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L. LEINDLER, RÓZSA PÉTER, A. RAPCSÁK, L. RÉDEI, B. SZ.-NAGY,
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REDIGIT

G. ALEXITS

SECRETARIUS REDACTIONIS

J. SZABADOS

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A CHAOTIC SPACE

By

E. S. BERNEY (Pocatello)

In this paper, assuming the Continuum Hypothesis, an example of a separable metric space is given which has cardinality c, which is linearly ordered (the linear order inducing the topology), and which has the following property; $p, q \in X$ such that $p \neq q$, implies there exist open sets U_p (containing p) and U_q (containing q) such that no non-empty open subset of U_p is homeomorphic to an open subset of U_q. A non-trivial topological space with this latter property is said to be *chaotic*. This result gives a positive answer to parts *A* and *B* of the following problem posed by E. D. NIX in [2, p. 957]: (A) Do chaotic spaces exist? (B) Do chaotic spaces of cardinality c exist? (C) Do there exist completely normal, connected, and locally connected chaotic spaces?

1. Preliminaries. Let I denote the unit interval. Let $\Delta = \{(x, x) : x \in I\}$. If A is a set, let card A denote the cardinality of A. Let $c = \text{card } I$ as well as the first ordinal of cardinality c. Let N denote the natural numbers. A subset A of a topological space will be called *semi-perfect* if $\bar{A}$ is perfect. Throughout this paper we assume the Continuum Hypothesis.

2. Results. THEOREM. *There exists a separable metric space X such that $X \subseteq I$,* card $X = c$, *and X is chaotic.*

To prove the Theorem, two lemmas are first established from which the Theorem immediately follows.

LEMMA I. *There exists a subset X of $I - \{0, 1\}$ such that*

1. *If A is a perfect subset of I, then* card $(A \cap X) = c$.
2. *There exists no pair of f and D such that:*
   i) $D \subseteq X$,
   ii) card $D = c$,
   iii) $f : D \to X$,
   iv) *and, either f is strictly increasing and $\{(x, f(x)) : x \in D\}$ is bounded away from Δ (i.e., $d[\{(x, f(x)) : x \in D\}, \Delta] > 0$) or f is strictly decreasing.*

PROOF. That such spaces exist with the perfect sets replaced by the set of open sets is demonstrated in [1]. The technique used here is a variation of the one in [1]. Let C, P, and G denote the set of closed subsets of I, non-empty perfect subsets of I, and non-empty open subsets of I, respectively. If $A \in C$, let $F(A)$ denote the set of all monotone functions on A into I whose point inverses are at most countable and which agree with Δ (the identity function) in at most a countable number of places. Note that card $F(A) \leqq c$ and card $C = c$. Therefore, letting $F = \cup \{F(A) : A \in C\}$,

card $(F) \leqq c$ (in fact, it is easily shown that card $F = c$). Let $\{f_\alpha : \alpha < c\}$, $\{P_\alpha : \alpha < c\}$, and $\{G_\alpha : \alpha < c\}$ be most economical well orderings of F, P, and G. Define a set X as follows. Select a point x_1 from G_1 such that $f_1(x_1) \neq x_1$. Let $A_1 = \{x_1\}$. Let $\beta < c$ and suppose A_α has been defined for $\alpha < \beta$ and that card $A_\alpha < c$. Let M_β denote the following set:

$$G_\beta - [\cup \{f_\alpha^{-1}(A_\tau) : \alpha \leqq \beta, \tau < \beta\} \cup \cup \{A_\alpha : \alpha < \beta\} \cup \cup \{f_\alpha(A_\tau) : \alpha \leqq \beta, \tau < \beta\}.$$

Let $A_\beta \subseteqq M_\beta$ such that: card $A_\beta < c$; if $\alpha < \beta$ and $P_\alpha \cap G_\beta \neq \varnothing$, then $P_\alpha \cap A_\beta \neq \varnothing$; and, if $\alpha \leqq \beta$, then $f_\alpha(A_\beta) \cap A_\beta = \varnothing$. Let $X = \cup \{A_\alpha : \alpha < c\}$. That A_β can be chosen to satisfy the preceding is demonstrated in the following.

Fewer than c points have been removed from G_β to get M_β. Note, if $P_\alpha \cap G_\beta \neq \varnothing$, then card $(P_\alpha \cap G_\beta) = c$ and hence, card $(P_\alpha \cap M_\beta) = c$. Let $B_\beta = \{\alpha : P_\alpha \cap G_\beta \neq \varnothing$ and $\alpha < \beta\}$. Note B_β is well ordered. Let α be the first element of B_β. Choose an element x_α in $P_\alpha \cap M_\beta$ such that $\gamma \leqq \beta$ implies $f_\gamma(x_\alpha) \neq x_\alpha$. If no such choice exists, then there exists a γ such that $\gamma \leqq \beta$ and f_γ is the identity function on a subset of $P_\alpha \cap M_\beta$ which has cardinality c. But $f \in F$ implies f agrees with Δ at "at most" a countable number of points, a contradiction. Let $\delta \in B_\beta$ and suppose x_γ has been chosen for each $\gamma < \delta$, $\gamma \in B_\beta$. Let S_β^δ denote the following set:

$[M_\beta \cap P_\delta] - [\cup \{f_\alpha^{-1}(x_\gamma) : \gamma \in B_\beta, \gamma < \delta,$ and $\alpha \leqq \beta\} \cup \{x_\gamma : \gamma \in B_\beta$ and $\gamma < \delta\} \cup \cup \{f_\alpha(x_\gamma) : \gamma \in B_\beta, \gamma < \delta$, and $\alpha \leqq \beta\}]$. Note that card $S_\beta^\delta = c$. Repeating the preceding argument, one can obtain $x_\delta \in S_\beta^\delta$ such that $\alpha \leqq \beta$ implies $f_\alpha(x_\delta) \neq x_\delta$. Let $A_\beta = \{x_\alpha : \alpha \in B_\beta\}$. Observe that card $(A_\beta) =$ card $B_\beta < c$.

It will now be shown that $P_\alpha \in P$ implies card $(P_\alpha \cap X) = c$. Let $P_\alpha \in P$, $G_\gamma \in G$, and $G_\gamma \cap P_\alpha \neq \varnothing$. Let R denote the following set: $\{\beta : G_\beta \in G, G_\beta \subseteqq G_\gamma, G_\beta \cap P_\alpha \neq \varnothing$, and $\beta > \max\{\gamma, \alpha\}\}$. Then card $R = c$. For each $\beta \in R$, $A_\beta \cap P_\alpha \neq \varnothing$. Since $\{A_\beta : \beta \in R\}$ is a disjoint collection of *non-empty* sets, card $(X \cap P_\alpha) = c$.

It will be shown that X has the other properties as asserted in Lemma I. Suppose there exists a pair f and D satisfying 2i—2iv of Lemma I. Since card $D = c$, c of the A_α's have a non-void intersection with D. Choose a point from each such A_α, say x_α, such that the point is in D. Let D' denote the set of points chosen. Then card $D' = c$ and f restricted to D' is strictly monotone. Assume f is strictly decreasing (the argument is similar if f is strictly increasing and bounded away from Δ). Extend f to $\overline{D'}$ as in the proof of the Lemma in [1]. Denote the extension by $\hat{f}$. It is easily checked that $\hat{f} \in F$. Therefore, for some ordinal α, $\hat{f} = f_\alpha$. Now there exists γ, $\beta > \alpha$ such that $x_\gamma \in A_\gamma \cap D'$ and $f_\alpha(x_\gamma) \in A_\beta$. If not, then $f_\alpha(\{x_\gamma \in D' : \gamma > \alpha\}) \subseteqq \cup \{A_\gamma : \gamma \leqq \alpha\}$. But card $\{x_\gamma \in D' : \gamma > \alpha\} >$ card $\cup \{A_\gamma : \gamma \leqq \alpha\}$ and f is a one to one function on D', a contradiction. Therefore, let γ, β be as asserted. If $\gamma > \beta$, then $x_\gamma \in f_\alpha^{-1}(A_\beta)$, contrary to the choice of A_γ. If $\gamma < \beta$, then $f_\alpha(x_\gamma) \in A_\beta$, contrary to the choice of A_β. Therefore, $\gamma = \beta$ and $f_\alpha(x_\beta) \in A_\beta$, a contradiction. Thus, the proof of Lemma 1 is complete.

LEMMA II. *Suppose X is a subspace of $I - \{0, 1\}$ satisfying parts* 1 *and* 2 *of Lemma* I. *Then X is chaotic.*

PROOF. Assume X is not chaotic. Then there exist points a and b in X; open sets U' (containing a), V' (containing b), and U' is bounded away from V'; and a homeomorphism f from an open subset U of U' onto an open subset V of V'. It will be shown that the previous assumption implies the existence of a subset D

of U such that card $D=c$ and f restricted to D is a strictly monotone function whose graph is bounded away from Δ, thus contradicting properties 2i—2iv in Lemma I. (The reason for choosing U' bounded away from V' is that any function g such that $g \subseteq U' \times V'$ is bounded away from Δ since $U' \times V'$ is bounded away from Δ.)

If $p \in I$ and $n \in N$, let $S(n, p) = \left\{x \in U : |x-p| < \frac{1}{n}\right\}$. Define

$X_1^1 = \{p \in U$: There exists $n_p \in N$ such that $t \in S(n_p, p)$ and $t<p$ implies $f(t)>f(p)\}$,

$X_1^2 = \{p \in U$: There exists $n_p \in N$ such that $t \in S(n_p, p)$ and $t>p$ implies $f(t)<f(p)\}$,

$X_2^1 = \{p \in U$: There exists $n_p \in N$ such that $t \in S(n_p, p)$ and $t<p$ implies $f(t)<f(p)\}$,

$X_2^2 = \{p \in U$: There exists $n_p \in N$ such that $t \in S(n_p, p)$ and $t>p$ implies $f(t)>f(p)\}$.

It will now be shown that X_j^i is not dense in any non void open interval of X contained in U. Suppose X_1^1 (similar arguments for X_j^i) is dense in B where B is some non empty open interval of X contained in U. Clearly (by properties guaranteed in part 1 of Lemma 1), every point of B is a limit point from below of X_1^1. Let $R = \{D | D \subseteq X_1^1 \cap B$ and f restricted to D is strictly decreasing$\}$. Partially order R by set inclusion (i.e., L precedes M iff $L \subseteq M$). Let R'' be a maximal chain. Let $R' = \cup R''$. Observe $R' \in R''$. Also $R' \neq \varnothing$ since $R \neq \{\varnothing\}$. We will now show every point of R' is a limit point of R' from below (hence, R' will be semi-perfect). Suppose $s \in R'$ and there exists $\delta > 0$ such that $t \in R'$ and $t<s$ implies $t+\delta \leqq s$. If $\langle t_i \rangle$ is in R', $t_i < s$, and $f(t_i) \to f(s)$, then, since f is a homeomorphism on U, $t_i \to s$, a contradiction. Therefore, there exists $\delta' > 0$ such that $t<s$ and $t \in R'$ implies $f(t) \geqq f(s)+\delta'$. By an earlier observation there exists $\langle q_i \rangle$ in X_1^1 such that $q_i < s$ and $q_i \to s$. Therefore, there exists i such that $|q_i - s| < \delta$, $q_i \in S(n_s, s)$, and $q_i \in B$. Now $t \in R'$ such that $t<s$ implies $f(t)>f(q_i)>f(s)$. Hence f is strictly decreasing on $R' \cup \{q_i\}$ contradicting the maximality of R'. Therefore, every point of R' is a limit point of R' from below and R' is semi-perfect. Thus, card $(\bar{R}' \cap B) = c$ since $\bar{R}'$ is perfect in I (part 1 of Lemma 1). Since f is a homeomorphism on $\bar{R}' \cap B$ and f is strictly decreasing on a dense subset of $\bar{R}' \cap B$ (eg., R'), we have f is strictly decreasing on $\bar{R}' \cap B$, a contradiction. Thus X_1^1 (X_j^i) is not dense in any non-void open interval of U. Thus, $X^1 = X_1^1 \cup X_2^1 \cup X_1^2 \cup X_2^2$ is nowhere dense in U. Choose B a non void open interval of X such that $B \cap X^1 = \varnothing$ and $B \subseteq U$. Let $R = \{D : D \subseteq B$ and f restricted to D is strictly decreasing$\}$. Partially order R by set inclusion and let R'' be a maximal chain. Let $R' = \cup R''$. Note that $R' \in R$ and R' is the maximal element of R''. It will be shown that R' is semi-perfect.

Let $q \in R'$ and suppose there exists $\delta > 0$ such that $t \in R'$ and $t<q$ implies $t+\delta \leqq q$. Since f is a homeomorphism, there exists $\delta' > 0$ such that $t \in R'$ and $t<q$ implies $f(t) \geqq f(q)+\delta'$. Also, since B is open, $q \in B$, and $B \cap X_2^1 = \varnothing$ one can construct a sequence $\langle q_i \rangle$ in B such that $q_i < q$, $f(q_i) > f(q)$ and $q_i \to q$. Thus, there exists q_i such that $|q_i - q| < \delta$ and $|f(q_i) - f(q)| < \delta'$. Now f restricted to $R' \cup \{q_i\}$ is strictly decreasing and $R' \cup \{q_i\} \subseteq B$, a contradiction to the maximality of R'. Thus R' is semi-perfect. Therefore, card $(\bar{R}' \cap U) = c$ and f restricted to $\bar{R}' \cap U$ is strictly decreasing, a contradiction to the properties 2i—2iv possessed by X. Thus, there

do not exist points a and b, open sets U', V', U, and V; and homeomorphism f as asserted. Q.E.D.

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References

[1] E. S. BERNEY, A regular Lindelöf semi-metric space which has no countable network, *Proc. of Amer. Math. Soc.*, **26**, **(2)** (1970).
[2] E. D. NIX, Chaotic Spaces, *Bull. Amer. Math. Soc.*, **76** **(5)** (1970), question 14, p. 975.

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A NOTE ON STRONG RADICALS[1]

By

T. ANDERSON (Vancouver)

1. In the general theory of Kurosh—Amitsur radicals the relation between the radical of a ring and the radical of its ideals has been satisfactorily described, even in the case of certain non-associative rings (see [1], [2], or [3]). On the other hand, very little is known about the connection between the radical of a ring and the radical of its subrings. This is not surprising since the notion of Kurosh—Amitsur radical is formulated entirely in the language of ideals and homomorphisms.

Evidently, to relate the radical of a ring A to the radical of one of its subrings S, the ideals of S must be determined to some extent by the ideals of A. In this context it is natural to consider the subring $S=eAe$, where e is an idempotent of A, since each ideal of eAe is of the form ePe for some ideal P of A (see [4, p. 379]). What we prove here is this: If β is a strong radical in the sense of DIVINSKY, KREMPA, and SULINSKI [5] and if e is an idempotent of a β-semi-simple ring A then eAe is β-semi-simple.

2. All rings are assumed to be associative and for a ring A, $\beta(A)$ will denote the β-radical of A. The radical β is called strong if $\beta(A)$ contains all the one-sided β-radical ideals of A.

For an idempotent e of A we have the Peirce decomposition $A = A_{11}+A_{10}+A_{01}+A_{00}$, where the sum is direct and $A_{ij}=\{x\in A \mid xe=jx \text{ and } ex=ix\}$ for $i, j=0, 1$. Moreover, we have the following multiplicative relations for the Peirce components.

$$A_{ij}A_{kl} = 0 \quad \text{for} \quad j \neq k \quad \text{and} \quad A_{ij}A_{jk} \subseteq A_{ik}. \tag{1}$$

In particular, $A_{11}=eAe$ is a subring of A.

THEOREM. *If A is semi-simple with respect to a strong radical β then eAe is also β-semi-simple for an idempotent e.*

PROOF. Let $M=\beta(eAe)$. Since M is an ideal of A_{11} it follows from (1) that $M+MA_{10}$ is a right ideal of A. To prove the theorem it will be sufficient to show that $M+MA_{10}$ is a β-ring.

We show $M+MA_{10}$ is a β-ring by showing each non-zero homomorphic image $M+MA_{10}/W$ contains a non-zero left β-ideal.

If $W \not\supseteq M$ then $M+MA_{10}/W$ contains the non-zero left ideal $W+M/W$. Since $W+M/W \cong M/W\cap M$, this left ideal is a homomorphic image of M, hence is a left β-ideal. Therefore we assume $W\supseteq M$. Then $Ma_{10}\not\subseteq W$ for some $a_{10}\in A_{10}$ and

[1] This research was partially supported by the National Research Council of Canada.

from (1) we have that $W+Ma_{10}/W$ is a non-zero left ideal of $M+MA_{10}/W$. We show $W+MA_{10}/W$ to be β-radical by establishing a homomorphism h from M onto $W+Ma_{10}/W$. We define h by mapping $m \in M$ onto the coset $W+ma_{10}$. Certainly h is linear and maps M onto $W+Ma_{10}/W$. For $m, m' \in M$, $h(mm') = W+$ $+(mm')a_{10} = W+m(m'a_{10})$. But $m \in W$, $m'a_{10} \in M+MA_{10}$, and W is an ideal of $M+MA_{10}$; hence $W+m(m'a_{10}) = W$. On the other hand, $h(m)h(m') = W+0$ because of (1). Thus $h(mm')=h(m)h(m')$, and h is a homomorphism as required.

In each case we have exhibited a non-zero left β-ideal of $M+MA_{10}/W$. Thus the theorem is proved.

We leave as an exercise the proof of the

COROLLARY. *If β is a strong radical and e is an idempotent of A then $\beta(eAe) \subseteq$ $\subseteq \beta(A) \cap eAe = e\beta(A)e$.*

REMARKS. It is an interesting question whether our theorem is true for arbitrary radicals; indeed, nothing seems to be known about the nil radical in this regard.

We also wonder whether the above theorem is valid for alternative rings. For such rings also have a Peirce decomposition, but not all the relations (1) are valid (see [6, Lemma 13]).

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References

[1] A. AMITSUR, A general theory of radicals II, *Amer. J. Math.*, **76** (1954), pp. 100—125.
[2] T. ANDERSON, N. DIVINSKY and A. SULINSKI, Hereditary radicals in associative and alternative rings, *Can. J. Math.*, **17** (1965), pp. 594—603.
[3] T. ANDERSON, Hereditary radicals and derivations of algebras, *Can. J. Math.*, **21** (1969), pp. 372—377.
[4] CURTIS and REINER, Representation theory of finite groups and associative algebras, *Interscience* (1962), New York.
[5] N. DIVINSKY, J. KREMPA and A. SULINSKI, Strong radical properties of alternative and associative rings, *J. Alg.*, **17** (1971), pp. 369—388.
[6] M. HUMM, On a class of right alternative rings without nilpotent ideals, *J. Alg.*, **5** (1967), pp. 164—174.

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EINE UNGLEICHUNG ZWISCHEN VOLUMEN, OBERFLÄCHE UND GITTERPUNKTANZAHL KONVEXER MENGEN IM R^3

Von

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$\mathfrak{K}^n$ sei die Menge der beschränkten konvexen Mengen mit inneren Punkten des n-dimensionalen euklidischen Raumes R^n.

Zu einem $K \in \mathfrak{K}^n$ sei $\overline{K}$ der zugehörige konvexe Körper, $V = V(K) = V(\overline{K})$ das Volumen, $F = F(K) = F(\overline{K})$ die Oberfläche und $G = G(K)$ die Anzahl der Gitterpunkte g (mit ganzzahligen Koordinaten) mit $g \in K$. Für den R^2 bewies M. NOSARZEWSKA [2]: $V - G < \frac{1}{2}F$, und $\frac{1}{2}$ ist optimal (d.h. $V - G < \alpha F$ gilt nur für $\alpha \geqq \frac{1}{2}$). Gelten Analoga für $n \geqq 3$, speziell für R^3? In dieser Arbeit wird gezeigt:

SATZ. *Im R^3 gilt $V - G < \frac{1}{2}F$, und der Faktor $\frac{1}{2}$ ist optimal.*[1]

Neben der Tatsache, daß überhaupt eine lineare Ungleichung zwischen V, F und G besteht und daß die Ungleichung keinerlei Homogenität bezüglich der Dimension aufweist, ist bemerkenswert, daß die Ungleichung im R^2 und im R^3 dieselbe ist.

Dies und einige weitere Indizien, auf die wir nicht eingehen, sprechen dafür, daß die Ungleichung für beliebige n gilt. In diesem Fall würde sie den folgenden Satz von HADWIGER [1] umfassen: $G = 0 \Rightarrow V < \frac{1}{2}F$ und $\frac{1}{2}$ ist optimal.

Für die Theorie der Überdeckungen ergibt sich aus dem obigen Satz die interessante

FOLGERUNG. *Ist m eine natürliche Zahl und $K \in \mathfrak{K}^3$ mit $V - \dfrac{F}{2} \geqq m - 1$, so ist $\{K + g \mid g \in \mathbf{Z}^3\}$ eine (mindestens) m-fache Überdeckung des R^3.*

Der Vollständigkeit halber seien noch zwei Arbeiten von REEVE [3], [4] erwähnt, in denen für Polytope des R^3, deren Ecken Gitterpunkte sind, eine lineare Gleichung zwischen Volumen, Anzahl der Gitterpunkte im Innern und Anzahl der Gitterpunkte auf dem Rand aufgestellt wird.

BEWEIS DES SATZES. (Die Vorüberlegungen gelten für beliebige $n \geqq 2$.) Sei $K \in \mathfrak{K}^n$. Der R^n werde in achsenparallele abgeschlossene Würfel W_i mit der Kantenlänge 1 und mit einem Gitterpunkt g_i als Mittelpunkt zerlegt. (Die auftretende Überschneidung benachbarter Würfel ist für das Folgende bedeutungslos.)

Sei $K_i = W_i \cap K$. Die konvexen Mengen K_i können in drei Klassen eingeteilt

[1] Wie wir inzwischen erfahren haben, hat W. M. Schmidt dasselbe Ergebnis erhalten und darüber auf der Tagung über Diskrete Geometrie in Oberwolfach im Juni 1971 vorgetragen.

werden, so daß sich nach passender Numerierung ergibt:

1. $V(K_i)>0$ und $g_i \in K_i$ $\quad (i = 1, \ldots, r)$
2. $V(K_i)>0$ und $g_i \notin K_i$ $\quad (i = r+1, \ldots, s)$
3. $V(K_i)=0$ $\quad (\Rightarrow g_i \notin K_i)$.

Damit ist $G(K)=r$ und

$$V(K) = \sum_{i=1}^{s} V(K_i) \leqq r + \sum_{i=r+1}^{s} V(K_i)$$

oder

$$V-G \leqq \sum_{i=r+1}^{s} V(K_i).$$

Sei $\partial K = \overline{K}-K^0$ der Rand von K und $\partial' K_i = \partial K \cap W_i^0$ der nicht auf Würfelseiten liegende Teil von ∂K_i. Weiter sei $F'(K_i)$ das $(n-1)$-dimensionale Volumen von $\partial' K_i$. Dann ist

$$F(K) \geqq \sum_{i=r+1}^{s} F'(K_i).$$

Mit dem folgenden Hilfssatz wird für $n=3$ gezeigt:

$$(1) \qquad \begin{cases} V(K_i) \geqq \frac{1}{2} F'(K_i) & (i = r+1, \ldots, s) \\ \text{und } < \text{ gilt für mindestens ein } i \in \{r+1, \ldots, s\}. \end{cases}$$

Damit ist $V(K)-G(K) < \frac{1}{2} F(K)$, falls $n=3$.

Daß der Faktor $\frac{1}{2}$ optimal ist, folgt aus Hadwigers Satz oder aus folgendem Gegenbeispiel:

Sei

$$K(q) = \{(x_1, \ldots, x_n) | 0 < x_1 < 1,\ 0 < x_i < q,\ i = 2, \ldots, n\}$$

also

$$V(K) = q^{n-1}, \quad F(K) = 2q^{n-1} + 2(n-1)q^{n-2}, \quad G(K) = 0.$$

Mit $q \to \infty$ folgt die Behauptung

Die mit dem Hilfssatz zu zeigende Aussage (1) ist bewegungsinvariant. Man kann daher statt W_i den Einheitswürfel wählen, für g_i den Punkt $m=(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ und dem Hilfssatz, der die eigentlichen Schwierigkeiten des Satzes birgt, die folgende Fassung geben:

HILFSSATZ. *Seien* $a_i \geqq 0$, $i=1, 2, 3$ *mit* $\sum_{i=1}^{3} a_i = 1$,

$$H = H(a_1, a_2, a_3) = \left\{ (x_1, x_2, x_3) \,\middle|\, \sum_{i=1}^{3} a_i x_i \leqq \tfrac{1}{2} \right\}$$

und $K' \in \mathfrak{K}^3$ *mit* $K' \subset H$. *Weiter sei* $W = \{(x_1, x_2, x_3) | 0 \leqq x_i \leqq 1,\ i=1, 2, 3\}$, $K = K' \cap W$, $\partial' K = \partial K' \cap W^0$ *und* $F'(K)$ *das 2-dimensionale Volumen von* $\partial' K$. *Dann gilt* $V(K) \leqq \frac{1}{2} F'(K)$ *und gleich genau dann, wenn* $\partial' K$ *in einer der drei Ebenen* $x_i = \frac{1}{2}$, $i=1, 2, 3$ *liegt.*

BEMERKUNG. K spielt hier die Rolle eines der K_i aus Klasse 2) des Satzes. Man erkennt, daß nicht für alle K_i Gleichheit gelten kann.

BEWEIS DES HILFSSATZES. Es genügt, sich auf kompakte K' bzw. K zu beschränken, so daß m als Randpunkt auftreten kann. Wegen $K \subset W \cap H$ ist $V(K) \leqq \frac{1}{2}$. Der Beweis des Hilfssatzes zerfällt in 10 Lemmata.

LEMMA 1. *Im Hilfssatz darf ohne Einschränkung K' symmetrisch zu den Ebenen $x_1=0$, $x_2=0$, $x_3=0$ angenommen werden.*

BEWEIS. Sei $\tilde{K}_i$ der zu K bzgl. der Ebene $x_i=0$ $(i=1, 2, 3)$ symmetrisch liegende Körper. Man symmetrisiere $K \cup \tilde{K}_i$ steinersch an der Ebene $x_i=0$ und nenne den in W liegenden Teil des entstandenen Körpers $\tilde{\tilde{K}}$ wiederum K. Dabei vergrößert sich $F'(K)$ nicht, während $V(K)$ konstant bleibt.

$\tilde{\tilde{K}}$ ist wieder konvex, da $\tilde{\tilde{K}}$ auch durch Symmetrisierung von K nebst anschließender Streckung um den Faktor 2 entsteht. Den aus K (nach der dreimaligen Symmetrisierung) durch Spiegelung an den Ebenen $x_1=0$, $x_2=0$, $x_3=0$ entstehenden konvexen Körper mit achtfachem Volumen von K denke man sich von vornherein als K' gegeben.

Es ist nun sinnvoll, die Typen von K danach zu klassifizieren, welche der acht Eckpunkte von W eine Umgebung U besitzen mit $U \cap W \subset K$. Die Eckpunkte von W seien wie folgt bezeichnet, so daß sich die Punkte 0 und d sowie e_i und d_i $(i=1, 2, 3)$ jeweils paarweise gegenüberliegen: $0=(0, 0, 0)$, $d=(1, 1, 1)$, $e_1=(1, 0, 0)$, $d_1=(0, 1, 1)$, $e_2=(0, 1, 0)$, $d_2=(1, 0, 1)$, $e_3=(0, 0, 1)$, $d_3=(1, 1, 0)$.

Da wegen $K' \subset H$ nicht zwei gegenüberliegende Ecken von W zugleich eine Umgebung besitzen können, deren Schnitt mit W in K enthalten ist, trifft dies für höchstens 4 Ecken zu, dabei stets für 0. Bis auf Koordinatenpermutation bleiben folgende Fälle, in denen eine solche Umgebung existiert:

I) 0
II) $0, e_1$
III) $0, e_1, e_2$
IV) $0, e_1, e_2, e_3$
V) $0, e_1, e_2, d_3$.

Die Beweise für diese 5 Fälle werden einzeln geführt und sind von unterschiedlicher Schwierigkeit. Aus den Lemmata 2, 3, 5, 8 und 10 folgt dann die Behauptung des Hilfssatzes.

LEMMA 2. *Im Fall* V) *gilt $V(K) \leqq \frac{1}{2} F'(K)$ und gleich genau dann, wenn $\partial' K$ in der Ebene $x_3=\frac{1}{2}$ liegt.*

BEWEIS. Es ist $V(K) \leqq \frac{1}{2}$ und gleich genau dann, wenn $\partial' K \subset \partial H$. Andererseits ist $F'(K) \geqq 1$ und gleich genau dann, wenn $\partial' K = \{(x_1, x_2, x_3) | 0 < x_i < 1,\ i=1, 2$ und $x_3=c\}$ mit $c \in (0, \frac{1}{2}]$. Nur für $c=\frac{1}{2}$ gilt in beiden Fällen das Gleichheitszeichen.

LEMMA 3. *Im Fall* I) *gilt $V(K) < \frac{1}{3} F'(K)$.*

BEWEIS. Hier ist $F(K')=8F'(K)$ und $V(K')=8V(K) \leqq 4$. Sei S eine Kugel mit Radius r und $V(S)=V(K')$. Dann ist $\frac{4}{3}\pi r^3 \leqq 4$ oder $r \leqq \sqrt[3]{\frac{3}{\pi}} < 1$, und mit der

Isoperimetrieeigenschaft folgt

$$\frac{V(K)}{F'(K)} = \frac{V(K')}{F(K')} \leqq \frac{V(S)}{F(S)} = \frac{r}{3} < \frac{1}{3}.$$

Für die nächsten Lemmata benötigen wir folgende

DEFINITION. Sei $f_i' = K \cap \{x_i = 1\}$, $l_i' = \overline{\partial' K} \cap \{x_i = 1\}$ $(i=1, 2, 3)$. Weiter sei f_i das 2-dimensionale Volumen von f_i', l_i die Länge von l_i'.

LEMMA 4. *Im Fall* II) *ist* $f_1 < \frac{1}{2} l_1$.

BEWEIS. Wegen $K \subset H$ und Lemma 1 ist $f_1 \leqq \frac{1}{2}$. $K' \cap \{x_1 = 1\}$ ist im Fall II) eine 2-dimensionale konvexe Menge mit dem Umfang $4l_1$ und dem Flächeninhalt $4f_1 \leqq 2$.

Sei S ein Kreis mit Radius r und Flächeninhalt $V(S) = 4f_1 \leqq 2$. Dann ist $\pi r^2 \leqq 2$ oder $r \leqq \sqrt{\frac{2}{\pi}} < 1$. Damit folgt mit der Isoperimetrieeigenschaft $\frac{f_1}{l_1} \leqq \frac{V(S)}{F(S)} =$

$$= \frac{r}{2} < \frac{1}{2}.$$

LEMMA 5. *Im Fall* II) *gilt* $V(K) < \frac{1}{2} F'(K)$.

BEWEIS. Es gilt $F'(K) \geqq \int_0^1 l(t)\,dt$ mit $l(t) =$ Länge von $\overline{\partial' K} \cap \{x_1 = t\}$. Wegen Lemma 1 und der Monotonie des Umfangs für konvexe Körper gilt $l(t) \geqq l(1) = l_1$, also folgt mit Lemma 4 $F'(K) \geqq l_1 > 2f_1$. Wie in Lemma 3 folgt $V(K') < \frac{1}{3} F(K')$ und wegen

$$F(K') = 8F'(K) + 8f_1 < 12F'(K)$$

ist $V(K) < \frac{1}{2} F'(K)$.

LEMMA 6. *In den Fällen* III) *und* IV) *kann ohne Einschränkung* $m = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \partial' K$ *angenommen werden.*

BEWEIS. a) *Fall* IV). Man ersetze K' durch den Körper $\tilde{K} = \lambda K' \cap \tilde{W}$, wobei $\tilde{W} = \{(x_1, x_2, x_3) \mid |x_i| \leqq 1, i = 1, 2, 3\}$ ist und $\lambda \geqq 1$ so bestimmt wird, daß $m \in \partial(\lambda K')$ gilt.

Im allgemeinen ändert sich H, aber K ist wieder von Typ IV). Wegen der Dilatation mit $\lambda \geqq 1$ hat man

$$\frac{V(\lambda K')}{F(\lambda K')} = \lambda \frac{V(K')}{F(K')} > \frac{V(K)}{F(K)}.$$

Für die 6 nicht zu $\tilde{W}$ gehörigen Teile von $\lambda K'$ schließt man wie in Lemma 5, so daß verbleibt, $V(\tilde{K} \cap W) < \frac{1}{2} F(\tilde{K} \cap W)$ zu zeigen. Dies ist nach Ersetzung von K' durch $\tilde{K}$ die Behauptung des Hilfssatzes im Fall IV) mit $m \in \partial' K$.

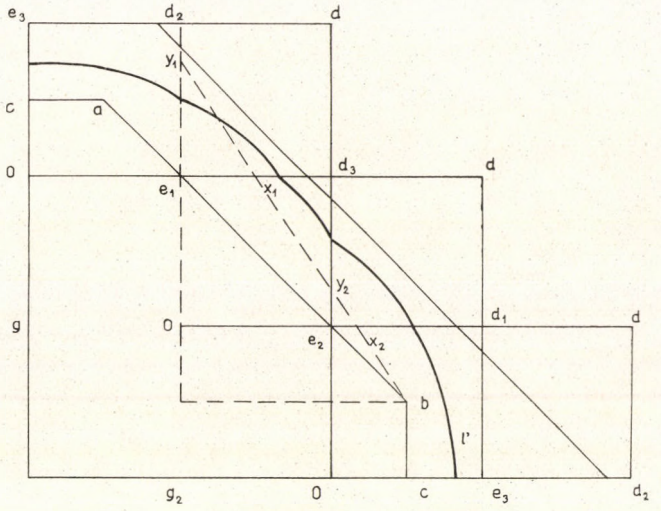
b) *Fall* III). Hier strecke man K in x_3-Richtung mit einem Faktor $\lambda \geqq 1$, so daß entweder Fall IV) entsteht oder m Randpunkt wird. Das Verhältnis $\frac{V(K)}{F(K)}$ wird bei dieser Streckung nicht kleiner.

LEMMA 7. *Sei* $l' = \overline{\partial' K} \cap \partial W$ *und* l *die Länge von* l'. *Dann gilt im Fall* IV): $l \geqq \sqrt{18} > 4$.

BEWEIS. Wie man leicht sieht, läßt sich die Oberfläche von W wie in der Abbildung abwickeln. Im Fall IV) verläuft l' über alle 6 Seitenflächen von W und schneidet z. B. die Strecke $\overline{e_3 d_2}$. Da l' eine geschlossene Kurve ist, folgt mit Pythagoras $l \geqq \sqrt{18} > 4$.

LEMMA 8. *Im Fall* IV) *gilt* $V(K) < \frac{1}{2} F'(K)$.

BEWEIS. Man betrachte alle aus W durch Dilatation (mit m als Ursprung) entstehenden Würfel $W(x)$ der Kantenlänge $2x$ $(0 < x \leqq \frac{1}{2})$. Für jedes $W(x)$ liegt



Fall IV) vor, da mit jeder Ecke von W auch die offene Verbindungsstrecke vom Mittelpunkt m zur Ecke entweder ganz zu K^0 oder ganz nicht zu K^0 gehört.

Die Überlegung von Lemma 7 auf den Würfel $W(x)$ übertragen liefert $l(x) \geqq \geqq \sqrt{18} \cdot 2x$, und für $F'(K)$ gilt die Abschätzung

$$F'(K) \geqq \int_0^{1/2} l(x)\, dx \geqq \frac{\sqrt{18}}{4} > 1.$$

LEMMA 9. *Sei* $l(x) = \overline{\partial' K} \cap W(x)$ *(vgl. Lemma* 7+8*) und* l_0 *eine Zahl mit* $l_0 < \dfrac{l(x)}{2x}$ *für alle* $x \in (0, \frac{1}{2}]$. *Dann gilt im Fall* III):

$$F'(K) > \frac{l_0}{4}.$$

BEWEIS. Wie in Lemma 8 betrachte man die Würfel $W(x)$. Es gilt wie dort die Abschätzung $F'(K) \geqq \int_0^{1/2} l(x)dx > \frac{l_0}{4}$.

LEMMA 10. *Im Fall* III) *gilt* $V(K) < \frac{1}{2}F'(K)$.

BEWEIS. Die Oberfläche von W werde wie in Lemma 7 abgewickelt (s. Abb.). Im Fall III) verläuft l' über 5 der 6 Seitenflächen von W und zwar nicht über die durch e_3, d_2, d, d_1 gehende Seitenfläche. l' schneidet die Strecke $\overline{0e_3}$.

Nach Lemma 6 gilt $m \in \partial' K$ und wegen der Symmetrie und Konvexität von K' verläuft l' nicht unterhalb der Punkte $a=(\frac{1}{2}, 0, \frac{1}{2})$, $b=(0, \frac{1}{2}, \frac{1}{2})$ und $c=(0, 0, \frac{1}{2})$.

Die kürzeste Verbindung ist jetzt $1+l(\overline{ab}) = 1+2\sqrt{2} > 3{,}8$. ($l(\overline{ab})$ bezeichne die Länge der Verbindungsstrecke von a nach b in der Abwicklung.) Für die Würfel $W(x)$ gilt die gleiche Überlegung: $l(x) > 3{,}8 \cdot 2x$. (Der Faktor 3,8 kann sogar durch 4 ersetzt werden, wenn für $W(x)$ der Fall IV) vorliegt: andere Fälle können nicht auftreten, vgl. Lemma 8.)

Mit Lemma 9 hat man $F'(K) > 0{,}95$.

Wir können nun annehmen, daß $f_1+f_2 > \frac{F'(K)}{2} > 0{,}475$ gilt. Andernfalls gilt analog zu Lemma 3: $V(K') < \frac{1}{3}F(K')$ und mit $3 \cdot 8V(K) = 3V(K') < F(K') = 8 \cdot (F'(K)+f_1+f_2) \leqq \frac{8 \cdot 3}{2} F'(K)$ ist alles bewiesen.

O. E. sei $f_1 \geqq f_2$, also $f_1 > \frac{1}{2} \cdot 0{,}475 = 0{,}2375$. Die Figur (s. Abb.) werde an den Geraden g_1 und g_2 gespiegelt. Man betrachte die konvexe Hülle (Fläche $4f''$, Umfang $4l''$) der durch $4l'$ umschlossenen Fläche. Dann gilt aufgrund der Isoperimetrieeigenschaft des Kreises $l \geqq l'' \geqq \frac{1}{4} \cdot 2\pi \cdot \sqrt{\frac{4f''}{\pi}} > 4$, sofern $f'' > \frac{16}{\pi}$ ist. Es ist (s. Abb.) $f'' \geqq 4{,}25+f_1+f_2+f > 4{,}725+f$, wobei f die Fläche des Dreiecks $e_1 b x_1$ minus der Fläche des Dreiecks $e_2 x_2 y_2$ ist.

f ist am kleinsten, wenn die Fläche des Dreiecks $e_1 x_1 y_1$ gleich 0,2375 ist. Nun ist die Fläche des Dreiecks $e_2 x_2 y_2$ gleich $\frac{1}{9}$ der Fläche des Dreiecks $e_1 x_1 y_1$, also $\frac{1}{9} \cdot 0{,}2375$. Weiter ist $\frac{1}{2} \cdot l(\overline{e_1 x_1}) \cdot l(\overline{e_1 y_1}) = 0{,}2375$ und

$$\frac{l(\overline{e_1 x_1})}{1{,}5} = \frac{l(\overline{e_1 y_1})}{l(\overline{e_1 y_1})+1{,}5}$$

also $l(\overline{e_1 x_1}) > 0{,}54$. Daraus folgt $f = \frac{1}{2}\left(\frac{3}{2}\sqrt{2}\right)\frac{l(\overline{e_1 x_1})}{\sqrt{2}} - \frac{1}{9} \cdot 0{,}2375$, $f > 0{,}375$, also $f'' > \frac{16}{\pi}$ und damit $l > 4$.

Die entsprechende Abschätzung $\frac{1}{2x} l(x) > 4$ ist ebenfalls richtig: Im möglicherweise auftretenden Fall IV) gilt diese Ungleichung entsprechend der Überlegung von Lemma 8. Im Fall III) lassen sich die letzten Überlegungen übertragen, denn

die Flächen auf $W(x)$, die f_1 und f_2 entsprechen, können sich wegen der Konvexität von K relativ zu den Seitenflächen von $W(x)$ nicht verkleinern.

Mit Lemma 9 folgt $F'(K)>1$ und wegen $V(K)\leqq\frac{1}{2}$ gilt auch im Fall III) $V(K)<\frac{1}{2}F'(K)$.

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Literaturverzeichnis

[1] H. HADWIGER, Volumen und Oberfläche eines Eikörpers, der keine Gitterpunkte überdeckt, *Math. Z.*, **116** (1970), S. 191—196.
[2] M. NOSARZEWSKA, Évaluation de la différence entre l'aire d'une région plane convexe et le nombre des points aux coordonnées entières couverts par elle, *Coll. Math.*, **1** (1948), S. 305—311.
[3] J. E. REEVE, On the volume of lattice polyhedra, *Proc. Lond. Math. Soc.*, (3), **7** (1957), S. 378—395.
[4] J. E. REEVE, Further note on the volume of lattice polyhedra, *Journ. Lond. Math. Soc.*, **34** (1959), S. 57—62.

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ON THE BOUNDARY OF ALGEBRAIC RADICALS IN TOPOLOGICAL SEMIGROUPS

By

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A topological semigroup is a semigroup with a Hausdorff topology in which multiplication is continuous in both variables. In what follows S will denote a topological semigroup. We let A be any subset of S. The algebraic radical of A is defined to be the set $\{x \in S | x^k \in A \text{ for some integer } k \geqq 1\}$ and is denoted by $R(A)$. In [3], K. P. SHUM and C. S. HOO studied some properties of the algebraic radicals of ideals in compact abelian semigroups. In this paper, we are going to give further investigations on the algebraic radical of an ideal A of S. We find that the boundary of the algebraic radical of an open ideal A of S is closely related with a compact group in S. But in general, the structure of such a compact group is still not known to the author. Here only the existence of a compact group on the boundary of $R(A)$ will be discussed.

We list here several definitions which will be frequently used in this paper.

Let A be any subset of S, then $J(A) = A \cup SA \cup AS \cup SAS$ is called the principal ideal generated by A. If S is abelian, then an ideal P of S is said to be prime if $ab \in P$ implies that $a \in P$ or $b \in P$. An ideal Q of S is said to be primary if $ab \in Q$ implies that $a \in Q$ or there exists an integer $k \geqq 1$ such that $b^k \in Q$. If S is a semigroup without zero, then S is simple if S does not properly contain an ideal. In an abelian semigroup S, an ideal Q of S is primary if and only if $R(Q)$ is a prime ideal of S, cf. [3].

Throughout, we use $\mathrm{cl}(C)$ to denote the closure of the set C, C' for the complement of C and $B(C)$ for the boundary of C. The reader is referred to [4] for other unstated definitions and notations.

THEOREM 1. *Let S be a connected abelian semigroup with unity and let A be an open ideal of S. Then A is a primary ideal if and only if the boundary $B(R(A))$ of $R(A)$ is a subsemigroup of S.*

In order to prove this theorem, the following lemma is needed.

LEMMA A. *If A is an open subset of a topological semigroup S, then $R(A)$ is also an open subset of S.*

PROOF. Take any $x \in R(A)$, then there is an integer $k \geqq 1$ such that $x^k \in A$. Let $\mu: \prod^k S \to S$ defined by $\mu(x_1, x_2, \ldots, x_k) = x_1 x_2 \ldots x_k$. Then μ is a continuous map and $\mu^{-1}(A)$ is an open neighbourhood of $x \ldots x$ (k terms). Let V be an open neighbourhood of x such that $\mu(V \times V \times \ldots \times V)$ (k terms) $\subseteq \mu\mu^{-1}(A)$. That is, $V^k \subset A$. This implies $V \subset R(A)$, so $R(A)$ is open.

We now prove Theorem 1.

Suppose that A is an open primary ideal of S, then by Lemma A, $R(A)$ is an open prime ideal of S. Clearly $R(A)^2 \subseteq R(A)$ and $(S-R(A))^2 \subseteq S-R(A)$ since $R(A)$ is a prime ideal of S. Thus $(B(R(A)))^2 \subseteq B(R(A))$. $B(R(A))$ is therefore a subsemigroup of S.

Conversely, suppose that $B(R(A))$ is a subsemigroup of S and suppose that $R(A)$ is not prime. Then we can take $a, b \in S-R(A)$ with $ab=ba \in R(A)$. As S has a unity, we have $Sa \cap R(A) \neq \emptyset$ and $Sa \cap (S-R(A)) \neq \emptyset$. So, it follows that $Sa \cap B(R(A)) \neq \emptyset$. Similarly, $bS \cap B(R(A)) \neq \emptyset$. Since $R(A)$ is open and S is connected, we have $\emptyset \neq [Sa \cap B(R(A))]. [bS \cap B(R(A))] \subseteq (SabS) \cap (B(R(A)))^2 \subseteq$ $\subseteq R(A) \cap B(R(A)) \neq \emptyset$, for $B(R(A))$ is a subsemigroup of S. This is a contradiction. Thus we prove that $R(A)$ must be prime, which implies that A is a primary ideal of S.

COROLLARY. *Let S be a connected abelian semigroup and A be an open ideal of S. Then $S-R(A)$ is a subsemigroup of S if and only if the boundary of $R(A)$ is a subsemigroup of S.*

The following theorem tells us that there always exists a compact group lying on the boundary of $R(A)$.

THEOREM 2. *Let S be a compact abelian semigroup and A be an open ideal of S. Then either $B(R(A)) \neq \emptyset$ or there exists a non-zero compact group which lies on the boundary of $R(A)$.*

In proving this theorem, the following lemma is crucial.

LEMMA B. *Let A be any open ideal of a compact abelian semigroup S. If B is an ideal which is not contained in $R(A)$, then B has an idempotent which is not in A.*

(This lemma was proved by K. P. SHUM in [3]. But for the sake of completeness and for the convenience of the reader, we provide a proof here.)

PROOF. Since A is an ideal of S, so is $R(A)$. As $B \nsubseteq R(A)$, there exists an element $b \in B$ such that $b \notin R(A)$. Now $J(b)=\{b\} \cup bS \subseteq B$, and $J(b)$ is compact for S is compact. So there exists an idempotent $e^2=e \in \Gamma(b)=\bigcup_{n=1}^{\infty} \text{cl}\,\{b^n\} \subseteq J(b) \subseteq B$ ([4], page 22). Suppose on the contrary that $e \in A$. Then $K(b)=e\Gamma(b) \subseteq A$ where $K(b)=$ $=\bigcap_{n=1} \text{cl}\,\{b^i | i \geq n\}$ ([4], page 25). Since A is open, we must have $b^n \in A$ for some integer $n \geq 1$. For otherwise, suppose $b^n \notin A$ for all integers $n \geq 1$. Then $b^n \in A'$ for all integers $n \geq 1$. Because A' is closed in S and hence compact, we have $\Gamma(b) \subseteq A'$, which implies $e \notin A$, a contradiction. Thus $b^n \in A$ implies $b \in R(A)$, which is impossible.

We now return to prove Theorem 2.

Since A is an open ideal of S, by Lemma A, $R(A)$ is also an open ideal of S. If $R(A)=S$, then clearly $B(R(A))=\emptyset$. Suppose that $B(R(A)) \neq \emptyset$, then $\text{cl}\,R(A)-$ $-R(A) \neq \emptyset$. Since the closure of an ideal is also an ideal of S, by Lemma B, there exists an idempotent $e^2=e \in \text{cl}\,R(A)-R(A)$. Since $R(A)$ is open, we have $B(R(A))=$ $=\text{cl}\,R(A) \cap \text{cl}\,(S-R(A))=\text{cl}\,R(A) \cap (S-R(A))$. Consequently, we obtain that $e^2=e \in B(R(A))$. Now, let the maximal subgroup of S containing the idempotent e be $H(e)$. $H(e)$ is compact because S is compact. As both $\text{cl}\,R(A)$ and $R(A)$ are ideals of S, we thus obtain that $H(e) \cap R(A)=\emptyset$ and $H(e) \subseteq \text{cl}\,R(A)$, which implies $H(e) \subseteq$ $\subseteq B(R(A))$, completing the proof.

COROLLARY. *Let the clan S (that is, a compact connected semigroup with unit) be contained in the Euclidean space E^n, $n \geqq 2$. If A is an ideal of S such that* cl $R(A)$ *includes the boundary of S, then $R(A)$ is a dense connected ideal of S (see* [5]*).*

PROOF. Let $B(S)$ be the boundary of S in E^n. By our hypothesis, $B(S) \subseteq$ cl $R(A)$. This implies that $B(S) \subseteq SB(S) \subseteq S$. cl $R(A) \subseteq$ cl $R(A)$, since S is a clan and cl $R(A)$ is an ideal of S. Then $i^*: H^{n-1}(S) \cong H^{n-1}$ cl $(R(A))$ since, by a famous result of A. D. WALLACE [5], $H^{n-1}(S, \text{cl } R(A)) = 0 = H^n(S, \text{cl } R(A))$. Hence by a result of A. D. WALLACE [5] again, cl $R(A) = S$, which means that $R(A)$ is dense in S. As S is connected, $R(A)$ is connected.

REMARK. The stipulation "A is an ideal" of S cannot be weakened. The following example shows that even A is an open subset of S but is not an ideal, our Theorem 2 fails to be true.

EXAMPLE 1. Let S be a comb space, that is,

$$S = \left\{0, \frac{1}{n}\right\} \times \underline{[0, 1]} \cup \underline{\left[0, \frac{1}{2}\right]} \times \{0\},$$

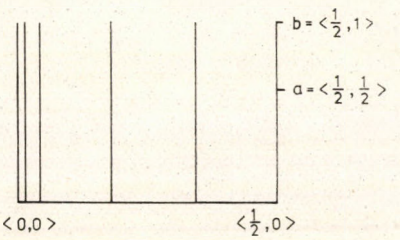


Fig. 1

for all $n = 1, 2, \ldots$ (see Figure 1), where the underlined brackets denote the intervals, and define multiplication in S by:

$$\langle x_1, y_1 \rangle * \langle x_2, y_2 \rangle = \langle x_1 x_2, \min \{y_1, y_2\} \rangle.$$

S is an abelian semigroup.

Now, let $A = \underline{(a, b)}$ where $a = \langle \frac{1}{2}, \frac{1}{2} \rangle$, $b = \langle \frac{1}{2}, 1 \rangle$, then $R(A) = \underline{(a, b)}$, $B(R(A)) = \{a\} \cup \{b\}$, which are not idempotents of S.

If the compact group G does not lie in the boundary of $R(A)$, then the compact group G must be contained in the ideal A. In fact, we have:

THEOREM 3. *Let S be an arbitrary topological semigroup and G be a compact group in S. Then G is entirely contained in the algebraic radical of an open subsemigroup A of S if and only if $G \cap A \neq \emptyset$.*

PROOF. Suppose $G \cap A \neq \emptyset$, then $G \cap A$ is an open subsemigroup of G since A is open in S. By F. B. WRIGHT [6], page 310, any open subsemigroup of a compact group is a closed subgroup, therefore $G \cap A$ is a closed neighbourhood of the identity

of the compact group G. Recalling a famous result in topological groups (see [4], Theorem 7.6, page 61), $G \cap A$ contains a normal subgroup N such that the quotient group G/N is finite. Let k be the order of the group G/N. Then for any $g \in G$, we have $g^k \in N \subseteq G \cap A \subseteq A$. This means that $g \in R(A)$. In other words, $G \subseteq R(A)$. Conversely, let us suppose that $G \subseteq R(A)$, then for every $g \in G$, $g \in R(A)$. Hence there exists an integer $k \geqq 1$ such that $g^k \in A$. Since G is a group, we have $g^k = g' \in G$, thus we prove that $G \cap A \neq \varnothing$.

COROLLARY. *In the above theorem, if A is an open ideal of S, then $G \subseteq A$ if $G \cap A \neq \varnothing$.*

PROOF. If A is assumed to be an ideal of S, then $G \cap A$ is an ideal of G. Because a group cannot properly contain an ideal, $G \cap A$ must be equal to G if non-empty. This implies that $G \subseteq A$.

The following example shows that the corollary of Theorem 3 is not true if A is not an ideal of S.

EXAMPLE 2. Let S be the real interval $[-1, 1]$. S is clearly a topological semigroup with the usual topology and usual multiplication. Let $A = (-1, 1]$, then $R(A) = [-1, 1]$ and $G = \{-1, 1\}$ is a compact group in S. One can observe that $G \subseteq R(A)$. But $G \not\subset A$.

Finally, we give a theorem which is a modification of the result of Theorem 2.

THEOREM 4. *Let G be a locally compact group with identity e. If A is an open subsemigroup of G such that $e \notin A$ but $e \in \mathrm{cl}_G(A)$, then the boundary of A in G is a topological group.*

(REMARK. The major technique used in proving this theorem was adapted from J. G. DOBBINS [1], page 30, who used similar methods to prove a theorem which appeared in his Ph. D. dissertation, 1969.)

PROOF. A locally compact group is well-known to be a topological group ([4], page 21), hence, the operation of translation and inversion are open functions in G. Let $A^{-1} = \{x^{-1} \in G | x \in A\}$, then A^{-1} is an open subset of G for A is open. Moreover, A is also a subsemigroup of G with $e \notin A$, hence, we must have $A \cap A^{-1} = \varnothing$ and $\mathrm{cl}_G A \cap A^{-1} = \varnothing$. The boundary of A in G is the set $B(A) = \mathrm{cl}_G A - A$, which is a locally compact set in G. G, being a group, every element of G must have a unique inverse in G. Clearly, $B(A) \cap A^{-1} = \varnothing$. Thus, for every element $x \in B(A)$, we must have $x^{-1} \in B(A)$. This fact implies that $B(A)$ is a group. Being an abstract group and a locally compact set, $B(A)$ is therefore a topological group ([4], page 21). The proof is completed.

We give here an example to illustrate the situations of Theorem 4.

EXAMPLE 3. Let G be the set of all pairs (x, y) of real numbers such that $x > 0$, with the usual plane topology as well as a multiplicative system with multiplication

$$(x_1, y_1) * (x_2, y_2) = (x_1 x_2, x_1 y_2 + y_1)$$

where the multiplication and addition in the parentheses at the right hand side

are the usual multiplication and addition. Let $A=\{(x, y)|x>0, y>0\}$. The boundary of A in G is the set $\{(x, 0)|x>0\}$, which is homeomorphic to $(0, \infty)$ with usual topology. It is easy to see that $B(A)$ is a topological group.

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References

[1] J. G. DOBBINS, *Ideals in Topological Semigroups*, Ph. D. dissertation, University of Kentucky (1969).
[2] E. HEWITT and K. A. ROSS, *Abstract Harmonic Analsys I*, Academic Press (New York, 1963).
[3] C. S. HOO and K.-P. SHUM, On algebraic radicals in mobs, *Colloquim Mathematicum Poland*, **25** (1972), pp. 25—35.
[4] A. B. PAALMAN-DE-MIRANDA, *Topological semigroups*, Mathematish Centrum (Amsterdam, 1964).
[5] A. D. WALLACE, Inverses in Euclidean mobs, *Math. J. Okayama Univ.*, **3** (1953), pp. 23—28.
[6] F. B. WRIGHT, Semigroups in compact groups, *Proc. Amer. Math. Soc.*, **7** (1956), pp. 309—311.

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Tomus 25 (1—2), (1974), pp. 21—30.

FINITE SYSTEMS OF FUNCTIONAL DIFFERENTIAL INEQUALITIES AND MINIMAX SOLUTIONS[1]

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1. Introduction

The notion of minimax solutions for an ordinary differential system is well known, see BURTON and WHYBURN [1]. In the present paper we introduce this notion for a functional differential system. With this end in view, some results on finite systems of functional differential inequalities are obtained. These are the natural extensions of many of our results in [4]. Finally, establishing the notion of minimax solutions, it is used to obtain a comparison theorem which is an extension of a similar one in [4].

2. Preliminaries

Let R^n denote the Euclidean n-space, I the interval $0 \leqq t < \infty$ and

$$C^n = C^n\{[-\tau, 0], R^n\}, \tau > 0,$$

the space of continuous functions from $[-\tau, 0]$ into R^n. For any vector $x \in R^n$, the i^{th} component will be denoted by x^i and its norm by

$$|x| = \left[\sum_{i=1}^{n} (x^i)^2\right]^{1/2}.$$

We define, for any $\Phi \in C^n$, the norm of Φ by

$$\|\Phi\| = \sup_{-\tau \leqq s \leqq 0} |\Phi(s)|.$$

Let C_H^n denote the set of those $\Phi \in C^n$ for which $\|\Phi\| \leqq H$. Clearly $C_H^n \subset C^n$. For any continuous function $x(u)$ from $[-\tau, \infty)$ into R^n, let, for each $t \geqq 0$, the symbol x_t denote the function

$$x_t = x_t(s) = x(t+s), \quad -\tau \leqq s \leqq 0.$$

Thus, for each $t \geqq 0$, $x_t \in C^n$. With these notations we shall consider the functional

[1] This paper forms one of the chapters in the author's Ph. D. thesis written with the guidance of Professor V. Lakshmikantham and submitted to the Marathwada University, Aurangabad (Maharashtra), India in March 1968.

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differential equation in the vector form

$$\dot{x} = f(t, x, x_t) \quad \left(\cdot \equiv \frac{d}{dt}\right) \tag{2.0}$$

where $f(t, x, \Psi)$ is a continuous functional from $I \times R^n \times C^n$ into R^n.

DEFINITION. Given an initial function $\Phi \in C_H^n$ at $t = t_0$, $t_0 \in I$, a function $x(t_0, \Phi)$ is said to be a solution of (2.0) on $[t_0, \infty)$ if the following conditions are satisfied:

(i) $x(t_0, \Phi)$ is defined and continuous on $[t_0, \infty)$ and $x_t(t_0, \Phi) \in C_H^n$ for $t \geqq t_0$;
(ii) $x_{t_0}(t_0, \Phi) = \Phi$;
(iii) the derivative of $x(t_0, \Phi)$ at t, $\dot{x}(t_0, \Phi)(t)$, exists for all $t \geqq t_0$ and satisfies (2.0) for $t \in [t_0, \infty)$.

For existence theorems for differential equations with retarded arguments we refer to DRIVER [2], LAKSHMIKANTHAM and SHENDGE [4], FRANKLIN [3]. Theorem 1 stated below asserts, in particular, the existence of solutions of (2.0) and the next Theorem 2 gives sufficient conditions for any solution of (2.0) to exist in the large. With a proper understanding of the norms involved, the proofs of these two theorems are essentially the same as those of Theorems 1 and 2 in our earlier paper [4]. Hence we omit the same.

THEOREM 1. *Let $f(t, x, \Psi)$ be defined and continuous on $I \times R^n \times C_H^n$. Then, for a given initial function $\Phi \in C_H^n$ at $t = t_0$, $t_0 \in I$, there exists an $\alpha > 0$, such that, there is a solution $x(t_0, \Phi)$ of (2.0) on $[t_0, t_0 + \alpha)$.*

THEOREM 2. *Let the function $g(t, u, v) \geqq 0$ be defined and continuous on $I \times R^+ \times R^+$, where R^+ is the nonnegative real line. Suppose also that g is nondecreasing in u and v for each $t \in I$. Assume that the maximal solution $r(t)$ of the ordinary differential equation*

$$\dot{r} = g(t, r, r), \quad r(t_0) = r_0, \quad r_0 \geqq 0,$$

exists for all $t \geqq t_0$, $t_0 \in I$. Suppose further that the functional $f(t, x, \Psi)$ of (2.0) satisfies the condition

$$|f(t, x, \Psi)| \leqq g(t, |x|, \|\Psi\|).$$

Then, the largest interval of the existence of any solution $x(t_0, \Psi)$ of (2.0) is $[t_0, \infty)$.

In what follows, for any continuous function $p(t)$ defined on I, either scalar or vector, we adopt the following notations for its Dini derivatives at t, $0 < t < \infty$.

$$D^+ p(t) = \limsup_{h \to 0^+} \frac{1}{h}[p(t+h) - p(t)], \quad D_+ p(t) = \liminf_{h \to 0^+} \frac{1}{h}[p(t+h) - p(t)],$$

$$D^- p(t) = \limsup_{h \to 0^-} \frac{1}{h}[p(t+h) - p(t)], \quad D_- p(t) = \liminf_{h \to 0^-} \frac{1}{h}[p(t+h) - p(t)].$$

When $D^+ p(t) = D_+ p(t)$, the right derivative of $p(t)$ at t will be denoted by $D_R p(t)$. Similarly $D_L p(t)$ denotes the left derivative.

We mention below two useful lemmas, without proofs. Although these are stated for scalar functions, one can observe that they are also true for vector functions.

LEMMA 1 (ZYGMUND). *Let J and J_0 denote the intervals $t_0 \leqq t < t_0 + a$, and $t_0 < t < t_0 + a$, $a > 0$ respectively. Suppose $q(t)$ is a continuous function from J into R and*

$$Dq(t) \leqq 0, \quad t \in J_0 - S$$

where D is any fixed Dini derivative and S is an at most countable subset of J. Then $q(t)$ is nonincreasing in t on J.

LEMMA 2. *Let $p(t)$ and $q(t)$ be continuous functions from J into R and for some fixed Dini derivative*

$$Dq(t) \leqq p(t), \quad t \in J_0 - S$$

where J_0 and S have the same meaning as in Lemma 1 stated above. Then

$$Dq(t) \leqq p(t), \quad t \in J.$$

3. Finite systems of functional differential inequalities

For the sake of convenience we shall suppose that the superscript i ranges over the integers $1, 2, \ldots, n$. Let $0 \leqq k \leqq n$. We suppose that the superscripts p and q range over the integers $1, 2, \ldots, k$ and $k+1, \ldots, n$ respectively. Also whenever vectorial inequalities will be used, it is to be understood that the same inequalities hold between their corresponding components.

We shall now state the definitions of under- and over-functions with respect to the functional differential system (2.0).

DEFINITION. Let $x(u)$ be a continuous function from $[t_0 - \tau, \infty)$ into R^n, $t_0 \in I$, possessing a right derivative $D_R x(t)$ for $t \in [t_0, \infty)$. If $x(u)$ satisfies the differential inequalities

$$D_R x^p(t) < f^p(t, x(t), x_t), \quad D_R x^q(t) > f^q(t, x(t), x_t) \tag{3.0}$$

for $t > t_0$, then it is said to be a k under $(n-k)$ over-function with respect to (2.0). On the other hand, if the same function $x(u)$ satisfies the inequalities (3.0) in the reversed form

$$D_R x^p(t) > f^p(t, x(t), x_t), \quad D_R x^q(t) < f^q(t, x(t), x_t)$$

for $t > t_0$, then it is said to be a k over $(n-k)$ under-function with respect to (2.0). It is evident that these definitions include the definitions of under- and over-functions as particular cases, namely for $k = n$ and $k = 0$, respectively.

In order to prove the desired results on systems of differential inequalities, we require the functional $f(t, x, \Psi)$ to satisfy certain monotonic properties. The same will now be defined.

DEFINITION. We shall say that the functional $f(t, x, \Psi)$ possesses a mixed quasi-monotone property if the following conditions hold:

(i) $f^p(t, x, \Psi)$ is nondecreasing in x^j, $j = 1, 2, \ldots, k$, $j \neq p$; nonincreasing in x^q for each (t, x) and $f^p(t, x, \Psi)$ is nondecreasing in Ψ^j, $j = 1, 2, \ldots, k$; nonincreasing in Ψ^q for each (t, x).

(ii) $f^q(t, x, \Psi)$ is nonincreasing in x^p, nondecreasing in x^j, $j=k+1, \ldots, n$, $j\neq q$, for each (t, x) and $f^q(t, x, \Psi)$ is nonincreasing in Ψ^p, nondecreasing in Ψ^j, $j=k+1, \ldots, n$, for each (t, x).

It is clear that the particular cases $k=n$ and $k=0$ in the mixed quasimonotone property correspond to quasimonotone nondecreasing and quasimonotone nonincreasing properties of $f(t, x, \Psi)$, respectively.

We are now in a position to state and prove the following theorem which is an extension of Theorem 3 in [4]. This theorem plays an essentially important role in the theory of finite systems of functional differential inequalities, since it is useful to establish many interesting results.

In what follows and for all future purposes whenever we write $\sigma_1 \leqq \sigma_2$ for any $\sigma_1, \sigma_2 \in C^n$, we mean $\sigma_1(s) \leqq \sigma_2(s)$, $-\tau \leqq s \leqq 0$.

THEOREM 3. (a) *Let the functional $f(t, x, \Psi)$ be defined and continuous on $I \times R^n \times C_H^n$ and satisfy the mixed quasimonotone property.*

(b) *Let $x(u)$ and $y(u)$ be continuous functions from $[t_0-\tau, \infty)$ into R^n, $t_0 \in I$, such that $(t, x(t), x_t)$ and $(t, y(t), y_t) \in I \times R^n \times C_H^n$.*

(c) *Assume further that*

$$x_{t_0}^p < y_{t_0}^p, \quad x_{t_0}^q > y_{t_0}^q \tag{3.1}$$

and for $t > t_0$ the following inequalities are satisfied:

$$D_- x^p(t) \leqq f^p(t, x(t), x_t), \tag{3.2}$$

$$D_- x^q(t) > f^q(t, x(t), x_t), \tag{3.3}$$

$$D_- y^p(t) > f^p(t, y(t), y_t), \tag{3.4}$$

$$D_- y^q(t) \leqq f^q(t, y(t), y_t). \tag{3.5}$$

Then for all $t \geqq t_0$ the inequalities

$$x^p(t) < y^p(t), \quad x^q(t) > y^q(t) \tag{3.6}$$

hold.

PROOF. Define for $u \in [t_0-\tau, \infty)$

$$m^p(u) = y^p(u) - x^q(u), \quad m^q(u) = x^q(u) - y^q(u). \tag{3.7}$$

It is clear from (3.1) that $m^i(t_0) > 0$. Suppose that the assertion (3.6) is not true. Then the set

$$Z = \bigcup_{i=1}^{n} \{t > t_0 : m^i(t) \leqq 0\}$$

is nonempty. Let $t^* = \inf Z$. Then $t^* > t_0$ and $t^* \in Z$ since Z is closed. Consequently there exists an integer j, $1 \leqq j \leqq n$, such that

$$m^j(t^*) = 0. \tag{3.8}$$

For, otherwise we have $m^j(t^*)<0$, which implies that $m^j(t)<0$ in a sufficiently small neighbourhood of t^* on its left, contradicting the definition of t^*. Now, for small $h<0$ we have

$$\frac{1}{h}[m^j(t^*+h)-m^j(t^*)] < 0$$

which implies that

$$D_- m^j(t^*) \leqq 0. \tag{3.9}$$

We also have by the definition of $m(u)$

$$m^i(t^*) \geqq 0, \qquad i \neq j. \tag{3.10}$$

There are two possibilities about j for which (3.8) holds. Namely (i) $1 \leqq j \leqq k$ and (ii) $k+1 \leqq j \leqq n$. In the case (i) the relations (3.7) and (3.9) imply $D_- y^j(t^*) \leqq \leqq D_- x^j(t^*)$, which, in view of (3.2) and (3.4), yields

$$f^j(t^*, y(t^*), y_{t^*}) < f^j(t^*, x(t^*), x_{t^*}). \tag{3.11}$$

Also the relations (3.8) and (3.10) imply $x_{t^*}<y_{t^*}$. This together with (3.8), (3.10) and the mixed quasimonotone property of $f(t, x, \Psi)$ gives

$$f^j(t^*, x(t^*), x_{t^*}) \leqq f^j(t^*, y(t^*), y_{t^*}). \tag{3.12}$$

The inequalities (3.11) and (3.12) imply the contradiction

$$f^j(t^*, x(t^*), x_{t^*}) < f^j(t^*, x(t^*), x_{t^*}).$$

Similarly in the case (ii) by using an argument as above we are led to the contradiction

$$f^j(t^*, y(t^*), y_{t^*}) > f^j(t^*, y(t^*), y_{t^*})$$

by using the relations (3.3), (3.5), (3.7), (3.8), (3.10) and the mixed quasimonotone property of $f(t, x, \Psi)$. Hence the set Z is empty and the assertion (3.6) holds.

The proof is complete.

COROLLARY 1. *Let the hypotheses* (a) *and* (b) *of Theorem* 3 *hold except that the mixed quasimonotone property of* $f(t, x, \Psi)$ *is replaced by the quasimonotone nondecreasing property. Assume further that*

$$D_- x(t) \leqq f(t, x(t), x_t), \quad D_- y(t) > f(t, y(t), y_t)$$

for $t>t_0$. *Then* $x(t)<y(t)$, $t \geqq t_0$ *whenever* $x_{t_0}<y_{t_0}$.

REMARK 1. It is easy to observe that the assertion of Theorem 3 holds good even if the inequalities (3.2), (3.3), (3.4) and (3.5) are replaced by

$$D_- x^p(t) < f^p(t, x(t), x_t), \quad D_- x^q(t) \geqq f^q(t, x(t), x_t),$$

$$D_- y^p(t) \geqq f^p(t, y(t), y_t), \quad D_- y^q(t) < f^q(t, y(t), y_t).$$

REMARK 2. In Theorem 3 and its Corollary 1 we can as well use any fixed Dini derivative D in place of D_-, the corresponding inequalities being satisfied only for $t \in (t_0, \infty) - S$ where S is an at most countable subset of (t_0, ∞). The assertions follow by using the Lemmas 1 and 2.

THEOREM 4. *Let $y(u)$ and $z(u)$ be k under $(n-k)$ over- and k over $(n-k)$ under-functions, respectively, with respect to the system* (2.0). *Assume that $f(t, x, \Psi)$ possesses the mixed quasimonotone property. Let $x(t_0, \Phi)$ be any solution of* (2.0) *existing on $[t_0, \infty)$ such that*

$$y_{t_0}^p < \Phi^p < z_{t_0}^p, \quad y_{t_0}^q > \Phi^q > z_{t_0}^q.$$

Then the inequalities

$$y^p(t) < x^p(t_0, \Phi)(t) < z^p(t), \quad y^q(t) > x^q(t_0, \Phi)(t) > z^q(t)$$

hold for all $t \geqq t_0$.

The proof follows by a repeated application of Theorem 3 and the Remarks 1 and 2. The details are omitted.

COROLLARY 2. *Let $y(u)$ and $z(u)$ be under- and over-functions with respect to* (2.0), *respectively. Assume that $f(t, x, \Psi)$ is quasimonotone nondecreasing. Let $x(t_0, \Phi)$ be any solution of* (2.0) *on $[t_0, \infty)$, such that $y_{t_0} < \Phi < z_{t_0}$. Then*

$$y(t) < x(t_0, \Phi)(t) < z(t), \quad t \in [t_0, \infty).$$

REMARK 3. Corollary 2 is an analogue of Theorem 4 in [4].

4. Minimax solutions and applications

DEFINITION. Let $r(t_0, \Phi)$ be a solution of the functional differential system (2.0) on $[t_0, \infty)$ such that for every solution $x(t_0, \Phi)$ of (2.0) on $[t_0, \infty)$, the inequalities

$$x^p(t_0, \Phi)(t) \leqq r^p(t_0, \Phi)(t), \quad x^q(t_0, \Phi)(t) \geqq r^q(t_0, \Phi)(t) \tag{4.0}$$

are satisfied for $t \in [t_0, \infty)$. In this case $r(t_0, \Phi)$ is called a k max $(n-k)$ mini-solution of (2.0). On the other hand if $r(t_0, \Phi)$ satisfies the inequalities (4.0) in the reversed form

$$x^p(t_0, \Phi)(t) \geqq r^p(t_0, \Phi)(t), \quad x^q(t_0, \Phi)(t) \leqq r^q(t_0, \Phi)(t)$$

for $t \in [t_0, \infty)$ then it is called a k mini $(n-k)$ max-solution. In either case $r(t_0, \Phi)$ is said to be a minimax solution of (2.0).

Clearly a k max $(n-k)$ mini-solution reduces to a maximal solution when $k=n$ and a minimal solution when $k=0$. Similarly a k mini $(n-k)$ max-solution reduces to a minimal or a maximal solution according as $k=n$ or $k=0$, respectively.

Since a minimax solution includes the maximal and minimal solutions as particular cases, we deal with the existence problem for minimax solutions only.

THEOREM 5. *Let $f(t, x, \Psi)$ be the same functional as in Theorem* 1. *Assume further that f possesses the mixed quasimonotone property. Then, for a given initial function $\Phi \in C_H^n$ at $t=t_0$, $t_0 \in I$, there exists an $\alpha_1 > 0$, such that there is a k max $(n-k)$ mini- and a k mini $(n-k)$ max-solution of* (2.0) *on $[t_0, t_0+\alpha_1)$.*

PROOF. Let a and b be positive real numbers such that $b \geqq 2H$. As in the proof of Theorem 1, see [4], define the set $\Omega \subset I \times R^n \times C_H^n$ by

$$\Omega = \left\{ (t, x, \Psi) : \begin{array}{l} \text{(i)} \quad t_0 \leqq t \leqq t_0 + a, \\ \text{(ii)} \quad |x - \Phi(0)| \leqq b, \ x \in R^n, \\ \text{(iii)} \quad \|\Psi - \Phi(0)\| \leqq b, \ \Psi \in \hat{C}_H^n \end{array} \right\} \tag{4.1}$$

where $\hat{C}_H^n \subset C_H^n$ is the set of functions which are Lipschitz continuous with a constant $M_1 > 0$. Since $f(t, x, \Psi)$ is continuous on Ω which is a compact set in virtue of the Ascoli—Arzela's theorem, it is bounded there. Let M_2 be the bound of f on Ω and let $M = \operatorname{Max}[M_1, M_2]$.

Let ε be a positive n-vector such that $|\varepsilon| \leqq \dfrac{b}{2}$. Consider

$$\begin{cases} \dot{x}^p(t) = f^p(t, x(t), x_t) + \varepsilon^p, & x_{t_0}^p(t_0, \Phi, \varepsilon) = \Phi^p + \varepsilon^p, \\ \dot{x}^q(t) = f^q(t, x(t), x_t) - \varepsilon^q, & x_{t_0}^q(t_0, \Phi, \varepsilon) = \Phi^q - \varepsilon^q. \end{cases} \tag{4.2}$$

Then $f_\varepsilon(t, x, \Psi) = f(t, x, \Psi) \pm \varepsilon$ is defined and continuous on

$$\Omega_\varepsilon = \left\{ (t, x, \Psi) : \begin{array}{l} \text{(i)} \quad t_0 \leqq t \leqq t_0 + a, \\ \text{(ii)} \quad \left|x - (\Phi(0) \pm \varepsilon)\right| \leqq \dfrac{b}{2}, \ x \in R^n, \\ \text{(iii)} \quad \left\|\Psi - (\Phi(0) \pm \varepsilon)\right\| \leqq \dfrac{b}{2}, \ \Psi \in \hat{C}_H^n \end{array} \right\}.$$

Clearly $\Omega_\varepsilon \subset \Omega$ and the bound of f_ε on Ω_ε is $M_2 + \dfrac{b}{2}$. Therefore, by Theorem 1, there is a solution $x(t_0, \Phi, \varepsilon)$ of (4.2) on $[t_0, t_0 + \alpha_1)$ where $\alpha_1 = \min\left(a, \dfrac{b}{2(2M+b)}\right)$. We show that the assertion of the theorem is true for this α_1.

For any n-vectors ε_1 and ε_2 such that $0 < \varepsilon_2 < \varepsilon_1 \leqq \varepsilon$, we have

$$x_{t_0}^p(t_0, \Phi, \varepsilon_2) < x_{t_0}^p(t_0, \Phi, \varepsilon_1), \quad x_{t_0}^q(t_0, \Phi, \varepsilon_2) > x_{t_0}^q(t_0, \Phi, \varepsilon_1) \tag{4.3}$$

and

$$\dot{x}^p(t_0, \Phi, \varepsilon_2)(t) = f^p\big(t, x(t_0, \Phi, \varepsilon_2)(t), x_t(t_0, \Phi, \varepsilon_2)\big) + \varepsilon_2^p, \tag{4.4}$$

$$\dot{x}^q(t_0, \Phi, \varepsilon_2)(t) = f^q\big(t, x(t_0, \Phi, \varepsilon_2)(t), x_t(t_0, \Phi, \varepsilon_2)\big) - \varepsilon_2^q, \tag{4.5}$$

$$\dot{x}^p(t_0, \Phi, \varepsilon_1)(t) > f^p\big(t, x(t_0, \Phi, \varepsilon_1)(t), x_t(t_0, \Phi, \varepsilon_1)\big) + \varepsilon_2^p, \tag{4.6}$$

$$\dot{x}^q(t_0, \Phi, \varepsilon_1)(t) < f^q\big(t, x(t_0, \Phi, \varepsilon_1)(t), x_t(t_0, \Phi, \varepsilon_1)\big) - \varepsilon_2^q. \tag{4.7}$$

A suitable application of Theorem 3 to (4.3), (4.4), (4.5), (4.6) and (4.7) now gives

$$x^p(t_0, \Phi, \varepsilon_2)(t) < x^p(t_0, \Phi, \varepsilon_1)(t), \quad x^q(t_0, \Phi, \varepsilon_2)(t) > x^q(t_0, \Phi, \varepsilon_1)(t),$$

for $t \in [t_0, t_0+\alpha_1)$. Observing that the family of functions $\{x(t_0, \Phi, \varepsilon)\}$, is equi-continuous and uniformly bounded, one can establish that for some sequence $\varepsilon_n \to 0$

$$\lim_{\varepsilon_n \to 0} x(t_0, \Phi, \varepsilon_n)(t) = r(t_0, \Phi)(t) \tag{4.8}$$

exists uniformly on $[t_0, t_0+\alpha_1]$ and that $r(t_0, \Phi)$ is a solution of (2.0) on $[t_0, t_0+\alpha_1)$.

We now show that $r(t_0, \Phi)$ is a k max $(n-k)$ mini-solution of (2.0) on $[t_0, t_0+\alpha_1)$. We have therefore to prove that the inequalities (4.0) are satisfied on this interval. Let $x(t_0, \Phi)$ be any solution of (2.0) on $[t_0, t_0+\alpha_1)$. Then we have

$$x^p_{t_0}(t_0, \Phi) < x^p_{t_0}(t_0, \Phi, \varepsilon), \quad x^q_{t_0}(t_0, \Phi) > x^q_{t_0}(t_0, \Phi, \varepsilon)$$

and

$$\dot{x}^p(t_0, \Phi)(t) < f^p\big(t, x(t_0, \Phi)(t), x_t(t_0, \Phi)\big) + \varepsilon^p,$$

$$\dot{x}^q(t_0, \Phi)(t) > f^q\big(t, x(t_0, \Phi)(t), x_t(t_0, \Phi)\big) - \varepsilon^p,$$

$$\dot{x}^p(t_0, \Phi, \varepsilon)(t) = f^p\big(t, x(t_0, \Phi, \varepsilon)(t), x_t(t_0, \Phi, \varepsilon)\big) + \varepsilon^p,$$

$$\dot{x}^q(t_0, \Phi, \varepsilon)(t) = f^q\big(t, x(t_0, \Phi, \varepsilon)(t), x_t(t_0, \Phi, \varepsilon)\big) - \varepsilon^q$$

for any positive n-vector ε, such that, $|\varepsilon| \leqq \dfrac{b}{2}$. An appropriate application of Theorem 3 now yields

$$x^p(t_0, \Phi)(t) < x^p(t_0, \Phi, \varepsilon)(t), \quad x^q(t_0, \Phi)(t) > x^q(t_0, \Phi, \varepsilon)(t)$$

for $t \in [t_0, t_0+\alpha_1)$. As a consequence of taking limit as $\varepsilon \to 0$, we obtain, by virtue of (4.8), the inequalities

$$\begin{cases} x^p(t_0, \Phi)(t) \leqq \lim\limits_{\varepsilon \to 0} x^p(t_0, \Phi, \varepsilon)(t) = r^p(t_0, \Phi)(t), \\ x^q(t_0, \Phi)(t) \geqq \lim\limits_{\varepsilon \to 0} x^q(t_0, \Phi, \varepsilon)(t) = r^q(t_0, \Phi)(t) \end{cases}$$

for $t \in [t_0, t_0+\alpha_1)$. This proves the desired result.

The existence of k mini $(n-k)$ max-solution of (2.0) can be established by changing the signs of ε in (4.2) and proceeding on similar lines as above. This completes the proof of Theorem 5.

As a consequence of the above existence theorem for minimax solutions and the extension Theorem 2, we have the following:

THEOREM 6. *Let $f(t, x, \Psi)$ be the same functional as in Theorem* 1 *and satisfy the mixed quasimonotone property. Assume further that the hypothesis of Theorem* 2 *holds. Then, for a given initial function $\Phi \in C^n_H$ at $t=t_0$, $t_0 \in I$, the system* (2.0) *has minimax solutions which exist for all $t \geqq t_0$.*

COROLLARY 3. *Let the hypothesis of Theorem* 6 *hold. Let J be the largest interval of the existence of k max $(n-k)$ mini-solution $r(t_0, \Phi)$ of* (2.0). *Suppose $[t_0, t^*]$ is a compact subinterval of J. Then for sufficiently small $\varepsilon > 0$, the k max $(n-k)$ mini-solution $r(t_0, \Phi, E)$ of the system* (4.2) *exists on $[t_0, t^*]$ and*

$$\lim_{\varepsilon \to 0} r(t_0, \Phi, \varepsilon) = r(t_0, \Phi)(t)$$

uniformly on $[t_0, t^*]$.

5. A comparison theorem

As an application of the minimax solutions, we shall now state the following comparison theorem.

THEOREM 7. *Let $f(t, x, \Psi)$ be the same functional as in Theorem* 1 *and satisfy the mixed quasimonotone property. Suppose $r(t_0, \Phi)$ is the k max $(n-k)$ mini-solution of* (2.0) *on* $[t_0, \infty)$.

Let $m(n)$ be a continuous function from $[t_0-\tau, \infty)$ into R^n such that

$$(t, m(t), m_t) \in I \times R^n \times C_H^n$$

and

$$m_{t_0}^p \leqq r_{t_0}^p(t_0, \Phi), \quad m_{t_0}^q \geqq r_{t_0}^q(t_0, \Phi). \tag{5.0}$$

Assume further that then inequalities

$$D_- m^p(t) \leqq f^p(t, m(t), m_t), \quad D_- m^q(t) \geqq f^q(t, m(t), m_t) \tag{5.1}$$

hold for $t > t_0$. Then the inequalities

$$m^p(t) \leqq r^p(t_0, \Phi)(t), \quad m^q(t) \geqq r^q(t_0, \Phi)(t) \tag{5.2}$$

hold for all $t \geqq t_0$.

PROOF. Let $t^* > t_0$. Then it follows from Corollary 3 that for all sufficiently small $\varepsilon > 0$, the k max $(n-k)$ mini-solution $r(t_0, \Phi, \varepsilon)$ of the system (4.2) exists on $[t_0, t^*]$ and is such that

$$\lim_{\varepsilon \to 0} r(t_0, \Phi, \varepsilon)(t) = r(t_0, \Phi)(t) \tag{5.3}$$

uniformly on $[t_0, t^*]$. Observe from (5.0) and (4.2) that

$$m_{t_0}^p < r_{t_0}^p(t_0, \Phi, \varepsilon), \quad m_{t_0}^q > r_{t_0}^q(t_0, \Phi, \varepsilon). \tag{5.4}$$

Theorem 3 can now be applied to the relations (5.4), (4.2) and (5.1) to yield

$$m^p(t) < r^p(t_0, \Phi, \varepsilon)(t), \quad m^q(t) > r^q(t_0, \Phi, \varepsilon)(t),$$

which, in view of (5.3), imply the conclusion (5.2). The proof of the theorem is complete.

REMARK 1. In Theorem 7 we can use any fixed Dini derivative D in place of D_-, the corresponding inequalities being satisfied on $(t_0, \infty) - S$ where S is an at most countable subset of (t_0, ∞). The conclusion follows by the use of Lemmas 1 and 2.

REMARK 2. If the inequalities (5.0) and (5.1) in Theorem 7 are assumed in the reverse form, the assertion (5.2) should be replaced by

$$m^p(t) \geqq \varphi^p(t_0, \Phi)(t), \quad m^q(t) \leqq \varphi^q(t_0, \Phi)(t)$$

where $\varphi(t_0, \Phi)$ is the k mini $(n-k)$ max-solution of (2.0).

The proof follows by obvious changes in the proof of Theorem 7.

COROLLARY 4. *Let the hypothesis of Theorem 7 hold except that the mixed quasimonotone property of f is replaced by the quasimonotone nondecreasing property. Let $r(t_0, \Phi)$ be the maximal solution of* (2.0) *on* $[t_0, \infty)$.

Let $m(u)$ be a continuous function from $[t_0-\tau, \infty)$ into R^n such that $(t, m(t), m_t) \in \in I \times R^n \times C_H^n$ and

$$m_{t_0} \leqq r_{t_0}(t_0, \Phi). \tag{5.5}$$

Assume further that for $t > t_0$ the inequality

$$D_- m(t) \leqq f(t, m(t), m_t) \tag{5.6}$$

holds. Then

$$m(t) \leqq r(t_0, \Phi)(t), \quad t \geqq t_0. \tag{5.7}$$

REMARK 3. In Corollary 4, we can replace D_- by any fixed Dini derivative D, the corresponding inequalities being satisfied on $(t_0, \infty) - S$ where S is an at most countable subset of (t_0, ∞). The conclusion (5.7) still follows by the use of Lemmas 1 and 2.

REMARK 4. If in Corollary 4, the inequalities (5.5) and (5.6) are satisfied in the reverse form, the conclusion (5.7) should be replaced by

$$m(t) \geqq \varrho(t_0, \Phi)(t), \quad t \geqq t_0,$$

where $\varrho(t_0, \Phi)$ is the minimal solution of (2.0) on $[t_0, \infty)$. This follows from Remark 2.

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References

[1] L. P. BURTON and W. M. WHYBURN, Minimax Solutions of Ordinary Differential Systems, *Proc. Amer. Math. Soc.*, **3** (1952), pp. 794—803.
[2] R. D. DRIVER, Existence Theory for a Delay-Differential System, *Contr. Diff. Equ.*, **1** (3) (1963), pp. 317—336.
[3] JOEL FRANKLIN, On the Existence of Solutions of Functional Differential Systems, *Proc. Amer. Math. Soc.*, **5** (1954), pp. 363—369.
[4] V. LAKSHMIKANTHAM and G. R. SHENDGE, Functional Differential Inequalities, *An. Acad. Brasil. Ci.*, **39** (1) (1967), pp. 7—14.

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ON THE LOWER RADICAL CONSTRUCTION

By

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Let $\mathcal{M}$ be a non-empty, homomorphically closed class of zero $(R^2=(0))$ rings. Put $\mathcal{M}_1=\mathcal{M}$, and for integers $i>1$, define $\mathcal{M}_i$ to be the class of all associative rings R such that every non-zero homomorphic image of R contains a non-zero ideal in $\mathcal{M}_{i-1}$. In this way we obtain an ascending chain $\mathcal{M}_1\subseteq\mathcal{M}_2\subseteq\ldots$, the union of which is the smallest radical class containing $\mathcal{M}$ [2]. We shall denote this radical class by $L(\mathcal{M})$, it is the *lower radical class determined by* $\mathcal{M}$.

GARDNER proves that $L(\mathcal{M})=\mathcal{M}_4$ [1], and in [2] an example is given of a class $\mathcal{M}$ of zero rings such that $L(\mathcal{M})\supsetneqq\mathcal{M}_2$. In this note we prove:

THEOREM. $L(\mathcal{M})=\mathcal{M}_3$.

PROOF.[1,2] Let k be an integer $\geqq 4$ and R be a ring in $\mathcal{M}_k$. There is a chain of subrings

$$A=A_1\subseteq A_2\subseteq\ldots\subseteq A_k=R$$

such that A_i is an ideal of A_{i+1} for $1\leqq i\leqq k-1$, and $A_i\in\mathcal{M}_i$ for $1\leqq i\leqq k$ [2]. Let $\bar{A}=A+AR+RA+RAR$ be the ideal of R which is generated by A. It is sufficient to prove that $\bar{A}\in\mathcal{M}_2$.

By repeated application of Andrunakievic's Lemma [2; Lemma 3] we obtain $\bar{A}^{3^{k-2}}\subseteq A$, and since $A^2=(0)$, $\bar{A}$ is nilpotent.

Let $\bar{A}/K$ be a non-zero homomorphic image of $\bar{A}$. Choose n minimal so that $\bar{A}^n\subseteq K$. Since $\bar{A}^{n-1}\nsubseteq K$, $(RA+A)\bar{A}^{n-2}\nsubseteq K$ (if $n=2$, let $\bar{A}^{n-2}=R^*$, the ring obtained by adjoining an identity to R in the usual way).

Choose an integer m, an element $r\in R$ and an element $a\in\bar{A}^{n-2}$ such that $(r+m)Aa\nsubseteq K$. Define a mapping

$$f: A\to\bar{A}/K$$

by

$$f(x)=(r+m)xa+K$$

for each $x\in A$. Clearly f behaves well with respect to addition. If $x_1, x_2\in A$, then

$$(r+m)x_1x_2a\in R^*A^2\bar{A}^{n-2}\subseteq\bar{A}^n\subseteq K$$

and

$$(r+m)x_1a(r+m)x_2a\in R^*A\bar{A}^{n-2}A\bar{A}^{n-2}\subseteq\bar{A}^n\subseteq K,$$

so $f(x_1x_2)=0=f(x_1)f(x_2)$. Therefore, f is a homomorphism.

[1] B. J. Gardner has observed that this proof establishes the theorem for any non-empty, homomorphically closed class $\mathcal{M}$ of nilpotent rings.

[2] This proof is patterned after the proof of Lemma 2 in [2].

The image $f[A]$ of A under f is contained in $(\bar{A}^{n-2}+K)/K$ and is consequently an ideal of $\bar{A}/K$. Since $\mathscr{M}_1$ is homomorphically closed, $f[A]\in\mathscr{M}_1$; and $f[A]\neq(0)$ because $(r+m)Aa\nsubseteq K$. Thus $\bar{A}/K$ has a non-zero ideal in $\mathscr{M}_1$.

Therefore, $\bar{A}\in\mathscr{M}_2$.

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References

[1] B. J. GARDNER, Radicals of abelian groups and associative rings, *Acta Math. Acad. Sci. Hung.*, **24** (1973), pp. 259—268.

[2] A. SULINSKI, R. ANDERSON and N. DIVINSKY, Lower radical properties for associative and alternative rings, *J. Lond. Math. Soc.*, **4** (1966), pp. 417—424.

Acat Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (1—2), (1974), pp. 33—45.

ON THE UNIFORM CONVERGENCE OF TRIGONOMETRIC INTERPOLATING POLYNOMIALS TO FUNCTIONS IN THE CLASS $C_T^*(\omega)$

By

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§ 1. Introduction. For what follows, we need the following definitions:

By $C_{2\pi}$ we mean the set of continuous 2π-periodic functions. Let $\omega(h)$ be an arbitrary modulus of continuity function, where $0 \leqq h \leqq 2\pi$. Then by $C_T^*(\omega)$ we mean the class

$$C^*(\omega) = \left\{ f \in C_{2\pi} \,\middle|\, \lim_{h \to 0^+} \frac{|f(x+h)-f(x)|}{\omega(h)} = 0 \text{ uniforrmly on } [0, 2\pi] \right\}.$$

(Note that if we denote by $\omega_f(h)$ the modulus of continuity of the continuous function f, then an equivalent formulation of $C_T^*(\omega)$ is $C_T^*(\omega) = \left\{ f \in C_{2\pi} \,\middle|\, \lim_{h \to 0^+} \frac{\omega_f(h)}{\omega(h)} = 0 \right\}$.)

By $\mathscr{P}_n^T$ we mean the set of trigonometric polynomials of order n.

Consider the infinite matrix

$$\begin{pmatrix} x_{0,0} & & & \\ x_{1,0} & x_{1,1} & x_{1,2} & \\ \dots\dots\dots\dots\dots\dots\dots\dots & & & \\ x_{n,0} & x_{n,1} & \dots & x_{n,2n} \\ \dots\dots\dots\dots\dots\dots\dots\dots & & & \end{pmatrix}$$

where

$$0 \leqq x_{k,0} < x_{k,1} < \dots < x_{k,2k} < 2\pi, \qquad k \in \mathbf{N}.$$

For $f \in C_{2\pi}$, by $L_n^T(f, X, x)$ we mean the trigonometric Lagrange interpolating polynomial for f associated with the nth row of nodes of X:

$$L_n^T(f, X, x) = \sum_{k=0}^{2n} f(x_{n,k}) t_{k,n}(x) \quad \text{wehere} \quad t_{k,n}(x) = t_k(x)$$

is the kth trigonometric basic polynomial for the set of nodes $\{x_{n,0}, \dots, x_{n,2n}\}$.

The sequence $\{L_n^T(f, X, x)\}_{n=1}^{\infty}$ of trigonometric Lagrange interpolating polynomials is called the *trigonometric interpolatory process* for the system of nodes X.

Finally, we define the *trigonometric Lebesgue function* to be $\lambda_n^T(x) = \sum_0^{2n} |t_k(x)|$, and the *trigonometric Lebesgue constant* to be $\lambda_n^T(X) = \max_{x \in [0, 2\pi]} \lambda_n^T(x)$.

§ 2. The statement of the problem. The question of concern to us here will be: given $f \in C_T^*(\omega)$ and a matrix of nodes X, under what conditions will the trigonometric interpolatory process converge uniformly to f? The purpose of this paper

will be to present necessary and sufficient conditions for the convergence, to show that the necessary condition cannot be strengthened and that the sufficient condition cannot be weakened. (For a study of the analogous results relating to interpolation by algebraic polynomials, see the paper of SZABADOS, [1].) We state then, the theorems we seek to prove:

THEOREM 1. (The sufficient condition.) *If* $\overline{\lim_{n\to\infty}}\ \omega\left(\frac{1}{n}\right)\lambda_n^T(X)<\infty$, *then for every* $f\in C_T^*(\omega)$, *we have* $\lim_{n\to\infty}\|f(x)-L_n^T(f, X, x)\|=0$.

REMARK. Since, for $P\in\mathscr{P}_n^T$ and $m\geqq n$ we have $L_m^T(P, X, x)\equiv P$, Theorem 1 follows trivially from Jackson's Theorem and the definition of $C_T^*(\omega)$.

THEOREM 2. (The necessary condition.) *If*

$$\overline{\lim_{n\to\infty}}\ \omega\left(\frac{1}{n\lambda_n^T(X)}\right)\lambda_n^T(X)=\infty, \tag{2.1}$$

then there exists some $f\in C_T^*(\omega)$ *such that*

$$\overline{\lim_{n\to\infty}}\ \|L_n^T(f, X, x)\|=\infty. \tag{2.2}$$

THEOREM 3. (The necessary condition cannot be strengthened.) *If*

$$\overline{\lim_{n\to\infty}}\ \omega\left(\frac{1}{n\lambda_n^T(X)}\right)\lambda_n^T(X)<\infty, \tag{2.3}$$

then there exists a matrix of nodes Y *such that*

$$\lambda_n^T(X)=\lambda_n^T(Y), \qquad n\in\mathbf{N} \tag{2.4}$$

and if $f\in C_T^*(\omega)$, *we have*

$$\lim_{n\to\infty}\|f(x)-L_n^T(f, Y, x)\|=0. \tag{2.5}$$

THEOREM 4. (The sufficient condition cannot be weakened.) *If* $C_T^*(\omega)$ *contains some non-constant function, and*

$$\overline{\lim_{n\to\infty}}\ \omega\left(\frac{1}{n}\right)\lambda_n^T(X)=\infty, \tag{2.6}$$

then there exists a matrix of nodes Y *for which* (2.4) *holds and a function* $f\in C_T^*(\omega)$ *such that*

$$\overline{\lim_{n\to\infty}}\ \|f(x)-L_n^T(f, Y, x)\|>0. \tag{2.7}$$

§ 3. Proof of Theorem 2. We will need the following lemma of LOZINSKI [1]: Designate by $\mathrm{Lip}^{(2\pi)}1$ the class of 2π-periodic functions in Lip 1. If $\lim_{h\to 0^+}\frac{h}{\omega(h)}>0$, then $C_T^*(\omega)$ contains *only* constant functions. If, on the other hand, $\lim_{h\to 0^+}\frac{h}{\omega(h)}=0$, then $\mathrm{Lip}^{(2\pi)}1\subset C_T^*(\omega)$.

To begin with, let $z_n \in [0, 2\pi]$ be the point which maximizes the trigonometric Lebesgue function, i.e.,

$$\lambda_n^T(X) = \sum_{i=0}^{2n} |t_i(z_n)| = \max_{x \in [0, 2\pi]} \sum_{i=0}^{2n} |t_i(x)|.$$

Next, we define $g_n(x) \in C_{2\pi}$, $n \in \mathbf{N}$, as follows: Let $x_{-1} = x_{2n} - 2\pi$ and $x_{2n+1} = x_0 + 2\pi$. Let $g_n(x_i) = \operatorname{sign} t_i(z_n)$, $0 \leqq i \leqq 2n$, $g_n(x_{-1}) = g_n(x_{2n}) = \operatorname{sign} t_{2n}(z_n)$, $g_n(x_{2n+1}) = g_n(x_0) = \operatorname{sign} t_0(z_n)$. Let $g_n(x)$ be linear between x_i, $-1 \leqq i \leqq 2n+1$. An easy computation shows that $g_n(0) = g_n(2\pi)$. Finally, restrict $g_n(x)$ to $[0, 2\pi]$.

We note that the slopes of the line segments connecting $(x_i, g_n(x_i))$ to $(x_{i+1}, g_n(x_{i+1}))$, $-1 \leqq i \leqq 2n$, are certainly each less than or equal to

$$\frac{2}{\min\limits_{-1 \leqq k \leqq 2n} (x_{k+1} - x_k)}.$$

Now for $-1 \leqq k \leqq 2n$, we have (cf. e.g. [5])

$$\min_{-1 \leqq k \leqq 2n} (x_{k+1} - x_k) \geqq \frac{1}{n\lambda_n^T(X)}.$$

Therefore the slopes of the line segments connecting $(x_i, g_n(x_i))$ to $(x_{i+1}, g_n(x_{i+1}))$ for $-1 \leqq i \leqq 2n$ are each less than or equal to $2n\lambda_n^T(X)$.

We can now summarize the properties of our functions $g_n(x) \in C_{2\pi}$:

$$|g_n(x)| \leqq 1, \quad n \in \mathbf{N} \tag{3.1}$$

from which immediately follows

$$|L_n^T(g_m, X, z_n)| \leqq \lambda_n^T(X), \qquad n, m \in \mathbf{N}. \tag{3.2}$$

Also,

$$L_n^T(g_n, X, z_n) = \sum_{i=0}^{2n} g_n(x_i) t_i(z_n) = \sum_{i=0}^{2n} (\operatorname{sign} t_i(z_n)) t_i(z_n) = \tag{3.3}$$

$$= \sum_{i=0}^{2n} |t_i(z_n)| = \lambda_n^T(X), \qquad n \in \mathbf{N},$$

$$g_n(x) \in \operatorname{Lip}^{(2\pi)} 1, \qquad n \in \mathbf{N} \tag{3.4}$$

and moreover,

$$|g_n(x+h) - g_n(x)| \leqq 2n\lambda_n^T(X) \cdot h \qquad (0 \leqq x \leqq x+h \leqq 2\pi). \tag{3.5}$$

By (2.1),

$$\frac{\dfrac{1}{n\lambda_n^T(X)}}{\omega\left(\dfrac{1}{n\lambda_n^T(X)}\right)} = \frac{1}{n} \cdot \frac{1}{\omega\left(\dfrac{1}{n\lambda_n^T(X)}\right)\lambda_n^T(X)} \to 0 \qquad (n \to \infty).$$

Hence

$$\lim_{h \to 0+} \frac{h}{\omega(h)} = 0$$

and therefore by Lozinski's Lemma, $\mathrm{Lip}^{(2\pi)}1 \subset C_T^*(\omega)$. Thus

$$g_n(x) \in C_T^*(\omega), \qquad n \in \mathbf{N}. \tag{3.6}$$

In proving Theorem 2, we can assume that

$$\overline{\lim_{n\to\infty}} \|L_n^T(g_m, X, x)\| < \infty, \qquad m \in \mathbf{N}, \tag{3.7}$$

since otherwise, the theorem is proved. Let us define an increasing sequence of natural numbers $\{n_i\}$, $i \in \mathbf{N}$, such that

$$\omega\left(\frac{1}{n_i \lambda_{n_i}^T(X)}\right) \lambda_{n_i}^T(X) \geqq i \cdot 2^i, \qquad i \in \mathbf{N}, \tag{3.8}$$

$$\omega\left(\frac{1}{n_{i+1} \lambda_{n_{i+1}}^T(X)}\right) \lambda_{n_{i+1}}^T(X) \geqq 2\omega\left(\frac{1}{n_i \lambda_{n_i}^T(X)}\right) \lambda_{n_i}^T(X), \qquad i \in \mathbf{N}, \tag{3.9}$$

$$\omega\left(\frac{1}{n_i \lambda_{n_i}^T(X)}\right) \geqq 2\omega\left(\frac{1}{n_{i+1} \lambda_{n_{i+1}}^T(X)}\right), \qquad i \in \mathbf{N}, \tag{3.10}$$

and

$$2^{-i}\omega\left(\frac{1}{n_i \lambda_{n_i}^T(X)}\right) \lambda_{n_i}^T(X) \geqq 3 \sum_{k=1}^{i-1} 2^{-k} \omega\left(\frac{1}{n_k \lambda_{n_k}^T(X)}\right) \|L_{n_i}^T(g_{n_k}, X, x)\|, \qquad i \geqq 2. \tag{3.11}$$

This construction is possible by (2.1), the fact that $\lim_{h\to 0^+} \omega(h) = 0$, and by (3.7).

Next, we define

$$f(x) = \sum_{i=1}^{\infty} 2^{-i} \omega\left(\frac{1}{n_i \lambda_{n_i}^T(X)}\right) g_{n_i}(x), \qquad x \in [0, 2\pi]. \tag{3.12}$$

Then in view of (3.1), f converges uniformly to a 2π-periodic continuous function.

Let j be the smallest index such that

$$n_j \lambda_{n_j}^T(X) h \geqq 1, \tag{3.13}$$

i.e., for $1 \leqq i \leqq j-1$, we have $n_i \lambda_{n_i}^T(X) h < 1$. Then for $1 \leqq i \leqq j-1$, in view of (3.5) and (3.9),

$$2^{-i}\omega\left(\frac{1}{n_i \lambda_{n_i}^T(X)}\right) |g_{n_i}(x+h) - g_{n_i}(x)| \leqq 2^{1-i}\omega\left(\frac{1}{n_i \lambda_{n_i}^T(X)}\right) n_i \lambda_{n_i}^T(X) h \leqq$$

$$\leqq 2^{2-j}\omega\left(\frac{1}{n_{j-1} \lambda_{n_{j-1}}^T(X)}\right) n_{j-1} \lambda_{n_{j-1}}^T(X) h \qquad (0 \leqq x \leqq x+h \leqq 2\pi).$$

Since

$$h\omega(h) \leqq 2H\omega(h), \qquad h \leqq H, \tag{3.14}$$

(see e. g., [2]) by (3.13) for

$$H = \left(\frac{1}{n_{j-1} \lambda_{n_{j-1}}^T(X)}\right),$$

we have

$$2^{-i}\omega\left(\frac{1}{n_i\lambda_{n_i}^T(X)}\right)|g_{n_i}(x+h)-g_{n_i}(x)| \leqq 2^{3-j}\omega(h) \tag{3.15}$$

$$(1 \leqq i \leqq j-1,\ 0 \leqq x \leqq x+h \geqq 2\pi).$$

If $i \geqq j$, from (3.1), (3.10) and (3.13) we have

(3.16)

$$2^{-i}\omega\left(\frac{1}{n_i\lambda_{n_i}^T(X)}\right)|g_{n_i}(x+h)-g_{n_i}(x)| \leqq 2^{1-i}2^{j-i}\omega\left(\frac{1}{n_j\lambda_{n_j}^T(X)}\right) \leqq 2^{j-2i+1}\omega(h)$$

$$(i \geqq j,\ 0 \leqq x \leqq x+h \leqq 2\pi).$$

Then (3.15), (3.16) and the definition (3.12) give

$$|f(x+h)-f(x)| \leqq \omega(h)\frac{8j}{2^j}+2^{j+1}\sum_{i=j}^{\infty}2^{-2i} = \omega(h)\cdot\frac{3j+1}{3\cdot 2^{j-3}} \qquad (0\leqq x\leqq x+h\leqq 2\pi).$$

In view of (3.13), $h\to 0$ which implies $j\to\infty$ which in turn implies

$$\lim_{h\to 0^+}\frac{|f(x+h)-f(x)|}{\omega(h)} \leqq \lim_{j\to\infty}\frac{3j+1}{3\cdot 2^{j-3}} = 0, \qquad 0\leqq x\leqq 2\pi.$$

I.e., $f(x)\in C_T^*(\omega)$. Since

$$L_{n_k}^T(f, X, z_{n_k}) = \sum_{i=1}^{\infty}2^{-i}\omega\left(\frac{1}{n_i\lambda_{n_i}^T(X)}\right)L_{n_k}^T(g_{n_i}, X, z_{n_k}), \qquad k\in\mathbf{N} \tag{3.17}$$

from (3.11) we have

$$\sum_{i=1}^{k-1}2^{-i}\omega\left(\frac{1}{n_i\lambda_{n_i}^T(X)}\right)|L_{n_k}^T(g_{n_i}, X, z_{n_k})| \leqq \frac{2^{-k}}{3}\omega\left(\frac{1}{n_k\lambda_{n_k}^T(X)}\right)\lambda_{n_k}^T(X). \tag{3.18}$$

By (3.3) we have

$$2^{-k}\omega\left(\frac{1}{n_k\lambda_{n_k}^T(X)}\right)L_{n_k}^T(g_{n_k}, X, z_{n_k}) = 2^{-k}\omega\left(\frac{1}{n_k\lambda_{n_k}^T(X)}\right)\lambda_{n_k}^T(X), \tag{3.19}$$

and by (3.2) and (3.10), we have

$$\sum_{i=k+1}^{\infty}2^{-i}\omega\left(\frac{1}{n_i\lambda_{n_i}^T(X)}\right)|L_{n_k}^T(g_{n_i}, X, z_{n_k})| \leqq \tag{3.20}$$

$$\leqq \lambda_{n_k}^T(X)\sum_{i=k+1}^{\infty}2^{k-2i}\omega\left(\frac{1}{n_k\lambda_{n_k}^T(X)}\right) = \frac{2^{-k}}{3}\omega\left(\frac{1}{n_k\lambda_{n_k}^T(X)}\right)\lambda_{n_k}^T(X).$$

Finally, (3.17)—(3.20) and (3.8) give

$$L_{n_k}(f, X, z_{n_k}) \geqq \frac{2^{-k}}{3}\omega\left(\frac{1}{n_k\lambda_{n_k}^T(X)}\right)\lambda_{n_k}^T(X) \geqq \frac{k}{3}.$$

So $\lim_{k\to\infty}L_{n_k}^T(f, X, z_{n_k})=\infty$, from which (2.2) follows and Theorem 2 is proved.

§ 4. Proof of Theorem 3. Let Z be the matrix of equidistant nodes (i.e., the nth row of Z is $\left\{x_i=\frac{2\pi i}{2n+1}\right\}_{i=0}^{2n}$) on $[0, 2\pi]$, the trigonometric analogue of the Chebyshev nodes for algebraic polynomials. Let $\tau=\{s_n\}$ be any sequence of real numbers subject to the condition that $0\leqq s_n<2$.

Using the matrix Z, we define (according to its nth row) a new matrix $u(\tau)$ as follows:

$$u(\tau)=\begin{cases} u_0=\dfrac{\pi s_n}{2n+1} \\ u_i=x_i, \quad 1\leqq i\leqq 2n \end{cases} \tag{4.1}$$

Four important properties of $u(\tau)$ are then obtained (see e.g. [3]):

Property 1: If $\lambda_n^T(X)\geqq\lambda_n^T(Z)$, then for properly chosen s_n, $\lambda_n^T(u(\tau))=\lambda_n^T(X)$.

