constant-preserving property, we have

THEOREM 1.4. *There exists a sequence of positive linear operators (from C^* into C^*) which is saturated under* (D2) *but not under* (D1).

PROOF. Let

$$R_n(f) = \begin{cases} \sigma_n(f) & \text{if } n \text{ is odd,} \\ \sigma_n(f) + f\left(\frac{\pi}{2}\right) & \text{if } n \text{ is even,} \end{cases} \tag{1.19}$$

where $\sigma_n(f)$ is the Fejér operator (see (1.3)). Then $\{R_n\}$ is (D2)-saturated with order $\{n^{-1}\}$. Indeed, if f=constant then $\|f-R_{2k+1}(f)\|=0$.

On the other hand, let $\|f-R_n(f)\|n=o(1)$ $(n=n_1, n_2, n_3, \ldots)$. If $f\left(\frac{\pi}{2}\right)=0$, then $f-R_n(f)=f-\sigma_n(f)$ but then we know that $f\equiv$constant. If $f\left(\frac{\pi}{2}\right)\neq 0$ then using that for any $f\in C^*$, $\|\sigma_n(f)-f\|\to 0$, we have to suppose that $n_i=2s_i+1$ $(i\geqq i_0)$, i.e. we have $\varliminf\limits_{n\to\infty} \|f-\sigma_n(f)\|n=0$. But then as we know $f\equiv$constant. So we proved (a2). To see (b), we consider the function $f_1(x)=\cos x$ for which $R_n(f_1; x)\equiv\sigma_n(f_1; x)$ $(n=1, 2, 3, \ldots)$. But $\{R_n\}$ is not (D1)-saturated. Namely, for any $f=c$ $(c\neq 0)$ and $\{\Phi_n\}$ $(\Phi_n>0, \lim\limits_{n\to\infty} \Phi_n=0)$ we get

$$\lim_{n=2k+1} \frac{\|f-R_n(f)\|}{\Phi_n} = 0 \quad \text{and} \quad \lim_{n=2k} \frac{\|f-R_n(f)\|}{\Phi_n} = \lim_{n=2k} \frac{|c|}{\Phi_n} = \infty.$$

So $\lim\limits_{n\to\infty} \dfrac{\|f-R_n(f)\|}{\Phi_n}$ does not exist.

At last I should like to thank Professor J. SZABADOS for his remarks at Theorems 1.3 and 1.4.

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ОБСЛУЖИВАНИЕ ТРЕБОВАНИЙ ДВУХ ТИПОВ

Л. ЛАКАТОШ (Будапешт)

1. Постановка задачи

При рассмотрении одной более общей задачи, поставленной И. Н. Коваленко, возникла необходимость исследования системы массового обслуживания, в которую могут поступать требования нескольких типов, но одинакового приоритета. В настоящей статье мы ограничиваемся случаем, когда у нас в обслуживании могут участвовать лишь требования двух типов, первого и второго соответственно. В такой ситуации естественно возникает вопрос, что именно подразумевать под состоянием системы: в данном случае мы изучаем те периоды в ходе обслуживания, которые начинаются обязательным наличием требований обоих типов. Приведем пример для выяснения смысла такого определения: если у нас идет обслуживание требования одного типа, то под занятостью будем понимать период от момента поступления требования другого типа (предпологая конечно, что в ходе данного цикла обслуживания до этого момента требования этого другого типа вообще не поступали) до момента полного освобождения системы от требований обоих типов. Отсюда ясно, что сам обслуживающий прибор может быть в занятом состоянии, а в тот же момент с нашей точки зрения система свободна. Чтобы различать два типа занятости, первый из них будем называть периодом полной занятости (занятость с нашей точки зрения), а второй просто периодом занятости (занятость с точки зрения обслуживающего прибора). Фактически здесь речь идет о системе обслуживания типа $M/G/1$ которую мы изучаем с помощью интегро-дифференциального уравнения, и при выводе формул мы будем следовать по описанной в [1] (см. гл. IV. п. 1) схеме, которая используется при получении интегро-дифференциального уравнения Такача. Перейдем к выводу интегро-дифференциального уравнения. Под циклом обслуживания в дальнейшем подразумеваем период физической занятости обслуживающего прибора.

Пусть $F(t, x)$ будет вероятность того, что оставшаяся до конца обслуживания работа меньше либо равна x, подразумевая здесь необходимую для обслуживания всех поступающих до t требований первого и второго типов работу (здесь мы конечно предпологаем согласно предыдущему, что в течении данного цикла обслуживания в системе имелись требования обоих типов, хотя может быть, что в данный момент у нас есть в наличии либо только первый, либо только второй). Событие, вероятность которого $F(t+h, x)$, может осуществиться несколькими несовместными способами:

1. В момент времени t у нас идет цикл обслуживания, в ходе которого мы имели требования обоих типов, за h новое требование не поступает, и до конца обслуживания осталась работа, меньше $x+h$. Вероятность такого

события:

$$[1-(\lambda_1+\lambda_2)h]F(t, x+h).$$

2. В момент времени t у нас идет цикл обслуживания вышеуказанного типа, до конца обслуживания осталась работа y, от t до $t+h$ поступает требование, для обслуживания которого требуется работа, меньше $x-y$. Так как в систему поступает пуассоновский поток с параметром $\lambda_1+\lambda_2$, а функция распределения длительности обслуживания очевидно равна

$$B(x) = \frac{\lambda_1}{\lambda_1+\lambda_2} B_1(x) + \frac{\lambda_2}{\lambda_1+\lambda_2} B_2(x),$$

где $B_i(x)$ — функция распределения длительности обслуживания требования i-го типа ($i=1, 2$), то вероятность этого события равна

$$(\lambda_1+\lambda_2) \int_0^x B(x-y)\, d_y F(t, y).$$

3. В предыдущих двух случаях мы предпологали наличие требований обоих типов в текущем цикле, а теперь рассмотрим случай когда до момента t мы имели дело лишь с требованиями, одного типа. С нашей точки зрения система в таком случае является свободной. Пусть, например, идет обслуживание требования первого типа, в течении данного цикла требования второго типа еще не поступили, а от t до $t+h$ такое требование поступает. Вероятность этого события

$$\lambda_2 h \int_0^x B_2(x-y)\, d_y G_1(t, y),$$

где $G_1(t, x)$ — вероятность того, что в ходе текущего цикла мы имели лишь требования первого типа, и в момент времени t для их обслуживания нужна еще работа, меньше x. Аналогично в случае, когда у нас до t в данном цикле были только лишь требования второго типа, имеет место

$$\lambda_1 h \int_0^x B_1(x-y)\, d_y G_2(t, y).$$

Определением функций $G_i(t, x)(i=1, 2)$ будем заниматься дальше. На осно вании вышесказанных наше уравнение выводится следующим образом. Имеем

$$F(t+h, x) = [1-(\lambda_1+\lambda_2)h]F(t, x+h) + (\lambda_1+\lambda_2) \int_0^x B(x-y)\, d_y F(t, y) +$$
$$+\lambda_1 h \int_0^x B_1(x-y)\, d_y G_2(t, y) + \lambda_2 h \int_0^x B_2(x-y)\, d_y G_1(t, y) + o(h)$$

и отсюда после очевидных преобразований получаем интегро-дифференциальное уравнение

$$\frac{\partial F(t, x)}{\partial t} = \frac{\partial F(t, x)}{\partial x} - (\lambda_1 + \lambda_2) F(t, x) + (\lambda_1 + \lambda_2) \int_0^x B(x-y) d_y F(t, y) +$$

$$+ \lambda_1 \int_0^x B_1(x-y) d_y G_2(t, y) + \lambda_2 \int_0^x B_2(x-y) d_y G_1(t, y).$$

2. Определение функций $G_i(t, x)$

В настоящем параграфе ограничимся выводом функции $G_1(t, x)$, а в силу одинаковой роли в системе требований обоих типов, эти утверждения дословно переносятся и для функции $G_2(t, x)$, нужно лишь поменять соответствующие индексы.

Если рассматриваем процесс обслуживания, то там очевидно чередуются периоды свободного состояния и занятости обслуживающего прибора (в данном случае под свободным состоянием мы подразумеваем, что в данный период обслуживающий прибор не обслуживает ни требований первого, ни требований второго типов). Периоды свободного состояния являются экспоненциально распределенными случайными величинами с параметром $\lambda_1 + \lambda_2$, а преобразование Лапласа—Стильтьеса функции распределения периода занятости является единственным аналитическим в [0, 1] решением функционального уравнения (см. например в [2]):

$$(1) \qquad a(s) = \sum_{i=1}^{2} \frac{\lambda_i}{\lambda_1 + \lambda_2} b_i[s + (\lambda_1 + \lambda_2) - (\lambda_1 + \lambda_2) a(s)],$$

где $a(s)$ есть искомое преобразование Лапласа—Стильтьеса, по которому функция распределения определяется однозначно. Кроме этого нам понадобится аналог этой формулы для системы, в которой обслуживаются лишь требования первого типа, в этом случае нужное нам $a_1(s)$ (и определяемая тем самым функция распределения периода занятости $A_1(x)$) определяется из функционального уравнения (см. в [1]):

$$(2) \qquad a_1(s) = b_1\big(s + \lambda_1 - \lambda_1 a_1(s)\big).$$

По определению функции $G_1(t, x)$ нам необходимо, чтобы в момент времени t у нас шел цикл обслуживания и в этом цикле были обслужены только требования первого типа, в ходе данного цикла до t требования второго типа вообще не поступали и остаток необходимого еще обслуживания был меньше x. Рассмотрим теперь рекуррентный поток с запаздыванием, определяемый функциями $E(x)$ и $C(x)$. Под рекуррентным потоком с запаздыванием понимаем процесс восстановления, для которого длина промежутка от начального момента до первого момента восстановления является случайной величиной с функцией распределения $E(x)$, а длины промежутков между всеми остальными соседними моментами восстановления являются одинаково распределенными случайными величинами с функцией распределения $C(x)$. В нашем случае $E(x)$

есть экспоненциальное распределение с параметром $\lambda_1+\lambda_2$, а $C(x)$ есть свертка двух функций распределения: периода занятости $A(x)$ и экспоненциального с параметром $\lambda_1+\lambda_2$. Таким образом мы определили процесс восстановления, для которого моментами восстановления являются моменты перехода обслуживающего прибора из свободного состояния в занятое. Событие, вероятность которого обозначается через $G_1(t, x)$, может осуществиться несколькими несовместными способами. Пусть $t_1, t_2, \ldots, t_i, \ldots$ будут моментами восстановления рекуррентного потока с запаздыванием, определенного с помощью $E(x)$ и $C(x)$, и из моментов восстановления t_n является последним до t. Для этого должна выполниться следующая цепочка неравенств: $t_n \leqq t < t_n+\eta < t+x$, где η есть период занятости системы при условии, что туда поступают только требования первого типа. Этим обеспечивается, чтобы в момент времени t на обслуживании находилось требование первого типа, все остальные присутствующие в системе требования также были первого типа и длительность периода до освобождения обслуживающего прибора была меньше x. Пусть $t_n=y$. В этом случае в момент времени y закончится свободное состояние, на обслуживание поступает требование, которое с вероятностью $\lambda_1/(\lambda_1+\lambda_2)$ будет первого типа; этим у нас начинается период занятости, длительность которого должна находиться между $t+x-y$ и $t-y$. Кроме того мы должны обеспечить, что от t_n до t в систему требования второго типа не поступили, вероятность чего $\exp[-\lambda_2(t-y)]$. На основании вышесказанных искомая нами вероятность равна

$$\frac{\lambda_1}{\lambda_1+\lambda_2}\int_0^t [A_1(t+x-y)-A_1(t-y)]e^{-\lambda_2(t-y)}dE_n(y),$$

где

$$E_n(x)=\int_0^x E_{n-1}(x-y)dC(y) \quad \text{и} \quad E(x)=1-e^{-(\lambda_1+\lambda_2)x},$$

а $A_1(x)$ определяется по (2). Суммированием по n от 1 до ∞ получаем

$$G_1(t, x)=\frac{\lambda_1}{\lambda_1+\lambda_2}\int_0^t [A_1(t+x-y)-A_1(t-y)]e^{-\lambda_2(t-y)}dZ(y),$$

где $Z(x)$ есть функция восстановления рекуррентного потока с запаздыванием, определенного функциями $E(x)$ и $C(x)$. Как мы уже сказали, аналогично выводится формула и для $G_2(t, x)$.

3. Определение вероятности свободного состояния

Рассматривая вывод интегро-дифференциального уравнения Такача по [1], мы можем видеть какую основопологающую роль играет в случае применения преобразования Лапласа—Стильтьеса знание вероятности свободного состояния системы в некоторый момент времени t $F(t, +0)$. Для этого в [1] выводится формула

$$F(t, +0)=e^{-\lambda t}+\int_0^t e^{-\lambda(t-\tau)}dH(\tau),$$

в которой $e^{-\lambda t}$ есть вероятность непоступления в систему требования, а $H(t)$ — функция восстановления процесса, для которого отрезок между двумя моментами восстановления складывается из двух случайных величин: одного экспоненциально распределенного и одного периода занятости. В дальнейшем мы опишем метод определения вероятности свободного состояния для нашей системы, мы увидим что принцип вывода не меняется, осложняется лишь немножко сам вывод ввиду более сложного определения состояний системы.

Как мы уже раньше говорили, под свободным состоянием подразумеваем, что в системе нет никакого требования или в данный момент идет цикл, в ходе которого до настоящего момента обслуживались лишь требования одного типа и требования другого типа вообще не поступали. Сначала определим вероятность того, что длительность свободного состояния меньше x. Для этого необходимо, чтобы в некоторый момент шел цикл обслуживания требования одного типа и в этот же момент поступило требование другого типа. До этого момента уже могло быть несколько циклов, складывающихся из одного периода занятости (за который обслуживались только требования одного и того же типа) и из одного экспоненциально распределенного (с параметром $\lambda_1+\lambda_2$) периода отсутствия требований обоих типов (здесь мы также имеем дело с рекуррентным потоком с запаздыванием, предпологая моментами восстановления моменты перехода из свободного состояния в занятое, таким образом распределение первого цикла экспоненциальное с параметром $\lambda_1+\lambda_2$).

Функция распределения периода занятости (за этот период могут обслуживаться требования лишь одного типа) равна

$$\bar{A}(x) = \frac{\lambda_1}{\lambda_1+\lambda_2} A_1(x) + \frac{\lambda_2}{\lambda_1+\lambda_2} A_2(x), \tag{3}$$

где $A_1(x)$ — функция распределения, преобразование Лапласа—Стильтьеса которой определяется из функционального уравнения (2), а для случая $i=2$ из аналогичной формулы для $a_2(s)$. Теперь рассмотрим тот случай, когда у нас присутствуют требования обоих типов. И здесь очевидно, что у нас начинается период занятости одного типа и в течении этого поступает требование другого типа. Так как период занятости с вероятностью $\lambda_1/(\lambda_1+\lambda_2)$ начинается поступлением требования первого типа и с вероятностью $\lambda_2/(\lambda_1+\lambda_2)$ поступлением второго, то функция распределения этого последнего периода

$$D(x) = \frac{\lambda_1}{\lambda_1+\lambda_2}(1-e^{-\lambda_2 x}) + \frac{\lambda_2}{\lambda_1+\lambda_2}(1-e^{-\lambda_1 x}).$$

Определим теперь вероятность того, что у нас будет именно n периодов, функция распределения которых есть свертка функции распределения (3) и экспоненциального с параметром $\lambda_1+\lambda_2$ распределения. Обозначив через $A_1(x)$ и $A_2(x)$ функции распределения уже упомянутого периода занятости (появляются и обслуживаются требования только одного типа), нам нужно, чтобы n раз требования другого типа не поступали, а в $n+1$-ый раз они обязательно поступили. Вероятности непоступления

$$\int_0^\infty e^{-\lambda_1 t} dA_2(t) \quad \text{и} \quad \int_0^\infty e^{-\lambda_2 t} dA_1(t)$$

соответственно, а общая вероятность непоступления требования другого типа

$$q = \frac{\lambda_1}{\lambda_1+\lambda_2}\, a_1(\lambda_2) + \frac{\lambda_2}{\lambda_1+\lambda_2}\, a_2(\lambda_1),$$

где $a_1(s)$ и $a_2(s)$ — преобразования Лапласа—Стильтьеса функций $A_1(x)$ и $A_2(x)$. А вероятность того, что в периоде занятости требования другого типа поступают

$$p = \frac{\lambda_1}{\lambda_1+\lambda_2}\,[1-a_1(\lambda_2)] + \frac{\lambda_2}{\lambda_1+\lambda_2}\,[1-a_2(\lambda_1)],$$

и в силу этого искомая вероятность равна qp^n, где q и p определяются из вышеприведенных формул.

На основании этого вероятность одновременного наличия в системе требований обоих типов раньше момента x определяется из

$$H(x) = \sum_{n=0}^{\infty} \{E(x) * [E(x) * \bar{A}(x)]^n * D(x)\} q^n p, \tag{4}$$

где $E(x)$ есть экспоненциальное с параметром $\lambda_1+\lambda_2$ распределение, $*$ означает операцию свертки, а n при квадратных скобках, что операцию свертки мы должны применять в n раз. Таким образом вероятность того, что период свободного состояния (т. е. период до одновременного присутствия требований обоих типов) больше x, равна $1-H(x)$.

Нам осталось определить функцию распределения периода между двумя моментами освобождения (подразумевая здесь под занятым состоянием период от момента одновременного наличия требований обоих типов до момента физического освобождения обслуживающего прибора). Этот период очевидно складывается из двух случайных величин: от момента окончания предыдущего периода занятости до момента одновременного присутствия требований обоих типов и периода одновременного присутствия требований двух типов до момента физического освобождения обслуживающего прибора. Первая случайная величина имеет функцию распределения $H(x)$, определенную по (4), а определением функции распределения второй случайной величины мы займемся теперь.

За этот период у нас обязательно обслуживаются требования обоих типов, и пусть у нас идет обслуживание требований первого типа (с момента начала физической эанятости обслуживающего прибора у нас появились только требования отого типа) и через время t после начала обслуживания поступает первое требование второго типа. Общее распределение периода занятости в случае требований двух типов есть $A(x)$ ($A(x)$ есть функция распределения, определенная по ее преобразованию Лапласа—Стильтьеса из функционального уравнения (1)). Так как мы должны обеспечить, что сам этот период был длиннее, чем отрезок до поступления нервого требовануя второго типа, мы пойдем следующим путем: по нашему предположению первое требование второго тина поступает через время t, так условная функция распределения (полны период заняточи меньше x, при условии, что первое требование второго типа поступает через время t после момента начала физической занятости) имеет вид

$$\frac{A(t+x)-A(t)}{1-A(t)},$$

а безусловная функция распределения

$$\lambda_2 \int_0^\infty \frac{A(t+x)-A(t)}{1-A(t)} e^{-\lambda_2 t} dt.$$

Можно вывести аналогичную формулу и в том случае, когда впервые поступают требования второго типа, и взяв их с соответствующими весами, получим функцию распределения периода занятости с нашей точки зрения:

$$\frac{\lambda_1 \lambda_2}{\lambda_1+\lambda_2} \int_0^\infty \frac{A(t+x)-A(t)}{1-A(t)} (e^{-\lambda_1 t}+e^{-\lambda_2 t})\, dt = Q(x).$$

Применяя операцию свертки к функциям $H(x)$ и $Q(x)$, получим функцию распределения периода между двумя моментами освобождения обслуживающего прибора. Вспомнив теперь формулу

$$F(t_1+0) = e^{-\lambda t} + \int_0^t e^{-\lambda(t-\tau)} dH(\tau),$$

можем заметить полную аналогию, только вместо $e^{-\lambda t}$ нужно брать $1-H(t)$, а вместо $H(\tau)$ функцию восстановления, для которой функция распределения между двумя моментами восстановления есть $H(x) * Q(x)$.

Необходимые для понимания данной статьи сведения содержатся в первых двух пунктах четвертой главы [1].

Автор считает своим приятным долгом выразить глубокую благодарность И. Н. Коваленко за постановку задачи.

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MULTIPLE PACKING AND COVERING OF THE PLANE WITH CIRCLES

By

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A system of open circles is said to form a *k-fold packing*, and a system of closed circles is said to form a *k-fold covering*, if each point of the plane belongs to at most and to at least k circles, respectively. The problems of finding the densest k-fold packing and the thinnest k-fold covering of the plane with equal circles has been solved long ago if $k=1$ (see e.g. [1]), but they seem to be hopelessly difficult when $k>1$. Let d_k be the supremum of the densities of all k-fold packings of the plane with equal circles. Similarly, let D_k be the infimum of the densities of all k-fold coverings of the plane with equal circles. Obviously, we have the trivial bounds

$$d_k \leqq k \leqq D_k.$$

But it is interesting to observe, that in spite of the fact that the theory of multiple packings and coverings has a rather wast literature (see [1] and [2], where further literature can be found), no non-trivial upper bounds of d_k and no non-trivial lower bounds of D_k were published as yet for $k>1$. In this paper we try to fill this gap by proving the following theorems:

THEOREM 1. *We have*

$$d_k \leqq \frac{\pi}{6}\cot\frac{\pi}{6k}.$$

THEOREM 2. *We have*

$$D_k \geqq \frac{\pi}{3}\operatorname{cosec}\frac{\pi}{3k}.$$

Observe that $\frac{\pi}{6k}\cot\frac{\pi}{6k}$ and $\frac{\pi}{3k}\operatorname{cosec}\frac{\pi}{3k}$ are equal to the density of a disc with respect to the circumscribed and inscribed regular $6k$-gon.

Compare these bounds for $k=2$ with the lower and upper bounds of HEPPES [3] and DANZER [4] obtained by special constructions: $1.854\ldots\leqq d_2\leqq 1.954\ldots$, $2.094\ldots\leqq D_2\leqq 2.347\ldots$.

Instead of the whole plane we shall consider packings and coverings of a finite domain D. We say that a set of circles forms a k-fold packing of D, if all circles are contained in D and each point of D belongs to at most k circles. Similarly, we say that a set of circles forms a k-fold covering of D, if each point of D belongs to at least k circles. In what follows we shall denote a domain and its area with the same symbol. We shall write $|S|$ for the number of elements of S.

We shall prove the following, more general theorems:

THEOREM 1*. *Let H be a convex hexagon and S a finite set of unit circles forming a k-fold packing of H. Then*

$$H \geqq |S| 6 \tan \frac{\pi}{6k}.$$

THEOREM 2*. *Let H be a convex hexagon and S a finite set of unit circles forming a k-fold covering of H. Then*

$$H \leqq |S| 3 \sin \frac{\pi}{3k}.$$

The special case of these theorems when $k=1$ is due to L. FEJES TÓTH [1].

The proof of both theorems rests on the investigation of certain polygons associated with the circles. It is clear, that we may assume in both cases without loss of generality that there are no coincident circles in S. We define the *k'th Dirichlet cell* D_C^k of the circle $C \in S$ with respect to H as the set of all points $P \in H$ such that there are at most $k-1$ centres of the circles of S nearer to P than the centre of C. It is easy to show that D_C^k is a simple polygon (in fact it is star-like with respect to the centre of C) and if the number of circles is at least k then these polygons cover H exactly k times. More precisely, each point $P \in H$ not belonging to the boundary of a polygon is an interior point of exactly k polygons.

Let p_C^k be the number of the angles of D_C^k less than π. We shall need the following

LEMMA. *Let S be a finite set of different unit circles such that the centre of any circle of S lies in H. Suppose that there are no four centres of the circles of S lying on a circle or on a line, no three centres have equal distances from a point of the boundary of H and no two centres have equal distances from a vertex of H. Then we have*

$$\sum_{C \in S} p_C^k \leqq 6k|S|.$$

Since the case $k \geqq |S|$ of the Lemma is trivial, we restrict ourselves to the case when $k < |S|$.

Consider any subset A of S. We associate with A the subset D_A of those points of H whose distance from the centre of any circle of A is less than or equal to their distance from the centre of any circle of $S-A$. (It may occur that D_A is empty, but it is easily seen that under the conditions of the Lemma any D_A which is not empty has interior points.) The definitions of D_C^k and D_A imply that for $|S|>k$

$$D_C^k = \bigcup_{A \ni C,\ |A|=k} D_A.$$

We claim that D_A is a convex polygon ($A \subset S$). To see this we observe that the domain D_A can be constructed in the following way: Consider a circle $C \in A$, and construct the first Dirichlet cells of the circles of $(S-A) \cup \{C\}$ with respect to H, obtaining a decomposition of H into convex polygons. If $D_C(A)$ denotes the cell belonging to the circle C in this decomposition, then we have

$$D_A = \bigcap_{C \in A} D_C(A).$$

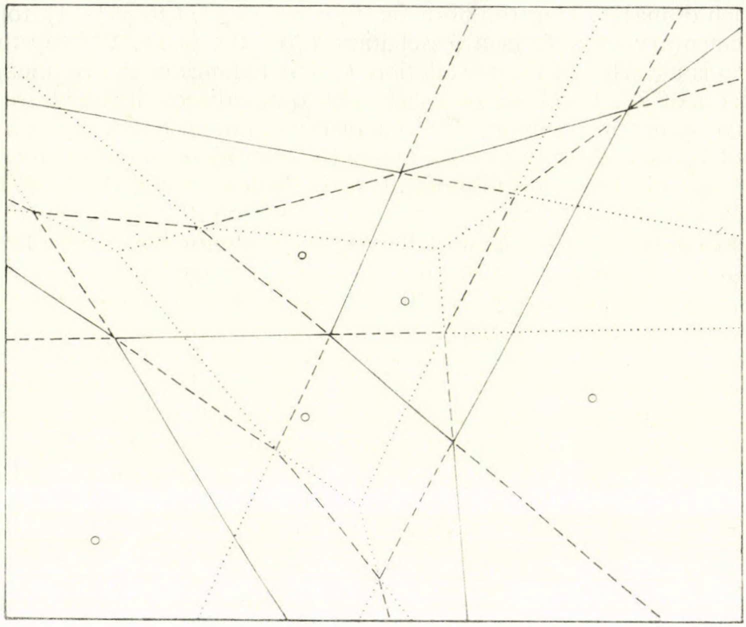
It is easy to see that each point in the interior of H lies either in the interior of exactly one of the domains D_A with $|A|=j$, or on the common boundary of two or more such domains. Thus the domains D_A with $|A|=j$ ($j \leqq |S|-1$), together with the complementary of H form a tessellation T_j of the plane. We say that a point of the plane is a vertex of the tessellation T_j if it belongs to the boundary of more than two faces of T_j. Two faces of T_j are said to be adjacent if they have more than one boundary-point in common. The sets of the common boundary-points of adjacent faces of T_j are called edges of T_j. Under the conditions of the Lemma the radical axis of each two circles of S is different. It easily follows that if the faces D_A and $D_{A'}$ of the tessellation T_j are adjacent then both $A-A'$ and $A'-A$ consists of one circle. Denote these circles by $(A-A')$ and $(A'-A)$. The common boundary of the domains D_A and $D_{A'}$ is a segment of the radical axis of these circles. This, along with the conditions of the Lemma imply that T_j is trihedral, i.e. in each vertex exactly three faces meet.

To obtain a deeper insight in the structure of the tessellations T_j, we observe that the domains D_A can be constructed by induction in the following way: If A consists of a single circle C then D_A is the first Dirichlet cell of C: $D_{\{C\}}=D_C^1$. Now suppose that the domains D_A with $|A| \leqq j$ are already constructed. Let A be a subset of S with $|A|=j$. Construct the first Dirichlet cells of all circles of $S-A$ with respect to D_A, obtaining a decomposition of D_A. In this decomposition let D_{AC} be the cell belonging to the circle C. If $B \subset S$ and $|B|=j+1$ then we have

$$D_B = \bigcap_{C \in B,\, A=B-\{C\}} D_{AC}.$$

Observe that if in the above decomposition of D_A into the Dirichlet cells of the circles of $S-A$ the cell D_{AC} is not empty then $C=(A'-A)$ for some subset A of S such that $|A|=|A'|=j$ and D_A and $D_{A'}$ are adjacent faces of the tessellation T_j. To see this assume that there is a circle C such that D_{AC} is not empty and there is no A satisfying the above conditions such that $C=(A'-A)$. Then the centre O of C cannot lie in D_A or in any face of T_j adjacent to D_A. Thus we can choose a point Q in the interior of D_{AC} such that the segment OQ intersects the interior of a domain $D_{A'}$ adjacent to D_A. It follows that the segment OQ contains a point of $D_{A'}$ which is nearer to O than to any centre of the circles of $S-A$. In particular, this point of D_A is nearer to O than to the centre of the circle $C'=(A'-A)$ which is impossible since we have assumed that $C \notin A'$.

If D_{A_1} and D_{A_2} are adjacent faces of T_j then the circles $C_1=(A_1-A_2)$ and $C_2=(A_2-A_1)$ are such that the domains $D_{A_1C_2}$ and $D_{A_2C_1}$ are non-empty and they are contained in the same face $D_B=D_{A_1\cup A_2}$ of the tessellation T_{j+1}. Suppose that the faces D_{A_1}, D_{A_2} and D_{A_3} of T_j meet in a vertex. Then we have $|A_1 \cup A_2 \cup A_3| = j+2$ or $|A_1 \cup A_2 \cup A_3|=j+1$. If $|A_1 \cup A_2 \cup A_3|=j+2$ then the six domains $D_{A_1(A_2-A_1)}$, $D_{A_2(A_1-A_2)}$, $D_{A_1(A_3-A_1)}$, $D_{A_3(A_1-A_3)}$, $D_{A_2(A_3-A_2)}$ and $D_{A_3(A_2-A_3)}$ are contained in three different faces of T_{j+1}, namely in $D_{A_1\cup A_2}$, $D_{A_1\cup A_3}$ and $D_{A_2\cup A_3}$. If $|A_1 \cup A_2 \cup A_3|=j+1$ then all these domains are contained in the domain $D_{A_1\cup A_2\cup A_3}$. It follows immediately that the vertices of T_j which are not vertices of T_{j-1} and do not lie on the boundary of H are precisely the common vertices of the tessellations T_j and T_{j+1} (see the figure, where the tessellation T_1, T_2 and T_3 are represented by full lines, broken lines and dotted lines, respectively). Thus using the notations



b_j = number of vertices of T_j lying on the boundary of H,
c_j = number of common vertices of T_j and T_{j-1},
d_j = number of those vertices of T_j which are not vertices of T_{j-1},
we have

(1) $$c_{j+1} = d_j - b_j \quad (j = 1, \ldots, |S|-1).$$

Let f_j and v_j be the number of faces and vertices of T_j, respectively. Since T_j is trihedral, we have in view of Euler's theorem

(2) $$v_j = 2f_j - 2 \quad (j = 1, \ldots, |S|-1).$$

Next we show that there is no closed polygonal line consisting of the edges of T_j joining common vertices of T_j and T_{j-1}. For suppose that Π is such a polygonal line. We may assume without loss of generality that Π is a simple polygon and there is no polygonal line between two vertices of Π consisting of edges of T_j joining common vertices of T_j and T_{j-1} lying in the region enclosed by Π. If Π is not the boundary of a face D_A of the tessellation T_j then there is a vertex W of T_j which is not a vertex of T_{j-1} lying in the region enclosed by Π. It follows by the construction of the tessellation T_j that the boundary of Π is contained in a face D_B of the tessellation T_{j+1}. Since D_B is a convex polygon, it follows that the whole interior of Π belongs to D_B. But the faces of T_j joining at the vertex W cannot be all contained in the same face D_B of T_{j+1}. Thus Π must be the boundary of a face D_A of the tessellation T_j. Let $D_{A_1}, \ldots, D_{A_n}$ be the faces of T_j adjacent to D_A. Then the centre of the circle $C = (A_1 - A) = (A_2 - A) = \ldots = (A_n - A)$ lies in the intersection of the convex angular regions bounded by the elongations of adjacent sides of Π. But this is impossible since the intersection of these angular regions at all vertices of D_A is empty.

Note that the same argument shows that there is no closed polygonal line Π such that all edges of Π are edges of T_j and all but one vertices of Π are common vertices of T_j and T_{j-1}.

Consider the graph G whose vertices are the vertices of T_j and whose edges are the edges of T_j not belonging to the boundary of H. Consider the family $\mathcal{F}$ of all subgraphs F of G which have the property that any two of their vertices can be connected by a path in the subgraph whose vertices are, with the possible exception of the endpoints of the path, all common vertices of T_j and T_{j-1} and which are maximal with respect to this property. Referring to the construction of T_j, we see that the faces of T_{j+1} other than the complementary of H correspond in a one-to-one way to the subgraphs $F\in\mathcal{F}$. The above considerations show that these subgraphs are trees. Let $n(F)$ and $m(F)$ be the number of vertices of F and the number of those vertices of F which are common vertices of T_j and T_{j-1}, respectively ($F\in\mathcal{F}$). Using the fact that T_j is trihedral and that F is a tree, we get $2=n(F)-m(F)$ ($F\in\mathcal{F}$). Adding these equalities for all subgraphs $F\in\mathcal{F}$, we obtain on the left hand side two times the number of faces of the tessellation T_{j+1} other than the complementary of H. On the right hand side each vertex of T_j which is not a vertex of T_{j-1} and does not lie on the boundary of H is counted exactly three times and each vertex of T_j which is also a vertex of T_{j-1} or lies on the boundary of H is counted exactly minus one times. Thus we have

$$2(f_{j+1}-1) = 3(d_j-b_j)-c_j-b_j,$$

i.e.

$$2f_{j+1} = 3d_j-2b_j-c_j+2. \tag{3}$$

Combining (1), (2) and (3) and keeping in view that

$$v_j = c_j+d_j \quad (j = 1, \ldots, |S|-1),$$

we obtain

$$d_{j+1} = 2d_j-d_{j-1}+b_{j-1}-b_j-2 \quad (j = 1, \ldots, |S|-1). \tag{4}$$

Using the initial values $c_0=b_0=0$, $c_1=v_1=2(|S|-1)$ and formula (4), we obtain by induction

$$d_k = 2k|S|-k(k+1)-\sum_{j=1}^{k-1} b_j. \tag{5}$$

A vertex of a k'th Dirichlet cell with a convex angle is either a vertex of T_k which is not a vertex of T_{k-1} or a vertex of the hexagon H. Each vertex of T_k which is not a vertex of T_{k-1} belongs to at most three of the k'th Dirichlet cells. Since by the assumption of the Lemma no edge of T_k emanates from a vertex of H, each vertex of H is a vertex of exactly k k'th Dirichlet cells. Thus we obtain by (5)

$$\sum_{C\in S} p_C^k \leqq 3d_k+6k \leqq 6k|S|.$$

This completes the proof of the Lemma.

The rest of the proof of Theorems 1* and 2* follows a known technique [1]. Clearly we may restrict ourselves in the proof of both theorems to the case when S satisfies all the conditions of the Lemma. We observe that if S forms a k-fold packing (covering) of H then D_C^k contains (is contained in) C. Obviously, if $C \subset D_C^k$ then there is a convex polygon $\bar{D}_C^k$ with at most p_C^k vertices such that $C \subset \bar{D}_C^k \subset D_C^k$. Similarly, if $C \supset D_C^k$ then there is a convex polygon $\underline{D}_C^k$ with at most p_C^k vertices such that $C \supset \underline{D}_C^k \supset D_C^k$. Thus we have

$$D_C^k \geqq \varphi(p_C^k), \quad \varphi(p) = p \tan \frac{\pi}{p}$$

in the case of a k-fold packing, and

$$D_C^k \leqq \psi(p_C^k), \quad \psi(p) = \frac{p}{2} \sin \frac{2\pi}{p}$$

in the case of a k-fold covering. Thus, referring to the fact that $\varphi(p)$ $(p \geqq 3)$ is a strictly decreasing convex function and $\psi(p)$ $(p \geqq 3)$ is a strictly increasing concave function, we have, by Jensen's inequality and the Lemma

$$kH = \sum_{C \in S} D_C^k \geqq \sum_{C \in S} \varphi(p_C^k) \geqq |S| \varphi(6k)$$

and

$$kH = \sum_{C \in S} D_C^k \leqq \sum_{C \in S} \psi(p_C^k) \leqq |S| \psi(6k),$$

respectively. This completes the proof of the theorems.

The notion of the k'th Dirichlet cell enables us to give non-trivial density-bounds for k-fold sphere-packings and sphere-coverings of the n-space for any $k \geqq 1$ and $n \geqq 2$. We shall come back to this problem in another paper.

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НЕРАВЕНСТВО ДЛЯ АЛГЕБРАИЧЕСКИХ МНОГОЧЛЕНОВ И ЗАВИСИМОСТЬ МЕЖДУ НАИЛУЧШИМИ СТЕПЕННЫМИ ПРИБЛИЖЕНИЯМИ $E(f)_{L_p}$ И $E(f)_{L_q}$ ФУНКЦИЙ $f(x)\in L_p$

ХО ТХО КАУ (Будапешт—Ханой)

Известно (см. [1], стр. 239), что если $Q_n(x)$ — произвольный алгебраический многочлен порядка $\leqq n$, то всегда имеет место неравенство:

$$\max_{-1\leqq x\leqq 1}|Q_n^{(k)}(x)|\leqq n^{2k}\max_{-1\leqq x\leqq 1}|Q_n(x)|. \tag{1}$$

Если k неограниченно возрастает, то следующая оценка (см. [1], стр. 241) будет личше, чем (1)

$$\max_{-1\leqq x\leqq 1}|Q_n^{(k)}(x)|\leqq \frac{1}{2}\left(\frac{4n^2}{k}\right)^k\max_{-1\leqq x\leqq 1}|Q_n(x)|. \tag{2}$$

В статье [2] Н. К. Бари доказано, что если $Q_n(x)$ — произвольный алгебраический многочлен, то

$$\|Q_n'(x)\|_{L_p[a,b]}\leqq L(a,b)n^2\|Q_n\|_{L_p[a,b]}, \tag{3}$$

где $L(a,b)$ константа, зависящая только от a и b, и $1\leqq p\leqq +\infty$. Отсюда, непосредственно вытекает, что

$$\|Q_n^{(k)}\|_{L_p[a,b]}\leqq C(a,b)n^{2k}\|Q_n\|_{L_p[a,b]}, \tag{4}$$

где $C(a,b)$ константа, зависящая только от a и b. (Это неравенство также фигурирует в статье [3].)

Кроме того в книге [1] (на стр. 251) А. Ф. Тимана доказано, что

$$\|Q_n\|_{L_q[a,b]}\leqq\left[\frac{2(p+1)}{b-a}\right]^{\frac{1}{p}-\frac{1}{q}}n^{2\left(\frac{1}{p}-\frac{1}{q}\right)}\|Q_n\|_{L_p[a,b]}. \tag{5}$$

Ниже (формула (18)) показываем, что в случае, если $p\leqq 8$, константу в оценке (5) можно улучшить.

Теорема 1. *Если $Q_n(x)$ — произвольный алгебраический многочлен порядка $\leqq n$, то всегда имеет место неравенство:*

$$\max_{-1\leqq x\leqq 1}|Q_n(x)|\leqq\left(\frac{nq_1+2}{2\sqrt{2}}\right)^{\frac{2}{p}}\|Q_n\|_{L_p[-1,1]}, \tag{6}$$

где q_1 наименьшее четное число, болшее либо равное p.

Доказательство. Пусть $\{p_n(x)\}$ $(n=0, 1, 2, \ldots)$ система ортонормальных полиномов Лежандра, т. е.

$$(7)\qquad \int_{-1}^{1} P_j(x)\cdot P_m(x)\,dx = \begin{cases} 0, & \text{если} \quad j \neq m, \\ 1, & \text{если} \quad j = m. \end{cases}$$

Если функция $f(x)$ на отрезке $[-1, 1]$ интегрируема и

$$(8)\qquad f(x) = c_0 p_0(x) + c_1 p_1(x) + \ldots + c_n p_n(x) + \ldots,$$

то

$$c_j = \int_{-1}^{1} f(t) P_j(t)\,dt.$$

Если $S_n(f; x)$ n-тая частичная сумма ряда (8), то

$$(9)\qquad S_n(f; x) = \int_{-1}^{1} f(t) K_n(x; t)\,dt,$$

где

$$K_n(x; t) = \sum_{j=0}^{n} P_j(t) P_j(x).$$

Специальным образом, если $f(x) \equiv Q_n(x)$ — произвольный алгебраический многочлен порядка $\leqq n$, то из (9) следует:

$$Q_n(x) = \int_{-1}^{1} Q_n(t) K_n(x; t)\,dt.$$

Отсюда и в силу неравенства Буняковского—Шварца непосредственно вытекает, что

$$(10)\qquad \max_{-1 \leqq x \leqq 1} |Q_n(x)| \leqq \max_x \|K_n(x; t)\|_{L_2[-1,1]} \cdot \|Q_n\|_{L_2[-1,1]}.$$

Здесь

$$\|K_n(x; t)\|_{L_2[-1,1]} = \left[\int_{-1}^{1} \left(\sum_{j=0}^{n} P_j(t) P_j(x)\right)^2 dt\right]^{\frac{1}{2}}.$$

Отсюда и из (7) следует:

$$\|K_n(x; t\|_{L_2[-1,1]} = \left(\sum_{j=0}^{n} P_j^2(x)\right)^{\frac{1}{2}}.$$

Известно (см. например, [4], стр. 300—301), что

$$\max_{-1 \leqq x \leqq 1} |P_j(x)| \leqq \sqrt{\frac{2j+1}{2}}.$$

Таким образом

$$\max_x \|K_n(x; (t)\|_{L_2[-1,1]} \leqq \frac{n+1}{\sqrt{2}}.$$

Отсюда и из (10) следует следующее неравенство:

$$\max_{-1\leqq x\leqq 1}|Q_n(x)|\leqq\frac{n+1}{\sqrt{2}}\|Q_n\|_{L_2[-1,1]}. \tag{11}$$

Если теперь применить неравенство (11) к полиному $\{Q_n(x)\}^{\frac{q_1}{2}}$, то получим:

$$\max_{-1\leqq x\leqq 1}|Q_n(x)|^{\frac{q_1}{2}}\leqq\frac{nq_1+2}{2\sqrt{2}}\Big(\int_{-1}^{1}|Q_n|^{q_1}\,dt\Big)^{\frac{1}{2}};$$

поэтому

$$\max_{-1\leqq x\leqq 1}|Q_n(x)|^{\frac{q_1}{2}}\leqq\frac{nq_1+2}{2\sqrt{2}}\Big(\int_{-1}^{1}|Q_n|^{q_1-p}|Q_n|^{p}\,dt\Big)^{\frac{1}{2}}\leqq$$

$$\leqq\frac{nq_1+2}{2\sqrt{2}}\max|Q_n|^{\frac{q_1-p}{2}}\Big(\int_{-1}^{1}|Q_n|^{p}\,dt\Big)^{\frac{1}{2}}.$$

Отсюда следует:

$$\max_{-1\leqq x\leqq 1}|Q_n(x)|^{\frac{p}{2}}\leqq\frac{nq_1+2}{2\sqrt{2}}\|Q_n\|_{L_p[-1,1]}^{\frac{p}{2}};$$

поэтому

$$\max_{-1\leqq x\leqq 1}|Q_n(x)|\leqq\left(\frac{nq_1+2}{2\sqrt{2}}\right)^{\frac{2}{p}}\|Q_n\|_{L_p[-1,1]}. \tag{12}$$

Этим теорема 1 доказана.

Замечание. Так как

$$\frac{nq_1+2}{2\sqrt{2}}\leqq\frac{q_1+1}{2\sqrt{2}}n\quad(n\geqq 2), \tag{13}$$

то из (12) следует неравенство:

$$\max_{-1\leqq x\leqq 1}|Q_n(x)|\leqq\left(\frac{q_1+1}{2\sqrt{2}}\right)^{\frac{2}{p}}n^{\frac{2}{p}}\|Q_n\|_{L_p[-1,1]}. \tag{14}$$

Следствие. *Если $Q_n(x)$ — произвольный алгебраический многочлен порядка $\leqq n$, то*

$$\max_{-1\leqq x\leqq 1}|Q_n^{(k)}(x)|\leqq\left(\frac{q_1+1}{2\sqrt{2}}\right)^{\frac{2}{p}}n^{2\left(k+\frac{1}{p}\right)}\|Q_n\|_{L_p[-1,1]}. \tag{15}$$

В самом деле, в силу неравенства (1) и неравенства (14):

$$\max_{-1\leqq x\leqq 1}|Q_n^{(k)}(x)|\leqq n^{2k}\max_{-1\leqq x\leqq 1}|Q_n(x)|\leqq\left(\frac{q_1+1}{2\sqrt{2}}\right)^{\frac{2}{p}}n^{2\left(k+\frac{1}{p}\right)}\|Q_n\|_{L_p[-1,1]}.$$

Заметим, что в случае, если k неограниченно возрастает (именно $k \geqq 4$), следующая оценка будет лучше, чем (15):

$$\max_{-1 \leqq x \leqq 1} |Q_n^{(k)}(x)| \leqq \frac{1}{2} \frac{4^k}{k^k} \left(\frac{q_1+1}{2\sqrt{2}} \right)^{\frac{2}{p}} n^{2\left(k+\frac{1}{p}\right)} \|Q_n\|_{L_p[-1,1]}. \tag{16}$$

Это неравенство нетрудно получить из неравенства (2) и (14).

Теорема 2. *Если $Q_n(x)$ — произвольный алгебраический многочлен порядка $\leqq n$ и $1 \leqq p \leqq q \leqq +\infty$, то*

$$\|Q_n\|_{L_q[-1,1]} \leqq \left(\frac{nq_1+2}{2\sqrt{2}} \right)^{2\left(\frac{1}{p}-\frac{1}{q}\right)} \|Q_n\|_{L_p[-1,1]}, \tag{17}$$

где q_1 — наименьшее четное число, большее либо равное p.

Доказательство. Так как $p \leqq q$, то

$$\int_{-1}^{1} |Q_n|^q \, dx = \int_{-1}^{1} |Q_n|^{q-p} |Q_n|^p \, dx \leqq \max_{-1 \leqq x \leqq 1} |Q_n|^{q-p} \int_{-1}^{1} |Q_n|^p \, dx.$$

Отсюда следует:

$$\|Q_n\|_{L_q[-1,1]} \leqq \max |Q_n|^{\frac{q-p}{q}} \|Q_n\|_{L_p[-1,1]}^{\frac{p}{q}}.$$

В силу теоремы 1

$$\|Q_n\|_{L_q[-1,1]} \leqq \left[\left(\frac{nq_1+2}{2\sqrt{2}} \right)^{\frac{2}{p}} \|Q_n\|_{L_p[-1,1]} \right]^{\frac{q-p}{q}} \|Q_n\|_{L_p[-1,1]}^{\frac{p}{q}},$$

следовательно

$$\|Q_n\|_{L_q[-1,1]} \leqq \left(\frac{nq_1+2}{2\sqrt{2}} \right)^{2\left(\frac{1}{p}-\frac{1}{q}\right)} \|Q_n\|_{L_p[-1,1]}. \tag{18}$$

Этим теорема 2 доказана.

Замечание. Так как

$$\frac{nq_1+2}{2\sqrt{2}} \leqq \frac{q_1+1}{2\sqrt{2}} n \quad (n \geqq 2),$$

то из теоремы 2 следует, что

$$\|Q_n\|_{L_q[-1,1]} \leqq \left(\frac{q_1+1}{2\sqrt{2}} \right)^{2\left(\frac{1}{p}-\frac{1}{q}\right)} n^{2\left(\frac{1}{p}-\frac{1}{q}\right)} \|Q_n\|_{L_p[-1,1]}. \tag{19}$$

Следствие. *Пусть $Q_n(x)$ — произвольный алгебраический многочлен порядка $\leqq n$ и $1 \leqq p \leqq q \leqq +\infty$, тогда*

$$\|Q_n^{(k)}\|_{L_q[-1,1]} \leqq C \left(\frac{q_1+1}{2\sqrt{2}} \right)^{2\left(\frac{1}{p}-\frac{1}{q}\right)} n^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)} \|Q_n\|_{L_p[-1,1]}, \tag{20}$$

где C — абсолютная константа.

В случае, если $k \geqq 4$, *следующая оценка лучше, чем оценка* (20):

(21) $$\|Q_n^{(k)}(x)\|_{L_q[-1,1]} \leqq C\left(\frac{q_1+1}{2\sqrt{2}}\right)^{2\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{4^k}{2k^k}\right)^{1-\frac{p}{q}} n^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)}\|Q_n\|_{L_p[-1,1]}.$$

Замечание. *Из* (20) *и* (21) *следует следующая оценка:*

(22) $$\|Q_n^{(k)}\|_{L_q[-1,1]} \leqq M n^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)}\|Q_n\|_{L_p[-1,1]},$$

где

$$M = \begin{cases} C\left(\dfrac{q_1+1}{2\sqrt{2}}\right)^{2\left(\frac{1}{p}-\frac{1}{q}\right)}, & \text{если} \quad k < 4, \\ C\left(\dfrac{q_1+1}{2\sqrt{2}}\right)^{2\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\dfrac{4^k}{2k^k}\right)^{1-\frac{p}{q}}, & \text{если} \quad k \geqq 4. \end{cases}$$

Доказательство. Неравенство (20) следует из неравенства (4) и (19). Для доказательства неравенства (21) заметим, что

$$\int_{-1}^{1} |Q_n^{(k)}|^q\,dx = \int_{-1}^{1} |Q_n^{(k)}|^p |Q_n^{(k)}|^{q-p}\,dx \leqq \max|Q_n^{(k)}(x)|^{q-p} \int_{-1}^{1} |Q_n^{(k)}|^p\,dx,$$

т. е.

$$\|Q_n^{(k)}\|_{L_q[-1,1]} \leqq \max|Q_n^{(k)}|^{\frac{q-p}{q}} \|Q_n^{(k)}\|_{L_p}^{\frac{p}{q}}.$$

Отсюда в силу неравенства (2), (4) и (14) следует:

$$\|Q_n^{(k)}\|_{L_q[-1,1]} \leqq \left[\frac{4^k n^{2k}}{2k^k}\left(\frac{q_1+1}{2\sqrt{2}}\right)^{\frac{2}{p}} n^{\frac{2}{p}}\|Q_n\|_{L_p[-1,1]}\right]^{\frac{q-p}{q}} C^{\frac{p}{q}} n^{\frac{2kp}{q}} \|Q_n\|_{L_p}^{\frac{p}{q}}.$$

Но так как $\dfrac{p}{q} \leqq 1$, то отсюда непосредственно вытекает, что

$$\|Q_n^{(k)}\|_{L_q[-1,1]} \leqq C\left(\frac{4^k}{2k^k}\right)^{1-\frac{p}{q}}\left(\frac{q_1+1}{2\sqrt{2}}\right)^{2\left(\frac{1}{p}-\frac{1}{q}\right)} n^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)}\|Q_n\|_{L_p[-1,1]}.$$

Этим неравенство (21) доказано.

Теорема 3. *Пусть* $1 \leqq p \leqq q \leqq +\infty$ *и функция* $f(x)$, *заданная на* $[-1, 1]$, *принадлежит* L_p, *т. е.*

$$\|f\|_{L_p[-1,1]} = \left(\int_{-1}^{1} |f(x)|^p\,dx\right)^{\frac{1}{p}} < +\infty.$$

Если наилучшее приближение $E_n(f)_{L_p}$ *обладает тем свойством, что при некотором* $k \geqq 0$

$$\sum_{n=1}^{\infty} n^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)-1} E_n(f)_{L_p} < +\infty,$$

то функция $f(x)$ почти везде имеет k-ую производную $f^{(k)}(x)\in L_q[-1,1]$ и справедливо неравенство:

$$E_n(f^{(k)})_{L_q[-1,1]}\leqq$$

$$\leqq M2^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)+1}\left[n^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)}E_n(f)_{L_p}+2^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)}\sum_{\nu=n+1}^{\infty}\nu^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)-1}E_\nu(f)_{L_p}\right],$$

где M константа, зависящая только от k, p и q (см. (22)). В частности, если $k=0$, то

$$E_n(f)_{L_q[-1,1]}\leqq L\left[n^{2\left(\frac{1}{p}-\frac{1}{q}\right)}E_n(f)_{L_p}+2^{2\left(\frac{1}{p}-\frac{1}{q}\right)}\sum_{\nu=n+1}^{\infty}\nu^{2\left(\frac{1}{p}-\frac{1}{q}\right)-1}E_\nu(f)_{L_p}\right],$$

где

$$L=2\left(\frac{q_1+1}{\sqrt{2}}\right)^{2\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

Доказательство. Пусть $E_n(f)_{L_p}$ обозначает наилучшее приближение функций $f(x)\in L_p$ посредством алгебраических многочленов порядка $\leqq n$ в метрике пространства L_p и $p_n(f;x)$ алгебраический многочлен, наименее уклоняющийся от $f(x)$ т. е.

$$E_n(f)_{L_p[-1,1]}=\|f-p_n(f;x)\|_{L_p[-1,1]}; \tag{24}$$

тогда в смысле сходимости в $L_p[-1,1]$ будет

$$f(x)=p_n(f;x)+\sum_{i=1}^{\infty}u_i(x), \tag{25}$$

где

$$u_i(x)=p_{2^in}(f;x)-p_{2^{i-1}n}(f;x).$$

Для доказательства того, что функция $f(x)$ имеет k-ую производную $f^{(k)}(x)\in\in L_q[-1,1]$, надо показать, что формулу (25) можно продифференцировать почленно k раз, т. е.

$$f^{(k)}(x)=p_n^{(k)}(f;x)+\sum_{i=1}^{\infty}u_i^{(k)}(x). \tag{26}$$

Для доказательства равенства (26) покажем, что ряд справа в (26) в смысле метрики $L_q[-1,1]$ сходится. Прежде всего, оценим $\|u_i(x)\|_{L_p[-1,1]}$. Имеем:

$$u_i(x)=P_{2^in}(f;x)-p_{2^{i-1}n}(f;x).$$

Отсюда

$$\|u_i(x)\|_{L_p[-1,1]}\leqq\|p_{2^in}-f\|_{L_p}+\|f-p_{2^{i-1}n}\|_{L_p}\leqq 2E_{2^{i-1}n}(f)_{L_p},$$

т. е.

$$\|u_i(x)\|_{L_p[-1,1]}\leqq 2E_{2^{i-1}n}(f)_{L_p[-1,1]}. \tag{27}$$

Оценим теперь $\|u_i^{(k)}(x)\|_{L_q}$.

Из (21) получим, что

$$\|u_i^{(k)}(x)\|_{L_q[-1,1]}\leqq M(2^in)^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)}\|u_i(x)\|_{L_p[-1,1]}. \tag{28}$$

Отсюда и из (27) вытекает, что

$$\|u_i^{(k)}(x)\|_{L_q[-1,1]} \leqq 2M(2^i n)^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)} E_{2^{i-1}n}(f)_{L_p[-1,1]}. \tag{29}$$

Таким образом

$$\begin{aligned}\sum_{i=1}^{\infty} \|u_i^{(k)}(x)\|_{L_q} &\leqq 2M \sum_{i=1}^{\infty} (2^i n)^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)} E_{2^{i-1}n}(f)_{L_p[-1,1]} \leqq \\ &\leqq 2M\left[(2n)^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)} E_n(f)_{L_p} + \sum_{i=1}^{\infty} (2^{i+1}n)^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)} E_{2^i n}(f)_{L_p[-1,1]}\right].\end{aligned} \tag{30}$$

Но при $k \geqq 0$, имеем:

$$(2^{i+1}n)^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)} E_{2^i n}(f)_{L_p} \leqq 4^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)} \sum_{\nu=2^{i-1}n+1}^{2^i n} \nu^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)-1} E_\nu(f)_{L_p}.$$

Благодаря этому неравенство (30) примет вид:

$$\sum_{i=1}^{\infty} \|u_i^{(k)}(x)\|_{L_q} \leqq$$

$$\leqq M2^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)+1}\left[n^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)} E_n(f)_{L_p} + 2^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)} \sum_{\nu=n+1}^{\infty} \nu^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)-1} E_\nu(f)_{L_p}\right].$$

Отсюда вытекает, что если ряд

$$\sum_{n=1}^{\infty} n^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)-1} E_n(f)_{L_p[-1,1]}$$

сходится, то ряд (26) сходится в смысле метрики $L_q[-1,1]$ к функции $f^{(k)}(x) \in L_q[-1,1]$ и кроме того получим следующую оценку:

$$E_n(f^{(k)})_{L_q[-1,1]} = \|f^{(k)} - p_n^{(k)}\|_{L_q[-1,1]} \leqq$$

$$\leqq M2^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)+1}\left[n^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)} E_n(f)_{L_p} + 2^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)} \sum_{\nu=n+1}^{\infty} \nu^{2\left(k+\frac{1}{p}-\frac{1}{q}\right)-1} E_\nu(f)_{L_p}\right],$$

что и требовалось доказать.

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THE DISTRIBUTION OF DIVISORS MOD 1

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1. Let d denote a general positive divisor of the natural number n and $\tau(n)$ the number of these divisors. For any real number x, let $\{x\}=x-[x]$ denote the fractional part of x.

Let

$$f_n(\alpha) = \frac{1}{\tau(n)} \sum_{\{\log d\} \leqq \alpha} 1. \tag{1.1}$$

R. R. HALL proved in his paper [1] that for almost all n (i. e. removing a set of zero density of integers n) the relation $f_n(\alpha) \to \alpha$ holds uniformly for $\alpha \in [0, 1]$. Furthermore he gave an estimation for the discrepancy

$$\Delta(n) \overset{\text{def}}{=} \sup_{0 \leqq \alpha < \beta < 1} |f_n(\beta) - f_n(\alpha) - (\beta - \alpha)|. \tag{1.2}$$

Namely, he proved the following assertion:

If $\lambda < \frac{1}{2}$, then for almost all n,

$$\Delta(n) \leqq \tau(n)^{-\lambda}. \tag{1.3}$$

Our aim is to improve (1.3). We shall prove the following

THEOREM. *If* $\lambda < \dfrac{\log \pi}{\log 2} - 1$, *then for almost all* n

$$\Delta(n) \leqq \tau(n)^{-\lambda}. \tag{1.4}$$

It seems to me probable that this result is not far from the best possible one.

2. In the proof of our theorem we need a result of ERDŐS and TURÁN [2] which we state now as

LEMMA 1. *Let* $x_1, x_2, \dots x_N$ *be any real numbers,*

$$f(x) = \frac{1}{N} \sum_{\{x_i\} \leqq x} 1, \quad S_m = \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i m x_j}.$$

Then if

$$\Delta = \sup_{0 \leqq \alpha < \beta \leqq 1} |f(\beta) - f(\alpha) - (\beta - \alpha)|$$

and T *is any positive integer,*

$$\Delta \ll \frac{1}{T} + \sum_{m=1}^{T} \frac{|S_m|}{m}. \tag{2.1}$$

The constant implied by Vinogradov's notation $\ll$ is independent of T and the numbers x_i.

We set

$$\tau(n, \theta) = \sum_{d|n} d^{i\theta}. \tag{2.2}$$

From Lemma 1 we get

$$\Delta(n) \ll \frac{1}{T} + \sum_{m=1}^{T} \frac{1}{m} \cdot |\tau(n, 2\pi m)|. \tag{2.3}$$

First we prove the following

LEMMA 2. *Let $f(n)$ be a multiplicative function, $f(n) \geqq 0$ for every n, and $f(p^\alpha) \leqq c\alpha$ for every prime power p^α. Then*

$$\sum_{n \leqq x} f(n) \leqq c_1 \frac{x}{\log x} \exp\left(\sum_{p \leqq x} \frac{f(p)}{p}\right), \tag{2.4}$$

where c_1 depends only on c.

Similar results have been given by many authors, perhaps (2.4) is not new. We give a proof for the sake of completeness. Let

$$A(x) = \sum_{n \leqq x} f(n), \quad B(x) = \sum_{n \leqq x} f(n) \cdot \log n.$$

From the non-negativity of f we get

$$A(x) - A(\sqrt{x}) \leqq \frac{2}{\log x} \cdot B(x).$$

From $f(p^\alpha) \leqq c\alpha$ we get easily that

$$A(\sqrt{x}) \ll x^{1/2+\varepsilon},$$

where ε is an arbitrary constant. Furthermore,

$$B(x) = \sum_{n \leqq x} f(n) \cdot \sum_{q^\alpha \| n} \log q^\alpha = \sum_{q^\alpha h \leqq x} f(h) f(q^\alpha) \log q^\alpha,$$

where in the last sum $(q^\alpha, h) = 1$ is assumed too.

Observing that from the conditions the relation

$$\sum_{q^\alpha \leqq y} f(q^\alpha) \log q^\alpha \ll y$$

immediately follows, we get

$$B(x) \ll x \sum_{h \leqq x} \frac{f(h)}{h}.$$

Taking into consideration that

$$\sum_{h \leqq x} \frac{f(h)}{h} \leqq \prod_{p \leqq x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right)$$

and that $f(p^\alpha) < c\alpha$, we get (2.4).

Let now γ be a constant in [0, 1] and consider the sum

$$C_\gamma(x) = \sum_{n\le x} \Delta(n)\cdot\tau(n)^\gamma. \tag{2.5}$$

By (2.3) we get

$$C_\gamma(x) \ll \frac{1}{T}\sum_{n\le x}\tau(n) + \sum_{m=1}^{T}\frac{1}{m}\sum_{n\le x}|\tau(n, 2\pi m)|\cdot\tau(n)^{\gamma-1}. \tag{2.6}$$

We take $T=[\log^2 x]$, and so for the first sum we have

$$\frac{1}{T}\sum_{n\le x}\tau(n) \ll \frac{x}{\log x}.$$

Now we shall give an upper estimation for the functions

$$D_m(x) = \sum_{n\le x}|\tau(n, 2\pi m)|\cdot\tau(n)^{\gamma-1}.$$

Let $f_m(n)=|\tau(n, 2\pi m)|\cdot\tau(n)^{\gamma-1}$. Then $0\le f_m(p^\alpha)\le\alpha+1\le 2\alpha$, and so the conditions of Lemma 2 are satisfied with $c=2$. We have $f_m(p)=2^\gamma|\cos 2\pi m\log p|$. We shall use the prime number theorem $\pi(u)=\mathrm{li}\, u+R(u)$ with the remainder term

$$R(u) \ll u\exp\left(-\sqrt{\log u}\right). \tag{2.7}$$

Let z be defined by the relation $\log z=(\log x)^{\varepsilon(x)}$, where $\varepsilon(x)$ tends to zero slowly. Then

$$\sum_{p<z}\frac{f_m(p)}{p} \ll \varepsilon(x)\log\log x.$$

Furthermore, we take

$$\sum_{z\le p\le x}\frac{f_m(p)}{p} = 2^\gamma\int_z^x\frac{|\cos 2\pi m\log u|}{u}\,d\,\mathrm{li}\, u + 2^\gamma\int_z^x\frac{|\cos 2\pi m\log u|}{u}\,dR(u) = 2^\gamma I_1 + 2^\gamma I_2.$$

We consider the first integral. By taking the substitution $m\log u=v$, we get

$$I_1 = \int_A^B\frac{|\cos 2\pi v|}{v}\,dv,$$

where $A=m\log z$, $B=m\log x$. Hence, by elementary calculations we get

$$I_1 = \frac{2}{\pi}\log\frac{B}{A}+O(1) \le \frac{2}{\pi}\log\log x+O(1).$$

Now we estimate I_2. By partial integration we obtain that

$$I_2 = \left[R(u)\frac{|\cos 2\pi m\log u|}{u}\right]_z^x - \int_z^x R(u)\left(\frac{|\cos 2\pi m\log u|}{u}\right)'du.$$

Observing that

$$\left|\left(\frac{|\cos 2\pi m\log u|}{u}\right)'\right| \ll \frac{m}{u^2},$$

we deduce straightaway that

$$I_2 \ll \exp(-\sqrt{\log z}) + m \int_z^x \exp(-\sqrt{\log u})\, du \ll$$

$$\ll m \log z \cdot \exp(-\sqrt{\log z}) \ll \exp\left(-\frac{1}{2}\sqrt{\log z} - 2\log\log x\right) = o(1),$$

if $\varepsilon(x)$ tends to zero sufficiently slowly. Thus we obtained that for an arbitrary positive $\varepsilon > 0$ the inequality

$$\sum_{p<x} \frac{f_m(p)}{p} \leqq (1+\varepsilon) \cdot \frac{2^{\gamma+1}}{\pi} \log\log x$$

holds uniformly for $m \leqq T$, when $x \geqq x_0(\varepsilon)$.

From (2.6) we deduce that

$$C_\gamma(x) \ll \frac{x}{\log x} + x \log\log x \cdot \exp\left(\left[(1+\varepsilon)\frac{2^{\gamma+1}}{\pi} - 1\right] \log\log x\right).$$

Let γ be chosen so that $2^{\gamma+1} < \pi$, and ε be so small that

$$(1+\varepsilon)\frac{2^{\gamma+1}}{\pi} - 1 < 0.$$

Hence we get $C_\gamma(y) \ll x \cdot (\log x)^{-\delta}$, $\delta > 0$ being a constant.

Let $M(x)$ denote the number of those integers $n \leqq x$ for which $\Delta(n)\tau(n)^\gamma \geqq 1$. Obviously we have

$$M(x) \leqq C(x) \ll x/(\log x)^\delta.$$

Hence we get our assertion immediately.

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A RESULT ON CONSECUTIVE PRIMES

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1. Let

(1.1) $$S(n) = \sum_{p<n} \frac{1}{n-p},$$

where p runs over the prime numbers. ERDŐS and DE BRUIJN proved (see [1]) that

(1.2) $$c_1 N \leqq \sum_{n \leqq N} S^2(n) \leqq c_2 N,$$

(1.3) $$c_1 N/\log N \leqq \sum_{p \leqq N} S(p) \leqq c_2 N/\log N,$$

(1.4) $$c_1 N/\log N \leqq \sum_{p \leqq N} S^2(p) \leqq c_2 N/\log N,$$

where c_1, c_2 are suitable constants.

The investigation of the sum (1.1) is interesting from the following point of view. From the relation $\limsup S(n) = \infty$ it would follow that

$$\liminf_{k \to \infty} \frac{p_{k+1} - p_k}{\log p_k} = 0,$$

where p_k denotes the k'th prime.

I proved the following theorems ([2], [3]) on the assumption of the density hypothesis for the Riemann ζ-function.

Let p, q be prime numbers, $\Lambda(n)$ denote the Mangoldt-function,

(1.5) $$S_1(n) = \sum_{m<n} \frac{\Lambda(m)}{n-m}.$$

Let $N(\sigma_0, T)$ be the number of zeros of $\zeta(s)$ in the rectangle $\sigma_0 \leqq \sigma \leqq 1$, $|t| \leqq T$ $(s = \sigma + it)$.

We assume that the density hypothesis holds in the following form:

(1.6) $$N(\sigma, T) < cT^{2(1-\sigma)} \log^2 T,$$

when $1/2 \leqq \sigma \leqq 1, \quad T > 0$.

Then the following relations hold:

$$\sum_{n\leqq N}(S_1(n)-\log n)^2 = O(N\log N\cdot(\log\log N)^2), \tag{1.7}$$

$$\sum_{n\leqq N}(S(n)-1)^2 = O\left(\frac{N}{\log N}\cdot(\log\log N)^2\right), \tag{1.8}$$

$$\sum_{q<N}|S(q)-1| = O\left(\frac{N}{\log^2 N}\cdot(\log\log N)^{3/2}\right), \tag{1.9}$$

q runs over the primes.

At the same time I was unable to prove that

$$\sum_{q<N}(S(q)-1)^2 = o(N/\log N). \tag{1.10}$$

Now we shall deduce that from (1.8).

For the completeness we give a proof for (1.7) too. (1.8) is an easy consequence of (1.7).

Let $z=x+iy,\quad 0<x<1/2,\quad -\pi<y<\pi,$

$$f_1(z) = \sum_{n=2}^{\infty}\Lambda(n)e^{-nz},\quad f_2(z) = \sum_{n=1}^{\infty}\frac{e^{-nz}}{n},\quad s(n) = \sum_{\nu=1}^{n}\frac{1}{\nu}. \tag{1.11}$$

Let

$$g(z) \overset{\text{def}}{=} \left(f_1(z)-\frac{1}{1-e^{-z}}\right)f_2(z) = \sum_{n=1}^{\infty}(S_1(n)-s(n))e^{-nz}. \tag{1.12}$$

By the Parseval-formula we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(z)|^2\,dy = \sum_{n=1}^{\infty}(S_1(n)-s(n))^2e^{-2nx}. \tag{1.13}$$

Let

$$T(z) = \sum_{\varrho} z^{-\varrho}\Gamma(\varrho), \tag{1.14}$$

where ϱ runs over the non-trivial zeros of the zeta-function. YU. V. LINNIK proved [4, 5] that

$$f_1(z) = \frac{1}{z}-T(z)+O\left(\log^3\frac{1}{x}\right), \tag{1.15}$$

further he deduced from (1.6), that

$$\int_{-\Delta}^{\Delta}|T(z)|^2\,dy = O\left(\frac{1}{x}\left(\log\frac{1}{x}\right)^{-1}\right), \tag{1.16}$$

where

$$\Delta = \left(\log\frac{1}{x}\right)^{-7}. \tag{1.17}$$

Since

$$\frac{1}{z}-\frac{1}{1-e^{-z}} = O(1)$$

for $0<x<1$, we get

(1.18)
$$\int_{-\Delta}^{\Delta} |g(z)|^2\,dy \leqq c_1 \int_{-\Delta}^{\Delta} |f_2(z)|^2\,dy + $$
$$+c_1\left(\log\frac{1}{x}\right)^6 \int_{-\Delta}^{\Delta} |f_2(z)|^2\,dy + c_1 \int_{-\Delta}^{\Delta} |T(z)|^2|f_2(z)|^2\,dy,$$

$c_1, c_2, \ldots$ denote suitable positive constants.

Furthermore,

$$f_2(z) = -\log(1-e^{-z}),$$

so

(1.19)
$$|f_2(z)|^2 = \frac{1}{4}\log^2(1-2e^{-x}\cos y+e^{-2x})+O(1),$$

and hence

$$|f_2(z)|^2 = O\left(\log^2\frac{1}{x}\right).$$

Using this and (1.16), from (1.18) we can deduce easily that

(1.20)
$$\int_{-\Delta}^{\Delta} |g(z)|^2\,dy = O\left(\frac{1}{x}\log\frac{1}{x}\right).$$

Furthermore, for $|y|\geqq\Delta$

(1.21)
$$|f_2(z)| = O\left(\log\log\frac{1}{x}\right),$$

as easily seen from (1.19). Thus

(1.22)
$$\int_{\Delta\leqq|y|\leqq\pi} |g(z)|^2\,dy \leqq c_2\left(\log\log\frac{1}{x}\right)^2\left\{\int_{-\pi}^{\pi}|f_1(z)|^2\,dy + \int_{-\pi}^{\pi}\left|\frac{1}{1-e^{-z}}\right|^2 dy\right\}.$$

Since

$$\int_{-\pi}^{\pi} |f_1(z)|^2\,dy = 2\pi\sum_{n=1}^{\infty}\Lambda^2(n)e^{-2nx} = O\left(\frac{1}{x}\log\frac{1}{x}\right),$$

$$\int_{-\pi}^{\pi} |1-e^{-z}|^{-2}\,dy = 2\pi\sum_{n=0}^{\infty}e^{-2nx} = O(x^{-1}),$$

thus

$$\int_{\Delta\leqq|y|\leqq\pi} |g(z)|^2\,dy = O\big(1/x\cdot(\log 1/x)(\log\log 1/x)^2\big).$$

Hence we get

$$\sum_{n=1}^{\infty}(S_1(n)-s(n))^2e^{-2nx} = O\left(\frac{1}{x}\cdot\left(\log\frac{1}{x}\right)\cdot\left(\log\log\frac{1}{x}\right)^2\right)$$

and by choosing $x = 1/N$

$$\sum_{n \leqq N} (S_1(n) - s(n))^2 = O(N \log N \cdot (\log \log N)^2).$$

Taking into account that $s(n) = \log n + O(1)$, we get (1.7).

The relation (1.8) almost immediately follows from (1.7).

2. Now we are dealing with the estimation of the sum

$$\sum_{q<N} (S(q) - 1)^2. \tag{2.1}$$

We shall use the following well known inequalities. Let $\pi(x)$ denote the number of primes in the interval $[1, x]$. Then

$$\pi(u+v) - \pi(u) \leqq c_1 \frac{v}{\log v} \quad (u, v \geqq 2) \tag{2.2}$$

(See e.g. [1].) For a positive integer k let $N(x, k)$ denote the number of solutions of the equation $p - q = k$ $(p, q \leqq x)$ where p, q run over the primes. Then

$$N(x, k) < c_2 \frac{x}{\log^2 x} \cdot \frac{k}{\varphi(k)}, \tag{2.3}$$

φ denotes the Euler-function.

Let $Q(x, k, l)$ denote the number of those primes $p \leqq x$, for which $p+k$ and $p+k+l$ are primes too. Then

$$Q(x, k, l) < c \frac{x}{\log^3 x} \lambda(k, l), \quad \lambda(k, l) = \prod_{p \mid kl(k-l)} \left(1 - \frac{1}{p}\right)^{-2}. \tag{2.4}$$

Let $y > 1, \quad 1 < z < y$,

$$R(n) = \sum_{n-z<p<n} \frac{1}{n-p} \tag{2.5}$$

$$T(n) = S(n) - R(n) = \sum_{p \leqq n-z} \frac{1}{n-p}. \tag{2.6}$$

First we estimate the sums

$$A(y) = \sum_{y \leqq n \leqq 2y} R^2(n), \tag{2.7}$$

$$B(y) = \sum_{y \leqq q \leqq 2y} R^2(q). \tag{2.8}$$

We have

$$R^2(n) \leqq \sum_{p<n} \frac{1}{(n-p)^2} + 2 \sum_{n-z \leqq p_1 < p_2 < n} \frac{1}{(n-p_1)(n-p_2)} = U_1(n) + 2U_2(n). \tag{2.9}$$

It is obvious that

$$\sum_{y \leqq n \leqq 2y} U_1(n) \leqq c\pi(y) = O(y/\log y).$$

Furthermore,

$$\sum_{y\leqq n\leqq 2y} U_2(n) \leqq \sum_{k\leqq z} N(2y,k) \sum_{v=1}^{z} \frac{1}{v(v+k)} \leqq$$

$$\leqq c \sum_{1\leqq k\leqq z} \frac{2y}{\log^2 y}\cdot\frac{k}{\varphi(k)}\frac{\log k}{k} = O\left(\frac{y}{\log^2 y}\cdot\log^2 z\right),$$

and so

$$A(y) = O(y/\log y)+O\big(y/\log^2 y\cdot(\log z)^2\big). \tag{2.10}$$

Now we estimate (2.7).

$$\sum_{y<q\leqq 2y} U_1(q) \leqq \sum_{y-z<p<q<2y} \frac{1}{(q-p)^2} \leqq \sum_{k<z} \frac{N(2y,k)}{k^2} = O(y/\log^2 y).$$

Furthermore, by (2.4)

$$\sum_{y<q<2y} U_2(q) \leqq \sum_{q-z\leqq p_1<p_2<q} \frac{1}{(q-p_1)(q-p_2)} \leqq$$

$$\leqq \sum_{k,\, l\leqq z} \frac{1}{k(k+l)} Q(2y,k,l) = O\left(\frac{y}{\log^3 y}\log^2 z\right).$$

Hence

$$B(y) = O\left(\frac{y}{\log^2 y}\right)+O\left(\frac{y}{\log^3 y}\log^2 z\right). \tag{2.11}$$

Let now H be chosen so that $1\leqq H\leqq z$, and let $1\leqq j\leqq z$. Then

$$T(n+j)-T(n) = \sum_{n-z<p<n+j-z} \frac{1}{n+j-p} + \sum_{p<n-z}\left(\frac{1}{n+j-p}-\frac{1}{n-p}\right) =$$

$$= D_1(n,j)-D_2(n,j),$$

$$D_2(n,j) = j\sum_{p<n-z}\frac{1}{(n-p)(n+j-p)}.$$

We observe that

$$D_1(n,j) \leqq \frac{1}{z}+\frac{1}{z+1}+\ldots+\frac{1}{z+j} = O(j/z),$$

$$D_2(n,j) \leqq j\sum_{p<n-z}\frac{1}{(n-p)^2} = O(j/z), \quad j\geqq 1,$$

whence $|T(n+j)-T(n)| = O(j/z)$ follows. Hence

$$\big|(T(q)-1)^2-(T(q+j)-1)^2\big| \leqq |T(q+j)-T(q)|\cdot\{T(q+j)+T(q)\} \leqq$$

$$\leqq c\frac{j}{z}(T(q)+T(q+j)) \leqq 2c\frac{j}{z}T(q)+O\left(\left(\frac{j}{z}\right)^2\right) = O\left(\frac{j}{z}\right)\cdot(T(q)+1).$$

Hence

$$L_1 \stackrel{\text{def}}{=} \sum_{y\leqq q\leqq 2y}(T(q)-1)^2 \leqq \frac{L_2}{H}+c\frac{H}{z}\sum_{y\leqq q\leqq 2y}(T(q)+1),$$

where

$$L_2=\sum_{j=1}^{H}\sum_{y\le q\le 2y}(T(q+j)-1)^2.$$

Observing that

$$\sum_{y\le q\le 2y}T(q)\le\sum\sum_{p<q<2y}\frac{1}{q-p}\le\sum_{k<2y}N(2y,k)\cdot\frac{1}{k}=O(y/\log y),$$

we get

$$L_1\le\frac{L_2}{H}+O\left(\frac{H}{z}\,y/\log y\right).$$

Now we estimate L_2.

$$L_2\le\sum_{y\le n\le 2n+H}(T(n)-1)^2\sum_{n-H\le p\le n}1\le\sum_{y\le n\le 2y+H}(T(n)-1)^2\{\pi(n)-\pi(n-H)\}.$$

Hence by (2.2)

$$L_2\le c\,\frac{H}{\log H}\sum_{y\le n\le 2y+H}(T(n)-1)^2.$$

From the inequalities

$$(S(q)-1)^2\le 2(T(q)-1)^2+2R^2(q),\quad (T(n)-1)^2\le 2(S(n)-1)^2+2R^2(n),$$

we deduce that

$$K(y)\overset{\mathrm{def}}{=}\sum_{y\le q\le 2y}(S(q)-1)^2\le\frac{c}{\log H}\Big\{\sum_{y\le n\le 2y+H}(S(n)-1)^2+A(y)+A(2y)\Big\}+$$

$$+B(y)+O\left(\frac{H}{z}\,\frac{y}{\log y}\right).$$

Using the inequalities (1.8), (2.10), (2.11), we have

$$K(y)\le\frac{c_1}{\log H}\left\{\frac{y}{\log^2 y}(\log\log y)^2+\frac{y}{\log y}+\frac{y}{\log^2 y}\log^2 z\right\}+$$

$$+c_2\frac{y}{\log^2 y}+c_3\frac{y}{\log^3 y}\log^2 z+c_4\frac{H}{z}\,\frac{y}{\log y}.$$

Let now $z=H^2$, $z=\exp\big((\log y)^{1/2}\log\log y\big)$. Then

$$K(y)\le c\,\frac{y(\log\log y)}{(\log y)^{3/2}}.$$

Since

$$\sum_{q\le N}(S(q)-1)^2\le\sum_{1\le j\le\frac{\log N}{\log 2}}K(N/2^j),$$

we get immediately that

$$\sum_{q\le N}(S(q)-1)^2=O\left(\frac{N\log\log N}{(\log N)^{3/2}}\right).$$

Thus we proved the following

THEOREM. *Assuming the validity of the density hypothesis in the form* (1.6),

$$\sum_{q\leqq N}(S(q)-1)^2 = O\left(\frac{N\log\log N}{(\log N)^{3/2}}\right).$$

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SATURATION OF CERTAIN OPERATOR-SEQUENCES

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1. Introduction

The approximation properties of convolution operators having positive kernels or kernels of finite oscillations have a well developed theory. We shall see that in many cases the above mentioned restrictions for the kernels can be omitted. Namely, we prove some convergence and saturation theorems for convolution-type operators where the number of oscillations of the kernels are not necessarily bounded when $n \to \infty$.

2. The notion of quasi-positive convolution operators

2.1. With the notations

$$l_n(t) = \frac{\sin \frac{2n+1}{2} t}{(2n+1) \sin \frac{t}{2}} \qquad (n = 1, 2, 3, \ldots), \tag{2.1}$$

$$\begin{cases} t_{k,n} = \dfrac{2k\pi}{2n+1} \\ t_{n+1,n} = \pi, \quad t_{-n-1,n} = -\pi \end{cases} \qquad (k = 0, \pm 1, \ldots, \pm n;\ n = 1, 2, 3, \ldots) \tag{2.2}$$

we can define the kernels $P_{n,\alpha}(t)$ and $Q_{n,\alpha}(t)$ for any real $\alpha > 1$ as

$$P_{n,\alpha}(t) \overset{\text{def}}{=} \frac{(2n+1)^\alpha}{2p_\alpha} |l_n(t)|^\alpha \quad (\alpha > 1), \tag{2.3}$$

$$Q_{n,\alpha}(t) \overset{\text{def}}{=} \begin{cases} \dfrac{(2n+1)^\alpha}{2q_\alpha} (-1)^k |l_n(t)|^\alpha & 0 \leqq t_{k,n} \leqq t \leqq t_{k+1,n} \leqq \pi, \\ Q_{n,\alpha}(-t) \quad \text{if} \quad t \leqq 0 \quad (\alpha > 1), \end{cases} \tag{2.4}$$

where

$$p_\alpha \overset{\text{def}}{=} \frac{(2n+1)^\alpha}{\pi} \int_{-\pi}^{\pi} |l_n(t)|^\alpha \, dt, \tag{2.5}$$

$$q_\alpha \overset{\text{def}}{=} 2 \frac{(2n+1)^\alpha}{\pi} \sum_{k=0}^{n} (-1)^k \int_{t_{k,n}}^{t_{k+1,n}} |l_n(t)|^\alpha \, dt \quad (\alpha > 1). \tag{2.6}$$

As we shall prove, for any fixed $\alpha>1$

$$p_\alpha \sim n^{\alpha-1}, \quad q_\alpha \sim n^{\alpha-1} \quad (\alpha > 1), \tag{2.7}$$

so our definitions have meaning (see Lemma 5.1). By these even, 2π-periodic and continuous kernels for any fixed $\alpha>1$ we introduce the operators $\{P_{n,\alpha}\}$ and $\{Q_{n,\alpha}\}$ as follows:

$$(P_{n,\alpha}*f)(x) \overset{\text{def}}{=} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) P_{n,\alpha}(t)\, dt \quad (\alpha > 1), \tag{2.8}$$

$$(Q_{n,\alpha}*f)(x) \overset{\text{def}}{=} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) Q_{n,\alpha}(t)\, dt \quad (\alpha > 1), \tag{2.9}$$

where f is 2π-periodic and continuous, i.e. $f\in\tilde{C}$.

By definition $(P_{n,\alpha}*1)(x)\equiv(Q_{n,\alpha}*1)(x)\equiv 1$, i.e. the kernels are normalized.

2.2. $P_{n,\alpha}(t)$ is positive. On the other hand, the number of oscillations of $Q_{n,\alpha}(t)$ is not bounded, when $n\to\infty$. These functions have some interesting properties if $\alpha>3$. Namely

$$\varliminf_{n\to\infty} \frac{\int_{-\pi}^{\pi} \sin^2\frac{t}{2}\, Q_{n,\alpha}(t)\, dt}{\int_{-\pi}^{\pi} \sin^2\frac{t}{2}\, |Q_{n,\alpha}(t)|\, dt} > 0 \quad (\alpha > 3) \tag{2.10}$$

(see Lemma 5.2). Further we have

$$\varlimsup_{n\to\infty} \int_{-\pi}^{\pi} |Q_{n,\alpha}(t)|\, dt < \infty \quad (\alpha > 1). \tag{2.11}$$

(see Lemma 5.1). We use these important properties at the following

2.3. Definition. Let $\{L_n\}$ be a sequence of convolution operators ($=$CO), i.e. for any $f\in\tilde{C}$

$$L_n(f;\, x) = (f*d\mu_n)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)\, d\mu_n(t) \qquad (n = 1, 2, 3, \ldots) \tag{2.12}$$

where $d\mu_n$ is even Borel measure on $[-\pi, \pi)$ with $L_n(1;\, x)\equiv 1$. If

(i) $\varlimsup_{n\to\infty} A_{0,n} = A_0 < \infty$ where $A_{0,n} \overset{\text{def}}{=} \int_{-\pi}^{\pi} |d\mu_n(t)|$,

(ii) $\varliminf_{n\to\infty} A_{1,n} = A_1 > 0$ where $A_{1,n} \overset{\text{def}}{=} \dfrac{\int_{-\pi}^{\pi} \sin^2\frac{t}{2}\, d\mu_n(t)}{\int_{-\pi}^{\pi} \sin^2\frac{t}{2}\, |d\mu_n(t)|}$

then we say that $\{L_n\}$ is a quasi-positive convolution operator ($=$QPCO)-sequence.

By this definition $\{Q_{n,\alpha}\}$ is a QPCO-sequence for $\alpha>3$.