Property 2:

$$\sum_{i=2}^{2n}\left|t_i(x, u(\tau))\right| = O(\ln n), \qquad 0\leqq x\leqq 2\pi. \tag{4.2}$$

Property 3:

$$\left|t_0(x, u(\tau))+t_1(x, u(\tau))\right| = O(1), \qquad 0\leqq x\leqq 2\pi. \tag{4.3}$$

Property 4: If

$$\max_{x\in[0,\,2\pi)}\left[\left|t_0(x, u(\tau))\right|+\left|t_1(x, u(\tau))\right|\right]\geqq\frac{1}{2}\lambda_n^T(u(\tau)), \tag{4.4}$$

then

$$x_1-u_0\leqq\frac{C}{n\lambda_n^T(u(\tau))}, \quad \text{where} \quad C=\frac{5\pi^3}{2}. \tag{4.5}$$

Before defining the matrix Y of Theorem 3, we establish that

$$\overline{\lim_{n\to\infty}}\,\omega\left(\frac{1}{n\lambda_n^T(X)}\right)\lambda_n^T(X)<\infty \quad \text{implies} \quad \overline{\lim_{n\to\infty}}\,\omega\left(\frac{1}{n}\right)\ln n<\infty. \tag{4.6}$$

This is the trigonometric analogon of a statement in [5] (p. 399, Remark 2), and it is easily seen.

Now we define the matrix Y by:

$$n\text{th row of } Y=\begin{cases} n\text{th row of } X \text{ if } \lambda_n^T(X)<\lambda_n^T(Z) \\ n\text{th row of } u(\tau) \text{ with the value of } s_n \text{ for which} \\ \lambda_n^T(u(\tau))=\lambda_n^T(X) \text{ if } \lambda_n^T(X)\geqq\lambda_n^T(Z). \end{cases}$$

(We have used Property 1 of $u(\tau)$ in defining Y.) We shall denote the nth row of nodes of Y by $\{y_i\}_{i=0}^{2n}$. Clearly, $\lambda_n^T(X)=\lambda_n^T(Y)$. We must show that (2.5) holds for every $f\in C_T^*(\omega)$.

First, let us consider those n's for which either

(i) $\lambda_n^T(X)<\lambda_n^T(Z)$, or
(ii) $\lambda_n^T(X)\geqq\lambda_n^T(Z)$ and (4.4) does not hold.

In case (i), $\lambda_n^T(Y)=\lambda_n^T(X)<\lambda_n^T(Z)=O(\ln n)$. Using the fact that (4.4) does not hold and Property 2 of $u(\tau)$ respectively, we have in case (ii),

$$\max_{x\in[0,\,2\pi]}\sum_0^{2n}|t_i(x,u(\tau))| = \max_{x\in[0,\,2\pi]}\left(\sum_0^1|t_i(x,u(\tau))|+\sum_2^{2n}|t_i(x,u(\tau))|\right)\leqq$$

$$\leqq \max_{x\in[0,\,2\pi]}\sum_0^1|t_i(x,u(\tau))|+\max_{x\in[0,\,2\pi]}\sum_2^{2n}|t_i(x,u(\tau))|\leqq\frac{1}{2}\lambda_n^T(u(\tau))+O(\ln n).$$

I. e.,

$$\lambda_n^T(u(\tau))<\frac{\lambda_n^T(u(\tau))}{2}+O(\ln n)$$

which implies $\lambda_n^T(u(\tau))=O(\ln n)$. Therefore, since (4.6) guarantees that

$$\overline{\lim_{n\to\infty}}\,\omega\left(\frac{1}{n}\right)\ln n<\infty,$$

and we have

$$\omega\left(\frac{1}{n}\right)\lambda_n^T(Y)=O\left(\omega\left(\frac{1}{n}\right)\ln n\right),$$

it follows that

$$\overline{\lim_{n\to\infty}}\,\omega\left(\frac{1}{n}\right)\lambda_n^T(Y)<\infty.$$

Hence by Theorem 1, (2.5) follows.

Next, we consider the remaining values of n, i.e., those n's for which $\lambda_n^T(X)\geqq\lambda_n^T(Z)$ and (4.4) holds. Let $m=[\sqrt{n}+1]$. For this m, let $P_m(x)\in\mathbf{P}_m^T$ be the trigonometric polynomial of best approximation to f on $[0, 2\pi]$. Therefore, for $n\geqq3$, we have

$$f(x)-L_n^T(f,Y,x)=f(x)-P_m(x)+L_n^T(P_m-f,Y,x)=$$

$$=f(x)-P_m(x)+\sum_{i=2}^{2n}[P_m(y_i)-f(y_i)]t_i(Y,x)+[P_m(y_0)-f(y_0)]\cdot[t_0(Y,x)+t_1(Y,x)]+$$

$$+[P_m(y_1)-P_m(y_0)+f(y_0)-f(y_1)]t_1(Y,x).$$

Therefore,

$$\|f(x)-L_n^T(f,Y,x)\|\leqq \tag{4.7}$$

$$\leqq\|f(x)-P_m(x)\|\cdot\left[1+\left\|\sum_{i=2}^{2n}|t_i(Y,x)|\right\|+\|t_0(Y,x)+t_1(Y,x)\|\right]+$$

$$+[|f(y_0)-f(y_1)|+|P_m(y_0)-P_m(y_1)|]\cdot\|t_1(Y,x)\|.$$

Now we observe that $|f(y_0)-f(y_1)|\leqq\omega_f(y_1-y_0)$ and since by Property 4 of $u(\tau)$ (here is where we use the fact that (4.4) holds), $y_1-y_0\leqq\frac{C}{n\lambda_n^T(Y)}$, it follows that

$$|f(y_0)-f(y_1)|\leqq\omega_f\left(\frac{C}{n\lambda_n^T(Y)}\right). \tag{4.8}$$

Also, observe that

$$|P_m(y_0)-P_m(y_1)|=|P_m'(\xi)|(y_1-y_0)\leqq m\|P_m(x)\|(y_1-y_0),\ \xi\in(y_0,y_1). \tag{4.9}$$

Therefore from (4.7), we have by Jackson's Theorem, Property 2 of $u(\tau)$, Property 3 of $u(\tau)$, (4.8) and (4.9) that

$$\|f(x)-L_n^T(f,Y,x)\|\leqq$$

$$\leqq O\left(\omega_f\left(\frac{1}{m}\right)\right)[1+O(\ln n)+O(1)]+\left[\omega_f\left(\frac{C}{n\lambda_n^T(Y)}\right)+m\|P_m(x)\|\cdot(y_1-y_0)\right]\lambda_n^T(Y)\leqq$$

$$\leqq O\left(\omega_f\left(\frac{1}{m}\right)\ln n\right)+O\left(\omega_f\left(\frac{1}{n\lambda_n^T(Y)}\right)+\frac{\alpha\cdot m}{n\lambda_n^T(Y)}\right)\lambda_n^T(Y)\leqq$$

$$\leqq O\left(\omega_f\left(\frac{1}{m}\right)\ln m\right)+O\left(\omega_f\left(\frac{1}{n\lambda_n^T(Y)}\right)\lambda_n^T(Y)\right)+O\left(\frac{1}{\sqrt{n}}\right)=$$

$$=O\left(\omega\left(\frac{1}{m}\right)\cdot\ln m\cdot\frac{\omega_f\left(\frac{1}{m}\right)}{\omega\left(\frac{1}{m}\right)}\right)+O\left(\frac{\omega_f\left(\frac{1}{n\lambda_n^T(Y)}\right)}{\omega\left(\frac{1}{n\lambda_n^T(Y)}\right)}\cdot\omega\left(\frac{1}{n\lambda_n^T(Y)}\right)\cdot\lambda_n^T(Y)\right)+O\left(\frac{1}{\sqrt{n}}\right)=$$

$$=O\left(\frac{\omega_f\left(\frac{1}{m}\right)}{\omega\left(\frac{1}{m}\right)}+\frac{\omega_f\left(\frac{1}{n\lambda_n^T(Y)}\right)}{\omega\left(\frac{1}{n\lambda_n^T(Y)}\right)}+\frac{1}{\sqrt{n}}\right)$$

where $\alpha=\max_{x\in[0,\,2\pi]}|P_m(x)|$. Hence (2.5) follows by the definition of $C_T^*(\omega)$. This completes the proof of Theorem 3.

§ 5. Proof of Theorem 4. To begin with, let $\{q_n\}$ be any sequence of real numbers such that $q_n\geqq1$. Let $V_k=\frac{x_k}{q_n}$, where $x_k=\frac{2\pi k}{2n+1}$, $0\leqq k\leqq2n$. Let $V(q)$ be the matrix of nodes with nth row $\{V_k\}_{k=0}^{2n}$. In particular, if $q_n=1$, $V(q)=Z$.

We have then, the following property of $V(q)$:

(5.1) If $\lambda_n^T(X)\geqq\lambda_n^T(Z)$, then for properly chosen q_n, $\lambda_n^T(X)=\lambda_n^T(V(q))$.

We break the proof of Theorem 4 into two main cases.

Case 1: $\overline{\lim}_{n\to\infty} \omega\left(\frac{1}{n}\right)\ln n=\infty$. Then by Lemma (4.7), $\overline{\lim}_{n\to\infty} \omega\left(\frac{1}{n\lambda_n^T(X)}\right)\lambda_n^T(X)=\infty$. Therefore, by Theorem 2, with $Y=X$, there exists $f\in C_T^*(\omega)$ such that

$$\overline{\lim_{n\to\infty}} \|L_n^T(f, X, x)\| = \infty.$$

(2.7) then follows.

Case 2: $\overline{\lim}_{n\to\infty} \omega\left(\frac{1}{n}\right)\ln n<\infty$. In this case, we define the matrix Y as follows:

$$(5.2)\qquad n\text{th row of } Y = \begin{cases} n\text{th row of } X \text{ if } \lambda_n^T(X) < \lambda_n^T(Z) \\ n\text{th row of } V(q) \text{ with the } q_n \\ \text{for which } \lambda_n^T(V(q)) = \lambda_n^T(X) \text{ if } \lambda_n^T(X) \geqq \lambda_n^T(Z). \end{cases}$$

Clearly, $\lambda_n^T(X) = \lambda_n^T(Y)$.

Suppose $\overline{\lim}_{n\to\infty} \omega\left(\frac{1}{n}\right)\ln n=L\geqq 0$. Then there exists $K>0$ such that

$$0 < \omega\left(\frac{1}{n}\right)\ln n < K, \qquad n\in\mathbf{N}.$$

Then $\dfrac{\lambda_n^T(X)}{\ln n} = \dfrac{\omega\left(\frac{1}{n}\right)\lambda_n^T(X)}{\omega\left(\frac{1}{n}\right)\ln n} > \dfrac{1}{K}\omega\left(\frac{1}{n}\right)\lambda_n^T(X)$, $n\in\mathbf{N}$. Therefore, by (2.6),

$$\overline{\lim_{n\to\infty}} \frac{\lambda_n^T(X)}{\ln n} = \infty.$$

Let $A>0$ be such that $\lambda_n^T(Z)\leqq A\ln n$ (see e.g. [4]). Now we can choose a subsequence of natural numbers $\{r_i\}$ such that for $i\geqq i_0$, $\lambda_{r_i}^T(X)\geqq A\ln r_i$. But then

$$(5.3)\qquad \lambda_{r_i}^T(X) \geqq \lambda_{r_i}^T(Z)$$

(i.e., $\lambda_{r_i}^T(Y)=\lambda_{r_i}^T(V(q))$, where the $V(q)$ is determined by (5.2)).

Now we examine the associated sequence $\{q_{r_i}\}$. One of two situations must arise:

(i) $\overline{\lim}_{i\to\infty} q_{r_i} > 1$.

We then choose a subsequence $\{s_i\}$ of $\{r_i\}$ such that $q_{s_i}\geqq d>1$. Then $0\leqq V = \dfrac{2\pi k}{2s_i+1}\cdot\dfrac{1}{q_{s_i}}\leqq\dfrac{2\pi}{d}<2\pi$, $0\leqq k\leqq 2s_i$. Define

$$f(x) = \begin{cases} 0, & 0 \leqq x \leqq \dfrac{2\pi}{d} \\ \dfrac{d}{\pi(d-1)}x - \dfrac{2}{d-1}, & \dfrac{2\pi}{d} < x \leqq \dfrac{\pi(d+1)}{d} \\ \dfrac{-d}{\pi(d-1)}x + \dfrac{2d}{d-1}, & \dfrac{\pi(d+1)}{d} < x \leqq 2\pi. \end{cases}$$

Then $f(x) \in C_{2\pi}$. Now $L_{s_i}^T(f, Y, x) = \sum_{k=0}^{2s_i}{}' f(V_k)t_k(x) \equiv 0$ since all the V_k's are in $\left[0, \frac{2\pi}{d}\right]$. However, $L_{s_i}^T(f, Y, x)$ does not converge uniformly to $f(x)$ on $[0, 2\pi]$ and in fact, fails to converge pointwise on $\left(\frac{2\pi}{d}, 2\pi\right)$.

The other situation which can occur is:

(ii) $\lim_{i\to\infty} q_{r_i} = 1$.

Let $V(q)$ be the matrix of (5.2). Let $z_n \in [0, 2\pi]$ be such that

$$\lambda_n^T(V(q)) = \max_{x\in[0,\,2\pi]} \sum_{i=0}^{2n} |t_i(V(q), x)| = \sum_{i=0}^{2n} |t_i(V(q), z_n)|.$$

Now define continuous functions $g_n(x)$ as follows:

$$g_n(V_i) = \operatorname{sign} t_i(V(q), z_n), \quad 0 \leqq i \leqq 2n.$$

If we let $V_{-1} = V_{2n} - 2\pi$, and $V_{2n+1} = V_0 + 2\pi = 2\pi$, we define

$$g_n(V_{-1}) = g_n(V_{2n}) = \operatorname{sign} t_{2n}(V(q), z_n),$$

$$g_n(V_{2n+1}) = g_n(V_0) = \operatorname{sign} t_0(V(q), z_n).$$

Let g_n be linear between V_i, $-1 \leqq i \leqq 2n+1$. Then $g_n(0) = g_n(2\pi)$. Restricting g_n to $[0, 2\pi]$, we have $g_n \in C_{2\pi}$.

We note that for $0 \leqq k \leqq 2n+1$,

$$V_{k+1} - V_k = \frac{1}{q_n} \cdot \frac{2\pi}{2n+1}.$$

Also,

$$V_{2n+1} - V_{2n} = V_0 - V_{-1} = 2\pi - V_{2n} = \frac{2\pi(2n+1)q_n - 4\pi n}{q_n(2n+1)} \geqq$$

$$\geqq \frac{2\pi(2n+1) - 4\pi n}{q_n(2n+1)} = \frac{2\pi}{q_n(2n+1)}.$$

Hence,

$$V_{j+1} - V_j = \min_{-1\leqq k\leqq 2n} \{V_{k+1} - V_k\} \geqq \frac{2\pi}{q_n(2n+1)}. \tag{5.4}$$

Our g_n's therefore possess the properties:

$$|g_n(x)| \leqq 1 \quad \text{on} \quad [0, 2\pi] \tag{5.5}$$

$$g_n(x) \in \operatorname{Lip}^{(2\pi)} 1 \subset C_T^*(\omega). \tag{5.6}$$

(Here we use the fact that $C_T^*(\omega)$ contains non-constant functions in applying Lozinski's Lemma.)

And by (5.4) we have

$$|g_n(x+h)-g_n(x)| \leqq \frac{2h}{V_{j+1}-V_j} \leqq \frac{q_n(2n+1)h}{\pi} \leqq \tag{5.7}$$

$$\leqq \frac{4 \cdot q_n \cdot n \cdot h}{\pi} \leqq \frac{8n}{\pi} h \quad \text{if} \quad q_n \leqq 2.$$

We can assume

$$\overline{\lim_{n\to\infty}} \|L_n^T(g_{r_i}, Y, x)\| < \infty, \quad i \in \mathbf{N} \tag{5.8}$$

since otherwise, the theorem is proved. Designate by $\{n_i\}$ that subsequence of $\{r_i\}$ for which:

$$q_{n_i} \leqq 2 \tag{5.9}$$

$$2\frac{n_i}{n_{i+1}} \leqq \frac{\omega\left(\frac{1}{n_{i+1}}\right)}{\omega\left(\frac{1}{n_i}\right)} \tag{5.10}$$

$$\omega\left(\frac{1}{n_{i+1}}\right) \leqq \frac{1}{3}\omega\left(\frac{1}{n_i}\right) \tag{5.11}$$

and

$$2^{-i}\omega\left(\frac{1}{n_i}\right)\lambda_{n_i}^T(Y) \geqq \max\left[i, 3\sum_{k=1}^{i-1} 2^{-k}\omega\left(\frac{1}{n_k}\right)\|L_{n_i}^T(g_{n_k}, Y, x)\|\right], \quad i \geqq 2. \tag{5.12}$$

This construction is possible since

$$\lim_{i\to\infty} q_{r_i} = 1, \quad \lim_{h\to 0^+} \frac{h}{\omega(h)} = 0, \quad \lim_{h\to 0^+} \omega(h) = 0,$$

and from the assumptions

$$\overline{\lim_{i\to\infty}} \omega\left(\frac{1}{r_i}\right)\lambda_{r_i}^T(X) = \infty \quad \text{and} \quad \overline{\lim_{n\to\infty}} \|L_n^T(g_{r_i}, Y, x)\| < \infty.$$

Now let

$$f(x) = \sum_{i=1}^{\infty} 2^{-i}\omega\left(\frac{1}{n_i}\right) g_{n_i}(x), \quad 0 \leqq x \leqq 2\pi.$$

This series converges uniformly for $x \in [0, 2\pi]$, and $f \in C_{2\pi}$.

We first show $f \in C_T^*(\omega)$. Designate by j the smallest index for which $n_j h \geqq 1$, if h is sufficiently small. Since $\frac{\omega(H)}{H} \leqq \frac{2\omega(h)}{h}$ for $h \leqq H$ and since $n_{j-1}h \leqq 1$, we have

$$\omega\left(\frac{1}{n_{j-1}}\right) n_{j-1} \leqq \frac{2\omega(h)}{h}. \tag{5.13}$$

From (5.11) we have

$$\omega\left(\frac{1}{n_i}\right) \leqq 2^{j-i}\omega\left(\frac{1}{n_j}\right) \tag{5.14}$$

and from (5.10) we have

$$n_i\omega\left(\frac{1}{n_i}\right) \leqq 2^{i-j+1}n_{j-1}\omega\left(\frac{1}{n_{j-1}}\right). \tag{5.15}$$

Then

$$|f(x+h)-f(x)| = \sum_{i=1}^{j-1} 2^{-i}\omega\left(\frac{1}{n_i}\right)|g_{n_i}(x+h)-g_{n_i}(x)| +$$

$$+\sum_{i=j}^{\infty} 2^{-i}\omega\left(\frac{1}{n_i}\right)|g_{n_i}(x+h)-g_{n_i}(x)| \leqq \sum_{i=1}^{j-1}\frac{1}{\pi}\cdot 2^{3-i}\omega\left(\frac{1}{n_i}\right)n_i h+\sum_{i=j}^{\infty}2^{1+j-2i}\omega\left(\frac{1}{n_i}\right)=$$

$$=\sum_{i=1}^{j-1}\frac{1}{\pi}\cdot 2^{3-i}\cdot 2^{i-j+1}\omega\left(\frac{1}{n_{j-1}}\right)n_{j-1}h+\sum_{i=j}^{\infty}2^{1+j-2i}\omega\left(\frac{1}{n_j}\right)\leqq$$

$$\leqq\frac{1}{\pi}\sum_{i=1}^{j-1}2^{4-j}\omega\left(\frac{1}{n_{j-1}}\right)n_{j-1}h+\omega(h)\sum_{i=j}^{\infty}2^{1+j-2i}\leqq\frac{1}{\pi}\sum_{i=1}^{j-1}2^{4-j}2\omega(h)+\frac{2^{3-j}}{3}\omega(h)\leqq$$

$$\leqq\frac{1}{\pi}\cdot j\cdot 2^{5-j}\omega(h)+\frac{2^{3-j}}{3}\omega(h)\leqq\frac{1}{3}\omega(h)(j+1)2^{5-j}=\frac{1}{3}\omega(h)\cdot\frac{j+1}{2^{j-5}}.$$

Then since $n_j h\geqq 1$, $h\to 0$ implies $j\to\infty$ and we have

$$\lim_{h\to 0}\frac{|f(x+h)-f(x)|}{\omega(h)}\leqq\lim_{j\to\infty}\frac{1}{3}\cdot\frac{j+1}{2^{j-5}}=0.$$

Therefore, $f\in C_T^*(\omega)$.

Finally, we examine

$$L_{n_i}^T(f, Y, z_{n_i}) = \sum_{k=1}^{\infty}2^{-k}\omega\left(\frac{1}{n_k}\right)L_{n_i}^T(g_{n_k}, Y, z_{n_i}).$$

We note that

$$2^{-i}\omega\left(\frac{1}{n_i}\right)L_{n_i}^T(g_{n_i}, Y, z_{n_i}) = 2^{-i}\omega\left(\frac{1}{n_i}\right)\lambda_{n_i}^T(Y), \tag{5.16}$$

and from (5.12),

$$\left|\sum_{k=1}^{i-1}2^{-k}\omega\left(\frac{1}{n_k}\right)L_{n_i}^T(g_{n_k}, Y, z_{n_i})\right|\leqq\frac{2^{-i}}{3}\omega\left(\frac{1}{n_i}\right)\lambda_{n_i}^T(Y). \tag{5.17}$$

From (5.5) and (5.11), follows

$$\left|\sum_{k=i+1}^{\infty}2^{-k}\omega\left(\frac{1}{n_k}\right)L_{n_i}^T(g_{n_k}, Y, z_{n_i})\right|\leqq\sum_{k=i+1}^{\infty}2^{-k}\omega\left(\frac{1}{n_k}\right)\lambda_{n_i}^T(Y)\leqq \tag{5.18}$$

$$\leqq\frac{1}{3}\omega\left(\frac{1}{n_i}\right)\lambda_{n_i}^T(Y)\cdot\sum_{k=i+1}^{\infty}2^{-k}=\frac{2^{-i}}{3}\omega\left(\frac{1}{n_i}\right)\lambda_{n_i}^T(Y).$$

Therefore (5.16)—(5.18) and (5.12) give

$$L_{n_i}^T(f, Y, z_{n_i}) = \sum_{k=1}^{i-1} 2^{-k}\omega\left(\frac{1}{n_k}\right) L_{n_i}^T(g_{n_k}, Y, z_{n_i}) +$$

$$+ 2^{-i}\omega\left(\frac{1}{n_i}\right) L_{n_i}^T(g_{n_i}, Y, z_{n_i}) + \sum_{k=i+1}^{\infty} 2^{-k}\omega\left(\frac{1}{n_k}\right) L_{n_i}^T(g_{n_k}, Y, z_{n_i}) \geqq$$

$$\geqq 2^{-i}\omega\left(\frac{1}{n_i}\right) L_{n_i}^T(g_{n_i}, Y, z_{n_i}) - \left|\sum_{k=1}^{i-1} 2^{-k}\omega\left(\frac{1}{n_k}\right) L_{n_i}^T(g_{n_k}, Y, z_{n_i})\right| -$$

$$- \left|\sum_{k=i+1}^{\infty} 2^{-k}\omega\left(\frac{1}{n_k}\right) L_{n_i}^T(g_{n_k}, Y, z_{n_i})\right| \geqq \frac{2^{-i}}{3}\omega\left(\frac{1}{n_i}\right)\lambda_{n_i}^T(Y) > \frac{i}{3}.$$

I.e., $L_{n_i}^T(f, Y, z_{n_i}) \to \infty$ $(i \to \infty)$ and hence $\overline{\lim_{n\to\infty}} \|L_n^T(f, Y, x)\| = \infty$, thus completing the proof of Theorem 4.

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References

[1] J. SZABADOS, On the convergence of Lagrange interpolating polynomials in a certain class of functions (in Russian), *Acta Math. Acad. Sci. Hung.* **17** (1966), pp. 31—44.
[2] A. F. TIMAN, *Theory of approximation of functions of a real variable,* Macmillan Co. (New York, 1963), 631 pp.
[3] O. KIS, On the convergence of interpolation procedures on certain function spaces (in Russian), *MTA Mat. Kut. Int. Közl.* (Proceedings of the Math. Inst. of the Hungarian Acad. Sci.), **7** (1962), pp. 95—111.
[4] I. P. NATANSON, *Constructive Function Theory,* Vol. III, Ungar (New York, 1964), 173 pp.
[5] O. KIS—J. SZABADOS, Remarks on the convergence of the Lagrange interpolation (in Russian), *Acta Math. Acad. Sci. Hung.*, **16** (1965), pp. 389—430.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (1—2), (1974), pp. 47—53.

SYNTOPOGENOUS SPACES

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Introduction

In CSÁSZÁR's book [1], a category of objects is defined which contains the categories of topological spaces, of uniform spaces, and others. There is developed a theory of "syntopogenous spaces", structures which may be specialised by the addition of suitable extra axioms into either topologies, uniformities, proximity structures, semi-uniformities or quasi-ordoform structures.

In what follows, a problem posed by CSÁSZÁR in [1] is stated and solved by what amounts to an application of the adjoint functor theorem of [2]. We change the setting of syntopogenous spaces to a more natural category theoretic one: we presuppose some familiarity with category theory and we shall occassionally cite [1] in place of a proof, otherwise this paper is substantially self contained.

§ 1. Preliminary definitions

1.1. REMARK. A topology on a set E determines a partial order L on the subsets of E with $A \mathsf{L} B$ iff $A \subset \text{int}(B)$, where int (B) is the interior of B in the given topology. We start by considering order relations and aim, in particular cases, to recover the topology.

1.2. DEFINITION. L is a relation on a set E (we abuse language, we mean L is a subset of $\mathscr{P}(E)\times\mathscr{P}(E)$) satisfying some of the following conditions.

T1 : $\varnothing \mathsf{L} \varnothing$ and $E \mathsf{L} E$,

T2 : $A \mathsf{L} B \Rightarrow A \subset B$,

T3 : $X \subset A \mathsf{L} B \subset Y \Rightarrow X \mathsf{L} Y$,

Q : $X \mathsf{L} Y$ and $A \mathsf{L} B \Rightarrow X \cap A \mathsf{L} Y \cap B$ and $X \cup A \mathsf{L} Y \cup B$,

P : $X_j \mathsf{L} Y_j$, $\forall_j \in J \Rightarrow \bigcup_J X_j \mathsf{L} \bigcup_J Y_j$ where J is any indexing set.

If the order satisfies T1 to T3 inclusive, it is called *semitopogenous*; if it also satisfies Q it is called a *topogenous* order, and if it satisfies P as well it is called a *perfect* topogenous order.

Given a set of topogenous orders (all in the same set E) we take their intersection and union in the obvious way.

$$A(\bigcap_J \mathsf{L}_j)B \quad \text{iff} \quad \forall_j \in J: A\mathsf{L}_j B,$$

$$A(\bigcup_J \mathsf{L}_j)B \quad \text{iff} \quad \exists_j \in J: A\mathsf{L}_j B.$$

It is simple to verify that the intersection of topogenous orders is topogenous, while the union is not, being normally only semitopogenous.

In what immediately follows we fix the set E unless explicitly stated to the contrary.

1.3. DEFINITION. Given a semitopogenous order L, we may 'operate on it' by an operator q to obtain L^q, a topogenous order, as follows.

$$X\mathsf{L}^q Y \text{ iff } \exists X_i,\ 1 \leqq i \leqq n,\ \exists Y_j,\ i \leqq j \leqq m: X = \bigcup_1^n X_i,\ Y = \bigcap_1^m Y_j \text{ and } \forall_{i,j},\ X_i \mathsf{L} Y_j.$$

Then it is trivial to verify that L^q is topogenous and is universal for L in this property; i.e. L^q is finer than L and if L is topogenous and finer than L, it is also finer than L^q.

1.4. DEFINITION. Given $f: E \to E'$ a set map and L a topogenous order on E then we may define $f\mathsf{L}$, a topogenous order on E' by $X, Y \subset E'$, $Xf\,\mathsf{L}\,Y$ iff $f^{-1}(X)\,\mathsf{L}\,f^{-1}(Y)$ and $X \subset Y$. Dually if L is a topogenous order on E', $f^{-1}\mathsf{L}$ is defined by $\forall X, Y \subset E$, $Xf^{-1}\,\mathsf{L}\,Y$ iff $fX\,\mathsf{L}\,E' \setminus f(E \setminus Y)$. Again this is a topogenous order.

1.5. DEFINITION. An order family is a set of topogenous orders, all on the same set E. A *syntopogeny* is an order family S satisfying

S1 : ${}^1\mathsf{L}$ and ${}^2\mathsf{L} \in S \Rightarrow \exists\, \mathsf{L} \in S$

$: X\,{}^1\mathsf{L}\,Y$ or $X\,{}^2\mathsf{L}\,Y \Rightarrow X\mathsf{L}Y$,

S2 : $\forall \mathsf{L} \in S\, \exists\, '\mathsf{L} \in S: X\mathsf{L}Y \Rightarrow \exists Z: X'\mathsf{L}Z'\mathsf{L}Y$.

A *syntopogenous space* is a set E together with a syntopogeny on E.

Note that any topological space is a syntopogenous space with the single order obtained from the open sets as described. This is clearly topogenous and indeed perfect. Furthermore S1 is satisfied trivially and for S2 we interpolate int (Y).

1.6. DEFINITION. Given $f: E \to E'$ and S' a syntopogeny on E', then $\{f^{-1}\mathsf{L}: \mathsf{L} \in S'\}$ is an order family on E. Furthermore it is a syntopogeny. We call it $f^{-1}(S')$. If, however, we have a syntopogeny S on E, the order family $f(S) = \{f\mathsf{L}: \mathsf{L} \in S\}$ is not in general a syntopogeny: it generally violates S2.

1.7. DEFINITION. Let $f: E, S \to E', S'$ be a set map between syntopogenous spaces. We shall say that f is continuous iff $f^{-1}(S')$ is refined by S in the sense that, for any order $\mathsf{L}' \in f^{-1}S'$, there is a (finer) order $\mathsf{L} \in S$ such that

$$\forall X, Y \subset E, \quad X\mathsf{L}'Y \Rightarrow X\mathsf{L}Y.$$

It is clear that the composites of continuous maps are continuous, and so we have a category, $\mathscr{S}$yn, the objects of which are syntopogenous spaces and the maps of which are continuous maps.

§ 2. Császár's problem, and related problems

2.1. REMARK. Given a collection $\{S_j\}_J$ of syntopogenies on a set E, we try to obtain a syntopogeny finer than every S_j and universal in this repsect, i.e. a l.u.b. in the set of syntopogenies partially ordered by coarseness.

The set union $\bigcup_J S_j$ is a natural candidate, but this is not a syntopogeny since it manifestly fails S1 in general: it does however satisfy S2.

We therefore adjoin to $\bigcup_J S_j$ orders that are unions of finite sets of orders out of $\bigcup_J S_j$; typically $L_\Gamma = \bigcup_\Gamma L_\gamma$ for some finite set Γ.

But these are not topogenous, so we q them, the revised $L_\Gamma = (\bigcup_\Gamma L_\gamma)^q$, Γ finite. The set of all such L_Γ is an order family containing $\bigcup_J S_j$ (Γ the singleton sets), satisfies S1 and S2, and by construction is universal in this respect.

We call the operator that takes q of all finite unions g and write:

$$\bigvee_J S_j = g(\bigcup_J S_j) \quad \text{or} \quad = (\bigcup_J (S_j))^g.$$

There are two noteworthy syntopogenies on any set E, both comprising a single topogenous order, both in fact topologies, viz the discrete and the indiscrete. Clearly the discrete syntopogeny is finer than all syntopogenies and the indiscrete syntopogeny is coarser than all syntopogenies.

It follows immediately that we can define $\bigwedge$ on the partially ordered set of syntopogenies on E by:

$$\bigwedge_J S_j = \bigvee_{S' \alpha S_j,\ \forall j} S',$$

where $S' \alpha S_j \forall_j$ means we take the join of all syntopogenies S' *coarser* than every syntopogeny S_j. Evidently we have a complete lattice. Császár (P105) notes that given a map $f: E,\ S \to E'$ we can define an image syntopogeny fS by

$$fS = \bigvee_{f^{-1} S' \alpha S} S'.$$

This makes f "just continuous". Is there an explicit form for fS in terms of f and S?

A related problem: we have the existence of $\bigwedge_J \{S_j\}$ guaranteed but only in a somewhat oblique way. Can we find an explicit construction for it?

Other problems: what is the role of the topologies among the syntopogenies, what are the category theoretic properties of $\mathscr{S}$yn? Is $\mathscr{S}$yn complete, cocomplete? Is it closed in the sense of admitting "function spaces" as objects? It is plain that the unstated, underlying problem is to find out if $\mathscr{S}$yn is a sufficiently 'nice' category to justify spending any time working inside it. While we do not presume to have a solution to this last problem, we are able to show that $\mathscr{S}$yn has some desirable features.

The operator g assigns to each order family satisfying S2 but not S1 a universal syntopogeny; we show that there is a well defined "dual" operator which we call h that assigns to each order family satisfying S1 but not S2 a universal syntopogeny. These operators are of course functors between the appropriate categories and it is convenient to construe them as such. This necessitates some definitons which follow in the next section.

§ 3. The category theoretic formulation

3.1. DEFINITION. Let O denote any order family (of topogenous orders), O' another. Then a set map $f:E,\ O\to E',O'$ is *continuous* when $f^{-1}(O')$ is coarser than O in the sense that for each $f^{-1}L'$ in $f^{-1}(O')$ there is a L in O such that

$$X,Y\subset E,\quad X(f^{-1}L')Y\Rightarrow XLY.$$

$f^{-1}(O')$ is of course a perfectly well defined order family, when O' satisfies SJ, so does $f^{-1}(O')$, for $J=1,2$.

Let $\mathscr{P}\mathrm{re}\mathscr{S}\mathrm{yn}$ denote the category having as objects sets with order families, and as maps the continuous maps.

Let $\mathscr{S}1$ denote the category having as objects sets with S1 order families and as maps the continuous maps, let $\mathscr{S}2$ denote the category having as objects sets with S2 order families and as maps the continuous maps. Then $\mathscr{S}1$ and $\mathscr{S}2$ are full subcategories of $\mathscr{P}\mathrm{re}\mathscr{S}\mathrm{yn}$ and $\mathscr{S}\mathrm{yn}$ is the "intersection" category.

There are, more formally, a collection of (faithful) forgetful functors. The category $\mathscr{T}\mathrm{op}$ of topological spaces and continuous maps is a subcategory of $\mathscr{S}\mathrm{yn}$ by another forgetful functor. We are concerned with the existence of adjoints to these functors.

3.2. REMARK. The operator g now determines a functor $\mathscr{G}:\mathscr{P}\mathrm{re}\mathscr{S}\mathrm{yn}\to\mathscr{S}1$ which assigns to $E,\ O$ the object $E,\ O^g$ where O^g is the S1 order family of topogenous orders $L_\Gamma=(\bigcup_\Gamma L_\gamma)^g$ over all finite sets Γ (of L_γ) in O. It is easy to see that if $f:E,O\to E',O'$ is a map in $\mathscr{P}\mathrm{re}\mathscr{S}\mathrm{yn}$ then $\mathscr{G}f$, the same set map, is a continuous map from E,O^g to $E',(O')^g$ in $\mathscr{S}1$.

If $\mathscr{F}1:\mathscr{S}1\to\mathscr{P}\mathrm{re}\mathscr{S}\mathrm{yn}$ denotes the forgetful functor we note that

(1) $\mathscr{G}\circ\mathscr{F}1$ is the identity on $\mathscr{S}1$,

(2) there is a natural transformation from $\mathscr{F}1\circ\mathscr{G}$ to the identity: for each set E, O any order family (O^g) is finer than O and so the identity map $E,(O^g)\to E,O$ is continuous.

3.3. DEFINITION. $\mathscr{A}$ is an order family on a set E, $L\in\mathscr{A}$ is *locally* S2 *in* $\mathscr{A}$ *at* A,B iff

(1) ALB,

(2) for every dyadic rational $\frac{p}{2^q}\in[0,1]$, $0\leqq p\leqq 2^q-1$, $q\in\mathbf{N}$ there is an order L_q in $\mathscr{A}$ and sets $\frac{C_p}{2^q}$, $\frac{C_{p+1}}{2^q}$, and $\frac{C_p}{2^q}L_q\frac{C_{p+1}}{2^q}$ $\forall p:0\leqq p\leqq 2^q-1$; with $C_0=A$, $C_1=B$, $L_0=L$.

3.4. REMARK. We shall call the interpolating sets together with the interpolating orders L_p an *insert* for L at A, B.

The process of "making an order family S2" we now regard as an inspection of each "piece" between A and B of L (with $A\,L\,B$) occuring in $\mathscr{A}$, and a rejection of those pieces not admitting inserts.

3.5. DEFINITION. The elementary topogenous order $\underset{AB}{L}$, $A\subset B\subset E$ is the topogenous order defined by

$$X \underset{AB}{L} Y \quad \text{iff} \quad X\subset A\subset B\subset Y \quad \text{or} \quad X=\varnothing \quad \text{or} \quad Y=E.$$

3.6. DEFINITION. For $\mathscr{A}$ an order family on E, let $S=\{L_0, L_1, L_2, \ldots\}_{\mathbf{N}}$ be a sequence of topogenous orders from $\mathscr{A}$. A *fragment of* S is a system of sets $\{C_d\}_D$ indexed by the dyadic rationals D in $[0, 1]$ such that $\forall n\in\mathbf{N}$, $\forall p: 0\leqq p\leqq 2^n-1$,

$$\frac{C_p}{2^n} L_n \frac{C_{(p+1)}}{2^n}; \quad \text{i.e. an insert for } L_0 \text{ at } C_0, C_1.$$

3.7. REMARK. All sequences in $\mathscr{A}$ have fragments: $C_d=\varnothing$ $\forall d\in D$ is one. If the sequence $S=(L_0, L_1, L_2, \ldots)$ admits a non-trivial fragment then so does any right segment of S (i.e. a subsequence of S containing L_{r+1} whenever it contains L_r).

3.8. DEFINITION. For a sequence S of elements of $\mathscr{A}$ let $\mathscr{F}_S$ denote the set of all fragments of S. If $f\in\mathscr{F}_S$ then let $\underset{f}{L}$ denote the elementary order $\underset{C_0,C_1}{L}$ where C_0 and C_1 are the obvious sets in f.

3.9. DEFINITION. For any order family $\mathscr{A}$, let $\mathscr{A}^h$ denote the order family given by

$$L\in\mathscr{A}^h \quad \text{iff} \quad L=\bigcup_{\mathscr{F}_S f} \underset{f}{L}$$

for some sequence S of elements of $\mathscr{A}$, for the set of all fragments $\mathscr{F}_S$ of S.

3.10. REMARK. If $\mathscr{A}$ is already S2 then whenever $L\in\mathscr{A}$ and $X\,L\,Y$ we have a sequence in $\mathscr{A}$ with $L=L_0$ so $\underset{XY}{L}$ is in $\mathscr{F}_S$, and we have such an elementary order whenever $X\,L\,Y$, hence $\mathscr{A}^h$ refines $\mathscr{A}$. Conversely suppose we have $L\in\mathscr{A}^h$ then there is a sequence $\{L_0, L_1, L_2, \ldots\}$ in $\mathscr{A}$ and whenever $X\,L\,Y$ we have a fragment $\{C_d\}_D$ with $C_0=X$ and $C_1=Y$ and $C_0\,L_0\,C_1$. Hence $\mathscr{A}$ refines $\mathscr{A}^h$ and we have shown that:

3.11. PROPOSITION. *If $\mathscr{A}$ is* S2 *then $\mathscr{A}^h$ is equivalent to $\mathscr{A}$ in the category $\mathscr{S}2$.*

§ 4. Solutions to some problems

4.1. PROPOSITION. *$\mathscr{A}^h$ is an* S2 *order family for $\mathscr{A}$ any order family.*

PROOF. First we must show that $\mathscr{A}^h$ is an order family, i.e. that $L\in\mathscr{A}^h$ is topogenous. It is a union of elementary topogenous orders, hence semitopogenous.

Let $X \mathsf{L} Y$ and $A \mathsf{L} B$ hold for $\mathsf{L} \in \mathscr{A}^h$, and let L be the union of elementary orders derived from the sequence $S = \{\mathsf{L}_0, \mathsf{L}_1, \mathsf{L}_2, \ldots\}$ with

$$X = C_0, \quad Y = C_1, \quad A = C_0^1, \quad B = C_1^1$$

and $\{C_j\}_D$, $\{C_j'\}_D$ the two fragments. It is plain that

$$\left(\frac{C_p}{2^q} \cap \frac{C_p^1}{2^q}\right) \mathsf{L}_q \left(\frac{C_{p+1}}{2^q} \cap \frac{C_{p+1}^1}{2^q}\right) \quad \forall q \in \mathbf{N}, \quad \forall p : 0 \leqq p \leqq 2^q - 1$$

defines an "intersection fragment". Similarly there is a union fragment and it follows immediately that L is topogenous.

$\mathscr{A}^h$ is S2: $\mathsf{L} \in \mathscr{A}^h$ is derived from a sequence starting with L_0; the sequence starting with L_1 clearly gives rise to an order $\mathsf{L}' \in \mathscr{A}^h$ and the result follows immediately.

4.2. REMARK. The following propositions require only routine verification of a somewhat tedious and unenlightening kind. We therefore omit the proofs.

4.3. PROPOSITION. *If $\mathscr{A} \in \mathscr{S}1$, $\mathscr{A}^h$ is a syntopogeny.*

4.4. PROPOSITION. *If $\mathscr{B} \in \mathscr{S}2$ and $\mathscr{B}$ is coarser than $\mathscr{A}$, then $\mathscr{B}$ is a coarser than $\mathscr{A}^h$.*

4.5. PROPOSITION. *There is a functor $\mathscr{H} : \mathscr{P}\mathrm{re}\mathscr{S}\mathrm{yn} \to \mathscr{S}2$; $\mathscr{H}(E, \mathscr{A}) = (E, \mathscr{A}^h)$. If $\mathscr{F}2 : \mathscr{S}2 \to \mathscr{P}\mathrm{re}\mathscr{S}\mathrm{yn}$ denotes the forgetful functor, then $\mathscr{H} \circ \mathscr{F}2$ is naturally equivalent to the identity and there is a natural transformation from the identity to $\mathscr{F}2 \circ \mathscr{H}$.*

$\mathscr{H}$ is a left adjoint of $\mathscr{F}2$ making $\mathscr{S}2$ a reflective subcategory of $\mathscr{P}\mathrm{re}\mathscr{S}\mathrm{yn}$.

4.6. REMARK. The duality here enables one to correctly infer that

$$\mathscr{H} \circ \mathscr{G} : \mathscr{P}\mathrm{re}\mathscr{S}\mathrm{yn} \to \mathscr{S}\mathrm{yn}$$

is *not* naturally equivalent to $\mathscr{G} \circ \mathscr{H}$; or in our terminology, g and h fail to commute. It is not hard to conclude that the lattice of syntopogenies is not distributive.

We may now readily solve the problem cited above due to Császár.

4.7. PROPOSITION. $fS = \bigvee_{f^{-1}S' \alpha S} S' \sim \{f\mathsf{L} : \mathsf{L} \in S\}^h$ *where α denotes the coarseness relation and $\sim$ the corresponding equivalence.*

PROOF. The result follows immediately from universality considerations.

4.8. PROPOSITION. *$\{S_j : j \in J \neq \varnothing\}$ a set of syntopogenies on a set E, let $\bigcap_J S_j$ denote the family of topogenous orders given by $\mathsf{L} \in \bigcap_J S_j$ iff $\mathsf{L} = \bigcap_J \mathsf{L}_j$, $\forall \mathsf{L}_j \in S_j$. Then $\bigcap_J S_j$ is not* S2 *but*

$$(\bigcap_J S_j)^h \sim \bigwedge S_j.$$

PROOF. Again an argument from universality suffices.

§ 5. Concluding remarks

We draw attention to the functor

$$T: \mathcal{S}\mathrm{yn} \to \mathcal{J}\mathrm{op}$$

which to the syntopogeny $\mathcal{S}$ on a set E assigns the topology $\mathcal{S}^{\mathrm{tp}}$ in the notation of [1]. It is stated in [1] that $\mathcal{S}^{\mathrm{tp}}$ is a topology finer than S and universal in this respect. This makes $\mathcal{J}\mathrm{op}$ a coreflective subcategory of $\mathcal{S}\mathrm{yn}$ (another way of saying the same thing). We note that $\mathcal{S}\mathrm{yn}$ is complete and cocomplete, by easy constructions. The order family operations of [1] are but functors under another name: there is no particular merit in rechristening them, but there would seem to be a uniform method of constructing them as adjoints to various forgetful functors.

I am obliged to the referee for pointing out to me the paper by HEUCHENNE, [3], which also treats of Császár's problem, and also for suggesting a simplification in the definition of a fragment which has, I believe, clarified the presentation. I wish to thank Professor C. T. C. Wall for drawing the problem to my attention.

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References

[1] Á. CSÁSZÁR, *Foundations of General Topology*, Pergamon, 1963.
[2] P. FREYD, *Abelian Categories*, Harper Row, 1964.
[3] C. HEUCHENNE, Étude comparée des topologies, proximités et uniformités par relations binaires, *Bull. Soc. Roy. Sci. Liège*, **35** (1966), pp. 417, 520.

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ОБОБЩЕНИЕ ТЕОРЕМЫ ОБ АНАЛИТИЧЕСКОМ ВИДЕ ХАРАКТЕРИСТИК ОБОБЩЕННЫХ ОДНОЛИНЕЙНЫХ СИСТЕМ МАССОВОГО ОБСЛУЖИВАНИЯ ДЛЯ ОДНОГО ТИПА ПРИОРИТЕТНЫХ СИСТЕМ

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I. Введение

В книге Б. В. Гнеденко и И. Н. Коваленко „Введение в теорию массового обслуживания" была рассмотрена и доказана теорема об аналитическом виде характеристик обобщенных однолинейных систем массового обслуживания, действие которых описывается кусочно-линейным марковским процессом. Напомним, что кусочно-линейным процессом называется марковский случайный процесс

$$\zeta(t) = \{\nu(t);\ \xi_1(t), \xi_2(t), \ldots, \xi_{|\nu(t)|}(t)\} \tag{1}$$

где $\nu(t) \in N$ (N конечное или счётное множество) дискретная компонента. Каждому $\nu \in N$ сопоставлено в соответствие неотрицательное целое число $|\nu|$. Для нулевого состояния $\nu = 0$: $|\nu| = 0$, для остальных $|\nu| > 0$. $\xi_j(t)$ — непрерывные компоненты. После некоторого начального момента $\zeta(t)$ изменяется в соответствии с статистическим законом: для каждого $\nu \neq 0$ задан постоянный вектор

$$\alpha_\nu = \{\alpha_{\nu_1}, \alpha_{\nu_2}, \ldots, \alpha_{\nu_{|\nu|}}\}.$$

В $\nu(t) = \nu$ переменные $\xi_j(t)$, $1 \leqq j \leqq |\nu|$ убывают со скоростями, равными соответственно

$$\frac{d\xi_j(t)}{dt} = -\alpha_{\nu_j}. \tag{2}$$

Если $\xi_j(t)$ обращается в нуль и при этом мы имеен состояние ν, то новое значение $\nu(t)$ становится равным μ с вероятностью $p_{\nu\mu}^{(j)}$, где $\{p_{\nu\mu}^{(j)}\}$ — распределение вероятностей на $N = \{\nu\}$. Обратившаяся в нуль компонента (1) вычёркивается, следовательно

$$|\mu| = |\nu| - 1.$$

Кроме изменений по (2) и скачков в моменты достижения переменными нулевого значения могут происходить спонтанные скачки $\zeta(t)$. За h система может с вероятностью $\lambda_{\nu\mu}h + o(h)$ перейти из ν в μ, где $|\mu| > |\nu|$, ($\lambda_\nu = \sum_\mu \lambda_{\nu\mu}$). При этом к $\xi_1(t), \xi_2(t), \ldots, \xi_{|\nu|}(t)$ справа приписываются неотрицательные переменные $\eta_1, \eta_2, \ldots, \eta_{|\mu|-|\nu|}$ независимые от течения процесса до рассматриваемого момента и распределенные по закону

$$H(x_1, x_2, \ldots) = \mathbf{P}\{\eta_1 < x_1, \eta_2 < x_2, \ldots\}.$$

В упомянутой книге доказывается для кусочно-линейных марковских процессов следующая эргодическая теорема: если множество состояний процесса $\nu(t)$ конечно и если для любого $\nu \neq 0$ справедливо неравенство

$$\sum_{j=1}^{|\nu|} \alpha_{\nu_j} > 0$$

причем все η_j имеют конечные математические ожидания, то существует эргодическое распределение процесса $\zeta(t)$

$$F(\nu, x) = \lim_{t\to\infty} F(\nu, t, x), \quad \sum_{\nu} \lim_{x_1, x_2, \ldots \to \infty} F(\nu, x) = 1,$$

где

$$F(\nu, t, x) = F(\nu, t;\ x_1, x_2, \ldots, x_{|\nu|}) = \mathbf{P}\{\nu(t) = \nu;\ \xi_j(t) < x_j,\ 1 \leqq j \leqq |\nu|\}.$$

По [1] мы предполагаем далее выполнение свойства упорядоченности, но в случае поступления требования более высокого приоритета, обслуживающий прибор сразу перейдет к обслуживанию этого требования. Пусть $\{H_r^k(x)\}$ — одномерные функции распределения; используя понятие „рациональной суперпозиции“ была доказана теорема: в сформулированных условиях существует эргодическое распределение процесса

$$p_\nu = \lim_{t\to\infty} \mathbf{P}\{\nu(t) = \nu\},$$

где $\{p_\nu\}$ — рациональные суперпозиции функций распределения $\{H_r^k(x)\}$ и параметров $\{\lambda_{\nu\mu}\}$, $\{\alpha_\nu\}$.

В [1] был заметен тот факт, что для некоторых типов систем массового обслуживания характеристики обслуживания выражаются с помощью рациональных функций от некоторых параметров и производных преобразования Лапласа в некоторых точках. Здесь же был поставлен вопрос: насколько широк класс систем массового обслуживания, для которых справедливо аналогичное обстоятельство. Оказалось, что такое утверждение остается в силе для обобщенных однолинейных систем, функционирование которых описывается кусочно-линейным марковским процессом.

Используя обозначения и результаты цитируемой статьии мы теперь обсудим, что можно ли перенести эти утверждения для некоторого специального типа приоритетной системы. При выводе формул мы ограничимся случаем, когда обслуживается требование более низкого приоритета и за время обслуживания поступает требование более высокого приоритета. Мы можем достичь дальнейшего сокращения, если функционирование обеих систем описывается кусочно-линейным процессом одинакового типа. Мы будем использовать обозначения, введенные в [1] (стр. 343—353).

2. Рассмотрение приоритетной системы

Примем обычный случай приоритетных систем при наличии двух приоритетов. Требования первого типа обслуживаются независимо от требований второго типа; если поступает требование первого типа, то обслуживающий прибор моментально перейдет к его обслуживанию, независимо от того, что он

был в состоянии покоя или обслуживал требование второго типа. Требование второго типа обслуживается, пока не поступает требование первого типа; если поступило, то система обслуживает его до конца, после этого продолжает обслуживание требования второго типа. Теперь зависимо от того, как продолжается обслуживание, будем иметь различные случаи. Так как обслуживание требований второго типа не оказывает влияния на обслуживание требования первого типа, так выведенные в [1] формулы остаются в силе для требований первого типа. Аналогичное утверждение имеет место для требований второго типа при условии, что за время его обслуживания требование первого типа не поступило.

Предположим теперь, что при обслуживании требований второго типа мы находимся на отрезке (ν, μ) (это означает, что обслуживание начинается в состоянии ν, и заканчивается в состоянии μ). Возможны два случая: 1. за время обслуживания требование первого типа не поступает — в этом случае обслуживание осуществляется так же, как для требований первого типа, 2. за время обслуживания требования второго типа поступило требование первого типа. В этом случае если уже выполненное обслуживание учитывается и во время обслуживания требования первого типа никаких изменений не будет, т. е. мы с вероятностью 1 останемся в состоянии, куда мы дошли, то характеристики обслуживания не меняются, лишь по оси времени мы смещаемся настолько, сколько времени было затречено на обслуживание требование первого типа.

Рассмотрим теперь более общую ситуацию. Пусть у нас заданы два объекта, которые требуют обслуживания, причем первый из них имеет более высокий приоритет, и их изменения могут быть описаны кусочно-линейным марковским процессом. Обслуживание требования первого типа осуществляется обычным образом, если поступает сигнал о потребности в обслуживании, мы сразу же перейдем к его обслуживанию. Таким образом характеристики обслуживания могут быть определены по формулам [1]. Перейдём теперь к требованиям второго типа. Пусть идет обслуживание требования второго типа и при этом находимся на отрезке (ν, μ). Если за время обслуживания поступает требование первого типа, то сразу перейдем к его обслуживанию и доводим до конца. После этого продолжим обслуживание требование второго типа, но уже затраченная работа не учитывается, а все начинается заново на отрезке (ν, μ).

Пусть $l_{\nu\mu}(x)$ означает условную вероятность того, что обслуживание, начавшееся в состоянии ν, закончится в состоянии μ при условии, что величина работы, необходимой для выполнения данного обслуживания, составляет x единиц. Как известно из [1], $l_{\nu\mu}(x)$ удовлетворяют системе дифференциальных уравнений:

$$l'_{\nu\mu}(x)+\frac{\lambda_\mu-L_{\mu\mu}}{\alpha_\mu}\,l_{\nu\mu}(x)=\sum_{\substack{|\mu'|=|\nu|\\ \mu'\neq\mu}}\frac{\lambda_{\mu'\mu}+L_{\mu'\mu}}{\alpha_{\mu'}}\,l_{\nu\mu'}(x)\quad(|\nu|-1\leqq|\mu|\leqq|\nu|)$$

при начальный условиях $l_{\nu\mu}(0)=\delta_{\nu\mu}$, где $\delta_{\nu\mu}$ — символ Кронекера. Таким образом

$$l_{\nu\mu}(x)=\sum_r a_{\nu\mu r}x^{b_{\nu\mu r}}\cdot\exp\left(-\varrho_{\nu\mu r}x\right) \tag{3}$$

где $b_{\nu\mu r}$ — неотрицательные целые числа.

Рассмотрим теперь, как можно определить $l_{\nu\mu}$. Для требований первого типа и для требований второго типа в том случае, когда не поступает требование первого типа, остается в силе формула, приведенная в [1]:

$$l_{\nu\mu} = \sum_{|\mu'|=|\nu|} p_{\mu'\mu} \int_0^\infty l_{\nu\mu'}(x)\, dH_\nu(x).$$

Пусть теперь за время обслуживания требования второго типа на отрезке (ν, μ) поступило требование первого типа, когда на отрезке (ν, μ) мы уже затратили на обслуживание работу u. Используя известное свойство показательного распределения (как известно по [1] длина промежутка между двумя моментами поступлений требований первого типа имеет показательное распределение и в этом случае распределение времени от некоторого промежуточного момента до поступления следующего требования первого типа также является показательным) имеем:

$$l_{\nu\mu} = \sum_{|\mu'|=|\nu|} p_{\mu'\mu} \int_0^\infty \left\{ \frac{\lambda_0}{\alpha_\nu} \int_0^x \exp\left(-\lambda_0 \frac{u}{\alpha_\nu}\right) du \right\} l_{\nu\mu'}(x)\, dH_\nu(x).$$

Здесь $\exp\left(-\lambda_0 \dfrac{u}{\alpha_\nu}\right)$ есть вероятность непоступления требования первого типа от начала цикла до момента $\dfrac{u}{\alpha_\nu}$, а $\lambda_0 du/\alpha_\nu$ вероятность поступления после этого требования первого типа.

$$\frac{\lambda_0}{\alpha_\nu} \int_0^x \exp\left(-\lambda_0 \frac{u}{\alpha_\nu}\right) du = 1 - \exp\left(-\lambda_0 \frac{x}{\alpha_\nu}\right).$$

Так как приведенное выше выражение представляет собой некоторую экспоненту, то для $l_{\nu\mu}$ и в случае поступления требования первого типа остается в силе представление

$$l_{\nu\mu} = \sum_{|\mu'|=|\nu|} p_{\mu'\mu} \sum_r a_{\nu\mu r} (-1)^{b_{\nu\mu r}} h_\nu^{(b_{\nu\mu r})}(\varrho_{\nu\mu r}).$$

Определим $t_{\nu\mu}^{\nu'} l_{\nu\mu}$ при условии, что поступило требование первого типа. В состоянии ν' мы можем быть до и после прерывания при обслуживании требования второго типа:

$$t_{\nu\mu}^{\nu'} l_{\nu\mu} = \sum_{|\mu'|=|\nu|} p_{\mu'\mu} \int_0^\infty dH_\nu(x) \left\{ \frac{\lambda_0}{\alpha_\nu} \int_0^x e^{-\lambda_0 \frac{u}{\alpha_\nu}} \int_0^u \left[\frac{1}{\alpha_{\nu'}} l_{\nu\nu'}(z) \times \right.\right.$$

$$\times \left(1 - \sum_{\substack{|\varepsilon|=|\nu|-1 \\ \varepsilon=\mu'}} l_{\nu'\varepsilon}(u-z)\right) + \sum_{|\nu_1|=|\nu_2|=|\nu|} \frac{L_{\nu_1\nu_2} T_{\nu_1\nu_2}^{\nu'}}{\alpha_{\nu_1}} l_{\nu\nu_1}(z) \left(1 - \sum_{\substack{|\varepsilon|=|\nu|-1 \\ \varepsilon=\mu'}} l_{\nu_2\varepsilon}(u-z)\right) \Bigg] dz\, du +$$

$$\left. + \int_0^\infty \left[\frac{1}{\alpha_{\nu'}} l_{\nu\nu'}(z) l_{\nu'\mu'}(x-z) + \sum_{|\nu_1|=|\nu_2|=|\nu|} \frac{L_{\nu_1\nu_2} T_{\nu_1\nu_2}^{\nu'}}{\alpha_{\nu_1}} l_{\nu\nu_1}(z) l_{\nu_2\mu'}(x-z) \right] dz \right\}.$$

Здесь мы используем то, с практической точки зрения весьма естественное предположение, что после выполнения работы u мы будем находиться в сос-

тоянии $\varepsilon|\varepsilon| \geqq |\nu|$, иначе наш цикл кончился бы после затрачивания работы меньше чем x (по условию необходимая для этого работа равна x), и переход в состояние μ, $|\mu|=|\nu|-1$ возможен лишь после выполнения некоторой работы.

Ввиду того, что $l_{\nu\mu}(x)$ имеет вид (3), $t^{\nu'}_{\nu\mu} l_{\nu\mu}$ будут суммой подобных функций и некоторых констант. Определим теперь $L_{\nu\mu}$ и $T^{\nu'}_{\nu\mu}$, при условии, что на протяжении цикла (ν, μ) поступило требование первого типа.

$$L_{\nu\mu} = \sum_{|\mu'|=|\nu|+1} \lambda_{\nu\mu'} \tilde{L}_{\mu'\mu},$$

где $\tilde{L}_{\mu'\mu}$ вероятность того, что цикл, начавшийся в состоянии μ', закончится переходом в μ, $|\mu|=|\nu|-1$. Для $\tilde{L}_{\nu\mu}$ имеем систему уравнений

$$\tilde{L}_{\nu\mu} = l_{\nu\mu} + \sum_{|\mu'|=|\nu|} l_{\nu\mu'} \tilde{L}_{\mu'\mu},$$

которая согласно теории марковских цепей имеет единственное решение. $\tilde{L}_{\nu\mu}$ определяются через $\{l_{\nu\mu'}\}$, следовательно имеют форму соответствующую условиям теоремы. Мы имеем также

$$\lambda_\nu T^{\nu'}_{\nu\mu} = \sum_{|\mu'|=|\nu|+1} \lambda_{\nu\mu'} \tilde{T}^{\nu'}_{\mu'\mu},$$

где постоянные определяются системой уравнений

$$\tilde{L}_{\nu\mu} \tilde{T}^{\nu'}_{\nu\mu} = l_{\nu\mu} t^{\nu'}_{\nu\mu} + \sum_{|\mu'|=|\nu|} [l_{\nu\mu'} \tilde{L}_{\mu'\mu 1} (t^{\nu'}_{\nu\mu'} + T^{\nu'}_{\mu' 1}) + l_{\nu\mu' 1} \tilde{L}_{\mu'\mu} (t^{\nu'}_{\nu\mu' 1} + T^{\nu'}_{\mu'\mu})], \tag{4}$$

где обозначения с дополнительным индексом 1 означают параметры при непоступлении требования первого типа. Формула (4) несколько отличается от формулы, приведенной в [1], так как поступление требования первого типа возможно либо между состояниями ν и μ', либо μ' и μ. Это связано с тем дополнительным предположением, что если одно требование первого типа в (ν, μ) поступило, то второго поступления уже не будет.

Таким образом мы получили аналогичный приведенным в [1] рекуррентный процесс и распространили утверждения теоремы [1] для описанного типа приоритетных систем.

3. Выводы

Характеристики обобщенных однолинейных систем массового обслуживания в настоящей статье были определены для того случая, когда на протяжении цикла обслуживания требования второго типа поступило требование первого типа. Требования первого типа обслуживаются, как будто требований второго типа не было, так выведенные в [1] формулы вполне определяют соответствующие характеристики. Для требований второго типа возможны два случая: требование первого типа не поступило или поступило. Так как мы уже знаем эргодическое распределение для требований первого типа так вероятность ему побывать в нулевом состоянии равна p_0, а вне этого состояния $1-p_0$. Мы определим характеристики в обоих случаях и умножив их соответственно на p_0 и $1-p_0$ и сложив их получим нужные нам характеристики. Если у нас требования по существу имеют одинаковый тип (интерпретации

кусочно-линейных марковских процессов совпадают, а только их приоритет различен), то мы можем ограничиться решением для требований первого типа, это перенесется без изменений для требований второго типа в случае, когда за время обслуживания требование первого типа не поступает. А для случая когда поступает, характеристики обслуживания определяются по формулам, приведенным в настоящей статье. Утверждения настоящей статьи могут быть перенесены также на такую систему, где за время прохождения цикла (ν, μ) будет не одно, а несколько прерываний.

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Литература

[1] Б. В. Гнеденко, И. Н. Коваленко, *Введение в теорию массового обслуживания*, изд. „Наука“, 1966.

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ON THE RATIONAL APPROXIMATION OF FUNCTIONS OF THE CLASS V_r

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Let us denote by V_r the class of functions, the r-th derivative of which is of bounded variation. TURÁN and SZÜSZ [1] have shown that the functions of the class V_1 can be approximated by means of rational functions of degree n better than by algebraic polynomials of the same degree. In [2] G. FREUD found the following estimation for the best uniform approximation $R_n(f)$ of the function f, defined on the interval [0, 1], by means of rational functions of degree n:

$$\sup_{V_0^1(f^{(r)})\leqq 1} R_n(f) = O\left(\frac{\ln^2 n}{n^{r+1}}\right) \tag{1}$$

where $V_0^1(g)$ denotes the variation of the function g on the interval [0, 1], $r>0$.

On the other hand it is easy to see that the right-hand side of (1) cannot be better than $O(1/n^{r+1})$. In this connection the question arises if the factor $\ln^2 n$ in the numerator of (1) can be removed.

In this note we shall obtain the following result: for every natural number $k>0$ the following estimate is valid:

$$R_n(f) \leqq C_{k,r} \frac{V_0^1(f^{(r)}) \overbrace{\ln \ln \ldots \ln}^{k} n}{n^{r+1}} \tag{2}$$

where $C_{k,r}$ is a constant, depending only on k and r.

The estimate (2) shows that the exact order in (1) is probably $O(V_0^1(f^{(r)})/n^{r+1})$.

The method of proof of (2) is a modification of the method of G. FREUD from [2].

Let R_n be the set of all rational functions of degree n and $R_n(f; \Delta)$ the best uniform approximation of the function f by means of elements of R_n in the interval $\Delta=[a, b]$:

$$R_n(f; \Delta) = \inf_{q \in R_n} \|f-q\|_{C[a, b]}.$$

We shall need the following

LEMMA 1. *Let*

$$\sup_{V_0^1(f^{(r)})\leqq 1} R_n(f; [0, 1]) = \Phi_n^r$$

Then

$$\sup_{V_a^b(f^{(r)})\leqq M} R_n(f; [a, b]) = (b-a)^r M \Phi_n^r.$$

PROOF. Let $V_0^b(f^{(r)}) \leqq M$, and consider the function $g(x)=b^{-r}M^{-1}f(bx)$. It is evident that $V_0^1(g^{(r)}) \leqq 1$ and consequently there exist rational functions $q_n \in R_n$ such that

$$\|g-q_n\|_{C[0,1]} \leqq \Phi_n^r.$$

Hence

$$\|g-q_n\|_{C[0,1]} = \max_{x\in[0,1]} |g(x)-q_n(x)| = \max_{x\in[0,b]} \left| \frac{f(x)}{b^r M} - q_n\left(\frac{x}{b}\right)\right| \leqq \Phi_n^r,$$

that is

$$\max_{x\in[0,b]} \left| f(x) - q_n\left(\frac{x}{b}\right) b^r M \right| \leqq b^r M \Phi_n^r. \tag{3}$$

From (3) follows

$$\sup_{V_0^b(f^{(r)})\leqq M} R_n(f;\ [0,b]) \leqq b^r M \Phi_n^r.$$

The converse inequality can be obtained in a similar way.

THEOREM 1. *Let*

$$\Phi_n^r = \sup_{V_0^1(f^{(r)})\leqq 1} R_n(f;\ [0,1]) \leqq \frac{\Psi_r(n)}{n^{r+1}}, \quad r>0.$$

Then

$$\Phi_n^r \leqq C_r \frac{\Psi_r(\ln^2 n)}{n^{r+1}}$$

where C_r is a constant, depending only on r.

PROOF. Let $V_0^1(f^{(r)}) \leqq 1$, and divide the interval $[0, 1]$ into at most $2m$ subintervals $[y_{i-1}, y_i]$, $i=1, \ldots, 2m$, with the following properties: in the interval $[y_{i-1}, y_i]$ we have either $y_i - y_{i-1} = \frac{1}{m^2}$ or $\frac{1}{m^2} \leqq y_i - y_{i-1} \leqq \frac{1}{m}$ and $V_{y_{i-1}}^{y_i}(f^{(r)}) \leqq \frac{1}{m}$.

Let us divide each interval of the second kind in 5 subintervals $[y_{i-1}, t_{i-1}]$, $[t_{i-1}, z_{i-1}]$, $[z_{i-1}, z_i]$, $[z_i, t_i]$ and $[t_i, y_i]$ in such a way that

$$t_{i-1}-y_{i-1} = z_{i-1}-t_{i-1} = t_i - z_i = y_i - t_i = \frac{1}{5m^2}.$$

Let the new intervals (which are at most $6m$) be

$$\Delta_i = [x_{i-1}, x_i], \qquad i = 1, \ldots, s, \quad s \leqq 6m.$$

Denote $\delta_i = \Delta_{i-1} \cup \Delta_i \cup \Delta_{i+1}$, $i=2, \ldots, s-1$. By construction, we have in the interval $\delta_i = [x_{i-2}, x_{i+1}]$ either

$$\frac{1}{m^2} \leqq x_{i+1} - x_{i-2} \leqq \frac{7}{5m^2}$$

or

$$\frac{1}{m^2} \leqq x_{i+1} - x_{i-2} \leqq \frac{1}{m} \quad \text{and} \quad V_{x_{i-2}}^{x_{i+1}}(f^{(r)}) \leqq \frac{1}{m}.$$

From Lemma 1 follows that there exist rational functions $q_{k,i} \in R_k$, $i=2, \ldots, s-1$, such that

$$\|f-q_{k,i}\|_{C[x_{i-2}, x_{i+1}]} \leqq \max\left\{\frac{1}{m^{r+1}}, \left(\frac{7}{5m^2}\right)^r\right\} \Phi_k^r \leqq \frac{C_r'}{m^{r+1}} \Phi_k^r. \tag{4}$$

From the result in [3] follows that there exist rational functions $\varphi_n(x)$ of degree $[C_r \ln^2 n]$, where C_r is a constant, depending only on r, such that

$$\begin{cases} |\varphi_n(x)| \leqq n^{-(2r+3)} & \text{for} \quad -1 \leqq x \leqq -n^{-(2r+3)} \\ 0 \leqq \varphi_n(x) \leqq 1 & \text{for} \quad |x| \leqq 1 \\ |1-\varphi_n(x)| \leqq n^{-(2r+3)} & \text{for} \quad n^{-(2r+3)} \leqq x \leqq 1. \end{cases} \tag{5}$$

Denote (see SZABADOS [4])

$$Q_i(x) = \frac{q_{k,i}(x)}{1+n^{-r-1} q_{k,i}^2(x)}, \qquad i = 2, \ldots, s-1.$$

Since $V_0^1(f^{(r)}) \leqq 1$, we can assume that $\|f\|_{C[0,1]} \leqq 1$. Consequently $\|q_{k,i}(x)\|_{C[\delta_i]} \leqq 2$, and we have

$$\|Q_i(x)-q_{k,i}(x)\|_{C[\delta_i]} = \left\| \frac{1}{n^{r+1}} \cdot \frac{q_{k,i}^3(x)}{1+\frac{1}{n^{r+1}} q_{k,i}^2(x)} \right\|_{C[\delta_i]} \leqq \frac{8}{n^{r+1}}. \tag{6}$$

On the other hand

$$\|Q_i(x)\|_{C[0,1]} \leqq n^{r+1}. \tag{7}$$

Let us consider the rational function

$$R(x) = Q_2(x) + \sum_{i=2}^{s-2} [Q_{i+1}(x)-Q_i(x)] \varphi_n(x-x_i).$$

$R(x)$ is of degree at most $(4k+[C_r \ln^2 n])s$.

Let $n > m$. For $x \in \Delta_j$, $2 \leqq j \leqq s-1$, we obtain from (5) and (7):

$$\left| \sum_{i=2}^{j-2} [Q_{i+1}(x)-Q_i(x)] \varphi_n(x-x_i) + Q_2(x) - Q_{j-1}(x) \right| \leqq \tag{8}$$

$$\leqq 2n^{r+1} \frac{(j-2)}{n^{2r+3}} = \frac{2(j-2)}{n^{r+2}}$$

and

$$\left| \sum_{i=j+1}^{s-2} [Q_{i+1}(x)-Q_i(x)] \varphi_n(x-x_i) \right| \leqq \frac{2(s-j+3)}{n^{r+2}}. \tag{9}$$

Denote $\alpha=\varphi_n(x-x_{j-1})$, $\beta=\varphi_n(x-x_j)$. From (5) follows that $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$.

For $n>m$ and $x\in\Delta_j$, $2\leqq j\leqq s-1$, we obtain (using (4)—(6), (8) and (9)):

$$|R(x)-f(x)| = \Bigg| \sum_{i=2}^{j-2} [Q_{i+1}(x)-Q_i(x)]\varphi_n(x-x_i)+Q_2(x)-$$

$$-Q_{j-1}(x)+Q_{j+1}(x)+[Q_j(x)-Q_{j-1}(x)]\varphi_n(x-x_{j-1})+$$

$$+[Q_{j+1}(x)-Q_j(x)]\varphi_n(x-x_j)+\sum_{i=j+1}^{s-2}[Q_{i+1}(x)-Q_i(x)]\varphi_n(x-x_i)-f(x)\Bigg| \leqq$$

$$\leqq \frac{2}{n^{r+1}}+|(1-\alpha)[Q_{j-1}(x)-q_{k,j-1}(x)]+(\alpha-\beta)[Q_j(x)-q_{k,j}(x)]+$$

$$+\beta[Q_{j+1}(x)-q_{k,j+1}(x)]|+|(1-\alpha)[q_{k,j-1}(x)-f(x)|+$$

$$+(\alpha-\beta)[q_{k,j}(x)-f(x)]+\beta[q_{k,j+1}(x)-f(x)]| \leqq$$

$$\leqq \frac{2}{n^{r+1}}+3\frac{8}{n^{r+1}}+C_r' m^{-r-1}\Phi_k^r.$$

Consequently, if $n>m$ and $x\in\Delta_j$, $2\leqq j\leqq s-1$, then

$$|R(x)-f(x)| \leqq \frac{26}{n^{r+1}}+C_r' m^{-r-1}\Phi_k^r. \tag{10}$$

If $n>m$ and $x\in\Delta_1$, we obtain from (4)—(9):

$$|R(x)-f(x)| \leqq |Q_2(x)-f(x)|+\Bigg|\sum_{i=2}^{s-1}[Q_{i+1}-Q_i]\varphi_n(x-x_i)\Bigg| \leqq \tag{11}$$

$$\leqq \frac{2}{n^{r+1}}+C_r' m^{-r-1}\Phi_k^r$$

and if $n>m$, $x\in\Delta_s$, we have

$$|R(x)-f(x)| \leqq \frac{2}{n^{r+1}}+|Q_{s-1}(x)-f(x)| \leqq \frac{2}{n^{r+1}}+C_r' m^{-r-1}\Phi_k^r. \tag{12}$$

From (10)—(12) we obtain if $n>m$

$$\|R(x)-f(x)\|_{C[0,1]} \leqq \frac{26}{n^{r+1}}+C_r' m^{-r-1}\Phi_k^r \tag{13}$$

and $R(x)$ is a rational function of degree at most $(4k+[C_r\ln^2 n])s$.

Setting in (13)

$$k=[\ln^2 n], \quad m=\left[\frac{n}{6(4k+[C_r\ln^2 n])}\right]$$

we obtain by the assumption of the theorem that

$$\|R(x)-f(x)\|_{C[0,1]} \leqq \tilde{C}_r\frac{\Psi_r(\ln^2 n)}{n^{r+1}} \tag{14}$$

where $\tilde{C}_r$ is a constant, depending only on r, and $R(x)$ is a rational function of degree n.

From (14) follows the statement of the theorem.

From the result of G. FREUD (1) and theorem 1 one immediately obtains

THEOREM 2. *For every integer $k>0$, $r>0$, there exists a constant $C_{k,r}$, depending only on k and r such that*

$$\sup_{V_0^1(f^{(r)})\leqq 1} R_n(f;\ [0,1]) \leqq C_{k,r}\frac{\overbrace{\ln\ln\ldots\ln}^{k} n}{n^{r+1}}.$$

From theorem 2 and lemma 1 follows the following

COROLLARY. *For every integer $k>0$, $r>0$, there exists a constant $C_{k,r}$, depending only on k and r such that*

$$R_n(f;\ [a,b]) \leqq C_{k,r}\frac{V_a^b(f^{(r)})\,(b-a)^r \overbrace{\ln\ldots\ln}^{k} n}{n^{r+1}}.$$

Let us mention at last that the method, used in the proof of Theorem 1, can be applied in other questions, connected with estimations of rational approximation of functions.

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References

]1] P. TURÁN, On the approximation of piecewise analytic functions by rational functions, *Современные проблемы теории аналитических функции* (Москва, 1966), Международная конф. по теории аналитических функции, Ереван, 1965.

[2] G. FREUD, Über die Approximation reeler Funktionen durch rationale gebrochene Funktionen, *Acta Math. Acad. Sci. Hung.*, **17** (1966), pp. 313—324.

[3] А. А. Гончар, Оценки роста рациональных функий и некоторые их приложении, *Мат. сборник*, **72 (114)** (1967), pp. 489—503.

[4] J. SZABADOS, Generalization of two theorems of G. Freud concerning the rational approximation, *Studia Sci. Math. Hung.*, **2** (1967), pp. 73—80.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (1—2), (1974), pp. 67—80.

ON CERTAIN LINEAR OPERATORS. VII
(A SUMMARY FROM NEW POINT OF VIEW. ESTIMATIONS FOR MECHANICAL QUADRATURES)

By

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To Professor P. Turán on his 60th birthday

1. Introduction

Let us consider the set of all continuous functions on the interval $[a, b]$ denoted by $C^{[a,b]}$ (or C) and the sequence of linear operators $T_n(f; x)$ mapping C into C. It is natural to investigate the convergence behaviour of this operator-sequence to $f(x)$. For example, we can ask the convergence property of $L_n(f; x)$, $S_n(f; x)$ or $H_n(f; x)$. ($L_n(f; x)$ is the n-th Lagrange interpolatory polynomial, $S_n(f; x)$ the n-th Fourier sum, $H_n(f; x)$ the n-th Hermite—Fejér interpolatory polynomial.)

A great number of authors investigated this question. A further problem is to estimate the difference $T_n(f; x)-f(x)$ by $\omega_m(f; t)$ and $\lambda_n(x)$, where $\omega_m(f; t)$ is the m-th modulus of smoothness of $f(x)$ and $\lambda_n(x)$ is the so-called Lebesgue function of the process. For example, we know that with the Chebyshev-nodes

$$|L_n(f; x)-f(x)| \leqq c_1 \ln n \cdot \omega_m\left(f; \frac{1}{n}\right) \qquad (n=2, 3, 4 \ldots), \tag{1.1}$$

and for Fourier-series

$$|S_n(f; x)-f(x)| \leqq c_2 \ln n \cdot \omega_m\left(f; \frac{1}{n}\right) \qquad (n=2, 3, 4 \ldots). \tag{1.2}$$

(Here and in what follows c_i denote absolute positive constants.)

Beside the convergence behaviour, the question of divergence is also very interesting, i.e. what kind of lower estimations exist, not for a function, but for a function-class or for special functions (see e.g. [1] and [2]).

Significant step was the application of $\omega(f; t)$ and $\lambda_n(x)$ for the lower estimation (e.g. [3], [4] and [5]). A special case of a theorem from [5] is the following

THEOREM 1.1 (O. KIS). *If the nodes are equidistant and $\omega(t)$ is a modulus of continuity such that* $\lim_{t=+0} (t/\omega(t))=0$ *then for a suitable function* $f(x) \in C(\omega)$

$$\overline{\lim_{n\to\infty}} \|L_n(f; x)-f(x)\| > \omega\left(\frac{1}{n}\right)\lambda_n. \tag{1.3}$$

(Here $C(\omega)=\{f(x); f(x)$ continuous on $[a, b]$ and $\omega(f; t)\leqq a(f)\omega(t)\}$. The $\|\cdot\|$ is the continuous norm, λ_n is the Lebesgue constant.)

The above mentioned papers [2]—[5] dealt with the interpolatory operators. In many papers I succeeded in generalizing these theorems for another operators and in proving some theorems of new type ([6]—[12]).

In the present paper I wish to give a survey of the common idea of these papers, to get a further generalization and investigate the mentioned papers from a new point of view for mechanical quadratures.

2. The common ideas

2.1. Let us denote by $\omega_m(t)$ a function such that

(i) $\omega_m(t)>0$ for $t>0$, $\omega_m(0)=0$, $\omega_m(T)\geqq\omega_m(t)$ for $T\geqq t$, $\omega_m(t)$ is a continuous function for $t\geqq 0$,

(ii) $\dfrac{t^m}{\omega_m(t)}$ is a monotone increasing function for $t\geqq 0$ ($m\geqq 0$ fixed integer number).

2.2. Let us denote by $C_P^{\{a,b\}}(\omega_m)$ the class of functions $f(x)$ having the following properties

(2.1) $f(x)$ is continuous on the corresponding interval $\{a, b\}$,

(2.2) $\omega_m(f; t)\leqq a_m(f)\omega_m(t)$ where $\omega_m(f; t)$ is the m-th modulus of smoothness of $f(x)$.

(2.3) If P is a set having the elements p_i where every p_i is a certain property, then $f(x)$ has these properties.

In many cases $P\equiv\varnothing$, but sometimes it has some elements. For example, we wish the 2π-periodicity of $f(x)$, or the existence of a continuous derivative, etc.

If $f(x)$ fulfils (2.1) and (2.2) we write $f(x)\in C^{\{a,b\}}(\omega_m)$, if $f(x)$ fulfils (2.1) and (2.3) we write $f(x)\in C_P^{\{a,b\}}$. (In many cases $U_n(f; x)$ are the unit operator E, i.e. $U_n(f;x)\equiv f(x)$ $(n=1,2,3,\ldots)$.

2.3. Let us denote by $\{T_n\}_{n=1}^{\infty}$ and $\{U_n\}_{n=1}^{\infty}$ linear (i.e. additive and continuous) operators on the set $C_P^{\{a,b\}}(\omega_m)$.

2.4. A. *Let us suppose that there exist certain functions* $g_n(x)$ $(n=1, 2, 3, \ldots)$ *for which we have the restrictions* (2.4)—(2.8):

$$g_n(x)\in C_P^{\{a,b\}}(\omega_m), \tag{2.4}$$

$$T_n(g_n; z_n)\geqq c_4\lambda_n(z_n) \quad \textit{for certain} \quad \{z_n\}_{n=1}^{\infty}\subset\{a, b\} \tag{2.5}$$

(*definition of* $\lambda_n(z_n)$),

$$|T_n(g_N; z_n)|\leqq c_5\lambda_n(z_n) \quad (N>n). \tag{2.6}$$

We mention that in almost every case we suppose

$$|g_n(x)|\leqq c_3 \quad \text{or} \quad |g_n(x)|\lesssim 1, \tag{2.7}$$ [1]

$$\omega_m(g_n; t)\leqq B_m t^m\delta_n^{-m} \tag{2.8}$$

where $\delta_n\searrow 0$ [2] for a certain $\{\delta_n\}$.

We can suppose that

$$|T_n(g_N; z_n)-U_n(g_N; z_n)|\leqq e_n\omega_m(\delta_n)\lambda_n(z_n) \quad (n>M(N)) \tag{2.9}$$

[1] $f_n(x)\lesssim g_n(x)$ if $f_n(x)=O(g_n(x))$; $f_n(x)\sim g_n(x)$ if $f_n(x)\lesssim g_n(x)$ and $g_n(x)\lesssim f_n(x)$ uniformly in x.

[2] $\delta_n\searrow 0$ means that $\delta_1\geqq\delta_2\geqq\delta_3\geqq\ldots\geqq\delta_n>0$ and $\lim_{n\to\infty}\delta_n=0$.

for certain $\{e_n\}_{n=1}^{\infty}$, $0<e_n\leqq 1$, and $\{\delta_n\}_{n=1}^{\infty}$, $\delta_n \searrow 0$, because otherwise our main theorem (2.20) would be true for $g_N(x)$.

2.5. By (i) and the restrictions for $\{\delta_n\}$ and $\{e_n\}$ we can define the sequence of indices $0<n_1<n_2<n_3<\ldots$ as follows.

$$\begin{cases} e_{n_1}\omega_m(\delta_{n_1}) \leqq q, \\ e_{n_{i+1}}\omega_m(\delta_{n_{i+1}}) \leqq q e_{n_i}\omega_m(\delta_{n_i}) & (i=1,2,3,\ldots),\ 0<q<1, \end{cases} \tag{2.10}$$

$$n_{i+1} > M(n_i) \qquad (i=1,2,3,\ldots). \tag{2.11}$$

2.6. Let

$$\tilde{f}(x) \equiv Q\sum_{i=1}^{\infty} e_{n_i}\omega_m(\delta_{n_i})\,g_{n_i}(x), \quad Q>0 \text{ (definition)}. \tag{2.12}$$

B. *Let us suppose, that* $\tilde{f}(x)\in C_P^{\{a,b\}}(\omega_m)$. We now prove a lower estimation. With a simple computation we have

$$T_{n_k}(\tilde{f};z_{n_k})-U_{n_k}(\tilde{f};z_{n_k}) = \tag{2.13}$$

$$= Q\sum_{i=1}^{\infty} e_{n_i}\omega_m(\delta_{n_i})\,[T_{n_k}(g_{n_i};z_{n_k})-U_{n_k}(g_{n_i};z_{n_k})] \geqq$$

$$\geqq Q[e_{n_k}\omega_m(\delta_{n_k})\,T_{n_k}(g_{n_k};z_{n_k})]-\sum_{i=1}^{k-1} e_{n_i}\omega_m(\delta_{n_i})\,|T_{n_k}(g_{n_i};z_{n_k})-U_{n_k}(g_{n_i};z_{n_k})|-$$

$$-\sum_{i=k+1}^{\infty} e_{n_i}\omega_m(\delta_{n_i})\,|T_{n_k}(g_{n_i};z_{n_k})|-\sum_{i=k}^{\infty} e_{n_i}\omega_m(\delta_{n_i})\,|U_{n_k}(g_{n_i};z_{n_k})|,$$

supposing that the difference (2.13) has meaning. But here by (2.5), further (2.9), (2.10), using (2.6), (2.10), and once again (2.10)

$$e_{n_k}\omega_m(\delta_{n_k})\,T_{n_k}(g_{n_k};z_{n_k}) \geqq c_4 e_{n_k}\omega_m(\delta_{n_k})\lambda_{n_k}(z_{n_k}) \qquad (k=1,2,3,\ldots), \tag{2.14}$$

$$\sum_{i=1}^{k-1} e_{n_i}\omega_m(\delta_{n_i})\,|T_{n_k}(g_{n_i};z_{n_k})-U_{n_k}(g_{n_i};z_{n_k})| \leqq \tag{2.15}$$

$$\leqq e_{n_k}\omega_m(\delta_{n_k})\lambda_{n_k}(z_{n_k})\sum_{i=1}^{k-1} e_{n_i}\omega_m(\delta_{n_i}) \leqq \frac{q}{1-q}\,e_{n_k}\omega_m(\delta_{n_k})\lambda_{n_k}(z_{n_k}) \qquad (k=1,2,3,\ldots),$$

$$\sum_{i=k+1}^{\infty} e_{n_i}\omega_m(\delta_{n_i})\,|T_{n_k}(g_{n_i};z_{n_k})| \leqq c_5\frac{q}{1-q}\,e_{n_k}\omega_m(\delta_{n_k})\lambda_{n_k}(z_{n_k}) \qquad (k=1,2,3,\ldots), \tag{2.16}$$

$$\sum_{i=k}^{\infty} e_{n_i}\omega_m(\delta_{n_i})\,|U_{n_k}(g_{n_i};z_{n_k})| \leqq e_{n_k}\omega_m(\delta_{n_k})\sum_{i=k}^{\infty} q^{i-k}|U_{n_k}(g_{n_i};z_{n_k})| \equiv \tag{2.17}$$

$$\equiv e_{n_k}\omega_m(\delta_{n_k})\,r(z_{n_k}) \qquad (k=1,2,3,\ldots) \text{ (definition of } r).$$

By (2.13)—(2.17) we have

$$T_{n_k}(\tilde{f};z_{n_k})-U_{n_k}(\tilde{f};z_{n_k}) \geqq e_{n_k}\omega_m(\delta_{n_k})\left[c_4 Q\lambda_{n_k}(z_{n_k})-(1+c_5)\frac{Qq}{1-q}\,\lambda_{n_k}(z_{n_k})-Qr(z_{n_k})\right].$$

Now let us suppose $[\cdots] > c_6 \lambda_{n_k}(z_{n_k})$ or, what is the same

$$\lambda_{n_k}(z_{n_k})\left[c_4 Q - c_6 - (1+c_5)\frac{Qq}{1-q}\right] > Qr(z_{n_k}) \qquad (k = 1, 2, 3, \ldots). \tag{2.18}$$

By an easy computation we can show that (2.18) is fulfilled by suitable large Q and small q if, in addition, the condition C below holds.

C. *We suppose that*

$$c_4 \lambda_{n_k}(z_{n_k}) > r(z_{n_k}) \equiv \sum_{i=k}^{\infty} q^{i-k} |U_{n_k}(g_{n_i}; z_{n_k})| \qquad (k = 1, 2, 3, \ldots). \tag{2.19}$$

Then by (2.13)—(2.19) we have our main result:

$$T_{n_k}(\tilde{f}; z_{n_k}) - U_{n_k}(\tilde{f}; z_{n_k}) > c_7 e_{n_k} \omega_m(\delta_{n_k}) \qquad (k = 1, 2, 3, \ldots).$$

Or, with $f(x) = \frac{1}{c_7}\tilde{f}(x)$ we can assert the following

MAIN THEOREM 2.1. *Let* $\{T_n\}_{n=1}^{\infty}$ *and* $\{U_n\}_{n=1}^{\infty}$ *be linear operators mapping* $C_P^{\{a,b\}}(\omega_m)$ *into* $C_P^{\{a,b\}}(\omega_m)$.[3] *If the conditions A, B and C are fulfilled with certain* $\{\delta_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$, $\{e_n\}_{n=1}^{\infty}$ *and* $\{\lambda_n(z_n)\}_{n=1}^{\infty}$ *then there exist an* $f(x) \in C_P^{\{a,b\}}(\omega_m)$ *and a sequence* $0 < n_1 < n_2 < n_3 < \ldots$ *such that*

$$T_{n_k}(f; z_{n_k}) - U_{n_k}(f; z_{n_k}) > e_{n_k} \omega_m(\delta_{n_k}) \lambda_{n_k}(z_{n_k}) \qquad (k = 1, 2, 3, \ldots), \tag{2.20}$$

if these differences have meaning.

3. On the conditions A, B and C

3.1. In this part we try to give same methods and restrictions to fulfil the conditions of Theorem 2.1, without any claim of completeness. Later we give some examples.

3.2. A. The first important question is the choice of δ_n and $\lambda_n(x)$. (Practically, we construct the functions $g_n(x)$ to these quantities.) In many cases we have a "direct" theorem (see (1.1) or (1.2)) from which we can usually determine δ_n and $\lambda_n(x)$.

Mostly δ_n is equal to the argument of ω_m in the direct theorem. Especially $\delta_n = d_n$ where d_n is the minimal distance of nodes e.g. at trigonometric interpolation: $\lambda_n(x)$ is the Lebesgue function of our process.

As for the sequence $\{z_n\}$, we can choose z_n such that $\lambda_n \equiv \lambda_n(z_n) = \max \lambda_n(x)$ when only the existence of $\{z_n\}$ is known; but sometimes we know that $\lambda_n(x_0) \sim \lambda_n$. Then, of course, $z_n = x_0$ $(n = 1, 2, 3, \ldots)$. In such a way we can get pointwise estimations.

The properties of the set P and the interval $\{a, b\}$ may be determined from the direct theorems also. E.g. if $f(x)$ is 2π-periodic and twice differentiable, it is natural that these properties are elements of P and $\{a, b\} = [0, 2\pi)$ (see [5], [6] and [7]).

[3] As above, the operator S is said to be linear if it is additive and continuous.

Now if we choose as $g_n(x)$ a spline function constructed from Hermite interpolatory polynomials defined on suitable nodes determined by direct theorems and the desired assertion, this function $g_n(x)$ generally fulfils condition A, (2.7) and (2.8). (As for $\{e_n\}$, see later.)

3.3. Let us see the condition B.

B1. *If we suppose that* $\tilde{f}(x) \in C_P^{\{a,b\}}$, (2.7), (2.8) *and*

$$\lim_{t=+0} \frac{t^m}{\omega_m(t)} = 0 \tag{iii}$$

then $\tilde{f}(x) \in C_P^{\{a,b\}}(\omega_m)$ *even if* $e_n = 1$ $(n = 1, 2, 3, \ldots)$, *i.e.* B *is fulfilled.* We note that (iii) precludes the case $\omega_m(t) \sim t^m$. Indeed, we can suppose for a suitable $\{n_i\}$ that

$$\sum_{i=1}^{j-1} \omega_m(\delta_{n_i}) \delta_{n_i}^{-m} \leqq \omega_m(\delta_{n_j}) \delta_{n_j}^{-m} \qquad (j = 2, 3, 4, \ldots) \tag{3.1}$$

(see (2.8)). Let

$$\delta_{n_{j+1}} < t \leqq \delta_{n_j} \quad (\text{if } \delta_{n_1} < t, \quad j = 0). \tag{3.2}$$

Then

$$\omega_m(\tilde{f}; t) \leqq Q \sum_{i=1}^{\infty} \omega_m(\delta_{n_i}) \omega_m(g_{n_i}; t) = Q \left(\sum_{i=1}^{j} + \sum_{i=j+1}^{\infty} \right). \tag{3.3}$$

But by (2.8), (3.1), (3.2) and (ii)

$$\sum_{i=1}^{j} \omega_m(\delta_{n_i}) \omega_m(g_{n_i}; t) \leqq B_m t^m \sum_{i=1}^{j} \omega_m(\delta_{n_i}) \delta_{n_i}^{-m} \leqq 2B_m t^m \omega_m(\delta_{n_j}) \delta_{n_j}^{-m} \leqq 2B_m \omega_m(t)$$

and by (2.7), (2.10) and (3.2)

$$\sum_{i=j+1}^{\infty} \omega_m(\delta_{n_i}) \omega_m(g_{n_i}; t) \leqq (c_3 \cdot 2)^m \sum_{i=j+1}^{\infty} \omega_m(\delta_{n_i}) \leqq (c_3 \cdot 2)^m \frac{\omega_m(\delta_{n_{j+1}})}{1-q} \leqq \frac{(c_3 \cdot 2)^m}{1-q} \omega_m(t),$$

fromwhere $\tilde{f}(x) \in C^{\{a,b\}}(\omega_m)$. But $\tilde{f}(x) \in C_P^{\{a,b\}}$, so B is valid.

B2. *If we suppose that* $\tilde{f}(x) \in C_P^{\{a,b\}}$, (2.7), (2. 8) *and*

$$\sum_{i=1}^{\infty} e_i < \infty \tag{3.4}$$

then $\tilde{f}(x) \in C_P^{\{a,b\}}(\omega_m)$, *i.e.* B *is fulfilled.* We note that our assertion (2.20) is weaker than in the case $e_n = 1$, but it is valid for $\omega_m(t) \sim t^m$. Indeed, as above, by (2.8), (ii), (3.2) and (3.4)

$$\sum_{i=1}^{j} e_{n_i} \omega_m(\delta_{n_i}) \omega_m(g_{n_i}; t) \leqq B_m \sum_{i=1}^{j} e_{n_i} \omega_m(\delta_{n_i}) \delta_{n_i}^{-m} t^m < B_m \omega_m(t) \sum_{i=1}^{\infty} e_{n_i},$$

and by (2.7), (2.10) and (3.2)

$$\sum_{i=j+1}^{\infty} e_{n_i} \omega_m(\delta_{n_i}) \omega_m(g_{n_i}; t) \leqq (c_3 \cdot 2)^m \sum_{i=j+1}^{\infty} e_{n_i} \omega_m(\delta_{n_i}) \leqq \frac{(c_3 \cdot 2)^m}{1-q} \omega_m(t).$$

3.4. Now let us consider condition C. First of all by (2.17) and (2.7)

$$0 \leqq r(z_{n_k}) \equiv \sum_{i=k}^{\infty} q^{i-k} |U_{n_k}(g_{n_i}; z_{n_k})| \leqq c_3 \sum_{i=k}^{\infty} q^{i-k} \|U_{n_k}\| \tag{3.5}$$

i.e., if $\|U_{n_k}\| \leqq K$ $(k = 1, 2, 3, \ldots)$ then

$$r(z_{n_k}) \leqq \frac{c_3 K}{1-q}.$$

Especially we have

C1. *If* $\|U_n\| \leqq K$ $(n=1, 2, \ldots)$, (2.7) *is valid and* $\limsup_{n\to\infty} \lambda_n(z_n) = \infty$ *then* $c_4 \lambda_{n_k}(z_{n_k}) > r(z_{n_k})$ *i.e.* C *is valid.*

Or

C2. *If* U_n *are equal to the unit operator* E, $c_4 \geqq c_3$, (2.7) *is valid and* $\limsup_{n\to\infty} \lambda_n(z_n) = 1+\varepsilon$ *then* C *holds.*

If we are unable to ensure C by $\limsup_{n\to\infty} \lambda_n(z_n)$ we may try another way. From (3.5), with a suitable q, we have

C3. *If the* U_n *are the unit operator* E *and* $|g_{n_k}(z_{n_k})| \leqq \alpha < c_4 \lambda_{n_k}(z_{n_k})$ *or*

$$|U_{n_k}(g_{n_k}; z_{n_k})| \leqq \alpha < c_4 \lambda_{n_k}(z_{n_k}) \qquad (k=1, 2, \ldots)$$

then condition C holds. Sometimes $\alpha = 0$.

As C3 shows, the functions $g_n(x)$ can satisfy condition C as well.

4. Some previously proved theorems

4.1. Let (with $x_k = x_{kn}$)

$$-1 \leqq x_1 < x_2 < \ldots < x_n \leqq 1 \qquad (n = 1, 2, 3, \ldots) \tag{4.1}$$

be an infinite point system. Let

$$L_n(f; x) = \sum_{k=1}^{n} f(x_{kn}) l_{kn}(x), \quad \lambda_n(x) = \sum_{k=1}^{n} |l_{kn}(x)|, \quad \lambda_n = \lambda_n(z_n) = \max_{-1 \leqq x \leqq 1} \lambda_n(x) \tag{4.2}$$

be the Lagrange interpolatory polynomial of degree $\leqq n-1$ of the function $f(x)$; where $l_{kn}(x)$ are the fundamental polynomials, $\lambda_n(x)$ are the Lebesgue functions, and λ_n are the Lebesgue constants. If

$$d_n = \min_{1 \leqq k \leqq n-1} |x_{k+1} - x_k| \qquad (n=1, 2, 3, \ldots), \quad d_n \searrow 0 \tag{4.3}$$

then we have

THEOREM 4.1 (O. KIS, [5]). *If* $\lim_{t=+0}(t^m/\omega_m(t))=0$ *then for a suitable* $f(x)\in C^{[-1,1]}(\omega_m)$ *and* $\{n_i\}$

$$\|L_{n_k}(f;x)-f(x)\|_{[-1,1]} > \omega_m(d_{n_k})\lambda_{n_k} \qquad (k=1,2,3,\ldots) \tag{4.4}$$

holds.

PROOF. First of all we construct the $g_n(x)$. Let

$$\begin{cases} g_n(x_k) = \operatorname{sign} l_k(z_n) & (k=1,2,\ldots,n) \\ g_n(x) = \begin{cases} g_n(x_1) & \text{if} \quad -1 \leqq x < x_1 \\ g_n(x_n) & \text{if} \quad 1 \geqq x > x_n. \end{cases} \end{cases} \tag{4.5}$$

Let us define $g_n(x)$ for $x\in(x_k, x_{k+1})$. In this interval let $g_n(x)$ be the Hermite interpolatory polynomial of degree $\leqq 2n-1$ which is equal to $g_n(x_k)$ or $g_n(x_{k+1})$ at the endpoints, respectively, and in these endpoints $g_n'(x_j)=g_n''(x_j)=\ldots=g_n^{(m-1)}(x_j)=0$ $(j=k, k+1)$. By an easy computation we can see that A holds with $\delta_n=d_n$, $e_n=1$, $c_3=c_4=c_5=1$, $U_n=E$ (see [5]).

B1 is a condition of our theorem, $\overline{\lim_{n\to\infty}}\,\lambda_n=\infty$ for any interpolatory process, so we have C1 also.

By A, B1 and C1 we can prove (4.4). Q.e.d.

4.2. Let us consider a pointwise theorem for a special Lagrange interpolatory process. Let

$$x_k = \cos\frac{2k-1}{2n}\pi \qquad (k=1,2,\ldots,n). \tag{4.6}$$

THEOREM 4.2 ([13]). *If* $\lim_{t=+0}(t^m/\omega_m(t))=0$ *then for a suitable* $f(x)\in C^{[-1,1]}(\omega_m)$

$$L_n(f;0)-f(0) > \log n\cdot\omega_m\left(\frac{1}{n}\right) \qquad (n=2,4,6,\ldots). \tag{4.7}$$

PROOF. Let $z_n=0$, $e_n=1$, $\delta_n=\dfrac{1}{n}$ $(n=1,2,3,\ldots)$,

$$\begin{cases} g_n(x_k) = \operatorname{sign} l_k(0) & \text{if} \quad \left[\dfrac{n}{4}\right] \leqq k \leqq \left[\dfrac{3n}{4}\right] \quad (n\geqq 8), \\ g_n(x_k) = 0 & \text{if} \quad \left[\dfrac{n}{4}\right] > k \quad \text{or} \quad \left[\dfrac{3n}{4}\right] < k \quad (n\geqq 8). \end{cases} \tag{4.8}$$

On the intervals $[-1, x_n]$, $[x_n, x_{n-1}]$, $\ldots$, $[x_2, x_1]$, $[x_1, 1]$, $g_n(x)$ is the above defined Hermite interpolatory polynomial.

But easy to see ([13], (7.2)) that

$$\lambda_n \sim \log n \sim \lambda_n(0) \sim L_n(g_n;0) \qquad (n=2,4,6,\ldots). \tag{4.9}$$

So by A, B1, and C1 we get (4.7). Q.e.d.

Now let us see a theorem for $\omega_m(t)\sim t^m$ with the nodes (4.6).

THEOREM 4.3 ([8] and [13]). *For every* $\varepsilon_n \searrow 0$ *we have a function* $f(x)\in C^{[-1,1]}(t^m)$ *and* $\{n_i\}$ *such that*

$$L_{n_k}(f;0)-f(0) > \varepsilon_{n_k}\frac{\log n_k}{n_k^m} \qquad (k=1,2,3,\ldots). \tag{4.10}$$

PROOF. Let $\varepsilon_{n_k}\leqq k^{-2}$ and $e_{n_k}\leqq\varepsilon_{n_k}$. The further steps are as at the theorem 4.2. Q.e.d.

Comparing these result to (1.1), we can see that in many cases we have the order of best lower estimation.

4.3. Let us prove a theorem for a Fourier series. If $S_n(f;x)$ is the n-th partial sum of the Fourier series of a 2π-periodic function $f(x)$ then let $P=\{f(x)$ is 2π-periodic$\}$.

THEOREM 4.4 ([9]). *If* $\lim\limits_{t=+0}\left(t^m/\omega_m(t)\right)=0$ *then for a suitable* $f(x)\in C^{[0,2\pi)}(\omega_m)$ *and* $\{n_i\}$

$$S_{n_k}(f;0)-f(0) > \log n_k\cdot\omega_m\left(\frac{1}{n_k}\right) \qquad (k=1,2,3,\ldots). \tag{4.11}$$

PROOF. Let $z_n=0$, $\delta_n=\dfrac{1}{n}$, $e_n=1$. The following formulae are well known:

$$S_n(f;x)=\frac{1}{\pi}\int_0^{2\pi} f(t)\frac{\sin\dfrac{2n+1}{2}(t-x)}{2\sin\dfrac{t-x}{2}}\,dt, \tag{4.12}$$

$$\lambda_n=\frac{1}{\pi}\int_0^{2\pi}|D_n(t)|\,dt=\frac{1}{\pi}\int_0^{2\pi}\left|\frac{\sin(2n+1)\dfrac{t}{2}}{2\sin\dfrac{t}{2}}\right|dt=\frac{4}{\pi^2}\ln n+O(1). \tag{4.13}$$

Let

$$\begin{cases} x_{kn}=\dfrac{2k\pi}{2n+1} & (n=1,2,3,\ldots;\ k=0,1,\ldots,2n);\\ d_n=\dfrac{2\pi}{2n+1} & (n=1,2,3,\ldots),\\ 0<s<0.5, \quad x_{in}^{+}=x_{in}+sd_n, \quad x_{in}^{-}=x_{in}-sd_n,\\ H_n=[0,x_{1n}^{-}]\cup[x_{1n}^{+},x_{2n}^{-}]\cup\ldots\cup[x_{2n-1,n}^{+},x_{2n,n}^{-}]\cup[x_{2n,n}^{+},2\pi]. \end{cases} \tag{4.14}$$

We now define the functions $g_n(x)$ as follows.

$$(4.15)\quad \begin{cases} g_n(x) = \operatorname{sign} D_n(x) \quad \text{for} \quad x \in H_n, \\ g_n(x) = g_n(x_{kn}^-)\left[1 - \dfrac{(2m-1)!!}{(2m-2)!!}\right] \displaystyle\int_{-1}^{2\frac{x-x_{kn}^-}{x_{kn}^+ - x_{kn}^-}-1} (1-t^2)^{m-1}\,dt \quad (x_{kn}^- \leqq x \leqq x_{kn}^+), \\ 2\pi\text{-periodic elsewhere.} \end{cases}$$

As above, we can get A, B1 and C1. Q.e.d.

4.4. I wish to mention that there are interesting examples for the application of C3 in [11], while in [12] we applied a sequence $\delta_n \searrow 0$ totally independent from the corresponding direct theorem.

5. New results for the quadrature procedures

Let

$$(5.1)\qquad a \leqq x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} \leqq b \qquad (n = 1, 2, 3, \ldots)$$

be an infinite triangular matrix (sometimes $a = -\infty$, $b = +\infty$, but x_{in} are finite), and

$$(5.2)\qquad A_{1n}, A_{2n}, \ldots, A_{nn} \qquad (n = 1, 2, 3, \ldots)$$

finite real numbers. If $f(x)$ is defined on $\{a, b\}$ we can consider the mechanical quadrature procedure

$$(5.3)\qquad Q_n(f) \equiv \sum_{k=1}^{n} A_{kn} f(x_{kn}) \qquad (n = 1, 2, 3, \ldots)$$

and its Lebesgue constants

$$(5.4)\qquad \lambda_n = \sum_{k=1}^{n} |A_{kn}| \qquad (n = 1, 2, 3, \ldots).$$

We shall use the natural restrictions

$$(5.5)\qquad |A_{kn}| > 0, \quad \lambda_n \geqq c > 0 \qquad (k = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots).$$

Let us consider an important special case. Let $u(x)$ be a real-valued non-decreasing bounded function having infinitely many different values and

$$(5.6)\qquad A_{kn} = \int_a^b l_{kn}(x)\,du(x) \qquad (k = 1, 2, \ldots, n)$$

where $l_{kn}(x)$ are the fundamental polynomials of the Lagrange interpolation belonging to (5.1). Then (5.3) is an *interpolatory* quadrature procedure.

It is a well-known fact that for interpolatory quadrature procedures (supposing that $|a|, |b| < \infty$)

$$\lim_{n\to\infty} Q_n(f) = \int_a^b f(x)\,du(x)^4 \tag{5.7}$$

holds for all continuous function $f(x)$ if and only if the sequence λ_n is bounded ([15], 15.2).

In their famous paper [4] P. ERDŐS and P. TURÁN raised the problem: what sort of condition must hold for the existence (5.7) for all $f(x) \in \mathrm{Lip}\ \alpha$. J. SZABADOS investigated this problem in connection with the class $C^{[-1,1]}(\omega_1)$ with $u(x) \equiv 1$ (see [14]). Now I wish to continue this investigation.

5.2. First of all we state a direct theorem for $a = -1$, $b = 1$.

THEOREM 5.1. *If $f(x) \in C^{[-1,1]}(\omega_m)$ then we have for every interpolatory quadrature procedure*

$$\left|Q_n(f) - \int_{-1}^{1} f(x)\,du(x)\right| = O(\lambda_n) E_{n-1}(f) \qquad (n = 1, 2, 3, \ldots). \tag{5.8}$$

PROOF. Let $f(x) \in C^{[-1,1]}(\omega_m)$ and p_{n-1} be the best approximating polynomial of $f(x)$ of degree $\leqq n-1$,

$$E_{n-1}(f) = \max_{-1 \leqq x \leqq 1} |f(x) - p_{n-1}(x)|.$$

By

$$E_{n-1}(f) \lesssim \omega_m\left(\frac{1}{n}\right)$$

and

$$Q_n(p_{n-1}) = \int_{-1}^{1} p_{n-1}(x)\,du(x) \tag{5.9}$$

we have

$$\left|Q_n(f) - \int_{-1}^{1} f(x)\,du(x)\right| \leqq |Q_n(f) - Q_n(p_{n-1})| + \int_{-1}^{1} |p_{n-1}(x) - f(x)|\,du(x) \leqq$$

$$\leqq E_{n-1}(f) \sum_{k=1}^{n} |A_{kn}| + \mu_0 E_{n-1}(f) = O(\lambda_n) E_{n-1}(f) \sim \lambda_n \omega_m\left(\frac{1}{n}\right).$$

5.3. Now we prove a general lower estimation for arbitrary mechanical quadrature procedures.

THEOREM 5.2. *If $\lim_{t=+0} (t^m/\omega_m(t)) = 0$ then there exist an $f(x) \in C^{[a,b]}(\omega_m)$ and a*

[4] We always suppose that

$$\int_a^b f(x)\,du(x), \quad \mu_i = \int_a^b x^i\,du(x) \qquad (i = 1, 2, 3, \ldots)$$

are finite values.