3. Theorems for quasi-positive convolution operators

It is the most important from our point of view that many well-known theorems proved for positive convolution operators (=PCO) can be obtained for quasi-positive ones.

3.1. We shall prove an estimation which has been known only for PCO (see [2], Corollary 2.1 and [6]).

THEOREM 3.1. *If* $\{L_n\}$ *is a* QPCO-*sequence (from* $\tilde{C}$ *into* $\tilde{C}$*) then for each* $f\in\tilde{C}$

(3.1) $$\|L_n(f;x)-f(x)\| \leqq c_1\omega_2\big(f;(1-\varrho_{1,n})^{1/2}\big) \qquad (n=1,2,3,\ldots).$$

Here as usual

(3.2) $$\begin{cases} \|f\| = \max\limits_{0\leqq x<2\pi} |f(x)|, \\ \omega_k(f;t) \text{ is the } k\text{-th modulus of smoothness of } f(x)\ (\omega_1\equiv\omega;\ k=1,2,3\ldots), \end{cases}$$

further

(3.3) $$\varrho_{k,n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos kt\, d\mu_n(t) \qquad (k=0,1,2,\ldots).$$

We note that by (ii) we have for $n\geqq n_0$

(3.4) $$1-\varrho_{1,n} = \frac{1}{\pi}\int_{-\pi}^{\pi}(1-\cos t)\,d\mu_n(t) = \frac{2}{\pi}\int_{-\pi}^{\pi}\sin^2\frac{t}{2}\,d\mu_n(t) > 0.$$

From Theorem 3.1 we get the analogon of the wellknown Korovkin-theorem.

COROLLARY 3.1. *If* $\{L_n\}$ *is a* QPCO-*sequence and*

(3.5) $$L_n(\cos t;x)\to\cos x \quad (n\to\infty) \quad \left(x \text{ is arbitrary fixed, } x\neq(2k+1)\frac{\pi}{2}\right),$$

then for each $f\in\tilde{C}$ *we have*

$$\|L_n(f;x)-f(x)\|\to 0 \quad (n\to\infty).$$

Indeed,

$$|\cos x - L_n(\cos t;x)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi}[\cos x-\cos(t+x)]\,d\mu_n(t)\right| =$$

$$= \left|\frac{1}{\pi}\int_{-\pi}^{\pi}(\cos x-\cos t\cos x+\sin t\sin x)\,d\mu_n(t)\right| = 2\frac{|\cos x|}{\pi}\int_{-\pi}^{\pi}\sin^2\frac{t}{2}\,d\mu_n(t) =$$

$$= |\cos x|(1-\varrho_{1,n}),$$

fromwhere $1-\varrho_{1,n}\to 0$.

3.2. To determine the saturation, its order and saturation class for QPCO we shall use the next theorems. They are similar to theorems for PCO and can be proved as their positive analogon completing by (i), (ii) and the ideas of 5.1. We shall sketch or sometimes omit the proofs.

3.2.1. First we recall a necessary and sufficient condition for a CO-sequence to be saturated (see [2], Theorems 3.1 and 3.2).

THEOREM 3.2. *Let* $\{L_n\}$ *be a* CO-*sequence. Then* $\{L_n\}$ *is saturated if and only if for some positive integer m*

$$\varliminf_{n\to\infty} \frac{|1-\varrho_{k,n}|}{|1-\varrho_{m,n}|} \overset{\text{def}}{=} \psi_k > 0 \qquad (k = 1, 2, 3, \dots) \tag{3.6}$$

and a saturation order is $\{|1-\varrho_{m,n}|\}$. *Further if especially*

$$\lim_{n\to\infty} \frac{1-\varrho_{k,n}}{1-\varrho_{1,n}} = \psi_k \neq 0 \qquad (k = 1, 2, 3, \dots) \tag{3.7}$$

then for any $f \in S(L_n)$

$$\sum_{k=1}^{\infty} \psi_k A_k(f;x) \in L_\infty \overset{\text{def}}{=} \{f;\ |f(x)| \leqq M \text{ a.e.}\}. \tag{3.8}$$

We remark that we use the definition of saturation given by [2], 3.1.5 and 3.1.6. [1]

Here $A_k(f;x) = a_k(f)\cos kx + b_k(f)\sin kx$ where $a_k(f)$ and $b_k(f)$ are the Fourier coefficients of f.

The proof of Theorem 3.2 runs on the same line as Theorems 3.1. and 3.2 from [2] using absolute value where it is necessary.

3.2.2. A well-known property of the PCO-sequences is that if $m=1$ then $\psi_k = O(k^2)$ (see (3.6)). But we can see that this is true for QPCO-sequences, too. Indeed, by (ii)

$$|1-\varrho_{k,n}| = \left|\frac{2}{\pi}\int_{-\pi}^{\pi}\sin^2\frac{kt}{2}\,d\mu_n(t)\right| \leqq \frac{2}{\pi}\int_{-\pi}^{\pi}\sin^2\frac{kt}{2}\,|d\mu_n(t)| \leqq$$

$$\leqq \frac{2}{\pi}k^2\int_{-\pi}^{\pi}\sin^2\frac{t}{2}\,|d\mu_n(t)| = \frac{2}{\pi^2}\frac{k^2}{A_{1,n}}\int_{-\pi}^{\pi}\sin^2\frac{t}{2}\,d\mu_n(t) = \frac{k^2}{A_{1,n}}(1-\varrho_{1,n})$$

as we stated.

Moreover we have the exact analogon of the theorem of A. H. TURECKII (see [2], Theorem 3.6).

THEOREM 3.3. *If* $\{L_n\}$ *is a* QPCO-*sequence for which*

$$\lim_{n\to\infty} \frac{1-\varrho_{k,n}}{1-\varrho_{1,n}} = k^2 \qquad (k = 1, 2, 3, \dots) \tag{3.9}$$

[1] DEFINITION. Let $\{L_n\}$ be a sequence of operators from $\tilde{C}$ into $\tilde{C}$ and $\{\Phi_n\}$ a sequence of positive real numbers which converge to 0. We say that $\{L_n\}$ is saturated with order $\{\Phi_n\}$ if the following two conditions are satisfied:

(a) If $f \in \tilde{C}$, then

$$\varliminf_{n\to\infty} \frac{\|f(x)-L_n(f;x)\|}{\Phi_n} = 0$$

if and only if f is a constant;

(b) There is a non-constant function $f_0 \in \tilde{C}$ for which

$$\|f_0(x)-L_n(f_0;x)\| = O(\Phi_n).$$

We denote the saturation class by $S(L_n)$ (=all functions f_0 which satisfy (b)).

then $\{L_n\}$ *is saturated with order* $\{1-\varrho_{1,n}\}$ *and*

$$S(L_n) = \{f;\ \sum_{k=1}^{\infty} k^2 A_k(f;\ x) \in L_\infty\} = \{f; f' \in \mathrm{Lip}\ 1\}.$$

Indeed, by Theorem 3.2 $\{L_n\}$ is saturated with order $\{1-\varrho_{1,n}\}$, and for $f\in S(L_n)$ $\sum_{k=1}^{\infty} k^2A_k(f;x)\in L_\infty$ i.e. $f'\in\mathrm{Lip}\ 1$. On the other hand if $f'\in\mathrm{Lip}\ 1$ then by (3.1) $\|f(x)-L_n(f;x)\|=O(1-\varrho_{1,n})$, i.e. $f\in S(L_n)$.

3.2.3. To state the equivalent formulations of Tureckii's condition (3.9) for CO-sequences, we assume that

$$\text{(iii)}\quad \lim_{n\to\infty} A_{2,n} = A_2 > 0 \quad \text{where} \quad A_{2,n} \overset{\text{def}}{=} \frac{\int_{-\pi}^{\pi} \sin^4 \frac{t}{2}\, d\mu_n(t)}{\int_{-\pi}^{\pi} \sin^4 \frac{t}{2}\, |d\mu_n(t)|}.$$

THEOREM 3.4. *If* $\{L_n\}$ *is a* CO-*sequence having* (ii) *and* (iii) *then the following are equivalent*

$$\text{(3.10)}\qquad \lim_{n\to\infty} \frac{1-\varrho_{k,n}}{1-\varrho_{1,n}} = k^2 \qquad (k = 1, 2, 3, \ldots),$$

$$\text{(3.11)}\qquad \lim_{n\to\infty} \frac{1-\varrho_{2,n}}{1-\varrho_{1,n}} = 4,$$

$$\text{(3.12)}\qquad \left|\int_{-\pi}^{\pi} \sin^4 \frac{t}{2}\, d\mu_n(t)\right| = |A_{2,n}| \int_{-\pi}^{\pi} \sin^4 \frac{t}{2}\, |d\mu_n(t)| = o(1-\varrho_{1,n}),$$

$$\text{(3.13)}\qquad \int_{\delta}^{\pi} |d\mu_n(t)| = o(1-\varrho_{1,n}) \quad \text{for each} \quad \delta > 0.$$

The theorem and its proof is similar to Theorem 3.8, [2]. In many cases the following statement — which can be proved like Theorem 3.4 — may be useful.

THEOREM 3.5. *If for a* CO-*sequence* (ii) *is valid then we have the following relation-chain for* $\gamma\geqq 0$, $0<\delta<\pi$

$$\text{(3.14)}\qquad \begin{cases} \int_{-\pi}^{\pi} \left|\sin \frac{t}{2}\right|^{\gamma} |d\mu_n(t)| = o(1-\varrho_{1,n}) \Rightarrow \int_{\delta}^{\pi} |d\mu_n(t)| = \\ = o(1-\varrho_{1,n}) \Rightarrow \lim_{n\to\infty} \frac{1-\varrho_{k,n}}{1-\varrho_{1,n}} = k^2 \qquad (k = 1, 2, 3, \ldots) \\ \Rightarrow \lim_{n\to\infty} \frac{1-\varrho_{2,n}}{1-\varrho_{1,n}} = 4. \end{cases}$$

(„$A\Rightarrow B\Rightarrow C$" means: A implies B; B implies C.)

4. Some applications

4.1. In this section we wish to apply our results for $\{Q_{n,\alpha}\}$ and $\{P_{n,\alpha}\}$. We remark that $\{Q_{n,\alpha}\}$ is a QPCO-sequence if $\alpha>3$ (see (2.10) and (2.11)), further $\{P_{n,\alpha}\}$ is PCO-, i.e., QPCO-sequence, too ($\alpha>1$). Moreover, $Q_{n,2s+1}(t)$ and $P_{n,2s}(t)$ are trigonometric polynomials of degree $(2s+1)n$ and $2sn$, respectively ($s=1, 2, 3, \ldots$).

THEOREM 4.1. *For any fixed* $\alpha>3$ *the* QPCO-*sequence* $\{Q_{n,\alpha}\}$ *and the* PCO-*sequence* $\{P_{n,\alpha}\}$ *are saturated with order* $\{n^{-2}\}$ *further*

$$S(Q_{n,\alpha}) = S(P_{n,\alpha}) = \{f; f\in \tilde{C} \quad \textit{and} \quad f'\in \mathrm{Lip}\,1\}. \tag{4.1}$$

As the rate of the convergence we have

$$|(Q_{n,\alpha}*f)(x)-f(x)| \leqq c_\alpha \omega_2\left(f; \frac{1}{n}\right), \tag{4.2}$$

$$|(P_{n,\alpha}*f)(x)-f(x)| \leqq d_\alpha \omega_2\left(f; \frac{1}{n}\right). \tag{4.3}$$

4.2. For $\alpha\leqq 3$ we have to refine our estimations.

THEOREM 4.2. *If* $2<\alpha\leqq 3$ *then the* CO-*sequence* $\{Q_{n,\alpha}\}$ *is saturated with order* $\{n^{-2}\}$ *further*

$$S(Q_{n,\alpha}) = \{f; f\in \tilde{C}, f'\in \mathrm{Lip}\,1\}. \tag{4.4}$$

We have

$$|(Q_{n,\alpha}*f)(x)-f(x)| \leqq \begin{cases} c_\alpha \omega\left(f; \dfrac{1}{n}\right) & \textit{for each} \quad f\in \tilde{C}, \\ \dfrac{e_\alpha}{n}\omega\left(f'; \dfrac{1}{n}\right) & \textit{if} \quad f'\in C'. \end{cases} \tag{4.5}$$

4.3. If $\alpha=3$ we have the following interesting saturation theorem for $\{P_{n,3}\}$.

THEOREM 4.3. *The* PCO-*sequence* $\{P_{n,3}\}$ *is saturated with order* $\left\{\dfrac{\log n}{n^2}\right\}$ *further*

$$S(P_{n,3}) = \{f; f\in \tilde{C}, f'\in \mathrm{Lip}\,1\}.$$

We have

$$|(P_{n,3}*f)(x)-f(x)| \leqq d_3 \omega_2\left(f; \frac{(\log n)^{1/2}}{n}\right) \qquad (n = 2, 3, 4, \ldots). \tag{4.6}$$

4.4. Let $m\geqq 3$ be a fixed integer, further

$$\begin{cases} V_{n,m}(t) = P_{n,m}(t), & V_{n,m} = P_{n,m} \quad \text{if } m \text{ is even}, \\ V_{n,m}(t) = Q_{n,m}(t), & V_{n,m} = Q_{n,m} \quad \text{if } m \text{ is odd}, \end{cases} \tag{4.7}$$

$$\varrho_{k,n}^{(m)} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos kt V_{n,m}(t)\,dt \quad (m\geqq 3). \tag{4.8}$$

Then for these trigonometric CO-sequences $\{V_{n,m}\}$ we have the following

THEOREM 4.4. *For any fixed positive integer m*

(4.9) $$\lim_{n\to\infty} n^2(1-\varrho_{k,n}^{(m)}) = -\frac{B_{2,m}}{A_{1,m}}k^2 \qquad (m \geqq 3;\ k = 1, 2, 3, \ldots),$$

where

(4.10) $$A_{1,m} \overset{\text{def}}{=} \frac{1}{(m-1)!}\sum_{i=0}^{\left[\frac{m}{2}\right]}(-1)^i\binom{m}{i}(m-2i)^{m-1} > 0,$$

(4.11) $$B_{2,m} \overset{\text{def}}{=} \frac{\binom{m-1}{2}}{(m-1)!}\sum_{i=0}^{\left[\frac{m}{2}\right]}(-1)^i\binom{m}{i}(m-2i)^{m-3} < 0.$$

(If $m \geqq 4$ is even, compare [3], Theorem 2 and Lemma 2.)

5. Proofs

5.1. PROOF OF THEOREM 3.1. We use the ideas of Theorem 2.4 from [6]. By (i), (ii) and (3.4)

$$|L_n(f; x)-f(x)| = \left|\frac{2}{\pi}\int_0^\pi [f(x+t)-2f(x)+f(x-t)]\,d\mu_n(t)\right| =$$

$$= O(1)\int_0^\pi \omega_2(f; t)|d\mu_n(t)| = O(1)\omega_2\big(f; (1-\varrho_{1,n})^{1/2}\big)\times$$

$$\times\int_0^\pi [1+(1-\varrho_{1,n})^{-1}t^2]\,|d\mu_n(t)| =$$

$$= O(1)\omega_2\big(f; (1-\varrho_{1,n})^{1/2}\big)\int_0^\pi \left[1+(1-\varrho_{1,n})^{-1}\sin^2\frac{t}{2}\right]|d\mu_n(t)| =$$

$$= O(1)\omega_2\big(f; (1-\varrho_{1,n})^{1/2}\big) \qquad (n \geqq n_0)$$

from where we get (3.1).

5.2. To go further we need some lemmas.

5.2.1. LEMMA 5.1.

(5.1) $$\int_{-\pi}^{\pi} |l_n(t)|^\alpha\,dt \sim \sum_{k=0}^{n}(-1)^k\int_{t_{k,n}}^{t_{k+1,n}} |l_n(t)|^\alpha\,dt \sim \frac{1}{n} \quad (\alpha > 1).$$

PROOF. We obtain after some transformations

$$\int_{-\pi}^{\pi} |l_n(t)|^\alpha dt = 2\int_0^{\pi} |l_n(t)|^\alpha dt = 2\left(\int_0^{t_1} + \int_{t_1}^{\pi}\right) \frac{\left|\sin \frac{2n+1}{2} t\right|^\alpha}{\left|(2n+1)\sin \frac{t}{2}\right|^\alpha} dt \sim$$

$$\sim \int_0^{t_1} \frac{n^\alpha t^\alpha}{n^\alpha t^\alpha} dt + \int_{t_1}^{\pi} |l_n(t)|^\alpha dt \sim \frac{1}{n}$$

because of

$$0 < \int_{t_1}^{\pi} |l_n(t)|^\alpha dt \leqq \frac{1}{n^\alpha} \int_{t_1}^{\pi} t^{-\alpha} dt \sim \frac{1}{n} \qquad (\alpha > 1;\ t_k \equiv t_{k,n}).$$

For the second relations we estimate as follows.

$$\int_0^{t_1} \frac{\sin^\alpha \frac{2n+1}{2} t}{(2n+1)^\alpha \sin^\alpha \frac{t}{2}} dt > \frac{1}{(2n+1)^\alpha}\left[\int_0^{\frac{\pi}{2n+1}} \frac{\left(\frac{2}{\pi}\frac{2n+1}{2} t\right)^\alpha}{\left(\frac{t}{2}\right)^\alpha} dt + \int_{\frac{\pi}{2n+1}}^{\frac{2\pi}{2n+1}} \frac{\sin^\alpha \frac{2n+1}{2} t}{\sin^\alpha \frac{t}{2}} dt\right] >$$

$$> \frac{1}{(2n+1)^\alpha}\left[2^\alpha \left(\frac{2n+1}{\pi}\right)^{\alpha-1} + \frac{1}{\sin^\alpha \frac{\pi}{2n+1}} \int_0^{\frac{\pi}{2n+1}} \sin^\alpha \frac{2n+1}{2} t\, dt\right].$$

Further

$$\int_{t_1}^{t_2} \frac{\left|\sin \frac{2n+1}{2} t\right|^\alpha}{\left|(2n+1)\sin \frac{t}{2}\right|^\alpha} dt < \frac{2}{(2n+1)^\alpha \sin^\alpha \frac{\pi}{2n+1}} \int_0^{\frac{\pi}{2n+1}} \sin^\alpha \frac{2n+1}{2} t\, dt.$$

So denoting

$$\int_{t_k}^{t_{k+1}} |l_n(t)|^\alpha dt \quad \text{by} \quad a_k,$$

we obtain

$$a_0 - a_1 > \frac{1}{(2n+1)^\alpha}\left[2^\alpha \left(\frac{2n+1}{\pi}\right)^{\alpha-1} - \frac{1}{\sin^\alpha \frac{\pi}{2n+1}} \int_0^{\frac{\pi}{2n+1}} \sin^\alpha \frac{2n+1}{2} t\, dt\right] >$$

$$> \frac{1}{(2n+1)^\alpha}\left[2^\alpha \left(\frac{2n+1}{\pi}\right)^{\alpha-1} - \left(\frac{2n+1}{2}\right)^\alpha \frac{2}{2n+1} \int_0^{\frac{\pi}{2}} \sin^\alpha x\, dx\right] >$$

$$> \frac{2}{2n+1}\left[\left(\frac{2}{\pi}\right)^{\alpha-1} - \frac{1}{2^\alpha}\right] \sim \frac{1}{n}.$$

Using that

$$\sum_{k=-(n+1)}^{n} a_k > \sum_{k=-(n+1)}^{n} (-1)^k a_k = 2\sum_{k=0}^{n} (-1)^k a_k > 2(a_0 - a_1)$$

we have (5.1).

5.2.2. LEMMA 5.2. *We have*

$$1-\gamma_{1,n}^{(\alpha)} \overset{\text{def}}{=} \frac{2}{\pi}\int_{-\pi}^{\pi} \sin^2\frac{t}{2}\, Q_{n,\alpha}(t)\,dt \sim \frac{1}{n^2} \quad \textit{if} \quad \alpha > 2; \tag{5.2}$$

$$1-\mathfrak{p}_{1,n}^{(\alpha)} \overset{\text{def}}{=} \frac{2}{\pi}\int_{-\pi}^{\pi} \sin^2\frac{t}{2}\, P_{n,\alpha}(t)\,dt \sim \begin{cases} \dfrac{1}{n^2} & \textit{if} \quad \alpha > 3, \\[2ex] \dfrac{\log n}{n^2} & \textit{if} \quad \alpha = 3, \\[2ex] \dfrac{1}{n^{\alpha-1}} & \textit{if} \quad 1 < \alpha < 3. \end{cases} \tag{5.3}$$

PROOF. We have

$$\frac{2}{\pi}\int_{-\pi}^{\pi} \sin^2\frac{t}{2}\, Q_{n,\alpha}(t)\,dt \sim$$

$$\sim \frac{1}{n^{\alpha-1}} \sum_{k=1}^{\left[\frac{n}{2}\right]} \int_{t_{2k}}^{t_{2k+1}} \sin^\alpha \frac{2n+1}{2}t \left[\frac{1}{\sin^{\alpha-2}\dfrac{t}{2}} - \frac{1}{\sin^{\alpha-2}\left(\dfrac{t}{2}+\dfrac{\pi}{2n+1}\right)}\right] dt +$$

$$+\frac{1}{n^2} \sim \frac{1}{n^\alpha}\sum_{k=1}^{n} \frac{1}{n}\frac{n^{\alpha-1}}{k^{\alpha-1}} + \frac{1}{n^2} \sim \frac{1}{n^2} \quad (\alpha > 2).$$

Here we used that for $\beta \neq 0$

$$\frac{1}{\sin^\beta \dfrac{t}{2}} - \frac{1}{\sin^\beta\left(\dfrac{t}{2}+\dfrac{\pi}{2n+1}\right)} = \frac{\beta \sin^{\beta-1}\xi \cos\xi \cdot \dfrac{\pi}{2n+1}}{\sin^\beta \dfrac{t}{2} \sin^\beta\left(\dfrac{t}{2}+\dfrac{\pi}{2n+1}\right)} \sim \frac{\operatorname{sg}\beta}{n} \cdot \frac{\cos\dfrac{t}{2}}{\sin^{\beta+1}\dfrac{t}{2}}$$

$$\left(0 < \frac{t_1}{2} \leqq \frac{t}{2} < \xi < \frac{t}{2} + \frac{\pi}{2n+1}\right).$$

For proving (5.3) we have

$$\frac{2}{\pi}\int_{-\pi}^{\pi} \sin^2\frac{t}{2}\, P_{n,\alpha}(t)\,dt \sim \frac{1}{n^{\alpha-1}}\left(\int_0^{t_1} + \sum_{k=1}^{n}\int_{t_k}^{t_{k+1}}\right) \frac{\left|\sin^\alpha \dfrac{2n+1}{2}t\right|}{\sin^{\alpha-2}\dfrac{t}{2}}\,dt \sim$$

$$\sim \frac{1}{n^{\alpha-1}}\left(\frac{1}{n}n^{\alpha-2} + \frac{1}{n}\sum_{k=1}^{n}\frac{n^{\alpha-2}}{k^{\alpha-2}}\right),$$

from where we get (5.3).

5.2.3. The following fact will be very useful.

LEMMA 5.3. *For any* $0<\delta<\pi$ *we have*

$$\int_\delta^\pi |Q_{n,\alpha}(t)|\,dt \sim \int_\delta^\pi P_{n,\alpha}(t)\,dt \sim \frac{1}{n^{\alpha-1}} \quad (\alpha>1). \tag{5.4}$$

Indeed, we have by (2.3), (2.4) and (2.7)

$$\int_\delta^\pi |Q_{n,\alpha}(t)|\,dt \sim \int_\delta^\pi P_{n,\alpha}(t)\,dt \sim \frac{1}{n^{\alpha-1}}\int_\delta^\pi \frac{\left|\sin^\alpha \frac{2n+1}{2}t\right|}{\sin^\alpha \frac{t}{2}}\,dt \sim \frac{1}{n^{\alpha-1}}.$$

5.3. PROOF OF THEOREM 4.1. By (5.2) and (5.3) $1-\gamma_{1,n}^{(\alpha)} \sim 1-\mathfrak{p}_{1,n}^{(\alpha)} \sim n^{-2}$. So using (5.4) we get

$$\int_\delta^\pi |Q_{n,\alpha}(t)|\,dt = o(n^{-2}), \qquad \int_\delta^\pi P_{n,\alpha}(t)\,dt = o(n^{-2})$$

i.e. by (3.14) and Theorem 3.3 we get the first part of our theorem. As for (4.2) and (4.3), we have to use the formulae (3.1), (5.2) and (5.3).

5.4. PROOF OF THEOREM 4.2. We want to prove that for a fixed k

$$\lim_{n\to\infty}\frac{1-\gamma_{k,n}^{(\alpha)}}{1-\gamma_{1,n}^{(\alpha)}} = k^2 \quad \text{for} \quad 2<\alpha\leqq 3,\ k=1,2,3,\dots. \tag{5.5}$$

We have the relation

$$\sin^2\frac{kt}{2} = \left(\frac{kt}{2}\right)^2 - \frac{1}{3}\cos\xi\left(\frac{kt}{2}\right)^4 \quad (0<\xi<kt). \tag{5.6}$$

So we have by (5.6)

$$1-\gamma_{k,n}^{(\alpha)} = \frac{2}{\pi}\int_{-\pi}^{\pi}\sin^2\frac{kt}{2}\,Q_{n,\alpha}(t)\,dt =$$

$$= \frac{4}{\pi}\int_0^\pi \frac{k^2t^2}{4}Q_{n,\alpha}(t)\,dt + O(1)\int_{-\pi}^{\pi} k^4t^4 Q_{n,\alpha}(t)\,dt \overset{\text{def}}{=} I_k + O(1)J_k.$$

As in the proof of Lemma 5.2, we have the relation $I_1 \sim \frac{1}{n^2}$. As for J_s, we obtain

$$J_s \sim \frac{1}{n^{\alpha-1}}\left(\int_0^{t_1} + \sum_{k=1}^{\left[\frac{n}{2}\right]}\int_{t_{2k}}^{t_{2k+1}}\right) t^4 \sin^\alpha\frac{2n+1}{2}t\left[\frac{1}{\sin^\alpha\frac{t}{2}} - \frac{1}{\sin^\alpha\left(\frac{t}{2}+\frac{\pi}{2n+1}\right)}\right]dt +$$

$$+\frac{1}{n^\alpha} \sim \frac{1}{n^{\alpha-1}}\sum_{k=1}^{n}\frac{k^4}{n^4}\frac{1}{n^2}\frac{n^{\alpha+1}}{k^{\alpha+1}} + \frac{1}{n^\alpha} \sim \frac{1}{n^4}\sum_{k=1}^{n}k^{3-\alpha} + \frac{1}{n^\alpha} \sim \frac{1}{n^\alpha} \qquad (s=1,2,3,\dots).$$

I.e. we have

$$\lim_{n\to\infty}\frac{1-\gamma_{k,n}^{(\alpha)}}{1-\gamma_{1,n}^{(\alpha)}}=\lim_{n\to\infty}\frac{k^2I_1+O(1)\,J_k}{I_1+O(1)\,J_1}=k^2.$$

By Theorem 3.2 $\{Q_{n,\alpha}\}$ is saturated with order $\{1-\gamma_{1,n}^{(\alpha)}\sim n^{-2}\}$. Further if $f\in S(Q_{n,\alpha})$ then $\sum k^2A_k(f;x)\in L_\infty$, i.e. $f'\in\text{Lip }1$. On the other hand supposing that (4.5) is true we get for $f'\in\text{Lip }1$ that $\|f(x)-(Q_{n,\alpha}*f)(x)\|=O(n^{-2})$, i.e. $f\in S(Q_{n,\alpha})$.

Let us see (4.5). By (2.7) we have by standard argument

$$\int_{t_k}^{t_{k+1}}|Q_{n,\alpha}(t)|\,dt\leqq\frac{c(\alpha)}{(k+1)^\alpha}\qquad(k=0,1,\ldots,n;\ \alpha>1).$$

So we have

$$|(Q_{n,\alpha}*f)(x)-f(x)|\leqq\frac{2}{\pi}\int_0^\pi\omega_2(f;t)\,|Q_{n,\alpha}(t)|\,dt=$$

$$=O(1)\int_0^\pi\omega(f;t)\,|Q_{n,\alpha}(t)|\,dt=O(1)\,\omega\left(f;\frac{1}{n}\right)\int_0^\pi(nt+1)\,|Q_{n,\alpha}(t)|\,dt=$$

$$=O(1)\,\omega\left(f;\frac{1}{n}\right)\sum_{k=1}^{n}\left(n\frac{k}{n}+1\right)k^{-\alpha}=O(1)\,\omega\left(f;\frac{1}{n}\right).$$

To prove the second relation, we have to use a more delicate estimation due to Dr. J. Szabados. We have

$$|f(x)-(Q_{n,\alpha}*f)(x)|=\frac{1}{\pi}\left|\int_0^\pi[f(x+t)-2f(x)+f(x-t)]\,Q_{n,\alpha}(t)\,dt\right|=$$

$$=\frac{1}{\pi}\left|\sum_{k=0}^{\left[\frac{n}{2}\right]}{}'\left(\int_{t_{2k}}^{t_{2k+1}}+\int_{t_{2k+1}}^{t_{2k+2}}\right)\right|^{2}=\frac{1}{\pi}\left|\sum_{k=0}^{\left[\frac{n}{2}\right]}{}''\int_{t_{2k}}^{t_{2k+1}}\Big\{[f(x+t)-2f(x)+f(x-t)]\,Q_{n,\alpha}(t)+\right.$$

$$\left.+\left[f\left(x+t+\frac{2\pi}{2n+1}\right)-2f(x)+f\left(x-t-\frac{2\pi}{2n+1}\right)\right]Q_{n,\alpha}\left(t+\frac{2\pi}{2n+1}\right)\Big\}\,dt\right|=$$

$$=\frac{1}{\pi}\sum_{k=0}^{\left[\frac{n}{2}\right]}{}'\int_{t_{2k}}^{t_{2k+1}}\Big\{\Big[f\left(x+t+\frac{2\pi}{2n+1}\right)-f(x+t)-f(x-t)+$$

$$+f\left(x-t-\frac{2\pi}{2n+1}\right)Q_{n,\alpha}\left(t+\frac{2\pi}{2n+1}\right)+$$

$$+[f(x+t)-2f(x)+f(x-t)]\left[Q_{n,\alpha}(t)+Q_{n,\alpha}\left(t+\frac{2\pi}{2n+1}\right)\right]\Big]\Big\}\,dt\Big|\leqq$$

As in Lemma 5.2, we have

$$Q_{n,\alpha}(t)+Q_{n,\alpha}\left(t+\frac{2\pi}{2n+1}\right)=O(1)\frac{n}{k^{\alpha+1}}\quad\text{if}\quad t_{2k}\leqq t\leqq t_{2k+1},\ k\geqq1.$$

[2] $\sum'$ means that sometimes the last term is superfluous.

So continuing

$$\leq \frac{c}{\pi} \sum_{k=1}^{\left[\frac{n}{2}\right]}{}' \int_{t_{2k}}^{t_{2k+1}} \left[\frac{2\pi}{2n+1} |f'(\xi)-f'(\eta)| \left|\varrho_{n,\alpha}\left(t+\frac{2\pi}{2n+1}\right)\right| + t\omega(f';t)\frac{n}{k^{\alpha+1}}\right] dt +$$

$$+O\left(\frac{1}{n}\omega\left(f';\frac{1}{n}\right)\right) \leq \frac{c}{n}\sum_{k=1}^{\left[\frac{n}{2}\right]}{}' \omega\left(f';\frac{k}{n}\right)\left[\int_{t_{2k}}^{t_{2k+1}} \left|\varrho_{n,\alpha}\left(t+\frac{2\pi}{2n+1}\right)\right| dt + \frac{1}{k^{\alpha+1}}\right] +$$

$$+O\left(\frac{1}{n}\omega\left(f';\frac{1}{n}\right)\right) \leq \frac{c}{n}\sum_{k=1}^{n}\omega\left(f';\frac{1}{n}\right)\frac{k+1}{k^{\alpha+1}} + O\left(\frac{1}{n}\omega\left(f';\frac{1}{n}\right)\right) \leq \frac{c}{n}\omega\left(f';\frac{1}{n}\right),$$

as we stated.

5.5 Proof of Theorem 4.3. By (5.3) $1-\mathfrak{p}_{1,n}^{(3)} \sim \log n \cdot n^{-2}$, further by (5.4)

$$\int_{\delta}^{\pi} P_{n,3}(t)\,dt = o(1-\mathfrak{p}_{1,n}^{(3)}).$$

I.e., by Theorems 3.4, 3.3 and 3.1 we get our statement.

5.6. Proof of Theorem 4.4. We wish to prove this theorem independently from the previous one. By (4.7), (4.8) and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} V_{n,m}(t)\,dt = 1 \quad (m \geq 3)$$

we have

$$V_{n,m}(t) = \frac{1}{2} + \sum_{k=1}^{mn} \varrho_{k,n}^{(m)} \cos kt \quad (m \geq 3).$$

To determine $\varrho_{k,n}^{(m)}$ we use the notions of [1], § 6. By (2.1) and the notation $z_1 = e^{it} = \cos t + i \sin t$ we get

$$(2n+1)\,l_m(t) = \sum_{k=-n}^{n} z_1^k = \frac{1}{z_1^n}\frac{1-z_1^{2n+1}}{1-z_1}$$

from where

$$[(2n+1)\,l_n(t)]^m = \sum_{k=-mn}^{mn} c_{k,n} z_1^k \qquad (m = 1, 2, 3, \ldots)$$

where $c_{k,m} = c_{-k,m}$ $(k = 1, 2, \ldots, m \cdot n)$. We can compute $c_{k,m}$ from the relation

$$(5.7) \quad \sum_{k=-mn}^{mn} c_{k,m} z^k = \frac{(1-z^{2n+1})^m}{z^{nm}} \sum_{s=0}^{\infty} \frac{(s+1)(s+2)\ldots(s+m-1)}{(m-1)!} z^s \quad (0 < |z| < 1).$$

To verify (5.7) we remark that

$$\frac{1}{(1-z)^m} = \sum_{s=0}^{\infty} \frac{(s+1)(s+2)\ldots(s+m-1)}{(m-1)!} z^s \quad (0 < |z| < 1).$$

Then, using e.g. continuity, we can have (3.5) for $|z| = 1$, which is the desired result. Now we wish to determine the number $c_{k,m}$. Comparing the coefficients in (5.7), we have

$$
(5.8)\quad \begin{cases} c_{k,m} = \dfrac{1}{(m-1)!} \sum\limits_{i=0}^{[\frac{m}{2}]}{}' (-1)^i \dbinom{m}{i} [(m-2i)\underbrace{n}_{1}+k-i+1][(m-2i)n+\underbrace{k}_{2}-i+2] \dots \\ \qquad\qquad \dots [(m-2i)n+\overbrace{k}^{m-1}-i+m-1] \\ \text{where } \sum' \text{ means that further condition for } i \text{ is that the factors} \\ \overset{1}{[\dots]}, \overset{2}{[\dots]}, \dots, \overset{m-1}{[\dots]} \text{ should be positive.} \end{cases}
$$

With another notations

$$
(5.9)\quad \begin{aligned} c_{k,m} &= A_{1,m}^{(k)} n^{m-1} + A_{2,m}^{(k)} n^{m-2} + \dots + A_{m,m}^{(k)} + \\ &+ B_{1,m}^{(k)} n^{m-2} k + B_{2,m}^{(k)} n^{m-3} k^2 + k^3 O(n^{m-4}) \quad (0 \leqq k \leqq mn;\ m \geqq 3). \end{aligned}
$$

By (5.8) we have that for any fixed $k \geqq 0$ there exists an $n \geqq n_0(k, m)$ such that

$$
A_{l,m}^{(k)} = A_{l,m}, \quad B_{j,m}^{(k)} = B_{j,m} \quad (m \geqq 3,\ 1 \leqq l \leqq m,\ 1 \leqq j \leqq 2,\ n \geqq n_0).
$$

E.g., we have by (5.8)

$$
A_{1,3} = 3, \quad A_{1,4} = \frac{16}{3}, \quad B_{1,3} = B_{1,4} = 0, \quad B_{2,3} = -1, \quad B_{2,4} = -2.
$$

But we need much more.

LEMMA 5.4. *We have*

$$
A_{1,m} > 0 \quad (m \geqq 3), \quad B_{1,m} = 0 \quad (m \geqq 3), \quad B_{2,m} < 0 \quad (m \geqq 3).
$$

For proving this lemma we have by (5.8)

$$
B_{1,m} = \frac{m-1}{(m-1)!} \sum_{i=0}^{[\frac{m}{2}]} (-1)^i \binom{m}{i} (m-2i)^{m-2}.
$$

But

$$
2 \sum_{i=0}^{[\frac{m}{2}]} (-1)^i \binom{m}{i} (m-2i)^{m-2} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m-2i)^{m-2}.
$$

According to Lemma 4 from [4], the last expression is equal to zero. To prove the remaining parts we need the formulae

$$
(5.10)\quad \int_0^\infty \frac{\sin^r x}{x^s} dx = \frac{1}{(s-1)!} \int_0^\infty \frac{d^{s-1} \sin^r x}{dx^{s-1}} \frac{dx}{x} \quad (r \geqq s \geqq 2),
$$

$$
(5.11)\quad \int_0^\infty \frac{\sin \alpha x}{x} dx = \operatorname{sg} \alpha \cdot \frac{\pi}{2},
$$

$$
(5.12)\quad \sin^{2\mu} x = \frac{(-1)^\mu}{2^{2\mu-1}} \left[\cos^2 \mu x - \binom{2\mu}{1} \cos(2\mu-2)x + \binom{2\mu}{2} \cos(2\mu-4)x + \dots + \frac{(-1)^\mu}{2} \binom{2\mu}{\mu} \right]
$$

(see [5], Part 497, 21; P. 497, 11; P. 461, 3 (a), respectively; here r, s and μ are integers, α is real). By (3.19) we get

$$\frac{d^{2\mu-3}\sin^{2\mu}x}{dx^{2\mu-3}} = \frac{-1}{2^{2\mu-1}}\sum_{i=0}^{\mu-1}(-1)^i\binom{2\mu}{i}(2\mu-2i)^{2\mu-3}\sin(2\mu-2i)x. \tag{5.13}$$

Further we have

$$\int_0^\infty \frac{\sin^r x}{x^s}\,dx > 0 \quad (r \geqq s \geqq 2).$$

Indeed,

$$\int_0^\infty \frac{\sin^r x}{x^s}\,dx = \sum_{k=0}^\infty (-1)^{kr}\int_{k\pi}^{(k+1)\pi}\frac{|\sin^r x|}{x^s}\,dx \overset{\text{def}}{=} \sum_{k=0}^\infty (-1)^{kr}a_k.$$

But obviously $a_0 > a_1 > \dots > a_k > \dots$; $a_i \to 0$. I.e., $I = \sum_{k=0}^\infty (-1)^{kr}a_k$ converges. So $I = \sum_{i=0}^\infty [a_{2i} + (-1)^r a_{2i+1}] > 0$, as we stated.

We have by (5.8)

$$A_{1,m} = \frac{1}{(m-1)!}\sum_{i=0}^{\left[\frac{m}{2}\right]}(-1)^i\binom{m}{i}(m-2i)^{m-1},$$

$$B_{2,m} = \frac{\binom{m-1}{2}}{(m-1)!}\sum_{i=0}^{\left[\frac{m}{2}\right]}(-1)^i\binom{m}{i}(m-2i)^{m-3}.$$

But by [5], P. 497, 21 (c) and (d)

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^m dx = \frac{\pi}{2^m}A_{1,m} \quad (m \geqq 2).$$

So we get that $A_{1,m} > 0$. Further, by [5], P. 497, (11) (a) we have for any odd $m \geqq 5$

$$\int_0^\infty \frac{\sin^m x}{x^{m-2}}\,dx = -\frac{\pi}{2^{m-1}}B_{2,m} \quad (m = 2q+1,\ m \geqq 5).$$

If m is even, $m \geqq 4$, then by (5.10)—(5.13) we have

$$\int_0^\infty \frac{\sin^m x}{x^{m-2}}\,dx = -\frac{\pi}{2^{m-1}}B_{2,m} \quad (m = 2q,\ m \geqq 4).$$

So we proved our lemma.

To go further let us notice that

$$\varrho_{k,n}^{(m)} = \frac{c_{k,m}}{c_{0,m}}.$$

So using our lemma we have

$$1-\varrho_{k,n}^{(m)} = 1-\frac{c_{k,m}}{c_{0,m}} = \frac{c_{0,m}-c_{k,m}}{c_{0,m}} = \frac{-B_{2,m}n^{m-3}k^2+k^3O(n^{m-4})}{A_{1,m}n^{m-1}+O(n^{m-2})}$$

from where we get (4.9). (4.10) and (4.11) were proved in the lemma.

6. Final remarks

6.1. Of course many further analogon theorems can be proved for QPCO. (For a good source see [2], Chapters 2, 3 and 4.) I should like to assert the following statement which also gives a good orientation of efficiency of special QPCO-sequences.

Let $d\mu_n(t)=T_n(t)dt$ where $T_n(t)$ is an even trigonometric polynomial of degree n. Then we have

THEOREM 6.1. *Let* $\{L_n\}$ *is a* QPCO-*sequence with* $d\mu_n(t)=T_n(t)dt$. *Moreover suppose that*

$$\varliminf_{n\to\infty} |1-\varrho_{k,n}|(1-\varrho_{1,n})^{-1} = \psi_k > 0 \qquad (k = 1, 2, 3, \dots).$$

If for $f\in\tilde{C}$

$$\varliminf_{n\to\infty} n^2\|f(x)-L_n(f;\, x)\| = 0,$$

then f is constant.

This is the analogon of P. C. Curtis' theorem ([2], Theorem 4.4). We sketch the proof. As in [2], Lemma 4.1, we get the relation

$$n^2(1-\varrho_{1,n}) \geqq 2c_0>0 \quad (n \geqq n_0).$$

But then

$$n^2|1-\varrho_{k,n}| \geqq \frac{n^2}{2}\psi_k(1-\varrho_{1,n}) \geqq c_0\psi_k > 0 \quad (n \geqq n_1(k)),$$

from where we get the theorem by the transform technique.

6.2. There are some possibilities to extend Definition 2.3.

a) As we know, the saturation is not always determined by $\{1-\varrho_{1,n}\}$. In many cases $S(L_n)$ depends on periodicity too, when we can use in (ii) $\{1-\varrho_{m,n}\}$ instead of $\{1-\varrho_{1,n}\}$. As for the corresponding theorem, see [2], 3.12.

b) Let $\{L_n\}$ be an arbitrary sequence of linear operators (from $\tilde{C}$ into $\tilde{C}$). Using the Riesz' theorem, $L_n=L_n^++L_n^-$ where $\{L_n^+\}$ and $\{-L_n^-\}$ are positive linear operators. If

(6.1) $\quad 0 < a_i \leqq A_i < \infty \quad$ where $\quad \varliminf_{n\to\infty} A_{i,n} = a_i, \quad \varlimsup_{n\to\infty} A_{i,n} = A_i \qquad (i = 0, 1, 2),$

$$A_{i,n} \overset{\text{def}}{=} \frac{\|L_n(e_i;\, x)-e_i(x)\|}{\|P_n(e_i;\, x)-e_i(x)\|} \qquad (i = 0, 1, 2)$$

(where $e_0(x)=1$, $e_1(x)=\sin x$, $e_2(x)=\cos x$; $P_n=L_n^+-L_n^-$) then we obtain a generalization of our Definition 2.3. Of course, we can apply this for non-periodic continuous functions with $e_i(x)=x^i$. By (6.1) we can prove, e.g., the analogon of the Bohman—Korovkin theorems (see [2], 2.2—2.6).

c) It may exist two sequences of linear operators $\{L_n\}$ and $\{P_n\}$ where P_n are positive, such that we have (6.1). Then we can say that L_n are quasi-P_n-positive. E.g. for $\alpha>2$ $Q_{n,\alpha}$ is quasi-$P_{n,\alpha+1}$-positive.

6.3 Let us suppose that $f\in\tilde{C}$ further $f''(x)$ exists and finite for a fixed x. Then we have

$$f(x+t)=f(x)+2f'(x)\sin\frac{t}{2}+2f''(x)\sin^2\frac{t}{2}+\alpha(t)\sin^2\frac{t}{2}$$

where $\lim\limits_{t=0}\alpha(t)=0$. So if L_n is a CO,

$$L_n(f;x)-f(x)=f''(x)(1-\varrho_{1,n})+\frac{1}{\pi}\int_{-\pi}^{\pi}\alpha(t)\sin^2\frac{t}{2}\,d\mu_n(t).$$

Supposing (ii) and $\int_{\delta}^{\pi}|d\mu_n(t)|=o(1-\varrho_{1,n})$ for any $\delta>0$, we have

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\alpha(t)\sin^2\frac{t}{2}\,d\mu_n(t)\right|\leqq\frac{2}{\pi}\int_{0}^{\delta}|\alpha(t)|\sin^2\frac{t}{2}\,|d\mu_n(t)|+\frac{2}{\pi}\int_{\delta}^{\pi}|d\mu_n(t)|\,dt\leqq\varepsilon(1-\varrho_{1,n})$$

i.e. we get the relation

$$\lim_{n\to\infty}\frac{L_n(f;x)-f(x)}{1-\varrho_{1,n}}=f''(x). \tag{6.2}$$

By (4.7), (4.9) and (6.2) we have

$$V_{n,m}(f;x)=f(x)-\frac{B_{2,m}}{A_{1,m}}\frac{f''(x)}{n^2}+o\left(\frac{1}{n^2}\right)\quad(m\geqq 3). \tag{6.3}$$

6.4. Y. Matsouka proved a similar saturation theorem and a formula corresponding to (6.3) (using the Dirichlet kernel as a starting point) but only for PCO-sequence where the kernels are even trigonometric polynomials ([3]). Our proof is different from his one.

6.5. We have some results for the remaining α, too. E.g., for $1<\alpha<3$ the PCO-sequence $\{P_{n,\alpha}\}$ is saturated with order $\{n^{1-\alpha}\}$; we have the estimations corresponding to Theorem 3.1. Moreover, for $1<\alpha\leqq 2$ the $\{Q_{n,\alpha}\}$-sequence is saturated with order $\{1-\gamma_{1,n}^{(\alpha)}\sim n^{-\alpha}\}$ and e.g. (4.2) is also valid. But we can determine only a part of the corresponding saturation class. We omit the details.

6.6. We can build examples for QPCO-sequences using e.g. the Fejér—Korovkin kernel

$$K_n(t)=\frac{1}{n+2}\left(\frac{\sin\dfrac{\pi}{n+2}\cos\dfrac{n+2}{2}t}{\cos t-\cos\dfrac{\pi}{n+2}}\right)^2$$

as a starting point.

6.7. We have begun to investigate the discretized convolution operators from the above points of view. We wish to return to this problem at another occasion.

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ON CONVOLUTION OPERATORS WITH KERNELS OF FINITE OSCILLATIONS

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1. Introduction

In the last thirteen years an extensive study has been made in the theory of approximation by operators having kernels of finite oscillations. Such kernels are obtained by multiplying non-negative factor-kernels by trigonometric polynomials of fixed degree. With these operators the degree of approximation can be increased, in contrast to the positive operators which can never have better degree than $O(n^{-2})$. A recent paper of J. C. HOFF [1] gives a general method for the above mentioned construction. In the trigonometric case, however, only two special factor-kernels (de la Vallée Poussin and generalized Jackson-kernels) are considered, and the rate of convergence is investigated only for one of them. Also the saturation problem is only partly solved (the direct theorem).

In this paper we give a general class of non-negative factor-kernels from which well-approximating operators can be built up. For all of these operators, the saturation problem will be completely solved as well. Moreover, we give an asymptotic relation, and investigate the structure of the oscillating part of the kernel constructed.

2. Construction of kernels of finite oscillations

The general class of non-negative factor-kernels is defined as follows. Let p be a positive integer, and $\{\varepsilon_n\}_{n=0}^{\infty}$ a sequence of positive real numbers such that

$$\lim_{n\to\infty} \varepsilon_n = 0. \tag{1}$$

The class $S_p(\{\varepsilon_n\})$ consists of all sequences of continuous 2π-periodic non-negative even functions $\{K_n(t)\}$ for which there exists a non-negative function $\varphi_p(x)$ defined on $[0, \infty)$ such that

$$\max_{0\leq x\leq \pi/\varepsilon_n} [(1+x^{4p-2})|K_n(\varepsilon_n x)-\varphi_p(x)|] = o(\varepsilon_n) \qquad (n\to\infty) \tag{2}$$

and

$$0 < \int_0^{\infty} x^{4p-2}\varphi_p(x)\,dx < \infty. \tag{3}$$

The function $\varphi_p(x)$ will be called *associated* to the sequence $\{K_n(t)\}$ of factor-kernels. In Section 6 we shall see that many non-negative kernels obtained from classical kernels belong to an $S_p(\{\varepsilon_n\})$ if the sequence $\{\varepsilon_n\}$ is properly chosen.

In order to build up kernels of finite oscillations for an arbitrary element of $S_p(\{\varepsilon_n\})$ we need two lemmas.

LEMMA 1. *Let* $\{K_n(t)\}\in S_p(\{\varepsilon_n\})$,

$$\mu_{k,n} = \frac{\varepsilon_n^{-k-1}}{\pi} \int_{-\pi}^{\pi} \left|\sin^k \frac{t}{2}\right| \cdot K_n(t)\, dt \qquad (k, n = 0, 1, \dots) \tag{4}$$

and

$$\mu_k = \frac{2^{1-k}}{\pi} \int_0^{\infty} x^k \varphi_p(x)\, dx \qquad (k = 0, 1, \dots, 4p-2). \tag{5}$$

Then

$$\lim_{n\to\infty} \mu_{k,n} = \mu_k \qquad (k = 0, 1, \dots, 4p-2) \tag{6}$$

and

$$\mu_{k,n} = o(\varepsilon_n^{4p-k-2}) \qquad (k \geqq 4p-1;\ n \to \infty). \tag{7}$$

PROOF. Using (1)—(3) and

$$\sin^k u = u^k + O(u^{k+2}) \tag{8}$$

we get

$$\mu_{k,n} = \frac{2\varepsilon_n^{-k-1}}{\pi} \int_0^{\pi} \sin^k \frac{t}{2} K_n(t)\, dt = \frac{2\varepsilon_n^{-k}}{\pi} \int_0^{\pi/\varepsilon_n} \sin^k \frac{\varepsilon_n x}{2} K_n(\varepsilon_n x)\, dx = \tag{9}$$

$$= \frac{2^{1-k}}{\pi} \int_0^{\pi/\varepsilon_n} x^k K_n(\varepsilon_n x)\, dx + O(\varepsilon_n^2) \int_0^{\pi/\varepsilon_n} x^{k+2} K_n(\varepsilon_n x)\, dx = \frac{2^{1-k}}{\pi} \int_0^{\pi/\varepsilon_n} x^k \varphi_p(x)\, dx +$$

$$+ \frac{2^{1-k}}{\pi} \int_0^{\pi/\varepsilon_n} x^k [K_n(\varepsilon_n x) - \varphi_p(x)]\, dx +$$

$$+ O(\varepsilon_n^2) \left\{ \int_0^{\pi/\varepsilon_n} x^{k+2} [K_n(\varepsilon_n x) - \varphi_p(x)]\, dx + \int_0^{\pi/\varepsilon_n} x^{k+2} \varphi_p(x)\, dx \right\} =$$

$$= \mu_k + o(1) + o(\varepsilon_n) \int_0^{\pi/\varepsilon_n} \frac{x^k\, dx}{1+x^{4p-2}} + o(\varepsilon_n^3) \int_0^{\pi/\varepsilon_n} \frac{x^{k+2}\, dx}{1+x^{4p-2}} + O(\varepsilon_n^2) \left(\int_0^{\pi/\sqrt{\varepsilon_n}} + \int_{\pi/\sqrt{\varepsilon_n}}^{\pi/\varepsilon_n} \right) x^{k+2} \times$$

$$\times \varphi_p(x)\, dx = \mu_k + o(1) + O(\varepsilon_n) \int_1^{\pi/\sqrt{\varepsilon_n}} x^{4p-2} \varphi_p(x)\, dx + O\left(\int_{\pi/\sqrt{\varepsilon_n}}^{\pi/\varepsilon_n} x^{4p-2} \varphi_p(x)\, dx \right) =$$

$$= \mu_k + o(1) \qquad (k = 0, 1, \dots, 4p-2).$$

Now, if $k \geqq 4p-1$ then similarly as in (9) we have

$$(10) \qquad 0 < \mu_{k,n} = O\left(\int_0^{\pi/\varepsilon_n} x^k K_n(\varepsilon_n x)\,dx\right) = O(\varepsilon_n^{4p-k-1}) \int_0^{\pi/\varepsilon_n} x^{4p-1} K_n(\varepsilon_n x)\,dx =$$

$$= O(\varepsilon_n^{4p-k-1}) \left\{\int_0^{\pi/\varepsilon_n} x^{4p-1}[K_n(\varepsilon_n x) - \varphi_p(x)]\,dx + \int_0^{\pi/\varepsilon_n} x^{4p-1}\varphi_p(x)\,dx\right\} =$$

$$= O(\varepsilon_n^{4p-k-1}) \left\{o(\varepsilon_n^{-1}) + \left(\int_0^{\pi/\sqrt{\varepsilon_n}} + \int_{\pi/\sqrt{\varepsilon_n}}^{\pi/\varepsilon_n}\right) x^{4p-1}\varphi_p(x)\,dx\right\} =$$

$$= O(\varepsilon_n^{4p-k-1}) \left\{o(\varepsilon_n^{-1}) + O(\varepsilon_n^{-\frac{1}{2}}) \int_0^{\pi/\sqrt{\varepsilon_n}} x^{4p-2}\varphi_p(x)\,dx + O(\varepsilon_n^{-1}) \int_{\pi/\sqrt{\varepsilon_n}}^{\pi/\varepsilon_n} x^{4p-2}\varphi_p(x)\,dx\right\} =$$

$$= o(\varepsilon_n^{4p-k-2}) \qquad (k \geqq 4p-1).$$

Q.E.D.

LEMMA 2. *For sufficiently large n's the system of equations*

$$(11) \qquad \sum_{k=0}^{p-1} \mu_{2j+2k,n}\lambda_{k,n} = \delta_{j,0} \qquad (j = 0, 1, \dots, p-1),$$

further the system of linear equations

$$(12) \qquad \sum_{k=0}^{p-1} \mu_{2j+2k}\lambda_k = \delta_{j,0} \qquad (j = 0, 1, \dots, p-1)$$

has a unique solution for which

$$(13) \qquad \lim_{n\to\infty} \lambda_{k,n} = \lambda_k \qquad (k = 0, 1, \dots, p-1)$$

and

$$(14) \qquad \lim_{n\to\infty} \sum_{k=0}^{p-1} \mu_{2p+2k,n}\lambda_{k,n} = \sum_{k=0}^{p-1} \mu_{2p+2k}\lambda_k \neq 0.$$

PROOF. First of all we prove that

$$(15) \qquad \begin{vmatrix} \mu_{2i} & \mu_{2i+2} \cdots \mu_{2j} \\ \mu_{2i+2} & \mu_{2i+4} \cdots \mu_{2j+2} \\ \vdots & \vdots \qquad \vdots \\ \mu_{2j} & \mu_{2j+2} \cdots \mu_{4j-2i} \end{vmatrix} \neq 0 \quad \text{if} \quad \max(0, 2j-2p+1) \leqq i \leqq j.$$

Namely, by (3), the linearly independent functions $\sqrt{\frac{2}{\pi}}\left(\frac{x}{2}\right)^l$ $(l=i, i+1, \dots, 2j-i)$ are elements of the space $L^2_{\varphi_p(x)}(0, \infty)$ and hence their Gram-determinant (15) is different from 0. But with $i=0$, $j=p-1$ this means that (11), and thus, by (6), also (12) for sufficiently large n's has a unique solution. (13) follows again from (6).

If (14) were zero, then the system of homogeneous linear equations

$$\sum_{k=0}^{p-1} \mu_{2j+2k}\bar{\lambda}_k = 0 \qquad (j = 1, 2, \dots, p)$$

would have a non-trivial solution (being the right hand side of the first equation in (12) 1), which contradicts (15) with $i=1, j=p$. Hence (14) holds. Q.E.D.

Now we are in the position as to define our kernel of finite oscillations. Let $\{K_n(t)\} \in S_p(\{\varepsilon_n\})$ and $\{\lambda_{k,n}\}_{k=0}^{p-1}$ the unique solution of (11). Consider

$$\text{(16)} \qquad \bar{K}_n(t) = K_n(t) \sum_{k=0}^{p-1} \lambda_{k,n} \varepsilon_n^{-2k-1} \sin^{2k} \frac{t}{2}.$$

This new kernel has some very important properties. First of all, it is normalized $\left(\frac{1}{\pi} \int_{-\pi}^{\pi} \bar{K}_n(t)\,dt = 1\right)$ and it is orthogonal to $\sin^{2j} \frac{t}{2}$ $(1 \leqq j \leqq p-1)$. This can be seen from

$$\text{(17)} \qquad m_{j,n} \overset{\text{def}}{=} \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^{2j} \frac{t}{2} \bar{K}_n(t)\,dt = \frac{1}{\pi} \sum_{k=0}^{p-1} \lambda_{k,n} \varepsilon_n^{-2k-1} \int_{-\pi}^{\pi} \sin^{2j+2k} \frac{t}{2} K_n(t)\,dt =$$

$$= \varepsilon_n^{2j} \sum_{k=0}^{p-1} \mu_{2j+2k,n} \lambda_{k,n} = \delta_{j,0} \qquad (j = 0, 1, \dots, p-1)$$

(see (16), (4) and (11)). As for the „absolute" moments $M_{j,n}$ of $\bar{K}_n(t)$we have by (16), (4), (6), (7) and (13)

$$\text{(18)} \qquad \begin{aligned} M_{j,n} &\overset{\text{def}}{=} \frac{1}{\pi} \int_{-\pi}^{\pi} \left|\sin^{j} \frac{t}{2} \bar{K}_n(t)\right| dt \leqq \frac{1}{\pi} \sum_{k=0}^{p-1} |\lambda_{k,n}|\, \varepsilon_n^{-2k-1} \int_{-\pi}^{\pi} \left|\sin^{j+2k} \frac{t}{2}\right| \cdot K_n(t)\,dt = \\ &= \varepsilon_n^{j} \sum_{k=0}^{p-1} \mu_{j+2k,n} |\lambda_{k,n}| = \begin{cases} O(\varepsilon_n^{j}) & \text{if} \quad 0 \leqq j \leqq 2p \\ o(\varepsilon_n^{2p}) & \text{if} \quad j \geqq 2p+1. \end{cases} \end{aligned}$$

3. The rate of convergence of operators with kernels of finite oscillations

At first we prove a theorem which shows how the convolution operators

$$\text{(19)} \qquad L_n(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \bar{K}_n(t)\,dt$$

approximate the function $f(x)$ which has some structural properties. Let $\omega(g, h)$ denote the modulus of continuity of the 2π-periodic continuous function $g (\in C_{2\pi})$ and let $\|g\|$ be the supremum norm of $g \in C_{2\pi}$.

THEOREM 1. (i) *If* $f^{(r)}(x) \in C_{2\pi}$ $(0 \leqq r \leqq 2p-1)$ *then*

$$\text{(20)} \qquad \|f(x) - L_n(f, x)\| = O(\varepsilon_n^{r})\, \omega(f^{(r)}, \varepsilon_n) \qquad (0 \leqq r \leqq 2p-1)$$

where the „O" sign depends only on f and p.

(ii) *If* $f^{(2p)}(x)\in C_{2\pi}$ *then for all* x

$$\lim_{n\to\infty}\frac{L_n(f,x)-f(x)}{\varepsilon_n^{2p}}=\frac{2^{2p-1}}{p}\left(\sum_{k=0}^{p-1}\mu_{2p+2k}\lambda_k\right)\left(\sum_{k=1}^{p}a_{k,p}f^{(2k)}(x)\right)\tag{21}$$

where

$$a_{k,p}=\frac{(-1)^{k+p}}{(2k)!}\left[x\binom{x+p-1}{2p-1}\right]_{x=0}^{(2k)}\qquad (k=1,2,\dots,p).$$

For the proof we need the following

LEMMA 3. *To each* k, $1\leqq k\leqq p$, *there exist real numbers* $b_{k,j}$ *and a continuous function* $\vartheta_{k,p}(t)$ *such that*

$$t^{2k}=\sum_{j=k}^{p}b_{k,j}\sin^{2j}\frac{t}{2}+\vartheta_{k,p}(t)\,t^{2p+2}\qquad (k=1,2,\dots,p).\tag{22}$$

(Compare also [1], Fact 5).

PROOF. First of all we have to prove that the system of linear equations

$$\sum_{j=k}^{l}b_{k,j}\left(\sin^{2j}\frac{t}{2}\right)_{t=0}^{(2l)}=(2k)!\,\delta_{k,l}\qquad (l=k,\dots,p)\tag{23}$$

has a unique solution. This follows from the fact that the determinant of this system is

$$\prod_{l=k}^{p}\left(\sin^{2l}\frac{t}{2}\right)_{t=0}^{(2l)}=\prod_{l=k}^{p}\frac{(2l)!}{2^{2l}}\neq 0.$$

Let

$$\Delta_{k,p}(t)=t^{2k}-\sum_{j=k}^{p}b_{k,j}\sin^{2j}\frac{t}{2}\tag{24}$$

then by $\left(\sin^{2j}\frac{t}{2}\right)_{t=0}^{(l)}=0$ $(l<2j)$ and (23) it follows that

$$\Delta_{k,p}^{(l)}(0)=0\qquad (l=0,1,\dots,2p+1).$$

Hence

$$\Delta_{k,p}(t)=\frac{\Delta_{k,p}^{(2p+2)}(\xi_t)}{(2p+2)!}\,t^{2p+2}\qquad (\xi_t\in(0,t))$$

which, together with (24), implies (22). Q.E.D.

PROOF OF THEOREM 1. We have the Taylor expansion

$$f(x+t)-f(x)=\sum_{k=1}^{r}\frac{f^{(k)}(x)}{k!}t^k+\frac{f^{(r)}(\eta)-f^{(r)}(x)}{r!}t^r\qquad (\eta\in(x,x+t))$$

(if $r=0$ then the right hand side sum is zero, and $\eta=x+t$). Substituting (22) we get

$$f(x+t)-f(x) = \sum_{k=0}^{\left[\frac{r-1}{2}\right]} \frac{f^{(2k+1)}(x)}{(2k+1)!} t^{2k+1} + \sum_{k=1}^{\left[\frac{r}{2}\right]} \frac{f^{(2k)}(x)}{(2k)!} \left(\sum_{j=k}^{p} b_{k,j} \sin^{2j} \frac{t}{2} + \vartheta_{k,p}(t) t^{2p+2} \right) + \\ + \frac{f^{(r)}(\eta)-f^{(r)}(x)}{r!} t^r \qquad (r \leqq 2p)$$

(in case $r=1$ the second sum is zero). Apply the operator (19) to both sides of this equality, and use (17) and (18):

$$L_n(f, x)-f(x) = m_{p,n} \sum_{k=1}^{\left[\frac{r}{2}\right]} \frac{f^{(2k)}(x)}{(2k)!} b_{k,p} + O(M_{2p+2,n}) + \\ + \frac{1}{\pi r!} \int_{-\pi}^{\pi} [f^{(r)}(\eta)-f^{(r)}(x)] t^r \bar{K}_n(t)\, dt. \tag{25}$$

Here the last integral is, by (18) again,

$$\frac{1}{\pi r!} \int_{-\pi}^{\pi} \omega(f^{(r)}, |t|) |t^r \bar{K}_n(t)|\, dt \leqq \frac{1}{\pi r!} \omega\left(f^{(r)}, \frac{M_{r+1,n}}{M_{r,n}}\right) \int_{-\pi}^{\pi} \left(1+\frac{M_{r,n}}{M_{r+1,n}}|t|\right) |t^r \bar{K}_n(t)|\, dt =$$

$$= O(M_{r,n})\, \omega\left(f^{(r)}, \frac{M_{r+1,n}}{M_{r,n}}\right) = \begin{cases} O(\varepsilon_n^r)\, \omega(f^{(r)}, \varepsilon_n) & \text{if} \quad 0 \leqq r \leqq 2p-1 \\ O(\varepsilon_n^{2p})\, \omega\big(f^{(2p)}, o(1)\big) = o(\varepsilon_n^{2p}) & \text{if} \quad r = 2p. \end{cases} \tag{26}$$

Thus if $0 \leqq r \leqq 2p-1$ then by (14)

$$m_{p,n} = \varepsilon_n^{2p} \sum_{k=0}^{p-1} \mu_{2p+2k,n} \lambda_{k,n} = O(\varepsilon_n^{2p}) \tag{27}$$

and we have from (25), using (18),

$$L_n(f, x)-f(x) = O(\varepsilon_n^{2p}) + o(\varepsilon_n^{2p}) + O(\varepsilon_n^r)\, \omega(f^{(r)}, \varepsilon_n) = O(\varepsilon_n^r)\, \omega(f^{(r)}, \varepsilon_n)$$

which proves (20).

Now let $r=2p$. Then (25)—(27) yield

$$\lim_{n\to\infty} \frac{L_n(f, x)-f(x)}{\varepsilon_n^{2p}} = \left(\sum_{k=0}^{p-1} \mu_{2p+2k} \lambda_k \right) \left(\sum_{k=1}^{p} \frac{f^{(2k)}(x)}{(2k)!} b_{k,p} \right). \tag{28}$$

Hence we have to prove that

$$b_{k,p} = \frac{(-1)^{k+p} 2^{2p-1}}{p} \left[x \binom{x+p-1}{2p-1} \right]_{x=0}^{(2k)} \qquad (k = 1, 2, \ldots, p). \tag{29}$$

For $k=p$ this gives $b_{p,p}=2^{2p}$ which is true, because of (22). Notice that if $f(x)$ is a trigonometric polynomial of degree at most $p-1$ then by (19) and (17)

$$L_n(f, x) = f(x). \tag{30}$$

Choose now $f_l(x)=\cos lx$ $(l=1, 2, \dots, p-1)$ then by (14), (28) and (30) we have for $x=0$

$$\sum_{k=1}^{p} \frac{(-1)^k l^{2k}}{(2k)!} b_{k,p} = 0 \qquad (l = 1, 2, \dots, p-1). \tag{31}$$

Consider this as a system of linear equations for the unknowns $\frac{(-1)^k}{(2k)!} b_{k,p}$ $(k=1, 2, \dots, p-1)$. This has a unique solution, because the determinant of the system is the Vandermonde-determinant of the elements $1^2, 2^2, \dots, (p-1)^2$. We show that the numbers (29) satisfy (31). Namely, the function

$$u_p(x) = \frac{2^{2p-1}(-1)^p}{p} x \binom{x+p-1}{2p-1}$$

is an even polynomial of degree $2p$, and by (29)

$$u_p(x) = \sum_{k=1}^{p} \frac{b_{k,p}(-1)^k}{(2k)!} x^{2k}.$$

But evidently $u_p(l)=0$ $(l=1, 2, \dots, p-1)$ which proves (31). Q.E.D.

4. The saturation problem

In order to solve the saturation problem of the operators (19), we need some auxiliary statements.

LEMMA 4. *If*

$$1-\cos jt = \sum_{l=1}^{j} \alpha_{j,l} \sin^{2l} \frac{t}{2} \qquad (j = 1, 2, \dots) \tag{32}$$

then

$$\alpha_{j,l} = (-1)^{l+1} \sum_{k=0}^{l} \binom{2j}{2k} \binom{j-k}{l-k} \qquad (l = 1, 2, \dots, j). \tag{33}$$

PROOF. Using twice the Newton's binomial formula we get

$$\cos jt = \operatorname{Re} \left(\cos\frac{t}{2} + i \sin\frac{t}{2}\right)^{2j} = \sum_{m=0}^{j} \binom{2j}{2m} (-1)^{j-m} \sin^{2j-2m}\frac{t}{2} \cos^{2m}\frac{t}{2} =$$

$$= \sum_{m=0}^{j} \binom{2j}{2m} (-1)^{j-m} \sin^{2j-2m}\frac{t}{2} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \sin^{2k}\frac{t}{2} =$$

$$= \sum_{m=0}^{j} \sum_{k=0}^{m} \binom{2j}{2m} \binom{m}{k} \left(-\sin^2\frac{t}{2}\right)^{j-m+k} = \sum_{l=0}^{j} \left(\sum_{k=0}^{j} \binom{2j}{2j+2k-2l} \binom{j+k-l}{k}\right) (-1)^l \sin^{2l}\frac{t}{2} =$$

$$= \sum_{l=0}^{j} (-1)^l \left(\sum_{k=0}^{l} \binom{2j}{2k} \binom{j-k}{l-k}\right) \sin^{2l}\frac{t}{2}.$$

Q.E.D.

LEMMA 5. *Let* $\{\overline{K}_n(t)\}_{n=0}^{\infty}$ *be a sequence of even, continuous, 2π-periodic functions,*

(34) $$\varrho_{0,n} = \frac{1}{2}, \quad \varrho_{j,n} = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos jt \overline{K}_n(t)\, dt \qquad (j = 1, 2, \dots)$$

and define the sequence $\{L_n\}_{n=0}^{\infty}$ *of operators by* (19). *Assume further that with* $p \geqq 1$ *we have*

(35) $$\lim_{n\to\infty} \frac{1-\varrho_{k,n}}{1-\varrho_{p,n}} \stackrel{\text{def}}{=} \psi_k \neq 0 \qquad (k = p, p+1, \dots).$$

Then

(i) $\lim\limits_{n\to\infty} \dfrac{\|f(x)-L_n(f,x)\|}{1-\varrho_{p,n}} = 0$ *implies that* $f(x)$ *is a trigonometric polynomial of degree at most* $p-1$; *and*

(ii) $\|f(x)-L_n(f,x)\| = O(1-\varrho_{p,n})$ *implies that* $\sum\limits_{j=p}^{\infty} \psi_j A_j(f,x) \in L_\infty$ (*=the space of a.e. bounded functions*) *where* $A_j(f,x) = a_j(f)\cos jx + b_j(f)\sin jx$ $(a_j(f), b_j(f)$ *are the Fourier coefficients of* $f(x))$.

This lemma is an exact analogon of Theorem 3.2 from [2], and the proof goes along the same lines except that instead of $j=0, 1, 2, \dots$ we now have $j=p, p+1, \dots$. We omit the details.

After these preliminaries we can state our saturation theorem.

THEOREM 2. *Let the operator-sequence* (19) *be obtained in the manner described in Section* 2. *Then*

(i) $\lim\limits_{n\to\infty} \varepsilon_n^{-2p} \|f(x)-L_n(f,x)\| = 0$ *if and only if* $f(x)$ *is a trigonometric polynomial of degree at most* $p-1$; *and*

(ii) $\|f(x)-L_n(f,x)\| = O(\varepsilon_n^{2p})$ *if and only if* $f^{(2p-1)} \in \text{Lip } 1$.

PROOF. In order to apply Lemma 5, we have to determine the ψ_k's defined in (35) for our particular operator. Let $j \geqq p$. Then by (34), (17), (32) and (11)

$$\varepsilon_n^{-2p}(1-\varrho_{j,n}) = \frac{\varepsilon_n^{-2p}}{\pi} \sum_{l=1}^{j} \alpha_{j,l} \int_{-\pi}^{\pi} \sin^{2l}\frac{t}{2}\, \overline{K}_n(t)\, dt = \varepsilon_n^{-2p} \sum_{l=1}^{j} \alpha_{j,l}\varepsilon_n^{2l} \sum_{k=0}^{p-1} \mu_{2l+2k,n}\lambda_{k,n} =$$

$$= \sum_{l=p}^{j} \alpha_{j,l}\varepsilon_n^{2l-2p} \sum_{k=0}^{p-1} \mu_{2l+2k,n}\lambda_{k,n} = \alpha_{j,p} \sum_{k=0}^{p-1} \mu_{2p+2k,n}\lambda_{k,n} + \sum_{l=p+1}^{j} \alpha_{j,l}\varepsilon_n^{2l-2p} \sum_{k=0}^{p-1} \mu_{2l+2k,n}\lambda_{k,n}.$$

Hence, by (7) and (13) we get

$$\left|\varepsilon_n^{-2p}(1-\varrho_{j,n}) - \alpha_{j,p}\sum_{k=0}^{p-1}\mu_{2p+2k,n}\lambda_{k,n}\right| \leqq \sum_{l=p+1}^{j} |\alpha_{j,l}|\varepsilon_n^{2l-2p} \sum_{k=0}^{p-1} \mu_{2l+2k,n}|\lambda_{k,n}| =$$

$$= \sum_{l=p+1}^{j} |\alpha_{j,l}|\varepsilon_n^{2l-2p} o(\varepsilon_n^{2p-2l}) = o(1) \qquad (j \geqq p).$$

Thus (1), (6) and (13) imply that

(36) $$\lim_{n\to\infty} \varepsilon_n^{-2p}(1-\varrho_{j,n}) = \alpha_{j,p} \sum_{k=0}^{p-1} \mu_{2p+2k}\lambda_k \qquad (j = p, p+1, \dots).$$

Here by (33) $\alpha_{j,p}\neq 0$ $(j\geqq p)$. Therefore (36) and (14) yield

$$\lim_{n\to\infty}\frac{1-\varrho_{j,n}}{1-\varrho_{p,n}}=\frac{\alpha_{j,p}}{\alpha_{p,p}}\neq 0 \qquad (j=p,\,p+1,\,\dots). \tag{37}$$

This shows that we may apply Lemma 5.

(i) If $f(x)$ is a trigonometric polynomial of degree $p-1$, then we have (30), which is even more than $\lim\limits_{n\to\infty}\varepsilon_n^{-2p}\|f-L_n(f)\|=0$. Conversely, if

$$\lim_{n\to\infty}\frac{\|f(x)-L_n(f,x)\|}{1-\varrho_{p,n}}=\left(\alpha_{p,p}\sum_{k=0}^{p-1}\mu_{2p+2k}\lambda_k\right)^{-1}\lim_{n\to\infty}\varepsilon_n^{-2p}\|f-L_n(f)\|=0$$

(by (36)), then according to Lemma 5, $f(x)$ is a trigonometric polynomial of degree at most $p-1$.

(ii) If $f^{(2p-1)}(x)\in\mathrm{Lip}\,1$ then by Theorem 1,

$$\|f(x)-L_n(f,x)\|=O(\varepsilon_n^{2p}). \tag{38}$$

Conversely, assume that (38) holds. Then Lemma 5 and (37) yield that

$$\sum_{j=p}^{\infty}\alpha_{j,p}A_j(f,x)\in L_\infty \tag{39}$$

i.e.

$$|\alpha_{j,p}|\cdot\|A_j(f,x)\|=O(1) \qquad (j\to\infty). \tag{40}$$

Using (33), an easy calculation shows that

$$\alpha_{j,p}=\frac{(-1)^{p+1}2^{2p}}{(2p)!}j^{2p}+\sum_{k=0}^{2p-2}\beta_{p,k}j^k \qquad (j\geqq p) \tag{41}$$

where the $\beta_{p,k}$'s do not depend on j. (40) and (41) show that $\|A_j(f,x)\|=O(j^{-2p})$. Hence the series

$$\sum_{j=p}^{\infty}j^kA_j(f,x) \qquad (k=0,1,\dots,2p-2)$$

uniformly converge, because they have a convergent numerical majorant. Thus

$$\sum_{j=p}^{\infty}\left(\sum_{k=0}^{2p-2}\beta_{p,k}j^k\right)A_j(f,x)\in C_{2\pi}$$

which, by (39) and (41), implies that $\sum\limits_{j=p}^{\infty}j^{2p}A_j(f,x)\in L_\infty$. This means that $f^{(2p)}(x)\in L_\infty$ i.e. $f^{(2p-1)}(x)\in\mathrm{Lip}\,1$. Q.E.D.

5. The structure of the oscillating part of $\overline{K}_n(t)$

Let

$$\psi_{p,n}(t)=\sum_{k=0}^{p-1}\lambda_{k,n}\varepsilon_n^{-2k-1}\sin^{2k}\frac{t}{2}$$

be the oscillating part of the kernel $\overline{K}_n(t)$. In order to get an insight into the structure of this trigonometric polynomial of degree $p-1$, we prove the following

THEOREM 3. *All the* $2p-2$ *roots* $\pm t_{1,n}, \pm t_{2,n}, \dots, \pm t_{p-1,n}$ *of* $\psi_{p,n}(t) \mod 2\pi$ *are real, simple, and*

$$\lim_{n\to\infty} \frac{t_{k,n}}{\varepsilon_n} = c_k \neq 0 \qquad (k = 1, 2, \dots, p-1)$$

where $c_j \neq c_k$ $(j \neq k)$.

PROOF. Let $x_n = \frac{2}{\varepsilon_n} \sin\frac{t}{2}$ then the roots of $\psi_{p,n}(t)$ transformed this way have to satisfy

$$\sum_{k=0}^{p-1} \lambda_{k,n} \left(\frac{x_n}{2}\right)^{2k} = 0.$$

But by (13), the roots of this equation tend to the roots of

$$\sum_{k=0}^{p-1} \lambda_k \left(\frac{x}{2}\right)^{2k} = 0.$$

Let

$$T_p(x) = x^2 \sum_{k=0}^{p-1} 2^{-k} \lambda_k x^{2k},$$

then by (5) and (12) we have

$$\int_0^\infty \varphi_p(x) T_p(x) x^{2j}\, dx = 0 \qquad (j = 0, 1, \dots, 2p-4). \tag{42}$$

In order to prove Theorem 3, first we have to show that all the roots of $T_p(x)$ different from zero are real and simple. Assume that α (and hence $-\alpha$) is a root of $T_p(x)$ of multiplicity $\geqq 2$. Then by (42) $T_p(x)$ is orthogonal to $T_p(x)(x^2-\alpha^2)^{-2}$, and therefore

$$\int_0^\infty \varphi_p(x) \frac{T_p(x)^2}{(x^2-\alpha^2)^2}\, dx = 0$$

which, by $\varphi_p(x) \geqq 0$, is impossible (by (12), not all the λ_k's are zero). In a similar manner we are led to a contradiction if we assume that $T_p(x)$ has a non-real root.

Thus, all we have to prove is that the zero is not a root of $T_p(x)x^{-2}$; in other words $\lambda_0 \neq 0$. But by (12) and (15)

$$\lambda_0 = \begin{vmatrix} \mu_4 & \mu_6 & \dots & \mu_{2p} \\ \mu_6 & \mu_8 & \dots & \mu_{2p+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2p} & \mu_{2p+2} & \dots & \mu_{4p-4} \end{vmatrix} \cdot \begin{vmatrix} \mu_0 & \mu_2 & \dots & \mu_{2p-2} \\ \mu_2 & \mu_4 & \dots & \mu_{2p} \\ \vdots & \vdots & & \vdots \\ \mu_{2p-2} & \mu_{2p} & \dots & \mu_{4p-4} \end{vmatrix}^{-1} \neq 0.$$

Q.E.D.

6. Applications

1. *Jackson-kernel.* Let

$$K_n(t) = \left(\frac{\sin \frac{nt}{2}}{n \sin \frac{t}{2}}\right)^{4p}$$

be the generalized Jackson-kernel. Choose $\varepsilon_n = \frac{2}{n}$ then $\{K_n(t)\} \in S_p(\{2/n\})$ with the associated function

$$\varphi_p(x) = \left(\frac{\sin x}{x}\right)^{4p}.$$

Namely

$$(1+x^{4p-2})|K_n(\varepsilon_n x) - \varphi_p(x)| = (1+x^{4p-2}) \sin^{4p} x \left| \frac{1}{\left(n \sin \frac{x}{n}\right)^{4p}} - \frac{1}{x^{4p}} \right| =$$

$$= O(1+x^{4p-2}) \sin^{4p} x \frac{x^{4p+2} n^{-2}}{x^{8p}} = O\left(\frac{1}{n^2}\right) = o(\varepsilon_n) \qquad \left(0 \leqq x \leqq \frac{\pi n}{2}\right).$$

Thus the corresponding operator (19) is saturated with $O(n^{-2p})$ (the saturation class is always the same specified in Theorem 2).

2. *Generalized de la Vallée Poussin kernel.* Let $q \geqq 1$ be an integer, and

$$K_n(t) = \left(1 - \sin^{2q} \frac{t}{2}\right)^n.$$

For $q=1$ this is the classical de la Vallée Poussin kernel (apart from a constant factor depending on n). Choose $\varepsilon_n = 2n^{-\frac{1}{2q}}$ and p arbitrary, then $\{K_n(t)\} \in S_p(\{2n^{-\frac{1}{2q}}\})$, with

$$\varphi_p(x) = e^{-x^{2q}}.$$

(Notice that $\varphi_p(x)$ does not depend on p). Namely

$$(1+x^{4p-2})|K_n(\varepsilon_n x) - \varphi_p(x)| = (1+x^{4p-2}) \left| \left(1 - \sin^{2q} \frac{\varepsilon_n x}{2}\right)^n - e^{-x^{2q}} \right| =$$

$$= \begin{cases} O(1+x^{4p-2}) \left| e^{n \log\left(1 - \sin^{2q} \frac{\varepsilon_n x}{2}\right)} - e^{-x^{2q}} \right| = O\left(\frac{1+x^{4p-2}}{e^{x^{2q}}}\right) \left| e^{O\left(\frac{x^{2q+2}}{n^{1/q}}\right)} - 1 \right| = \\ \qquad = O\left(\frac{\log^{2q+2} n}{n^{1/q}}\right) = o(\varepsilon_n) \quad \text{if} \quad 0 \leqq x \leqq \log n; \\ O\left(n^{\frac{2p-1}{q}}\right) \left[\left(1 - \frac{\log^{2q} n}{\pi^{2q} n}\right)^n + e^{-\log^{2q} n} \right] = o(\varepsilon_n) \quad \text{if} \quad \log n \leqq x \leqq \pi/\varepsilon_n. \end{cases}$$

Thus we have the following interesting result:

Given an arbitrary positive rational number $\varrho = \frac{p}{q}$, *there exist a sequence of operators* $\{L_n\}$ *of finite oscillations which are trigonometric polynomials of degree at most* $nq+p-1$, *are saturated with* $O(n^{-\varrho})$ *and have saturation class* $\{f; f^{(2p-1)} \in \text{Lip } 1\}$.

3. *Fejér—Korovkin operators.* The kernel

$$K_n(t) = \left(\frac{\cos \frac{nt}{2}}{n^2 \left(\cos t - \cos \frac{\pi}{n} \right)} \right)^{2p}$$

is the p^{th} power of the Fejér—Korovkin kernel (apart from a constant; see e.g. [2], p. 105). We show that $\{K_n(t)\} \in S_p(\{1/n\})$ with the associated function

$$\varphi_p(x) = \left(\frac{2 \cos \frac{x}{2}}{x^2 - \pi^2} \right)^{2p}.$$

Namely, by

$$0 \leqq \varphi_p(x) = O\left(\frac{1}{1+x^{4p}} \right) \qquad (0 \leqq x < \infty)$$

we have $\left(\varepsilon_n = \frac{1}{n} \right)$

$$(1+x^{4p-2})|K_n(\varepsilon_n x) - \varphi_p(x)| = (1+x^{4p-2}) \cos^{2p} \frac{x}{2} \cdot \left| \frac{1}{\left(2n^2 \sin \frac{x-\pi}{2n} \sin \frac{x+\pi}{2n} \right)^2} - \right.$$

$$\left. - \left(\frac{2}{x^2-\pi^2} \right)^{2p} \right| = O(1+x^{4p-2}) \frac{x^2}{n^2} \left(\frac{\cos \frac{x}{2}}{n^2 \sin \frac{x-\pi}{2n} \sin \frac{x+\pi}{2n}} \right)^{2p} \qquad (0 \leqq x \leqq n\pi).$$

Now, if $0 \leqq x \leqq 2\pi$ then this is

$$\frac{O\left(\frac{1}{n^2} \right)}{\left(n \sin \frac{x+\pi}{2n} \right)^{2p}} = O\left(\frac{1}{n^2} \right).$$

If $2\pi \leqq x \leqq n\pi$ then we get

$$O\left(\frac{1}{n^2} \right) \frac{x^{4p}}{(x^2-\pi^2)^{2p}} = O\left(\frac{1}{n^2} \right).$$

The corresponding operators (19) are saturated with $O(n^{-2p})$.

7. Remarks

1. It would have been easy to construct kernels $\bar{K}_n(t)$ which are not trigonometric polynomials, but these cases are less interesting. If $K_n(t)$ is a trigonometric polynomial of degree m then $\bar{K}_n(t)$ is of degree $m+p-1$.

2. A different approach to the problem was given by A. I. KOVALENKO [3]. He determined the oscillating part in (16) through its roots, using a certain set of non-negative factor-kernels $K_n(t)$, and gave asymptotic formulas similar to (21), with $\varepsilon_n = n^{-1}$. The condition imposed on the factor kernel and on the roots of the oscillating part are complicated. Nevertheless, in his formula only $f^{(2p)}(x)$ appears on the right hand side of (21). This could be achieved by our construction, too, if we considered

$$\mu_{j,k,n} = \frac{\varepsilon_n^{-k-j-1}}{\pi} \int_{-\pi}^{\pi} \left| t^j \sin^k \frac{t}{2} \right| \cdot K_n(t)\, dt$$

instead of (4), but then we should loose the nice property of (19) that it reproduces trigonometric polynomials of degree at most $p-1$.

3. By (17), if the kernel (16) is a trigonometric polynomial of degree m, then

$$\bar{K}_n(t) = D_{p-1}(t) + \sum_{j=p}^{m} \varrho_{j,n} \cos jt$$

where $D_{p-1}(t)$ is the Dirichlet-kernel of degree $p-1$.

4. In case $p=1$ we get the following interesting result for positive operators, from Theorems 1 and 2:

COROLLARY *If $\{K_n(t)\}$ is a sequence of nonnegative, even, continuous, 2π-periodic functions such that for a suitable sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ $(n \to \infty)$ and a nonnegative function $\varphi_1(x)$ we have*

$$\max_{0 \leq x \leq \pi/\varepsilon_n} [(1+x^2)|K_n(\varepsilon_n x) - \varphi_1(x)|] = o(\varepsilon_n)$$

and

$$0 < \int_0^\infty x^2 \varphi_1(x)\, dx < \infty,$$

then the sequence of positive linear operators

$$L_n(f, x) = \frac{\int_{-\pi}^{\pi} f(x+t) K_n(t)\, dt}{\int_{-\pi}^{\pi} K_n(t)\, dt} \qquad (n = 0, 1, \ldots)$$

has the following properties:

(i) $\|f(x)-L_n(f,x)\| = \begin{cases} O(\omega(f,\varepsilon_n)) & \text{if } f(x)\in C_{2\pi} \\ O(\varepsilon_n\omega(f',\varepsilon_n)) & \text{if } f'(x)\in C_{2\pi} \end{cases}$

(ii) $$\lim_{n\to\infty}\frac{L_n(f,x)-f(x)}{\varepsilon_n^2} = \frac{\int_0^\infty u^2\varphi_1(u)\,du}{\int_0^\infty \varphi_1(u)\,du} f''(x) \qquad (f''\in C_{2\pi},\ -\infty<x<\infty)$$

(iii) $\varliminf_{n\to\infty} \varepsilon_n^{-2}\|L_n(f,x)-f(x)\| = 0$ *if and only if* $f=\text{const.}$

(iv) $\|L_n(f,x)-f(x)\| = O(\varepsilon_n^2)$ *if and only if* $f'\in \text{Lip}\,1$.

If we apply (ii) to the nonnegative factor-kernels of the previous section then we can easily get more or less well-known asymptotic formulas for classical operators.

5. When $p\geqq 2$, the construction of the kernel (16) and calculating the right hand side of (21) becomes tedious. E.g. for $p=2$ we get

$$\lim_{n\to\infty}\frac{L_n(f,x)-f(x)}{\varepsilon_n^4} = -\frac{2}{3}\cdot\frac{\mu_4^2-\mu_2\mu_6}{\mu_2^2-\mu_0\mu_4}[f^{(2)}(x)+f^{(4)}(x)] \qquad (f^{(4)}\in C_{2\pi}).$$

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ON COMMUTATOR SUBGROUPS

By
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1. Introduction

In this paper we consider only finite groups. A finite p-group P is said to be of sectional rank less than or equal to k if given any two normal subgroups R, S of P such that $S \geqq R$ and S/R is elementary abelian, the order of S/R is less than or equal to p^k.

A group G is said to be characteristically solvable if there exists a series $E=G_0< <\dots<G_n=G$ of characteristic subgroups of G such that G_i/G_{i-1} is of prime order for $i=1, \dots, n$. Characteristic solvability is obviously more than supersolvability.

The aim of this paper is to prove the following:

THEOREM. *Given a 2-group G of sectional rank 2, assume that G is isomorphic to the derived subgroup of a finite group H. Then either G is the quaternion-group of order 8 or*

$$G = \langle a, b | a^{2^m} = b^{2^{n+k}} = 1, [a, b] = b^{2^n} \rangle, \quad \textit{where} \quad 2k \leqq m \leqq n$$

or G is abelian.

Our notation is that of [1].

The proof is based on the following theorem of BLACKBURN [2]: Given a p-group P assume that its derived subgroup P' can be generated by two elements. Then either P' is abelian or

$$P' = \langle a, b | a^{2^m} = b^{2^{n+k}} = 1, [a, b] = b^{2^n} \rangle, \quad \text{where} \quad 2k \leqq m \leqq n.$$

We shall make use of a theorem of BLACKBURN [2] asserting that if P is a p-group and both P and P' can be generated by two elements then P' is abelian.

2. The proof of the theorem

First we prove a lemma.

LEMMA. *Let P be a p-group of sectional rank 2. Then P is either characteristically solvable or it is the direct product of two cyclic groups of equal orders or it is isomorphic either to the quaternion-group of order 8 or to $M(p)$, the non-abelian group of order p^3 and of exponent p, p odd.*

PROOF. For abelian groups the statement of the lemma can be checked easily so we may apply induction on $|P|$ and assume $P' \neq E$.

I. First we show that there exists a characteristic subgroup R of P such that $R \leqq P'$, $|P':R|=p$. This is obvious if P' is characteristically solvable. If this is not the case then $\Phi(P')$ is of index p^2 in P'. Let us denote $\Phi(P')$ by F. $(P/F)'=P'/F$ is a normal subgroup of order p^2 in P/F. Hence $Z(P/F)$ intersects it non-trivially. This intersection is a characteristic subgroup of P/F whence its full inverse image with respect to the natural homomorphism $P \to P/F$ is a characteristic subgroup of P. By way of contradiction we may assume that it is not of prime index in P' i.e. $Z(P/F)$ contains P'/F. As $P/\Phi(P)$ is elementary abelian so it is of order less than or equal to p^2 i.e. P can be generated by two elements a, b. Then we have ($\bar{a}$, $\bar{b}$ denote aF/F, bF/F, respectively): $P'/F=\langle[\bar{a}, \bar{b}]^c, c \in P/F\rangle=\langle[\bar{a}, \bar{b}]\rangle$ (since $[\bar{a}, \bar{b}] \in Z(P/F)$) and consequently $\langle[\bar{a}, \bar{b}]^p\rangle$ is of order p and we can take its inverse image in P to be R.

II. R is of index p in P', both of these groups are abelian. Hence at least one of them is not the product of two cyclic groups of equal orders which entails that this one is characteristically solvable. As a consequence there exists a series $E=H_0 \leqq \ldots \leqq H_k=P'$, H_i is characteristic in P, $|H_i/H_{i-1}|=p$ $i=1, 2, \ldots, k$.

We distinguish two cases:

a) $k \geqq 2$. P/H_1 is a non-abelian p-group satisfying the assumptions of the lemma and $|P/H_1|<|P|$. Hence P/H_1 is either characteristically solvable and then so is P or P/H_1 is one of the exceptional groups of order p^3. We may assume that this latter is the case. Then P is of order p^4 while P' is of order p^2. $P' \leqq \Phi(P)$ implies that $P'Z(P)$ is a proper characteristic subgroup of P. If it properly contained P' then the characteristic solvability of P would follow. Hence we may assume $Z(P) \leqq P'$. $P/C_P(P')$ is isomorphic to a p-subgroup of the automorphism-group of P', a p-group of order p^2. Hence $|P/C_P(P')|$ divides p. If $C_P(P')$ is a proper subgroup of P then the characteristic solvability of P follows. By way of contradiction we may assume $C_P(P')=P$ i.e. $P'=Z(P)$. Let P be generated by the two elements a, b. We have, as above $P'=\langle[a, b]^c|c \in P\rangle=\langle[a, b]\rangle$. As $P/\Phi(P)$ is elementary abelian of order p^2 we have $P'=\Phi(P)$, consequently $a^p \in P'=Z(P)$. But we have $[a^p, b]=[a, b]^p \neq 1$ which is a contradiction.

b) $k=1$. We have $|P'|=p$ and consequently $Z(P) \geqq P'$. If equality holds then we have for any two elements a, b of P $[a^p, b]=[a, b]^p=1$ which implies $a^p \in Z(P)$ for any $a \in P$. Hence we have $P/Z(P)$ is elementary abelian. As p is of sectional rank 2 and as it is non-commutative so we have $|P|=p^3$. For groups of order p^3 one can easily check the lemma. The only remaining case is $Z(P) \gneqq P'$. $Z(P)$ and $Z(P)/P'$ are both commutative and the order of one of them is not a square. Hence this one is characteristically solvable. Consequently we can find a characteristic subgroup T of P in which P' has prime index. P/P' and P/T are both abelian whence at least one of them is characteristically solvable and, consequently, so is P too. Q.e.d.

Now we shall deduce the theorem from the lemma. Let $H'=G$ where G is a 2-group of sectional rank 2 and let us further suppose that G is neither commutative nor isomorphic with the quaternion group of order 8. Then G is characteristically solvable which entails the supersolvability of H. As a consequence H has a normal 2-complement K. Hence $(H/K)'=GK/K \cong G$, which means that G is isomorphic to the derived subgroup of a finite 2-group and the statement of the theorem follows from the above mentioned result of Blackburn [2].

On the other hand these groups can be obtained as commutator subgroups of finite groups. This has been proved by Blackburn [2] except for the quaternion-group which is the derived subgroup of $SL(2, 3)$. Q.e.d.

As a corollary we have that the groups D_k (the dihedral group of order 2^k), S_k (the semidihedral group of order 2^k, $k \geqq 4$), Q_k (the generalized quaternion-group of order 2^k, $k \geqq 4$) and $M(k, 2)$, $k \geqq 3$ cannot be obtained as the commutator subgroup of a finite group.

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A MÜNTZ-TYPE PROBLEM FOR RATIONAL APPROXIMATION

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Let $\{n_j\}_{j=1}^{\infty}$ denote a sequence of real exponents for which

$$(1)\qquad 0 = n_1 < n_2 < n_3 < \dots, \quad \lim_{j\to\infty} n_j = +\infty.$$

As the well-known theorem of G. M. Müntz states the linear combinations of the functions x^{n_j} ($j=1, 2, \dots$) form a dense set in $C[0, 1]$ with respect to the maximum norm if and only if $\sum_{j=2}^{\infty} \frac{1}{n_j}$ diverges.

In 1974, D. Newman raised the following question: for which sequences $\{n_j\}_{j=1}^{\infty}$ is the set of quotients

$$(2)\qquad h(x) = \frac{\sum_{j=1}^{s} a_j \cdot x^{n_j}}{\sum_{j=1}^{s} b_j \cdot x^{n_j}}$$

($s>0$ integer, a_j and b_j ($j=1, 2, \dots, s$) arbitrary constants) dense in $C[0, 1]$ (considering only those $h(x)$'s which are continuous for $0\leqq x\leqq 1$)? Now we shall answer this question by proving the following theorem:

THEOREM. *For each sequence of exponents satisfying* (1) *the set of the quotients considered in* (2) *is dense in* $C[0, 1]$ *concerning the maximum norm.*

PROOF. Let us suppose that $f\in C[0, 1]$, $\varepsilon>0$ is arbitrary and $k=k(\varepsilon)>1$ is an integer with the property that

$$(3)\qquad |f(x)-f(y)| < \varepsilon \quad \text{if} \quad 0 \leqq x < y \leqq 1, \quad y-x \leqq \frac{1}{k}.$$

Then the subsequence $0=m_1<m_2<\dots<m_{2k-2}$ of $\{n_j\}_{j=1}^{\infty}$ can be chosen so that

$$(4)\qquad \left| f(x) - \frac{\sum_{i=1}^{2k-2} f(x_i)\left(\frac{x}{x_i}\right)^{m_i}}{\sum_{i=1}^{2k-2}\left(\frac{x}{x_i}\right)^{m_i}} \right| < \varepsilon$$

holds for all $0 \leqq x \leqq 1$, where $x_i = i/2k$ $(i = 1, 2, \ldots, 2k)$. We shall define the values $m_i = n_{j_i}$ successively so that the functions

$$h_q(x) = \frac{\sum_{i=1}^{q} f(x_i)\left(\frac{x}{x_i}\right)^{m_i}}{\sum_{i=1}^{q}\left(\frac{x}{x_i}\right)^{m_i}} = \frac{P_q(x)}{Q_q(x)} \qquad (q = 1, 2, \ldots, 2k-2) \tag{5}$$

will satisfy the inequalities

$$|f(x) - h_q(x)| < \varepsilon \quad \text{if} \quad 0 \leqq x \leqq x_{q+2}. \tag{6}$$

For $q = 2k-2$, (6) gives (4) which proves our theorem.

We have $h_1(x) \equiv f(x_1)$ by (5) and it follows from (3) that

$$|f(x) - h_1(x)| = |f(x) - f(x_1)| < \varepsilon \quad \text{if} \quad 0 \leqq x \leqq x_3$$

thus (6) is true for $q = 1$. Let us suppose that $0 = m_1 < m_2 \ldots < m_q$ $(1 \leqq q < 2k-2)$ have been already defined and (6) holds for this q. Now we show that if we choose $m_{q+1} = n_{j_{q+1}}$ large enough then replacing q by $q+1$, (6) remains valid. We get from (5)

$$h_{q+1}(x) = \frac{P_{q+1}(x)}{Q_{q+1}(x)} = \frac{P_q(x) + f(x_{q+1}) \cdot \left(\frac{x}{x_{q+1}}\right)^{m_{q+1}}}{Q_q(x) + \left(\frac{x}{x_{q+1}}\right)^{m_{q+1}}}. \tag{7}$$

$(x/x_{q+1})^m$ tends to zero uniformly in the interval $0 \leqq x \leqq x_q$ as m tends to $+\infty$ and $Q_q(x) \geqq Q_1(x) \equiv 1$ holds everywhere. Therefore, if m_{q+1} is large enough, $|h_{q+1}(x) - h_q(x)|$ will be uniformly as small as we need it in $0 \leqq x \leqq x_q$ thus here

$$|h_{q+1}(x) - f(x)| \leqq |h_{q+1}(x) - h_q(x)| + |h_q(x) - f(x)| < \varepsilon$$

will be valid (we used (6)). (7) gives that $h_{q+1}(x)$ will be close to the number $f(x_{q+1})$ uniformly in the interval $x_{q+2} \leqq x \leqq x_{q+3}$ if m_{q+1} is large enough because $P_q(x)$ and $Q_q(x)$ are bounded functions and $(x/x_{q+1})^m$ tends to $+\infty$ uniformly as $m \to +\infty$. Thus it follows from $|x - x_{q+1}| \leqq 1/k$ by (3) that here again

$$|h_{q+1}(x) - f(x)| \leqq |h_{q+1}(x) - f(x_{q+1})| + |f(x_{q+1}) - f(x)| < \varepsilon$$

will be true. In case $x_q \leqq x \leqq x_{q+2}$ we take into consideration that the quotients

$$h_q(x) = \frac{P_q(x)}{Q_q(x)} = \frac{a}{b} \quad \text{and} \quad f(x_{q+1}) = \frac{f(x_{q+1})\left(\frac{x}{x_{q+1}}\right)^{m_{q+1}}}{\left(\frac{x}{x_{q+1}}\right)^{m_{q+1}}} = \frac{c}{d}$$

have positive denominators therefore the number

$$h_{q+1}(x) = \frac{a+c}{b+d}$$

is between the values $h_q(x)$ and $f(x_{q+1})$. Thus by using the relations $|x-x_{q+1}|<1/k$, (3) and (6) we get for $x_q \leqq x \leqq x_{q+2}$

$$|h_{q+1}(x)-f(x)| \leqq \max \{|h_q(x)-f(x)|, \quad |f(x_{q+1})-f(x)|\} < \varepsilon$$

which completes the proof.

NOTES. 1. Let $\{f_j(x)\}_{j=1}^{\infty}$ denote a sequence of functions which are positive-valued and monotonically increasing in [0, 1]. Let us suppose that for each fixed pair $0 \leqq x < y \leqq 1$

$$\lim_{j\to\infty} \frac{f_j(y)}{f_j(x)} = +\infty$$

holds. Then (as Prof. D. Newman has noticed) similarly to our theorem it can be seen that each $f \in C[0, 1]$ can be uniformly approximated by the quotients

$$\frac{\sum_{j=1}^{s} a_j \cdot f_j(x)}{\sum_{j=1}^{s} b_j \cdot f_j(x)}.$$

2. In the case when we give two different sequences of exponents satisfying (1) $\{m_j\}_{j=1}^{\infty}$ for the polynomial of the numerator and $\{n_j\}_{j=1}^{\infty}$ for the denominator of the rational function the set of the quotients is not always dense in $C[0, 1]$. A simple modification of the proof of our theorem gives that the assumption $|m_j-n_j|<K$ (where K does not depend on j) is enough for the density.

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SEPARATION OF SETS BY DERIVATIVES

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1. Introduction

Let $\mathcal{D}$ denote the family of derivatives defined on [0, 1] i.e. let $f \in \mathcal{D}$ if and only if there exists a function $F(x)$ on [0, 1] having a finite derivative everywhere and $F'(x)=f(x)$.

Our aim is to characterize the pairs, $X_1, X_2 \subset [0, 1]$ separable in the family $\mathcal{D}$. Theorems (2.2) and (2.3) give necessary and sufficient conditions imposed on X_1, X_2 for the existence $f \in \mathcal{D}$ with $f(x)=0$ if $x \in X_1$ and $f(x)=1$ if $x \in X_2$.

For the sets $X \subset [0, 1]$ we introduce the operation $T_{\mathcal{D}}(X)$ as follows:

$$T_{\mathcal{D}}(X) = \{x \in [0, 1] : f \in \mathcal{D},\ f(y) = 0 \quad \text{for} \quad y \in X \Rightarrow f(x) = 0\}.$$

In other words $T_{\mathcal{D}}(X)$ is the intersection of all $\mathcal{D}$-zero-sets containing X. (H is said to be a $\mathcal{D}$-zero-set if $H=\{x : f(x)=0\}$ for a suitable $f \in \mathcal{D}$).

It is easy to verify that $T_{\mathcal{D}}$ possesses all the properties of a usual closure operation except the additivity (see [1]). For this reason we call $T_{\mathcal{D}}(x)$ the $\mathcal{D}$-closure of X and the set $X \subset [0, 1]$ is said to be $\mathcal{D}$-closed if $T_{\mathcal{D}}(X)=X$. Every $\mathcal{D}$-zero-set is obviously $\mathcal{D}$-closed. In order to describe the operation $T_{\mathcal{D}}$ we define the following notions.

An interval I is said to be a *quasi-interval of* X if $\underline{\lambda}\,(I \setminus X)=0$ where $\underline{\lambda}$ denotes the Lebesgue inner measure. x_0 is called a ***strong limit point*** (s.1. point) *of* X if there exist $c>0$ and a sequence I_n of quasi-intervals of X such that $I_n \to x_0$ and

$$|I_n| > c \cdot \operatorname{dist}(I_n, x_0) \qquad (n = 1,2, \ldots).$$

($|I|$ denotes the length of I, $I_n \to x_0$ means that the endpoints of I_n converge to x_0).

For any $X \subset [0, 1]$ let $C_{\mathcal{D}}(X)$ denote the set of s.l. points of X.

We shall prove that, for every $X \subset [0, 1]$, $T_{\mathcal{D}}(X)=X \cup C_{\mathcal{D}}(X)$ holds (Theorem (3.8)). That is, if $T_{\mathcal{D}}(X)$ is considered as a certain closure of X, then $C_{\mathcal{D}}(X)$ corresponds to the derived set of X.

First of all we prove

PROPOSITION (1.1). *For every* $X \subset [0, 1]$ *we have* $T_{\mathcal{D}}(X) \supset X \cup C_{\mathcal{D}}(X)$.

PROOF. $T_{\mathcal{D}}(X) \supset X$ is trivial. Let $x_0 \in C_{\mathcal{D}}(X)$, then there exist $c>0$ and a sequence of intervals I_n such that $I_n=[a_n, b_n] \to x_0$, $\underline{\lambda}\,(I_n \setminus X)=0$ and $|I_n|>c \cdot \operatorname{dist}(I_n, x_0)$ $(n=1, 2, \ldots)$. We have to prove that if $f \in \mathcal{D}$ and $f(x)=0$ for $x \in X$ then $f(x_0)=0$. Let $F(x)$ be a primitive of f. Since $\underline{\lambda}\,(I_n \setminus X)=0$ and f is measurable we have $F'(x)=0$ a.e. in I_n. F is differentiable everywhere hence F is constant in I_n (see e. g. [2], Theorem 6.6, p. 143), thus $F(a_n)=F(b_n)$. Now

$$F(x) = F(x_0)+f(x_0)(x-x_0)+\varepsilon(x)(x-x_0),$$

where $\lim_{x\to x_0} \varepsilon(x)=0$, hence

$$F(b_n) = F(x_0)+f(x_0)(b_n-x_0)+\varepsilon(b_n)(b_n-x_0)$$

and

$$F(a_n) = F(x_0)+f(x_0)(a_n-x_0)+\varepsilon(a_n)(a_n-x_0).$$

By subtraction

$$0 = f(x_0)(b_n-a_n)+\varepsilon(b_n)(b_n-a_n)+\big(\varepsilon(b_n)-\varepsilon(a_n)\big)(a_n-x_0).$$

Thus

$$|f(x_0)| = \left|-\varepsilon(b_n)+\big(\varepsilon(a_n)-\varepsilon(b_n)\big)\frac{a_n-x_0}{b_n-a_n}\right|.$$

By $\lim_{n\to\infty}\varepsilon(a_n)=\lim_{n\to\infty}\varepsilon(b_n)=0$ and

$$\left|\frac{a_n-x_0}{b_n-a_n}\right| \leqq \frac{\operatorname{dist}(I_n, x_0)+|I_n|}{|I_n|} < \frac{1}{c}+1$$

we have $f(x_0)=0$, q.e.d.

2. The separation theorem

THEOREM (2.1). *Let H_1, H_2 be disjoint G_δ sets in* [0, 1]. *Suppose $C_{\mathscr{D}}(H_i)\subset H_i$ $(i=1, 2)$. Then H_1, H_2 are separable in $\mathscr{D}$.*

This result implies

THEOREM (2.2). *X_1 and X_2 can be separated by a derivative if and only if they can be separated by disjoint G_δ and $\mathscr{D}$-closed sets.*

PROOF. If $X_1\subset A$ and $X_2\subset B$ where A and B are disjoint G_δ and $\mathscr{D}$-closed then by (1.1) $A=T_{\mathscr{D}}(A)\supset C_{\mathscr{D}}(A)$ and $B=T_{\mathscr{D}}(B)\supset C_{\mathscr{D}}(B)$. Thus by (2.1) A and B are separable by a derivative which obviously separates X_1 and X_2 as well. On the other hand, if $f(x)\in\mathscr{D}$ separates X_1 and X_2 then $A=\{x: f(x)=0\}$ and $B=\{x: f(x)-1=0\}$ are disjoint and G_δ $\mathscr{D}$-zero-sets containing X_1 and X_2 respectively.

The same argument proves the following

THEOREM (2.3). *X_1 and X_2 can be separated by a derivative if and only if they can be separated by disjoint $\mathscr{D}$-zero-sets.*

Compared with Theorem (2.2) the condition in (2.3) is (apparently) a stronger one because a $\mathscr{D}$-closed G_δ set is not necessarily a $\mathscr{D}$-zero-set (see [3]).

We remark that a particular case of our Theorem (2.1) was proved in [4]: if H_1 and H_2 are disjoint G_δ sets each having metrically dense complements (i.e. $\lambda(I\setminus H_i)>0$ $(i=1, 2)$ for every open interval I) then H_1 and H_2 can be separated by derivatives.

In fact, in this case H_1 and H_2 have no quasi-intervals that is $C_{\mathscr{D}}(H_1)=C_{\mathscr{D}}(H_2)=\emptyset$. In particular H_1 and H_2 contain their s.1. points and (2.1) is applicable.

PROOF OF (2.1). Since H_1 and H_2 are disjoint G_δ sets we can find F_σ sets K_1, K_2 such that

(2.4) $$H_1\subset K_1,\quad H_2\subset K_2,\quad K_1\cap K_2=\emptyset,\quad K_1\cup K_2=[0, 1].$$

(See e.g. [5] p. 257.) We put

$$E = (\overline{H}_1 \setminus \operatorname{int} \overline{H}_1) \cup (\overline{H}_2 \setminus \operatorname{int} \overline{H}_2). \tag{2.5}$$

E is a nowhere dense closed set. Let J_k denote the sequence of intervals contiguous to E. For every J_k, either $J_k \cap H_1 = \emptyset$ or $J_k \cap H_2 = \emptyset$ holds. In fact if $J_k \cap H_i \neq \emptyset$ then H_i is everywhere dense in J_k (otherwise J_k would contain a point $x \in \overline{H}_i \setminus \operatorname{int} \overline{H}_i$). Everywhere dense G_δ sets have nonempty intersection hence $J_k \cap H_1 \neq \emptyset$, $J_k \cap H_2 \neq \emptyset$ imply $H_1 \cap H_2 \neq \emptyset$, a contradiction.

Let

$$E_1 = E \cap K_1, \quad E_2 = E \cap K_2. \tag{2.6}$$

Then E_1, E_2 are nowhere dense F_σ sets hence they can be represented in the form

$$E_1 = \bigcup_{k=1}^{\infty} F_{2k-1}, \quad E_2 = \bigcup_{k=1}^{\infty} F_{2k} \tag{2.7}$$

where F_n are pairwise disjoint and nowhere dense closed sets. (See [6], p. 197.) By (2.4) and (2.6) we have

$$E = E_1 \cup E_2 = \bigcup_{n=1}^{\infty} F_n. \tag{2.8}$$

We outline here the main idea of the proof. We are going to construct two sequences $t_n(x)$, $T_n(x)$ of functions defined on $[0, 1]$ such that for every $x \in [0, 1]$

$$t_1(x) \leqq t_2(x) \leqq \ldots \leqq t_n(x) \leqq \ldots \leqq T_n(x) \leqq \ldots \leqq T_2(x) \leqq T_1(x)$$

and with the property that whenever a function $f(x)$ satisfies $t_n(x) \leqq f(x) \leqq T_n(x)$ $(x \in [0, 1])$ then $f(x)$ is differentiable in the points of the set $Z_n = \{x : t_n(x) = T_n(x)\}$ and

$$f'(x) = \begin{cases} 0 & \text{if} \quad x \in H_1, \\ 1 & \text{if} \quad x \in H_2. \end{cases}$$

The sets Z_n should be chosen according to the conditions

a) $Z_n \supset F_n \quad (n = 1, 2, \ldots)$ and

b) $Z = \bigcup_{n=1}^{\infty} Z_n$ is closed.

Then for $x \in Z_n$ we define $f(x) = t_n(x) = T_n(x)$ and the definition of $f(x)$ can be easily extended to the intervals contiguous to Z such that $f(x)$ has a derivative everywhere and $f'(x)$ separates H_1 and H_2.

The lemmas (2.15), (2.22) and (2.28), (2.29) prepare the construction of the pair $t_n(x)$, $T_n(x)$ for odd and even n, respectively. To prove them we need the following lemmas (2.9), (2.10), (2.12), and (2.13).

LEMMA (2.9). *Let H_1 and H_2 be disjoint G_δ sets in $[a, b]$. Suppose $C_{\mathscr{D}}(H_i) \subset H_i$ and $H_i \neq [a, b]$ $(i = 1, 2)$. Then there exists an open interval $I \subset (a, b)$ such that $\lambda(I \cap H_i) < |I|$ $(i = 1, 2)$ and either $I \cap H_1 = \emptyset$ or $I \cap H_2 = \emptyset$.*

PROOF. If $G=\operatorname{int} H_1 \cup \operatorname{int} H_2$ then $G \neq (a, b)$. H_1 and H_2 cannot be everywhere dense in $Z=[a, b] \setminus G$ because they are G_δ sets and $H_1 \cap H_2 = \emptyset$. Hence there exists an open interval $I \subset (a, b)$ such that $Z \cap I \neq \emptyset$ and, say, $Z \cap I \cap H_1 = \emptyset$. If $I \cap H_1 \neq \emptyset$ then there exists a component (c, d) of G such that $(c, d) \subset H_1$ and $(c, d) \cap I \neq \emptyset$. Since c and d are s. l. points of H_1, $c, d \in H_1$. If $c \in I$ then $c \in Z \cap I \cap H_1$ which is impossible. Thus $I \cap H_1 = \emptyset$.

Suppose $\lambda(I \cap H_2) = |I|$. Then I is a quasi-interval of H_2 hence $\bar{I} \subset H_2$, $I \subset G$, $I \cap Z = \emptyset$, a contradiction. Thus we have $\lambda(I \cap H_2) < |I|$, q.e.d.

LEMMA (2.10). *Let $H \subset [a, b]$ be measurable with $\lambda([a, b] \setminus H) > 0$ and let A, B be given real numbers. Then there exists a monotone, everywhere differentiable function $F(x)$ on $[a, b]$ satisfying*

$$F(a) = A, \quad F(b) = B, \quad F'(x) = 0 \quad (x \in H). \tag{2.11}$$

For the proof see [1], Lemma 3.1.3.

LEMMA (2.12). *Let the open set $G \subset (a, b)$ and $\varepsilon > 0$ be given. Then there exists a subdivision $a = x_0 < x_1 < \ldots < x_n = b$ of the interval $[a, b]$ such that for every $1 \leq k \leq n$ $x_k \notin G$ and either (x_{k-1}, x_k) is a component of G or $x_k - x_{k-1} < \varepsilon$.*

PROOF. Let $a = y_0 < y_1 < \ldots < y_k = b$ be a subdivision of (a, b) finer that $\frac{\varepsilon}{2}$. Take one point from each set $(y_{i-1}, y_i) \setminus G$ (if it is nonempty), let $z_1, z_2, \ldots, z_l$ denote these points. G has only a finite number of components of length not smaller than ε. Let $u_1, u_2, \ldots, u_m$ be the endpoints of these components. It can be easily seen that the points z_i, u_j and a, b form a suitable subdivision of $[a, b]$.

LEMMA (2.13). *Let $G \subset (a, b)$ be open, $F \subset [a, b]$ be closed and suppose $G \cap F = \emptyset$. Let (α_k, β_k) denote the sequence of the intervals contiguous to F in $[a, b]$. Then for every k there exists a strictly monotone sequence $\{x_n^{(k)}\}_{n=u_k}^{v_k} \subset [\alpha_k, \beta_k] \setminus G$ such that*

(i) if (α_k, γ_k) is a component of G for a suitable γ_k then $u_k = 0$ and $x_0^{(k)} = \alpha_k$, $x_1^{(k)} = \gamma_k$. If (α_k, y) is not a component of G for any y then $u_k = -\infty$ and $\lim_{n \to -\infty} x_n^{(k)} = \alpha_k$. Similarly, if (δ_k, β_k) is a component of G then v_k is finite and $x_{v_k-1}^{(k)} = \delta_k$, $x_{v_k}^{(k)} = \beta_k$, otherwise $v_k = \infty$ and $\lim_{n \to \infty} x_n^{(k)} = \beta_k$.

(ii) If $a \notin F$ then the interval $[a, \beta_1)$ is contiguous to F for a suitable β_1. For the sequence $\{x_n^{(1)}\}_{u_1}^{v_1} \subset [a, \beta_1]$ let $u_1 = 0$ and $x_0^{(1)} = a$ even if (a, y) is not a component of G for any y. Similarly at b.

(iii) If the subintervals $I_i = (x_{n_i-1}^{(k_i)}, x_{n_i}^{(k_i)})$ are not components of G and $I_i \to x_0$ where $x_0 \in F$ then

$$\lim_{i \to \infty} \frac{|I_i|}{\operatorname{dist}(I_i, x_0)} = 0.$$

PROOF. For a fixed k we can choose a strictly monotone sequence $\{y_n^{(k)}\}$ from $[\alpha_k, \beta_k] \setminus G$ satisfying (i) and (ii). Apply Lemma (2.12) for the open set $G \cap (y_{n-1}^{(k)}, y_n^{(k)})$ and $\varepsilon = [\min (y_{n-1}^{(k)} - \alpha_k, \beta_k - y_n^{(k)})]^2$ in the interval $(y_{n-1}^{(k)}, y_n^{(k)})$ for every n which occurs among the indices of $\{y_n^{(k)}\}$. The points obtained and the $y_n^{(k)}$'s will compose the sequence $\{x_n^{(k)}\}$.

$\{x_n^{(k)}\}_{u_k}^{v_k}$ satisfies (i) and (ii) and if $x_0 \in F$ and $I = (x_{n-1}^{(k)}, x_n^{(k)})$ is not a component of G then $|I| < (x_{n-1}^{(k)} - x_0)^2$. Consequently if $I_i \searrow x_0$ where the intervals $I_i =$

$=(x^{(k_i)}_{n_i-1}, x^{(k_i)}_{n_i})$ are not components of G then

$$\frac{|I_i|}{x^{(k_i)}_{n_i-1}-x_0} < \frac{(x^{(k_i)}_{n_i-1}-x_0)^2}{x^{(k_i)}_{n_i-1}-x_0} = x^{(k_i)}_{n_i-1}-x_0 \to 0,$$

q.e.d.

REMARK (2.14). It is obvious from the construction that for every $\eta>0$ the sequences can be given such that besides (i), (ii), and (iii) even the following assertion holds:

If an interval $(x^{(k)}_{n-1}, x^{(k)}_n)$ *is not a component of* G *then* $x^{(k)}_n - x^{(k)}_{n-1} < \eta$. *This can be achieved if we apply Lemma* (2.12) *with* $\varepsilon_1 = \min(\eta, \varepsilon)$ *instead of* ε.

LEMMA (2.15). *Let* H_1 *and* H_2 *be disjoint* G_δ *subsets of* $[a, b]$ *and suppose* $C_{\mathscr{D}}(H_i) \subset \subset H_i$ $(i=1, 2)$. *Let the closed set* $F \subset [a, b]$ *be disjoint from* H_2 *and let* $\vartheta>0$ *be arbitrary. Then there exist two functions* $l(x)$ *and* $L(x)$ *on* $[a, b]$ *such that*

a) $-2(b-a) \leqq l(x) \leqq L(x) \leqq 2(b-a)$ *for every* $x \in [a, b]$,

b) *the set* $U = \{x : l(x) = L(x)\}$ *is closed,*

c) $F \cup \{a, b\} \subset U$; *every point of* $F \cap (a, b)$ *is a bilateral accumulation point of* U; *if* $a \in F$ *or* $b \in F$ *then* a *is a right-hand side and* b *is a left-hand side accumulation point of* U, *respectively. If* x *is a unilateral (and not bilateral) accumulation point of* U *then* $x \in H_1 \cup H_2 \cup F$.

If the interval (c, d) *is contiguous to* U *then*

d) $d-c<\vartheta$,

e) $(c, d) \not\subset H_1$, $(c, d) \not\subset H_2$,

f) if $l(c)=L(c)=A$ *and* $l(d)=L(d)=B$ *then* $l(x)=\min(A, B)-6(d-c)$ *and* $L(x)=\max(A, B)+6(d-c)$ *for every* $x \in (c, d)$.

g) *If the function* $f(x)$ *defined on* $[a, b]$ *satisfies* $l(x) \leqq f(x) \leqq L(x)$ $(x \in [a, b])$ *and the point* x_0 *is a bilateral accumulation point of* U *then* $f(x)$ *is differentiable at* x_0 *and*

$$(*) \qquad \begin{aligned} f'(x_0) &= 0 \quad \text{if} \quad x_0 \in H_1 \cup F \text{ and} \\ f'(x_0) &= 1 \quad \text{if} \quad x_0 \in H_2. \end{aligned}$$

If x_0 *is a right-hand side (left-hand side) accumulation point of* U *then* $f(x)$ *has a right-hand side (left-hand side) derivative at* x_0 *and* $(*)$ *holds for* $f'_+(x_0)$ *and* $f'_-(x_0)$, *respectively.*

PROOF. We put $G_1 = \operatorname{int} H_1 \setminus F$ and $G_2 = \operatorname{int} H_2$ and apply Lemma (2.13) for the sets $G = G_1 \cup G_2$ and F. An interval[1] $J = (x^{(k)}_{n-1}, x^{(k)}_n)$ is said to be of type (i), (ii), and (iii) if

(i) J is a component of G_1;

(ii) J is a component of G_2;

(iii) J is a component neither of G_1 nor G_2.

Now we define the functions $l(x)$ and $L(x)$ as follows.

$$(2.16) \qquad l(x) = L(x) = 0 \quad \text{if}$$

a') $x \in F$

[1] In the course of the proof the term 'interval' will mean an interval of the form $(x^{(k)}_{n-1}, x^{(k)}_n)$.

or

b') $x \in \bar{J}$ where J is of type (i)

or

c') x is the common endpoint of two adjacent intervals of type (iii).

If the interval $J=(c, d)$ is of type (ii) then

$$l(x) = L(x) = x - \frac{c+d}{2} \tag{2.17}$$

for every $x \in [c, d]$. $l(a)$ and $L(a)$ have been defined if $a \in F$ or if $a \notin F$ and the interval $[a, x_1^{(1)})$ (see (2.13) (ii)) is of type (i) or (ii). If $[a, x_1^{(1)})$ is of type (iii) then we define $l(a)=L(a)=0$. The definition is similar at b.

If $J=(c, d)$ is of type (iii) and $l(c)=L(c)=A$, $l(d)=L(d)=B$ ($l(x)$ and $L(x)$ have been defined at c and d!) then let

$$l(x) = \min(A, B) - 6(d-c) \quad \text{and} \tag{2.18}$$

$$L(x) = \max(A, B) + 6(d-c) \quad \text{for every} \quad x \in (c, d). \tag{2.19}$$

This definition of $l(x)$ and $L(x)$ is unambiguous because it is impossible that two intervals J_1 and J_2 are adjacent and J_1 is of type (ii) and J_2 is of type (i) or (ii). In fact, if J_1 and J_2 are adjacent and of type (ii) then their common endpoint is a s.1. point and hence a point of H_2. Therefore J_1 cannot be a component of G_2, a contradiction. If J_2 is of type (i) then the common endpoint of J_1 and J_2 is a s.1. point both of H_1 and H_2 hence it is a common point of H_1 and H_2 which is a contradiction again.

According to the Remark (2.14) we may suppose that the length of every interval of type (iii) is shorter than $\eta = \min\left(\vartheta, \frac{b-a}{4}\right)$. Hence

$$|l(x)| \leq \frac{b-a}{2} + 6 \cdot \frac{b-a}{4} = 2(b-a)$$

and similarly $|L(x)| \leq 2(b-a)$. Therefore a) and d) hold.

Since $[a, b] \setminus U$ is the union of the intervals of type (iii), hence the statements b), c), e) and f) can be easily seen as well.

Suppose that he function $f(x)$ satisfies $l(x) \leq f(x) \leq L(x)$ on $[a, b]$. If x_0 is a bilateral accumulation point of U then either x_0 is a point of an interval of type (i) or (ii) or $x_0 \in F$. In the first case $(*)$ obviously follows from (2.16) b') or from (2.17). If $x_0 \in F$ then by Lemma (2.13) (iii)

$$\lim_{i \to \infty} \frac{|I_i|}{\operatorname{dist}(I_i, x_0)} = 0$$

holds for every sequence of intervals I_i of type (iii) and converging to x_0. Since $x_0 \notin H_2$ and $C_{\mathscr{D}}(H_2) \subset H_2$ the same is true if the intervals I_i are of type (ii).

We have to prove $f'(x_0)=0$ that is

$$\frac{f(x_i)-f(x_0)}{x_i - x_0} \to 0 \qquad (x_i \to x_0). \tag{2.20}$$

Suppose $x_i \to x_0$, $x_i \in \bar{I}_i$ where I_i is of type (ii). Then $x_0 \notin \bar{I}_i$ and hence $I_i \to x_0$. Consequently

$$\left|\frac{f(x_i)-f(x_0)}{x_i-x_0}\right| = \frac{|f(x_i)|}{|x_i-x_0|} \le \frac{\frac{1}{2}|I_i|}{\operatorname{dist}(I_i, x_0)} \to 0.$$

Now suppose $x_i \to x_0$, $x_i \in I_i = (c_i, d_i)$ where I_i is of type (iii). Then $I_i \to x_0$ again and by (2.18) and (2.19)

$$(2.21) \qquad |f(x_i)| \le \max\left(|f(c_i)|, |f(d_i)|\right) + 6(d_i - c_i) \le |f(c_i)| + |f(d_i)| + 6|I_i|.$$

Now either $f(c_i)=0$ or the preceding interval $J_i=(e_i, c_i)$ is of type (ii). In the latter case

$$\frac{|f(c_i)|}{|x_i-x_0|} \le \frac{|f(c_i)|}{|c_i-x_0|}\cdot\left|\frac{c_i-x_0}{x_i-x_0}\right| \le \frac{\frac{1}{2}|J_i|}{\operatorname{dist}(J_i, x_0)}\left|1+\frac{c_i-x_i}{x_i-x_0}\right| \to 0$$

since $\left|\frac{c_i-x_i}{x_i-x_0}\right| \le \frac{|I_i|}{\operatorname{dist}(I_i, x_0)} \to 0$. In both cases $\frac{f(c_i)}{x_i-x_0} \to 0$ and similarly $\frac{f(d_i)}{x_i-x_0} \to 0$. By (22.1)

$$\left|\frac{f(x_i)-f(x_0)}{x_i-x_0}\right| \le \left|\frac{f(c_i)}{x_i-x_0}\right| + \left|\frac{f(d_i)}{x_i-x_0}\right| + 6\cdot\frac{|I_i|}{\operatorname{dist}(I_i, x_0)} \to 0.$$

Since for every x either $f(x)=0$ or $x \in \bar{I}$ where I is of type (ii) or $x \in I$ where I is of type (iii) we have (2.20) for every $x_i \to x_0$. That is $(*)$ holds.

If x_0 is a right-hand side accumulation point of U then either $x_0 \in F$ (when $(*)$ holds) or $x_0 \in \bar{J}$ where the interval J is of type (i) or (ii). In this case the statement $(*)$ for $f'_+(x_0)$ immediately follows from (2.16) b') and (2.17). This completes the proof of Lemma (2.15).

LEMMA (2.22). *Let H_1 and H_2 be disjoint G_δ subsets of $[a, b]$ and suppose $C_{\mathscr{D}}(H_i) \subset H_i$ and $H_i \ne [a, b]$ $(i=1, 2)$, moreover suppose that $H_1 \cup H_2$ is everywhere dense in $[a, b]$. Let $F \subset [a, b]$ be a nowhere dense closed set such that $F \cap H_2 = \emptyset$ and*

$$(2.23) \qquad F \cap (a, b) \subset (\bar{H}_1 \setminus \operatorname{int} \bar{H}_1) \cup (\bar{H}_2 \setminus \operatorname{int} \bar{H}_2).$$

Let the arbitrary real numbers $\vartheta > 0$ and A_0, B_0 be given. Then there exist two functions $l(x)$ and $L(x)$ defined on $[a, b]$ such that besides the statements b), c), d), e), f), *and* g) *of Lemma* (2.15) *even*

$$(2.24) \qquad l(a) = L(a) = A_0,$$

$$(2.25) \qquad l(b) = L(b) = B_0, \quad \textit{and}$$

$$(2.26) \qquad \min(A_0, B_0) - 6(b-a) \le l(x) \le L(x) \le \max(A_0, B_0) + 6(b-a)$$

hold for every $x \in [a, b]$.

Proof. By Lemma (2.9) there exists an open interval $I\subset[a, b]$ such that $I\cap H_1=\emptyset$ or $I\cap H_2=\emptyset$ and $\lambda(I\setminus H_i)>0$ $(i=1, 2)$. We may suppose $I\cap F=\emptyset$. In fact, if $I\cap F\neq\emptyset$ and $I\setminus F=\bigcup_{k=1}^{\infty} I_k$ where I_k are disjoint open intervals then we can choose one of the I_k's instead of I. Namely if (say) $I\cap H_1=\emptyset$ then H_2 cannot be everywhere dense in each I_k beacuse otherwise H_2 would be everywhere dense in I and $I\cap(\bar{H}_2\setminus\operatorname{int}\bar{H}_2)=\emptyset$ would hold, hence by (2.23) $I\cap F=\emptyset$ which is a contradiction. If H_2 is not everywhere dense in I_k then we can choose I_k instead of I.

a) Suppose first $I\cap H_2=\emptyset$. Since $H_1\cup H_2$ is everywhere dense in $[a, b]$ there exist $\alpha, \beta\in H_1\cap I$ such that $\lambda\big((a, \beta)\setminus H_1\big)>0$. We apply Lemma (2.15) for the sets $H_1\cap[a, \alpha]$, $H_2\cap[a, \alpha]$ and $F\cap[a, \alpha]$ in the interval $[a, \alpha]$ and let the functions obtained be denoted by $l_1(x)$ and $L_1(x)$. We construct the functions $l_2(x)$ and $L_2(x)$ on $[\beta, b]$ analogously.

Now we define the functions $l(x)$ and $L(x)$ by

(2.27) $$l(x) = l_1(x)+A_0-l_1(a) \quad\text{and}\quad L(x) = L_1(x)+A_0-l_1(a)$$

if $x\in[a, \alpha]$;

$$l(x) = l_2(x)+B_0-l_2(b) \quad\text{and}\quad L(x) = L_2(x)+B_0-l_2(b)$$

if $x\in[\beta, b]$. In order to define $l(x)$ and $L(x)$ in the interval $[\alpha, \beta]$ we apply Lemma (2.10) for the set $H_1\cap[\alpha, \beta]$ and the numbers $A=l_1(\alpha)+A_0-l_1(a)$ and $B=l_2(\beta)+B_0-l_2(b)$ in the interval $[\alpha, \beta]$. Then we have a monotone, everywhere differentiable function $F(x)$ satisfying $F'(x)=0$ $(x\in H_1\cap[\alpha, \beta])$ and

$$F(\alpha) = l_1(\alpha)+A_0-l_1(a), \quad F(\beta) = l_2(\beta)+B_0-l_2(b).$$

We put $l(x)=L(x)=F(x)$ for $x\in[\alpha, \beta]$.

It is easy to see that $l(x)$ and $L(x)$ satisfy the statements (2.15) b) to g). Moreover by (2.27) $l(a)=L(a)=A_0$ and $l(b)=L(b)=B_0$ (since $l_1(a)=L_1(a)$ and $l_2(b)=L_2(b)$) that is (2.24) and (2.25) hold.

By (2.15) a) $|l_1(x)|\leqq 2(\alpha-a)\leqq 2(b-a)$ if $x\in[a, \alpha]$ and $|l_2(x)|\leqq 2(b-\beta)\leqq 2(b-a)$ if $x\in[\beta, b]$ hence $l(x)\geqq\min\big(A_0-4(b-a),\ B_0-4(b-a)\big)=\min(A_0, B_0)-4(b-a)>\min(A_0, B_0)-6(b-a)$. Similarly $L(x)\leqq\max(A_0, B_0)+6(b-a)$ if $x\in[a, \alpha]$ or $x\in[\beta, b]$. Since $F(x)$ is monotone, these inequalities are valid in $[\alpha, \beta]$ too. Thus (2.26) holds.

b) Suppose now $I\cap H_1=\emptyset$. An argument similar to that of case a) shows that there exist $\alpha, \beta\in H_2\cap I$ such that $\lambda\big((\alpha, \beta)\setminus H_2\big)>0$. Applying Lemma (2.15) for the sets $H_1\cap[a, \alpha]$, $H_2\cap[a, \alpha]$ and $F\cap[a, \alpha]$ we obtain the functions $l_1(x)$ and $L_1(x)$ in $[a, \alpha]$. In the same way we get $l_2(x)$ and $L_2(x)$ in $[\beta, b]$. Let

$$l(x) = l_1(x)+A_0-l_1(a) \quad\text{and}\quad L(x) = L_1(x)+A_0-l_1(a),$$

if $x\in[a, \alpha]$;

$$l(x) = l_2(x)+B_0-l_2(b) \quad\text{and}\quad L(x) = L_2(x)+B_0-l_2(b),$$

if $x\in[\beta, b]$. Applying Lemma (2.10) for the set $H_2\cap[\alpha, \beta]$ and $A=l_1(\alpha)+A_0-l_1(a)$, $B=l_2(\beta)+B_0-l_2(b)-(\beta-\alpha)$ we have the function $F(x)$ in $[\alpha, \beta]$. Let $l(x)=L(x)=F(x)+x-\alpha$, if $x\in[\alpha, \beta]$.

It is easy to verify that all the statements of (2.15) (except a)) are valid for $l(x)$ and $L(x)$. Since $|l_1(x)|\leqq 2(b-a)$ if $x\in[a, \alpha]$ and $|l_2(x)|\leqq 2(b-a)$ if $x\in[\beta, b]$ hence

(2.26) holds for every $x\in[a,\alpha]\cup[\beta,b]$. If $x\in[\alpha,\beta]$ then $F(x)\leqq\max\big(l_1(\alpha)+A_0-l_1(a), l_2(\beta)+B_0-l_2(b)-(\beta-\alpha)\big)\leqq\max\big(A_0+4(b-a), B_0+5(b-a)\big)\leqq\max(A_0,B_0)+5(b-a)$. Thus $F(x)+x-\alpha\leqq\max(A_0,B_0)+6(b-a)$. Similarly $F(x)+x-\alpha\geqq\min(A_0,B_0)-6(b-a)$ that is (2.26) holds everywhere, q.e.d.

LEMMA (2.28). *Let H_1 and H_2 be disjoint G_δ subsets of $[a,b]$ and suppose $C_{\mathscr{D}}(H_i)\subset H_i$ $(i=1,2)$. Let the closed set $F\subset[a,b]$ be disjoint from H_1 and let $\vartheta>0$ be arbitrary. Then there exist two functions $l(x)$ and $L(x)$ defined on $[a,b]$ satisfying the statements* (2.15) a) *to* f) *and the following one:*

g*) *If the function $f(x)$ defined on $[a,b]$ satisfies $l(x)\leqq f(x)\leqq L(x)$ and the point x_0 is a bilateral accumulation point of U then $f(x)$ is differentiable at x_0 and*

$$(**)\qquad f'(x_0)=\begin{cases}0, & \text{if } x\in H_1\\ 1 & \text{if } x\in H_2\cup F.\end{cases}$$

If x_0 is a right-hand side (left-hand side) accumulation point of U then $f(x)$ has a right-hand side (left-hand side) derivative at x_0 and $(**)$ holds for $f'_+(x_0)$ and $f'_-(x_0)$, respectively.

PROOF. We put $G_1=\operatorname{int}H_1$, $G_2=\operatorname{int}H_2\setminus F$ and apply Lemma (2.13) for the open set $G=G_1\cup G_2$ and the closed set F. We obtain subintervals of the form $(x_{n-1}^{(k)}, x_n^{(k)})$ and classify them in the same way as in the proof of Lemma (2.15). We define the functions $l(x)$ and $L(x)$ as follows: $l(x)=L(x)=x-a$, if $x\in F$ or $x\in\bar J$ where J is of type (ii) or x is the common endpoint of two adjacent intervals of type (iii). $l(x)=L(x)=\frac{c+d}{2}-a$ for every $x\in[c,d]$ if (c,d) is of type (i). If $l(x)$ and $L(x)$ have not been defined so far at a or b then let $l(a)=L(a)=0$ and $l(b)=L(b)=b-a$, respectively. Finally, if (c,d) is of type (iii) and $l(c)=L(c)=A$, $l(d)=L(d)=B$ ($l(x)$ and $L(x)$ have been defined at c and d!) then let

$$l(x)=\min(A,B)-6(d-c)\quad\text{and}\quad L(x)=\max(A,B)+6(d-c)$$

for every $x\in(c,d)$. We can show in the same way as in the proof of Lemma (2.15) that the statements (2.15) a) to f) hold. The only difference is that the intervals of type (iii) must be chosen of length shorter than $\eta=\min\left(\vartheta,\frac{b-a}{6}\right)$. Suppose that the function $f(x)$ satisfies $l(x)\leqq f(x)\leqq L(x)$ on $[a,b]$. If x_0 is a bilateral accumulation point of U then either x_0 is a point of an interval of type (i) or (ii) or $x_0\in F$. In the first case $(**)$ obviously holds. If $x_0\in F$ then by Lemma (2.13) (iii)

$$\lim_{i\to\infty}\frac{|I_i|}{\operatorname{dist}(I_i,x_0)}=0$$

holds for every sequence of intervals I_i of type (iii) and converging to x_0. Since $x_0\notin H_1$ and $C_{\mathscr{D}}(H_1)\subset H_1$ the same is true if the intervals I_i are of type (i).

We have to show $f'(x_0)=1$. Suppose $x_i\to x_0$, $x_i\in\bar I_i=[c_i,d_i]$ where I_i is of type (i). Then $I_i\to x_0$, $f(x_i)=\frac{c_i+d_i}{2}-a$ and $f(x_0)=x_0-a$ so that

$$\left|\frac{f(x_i)-f(x_0)}{x_i-x_0}-1\right|=\left|\frac{\frac{c_i+d_i}{2}-x_i}{x_i-x_0}\right|\leqq\frac{|I_i|}{\operatorname{dist}(I_i,x_0)}\to 0.$$

Now suppose $x_i \to x_0$, $x_i \in I_i = (c_i, d_i)$ where I_i is of type (iii). Then $I_i \to x_0$ again and by the definition of $l(x)$ and $L(x)$ we have

$$\min\left(f(c_i), f(d_i)\right) - 6|I_i| \leqq f(x_i) \leqq \max\left(f(c_i), f(d_i)\right) + 6|I_i|.$$

Thus in order to show $\frac{f(x_i)-f(x_0)}{x_i-x_0} \to 1$ it is enough to prove $\frac{f(c_i)-f(x_0)}{x_i-x_0} \to 1$, $\frac{f(d_i)-f(x_0)}{x_i-x_0} \to 1$ and $\frac{|I_i|}{x_i-x_0} \to 0$.

Now either $f(c_i) = c_i - a$ or the preceding interval $J_i = (e_i, c_i)$ is of type (i) and thus $f(c_i) = \frac{e_i+c_i}{2} - a$. In the first case

$$\left|\frac{f(c_i)-f(x_0)}{x_i-x_0} - 1\right| = \left|\frac{c_i - x_i}{x_i - x_0}\right| \leqq \frac{|I_i|}{\operatorname{dist}(I_i, x_0)} \to 0.$$

In the second case

$$\left|\frac{f(c_i)-f(x_0)}{x_i-x_0} - 1\right| = \left|\frac{\frac{1}{2}(e_i+c_i)-x_i}{x_i-x_0}\right| \leqq \frac{|J_i|+|I_i|}{|x_i-x_0|} = \frac{|J_i|}{|c_i-x_0|}\left|\frac{c_i-x_0}{x_i-x_0}\right| + \frac{|I_i|}{|x_i-x_0|} \leqq$$

$$\leqq \frac{|J_i|}{\operatorname{dist}(J_i, x_0)}\left|1+\frac{c_i-x_i}{x_i-x_0}\right| + \frac{|I_i|}{|x_i-x_0|} \to 0$$

since $J_i \to x_0$. Hence in both cases $\frac{f(c_i)-f(x_0)}{x_i-x_0} \to 1$ and similarly $\frac{f(d_i)-f(x_0)}{x_i-x_0} \to 1$. Finally

$$\frac{|I_i|}{|x_i-x_0|} \leqq \frac{|I_i|}{\operatorname{dist}(I_i, x_0)} \to 0.$$

Since for every x either $f(x) = x - a$ or $x \in \bar{I}$ where I is of type (i) or $x \in I$ where I is of type (iii) we have $f'(x_0) = 1$ that is $(**)$ holds.

If x_0 is a right-hand side accumulation point of U then either $x_0 \in F$ (when $(**)$ holds) or $x_0 \in \bar{J}$ where the interval $\bar{J}$ is of type (i) or (ii). In this case the statement $(**)$ for $f'_+(x_0)$ immediately follows from the definition of $l(x)$ and $L(x)$. This completes the proof of Lemma (2.28).

LEMMA (2.29). *Let H_1 and H_2 be disjoint G_δ sets in* $[a, b]$ *and suppose that $H_i \neq [a, b]$ and $C_{\mathcal{D}}(H_i) \subset H_i$ $(i=1, 2)$. In addition suppose that $H_1 \cup H_2$ is everywhere dense in* $[a, b]$. *Let $F \subset [a, b]$ be a nowhere dense closed set such that $F \cap H_1 = \emptyset$ and* (2.23) *holds. Let the arbitrary real numbers $\vartheta > 0$ and A_0, B_0 be given. Then there exist two fuctions $l(x)$ and $L(x)$ defined on* $[a, b]$ *such that the statements* (2.15) b) *to* f), (2.28) g*), *and* (2.24), (2.25), (2.26) *hold.*

This can be proved in the same way as in the case of Lemma (2.22).

The following step of the proof of Theorem (2.1) is the construction of the functions $t_n(x)$ and $T_n(x)$.

Consider the sets H_1 and H_2 given in Theorem (2.1). We can suppose $H_i \neq [0, 1]$ ($i=1, 2$) since otherwise the theorem is trivial. Moreover we can suppose that $H_1 \cup H_2$ is everywhere dense in [0,1]. In fact if this is false then take the open set $G=(0,1)\setminus \setminus \overline{H_1 \cup H_2}$. Let $P \subset G$ be a G_δ set of measure zero and everywhere dense in G and put $H_1'=H_1 \cup P$. Then H_1' and H_2 are disjoint G_δ sets and $C_{\mathcal{D}}(H_1') \subset H_1'$ since every quasi-interval of H_1' is a quasi-interval of H_1 as well so that $C_{\mathcal{D}}(H_1') \subset C_{\mathcal{D}}(H_1) \subset \subset H_1 \subset H_1'$.

We define the functions $t_n(x)$ and $T_n(x)$ by induction. Apply Lemma (2.15) for the sets H_1, H_2 and F_1 and for $\vartheta=1$. The lemma is applicable because $F_1 \subset E_1$ and $E_1 \cap H_2 = \emptyset$ so that $F_1 \cap H_2 = \emptyset$. Let $t_1(x)=l(x)$ and $T_1(x)=L(x)$ where $l(x)$ and $L(x)$ are the functions obtained in Lemma (2.15).

Suppose we have defined $t_i(x)$ and $T_i(x)$ for $1 \leqq i \leqq n$ such that

(2.30) (a) $t_i(x) \leqq t_{i+1}(x) \leqq T_{i+1}(x) \leqq T_i(x)$ for every $x \in [0, 1]$ and $1 \leqq i < n$;

(b) the set $Z_n = \{x;\, t_n(x) = T_n(x)\}$ is closed;

(c) $\bigcup_{i=1}^{n} F_i \cup \{0, 1\} \subset Z_n$; every point of $\bigcup_{i=1}^{n} F_i \cap (0, 1)$ is a bilateral accumulation point of Z_n; if $0 \in \bigcup_{i=1}^{n} F_i$ or $1 \in \bigcup_{i=1}^{n} F_i$ then 0 is a right-hand side and 1 is a left-hand side accumulation point of Z_n, respectively. If x is a unilateral accumulation point of Z_n then

$$x \in H_1 \cup H_2 \cup \bigcup_{i=1}^{n} F_i.$$

For every interval (a, b) contiguous to Z_n

(d) either (a, b) is contiguous to Z_{n-1} as well or $b-a<\frac{1}{n}$;

(e) $(a, b) \not\subset H_1, \quad (a, b) \not\subset H_2$;

(f) $t_n(x) = \min(A, B) - 6(b-a)$ and $T_n(x) = \max(A, B) + 6(b-a)$ for every $x \in (a, b)$ where $A = t_n(a) = T_n(a)$ and $B = t_n(b) = T_n(b)$; finally

(g) if the function $f(x)$ defined on [0, 1] satisfies $t_n(x) \leqq f(x) \leqq T_n(x)$ on [0, 1] and if x_0 is a bilateral (unilateral) accumulation point of Z_n then $f(x)$ has a derivative (unilateral derivative) at x_0 and

$$(***) \qquad f'(x_0) = \begin{cases} 0 & \text{if } x_0 \in H_1 \\ 1 & \text{if } x_0 \in H_2 \end{cases}$$

holds (for the suitable unilateral derivative at x_0).

It is easy to see that (2.30) (a) to (g) are valid for $n=1$. Let

$$(2.31) \qquad t_{n+1}(x) = T_{n+1}(x) = t_n(x) \quad \big(=T_n(x)\big) \quad \text{for every} \quad x \in Z_n.$$

Let (a, b) be an arbitrary interval contiguous to Z_n. *If* $[a, b] \cap F_{n+1} = \emptyset$ then let

$$(2.32) \qquad t_{n+1}(x) = t_n(x) \quad \text{and} \quad T_{n+1}(x) = T_n(x) \quad \text{for every} \quad x \in (a, b).$$

If $[a, b] \cap F_{n+1} \neq \emptyset$ then we apply Lemma (2.22) or Lemma (2.29) for the sets $H_1 \cap [a, b]$, $H_2 \cap [a, b]$ and $F_{n+1} \cap [a, b]$ according to that n is even or odd.

These lemmas are applicable because by (2.30) (e) $H_1\cap[a,b]\neq[a,b]$ and $H_2\cap[a,b]\neq[a,b]$; (2.23) follows from (2.5), (2.6), and (2.7), furthermore by (2.4), (2.6), and (2.7) we have $F_{n+1}\cap H_2=\emptyset$ if n is even and $F_{n+1}\cap H_1=\emptyset$ if n is odd.

Now we apply the suitable lemma with $\vartheta=\frac{1}{n+1}$, $A_0=t_n(a)=T_n(a)$ and $B_0==t_n(b)=T_n(b)$ and put

$$(2.33)\qquad t_{n+1}(x)=l(x) \quad\text{and}\quad T_{n+1}(x)=L(x) \quad \text{for every} \quad x\in(a,b)$$

where $l(x)$ and $L(x)$ are the functions constructed in the suitable lemma.

We show that (2.30) (a) to (g) are valid for $t_{n+1}(x)$ and $T_{n+1}(x)$. The statement (a) follows from (2.31) if $x\in Z_n$ and from (2.30) (f) and (2.26) if $x\notin Z_n$. The statement (b) is obvious from (2.15) b) and from the fact that Z_n is closed. The proof of (c): $\bigcup_{i=1}^{n+1} F_i\cup\{0,1\}\subset Z_{n+1}$ is trivial from $\bigcup_{i=1}^{n} F_i\cup\{0,1\}\subset Z_n$ and from (2.15) c). Since $Z_n\subset Z_{n+1}$ hence every point of $\bigcup_{i=1}^{n} F_i\cap(0,1)$ is a bilateral accumulation point of Z_{n+1}. If $x\in F_{n+1}\cap(0,1)$ then this follows from (2.15) c). If x is a unilateral accumulation point of Z_{n+1} then either it is one of Z_n when (2.15) c) and the statement (2.30) (c) for $t_n(x)$ and $T_n(x)$ imply $x\in H_1\cup H_2\cup\bigcup_{i=1}^{n+1}F_i$ or not when (2.15) c) implies $x\in H_1\cup H_2\cup\bigcup_{i=1}^{n+1}F_i$ again.

(2.30) (d) follows from (2.15) d) and $\vartheta=\frac{1}{n+1}$. (e) and (f) are consequences of (2.15) e), (2.15) f), and (2.33).

Finally we prove (2.30) (g). Suppose $t_{n+1}(x)\leqq f(x)\leqq T_{n+1}(x)$ and let x be a bilateral accumulation point of Z_{n+1}. The statement is trivial if x is a bilateral accumulation point of Z_n, too. If x is a left-hand side but not a right-hand side accumulation point of Z_n then there exists $y>x$ such that (x,y) is an interval contiguous to Z_n. (2.30) (c) and (g) imply that $x\in H_1\cup H_2\cup\bigcup_{i=1}^{n}F_i$ furthermore that $f'_-(x)$ exists. Since x is a unilateral accumulation point of $Z_{n+1}\cap[x,y]$ (2.15) c) implies $x\in H_1\cup H_2\cup F_{n+1}$. Hence $x\in H_1\cup H_2$ because $\left(\bigcup_{i=1}^{n}F_i\right)\cap F_{n+1}=\emptyset$. Consequently $f'_+(x)=f'_-(x)=0$ or 1 according to $x\in H_1$ or $x\in H_2$. The proof is similar when x is an isolated point of Z_n. If $x\in Z_{n+1}\setminus Z_n$ then (2.15) g) implies (2.30) (g). This argument can be repeated if x is a unilateral accumulation point of Z_{n+1}.

Now we have defined $t_n(x)$ and $T_n(x)$ for $n=1,2,\ldots$. Let Q_n be a family of intervals such that $(a,b)\in Q_n$ if and only if (a,b) is an interval contiguous to Z_n but not contiguous to Z_{n-1} and $[a,b]\cap E=\emptyset$.

$\bigcup_{n=1}^{\infty}Q_n$ is obviously a set of disjoint open intervals.

LEMMA (2.34). $Z=\bigcup_{n=1}^{\infty}Z_n=[0,1]\setminus\cup\{I: I\in\bigcup_{n=1}^{\infty}Q_n\}$.

PROOF. $[\bigcup\{I: I\in \bigcup_{n=1}^{\infty} Q_n\}]\cap[\bigcup_{n=1}^{\infty} Z_n]=\emptyset$ because if $I\in Q_n$ then $\bar{I}\cap E=\emptyset$ thus $\bar{I}\cap F_k=\emptyset$ for every $k>n$ so that $I\cap Z_k=\emptyset$ for every $k>n$.

Suppose $x\notin \bigcup_{n=1}^{\infty} Z_n$, we have to prove $x\in I$ for a suitable $I\in Q_n$. By (2.30) (c) and (2.8) we have $E\subset \bigcup_{n=1}^{\infty} Z_n$ hence $x\notin E$ that is $\varrho=\operatorname{dist}(x, E)>0$. Denote (a_n, b_n) the interval contiguous to Z_n containing x. If $b_n-a_n<\varrho$ then by $x\in(a_n, b_n)$ we have $[a_n, b_n]\cap E=\emptyset$, $(a_n, b_n)\in \bigcup_{n=1}^{\infty} Q_n$ and the lemma is proved. If $b_n-a_n\geqq\varrho$ for every n then by (2.30) (d) we have $(a_n, b_n)=(a_N, b_N)$ for every $n\geqq N$. Hence $[a_N, b_N]\cap E=\emptyset$ because otherwise $[a_N, b_N]\cap F_k\neq\emptyset$ would hold for a suitable $k>N$ $\left([a_N, b_N]\cap \bigcup_{i=1}^{N} F_i=\emptyset\right.$ by (2.30) (c)$\left.\right)$ which implies $(a_k, b_k)\subsetneqq(a_{k-1}, b_{k-1})$, a contradiction. Thus $(a_N, b_N)\in \bigcup_{n=1}^{\infty} Q_n$, q.e.d.

Lemma (2.34) implies that $Z=\bigcup_{n=1}^{\infty} Z_n$ is closed. We remark that if x is a right-hand side accumulation point of Z then x is a right-hand side accumulation point of Z_n too, for a suitable n. In fact, if $x\notin E$ and for every $n\geqq n_0$ there were $y_n>x$ such that (x, y_n) is an interval contiguous to Z_n then in the same way as in the proof of Lemma (2.34) we could prove that $(x, y_n)\in Q_n$ for a suitable n. But this is impossible because x is a right-hand side accumulation point of Z. The case $x\in E$ is trivial from (2.30) (c).

Now we are going to define the function $f(x)$. Let

$$f(x)=t_n(x)\quad \left(=T_n(x)=t_k(x)=T_k(x)\quad \text{for}\quad k\geqq n\right) \tag{2.35}$$

for every $x\in Z_n$.

Let $(a, b)\in Q_n$ be arbitrary. (2.30) (e) implies that

$$\lambda\left((a, b)\setminus H_i\right)>0 \qquad (i=1,2) \tag{2.36}$$

or else (a, b) would be a quasi-interval of H_i so that $[a, b]\subset C_{\mathcal{D}}(H_i)\subset H_i$ would hold, which is impossible. $[a, b]\cap E=\emptyset$ implies $[a, b]\subset J_k$ for a suitable k. As we have shown either $J_k\cap H_1=\emptyset$ or $J_k\cap H_2=\emptyset$ holds. Suppose first $J_k\cap H_2=\emptyset$. Apply Lemma (2.10) for the set $[H_1\cap(a, b)]\cup\{a, b\}$ and for $A=f(a)$ $\left(=t_n(a)=T_n(a)\right)$, $B=f(b)$ $\left(=t_n(b)=T_n(b)\right)$ in $[a, b]$. Let

$$f(x)=F(x)\quad \text{for every}\quad x\in[a, b] \tag{2.37}$$

if $F(x)$ is the function constructed in Lemma (2.10). Then $f(x)$ is differentiable in $[a, b]$ and

$$f'(x)=0 \quad \text{if}\quad x\in H_1\cap(a, b) \quad \text{moreover}\quad f'_+(a)=f'_-(b)=0. \tag{2.38}$$

Now suppose $J_k \cap H_2 \neq \emptyset$, then $J_k \cap H_1 = \emptyset$. Apply Lemma (2.10) again for the set $[H_2 \cap (a, b)] \cup \{a, b\}$ and for $A = f(a)$, $B = f(b) - (b-a)$ in $[a, b]$. Let

(2.39) $$f(x) = F(x) + x - a \text{ if } x \in [a, b]$$

where $F(x)$ is the function constructed in the Lemma. Then $f(x)$ is differentiable in $[a, b]$ and

(2.40) $$f'(x) = 1, \text{ if } x \in H_2 \cap (a, b) \text{ moreover } f'_+(a) = f'_-(b) = 1.$$

(2.35), (2.37), and (2.39) define $f(x)$ on Z and on every interval contiguous to Z i.e. on $[0, 1]$. We prove that $f(x)$ is differentiable on $[0, 1]$ and

(2.41) $$f'(x) = \begin{cases} 0 & \text{if } x \in H_1, \\ 1 & \text{if } x \in H_2. \end{cases}$$

First we show that

(2.42) $$t_n(x) \leqq f(x) \leqq T_n(x) \quad \text{for every} \quad x \in [0, 1] \quad \text{and} \quad n = 1, 2, \ldots.$$

This is obvious if $x \in Z$. If $x \in I = (a, b)$ where $I \in Q_n$; $I \subset J_k$, $J_k \cap H_2 = \emptyset$ then by (2.37) $f(x)$ is monotone on $[a, b]$ thus $\min(A, B) \leqq f(x) \leqq \max(A, B)$ if $x \in [a, b]$. Comparing with (2.30) (f) and (2.32) we have (2.42). If $J_k \cap H_2 \neq \emptyset$ then by (2.39) $\min(A, B) - (b-a) \leqq F(x) \leqq \max(A, B)$ thus $\min(A, B) - (b-a) \leqq f(x) \leqq$ $\leqq \max(A, B) + (b-a)$ on $[a, b]$ which implies (2.42) again.

Suppose that x is a bilateral accumulation point of Z. As we have remarked this implies that x is a left-hand side and right-hand side accumulation point of Z_n and Z_m, respectively, for suitable n and m. Hence x is a bilateral accumulation point of $Z_{\max(n, m)}$ and by (2.42) and (2.30) (g) f is differentiable at x and (2.41) holds.

Suppose that x is a left-hand side but not a right-hand side accumulation point of Z. In this case $(x, b) \in Q_n$ for a suitable $b > x$. Then $[x, b] \subset J_k$ for a suitable k. If $J_k \cap H_2 = \emptyset$ then by (2.38) $f'_+(x)$ exists and vanishes. Since x is a left-hand side accumulation point of Z_m for at least one m hence by (2.30) (c) we have $x \in H_1 \cup$ $\cup H_2 \cup \bigcup_{i=1}^{m} F_i$. But $x \notin H_2 \cup \bigcup_{i=1}^{m} F_i$ thus $x \in H_1$. (2.42) and (2.30) (g) imply $f'_-(x) = 0$ hence (2.41) holds again. We can argue similarly if

$$J_k \cap H_2 \neq \emptyset \quad \text{or} \quad (b, x) \in Q_n.$$

If x is an isolated point of Z then $(a, x) \in Q_n$ and $(x, b) \in Q_m$ for suitable a, b, n, m. Since $[a, x] \cap E = \emptyset$ and $[x, b] \cap E = \emptyset$ we have $[a, b] \subset J_k$. Hence by (2.38) and (2.40) $f'_+(x) = f'_-(x) = 0$ or 1 according to $J_k \cap H_2 = \emptyset$ or $J_k \cap H_2 \neq \emptyset$.

Finally if $x \notin Z$ then $x \in (a, b)$ where $(a, b) \in Q_n$ and (2.38) or (2.40) imply (2.41) again. This completes the proof of (2.1).

3. The operations $C_{\mathscr{D}}$ and $T_{\mathscr{D}}$

LEMMA (3.1). *Let $X \subset [0, 1]$ be arbitrary and let G denote the union of all open quasi-intervals of X. Then for every open interval I either $I \subset G$ or $\lambda(I \setminus G) > 0$ furthermore I is a quasi-interval of G if and only if I is a quasi-interval of X. Consequently $C_{\mathscr{D}}(G) = C_{\mathscr{D}}(X)$.*

PROOF. First we prove that the set of the quasi-intervals of G and those of X coincide. If I is a quasi-interval of X then $\operatorname{int} I \subset G$ hence I is a quasi-interval of G as well.

If $(a, b) \subset G$ then by the Borel covering theorem $[a+\varepsilon, b-\varepsilon]$ can be covered by a finite number of quasi-intervals of X for every $\varepsilon > 0$. This easily implies that (a, b) is a quasi-interval of X, too. Consequently for every component I_k of G $\underline{\lambda}(I_k \setminus X) = 0$ hence $\underline{\lambda}(G \setminus X) = 0$.

If I is an arbitrary quasi-interval of G then $\lambda(I \setminus G) = 0$ thus from $I \setminus X \subset (I \setminus G) \cup \cup (G \setminus X)$ we have $\underline{\lambda}(I \setminus X) = 0$ i.e. I is a quasi-interval of X.

Let I be an open interval. If $\lambda(I \setminus G) = 0$ then I is a quasi-interval of G hence I is a quasi-interval of X thus, by the definition of G, $I \subset G$, q.e.d.

LEMMA (3.2). *Let $G \subset [0, 1]$ be open and suppose that for every open interval I either $I \subset G$ or $\lambda(I \setminus G) > 0$ holds. Then*

$$C_{\mathscr{D}}(G) = \bigcup_{n=1}^{\infty} \bar{I}_n \cup \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N+1}^{\infty} I_n^k \tag{3.3}$$

where $I_n = (a_n, b_n)$ denotes the components of G and

$$I_n^k \overset{\text{def}}{=} \left\{ x : \left| x - \frac{a_n + b_n}{2} \right| < k \frac{b_n - a_n}{2} \right\}$$

that is I_n^k is obtained from I_n by enlarging it k times from its middle point.

PROOF. Let B denote the right-hand side of (3.3). Suppose $x_0 \in C_{\mathscr{D}}(G)$. Then there exist $c > 0$ and $(a_i, b_i) \subset G$ such that $a_i \to x_0$, $b_i \to x_0$ and

$$b_i - a_i > c \operatorname{dist}((a_i, b_i), x_0). \tag{3.4}$$

Let I_{n_i} denote the component of G containing (a_i, b_i). If $I_{n_i} = I_{n_0}$ holds for infinitely many i then $\operatorname{dist}(I_{n_0}, x_0) = 0$, $x_0 \in \bar{I}_{n_0}$ and thus $x_0 \in B$. If there are infinitely many different intervals in the sequence $\{I_{n_i}\}$ then by (3.4) we have $|I_{n_i}| > c$ dist (I_{n_i}, x_0) thus $x_0 \in I_{n_i}^{1+\frac{2}{c}}$. Consequently, for $k > 1 + \dfrac{2}{c}$, $x_0 \in I_n^k$ holds for infinitely many n that is $x_0 \in \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N+1}^{\infty} I_n^k$, $x_0 \in B$.

Now suppose $x_0 \in B$. If $x_0 \in \bar{I}_n$ for some n then obviously $x_0 \in C_{\mathscr{D}}(G)$. If there exists a k for which $x_0 \in \bigcap_{N=1}^{\infty} \bigcup_{n=N+1}^{\infty} I_n^k$ then $x_0 \in I_n^k$ holds for infinitely many n, let $x_0 \in I_{n_i}^k = (a_{n_i}, b_{n_i})^k$ $(i = 1, 2, \ldots)$. Hence

$$\left| x_0 - \frac{a_{n_i} + b_{n_i}}{2} \right| < k \cdot \frac{b_{n_i} - a_{n_i}}{2}$$

thus $b_{n_i}-a_{n_i}>\frac{2}{k}\operatorname{dist}(I_{n_i}, x_0)$. Since $|I_n|\to 0$ as $n\to\infty$ hence $a_{n_i}\to x_0$ and $b_{n_i}\to x_0$. That is the definition of the s.1. point is fulfilled for x_0 with $c=\frac{2}{k}$; $x_0\in C_{\mathcal{D}}(G)$, q.e.d.

PROPOSITION (3.5). *For every $X\subset[0, 1]$, $C_{\mathcal{D}}(X)\subset C_{\mathcal{A}}(X)$ holds where $C_{\mathcal{A}}(X)$ denotes the set of all points x at which the upper density of X is positive. (See* [1].*)*

PROOF. Suppose $x_0\in C_{\mathcal{D}}(X)$ then there exist $c>0$ and $I_n\to x_0$ such that $\underline{\lambda}(I_n\setminus X)=0$ and $|I_n|>c\cdot\operatorname{dist}(I_n, x_0)$ holds for every n. Thus $\delta_n=|I_n|+\operatorname{dist}(I_n, x_0)\to 0$ as $n\to\infty$ and $I_n\subset[x_0-\delta_n, x_0+\delta_n]$. Thus (if $\bar{d}(X, x_0)$ denotes the upper density of X at x_0)

$$\bar{d}(X, x_0) \geqq \overline{\lim_{n\to\infty}}\, \frac{\lambda(X\cap[x_0-\delta_n, x_0+\delta_n])}{2\delta_n} \geqq \overline{\lim_{n\to\infty}}\, \frac{|I_n|}{2|I_n|+2\operatorname{dist}(I_n, x_0)} \geqq$$

$$\geqq \frac{1}{2+\frac{2}{c}} > 0$$

hence $x_0\in C_{\mathcal{A}}(X)$.

PROPOSITION (3.6). *For every measurable set $X\subset[0, 1]$, $\lambda(C_{\mathcal{D}}(X)\setminus X)=0$ holds.*

PROOF. According to Lebesgue's density theorem $\lambda(C_{\mathcal{A}}(X)\setminus X)=0$. Thus Proposition (3.5) implies (3.6).

THEOREM (3.7). *$C_{\mathcal{D}}(X)$ is Borel-measurable ($G_{\delta\sigma}$) for every $X\subset[0, 1]$.*

PROOF. Let G be the open set defined in Lemma (3.1). Then by Lemma (3.1) $C_{\mathcal{D}}(X)=C_{\mathcal{D}}(G)$ and (3.3) shows that $C_{\mathcal{D}}(G)$ is a $G_{\delta\sigma}$-set.

THEOREM (3.8). *For every $X\subset[0, 1]$, $T_{\mathcal{D}}(X)=X\cup C_{\mathcal{D}}(X)$.*

PROOF. Since $T_{\mathcal{D}}(X)\supset X\cup C_{\mathcal{D}}(X)$ was shown in (1.1) we have to prove only $T_{\mathcal{D}}(X)\subset X\cup C_{\mathcal{D}}(X)$. Suppose $x_0\notin X\cup C_{\mathcal{D}}(X)$. In order to prove $x_0\notin T_{\mathcal{D}}(X)$ we have to find a function $f(x)\in\mathcal{D}$ for which $f(x)=0$ if $x\in X$ and $f(x_0)\neq 0$. Let $X'\supset X$ be measurable and such that $\underline{\lambda}(X'\setminus X)=0$ and $x_0\notin X'$. Then every quasi-interval of X' is a quasi-interval of X hence $C_{\mathcal{D}}(X')=C_{\mathcal{D}}(X)$. By Proposition (3.6) $C_{\mathcal{D}}(X')\setminus X'$ is of measure zero thus every quasi-interval of $X'\cup C_{\mathcal{D}}(X')$ is a quasi-interval of X, too. Theorem (3.7) asserts that $C_{\mathcal{D}}(X')$ is measurable hence $X'\cup C_{\mathcal{D}}(X')$ is measurable as well. Thus there exists $Y\supset X'\cup C_{\mathcal{D}}(X')$ such that Y is G_δ, $x_0\notin Y$ and $\lambda(Y\setminus(X'\cup C_{\mathcal{D}}(X')))=0$. Then every quasi-interval of Y is also a quasi-interval fo X that is $C_{\mathcal{D}}(Y)\subset C_{\mathcal{D}}(X')\subset Y$. Apply Theorem (2.1) for the sets Y and $\{x_0\}$, then we get a function $f(x)\in\mathcal{D}$ for which $f(x)=0$ if $x\in Y$ and $f(x_0)=1$. Since $X\subset Y$ we have $x_0\notin T_{\mathcal{D}}(X)$, q.e.d.

REMARK. Making use of the notions of [1], Theorem (2.2) can be formulated as follows: the class $\mathcal{D}$ has property S_2. We mention that by the aid of Lemma (3.2) one can construct two disjoint open sets G_1 and G_2 in such a way that

$$T_{\mathcal{D}}(G_1)\cap T_{\mathcal{D}}(G_2) = \emptyset$$

but G_1 and G_2 are not separable by a derivative. That is $\mathcal{D}$ does not possess property S_3.

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ON A PROBLEM OF R. DEVORE

By

J. SZABADOS (Budapest)

In [1], p. 581, R. DEVORE raised the following problem: "Do there exist polynomials $p_0, p_1, p_2, \ldots, p_n$ of degree $\leqq n$ such that the linear operators

$$L_n(f, x) = \sum_{k=0}^{n} f(k/n) p_k(x) \tag{1}$$

provide the Jackson estimates

$$\|f - L_n(f)\| = O(\omega(f, n^{-1})), \tag{2}$$

where $\|.\|$ is the sup norm on [0, 1]?"

In what follows, we give an answer for this question in the affirmative. For technical reasons, instead of [0, 1] we will consider the interval $[-1/4, 1/4]$, and the equidistant nodes

$$x_k = x_{k,n} = \frac{k}{4n} \qquad (k = -n, -n+1, \ldots, n).$$

Of course, this does not mean any restriction, because by proper linear transformation any finite interval can be mapped into another one.

Our method of proving (1)—(2) consists of making use of the intermediate approximating rational functions

$$R_{n,0}(f, x) = \frac{\sum_{k=-n}^{n} f(x_k)(x-x_k)^{-4}}{\sum_{k=-n}^{n} (x-x_k)^{-4}}, \tag{3}$$

and the using a standard polynomial aproximation to this rational function. The use of this type of rational functions as approximating means has been recognized by J. BALÁZS [2]. We shall need the following extension of this rational function to the interval $[-1/2, 1/2]$:

$$\bar{R}_{n,0}(f, x) = \begin{cases} R_{n,0}(f, -1/4) = f(-1/4) & \text{if} \quad -1/2 \leqq x \leqq -1/4 \\ R_{n,0}(f, x) & \text{if} \quad -1/4 \leqq x \leqq 1/4 \\ R_{n,0}(f, 1/4) = f(1/4) & \text{if} \quad 1/4 \leqq x \leqq 1/2. \end{cases} \tag{4}$$

Further let (cf. BOJANIC—DEVORE [3])

$$P_n(x) = c_n \left(\frac{\cos(2n \arccos x)}{x^2 - \sin^2 \frac{\pi}{4n}} \right)^2$$

where c_n is chosen such that $\int_{-1}^{1} P_n(x)\,dx = 1$. Evidently, $P_n(x)$ is a polynomial of degree $4n-4$. Using this polynomial as a kernel, BOJANIC and DEVORE defined the linear operator

$$K_n(g, x) = \int_{-1/2}^{1/2} [g(t) - g(0)] P_n(t-x)\,dt + g(0) \tag{5}$$

for all $g(x) \in C[-1/2, 1/2]$. Then (see [3])

$$\|g - K_n(g)\| \leqq 4\omega(g, n^{-1}) \qquad (n = 3, 4, \ldots). \tag{6}$$

Here and in what follows all norms mean supremum norm on the interval $[-1/4, 1/4]$. We are going to prove the following

THEOREM. *If* $f(x) \in C[-1/4, 1/4]$ *then*

$$\|f(x) - K_n(\bar{R}_{n,0}(f), x)\| \leqq 3605 \cdot \omega(f, n^{-1}) \qquad (n = 3, 4, \ldots). \tag{7}$$

In order to see that this is, indeed, a solution for the DEVORE's problem, let us write (5) in the form

$$K_n(\bar{R}_{n,0}(f), x) = \bar{R}_{n,0}(f, 0) + \int_{-1/2}^{1/2} [\bar{R}_{n,0}(f, t) - \bar{R}_{n,0}(f, 0)] P_n(t-x)\,dt =$$

$$= f(0)\left[1 - \int_{-1/2}^{1/2} P_n(t-x)\,dt\right] + f(-1/4) \int_{-1/2}^{-1/4} P_n(t-x)\,dt +$$

$$+ \sum_{k=-n}^{n} f(x_k) \int_{-1/4}^{1/4} \frac{P_n(t-x)\,dt}{(t-x_k)^4 \sum_{j=-n}^{n} (t-x_j)^{-4}} + f(1/4) \int_{1/4}^{1/2} P_n(t-x)\,dt,$$

where we made use of (3) and (4).