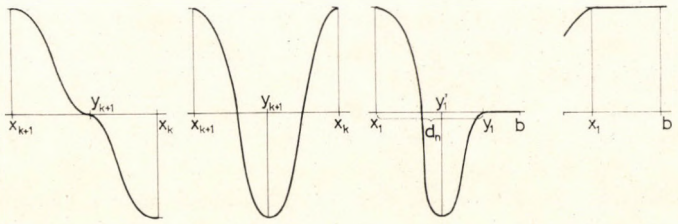
sequence $\{n_i\}$ *such that*

$$Q_{n_k}(f) - \int_a^b f(x)\,du(x) > \lambda_{n_k}\omega_m(d_{n_k}(u)) \qquad (k = 1, 2, 3, \ldots).^{5} \tag{5.10}$$

PROOF. We shall apply our Theorem 2.1. For this aim let $T_n(f; x)=Q_n(f)$, $U_n(f; x)=\int_a^b f(x)\,du(x)$. Let $\delta_n=d_n(u)$ $e_n=1$, z_n is arbitrary from $\{a, b\}$.

For C3 let us define the $g_n(x)$ $(n\geqq 4)$ on the points $a\leqq y_{n+1}<x_n<y_n<x_{n-1}<\ldots$ $\ldots<x_2<y_2<x_n<y_1\leqq b$ as follows (if $a=x_n$ then $y_{n+1}=x_n$; similarly at b).

(5.11)



a) *b)* *c)* *d)*

I.e. let

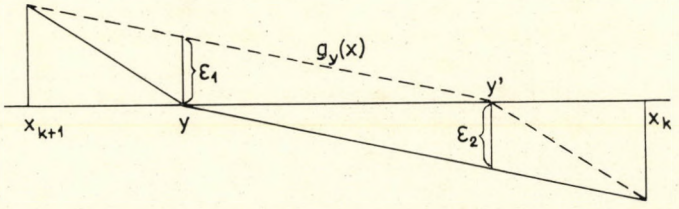
$$\begin{cases} g_n(x_k) = \operatorname{sign} A_k & (k = 1, 2, \ldots, n) \\ g_n'(x_k) = g_n''(x_k) = \cdots = g^{(m-1)}(x_k) = 0 & (k = 1, 2, \ldots, n). \end{cases} \tag{5.12}$$

a) If $1<k<n-1$ and $g_n(x_k)=-g_n(x_{k+1})$ then let $g_n(y_{k+1})=g_n'(y_{k+1})=\ldots=$ $=g_n^{(m-1)}(y_{k+1})=0$, on the interval $[x_{k+1}, y_{k+1}]$ (or on $[y_{k+1}, x_k]$) the Hermite interpolatory polynomial of degree $\leqq 2m-1$ with such y_{k+1} that

$$\int_{x_{k+1}}^{x_k} g_n(x)\,du(x) = 0.^{6} \tag{5.13}$$

[5] $d_n(u)$=the minimal length of the intervals at (5.11) a), b) and c) where $g_n(x)\not\equiv 0$; as d_n, see (4.3). Obviously $d_n(u)\leqq d_n$. As before we suppose that $d_n\searrow 0$.

[6] To prove the existence of such an y we show that the integral is a continuous function of y. For the sake of simplicity let $m=1$. Then we have



$$\left|\int_{x_{k+1}}^{x_k} g_y(x)\,du(x) - \int_{x_{k+1}}^{x_k} g_{y'}(x)\,du(x)\right| \leqq \max\left[\varepsilon_1 \int_{x_{k+1}}^{y'} du(x),\ \varepsilon_2 \int_{y}^{x_k} du(x)\right] < \varepsilon_3\mu_0.$$

But $\varepsilon_3\mu_0<\varepsilon$ if $\delta=y'-y$ is small enough. Let us consider that the quotient $(x_k-y_{k+1})/(y_{k+1}-x_{k+1})$ depends only on $du(x)$.

b) Let $1<k<n-1$ and $g_n(x_k)=g_n(x_{k+1})$. The only difference is that now $g_n(y_{k+1})=-\operatorname{sign} g_n(x_k)$.

c) Let $k=1$ (or $k=n$) and $b-x_1\geqq d_n$. We can choose $y_1=x_1+d_n$ and afterwards y_1' in accordance a), b) and (5.13).

d) Let $k=1$ (or $k=n$) and $x_1+d_n>b$. Then e.g. $y_1=b$, $g_n(x)=\operatorname{sign} g_n(x_1)$ if $x\geqq x_1$. So

$$\int_{x_1}^{b} g_n(x)\,du(x) \leqq \mu_0 d_n \qquad (n=1, 2, 3, \ldots). \tag{5.14}$$

By (5.11)—(5.14) we can prove A with $c_3=c_4=c_5=1$. Further by (5.13), (5.14), (4.3) and (5.5)

$$U_n(g_n) = \left|\int_a^b g_n(x)\,du(x)\right| = \sum_{k=1}^{n-1}\left|\int_{x_k}^{x_{k+1}} g_n(x)\,du(x)\right| \leqq 2\mu_0 d_n. \tag{5.15}$$

Hence $r_n\leqq 2\mu_0 d_n\dfrac{1}{1-q}<\lambda_n$ $(n>n_0)$, i.e. C is fulfilled. The condition (iii) gives B1, so our theorem is completely proved. Q.e.d.

5.4. Let us consider a very important special case, the Gauss—Jacobi interpolatory quadrature formula, where

$$du(x) = (1-x)^{\alpha}(1+x)^{\beta}\,dx \qquad (\alpha, \beta > -1,\ a=-b=-1), \tag{5.16}$$

(5.17)

$$x_k = x_{kn}^{(\alpha,\beta)} = \cos\vartheta_{kn}^{(\alpha,\beta)} \sim \cos\frac{k\pi}{n} \qquad (k=0, 1, \ldots, n+1;\ x_0=+1,\ x_{n+1}=-1),$$

$$A_k = A_{kn}^{(\alpha,\beta)} \geqq 0 \quad (k=1, 2, \ldots, n); \quad A_k \sim \vartheta_k^{2\alpha+1} n^{-1} \quad (0<\vartheta_k\leqq\pi-\varepsilon) \tag{5.18}$$

(see [15], 15.3 and [16]). Our direct theorem (considering that, by (5.18),

$$\lambda_n = \sum_{k=1}^{n} |A_k| \equiv \sum_{k=1}^{n} A_k = \mu_0), \tag{5.19}$$

is the following

THEOREM 5. 3. *If* $f(x)\in C^{[-1,1]}(\omega_m)$ *then for* $\alpha>-1$, $\beta>-1$

$$\left|Q_n(f) - \int_{-1}^{1} f(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx\right| \leqq 4E_{2n-1}(f). \tag{5.20}$$

PROOF. The only difference from Theorem 5.1 is that for the Gauss—Jacobi quadrature formula

$$Q_n(p_{2n-1}) = \int_{-1}^{1} p_{2n-1}(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

Q.e.d.

Let us consider a lower estimation for the Gauss—Jacobi quadrature procedure. We prove the following

THEOREM 5.4. *If* $\lim_{t=+0}(t^m/\omega_m(t))=0$ *then there exist an* $f(x)\in C^{[-1,1]}(\omega_m)$ *and a sequence* $\{n_i\}$ *such that*

$$(5.21)\quad Q_{n_k}(f)-\int_{-1}^{1} f(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx > \omega_m\left(\frac{1}{n_k}\right) \qquad (k=1,2,3,\ldots).$$

PROOF. The ideas are similar to that of Theorem 5.2, but now $d_n\sim n^{-2}$. So we have to choose $g_n(x)$ a slightly different. For this aim let

$$(5.22)\quad \begin{cases} g_n(x_k)=\operatorname{sign} A_k(=1) & \text{for } \left[\frac{n}{4}\right]\leqq k\leqq\left[\frac{3n}{4}\right], \\ g_n(x_k)=0 & \text{for } 1\leqq k<\left[\frac{n}{4}\right] \text{ or } \left[\frac{3n}{4}\right]<k\leqq n \quad (n\geqq n_0). \end{cases}$$

We may define $g_n(x)$ on the intervals $[x_{k+1}, x_k]$ as in Theorem 5.2. Let

$$(5.23)\quad \delta_n=\min_{\left[\frac{n}{4}\right]\leqq k\leqq\left[\frac{3n}{4}\right]}(x_k-x_{k+1}) \quad (n\geqq n_0).$$

Then by (5.18) $Q_n(g_n)\geqq c(\alpha,\beta)\lambda_n$ so A is valid (see (5.23)). As for C, we remark that by (5.22) and (5.13) $Ug_n=0$, so (2.19) is true.

5.5. At last we give a remark for the Newton—Cotes interpolatory quadrature formulae. (As it is well-known, here $u(x)=x$, and the x_k's are equidistant, $x_1=b$, $x_n=a$.) R. O. Kuzmin proved that in this case

$$(5.24)\quad \sum_{k=1}^{2s+1}|A_{k,n}|>|A_{s+1,n}|>c\cdot\frac{2^{2n-1}}{n^3\sqrt{n}} \qquad (n=1,3,5,\ldots;\ a=-1,\ b=1)$$

(see [17]). By (5.24) we have

THEOREM 5.5. *If* $\lim_{t=+0}(t^m/\omega_m(t))=0$ *then there exists an* $f(x)\in C^{[-1,1]}(\omega_m)$ *such that*

$$(5.25)\quad \lim_{n\to\infty} Q_n(f)=\infty \qquad (n=1,3,5,\ldots)$$

where Q_n *is the Newton—Cotes interpolatory quadrature procedure.*

PROOF. By (5.10) and (5.24)

$$\overline{\lim_{n\to\infty}}\left|Q_n(f)-\int_{-1}^{1} f(x)\,dx\right|=\infty,$$

but $\left|\int_{-1}^{1} f(x)\,dx\right|\leqq K$. Q.e.d.

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References

[1] H. LEBESGUE, Sur la representation trigonométrique approchée des fonctions satisfaisant á une condition de Lipschitz, *Bull. S. M. F.*, **38** (1910), pp. 184—210.
[2] G. FABER, Über die interpolatorische Darstellung stetiger Funktionen, *Jahresber. der Deutschen Math. Ver.*, **23** (1914), pp. 192—210.
[3] С. М. Лозинский, Пространства $\tilde{C}_\omega$ и $\tilde{C}^*_\omega$ и сходимость интерполяционных процессов в них, *DAH*, **59** (1948) pp. 1389—1392.
[4] P. ERDŐS, P. TURÁN, On the role of the Lebesgue functions in the theory of Lagrange interpolation, *Acta Math. Acad. Sci. Hung.*, **6** (1955), pp. 47—65.
[5] О. Киш, Замечания о порядке погрешности интерполяции, *ibid.*, **20** (1969), pp. 339—346.
[6] P. O. H. VÉRTESI, On the divergence of the sequence of linear operators, *ibid.*, **20** (1969), pp. 399—408.
[7] O. KIS, P. O. H. VÉRTESI, On certain linear operators. I, *ibid.*, **22** (1971), pp. 65—71.
[8] P. O. H. VÉRTESI, On certain linear operators. II, *ibid.*, **22** (1971), pp. 251—255.
[9] P. O. H. VÉRTESI, On certain linear operators. III, *ibid.*, **23** (1972), pp. 109—112.
[10] P. O. H. VÉRTESI, On certain linear operators. IV, *ibid.*, **23** (1972), pp. 115—125.
[11] P. O. H. VÉRTESI, On certain linear operators. V, *ibid.*, **23** (1972), pp. 433—437.
[12] P. O. H. VÉRTESI, On certain linear operators. VI, *ibid.*, **24** (1973), pp. 423—427.
[13] P. O. H. VÉRTESI, Lower estimations for some interpolating processes, *Studia Sci. Math. Hung.*, **5** (1970), pp. 401—410.
[14] J. SZABADOS, On the convergence of the interpolatory quadrature procedures in certain classes of functions, *Acta Math. Acad. Sci. Hung.*, **18** (1967), pp. 97—111.
[15] G. SZEGŐ, *Orthogonal polynomials* (New York, 1959).
[16] Г. И. Натансон, Двухсторонная оценка функции Лебега интерполяционного процесса Лагранжа с узлами Якоби, *Изв. высш. учеб. зав.*, **11(66)** (1967), pp. 67—74.
[17] I. P. NATANSON, Constructive theory of functions (FUPC, New York, 1964).

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ON TAYLOR SERIES ABSOLUTELY CONVERGENT ON THE CIRCUMFERENCE OF THE CIRCLE OF CONVERGENCE. III

By

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Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

be regular for $|z|<1$ and substitute for z a conformal mapping of $|w|<1$ onto $|z|<1$ different from a rotation

$$z = T(w) = \varrho \frac{w-\xi}{1-w\bar{\xi}} \quad (|\varrho| = 1,\ 0 < |\xi| < 1),$$

$$f(T(w)) = \sum_{n=0}^{\infty} b_n w^n \quad (|w| < 1).$$

L. ALPÁR [1] proved that the absolute convergence of $\sum_{k=0}^{\infty} a_k z^k$ on the periphery is not necessarily inherited by $\sum_{n=0}^{\infty} b_n w^n$. In Parts I and II ([2], [3]) of this series we imposed extra conditions on the transformed and the original series, respectively, and obtained some positive and negative results. Here we are concerned with the behaviour of the function.

If $f(z)$ can be extended continuously to the closed $|z|\leqq 1$ and $f(e^{it})$ as a function of the real t satisfies a uniform Lipschitz condition with exponent greater than 1/2, then, owing to a theorem of S. Bernstein (see [4], Ch. VI, Theorem (3.1)), $\sum_{k=0}^{\infty} |a_k| < +\infty$. In this case, $T(w)$ being analytic on the periphery $|w|=1$, $f(T(w))$ has the same property, hence also $\sum_{n=0}^{\infty} |b_n| < +\infty$. We show that 1/2 is best possible here.

THEOREM 1. *There exists a series*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

with $\sum_{k=0}^{\infty} |a_k| < +\infty$, *such that* $f(e^{it})$ *satisfies a uniform Lipschitz condition of exponent* 1/2, *yet*

$$\sum_{n=0}^{\infty} |b_n| = +\infty.$$ [1]

[1] In its final form, the corresponding problem for ordinary convergence was solved by K. H. INDLEKOFER [5], completing work by J. CLUNIE [6] and W. SCHWARTZ [7].

We are unable to produce such a series and that might be quite difficult. However, only a minor improvement can be expected from a direct construction (see the remark at the end of the paper) and our proof will be an existence proof based on simple probabilistic principles.

With a given sequence of complex numbers α_k, $k=0, 1, \ldots$, let $a_k=\pm\alpha_k$ be independent random variables assuming $+\alpha_k$ and $-\alpha_k$ with probability 1/2. In purely analytic form we can say that $a_k=r_k(\omega)\alpha_k$ $(0\leqq\omega\leqq 1)$ where $r_k(\omega)$ is the kth Rademacher function. "With probability 1" means then that the statement in question holds for ω outside a set of measure 0.

We can again consider the transformed series with coefficients $b_n=b_n(\omega)$. We begin by proving the following general result.

THEOREM 2. *Let the random power series*

$$f(z)=\sum_{k=0}^{\infty}\pm\alpha_k z^k$$

converge for $|z|<1$.

(i) *If*

$$\sum_{l=0}^{\infty}\sqrt{2^l\sum_{2^l<k\leqq 2^{l+1}}|\alpha_k|^2}<+\infty,$$

then

$$\sum_{n=0}^{\infty}|b_n|<+\infty$$

in each case.

(ii) *If*

$$\sum_{l=0}^{\infty}\sqrt{2^l\sum_{2^l<k\leqq 2^{l+1}}|\alpha_k|^2}=+\infty,$$

then

$$\sum_{n=0}^{\infty}|b_n|=+\infty$$

with probability 1.

We remark that the number 2 has no special significance in the conditions. It could be replaced, simultaneously at all its appearences, by any real number greater than 1. (See the proof.) The conditions are also independent of the transformation $T(w)$. E.g. in (ii), for each $T(w)$ the event $\sum_{n=0}^{\infty}|b_n|=+\infty$ has probability 1 but this event might depend on $T(w)$. Is it really the case or with probability 1 $\sum_{n=0}^{\infty}|b_n|=+\infty$ for all $T(w)$?

(i) will not be needed for the proof of Theorem 1 and we only remark that it can be got out of the proof of Theorem 1 of Part II [3]. On the other hand, (ii) could be used to produce a new proof for Theorem 2 of Part II.

PROOF OF THEOREM 2, (ii). Putting the Taylor series of $T(w)$ into that of $f(z)$ we find

$$b_n = b_n(\omega) = \sum_{k=0}^{\infty} t_{kn} a_k = \sum_{k=0}^{\infty} t_{kn} r_k(\omega) \alpha_k,$$

where t_{kn} is the nth coefficient of the kth power of $T(w)$; the series converges absolutely.

Suppose now, contrary to what we have to prove, that $\sum_{n=0}^{\infty} |b_n(\omega)|$ is finite on a set of positive measure. Then we can find a subset of positive measure, we call it E, where the sum is even bounded, hence also integrable:

$$\int_E \sum_{n=0}^{\infty} |b_n(\omega)|\, d\omega = \sum_{n=0}^{\infty} \int_E |b_n(\omega)|\, d\omega < +\infty. \tag{1}$$

For each n, $b_n(\omega)$ is the sum of a Rademacher series with coefficients $t_{kn}\alpha_k$ and as is well known (see [4], Ch. V, Theorem (8.4), lemma (8.3) and footnote [2]), its L_p norms with respect to sets of positive measure are all, up to a constant factor, depending only on the set and p, about the same, provided that the first few Rademacher functions are absent from the series, in dependence on the given set. This can be achieved by turning the first few α_k into 0; since for each k $T^k(\omega)$ has an absolutely convergent Taylor series in $|w| \leqq 1$, the validity of (1) is not effected. What we actually need is

$$\int_E |b_n(\omega)|\, d\omega \geqq c_1 \sqrt{\int_0^1 |b_k(\omega)|^2\, d\omega} = c_1 \sqrt{\sum_{k=0}^{\infty} |t_{kn}\alpha_k|^2}$$

with a positive $c_1 = c_1(E)$.[2] Combining with (1)

$$\sum_{n=0}^{\infty} \sqrt{\sum_{k=0}^{\infty} |t_{kn}\alpha_k|^2} < +\infty. \tag{2}$$

In order to eliminate t_{kn} we are now making use of some of its estimations by L. ALPÁR [1] as given (with a much shorter proof) in [8], Lemma 4.[3]

With an $N \geqq 1$ let us first restrict ourselves to the range $aN < n \leqq bN$, keeping only $cN < k \leqq dN$ in the inner sum; a, b, c, d are positive constants to be determined later. We need the estimation

$$|t_{kn}| \leqq \frac{c_2(\delta)}{\sqrt{n}} \leqq \frac{c_2(\delta)}{\sqrt{aN}} \quad \text{for} \quad (p+\delta)n \leqq k \leqq (q-\delta)n$$

where $p = \dfrac{1-|\xi|}{1+|\xi|}$, $q = \dfrac{1}{p} = \dfrac{1+|\xi|}{1-|\xi|}$ (see the definition of $T(w)$) and $\delta > 0$ arbitrary. Its

[2] This, although not stated explicitly in [4], is implicit in the proofs. In the proof of the first inequality of Theorem (8.4) one should replace [0, 1] by the subset E. One needs a lower estimate for $\int_E |\ |^2$ supplied by Lemma (8.3) and an upper estimate for $\int_E |\ |^\alpha \leqq \int_0^1 |\ |^\alpha$ given by the second inequality of Theorem (8.4).

[3] Our t_{kn} is $\bar{\sigma}_{nk}$ of formula (3) of that paper with $P(\vartheta) \equiv 1$.

condition is fulfilled for each of our k, n if

$$p < \frac{c}{b}, \quad \frac{d}{a} < q \tag{3}$$

$\left(\text{with } \delta = \min\left(\frac{c}{b}-p,\ q-\frac{d}{a}\right)\right)$ and then we have

$$\sqrt{\sum_{k=0}^{\infty} |t_{kn}\alpha_k|^2} \geqq \frac{\sum\limits_{cN<k\leqq dN} |t_{kn}\alpha_k|^2}{\sqrt{\sum\limits_{cN<k\leqq dN} |t_{kn}\alpha_k|^2}} \geqq \frac{\sqrt{aN}}{c_2(\delta)} \frac{\sum\limits_{cN<k\leqq dN} |t_{kn}\alpha_k|^2}{\sqrt{\sum\limits_{cN<k\leqq dN} |\alpha_k^2|}}.$$

Summing for our n

$$\sum_{aN<n\leqq bN} \sum_{cN<k\leqq dN} |t_{kn}\alpha_k|^2 = \sum_{cN<k\leqq dN} |\alpha_k|^2 \sum_{aN<n\leqq bN} |t_{kn}|^2.$$

The inner sum, when extended over all n, is by Parseval's equality, 1, since $|T^k(\omega)|=1$ for $|w|=1$. If we can show that also

$$\sum_{aN<n\leqq bN} |t_{kn}|^2 \geqq c_3 > 0, \tag{4}$$

then we have, combined with the previous line

$$\sum_{aN<n\leqq bN} \sqrt{\sum_{k=0}^{\infty} |t_{kn}\alpha_k|^2} \geqq \frac{\sqrt{aN}}{c_2(\delta)} \frac{c_3 \sum\limits_{cN<k\leqq dN} |\alpha_k|^2}{\sqrt{\sum\limits_{cN<k\leqq dN} |\alpha_k|^2}} = c_4(\delta) \sqrt{N \sum_{cN<k\leqq dN} |\alpha_k|^2}. \tag{5}$$

It would not be difficult to prove (4) for any $a<b$ satisfying (3). However, by further restricting a, b, c, d we can get in a simpler way

$$\sum_{n\leqq aN} + \sum_{n>bN} |t_{kn}|^2 \leqq \frac{1}{2} \tag{6}$$

for N large enough and this, according to the above, implies (4) with $c_3=1-1/2=1/2$.

Let $m=m(k, n, p, q)=\min(n-pk, qk-n)$. From the reference given we also know

$$|t_{kn}| \leqq \frac{c_4}{-m} \qquad \text{for} \quad m \leqq -k^{1/3}$$

$$|t_{kn}| \leqq \frac{c_4}{k^{1/3}} \qquad \text{for} \quad -k^{1/3} < m \leqq k^{1/3}$$

$$|t_{kn}| \leqq \frac{c_4}{k^{1/4} m^{1/4}} \qquad \text{for} \quad k^{1/3} \leqq m.$$

Hence

$$\sum_{\substack{n \\ m \leqq -k^{1/3}}} |t_{kn}|^2 \leqq 2c_4^2 \sum_{-\infty < m \leqq -k^{1/3}} \frac{1}{m^2} \leqq \frac{c_5}{k^{1/3}},$$

$$\sum_{\substack{n \\ -k^{1/3} < m \leqq k^{1/3}}} |t_{kn}|^2 \leqq \frac{2c_4^2}{k^{2/3}} \sum_{-k^{1/3} < m \leqq -k^{1/3}} 1 \leqq \frac{c_6}{k^{1/3}}$$

$$\sum_{\substack{n \\ k^{1/3} < m \leqq M}} |t_{kn}|^2 \leqq \frac{2c_4^2}{k^{1/2}} \sum_{k^{1/3} < m \leqq M} \frac{1}{m^{1/2}} \leqq c_7 \frac{M^{1/2}}{k^{1/2}}$$

and with $M = \eta k$ in the last sum we can fix η sufficiently small, in order to make the three contributions together less than $1/2$ for k, i.e. N large enough. Taking into account the definition of m, we have proved

$$\sum_{n \leqq (p+\eta)k} + \sum_{n \geqq (q-\eta)k} |t_{kn}|^2 \leqq \frac{1}{2}.$$

This will suffice for (6) if for our k's $aN \leqq (p+\eta)k$, $bN \geqq (q-\eta)k$, i.e. $a \leqq (p+\eta)c$, $b \geqq (q-\eta)d$ hold. Putting $c = 1-\varepsilon$, $d = 1+\varepsilon$ this and condition (3) become

$$\frac{1+\varepsilon}{q} = (1+\varepsilon)p < a \leqq (p+\eta)(1-\varepsilon),$$

$$(q-\eta)(1+\varepsilon) \leqq b < \frac{1-\varepsilon}{p} = (1-\varepsilon)q$$

and these can be fulfilled with a sufficiently small $\varepsilon > 0$, $\eta > 0$ being fixed.

With the constants so specified we have completed the proof of(5) for N large enough. Let $\varrho = \frac{d}{c}$ (>1), $N = \frac{1}{c}\varrho^r$ $(r = 0, 1, \ldots)$. Summing the inequality for r, the left hand side converges owing to (2), since each n is covered a bounded number of times by the intervals $\left(\frac{a}{c}\varrho^r, \frac{b}{c}\varrho^r\right)$ $(r = 0, 1, \ldots)$, implying

$$\sum_{r=0}^{\infty} \sqrt{\varrho^r \sum_{\varrho^r < k \leqq \varrho^{r+1}} |\alpha_k|^2} < +\infty.$$

This is a contradiction if we can replace ϱ by 2.

Denoting by $\sum_r{}_l$ summation over those r for which $(\varrho^r, \varrho^{r+1}]$ intersects $(2^l, 2^{l+1}]$ we have

$$2^l \sum_{2^l < k \leqq 2^{l+1}} |\alpha_k|^2 \leqq 2^l \sum_r{}_l \sum_{\varrho^r < k \leqq \varrho^{r+1}}' |\alpha_k|^2 \leqq \varrho \sum_r{}_l \varrho^r \sum_{\varrho^r < k \leqq \varrho^{r+1}} |\alpha_k|^2$$

and by an elementary inequality

$$\sqrt{2^l \sum_{2^l < k \leqq 2^{l+1}} |\alpha_k|^2} \leqq \sqrt{\varrho} \sum_r{}_l \sqrt{\varrho^r \sum_{\varrho^r < k \leqq \varrho^{r+1}} |\alpha_k|^2}.$$

As there are only a bounded number of intervals $(2^l, 2^{l+1}]$ meeting $(\varrho^r, \varrho^{r+1}]$, summing over $l=0, 1, 2, \ldots$, we get

$$\sum_{l=0}^{\infty} \sqrt{2^l \sum_{2^l<k\leqq 2^{l+1}} |\alpha_k|^2} \leqq c_8 \sum_{r=0}^{\infty} \sqrt{\varrho^r \sum_{\varrho^r<k\leqq\varrho^{r+1}} |\alpha_k|^2}\,.$$

Q.e.d.

PROOF OF THEOREM 1. For $2^l<k\leqq 2^l+m_l$ $(l=1, 2, \ldots, m_l\leqq 2^l)$ let α_k be constant $c_l>0$ and 0 for k outside these intervals. The random power series $f(z)=\sum_{k=0}^{\infty} \pm\alpha_k z^a$ converges absolutely for $|z|\leqq 1$ if

$$\sum_{l=1}^{\infty} c_l \sum_{2^l<k\leqq 2^l+m_l} 1 = \sum_{l=1}^{\infty} c_l m_l < +\infty$$

e.g. choosing $c_l=\dfrac{1}{m_l l^{1+\varepsilon}}$. The transformed series diverges absolutely for $|w|=1$ with probability 1 provided that

$$\sum_{l=1}^{\infty} \sqrt{2^l \sum_{2^l<k\leqq 2^l+m_l} \alpha_k^2} = \sum_{l=1}^{\infty} c_l \sqrt{2^l m_l} = \sum_{l=1}^{\infty} \frac{\sqrt{2^l}}{\sqrt{m_l}\, l^{1+\varepsilon}} = +\infty$$

by Theorem 2, (ii). This will be fulfilled by the choice $m_l=\left[\dfrac{2^l}{l^{2\varepsilon}}\right]+1$ in which case

$$c_l = \frac{1}{m_l l^{1+\varepsilon}} \leqq \frac{1}{2^l l^{1-\varepsilon}}$$

and to prove the existence of (a multitude of) functions required by our theorem it remains to show that $f(z)$ satisfies, with probability 1, a Lipschitz condition of exponent 1/2. Now,

$$f(z) = \sum_{l=1}^{\infty} c_l \sum_{2^l<k\leqq 2^l+m_l} r_k(\omega) z^k \overset{\text{def}}{=} \sum_{l=1}^{\infty} P_l(z)$$

and we have to estimate $f(e^{it_1})-f(e^{it_2})$ for $|t_1-t_2|\leqq\delta$. For the partial sums of the random series $\sum_{k=0}^{\infty} r_k(\omega)z^k$ one has, by a result of SALEM and ZYGMUND ([9])

$$\max_{|z|\leqq 1} \left| \sum_{k=0}^{n} r_k(\omega) z^k \right| = O\left(\sqrt{n \log n}\right)$$

with probability 1, the constant involved in $O(\,)$ might depend on ω. In such a case, applying it with $n=2^l+m_l$ $(\leqq 2^{l+1})$ and $n=2^l$ we get

$$\max_{|z|\leqq 1} |P_l(z)| = O\left(c_l \sqrt{2^l l}\right) = O\left(\frac{1}{2^{l/2} l^{(1/2)-\varepsilon}}\right)$$

and also, since $P_l(z)$ is a polynomial of degree less than 2^{l+1}, using a theorem of S. BERNSTEIN (see [4], Ch. X, Theorem (3.13))

$$|P_l(e^{it_1})-P_l(e^{it_2})| \leqq |t_1-t_2| \max_{|z|\leqq 1} |P_l'(z)| \leqq \delta 2^{l+1} \max_{|z|\leqq 1} |P_l(z)| = O\left(\frac{\delta 2^{l/2}}{l^{(1/2)-\varepsilon}}\right).$$

In the expression

$$f(e^{it_1})-f(e^{it_2}) = \sum_{l=1}^{\infty} \{P_l(e^{it_1})-P_l(e^{it_2})\}$$

we use for $l>L=L(\delta)$ for both t_1 and t_2 the first of these and for $l\leqq L$ the second, giving

$$f(e^{it_1})-f(e^{it_2}) = O(\delta)\sum_{l\leqq L}\frac{2^{l/2}}{l^{(1/2)-\varepsilon}}+O(1)\sum_{l>L}\frac{1}{2^{l/2}\,l^{(1/2)-\varepsilon}} =$$

$$= O\left(\delta\,\frac{2^{L/2}}{L^{(1/2)-\varepsilon}}\right)+O\left(\frac{1}{2^{L/2}\,L^{(1/2)-\varepsilon}}\right).$$

The optimal choice of L is $2^L=\frac{1}{\delta}$ and we get

$$f(e^{it_1})-f(e^{it_2}) = O\left(\frac{\sqrt{\delta}}{\log^{(1/2)-\varepsilon}\frac{1}{\delta}}\right) \qquad (|t_1-t_2|\leqq\delta\to 0)$$

which is slightly more than what we had to prove.

(Our probabilistic proof cannot, however, yield the sharp exponent of $\log\frac{1}{\delta}$ which is likely to be 1.)

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References

[1] L. ALPÁR, Sur certaines transformées des séries de puissance absolument convergentes sur la frontière de leur cercle de convergence, *Magyar Tud. Akad. Mat. Kut. Int. Közl.*, **7** (1962), pp. 287—316.

[2] G. HALÁSZ, On Taylor series absolutely convergent on the circumference of the circle of convergence, I, *Publ. Math. Debrecen*, **14** (1967), pp. 63—68.

[3] G. HALÁSZ, On Taylor series absolutely convergent on the circumference of the circle of convergence, II, *Publ. Math. Debrecen*, **15** (1968), pp. 23—31.

[4] A. ZYGMUND, *Trigonometric Series*, I (Cambridge, 1959).

[5] K.-H. INDLEKOFER, Bemerkungen über äquivalente Potenzreihen von Funktionen mit einem gewissen Stetigkeitsmodul, *Monatschefte für Math.*, **76** (1972), pp. 124—129.

[6] J. CLUNIE, On equivalent power series, *Acta Math. Acad. Sci. Hung.*, **18** (1967), pp. 165—169.

[7] W. SCHWARZ, Bemerkungen zu einem Satz der Herren Turán und Clunie II, *Publ. Math. Debrecen*, **18** (1971).

[8] G. HALÁSZ, On the behaviour of Taylor series under conformal mappings of the circle of convergence, *Studia Sci. Math. Hung.*, **1** (1966), pp. 389—401.

[9] R. SALEM and A. ZYGMUND, Some properties of trigonometric series whose terms have random signs, *Acta Math.*, **91** (1954), pp. 245—301.

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A REMARK ON CONTRACTION OF GRAPHS WITH GIVEN GROUP

By

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1. In this paper we are going to improve a result of the author [2] concerning the relationship between the automorphism group and connected homomorphic images of graphs. The proof is a modification of the original one; we do not repeat those parts which remain unchanged.

2. We deal only with graphs without loops and multiple edges. $V(X)$ denotes the set of vertices, $E(X)$ the set of edges of X.

Let X and Y be graphs. A mapping $f: V(X) \to V(Y)$ of the vertex set of X onto the vertex set of Y is called connected homomorphism (or contraction) if

a) $[y_1, y_2] \in E(Y) \Leftrightarrow y_1 \neq y_2$ and there is an edge $[x_1, x_2]$ in X for which $f(x_1) = y_1$, $f(x_2) = y_2$;

b) for any $y \in V(Y)$ the full subgraph of X on the subset $f^{-1}(y)$ of $V(X)$ is connected.

The graph X is said to be homomorphic to the graph Z, denoted by $X \succ Z$, if Z is a subgraph of some $f(X)$ where f is a connected homomorphism.

$A(X)$ denotes the automorphism group of X. For definitions of undefined terms we refer to [3].

3. The main result of [2] is the following:

[2, Theorem 3.1]. *Given a class $\mathscr{C}$ of (not necessarily finite) graphs, closed under connected homomorphisms and forming subgraphs, assume that for any finite group G there exists a graph in $\mathscr{C}$, the automorphism group of which is isomorphic to G. Then $\mathscr{C}$ contains all finite graphs.*

The more detailed form of the above theorem, [2, Theorem 3.1'] states that if $A(X)$ is a simple group, containing a subgroup isomorphic to Z_p^n, the direct product of n copies of the cyclic group of order p where p is an odd prime number, p and n are large enough, then X is homomorphic to some large complete graph. In this section we are going to improve this result by showing that n does not need to be large.

THEOREM 1. *Let G be a finite simple group containing a subgroup isomorphic to $Z_p \times Z_p \times Z_p$. Assume that p is an odd prime number, $p \geqq 2^{3N-7}$ $(N \geqq 3)$. Then any graph, the automorphism group of which is isomorphic to G, is homomorphic to K_N, the complete graph on N vertices.*

Analogously, we improve [2, Theorem 3.3]. Let G denote the same group as in Theorem 1.

THEOREM 2. *Assume that the automorphism group of the connected graph X contains a subgroup isomorphic to G, acting fixed-point-free on $V(X)$. Then X is homomorphic to K_N.*

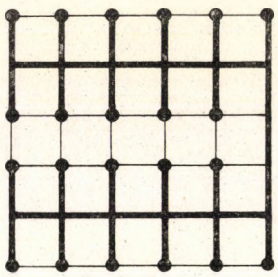
4. To prove these theorems we need a lemma. Let $\times$ denote the Cartesian product of graphs (see [4], [5], [1]), and P_N the path of length $N-1$.

LEMMA. *$P_N \times P_N \times P_N \succ K_N$ holds for any $N \geqq 1$.*

PROOF. Let $V(P_N)=\{1, \ldots, N\}$, $E(P_N)=\{[i, i+1]: i=1, \ldots, N-1\}$.

For $N \leqq 3$ the statement is obvious. Let $N \geqq 4$. One can easily find a spanning tree T in $P_N \times P_N$, containing at least $(N-1)\cdot\left[\frac{2}{3}N\right]$ end vertices. (See the figure, $N=6$.) As $\left[\frac{2}{3}N\right] \geqq \frac{N}{2}$ $(N \geqq 4)$, we have at least $\binom{N}{2}$ end vertices in T. Let f denote a one-to-one function which maps $E(K_N)$ into the set of end vertices of T.

We define the graph Z by contracting all edges of $Y=P_N \times P_N \times P_N$ of type $[(a, b, c), (a', b', c')]$ where $a=a'$ and $[(b, c), (b', c')]$ is an edge of T which is not incident with the set $f(E(K_N))$. Let $g(a)$ denote the image of $(a, b, c) \in V(Y)$ under this contraction. Here $(b, c) \notin f(E(K_N))$ is assumed, thus $g(a)$ is well-defined.



The tree T in $P_6 \times P_6$

Z contains a subgraph, homomorphic to K_N:

$g(i)$ and $g(j)$ $(1 \leqq i < j \leqq N)$ are connected by a path through the vertices $\{(a, b, c): i \leqq a \leqq j,\ (b, c) = f((i, j))\}$. As f is one-to-one, these paths are pairwise disjoint (apart from their end vertices $g(i): i=1, \ldots, N$). This proves the lemma.

PROOF OF THEOREMS 1 AND 2. The proof goes on like in [2]. Change must be made only in section C) of the proof mentioned. The following statement has been proved there: *The Cayley colour-graph $X_{H,S}$ is homomorphic to K_N for any generating system S of the group H where H is isomorphic to Z_p^n, $n \geqq 2^{N-3}$, p a prime number, $p \geqq 3$.* We assert that the same holds for $n \geqq 3$, $p \geqq N$. As in Theorems 1 and 2 $p \geqq 2^{3N-7} \geqq N$ $(N \geqq 3)$, this yields the improvement.

$X_{H,S}$ is defined by

$$V(X_{H,S}) = H, \quad E(X_{H,S}) = \{[h, sh]: h \in H, s \in S, s \neq 1\}.$$

It is easily seen [1, Proposition 2] that $X_{H,S}$ contains a subgraph isomorphic to $C_p \times \ldots \times C_p$ (n-times) where C_p denotes the cycle of length p. Hence $X_{H,S}$ contains also a subgraph isomorphic to $P_N \times P_N \times P_N$ (as $p \geqq N$, $n \geqq 3$). An application of the Lemma completes the proof.

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References

[1] L. BABAI, Groups of graphs on given surfaces, *Acta Math. Acad. Sci. Hung.*, **24** (1973), pp. 215—221.
[2] L. BABAI, Automorphism groups of graphs and edge contraction, *Discrete Math.*, to appear.
[3] F. HARARY, *Graph theory*, Addison-Wesley (Reading, 1969).
[4] G. SABIDUSSI, Graph multiplication, *Math. Zeitschrift*, **72** (1960), pp. 446—457.
[5] G. SABIDUSSI, Graphs with given group and given graph theoretical properties, *Canad. J. Math.*, **9** (1957), pp. 515—525.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (1—2), (1974), pp. 93—98.

NOTE ÜBER KANTENSCHNITTVERBÄNDE IN GRAPHEN

Von

F. ESCALANTE (Mexico) und T. GALLAI (Budapest)

G sei ein schlichter, ungerichteter, zusammenhängender Graph; $E(G)$ bzw. $K(G)$ bezeichne die Ecken- bzw. Kantenmenge von G, und für $x, y \in E(G)$ bezeichne (x, y) die Kante mit Endecken x, y. Es seien a, b zwei verschiedene Ecken von G. Eine Teilmenge T der Ecken-(Kanten-)Menge von G heißt ein *a, b-Ecken-(Kanten-) Schnitt*, wenn T, aber keine echte Teilmenge T' von T, die Ecken a und b trennt, d.h., wenn in $G-T$ a und b zu verschiedenen Komponenten gehören, und in $G-T'$ stets ein a, b-Weg existiert. Ecken-(Kanten-)Schnitte mit kleinster Kardinalzahl heißen *minimal*.

Sind S, T a, b-Ecken-(Kanten-)Schnitte und ordnet man sie so, daß $S \leqq T$ genau dann gilt, wenn kein a, b-Weg zuerst T, dann S trifft, so bilden die a, b-Ecken- bzw. Kantenschnitte bezüglich dieser Teilordnung einen vollständigen Verband $V_1(G; a, b)$ bzw. $V_2(G; a, b)$, während im Falle endlicher a, b-Mengerschen Zahl die minimalen a, b-Ecken- bzw. Kantenschnitte einen endlichen distributiven Verband $U_1(G; a, b)$ bzw. $U_2(G; a, b)$ bilden (zum Beweise vgl. etwa [1], Sätze 1, 3 und 7). Ein vollständiger Verband bzw. ein endlicher distributiver Verband V heißt *V_i- bzw. U_i-darstellbar*, $i = 1, 2$, wenn es einen schlichten Graphen (d.h. ohne Schlingen und Mehrfachkanten) G und $a, b \in E(G)$ gibt derart, daß $V_i(G; a, b) \simeq V$ bzw. $U_i(G; a, b) \simeq V$.[1] In analoger Weise lassen sich diese Definitionen auf gerichtete Graphen übertragen.

In [1] wurde gezeigt:

(1) Jeder vollständige Verband ist V_1-darstellbar;

(2) Jeder endliche distributive Verband ist U_1-darstellbar;

(3) Nicht jeder vollständige Verband ist V_2-darstellbar (etwa der Verband, der aus den Ketten $x < u < v < y$ und $x < w < y$ besteht; vgl. [1], Satz 8. Man kann sogar zeigen, daß in jedem $V_2(G; a, b)$ die Ketten zwischen je zwei vergleichbaren Elementen gleichlang sein müssen. Es wird vom ersten Verfasser geplant, den Beweis dieses und anderer Sachverhalten in einer weiteren Note zusammenzufassen).

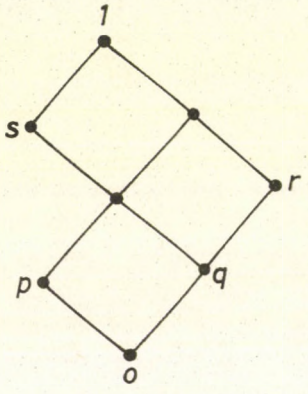
Das Ziel dieser Note ist es zu zeigen, daß jeder endliche distributive Verband U_2-darstellbar ist; damit bleibt nur die Frage nach der V_2-Darstellbarkeit offen.

Wir beweisen zunächst:

SATZ 1. *Zu jedem endlichen distributiven Verband $(V, \leqq)$ gibt es einen (nicht notwendigerweise schlichten) Graphen G und a, b in $E(G)$ derart, daß $U_2(G; a, b) \simeq V$.*

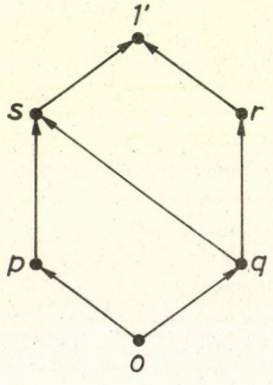
[1] Eigentlich kann man von der Forderung absehen, daß die Graphen schlicht sein müssen, wie es offenbar im Falle der Eckenschnitte und der allgemeinen Kantenschnitte ist, und wie wir in einem kommenden Lemma für den Fall der minimalen Kantenschnitte andeuten werden.

BEWEIS. 0 und 1 bezeichnen das Null- bzw. Einselement von V und sei I die Menge aller Vereinigung-irreduziblen Elemente von V, d.h., $I=\{0\}\cup\{x\in V|x$ hat genau einen unteren Nachbarn$\}$. Sei $1'\notin V$ und sei $(I\cup\{1'\}, \leqq)$ die Teilordnung, deren Einschränkung auf I die von $(V, \leqq)$ induzierte Teilordnung ist, und für die die zusätzliche Vorschrift gilt: Für alle $x\in I$ ist $x<1'$.[2] Der gerichtete Graph $\vec{G}_0$ stelle das Hassediagramm der Teilordnung $(I\cup\{1'\}, \leqq)$ dar, d.h.: $E(\vec{G}_0)=I\cup\{1'\}$ und zwei verschiedene Ecken $x, y\in E(\vec{G}_0)$ sind in $\vec{G}_0$ genau dann durch eine gerichtete Kante (x, y)[3] verbunden, wenn y oberer Nachbar von x ist, also wenn $x<y$ und für kein $z\in E(\vec{G}_0)$ gilt $x<z<y$ (vgl. Fig. 1a, 1b).



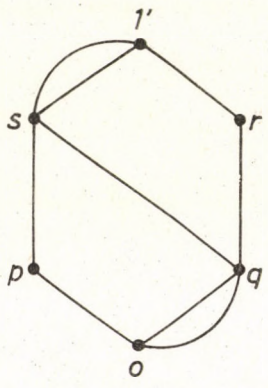
V

Fig. 1a



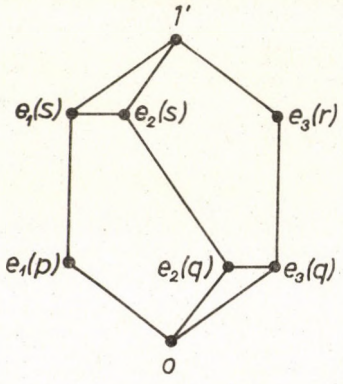
$\vec{G}_0$

Fig. 1b



G

Fig. 1c



G'

Fig. 1d

[2] $x<y$ bedeutet: $x\leqq y$ und $x\neq y$.

[3] Wir bezeichnen in gleicher Weise gerichtete wie ungerichtete Kanten, wenn keine Verwirrung entstehen kann. Bei einer gerichteten Kante (x, y) soll x als Anfangs-, y als Endecke aufgefaßt werden.

Im folgenden werden wir gelegentlich $\vec{G}_0$ mit der Teilordnung $(I \cup \{1'\}, \leqq)$ identifizieren, so daß die Ecken von $\vec{G}_0$ mit ihrer Ordnung $\leqq$ aufzufassen sind.

Für $x \in E(\vec{G}_0)$ bezeichne $O(x)$ bzw. $U(x)$ die Menge aller oberen bzw. unteren Nachbarn von x. Gilt für ein $x \in E(\vec{G}_0) - \{0, 1'\}$ die Ungleichheit $d := |O(x)| - |U(x)| \neq 0$ ($|M|$ bezeichnet die Mächtigkeit der Menge M), so verbinden wir, falls $d > 0$, 0 mit x durch d neue gerichtete $(0, x)$-Kanten; ist dagegen $d < 0$, so fügen wir $-d$ neue gerichtete $(x, 1')$-Kanten hinzu.[4]

In dieser Weise erhält man einen zusammenhängenden, gerichteten Graphen $\vec{G}$ mit folgenden Eigenschaften:

(i) $\vec{G}$ ist endlich;

(ii) $\vec{G}$ ist azyklisch;

(iii) Für jede Ecke $x \neq 0, 1'$ ist die Anzahl der hinlaufenden Kanten gleich der Anzahl der auslaufenden Kanten; mit 0 inzidieren nur aus-, mit $1'$ nur hinlaufende Kanten.

(iv) $\vec{G}$ läßt sich in gerichtete, kantendisjunkte 0, $1'$-Wege zerlegen (denn jeder gerichtete 0, $1'$-Weg $\vec{W}$ ist endlich und der Graph $\vec{G} - \vec{W}$ besitzt offenbar auch die Eigenschaften (i), (ii), (iii)).

Es sei nun $\vec{G}$ in die kantendisjunkten (gerichteten) 0, $1'$-Wege $\vec{W}_1, \ldots, \vec{W}_n$ $(n \geqq 1)$ zerlegt. Jeder 0, $1'$-Schnitt S enthält mindestens eine Kante von jedem $\vec{W}_i$, also $|S| \geqq n$. Die n mit 0 inzidierenden Kanten bilden einen 0, $1'$-Schnitt S_0 mit Mächtigkeit n. Daher folgt, daß S_0 minimal ist, und daß jeder minimale 0, $1'$-Schnitt genau eine Kante von jedem $\vec{W}_i$ enthält. Da der Weg $\vec{W}_1$ beliebig definiert werden kann, können wir folgern:

a) Jeder gerichtete 0, $1'$-Weg hat genau eine Kante mit jedem minimalen 0, $1'$-Schnitt gemeinsam.

Wir bemerken weiter:

b) In $\vec{G}$ (wie in jedem azyklischen Graphen) ist jede gerichtete Kantenfolge ein Weg.

c) Für $x, y \in E(\vec{G})$ gilt: $x < y \Leftrightarrow$ Es gibt in $\vec{G}$ einen gerichteten x, y-Weg.

Läßt man die Orientierung der Kanten von $\vec{G}$ außer acht, so erhält man einen ungerichteten Graphen G (Fig. 1c). Bezeichnet W_i den in dieser Weise aus $\vec{W}_i$ erhaltenen Weg, so ist G die kantendisjunkte Vereinigung der 0, $1'$-Wege $W_1, \ldots, W_n$, und die Mächtigkeit der minimalen Schnitte in G ist ebenfalls n.

Wir wollen zeigen, daß $U_2(G; 0, 1') \simeq V$, was wir mit Hilfe des Graphen $\vec{G}$ durchführen werden.

Eine nichtleere Menge $A \subseteqq E(G) - \{1'\}$ (und entsprechend in $\vec{G}$) heiße *nach unten abgeschlossen*, wenn gilt: $x \in A \Rightarrow y \in A$ für alle $y \leqq x$. Ferner bezeichne K_A die Menge aller Kanten $(x, y) \in K(G)$ mit $x \in A$ und $y \in E(G) - A$. Wir zeigen:

(1) Ist A nach unten abgeschlossen, so ist K_A ein minimaler 0, $1'$-Schnitt von $\vec{G}$. Denn: Aus $0 \in A$ und $1' \in E(\vec{G}) - A$ folgt, daß jeder gerichtete 0, $1'$-Weg $\vec{W}$ eine Kante von K_A enthält. K_A trennt 0 und $1'$. Ferner kann $\vec{W}$ nicht mehr als eine Kante von K_A enthalten. Denn wären $(u, v), (u', v') \in K(\vec{W}) \cap K_A$, in der Reihenfolge von 0 nach $1'$ angegeben, so wäre $v \leqq u'$, $v \notin A$, $u' \in A$, was ein Widerspruch

[4] Erhält ein Graph Mehrfachkanten, so sollen jene Kanten, die die Ecken x und y verbinden *(x, y)-Kanten* heißen. Die für eine bestimmte Kante eingeführte Bezeichnung (x, y) ist in diesem Falle im allgemeinen nicht eindeutig. In unserem Beweis verursacht jedoch diese Bezeichnungsweise kein Mißverständnis, sie soll daher auch weiterhin benutzt werden.

zur Abgeschlossenheit von A wäre. Jedes $\vec{W}_i$ enthält also genau eine Kante von K_A, es gilt also $|K_A|=n$.

Da K_A die Ecken 0 und 1′ auch in G trennt, so ist K_A auch ein minimaler Schnitt von G.

(2) Jeder minimale 0, 1′-Schnitt hat die Form K_A. Denn sei

$$S:=\{(x_1, y_1), \ldots, (x_n, y_n)\}$$

ein minimaler Schnitt von $\vec{G}$ und sei $A=\bigcup_{i=1}^{n}\{z\in E(\vec{G})|z\leqq x_i\}$. A ist (nach unten) abgeschlossen. Wir zeigen, daß $y_i\notin A$ für alle $i=1, \ldots, n$. Wäre nämlich $y_i\leqq x_j$ (natürlich $i\neq j$), dann wäre $x_i<y_i\leqq x_j<y_j$ und, nach b) und c) oben, gäbe es einen gerichteten 0, 1′-Weg (die Zusammensetzung von 0, x_i-, y_i, x_j-, y_j, 1′-Wegen und den Kanten (x_i, y_i), (x_j, y_j)), der zwei Kanten eines einzigen minimalen Schnittes enthielte, entgegen der Bemerkung a). Daher ist $S\subseteqq K_A$. Dies und $|K_A|=n$ ergeben $S=K_A$.

Es sei nun S ein minimaler Schnitt von G. S trennt 0 und 1′ auch in $\vec{G}$ und wegen $|S|=n$ ist dann S auch in $\vec{G}$ ein minimaler Schnitt.

Ω bezeichne die Familie aller nach unten abgeschlossenen Mengen von $E(G)-\{1'\}$ und sei $(\Omega, \subseteqq)$ die bezüglich Inklusion definierte Teilordnung auf Ω. $(\Omega, \subseteqq)$ ist ein distributiver Verband. Es gilt:

(3) $(\Omega, \subseteqq)\simeq U_2(\vec{G}; 0, 1')$, $(\Omega, \subseteqq)\simeq U_2(G; 0, 1')$.

Beweis. Wir behaupten, daß die Isomorphien durch die Abbildung $A\to K_A$ hergestellt werden.

a) Es seien $A_1, A_2\in\Omega$ mit $A_1\subseteqq A_2$. W sei ein beliebiger ungerichteter (bzw. gerichteter) 0, 1′-Weg von G (bzw. $\vec{G}$). Es sei (x_2, y_2) die erste Kante von K_{A_2}, die W, in seinem Lauf von 0 nach 1′, zuerst trifft. Ist $x_2\in A_1$, so gilt $(x_2, y_2)\in K_{A_1}$; ist dagegen $x_2\in A_2-A_1$, so enthält der zwischen 0 und x_2 liegende Teil von W eine Kante von K_{A_1}. Also trifft W den Schnitt K_{A_2} nicht vor K_{A_1}, d.h., $K_{A_1}\leqq K_{A_2}$ sowohl in $\vec{G}$ als in G.

b) Gilt $K_{A_1}\leqq K_{A_2}$ in G, so gilt es auch in $\vec{G}$. Nehmen wir also an, daß $K_{A_1}\leqq K_{A_2}$ in $\vec{G}$ gilt. Nach (1) sind K_{A_1} und K_{A_2} minimale 0, 1′-Schnitte. Wir setzen $K_{A_j}=\{(x_{j1}, y_{j1}), \ldots, (x_{jn}, y_{jn})\}$, $j=1, 2$. $\vec{G}$ sei in die kantendisjunkten Wege $\vec{W}_1, \ldots, \vec{W}_n$ zerlegt. Jedes $\vec{W}_i$ enthält genau eine Kante von jedem K_{A_j}, so daß man $(x_{ji}, y_{ji})\in\vec{W}_i$, $(j=1, 2; i=1, \ldots, n)$ annehmen kann. Aus $K_{A_1}\leqq K_{A_2}$ folgt $x_{1i}\leqq x_{2i}$ $(i=1, \ldots, n)$. Nach (2) ist jedoch $A_j=\bigcup_{i=1}^{n}\{z\in E(\vec{G})|z\leqq x_{ji}\}$, $j=1, 2$, also $A_1\subseteqq A_2$.

Schließlich gilt (vgl. etwa [3; Seite 186]), da Ω die Familie aller nach unten abgeschlossenen Teilmengen von $I\subseteqq V$ ist:

(4) $V\simeq(\Omega, \subseteqq)$.

Aus (3) und (4) ergibt sich also die Behauptung von Satz 1.

Als Letztes wollen wir zeigen:

Lemma. *Der Graph G sei in n kantendisjunkte a, b-Wege zerlegbar, $a, b\in E(G)$. In G seien Mehrfachkanten vorhanden. Dann existiert ein schlichter Graph G' und $c, d\in E(G')$, so daß $U_2(G'; c, d)\simeq U_2(G; a, b)$.*

BEWEIS. $W_1, \ldots, W_n$ seien die a, b-Wege, deren Vereinigung G ergibt. Wir konstruieren den Graphen G' folgendermaßen: Jede Ecke $x \neq a, b$ von G, die zu $m \geqq 1$ Wegen $W_{i_1}, \ldots, W_{i_m}$ gehört wird ersetzt durch einen vollständigen m-Graphen mit Ecken $e_{i_1}(x), \ldots, e_{i_m}(x)$ und derart, daß $e_{i_r}(x) \in W_{i_s} \Leftrightarrow r = s$; a und b bleiben unverändert. Damit haben wir alle Ecken von G (bis auf a und b) entweder tatsächlich ersetzt oder einfach umbenannt (falls x nur auf einem Weg liegt) und die n kantendisjunkten Wege zu n eckendisjunkten Wegen gemacht (vgl. Fig. 1d). Offenbar enthält der so konstruierte Graph G' keine Mehrfachkanten. Wir brauchen nur zu zeigen, daß der Verband der minimalen Kantenschnitte unverändert geblieben ist. Dazu führen wir ein, zur Vereinfachung der Sprache, die injektive Abbildung

$$f: K(G) \to K(G')$$

vermöge

$$f(k) := (e_i(x), e_i(y)),$$

wobei k eine Kante mit Endecken x, y ist, und die auf dem Weg W_i liegt; falls z. B. $x = a$, so schreiben wir $f(k) = (a, e_i(y))$, u.s.w. Für $A \subseteqq K(G)$ bezeichnen wir mit $f(A)$ die Menge $\{f(x) | x \in A\}$.[5] f^{-1} ist also auf $M := \bigcup_{i=1}^{n} f(W_i)$ definiert, und wir bemerken, daß, wenn W' ein $e_i(x), e_j(y)$-Weg ist, so gibt es in G einen x, y-Kantenzug (im allgemeinen aber keinen x, y-Weg!) K mit $f(K) = W' \cap M$.

Ferner, für jedes $S \in U_2(G; a, b)$ ist $f(S) \in U_2(G'; a, b)$, denn: (i) $|f(S)| = n$; und (ii) sei W' ein a, b-Weg in G'; dann gilt für den a, b-Kantenzug K mit $f(K) = W' \cap M$, daß $K \cap S \neq \varnothing$ ist, also ist $f(K) \cap f(S) \neq \varnothing$, und da $f(K) \subseteqq W'$, so ist recht $W' \cap f(S) \neq \varnothing$.

Umgekehrt, ist S' ein minimaler a, b-Schnitt in G', so muß S' in M enthalten sein (weil $|S'| = n$); wäre für einen a, b-Weg W in G etwa $f^{-1}(S') \cap W = \varnothing$, so müßte $S' \cap f(W) = \varnothing$ gelten; aber $f(W)$ könnte durch Adjunktion von Kanten aus $K(G') - M$ zu einem a, b-Weg in G' gemacht werden, der mit S' einen leeren Durchschnitt hätte. Widerspruch. Also muß $f^{-1}(S') \in U_2(G; a, b)$ sein. Schließlich, da wir offenbar $S \subseteqq T \Leftrightarrow f(S) \subseteqq f(T)$, für je zwei $S, T \in U_2(G; a, b)$ haben, so folgt unsere Behauptung $U_2(G'; a, b) \simeq U_2(G; a, b)$.

Satz 1 zusammen mit dem Lemma liefern das Ziel dieser Note:

SATZ 2. *Jeder endliche distributive Verband ist U_2-darstellbar.*[6]

(Eingegangen am 22. Februar 1973.)

[5] Bezeichnet W einen Weg (oder einen Kantenzug), so wird, wenn kein Mißverständnis entstehen kann, $K(W)$ einfach durch W ersetzt.

[6] Man kann sogar zeigen: Sei H ein Graph, der möglicherweise Mehrfachkanten enthält und $c, d \in E(H)$ zwei Ecken mit Kanten-Mengerschen Zahl n. Dann gibt es einen Graphen G und $a, b \in E(G)$, derart, daß G aus genau n kantendisjunkten a, b-Wegen besteht und $U_2(G; a, b) \simeq U_2(H; c, d)$. Damit und mit Hilfe des obigen Lemmas könnte man von der Forderung über die Schlichten Graphen bei der Definition der U_2-Darstellbarkeit absehen. Für die Zwecke dieser Note genügt aber das Lemma.

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Literaturverzeichnis

[1] F. ESCALANTE, Schnittverbände in Graphen, *Abh. Math. Seminar Hamburg*, **38** (1972), S. 199—220.
[2] O. ORE, *Theory of Graphs.* Amer. Math. Soc. Colloq. Publ. 38 (Providence, 1962).
[3] G. ROTA and L. H. HARPER, *Matching Theory, an Introduction.* Advances in Probability, Vol 1, 1971, Marcel Dekker Inc., New York, pp. 170—215.

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ON ESTIMATIONS OF THE GREATEST ZEROES OF ORTHOGONAL POLYNOMIALS

By
G. FREUD (Budapest)

1. Let $w(x)$ be a nonnegative $\mathscr{L}$-integrable weight in $(-\infty, \infty)$, for which

$$\int_{-\infty}^{\infty} x^{2n} w(x)\,dx < \infty \qquad (n = 1, 2, \ldots).$$

To such a weight there exists a uniquely determined sequence of orthogonal polynomials $\{p_n(w; x)\}$ where $p_n(w; x)$ is of degree n,

$$p_n(w; x) = \gamma_n(w)x^n + \ldots, \qquad \gamma_n(w) > 0$$

and

$$\int_{-\infty}^{\infty} p_n(w; x)p_m(w; x)w(x)\,dx = \begin{cases} 1 & (m=n) \\ 0 & (m \neq n) \end{cases}.$$

As is wellknown (see e.g. [2], Theorem I.2.2) all zeroes of $p_n(w; x)$ are real and simple. We denote the greatest zero of $p_n(w; x)$ by $x_{1n}(w)$, and the smallest zero by $x_{nn}(w)$. The present paper is devoted to an estimate of $x_{1n}(w)$ and $X_n(w) = \max(x_{1n}, -x_{nn})$ (see Theorems 1 and 2 below).

In our paper [3] we discovered the following relation among $\{x_{1n}(w)\}$ and the sequence of leading coefficients $\{\gamma_\nu(w)\}$:

THEOREM A. *Let*

$$\Gamma_n(w) = \max_{1 \leq \nu \leq n-1} \frac{\gamma_{\nu-1}(w)}{\gamma_\nu(w)} \tag{1}$$

then we have for every even w

$$\Gamma_n(w) \leqq x_{1n}(w) \leqq 2\Gamma_n(w). \tag{2}$$

In what follows let $c_1, c_2, \ldots$ be positive numerical constants depending on w only.

We apply first Theorem A by connecting it to a later result of the author (see [4]):
Let

$$W_Q(x) = \exp\{-2Q(x)\} \tag{3}$$

(∗) *where* $Q(t)$ *is an even, positive and in* $(0, \infty)$ *continuously differentiable function for which* $Q'(t)$ *is increasing for* $t > 0$.

Consequently $tQ'(t)$ is increasing for $0<t<\infty$ from 0 to ∞. Thus the sequence $\{q_n\}$ is welldefined as the positive solutions of the equations

$$q_n Q'(q_n) = n \qquad (n=1, 2, \ldots). \tag{4}$$

Under the conditions (3), (*) we proved in [4], Lemma 2.7 that

$$\frac{\gamma_{\nu-1}(W_Q)}{\gamma_\nu(W_Q)} \leqq c_1 q_{2\nu} \qquad (\nu = 1, 2, \ldots). \tag{5}$$

To apply our results in [4], we need some Lemmas.

LEMMA 1. *The leading coefficients $\gamma_\nu(w)$ can be characterized by the following extremum problem: Denoting by $\mathcal{P}_n$ the set of polynomials of degree not exceeding n we have*

$$\gamma_n^{-2}(w) = \min_{q \in \mathcal{P}_{n-1}} \int_{-\infty}^{\infty} [x^n + q(x)]^2 w(x)\, dx. \tag{6}$$

(See e.g. G. SZEGŐ [7], Theorem 3.1.2).

The following lemma is a direct consequence of (6):

LEMMA 2. *Under the assumption*

$$w_1(x) \leqq c_2 w_2(x) \quad (-\infty < x < \infty) \tag{7}$$

we have

$$\gamma_\nu(w_2) \leqq c_3 \gamma_\nu(w_1) \qquad (\nu = 1, 2, \ldots). \tag{8}$$

LEMMA 3. *We have under the conditions (*) and* (3)

$$\gamma_n(w_Q) \leqq c_4 e^{c_5 n + Q(q_n)} q_n^{-n-1/2}. \tag{9}$$

PROOF. Let

$$w_{Q,n}^2(x) = \begin{cases} W_Q(x) & (|x| \leqq q_n) \\ 0 & (|x| > q_n) \end{cases}$$

then by [4], Lemma 2.2

$$\gamma_\nu(w_{Qn}^2) \leqq c_6 e^{c_7 n + Q(q_n)} q_n^{-n-1/2}. \tag{10}$$

Since $W_Q(x) \leqq w_{Qn}^2(x)$ $(-\infty < x < \infty)$ by Lemma 2 (9) is consequence of (10) q.e.d.

LEMMA 4. *We have under the conditions* (*), (3)

$$c_8 q_{2n} \leqq \Gamma_n(W_Q) \leqq c_4 q_{2n} \qquad (n = 1, 2, \ldots). \tag{11}$$

PROOF. Let us observe that the sequences q_n and $n/q_n = Q'(q_n)$ are increasing. Thus the right hand side of (11) is implied by (1) and (5).

We have by virtue of (9)

$$\Gamma_{n+1}(W_Q) \geqq \left(\frac{\gamma_0(W_Q)}{\gamma_1(W_Q)} \frac{\gamma_1(W_Q)}{\gamma_2(W_Q)} \cdots \frac{\gamma_{n-1}(W_Q)}{\gamma_n(W_Q)} \right)^{1/n} = \tag{12}$$

$$= [\gamma_0(W_Q)]^{1/n} [\gamma_n(W_Q)]^{-1/n} \geqq [\gamma_0(W_Q)]^{1/n} c_6^{-1/n} e^{-c_7} e^{-Q(q_n)/n} q_n^{1+1/2n}.$$

Since $Q'(t)$ is increasing, by $(*)$

$$0 < Q(q_n) = Q(0) + \int_0^{q_n} Q'(t)\,dt \leqq Q(0) + q_n Q'(q_n) = Q(0) + n$$

so that

$$e^{Q(q_n)/n} \leqq c_9 \tag{13}$$

Since $\{q_n\}$ is increasing to ∞ and $\{q_n/n\}$ is decreasing to 0 we have

$$\lim_{n\to\infty} \sqrt[n]{q_n} = 1. \tag{14}$$

We infer from (12), (13) and (14) that

$$\Gamma_{n+1}(W_Q) \geqq c_{10} q_n. \tag{15}$$

By application of $\frac{q_{n-1}}{n-1} \geqq \frac{q_{2n}}{2n}$ the left hand side of (11) turns to be a consequence of (15), q.e.d.

LEMMA 5. *If Q is even and satisfies* (3), $(*)$ *then*

$$c_{11} q_{2n} \leqq x_{1n}(W_Q) \leqq c_{12} q_{2n}. \tag{16}$$

PROOF. Since W_Q is even, (16) is a consequence of Lemma 4 and Theorem A. Let

$$X_n(w) = \max\{x_{1n}(w), -x_{nn}(w)\}. \tag{17}$$

LEMMA 6. *We have*

$$X_n^2(w) \geqq \max_{q\in\mathcal{P}_{n-2}} \frac{\int_{-\infty}^{\infty} x^2 q^2(x) w(x)\,dx}{\int_{-\infty}^{\infty} q^2(x) w(x)\,dx}. \tag{18}$$

PROOF. Since $q^2(x)$ as well as $x^2 q^2(x)$ belong to $\mathcal{P}_{2n-2} \subset \mathcal{P}_{2n-1}$ we can apply the Gauss—Jacobi quadrature formula with the nodes $x_{kn}(w)$ (see e.g. [2], I (3.9)) in calculating the integrals on the right of (18):

$$\frac{\int_{-\infty}^{\infty} x^2 q^2(x) w(x)\,dx}{\int_{-\infty}^{\infty} q^2(x) w(x)\,dx} = \frac{\sum_{k=1}^{n} \lambda_{kn} x_{kn}^2 q^2(x_{kn})}{\sum_{k=1}^{n} \lambda_{kn} q^2(x_{kn})} \leqq X_n^2(w).$$

q.e.d.

LEMMA 7. *Under the condition*

$$c_{13} w_1(x) \leqq w_2(x) \leqq c_{14} w_1(x) \tag{19}$$

we have

$$\frac{c_{13}^{1/2}}{c_{14}^{1/2}} X_{n-1}(w_1) \leqq X_n(w_2) \leqq \frac{c_{14}^{1/2}}{c_{13}^{1/2}} X_{n+1}(w_1). \tag{20}$$

PROOF. By a classical result of P. L. CHEBYCHEV [1] (see e.g. G. SZEGŐ [7], 7.72) we have

$$x_{1n}(w) = \max_{q \in \mathcal{P}_{n-1}} \frac{\int_{-\infty}^{\infty} x q^2(x) w(x)\, dx}{\int_{-\infty}^{\infty} q^2(x) w(x)\, dx} \tag{21a}$$

and by symmetry

$$x_{mn}(w) = \min_{q \in \mathcal{P}_{n-1}} \frac{\int_{-\infty}^{\infty} x q^2(x) w(x)\, dx}{\int_{-\infty}^{\infty} q^2(x) w(x)\, dx}. \tag{21b}$$

Consequently

$$X_n(w) = \max_{q \in \mathcal{P}_{n-1}} \frac{\left| \int_{-\infty}^{\infty} x q^2(x) w(x)\, dx \right|}{\int_{-\infty}^{\infty} q^2(x) w(x)\, dx}. \tag{22}$$

By (22), (19) and Lemma 6, applying Schwartz's inequality

$$X_n^2(w_2) \leqq \max_{q \in \mathcal{P}_{n-1}} \frac{\int_{-\infty}^{\infty} x^2 q^2(x) w_2(x)\, dx}{\int_{-\infty}^{\infty} q^2(x) w_2(x)\, dx} \leqq$$

$$\leqq \frac{c_{14}}{c_{13}} \max_{q \in \mathcal{P}_{n-1}} \frac{\int_{-\infty}^{\infty} x^2 q^2(x) w_1(x)\, dx}{\int_{-\infty}^{\infty} q^2(x) w_1(x)\, dx} \leqq \frac{c_{14}}{c_{13}} X_{n+1}^2(w_1).$$

This proves the validity of the right inequality in (20). The left inequality in (20) obtained by interchanging the role of w_1 and w_2, q.e.d.

THEOREM 1. *Let Q satisfy* (3), (∗) *and*

$$c_{15} W_Q(x) \leqq w(x) \leqq c_{16} W_Q(x) \tag{23}$$

then

$$c_{17} q_{2n} \leqq X_n(w) \leqq c_{18} q_{2n}. \tag{24}$$

Let us observe that $w(x)$ is not necessarily even.

PROOF. By Lemma 7

$$c_{19}X_{1,n-1}(W_Q) \leqq X_n(w) \leqq c_{20}X_{1,n+1}(W_Q)$$

and by Lemma 5

$$c_{21}q_{2n-2} \leqq X_n(w) \leqq c_{22}q_{2n+2} \tag{25}$$

Since $\{q_n\}$ is increasing and $\{q_n/n\}$ is decreasing, (25) implies (24), q.e.d.

2. It is quite probable that Theorem 1 is valid for a far more extended class of weights. In particular we use this occassion to state the following

HYPOTHESIS. *Let $Q(x)$ be even differentiable for $x \in (0, \infty)$ and let $xQ'(x)$ be increasing. Then* (16) *is valid.*

Let us observe that under the conditions of our hypothesis the sequence $\{q_n\}$ is welldefined. This part 2 of our paper is devoted to provide some useful but more special cases for which a complete proof is available.

THEOREM B. *Let*

$$w_{\alpha\beta}(x) = |x|^{\alpha}e^{-|x|^{\beta}} \quad (\alpha > -1,\ \beta > 0) \tag{26}$$

then

$$c_{23}n^{1/\beta} \leqq x_{1n}(w_{\alpha\beta}) \leqq c_{24}n^{1/\beta}. \tag{27}$$

Theorem B was proved by G. P. NÉVAI [6]; for the special case $\alpha \geqq 0$ and β is an even natural integer, (24) was proved earlier by the author [3].

LEMMA 8. *Let w_1 and w_2 be two even weight functions and let $w_2(x)/w_1(x)$ be non-decreasing for $x>0$ then*

$$x_{1n}(w_1) \leqq x_{1n}(w_2) \qquad (n = 1, 2, \ldots). \tag{28}$$

PROOF. Seting $v_i(x) = x^{1/2}w_i(x^{1/2})$ and $V_i(x) = x^{-1/2}w_i(x^{1/2})$ for $x \geqq 0$ and $v_i(x) = V_i(x) = 0$ for $x < 0$ resp., we have by elementary verification

$$p_{2n}(w_i; x) = p_n(V_i; x^2) \qquad (i = 1, 2;\ n = 0, 1, \ldots) \tag{29a}$$

and

$$p_{2n+1}(w_i; x) = xp_n(v_i; x^2) \qquad (i = 1\ 2;\ n = 0, 1, \ldots) \tag{29b}$$

i.e.

$$x_{1,2n}(w_i) = x_{1n}^{1/2}(V_i) \qquad (i = 1, 2;\ n = 1, 2, \ldots) \tag{30a}$$

and

$$x_{1,2n+1}(w_i) = x_{1n}^{1/2}(v_i) \qquad (i = 1, 2;\ n = 1, 2, \ldots). \tag{30b}$$

Since $v_2(x)/v_1(x)$ as well as $V_2(x)/V_1(x)$ are, by assumption, nondecreasing on their whole common support $[0, \infty)$, we have by a theorem of A. A. MARKOV (see part 3 of this paper)

$$x_{1n}(v_1) \leqq x_{1n}(v_2), \quad x_{1n}(V_1) \leqq x_{1n}(V_2) \qquad (n = 1, 2, \ldots). \tag{31}$$

From (30a), (30b) and (31) follows (28), q.e.d.

LEMMA 9. *Let $W(x)$ be an even weight function such that there exist numbers $\beta>0$, $\alpha_2>\alpha_1>-1$ for which $w_{\alpha_1\beta}^{-1}(x)W(x)$ is nondecreasing for $x>0$ and $w_{\alpha_2\beta}^{-1}(x)W(x)$ is nonincreasing for $x>0$ then*

$$c_{25}n^{1/\beta} \leqq x_{1n}(W) \leqq c_{26}n^{1/\beta}. \tag{32}$$

PROOF. Lemma 9 is a consequence of Theorem B and Lemma 8.

THEOREM 2. *Let $W(x)$ satisfy the conditions of Lemma 9 and*

$$c_{27}W(x) \leqq w(x) \leqq c_{28}W(x) \tag{33}$$

then

$$c_{29}n^{1/\beta} \leqq X_n(w) \leqq c_{30}n^{1/\beta}. \tag{34}$$

PROOF. (31) is a consequence of Lemma 7 and Lemma 9.

3. We applied in this paper A. A. Markov's comparison theorem [5]. We applied also the same theorem at other occassions to weights with infinite support which were not necessarily continuous over their support. Since Markov's theorem is stated for compact support and continuous weights both in the original publication [5] and in the alternative proof in Szegő's book [7], it seems to be of absolute necessity to repeat Szegő's proof in such a form that all unnecessary restrictions are eliminated. We denote — as usual — by $x_{kn}(w)$ $(k=1, 2, \ldots, n)$ the zeroes of $p_n(w; x)$ in decreasing order.

THEOREM C. *Let w_0 and w_1 be two weight functions and let $w_1(x)/w_0(x)$ be nondecreasing on the union of the supports of w_0, w_1. Then*

$$x_{kn}(w_1) \geqq x_{kn}(w_0) \qquad (k = 1, 2, \ldots, n) \tag{36}$$

PROOF. Let $w_\tau(x)=(1-\tau)w_0(x)+\tau w_1(x)$. We consider the moments

$$\mu_n(w_\tau) = \int_{-\infty}^{\infty} x^n w_\tau(x)\,dx = (1-\tau)\mu_n(w_0)+\tau\mu_n(w_1). \tag{37}$$

Then $p_n(w_\tau; x)$ is proportional to the determinant

$$A_n p_n(w_\tau; x) = \begin{vmatrix} \mu_0(w_\tau) & \mu_1(w_\tau) & \ldots & \mu_n(w_\tau) \\ \mu_1(w_\tau) & \mu_2(w_\tau) & \ldots & \mu_{n+1}(w_\tau) \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ \mu_{n-1}(w_\tau) & \mu_n(w_\tau) & \ldots & \mu_{2n-1}(w_\tau) \\ 1 & x & \ldots & x^n \end{vmatrix} \tag{38}$$

where $A_n(w_\tau) \neq 0$ does not depend on x (see e.g. [2] § II. 1). Consequently, $x_{kn}(w_\tau)$ $(k=1, 2, \ldots, n)$ are the zeroes of the determinant (38) and we see by (37) that this determinant is a polynomial in τ. We infer that $x_{kn}(w_\tau)$ $(k=1, 2, \ldots, n;\ n=1, 2, \ldots)$ are algebraic functions of the parameter τ. As such they admit a local expansion of the form

$$x_{kn}(\tau) = x_{kn}(\tau_0) + \sum_{r=1}^{\infty} b_r(\tau-\tau_0)^{r/q} \tag{39}$$

for every $\tau_0 \in (0, 1)$ where $q=q(\tau_0)$ is a positive integer. Since $p_n(w_\tau; x)$ has the exact degree n for every $\tau \in [0, 1]$, all zeroes are finite for every value of τ and thus no term with negative r is possible in the expansion (39). We now recall the fact that all zeroes $x_{kn}(\tau)$ are real for every $0 \leqq \tau \leqq 1$. Consequently we must have $q(\tau_0)=1$ for every $0<\tau_0<1$. This implies that $x_{kn}(w_\tau)$ $(k=1, 2, \ldots, n)$ are differentiable functions of the parameter τ for $0<\tau<1$ and — since the zeroes of an algebraic equation depend continuously on their coefficients — the $x_{kn}(w_\tau)$ are continuous functions of τ in $0 \leqq \tau \leqq 1$.

By the Gauss—Jacobi quadrature formula we have

$$\int_{-\infty}^{\infty} q(x) W_\tau(x)\, dx = \sum_{k=1}^{\infty} \lambda_{kn}(w_\tau)\, q[x_{kn}(w_\tau)] \tag{40}$$

for every $q \in \mathscr{P}_{2n-1}$, where $\lambda_{kn}(w_\tau)$ are the corresponding Christoffel numbers. We fix q and differentiate (41) with respect to τ:

$$\int_{-\infty}^{\infty} q(x)[W_1(x) - W_0(x)]\, dx = \sum_{k=1}^{\infty} \frac{d}{d\tau}[\lambda_{kn}(w_\tau)] \cdot q[x_{kn}(w_\tau)] + \tag{41}$$

$$+ \sum_{k=1}^{n} \lambda_{kn}(w_\tau) q'[x_{kn}(w_\tau)] \frac{dx_{kn}}{d\tau} \qquad (0 < \tau < 1).$$

To verify (41) we only have to prove that $\lambda_{kn}(w_\tau)$ is a differentiable function of τ. We know that (see [2] I. (47))

$$\lambda_{kn}(w_\tau) = \lambda_n(w_\tau; x_{kn}(w_\tau)) \tag{42}$$

where

$$\lambda_n^{-1}(w_\tau; x) = \sum_{\nu=0}^{n-1} p_\nu^2(w_\tau; x). \tag{43}$$

Calculating the coefficients A_n figuring in (38) from the condition that $\int p_n^2(w_\tau) w_\tau\, dx = 1$ we see from (43) that $\lambda_n(w_\tau; x)$ is a positive rational fractional function of τ. Since $\lambda_n(w_\tau; x)$ is finite for every $0 \leqq \tau \leqq 1$ the denominator of this fraction is nonvanishing for $0 \leqq \tau \leqq 1$. Since $\lambda_n(w_\tau; x) > 0$ we infer that for fixed values of x, $\lambda_n(w_\tau; x)$ is a differentiable function of τ. By (43), $\lambda_n(w_\tau; x)$ is also a nonvanishing rational function of x, i.e. a differentiable function of x. In virtue of (42) and the differentiability of $x_{kn}(\tau)$ with respect to τ, $\lambda_{kn}(w_\tau)$ is a differentiable function of τ for $0<\tau<1$.

We have proved that (41) is valid for every $q \in \mathcal{P}_{2n-1}$ and every $0<\tau<1$. Let us fix τ and set in (41)

$$q(x) = \frac{p_n^2(w_\tau; x)}{x - x_{kn}(w_\tau)} \in \mathcal{P}_{2n-1}.$$

We obtain

$$(44) \quad I_1 = \int_{-\infty}^{\infty} \frac{p_n^2(w_\tau; x)}{x - x_{kn}(w_\tau)} [w_1(x) - w_0(x)]\, dx = \lambda_{kn}(w_\tau) \{p_n'[w_\tau; x_{kn}(w_\tau)]\}^2 \frac{dx_{kn}}{d\tau}.$$

In view of the fact that

$$\frac{p_n(w_\tau; x)}{x - x_{kn}(w_\tau)} \in \mathcal{P}_{n-1}$$

is orthogonal to $p_n(w_\tau; x)$, we have

$$(45) \qquad I_2 = \int_{-\infty}^{\infty} \frac{p_n^2(w_\tau; x)}{x - x_{kn}(w_\tau)} w_\tau(x)\, dx = 0.$$

By (44) and (45)

(46)

$$I_1 - \frac{w_1(x_{kn}) - w_0(x_{kn})}{w_\tau(x_{kn})} I_2 = \int_{-\infty}^{\infty} \left\{ \frac{1}{x - x_{kn}(w_\tau)} \left[\frac{w_1(x) - w_0(x)}{w_\tau(x)} - \frac{w_1(x_{kn}) - w_0(x_{kn})}{w_\tau(x_{kn})} \right] \right\} \cdot$$

$$\cdot p_n^2(w_\tau; x) w_\tau(x)\, dx = \lambda_{kn}(w_\tau) \{p_n'[w_\tau; x_{kn}(w_\tau)]\}^2 \frac{d}{d\tau} x_{kn}(w_\tau).$$

Now w_1/w_0 is nondecreasing and at the same time

$$\frac{w_1(x) - w_0(x)}{w_\tau(x)} = \tau^{-1} - \frac{\tau^{-1}}{1 - \tau + \tau[w_1(x)/w_0(x)]}$$

is also a nondecreasing function. Thus the expression in curly brackets in the integrand of (46) is nonnegative. Since $\lambda_{kn}(w_\tau) > 0$ we have by (46)

$$\frac{d}{d\tau} x_{kn}(w_\tau) \geqq 0 \quad (0 < \tau < 1)$$

and consequently

$$x_{kn}(w_1) \geqq x_{kn}(w_0)$$

q.e.d.

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References

[1] П. Л. Чебышев, Об отклонении двух интегралов, распространенных на одни и те же величины переменной, *Сочинения* **3**, pp. 132—156.

[2] G. FREUD, *Orthogonale Polynome*. Joint edition of Akadémiai Kiadó, Birkhäuser Verl. and Deutscher Verl. der W., 1969. English translation by I. Földes (Akadémiai Kiadó—Pergamon Press, 1971).

[3] G. FREUD, On the greatest zero of an orthogonal polynomial, I, *Acta Sci. Math. Szeged*, **34** (1973), pp. 91—97.

[4] G. FREUD, On polynomial approximation with respect to general weight functions. In *Proceedings of the "Conference on functional analysis and its applications (Madras, 1973)"*. Editors: H. G. Garnir, K. R. Unni, T. H. Wiliamson (Springer Verl., 1974).

[5] А. А. Марков, О корнях некоторых уравнений, II, *Избр. Труды*, pp. 44—50.

[6]. G. P. NÉVAI, Многочлены, ортонормальные на вещественной оси с весом $|x|^{\alpha}e^{-|x|^{\beta}}$, *Acta Math. Acad. Sci. Hung.*, **24** (1973), pp. 335—342.

[7] G. SZEGŐ, *Orthogonal Polynomials*, American Math. Soc. Coll. Publ. Second edition, 1959.

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EXTENSION OF THE DIRICHLET—JORDAN CRITERION TO A GENERAL CLASS OF ORTHOGONAL POLYNOMIAL EXPANSIONS

By

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1. Introduction. The Dirichlet—Jordan convergence criterion states that the trigonometric Fourier series of a 2π periodic function $f(x)$ having bounded variation converges for every x to $\frac{1}{2}[f(x+0)+f(x-0)]$ and this convergence is uniform on every closed interval over which f is continuous (see e.g. A. ZYGMUND [24], N. K. BARI [1]).

In the present paper we treat the extension of the criterion to orthogonal polynomial series where the weight has infinite support. Let $w(x)$ be a weight function i.e. a nonnegative $\mathscr{L}$-integrable function for which the moments

$$\mu_{2n}(w) = \int_{-\infty}^{\infty} x^{2n} w(x)\, dx \qquad (n = 0, 1, 2, \ldots)$$

are finite. For such a weight we define the set $\{p_n(w; x)\}$ of orthonormal polynomials: $P_n(w; x)$ is a polynomial of precise degree n and we have

$$\int_{-\infty}^{\infty} p_n(w; x) p_m(w; x) w(x)\, dx = \begin{cases} 1 & (m = n) \\ 0 & (m \neq n). \end{cases}$$

Under the additional condition that the leading coefficients are positive the sequence $\{p_n(w; x)\}$ is uniquely determined (see e.g. [3]).

The orthogonal polynomial expansion of a function $f(x)$ is defined by [1]

$$(1.1) \qquad f(x) \sim \sum_{\nu=0}^{\infty} a_\nu(w; f) p_\nu(w; x), \quad a_\nu(w; f) = \int_{-\infty}^{\infty} f(t) p_\nu(w; t) w(t)\, dt.$$

For the proof of the Dirichlet—Jordan criterion two classical arguments are wellknown. The first is the original reasoning of L. Dirichlet (see e.g. [24]): It represents the partial sums with the aid of the "Dirichlet kernel" and applies the second mean value theorem of integration. An alternative method was initiated by L. FEJÉR [2] and G. H. HARDY [18]. The starting point is Fejér's summability theorem. It states that the Fourier series of a function $f(x)$ [2] is $(C, 1)$-summable to $\frac{1}{2}[f(x+0)+f(x-0)]$

[1] If the support of w is finite it is natural to assume that (1.1) has a meaning if $fw \in \mathscr{L}$. If the support of w is infinite, several forms of existence conditions are suitable. The condition which turned to be the most natural in our investigations is $fw^{1/2} \in \mathscr{L}_p$ for some $1 \leqq p \leqq \infty$. If the weight w satisfies for every natural n $x^n w^{1/2} \in \mathscr{L}_q$ for the conjugate exponent q then (1) is welldefined.

[2] L. Fejér stated his summability theorem in his original paper [2] for bounded and Riemann-integrable functions. This restricted version of Fejér's theorem is sufficient in connection to its application for the proof of the Dirichlet—Jordan criterion. The theorem was later generalized to arbitrary $\mathscr{L}$-integrable functions by H. LEBESGUE (see e.g. [20]).

at every point where $f(x+0)$ and $f(x-0)$ exists and uniformly $(C, 1)$-summable to $f(x)$ in every closed interval where $f(x)$ is continuous. It was observed that for a function $f(x)$ of bounded variation the nth term of the Fourier series is of order $O(n^{-1})$. (See e.g. N. K. BARI [1].) The Dirichlet—Jordan convergence criterion is deduced from this observation by reference to their summability theorem and to G. H. HARDY's tauberian theorem ([17], see also [18]).

As one of the most important applications of the Fejér—Hardy argument, we mention the extension of the Dirichlet—Jordan criterion to Sturm—Liouville expansions, treated in the book of E. C. TITCHMARSH [23]. A third classical argument to extend the Dirichlet—Jordan criterion to (1.1) is to apply equiconvergence theorems. This idea is due to E. W. HOBSON [19] and A. HAAR [16]. Making use of their idea G. SZEGŐ [22] (see also [21]) found equiconvergence theorems between orthogonal polynomial expansions and trigonometric Fourier series for extended classes of weights with compact support, which imply the validity of the Dirichlet—Jordan criterion to this expansions. Szegő's "local" equiconvergence theorem was refined by the author [6] (see [3], Theorem V.7.4). Moreover we observed that the technique applied to prove equiconvergence theorems yields a far more general result if we treat *ab initio*, functions of bounded variation only (resp. functions of the slightly more general Lip $(1/2, 2)$ class):

THEOREM A. (See [3], Theorem V.7.5.) *Let* $w(x)$ *be a weight with support* $[-1, 1]$ *satisfying*

$$0 < w(x) \leqq \frac{M}{\sqrt{1-x^2}}.$$

Then the orthogonal polynomial expansion of a function $f(x)$ *of bounded variation with respect to this weight converges to* $\frac{1}{2}[f(x+0)+f(x-0)]$ *at every point* $x \in (-1, 1)$ *where* $w(x)$ *is positive and continuous; the convergence is uniform in every closed subinterval of* $(-1, 1)$ *where* f *is continuous and* w *is positive and continuous.*

The idea to prove the criterion via $(C, 1)$-summability and tauberian theorems was first exploited by the author [3] in connection with orthogonal polynomial series (1.1). The apparent reason that no earlier attempts where made is that this happened after the author found his summability theorem [4] (see also [3], IV.3) which was sufficiently general to take it as a starting point of such an investigation.

Our result which was proved first by us through Dirichlet's argument reads as follows:

THEOREM B. (See [5] and [3], Theorem IV.5.7.) *Assuming that* w *satisfies the conditions*

a) $w(x) \leqq M \quad (x \in [a, b] \subset (-1, 1))$

b) *there exists a polynomial* $q(x)$ *so that the sequence* $\{q(x)p_n(w; x)\}$ *is uniformly bounded in* $[-1, 1]$,

the expansion (1.1) *of a function* $f(x)$ *of bounded variation which is continuous in* $[a, b]$ *converges uniformly to* $f(x)$ *in every closed subinterval of* (a, b).

Under the same assumption that f is not only of bounded variation but continuous in $[-1, 1]$ a more flexible device was found by the author: We compare the partial sums and the shifted arithmetic (de la Vallée Poussin) means of the orthogonal polynomial expansion:

THEOREM C. (See [7], [3] and [3], Theorem IV.5.2.). *Let w have $[-1, 1]$ as support, $w(x) \leqq M(1-x)^{-1/2}$ $(-1<x<1)$ and $w(x) \geqq m>0$ $(x \in [a, b] \subset (-1, 1))$, let further f be continuous and of bounded variation in $[-1, 1]$. Then the orthonormal polynomial expansion* (1.1) *of f with respect to the weight converges uniformly in every closed subinterval of (a, b).*

In our present paper we extend Theorem C to a general class of weights with infinite support. In the proof we are going to apply our argument (i.e. comparing partial sums with delated means) but we need to combine it to some recent deep results ([8], [10], [12], [13]) about weighted polynomial approximation on the whole real line. The following is a typical consequence of the result we are going to prove:

THEOREM D. *Let w have support $(-\infty, \infty)$ and*

$$w(x) = (1+|x|)^{\beta} e^{-|x|^{\alpha}} \tag{1.2}$$

where either $\alpha>2$ and β is an arbitrary real number, or $\alpha=2$ and $\beta \geqq 0$. Further let $f(x)$ be continuous and of bounded variation over every finite segment; moreover

$$\int_{-\infty}^{\infty} (1+|x|)^{\beta/2} e^{-(1/2)|x|^{\alpha}} |df(x)| < \infty.$$

Then the partial sums

$$s_n(w; f; x) = \sum_{\nu=0}^{n-1} a_\nu(w; f) p_\nu(w; x) \tag{1.3}$$

of (1.1) *satisfy*

$$\lim_{n\to\infty} [w(x)]^{1/2} |f(x) - s_n(w; f; x)| = 0 \tag{1.3}$$

uniformly with respect to $x \in (-\infty, \infty)$.

In particular, the expansion (1.1) *converges uniformly to $f(x)$ on every finite interval.*

To give the reader a clear view concerning the depth of this result let us observe the following: Taking $\alpha=2$ and $\beta=0$, we obtain — as a special case of Theorem D —

THEOREM D_1. *Let $f(x)$ be continuous in $(-\infty, \infty)$, let it have a bounded variation on every finite interval and let*

$$\int_{-\infty}^{\infty} e^{-x^2/2} |df(x)| < \infty.$$

Then the expansion of $f(x)$ in Hermite polynomials converges uniformly to $f(x)$ on every finite interval and the partial sums $S_n(f; x)$ of the Hermite polynomial expansion

[3] As was pointed out in his Zentralblatt review by L. LEINDLER, the localized version of our theorem in [7] is valid only under the additional assumption that the sequence of orthonormal polynomials is uniformly bounded on the same interval where the expansion should converge.

By this correct remark the problem of localization along the lines indicated in our paper [7] is reduced to the unproved conjecture of V. A. STEKLOV. This conjecture states that $w(x) \geqq m>0$ $(a<x<b)$ implies that the sequence of orthonormal polynomials $\{p_n(w; x)\}$ is uniformly bounded over every closed subinterval of (a, b). In turn, Theorem B is capable of localization (see [3], Chapter IV).

of $f(x)$ satisfy

$$\lim_{n\to\infty} e^{-x^2/2}|f(x)-S_n(f;x)| = 0 \tag{1.4}$$

uniformly with respect to $x\in(-\infty, \infty)$.

Even this special case (1.4) of Theorem D is new. Let us emphasize the fact that Theorem D_1 is only a special case of Theorem D and that — in contrast to Hermite polynomials — on orthogonal polynomials with respect to the weight (1.2) only some rudimental facts are known.

2. Strongly regular weights. The concept of "regular weight" was introduced in our paper [14]. Let $\gamma_n(w)$ be the leading coefficient of $p_n(w; x)$ i.e.

$$p_n(w; x) = \gamma_n(w)x^n + \ldots; \quad \gamma_n(w) > 0$$

and let (see [15])

$$\Gamma_n(w) = \max_{1\leqq\nu\leqq n-1} \frac{\gamma_{\nu-1}(w)}{\gamma_\nu(w)}. \tag{2.1}$$

Beside the quantities $\Gamma_n(w)$ we consider the Christoffel functions

$$\lambda_n(w; x) = \left\{\sum_{k=0}^{n-1} p_k^2(w; x)\right\}^{-1} \qquad (n = 1, 2, \ldots) \tag{2.2}$$

as a second characteristic of the weight w.

DEFINITION 2.1. The weight w is called "regular" iff it is continuous, it has support $(-\infty, \infty)$ and there exists a constant[4] c_0 so that

$$n\lambda_n(w; x) \geqq c_0\Gamma_n(w)w(x) \qquad (-\infty < x < \infty;\ n=1, 2, \ldots) \tag{2.3}$$

is valid and we have $\Gamma_n(w)/n\to 0$ as $n\to\infty$.

We created the concept of strong regularity in connection with our research on weighted polynomial approximation. For an arbitrary weight $W(x)$ we define the sequence of minimal weighted $\mathscr{L}_1$-deviation from polynomials of a function satisfying $fW\in\mathscr{L}$ by

$$\varepsilon_n^{(1)}(W; f) = \inf_{q\in\mathscr{P}_n} \int_{-\infty}^{\infty} |f(x)-q(x)|W(x)\,dx \qquad (n=1, 2, \ldots). \tag{2.4}$$

Of particular interest are the functions

$$\Gamma_\xi(x) = \begin{cases} 0 & (x<\xi) \\ 1 & (x\geqq\xi). \end{cases} \tag{2.5}$$

[4] Here and in what follows we denote by $c_0, c_1, \ldots$ positive numbers the values of which depend only on the choice of the corresponding weight functions.

DEFINITION 2.2. *We call the weight* $w(x)=W^2(x)$ *"strongly regular" iff it is regular, continuous and positive in* $(-\infty, \infty)$ *and it satisfies the inequalities*

$$\varepsilon_n^{(1)}(w^{1/2}; \Gamma_\xi) \leqq c_1 \frac{\Gamma_n(w)}{n} w^{1/2}(\xi) \tag{2.6}$$

i.e.

$$\varepsilon_n^{(1)}(W; \Gamma_\xi) \leqq c_1 \frac{\Gamma_n(W^2)}{n} W(\xi) \tag{2.7}$$

for every real ξ *and every natural* n.

Strongly regular weight were investigated implicity in [8], [10], [12] and [13]. Without explicitly introducing this term, we proved in these papers that the following weights are strongly regular:

a) The weights

$$w_{\alpha\beta}(x) = (1+|x|^\alpha)^{\beta/\alpha} e^{-|x|^\alpha} \tag{2.8}$$

are strongly regular under the condition that $\alpha=2$ and $\beta \geqq 0$ (see [10]) and also under the condition that $\alpha>2$ and β is an arbitrary real number [12].

b) (See [13].) Let

$$w_Q(x) = e^{-2Q(x)} \tag{2.9}$$

where $Q(x)$ is an even positive function which is twice continuously differentiable in $(0, \infty)$ and satisfies for $x>c_2$ the following conditions (i), (ii) and (iii):

(i) $$0 < Q''(t) < (1+c_3)Q''(x) \quad (c_2 < t < x)$$

(ii) $$Q''(2t) \geqq (1+c_4)Q''(t) \quad (t > c_2)$$

(iii) $$t\frac{Q''(t)}{Q'(t)} \leqq c_5 \quad (t > c_2).$$

We observe that the second case (i.e. $\alpha>2$) under a) is included in b). An easy consequence of the definition is

THEOREM 2.1. *If* w_1 *is strongly regular and*

$$c_6 w_1(x) \leqq w_2(x) \leqq c_7 w_1(x) \quad (-\infty < x < \infty) \tag{2.10}$$

then also $w_2(x)$ *is strongly regular.*

PROOF. It was proved in [14] (Theorem 2.4) that (2.10) and the regularity of w_1 imply the regularity of w_2. From the definition (2.4) and (2.10) we infer that

$$c_6^{1/2}\varepsilon_n^{(1)}(w_1^{1/2}; \Gamma_\xi) \leqq \varepsilon_n^{(1)}(w_2^{1/2}; \Gamma_\xi) \leqq c_7^{1/2}\varepsilon_n^{(1)}(w_1^{1/2}; \Gamma_\xi). \tag{2.11}$$

We proved in [14], (2.1) that (2.10) implies

$$\frac{c_6^{1/2}}{c_7^{1/2}}\Gamma_n(w_1) \leqq \Gamma_n(w_2) \leqq \frac{c_7^{1/2}}{c_6^{1/2}}\Gamma_n(w_1). \tag{2.12}$$

We see from (2.11) and (2.12) that the relation (2.6) is satisfied for $w=w_2$ iff it is satisfied for $w=w_1$. Consequently w_2 is regular at the same time as w_1, q.e.d.

REMARK 2.1. Since the weights (2.8) are strongly regular for $\alpha=2$ and $\beta\geqq 0$ resp. $\alpha>2$ and $-\infty<\beta<\infty$ we infer from Theorem 2.1 that also the weights (1.2) are strongly regular for the same range of α and β.

The following Theorem 2.2 is a useful tool to prove strong regularity of weights.

Let w_Q be a weight represented in the form (2.9) where $Q(x)$ is even and continuously differentiable for $x\in(0, \infty)$. Further let $xQ'(x)$ be increasing for $x>0$. For every such $Q(x)$ defined in [8], there exists the sequence $\{q_n\}$ as the unique positive solutions of the equations

$$q_n Q'(q_n) = n \qquad (n = 1, 2, \ldots). \tag{2.13}$$

THEOREM 2.2. *Let $Q(x)$ be even and for $x>0$ continuously differentiable. Further let $Q'(x)$ be increasing for $x>0$ and*

$$Q'(2x) < c_8 Q'(x) \tag{2.14}$$

and finally, let us assume that the weight

$$W_Q(x) = e^{-Q(x)} \tag{2.15}$$

satisfies

$$\lambda_n(W_Q; \xi) \leqq c_{10} \frac{q_{2n}}{n} W_Q(\xi) \quad (|\xi| \leqq c_{11} q_{2n}) \tag{2.16}$$

for every natural n. Then we have

$$\varepsilon_n(W_Q; \Gamma_\xi) \leqq c_{12} \frac{q_{2n}}{n} W_Q(\xi). \tag{2.17}$$

REMARK 2.2. We proved in [15] that under the assumption that $Q'(t)$ is increasing for $t>0$ we have

$$q_{2n} \leqq c_{13} \Gamma_n(W_Q^2). \tag{2.18}$$

Thus we infer from (2.17) that (2.7) and the equivalent statement (2.6) are valid for $W=W_Q$ i.e. $w=w_Q$.

PROOF OF THEOREM 2.2. By the Markov—Stieltjes construction (see [3], § 1.5), for every real ξ there exist polynomials $\varphi_n(x)$ and $\Phi_n(x)$ of degree less than $2n$ satisfying

$$\varphi_n(x) \leqq \Gamma_\xi(x) \leqq \Phi_n(x) \tag{2.19}$$

and

$$\int_{-\infty}^{\infty} [\Phi_n(x) - \varphi_n(x)] W_Q(x)\, dx = \lambda_n(W_Q; \xi). \tag{2.20}$$

Consequently

$$\varepsilon_n(W_Q; \Gamma_\xi) \leqq \int_{-\infty}^{\infty} [\Gamma_\xi(x) - \varphi_n(x)]\, W_Q(x)\, dx \leqq \int_{-\infty}^{\infty} [\Phi_n(x) - \varphi_n(x)] W_Q(x)\, dx = \lambda_n(W_Q; \xi). \tag{2.21}$$

We infer from (2.16) and (2.21) that (2.17) holds for even n and $|\xi| \leqq c_{11} q_{2n}$. Since q_n is increasing and q_n/n decreasing, (2.17) is valid for every natural n and $|\xi| \leqq c_{14} q_{2n}$. For $\xi > c_{14} q_{2n}$ we approximate Γ_ξ by $q_0(x) \equiv 0$. Since $Q'(x)$ is increasing we have

$$\int_{-\infty}^{\infty} |\Gamma_\xi(x) - q_0(x)| W_Q(x)\, dx = \int_{\xi}^{\infty} e^{-Q(x)}\, dx \leqq \tag{2.22}$$

$$\leqq \frac{1}{Q'(\xi)} \int_{\xi}^{\infty} Q'(x) e^{-Q(x)}\, dx \leqq \frac{1}{Q'(c_{14} q_{2n})}\, e^{-Q(\xi)}.$$

By (2.14) we have

$$\frac{1}{Q'(c_{14} q_{2n})} \leqq \frac{c_{15}}{Q'(q_{2n})} = c_{15} \frac{q_{2n}}{n} \tag{2.19}$$

and in consequence of (2.18) and (2.19), the inequality (2.17) is valid for $\xi > c_{14} q_{2n}$. By symmetry it holds also for $\xi < -c_{14} q_{2n}$ i.e. it is valid for every real ξ, q.e.d.

3. Proof of the Dirichlet—Jordan criterion.

THEOREM 3.1. *Let $w(x)$ be continuous and positive for $x \in (-\infty, \infty)$, let $(1+|x|^n) w(x) \in \mathscr{L}$ and let $\{\eta_n\}$ be a null sequence for which*

$$\varepsilon_n^{(1)}(W; \Gamma_\xi) \leqq \eta_n W(\xi) \quad (-\infty < \xi < \infty). \tag{3.1}$$

Then for every function $f(x)$ which satisfies $fW \in \mathscr{L}$ and is of bounded variation on every finite interval, we have

$$\varepsilon_n^{(1)}(W; f) \leqq \eta_n \int_{-\infty}^{\infty} W(\xi)\, |df(\xi)|. \tag{3.2}$$

Theorem 2.3 is obtained by the duality argument applied first in [9] and later in several papers of the author concerning weighted polynomial approximation (see [10], [11], [12], [13]; in particular the proof of Lemma 5.2, and Theorem 5.3 in [13]). Since there is some change in the argument we give the proof in an Appendix to the present paper.

REMARK 3.1. We see from Theorem 3.1 and (2.7) that if $w(x) = W^2(x)$ is strongly regular then

$$\varepsilon_n^{(1)}(W; f) \leqq c_1 \frac{\Gamma_n(W^2)}{n} \int_{-\infty}^{\infty} W(\xi)\, |df(\xi)|. \tag{3.2a}$$

THEOREM 3.2. *If the weight satisfying $w = W^2$ is regular, then for the partial sums* (1.3) *of* (1.2) *we have*

$$W(x)\, n^{-1} \sum_{m=1}^{n} |s_m(w; f; x)| \leqq c_{16} \sup_{-\infty < x < \infty} |W(x) f(x)|. \tag{3.3}$$

Apart from notation, the proof of Theorem 2.4 is exactly the same as the proof of Theorem 4.1 in [13][5] (see also [14], (3.13)).

[5] In the proof of Theorem 4.1 we must replace q_{2n} by $\Gamma_n(w_Q)$ and the inequality [13], (3.25) applied in the course of proving [13], Theorem 4.1 by assumption (2.3).

We introduce the shifted arithmetic (de la Vallée Poussin) means

$$v_{2n}(w;f;x) = \frac{1}{n} \sum_{m=n+1}^{2n} s_m(w;f;x). \tag{3.4}$$

As a consequence of (3.3) we have

$$W(x)\,|v_{2n}(w;f;x)| \leqq 3c_{16} \sup_{-\infty<x<\infty} |W(x)f(x)|. \tag{3.5}$$

Thus by construction

$$s_m(W;r;x) \equiv r(x) \quad (r\in\mathscr{P}_n,\ m \geqq n) \tag{3.6}$$

and from (3.4) and (3.6) we see that for every natural n

$$v_{2n}(w;r;x) = r(x) \quad (r\in\mathscr{P}_n). \tag{3.7}$$

We introduce for nonnegative weights w the quantities

$$\varepsilon_n^{(\infty)}(W;f) = \inf_{q\in\mathscr{P}_n} \operatorname{vrai\,max}_{-\infty<x<\infty} [W(x)\,|f(x)-q(x)|]. \tag{3.8}$$

THEOREM 3.3. *Let $w=W^2$ be strongly regular and let $f(x)$ be an integral function of $f', f'W\in\mathscr{L}_\infty$. Then we have*

$$\varepsilon_n^{(\infty)}(W;f) \leqq c_{17}\frac{\Gamma_n(W^2)}{n} \operatorname{vrai\,max}_{-\infty<x<\infty} [W(x)\,|f'(x)|]. \tag{3.9}$$

Theorem 3.3 is a consequence of Remark 3.1 and the inequality (3.5). The details of proof are exactly the same as in our earlier papers ([9], [10], [12], [13]) so we omit the details.[6]

LEMMA 3.4. *Let W^2 be strongly regular and let $f(x)$ be continuous for every $x\in(-\infty,\infty)$ and*

$$\lim_{|x|\to\infty} W(x)f(x) = 0 \tag{3.10}$$

then we have

$$\lim_{n\to\infty} \varepsilon_n^{(\infty)}(W;f) = 0. \tag{3.11}$$

PROOF. As a consequence of our assumptions, for every $\varepsilon>0$ there exists a continuously differentiable function $F(x)$ with finite support satisfying

$$W(x)\,|f(x)-F(x)| = \varepsilon \quad (-\infty < x < \infty). \tag{3.12}$$

Consequently

$$\varepsilon_n^{(\infty)}(W;f) \leqq \varepsilon + \varepsilon_n^{(\infty)}(W;F). \tag{3.13}$$

[6] As a matter of fact, we proved also that the Jackson-type theorems discussed in our papers above have a common generalization for square roots of strongly regular weights. We hope to return to this point with some additional results in a subsequent paper.

By Theorem 3. 3

$$\lim_{n\to\infty} \varepsilon_n^{(\infty)}(W;F) = 0. \tag{3.14}$$

From (3.13) and (3.14)

$$\overline{\lim_{n\to\infty}}\, \varepsilon_n^{(\infty)}(W;f) \leqq \varepsilon. \tag{3.15}$$

Since $\varepsilon>0$ was arbitrary (3.15) implies (3.11), q.e.d.

LEMMA 3.5. *For every regular* $w=W^2$ *and every* f *satisfying* $fW\in\mathscr{L}$ *we have*

$$\int_{-\infty}^{\infty} |f(x)-v_{2n}(w;f;x)|\,W(x)\,dx \leqq c_{18}\,\varepsilon_n^{(1)}(W;f) \tag{3.16}$$

and

$$W(x)\,|f(x)-v_{2n}(w;f;x)| \leqq c_{18}\,\varepsilon_n^{(\infty)}(W;f). \tag{3.17}$$

for every continuous f *satisfying* (3.10), *respectively.*

PROOF. Applying the same reasoning as in [9], we infer from (3.5) by an adjoint space argument that

$$\left|\int_{-\infty}^{\infty} v_{2n}(w;f;x)W(x)\,dx\right| \leqq 2c_{16}\int_{-\infty}^{\infty} |f(x)|W(x)\,dx. \tag{3.18}$$

Let us denote by $\|\cdot\|_p$ the $\mathscr{L}_p(-\infty,\infty)$-norm, further let $\{r_n^{(1)}\in\mathscr{P}_n\}$ and $\{r_n^{(\infty)}\in\mathscr{P}_n\}$ polynomial sequences satisfying

$$\|W(f-r_n^{(p)})\|_p \leqq 2\varepsilon_n^{(p)}(W;f). \tag{3.19}$$

Applying (3.7) we infer from (3.5) resp. (3.18) and (3.19) that for $p=1$ and $p=\infty$.

$$\|W(x)[f(x)-v_{2n}(w;f;x)]\|_p \leqq \tag{3.20}$$

$$\leqq \|W(x)[f(x)-r_n^{(p)}(x)]\|_p + \|W(x)v_{2n}(w;f-r_n^{(p)};x)\|_p \leqq$$

$$\leqq (2+4c_{16})\,\|W(x)[f(x)-r_n^{(p)}(x)]\|_p \leqq c_{18}\,\varepsilon_n^{(p)}(W;f)$$

q.e.d.