For the proof of the theorem we need some auxiliary statements.

LEMMA 1. *Let* $f^{(r)}(x) \in C[-1/4, 1/4]$ $(r \geqq 0)$ *and* $s = 2[r/2] + 4$. *Then for the rational functions*

$$R_{n,r}(f, x) = \frac{\sum_{k=-n}^{n} \sum_{j=0}^{r} \frac{f^{(j)}(x_k)}{j!} (x-x_k)^{j-s}}{\sum_{k=-n}^{n} (x-x_k)^{-s}} \tag{8}$$

of degree $\leqq 2sn + r$ *we have*

$$\|f(x) - R_{n,r}(f, x)\| \leqq 5n^{-r}\omega(f^{(r)}, n^{-1}) \qquad (n = 1, 2, \ldots). \tag{9}$$

We shall use this lemma only with $r = 0$; nevertheless, this general result is interesting in itself, and can be a starting point of further investigations (see the problems proposed at the end of this paper). Lemma 1 is a generalization of the result of BALÁZS [2].

PROOF OF LEMMA 1. Let i be an index such that

$$|x-x_i| = \min_{|k|\le n} |x-x_k| \leqq \frac{1}{8n}. \tag{10}$$

Then evidently

$$\frac{|i-k|}{8n} \leqq |x-x_k| \leqq \frac{|i-k|}{2n} \qquad (i \neq k). \tag{11}$$

Thus we get by (8), $s-r\geqq 3$ and the remainder of the Taylor series

$$|f(x)-R_{n,r}(f,x)| = \frac{\left|\sum_{k=-n}^{n}(x-x_k)^{-s}\left[f(x)-\sum_{j=0}^{r}\frac{f^{(j)}(x_k)}{j!}(x-x_k)^j\right]\right|}{\sum_{k=-n}^{n}(x-x_k)^{-s}} \leqq$$

$$\leqq (x-x_i)^s\left|\sum_{k=-n}^{n}[f^{(r)}(\xi_k)-f^{(r)}(x_k)](x-x_k)^{r-s}\right| \leqq$$

$$\leqq (x-x_i)^s \sum_{k=-n}^{n}\omega(f^{(r)}, |x-x_k|)|x-x_k|^{r-s} \leqq$$

$$\leqq |x-x_i|^r\omega(f^{(r)}, |x-x_i|)+(8n)^{-s}\sum_{\substack{k=-n\\k\neq i}}^{n}\omega\left(f^{(r)}, \frac{|i-k|}{2n}\right)\left(\frac{8n}{|i-k|}\right)^{s-r} \leqq$$

$$\leqq (8n)^{-r}\omega(f^{(r)}, n^{-1})\left[1+\sum_{\substack{k=-n\\k\neq i}}^{n}|i-k|^{-2}\right] \leqq 5n^{-r}\omega(f^{(r)}, n^{-1})$$

$$(\xi_k\in(x_k, x),\ k=-n, \dots, n).$$

Q.E.D.

REMARK. Notice that $R_{n,r}(f, k/n)=f(k/n)$ $(k=-n, \dots, n)$, i.e. we constructed *interpolating* rational functions realizing the Jackson order of approximation.

LEMMA 2. *If* $f(x)\in C[-1/4, 1/4]$ *then*

$$\|R'_{n,0}(f,x)\| \leqq 900\cdot n\omega(f, n^{-1}) \qquad (n=1, 2, \dots).$$

PROOF. We have by (3), (10) and (11)

$$|R'_{n,0}(f,x)| =$$

$$= \frac{\left| -4 \sum_{k=-n}^{n} f(x_k)(x-x_k)^{-5} \sum_{k=-n}^{n} (x-x_k)^{-4} + 4 \sum_{k=-n}^{n} f(x_k)(x-x_k)^{-4} \sum_{k=-n}^{n} (x-x_k)^{-5} \right|}{\left(\sum_{k=-n}^{n} (x-x_k)^{-4} \right)^2} =$$

$$= 2 \frac{\left| \sum_{k=-n}^{n} (x-x_k)^{-5} \sum_{j=-n}^{n} [f(x_k)-f(x_j)](x_k-x_j)(x-x_j)^{-5} \right|}{\left(\sum_{j=-n}^{n} (x-x_j)^{-4} \right)^2} \leqq$$

$$\leqq 2(x-x_i)^8 \sum_{k=-n}^{n} |x-x_k|^{-5} \frac{1}{4n} \omega(f, n^{-1}) \sum_{\substack{j=-n \\ j\neq k}}^{n} \frac{|j-k|^2}{|x-x_j|^5} \leqq$$

$$\leqq \frac{(8n)^{-3}\omega(f,n^{-1})}{2n} \cdot \left\{ \sum_{\substack{j=-n \\ j\neq i}}^{n} \left(\frac{8n}{|j-i|} \right)^5 |j-i|^2 + \sum_{\substack{k=-n \\ k\neq i}}^{n} \left(\frac{8n}{|i-k|} \right)^5 \left[|i-k|^2 + \sum_{\substack{j=-n \\ j\neq i}}^{n} \frac{|j-k|^2}{|j-i|^5} \right] \right\} \leqq$$

$$\leqq 32n\omega(f,n^{-1}) \left\{ \sum_{\substack{j=-n \\ j\neq i}}^{n} |j-i|^{-3} + \sum_{\substack{k=-n \\ k\neq i}}^{n} |i-k|^{-3} \left[1 + 4 \sum_{\substack{j=-n \\ j\neq i}}^{n} |j-i|^{-3} \right] \right\} \leqq 900n\omega(f,n^{-1}),$$

Q.E.D.

PROOF OF THE THEOREM. Lemma 2 and (4) imply that

$$(12 \qquad \omega(\bar{R}_{n,0}(f), n^{-1}) = \omega(R_{n,0}(f), n^{-1}) \leqq n^{-1} \|R'_{n,0}(f)\| \leqq 900 \cdot \omega(f, n^{-1}).$$

Apply now (5)—(6) for $g(x)=\bar{R}_{n,0}(f,x)$, then by (4) and (12) we obtain

$$\|R_{n,0}(f)-K_n(\bar{R}_{n,0}(f))\| = \|\bar{R}_{n,0}(f)-K_n(\bar{R}_{n,0}(f))\| \leqq 3600\omega(f,n^{-1}),$$

which together with (9) (in case $r=0$) yields (7). Q.E.D.

REMARK. Instead of (5), we could use any constructive method of proof of the Jackson theorem.

Having this theorem, it is easy to show, by standard arguments, the following more general

COROLLARY. *If $f^{(r)}(x) \in C[-1/4, 1/4]$ then there exist polynomials $p_{j,k,r}(x)$ of degree $\leqq n$ such that*

$$(13) \qquad \left\| f(x) - \sum_{k=-n}^{n} \sum_{j=0}^{r} f^{(j)}(x_k) p_{j,k,r}(x) \right\| = O(n^{-r}) \omega(f^{(r)}, n^{-1}).$$

For $r=0$ this is already proved above. Assume that it holds for $r-1$, and prove that it holds for r, too. By the theorem we have

$$\left\| f^{(r)}(x) - \sum_{k=-n}^{n} f^{(r)}(x_k) p_{0,k,0}(x) \right\| = O(\omega(f^{(r)}, n^{-1})).$$

Let $q_k(x)$ be a polynomial such that $q_k^{(r)}(x)=p_{0,k,0}(x)$. Then for the function

$$\varphi(x)=f(x)-\sum_{k=-n}^{n} f^{(r)}(x_k)q_k(x)$$

we have $\|\varphi^{(r)}(x)\|=O\left(\omega(f^{(r)}, n^{-1})\right)$. Thus, applying (13) with $r-1$ and $\varphi(x)$ instead of $f(x)$ we get

$$\left\| f(x)-\sum_{k=-n}^{n}\left\{\sum_{j=0}^{r-1} f^{(j)}(x_k)p_{j,k,r-1}(x)+f^{(r)}(x_k)\left[q_k(x)-\sum_{l=-n}^{n}\sum_{j=0}^{r-1} q_k^{(j)}(x_l)p_{j,l,r-1}(x)\right]\right\}\right\|=$$

$$=\left\|\varphi(x)-\sum_{k=-n}^{n}\sum_{j=0}^{r-1}\varphi^{(j)}(x_k)p_{j,k,r-1}(x)\right\|=$$

$$=O(n^{-r+1})\omega(\varphi^{(r-1)}, n^{-1})=O(n^{-r})\|\varphi^{(r)}\|=O(n^{-r})\omega(f^{(r)}, n^{-1}),$$

which proves the corollary, being the approximating polynomial thus constructed of the structure like in (13).

PROBLEMS. 1. Is it possible to improve the Jackson order $n^{-r}\omega(f^{(r)}, n^{-1})$ to the Timan—Telyakowski pointwise estimate $\left(\sqrt{1-x^2}/n\right)^r\omega\left(f^{(r)}, \sqrt{1-x^2}/n\right)$ (on the interval $[-1, 1]$, say) by operators like in (13)?

2. Do there exist operators with the property (13), for which $L_n^{(j)}(f, k/n)=$ $=f^{(j)}(k/n)$ $(j=0, \dots, r;\ k=-n, \dots, n)$? Even the possibility of uniform convergence of this kind of operators seems to be doubtful.

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TOPOLOGICAL COORDINATIZATIONS AND DUALITIES OF BROUWER LATTICES

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Introduction. If X is any topological space, the lattice $\mathcal{O}(X)$ of open sets of X is a complete, Brouwer lattice in which each element <1 is a meet of prime elements. Conversely, if L has these properties then there is a T_0-space X so that $L\simeq\mathcal{O}(X)$. We characterize all the T_0-spaces that arise in this manner in terms of subsets of the set of primes of L. Some special results are obtained, concerning lattices which can be realized as lattices of open sets of topological spaces with some restrictions, such as separation axioms.

Notation and terminology. If A and B are subsets of a set X, $(A\subset B)\, A\subseteqq B$ denotes (proper) containment of A in B. $A\setminus B$ is the complement of B in A.

Our lattice theoretic terminology is standard for the most part, except where specifically noted that it is not. The topological terms used in the sequel are quite familiar and standard. One exception: compactness of a set refers to the covering property only and does not imply the Hausdorff separation axiom.

The author wishes to thank Professor Jurgen Schmidt for his kind suggestions. Also it is absolutely essential to acknowledge the many contributions to this theory, in this or other language; in a modest bibliography it is impossible to single out every individual. Some of the works we do quote contain rather extensive references, notably [6] and [9]. The reader is heartily recommended to consult these two sources.

Let us say that a lattice L is *semiprime* if it is complete and each, $x\in L$, $x<1$ is the (possibly infinite) meet of prime elements of L. ($x\in L$ is *prime* if $x\geqq a\wedge b$ implies $x\geqq a$ or $x\geqq b$.) A lattice is *Brouwer* if for each $x, y\in L$ the set $\{a\in L|a\wedge x\leqq y\}$ has a unique largest element. It is well known that if L satisfies the distributive law:

$$a\wedge(\vee_\lambda x_\lambda) = \vee_\lambda(a\wedge x_\lambda),\ a, x_\lambda\in L,\ (\lambda\in\Lambda),$$

then L is Brouwer, and if L is complete the converse is true (see [1]).

Throughout, $\mathcal{O}(X)$ denotes the lattice of open subsets of a topological space X. It is a complete lattice when we think of arbitrary meets as the interior of an intersection of open sets. It is then clear that $\mathcal{O}(X)$ is a Brouwer lattice.

LEMMA. *For any topological space X the lattice $\mathcal{O}(X)$ is semiprime.*

PROOF. For each $x\in X$ let U_x be the largest open set missing x (U_x is the complement of the closure of x). Then U_x is prime in $\mathcal{O}(X)$; further, if V is an open set, $V=\bigcap\{U_x|x\notin V\}$.

If L is a lattice, we call the topological space X a *coordinatization* of L if $L\simeq\mathcal{O}(X)$. We have established that lattices admitting a coordinatization are semiprime Brouwer lattices.

Conversely, let us suppose L is a semiprime Brouwer lattice. Let $\mathfrak{P}(L)$ be the set of prime elements of L. For each $x \in L$ let $P(x)=\{p \in \mathfrak{P}(L) | p \not\geqq x\}$. Then the sets $P(x)$, for $x \in L$, form the open sets for a topology on $\mathfrak{P}(L)$, and the mapping $x \to P(x)$ is a lattice isomorphism. (It is one to one since L is semiprime.) Thus $\mathfrak{P}(L)$ with this "hull kernel" topology coordinatizes L.

Moreover, $\mathfrak{P}(L)$ is a T_0-space; if $p \neq q$ are primes of L, we may assume without loss of generality that $p \not\geqq q$. Then $q \in P(p)$ but $p \notin P(p)$.

Following HOFMANN and KEIMEL [6], call a topological space X *spectral* if X is T_0 and in $\mathcal{O}(X)$ every prime element is of the form U_x for some $x \in X$; (recall $U_x = X \setminus \overline{\{x\}}$). With a little more discussion we have the following result.

THEOREM 1. *If L is a semiprime Brouwer lattice then it has a spectral coordinatization. This spectral coordinatization is unique up to homeomorphism.*

PROOF. We prove that $\mathfrak{P}(L)$ is spectral. This follows from the fact that $P(x)$ is a prime open set if and only if $x \in \mathfrak{P}(L)$, in which case $P(x)=\mathfrak{P}(L) \setminus \{p \in \mathfrak{P}(L) | p \geqq x\}$, and $\{p \in \mathfrak{P}(L) | p \geqq x\}$ is the closure of x in $\mathfrak{P}(L)$.

The uniqueness follows from theorem 4.17 in [6].

In [6] HOFMANN and KEIMEL prove (theorem 4.17) that the pair of functors $(\mathcal{O}, \mathfrak{P})$ establishes a duality between the category of spectral spaces and continuous mappings, and the category of semiprime Brouwer lattices and *join-complete* lattice homomorphisms; i.e. lattice homomorphism preserving 0, 1 and all joins.

We proceed now to characterize all the T_0-coordinatizations of a fixed semiprime Brouwer lattice L. Recall that $t \in L$ is *meet irreducible* if $t = \bigwedge_\lambda x_\lambda$ implies that $t = x_\mu$ for some index μ. $\mathfrak{J}(L)$ will denote the set of meet irreducibles of L; clearly $\mathfrak{J}(L) \subseteqq \mathfrak{P}(L)$.

Suppose $L \simeq \mathcal{O}(X)$, where X is a T_0-space. Let $\alpha: \mathcal{O}(X) \to L$ be a lattice isomorphism, and $\mathfrak{B} = \{\alpha(U_x) | x \in X\}$. One easily verifies that $\mathfrak{B} \subseteqq \mathfrak{P}(L)$ and $\mathfrak{J}(L) \subseteqq \mathfrak{B}$; (the latter containment comes about since each meet irreducible open set is of the form U_x for some $x \in X$). Topologize $\mathfrak{B}$ with the subspace topology of $\mathfrak{P}(L)$, and let $B(x) = P(x) \cap \mathfrak{B}$, for all $x \in L$. Define a mapping $\Phi: X \to \mathfrak{B}$ by $\Phi(X) = \alpha(U_x)$; since X is T_0 Φ is a bijection. If $A \subseteqq X$ and A is open then $\Phi(A) = \{\alpha(U_x) | x \in A\} = \{\alpha(U_x) | A \subsetneqq U_x\} = B(\alpha(A))$, so α is open. Conversely, if $\Phi(C) = B(\alpha(V))$ for some open set V then one can easily show that $C = V$, i.e. α is continuous. We conclude that X is homeomorphic to $\mathfrak{B}$.

We summarize

THEOREM 2. *The T_0-coordinatizations of L are obtained (up to homeomorphisms) from subsets $\mathfrak{B}$ of $\mathfrak{P}(L)$ that contain $\mathfrak{J}(L)$.*

Warning: This does not say that each such subset produces a coordinatization. If L is the usual closed unit interval [0, 1] with the natural ordering of real numbers, then $\mathfrak{J}(L) = \emptyset$. One can also show that L has no minimal coordinatizations, in the sense that if $\mathfrak{B} \subseteqq \mathfrak{P}(L)$ coordinatizes L so does $\mathfrak{B} \setminus \{p\}$ for each $p \in \mathfrak{B}$.

Of course, when $\mathfrak{J}(L)$ does coordinatize L it is the smallest. This coordinatization is rather special; let us say that L is *representable* if each $x < 1$ is the meet of meet irreducible elements of L. CONRAD proved in [3] that a representable lattice is complete.

If X is a topological space we say X is t_1 if for each $x \in X$, $\overline{\{x\}} \setminus \{x\}$ is closed. It is easy to verify that X is a t_1-space if and only if $U_x \cup \{x\}$ is open for each $x \in X$.

BRUNS [2] first dealt with this separation axiom, and called it $T_{1/2}$; for typographical reasons we shall keep the name t_1. It is easy to show that $T_1 \to t_1 \to T_0$, and no reverse implication hold (see [2]).

THEOREM 3. *If L is a representable Brouwer lattice then $\mathfrak{J}(L)$ (with the subspace topology of $\mathfrak{P}(L)$) is a t_1-coordinatization of L. Up to homeomorphism it is the unique t_1-coordinatization.*

PROOF. From previous similar considerations it is clear that $\mathfrak{J}(L)$ coordinatizes L. Now select a point $t \in \mathfrak{J}(L)$; since t is meet irreducible it has a cover t^*; i.e. the meet of all the elements of L that properly exceed t. $\|\overline{\{t\}}\| = \{s \in \mathfrak{J}(L) | s \geqq t\}$ so $\overline{\{t\}} \setminus \{t\} = \{s \in \mathfrak{J}(L) | s > t\} = \overline{\{t^*\}}$; it follows that $\mathfrak{J}(L)$ is t_1.

If X is any t_1-coordinatization of L and $\alpha \colon \mathcal{O}(X) \to L$ is a lattice isomorphism, we claim that $\mathfrak{J}(L) = \{\alpha(U_x) | x \in X\}$. Since X is t_1 every U_x is a meet irreducible open set; $U_x \cup \{x\}$ is its cover, thus establishing our claim. Now, using the proof of theorem 2 we conclude that X is homeomorphic to $\mathfrak{J}(L)$.

REMARK. This is perhaps a good place to point out that these first three theorems were in essence proved by BRUNS in [2] for algebraic, Brouwer lattices. In his language the lattices were realized as lattices of ideals of distributive semilattices (i.e. semilattices with a distributive lattice of ideals). The assumptions of algebraic character are obviously not needed, as has been demonstrated above. We shall specialize to algebraic lattices later.

Once again we should comment on dualities: let $\mathcal{T}_1$ be the category of t_1-spaces with certain morphisms which we shall describe soon, and $\mathcal{R}$ be the category of representable Brouwer lattices with morphisms we now proceed to describe.

Let L and M be representable, Brouwer lattices and $\alpha \colon L \to M$ be a join-complete lattice homomorphism. For each $a \in M$ let $\alpha^*(a) = \bigvee \{x \in L | \alpha x \leqq a\}$. This defines an order preserving map $\alpha^* \colon M \to L$ satisfying: $\alpha x \leqq a$ if and only if $x \leqq \alpha^* a$. What we require of α is that α^* take $\mathfrak{J}(M)$ into $\mathfrak{J}(L)$, and we denote its restriction to $\mathfrak{J}(M)$ by $\mathfrak{J}(\alpha)$. It is easy to verify that $\mathfrak{J}(\alpha)$ is continuous. So $\mathcal{R}$ is the category with the prescribed objects and *these* join-complete lattice homomorphisms; let us call them *strongly reversible.*

For $\mathcal{T}_1$ take all t_1-spaces with continuous mappings $\Phi \colon X \to Y$ having the property that if $\mathcal{O}(\Phi) \colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is defined by $\mathcal{O}(\Phi)(V) = \Phi^{-1}(V)$ then $\mathcal{O}(\Phi)$ is a strongly reversible lattice homomorphism; we call these continuous maps *strongly reversible* as well.

With these rather restricted classes of morphisms we have:

THEOREM 4. *The categories $\mathcal{R}$ and $\mathcal{T}_1$ are dual under the fuctor pair $(\mathcal{O}, \mathfrak{J})$.*

PROOF. All that is required is to verify that $\mathcal{O}$ and $\mathfrak{J}$ do take the respective classes of morphisms into each other. If $\alpha \colon L \to M$ is a strongly reversible morphism of $\mathcal{R}$ we have induced maps $\mathfrak{J}(\alpha) \colon \mathfrak{J}(M) \to \mathfrak{J}(L)$ and $\mathcal{O}(\mathfrak{J}(\alpha)) \colon \mathcal{O}(\mathfrak{J}(L)) \to \mathcal{O}(\mathfrak{J}(M))$. The reader may easily verify the commutativity of the diagram below:

$$\begin{array}{ccc} L & \xrightarrow{\ I\ } & \mathcal{O}(\mathfrak{J}(L)) \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{O}(\mathfrak{J}(\alpha))} \\ M & \xrightarrow{\ I\ } & \mathcal{O}(\mathfrak{J}(M)), \end{array}$$

where $I(x)=P(x)\cap\mathfrak{I}(L)$, for $x\in L$. Now since α is strongly reversible so is $\mathcal{O}(\mathfrak{I}(\alpha))$, and hence $\mathfrak{I}(\alpha)$ is a strongly reversible continuous map.

By checking a similar argument the reader may verify that if Φ is a strongly reversible continuous mapping then $\mathcal{O}(\Phi)$ is strongly reversible.

We collect a few easy consequences of the above results:

1. If a (representable) lattice has a T_1-coordinatization it is unique, and coincides with the $\mathfrak{I}(L)$-coordinatization. L has such a coordinatization if and only if it is Brouwer and each meet irreducible element is maximal.

2. If L has a spectral t_1-coordinatization then $\mathfrak{I}(L)=\mathfrak{P}(L)$ and so it is the unique T_0-coordinatization of L.

3. A Hausdorff space is necessarily spectral; so if L has a T_2-coordinatization it is unique and $\mathfrak{I}(L)=\mathfrak{P}(L)$; further, every prime is maximal.

Now let us turn to algebraic lattices: suppose L is a complete lattice; $a\in L$ is *compact* if $a\leqq\bigvee\{x_i|i\in I\}$ implies that $a\leqq x_{i_1}\vee\ldots\vee x_{i_n}$, for some $i_1,\ldots,i_n\in I$. L is *algebraic* if each $x\in L$ is the join of compact elements. For various characterizations of algebraic lattices we refer the reader to [1] or [5], or to the summary in [7].

It is well known that algebraic lattices are representable (see [7]), so our previous results apply to algebraic Brouwer lattices. If L is an algebraic Brouwer lattice, and we coordinatize by some subspace $\mathfrak{B}$ of $\mathfrak{P}(L)$ containing $\mathfrak{I}(L)$, then $\mathfrak{B}$ has a base for the topology consisting of compact (open) sets; namely, the sets $B(x)$ where $x\in L$ is compact.

The next lemma is well known; a proof may be found in [7].

LEMMA. *If L is an algebraic lattice and c, $x\in L$, with $c\not\leqq x$ and c compact, then there is a meet irreducible $p\geqq x$ which is maximal with respect to x not exceeding c.*

COROLLARY. *Let L be an algebraic, Brouwer lattice; then $\mathfrak{I}(L)$ is dense in $\mathfrak{P}(L)$.*

PROOF. Pick $p_0\in\mathfrak{P}(L)$ and let $P(c)$ be a compact neighbourhood of p_0. By the lemma there is a meet irreducible $t\geqq p_0$ such that $t\in P(c)$. This suffices to show that $\mathfrak{I}(L)$ is dense in $\mathfrak{P}(L)$.

COMMENT. What is essential in the above corollary is that $\mathfrak{I}(L)$ is cofinal in $\mathfrak{P}(L)$ in a very strong sense. Thus in the case of an algebraic Brouwer lattice L the T_0-coordinatizations of L form a system of dense subspaces of $\mathfrak{P}(L)$.

We shall conclude the paper with lattices having rather special coordinatizations.

A. *L is algebraic, and has a T_2-coordinatization.* Any such coordinatization is essentially the $\mathfrak{I}(L)$-coordinatization. Also, $\mathfrak{I}(L)$ has a base of compact, open sets, and each of these is closed since $\mathfrak{I}(L)$ is T_2. Hence, for each compact element $c\in L$ there is an $x\in L$ such that $I(c)$ and $I(x)$ are complementary sets; that is, $c\wedge x=0$ and $c\vee x=1$. Thus each compact element is (uniquely) complemented; an algebraic Brouwer lattice with this property is said to have the *compact splitting property (CSP)*. In [7], theorem 2.4, the author showed that L has the CSP if and only if every prime of L is maximal and the meet of two compact elements is compact. We have therefore proved part of:

THEOREM 5. *If L is an algebraic Brouwer lattice having a Hausdorff coordinatization, then it is the only T_0-coordinatization, and it is spectral. L has a T_2-coordinatization if and only if L has the CSP.*

PROOF. We need only prove the last sentence, and there only the sufficiency. So if L has the CSP and p, q are distinct meet irreducibles, we may assume without loss of generality that $c \leqq p$ yet $c \nleqq q$, for some compact element c of L. Let d be the complement of c; by theorem 2.4 in [7] p (and q) is a minimal prime, so $d \nleqq p$. But this says that $p \in I(d)$, $q \in I(c)$ and $I(c) \cap I(d) = \emptyset$; hence $\mathfrak{I}(L)$ is Hausdorff.

REMARK. Requiring compact elements to be complemented does not mean that their complements are themselves compact. If $1 \in L$ is compact then the complements of compact elements are compact, and the sublattice of compact elements is a Boolean algebra (all in the presence of CSP).

Notice also that under the duality following theorem 1 (or that of theorem 4, since they agree when L has the CSP), the category $\mathscr{CSP}$ of algebraic Brouwer lattices with the CSP together with all join-complete lattice homomorphisms is dual to the category $\mathscr{T}_2$ of Hausdorff spaces with a base of compact, open sets, together with all continuous mappings. If we then specialize to those lattices in $\mathscr{CSP}$ with 1 compact, the duality reduces to the classical Boolean duality.

One further comment: in [7], theorem 3.2, the author showed that the algebraic Brouwer lattices with CSP are precisely those arising from hyperarchimedean l-groups as the lattices of all l-ideals of such a group.

In any Brouwer lattice L for each $x \in L$ there is a unique element $x' \in L$ such that $x \wedge a = 0$ if and only if $a \leqq x'$. The assignment $x \to x'$ defines an auto-Galois-connection on L, and so the mapping $x \to x''$ is a closure operator; see [1]. The set $\mathscr{P}(L)$ of closed elements is a Boolean algebra in which meets agree with those in L. We shall refer to $\mathscr{P}(L)$ as the *Boolean algebra of polars of L*, and to its elements as *polars*. In general $\mathscr{P}(L)$ is not a sublattice of L, but it is if and only if $x' \vee x'' = 1$ for all $x \in L$; see [1].

B. *L (not necessarily algebraic) has an extremally disconnected coordinatization.* Recall that a space X is *extremally disconnected* if and only if the closure of an open set is open. If $\mathfrak{B} \subseteqq \mathfrak{P}(L)$ coordinatizes L it is easy to verify that the closure of the open set $B(x)$ is $\mathfrak{B} \setminus B(x')$. Thus if $\mathfrak{B}$ is extremally disconnected this closure is $B(x'')$ and so $x' \vee x'' = 1$, i.e. the Boolean algebra of polars is a sublattice of L.

Conversely, suppose $x' \vee x'' = 1$ for all $x \in L$; then for any coordinatization $\mathfrak{B} \subseteqq \mathfrak{P}(L)$, $B(x'')$ is the closure of $B(x)$, i.e. $\mathfrak{B}$ is extremally disconnected. We therefore have:

THEOREM 6. *If the lattice L has an extremally disconnected coordinatization then every T_0-coordinatization is extremally disconnected. L has such a coordinatization if and only if it is a semiprime Brouwer lattice in which $\mathscr{P}(L)$ is a sublattice of L.*

Now let us briefly recall that in the classical Boolean duality the category $\mathscr{B}$ of Boolean algebras and Boolean homomorphisms is dual to the category $\hat{\mathscr{B}}$ of compact, totally disconnected Hausdorff spaces and all continuous mappings. In this duality the complete Boolean algebras correspond to the the compact, extremally disconnected Hausdorff spaces; that is, the so-called *Stone spaces*.

On the other hand, if $\mathscr{S}$ is the full subcategory of $\hat{\mathscr{B}}$ consisting of the Stone spaces, and X is a Stone space, then $L = \mathscr{O}(X)$ is an algebraic Brouwer lattice in which 1 is compact, having the CSP, and such that $x' \vee x'' = 1$, for all $x \in L$; further $\mathfrak{I}(L)$ is homeomorphic to X. Let us call these algebraic Brouwer lattices *generalized fields* *of sets*. Conversely if L is a generalized field of sets, then $\mathfrak{I}(L)$ is in $\mathcal{S}$ and $L \simeq \mathcal{O}(\mathfrak{I}(L))$. We therefore have:

THEOREM 7. *The category $\mathcal{GF}$ of all generalized fields of sets together with all join-complete lattice homomorphisms is dual to the category $\mathcal{S}$ of Stone spaces and continuous mappings.*

COROLLARY. *The category $c\mathcal{B}$ of complete Boolean algebras with all Boolean homomorphisms is equivalent to the category $\mathcal{GF}$ of generalized fields of sets.*

PROOF. Piece Boolean duality to the duality of theorem 7.

(REMARK. It is interesting to note that the functor that assigns to a generalized field of sets L its Boolean algebra of polars $\mathcal{P}(L)$ accomplishes this equivalence of categories. For a given complete Boolean algebra B, the associated generalized field of sets is obtained by taking first the Stone space of B and then the lattice of open sets of that; B is then isomorphic to the Boolean algebra of polars of this lattice.)

Note the following as well: if L is a generalized field of sets, its sublattice C of compact elements is a Boolean algebra, and is complete since $\mathfrak{I}(L)$ is a Stone space. Glivenko's theorem (see [1]) identifies $\mathcal{P}(L)$ as the Dedekind—MacNeille completion of C. For a generalized field of sets then, C coincides with $\mathcal{P}(L)$; i.e. every polar is compact.

Our last result concerns discrete coordinatizations. Obviously, a lattice L has a discrete coordinatization if and only if it is a complete field of sets. If L is an algebraic Brouwer lattice, a sufficient condition is that L be a Boolean algebra. For then it has the CSP and so the $\mathfrak{I}(L)$-coordinatization is certainly T_1. Hence $\mathfrak{I}(L)\setminus\{t\}$ is open for each $t\in\mathfrak{I}(L)$, and since L is Boolean $\{t\}$ is open as well; it follows that $\mathfrak{I}(L)$ is discrete. Thus:

THEOREM 8. *The algebraic Brouwer lattice L has a discrete coordinatization if and only if it is Boolean. In such a case L is necessarily a complete field of sets.*

We note in closing that theorem 8 is partly the reason why we have referred to the lattices of theorem 7 as generalized fields of sets; admittedly though, if L is a generalized field of sets which is also Boolean, then L is finite (and a complete field of sets).

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SUR LES ANNEAUX PRINCIPAUX

Par

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A est un *anneau principal* s'il est commutatif, unitaire et si tous ses idéaux sont des idéaux principaux.

Tout anneau principal, est, de façon unique, produit direct de $k \geqq 0$ anneaux principaux intègres et de $h \geqq 0$ anneaux principaux spéciaux [9]. Le couple (k, h) est appelé *le type de A*.

Au premier paragraphe, nous examinons, pour un anneau principal quelconque, ce qui deviennent quelques propriétés classiques des anneaux principaux intègres: dimensions de Krull, spectre et spectre maximal, atomicité, décomposition des A-modules, représentation de A en sommes sous-directes, recherche des anneaux principaux sous-directement irréductibles, etc. Les résultats sont donnés en fonction du type de l'anneau principal considéré.

Toujours par analogie avec le cas des anneaux intègres, nous étudions, au deuxième paragraphe les modules de torsion sur un anneau principal. Nous donnons un théorème de décomposition de ces modules et nous précisons ce résultat en fonction du type de l'anneau A.

Au dernier paragraphe, nous présentons une généralisation de la notion de module libre sur un anneau intègre: les modules *quasi-libres*. Nous montrons que si M est un module libre sur un anneau principal, A, tous ses sous-modules sont quasi-libres. Nous en déduisons une condition nécessaire et suffisante pour qu'un anneau de Bezout soit principal, et nous montrons que sur un anneau principal, tout module projectif est quasi-libre.

§ 1. Divisibilite dans les anneaux principaux

Les anneaux considérés sont commutatifs unitaires. Si A est un anneau, on désigne par $\mathcal{U}(A)$ le groupe de ses unités, par $\operatorname{div}(A)$ l'ensemble de ses diviseurs de zéro, $\operatorname{rad}(A)$ son radical de Jacobson, $\operatorname{spec}(A)$ son spectre premier et $\max(A)$ son spectre maximal.

On dit que A est présimplifiable [4] si $\operatorname{div}(A) \subseteqq 1-\mathcal{U}(A)$. On appelle *anneau principal* tout anneau, non nécessairement intègre, dont tout idéal est principal.

On dit que A est un *anneau de Bezout* si la somme de deux idéaux principaux est un idéal principal, (ce qui équivaut à dire que tout idéal de type fini est principal). A est un *anneau de valuation* si l'ensemble de ses idéaux principaux est totalement ordonné par inclusion.

1—1. PROPOSITION. a) *A est un anneau de valuation si et seulement si A est un anneau de Bezout local.*

b) *A est un anneau de valuation n'ayant qu'un nombre fini d'idéaux principaux si et seulement si A est un anneau principal spécial* [9].

DÉMONSTRATION. a) $*$ Si A est un anneau de valuation, A est trivialement un anneau de Bezout. Il est local car si $a, b \notin \mathcal{U}(A)$ et si $a-b \in \mathcal{U}(A)$, $aA+bA=A$ entraine $aA=A$ ou $bA=A$ ce qui est absurde.

$*$ Si A est un anneau de Bezout local, soient a et b deux éléments de A; on peut les supposer non nuls. D'une part, $aA+bA=kA$; donc $k=a\alpha+b\beta$. Mais $a=ka'$ et $b=kb'$ donc $k=ka'\alpha+kb'\beta$. Or $k\neq p$ et A local est présimplifiable ([4] 1—1). On a donc $a'\alpha+b'\beta=u\in\mathcal{U}(A)$.

D'autre part, A est local; donc

$$a'\alpha+b'\beta\in\mathcal{U}(A) \Rightarrow a'\in\mathcal{U}(A) \text{ ou } b'\in\mathcal{U}(A)$$

alors $aA\subseteq kA$ ou $bA\subseteq kA$ soit $bA\subseteq aA$ ou $aA\subseteq bA$.

b) La condition est évidemment suffisante. Montrons qu'elle est nécessaire. Soit I un idéal de A; $I=\sum_{x\in A} xA$. Puisque A est un anneau de Bezout et ne possède qu'un nombre fini d'idéaux principaux, I est un idéal principal. Donc A est principal. Puisque A n'a qu'un nombre fini d'idéaux, A est principal spécial.

1—2. LEMME ([4] 4—3). *Tout anneau principal est, de façon unique, produit direct fini d'anneaux principaux présimplifiables.*

Puisque tout anneau principal est de façon unique produit direct fini d'anneaux principaux présimplifiables, on peut parler de la décomposition canonique d'un anneau principal en produit direct d'anneaux principaux présimplifiables, qui sont des anneaux intègres ou des anneaux principaux spéciaux. *Dans ce qui suit, on considère toujours un corps comme un anneau principal spécial et non pas comme un anneau principal intègre.*

Soit A un anneau principal; A est produit direct (de façon unique) de k anneaux principaux intègres et de h anneaux principaux spéciaux. Le couple (k, h) est appellé *le type de l'anneau principal A.*

1—3. PROPOSITION. *Si S-est-une partie multiplicative d'un anneau principal A de type (k, h) $S^{-1}A$ est un anneau principal de type (k', h') avec $k'\leqq k$ et $h'\leqq h+(k-k')$.*

DÉMONSTRATION. Si $A=\prod_{i=1}^{n} A_i$, $S^{-1}A$ peut se mettre sous la forme $\prod_{i=1}^{n} S_i^{-1}A_i$. Si A_i est intègre, $S_i^{-1}A_i$ est intègre ou un corps. Si A_i est principal spécial, il n'a que deux anneaux de fractions: A_i et 0. Notre assertion en découle immédiatement.

1—4. PROPOSITION. *La dimension de Krull d'un anneau principal A est* $\leqq 1$; *A est de dimension* 1 *si et seulement s'il est de type (k, h) avec $k\geqq 1$.*

DÉMONSTRATION. La dimension d'un produit direct d'anneaux est le maximum des dimensions de ses facteurs; tout anneau principal intègre est de dimension 1 [6] et tout anneau principal spécial est de dimension nulle.

1—5. PROPOSITION. *Pour un anneau A, les assertions suivantes sont équivalentes:*

(i) *A est principal de type $(0, h)$.*

(ii) *A est principal artinien.*

(iii) *Tout A-module est somme directe de A-modules indécomposables.*

DÉMONSTRATION. (ii)⇔(iii) d'après ([8] Th. 3); (i) entraine trivialement (ii). (ii)⇔(i): Si $A=\prod_{i=1}^{n} A_i$ est artinien, il en est de même de chaque A_i. Comme A_i est principal présimplifiable, A_i est soit intègre, donc un corps, soit principal spécial.

1–6. Un anneau A est *atomique* [4] si tout élément non nul et non inversible est produit fini d'éléments irréductibles. *Un anneau principal est atomique si et seulement s'il* est de type (1, 0) ou (0, h) ([4] 4—3). C'est le cas, par exemple, de tout anneau euclidien ([4] 4—6).

Si A est un anneau principal atomique, si $x \in A^*$, l'ensemble des idéaux contenant xA est fini et x n'a, à une association près, qu'un nombre fini de diviseurs irréductibles; ceci est classique lorsque A est de type (1, 0). C'est évident si A est de type (0, h) car A n'a alors qu'un nombre fini d'idéaux.

1–7. REMARQES. a) Si A est un anneau principal atomique, si $p \in A^* - \mathcal{U}(A)$, d'après [4], les assertions suivantes sont équivalentes:

pA est premier; pA est maximal; p est irréductible.

b) Si A est de type (1, 0), $\operatorname{spec}(A) = \max(A)$ {0}; si A est de type (0, 1), $\operatorname{spec}(A) = \max(A)$. Par passage ou produit direct, on peut ainsi décrire $\operatorname{spec}(A)$ et $\max(A)$ pour tout anneau principal A.

c) De la même façon, on détermine $\operatorname{rad}(A)$ et $\operatorname{nil}(A)$. Ainsi, si A est de type (0, h), $\operatorname{nil}(A) = \operatorname{rad}(A)$.

1–8. PROPOSITION. *Soit A un anneau atomique non intègre. A est principal si et seulement si les idéaux premiers minimaux non nuls sont les idéaux engendrés par les éléments irréductibles.*

DÉMONSTRATION. Condition nécessaire: si A est principal, d'après 1—7, tout idéal premier non nul est engendré par un élément irréductible. Réciproquement, si p est irréductible non nul, d'après 1—7, pA est maximal; donc pA est aussi minimal parmi les idéaux premiers non nuls.

Condition suffisante: d'après [1] Th. 3, A est un anneau D-atomique puisqu'il est non intègre, d'après ([4] Th. 4—6), il est principal.

1–9. On dit qu'un anneau A est *sous directement irréductible* [5] si l'intersection de ses idéaux non nuls est un idéal non nul.

Un tel anneau est nécessairement indécomposable en produit direct; donc un anneau principal sous directement irréductible est de type (1, 0) ou (0, 1). Réciproquement, un anneau principal de type (1, 0) n'est pas nécessairement sous directement irréductible (c'est le cas de **Z**); mais tout anneau principal de type (0, 1) est sous directement irréductible.

1–10. LEMME. *Pour un idéal I d'un anneau principal, A les assertions suivantes sont équivalentes:*

(i) *I est primaire*

(ii) *A/I est présimplifiable*

(iii) *$\sqrt{I}$ est premier.*

DÉMONSTRATION. (i)⇔(ii) d'après ([4] 4—7). (i)⇔(iii) d'après ([4] 4—4) pour un anneau présimplifiable, et par passage au produit direct sinon.

1—11. PROPOSITION. *Soit I un idéal d'un anneau principal A.*

(i) *Si A/I est sous directement irréductible, I est primaire.*

(ii) *Si I est primaire, pour que A/I soit sous-directement irréductible, il suffit que I soit non premier, ou premier non nul avec A atomique.*

DÉMONSTRATION. (i) Si A/I est sous-directement irréductible, d'après 1—9, A/I est présimplifiable, donc I est primaire d'après 1—10.

(ii) Soit I un idéal primaire non premier de A; A/I est un anneau principal présimplifiable d'après 1—10, et non intègre. A/I est de type (0, 1), donc sous directement irréductible d'après 1—9. Si I est un idéal premier non nul, d'après 1—7, I est un idéal maximal; A/I est un corps, donc encore un anneau principal de type (0, 1).

1—12. Soit $(I_k)_{k\in K}$ les idéaux non nuls d'un anneau A; *A admet une représentation comme somme sous-directe* des anneaux $(A/I_k)_{k\in K}$ si [5] $\bigcap_{k\in K} I_k=(0)$.

Z admet une représentation comme somme sous-directe de corps et également une représentation comme somme sous directe d'anneaux sous-directement irréductibles qui ne sont pas des corps. On examine ici le cas d'un anneau principal.

1—13. PROPOSITION. *Pour un anneau principal A, les assertions suivantes sont équivalentes:*

(i) *A est sans radical* [3]

(ii) *A est de type (k, h) avec A_i non semi-local pour $1\leqq i\leqq k$ et A_i corps pour $k+1\leqq i\leqq k+h$.*

(iii) *A possède une représentation comme somme sous-directe de corps.*

DÉMONSTRATION. L'équivalence de (i) et (ii) vient des remarques suivantes: si $A=\prod_{i=1}^{n} A_i$, alors rad $(A)=\prod_{i=1}^{n}$ rad (A_i); lorsque A_i est intègre, A_i est sans radical si et seulement si il est non semi-local ou un corps [3]; un anneau principal spécial qui n'est pas un corps à son radical non réduit à zéro.

(i)⇒(iii) Si $(I_k)_{k\in K}$ est la famille des idéaux maximaux de A, puisque rad $(A)=0$, A admet une représentation comme somme sous-directe des $(A/I_k)_{k\in I}$.

(iii)⇒(i) Si A admet une représentation comme somme sous-directe de corps A/I_k, les I_k sont des idéaux maximaux de $A=\prod_{i=1}^{n} A_i$. I_k est de la forme $I_k=A_1\times\dots$ $\dots\times A_{j-1}\times J_{k,j}\times A_{j+1}\times\dots\times A_n$ où $J_{k;j}$ est un idéal maximal de A_j. Puisque $\bigcap_k I_k=0$, on a, pour tout $j=1,\dots,n$ $\bigcap_k J_{k,j}=0$ et à fortiori rad $(A_j)=0$. Donc rad $(A)=0$.

1—14. PROPOSITION. *Tout anneau principal de type $(k, 0)$ possède une représentation comme somme sous-directe, d'anneaux sous-directement irréductibles qui ne sont pas des corps.*

DÉMONSTRATION. Si $A=\prod_{i=1}^{k} A_i$, si p_i est un irréductible de A_i et si, pour $j\geqq 2$ on pose J_{ij} est un idéal primaire non premier de A_i. Alors $I_{ij}=A_1\times\dots\times A_{j-1}\times$ $\times J_{ij}\times A_{j+1}\times\dots\times A_k$ est un idéal primaire non premier de A. D'après 1—11, A/I_{ij}

est sous-directement irréductible; il n'est pas un corps car il est non intègre. Enfin:

$$\bigcap_{i,j} I_{ij} = \prod_{i=1}^{k} \Big[\bigcap_{j\geqq 2} p_i^k A_i\Big] = (0).$$

1—15. Soit A un anneau principal; on dit que $x=(x_1, \dots, x_n)\in \prod_{i=1}^{n} A_i$ est *sans facteur multiple*, si, pour tout $i=1, \dots, n$, x_i est sans facteur multiple dans A_i. Cela veut dire que si A_i est un corps, $x_i=0$ et si A_i n'est pas un corps, x_i est un élément non nul de A_i qui n'est pas divisible par une puissance $\geqq 2$ d'un irréductible de A_i.

On sait que les assertions suivantes sont équivalentes:

$\mathbf{Z}/_n\mathbf{Z}$ est sans radical; $\mathbf{Z}/_n\mathbf{Z}$ est héréditaire; n est sans facteur multiple.

1—16. PROPOSITION. *Pour un élément x d'un anneau principal A, les assertions suivantes sont équivalentes:*

(i) *x est sans facteur multiple.*

(ii) *A/xA est sans radical.*

DÉMONSTRATION. (i)⇒(ii) Si x est sans facteur multiple dans $A=\prod_{i=1}^{n} A_i$, pour tout $i=1, \dots, n$, x_i est sans facteur multiple dans A_i. Deux cas sont possibles:

— A_i est intègre; alors A_i/x_iA_i est sans radical d'après [3].

— A_i est principal spécial; si p_i est son unique élément irréductible, comme x_i est sans facteur multiple, $x_iA_i=p_iA_i$ est un idéal maximal de A_i, A_i/x_iA_i est un corps donc est sans radical.

(ii)⇒(i) Si A/xA est sans-radical, il en est de même de A_i/x_iA_i pour tout $i=1, \dots$ $\dots, n$. Si A_i est intègre, x_i est sans facteurs multiple d'après [3]. Sinon, A_i est principal spécial et pour que A_i/x_iA_i soit sans-radical il faut que A_i/x_iA_i soit un corps, c'est-à-dire que x_i soit irréductible dans A_i: donc x_i est encore sans-facteur multiple dans A_i.

1—17. PROPOSITION. *Soit $x=(x_1, \dots, x_k)$ un élément d'un anneau principal A de type $(k, 0)$ tel que pour tout $i=1, \dots, k$ on ait $x_i\neq 0$. Pour que A/xA soit héréditaire, il faut que x soit sans facteur multiple.*

DÉMONSTRATION. Soit $x=(x_1, \dots, x_n)\in \prod_{i=1}^{n} A_i$ et $I=I_1\times I_2\times\dots\times I_n$ un idéal de A/xA. I est un (A/xA)-module projectif si et seulement si pour tout $i=1, \dots, n$ I_i, considéré comme (A/xA)-module, est projectif.

Par définition des éléments sans facteur multiple, puisque A est de type $(k, 0)$, on peut supposer les x_i non nuls. Donc x_i peut s'écrire $x_i=\prod_{j=1}^{m_i} p_{ij}^{\alpha_{ij}}$ où p_{ij} est un irréductible de A_i. Alors, comme pour $j\neq k$

$$p_{ij}^{\alpha_{ij}}A_i+p_{i_k}^{\alpha_{ik}}A_i = A_i$$

on a, d'après [3]:

$$A_i/x_iA_i = \prod_{j=1}^{m_i} A_i/p_{ij}^{\alpha_{ij}}A_i$$

et $I_i = \prod_{j=1}^{m_i} I_{ij}$ où I_{ij} est un idéal de $A_i/p_{ij}^{\alpha_{ij}} A_i$. I_{ij} peut être considéré comme un (A/xA)-module et I est projectif si et seulement si tous les I_{ij} sont projectifs.

Poisque I_{ij} est un idèal de $A_i/p_{i_\alpha}^{\alpha_{ij}} A_i$, il est de la forme

$$I_{ij} = p_{ij}^{\beta_{ij}} A_i/p_{ij}^{\alpha_{ij}} A_i \quad \text{avec} \quad \beta_{ij} \leqq \alpha_{ij}.$$

Si I_{ij} est projectif, la suite exacte

$$0 \to p_{ij}^{\alpha_{ij}} A_i \to p_{ij}^{\beta_{ij}} A_i \to p_{ij}^{\alpha_{ij}} A_i/p_{ij}^{\alpha_{ij}} A_i \to 0$$

est une suite exacte scindée; donc

$$p_{ij}^{\beta_{ij}} A_i = p_{ij}^{\alpha_{ij}} \oplus (p_{ij}^{\beta_{ij}} A_i/p_{ij}^{\alpha_{ij}} A_i).$$

Dans le premier membre, comme A_i est intègre, il n'existe pas d'élément $a \neq 0$ tel que $p_{ij}^{\gamma} \cdot a = 0$. Pour qu'il en soit ainsi au second membre, il faut que $p_{ij}^{\beta_{ij}} A_i/p_{ij}^{\alpha_{ij}} A_i = 0$ et ceci quelque soit l'idéal I_{ij}, c'est-à-dire quelque soit $\beta_{ij} \leqq \alpha_{ij}$. Il faut donc que $\alpha_{ij} = 1$.

§ 2. Modules de torsion sur un anneau principal

Dans ce paragraphe, A est un anneau principal de type (k, h) et $\prod_{i=1}^{k+h} A_i$ sa décomposition canonique.

Si $a \in A$ est nilpotent, on note r_a son indice de nilpotence; sinon, on pose $r_a = \infty$. Si M est un A-module, $M(a)$ désigne le noyau de l'andomorphisme de M: $x \mapsto ax$.

Il est clair que $a|b \Rightarrow M(a) \subseteqq M(b)$ et $1 \leqq n \leqq m \Rightarrow M(a^n) \subseteqq M(a^m)$. $M_a = \bigcup_{n=1}^{\overline{r-1}_a} M(a^n)$ est un sous-module de M et:

$$x \in M_a \Leftrightarrow \exists n \in \mathbf{N}, \quad 0 < n < r_a \quad \text{et} \quad a^n x = 0.$$

Si N est un sous-module de M, $N_a = M_a \cap N$.

Si p est un élément irréductible de A, si $M = M_p$, on dit que M est un *p-module* celà veut dire que:

$$\forall x \in M, \quad \exists n \in \mathbf{N}, \quad 0 < n < r_p \quad \text{et} \quad p^n x = 0.$$

M_p est toujours un p-module ainsi que $A/p^k A$ si $k \geqq 1$.

2—1. LEMME. *Soit A un anneau principal de type* $(0, h)$. *Si p est un irréductible de A et si $p^\alpha A \subseteqq xA$, il existe $\beta \leqq \alpha$ tel que $xA = p^\alpha A$.*

DÉMONSTRATION. On peut supposer que $p = (p_1, u_2, \dots, u_h)$ où p_1 est irréductible dans A_1 et $u_i \in \mathcal{U}(A_i)$. Alors, pour $i \geqq 2$ $p^\alpha A \subseteqq xA \Rightarrow x_i \in \mathcal{U}(A_i)$. Comme A_1 est principal spécial puisque $p_1^\alpha A_1 \subseteqq x_1 A_1$, x_1 est de la forme p_1^β avec $\beta \leqq \alpha$ et $xA = p^\alpha A$.

2—2. LEMME. *Soit A un anneau principal de type* $(0, h)$. *Si p est irréductible de A et M un A-module alors:*

$$\text{Ann}\,(M_p) = p^s A \quad \textit{avec} \quad s < r_p.$$

DÉMONSTRATION. Puisque A est de type $(0, h)$, on a $r<\infty$. Donc $M_p=M(p^{\frac{r-1}{a}})$; alors pour tout $x\in M_p$, $p^{\frac{r-1}{p}}\cdot x=0$; $p^{\frac{r-1}{p}}\in \mathrm{Ann}\,(M_p)$ et d'après 2—1, il existe $s<r_p$ tel que $p^s A=\mathrm{Ann}\,(M_p)$.

2—3. On appelle *Base* d'un anneau tout système représentatif d'éléments irréductibles de cet anneau.

Si A est un anneau principal, si $a\in \prod_{i=1}^{k+h} A_i^*$, si B est une base de A, a peut s'écrire, de façon unique, à une unité prés $a=u \prod_{p\in B} p^{n_p}$, avec $u\in\mathcal{U}(A)$ et $n_p\in\mathbf{N}$. One pose alors $v_p(a)=n_p$ et a s'écrit $a=u\prod_{p\in B} p^{v_p(a)}$.

2—4. PROPOSITION. *Soit B une base d'un anneau principal A de type (k, h), $\prod_{i=1}^{k+h} A_i$ sa décomposition canonique, M un A-module de torsion et $a=u\prod_{p\in B} p^{v_p(a)}$ un élément de $\prod_{i=1}^{k+h} A_i$. Si $a_q=\prod_{p\neq q} p^{v_p(a)}$ alors:*

(i) $M(a)=\bigoplus_{p\in B} M(p^{v_p(a)})$ *et la projection de $M(a)$ dans $M(p^{v_p(a)})$ est une homothétie.*

(ii) $M(p^{v_p(a)})=a_q M(a)$.

(iii) *Pour que $M(p^{v_p(a)})=M(a)\cap M_p$, il suffit que A soit de type $(0, h)$ ou $(1, 0)$ c'est-à-dire atomique.*

DÉMONSTRATION. (i) D'une part si $\alpha A+\beta A=A$ alors $M(\alpha\beta)=M(\alpha)\oplus M(\beta)$. D'autre part, si p et q sont deux éléments distincts de B, on a $pA+qA=A$. D'après [3]; celà entraine $p^{v_p(a)}A+q^{v_q(a)}A=A$. Ces deux remarques permettent d'affirmer que

$$M(a) = \bigoplus_{p\in B} M(p^{v_p(a)}),$$

et l'application projection de $M(a)$ dans $M(p^{v_p(a)})$ est une homothétie.

(ii) Posons $N_p=a_p M(a)$. Si $x=a_p y\in N_p$, avec $y\in M(a)$, $p^{v_p(a)}\cdot x=ay=0$; donc $x\in M(p^{v_p(a)})$. Par conséquent: $N_p\subseteqq M(p^{v_p(a)})$, $\sum_{p\in B} N_p\subseteqq \bigoplus_{p\in B} M(p^{v_p(a)}) = M(a)$. Pour montrer que $N_p=M(p^{v_p(a)})$, il nous suffit de prouver que $M(a)\subseteqq \sum_{p\in B} N_p$.

Soit $a=up_1^{n_1}\cdot p_2^{n_2}\dots p_s^{n_s}$ une factorisation de a en éléments irréductibles. Soit $d\in A$ tel que $dA=\sum_{i=1}^{s} a_{p_i}A$.

Si $d=(d_1, \dots, d_{k+h})$ et $a=(a_1, \dots, a_{k+h})$, on a $aA\subseteqq a_{p_i}A\subseteqq dA$; donc pour $1\leqq i\leqq k+h$, $a_iA_i\subseteqq d_iA_i$.

Si $d\notin\mathcal{U}(A)$, il existe λ tel que $d_\lambda\notin\mathcal{U}(A_\lambda)$. Puisque $a_\lambda\neq 0$, on a $d_\lambda\neq 0$ et $d_\lambda\notin\mathcal{U}(A)$. A_λ est un anneau à factorisation unique [4]. Soit ϱ_λ un diviseur irréductible de d_λ. C'est un diviseur de a_λ. L'élément $\varrho=(1, 1, \dots, 1, \varrho_\lambda, 1, \dots, 1)$ est un diviseur irréductible de a. On peut supposer, par exemple que $\varrho=p_1$. Alors, p_1 divise d, donc a_{p_1}. Or, d'après, [3] et par définition de a_{p_1}, $p_1A+a_{p_1}A=A$, d'où une contradiction.

Nécessairement, $d\in\mathcal{U}(A)$; $A=\sum_{i=1}^{s} a_{p_i}A$ et $1=\sum_{i=1}^{s} U_i a_{p_i}$; donc

$$x=\sum_{i=1}^{s} b_i a_{p_i} x\in \sum_{i=1}^{s} a_{p_i} M(a)=\sum_{p\in B} N_p.$$

(iii) Si A est de type $(1,0)$, notre assertion est démontrée dans [3]. Nous pouvons supposer A de type $(0,h)$. On sait que $M(a)=M(a_p)\oplus M(p^{v_p(a)})$. Donc si $x\in M(a)\cap M_p$, x peut s'écrire $x=y+z$ avec $y\in M(p^{v_p(a)})$ et $z\in M(a_p)$. Puisque $x\in M_p$, il existe $k\in\mathbf{N}$ tel que $0\leqq k<r_p$ et $p^k x=0$. Alors, $p^k y+p^k z=0$ entraîne $p^k z=0$; puisque $y\in M(p^{v_p(a)})$, pour montrer que $x\in M(p^{v_p(a)})$, il nous suffit de prouver que $z=0$.

Puisque $pA+a_pA=A$, d'après [3], $p^kA+a_pA=A$; il existe $r,s\in A$ tels que $1=rp^k+sa_p$; alors $z=rp^kz+sa_pz$; puisque $z\in M(a_p)$ et $p^kz=0$, on a $z=0$.

2—5. Définition. Soit M un module sur $A=\prod_{i=1}^{n} A_i$. On dit que *M est de torsion sur chaque A_i* si λ_i désignant l'injection canonique $A_i\to A$;

$$\forall x\in M\quad \exists i=1,\dots,n\quad \exists a_i\in A_i^*\quad \text{tel que}\quad \lambda_i(a_i)x=0$$

c'est-à-dire si M est un A_i-pseudo-module [11] de torsion pour la structure définie canoniquement par chaque $A_i\to A$.

2—6. Lemme. *Si M est un module de torsion sur chaque A_i et si aucun A_i n'est un corps pour toute famille $(x_j)_{1\leqq j\leqq m}$ d'éléments de M, il existe $c\in\prod_{i=1}^{n} A_i^*$ tel que pour tout j: $x_j\in M(c)$.*

Démonstration. Puisque M est de torsion sur chaque A_i:

$$\forall j=1,\dots,m\ \forall i=1,\dots,n\ \exists a_{ij}\in A_i^*\quad \text{tel que}\quad \lambda_i(a_{ij})x_j=0.$$

Soit $a_j=(a_{1j},a_{2j},\dots,a_{nj})=\sum_{i=1}^{n}\lambda_i(a_{ij})\in\prod_{i=1}^{n} A_i^*$. On a $a_jx_j=0$.

a) Si A_i est intègre, on a $\prod_{j=1}^{m} a_{ij}\in A_i^*$. On pose alors:

$$c_i=\prod_{j=1}^{m} a_{ij}.$$

b) Si A_i est principal spécial, soit q_i sont unique élément irréductible et r_i l'indice de nilpotence de q_i. Comme aucun A_i n'est un corps, $q_i\neq 0$. On pose $c_i=q_i^{r_i-1}\in A_i^*$. Il existe $u_i\in\mathcal{U}(A_i)$ et $s_{ij}<r_j$ tels que $a_{ij}=u_iq_i^{s_{ij}}$; donc:

$$c_i=u_i^{-1}\cdot a_{ij}\cdot q_i^{r_i-1-s_{ij}}.$$

λ_i n'est pas nécessairement un homomorphisme d'anneaux; toutefois, il est multiplicatif; par conséquent:

$$\lambda_i(c_i)x_j=[\lambda_i(u_i^{-1}\cdot q_i^{r_i-1-s_{ij}})\lambda_i(a_{ij})]x_j=\lambda_i(u_i^{-i}\cdot q_i^{r_i-1-s_{ij}})[\lambda_i(a_{ij})x_j]=0.$$

Il suffit alors de poser $c=(c_1,\dots,c_n)$.

2—7. Proposition. *Soit A un anneau principal de type (k, h), $\prod_{i=1}^{n} A_i$ sa décomposition en anneau principaux présimplifiables, M un A-module de torsion sur chaque A_i et B une base de A.*

(i) $M = \sum_{p \in B} M_p$.

(ii) *Pour que cette somme soit directe, il suffit que A soit de type* $(1, 0)$ *ou de type* $(0, h)$, *aucun A_i n'étant un corps.*

Démonstration. (i) Puisque M est un A-module de torsion sur chaque A_i pour tout $x \in M$, il existe $a \in \prod_{i=1}^{n} A_i^*$ tel que $ax=0$. Donc $x \in M(a)$. D'après 2—4, $M(a) = \bigoplus_{p \in B} M(p^{v_p(a)})$. Donc $x = \sum_{p \in B} x_p$ avec $x_p \in M(p^{v_p(a)}) \subseteqq M_p$. (Les sommes considérées sont à support fini.) Donc: $M(a) \sum_{p \leqq B} M_p$.

(ii) Si A est de type $(1, 0)$, la démonstration est dans [3]. Supposons que A soit de type $(0, h)$ et qu'aucun A_i ne soit un corps. Si $\sum_{p \in B} x_p = \sum_{p \in B} y_p$ avec $x_p, y_p \in M_p$, les sommes étant à support fini, d'après 2—5, il existe $c \in \prod_{i=1}^{n} A_i$ tel que pour tout $p \in B$, on ait $x_p \in M(c)$ et $y_p \in M(c)$.

Puisque A est de type $(0, h)$, d'après 2—4: $x_p \in M(a) \cap M_p = M(p^{v_p(a)})$ et toujours d'après 2—4: $\sum_{p \in B} x_p = \sum_{p \in B} y_p$ entraine, pour tour $p \in B$ que $x_p = y_p$.

2—8. Conséquence immédiate: sous les hypothèses de 2—6, si N est un sous-module de M, on a: $N = \sum_{p \in B} (N \cap M_p)$; cette somme est directe si et seulement si A est de type $(1, 0)$ ou de type $(0, h)$, aucun A_i n'étant un corps.

2—9. Si $pA = qA$, alors $M_p = M_q$; M_p ne dépend que de pA. On l'appelle une *p-composante de M* et $\sum_{p \in B} M_p$ est la décomposition de M en p-composantes.

Si M est un module de torsion sur chaque A_i, les p-composantes de M sont nulles à l'exeption d'un nombre fini d'entre elles et l'application qui à tout $x \in M$ fait correspondre sa composante dans M_p est une homothétie.

§ 3. Modules libres sur un anneau principal

3—1. Définition. Soit A un anneau et M un A-module; une famille $(x_i)_{i \in I}$ d'éléments de M est *quasi-libre* si pour toute partie finie J de I:

$$\sum_{j \in J} a_j x_j = 0 \quad \forall j \in J \quad a_j \in \operatorname{div}(A).$$

Si une famille quasi-libre engendre M, on dit qu'elle constitue une *quasi-base* de M et que M est un *module quasi-libre.*

3—2. Exemples. (i) Tout module libre est un module quasi-libre.

(ii) Pour un anneau intègre les notions de module libre et de module quasi-libre sont équivalentes.

(iii) Soit $p>0$ un entier premier, $n\geqq 2$ un entier, $A=\mathbf{Z}/p^n\mathbf{Z}$ et $I=p^kA$ avec $0<k<n$. $\{p^k\}$ est une quasi-base de I, mais n'est pas une base. En outre, I n'est pas un module libre mais il est quasi-libre.

(iv) Si A est un anneau intègre non principal et I un idéal de A non principal I n'est pas un module quasi-libre.

3—3. Remarques. (i) Si $M \bigoplus_{i\in I} M_i$, si chaque M_i est un A-module quasi-libre, M est également quasi-libre.

(ii) Sur un anneau local noéthérien, tout module projectif est libre [2].

3—2 (iii) prouve donc *qu'un module quasi-libre n'est pas nécessairement projectif,* même si A est principal (présimplifiable ou non),

3—4. Proposition. *Soit M un module libre sur un anneau principal A. Tout sous-module de M est quasi-libre.*

Démonstration. Soit $(e_i)_{i\in I}$ une base de M; si J est une partie de I, on note M_J le module engendré par $(e_i)_{i\in J}$ dans M. Soit M' un sous-module de M. On note F l'ensemble des couples (J, B) tels que $M'\cap M_J$ est un A-module quasi-libre de quasi-base B. On ordonne F par l'inclusion sur chaque composante. F est non vide.

F est inductif: soit $(J_k, B_k)_{k\in K}$ une chaine de F. On pose $J=\bigcup_{k\in K} J_k$ et $B=\bigcup_{k\in K} B_k$. Alors, $M'\cap M_J$ est un A-module quasi-libre de quasi-base B; en effet:

Si $x\in M'\cap M_J$ alors, $x\in M_J$ donc $x=\sum_{i\in J'} a_i e_i$ où J' est une partie finie de J. Comme les $(B_k)_{k\in K}$ forment une chaine, x est combinaison linéaire d'éléments de B.

Si $\sum_{b\in B} a_b\cdot b=0$, cette somme étant à support fini, comme les B_k forment une chaine, il existe k_0 tel que tous les b apparaissant dans cette somme avec un coefficient non nul soient dans B_{k_0}; comme B_{k_0} est quasi-libre $\sum_{b\in B_{k_0}} a_i\cdot b=0$ entraine, pour tout b $a_b\in \operatorname{div}(A)$.

Soit (J_0, B_0) un élément maximal de F. Si $I=J_0$, $M'=M'\cap M_{J_0}$ est quasi-libre de quasi-base B_0 et notre assertion est démontrée. Supposons le contraire et soit $i_0\in I-J_0$. Posons $J'=J_0\cup\{i_0\}$.

Soit P l'ensemble des éléments a de A tels qu'il existe $a_{i_1}, a_{i_2}, \dots, a_{i_n}$ dans A, $i_j\neq i_0$ et $ae_{i_0}+\sum_{j=1}^{n} a_{i_j}e_{i_j}\in M'\cap M_{J'}$. Deux cas sont possibles.

1) $P=\{0\}$. Alors, soit $x\in M'\cap M_{J'}$. D'une part, x peut s'ècrire $x=a_{i_0}e_{i_0}+ +\sum_{i\neq i_0} a_i e_i$. Puisque $P=\{0\}$, quelque soit l'ècriture de x sous cette forme, on a $a_{i_0}=0$; donc $x=\sum_{i\in J} a_i e_i\in M'\cap M_{J_0}$. Par conséquent, $M'\cap M_{J'}=M'\cap J_0$ est quasi-libre de quasi-base B_0; $(J', B_0)\in F$ ce qui est absurde.

2) $P\neq\{0\}$. Il est clair que P est un idéal de A (vérification immédiate). P est donc principal. Soit d_0 un générateur de P. Par définition de P, il existe des d_i dans A tels que $x_0=d_0e_{i_0}+\sum_{i\neq i_0} d_i e_i\in M'\cap M_{J'}$.

Soit $x\in M'\cap M_{J'}$; x peut s'écrire $x=ae_{i_0}+\sum_{i\neq i_0} a_i e_i$. Par définition de P, $a\in P$, donc il existe $\mu\in A$ tel que $a=\mu d_0$. Puisque $x-\mu x_0=\sum_{i\neq i_0}(a_i-\mu d_i)e_i\in M'\cap M_{J'}$, on a $x-\mu x_0\in M'\cap M_{J_0}$.

Alors: $x-\mu x_0=\sum_{b\in B_0} c_b\cdot b$ et $x=\mu x_0+\sum_{b\in B_0} c_b\cdot b$; donc $B_0 u\{x_0\}$ engendre $M'\cap M_{J'}$.

$B_0\cup\{x_0\}$ est quasi-libre: si $ax_0+\sum_{b\in B_0} c_b\cdot b=0$, puisque B_0 est une quasi-base de $M'\cap M_{J_0}$, on a $y=\sum_{b\in B_0} c_b\cdot b\in M'\cap M_{J_0}$ qui peut s'ècrire $y=\sum_{i\in J_0} a_i\cdot e_i$ et ax_0+ $+\sum_{i\neq i_0} a_i e_i=0$. Comme $x_0=s_0 e_0+\sum_{i\neq i_0} d_i e_i$, on obtient:

$$ad_0 e_{i_0}+\sum_{i\neq i_0}(a_i+d_i)e_i = 0.$$

Comme $(e_i)_{i\in I}$ est une base de M, on a nécessairement $ad_0=0$; donc $a\in\operatorname{div}(A)$.

D'autre part: $ax_0+\sum_{i\neq i_0} a_i e_i=0$ entraine $\sum_{i\neq i_0} d_0 a_i e_i=0$. Donc pour tout $i\in I\, d_0 a_i=0$, c'est-à-dire $a_i\in\operatorname{div}(A)$. Alors: $(J', B_0\cup\{x_0\})\in F$, ce qui est contradictoire; d'où notre assertion.

3—5. Remarque. Nous ignorons si cette propriété est caractéristique des anneaux principaux; toutefois, il est clair que si A est un anneau tel que tout sous-module d'un A-module libre est quasi-libre, alors A est un anneau noethérien.

3—6. Corollaire. *Soit A un anneau principal.*

(i) *Pour tout $n\geqq 1$, tout sous-module de A^n est quasi-libre.*

(ii) *Tout idéal de A est quasi-libre.*

(iii) *Tout A-module projectif est quasi-libre.*

Démonstration. (i) et (ii): conséquence immédiate de 3—4; ainsi que (iii) puisque tout module projectif est facteur direct d'un module libre.

Le résultat suivant apporte une réponse partielle au problème posé en 3—5.

3—7. Proposition. *Pour, un anneau de Bezout A, les assertions suivantes sont équivalentes:*

(i) *A est principal.*

(ii) *Tout sous-module d'un A-module libre est quasi-libre.*

Démonstration. (i)⇒(ii) d'après 3—4 et (ii)⇒(i) d'après la définition d'un anneau de Bezout et d'un module quasi-libre.

3—8. Définition. Soit M un A-module; s'il existe un entier $n\geqq 1$ tel que toute famille quasi-libre a $m\leqq n$ éléments, on dit que M est de *rang fini* et le plus petit entier $n\geqq 1$ à avoir cette propriété est le *rang de M*. Si M est un anneau intègre, on retrouve la notion classique [3].

3—9. Proposition. *Soit M un A-module libre de rang fini n sur un anneau principal A. Tout sous-module de M est quasi-libre de rang fini $m\leqq n$.*

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ON COMPLETE METRIZABILITY OF SOME TOPOLOGICAL SPACES

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1. Introduction

If X is a topological space and $\beta=(B_s)_{s\in S}$ is a covering of this space, a subset M of X is said to be of diameter less than β provided that there exists some $s\in S$ such that $M\subset \bar{B}_s$; we write then $\delta(M)<\beta$. A countable family $(\beta_n)_{n\geqq 1}$ of open coverings of a topological space X is said to have the L-property, if for every countable family $\mathcal{F}=(F_n)_{n\geqq 1}$ of decreasing and non-empty closed sets of X such that $\delta(F_n)<\beta_n$, $\bigcap_n F_n$ is exactly one point of X. We say in this case that X is an L-space.

It is shown in this paper that a paracompact space X is completely metrizable iff it is an L-space. If X is a topological space with the Lindelöf property, then the same characterization of complete metrizability holds true. It follows that a K_σ-space (countable union of open relatively compact subsets) is completely metrizable iff its diagonal Δ is a G_δ set in $X\times X$, and that every K_σ-space which is semimetrizable is metrizable. We show finally that the continuous image of a separable metric space onto a locally compact space is also separable and (completely) metrizable.

2. Complete metrizability of Lindelöf spaces

The following remark is useful for further proofs.

REMARK 2.1. *Every L-space is T_1-regular and first countable.*

PROOF. Let X be a topological space and let $(\beta_n)_{n\geqq 1}$ be a sequence of open coverings of X which has the L-property. For every $x\in X$ and every $n\geqq 1$ select $V_n\in\beta_n$ such that $x\in V_n$. Call $W_n=\bigcap_{k=1}^{n} V_k$. Notice that $\bigcap_n W_n=\bigcap_n \bar{W}_n=\{x\}$: for the family $(\bar{W}_n)_{n\geqq 1}$ is a decreasing sequence of non-empty closed sets of X and $\delta(\bar{W}_n)<\beta_n$, so that $\bigcap_n \bar{W}_n$ is exactly one point of X. Thus X is a T_1-space. To finish the proof we show that, for every open set V containing $x\in X$, there is some $n\geqq 1$ for which $x\in\bar{W}_n\subset V$. Suppose not i.e., for every $n\geqq 1$, $F_n=\bar{W}_n\cap(X-V)\neq\emptyset$; since $F_{n+1}\subset F_n$ and $\delta(F_n)<\beta_n$, then one has simultaneously $\bigcap_n F_n=\{y\}$ with $y\neq x$ and $\bigcap_n F_n\subset\bigcap \bar{W}_n=\{x\}$, which is impossible.

We now show the following metrization theorem for paracompact spaces:

THEOREM 2.2. *Let X be a paracompact (nat necessarily Hausdorff) space; then X is completely metrizable iff X is an L-space.*

PROOF. The necessity is a classical result for complete metric spaces. We show only that the condition is sufficient. Let $(\alpha_n)_{n\geqq 1}$ be a sequence of open coverings of a paracompact space X having the L-property. Let α_n^*, for every n, be a locally finite open cover of X which refines α_n. Call β_n the family of the sets having the form $\bigcap_{k=1}^{n} A_k$ with $A_k \in \alpha_k^*$. Notice that β_n is also an open locally finite covering of X. By the NAGATA—SMIRNOV theorem [1] and [2], for metrizability of X it suffices that $\beta = \bigcup_n \beta_n$ is a base for X. Let $x \in X$ and V an open set containing x. For every $n \geqq 1$, there exists $A_n \in \alpha_n^*$ and $V_n \in \alpha_n$ such that $x \in A_n \subset V_n$. Call $B_n = \bigcap_{k=1}^{n} A_k$. Now $B_n \in \beta_n$ and the same argument of Remark 2.1 shows that $x \in \bar{B}_n \subset V$ for some $n \geqq 1$, and hence β is a base for X.

To finish the proof we show that X is complete for some compatible metric. Let $\hat{X}$ be the completion space of X. By [3], it suffices to show that X is a G_δ set in $\hat{X}$. Write, for $n \geqq 1$, $\beta_n = \{B_s^n\}_{s \in S_n}$ and let W_s^n be an open set in $\hat{X}$ such that $B_s^n = W_s^n \cap X$. We will show that $X = \bigcap_n \left(\bigcup_{s \in S_n} W_s^n \right)$. Evidently $X \subset \bigcap_n \left(\bigcup_{s \in S_n} W_s^n \right)$. Let now $x \in \bigcap_n \left(\bigcup_{s \in S_n} W_s^n \right)$ and denote, for every integer m, by V_m the closed sphere in $\hat{X}$ of centre x and radius $1/m$. For every $n \geqq 1$ there exists $s(n) \in S_n$ and an integer $m(n)$, $m(n) \geqq n$, such that $x \in V_{m(n)} \subset W_{s(n)}^n$. So $V_{m(n)} \cap X \subset W_{s(n)}^n \cap X = B_{s(n)}^n \subset \bar{B}_{s(n)}^n$. Without loss of generality one can suppose that $m(n) \leqq m(n+1)$ for every n. Call $F_n = V_{m(n)} \cap X$ and notice that for any n, $F_n \neq \emptyset$ (since $\bar{X} = \hat{X}$) and $F_{n+1} \subset F_n$. The family $(\beta_n)_{n \geqq 1}$ of open covers of X has also the L-property and $\delta(F_n) < \beta_n$, so $\bigcap_n (V_{m(n)} \cap X)$ is a single point of X which is necessarily x since $\bigcap_n V_{m(n)} = \{x\}$, and consequently $x \in X$.

Since every T_1-regular Lindelöf space is paracompact, and since every L-space is T_1-regular, one concludes the following metrization theorem for Lindelöf spaces.

THEOREM 2.3. *Let X be a Lindelöf space. Then X is completely metrizable iff X is an L-space.*

3. Application on metrizability of K_σ-spaces

DEFINITION 3.1. *A topological space X is called a K_σ-space, if $X = \bigcup_n W_n$ where W_n is an open relatively compact subset of X.*

Clearly every K_σ-space has the Lindelöf property.

It is well known [4] that every compact space X having its diagonal Δ as a G_δ set in $X \times X$ is metrizable. Using theorem 2.3, we now show that this result holds true for a wider class of topological spaces, precisely, K_σ-spaces.

THEOREM 3.2. *If X is a K_σ-space, then X is metrizable iff the diagonal Δ of X is a G_δ set in $X \times X$.*

PROOF. Let $X = \bigcup_i W_i$ be a K_σ-space, where W_i is an open set in X and $\bar{W}_i$ compact for every $i \geqq 1$. Suppose that $\Delta = \bigcap G_n$ where G_n is open in $X \times X$. For each $x \in X$ and each $n \geqq 1$, there exists an open set $A_n(x)$ containing x and such that $A_n(x) \times A_n(x) \subset G_n$; there exists also some $i(x) \geqq 1$ for which $x \in W_{i(x)}$. Call β_n the family of all sets of X of the form $A_n(x) \cap W_{i(x)}$. Since every K_σ-space is Lindelöf, it suffices to show that $(\beta_n)_{n \geqq 1}$ has the L-property.

Let $(F_n)_{n\geqq 1}$ be a decreasing sequence of non-empty closed sets of X such that $\delta(F_n)<\beta_n$ for every $n\geqq 1$. Notice that, for some $x\in X$, one has $F_n\subset\overline{A_1(x)\cap W_{i(x)}}\subset\subset\overline{W}_{i(x)}$ for every $n\geqq 1$. Since $\overline{W}_{i(x)}$ is compact, then $\bigcap_n F_n\neq\emptyset$. If for $x, y\in X$ we have $x, y\in\bigcap_n F_n$, then there exists a sequence (x_n) of points of X such that

$$(x, y)\in F_n\times F_n\subset A_n(x_n)\times A_n(x_n)\subset G_n$$

for every $n\geqq 1$, so that $(x, y)\in\bigcap_n G_n$ and this implies that $x=y$. Thus $\bigcap_n F_n$ is exactly one point of X.

4. Metrizability of continuous images of separable metric spaces

DEFINITION 4.1. A semi-metric on a set X is a function $s\colon X\times X\to R^+$ satisfying the following conditions: for all $x, y\in X$

a) $s(x, y)=0$ iff $x=y$, and

b) $s(x, y)=s(y, x)$.

One can define open sets in a semi-metric space as if s were a metric, and the result is a topology on X.

Topological spaces that are semi-metrizable are characterized by [5] as follows:

THEOREM 4.2. *A Hausdorff space X is semi-metrizable if and only if there is a collection $\{V_n(x)\colon x\in X, n=1, 2, \dots\}$ of open sets of X such that*

a) *For each $x\in X$, $\{V_n(x)\colon n=1, 2, \dots\}$ is a local base for x, and*

b) *If $y\in X$ and, for each n, $x_n\in X$ such that $y\in V_n(x_n)$, then (x_n) converges to y.*

The following remark is useful:

REMARK 4.3. *If X is a Hausdorff semi-metric space, then the diagonal Δ is a G_δ set in $X\times X$.*

For a proof, it is sufficient to consider $G_n=\bigcup_{x\in X}(V_n(x)\times V_n(x))$, where $\{V_n(x)\}$ satisfies a) and b) of theorem 4.2.

THEOREM 4.4. *Every Hausdorff K_σ-space which is semi-metrizable is metrizable (completely).*

DEFINITION 4.5. A T_1-regular space which is the continuous image of a separable metric space is called cosmic space.

Cosmic spaces are clearly separable and Lindelöf thus paracompact. A first countable cosmic space is semi-metrizable (see [6]).

REMARK 4.6. *A cosmic space is perfectly normal.*

In fact by [7] every hereditarily paracompact and separable space is perfectly normal, and so it suffices to show that a cosmic space is hereditarily paracompact.

Let Y' be a subspace of a cosmic space Y. By assumption there exists a separable metric space X and a continuous mapping $f\colon X\to Y$ which is onto. If $X'=f^{-1}(Y')$, then X' is clearly a separable metric subspace of X and $g=f|X'$ is a continuous mapping of X' onto Y', so that Y' is also a cosmic space and thus paracompact.

The following theorem is proved now.

THEOREM 4.7. *Let X be a separable metric space and f a continuous mapping of X onto a Hausdorff locally compact space Y. Then Y is a separable metric space.*

PROOF. Y is Lindelöf and thus a K_σ-space. To show metrizability of Y, by theorem 3.2 it suffices that the diagonal Δ is a G_δ in $X\times X$, and for this it suffices that Y is first countable (4.5 and 4.3). Let $x\in Y$. Since Y is a cosmic space, it is perfectly normal and thus there exists a sequence $(V_n)_{n\geqq 1}$ of open sets such that $\{x\}=\bigcap_n V_n$. Since Y is regular, then for every $n\geqq 1$, there exists an open set H_n such that $x\in H_n\subset \overline{H}_n\subset V_n$. Let K be a compact neighbourhood of x and call $W_n=\left(\bigcap_{i=1}^{n} H_i\right)\cap K$. Notice that $\{x\}=\bigcap_n \overline{W}_n$ and $\overline{W}_n\subset K$ for every $n\geqq 1$. Let W be an arbitrary open set containing x. If $F_n=\overline{W}_n\cap(X-W)\neq\emptyset$ for every n, then $(F_n)_{n\geqq 1}$ is a decreasing sequence of non-empty closed sets in K, and thus $\bigcap_n F_n\neq\emptyset$ which is impossible. Hence $x\in\overline{W}_n\subset W$ for some n and this completes the proof.

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ORTHOGONALE POLYNOME, DIE EINER REKURSIONSGLEICHUNG MIT KONSTANTEN POLYNOMKOEFFIZIENTEN GENÜGEN

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1. Problemstellung

Auf dem Raume $\mathcal{P}$ aller reellen Polynome werden Differenzengleichungen der Gestalt

$$(1.1)\qquad \sum_{\nu=0}^{k} a_\nu p_{n-\nu} = 0 \quad (n = k, k+1, \dots)$$

mit einer Ordnung $k \geqq 1$ und Koeffizienten a_ν mit

$$(1.2)\qquad a_\nu \in \mathcal{P}_\nu \quad (\nu = 0, 1, \dots, k), \quad a_0 = 1$$

betrachtet. Dabei bezeichne $\mathcal{P}_\nu$ den Raum aller reellen Polynome höchstens ν-ten Grades; speziell wird $\mathcal{P}_{-1} = \{0\}$ gesetzt. Als Polynomfolgen (im engeren Sinne) werden die Elemente aus $\mathcal{P}_0 \times \mathcal{P}_1 \times \dots$ bezeichnet. Eine Polynomfolge, welche (1.1) erfüllt, heiße Lösung. Eine Lösung, welche bezüglich irgendeines Skalarproduktes auf $\mathcal{P}$ ein Orthogonalsystem bildet, heiße Orthogonallösung.

Es ist bekannt, daß die spezielle Differenzengleichung

$$(1.3)\qquad p_n - U_1 p_{n-1} + p_{n-2} = 0 \quad (n = 2, 3, \dots)$$

die beiden Tschebyscheff-Systeme $\{T_n\}$ und $\{U_n\}$ als Orthogonallösungen besitzt, während die Theorie die Existenz wenigstens einer solchen Lösung aussagt (FREUD [2], p. 64). Allgemein gilt der folgende

SATZ 1.1 (Orthogonallösungen 2. Ordnung). *Sei* $k = 2$. *Die Differenzengleichung* (1.1), (1.2) *besitzt genau dann eine Orthogonallösung zu einem assoziativen Skalarprodukt, wenn* $a_1 \in \mathcal{P}_1 \setminus \mathcal{P}_0$, $a_2 \in \mathcal{P}_0 \setminus \mathcal{P}_{-1}$ *gilt und dabei* a_2 *positiv ist.*

(Definition des assoziativen Skalarproduktes s. unten, Beweis s. Abschnitt 2). Eine Verallgemeinerung dieses Satzes auf den Fall $k \geqq 3$ ist unbekannt, im Hinblick auf die Bedeutung rekursiver Polynomsysteme mit einer Orthogonalitätseigenschaft für die moderne Rechentechnik (s. FOX and PARKER [1]) wären Ergebnisse in dieser Richtung jedoch sehr wichtig. In Abschnitt 2 werden wir nur einige notwendige Bedingungen für die Existenz orthogonaler Lösungen aufstellen, welche jedoch gestatten, das Problem im Falle der Ordnung 2 zu behandeln. Der Hauptteil dieser Arbeit ist jedoch der Frage gewidmet, ob die Differenzengleichung (1.3) außer den beiden Tschebyscheff-Systemen noch weitere Orthogonallösungen besitzt und wie alle diese gegebenenfalls aussehen. Diese Untersuchungen erfordern die Lösung eines Momentenproblems und erfordern eine gewisse Beschränkung hinsichtlich der zuzulassenden Skalarprodukte.

DEFINITION. Ein Skalarprodukt auf $\mathscr{P}$ heiße assoziativ, wenn

$$\langle fg, h\rangle = \langle f, gh\rangle \quad (f, g, h\in\mathscr{P})$$

gilt. Ein assoziatives Skalarprodukt heißt normal, wenn das durch $I(f):=\langle f, 1\rangle$ auf $\mathscr{P}$ definierte lineare Funktional im Sinne der durch

$$\|f\| := \max\{|f(x)|: -1\leqq x\leqq 1\}$$

auf $\mathscr{P}$ gegebenen Norm beschränkt ist.