REMARK 3.2. As a matter of fact, (3.20) holds for every $1\leqq p\leqq\infty$. This is proved by applying the Riesz—Thorin interpolation theorem to (3.5) and (3.18) and then using (3.19) and (3.20) for an arbitrary $p\in[1,\infty)$. We do not make any use of this remark in our present paper.

LEMMA 3.6. *Let $w=W^2$ be strongly regular and let f be a continuous function on $(-\infty, \infty)$ which is of bounded variation over every finite segment, moreover it satisfies* (3.10)[7] *and*

$$\int_{-\infty}^{\infty} W(x)\,|df(x)| < \infty \tag{3.21}$$

then the coefficients of the orthogonal expansion (1.1) *satisfy*

$$\sum_{\nu=n}^{\infty} a_\nu^2(W;f) = o(1)\frac{\Gamma_n(w)}{n}. \tag{3.22}$$

PROOF. Since $v_{2N}(w;f;x)\in\mathscr{P}_{2N}$ we have by Bessel's inequality

$$\sum_{\nu=2N+1}^{\infty} a_\nu^2(w;f) \leqq \int_{-\infty}^{\infty} |f(x)-v_{2N}(w;f;x)|^2 W^2(x)\,dx \leqq \tag{3.23}$$

$$\leqq \|W(x)[f(x)-v_{2N}(w;f;x)]\|_\infty \|W(x)[f(x)-v_{2N}(w;f;x)]\|_1$$

we obtain from (3. 23) and Lemma 3. 5

$$\sum_{\nu=2N+1}^{\infty} a_\nu^2(w;f) \leqq o(1)\frac{\Gamma_N(w)}{N}. \tag{3.22a}$$

Considering the facts that the sum of the left of (3.22) is decreasing and that $\Gamma_N(w)$ is increasing (see (2.1)) we see that (3.22) is a consequence of (3.22a), q.e.d.

THEOREM 3.7. *Under the conditions of Lemma* 3.6 *we have*

$$\lim_{n\to\infty} W(x)\,|f(x)-s_n(w;f;x)| = 0 \tag{3.24}$$

uniformly in $x\in(-\infty, \infty)$.

PROOF. By (3.4)

$$v_{2n}(w;f;x) = s_n(w;f;x) + \sum_{\nu=n}^{2n-1}\left(2-\frac{\nu}{n}\right)a_\nu(w;f)p_\nu(w;x) \tag{3.25}$$

i.e.

$$|v_{2n}(w;f;x)-s_n(w;f;x)| \leqq \sum_{\nu=n}^{2n-1} |a_\nu(w;f)|\,|p_\nu(w;x)|. \tag{3.26}$$

[7] If $W(x)$ is a decreasing function of $|x|$ for sufficiently large $|x|$, then (3.10) is a consequence of (3.21). Namely, take $w>0$ so large that

$$\int_{\omega}^{\infty} W(t)\,|df(t)| \leqq \varepsilon$$

thus

$$|f(x)| \leqq f(\omega) + \int_{\omega}^{x} |df(t)| \leqq |f(\omega)| + W^{-1}(x)\varepsilon$$

so that

$$\lim_{x\to\infty} W(x)f(x) = 0$$

and similarly

$$\lim_{x\to-\infty} W(x)f(x) = 0.$$

By Schwarz's inequality, applying (3.22), (2.2) and (2.3) we infer

$$|v_{2n}(w;f;x)-s_n(w;f;x)|^2 \leqq \sum_{\nu=n}^{\infty} a_\nu^2(w;f) \sum_{\nu=0}^{2n-1} p_\nu^2(w;x) =$$

$$= o(1)\frac{\Gamma_n(w)}{n}\frac{2n}{\Gamma_{2n}(w)} w^{-1}(x) = o(1)W^{-2}(x)$$

i.e. we have

$$\lim_{n\to\infty} W(x)|v_{2n}(w;f;x)-s_n(w;f;x)| = 0 \tag{3.27}$$

uniformly with respect to $x\in(-\infty,\infty)$. (3.24) is a consequence of (3.17), (3.11) and (3.27), q.e.d.

4. Appendix. Proof of Theorem 3.1. LEMMA 4.1. *Let* $0<W(x)\in C(-\infty,\infty)$ *satisfy* (3.1) *and* $(1+|x|^n)W(x)\in\mathscr{L}$ *further let* $gW\in\dot{\mathscr{L}}_\infty$ *and*

$$\operatorname*{vrai\,max}_{-\infty<x<\infty} W(x)\,|g(x)| \leqq 1 \tag{4.1}$$

and for every $r\in\mathscr{P}_n$ *let*

$$\int_{-\infty}^{\infty} g(x)r(x)W^2(x)\,dx = 0 \quad (r\in\mathscr{P}_n) \tag{4.2}$$

then

$$\Big|\int_x^{\infty} g(t)W^2(t)\,dt\Big| \leqq \eta_n W(x) \quad (-\infty<x<\infty). \tag{4.3}$$

PROOF. We have for every $r\in\mathscr{P}_n$

$$\int_x^{\infty} g(t)W^2(t)\,dt = \int_{-\infty}^{\infty} g(t)[\Gamma_x(t)-r(t)]W^2(t)\,dt$$

consequently

$$\Big|\int_x^{\infty} g(t)W^2(t)\,dt\Big| \leqq \inf_{r\in\mathscr{P}_n}\int_{-\infty}^{\infty}|\Gamma_x(t)-r(t)|W(t)\,dt = \varepsilon_n^{(1)}(W;\Gamma_x) \leqq \eta_n W(x).$$

LEMMA 4.2. *If* $r\in\mathscr{P}_n$ *satisfies*

$$\int_{-\infty}^{\infty}|r(x)|W(x)\,dx \leqq K \tag{4.4}$$

for some fixed $K>0$ *then all coefficients of* $r(x)$ *are smaller in modulus than* $A_1(n)K$.

PROOF. Let $W_1(x)=W(x)e^{-|x|}$. Representing $r(x)$ as

$$r(x) = \sum_{\nu=0}^{n} a_\nu(W_1;r)p_\nu(W_1;x)$$

we have

$$|a_\nu(W_1; r)| = \left| \int_{-\infty}^{\infty} r(x) p_\nu(W_1; x) W_1(x)\, dx \right| \leqq$$

$$\leqq \int_{-\infty}^{\infty} |r(x)| W(x)\, dx \max_{-\infty < x < \infty} \{|p_\nu(W_1; x)| e^{-|x|}\}$$

and our assertion follows, q.e.d.

Let us denote by R_n the set of measurable functions satisfying (4.1) and (4.2). Then by Nikolski's duality lemma (see e.g. [10], Lemma 3.2)

$$\varepsilon_n^{(1)}(W; f) = \sup_{g \in R_n} \int_{-\infty}^{\infty} f(x) g(x) W^2(x)\, dx. \tag{4.4}$$

Let us assume for the moment that $f(x)$ is of bounded variation and vanishes for sufficiently large $|x|$. Setting

$$\mathcal{G}(x) = \int_x^{\infty} g(t) W^2(t)\, dt,$$

we obtain by partial integration

$$\int_{-\infty}^{\infty} f(x) g(x) W^2(x)\, dx = \int_{-\infty}^{\infty} \mathcal{G}(\xi)\, df(\xi). \tag{4.5}$$

By (4.4), (4.5) and Lemma 4.1 we see that under our additional assumption concerning $f(x)$

$$\varepsilon_n^{(1)}(W; f) \leqq \eta_n \int_{-\infty}^{\infty} W(\xi) |df(\xi)| \tag{4.6}$$

is valid.

Let now f be an arbitrary function which satisfies the conditions of Theorem 3.1. From $fW \in \mathcal{L}_1$ and the positivity and continuity of W we infer that there exist sequences $x_{N1} \to \infty$ and $x_{N2} \to -\infty$ so that

$$\lim_{N \to \infty} f(x_{N,i}) W(x_{N,i}) = 0. \tag{4.7}$$

Let us define

$$f_N(x) = \begin{cases} f(x) & (x \in [x_{N,2}, x_{N,1}]) \\ 0 & (x \notin [x_{N,2}, x_{N,1}]). \end{cases}$$

Since $f_N(x)$ vanishes for sufficiently large $|x|$, by (4.6) there exists a polynomial $p_{n,N}$ satisfying

$$\int_{-\infty}^{\infty} |f_N(x) - p_{n,N}(x)| W(x)\, dx \leqq \eta_n \int_{-\infty}^{\infty} W(\xi) |df_N(\xi)| \leqq \tag{4.8}$$

$$\leqq \eta_n \left[\int_{-\infty}^{\infty} W(\xi) |df(\xi)| + W(x_{N,1}) |f(x_{N,1})| + W(x_{N,2}) |f(x_{N,2})| \right].$$

By (4.8) the integrals $\int |p_{n,N}(x)|W(x)dx$ are uniformly bounded. By Lemma 4.2 there exists a subsequence p_{n,N_k} converging to a $p_n \in \mathscr{P}_n$.

Moreover, by Lemma 4.2 the polynomials p_{n,N_k} have a common majorant $A_2(n)(1+|x|^n)$.

Consequently

$$|f_N(x)-p_{n,N}(x)|\,W(x) \leqq \{|f(x)|+c_{20}(1+|x|^n)\}W(x) \in \mathscr{L}.$$

By Lebesgue's dominated convergence theorem we infer from (4.8) and (4.7)

$$\int_{-\infty}^{\infty} |f(x)-p_n(x)|\,W(x)\,dx =$$

$$= \lim_{k\to\infty} \int_{-\infty}^{\infty} |f_{N_k}(x)-p_{n,N_k}(x)|\,W(x)\,dx \leqq \eta_n \int_{-\infty}^{\infty} W(\xi)\,|df(\xi)|,$$

q.e.d.

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References

[1] Н. К. Бари, Тригонометрические ряди. Москва, 1961.
[2] L. FEJÉR, Sur les fonctions bornées et integrables, *Comptes Rendus de l'Acad. de Sci. (Paris)*, **131** (1900), pp. 984—987.
[3] G. FREUD, Orthogonale Polynome. Joint edition of Akadémiai Kiadó (Budapest), Birkhauser Verl. (Basel) and Deutscher Verl. der Wiss. (Berlin — DDR). English translation by L. Földes: Orthogonal Polynomials, Pergamon Press, 1971.
[4] G. FREUD, Über die starke (C, 1)-Summierbarkeit von Orthogonalen Polynomreihen, *Acta Math. Acad. Sci. Hung.*, **3** (1952), pp. 83—88.
[5] G. FREUD, Über die Konvergenz von orthogonalen Polynomreihen, *Acta Math. Acad. Sci. Hung.*, **3** (1952), pp. 89—98.
[6] G. FREUD, On orthogonal polynomials (in Hungarian) Thesis, (Budapest, 1956).
[7] G. FREUD, Über die Konvergenz der Orthogonalpolynomreihe einer Funktion mit beschränkter Variation, *Arch. Math. (Brno)*, **1** (1965), pp. 247—250.
[8] G. FREUD, On weighted polynomial approximation on the whole real axis, *Acta Math. Acad. Sci. Hung.*, **20** (1969), pp. 223—225.
[9] Г. Фройд, Об аппроксимации с весом многочленами на действительной оси, *Доклади Ак. Наук СССР*, **191** (1970), pp. 293—294. English translation: On approximation with weight by polynomials on the real line, *Soviet Math. Dokladii*, **11** (1970), pp. 370—371.
[10] G. FREUD, A contribution to the problem of weighted polynomial approximation, ISNM vol. 20 "*Linear Operators and approximation theory*" (conference report Oberwolfach, 1970). Editors: P. L. Butzer, J. P. Kahane and B. Sz.-Nagy. Birkhäuser Verl. (Basel, 1972).
[11] G. FREUD, On weighted L_1-approximation by polynomials, *Studia Math.*, **46** (1973).

[12] G. FREUD, Investigations on weighted approximation by polynomials, *Studia Sci. Math. Hung.*
[13] G. FREUD, On polynomial approximation with respect to general weight functions. *Functional Analysis and its Applications.* (Conferenc report, Madras, 1973.)
[14] G. FREUD, On the extension of the Fejér—Lebesgue theorem to orthogonal polynomial series Iliev Festschrift, Sofia.
[15] G. FREUD, On estimations of the greatest zeroes of orthogonal polynomials, *Acta Math. Acad. Sci. Hung.*, **25** (1974), pp. 99—107.
[16] A. HAAR, Reihenentwicklungen nach Legendreschen Polynomen, *Mathem. Annalen,* **78** (1917), pp. 121—136.
[17] G. H. HARDY, *Divergent series.* Clarendon Press (Oxford, 1949).
[18] G. H. HARDY, Theorems relating to the summability and convergence of slowly oscillating series, *Proc. London Math. Soc.,* (2) **9** (1910), pp. 301—320.
[19] G. W. HOBSON, On the representation of a summable function by a series of finite polynomials, *Proceedings London Math. Soc.,* (2) **12** (1913), pp. 156—173.
[20] H. LEBESGUE, *Lecons sur les series trigonometriques.* Gauthier Villars (Paris, 1906).
[21] G. SZEGŐ, *Orthogonal polynomials.* 2nd ed. Amer. Math. Soc. (New York, 1959).
[22] G. SZEGŐ, Über die Entwicklung einer willkürlichen Funktion nach den Polynomen eines Orthogonalsystems, *Mathem. Zeitschr.,* **12** (1921), pp. 61—94.
[23] E. C. TITCHMARSCH, *Eigenfunction expansions associated with second-order differential equations,* I. 2ed. Clarendon Press (Oxford, 1962).
[24] A. ZYGMUND, *Trigonometric series,* I. The University Press (Cambridge, 1959).

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Tomus 25 (1—2), (1974), pp. 123—144.

ЗАМЕЧАНИЯ ОБ ИНТЕРПОЛИРОВАНИИ

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§ 1. Введение

Пусть $w(x) \geqq 0$ некоторая (дифференциальная) весовая функция, определенная на всей вещественной оси. Обозначим через $p_n(w; x)$ $(=\gamma_n(w)x^n+\ldots)$ $(n=0, 1, 2, \ldots; \gamma_n(w)>0)$ многочлены, ортонормальные с весом $w(x)$, значит

$$\int_{-\infty}^{\infty} p_n(w; x)p_m(w; x)w(x)\,dx = \begin{cases} 0, & \text{при } n \neq m \\ 1, & \text{при } n = m \end{cases} \quad (n, m=0, 1, 2, \ldots).$$

Пусть далее

$$x_{1n}(w) > x_{2n}(w) > \ldots > x_{nn}(w) \tag{1}$$

корни n-го ортонормированного многочлена $p_n(w; x)$, а $\lambda_n(w; x)$ $(n=1, 2, \ldots)$ функции Кристоффеля, индуцированные при помощи веса $w(x)$, значит

$$\lambda_n(w; x) = \left[\sum_{k=0}^{n-1} p_k^2(w; x)\right]^{-1} \qquad (n = 1, 2, \ldots). \tag{2}$$

Пусть (a, b) некоторый (конечный или бесконечный) интервал, содержащий носитель веса $w(x)$. Обозначим через $L_n(w, x, f)$ $(n=1, 2, \ldots)$ интерполяционные многочлены Лагранжа степени не выше $n-1$, совпадающие с заданной функцией $f(x)$, определенной на интервале (a, b), в узлах (1), то есть

$$L_n(w, x, f) = \sum_{k=1}^{n} f\big(x_{kn}(w)\big)\, l_{kn}(w; x), \tag{3}$$

где

$$l_{kn}(w; x) = \frac{p_n(w; x)}{p_n(w; x_{kn}(w))'(x - x_{kn}(w))} \quad (k=1, 2, \ldots, n) \tag{4}$$

фундаментальные многочлены Лагранжева интерполирования по узлам (1). Пусть далее

$$L_n(w; x) = \sum_{k=1}^{n} |l_{kn}(w; x)| \quad (n=1, 2, \ldots) \tag{5}$$

функции Лебега интерполяционного процесса Лагранжа (3).

Для классических весовых функций мы вводим следующие обозначения. Пусть

$$w_1(x) \equiv w_1^{(\beta, \gamma)}(x) = \begin{cases} 0, & \text{при } x \leqq -1 \\ (1-x)^\beta (1+x)^\gamma, & \text{при } -1 < x < 1 \\ 0, & \text{при } x \geqq 1 \end{cases}$$

($\beta(>-1)$ и $\gamma(>-1)$ фиксированные, вещественные параметры) вес Якоби,

$$w_2(x) = e^{-x^2} \quad (-\infty < x < \infty)$$

вес Эрмита и

$$w_3(x) \equiv w_3^{(\beta)}(x) = \begin{cases} 0, & \text{при} \quad x \leqq 0 \\ x^\beta e^{-x}, & \text{при} \quad x > 0 \end{cases}$$

($\beta(>-1)$ фиксированный, вещественный параметр) вес Лагерра.

Основной целью настоящей работы является изучение — при помощи одного нового метода, введенного нами в работе [21] — сходимости и скорости сходимости интерполяционных многочленов Лагранжа $L_n(w_i, x, f)$ $(i=1, 2, 3)$ в зависимости от различных структурных свойств индуцирующей функции $f(x)$. А именно, мы докажем следующие теоремы.

Теорема 1. *Пусть функция $f(x)$ непрерывна на отрезке $[-1, 1]$. Если отрезок $[a, b]$ $(\subset(-1, 1))$ и число $\varepsilon(>0)$ таковы, что $-1<a-\varepsilon<b+\varepsilon<1$, то равномерно для всех x из отрезка $[a, b]$ выполняется оценка*

$$(6) \quad \left| L_n(w_1, x, f) - f(x) - \frac{1}{2} \sum_{|x-x_{kn}(w_1)|<\varepsilon} \left[f\left(x_{kn}(w_1)\right) - f\left(x_{k-1,n}(w_1)\right) \right] l_{kn}(w_1; x) \right| =$$

$$= O\left[\omega\left(f; \frac{\log n}{n} \right) \right] \quad (a \leqq x \leqq b, \quad n = 1, 2, \ldots).$$

Теорема 2. *Пусть функция $f(x)$ равномерно непрерывна на всей вещественной оси. Если ε произвольное, но фиксированное, положительное число, то равномерно по x имеет место оценка*

(7)

$$e^{-\frac{x^2}{2}} \left| L_n(w_2, x, f) - f(x) - \frac{1}{2} \sum_{|x-x_{kn}(w_2)|<\varepsilon} \left[f\left(x_{kn}(w_2)\right) - f\left(x_{k-1,n}(w_2)\right) \right] l_{kn}(w_2; x) \right| =$$

$$= O\left[\omega\left(f; \frac{\log n}{\sqrt{n}} \right) \right] \quad (-\infty < x < \infty, \quad n = 1, 2, \ldots).$$

Теорема 2 есть обобщение одного нашего результата [21], согласно которому левая часть в (7) есть величина, порядка $o(1)$, если $f(x)$ не только равномерно непрерывна, но и ограничена на всей оси.

Теорема 3. *Пусть функция $f(x)$ $(0 \leqq x < \infty)$ такова, что функция $g(x) \equiv f(x^2)$ равномерно непрерывна на неотрицательной полуоси. Если конечный отрезок $[a, b]$ $(\subset(0, \infty))$ и число $\varepsilon(>0)$ таковы, что $0<a-\varepsilon$, то равномерно по всем x, принадлежащим к отрезку $[a, b]$, справедливо соотношение*

$$(8) \quad \left| L_n(w_3, x, f) - f(x) - \frac{1}{2} \sum_{|x-x_{kn}(w_3)|<\varepsilon} \left[f\left(x_{kn}(w_3)\right) - f\left(x_{k-1,n}(w_3)\right) \right] l_{kn}(w_3; x) \right| =$$

$$= O\left[\bar{\omega}\left(f; \frac{\log n}{\sqrt{n}} \right) \right] \quad (a \leqq x \leqq b, \quad n = 1, 2, \ldots),$$

где $\bar{\omega}(f; \delta)$ модуль непрерывности функции $f(x^2)$.

В работе [25] мы показали, что если ограниченная функция $f(x)$ удовлетворяет условиям Теоремы 3, то левая часть в формуле (8) есть $o(1)$.

Будем говорить, что функция $f(x)$ *удовлетворяет одностороннему условию Дини—Липшица на (конечном или бесконечном) интервале* (a, b), *если*

$$f(x+\delta)-f(x) \leqq -\frac{v(\delta)}{|\log \delta|} \quad (a < x \leqq x+\delta < b), \tag{9}$$

где $v(\delta)$ *такая неубывающая функция, для которой*

$$\lim_{\delta \to +0} v(\delta) = 0.$$

Теорема 4. *Пусть, функция* $f(x)$ *непрерывна на отрезке* $[-1, 1]$. *Если на интервале* (a, b) $(\subset(-1, 1))$ $f(x)$ *удовлетворяет одностороннему условию Дини—Липшица* (9), *то внутри интервала* (a, b) *интерполяционные многочлены* $L_n(w_1, x, f)$ *равномерно сходятся к функции* $f(x)$ *при* $n \to \infty$.[1]

Теорема 4 широко обобщает один результат Г. Сеге [30], согласно которому выполнение классического условия Дини—Липшица достаточно для равномерной сходимости многочленов $L_n(w_1, x, f)$.

Теорема 5. *Пусть функция* $f(x)$ *равномерно непрерывна на всей вещественной оси. Если функция* $f(x)$ *удовлетворяет одностороннему условию Дини—Липшица* (9) *на некотором конечном или бесконечном интервале* (a, b), *то для всех фиксированных* ε, *удовлетворяющих условию* $0 < \varepsilon < \frac{b-a}{2}$, *имеет место предельное соотношение*

$$\lim_{n \to \infty} \sup_{a+\varepsilon \leq x \leq b-\varepsilon} e^{-\frac{x^2}{2}} |f(x) - L_n(w_2, x, f)| = 0. \tag{10}$$

Частный случай Теоремы 5, когда функция $f(x)$ удовлетворяет классическому условию Дини—Липшица, был доказан Г. Фройдом в работе [7.]

Теорема 6. *Пусть функция* $f(x)$ $(0 \leqq x < \infty)$ *такова, что функция* $g(x) \equiv f(x^2)$ *равномерно непрерывна на интервале* $[0, \infty)$. *Если на конечном интервале* (a, b) $(\subset(0, \infty))$ *функция* $f(x)$ *удовлетворяет одностороннему условию Дини—Липшица* (9), *то внутри интервала* (a, b) *равномерно по* x *выполняется соотношение*

$$\lim_{n \to \infty} |f(x) - L_n(w_3, x, f)| = 0. \tag{11}$$

Следствие. *Если функция* $f(x)$ *из Теоремы* 6 *такова, что*

$$\omega(f; \delta) = o(|\log \delta|)$$

то имеет место (11).

[1] Значит сходимость равномерна на всех отрезках принадлежащих к (a, b).

Более слабое утверждение, что если

$$\omega(g;\delta) = o(|\log\delta|),$$

то выполняется соотношение (11), было доказано нами в работе [19].

Отметим, что смысл введения понятия одностороннего условия Дини—Липшица возник у нас впервые в работе [26].

Пусть $\Phi(\xi)$ $(\xi \geqq 0)$ вещественнозначная функция, обладающая свойствами

$$\Phi(0) = 0, \quad \Phi(\xi_1) \leqq \Phi(\xi_2) \quad (\xi_1 < \xi_2) \tag{12}$$

и

$$\int_0^1 |\log \Phi(\xi)|\, d\xi < \infty. \tag{13}$$

Функция $f(x)$, определенная на некотором конечном или бесконечном интервале (a, b), *имеет ограниченное* Φ*-изменение,* то есть $f \in V_\Phi(a, b)$, *если*

$$v_\Phi(f) = \sup_\tau \sum_i \Phi(|f(b_i) - f(a_i)|) < \infty,$$

где τ *произвольная конечная система попарно непересекающихся интервалов* (a_i, b_i), *принадлежащих к* (a, b) (см. Л. С. Юнг [32]).

Теорема 7. *Если функция* $f(x)$ *непрерывна на отрезке* $[-1, 1]$ *и имеет ограниченное* Φ*-изменение на интервале* (a, b) $(\subset(-1, 1))$, *то равномерно внутри интервала* (a, b) *мы имеем оценку*

$$|f(x) - L_n(w_1, x, f)| = O\left[\int_0^{\omega\left(f;\frac{1}{n}\right)} \log\frac{2v_\Phi(f)}{\Phi(\xi)}\, d\xi + \omega\left(f;\frac{\log n}{n}\right)\right] \tag{14}$$

$$(x \in [a_1, b_1] \subset (a, b); \quad n = 1, 2, \ldots),$$

значит интерполяционный процесс Лагранжа $L_n(w_1, x, f)$ *равномерно сходится к функции* $f(x)$ *внутри* (a, b).

Сходимость интерполяционных многочленов Лагранжа $L_n(w_1, x, f)$ для функций, имеющих ограниченное изменение (то есть в случае $\Phi(\xi) \equiv \xi$) была доказана Я. Л. Геронимусом [13] (см. еще В. И. Крылов [14]).

В случае $\Phi(\xi) \equiv \xi^p$ $(1 \leqq p < \infty)$, значит в случае функций с p ограниченным изменением соотношение (14) превращается в оценку

$$|f(x) - L_n(w_1, x, f)| = O\left[\log\frac{2v_\Phi(f)}{\omega\left(f;\frac{1}{n}\right)}\,\omega\left(f;\frac{1}{n}\right) + \omega\left(f;\frac{\log n}{n}\right)\right]$$

$$(\Phi(\xi) \equiv \xi^p; \quad n = 1, 2, \ldots).$$

Теорема 8. *Если функция $f(x)$ равномерно непрерывна на всей вещественной оси и $f\in V_\Phi(a, b)$, где $(a, b)\subseteqq(-\infty, \infty)$, то при всех $\varepsilon\left(0<\varepsilon<\frac{b-a}{2}\right)$ имеет место оценка*

$$\sup_{a+\varepsilon<x<b-\varepsilon} e^{-\frac{x^2}{2}}|f(x)-L_n(w_2, x, f)| = \tag{15}$$

$$= O\left[\int_0^{\omega\left(f;\frac{1}{\sqrt{n}}\right)} \log\frac{2v_\Phi(f)}{\Phi(\xi)}\,d\xi+\omega\left(f;\frac{\log n}{\sqrt{n}}\right)\right] \qquad (n = 1, 2, \ldots).$$

Теорема 9. *Если функция $f(x)$ $(0\leqq x+\infty)$ такова, что функция $g(x)\equiv f(x^2)$ равномерно непрерывна на $[0, \infty)$ и $f\in V_\Phi(a, b)$, где (a, b) $(\subset(0, \infty))$ некоторый конечный интервал, то равномерно внутри интервала (a, b) имеет место оценка*

$$|f(x)-L_n(w_3, x, f)| = O\left[\int_0^{\omega\left(f;\frac{1}{\sqrt{n}}\right)} \log\frac{2v_\Phi(f)}{\Phi(\xi)}\,d\xi+\bar{\omega}\left(f;\frac{\log n}{\sqrt{n}}\right)\right] \tag{16}$$

$$(x\in[a_1, b_1]\subset(a, b); \quad n = 1, 2, \ldots),$$

где $\bar{\omega}(f;\ \delta)$ модуль непрерывности функции $f(x^2)$.

Частный случай теорем 8 и 9 $\Phi(\xi)\equiv\xi$ был изучен нами в работах [21] и [25], где мы установили сходимость интерполяционных многочленов Лагранжа для функций, имеющих ограниченное изменение.

Отметим, что в случае $\Phi(\xi)\equiv\xi^p$ $(1\leqq p<\infty)$ интегралы, стоящие в правых частях соотношений (15) и (16) легко оценить. Оба имеют порядок

$$O\left[\log\frac{2v_\Phi(f)}{\omega\left(f;\frac{1}{\sqrt{n}}\right)}\,\omega\left(f;\frac{1}{\sqrt{n}}\right)\right].$$

Пусть весовая функция $w(x)$ с носителем $[-1, 1]$ такова, что на некотором отрезке $[a^*, b^*]$ $(\subset(-1, 1))$ выполняются условия

$$|p_n(w; x)| \leqq C_1 \quad (a^* \leqq x \leqq b^*,\ n = 1, 2, \ldots) \tag{17}$$

и

$$w(x) \leqq C_2 \quad (a^* \leqq x \leqq b^*), \tag{18}$$

где C_1 и C_2 некоторые конечные константы.

Теорема 10. *Пусть функция $f(x)$ непрерывна на отрезке $[-1, 1]$. Если точка x из интервала (a^*, b^*) и число $\varepsilon(>0)$ таковы, что $a^*<x-\varepsilon<x+\varepsilon<b^*$, то*

$$\left|L_n(w, x, f)-f(x)-\frac{1}{2}\sum_{|x-x_{kn}(w)|<\varepsilon}[f(x_{kn}(w))-f(x_{k-1,n}(w))]\,l_{kn}(w; x)\right| = \tag{19}$$

$$= \sum_{k=1}^{n-1}|l_{kn}(w; x)+l_{k+1,n}(w; x)|\,O\left[\omega\left(f;\frac{\log n}{n}\right)\right] \qquad (n = 1, 2, \ldots).$$

Теорема 11. *Пусть функция $f(x)$ непрерывна на отрезке $[-1, 1]$ и на интервале (a, b) $(\subset (a^*, b^*))$ удовлетворяет одностороннему условию Дини—Липшица* (9). *Если на $[a_1, b_1]$ из (a, b) имеет место соотношение*

$$\lim_{n\to\infty} \max_{a_1 \leqq x \leqq b_1} \sum_{k=1}^{n-1} |l_{kn}(w; x) + l_{k+1,n}(w; x)| \, \omega\left(f; \frac{\log n}{n}\right) = 0, \tag{20}$$

то тогда

$$\lim_{n\to\infty} L_n(w, x, f) = f(x) \quad (x \in [a_1, b_1]), \tag{21}$$

причем сходимость равномерно на отрезке $[a_1, b_1]$.

Теорема 12. *Если функция $f(x)$ непрерывна на отрезке $[-1, 1]$ и $f \in V_\Phi(a, b)$ $(a^* \leqq a < b \leqq b^*)$, то равномерно для всех отрезков $[a_1, b_1]$ из (a, b) мы имеем оценку*

$$|f(x) - L_n(w, x, f)| = \tag{22}$$

$$= O\left[\int_0^{\omega\left(f; \frac{1}{n}\right)} \log \frac{2v_\Phi(f)}{\Phi(\xi)} \, d\xi + \max_{a_1 \leqq x \leqq b_1} \sum_{k=1}^{n-1} |l_{kn}(w; x) + l_{k+1,n}(w; x)| \, \omega\left(f; \frac{\log n}{n}\right)\right],$$

значит при выполнении условия (20) *интерполяционные многочлены Лагранжа $L_n(w, x, f)$ равномерно на $[a_1, b_1]$ сходятся к функции $f(x)$ при $n \to \infty$.*

Мы думаем, что выражения

$$\hat{L}_n(w; x) = \sum_{k=1}^{n-1} |l_{kn}(w; x) + l_{k+1,n}(w; x)| \qquad (n = 1, 2, \ldots),$$

которые имеют вид, похожий к функциям Лебега $L_n(w, x)$, будут иметь немалостепенное значение в дальнейших исследованиях по теории интерполирования.

Наряду с $L_n(w, x, f)$ мы будем также рассматривать тригонометрические интерполяционные многочлены $\tilde{S}_n(x, f)$ $(n = 1, 2, \ldots)$, совпадающие с 2π-периодической функцией $f(x)$ в узлах

$$x_{kn} = \frac{2k\pi}{2n+1} \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{23}$$

Известно, что

$$\tilde{S}_n(x, f) = \frac{2}{2n+1} \sum_{k=-n}^{n} f(x_{kn}) \, D_n(x - x_{kn}), \tag{24}$$

где

$$D_n(t) = \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} \tag{25}$$

ядро Дирихле n-го порядка.

Обозначим через[2]

$$\Delta_\delta^R f(x) = 2^{-R} \sum_{k=0}^{R} (-1)^k \binom{R}{k} f(x+k\delta) \qquad (R = 1, 2, \ldots)$$

R-ую (модифицированную) разность функции $f(x)$ с шагом δ, а через

$$\omega_R(f; t) = \max_{\substack{|\delta| \leqq t \\ |x| \leqq \pi}} |\Delta_\delta^R f(x)| \qquad (t \geqq 0; \quad R = 1, 2, \ldots)$$

(модифицированный) модуль гладкости порядка R функции $f(x)$.

Пусть $V_{\Phi, R}(a, b)$ $(R=1, 2, \ldots;\ (a, b) \subseteqq (-2\pi, 2\pi)$; функция $\Phi(\xi)$ $(\xi \geqq 0)$ удовлетворяет условиям (12) и (13)) *класс тех* 2π*-периодических функций* $f(x)$, *для которых*

$$v_{\Phi, R}(f) = \sup_n \sum_{a \leqq x_{kn} \leqq b} \Phi\left(|\Delta^R_{\frac{2\pi}{2n+1}} f(x_{kn})|\right) < \infty \qquad \left(x_{kn} = \frac{2k\pi}{2n+1}\right).$$

Теорема 13. *Если непрерывная функция* $f(x)$ *при некотором натуральном* R *содержится в классе* $V_{\Phi, R}(a, b)$, *то имеет место оценка*

$$|f(x) - \tilde{S}_n(x, f)| = O\left[\int_0^{\omega_R\left(f; \frac{1}{n}\right)} \log \frac{2v_{\Phi, R}(f)}{\Phi(\xi)}\, d\xi\right] \qquad (a < x < b,\ n = 1, 2, \ldots), \tag{26}$$

которая выполняется равномерно внутри интервала (a, b). *Таким образом полиномы* $\tilde{S}_n(x, f)$ *равномерно внутри* (a, b) *сходятся к функции* $f(x)$ *при* $n \to \infty$.

Несколько частных случаев настоящей теоремы были доказаны нами в работах [18], [20] и [22] (см. еще [23]).

Пусть $f(x)$ непрерывная функция пернода 2π. Обозначим через

$$\bar{f}(x) = -\frac{1}{\pi} \int_0^{\pi} \frac{f(x+t) - f(x-t)}{2 \operatorname{tg} \frac{t}{2}}\, dt$$

функцию, тригонометрически сопряженную к функции $f(x)$. Пусть далее, $\bar{\tilde{S}}_n(x, f)$ есть полином, тригонометрически сопряженный к интерполяционному полиному $\tilde{S}_n(x, f)$.

Теорема 14. *Если функция* $f(x)$ *удовлетворяет условиям Теоремы* 13, *а* $\bar{f}(x)$ *также непрерывна, то тогда при* $x \in [a_1, b_1] \subset (a, b)$ *мы имеем равномерную оценку*

$$|\bar{f}(x) - \bar{\tilde{S}}_n(x, f)| = O\left[\int_0^{\omega_R\left(f; \frac{1}{n}\right)} \log \frac{2v_{\Phi, R}(f)}{\Phi(\xi)}\, d\xi + \omega_{R'}\left(\bar{f}; \frac{1}{n}\right)\right] \qquad (a_1 \leqq x \leqq b_1)$$

[2] Мы ввели множитель $(-2)^{-R}$ из соображений, рассмотренных в [27].

($n=1, 2, \ldots$; $R'(\geqq R)$ *произвольное, но фиксированное натуральное число*), *в частности, если* $f\in \mathrm{Lip}\,\alpha$ $(0<\alpha\leqq 1)$ *то тогда*

$$|\bar{f}(x)-\bar{\bar{S}}_n(x,f)| = O\left[\int_0^{\frac{1}{n^\alpha}} \log\frac{2v_{\Phi,R}(f)}{\Phi(\xi)}\,d\xi\right] \quad (a_1\leqq x\leqq b_1,\ n=1,2,\ldots).$$

Следует подчеркнуть, что при изучении классов $V_\Phi(a,b)$ и $V_{\Phi,R}(a,b)$ нам не понадобилось понятие сопряженной функции в смысле Юнга, которое у разных авторов всегда появилось (см. например в книге Н. К. Бари [3]).

§ 2. Доказательства

Лемма 1. *Пусть* $[a,b]$ *произвольный, но фиксированный отрезок, содержащийся в интервале* $(-1,1)$. *Равномерно по* x *из* $[a,b]$ *имеет место оценка*

$$\sum_{k=1}^{n} |l_{kn}(w_1;\ x)+l_{k+1,n}(w_1;\ x)| = O(1) \tag{27}$$

$$(a\leqq x\leqq b,\ n=1,2,\ldots;\ l_{n+1,n}(w_1,x)\equiv 0).$$

Доказательство. Пусть δ достаточно малое положительное число. Согласно формулам (14.4.4) и (14.4.5) из книги Г. Сеге [30] мы имеем

$$\sum_{|x-x_{kn}|>\delta} |l_{kn}(w_1,x)| = O(1) \quad (a\leqq x\leqq b) \tag{28}$$

и

$$|l_{kn}(w_1,x)| = O(1) \quad (a\leqq x\leqq b;\ |x-x_{kn}|\leqq\delta), \tag{29}$$

значит

$$\sum_{k=1}^{n} |l_{kn}(w_1;\ x)+l_{k+1,n}(w_1;\ x)| = \tag{30}$$

$$= \sum_{x-\delta\leqq x_{kn}(w_1)\leqq x-\frac{1}{n}} |l_{kn}(w_1;\ x)+l_{k+1,n}(w_1;\ x)| +$$

$$+ \sum_{x+\frac{1}{n}\leqq x_{k+1,n}(w_1)\leqq x+\delta} |l_{kn}(w_1;\ x)+l_{k+1,n}(w_1;\ x)| + O(1) \quad (a\leqq x\leqq b),$$

ибо

$$\theta_{k+1,n}-\theta_{kn}\sim\frac{1}{n} \qquad (k=0,1,\ldots,n) \tag{31}$$

$$(\cos\theta_{kn}=x_{kn}(w_1),\quad n=1,2,\ldots,\quad \theta_{0n}=1,\quad \theta_{n+1,n}=-1),$$

как показал Г. И. Натансон в работе [16].[3] Мы будем оценивать лишь вторую сумму, стоящую на правой части (30), первая же оценивается аналогичным образом. Из формул (1) и (4), далее из соотношений (4.3.4) и (15.3.1) книги Г. Сеге [30] вытекает, что

$$l_{kn}(w_1, x) = (-1)^{k+1} \frac{p_n(w_1;\ x)}{\sqrt{2n+\beta+\gamma+1}} \frac{\sqrt{\lambda_{kn}(w_1)(1-x_{kn}^2(w_1))}}{x-x_{kn}(w_1)} \tag{32}$$

$$\left(n = 1, 2, \ldots;\quad k = 1, 2, \ldots, n;\quad \lambda_{kn}(w_1) \equiv \lambda_n(w_1;\ x_{kn}(w_1))\right).$$

Отсюда

$$\sum_{x+\frac{1}{n} \leqq x_{k+1,n}(w_1) \leqq x+\delta} |l_{kn}(w_1;\ x) + l_{k+1,n}(w_1;\ x)| =$$

$$= \frac{p_n(w_1, x)}{\sqrt{2n+\beta+\gamma+1}} \sum_{x+\frac{1}{n} \leqq x_{k+1,n}(w_1) \leqq x+\delta} \left| \frac{\sqrt{\lambda_{kn}(w_1)(1-x_{kn}^2(w_1))}}{x_{kn}(w_1)-x} - \right.$$

$$\left. - \frac{\sqrt{\lambda_{k+1,n}(w_1)(1-x_{k+1,n}^2(w_1))}}{x_{k+1,n}(w_1)-x} \right| =$$

$$= O\left(\frac{1}{\sqrt{n}}\right) \sum_{x+\frac{1}{n} \leqq x_{k+1,n}(w_1) \leqq x+\delta} \left| \frac{\sqrt{\lambda_{kn}(w_1)(1-x_{kn}^2(w_1))}}{x_{kn}(w_1)-x} - \frac{\sqrt{\lambda_{k+1,n}(w_1)(1-x_{k+1,n}^2(w_1))}}{x_{k+1,n}(w_1)-x} \right|$$

$$(a \leqq x \leqq b),$$

так как ортонормированные многочлены Якоби внутри интервала ортогональности ограничены в совокупности по абсолютной величине (см. Г. Сеге [30], (4.3.4) и (7.32.5)). Значит

$$\sum_{x+\frac{1}{n} \leqq x_{k+1,n}(w_1) \leqq x+\delta} |l_{kn}(w_1;\ x) + l_{k+1,n}(w_1;\ x)| = \tag{33}$$

$$= O\left(\frac{1}{\sqrt{n}}\right) \left[\sum_{x+\frac{1}{n} \leqq x_{k+1,n}(w_1) \leqq x+\delta} \sqrt{\lambda_{kn}(w_1)} \left| \frac{\sqrt{1-x_{kn}^2(w_1)}}{x_{kn}(w_1)-x} - \frac{\sqrt{1-x_{k+1,n}^2(w_1)}}{x_{k+1,n}(w_1)-x} \right| + \right.$$

$$\left. + \sum_{x+\frac{1}{n} \leqq x_{k+1,n}(w_1) \leqq x+\delta} \frac{\sqrt{1-x_{k+1,n}^2(w_1)}}{x_{k+1,n}(w_1)-x} \left| \sqrt{\lambda_{kn}(w_1)} - \sqrt{\lambda_{k+1,n}(w_1)} \right| \right] \quad (a \leqq x \leqq b).$$

В силу того, что выражение $\sqrt{\lambda_{kn}(w_1)} - \sqrt{\lambda_{k+1,n}(w_1)}$ меняет своего знака лишь конечное число раз (см. Г. Сеге [30], 15.3), мы получим, при помощи суммирования по частям, что вторая сумма, стоящая в правой части выражения

[3] Недавно Г. Фройд и А. Шарма [11] независимо от Г. И. Натансона также доказали соотношение (31).

(33) имеет порядок первой суммы в той же части (33). Отсюда

$$(34)\qquad \sum_{x+\frac{1}{n}\leqq x_{k+1,n}(w_1)\leqq x+\delta} |l_{kn}(w_1;\ x)+l_{k+1,n}(w_1;\ x)| =$$

$$= O\left(\frac{1}{\sqrt{n}}\right) \sum_{x+\frac{1}{n}\leqq x_{k+1,n}(w_1)\leqq x+\delta} \sqrt{\lambda_{kn}(w_1)}\left|\frac{\sqrt{1-x_{kn}^2(w_1)}}{x_{kn}(w_1)-x}-\frac{\sqrt{1-x_{k+1,n}^2(w_1)}}{x_{k+1,n}(w_1)-x}\right| =$$

$$= O\left(\frac{1}{\sqrt{n}}\right) \sum_{x+\frac{1}{n}\leqq x_{k+1,n}(w_1)\leqq x+\delta} \sqrt{\lambda_{kn}(w_1)\left(1-x_{kn}^2(w_1)\right)}\left|\frac{1}{x_{kn}(w_1)-x}-\frac{1}{x_{k+1,n}(w_1)-x}\right| +$$

$$+ O\left(\frac{1}{\sqrt{n}}\right) \sum_{x+\frac{1}{n}\leqq x_{k+1,n}(w_1)\leqq x+\delta} \sqrt{\lambda_{kn}(w_1)}\left|\frac{\sqrt{1-x_{kn}^2(w_1)}-\sqrt{1-x_{k+1,n}^2(w_1)}}{x_{k+1,n}(w_1)-x}\right| =$$

$$= O\left(\frac{1}{\sqrt{n}}\right) \sum_{x+\frac{1}{n}\leqq x_{k+1,n}(w_1)\leqq x+\delta} \sqrt{\lambda_{kn}(w_1)}\,\frac{x_{kn}(w_1)-x_{k+1,n}(w_1)}{\left(x_{kn}(w_1)-x\right)\left(x_{k+1,n}(w_1)-x\right)} +$$

$$+ O\left(\frac{1}{\sqrt{n}}\right) \sum_{x+\frac{1}{n}\leqq x_{k+1,n}(w_1)\leqq x+\delta} \sqrt{\lambda_{kn}(w_1)}\,\frac{x_{kn}(w_1)-x_{k+1,n}(w_1)}{x_{k+1,n}(w_1)-x} \quad (a\leqq x\leqq b),$$

значит — имея в виду соотношениа (31) и

$$(35)\qquad \lambda_n(w_1;\ \xi) = O\left(\frac{1}{n}\right) \quad (a'\leqq \xi\leqq b'),$$

где $[a', b']$ произвольный, но фиксированный отрезок, принадлежащий к интервалу $(-1, 1)$ (см. например Г. П. Неваи [24]) —

$$(36)\qquad \sum_{x+\frac{1}{n}\leqq x_{k+1,n}(w_1)\leqq x+\delta} |l_{kn}(w_1;\ x)+l_{k+1,n}(w_1;\ x)| =$$

$$= O\left(\frac{1}{n^2}\right) \sum_{x+\frac{1}{n}\leqq x_{k+1,n}(w_1)\leqq x+\delta} \frac{1}{\left(x_{kn}(w_1)-x\right)^2} +$$

$$+ O\left(\frac{1}{n^2}\right) \sum_{x+\frac{1}{n}\leqq x_{k+1,n}(w_1)\leqq x+\delta} \frac{1}{x_{kn}(w_1)-x} = O(1)+O\left(\frac{\log n}{n}\right) = O(1) \quad (a\leqq x\leqq b).$$

(27) следует теперь из оценок (30) и (36).

Доказательство Теоремы 1. Пусть $m=\left[\frac{n}{\log n}\right]$. При помощи теоремы

Джексона мы подберем полином $P_m(x)$ степени не выше m, для которого

$$(37) \qquad |f(x)-P_m(x)| = O\left[\omega\left(f;\frac{1}{m}\right)\right] = O\left[\omega\left(f;\frac{\log n}{n}\right)\right] \quad (-1 \leqq x \leqq 1).$$

Согласно одному неравенству С. Б. Стечкина [31] и М. Заманского [33] мы имеем

$$|P'_m(x)| = O\left[m\omega\left(P_m;\frac{1}{m}\right)\right] \quad (x\in[a', b']\subset(-1, 1)),$$

значит

$$|P'_m(x)| = O\left[m\omega\left(P_m-f;\frac{1}{m}\right)+m\omega\left(f;\frac{1}{m}\right)\right] =$$

$$= O\left[m\max_{|x|\leqq 1}|f(x)-P_m(x)|+m\omega\left(f;\frac{1}{m}\right)\right] \quad (x\in[a', b']\subset(-1, 1)),$$

то есть

$$(38) \qquad |P'_m(x)| = O\left[\frac{n}{\log n}\,\omega\left(f;\frac{\log n}{n}\right)\right] \quad (x\in[a', b']\subset(-1, 1)).$$

Из соотношения (37) следует, что

$$(39) \quad L_n(w_1,\ x,\ f)-f(x) = L_n(w_1,\ x,\ f-P_m)+O\left[\omega\left(f;\frac{\log n}{n}\right)\right] \quad (a \leqq x \leqq b).$$

Обозначив через $f(x_{0n}(w_1))=P_m(x_{0n}(w_1))=0$ и $l_{n+1,n}(w_1;\ x)\equiv 0$ мы получим с помощью элементарного преобразования, что

$$(40) \qquad L_n(w_1,\ x,\ f-P_m) =$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left[\left(f(x_{kn}(w_1))-P_m(x_{kn}(w_1))\right)-\left(f(x_{k-1,n}(w_1))-P_m(x_{k-1,n}(w_1))\right)\right]l_{kn}(w_1;\ x)+$$

$$+\frac{1}{2}\sum_{k=1}^{n}\left[f(x_{kn}(w_1))-P_m(x_{kn}(w_1))\right]\left[l_{kn}(w_1;\,x)+l_{k+1,n}(w_1;\ x)\right].$$

Теперь мы воспользуемся Леммой 1 и из соотношений (37), (39) и (40) выводим, что

$$(41) \qquad L_n(w_1,\ x,\ f-P_m) =$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left[\left(f(x_{kn}(w_1))-P_m(x_{kn}(w_1))\right)-\left(f(x_{k-1,n}(w_1))-P_m(x_{k-1,n}(w_1))\right)\right]l_{kn}(w_1;\ x)+$$

$$+O\left[\omega\left(f;\frac{\log n}{n}\right)\right] \quad (a \leqq x \leqq b).$$

Оценки (28,) (37) и (41) вместе показывают, что

$$L_n(w_1,\ x,\ f-P_m) = \tag{42}$$

$$= \frac{1}{2} \sum_{|x-x_{kn}(w_1)|<\varepsilon} \left[f\left(x_{kn}(w_1)\right)-f\left(x_{k-1,n}(w_1)\right)\right] l_{kn}(w_1;\ x)+$$

$$+\frac{1}{2} \sum_{|x-x_{kn}(w_1)|<\varepsilon} \left[P_m\left(x_{kn}(w_1)\right)-P_m\left(x_{k-1,n}(w_1)\right)\right] l_{kn}(w_1;\ x)+$$

$$+O\left[\omega\left(f;\ \frac{\log n}{n}\right)\right] \quad (a \leqq x \leqq b).$$

Далее, так как

$$\sum_{|x-x_{kn}(w_1)|<\varepsilon} |l_{kn}(w_1;\ x)| = O(\log n) \quad (a \leqq x \leqq b) \tag{43}$$

(см. Г. Сеге [30], (14.4.6)), то в силу (31) и (38) мы имеем

$$\left| \sum_{|x-x_{kn}(w_1)|<\varepsilon} \left[P_m\left(x_{kn}(w_1)\right)-P_m\left(x_{k-1,n}(w_1)\right)\right] l_{kn}(w_1;\ x)\right| = \tag{44}$$

$$= O\left[\frac{\log n}{n} \max_{|x-t|\leqq\varepsilon} |P_m'(t)|\right] = O\left[\omega\left(f;\ \frac{\log n}{n}\right)\right] \quad (a \leqq x \leqq b).$$

Из соотношений (42) и (44) следует, что

$$L_n(w_1,\ x,\ f-P_m) =$$

$$= \frac{1}{2} \sum_{|x-x_{kn}(w_1)|<\varepsilon} \left[f\left(x_{kn}(w_1)\right)-f\left(x_{k-1,n}(w_1)\right)\right] l_{kn}(w_1;\ x)+$$

$$+O\left[\omega\left(f;\ \frac{\log n}{n}\right)\right] \quad (a \leqq x \leqq b),$$

что вместе с (39) доказывает Теорему 1.

Лемма 2. *Равномерно по всем x из интервала $(-\infty, \infty)$ имеет место соотношение*

$$e^{-\frac{x^2}{2}} \sum_{k=1}^{n} |l_{kn}(w_2;\ x)+l_{k+1,n}(w_2;\ x)| = O(1) \tag{45}$$

$$(-\infty < x < \infty, \quad n = 1, 2, \ldots; \quad l_{n+1,n}(w_2;\ x) \equiv 0).$$

Доказательство. Если $|x| \geqq n^{1/4}$, то — как показал Г. Фройд в работе [7] — справедлива оценка

$$e^{-\frac{x^2}{2}} L_n(w_2, x) = O(1) \quad (|x| \geqq n^{1/4}, \quad n = 1, 2, \ldots), \tag{46}$$

значит тем более имеет место (45). Для случая $|x| \leqq n^{1/4}$ выполнение оценки (45) было показано нами в работе [21].

Доказательство Теоремы 2. Нам следует различать два подслучая.

а) Пусть $|x| \leqq n^{1/4}$. В этом случае — применяя теорему Джексона к отрезку $[-2\sqrt{n}, 2\sqrt{n}]$ — мы возьмем многочлен $P_m(x)$ степени не выше $m = \left[\frac{n}{\log n}\right]$, который удовлетворяет неравенству

$$|f(x) - P_m(x)| = O\left[\omega\left(f; \frac{\log n}{\sqrt{n}}\right)\right] \quad (|x| \leqq 2\sqrt{n}),$$

а для производной которого по уже цитированным результатам С. Б. Стечкина и. М. Заманского мы имеем

$$|P_m'(x)| = O\left[\frac{\sqrt{n}}{\log n}\omega\left(f; \frac{\log n}{\sqrt{n}}\right)\right] \quad (|x| \leqq \sqrt{2n+1}).$$

В дальнейшем схема доказательства в точности совпадает со схемой доказательства Теоремы 1; при этом вместо (28) следует воспользоваться оценкой

$$e^{-\frac{x^2}{2}} \sum_{|x - x_{kn}(w_2)| > \delta} |l_{kn}(w_2; x)| = O(1) \quad (|x| \leqq \sqrt{n}), \tag{47}$$

вместо (31) оценкой

$$x_{kn}(x_2) - x_{k+1,n}(w_2) \sim \frac{1}{\sqrt{n}} \quad (|x_{kn}(w_2)| \leqq \sqrt{n}), \tag{48}$$

вместо (43) оценкой

$$e^{-\frac{x^2}{2}} \sum_{|x - x_{kn}(w_2)| < \varepsilon} |l_{kn}(w_2; x)| = O(\log n) \quad (|x| \leqq \sqrt{n}), \tag{49}$$

а вместо Леммы 1 Леммой 2. Соотношения (47)—(49) доказаны в работе Г. Фройда [7].

б) Пусть теперь $|x| \geqq n^{1/4}$. Прежде всего покажем, что

$$e^{-\frac{x^2}{2}} \sum_{|x - x_{kn}(w_2)| < \varepsilon} |l_{kn}(w_2; x)| = O(e^{-n^{1/4}}) \quad (|x| \leqq n^{1/4}, \quad n = 1, 2, \ldots). \tag{50}$$

Для доказательства (50) нам понадобятся соотношения:

$$\sum_{k=1}^{n} \lambda_{kn}(w_2) e^{\frac{x_{kn}^2(w_2)}{2}} \leqq \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \quad (n = 1, 2, \ldots) \tag{51}$$

(см. Й. Балаж—П. Туран [2]) и

$$e^{-x^2} \sum_{k=1}^{n} \frac{l_{kn}^2(w_2; x)}{\lambda_{kn}(w_2)} = O(n^{1/2}) \quad (-\infty < x < \infty, \quad n = 1, 2, \ldots) \tag{52}$$

(см. Г. Фройд [21]). В силу (51 и (52) мы имеем

$$\sum_{k=1}^{n} |l_{kn}(w_2; e)| e^{\frac{x_{kn}^2(w_2)}{4}} \leqq \left[\sum_{k=1}^{n} \frac{l_{kn}^2(w_2; x)}{\lambda_{kn}(w_2)}\right]^{1/2} \left[\sum_{k=1}^{n} \lambda_{kn}(w_2) e^{\frac{x_{kn}^2(w_2)}{2}}\right]^{1/2} =$$

$$= e^{\frac{x^2}{2}} O\left(n^{\frac{1}{4}}\right) \qquad (-\infty < x < \infty)$$

значит тем более

$$\sum_{|x_{kn}(w_2)| > \frac{n^{1/4}}{2}} |l_{kn}(w_2; x)| = e^{\frac{x^2}{2}} O\left(n^{\frac{1}{4}} e^{-\frac{\sqrt{n}}{16}}\right) \quad (-\infty < x < \infty),$$

откуда легко следует соотношение (50).

Из оценок

$$\max_{1 \leqq k \leqq n-1} [x_{kn}(w_2) - x_{k+1,n}(w_2)] = O\left(n^{-1/6}\right)$$

(см. Г. Сеге [30], Задача 35) и (50) сразу вытекает, что

$$(53) \qquad e^{-\frac{x^2}{2}} \Big| \sum_{|x - x_{kn}(w_2)| < \varepsilon} [f(x_{kn}(w_2)) - f(x_{k-1,n}(w_2))] l_{kn}(w_2; x) \Big| =$$

$$= O\left[e^{-n^{1/4}} \omega(f; n^{-1/6})\right] = O\left[\omega\left(f; \frac{\log n}{\sqrt{n}}\right)\right] \qquad (|x| \geqq n^{1/4},\ n = 1, 2, \ldots).$$

Далее к функции $f(x)$, в силу одной теоремы Г. Фройда [8], при всех натуральных n можно подобрать многочлены $P_n(x)$ степени не выше $n-1$, для которых

$$(54) \qquad |f(x) - P_n(x)| =$$

$$= \begin{cases} O\left[\omega\left(f; \frac{1}{\sqrt{n}}\right)\right], & \text{при} \quad |x| \leqq 4\sqrt{n} \\ e^{\frac{x^2}{2}} O\left[\omega\left(f; \frac{1}{\sqrt{n}}\right) + e^{-cn}\left(\max_{|t| \leqq 1} |f(t)| + \omega(f; 1)\right)\right], & \text{при} \quad -\infty < x < \infty, \end{cases}$$

где $c(>0)$ некоторая абсолютная константа. Известно, что корни многочлена $p_n(w_2; x)$ по абсолютной величине не превосходят $\sqrt{2n+1}$ (см. Г. Сеге [30], (6.32.6)), значит в силу (46) и (53) мы имеем следующую цепочку соотношений:

$$(55) \qquad |L_n(w_2, x, f) - f(x)| = |L_n(w_2, x, f - P_n) - [f(x) - P_n(x)]| \leqq$$

$$\leqq e^{\frac{x^2}{2}} O\left[\max_{|t| \leqq \sqrt{2n+1}} |f(t) - P_n(t)|\right] + |f(x) - P_n(x)| =$$

$$= e^{\frac{x^2}{2}} O\left[\omega\left(f; \frac{1}{\sqrt{n}}\right)\right] \qquad \left(|x| \geqq n^{\frac{1}{4}}\right),$$

ибо выражение e^{-cn} асимптотически меньше, чем $\omega\left(f; \frac{1}{\sqrt{n}}\right)$.

Оценки (53) и (55) показывают, что мы доказали Теорему 2 и для случая $|x| \geqq n^{1/4}$.

Лемма 3. *Если* $[a, b] \subset (0, \infty)$, *то равномерно по всем* x, *принадлежащим к отрезку* $[a, b]$ *мы имеем*

$$\sum_{k=1}^{n} |l_{kn}(w_3; x) + l_{k+1,n}(w_3; x)| = O(1) \tag{56}$$

$$(a \leqq x \leqq b, \quad n = 1, 2, \ldots; \quad l_{n+1,n}(w_3; x) \equiv 0).$$

Лемма 3 была доказана нами в работе [25].

Доказательство Теоремы 3 можно проводить методом, применяемым при доказательстве Теоремы 1. При этом нам следует применять теорему Джексона не к функции $f(x)$ на отрезке $[0, 5n]$ (этот отрезок содержит все корни многочлена $p_n(w_3; x)$), а к функции $g(x) \equiv f(x^2)$ на отрезке $[-\sqrt{5n}, \sqrt{5n}]$. Далее оценки (28), (31) и (43) заменяются соответственно оценками

$$\sum_{|x - x_{kn}(w_3)| > \varepsilon} |l_{kn}(w_3; x)| = O(1) \quad (a \leqq x \leqq b), \tag{57}$$

$$x_{kn}(w_3) - x_{k+1,n}(w_3) \sim \frac{1}{\sqrt{n}} \quad (a \leqq x_{kn}(w_3) \leqq b) \tag{58}$$

и

$$\sum_{|x - x_{kn}(w_3)| < \varepsilon} |l_{kn}(w_3; x)| = O(\log n) \quad (a \leqq x \leqq b). \tag{59}$$

(57) и (59) были доказаны в нашей работе [19], а (58) в работе Г. Фройда [9].

Доказательство Теоремы 4. Если $a < a_1 < b_1 < b$ и ε как угодно малое, но достаточно малое, положительное число, то в силу Теоремы 1 мы имеем

$$|f(x) - L_n(w_1, x, f)| \leqq$$

$$\leqq \frac{1}{2} \Big| \sum_{|x - x_{kn}(w_1)| < \varepsilon} [f(x_{kn}(w_1)) - f(x_{k-1,n}(w_1))] l_{kn}(w_1; x) \Big| +$$

$$+ O\left[\omega\left(f; \frac{\log n}{n}\right)\right] \qquad (a_1 \leqq x \leqq b_1, \ n = 1, 2, \ldots),$$

где константа, входящая в „O"-член зависит лишь от числа ε и расстояния

отрезка $[a, b]$ от точек -1 и 1. Значит в силу условия (9)

$$|f(x)-L_n(w_1, x, f)| \leqq \tag{60}$$

$$\leqq \frac{1}{2} \sum_{|x-x_{kn}(w_1)|<\varepsilon} \left[f\left(x_{k-1,n}(w_1)\right) - f\left(x_{kn}(w_1)\right) + \frac{v\left(x_{k-1,n}(w_1)-x_{kn}(w_1)\right)}{\left|\log\left(x_{k-1,n}(w_1)-x_{kn}(w_1)\right)\right|} \right] \times$$

$$\times |l_{kn}(w_1; x)| + \max_k \left\{ \frac{v\left(x_{k-1,n}(w_1)-x_{kn}(w_1)\right)}{\left|\log\left(x_{k-1,n}(w_1)-x_{kn}(w_1)\right)\right|} \right\} L_n(w_1, x) +$$

$$+ O\left[\omega\left(f; \frac{\log n}{n}\right)\right] \quad (a_1 \leqq x \leqq b_1).$$

Из соотношений (28), (43), (31) и (60) вытекает, что

$$|f(x)-L_n(w_1, x, f)| \leqq \tag{61}$$

$$\leqq \left| \frac{1}{2} \sum_{|x-x_{kn}(w_1)|<\varepsilon} \left[f\left(x_{k-1,n}(w_1)\right) - f\left(x_{kn}(w_1)\right) \right] |l_{kn}(w_1; x)| \right| +$$

$$+ O\left[v\left(\frac{c(n)}{n}\right)\right] + O\left[\omega\left(f; \frac{\log n}{n}\right)\right] \qquad (a_1 \leqq x \leqq b_1, \quad n = 1, 2, \ldots),$$

где последовательность $c(n)$ ограничена и снизу и сверху положительными константами. Из оценки (61) при помощи неравенства $\big||\alpha|-|\beta|\big| \leqq |\alpha+\beta|$ и суммирования по частям получим, что

$$|f(x)-L_n(w_1, x, f)| \leqq$$

$$\leqq \frac{1}{2} \sum_{|x-x_{kn}(w_1)|<\varepsilon} \left| f\left(x_{kn}(w_1)\right) - f(x) \right| \left| l_{kn}(w_1; x) + l_{k-1,n}(w_1; x) \right| +$$

$$+ O\left[v\left(\frac{c(n)}{n}\right)\right] + O\left[\omega\left(f; \frac{\log n}{n}\right)\right] \quad (a_1 \leqq x \leqq b_1),$$

значит применяя Лемму 1 мы имеем

$$|f(x)-L_n(w_1, x, f)| =$$

$$= O(1)\omega(f; \varepsilon) + O\left[v\left(\frac{c(n)}{n}\right)\right] + O\left[\omega\left(f; \frac{\log n}{n}\right)\right] \qquad (a_1 \leqq x \leqq b_1, \quad n = 1, 2, \ldots).$$

В последней оценке число ε может быть как угодно малым, таким образом Теорема 4 установлена.

Если интервал (a, b), участвующий в формулировке Теоремы 5, конечный, то Теорема 5 доказывается дословно также, как и Теорема 4. Если же интервал (a, b) бесконечный, то доказательство Теоремы 5 разбивается на два пункта:

а) $x \in (a+\varepsilon, b-\varepsilon) \cap [-n^{1/4}, n^{1/4}]$

и

б) $x \in (a+\varepsilon, b-\varepsilon)$ и одновременно $|x| > n^{1/4}$.

В случае а) доказательство не отличается от доказательства Теоремы 4. Во втором же случае нужно воспользоваться оценкой (55), из которой немедленно вытекает требуемый результат.

Теорема 6 доказывается также, как и Теорема 4.

Для дальнейшего нам понадобится следующее утверждение, которое в сущности доказано в работе К. И. Осколкова [28]. Этот результат был получен Осколковым по пути доказательства одной теоремы, связанной с тригонометрическими рядами Фурье, однако он применим при существенно более общей ситуации. Хотя доказательство этого утверждения есть почти дословное повторение рассуждений К. И. Осколкова, мы его проводим. Это мы сделаем с той целью, потому что нам кажется, что это утверждение в будущем будет часто применяться в различных вопросах теории аппроксимации.

Лемма 4. *Пусть функция* $\Phi(\xi)$ $(\xi \geqq 0)$ *удовлетворяет условиям* $\Phi(0)=0$, $\Phi(\xi_1) \leqq \Phi(\xi_2)$ $(\xi_1 < \xi_2)$ *и*

$$\int_0^1 |\log \Phi(\xi)|\, d\xi < \infty. \tag{62}$$

Если $a_{kn}(\geqq 0)$ $(n=1, 2, \ldots;\ k=1, 2, \ldots, n)$ *такая треугольная матрица чисел, что*

$$v = \sup_n \sum_{k=1}^{n} \Phi(a_{kn}) < \infty, \tag{63}$$

то тогда

$$\sum_{k=1}^{n} \frac{a_{kn}}{k} = O\left[\int_0^{a_n} \log \frac{2v}{\Phi(\xi)}\, d\xi\right] \qquad (n = 1, 2, \ldots). \tag{64}$$

Доказательство. Обозначим через $\chi_{kn}(\xi)$ характеристическую функцию интервала $[0, a_{kn})$ и пусть

$$M_{kn}(\xi) = 1 + \sum_{i=1}^{k-1} \chi_{in}(\xi) \qquad (k = 1, 2, \ldots, n+1).$$

В силу того, что $1 \leqq M_{kn}(\xi) \leqq k$ и

$$x \leqq \frac{1}{\log 2} \log (1+x) \quad (0 \leqq x \leqq 1)$$

мы имеем соотношения

$$\sum_{k=1}^{n} \frac{a_{kn}}{k} = \int_0^{a_n} \sum_{k=1}^{n} \frac{\chi_{kn}(\xi)}{k}\, d\xi \leqq \tag{65}$$

$$\leqq \frac{1}{\log 2} \int_0^{a_n} \sum_{k=1}^{n} \log\left[1 + \frac{\chi_{kn}(\xi)}{M_{kn}(\xi)}\right] d\xi = \frac{1}{\log 2} \int_0^{a_n} \log M_{n+1,n}(\xi)\, d\xi.$$

Далее, согласно (63),

$$\frac{1}{a_n}\int_0^{a_n}\Phi(\xi)M_{n+1,n}(\xi)\,d\xi \leqq \Phi(a_n)+\frac{1}{a_n}\sum_{k=1}^{n}\Phi(a_{kn})a_{kn} \leqq 2v,$$

значит, в силу неравенства Йенсена (см. А. Зигмунд [35], 1 (10.8))

$$\int_0^{a_n}\log\left[M_{n+1,n}(\xi)\Phi(\xi)\right]d\xi \leqq a_n\log 2v. \tag{66}$$

Резумируя соотношения (65) и (66) мы получим, что

$$\sum_{k=1}^{n}\frac{a_{kn}}{k} \leqq \frac{1}{\log 2}\int_0^{a_n}\log\frac{2v}{\Phi(\xi)}d\xi+\frac{1}{\log 2}\int_0^{a_n}\log\left[\frac{M_{n+1,n}(\xi)\Phi(\xi)}{2v}\right]d\xi =$$

$$\leqq \frac{1}{\log 2}\int_0^{a_n}\log\frac{2v}{\Phi(\xi)}d\xi = O\left[\int_0^{a_n}\log\frac{2v}{\Phi(\xi)}d\xi\right].$$

Лемма 4 полностью установлена.

Доказательство Теоремы 7. В силу Теоремы 1 нам нужно оценить лишь выражение

$$\left|\sum_{|x-x_{kn}(w_1)|<\varepsilon}\left[f\left(x_{kn}(w_1)\right)-f\left(x_{k-1,n}(w_1)\right)\right]l_{kn}(w_1;x)\right| \quad (a_1 \leqq x \leqq b_1),$$

где $[a_1, b_1]\subset(a, b)$. Далее из соотношений (31), (32) и (35) следует, что

$$|l_{kn}(w_1;\ x)| \leqq C\frac{1}{|k-k_0|+1} \quad (a_1 \leqq x \leqq b_1;\ |x-x_{kn}(w_1)|<\varepsilon),$$

где k_0 индекс корня $x_{kn}(w_1)$ ближащего к точке x, а константа C не зависит от x и n. Обозначив через a_{kn} выражение $\left|f\left(x_{kn}(w_1)\right)-f\left(x_{k-1,n}(w_1)\right)\right|$ мы увидим, что Теорема 7 следует непосредственно из Леммы 4.

В точности также доказываются и Теоремы 8 и 9, при этом при доказательстве Теоремы 8 в случае $|x| \geqq n^{1/4}$ следует воспользоваться оценкой (55).

Доказательство Теоремы 10. Прежде всего отметим, что из условия (17) вытекает, что

$$C_3 \leqq w(x) \quad (a^* \leqq x \leqq b^*), \tag{67}$$

где $C_3>0$ (см. Я. Л. Геронимус [12]). В силу неравенств (18) и (67) мы имеем

$$x_{kn}(w)-x_{k-1,n}(w) \sim \frac{1}{n} \qquad \left(a^*+\delta \leqq x_{kn}(w) \leqq b^*-\delta\right) \tag{68}$$

(см. П. Эрдеш—П. Туран [5]), где δ как угодно малое, но фиксированное,

положительное число. Далее, как показал Г. Фройд в работе [6], из условий (17), (18) и (67) следуют оценки[4]

$$\sum_{|x-x_{kn}(w)|>\varepsilon} |l_{kn}(w; x)| = O(1) \quad (a^*+\varepsilon<x<b^*-\varepsilon) \tag{69}$$

и

$$\sum_{|x-x_{kn}(w)|<\varepsilon} |l_{kn}(w; x)| = O(\log n) \quad (a^*+\varepsilon < x < b^*-\varepsilon). \tag{70}$$

Таким образом для доказательства Теоремы 10 схема доказательства Теоремы 1 может быть применена. Это мы предоставляем Читателю.

Теорема 11 доказывается как Теорема 4, а Теорема 12 как Теорема 7. По пути доказательства Теоремы 12 следует воспользоваться оценкой

$$|l_{kn}(w; x)| \leqq C \frac{1}{|k-k_0|+1} \quad (a_1 \leqq x \leqq b_1;\ |x-x_{kn}(w)| < \varepsilon)$$

(k_0 индекс ближащего к точке x корня $x_{kn}(w)$), доказанной Г. Фройдом в работе [6], которая справедлива при условиях (17), (18) и (67).

Теорема 13 есть непосредственное следствие Леммы 4 и представления

$$\tilde{S}_n(x, f) - f(x) = \tilde{S}_n(x, \Delta^R_{\frac{2\pi}{2n+1}}) + O\left[\omega_R\left(f; \frac{1}{n}\right)\right] \qquad (n = 1, 2, \ldots),$$

которое выполняется при всех фиксированных, натуральных чисел R (см. [17]).

Доказательство Теоремы 14. Для краткости записи обозначим через

$$\|f\| = \max_{|x|\leqq\pi} |f(x)|.$$

Пусть

$$S_n(x, f) = \frac{a_0(f)}{2} + \sum_{k=1}^{n} (a_k(f) \cos kx + b_k(f) \sin kx) \equiv \sum_{k=0}^{n} A_k(x, f)$$

n-ая частная сумма тригонометрического ряда Фурье функции $f(x)$,

$$\bar{S}_n(x, f) = \sum_{k=1}^{n} (-a_k(f) \sin kx + b_k(f) \cos kx) \equiv \sum_{k=1}^{n} \bar{A}_k(x, f)$$

n-ая частная сумма ряда Фурье сопряженной функции $\bar{f}(x)$, а

$$\bar{\tilde{S}}_n(x, f) = \sum_{k=1}^{n} (-a_{kn}(f) \sin kx + b_{kn}(f) \cos kx) \equiv \sum_{k=1}^{n} \bar{\tilde{A}}_{kn}(x, f)$$

полином, тригонометрически сопряженный к $\tilde{S}_n(x, f)$ ($n=1, 2, \ldots$). Здесь $a_k(f)$ и $b_k(f)$ коэффициенты Фурье фурнкции $f(x)$, а $a_{kn}(f)$ и $b_{kn}(f)$ коэффициенты Фурье-Лагранжа (относительно этих последних см. А. Зигмунд [35], X. 4).

[4] Относительно (69) и (70) см. еще Г. Фройд [10].

Сначала покажем, что если и $f(x)$ и $\bar{f}(x)$ обе непрерывны, то при всех фиксированных $R(=1, 2, \ldots)$ имеет место оценка

$$(71)\quad \|\bar{\tilde{S}}_n(f)-\bar{f}\| \leqq \|\tilde{S}_n(f)-f\| + O_R\left[\omega_R\left(f;\frac{1}{n}\right)+\omega_R\left(\bar{f};\frac{1}{n}\right)\right] \qquad (n=1, 2, \ldots).$$

Ясно, что (71) достаточно показать только для нечетных значений числа R. Для этого отметим, что

$$\bar{\tilde{S}}_n(x, f)-\bar{f}(x) = \sum_{k=1}^{n}\left[1-\left(\frac{k}{n}\right)^R\right]\bar{\tilde{A}}_{kn}(x, f)-\bar{f}(x)+\sum_{k=1}^{n}\left(\frac{k}{n}\right)^R\bar{\tilde{A}}_n(x, f),$$

значит

$$(72)\quad \|\bar{\tilde{S}}_n(f)-\bar{f}\| \leqq \left\|\sum_{k=1}^{n}\left[1-\left(\frac{k}{n}\right)^R\right]\bar{\tilde{A}}_{kn}(f)-\bar{f}\right\|+\frac{1}{n^R}\|\tilde{S}_n^{(R)}(f)\|.$$

Из одного неравенства С. Б. Стечкина [31] следует, что

$$\frac{1}{n^R}\|\tilde{S}_n^{(R)}(f)\| \leqq \omega_R\left(\tilde{S}_n(f);\frac{\pi}{n}\right),$$

значит

$$(73)\quad \frac{1}{n^R}\|\tilde{S}_n^{(R)}(f)\| \leqq \|\tilde{S}_n(f)-f\|+\omega\left(f;\frac{\pi}{n}\right).$$

Далее в силу результатов Д. Л. Бермана [4] мы имеем

$$\left\|\sum_{k=1}^{n}\left[1-\left(\frac{k}{n}\right)^R\right]\bar{\tilde{A}}_{kn}(f)-\bar{f}\right\| \leqq C_n\left\|\sum_{k=1}^{n}\left[1-\left(\frac{k}{n}\right)^R\right]\bar{A}_k(f)-\bar{f}\right\|,$$

$C_n=1+C_n^1+C_n^2$, где C_n^1 есть константы Лебега процесса $\sum_{k=1}^{n}\left[1-\left(\frac{k}{n}\right)^R\right]\bar{A}_k(x, f)$, а C_n^2 есть константы Лебега процесса $\sum_{k=1}^{n}\left[1-\left(\frac{k}{n}\right)^R\right]\bar{\tilde{A}}_{kn}(x, f)$.[5] Как показали Й. Марцинкевич и А. Зигмунд в работе [15] мы имеем $C_n^1=O(1)$ и $C_n^2=O(1)$, следовательно — согласно одной теореме В. В. Жука [36] —

$$(74)\quad \left\|\sum_{k=1}^{n}\left[1-\left(\frac{k}{n}\right)^R\right]\bar{\tilde{A}}_{kn}(f)-\bar{f}\right\| = O\left[\omega_R\left(f;\frac{1}{n}\right)+\omega_R\left(\bar{f};\frac{1}{n}\right)\right].$$

(71) вытекает теперь из соотношений (72)—(74).

Первая часть Теоремы 14 немедленно следует из Теоремы 13 и неравенства (71).

Если же $f\in\mathrm{Lip}\,\alpha$ $(0<\alpha<1)$ то по теореме И. И. Привалова [29] $\bar{f}\in\mathrm{Lip}\,\alpha$ также. Если же $f\in\mathrm{Lip}\,1$ то по результатам Г. Алексича [1] мы имеем $E_n(\bar{f})=O\left(\frac{1}{n}\right)$, где $E_n(\bar{f})$ мера наилучшего равномерного приближения функции

[5] Здесь речь идет о суммах Рисса.

$\bar{f}(x)$ при помощи тригонометрических полиномов порядка не выше n, таким образом по теореме А. Зигмунда [34] $\omega_2(\bar{f}; \delta)=O(\delta)$. Следовательно при $f\in \mathrm{Lip}\,\alpha\,(0<\alpha\leqq 1)$ неравенство (71) превращается в

$$\|\bar{\bar{S}}_n(f)-\bar{f}\| \leqq \|\bar{S}_n(f)-f\|+O\left(\frac{1}{n^\alpha}\right) \qquad (n=1, 2, \ldots), \tag{75}$$

что вместе с Теоремой 13 доказывает справедливость второй части Теоремы 14.

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Литература

[1] Gy. Alexits, A Fourier-sor Cesáro-közepeivel való approximäció nagyságrendjéről, *Matematikai és Fizikai Lapok*, **48** (1941), стр. 410—422 (по венгерски).
[2] J. Balázs—P. Turán, Notes on interpolation. VIII, *Acta Math. Acad. Sci. Hung.*, **12** (1961), стр. 469—474.
[3] Н. К. Бари, *Тригонометрические ряды*, ГИФМЛ (Москва, 1961).
[4] Д. Л. Берман, Некоторые неравенства и их применения в теории интерполяции, *Исследования по совр. пробл.* КТФ, ГИФМЛ (Москва, 1961), стр. 116—122.
[5] P. Erdős—P. Turán, On interpolation III, *Annals Math.*, **41** (1940), стр. 510—553.
[6] G. Freud, Über die Lebesgueschen Funktionen der Lagrangeschen Interpolation, *Acta Math. Acad. Sci. Hung.*, **4** (1953), стр. 137—142.
[7] G. Freud, Lagrangesche Interpolation über die Nullstellen der Hermiteschen Orthogonalpolynome, *Studia Sci. Math. Hung.*, **4** (1968), стр. 179—190.
[8] G. Freud, On weighted polynomial approximation on the whole real axis, *Acta Math. Acad. Sci. Hung.*, **20** (1969), стр. 223—225.
[9] Г., Фройд, Об одном классе ортогональных многочленов, *Мат. Заметки*, **9** (1971), стр. 511—520.
[10] G. Freud, *Orthogonal polynomials*, Akadémiai Kiadó (Budapest, 1971).
[11] G. Freud—A. Sharma, Some good sequences of interpolatory polynomials, *Canad. Journ. of Math.*
[12] Я. Л. Геронимус, О некоторых основных неравенствах в теории ортогональных многочленов, *Доклады АН СССР*, **140** (1961), стр. 1002—1004.
[13] Я. Л. Геронимус, О сходимости интерполяционного процесса Лагранжа с узлами в корях ортогональных многочленов, *Известия АН СССР (серия матем.)*, **27** (1963), стр. 529—560.
[14] В. И. Крылов, Сходимость алгебраического интерполирования по корням многоченов Чебышева для абсолютно непрерывных функций и функций с ограниченным изменением, *Доклады АН СССР*, **107** (1956), стр. 362—365.
[15] J. Marcinkiewicz—A. Zygmund, Mean values of trigonometric polynomials, *Fund. Math.*, **28** (1937), стр. 131—166.
[16] Г. И. Натансон, Двусторонная оценка функции Лебега интерполяционного процесса Лагранжа с узлами Якоби, *Известия Высших Уч. зав.*, **11(66)** (1967), стр. 67—74.
[17] Г. П. Неваи, Об отклонении тригонометрических интерполяционных сумм, *Acta Math. Acad. Sci. Hung.*, **23** (1972), стр. 203—205.
[18] Г. П. Неваи, Асимптотическая формула для отклонения тригонометрических интерполяционных сумм, *Studia Sci. Math. Hung.*, **7** (1972), стр. 391—394.
[19] G. P. Névai, A Laguerre-polinomok gyökein alapuló Lagrange-féle interpolációról, *Mat. Lapok*, **22** (1—2) (1971), стр. 149—164 (по венгерски).

[20] G. P. NÉVAI, Az ekvidisztáns csomópontokon alapuló trigonometrikus interpolációról, *MTA III. Osztályának Közleményei* **21** (1973), стр. 449—484 (по венгерски).
[21] Г. П. Неваи, Об интерполяционном процессе Лагранжа с узлами в корнях многочленов Эрмита, *Acta Math. Acad. Sci. Hung.*, **24** (1973), стр. 209—213.
[22] Г. П. Неваи, Замечания о тригонометрическом интерполировании и суммах Фурье, *Studia Sci. Math. Hung.*
[23] G. P. NÉVAI, Trigonometric interpolation by equidistant nodes, *Studia Math.*
[24] Г. П. Неваи, Многочлены, ортормальные на вещественной оси с весом $|x|^{\alpha}e^{-|x|^{\beta}}$. I, *Acta Math. Acad. Sci. Hung.*, **24** (1973), стр. 335—342.
[25] Г. П. Неваи, О сходимости Лагранжева интерполирования по узлам Лагерра, *Publicationes Math.*, **20** (1973), стр. 235—239.
[26] Г. П. Неваи, О признаке Дини—Липшица, *Acta Math. Acad. Sci. Hung.*, **24** (1973), стр. 349—351.
[27] Г. П. Неваи, Приближение периодических функций многих переменных. I, *Acta Math. Acad. Sci. Hung.*, **24** (1973), стр. 407—416.
[28] К. И. Осколков, Обобщенная вариация, индикатриса Банаха и равномерная сходимость раядов Фурье, *Мат. Заметки*, **12** (1972), стр. 313—324.
[29] I. I. PRIVALOFF, Sur les fonctions conjugées, *Bull. de la Soc. Math. de France*, **44** (1916), стр. 100—103.
[30] G. SZEGŐ, *Orthogonal polynomials*, Amer. Math. Soc. Coll. Publ., vol. XXIII, Providence, Rhode Island, rev. ed., 1967.
[31] С. Б. Стечкин, Обобщение некоторых неравентсв С. Н. Бернштейна, *Доклады АН СССР*, **60** (1948), стр. 1511—1514.
[32] L. C. YOUNG, Sur une généralisation de la notion de variation de puissance *p*-iéme bornée au sens de M. Wiener, et sur la convergence des séries de Fourier, *Comptes Rendus*, **204** (1937), стр. 470—472.
[33] M. ZAMANSKY, *Classes de saturation de certain procédés d'approximation des séries de Fourier des fonctions continues et applications a quelques problémes d'approximation*, Gauthier—Villars, Impriemur-éditeur (Paris, 1949).
[34] A. ZYGMUND, Smooth functions, *Duke Math. Journal*, **12** (1945), стр. 47—76.
[35] A. ZYGMUND, *Trigonometric series*, vol. I—II, University Press (Cambridge, 1959).
[36] В. В. Жук, О приближении 2π-периодической функции линейным оператором, *Исследования по некоторым проблемам КТФ* (Ленинград, 1965), стр. 93—115.

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A NEW PROOF OF THE RIEMANN'S LOCALIZATION PRINCIPLE

By

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The well-known localization theorem of Riemann states that if a 2π *periodic function is integrable over the segment* $[-\pi, \pi]$ *then the convergence of its trigonometric Fourier series at an arbitrary point* x *depends only on the behaviour of the function in the neighbourhood of the point* x, i.e. for every $f \in L_1[-\pi, \pi]$ and $\delta > 0$ we have

$$(*) \qquad \lim_{n\to\infty} \frac{1}{\pi} \int_{\delta \leqq |t| \leqq \pi} f(t+x) D_n(t)\, dt = 0 \qquad \left(D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} \right),$$

uniformly in x.

We now give a simple proof for this statement.

Indeed, we get by a simple computation

$$\frac{1}{\pi} \int_{\delta \leqq |t| \leqq \pi} f(t+x) D_n(t)\, dt = \frac{1}{2\pi} \int_{\delta \leqq |t| \leqq \pi} \left[f(t+x) - f\left(t+x+\frac{2\pi}{2n+1}\right) \right] D_n(t)\, dt +$$

$$+ \frac{1}{2\pi} \int_{\delta \leqq |t| \leqq \pi} f(t+x) \left[D_n(t) + D_n\left(t-\frac{2\pi}{2n+1}\right) \right] dt +$$

$$+ \left\{ \frac{1}{2\pi} \int_{-\delta}^{-\delta+\frac{2\pi}{2n+1}} - \frac{1}{2\pi} \int_{\delta}^{\delta+\frac{2\pi}{2n+1}} \right\} f(t+x) D_n\left(t-\frac{2\pi}{2n+1}\right) dt = A + B + C.$$

It is clear that $A = \frac{1}{\delta} O\left[\omega\left(f; \frac{1}{n}\right)_{L_1}\right]$. Further one sees easily from the theory of Bernstein—Rogosinski sums that

$$D_n(t) + D_n\left(t-\frac{2\pi}{2n+1}\right) = O\left(\frac{n}{1+n^2t^2}\right) \qquad (|t| \leqq \pi),$$

hence $B = \frac{1}{\delta^2} \|f\|_{L_1} O\left(\frac{1}{n}\right)$. We have $C = \frac{1}{\delta} o(1)$ as a consequence of the absolute continuity of the Lebesgue integral. So the relation $(*)$ is fulfilled.

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STABLE INTERPOLATION ON AN INFINITE INTERVAL

By
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1. The problems of stable interpolation on an infinite interval were dealt by E. EGERVÁRY and P. TURÁN in their paper [1]. Let

$$0 \leqq \xi_0 < \xi_1 < \ldots < \xi_n \tag{1.1}$$

be $n+1$ points in $[0, \infty)$ and let the fundamental interpolation polynomials belonging to these points be

$$q_0(x), q_1(x), \ldots, q_n(x) \tag{1.2}$$

for which

$$q_i(\xi_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \qquad (i, j = 0, 1, \ldots, n) \tag{1.3}$$

holds. Moreover let y_i $(i=0, 1, \ldots, n)$ be arbitrary real numbers, then

$$Q_n(x) \overset{\text{def}}{=} \sum_{i=0}^{n} y_i q_i(x) \tag{1.4}$$

is obviously an interpolation polynomial, i.e.

$$Q_n(\xi_i) = y_i \qquad (i = 0, 1, \ldots, n). \tag{1.5}$$

In [1] the authors were looking for the economical stable interpolation polynomial for which, on the one hand, the following inequality is fulfilled:

$$0 \leqq e^{-x} \sum_{i=0}^{n} (y_i - y_i^*) q_i(x) \leqq \max_i \{e^{-\xi_i}(y_i - y_i^*)\} \quad (x \in [0, \infty)) \tag{1.6}$$

where $y_i \geqq y_i^*$ $(i=0, 1, \ldots, n)$ are arbitrary real numbers; on the other hand

$$\deg Q_n(x) \overset{\text{def}}{=} \sum_{i=0}^{n} \deg \{y_i q_i(x)\} \tag{1.7}$$

is minimal. The quoted authors proved that (1.6) and (1.7) are fulfilled simultaneously if and only if

$$q_0(x) = L_n(x)^2, \quad q_i(x) = \frac{x}{\xi_i} l_i^2(x) \qquad (i = 1, 2, \ldots, n) \tag{1.8}$$

where $L_n(x)$ denotes the Laguerre polynomial of degree n, normalized to $L_n(0)=1$, ξ_i $(i=1, 2, \ldots, n)$ denote the roots of the polynomial $L_n(x)$, and

$$l_i(x) = \frac{L_n(x)}{L_n'(\xi_i)(x-\xi_i)}.$$

In the case of the fundamental interpolation polynomials (1.8) J. BALÁZS and P. TURÁN [2] proved the following convergence theorem:

THEOREM. *If $f(x)$ is continuous and bounded in $[0, \infty)$ and ω is an arbitrary large positive number, then $Q_n(f)$ tends to the function $f(x)$ in $[0, \infty)$ and this convergence is uniform in $[0, \omega]$ where*

$$Q_n(f) \overset{\text{def}}{=} \sum_{i=0}^{n} f(\xi_i) q_i(x).$$

2. In the present paper we are going to define a stable interpolation process in the case the $n+1$ points

$$0 = x_0 < x_1 < \ldots < x_n \tag{2.1}$$

are given in $[0, \infty)$; the fundamental interpolation polynomials of the first kind belonging to these are

$$r_0(x), r_1(x), \ldots, r_n(x) \tag{2.2}$$

i.e. for these

$$r_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}, \qquad (i, j = 0, 1, \ldots, n) \tag{2.3}$$

and for the derivatives

$$r_i'(0) = 0 \qquad (i = 0, 1, \ldots, n) \tag{2.4}$$

hold. Furthermore we introduce $\varrho(x)$, the so called fundamental interpolation polynomial of the second kind, i.e. for which

$$\varrho(x_i) = 0 \qquad (i = 0, 1, \ldots, n) \tag{2.5}$$

holds and for the derivative

$$\varrho'(x_i) = \begin{cases} 0, & i \neq 0 \\ 1, & i = 0 \end{cases} \qquad (i=0, 1, \ldots, n) \tag{2.6}$$

is fulfilled.

It is obvious that if y_i $(i=0, 1, \ldots, n)$ and y' are arbitrary real numbers, then

$$R_n(x) \overset{\text{def}}{=} \sum_{i=0}^{n} y_i r_i(x) + y' \varrho(x) \tag{2.7}$$

is an interpolation polynomial for which

$$R_n(x_i) = y_i \qquad (i = 0, 1, \ldots, n) \tag{2.8}$$

and for its derivative

$$R_n'(x_0) = R_n'(0) = y' \tag{2.9}$$

hold.

We shall look for the stable and economical interpolation polynomial, i.e. for which, on the one hand, the following inequality is fulfilled:

$$0 \leqq e^{-x}\left\{\sum_{i=0}^{n}(y_i-y_i^*)r_i(x)+(y'-y^{*\prime})\varrho(x)\right\} \leqq \tag{2.10}$$

$$\leqq \max\left\{\max_i [e^{-x_i}(y_i-y_i^*)],\ y'-y^{*\prime}\right\} \quad (x\in[0,\infty))$$

where $y_i \geqq y_i^*$ $(i=0, 1, \ldots, n)$ and $y' \geqq y^{*\prime}$ are arbitrary real numbers; on the other hand

$$\deg R_n(x) \overset{\text{def}}{=} \sum_{i=0}^{n} \deg\{y_i r_i(x)\} + \deg\{y'\varrho(x)\} \tag{2.11}$$

is minimal. We are going to prove the following theorems:

THEOREM I. *The inequality* (2.10) *and the relation* (2.11) *are satisfied simultaneously if and only if the fundamental points* (2.1) *are* $x_0=0$ *and the roots of the generalized Laguerre polynomial* $L_n^{(1)}(x)$ *of degree* n*; the fundamental interpolation polynomial of the first kind under* (2.2) *are*

(2.12)

$$r_0(x) = (1+nx)\frac{[L_n^{(1)}(x)]^2}{(n+1)^2}; \quad r_i(x) = \frac{x^2}{x_i^2}\left[\frac{L_n^{(1)}(x)}{L_n^{(1)\prime}(x_i)\,(x-x_i)}\right]^2 \qquad (i=1, 2, \ldots, n)$$

and the fundamental polynomial of the second kind is

$$\varrho(x) = x\frac{[L_n^{(1)}(x)]^2}{(n+1)^2}. \tag{2.13}$$

Let $f(x)$ be a real valued function defined in $[0, \infty)$ and $y'=y_n'$ an arbitrary real number, then

$$R_n(f) \overset{\text{def}}{=} \sum_{i=0}^{n} f(x_i)r_i(x) + y'\varrho(x) \tag{2.14}$$

is the interpolation polynomial belonging to $f(x)$; especially it may also be $y'=f'(0)$ if $f'(0)$ exists.

Theorem I shows that prescribing the value of the derivative of the function only in one fundamental point, the stable and economical interpolation process is attained by entirely different fundamental points compared with those in EGERVÁRY and TURÁN's note [1].

We are going to prove the following convergence theorem:

THEOREM II. *If* $f(x)$ *is continuous in* $[0, \infty)$, *further* $f(x)=O(e^{\beta x})$ *if* $x\to\infty$ *and* β *is an arbitrary fixed number for which* $0<\beta<1$ *and* $y'=o(n^{\frac{1}{2}})$ *hold, then in the case of the fundamental interpolation polynomials* (2.12) *and* (2.13) $R(f)$ *converges to* $f(x)$ *for every* $x\geqq 0$ *and this convergence is uniform in every interval* $[0, \omega]$, *where* ω *is an arbitrary positive real number.*

In the convergence theorem proved by BALÁZS and TURÁN [2] it was assumed that the function $f(x)$ is continuous and bounded in $[0, \infty)$. The authors noted without proof that their convergence theorem is by all means true in the case when instead of the boundedness of $f(x)$, the condition $f(x)=O(x^m)$ (if $x\to\infty$ and m is an arbitrary fixed number) is prescribed. We should like to remark that, similarly to Theorem II of this paper, the theorem proved by Balázs and Turán is also true in the case when instead of the boundedness of $f(x)$ resp. $f(x)=O(x^m)$ we prescribe the weaker condition $f(x)=O(e^{\beta x})$. This is ascertained in the course of proving Theorem II.

3. To prove the theorems we shall need the following, known relations (cf. SZEGŐ's book [3]).

The polynomials

$$L_n^{(\alpha)}(x) = \frac{1}{n!}(e^{-x}x^{n+\alpha})^{(n)}e^x x^{-\alpha} \qquad (n = 0, 1, \ldots) \tag{3.1}$$

where $\alpha > -1$, are called "generalized Laguerre polynomials". It is well known that the polynomials $L_n^{(\alpha)}(x)$ form an orthogonal system concerning the weight function $x^\alpha e^x$ in $[0, \infty)$, furthermore that the roots of these polynomials are real, simple and positive.

The polynomials $y_n(x)=L_n^{(\alpha)}(x)$ satisfy (SZEGŐ [3], (5.1.2)) the differential equation

$$xy_n''(x)+(\alpha+1-x)y_n'(x)+ny_n(x) = 0. \tag{3.2}$$

The following equalities are valid.

$$\frac{d}{dx}L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x) \quad ([3], (5.1.14)) \tag{3.3}$$

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} \quad ([3], (5.1.7)) \tag{3.4}$$

$$\sum_{i=1}^{n} x_i^{-1}\{L_n^{(\alpha)\prime}(x_i)\}^{-2} = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \quad ([3], (14.7.4)), \tag{3.5}$$

where x_i $(i=1, 2, \ldots, n)$ denote the roots of the polynomial $L_n^{(\alpha)}(x)$;

$$\sum_{i=1}^{n} h_i(x) = \sum_{i=1}^{n}\left[1-\frac{x_i-\alpha-1}{x_i}(x-x_i)\right]l_i^2(x) \equiv 1 \tag{3.6}$$

([3], (14.1.11)) where

$$l_i(x) = \frac{y_n(x)}{y_n'(x_i)\,(x-x_i)} \qquad (i = 1, 2, \ldots, n).$$

It is evident that

$$l_i(x_j) = \begin{cases} 0, & i\neq j \\ 1, & i=j \end{cases} \quad (i = 1, 2, \ldots, n). \tag{3.7}$$

and it follows from a simple calculation by (3.2) that

$$l_i'(x_i) = \frac{y_n''(x_i)}{2y_n'(x_i)} = \frac{x_i-\alpha-1}{x_i} \qquad (i = 1, 2, \ldots, n). \tag{3.8}$$

If

$$M_n \overset{\text{def}}{=} \max_{0<x\leqq\omega} e^{\frac{x}{2}} x^{\frac{\alpha}{2}+\frac{1}{4}} |L_n^{(\alpha)}(x)|$$

then

$$\lim_{n\to\infty} n^{-\frac{\alpha}{2}+\frac{1}{4}} M_n = \text{const.}, \qquad \left(\alpha > -\frac{1}{2}\right) \tag{3.9}$$

(see [3], Theorem (7.6.5)).

If the continuity module of the function $f(x)$ is denoted by $k_{2\omega}$ in the closed interval $[0, 2\omega]$, $\omega>0$, and λ, δ are two arbitrary positive numbers for which $\lambda\delta\leqq 2\omega$ holds, then the inequality

$$k_{2\omega}(\lambda\delta) \leqq (\lambda+1)\, k_{2\omega}(\delta) \tag{3.10}$$

is true. This is well known.

4. In what follows we are going to prove Theorem I. The method of proof is similar to that used by EGERVÁRY and TURÁN [1] proving the stability and economical character of their interpolation process in the case $\alpha=0$. We assume that there exists a stable interpolation process of degree not higher than $2n+1$.

First we shall consider the fundamental point $x_0=0$. Let us choose especially the values of y_i, y_i^* $(i=0, 1, \ldots, n)$ and $y', y^{*\prime}$, in the following way: $y_0=1$, $y_i=y_i^*=0$ $(i=1, 2, \ldots, n)$ and $y_0^*=y'=y^{*\prime}=0$, then by (2.10) the inequality

$$0 \leqq e^{-x} r_0(x) \leqq 1 \tag{4.1}$$

is satisfied. Let

$$\omega_n(x) \overset{\text{def}}{=} \prod_{i=1}^{n} (x-x_i) \tag{4.2}$$

then, as $x_i>0$ $(i=1, 2, \ldots, n)$, so by (2.3), (2.4), (2.11) and (4.1) the form of $r_0(x)$ must be

$$r_0(x) = (ax+b)\omega_n(x)^2. \tag{4.3}$$

It follows from (4.3) on the basis of (2.3), (2.4) by simple calculation that

$$r_0(x) = \left(1-\frac{2\omega_n'(0)}{\omega_n(0)}\, x\right) \frac{\omega_n(x)^2}{\omega_n(0)^2} \tag{4.4}$$

holds.

Let us consider now the case when $1\leqq i\leqq n$. If $y_i=1$, $y_i^*=0$, $y_j=y_j^*=y'=y^{*\prime}=0$ $(j=0, 1, 2, \ldots, i-1, i+1, \ldots, n)$, then by (2.10) the inequality

$$0 \leqq e^{-x+x_i} r_i(x) \leqq 1 \tag{4.5}$$

is satisfied. Considering (2.3), (2.4), (2.11), (4.2) and (4.5), on the one hand

$$r_i(x) = \frac{x^2}{x_i^2}\left(\frac{\omega_n(x)}{\omega_n'(x_i)(x-x_i)}\right)^2 = \frac{x^2}{x_i^2} l_i^2(x) \tag{4.6}$$

is satisfied, on the other hand, by (2.3) and (4.5) $\{e^{-x+x_i} r_i(x)\}_{x=x_i} = 1$ i.e. $\{e^{-x+x_i} r_i(x)\}'_{x=x_i} = 0$ must hold, and considering (4.6), from this the equality

$$x_i \omega_n''(x_i) + (2-x_i)\omega_n'(x_i) = 0 \qquad (i = 1, 2, \ldots, n)$$

can be derived, i.e. since $\omega_n(x)$ is a polynomial of degree exactly n, there exist a constant C, for which the identity

$$x\omega_n''(x) + (2-x)\omega_n'(x) = C\omega_n(x)$$

holds. However, $C = -n$, and by SZEGŐ [3], (5.1.3),

$$\omega_n(x) = (-1)^n n!\, L_n^{(1)}(x) \tag{4.7}$$

i.e. the fundamental interpolation points required must be $x_0 = 0$ and the roots of the Laguerre polynomial $L_n^{(1)}(x)$ of degree n. So the required fundamental interpolation polynomials $r_i(x)$ $(i = 0, 1, 2, \ldots, n)$ and $\varrho(x)$ — by reason of (4.4), (4.6) and (4.7) taking into account (3.3) and (3.4) as well — may be only those defined by (2.12) and (2.13).

Now we are going to prove that with this choice of the fundamental interpolation polynomials, the inequality (2.10) is satisfied. For this we need the following lemma.

LEMMA 4.1. *If* x_i $(i = 1, 2, \ldots, n)$ *denote the roots of the Laguerre polynomials* $L_n^{(1)}(x)$, *then*

$$V_n(x) \overset{\text{def}}{=} e^x - (1+nx)\frac{[L_n^{(1)}(x)]^2}{(n+1)^2} - \sum_{i=1}^{n} e^{x_i}\frac{x^2}{x_i^2} l_i^2(x) - x\frac{[L_n^{(1)}(x)]^2}{(n+1)^2} \geqq 0 \tag{4.8}$$

$$(x \geqq 0;\ n = 1, 2, \ldots)$$

holds, where $l_i(x)$ *denotes the Lagrange fundamental interpolation polynomial belonging to the roots of the polynomial* $L_n^{(1)}(x)$ *as to fundamental points.*

PROOF. By (4.8) and (3.4), obviously $V_n(0) = V_n(x_i) = 0$ $(i = 1, 2, \ldots, n)$, on the other hand $V_n'(0) = V_n'(x_i) = 0$ by (3.2), (3.3), (3.4), (3.8). This means that the function $V_n(x)$ has at least $2n+2$ zeros in $[0, \infty)$. So if there were one point $\xi > 0$, in which $V_n(\xi) < 0$, then — as $V_n(0) = 0$, $V_n(\infty) = \infty$ — the function $V_n(x)$ would have at least one more, i.e. altogether at least $2n+3$ zeros in $[0, \infty)$, and in this case, according to Rolle's theorem, the function $V_n^{(2n+2)}(x)$ would have at least one positive zero. However, this leads to a contradiction, since by (4.8)

$$V_n^{(2n+2)}(x) = e^x > 0.$$

So the lemma is proved.

The following two corollaries are easy consequence of (4.8):

COROLLARY 4.1.

$$e^{-x}\left\{\sum_{i=0}^{n} e^{x_i} r_i(x) + \varrho(x)\right\} \leqq 1 \qquad (x \geqq 0;\ n = 1, 2, \ldots). \tag{4.9}$$

COROLLARY 4.2.

$$|L_n^{(1)}(x)| \leqq \frac{n+1}{\sqrt{1+(n+1)x}} e^{\frac{x}{2}} \qquad (x \geqq 0;\ n = 1, 2, \ldots) \tag{4.10}$$

which turns to equality in case $x=0$.

We remark that SZEGŐ (see [3], (7.21.3)) proved that if $L_n^{(0)}(x) \overset{\text{def}}{=} L_n(x)$, then

$$|L_n(x)| \leqq e^{\frac{x}{2}} \qquad (x \geqq 0;\ n = 1, 2, \ldots).$$

This inequality was proved by Egerváry and Turán in a simple way, similarly to (4.10).

Furthermore we note, that the idea of proof of Lemma 4.1 is the same as that of EGERVÁRY and TURÁN [1] in proving an inequality concerning the polynomials $L_n(x)$.

Applying the inequality (4.9), by reason of (2.12), (2.13) the inequality (2.10), that is the stability of the interpolation process is readily seen.

5. For the proof of Theorem II we need some simple lemmas.

LEMMA 5.1.

$$0 \leqq \varrho(x) \leqq c_1 (n+1)^{-\frac{1}{2}} \qquad (0 \leqq x \leqq \omega;\ n = 1, 2, \ldots) \tag{5.1}$$

where c_1, *and further on* $c_2, c_3, \ldots$ *depend only on* ω.

PROOF. If $0 \leqq x \leqq \frac{1}{\sqrt{n+1}}$, then by (2.13) and (4.10)

$$0 \leqq \varrho(x) \leqq x \frac{e^x}{1+(n+1)x} \leqq \frac{1}{\sqrt{n+1}} e^x. \tag{5.2}$$

If $x > \frac{1}{\sqrt{n+1}}$, then by (2.13) and (4.10)

$$0 \leqq \varrho(x) \leqq x \frac{e^x}{1+(n+1)x} \leqq \frac{x}{\sqrt{n+1}} e^x. \tag{5.3}$$

(5.2) and (5.3) give the inequality (5.1) which was to be proved.

LEMMA 5.2.

$$\left|1 - \sum_{i=0}^{n} r_i(x)\right| \leqq c_2 n^{-\frac{1}{4}} \qquad (0 \leqq x \leqq \omega;\ n = 1, 2, \ldots). \tag{5.4}$$

PROOF. If $G(x) \overset{\text{def}}{=} 1 - \sum_{i=0}^{n} r_i(x)$, then by (2.12) and (2.4) obviously $G(0) = G'(0) = G(x_i) = 0$ $(i = 1, 2, \ldots, n)$, so

$$G(x) = x^2 L_n^{(1)}(x) g_{n-1}(x) \tag{5.5}$$

where $g_{n-1}(x)$ is a polynomial of degree at most $n-1$ for which by (2.12), (3.7), (3.8) and $\alpha=1$, it follows from (5.5) that

$$G'(x_i) = -\frac{2}{x_i} - 2l_i'(x_i) = -\frac{2}{x_i} - \frac{x_i - 2}{x_i} = -1 = x_i^2 L_n^{(1)\prime}(x_i) g_{n-1}(x_i)$$

se obviously

$$G(x) = x^2 L_n^{(1)}(x) \sum_{i=1}^{n} g_{n-1}(x_i) l_i(x) = \tag{5.6}$$

$$= x^2 L_n^{(1)}(x) \sum_{i=1}^{n} \frac{-1}{x_i^2 L_n^{(1)\prime}(x_i)} l_i(x) = x^2 L_n^{(1)}(x) \left\{ \sum_{|x-x_i| \leqq n^{-\delta}} + \sum_{|x-x_i| > n^{-\delta}} \right\} =$$

$$= x^2 L_n^{(1)}(x) \{A + B\}.$$

By (5.6), (2.12) and (4.9) obviously

$$|x^2 L_n^{(1)}(x) A| \leqq \sum_{|x-x_i| \leqq n^{-\delta}} \frac{x^2}{x_i^2} |x - x_i| l_i(x) \leqq n^{-\delta} \sum_{|x-x_i| \leqq n^{-\delta}} r_i(x) \leqq \tag{5.7}$$

$$\leqq n^{-\delta} \sum_{i=1}^{n} r_i(x) \leqq n^{-\delta} e^{\omega}.$$

From (3.6), in case of $x=0$, $\alpha=1$, we get

$$\sum_{i=1}^{n} \frac{1}{x_i^2 L_n^{(1)\prime}(x_i)^2} = \sum_{i=1}^{n} \frac{1}{x_i L_n^{(1)\prime}(x_i)^2} - \frac{1}{L_n^{(1)}(0)^2}. \tag{5.8}$$

From (5.8), on the basis of (3.4) and (3.5), for the case $\alpha=1$ the equality

$$\sum_{i=1}^{n} \frac{1}{x_i^2 L_n^{(1)\prime}(x_i)^2} = \frac{\Gamma(2)}{n+1} - \frac{1}{(n+1)^2} \tag{5.9}$$

can be deduced. By (5.6), (5.9) and (3.9), in the case $\alpha=1$ for every fixed value of ω

$$|x L_n^{(1)}(x) B| \leqq n^{\delta} x^2 L_n^{(1)}(x)^2 \sum_{|x-x_i| > n^{\delta}} \frac{1}{x_i^2 L_n^{(1)\prime}(x_i)^2} \leqq \tag{5.10}$$

$$\leqq n^{\delta} x^2 L_n^{(1)}(x)^2 \sum_{|x-x_i| > n^{\delta}} \frac{1}{x_i^2 L_n^{(1)\prime}(x_i)^2} \leqq$$

$$\leqq n^{\delta} x^{\frac{1}{2}} e^x M_n^2 \left(\frac{\Gamma(2)}{n+1} - \frac{1}{(n+1)^2} \right) \leqq c_2' \frac{1}{n^{(1/2)-\delta}}.$$

By (5.7) and (5.10), we get the inequality (5.4) in case $\delta = \frac{1}{4}$.

LEMMA 5.3.

$$C \overset{\text{def}}{=} \sum_{i=0}^{n} |x-x_i| r_i(x) \leqq c_3 n^{-\frac{1}{4}} \qquad (0 \leqq x \leqq \omega;\ n = 1, 2, \ldots). \tag{5.11}$$

PROOF. On the basis of (2.12), (2.13),

$$C \leqq (1+nx)x \frac{L_n^{(1)}(x)^2}{(n+1)^2} + n^{-\delta} \sum_{|x-x_i| \leqq n^{-\delta}} r_i(x) + n^{\delta} x^2 L_n^{(1)}(x)^2 \sum_{|x-x_i|>n^{-\delta}} \frac{1}{x_i^2 L_n^{(1)\prime}(x_i)^2} \leqq$$

$$\leqq x \frac{L_n^{(1)}(x)^2}{(n+1)^2} + \frac{x^2 L_n^{(1)}(x)^2}{n+1} + n^{-\delta} \sum_{i=1}^{n} r_i(x) + n^{\delta} x^2 L_n^{(1)}(x)^2 \sum_{i=1}^{n} \frac{1}{x_i^2 L_n^{(1)\prime}(x_i)^2}$$

and from this in case $\delta = \frac{1}{4}$ by (5.1), (3.9), (5.7), (5.10) we get $C \leqq c_3 n^{-\frac{1}{4}}$ which was to be proved.

The next lemma is proved similarly to the proof of the inequality of BALÁZS and TURÁN's lemma [4].

LEMMA 5.4.

$$\sum_{i=1}^{n} e^{\beta x_i} \frac{1}{x_i L_n^{(2)\prime}(x_i)^2} \leqq c_4 (n+1)^{-1} \qquad (n = 1, 2, \ldots) \tag{5.12}$$

where $0 < \beta < 1$ *is an arbitrary fixed number.*

PROOF. With the method used in the proof of (4.11), it is readily seen that

$$e^{\beta x} - \sum_{i=1}^{n} e^{\beta x_i} l_i(x) - \sum_{i=1}^{n} s(x_i)(x-x_i) l_i^2(x) \geqq 0 \qquad (n = 1, 2, \ldots) \tag{5.13}$$

where $l_i(x)$ is the Lagrange fundamental interpolation polynomial based on the roots of $L_n^{(1)}(x)$, and

$$s(x_i) = \beta e^{\beta x_i} - \sum_{i=1}^{n} e^{\beta x_i} l_i'(x_i) \qquad (i = 1, 2, \ldots, n).$$

Considering the orthogonality of the polynomials $L_n^{(1)}(x)$, from (5.13) we have

$$\int_0^{\infty} x e^{(\beta-1)x} dx \geqq \sum_{i=1}^{n} e^{\beta x_i} \int_0^{\infty} l_i(x) x e^{-x} dx \geqq \frac{\Gamma(n+2)}{\Gamma(n+1)} \sum_{i=1}^{n} e^{\beta x_i} \frac{1}{x_i L_n^{(1)\prime}(x_i)^2}$$

by reason of SZEGŐ [3], (15.3.5), from which (5.12) follows.

LEMMA 5.5.

$$D \overset{\text{def}}{=} \sum_{x_i > 2\omega} e^{\beta x_i} r_i(x) \leqq c_5 n^{-\frac{1}{2}} \qquad (0 \leqq x \leqq \omega;\ n = 1, 2, \ldots). \tag{5.14}$$

PROOF. On the basis of (2.12), (3.9) and (5.12)

$$D \leqq \frac{1}{\omega^2} x^2 L_n^{(1)}(x)^2 \sum_{\substack{i \\ x_i > 2\omega}} e^{\beta x_i} \frac{1}{x_i L_n^{(1)\prime}(x_i)^2} \leqq c_5 n^{-\frac{1}{2}}.$$

So (5.14) is proved.

We remark that this lemma is valid also in the case when the fundamental points of the interpolation are the roots of the Laguerre polynomials having the parameter $\alpha=0$ and $x_0=0$. So Lemma 3 in BALÁZS—TURÁN's paper [2] can be replaced by our lemma containing a stronger statement in the proof of their convergence theorem. This fact guarantees that instead of the boundedness of $f(x)$ or $f(x)=$ $=O(x^m)$ $(x\to\infty, m>0)$ the condition $f(x)=O(e^{\beta x})$ $(x\to\infty, 0<\beta<1$ fixed) can be assumed.

6. Now we have come to prove Theorem II.

$$\Delta_n(x) \overset{\text{def}}{=} |R_n(x)-f(x)| = \left| \sum_{i=0}^{n} [f(x_{in})-f(x)] r_{in}(x) - \right. \tag{6.1}$$

$$\left. -f(x)\left[1-\sum_{i=0}^{n} r_{in}(x)\right] + y' \varrho(x) \right|,$$

and from this, considering (5.1), as $x \geqq 0$, and by the conditions mentioned in Theorem II concerning $f(x)$, we have

$$\Delta_n(x) \leqq \sum_{\substack{i \\ x_{in} \leqq 2\omega}} |f(x_{in})-f(x)| r_{in}(x) + M_f' \sum_{\substack{i \\ x_{in} > 2\omega}} e^{\beta x_{in}} r_{in}(x) + \tag{6.2}$$

$$+ M_f \left| 1 - \sum_{i=0}^{n} r_{in}(x) \right| + y' c_1 (n+1)^{-\frac{1}{2}}$$

where

$$M_f \overset{\text{def}}{=} \max_{0 \leqq x \leqq \omega} |f(x)|, \quad M_f' \overset{\text{def}}{=} \sup_{0 \leqq x < \infty} 2|f(x)| e^{-\beta x}. \tag{6.3}$$

By (3.10), if $k_{2\omega}$ denotes the continuity modulus of $f(x)$ in $[0, 2\omega]$, then by

$$|f(x_{in})-f(x)| \leqq k_{2\omega}(n^{-\frac{1}{4}} n^{\frac{1}{4}} |x_{in}-x|) \leqq k_{2\omega}(n^{-\frac{1}{4}}) \{n^{\frac{1}{4}} |x_{in}-x|+1\}$$

and by (5.11), (4.9) we get

$$\sum_{\substack{i \\ x_{in} \leqq 2\omega}} |f(x_{in})-f(x)| r_{in}(x) \leqq k_{2\omega}(n^{-\frac{1}{4}}) \{c_3 + e^{\omega}\}. \tag{6.4}$$

By reason of (6.4), (5.14), (5.4), (6.3) and $y'=o(n^{\frac{1}{2}})$

$$\Delta_n(x) \leqq c_6 k_{2\omega}(n^{-\frac{1}{4}}) + c_5 M_f' n^{-\frac{1}{2}} + c_2 M_f n^{-\frac{1}{4}} + c_1 y'(n+1)^{-\frac{1}{2}}.$$

Hence $\Delta_n(x)=o(1)$ and this completes the proof of Theorem II.

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References

[1] E. EGERVÁRY and P. TURÁN, Notes on interpolation. VI, *Acta Math. Acad. Sci. Hung.*, **10** (1959), pp. 55—63.
[2] J. BALÁZS and P. TURÁN, Notes on interpolation. VII, *Acta Math. Acad. Sci. Hung.*, **10** (1959), pp. 63—68.
[3] G. SZEGŐ, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ. (New York, 1959).
[4] J. BALÁZS and P. TURÁN, Notes on interpolation. VIII, *Acta Math. Acad. Sci. Hung.*, **12** (1961), pp. 469—474.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (1—2), (1974), pp. 159—164.

ON SOME INTERPOLATORY PROCEDURES BASED ON THE ROOTS OF UNITY

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We shall be concerned with the exact error-estimates of some interpolatory procedures based on the uniformly distributed nodes on the unit circle of the complex domain, in a certain class of functions. The common feature of these interpolatory procedures is that the order of magnitude of their Lebesgue-constant is $O(\log n)$.

Let $\omega(h)$ be an arbitrary modulus of continuity, and denote by $AC(\omega)$ that class of functions $f(x)$ which are analytic in $|z|<1$, and whose modulus of continuity $\omega(f, h)$ in $|z|\leqq 1$ is of order $O(\omega(h))$ (as $h\to+0$). For these functions we consider the following interpolatory procedures:

$$I_{n,j}(f, z)=\sum_{k=1}^{n} f(z_k)A_{k,j}(z) \qquad (n=1, 2, \ldots;\ j=1, 2, 3)$$

where

$$z_k=e^{it_k}, \quad t_k=\frac{(2k-1)\pi}{n} \qquad (k=1, 2, \ldots, n) \tag{1}$$

and

$$A_{k,1}(z)=l_k(z)=\frac{z_k}{n}\frac{z^n+1}{z-z_k} \tag{2}$$

$$A_{k,2}(z)=\left[1-\frac{n-1}{z_k}(z-z_k)\right]l_k(z)^2 \tag{3}$$

$$A_{k,3}(z)=l_k(z)+\frac{1}{2n}\left(z^{\frac{n+1}{2}}+z^{\frac{n-1}{2}}\right)\int_1^z t^{\frac{n+1}{2}}l_k''(t)\,dt. \tag{4}$$

The linear operators $I_{n,j}(f, z)$ have the following properties: $\deg I_{n,1}(f, z)\leqq n-1$; $I_{n,1}(f, z_k)=f(z_k)$ $(k=1, 2, \ldots, n$; Lagrange interpolation); $\deg I_{n,2}(f, z)\leqq 2n-1$; $I_{n,2}(f, z_k)=f(z_k)$, $I'_{n,2}(f, z_k)=0$ $(k=1, 2, \ldots, n$; Hermite—Fejér interpolation); $\deg I_{n,3}(f, z)\leqq 2n-1$; $I_{n,3}(f, z_k)=f(z_k)$, $I''_{n,3}(f, z_k)=0$ $(k=1, 2, \ldots, n$; (0, 2) interpolation).

How these operators approximate a function $f(z)\in AC(\omega)$? We state the following

THEOREM 1. *If* $f(z)\in AC(\omega)$ *then*

$$\max_{|z|=1}|f(z)-I_{n,j}(f, z)|=O\left(\omega\left(\frac{1}{n}\right)\log n\right) \qquad (j=1, 2, 3). \tag{5}$$

All these estimates follow from the fact that

$$\lambda_{n,j} = \max_{|z|=1} \sum_{k=1}^{n} |A_{k,j}(z)| \leqq 4 \log n \qquad (n \geqq 2).^{1} \tag{6}$$

(For $j=1$ see ALPER [1], $j=2$ GAIER [2], $j=3$ KIS [3].) The statement itself is not new for $j=1$ and 3 (cf. [1] and [3]). In case $j=2$, it follows from the identity

$$I_{n,2}(f, z) = I_{n,1}(f, z) + \frac{z(z^n+1)}{n} I'_{n,1}(f, z)$$

which is easily verified. Namely, using the estimate

$$|p'_n(z)| = O(n)\omega\left(p_n, \frac{1}{n}\right)$$

($|z|=1$, p_n a polynomial of degree $\leqq n$) which is an analog of a result of STEČKIN [4], one obtains

$$|f(z)-I_{n,2}(f, z)| \leqq |f(z)-I_{n,1}(f, z)| + \frac{2}{n}|I'_{n,1}(f, z)| =$$

$$= O\left(\omega\left(\frac{1}{n}\right)\right)\log n + O\left(\omega\left(I_{n,1}, \frac{1}{n}\right)\right) =$$

$$= O\left(\omega\left(\frac{1}{n}\right)\log n\right) + O\left(\omega\left(f, \frac{1}{n}\right)\right) = O\left(\omega\left(\frac{1}{n}\right)\log n\right).$$

The main purpose of this paper is to prove that the estimate (5) cannot be improved for the whole class $AC(\omega)$. This is expressed in the next

THEOREM 2. *To every $\omega(h)$, there exists a function $f(z) \in AC(\omega)$ such that*

$$\limsup_{n\to\infty} \frac{|f(1)-I_{n,j}(f, 1)|}{\omega\left(\frac{1}{n}\right)\log n} > 0 \qquad (j = 1, 2, 3). \tag{7}$$

COROLLARY. *The necessary and sufficient condition for*

$$\lim_{n\to\infty} \max_{|z|=1} |f(z)-I_{n,j}(f, z)| = 0$$

for all $f(z) \in AC(\omega)$ is

$$\lim_{n\to\infty} \omega\left(\frac{1}{n}\right)\log n = 0. \tag{8}$$

[1] In what follows, whenever j appears in a formula, it is always meant that $j=1, 2, 3$.

PROOF OF THEOREM 2.[2] We define a sequence of integers $N_m=8^{n_m}+1$, $n_1<n_2<\ldots$ such that

$$\omega\left(\frac{1}{N_m}\right)\leqq q<1$$

and

$$\omega\left(\frac{1}{N_{m+1}}\right)\leqq q\omega\left(\frac{1}{N_m}\right) \tag{9}$$

($0<q<1$ will be determined later) and

$$n_{m+1}\geqq 2n_m+2. \tag{10}$$

More precisely, there will be one more condition on n_m, but before stating this, we have to define certain polynomials.

Let

$$w_k=e^{i\arg A_{k,j}(1)} \qquad (2^{n_m}\leqq k\leqq 4^{n_m}), \tag{11}$$

$$p_m(z)=\sum_{k=2^{n_m}}^{4^{n_m}} w_k l_k(z)^2 \tag{12}$$

where (see (2)—(4))

$$l_k(z)=\frac{z_k}{N_m}\frac{z^{N_m}+1}{z-z_k} \qquad (z_k^{N_m}+1=0,\ k=1,2,\ldots,N_m).$$

Now the above mentioned condition on n_m is

$$|p_l(1)-I_{N_m,j}(p_l,1)|\leqq\omega\left(\frac{1}{N_m}\right)n_m \qquad (1\leqq l\leqq m). \tag{13}$$

This can be assumed without loss of generality. Namely, if this were false for a fixed l and infinitely many m's then by $p_l(z)\in AC(\omega)$ the theorem would be proved.

We mention some properties of $p_m(z)$. Evidently

$$p_m(z_k)=\begin{cases} w_k & \text{if } 2^{n_m}\leqq k\leqq 4^{n_m}\\ 0 & \text{if } 1\leqq k\leqq 2^{n_m} \text{ or } 4^{n_m}\leqq k\leqq N_m\end{cases} \tag{14}$$

and with the notation $z=e^{it}$ $(0\leqq t<2\pi)$

$$|p_m(z)|\leqq\sum_{k=2^{n_m}}^{4^{n_m}}|l_k(z)|^2=\sum_{k=2^{n_m}}^{4^{n_m}}N_m^{-2}\left|\frac{e^{itN_m}+1}{e^{it}-e^{it_k}}\right|^2\leqq\sum_{k=1}^{N_m}\left(\frac{\sin N_m\dfrac{t-t_k}{2}}{N_m\sin\dfrac{t-t_k}{2}}\right)^2 \qquad (|z|=1).$$

Here the right hand side less one is a trigonometric polynomial of degree $\leqq N_m-1$ which has N_m roots t_k $(k=1,\ldots,N_m)$ of multiplicity 2. Thus it must be identically 0, i.e.

$$\max_{|z|=1}|p_m(z)|=1 \qquad (m=1,2,\ldots). \tag{15}$$

[2] The proof can be essentially simplified when $\lim_{h\to+0} h/\omega(h)=0$ i.e. when $AC(\omega)$ is not identical to the class of analytic functions belonging to Lip 1 on $|z|=1$.

Hence a rough estimate for the derivative can be obtained by the Bernstein inequality:

$$\max_{|z|=1} |p_m'(z)| \leqq 2(N_m-1) = 2\cdot 8^{n_m}. \tag{16}$$

For certain values of $z=e^{it}$, we need a finer estimate. Assume that

$$t \notin [\pi 4^{-n_m},\ 3\pi\cdot 2^{-n_m}] = J_m. \tag{17}$$

Then we have by (12)

$$|p_m'(z)| \leqq \frac{2}{N_m^2} \sum_{k=2^{n_m}}^{4^{n_m}} \left|\frac{z^{N_m}+1}{z-z_k}\right| \cdot \left|\frac{N_m z^{N_m-1}(z-z_k)-(z^{N_m}+1)}{(z-z_k)^2}\right|.$$

Here

$$|z-z_k| = 2\sin\frac{|t-t_k|}{2} \geqq$$

$$\geqq \begin{cases} \dfrac{2}{\pi}|t-t_k| \geqq \begin{cases} \dfrac{2(2k-1-2^{n_m})}{N_m} & \text{if } 0 \leqq t \leqq \pi\cdot 4^{-n_m},\ |t-t_k| \leqq \pi \\ \dfrac{2(3\cdot 4^{n_m}-2k+1)}{N_m} & \text{if } 3\pi\cdot 2^{-n_m} \leqq t,\ |t-t_k| \leqq \pi \end{cases} \\ \dfrac{2}{\pi}|2\pi+t_k-t| \geqq \dfrac{2}{\pi}t_k = \dfrac{2(2k-1)}{N_m} \quad \text{if } t-t_k > \pi. \end{cases}$$

Thus

$$|p_m'(z)| \leqq \frac{8}{N_m} \sum_{k=2^{n_m}}^{4^{n_m}} \frac{1}{|z-z_k|^2} \leqq \frac{8}{N_m}\left(\sum_{\substack{2^{n_m}\leqq k\leqq 4^{n_m} \\ |t-t_k|\leqq\pi}} + \sum_{\substack{2^{n_m}\leqq k\leqq 4^{n_m} \\ t-t_k\geqq\pi}} \right) \leqq$$

$$\leqq \begin{cases} 2N_m \displaystyle\sum_{k=2^{n_m}}^{4^{n_m}} \left[\frac{1}{(2k-1-2^{n_m})^2}+\frac{1}{(2k-1)^2}\right] \leqq 2\cdot 4^{n_m} & (|t-t_k| \leqq \pi) \\ 2N_m \displaystyle\sum_{k=2^{n_m}}^{4^{n_m}} \frac{1}{(2k-1)^2} \leqq 4^{n_m} & (t-t_k \geqq \pi) \end{cases}$$

i.e.

$$|p_m'(e^{it})| \leqq 2\cdot 4^{n_m} \qquad (t \notin J_m). \tag{18}$$

Now let

$$f(z) = \sum_{m=1}^{\infty} \omega\left(\frac{1}{N_m}\right) p_m(z) \qquad (|z| \leqq 1). \tag{19}$$

The series on the right hand side uniformly converges in $|z|\leqq 1$ by (15) and (9), thus $f(z)$ is analytic in $|z|<1$. We show that $f(z)\in AC(\omega)$. Let $|z_1|=|z_2|=1$ and l such that

$$\frac{1}{N_{l+1}} \leqq |\arg z_1 - \arg z_2| < \frac{1}{N_l}. \tag{20}$$

We have

$$|f(z_1)-f(z_2)| \leqq \left(\sum_{m=1}^{\infty}+\sum_{m=l+1}^{\infty}\right)\omega\left(\frac{1}{N_l}\right)|p_m(z_1)-p_m(z_2)|, \tag{21}$$

and

$$|p_m(z_1)-p_m(z_2)| = |p_m'(u_m)|\,|\arg z_1-\arg z_2| \qquad (1 \leqq m \leqq l)$$

where

$$|u_m| = 1, \quad \arg u_m \in [\arg z_1, \arg z_2] = J.$$

By (17) and (20),

$$|J_m| = \frac{3\cdot 4^{n_m}-2^{n_m}}{N_m}\pi > \frac{1}{N_m} > \frac{1}{N_l} > |\arg z_1-\arg z_2| \qquad (1 \leqq m \leqq l),$$

moreover the intervals $J_1, J_2, \ldots$ have no point in common, by (10). Thus there are at most two J_m's, $m \leqq l$ having points in J. For these we apply (18) and get

$$\omega\left(\frac{1}{N_m}\right)|p_m(z_1)-p_m(z_2)| \leqq 2N_m|\arg z_1-\arg z_2|\,\omega\left(\frac{1}{N_m}\right) \leqq 8\omega(|z_1-z_2|)$$ [3]

and for the remaining m's ($m \leqq l$)

$$\omega\left(\frac{1}{N_m}\right)|p_m(z_1)-p_m(z_2)| \leqq 2\cdot 4^{n_m}|z_1-z_2|\,\omega\left(\frac{1}{N_m}\right) \leqq 8\cdot 2^{-n_m}\omega(|z_1-z_2|)$$

by (16). Hence we have from (21), (9) and (20)

$$|f(z_1)-f(z_2)| \leqq 8\omega(|z_1-z_2|)\left(2+\sum_{m=1}^{l}2^{-n_m}\right)+2\sum_{m=l+1}^{\infty}\omega\left(\frac{1}{N_m}\right) \leqq$$

$$\leqq 17\omega(|z_1-z_2|)+2\omega\left(\frac{1}{N_{l+1}}\right)\sum_{m=l+1}^{\infty}q^{m-l} \leqq$$

$$\leqq 17\omega(|z_1-z_2|)+4\omega(|z_1-z_2|)\frac{q}{1-q} = \frac{10-8q}{1-q}\omega(|z_1-z_2|).$$

Now we have by (2), (4) and (1)

$$\sum_{k=2^{n_m}}^{4^{n_m}}|A_{k,j}(1)| = \sum_{k=2^{n_m}}^{4^{n_m}}|l_k(1)| = \frac{2}{N_m}\sum_{k=2^{n_m}}^{4^{n_m}}\frac{1}{|1-z_k|} =$$

$$= \frac{1}{N_m}\sum_{k=2^{n_m}}^{4^{n_m}}\frac{1}{\sin\frac{t_k}{2}} \geqq \frac{2}{N_m}\sum_{k=2^{n_m}}^{4^{n_m}}\frac{N_m}{(2k-1)\pi} =$$

$$= \frac{1}{\pi}\sum_{k=2^{n_m}}^{4^{n_m}}\frac{1}{k-\frac{1}{2}} \geqq \frac{1}{4}\log 2^{n_m} \geqq \frac{n_m}{8} \qquad \text{(for } j = 1 \text{ and } 3),$$

[3] Here we have used the inequalities $T^{-1}\omega(T) \leqq 2t^{-1}\omega(t)$ $(0<t\leqq T)$ and $|\arg z_1-\arg z_2| \leqq \frac{\pi}{2}|z_1-z_2|$.

further by (3)

$$\sum_{k=2^{n_m}}^{4^{n_m}} |A_{k,2}(1)| \geqq \frac{4}{N_m} \sum_{k=2^{n_m}}^{4^{n_m}} \frac{1}{|1-z_k|} - \sum_{k=1}^{N_m} |l_k(1)|^2 \geqq \frac{n_m}{8}.$$

Thus by (14) and (11) we get

$$(22) \quad I_{N_m,j}(p_m, 1) = \sum_{k=1}^{N_m} p_m(z_k) A_{k,j}(1) = \sum_{k=2^{n_m}}^{4^{n_m}} w_k A_{k,j}(1) = \sum_{k=2^{n_m}}^{4^{n_m}} |A_{k,j}(1)| \geqq \frac{n_m}{8}.$$

Finally, (19), (22), (13), (15), (6) and (9) yields

$$I_{N_m,j}(f, 1) - f(1) \geqq I_{N_m,j}(p_m, 1)\omega\left(\frac{1}{N_m}\right) - \sum_{l=1}^{m-1} \omega\left(\frac{1}{N_l}\right) |I_{N_m,j}(p_l, 1) - p_l(1)| -$$

$$- \sum_{l=m}^{\infty} \omega\left(\frac{1}{N_l}\right) |p_l(1)| - \sum_{l=m+1}^{\infty} \omega\left(\frac{1}{N_l}\right) |I_{N_m,j}(p_l, 1)| \geqq$$

$$\geqq \frac{n_m}{8}\omega\left(\frac{1}{N_m}\right) - \omega\left(\frac{1}{N_m}\right) n_m \sum_{l=1}^{m-1} \omega\left(\frac{1}{N_l}\right) - \sum_{l=m}^{\infty} \omega\left(\frac{1}{N_l}\right) - \lambda_{N_m,j} \sum_{l=m+1}^{\infty} \omega\left(\frac{1}{N_l}\right) \geqq$$

$$\geqq \omega\left(\frac{1}{N_l}\right)\left[n_m\left(\frac{1}{8} - \frac{q}{1-q}\right) - \frac{1}{1-q} - n_m 12(\log 2)\frac{q}{1-q}\right] \geqq$$

$$\geqq \omega\left(\frac{1}{N_m}\right) n_m \left(\frac{1}{4} - \frac{9q}{1-q}\right).$$

Choosing $0 < q < \frac{1}{37}$, we have (7).

In order to prove the Corollary, we note that the sufficiency of (8) is evident. As for the necessity, we only have to choose the sequence $n_1 < n_2 < \ldots$ in the above proof such that

$$\limsup_{m \to \infty} \omega\left(\frac{1}{N_m}\right) n_m > 0.$$

This can be done if (8) does not hold.

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References

[1] S. J. ALPER, On the convergence of the Lagrange interpolating polynomials in the complex domain, *Uspehi Mat. Nauk USSR*, **11** (5) (71) (1956), pp. 44—50 (Russian).

[2] D. GAIER, Über Interpolation in regelmäßig verteilten Punkten mit Nebenbedingungen, *Mat. Zeitschr.*, **61** (1954), pp. 119—133.

[3] O. KIS, Remarks on interpolation, *Acta Math. Acad. Sci. Hung.*, **11** (1960), pp. 49—64 (Russian).

[4] S. B. STEČKIN, Generalization of some inequalities due to S. N. Bernstein, *Dokl. Akad. Nauk USSR*, **60** (1948), pp. 1511—1514 (Russian).

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Tomus 25 (1—2), (1974), pp. 165—170.

ON A GENERAL LOCALIZATION THEOREM AND SOME APPLICATIONS IN THE THEORY OF RATIONAL APPROXIMATION

By

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In this note we present a general localization theorem in connection with the rational approximation. This theorem is a generalization of results of G. FREUD [1] and G. FREUD—J. SZABADOS [2], and is an improvement of an earlier localization theorem (cf. J. SZABADOS [3]). As applications, we shall generalize a recent result of V. A. POPOV [4], and some other theorems (cf. BULANOV [5], GONČAR [6]).

1. The localization theorem

Let

$$(1) \qquad 0 = \xi_{n,0} < \xi_{n,1} < \ldots < \xi_{n,s_n} = 1, \quad I_{n,i} = [\xi_{n,i-1}, \xi_{n,i}] \qquad (i = 1, 2, \ldots, s_n)$$

be a given subdivision of the interval $[0, 1]$ for all $n=0, 1, 2, \ldots$. The number of subintervals s_n is arbitrary: it may be bounded, but also may tend to infinity as $n \to \infty$, in a certain order (see later). Further let $C=\{f(x); f(x)$ is continuous in $[0, 1]\}$, and denote by $\omega_f(h)$ the modulus of continuity of an $f(x) \in C$. $\mathcal{R}_n$ will denote the set of all rational functions whose denominators and numerators are polynomials of degree at most n. Finally, let

$$R_n(f; [a, b]) = \inf_{r_n \in \mathcal{R}_n} \max_{a \leqq x \leqq b} |f(x) - r_n(x)| \qquad (f(x) \in C, \ [a, b] \subseteq [0, 1], \ n = 0, 1, 2, \ldots)$$

be the best rational approximation of $f(x)$ in the interval $[a, b]$.

THEOREM 1. *If*

$$(2) \qquad \lim_{n \to \infty} s_n e^{-1/2\sqrt{n}} = 0$$

then we have

$$R_{4ns_n}(f; [0, 1]) \leqq 5 \max_{1 \leqq i \leqq s_n} R_n(f; I_{n,i}) + \left(5 + \frac{4}{\omega_f(1)}\right) \omega_f\left(s_n e^{-1/2\sqrt{n}}\right)$$

for all $f(x) \in C$, $f(x) \neq$ const. *and sufficiently large* n*'s.*

REMARK. Roughly speaking, the theorem states that the approximation by rational splines is about the same as approximation by rational functions. Note that the situation is quite different if we consider polynomials and polynomial splines.

PROOF OF THEOREM 1. In order to make the following construction unique, we define

$$f(x) = \begin{cases} f(0) & \text{if } x \leqq 0 \\ f(1) & \text{if } x \geqq 1. \end{cases}$$

This does not change the modulus of continuity of $f(x)$.

We shall construct a new subdivision of [0, 1] by the help of (1). Starting with $i=1$, leave out $\xi_{n,i}$ if $|I_{n,i}|<2e^{-\sqrt{n}}$, and denote it by $\eta_{n,j}$ otherwise, until reaching $i=s_n$.[1] The resulting new subdivision

$$0 = \eta_{n,0} < \eta_{n,1} < \ldots < \eta_{n,t_n} = 1, \quad J_{n,j} = [\eta_{n,j-1}, \eta_{n,j}] \qquad (j = 1, 2, \ldots, t_n)$$

has the following properties:

$$|J_{n,j}| \geqq 2e^{-\sqrt{n}} \qquad (j = 1, 2, \ldots, t_n;\ t_n \leqq s_n)$$

and

$$R_{2n}(f; [\eta_{n,j-1} - e^{-\sqrt{n}}, \eta_{n,j} + e^{-\sqrt{n}}]) \leqq R_n(f; I_{n,k_j}) + \omega_f(s_n e^{-\sqrt{n}}) + 2\|f\|^3 e^{-\sqrt{n}} \tag{3}$$

$$(|I_{n,k_j}| \geqq 2e^{-\sqrt{n}},\ j = 1, 2, \ldots, t_n)$$

with a properly chosen k_j, $1 \leqq k_j \leqq s_n$.

For the proof of the latter statement, observe that by the construction, to each j, $1 \leqq j \leqq t_n$, there exists a k_j, $1 \leqq k_j \leqq s_n$ such that $|I_{n,k_j}| \geqq 2e^{-\sqrt{n}}$, $I_{n,k_j} \subseteqq J_{n,j}$, and a rational function $q_{n,k_j}(x) \in \mathscr{R}_n$ such that

$$\max_{x \in I_{n,k_j}} |f(x) - q_{n,k_j}(x)| = R_n(f; I_{n,k_j}) \qquad (j = 1, \ldots, t_n). \tag{4}$$

Consider the linear transformation

$$X = \frac{\xi_{n,k_j} - \xi_{n,k_j-1}}{\eta_{n,j} - \eta_{n,j-1} + 2e^{-\sqrt{n}}} (x - \eta_{n,j-1} + e^{-\sqrt{n}}) + \xi_{n,k_j-1} \qquad (j = 1, \ldots, t_n)$$

of the variable x, and define

$$r_{n,j}(x) = \frac{q_{n,k_j}(X)}{1 + e^{-\sqrt{n}}\, q_{n,k_j}(X)^2} \qquad (j = 1, \ldots, t_n). \tag{5}$$

Evidently $r_{n,j}(x) \in \mathscr{R}_{2n}$. Moreover

$$\sup_{-\infty < x < \infty} |r_{n,j}(x)| \leqq \frac{1}{2} e^{1/2\sqrt{n}} \qquad (j = 1, \ldots, t_n). \tag{6}$$

The mapping $x \to X$ has the property $[\eta_{n,j-1} - e^{-\sqrt{n}}, \eta_{n,j} + e^{-\sqrt{n}}] \to I_{n,k_j}$ and therefore

[1] By (2), we may assume that for sufficienthy large n's there exists an i such that $|I_{n,i}| \geqq 2e^{-\sqrt{n}}$. If an $\eta_{n,j}$ happens to be such that for all $\xi_{n,i}, \xi_{n,i} > \eta_{n,j}$, the relation $|I_{n,i}| < 2e^{-\sqrt{n}}$ holds, then leave out this $\eta_{n,j}$ and let $\eta_{n,j} = 1$.

by (4) and (5) we get for sufficiently large n's

$$(7)\qquad |f(x)-r_{n,j}(x)| \leqq |f(x)-f(X)|+|f(X)-q_{n,j}(X)|+|q_{n,j}(X)-r_{n,j}(x)| \leqq$$

$$\leqq \omega_f(|x-X|)+R_n(f;I_{n,k_j})+e^{-\sqrt{n}}\frac{|q_{n,j}(X)|^3}{1+e^{-\sqrt{n}}q_{n,j}(X)^2} \leqq$$

$$\leqq R_n(f;I_{n,k_j})+\omega_f\left(\max\left(\xi_{n,k_j-1}-\eta_{n,j-1}+e^{-\sqrt{n}},\ \eta_{n,j}-e^{-\sqrt{n}}-\xi_{n,k_j}\right)\right)+$$

$$+e^{-\sqrt{n}}\max_{x\in I_{n,k_j}}|q_{n,j}(x)|^3 \leqq R_n(f;I_{n,k_j})+\omega_f\left(s_n e^{-\sqrt{n}}\right)+2\|f\|^3 e^{-\sqrt{n}}$$

$$\left(\eta_{n,j-1}-e^{-\sqrt{n}} \leqq x \leqq \eta_{n,j}+e^{-\sqrt{n}}\right)$$

which proves (3).

Now let $\varphi_n(x)\in\mathscr{R}_n$ be that even rational function for which

$$|\varphi_n(x)| \leqq 1 \quad \text{if} \quad |x| \leqq e^{-\sqrt{n}}$$

and

$$|\operatorname{sign} x-\varphi_n(x)| \leqq 3e^{-\sqrt{n}} \quad \text{if} \quad e^{-\sqrt{n}} \leqq |x| \leqq 1$$

(cf. NEWMAN [7] or GONČAR [8]). Hence for the rational function

$$\psi_{n,j}(x)=\frac{1}{2}[\varphi_n(x-\eta_{n,j-1})+\varphi_n(\eta_{n,j}-x)]$$

of degree $\leqq 2n$ we have

$$(8)\qquad |\psi_{n,j}(x)| \leqq \begin{cases} 3e^{-\sqrt{n}} & \text{if } 0 \leqq x \leqq \eta_{n,j-1}-e^{-\sqrt{n}} \text{ or } \eta_{n,j}+e^{-\sqrt{n}} \leqq x \leqq 1\,[2] \\ 2 & \text{if } |x-\eta_{n,j-1}| \leqq e^{-\sqrt{n}} \text{ or } |x-\eta_{n,j}| \leqq e^{-\sqrt{n}}. \end{cases}$$

$$(9)\qquad |1-\psi_{n,j}(x)| \leqq 3e^{-\sqrt{n}} \quad \text{if} \quad \eta_{n,j-1}+e^{-\sqrt{n}} \leqq x \leqq \eta_{n,j}-e^{-\sqrt{n}}.$$

Moreover

$$\psi_{n,j}(x)+\psi_{n,j+1}(x)=\frac{1}{2}[\varphi_n(x-\eta_{n,j-1})+\varphi_n(\eta_{n,j+1}-x)]$$

and thus

$$(10)\qquad |1-\psi_{n,j}(x)-\psi_{n,j+1}(x)| \leqq 3e^{-\sqrt{n}} \quad \text{if} \quad \eta_{n,j-1}+e^{-\sqrt{n}} \leqq x \leqq \eta_{n,j+1}-e^{-\sqrt{n}}.$$

Now consider the rational function

$$r_n(x)=\sum_{j=1}^{t_n} r_{n,j}(x)\psi_{n,j}(x)$$

of degree $\leqq 4nt_n \leqq 4ns_n$. Assume first that $\eta_{n,l-1}+e^{-\sqrt{n}} \leqq x \leqq \eta_{n,l}-e^{-\sqrt{n}}$, then by (7),

[2] In case $j=1$ or $j=t_n$ the first or the second possibility does not occur, respectively.

(9), (6), (8) and (2) we get for sufficiently large n's

$$|f(x)-r_n(x)| \leqq |f(x)-r_{n,l}(x)|+|r_{n,l}(x)|\,|1-\psi_{n,l}(x)|+$$

$$+\sum_{\substack{j=1\\ j\neq l}}^{t_n} |r_{n,j}(x)\psi_{n,j}(x)| \leqq R_n(f; I_{n,k_l})+\omega_f\left(s_n e^{-\sqrt{n}}\right)+2\|f\|^3 e^{-\sqrt{n}}+$$

$$+4\|f\| e^{-\sqrt{n}}+\frac{1}{2} e^{1/2\sqrt{n}}\, t_n 3e^{-\sqrt{n}} \leqq R_n(f; I_{n,k_l})+\omega_f\left(s_n e^{-\sqrt{n}}\right)+2s_n e^{-1/2\sqrt{n}}$$

$$\left(\eta_{n,l-1}+e^{-\sqrt{n}} \leqq x \leqq \eta_{n,l}-e^{-\sqrt{n}}\right).$$

Now let $\eta_{n,l}-e^{-\sqrt{n}} \leqq x \leqq \eta_{n,l}$. Then we obtain by (7), (10), (8), (6) and (2) for sufficiently large n's

$$|f(x)-r_n(x)| \leqq |f(x)-r_{n,l}(x)|+|r_{n,l}(x)|\,|1-\psi_{n,l}(x)-\psi_{n,l+1}(x)|+$$

$$+|\psi_{n,l+1}(x)|\left(|r_{n,l}(x)-f(x)|+|f(x)-r_{n,l+1}(x)|\right)+$$

$$+\sum_{\substack{j=1\\ j\neq l,l+1}}^{t_n} |r_{n,j}(x)\psi_{n,j}(x)| \leqq R_n(f; I_{n,k_l})+\omega_f\left(s_n e^{-\sqrt{n}}\right)+$$

$$+2\|f\|^3 e^{-\sqrt{n}}+2\left[R_n(f; I_{n,k_l})+R_n(f; I_{n,k_{l+1}})+2\omega_f\left(s_n e^{-\sqrt{n}}\right)+4\|f\|^3 e^{-\sqrt{n}}\right]+$$

$$+\frac{3}{2} t_n e^{-1/2\sqrt{n}} \leqq 5 \max_{1\leqq l\leqq t_n} R_n(f; I_{n,k_l})+5\omega_f\left(s_n e^{-\sqrt{n}}\right)+2s_n e^{-1/2\sqrt{n}} \leqq$$

$$\leqq 5 \max_{1\leqq l\leqq s_n} R_n(f; I_{n,l})+\left(5+\frac{1}{\omega_f(1)}\right)\omega_f\left(s_n e^{-1/2\sqrt{n}}\right)$$

$$\left(\eta_{n,l}-e^{-\sqrt{n}} \leqq x \leqq \eta_{n,l}\right).$$

The case $\eta_{n,l-1} \leqq x \leqq \eta_{n,l-1}+e^{-\sqrt{n}}$ is quite similar, and the proof is completed.

2. Applications

The first author proved [4] that if $f^{(r)}(x)$ is of bounded variation then

$$R_n(f; [0, 1]) \leqq c_1 V_0^1(f^{(r)}) \frac{\log \ldots \log n}{n^{r+1}} \tag{11}$$

where $c_1>0$ depends only on the number of iterations in $\log \ldots \log n$ (this number is arbitrary but fixed). Applying our Theorem 1 we easily get the following

THEOREM 2. *Let $f\in C$ and assume that we have a fixed subdivision*

$$0=\xi_0<\xi_1<\ldots<\xi_s=1, \quad I_i=[\xi_{i-1}, \xi_i] \qquad (i=1, \ldots, s) \tag{12}$$

of the interval $[0, 1]$ *such that* $f^{(r)}(x)$ *exists and is of bounded variation in each* I_i $(i=1, \ldots, s)$. *Then*

$$R_n(f; [0, 1]) \leqq c_2 \max_{1 \leqq i \leqq s} V_{\xi_{i-1}}^{\xi_i}(f^{(r)}) \frac{\log \ldots \log n}{n^{r+1}} + c_3 \omega_f\left(se^{-\frac{1}{2}\sqrt{\frac{n}{4s}}}\right).$$

As for convex functions, BULANOV [5] proved an estimate of type (13). By Theorem 1 we have at once

THEOREM 3. *If* $f(x)$ *is piecewise convex in* $[0, 1]$, *then*

$$R_n(f; [0, 1]) = O\left(\frac{\log^2 n}{n}\right)\left[\omega_f(\lambda) + \frac{\log^2 n}{n} \log \frac{1}{\lambda}\right] + c_4 \omega_f\left(e^{-c_5\sqrt{n}}\right) \tag{13}$$

$(0 < \lambda < 1$ *arbitrary*$)$.

I.e., we have at least

$$R_n(f; [0, 1]) = O\left(\frac{\log^2 n}{n}\right) \omega_f\left(\frac{\log^2 n}{n}\right) + c_4 \omega_f\left(e^{-c_5\sqrt{n}}\right)$$

but

$$R_n(f; [0, 1]) = O\left(\frac{\log^5 n}{n^2}\right)$$

when $f \in \mathrm{Lip}\, \alpha$, $0 < \alpha \leqq 1$.

GONČAR [6] proved that if $f(z)$ is analytic in $|z-1|<1$ and bounded in $|z-1| \leqq 1$ then

$$R_n(f; [0, 1]) = O(\omega_f(e^{-\lambda}) + \lambda^2 e^{-c(n/\lambda)}) \quad (0 < \lambda < n \text{ arbitrary}).$$

Again, by Theorem 1, we could easily get an estimate for the corresponding functions piecewise analytic in open circles, but this has already been obtained earlier (cf. SZABADOS [9]).

The result of TURÁN and SZÜSZ [10] about functions which are piecewise analytic in closed domains also follows from our Theorem 1, but it follows from FREUD [1] and SZABADOS [9] as well.

In all these applications we worked with the fixed nodes (12). As an application for varying nodes, we could prove the result of POPOV cited under (11).

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References

[1] G. FREUD, Über die Approximation reeller Funktionen durch rationale gebrochene Funktionen, *Acta Math. Acad. Sci. Hung.*, **17** (1966), pp. 313—324.

[2] G. FREUD—J. SZABADOS, Rational approximation to x^α, *Acta Math. Acad. Sci. Hung.*, **18** (1967), pp. 393—399.

[3] J. SZABADOS, Generalization of two theorems of G. Freud concerning the rational approximation, *Studia Sci. Math. Hung.*, **2** (1967), pp. 73—80.
[4] V. A. POPOV, On rational approximation of functions in the class V_r, *Acta Math. Acad. Sci. Hung.*, **25** (1974), pp. 61—65.
[5] A. P. BULANOV, Rational approximation of convex functions with given modulus of continuity (Russian), *Mat. Sbornik*, **84 (126):3** (1971), pp. 476—494.
[6] A. A. GONČAR, On the degree of rational approximation of continuous functions with characteristic singularity (Russian), *Mat. Sbornik*, **73 (115)** (1967), pp. 630—638.
[7] D. J. NEWMAN, Rational approximation to $|x|$, *Michigan Mat. Journal*, **11** (1964), pp. 11—14.
[8] A. A. GONČAR, Estimates for the growth of rational functions and some applications (Russian), *Mat. Sbornik*, **72 (114):3** (1967), pp. 489—503.
[9] J. SZABADOS, Rational approximation of analytic functions with finite number of singularities on the real axis, *Acta Math. Acad. Sci. Hung.*, **20** (1969), pp. 159—167.
[10] P. SZÜSZ—P. TURÁN, On the constructive theory of functions III, *Studia Sci. Math. Hung.*, **1** (1966), pp. 315—322.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (1—2), (1974), pp. 171—187.

ON CERTAIN LINEAR OPERATORS. VIII

(FUNCTIONS WHERE $\omega_m(f;t)=O[\omega_m(t)]$ INCLUDING $\omega_m(t)=t^m$ ALSO. FUNCTIONS HAVING BOUNDED DERIVATIVES)

By

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Introduction

This paper consists of two parts. In the first part (§§. 1—2) we give some lower estimations for the deviation of $L_n(f;x)$, $H_n(f;x)$ or another linear operators from $f(x)$ if $\omega_m(f;t)=O[\omega_m(t)]$ allowing $\omega_m(t)=t^m$ also, by a general theorem proved in [3] (as for the definition, see below). In the second part (§§. 3—4) we prove a general theorem and give some applications for the above mentioned deviations for functions having bounded derivatives.

PART I

1. A general theorem

1.1. It is an important question in the approximation theory the investigation of the difference of the n-th Lagrange interpolatory polynomials $L_n(f;x)$ (or the n-th Hermite—Fejér polynomials $H_n(f;x)$, etc.) from the function $f(x)$. In connection with these we proved the theorem as follows (see [3]).

1.2. Let us denote by $\omega_m(t)$ a function for which

(i) $\omega_m(t)>0$ for $t>0$, $\omega_m(0)=0$, $\omega_m(T)\geqq\omega_m(t)$ for $T\geqq t$, $\omega_m(t)$ is a continuous function for $t\geqq 0$;

(ii) $\dfrac{t^m}{\omega_m(t)}$ is a monotone increasing function for $t\geqq 0$ ($m\geqq 1$ is a fixed integer).

Let us denote by $C_P^{\{a,b\}}(\omega_m)$ the class of functions continuous on $\{a,b\}$ such that

$$\omega_m(f;t)\leqq a_m(f)\omega_m(t) \tag{1.1}$$

where $\omega_m(f;t)$ is the m-th modulus of smoothness of $f(x)$, P is a set of certain restrictions for $f(x)$.

Let us denote by $\{T_n\}_{n=1}^{\infty}$ and $\{U_n\}_{n=1}^{\infty}$ linear (i.e. additive and continuous) operators on the set $C_P^{\{a,b\}}(\omega_m)$.

Let us suppose the following restrictions.

A. There exist certain functions $g_n(x)$ $(n=1,2,3,\ldots)$ such that

$$g_n(x)\in C_P^{\{a,b\}}(\omega_m), \tag{α1}$$

$$T_n(g_n;z_n)\geqq c_4\lambda_n(z_n) \quad \text{for certain} \quad \{z_n\}\subset\{a,b\} \tag{α2}$$

(definition of $\lambda_n(z_n)$), [1]

$$|T_n(g_N;z_n)|\leqq c_5\lambda_n(z_n) \quad (N>n); \tag{α3}$$

[1] Any constant c or c_i is positive.

B. $\tilde{f}(x)\in C_P^{\{a,b\}}(\omega_m)$ where for certain $0<n_1<n_2<\dots$ ([3], (2.10) and (2.11))

$$(\beta 1)\qquad \tilde{f}(x)\overset{\text{def}}{=\!=} Q\sum_{i=1}^{\infty} e_{n_i}\omega_m(\delta_{n_i})g_{n_i}(x)\quad (Q>0)$$

where $0<e_n\leqq 1$ and $\delta_n\searrow 0$ (i.e. $\delta_1\geqq\delta_2\geqq\dots\geqq\delta_n$, $\lim\limits_{n\to\infty}\delta_n=0$);

C. for certain $0<q<1$ and $c_{4-}<c_4$

$$(\gamma_1)\qquad c_{4-}\lambda_{n_k}(z_{n_k})>r(z_{n_k})\overset{\text{def}}{=\!=}\sum_{i=k}^{\infty} q^{i-k}|U_{n_k}(g_{n_i};z_{n_k})|\qquad (k=1,2,3,\dots).$$

Now we can state the following

THEOREM 1.1 ([3]). *If the conditions A, B and C are satisfied then there exist an* $f(x)\in C_P^{\{a,b\}}(\omega_m)$ *and* $\{n_i\}_{i=1}^{\infty}$ *such that*

$$(1.2)\qquad T_{n_k}(f;z_{n_k})-U_{n_k}(f;z_{n_k})>e_{n_k}\omega_m(\delta_{n_k})\lambda_{n_k}(z_{n_k})\qquad (k=1,2,3,\dots)$$

if these differences have meaning.

2. Applications for $\omega_m(t)=t^m$

2.1. In the paper [3] we showed some application of Theorem 1.1, but we did not obtain the best results for $\omega_m(t)=t^m$. Our present aim is the improvement of these results in certain cases by helping of the "separating" of $g_n(x)$ which was applied for Fourier series in [1] and [2] by H. LEBESGUE and later by K. I. OSKOLKOV.

2.2. Let us consider the Lagrange interpolatory process for the Chebyshev roots, i.e.

$$(2.1)\quad L_n(f;x)=\sum_{k=1}^{n} f(x_{kn})l_{kn}(x),\quad x_{kn}=\cos\vartheta_{kn}=\cos\frac{2k-1}{2n}\pi\qquad (k=1,2,\dots,n)$$

where $l_{kn}(x)$ are the fundamental polynomials. For the Lebesgue functions and the Lebesgue numbers

$$(2.2)\qquad \lambda_n(x)=\sum_{k=1}^{n}|l_{kn}(x)|,\quad \lambda_n=\max_{-1\leqq x\leqq 1}\lambda_n(x)=\frac{2}{\pi}\log n+O(1).$$

holds.

It is well known that for $f(x)\in C^{[-1,1]}$

$$(2.3)\qquad |L_n(f;x)-f(x)|\leqq$$

$$\leqq[\lambda_n(x)+1]E_{n-1}(f)\leqq c[\lambda_n(x)+1]\omega_m\left(f;\frac{1}{n}\right)\qquad (m=1,2,3,\dots).$$ [2]

We now prove

[2] Here $E_n(f)=E_n(f;-1,1)=\max\limits_{-1\leqq x\leqq 1}|f(x)-P_n(f;x)|$ where $P_n(f;x)$ is the uniformly best approximating polynomial of $f(x)$ of degree $\leqq n$.

THEOREM 2.1. *If x^* is an arbitrary point from $(-1, 1)$ then there exist an $f(x) \in C^{[-1,1]}(\omega_m)$ and $0<n_1<n_2<\dots$ such that*

$$L_n(f; x^*)-f(x^*) > \log n\, \omega_m\left(\frac{1}{n}\right) \qquad (n = n_1, n_2, n_3, \dots;\ x^* \neq \pm 1) \tag{2.4}$$

where $L_n(f; x)$ is defined by (2.1).

PROOF. First of all a remark. It is well known that

$$l_{kn}(x) = \frac{(-1)^{k-1}}{n} \frac{\sin \vartheta_k \cos n\vartheta}{\cos \vartheta - \cos \vartheta_k} \qquad (k = 1, 2, \dots, n). \tag{2.5}$$

Let us suppose that e.g. $x^* \geqq 0$. Using the estimations and ideas of [4], 6 (see further [5], 3.8 and 3.9) we can choose an index-set I such that (with $\vartheta_{j-1,n} \leqq \vartheta^* \leqq \vartheta_{jn}$, $\vartheta_{0n}=1$, $\vartheta_{n+1,n}=-1$, $|k-j|=i$)

$$|l_{kn}(x^*)| \sim \begin{cases} \dfrac{1}{i} & x^* \neq 1,\ x_{kn} \in [a, b] \subset (-1, 1),\ n \in I,\ k \neq j,\ j-1\ ^3 \\ \dfrac{1}{i} & x^* = 1,\ -1+\varepsilon \leqq x_{kn} < 1. \end{cases} \tag{2.6}$$

Indeed, for $x^* \neq 1$ (2.6) was essentially proved in [4], but if $x^*=1$, we can obtain (2.6) by (2.5) and (2.1). (We can obtain an analogous result for $x^* \leqq 0$.)

Now we define the function $g_n(x)$ as follows.

$$g_n(x_{kn}) = \begin{cases} \operatorname{sg} l_{kn}(x^*) & \text{for } \left[\sqrt[3]{n}\right] \leqq i \leqq \left[\sqrt{n}\right], \\ 0 & \text{for } 1 \leqq i < \left[\sqrt[3]{n}\right] \text{ or } \left[\sqrt{n}\right] < i \quad (n > n_0). \end{cases} \tag{2.7}$$

In the interval $[x_{k+1,n}; x_{kn}]$ let $g_n(x)$ be the Hermite interpolating polynomial of degree $\leqq 2m-1$ which is equal to $g_n(x_{kn})$ or $g_n(x_{k+1,n})$ at the endpoints, respectively, further $g_n'(x_{kn}) = g_n''(x_{kn}) = \dots = g_n^{(m-1)}(x_{kn}) = g_n'(x_{k+1,n}) = g_n''(x_{k+1,n}) = \dots = g_n^{(m-1)}(x_{k+1,n})$. Obviously

$$g_n(x) = 0 \quad \text{if} \quad x \notin \Delta_n \tag{2.8}$$

where

$$\Delta_n = (x_{[\sqrt{n}]+j(n)+1,n};\ x_{[\sqrt[3]{n}]+j(n)-1,n}) \stackrel{\text{def}}{=} (a_n, b_n).$$

Using the definition of $g_n(x)$, further that d_n (the minimal distance of the nodes where $g_n(x) \neq 0$) $\geqq cn^{-1}$ (here we used that x^* is an inner point) we obtain

$$\omega_m(g_n; t) \leqq B_m n^m t^m, \quad |g_n(x)| \leqq 1 \tag{2.9}$$

(see e.g. [6]).

Let us choose a sequence $\{n_r\}$ such that $\{n_r\} \subset I$ and

$$b_{n_r} < a_{n_r+1} \qquad (r = 1, 2, 3, \dots). \tag{2.10}$$

[3] $f_n \sim g_n$ if $f_n = O(g_n)$ and $g_n = O(f_n)$.

Then we have our "separating relation" as follows

$$\Delta_{n_s} \cap \Delta_{n_t} = \varnothing \qquad (s \neq t). \tag{2.11}$$

For B we prove a lemma which is generalization of Lemma 1 from [2].

LEMMA 2.1. *If* $|g_n(x)| \leqq c$, $\omega_m(g_n; t) \leqq B_m \delta_n^{-m} t^m$ *with any* $\delta_n > 0$, $\delta_n \searrow 0$ *and* $g_n(x) = 0$ *for* $x \notin \Delta_n$, *further* $\Delta_i \cap \Delta_j = \varnothing$ $(i \neq j)$ *then* $\tilde{f}(x) \in C^{[-1,1]}(\omega_m)$ *with*

$$\tilde{f}(x) = Q \sum_{n=1}^{\infty} \omega_m(\delta_n) g_n(x).$$

PROOF. Using that in calculating $\omega_m(\tilde{f}; t)$, at most $m+1$ terms from $\tilde{f}$ will not vanish, we have (for simplicity we assume these to be the first $m+1$ terms)

$$\omega_m(\tilde{f}; t) \leqq Q \sum_{i=1}^{m+1} \omega_m(\delta_i) \omega_m(t; g_i).$$

Let us suppose that $t \leqq \min_{1 \leqq i \leqq m+1} (\delta_i)$. Then using (ii) we get

$$\omega_m(\tilde{f}; t) \leqq Q \sum_{i=1}^{m+1} \omega_m(\delta_i) B_m \delta_i^{-m} t^m \leqq (m+1) Q B_m \omega_m(t).$$

If e.g. $\delta_1, \delta_2, \ldots, \delta_k > t \geqq \delta_{k+1}, \delta_{k+2}, \ldots, \delta_{m+1}$ we obtain by (ii) and the monotonicity of $\omega_m(t)$

$$\omega_m(\tilde{f}; t) \leqq Q \sum_{i=1}^{m+1} \omega_m(\delta_i) \omega_m(g_i; t) \leqq [kQB_m + (m-k+1) Q c 2^m] \omega_m(t),$$

as stated.

Now we show that

$$L_{n_k}(g_{n_k}; x^*) \sim \log n_k \qquad (k = 1, 2, 3, \ldots). \tag{2.12}$$

Indeed, by (2.6) and (2.7) we have

$$L_{n_k}(g_{n_k}; x^*) = \sum_{r=1}^{n_k} g_{n_k}(x_r) l_r(x^*) = \tag{2.13}$$

$$= \sum_{x_r \in \Delta_{n_k}} \frac{1}{i} \sim \log \sqrt{n_k} - \log \sqrt[3]{n_k} \sim \log n_k \qquad (k = 1, 2, 3, \ldots).$$

If we apply our Theorem 1.1 with $T_n = L_n$, $U_n = E$ (E is the unit operator), $e_n = 1$, $\delta_n = d_n \sim n^{-1}$, $z_n = x^*$ and $\lambda_n(z_n) \sim \log n$ we can prove ($\alpha 1$) by (2.9); ($\alpha 2$) by (2.12); ($\alpha 3$) by (2.2) and (2.9); B by Lemma 2.1; ($\gamma 1$) by (2.8) ($r(x^*) = 0$ because of $x^* \notin \Delta_{n_k}$). So we perfectly proved our assertion for $x^* \geqq 0$. The case $x^* \leqq 0$ is similar.

2.3. Let us consider the situation at the end-points. With a modification of our original proof we can obtain an interesting improvement of Theorem 5.1 from [4].

THEOREM 2.2. *For any* $\varepsilon > 0$ *there exist an* $f(x) \in C^{[-1,1]}(\omega_m)$ *and* $0 < n_1 < n_2 < \ldots$ *such that*

$$L_n(f; 1) - f(1) > \log n \, \omega_m\left(\frac{1}{n^{1+\varepsilon}}\right) \qquad (n = n_1, n_2, n_3, \ldots) \tag{2.14}$$

where $L_n(f; x)$ *is defined by* (2.1).[4]

[4] Of course, we can prove (2.14) for $x = -1$.

PROOF. We can use the ideas of the above proof. The only modification is that for $\Delta_n=(a_n, b_n)$

$$a_n = x_{[n^{1-\varrho}],n} \quad b_n = x_{[n^{1-2\varrho}],n} \quad (n > n_0,\ 0 < \varrho < 1). \tag{2.15}$$

So

$$L_{n_k}(g_{n_k}; 1) = \sum_{x_r \in \Delta_{n_k}} \frac{1}{r} \sim \log n_k^{1-\varrho} - \log n_k^{1-2\varrho} = \varrho \log n_k \qquad (k = 1, 2, 3, \ldots), \tag{2.16}$$

further for $\delta_n \sim d_n$ we have

$$d_n = x_{[n^{1-2\varrho}]+1,n} - x_{[n^{1-2\varrho}],n} = \cos\frac{2[n^{1-2\varrho}]+1}{2n}\pi - \cos\frac{2[n^{1-2\varrho}]-1}{2n}\pi = \tag{2.18}$$

$$= 2\sin\frac{[n^{1-2\varrho}]}{n}\pi \sin\frac{1}{2n}\pi \sim \sin\frac{1}{n^{2\varrho}}\sin\frac{1}{n} \sim \frac{1}{n^{1+2\varrho}}.$$

With $2\varrho=\varepsilon$ we can prove (2.14).

2.4. If $\omega_m(t)$ is a "good" function then the loss of ε in (2.14) is significant. For this aim we give another estimation at the end-points.

THEOREM 2.3. *For every sequence* $\{R_n\}$ *where* $\overline{\lim_{n\to\infty}} R_n=\infty$ *there exist an* $f(x)\in C^{[-1,1]}(\omega_m)$ *and* $0<n_1<n_2<\ldots$ *such that*

$$L_n(f; 1)-f(1) > \omega_m\left(\frac{1}{R_n n}\right) \qquad (n = n_1, n_2, n_3, \ldots). \tag{2.18}$$

PROOF. Let us choose the sequences $\{M_n\}\subset\{R_j\}$ such that $M_i \leqq M_{i+1}$, $\lim_{n\to\infty} M_n=\infty$, $M_n=o(n)$ and let $M_n=2m_n$. Then with $a_n=x_{\frac{n}{m_n},n}$ and $b_n=x_{\frac{n}{m_n},n}$ we can satisfy (2.11), further

$$L_{n_k}(g_{n_k}; 1) = \sum_{x_r \in \Delta_{n_k}} \frac{1}{r} \geqq \log\frac{n}{m_n} - \log\frac{n}{M_n} = \log\frac{M_n}{m_n} = \log 2$$

and

$$\delta_n \sim d_n \sim \sin\frac{n}{M_n n}\sin\frac{1}{n} \sim \frac{1}{M_n n} \qquad (n = n_1, n_2, n_3, \ldots).$$

(If $n=O(R_n)$, see [4], (5.2).)

2.5. Now we consider the Lagrange interpolatory process on the Jacobi nodes, i.e. for the roots

$$-1 < x_{nn}^{(\alpha,\beta)} < x_{n-1,n}^{(\alpha,\beta)} < \ldots < x_{2n}^{(\alpha,\beta)} < x_{1n}^{(\alpha,\beta)} < 1 \tag{2.19}$$

of

(2.20)

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}] \quad (\alpha, \beta > -1).$$

We have (with $x = \cos\vartheta$ and $x_{kn}^{(\alpha,\beta)} = x_{kn} = \cos\vartheta_{kn}$)

$$P_n^{(\alpha,\beta)}(x) = O(n^{-1/2}) \quad \text{for every fixed} \quad [a,b] \subset (-1,1), \tag{2.21}$$

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{\alpha} \sim n^\alpha, \quad P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x), \tag{2.22}$$

$$|P_n'^{(\alpha,\beta)}(x_{kn}^{(\alpha,\beta)})| \sim k^{-\alpha-3/2} n^{\alpha+2} \qquad (0 < \vartheta_{kn}^{(\alpha,\beta)} \leqq \pi - \varepsilon), \tag{2.23}$$

(2.24)

$$\begin{cases} P_n^{(\alpha,\beta)}(\cos\vartheta) = n^{-1/2} k(\vartheta) \cos(N\vartheta + \gamma) + O(n^{-3/2}) \text{ uniformly for } \varepsilon \leqq \vartheta \leqq \pi - \varepsilon \\ k(\vartheta) = \pi^{-1/2} \left(\sin\frac{\vartheta}{2}\right)^{-\alpha-1/2} \left(\cos\frac{\vartheta}{2}\right)^{-\beta-1/2}, \\ N = n + \dfrac{\alpha+\beta+1}{2}, \quad \gamma = -\left(\alpha + \dfrac{1}{2}\right)\dfrac{\pi}{2} \end{cases}$$

(see [7], (7.32.5), (4.1.1), (4.1.3), (8.9.2) and (8.21.10)), further

$$\frac{c_1}{n} \leqq \vartheta_{k+1,n} - \vartheta_{k,n} \leqq \frac{c_2}{n} \qquad (k = 0, 1, \ldots, n) \tag{2.25}$$

with $0 < c_1 = c_1(\alpha,\beta)$, $c_2 = c_2(\alpha,\beta)$ and $\vartheta_{0n} = 0$, $\vartheta_{n+1,n} = \pi$,

$$|x - x_{kn}| \sim n^{-2} |j^2 - k^2| \quad \text{if} \quad x \in [x_{j+1}, x_j], \qquad k = 0, 1, \ldots, n+1; \; k \neq j, j+1 \tag{2.26}$$

([8], Lemmas 1 and 2).

Using (2.24) for every fixed $x^* \in (-1, 1)$ we can choose an index set I such that

$$|P_n^{(\alpha,\beta)}(x^*)| \sim n^{-1/2} \qquad (n \in I, \; x^* \in (-1,1)). \tag{2.27}$$

Then by (2.27), (2.23) and (2.26)

$$|l_{kn}^{(\alpha,\beta)}(x^*)| = \left| \frac{P_n^{(\alpha,\beta)}(x^*)}{P_n'^{(\alpha,\beta)}(x_{kn})(x^* - x_{kn})} \right| \sim \frac{n^{-1/2}}{k^{-\alpha-3/2} n^{\alpha+2} |j-k||j+k| n^{-2}} \qquad (n \in I). \tag{2.28}$$

But if $x_{kn} \in \Delta_n$ (see (2.8)) then $k \sim n$, $j \sim n$ so

$$|l_{kn}^{(\alpha,\beta)}(x^*)| \sim \frac{n^{-1/2}}{n^{-\alpha-3/2} n^{\alpha+2} i n n^{-2}} = \frac{1}{i} \quad (n \in I, \; x^* \in (-1,1), \; x_{kn} \in \Delta_n). \tag{2.29}$$

Now if we use the same ideas as in Part 2.2, we get for the $L_n^{(\alpha,\beta)}$ Lagrange-interpolatory process the following

THEOREM 2.4. *If x^* is an arbitrary point from $(-1, 1)$ then there exist an $f(x) \in C^{[-1,1]}(\omega_m)$ and $0 < n_1 < n_2 < \ldots$ such that*

$$L_n^{(\alpha,\beta)}(f; x^*) - f(x^*) > \log n \, \omega_m\left(\frac{1}{n}\right) \qquad (n = n_1, n_2, n_3, \ldots; \; x^* \neq \pm 1; \; \alpha, \beta > -1). \tag{2.30}$$

2.6. Let us consider the endpoints. By (2.22), (2.23) and (2.26)

$$l_{kn}^{(\alpha,\beta)}(1)\sim\frac{n^{\alpha}i^{\alpha+3/2}n^{2}}{n^{\alpha+2}i^{2}}\sim i^{\alpha-1/2}\qquad(0<\vartheta_{kn}\leqq\pi-\varepsilon). \tag{2.31}$$

We have, as above, the following statement:

THEOREM 2.5. *For any sequence* $\{R_n\}$ *where* $\overline{\lim_{n\to\infty}}\,R_n=\infty$ *there exist an* $f(x)\in C^{[-1,1]}(\omega_m)$ *and* $0<n_1<n_2<\ldots$ *such that*

$$L_n^{(\alpha,\beta)}(f;1)-f(1)>\left(\frac{n}{R_n}\right)^{\alpha+1/2}\omega_m\left(\frac{1}{R_n n}\right)\qquad(n=n_1,n_2,n_3,\ldots). \tag{2.32}$$

PROOF. We have already proved (2.32) for $\alpha=\beta=-1/2$ in (2.18). If $\alpha\neq-1/2$ we have by (2.31) as above

$$L_{n_k}(g_{n_k};1)=\sum_{x_r\in\Delta_{n_k}}i^{\alpha-1/2}\sim\left(\frac{n}{M_n}\right)^{\alpha+1/2}\qquad(n=n_1,n_2,n_3,\ldots) \tag{2.33}$$

and $\delta_n=d_n\sim(M_n n)^{-1}$. So we have (2.32).

2.7. For the trigonometric cases we can obtain similar theorems. E.g., let us consider the trigonometric Hermite—Fejér interpolating process

$$H_n^*(f;x)=\sum_{k=1}^{n}[f(x_{kn})A_k^{(n)}(x)+f'(x_{kn})B_k^{(n)}(x)], \tag{2.34}$$

$$x_{kn}=\frac{2k\pi}{n+1}\qquad(k=0,1,\ldots,n)$$

for 2π-periodic functions $f(x)$ having continuous derivatives $f'(x)$. If further $f(x)\in C(\omega_m)$ we denote this class of functions by $\tilde{C}_1(\omega_m)$ (see [9]).

We proved that for every $f(x)\in\tilde{C}_1(\omega_m)$

$$|H_n^*(f;x)-f(x)|\leqq b_m(f)\log n\,\omega_m\left(\frac{1}{n}\right)\qquad(n=2,3,4,\ldots). \tag{2.35}$$

By the above applied ideas and the method of [9] we can prove

THEOREM 2.6. *For any* $x^*\neq2l\pi$ $(l=0,\pm1,\pm2,\ldots)$ *there exist an* $f(x)\in\tilde{C}_1(\omega_m)$ *and* $0<n_1<n_2<\ldots$ *such that*

$$H_n^*(f;x^*)-f(x^*)>\log n\,\omega_m\left(\frac{1}{n}\right)\qquad(n=n_1,n_2,n_3,\ldots). \tag{2.36}$$

2.8. REMARKS. 2.8.1. By the condition

$$\text{(iii)}\qquad\lim_{t=+0}\frac{t^m}{\omega_m(t)}=0$$

we can ensure B, i.e. now (2.18) and (2.32) are true with $R_n=1$.

2.8.2. We can obtain theorems for the so called lacunary interpolations also (see e.g. [9], § 6).

2.9. Now we consider a special case in connection with mechanical quadrature, the Newton—Cotes formula on $[a, b]$. Let for some $f(x)$

$$Q_n(f) = \sum_{k=0}^{n} A_{kn} f(x_{kn}) \tag{2.37}$$

with $x_{kn} = a + kh$ and

$$A_{kn} = \int_a^b l_{kn}(x)\,dx \qquad (k = 0, 1, \ldots, n)$$

where $l_{kn}(x)$ are the fundamental Lagrange polynomials of degree n, $h=(b-a)n^{-1}$. Q_n is the so called Newton—Cotes mechanical quadrature process.

We have for our interpolatory quadrature process

$$\left|Q_n(f) - \int_a^b f(x)\,dx\right| = O(\lambda_n)\omega_m\left(f; \frac{1}{n}\right) \qquad (n = 1, 2, 3, \ldots)$$

where

$$\lambda_n = \sum_{k=0}^{n} |A_{kn}|. \tag{2.38}$$

(see e.g. [3], 5.2).

Now we give an improvement of Theorem 5.5 from [3].

THEOREM 2.7. *For any $\omega_m(t)$ there exist an $f(x) \in C^{[-1,1]}(\omega_m)$ such that*

$$\overline{\lim_{n\to\infty}} |Q_n(f)| = \infty. \tag{2.39}$$

PROOF. We proved our assertion if $\lim_{t\to+0} (t^m/\omega_m(t))=0$ (see [3], Theorem 5.5) so we can suppose $\omega_m(t)=t^m$. Further we have the following formulae

$$B_{kn} = \frac{(-1)^{k-1} n!}{k!(n-k)!\, n \log^2 n}\left[\frac{1}{k} + \frac{(-1)^n}{n-k}\right]\left[1 + O\left(\frac{1}{\log^2 n}\right)\right] \qquad (1 \leqq k \leqq n-1), \tag{2.40}$$

$$B_{0n} = B_{nn} = \frac{1}{n \log n}\left[1 + O\left(\frac{1}{\log^2 n}\right)\right] \tag{2.41}$$

where $B_{kn} = (b-a)^{-1} A_{kn}$ (see [11], 6/2).

Hence for suitable n we have by the Stirling formula

$$|B_{n/2,n}| \sim 4\,\frac{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}}{\left(\frac{n}{2e}\right)^n \pi n n^2 \log^2 n} = \sqrt{\frac{2}{\pi}}\,\frac{2^{n+2}}{n^{5/2}\log^2 n} \qquad (n = 2, 4, 6, \ldots). \tag{2.42}$$

Now let

$$g(x_{kn}) = \begin{cases} \operatorname{sg} B_{n/2,n} & \text{if } k = \dfrac{n}{2} \\ 0 & \text{if } k \neq \dfrac{n}{2} \end{cases} \qquad (n = 2, 4, 6, \ldots),$$

in the interval $[x_{k+1,n}; x_{kn}]$ the Hermite interpolatory polynomial of degree $\leqq 2m-1$ as defined e.g. in 2.2. Then we apply our Theorem 1.1 with $T_n = Q_n$, $U_n(f) = \int_a^b f(x)\,dx$, $e_n = \left(\frac{2}{3}\right)^n$, z_n are arbitrary, $\delta_n \sim \frac{1}{n}$ and $\lambda_n(z_n) = |B_{n/2,n}|(b-a)$. So obviously we have ($\alpha 1$), ($\alpha 2$), ($\alpha 3$) and ($\gamma 1$) if $n \geqq n_0$ and n is even. Further with $f(x) = Q \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{n_i} n_i^{-m} g_{n_i}(x)$ we have

$$\omega_m(f; t) \leqq \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{n_i} n_i^{-m} \omega_m(g_{n_i}; t).$$

Let

$$\frac{1}{n_{j+1}} < t \leqq \frac{1}{n_j}.$$

By (2.9), (ii) and (i) we obtain

$$\sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{n_i} n_i^{-m} \omega_m(g_{n_i}; t) \leqq B_m \sum_{i=1}^{j} \left(\frac{2}{3}\right)^{n_i} \frac{n_i^m}{n_i^m} t^m + 2^m \sum_{i=j+1}^{\infty} \left(\frac{2}{3}\right)^{n_i} \frac{1}{n_i^m} \leqq c_m t^m$$

from where we get B.

By (1.2)

$$Q_{n_k}(f) - \int_a^b f(x)\,dx > \left(\frac{2}{3}\right)^{n_k} \frac{1}{n_k^m} \sqrt{\frac{2}{\pi}} \frac{2^{n_k+2}}{n_k^{5/2} \log^2 n_k} \qquad (k = 1, 2, 3, \ldots)$$

hence using $\int_a^b f(x)\,dx = O(1)$, we get (2.39).

PART II

3. A general theorem for functions having bounded derivatives

3.1. If we consider the Lagrange interpolatory process for the Chebyshev nodes (see 2.2) we obtain from (2.2) and (2.3) that

$$\|L_n(f; x)\|_{[-1,1]} = o(\log n) \quad (n \to \infty) \ ^5$$

for any $f(x) \in C^{[-1,1]}$.

[5] $\|g(x)\|_{[a,b]} = \max_{a \leqq x \leqq b} |g(x)|$.

Now we can arise the question whether we can substitute o by O for some $f(x)$ which is not continuous but bounded only. The analogous problem for Fourier series was solved by OSKOLKOV in a theorem from [10].

The aim of this part is to obtain analogous results for linear operators defined on boundedly differentiable functions. Our method is a modification of the idea of the paper [3] and the Part I.

3.2. Let us denote by $B_{m,P}^{\{a,b\}}$ the set of functions $f(x)$ for which $f^{(m)}(x)$ exists and is bounded in $\{a, b\}$. P is a set of properties owed by $f(x)$. Here $m=0, 1, 2, \ldots$; if $m=0$ or $P=\varnothing$, we omit these notations. E.g. $B^{\{a,b\}}$ is the set of the (generally not uniformly) bounded functions on $\{a, b\}$.

3.3. Let us denote by $\{T_n\}_{n=1}^{\infty}$ and $\{U_n\}_{n=1}^{\infty}$ linear (i.e. additive and continuous) operators defined on the set $B_{m,P}^{\{a,b\}}$. In many cases the U_n's are the unit operator E $(n=1, 2, 3, \ldots)$.

3.4. A*. *Let us suppose that there exist certain functions* $g_n(x)$ $(n=1, 2, 3, \ldots)$ *having the restrictions* (a1) *and* (a2)

(a1) $g_n(x) \in B_{m,P}^{\{a,b\}}$

(a2) $T_n(g_n; z_n) \geqq c_4 \lambda_n(z_n)$ *for certain* $\{z_n\}_{n=1}^{\infty} \subset \{a, b\}$ (*definition of* $\lambda_n(z_n)$).

We can suppose that for certain $\{e_n\}_{n=1}^{\infty}$ $(0 < e_n \leqq 1,\ e_{n+1} \leqq e_n)$ and $\{\delta_n\}_{n=1}^{\infty}$ $(0 < \delta_n < 1,\ \delta_n \searrow 0)$ we have

$$|T_n(g_N; z_n) - U_n(g_N; z_n)| \leqq \eta^N e_n \delta_n^m \lambda_n(z_n) \quad (n > M(N),\ 0 < \eta < 1). \tag{3.1}$$

The estimation (3.1) is true otherwise our assertion (3.8) would be true with $f(x) = \eta^{-N} g_N(x)$ or $f(x) = -\eta^{-N} g_N(x)$.

3.5. B*. *Let us suppose that* $\tilde{f}(x) \in B_{m,P}^{\{a,b\}}$ *where*

$$\tilde{f}(x) \overset{\text{def}}{=} Q \sum_{i=1}^{\infty} e_{n_i} \delta_{n_i}^m g_{n_i}(x) \quad \text{with} \quad n_{i+1} > M(n_i). \tag{b1}$$

Now we prove a lower estimation. With a simple computation

$$T_{n_k}(\tilde{f}; z_{n_k}) - U_{n_k}(\tilde{f}; z_{n_k}) = Q \sum_{i=1}^{\infty} e_{n_i} \delta_{n_i}^m [T_{n_k}(g_{n_i}; z_{n_k}) - U_{n_k}(g_{n_i}; z_{n_k})] \geqq \tag{3.2}$$

$$\geqq Q[e_{n_k} \delta_{n_k}^m T_{n_k}(g_{n_k}; z_{n_k}) - \sum_{i=1}^{k-1} e_{n_i} \delta_{n_i}^m |T_{n_k}(g_{n_i}; z_{n_k}) - U_{n_k}(g_{n_i}; z_{n_k})| -$$

$$- \sum_{i=k+1}^{\infty} e_{n_i} \delta_{n_i}^m |T_{n_k}(g_{n_i}; z_{n_k})| - \sum_{i=k}^{\infty} e_{n_i} \delta_{n_t}^m |U_{n_k}(g_{n_i}; z_{n_k})|,$$

supposing that the difference (3.2) has meaning.

But here, by (a2) and (3.1), we have

$$e_{n_k}\delta_{n_k}^m T_{n_k}(g_{n_k}; z_{n_k}) \geqq c_4 e_{n_k}\delta_{n_k}^m \lambda_{n_k}(z_{n_k}) \qquad (k = 1, 2, 3, \ldots), \tag{3.3}$$

$$\sum_{i=1}^{k-1} e_{n_i}\delta_{n_i}^m |T_{n_k}(g_{n_i}; z_{n_k}) - U_{n_k}(g_{n_i}; z_{n_k})| \leqq \tag{3.4}$$

$$\leqq e_{n_k}\delta_{n_k}^m \lambda_{n_k}(z_{n_k}) \sum_{i=1}^{k-1} e_{n_i}\delta_{n_i}^m \eta^{n_i} \leqq e_{n_k}\delta_{n_k}^m \lambda_{n_k}(z_{n_k}) \frac{\eta}{1-\eta}.$$

So we obtain

$$T_{n_k}(\tilde{f}; z_{n_k}) - U_{n_k}(\tilde{f}; z_{n_k}) \geqq$$

$$\geqq e_{n_k}\delta_{n_k}^m \left\{ c_4 Q\lambda_{n_k}(z_{n_k}) - \frac{Q\eta}{1-\eta}\lambda_{n_k}(z_{n_k}) - Q[S(z_{n_k}) + R(z_{n_k})] \right\}$$

where

$$S(z_{n_k}) \overset{\text{def}}{=} \sum_{i=k+1}^{\infty} e_{n_i}\delta_{n_i}^m |T_{n_k}(g_{n_i}; z_{n_k})|, \tag{3.5}$$

$$R(z_{n_k}) \overset{\text{def}}{=} \sum_{i=k}^{\infty} e_{n_i}\delta_{n_i}^m |U_{n_k}(g_{n_i}; z_{n_k})|. \tag{3.6}$$

Let us suppose that $\{\ldots\} > c_8\lambda_{n_k}(z_{n_k})$ or, what is the same

$$\lambda_{n_k}(z_{n_k})\left(c_4 Q - c_8 - \frac{Q\eta}{1-\eta} \right) > Q[S(z_{n_k}) + R(z_{n_k})]. \tag{3.7}$$

With an easy computation we obtain that (3.7) is valid with a sufficiently large Q and small η if, in addition, the following holds:

3.6. C*. *We suppose that for certain* $c_{4-} < c_4$

$$c_{4-}\lambda_{n_k}(z_{n_k}) > S(z_{n_k}) + R(z_{n_k}) \geqq 0 \qquad (k = 1, 2, 3, \ldots). \tag{c1}$$

Then we have with $f(x) = c_8^{-1}\tilde{f}(x)$

THEOREM 3.1. *Let* $\{T_n\}$ *and* $\{U_n\}$ *be linear operators defined on* $B_{m,P}^{\{a,b\}}$ *(*$m \geqq 0$ *a fixed integer). If the conditions* A*, B* *and* C* *are satisfied with certain* $\{\delta_n\}$, $\{z_n\}$, $\{\lambda_n(z_n)\}$ *and* $\{e_n\}$ *then there exist an* $f(x) \in B_{m,P}^{\{a,b\}}$ *and a sequence* $0 < n_1 < n_2 < \ldots$ *such that*

$$T_n(f; z_n) - U_n(f; z_n) > e_n\delta_n^m\lambda_n(z_n) \qquad (n = n_1, n_2, n_3, \ldots), \tag{3.8}$$

supposing that these differences have meaning.

3.7. If $m \geqq 1$ and

$$|T_n(g_N; z_n)| \leqq c_5\lambda_n(z_n) \quad (N > n) \tag{a3}$$

then we have with suitable $\{n_i\}$ for any fixed $0<q<1$

$$\sum_{i=k+1}^{\infty} e_{n_i}\delta_{n_i}^m|T_{n_k}(g_{n_i};z_{n_k})| \leqq \lambda_{n_k}(z_{n_k})\delta_{n_k}^m \sum_{i=k+1}^{\infty} e_{n_i}q^{i-k} \leqq \lambda_{n_k}(z_{n_k})\delta_{n_k}^m e_{n_k}\frac{q}{1-q}$$

from where we can prove our theorem if

C1*. *We suppose* (a3) *and for certain* $c_{4-}<c_4$

$$\text{(c2)} \qquad c_{4-}\lambda_{n_k}(z_{n_k}) > R(z_{n_k}) \qquad (k=1,2,3,\ldots).$$

I.e. we have

THEOREM 3.2. *If* $m\geqq 1$ *then supposing* A*, B* *and* C1* *we have for suitable* $f(x)\in B_{m,P}^{\{a,b\}}$ *and* $\{n_i\}$

$$\text{(3.9)} \qquad T_n(f;z_n)-U_n(f;z_n) > e_n\delta_n^m\lambda_n(z_n) \qquad (n=n_1,n_2,n_3,\ldots)$$

if these differences have meaning.

4. Some applications of the general theorem

4.1. First of all we prove a theorem for the process $L_n^{(\alpha,\beta)}$ (see 2.3).

THEOREM 4.1. *If* x^* *is an arbitrary point from* $(-1,1)$ *then there exist an* $f(x)\in B_m^{[-1,1]}$ *and* $0<n_1<n_2<\ldots$ *such that*

$$\text{(4.1)} \qquad L_n^{(\alpha,\beta)}(f;x^*)-f(x^*) > \frac{\log n}{n^m} \qquad (n=n_1,n_2,n_3,\ldots;\ x^*\neq\pm 1).$$

PROOF. Let us define $g_n(x)$ for fixed m as in (2.2) for $(m+1)$, but in the definition of n_i we suppose not only (2.10) but

$$\text{(4.2)} \qquad x_{j(n_i),n_i} < a_{n_{i+1}}$$

also.

Now we prove a lemma for $g_n(x)$ defined on a slightly more general $\Delta_n=(a_n,b_n)$, $n\geqq n_0$.

LEMMA 4.1. *If* $m\geqq 1$ *then let* $\delta_n=\min_{x_{k+1,n},\,x_{kn}\in\Delta_n}(x_{k+1,n}-x_{kn})$ *for* $n\geqq n_0$. *(I.e., we have* $\delta_n\searrow 0$ *and* $\omega(g_n^{(m)};t)\leqq F_{m+1}\delta_n^{-m-1}t$*). Now supposing only* (2.11) *for* Δ_{n_i} *we have* $f(x)\in B_m^{[-1,1]}$ *with*

$$f(x)=Q\sum_{i=1}^{\infty}\delta_{n_i}^m g_{n_i}(x).$$

PROOF. If $x\notin\bigcup[\Delta_j]\cup\{x^*\}$ then we prove that $g_s^{(k)}(x)=0$ $(s=1,2,3,\ldots;\ k=0,1,2,\ldots)$ and $f^{(k)}(x)=0$ $(k=0,1,2,\ldots)$.[6] Indeed, $f(x)=0$ because of $g_n(x)=0$

[6] For the sake of simplicity sometimes we omit the double indices.

if $x \notin \Delta_n$. Further by the notation

$$D_{h_1} g(x) = D_{h_1}^1 g(x) = \frac{g(x+h_1)-g(x)}{h_1},$$

$$D_{h_k}^k g(x) = D_{h_k} D_{h_{k-1}}^{k-1} g(x) \quad \text{with} \quad \mathbf{h}_j = (h_1, h_2, \ldots, h_j)$$

we obtain that $D_{h_s}^s g_n(x)=0$ if $|h_i|$ are small enough, from where we get our assertion.

If $x \in [\Delta_l]$ but $x \neq x^*$ then at first we suppose $[\Delta_l] \cap [\Delta_i] = \varnothing$ $(i=l-1, l+1)$. But now using that $g_l^{(m)}(x)$ exists we obtain the existence of $f^{(m)}(x)$. If $x=a_l=b_{l-1}$ or $x=b_l=a_{l+1}$ the only difference is that $D_h^m f(x)$ consists of two terms.

To prove that $f^{(m)}(x) \in B^{[-1,1]}$ we verify that

$$\omega(f^{(m)}; t) \leqq 4Q \max(F_{m+1}, 2^{m+1}) \quad (x \in [\Delta_l],\ x \neq x^*).$$

Indeed, we have with the notations of Lemma 2.1

$$\omega(f^{(m)}; t) \leqq Q \sum_{i=1}^{4} \delta_i^m \omega(g_i^{(m)}; t) \leqq$$

$$\leqq QF_{m+1}\delta_1^m \delta_1^{-m-1} t + 3Q2^{m+1} \leqq Q(F_{m+1} + 3 \cdot 2^{m+1})$$

if, e.g., $\delta_2, \delta_3, \delta_4 \leqq t < \delta_1$.

Finally let $x=x^*$. Then using that $x^* \notin \cup [\Delta_i]$ we can prove that $g_k^{(s)}(x^*)=0$ $(k=1, 2, 3, \ldots;\ s=0, 1, 2, \ldots)$. By this we shall prove that

$$f^{(k)}(x^*) = 0 \qquad (k = 0, 1, \ldots, m).$$

Indeed, $f(x^*)=0$ by definition, further if e.g. $\Delta_n = \sum_{i=1}^{p} \Delta^i$ where Δ^i is open and $|\Delta^i| = t_i$ then we have (if u is small enough)

$$\omega_r(g_n; u; \Delta^i) = \omega_r\left(g_n; u\frac{t_j}{t_i}; \Delta^j\right) \qquad (i, j \neq 1, p;\ r = 1, 2, 3, \ldots).$$

Considering that if e.g. $t_1 = t_1$ $(p > 2)$ then $\omega_r(g_n; u; \Delta_1^1) \leqq \omega_r(g_n; u; \Delta^2)$ we have that

$$\omega_r(g_n; u; I_n) \geqq \omega_r(g_n; u; \Delta^i) \qquad (i = 1, 2, \ldots, p;\ r = 1, 2, 3, \ldots).$$ [7]

But then $|g_n^{(r)}(x)|$ $(r=1, 2, 3, \ldots)$ *attains its biggest value in* I_n.

Now

$$(4.3) \quad g_n(x) = \tau_{2m+1}\left(\frac{x}{\delta_n}\right)^{2m+1} + \tau_{2m-1}\left(\frac{x}{\delta_n}\right)^{2m-1} + \ldots + \tau_1\left(\frac{x}{\delta_n}\right) \quad \text{for} \quad x \in [I_n]$$

if we transform the zero point to the midpoint of I_n. Here τ_i are independent of I_n. (We can obtain this form e.g. from the formula (30) in [6].) But then $f^{(r)}(x^*)=$

[7] Here $|I_n| = \delta_n$.

$= 0 (0 \leqq r \leqq m)$ because of

$$|h^{-1}[f^{(k)}(x^*-h)-f^{(k)}(x^*)]| = |h^{-1}\delta_{n(h)}^m g_{n(h)}^{(k)}(x^*-h)| \leqq$$

$$\leqq c_m h^{-1}\delta_{n(h)}^m \delta_{n(h)}^{-k} = c_m \frac{\delta_{n(h)}^{m-k}}{h} \xrightarrow[h\to 0]{} 0 \qquad (k = 0, 1, \ldots, m-1)$$

considering that $\delta_{n(h)} h^{-1} \xrightarrow[h\to 0]{} 0$. (We remark that if $g_n(x)$ is defined e.g. by (2.8) then $h \sim \frac{\sqrt[3]{n}}{n}$ and $\delta_{n(h)} \sim n^{-1}$.)

So we perfectly proved our lemma if $m \geqq 1$.

If $m=0$ we have a stronger statement and I think it is worth to formulate it distinctly.

LEMMA 4.2. *If* $|h_n(x)| \leqq c$ *for* $x \in [-1, 1]$ *and* $h_n(x)=0$ *for* $x \notin \chi_n \subset [-1, 1]$, *further with a fixed* s $\chi_{i_1} \cap \chi_{i_2} \cap \ldots \cap \chi_{i_{s+1}} = \varnothing$ *then* $f(x) \in B^{[-1,1]}$ *with* $f(x) = Q \sum_{n=1}^{\infty} h_n(x)$.

PROOF. If $x \notin \bigcup \chi_n$ then $f(x)=0$; if $x \in \chi_k$ then $|f(x)| \leqq sQc$ as it was stated.

Now we can prove our theorem. Applying Theorem 3.1 with $T_n = L_n^{(\alpha,\beta)}$, $U_n = E$, $e_n = 1$, $n^{-1} = \delta_n \sim d_n$, $z_n = x^*$ and $\lambda_n(z_n) \sim \log n$, we can verify the condition (a1) by (2.9), (2.25) and (2.26) (replacing m by $(m+1)$), (a2) by (2.29) and (2.13). As for the condition B* we can apply Lemma 4.1.

At last using (2.8), (2.10) and (4.2) we have that $S(z_{n_k}) = R(z_{n_k}) = 0$ i.e. (c1) is true.

4.2. We can formulate some assertions for the endpoints also. Some of these are expressed in Theorems 4.2—4.4.

THEOREM 4.2. *For every sequence* $\{R_n\}$ *where* $\overline{\lim_{n\to\infty}} R_n = \infty$ *there exist an* $f(x) \in B_m^{[-1,1]}$ *and* $0 < n_1 < n_2 < \ldots$ *such that*

$$L_n^{(\alpha,\beta)}(f; 1) - f(1) > \frac{n^{\alpha+1/2-m}}{R_n^{\alpha+1/2+m}} \qquad (n = n_1, n_2, \ldots). \tag{4.4}$$

PROOF. We have to use the ideas of 2.4.

4.3. If $\alpha = \beta = -1/2$ and $m=0$ we have a better estimation than (4.4), by using the idea of 2.3. Uniting this result with (4.1) we obtain

THEOREM 4.3. *For any point* $x^* \in [-1, 1]$ *there exist an* $f(x) \in B^{[-1,1]}$ *and* $0 < n_1 < n_2 < \ldots$ *such that*

$$L_n^{(-1/2,-1/2)}(f; x^*) - f(x^*) > \log n \qquad (n = n_1, n_2, n_3, \ldots). \tag{4.5}$$

4.4. For the H_n^* process (see 2.5) we can prove

THEOREM 4.4. *For any* $x^* \neq 2l\pi$ $(l = 0, \pm 1, \pm 2, \ldots)$ *there exist a* 2π*-periodic* $f(x) \in B_m^{[0,2\pi)}$ $(m \geqq 1)$ *and* $0 < n_1 < n_2 < \ldots$ *such that*

$$H_n^*(f; x^*) - f(x^*) > \frac{\log n}{n^m} \qquad (n = n_1, n_2, n_3, \ldots). \tag{4.6}$$

4.5. REMARKS. 4.5.1. By the condition (iii) (see (2.6.1)) we can prove some new estimations for the endpoints.

4.5.2. We can obtain theorems for the lacunary interpolation also (see [9], § 6).

4.5.3 By the above methods we can prove a theorem which is an improvement of a statement from [10].

THEOREM 4.5. *For any point* $x^* \in [0, 2\pi)$ *there exist a* 2π*-periodic* $f(x) \in B_m^{[0, 2\pi)}$ *and* $0 < n_1 < n_2 < \dots$ *such that*

$$S_n(f; x^*) - f(x^*) > \frac{\log n}{n^m} \qquad (n = n_1, n_2, n_3, \dots)$$

where S_n *is the n-th partial sum of the Fourier series of* $f(x)$.

We note that in [10] it was proved that for any $\{n_i\}$ there exist an $f(x) \in B^{[0, 2\pi)}$ and x^* such that

$$\overline{\lim_{i \to \infty}} \frac{S_{n_i}(f; x^*)}{\log n_i} > 1.^8$$

4.7. Now let us make some remarks for the complex case. If we consider $K_R = \{z; |z| < R\}$ instead of $\{a, b\}$ we can define $B_m^{K_R}$ as the set of functions $f(z)$ for which $f(z)$ is analytic in the open circle K_R and $f^{(m)}(z) \in B$ if z is on the contour.

Now if we use absolute value where it is necessary we can prove the complex analogy of Theorems 3.1 and 3.2.

Let us investigate the complex Lagrange interpolatory process on K_1:

$$L_n(f; z) = \sum_{k=1}^{n} f(z_{kn}) l_{kn}(z) \tag{4.7}$$

with

$$l_{kn}(z) = \frac{z^n + 1}{z_{kn} - 1} \frac{z_{kn}}{n}, \quad z_{kn} = e^{i \frac{2k-1}{2n} \pi}.$$

Then, as it is well known,

$$\lambda_n = \max_{|z| \leq 1} \sum_{k=1}^{n} |l_{kn}(z)| \sim \log n. \tag{4.8}$$

(see [12], 6 § 1). By (4.7) and (4.8) we can easily prove that

$$|L_n(f; z) - f(z)| \leq O(1) \log n \quad \text{if} \quad |z| \leq 1, \quad f(z) \in B^{k_1}.$$

On the other hand, with a new $g_n(x)$, we prove

THEOREM 4.6. *There exist an* $f(x) \in B^{K_1}$ *and a sequence* $0 < n_1 < n_2 < \dots$ *such that*

$$|L_n(f; 1) - f(1)| > \log n \qquad (n = n_1, n_2, n_3, \dots). \tag{4.9}$$

[8] We remark that now $S_n(g_n; x) \sim \log n$ for every n, so our theorem is an improvement indeed.

PROOF. By a simple computation we have

$$\sum_{k=1}^{n} |l_{kn}(1)| = \frac{2}{n} \sum_{k=1}^{n} \frac{1}{|z_{kn}-1|} > \frac{1}{n} \sum_{k=1}^{[n/4]} \frac{1}{\sin \frac{2k-1}{n}\pi} \sim \log n .$$

Now let

$$g_n(z_{kn}) = \begin{cases} e^{-i \arg l_{kn}(1)} & \text{if} \quad k \leqq \left[\frac{n}{4}\right], \\ 0 \quad \text{elsewhere.} \end{cases} \tag{4.8}$$

Then obviously

$$L_n(g_n; 1) \sim \log n, \quad |g_n(z)| \leqq 1 \quad \text{if} \quad |z| \leqq 1. \tag{4.9}$$

So we can prove A*. (Now $T_n = L_n$, $U_n = E$, $\lambda_n \sim \log n$, $e_n = 1$, $m = 0$.) To obtain B*, we have to apply the complex analogy of Lemma 4.2. For this aim let $n_k = 2^k$. Then $(\bigcup_k z_{kn_i}) \cap (\bigcup_l z_{ln_j}) = \varnothing$ if $i \neq j$ so we can apply the lemma.

To prove C*, we remark that $L_{n_k}(g_{n_i}; 1) = 0$ if $k \neq i$ further $g_n(1) = 0$ $(n = 1, 2, 3, \ldots)$.

So we perfectly proved our theorem applying Theorem 3.1.

4.7.1. We can prove (4.9) for any fixed z if $|z| \leqq 1$.

4.8. Now we consider the mechanical quadrature, more exactly the Gauss—Jacobi interpolatory quadrature formulae where for $\alpha, \beta > -1$

$$\begin{cases} T_n^{(\alpha,\beta)}(f) = Q_n^{(\alpha,\beta)}(f) = \sum_{k=1}^{n} A_{kn}^{(\alpha,\beta)} f(x_{kn}^{(\alpha,\beta)}), \\ U_n^{(\alpha,\beta)}(f) = \int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta} dx, \\ x_{kn}^{(\alpha,\beta)} = \cos \vartheta_{kn}^{(\alpha,\beta)} \sim \cos \dfrac{k\pi}{n} \qquad (k = 0, 1, \ldots, n+1; \ x_0 = 1, \ x_{n+1} = -1), \\ A_{kn}^{(\alpha,\beta)} = \int_{-1}^{1} l_{kn}^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta} dx > 0; \\ A_{kn}^{(\alpha,\beta)} \sim \vartheta_k^{2\alpha+1} n^{-1} \quad (0 < \vartheta_k^{(\alpha,\beta)} \leqq \pi - \varepsilon). \end{cases} \tag{4.10}$$

Here $l_{kn}^{(\alpha,\beta)}(x)$ are the fundamental Lagrange interpolatory polynomials of degree $n-1$ defined on $x_{1n}^{(\alpha,\beta)}, x_{2n}^{(\alpha,\beta)}, \ldots, x_{nn}^{(\alpha,\beta)}$ (see [7], 15.3 and [8]).

By a simple computation we have with $|f(x)| \leqq M$

$$\left| Q_n^{(\alpha,\beta)}(f) - \int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta} dx \right| \leqq M \sum_{k=1}^{n} A_{kn}^{(\alpha,\beta)} + 2M = 4M. \tag{4.11}$$

On the other hand, we obtain

THEOREM 4.7. *For any fixed* $\alpha, \beta > -1$ *there exist an* $f(x) \in B^{[-1,1]}$ *and* $0 < n_1 < n_2 < \ldots$ *such that*

$$Q_n^{(\alpha,\beta)}(f) - \int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta} dx > 1 \qquad (n = n_1, n_2, n_3, \ldots). \tag{4.12}$$

PROOF. For a fixed n_0 we define $g_{n_0}(x)$ as follows.

$$g_{n_0}(x_{kn_0}^{(\alpha,\beta)}) = \begin{cases} 1 & \text{if } \left[\frac{n_0}{4}\right] \leqq k \leqq \left[\frac{3n_0}{4}\right], \\ 0 & \text{elsewhere;} \end{cases}$$

further let

$$g_{n_{r+1}}(x_{kn_{r+1}}^{(\alpha,\beta)}) = \begin{cases} 1 & \text{if } \left[\frac{n_{r+1}}{4}\right] \leqq k \leqq \left[\frac{3n_{r+1}}{4}\right] \text{ and } x_{kn_{r+1}} \notin \bigcup_{\substack{1 \leqq k \leqq h_i \\ 0 \leqq i \leqq r}} x_{kn_i} \\ 0 & \text{elsewhere.} \end{cases}$$

Then $g_{n_{r+1}}$ equals to 1 at least in $\left(\left[\frac{n_{r+1}}{2}\right] - \sum_{i=0}^{r} n_r\right) \equiv D_r$ points. But we can choose n_{r+1} such that $D_r \geqq \frac{n_{r+1}}{4}$. So for $g_{n_k}(x)$

$$Q_{n_k}(g_{n_k}) \geqq \frac{1}{2}, \qquad Q_{n_k}(g_{n_i}) = 0 \quad (i \neq k), \quad U(g_{n_k}) \equiv 0.$$

Obviously $g_{n_i}(x) \cdot g_{n_j}(x) = 0$ $(i \neq j)$ for all $x \in [-1, 1]$, hence $\tilde{f} \in B^{[-1,1]}$. So we perfectly proved (4.12).

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References

[1] H. LEBESGUE, Sur la représentation trigonométrique approchée des fonctions satisfaisant a une condition de Lipschitz, *Bull. Soc. Math. France,* **38** (1910), pp. 184—210.
[2] K. I. OSKOLKOV, Improvement of an estimation of Lebesgue ..., *Trudi Matem. in.-ta im. Steklova,* **CXII** (1971), pp. 337—345 (Russian).
[3] P. O. H. VÉRTESI, On certain linear operators. VII, *Acta Math. Acad. Sci. Hung.,* **25** (1974), pp. 67—80.
[4] P. O. H. VÉRTESI, Lower estimations for some interpolating processes, *Studia Sci. Math. Hung.,* **5** (1970), pp. 401—410.
[5] A. H. TURECKI, *Interpolation Theory through Problems*, Minsk, 1968 (Russian).
[6] O. KIS, Remarks on the order of convergence of the Lagrange interpolation, *Annales Univ. Sci. Budapest,* **11** (1968), pp. 27—40 (Russian).
[7] G. SZEGŐ, *Orthogonal Polynomials* (New York, 1959).
[8] G. I. NATANSON, Two-sided estimate for the Lebesgue-function of the Lagrange interpolation with Jacobi nodes, *Izv. Vyss. Ucebn. Zaved. Matematika,* **11** (1967), pp. 67—74 (Russian).
[9] P. O. H. VÉRTESI, On certain linear operators. IV, *Acta Math. Acad. Sci. Hung.,* **23** (1972), pp. 115—125.
[10] K. I. OSKOLKOV, Subsequences of Fourier-sum of functions ..., *Mat. Sbor.,* **88 (130) 3 (7)** (1972), pp. 447—469 (Russian).
[11] N. I. KRYLOV, *Approximate Calculation of Integrals* (Macmillan Co. New York, 1962).
[12] V. I. SMIRNOV—N. A. LEBEDEV, Constructive theory of functions of complex variable, Moscow—Leningrad, 1964 (Russian).
[13] A. F. TIMAN, *Theory of approximation of functions of a real variable* (Moscow, 1960) (Russian).

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Tomus 25 (1—2), (1974), pp. 189—212.

BAIRE 1 FUNCTIONS, APPROXIMATELY CONTINUOUS FUNCTIONS AND DERIVATIVES

By

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Dedicated to Professor Ákos Császár on the occasion of his 50th birthday

1. Introduction

The main problems dealt with in this paper are to find the subsets $H \subset [0, 1]$ such that every Baire 1 function restricted to H can be extended to $[0, 1]$ resulting in

(i) an approximately continuous function,

(ii) a derivative.

In both cases $\lambda(H)=0$ is the necessary and sufficient condition. However, in case (i) the matter can be settled by topological methods since the class of approximately continuous functions is exactly the class of continuous functions if $[0, 1]$ is endowed with a suitable topology. No such topology exists for derivatives. Hence our argument is more complicated in the discussion of (ii). Among these results some applications and problems are stated as well.[1]

2. Preliminary remarks and notations

Let $H \subset [0, 1]$. As usual, we denote by $\mathscr{C}(H)$ the class of continuous functions defined on H.

DEFINITION 2.1 ([1]). A finite valued real function $f(x)$ defined on H is said to belong to the first class of Baire $(f \in \mathscr{B}^1(H))$ if it is the pointwise limit of a sequence from $\mathscr{C}(H)$. For the sake of brevity we put $\mathscr{B}^1 = \mathscr{B}^1([0, 1])$.

All the properties of Baire 1 functions used without proof can be found in any textbook containing the Baire classification (see e.g. [1]). The most important is the characterization by the level-sets:

$f \in \mathscr{B}^1$ if and only if $\{f<c\}$ and $\{f>c\}$ are of type F_σ for every real c (or, equivalently $\{f \leqq c\}$ and $\{f \geqq c\}$ are G_δ sets for every c).

Another remarkable fact is that the set of points of continuity of $f \in \mathscr{B}^1$ is everywhere dense in $[0, 1]$.

DEFINITION 2.2 ([2]). A finite valued real function $f(x)$ defined on $[0, 1]$ is said to be approximately continuous (app. c.) in $x_0 \in [0, 1]$ if, for every $\varepsilon>0$, the set $\{x: |f(x)-f(x_0)|<\varepsilon\}$ has inner density 1 at x_0.

[1] *Added in proof (February 1, 1974).* After our paper was sent to the print we have learned from a letter of Prof. J. S. Lipiński that, the problem whether every Baire 1 function restricted to a set of measure zero has approximately continuous extension was raised also by him. He kindly called our attention to the fact that our Theorem 3.2 implies a positive answer to another problem published by him: assuming the continuum hypothesis our extension theorem implies that every Baire 1 function is the limit of a transfinite sequence of approximately continuous functions (see J. S. LIPIŃSKI, On transfinite sequences of approximately continuous functions, *Bulletin de l'Acad. Pol. des Sci.*, **XXI, 9** (1973), pp. 817—821).

(In the definition of the density at the end points 0 and 1 the one sided neighbourhoods are to be considered only.) $f(x)$ is called approximately continuous on $[0, 1]$ if it is app. c. in every $x_0 \in [0, 1]$, and the class of such functions is denoted by $\mathscr{A}$.

DEFINITION 2.3 ([3]). A set $H \subset [0, 1]$ is said to be D-open if H has inner density 1 in every $x \in H$.

It is known ([3]) that the D-open sets form a topology (denoted usually also by D) on $[0, 1]$ and $f \in \mathscr{A}$ if and only if f is D-continuous on $[0, 1]$. It is also known that the D-open sets are measurable as well. This obviously implies that the elements of $\mathscr{A}$ are measurable functions. Moreover the inclusion

$$\mathscr{A} \subset \mathscr{B}^1 \tag{2.1}$$

holds too ([4]). It should be realized that every set of measure zero is D-closed.

The notations below are frequently used in the sequel.

— $\mathscr{D}(H)$ denotes the family of finite valued derivatives defined on H, i.e. $f \in \mathscr{D}(H)$ if and only if there exists $F(x)$ defined on H with

$$f(x) = \lim_{\substack{t \to x \\ t \in H}} \frac{F(t) - F(x)}{t - x} \quad (x \in H).$$

Hence $\mathscr{D} = \mathscr{D}([0, 1])$ is the set of ordinary derivatives.

— $f|_H(x)$ denotes the restriction of f to H. If $\mathscr{F}$ is any family of functions then

$$\mathscr{F}_H = \{f|_H : f \in \mathscr{F}\}$$

and

$$b\mathscr{F} = \{f : f \in \mathscr{F},\ f \text{ is bounded}\}.$$

($\mathscr{D}(H)$ and $\mathscr{D}_H$, $\mathscr{B}^1(H)$ and $\mathscr{B}^1_H$ should be carefully distinguished. Some relations between these families will be discussed later.)

— $\lambda(H)$ denotes Lebesgue's (outer) measure of $H \subset [0, 1]$; $\operatorname{cl} H$ and $\operatorname{int} H$ are the closure and the interior of H respectively; $\overline{H}$ denotes the complement $\overline{H} = [0, 1] \setminus H$.

Finally we remark that $\mathscr{D} \subset \mathscr{B}^1$ is obvious and $b\mathscr{A} \subset \mathscr{D}$ follows immediately by differentiation of the integral of $f \in b\mathscr{A}$.

3. Approximately continuous functions

This section contains our result concerning app. c. functions. The proof relies upon an important theorem of ZAHORSKI. This is

LEMMA 3.1 ([5], p. 29, Lemme 12). *If $H_1, H_2 \subset [0, 1]$ are G_δ and D-closed sets, and $H_1 \cap H_2 = \varnothing$, then there exists a function $f \in \mathscr{A}$ such that $f(x) = 0$ $(x \in H_1)$, $f(x) = 1$ $(x \in H_2)$ and $0 < f(x) < 1$ $(x \notin H_1 \cup H_2)$.*

(In the work [5] the F_σ and D-open sets are called M_5 sets.)

THEOREM 3.2. *The conditions*

(i) $\lambda(H) = 0$,

(ii) $b\mathscr{A}_H = b\mathscr{B}^1_H$,

(iii) $\mathscr{A}_H = \mathscr{B}^1_H$

are equivalent to each other.

We remark that $f \in \mathscr{A}$ ($f \in \mathscr{B}^1$) implies

$$f_n = \min(\max(f, -n), n) \in \mathscr{A}$$

($f_n \in \mathscr{B}^1$). Hence $(b\mathscr{A})_H = b\mathscr{A}_H$ and $(b\mathscr{B}^1)_H = b\mathscr{B}^1_H$, i.e. (ii) could be replaced by $(b\mathscr{A})_H = (b\mathscr{B}^1)_H$.

PROOF. (i)⇒(ii). Suppose $f \in b\mathscr{B}^1$, say $-1 \leqq f \leqq 1$. We are to prove $f|_H \in b\mathscr{A}_H$. If $\lambda(H)=0$ then there exists $H' \supset H$ such that H' is of type G_δ and also $\lambda(H')=0$. If $f|_{H'}$ has an extension in $b\mathscr{A}$ then a fortiori so does $f|_H$, thus we can suppose that H is a G_δ-set. Now we apply the method widely known from the proof of the general Tietze extension theorem. Put

$$H_1^* = \left\{x : f(x) \leqq -\frac{1}{3}\right\}, \quad H_2^* = \left\{x : f(x) \geqq \frac{1}{3}\right\}.$$

Being level sets of a Baire 1 function H_1^* and H_2^* are G_δ sets and clearly $H_1^* \cap H_2^* = \varnothing$. Hence the sets $H_1 = H \cap H_1^*$, $H_2 = H \cap H_2^*$ are also disjoint G_δ sets and being of measure zero they are D-closed too. By Lemma 3.1 we can choose a separating function $\varphi_1 \in \mathscr{A}$ with

$$\varphi_1(x) = -\frac{1}{3} \quad (x \in H_1), \quad \varphi_1(x) = \frac{1}{3} \quad (x \in H_2), \quad -\frac{1}{3} \leqq \varphi_1(x) \leqq \frac{1}{3} \quad \text{otherwise.}$$

Putting $f_1 = f - \varphi_1$ we have

$$-\frac{2}{3} \leqq f_1|_H \leqq \frac{2}{3}.$$

Suppose that $\varphi_1, \ldots, \varphi_n$ have already been defined such that

$$\varphi_k \in \mathscr{A}, \qquad |\varphi_k| \leqq \frac{1}{3}\left(\frac{2}{3}\right)^{k-1} \qquad (k = 1, \ldots, n)$$

and, for $f_n = f - \sum_{k=1}^{n} \varphi_k$,

$$-\left(\frac{2}{3}\right)^n \leqq f_n|_H \leqq \left(\frac{2}{3}\right)^n \tag{3.1}$$

holds. Applying Lemma 3.1 let φ_{n+1} be an app. c. function separating the sets

$$H_1^n = H \cap \left\{x : f_n(x) \leqq -\frac{1}{3}\left(\frac{2}{3}\right)^n\right\}, \quad H_2^n = H \cap \left\{x : f_n(x) \geqq \frac{1}{3}\left(\frac{2}{3}\right)^n\right\}$$

such that

$$\varphi_{n+1}(x) = \begin{cases} -\frac{1}{3}\left(\frac{2}{3}\right)^n & (x \in H_1^n), \\ \frac{1}{3}\left(\frac{2}{3}\right)^n & (x \in H_2^n), \end{cases}$$

$$|\varphi_{n+1}(x)| \leqq \frac{1}{3}\left(\frac{2}{3}\right)^n \quad \text{otherwise.}$$

By this construction we obviously have

$$-\left(\frac{2}{3}\right)^{n+1} \leqq f_{n+1}|_H \leqq \left(\frac{2}{3}\right)^{n+1}$$

where $f_{n+1} = f_n - \varphi_{n+1}$. We obtain the sequence $\varphi_1, \ldots, \varphi_n, \ldots$ by induction and put

$$g(x) = \sum_{k=1}^{\infty} \varphi_k(x). \tag{3.2}$$

Since $|\varphi_k| \leqq \frac{1}{3}(\frac{2}{3})^{k-1}$ $(k=1, 2, \ldots)$, the series in (3.2) is uniformly convergent, hence $g \in b\mathscr{A}$, $|g| \leqq 1$. By (3.1) we have $g|_H = f|_H$, i.e. $b\mathscr{A}_H \ni f|_H$. Thus $b\mathscr{B}_H^1 \subset b\mathscr{A}_H$ is proved. $b\mathscr{A}_H \subset b\mathscr{B}_H^1$ is obvious from the remark (2.1) made after Definition 2.3, thus (ii) is verified.