Mit Hilfe der Sätze von Hahn—Banach und Riesz kann man zeigen, daß ein normales Skalarprodukt auf $\mathscr{P}$ stets stetig nach $C[-1, 1]$ fortgesetzt werden kann und dort eine Darstellung der Gestalt

$$\langle u, v\rangle = \int_{-1}^{1} u(x)v(x)\,dg(x)$$

mit einem monoton nicht fallenden, beschränkten Integrator besitzt. Insbesondere ist dann die Fortsetzung von I positiv. Mit den eingeführten Begriffen gilt nun der

SATZ 1.2 (Orthogonallösungen 2. Ordnung zu normalem Skalarprodukt). *Die Polynomfolge $\{P_n\}$ ist genau dann eine Orthogonallösung der Differenzengleichung* (1.3) *zu einem normalen Skalarprodukt, wenn folgendes gilt:*

(i) $P_n=\alpha U_n+\beta U_{n-1}+\gamma U_{n-2}$ *für* $n=0, 1, \dots$ *mit* $\alpha\neq 0$, $U_{-1}=0$, $U_{-2}=-1$;

(ii) *Die Nullstellen des Polynoms* $\alpha t^2+\beta t+\gamma=\alpha(t-s_1)(t-s_2)$ *erfüllen eine der beiden folgenden Bedingungen:*

(a) $$s_1 = \bar{s}_2, \quad |s_1| = |s_2|<1,$$

(b) $$-1 \leqq s_1<s_2 \leqq 1$$

(bei geeigneter Numerierung).

Gegebenenfalls hat das zugehörige Skalarprodukt die Gestalt

$$\langle u, v\rangle = \langle 1, 1\rangle \int_{-1}^{1} u(x)v(x)K(x)\sqrt{1-x^2}\,dx$$

mit dem Kern

$$K(x) = \frac{2}{\pi}\cdot\frac{1-\gamma}{(1-\gamma)^2+\beta^2+2\beta(1+\gamma)x+4\gamma x^2}.$$

(Beweis s. Abschnitt 3.) Setzt man für $\gamma\neq 1$ und $n=0, 1, \dots$

$$Q_n^{(\beta,\gamma)} := \frac{1}{1-\gamma}(U_n+\beta U_{n-1}+\gamma U_{n-2}),$$

so ergibt sich aus Satz 1.2, daß (bei Zulassung beliebiger Normierungen) im wesentlichen nur vier Systeme von Jacobi-Polynomen als Lösung von (1.3) auftreten. Es sind dies die Orthogonallösungen

$$Q_n^{(0,0)} = U_n, \quad Q_n^{(1,0)} = U_n+U_{n-1} = c_n P_n^{\left(\frac{1}{2}, -\frac{1}{2}\right)},$$

$$Q_n^{(0,-1)} = T_n, \quad Q_n^{(-1,0)} = U_n-U_{n-1} = c_n P_n^{\left(-\frac{1}{2}, \frac{1}{2}\right)}$$

und alle Vielfachen von ihnen.

2. Normierungen und notwendige Bedingungen für die Existenz von Orthogonallösungen

a) Betrachtet wird eine Differenzengleichung (1.1). Dabei werde angenommen, daß $a_1 \notin \mathcal{P}_0$ gilt. Es sei τ der Automorphismus von $\mathcal{P}$ mit der Eigenschaft

$$q = \tau p \Leftrightarrow q\left(-\frac{1}{2}a_1(x)\right) = p(x).$$

Ist dann $\{P_n\}$ eine Orthogonallösung von (1.1), so ist $\{\tau P_n\}$ eine Orthogonallösung einer entsprechenden Differenzengleichung der besonderen Gestalt

$$P_n - U_1 P_{n-1} + \ldots = 0 \quad (n = k, k+1, \ldots).$$

Es kann also o. B. d. A. angenommen werden, daß

(2.0) $$a_1 \in \mathcal{P}_0 \text{ oder } a_1 = -U_1$$

gilt.

b) Differenzengleichungen mit einer Orthogonallösung zu einem assoziativen Skalarprodukt unterliegen gewissen Bedingungen:

Lemma 2.1 (notwendige Bedingung). *Die Differenzengleichung* (1.1), (1.2) *besitze eine Orthogonallösung zu einem assoziativen Skalarprodukt. Dann gilt* $a_k \in \mathcal{P}_{k-2}$. *Insbesondere besitzt die Differenzengleichung im Falle* $k=1$ *keine derartige Orthogonallösung.*

Beweis. Sei $\{P_n\}$ eine Orthogonallösung von (1.1) zum assoziativen Skalarprodukt $\langle .,. \rangle$. Dann gilt

(2.1) $$\sum_{\nu=0}^{k} \langle a_\nu P_m, P_{n-\nu} \rangle = 0 \quad \text{für} \quad n-2k \leqq m \leqq n,\ n \geqq k$$

wenn vorrübergehend $P_{-n}=0$ für $n=1, 2, \ldots$ gesetzt wird. Für $m=0$ und $n=2k$ bzw. $n=2k-1$ folgt aus (2.1) bei Berücksichtigung von (1.2) das Bestehen der Gleichungen $\langle a_k, P_k \rangle = 0$, $\langle a_k, P_{k-1} \rangle = 0$. Daraus ergibt sich die Behauptung.

Weiterhin sei $\{P_n\}$ eine Orthogonallösung zu einem assoziativen Skalarprodukt, und es mögen folgende Entwicklungen bestehen:

$$P_m(x) = C_m x^m + \ldots \quad (m = 0, 1, \ldots),$$

$$a_{k-1}(x) = -Bx^{k-1} + \ldots, \quad a_k(x) = Ax^{k-2} + \ldots$$

für $k \geqq 2$. Für $n \geqq 2k-2$ und $m = n-2k+2 \geqq 0$ folgt nunmehr aus (2.1) die Beziehung

$$\langle a_k P_m, P_{n-k} \rangle + \langle a_{k-1} P_m, P_{n-k+1} \rangle = 0,$$

und hieraus ergibt sich die Beziehung

$$\frac{A}{C_{n-k}} \langle P_{n-k}, P_{n-k} \rangle - \frac{B}{C_{n-k+1}} \langle P_{n-k+1}, P_{n-k+1} \rangle = 0$$

für $n = 2k-2, 2k-1, \ldots$. Damit erhalten wir das

LEMMA 2.2. *Sei P_n eine Orthogonallösung von* (1.1) *zu einem assoziativen Skalarprodukt. Dann gilt eine der beiden folgenden Aussagen:*

(i) $A = B = 0$;

(ii) $A \neq 0 \neq B$, $\dfrac{\langle P_m, P_m\rangle}{\langle P_{m-1}, P_{m-1}\rangle} = \dfrac{A}{B}\cdot\dfrac{C_m}{C_{m-1}}$ $(m = k-1, k, \ldots)$.

c) Die Ergebnisse dieses Abschnittes enthalten bereits den Beweis zu Satz 1.1. Vorgelegt sei nämlich eine Differenzengleichung (1.1), (1.2) mit $k=2$. Hinsichtlich der Lösbarkeit dieser Gleichung durch Orthogonallösungen kann o. B. d. A. die Annahme (2.0) getroffen werden. Es wird jetzt angenommen, die Differenzengleichung besitze eine Orthogonallösung zu einem assoziativen Skalarprodukt. Nach Satz 2.1 ist dann a_2 eine Konstante und ein Gradvergleich in der Differenzengleichung zeigt, daß nicht auch noch a_1 eine Konstante sein kann. Also hat die Differenzengleichung für $p_n = P_n$ die Gestalt

$$P_n - U_1 P_{n-1} + AP_{n-2} = 0 \quad (n = 2, 3, \ldots). \tag{2.2}$$

Damit erhält man für die Leitkoeffizienten der P_n die Beziehung $C_n = 2^{n-1}C_1$ $(n = 1, 2, \ldots)$, und aus dem Lemma 2.2 folgt mit $B=2$ sofort die Ungleichung $A>0$. Damit ist die Notwendigkeit der Bedingung von Satz 1.1 bewiesen. Daß diese Bedingung auch hinreichend ist, findet man bei FREUD [2]. Damit ist Satz 1.1 bewiesen.

3. Die Orthogonallösungen zweiter Ordnung

Dieser Abschnitt soll eine Übersicht über die Orthogonallösungen der Differenzengleichungen (1.1) der Ordnung $k=2$ geben. Nach den Überlegungen von Abschnitt 2 genügt es, alle Differenzengleichungen der Gestalt

$$p_n - U_1 p_{n-1} + c^2 p_{n-2} = 0 \quad (n = 2, 3, \ldots), \quad c \neq 0$$

auf ihre Orthogonallösungen hin zu untersuchen, wenn das Skalarprodukt als assoziativ vorausgesetzt wird, was hiermit geschieht. Mit Hilfe eines weiteren Automorphismus von $\mathcal{P}$ kann dabei noch die Konstante zu $c=1$ normiert werden. Dies erkennt man bei Benutzung der Substitution $P_n(x) = c^n p_n(x/c)$. Damit ist die Differenzengleichung (1.3) als Prototyp einer Gleichung 2. Ordnung mit Orthogonallösungen zu assoziativen Skalarprodukten erkannt, aus deren Orthogonallösungen alle übrigen Orthogonallösungen 2. Ordnung durch Substitutionen gewonnen werden können.

Wir treffen weiterhin die Definitionen $U_{-1} := 0$, $U_{-2} := -1$. Dann bilden nämlich die Folgen $\{U_n\}$, $\{U_{n-1}\}$, $\{U_{n-2}\}$ eine Basis für den linearen Raum aller Polynomlösungen der Differenzengleichung (1.3). Sodann beweisen wir den

SATZ 3.1 (Kennzeichnung der orthogonalen Polynomlösungen). *Sei $P_0 \in \mathcal{P}_0$, $P_1 \in \mathcal{P}_1$ und $\{P_n\}$ die durch P_0, P_1 eindeutig bestimmte Polynomlösung von* (1.3). *Ferner sei $\langle .,. \rangle$ ein assoziatives Skalarprodukt auf $\mathcal{P}$. Dann gilt: Genau dann ist $\{P_n\}$ ein Orthogonalsystem bezüglich dieses Skalarproduktes, wenn die beiden Bedingungen erfüllt sind:*

(i) $P_0 \notin \mathcal{P}_{-1}$, $P_1 \notin \mathcal{P}_0$,

(ii) $\langle P_n, 1\rangle = 0$ *für* $n = 1, 2, \ldots$.

BEWEIS. Die Notwendigkeit der angegebenen Bedingungen liegt auf der Hand. Es werden jetzt (i) und (ii) vorausgesetzt. Dann hat P_n zunächst die Darstellung

$$P_n = \alpha U_n + \beta U_{n-1} + \gamma U_{n-2} \quad (n = 0, 1, \dots). \tag{3.1}$$

Aus (i) folgt dabei $\alpha \neq 0$. Somit gilt $P_n \neq 0$ $(n=0, 1, \dots)$. Um die Orthogonalität der P_n zu beweisen, genügt es zu zeigen, daß

$$\langle P_n, U_1^m \rangle = 0 \quad \text{für} \quad 0 \leqq m < n \tag{3.2}$$

gilt. Dieser Nachweis wird durch vollständige Induktion nach $m+n$ geführt. Offenbar ist (3.2) nach Voraussetzung (ii) wahr für $m+n=1$. Es sei jetzt $r>1$, und es wird angenommen, daß (3.2) wahr sei für $m+n<r$. Dann ist (3.2) auch wahr für $m+n=r$, $m=0$ wegen (ii). Die Beziehung bleibt zu beweisen für $1 \leqq m < n$, $m+n=r$. Unter dieser Bedingung findet man nun unter Benutzung von (1.3), daß

$$\langle P_n, U_1^m \rangle = \langle U_1 P_n, U_1^{m-1} \rangle = \langle P_{n+1} + P_{n-1}, U_1^{m-1} \rangle = \langle P_{n+1}, U_1^{m-1} \rangle$$

gilt. Eventuelle Wiederholung des Schlusses führt zuletzt auf die Kette

$$\langle P_n, U_1^m \rangle = \langle P_{n+1}, U_1^{m-1} \rangle = \dots = \langle P_{n+m}, 1 \rangle = 0,$$

so daß (3.2) auch für $m+n=r$ gilt, w.z.b.w.

Aus Satz 3.1 folgt auf sehr einfache Weise das

KOROLLAR. *Das System $\{U_n\}$ ist orthogonal bezüglich des Skalarprodukts*

$$\langle f, g \rangle := \int_{-1}^{1} f(x) g(x) \sqrt{1-x^2}\, dx.$$

Zum Beweis benutze man die Beziehung

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{t^2-2xt+1} dx = \sum_{n=0}^{\infty} t^n \langle U_n, 1 \rangle \quad (|t| < 1).$$

Nach Satz 3.1 ist nur zu zeigen, daß die auf der linken Seite stehende Funktion eine Konstante ist, was nach der Substitution $x=\frac{1}{2}(z+z^{-1})$ mit Hilfe des Residuensatzes leicht geschehen kann.

Im weiteren setzen wir voraus, daß $\langle .,. \rangle$ ein normales Skalarprodukt ist und daß bezüglich dieses Skalarproduktes $\{P_n\}$ eine Ortogonallösung der Differenzengleichung (1.3) ist. Es gilt dann wieder (3.1) mit einem $\alpha \neq 0$. Somit ist die Bedingung (i) von Satz 1.2 erfüllt. Für das Weitere kann o.B.d.A. angenommen werden, daß

$$\alpha = 1 \tag{3.3}$$

gilt. Betrachtet werden nun die Polynome

$$g(t) := t^2 + \beta t + \gamma = (t-s_1)(t-s_2)$$

bzw.

$$h(t) := 1 + \beta t + \gamma t^2 = (1-ts_1)(1-ts_2).$$

Mithilfe der erzeugenden Relation der U_n erkennt man, daß die P_n wegen (3.1), (3.3) die erzeugende Funktion

$$\frac{h(t)}{t^2-2xt+1} = 1+\sum_{n=1}^{\infty} t^n P_n(x) \quad (-1 \leqq x \leqq 1, |t| < 1)$$

besitzen, wobei die Reihe bei festem t gleichmäßig bezüglich x konvergiert. Wird nun $I(f)$ wie in Abschnitt 1 definiert und nach $C[-1, 1]$ fortgesetzt, so folgt daraus die Beziehung

$$(3.4) \qquad h(t)I_x\left(\frac{1}{t^2-2xt+1}\right) = I(1) > 0 \quad (-1 < t < 1).$$

Hier soll der Index von I_x daraufhinweisen, daß das Argument als Funktion von x zu verstehen ist. Diese Funktion ist offenbar positiv. Da auch I positiv ist, folgt somit

$$(3.5) \qquad h(t) > 0 \quad (-1 < t < 1).$$

Daraus erhält man

$$(3.6) \qquad \gamma \geqq |\beta|-1.$$

Insbesondere ist also $\gamma \geqq -1$. Wir verfolgen die Annahme, es sei $\gamma > 1$, und führen diese zum Widerspruch: Aus (3.4) folgt zunächst bei Differentiation die Beziehung

$$(3.7) \qquad I_x\left(\frac{F(t)+xG(t)}{(t^2-2xt+1)^2}\right) = 0 \quad (-1 < t < 1)$$

mit

$$F(t) = \beta(1-t^2)+2(\gamma-1)t, \quad G(t) = 2(1-\gamma t^2).$$

Es sei jetzt τ die positive Nullstelle von G. Nach Vorraussetzung gilt $0<\tau<1$. Aus (3.7) folgt dann wegen der Positivität von I sofort

$$F(\tau) = F(-\tau) = 0, \quad 0 = F(\tau)-F(-\tau) = 4(\gamma-1)\tau > 0,$$

womit der gewünschte Widerspruch erzielt ist. Berücksichtigen wir noch die Beziehung $0 \neq P_0 = 1-\gamma$, so erhalten wir schließlich

$$(3.8) \qquad -1 \leqq \gamma < 1.$$

Wir betrachten jetzt die Nullstellen von g. Wegen (3.5) liegen diese sicher auf keinem der Intervalle $(-\infty, -1)$, $(1, +\infty)$. Entweder, es gilt nun $s_1 = \bar{s}_2$. Dann ist die Bedingung (a) in Satz 1.2 wegen (3.8) erfüllt. Oder s_1 und s_2 sind reell und von einander verschieden. Dann gilt, wie wir eben sahen, die Bedingung (b) von Satz 1.2. Also ist auch die Bedingung (ii) dieses Satzes erfüllt. Damit ist Satz 1.2 in der einen Richtung bewiesen.

Es wird jetzt vorausgesetzt, die Voraussetzungen (i) und (ii) von Satz 1.2 seien erfüllt. O. B. d. A. kann dabei $\alpha=1$ vorausgesetzt werden. Die P_n sind dann eine Polynomlösung von (1.3). Es bleibt ein normales Skalarprodukt zu bestimmen, bezüglich dessen die P_n ein Orthogonalsystem bilden. Unter der Annahme, daß es ein solches Skalarprodukt gibt, läßt sich dieses wie folgt bestimmen:

Das Skalarprodukt kann durch seine Momente $\mu_n=\langle x^n, 1\rangle$ $(n=0, 1, \ldots)$ beschrieben werden. Entwickelt man nun die Monome in der Form

(3.9) $$x^n = \sum_{\nu=0}^{\infty} b_\nu^{(n)} P_\nu(x), \quad b_\nu^{(n)} = 0 \quad \text{für} \quad \nu > n,$$

so lassen sich die Momente mit Hilfe der Entwicklungskoeffizienten in der Form

(3.10) $$\mu_n = b_0^{(n)}\langle P_0, 1\rangle = (1-\gamma) b_0^{(n)} \mu_0$$

ausdrücken. Entwickelt man andrerseits die Monome in der Form

(3.11) $$x^n = \sum_{\nu=0}^{\infty} a_\nu^{(n)} U_\nu(x), \quad a_\nu^{(n)} = \frac{2}{\pi} \int_{-1}^{1} x^n U_\nu(x) \sqrt{1-x^2}\, dx,$$

so ergibt sich aus (3.11) und (3.9) nach Einsetzen von (3.1) mit $\alpha=1$ bei Koeffizientenvergleich das folgende Gleichungssystem:

$$\begin{pmatrix} 1 & \beta & \gamma & & 0 \\ & 1 & \beta & \gamma & \\ & & \ddots & \ddots & \gamma \\ & 0 & & \ddots & \beta \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu_n \\ \mu_0 \\ b_1^{(n)} \\ \vdots \\ b_n^{(n)} \end{pmatrix} = \begin{pmatrix} a_0^{(n)} \\ a_1^{(n)} \\ \vdots \\ a_n^{(n)} \end{pmatrix} \quad (n = 1, 2, \ldots).$$

Die ν-reihigen Adjunkten

$$\Delta_\nu := (-1)^\nu \begin{vmatrix} \beta & \gamma & & 0 \\ 1 & \beta & \gamma & \\ & 1 & \ddots & \ddots \\ & & \ddots & \ddots & \gamma \\ 0 & & & 1 & \beta \end{vmatrix}_\nu \quad (\nu = 1, 2, \ldots)$$

der auftretenden Matrix genügen der Differenzengleichung

$$\Delta_\nu + \beta\Delta_{\nu-1} + \gamma\Delta_{\nu-2} = 0 \quad (\nu = 1, 2, \ldots),$$

wenn zusätzlich $\Delta_0:=1$, $\Delta_{-1}:=0$ gesetzt wird. Da diese Differenzengleichung offenbar g zum charakteristischen Polynom hat, erhält man leicht die folgende Beziehung:

(3.12) $$\Delta_\nu = \begin{cases} \dfrac{s_1^{\nu+1} - s_2^{\nu+1}}{s_1 - s_2} & \text{für} \quad s_1 \neq s_2 \\ (\nu+1) s^\nu & \text{für} \quad s_1 \neq s_2 = s \end{cases} \quad (\nu = -1, 0, 1, \ldots).$$

Damit folgt aus dem linearen Gleichungssystem für die Momente die Darstellung

(3.13) $$\mu_n = \mu_0 \sum_{\nu=0}^{n} \Delta_\nu a_\nu^{(n)} \quad (n = 0, 1, \ldots).$$

Die weiteren Untersuchungen machen eine Fallunterscheidung erforderlich. Wir begnügen uns, den Fall $s_1 \neq s_2$, $|s_1|<1$, $|s_2|<1$ zu erörtern. In diesem Fall erhält man aus (3.13) unter Benutzung von (3.12) die Beziehung

$$\mu_n = \frac{2}{\pi}\mu_0 \int_{-1}^{1} x^n \sqrt{1-x^2} \left\{\frac{s_1}{s_1-s_2}\sum_{\nu=0}^{n} s_1^\nu U_\nu(x) - \frac{s_2}{s_1-s_2}\sum_{\nu=0}^{n} s_2^\nu U_\nu(x)\right\} dx.$$

Unter Berücksichtigung der Orthogonalrelation der U_n können die auftretenden Summen durch die entsprechenden unendlichen Reihen ersetzt werden, da diese gleichmäßig konvergieren. Mit Hilfe der erzeugenden Funktion der U_n erhält man so die Beziehung

$$\mu_n = \mu_0 \int_{-1}^{1} x^n K(x) \sqrt{1-x^2}\, dx \tag{3.14}$$

mit

$$K(x) = \frac{1}{\pi(s_1-s_2)}\left\{\frac{1}{\sigma_1-x} - \frac{1}{\sigma_2-x}\right\}, \quad \sigma_j = \frac{1}{2}(s_j+s_j^{-1}) \quad (j = 1, 2).$$

Drückt man die s_j durch β und γ aus, so erkennt man, daß K mit dem in Satz 1.2 angegebenen Kern übereinstimmt. Die Überlegungen zu den übrigen unter die Bedingung (ii) von Satz 1.2 fallenden Fällen laufen ganz entsprechend. Offenbar ist K im Integrationsintervall stets positiv, so daß in der in Satz 1.2 angegebenen Weise ein Skalarprodukt definiert wird. Es bleibt zu zeigen, daß die P_n in Bezug auf dieses Skalarprodukt auch tatsächlich orthogonal sind. Nach Satz 3.1 genügt es dabei, das Bestehen der Beziehungen

$$\langle P_n, 1\rangle = 0 \quad (n = 1, 2, \ldots) \tag{3.15}$$

nachzuweisen. Dies kann in einer Reihe von Fallunterscheidungen auf elementarem Wege geschehen. Damit ist Satz 1.2 bewiesen.

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VERTEX-CRITICAL GRAPHS OF GIVEN DIAMETER

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The graphs considered in this paper are undirected, finite, without loops or multiple edges. Various results were published about graphs with given diameter (cf. [1], [2], [6], [7]). Edge-critical graphs of given diameter were studied e.g. in [3], [4], [8]. Here we shall prove some theorems about existence and basic properties of vertex-critical graphs of given diameter.

1. Notation and notions. Let G be a graph. Then $V(G)$ will denote the vertex set of G, $E(G)$ the edge set of G, $d_G(u, v)$ the distance between the vertices $u, v \in V(G)$ in G, $d(G)$ the diameter of G, $\deg_G u$ the degree of the vertex u in G, $N_G(u)$ the neighbourhood of the vertex u, $\delta(G)$ the minimum degree of G, $\Delta(G)$ the maximum degree of G, $\varkappa(G)$ the vertex-connectivity of G, $\lambda(G)$ the edge-connectivity of G and $\bar{G}$ the complement of G. In addition, we denote by $|A|$ the cardinality of a set A, by K_p the complete graph with p vertices, by C_p the circuit of length p and by $[x]$ the greatest integer not exceeding the real number x.

Let $G-x$ be the graph obtained from G by deleting the edge x of G. Let $G-A$, where $A \subset V(G)$, be the graph obtained from G by deleting every vertex $v \in A$ and all edges incidental with it. If $A=\{w\}$, then we write $G-A=G-w$. Definitions not included here can be found in [5].

If $x \in E(G)$ then $d(G-x) \geqq d(G)$. This inequality does not hold for all vertices e.g. in the graph Q on Fig. 1,

$$d(Q-a) < d(Q), \quad d(Q-b) = d(Q) \quad \text{and} \quad d(Q-c) > d(Q).$$

An edge x of G is called critical if $d(G-x) > d(G)$. A vertex v of G is called critical if $d(G-v) \neq d(G)$. A vertex u of G is called decreasing if $d(G-u) < d(G)$.

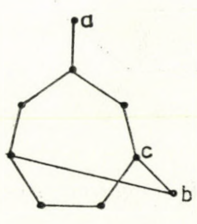

Fig. 1

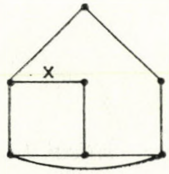

Fig. 2

Fig. 3

DEFINITION 1. A graph G is called:
a) e-critical, if $d(G-x) > d(G)$ for every edge x of G.

b) v-critical, if $d(G-v)\neq d(G)$ for every vertex v of G.
c) v_0-critical, if $d(G-v)>d(G)$ for every vertex v of G.
d) v_1-critical, if $d(G-v)<d(G)$ for every vertex v of G.

One can easily verify that the graph Q on Fig. 2 is v-critical but not e-critical; the graph $Q-x$ is e-critical and v-critical and the graph C_4 is e-critical but not v-critical. We note that v_0-critical graphs are defined in [4] and [8], where some basic properties are given.

2. Simple results. One can verify the following sufficient conditions for a graph to be v-critical.

LEMMA 1. *Let G be a graph of diameter $d\geqq 2$ containing a circuit. Let $u\in V(G)$. Then,*

a) *if* $\deg u\geqq 2$ *and u does not belong to any circuit of length $r\leqq d+2$, then the vertex u is critical;*

b) *if $\delta(G)\geqq 2$ and the girth of G is at least $d+3$, then G is v-critical.*

We note that the graph C_4 is not v-critical and contains a $(d+2)$-angle; the graph Q on Fig. 3 is not v-critical, has girth $k=d+3$ and $\deg a=1$. Hence the assumptions of Lemma 1 cannot be improved.

LEMMA 2. *Let G be a connected graph and let A be the set of its decreasing vertices. Then $|A|\leqq 2$ and moreover $d(G)-2\leqq d(G-A)\leqq d(G)-1$ for $A\neq\emptyset$.*

PROOF. Let $d(G)=d$ and $a\in A$. Then $d_G(u, v)<d$ for every $u, v\in V(G)-\{a\}$, because $d(G-a)<d$. So if $d_G(u, v)=d$ for some $u, v\in V(G)$, then one of the vertices u, v is equal to a. Thus if there exists a vertex $b\in A$, $b\neq a$, then $d(a, b)=d$. If a vertex $c\in A$, $c\neq a$, $c\neq b$ exists, then as $a, b\in V(G)-\{c\}$ we would have $d(a, b)<d$, which is impossible. Hence $|A|\leqq 2$ holds. After deleting at most two vertices from G the length of every path in G will decrease at most by two. So the inequalities hold. Graphs on Fig. 4 and Fig. 5, respectively show that both bounds can be reached. So the lemma holds.

Fig. 4

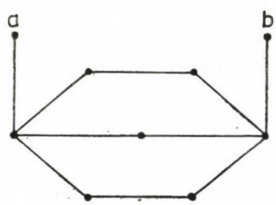

Fig. 5

COROLLARY 1. *A graph G is v_1-critical if and only if it is isomorphic to either the graph K_2 or the graph $\overline{K}_2$.*

In the next section we shall show that there exist many v_0-critical graphs of diameter $d\geqq 2$ and also many v-critical, but not v_0-critical graphs of diameter $d\geqq 4$.

3. Existence theorems. The following lemma is proved in [4] and [3].

LEMMA 3. *Let G be a graph and let $d \geqq 2$ be an integer. Then there exists an e-critical, v_0-critical graph Q of diameter d containing the graph G as an induced subgraph.*

For diameter two we shall prove a stronger assertion than in Lemma 3, where $\delta(Q)=\varkappa(Q)=2$.

THEOREM 1. *Let G be a graph and let $k \geqq 2$ an integer. Then there exists an e-critical, v_0-critical graph Q of diameter two containing the graph G as an induced subgraph such that $\delta(Q)=\varkappa(Q)=k$.*

PROOF. If $k=2$, then the graph Q of diameter two constructed in Lemma 3 is an appropriate graph.

Let $k \geqq 3$, and G_1 the graph that arises from G by adding one isolated vertex v and further $k-2-|V(G)|$ isolated vertices, if $|V(G)|<k-2$. Let G_2 be a graph constructed from G_1 by adding one new vertex w that is adjacent with all vertices of the graph G_1. Hence $|V(G_2)|=\max(k-1, |V(G)|+2)$. Let G_2' be a duplicate of G_2, whereby if $u \in V(G_2)$ then u' is the vertex of G_2 associated with u. From the graphs G_2, G_2' we construct a graph R by adding one new vertex z and the following edges:

a) we join every vertex $x \in V(G_2)$, $x \neq w$ with the set of vertices $A=V(G_2')-N_{G_2'}(x')$.

b) we join the vertex z with the vertices w, w' and if $k \geqq 4$ then also with the vertex v and with some $k-4$ vertices from the set $(V(G_2) \cup V(G_2'))-\{w, w', v\}$. The graph R does not contain other vertices and edges.

One can verify that $d(R)=2$ and the graph G is an induced subgraph of R. We have $|V(R)|=2|V(G_2)|+1$. The degree of the vertex z is $k-1$ and the degrees of the other vertices of the graph R are either $|V(G_2)|$ or $|V(G_2)|+1$. So we have $\delta(R)=k-1$ and $\delta(\bar{R}) \geqq k-1$. Let $\bar{R}$ be the complement of R that is vertex disjoint with the graph R, whereby the associated vertices we denote by $x, \bar{x}$, where $x \in V(R)$.

Let $k=3$. Then $d(\bar{R})=2$ because $\bar{R}$ contains the edges (z, x), (z, x'), (w, x') and (w', x) for every $x \in V(G_2)-\{w\}$. We construct the graph Q as follows: $V(Q)=V(R) \cup V(\bar{R})$; $E(Q)=E(R) \cup E(\bar{R}) \cup \bigcup_{x \in V(R)} \{(x, \bar{x})\}$.

Let $k \geqq 4$. Then one can see that $d(\bar{R})=3$, e.g. $d_{\bar{R}}(\bar{v}, \bar{v}')=3$. We put $V(Q)=V(R) \cup V(\bar{R}) \cup \{c\}$; $E(Q)=E(R) \cup E(\bar{R}) \cup \bigcup_{x \in V(R)} \{(x, \bar{x}), (\bar{x}, c)\}$.

One can verify that $d(Q)=2$, $\delta(Q)=k$ and that G is an induced subgraph of the graph Q. It can be seen, by investigation of separate cases, that between any two different vertices of Q there exist at least k vertex-disjoint paths. So from this fact and the equality $\delta(Q)=k$ it follows $\varkappa(Q)=k$.

The graph Q is e-critical, because after deleting the edge:

$(x, y) \in E(R)$ we have $d(\bar{x}, y)>2$;
$(\bar{x}, \bar{y}) \in E(\bar{R})$ we have $d(x, \bar{y})>2$;
$(x, \bar{x}) \in E(Q)$ we have $d(\bar{y}, x)>2$ for any $\bar{y} \in N_{\bar{R}}(\bar{x})$ and
$(c, \bar{x})$ for $k \geqq 4$, $\bar{x} \in V(\bar{R})$ we have $d(c, x)>2$.
The graph Q is v_0-critical, because after removing the vertex:
$x \in V(R)$ we have $d(y, \bar{x})>2$ for every $y \in N_R(x)$;
$\bar{x} \in V(\bar{R})$ we have $d(\bar{y}, x)>2$ for every $\bar{y} \in N_{\bar{R}}(\bar{x})$;
c, for $k \geqq 4$, we have $d(\bar{v}, \bar{v}')>2$.

This completes the proof.

THEOREM 2. *Let G be a graph that is v-critical, but not v_0-critical. Let $d(G)=d$ be equal to either* 1, 2 *or* 3. *Then G is a path of length d.*

PROOF. Let A be the set of decreasing vertices, and let $a \in A$. Let us define $Z_i = \{z | z \in V(G) \wedge d_G(a, z) = i\}$, for $i=1, 2, \dots, d$. If $d=1$ then obviously $G=P_1$. If $d=2$ then the graph $G-a$ is a complete graph and then $|Z_2|=1$ and also $|Z_1|=1$, because otherwise G would not be v-critical. So $G=P_2$.

Let $d=3$ and $R=G-a$. Then $d(R)=2$. Let $w \in Z_3$. If $d(G-w)<3$, then $Z_3=\{w\}$, because $w \in A$ thus $d(u, v)<3$ for every $u, v \in V(G)-\{a, w\}$.

Let $d(G-w)>3$. Then $N_G(a) \cap N_G(w) = \emptyset$, because $d_G(a, w)=3$. From the properties of the graph $R-w$ we shall show that $Z_3=\{w\}$ holds.

Assertion 1. Let $d_{R-w}(x, y) \geqq 3$ hold for some $x, y \in V(R-w)$. Then $(x, w) \in E(R)$ and $(y, w) \in E(R)$ hold, because otherwise we would have $d_R(x, y)>2$.

1) Let $d(R-w) \geqq 5$. Then the eccentricity $e_{R-w}(x)$ of every vertex $x \in V(R-w)$ satisfies $e_{R-w}(x) \geqq 3$ because if $e_{R-w}(z) \leqq 2$ for some $z \in V(R-w)$, then $d(R-w) \leqq 4$, a contradiction. Hence for every $x \in V(R-w)$ there exists $y \in V(R-w)$ such that $d(x, y) \geqq 3$ and then according to Assertion 1, $(x, w) \in E(R)$. Hence $N_R(w)=V(R)-\{w\}=V(G)-\{a, w\}$, which is impossible because $d_G(a, w)=3$.

2) Let $d(R-w)=4$. Let $P=(z_1, z_2, z_3, z_4, z_5)$ be a path of the graph $R-w$ of length 4. From Assertion 1 it follows that $(z_i, w) \in E(R)$ for $i=1, 2, 4, 5$. For every $x \in N_{R-w}(z_1)$ we have $d_{R-w}(x, z_5) \geqq 3$, because otherwise we would have $d_{R-w}(z_1, z_5) \leqq 3$. According to Assertion 1, $(x, w) \in E(R)$ and then $N_R(z_1) \subset N_R(w)$ and also $N_G(z_1) \cap N_G(a) = \emptyset$ holds. So $z_1 \in Z_3$ and $d(R-z_1)=3$, because $d(R)=2$ and $N_R(z_1) \subset N_R(w)$. Hence $d(G-z_1)=3$, because $d_{G-z_1}(a, w)=3$ and $d_{G-z_1}(a, x) \leqq 3$ for every $x \in V(G-z_1)$. So G is not v-critical, a contradiction.

3) If $d(R-w) \leqq 3$, then one can verify that $d(G-w) \leqq 3$ holds, what contradicts the assumption $d(G-w)>3$. Hence we have $Z_3=\{w\}$ and $d(G-w) \leqq 2$.

Now we shall prove that $|Z_2|=1$. Let $z \in Z_2$. If $d(R-z) \leqq 3$, then one can verify that $d(G-z) \leqq 3$, which cannot happen because $z \notin A$. If $d(R-z)=4$, then there exists a path $P(y_1, y_2, y_3, y_4, y_5)$ of length 4 in the graph $R-z$. Then one can prove that $N_R(y_1) \subset N_R(z)$ holds analogously as in part 2) of this proof. Hence $d(R-y_1)=d(R)=2$. Then it can be easily seen that $d(G-y_1) \leqq 3$, which cannot happen, because $y_1 \notin A$. If $d(R-z) \geqq 5$ then according to part 1) of this proof $(z, x) \in E(R)$ for every $x \in V(R)$, $x \neq z$, i.e. $N_R(z)=V(R)-\{z\}$. So $(z, w) \in E(R)$ and thus $Z_2=N(w)$ because z is an arbitrary vertex of Z_2.

If there exists $y \in Z_2$, $y \neq z$ then one can prove, analogously as for the vertex z, that $d(R-y) \leqq 3$ and $d(R-y)=4$ can not happen. If $d(R-y) \geqq 5$, then $N_R(y)=V(R)-\{y\}$ owing to the same reasoning as for the vertex z. Hence $N_R(z)=N_R(y)$ and thus $d(R-y)=2$, which is impossible. Hence $|Z_2|=1$. From this and the v-criticity of G it can be easily verified that $|Z_1|=1$. The theorem follows.

THEOREM 3. *Let G be a graph and $d \geqq 4$ an integer. Then there exists a graph of diameter d that is v-critical, but not v_0-critical and contains the graph G as an induced subgraph.*

PROOF. Let R denote the graph that arises from G by adding one new vertex a and edges (a, x) for all $x \in V(G)$. It is clear that $1 \leqq d(R) \leqq 2$. Let $p=|V(R)|$. Let us set $V(Q)=\bigcup_{i=1}^{d-1} A_i \cup \{w, z\}$, where $A_i=\{a_k^i\}_{k=1}^p$, $A_1=V(R)$ and the sets A_i are mutually disjoint. The subgraphs of Q induced by the sets A_1, A_{d-1} and A_i, $i=2, 3, \ldots, \ldots, d-2$ are the graph R, the graph $\bar{R}$ and the graph not containing any edge, respectively. The graph Q contains also the following edges:

(a_k^i, a_k^{i+1}), for $i=1, 2, \ldots, d-2$; $k=1, 2, \ldots, p$;

(a_k^{d-1}, w) for $k=1, 2, \ldots, p$; the edge (w, z) and no other vertices and edges.

One can see that $d(Q)=d$ and that G is an induced subgraph of Q. The graph Q is not v_0-critical, because $d(Q-z)<d(Q)$, which can be verified by investigating separate cases and using that the subgraph of Q induced by the set A_{d-1} is $\bar{R}$. The graph Q is v-critical, because after deleting the vertex

w, the graph $Q-w$ is not connected;

a_k^i, for $k=1, 2, \ldots, p$ and $i=2, 3, \ldots, d-1$ we have $d(z, a_k^1)>d$;

a_k^1, for $k=1, 2, \ldots, p$ we have $d(a_k^2, x)>d$ for any $x\in N_R(a_k^1)$.

(Such a vertex x exists as $1\leqq d(R)\leqq 2$.) So the theorem holds.

4. Further results. As usual, we define a branch at a cutpoint u of a graph G as a maximal subgraph of G containing u in which u is not a cutpoint. The length of a branch Q at u is the excentricity of u in the graph Q, $e_Q(u)$.

THEOREM 4. *Let G be a v-critical graph of diameter d, $d\geqq 2$. Then G is either a path of length d, or a block with at most two branches which are paths, the length of each of them is at most k, where*

$$k=\begin{cases}\left[\dfrac{d}{2}\right]-1, & \text{if either } d=3 \text{ or } d\geqq 2,\ d \text{ even}.\\[2ex] \left[\dfrac{d}{2}\right], & \text{if } d\geqq 5,\ d \text{ odd}.\end{cases}$$

PROOF. It can be verified that if G does not contain a circuit then it is a path of length d.

Let G contain at least one circuit and let B be a block of G that contains some circuit. Then either $G=B$ or at least one vertex of B is a cutpoint. Let C be the set of cutpoints of G that belong to B and let A be the set of decreasing vertices of G. It is clear that $A\cap C=\emptyset$.

Let $w\in C$. Let us decompose the graph G in two connected subgraphs $G_1=G_1(w)$, $G_2=G_2(w)$ in such a way that they have only the vertex w in common. At least one of the excentricities $e_{G_1}(w)$, $e_{G_2}(w)$ is not greater than $\left[\frac{d}{2}\right]$. Let us consider the case when $\left[\frac{d}{2}\right]\geqq e_{G_2}(w)=d(w, z)$, for some $z\in V(G_2)$. Then $d(G_2)\leqq d$, $d(G_2-z)\leqq d$ and hence $d(G-z)\leqq d$ holds, as $d(G)=d$. So $d(G-z)<d$, $z\in A$ holds, as G is v-critical. So we proved that if $C\neq\emptyset$ then $A\neq\emptyset$.

We shall show that no vertex $x\in V(G_2)$, $x\neq z$ belongs to A. If $x\in A\cap V(G_2)$, $x\neq z$ existed, then $d_G(z, x)=d$ would hold and for any other pair $a, b\in V(G)$ we would have $d_G(a, b)<d$. So $d(w, z)=d(w, x)=\left[\frac{d}{2}\right]=\frac{d}{2}$. But for $y\in V(G_1)$ such that $d(w, y)=e_{G_1}(w)$ we have $d(z, y)=d$, because $e_{G_1}(w)\geqq\left[\frac{d}{2}\right]$, $d(z, y)=e_{G_1}(w)+e_{G_2}(w)$ holds. This is a contradiction. So if $x\in V(G_2)$, $x\neq z$ then $x\notin A$.

Let $P(w, z)$ be a shortest path between the vertices w, z in the graph G. If there exist vertices of G_2 not belonging to the path $P(w, z)$ then we choose from them some vertex x which is the farthest from w and then analogously as in the previous part of this proof we can verify that $d(G_2-x)\leqq d$ and $d(G-x)\leqq d$, which contradicts the v-criticity of G. So the subgraph G_2 equals the path $P(w, z)$ and thus G_1 contains B.

The length k of the path $P(w, z)$ is not greater than $\left[\frac{d}{2}\right]$. If $d=3$ and G is not a path then according to Theorem 2 the graph G is v_0-critical. Hence $A=\emptyset$ and thus $C=\emptyset$. So the graph G is a block and $k=0=\left[\frac{d}{2}\right]-1$.

Let d be even, $d\geqq 2$. Assume $d(w, z)=\left[\frac{d}{2}\right]=k$. Then $e_{G_1}(w)=\frac{d}{2}$. If $A\cap V(G_1)=\emptyset$ holds, then after deleting any vertex $x\in V(G_1)$ that is the farthest from w we get $d(G_1-x)\leqq 2\cdot\frac{d}{2}$ and also $d(G-x)\leqq d$, which is impossible. Hence $k\leqq\left[\frac{d}{2}\right]-1$. If there exists $y\in A\cap V(G_1)$ then $d(y, z)=d$; $d(y, w)=\frac{d}{2}$ and for every $x\in V(G_1)-\{y\}$ we have $d(x, w)\leqq\left[\frac{d}{2}\right]-1$. Let $P(y, w)$ be a shortest path between the vertices y, w. Let the vertex u of G_1 not belong to $P(y, w)$ and let it be the farthest from w. Then we have $d(G_1-u)\leqq d$ and $d(G-u)\leqq d$, which is impossible. So $k\leqq\left[\frac{d}{2}\right]-1$ holds.

If there exists $v\in C$, $v\neq w$ then analogously as in the previous case, we define the subgraphs $G_1(v)$, $G_2(v)$, then we prove that $G_2(v)$ is a path $P(v, z_1)$, where $z_1\in V(G_2(v))$, $z_1\neq z$, $z_1\in A$ and then we estimate its length.

If a vertex $u\in C$, $u\neq v$, $u\neq w$ would exist, then as above we define the subgraphs $G_1(u)$, $G_2(u)$ and then we prove that $G_2(u)$ is a path $P(u, z_2)$, for some vertex $z_2\neq z_1$, $z_2\neq z$, $z_2\in A$. This contradicts the inequality $|A|\leqq 2$ proved in Lemma 2. This completes the proof.

From the proof of Theorem 4 immediately follows:

COROLLARY 2. *Every v_0-critical graph is a block.*

The graph Q on Fig. 6 is v-critical but not v_0-critical and it is a block. The graphs $Q-a$, $Q-b$ have the same properties.

REMARK 1. The estimate of the length of the branches, which are paths for v-critical graphs can be reached. The required graph for $d=7$ is on Fig. 7. For $d\geqq 9$, d odd, it can be constructed analogously, where $d(a, w)=\left[\frac{d}{2}\right]$, $d(w, z)=d-\left[\frac{d}{2}\right]$ and the sets X_i, $1\leqq i\leqq d-\left[\frac{d}{2}\right]$ will contain $d+3$ vertices. The graphs C_5, C_6 are v-critical of diameter two, three, respectively.

If d is even, $d\geqq 2$, then the graph that arises from the circuit C_{d+3} by adding a path of length $\left[\frac{d}{2}\right]-1$ as a branch is v-critical of diameter d and contains a branch of length $\left[\frac{d}{2}\right]-1$.

COROLLARY 3. *Let $d \geqq 2$ be an integer. Let G be a v-critical graph of diameter d. Then if at least one of the numbers $\varkappa(G)$, $\lambda(G)$, $\delta(G)$ is equal to one, then $\varkappa(G) = \lambda(G) = \delta(G) = 1$.*

PROOF. Trivially, $1 \leqq \varkappa(G) \leqq \lambda(G) \leqq \delta(G)$. If we suppose that $\varkappa(G)=1$ and the vertex u is a cutpoint, then at least one of the branches at u is a path, by Theorem 4. Thus $\delta(G)=1$ and hence $\lambda(G)=1$. Q.E.D.

Fig. 6

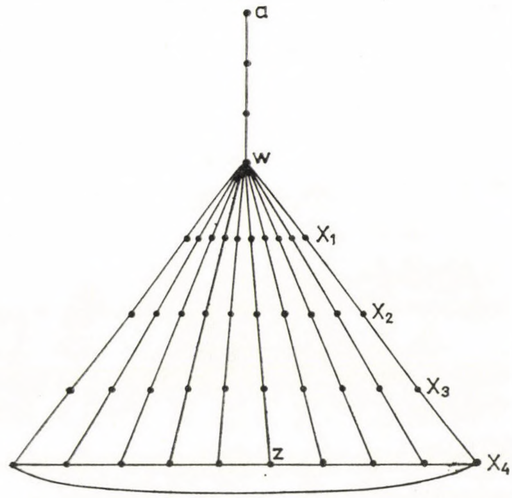

Fig. 7

COROLLARY 4. *If G is a v-critical graph of diameter two and at least one of the numbers $\varkappa(G)$, $\lambda(G)$, $\delta(G)$ is equal to one, then G is a path of length two.*

REMARK 2. Let G be a v-critical graph and let A be the set of decreasing vertices of G. Then one can see:

1) if $u \in V(G) - A$ then $\deg u \geqq 2$, because otherwise we would have $d(G-u) \leqq d(G)$.

2) at most two vertices of G have degree one, because $|A| \leqq 2$.

3) if G is v_0-critical, then $\delta(G) \geqq 2$.

THOEREM 5. *Let $d \geqq 2$ be an integer. Let G be a v-critical graph of diameter d with p-vertices. Then* $\delta(G) \leqq \left[\frac{p-d+1}{2}\right]$.

PROOF. If $\delta(G)=1$, then the inequality holds, because $p \geqq d+1$.

Let $\delta(G) \geqq 2$. Then $\lambda(G) \geqq 2$ and $\varkappa(G) \geqq 2$ by Corollary 3. So the graph $G-x$ is connected for every $x \in V(G)$. There exists a vertex z of G such that $d(G-z) > d(G)$, because $p \geqq d+1 \geqq 3$ and G contains at most two decreasing vertices. Let $R = G-z$ and let $d_R(u, v) = d(R)$. Then $N_R(u) \cap N_R(v) = \emptyset$ as $d(R) > d(G) \geqq 2$. We have $\deg_R u \geqq \deg_G u - 1$, $\deg_R v \geqq \deg_G v - 1$. Adding these inequalities we get $\deg_R u + \deg_R v + 2 \geqq \deg_G u + \deg_G v$.

Let $P(u, v)$ be a shortest path between the vertices u, v in the graph R and let M be the set of its vertices. The length of the path $P(u, v)$ is at least $d+1$ and hence $|M| \geqq d+2$. The path $P(u, v)$ contains exactly one vertex of the sets $N_R(u)$ and $N_R(v)$. So we have $p=|V(G)| \geqq |M|+|N_R(u)|+|N_R(v)|-2+|\{z\}| \geqq \deg_R u+\deg_R v+d-1 \geqq \geqq \deg_G u+\deg_G v+d-1$. So $\deg_G u+\deg_G v \leqq p-d+1$ and then either $\deg_G u$ or $\deg_G v$ is not greater than $\left[\frac{p-d+1}{2}\right]$. Thus $\delta(G) \leqq \left[\frac{p-d+1}{2}\right]$ and the theorem follows.

This bound is reached for the graphs C_5, C_6. It is an improvement of the inequality $\delta(G) \leqq \frac{p-1}{2}$ which is proved for v_0-critical graphs in [8]. Finding estimates for other invariants, e.g. for the number of edges, of v-critical graphs seems to be a more difficult problem.

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Von

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Gegeben seien zwei kompakte Intervalle $K=[a, b]$ und $J=[c, d]$ sowie m paarweise verschiedene Punkte $x_1, \dots, x_m$ in K und m Funktionen $h_k \in \mathscr{C}_{\mathbf{R}}(J)$, $k=1, \dots, m$. (Mit $\mathscr{C}_{\mathbf{R}}(K)$ bzw. $\mathscr{C}_{\mathbf{R}}(J)$ bezeichnen wir den Banach-Raum der reellwertigen, auf K bzw. J definierten, stetigen Funktionen unter der Čebyšev-Norm.) Dann wird ein Interpolationsoperator $L_m: \mathscr{C}_{\mathbf{R}}(K) \to \mathscr{C}_{\mathbf{R}}(J)$ definiert durch

$$(L_m f)(x) = \sum_{k=1}^{m} h_k(x) f(x_k)$$

(vgl. R. A. DEVORE [1]). Sind die Funktionen h_k nichtnegativ, so entsteht ein positiver linearer Operator L_m. Für $K=J=[-1, 1]$ erhalten wir z. B. einen positiven Interpolationsoperator, wenn $x_1, \dots, x_m$ die Nullstellen des Jacobi-Polynoms $J_m^{(\alpha, \beta)}$ mit $-1<\alpha \leqq 0$, $-1<\beta \leqq 0$ sind, und wenn $L_m f$ das eindeutig bestimmte Hermite—Fejér-Polynom vom Grad $\leqq 2m-1$ ist, für welches gilt:

$$(L_m f)(x_k) = f(x_k), \quad (L_m f)'(x_k) = 0 \quad (k = 1, \dots, m)$$

(vgl. I. P. NATANSON [3]).

Wir wollen nun zu bestimmten Knoten $x_1, \dots, x_m$ im Intervall $K=J=[-1, 1]$ *den* Interpolationsoperator L_m auf Positivität untersuchen, der dadurch entsteht, daß jedem $f \in \mathscr{C}_{\mathbf{R}}(K)$ das eindeutig bestimmte Polynom vom Grad $\leqq 4m-1$ mit

$$(1) \qquad (L_m f)(x_k) = f(x_k), \quad (L_m f)^{(n)}(x_k) = 0, \quad (k = 1, \dots, m, \quad n = 1, 2, 3)$$

zugeordnet wird. Wie wählen als Knoten $x_1, \dots, x_m$ die Nullstellen des m-ten Čebyšev-Polynoms $T_m(x) = \cos(m \cdot \arccos x)$. Laut M. MÜLLER [2] läßt sich das Polynom $L_m f$ in der Form

$$(L_m f)(x) = \sum_{k=1}^{m} a_k(x) l_k^4(x) f(x_k)$$

darstellen, wo

$$(2) \quad a_k(x) = \frac{1}{6(1-x_k^2)^2}\{6(1-x_k x)^2 + [(4m^2-1)(1-x_k x)-3](x-x_k)^2\} \quad (k=1,\dots,m)$$

und l_k das Lagrange-Grundpolynom mit

$$l_k(x) = \frac{T_m(x)}{T_m'(x_k)(x-x_k)}$$

ist. Es ist leicht zu sehen, daß dieser Operator die Eigenschaften (1) erfüllt. Der folgende Satz zeigt, daß der hier konstruierte Interpolationsoperator positiv ist.

SATZ 1. *Sind $x_1, \dots, x_m$ die Nullstellen des m-ten Čebyšev-Polynoms T_m, so gilt für alle $k=1, \dots, m$ und alle $x \in K=[-1, 1]$:*

$$a_k(x) \geqq \frac{1}{8}.$$

BEWEIS. Wir betrachten die in (2) auftretenden Funktionen, für die folgende Beziehungen erfüllt sind:

$$(1-x_k x)^2 \geqq (x-x_k)^2 \quad \text{für alle} \quad x \in K,$$

$$(4m^2-1)(1-x_k x)-3 \geqq -3 \quad \text{für alle} \quad x \in K.$$

Damit ist für alle $x \in K$

$$6(1-x_k x)^2+\big((4m^2-1)\cdot(1-x_k x)-3\big)(x-x_k)^2 \geqq 3(1-x_k x)^2,$$

also

$$a_k(x) \geqq \frac{(1-x_k x)^2}{2(1-x_k^2)^2} \geqq \frac{(1-x_k)^2}{2(1-x_k^2)^2} = \frac{(1-x_k)^2}{2(1-x_k)^2(1+x_k)^2} = \frac{1}{2(1+x_k)^2} \geqq \frac{1}{8}.$$

Wir wollen nun die Frage untersuchen, ob für die Folge $(L_m f)_{m \in \mathbf{N}}$ von Interpolationspolynomen gilt:

$$\lim_{m \to \infty} \|f-L_m f\| = 0.$$

Dabei deutet $\|.\|$ die Čebyšev-Norm im Raum $\mathscr{C}_{\mathbf{R}}(K)$ an. Wir benutzen eine allgemeine Aussage über die Konvergenzgüte einer Folge positiver linearer Operatoren von $\mathscr{C}_{\mathbf{R}}(K)$ nach $\mathscr{C}_{\mathbf{R}}(J)$ (siehe R. A. DEVORE [1; S.28]). Danach gilt für alle $f \in \mathscr{C}_{\mathbf{R}}(K)$ und alle $x \in K$:

$$|f(x)-(L_m f)(x)| \leqq 2\omega(f, \alpha_m(x)).$$

Hierbei ist ω: $\mathscr{C}_{\mathbf{R}}(K) \times \mathbf{R}_+^* \ni (f, \delta) \mapsto \omega(f, \delta)$ der Stetigkeitsmodul, $\alpha_m^2(x) = \big(L_m\big((\pi_1-x)^2\big)\big)(x)$ und π_1 die Identität auf K. ($\mathbf{R}_+^* = \{r \in \mathbf{R};\ r > 0\}$.)

SATZ 2. *Für alle $f \in \mathscr{C}_{\mathbf{R}}(K)$, alle $m \in \mathbf{N}$ und alle $x \in K$ gilt:*

$$|f(x)-(L_m f)(x)| \leqq 2\omega\left(f, |T_m(x)| \left(2m^{-2}+\frac{2}{3}m^{-3}(m^2-1)T_m^2(x)\right)^{1/2}\right).$$

BEWEIS. Bekanntlich läßt sich das Lagrange-Grundpolynom bezüglich der Nullstellen des m-ten Čebyšev-Polynoms T_m in der Form

$$l_k(x) = \frac{(-1)^{k-1} T_m(x)}{m(x-x_k)} (1-x_k^2)^{1/2}$$

darstellen. Damit ist

$$(L_m f)(x) = m^{-4} \sum_{k=1}^{m} \frac{(1-x_k x)^2}{(x-x_k)^4} T_m^4(x) f(x_k) +$$

$$+m^{-4} \sum_{k=1}^{m} \frac{1}{6}\big((4m^2-1)(1-x_k x)-3\big) \frac{T_m^4(x)}{(x-x_k)^2} f(x_k).$$

Für $\alpha_m^2(x)$ erhalten wir daraus:

$$\alpha_m^2(x) = m^{-2}T_m^2(x)\sum_{k=1}^{m}(1-x_kx)\frac{1-x_kx}{m^2(x-x_k)^2}T_m^2(x)+$$
$$+m^{-4}\sum_{k=1}^{m}\frac{1}{6}\left((4m^2-1)(1-x_kx)-3\right)T_m^4(x).$$

Wegen

$$\sum_{k=1}^{m}\frac{1-x_kx}{m^2(x-x_k)^2}T_m^2(x) = 1 \quad \text{und} \quad \sum_{k=1}^{m}(1-x_kx) = m$$

können wir α_m^2 folgendermaßen abschätzen:

$$\alpha_m^2(x) \leqq 2m^{-2}T_m^2(x)+m^{-4}T_m^4(x)\left(\frac{1}{6}(4m^2-1)m-\frac{1}{2}m\right) =$$
$$= 2m^{-2}T_m^2(x)+\frac{2}{3}m^{-3}(m^2-1)T_m^4(x).$$

Aufgrund der Monotonie des Stetigkeitsmoduls im zweiten Argument folgt daraus die Behauptung.

KOROLLAR. *Für alle $f\in\mathscr{C}_{\mathbf{R}}(K)$ und alle $m\in\mathbf{N}$ gilt:*

$$\|f-L_mf\| \leqq 5\omega\left(f,\frac{1}{\sqrt{m}}\right).$$

BEWEIS. Aus Satz 2 folgt sofort

$$\|f-L_mf\| \leqq 2\omega\left(f,\left(\frac{2}{3}m^{-3}(m^2+3m-1)\right)^{1/2}\right) \leqq$$
$$\leqq 2\omega\left(f,\left(\frac{2}{3}m^{-3}3m^2\right)^{1/2}\right) = 2\omega\left(f,\sqrt{2}m^{-1/2}\right) \leqq 5\omega\left(f,\frac{1}{\sqrt{m}}\right).$$

D. D. STANCU [4] erhält für das Interpolationspolynom L_mf ähnliche Fehlerabschätzungen wie im Korollar.

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RELATIONS BETWEEN SUMMABILITY OF FUNCTIONS AND THEIR FOURIER SERIES

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1. Introduction and announcement of the results

Let the function f have period 1, let f be integrable on [0, 1] and let $c_n = c_n(f)$, $n \in \mathbf{Z}$, be its complex Fourier coefficients. The sequence $(c_n^*)_0^\infty$ is the sequence $(|c_n|)_{-\infty}^{\infty}$ rearranged so that $c_0^* \geqq c_1^* \geqq c_2^* \geqq \dots$ and $f^*(x)$ denotes the function equimeasurable with $|f(x)|$ and nonincreasing.

STEIN [11, p. 490], has proved the following theorem.

THEOREM I. *Let α, β and γ be real numbers such that $-1 < \alpha \leqq \min(\beta-2, 0)$ and $\gamma \leqq \beta$, Then*

$$\left(\sum_{n=1}^{\infty} (c_n^*)^\beta n^\alpha\right)^{1/\beta} \leqq K(\alpha, \beta, \gamma)\Big(\int_0^1 (f^*(x))^\gamma x^{(\beta-\alpha-1)\gamma/\beta-1}\, dx\Big)^{1/\gamma}.$$

The notation $K(\alpha, \beta, \gamma)$ stands for a constant depending at most on α, β and γ. The theorem of Stein contains some well-known results of HAUSDORFF—YOUNG (see [14, Vol II, p. 101]). HARDY—LITTLEWOOD (see [14, Vol II, p. 123]), and PITT (see [9, p. 747]). Furthermore, WIK and the present author, [8, p. 298], have stated.

THEOREM II. *If α is a real number satisfying $-1 < \alpha < -1/2$, and if there exists a nonnegative and continuous function h on $[0, \infty[$ such that $h(t)a^t$ is an increasing function of t for some $a \in \mathbf{R}_+$, $1/h(x)$ is integrable and*

$$\int_0^1 |f(x)|^{-1/\alpha}\big(h(\log^+|f(x)|)^{-1-1/\alpha}\big)\, dx < \infty,$$

then

$$\sum_{n \neq 0} |c_n(f)|\,|n|^\alpha < \infty.$$

Our first purpose is to state a theorem containing Theorems I and II as special cases but first we need a definition. We say that a nonnegative function λ on $[1, \infty]$ belongs to the class $Q(a_0, b_0)$ if, for some real numbers $a < a_0$ and $b > b_0$, $\lambda(t)t^a$ is an increasing and $\lambda(t)\, t^b$ is a decreasing function of t.

THEOREM 1. *Let $\beta > 0$ and let λ be a nonnegative function on $[0, \infty[$ which is constant on [0, 1].*

(a) *If $\lambda \in Q(1, 1-\beta/2)$ then*

$$\sum_{n=0}^{\infty} (c_n^*)^\beta \lambda(n) \leqq K(\beta, \lambda) \int_0^1 (f^*(x))^\beta x^{\beta-2} \lambda\left(\frac{1}{x}\right) dx. \tag{1.1}$$

(b) *If* $\lambda \in Q(1-\beta/2,\ 1-\beta)$ *and if* c_n *are complex numbers such that* $c_n \to 0$ *as* $n \to \pm\infty$ *and* $\sum_{n=0}^{\infty} (c_n^*)^\beta \lambda(n) < \infty$, *then there exists a function* f *with* c_n *as Fourier coefficients and such that*

$$\int_0^1 (f^*(x))^\beta x^{\beta-2} \lambda\left(\frac{1}{x}\right) dx \leqq K(\beta, \lambda) \sum_{n=0}^{\infty} (c_n^*)^\beta \lambda(n). \tag{1.2}$$

We note that

(i) if $\gamma \leqq \beta$ then

$$\Big(\int_0^1 (f^*(x))^\beta x^{\beta-\alpha-2}\, dx\Big)^{1/\beta} \leqq K\left(\sum_{l=1}^{\infty} (f^*(2^{-l}))^\beta 2^{l(1+\alpha-\beta)}\right)^{1/\beta} \leqq$$

$$\leqq K\left(\sum_{l=1}^{\infty} (f^*(2^{-l}))^\gamma 2^{l(1+\alpha-\beta)\gamma/\beta}\right)^{1/\gamma} \leqq K\Big(\int_0^1 (f^*(x))^\gamma x^{(\beta-\alpha-1)\gamma/\beta-1}\, dx\Big)^{1/\gamma}$$

and, thus, according to the fact that Theorem 1(a) can be applied with $\lambda(t) = t^\alpha$, $-1 < \alpha < \beta/2 - 1$, we find that Theorem I holds in a wider range of parameters than stated.

(ii) by applying Theorem 1(a) with $\beta = 1$ and $\lambda(t) = t^\alpha$, $-1 < \alpha - 1/2$, and by using our Theorem 7 we see that Theorem 1 generalizes Theorem II.

(iii) Theorem 1(a) does not hold in general for example when $\lambda(t) = t^a, \alpha > \beta/2 - 1$ or $\alpha = \beta/2 - 1$, $0 < \beta < 2$ (see [10, Ch IV:6] or [14, Vol I, p. 225]) and Theorem 1(b) fails when $\lambda(t) = t^\alpha$, $\alpha < \beta/2 - 1$ or $\alpha = \beta/2 - 1, \beta > 2$ (see [4, p. 41] or [14, Vol I, p. 215]).

THEOREM 2. *Let* $\beta > 0$ *and let* λ *be a nonnegative function on* $[0, \infty[$ *which is constant on* $[0, 1]$ *and* $\lambda \in Q(1, 1-\beta)$.

(a) *If* f *is a nonnegative function on* $[0, 1]$ *such that*

$$f^*(x) \leqq A x^{-1} \int_0^x f(t)\, dt,$$

then (1.2) *holds.*

(b) *If* f *is nonnegative, even and nonincreasing in* $[0, 1/2]$ *then* (1.1) *is satisfied.*

(c) *If* (c_n), $n \in \mathbf{Z}$, *are nonnegative and if*

$$c_n^* \leqq A\Big(n^{-1} \sum_{|k| \leqq n} c_k\Big)$$

then (1.1) *holds.*

(d) *If* $a_1 \geqq a_2 \geqq \ldots \to 0$ *and if* $f(x) = \sum_{n=1}^{\infty} a_n \cos 2\pi n x$ *then* (1.2) *is satisfied.*

In particular Theorem 2 contains the fact that if $\beta > 0$ and if $\lambda \in Q(1, 1-\beta)$ then the conditions

$$\int_0^1 (f^*(x))^\beta x^{\beta-2} \lambda\left(\frac{1}{x}\right) dx < \infty \quad \text{and} \quad \sum_1^{\infty} (c_n^*)^\beta \lambda(n) < \infty \tag{1.3}$$

are equivalent in the classes of functions considered in (b) and (d). By analysing the proof of Theorem 2 it is easy to deduce that if $\beta \geqq 1$, then, in the same classes of functions, also the conditions

$$\int_0^1 |f(x)|^\beta x^{\beta-2} \lambda\left(\frac{1}{x}\right) dx < \infty \quad \text{and} \quad \sum_{-\infty}^{\infty} |c_n|^\beta \lambda(|n|) < \infty$$

are equivalent to the conditions in (1.3). Thus Theorem 2 can be considered as a generalization of some well-known theorems by SZ.-NAGY [7, p. 121] ($\beta=1$) and BOAS [1, p. 35] ($\beta>1$).

By combining Theorem 1 with Theorem 2(a) and (b) we obtain

COROLLARY 3. *Let $\beta>0$, let g be an integrable function on* $[0, 1]$ *and let λ be a nonnegative function on* $[1, \infty[$.

(i) *If $\lambda \in Q(1, 1-\beta/2)$ then*

$$\sum_{n=1}^{\infty} (c_n^*(f))^\beta \lambda(n) < \infty \tag{1.4}$$

for every function f which is equidistributed with g if and only if

$$\int_0^1 (g^*(x))^\beta x^{\beta-2} \lambda\left(\frac{1}{x}\right) dx < \infty. \tag{1.5}$$

(ii) *If $\lambda \in Q(1-\beta/2, 1-\beta)$ then* (1.4) *holds for some function f which is equidistributed with g if and only if* (1.5) *is satisfied.*

For the case $\lambda(t)\equiv 1$ Corollary 3 reduces to a well-known theorem by HARDY and LITTLEWOOD (see [2, p. 9] or [14, Vol II, p. 130]). We also note that by applying Corollary 3 with $\lambda(t)=t^\alpha$, $-1<\alpha<\beta-1$, we obtain a result which is similar to what HUNT [3, p. 270] has proved for Fourier transforms on $\mathbf{R}$.

The following dual version of Corollary 3 follows by combining Theorem 1 with Theorem 2(c) and (d).

COROLLARY 4. *Let $\beta>0$, let λ be a nonnegative function on* $[1, \infty[$, *and let c_n, $n \in \mathbf{Z}$, be complex numbers such that $c_n \to 0$ as $n \to \pm\infty$.*

(i) *If $\lambda \in Q(1-\beta/2,\ 1-\beta)$ then a necessary and sufficient condition that (c_n) should be, for every variation of their arguments and arrangement, the Fourier coefficients of a function f satisfying*

$$\int_0^1 (f^*(x))^\beta x^{\beta-2} \lambda\left(\frac{1}{x}\right) dx < \infty \tag{1.6}$$

is that

$$\sum_{n=1}^{\infty} (c_n^*)^\beta \lambda(n) < \infty. \tag{1.7}$$

(ii) *If $\lambda \in Q(1, 1-\beta/2)$ then a necessary and sufficient condition that (c_n) should be, for some variation of their arguments and arrangement, the Fourier coefficients of a function f satisfying* (1.6) *is that* (1.7) *holds.*

In particular when $\lambda(t)=t^{\beta-2}$ we obtain a well-known result of HARDY and LITTLEWOOD (see [2, p. 3] or [14, Vol II, p. 128]).

Next we observe that it is easy to find examples showing that Theorem 1(a) ceases to be true for instance when $\lambda(t)=t^{-1}$ (see e.g. ZYGMUND [14. Vol I, p. 189]). We shall now state a theorem for $\beta\geqq 1$ which contains this exceptional case.

THEOREM 5. *Let $\beta\geqq 1$ and let μ be a nonnegative function on* $[1, \infty[$ *which is constant on* $[1, 2]$ *and, for some real number b_0, $\mu(2^t)\in Q(1, b_0)$. Then*

$$\sum_{n=1}^{\infty}(c_n^*)^\beta n^{-1}\mu(n) \leqq K(\beta,\mu)\int_0^1 (f^*(x))^\beta(\log^+ f^*(x))^\beta \mu(f^*(x))x^{\beta-1}\,dx + K(\beta,\mu).$$

By choosing $\mu(t)=(\log t)^\delta$, $\delta>-1$, we obtain the following generalization of a well-known estimate due to ZYGMUND (see [13, p. 297] and [14, Vol II, p. 158])

$$\sum_{1}^{\infty}(c_n^*)^\beta n^{-1}(\log(n+1))^\delta \leqq K(\beta,\delta)\int_0^1 (f^*(x))^\beta(\log^+ f^*(x))^{\beta+\delta}x^{\beta-1}\,dx + K(\beta,\delta).$$

According to elementary examples we see that the exponent $\beta+\delta$ on the logarithm is the best possible and that the inequality does not hold for $\delta=-1$ (see [14, Vol I, p. 189]) but by following the proof of Theorem 5 we can deduce

$$\sum_{n=1}^{\infty}(c_n^*)^\beta (n\log(n+1))^{-1} \leqq$$
$$\leqq K(\beta)\int_0^1 (f^*(x))^\beta(\log^+ f^*(x))^{\beta-1}(\log^+\log^+ f^*(x))^\beta x^{\beta-1}\,dx + K(\beta).$$

The exponent β on $\log^+\log^+$ can not be replaced by any smaller number (see [14, Vol I, p. 189]).

For the sake of completeness we shall also state a theorem which, in a way, is dual to Theorem 5.

THEOREM 6. *Let $\beta\geqq 1$ and let μ be a positive function on* $[0, \infty[$ *which is constant on* $[0, 2]$ *and, for some real number a_0, $\mu(2^t)\in Q(a_0, 1)$. Then*

$$\int_0^1 (f^*(x))^\beta x^{-1}\mu\left(\frac{1}{x}\right)dx \leqq K(\beta,\mu)\sum_{n=0}^{\infty}(c_n^*)^\beta\left(\log^+\frac{1}{c_n^*}\right)^\beta \mu\left(\frac{1}{c_n^*}\right)(n+1)^{\beta-1} + K(\beta,\mu).$$

To see that the exponent β on the logarithm is the best possible suitable examples can be found in [14, Vol I p. 188].

In our next theorem we shall characterize a special case of the Lorentz spaces (for the definition see [6, p. 37]).

THEOREM 7. *Let α and β be real numbers such that $\beta>0$ and $\beta-2<\alpha<\beta-1$ and let f be a measurable function on* $[0, 1]$. *Then the following conditions are equivalent:*

I. $\int_0^1 (f^*(x))^\beta x^{\beta-\alpha-2}\,dx < \infty.$

II. *There exists a nonnegative and continuous function h on* $[0, \infty]$ *such that*
(a) $h(t)\,a^t$ *is a decreasing or an increasing function of t for some* $a \in \mathbf{R}_+$.

(b) $\int_0^\infty \frac{1}{h(t)}\,dt < \infty$

and

(c) $\int_0^1 |f(x)|^{\frac{\beta}{\beta-\alpha-1}} \left(h(\log^+|f(x)|)\right)^{\frac{2-\beta+\alpha}{\beta-\alpha-1}} dx < \infty.$

If $\beta > 0$ and $-1 < \alpha < \beta/2 - 1$ then it is easy to deduce from Hausdorff—Young's theorem and elementary inequalities that the series $\sum_1^\infty (c_n^*(f))^\beta n^\alpha$ converges if $f \in L^p[0, 1]$ for some $p > \beta/(\beta-\alpha-1)$. (For the case $\alpha \leqq \beta - 2$ we can even have $p = \beta/(\beta-\alpha-1)$.) By combining Theorem 1 (a) and Theorem 7 we obtain a refinement of this statement namely, if $\beta > 0$, $-1 < \alpha < \beta/2 - 1$, and $\alpha \geqq \beta - 2$ and if there exists a function h satisfying the conditions in II then the series $\sum_1^\infty (c_n^*)^\beta n^\alpha$ is convergent. This statement is sharp in the following sense.

THEOREM 8. *Let α and β be real numbers satisfying $\beta > 0$ and* $\max(\beta-2, -1) < \alpha < \beta - 1$ *and let h be a nonnegative continuous function on* $[0, \infty[$ *such that, for some* δ, $0 < \delta < \beta$, $h({}^2\log^+ t) \in Q(\delta, -\delta)$ *and* $\int_0^\infty \frac{1}{h(t)}\,dt = \infty$. *Then there exists a nonnegative function f such that*

$$\int_0^1 |f(x)|^{\frac{\beta}{\beta-\alpha-1}} \left(h(\log^+|f(x)|)\right)^{\frac{2-\beta-\alpha}{\beta-\alpha-1}} dx < \infty$$

but

$$\sum_{n \neq 0} |c_n|^\beta |n|^\alpha = \infty.$$

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2. Proofs of the theorems

We need the following lemmas.

LEMMA 1. *Let $(a_l)_0^\infty$ and $(b_l)_0^\infty$ be positive sequences and let $p \geqq 1$. Then*

$$\sum_{l=0}^\infty a_l \left(\sum_{n=0}^l b_n\right)^p \leqq p^p \sum_{l=0}^\infty a_l^{1-p} \left(\sum_{n=l}^\infty a_n\right)^p b_l^p \tag{2.1}$$

and

$$\sum_{l=0}^\infty a_l \left(\sum_{n=l}^\infty b_n\right)^p \leqq p^p \sum_{l=0}^\infty a_l^{1-p} \left(\sum_{n=0}^l a_n\right)^p b_l^p. \tag{2.2}$$

PROOF. A proof can be found in [5, p. 279].

LEMMA 2. *Let δ be a positive number and let g be a positive, integrable function on $[0, 1[$ such that $g(x)x^a$ is decreasing or increasing for some real number a. Then there exists an integrable function $g_1(x) \geqq g(x)$ such that $x^{1+\delta}g_1(x)$ is increasing and $x^{1-\delta}g_1(x)$ is decreasing.*

LEMMA 3. *Let δ be a positive number and let g be a positive, integrable function on $[0, \infty[$ such that $a^x g(x)$ is decreasing or increasing for some $a>0$. Then there exists an integrable function $g_2(x) \geqq g(x)$ such that $2^{\delta x}g_2(x)$ is nondecreasing and $2^{-\delta x}g_2(x)$ is nonincreasing.*

PROOF. Proofs of Lemmas 2 and 3 are given in [8, p. 293—294] but the proofs can be carried out even simpler by convolutions.

PROOF OF THEOREM 1(a). For $l=0, 1, 2, \ldots$ we put $f^*(2^{-l})=N_l$. Let E_l, $l=0, 1, 2, \ldots$, be measurable sets with $m(E_l)=2^{-l}$ and $\{x \,||\, f(x)|>N_l\} \subset E_l \subset \{x \,||\, f(x)| \geqq N_l\}$. We put

$$f_l(x) = \begin{cases} f(x) & \text{on } E_l \\ 0 & \text{elsewhere.} \end{cases}$$

For $n=0, \pm 1, \pm 2, \ldots$ we have

$$(2.3) \qquad |c_n(f_l)| = \left|\int_0^1 f_l(x)e^{-in2\pi x}\,dx\right| \leqq \int_0^1 |f_l(x)|\,dx = \int_0^{2^{-l}} f^*(x)\,dx.$$

Moreover, by Parseval's relation,

$$(2.4) \qquad \sum_{n=-\infty}^{\infty} |c_n(f-f_l)|^2 = \int_0^1 |f(x)-f_l(x)|^2\,dx = \int_{2^{-l}}^1 (f^*(x))^2\,dx.$$

We combine (2.3) and (2.4) and obtain

$$(2.5) \qquad \sum_{n=2^l}^{2^{l+1}} (c_n^*(f))^2 \leqq 2\sum_{n=2^l}^{2^{l+1}} |c_n^*(f-f_l)|^2 + 2\sum_{n=2^l}^{2^{l+1}} |c_n^*(f_l)|^2 \leqq$$

$$\leqq 2\int_{2^{-l}}^1 (f^*(x))^2\,dx + 2^{l+1}\Big(\int_0^{2^{-l}} f^*(x)\,dx\Big)^2.$$

From now on by K we shall mean a constant depending at most on β and λ but not necessarily the same on all occasions. By using (2.5) and the growth properties of λ, f^* and (c_n^*) we find after some calculations

$$(2.6) \qquad \sum_{2^{l+1}}^{2^{l+2}} (c_n^*(f))^\beta \lambda(n) \leqq K\left\{2^{(1-\beta/2)l}\lambda(2^l)\left(\sum_{n=0}^{l} (f^*(2^{-n}))^2 2^{-n}\right)^{\beta/2} + \right.$$

$$\left. + 2^l \lambda(2^l)\left(\sum_{n=l}^{\infty} (f^*(2^{-n})2^{-n}\right)^{\beta}\right\}.$$

We put, for $l=0, 1, 2, \ldots$, $a_l=\lambda(2^l)2^{l(1-\beta/2)}$ and $b_l=(f^*(2^{-l}))^2 2^{-l}$. According to the hypothesis $\lambda \in Q(-, 1-\beta/2)$ there exists a real number b, $b>1-\beta/2$, so that the terms of the sequence $(a_l)_0^\infty$ decrease with a ratio at least as large as that of a geometric sequence with ratio $2^{1-\beta/2-b}$. Therefore, by applying (2.1) in Lemma 1, we find, for $\beta \geqq 2$,

$$(2.7) \qquad \sum_{l=0}^{\infty} \lambda(2^l) 2^{l(1-\beta/2)} \left[\sum_{n=0}^{l} (f^*(2^{-n}))^2 2^{-n} \right]^{\beta/2} \leqq K \sum_{l=0}^{\infty} (f^*(2^{-l}))^\beta \lambda(2^l) 2^{l(1-\beta)}.$$

For $\beta < 2$ we first make the trivial estimate

$$\left(\sum_{n=0}^{l} b_n \right)^{\beta/2} \leqq \sum_{n=0}^{l} b_n^{\beta/2}$$

and then we apply (2.1) with $p=1$ to obtain (2.7). Analogously, by the assumption $\lambda \in Q(1, -)$, we see that, for $l=0, 1, 2, \ldots$

$$\frac{\lambda(2^{l+1}) 2^{l+1}}{\lambda(2^l) 2^l} \geqq K > 1,$$

and, thus, by (2.2)

$$(2.8) \qquad \sum_{l=0}^{\infty} 2^l \lambda(2^l) \left(\sum_{n=l}^{\infty} f^*(2^{-n}) 2^{-n} \right)^\beta \leqq K \sum_{l=0}^{\infty} (f^*(2^{-l}))^\beta \lambda(2^l)\, 2^{l(1-\beta)}.$$

As it is easy to verify that for every n and $\lambda \in Q(1, b)$, $b \in \mathbf{R}$,

$$c_n^* \leqq K \left(\int_0^1 (f^*(x))^\beta x^{\beta-2} \lambda\left(\frac{1}{x}\right) dx \right)^{1/\beta}$$

we can use (2.6)—(2.8) and to find

$$\sum_{n=0}^{\infty} (c_n^*)^\beta \lambda(n) \leqq K \sum_{l=0}^{\infty} (f^*(2^{-l}))^\beta \lambda(2^l) 2^{l(1-\beta)}.$$

This inequality is equivalent to (1.1) so the proof is complete.

PROOF OF THEOREM 1(b). The hypothesis $\sum_0^\infty (c_n^*)^\beta \lambda(n) < \infty$, $\lambda \in Q(1-\beta/2, 1-\beta)$, implies that $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$ and the existence of the function f is clear. By a personal communication I. Wik has pointed out to me that the basic estimate (2.5) in the proof of Theorem 1(a) can be generalized to every local compact Abelian group $\mathbf{G}$ in the following way. For every integrable function f on $\mathbf{G}$ we define the function f^* on $[0, \infty[$ as the inverse of the distribution function, modified in a suitable way. Analogously, for the Fourier transform on the dual group $\hat{\mathbf{G}}$ we define $\hat{f}^*$ on $[0, \infty[$. By arguing as in the proof above we find, for $l \in \mathbf{Z}$,

$$(\hat{f}^*(2^{-l}))^2 \leqq 2\left(\left(\sum_{n=-\infty}^{l} 2^n f^*(2^n) \right)^2 + 2^l \sum_{l}^{\infty} 2^n (f^*(2^n))^2 \right).$$

In particular, for the case $\mathbf{G}=\mathbf{Z}$, $\hat{\mathbf{G}}=\mathbf{R}/2\pi z$ we obtain, for $l=0, 1, 2, \ldots$

$$(f^*(2^{-l}))^2 \leqq 2\left(\sum_{n=0}^{l} c_{2^n}^* 2^n\right)^2 + 2^{l+1}\sum_{n=l}^{\infty}(c_{2^n}^*)^2 2^n.$$

Hence

$$(f^*(2^l))^\beta 2^{l(1-\beta)}\lambda(2^l) \leqq K\left(2^{l(1-\beta)}\lambda(2^l)\left(\sum_{n=0}^{l} c_{2^n}^* 2^n\right)^\beta + 2^{l(1-\beta/2)}\lambda(2^l)\left(\sum_{n=l}^{\infty}(c_{2^n}^*)^2 2^n\right)^{\beta/2}\right).$$

By this estimate, Lemma 1 and the condition $\lambda\in Q(1-\beta/2, 1-\beta)$ the rest of the proof can be carried out just as in the proof of (a).

Proof of Theorem 2(a). The Fejér kernel

$$F_N(t) = \sum_{|k|\leqq N}\left(1-\frac{|k|}{N+1}\right)e^{2\pi ikt} = \left(\frac{\sin\pi(N+1)t}{\sin\pi t}\right)^2\frac{1}{N+1}$$

satisfies, for $0<|t|\leqq 1/2(N+1)$,

$$F_N(t) \geqq \left(\frac{2(N+1)t}{\pi t}\right)^2\frac{1}{N+1} = \left(\frac{2}{\pi}\right)^2(N+1).$$

Therefore, by the assumption $f\geqq 0$,

$$\left(\frac{2}{\pi}\right)^2(N+1)\int_0^{1/2(N+1)} f(t)\,dt \leqq F_N * f(0) \leqq \sum_{|k|\leqq N}|c_k|$$

and we find

$$\int_0^1\left(\int_0^x f(t)\,dt\right)^\beta x^{-2}\lambda\left(\frac{1}{x}\right)dx \leqq K\sum_{l=0}^{\infty}\left(\int_0^{2^{-l}} f(t)\,dt\right)^\beta 2^l\lambda(2^l) \leqq \tag{2.9}$$

$$\leqq K\sum_{l=0}^{\infty}\left(\sum_{|k|\leqq 2^l} c_k\right)^\beta 2^{l(1-\beta)}\lambda(2^l) \leqq K\sum_{l=0}^{\infty}\left(\sum_{k=0}^{2^l} c_k^*\right)^\beta 2^{l(1-\beta)}\lambda(2^l).$$

Since $\lambda\in Q(1, 1-\beta)$ we have, for $l=0, 1, 2, \ldots,$

$$\frac{2^{(l+1)(1-\beta)}\lambda(2^{l+1})}{2^{l(1-\beta)}\lambda(2^l)} \geqq K > 1$$

and, thus, we can apply (2.1) to obtain, for $\beta\geqq 1$,

$$\sum_{l=0}^{\infty}\left(\sum_{k=0}^{2^l} c_k^*\right)^\beta 2^{l(1-\beta)}\lambda(2^l) \leqq K\sum_{l=1}^{\infty}\left(\sum_{2^{l-1}}^{2^l} c_k^*\right)^\beta 2^{l(1-\beta)}\lambda(2^l) + \tag{2.10}$$

$$+(c_0^*)^\beta\lambda(1) \leqq K\sum_{n=0}^{\infty}(c_n^*)^\beta\lambda(n).$$

When $\beta<1$ we first make the estimate

$$\left(\sum_0^{2^l} c_k^*\right)^\beta \leqq \sum_{n=0}^{l}\left(\sum_{2^{n-1}}^{2^n} c_k^*\right)^\beta + (c_0^*)^\beta$$

and then we apply (2.1) with $p=1$ to obtain (2.10). According to the hypothesis $f^*(x)\leqq Ax^{-1}\int_0^x f(t)\,dt$ the desired result follows by combining (2.9) and (2.10).

PROOF OF THEOREM 2(b). Following ZYGMUND [14, Vol II, p. 130], we have, for $n\neq 0$,

$$|c_n|\leqq A\int_0^{1/2|n|} f(x)\,dx.$$

The sequence $\Big(\int_0^{1/2n} f(x)\,dx\Big)_1^\infty$ is nonincreasing and, thus, for $n>0$,

$$c_n^*\leqq A\Big(\int_0^{1/2n} f(x)\,dx\Big)^* = A\int_0^{1/2n} f(x)\,dx = A\int_0^{1/2n} f^*(x)\,dx.$$

Therefore, by the hypothesis $\lambda\in Q(1,1-\beta)$, we can, analogously to the proof of (a), apply (2.2) in Lemma 1 and complete the proof with the following estimates

$$\sum_{n=1}^\infty (c_n^*)^\beta\lambda(n)\leqq K\sum_{l=0}^\infty (c_{2^l}^*)^\beta\lambda(2^l)2^l\leqq K\sum_{l=0}^\infty \lambda(2^l)2^l\left(\sum_{n=l}^\infty f^*(2^{-n})2^{-n}\right)^\beta\leqq$$

$$\leqq K\sum_{l=0}^\infty (f^*(2^{-l}))^\beta 2^{l(1-\beta)}\lambda(2^l)\leqq K\int_0^1 (f^*(x))^\beta x^{\beta-2}\lambda\left(\frac{1}{x}\right)dx.$$

PROOF OF THEOREM 2(c). Consider the triangular kernels, $P_N(x)$, defined by $P_N(0)=2N$, $P_N(x)=0$ for $|x|>1/2N$ and linear elsewhere. An easy calculation shows that

$$c_k(P_N)=\begin{cases}\dfrac{4N^2}{\pi^2k^2}\left(\sin\dfrac{\pi k}{2N}\right)^2 & \text{for } k\neq 0\\ \quad 1 & \text{for } k=0.\end{cases}$$

Therefore, by Parseval's relation, we obtain

$$\int_{-1/2}^{1/2} f(t)\,P_N(t)\,dt=\frac{4N^2}{\pi^2}\sum_{-\infty}^{\infty} c_k k^{-2}\left(\sin\frac{\pi k}{2N}\right)^2,$$

where the term corresponding to $k=0$ beeing understood to mean c_0. Hence

$$\sum_{|k|\leqq N} c_k\leqq\frac{\pi^2}{4}\int_{-1/2}^{1/2} f(t)P_N(t)\,dt\leqq\frac{\pi^2}{2}N\int_{-1/2N}^{1/2N} f(t)\,dt\leqq\frac{\pi^2}{2}N\int_0^{1/N} f^*(t)\,dt.$$

According to the hypothesis

$$c_n^*\leqq An^{-1}\sum_{|k|\leqq n}|c_k|$$

and $\lambda\in Q(1,1-\beta)$ we can argue just as in the proof of (b) to complete the proof.