(ii)$\Rightarrow$(i). Suppose indirectly $\lambda(H) > 0$. According to Lebesgue's density theorem all the points of H are of outer density 1 apart from a subset of measure zero. Put

$$f(x) = \begin{cases} 1 & (x = x_0), \\ 0 & (x \neq x_0) \end{cases}$$

where in $x_0 \in H$ the outer density of H is equal to 1. $f \in b\mathscr{B}^1$ is obvious and it is also obvious that $f|_H$ has no extension in $\mathscr{A}$.

(ii)$\Rightarrow$(iii). By the preceding step we have $\lambda(H) = 0$. Without loss of generality we can assume again that H is of type G_δ. Let $f \in \mathscr{B}^1$ and denote $f^* = \operatorname{arc\,tg} f$. Obviously $f^* \in b\mathscr{B}^1$. By (ii) $f^*|_H$ can be extended to an app. c. function $g^* \in b\mathscr{A}$ (actually we need only $g^* \in \mathscr{A}$). The set

$$H_3 = \left\{x : |g^*(x)| \geqq \frac{\pi}{2}\right\}$$

is G_δ and D-closed because of $g^* \in \mathscr{A} \subset \mathscr{B}^1$. By the definition of f^* we have $H_3 \cap H = \varnothing$. Referring to Lemma 3.1 there exists a function $h \in \mathscr{A}$ such that

$$h(x) = \begin{cases} 1 & (x \in H), \\ 0 & (x \in H_3), \end{cases}$$

$$0 < h(x) < 1 \quad \text{otherwise.}$$

Hence $h(x)g^*(x) \in \mathscr{A}$, $-\frac{\pi}{2} < h(x)g^*(x) < \frac{\pi}{2}$ and

$$h(x)g^*(x) = f^*(x) \quad (x \in H).$$

Now $g(x)=\operatorname{tg}\big(h(x)g^*(x)\big)$ provides the desired extension of $f|_H$. Hereby we obtain $\mathscr{B}_H^1\subset\mathscr{A}_H$, i.e. $\mathscr{B}_H^1=\mathscr{A}_H$.

The proof of (ii)$\Rightarrow$(i) gives also (iii)$\Rightarrow$(i) hence the proof of the theorem is completed.

4. Derivatives

In this section we develop another kind of extension method called "strip" method. By a strip we mean a pair of continuous functions $s_1(x)\leqq s_2(x)$. The strip method is based upon the following simple observation: if $s_1(x_0)=s_2(x_0)$ and $s_1'(x_0)=s_2'(x_0)$ then any function $s(x)$ between s_1 and s_2 also has a derivative at x_0 and $s'(x_0)=s_1'(x_0)$. The easy proofs of the following simple lemmas are left to the reader.

LEMMA 4.1. *If $g(x)$ and $h(x)$ are continuous in (a, b) (in $[a, b]$) and $g(x)<h(x)$ for $a<x<b$ (for $a\leqq x\leqq b$), then there exists a function $F(x)$ continuously differentiable in (a, b) ($[a, b]$) such that $g(x)<F(x)<h(x)$. If furthermore in the case of $[a, b]$ $g(a)<A<h(a)$ $\big(g(b)<B<h(b)\big)$ holds then $F(x)$ can be chosen to fulfil arbitrary initial conditions $F(a)=A$, $F'(a)=\alpha$ $\big(F(b)=B, F'(b)=\beta\big)$.*

LEMMA 4.2. *If $g(x)\leqq f(x)\leqq h(x)$ holds for $x_0-\delta<x<x_0+\delta$ and $g(x_0)=h(x_0)$, $g'(x_0)=h'(x_0)$ then $f'(x_0)$ also exists and $f'(x_0)=g'(x_0)$.*

LEMMA 4.3. *For an arbitrary subset $\emptyset\neq E\subset(-\infty, +\infty)$ we denote*

$$h(x, E) = [\inf\{|x-y| : y\in E\}]^2. \tag{4.1}$$

Then $h(x)=h(x, E)$ is continuous. Moreover it has a derivative in the points of cl E *and*

$$h(x) = h'(x) = 0 \quad (x\in \operatorname{cl} E).$$

LEMMA 4.4. *Suppose that $H\subset(a, b)$ ($H\subset[a, b]$) is nowhere dense and closed. If $f\in\mathscr{D}_H$ and $g(x)<h(x)$ are continuous on (a, b) (on $[a, b]$) then there exists a primitive $F(x)$ such that*

$F'(x)=f(x)$ $(x\in H)$,
$F'(x)$ *is continuous in* $\overline{H}=(a, b)\setminus H$ $(\overline{H}=[a, b]\setminus H)$,
$g(x)<F(x)<h(x)$.

PROOF. If H is empty then the assertion reduces to that of Lemma 4.1. Hence we suppose $H\neq\emptyset$. We consider first the case of the open interval (a, b). Let f^* be a derivative in (a, b) with $f^*|_H=f$ and denote by $G_1(x)$ an arbitrary primitive of f^*. Apply Lemma 4.1 to the functions $G_1(x)$ and $G_1(x)+h(x, H)$ on every interval contiguous to H. Hence we obtain a function $G(x)$ continuously differentiable on $\overline{H}=(a, b)\setminus H$ and $G_1(x)<G(x)<G_1(x)+h(x, H)$ $(x\in\overline{H})$. Let $G(x)=G_1(x)$ for $x\in H$, then by Lemmas 4.2 and 4.3, $G'(x)=G_1'(x)=f(x)$ $(x\in H)$. Choose now a subinterval $[c, d]\subset(a, b)$ with $H\subset(c, d)$ and denote

$$m = \min_{c\leqq x\leqq d}\big(h(x)-g(x)\big) > 0.$$

By the continuity of $G(x)$ and $d(x)=\frac{1}{2}(g(x)+h(x))$ we can find $\delta>0$ such that

$$|(G(y)-d(y))-(G(x)-d(x))|<\frac{m}{2} \quad (x, y\in[c, d], |x-y|<\delta). \tag{4.2}$$

We then take a finite system of open intervals $I_k=(a_k,b_k)$ $(k=1, \ldots, n)$ such that

$$b_k-a_k<\delta, \quad c\leqq a_k<b_k\leqq d \qquad (k=1, \ldots, n),$$

$$H\subset\bigcup_{k=1}^{n} I_k, \quad [a_k, b_k]\cap[a_j, b_j]=\varnothing \quad (k\neq j).$$

Put

$$F(x)=G(x)-G(a_k)+d(a_k) \quad (a_k\leqq x\leqq b_k), \qquad k=1, \ldots, n.$$

If $x\in\bigcup_{k=1}^{n} I_k$ then $F'(x)=G'(x)$, in particular $F'(x)=f(x)$ $(x\in H)$. By (4.2) the inequality $g(x)<F(x)<h(x)$ is obvious on $\bigcup_{k=1}^{n}\operatorname{cl} I_k$. To give a suitable definition of $F(x)$ on the remaining parts of (a, b) we refer the reader to Lemma 4.1.

Now we prove the assertion when the underlying interval is closed. Suppose first $H\subset(a, b)$. Then we take $a<c<d<b$ such that $H\subset[c, d]$ and construct $F_1(x)$ which is continuously differentiable in $(a, b)\setminus H$ and satisfies $g(x)<F_1(x)<h(x)$ in (a, b), $F_1'(x)=f(x)$ for $x\in H$, and finally we modify $F_1(x)$ in $[a, c]$ and $[d, b]$ by means of Lemma (4.1) in order to obtain the function $F(x)$ desired. In this case it is also possible to prescribe the values $F(a)$, $F'(a)$, $F(b)$, $F'(b)$ (fulfilling of course the conditions $g(a)<F(a)<h(a)$, $g(b)<F(b)<g(b)$). Let us now drop the condition $H\subset(a, b)$. Assume $a, b\in H$. Since H is nowhere dense in $[a, b]$ we can choose the sequences $a_1>a_2>\ldots$ and $a_1<b_1<b_2<\ldots$ such that $a_n\to a$, $b_n\to b$, $a_n, b_n\notin H$. Let $E=\{a, b\}$. Applying Lemma 4.1 we can find $F_1(x)$ such that $F_1(x)$ is continuously differentiable in $[a, b]$, $g(x)<F_1(x)<h(x)$ $(x\in[a, b])$, $F_1'(a)=f(a)$, $F_1'(b)=f(b)$. If $\delta>0$ is small enough then $F_1(x)+\delta h(x, E)<h(x)$ holds in $[a, b]$ and now we apply the case proved above for each of the intervals

$$[a_1, b_1], [a_{n+1}, a_n], [b_n, b_{n+1}] \qquad (n=1, 2, \ldots)$$

and the functions

$$g(x):=F_1(x), \quad h(x):=F_1(x)+\delta h(x, E)$$

by prescribing the value 0 for the derivative $F'(x)$ at the extremities of the intervals in question. Hence a function $F(x)$ is obtained in (a, b) and supplying its definition by $F(a)=F_1(a)$, $F(b)=F_1(b)$ we obtain our required function.

If $a\in H$, $b\notin H$ or $a\notin H$, $b\in H$ then a combination of the above methods furnishes the function required.

The next theorem is one of the main tools in the proof of our extension theorem on derivatives.

THEOREM 4.5. *Let $E_k\subset[0, 1]$ be nowhere dense and closed $(k=1, \ldots)$, $E_k\cap E_j=\varnothing$ $(k\neq j)$. If $f_k\in\mathscr{D}_{E_k}$ $(k=1, 2, \ldots)$ then there exists a continuous function $F(x)$ in $[0, 1]$ with*

$$F'(x)=f(x) \quad (x\in E)$$

where $E=\bigcup_{k=1}^{\infty} E_k$ *and* f *is defined on* E *by* $f|_{E_k}=f_k$. *If furthermore* E *is closed then* $f\in\mathscr{D}_E$, *i.e. in this case* $F(x)$ *can be chosen to have derivative everywhere in* $[0, 1]$, *moreover the derivative* $F'(x)$ *is continuous in* $\bar{E}=[0, 1]\setminus E$.

PROOF. The proof is carried out by the strip method. We are going to define a sequence $F_1, \ldots, F_n, \ldots$ by induction. Let $F_1(x)$ denote an arbitrary primitive of f_1^*, where $f_1^*\in\mathscr{D}$ and $f_1^*|_{E_1}=f_1$. Suppose that $F_n(x)$ has already been defined with the properties

(i) $F_n(x)$ is continuous in $[0, 1]$,

(ii) $F_n'(x)$ exist on $A_n=\bigcup_{k=1}^{n} E_k$ and $F_n'(x)=f_k(x)\quad (x\in E_k,\ 1\leqq k\leqq n)$.

Put $F_n^*(x)=F_n(x)+\frac{1}{2^{n+1}}h(x, A_n)$ (see the definition of $h(x, \cdot)$ in (4.1)). Let $\{(a_k, b_k)\}_{k=1}^{\infty}$ denote the sequence of intervals contiguous to the closed set A_n ($[0, b_k)$ or $(a_k, 1]$ may occur here, too). Applying Lemma 4.4 with

$$H=E_{n+1}\cap(a_k, b_k)\subset(a_k, b_k),\quad f=f_{n+1}|_H,\quad g(x)=F_n(x),\quad h(x)=F_n^*(x),$$

we obtain a differentiable function $F_{n+1}^{(k)}$ on (a_k, b_k) with

$$(F_{n+1}^{(k)})'=f_{n+1}\quad (x\in E_{n+1}\cap(a_k, b_k)). \tag{4.3}$$

Now we define

$$F_{n+1}(x)=\begin{cases}F_n(x) & (x\in A_n),\\ F_{n+1}^{(k)}(x) & (x\in(a_k, b_k),\ k=1, 2, \ldots).\end{cases}$$

Making use of

$$F_n(x)\leqq F_{n+1}(x)\leqq F_n(x)+\frac{1}{2^{n+1}}h(x, A_n)\leqq F_n(x)+h(x, A_n) \tag{4.4}$$

by Lemma 4.3 and 4.2 we get at once

$$F_{n+1}'(x)=F_n'(x)\quad (x\in A_n).$$

Hence by (4.3) the conditions (i) and (ii) are verified also for F_{n+1}. Observe that $0\leqq F_{n+1}-F_n\leqq\frac{1}{2^{n+1}}$ $(n=1, 2, \ldots)$. Hence $F_n(x)\to G(x)$ uniformly holds on $[0, 1]$. In particular $G(x)$ is continuous as well. We have to prove $G'(x)=f(x)$ $(x\in E)$. $A_{n+1}\supset A_n$ obviously implies $h(x, A_{n+1})\leqq h(x, A_n)$ $(n=1, 2, \ldots)$. Hence by repeated use of

$$F_n\leqq F_{n+1}\leqq F_n+\frac{1}{2^{n+1}}h(x, A_n)$$

we obtain

$$F_n\leqq F_{n+k}\leqq F_n+\sum_{m=n+1}^{n+k}\frac{1}{2^m}h(x, A_{m-1})\leqq F_n+h(x, A_n)\sum_{m=n+1}^{\infty}\frac{1}{2^m}\leqq F_n+h(x, A_n).$$

By $k \to +\infty$

$$F_n \leqq G \leqq F_n + h(x, A_n)$$

holds. Referring to Lemmas 4.3, 4.2 and (ii) we have

$$G'(x) = F_n'(x) = f(x) \qquad (x \in A_n),\ n = 1, 2, \ldots.$$

Taking $F(x) = G(x)$ the first part of our assertion is completely proved.

If E is closed then consider the continuous functions $G(x)$ and $G(x)+h(x, E)$ and apply Lemma 4.1 for every interval contiguous to E. This gives a continuously differentiable function G_1 between G and $G+h(x, E)$ defined on $[0, 1] \setminus E$, and let

$$F(x) = \begin{cases} G(x) & (x \in E), \\ G_1(x) & (x \notin E). \end{cases}$$

By Lemmas 4.3 and 4.2 $F(x)$ has all the required properties.

COROLLARY 4.6. *Let* E_k *and* $E = \bigcup_{k=1}^{\infty} E_k$ *be closed and nowhere dense in* $[0, 1]$, $E_k \cap E_j = \varnothing$ *for* $k \neq j$. *If* $f(x)$ *is defined on* E *and* $f|_{E_k}$ *is continuous for* $k = 1, 2, \ldots$ *then* $f \in \mathscr{D}_E$.

PROOF. Since $f|_{E_k}$ is continuous and E_k is closed there exists a continuous function f_k^* defined on $[0, 1]$ for which $f|_{E_k} = f_k^*|_{E_k}$, hence $f|_{E_k} \in \mathscr{D}_{E_k}$. Applying the second part of Theorem 4.5 we have $f \in \mathscr{D}_E$.

COROLLARY 4.7. *If* $f|_{E_k}$ *is constant for* $k = 1, 2, \ldots$ *(and* E_k *satisfies the conditions as above) then* $f \in \mathscr{D}_E$.

LEMMA 4.8. *Let* $E \subset [0, 1]$ *be a nowhere dense closed set and* $f \in \mathscr{B}_E^1$. *Then for every* $\varepsilon > 0$ *there exists a disjoint decomposition* $E = \bigcup_{k=1}^{\infty} E_k$ *of* E *by closed sets* E_k *and a function* $f_\varepsilon(x)$ *defined on* E *such that*

$$|f(x) - f_\varepsilon(x)| < \varepsilon \quad (x \in E)$$

and $f_\varepsilon|_{E_k}$ *is constant* $(k = 1, 2, \ldots)$.

PROOF. Put $H_i = \{x \in E : (i-1)\varepsilon < f(x) < (i+1)\varepsilon\}$. Since $f \in \mathscr{B}_E^1$ and E is closed the sets H_i are of type F_σ and obviously $E = \bigcup_{i=-\infty}^{\infty} H_i$.

It is an elementary fact that for an arbitrary countable union $Y = \bigcup_{j=1}^{\infty} Y_j$ of F_σ-sets there exist F_σ subsets $Y_j' \subset Y_j$ such that $Y = \bigcup_{j=1}^{\infty} Y_j'$ is a disjoint decomposition of Y. Applying this remark to our case we obtain

$$E = \bigcup_{j=-\infty}^{\infty} H_j', \quad H_j' \subset H_j, \quad H_j' \cap H_k' = \varnothing$$

for $j \neq k$ and the H_j' are of type F_σ.

Let $f_\varepsilon(x)=i\varepsilon$ $(x\in H_i')$. Then $|f(x)-f_\varepsilon(x)|<\varepsilon$ trivially holds on E. Since $f_\varepsilon|_{H_i'}$ is constant we have to prove that H_i' can be given in the form $H_i'=\bigcup_{k=1}^{\infty} E_i^k$ where E_i^k are pairwise disjoint closed sets $(-\infty<i<\infty)$.

Let H be an arbitrary nowhere dense F_σ-set. Then $H=\bigcup_{k=1}^{\infty} R_k$ where R_k is closed and put

$$R_k' = R_k \setminus \bigcup_{l=1}^{k-1} R_l.$$

Then $H=\bigcup_{k=1}^{\infty} R_k'$, $R_k'\cap R_j'=\varnothing$ for $k\neq j$ and $R_k'=R_k\cap G_k$, where G_k is open. Let $G_k=\bigcup_{l=1}^{\infty}(a_l, b_l)$, where the open intervals (a_l, b_l) are pairwise disjoint.

Then $R_k'=\bigcup_{l=1}^{\infty}(R_k\cap(a_l, b_l))$. We are going to give a decomposition of $R_k\cap(a_l, b_l)$ into disjoint closed sets.

Since R_k is a nowhere dense set there exist two sequences $\{x_n\}$ and $\{y_n\}$ from $(a_l, b_l)\setminus R_k$ such that

$$x_n \searrow a_l, \quad y_n \nearrow b_l, \quad x_1 < y_1.$$

Then

$$R_k\cap(a_l, b_l) = \bigcup_{n=1}^{\infty}(R_k\cap[x_{n+1}, x_n])\cup(R_k\cap[x_1, y_1])\cup\bigcup_{n=1}^{\infty}(R_k\cap[y_n, y_{n+1}])$$

is the required decomposition. This proves the lemma.

We note that (0, 1) is an F_σ-set but it cannot be represented as a countable union of disjoint closed sets. This shows that it was essential to suppose for E to be nowhere dense.

COROLLARY 4.9. *Let $E\subset[0, 1]$ be a nowhere dense closed set and $f\in\mathring{\mathscr{B}}_E^1$. Then f is a uniform limit of a sequence from $\mathscr{D}_E$.*

PROOF. $f(x)=\lim_{n\to\infty} f_{1/n}(x)$ where $f_{1/n}$ is the function constructed for $\varepsilon=\frac{1}{n}$ in Lemma 4.8. By Corollary 4.7 $f_{1/n}\in\mathscr{D}_E$.

Our next theorem may have some interest in itself. In particular it implies that the primitives of the so called Pompeiu functions ([6]) are everywhere dense in $\mathscr{C}([0, 1])$ ($\mathscr{C}([a, b])$ is supposed to be endowed with the sup-norm topology).

THEOREM 4.10. *Let $H\subset[a, b]$ be measurable and $\mathscr{F}(H)=\{f: f'(x)$ exists for $x\in[a, b]$ and $f'(x)=0$ $(x\in H)\}$. $\mathscr{F}(H)$ is everywhere dense in $\mathscr{C}([a, b])$ if and only if $\bar{H}=[a, b]\setminus H$ is metrically dense in $[a, b]$, i.e. $\lambda(\bar{H}\cap I)>0$ for every subinterval $I\subset[a, b]$.*

PROOF. Suppose that $\mathscr{F}(H)$ is everywhere dense in $\mathscr{C}([a, b])$ and let $I\subset[a, b]$ be an arbitrary subinterval. Let $2\varepsilon<\lambda(I)$ and choose $f\in\mathscr{F}(H)$ such that

$$|f(x)-x|<\varepsilon \quad (x\in[a, b]).$$

Then $f(x)$ is not constant in I. It is known ([2]) that if the derivative of an everywhere differentiable function vanishes almost everywhere in an interval I then the function is constant on I. Hence

$$\lambda(\overline{H} \cap I) \geqq \lambda(\{x : x \in I,\ f'(x) \neq 0\}) > 0,$$

i.e. $\overline{H}$ is metrically dense in $[a, b]$.

As a first step in the converse direction we are going to prove that if $\overline{H}$ is metrically dense in $[a, b]$ and the real numbers y_1, y_2 are given in advance then $\mathscr{F}(H)$ contains an element $f \in \mathscr{F}(H)$ such that f is *monotone* in $[a, b]$, $f(a)=y_1$, $f(b)=y_2$, $f'(a)=f'(b)=0$. Let H_1 denote the D-closure of H. By Lebesgue's density theorem $\lambda(H_1)=\lambda(H)$. Let $K \supset H_1$ be a G_δ set with $\lambda(K)=\lambda(H_1)$. If necessary we replace K by $K \cup \{a\} \cup \{b\}$. By $\lambda(K \setminus H_1)=0$ the set K is also D-closed and by $\lambda(K \setminus H)=0$ the set K is also metrically dense in $[a, b]$. In particular $\lambda(K) < b-a$. Choose now an arbitrary point $t \in (a, b) \setminus K$ and apply Lemma 3.1 to the sets K and $\{t\}$. Hence we obtain a function $g \in b\mathscr{A}$ such that

$$g(x) = 0 \quad (x \in K), \qquad g(t) = 1, \quad 0 < g(x) < 1 \text{ otherwise.} \tag{4.5}$$

Since $g \in b\mathscr{A}$ the function $f_1(x) = \int_a^x g(u)\,du$ is differentiable in $[a, b]$ and $f_1'(x)=g(x)$ $(x \in [a, b])$. Obviously $f_1(x)$ is not constant $(f_1'(t)=1)$ and increasing $(f_1'(x) \geqq 0)$. Thus

$$f(x) = y_1 + (y_2 - y_1) f_1(x) \frac{1}{f_1(b)}$$

satisfies all the required conditions.

Now we can easily complete the proof. Suppose that $\overline{H}$ is metrically dense in $[a, b]$, let $g \in \mathscr{C}([a, b])$ and $\varepsilon > 0$ be given. We choose $\delta > 0$ so small that $|x-y| < \delta$ implies $|g(x)-g(y)| < \frac{\varepsilon}{2}$ and then we divide $[a, b]$ by $a_0 = x_0 < x_1 < \ldots < x_n = b$ such that $|x_i - x_{i-1}| < \delta$ $(i=1, \ldots, n)$. Apply the preceding construction to every subinterval $[x_{i-1}, x_i]$. This yields monotone functions $f_i(x)$ such that

$$f_i(x_{i-1}) = g(x_{i-1}), \quad f_i(x_i) = g(x_i), \quad f_i'(x_{i-1}) = f_i'(x_i) = 0$$

and even $f_i'(x)=0$ $(x \in H \cap [x_{i-1}, x_i])$. Then taking $f(x)=f_i(x)$ if $x \in [x_{i-1}, x_i]$ for $x \in [a, b]$ we have $f \in \mathscr{F}(H)$ and

$$|f(x)-g(x)| \leqq |f_i(x)-f_i(x_{i-1})| + |f_i(x_{i-1})-g(x)| \leqq$$

$$\leqq |f_i(x_i)-f_i(x_{i-1})| + |f_i(x_{i-1})-g(x)| =$$

$$= |g(x_i)-g(x_{i-1})| + |g(x_{i-1})-g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where i is chosen to fulfil $x \in [x_{i-1}, x_i]$. Hence the proof is complete and even $f(a)=g(a)$, $f(b)=g(b)$, $f'(a)=f'(b)=0$ holds too.

COROLLARY 4.11. *If for* $H\subset[a, b]$ $\overline{H}$ *is metrically dense and measurable and the continuous functions* $g(x)$, $h(x)$ *satisfy* $g(x)<h(x)$ $(x\in[a, b])$ *then there exists* $F\in\mathscr{F}(H)$ *with* $g(x)<F(x)<h(x)$ *and*

$$F(a)=\frac{1}{2}(g(a)+h(a)),\quad F(b)=\frac{1}{2}(g(b)+h(b)),\quad F'(a)=F'(b)=0.$$

PROOF. This is obvious by the preceding proof if

$$\left|F(x)-\frac{1}{2}(g(x)+h(x))\right|<\varepsilon$$

and $\varepsilon>0$ is small enough.

COROLLARY 4.12. *Let* $H\subset(a, b)$, $\overline{H}=(a, b)\setminus H$ *be metrically dense in* (a, b), $g, h\in\mathscr{C}((a, b))$, $g(x)<h(x)$ $(x\in(a, b))$. *Then there exists a differentiable function in* (a, b) *such that*

(i) $F'(x)=0$ $(x\in H)$,

(ii) $g(x)<F(x)<h(x)$.

PROOF. Consider an increasing sequence of intervals

$$I_n=[a_n, b_n]\subset(a_{n+1}, b_{n+1})$$

with $a_n\to a$, $b_n\to b$. Being $\overline{H}=(a, b)\setminus H$ metrically dense in (a, b), we can apply Corollary 4.11 for every closed subinterval $[c, d]\subset(a, b)$. Denote $[c, d]$ any of the intervals $[a_1, b_1]$, $[b_n, b_{n+1}]$, $[a_{n+1}, a_n]$ $(n=1, 2, \ldots)$ and apply Corollary 4.11. The initial conditions

$$F(c)=\frac{1}{2}(g(c)+h(c)),\quad F(d)=\frac{1}{2}(g(d)+h(d)),\quad F'(c)=F'(d)=0$$

make the definition of $F(x)$ unambiguous all over the interval (a, b).

Now we are going to state and prove our principal results on derivatives. Analogously to Theorem 3.2 $\lambda(H)=0$ is necessary and sufficient for $\mathscr{B}_H^1=\mathscr{D}_H$. Before giving the details it is worthwhile to make some comparison with the case of app. c. functions. If $f\in b\mathscr{B}_H^1$ then $f\in\mathscr{D}_H$ is trivially implied by Theorem 3.2 because of $b\mathscr{A}\subset\mathscr{D}$ $(\lambda(H)=0$ is assumed). Lemma 4.13 below gives a stronger result where local boundedness is supposed only. As we saw the unbounded case had a very natural reduction to the bounded one when we wanted the extension to belong to $\mathscr{A}$. This reduction cannot be applied now since $f\in\mathscr{D}$ no longer implies arc tg $f\in\mathscr{D}$. This is the point where Corollary 4.9 contributes essentially to the solution: if H is perfect then $f\in\mathscr{B}_H^1$ can be approximated uniformly by $g\in\mathscr{D}_H$. Hence $f-g\in b\mathscr{B}_H^1$ and now Theorem 3.2 applies. The technicalities involved in the proof are to overcome the difficulties when H is not perfect.

LEMMA 4.13. *Let* $H\subset N\subset[0, 1]$, $\lambda(H)=0$ *and* N *be open. Suppose that* $f\in\mathscr{B}^1$ *is locally bounded in* N *(for every* $x\in N$ $f|_{(x-\delta, x+\delta)}$ *is bounded if* $\delta>0$ *is small enough). Then* $f|_H\in\mathscr{D}_H$ *and moreover the primitive* $F(x)$ *(i.e.* $F'(x)$ *exists everywhere and* $F'|_H=f|_H$*) can be chosen to satisfy* $0\leqq F(x)\leqq h(x, \overline{N})$.

PROOF. We assume that H is everywhere dense in N (this causes no loss of generality). Let $(a, b)\subset N$ denote an arbitrary component of N. Consider a sequence of increasing intervals $[c_n, d_n]\subset(c_{n+1}, d_{n+1})\subset(a, b)$, $n=1, 2, \ldots$ such that $c_n, d_n\in H$ $(n=1, 2, \ldots)$. If $[c, d]$ denotes any of the intervals $[c_1, d_1]$, $[d_n, d_{n+1}]$, $[c_{n+1}, c_n]$ $(n=1, 2, \ldots)$ then f is bounded in $[c, d]$. By Theorem 3.2 there exists a bounded app. c. function $g(x)$ which satisfies $g|_{H\cap[c,d]}=f|_{H\cap[c,d]}$. In particular $g(c)=f(c)$, $g(d)=f(d)$. If $[c, d]$ runs over the intervals indicated above we obtain the function $g(x)$ unambiguously defined on the whole interval (a, b) such that g is bounded and app. c. in every closed subinterval $[a', b']\subset(a, b)$. Hence

$$G(x)=\int_{x_0}^{x} g(t)\,dt \quad (x\in(a, b))$$

is differentiable in (a, b) and $G'(x)=g(x)$ $(x_0\in(a, b)$ is fixed arbitrarily). Hereby we have

$$G'(x)=f(x) \quad (x\in H\cap(a, b)).$$

Referring to Corollary 4.12 we find a differentiable function $G_1(x)$ defined in (a, b) such that $G_1'(x)=0$ $(x\in H\cap(a, b))$ and $G(x)-h(x, \overline{N})<G_1(x)<G(x)$. Hence for $F(x)=G(x)-G_1(x)$ we have

$$F'(x)=f(x) \quad (x\in H\cap(a, b)), \quad 0<F(x)<h(x, \overline{N}) \quad (x\in(a, b)).$$

Let $F(x)$ be defined in every component (a, b) in this manner and finally put $F(x)=0$ $(x\in\overline{N})$. By Lemmas 4.2 and 4.3 $F(x)$ fulfils all the requirements.

THEOREM 4.14. *Let* $H\subset[0, 1]$, $\lambda(H)=0$. *Then*

$$\mathscr{B}_H^1=\mathscr{D}_H.$$

PROOF. We have to prove only $\mathscr{B}_H^1\subset\mathscr{D}_H$. Let $f\in\mathscr{B}^1$ be arbitrary. We define

$$N=\{x: f \text{ is locally bounded in } x\}.$$

N is obviously open and since it contains the continuity points of f, N is also everywhere dense in $[0, 1]$. Thus $\overline{N}$ is nowhere dense and closed and applying Corollary 4.9 we can find the differentiable function $G(x)$ such that

$$|G'(x)-f(x)|<1 \quad (x\in\overline{N}).$$

By Lemma 4.4 we can assume that $G'(x)$ is also continuous in N. By Theorem 3.2 there exists $A(x)$ such that $A'\in b\mathscr{A}$ and

$$A'(x)=f(x)-G'(x) \quad (x\in H\cap\overline{N}). \tag{4.6}$$

Since the function $f(x)-G'(x)-A'(x)$ is locally bounded in N, by Lemma 4.13 there exists an $F(x)$ such that

$$F'(x)=f(x)-G'(x)-A'(x) \quad (x\in H\cap N) \tag{4.7}$$

and $0\leqq F(x)\leqq h(x, \overline{N})$. In particular

$$F'(x)=0 \quad (x\in\overline{N}). \tag{4.8}$$

By (4.6), (4.7) and (4.8) we have

$$f(x) = \big(G(x)+A(x)+F(x)\big)' \qquad (x\in H)$$

i.e. $f|_H \in \mathscr{D}_H$ and the proof is completed.

Our next aim is to prove that $\lambda(H)=0$ is also necessary for $\mathscr{D}_H=\mathscr{B}_H^1$. We might give at once a function as a counterexample for $\lambda(H)>0$. However the structure of this function is — in some extent — characteristic for every derivative. Hence it seems to have more use of giving it as a structure theorem. Preparing this we need

LEMMA 4.15. *Let $F(x)$ be differentiable in $[a, b]$. Then there exists a differentiable function $F_1(x)$ such that*

(i) $F_1'(a) = F'(a)$, $F_1'(b) = F'(b)$,

(ii) $|F_1'(x)| \leqq 1+\max\left(|F'(a)|, |F'(b)|, \left|\frac{F(b)-F(a)}{b-a}\right|\right)$ $(a \leqq x \leqq b)$,

(iii) *the graph of F_1 is in the convex hull of the graph of F.*

PROOF. Let K denote the convex hull of the graph $\{(x, F(x)): a\leqq x\leqq b\}$. Put

$$\varphi(x) = \min\{y:(x, y)\in K\}, \quad \psi(x) = \max\{y:(x, y)\in K\}.$$

It is easy to see that φ and ψ are convex and concave functions in $[a, b]$, respectively, and $\varphi\leqq F\leqq\psi$ trivially holds in $[a, b]$. We can easily verify that φ and ψ are also differentiable in $[a, b]$. In fact, let $x_0\in(a, b)$ be arbitrary. (In the endpoints a, b the convexity of φ implies the existence of the one-sided derivatives.) If $F(x_0)>\varphi(x_0)$, then $\varphi(x)$ is linear in a neighbourhood of x_0. If $F(x_0)=\varphi(x_0)$ then

$$\frac{F(x)-F(x_0)}{x-x_0} \underset{\geqq}{\leqq} \frac{\varphi(x)-\varphi(x_0)}{x-x_0} \quad \text{according to} \quad x \underset{>}{\leqq} x_0.$$

Hence $\varphi'_+(x_0)\leqq F'(x_0)\leqq\varphi'_-(x_0)$, where $+$ and $-$ denote the right hand side and left hand side derivatives of φ, respectively. By the convexity of φ we have $\varphi'_-(x_0)\leqq\leqq\varphi'_+(x_0)$, thus $\varphi'(x_0)=F'(x_0)$. An analogous argument yields the existence of $\psi'(x_0)$. Now we put

$$K_1(\varepsilon) = \{(x, y)\in K: x\leqq a+\varepsilon\}, \quad K_2(\varepsilon) = \{(x, y)\in K: x\geqq b-\varepsilon\}.$$

If $\varepsilon>0$ is small enough then, for any choice of $P\in K_1(\varepsilon)$ and $Q\in K_2(\varepsilon)$,

(4.9) the slope of the segment $\overline{PQ}$ does not exceed $\frac{1}{2}+\left|\frac{F(b)-F(a)}{b-a}\right|$.

Suppose that $\varphi'(a)<F'(a)<\psi'(a)$. Then the line $l(x)=F'(a)(x-a)+F(a)$ enters the set K for $a\leqq x\leqq a+\varepsilon_1$. Beside (4.9) we choose $\varepsilon>0$ such that $\varepsilon\leqq\varepsilon_1$ and define

$$G(x) = l(x) \quad (x\in[a, a+\varepsilon]).$$

If either $\varphi'(a)=F'(a)$ or $\psi'(a)=F'(a)$ holds, say $\varphi'(a)=F'(a)$, then we put

$$G(x) = \varphi(x) \quad (x\in[a, a+\varepsilon])$$

where $\varepsilon>0$ is chosen according to (4.9).

We define $G(x)$ analogously in a small interval $[b-\varepsilon, b]$. Finally we extend $G(x)$ linearly to $[a+\varepsilon, b-\varepsilon]$ by connecting the points $P=(a+\varepsilon, G(a+\varepsilon))$, $Q=(b-\varepsilon, G(b-\varepsilon))$. Since $\varphi'(x)$ is monotone and Darboux, it is also continuous, hence if $\varepsilon>0$ is small enough, the construction yields

$$|G'(x)| \leqq M \quad (a \leqq x \leqq b)$$

where

$$M = \frac{1}{2} + \max\left(|F'(a)|, |F'(b)|, \left|\frac{F(b)-F(a)}{b-a}\right|\right).$$

(This estimation in $a+\varepsilon$ and $b-\varepsilon$ concerns the onesided derivatives of $G(x)$.)

In order to get $F_1(x)$ we have to smooth $G(x)$ in $a+\varepsilon$ and $b-\varepsilon$. We define

$$G_1(x) = \begin{cases} G(x) & \left(a \leqq x \leqq a+\frac{\varepsilon}{2}\right) \\ G(x)+\eta\left(x-a-\frac{\varepsilon}{2}\right)^2 & \left(a+\frac{\varepsilon}{2} \leqq x \leqq a+\varepsilon\right) \end{cases}$$

then we proceed $G_1(x)$ continuously with a straight line having the slope $G'_-(a+\varepsilon)+\eta\varepsilon$, i.e. the left hand side derivative of G_1 in $a+\varepsilon$. If $|\eta|$ is small enough and sign η is properly chosen then the graph of $G_1(x)$ will intersect that of $G(x)$ within the interval $\left(a+\varepsilon, \frac{a+b}{2}\right)$ and {the graph of $G_1(x)$}$\subset K$. We modify $G(x)$ similarly in a neighbourhood of $b-\varepsilon$. Hence the break points of G are shifted to the right and to the left and — that was aimed — the breaking is at the junction point of linear segments. Hence by the aid of small circular arcs these breakings can be smoothed and by this process the change of the derivative obviously will not exceed $\frac{1}{2}$. Having this carried out we denote the result by $F_1(x)$ and it fulfils our requirements.

LEMMA 4.16. *Let $F(x)$ be differentiable in $[0, 1]$ and $H\subset[0, 1]$ be perfect. If $|F(x)-F(y)|\leqq K|x-y|$ for every $x, y\in H$, then there exists a differentiable function $F_1(x)$ such that $F(x)=F_1(x)$, $F'(x)=F_1'(x)$ $(x\in H)$ and $|F_1(x)-F_1(y)|\leqq(K+1)|x-y|$ $(x, y\in[0, 1])$.*

PROOF. Let $F_1(x)=F(x)$ for $x\in H$ and apply Lemma 4.15 for every interval $[a, b]$ contiguous to H. By (i) and (iii) of Lemma 4.15 $F_1'(x)=F'(x)$ $(x\in H)$. In fact, if $x_0\in H$ and $\varepsilon>0$, we consider the triangle T bounded by the straight lines

$$y = (F'(x_0)+\varepsilon)(x-x_0)+F(x_0), \quad y = (F'(x_0)-\varepsilon)(x-x_0)+F(x_0), \quad x = x_0+\delta.$$

If $\delta=\delta(\varepsilon)$ is small enough then the graph of $F(x)$

$$\{(x, F(x)) : x_0 \leqq x \leqq x_0+\delta\}$$

lies in T. Hence by (iii) of Lemma 4.15 also the graph $\{(x, F_1(x)) : x_0\leqq x\leqq x_0+\delta'\}$ is contained in T whenever $0<\delta'\leqq\delta$ and $x_0+\delta'\in H$. (If $(x_0, x_0+\delta)\subset\overline{H}$ we have only to refer to (i) of Lemma 4.15.) This gives $F_{1+}'(x_0)=F'(x_0)$ and a similar argument applies for the left hand side derivative.

By Lemma 4.15 $F_1'(x)$ exists in $\overline{H}$ and since the Lipschitz condition on $F(x)$ implies $|F'(x)| \leqq K\,(x \in H)$, we have also $|F_1'(x)| \leqq K\,(x \in H)$ and hence by Lemma 4.15 $|F_1'(x)| \leqq K+1 \quad (x \in [0, 1])$.

LEMMA 4.17. *Let $H \subset [0, 1]$ be perfect, $\neq \varnothing$ and let F be differentiable in* $[0, 1]$. *Then there exists an open interval I and $K>0$ such that $I \cap H \neq \varnothing$ and*

$$\left|\frac{F(x)-F(y)}{x-y}\right| \leqq K \quad (x, y \in I \cap H).$$

PROOF. Suppose indirectly that the assertion is false. Then there exist $x_1, y_1 \in (0, 1)$ with $x_1 < y_1$, $x_1, y_1 \in H$ and

$$\left|\frac{F(x_1)-F(y_1)}{x_1-y_1}\right| > 1.$$

By the continuity of F we can suppose that x_1 and y_1 (and also the points x_k, y_k selected below) are not endpoints of any interval contiguous to H.

If the intervals $I_k=(x_k, y_k)$ have already been defined for $k=1, \ldots, n$ such that $[x_{k+1}, y_{k+1}] \subset (x_k, y_k)$ and

$$\left|\frac{F(x_k)-F(y_k)}{x_k-y_k}\right| > k$$

then we apply our indirect assumptions for I_n and we select $x_{n+1}, y_{n+1} \in H \cap (x_n, y_n)$ with

$$\left|\frac{F(x_{n+1})-F(y_{n+1})}{x_{n+1}-y_{n+1}}\right| > n+1.$$

Let $x = \lim x_n$ and $y = \lim y_n$. Then

$$+\infty = \lim_{n\to\infty}\left|\frac{F(x_n)-F(y_n)}{x_n-y_n}\right| = \begin{cases} \left|\dfrac{F(x)-F(y)}{x-y}\right| & \text{if} \quad x < y \\ |F'(x)| & \text{if} \quad x = y \end{cases}$$

and both cases yield a contradiction.

THEOREM 4.18. *Let $f(x)$ be defined in* $[0, 1]$ *and denote*

$$A_f = \left\{x : \lim_{t\to x} \operatorname{app} f(t) \text{ exists and } \neq f(x)\right\}.$$

Let $Z \subset [0, 1]$ be closed and denote by U the interior of Z in the topology D (i.e. U consists of those points of Z where Z is of density 1).

If $f \in \mathcal{D}$ then $U \cap A_f$ is nowhere dense in Z. In particular the set

$$S_f = \{x \in A_f : \operatorname{cl} A_f \text{ has density } 1 \text{ in } x\}$$

is nowhere dense in $\operatorname{cl} A_f$.

PROOF. Note that if $f \in b\mathscr{D}$ then $A_f = \emptyset$ (more generally, if for an interval I, $f \in b\mathscr{D}_I$ then $A_f \cap I = \emptyset$). If $\lambda(Z)=0$ then $U=\emptyset \Rightarrow U \cap A_f = \emptyset$ and we have nothing to prove. Suppose that $\lambda(Z)>0$ and denote

$$N = \bigcup\{(a, b) : \lambda(Z \cap (a, b)) = 0,\ a, b \text{ rational}\}.$$

N is open and $\lambda(N \cap Z)=0$. Hence

$$H = Z \setminus N$$

is a non-empty perfect set with the property that if an open interval I meets H then also $\lambda(H \cap I)>0$ holds. Clearly $U \cap N = \emptyset$ (i.e. $(U \cap A_f) \cap N = \emptyset$) therefore we are to prove that $U \cap A_f$ is nowhere dense in H.

Suppose indirectly for an open interval J the relations

$$J \cap H \neq \emptyset, \quad \operatorname{cl}(U \cap A_f) \supset J \cap H.$$

Applying Lemma 4.17 we find an interval $I \subset J$ and $K>0$ such that $I \cap H \neq \emptyset$ and

$$\left|\frac{F(x)-F(y)}{x-y}\right| \leqq K \quad (x, y \in I \cap H)$$

where $F(x)$ denotes a primitive of f. By Lemma 4.16 (applied for cl I instead of [0, 1]) we obtain $F_1(x)$ such that F_1 is differentiable in I and

$$F_1(x) = F(x) \quad (x \in I \cap H), \tag{4.10}$$

$$f_1(x) = F_1'(x) = F'(x) = f(x) \quad (x \in I \cap H), \tag{4.11}$$

$$\left|\frac{F_1(x)-F_1(y)}{x-y}\right| \leqq K+1 \quad (x, y \in I). \tag{4.12}$$

By (4.12) $f_1 \in b\mathscr{D}_I$, hence $A_{f_1}\ (=A_{f_1} \cap I) = \emptyset$. By the indirect assumption $(U \cap A_f) \cap (I \cap H) \neq \emptyset$, say $x_0 \in U \cap A_f \cap I \cap H$. Since x_0 is of density 1 in Z it is also a density point in $H \cap I$ because of $\lambda(N \cap Z)=0$. Hence by (4.11) $\lim\limits_{t \to x_0} \operatorname{app} f_1(t) = \lim\limits_{t \to x_0} \operatorname{app} f(t)$, and also $f_1(x_0)=f(x_0)$. But $x_0 \in A_f$ thus we have $x_0 \in A_{f_1}$ which contradicts $A_{f_1} = \emptyset$.

THEOREM 4.19. *Let* $H \subset [0, 1]$. $\mathscr{D}_H = \mathscr{B}_H^1$ *if and only if* $\lambda(H)=0$.

PROOF. The "if" part was stated and proved in Theorem 4.14. Let $\lambda(H)>0$ and denote by M a measurable hull of H (in particular if $M_1 \subset M$, M_1 is measurable and $\lambda(M_1)>0$ then $H \cap M_1 \neq \emptyset$). Let $P \subset M$ denote a nowhere dense perfect subset such that $\lambda(P)>0$ and whenever an open interval I meets P then also $\lambda(I \cap P)>0$ holds. Let Q consist of the points of density 1 from P. If for an open interval I we have $I \cap Q \neq \emptyset$ then also $\lambda(I \cap Q)>0$ and this implies $I \cap Q \cap H \neq \emptyset$, i.e. $Q \cap H$ is everywhere dense in Q. Hence we can find a sequence $x_1, \ldots, x_n, \ldots$ from $H \cap Q$ which is everywhere dense in Q and thus also in P. Put

$$f(x) = \begin{cases} \dfrac{1}{n}, & \text{if } x = x_n, \\ 0, & \text{if } x \neq x_n, \quad n = 1, 2, \ldots. \end{cases}$$

Obviously $f\in\mathscr{B}^1$, i.e. $f|_H\in\mathscr{B}^1_H$. We are going to show that $f|_H\notin\mathscr{D}_H$. Suppose that $f|_H\in\mathscr{D}_H$ and denote by $g\in\mathscr{D}$ an extension of $f|_H$. The set

$$R=\{x: f(x)-g(x)=0\}$$

is measurable and $R\supset H$.

This implies that the measurable hull M is almost contained in R and hence we have

$$R\supset P_1 \tag{4.13}$$

where $P_1\subset P$ and $\lambda(P\setminus P_1)=0$. Since $g|_P\in\mathscr{D}_P$, by Lemma 4.4 there exists an extension of $g|_P$ denoted by g^* such that $g^*\in\mathscr{D}$ and g^* is continuous in $[0,1]\setminus P$. Hence $A_{g^*}\subset P$ (see the definition of A_{g^*} in Theorem 4.18). We prove that $x_n\in A_{g^*}$, $n=1,2,\dots$. In fact, since $x_n\in P$, we know $g^*(x_n)=g(x_n)$. But also $x_n\in H$ holds, thus $g(x_n)=f(x_n)=\frac{1}{n}$, i.e. $g^*(x_n)=\frac{1}{n}$. Furthermore x_n is of density 1 in P hence also in P_1, hence $\lim\limits_{t\to x_n}\operatorname{app} g^*(t)=\lim\limits_{t\to x_n,\ t\in P_1}\operatorname{app}\ g^*(t)$. If $t\in P_1$ then $g^*(t)=g(t)$, and by (4.13) $g(t)=f(t)$. Since obviously $\lim\limits_{t\to x_n}\operatorname{app} f(t)=0$, we have

$$\lim_{t\to x_n}\operatorname{app} g^*(t)=0\neq g^*(x_n)=\frac{1}{n}$$

and thus $x_n\in A_{g^*}$, $n=1,2,\dots$. By the choice of $\{x_n\}$ cl $A_{g^*}=P$ is proved. But making use of the density property of x_n again we find cl $S_{g^*}=$cl A_{g^*}. This is in contradiction with Theorem 4.18.

5. Applications and problems

5.1. The "usual" extension processes are all "normpreserving", i.e. for bounded functions they provide an extension with the same bound. Such methods necessarily fail to work in the case of derivatives.

Let H be a set of Cantor type, $\lambda(H)>0$ and let x_0 be a density point of H. Put

$$f(x)=\begin{cases}1 & (x=x_0),\\ 0 & (x\neq x_0,\ x\in H).\end{cases}$$

According to Corollary 4.7, $f\in\mathscr{D}_H$. Although $|f|\leqq 1$, $f(x)$ has only unbounded extensions f^* in $\mathscr{D}$, since a bounded derivative cannot have a removable approximate discontinuity.

PROBLEM 5.1.1. Let H be nowhere dense and perfect, $\lambda(H)>0$, $f\in b\mathscr{D}_H$. Does there exist an integrable extension of f in $\mathscr{D}$?[1]

PROBLEM 5.1.2. Let H be as above. Are the functions $f\in\mathscr{B}^1_H$ "almost" extendible? By this we mean the existence of a subset $H^*\subset H$ such that $\lambda(H\setminus H^*)=0$ and $f|_{H^*}\in\mathscr{D}_{H^*}$ (the set H^* may depend on f).[2]

We remark that the counterexample constructed in Theorem 4.19 is almost extendible.

[1] An affirmative answer has been given by A. M. BRUCKNER.

[2] This problem has recently been answered by the first author in the negative.

PROBLEM 5.1.3. Let $\lambda(H)=0$. We know that $f\in\mathscr{B}_H^1$ is extendible both in $\mathscr{A}$ and $\mathscr{D}$. Has it an extension in $\mathscr{A}\cap\mathscr{D}$?

5.2. By Corollary 4.9 we know that if H is nowhere dense and perfect then $f\in\mathscr{B}_H^1$ is the uniform limit of a sequence from $\mathscr{D}_H$, i.e. the closure of $\mathscr{D}_H$ with respect to the uniform convergence is $\mathscr{B}_H^1$. Hence we have

COROLLARY 5.2.1. *If the nowhere dense perfect set H is of positive measure then $\mathscr{D}_H$ is not uniformly closed.*

In fact, by Theorem 4.19 $\mathscr{D}_H\neq\mathscr{B}_H^1$.
On the other hand $\mathscr{D}_H$ is uniformly closed if

— H is an interval (obvious),
— $\lambda(H)=0$ (by Theorem 4.14 $\mathscr{D}_H=\mathscr{B}_H^1$).

PROBLEM 5.2.2. Characterize the sets H for which $\mathscr{D}_H$ is uniformly closed.
(It can be easily verified that $\mathscr{B}_H^1$ is uniformly closed. Suppose $f_n\to f$ uniformly on $H, f_n\in\mathscr{B}_H^1$. We can assume that $|f_n-f|<\frac{1}{2^{n+2}}$ $\left(\text{hence } |f_n-f_{n-1}|<\frac{1}{2^n}\right)$. $f_n-f_{n-1}\in$ $\in\mathscr{B}_H^1$ and we can choose an extension $(f_n-f_{n-1})^*\in\mathscr{B}_H^1$ such that

$$|(f_n-f_{n-1})^*|\leqq\frac{1}{2^n}.$$

Hence the series

$$f_1^*+\sum_{n=2}^{\infty}(f_n-f_{n-1})^*$$

is uniformly convergent, thus it represents a Baire 1 extension of f.)

5.3. COROLLARY 5.3.1. *If $\lambda(H)=0$ and $f\in\mathscr{C}(H)$ then $f\in\mathscr{A}_H$ and $f\in\mathscr{D}_H$.*

PROOF, We have only to prove $\mathscr{C}(H)\subset\mathscr{B}_H^1$ for any H. Let $f\in\mathscr{C}(H)$ and suppose $f\geqq 0$. For $x\in\operatorname{cl}H$ put

$$f^*(x)=\max\{f(x),\ \overline{\lim}f(t)\ \ (t\to x,\ t\in H)\},$$

i.e. f^* is the upper boundary function of f.
Let

$$g(x)=\begin{cases}f^*(x) & \text{if } x\in\operatorname{cl}H \text{ and } f^*(x)<+\infty,\\ 0 & \text{otherwise.}\end{cases}$$

By $f\in\mathscr{C}(H)$ we have

$$g(x)=f^*(x)=f(x)\quad(x\in H).$$

We prove $g\in\mathscr{B}^1$. Let

$$H_1=\{x:f^*(x)=+\infty\}.$$

H_1 is closed obviously. $\{x:g(x)>c\}=[0,1]$ if $c<0$. Let $c\geqq 0$, then by

$$\{g>c\}=\{f^*>c\}\setminus H_1=\bigcup_{n=1}^{\infty}\left(\left\{f^*\geqq c+\frac{1}{n}\right\}\setminus H_1\right)$$

$\{x:g(x)>c\}$ is an F_σ set.

Consider the sets $\{x: g(x)<c\}$. If $c\leqq 0$, then this is the empty set. Let $c>0$, then

$$\{g<c\} = H_1 \cup ([0,1] \setminus \operatorname{cl} H) \cup \{f^*<c\}.$$

Now $\{x: f^*(x)<c\}$ is relatively open in the closed set cl H hence $\{x: g(x)<c\}$ is F_σ. This gives $g\in\mathscr{B}^1$ and $g|_H = f\in\mathscr{B}^1_H$.

If $f\in\mathscr{C}(H)$ is arbitrary, then

$$f(x) = \max\big(f(x), 0\big) + \min\big(f(x), 0\big) = f_+(x) + f_-(x)$$

and the previous argument applies for f_+ and $-f_-$.

Suppose that $H\subset[0,1]$ is closed and $f\in\mathscr{B}^1(H)$. Then treating the level sets as above one can easily prove that $g\in\mathscr{B}^1$, where

$$g(x) = \begin{cases} f(x) & (x\in H), \\ 0 & (x\notin H). \end{cases}$$

Thus we obtain

LEMMA 5.3.2. *If $H\subset[0,1]$ is closed then*

$$\mathscr{B}^1_H = \mathscr{B}^1(H).$$

5.4. Our next aim is to show that in some sense $\mathscr{A}$ and $\mathscr{D}$ are "dense" in $\mathscr{B}^1$.

DEFINITION 5.4.1. The functions $f, g\in\mathscr{B}^1$ are said to be equivalent ($f\sim g$) if $f(x)=g(x)$ holds on a subset H everywhere dense in $[0,1]$.

LEMMA 5.4.2. "$\sim$" *is an equivalence relation on $\mathscr{B}^1$. The equivalence classes (e.c.) are closed with respect to the uniform convergence. In $b\mathscr{B}^1$ the e.c. are nowhere dense subsets ($b\mathscr{B}^1$ is endowed with the sup norm topology).*

PROOF. $f\sim f$ and $f\sim g \Rightarrow g\sim f$ are fulfilled obviously. To prove the transitivity we properly need that $\sim$ is defined on $\mathscr{B}^1$. On wider classes our relation fails to be transitive. If $f\in\mathscr{B}^1$ then $\{x: f(x)=c\}$ is of type G_δ. Hence $f\sim g$ if and only if $f(x)-g(x)=0$ holds in an everywhere dense G_δ set. It is well known that if $H_1, H_2, \ldots$ are everywhere dense G_δ sets then also $\bigcap_{k=1}^{\infty} H_k$ is everywhere dense and G_δ. This proves at once the transitivity and also our assertion concerning the uniform convergence.

Let E be an e.c. in $b\mathscr{B}^1$ and let $G\subset b\mathscr{B}^1$ be an arbitrary open set. If $f\in G$ and $\varepsilon>0$ is small enough then $f+\varepsilon\in G$ as well. Obviously $f\sim f+\varepsilon$ cannot hold, hence $G\not\subset E$, i.e. the interior of E is empty. E being closed it is nowhere dense in fact.

THEOREM 5.4.3. *Every e.c. in $\mathscr{B}^1$ (in $b\mathscr{B}^1$) has representatives both in $\mathscr{A}$ and $\mathscr{D}$ (in $b\mathscr{A}$).*

PROOF. Let $H\subset[0,1]$ be a countable and everywhere dense set, $f\in\mathscr{B}^1$ ($f\in b\mathscr{B}^1$). According to Theorems 3.2 and 4.14 $f|_H$ has extensions both in $\mathscr{A}$ and $\mathscr{D}$ (in $b\mathscr{A}$) and these extensions are equivalent to f.

5.5. We are going to investigate the class $\mathscr{D}(H)$. Recall the definition: $f\in\mathscr{D}(H)$ means the existence of F defined on H such that $f(x_0) = \lim_{x\to x_0} \dfrac{F(x)-F(x_0)}{x-x_0}$ $(x, x_0\in H)$.

It is natural to ask whether $\mathscr{D}(H)=\mathscr{D}_H$. For perfect subsets $H\subset[0, 1]$ one expects the answer to be affirmative, nevertheless we were unable to find a direct proof avoiding the extension theorems.

LEMMA 5.5.1. *Let $H\subset[0, 1]$ be dense in itself. Then $\mathscr{D}(H)\subset\mathscr{B}^1(H)$.*

PROOF. First suppose that $\overline{H}$ is everywhere dense in $[0, 1]$. Let $f\in\mathscr{D}(H)$ and denote F a primitive of f. Take $0=x_0<x_1<\ldots<x_{k+1}=1$ such that

$$x_i\in\overline{H} \qquad (i=1, \ldots, k), \qquad x_i-x_{i-1}<\frac{1}{n} \qquad (i=1, \ldots, k+1). \tag{5.1}$$

Let ξ_i be an arbitrary point of $(x_{i-1}, x_i)\cap H$ (if this intersection is not empty) and for $x\in H$ denote

$$h_n(x)=\xi_i-x \qquad (x\in(x_{i-1}, x_i)\cap H). \tag{5.2}$$

Now $h_n\in\mathscr{C}(H)$ and hence also $f_n\in\mathscr{C}(H)$, where

$$f_n(x)=\begin{cases}\dfrac{F(x+h_n(x))-F(x)}{h_n(x)} & (x\neq\xi_i),\\ f(\xi_i) & (x=\xi_i,\ i=1, \ldots, k+1).\end{cases} \tag{5.3}$$

By the definition of F we have $\lim\limits_{n\to\infty} f_n(x)=f(x)$ $(x\in H)$ hence $f\in\mathscr{B}^1(H)$.

If $\operatorname{int} H\neq\varnothing$ then we have to modify the definition of $h_n(x)$.

Consider the components of $\operatorname{int} H$. Only a finite number of components, say $I_1, \ldots, I_N$ can have length $\geqq\frac{1}{n}$. Let $I_k=(a_k, b_k)$, and $b_k\leqq a_{k+1}$ $(k=1, \ldots, N-1)$, then we define $h_n(x)$ in $\operatorname{cl} I_k$ by connecting linearly the points $\left(a_k, \frac{1}{3n}\right)$, $\left(c_k-\frac{1}{3n}, \frac{1}{3n}\right)$, $\left(c_k+\frac{1}{3n}, \frac{-1}{3n}\right)$, $\left(b_k, \frac{-1}{3n}\right)$, where $c_k=\frac{1}{2}(a_k+b_k)$. We extend the definition of $h_n(x)$ to the remaining intervals (b_k, a_{k+1}) as follows (also $[0, a_1)$ and $(b_N, 1]$ may occur here). Each of the remaining intervals can be divided by suitable points

$$x_0=b_k<x_1<x_2<\cdots<x_p<a_{k+1}=x_{p+1}$$

such that

$$x_i\in\overline{H} \qquad (i=1, \ldots, p), \qquad x_{i+1}-x_i<\frac{1}{n} \qquad (i=0, \ldots, p-1).$$

Now we define $h_n(x)$ on $[b_k, x_1]$ and $[x_p, a_{k+1}]$ by connecting linearly the points $\left(x_1, -\frac{1}{n}\right)$, $\left(b_k, -\frac{1}{3n}\right)$ and $\left(x_p, \frac{1}{n}\right)$, $\left(a_{k+1}, \frac{1}{3n}\right)$ respectively. On the intervals (x_i, x_{i+1}), $i=1, 2, \ldots, p-1$ we define $h_n(x)$ by (5.2). (The peculiar definition of $h_n(x)$ in the first and last interval should be replaced by (5.2) if the underlying interval is $[0, a_1)$ or $(b_N, 1]$ respectively.)

Now it is obvious that $h_n(x)$ and

$$f_n(x) = \begin{cases} \dfrac{F(x+h_n(x))-F(x)}{h_n(x)}, & h_n(x) \neq 0 \\ f(x), & h_n(x) = 0 \end{cases} \tag{5.4}$$

are continuous and $f_n(x) \to f(x)$ $(x \in H)$.

THEOREM 5.5.2. *If $H \subset [0, 1]$ is perfect, then*

$$\mathscr{D}(H) = \mathscr{D}_H.$$

PROOF. The relation $\mathscr{D}_H \subset \mathscr{D}(H)$ is trivial. Turning to the non-trivial inclusion $\mathscr{D}(H) \subset \mathscr{D}_H$ we remark that by Lemma 5.5.1 $\mathscr{D}(H) \subset \mathscr{B}^1(H)$ and by Lemma 5.3.2 $\mathscr{B}^1(H) = \mathscr{B}^1_H$, i.e. $\mathscr{D}(H) \subset \mathscr{B}^1_H$. This gives the possibility to apply the extension theorems. Let H_1 denote the set of points where H has density 1, $H_2 = H \setminus H_1$. By the density theorem $\lambda(H_2) = 0$. Let $f \in \mathscr{D}(H)$, denote by $F(x)$ a primitive of f on H. As we have mentioned $f \in \mathscr{B}^1_H$, therefore $f|_{H_2} \in \mathscr{B}^1_{H_2}$. By Theorem 4.14 $f|_{H_2} \in \mathscr{D}_{H_2}$. We denote by $G(x)$ a differentiable function such that $G'|_{H_2} = f|_{H_2}$. Put

$$F_1(x) = F(x) - G(x) \quad (x \in H), \quad f_1(x) = F_1'(x) = F'(x) - G'(x) \quad (x \in H)$$

where the limit in the differentiation is restricted to H. Hence $f_1 \in \mathscr{D}(H)$ and

$$f_1(x) = 0 \quad (x \in H_2). \tag{5.4}$$

Now we are going to extend $F_1(x)$ to obtain a differentiable function on the whole interval $[0, 1]$.

If $I = (a, b)$ is any interval and α, β are arbitrary real numbers we denote

$$p(x) = p(x; I, \alpha, \beta) = \alpha + \frac{6(\beta-\alpha)}{(b-a)^3} \int_a^x (t-a)(b-t)\, dt.$$

We need the following properties of $p(x)$:

(i) $p(a) = \alpha$, $p(b) = \beta$,

(ii) $p'(a) = p'(b) = 0$,

(iii) the graph of $p(x)$ lies in the polygon with vertices in order (a, α), (b, β), $\left(\dfrac{a+b}{2}, \beta\right)$, $\left(\dfrac{a+b}{2}, \alpha\right)$, (a, α) (see figure 1). Let $\{I_k\} = \{(a_k, b_k)\}$ be the sequence of intervals contiguous to H, and $c_k = \dfrac{a_k+b_k}{2}$. Put

$$F_2(x) = \begin{cases} F_1(x) & (x \in H), \\ p(x; I_k, F_1(a_k), F_1(b_k)) & (x \in I_k,\ k = 1, 2, \ldots). \end{cases}$$

We are going to verify that $F_2'(x)$ exists everywhere. $F_2'(x)$ obviously exists in $x \in \bigcup_{k=1}^{\infty} I_k$ and even for $x=a_k$ from the right and for $x=b_k$ from the left and by (ii) these one-sided derivatives are equal to $f_1(x)=0$. Let $x_0 \in H$, $x \to x_0$, and suppose that (x_0, x)

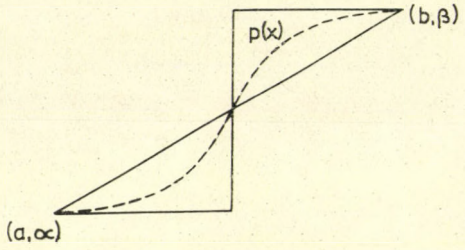


Fig. 1

contains an infinite subsequence $I_{k_1}, I_{k_2}, \ldots$ for every $x > x_0$ (there is no loss of generality in assuming $x > x_0$). If $x \to x_0$ and $x \in H$ then

$$\frac{F_2(x)-F_2(x_0)}{x-x_0} = \frac{F_1(x)-F_1(x_0)}{x-x_0} \to f_1(x_0).$$

If $x_0 < a_k \leqq x \leqq b_k$, then according to (iii) $\dfrac{F_2(x)-F_2(x_0)}{x-x_0}$ is majorized by the maximal term of the four ratios

$$(5.5) \qquad \frac{F_2(a_k)-F_2(x_0)}{a_k-x_0}, \quad \frac{F_2(b_k)-F_2(x_0)}{b_k-x_0}, \quad \frac{F_2(b_k)-F_2(x_0)}{c_k-x_0}, \quad \frac{F_2(a_k)-F_2(x_0)}{c_k-x_0}$$

and the symmetrical assertion holds for $\min_{a_k \leqq x \leqq b_k} \dfrac{F_2(x)-F_2(x_0)}{x-x_0}$. Hence it is enough to prove that all the ratios in (5.5) tend to $F_1'(x_0)=f_1(x_0)$ if $b_k \to x_0$. This obviously holds for the first two of them. As for the third we put

$$(5.6) \qquad \frac{F_2(b_k)-F_2(x_0)}{c_k-x_0} = \frac{F_2(b_k)-F_2(x_0)}{b_k-x_0} \cdot \frac{b_k-x_0}{c_k-x_0} \overset{\text{def}}{=} \varphi_1 \cdot \varphi_2.$$

Two cases are to be distinguished.

1) $x_0 \in H_1$. x_0 being a point of density we have $\varphi_2 \to 1$ and hence by $\varphi_1 \to f_1(x_0)$ we obtain

$$\frac{F_2(b_k)-F_2(x_0)}{c_k-x_0} \to f_1(x_0).$$

2) $x_0 \in H_2$. Then $f_1(x_0)=0$, i.e. $\varphi_1 \to 0$. By

$$\frac{b_k-x_0}{c_k-x_0} \leqq \frac{b_k-a_k}{c_k-a_k} = 2$$

the estimation

$$\left|\frac{F_2(b_k)-F_2(x_0)}{c_k-x_0}\right| \leqq 2|\varphi_1|$$

yields $\frac{F_2(b_k)-F_2(x_0)}{c_k-x_0} \to f_1(x_0)$ again.

A similar argument applies for the last ratio $\frac{F_2(a_k)-F_2(x_0)}{c_k-x_0}$. Hence we have proved that F_2 is differentiable and

$$F_2'(x) = F_1'(x) = f_1(x) \quad \text{for} \quad x \in H.$$

Finally by $F_3 = F_2 + G$ we obtain the function required by the theorem.

We rewrite the assertion of the preceding theorem as an extension theorem concerning differentiable functions. Both versions can be considered as analogues of Tietze's theorem.

THEOREM 5.5.3. *Let F be defined on the perfect set $H \subset [0, 1]$. If F is differentiable in H then it has a differentiable extension on* $[0, 1]$. *If*

$$|F(x)-F(y)| \leqq K|x-y| \quad (x, y \in H)$$

then such estimation holds for a suitably extended function as well if K is replaced by $K+1$.

PROOF. We apply the *construction* given in the preceding proof for $f(x) = F'(x) \in \mathscr{D}(H)$ and by $F = G + F_2$ we obtain the extension. Then the assertion on Lipschitz condition follows by Lemma 4.16.

COROLLARY 5.5.4. *It is well known that $|F(x)-F(y)| \leqq K|x-y|$ and $|F'(x)| \leqq K$ are equivalent conditions if $F(x)$ is differentiable in an interval. This equivalence no longer holds for differentiable functions defined on perfect sets.*

In fact, let f denote the function as in 5.1, let F be an arbitrary primitive of f in H. Then

$$|f(x)| = |F'(x)| \leqq 1 \quad (x \in H).$$

If $F(x)$ satisfied

$$|F(x)-F(y)| \leqq K|x-y| \quad (x, y \in H),$$

then by Theorem 5.5.3 $F(x)$ would possess an extension $G(x)$ with

$$|G(x)-G(y)| \leqq (K+1)|x-y|$$

and hence $G'(x)$ would provide a bounded extension of f. As it was remarked in 5.1, this is impossible.

5.6. Finally it is worth while to remark that neither $\mathscr{D}(H) = \mathscr{D}_H$ nor $\mathscr{B}^1(H) = \mathscr{B}_H^1$ holds any longer if the perfectness condition on H is dropped.

For example if $H = \{r_1, \ldots\}$ is the set of rationals and $\{d_1, \ldots\}$ is *any* sequence of real numbers, then by Theorem 4.5 there exists a function $F(x)$ continuous on $[0, 1]$ such that $F'(r_n) = d_n$ $(n = 1, 2, \ldots)$. In this case $\mathscr{D}(H)$ contains *every* real func-

tion defined on H. By Lemma 5.5.1 the same holds for $\mathscr{B}^1(H)$, but the function

$$f(x) = \begin{cases} 0, & x = \dfrac{p}{2q} \\ 1, & x = \dfrac{p}{2q+1} \end{cases}$$

does not belong to $\mathscr{B}^1_H$.

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References

[1] I. P. NATANSON, *Theorie der Funktionen einer reellen Veränderlichen.* Akademie Verlag (Berlin, 1961).
[2] S. SAKS, *Theory of the Integral.* Dover Publ., New York.
[3] O. HAUPT, CHR. PAUC, La topologie approximative de Denjoy envisagée comme vraie topologie, *Comptes Rendus Paris,* **234** (1952), pp. 390—392.
[4] G. PETRUSKA—M. LACZKOVICH, A theorem on approximately continuous functions, *Acta Math. Acad. Sci. Hung.*, **24** (1973), pp. 383—387.
[5] Z. ZAHORSKI, Sur la première dérivée, *Trans. Amer. Math. Soc.*, **69** (1950), pp. 1—54.
[6] S. MARCUS, Sur les dérivées dont les zéros forment un ensemble frontière partout dense, *Rend. Circ. Mat. Palermo,* **(2) 12** (1963), pp. 1—36.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (1—2), (1974), pp. 213—221.

AN ESTIMATE OF THE ERROR OF PADÉ APPROXIMANTS

By

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1. Introduction

In our present paper we give an error estimate for the Padé approximants of functions $F(z)$ representable as Stieltjes transforms of a nonnegative measure on the real line. We discuss our result for the case of a finite support but our method remains applicable even if the support is infinite. The essential argument is an estimate for orthogonal polynomials outside their support. Without loss of generality we can assume that the support of the representing measure $d\alpha$ is contained in $[-1, 1]$:

$$F(d\alpha; z) = \int_{-1}^{1} \frac{d\alpha(t)}{z-t} \quad (z \notin [-1, 1]). \tag{1}$$

We infer from (1) that $F(z)$ is analytic outside $[-1, 1]$ including the point $z=\infty$ and clearly

$$\lim_{|z|\to\infty} zF(z) = \int_{-1}^{1} d\alpha(t) = \mu_0(d\alpha). \tag{2}$$

The Padé approximant[1] of degree n around $z=\infty$, denoted by $\varrho_n(z)$, is the uniquely determined rational function of degree[2] n for which the first $2n+1$ coefficients of the development in powers of z^{-1} are the same as those of $F(z)$.

In our present paper we prove a rather simple but fairly precise estimate for $|F(z)-\varrho_n(z)|$ which is applicable, for every $z \notin [-1, 1]$.

2. A formula for the remainder of the Padé approximation

Let

$$Q_n(d\alpha; f) = \sum_{k=1}^{n} \lambda_{kn}(d\alpha) f(x_{kn}) \tag{3}$$

be the sequence of Gauss—Jacobi quadrature formulas with respect to the measure $d\alpha$. (See e.g. G. FREUD [2], G. SZEGŐ [8].) Here $x_{kn}=x_{kn}(d\alpha)$ are the zeroes of the n-th orthonormal polynomial $p_n(d\alpha; x)$.

[1] For a general treatise concerning Padé approximation we refer to H. S. WALL [9].

[2] A rational function of degree n is by definition the quotient of two polynomials of degree at most n.

By a result of A. A. MARKOV [5] (see also G. SZEGŐ [8], § 3.5) the Padé approximants of degree n, around $z=\infty$ can be represented as

$$\varrho_n(d\alpha; z) = Q_n(d\alpha; f_z) = \sum_{k=1}^{n} \lambda_{kn}(z-x_{kn})^{-1}; \quad f_z(t) = (z-t)^{-1}. \tag{4}$$

We denote by $\gamma_n(d\alpha)$ the leading coefficient of $p_n(d\alpha; x)$, i.e.

$$p_n(d\alpha; x) = \gamma_n(d\alpha) \prod_{k=1}^{n} (x-x_{kn}); \quad \gamma_n(d\alpha) > 0. \tag{5}$$

Every $x_{kn}(d\alpha)$ is situated in $(-1, 1)$.

Beside the zeroes $\{x_{kn}\}$ we will consider also the leading coefficient $\gamma_n(d\alpha)>0$ and the Christoffel numbers $\lambda_{kn}(d\alpha)$ $(k=1, 2, \ldots, n)$ of the quadrature formula (3).

We are going to prove estimates of $|F(z)-\varrho_n(z)|$ where $\varrho_n(z)$ is defined by (4). From our estimate we obtain an idependent proof of A. A. Markov's result, i.e. that the rational function $\varrho_n(z)$ defined by (4) is identical with the Padé approximant of degree n.

LEMMA 1. *We have*

$$\varrho_{n+1}(z)-\varrho_n(z) = \frac{\gamma_{n+1}(d\alpha)}{\gamma_n(d\alpha)} \frac{1}{p_n(d\alpha; z)p_{n+1}(d\alpha; z)}. \tag{6}$$

PROOF. (6) is implied by the results of A. A. MARKOV [5]. We gave an elementary proof of (6) in our paper [3], page 115.

By (1) and (4) we infer from classical results of A. A. STEKLOV [7] and L. FEJÉR [1] (see also G. PÓLYA [6] and [2], III.1 and [8], § 15.2) that

$$\lim_{n\to\infty} \varrho_n(z) = \lim_{n\to\infty} Q_n(d\alpha; f_z) = F(z) \quad (z \notin [-1, 1]). \tag{7}$$

From (6) and (7) we have

$$F(d\alpha; z)-\varrho_n(d\alpha; z) = \sum_{\nu=n}^{\infty} \frac{\gamma_{\nu+1}(d\alpha)}{\gamma_\nu(d\alpha)} \frac{1}{p_\nu(d\alpha; z)p_{\nu+1}(d\alpha; z)}. \tag{8}$$

We obtain our main result by combining (8) with an elementary estimate of $p_\nu^{-1}(d\alpha; z)$.

3. Formulation and proof of the error estimate

Let $\Delta(z)$ be the distance of z from the segment $[-1, 1]$ and

$$\varrho = \varrho(z) = \left|z+\sqrt{z^2-1}\right| > 1 \quad (z \notin [-1, 1]). \tag{9}$$

The orthonormal polynomial with respect to the weight $w_0(x) = (1-x^2)^{-1/2}$ $(-1 < x < 1)$ is

$$p_\nu(w_0; z) = \sqrt{\frac{2}{\pi}}\, T_\nu(z) = \frac{1}{\sqrt{2\pi}} \left\{\left(z+\sqrt{z^2-1}\right)^\nu + \left(z-\sqrt{z^2-1}\right)^\nu\right\} \tag{10}$$

where $T_\nu(z)$ is the Chebyshev polynomial of the first kind of degree n.

Since $\left(z+\sqrt{z^2-1}\right)\left(z-\sqrt{z^2-1}\right)=1$ we have

$$|T_\nu(z)| > \frac{1}{2}\{[\varrho(z)]^\nu - [\varrho(z)]^{-\nu}\} \qquad (\nu = 1, 2, \ldots). \tag{11}$$

LEMMA 2. *We have*

$$\frac{1}{|p_\nu(d\alpha; z)|} \leqq \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} \frac{\sqrt{\mu_0(d\alpha)}}{\Delta(z)|T_{\nu-1}(z)|}. \tag{12}$$

PROOF. By [2], Chapter III, formula (7.7) we have

$$\left|\frac{p_{\nu-1}(w_0; z)}{p_\nu(d\alpha; z)}\right| \leqq \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} \frac{1}{\Delta(z)} \left\{\int_{-1}^{1} p_{\nu-1}^2(w_0; x)\, d\alpha(x)\right\}^{1/2}$$

i.e. by (10)

$$\left|\frac{T_{\nu-1}(z)}{p_\nu(d\alpha; z)}\right| \leqq \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} \frac{1}{\Delta(z)} \left\{\int_{-1}^{1} T_{\nu-1}^2(x)\, d\alpha(x)\right\}^{1/2}. \tag{13}$$

Since $|T_n(x)| \leqq 1$ for $x \in [-1, 1]$ we obtain (12) from (13), q.e.d.

LEMMA 3. *We have for every $d\alpha$ with support contained in $[-1, 1]$*

$$\frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} \leqq 1 \qquad (\nu = 1, 2, \ldots) \tag{14}$$

and under the additional condition $\log \alpha'(\cos\theta) \in L$ we have

$$\lim_{n\to\infty} \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} = \frac{1}{2}. \tag{15}$$

PROOF. Both relations are wellknown. (13) is proved by the following formula (see e.g. [2], § I.2):

$$\frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} = \int_{-1}^{1} x p_{\nu-1}(d\alpha; x) p_\nu(d\alpha; x)\, d\alpha(x) \leqq \int_{-1}^{1} |p_{\nu-1}(d\alpha; x) p_\nu(d\alpha; x)|\, d\alpha(x) \leqq$$

$$\leqq \left\{\int_{-1}^{1} p_{\nu-1}^2(d\alpha; x)\, d\alpha(x) \int_{-1}^{1} p_\nu^2(d\alpha; x)\, d\alpha(x)\right\}^{1/2} = 1.$$

(15) is a consequence of a generalized form of Theorem 12.1.2 in G. SZEGŐ [8]; see e.g. [2], § V.5, Table I and JA. L. GERONIMUS [4].

THEOREM 1. *We have*

$$|F(d\alpha; z) - \varrho_n(d\alpha; z)| \leqq \frac{\mu_0(d\alpha)}{[\Delta(z)]^2} \sum_{\nu=n}^{\infty} \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} \frac{1}{|T_{\nu-1}(z) T_\nu(z)|} =$$

$$= \frac{4\mu_0(d\alpha)}{[\Delta(z)]^2} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \left\{[\varrho(z)]^{-2n+1} + O\left([\varrho(z)]^{-2n}\right)\right\} \tag{16}$$

where the last estimate is uniform form $\varrho(z) \geqq \varrho_0 > 1$ *also with respect to the choice of* $d\alpha$.

REMARK 1. Combining (16) and (15) we see that under the condition $\log \alpha'(\cos\theta) \in L$ we have

$$|F(z) - \varrho_n(z)| \leqq 2\frac{\mu_0(d\alpha)}{[\Delta(z)]^2}(1+\varepsilon_n)\{[\varrho(z)]^{-2n+1} + O([\varrho(z)]^{-2n})\}. \tag{17}$$

For the weight $w_0(x)$ we get from (8) and (10)

$$F(w_0; z) - \varrho_n(w_0; z) = \pi \sum_{\nu=n}^{\infty} \frac{1}{T_\nu(z) T_{\nu+1}(z)} = \frac{4\pi}{[\varrho(z)]^{2n+1}} + O([\varrho(z)]^{-2n-2}). \tag{18}$$

Since for $|z| \to \infty$ we have $\Delta(z) \approx \frac{1}{2}\varrho(z)$ and $\mu_0(w_0) = \pi$ we see from (18) *that the factor two in* (17) *cannot be replaced by a number smaller than one.*

REMARK 2. By (16) we have for fixed n and $|z| \to \infty$

$$F(d\alpha; z) - \varrho_n(d\alpha; z) = O(|z|^{-2n-1}). \tag{19}$$

This estimate gives the new proof announced in part 2 of this paper that the expression $\varrho_n(d\alpha; z)$ defined by (4) is the n-th degree Padé approximant around $z = \infty$ of $F(d\alpha; z)$.

REMARK 3. Combining (14) and the first part of (16) we obtain

$$|F(d\alpha; z) - \varrho_n(d\alpha; z)| \leqq \tag{20}$$

$$\leqq \frac{\mu_0(d\alpha)}{[\Delta(z)]^2}\left\{\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\frac{1}{|T_{n-1}(z)T_n(z)|} + \eta_{n+1}(z)\right\} \leqq \frac{\mu_0(d\alpha)}{[\Delta(z)]^2}\eta_n(z).$$

Here the functions

$$\eta_n(z) = \sum_{\nu=n}^{\infty} \frac{1}{|T_{\nu-1}(z)T_\nu(z)|} \tag{21}$$

are easily calculated and independent of $d\alpha$.

In particular if $z = x \notin [-1, 1]$ is real we have by (10)

$$\eta_n(x) \leqq 4\sum_{\nu=n}^{\infty} [\varrho(x)]^{-2\nu+1} = 4[1 - \varrho^{-2}(x)]^{-1}[\varrho(x)]^{-2n+1}. \tag{22}$$

PROOF OF THEOREM 1. The first part of (16) follows from (6) and (12). The second part of (16) was proved in a more precise form in Remark 3.

4. On the outer asymptotic of orthogonal polynomials

In the present part 4 we give a refinement of Lemma 2. We consider the Lagrange interpolation formula over the zeroes of $p_\nu(d\alpha; x)$ as nodes. The fundamental functions of Lagrange interpolation can be represented as

$$l_{k\nu}(d\alpha; x) = \lambda_{k\nu}(d\alpha)\frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)}\frac{p_{\nu-1}(d\alpha; x_{kn})p_\nu(d\alpha; x)}{x-x_{k\nu}}. \tag{23}$$

We express the Chebychev polynomial $T_{\nu-1}(x)$ by the Lagrange interpolation formula:

$$T_{\nu-1}(z) = \sum_{k=1}^{\nu} l_{k\nu}(d\alpha; z)T_{\nu-1}(x_{k\nu}) = \tag{24}$$

$$= \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)}p_\nu(d\alpha; z)\sum_{k=1}^{\nu}\lambda_{k\nu}(d\alpha)p_{\nu-1}(d\alpha; x_{k\nu})T_{\nu-1}(x_{k\nu})(z-x_{k\nu})^{-1} =$$

$$= \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)}p_\nu(d\alpha; z)\left\{\frac{1}{z}\sum_{k=1}^{\nu}\lambda_{k\nu}(d\alpha)p_{\nu-1}(d\alpha; x_{k\nu})T_{\nu-1}(x_{k\nu}) + \right.$$

$$\left. + \frac{1}{z}\sum_{k=1}^{\nu}\lambda_{k\nu}(d\alpha)p_{\nu-1}(d\alpha; x_{k\nu})T_{\nu-1}(x_{k\nu})x_{k\nu}(z-x_{k\nu})^{-1}\right\} =$$

$$= \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)}p_\nu(d\alpha; z)\left\{\frac{1}{z}\sum_{k=1}^{\nu}\lambda_{k\nu}(d\alpha)p_{\nu-1}(d\alpha; x_{k\nu})T_{\nu-1}(x_{k\nu}) + \right.$$

$$+ \frac{1}{z^2}\sum_{k=1}^{\nu}\lambda_{k\nu}(d\alpha)p_{\nu-1}(d\alpha; x_{k\nu})T_{\nu-1}(x_{k\nu})x_{k\nu} +$$

$$\left. + \frac{1}{z^2}\sum_{k=1}^{\nu}\lambda_{k\nu}(d\alpha)p_{\nu-1}(d\alpha; x_{k\nu})T_{\nu-1}(x_{k\nu})x_{k\nu}^2(z-x_{k\nu})^{-1}\right\}.$$

We now apply the fact that for every polynomial q of degree at most $2\nu-1$ we have

$$Q_\nu(d\alpha; q) = \int_{-1}^{1} q(x)\,d\alpha(x)$$

so that

$$A_\nu = \sum_{k=1}^{\nu}\lambda_{k\nu}(d\alpha)p_{\nu-1}(d\alpha; x_{k\nu})T_{\nu-1}(x_{k\nu}) = \tag{25}$$

$$= \int_{-1}^{1} p_{\nu-1}(d\alpha; x)T_{\nu-1}(x)\,d\alpha(x) = \frac{2^{\nu-2}}{\gamma_{\nu-1}(d\alpha)}$$

and [3]

$$B_\nu = \sum_{k=1}^{\nu} \lambda_{k\nu}(d\alpha) p_{\nu-1}(d\alpha; x_{k\nu}) T_{\nu-1}(x_{k\nu}) x_{k\nu} = \tag{26}$$

$$= \int_{-1}^{1} p_{\nu-1}(d\alpha; x) T_{\nu-1}(x) x \, d\alpha(x) = -\frac{2^{\nu-2}}{\gamma_\nu(d\alpha)} \frac{\gamma_\nu^{(1)}(d\alpha)}{\gamma_{\nu-1}(d\alpha)},$$

where $\gamma_\nu^{(1)}(d\alpha)$ is the coefficient of $x^{\nu-1}$ in $p_\nu(d\alpha; x)$.

THEOREM 2. *We have under the only condition that the support of $d\alpha$ is contained in* $[-1, 1]$

$$\left| \frac{zT_{\nu-1}(z)}{p_\nu(d\alpha; z)} - \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} A_\nu \right| \leqq \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} \frac{\sqrt{\mu_2(d\alpha)}}{\Delta(z)} \qquad (z \notin [-1, 1];\ \nu = 2, 3, \ldots) \tag{27}$$

and

$$\left| \frac{zT_{\nu-1}(z)}{p_\nu(d\alpha; z)} - \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} A_\nu - \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} B_\nu z^{-1} \right| \leqq \tag{28}$$

$$\leqq \frac{\gamma_{\nu-1}(d\alpha)}{\gamma_\nu(d\alpha)} \frac{1}{\Delta(z)|z|} \sqrt{\mu_4(d\alpha)} \qquad (z \notin [-1, 1];\ \nu = 3, 4, \ldots)$$

where A_ν, B_ν are defined by (25), (26) *and*

$$\mu_k(d\alpha) = \int_{-1}^{1} x^k \, d\alpha(x). \tag{29}$$

REMARKS. a) If the measure $d\alpha$ is distributed symmetrically around the origin then $B_\nu = 0$.

b) We have by (25) and (26)

$$0 < A_\nu \leqq \int_{-1}^{1} |p_{\nu-1}(d\alpha; x)| \, d\alpha(x) \leqq \sqrt{\mu_0(d\alpha)} \tag{30}$$

and

$$|B_\nu| \leqq \int_{-1}^{1} |p_{\nu-1}(d\alpha; x)| \, |x| \, d\alpha(x) \leqq \sqrt{\mu_2(d\alpha)}. \tag{31}$$

[3] The right hand side of (25) and (26) is obtained as follows:
We have

$$T_{\nu-1}(x) = 2^{\nu-2} x^{\nu-1} + a_{\nu-1} x^{\nu-3} + \ldots$$

so that by comparing coefficients

$$T_{\nu-1}(x) = [2^{\nu-2}/\gamma_{\nu-1}(d\alpha)] p_{\nu-1}(d\alpha; x) + Q_{\nu-2}(x)$$

resp.

$$xT_{\nu-1}(x) = [2^{\nu-2}/\gamma_\nu(d\alpha)] p_\nu(d\alpha; x) - [2^{\nu-2}/\gamma_\nu(d\alpha)] [\gamma_\nu^{(1)}(d\alpha)/\gamma_{\nu-1}(d\alpha)] p_{\nu-1}(d\alpha; x) + q_{\nu-2}(x),$$

where $Q_{\nu-2}$ and $q_{\nu-2}$ are polynomials of degree at most $\nu-2$. Hence (25) and (26) follow from the orthonormal relations for the sequence $\{p_n(d\alpha; x)\}$.

c) The moments $\{\mu_r(d\alpha)\}$ can be characterized as the coefficients in the Laurent series

$$F(d\alpha; z) = \sum_{r=0}^{\infty} \mu_r(d\alpha) z^{-r-1}.$$

PROOF OF THEOREM 2. By $x_{kv} \in (-1, 1)$ we have $|T_{v-1}(x_{kv})| \leqq 1$, and consequently

$$\left| \sum_{k=1}^{v} \lambda_{kv}(d\alpha) p_{v-1}(d\alpha; x_{kv}) T_{v-1}(x_{kv}) x_{kv} (z - x_{kv})^{-1} \right| \leqq \tag{32}$$

$$\leqq \frac{1}{\Delta(z)} \sum_{k=1}^{v} \lambda_{kv}(d\alpha) |p_{v-1}(d\alpha; x_{kv}) x_{kv}| \leqq$$

$$\leqq \frac{1}{\Delta(z)} \left\{ \sum_{k=1}^{v} \lambda_{kv}(d\alpha) p_{v-1}^2(d\alpha; x_{kv}) \sum_{k=1}^{v} \lambda_{kv}(d\alpha) x_{kv}^2 \right\}^{1/2} = \frac{1}{\Delta(z)} \sqrt{\mu_2(d\alpha)}$$

and

$$\left| \sum_{k=1}^{v} \lambda_{kv}(d\alpha) p_{v-1}(d\alpha; x_{kv}) T_{v-1}(x_{kv}) x_{kv}^2 (z - x_{kv})^{-1} \right| \leqq \tag{33}$$

$$\leqq \frac{1}{\Delta(z)} \sum_{k=1}^{v} \lambda_{kv}(d\alpha) |p_{v-1}(d\alpha; x_{kv})| x_{kv}^2 \leqq$$

$$\leqq \frac{1}{\Delta(z)} \left\{ \sum_{k=1}^{v} \lambda_{kv}(d\alpha) p_{v-1}^2(d\alpha; x_{kv}) \sum_{k=1}^{v} \lambda_{kv}(d\alpha) x_{kv}^4 \right\}^{1/2} =$$

$$= \frac{1}{\Delta(z)} \left\{ \int_{-1}^{1} p_{v-1}^2(d\alpha; x_{kv})\, d\alpha(x) \int_{-1}^{1} x^4\, d\alpha(x) \right\}^{1/2} = \frac{\sqrt{\mu_4(d\alpha)}}{\Delta(z)}.$$

Now from the first part of (24) combined with (25) and (32) follows (27), moreover from the last part of (24) combined with (25), (26) and (33) follows (28), q.e.d.

5. Refinement of the error estimate

THEOREM 3. *We have for arbitrary $d\alpha$ with support contained in* $[-1, 1]$

$$\left| F(d\alpha; z) - \varrho_n(d\alpha; z) - \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} A_n A_{n+1} \frac{1}{z^2 T_{n-1}(z) T_n(z)} \right| \leqq \tag{34}$$

$$\leqq 2 \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \frac{\sqrt{\mu_0(d\alpha)\mu_2(d\alpha)}}{[\Delta(z)]^2 |z T_{n-1}(z) T_n(z)|} + \frac{\mu_0(d\alpha)}{[\Delta(d\alpha)]^2} \eta_{n+1}(z)$$

and if the measure $d\alpha$ is even and has a support contained in $[-1, 1]$ *then we have*

$$\left| F(d\alpha; z) - \varrho_n(d\alpha; z) - \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} A_n A_{n+1} \frac{1}{z^2 T_{n-1}(z) T_n(z)} \right| \leqq \tag{35}$$

$$\leqq 2 \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \frac{\sqrt{\mu_0(d\alpha)\mu_4(d\alpha)}}{[\Delta(z)]^2 |z^2 T_{n-1}(z) T_n(z)|} + \frac{\mu_0(d\alpha)}{[\Delta(z)]^2} \eta_{n+1}(z)$$

$$(z \notin [-1, 1];\ v = 3, 4, \ldots).$$

REMARK 1. For $|z|\to\infty$ the error term in (34) is $O(|z|^{-2n-2})$ and the error term in (35) is $O(|z|^{-2n-3})$.

REMARK 2. We can combine both estimates with the inequalities (14) and (30).

PROOF OF THEOREM 3. In the elementary inequality

$$|a_n a_{n+1}-b_n b_{n+1}| \leqq |a_n|\,|a_{n+1}-b_{n+1}|+|b_{n+1}|\,|a_n-b_n|$$

we substitue

$$a_n=\frac{1}{p_n(d\alpha;z)} \quad \text{and} \quad b_n=\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}A_n\frac{1}{zT_{n-1}(z)}$$

so that by Lemma 2

$$|a_n| \leqq \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\frac{\sqrt{\mu_0(d\alpha)}}{\Delta(z)|T_{n-1}(z)|}$$

and by Theorem 2

$$|a_n-b_n| \leqq \begin{cases} \dfrac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\dfrac{\sqrt{\mu_2(d\alpha)}}{\Delta(z)}\dfrac{1}{|zT_{n-1}(z)|} & \text{for arbitrary } d\alpha \\ \dfrac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\dfrac{\sqrt{\mu_4(d\alpha)}}{\Delta(z)}\dfrac{1}{|z^2T_{n-1}(z)|} & \text{for even } d\alpha. \end{cases}$$

In this way we obtain considering also (30) and $\Delta(z)\leqq|z|$ for arbitrary $d\alpha$

$$\text{(36)} \qquad \left|\frac{1}{p_n(d\alpha;z)p_{n+1}(d\alpha;z)}-\frac{\gamma_{n-1}(d\alpha)}{\gamma_{n+1}(d\alpha)}A_nA_{n+1}\frac{1}{z^2T_{n-1}(z)T_n(z)}\right| \leqq$$

$$\leqq \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\frac{\sqrt{\mu_0(d\alpha)}}{\Delta(z)|T_{n-1}(z)|}\frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)}\frac{\sqrt{\mu_2(d\alpha)}}{\Delta(z)|zT_n(z)|}+$$

$$+\frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)}A_{n+1}\frac{1}{|zT_n(z)|}\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\frac{\sqrt{\mu_2(d\alpha)}}{\Delta(z)}\frac{1}{|zT_{n-1}(z)|} \leqq$$

$$\leqq 2\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\frac{\sqrt{\mu_0(d\alpha)\mu_2(d\alpha)}}{[\Delta(z)]^2|zT_{n-1}(z)T_n(z)|}$$

resp. for even $d\alpha$ in an analogous way

$$\text{(37)} \qquad \left|\frac{1}{p_n(d\alpha;z)p_{n+1}(d\alpha;z)}-\frac{\gamma_{n-1}(d\alpha)}{\gamma_{n+1}(d\alpha)}A_nA_{n+1}\frac{1}{z^2T_{n-1}(z)T_n(z)}\right| \leqq$$

$$\leqq 2\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\frac{\sqrt{\mu_0(d\alpha)\mu_4(d\alpha)}}{[\Delta(z)]^2|z^2T_{n-1}(z)T_n(z)|}.$$

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From (8), (12), (14) and (21) we get

$$(38)\qquad \left| F(d\alpha; z) - \varrho_n(d\alpha; z) - \frac{\gamma_{n+1}(d\alpha)}{\gamma_n(d\alpha)} \frac{1}{p_n(d\alpha; z) p_{n+1}(d\alpha; z)} \right| \leqq \frac{\mu_0(d\alpha)}{[\Delta(z)]^2} \eta_{n+1}(z).$$

Combining (36) and (38) resp. (37) and (38) we obtain (34) resp. (35), q.e.d.

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Quotations

[1] L. FEJÉR, Mechanische Quadraturen mit positiven Cotesschen Zahlen, *Math. Zeitschr.*, **37** (1933), pp. 287—309.
[2] G. FREUD, *Orthogonale Polynome*. Birkhäuser Verl. (Basel, 1969). English translation by L. Földes, Pergamon Press, 1971.
[3] G. FREUD, Error estimates for Gauss—Jacobi quadrature formulae in "*Topics in Numerical Analysis*" editor J. Miller (Conference report Dublin 1972) Academic Press (London) 1973, pp. 113—121.
[4] Я. Л. Геронимус, *Многочлены, ортогональные на окружности и на отрезке* (Москва, 1958).

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FINITE GROUPS WITH FEW NON-LINEAR IRREDUCIBLE CHARACTERS

By
J. PELIKÁN (Budapest)

If we denote the number of conjugate classes of a finite group G by k, then we have $k \geqq |G:G'|$ and the equality sign holds if and only if G is abelian. In fact, if two elements of G are conjugate then they are obviously in the same coset of G' and G' itself consists of at least two conjugate classes unless $G'=1$. The non-abelian groups which have in a sense the smallest possible number of conjugate classes are those with $k=|G:G'|+1$. Our aim in this paper is to give a full description of these groups. These are at the same time the non-abelian groups whose character tables can be constructed the most easily, as they have exactly one non-linear irreducible character.

We also show that if G has at most 3 non-linear irreducible characters then G is solvable. This result cannot be improved as A_5 has 4 non-linear irreducible characters.

As the method of proof will be purely group-theoretic we prefer the formulation of the theorems given below to the one in terms of the number of non-linear irreducible characters.

THEOREM 1. *For a finite group G, $k=|G:G'|+1$ if and only if G is one of the following groups (k denotes the number of conjugate classes of G):*

1. $G=L_2(q)$, *the group of all mappings* $x \to ax+b$ $(a, b \in GF(q), a \neq 0)$ *of* $GF(q)$ *onto itself.* $(q>2$ *a prime-power,* $|G|=q(q-1).)$

2. $G=D_4 \curlyvee D_4 \curlyvee \ldots \curlyvee D_4$, *the direct product with amalgamated centres of m copies of the dihedral group* $D_4 (|G|=2^{2m+1})$.

3. $G=D_4 \curlyvee \ldots \curlyvee D_4 \curlyvee Q$, *the direct product with amalgamated centres of $m-1$ copies of D_4 and one quaternion group Q* $(|G|=2^{m+1})$.

PROOF. $k=|G:G'|+1$ implies $|G'|>1$. Thus G' consists of two conjugate classes: 1 and $G'-1$, while each coset of G' is a complete conjugate class. Hence G' has a transitive group of automorphisms so it is elementary abelian, $|G'|=p^l=q$. We see in particular that G is solvable.

Now we distinguish two cases.

Case A: $Z(G) \neq 1$.

Since each element of $Z(G)$ is a conjugate class in itself, we have $G'=Z(G)$, $|G'|=2$. Thus G is nilpotent so

$$G = \prod_{i=1}^{h} P_i$$

is the direct product of its Sylow subgroups. Now

$$Z(G) = \prod_{i=1}^{h} Z(P_i),$$

which implies that G must be a 2-group.

Consider now an $a \in G$, $a \notin G'$. Suppose $a^2 \notin G'$. Then denoting by b the other element in the coset $a^2 G'$, we have $y^{-1} a^2 y = b$ for a suitable $y \in G$. Thus $(y^{-1} a y)^2 = b$, implying $y^{-1} a y \neq a$ so $y^{-1} a y = a_1$ is the other element of aG'. But $a_1^2 = (az)^2 = a^2 z^2 = a^2$ because of $z \in G' = Z(G)$. The contradiction shows $a^2 \in G'$. Thus $\Phi(G) = G^2 = G'$ (cf. [3], p. 272) and G is an extra-special 2-group. This implies that G is one of the groups described in 2. and 3. of the theorem (cf. [3], p. 355).

On the other hand, $G' = Z(G)$, $|G'| = 2$ implies $k = |G : G'| + 1$ since each $a \notin Z(G)$ has at least two conjugates.

Case B: $Z(G) = 1$.

In this case necessarily $|G'| > 2$. For any $x \in G'$, $x \neq 1$ we have $|G : N_G(x)| = q - 1$; for $y \in G - G'$ $|G : N_G(y)| = q$.

Let P be a Sylow p-subgroup of G containing G'. Clearly, $P \lhd G$, so P is the only Sylow p-subgroup of G.

For any $x \in G'$, $x \neq 1$ we have $N_G(x) \subseteq P$, as $y \in N_G(x)$, $o(y) = r$, r a prime different from p would imply $o(xy) = rp \neq o(y)$ contradicting the assumption that all elements of the coset $G'y$ are conjugate.

Assume $x \in P$, $x \notin G'$. Then $|G : P| \,||\, |G : N_G(x)| = q$. As $(p, |G : P|) = 1$ we have $G = P$, thus $Z(G) \neq 1$, contradiction. Hence $P = G'$. $x \in G'$ implies $G' \subseteq N_G(x) \subseteq P$, consequently $N_G(x) = G'$. Thus $|G : G'| = q - 1$ and $|G| = q(q-1)$.

Now let H be a subgroup of G with $|H| = q - 1$ (e.g. $H = N_G(y)$ for any $y \in G - G'$). Let $x = hg \in G$ with $h \in H$, $g \in G'$. Suppose $x \notin H$ and

$$y \in x^{-1} H x \cap H = g^{-1} H g \cap H.$$

As y is the only element in $H \cap yG'$ we have $g^{-1} y g = y$. Thus $y \in N_G(g) = G'$ and so $y = 1$. This shows that G is a Frobenius group and so it is isomorphic to a sharply 2-transitive permutation group of degree q. Any such group is the group $G = L_2(F)$ of permutations of the form $x \to ax + b$ ($a, b \in F$, $a \neq 0$) where F is a finite near-field of order q (cf. e.g. [1] p. 401). Now if $\alpha : x \to ax + b$, $\beta : x \to cx + d$ are two elements of $L_2(F)$ then

$$\alpha^{-1} \beta^{-1} \alpha \beta : x \to cac^{-1} a^{-1} x + (-cac^{-1} a^{-1} b - cac^{-1} d + cb + d).$$

Substituting $a = 1$ we see that

$$x \to x + (cb - b)$$

is in G' thus G' contains all permutations of the form

$$x \to x + f \quad (f \in F).$$

These permutations form a subgroup of order q so this must be the whole G'. Consequently $cac^{-1} a^{-1} = 1$ for any $a, c \in F$, $a, c \neq 0$, i.e. the multiplicative group of F is commutative and so $F = GF(q)$. Thus $G = L_2(q)$ as we have stated in 1. of the theorem.

It is easy to check directly that for $G=L_2(q)$ we have indeed $k=|G:G'|+1$. This completes the proof.

LEMMA. *Let us call two elements a, b of a finite group G equivalent if there is a* $\varphi \in \mathrm{Aut}(G)$ *such that* $a\varphi=b$. *Suppose that G has* 4 *equivalence classes with respect to this equivalence relation (one of them being the identity element). Then G is either solvable or* $G \simeq A_5$.

PROOF. All the elements of an equivalence class have the same order. Thus either $|G|$ is divisible only by two primes and then G is solvable or $|G|=p^a q^b r^c$ ($p<q<r$ primes) and the elements in the 3 non-identity equivalence classes have orders p, q and r, resp. Suppose G is not solvable. Then G is characteristically simple since a non-trivial characteristic subgroup H would consist of 1 and one or two non-identity equivalence classes so both $|H|$ and $|G/H|$ would have only two prime divisors implying the solvability of G. Thus G is the direct product of s copies of a simple group G_0, $|G_0|=p^{\frac{a}{s}} q^{\frac{b}{s}} r^{\frac{c}{s}}$. But $G_0 \times G_0$ contains an element of order pq so we have $s=1$ and G simple. By the Feit—Thompson theorem we have $p=2$. A Sylow 2-subgroup of G has only elements of order 2 so it is elementary abelian. Now we can apply a theorem of HERZOG [2] stating that if G is simple, $|G|=2^a q^b r^c$ and the Sylow 2-subgroups of G are abelian then $G \simeq A_5$ or $G \simeq \mathrm{PSL}(2, 8)$. But PSL (2, 8) contains an element of order 9 which completes the proof.

REMARK. We could have completed the proof by using a theorem of ITO [4] instead of the theorem of Herzog, too.

THEOREM 2. *If* $k \leqq |G:G'|+3$ *for a finite group G then G is solvable.*

PROOF. If $k \leqq |G:G'|+2$ then G' contains at most 3 equivalence classes (of G') in the sense of the previous lemma. In particular there are at most two primes dividing the order of G' so applying the theorem of Burnside we see that G is solvable.

If $k=|G:G'|+3$ then by the previous considerations we have to consider only the case when G' contains 4 equivalence classes (one of them being the identity element) and $|G'|=p^a q^b r^c$. By the lemma we have either G' solvable or $G' \simeq A_5$. In the latter case consider the subgroup $A \subseteq \mathrm{Aut}(G')$ of those automorphisms which arise when we restrict the inner automorphisms of G to G'. $\mathrm{Aut}(G') \simeq S_5$, $\mathrm{Inn}(G') \simeq A_5$ so either $A \simeq A_5$ or $A \simeq S_5$.

It is obvious that any factor group of a group with at most 3 non-linear irreducible characters itself has at most 3 non-linear irreducible characters. Now $A \simeq G/C_G(G')$ but A_5 and S_5 both have at least 4 non-linear irreducible characters. Thus the case $G' \simeq A_5$ can not occur and this completes the proof.

The author is indebted to L. BABAI for suggesting some simplifications of the proofs.

(Received 21 May 1973)

Note added in proof (4 February 1974). I have learnt only recently, that finite groups with exactly one non-linear irreducible character had been characterized by G. SEITZ: Finite groups having only one irreducible representation of degree greater than one, *Proc. Amer. Math. Soc.*, **19** (1968), pp. 459—461, similarly to the above Theorem 1. Our proof gives slightly more, asserting $\Phi(G) = G'$ in case A, which yields an explicit description of the groups under 2 and 3. The proofs in case B do not coincide either.

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References

[1] R. D. CARMICHAEL, *Introduction to the theory of groups of finite order.* New York: Dover Publ. Inc. 1956.
[2] M. HERZOG, On finite simple groups of order divisible by three primes only, *J. Algebra,* **10** (1968), pp. 383—388.
[3] B. HUPPERT, *Endliche Gruppen I.* Berlin—Heidelberg—New York: Springer 1967.
[4] N. ITO, On finite groups with given conjugate types III, *Math. Z.*, **117** (1970), pp. 267—271.

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ON LIMIT DISTRIBUTIONS OF SEQUENCES OF RANDOM VARIABLES WITH RANDOM INDICES

By
S. CSÖRGŐ (Szeged)

§. 1

Throughout this paper let $\{\xi_n\}$ be a sequence of independent, identically distributed random variables (r.v.'s) having mean value 0 and variance 1, defined on a probability space $\{\Omega, \mathscr{A}, \mathbf{P}\}$ and let $\{\nu_n\}$ denote a sequence of positive, integer-valued r.v.'s on $\{\Omega, \mathscr{A}, \mathbf{P}\}$ such that

$$\frac{\nu_n}{n} \xrightarrow{\mathbf{P}} \nu,$$

($\xrightarrow{\mathbf{P}}$ denotes convergence in probability) where ν is a r.v. satisfying $\mathbf{P}\{\nu \leqq 0\} = 0$. It is known that the distribution of $S_{\nu_n}/\sqrt{\nu_n}$ (where $S_k = \xi_1 + \ldots + \xi_k$) converges to the standard normal distribution. This proposition was proved first (as a special case of a more general theorem) by ANSCOMBE [1] in the special case when ν is constant with probability 1. RÉNYI [2] extended this result to discrete ν, later on MOGYORÓDI [8] and BLUM, HANSON and ROSENBLATT [3], independently, obtained a proof for arbitrary positive ν. Finally GUIASU [5] has proved a general theorem (see also [4]), which contains all the formerly mentioned ones, and we record it here in a slightly modified form (see [16]).

THEOREM A. *Let $a_n > 0$ and b_n sequences of constants and let $\{(x_n - b_n)/a_n\}$ be a mixing sequence of r.v.'s with the distribution function F, in the sense of* RÉNYI*'s definition* [14], *satisfying also the following Anscombe type condition, that is, for arbitrary positive ε and η there exist a positive c and an integer n_0 such that*

$$(*) \qquad \mathbf{P}\Big\{\max_{n(1-c) \leqq m \leqq n(1+c)} |x_n - x_m| > \varepsilon a_n \,|\, A\Big\} < \eta$$

for $n > n_0$. Here A is some event of positive probability in the σ-algebra generated by ν. Then the limit distribution of $\{(x_{\nu_n} - b_{\nu_n})/a_{\nu_n}\}$ is F.

SREEHARI [15] (with the above given ξ_n, S_n and ν_n) proved this "randomised" version of the first two ERDŐS—KAC limit distribution theorems [6], concerning $\{n^{-\frac{1}{2}} \max_{1 \leqq k \leqq n} S_k\}$ and $\{n^{-\frac{1}{2}} \max_{1 \leqq k \leqq n} |S_k|\}$, while BARNDORFF-NIELSEN [2] proved the same type generalization of Gnedenko's theorem for $\{(\max(\xi_1, \ldots, \xi_n) - b_n)/a_n\}$. Also, MOGYORÓDI [7] gave a generalization of the results of [2] proving a stability property for this last sequence. The general method of these three papers can be summarized as follows. All three authors prove the mixing property for their respective sequences, and then, by direct computations, obtain the quoted results. However, the Anscombe condition $(*)$ can also be verified for these sequences

in question (see [5]), and thus these results are special cases of Theorem A. It is therefore of interest to investigate what enables us to pay no attention to Anscombe's condition. It will be shown in this paper that, besides a sufficiently wide restriction on the behaviour of the constants, mixing and monotonicity are the crucial properties of these sequences, so the monotonicity part is the one which, in these cases, takes over the role of Anscombe's condition. In applications it is also easier to observe the former then to check the latter, in general.

We also note here that, in general, Anscombe's condition and the requirement of monotonicity with the condition below on the constants do not seem to provide information about each other, however, in Sreehari's case the Anscombe condition is seen to be true via the increasing monotonicity of the sequences involved.

§. 2

THEOREM 1. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of constants,* $a_n>0$, *and* F *a distribution function, and let* $\{X_n\}$ *be a sequence of r.v.'s defined on* $\{\Omega, \mathscr{A}, \mathbf{P}\}$. *Let us suppose that the following conditions hold.*

1) *The sequence* $\{(X_n-b_n)/a_n\}$ *is mixing with* F, *that is for every* $A\in\mathscr{A}$ *of positive probability*

$$\lim_{n\to\infty} \mathbf{P}\{X_n < a_n y+b_n \mid A\} = F(y),$$

at every continuity point y *of* F.

2) *The sequence* $\{X_n\}$ *is monotonically increasing with probability* 1, *that is, for every* n

$$\mathbf{P}\{X_n \leqq X_{n+1}\} = 1.$$

3) $\{a_n\}$ *and* $\{b_n\}$ *are such that if* $0<p_n$ *and* $p_n\to p>0$, *then*

$$\frac{a_n}{a_{[np_n]}} \to f(p) \quad \text{and} \quad \frac{b_{[np_n]}-b_n}{a_{[np_n]}} \to g(p),$$

where $0\neq f(p)$ *and* $g(p)$ *are both continuous and monotonic functions of* p *for* $p>0$. *Then* $\{(X_{\nu_n}-b_{\nu_n})/a_{\nu_n}\}$ *is also mixing with* F.

REMARK 1. Theorem 1 is also valid if $\{X_n\}$ monotonically decreases with probability 1.

REMARK 2. Theorem 1 is also valid if the condition on $\{\nu_n\}$ is that $\nu_n/\omega(n) \xrightarrow{\mathbf{P}} \nu$, where the function $\omega(n)$ increases monotonically to ∞.

REMARK 3. If, for instance, $a_n=n^\alpha L(n)$ is slowly oscillating and $b_n\equiv 0$, then condition 3) is satisfied. We will need the following result of MOGYORÓDI (Lemma 2 in [7]):

LEMMA. *If condition* 3) *of Theorem* 1 *is satisfied, then*

$$\frac{a_n}{a_{\nu_n}} \xrightarrow{\mathbf{P}} f(\nu) \quad \text{and} \quad \frac{b_{\nu_n}-b_n}{a_{\nu_n}} \xrightarrow{\mathbf{P}} g(\nu).$$

PROOF OF THEOREM 1. This proof is essentially the same as that of Theorem 3 of MOGYORÓDI [7], so we give only those details where it slightly differs from that. For an arbitrary small $\varepsilon>0$ choose $0<p_1<p_2$ so that $\mathbf{P}\{p_1\leqq\nu<p_2\}>1-\varepsilon$. For an arbitrary $\delta>0$ there is a $\varrho>0$ such that if $|x-y|<\varrho$ ($x, y\in[p_1, p_2]$) then $\left|\frac{f(x)}{f(y)}-1\right|<\delta$ and $|g(x)-g(y)|<\delta$. Let us divide the interval $[p_1, p_2]$ by the subdivision points $p_1=Z_0<Z_1<\ldots<Z_k=p_2$ such that $\max\limits_{1\leqq i\leqq k}(Z_i-Z_{i-1})<\varrho$ and all the Z_i's are continuity points of the distribution function of ν. Using condition 2), for an arbitrary event A, we have for sufficiently large n

$$-\varepsilon+\sum_{i=1}^{k}\mathbf{P}\left\{\frac{X_{[nZ_i]}-b_{[nZ_i]}}{a_{[nZ_i]}}\,\frac{a_{[nZ_i]}}{a_{\nu_n}}-\frac{b_{\nu_n}-b_{[nZ_i]}}{a_{\nu_n}}<y,\ A,\ Z_{i-1}\leqq\nu<Z_i\right\}\leqq$$

$$\leqq\mathbf{P}\{X_{\nu_n}<a_{\nu_n}y+b_{\nu_n},\ A\}\leqq$$

$$\leqq\sum_{i=1}^{k}\mathbf{P}\left\{\frac{X_{[nZ_{i-1}]}-b_{[nZ_{i-1}]}}{a_{[nZ_{i-1}]}}\,\frac{a_{[nZ_{i-1}]}}{a_{\nu_n}}-\frac{b_{\nu_n}-b_{[nZ_{i-1}]}}{a_{\nu_n}}<y,\ A,\ Z_{i-1}\leqq\nu<Z_i\right\}+\varepsilon.$$

By the Lemma and the choice of the points Z_i we have with a probability exceeding $1-\varepsilon$ for sufficiently large n and every $1\leqq i\leqq k$

$$1-2\delta\leqq\frac{a_{[nZ_i]}}{a_{\nu_n}},\quad\frac{a_{[nZ_{i-1}]}}{a_{\nu_n}}\leqq1+2\delta$$

and

$$-2\delta(1+K)\leqq\frac{b_{\nu_n}-b_{[nZ_i]}}{a_{\nu_n}},\quad\frac{b_{\nu_n}-b_{[nZ_{i-1}]}}{a_{\nu_n}}\leqq2\delta(1+K),$$

where $K=\max(|g(p_1)|, |g(p_2)|)$. By these relations, using the mixing property 1) we obtain

$$-2\varepsilon+F\left(\frac{y-2\delta(1+K)}{1\pm2\delta}\right)\mathbf{P}\{A, p_1\leqq\nu<p_2\}\leqq$$

$$\leqq\liminf_{n\to\infty}\mathbf{P}\{X_{\nu_n}<a_{\nu_n}y+b_{\nu_n},A\}\leqq\limsup_{n\to\infty}\mathbf{P}\{X_{\nu_n}<a_{\nu_n}y+b_{\nu_n},A\}\leqq$$

$$\leqq F\left(\frac{y+2\delta(1+K)}{1\mp2\delta}\right)\mathbf{P}\{A, p_1\leqq\nu<p_2\}+2\varepsilon,$$

and as ε and δ are arbitrary, this proves Theorem 1.

As known, we may define stable sequences of r.v.'s as follows (see [13] or [7, 10]).

DEFINITION. $\{Y_n\}$ is stable with local density $\alpha_x(\omega)$ at x if for every event B we have

$$\lim_{n\to\infty}\mathbf{P}\{Y_n<x,\ B\}=\int_B\alpha_x(\omega)\,d\mathbf{P}(\omega).$$

We quote here another result of MOGYORÓDI [7, 10].

THEOREM B. *Let $\{X_n\}$ be a mixing sequence of r.v.'s with limit distribution function F and $\{\xi_n\}$, $\{\eta_n\}$ be sequences of r.v.'s for which $\xi_n \xrightarrow{\mathbf{P}} \xi$, $\eta_n \xrightarrow{\mathbf{P}} \eta$, where ξ and η are arbitrary r.v.'s. If $g(x, y, z)$ is a continuous function of three variables then the sequence $g(X_n, \xi_n, \eta_n)$ of r.v.'s is stable with local density*

$$\alpha_x(\omega) = \int\limits_{\{g(y, \xi(\omega), \eta(\omega)) < x\}} dF(y).$$

THEOREM 2. *Let all the suppositions of Theorem 1 be satisfied and further (and throughout the rest of this note) let $\{\mu_n\}$ be another sequence of positive integer-valued r.v.'s on $\{\Omega, \mathcal{A}, \mathbf{P}\}$ for which $\frac{\mu_n}{n} \xrightarrow{\mathbf{P}} \mu$, where μ is a positive r.v. Then the sequence $\{(X_{\nu_n} - b_{\mu_n})/a_{\mu_n}\}$ is stable with local density*

$$F\left(x \frac{f(v)}{f(\mu)} + g(\mu) \frac{f(v)}{f(\mu)} - g(v)\right).$$

PROOF. By Theorem 1 $\{(X_{\nu_n} - b_{\nu_n})/a_{\nu_n}\}$ is mixing with F, so we can apply Theorem B for the sequence

$$\frac{X_{\nu_n} - b_{\mu_n}}{a_{\mu_n}} = \frac{X_{\nu_n} - b_{\nu_n}}{a_{\nu_n}} \frac{a_{\nu_n}}{a_{\mu_n}} - \frac{b_{\mu_n} - b_{\nu_n}}{a_{\mu_n}}$$

with the function $g(x, y, z) = xy - z$ and get the required result.

COROLLARY 1. *Let $\mu_n \equiv n$ and $B = \Omega$ in Theorem 2. Then*

$$\lim_{n \to \infty} \mathbf{P}\{X_{\nu_n} < a_n y + b_n\} = \int\limits_0^\infty F\left(y \frac{f(t)}{f(1)} + g(1) \frac{f(t)}{f(1)} - g(t)\right) d\mathbf{P}(\nu < t).$$

COROLLARY 2. *Let $\nu_n \equiv n$ and $B = \Omega$ in Theorem 2. Then*

$$\lim_{n \to \infty} \mathbf{P}\{X_n < a_{\mu_n} y + b_{\mu_n}\} = \int\limits_0^\infty F\left(y \frac{f(1)}{f(t)} + g(t) \frac{f(1)}{f(t)} - g(1)\right) d\mathbf{P}(\mu < t).$$

§. 3

Now the results of MOGYORÓDI in [7] and consequently that of BARNDORFF—NIELSEN [2] concerning $(\max(\xi_1, \ldots, \xi_n) - b_n)/a_n$ are special cases of Theorem 2. Also, all the results of SREEHARI [15] and the random indexed analogues of the other two ERDŐS—KAC theorems in terms of the more general notion of stability are immediate consequences of Theorem 2, as the following theorem states.

THEOREM 3. *The sequences of r.v.'s*

(1) $\left\{\mu_n^{-\frac{1}{2}} \max_{1 \le k \le \nu_n} S_k\right\}$, (2) $\left\{\mu_n^{-\frac{1}{2}} \max_{1 \le k \le \nu_n} |S_k|\right\}$,

(3) $\{\mu_n^{-2}(S_1^2 + S_2^2 + \ldots + S_{\nu_n}^2)\}$ *and* (4) $\left\{\mu_n^{-\frac{3}{2}}(|S_1| + |S_2| + \ldots + |S_{\nu_n}|)\right\}$

are stable with local densities $\sigma_1\left(x\left(\frac{\mu}{\nu}\right)^{\frac{1}{2}}\right)$, $\sigma_2\left(x\left(\frac{\mu}{\nu}\right)^{\frac{1}{2}}\right)$, $\sigma_3\left(x\left(\frac{\mu}{\nu}\right)^{2}\right)$ *and* $\sigma_4\left(x\left(\frac{\mu}{\nu}\right)^{\frac{3}{2}}\right)$, *respectively* (*for the distribution functions* σ_1, σ_2, σ_3 *and* σ_4 *see* [6]).

COROLLARY 3. *For all the four cases Theorem* 3 *contains the assertions of the type of Corollaries* 1 *and* 2 *and also that for* $\mu_n \equiv \nu_n$ *the sequence* (*i*) *is mixing with* σ_i $(i=1, 2, 3, 4)$.

PROOF OF THEOREM 3. As the constants $a_n=n^{\frac{1}{2}}$ in cases of (1) and (2), $a_n=n^2$ of (3) and $a_n=n^{\frac{3}{2}}$ of (4) with $b_n\equiv 0$ in all cases satisfy condition 2) of Theorem 2 with $f(p)=p^{-\frac{1}{2}}$, $f(p)=p^{-2}$, $f(p)=p^{-\frac{3}{2}}$ resp. and $g(p)\equiv 0$ in all cases and as all sequences of r.v.'s involved are monotonically increasing we have only to prove mixing. For (1) and (2) it is proved by SREEHARI [15]. For (3) it can be done as follows. For any increasing sequence of positive integers $\{k_n\}$, $k_n \leqq n$, the event

$$A_n^{(x)} = \{(S_{k_n+1}-S_{k_n})^2+\ldots+(S_n-S_{k_n})^2 < n^2x\} = \{S_1^2+\ldots+S_n^2-Z_{k_n} < n^2x\}$$

(where $Z_{k_n} = S_1^2+\ldots+S_{k_n}^2+(n-k_n)S_{k_n}^2-2\sum_{i=k_n+1}^{n} S_iS_{k_n}$) depends only on $\xi_{k_n+1}, \ldots, \xi_n$. Thus, by Lemma 3 of [2], $\mathbf{P}\{A_n^{(x)}|A\}-\mathbf{P}\{A_n^{(x)}\}\to 0$, as $n\to\infty$, for any $A\in\mathscr{A}$ having positive probability. At the same time, if $k_n^6/n\to 0$ then $\mathbf{P}\{D_n\}=\mathbf{P}\{Z_{k_n}>n^2\varepsilon\}\to 0$ $(n\to\infty$, ε arbitrary positive), as a simple calculation shows, so for such $\{k_n\}$ we have $\mathbf{P}\{A_n^{(x)}\}\to\sigma_3(X)$. We need then only to prove

$$\mathbf{P}\{S_1^2+\ldots+S_n^2 < n^2x|A\}-\mathbf{P}\{A_n^{(x)}|A\} \to 0,$$

which follows by observing that

$$\mathbf{P}\{S_1^2+\ldots+S_n^2 < n^2x,\ \overline{A_n^{(x)}}\} \leqq$$

$$\leqq \mathbf{P}\{D_n\}+\mathbf{P}\{S_1^2+\ldots+S_n^2 < n^2x,\ S_1^2+\ldots+S_n^2 \geqq n^2x+Z_{k_n},\ \overline{D_n}\} \leqq$$

$$\leqq \mathbf{P}\{D_n\}+\mathbf{P}\left\{x-\varepsilon \leqq \frac{S_1^2+\ldots+S_n^2}{n^2} < x\right\} \to \sigma_3(x)-\sigma_3(x-\varepsilon)$$

and the continuity of $\sigma_3(x)$. In case (4) the mixing property can be similarly proved.

REMARK 4. Knowing the way of proof of Rényi's basic results concerning the limit distributions for empirical distribution functions [11], in order to obtain random sample size analogues, one thinks of direct applications of Theorem A or 1. For example Rényi reduced the problem of determining the limit distribution of

$$n^{\frac{1}{2}} \sup_{a\leqq F(x)} \frac{F_n(x)-F(x)}{F(x)} = R_n$$

to that of

$$R_n^* = n^{\frac{1}{2}} \max_{1\leqq k\leqq n(1-a)+1} \sum_{i=1}^{k} \frac{\delta_i-1}{n+1-i},$$

where the δ_i's are independent, exponentially distributed r.v.'s. To determine now the limit distribution of R_{ν_n} one can easily repeat Rényi's proof to obtain that it is

enough again to deal with $R^*_{\nu_n}$. The sequence $R^*_{\nu_n}$ of r.v.'s is clearly mixing but not monotonic, and I have not managed to verify that it satisfies the Anscombe's condition. This randomized version in question, among related ones, is proved in [16] by means of Theorem A and the so-called invariance methods.

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References

[1] F. J. ANSCOMBE, Large-sample theory of sequential estimation, *Proc. Cambridge Philos. Soc.*, **48** (1952), pp. 600—607.

[2] O. BARNDORFF-NIELSEN, On the limit distribution of the maximum of a random number of independent random variables, *Acta Math. Acad. Sci. Hung.*, **15** (1964), pp. 399—403.

[3] J. BLUM, D. HANSON and J. ROSENBLATT, On the central limit theorem for the sum of a random number of independent random variables, *Zeitschrift für Wahrsch. und Verw. Gebiete*, (1963), pp. 389—393.

[4] M. CSÖRGŐ and S. CSÖRGŐ, On weak convergence of randomly selected partial sums, *Acta Sci. Math. Szeged*, **34** (1973), pp. 53—60.

[5] S. GUIASU, On the asymptotic distribution of the sequences of random variables with random indices, *Ann. Math. Statist.*, **42** (1971), pp. 2018—2028.

[6] P. ERDŐS and M. KAC, On certain limit theorems of the theory of probability, *Bull. of Amer. Math. Soc.*, **52** (1946), pp. 292—302.

[7] J. MOGYORÓDI, On the limit distribution of the maximal term of a random sample of random size (in Hungarian), *MTA III. Oszt. Közl.*, **17** (1967), pp. 75—83.

[8] J. MOGYORÓDI, A central limit theorem for the sum of a random number of independent random variables, *Publ. Math. Inst. Hung. Acad. Sci.*, **7A** (1962), pp. 409—424.

[9] J. MOGYORÓDI, Limit distributions for sequences of random variables with random indices, *Trans. Fourth Prague Conf.* (Prague, 1965), pp. 463—470.

[10] J. MOGYORÓDI, A remark on the stable sequences of random variables and a limit distribution theorem for the sum of independent random variables, *Acta Math. Acad. Sci. Hung.*, **17** (1966), pp. 401—409.

[11] A. RÉNYI, On the theory of order statistics, *Acta Math. Acad. Sci. Hung.*, **4** (1953), pp. 191—227.

[12] A. RÉNYI, On the central limit theorem for the sum of a random number of independent random variables, *Acta Math. Acad. Sci. Hung.*, **11** (1960), pp. 383—393.

[13] A. RÉNYI, On stable sequences of events, *Sankhyā*, **25A** (1963), pp. 293—302.

[14] A. RÉNYI and P. RÉVÉSZ, On mixing sequences of random variables, *Acta Math. Acad. Sci. Hung.*, **19** (1958), pp. 383—393.

[15] M. SREEHARI, A limit theorem for the maximum of cumulative sums, *Acta Math. Acad. Sci. Hung.*, **19** (1968), pp. 117—120.

[16] S. CSÖRGŐ, On weak convergence of the empirical process with random sample size, *Acta Sci. Math. Szeged*, **36** (1974), pp. 17—25.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (3—4), (1974), pp. 233—243.

ZUM WERTEBEREICH EINER DIOPHANTISCHEN APPROXIMATIONS-FUNKTION

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1. Einleitung

Wie üblich sei $\mathbf{R}$ die Menge der reellen, $\mathbf{Q}$ der rationalen, $\mathbf{Z}$ der ganzen und $\mathbf{N}$ der natürlichen Zahlen. Weiter sei zu einem $\xi\in\mathbf{R}$ und $q\in\mathbf{N}$: $\|q\xi\| = \min\limits_{p\in\mathbf{Z}} |q\xi-p|$.
In der vorliegenden Arbeit wird der Wertebereich der Funktion

$$v(\xi) = \liminf_{q\to\infty} q\|q\xi\| \quad \text{bzw.} \quad M(\xi) = 1/v(\xi)$$

untersucht. (Für $\xi\in\mathbf{Q}$ ist $v(\xi)=0$. Es genügt also, sich auf $\xi\in\mathbf{R}\setminus\mathbf{Q}$ zu beschränken.)

Nach HURWITZ (siehe [3], Seite 11) gilt für jedes $\xi\in\mathbf{R}$: $M(\xi)\geqq\sqrt{5}$, und nach Markoff (siehe [3], Seite 18 folgende) bilden die Werte von $M(\xi)$ im abgeschlossenen Intervall $\{\sqrt{5}, 3\}$ eine unendliche gegen 3 konvergierende Folge. Außerdem gilt $M(\xi)=3$ für überabzählbar viele $\xi\in\mathbf{R}$. Nach PERRON [1] liegen in den offenen Intervallen $(\sqrt{12}, \sqrt{13})$ und $\left(\sqrt{13}, \frac{9\cdot\sqrt{3}+65}{22}\right)$ keine Werte von $M(\xi)$, außerdem gilt $M(\xi)=\sqrt{13}$ nur für abzählbar viele $\xi\in\mathbf{R}$, während $M(\xi)=\sqrt{12}=2\cdot\sqrt{3}$ und $M(\xi)=\frac{9\cdot\sqrt{3}+65}{22}$ für überabzählbar viele $\xi\in\mathbf{R}$ gilt. Über den Wertebereich von $M(\xi)$ in $(3, 2\cdot\sqrt{3})$ ist bisher nichts bekannt. Wir zeigen in dieser Arbeit

SATZ 1. *In $(3, \sqrt{12})$ gibt es abzählbar viele offene disjunkte Intervalle, in denen keine Funktionswerte von $M(\xi)$ liegen.*

SATZ 2. *In $(3, \sqrt{12})$ liegen mindestens abzählbar viele Häufungspunkte von Funktionswerten von $M(\xi)$.*

2. Kettenbrüche

Eine zentrale Rolle bei der Bestimmung der Funktionswerte von $M(\xi)$ spielt die Theorie der Kettenbrüche, denn nach [1] gilt

$$M(\xi) = \limsup_{q\in\mathbf{N}} (q\|q\xi\|)^{-1} = \limsup_{n\to\infty} \varrho_n,$$

mit

$$\varrho_n = (\xi_n+\eta_{n-1}), \quad \xi = [a_0, a_1, \ldots], \quad \xi_n = [a_n, a_{n+1}, \ldots]$$

und

$$\eta_{n-1} = [0, a_{n-1}, a_{n-2}, \ldots, a_0].$$

Dabei bezeichnet $[a_0, a_1, a_2, \ldots]$ einen regelmäßigen Kettenbruch, und um Verwechslungen auszuschließen, bezeichnen wir abgeschlossene Intervalle mit $\{x, x+a\}$. In dieser Arbeit werden beschränkte Kettenbrüche, d. h. Kettenbrüche mit $a_i \leqq k$, $k \in \mathbf{N}$ für alle $i \in \mathbf{N} \cup \{0\}$ und periodische Kettenbrüche, d. h. solche mit $a_{i+k} = a_i$ für alle $i \in \mathbf{N} \cup \{0\}$ und $i \geqq l$ für ein $l \in \mathbf{N} \cup \{0\}$ mit festem $k \in \mathbf{N} \cup \{0\}$, also $\xi = [a_0, a_1, \ldots, a_i, \ldots, a_{i+k-1}, a_i, \ldots] = [a_0, \ldots, \overline{a_i, \ldots, a_{i+k-1}}]$, eine besondere Rolle spielen. Ferner benötigen wir drei Ergebnisse aus der Theorie der Kettenbrüche:

Für $\xi = [a_0, a_1, \ldots]$, $\xi_n = [a_n, a_{n+1}, \ldots]$ und die Folge $\{x_\nu / y_\nu\}_{\nu=0}^{\infty}$ der Näherungsbrüche von ξ gilt

$$(1)\quad \xi = (x_n \xi_{n+1} + x_{n-1}) / (y_n \xi_{n+1} + y_{n-1}),$$

$$(2)\quad x_n y_{n-1} - x_{n-1} y_n = (-1)^{n-1},$$

für $\xi = [a_0, a_1, \ldots]$, $\xi' = [b_0, b_1, \ldots]$, $a_i = b_i$ für alle $0 \leqq i < n$, $n \in \mathbf{N}$ und $a_n < b_n$ gilt $\xi < \xi'$, wenn n gerade ist und $\xi > \xi'$, wenn n ungerade ist.

Nach [3], Seite 9 haben zueinander äquivalente Zahlen ξ und ξ' (Schreibw.: $\xi \cong \xi'$) von einer bestimmten Stelle an die gleiche Kettenbruchentwicklung, und es gilt $M(\xi) = M(\xi')$.

DEFINITION 1. $F(n) := \{\xi \mid \xi = [a_0, a_1, \ldots],\ a_i \leqq n \text{ für alle } i \in \mathbf{N} \cup \{0\}\}$ heißt die Menge der durch n beschränkten Kettenbrüche.

LEMMA 1. *Gilt* $M(\xi) \leqq \sqrt{12}$, *dann folgt* $\xi \in F(2)$.

BEWEIS. Nach [1] Seiten 7/11 gilt $M(\xi) \leqq \sqrt{b^2+4b}$ und $M(\xi) \geqq \sqrt{b^2+4}$, wenn es in der Kettenbruchentwicklung von $\xi = [a_0, a_1, \ldots]$ unendlich viele a_ν gibt mit $a_\nu \geqq b$. Für $\xi \in F(n)$, $n > 2$ aber $\xi \notin F(2)$ ist dann $M(\xi) \geqq \sqrt{n^2+4} \geqq \sqrt{13}$ und für $\xi \in F(2)$ folgt $M(\xi) \leqq \sqrt{12}$.

3. Bestimmung der Lücken und Häufungspunkte

DEFINITION 2. Für $\lambda \in \mathbf{N} \cup \{0\}$ sei

$L_\lambda := 1, 2, 1, \ldots$ eine Folge von λ Zahlen, die mit 1 beginnt und bei der sich 1 und 2 abwechseln,

$L'_\lambda := 2, 1, 2, \ldots$ eine Folge von λ Zahlen, die mit 2 beginnt und bei der sich 2 und 1 abwechseln und

$L''_\lambda := \ldots, 1, 2, 1$ eine Folge von λ Zahlen, bei der sich 1 und 2 abwechseln und die immer auf 1 endet.

Gibt es in der Kettenbruchentwicklung von ξ eine Folge von L_{λ_i} oder L'_{λ_i} mit $\lambda_i \to \infty$, dann gilt $M(\xi) = [\overline{2, 1}] + [0, \overline{1, 2}] = \sqrt{12}$ ([1], Seite 7 (A. 8)). Tritt L_λ oder L'_λ in der Kettenbruchentwicklung von ξ für alle $\lambda \leqq N$, $N \in \mathbf{N}$, nur endlich oft auf, dann ist entweder $\xi \cong [\overline{1}]$ oder $\xi \cong [\overline{2}]$, und es gilt $M(\xi) = \sqrt{5}$ bzw. $M(\xi) = \sqrt{8}$ ([3], Seite 11). Für $\xi \ncong [\overline{1}]$, $\xi \ncong [\overline{2}]$ und $\xi \ncong \xi'$ mit $M(\xi') = \sqrt{12}$ gibt es deshalb ein $\lambda > 1$, so daß L_λ oder L'_λ unendlich oft auftreten aber L_μ oder L'_μ nur endlich oft für $\mu > \lambda$. Wir betrachten nun Ausdrücke der Form $L''_k, 2, L_j$ und $L''_j, 2, L_k$ mit $0 \leqq j \leqq k$, $k - j \leqq 2$, die unendlich oft auftreten.

Für λ gerade sei $j:=\lambda/2-1$, $k:=\lambda/2$. Ist λ ungerade, $(\lambda-1)/4\in\mathbf{N}$ und tritt L'_λ höchstens endlich oft auf, oder ist λ ungerade, $(\lambda+1)/4\in\mathbf{N}$ und tritt L_λ höchstens endlich oft auf, dann sei $j:=(\lambda-3)/2$, $k:=(\lambda+1)/2$. Sonst sei für λ ungerade $j:=k:=(\lambda-1)/2$. Für $\xi\cong[\overline{1}]$ oder $\xi\cong[\overline{2}]$ sei $j:=k:=0$. Tritt eine Folge von L_{λ_i} oder L'_{λ_i} mit $\lambda_i\to\infty$ auf, also $M(\xi)=\sqrt{12}$, dann sei $j:=\infty$.

Somit können wir aus der Kettenbruchentwicklung von $\xi\in F(2)$ eindeutig j bestimmen und damit $F(2)$ vollständig in Klassen G_j mit $j\in\mathbf{N}\cup\{0\}\cup\{\infty\}$ zerlegen. Für jede dieser Klassen G_j wollen wir nun eine untere und eine obere Schranke der Funktionswerte von $M(\xi)$ bestimmen. Dazu benötigen wir folgendes Lemma.

LEMMA 2. a) *Sei* $\alpha_j:=[\overline{L''_j, 2, L_j}]$, $j\geqq 1$, *dann gilt* $\alpha_j\in G_j$ *und* $M(\alpha_j)=$ $=[\overline{2, L_j, L''_j}]+[0, \overline{L_j, L''_j, 2}]$.

b) *Sei* $\beta_j:=[\overline{L'_j, 2, L_{j+2}, 2}]$ *für* j *gerade und* $\beta_j:=[\overline{L_j, 2, L_{j+2}, 1}]$ *für* j *ungerade, dann gilt* $\beta_j\in G_j$ *und* $M(\beta_j)=[\overline{2, L_{j+2}, 2, L'_j}]+[0, \overline{L_j, 2, L'_{j+2}, 2}]$ *für* j *gerade bzw.* $M(\beta_j)=[\overline{2, L_{j+2}, 1, L_j}]+[0, \overline{L_j, 1, L_{j+2}, 2}]$ *für* j *ungerade.*

BEWEIS. a) $\alpha_j=[\overline{L''_j, 2, L_j}]$, $j\geqq 1$. Für gerades j tritt L'_λ, $\lambda=2j+1$, also $(\lambda-1)/4\in\mathbf{N}$, unendlich oft auf. Deshalb gilt $\alpha_j\in G_{j'}$ mit $j'=(\lambda-1)/2=j$ oder $\alpha_j\in G_j$. Für ungerades j tritt L_λ, $\lambda=2j+1$, also $(\lambda+1)/4\in\mathbf{N}$, unendlich oft auf. Damit gilt $\alpha_j\in G_{j'}$ mit $j'=(\lambda-1)/2=j$ oder $\alpha_j\in G_j$. Sei $M'_i:=[L_i, \alpha_j]+[0, L'_{2j+1-i}, \alpha_j]$ für $i+j$ gerade, $1\leqq i\leqq 2j+1$, und $M_i:=[L'_i, \alpha_j]+[0, L_{2j+1-i}, \alpha_j]$ für $i+j$ ungerade $1\leqq i\leqq 2j+1$. Da die M_i und M'_i alle möglichen Häufungspunkte der ϱ_n bei der Kettenbruchentwicklung von ξ sind, gilt $M(\alpha_j)=\max\limits_i(M_i, M'_i)$. Wegen $j\geqq 1$ ist aber $M_i=[2, 1, \ldots]+[0, 1, \ldots]>[2, 1, 1]+[0, 1, 1]=3$, $i\geqq 1$, und wegen $M'_i=$ $=[1, \ldots]+[0, \ldots]<3$ gilt $M'_i<M(\alpha_j)$.

Wir brauchen also nur das Maximum der M_i zu bestimmen. Für $j=1$ ist $i=2$ der einzige mögliche Wert von i mit $i+j$ ungerade, also $M(\alpha_j)=M_2$, und damit folgt die Behauptung. Sei nun $j\geqq 2$. Wir wollen zeigen, daß für $1\leqq i\leqq j-1$, $i+j$ ungerade, gilt $M_i<M_{i+2}$:

$$M_i<M_{i+2}\Leftrightarrow[0, L_{i-1}, \alpha_j]+[0, L_{2j+1-i}, \alpha_j]<[0, L_{i+1}, \alpha_j]+[0, L_{2j-1-i}, \alpha_j].$$

Mit

$$x:=[0, L_{i-1}, \alpha_j],\quad y:=[0, L_{2j-i-1}, \alpha_j],$$

$$[0, L_{i+1}, \alpha_j]=[0, 1, 2, L_{i-1}, \alpha_j]=[0, 1, 2, x]=(2x+1)/(3x+1),$$

$$[0, L_{2j+1-i}, \alpha_j]=[0, 1, 2, L_{2j-1-i}, \alpha_j]=[0, 1, 2, y]=(2y+1)/(3y+1)$$

nach (1) folgt dann

$$M_i<M_{i+2}\Leftrightarrow x+\frac{2y+1}{3y+1}<y+\frac{2x+1}{3x+1}\Leftrightarrow 9xy\cdot(x-y)+3(x^2-y^2)+2(x-y)<0,$$

und wegen $1\leqq i\leqq j-1$ gilt $x\leqq[0, L_{j-2}, \alpha_j]<[0, L_j, \alpha_j]\leqq y$, also $M_i<M_{i+2}$. Für $j+1\leqq i\leqq 2j-1$ gilt $x\geqq[0, L_j, \alpha_j]>[0, L_{j-2}, \alpha_j]\geqq y$, und damit folgt dann $M_i>M_{i+2}$. Es gilt also $M(\alpha_j)=\max\limits_i M_i=M_{j+1}$ und damit die Behauptung.

b) I. j gerade, d. h. $\beta_j=[\overline{L'_j, 2, L_{j+2}, 2}]$.

L'_λ mit $\lambda=2j+3$, also $(\lambda+1)/4\in\mathbf{N}$, kommt unendlich oft vor, aber nicht L_λ, d. h. $\beta_j\in G_{j'}$ mit $j'=(\lambda-3)/2=j$, also $\beta_j\in G_j$. Seien

$$M_0 := [2, \beta_j]+[0, \beta_j],$$

$$M_i := [L_i, 2, \beta_j]+[0, L'_{2j+3-i}, 2, \beta_j], \quad 2\leqq i\leqq 2j+3, \quad i \text{ gerade},$$

$$M_i := [L'_i, 2, \beta_j]+[0, L_{2j+3-i}, 2, \beta_j], \quad 1\leqq i\leqq 2j+3, \quad i \text{ ungerade}.$$

Dann gilt $M(\beta_j)=\max\limits_{0\leqq i\leqq 2j+3} M_i$.

Sei $j=0$. Dann gilt

$$M_0 = \overline{[2, 2, 1, 2]}+[0, \overline{2, 1, 2, 2}] < \overline{[2, 1, 2, 2]}+[0, \overline{2, 2, 1, 2}] = M_1 = M_3$$

und

$$M_2 = [1, \ldots]+[0, \ldots] < 3 < M_1, \quad \text{also} \quad M(\beta_j) = M_1$$

und damit die Behauptung.

Für $j>0$ gilt $M(\beta_j)\geqq[2, 1, \ldots]+[0, 1, \ldots]>3$ und damit $M_i<3<M(\beta_j)$ für i gerade, $0\leqq i\leqq 2j+3$. Daraus folgt $M(\beta)=\max\limits_{1\leqq i\leqq 2j+3,\, i \text{ ungerade}} M_i$.

Für $1\leqq i\leqq j-1$, i ungerade, gilt $M_i<M_{i+2}$, denn mit

$$x := [0, L_{i-1}, 2, \beta_j], \quad y := [0, L_{2j+1-i}, 2, \beta_j],$$

$$[0, L_{i+1}, 2, \beta_j] = [0, 1, 2, x] = (2x+1)/(3x+1),$$

$$[0, L_{2j+3-i}, 2, \beta_j] = [0, 1, 2, y] = (2y+1)/(3y+1)$$

gilt

$$M_i < M_{i+2} \Leftrightarrow [0, L_{i-1}, 2, \beta_j]+[0, L_{2j+3-i}, 2, \beta_j] < [0, L_{i+1}, 2, \beta_j]+$$

$$+[0, L_{2j+1-i}, 2, \beta_j] \Leftrightarrow x+\frac{2y+1}{3y+1} < y+\frac{2x+1}{3x+1},$$

und damit also $M_i<M_{i+2}$ wegen $1\leqq i\leqq j-1$, i ungerade und $x\leqq[0, L_{j-2}, 2, \beta_i]<$ $<[0, L_{j+2}, 2, \beta_j]\leqq y$. Außerdem gilt

$$M_{j+1} = [L'_{j+1}, 2, \beta_j]+[0, L_{j+2}, 2, \beta_j] = [0, L_j, 2, \beta_j]+[L'_{j+3}, 2, \beta_j] = M_{j+3},$$

und für $j+3\leqq i\leqq 2j+1$ gilt $x\geqq[0, L_{j+2}, 2, \beta_j]>[0, L_j, 2, \beta_j]\geqq y$ und damit $M_i>M_{i+2}$. Daraus folgt also $M(\beta_j)=M_{j+3}$ und damit die Behauptung.

II. j ungerade, d. h. $\beta_j=\overline{[L_j, 2, L_{j+2}, 1]}$.

L_λ mit $\lambda=2j+3$, also $(\lambda-1)/4\in\mathbf{N}$, tritt unendlich oft auf aber nicht L'_λ, d. h. $\beta_j\in G_{j'}$ mit $j'=(\lambda-3)/2=j$ oder $\beta_j\in G_j$. Seien jetzt

$$M_0 := [1, \beta_j]+[0, \beta_j],$$

$$M_i := [L_i, 1, \beta_j]+[0, L'_{2j+3-i}, 1, \beta_j] \quad \text{für} \quad 1\leqq i\leqq 2j+3, \quad i \text{ ungerade}$$

$$M_i := [L'_i, 1, \beta_j]+[0, L_{2j+3-i}, 1, \beta_j] \quad \text{für} \quad 1\leqq i\leqq 2j+3, \quad i \text{ gerade}.$$

Es gilt wieder $M(\beta_j) = \max\limits_{0 \leqq i \leqq 2j+3} M_i$, und wegen $j \geqq 1$

$$M(\beta_j) \geqq [2, 1, \ldots] + [0, 1, \ldots] > 3,$$

$M_i = [1, \ldots] + [0, \ldots] < 3$ für $1 \leqq i \leqq 2j+3$, i ungerade und $i = 0$ gilt

$$M(\beta_j) = \max_{2 \leqq i \leqq 2j+2,\, i \text{ gerade}} M_i.$$

Sei nun $x := [0, L_{i-1}, 1, \beta_j]$, $y := [0, L_{2j+1-i}, 1, \beta_j]$, dann gilt wie bei I $M_i < M_{i+2}$ für $1 \leqq i \leqq j-1$, i gerade, $M_{j+3} = M_{j+1}$ und $M_i > M_{i+2}$ für $j+3 \leqq i \leqq 2j+1$, i gerade, das heißt $M(\beta_j) = M_{j+3}$, und damit gilt die Behauptung.

Wir wollen nun zeigen, daß $M(\alpha_j)$ untere Schranke der Funktionswerte von $M(\xi)$ für alle $\xi \in G_j$ ist mit $j \geqq 1$.

LEMMA 3. *Für alle* $\xi \in G_j$, $j \in \mathbf{N}$, *gilt* $M(\xi) \geqq M(\alpha_j)$.

BEWEIS. (A) Ist $\xi \cong \alpha_j$, dann gilt $M(\xi) = M(\alpha_j)$.

(B) Sei $\xi \ncong \alpha_j$, d. h. der Zahlenkomplex

$$b_m, b_{m-1}, \ldots, b_1, L_j'', 2, L_j, c_1, \ldots, c_{n-1}, c_n$$

tritt in der Kettenbruchenwicklung von ξ unendlich oft auf, wobei $b_{m-1}, \ldots, b_1$, L_j'', 2, L_j, $c_1, \ldots, c_{n-1}$ unendlich oft Teil der Kettenbruchentwicklung von α_j ist, aber nicht $b_m, b_{m-1}, \ldots, c_{n-1}$ oder $b_{m-1}, \ldots, c_{n-1}, c_n$. Ist $\xi = [a_0, a_1, \ldots]$, dann gibt es also eine unendliche Folge natürlicher Zahlen $\{l_i, i \in \mathbf{N}\}$ mit

$$[a_{l_i-j-m}, \ldots, a_{l_i}, \ldots, a_{l_i+j+n}] = [b_m, \ldots, 2, \ldots, b_n].$$

Ohne Beschränkung der Allgemeinheit sei $n \leqq 2j+1$. Dann gibt es ein $k \in \mathbf{N} \cup \{0\}$ mit $k \cdot (2j+1) < m \leqq (k+1) \cdot (2j+1)$.

(a) j ungerade. Dann ist

$$\varrho_{l_i} = [2, L_j, L_{n-1}, x'] + [0, L_j, \underbrace{L_{2j+1}}_{k\text{-mal}}, L_{q-1}, z']$$

und

$$M(\alpha_j) = [2, L_j, L_{n-1}, x] + [0, L_j, \underbrace{L_{2j+1}}_{k\text{-mal}}, L_{q-1}, z]$$

mit

$$q := m - k \cdot (2j+1), \quad x := [L_{2j+2-n}'', \overline{L_{2j+1}}], \quad z := [L_{2j+2-q}'', \overline{L_{2j+1}}],$$

$$x' := \xi_{l_i+j+n}, \quad z' := (\eta_{l_i-j-(2j+1)k-q})^{-1}.$$

Wir wollen zeigen, daß für alle $i \in \mathbf{N}$ gilt $\varrho_{l_i} \geqq M(\alpha_j)$. Sei nun

$$D_1 := [0, L_j, L_{n-1}, x'] - [0, L_j, L_{n-1}, x]$$

und

$$D_2 := [0, L_j, \underbrace{L_{2j+1}}_{k\text{-mal}}, L_{q-1}, z] - [0, L_j, \underbrace{L_{2j+1}}_{k\text{-mal}}, L_{q-1}, z'],$$

dann gilt $\varrho_{l_i} \geqq M(\alpha_j) \Leftrightarrow D_1 \geqq D_2$. Für k gerade gilt $D_2 < 0 < D_1$, also die Behauptung. Sei k also ungerade. Seien x_r/y_r und x_{r-1}/y_{r-1} die letzten beiden Näherungsbrüche von $[0, L_j, L_{n-1}]$. Mit $r = j+n-1$, $\varepsilon := y_{r-1}/y_r > 0$, $a=1$ für n ungerade, $a=2$ für n gerade,

$$[0, L_j, L_{n-1}, x'] = \frac{x'x_r + x_{r-1}}{x'y_r + y_{r-1}}, \quad [0, L_j, L_{n-1}, x] = \frac{xx_r + x_{r-1}}{xy_r + y_{r-1}},$$

$$[0, L_j, L_{n-1}, a, y] = \frac{\left(a+\frac{1}{y}\right)x_r + x_{r-1}}{\left(a+\frac{1}{y}\right)y_r + y_{r-1}}, \quad [0, L_j, L_{n-1}, a, y'] = \frac{\left(a+\frac{1}{y'}\right)x_r + x_{r-1}}{\left(a+\frac{1}{y'}\right)y_r + y_{r-1}}$$

wegen (1) mit

$$y := [L''_{2j+1-n}, \underbrace{L_{2j+1}}_{(k-1)\text{-mal}}, L_{q-1}, z], \quad y' := [L''_{2j+1-n}, \underbrace{L_{2j+1}}_{(k-1)\text{-mal}}, L_{q-1}, z']$$

gilt mit $x_r y_{r-1} - x_{r-1} y_r = (-1)^{r-1}$ wegen (2)

$$D_1 = \frac{x'x_r + x_{r-1}}{x'y_r + y_{r-1}} - \frac{xx_r + x_{r-1}}{xy_r + y_{r-1}} = \frac{(x'-x)(-1)^{r-1}}{y_r^2(x'+\varepsilon)(x+\varepsilon)} \geqq 0$$

und

$$D_2 = \frac{\left(a+\frac{1}{y}\right)x_r + x_{r-1}}{\left(a+\frac{1}{y}\right)y_r + y_{r-1}} - \frac{\left(a+\frac{1}{y'}\right)x_r + x_{r-1}}{\left(a+\frac{1}{y'}\right)y_r + y_{r-1}} = \frac{\left(\frac{1}{y}-\frac{1}{y'}\right)(-1)^{r-1}}{y_r^2\left(a+\frac{1}{y}+\varepsilon\right)\left(a+\frac{1}{y'}+\varepsilon\right)} > 0$$

$$\frac{D_1}{D_2} = \frac{(x'-x)\left(a+\frac{1}{y}+\varepsilon\right)\left(a+\frac{1}{y'}+\varepsilon\right)}{\left(\frac{1}{y}-\frac{1}{y'}\right)(x'+\varepsilon)(x+\varepsilon)} \geqq 0.$$

(α) Für n ungerade ist $a=1$, $x=[1, \ldots]$, $x'=[2, \ldots]$ und D_1/D_2 ist monoton steigend in x' und $1/y'$ und monoton fallend in x und $1/y$. Aus

$$x' \geqq [2, \overline{2, 1}] = \frac{3}{2} + \frac{1}{2}\sqrt{3}, \quad \frac{1}{y'} \geqq [0, \overline{2, 1}] = -\frac{1}{2} + \frac{1}{2}\sqrt{3},$$

$$x \leqq [1, \overline{1, 2}] = \sqrt{3}, \qquad \frac{1}{y} \leqq [0, \overline{1, 2}] = \sqrt{3} - 1$$

folgt

$$\frac{D_1}{D_2} \geqq \frac{\left(\frac{3}{2}-\frac{1}{2}\sqrt{3}\right)(\sqrt{3}+\varepsilon)\left(\frac{1}{2}\sqrt{3}+\frac{1}{2}+\varepsilon\right)}{\left(\frac{1}{2}\sqrt{3}-\frac{1}{2}\right)(\sqrt{3}+\varepsilon)\left(\frac{3}{2}+\frac{1}{2}\sqrt{3}+\varepsilon\right)} = \frac{3+\sqrt{3}+2\varepsilon\sqrt{3}}{3+\sqrt{3}+2\varepsilon} > 1,$$

also $D_1 > D_2$.

(β) Für n gerade ist $a=2$, $x=[2, \ldots]$, $x'=[1, \ldots]$ und D_1/D_2 ist monoton steigend in x und $1/y$ und monoton fallend in x' und $1/y'$, und wie bei (α) folgt $D_1>D_2$.

(b) j gerade. Ist

$$\varrho_{l_i} = [2, L_j, L'_{n-1}, x'] + [0, L_j, \underbrace{L'_{2j+1}}_{k\text{-mal}}, L'_{q-1}, z'],$$

x' und z' wie in (a),

$$x := [L''_{2j+1-n}, 2, \overline{L'_{2j+1}}], \quad z := [L''_{2j+1-q}, 2, \overline{L'_{2j+1}}],$$

$$D_1 := [0, L_j, L'_{n-1}, x'] - [0, L_j, L'_{n-1}, x],$$

$$D_2 := [0, L_j, \underbrace{L'_{2j+1}}_{k\text{-mal}}, L'_{q-1}, z] - [0, L_j, \underbrace{L'_{2j+1}}_{k\text{-mal}}, L'_{q-1}, z'],$$

dann gilt $M(\alpha_j) = [2, L_j, L'_{n-1}, x] + [0, L_j, \underbrace{L'_{2j+1}}_{k\text{-mal}}, L'_{q-1}, z]$ und $M(\xi) \geqq M(\alpha_j)$ für $\varrho_{l_i} > M(\alpha_j)$ bzw. $D_1 > D_2$, und das folgt wie bei (a). Damit ist Lemma 3 bewiesen.

Nach der unteren Schranke für $M(\xi)$, $\xi \in G_j$, $j \in \mathbf{N}$, wollen wir eine obere Schranke der Funktionswerte von $M(\xi)$ für alle $\xi \in G_j$, $j \in \mathbf{N} \cup \{0\}$, bestimmen:

LEMMA 4. *Für alle $\xi \in G_j$ gilt $M(\xi) \leqq M(\beta_j)$ für $j \in \mathbf{N} \cup \{0\}$.*

BEWEIS. Für $\xi = [a_0, a_1, a_2, \ldots] \neq \beta_j$ wollen wir zeigen, daß für alle $\nu \in \mathbf{N}$ außer höchstens endlich vielen

$$\varrho_\nu = [a_\nu, a_{\nu+1}, \ldots] + [0, a_{\nu-1}, \ldots] = \xi_\nu + \eta_{\nu-1} \leqq M(\beta_j),$$

also $\limsup_{\nu \in \mathbf{N}} \varrho_\nu \leqq M(\beta_j)$ gilt. Aus Lemma 2, Beweis, folgt $M(\beta_j) > 3$. Für $a_\nu = 1$ gilt aber $\varrho_\nu = [1, \ldots] + [0, \ldots] < 3$, und für $a_{\nu-1} = a_\nu = a_{\nu+1} = 2$ gilt $\varrho_\nu = [2, 2, \ldots] + [0, 2, \ldots] < [2, 2] + [0, 2] = 3$ und damit die Behauptung.

Wir betrachten im folgenden den Fall $a_\nu = 2$, a_ν aus L_k oder L'_k. Wegen $\xi \in G_j$, $j \neq \infty$, das heißt $M(\xi) \neq \sqrt{12}$, gibt es höchstens endlich viele L_k oder L'_k mit $k > 2j+3$. Es gibt also ein ν_0, so daß in der Kettenbruchentwicklung von ξ_{ν_0} nur L_k oder L'_k mit $k \leqq 2j+3$ auftreten. Da bei der Bildung des $\limsup_{\nu \in \mathbf{N}}$ endlich viele ϱ_ν keine Rolle spielen, betrachten wir im folgenden nur ϱ_ν, $\nu \geqq \nu_0$.

(A) Sei j gerade, also $M(\beta_j) = [\overline{2, L_{j+2}, 2, L'_j}] + [0, L_j, 2, L'_{j+2}, 2]$ und seien M_i, $1 \leqq i \leqq 2j+3$, i ungerade, wie im Beweis von Lemma 2, b I. L_λ, $\lambda = 2j+3$, tritt höchstens endlich oft auf, denn sonst wäre $\xi \in G_{j'}$ mit $j' = (\lambda-1)/2 = j+1$, also $\xi \notin G_j$. Wir betrachten also nur L_k, $k < 2j+3$, und L'_k, $k \leqq 2j+3$.

(1) $k = 2j+3$: $\xi_\nu = [2, L_{i-1}, 2, \xi_{\nu+i+1}]$, $\eta_{\nu-1} = [0, L_{2j+3-i}, 2, x]$ mit $x = (\eta_{\nu+i-2j-5})^{-1}$, $1 \leqq i \leqq 2j+3$, i ungerade. Ist $\xi_{\nu+i+1} \neq \beta_j$, so gibt es folgende Möglichkeiten:

(a) $\xi_{\nu+i+1}$ und β_j unterscheiden sich an einer geraden Stelle, z. B. der l-ten Stelle, und sind an den ersten $(l-1)$ Stellen gleich. Daraus folgt $a_{\nu+i+l+1} = 1$ und damit $\xi_{\nu+i+1} < \beta_j$. Da $\xi_{\nu+i+1}$ an einer geraden Stelle von ξ_ν steht, gilt $\xi_\nu < [2, L_{i-1}, 2, \beta_j]$.

(b) Ist l ungerade, $l \neq k'(2j+4)-1$, $k' \in \mathbf{N}$, dann gilt $a_{\nu+i+l+1} = 2$ und damit $\xi_{\nu+i+1} < \beta_j$ und $\xi_\nu < [2, L_{i-1}, 2, \beta_j]$.

(c) Ist $l = k'(2j+4)-1$, $k' \in \mathbf{N}$, dann gilt $a_{\nu+i+l+1}=1$, das heißt in $\xi_{\nu+i+1}$ kommt L_k' mit $k>2j+4$ vor, und das hatten wir ausgeschlossen.

Für $x \neq \beta_j$ gibt es folgende Fälle:

(d) Sind x und β_j an den ersten $l-1$ Stellen gleich und unterscheiden sich an der l-ten Stelle (wieder bei der 0-ten Stelle angefangen), l gerade, dann gilt $a_{\nu+i-2j-5-l}=1$, und damit folgt $x<\beta_j$ und $\eta_{\nu-1}<[0, L_{2j+3-i}, 2, \beta_j]$.

(e) Ist l ungerade, $l \neq k'(2j+4)-1$, $k' \in \mathbf{N}$, dann gilt $a_{\nu+i-2j-5-l}=2$, und damit $x<\beta_j$ oder $\eta_{\nu-1}<[0, L_{2j+3-i}, 2, \beta_j]$.

(f) Ist l ungerade, $l=k'(2j+4)-1$, $k' \in \mathbf{N}$, dann gilt $a_{\nu+i-2j-5-l}=1$. Da $\xi \ncong \beta_j$ und da nur endlich viele L_k und L_k' mit $k>2j+3$ vorkommen, gibt es ein ν_1, so daß dieser Fall für $\nu>\nu_1$ nicht auftritt.

(2) $k<2j+3$:

(a) k ungerade, das heißt $k \leqq 2j+1$:

(α) a_ν aus L_k: $\xi_\nu=[2, L_i, 1, \xi_{\nu+i+2}]$, $\eta_{\nu-1}=[0, L_{k-i-1}, 1, x]$, $x=(\eta_{\nu+i-k-1})^{-1}$, i ungerade, $1 \leqq i \leqq k-2$. Dann gilt $\xi_\nu<[2, L_{i+1}, 2, \beta_j]$ und $\eta_{\nu-1}<[0, L_{2j+1-i}, 2, \beta_j]$.

(β) a_ν aus L_k': $\xi_\nu=[2, L_{i-1}, 2, \xi_{\nu+i+1}]$, $\eta_{\nu-1}=[0, L_{k-i}, 2, x]$, $x=(\eta_{\nu+i-k-2})^{-1}$, i ungerade, $1 \leqq i \leqq k$. Dann gilt $\xi_\nu \leqq [2, L_{i-1}, 2, \beta_j]$ wie bei (1) (a)—(c) und $\eta_{\nu-1}<[0, L_{2j+3-i}, 2, \beta_j]$, also $\varrho_\nu < M_i \leqq M(\beta_j)$.

(b) k gerade, das heißt $k \leqq 2j+2$.

(α) a_ν aus L_k: $\xi_\nu=[2, L_{i-1}, 2, \xi_{\nu+i+1}]$ und $\eta_{\nu-1}=[0, L_{k-i}, 1, x]$, $x=(\eta_{\nu+i-k-2})^{-1}$, i ungerade, $1 \leqq i \leqq k-1$. Wie bei (2)(a)(β) gilt $\xi_\nu \leqq [2, L_{i-1}, 2, \beta_j]$ und $\eta_{\nu-1}<$ $<[0, L_{2j+3-i}, 2, \beta_j]$, also $\varrho_\nu < M_i \leqq M(\beta_j)$.

(β) a_ν aus L_k': $\xi_\nu=[2, L_i, 1, \xi_{\nu+i+2}]$, $\eta_{\nu-1}=[0, L_{k-i-1}, 2, x]$, $x=(\eta_{\nu+i-k-1})^{-1}$, i ungerade, $1 \leqq i \leqq k-1$. Dann gilt $\xi_\nu<[2, L_{i+1}, 2, \beta_j]$ und wie bei Fall (1) (d)—(f) $\eta_{\nu-1}<[0, L_{2j+1-i}, 2, \beta_j]$ für $\nu>\nu_1$, also $\varrho_\nu < M_i \leqq M(\beta_j)$.

Für j gerade gibt es also ein ν_1, so daß für $\nu \geqq \nu_1$ $M(\beta_j) \geqq \varrho_\nu$, und damit ist die Behauptung bewiesen.

(B) Sei j ungerade, also $M(\beta_j)=\overline{[2, L_{j+2}, 1, L_j]}+[0, \overline{L_j, 1, L_{j+2}, 2}]$ und seien M_i, $1 \leqq i \leqq 2j+3$, i gerade, wie im Beweis von Lemma 2, Teil b II. Betrachtet man hier L_k, $k \leqq 2j+3$ und L_k', $k<2j+3$, dann ergeben sich entsprechende Fallunterscheidungen wie in Teil (A), und damit ist die Behauptung auch für ungerades j bewiesen.

Das erste Hauptergebnis unserer Arbeit lautet dann:

SATZ 1. *Sei* $\alpha_j:=\overline{[L_j'', 2, L_j]}$, $j \geqq 1$, $\beta_j:=\overline{[L_j', 2, L_{j+2}, 2]}$ *für gerade* j *und* $\beta_j:=$ $=\overline{[L_j, 2, L_{j+2}, 1]}$ *für ungerade* j, *dann nimmt die Funktion* $M(\xi)$ *in den abzählbar vielen offenen disjunkten Teilintervallen* $(M(\beta_j), M(\alpha_{j+1}))$, $j \in \mathbf{N} \cup \{0\}$, *des Intervalls* $\{3, 2\sqrt{3}\}$ *keinen Wert an.*

BEWEIS. Da wir $F(2)$ vollständig in Klassen G_j zerlegt haben und nach Lemma 1 für alle $M(\xi) \leqq \sqrt{12}$ gilt $\xi \in F(2)$, genügt es zu zeigen

$$M(\alpha_{j+1}) > M(\beta_j), \quad j \in \mathbf{N} \cup \{0\},$$

denn nach Lemma 3 gilt $M(\xi) \geqq M(\alpha_j)$ für alle $\xi \in G_j$ und nach Lemma 4 $M(\xi) \leqq$ $\leqq M(\beta_j)$ für alle $\xi \in G_j$.

a) j gerade:

$$M(\beta_j) = [\overline{2, L_{j+2}, 2, L'_j}] + [0, \overline{L_j, 2, L'_{j+2}, 2}]$$

und

$$M(\alpha_{j+1}) = [\overline{2, L_{j+1}, L''_{j+1}}] + [0, \overline{L_{j+1}, L''_{j+1}, 2}].$$

Nach einer Beweisidee von HALL [4], Seite 972 folgende setzen wir

$$u := [0, \overline{L_{j+2}, 2, L'_j, 2}], \quad v := [0, \overline{L_j, 2, L'_{j+2}, 2}], \quad w := [0, \overline{L_{j+1}, L''_{j+1}, 2}].$$

Dann gilt

$$M(\beta_j) < M(\alpha_{j+1}) \Leftrightarrow 2+u+v < 2+2w \Leftrightarrow (u-v)/(u-w) > 2.$$

Seien x_j/y_j und x_{j-1}/y_{j-1} die letzten beiden Näherungsbrüche von $[0, L_j]$, dann gilt nach (1)

$$[1, 2, 2, \beta_j] = (7\beta_j+3)/(5\beta_j+2), \quad [2, \beta_j] = (2\beta_j+1)/(\beta_j),$$

$$[1, \alpha_{j+1}] = (\alpha_{j+1}+1)/(\alpha_{j+1})$$

oder

$$u = \left[0, L_j, \frac{7\beta_j+3}{5\beta_j+2}\right], \quad v = \left[0, L_j, \frac{2\beta_j+1}{\beta_j}\right], \quad w = \left[0, L_j, \frac{\alpha_{j+1}+1}{\alpha_{j+1}}\right].$$

Mit $\varepsilon = y_{j-1}/y_j$ und (2) folgt wieder aus (1)

$$u-v = \frac{(2\beta_j+1)(5\beta_j+2)-(7\beta_j+3)\beta_j}{y_j^2(7\beta_j+3+5\varepsilon\beta_j+2\varepsilon)(2\beta_j+1+\varepsilon\beta_j)} =$$

$$= \frac{3\beta_j^2+6\beta_j+2}{y_j^2(7\beta_j+3+5\varepsilon\beta_j+2\varepsilon)(2\beta_j+1+\varepsilon\beta_j)},$$

$$u-w = \frac{(\alpha_{j+1}+1)(5\beta_j+2)-(7\beta_j+3)\alpha_{j+1}}{y_j^2(7\beta_j+3+5\varepsilon\beta_j+2\varepsilon)(\alpha_{j+1}+1+\varepsilon\alpha_{j+1})} =$$

$$= \frac{5\beta_j+2-2\alpha_{j+1}\beta_j-\alpha_{j+1}}{y_j^2(7\beta_j+3+5\varepsilon\beta_j+2\varepsilon)(\alpha_{j+1}+1+\varepsilon\alpha_{j+1})}$$

und damit

$$\frac{u-v}{u-w} = \frac{(3\beta_j^2+6\beta_j+2)(\alpha_{j+1}+1+\varepsilon\alpha_{j+1})}{(2\beta_j+1+\varepsilon\beta_j)(5\beta_j+2-2\alpha_{j+1}\beta_j-\alpha_{j+1})} =$$

$$= \frac{3\alpha_{j+1}\beta_j^2+6\alpha_{j+1}\beta_j+2\alpha_{j+1}+3\beta_j^2+6\beta_j+2+3\varepsilon\alpha_{j+1}\beta_j^2+6\varepsilon\alpha_{j+1}\beta_j+2\varepsilon\alpha_{j+1}}{10\beta_j^2+9\beta_j-4\alpha_{j+1}\beta_j^2-4\alpha_{j+1}\beta_j+2-\alpha_{j+1}+5\varepsilon\beta_j^2+2\varepsilon\beta_j-2\varepsilon\alpha_{j+1}\beta_j^2-\varepsilon\alpha_{j+1}\beta_j}.$$

Durch Multiplikation mit dem Nenner und Umordnung erhält man

$$(u-v)/(u-w) > 2 \Leftrightarrow \beta_j^2(11\alpha_{j+1}-17)+\beta_j(14\alpha_{j+1}-12)+4\alpha_{j+1}-2+$$

$$+\varepsilon\big(\beta_j^2(7\alpha_{j+1}-10)+\beta_j(8\alpha_{j+1}-4)+2\alpha_{j+1}\big) > 0,$$

und das ist wegen $\alpha_{j+1}>[1, 2, 1]=4/3$, $\beta_j<[2, 1, 2, 1]=11/4$ erfüllt. Damit ist die Behauptung für gerade j bewiesen.

b) j ungerade:

$$M(\beta_j) = [\overline{2, L_{j+2}, 1, L_j}]+[0, \overline{L_j, 1, L_{j+2}, 2}]$$

und

$$M(\alpha_{j+1}) = [\overline{2, L_{j+1}, L''_{j+1}}]+[0, \overline{L_{j+1}, L''_{j+1}, 2}].$$

Sei

$$u := [0, \overline{L_{j+2}, 1, L_j, 2}] \quad v := [0, \overline{L_j, 1, L_{j+2}, 2}]$$

und w wie in a), dann folgt wie in a) $M(\beta_j)<M(\alpha_{j+1})\Leftrightarrow(u-v)/(u-w)>2$ und nach gleicher Rechnung wie in a) $(u-v)/(u-w)>2 \Leftrightarrow 4\alpha_{j+1}(\beta_j^2-\beta_j-1)+$ $+(7\beta_j^2+6\beta_j+1)+\varepsilon\big(\alpha_{j+1}(\beta_j^2-2\beta_j-1)+(4\beta_j^2+2\beta_j)\big)>0$, und das ist mit $\beta_j>4/3$ und $\alpha_{j+1}<3$ erfüllt. Damit ist Satz 1 vollständig bewiesen.

Die Folge der $M(\beta_j)$ und $M(\alpha_{j+1})$ konvergiert gegen $\sqrt{12}$. Wir wollen nun in $\{3, M(\beta_0)\}$ abzählbar viele Häufungspunkte der Funktionswerte von $M(\xi)$ angeben und betrachten dazu $\xi\in G_0$. Entsprechende Konstruktionen kann man für alle Intervalle $\{M(\alpha_j), M(\beta_j)\}$ durchführen, aber man erhält wieder nur abzählbar viele Häufungspunkte, also keine wesentlich andere Aussage. Das zweite Hauptergebnis unserer Arbeit lautet dann:

SATZ 2. *Im Intervall* $\{3, 2\sqrt{3}\}$ *gibt es mindestens abzählbar viele Häufungspunkte, die selbst wieder Funktionswerte von* $M(\xi)$ *sind.*

BEWEIS. Wie schon gesagt, beschränken wir uns auf das Intervall $\{3, M(\beta_0)\}=$ $=\{3, 4\sqrt{30}/7\}$. Sei nun $\xi^{(n,m)} := [\overline{\underbrace{1, \ldots, 1}_{n\text{-stück}}, 2, \underbrace{2, \ldots, 2}_{m\text{-stück}}}]$, dann gilt $\xi^{(n,m)}\in G_0$ und wie bei Lemma 2

$$M(\xi^{(n,m)}) = [2, \underbrace{2, \ldots, 2}_{m\text{-stück}}, \xi^{(n,m)}]+[0, \xi^{(n,m)}].$$

Für $m\in\mathbf{N}$ sei

$$M^{(m)} := \lim_{n\to\infty} M(\xi^{(n,m)}) = \lim_{n\to\infty}([2, \underbrace{2, \ldots, 2}_{m\text{-stück}}, \xi^{(n,m)}])+\lim_{n\to\infty}([0, \xi^{(n,m)}]) =$$

$$= [2, \underbrace{2, \ldots, 2}_{m\text{-stück}}, \bar{1}]+[0, \bar{1}]$$

und für $n\in\mathbf{N}$ sei

$$M^{(n)'} := \lim_{m\to\infty} M(\xi^{(n,m)}) = [\bar{2}]+[0, \underbrace{1, \ldots, 1}_{n\text{-stück}}, \bar{2}].$$

Die Werte $M^{(m)}$ und $M^{(n)'}$ werden auch angenommen, denn für beliebiges $m \in \mathbf{N}$ gilt zum Beispiel für

$$\xi := [2, \underbrace{2, \ldots, 2}_{m\text{-stück}}, 1, 1, \underbrace{2, \ldots, 2}_{m\text{-stück}}, 1, 1, 1, 1, \underbrace{2, \ldots, 2}_{m\text{-stück}}, 1, 1, 1, 1, 1, 1, 2, \ldots]$$

$M(\xi) = M^{(m)}$. Analog gilt zum Beispiel für $n \in \mathbf{N}$ und

$$\xi := [\underbrace{1, \ldots, 1}_{n\text{-stück}}, 2, 2, 2, \underbrace{1, \ldots, 1}_{n\text{-stück}}, 2, 2, 2, 2, 2, 2, \underbrace{1, \ldots, 1}_{n\text{-stück}}, \ldots]$$

$M(\xi) = M^{(n)'}$. Damit ist Satz 2 bewiesen.

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Literaturverzeichnis

[1] O. PERRON, *Über die Approximation einer irrationalen Zahl durch rationale,* S.—B.—Heidelberg Akademie der Wissenschaften (1921), Abh. 4/8.
[2] O. PERRON, *Kettenbrüche,* Teubner Verlag (Berlin—Leipzig, 1929), 3. Aufl. 1954, Stuttgart.
[3] J. W. S. CASSELS, *An introduction to diophantine approximation,* Cambridge University Press No. 45 (1965).
[4] M. HALL, JR., On the sum and product of continued fractions, *Annals of Mathematics,* **48** (1947).

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STRONGLY HAUSDORFF SPACES

By

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1. Introduction. In [HJ, J] HAJNAL and JUHÁSZ introduced a separation axiom strictly between Hausdorff and Urysohn, called strongly Hausdorff, and used it to investigate the cardinality of discrete subsets of Hausdorff spaces. In response to a remark in [J], a simple example of a Hausdorff space which is not strongly Hausdorff is provided in Section 2 of this paper. Since an Urysohn, minimal Hausdorff space is compact, it is natural to inquire whether a strongly Hausdorff, minimal Hausdorff space is compact. A negative answer is provided in Section 3; also, in Section 3, minimal strongly Hausdorff (resp. strongly Hausdorff-closed) spaces are characterized as strongly Hausdorff spaces which are minimal Hausdorff (resp. H-closed).

A Hausdorff space X is defined to be *strongly Hausdorff* if for each infinite subset $A \subseteq X$, there is a sequence $\{U_n : n \in \mathbf{N}\}$ of pairwise disjoint open sets such that $A \cap U_n \neq \emptyset$ for each $n \in \mathbf{N}$. A space is *Urysohn* if every pair of distinct points have disjoint closed neighbourhoods. The symbols $\mathbf{N}$ and $\mathbf{Q}$ are used to denote the set of positive integers and rational numbers, respectively.

A few of the basic properties of strongly Hausdorff are listed below:

(1.1) [HJ] *A Urysohn space is strongly Hausdorff.*

(1.2) *The strongly Hausdorff property is hereditary.*

(1.3) *The strongly Hausdorff property is expansive,* i.e., if (X, τ) is strongly Hausdorff and σ is a topology on X such that $\sigma \supseteq \tau$, then (X, σ) is strongly Hausdorff.

A subset A of a space (X, τ) is called *regular-open* if $A = \mathrm{int}_X \mathrm{cl}_X A$; the set of regular-open subsets of a space forms a basis for a topology [B, p. 138] denoted as τ_s. τ_s is called the *semiregularization* of τ. A space (X, τ) is *semiregular* if $\tau = \tau_s$; in particular, (X, τ_s) is semiregular.

(1.4) *If (X, τ) is strongly Hausdorff, then so is (X, τ_s).*

(1.5) *The product of nonempty spaces is strongly Hausdorff if and only if each coordinate space is strongly Hausdorff.*

PROOF. Suppose the product of a family of nonempty spaces is strongly Hausdorff. Since each coordinate space is homeomorphic to a subspace of the product, then it follows by (1.2) that each coordinate space is strongly Hausdorff. Conversely, suppose $\{X_a : a \in A\}$ is a family of nonempty strongly Hausdorff spaces. ΠX_a is Hausdorff. Let $B \subseteq \Pi X_a$ be an infinite subset. If the projection of B to some coordinate space is infinite, then it is straightforward to find a sequence of pairwise disjoint, open sets each meeting B. Suppose the projection of B to each coordinate space is finite. Tehere is a coordinate $a_1 \in A$ such that the a_1-projection (denoted by p_1) of B contains at least two points, say b_1 and b_2, where $p_1^{-1}(b_1) \cap B$ is infinite. There are disjoint open sets U_1 of b_1 and U_2 of b_2 in the a_1-coordinate space. Let $V_1 = p_1^{-1}(U_2)$.

There is $a_2 \in A \setminus \{a_1\}$ such that the a_2-projection (denoted by p_2) of $p_1^{-1}(b_1) \cap B$ contains at least two points, say c_1 and c_2, where $p_2^{-1}(c_1) \cap p_1^{-1}(b_1) \cap B$ is infinite. There are disjoint open sets W_1 of c_1 and W_2 of c_2 in the a_2-coordinate space. Let $V_2 = p_1^{-1}(U_1) \cap p_2^{-1}(W_2)$. Continue by induction, obtaining a sequence $\{V_n : n \in \mathbf{N}\}$ of pairwise disjoint open sets such that $V_n \cap B \neq \emptyset$.

2. A Hausdorff space which is not strongly Hausdorff. This example was inspired by the corresponding example in [HJ]. Let $X = \{(n, n) : n \in \mathbf{N}\} \cup \left\{\left(y, n - \frac{1}{m}\right) \in \right.$ $\left. \in \mathbf{Q} \times \mathbf{Q} : n, m \in \mathbf{N}, m \neq 1\right\}$. The topology is generated by the open basis $\{\{(a, b)\} :$ $(a, b) \in X, a \neq b\} \cup \{U(n, r) : n, r \in \mathbf{N}\}$ where $U(n, r) = \{(n, n)\} \cup \left\{\left(y, n - \frac{1}{m}\right) : m > r\right\} \cup$ $\cup \{(n, y) : y > r\}$. Clearly, diagonal and nondiagonal points and distinct nondiagonal points can be separated by disjoint open sets. If $n < m$, then $U(n, m) \cap U(m, 1) = \emptyset$. So, X is Hausdorff.

Let $A = \{(n, n) : n \in \mathbf{N}\}$. Assume there is a sequence of pairwise disjoint sets $\{U_n : n \in \mathbf{N}\}$ such that $(i_n, i_n) \in U_n \cap A$. There is $r \in \mathbf{N}$ such that $U(i_1, r) \subseteq U_1$. There is $n \in \mathbf{N}$ such that $i_n > \max\{r, i_1\}$, and there is $s \in \mathbf{N}$ such that $V(i_n, s) \subseteq U_n$. But $\left(i_1, i_n - \frac{1}{s+1}\right) \in V(i_1, r) \cap V(i_n, s) \subseteq U_1 \cap U_n$, a contradiction. Hence, X is not strongly Hausdorff.

3. Strongly Hausdorff-closed and minimal strongly Hausdorff spaces. Let P be be a topological property. A space X with property P is called *P-closed* (resp. *minimal P*) if X is a closed subspace in every P-space in which it is embedded (resp. X has no strictly coarser P-topologies). The term "Hausdorff-closed" is shortened to "H-closed". Much of the known results about P-closed and minimal P spaces are presented in [BPS].

THEOREM 3.1. *A space is strongly Hausdorff-closed if and only if it is strongly Hausdorff and H-closed.*

PROOF. Suppose X is strongly Hausdorff-closed and a subspace of a Hausdorff space Y. To show X is closed in Y we can assume without loss of generality, that $Y \setminus X$ is a singleton. However, a one-point, Hausdorff extension of a strongly Hausdorff space is clearly strongly Hausdorff. So, it follows that X is closed in Y. The proof of the converse is obvious.

The following facts follow from the corresponding statements about H-closed spaces [B, p. 145—146] and minimal Hausdorff spaces [B, p. 146—147] in conjunction with (1.4), (1.5) and Theorem 3.1.

(3.2) *A space (X, τ) is strongly Hausdorff-closed if and only if (X, τ_s) is minimal strongly Hausdorff.*

(3.3) *A space is minimal strongly Hausdorff if and only if it is strongly Hausdorff and minimal Hausdorff.*

(3.4) *A product of nonempty spaces is strongly Hausdorff-closed (resp. minimal strongly Hausdorff) if and only if each coordinate space is strongly Hausdorff-closed (resp. minimal strongly Hausdorff).*

A filter on a space X is called *open* if it is generated by a filter base of open sets. An *open ultrafilter* on a space X is a maximal element in the set of all open filters on X partially ordered by set inclusion. A standard argument using Zorn's Lemma shows that an open filter is contained in some open ultrafilter. The *adherence* of a filter $\mathscr{F}$ on a space X is $\bigcap\{\mathrm{cl}\, F: F\in\mathscr{F}\}$ and denoted as $a(\mathscr{F})$. A filter with void adherence is called *free*, otherwise it is called *fixed*. The neighbourhood filter of a point x is denoted as $\mathscr{N}_x$.

THEOREM 3.5. *A strongly Hausdorff space can be densely embedded in a strongly Hausdorff-closed space.*

PROOF. Let X be a strongly Hausdorff, and let $\varkappa X$ be the KATĚTOV H-closed extension of X, cf. [K]. $\varkappa X$ is the set $X\cup\{\mathscr{U}: \mathscr{U}$ is free open ultrafilter on $X\}$ with the topology generated by the open basis $\{U: U$ open in $X\}\cup\{U\cup\{\mathscr{U}\}: U$ open in $X, \mathscr{U}\in\varkappa X\setminus X, U\in\mathscr{U}\}$. By Theorem 3.1, it suffices to show that $\varkappa X$ is strongly Hausdorff. Let A be an infinite subset of X. Since X is open in X and X is strongly Hausdorff, it is easy to reduce the problem to the case $A\subseteq\varkappa X\setminus X$. Let θX be the Stone—Čech compactification of the absolute of X [IF], i.e., θX is the set $\{\mathscr{U}: \mathscr{U}$ is open ultrafilter on $X\}$ with the topology generated by the open basis $\{o(U): U$ open in $X\}$ where $o(U)=\{\mathscr{U}\in\theta X: U\in\mathscr{U}\}$. Now $A\subseteq\theta X$, and by (1.1) there is a sequence $\{o(U_n): n\in\mathbf{N}\}$ of pairwise disjoint open sets such that $o(U_n)\cap A\neq\emptyset$. Let $\mathscr{U}_n\in o(U_n)\cap A$ and $V_n=U_n\cup\{\mathscr{U}_n\}$. Since V_n is open in $\varkappa X$, it remains to show that for $n\neq m$, $U_n\cap U_m=\emptyset$. Assume there is a point $p\in U_n\cap U_m$. Then $\mathscr{N}_p$ is contained in some open ultrafilter $\mathscr{U}_p$. But $\mathscr{U}_p\in o(U_n)\cap o(U_m)$, a contradiction.

(3.6) *A strongly Hausdorff space can be densely embedded in a minimal strongly Hausdorff space if and only if it is semiregular.*

PROOF. Follows from proof of Corollary 5.6 in [PT] and Theorem 3.5.

THEOREM 3.7. *A strongly Hausdorff space X can be embedded as a closed, nowhere dense subset of a minimal strongly Hausdorff space.*

PROOF. X can be embedded as a closed, nowhere dense subspace of a semiregular space Y [W, Exercise 14E3]; it is straightforward to show that Y is strongly Hausdorff when X is. Since $Y\setminus X$ is open and dense in Y, then $Y\setminus X$ is a member of every open ultrafilter on Y; so, X is a closed (and nowhere dense) subset of $\varkappa Y$. By the proof of Theorem 3.5, $\varkappa Y$ is strongly Hausdorff. Now $\varkappa Y$ can be embedded as a closed, nowhere dense subset of a strongly Hausdorff, semiregular space Z. By (3.6), Z can be densely embedded in a minimal strongly Hausdorff space W. Since $\varkappa Y$ is H-closed, $\varkappa Y$ is closed in W. Thus, X is a closed, nowhere dense subset of W.

A space by Urysohn (cf. Example 3.14 in [BPS]) is minimal strongly Hausdorff but not compact. We conclude this section with a minimal Hausdorff space which is not strongly Hausdorff. Let X denote the set $\left\{\left(k, n-\frac{1}{m}\right): k, n, m\in\mathbf{N},\ k\leqq n,\ m\neq 1\right\}\cup$ $\cup\{(k, -n): k, n\in\mathbf{N}\}$ and Y the set $X\cup\{(n, n): n\in\mathbf{N}\}$ plus two points a and b. A set $U\subseteq Y$ is open if

(i) $(n, n) \in U$ implies $\left\{\left(k, n-\frac{1}{m}\right) \in X : m > r\right\} \cup \{(n, y) \in X : |y| > r\} \subseteq U$ for some $r \in \mathbf{N}$,

(ii) $a \in U$ implies $\left\{\left(k, n-\frac{1}{m}\right) \in X : k > r\right\} \subseteq U$ for some $r \in \mathbf{N}$ and

(iii) $b \in U$ implies $\{(k, -n) \in X : k > r\} \subseteq U$ for some $r \in \mathbf{N}$. The space Y has the desired properties.

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References

[B] N. BOURBAKI, *General Topology,* Part 1, Addison-Wesley (Massachusetts, 1966).

[BPS] M. P. BERRI, J. R. PORTER and R. M. STEPHENSON, JR., A survey of minimal topological spaces, General Topology and its Relations to Modern Analysis and Algebra, *Proc. Kanpur Topology Conf.* 1968, Academic Press 1970, pp. 93—114.

[HJ] A. HAJNAL and I. JUHÁSZ, Some remarks on a property of topological cardinal functions, *Acta Math. Acad. Sci. Hung.*, **20** (1969), pp. 25—37.

[IF] S. ILIADIS and S. FOMIN, The method of centered systems in the theory of topological spaces, *Uspekhi Mat. Nauk*, **21** (1966), pp. 47—76; *Russian Math. Surveys,* **21** (1966), pp. 37—62.

[J] I. JUHÁSZ, On two properties between T_2 and T_3, *Proc. Symp. Topology,* Herceg-Novi, 1968, Belgrade, Yugoslavia, 1969, p. 210.

[K] M. KATĚTOV, Über H-abgeschlossene und bikompakt Räume, *Časopis pěst. mat.,* **69** (1940), pp. 36—49.

[PT] J. PORTER and J. THOMAS, On H-closed and minimal Hausdorff spaces, *Trans. Amer. Math. Soc.,* **138** (1969), pp. 159—170.

[W] S. WILLARD, *General Topology,* Addison-Wesley (Reading, Massachusetts, 1970).

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ON THE MEASURABLE SOLUTIONS OF THE FUNCTIONAL EQUATION

$$\sum_{i=1}^{2}\sum_{j=1}^{3} F_{i,j}(p_i q_j) = \sum_{i=1}^{2} G_i(p_i) + \sum_{j=1}^{3} H_j(q_j)$$

By

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1. Introduction

Let us denote by $\Delta_n := \{(p_1, p_2, \ldots, p_n)\colon p_i \geqq 0,\ i=1, 2, \ldots, n \text{ and } \sum_{i=1}^{n} p_i = 1\}$ the set of all complete finite n-ary probability distributions ($n=1, 2, \ldots$). In 1960 T. W. CHAUNDY and J. B. MCLEOD [2] gave the following result.

THEOREM 1. *The continuous solutions* $F\colon [0, 1] \to R$ *of the functional equation*

$$\begin{cases} \sum_{i=1}^{n}\sum_{j=1}^{m} F(p_i q_j) = \sum_{i=1}^{n} F(p_i) + \sum_{j=1}^{m} F(q_j) \\ (p_1, p_2, \ldots, p_n) \in \Delta_n, \quad (q_1, q_2, \ldots, q_m) \in \Delta_m, \quad n, m = 1, 2, \ldots \end{cases} \tag{1.1}$$

are given by

$$F(x) = \begin{cases} ax \log x & \text{for} \quad x \in]0, 1] \\ a0 \log 0 := 0 & \text{for} \quad x = 0 \end{cases} \tag{1.2}$$

where a is an arbitrary constant.

A simple proof of Theorem 1 was given by PL. KANNAPPAN [4] in 1972. In 1963 J. ACZÉL and Z. DARÓCZY [1] proved that the result remains true if the functional equation (1.1) is supposed to hold for $n=m$, $n=1, 2, \ldots$. In 1971 this was further generalized by DARÓCZY [3], based on a result of P. M. LEE [5] in 1964, to the following:

THEOREM 2. *The measurable solutions of the functional equation* (1.1) *for* $(n, m)=(1, 1)$ *and* $(2, 3)$ *are given by* (1.2).

In the next section we shall generalize these results by finding all measurable solutions of the functional equation (1.1) taking $(n, m)=(2, 3)$ alone, and we allow the functions for the eleven terms appearing in (1.1) to be all different.

2. Measurable solutions of some functional equations

We prepare our main theorem by a sequence of propositions.

PROPOSITION 1 (by DARÓCZY, cf. [3]). *The measurable solutions of the functional equation*

$$\begin{cases} F(q) - F(pq) - F(q - pq) = A(p, q) \quad p, q \in [0, 1] \\ q \to A(p, q) \ \textit{additive} \\ F(1) = 0 \end{cases} \tag{2.1}$$

are given by

$$\begin{cases} F(p) = ap\log p \\ A(p,q) = a[-p\log p-(1-p)\log(1-p)]q. \end{cases} \tag{2.2}$$

PROPOSITION 2. *The measurable solutions of the functional equation*

$$\begin{cases} F(q)-F(pq)-F(q-pq) = A(p,q) \\ q \to A(p,q) \text{ additive} \end{cases} \tag{2.3}$$

are given by

$$\begin{cases} F(p) = ap\log p+bp \\ A(p,q) = a[-p\log p-(1-p)\log(1-p)]q. \end{cases} \tag{2.4}$$

PROOF. Let $G(p):=F(p)-F(1)p$. The measurable function G satisfies

$$\begin{cases} G(q)-G(pq)-G(q-pq) = A(p,q) \\ q \to A(p,q) \text{ additive} \\ G(1) = 0 \end{cases}$$

and it follows from Proposition 1 that

$$\begin{cases} G(p) = ap\log p \\ A(p,q) = a[-p\log p-(1-p)\log(1-p)]q. \end{cases}$$

This proves (2.4) with $b=F(1)$. The converse part is trivial.

PROPOSITION 3. *The measurable solutions of the functional equation*

$$\begin{cases} F(q)-F(pq)-F(q-pq) = A(p,q)+B(p) \\ q \to A(p,q) \text{ additive} \end{cases} \tag{2.5}$$

are given by

$$\begin{cases} F(p) = ap\log p+bp+c \\ A(p,q) = a[-p\log p-(1-p)\log(1-p)]q \\ B(p) = -c. \end{cases} \tag{2.6}$$

PROOF. Putting in (2.5) $q=0$,

$$B(p) = -F(0) =: -c$$

follows. The function $G(p):=F(p)-c$ now satisfies

$$\begin{cases} G(q)-G(pq)-G(q-pq) = A(p,q) \\ q \to A(p,q) \text{ additive} \end{cases}$$

and Proposition 2 leads to

$$\begin{cases} G(p) = ap\log p+bp \\ A(p,q) = a[-p\log p-(1-p)\log(1-p)]q. \end{cases}$$

PROPOSITION 4. *The measurable solutions of the functional equation*

$$\begin{cases} F_1(q)-F_2(pq)-F_3(q-pq) = A(p,q)+B(p) \\ q \to A(p,q) \text{ additive} \end{cases} \tag{2.7}$$

are given by

$$\begin{cases} F_1(p) = ap\log p+b_1p+c_1 \\ F_2(p) = ap\log p+b_2p+c_2 \\ F_3(p) = ap\log p+b_3p+c_3 \\ A(p,q) = a[-p\log p-(1-p)\log(1-p)]q+(b_3-b_2)pq+(b_1-b_3)q \\ B(p) = c_1-c_2-c_3. \end{cases} \tag{2.8}$$

PROOF. Putting in (2.7) $q=0$, $p=1$ and $p=0$, we have

$$B(p) = F_1(0)-F_2(0)-F_3(0), \tag{2.9}$$

$$F_2(q) = F_1(q)-A(1,q)-F_1(0)+F_2(0), \tag{2.10}$$

and

$$F_3(q) = F_1(q)-A(0,q)-F_1(0)+F_3(0), \tag{2.11}$$

respectively. We replace F_2 and F_3 in (2.7) by means of (2.10) and (2.11) to get

$$\begin{cases} F_1(q)-F_1(pq)-F_1(q-pq) = A(p,q)-A(1,pq)-A(0,q-pq)-F_1(0) \\ \text{with } q \to A_1(p,q) := A(p,q)-A(1,pq)-A(0,q-pq) \text{ additive.} \end{cases}$$

Hence by Proposition 3 it follows

$$\begin{cases} F_1(p) = ap\log p+b_1p+F_1(0) \\ A_1(p,q) = a[-p\log p-(1-p)\log(1-p)]q. \end{cases}$$

If we put $b_2p := b_1p-A(1,p)$ and $b_3p := b_1p-A(0,p)$ the above equalities can be written as

$$\begin{cases} F_1(p) = ap\log p+b_1p+F_1(0) \\ A(p,q) = a[-p\log p-(1-p)\log(1-p)]q+A(1,pq)+A(0,q-pq) = \\ \qquad = a[-p\log p-(1-p)\log(1-p)]q+(b_1-b_2)pq+(b_1-b_3)(q-pq) = \\ \qquad = a[-p\log p-(1-p)\log(1-p)]q+(b_3-b_2)pq+(b_1-b_3)q. \end{cases} \tag{2.12}$$

From (2.10) and (2.12) it follows

$$F_2(p) = ap\log p+b_1p-A(1,p)+F_2(0) = ap\log p+b_2p+F_2(0). \tag{2.13}$$

From (2.11) and (2.12) it follows

$$F_3(p) = ap\log p+b_1p-A(0,p)+F_3(0) = ap\log p+b_3p+F_3(0). \tag{2.14}$$

This completes the proof of (2.8) with $c_i := F_i(0)$.

PROPOSITION 5. *The measurable solutions of the functional equation*

$$[F_1(q_1)-F_2(pq_1)-F_3(q_1-pq_1)]+[F_4(q_2)-F_5(pq_2)-F_6(q_2-pq_2)] = \tag{2.15}$$
$$= A(p, q_1+q_2) \qquad (p\in[0, 1],\ q_1,\ q_2,\ q_1+q_2\in[0, 1])$$

are given by

$$\begin{cases} F_i(p) = ap\log p+b_ip+c_i \quad \textit{for} \quad i=1, 2, 3 \\ F_i(p) = ap\log p+(b_{i-3}+d)p+c_i \quad \textit{for} \quad i=4, 5, 6 \\ A(p, q) = a[-p\log p-(1-p)\log(1-p)]q+(b_3-b_2)pq+ \\ \qquad +(b_1-b_3)q+c_1-c_2-c_3+c_4-c_5-c_6. \end{cases} \tag{2.16}$$

PROOF. We put in (2.15) $q_2=0$ and $q_1=0$ to get

$$F_1(q_1)-F_2(pq_1)-F_3(q_1-pq_1) = A(p, q_1)-F_4(0)+F_5(0)+F_6(0) \tag{2.17}$$

and

$$F_4(q_2)-F_5(pq_2)-F_6(q_2-pq_2) = A(p, q_2)-F_1(0)+F_2(0)+F_3(0), \tag{2.18}$$

respectively. Adding (2.17) to (2.18) and comparing it with (2.15) we have

$$A(p, q_1)+A(p, q_2) = A(p, q_1+q_2)+F_1(0)-F_2(0)-F_3(0)+F_4(0)-F_5(0)-F_6(0)$$

i.e.

$$q \to A_1(p, q) := A(p, q)-F_1(0)+F_2(0)+F_3(0)-F_4(0)+F_5(0)+F_6(0) \tag{2.19}$$

is additive.

We can now apply Proposition 4 to (2.17) and (2.19) and obtain

$$\begin{cases} F_1(p) = ap\log p+b_1p+c_1 \\ F_2(p) = ap\log p+b_2p+c_2 \\ F_3(p) = ap\log p+b_3p+c_3 \\ A(p, q) = a[-p\log p-(1-p)\log(1-p)]q+(b_3-b_2)pq+(b_1-b_3)q+ \\ \qquad +F_1(0)-F_2(0)-F_3(0)+F_4(0)-F_5(0)-F_6(0). \end{cases} \tag{2.20}$$

Similarly, from (2.18) and (2.19) we obtain

$$\begin{cases} F_4(p) = a'p\log p+b_4x+c_4 \\ F_5(p) = a'p\log p+b_5x+c_5 \\ F_6(p) = a'p\log p+b_6x+c_6 \\ A(p, q) = a'[-p\log p-(1-p)\log(1-p)]q+(b_6-b_5)pq+(b_4-b_6)q+ \\ \qquad +F_1(0)-F_2(0)-F_3(0)+F_4(0)-F_5(0)-F_6(0). \end{cases} \tag{2.21}$$

We compare the expression for A in (2.20) with that in (2.21) and get

$$a=a',\ b_3-b_2=b_6-b_5 \quad \text{and} \quad b_1-b_3=b_4-b_6. \tag{2.22}$$

Equations (2.20), (2.21) and (2.22) give (2.16). The converse is trivial.

We are now ready to give our main result.

THEOREM 3. *The measurable solutions of the functional equation*

$$\begin{cases} \sum_{i=1}^{2}\sum_{j=1}^{3} F_{i,j}(p_i q_j) = \sum_{i=1}^{2} G_i(p_i) + \sum_{j=1}^{3} H_j(q_j) \\ (p_1, p_2) \in \Delta_2,\ (q_1, q_2, q_3) \in \Delta_3 \end{cases} \tag{2.23}$$

are given by

$$\begin{cases} H_1(q) = aq\log q + b_1 q + c_1 \\ H_2(q) = aq\log q + (b_1+d)q + c_4 \\ H_3(q) = aq\log q + (b_1+e)q + c_7 \\ F_{1,1}(p) = ap\log p + b_2 p + c_2 \\ F_{1,2}(p) = ap\log p + (b_2+d)p + c_5 \\ F_{1,3}(p) = ap\log p + (b_2+e)p + c_8 \\ F_{2,1}(p) = ap\log p + b_3 p + c_3 \\ F_{2,2}(p) = ap\log p + (b_3+d)p + c_6 \\ F_{2,3}(p) = ap\log p + (b_3+e)p + c_9 \\ G_1(p) = g(p) \\ G_2(p) = -g(1-p) + a[p\log p + (1-p)\log(1-p)] + (b_3-b_2)p + (b_2-b_1) - \\ \qquad -c_1 + c_2 + c_3 - c_4 + c_5 + c_6 - c_7 + c_8 + c_9 \end{cases} \tag{2.24}$$

where $a, b_1, b_2, b_3, c_1, c_2, \ldots, c_9, d, e$ *are arbitrary constants and* g *is an arbitrary measurable function.*

PROOF. Let us put $p_1 = p$, $p_2 = 1-p$ in (2.23) and rewrite it as

$$\begin{aligned} &H_1(q_1) - F_{1,1}(pq_1) - F_{2,1}(q_1 - pq_1) + H_2(q_2) - F_{1,2}(pq_2) - F_{2,2}(q_2 - pq_2) = \\ &= F_{1,3}(pq_3) + F_{2,3}(q_3 - pq_3) - H_3(q_3) - G_1(p) - G_2(1-p) =: A(p, 1-q_3) \quad \text{say.} \end{aligned} \tag{2.25}$$

From the identity

$$H_1(q_1) - F_{1,1}(pq_1) - F_{2,1}(q_1 - pq_1) + H_2(q_2) - F_{1,2}(pq_2) - F_{2,2}(q_2 - pq_2) = A(p, q_1 + q_2)$$

we deduce from Proposition 5 that

$$\begin{cases} H_1(q) = aq\log q + b_1 q + c_1 \\ H_2(q) = aq\log q + (b_1+d)q + c_4 \\ F_{1,1}(p) = ap\log p + b_2 p + c_2 \\ F_{1,2}(p) = ap\log p + (b_2+d)p + c_5 \\ F_{2,1}(p) = ap\log p + b_3 p + c_3 \\ F_{2,2}(p) = ap\log p + (b_3+d)p + c_6 \\ A(p,q) = a[-p\log p - (1-p)\log(1-p)]q + (b_3-b_2)pq + (b_1-b_3)q + \\ \qquad + c_1 - c_2 - c_3 + c_4 - c_5 - c_6. \end{cases} \tag{2.26}$$

From (2.25) we have

$$F_{1,3}(pq_3)+F_{2,3}(q_3-pq_3)-H_3(q_3)-G_1(p)-G_2(1-p)=A(p,1-q_3)$$

and with the last identity in (2.26) we can rewrite it as

$$\text{(2.27)}\quad \begin{cases} H_3(q_3)-F_{1,3}(pq_3)-F_{2,3}(q_3-pq_3)= \\ =a[-p\log p-(1-p)\log(1-p)]q_3+(b_3-b_2)pq_3+(b_1-b_3)q_3- \\ -a[-p\log p-(1-p)\log(1-p)]-(b_3-b_2)p-(b_1-b_3)- \\ -c_1+c_2+c_3-c_4+c_5+c_6-G_1(p)-G_2(1-p). \end{cases}$$

We apply Proposition 4 to (2.27) and deduce that

$$\text{(2.28)}\quad \begin{cases} H_3(q) &= aq\log q+b_7q+c_7 \\ F_{1,3}(p) &= ap\log p+b_8p+c_8 \\ F_{2,3}(p) &= ap\log p+b_9p+c_9 \\ b_9-b_8 &= b_3-b_2; \\ b_7-b_9 &= b_1-b_3-a[-p\log p-(1-p)\log(1-p)]-(b_3-b_2)p-(b_1-b_3)- \\ -c_1+c_2+c_3-c_4+c_5+c_6-G_1(p)-G_2(1-p) &= c_7-c_8-c_9. \end{cases}$$

Equation (2.24) follows from (2.26) and (2.28). In fact it is easy to show that functions given by (2.24) satisfy (2.23).

3. Remark

The general measurable solutions of the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m}F_{i,j}(p_iq_j)=\sum_{i=1}^{n}G_i(p_i)+\sum_{j=1}^{m}H_j(q_j)$$

$$(p_1,p_2,\ldots,p_n)\in\Delta_n,\quad (q_1,q_2,\ldots,q_m)\in\Delta_m$$

for given integers $n, m\geqq 2$ with $n+m\geqq 5$ can be found easily if we equate some of the p_i's and q_j's to zero and reduce the equation to (2.23) giving the solutions of eleven functions at a time.

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References

[1] J. Aczél und Z. Daróczy, Charakterisierung der Entropien positiver Ordnung und der Shannonschen Entropie, *Acta Math. Acad. Sci. Hung.*, **14** (1963), pp. 95—121.
[2] T. W. Chaundy and J. B. McLeod, On a functional equation, *Edinburgh Math. Notes,* **43** (1960), pp. 7—8.
[3] Z. Daróczy, On the measurable solutions of a functional equation, *Acta Math. Acad. Sci. Hung.*, **22** (1971), pp. 11—14.
[4] Pl. Kannappan, On Shannon's entropy, directed divergence and inaccuracy, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, **22** (1972), pp. 95—100.
[5] P. M. Lee, On the axioms of information theory, *Ann. Math. Statis.*, **35** (1964), pp. 415—418.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (3—4), (1974), pp. 255—262.

A HOMOLOGICAL CHARACTERIZATION OF LARGE SUBGROUPS

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1. Introduction

A large subgroup of a p-primary abelian group must, together with any basic subgroup, generate the group. This fruitful idea, originated by R. S. PIERCE (see [8]), for the purpose of "getting a handle" on the homomorphism groups of primary groups, has resulted in some very beautiful discoveries. We mention here only the remarkable theorem of CUTLER and STRINGALL [3] that the completion with respect to the topology generated by the large subgroups is precisely the torsion completion with respect to the p-adic topology. For other interesting work on large subgroups see MEGIBBEN [6], and especially the recent paper of BENABDALLAH, EISENSTADT, IRWIN, and POLUIANOV [1].

Our initial sentence above serves to define the notion of a large subgroup only if we further demand that a large subgroup be fully invariant, and it is this aspect which we wish to investigate below. In particular we will show that all large subgroups may be described as *functorial subgroups*, i.e. as *preradicals* in the sense of NUNKE [7]. Our motivation in this work is to provide a way of looking at large subgroups from the homological point of view, with the idea in mind of employing these powerful methods to shed some light on yet another aspect of the theory of abelian groups.

Our point of departure will be the concept of a "representable subfunctor of the identity" as introduced by NUNKE [7]. By identifying an abelian group G with the homomorphism group $\mathrm{Hom}(Z, G)$, Z being the additive group of rational integers, Nunke was able to represent certain fully invariant subgroups of G in the form

$$\mathrm{Im}\left(\mathrm{Hom}(M, G) \xrightarrow{\mathrm{Hom}(\alpha, 1)} \mathrm{Hom}(Z, G)\right)$$

where $\alpha: Z \to M$ is a monomorphism and 1 is the identity map on G. We shall find it convenient to look at the adjoint situation and consider the functors Tor and $\otimes$. For the basic concepts of homological algebra the reader should consult FUCHS [5], or the standard reference CARTAN and EILENBERG [2].

It does not seem widely known that the germinal idea underlying the concept and characterization of large subgroups appeared in a fecund article of SHIFFMAN [9], in 1940.** It might also be well to point out that the word "large" seems to be employed in algebra generally to indicate large cardinality, for example of a subgroup, but

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** The author is indebted to Professor Cutler for this observation.

its use in our specialized sense has been firmly established in abelian groups from the time of its introduction by R. S. PIERCE [8].

As we shall be concerned with extensions $K \rightarrowtail F \twoheadrightarrow H$ where K is free and H is torsion, it might be worthwhile to recall the result of R. G. Swan that even in the non-abelian case, if F is torsion-free and K is free and of finite index, then F must in fact be free. In the abelian case, Phillip Griffith was able to obtain a considerable strengthening by replacing the hypothesis that K be of finite index with the assumption that H be a weak S_p-projective for some reduced cotorsion functor S_p, in the sense of NUNKE [7]. Combining this with the recent result of P. Dubois that if for subgroups K_1 and K_2 of the abelian group F, we have both F/K_1 and F/K_2 weak S_p-projective, then F/K_{12} is weak S_p-projective where K_{12} is the intersection of K_1 and K_2, we have the following:

THEOREM. *If F is a torsion free abelian group such that F contains subgroups K_1 and K_2 whose intersection is free and such that both F/K_1 and F/K_2 are weak S_p-projective for some reduced cotorsion functor S_p, then F is free.*

It should be noted that although arbitrary direct sums of reduced countable p-primary abelian groups are weak S_p-projectives, Paul Dubois has given an example to show that it does not follow that if both F/K_1 and F/K_2 are direct sums of countable groups, that F/K_{12} must be a direct sum of countable groups. In fact F/K_{12} need not even be summable. (See P. DUBOIS, An example in d.s.c. abelian groups, *Notices of the A. M. S.*, January 1972.)

In what follows, we have by no means been as general as possible, and the reader will have no trouble in going from large subgroups to wider classes of fully invariant subgroups by considering different extensions of the integers than the special case considered here. In particular we leave the more interesting extension to torsion free abelian groups of rank one to a later paper.

In the final section of this article we investigate the connections between a special case of F-purity as discussed in FUCHS [5], and in fact achieve a complete characterization of the F-pure short exact sequences in the case that F represents a functorial large subgroup.

2. Definitions

The following definitions and conventions will be employed throughout the remainder of this paper. We will only concern ourselves with abelian groups, and therefore "group" will mean "abelian group". We will discuss p-primary abelian groups, where p is a prime number which is fixed throughout. Z will denote the additive group of the rational integers and Q the additive group of rational numbers. We will denote by $Z(p^\infty)$ the p-primary component of Q/Z. For an arbitrary group G, G_t will denote its torsion subgroup; and G_m^n will mean the subgroup of all elements of order less than or equal to p^n of the subgroup of G consisting of all elements whose height in G is greater or equal to p^m, where G, is assumed p-primary, and n and m are non-negative integers.

For groups G and H, $G+H$ will denote the lattice sum and $G \oplus H$ the direct sum. For a family of groups, we will employ $\sum$ for the lattice sum and retain $\oplus$ for the direct sum. By $h_p(x)$ we shall mean the exponential p-height of x in a group

G determined in each case by the context. For groups G and H and a homomorphism f mapping G into H we will write $f:G\rightarrowtail H$ when f is injective and we will write $f:G\twoheadrightarrow G$ when f is surjective.

M_m^n will denote the group formed by adjoining a single element z_m^n to Z subject to the relation $p^{n+m}z_m^n=p^{m_1}$ where 1 is the distinguished generator of Z. The induced short-exact sequence formed thereby, $Z\overset{\alpha}{\rightarrowtail}M_m^n\overset{\beta}{\twoheadrightarrow}H_m^n$ will be denoted by c_m^n. When it is not obvious from the context we write α_m^n, and β_m^n to denote these maps.

The reader should observe that we have indulged in ambiguity in defining M_m^n in such a way as to conflict with our notation, $G_m^n=p^mG[p^n]$, but as we have limited our use of M to this one immutable case, *cavaet scholar*.

We shall write $c\in\in \operatorname{Ext}(B, A)$ to mean that the short exact sequence c is a member of one of the equivalence classes which comprise $\operatorname{Ext}(B, A)$. For any ordinal λ, $p^\lambda \operatorname{Ext}(B, A)$ is defined recursively as the intersection of all of the $p^\zeta \operatorname{Ext}(B, A)$ with $\zeta<\lambda$, in the case that λ is a limit ordinal, and as $p(p^{\lambda-1}\operatorname{Ext}(B, A))$ otherwise. $p\operatorname{Ext}(B, A)$ of course is $\{\Gamma|\Gamma=p\chi, \chi\in \operatorname{Ext}(B, A)\}$.

We will denote the exponential order of an element a of a p-primary abelian group A by $O(a)$ or by $O_A(a)$, if the context is ambiguous.

Our notation for the direct limit will be $\varinjlim$, however for the purpose at hand we might just as well use union.

We shall follow the rather unorthodox (but sensible) procedure of numbering theorems quoted from other sources with their number in the original work, together with a reference to the original.

NUNKE [7] defines as a subfunctor of the identity on abelian groups any covariant functor assigning to each group a subgroup of itself which in addition preserves monomorphisms in the sense that if S is our functor and $f\in \operatorname{Hom}(A, B)$ for arbitrary abelian groups A and B, then $fS(A)$ is included in $S(B)$. We shall be interested in (to use the current terminology) preradicals S as defined immediately above which have the stronger property of assigning to every p-primary abelian group a "large" subgroup of itself, where a "large subgroup" as defined in PIERCE [8], is a fully invariant subgroup that together with any basic subgroup generates the group. We shall call such preradicals, "large preradicals".

3. Preradicals

In this section we will introduce preradicals which in some sense are dual to the "representable subfunctors of the identity" defined by NUNKE in [7]. However, our preradicals are distinct from the "representable subfunctors of the identity" and this will also be indicated. We will also concern ourselves with the interconnections between our preradicals and their computation in some simple cases. This is in the way of a prelude to Theorem 1 wherein we construct all of the large subgroups of a group by suitably combining these special cases.

For an arbitrary group G we define:

$$J(c)G = \ker(\alpha\otimes l_G)$$

for exact $c:Z\overset{\alpha}{\rightarrowtail}M\overset{\beta}{\twoheadrightarrow}H$ and where $l_G:G\rightarrowtail\twoheadrightarrow G$ denotes the identity map.

PROPOSITION 1. *$J(c)$ is a preradical on abelian groups.*

The proof follows easily once we identify G with $Z \otimes G$.

For any group G and any short-exact sequence $A \overset{\Phi}{\rightarrowtail} B \overset{\Psi}{\twoheadrightarrow} Q/Z$, \$, we define:

$$T(\$)G = \text{im}\,(\text{Tor}\,(\psi, l_G)).$$

PROPOSITION 2. *$T(\$)$ is a preradical on abelian groups.*

Again by making the identification, $G_t = (\text{Tor}\, Q/Z, G)$, the proof is routine.

In the torsion case T and J are identical. We prove the implication in one direction and leave the analogous other case to the reader.

PROPOSITION 3. *For any $c \in \in \text{Ext}\,(H, Z)$, $c: Z \rightarrowtail M \twoheadrightarrow H$, there exists a $\$ \in \in \text{Ext}\,(Q/Z, M)$ such that $J(c)_t = T(\$)$; and conversely given any $\$ \in \in \text{Ext}\,(Q/Z, M)$, $\$: M \rightarrowtail H \twoheadrightarrow Q/Z$, there exists a $c \in \in \text{Ext}\,(H, Z)$ such that $J(c)_t = T(\$)$.*

PROOF. Here we are denoting by $J(c)_t$ the composite preradical formed from $J(c)$ and the preradical which assigns to every group its torsion subgroup. Let us call the short exact sequence, with all maps natural, $Z \rightarrowtail Q \twoheadrightarrow Q/Z$, @. Next form the commutative pushout diagram

$$\begin{array}{ccccc} @: & Z & \rightarrowtail & Q & \twoheadrightarrow & Q/Z \\ & {\scriptstyle\alpha}\downarrow & & \downarrow & & \| \\ \alpha@: & M & \rightarrowtail & N & \twoheadrightarrow & Q/Z \\ & \downarrow & & & & \\ & H & & & & \end{array}$$

This gives rise to a diagram, which is commutative:

$$\begin{array}{ccc} \text{Tor}\,(Q/Z, G) & \simeq & Z \otimes G_t \\ \| & & \downarrow{\scriptstyle \alpha \otimes l_G} \\ \text{Tor}\,(Q/Z, G) & \xrightarrow[\partial]{} & M \otimes G_t \end{array}$$

which gives, by setting $\$ = \alpha c$; $T(\$)G = \ker(\partial) = \ker(\alpha \otimes l_G) = (J(@)G)_t$.

To see that T is not a "representable subfunctor of the identity" in the sense of NUNKE [7], recall that for each such subfunctor and each group G, the image of G must contain its divisible subgroup. Then taking $@: Z \rightarrowtail Q \twoheadrightarrow Q/Z$ as above it is easily seen that $T(@)G = 0$ for all groups G, so we must conclude that T is not a "representable subfunctor of the identity".

We next proceed to calculate $J(c)$ in the case where c has one of the groups M_m^n as its right hand member, i.e. we take c to be the c_m^n defined above, in Section 2.

PROPOSITION 4. *$J(c_m^n)G = G_m^n$, for any p-primary abelian group G.*

PROOF. Let G be an arbitrary p-primary abelian group, and let J, c_m^n, α_m^n, M_m^n, etc. be as defined above, then:

$$J(c_m^n)G = \ker(\alpha \otimes 1_G).$$

Referring to the result of Dieudonné quoted in Section 1, (Theorem 5 of DIEUDONNÉ [4]), we proceed to demonstrate the inclusion of the set $J(c_m^n)G$ in the set G_m^n, where

as above, we identify G with $Z\otimes G$. The other inclusion is obvious from the elementary properties of the tensor product, and we omit it.

Let us then choose x arbitrarily in $J(c_m^n)G$, $x\neq 0$. We may represent x with $1\otimes g$, for some g an element of G, where the tensoring is done in $Z\otimes G$. Then with the tensoring accomplished in $M_m^n\otimes G$, we have $\alpha_m^n(1)\otimes g=0$. Let $\langle\theta_0, 1+\theta_1, 2+\theta_2, \ldots, \eta+\theta, \ldots\rangle$ be the exponential height sequence for $\alpha_m^n(1)$ defined by Dieudonné in DIEUDONNÉ [4], i.e. $\eta+\theta_\eta$ is the exponential height of $p^\eta\alpha_m^n(1)$ in $M_m^n\otimes G$. In the present instance, this sequence becomes (fixing n and m): $\langle 0, 1+0, 2+0, \ldots, \ldots, (m-1)+0, m+n, (m+1)+n, (m+2)+n, \ldots, (m+f)+n, \ldots\rangle$.

This yields, again referring to DIEUDONNÉ [4],

$$0 = \max\left(\min\left(O_G(g)-\theta_0, \varrho\right), 0\right)$$

where

$$\theta_0 = \begin{cases} 0 & \text{if } m>0 \\ n & \text{if } m=0 \end{cases}$$

and where ϱ is the smallest of the numbers

$$\max\left(O_G(g)-\theta_n, \eta-h_p(g)\right)$$

where the maximum is computed for all values of $\eta\geqq 0$. But then

$$\min\left(O_G(g)-\theta_0, \varrho\right)\leqq 0.$$

Consider first the case in which $m=0$. We may then view $M_m^n=M_0^n$ as another copy of Z, in which case $\alpha(z)=p^n z$ for all $z\in Z$. But this induces $\alpha\otimes 1_G(y)=p^n y$ for all $y\in Z\otimes G$, so that $\ker(\alpha\otimes 1_G)=p^0G[p^n]$, as required by the statement of the proposition.