PROOF OF THEOREM 2(d). Following ZYGMUND [14, Vol II, p. 129], we observe that, in the class of functions considered,

$$|f(x)| \leqq 2h(x) \quad \text{where} \quad h(x) = \sum_{1}^{2^l} a_n \quad \text{for} \quad 2^{-l-1} \leqq x < 2^{-l}, \quad l = 1, 2, \ldots.$$

The function h is nonincreasing and therefore

$$f^*(2^{-l}) \leqq 2 \sum_{n=1}^{2^l} a_n \leqq 2 \sum_{n=0}^{2^l} c_n^*.$$

Since we have assumed that $\lambda \in Q(1, 1-\beta)$ the theorem follows from this estimate and (2.10).

PROOF OF THEOREM 5. The condition $\lambda(2^t) \in Q(1, b_0)$ implies that there exist real numbers a, $a<1$, and b such that, for $2^{m-1} \leqq n \leqq 2^m$, $m=1, 2, \ldots$,

$$\min(2^b, 1)\mu(2^{2^m}) \leqq \mu(2^n) \leqq \max(2^a, 1)\mu(2^{2^m}) \tag{2.11}$$

and

$$\mu(2^n) \geqq \min(2^{-a}, 1)\mu(2^{2^{m-1}}). \tag{2.12}$$

In particular we see, by (2.12), that the terms of the sequence $(2^m \mu(2^{2^m}))$ increase with a ratio at least as large as that of some geometric sequence with ratio $\min(2^{1-a}, 1) > 1$. Therefore, by (2.11), we find, for $2^{s-1} \leqq l \leqq 2^s$, $s=1, 2, \ldots$,

$$\sum_{1}^{l} \mu(2^n) \leqq \sum_{1}^{2^s} \mu(2^n) \leqq \sum_{m=1}^{s} \sum_{2^{m-1}}^{2^m} \mu(2^n) \leqq K \sum_{m=1}^{s} 2^m \mu(2^{2^m}) \leqq \tag{2.13}$$

$$\leqq K 2^s \mu(2^{2^s}) \leqq K l \mu(2^l).$$

An application of (2.6) and (2.7) with $\lambda(t) = \mu(t)/t$ shows that

$$\sum_{n=1}^{\infty} (c_n^*)^\beta \mu(n)/n \leqq K \int_0^1 \Big(\int_0^x f^*(t)\,dt\Big)^\beta x^{-1} \mu\left(\frac{1}{x}\right) dx. \tag{2.14}$$

Now we use the growth properties of μ, (2.2), (2.13) and (2.14) and obtain

$$\sum_{n=1}^{\infty} (c_n^*)^\beta \mu(n)/n \leqq K \sum_{l=0}^{\infty} \mu(2^l) \left(\sum_{n=l}^{\infty} f^*(2^{-n}) 2^{-n}\right)^\beta \leqq \tag{2.15}$$

$$\leqq K \sum_{l=0}^{\infty} (f^*(2^{-l}))^\beta 2^{-l\beta} \left(\sum_{n=0}^{l} \mu(2^n)\right)^\beta (\mu(2^l))^{1-\beta} \leqq$$

$$\leqq K \sum_{l=0}^{\infty} (f^*(2^{-l}))^\beta 2^{-l\beta} l^\beta \mu(2^l) \leqq K \int_0^1 (f^*(x))^\beta \left(\log \frac{1}{x}\right)^\beta \mu\left(\frac{1}{x}\right) x^{\beta-1}\,dx.$$

We put $M_1 = \{x \mid 0 \leqq x \leqq 1,\ f^*(x) \geqq 1/x\}$, $M_2 = \{x \mid 0 \leqq x \leqq 1, f^*(x) \leqq x^{-1/2}\}$ and $M_3 = [0, 1] \setminus M_1 \cup M_2$. The function $(\log t)^\beta \mu(t)$ is increasing and, thus, for $x \in M_1$,

$$\left(\log \frac{1}{x}\right)^\beta \mu\left(\frac{1}{x}\right) \leqq (\log f^*(x))^\beta \mu(f^*(x)). \tag{2.16}$$

Moreover, for $x \in M_3$,

$$\frac{1}{2}\log\frac{1}{x} \leqq \log f^*(x) \leqq \log\frac{1}{x}. \tag{2.17}$$

Therefore the condition $\lambda \in Q(-, b_0)$ implies that there exists b such that

$$\mu\left(\frac{1}{x}\right) \leqq \left(\frac{\log f^*(x)}{\log\frac{1}{x}}\right)^b \mu(f^*(x)) \leqq \max(2^{-b}, 1)\mu(f^*(x)). \tag{2.18}$$

Finally we observe that the contribution to the integral in (2.15) from the set M_2 is less than a constant (not depending on f) so the proof follows when combining (2.15)—(2.18).

Proof of Theorem 6. Since $\lambda(2^t) \in Q(a_0, 1)$ the inequality

$$\sum_l^\infty \mu(2^n) \leqq Kl\mu(2^l)$$

follows by using similar arguments as in the proof of Theorem 5. Furthermore, from the proof of Theorem 1 (b) it is easy to deduce

$$\int_0^1 (f^*(x))x^{-1}\mu\left(\frac{1}{x}\right)dx \leqq K\sum_{l=0}^\infty \Big(\sum_{|n|\leqq l} c_n^*\Big)^\beta (l+1)^{-1}\mu(l).$$

In view of these estimates and (2.1) the proof is similar to the proof of Theorem 5 so we omit the details.

Proof of Theorem 7. First we assume that II is satisfied. We choose δ, $0<\delta<\beta/(2-\beta+\alpha)$, and apply Lemma 3 to the function $g(x)=\frac{1}{h(x)}$. The modified function $h_2(x)=\frac{1}{g_2(x)}$ satisfies

(a_1) $h_2(x)2^{\delta x}$ is nondecreasing, and

(a_2) $h_2(x)2^{-\delta x}$ is nonincreasing.

It is easily seen that our modified function satisfies (b) and (c) too, so we can, without loss of generality, assume that our function h satisfies (a_1) and (a_2). We observe that the choice of δ makes the function $t^\beta(h(\log t))^{2-\beta+\alpha}$ increasing on $[1, \infty]$. Therefore (c) implies that the series

$$\sum_{k=1}^\infty \left(2^{k\beta}(h(k))^{2-\beta+\alpha}\right)^{\frac{1}{\beta-\alpha-1}}(\alpha_k-\alpha_{k-1}) \tag{2.19}$$

converges, where α_k, $k=1, 2, \ldots$, are the least real numbers such that $f^*(\alpha_k)=2^k$. We make Abelian transformation on the series (2.19) to find that this series converges exactly when

$$\sum_{k=1}^\infty \left(2^{k\beta}(h(k))^{2-\beta+\alpha}\right)^{\frac{1}{\beta-\alpha-1}}\alpha_k < \infty. \tag{2.20}$$

We also note that

$$(2.21)\quad \int_0^1 (f^*(x))^\beta x^{\beta-\alpha-2}\,dx \leqq \sum_{k=1}^{\infty} 2^{k\beta} \int_{\alpha_{k-1}}^{\alpha_k} x^{\beta-\alpha-2}\,dx + A \leqq A \sum_{k=1}^{\infty} 2^{k\beta}\alpha_k^{\beta-\alpha-1} + A$$

and, by Hölder's inequality,

$$(2.22)\quad \sum_{k=1}^{\infty} 2^{k\beta}\alpha_k^{\beta-\alpha-1} \leqq \left(\sum_{k=1}^{\infty} \left(2^{k\beta}(h(k))^{2-\beta+\alpha}\right)^{\frac{1}{\beta-\alpha-1}}\alpha_k\right)^{\beta-\alpha-1} \left(\sum_{k=1}^{\infty} \frac{1}{h(k)}\right)^{2-\beta+\alpha}.$$

The growth and integrability conditions on h imply that the series $\sum_{k=1}^{\infty} \frac{1}{h(k)}$ converges and, thus, by (2.20)—(2.22), we see that I is satisfied. On the other hand, if I holds, and if ε is a real number such that $0<\varepsilon<\beta-\alpha-1$, then according to Lemma 2, there exists a function $g_1 \geqq 1$ such that $f^*(x) \leqq g_1(x)$,

$$(2.23)\quad x^{\beta-\alpha-1+\varepsilon}(g_1(x))^\beta \text{ is nondecreasing,}$$

$$(2.24)\quad x^{\beta-\alpha-1-\varepsilon}(g_1(x))^\beta \text{ is nonincreasing}$$

and

$$\int_0^1 (g_1(x))^\beta x^{\beta-\alpha-2}\,dx < \infty.$$

The last condition is equivalent to

$$(2.25)\quad \sum_{k=1}^{\infty} (g_1(2^{-k}))^\beta 2^{k(1-\beta+\alpha)} < \infty.$$

We put $a_k = g_1(2^{-k})2^{k(1-\beta+\alpha)}$ and define the function h at the points $x_k = \log g_1(2^{-k})$ by $h(x_k) = 1/a_k$, $k=0, 1, 2, \ldots$. It follows from (2.23) and (2.24) that

$$g_1(2^{-k})2^{\frac{(\beta-\alpha-1-\varepsilon)t}{\beta}} \leqq g_1(2^{-k-t}) \leqq g_1(2^{-k})2^{\frac{(\beta-\alpha-1+\varepsilon)t}{\beta}}, \quad 0 \leqq t \leqq 1.$$

Therefore

$$(2.26)\quad 0 < \frac{\beta-\alpha-1-\varepsilon}{\beta} \leqq x_{k-1} - x_k \leqq \frac{\beta-\alpha-1+\varepsilon}{\beta}$$

and

$$2^{-\varepsilon} \leqq \frac{h(x_k)}{h(x_{k+1})} = \frac{g_1^\beta(2^{-k-1})}{g_1^\beta(2^{-k})2^{\beta-\alpha-1}} \leqq 2^\varepsilon.$$

Thus h can be extended to a continuous function, defined for $x \geqq 0$, and satisfying (a_1) and (a_2) with $\delta = \varepsilon\left(\beta/(\beta-1-\alpha+\varepsilon)+1\right)$. By (2.25) the series

$$\sum_{k=1}^{\infty} \frac{1}{h(x_k)} = \sum_{k=1}^{\infty} a_k$$

is convergent and from the right hand inequality of (2.26) and the growth properties of h we conclude that $\frac{1}{h(x)}$ is integrable so (b) is satisfied. In view of the growth properties of g_1 and h we find

$$\int_0^1 (g_1(x))^{\frac{\beta}{\beta-\alpha-1}} (h(\log^+ g_1(x)))^{\frac{2-\beta+\alpha}{\beta-\alpha-1}} dx \leqq A \sum_{k=0}^{\infty} (g_1(2^{-k}))^{\frac{\beta}{\beta-\alpha-1}} (h(x_k))^{\frac{2-\beta+\alpha}{\beta-\alpha-1}} 2^{-k} =$$
$$= A \sum_{k=0}^{\infty} a_k.$$

(We can for example choose $A=2^{\frac{\varepsilon\beta}{\beta-\alpha-1}+\frac{\delta(2-\beta+\alpha)}{\beta-\alpha-1}}$.) We have just proved that the condition (c) is satisfied by our auxiliary function g_1 and by choosing ε small enough we obtain that $0<\delta<\beta/(2-\beta+\alpha)$ so that the function $t^{\beta}(h(\log^+ t))^{2-\beta+\alpha}$ is increasing and, thus, (c) is also satisfied by f^*. The proof is complete.

PROOF OF THEOREM 8. The proof is rather technical and tedious but can be carried out in a similar way as the proof of the special case $\beta=1$ which previously has been proved in [8, p. 308] so we shall omit the details. We only note that our function f is defined by

$$f(x) = \begin{cases} (h(x_{p_\nu}))^{-1} c^{p_\nu(\beta-\alpha-1)/\beta} & \text{on} \quad I_k = [c^{-p_\nu}, c^{-p_\nu}(1+b_\nu)], \quad \nu = 2, 3, \dots \\ 0 \quad \text{elsewhere} \end{cases}$$

where $c>2$ and (p_ν), (x_{p_ν}) and (b_ν) are sequences satisfying certain properties e.g. $p_\nu \in \mathbf{Z}_+$, $p_{\nu+1} \geqq p_\nu$, $x_{p_{\nu+1}} - x_{p_\nu} \geqq K > 0$, $0<b_\nu<1$, $\sum_{\nu=2}^{\infty} b_\nu/h(x_{p_\nu}) < \infty$, and $\sum_{\nu=2}^{\infty} b_\nu^{\beta-\alpha-1}/h(x_{p_\nu}) = \infty$. The proof is based on a generalized form of Lemma 2.4 in p. 294] and a lemma by WIK [12, p. 75].

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ON THE EVOLUTION OF RANDOM GRAPHS OVER EXPANDING SQUARE LATTICES

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1. Introductory remarks

The random graphs studied in ERDŐS—RÉNYI [2] are "unrestricted" ones in the sense that any two of their vertices can be connected by an edge. Below we introduce two types of "restricted" random graphs the edges of which belong to a special lattice. For simplicity we shall study only the case of (simple) two-dimensional square lattices. It can be easily seen that slight modifications of our arguments will yield similar results for higher-dimensional square lattices and also for other types of lattices (e.g. triangular, hexagonal, Kagomé). To some extent our study is motivated by some problems in crystal physics (compare e.g. the excellent survey article of KASTELEYN [6]). In the sequel all graphs in consideration are finite, undirected and have neither loops nor multiple edges.

Let $L_{m,n}$ denote the following two-dimensional (simple) square lattice: The set $V(L_{m,n})$ of vertices (points) of $L_{m,n}$ is the set of all ordered pairs (i,j) of natural numbers i,j such that $1\leqq i\leqq m$, $1\leqq j\leqq n$, and two vertices of $L_{m,n}$ are called *adjacent* (i.e. are connected by a horizontal or vertical edge) if and only if their distance in the Cartesian plane is 1 (observe that the number of all possible edges in $L_{m,n}$ is equal to $2mn-(m+n)$, $m, n=1, 2, \ldots$). Using $L_{m,n}$ we can define two types of random graphs which we denote by $\Gamma_{m,n,N}$ and $\Gamma_{m,n,p}$, respectively. The random graphs $\Gamma_{m,n,N}$ and $\Gamma_{m,n,p}$ both have $V(L_{m,n})$ as set of vertices; their sets of edges are contained in the set of edges of $L_{m,n}$. The number of edges of $\Gamma_{m,n,N}$ is equal to $N=N(m,n)$ $(0\leqq N\leqq 2mn-(m+n))$ where N depends in a deterministic way on m and n. We assume that these N edges are chosen at random among the $2mn-(m+n)$ edges of $L_{m,n}$ in such a way that all possible choices have the same probability, namely $\binom{2mn-(m+n)}{N}^{-1}$ (a similar "random mechanism" was introduced in ERDŐS—RÉNYI [2]). In $\Gamma_{m,n,p}$ each of the possible $2mn-(m+n)$ edges occurs with probability $p=p(m,n)$ $(0\leqq p\leqq 1)$ where p depends in a deterministic way on m and n, and any edge is chosen independently of any other edge. Therefore the number of edges of $\Gamma_{m,n,p}$ is a random variable with expectation $(2mn-(m+n))p(m,n)$. If $\mathcal{F}$ is any family of graphs, by $\mathbf{P}\{\Gamma_{m,n,N}\in\mathcal{F}\}=\mathbf{P}_{m,n,N}\{\Gamma_{m,n,N}\in\mathcal{F}\}$ $(\mathbf{P}\{\Gamma_{m,n,p}\in\mathcal{F}\}=\mathbf{P}_{m,n,p}\{\Gamma_{m,n,p}\in\mathcal{F}\})$ we denote the probability that $\Gamma_{m,n,N}(\Gamma_{m,n,p})$ is an element of $\mathcal{F}$. We write $\mathbf{E}(\xi)$ $(\mathbf{V}(\xi))$ for the expectation (variance) of a random variable ξ, taken with respect to $\mathbf{P}_{m,n,N}$ or $\mathbf{P}_{m,n,p}$ (it will be clear from the context which of the two probabilites is meant).

We introduce two different types of threshold functions (see also ERDŐS—RÉNYI [2, p. 19]). Let $\mathcal{F}$ denote any family of graphs. A positive function $T_1=T_1(m,n)$

for which $\lim_{m,n\to\infty} T_1(m,n)=\infty$ is called a *N-threshold function* (of $\mathscr{F}$) if

$$(1.1)\qquad \lim_{m,n\to\infty} \mathbf{P}_{m,n,N}\{\Gamma_{m,n,N}\in\mathscr{F}\} = \begin{cases} 0 & \text{if} \quad \lim_{m,n\to\infty} \dfrac{N(m,n)}{T_1(m,n)} = 0 \\ 1 & \text{if} \quad \lim_{m,n\to\infty} \dfrac{N(m,n)}{T_1(m,n)} = \infty. \end{cases}$$

If additionally there exists a probability distribution function F such that for all x $(0<x<\infty)$ being continuity points of F

$$(1.2)\qquad \lim_{m,n\to\infty} \mathbf{P}_{m,n,N}\{\Gamma_{m,n,N}\in\mathscr{F}\} = F(x) \quad \text{if} \quad \lim_{m,n\to\infty} \frac{N(m,n)}{T_1(m,n)} = x,$$

then F is called a *N-threshold distribution function* (of $\mathscr{F}$). A positive function $T_2= =T_2(m,n)$ for which $\lim_{m,n\to\infty} T_2(m,n)=\infty$ is called a *p-threshold function* (of $\mathscr{F}$) if

$$(1.3)\qquad \lim_{m,n\to\infty} \mathbf{P}_{m,n,p}\{\Gamma_{m,n,p}\in\mathscr{F}\} = \begin{cases} 0 & \text{if} \quad \lim_{m,n\to\infty} p(m,n)T_2(m,n) = 0 \\ 1 & \text{if} \quad \lim_{m,n\to\infty} p(m,n)T_2(m,n) = \infty. \end{cases}$$

The definition of a p-threshold distribution function is similar to that of a N-threshold distribution function.

It will be shown below (Theorems 1 and 2 of section 2) that for many families $\mathscr{F}$ of graphs there exist N- and p-threshold functions. It is surprising that in all these cases the threshold functions depend only on the number of edges but not on the number of vertices of the graphs contained in $\mathscr{F}$. This is in sharp contrast to the random graphs considered in ERDŐS—RÉNYI [2].

Let G be any graph. A graph G' is called a *subgraph* of G (HARARY [5, p. 11]) if the set of vertices and the set of edges of G' are subsets of the corresponding sets of G. If $\mathscr{F}$ is any set of graphs then let $\mathscr{F}_{m,n}$ denote the set of all subgraphs of $L_{m,n}$ which are isomorphic to a graph of $\mathscr{F}$. We write $\hat{\mathscr{F}}_{m,n}$ for the set of all spanning subgraphs of $L_{m,n}$ which contain some graph of $\mathscr{F}_{m,n}$. Put $\mathscr{F}_0=\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\hat{\mathscr{F}}_{m,n}$. Call two graphs G_1, G_2 of $\mathscr{F}_{m,n}$ *τ-equivalent* if G_1 can be carried into G_2 by a translation. Let $t(\mathscr{F}_{m,n})$ denote the number of equivalence classes in $\mathscr{F}_{m,n}$ with respect to τ-equivalence. We write $\xi_{m,n,N}$ $(\tilde{\xi}_{m,n,N})$ for the number of graphs in $\mathscr{F}_{m,n}$ which are subgraphs (isolated subgraphs) of $\Gamma_{m,n,N}$. (A subgraph G' of a graph G is called *isolated* if G' is a union of connected components of G.) In the same way we introduce $\xi_{m,n,p}$ and $\tilde{\xi}_{m,n,p}$ for the random graphs $\Gamma_{m,n,p}$. We say that a family $\mathscr{F}$ of graphs has *property* $(\mathrm{A})_k$ $(k\geqq 1)$ in case all graphs in $\mathscr{F}$ are connected, each graph in $\mathscr{F}$ has k edges, and $\mathscr{F}_{m,n}\neq\emptyset$ for some m and n. If $\mathscr{F}$ is a family of graphs, having property $(\mathrm{A})_k$ we put $t=t(\mathscr{F}_0)=\lim_{m,n\to\infty} t(\mathscr{F}_{m,n})$. In the sequel C, C_1, C_2 denote suitable positive constants not depending on (m,n).

As we mentioned above, in section 2 we show that for many families $\mathscr{F}$ of graphs there exist N- and p-threshold functions. In section 3 we derive some limit theorems for the random variables $\xi_{m,n,N}$ and $\xi_{m,n,p}$.

We would like to remark that the methods used for the proofs are mainly those of ERDŐS—RÉNYI [2].

2. Existence of threshold functions

We begin with

THEOREM 1. *Let $\mathscr{F}$ be a family of graphs, having property* $(\mathrm{A})_k$ *for some* $k \geqq 2$. *Then*

$$T_1(m, n) = (mn)^{\frac{k-1}{k}} \tag{2.1}$$

is a N-threshold function of $\mathscr{F}_0$.

PROOF. Suppose that $t(\mathscr{F}_{m,n}) = t$ for all sufficiently large m's and n's. Then

$$\mathbf{P}_{m,n,N}\{\xi_{m,n,N} \geqq 1\} \leqq \mathbf{E}(\xi_{m,n,N}) \leqq$$

$$\leqq mnt \frac{\binom{2mn-(m+n)-k}{N-k}}{\binom{2mn-(m+n)}{N}} \leqq \frac{t}{2^k}\left(\frac{N}{T_1}\right)^k \frac{1}{\left(1-\frac{m+n}{mn}\right)^k}.$$

Hence $\mathbf{P}_{m,n,N}\{\xi_{m,n,N} \geqq 1\} = o(1)$ if $\frac{N}{T_1} = o(1)$ (the Landau symbols $o(1)$ and $O(1)$ will be used only for the limit $m, n \to \infty$). On the other hand,

$$\mathbf{E}(\xi_{m,n,N}) \geqq (m-C)(n-C)t \frac{\binom{2mn-(m+n)-k}{N-k}}{\binom{2mn-(m+n)}{N}} \geqq t(m-C)(n-C)\left(\frac{N-k}{2mn}\right)^k$$

and therefore

$$\mathbf{E}(\xi_{m,n,N}) \geqq \frac{t}{2^k}\left(1-\frac{C}{m}\right)\left(1-\frac{C}{n}\right)\left(\frac{N}{T_1}\right)^k\left(1-\frac{k}{N}\right)^k. \tag{2.2}$$

For the variance $\mathbf{V}(\xi_{m,n,N}) = \mathbf{E}(\xi^2_{m,n,N}) - \mathbf{E}^2(\xi_{m,n,N})$ we get using (2.2)

$$\mathbf{V}(\xi_{m,n,N}) \leqq \mathbf{E}(\xi^2_{m,n,N}) - \frac{t^2}{4^k}\left(1-\frac{C}{m}\right)^2\left(1-\frac{C}{n}\right)^2\left(\frac{N}{T_1}\right)^{2k}\left(1-\frac{k}{N}\right)^{2k}. \tag{2.3}$$

We introduce the random variables $I(S) = I_{m,n,N}(S)$, $S \in \mathscr{F}_{m,n}$, being defined as

$$I(S) = I_{m,n,N}(S) = \begin{cases} 1 & \text{if } S \text{ is a subgraph of } \Gamma_{m,n,N} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Therefore

$$\mathbf{E}(\xi^2_{m,n,N}) = \mathbf{E}(\xi_{m,n,N}) + \sum_{\substack{S_1 \neq S_2 \\ S_1, S_2 \in \mathscr{F}_{m,n}}} \mathbf{E}\big(I(S_1)\cdot I(S_2)\big). \tag{2.5}$$

We put

$$\sum_{\substack{S_1 \neq S_2 \\ S_1, S_2 \in \mathscr{F}_{m,n}}} \mathbf{E}\big(I(S_1)\cdot I(S_2)\big) = \Sigma' + \sum_{l=0}^{k-1} \Sigma''_l \tag{2.6}$$

the summation in $\sum'$ being taken over all ordered pairs (S_1, S_2) $(S_1, S_2 \in \mathcal{F}_{m,n})$ such that S_1 and S_2 have no vertex in common, whereas in $\sum_l''$ we sum over all ordered pairs (S_1, S_2) $(S_1, S_2 \in \mathcal{F}_{m,n})$ such that S_1 and S_2 have at least one vertex and exactly l $(0 \leqq l \leqq k-1)$ edges in common. Define (for the moment) $\mathcal{F}_{m,n,l}$ as the set of all graphs $S_1 \cup S_2$ $(S_1, S_2 \in \mathcal{F}_{m,n})$ such that S_1 and S_2 have at least one vertex and exactly $l (0 \leqq l \leqq k-1)$ edges in common. The connectedness of the graphs in $\mathcal{F}_{m,n}$ implies that $t(\mathcal{F}_{m,n,l})$ for fixed l is bounded. Let $t(\mathcal{F}_{m,n,l}) \leqq t_0 < \infty$ for all $m, n \geqq 1$ and $0 \leqq l \leqq k-1$. Obviously

$$\sum' \leqq (mn)^2 t^2(\mathcal{F}_{m,n}) \frac{\binom{2mn-(m+n)-2k}{N-2k}}{\binom{2mn-(m+n)}{N}} \leqq \frac{t^2(mnN^k)^2}{(2mn-(m+n)-2k)^{2k}} \leqq$$

$$\leqq (1+o(1)) \frac{t^2}{4^k} \left(\frac{N}{T_1}\right)^{2k}.$$

Furthermore for $0 \leqq l \leqq k-1$ we have

$$\sum_l'' \leqq mnt_0 \frac{\binom{2mn-(m+n)-(2k-l)}{N-(2k-l)}}{\binom{2mn-(m+n)}{N}} \leqq (1+o(1)) \frac{t_0}{2^k} \left(\frac{N}{T_1}\right)^{2k} \frac{(mn)^{l-1}}{N^l}.$$

Taking into account (2.3), (2.5) and (2.6) we therefore arrive at

$$\mathbf{V}(\xi_{m,n,N}) \leqq o(1) \left(\frac{N}{T_1}\right)^{2k} + (1+o(1)) \frac{t}{2^k} \left(\frac{N}{T_1}\right)^{k} + (1+o(1)) \frac{kt_0}{2^k} \left(\frac{N}{T_1}\right)^{2k} \left(\frac{1}{mn} + \frac{1}{T_1}\right).$$

Thus

$$\mathbf{V}(\xi_{m,n,N}) = o\left(\left(\frac{N}{T_1}\right)^{2k}\right) \quad \text{if} \quad \frac{N(m,n)}{T_1(m,n)} \to \infty \quad \text{for} \quad m, n \to \infty. \tag{2.7}$$

For the rest of the proof we assume that $\lim\limits_{m,n\to\infty} \frac{N(m,n)}{T_1(m,n)} = \infty$. Application of Tchebychev's inequality — together with (2.2) and (2.7) — gives

$$\mathbf{P}\left\{|\xi_{m,n,N} - \mathbf{E}(\xi_{m,n,N})| > \frac{1}{2}\mathbf{E}(\xi_{m,n,N})\right\} \leqq \frac{4\mathbf{V}(\xi_{m,n,N})}{\mathbf{E}^2(\xi_{m,n,N})} = o(1).$$

This shows that $\lim\limits_{m,n\to\infty} \mathbf{P}\left\{\xi_{m,n,N} \geqq \frac{1}{2}\mathbf{E}(\xi_{m,n,N})\right\} = 1$. But $\lim\limits_{m,n\to\infty} \mathbf{E}(\xi_{m,n,N}) = \infty$. Therefore T_1 is a threshold function of $\mathcal{F}_0$.

For the random graphs $\Gamma_{m,n,p}$ we have

THEOREM 2. *Let $\mathcal{F}$ be a family of graphs, having property* (A)$_k$ *for some* $k \geqq 1$. *Then*

$$T_2(m,n) = (mn)^{1/k} \tag{2.8}$$

is a p-threshold function of $\mathcal{F}_0$.

REMARK. Let $\mathcal{F}$ be a family of graphs, having property $(A)_k$ for some $k \geqq 2$. Then Theorems 1 and 2 show that it is possible to choose a N-threshold function T_1 (of $\mathcal{F}_0$) and a p-threshold function T_2 (of $\mathcal{F}_0$) in such a way that

$$T_1(m, n) \cdot T_2(m, n) = mn, \quad m, n \geqq 1. \tag{2.9}$$

PROOF OF THEOREM 2. The proof is analogous to that of Theorem 1. Obviously $\mathbf{P}_{m,n,p}\{\Gamma_{m,n,p} \in \mathcal{F}_{m,n}\} \leqq t(\mathcal{F}_{m,n}) mnp^k \leqq t(pT_2)^k$. Hence $\mathbf{P}_{m,n,p}\{\Gamma_{m,n,p} \in \mathcal{F}_{m,n}\} = o(1)$ if $pT_2 = o(1)$. On the other hand

$$t(m-C)(n-C)p^k \leqq \mathbf{E}(\xi_{m,n,p}) \leqq tmnp^k. \tag{2.10}$$

Using relations similar to (2.5) and (2.6) we get (t_0 having the same meaning as in the proof of Theorem 1)

$$\mathbf{E}(\xi^2_{m,n,p}) \leqq \mathbf{E}(\xi_{m,n,p}) + t^2(mn)^2 p^{2k} + t_0 mn \sum_{l=0}^{k-1} p^{2k-l}$$

and therefore

$$\mathbf{V}(\xi_{m,n,p}) \leqq o(1)(mn)^2 p^{2k} + tmnp^k + t_0 mn \sum_{l=0}^{k-1} p^{2k-l} \leqq \left[o(1) + \frac{t + t_0 k}{(pT_2)^k}\right](pT_2)^{2k}.$$

This shows

$$\mathbf{V}(\xi_{m,n,p}) = o\big((pT_2)^{2k}\big) \quad \text{if} \quad p(m,n)T_2(m,n) \to \infty \quad \text{for} \quad m, n \to \infty. \tag{2.11}$$

Observing (2.10) we arrive at

$$\mathbf{P}\left\{|\xi_{m,n,p} - \mathbf{E}(\xi_{m,n,p})| > \frac{1}{2}\mathbf{E}(\xi_{m,n,p})\right\} \leqq \frac{4\mathbf{V}(\xi_{m,n,p})}{\mathbf{E}^2(\xi_{m,n,p})} = o(1)$$

if $\lim_{m,n\to\infty} p(m,n)T_2(m,n) = \infty$. The proof is now finished in the same way as in Theorem 1.

3. Some limit theorems

Let $\mathcal{F}$ be a family of graphs, having property $(A)_k$ for some $k \geqq 2$. It is then natural to ask what is happening if

$$\lim_{m,n\to\infty} \frac{N(m,n)}{(mn)^{\frac{k-1}{k}}} = \varrho, \quad 0 < \varrho < \infty. \tag{3.1}$$

To treat this case we need the following result which is a slight variant of a theorem proved in ERDŐS—RÉNYI [2, p. 27].

LEMMA 1. *Let* $\{\eta_{j1}, \eta_{j2}, \ldots, \eta_{jl_j}\}$ $(j \geqq 1)$ *be families of random variables taking on the values* 0 *and* 1 *only. Put*

$$A_r(j) = \sum_{1 \leqq i_1 < i_2 < \ldots < i_r \leqq l_j} \mathbf{E}(\eta_{ji_1}\eta_{ji_2} \ldots \eta_{ji_r})$$

the summation being extended over all combinations of the numbers $1, 2, \ldots, l_j$ *of order* r *without repetitions. Suppose that there exists a function* $u = u(j)$ *for which* $\lim_{j\to\infty} u(j) =$

$= \infty$ *such that*

(3.2) $$\lim_{j\to\infty} A_r(j) = \frac{\lambda^r}{r!} \quad \textit{uniformly in } r, \quad 1 \leqq r \leqq u(j).$$

If $\lambda > 0$ *and if the expressions* $A_r(j)$, $1 \leqq r \leqq l_j$, $j = 1, 2, \dots$, *are uniformly bounded, then*

(3.3) $$\lim_{j\to\infty} \mathbf{P}\left\{\sum_{i=1}^{l_j} \eta_{ji} = s\right\} = \frac{\lambda^s e^{-\lambda}}{s!}, \quad s = 0, 1, \dots .$$

Now we can prove

THEOREM 3. *Let* $\mathscr{F}$ *be a family of graphs, having property* $(\mathrm{A})_k$ *for some* $k \geqq 2$. *If*

(3.4) $$\lim_{m,n\to\infty} \frac{N(m,n)}{(mn)^{\frac{k-1}{k}}} = \varrho \quad (0 < \varrho < \infty)$$

then we have

(3.5) $$\lim_{m,n\to\infty} \mathbf{P}_{m,n,N}\{\xi_{m,n,N} = s\} = \frac{1}{s!}\left[t(\mathscr{F}_0)\left(\frac{\varrho}{2}\right)^k\right]^s e^{-t(\mathscr{F}_0)\left(\frac{\varrho}{2}\right)^k}, \quad s = 0, 1\ 2, \dots$$

i.e. $\xi_{m,n,N}$ *asymptotically has a Poisson distribution with mean* $t(\mathscr{F}_0)\left(\frac{\varrho}{2}\right)^k$.

REMARK. The following proof shows that (3.5) remains true if $\xi_{m,n,N}$ is replaced by $\tilde{\xi}_{m,n,N}$.

PROOF OF THEOREM 3. It suffices to prove (3.5) for $\tilde{\xi}_{m,n,N}$ instead of $\xi_{m,n,N}$, for, if a graph of $\mathscr{F}_{m,n}$ is a subgraph of $\Gamma_{m,n,N}$, which is not isolated, then $\Gamma_{m,n,N}$ contains a connected subgraph having $k+1$ edges. The last event has probability $o(1)$ which can be deduced from Theorem 1, because (3.4) implies $\lim_{m,n\to\infty} \frac{N(m,n)}{(mn)^{\frac{k}{k+1}}} = 0$. For $S \in \mathscr{F}_{m,n}$ we define

(3.6) $$I(S) = I_{m,n,N}(S) = \begin{cases} 1 & \text{if } S \text{ is an isolated subgraph of } \Gamma_{m,n,N} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

(3.7) $$\tilde{\xi}_{m,n,N} = \sum_{S \in \mathscr{F}_{m,n}} I(S).$$

If $S_1, \dots, S_r$ are different graphs of $\mathscr{F}_{m,n}$ having pairwise no vertices in common, then we get

$$\mathbf{E}(I(S_1) \dots I(S_r)) \leqq \frac{\binom{2mn-(m+n)-rk}{N-rk}}{\binom{2mn-(m+n)}{N}} \leqq \left(\frac{N}{2mn}\right)^{rk} \frac{1}{\left(1 - \frac{m+n+rk}{2mn}\right)^{rk}}.$$

Put $u(m,n) = \log(\min(m,n))$ ("log" denotig natural logarithm). Observing that

$-2x \leqq \log(1-x) \leqq -\frac{1}{2}x$, $0 \leqq x \leqq \frac{1}{2}$, we arrive at

$$(3.8)\quad \mathbf{E}\big(I(S_1)\dots I(S_r)\big) \leqq \big(1+o(1)\big)\left(\frac{N}{2mn}\right)^{rk} \quad \text{uniformly in } r, \quad 1 \leqq r \leqq u(m,n).$$

On the other hand, if $\lim_{m,n\to\infty} N(m,n)=\infty$ and $1 \leqq r \leqq u(m,n)$ we have (C depending only on $\mathscr{F}$)

$$\mathbf{E}\big(I(S_1)\dots I(S_r)\big) \geqq \frac{\binom{2mn-(m+n)-rk-Cr}{N-rk}}{\binom{2mn-(m+n)}{N}} \geqq$$

$$\geqq \left(\frac{N}{2mn}\right)^{rk} e^{-\frac{(rk)^2}{N}} \frac{\left(1-\frac{m+n+N}{2mn}\right)\dots\left(1-\frac{m+n+N+Cr-1}{2mn}\right)}{\left(1-\frac{m+n}{2mn}\right)\dots\left(1-\frac{m+n+rk+Cr-1}{2mn}\right)}.$$

Therefore, using (3.8) we get

$$(3.9)\quad \mathbf{E}\big(I(S_1)\dots I(S_r)\big) = \big(1+o(1)\big)\left(\frac{N}{2mn}\right)^{rk} \quad \text{uniformly in } r, \quad 1 \leqq r \leqq u(m,n)$$

if $\lim_{m,n\to\infty} N(m,n)=\infty$ and $\frac{N(m,n)\cdot u(m,n)}{mn}=o(1)$. The relation (3.9) implies

$$(3.10)\quad \sum \mathbf{E}\big(I(S_1)\dots I(S_r)\big) \leqq \frac{1+o(1)}{r!}\big(t(\mathscr{F}_0)\big)^r \left(\frac{N}{2mn}\right)^{rk} (mn)^r \quad \text{uniformly in } r,$$
$$1 \leqq r \leqq u(m,n)$$

(the summation is extended over all combinations of different graphs in $\mathscr{F}_{m,n}$ of order r). Similarly

$$(3.11)\quad \sum \mathbf{E}\big(I(S_1)\dots I(S_r)\big) \geqq \frac{1+o(1)}{r!}\big(t(\mathscr{F}_0)\big)^r \left(\frac{N}{2mn}\right)^{rk} (m-rC_1)^r(n-rC_1)^r$$

(uniformly in r, $1 \leqq r \leqq u(m,n)$). Thus (3.4), (3.10) and (3.11) imply

$$(3.12)\quad \lim_{m,n\to\infty} \sum \mathbf{E}\big(I(S_1)\dots I(S_r)\big) = \frac{1}{r!}\left[t(\mathscr{F}_0)\left(\frac{\varrho}{2}\right)^k\right]^r \text{uniformly in } r, \quad 1 \leqq r \leqq u(m,n).$$

It can be easily seen that $\sum \mathbf{E}\big(I(S_1)\dots I(S_r)\big)$, $r=1,2,\dots$ is uniformly bounded. The desired result is therefore a consequence of Lemma 1.

COROLLARY 1. *The function F defined by*

$$(3.13)\qquad F(x) = \begin{cases} 1-\exp\left[-t(\mathscr{F}_0)\left(\frac{x}{2}\right)^k\right], & x \geqq 0 \\ 0, & x < 0 \end{cases}$$

is a N-threshold distribution function of $\mathscr{F}_0$.

PROOF. If (3.4) holds then according to Theorem 3 we get

$$\lim_{m,n\to\infty} \mathbf{P}\{\xi_{m,n,N} \geqq 1\} = 1 - \lim_{m,n\to\infty} \mathbf{P}\{\xi_{m,n,N} = 0\} = 1 - \exp\left[-t(\mathscr{F}_0)\left(\frac{\varrho}{2}\right)^k\right].$$

EXAMPLES. Let $\mathscr{T}_k$ $(k \geqq 1)$ denote the family of all trees having k edges. We get $t((\mathscr{T}_1)_0)=2$, $t((\mathscr{T}_2)_0)=6$, $t((\mathscr{T}_3)_0)=22$, $t((\mathscr{T}_4)_0)=87$, $t((\mathscr{F}_5)_0)=364$.

If $\mathscr{C}_k$ $(k \geqq 4)$ denotes the family of all circles of order k, then we have $t((\mathscr{C}_{2k+1})_0)=0$, $k=2, 3, \dots$, and $t((\mathscr{C}_4)_0)=1$, $t((\mathscr{C}_6)_0)=2$, $t((\mathscr{C}_8)_0)=7$, $t((\mathscr{C}_{10})_0)=28$, $t((\mathscr{C}_{12})_0)=134$. It seems to be an unsolved problem in enumerative graph theory to give explicit formulas for $t((\mathscr{T}_k)_0)$ and $t((\mathscr{C}_{2k})_0)$, $k=2, 3, \dots$.

Call a property or family of graphs *isotone (antitone)* if any graph obtained by adding (deleting) any edge of a given graph with this property (the set of vertices being fixed) has this property, too. Then we have the almost trivial result (its proof being omitted)

LEMMA 2. *If $\mathscr{G}_{m,n}$ is an isotone family of subgraphs of $L_{m,n}$, all having $V(L_{m,n})$ as their set of vertices, then we have for* $0 \leqq N \leqq N' \leqq 2mn-(m+n)$

$$\mathbf{P}_{m,n,N}\{\Gamma_{m,n,N} \in \mathscr{G}_{m,n}\} \leqq \mathbf{P}_{m,n,N'}\{\Gamma_{m,n,N'} \in \mathscr{G}_{m,n}\}. \tag{3.14}$$

Similarly, if $\mathscr{G}_{m,n}$ is antitone then the sense of the inequality (3.14) *has to be reversed.*

It can be easily seen that Theorem 3 — together with Lemma 2 — implies Theorem 1.

The following result is the analogon of Theorem 3 for the random graphs $\Gamma_{m,n,p}$.

THEOREM 4. *Let $\mathscr{F}$ be a family of graphs, having property* $(\mathrm{A})_k$ *for some* $k \geqq 1$. *If*

$$\lim_{m,n\to\infty} p(m,n)(mn)^{1/k} = \varrho \quad (0 < \varrho < \infty) \tag{3.15}$$

then

$$\lim_{m,n\to\infty} \mathbf{P}_{m,n,p}\{\xi_{m,n,p} = s\} = \frac{1}{s!}[t(\mathscr{F}_0)\varrho^k]^s e^{-t(\mathscr{F}_0)\varrho^k}, \quad s = 0, 1, 2, \dots. \tag{3.16}$$

PROOF. Using Theorem 2, it can be easily seen that ifs suffices to prove (3.16) for the random variables $\tilde{\xi}_{m,n,p}$ instead of $\xi_{m,n,p}$. Put for $S \in \mathscr{F}_{m,n}$

$$I(S) = I_{m,n,p}(S) = \begin{cases} 1 & \text{if } S \text{ is an isolated subgraph of } \Gamma_{m,n,p} \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

Clearly

$$\tilde{\xi}_{m,n,p} = \sum_{S \in \mathscr{F}_{m,n}} I(S). \tag{3.18}$$

On the other hand

$$\sum \mathbf{E}(I(S_1) \dots I(S_r)) \leqq \frac{1}{r!}(t(\mathscr{F}_0)mnp^k)^r, \quad r = 1, 2, \dots \tag{3.19}$$

(the summation being the same as that occuring in (3.10)). There exists an integer

$k_0>0$ not depending on (m, n) such that to any graph $G \in \mathcal{F}_{m,n}$ there exist at most k_0 edges not belonging to G which are incident with a vertex of G. Then

(3.20)

$$\sum \mathbf{E}(I(S_1)\ldots I(S_r)) \geqq \frac{1}{r!}t^r(\mathcal{F}_0)(m-rC)^r(n-rC)^r[p^k(1-p)^{k_0}]^r, \quad 1 \leqq r \leqq \frac{1}{C}\min(m,n).$$

This can be seen as follows. The properties of $\mathcal{F}$ imply that there exists a natural number $K \geqq 2$ (not depending on (m, n)) such that the set of vertices of each graph in $\mathcal{F}_{m,n}$ is contained in a suitable square array $Q \subset V(L_{m,n})$ (being oriented parallel to the coordinate axes) each "side" of which is consisting of K points. Then for the number $N_r(m, n)$ of all ordered r-tuples of different such square arrays Q which have pairwise at most "side points" in common, we get

$$N_r(m, n) \geqq \prod_{i=0}^{r-1}(m-((2i+1)K-3i-1))(n-((2i+1)K-3i-1)) \tag{3.21}$$

for all $r \geqq 1$ such that all $2r$ factors on the right side of the inequality (3.21) are non-negative. Clearly (3.21) implies (3.20). On the other hand (3.20) implies

$$\sum \mathbf{E}(I(S_1)\ldots I(S_r)) \geqq \frac{1}{r!}\left(1-r^2C\left(\frac{1}{m}+\frac{1}{n}\right)\right)[t(\mathcal{F}_0)mnp^k(1-p)^{k_0}]^r, \quad 1 \leqq r \leqq \frac{1}{C}\min(m,n).$$

Put $u(m, n) = \log(\min(m, n))$. The last inequality — together with (3.19) and (3.15) — leads to

$$\lim_{m,n\to\infty} \sum \mathbf{E}(I(S_1)\ldots I(S_r)) = \frac{1}{r!}[t(\mathcal{F}_0)\varrho^k]^r \quad \text{uniformly in } r, \quad 1 \leqq r \leqq u(m,n). \tag{3.22}$$

The uniform boundedness of $\sum \mathbf{E}(I(S_1)\ldots I(S_r))$, $r=1, 2, \ldots$ is immediately seen from (3.19) and (3.15). The desired result is therefore a consequence of Lemma 1.

COROLLARY 2. *The function F defined by*

$$F(x) = \begin{cases} 1-\exp[-t(\mathcal{F}_0)x^k], & x \geqq 0 \\ 0, & x < 0 \end{cases} \tag{3.23}$$

is a p-threshold distribution function of $\mathcal{F}_0$.

REMARK. Comparison of Theorems 3 and 4 shows that $\Gamma_{m,n,N}$ "formally" can be considered as a random graph $\Gamma_{m,n,\tilde{p}}$, $\tilde{p}$ given by

$$\tilde{p} = \tilde{p}(m, n) = \frac{N(m, n)}{mn\left(2-\frac{1}{m}-\frac{1}{n}\right)} \tag{3.24}$$

whereas in some situations $\Gamma_{m,n,p}$ "formally" can be considered as a random graph $\Gamma_{m,n,\tilde{N}}$ with $\tilde{N}$ given by

$$\tilde{N} = (2mn-(m+n))p(m, n) \tag{3.25}$$

(this being the expectation mentioned in section 1).

Now let $\mathcal{F}$ be a family of graphs, having property $(\mathrm{A})_k$ for some $k \geqq 2$. If (3.4) holds, Theorem 3 says that $\xi_{m,n,N}$ and $\tilde{\xi}_{m,n,N}$ asymptotically have a Poisson distribution with mean $t(\mathcal{F}_0)\left(\frac{\varrho}{2}\right)^k$. It is well known (see e.g. FELLER [3, p. 194]) that if the random variables X_λ $(\lambda > 0)$ have a Poisson distribution with mean λ, then the normalized variables $\frac{X_\lambda - \lambda}{\sqrt{\lambda}}$ asymptotically (i.e. for $\lambda \to \infty$) have a normal distribution with mean 0 and variance 1. Therefore we are led to consider the case

$$\lim_{m,n\to\infty} \frac{N(m,n)}{(mn)^{\frac{k-1}{k}}} = \infty. \tag{3.26}$$

We shall show below (Theorem 5) that if in (3.26) the convergence to infinity is not too quick, then $\xi_{m,n,N}$ (after having been suitably normalized) asymptotically has a normal distribution with mean 0 and variance 1. From the proof of Theorem 3 follows: for fixed $r \geqq 1$

$$\sum \mathbf{E}(I(S_1) \dots I(S_r)) = \frac{1}{r!}\left(1 + O\left(\frac{1}{m} + \frac{1}{n} + \frac{1}{N} + \frac{N}{mn}\right)\right) a^r(m,n,N) \text{ if } N = o(mn) \tag{3.27}$$

$(I(S),\ S \in \mathcal{F}_{m,n}$ being defined by (3.6)) if we put

$$a = a(m,n,N) = t(\mathcal{F}_0)\, mn \left(\frac{N}{2mn}\right)^k. \tag{3.28}$$

This implies

$$\mathbf{E}(\tilde{\xi}_{m,n,N}) \sim a(m,n,N) \quad \text{if} \quad N = o(mn). \tag{3.29}$$

The relation (3.29) — together with the above considerations — suggests to consider the normalized random variables

$$\frac{\tilde{\xi}_{m,n,N} - a(m,n,N)}{\sqrt{a(m,n,N)}}.$$

We have

THEOREM 5. *Let $\mathcal{F}$ be a family of graphs, having property $(\mathrm{A})_k$ for some $k \geqq 2$. If*

$$\lim_{m,n\to\infty} \frac{N(m,n)}{(mn)^{\frac{k-1}{k}}} = \infty \tag{3.30}$$

and

$$\lim_{m,n\to\infty} \frac{N(m,n)}{(mn)^{\frac{k-1}{k}} [\min(m,n)]^\delta} = 0 \quad \text{for all} \quad \delta > 0 \tag{3.31}$$

then

$$\lim_{m,n\to\infty} \mathbf{P}_{m,n,N}\left\{\frac{\xi_{m,n,N} - a(m,n,N)}{\sqrt{a(m,n,N)}} \leqq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \quad -\infty < x < \infty. \tag{3.32}$$

PROOF. It suffices to prove (3.32) for $\tilde{\xi}_{m,n,N}$ instead of $\xi_{m,n,N}$. We use the method of ERDŐS—RÉNYI [2]. The normal distribution $N(0,1)$ being uniquely determined by its moments (this follows e.g. from a general result of Carleman, see FELLER

[4, p. 228]) it suffices for the proof of (3.32) to show that the moments

$$\mathbf{E}\left(\left(\frac{\xi_{m,n,N}-a(m,n,N)}{\sqrt{a(m,n,N)}}\right)^r\right), \quad r = 1, 2, \dots$$

converge to the corresponding moments of $N(0, 1)$ (compare LOÉVE [7, p. 185]). Now (3.27) implies

$$\text{(3.33)} \quad \mathbf{E}(\xi^r_{m,n,N}) = \left(1+O\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{N}+\frac{N}{mn}\right)\right)\sum_{j=1}^{r} S(r,j)\,a^j(m,n,N), \quad r = 1, 2, \dots$$

where the $S(r,j)$ denote the Stirling numbers of the second kind (see COMTET [1, p. 39]). But (ERDŐS—RÉNYI [2, p. 33])

$$\text{(3.34)} \quad \sum_{j=1}^{r} S(r,j)x^j = \sum_{j=0}^{\infty}\frac{x^j}{j!}e^{-x}j^r, \quad r = 1, 2, \dots$$

and therefore (3.33) leads to

$$\text{(3.35)} \quad \mathbf{E}(\xi^r_{m,n,N}) = \left(1+O\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{N}+\frac{N}{mn}\right)\right)\sum_{j=0}^{\infty}\frac{a^j}{j!}e^{-a}j^r, \quad r = 0, 1, 2, \dots.$$

Using (3.35) and (3.34) we get

$$\mathbf{E}\left(\left(\frac{\xi_{m,n,N}-a(m,n,N)}{\sqrt{a(m,n,N)}}\right)^r\right) \leqq \frac{1-C_1\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{N}+\frac{N}{mn}\right)}{a^{r/2}}\sum_{j=0}^{\infty}\frac{a^j}{j!}e^{-a}(j-a)^r +$$

$$+C_2\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{N}+\frac{N}{mn}\right)a^{r/2}\sum_{j=1}^{r}S(r,j)a^j, \quad r = 1, 2, \dots.$$

Because of

$$\text{(3.36)} \quad \lim_{x\to\infty}\frac{1}{x^{r/2}}\sum_{j=0}^{\infty}\frac{x^j}{j!}e^{-x}(j-x)^r = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}u^r e^{-u^2/2}\,du, \quad r = 1, 2, \dots$$

(ERDŐS—RÉNYI [2, p. 33]) and the well known relation

$$\text{(3.37)} \quad \sum_{j=1}^{r} S(r,j)x(x-1)\dots(x-j+1) = x^r, \quad r = 1, 2, \dots$$

(COMTET [1, p 41]) the above inequalities — together with (3.30) and (3.28) — imply

$$\text{(3.38)} \quad \mathbf{E}\left(\left(\frac{\xi_{m,n,N}-a(m,n,N)}{\sqrt{a(m,n,N)}}\right)^r\right) \leqq \frac{1+o(1)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}u^r e^{-u^2/2}\,du +$$

$$+C\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{N}+\frac{N}{mn}\right)a^{3r/2}, \quad r = 1, 2, \dots,$$

Hence according to (3.30) and (3.31) we arrive at

$$\limsup_{m,n\to\infty}\mathbf{E}\left(\left(\frac{\xi_{m,n,N}-a(m,n,N)}{\sqrt{a(m,n,N)}}\right)^r\right) \leqq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}u^r e^{-u^2/2}\,du, \quad r = 1, 2, \dots.$$

A similar consideration gives

$$\liminf_{m,n\to\infty} \mathbf{E}\left(\left(\frac{\tilde{\xi}_{m,n,N}-a(m,n,N)}{\sqrt{a(m,n,N)}}\right)^r\right) \geqq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u^r e^{-u^2/2}\,du, \quad r=1,2,\ldots$$

from which the desired result follows.

Similarly we can prove for the random graphs $\Gamma_{m,n,p}$

THEOREM 6. *Let $\mathcal{F}$ be a family of graphs, having property* $(\mathrm{A})_k$ *for some* $k \geqq 1$. ***Put***

(3.39) $$\tilde{a} = \tilde{a}(m, n\, p) = t(\mathcal{F}_0)\, mnp^k.$$

If

(3.40) $$\lim_{m,n\to\infty} p(m,n)(mn)^{1/k} = \infty$$

and

(3.41) $$\lim_{m,n\to\infty} \frac{p(m,n)(mn)^{1/k}}{[\min(m,n)]^\delta} = 0 \quad \textit{for all} \quad \delta > 0$$

then

(3.42) $$\lim_{m,n\to\infty} \mathbf{P}_{m,n,p}\left\{\frac{\xi_{m,n,p}-\tilde{a}(m,n,p)}{\sqrt{\tilde{a}(m,n,p)}} \leqq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\,du, \quad -\infty < x < \infty.$$

PROOF. Instead of (3.33) we now have

$$\mathbf{E}(\tilde{\xi}^r_{m,n,p}) = \left(1+O\left(\frac{1}{m}+\frac{1}{n}+p\right)\right) \sum_{j=1}^{r} S(r,j)\,\tilde{a}^j(m,n,p), \quad r=1,2,\ldots$$

and the rest of the proof is similar to that of Theorem 5.

After having developed some fundamental properties of the random graphs $\Gamma_{m,n,N}$ and $\Gamma_{m,n,p}$ we intend to investigate these random graphs further in a subsequent paper.

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SOME MONOTONICITY PROPERTIES OF POLYNOMIALS WITH EQUALLY SPACED ZEROS[1]

By

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To Paul Erdős on his 60th birthday

1. Introduction. Concerning an arbitrary polynomial with simple, equally spaced and exclusively real zeros, PAUL ERDŐS conjectured and ELEMÉR BÁLINT proved [1] that the distances between consecutive zeros of its first derivative *increase* when measured outward from the centre of symmetry of its zeros.

The present note establishes in § § 2—4 other monotonicity properties associated with such a polynomial. The concluding § 5 (appropriate to, perhaps even required of, a paper dedicated to Paul Erdős) raises some open questions and advances some conjectures which, if verified, would generalize substantially both the Erdős—Bálint result and the present ones.

Here it is shown that the areas and maxima of the absolute values of consecutive arches of the graph, as well as the magnitudes of the slopes at consecutive zeros, increase outward from the centre of symmetry of the zeros. A zero of fixed rank of the derivative moves closer to the centre as new zeros are adjoined. Various related properties are indicated as they arise.

The proofs are quite simple. To facilitate their presentation, the polynomials in question are written in the respective forms

$$p_n(x) = x \prod_{k=1}^{n} (x^2-k^2), \quad n = 1, 2, \dots, \tag{1}$$

and

$$q_0(x) = x(x-1), \quad q_n(x) = (x-n-1)p_n(x), \quad n = 1, 2, \dots, \tag{2}$$

by change of scale, depending on whether the degree is odd, $2n+1$, $n=1, 2, \dots$, or even, $2n+2, n=0, 1, \dots$.

In form (1), the centre of symmetry of the zeros is at the origin, in (2) at $\frac{1}{2}$. The results can therefore be stated in terms of positive x, say, the symmetry permitting automatic formulations for negative x.

2. Monotonicity of areas and extrema. This is susceptible to two interpretations. One of them prescribes the (degree of the) polynomial, in which case it is required to show that the areas and maxima under thd graph of the absolute value of the fixed polynomial increase, respectively, in the progression from one arch to the next, for $x \geqq 0$, with an analogous result for $x \leqq 0$. This result does *not* depend on the normalizations (1) and (2).

[1] This note was done while on sabbatical leave from York University, assisted by a Canada Council Leave Fellowship. It was read at the International Colloquium on Infinite and Finite Sets, Keszthely, 1973, in honour of Paul Erdős's 60th birthday.

On the other hand, the other monotonicity discussed does depend on the coordinate scale. It arises from considering the absolute value of a fixed arch of a polynomial in the system (1)—(2) and then proving that its area and its maximum increase as the degree of the defining polynomial in (1)—(2) increases.

The first version is contained in the following more precise assertion:

(I)[2] *If x is not an integer, and if* $0<x<n$, *then*

$$|p_n(x+1)| > |p_n(x)| \quad \text{and} \quad |q_n(x+1)| > |q_n(x)|, \tag{3}$$

so that the areas and maxima under the successive arches of $|p_n(x)|$ *and* $|q_n(x)|$ *form respective increasing sequences,* $x>0$.

PROOF.

$$p_n(x+1) = (x+1) \prod_{k=1}^{n} [(x+1)^2-k^2] =$$

$$= x(x^2-1) \dots (x^2-n^2)(x+n+1)/(x-n) = [(x+n+1)/(x-n)]p_n(x).$$

Clearly, $x+n+1>n-x$, $0<x<n$, while $p_n(x)\neq 0$, $p_n(x+1)\neq 0$ for x non-integral. This establishes (I) for the polynomials $p_n(x)$.

For $q_n(x)$, a similar calculation shows that

$$q_n(x+1) = [(x+n+1)/(x-n-1)]\,q_n(x)$$

and the result follows as for $p_n(x)$.

Correspondingly, the second version is contained in the following assertion:

(II) *For* $0<x<n$ *and x not an integer,*

$$|p_{n+1}(x)| > (n+1)|q_n(x)| > (n+1)|p_n(x)|. \tag{4}$$

Hence, increasing the degree of a polynomial in the system (1)—(2) *increases the area and the maximum of the absolute value of the polynomial, in each fixed arch.*

PROOF. $p_{n+1}(x)=(x+n+1)q_n(x)=(x+n+1)(x-n-1)p_n(x)$, while $|x+n+1|= =x+n+1>n+1$, $|x-n-1|=n+1-x>1$, $0<x<n$, making the assertions obvious.

3. Monotonicity of the zeros of the derivatives with respect to degree. BÁLINT's verification [1] of Erdős' conjecture establishes a monotonicity property of the zeros of the derivative of a *fixed* polynomial of the form (1)—(2). The sequence he obtains results from going from one arch of the graph to the next. In this sense, statement (I) above is an analogue of Bálint's result.

There is a corresponding analogue of (II), sharing with (II) the drawback of being somewhat dependent on the scale.

To formulate these results, *let the jth positive zero of $p_n'(x)$ be denoted by x_{nj}'*, $j=1, 2, \dots, n$, *of $q_n'(x)$ by ξ_{nj}'*, $j=1, 2, \dots, n+1$.

It might be well to observe that there are indeed n and $n+1$, respectively, of these positive zeros (also n negative ones) and that $j-1<x_{nj}'$, $\xi_{nj}'<j$, $j=1, \dots, n$, with $n<\xi_{n,n+1}'<n+1$. (Any polynomial with exclusively real zeros has precisely one non-zero extremum in the interior of each arch of its graph; the degree of its derivative

[2] D. J. Newman has mentioned that parts of (I) are implicit in [2, esp. p. 131ff].

is one less than the degree of the given polynomial and so there is one zero of the derivative in each arch. This holds even if, unlike here, some of the zeros are multiple.)

Various monotonicity properties of the quantities x'_{nj}, ξ'_{nj} follow:

First,

(5) $$\xi'_{n1} = \tfrac{1}{2}, \quad n = 0, 1, \dots.$$

Proof is by induction. The result is obvious for $n=0$, since $q_0(x)=x(x-1)$. From the definition $q_{n+1}(x)=(x-n-2)(x+n+1)q_n(x)$, so that

(6) $$q'_{n+1}(x) = (x-n-2)(x+n+1)q'_n(x)+(2x-1)q_n(x),$$

whence $q'_{n+1}(\tfrac{1}{2}) = 0$ whenever $q'_n(\tfrac{1}{2}) = 0$.

Next,

(7) $$\xi'_{nj} > \xi'_{n+1,j}, \quad j = 2, \dots, n+1.$$

Proof. From (6), $q'_{n+1}(\xi'_{nj})=(2\xi'_{nj}-1)q_n(\xi'_{nj})$. Now, $q_{n+1}(x)$ and $q_n(x)$ are of opposite signs in $j-1<x<j$, say, negative and positive, respectively, while $\xi'_{nj}>\tfrac{1}{2}$, $j=2, \dots, n+1$. Thus $q'_{n+1}(\xi'_{nj})>0$, $j=2, \dots, n+1$, so that the negative function $q_{n+1}(x)$, having decreased from 0 at $x=j-1$, is increasing at $x=\xi'_{nj}$. This means that at $x=\xi'_{nj}$ the function has passed its minimum point, achieved uniquely at $x=\xi'_{n+1,j}$ and the proof is complete.

Next we show

(8) $$x'_{n+1,j} > \xi'_{nj}, \quad j = 1, 2, \dots, n.$$

The proof of (8) is similar to that of (7). Here $p_{n+1}(x)=(x+n+1)q_n(x)$, so that $p_{n+1}(x)$ and $q_n(x)$ are of the same sign, say positive, for $j-1<x<j$. Hence $p'_{n+1}(\xi'_{nj})= =q_n(\xi'_{nj})>0$ and so $p_{n+1}(x)$ achieves its maximum (at $x=x'_{n+1,j}$) after $x=\xi'_{nj}$, as asserted.

Additionally, we have

(9) $$x'_{nj} > x'_{n+1,j}, \quad j = 1, \dots, n.$$

To establish (9), we write $p_{n+1}(x)=(x^2-[n+1]^2)p_n(x)$, and note that $p_n(x)$ and $p_{n+1}(x)$ are of opposite signs in $j-1<x<j$, $j=1, \dots, n$, say positive and negative, respectively. Then $p'_{n+1}(x'_{nj})=2x'_{nj}p_n(x'_{nj})>0$ and the proof can be concluded as for (7).

Combining (8) and (9) gives

(10) $$x'_{nj} > x'_{n+1,j} > \xi'_{nj}, \quad j = 1, 2, \dots, n.$$

With (7) this can be written as

(11) $$x'_{nj} > x'_{n+1,j} > \xi'_{nj} > \xi'_{n+1,j}, \quad j = 2, \dots, n.$$

In particular, $x'_{n1}>\tfrac{1}{2}$ for all n. This shows that the first positive zero of $p'_n(x)$ lies closer to the right endpoint of the first arch of $p_n(x)$ than to the left. This is true for *all* arches of the graph of $p_n(x)$ for $x>0$, and of $q_n(x)$ for $x>1$, i.e.,

(12) $$x'_{nj} > j-\tfrac{1}{2}, \quad j = 1, 2, \dots, n; \quad \xi'_{nj} > j-\tfrac{1}{2}, \quad j = 2, 3, \dots, n+1.$$

BÁLINT showed this [1, (I), p. 35] in a different notation. He proved [1, (II), p. 36] also that

(13) $$x'_{n,j+1}-x'_{nj}>1,\quad \xi'_{n,j+1}-\xi'_{nj}>1,\quad j=1,2,\dots,n.$$

In view of (5) and (10), a simple induction makes it clear that (13) implies (12). It suffices to establish the ξ-inequality in (12), since the x-inequality follows then from (10) except when $j=1$; in that case, it is a consequence of (5).

Here another proof of (13), and hence also of (12), is presented. It follows the lines of the proof of our (I), above.

From that proof we see that

$$p'_n(x+1)=[(x+n+1)/(x-n)]p'_n(x)-(2n+1)(x-n)^{-2}p_n(x),$$

so that

$$p'_n(x'_{nj}+1)=-(2n+1)(x'_{nj}-n)^{-2}p_n(x'_{nj}).$$

Now, the consecutive arches of the graph of $p_n(x)$ lie on opposite sides of the x-axis. If, e.g., the arch from $(j-1, 0)$ to $(j, 0)$ is above the axis, then the next arch is below the axis. In this case, $p'_n(x'_{nj}+1)<0$; i.e., $p_n(x)$ is still decreasing at $x=x'_{nj}+1$, so that $x'_{n,j+1}>x'_{nj}+1$, as asserted in (13).

The arguments for the other case, and also for $q_n(x)$, are the same, obvious changes aside. Thus, (13) is proved and, with it, (12).

The inequalities (12) can be supplemented and shown to be "best possible" in a certain sence. The precise statement follows.

(14) $$x'_{nj}\downarrow j-\tfrac{1}{2},\quad j=1,2,\dots;\quad \xi'_{nj}\downarrow j-\tfrac{1}{2},\quad j=2,3,\dots\quad (n\to\infty).$$

From (7) and (10) we see that it is enough to prove that

$$x'_j=\lim_{n\to\infty}x'_{nj}\quad\text{and}\quad \xi'_j=\lim_{n\to\infty}\xi'_{nj},$$

exist, and that $x'_j=\xi'_j=j-\frac{1}{2}$, $j=1,2,\dots$.

Of course, ξ'_1 exists and equals $\frac{1}{2}$, from (5), but this is not germane to (14). That the relevant limits x'_j, ξ'_j exist follows at once from (9) and (7), respectively.

Thus, it remains to show that $x'_j=j-\frac{1}{2}$, $j=1,2,\dots$, and that $\xi'_j=j-\frac{1}{2}$, $j=2,3,\dots$.

To verify this for x'_j, we define

$$P_n(x)\equiv\frac{(-1)^n\pi p_n(x)}{(n!)^2}=\pi x\prod_{k=1}^{n}\left(1-\frac{x^2}{k^2}\right),$$

a polynomial having the same zeros x'_{nj}, $j=1,\dots,n$, for its derivative as does $p_n(x)$. It is well-known that

$$\lim_{n\to\infty}P_n(x)=\sin\pi x,$$

uniformly in each finite interval.

From the uniformity of this convergence, it follows that

$$\lim_{n\to\infty}P_n(x'_{nj})=\sin\pi x'_j,\quad j=1,2,\dots.$$

On the other hand, it is easy to show that $x=x'_j$ yields the unique extremum of $\sin \pi x$ in $j-1<x<j$, since $P_n(x'_{nj})$ is the unique extremum of $P_n(x)$ in that interval, for each j and n, $j \leq n$.

Hence, $x'_j=j-\frac{1}{2}$, $j=1, 2, \dots$. Similar reasoning applied to $q_n(x)$ shows that $\xi'_j=j-\frac{1}{2}$, $j=2, 3, \dots$.

The proof of (14) is now complete.

REMARKS. (i) The first part of (14) can be rephrased to state that *the* jth *positive zero of the derivative of the partial product* $P_n(x)$ *of* $\sin \pi x$ *decreases to* $j-\frac{1}{2}$, *the* jth *positive zero of the derivative of* $\sin \pi x$, as $n \to \infty$, $j=1, 2, \dots$.

(ii) A corollary of (14) is

$$\lim_{n\to\infty} (x'_{n,j+1}-x'_{nj}) = \lim_{n\to\infty} (\xi'_{n,j+1}-\xi'_{nj}) = 1, \quad j = 1, 2, \dots, \tag{15}$$

i.e., the left members of the inequalities (13) approach the respective right members as $n \to \infty$. Thus (13) raises the question, unanswered here, as to whether either convergence in (15) is monotonic. Such monotonicity would be with respect to n.

BÁLINT's result [1, (II), p. 36], verifying Erdős' conjecture, establishes monotonicity in j, $j=1, 2, \dots, n$, for each fixed n, of the differences whose limits appear in (15). His result can be written as

$$\Delta^2 x'_{nj} = x'_{n,j+2}-2x'_{n,j+1}+x'_{nj} > 0, \quad j = 1, \dots, n,$$

with a corresponding inequality for the ξ's. Again, (14) implies that the left members of these inequalities converge to the right members as $n \to \infty$, giving rise to the question of monotonicity of convergence.

(iii) The location of the zeros of the derivative moves, as in (7) and (9), toward the centre of symmetry of the zeros of *any* polynomial $P(x)$ having exclusively real zeros, even if not simple or equally spaced, when a zero larger than the ones already under consideration is adjoined (i.e., with $Q(x)=(x-\alpha)P(x)$, where α is not less than any zero of $P(x)$).

4. Monotonicity of slopes at the zeros. Another readily established monotonicity property is associated with $\{|p'_n(j)|\}$, $j=0, 1, \dots, n$ and a similar one with $\{|q'_n(j)|\}$, $j=1, 2, \dots, n+1$, i.e., the numerical values of the slopes at the zeros.

First, we note

$$|p'_n(j)| = (n+j)!(n-j)!, \quad j = 0, 1, \dots, n, \tag{16}$$

where, as usual, $0!=1$.

This implies that *the sequence* $\{|p'_n(j)|\}$, $j=0, 1, \dots, n$, *increases for each fixed* n, since, assuming (16),

$$|p'_n(j+1)|-|p'_n(j)| = (2j+1)(n+j)!(n-j-1)! > 0, \quad j=0, 1, \dots, n-1. \tag{17}$$

To prove (16), Leibniz's formula for the differentiation of a product is applied to (1). This gives

$$p'_n(j) = 2j^2 \prod_{k=1}^{j-1} (j^2-k^2) \prod_{k=j+1}^{n} (j^2-k^2), \quad j = 2, \dots, n-1,$$

since all the other terms of the derivative have a zero factor.

Now,

$$(n+j)!(n-j)! = [(n+j)(n-1+j)\dots(n-\{n-j-1\}+j)][(2j)!]\times$$

$$\times[(n-j)(n-1-j)\dots(n-\{n-j-1\}-j)] = (2j)!\prod_{k=j+1}^{n}(k^2-j^2).$$

Furthermore,

$$2j^2\prod_{k=1}^{j-1}(j^2-k^2) = 2j^2[(j+1)\dots(j+\{j-1\})]\times[(j-1)\dots(j-\{j-1\})] = (2j)!,$$

and (16) is established (hence also (17)), except for the cases $j=0, 1, n$, not covered by the formal expression given for $p_n'(j)$. However, these are easily verified directly.

The corresponding results for $q_n(x)$ follow from those for $p_n(x)$, since $q_n(x)= =(x-n-1)p_n(x)$ so that $q_n'(x)=(x-n-1)p_n'(x)+p_n(x)$. They are:

(18) $$|q_n'(j)| = (n+j)!(n+1-j)!,\quad j=0, 1, \dots, n+1$$

and

(19) $$|q_n'(j+1)|-|q_n'(j)| = (2j)(n+j)!(n-j)! > 0,\quad j=1, \dots, n.$$

For $j=0, 1, \dots, n$,

$$|q_n'(j)| = |(j-n-1)||p_n'(j)| = (n+1-j)(n+j)!(n-j)! = (n+j)!(n+1-j)!,$$

as asserted in (18). In the remaining case, $j=n+1$, and

$$q_n'(n+1) = p_n(n+1) = (n+1)\prod_{k=1}^{n}(\{n+1\}^2-k^2) =$$

$$= (n+1)(n+1+1)(n+1+2)\dots(n+1+n)\times(n+1-1)(n+1-2)\dots(n+1-n) =$$

$$= (2n+1)!,$$

completing the verification of (18).

(19) is an immediate consequence. Thus, *the sequence* $\{|q_n'(j)|\}$ *increases,* $j=1, 2, \dots, n+1$, *while* $|q_n'(0)|=|q_n'(1)|=n!(n+1)!$, *for each fixed n.*

More is true:

(III) *The sequences* $\{|p_n'(j)|\}$, $j=0, \dots, n$, *and* $\{|q_n'(j)|\}$, $j=1, \dots, n+1$, *are each absolutely monotonic.*

(An absolutely monotonic sequence $\{a_j\}$, $j=0, 1, \dots$, is one all of whose defined differences are non-negative, i.e. $\Delta^m a_j \geqq 0$, $m=0, 1, \dots$, $j=0, 1, \dots$, where $\Delta^0 a_j=a_j$ and $\Delta^{m+1}a_j=\Delta^m a_{j+1}-\Delta^m a_j$.)

Thus, to establish (III) for $\{|p_n'(j)|\}$, $j=0, \dots, n$, it is required to verify

(20) $$\Delta^m\{(n+j)!(n-j)!\} \geqq 0,\quad j=0, \dots, n;\quad m=0, \dots, n-j.$$

Actually, strict positivity will be shown. This will be done by demonstrating for these j and m that

$$\Delta^m\{(n+j)!(n-j)!\} = (n+j)!(n-j-m)!R_n(j, m),$$

where $R_n(j, m)$ is a polynomial in j, m, n with exclusively nonnegative integer coefficients, not all zero.

Proof is by induction. The result is obvious for $n=0$, and is contained in (17) for $n=1$. Let it be assumed now for arbitrary m, $0<m<n-j$, and consider

$$\Delta^{m+1}\{(n+j)!(n-j)!\} =$$
$$= (n+j+1)!(n-j-m-1)!R_n(j+1, m)-(n+j)!(n-j-m)!R_n(j, m) =$$
$$= (n+j)!(n-j-m-1)![n\{R_n(j+1, m)-R_n(j, m)\}+$$
$$+(j+1)R_n(j+1, m)+(j+m)R_n(j, m)].$$

The expression in brackets is clearly a polynomial of the type described, as a consequence of the induction hypothesis, and so (20) holds.

A corresponding result for $\{|q_n'(j)|\}$, $j=1, \ldots, n+1$, can be derived in the same fashion, completing the proof of (III).

REMARKS. (i) Similar results hold for the sequences $\{|p_n'(j)|\}$, $\{|q_n'(j)|\}$ considered for each fixed j, as n varies.

(ii) D. J. Newman has called to my attention that, by utilizing the beta function, $\Delta^m\{(n+j)!(n-j)!\}$ can be represented explicitly as a definite integral which makes (20) obvious.

5. Open problems. Two unanswered questions are asked in § 3, Remark (ii). They deal with whether the convergence which occurs as $n\to\infty$ for certain first and second differences taken with respect to j is monotonic, in that context decreasing. A number of other results above suggest similar problems which the reader can formulate explicitly.

In another direction, it may be of interest to look at the behaviour of the zeros of $p_n''(x)$ and $q_n''(x)$, say x_{nj}'' and ξ_{nj}'', respectively, since these are of geometric significance, as well as higher order derivatives generally.

There arises also the problem of weakening hypotheses and/or strengthening conclusions. Some possibilities appear to be present in terms of higher differences, as defined in connection with (III).

Mr. Anastase Mastoras, in a paper written for a course taught by Dr. Marian Shepherd at York University, has been kind enough to make extensive numerical calculations of the roots of the first two derivatives of $p_n(x)$ and $q_n(x)$, and then to calculate appropriate differences relevant to the foregoing questions. His computations suggest that

$$(21)\qquad \begin{cases}(-1)^m\Delta^m x_{nj}' \geqq 0; & m=2, 3, \ldots, n-1,\\ (-1)^m\Delta^m x_{nj}'' \geqq 0; & m=2, 3, \ldots, n-2,\end{cases}$$

and similarly for ξ_{nj}', ξ_{nj}'', where the differencing is done with respect to j. For x_{nj}' and ξ_{nj}', and $m=2$, this is the substance of the Erdős—Bálint result.

His calculations were performed for polynomials of degrees 5, 10, 15, 20, 50, 101, 102, 307, 1000. In the notation of (1) and (2), this means for $p_n(x)$ with $n=2, 7, 50, 153$, and for $q_n(x)$ with $n=4, 9, 24, 50, 499$.

Some of his calculations fail to conform with (21), but only for entries combining high degree of the polynomial, high rank of the zero and at least moderately high order of differencing. Thus, these conflicts may well be due to round-off error, and so it would still be reasonable to conjecture the truth of (21).

Thanks are extended also to Professor Irene Gargantini of the University of Western Ontario for her interest and advice.

If (21) is indeed true for x''_{nj} and ξ''_{nj}, this would suggest a conjecture in which the hypotheses on $p_n(x)$, $q_n(x)$ are weakened, with the conclusion still strengthened to (21). It would be reasonable to assume that x_{nj}, ξ_{nj}, instead of being equally spaced, satisfy $(-1)^m \Delta^m x_{nj} \geqq 0$, $m=2, 3, \ldots, n$, while, of course $\Delta x_{nj} > 0$ (differencing with respect to j), and to expect that, still, $(-1)^m \Delta^m x'_{nj} \geqq 0$, $m=2, 3, \ldots, n$, with the result true also when x is replaced by ξ throughout.

If this conjecture be proved as stated, then the truth of the first part of (21), i.e., $(-1)^m \Delta^m x'_{nj} \geqq 0$, $(-1)^m \Delta^m \xi'_{nj} \geqq 0$, would automatically imply not only the second part of (21), but also the analogous result for all relevant higher derivatives.

Analogous results may hold also for the sequences arising in (13) and (14), where the differencing would be with respect to n rather than j.

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A RANDOM FUNCTIONAL CENTRAL LIMIT THEOREM FOR MARTINGALES

By

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1. Introduction. The last decade has witnessed great developments in the area of martingale central limit theorems (CLT). A recent paper by B. M. BROWN [3] may be referred to for a brief outline of the historical development, and a good bibliography. It may be mentioned that the most general types of results in this direction provide not only a proof of the classical Lindeberg—Feller CLT for martingales, but also guarantee the weak convergence of all finite dimensional distributions of an a.e. sample continuous stochastic process to those of a Wiener process. A *functional CLT* (also known as an *invariance principle*) is proved which says that the distributions of the said process converge weakly to a Wiener measure on $C[0, 1]$.

Functional CLT's were proved for martingales under the stationarity and ergodicity assumptions by BILLINGSLEY [1], [2] and IBRAGIMOV [6]. These conditions were relaxed and replaced by a Lindeberg-type condition by BROWN [3]. The present paper extends Brown's results to a martingale sequence with random indices, proving a functional CLT. The main results are given in section 2. Classical random CLT's for martingales are proved by CSORGO [4] and PRAKASA RAO [7]. We shall see at the end of section 2 that conditions imposed by them for proving the CLT imply ours, and are, in fact, much more restrictive.

2. The main results. We adopt the same notations as BROWN's [3]. Let $\{S_n, \mathscr{F}_n, n \geqq 1\}$ be a martingale sequence on the probability space $(\Omega, \mathscr{F}, \mathbf{P})$ with $S_0=0$. Define $X_n=S_n-S_{n-1}, n\geqq 1$. $\mathscr{F}_0$ need not be the trivial σ-field $\{\emptyset, \Omega\}$. Let $\mathbf{E}_{j-1}(Y)=\mathbf{E}(Y/\mathscr{F}_{j-1})$. Define

(2.1) $$\sigma_j^2 = \mathbf{E}_{j-1}(X_j^2), \qquad j \geqq 1,$$

(2.2) $$V_n^2 = \sum_{j=1}^{n} \sigma_j^2, \qquad n \geqq 1,$$

(2.3) $$s_n^2 = \mathbf{E}(V_n^2) = \mathbf{E}(S_n^2), \quad n \geqq 1.$$

We assume the following two conditions are satisfied:

(2.4) $$V_n^2 s_n^{-2} \xrightarrow{\mathbf{P}} 1 \quad \text{as} \quad n \to \infty$$

(2.5) $$s_n^{-2} \sum_{j=1}^{n} \mathbf{E}[X_j^2 I(|X_j| \geqq \varepsilon s_n)] \to 0,$$

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as $n\to\infty$ for all $\varepsilon>0$, where $I(B)$ denotes the indicator function of B. Martingale functional CLT's were proved by BROWN [3] under (2.4) and (2.5). Under the same conditions, we shall prove here the following two theorems, the second one giving in fact a random functional CLT for martingales. With this end, first define the process

$$\xi_n(t) = s_n^{-1}(S_r + X_{r+1}(ts_n^2 - s_r^2)/(s_{r+1}^2 - s_r^2)), \tag{2.6}$$

for $0\leqq t\leqq 1$, and $s_n^{-2}s_r^2 \leqq t \leqq s_n^{-2}s_{r+1}^2$, $r=0, 1, \dots, n-1$, $s_0=0$. Then we have the following two theorems.

THEOREM 1. *Let $B\in\mathcal{F}_k$ and $\mathbf{P}(B)>0$. Then under* (2.4) *and* (2.5) *one has*

$$\lim_{n\to\infty} \mathbf{P}(s_n^{-1}S_n \leqq x|B) = \Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}y^2)\,dy,$$

for all x. Further, all the finite dimensional distributions of $\xi_n(t)$ converge weakly under the measure $\mathbf{P}_B$ to the finite dimensional distributions of the Wiener measure, where

$$\mathbf{P}_B(A) = \mathbf{P}(A|B) \quad \textit{for any} \quad A\in\mathcal{F}. \tag{2.7}$$

THEOREM 2. *Let $\{\nu_n\}$ be a sequence of positive integer valued random variables (rv's) defined on $(\Omega, \mathcal{F})$. Also, let there exist a sequence $\{a_n\}$ of positive integers such that $a_n\to\infty$ as $n\to\infty$, and*

$$s_{\nu_n}^2/s_{a_n}^2 \overset{\mathbf{P}}{\to} \lambda, \tag{2.8}$$

for some positive rv λ. Then under (2.4) *and* (2.5) *the process $\{\xi_{\nu_n}(t): 0\leqq t\leqq 1\}$ converges weakly to the Wiener measure.*

Theorem 1 ensures a Rényi type mixing condition (see RÉNYI [8]). This is the major tool used in proving theorem 2 along the lines of BILLINGSLEY [2].

PROOF OF THEOREM 1. Let $B\in\mathcal{F}_k$ and $\mathbf{P}(B)>0$. Define

$$S_n^B = S_n \quad \text{if} \quad n\geqq k+1, \quad S_k^B = 0, \quad X_{n,B} = S_n^B - S_{n-1}^B \quad \text{for} \quad n\geqq k+1, \tag{2.9}$$

so that $X_{n,B}=X_n$ if $n\geqq k+2$, $X_{k+1,B}=S_{k+1}$. To prove the theorem, first observe that $\{S_n^B, \mathcal{F}_n, n\geqq k+1\}$ is a martingale on $(\Omega, \mathcal{F}, \mathbf{P}_B)$. To see this, note that with the use of the notation $\mathbf{E}_B(f)$ for $\int f\,d\mathbf{P}_B$, one has for any $A\in\mathcal{F}$, and for any $n\geqq k+2$,

$$\int_A \mathbf{E}_B(X_n^B|\mathcal{F}_{n-1})\,d\mathbf{P}_B = \int_A \mathbf{E}_B(X_n|\mathcal{F}_{n-1})\,d\mathbf{P}_B = \int_A X_n\,d\mathbf{P}_B = \tag{2.10}$$

$$= (\mathbf{P}(B))^{-1} \int_{A\cap B} X_n\,d\mathbf{P} = (\mathbf{P}(B))^{-1} \int_{A\cap B} \mathbf{E}(X_n|\mathcal{F}_{n-1})\,d\mathbf{P} =$$

$$= (\mathbf{P}(B))^{-1} \int_A I_B \mathbf{E}(X_n|\mathcal{F}_{n-1})\,d\mathbf{P}.$$

But $B\in\mathcal{F}_k\subset\mathcal{F}_{n-1}\,(n\geqq k+2) \Rightarrow I_B\mathbf{E}(X_n|\mathcal{F}_{n-1})$ is $\mathcal{F}_{n-1}$ measurable. Hence

$$\mathbf{E}_B(X_n|\mathcal{F}_{n-1}) = I(B)\mathbf{E}(X_n|\mathcal{F}_{n-1}) = 0 \quad \text{a.e.} \quad [\mathbf{P}_B].$$

Define now $\sigma^2_{n,B}=\mathbf{E}_B(X^2_{n,B}|\mathscr{F}_{n-1})$, $n\geqq k+1$. Proceeding as in the earlier paragraph one gets

(2.11) $$\sigma^2_{n,B} = I(B)\mathbf{E}(X^2_n|\mathscr{F}_{n-1}) \quad \text{a.e} \quad [\mathbf{P}_B].$$

Let $V^2_{n,B}=\sum_{j=k+1}^{n}\sigma^2_{j,B}$ for $n\geqq k+1$. Then one gets

(2.12) $$V^2_{n,B} = I(B)\sum_{j=k+2}^{n}\mathbf{E}(X^2_j|\mathscr{F}_{j-1})+\sigma^2_{k+1,B} \quad \text{a.e.} \quad [\mathbf{P}_B]$$
$$= I(B)V^2_n - I(B)\sum_{j=1}^{k+1}\mathbf{E}(X^2_j|\mathscr{F}_{j-1})+\sigma^2_{k+1,B} \quad \text{a.e.} \quad [\mathbf{P}_B[.$$

Hence, $V^2_{n,B}/s^2_n \overset{\mathbf{P}_B}{\to} 1$ as $n\to\infty$ and proceeding as in lemma 1 of Brown [3]

(2.13) $$\mathbf{E}_B(V^2_{n,B}/s^2_n) \to 1 \quad \text{as} \quad n\to\infty.$$

This leads to

(2.14) $$V^2_{n,B}(\mathbf{E}_B V^2_{n,B})^{-1} \overset{\mathbf{P}_B}{\to} 1 \quad \text{as} \quad n\to\infty.$$

Again the Lindeberg condition for the sequence $\{X_{n,B}, n\geqq k+1\}$ of rv's namely

(2.15) $$(\mathbf{E}_B V^2_{n,B})^{-1}\sum_{j=k+1}^{n}\mathbf{E}_B[X^2_{j,B}I(|X_{j,B}| > \varepsilon\mathbf{E}_B^{\frac{1}{2}}(V^2_{n,B}))] \to 0$$

as $n\to\infty$ for all $\varepsilon>0$, follows from the definition of $X_{j,B}$'s ($j\geqq k+1$), (2.5) and (2.13). Hence, by theorem 2 of Brown [3] one has

(2.16) $$\lim_{n\to\infty}\mathbf{P}_B((\mathbf{E}_B V^2_{n,B})^{-\frac{1}{2}}S_{n,B}\leqq x) = \Phi(x).$$

(2.9), (2.13) and (2.16) now lead to

(2.17) $$\lim_{n\to\infty}\mathbf{P}(s_n^{-1}S_n \leqq x|B) = \Phi(x).$$

From the same theorem of Brown, it follows that the finite dimensional distributions of $\xi_n(t)$ converge weakly under the measure $\mathbf{P}_B$ to the corresponding finite dimensional distributions of the Wiener measure. This completes the proof of Theorem 1.

Remark 1. It is also possible to have a result similar as Brown's theorem 3. This essentially says that if $\{D, \mathscr{D}, \mathbf{W}\}$ is the probability space, where $D=D[0, 1]$, $\mathscr{D}$ is the Borel σ-field generated by open sets in D, and $\mathbf{W}$ is Wiener measure on $D[0, 1]$, then $\mathbf{P}(\xi_n\in A|B)\to\mathbf{W}(A)$, for all $\mathbf{W}$-continuity sets A in D and $B\in\mathbf{U}$, where $\mathbf{U}=\{B:B\in\mathscr{F}_k$ for some $k\geqq 1$ and $\mathbf{P}(B)>0\}$. Note that Brown's result is for all $\mathbf{W}'$-continuity sets A in $C=C[0, 1]$, where $\mathbf{W}'$ is Wiener measure on C, but the extension is trivial since $\mathbf{W}(D-C)=0$.

Proof of Theorem 2. We proceed analogously as theorem 10.3 and 17.2 of Billingsley [3]. First, let p_n be a sequence of real numbers such that

$$s_{p_n}\to\infty \text{ and } s_{p_n}/s_n\to 0 \quad \text{as} \quad n\to\infty.$$

Defining now $Y_n(t)=0$ if $ts_n^2<s_{p_n}^2$ and $=s_n^{-1}(S_r-S_{p_n})$ if $s_{p_n}^2\leqq s_r^2\leqq ts_n^2\leqq s_{r+1}^2$ for all $r\geqq p_n$, one gets from (2.6)

$$(2.18)\qquad |\xi_n(t)-Y_n(t)| = \begin{cases} |\xi_n(t)| \quad \text{if} \quad ts_n^2 < s_{p_n}^2 \\ s_n^{-1}|S_{p_n}+X_{r+1}(ts_r^2-s_n^2)/(s_{r+1}^2-s_r^2)| \\ \text{if} \quad s_{p_n}^2 \leqq s_r^2 \leqq ts_n^2 \leqq s_{r+1}^2 \quad \text{for all} \quad r \geqq p_n. \end{cases}$$

Now from (2.6)

$$(2.19)\qquad \sup_{0\leqq t\leqq s_{p_n}^2} |\xi_n(t)| \leqq s_n^{-1} \max_{1\leqq i\leqq p_n} (|S_i|+|X_i|) \leqq 3s_n^{-1} \max_{1\leqq i\leqq p_n} |S_i|,$$

using $X_i=S_i-S_{i-1}$ $(1\leqq i\leqq p_n)$. Thus, from (2.18),

$$(2.20)\qquad \sup_t |\xi_n(t)-Y_n(t)| \leqq 3s_n^{-1} \max_{1\leqq i\leqq p_n} |S_i|+s_n^{-1} \max_{1\leqq r\leqq n} |X_r|.$$

The Kolmogorov inequality for martingales gives

$$(2.21)\qquad \mathbf{P}\left\{\max_{1\leqq i\leqq p_n} |S_i| > \varepsilon s_n\right\} \leqq \varepsilon^{-2}s_n^{-2}s_{p_n}^2 \to 0$$

as $n\to\infty$. Also, from (2.5), $s_n^{-1} \max_{1\leqq r\leqq n} |X_r| \xrightarrow{\mathbf{P}} 0$ as $n\to\infty$. (2.20) and (2.21) now give

$$(2.22)\qquad \sup_t |\xi_n(t)-Y_n(t)| \xrightarrow{\mathbf{P}} 0 \text{ as } n \to \infty.$$

By virtue of theorem 1 (the remarks following it), (2.22) and theorem 3 of Brown, it follows that

$$(2.23)\qquad |\mathbf{P}\{(Y_n\in A)\cap B\}-\mathbf{P}(Y_n\in A)\mathbf{P}(B)| \to 0$$

as $n\to\infty$ for all $B\in\mathbf{U}$ and for all $\mathbf{W}$-continuity sets A in D.

Next we proceed as theorem 17.2 of Billingsley, changing the definitions of $\Phi_n(t, w)$ in his (17.16) by

$$\Phi_n(t, w) = ts_{v_n(w)}^2/s_{a_n}^2 \quad \text{if} \quad s_{v_n(w)}^2/s_{a_n}^2 \leqq 1$$

and $t\theta$ otherwise. This leads to the result.

REMARK 2. Condition (2.8) seems to be more involved than the usual condition

$$(2.24)\qquad v_n/a_n \xrightarrow{\mathbf{P}} \lambda, \quad \text{as} \quad n \to \infty,$$

where λ is a positive rv. It is easy to check that for a stationary sequence $\{X_i\}$ of rv's with $\mathbf{E}(X_1)=0$, $\mathbf{E}(X_1^2)=\sigma^2$, (2.8) in fact reduces to (2.24). However, (2.24) along with (2.4) and (2.5) will not lead to theorem 2 in general. The following example illustrates this.

Let $X_1, X_2, \ldots$ be independent normal variables with zero means, and $\mathbf{V}(X_1)=$ $=\mathbf{V}(X_2)=1$, $\mathbf{V}(X_i)=\exp(i/\log i)-\exp((i-1)/\log(i-1))$ for $i\geqq 3$. Then,

$$(2.25)\qquad s_n^2 = \mathbf{V}(S_n) = \exp(n/\log n)-\exp(2/\log 2)+2,$$

for $n\geqq 3$. Define $\xi_n(t)$ $(0\leqq t\leqq 1)$ as in (2.6). Using $n/\log n-(n-1)/\log(n-1)\to 0$ as $n\to\infty$, it is easy to check that $\mathbf{V}(X_n)/s_n^2\to 0$ as $n\to\infty$. In this case $V_n^2 s_n^{-2}=1$ for all

$n \geqq 1$ so that (2.4) is satisfied. Also, appealing to theorem 2, p. 492 of FELLER [5] we find that (2.5) is satisfied. Hence, from BROWN's [3] result, ξ_n converges weakly in $C[0, 1]$ to W, the standard Brownian motion process.

Define now

$$(2.26) \qquad v_n = \max \{j \leqq n : S_j \geqq 0\}, \quad n \geqq 1;$$

$$(2.27) \qquad m_n = [n - (\log n)^2], \quad n \geqq 1,$$

$[u]$ denoting the integer part of u. Then, since, $\log (s_{m_n}^2 / s_n^2) \sim -\log n$ (by $a_n \sim b_n$ we mean $a_n/b_n \to 1$ as $n \to \infty$), it follows that

$$(2.28) \qquad s_{m_n}^2 / s_n^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Note now that using (2.28) and the weak convergence of ξ_n to W in $C[0, 1]$,

$$(2.29) \; \mathbf{P}(v_n < m_n) = \mathbf{P}\Big\{ \sup_{s_{m_n}^2 s_n^{-2} \leqq t \leqq 1} W_n(t) < 0 \Big\} \to \mathbf{P}\Big\{ \sup_{0 \leqq t \leqq 1} W(t) < 0 \Big\} = 0 \quad \text{as} \quad n \to \infty.$$

From the definition of m_n in (2.27), it follows now that

$$(2.30) \qquad v_n / n \overset{\mathbf{P}}{\to} 1.$$

However, $\xi_{v_n}(1) = s_{v_n}^{-1} S_{v_n} \geqq 0$ for all n, so that ξ_{v_n} does not converge weakly in $C[0, 1]$ to W.

REMARK 3. CSORGO [4] proved a random CLT for a sequence of martingales with $\mathbf{E}(X_1^2) = \sigma_1^2$, $\mathbf{E}(X_k^2 | \mathcal{F}_{k-1}) = \sigma_1^2$ for all $k \geqq 2$. It is easy to see then that $V_n^2 / s_n^2 = 1$ for all n so that (2.4) is automatically satisfied. Also, $s_n^{-2} \max_{1 \leqq j \leqq n} \sigma_j^2 = n^{-1} \to 0$ as $n \to \infty$. Further, defining

$$(2.24) \qquad \varphi_j(t) = \mathbf{E}[\exp(itX_j) | \mathcal{F}_{j-1} [= \mathbf{E}_{j-1}[\exp(itX_j)]], \quad j \geqq 1,$$

$$(2.25) \qquad f_n(t) = \prod_{j=1}^{n} \varphi_j(t/s_n), \quad n \geqq 1,$$

one gets $f_n(t) = \left(1 - \dfrac{t^2}{2n} + o(n^{-1})\right)^n$, so that $\log f_n(t) \to -\frac{1}{2} t^2$ as $n \to \infty$. It follows now from theorem 1 of BROWN [3] that (2.5) holds. Thus Csorgo's assumptions imply ours.

REMARK 4. PRAKASA RAO [7] proved a random CLT under stationarity and ergodicity conditions of BILLINGSLEY [1], [2] along with the strong mixing condition

$$(2.26) \qquad |\mathbf{P}(B|A) - \mathbf{P}(B)| \leqq \psi(n),$$

if $A \in \mathcal{F}_l$, $B \in$ the σ-algebra generated by $(X_{l+n}, X_{l+n+1}, \ldots)$ for a fixed l, where $1 \geqq \psi(1) \geqq \psi(2) \geqq \ldots$, $\lim_{n \to \infty} \psi(n) = 0$. One can see easily that stationarity and ergodicity imply (2.5) and (2.4), where (2.26) is redundant.

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FORMULAS RELATING THE AREA OF THE SURFACE OF A DOMAIN TO THE CURVATURE OF SURFACES FILLING THE DOMAIN

By
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ADLER [1] derived several formulas of the type described in the title for domains in Euclidean spaces of two or three dimensions. He also supplied a clever proof of a formula for planar domains which allows the family of curves to be fractured along a "fault". We think that his "unfractured" formulas are just integrated forms of the first variation formula for the area of a hypersurface. Such formulas can be given readily in any number of dimensions on any Riemannian manifold. It is not much more difficult to give a many-dimensional formula when "faults" are present. Formulas of this kind are most easily treated using the exterior differential calculus of E. Cartan. We follow the notation of FLANDERS [2].

Let M be an oriented smooth Riemannian manifold of dimension $n+1$ and suppose $W: M\to\mathbf{R}$ to be a smooth function. Set $\Omega=\{0\leqq W\leqq 1\}\subset M$ and suppose first of all that W has no singularities on $\bar{\Omega}$. Take local positively-oriented orthonormal frame fields $\mathbf{e}_1, \dots, \mathbf{e}_{n+1}$ such that $\mathbf{e}_{n+1}$ is perpendicular to the level surfaces $W=$constant. Let $\sigma_1, \dots, \sigma_{n+1}$ be the dual solder one-forms with corresponding connection one-forms $\omega_{ij}(1\leqq i\leqq n+1,\ 1\leqq j\leqq n+1)$. Define g by $dW=g\sigma_{n+1}$. The non-singularity of W on $\bar{\Omega}$ is equivalent to the non-vanishing of g on $\bar{\Omega}$. Finally, let $*$ be the Hodge duality operator.

From $dW=g\sigma_{n+1}$, we get $*dW=g\sigma_1 \dots \sigma_n$ (exterior product). Then

$$d*dW= dg(\sigma_1 \dots \sigma_n)+g\sum_{i=1}^{n}(-1)^{i-1}\sigma_1 \dots d\sigma_i \dots \sigma_n.$$

Introduce the second fundamental form coefficients (a_{ik}) by the equation

$$\omega_{i,n+1}=\sum_{k=1}^{n} a_{ik}\sigma_k.$$

The first Cartan structure equation is

$$d\sigma_i=\sum_{j=1}^{n+1}\omega_{ij}\sigma_j.$$

We want that term in $d\sigma_i$ containing σ_i and σ_{n+1} since all others contribute 0 to the sum for $d*dW$. Because ω_{ij} does depend on all the σ's in general, we must take the term with $j=n+1$, and find the ith summand to be

$$(-1)^{i-1}\sigma_1 \dots \sigma_{i-1}\left(\sum_{k=1}^{n} a_{ik}\sigma_k\right)\sigma_{n+1}\sigma_{i+1}\dots\sigma_n=(-1)^{n-1}a_{ii}\sigma_1\dots\sigma_{n+1}.$$

Also, if $dg=\sum_{k=1}^{n+1} g_k\sigma_k$, then

$$(1)\qquad d*dW=\left[(-1)^n g_{n+1}+(-1)^{n-1}g\sum_{i=1}^{n} a_{ii}\right]\sigma_1\dots\sigma_{n+1}.$$

The element of area on a level surface is $\sigma_1\dots\sigma_n$, and

$$(2)\qquad d(\sigma_1\dots\sigma_n)=d\left(\frac{1}{g}g\sigma_1\dots\sigma_n\right)=(-1)^{n+1}\frac{g_{n+1}}{g}\sigma_1\dots\sigma_{n+1}+\frac{1}{g}d*dW.$$

Form the combination $\frac{1}{g}(1)+(2)$, cancel like terms, and get

$$d(\sigma_1\dots\sigma_n)=\left[(-1)^{n-1}\sum_{i=1}^{n} a_{ii}\right]\sigma_1\dots\sigma_{n+1}.$$

If we integrate this over Ω, the right-hand side gives

$$\int_{\Omega}(-1)^{n-1}\sum_{i=1}^{n} a_{ii}\,dV.$$

By Stokes' Theorem, the left-hand side integrates to

$$\int_{\partial\Omega}\sigma_1\dots\sigma_n=\text{area}_1-\text{area}_0.$$

The quantity $(-1)^n\sum_{i=1}^{n}$ is to be identified with Adler's $K(P)$, and is the signed mean curvature. (If $n=2$, the minus sign makes up for the opposite orientations of e_2 and the Frenet normal.) Here we have Adler's formulas (1) and (2) proved with less computation and in greater generality.

It is interesting to note that the underlying differential (unintegrated) formula (2) has been used by pure and applied mathematicians for quite a long time. For example, it has been known to geodesists for about a century under the name of Bruns' Formula [5: *p.* 80].

To get the usual variational formula, take the interval $[0, \varepsilon]$ instead of $[0, 1]$. Then

$$\text{area}_\varepsilon-\text{area}_0=\int_{0\le W\le\varepsilon}[(-1)^{n-1}\sum a_{ii}]\sigma_1\dots\sigma_{n+1}.$$

For small ε, $\{0\le W\le\varepsilon\}$ is a product set, and Fubini's Theorem allows integration over the normal direction (the σ_{n+1} integral) first. Dividing by ε and letting $\varepsilon\to 0$, we get the usual formula for variation of surface area:

$$\frac{d}{d\varepsilon}\text{area}_\varepsilon|_{\varepsilon=0}=\int_{W=0}[(-1)^{n-1}\sum a_{ii}]\sigma_1\dots\sigma_n.$$

Now for the singularities. In the manifold M of dimension n, let C be a finite union of disjoint, embedded, compact, at most n-dimensional, smooth submanifolds with boundary. Neither C nor ∂C has to be connected (Fig. 1). (We would like to use a finite differentiable *complex* as singularity set, but there seem to be rather grave difficulties in defining a normal bundle in all but the most "obvious" simple cases.) Suppose W: $M \to \mathbf{R}$ and that C is contained in $\{0 < W < 1\}$. Let $\Omega = \{0 \leqq W \leqq 1\}$. Assume W is differentiable in $\Omega - C$ and is "almost smooth" on Ω in the following sense. If a component of C has codimension at least 2, then W is to be continuous across that component in the usual sense with one-sided derivatives at W along any transversal smooth curve in Ω. If a component has codimension 1, then its relative interior locally separates M, and W is to be one-sidedly differentiable on any smooth curve in Ω at points of that component (with obvious one-sided differentiability at boundary points). Finally assume that no point in the relative interior of a codimension one component is on the boundary of two different levels on the same side (Fig. 2 shows a forbidden case.)

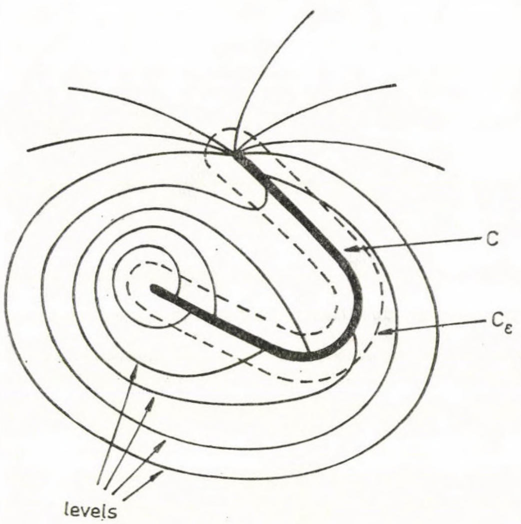

Fig. 1

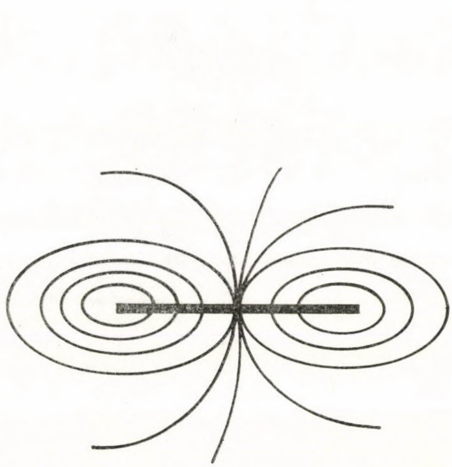

Fig. 2

Now pick an $\varepsilon > 0$, so small that the set C_ε of points in M at Riemannian distance ε from C forms a submanifold of M, which will have full smoothness except at finitely many subsets of lower dimension on which it may be only of class C^1 (cf. [4]). Since C_ε has dimension n and clearly has an "inside" (containing C), it divides M and has an "outside". Let Ω_ε be the closed set consisting of those points of Ω outside the inside of C_ε. With the same Cartan apparatus as before, we obtain

$$\int_{\Omega_\varepsilon} \left[(-1)^{n-1} \sum a_{ii}\right] dV = \text{area}_1 - \text{area}_\varepsilon + \int_{C_\varepsilon} \sigma_1 \dots \sigma_n. \tag{3}$$

We have to explain the last integral. Give C_ε positive orientation in the frame ordering (ordered bases for tangent vectors, normal pointing to the outside). Exactly which ordered basis of tangent vectors to take will be specified in the next paragraph.

By $\sigma_1 \ldots \sigma_n$, we intend the form obtained by pulling back $\sigma_1 \ldots \sigma_n$ from M to C_ε by the inclusion map of C_ε into M.

To compute the last integral, we specialize the frames $\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}$. Choose them so that when the base point approaches C_ε, the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ become tangent to C_ε (and then furnish the ordered basis required above). A geodesic normal to C will be normal to C_ε, by Gauss' Lemma. It is easy to see that, up to higher order in ε as $\varepsilon \to 0$, there is a relation holding on C_ε between the area element dC_ε and the pull back $\sigma_1 \ldots \sigma_n$ given by $\sigma_1 \ldots \sigma_n = (\cos \alpha) dC_\varepsilon$, where α is the angle between $\mathbf{e}_{n+1}$ and the outer normal to C_ε. Because the level surfaces have sufficiently nice behaviour as they approach the singular set C, a measure $d\mu$ is induced on the augmented unit normal bundle ν_+ of C (augmented, in the sense of [4]) in the following way. Over the relative interior of a component of C, the measure is obtained as the limit as $\varepsilon \to 0$ of the measure obtained by normalizing dC_ε by dividing by the area of the normal ε-sphere. Over boundaries, the normalizing divisor is the area of an ε-hemispherical cap. For every unit vector $\mathbf{v}$ normal to C, there will be an angle $\alpha(\mathbf{v})$ determined as $\varepsilon \to 0$, just by taking the limit of the angle α at points on the normal geodesic to C in the direction $\mathbf{v}$. Since C has measure zezo in M and $\text{area}_1 = \text{area}(\partial\Omega)$, we get

$$\int_\Omega [(-1)^{n-1} \sum a_{ii}]\, dV = \text{area}\,(\partial\Omega) + \int_{\nu_+} \cos \alpha(\mathbf{v})\, d\mu(\mathbf{v}). \tag{4}$$

The formula (3) of Adler's paper is a special case of our formula (4). In Adler's formula, C is a single one-simplex. At a point of the relative interior of C, there are two opposite unit normals, and the angles α are expressed in Adler's notation by $\frac{1}{2}\varphi \pm \psi$. Since $\cos(\frac{1}{2}\varphi + \psi) + \cos(\frac{1}{2}\varphi - \psi) = 2\cos\frac{1}{2}\varphi \cos\psi$, we obtain Adler's corrective term, with due regard to the meaning of integration with respect to $d\mu$. The "hemispheres" at the ends of C give no contribution in the limit because of the very nice behaviour Adler assumes for W at the ends of C.

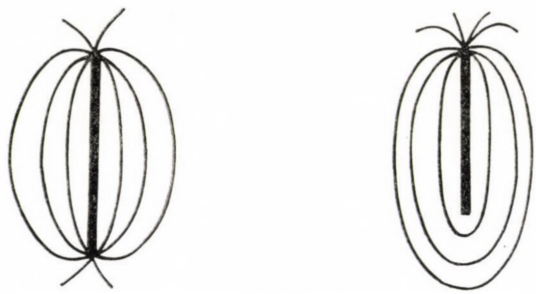

Fig. 3

It does seem necessary to use the augmented normal bundle since the level surfaces might have behaviour like that shown in Fig. 3. The measure $d\mu$ could be given *explicitly* in terms of the second fundamental forms of C and ∂C, along the lines set out in [3]. Finally, we reiterate that the method applies *ad hoc* to many cases where C has nastier singularities, but describing a larger class of C is a correspondingly nastier problem.

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A NOTE ON ONE-DIMENSIONAL f-EXPANSIONS

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§1. Let f be a decreasing function satisfying the following conditions:

A1) $f(1)=1$

A2) $f(t)$ is continuous and strictly decreasing to zero in $[1, T]$ and $f(t)=0$ for $t \geqq T$ where $2<T\leqq +\infty$. If $T=+\infty$ this is to mean $f(t)\to 0$ as $t\to +\infty$.

A3) $|f(t_2)-f(t_1)|\leqq |t_2-t_1|$ when $1\leqq t_1<t_2$ and there is a constant $\lambda\in(0, 1)$ such that $|f(t_2)-f(t_1)|\leqq \lambda |t_2-t_1|$ when $1+f(2)<t_1<t_2$.

BISSINGER [1] and RÉNYI [2] have shown that if f is so defined and if x is any real number then the expansion of x by the algorithm

$$\varepsilon_0(x) = [x], \quad r_0(x) = \{x\}$$

$$\varepsilon_{n+1}(x) = [\varphi(r_n(x))], \quad r_{n+1}(x) = \{\varphi(r_n(x))\}$$

is convergent. Here $[z]$ and $\{z\}$ denote the integral and fractional parts of z, respectively, and Φ is the function inverse to f. Hence any real number x has the "f-expansion"

$$x = \varepsilon_0(x)+f(\varepsilon_1(x)+f(\varepsilon_2(x)+\dots)\dots).$$

When either $2<T<+\infty$ and T is integral or when $T=+\infty$ Rényi has called this an f-expansion with "independent digits".

If $f_n(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x)+t)$ is written to mean $f(\varepsilon_1(x)+f(\varepsilon_2(x)+\dots+ +f(\varepsilon_n(x)+t)\dots)$ and if

$$H_n(x, t) \equiv \frac{d}{dt} f_n(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x)+t)$$

Rényi has imposed the regularity condition

$$\text{C)} \qquad \frac{\operatorname*{ess\,sup}_{0<t<1} |H_n(x, t)|}{\operatorname*{ess\,inf}_{0<t<1} |H_n(x, t)|} \leqq C$$

where C depends neither on x nor on n, and has proved the following theorem.

THEOREM (Rényi). *If f satisfies* A1), A2), A3) *and* C) *and if the digits are independent then for any $g\in L(0, 1)$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(r_k(x)) = M(g) \quad a.e. \ in \ (0, 1).$$

Here $M(g)$ is a finite constant which can be written as

$$M(g) = \int_0^1 g(x)h(x)\,dx$$

where $h(x)$ is a measurable function, depending only on $f(x)$ and satisfying $1/C \leqq h(x) \leqq C$ where C is the constant of condition C). The measure

$$\nu(E) = \int_E h(x)\,dx$$

is invariat with respect to the transformation $T(x) = \{\varphi(x)\}$ $(0<x<1)$.

Rényi also considered a similar class of increasing functions and proved a corresponding theorem but this will not concern us. Subsequent studies of these expansions have been made by ROHLIN [3], VINH-HIEN [4] and REZNIK [5] and analogues in n dimensions have been studied by several authors. An extensive bibliography is to be found in [6].

In all of these cases the regularity condition C) (or its higher-dimensional analogue) has been used and it is not at all clear how it can be lightened. Now this condition C) is clearly difficult to check in nearly every case so that examples of the theory are not easy to find. Unless one can find conditions which will imply C) it will be necessary to find an explicit formula for $H_n(x, t)$ and usually this will be impossible. Assuming throughout that the conditions A1), A2) and A3) are satisfied it is the purpose of the present note to show that C) is implied by the following two conditions:

A4) $f(t)$ is convex and $f(n+f(t))$ is concave in $(1, \infty)$ for each admissible digit n.

D) If A denotes the set of admissible digits then

$$\frac{\operatorname*{ess\,sup}_{0<t<1} \left| \frac{d}{dt} f(n+t) \right|}{\operatorname*{ess\,inf}_{0<t<1} \left| \frac{d}{dt} f(n+t) \right|} \leqq D$$

where D does not depend on $n \in A$.

§2. In this section we state two lemmas. The proof of the former is elementary and so shall not be given.

LEMMA 1. *Let $G(x)$ be convex decreasing or concave increasing on an interval containing a, b, c where $a<b<c$. Then*

$$\frac{G(b)-G(c)}{G(a)-G(c)} \leqq \frac{b-c}{a-c}.$$

If either of the two conditions on G still applies but $a>b>c$ then

$$\frac{G(b)-G(c)}{G(a)-G(c)} \geqq \frac{b-c}{a-c}.$$

It will be convenient to say that a real number γ "lies between" α and β if $\alpha \leqq \gamma \leqq \beta$ or $\alpha \geqq \gamma \geqq \beta$.

LEMMA 2. *Let f satisfy condition* A4). *Let $0 \leqq a < b < c$ or $a > b > c \geqq 0$ and let $b_1, b_2, \dots b_n$ denote any admissible digits. Then if $n \geqq 1$*

$$\frac{f_n(b_1, b_2, \dots, b_n+b) - f_n(b_1, b_2, \dots, b_n+c)}{f_n(b_1, b_2, \dots, b_n+a) - f_n(b_1, b_2, \dots, b_n+c)}$$

lies between $\dfrac{b-c}{a-c}$ $(=p$, *say*) *and* $\dfrac{f(b_n+b)-f(b_n+c)}{f(b_n+a)-f(b_n+c)}$ $(=q$, *say*).

If $a<b<c$ then $p \geqq q$ whilst if $a>b>c$ then $p \leqq q$.

PROOF OF LEMMA 2. We proceed by induction on n. Since $f_1(b_1+t) \equiv f(b_1+t)$ the result is trivially true when $n=1$. Next suppose that $n=2$ and consider

$$\frac{f\big(b_1+f(b_2+b)\big)-f\big(b_1+f(b_2+c)\big)}{f\big(b_1+f(b_2+a)\big)-f\big(b_1+f(b_2+c)\big)}.$$

If $a<b<c$ we have $f(b_2+a)>f(b_2+b)>f(b_2+c)$ and if $a>b>c$ we have $f(b_2+a)< <f(b_2+b)<f(b_2+c)$. Since $f(t)$ is convex decreasing and $f\big(b_1+f(t)\big)$ is concave increasing we find in either case, by Lemma 1, that the above quotient lies between $\dfrac{b-c}{a-c}$ and $\dfrac{f(b_2+b)-f(b_2+c)}{f(b_2+a)-f(b_2+c)}$.

Next suppose that the result has been proved for $n=m-1$, $n=m$ and consider $(m \geqq 2)$

$$A_{m+1} = \frac{f_{m+1}(b_1, b_2, \dots, b_{m+1}+b) - f_{m+1}(b_1, b_2, \dots, b_{m+1}+c)}{f_{m+1}(b_1, b_2, \dots, b_{m+1}+a) - f_{m+1}(b_1, b_2, \dots, b_{m+1}+c)} = \tag{2.1}$$

$$= \frac{f\big(b_1+f(b_2+B)\big)-f\big(b_1+f(b_2+C)\big)}{f\big(b_1+f(b_2+A)\big)-f\big(b_1+f(b_2+C)\big)} \quad \text{(say)}.$$

Since $f_{m-1}(b_3, b_4, \dots, b_{m+1}+t)$ is increasing or decreasing according as m is odd or even respectively then

$$A > B > C \quad \text{and} \quad f(b_2+A) < f(b_2+B) < f(b_2+C) \tag{2.2}$$

or

$$A < B < C \quad \text{and} \quad f(b_2+A) > f(b_2+B) > f(b_2+C). \tag{2.3}$$

Since $f(t)$ is convex decreasing and $f\big(b_1+f(t)\big)$ is concave increasing we apply Lemma 1 to (2.1) and find that in either of the cases (2.2) or (2.3) A_{m+1} lies between

$$\frac{B-C}{A-C} \quad \text{and} \quad \frac{f(b_2+B)-f(b_2+C)}{f(b_2+A)-f(b_2+C)}.$$

Now each of these is a quotient of the type being considered involving $m-1$ and m digits respectively. In each case the "last" digit is b_{m+1} and so by the induction hypothesis each lies between

$$\frac{b-c}{a-c} \quad \text{and} \quad \frac{f(b_{m+1}+b)-f(b_{m+1}+c)}{f(b_{m+1}+a)-f(b_{m+1}+c)}.$$

Hence A_{m+1} lies between these also and the induction is complete. To conclude the proof of Lemma 2 we note that the assertion in the last line of the lemma is a simple consequence of Lemma 1.

§3. In this section we apply Lemma 2 to show that conditions A4) and D) imply Rényi's condition C). It will be convenient to write

$$H_n(x, t) = \frac{d}{dt} G_n(x, t) \quad \text{where} \quad G_n(x, t) = f_n(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x)+t).$$

Let $0<a<b<1$ and consider

$$\frac{G_n(x,b)-G_n(x,a)}{G_n(x,1)-G_n(x,0)} = \frac{G_n(x,b)-G_n(x,a)}{G_n(x,1)-G_n(x,a)} \cdot \frac{G_n(x,1)-G_n(x,a)}{G_n(x,1)-G_n(x,0)} = P \cdot Q \quad \text{(say)}.$$

Now $1>b>a$ and $0<a<1$ and so by Lemma 2

$$\frac{b-a}{1-a} \leqq P \leqq \frac{f(\varepsilon_n(x)+b)-f(\varepsilon_n(x)+a)}{f(\varepsilon_n(x)+1)-f(\varepsilon_n(x)+a)} \tag{2.4}$$

and

$$\frac{f(\varepsilon_n(x)+1)-f(\varepsilon_n(x)+a)}{f(\varepsilon_n(x)+1)-f(\varepsilon_n(x)+0)} \leqq Q \leqq 1-a. \tag{2.5}$$

Note that all of these values are positive. Now by virtue of condition D)

$$\frac{f(\varepsilon_n(x)+b)-f(\varepsilon_n(x)+a)}{f(\varepsilon_n(x)+1)-f(\varepsilon_n(x)+a)} \cdot (1-a) \leqq (b-a)D$$

and

$$\frac{f(\varepsilon_n(x)+1)-f(\varepsilon_n(x)+a)}{f(\varepsilon_n(x)+1)-f(\varepsilon_n(x)+0)} \frac{b-a}{1-a} \geqq (b-a)\frac{1}{D}.$$

Hence by (2.4) and (2.5) we have

$$(b-a)\frac{1}{D} \leqq \frac{G_n(x,b)-G_n(x,a)}{G_n(x,1)-G_n(x,0)} \leqq (b-a)D$$

so that

$$|G_n(x,1)-G_n(x,0)|\frac{1}{D} \leqq \left|\frac{G_n(x,b)-G_n(x,a)}{b-a}\right| \leqq |G_n(x,1)-G_n(x,0)|D.$$

Now for each x, $H_n(x,t)$ is defined for almost all t. For such a value of t put $a=t$ and let $b \to t$ and we get

$$|G_n(x,1)-G_n(x,0)|\frac{1}{D} \leqq |H_n(x,t)| \leqq |G_n(x,1)-G_n(x,0)|D.$$

Hence for any x we find that

$$\frac{\operatorname*{ess\,sup}_{0<t<1} |H_n(x,t)|}{\operatorname*{ess\,inf}_{0<t<1} |H_n(x,t)|} \leqq D^2$$

and D is clearly independent of x and n. Hence condition C) is satisfied with $C=D^2$ and the proof of our assertion is complete.

§4. Examples of functions satisfying A1), A2), A3), A4) and D) are not difficult to find. Consider, for instance, the solution of the differential equation

$$\frac{d^2}{dx^2}f(x) = \frac{\alpha+1}{2}[f(x)]^\alpha$$

satisfying $f(1)=1, f'(1)=-1$. If we take $\alpha \geqq 1$ we find

$$f(x) = \begin{cases} e^{1-x} & (\alpha = 1) \\ \left[\dfrac{2}{(\alpha-1)x+3-\alpha}\right]^{\frac{2}{\alpha-1}} & (\alpha > 1). \end{cases}$$

It is easily verified, using the differential equation when necessary, that all of the conditions are satisfied. In fact one finds that one can take

$$\lambda = f(1+f(2)) \quad \text{and} \quad D = \left[\frac{f(1)}{f(2)}\right]^{\frac{\alpha+1}{2}}.$$

The case $\alpha=3$ is $f(x)=\dfrac{1}{x}$ which yields the case of continued fractions.

Clearly it was because we assumed $f(t)$ convex and $f(n+f(t))$ concave that we obtained inequalities on two sides for use in our proof of Lemma 2. In the case in which f is an increasing function, which Rényi also treated, one might attempt to consider the hypotheses $f(t)$ concave, $f(n+f(t))$ convex but in fact this cannot happen for an increasing $f(t)$ unless $f(t)$ is a linear function. Such functions are not in the class treated by Rényi and we have been unable to consider the case of $f(t)$ increasing.

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RADICALS AND FUNCTIONAL REPRESENTATIONS

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Let A be an associative ring with identity which is $\mathcal{N}_g$-semisimple (that is, A has no nonzero nilpotent elements). KOH [3] has shown that A is isomorphic to the ring of all global sections of a sheaf over the space of prime ideals of A. The stalk at the prime ideal P of A is A/O_P where $O_P=\{a\in A: ax=0 \text{ for some } x\in A\setminus P\}$. Let $\mathcal{R}$ be a radical class. If each stalk of Koh's sheaf is $\mathcal{R}$-semisimple, then A is $\mathcal{R}$-semisimple because $\bigcap\{O_P: P \text{ is a prime ideal of } A\}=(0)$, and so A is isomorphic to a subdirect product of the $\mathcal{R}$-semisimple stalks. In this paper we consider the converse problem: given an $\mathcal{R}$-semisimple ring A, are all the stalks of Koh's sheaf $\mathcal{R}$-semisimple?

Definitions and basic results from radical theory can be found in [2].

A prime ideal P of a ring A is a *minimal prime* if no ideal of A properly contained in P is prime. If $\mathcal{R}$ is a radical class and A is a ring such that A/P is $\mathcal{R}$-semisimple for every minimal prime ideal P of A, we shall say that the pair $(A, \mathcal{R})$ has *m.p.c. (the minimal prime condition).*

PROPOSITION 1. *Let $\mathcal{R}$ be a radical class containing $\mathcal{N}_g$ and A an $\mathcal{R}$-semisimple ring. The stalks of Koh's sheaf are $\mathcal{R}$-semisimple if and only if $(A, \mathcal{R})$ has m.p.c.*

PROOF. CORNISH [1] and SHIN [4] have shown that for each prime ideal P of A, $O_P=\bigcap\{Q: Q \text{ is a minimal prime ideal of } A \text{ and } Q\subseteq P\}$.

Suppose that the stalks of Koh's sheaf are $\mathcal{R}$-semisimple. If P is a minimal prime ideal of A, then $O_P=P$ and so A/P is $\mathcal{R}$-semisimple. Thus $(A, \mathcal{R})$ has m.p.c.

Conversely, suppose that $(A, \mathcal{R})$ has m.p.c. Then each stalk A/O_P of Koh's sheaf is $\mathcal{R}$-semisimple because it is isomorphic to a subdirect product of the family $\{A/Q: Q \text{ is a minimal prime ideal of } A \text{ and } Q\subseteq P\}$ of $\mathcal{R}$-semisimple rings.

PROPOSITION 2. *Let $\mathcal{R}$ be a hereditary radical class containing $\mathcal{N}_g$ and A an $\mathcal{R}$-semisimple ring. Suppose that whenever $\{I_\lambda: \lambda\in\Lambda\}$ is an ascending chain of ideals of A such that A/I_λ is $\mathcal{R}$-semisimple for all $\lambda\in\Lambda$, then $A/\bigcup\{I_\lambda: \lambda\in\Lambda\}$ is also $\mathcal{R}$-semisimple. Then the stalks of Koh's sheaf are $\mathcal{R}$-semisimple.*

PROOF. We verify that the pair $(A, \mathcal{R})$ has m.p.c. Let P be a minimal prime ideal of A. Use Zorn's Lemma to choose an ideal $Q\subseteq P$ such that A/Q is $\mathcal{R}$-semisimple and Q is maximal with this property.

If A/Q is not prime there is an ideal $I\supsetneqq Q$ such that $(I/Q)^*=\{\bar{a}\in A/Q: \bar{a}(I/Q)= =\bar{0}\}\neq\{\bar{0}\}$. Let $I_1/Q=(I/Q)^*$ and $I_2/Q=(I_1/Q)^*$. Since A/Q is $\mathcal{N}_g$-semisimple I_1 and

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I_2 are ideals of A such that $I_1 \cap I_2 \subseteq Q$. Moreover, both I_1 and I_2 properly contain Q. The ideal I_1+I_2/I_2 of A/I_2 is essential and $I_1+I_2/I_2 \cong I_1/I_1 \cap I_2 = I_1/Q$ is $\mathcal{R}$-semisimple. Thus, since $\mathcal{R}$ is hereditary, A/I_2 is $\mathcal{R}$-semisimple. Similarly, A/I_1 is $\mathcal{R}$-semisimple. Since $I_1 \cap I_2 \subseteq Q \subseteq P$, either $I_1 \subseteq P$ or $I_2 \subseteq P$. This contradicts the maximality of Q, so Q is prime.

Since P is a minimal prime, $Q=P$ and so A/P is $\mathcal{R}$-semisimple. Thus $(A, \mathcal{R})$ has m.p.c.

COROLLARY 3. *If $\mathcal{R}$ is a hereditary radical class containing $\mathcal{N}_g$ and A is an $\mathcal{R}$-semisimple ring which has the ascending chain condition on ideals I such that A/I is $\mathcal{R}$-semisimple, then the stalks of Koh's sheaf are $\mathcal{R}$-semisimple.*

A radical class $\mathcal{R}$ is *elementarily strict* if: a ring A is $\mathcal{R}$ semisimple if and only if every subring of A which is generated by one element is $\mathcal{R}$-semisimple.

THEOREM 4. *Let $\mathcal{R}$ be a hereditary elementarily strict radical class containing $\mathcal{N}_g$. A ring A is $\mathcal{R}$-semisimple if and only if every stalk of Koh's sheaf is $\mathcal{R}$-semisimple.*

PROOF. We have already noted that if the stalks of Koh's sheaf are $\mathcal{R}$-semisimple, then A is $\mathcal{R}$-semisimple.

To establish the converse we will check that the condition of Proposition 2 is satisfied. Let $\{I_\lambda : \lambda \in \Lambda\}$ be an ascending chain of ideals of A such that A/I_λ is $\mathcal{R}$-semisimple for all $\lambda \in \Lambda$. Let $I = \cup \{I_\lambda : \lambda \in \Lambda\}$. If A/I is not $\mathcal{R}$-semisimple, then, because $\mathcal{R}$ is elementarily strict, there is some $x \in A$ such that $(\langle x \rangle + I)/I$ is not $\mathcal{R}$-semisimple (here $\langle x \rangle$ denotes the subring of A which is generated by x). Because $\langle x \rangle$ satisfies the ascending chain condition there is a $\bar{\lambda} \in \Lambda$ such that $\langle x \rangle \cap I = \langle x \rangle \cap I_{\bar{\lambda}}$. Thus $(\langle x \rangle + I)/I \cong \langle x \rangle/(\langle x \rangle \cap I) = \langle x \rangle/(\langle x \rangle \cap I_{\bar{\lambda}}) \cong (\langle x \rangle + I_{\bar{\lambda}})/I_{\bar{\lambda}}$ is $\mathcal{R}$-semisimple because $A/I_{\bar{\lambda}}$ is $\mathcal{R}$-semisimple and $\mathcal{R}$ is elementarily strict. This contradiction shows that A/I is $\mathcal{R}$-semisimple.

COROLLARY 5. *Let $\mathcal{R}$ be a hereditary elementarily strict radical class containing $\mathcal{N}_g$. An $\mathcal{R}$-semisimple ring with identity is isomorphic to the ring of all global sections of Koh's sheaf and the stalks of this sheaf are $\mathcal{R}$-semisimple.*

COROLLARY 6. *Every hereditary elementarily strict radical class containing $\mathcal{N}_g$ is special.*

PROOF. Any supernilpotent radical class $\mathcal{R}$ with the property that $(A, \mathcal{R})$ has m.p.c. for all $\mathcal{R}$-semisimple rings A is special.

We conclude with some examples of hereditary elementarily strict radical classes containing $\mathcal{N}_g$. In each case we describe the associated class of semisimple rings.

1. The class of $\mathcal{N}_g$-semisimple rings.
2. For each integer $n \geqq 2$, the class $\mathcal{K}_n$ of all rings A such that $x^n = x$ for each $x \in A$ (see [5] and [6]).
3. The class of all $\mathcal{N}_g$-semisimple rings of characteristic zéro.

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ON THE APPROXIMATION AND SATURATION OF PERIODIC CONTINUOUS FUNCTIONS BY CERTAIN TRIGONOMETRIC INTERPOLATION POLYNOMIALS

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1. Introduction. Let $f(x)$ be a continuous periodic function with period 2π, E_n be the set of equidistant nodal points $t_j=2\pi j/(n+1)$ $(j=0, 1, \ldots, n)$, and $\omega_{n+1}(t)$ be a step function which has jumps $2\pi/(n+1)$ at the points of E_n and is constant in the interior of each interval (t_j, t_{j+1}). For these notations and fundamental properties of trigonometric interpolation, we refer to ZYGMUND's book [10, X]. Let $\{\alpha_{n,k}\}=\{\alpha_k\}$ $(k=1, 2, \ldots, n)$ be a triangular matrix of order n and set

$$K_n(t) = \frac{1}{2}\left\{1+\sum_{k=1}^{n} \alpha_k(e^{ikt}+e^{-ikt})\right\} = \frac{1}{2}+\sum_{k=1}^{n} \alpha_k \cos kt. \tag{1}$$

We consider the trigonometric polynomial

$$P_n f = \frac{1}{\pi}\int_0^{2\pi} f(t) K_n(x-t)\, d\omega_{n+1}(t) = \frac{2}{n+1}\sum_{j=0}^{n} f(t_j) K_n(x-t_j). \tag{2}$$

If $K_n(t)$ is the Fejér's kernel, then $P_n f$ is the Jackson polynomial of $f(x)$ and this polynomial is uniquely characterized by the interpolatory property $(P_n f)(t_j)=f(t_j)$, $(P_n f)'(t_j)=0$ $(j=0, 1, \ldots, n)$. For the approximation and saturation properties of the Jackson polynomial, V. F. VLASOV [8] and J. SZABADOS [5] gave a complete solution independently. Generalizing the Jackson polynomial, for a positive integer α, set

$$\alpha_k = \alpha_{n,k} = \frac{(n+1-k)^\alpha}{(n+1-k)^\alpha+k^\alpha}$$

in (1). These polynomials have been introduced by O. KIS [2], A. SHARMA and A. K. VARMA [3] and A. K. VARMA [6]. They and A. K. VARMA [7] investigated the approximation by these polynomials. If α is an odd integer, these polynomials are uniquely characterized by $(P_n f)(t_j)=f(t_j)$, $(P_n f)^{(\alpha)}(t_j)=0$ $(j=0, 1, \ldots, n)$. But if α is an even integer, then they only satisfy $(P_n f)(t_j)=f(t_j)$ $(j=0, 1, \ldots, n)$. Nevertheless these polynomials have a comparatively good approximation property. In the present note, we give a complete solution of the approximation and saturation problem by these polynomials. Our paper is heavily connected with SZABADOS' paper [5].

2. A general formula. LEMMA 1. *For any trigonometric polynomial*

$$t_n(x) = \frac{1}{2}a_0+\sum_{k=1}^{n} (a_k \cos kx+b_k \sin kx) = \sum_{k=0}^{n} A_k(x)$$

of order n, the trigonometric polynomial defined by (2) *has the property*

$$(3)\ (P_n t_n)(x) = \sum_{k=0}^{n} \alpha_k A_k(x) + \cos(n+1)x \sum_{k=0}^{n} \alpha_{n+1-k} A_k(x) + \sin(n+1)x \sum_{k=1}^{n} \alpha_{n+1-k} B_k(x)$$

where $B_k(x) = a_k \sin kx - b_k \cos kx$.

PROOF. For any integer p, q

$$(4)\ \frac{1}{\pi}\int_0^{2\pi} e^{iqt}\, d\omega_p(t) = \frac{2}{p}\sum_{j=0}^{p-1} e^{i2\pi jq/p} = \frac{2}{p}(e^{i2\pi q}-1)/(e^{\frac{i2\pi q}{p}}-1) = \begin{cases} 2 & \text{if } p/q \text{ integer} \\ 0 & \text{otherwise.} \end{cases}$$

For $f(t) = e^{imt}(0 < m \leqq n)$, we have

$$(P_n e^{imt})(x) = \frac{1}{\pi}\int_0^{2\pi} e^{imt} K_n(t-x)\, d\omega_{n+1}(t) =$$

$$= \sum_{k=0}^{n} \frac{\alpha_k}{2\pi}\int_0^{2\pi} e^{imt}(e^{ik(t-x)} + e^{-ik(t-x)})\, d\omega_{n+1}(t).$$

Since $p = n+1$, $0 < m+k \leqq 2n$ and $-n \leqq m-k \leqq n$, the nonvanishing terms in the last formula are $k = m$ and $k = n+1-m$ by (4). Hence we have

$$(P_n e^{imt})(x) = \alpha_m e^{imx} + \alpha_{n+1-m} e^{-i(n+1-m)x}.$$

Similarly for the negative index

$$(P_n e^{-imt})(x) = \alpha_m e^{-imx} + \alpha_{n+1-m} e^{i(n+1-m)x}.$$

We write these in real form,

$$(P_n \cos mt)(x) = \alpha_m \cos mx + \alpha_{n+1-m}\cos(n+1-m)x,$$

$$(P_n \sin mt)(x) = \alpha_m \sin mx - \alpha_{n+1-m}\sin(n+1-m)x.$$

Collecting these estimates, we get the formula (3).

Using the kernel (1), we set

$$(5)\qquad Q_n f = \frac{1}{\pi}\int_0^{2\pi} f(t) K_n(x-t)\, dt.$$

COROLLARY 1. *If we take*

$$(6)\qquad \alpha_k = \frac{(n+1-k)^\alpha}{(n+1-k)^\alpha + k^\alpha},$$

then

$$(P_n t_n)(x) = (Q_n t_n)(x) + \cos(n+1)x \sum_{k=1}^{n} \frac{1}{(n+1-k)^\alpha + k^\alpha} k^\alpha A_k(x) + \tag{7}$$

$$+ \sin(n+1)x \sum_{k=1}^{n} \frac{1}{(n+1-k)^\alpha + k^\alpha} k^\alpha B_k(x) =$$

$$= t_n(x) + \{\cos(n+1)x - 1\} \sum_{k=1}^{n} \frac{1}{(n+1-k)^\alpha + k^\alpha} k^\alpha A_k(x) +$$

$$+ \sin(n+1)x \sum_{k=1}^{n} \frac{1}{(n+1-k)^\alpha + k^\alpha} k^\alpha B_k(x).$$

3. The direct theorem. From now on, we only consider the case (6) and both $P_n f$ and $Q_n f$ mean the approximation process with this special kernel. At first we compare the approximation degree by $Q_n f$ with that of the typical mean of Fourier series, that is, with the matrix

$$\beta_k = 1 - \left(\frac{k}{n+1}\right)^\alpha = \frac{(n+1)^\alpha - k^\alpha}{(n+1)^\alpha}.$$

Then we have

$$1 - \alpha_k = \frac{\left(\frac{k}{n+1}\right)^\alpha}{\left(1 - \frac{k}{n+1}\right)^\alpha + \left(\frac{k}{n+1}\right)^\alpha} \quad (k < n+1); \quad = 1 \quad (k \geqq n+1)$$

$$1 - \beta_k = \left(\frac{k}{n+1}\right)^\alpha \quad (k < n+1); \quad = 1 \quad (k \geqq n+1).$$

So if we set

$$\alpha(x) = \frac{x^\alpha}{(1-x)^\alpha + x^\alpha} \quad (0 \leqq x \leqq 1); \quad = 1 \quad (x \geqq 1),$$

$$\beta(x) = x^\alpha \quad (0 \leqq x \leqq 1); \quad = 1 \quad (x \geqq 1),$$

then it is easy to see that both $\alpha(x)/\beta(x)$ and $\beta(x)/\alpha(x)$ are Fourier—Stieltjes transform of bounded measure. From the Poisson summation formula and Fourier multiplier theorem, we can conclude that the two approximation processes have the same order. For the proof of these facts, we refer to Chapter 6 of the book of P. L. BUTZER and R. J. NESSEL [1]. If the α-th derivative of $f(x)$ or $\tilde{f}(x)$ is bounded almost everopvhere, then the order of approximation by the typical means is $O(n^{-\alpha})$ according to α is even or odd, respectively, by ZYGMUND's result [9]. So the situation is the same for $Q_n f$. In the following, the norm of $A(\|A\|)$ means the uniform norm if A is a continuous function, the ess. sup norm if A is an essentially bounded function and the operator norm if A is an operator. If both $f^{(\alpha)}(x)$ and $\tilde{f}^{(\alpha)}(x)$ are bounded almost everywhere, then in the first formula of (7) we set $t_n(x)=(Q_nf)(x)$ and

$$\|f-P_nf\| \leqq \|f-Q_nf\|+\|Q_n(f-Q_nf)\|+\|Q_n(Q_nf)-P_n(Q_nf)\|+\|P_n(Q_nf-f)\| \leqq$$
$$\leqq \|f-Q_nf\|+\|Q_n\|\cdot\|f-Q_nf\|+\|P_n\|\cdot\|f-Q_nf\|+$$
$$+\left\|\sum_{k=1}^{n}\frac{1}{(n+1-k)^\alpha+k^\alpha}\,\frac{(n+1-k)^\alpha}{(n+1-k)^\alpha+k^\alpha}\,k^\alpha A_k(x)\right\|+$$
$$+\left\|\sum_{k=1}^{n}\frac{1}{(n+1-k)^\alpha+k^\alpha}\,\frac{(n+1-k)^\alpha}{(n+1-k)^\alpha+k^\alpha}\,k^\alpha B_k(x)\right\| \leqq$$
$$\leqq C_1\|f-Q_nf\|+C_2n^{-\alpha}(\|f^{(\alpha)}\|+\|\tilde f^{(\alpha)}\|) = O(n^{-\alpha}),$$

because $\|P_n\|=O(1)$ and $\|Q_n\|=O(1)$ by SHARMA and VARMA [3] and VARMA [5]. Hence we have the following theorem.

THEOREM 1. *If both $f^{(\alpha)}(x)$ and $\tilde f^{(\alpha)}(x)$ are bounded almost everywhere, then $\|f-P_nf\|=O(n^{-\alpha})$.*

4. The inverse theorem. THEOREM 2. *If $\|f-P_nf\|=O(n^{-\alpha})$ then both $f^{(\alpha)}(x)$ and $\tilde f^{(\alpha)}(x)$ are bounded almost everywhere.*

PROOF. Let in the second formula of (7) $t_n(x)$ be a polynomial of order ν ($\nu\leqq n$), change notation from n to m and from ν to n, and set

$$t_n(x) = (P_nf)(x) = \sum_{k=0}^{n} A_{n,k}(x),$$

then

$$P_m(P_nf) = (P_nf)(x)+\{\cos(m+1)x-1\}\sum_{k=1}^{n}\frac{1}{(m+1-k)^\alpha+k^\alpha}\,k^\alpha A_{n,k}(x)+$$
$$+\sin(m+1)x\sum_{k=1}^{n}\frac{1}{(m+1-k)^\alpha+k^\alpha}\,k^\alpha B_{n,k}(x) =$$
$$= (P_nf)(x)+\{\cos(m+1)x-1\}p_n(x)+\sin(m+1)x\,q_n(x),$$

say. From the hypothesis $\|f-P_nf\|=O(n^{-\alpha})$ and if $m=cn$ where c is an integral constant greater than 1, then

$$\|P_m(P_nf)-P_nf\| \leqq \|P_m(P_nf-f)\|+\|P_mf-f\|+\|f-P_nf\| = O(n^{-\alpha}).$$

Hence we have

$$\|\{\cos(m+1)x-1\}p_n(x)+\sin(m+1)x\,q_n(x)\| = O(n^{-\alpha}).$$

Let $M_n=\|p_n(x)\|$ and $|p_n(x_n)|=M_n$. By the mean value theorem and Bernstein's inequality, we have

$$|p_n(x_n-h)-p_n(x_n)| = |hp_n'(\xi_n)| \leqq nh\|p_n(x)\| = nhM_n,$$

where $x_n-h\leqq\xi_n\leqq x_n$. If we take $2nh\leqq1$, then $|p_n(x_n-h)|\geqq M_n/2$. The situation is the same to the right of x_n. So in the interval $\left(x_n-\frac{1}{2n},\ x_n+\frac{1}{2n}\right)$, $|p_n(x)|\geqq\frac{M_n}{2}$.

However

$$\cos(m+1)x-1 = -2\sin^2\frac{m+1}{2}x, \quad \sin(m+1)x = 2\sin\frac{m+1}{2}x\cos\frac{m+1}{2}x$$

and the points where $\cos\frac{m+1}{2}x=0$ appear with period $\frac{2\pi}{m+1}$. If we take $\frac{2\pi}{m+1}<\frac{1}{n}$, (for example, $m=8n$) then the interval $\left(x_n-\frac{1}{2n}, x_n+\frac{1}{2n}\right)$ contains a point x_0 such that $\cos\frac{m+1}{2}x_0=0$. At this point $\left|\sin\frac{m+1}{2}x_0\right|=1$. Hence $n^\alpha M_n \geqq n^\alpha|p_n(x_0)|=O(1)$. Hence $n^\alpha\|p_n(x)\|=O(1)$. Accordingly $\|\sin(m+1)xq_n(x)\|=O(n^{-\alpha})$. From the same reasoning above, we have $n^\alpha\|q_n(x)\|=O(1)$. That is to say, we have

$$\left\|\sum_{k=1}^{n}\frac{n^\alpha}{(8n+1-k)^\alpha+k^\alpha}k^\alpha A_{n,k}(x)\right\| = O(1),$$

$$\left\|\sum_{k=1}^{n}\frac{n^\alpha}{(8n+1-k)^\alpha+k^\alpha}k^\alpha B_{n,k}(x)\right\| = O(1).$$

From the formula

$$\frac{1}{\pi}\int_0^{2\pi} f(t)K_n(t-x)\,d\omega_{n+1}(t) = \sum_{k=0}^{n} A_{n,k}(x),$$

$$A_{n,k}(x)=\alpha_{n,k}\frac{1}{\pi}\int_0^{2\pi} f(t)\cos k(x-t)\,d\omega_{n+1}(t)=\frac{(n+1-k)^\alpha}{(n+1-k)^\alpha+k^\alpha}\{a_{n,k}\cos kx+b_{n,k}\sin kx\}$$

where

$$a_{n,k} = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\,d\omega_{n+1}(t), \qquad b_{n,k} = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\,d\omega_{n+1}(t)$$

which are the n-th Riemann sums to the integrals which determine the Fourier coefficients

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\,dt, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\,dt,$$

respectively. Thus, from (8), (9) and the well known weak* compactness argument (see BUTZER and NESSEL [1, p. 437]) we get the theorem.

REMARK. If $\|f-P_nf\|=o(n^{-\alpha})$, then f is a constant.

5. The approximation theorem. From Theorem 1, we get the following theorem by familiar STEČKIN's argument [4].

THEOREM 3. *If f and $\tilde{f}$ are continuous, then*

$$\|f-P_nf\| = 0\left\{(\omega_\alpha f)\left(\frac{1}{n}\right)+(\omega_\alpha\tilde{f})\left(\frac{1}{n}\right)\right\}$$

where ω_α means the modulus of smoothness of order α.

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VERALLGEMEINERTE WALSH—FOURIERREIHEN. II

Von

P. SIMON (Budapest)

Herrn Prof. K. Tandori zum 50. Geburtstag gewidmet

Einleitung

In dieser Arbeit beschäftigen wir uns mit der Konvergenz in der Norm $\|\dots\|_p$ $(1 \leqq p \leqq \infty)$ der nach dem Vilenkinschen orthonormierten System [8] fortschreitenden Fourier-Entwicklung.

Es ist wohlbekannt, daß die trigonometrische Fourierreihe einer nach 2π periodischen Funktion $f \in L^p\,[0, 2\pi)$ $(1 < p < \infty)$ in der Norm $\|\dots\|_p$ konvergiert und in den Fällen $p=1, \infty$ diese Behauptung falsch ist [11]. Das Analogon dieses Satzes für das Vilenkinsche System wurde bis jetzt nur unter gewissen Bedingungen bewiesen (das sog. „beschränkte Vilenkinsche System") [2], [9], [10]. Im folgenden werden wir den Satz ohne diese Bedingungen beweisen.

Zum Beweis konstruieren wir einen beschränkten linearen Operator, ein Analogon der trigonometrischen Konjugierung [11]. Es ist bekannt, daß die sog. konjugierte Funktion in der Theorie der trigonometrischen Fourierreihen eine große Rolle spielt [11], [12]. Man kann zum Beispiel mit Hilfe der konjugierten Funktion die Konvergenz in der Norm $\|\dots\|_p$ der Fourierreihe einer Funktion $f \in L^p\,[0, 2\pi)$ $(1 < p < \infty)$ einfach beweisen [11]. In dieser Arbeit wird gezeigt, daß das Analogon dieses Beweises auch für das Vilenkinsche System — als eine Anwendung des von uns zu konstruierenden Konjugierungsbegriffs — zu gebrauchen ist.

Wir bemerken, daß man den Hauptsatz unserer Arbeit auch dnrch Anwendung des Begriffs vom bedingten Erwartungswert in der Theorie der Orthogonalreihen beweisen kann. Dass folgt aus den sehr allgemeinen Sätzen von F. SCHIPP [5], [6] bezüglich multiplikativer Systeme.

Ich möchte Herrn Professor F. Schipp für seine wertvollen Ratschläge bei der Fertigstellung dieser Arbeit meinen aufrichtigen Dank aussprechen.

§ 1.

In diesem Paragraphen führen wir einige Bezeichnungen ein und formulieren die wichtigsten Definitionen. Es sei

(1) $$m = (m_0, m_1, \dots, m_k, \dots) \quad (2 \leqq m_k,\ m_k \in \mathbf{N},\ k \in \mathbf{N} := \{0, 1, \dots\})$$

eine Folge von natürlichen Zahlen und bezeichnen wir mit

(2) $$Z_{m_k} := \{0, 1, \dots, m_k - 1\} \quad (m_k \in \mathbf{N},\ m_k \geqq 2,\ k \in \mathbf{N})$$

die m_k-te diskrete zyklische Gruppe. Wir betrachten die komplette direkte Summe G_m der Gruppen (2), d. h.

(3) $$G_m := Z_{m_0} \oplus Z_{m_1} \oplus \dots \oplus Z_{m_k} \oplus \dots$$

dann wird G_m zu einer kompakten Abelschen Gruppe. Wir werden die Gruppenoperation in G_m mit $\overset{\circ}{+}$, die inverse Operation mit $\overset{\circ}{-}$ bezeichnen. Es sei

$$(4)\qquad I_n(x) := \{y: y = (x_0, \ldots, x_{n-1}, y_n, \ldots) | y \in G_m\} \quad (n \in \mathbf{N}, \quad x \in G_m)$$

die „n-te Umgebung" des Elements $x \in G_m$. Man kann leicht verifizieren, daß die durch die Umgebungsbasis von $0 \in G_m$ induzierte Topologie mit die obige Topologie von G_m übereinstimmt. Mit der Bezeichnung $\Gamma(m)$ für das Charaktersystem von G_m kann man $\Gamma(m)$ folgenderweise darstellen [8]. Es seien $n \in \mathbf{N}$ und

$$(5)\qquad r_n(x) := \exp \frac{2\pi x_n i}{m_n} \quad \left(x \in G_m, \ i := \sqrt{-1}\right).$$

Wenn

$$(6)\qquad M_0 := 1, \quad M_1 := m_0, \ldots, M_k := m_{k-1} M_{k-1}, \ldots \quad (k \in \mathbf{P} := \{1, 2, \ldots\})$$

sind, kann man jede natürliche Zahl n in der folgenden Gestalt eindeutig darstellen:

$$(7)\qquad n = \sum_{k=0}^{\infty} n_k M_k \quad (n_k \in \mathbf{N}, \ 0 \leqq n_k < m_k).$$

Mit Hilfe der Darstellung (7) kann man die endlichen Produkte der Funktionen (5) folgenderweise ordnen:

$$(8)\qquad \psi_0 \equiv 1, \quad \psi_n = \prod_{k=0}^{\infty} (r_k)^{n_k} \quad (n \in \mathbf{N}),$$

wo die n_k die unter (7) definierten Zahlen sind.

Man kann beweisen [8], daß das System (8) das Charaktersystem von G_m ist, und $\Gamma(m)$ ist ein vollständiges orthonormiertes System auf der Gruppe G_m bezüglich des Haarschen Maßes.

Wir bemerken, daß die Funktionen unter (5) im Fall $m_n = 2$ ($n \in \mathbf{N}$) die Rademacherschen Funktionen sind und das System unter (8) dasjenige von Walsh—Paley [1], [10] ist.

Es sei $f \in L^1(G_m)$ und bezeichnen wir mit $S(f)$, bzw. mit $S_n(f)$ ($n \in \mathbf{N}$) die Fourierreihe von f nach dem System (8), bzw. die n-te Partialsumme dieser Reihe (im weiteren reden wir nur kurz über „Fourierreihe"), d. h.

$$\hat{f}(n) := \int f(t) \overline{\psi_n(t)} \, dt, \qquad S(f) := \sum_{k=0}^{\infty} \hat{f}(k) \psi_k,$$

$$(9)\qquad S_0(f) \equiv 0, \quad S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k \quad \left(f \in L^1(G_m), \ n \in \mathbf{N}\right).$$

Führen wir noch die folgenden sog. Dirichletschen Kernfunktionen

$$(10)\qquad D_0 \equiv 0, \quad D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbf{P}),$$

ein, so ergibt sich

$$S_n(f)(x) = \int_{G_m} f(t) D_n(x \dot{-} t)\, dt = (f * D_n)(x) \quad (x \in G_m) \tag{11}$$

($f * g$ bedeutet die Faltung von $f, g \in L^1(G_m)$).

Wir haben in [7] die Darstellung

$$D_n = \psi_n \sum_{k=0}^{\infty} \left(\sum_{j=m_k-n_k}^{m_k-1} (r_k)^j \right) D_{M_k} = \psi_n \sum_{k=0}^{\infty} \left(\sum_{j=1}^{n_k} (r_k)^j \right) D_{M_k} \quad (n \in \mathbf{P}), \tag{12}$$

der Funktion D_n erhalten, wo die n_k die unter (7) definierten Zahlen sind. Es sei $\oplus$ die verallgemeinerte FINE-sche Operation [1], d. h.

$$n \oplus k := \sum_{j=0}^{\infty} [n_j + k_j (\operatorname{mod} m_j)] M_j \quad (n, k \in \mathbf{N}), \tag{13}$$

wo sich die Bedeutung von n_j, k_j aus (7) ergibt. Offensichtlich wird $\mathbf{N}$ mit der Operation $\oplus$ zu einer Abelschen Gruppe.

§ 2.

Im folgenden beschäftigen wir uns mit der Konvergenz in der Norm $\|\ldots\|_p$ der Fourierreihe von $f \in L^p(G_m)$ $(1 \leqq p \leqq \infty)$. Bis jetzt untersuchte man dieses Problem nur in jenem Fall, daß die Folge unter (1) beschränkt ist [2], [10].

Wir werden auch das Analogon vom trigonometrischen konjugierten Operator [11] für die Fourierreihe (9) definieren und das Äquivalent des bekannten Satzes von M. RIESZ [11] beweisen. Als Anwendung des letzteren Satzes wird dann der Hauptsatz dieser Arbeit gewonnen:

SATZ 1. *Für beliebige Folge vom Typ* (1) *gilt die folgende Behauptung:*

$$\lim \|S_n(f) - f\|_p = 0 \quad \left(n \to \infty,\ f \in L^p(G_m),\ 1 < p < \infty\right).$$

In den Fällen $p=1$, ∞ gilt die vorige Behauptung nicht. Wenn $p=\infty$ ist, können wir einfach nach der Formel (16) in [7] eine solche Funktion in $L^\infty(G_m)$ konstruieren, für die $\lim \|S_n(f)-f\|_\infty = 0$ $(n \to \infty)$ nicht gilt. Für $p=1$ ergibt sich ein ähnliches Beispiel mit Hilfe der Lebesgueschen Funktionen von $\Gamma(m)$ [8].

Wir definieren den Operator der Konjugierung für die Fourierreihen (9) folgenderweise. Es sei

$$\Delta_k := \begin{cases} 1 & (m_k = 2) \\ \left[\dfrac{m_k - 1}{2}\right] & (m_k > 2) \end{cases} \quad (k \in \mathbf{N})$$

($[\alpha]$ ist der ganze Teil von $\alpha \in \mathbf{R}$) und bezeichnen mit L_k die folgende Funktion:

$$L_k := -\sum_{j=1}^{\Delta_k} (r_k)^j + \sum_{j=\Delta_k+1}^{m_k-1} (r_k)^j.$$

Dann haben wir je nach der Parität der Zahl m_k die folgenden Aussagen über den Wert $L_k(x)$ $(x \in G_m)$:

a) $$L_k(x) = 1 - \frac{1}{i} \frac{(-1)^{x_k}}{\sin \frac{\pi x_k}{m_k}} + \frac{\exp\left(-\frac{\pi x_k i}{m_k}\right)}{i} \left(\sin \frac{x_k \pi}{m_k}\right)^{-1}$$

$\left(x_k \neq 0, m_k \equiv 1(2), i = \sqrt{-1}\right)$;

b) ist $m_k \equiv 0(2)$ $(m_k > 2)$, so gilt

$$L_k(x) = 1 + \frac{1}{i} \frac{\exp\left(-\frac{\pi x_k i}{m_k}\right)[1-(-1)^{x_k}]}{\sin \frac{\pi x_k}{m_k}} \qquad (x_k \neq 0);$$

c) im Falle $m_k = 2$ haben wir $L_k(x) = -(-1)^{x_k}$ $(x \in G_m)$.

Wir benutzen die jetzt eingeführten Bezeichnungen und definieren eine Operatorenfolge $\tilde{T}_n$ $(n \in \mathbf{N})$:

(14) $$\tilde{T}_n f := f * \left(\sum_{k=0}^{n} L_k D_{M_k}\right) =: f * \tilde{D}_n \qquad \left(f \in L^1(G_m), n \in \mathbf{N}\right).$$

Dann gilt der folgende

SATZ 2. *Die Operatorenfolge* $\tilde{T}_n$ $(n \in \mathbf{N})$ *ist in* n *vom gleichmäßigen Typ* (p, p) $(1 < p < \infty)$, *d. h. es gibt eine nur von* p *abhängige Konstante* $A(p)$ *derart, daß* $\|\tilde{T}_n f\|_p \leqq A(p) \|f\|_p$ *für alle* $f \in L^p(G_m)$ $(1 < p < \infty)$ *und* $n \in \mathbf{N}$ *gilt.*

Da die Menge der „Polynome“ $\left\{\sum_{k=0}^{n} c_k \psi_k \colon c_k \in \mathbf{R},\ n \in 1, 2, \ldots\right\}$ in $L^p(G_m)$ $(1 \leqq p < \infty)$ in bezug auf die $\|\ldots\|_p$ überall dicht ist, konvergiert die Folge $\tilde{T}_n f$ $(n \to \infty)$ — nach dem Banach-Steinhausschen Satz und dem vorigen Satz 2 — in der Norm $\|\ldots\|_p$ für jede Funktion $f \in L^p(G_m)$ $(1 < p < \infty)$. Mit der Bezeichnung $\tilde{T}f$ für diesen Limes ist $\tilde{T}\colon L^p(G_m) \to L^p(G_m)$ $(1 < p < \infty)$ ein beschränkter linearer Operator.

DEFINITION 1. $\tilde{T}$ wird der Operator der Konjugierung bezüglich des Systems $\Gamma(m)$ genannt.

Wir werden noch gewisse „Drehungstransformationen“ brauchen, die im folgenden definiert werden. Es sei $n = \sum_{k=0}^{\infty} n_k M_k$ eine festgesetzte natürliche Zahl und wir betrachten die Kernfunktionen

$$\Delta_k' := \sum_{\substack{s=1 \\ s \neq m_k - n_k - 1}}^{m_k - 1} \sum_{j=sM_k}^{(s+1)M_k - 1} \psi_j; \quad \Delta_k'' := \sum_{\substack{s=1 \\ s \neq n_k + 1}}^{m_k - 1} \sum_{j=sM_k}^{(s+1)M_k - 1} \psi_j \quad (k \in \mathbf{N}).$$

Mit Hilfe dieser Funktionen konstruieren wir die Operatoren

$$(15)\qquad T_M^{n^+}f:=\sum_{k=0}^{M}(f*\Delta_k')(r_k)^{n_k+1},$$

$$(16)\qquad T_M^{n^-}f:=\sum_{k=0}^{M}(f*\Delta_k'')(\bar{r}_k)^{n_k+1}\quad (f\in L^1(G_m),\ M\in\mathbf{N}).$$

Dann gilt der folgende

SATZ 3. *Die unter* (15) *und* (16) *definierten Operatoren sind in n, M vom gleichmäßigen Typ* (p,p) $(1<p<\infty;\ n,M\in\mathbf{N})$.

Wir bezeichnen mit $T^{n^+}f$ bzw. mit $T^{n^-}f$ den Limes in der Norm $\|\ldots\|_p$ $(1<p<\infty)$ von $T_M^{n^+}f$ bzw. von $T_M^{n^-}f$ $(M\to\infty)$, der nach dem Banach—Steinhausschen Satz und nach Satz 3 für jede beliebige Funktion $f\in L^p(G_m)$ $(1<p<\infty)$ existiert. Dann sind die Operatoren $T^{n^+}, T^{n^-}: L^p(G_m)\to L^p(G_m)$ $(1<p<\infty)$ beschränkte lineare Operatoren $(n\in\mathbf{N})$.

DEFINITION 2. T^{n^+} und T^{n^-} $(n\in\mathbf{N})$ werden die „Drehungstransformationen" bezüglich des Systems $\Gamma(m)$ genannt.

§ 3.

Zum Beweis der Sätze werden wir einige Hilfssätze benutzen. Es sei $I_n(x,k):=\{(x_0,\ldots,x_{n-1},k,y_{n+1},y_{n+2},\ldots)\in G_m\,|\,y\in G_m\}$ $(k\in Z_{m_n},\ n\in\mathbf{N})$. Für jedes $x\in G_m$ und jedes $n\in\mathbf{N}$ betrachten wir die Mengen

$$K_n^1(x):=\begin{cases}\bigcup\limits_{k=0}^{\left[\frac{m_n}{2}\right]-1} I_n(x,k) & (x_n<[m_n/2])\\ \bigcup\limits_{k=\left[\frac{m_n}{2}\right]}^{m_n-1} I_n(x,k) & (x_n\geqq[m_n/2])\end{cases}$$

bzw., wenn $K_n^j(x)=\bigcup\limits_{k=k_j}^{h_j} I_n(x,k)$ ist, die Mengen

$$K_n^{j+1}(x):=\begin{cases}\bigcup\limits_{k=k_j}^{\left[\frac{k_j+h_j}{2}\right]-1} I_n(x,k) & (x_n<[(k_j+h_j)/2])\\ \bigcup\limits_{k=\left[\frac{k_j+h_j}{2}\right]}^{h_j} I_n(x,k) & (x_n\geqq[(k_j+h_j)/2])\end{cases}\qquad (k_j,h_j,j\in\mathbf{P},\ k_j\leqq h_j).$$

Dann gibt es für beliebig festgesetzte $x\in G_m$ und $n\in\mathbf{N}$ eine Zahl $j_n\in\mathbf{P}$ derart, daß $K_n^{j_n}(x)=I_{n+1}(x)$ ist. Für die Mengen $K_n^j(x)$ $(n\in\mathbf{N},\ x\in G_m,\ j=1,\ldots,j_n-1)$ ergibt sich $1\leqq\dfrac{\operatorname{mes}K_n^j(x)}{\operatorname{mes}K_n^{j+1}(x)}\leqq 3$. Ordnen wir die Mengen $K_n^j(x)$ für jedes festgesetzte $x\in G_m$ nach der lexikographischen Ordnung der Menge der Paare (n,j), so ergibt sich je eine

Mengenfolge $K_n(x)$ und es gelten die folgenden Behauptungen. $K_n(x)$ nimmt monoton ab und lim $K_n(x)=\{x\}$ $(n\to\infty)$, bzw.

$$1 \leqq \frac{\operatorname{mes} K_n(x)}{\operatorname{mes} K_{n+1}(x)} \leqq 3 \quad (n\in\mathbf{N}).$$

Mit Hilfe der vorigen Mengen $K_n(x)$ $(n\in\mathbf{N},\ x\in G_m)$ formulieren wir das sog. HÖRMANDERsche Zerlegungslemma [3], das wir in der folgenden modifizierten Gestalt oft verwenden werden.

LEMMA 1. *Es seien* $f\in L^1(G_m)$ *und* $y>\|f\|_1$. *Dann existieren die Zerlegungen* $G_m=F\cup\bar{F}$ $(\bar{F}:=G_m\setminus F)$ *und* $f=f_0+w$ *derart, daß die folgenden Behauptungen gelten:*

1) $F=\bigcup_{k=1}^{\infty} K_{n_k}(x_k)=\bigcup_{k=1}^{\infty}\left(K'_{n_k}(x_k)\cup K''_{n_k}(x_k)\right)=:\bigcup_{k=1}^{\infty} K_{n_k}\ (K_{n_k}\cap K_{n_{k'}}=\emptyset\ (k\neq k'))$,

wo im Fall $K_{n_k}=\bigcup_{h=a}^{b} I_j(x_k,h)$ *die Mengen* $K'_{n_k}(x_k)=:K'_{n_k}:=\bigcup_{h=a}^{b} I_j(x_k,h)\ (h\equiv 0(2))$, $K''_{n_k}(x_k)=:K''_{n_k}:=\bigcup_{h=a}^{b} I_j(x_{k'}\,h)\ (h\equiv 1(2))$ *sind,*

2) $\|f_0\|_\infty\leqq 6y$ *und* $\|f_0\|_1\leqq 2\|f\|_1$,

3) $w=\sum_{k=1}^{\infty} f_k$ *und* $\operatorname{supp} f_k\subset K_{n_k}$, $\int_{K'_{n_k}} f_k=\int_{K''_{n_k}} f_k=0$, $1/|K_{n_k}|\int_{K_{n_k}}|f_k|\leqq 12y$ $(|K_{n_k}|:=$ $:=\operatorname{mes} K_{n_k})$, $\|w\|_1\leqq 12\|f\|_1$,

4) $\operatorname{mes} F\leqq \|f\|_1/y$.

Zur Formulierung von Lemma 2 benötigen wir die folgenden Bezeichnungen:

$$Z_k^n=\{a,a+1,\dots,a+n-1\}\subset Z_{m_k},\quad b:=a+[n/2]\quad (n\in\mathbf{P},\ k\in\mathbf{N}),$$

$$DZ_k^n:=\bigcup_{j=0}^{2\left[\frac{n}{2}\right]+n-1}\{(a^*+j)(\operatorname{mod} m_k)\}\quad (a^*:=a-[n/2],\ k\in\mathbf{N},\ n\in\mathbf{P}),$$

$$H_k^n(x):=\bigcup_{j\in Z_k^n} I_k(x,j),\quad H_{k0}^n(x):=\bigcup_{\substack{j\in Z_k^n\\ j\equiv 0(2)}} I_k(x,j),\quad H_{k1}^n(x):=\bigcup_{\substack{j\in Z_k^n\\ j\equiv 1(2)}} I_k(x,j),$$

$$DH_k^n(x):=\bigcup_{j\in DZ_k^n} I_k(x,j)\quad (n\in\mathbf{P},\ k\in\mathbf{N},\ x\in G_m).$$

Dann gilt das folgende

LEMMA 2. *Wir nehmen an, daß die nachstehenden Operatorenfolge* $T_n\colon L^1(G_m)\to L^1(G_m)$ $(n\in\mathbf{N})$ *die folgenden Eigenschaften besitzt:*

(a) $T_nf=\sum_{k=0}^{n}[f*(\Phi_kD_{M_k})]\varphi_k$ $(n\in\mathbf{N};\ \Phi_k,\varphi_k\in L^\infty(G_m),\ |\varphi_k|\leqq 1,\ f\in L^1(G_m))$,

(b) $\|T_nf\|_2\leqq A\|f\|_2$ $(n\in\mathbf{N},\ f\in L^2(G_m)$, *A ist eine absolute Konstante*),

(c) Φ_k *ist auf den Mengen* $I_j(x)$ $(j, k\in\mathbf{N}, j>k, x\in G_m)$ *konstant,*

(d) *es gibt für jedes Tripel* $k\in\mathbf{N}$, $n\in\mathbf{P}$, $y\in G_m$ *solche nur von* n, k *und von* $x, y\in G_m$ *abhängige Funktionen* $\Phi_k^{(i)}$ $(i=0, 1)$, *daß*

$$\frac{1}{|I_k(y)|}\int\limits_{I_k(y)\setminus DH_k^n(y)} |\Phi_k(x\dot{-}t)-\Phi_k^{(i)}(x)|\,dx\leqq B<\infty$$

$\big(t\in H_{ki}^n(y), i=0, 1$; *B ist eine absolute Konstante*$\big)$.

Dann existiert eine absolute Konstante $C>0$ *derart, daß*

$$\text{mes }\{x:|(T_nf)(x)|>y\}\leqq C\|f\|_1/y\ \ \big(y>0, n\in\mathbf{N}, f\in L^1(G_m)\big).$$

Beweis von Lemma 2. Es seien $y>0, f\in L^1(G_m)$. Wir dürfen evident voraussetzen, daß $y>\|f\|_1$. Nach Lemma 1 ergibt sich die Zerlegung $f=f_0+\sum_{k=1}^{\infty}f_k$, $G_m=F\cup\bar{F}$, für die die in Lemma 1 ausgesagten Behauptungen gelten. Wegen der Eigenschaften (a) und (b) der Operatoren $T_n(n\in\mathbf{N})$ haben wir

$$T_nf=T_nf_0+\sum_{k=1}^{\infty}T_nf_k.$$

Es seien in Lemma 1 $\operatorname{supp} f_k=K_k=H_{s_k}^{j_k}(\xi_k)$, $K_k'=H_{s_k0}^{j_k}(\xi_k)$, $K_k''=H_{s_k1}^{j_k}(\xi_k)$ $(k, j_k\in\mathbf{P}, s_k\in\mathbf{N}, \xi_k\in G_m)$, und betrachten wir die folgenden Mengen:

$$DF:=\bigcup_{k\in\mathbf{P}}DH_{s_k}^{j_k}(\xi_k),\quad \overline{DF}:=G_m\setminus DF.$$

Da

$$\{\text{mes }x:|(T_nf)(x)|>y\}\leqq\text{mes }\{x:|(T_nf_0)(x)|>y/2\}+\text{mes }\{x:x\in DF, |(T_nw)(x)|>y/2\}+$$

$$+\text{mes }\{x:x\in\overline{DF}, |(T_nw)(x)|>y/2\}=:E_n(1)+E_n(2)+E_n(3)\quad(n\in\mathbf{N})$$

ist, brauchen wir nur zu beweisen, daß $E_n(i)=O(1)\|f\|_1/y$ $(i=1, 2, 3;\ n\in\mathbf{N})$ besteht.

1) Aus den Definitionen der Menge $DF\subset G_m$ folgt nach Lemma 1

$$E_n(2)\leqq 2\text{ mes }F\leqq 2\|f\|_1/y\quad(n\in\mathbf{N}).$$

2) Nach der Eigenschaft (b) der Operatoren T_n $(n\in\mathbf{N})$ und nach Lemma 1 erhalten wir die Abschätzung

$$E_n(1)\leqq 4/y^2\|T_nf_0\|_2^2\leqq 4A\|f_0\|_2^2/y^2\leqq 4A/y^2\int\limits_{G_m}|f_0||f_0|\leqq 24A/y\|f_0\|_1\leqq$$

$$\leqq 48A\|f\|_1/y\quad(n\in\mathbf{N}).$$

3) Für T_nf_h $(h\in\mathbf{P}$, n ist fixiert) ergibt sich aus der Definition von T_n,

aus supp $D_{M_k}=I_k$ $(k\in\mathbf{N})$ und aus Lemma 1

$$(T_nf_h)(x) = \sum_{k=0}^{n}\Big[\int_{G_m} f_h(t)\,\Phi_k(x\dot{-}t)\,D_{M_k}(x\dot{-}t)\,dt\Big]\,\varphi_k(x) =$$

$$= \sum_{k=0}^{n}\Big[\int_{H_{s_h}^{j_h}(\xi_h)\cap I_k(x)} f_h(t)\,\Phi_k(x\dot{-}t)\,D_{M_k}(x\dot{-}t)\,dt\Big]\,\varphi_k(x) =$$

$$= \sum_{k=0}^{n} M_k\Big[\int_{H_{s_h}^{j_h}(\xi_h)\cap I_k(x)} f_h(t)\,\Phi_k(x\dot{-}t)\,dt\Big]\,\varphi_k(x) \qquad (x\in G_m,\ h\in\mathbf{P}).$$

Da $I_n(x)\cap I_m(y)=\emptyset$ oder $I_n(x)\supseteq I_m(y)$ $(x, y\in G_m,\ n, m\in\mathbf{N},\ m\leqq n)$ ist, gelten für jedes festgesetzte $x\in\overline{DF}$ die folgenden Aussagen:

a) Bei $0\leqq k<s_h$ $(k\in\mathbf{N})$ ist $I_k(x)\cap H_{s_h}^{j_h}(\xi_h)=\emptyset$ oder $H_{s_h}^{j_h}(\xi_h)$ $(h\in\mathbf{P})$. Da $\Phi_k(x\dot{-}t)$ nach der Bedingung (c) auf der Menge $H_{s_h}^{j_h}(\xi_h)$ (in t) konstant ist, gilt

$$\int_{I_k(x)\cap H_{s_h}^{j_h}(\xi_h)} f_h(t)\,\Phi_k(x\dot{-}t)\,dt = 0.$$

b) Ist $s_h<k\leqq n$ oder $s_h>n$ $(h\in\mathbf{P})$, so ist für $x\notin F$ $H_{s_h}^{j_h}(\xi_h)\cap I_k(x)=\emptyset$.

Das Obige zusammenfassend können wir folgendes sagen: für jedes $x\in\overline{DF}$ gelten entweder $(T_nf_h)(x)=0$, oder $s_h\leqq n$, $x\in I_{s_h}(\xi_h)\setminus DH_{s_h}^{j_h}(\xi_h)$ und

$$(T_nf_h)(x) = M_{s_h}\Big[\int_{H_{s_h}^{j_h}(\xi_h)} f_h(t)\,\Phi_{s_h}(x\dot{-}t)\,dt\Big]\,\varphi_{s_h}(x) =$$

$$= M_{s_h}\Big\{\int_{H_{s_h0}^{j_h}(\xi_h)} f_h(t)\,[\Phi_{s_h}(x\dot{-}t)-\Phi_{s_h}^{(0)}(x)]\,dt\Big\}\,\varphi_{s_h}(x)+$$

$$+ M_{s_h}\Big\{\int_{H_{s_h1}^{j_h}(\xi_h)} f_h(t)\,[\Phi_{s_h}(x\dot{-}t)-\Phi_{s_h}^{(1)}(x)]\,dt\Big\}\,\varphi_{s_h}(x).$$

(In der letzten Gleichung haben wir diejenige Behauptung von Lemma 1 gebraucht, nach der

$$\int_{H_{s_hi}^{j_h}(\xi_h)} f_h(t)\,dt = 0 \quad (i = 0, 1).)$$

Wenn wir Lemma 1 und die Bedingung (d) gebrauchen, ergibt sich die folgende Ungleichung:

$$\int_{\overline{DF}} |(T_nf_h)(x)|\,dx \leqq \frac{1}{|I_{s_h}(\xi_h)|}\int_{\overline{DF}}\Big|\sum_{i=0}^{1}\Big\{\int_{H_{s_hi}^{j_h}(\xi_h)} f_h(t)\,[\Phi_{s_h}(x\dot{-}t)-\Phi_{s_h}^{(i)}(x)]\,dt\Big\}\,\varphi_{s_h}(x)\Big|\,dx \leqq$$

$$\leqq \int_{I_{s_h}(\xi_h)\setminus DH_{s_h}^{j_h}(\xi_h)} \frac{1}{|I_{s_h}(\xi_h)|}\Big|\sum_{i=0}^{1}\int_{H_{s_hi}^{j_h}(\xi_h)} f_h(t)\,[\Phi_{s_h}(x\dot{-}t)-\Phi_{s_h}^{(i)}(x)]\,\varphi_{s_h}(x)\,dt\Big|\,dx \leqq$$

$$\leqq \sum_{i=0}^{1}\int_{H_{s_hi}^{j_h}(\xi_h)} |f_h(t)|\Big\{\frac{1}{|I_{s_h}(\xi_h)|}\int_{I_{s_h}(\xi_h)\setminus DH_{s_h}^{j_h}(\xi_h)} |[\Phi_{s_h}(x\dot{-}t)-\Phi_{s_h}^{(i)}(x)]\,\varphi_{s_h}(x)|\,dx\Big\}\,dt \leqq B\|f_h\|_1$$

$$(n\in\mathbf{N},\ h\in\mathbf{P}).$$

Daraus können wir leicht eine Abschätzung für $E_n(3)$ $(n\in\mathbf{N})$ feststellen, nämlich

$$E_n(3) \leqq 2/y \int_{\overline{D}} |(T_n w)(x)|\,dx \leqq 2/y \sum_{h=1}^{\infty} \int_{\overline{DF}} |(T_n f_h)(x)|\,dx = 2B/y \sum_{h=1}^{\infty} \|f_h\|_1 .$$

Nach dieser Abschätzung und nach Lemma 1 besteht $E_n(3) \leqq 24B/y\|f\|_1$ $(n\in\mathbf{N})$. Die Konstante $C=2+48A+24B$ genügt demnach der Behauptung von Lemma 2 und damit haben wir Lemma 2 bewiesen.

LEMMA 3. *Die Operatoren $\tilde{T}_n$ $(n\in\mathbf{N})$ sind in n von gleichmäßigem schwachem Typ* (1, 1), *d. h. es gibt eine absolute Konstante $C>0$ derart, daß die folgende Behauptung gilt:*

$$\operatorname{mes}\{x : x\in G_m,\ |(\tilde{T}_n f)(x)| > y\} \leqq C\|f\|_1/y \quad (f\in L^1(G_m),\ y>0,\ n\in\mathbf{N}).$$

BEWEIS. Es genügt zu prüfen, daß die Operatoren $\tilde{T}_n$ $(n\in\mathbf{N})$ den Bedingungen von Lemma 2 genügen. Mit der Wahl $\Phi_k \equiv L_k$, $\varphi_k \equiv 1$ $(k\in\mathbf{N})$ ist das für (a) evident. Nach der Besselschen Ungleichung erfüllt sich auch die Bedingung (b). Da der Wert $L_k(x)$ $(k\in\mathbf{N}, x\in G_m)$ nach (5) nur von x_k abhängt, erfüllt sich auch (c). Wir beweisen, daß die Bedingung (d) auch für die Funktionen L_k $(k\in\mathbf{N})$ besteht.

a) Für $m_k=2$ haben wir $L_k(x)=-(-1)^{x_k}$ $(x\in G_m)$ und mit der Wahl $\Phi_k^{(0)} \equiv \Phi_k^{(1)} \equiv 0$ erfüllt sich die Bedingung (d).

b) Wenn $m_k \equiv 1(2)$ ist, so sei für jedes 4-Tupel $y\in G_m$, $n\in\mathbf{P}$, $k\in\mathbf{N}$, $Z_k^n \subset Z_{m_k}$ (vgl. die vorige Bezeichnungen)

$$\Phi_k^{(0)}(x) := \begin{cases} 1-\dfrac{1}{i}\dfrac{(-1)^{x_k+1}}{\sin\dfrac{\pi(x_k-b)}{m_k}} + \dfrac{\exp\left(-\dfrac{\pi(x_k-b)}{m_k}i\right)}{i\sin\dfrac{\pi(x_k-b)}{m_k}} & (x_k < a) \\ & (x\in I_k(y)\setminus DH_k^n(y),\ i:=\sqrt{-1}) \\ 1-\dfrac{1}{i}\dfrac{(-1)^{x_k}}{\sin\dfrac{\pi(x_k-b)}{m_k}} + \dfrac{\exp\left(-\dfrac{\pi(x_k-b)}{m_k}i\right)}{i\sin\dfrac{\pi(x_k-b)}{m_k}} & (x_k > a+n-1) \end{cases}$$

bzw.

$$\Phi_k^{(1)}(x) := \begin{cases} 1-\dfrac{1}{i}\dfrac{(-1)^{x_k}}{\sin\dfrac{\pi(x_k-b)}{m_k}} + \dfrac{1}{i}\dfrac{\exp\left(-\dfrac{\pi(x_k-b)}{m_k}i\right)}{\sin\dfrac{\pi(x_k-b)}{m_k}} & (x_k < a) \\ & (x\in I_k(y)\setminus DH_k^n(y)) \\ 1-\dfrac{1}{i}\dfrac{(-1)^{x_k+1}}{\sin\dfrac{\pi(x_k-b)}{m_k}} + \dfrac{1}{i}\dfrac{\exp\left(-\dfrac{\pi(x_k-b)}{m_k}i\right)}{\sin\dfrac{\pi(x_k-b)}{m_k}} & (x_k > a+n-1) \end{cases}$$

und $\Phi_k^{(0)}(x) = \Phi_k^{(1)}(x) = 0$ $\big(x\notin I_k(y)\setminus DH_k^n(y)\big)$. Da $\dfrac{|t_k-b|}{|x_k-b|} \leqq \dfrac{1}{2}$, ist es leicht zu verifizieren, daß

$$|\Phi_k(x \dot{-} t) - \Phi_k^{(i)}(x)| \leqq 2 \left| \frac{1}{\sin \frac{\pi(x_k - t_k)}{m_k}} - \frac{1}{\sin \frac{\pi(x_k - b)}{m_k}} \right| +$$

$$+ \frac{\left| \exp\left(-\frac{\pi(x_k - t_k)}{m_k} i\right) - \exp\left(-\frac{\pi(x_k - b)}{m_k} i\right) \right|}{\left| \sin \frac{\pi(x_k - b)}{m_k} \right|} =$$

$$= 2 \left| \frac{1}{\sin \frac{\pi(x_k - t_k)}{m_k}} - \frac{1}{\sin \frac{\pi(x_k - b)}{m_k}} \right| + O(1) \quad (t \in H_{ki}^n(y);\ i = 0, 1;\ (k \in \mathbf{N})).$$

c) Ist $m_n \equiv 0(2)$ $(m_k > 2)$, so betrachten wir die nachstehenden Funktionen (vgl. die vorigen Bezeichnungen):

$$\Phi_k^{(0)}(x) := 1 + \frac{1}{i} \frac{1 - (-1)^{x_k}}{\sin \frac{\pi(x_k - b)}{m_k}} \cdot \exp\left(-\frac{\pi(x_k - b) i}{m_k}\right)$$

$$(x \in I_k(y) \setminus DH_k^n(y)),$$

$$\Phi_k^{(1)}(x) := 1 + \frac{1}{i} \frac{1 - (-1)^{x_k + 1}}{\sin \frac{\pi(x_k - b)}{m_k}} \cdot \exp\left(-\frac{\pi(x_k - b) i}{m_k}\right)$$

$\Phi_k^{(0)}(x) = \Phi_k^{(1)}(x) = 0$ $(x \notin I_k(y) \setminus DH_k^n(y))$. Eine ähnliche Rechnung wie früher ergibt

$$|\Phi_k(x \dot{-} t) - \Phi_k^{(i)}(x)| \leqq 2 \left| \frac{1}{\sin \frac{\pi(x_k - t_k)}{m_k}} - \frac{1}{\sin \frac{\pi(x_k - b)}{m_k}} \right| + O(1) =:$$

$$=: 2F_k(x, t) + O(1) \quad (t \in H_{ki}^n(y),\ i = 0, 1,\ k \in \mathbf{N}).$$

Wir sollen nun die Existenz einer solchen absoluten Konstante $B > 0$ beweisen, für die

$$\frac{1}{|I_k(u)|} \int_{I_k(u) \setminus DH_k^n(u)} F_k(x, t)\, dx \leqq B < \infty \quad (u \in G_m,\ n \in \mathbf{P},\ k \in \mathbf{N},\ t \in H_k^n(u))$$

ist. Zum Beweis dieser Behauptung schreiben wir das in Frage stehende Integral in die folgende Gestalt

$$M_k \sum_{j \in Z_{m_k} \setminus DZ_k^n} \int_{I_k(u, j)} F_k(x, t)\, dx = \frac{1}{m_k} \sum_j \left| \frac{1}{\sin \frac{\pi(j - t_k)}{m_k}} - \frac{1}{\sin \frac{\pi(j - b)}{m_k}} \right|.$$

Wir können ohne Beschränkung der Allgemeinheit annehmen, daß $0 \leqq a^* < a + [n/2] + n - 1 \leqq m_k - 1$ (vgl. die vorigen Bezeichnungen). Dann schreiben wir die letzte Summe als die Summe von $\sum^{(1)}$ und $\sum^{(2)}$, wo sich $\sum^{(1)}$ bzw. $\sum^{(2)}$ auf die $0 \leqq j < a^*$ bzw. auf die $a+[n/2]+n-1<j \leqq m_k-1$ bezieht.

Zuerst fassen wir $\sum^{(1)}$ ins Auge. Wenn $b \leqq t_k$ und $d := t_k - b$ sind (vgl. die Bezeichnungen), ist

$$\sum^{(1)} = \frac{1}{m_k} \sum_{z=2\left[\frac{n}{2}\right]}^{b} \left| \frac{1}{\sin \dfrac{\pi(d+z)}{m_k}} - \frac{1}{\sin \dfrac{\pi z}{m_k}} \right| = \sum_{z+\left[\frac{n}{2}\right] \leqq \left[\frac{m_k}{2}\right]} + \sum_{z > \left[\frac{m_k}{2}\right]} + \sum_{\text{sonst}} =:$$

$$=: \sum' + \sum'' + \sum'''.$$

Es gibt in der Summe $\sum'''$ höchtens d Summanden, woraus

$$\sum''' \leqq \frac{1}{m_k} \frac{2\left[\dfrac{n}{2}\right]}{\sin \dfrac{\pi 2[n/2]}{m_k}} \leqq 1/2$$

folgt. Nach der Definition von $\sum'$ gilt

$$\sum' = 1/m_k \sum_z{}' \left(\frac{1}{\sin \dfrac{\pi z}{m_k}} - \frac{1}{\sin \dfrac{\pi(d+z)}{m_k}} \right) \leqq 1/2$$

und ebenso ergibt sich $\sum'' \leqq 1/2$.

Im Fall $t_k < b$, bzw. für die Summe $\sum^{(2)}$ kann man ähnlicherweise eine obere Abschätzung feststellen. Damit haben wir Lemma 3 bewiesen.

LEMMA 4. *Die Operatoren $T_M^{n^+}$, $T_M^{n^-}$ sind in n, M von gleichmäßigem schwachem Typ* (1, 1) $(n, M \in \mathbf{N})$.

BEWEIS. Wir werden die Behauptung von Lemma 4 nur für $T_M^{n^+}$ beweisen und bemerken, daß man den Beweis für $T_M^{n^-}$ ähnlich führen kann. Es sei

$$\Phi_k := \sum_{\substack{s=1 \\ s \neq m_k - n_k - 1}}^{m_k - 1} \Psi_{sM_k} =: \sum_s{}^* \Psi_{sM_k}, \quad \varphi_k := (r_k)^{n_k+1} \quad \left(n = \sum_{j=0}^{\infty} n_j M_j \in \mathbf{N},\ k \in \mathbf{N}\right).$$

Dann sind die Bedingungen (a) und (c) von Lemma 2 evident erfüllt. Die Bedingung (b) erfüllt sich wegen der Besselschen Ungleichung. Zur Erfüllung von (d) seien $\Phi_k^{(0)} \equiv \Phi_k^{(1)} \equiv 0$. Da

$$\Phi_k(x) = \sum_s{}^* [r_k(x)]^s = \sum_s{}^* \exp \frac{2\pi x_k s i}{m_k} = \begin{cases} 1 & (n_k = m_k - 1) \\ -1 - [r_k(x)]^{n_k+1} & (n_k \neq m_k - 1,\ x_k \neq 0) \end{cases}$$

ist, besteht

$$\frac{1}{|I_k(u)|} \int_{I_k(u) \setminus DH_k^j(u)} |\Phi_k(x \dot{-} t)|\, dx < 2 \quad (u \in G_m,\ k \in \mathbf{N}, j \in \mathbf{P},\ t \in H_k^j(u)).$$

Damit ist Lemma 4 bewiesen.

§ 4.

Beweis der Sätze 2 und 3. Da die Operatoren $\tilde{T}_n$, $T_M^{n^+}$, $T_M^{n^-}$ ($n, M\in\mathbf{N}$) nach der Besselschen Ungleichung in n, M vom gleichmäßigen Typ (2, 2) sind, ergeben sich die Behauptungen der Sätze 2 und 3 im Fall $1<p<2$ aus Lemma 3 und 4 nach dem Marcinkiewiczschen Interpolationssatz [11]. Im Fall $2\leqq p<\infty$ benutzen wir zum Beweis der Sätze 2 und 3 das sog. konjugierte Verfahren [2].

Im folgenden werden wir die Partialsumme $S_n(P)$ ($n\in\mathbf{N}$) eines beliebigen Polynoms $P=\sum_{k=0}^{\infty} c_k\psi_k$ mit Hilfe der Operatoren $\tilde{T}$, T^{n^+}, T^{n^-} darstellen. Es sei

$$S_n^*(f):=\sum_{k=0}^{\infty}\int_{G_m}\overline{\Psi_n(t)}f(t)\sum_{j=m_k-n_k}^{m_k-1}[r_k(x\dot{-}t)]^j D_{M_k}(x\dot{-}t)\,dt$$

$$\left(f\in L^1(G_m),\ n=\sum_{k=0}^{\infty}n_k M_k\in\mathbf{N}\right);$$

dann ist $S_n(f)=\psi_n S_n^*(f\overline{\Psi}_n)$. Wir nehmen an, daß für die natürliche Zahl n die folgende Bedingung

$$0\leqq n_k\leqq \Delta_k\quad (k\in\mathbf{N}) \tag{17}$$

gilt, und beachten die folgenden für die Polynome definierten Operatoren T^* und T_n^{**}:

$$T^*P:=\sum_{k=0}^{\infty}\sum_{j=\Delta_k+1}^{m_k-1}\sum_{l=jM_k}^{(j+1)M_k-1}c_l\Psi_l+\frac{1}{2}c_0\Psi_0;$$

$$T_n^{**}P:=\sum_{k=0}^{\infty}\left(\sum_{j=m_k-n_k}^{m_k-1}\sum_{l=jM_k}^{(j+1)M_k-1}+\sum_{j=1}^{\Delta_k-n_k-1}\sum_{l=jM_k}^{(j+1)M_k-1}\right)c_l\Psi_l.$$

Dann ergeben sich die nachstehenden Darstellungen für die T^* und T_n^{**} ($n\in\mathbf{N}$):

$$T^*P=1/2(\tilde{T}P+P), \tag{18}$$

$$T_n^{**}P=1/2\{T^{n^-}[T^{n^+}P-\tilde{T}(T^{n^+}P)]\}. \tag{19}$$

Andererseits ist es leicht zu verifizieren, daß

$$S_n^*(P)=T^*(T_n^{**}P). \tag{20}$$

Wenn sich die Bedingung (17) für die natürliche Zahl n nicht erfüllt, schreiben wir $S_n^*(P)$ in der Gestalt

$$S_n^*(P)=\sum_{k=0}^{\infty}P*\left(\sum_{j=m_k-n_k}^{m_k-1}r_k^jD_{M_k}\right)=\sum_{n_k\leqq\Delta_k}P*\left(\sum_{j=m_k-n_k}^{m_k-1}r_k^jD_{M_k}\right)+$$
$$+\sum_{n_k>\Delta_k}P*\left(\sum_{j=1}^{m_k-1}r_k^jD_{M_k}\right)-\sum_{n_k>\Delta_k}P*\left(\sum_{j=1}^{m_k-n_k-1}r_k^jD_{M_k}\right).$$

Es seien

$$n_1:=\sum_{n_k\leqq\Delta_k}n_kM_k,\quad n_2:=\sum_{n_k>\Delta_k}(m_k-n_k-1)M_k,\quad n_3:=\sum_{n_k>\Delta_k}(m_k-1)M_k,$$

womit sich $S_n^*(P)=S_{n_1}^*(P)+S_{n_3}^*(P)-\overline{S_{n_2}^*(\overline{P})}$ ergibt. Da sich die Bedingung (17) für die Zahlen n_1, n_2 erfüllt, erhalten wir nach (20) die Darstellung

$$(21) \qquad S_n^*(P) = T^*(T_{n_1}^{**}P)-\overline{T^*(T_{n_2}^{**}\overline{P})}+S_{n_3}^*(P).$$

Wir bemerken, daß der Operator $S_{n_3}^*$ evident in n vom gleichmässigen Typ (p,p) $(1<p<\infty;\ n\in\mathbf{N})$ ist.

Beweis von Satz 1. Es sei n eine fixierte beliebige natürliche Zahl. Nach den Sätzen 2 und 3 und nach (18), (19) und (21) existiert eine nur von p abhängige Konstante $C(p)>0$ $(1<p<\infty)$ derart, daß $\|S_n^*(P)\|_p\leqq C(p)\|P\|_p$ für ein beliebiges Polynom gilt. Da die Menge der Polynome in $L^p(G_m)$ nach der Norm $\|\ldots\|_p$ überall dicht ist $(1\leqq p<\infty)$, erfüllt sich $\|S_n^*(f)\|_p\leqq C(p)\|f\|_p$ gleichzeitig auch für beliebige $f\in L^p(G_m)$ $(1<p<\infty)$. Daraus ergibt sich — nach der Definition von $S_n^*-\|S_n(f)\|_p=$ $=\|S_n^*(f\overline{\Psi}_n)\|_p\leqq C(p)\|f\|_p$ $(f\in L^p(G_m), 1<p<\infty, n\in\mathbf{N})$. Benutzen wir jetzt den Banach—Steinhausschen Satz, so ist damit Satz 1 bewiesen.

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ГРУППЫ КОБОРДИЗМОВ l-ПОГРУЖЕНИЙ. I

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А) ГОМОТОПИЧЕСКАЯ ПРЕДСТАВИМОСТЬ ГРУППЫ КОБОРДИЗМОВ ПОГРУЖЕНИЙ С ПРЕДПИСАННОЙ КРАТНОСТЬЮ САМОПЕРЕСЕЧЕНИЙ

Введение

Известна фундаментальная роль пространств Тома в теориях кобордизмов. В статье мы изложим обобщение пространства Тома, позволяющее изучить кобордизмы l-погружений методами алгебраической топологии (определение l-погружения см. ниже или в [2]).

Существуют два типа групп кобордизмов вложений:

1) группа кобордизмов n-мерных многообразий, впоженных в $n+k$-мерную сферу (рассмотренная Томом [1]), изоморфна группе $\pi_{n+k}(MO(k))$ ($MO(k)$ — это пространство Тома группы $O(k)$);

2) группа кобордизмов пар (M^n, N^{n+k}) (где M^n и N^{n+k} многообразия размерности n и $n+k$ соответственно, причем M^n вложено в N^{n+k}) была рассмотрена Уоллом [6]. Эта группа изоморфна группе бордизмов пространства Тома: $\mathfrak{N}_{n+k}(MO(k))$.

В литературе рассматривались аналоги этих групп и для погружений:

1) группа кобордизмов n-мерных многообразий, погруженных в сферу S^{n+k}, изоморфна стабильной гомотопической группе пространства Тома $\Pi^s_{n+k}(MO(k))$ (см. Уэллс [7]);

2) группа кобордизмов пар (M^n, N^{n+k}) в случае, когда M^n погружено в N^{n+k}, рассмотрена Швейцером [8].

Он показал, что эта группа изоморфна группе бордизмов пространства $\Omega^\infty S^\infty MO(k)$, где Ω^∞ есть бесконечно кратное пространство петель, а S^∞ — бесконечно кратная надстройка.

Мы будем рассматривать аналоги этих двух типов групп кобордизмов для так называемых l-погружений. Понятие l-погружения является обобщением понятий вложения и погружения и принадлежит Ушида (см. [2]).

Определение. Пусть l — натуральное число. Погружение (общего положения) называется l-погружением, если оно не имеет $l+1$ кратных точек.

Ясно, что 1-погружение это то же самое, что вложение, а для достаточно больших l (при фиксированных размерности и коразмерности погружения) понятие l-погружения совпадает с понятием погружения (говоря об l-погружении, мы будем допускать и значения $l=\infty$, понимая под ∞-погружением просто погружение).

Для любого l можно определить группы кобордизмов l-погружений первого и второго типов, т.е.:

1) группу кобордизмов n-мерных многообразий l-погруженных в сферу S^{n+k} (эту группу мы будем называть просто группой кобордизмов l-погружений). (Обозначение: $G^l(n, k)$.)

2) группу кобордизмов пар (M^n, N^{n+k}), где M^n l-погружено в N^{n+k} (эту группу мы будем называть группой Ушида). (Обозначение: $C(n, k; l)$.)

Ушида в своих работах исследовал вторую из этих групп и получил некоторую информацию о ней. Однако его методы чисто геометрические, необходимого здесь аналога пространства Тома он не построил.

В статье мы изложим конструкцию такого пространства, обозначаемого нами через $\Gamma_l MO(k)$. Основная идея конструкции принадлежит М. Л. Громову.

Мы покажем, что:

1) группа кобордизмов l-погружений n-мерных многообразий в $n+k$-мерную сферу изоморфна группе $\pi_{n+k}(\Gamma_l MO(k))$;

2) группа кобордизмов пар (M^n, N^{n+k}), где M^n l-погружено в N^{n+k}, изоморфна группе бордизмов $\mathfrak{N}_{n+k}(\Gamma_l MO(k))$.

С помощью этих изоморфизмов мы полностью вычислим группы Ушида и получим некоторую информацию о группах кобордизмов l-погружений первого типа.

Будут также рассмотрены соответствующие варианты этих групп для ориентированных многообразий.

Подробное содержание статьи следующее.

В первом параграфе части I мы определим некоторый класс погружений, так называемые разделяющиеся погружения. Они будут играть вспомогательную роль в изучении l-погружений.

Во втором параграфе I мы опишем конструкцию пространства $\Gamma_l MO(k)$. Однако перед этим мы опишем пространство, которое будет играть аналогичную поль при изучении разделяющихся погружений. Это пространство будет обозначаться через $\bar{\Gamma}_l MO(k)$. После описания конструкций пространств $\bar{\Gamma}_l MO(k)$ и $\Gamma_l MO(k)$ мы укажем на интересную связь этих конструкций с функтором Джеймса X_∞ [9] и функтором Баррата—Эклеса ΓX [10], которые дают для любого пространства X пространства ΩSX и $\Omega^\infty S^\infty X$ соответственно. Упомянутая связь заключается в том, что $\bar{\Gamma}_\infty MO(k) = MO(k)_\infty$ и $\Gamma_\infty MO(k) = \Gamma MO(k)$. Вытекающие из этих равенств изоморфизмы групп $\pi_n(\bar{\Gamma}_\infty MO(k)) \approx \pi_{n+1}(SMO(k))$ и $\pi_n(\Gamma_\infty MO(k)) \approx \pi_n^s(MO(k))$ проинтерпретируем как некоторые ослабленные варианты теоремы Хирша.

§ 1. Вводные замечания

Через $O^{(l)}(k)$ обозначим сплетение группы $O(k)$ с симметрической группой $S(l)$, т.е. существует короткая точная последовательность

$$(*) \qquad 0 \to \underbrace{O(k) \oplus \ldots \oplus O(k)}_{l\text{-раз}} \to O^{(l)}(k) \to S(l) \to 0$$

причем $S(l)$ действует на $O(k) \oplus \ldots \oplus O(k)$, переставляя слагаемые.

Замечание. Точная последовательность $(*)$ индуцирует

а) накрытие $B[O(k) \oplus \ldots \oplus O(k)] \to BO^{(l)}(k)$ с группой $S(l)$

б) расслоение $BO^{(l)}(k) \xrightarrow{B[O(k) \oplus \ldots \oplus O(k)]} BS(l)$

(где для любой группы G через BG обозначается база универсального G-расслоения).

Связь группы $O^{(l)}(k)$ с погружениями. Ясно, что если имеется произвольное погружение $f\colon M^n \to N^{n+k}$ с коразмерностью k, то нормальное расслоение к многообразию l-кратных точек в N^{n+k} (погруженному не обязательно вложенному в N^{n+k}) допускает в качестве структурной группы группу $O^{(l)}(k)$. При этом каждый замкнутый путь в многообразии l-кратных точек определяет элемент группы $S(l)$ следующим образом:

Нормальный слой к многообразию l-кратных точек (это есть lk-мерный шар D^{lk}) раскладывается самим погружением в произведение l штук k-мерных шаров. При этом выбор порядка сомножителей в слое над начальной точкой замкнутого пути определяет порядок сомножителей в слое над любой другой точкой пути. Однако, обойдя путь и возвратившись в начальную точку, мы, вообще говоря, получаем новый, отличающийся от исходного порядка сомножителей. Искомый элемент группы перестановок $S(l)$ — это та единственная перестановка, которая переводит исходное упорядочение в полученное. Этот элемент определен лишь с точностью до внутреннего автоморфизма $S(l)$, поскольку он зависит от выбора исходного упорядочения в слое над начальной точкой.

Определение. Если каждому замкнутому пути многообразия l-кратных точек сопоставляется описанным выше способом тривиальная перестановка, то погружение будем называть разделяющимся.

Определение группы кобордизмов l-погружений. Определение. l-погружения $f\colon M^n \to S^{n+k}$ и $f_1\colon M_1^n \to S^{n+k}$ назовем кобордантными, если существуют: a) многообразие N^{n+1} с краем $\partial N^{n+1} = M^n \cup M_1^n$ и b) его l-погружение g в $S^{n+k} \times I$, совпадающее на краю отображениями f и f_1, т.е. $g|_{M^n} = f$ и $g|_{M_1^n} = f_1$. Кроме того, образ отображения g должен быть трансверсальным к краям цилиндра $S^{n+k} \times I$.

Классы эквивалентности образуют группу относительно обычного сложе ния (несвязного объединения) погружений в сферу. Эта группа и есть $G^l(n, k)$.

Если в определении все погружения заменить на разделяющиеся погружения, то получим определение группы кобордизмов разделяющихся l-погружений, обозначаемой через $\bar{G}^l(n, k)$.

Можно рассматривать и ориентированный и оснащенный варианты этих групп (все многообразия должны быть ориентированными, соответственно оснащенными). Эти группы мы обозначим через $G^l_\Omega(n, k)$, $\bar{G}^l_\Omega(n, k)$ и $G^l_{fr}(n, k)$, $\bar{G}^l_{fr}(n, k)$ соответственно.

§ 2. Конструкция аналога пространства Тома для разделяющихся l-погружений

Наша задача в этом разделе — сконструировать некоторое пространство $\bar{\Gamma}_l MO(k)$, для которого $\pi_n(\bar{\Gamma}_l MO(k)) \approx \bar{G}^l(n-k, k)$. Для $l=1$ $\bar{\Gamma}_1 MO(k)$ пусть будет $MO(k)$. Для $l=2$ конструкция пространства $\bar{\Gamma}_2 MO(k)$ такова. Пусть $f\colon M^{n-k} \to S^n$ разделяющееся 2-погружение. Многообразие двойных точек обозначим через V^{n-2k} (тогда $V^{n-2k} \subset S^n$). Пусть T^n — нормальная трубчатая окрестность многообразия V^{n-2k}. Пусть N^n есть ε-окрестность образа погружения f.

Отобразим прежде всего расслоение $T^n \to V^{n-2k}$ в универсальное $O(k) \oplus O(k)$ расслоение, которое мы обозначим через γ_{k+k}

$$\begin{array}{ccc} T^n & \xrightarrow{\psi} & E(\gamma_{k+k}) \\ \downarrow & & \downarrow \\ V^{n-2k} & \longrightarrow & B(\gamma_{k+k}) \end{array}$$

Отображение $T^n \to E(\gamma_{k+k})$ обозначим через ψ. Часть образа погружения f, лежащая вне T^n, уже является вложенным подмногообразием в $S^n \setminus T^n$. Поэтому $S^n \setminus T^n$ можно отобразить обычным способом в $MO(k)$:

$$\varphi : (S^n \setminus T^n) \to MO(k), \quad \text{причем} \quad \varphi^{-1}(BO(k)) = f(M^{n-k}) \cap (S^n \setminus T^n).$$

Итак, мы имеем два непрерывных отображения ψ: $T^n \to E(\gamma_{k+k})$ и φ: $(S^n \setminus T^n) \to MO(k)$. Чтобы из них получить одно единое непрерывное отображение всей сферы S^n в какое-либо пространство X, мы должны склеить пространства $E(\gamma_{k+k})$ и $MO(k)$, т.е. X будет $E(\gamma_{k+k}) \bigcup_r MO(k)$ (пространство X будет кандидатом в пространство $\Gamma_2 MO(k)$). Здесь r есть некоторое склеивающее отображение, которое мы опишем следующим образом:

Расслоение γ_{k+k} раскладывается в ушиневскую сумму двух $O(k)$ расслоений, т.е. каждый слой расслоения γ_{k+k} разложен в сумму $R^{2k} = R_1^k \oplus R_2^k$.

Зафиксируем какую-нибудь риманову метрику на слоях γ_{k+k}. Рассмотрим в каждом слое такие векторы x, для которых при разложении $R^{2k} = R_1^k \oplus R_2^k$ норма обеих компонент не превосходит единицу, и возьмем объединение таких векторов по всем слоям. Обозначим это объединение через $\bar{E}$.

Значит

$$\bar{E} = \{x \in E(\gamma_{k+k}) |\, \|x_1\| \leqq 1 \quad \text{и} \quad \|x_2\| \leqq 1\}.$$

Введем еще следующие обозначения:

$$\partial\bar{E} = \{x |\, \|x_1\| = 1 \quad \text{или} \quad \|x_2\| = 1\}$$

$$S = \{x |\, \|x_1\| = 1 \quad \text{и} \quad \|x_2\| = 1\}$$

$$Z = \{x |\, \|x\| = 1 \quad \text{и} \quad (\|x_1\| = 0 \text{ или } \|x_2\| = 0)\}.$$

Легко видеть, что пространство $\partial\bar{E} \setminus S$ расслаивается над Z, причем слоем является k-мерный открытый шар. Это расслоение можно отобразить в универсальное. Обозначим это отображение через r'. Значит r': $(\partial\bar{E} \setminus S) \to EO(k)$. Пространство $MO(k)$ является одноточечной компактификацией пространства $EO(k)$ и поэтому $EO(k) \subset MO(k)$.

Отображение r': $(\partial\bar{E} \setminus S) \to EO(k) \subset MO(k)$ мы можем распространить на все $\partial\bar{E}$, отображая S в особую точку пространства $MO(k)$. Полученное отображение $\partial\bar{E} \to MO(k)$ и есть приклеивающее отображение r.

Пространство $\bar{E} \bigcup_r MO(k)$ является искомым аналогом пространства Тома для разделяющихся 2-погружений, т.е. $\bar{\Gamma}_2 MO(k)$.

Чтобы получить конструкцию для $l = 3$ подобно предыдущему мы должны приклеить пространство $\bar{E}[O(k) \oplus O(k) \oplus O(k)]$ вдоль его границы к пространству $\bar{\Gamma}_2 MO(k)$.

Через $\bar{E}[\underbrace{O(k)\oplus \dots \oplus O(k)}_{l\text{-раз}}]$ обозначается подпространство тотального пространства универсального векторного $\underbrace{O(k)\oplus \dots \oplus O(k)}_{l\text{-раз}}$ расслоения $E[O(k)\oplus \dots \dots \oplus O(k)]$, состоящее из объединения таких векторов x всех слоев, для которых все компоненты $x_1, \dots, x_l$ (при разложении слоя $R^{lk}=R_1^k\oplus \dots \oplus R_l^k$) имеют длину, не превосходящую единицы. Значит:

$$\bar{E}[O(k)\oplus \dots \oplus O(k)] =: \{x\in E[O(k)\oplus \dots \oplus O(k)] \mid \forall \|x_i\| \leqq 1,\ i=1, \dots, l\}.$$

Замечание. Пусть

$$\partial \bar{E}_2 = \{x \mid \|x_1\|=1 \quad \text{или} \quad \|x_2\|=1 \quad \text{или} \quad \|x_3\|=1\},$$

$S_2 = \{x \mid$ по крайней мере две компоненты вектора x имеют единичную норму$\}$.

Пространство $\partial \bar{E}_2 \setminus S_2$ будет тотальным пространством некоторого $O(k)\oplus O(k)$ расслоения. Выбор порядка слагаемых в разложении расслоения $E[O(k)\oplus \oplus O(k)\oplus O(k)]$ естественным образом индуцирует упорядочение слагаемых расслоения $\partial \bar{E}_2 \setminus S_2$. Приклеивающее отображение $(\partial \bar{E}_2 \setminus S_2)\to E[O(k)\oplus O(k)]$ должно сохранить порядок слагаемых.

Вообще, чтобы получить пространство $\bar{\Gamma}_l MO(k)$, нужно приклеить к пространству $\bar{\Gamma}_{l-1} MO(k)$ пространство $\bar{E}[\underbrace{O(k)\oplus \dots \oplus O(k)}_{l\text{-раз}}]$ по границе способом, аналогичным к описанному выше. При этом замечание, аналогичное сделанному для $l=3$, должно быть соблюдено.

Описание конструкции закончено.

Теорема 1. $\bar{G}^l(n-k, k) \approx \pi_n(\bar{\Gamma}_l MO(k))$.

Доказательство. Возможность построения отображения, сопоставляющего каждому разделяющемуся l-погружению отображение $S^n \to \bar{\Gamma}_l MO(k)$ следует из самой конструкции пространства $\bar{\Gamma}_l MO(k)$. Поскольку точно так же пленке в $S^n \times I$, осуществляющей кобордизм l-погружений, можно сопоставить отображение $S^n\times I \to \bar{\Gamma}_l MO(k)$, то получаем, что кобордантным l-погружениям соответствуют гомотопные отображения $S^n \to \bar{\Gamma}_l MO(k)$, т.е. мы получаем отображение $\bar{G}^l(n-k, k)\to \pi_n(\bar{\Gamma}_l MO(k))$, которое, как легко видеть, является гомоморфизмом групп. Поскольку конструкция отображения $\pi_n(\bar{\Gamma}_l MO(k))\to \to \bar{G}^l(n-k, k)$ несколько нестандартна, то мы ее опишем подробнее. Это отображение при $l=1$ известно, а для $l>2$ будет полностью аналогично случаю $l=2$. Поэтому опишем его лишь для $l=2$.

Итак, пусть $\alpha \in \pi_n(\bar{\Gamma}_2 MO(k))$ и $f\colon S^n \to \bar{\Gamma}_2 MO(k)$ произвольный представитель класса α.

Из конструкции следует, что $E[O(k)\oplus O(k)]\subset \bar{\Gamma}_2 MO(k)$.

Обозначим через $S(t)$ и $E(t)$ (где $0<t<1$) подпространство из $\bar{E}[O(k)\oplus \oplus O(k)]$, состоящее из объединения сфер (соответственно шаров) радиуса t всех слоев, т.е.

$$S(t) =: \{x\in \bar{E}[O(k)\oplus O(k)] \mid \|x\|=t\}, \quad E(t) =: \{x\in \bar{E}[O(k)\oplus O(k)] \mid \|x\| \leqq t\}.$$

Заметим, что $S(t)$ имеет в $\bar{E}[O(k)\oplus O(k)]$ коразмерность 1 и расслаивается над $B[O(k)\oplus O(k)]$ со слоем S^{k-1}. Можно считать, что $f\colon S^n\to\bar{\Gamma}_2MO(k)$ трансверсально к $S\left(\frac{1}{2}\right)$. Поэтому $f^{-1}\left(S\left(\frac{1}{2}\right)\right)$ есть $(n-1)$-мерное подмногообразие сферы S^n. Обозначим его через ∂N. Оно разрезает сферу на две части, которые мы будем обозначать через N_0 и N_1, причем $f^{-1}\left(E\left(\frac{1}{2}\right)\right)=N_0$ (тогда $\partial N_0=\partial N_1=$ $=\partial N$).

Обозначим через Y_1 и Y_2 подмножества в $E\left(\frac{1}{2}\right)$:

$$Y_1=\{x|\,\|x_2\|=0\},\quad Y_2=\{x|\,\|x_1\|=0\}.$$

Сужения $f|_{\partial N}$ и $f|_{N_0}$ можно сделать трансверсальными на Y_1 и Y_2, а полученное новое отображение, которое мы также будем обозначать через $f|_{N_0}$, можно распространить до отображения всей сферы в $\bar{\Gamma}_2MO(k)$. Это новое отображение по-прежнему обозначим через f. При этом можем считать, что по-прежнему

$$f(N_0)\subset E\left(\frac{1}{2}\right)\quad\text{и}\quad f(N_1)\subset\bar{\Gamma}_2MO(k)\diagdown E\left(\frac{1}{2}\right).$$

Обозначим через E_1 пространство $\bar{E}[O(k)\oplus O(k)]\diagdown E\left(\frac{1}{2}\right)$.

Заметим, что существует ретракция $\varrho\colon E_1\to\partial\bar{E}[O(k)\oplus O(k)]$.

Обозначим суперпозицию ретракции ϱ с приклеивающим отображением $r\colon\partial\bar{E}[O(k)\oplus O(k)]\to MO(k)$ через θ (значит $\theta=r\circ\varrho$). Суперпозиция $\theta\circ f$ отображает многообразие N_1 в пространство $MO(k)$, причем $\theta\circ f|_{\partial N}$ уже трансверсально к $BO(k)$. Поэтому отображение $\theta\circ f|_{N_1}$ можно сделать трансверсальным к $BO(k)$, не меняя его при этом в окрестности края N_1. Полученное новое отображение тоже будем обозначать через $\theta\circ f|_{N_1}$.

Объединение прообразов $(f|_{N_0})^{-1}(Y_1\cup Y_2)$ и $(\theta\circ f|_{N_1})^{-1}(BO(k))$ будет образом некоторого 2-погружения. Его класс мы и сопоставим злементу α из $\pi_n(\bar{\Gamma}_2MO(k))$.

§ 3.

В этом параграфе укажем на связь конструкции пространства $\bar{\Gamma}_lMO(k)$ с конструкцией Джеймса [9] приведенного произведения.

Прежде всего напомню конструкцию Джеймса.

Джеймс в 1959 году построил функтор, сопоставляющий каждому топологическому пространству X с отмеченной точкой $(*)$ топологический моноид X_∞, который является гомотопически эквивалентным пространству петель надстройки над X, т.е. $X_\infty\cong\Omega SX$. Конструкция топологического моноида X_∞ следующая.

Возьмем несвязное объединение $\bigcup\limits_{i=1}^{\infty}X^i$ (где $X^i=\underbrace{X\times\ldots\times X}_{i\text{-раз}}$) и профакторизуем его по следующему отношению эквивалентности: Две последовательности (не обязательно одинаковой длины) эквивалентны, если при вычеркивании отмеченных точек у обеих последовательностей в итоге останется одна и та же последовательность. По-другому можно сказать, что X_∞ есть свободный моноид с единицей, порожденный точками пространства X, причем единицей служит отмеченная точка пространства X.

Мы определим еще и пространство X_l. Оно состоит из тех слов свободного моноида X_∞, которые имеют длину, не превосходяшую l. То есть X_l получится, если по описанному отношению эквивалентности профакторизовать не бесконечное объединение $\bigcup_{i=1}^{\infty} X^i$, а лишь объединение по $i \leqq l$.

Утверждение 1. $MO(k)_l = \bar{\Gamma}_l MO(k)$.

Доказательство. Из конструкции $\bar{\Gamma}_l MO(k)$ легко видеть, что это пространство можно было бы получить и следующим образом.

Возьмем дизъюнктивное объединение

$$\bigcup_{i=1}^{l} \underbrace{EO(k) \times \ldots \times EO(k)}_{i\text{-раз}}$$

(где $EO(k)$ универсальное расслоение, где слои — замкнутые единичные шары), и профакторизуем по следующему отношению эквивалентности.

Пусть $(x_1, \ldots, x_r, \ldots, x_i)$ точка из i-ого члена объединения (где $1 \leqq r \leqq i$) и $x_r \in \partial EO(k)$. Тогда эту точку считаем эквивалентной точке $(i-1)$-ого члена, равной $(x_1, \ldots, x_{r-1}, x_{r+1}, x_{r+2}, \ldots, x_i)$.

Разложим это отношение эквивалентности в композицию двух других отношений эквивалентности:

1) $(x_1, \ldots, x_i) \underset{1}{\sim} (x'_1, \ldots, x'_i)$, если существует число r $(1 \leqq r \leqq i)$ такое, что:

a) $\|x_r\| = \|x'_r\| = 1$

b) $(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots x_i) = (x'_1, \ldots, x'_{r-1}, x'_{r+1}, \ldots, x'_i)$

2) $(x_1, \ldots, x_i) \underset{2}{\sim} (y_1, \ldots, y_j)$, если при вычерчивании точек, имеющих единичную норму, они станут равными.

Ясно, что после факторизации по отношению эквивалентности 1 мы получим объединение $\bigcup_{i=1}^{l} \big(MO(k)\big)^i$, а отношение эквивалентности 2 превратится в отношение эквивалентности из определения конструкции Джеймса. Доказательство закончено.

§ 4. Конструкция аналога пространства Тома для произвольных l-погружений

Конструкция пространства $\Gamma_l MO(k)$, для которого имеет место изоморфизм $\pi_n\big(\Gamma_l MO(k)\big) \approx G^l(n-k, k)$ отличается от конструкции пространства $\bar{\Gamma}_l MO(k)$ лишь тем, что вместо того, чтобы приклеить $\bar{E}[O(k) \oplus \ldots \oplus O(k)]$ по своей границе к $\bar{\Gamma}_{l-1} MO(k)$ нужно приклеить $\bar{E}O^{(l)}(k)$ по своей границе к $\bar{\Gamma}_{l-1} MO(k)$. (Пространство $\bar{E}O^{(l)}(k)$ определяется аналогично пространству $\bar{E}[O(k) \oplus \ldots \oplus O(k)]$ т.е.: $\bar{E}O^{(l)}(k) =$ объединение тех векторов из каждого слоя, которые при локальном разложении слоя в произведение $R^{lk}=R^k\oplus\ldots\oplus R^k$ имеют компоненты, не превосходящие единицы.)

Доказательство изоморфизма $\pi_n(\Gamma_l MO(k))\approx G^l(n-k, k)$ аналогично доказательству изоморфизма $\pi_n(\bar{\Gamma}_l MO(k))\approx \bar{G}^l(n-k, k)$.

Итак имеет место

Теорема 2. $\pi_n(\Gamma_l(MO(k))\approx G^l(n-k, k)$.

§ 5. Связь пространства $\Gamma_l MO(k)$ с Γ функтором Баррата—Эклеса

Баррат и Эклес, продолжив деятельность Джеймса, построили некоторый функтор Γ, сопоставляющий каждому пространству X пространство ΓX, которое гомотопически эквивалентно пространству $\Omega^\infty S^\infty X$.

Напомню определение пространства ΓX.

Пусть m, n — натуральные числа причем $m\leqq n$ и α монотонное отображение $\{1, \ldots, m\}\xrightarrow{\alpha}\{1, \ldots, n\}$. Пусть σ — элемент n-ой симметрической группы $S(n)$. Обозначим через $\alpha^*(\sigma)$ тот единственный элемент из $S(m)$, для которого композиция отображений

$$\{1, \ldots, m\}\xrightarrow{\alpha}\{1, \ldots, n\}\xrightarrow{\sigma}\{1, \ldots, n\}\xrightarrow{\text{приведение на образ } \sigma\circ\alpha}\{\sigma\alpha(1), \ldots, \sigma\alpha(m)\}\to$$
$$\xrightarrow{\alpha^*(\sigma)}\{1, \ldots, m\}$$

совпадает с тождественным. Тем самым мы определили отображение $\alpha^*\colon S(n)\to\to S(m)$ (см. [3]).

Пусть $WS(n)$ — ациклический свободный $S(n)$-комплекс. Его можно определить как бесконечное соединение: $S(n)*S(n)*\ldots$. Поэтому α^* индуцирует отображение $\alpha^*\colon WS(n)\to WS(m)$. Возьмем дизъюнктивное объединение

$$\bigcup_{i=1}^{\infty} WS(i)\times X^i \quad \text{где} \quad X^i=\underbrace{X\times\ldots\times X}_{i\text{-множителей}}$$

и произведем отождествление.

A) $(w_i, (x_1, \ldots, x_i))=(w_i\sigma, (x_1, \ldots, x_i)\sigma)$ для $\sigma\in S(i)$, $w_i\in WS(i)$

B) $(w_n, (x_1, \ldots, x_n))=(\alpha^* w_n, (x_1, \ldots, x_n)\alpha^*)$ для любого такого монотонного отображения $\alpha\colon\{1, \ldots, m\}\to\{1, \ldots, n\}$, для которого x_j есть отмеченная точка пространства X при $j\notin \operatorname{Im}\alpha$. Полученное пространство есть ΓX.

Определим пространство $\Gamma_l[X]$ как факторпространство *конечного* объединения $\bigcup_{i=1}^{l} WS(i)\times X^i$ по отношениям A и B.

Утверждение 2. $\Gamma_l[MO(k)]=\Gamma_l MO(k)$.

(Слева стоит результат применения функтора Γ_l к пространству $MO(k)$, а справа — классифицирующее пространство l-погружений.)

Доказательство аналогично доказательству утверждения 1. Заметим только, что поскольку существует расслоение

$$EO^{(l)}(k)\xrightarrow{E[O(k)\oplus\ldots\oplus O(k)]}BS(l),$$

то пространство $EO^{(l)}(k)$ можно представить в виде

$$EO^{(l)}(k) = WS(l) \underset{S(l)}{\times} E[O(k) \oplus \ldots \oplus O(k)] = WS(l) \underset{S(l)}{\times} \big(EO(k) \times \ldots \times EO(k)\big).$$

Ясно, что при приклеивании границы $EO^{(l)}(k)$ к $\Gamma_{l-1}MO(k)$ точки $(w, x_1, \ldots, x_l)$ и $(w, x_1', \ldots, x_l')$ отображаются в одну и ту же точку, если существует число r $(1 \leqq r \leqq l)$ такое, что $\|x_r\| = \|x_r'\| = 1$, а для всех остальных индексов j, $x_j = x_j'$. Поэтому можно сначала каждый множитель $EO(k)$ профакторизовать по своей границе, т.е. рассмотреть пространство

$$E_l = WS(l) \underset{S(l)}{\times} (MO(k) \times \ldots \times MO(k))$$

и приклеить его к $\Gamma_{l-1} MO(k)$. А при конструкции $\Gamma_l[MO(k)]$ после факторизации по отношению эквивалентности А мы получим пространство E_l, а факторизация В дает как раз приклеивающее отображение.

Замечание. Этим утверждением объясняется наш выбор для обозначения классифицирующего пространства $\Gamma_l MO(k)$.

§ 6. Теорема Хирша — геометрический смысл формул Джеймса и Баррата—Эклеса

Мы видели, что группы $\pi_{n+k}\big(MO(k)_\infty\big)$ и $\pi_{n+k}\big(\Gamma MO(k)\big)$ имеют геометрический смысл: они изоморфны группам кобордизмов $\bar{G}^\infty(n, k)$ и $G^\infty(n, k)$. Естественно возникает вопрос: какой же геометрический смысл имеют гомотопические эквивалентности:

a) $MO(k)_\infty \cong \Omega SMO(k)$

b) $\Gamma MO(k) \cong \Omega^\infty S^\infty MO(k)$.

Из них вытекают изоморфизмы

a) $\pi_n\big(MO(k)_\infty\big) \approx \pi_{n+1}\big(SMO(k)\big)$

b) $\pi_n\big(\Gamma MO(k)\big) \approx \pi_n^s\big(MO(k)\big)$

то есть

a′) $\bar{G}^\infty(n-k, k) \approx \pi_{n+1}\big(SMO(k)\big)$

b′) $G^\infty(n-k, k) \approx \pi_{n+N}\big(S^N MO(k)\big)$ где $N \gg n, k$.

Эти изоморфизмы равносильны теореме Хирша в ослабленной форме. Имеется в виду следующая теорема Хирша. Если погружение (коразмерности больше единицы) многообразия в эвклидово пространство обладает нормальным полем, то оно регулярно гомотопно погружению в эвклидово пространство на единицу меньшей размерности. Ослабление заключается в том, что вместо регулярной гомотопности утверждается лишь кобордантность такому погружению.

Для изоморфизма b′) доказательство его эквивалентности слабой теореме Хирша содержится в работе Уэллса [7].

Чтобы понять изоморфизм a′), нужно иметь в виду следующее:

Замечание. Пусть $f\colon M^{n-k}\to R^n$ некоторое погружение. Тогда и только тогда существует функция $h\colon M^{n-k}\to R'$, для которой $f\times h\colon M^{n-k}\to R^n\times R'$ есть вложение, если f есть разделяющееся погружение.

Итак, группа кобордизмов $(n-k)$-мерных разделяющихся погружений в сферу S^n изоморфна группе кобордизмов $(n-k)$-мерных вложений с нормальным 1 полем в сферу S^{n+1}, что и выражается изоморфизмом a′).

Б) ГРУППЫ $G^l(n, k)$

В данной части работы мы получаем некоторую информацию о группах кобордизмов l-погружений (определенных в предыдущем части) исходя из построенного там гомотопического представления.

Напомню, что в предыдущем части построены пространства $\bar{\Gamma}_l MO(k)$ и $\Gamma_l MO(k)$ со свойствами: $\bar{G}^l(n, k)\approx\pi_{n+k}(\bar{\Gamma}_l MO(k))$ и $G^l(n, k)\approx\pi_{n+k}(\Gamma_l MO(k))$.

Напомню также, что $\bar{\Gamma}_l MO(k)$ есть «подпространство слов длины не больше l» в пространстве $MO(k)_\infty$, где $MO(k)$ есть пространство Тома, а X_∞ есть функтор Джеймса, дающий ΩSX.

Пространство $\Gamma_l MO(k)$ есть «подпространство слов длины не больше l» в пространстве $\Gamma MO(k)$ где Γ есть Γ-функтор Баррата и Эклеса.

Теорема 3. a) *Пусть p — простое число. Если k четно и выполняются неравенства*

1. $k \geqq \dfrac{p-1}{4}$,

2. $n < k+2p^2-2p-1$,

то $G^l(n, k)\otimes Z_p = 0$ при любом l.

b) *Если k четно и*

1. $k \geqq \dfrac{p-1}{4}$,

2. $n = k+2p^2-2p-1$,

то существует l_0 такое, что:

1) *для любого l больше, либо равного l_0 группа $G^l(n, k)$ будет иметь нетривиальную p-компоненту;*

2) *для любого l меньшего l_0 группа $G^l(n, k)$ имеет тривиальную p-компоненту.*

Доказательство. a) Известно, что

$$\sum_{n=0}^{k+2p^2-2p-2} G^\infty(n, k)\otimes Z_p = 0 \quad \text{(см. Уэллс [7]).}$$

Значит

$$\sum_{j=0}^{2k+2p^2-2p-2} \overline{H}_j(\Gamma MO(k))\otimes Z_p = 0,$$

и для любого l

$$\sum_{j=0}^{2k+2p^2-2p-2} \overline{H}_j(\Gamma_l MO(k))\otimes Z_p = 0.$$

Значит

$$\sum_{j=0}^{2k+2p^2-2p-2} \pi_j(\Gamma_l MO(k))\otimes Z_p = 0, \quad \text{т. е.} \quad \sum_{n=0}^{k+2p^2-2p-2} G^l(n,k)\otimes Z_p = 0$$

что и требовалось доказать.

b) Известно, что $G^\infty(k+2p^2-2p-1, k)\otimes Z_p \neq 0$ (см. Уэллс [7]). Значит

$$\sum_{j=0}^{2k+2p^2-2p-1} \pi_j(\Gamma MO(k))\otimes Z_p \neq 0$$

и поэтому

$$\sum_{j=0}^{2k+2p^2-2p-1} H_j(\Gamma MO(k))\otimes Z_p \neq 0.$$

Значит по теореме Баррата—Эклеса [4] существует l_0 такое, что

$$\sum_{j=0}^{2k+2p^2-2p-1} H_j(\Gamma_l MO(k))\otimes Z_p \neq 0 \quad \text{при} \quad l \geqq l_0,$$

а при $l < l_0$ эта сумма равна нулю. Тогда при $l<l_0$

$$\sum_{j=0}^{2k+2p^2-2p-1} \pi_j(\Gamma_l MO(k))\otimes Z_p = 0 \quad \text{т. е.} \quad \sum_{n=0}^{k+2p^2-2p-1} G^l(n,k)\otimes Z_p = 0.$$

А для $l \geqq l_0$ мы будем иметь

$$\sum_{j=0}^{2k+2p^2-2p-1} H_j(\Gamma_l MO(k))\otimes Z_p \neq 0$$

значит

$$\sum_{j=0}^{2k+2p^2-2p-1} \pi_j(\Gamma_l MO(k))\otimes Z_p \neq 0 \quad \text{т. е.} \quad \sum_{n=0}^{2k+2p^2-2p-1} G^l(n,k)\otimes Z_p \neq 0.$$

Однако по а)

$$\sum_{n=0}^{k+2p^2-2p-2} G^l(n,k)\otimes Z_p = 0.$$

Поэтому $G^l(k+2p^2-2p-1, k)\otimes Z_p \neq 0$.

Теорема 4. *$G^l(n, 2k+1)$ 2-примарна при любом n и $k>0$.*

Замечание. Теорему можно вывести из результата Уэллса, утверждающего 2-примарность групп $G^\infty(n, 2k+1)$. Рассуждения при этом будут такими же, как в предыдущем доказательстве. Однако здесь я приведу другое доказательство, независимое от результатов Уэллса.

Доказательство. Утверждение теоремы следует из рассмотрения спектральной последовательности расслоения

$$\left(EO^{(l)}(k), \partial EO^{(l)}(k)\right) \xrightarrow{E[O(k)\oplus\ldots\oplus O(k)],\, \partial E[O(k)\oplus\ldots\oplus O(k)]} BS(l)$$

второй член которой есть группа $E_2 = H_*\left(BS(l), H_*(E, \partial E; Z_p)\right)$. А так как эта группа изоморфна группе $H_*\left(BS(l), H_*(MO(k)\wedge\ldots\wedge MO(k); Z_p)\right)$ а группа $H_*\left(MO(k)\wedge\ldots\wedge MO(k); Z_p\right) = 0$, то и она равна нулю. Значит

$$H_*\left(EO^{(l)}(k), \partial EO^{(l)}(k); Z_p\right) = H_*\left(\Gamma_l MO(k), \Gamma_{l-1} MO(k); Z_p\right) = 0.$$

Отсюда индукцией по l следует утверждение по теореме Уайтхеда (эта теорема применима, так как при $k>0$ $\Gamma_l MO(2k+1)$ односвязно).

Теорема 5. *При четном k и $n<3k-2$ для $l \geqq 2$*

$$\operatorname{rang} G^l(n, k) = \pi_k\left(\frac{n-k}{4}\right)$$

(*где $\pi_k(x)=0$ если x не целое натуральное число, и $\pi_k(x)$ число разбиений x в сумму натуральных чисел, не превосходящих k, если x натуральное число*).

Замечание. Для заданной области значений чисел n и k по размерностным соображениям имеют место изоморфизмы $G^3(n, k) \approx G^4(n, k) \approx \ldots \approx \approx G^\infty(n, k)$. Наш результат для ранга этих групп при $l \geqq 3$, как легко видеть, согласуется с теоремой Уэллса о ранге групп (см. теорему 5 из работы [7]). Однако изоморфность групп $G^2(n, k)\otimes Q$ и $G^3(n, k)\otimes Q$ из размерностных соображений не следует и составляет содержательную часть нашего утверждения. Наше доказательство будет независимым от результатов Уэллса.

Доказательство. Воспользуемся следующей теоремой Серра.

Если группы гомологий односвязного пространства X конечны до размерности r, то ранги групп $\pi_n(X)$ и $H_n(x)$ одинаковы при $n \leqq 2r$. Относительные гомологические группы $H_n\left(\Gamma_l MO(k), \Gamma_{l-1} MO(k)\right)$ конечны при $n<2lk$, а группы $H_n\left(\Gamma_1 MO(k)\right) = H_n\left(MO(k)\right)$ конечны при $n<2k$. Поэтому индукцией по l получается, что $H_n\left(\Gamma_l MO(k)\right)$ конечна при $n<2k$. Поэтому по упомянутой теореме Серра получаем:

$$\pi_n\left(\Gamma_l MO(k)\right)\otimes Q \approx H_n\left(\Gamma_l MO(k)\right)\otimes Q \quad \text{при} \quad n \leqq 4k-2.$$

Но ранг группы $H_n\left(\Gamma_l MO(k)\right)$ уже легко вычислить.

Пусть $P(X)$ означает для любого пространства X его приведенный ряд Пуанкаре над полем Q. Тогда

$$P\left(BO(k)\right) = \sum_{i=1}^{\infty} \pi_k\left(\frac{i}{4}\right)\cdot t^i.$$

Обозначим этот ряд через $f(t)$. Отсюда $P(MO(k)) = f(t)\cdot t^{2k}$. Поэтому

$$P(MO^{(j)}(k)) = P(\underbrace{MO(k)\wedge\ldots\wedge MO(k)}_{j\text{-раз}}) = f(t)^j\cdot t^{2kj}.$$

Нас интересует коэффициент при t^{n+k}, где $n<3k-2$. Поэтому нужно рассматривать лишь первые два слагаемых этой суммы. Первое слагаемое имеет при t^{n+k} коэффициент равный $\pi_k\left(\frac{n-k}{4}\right)$. Второе слагаемое имеет коэффициент при t^{n+k} равный

$$(*)\qquad \sum_{\substack{(i,j)\\ i+j=n+k}} \pi_k\left(\frac{i-2k}{4}\right)\cdot\pi_k\left(\frac{j-2k}{4}\right).$$

Однако каждый член в сумме $(*)$ равен нулю при $n<3k-2$, поскольку либо $i-2k<0$, либо $j-2k<0$.

Итак, ранг $\pi_{n+k}(\Gamma_l MO(k))=$ ранг $G^l(n,k)=\pi_k\left(\frac{n-k}{4}\right)$ если k четно и $n<3k-2$.

Теорема 6. *Ранги групп $G^l(n,k)$ и $G^{l-1}(n,k)$ равны при $n<l(2k-1)$.*

Доказательство. Группа гомологий $H_n(\Gamma_l MO(k)/\Gamma_{l-1}MO(k);Q) = H_n(MO^{(l)}(k);Q)=H_n(MO(k)\wedge\ldots\wedge MO(k);Q)$ равна нулю при $n\leqq l(2k-1)$. Поэтому гомоморфизм $H_n(\Gamma_{l-1}MO(k);Q)\to H_n(\Gamma_l MO(k);Q)$, индуцированный вложением $\Gamma_{l-1}MO(k)\subset\Gamma_l MO(k)$, является изоморфизмом для $n<l(2k-1)$. По теореме Уайтхеда $\pi_n(\Gamma_{l-1}MO(k))\otimes Q\to\pi_n(\Gamma_l MO(k))\otimes Q$ также изоморфизм при $n<(2k-1)\cdot l$. Доказательство закончено.

Замечание. a) По только что доказанной теореме $G^l(n,k)\otimes Q\approx G^\infty(n,k)\otimes Q$ для достаточно больших l, начиная примерно с $l\approx\frac{n}{2k}$. Размерностные соображения дают аналогичное утверждение лишь начиная примерно с $l\approx\frac{n}{k}$.

b) Ранги групп $G^\infty(n,k)$ известны (см. Уэллс [7]).

Для нечетного k $G^\infty(n,k)\otimes Q=0$.

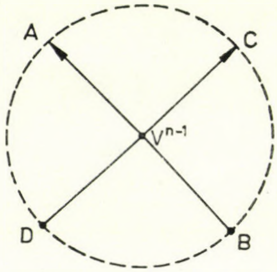
Для четного k ранг $G^\infty(n,k)$ равен числу разбиений числа $\frac{n-k}{4}$ в сумму не более чем $\frac{k}{2}$ слагаемых, не превосходящих числа k.

Теорема 7. $G^2_\Omega(n,1)\approx\pi_{n+1}(S'\vee SP^\infty R)$ *где $SP_\infty R$ есть надстройка над бесконечномерным проективным пространством.*

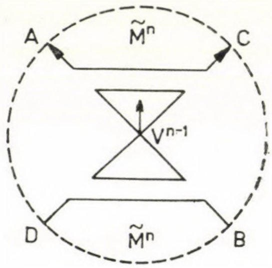
Доказательство. Рассмотрим вещественное двумерное представление $\varrho\colon Z_2\to O(2)$ группы Z_2, при котором нетривиальный элемент — это отражение плоскости R^2 на ось y. Пространство Тома, соответствующее этому представлению, обозначим через M_ϱ.

Очевидно, что $M_\varrho\cong SP_\infty R$. Кроме того, известно, что $S'\cong MSO(1)$. Итак, нужно доказать: $G^2_\Omega(n,1)\approx\pi_{n+1}(MSO(1)\vee M_\varrho)$.

Пусть $f: M^n \to R^{n+1}$ есть 2-погружение (M^n ориентировано). Пусть V^{n-1} многообразие двойных точек. Слой трубчатой окрестности V^{n-1} выглядит так:



где отрезки AB и CD суть пересечения слоя с образом погружения, а стрелки показывают нормальное поле погружения, задающее ориентацию M^n. В каждом слое эту фигуру заменим на следующую:



Нетрудно видеть, что после такой замены получим новое 2-погружение, которое

1) кобордантно исходному погружению f

2) характеризуется двумя вложениями:

a) одним n-мерным (см. $\tilde{M}^n$ на рисунке)

b) одним $(n-1)$-мерным с нормальным полем (см. V^{n-1}). Отсюда легко следует изоморфизм $G^2_\Omega(n, 1) \approx \pi_{n+1}(MSO(1) \vee M_\varrho)$.

Следствие 1. *Группы $G^2_\Omega(n, 1)$ 2-примарны.*

Доказательство. Универсальным накрывающим для пространства $S' \vee SP_\infty R$ является бесконечный букет $SP_\infty R \vee SP_\infty R \vee \ldots$. Его гомологические группы 2-примарны и значит гомотопические группы также.

Следствие 2. *$G^2_\Omega(2, 1) \approx Z_2^\infty$, где Z_2^∞ — бесконечная сумма групп Z_2.*

Доказательство. Будем пользоваться следующими результатами Милнора об F-функторе (см. [10]).

1) Для любого пространства A, сопоставленное ему F-функтором пространство $F(A)$ гомотопически эквивалентно пространству ΩSA, т. е. $F(A) \cong \Omega SA$.

2. $F(A\vee B)=\prod\limits_{i,j}F(A^{(i)}\wedge B^{(j)})$, где $A^{(i)}$ и $B^{(j)}$ — это приведенные степени $\underbrace{A\wedge\ldots\wedge A}_{i\text{-раз}}$ и $\underbrace{B\wedge\ldots\wedge B}_{j\text{-раз}}$. Пусть $A=S^0$ — нульмерная сфера, а $B=P_\infty R$. Тогда $\pi_{n+2}\big(F(A\vee B)\big)\approx\pi_{n+1}(S'\vee SP_\infty R)$. С другой стороны, эта же группа изоморфна произведению групп

$$\prod_i \pi_{n+2}\big(F(\underbrace{P_\infty R\wedge\ldots\wedge P_\infty R}_{i\text{-раз}})\big).$$

Покажем, что при $n=2$ каждая из групп $\pi_{n+2}\big(F(\overbrace{P_\infty R\wedge\ldots\wedge P_\infty R}^{i\text{-раз}})\big)$, $i=1,2,\ldots,$ имеет не более двух элементов, причем $\pi_4(FP_\infty R)=Z_2$. Из этого будет следовать, что группа изоморфна одной из следующий групп:

1) тривиальная группа
2) Z_2^k, где k — любое натуральное
3) Z_2^∞.

Однако группа $\pi_3(S'\vee SP_\infty R)$ содержит подгуппу Z_2^∞. Действитепьно, $\pi_3(S'\vee SP_\infty R)=\pi_3(SP_\infty R\vee SP_\infty R\vee\ldots)$, и значит содержит подгруппу вида $\pi_3(SP_\infty R)\oplus\pi_3(SP_\infty R)\oplus\ldots$ т. е. группу Z_2^∞. Значит $G_\Omega^2(2,1)=Z_2^\infty$.

Итак, осталось доказать лемму:

Лемма 1.

a) $\pi_4\big(F(P_\infty R)\big)=\pi_3(SP_\infty R)=Z_2$

b) $\pi_4\big(F(P_\infty R\wedge P_\infty R)\big)=\pi_3(SP_\infty R\wedge P_\infty R)=Z_2$

c) $\pi_4\big(F(\underbrace{P_\infty R\wedge\ldots\wedge P_\infty R}_{i\text{-раз}})\big)=0$ *если* $i>2$.

Доказательство утверждения а). $\pi_1(SP_\infty R)=0$, $\pi_2(SP_\infty R)=H_2(SP_\infty R)=Z_2$. Нам нужно вычислить группы $H_3(SP^\infty R|_2;Z_2)$, где $SP_\infty R|_2$ есть второе убивающее пространство для $SP_\infty R$.

Рассмотрим для этого стандартное расслоение

$$SP_\infty R|_2\xrightarrow{K(Z_2,1)}SP_\infty R.$$

Второй член соотвествующей когомологической спектральной последовательности выглядит так:

α^3	0	$\alpha^3\otimes e$	$\alpha^3\otimes f$
α^2	0	$\alpha^2\otimes e$	$\alpha^2\otimes f$
α	0	$\alpha\otimes e$	$\alpha\otimes f$
1	0	e	f

Из определения убивающего пространства следует, что $\tau(\alpha)=e$, где τ — трансгрессия. Поскольку $\alpha^2=Sq'\alpha$ из трансгрессивности элемента α следует трансгрессивность α^2, т.е. $d_3(\alpha^2)=f$. Итак, f исчезнет.

Остается ли элемент $\alpha\otimes e$? Он исчезнет тогда и только тогда, если: 1. либо $d_2(\alpha\otimes e)\neq 0$, 2. либо $d_2(\alpha^2)=\alpha\otimes e$. Покажем, что ни то, ни другое не имеет места.

$$d_2(\alpha\otimes e) = d_2(\alpha)\cdot e+\alpha\cdot d_2(e) = e\cdot e+\alpha\cdot O = e^2 = 0$$

(поскольку произведение в когомологиях надстройки тривиально). $d_2(\alpha^2)= =2\alpha\cdot d_2(\alpha)=0$, поскольку коэффициенты из Z_2, значит элемент $\alpha\otimes e$ остается в члене E_∞. Элемент α^3 исчезнет, так как $d_2(\alpha^3)=3\alpha^2 d_2\alpha=\alpha^2\otimes e\neq 0$. Значит остается только один элемент, т.е. $\pi_3(SP_\infty R)=Z_2$.

Доказательство утверждений b) и c). Пространство $S(\underbrace{P_\infty R\wedge\ldots\wedge P_\infty R}_{i\text{-раз}})$ является пространством Тома для подгруппы $O(i+1)$, состоящей из диагональных матриц вида

$$\begin{pmatrix} 1 & & & \\ & \varepsilon_1 & & 0 \\ & & \ddots & \\ & 0 & & \varepsilon_i \end{pmatrix}$$

где каждое из чисел $\varepsilon_1, \ldots, \varepsilon_i$ равно $+1$ или -1. Поэтому группу $\pi_{n+1}(SP_\infty R\wedge\ldots \ldots\wedge P_\infty R)$ можно интерпретировать как группу кобордизмов $(n-i)$-мерных многообразий, вложенных в сферу S^{n+1}, нормальное расслоение которых разложено в прямую сумму одномерных расслоений, одно из которых ориентированно.

Утверждения b) и c) получаются отсюда немедленно.

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GENERALIZATIONS OF THEOREMS OF KATONA AND MILNER

By
P. FRANKL (Budapest)

1. Introduction. GY. KATONA [1] proved the following

THEOREM Ia. *If* $1 \leqq g < h$, $1 \leqq k \leqq h$ *and* $g+k \geqq h$ *(g, h, k are integers) and* $\mathscr{A}$ *is an h-uniform hypergraph such that any two edges of* $\mathscr{A}$ *have at least k vertices in common then*

$$|\mathscr{A}^g| \geqq |\mathscr{A}| \frac{\binom{2h-k}{g}}{\binom{2h-k}{h}}.$$

($\mathscr{A}^g$ is a g-uniform hypergraph, its edges are exactly those g-sets which are contained in at least one edge of $\mathscr{A}$. $|\mathscr{A}|$ denotes the number of edges of $\mathscr{A}$.)

We prove the following generalization of Theorem Ia.

THEOREM Ib. *Let* g_1, g_2, h_1, h_2, k *be integers satisfying* $1 \leqq g_i < h_i$, $1 \leqq k \leqq h_i$, $k+g_i \geqq h_i$ *for* $i=1, 2$. *If we are given two non-empty hypergraphs,* $\mathscr{A}_1$ *is* h_1*-uniform,* $\mathscr{A}_2$ *is* h_2*-uniform such that whenever* A_1, A_2 *are edges of* $\mathscr{A}_1$ *and* $\mathscr{A}_2$*, resp. then* $|A_1 \cap A_2| \geqq k$*, then either*

$$|\mathscr{A}_1^{g_1}| \geqq |\mathscr{A}| \frac{\binom{2h_1-k}{g_1}}{\binom{2h_1-k}{h_1}} \tag{1}$$

or

$$|\mathscr{A}_2^{g_2}| > |\mathscr{A}_2| \frac{\binom{2h_2-k}{g_2}}{\binom{2h_2-k}{h_2}}. \tag{2}$$

Theorem Ib is a generalization of Theorem Ia, indeed, as we have to set $\mathscr{A}_1 = \mathscr{A}_2$ only.

Using Theorem Ia, E. C. Milner proved the following

THEOREM IIa. *Let X be an n-element set and* $\mathscr{F}$ *a family of subsets of X. Let us suppose that* $\mathscr{F}$ *is a Sperner-family i.e. for* $F, G \in \mathscr{F}$, $F \not\subset G$. *Suppose further that any two sets belonging to* $\mathscr{F}$ *have at least k elements in common. Then*

$$|\mathscr{F}| \leqq \binom{n}{\left[\frac{n+k+1}{2}\right]}.$$

In this paper we prove the following generalization of this theorem:

THEOREM IIb. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two Sperner-families on an n-element set X such that for $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$ we have $|A_1 \cap A_2| \geqq k$ then either*

$$(3) \qquad |\mathcal{A}_1| \leqq \binom{n}{\left[\frac{n+k+1}{2}\right]}$$

or

$$(4) \qquad |\mathcal{A}_2| \leqq \binom{n}{\left[\frac{n+k+1}{2}\right]}.$$

A similar type of generalization of the Erdős—Ko—Rado theorem was proven by D. J. KLEITMAN [3].

2. Proof of Theorem Ib. Let $X=\{1, 2, \dots, n\}$ be the join of the edges of the hypergraphs $\mathcal{A}_1$ and $\mathcal{A}_2$. Then it can be easily verified that

$$|\mathcal{A}_i^{g_i}| \geqq |\mathcal{A}| \frac{\binom{n}{g_i}}{\binom{n}{h_i}} \quad (i = 1, 2).$$

For $0<i<j\leqq n$ we define the following operation (applied first by Erdős, Ko, Rado):

$$K_{i,j}(A_t) = \begin{Bmatrix} (A_t-j)\cup i & \text{if} \quad j\in A_t \not\ni i,\ (A_t-j)\cup i \notin \mathcal{A}_t \\ A_t & \text{otherwise} \end{Bmatrix} \quad (A_t \in \mathcal{A}_t,\ t = 1, 2).$$

It is left to the reader to prove that the hypergraphs $K_{i,j}(\mathcal{A}_t)$ $(t=1, 2)$ satisfy the assumptions of the theorem and moreover

$$K_{i,j}(\mathcal{A}_t^{g_t}) \supseteqq (K_{i,j}(\mathcal{A}_t))^{g_t}.$$

Hence we may apply the K-operation several times until we get two hypergraphs $\mathcal{B}_1$ and $\mathcal{B}_2$ such that for $0<i<j\leqq n$ and $t=1, 2$ $K_{i,j}(\mathcal{B}_t)=\mathcal{B}_t$ i.e. whenever $i\notin B_t$, $j\in B_t\in\mathcal{B}_t$ then $(B_t-j)\cup i\in\mathcal{B}_t$. The assertion of the theorem trivially holds if h_1+ $+h_2-k\geqq n$ as for $g_i<h_i$ the function $\frac{\binom{m}{g_i}}{\binom{m}{h_i}}$ is monotonically decreasing in m, and it can be easily checked if either g_1 or g_2 equals to k. Hence we may suppose $g_1, g_2>k$ and $h_1+h_2-k<n$. We apply induction on n.

Let us define the following four hypergraphs on $\{1, \dots, n-1\}$:

$$\mathcal{C}_t = \{B_t\in\mathcal{B}_t | n\notin B_t\}, \quad \mathcal{D}_t = \{B_t-n | n\in B_t\in\mathcal{B}_t\} \quad (t = 1, 2).$$

The following properties of these hypergraphs can be easily checked:

(i) $\mathscr{C}_t$ is h_t-uniform while $\mathscr{D}_t$ is (h_t-1)-uniform;

(ii) $\mathscr{B}_t^{g_t} \supseteqq \mathscr{C}_t^{g_t} \cup \{n \cup D | D \in \mathscr{D}_t^{g_t-1}\}$;

(iii) if $F_1 \in \mathscr{C}_1 \cup \mathscr{D}_1$ and $F_2 \in \mathscr{C}_2 \cup \mathscr{D}_2$ then $|F_1 \cap F_2| \geqq k$.

(i) and (ii) are trivial while (iii) follows from $h_1+h_2-k<n$. Hence we may apply the induction hypothesis to the pairs

$$(\mathscr{C}_1, \mathscr{C}_2),\ (\mathscr{C}_1, \mathscr{D}_2),\ (\mathscr{D}_1, \mathscr{C}_2) \quad \text{and} \quad (\mathscr{D}_1, \mathscr{D}_2).$$

So we obtain:

$$\text{either} \quad |\mathscr{C}_1^{g_1}| \geqq \frac{\binom{2h_1-k}{g_1}}{\binom{2h_1-k}{h_1}} |\mathscr{C}_1| \quad \text{or} \quad |\mathscr{C}_2^{g_2}| > \frac{\binom{2h_2-k}{g_2}}{\binom{2h_2-k}{h_2}} |\mathscr{C}_2|,$$

$$\text{and either} \quad |\mathscr{C}_t^{g_t}| \geqq \frac{\binom{2h_t-k}{g_t}}{\binom{2h_t-k}{h_t}} |\mathscr{C}_t| \quad \text{or} \quad |\mathscr{D}_{3-t}^{g_{3-t}-1}| \geqq \frac{\binom{2h_{3-t}-2-k}{g_{3-t}-1}}{\binom{2h_{3-t}-2-k}{h_{3-t}-1}} |\mathscr{D}_{3-t}|,$$

$$\text{and either} \quad |\mathscr{D}_1^{g_1-1}| \geqq \frac{\binom{2(h_1-1)-k}{g_1-1}}{\binom{2(h_1-1)-k}{h_1-1}} |\mathscr{D}_1| \quad \text{or} \quad |\mathscr{D}_2^{g_2-1}| \geqq \frac{\binom{2(h_2-1)-k}{g_2-1}}{\binom{2(h_2-1)-k}{h_2-1}} |\mathscr{D}_2|.$$

(In the last two rows we had to allow equality too as it can happen that one of the $\mathscr{D}_t$'s is empty.)

$$\frac{\binom{2(h_t-1)-k}{g_t-1}}{\binom{2(h_t-1)-k}{ht-1}} : \frac{\binom{2h_t-k}{g_{t^*}}}{\binom{2h_t-k}{h_t}} = \frac{g_t(2h_t-k-g_t)}{h_t(h_t-k)} \geqq 1 \quad \text{as} \quad h_t > g_t \geqq h_t-k \quad (t = 1, 2).$$

Hence using the above inequalities and (ii) the statement of the theorem follows.

REMARK. A more careful examination shows that

$$|\mathscr{A}_t^{g_t}| \leqq |\mathscr{A}_t| \frac{\binom{2h_t-k}{g_t}}{\binom{2h_t-k}{h_t}}$$

holds for both $t=1$ and $t=2$ iff $h_1=h_2$;

$$\Big| \bigcup_{A_t \in \mathscr{A}_t} A_t = X \Big| = 2h_t-k; \quad \mathscr{A}_t = \{A \subset X \,|\, |A| = h_t\}.$$

3. Proof of Theorem IIb. Let us define

$$\mathscr{B}_t = \{X - A_t | A_t \in \mathscr{A}_t\} \quad (t = 1,2).$$

Then for $B_t \in \mathscr{B}_t$ we have

(5) $$|B_1 \cup B_2| \leqq n-k.$$

Let us further define

$$M_t = \max_{B_t \in \mathscr{B}_t} |B_t|, \quad \mathscr{C}_t = \{B_t \in \mathscr{B}_t \,|\, |B_t| = M_t\} \quad (t = 1, 2).$$

By (5) for $C_t \in \mathscr{C}_t$ we have

(6) $$|C_1 \cap C_2| \geqq M_1 + M_2 - (n-k).$$

We may suppose that we have a counter-example in which $M_1 + M_2$ is minimal. However if $M_t \leqq \frac{n-k}{2}$ then by Lubell's inequality it follows:

$$1 \geqq \sum_{B_t \in \mathscr{B}_t} \frac{1}{\binom{n}{|B_t|}} \geqq \sum_{B_t \in \mathscr{B}_t} \frac{1}{\binom{n}{M_t}} \geqq \frac{|\mathscr{B}_t|}{\binom{n}{\left[\frac{n-k}{2}\right]}} \quad \text{i.e.} \quad |\mathscr{B}_t| \leqq \binom{n}{\left[\frac{n-k}{2}\right]}.$$

Hence we may assume that for both $t=1$ and $t=2$ $M_t > \frac{n-k}{2}$. By (6) we have: $|C_1 \cap C_2| \geqq 1$ $(C_t \in \mathscr{C}_t)$. It follows from Theorem Ib that for either $t=1$ or $t=2$ we have:

(7) $$|\mathscr{C}_t^{M_t - 1}| \geqq |\mathscr{C}_t|.$$

By symmetry reasons we may suppose that (7) holds for $t=1$. It can be easily checked that the following two systems of sets also satisfy the assumptions of the theorem:

$$\mathscr{A}_2, \quad \mathscr{A}_1' = \{X - B_1 | B_1 \in \mathscr{B}_1 - \mathscr{C}_1\} \cup \{X - C | C \in \mathscr{C}_1^{M_1 - 1}\}.$$

$|\mathscr{A}_1'| \geqq |\mathscr{A}_1|$ whence we have constructed a new counter-example contradicting the minimality of $M_1 + M_2$, q.e.d.

REMARK. Using the assertion of the remark after the proof of the first theorem it can be easily shown that if $|\mathscr{A}_t| \geqq \binom{n}{\left[\frac{n+k+1}{2}\right]}$ for both $t=1$ and $t=2$ and a, $n+k$ are even then

$$\mathscr{A}_1 = \mathscr{A}_2 = \left\{A \subset X \,\middle|\, |A| = \frac{n+k}{2}\right\};$$

if b, $n+k$ is odd then for either $t=1$ or $t=2$

$$\mathscr{A}_t = \left\{A \subset X \,\middle|\, |A| = \frac{n+k+1}{2}\right\}$$

or there exists $Y \subset X$, $|Y| = k$ such that

$$\mathcal{A}_t = \left\{A \subset X \,\middle|\, |A| = \frac{n+k+1}{2},\ A \not\supset Y\right\} \cup \left\{A \subset X \,\middle|\, |A| = \frac{n+k-1}{2},\ A \supset Y\right\}.$$

4. An open question and a weaker result. GY. KATONA (1) proved the following

THEOREM IIIa. *Let X be a finite set of cardinality n and suppose $\mathcal{A}$ is a family of subsets of X such that any two elements of $\mathcal{A}$ have at least $k>0$ elements in common. Then $|\mathcal{A}| \leqq f_k(n)$ where*

$$f_k(n) = \begin{cases} \displaystyle\sum_{0 \leqq i \leqq \frac{n-k}{2}} \binom{n}{i} & \text{if } n-k \text{ is even} \\ \displaystyle\sum_{0 \leqq i \leqq \frac{n-k-1}{2}} \binom{n}{i} + \binom{n-1}{\frac{n-k-1}{2}} & \text{if } n-k \text{ is odd.} \end{cases}$$

It is natural to ask whether the following is true:

CONJECTURE.[1] Let X be a finite set of cardinality n. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be families of subsets of X such that for $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$ we have $|A_1 \cap A_2| \geqq k > 0$. Then either $|\mathcal{A}_1| \leqq f_k(n)$ or $|\mathcal{A}_2| \leqq f_k(n)$.

I can prove the following weakening of this conjecture:

Let X be a finite set, $|X| = n$. Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ be families of subsets of X such that whenever $A \in \mathcal{A}_s$, $B \in \mathcal{A}_t$ $1 \leqq s < t \leqq 3$ then $|A \cap B| \geqq k > 0$ holds. Then either $|\mathcal{A}_1| \leqq f_k(n)$ or $\min(|\mathcal{A}_2|, |\mathcal{A}_3|) < f_k(n)$.

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[1] This conjecture has been proved by R. Ahlswede and Gy. Katona.

INDEX

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[1] G. SZEGŐ, *Orthogonal polynomials*, AMS Coll. Publ. Vol. XXIII (Providence, 1939).
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