We may therefore assume that $m\neq 0$. This assures that $\theta_0=0$, so that continuing our previous reasoning gives

$$\min\left(O_G(g), \varrho\right)\leqq 0$$

from which we see that either $O_G(g)=0$, or $\varrho\leqq 0$. As $g\neq 0$, it must be the case that $\varrho\leqq 0$, so that as

$$\theta_\eta = \begin{cases} 0 \text{ if } n<m \\ n \text{ otherwise,} \end{cases}$$

then

$$0\geqq\varrho = \min\left(\max\left(O_G(g), -h_p(g)\right), \max\left(O_G(g)-n, m-h_p(g)\right)\right) =$$

$$= \min\left(O_G(g), \max\left(O_G(g)-n, m-h_p(g)\right)\right).$$

Since $O_G(g)>0$, we have that

$$0\geqq\max\left(O_G(g)-n, m-h_p(g)\right)$$

so that $O_G(g)\leqq n$ and $h_p(g)\geqq m$ and hence $x=1\otimes g\in G_m^n$ as required by the theorem.

We now wish to put together our characterization of the large subgroups $G_m^n=J(c_m^n)G$ with the well known fact that all large subgroups are generated by a collection of G_n^m's. In particular we want to show the existence of a short exact sequence $Z\overset{\alpha}{\rightarrowtail}M\twoheadrightarrow H$ for each large subgroup L of G such that $L=\ker(1_G\otimes\alpha)$. In order to do this we shall employ the following characterization of large subgroups, given in [1].

THEOREM 2.11 (BENABDALLAH, EISENSTADT, IRWIN, and POLUIANOV). *The subgroup L of the p-primary abelian group G is large if and only if $L=\sum_{k=1}^{\infty} G_{n_k}^{k-n_k}$ where*

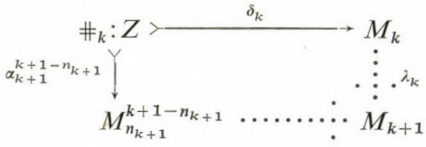
1) $n_k \leqq k$ *for all* $k \in Z^+$
2) $n_k \leqq n_{k+1} \leqq n_k+1$ *for all* $k \in Z^+$
3) *If the group G is unbounded, then so is the sequence*

$$\langle 1-n_1,\ 2-n_2,\ 3-n_3,\ \ldots\rangle.$$

The Ulm invariants of L are given by $f_L(n)=\sum_k (f_G(k-1)\colon k-n_k-1=n$ for all non-negative integers n.

Let us define for each $k \in Z^+$ the group generated by the first k groups in the theorem above, i.e. $F_k=\sum_{j=1}^{k} G_{n_j}^{j-n_j}$ and inclusion maps $f_k\colon F_k \rightarrowtail F_{k+1}$ so that $L=\varinjlim F_k$. By defining short exact sequences $\#_k : Z \overset{\delta}{\rightarrowtail} M_k \overset{\xi}{\twoheadrightarrow} H_k$ as follows we obtain the directed system of short exact sequences given by $\partial_k\colon \#_k \to \#_{k+1}$ where:

1) $\#_1=c_{n_1}^{1-n_1}$,
2) Given $\#_k$, we form $\#_{k+1}$ as the following pushout:



where we define $\delta_{k+1}=\lambda_k\delta_k$, ξ_{k+1} in the obvious way, and $\partial_k=(l_Z, \lambda_k, \sigma_k)$, where σ_k is induced by λ_k, $\sigma_k\colon H_k \to H_{k+1}$. This yields as kernels are preserved by direct limits, $L=\varinjlim F_k \rightarrowtail Z \otimes G \to \varinjlim M_k \otimes G$, exact at $Z \otimes G$, so with the definition $\varinjlim \#\colon Z \rightarrowtail \varinjlim M_k \twoheadrightarrow \varinjlim H_k$, we obtain $L=J(\varinjlim \#)$, which we state as a theorem.

THEOREM 1. *Given a p-primary abelian group G and a large subgroup L of G, there exists an extension $\#$ of Z by a p-primary group such that $L=J(\#)$.*

We have therefore accomplished our goal of finding a kernel representation for all the large subgroups of a p-primary abelian group and may therefore include them in the class of functorial subgroups discussed in FUCHS [5].

4. L-purity

For a preradical F, FUCHS [5] calls any short exact sequence of abelian groups *F-pure* whenever the sequence remains exact after application of F, and poses the problem of developing a theory for F-purity. While we will have more to say about the general case in a subsequent paper (See J. WINTHROP, The homological characterization of modules, *Notices of the A.M.S.*, April 1972), we content ourselves here with the special instance in which $F=J(c)$ represents a large preradical.

We quote the following lemmas concerning preradicals and large subgroups without proof. In each instance either a reference is provided or else the proof is of a totally routine nature and is therefore omitted.

LEMMA 1. *Let S be any preradical defined on abelian groups, then the short exact sequence $@: A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\twoheadrightarrow} C$ gives rise to the (left) exact sequence $S(@): S(A) \rightarrowtail S(B) \to S(C)$ with maps the restrictions of those of $@$ if and only if $\alpha S(A) = \alpha A \cap S(B)$.*

LEMMA 2. *With the definitions and setting of Lemma 1, S is a right exact functor if and only if $\beta^{-1} S(C) = S(B)$.*

LEMMA 3. *With S as in the preceding lemmas, and $\# : W \overset{\sigma}{\rightarrowtail} X \twoheadrightarrow Y \in \in$ Pext (Y, W), we have that S large entails that $\sigma S(W) = \sigma W \cap S(X)$.*

(This follows easily from Theorem 2.9 of [1].)

Let us now fix $L = J(c)$ to be large for $c: Z \overset{\alpha}{\rightarrowtail} M \overset{\beta}{\twoheadrightarrow} H$. We have then the following:

PROPOSITION 4. *L is left exact on pure extensions.*

PROOF. This follows readily from Lemmas 1 and 3 above.

It now seems appropriate to introduce the following terminology: We shall say that a short exact sequence $\#$ is "weakly L-pure" in the case that it remains at least half exact (in the usual sense of FUCHS [5]) after the application of any large preradical. Thus we may rephrase Proposition 4 to say that any pure short exact sequence is weakly L-pure. In case $\#$ remains exact after the application of all large preradicals, we follow FUCHS [5] in terming $\#$ merely "L-pure".

The question naturally arises as to whether or not all pure short exact sequences are in fact L-pure. We give an argument that this is indeed the case, based on the following lemma:

LEMMA 4 (YAHYA, see FUCHS [5] p. 124). *Let F be a covariant additive functor on abelian groups which commutes with direct limits, then the image under F of any pure exact sequence is again a pure exact sequence.*

Combining this with the well known fact that $\otimes$ commutes with direct limits, we may obtain:

THEOREM 2. *All pure short exact sequences of abelian groups are L-pure.*

PROOF. As L takes maps into their restrictions, L is additive. In view of Lemma 4 it clearly suffices to show that L commutes with direct limits. Recalling that direct limits are exactness preserving and considering the direct system, for G_i p-primary

$$f_i^j : G_i \rightarrowtail G_j \quad (i, j \in I)$$

and index set I, after application of L we obtain the system

$$\begin{array}{ccc} L(G_i) \rightarrowtail Z \otimes G_i & \longrightarrow & M \otimes G_i \\ \quad {\scriptstyle 1_Z \otimes f_i^j} \downarrow & & \downarrow {\scriptstyle 1_M \otimes f_i^j} \\ L(G_j) \rightarrowtail Z \otimes G_j & \longrightarrow & M \otimes G_j \end{array}$$

which becomes in the limit, $L(\varinjlim G_i) \rightarrowtail Z \otimes \varinjlim G_i \to M \otimes \varinjlim G_i$.

By Theorem 29.1 of FUCHS [5] we see that a sequence $@: A \rightarrowtail B \twoheadrightarrow C$ is p-pure if and only if $p^n A \rightarrowtail p^n B \twoheadrightarrow p^n C$ is exact for each non-negative integer n. However as this sequence is just $J(c)@$ for some c, if we make the assumption that $@$ is L-pure, we have immediately that $@$ is pure, and we may therefore state

THEOREM 3. *A short exact sequence of abelian groups is pure if and only if it is L-pure.*

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References

[1] K. M. BENABDALLAH, B. J. EISENSTADT, J. M. IRWIN and E. W. POLUIANOV, The structure of large subgroups of primary abelian groups, *Acta Math. Acad. Sci. Hung.*, **21** (1970), pp. 421—435.
[2] H. CARTAN and S. EILENBERG, *Homological algebra* (Princeton, 1956).
[3] D. O. CUTLER and R. STRINGALL, A topology for primary abelian groups, *Études sur les groupes abéliens* (Paris, 1968).
[4] J. DIEUDONNÉ, Sur les produits tensoriels, *Ann. Sci. École Norm. Sup.*, **64** (1948), pp. 101—117.
[5] L. FUCHS, *Infinite abelian groups,* Academic Press (New York, 1970).
[6] C. MEGIBBEN, Large subgroups and small homomorphisms, *Michigan Math. Journal,* **13** (1966), pp. 153—160.
[7] J. NUNKE, Purity and subfunctors of the identity, *Topics in abelian groups,* ed. by John. M. Irwin and Elbert A. Walker, Chicago, Scott Foresman, 1963.
[8] R. S. PIERCE, Homomorphisms of primary abelian groups, *Topics in Abelian Groups,* ed. by John M. Irwin and Elbert A. Walker, Chicago, Scott Foresman, 1963.
[9] M. SHIFFMAN, The ring of automorphisms of an abelian group, *Duke Math. Journal,* **6** (1940), pp. 579—597.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (3—4), (1974), pp. 263—268.

SOME REMARKS ON RADICALS OF RINGS WITH CHAIN CONDITIONS

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Introduction

DIVINSKY [2] has described those general radicals which have the classical effect on artinian rings. The smallest such radical class is (clearly) the lower radical class $\mathscr{D}$ defined by the class of nilpotent rings which are (Jacobson) radicals of artinian rings. Divinsky proved that $\mathscr{D}$ properly contains the lower radical class determined by all simple zerorings and is properly contained in the Baer lower radical class $\mathscr{B}$, the latter inequality being demonstrated by means of a moderately complicated example. Using a result from [6] we prove in §1 that the rings in $\mathscr{D}$ are characterized by their membership of $\mathscr{B}$ and a condition on their additive groups. From this it follows that while not hereditary, $\mathscr{D}$ has the property that pure ideals of $\mathscr{D}$-rings are $\mathscr{D}$-rings.

In §2 the effect on artinian rings of a fairly large selection of radicals is considered. It is shown that every hereditary radical coincides on artinian rings with the lower radical determined by a class of rings selected from the following: full matrix rings over division rings, simple zerorings and the zeroring on the group of integers. For radicals smaller than the upper radical determined by all full matrix rings over division rings there is an analogous result, using zerorings on prime-order-cyclic and quasicyclic groups and the group of rational numbers. The latter result does not provide a characterization of all radical subclasses of $\mathscr{D}$.

In §3 we introduce the class of *unequivocal* rings, the rings R for which $\mathscr{R}(R)=R$ or 0 for every radical class $\mathscr{R}$. Every radical partitions the unequivocal rings and certain partitions define upper and lower radicals as happens with the simple rings. It is not known whether the upper and lower radical defined by such a partition can be distinct; we prove, however, that if the lower class of the partition contains all simple zerorings and the zeroring on the group of rational numbers, then the upper and lower radicals coincide for artinian rings.

Rings with minimum condition on principal left ideals are considered in §4. For such a ring R, we show that $\mathscr{D}(R)$ coincides with the Jacobson radical $\mathscr{J}(R)$, thereby answering a question posed by SZÁSZ [13].

For the language of general radical theory we refer the reader to DIVINSKY's book [3]. We generally find it more convenient to refer to radical *classes* rather than *properties*, but otherwise our departures from Divinsky's usage are limited to a few pieces of notation which are included in the following list.

Notation

R^+	additive group of a ring R
G^0	zeroring on an abelian group G
Z	group of integers

Q group of rational numbers
$Z(p)$ cyclic group of order p
$Z(p^\infty)$ quasicyclic p-group
G_t maximum torsion subgroup of an abelian group G
$\oplus$ direct sum (of abelian groups or rings)
$L(\mathscr{C})$ lower radical class determined by a class $\mathscr{C}$
$\mathscr{D}$ lower radical class determined by the class of (Jacobson) radicals of artinian rings
$\mathscr{J}$ Jacobson radical class
$\mathscr{N}$ nil radical class
$\mathscr{B}$ Baer lower (=prime) radical class

A *p-ring* is a ring R for which R^+ is a p-group. All rings considered are associative.

1. The Divinsky radical

The following result is mentioned in [4] in conjunction with the proof of an analogous result for arbitrary nilpotent ideals. We include a proof here for the sake of completeness only.

PROPOSITION 1.1. *Let R be a ring for which R^+ is torsion-free. Then $\mathscr{N}(R)^+$ is a pure subgroup of R^+.*

PROOF. Let $I=\{x\in R|nx\in\mathscr{N}(R)$ for some integer $n\neq 0\}$. Then I is an ideal of R. If $x\in I$ and $nx\in\mathscr{N}(R)$, then $n^m x^m=(nx)^m=0$ for some m, so $x^m=0$ and I is a nil ideal. Thus $I=\mathscr{N}(R)$ and $\mathscr{N}(R)^+$ is a pure subgroup.

Note that the analogous statement for the Jacobson radical is false: consider for example the ring of rational numbers with odd denominators.

We can now characterize the radical class $\mathscr{D}$.

THEOREM 1.2. *The following classes are equal*
(i) $\mathscr{D}$
(ii) $L(\{Q^0\}\cup\{Z(p)^0|p$ *prime*$\})$
(iii) $\{R\in\mathscr{B}|R^+/(R^+)_t$ *is divisible*$\}$.

PROOF. Let R be the ring whose elements are ordered pairs of rational numbers, and whose operations are defined by

$$(a, b)+(c, d) = (a+c, b+d), \quad (a, b)(c, d) = (ad+bc, bd).$$

R is commutative with just three ideals, the maximal nilpotent one being isomorphic to Q^0. Since also each $[Z(p)]^0$ belongs to $\mathscr{D}$, we have (i)$\supseteq$(ii). The lower radical class of abelian groups determined by $\{Q\}\cup\{Z(p)|p$ prime$\}$ consists of those groups G for which G/G_t is divisible, and is closed with respect to pure subgroups (see [5], especially Theorem 4.2 and Proposition 4.6). Thus the equality (ii)=(iii) follows from Theorem 4.2 of [6]. Finally, any artinian ring R has the form $R_0\oplus R_1\oplus\ldots\oplus R_n$ (ring direct sum) where all factors are artinian rings, R_0^+ is torsion-free and divisible and each of $R_1^+, \ldots, R_n^+$ is primary ([4], [14]), so $\mathscr{J}(R)=\mathscr{J}(R_0)\oplus\mathscr{J}(R_1)\oplus\ldots\oplus\mathscr{J}(R_n)$. By Proposition 1.1, $\mathscr{J}(R_0)^+=\mathscr{N}(R_0)^+$ is divisible, so (i)$\subseteq$(iii).

COROLLARY 1.3. *$\mathscr{D}$ is not hereditary, but pure ideals of $\mathscr{D}$-rings are $\mathscr{D}$-rings.*

2. Some radicals of artinian rings

Because of the difficulty of obtaining a complete classification of all radical classes, it seems reasonable to adopt a more modest aim, and to proceed as follows: Select a class $\mathscr{K}$ of rings and try to find a collection Ω of radical classes, described fairly explicitly, so that every radical class has the same effect on rings in $\mathscr{K}$ as a radical class belonging to Ω. For the class of artinian rings, this aim will be partially realized through the next few results.

Let R be an artinian ring. If R has an ideal S_1 which is isomorphic to a full matrix ring over a division ring, there is a (ring) direct sum decomposition $R=S_1\oplus R'$. A finite number of repetitions produces a decomposition

$$R = S_1\oplus \ldots \oplus S_n\oplus \bar{R},$$

where each S_i is isomorphic to a full matrix ring over a division ring and $\bar{R}$ has no ideals of this form. Let I denote the maximum nilpotent ideal of $\bar{R}$. Then I is an essential ideal, since if $J\neq 0$ is an ideal of $\bar{R}$ with $J\cap I=0$, then J is isomorphic to an ideal of $\bar{R}/I$ and hence $\bar{R}=J\oplus K$ for some ideal K and J is isomorphic to a direct sum of matrix rings over division rings. By now applying the splitting theorem of SZÁSZ [14] and the primary decomposition to $\bar{R}$, we obtain

PROPOSITION 2.1. *Every artinian ring R has a direct decomposition*

$$(*) \qquad R = S_1\oplus \ldots \oplus S_n\oplus \bar{R}_0\oplus \bigoplus_p \bar{R}_p,\ p \text{ prime},$$

where $S_1, \ldots, S_n$ are isomorphic to full matrix rings over division rings, $\bar{R}_0$ and the $\bar{R}_p$ have essential maximum nilpotent ideals, $\bar{R}_0^+$ is torsion-free and divisible, $\bar{R}_p^+$ is a p-group and $R_p=0$ for almost all values of p.

If R is artinian and $\mathscr{R}$ is a hereditary radical class, then using $(*)$, we obtain

$$\mathscr{R}(R) = \mathscr{R}(S_1)\oplus \ldots \oplus \mathscr{R}(S_n)\oplus \mathscr{R}(\bar{R}_0)\oplus \bigoplus_p \mathscr{R}(\bar{R}_p).$$

Let us consider the possible values of the summands in this equation. $\mathscr{R}(S_i) = S_i$ or $0, i = 1, \ldots, n$. Let $A_0 = \mathscr{R}(\bar{R}_0)\cap \mathscr{J}(\bar{R}_0)$. Then A_0 is nilpotent and A_0^+ is torsion-free. If $A_0\neq 0$, then Z^0 is isomorphic to an ideal of A_0 and hence belongs to $\mathscr{R}$, so $\mathscr{B}\subseteqq\mathscr{R}$ ([8], [3] p. 43) and in particular $\mathscr{J}(\bar{R}_0)=\mathscr{B}(\bar{R}_0)\subseteqq\mathscr{R}(\bar{R}_0)$. Thus there are full matrix rings $B_1, \ldots, B_k$ over division rings such that

$$\mathscr{R}(\bar{R}_0)/\mathscr{J}(\bar{R}_0) = \mathscr{R}\big(\bar{R}_0/\mathscr{J}(\bar{R}_0)\big) \cong B_1\oplus \ldots \oplus B_k$$

whence it follows that

$$\mathscr{R}(\bar{R}_0) = L(\{Z^0, B_1, \ldots, B_k\})(\bar{R}_0).$$

If $A_0=0$, then $\mathscr{R}(\bar{R}_0)=0$. If $A_p=\mathscr{R}(\bar{R}_p)\cap\mathscr{J}(\bar{R}_p)\neq 0$ for some p, then $\mathscr{R}$ contains a nilpetent p-ring and hence all p-rings in $\mathscr{B}$ ([1], Lemma 1 and [6], Lemma 3.1) so as above

$$\mathscr{R}(\bar{R}_p) = L\big(\{Z(p)^0, C_1, \ldots, C_m\}\big)(\bar{R}_p)$$

for some full matrix rings $C_1, \ldots, C_m$ over division rings. If $A_p=0$, then $\mathscr{R}(\bar{R}_p)=0$.

It is now a routine matter to complete the proof of

PROPOSITION 2.2. *Let $\mathscr{R}$ be a hereditary radical class, $\mathscr{C}=\mathscr{R}\cap(\{Z^0\}\cup\{[Z(p)]^0|p$ prime$\}\cup\{S|S$ is a full matrix ring over a division ring$\})$. Then $\mathscr{R}(R)=L(\mathscr{C})(R)$ for every artinian ring R.*

Using Theorem 1.2 we can similarly describe the effect on artinian rings of radical classes smaller than $\mathscr{D}$ (or, equivalently, those for which all full matrix rings over division rings are semi-simple [2]). If $\mathscr{R}$ is such a class and R is artinian, then in terms of the representation $(*)$ of R, we have

$$\mathscr{R}(R) = \mathscr{R}(\overline{R}_0)\oplus\bigoplus_p \mathscr{R}(\overline{R}_p) = \mathscr{R}\big(\mathscr{J}(\overline{R}_0)\big)\oplus\bigoplus_p \mathscr{R}\big(\mathscr{J}(\overline{R}_p)\big).$$

For each p, $\mathscr{R}\big(\mathscr{J}(\overline{R}_p)\big)=\mathscr{J}(\overline{R}_p)$, its maximum divisible ideal or 0 ([6], Theorem 3.3). Let $I=\mathscr{R}\big(\mathscr{J}(\overline{R}_0)\big)$. We shall show that $Q^0\in\mathscr{R}$ if $I\neq 0$. As a first step we shall prove that I/I^2 has elements of infinite order. Suppose $(I/I^2)^+$ is a torsion group and $I^{k+1}=0\neq I^k$. For $x_1,\ldots,x_k\in I$, let $n_1,\ldots,n_k$ be their orders modulo I^2. Then $(n_1\ldots n_k)(x_1\ldots x_k)\in I^{2k}=0$, so $x_1\ldots x_k=0$, I^+ being torsion-free, i.e. $I^k=0$. Hence I has a homomorphic image $K\neq 0$ such that K^+ is torsion-free and $K^2=0$. K of course belongs to $\mathscr{R}$. By embedding K^+ in a minimal divisible extension, we obtain an exact sequence

$$0\to K\to D\to E\to 0$$

of zerorings, where D^+ is torsion-free divisible and E^+ is a divisible torsion group. Now E belongs to $\mathscr{R}$. (This follows from [5], Proposition 2.1 *via* considerations in [6]). Hence $\mathscr{R}$ contains D, Q^0 and therefore all $\mathscr{B}$-rings with divisible additive groups ([6], Theorem 4.2 again); in particular, $\mathscr{J}(\overline{R}_0)\in\mathscr{R}$. Thus, using Theorem 3.3 of [6], we obtain

PROPOSITION 2.3. *Let $\mathscr{R}$ be a radical subclass of $\mathscr{D}$. If $Q^0\in\mathscr{R}$, then*

$$\mathscr{R}(R) = L\big(\{Q^0\}\cup\{[Z(p)]^0|p \text{ prime},\ [Z(p)]^0\in\mathscr{R}\}\big)(R)$$

for every artinian ring R. If $Q^0\notin\mathscr{R}$, let

$$P_1 = \{p \text{ prime}|[Z(p)]^0\in\mathscr{R}\},$$

$$P_2 = \{p \text{ prime}|[Z(p^\infty)]^0\in\mathscr{R}, [Z(p)]^0\notin\mathscr{R}\}.$$

Then $\mathscr{R}(R)=L\big(\{[Z(p)]^0|p\in P_1\}\cup\{[Z(p^\infty)]^0|p\in P_2\}\big)(R)$ for every artinian ring R.

It should be noted that Proposition 2.3 does not provide a complete list of all radical subclasses of $\mathscr{D}$; for example there are some which do not contain any nilpotent rings ($\neq 0$), (see [6]).

3. A class of rings

SULIŃSKI [11] has considered radicals in the category of finite-dimensional associative algebras over a field, proving (Teorema 5) that any partition of the simple objects for which the algebra with trivial multiplication is in the lower class determines a unique radical, and observing that the same theorem is valid for artinian rings. With the obvious interpretation, the latter statement is incorrect. (Since artinian rings can have non-artinian radicals, the class of artinian rings does not generate

a category in which radical theory is viable.) For if R is an artinian ring with R^+ torsion-free and $\mathscr{J}(R)\neq 0$, then R is semi-simple with respect to the lower radical class determined by all simple zerorings, but not with respect to the upper radical class determined by all idempotent simple rings. There is a related theorem involving a class of rings which we now introduce.

A ring R will be called *unequivocal* if $\mathscr{R}(R)=R$ or 0 for every radical class $\mathscr{R}$. Examples of unequivocal rings are simple rings and the rings Q^0, $[Z(p^\infty)]^0$. Every radical property defines a partition of the class of unequivocal rings in the same way as for the class of simple rings, but the converse does not hold; for example it is not possible for $[Z(p)]^0$ to be radical and $[Z(p^\infty)]^0$ to be semi-simple. An *allowable partition* $(\mathscr{L}, \mathscr{U})$ of the unequivocal rings is a partition into two classes $\mathscr{L}$ and $\mathscr{U}$ (the *lower* and *upper classes* of the partition) such that every ring in $\mathscr{U}$ is $L(\mathscr{L})$-semi-simple. Since every class $\mathscr{C}$ of associative rings determines an upper radical $U(\mathscr{C})$, [9], every allowable partition $(\mathscr{L}, \mathscr{U})$ defines both an upper and a lower radical class.

PROPOSITION 3.1. *Let $(\mathscr{L}, \mathscr{U})$ be an allowable partition of the class of unequivocal rings for which $\mathscr{L}$ contains Q^0 and all simple zerorings. Then $L(\mathscr{L})(R)=U(\mathscr{U})(R)$ for all artinian rings R.*

PROOF. By Theorem 1.2, we have $\mathscr{D}\subseteq L(\mathscr{L})$ and hence

$$\mathscr{J}(R)\subseteq L(\mathscr{L})(R)\subseteq U(\mathscr{U})(R),$$

so

$$U(\mathscr{U})(R)/\mathscr{J}(R)=U(\mathscr{U})\big(R/\mathscr{J}(R)\big)=S_1\oplus\ldots\oplus S_n$$

where the S_i are isomorphic to full matrix rings over division rings, hence unequivocal, and not members of $\mathscr{U}$. Therefore $S_1, \ldots, S_n\in\mathscr{L}$ and $U(\mathscr{U})(R)\in L(\mathscr{L})$. This proves the result.

PROBLEM 1. Does there exist an allowable partition $(\mathscr{L}, \mathscr{U})$ of the unequivocal associative rings for which $L(\mathscr{L})\neq U(\mathscr{U})$?

There is an analogue of Problem 1 for any situation where radicals are defined. However in some such situations ([7], [10]) not every class of objects determines an upper radical. Thus a further question arises:

PROBLEM 2. Does an allowable partition of the unequivocal objects always define an upper radical?

4. Rings with minimum condition on principal left ideals

We now turn our attention to rings with minimum condition for principal left ideals, which for brevity we shall call MPL-*rings*. SZÁSZ ([12] Satz 5.1) has proved that $\mathscr{B}(R)=\mathscr{J}(R)$ for every MPL-ring R. The same author ([13], p. 459) asked whether the lower radical class $\mathscr{M}$ determined by Jacobson radicals of MPL-rings coincides with $\mathscr{D}$. (Clearly $\mathscr{D}\subseteq\mathscr{M}$.) Since $\mathscr{M}\subseteq\mathscr{B}$, Theorem 1.2 implies that an affirmative answer to the question will follow if it can be shown that the Jacobson radical of every MPL-ring has an additive group G such that G/G_t is divisible. This we shall now prove.

THEOREM 4.1. *If R is an* MPL*-ring then $\mathscr{J}(R)^+$ is the direct sum of a torsion group and a torsion-free divisible group.*

PROOF. $R^+ = A \oplus B$, where A is a torsion group and B is torsion-free and divisible ([12] Satz 3.1).

The maximum divisible torsion subgroup I^+ of R^+ is the additive group of an ideal I of R, and $I^+ \subseteq A$, so $(R/I)^+ \cong A' \oplus B$, where $A' = A/I^+$ is a reduced torsion group. Thus $R/I = R_1 \oplus R_2$ (ring direct sum) where R_1^+ is a torsion group and R_2^+ is torsion-free and divisible, so that $\mathscr{N}(R/I) = \mathscr{N}(R_1) \oplus \mathscr{N}(R_2)$ (ring direct sum) where $\mathscr{N}(R_1)^+$ is a torsion group, and by Proposition 1.1 $\mathscr{N}(R_2)^+$ is torsion-free divisible. Now I is a zeroring, so $I \subseteq \mathscr{N}(R)$, whence $\mathscr{N}(R/I) = \mathscr{N}(R)/I$. Thus there is a split exact sequence

$$0 \to I^+ \to \mathscr{N}(R)^+ \to \mathscr{N}(R_1) \oplus \mathscr{N}(R_2) \to 0$$

of abelian groups, so $\mathscr{J}(R)^+ = \mathscr{N}(R)^+$ is as required.

COROLLARY 4.2. $\mathscr{D} = \mathscr{M}$.

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Added in proof (25 September 1974). The answer to Problem 1 is „yes". (N. DIVINSKY, *Unequivocal rings*, to appear.)

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References

[1] E. P. ARMENDARIZ—W. G. LEAVITT, The hereditary property in the lower radical construction, *Canad. J. Math.*, **20** (1968), pp. 474—476.
[2] N. DIVINSKY, General radicals that coincide with the classical radical on rings with D. C. C., *Canad. J. Math.*, **13** (1961), pp. 639—644.
[3] N. J. DIVINSKY, *Rings and Radicals,* Allen and Unwin (London, 1965).
[4] L. FUCHS, Ringe und ihre additive Gruppe, *Publ. Math. Debrecen,* **4** (1956), pp. 488—508.
[5] B. J. GARDNER, Torsion classes and pure subgroups, *Pacific J. Math.*, **33** (1970), pp. 109—116.
[6] B. J. GARDNER, Radicals of abelian groups and associative rings, *Acta Math. Acad. Sci. Hung.*, **24** (1973), pp. 259—268.
[7] T. L. JENKINS—D. KREILING, Semisimple classes and upper-type radical classes of narings, *Proc. Amer. Math. Soc.*, **26** (1970), pp. 378—382.
[8] А. Г. Курош, Радикалы колец и алгебр, *Матем. Сб.*, **33** (1953), pp. 13—26.
[9] Y.-L. LEE, On the construction of upper radical properties, *Proc. Amer. Math. Soc.*, **19** (1968), pp. 1165—1166.
[10] Ю. М. Рябухин, Радикалы в Ω-группах, I, *Матем. исслед. Кишинев*, **3** (2) (1968), pp. 123—160.
[11] А. Сулиньский, Некоторые вопросы общей теории радикалов, *Матем. Сб.*, **44** (1958), pp. 273—286.
[12] F. Szász, Über Ringe mit Minimalbedingung für Hauptrechtsideale. II, *Acta Math. Acad. Sci. Hung.*, **12** (1961), pp. 417—439.
[13] F. Szász, Über Ringe mit Minimalbedingung für Hauptrechtsideale. III, *Acta Math. Acad. Sci. Hung.*, **14** (1963), pp. 447—461.
[14] F. Szász, Über Artinsche Ringe, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, **11** (1963), pp. 351—354.

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Tomrs 25 (3—4), (1974), pp. 269—277.

ENUMERATION OF PERMUTATIONS BY RISES, FALLS, RISING MAXIMA AND FALLING MAXIMA

By
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Let $A(r, s)$ denote the number of permutations with $r+1$ rises and $s+1$ falls, where a conventional rise on the left and a conventional fall on the right are counted. It is known that [2]

$$\sum_{r,s=0}^{\infty} A(r, s)\frac{x^r y^s}{(r+s+1)!} = \frac{e^x - e^y}{xe^y - ye^x} = F(x, y).$$

Moreover if $P(r, s, k)$ denotes the number of permutations with r rises, s falls and k maxima, it has been proved that

$$\sum_{r,s=0}^{\infty}\sum_{k=0}^{\min(r,s)} P(r+1, s+1, k+1)\frac{x^r y^s z^k}{(r+s+1)!} = F(U, V),$$

where

$$U = \frac{1}{2}\left[x+y+\sqrt{(x+y)^2-4xyz}\right], \quad V = \frac{1}{2}\left[x+y-\sqrt{(x+y)^2-4xyz}\right].$$

In the present paper, recurrences and generating functions are obtained for the number of permutations with a given number of rises, falls, rising maxima and falling maxima.

1. Let

$$\pi = (a_1, a_2, \ldots, a_n) \tag{1.1}$$

denote an arbitrary permutation of $Z_n = \{1, 2, \ldots, n\}$. By a *rise* is meant a pair of consecutive elements a_i, a_{i+1} with $a_i < a_{i+1}$. By a *fall* is meant a pair of consecutive elements a_i, a_{i+1} with $a_i > a_{i+1}$. In addition we count a conventional rise to the left of a_1 and a conventional fall to the right of a_n. It is convenient to set

$$\pi' = (0, a_1, a_2, \ldots, a_n, 0) \tag{1.2}$$

and count rises and falls of π'. Next a *maximum* of π is a triple of consecutive elements a_i, a_{i+1}, a_{i+2} with $a_i < a_{i+1}$, $a_{i+2} < a_{i+1}$. The maximum is called a *rising* maximum if $a_i < a_{i+2}$; it is called a *falling* maximum if $a_i > a_{i+2}$. Moreover, if $a_1 > a_2$, we count a rising maximum at a_1, a_2; if $a_{n-1} < a_n$, we count a falling maximum at a_{n-1}, a_n. As in the case of rises and falls, the scheme (1.2) is helpful in visualizing maxima.

With every maximum is associated one rise and one fall; we may call these *improper* rises and falls. All other rises and falls are *proper*.

To illustrate these definitions, the permutation (21534) has three rises, three falls and three maxima; all the rises and falls are improper; the first two maxima

are rising while the third is falling. The permutation (21543) has two rises, four falls and two maxima; both rises and two of the falls are improper; both maxima are rising.

The case $n=1$ is exceptional. For the permutation (1) the single maximum of $\pi'=(010)$ is neither rising nor falling.

Let $P(r, s, j, k)$ denote the number of permutations of Z_n with r rises, s falls, j rising maxima and k falling maxima. Clearly

$$r+s = n+1. \tag{1.3}$$

Let π denote an arbitrary permutation of Z_n with r rises, s falls, j rising maxima and k falling maxima. Consider the effect of inserting the element $n+1$ in π. It will be convenient to let r' denote the number of proper rises and s' the number of proper falls in π. It is evident that

$$r = r'+j+k, \quad s = s'+j+k. \tag{1.4}$$

If $n+1$ is inserted in a proper rise we have

$$r' \to r'-1, \quad s' \to s', \quad j \to j+1, \quad k \to k, \quad r \to r, \quad s \to s+1; \tag{1.5}$$

in a proper fall

$$r' \to r', \quad s' \to s'-1, \quad j \to j, \quad k \to k+1, \quad r \to r+1, \quad s \to s; \tag{1.6}$$

in an improper rise (rising maximum)

$$r' \to r', \quad s' \to s'+1, \quad j \to j, \quad k \to k, \quad r \to r, \quad s \to s+1; \tag{1.7}$$

in an improper fall (rising maximum)

$$r' \to r'+1, \quad s' \to s', \quad j \to j-1, \quad k \to k+1, \quad r \to r+1, \quad s \to s; \tag{1.8}$$

in an improper rise (falling maximum)

$$r' \to r', \quad s' \to s+1, \quad j \to j+1, \quad k \to k-1, \quad r \to r, \quad s \to s+1; \tag{1.9}$$

in an improper fall (falling maximum)

$$r' \to r'+1, \quad s' \to s, \quad j \to j, \quad k \to k, \quad r \to r+1, \quad s \to s. \tag{1.10}$$

Combining (1.5), (1.6), (1.7), (1.8), (1.9), (1.10), we get the recurrence

$$\begin{aligned} P(r, s, j, k) = {} & (r-j-k+1)P(r, s-1, j-1, k)+(s-j-k+1)P(r-1, s, j, k-1)+ \\ & +jP(r, s-1, j, k)+jP(r-1, s, j+1, k-1)+ \\ & +kP(r-1, s, j, k)+kP(r, s-1, j-1, k+1). \end{aligned} \tag{1.11}$$

Put

$$P(r, s, m) = \sum_{j+k=m} P(r, s, j, k), \tag{1.12}$$

so that $P(r, s, m)$ is equal to the number of permutations with r rises, s falls and m maxima. Then (1.11) implies

$$P(r, s, m) = (r-m+1)P(r, s-1, m-1)+(s-m+1)P(r-1, s, m-1)+ \tag{1.13}$$
$$+mP(r, s-1, m)+mP(r-1, s, m).$$

This is a known result [2, (5.9)], thus furnishing a partial check on (1.11).

2. We next obtain a generating function for $P(r, s, j, k)$. First we introduce four functions $R(z)$, $S(z)$, $J(z)$, $K(z)$ satisfying the following system of differential equations:

$$\begin{cases} R' = J; \quad S' = K; \quad J' = K' = JS+KR \\ R(0) = x, \quad S(0) = y, \quad J(0) = u, \quad K(0) = v; \end{cases} \tag{2.1}$$

the primes denote differentiation with respect to z.

It follows from (2.1) that

$$J' = K' = R'S+RS' = (RS)',$$

so that

$$J = RS+c_0, \quad K = RS+c_1, \tag{2.2}$$

where c_0, c_1 are independent of z. Hence

$$R' = RS+c_0, \quad S' = RS+c_1. \tag{2.3}$$

This gives

$$S = R+(c_1-c_0)z+c_2, \tag{2.4}$$

where c_2 is independent of z. Now substituting in the first of (2.3), we get

$$R' = R\big(R+(c_1-c_0)z+c_2\big)+c_0. \tag{2.5}$$

Put

$$R = -Q'/Q. \tag{2.6}$$

Then (2.5) becomes

$$Q''-\big((c_1-c_0)z+c_2\big)Q'+c_0Q = 0. \tag{2.7}$$

Now let

$$Q = \sum_{n=0}^{\infty} A_n z^n/n!. \tag{2.8}$$

Then, by (2.7),

$$A_{n+2}-c_2A_{n+1}+\big((c_0-c_1)n+c_0\big)A_n = 0. \tag{2.9}$$

In view of (2.6), we may take $A_0=1$; A_1 will be determined presently.

It follows from (2.1), (2.2), (2.3) and (2.4) that

$$J(0) = R(0)S(0)+c_0, \quad K(0) = R(0)S(0)+c_1, \quad S(0) = R(0)+c_2.$$

Thus $c_1-c_0 = K(0-J(0)$ and

$$c_0-\frac{1}{4}c_2^2 = J(0)+R(0)S(0)-\frac{1}{4}\big(S(0)-R(0)\big)^2 = J(0)-\frac{1}{4}\big(R(0)+S(0)\big)^2.$$

We have, by (2.1),

$$x = R(0), \quad y = S(0), \quad u = J(0), \quad v = K(0). \tag{2.10}$$

Then we see that

$$c_0 - c_1 = u - v, \quad c_2 = y - x, \quad c_0 = u - xy \tag{2.11}$$

and

$$c_0 = \frac{1}{4}c_2^2 = u - \frac{1}{4}(x+y)^2. \tag{2.12}$$

Thus the recurrence (2.9) becomes

$$A_{n+2} + (x-y)A_{n+1} + \big((u-v)n + u - xy\big)A_n = 0. \tag{2.13}$$

It follows from (2.6) and (2.10) that

$$A_1 = -x. \tag{2.14}$$

Now we show how the generating function for $P(r, s, j, k)$ may be obtained from the functions R, S, J and K satisfying (2.1). We associate with a permutation π of Z_n having r' proper rises, s' proper falls, j rising maxima and k falling maxima the function

$$R(z)^{r'} S(z)^{s'} J(z)^j K(z)^k.$$

Then the derivative of this function is the sum of the $n+1$ functions associated with the $n+1$ permutations π' of Z_{n+1} obtained by inserting the number $n+1$ in the $n+1$ possible places of π.

For example, if $n+1$ is inserted in any of the r' proper rises, we obtain the r' functions

$$r' R(z)^{r'-1} S(z)^{s'} J(z)^{j+1} K(z)^k.$$

To avoid the exceptional case $n=1$, consider the two permutations of Z_2. Associated with them are the two functions RK and JS. Hence

$$R(0)K(0) + J(0)S(0) = xv + uy$$

enumerates these permutations by proper rises, proper falls, rising maxima and falling maxima. As described above

$$\frac{d}{dz}\big(R(z)K(z) + J(z)S(z)\big)\Big|_{z=0}$$

enumerates the permutations of Z_3,

$$\frac{d^2}{dz^2}\big(R(z)K(z) + J(z)S(z)\big)\Big|_{z=0}$$

enumerates the permutation of Z_4, etc.

Let

$$f(z) = R''(z) = R(z)K(z) + J(z)S(z).$$

Then by Taylor's theorem and the remarks above, we have

$$R(z) = R(0)+R'(0)z+\sum_{n=0}^{\infty} R^{(n)}(0)z^n/n! = R(0)+R'(0)z+\sum_{n=2}^{\infty} f^{(n-2)}(0)z^n/n! =$$

$$= R(0)+R'(0)z+\sum_{n\geqq 2}\ \sum_{r',s',j,k} A_n(r', s', j, k)x^{r'}y^{s'}u^j v^k z^n/n!$$

where $A_n(r', s', j, k)$ is the number of permutations of Z_n with r' *proper* rises, s' *proper* falls, j rising maxima and k falling maxima. Now, to enumerate rises, falls and rising and falling maxima, we replace u, v by xyu and xyv respectively, in $R(z)=R(x, y, u, v; z)$ to obtain a function $\bar{R}$. We may now state

THEOREM 1. *The generating function*

$$\bar{R}(z) = x+xyuz+\sum_{\substack{r+s-1\geqq 2\\ j,k}} P(r, s, j, k)x^r y^s u^j v^k \frac{z^{r+s-1}}{(r+s-1)!} \tag{2.15}$$

is evaluated by

$$\bar{R}(z) = -\sum_{n=0}^{\infty}\frac{\bar{A}_{n+1}Z^n}{n!}\Big/\sum_{n=0}^{\infty}\frac{\bar{A}_n Z^n}{n!} = -Q'(z)/Q(z) \tag{2.16}$$

where

$$\begin{cases}\bar{A}_0 = 1, \quad \bar{A}_1 = -x \quad \textit{and}\\ \bar{A}_{n+2}+(x-y)\bar{A}_{n+1}+xy\big((u-v)n+u-1\big)\bar{A}_n = 0 \qquad (n = 0, 1, 2, \ldots).\end{cases} \tag{2.17}$$

That is, $Q(z)$ satisfies the differential equation

$$Q''(z)-\big((xyv-xyu)z+(y-x)\big)Q'(z)+(xyu-xy)Q(z) = 0.$$

It follows from (2.17) that

$$\bar{A}_2 = x^2-xyu, \quad \bar{A}_3 = (3x-y)xyu-x^2y^u-x^3.$$

It can be verified, using these values that

(2.18)

$$\bar{R}(z) = x+xyuz+(xy^2u+x^2yv)\frac{z^2}{2!}+\big(x^3yv+x^2y^2(u+2uv+v)+xy^3u\big)\frac{z^3}{3!}+\ldots.$$

Note that the term $xyuz$ on the right of (2.18) suggests that we are counting the single maximum of $\pi'=(010)$ as a rising maximum. On the other hand, since by (2.3) and (2.11)

$$\bar{S}(z) = y+uyvz+(xy^2u+x^2yv)\frac{z^2}{2!}+\ldots \tag{2.19}$$

we are now counting this maximum as a falling maximum. Strictly speaking, as noted above, this maximum is neither rising nor falling.

3. We shall now obtain some specialized but more explicit results. In the first place, we take $u=v$, so that we are enumerating rises, falls and maxima. Put

$$P(r, s, m) = \sum_{j+k=m} P(r, s, j+k). \tag{3.1}$$

Since $u=v$, (2.17) becomes

$$\bar{A}_{n+2}+(x-y)\bar{A}_{n+1}+xy(u-1)\bar{A}_n=0. \tag{3.2}$$

Since $\bar{A}_0=1$, $\bar{A}_1=-x$, we easily verifiy that (3.2) implies

$$\bar{A}_n=A\alpha^n+B\beta^n, \tag{3.3}$$

where

$$\alpha=\frac{1}{2}\left[-x+y+\sqrt{(x+y)^2-4xyu}\right],\quad \beta=\frac{1}{2}\left[-x+y-\sqrt{(x+y)^2-4xyu}\right].$$

Thus (2.16) becomes, after simplification,

$$\bar{R}(z)=x+xyu\frac{e^{Uz}-e^{Vz}}{Ue^{Vz}-Ve^{Uz}}, \tag{3.4}$$

where

$$U=\frac{1}{2}\left(x+y+\sqrt{(x+y)^2-4xyu}\right),\quad V=\frac{1}{2}\left(x+y-\sqrt{(x+y)^2-4xyu}\right). \tag{3.5}$$

We may state

THEOREM 2. *The generating function*

$$\bar{R}(z)=\sum_{r,s,m}P(r,s,m)x^r y^s u^m\frac{z^{r+s-1}}{(r+s-1)!} \tag{3.6}$$

is evaluated by (3.4) *and* (3.5).

This result has been obtained previously [2, Th. 2] by a different method.

4. Let us now take $x=y$, so that we are enumerating permutations of Z_n with a given number of rising and of falling maxima. Put

$$Q(n,j,k)=\sum_{r+s=n+1}P(r,s,j,k), \tag{4.1}$$

so that $Q(n,j,k)$ is the number of permutations of Z_n with j rising and k falling maxima.

Clearly (2.17) reduces to

$$\bar{A}_{n+2}+x^2\big((u-v)n+u-1\big)\bar{A}_n=0 \tag{4.2}$$

with the initial conditions $A_0=1$, $A_1=-x$. It follows from (4.2) that

$$\bar{A}_{2n}=(-1)^n x^{2n}\prod_{j=0}^{n-1}[2j(u-v)+u-1] \tag{4.3}$$

and

$$\bar{A}_{2n+1}=(-1)^{n+1}x^{2n+1}\prod_{j=0}^{n-1}[(2j+1)(u-v)+u-1]. \tag{4.4}$$

THEOREM 3. *We have*

$$\sum_{n,j,k} Q(n,j,k)\frac{x^n}{n!}u^r v^s = -\frac{\sum_0^\infty \frac{1}{n!}\bar{A}_{n+1}}{\sum_0^\infty \frac{1}{n!}\bar{A}_n}, \tag{4.5}$$

where $\bar{A}_n$ *is given by* (4.3) *and* (4.4).

In particular, when $u=v$, (4.3) and (4.4) reduce to

$$\bar{A}_{2n} = x^{2n}(1-u)^n, \quad \bar{A}_{2n+1} = -x^{2n+1}(1-u)^n,$$

so that

$$\sum_{n=0}^\infty \frac{1}{n!}\bar{A}_n = \sum_{n=0}^\infty \frac{x^{2n}(1-u)^n}{(2n)!} - \sum_{n=0}^\infty \frac{x^{2n+1}(1-u)^n}{(2n+1)!} =$$

$$= \cosh x\sqrt{1-u} - \frac{1}{\sqrt{1-u}}\sinh x\sqrt{1-u},$$

$$\sum_{n=0}^\infty \frac{1}{n!}\bar{A}_{n+1} = -x\sum_{n=0}^\infty \frac{x^{2n}(1-u)^n}{(2n)!} + \sum_{n=0}^\infty \frac{x^{2n+2}(1-u)^{n+1}}{(2n+1)!} =$$

$$= -\cosh x\sqrt{1-u} + x\sqrt{1-u}\sinh x\sqrt{1-u}.$$

We find that

$$-\frac{\sum_0^\infty \frac{1}{n!}\bar{A}_{n+1}}{\sum_0^\infty \frac{1}{n!}\bar{A}_n} = x + \frac{xu\left(e^{x\sqrt{1-u}} - e^{-x\sqrt{1-u}}\right)}{\left(1+\sqrt{1-u}\right)e^{x\sqrt{1-u}} - \left(1-\sqrt{1-u}\right)e^{-x\sqrt{1-u}}}, \tag{4.6}$$

in agreement with (3.4).

5. Another special case of Theorem 1 of some interest is the enumeration of permutations without proper falls (rises). To enumerate permutations without proper rises we replace u, v by $x^{-1}u$, $x^{-1}v$ and then put $x=0$; to enumerate permutations without proper falls we replace u, v by $y^{-1}u$, $y^{-1}v$ and then put $y=0$. To enumerate permutations with neither rises nor proper falls we replace u, v by $x^{-1}y^{-1}u$, $x^{-1}y^{-1}v$ and then put $x=y=0$.

The recurrence (2.17) becomes

$$\bar{A}_{n+2} - y\bar{A}_{n+1} + y\big((u-v)n+u\big)\bar{A}_n = 0, \tag{5.1}$$

$$\bar{A}_{n+2} + x\bar{A}_{n+1} + x\big((u-v)n+u\big)A_n = 0, \tag{5.2}$$

$$\bar{A}_{n+2} + \big((u-v)n+u\big)\bar{A}_n = 0, \tag{5.3}$$

corresponding to each of the special cases. The initial conditions are

$$\bar{A}_0 = 1, \quad \bar{A}_1 = 0, \tag{5.1}'$$

$$\bar{A}_0 = 1, \quad \bar{A}_1 = -x, \tag{5.2}'$$

$$\bar{A}_0 = 1, \quad \bar{A}_1 = 0. \tag{5.3}'$$

A permutation with neither proper rises nor falls is evidently a *down-up* permutation:



The recurrence (5.3) implies

$$\bar{A}_{2n} = (-1)^n \sum_{j=0}^{n-1} \bigl(2j(u-v)+u\bigr). \tag{5.4}$$

Hence, if $A(j, k)$ denotes the number of down-up permutations of $Z_{2j+2k-1}$ with j rising and k falling maxima, we have

$$\sum_{j+k>0} A(j,k)\frac{u^j v^k z^{2j+2k-1}}{(2j+2k-1)!} = \frac{-\sum_0^\infty \bar{A}_{2n+2}\dfrac{z^{2n+1}}{(2n+1)!}}{\sum_0^\infty \bar{A}_{2n}\dfrac{z^{2n}}{(2n)!}}, \tag{5.5}$$

where $\bar{A}_{2n}$ is given by (5.4).

We remark that

$$A(j,k) = P(j+k, j+k, j, k). \tag{5.6}$$

Corresponding to down-up permutations of type



we put

$$A_1(j,k) = P(j+k, j+k+1, j, k), \qquad A_1(0,0)=1; \tag{5.7}$$

for permutations of this kind we have exactly one proper fall.

It can be shown (by deleting the smallest element in the permutation) that $A_1(j, k)$ satisfies the recurrence

$$A_1(r,s) = \sum_{j+k>0} \binom{2r+2s-1}{2j+2k-1} A(j,k)A_1(r-j, s-k). \tag{5.8}$$

If we put

$$\Phi = \sum_{r,s=0}^{\infty} A_1(r,s)\frac{u^r v^s z^{2r+2s}}{(2r+2s)!},$$

it is easily seen that (5.8) implies

$$\frac{\Phi'}{\Phi} = \sum_{j+k>0} A(j,k)\frac{u^j v^k z^{2j+2k-1}}{(2j+2k-1)!}. \tag{5.9}$$

Hence, by (5.5), we have

$$\Phi = \left\{\sum_0^\infty A_{2n}\frac{z^{2n}}{(2n)!}\right\}^{-1}. \tag{5.10}$$

Generating functions equivalent to (5.5) and (5.10) were obtained by a different method in [1].

6. The problem discussed in the present paper can be refined in the following way. Let $\pi=(a_1, a_2, \ldots, a_n)$ denote an arbitrary permutation of Z_n and as above put $\pi'=(0, a_1, a_2, \ldots, a_n, 0)$. Consider a triple of consecutive elements (a_1, a_{i+1}, a_{i+2}), where either a_i or a_{i+2} may be the 0 element. For $n>1$ we may associate with this triple a permutation of Z_3. Thus in effect the present paper is concerned with the enumeration of permutations of Z_n in terms of the permutations (123), (132), (231), (321); the permutations (213) and (312) are neglected. Unfortunately it is not clear how to include these permutations.

The more general question suggested by the above is the enumeration of the permutations of Z_n in terms of the permutations of Z_m, where $3\leqq m\leqq n$. The case $m=2$ is simply the enumeration of permutations by rises and falls.

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References

[1] L. CARLITZ, Enumeration of up-down permutations by number of rises, *Pacific Journal of Mathematics*, **265** (1974), pp. 110—137.

[2] L. CARLITZ and RICHARD SCOVILLE, Generalized Eulerian numbers: combinatorial applications, *Journal für die reine und angewandte Mathematik*, **45** (1973), pp. 49—58.

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ON THE FACTORISATION OF FINITE ABELIAN GROUPS. III

By

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In Memoriam, György Hajós

1. Introduction

In a previous paper [7] it was shown that if in a factorisation of a finite cyclic group G one factor has order a power of a prime p then one of the two factors is periodic. This proved and generalised a conjecture of DE BRUIJN [2]. The main purpose of this paper is to further generalise this theorem by showing that the conclusion holds whenever G is a finite abelian group with cyclic p-component. Generalisations of this to cases involving more than two factors and to certain infinite groups are then deduced.

The problem of the factorisation of finite abelian groups arose from the solution by HAJÓS [3] of a classical problem of Minkowski. The definitions and notations used in this paper may be found in [7]. In the cyclic case considered there, it was not necessary to use representation theory formally, though the work involving roots of unity is essentially of this type. For the generalisation given here we make use of certain elementary facts about representations of finite abelian groups.

If $G=AB$ is a factorisation of a finite abelian group G and $\mathbf{M}$ is a representation then it follows easily that

$$\sum_{a\in A}\mathbf{M}(a)\sum_{b\in B}\mathbf{M}(b)=\sum_{g\in G}\mathbf{M}(g).$$

Every irreducible representation is of degree 1, since G is abelian, and for such a representation, other than the identity, we have

$$\sum_{g\in G}\mathbf{M}(g)=0 \quad \text{and so either} \quad \sum_{a\in A}\mathbf{M}(a)=0 \quad \text{or} \quad \sum_{b\in B}\mathbf{M}(b)=0.$$

The number of irreducible representations is equal to the order of the group G, and the intersection of their kernels is the identity subgroup. If a matrix M is formed whose general term is $\mathbf{M}_i(g_j)$ where $\mathbf{M}_i$ is the i-th irreducible representation and g_j the j-th element of G then M is a non-singular matrix, since the standard orthogonality relations show that the rows are independent. In particular if $\sum_j x_j\mathbf{M}_i(g_j)=0$ for all these representations $\mathbf{M}_i$ it follows that $x_j=0$ for all j.

Throughout the paper p denotes a prime number; all representations referred to are of degree 1; the cyclotomic polynomial whose roots are the d-th primitive roots of unity is denoted by $F_d(x)$. We use the fact that $F_d(x)$ is irreducible over the field of m-th roots of unity whenever m and d are relatively prime. This was proved in [7], Theorem 1, but in fact is a very much earlier result of KRONECKER [5].

2. Principal theorem

LEMMA. *Let G be a finite abelian group whose p-component H is cyclic. Let T be a non-identity subgroup of G such that $T\cap H=\{e\}$. Let $B=\{b_1, \ldots, b_n\}$ be a sequence of elements of G such that for every representation $\mathbf{M}$ of G, with $T\not\subset \ker \mathbf{M}$ and $H\cap\ker \mathbf{M}=\{e\}$, we have $\sum\mathbf{M}(b_j)=0$. Then B splits as a disjoint union of two subsequences of elements of G such that the elements of G of order p are periods of the first subsequence and the elements of T are periods of the second subsequence.*

PROOF. The proof is by induction on the order of T. Assume first that T has prime order q. Let K be the complementary subgroup to H in G. Then T is contained in a cyclic subgroup U of order q^μ such that U is a direct summand of K. Let h be a generator of H and u a generator of U. Let H have order p^λ and let ϱ, σ be primitive roots of unity of order p^λ, q^μ respectively. Let V be a complementary subgroup to U in K. Let $\mathbf{M}$ be a representation of G such that $\mathbf{M}(h)=\varrho$, $\mathbf{M}(u)=\sigma$. Then $\mathbf{M}$ satisfies the conditions of the theorem. Further, since V is a direct summand of G, every representation of V may be obtained as the restriction of such a representation of G.

For each $b_j\in B$, let $b_j=h^{f_j}u^{d_j}v_j$, where $0\leqq f_j<p^\lambda$, $0\leqq d_j<q^\mu$, $v_j\in V$. Then for every representation $\mathbf{M}$ of V we have

$$\sum \varrho^{f_j}\sigma^{d_j}\mathbf{M}(v_j) = 0.$$

It follows that, for each $v\in V$, $\sum\varrho^{f_j}\sigma^{d_j}=0$, where the summation is taken over all j with $v_j=v$. Let k_j be defined by $0\leqq k_j<p^\lambda q^\mu$, $k_j\equiv f_j \pmod{p^\lambda}$, $k_j\equiv d_j \pmod{q^\mu}$. Then $\sum_{v_j=v}(\varrho\sigma)^{k_j}=0$. Since $\varrho\sigma$ is a primitive $p^\lambda q^\mu$-th root of unity it follows that $F_{p^\lambda q^\mu}(x)$ divides $\sum_{v_j=v} x^{k_j}$, for each $v\in V$. Now applying Theorem 2 of DE BRUIJN [2] we have

$$\sum_{v_j=v} x^{k_j} = P_v(x)\frac{x^{p^\lambda q^\mu}-1}{x^{p^{\lambda-1}q^\mu}-1} + Q_v(x)\frac{x^{p^\lambda q^\mu}-1}{x^{p^\lambda q^{\mu-1}}-1},$$

where $P_v(x)$, $Q_v(x)$ have non-negative integral coefficients. This expression of $\sum_{v_j=v} x^k$ as a sum of two polynomials with non-negative integral coefficients splits the numbers k_j such that $v_j=v$ into two sequences such that $p^{\lambda-1}q^\mu$ is an additive period of the first modulo $p^\lambda q^\mu$, and $p^\lambda q^{\mu-1}$ is an additive period of the second modulo $p^\lambda q^\mu$. Choosing the corresponding elements b_j in B, B is a union of two sequences with $(hu)^{p^{\lambda-1}q^\mu}$, of order p, as a period of the first, and $(hu)^{p^\lambda q^{\mu-1}}$, which is a generator of T, as a period of the second. Thus the Lemma is true if T has order q.

Now choose a general T satisfying the required conditions and make the inductive assumption that the result holds for proper subgroups of T. Let F be a subgroup of T of prime index q. If $F\not\subset\ker\mathbf{M}$ then $T\not\subset\ker\mathbf{M}$. Hence the conditions of the Lemma hold for F and so by the inductive assumption B is a union of two sequences, B_1 and B_2 such that the subgroups H_1 of H of order p is a factor of B_1 and F is a factor of B_2. $B=B_1\cup B_2$, where $B_1=H_1C_1$, $B_2=FC_2$.

Let $\bar{B}$ be the corresponding sequence of elements $\bar{b}_j$, where $\bar{b}_j=b_jF$ belongs to $G/F=\bar{G}$. If $\mathbf{M}$ is a representation of G such that $F\subset\ker\mathbf{M}$, but $T\not\subset\ker\mathbf{M}$, then $\mathbf{M}$ induces a representation $\overline{\mathbf{M}}$ of G/F such that $\overline{T}=T/F\not\subset\ker\overline{\mathbf{M}}$. Hence the

conditions of the Lemma hold for $\overline{G}$ and $\overline{T}$. $\overline{T}$ has order q and by the above result $\overline{B}$ is a union of two sequences such that $\overline{H}_1=(H_1F)/F$ is a factor of the first and $\overline{T}$ is a factor of the second. Since $\overline{B}=\overline{B}_1\cup\overline{B}_2$ and H_1 is a factor of B_1 we have $\overline{H}_1$ is a period of $\overline{B}_1$ and hence $\overline{B}_2$ is a union of two sequences $\overline{D}_1$ and $\overline{D}_2$ which have $\overline{H}_1$ and $\overline{T}$ as factors respectively. However F is already a factor of B_2. It follows that H_1 is a factor of D_1 and T is a factor of D_2. Then $B=(B_1\cup D_1)\cup D_2$ is the required splitting of B. The Lemma follows by induction.

We are now ready to prove the generalisation of Theorem 2 of [7] as follows.

THEOREM 1. *Let G be a finite abelian group whose p-component is cyclic. If $AB=G$ where the order of A is a power of p then either A or B is periodic.*

PROOF. Let H be the p-component of G and let K be the complementary subgroup to H. Let H have order p^λ and let h be a generator of H. Let ϱ be a primitive p^λ-th root of unity. Let $A=\cup h^{e_i}a_i$, $B=\cup h^{f_j}b_j$, where $0\leqq e_i, f_j<p^\lambda$ and $a_i, b_j\in K$.

Since H is cyclic the arguments used in the proof of Theorem 2 of [7] about the exponents of the generator of H occurring in A and B still apply. Thus the integers e_i are distinct.

Let $\mathbf{M}$ be a representation of G such that $\mathbf{M}(h)=\varrho$. Then $H\cap\ker\mathbf{M}=\{e\}$. From $AB=G$ it follows that $\sum\varrho^{e_i}\mathbf{M}(a_i)\cdot\sum\varrho^{f_j}\mathbf{M}(b_j)=0$. The restriction of $\mathbf{M}$ to K is a representation of K and every representation of K occurs in this way since H and K are complementary direct summands. Thus for every representation $\mathbf{M}$ of K we have either $\sum\varrho^{e_i}\mathbf{M}(a_i)=0$ or $\sum\varrho^{f_j}\mathbf{M}(b_j)=0$.

Suppose first that $\sum\varrho^{f_j}\mathbf{M}(b_j)=0$ for every representation $\mathbf{M}$ of K. Then for each $k\in K$ it follows that $\sum\varrho^{f_j}=0$, where the summation is taken over all j such that $b_j=k$. Then $F_{p^\lambda}(x)$ divides $\sum_{b_j=k}x^{f_j}$. Since $0\leqq f_j<p^\lambda$ it follows that the other factor has degree less than $p^{\lambda-1}$ and since $F_{p^\lambda}(x)=1+x^{p^{\lambda-1}}+\ldots+x^{(p-1)p^{\lambda-1}}$ it follows that x^r, $x^{r+p^{\lambda-1}}$, ..., $x^{r+(p-1)p^{\lambda-1}}$ occur the same number of times in $\sum_{b_j=k}x^{f_j}$. Therefore $h^{p^{\lambda-1}}$ is a period of B.

We may now assume that $\sum\varrho^{e_i}\mathbf{M}(a_i)=0$ for at least one representation $\mathbf{M}$ of K. Since the numbers $\mathbf{M}(a_i)$ belong to the field of m-th roots of unity, where m is the exponent of K, and so is relatively prime to p^λ, it follows that $F_{p^\lambda}(x)$ divides $\sum x^{e_i}\mathbf{M}(a_i)$. As above the coefficients of x^{e_r} and x^{e_s} are equal whenever $e_r\equiv e_s\pmod{p^{\lambda-1}}$. Since the numbers e_i are distinct it follows that each such term occurs once only, or else not at all, in $\sum x^{e_i}\mathbf{M}(a_i)$. Therefore $F_{p^\lambda}(x)$ divides $\sum x^{e_i}$ and $e_r\equiv e_s\pmod{p^{\lambda-1}}$ implies $\mathbf{M}(a_r)=\mathbf{M}(a_s)$, i.e. $a_ra_s^{-1}\in\ker\mathbf{M}$. Conversely if $F_{p^\lambda}(x)$ divides $\sum x^{e_i}$ and if $e_r\equiv e_s\pmod{p^{\lambda-1}}$ implies $a_r^{-1}a_s\in\ker\mathbf{M}$, then $\sum\varrho^{e_i}\mathbf{M}(a_i)=0$. Thus if $\sum\varrho^{e_i}\mathbf{M}(a_i)=0$ and $\mathbf{N}$ is a representation of K such that $\ker\mathbf{M}\subset\ker\mathbf{N}$ then $\sum\varrho^{e_i}\mathbf{N}(a_i)=0$. Further if $\mathbf{M}_1$, $\mathbf{M}_2$ are representations such that $\sum\varrho^{e_i}\mathbf{M}_1(a_i)=$ $=\sum\varrho^{e_i}\mathbf{M}_2(a_i)=0$ and if $\mathbf{N}$ is a representation such that $\ker\mathbf{N}\supset\ker\mathbf{M}_1\cap\ker\mathbf{M}_2$ it follows that $\sum\varrho^{e_i}\mathbf{N}(a_i)=0$. It follows, taking the intersection of the kernels of all such representations, that K contains a subgroup T such that $\sum\varrho^{e_i}\mathbf{M}(a_i)=0$ if and only if $\ker\mathbf{M}\supset T$.

If $T=\{e\}$ then $\sum\varrho^{e_i}\mathbf{M}(a_i)=0$ for all representations $\mathbf{M}$ of K and it follows, as above for B, that $h^{p^{\lambda-1}}$ is a period of A.

Assume now that $T\neq\{e\}$. Then $e_r\equiv e_s\pmod{p^{\lambda-1}}$ implies $a_ra_s^{-1}\in\ker\mathbf{M}$ for every representation $\mathbf{M}$ such that $\ker\mathbf{M}\supset T$ and hence implies $a_ra_s^{-1}\in T$. If $\mathbf{M}$ is

a representation of K such that $\ker \mathbf{M} \supset T$ $\sum \varrho^{e_i} \mathbf{M}(a_i) \neq 0$ and so $\sum \varrho^{f_j} \mathbf{M}(b_j) = 0$. It follows from the Lemma that B is a disjoint union of two sets B_1, B_2 such that $B_1 = H_1 C$ and $B_2 = TD$, where H_1 is the subgroup of H of order p. ATD is a union of cosets of T. Since $AB = G$ it follows that AH_1C is also a union of cosets of T. Suppose $h^{e_r} a_r h_1 c \in gT$, where $h_1 \in H_1$, $c \in C$. Let $e_r \equiv e_s \pmod{p^{\lambda-1}}$. Then $h^{e_r - e_s} \in H_1$ and $a_r a_s^{-1} \in T$. It follows that $h^{e_s} a_s h_1 h^{e_r - e_s} c$ is an element of AH_1C also belonging to gT. Given r we can find s in p ways (including $r = s$) with $e_r \equiv e_s \pmod{p^{\lambda-1}}$. It follows that elements of AH_1C belonging to a given coset of T occur in disjoint sets of p elements. However this is impossible as the order of T divides the order of K and so is not divisible by p. Therefore C is empty and so $B = TD$ and is periodic.

This completes the proof.

It is not in general possible to drop the condition that the p-component be cyclic from the above theorem. In Lemma 3 of [7] it is shown that if A has order 2 or 3 then one factor must be periodic, but for any order larger than 3, whether a prime power or not, factorisations have been shown to exist ([1], [4]) in which neither factor is periodic. For general primes p, $p > 3$, the only non-cyclic p-group which is "good" is the direct product of two cyclic groups of prime order. This group has been proved to be "good" by RÉDEI [6]. However DE BRUIJN [1] has shown that if a group G contains this group as a proper subgroup, of index m say, then G admits a "bad" factorisation $G = AB$ with A of order p and B of order pm. Thus the condition that the p-component is cyclic cannot even be weakened to the condition that the p-component is "good". Similar remarks apply for $p = 2, 3$, if the p-component is non-cyclic, A has order 2^k, 3^k, where $k > 1$, and certain restrictions are placed on m to allow Theorem 1 of DE BRUIJN [1] to be applied.

Theorem 1 can be applied to many, but not all, of the cases arising in the proofs in [9] that certain non-cyclic groups are "good". Thus many of these proofs can be shortened by its use.

When the group of periods of the periodic factor is factored out in the factorisations arising in Theorem 1 the resulting factorisation of the quotient group is again of the same type. Hence Theorem 1 may be applied repeatedly and so all factorisations of this type are obtained. The formulae in section 5 of [7] will give all such factorisations. Certain restrictions must be placed on the orders of successive subgroups to ensure that A has prime power order.

3. Factorisations involving more than two factors

Theorem 1 may be extended to factorizations involving more than two factors in the same way that Theorem 2 of [10] is an extension of Theorem 2 of [1] for cyclic groups. As in that case it is necessary to examine the structure of the proof of Theorem 1 rather than just to use the result itself.

THEOREM 2. *Let G be a finite abelian group whose p-component is cyclic. If $G = A_1 A_2 \dots A_k B$, where each A_i has order a power of p, then one of the factors is periodic.*

PROOF. Let $A = A_1 A_2 \dots A_k$. Then A has order a power of p and $G = AB$. By Theorem 1 either A or B is periodic. If B is periodic we are finished. If not, then the

proof of Theorem 1 gives information connecting A and certain representations of G. Let us adopt the notations used in Theorem 1. Then, given that B is not periodic, we have $\sum \varrho^{e_i}\mathbf{M}(a_i)=0$ for every representation $\mathbf{M}$ of K. For $r=1, \ldots, k$ let $A_r = \bigcup_i h^{e_{ri}}a_{ri}$, where $0 \leqq e_{ri} < p^\lambda$ and $a_{ri} \in K$. As in previous proofs, from considerations of order and $A_1 \ldots A_k = A$, it follows that $F_{p^\lambda}(x)$ divides exactly one of the k polynomials $\sum_i x^{e_{ri}}$. Also it follows from $A=A_1 \ldots A_k$ that

$$0 = \sum \varrho^{e_i}\mathbf{M}(a_i) = \prod_{r=1}^{k} \left(\sum_i \varrho^{e_{ri}}\mathbf{M}(a_{ri})\right).$$

Hence for each representation $\mathbf{M}$ of K there exists r with $\sum_i \varrho^{e_{ri}}\mathbf{M}(a_{ri})=0$. However, as in the proof of Theorem 1, this implies $F_{p^\lambda}(x)$ divides $\sum x^{e_{ri}}$. Therefore the value of r is independent of $\mathbf{M}$. It then follows that $h^{p^{\lambda-1}}$ is a period of A_r for this value of r.

Again factoring out the subgroup of periods of the periodic factor leads to the same type of factorisation of the factor group. Hence the formulae in section 3 of [10] give all these factorisations, where again suitable restrictions are made to ensure that the appropriate factors have order a power of p.

4. Extensions to infinite groups

In [8] a generalisation of Theorem 2 of [7] is given to certain infinite groups. In a similar way we can generalise Theorem 1 and also Theorem 2. Since only the p-component is restricted to be cyclic, the restriction on the type of infinite group is very much weaker than that needed in Theorem 12 of [8].

THEOREM 3. *Let G be a torsion abelian group whose p-component is either cyclic or quasi-cyclic. If $G=A_1 \ldots A_kB$ where each A_r is finite and has order a power of p then one of the factors is periodic.*

PROOF. Let F be the subgroup of G generated by $A=A_1 \ldots A_k$. Since G is a torsion group and A is finite, F is finite. The p-component of F is a finite subgroup of a cyclic or quasi-cyclic group and so is itself cyclic. Let gF be a coset of F in G. Since $A \subset F$ we have $ab \in gF$, where $a \in A$, $b \in B$, if and only if $b \in gF$. Hence from $AB=G$ it follows that $A(B \cap gF)=gF$. Thus $(g^{-1}A)(B \cap gF)=F$. From $g^{-1}A_1A_2 \ldots A_k(B \cap gF)=F$ it follows from Theorem 2 that one of the factors is periodic. Clearly A_1 is periodic if and only if $g^{-1}A_1$ is periodic. Thus, if no factor A_r is periodic, $B \cap gF$ is periodic for each $g \in G$. Further an examination of the proof of Theorem 2 shows that if no A_r is periodic then A is not periodic. From $(g^{-1}A)(B \cap gF)=F$, an examination of the proof of Theorem 1 shows that if A is not periodic then there is a fixed element of F, depending only on A, which is a period of $B \cap gF$. Thus if no subset A_r is periodic there is a fixed element of F which is a period of $B \cap gF$ for all cosets gF. Since B is the union of these sets it follows that this element is a period of B.

Again factoring out the subgroup of periods of one factor leads to a factorisation of the quotient group of the same type. Since all the sets except B are finite and

subsets derived from B are not periodic in immediately successive stages, this process will terminate after a finite number of stages. So the formulae of section 3 of [10] give all such factorisations, or at least they would do so if all possible chains of subgroups of G were known.

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References

[1] N. G. DE BRUIJN, On the factorization of finite abelian groups, *Nederl. Akad. Wetensch. Indag. Math.*, **15** (1953), pp. 258—264.
[2] N. G. DE BRUIJN, On the factorization of cyclic groups, *Nederl. Akad. Wetensch. Indag. Math.*, **15** (1953), pp. 370—377.
[3] G. HAJÓS, Über einfache und mehrfache Bedeckung des n-dimensionalen Raumes mit einem Würfelgitter, *Math. Z.*, **47** (1941), pp. 427—467.
[4] G. HAJÓS, Sur le problème de factorisation des groupes cycliques, *Acta Math. Acad. Sci. Hung.*, **1** (1950), pp. 189—195.
[5] L. KRONECKER, Mémoire sur les facteurs irréductibles de l'expression $x^n - 1$, *Liouville, Journal de mathématiques pures et appliquées*, **I 19** (1854), pp. 177—192. *Werke* I pp. 75—93 (Leipzig, 1895).
[6] L. RÉDEI, Zwei Lückensatze über Polynome in endlichen Primkörpern mit Anwendungen auf die endlichen Abelschen Gruppen und die Gaussischen Summen, *Acta Math.*, **79** (1947), pp. 273—290.
[7] A. D. SANDS, On the factorisation of finite abelian groups, *Acta Math. Acad. Sci. Hung.*, **8** (1957), pp. 65—86.
[8] A. D. SANDS, The factorisation of abelian groups, *Quart. J. Math. (Oxford)*, (2) **10** (1959), pp. 81—91.
[9] A. D. SANDS, On the factorisation of finite abelian groups. II, *Acta Math. Acad. Sci. Hung.*, **13** (1962), pp. 153—169.
[10] A. D. SANDS, Factorization of cyclic groups, *Proc. Coll. Abelian Groups (Tihany)*, Budapest (1964), pp. 139—146.

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ON METRIZABILITY OF COUNTABLY COMPACT SPACES

By
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1. Introduction

Let (A_n) be a sequence of subsets of a topological space X. Denote by $\operatorname{Li} A_n$ the set of all points $x \in X$ such that every neighbourhood of x meets all but finitely many A_n. Notice that from the definition of $\operatorname{Li} A_n$, one has:

$$\operatorname{Li} A_n = \operatorname{Li} \overline{A}_n = \overline{\operatorname{Li} A_n}.$$

(For more details see C. KURATOWSKI [1].)

By definition, a sequence (A_n) of subsets of X will be said to be convergent if $\operatorname{Li} A_n$ exists and is reduced to a point. A convergent sequence (A_n) of subsets of X is uniformly convergent if each neighbourhood of $\{x\} = \operatorname{Li} A_n$ contains all but finitely many A_n. A space X is a C-space iff there exists a decreasing sequence $(\mathcal{U}_n)$ of open covers of X such that every sequence (A_n), with $A_n \in \mathcal{U}_n$ contains a convergent subsequence.

A space X is a CM-space iff there exists a decreasing sequence $(\mathcal{U}_n)$ of open covers of X such that every sequence (A_n), with $A_n \in \mathcal{U}_n$ contains a uniformly convergent subsequence.

It is the purpose of this paper to show that a countably compact space X is metrizable iff X is a C-space. It follows that a space X is compact and metrizable iff X is a CM-space.

2. Metrizability of countably compact spaces

The following two theorems are useful for the proof:

THEOREM 1. *Let X be a countably compact space, $x \in X$. If (A_n) is a sequence of open sets in X such that $\bigcap_n \overline{A}_n = \{x\}$, then the family $\{A_n\}_{n \geq 1}$ is a local subbase at x* (cf. [2]).

THEOREM 2 (Frink's metrization theorem [3]). *A T_1-space X is metrizable iff there is a local base $\{U_n(x)\}_{n \geq 1}$ at each $x \in X$ such that:*

a) $U_1(x) \supset U_2(x) \supset \ldots$,

b) *For each $x \in X$ and each $n \geq 1$, there is some $m > n$ such that:*

$$\text{if } y \in X \text{ and } U_m(x) \cap U_m(y) \neq \varnothing \text{ then } U_m(y) \subset U_n(x).$$

The following theorem is proved now:

THEOREM 3. *If X is a countably compact C-space, then X is a first countable T_1-space.*

PROOF. Let $x\in X$. For each $n\geqq 1$ select $V_n\in\mathscr{U}_n$ such that $x\in V_n$, and let (V_{n_k}) be a convergent subsequence of (V_n). Call $A_k=V_{n_k}$. Obviously $A_k\in\mathscr{U}_{n_k}\subset\mathscr{U}_k$, and $\{x\}=\mathrm{Li}\, A_k=\mathrm{Li}\,\overline{A}_k=\overline{\mathrm{Li}\, A_k}=\overline{\{x\}}$. This shows first that X is a T_1-space.

Notice that $\bigcap_k \overline{A}_k=\{x\}$. For if not, there will be some $y\in\bigcap_k \overline{A}_k$ with $y\neq x$, so that $y\in\mathrm{Li}\,\overline{A}_k=\{x\}$ which is impossible.

By theorem 1, the family $\{A_k\}_{k\geqq 1}$ is a local subbase at x and consequently X is first countable.

We show now the following metrization theorem:

THEOREM 4. *Let X be a countably compact space. Then X is metrizable iff X is a C-space.*

PROOF. *Necessity.* Let X be a countably compact space, and let d be a distance on X that generates its topology (hence X is compact). Denote by $B(x, r)$ the open ball of centre x and radius r. It is not difficult to show that the sequence $(\mathscr{U}_n)$ of open covers of X defined by:

$$\mathscr{U}_n=\{B(x, 1/i)\colon x\in X,\ i=n, n+1, n+2, \ldots\}\quad (n\geqq 1)$$

satisfies the required condition.

Sufficiency. Let X be a countably compact C-space. For each $x\in X$ and each $n\geqq 1$ select $A_n(x)\in\mathscr{U}_n$ so that $\{x\}=\mathrm{Li}\, A_n(x)$. Denote $A_n^*(x)=\bigcap_{k=1}^{n} A_k(x)$. By theorem 3, the family $\{A_n^*(x)\}_{n\geqq 1}$ is a decreasing local base at x. It suffices now to prove that the condition b) of theorem 2 holds. We argue by contradiction. Suppose for some $n\geqq 1$, there exists a sequence (y_m) of points of X such that, for every $m>n$:

$$A_m^*(x)\cap A_m^*(y_m)\neq\varnothing\quad\text{and}\quad A_m^*(y_m)\not\subset A_n^*(x).$$

Select $x_m\in A_m^*(x)\cap A_m^*(y_m)$ and $z_m\in A_m^*(y_m)\setminus A_n^*(x)$. Since X is first countable and countably compact, then X is sequentially compact and consequently $X\times X$ is also sequentially compact. Thus the sequence $(\langle x_m, z_m\rangle)$ of points of $X\times X$ contains a convergent subsequence $(\langle x_{m_k}, z_{m_k}\rangle)$. Clearly (x_{m_k}) vonverges to x and (z_{m_k}) converges to some $z\in\overline{X\setminus A_n^*(x)}=X\setminus A_n^*(x)$, so that $z\neq x$. Call

$$a_k=x_{m_k},\quad b_k=z_{m_k},\quad B_k=A_{m_k}(y_{m_k}),\quad B_k^*=A_{m_k}^*(y_{m_k}).$$

Since $B_k\in\mathscr{U}_{m_k}\subset\mathscr{U}_k$, let (B_{k_j}) be a convergent subsequence of (B_k). Notice that $a_{k_j}\in B_{k_j}^*\subset B_{k_j}$ and (a_{k_j}) converges to x. Consequently $\{x\}=\mathrm{Li}\, B_{k_j}^*=\mathrm{Li}\, B_{k_j}$. But since $b_{k_j}\in B_{k_j}$ and (b_{k_j}) converges to z, then $z\in\mathrm{Li}\, B_{k_j}=\{x\}$ which is impossible.

3. Characterization of compact and metrizable spaces

As a corollary of theorem 4, one can give the following characterization for the spaces which are compact and metrizable:

THEOREM 5. *A topological space X is compact and metrizable iff X is a CM-space.*

PROOF. *Necessity*. If X is a compact and metrizable space, then the same family of open covers considered in theorem 4 satisfies the required condition.

Sufficiency. Notice that if X is a CM-space, then X is countably compact: For if D is an infinite subset of X, select a sequence (x_n) of distinct elements of D and let $A_n \in \mathcal{U}_n$ so that $x_n \in A_n$ for every n. The sequence (A_n) contains a uniformly convergent subsequence (A_{n_k}). If $\{x\} = \mathrm{Li}\, A_{n_k}$, then x is an accumulation point of D. Since every CM-space is a C-space, then by theorem 4, X is metrizable and thus compact.

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References

[1] C. KURATOWSKI, *Topologie,* Volume I (1958), p. 241.
[2] S. WILLARD, *General Topology,* Addison-Wesley, (1970), 8708, p. 125.
[3] A. H. FRINK, Distance Functions and the Metrization Problem, *Bull. Amer. Math. Soc.,* **43** (1937), pp. 133—142.

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ON PSEUDOCOMPACT AND PERFECTLY NORMAL SPACES

By

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It is shown in this paper that a pseudocompact space is perfectly normal if and only if every closed subset of that space has a countable local base. It follows that, for topological spaces such that each closed subset has a countable local base, the notions of pseudo-compacity and countable compacity are equivalent.

A. Notations. Recall that a normal space X is perfectly normal if each closed subset of X is a G_δ.

A completely regular space X is pseudocompact if every real valued continuous function on X is bounded.

A Hausdorff space X is countably compact if each infinite subset of X has at least one accumulation point. Notice that every normal and pseudocompact space is countably compact [1].

B. Theorems. In [2] the following characterization theorem of perfectly normal spaces is proved:

THEOREM 1. *A T_1-space X is perfectly normal if and only if for every closed subset $A \subseteq X$ there exists a continuous function $f: X \to [0, 1]$ such that $f^{-1}(0) = A$.*

In [3] the following theorem is proved:

THEOREM 2. *A completely regular space X is pseudocompact if and only if for an arbitrary countable family of open subsets $\{G_n\}_{n \geq 1}$ of X which has the finite intersection property, the inequality $\bigcap_{n \geq 1} \overline{G}_n \neq \emptyset$ holds.*

The following theorem is proved now:

THEOREM 3. *A pseudocompact space X is perfectly normal if and only if each closed subset of X has a countable local base.*

Let X be a pseudocompact and perfectly normal space, and let A be a closed subset of X. By theorem 1, there exists a continuous mapping $f: X \to [0, 1]$ such that $f^{-1}(0) = A$. For every integer n, $n \geq 1$, denote $V_n = f^{-1}[0, 1/n[$. It is sufficient to show that the family $\{V_n\}_{n \geq 1}$ is a local base for A. In fact let V be an open subset of X such that $A \subset V$; if $V_n \not\subset V$ for every n, select $x_n \in V_n \setminus V$. The space X, being normal and pseudocompact, is necessarily countably compact, and consequently the sequence (x_n) has at least one cluster point x_0 which belongs to $X \setminus V$. But since $x_0 \notin A$ then $f(x_0) > 0$ and by the continuity of f there exists an open neighbourhood W of x_0 and an $\varepsilon > 0$ such that $f(x) > \varepsilon$ for every $x \in W$. The point x_0 being a cluster point

for (x_n), there will be a subsequence (x_{n_i}) of (x_n) such that $x_{n_i} \in W$ for every $i=1, 2, \ldots$. Let j be an integer such that $1/n_j < \varepsilon$; then $f(x_{n_j}) < 1/n_j < \varepsilon$ and $f(x_{n_j}) > \varepsilon$ which is a contradiction.

Conversely, let X be a pseudocompact space for which each closed subset A has a countable local base $\{V_n(A)\}_{n\geq 1}$ constituted of open subsets. Without loss of generality, one can suppose that $V_{n+1}(A) \subseteq V_n(A)$ for every n. Notice that if A is closed, then by the regularity of X $A = \bigcap_{n\geq 1} V_n(A)$, hence a G_δ.

Let A and B be two closed and disjoint subsets of X. It is sufficient to show that there exists an n such that $V_n(A) \cap V_n(B) = \varnothing$. If not, then $G_n = V_n(A) \cap V_n(B)$ is not empty and such that $G_{n+1} \subseteq G_n$ for every n; hence, by theorem 2 there exists an $x \in X$ such that $x \in \bigcap_{n\geq 1} \overline{G}_n$. Now $x \in A$, for if $x \notin A$ there will be open subsets V and W such that $x \in W$, $A \subseteq V$ and $V \cap W = \varnothing$. Let m be an integer such that $V_m(A) \subseteq V$; then $W \cap V_m(A) = \varnothing$ and this means that $x \notin \overline{V_m(A)} \supseteq \overline{G}_m$, which is a contradiction. By a similar proof the point x is also in B, and consequently $x \in A \cap B$, which is contrary to the hypothesis.

The following two theorems are immediate corollaries:

THEOREM 4. *Every pseudocompact and perfectly normal space is first countable.*

THEOREM 5. *Let X be a completely regular space for which every closed subset has a countable local base. Then X is pseudocompact if and only if it is countably compact.*

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References

[1] E. HEWITT, Rings of real-valued continuous functions, I, *Trans. Amer. Math. Soc.*, **64** (1948), pp. 45—99.
[2] É. ČECH, Sur la dimension des espaces parfaitement normaux, *Bull. Intern. Acad. Tchèque. Sci.*, **33** (1932), pp. 38—55.
[3] J. COLMEZ, Sur les espaces précompacts, *C. R. Acad. Paris*, **233** (1951), pp. 1552—1553.

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EXTREMAL PROPERTIES OF PADÉ QUOTIENTS

By

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In this note we extend a result of Stieltjes concerning extremal properties of convergents of certain continued fractions, establish a corresponding result for Padé quotients generated by certain series, and describe the numerical behaviour of the associated Padé tables.

Let the function $\sigma(\varrho)$ be bounded and nondecreasing for $-\infty \leqq \alpha < \beta \leqq \infty$ and such that the moments $t_\nu = (-1)^\nu \int_\alpha^\beta \varrho^\nu \, d\sigma(\varrho)$ $(\nu = 0, 1, \ldots)$ exist; assume that $\sigma(\varrho)$ is not a simple step function with a finite number of salti in the range $\alpha \leqq \varrho \leqq \beta$. For future use we adopt the notations $\sigma \in \mathfrak{D}_\alpha^\beta$ to describe such a function and $\mathfrak{MS}\{\sigma \in \mathfrak{D}_\alpha^\beta\} \equiv \{t_\nu\}$ to define the moment sequence generated by it. A nonterminating continued fraction ([1], [2]) whose successive convergents are $C_0(\lambda) = 0$ and

$$(1) \qquad C_r(\lambda) = \frac{v_1}{\lambda + w_1 +} \frac{v_2}{\lambda + w_2 +} \cdots \frac{v_r}{\lambda + w_r} \quad (r = 1, 2, \ldots)$$

may be derived ([3], [4]) from the series $\mathscr{F}(\lambda) = \sum t_\nu \lambda^{-\nu-1}$ (here and later summation from $\nu = 0$ to $\nu = \infty$ is denoted by the unembellished sign $\sum$) and if $p_r(\lambda) = \sum_{\nu=0}^{r} k_{r,\nu} \lambda^\nu$ $(k_{r,r} = 1;\ r = 0, 1, \ldots)$ are the orthogonal polynomials derived from the moments $\{t_\nu\}$ and $q_r(\lambda) = \int_\alpha^\beta \{p_r(\lambda) - p_r(-\varrho)\} d\sigma(\varrho)/(\lambda + \varrho)$ $(r = 0, 1, \ldots)$ are the associated orthogonal polynomials derived from the same set of moments, then $C_r(\lambda) = q_r(\lambda)/p_r(\lambda)$ $(r = 0, 1, \ldots)$. We adopt the notation $\mathscr{A}\{\mathscr{F}(\lambda)\}_r = C_r(\lambda)$ $(r = 0, 1, \ldots)$ to describe the convergents and their provenance.

Denote the set of real numbers $X_1, X_2, \ldots, X_r$ by $\mathfrak{X}$, and set

$$\int_\alpha^\beta \left\{1 + \sum_{\nu=1}^{r} X_\nu (\lambda + \varrho)^\nu\right\}^2 \frac{\varrho^m \, d\sigma(\varrho)}{\lambda + \varrho} = \mathsf{M}[\mathfrak{X} : \sigma;\ r, m]_\alpha^\beta.$$

It is easily shown that when $\lambda \in (-\alpha, \infty)$ (we take this interval to be empty when $\alpha = -\infty$, and adopt a similar convention throughout) $\mathsf{M}[\mathfrak{X} : \sigma; r, 0]_\alpha^\beta$ attains a minimum when $1 + \sum_{\nu=1}^{r} X_\nu (\lambda + \varrho)^\nu = p_r(-\varrho)/p_r(\lambda)$ $(r = 0, 1, \ldots)$. Furthermore, as is easily verified,

$$\int_\alpha^\beta \left\{\frac{p_r(-\varrho)}{p_r(\lambda)}\right\}^2 \frac{d\sigma(\varrho)}{\lambda + \varrho} = S(\lambda) - \frac{1}{p_r(\lambda)} \int_\alpha^\beta \frac{p_r(\lambda) - p_r(-\varrho)}{\lambda + \varrho} \, d\sigma(\varrho)$$

where we set

$$\mathscr{ST}[\sigma:\lambda]_\alpha^\beta = \int_\alpha^\beta \frac{d\sigma(\varrho)}{\lambda+\varrho}$$

and $S(\lambda)=\mathscr{ST}[\sigma:\lambda]_\alpha^\beta$.

A similar result holds when $\lambda\in(-\infty, -\beta)$. Taking the symbol min to mean the minimum as the members of $\mathfrak{X}$ are allowed to range over all real values, we have

THEOREM 1. *Let* $-\infty \leqq \alpha < \beta \leqq \infty$, $\mathfrak{MS}\{\sigma\in\mathfrak{D}_\alpha^\beta\} \equiv \{t_\nu\}$, $C_r(\lambda) = \mathscr{A}\{\sum t_\nu \lambda^{-\nu-1}\}_r$ $(r=0,1,\ldots)$ *and* $S(\lambda)=\mathscr{ST}[\sigma:\lambda]_\alpha^\beta$. *Subject to the attached conditions we have*

$$S(\lambda)-C_r(\lambda) = \min \mathsf{M}[\mathfrak{X}:\sigma;\, r,\, 0]_\alpha^\beta > 0 \quad (\lambda\in(-\alpha, \infty))$$

$$C_r(\lambda)-S(\lambda) = \min\{-\mathsf{M}[\mathfrak{X}:\sigma;\, r,\, 0]_\alpha^\beta\} > 0 \quad (\lambda\in(-\infty, -\beta))$$

for $r=0, 1, \ldots$.

If $\alpha\geqq 0$ or $\beta\leqq 0$, $\mathscr{F}(\lambda)$ also generates a corresponding continued fraction [1], [2] whose successive convergents are $\hat{C}_0(\lambda)=0$ and

$$(2) \qquad \hat{C}_{2r-1}(\lambda) = \frac{u_1}{\lambda+}\,\frac{u_2}{1+}\,\cdots\,\frac{u_{2r-2}}{1+}\,\frac{u_{2r-1}}{\lambda}, \quad \hat{C}_{2r}(\lambda) = \frac{u_1}{\lambda+}\,\frac{u_2}{1+}\,\cdots\,\frac{u_{2r-1}}{\lambda+}\,\frac{u_{2r}}{1}$$

$$(r = 1, 2, \ldots)$$

([5]). We then write $\mathscr{C}\{\mathscr{F}(\lambda)\}_r = \hat{C}_r(\lambda)$ $(r=0, 1, \ldots)$. When $\beta\leqq 0$, the series $\mathscr{F}(\lambda)$ may, by replacing ϱ by $-\varrho$ and λ by $-\lambda$, easily be transformed into an equivalent form for which $\alpha\geqq 0$; we therefore confine our attention to the latter case. In the notations of formulas (1) and (2), $C_r(\lambda)=\hat{C}_{2r}(\lambda)$ $(r=1, 2, \ldots)$; Theorem 1 may be presented in terms of the even order convergents $\{\hat{C}_{2r}(\lambda)\}$. Concerning the odd order convergents, we have

THEOREM 2. *Let* $0\leqq\alpha<\beta\leqq\infty$, $\mathfrak{MS}\{\sigma\in\mathfrak{D}_\alpha^\beta\}\equiv\{t_\nu\}$, $\hat{C}_r(\lambda)=\mathscr{C}\{\sum t_\nu\lambda^{-\nu-1}\}_r$ $(r=0, 1, \ldots)$ *and* $S(\lambda)=\mathscr{ST}[\sigma;\, \lambda]_\alpha^\beta$. *Subject to the attached conditions, we have*

$$\hat{C}_{2r+1}(\lambda) - S(\lambda) = \min \mathsf{M}[\mathfrak{X}:\sigma;\, r,\, 1]_\alpha^\beta > 0, \quad (\lambda\in(-\alpha, \infty))$$

$$S(\lambda) - \hat{C}_{2r+1}(\lambda) = \min\{-\mathsf{M}[\mathfrak{X}:\sigma;\, r,\, 1]_\alpha^\beta\} > 0 \quad (\lambda\in(-\infty, -\beta))$$

for $r=0, 1, \ldots$.

(We emphasize that the methods of proof for the above theorems are entirely due to STIELTJES [5] who derived the results of Theorems 1 and 2 for $\alpha=0$, $\beta=\infty$. We have merely extended the intervals over which these results hold.)

The symbol $\mathfrak{P}\{\sum t_\nu z^\nu\}\equiv P_{i,j}(z)$ is used to indicate that the set of Padé quotients ([6], [1], [2]) $P_{i,j}(z)$ $(i, j=0, 1, \ldots)$ is derived from the series $\sum t_\nu z^\nu$. If $-\infty\leqq\alpha<\beta\leqq\infty$, $\sigma\in\mathfrak{D}_\alpha^\beta$, $0\leqq m<\infty$, and $d\sigma_{2m}(\varrho)=\varrho^{2m}d\sigma(\varrho)(\alpha\leqq\varrho\leqq\beta)$, then $\sigma_{2m}\in\mathfrak{D}_\alpha^\beta$. Furthermore, if

$\mathfrak{MS}\{\sigma\in\mathfrak{D}_\alpha^\beta\}\equiv\{t_\nu\}$, $\mathscr{A}\{\sum t_{2m+\nu}\lambda^{-\nu-1}\}_r=C_r^{(2m)}(\lambda)$ $(r=0,1,\ldots)$ and $\mathfrak{P}\{\sum t_\nu z^\nu\}\equiv\{P_{i,j}(z)\}$, then

(3)
$$P_{r,2m+r-1}(z)=\lambda\left\{\sum_{\nu=0}^{2m-1}t_\nu\lambda^{-\nu-1}+\lambda^{-2m}C_r^{(2m)}(\lambda)\right\},\quad \lambda=z^{-1}\quad (m,r=0,1,\ldots).$$

(For convenience, we append the quotient $P_{0,-1}(z)=0$ to the Padé table.) If $S(\lambda)=\mathscr{ST}[\sigma;\lambda]_\alpha^\beta$ and

(4)
$$S(\lambda)=\cfrac{t_0}{\lambda-\cfrac{t_1}{t_0}-\tilde{S}(\lambda)},\quad \tilde{S}(\lambda)=\int_\alpha^\beta\frac{d\tilde{\sigma}(\varrho)}{\lambda+\varrho}$$

then $\tilde{\sigma}\in\mathfrak{D}_\alpha^\beta$ ([7]). If $\mathfrak{MS}\{\tilde{\sigma}\}\equiv\{\tilde{t}_\nu\}$ and $\mathscr{A}\{\sum\tilde{t}_{2m+\nu}\lambda^{-\nu-1}\}_r=\tilde{C}_r^{(2m)}(\lambda)$ $(r=0,1,\ldots)$, we have

(5)
$$P_{2m+r+1,r}(z)=\cfrac{t_0\lambda}{\lambda-\cfrac{t_1}{t_0}-\sum_{\nu=0}^{2m-1}\tilde{t}_\nu\lambda^{-\nu-1}-\lambda^{-2m}\tilde{C}_r^{(2m)}(\lambda)},\quad \lambda=z^{-1}\quad (m,r=0,1,\ldots).$$

We set

$$\int_\alpha^\beta\frac{d\sigma(\varrho)}{1+z\varrho}=\mathfrak{st}[\sigma:z]_\alpha^\beta.$$

Using the above results and Theorem 1, and setting

(6)
$$\int_\alpha^\beta\left\{1+\sum_{\nu=1}^{r}X_\nu(1+z\varrho)^\nu\right\}^2\frac{\varrho^m d\sigma(\varrho)}{1+z\varrho}=\mu[\mathfrak{X}:\sigma;r,m]_\alpha^\beta$$

we derive

THEOREM 3. *Let* $-\infty\leqq\alpha<0<\beta\leqq\infty$, $-\beta^{-1}<z<-\alpha^{-1}$, $\mathfrak{MS}\{\sigma\in\mathfrak{D}_\alpha^\beta\}\equiv\{t_\nu\}$, $\mathfrak{P}\{\sum t_\nu z^\nu\}\equiv\{P_{i,j}(z)\}$ *and* $f(z)=\mathfrak{st}[\sigma:z]_\alpha^\beta$. *In the notation of formula* (6) *we have*

(7)
$$\begin{aligned}f(z)-P_{r,2m+r-1}(z)&=z^{2m}\min\mu[\mathfrak{X}:\sigma;r,2m]_\alpha^\beta>0\\ t_0\{P_{2m+r+1,r}(z)^{-1}-f(z)^{-1}\}&=z^{2m+2}\min\mu[\mathfrak{X}:\tilde{\sigma};r,2m]_\alpha^\beta>0\end{aligned}$$

for $m,r=0,1,\ldots$.

Again it suffices, when $\alpha\geqq0$ or $\beta\leqq0$, to consider the former case alone. Relationships of the form (3) now also hold with $2m$ replaced by $2m+1$. If

(8)
$$\mathscr{ST}[\sigma:\lambda]_\alpha^\beta=\frac{t_0}{\lambda\{1+\hat{S}(\lambda)\}}\quad(\sigma\in\mathfrak{D}_\alpha^\beta),\quad \hat{S}(\lambda)=\int_\alpha^\beta\frac{d\hat{\sigma}(\varrho)}{\lambda+\varrho},$$

then $\hat{\sigma}\in\mathfrak{D}_\alpha^\beta$([8], [9]), and relationships similar to (5) hold. We derive

THEOREM 4. *Let* $0\leqq\alpha<\beta\leqq\infty$, $\mathfrak{MS}\{\sigma\in\mathfrak{D}_\alpha^\beta\}\equiv\{t_\nu\}$, $\mathfrak{P}\{\sum t_\nu z^\nu\}\equiv\{P_{i,j}(z)\}$, $f(z)=\mathfrak{st}[\sigma:z]_\alpha^\beta$, *and* $\hat{\sigma}$ *be defined by formulae* (8). *If* $z\in(-\beta^{-1},\infty)$, *then* $0<f(z)<\infty$ *and*

$$\begin{aligned}(-z)^{-m}\{f(z)-P_{r,m+r-1}(z)\}&=\min\mu[\mathfrak{X}:\sigma;r,m]_\alpha^\beta>0\\ t_0(-z)^{-m-1}\{P_{m+r,r}(z)^{-1}-f(z)^{-1}\}&=\min\mu[\mathfrak{X}:\hat{\sigma};r,m]_\alpha^\beta>0\end{aligned}$$

and if $z\in(-\infty, -\alpha^{-1})$ *then* $-\infty<f(z)<0$ *and*

$$(-z)^{-m}\{P_{r,m+r-1}(z)-f(z)\} = \min\{-\mu[\mathfrak{X}:\sigma; r, m]_\alpha^\beta\} > 0$$

$$t_0(-z)^{-m-1}\{f(z)^{-1}-P_{m+r,r}(z)^{-1}\} = \min\{-\mu[\mathfrak{X}:\hat{\sigma}; r, m]_\alpha^\beta\} > 0$$

with $m, r=0, 1, \ldots$ *in all cases.*

When $-\infty\leqq\alpha<\beta\leqq\infty$, $\mathfrak{MS}\{\sigma\in\mathfrak{D}_\alpha^\beta\}\equiv\{t_\nu\}$ and $\mathfrak{P}\{\sum t_\nu z^\nu\}\equiv\{P_{i,j}(z)\}$, all quotients of the form $P_{2i,2j+1}(z)$, $P_{2i+1,2j}(z)$ $(i, j=0, 1, \ldots)$ are distinct, except that one or more groups of at most two quotients of the form $P_{i'+1,j'}(z)$, $P_{i',j'+1}(z)$ may be identical ([7]). For a fixed finite integer $m\geqq0$, $\mu[\mathfrak{X}:\sigma; r, 2m]_\alpha^\beta$ has one more diposable parameter than $\mu[\mathfrak{X}:\sigma; r-1, 2m]_\alpha^\beta$ $(1\leqq r<\infty)$; thus, if $\alpha<0<\beta$, $z\in(-\beta^{-1}, -\alpha^{-1})$, then $\min\mu[\mathfrak{X}:\sigma; r, 2m]_\alpha^\beta\leqq\min\mu[\mathfrak{X}:\sigma; r-1, 2m]_\alpha^\beta$. But when $z\neq0$, distinct neighbouring Padé quotients have distinct values; thus when $z(\neq0)\in(-\beta^{-1}, -\alpha^{-1})$, formula (7) yields $0<f(z)-P_{r,2m+r-1}(z)<f(z)-P_{r-1,2m+r-2}(z)$ $(m\geqq0, r\geqq1)$. Formula (7) concerns a minimum over all polynomials $p_r(\varrho)$ of the r^{th} degree in ϱ such that $\{p_r(-z^{-1})\}^2=1$. Formula (7) with r replaced by $r-1$ and m replaced by $m+1$ concerns a corresponding minimum with the same weight function $\varrho^{2m}d\sigma(\varrho)/(1+z\varrho)$, over all polynomials $\hat{p}_r(\varrho) = z\varrho+\sum_{\nu=0}^{r-1} X_\nu z\varrho(1+z\varrho)^\nu$ of the r^{th} degree in ϱ such that $\{\hat{p}_r(-z^{-1})\}^2=1$. This latter formula has one disposable parameter less than the original formula, and hence, when $z(\neq0)\in(-\beta^{-1}, -\alpha^{-1})$ we deduce that $0<f(z)-P_{r,2m+r-1}(z)\leqq f(z)-P_{r-1,2m+r}(z)$ $(m\geqq0, r\geqq1)$.

If the terminating or nonterminating sequence $a_r(r=0, 1, \ldots, r')$ is monotonically increasing [decreasing] with upper [lower] bound A, we write $a_r|_{r=0}^{r'}\in\mathfrak{MT}\{A\}[a_r|_{r=0}^{r'}\in\mathfrak{MD}\{A\}]$; if, however, this sequence is nondecreasing with no three members equal, we use the symbol $a_r|_{r=0}^{r'}\in\mathfrak{MT}'\{A\}$. Using this notation, we give

THEOREM 5. *Let* $-\infty\leqq\alpha<0<\beta\leqq\infty$, $z(\neq0)\in(-\beta^{-1}, -\alpha^{-1})$, $\mathfrak{MS}\{\sigma\in\mathfrak{D}_\alpha^\beta\}\equiv\{t_\nu\}$, $\mathfrak{P}\{\sum t_\nu z^\nu\}\equiv\{P_{i,j}(z)\}$ *and* $f(z)=\mathfrak{st}[\sigma:z]_\alpha^\beta$; *let the finite integer* $m\geqq0$ *be fixed. Then* $0<f(z)<\infty$,

$$P_{2i,2j+1}(z), \quad P_{2i+1,2j}(z) < f(z) \quad (i, j = 0, 1, \ldots)$$

and

$$P_{r,2m+r-1}(z)|_{r=0}^{\infty}\in\mathfrak{MT}\{f(z)\}, \quad P_{2m+r+1,r}(z)|_{r=0}^{\infty}\in\mathfrak{MT}\{f(z)\},$$

$$P_{r,2m-r+1}(z)|_{r=0}^{m+1}\in\mathfrak{MT}'\{P_{m+1,m}(z)\}, \; P_{2m-r+1,r}(z)|_{r=0}^{m}\in\mathfrak{MT}'\{P_{m+1,m}(z)\}.$$

The position of the Padé quotient $P_{i,j}(z)$ may be referred to coordinates in an x', y'-plane, and the values of $f(z)$ and the $\{P_{i,j}(z)\}$ may be measured in the z'-direction, x', y' and z' being rectangular coordinates. The results of Theorem 5 indicate that, viewed from the positive z'-direction, the quotients $\{P_{2i,2j+1}(z)\}$ and $\{P_{2i+1,2j}(z)\}$ $(i, j=0, 1, \ldots)$ lie on an upturned hull, with the quotients $P_{m+1,m}(z)$ $(m=0, 1, \ldots)$ lying on the keel (the hull is in general assymetric). If $|z\alpha|, |z\beta|<1$, the sequences $P_{0,m}(z)$ $(m=0, 1, \ldots)$, $P_{m,0}(z)$ $(m=0, 1, \ldots)$ converge to $f(z)$: the hull has a pronounced prow. If $|z\alpha|>1$ or $|z\beta|>1$, and $\sigma(\alpha)<\sigma(\alpha+)$, $\sigma(\beta)>\sigma(\beta-)$, the sequence $P_{0,m}(z)$ $(m=0, 1, \ldots)$ diverges; if $\sigma(\alpha+\varrho)<\sigma(\alpha+\varrho+)$, $\sigma(\beta-\varrho-)<\sigma(\beta-\varrho)$ for all ϱ in all interval such as $0<\varrho\leqq\delta$ $(\delta>0)$, then $P_{m,0}(z)$ diverges also: in this case the hull has a receding prow. In all cases, the plane $z'=f(z)$ lies above the hull.

By methods similar to those used in the proof of Theorem 5, we derive

THEOREM 6. *Let* $0 \leqq \alpha < \beta \leqq \infty$, $\mathfrak{MS}\{\sigma \in \mathfrak{D}_\alpha^\beta\} \equiv \{t_\nu\}$, $\mathfrak{P}\{\sum t_\nu z^\nu\} \equiv \{P_{i,j}(z)\}$ *and* $f(z) = \mathfrak{St}[\sigma : z]_\alpha^\beta$.

i) *If* $z \in (0, \infty)$, *then* $0 < f(z) < \infty$,

$$(-1)^m\{f(z) - P_{r,m+r-1}(z)\} > 0, \quad (-1)^m\{f(z)^{-1} - P_{m+r,r}(z)^{-1}\} > 0$$

$$(m, r = 0, 1, \ldots)$$

and for $0 \leqq m < \infty$

$$P_{r,2m+r-1}(z)|_{r=0}^{\infty} \in \mathfrak{MT}\{f(z)\}, \quad P_{r,2m+r}(z)|_{r=0}^{\infty} \in \mathfrak{MD}\{f(z)\},$$

$$P_{2m+r,r}(z)^{-1}|_{r=0}^{\infty} \in \mathfrak{MT}\{f(z)^{-1}\}, \quad P_{2m+r+1,r}(z)|_{r=0}^{\infty} \in \mathfrak{MT}\{f(z)\},$$

$$P_{r,2m-r}(z)|_{r=0}^{m} \in \mathfrak{MD}\{P_{m,m}(z)\}, \quad P_{r,2m-r+1}(z)|_{r=0}^{m+1} \in \mathfrak{MT}\{P_{m+1,m}(z)\},$$

$$P_{2m-r,r}(z)^{-1}|_{r=0}^{m} \in \mathfrak{MT}\{P_{m,m}(z)^{-1}\}, \quad P_{2m-r+1,r}(z)|_{r=0}^{m} \in \mathfrak{MT}\{P_{m+1,m}(z)\}.$$

ii) *If* $z \in (-\beta^{-1}, 0)$, *then* $0 < f(z) < \infty$,

$$0 < P_{i,j}(z) < f(z) \quad (i, j = 0, 1, \ldots)$$

and for $0 \leqq m < \infty$

$$P_{r,m+r}(z)|_{r=0}^{\infty} \in \mathfrak{MT}\{f(z)\}, \quad P_{m+r,r}(z) \in \mathfrak{MT}\{f(z)\},$$

$$P_{r,2m-r}(z)|_{r=0}^{m} \in \mathfrak{MT}\{P_{m,m}(z)\}, \quad P_{r,2m-r+1}(z)|_{r=0}^{m+1} \in \mathfrak{MT}\{P_{m+1,m}(z)\},$$

$$P_{2m-r,r}(z)|_{r=0}^{m} \in \mathfrak{MT}\{P_{m,m}(z)\}, \quad P_{2m-r+1,r}(z)\,{}_{r=0}^{m} \in \mathfrak{MT}\{P_{m+1,m}(z)\}.$$

iii) *If* $z \in (-\infty, -\alpha^{-1})$, *then* $-\infty < f(z) < 0$,

$$P_{i,j}(z) > f(z) \quad (i, j = 0, 1, \ldots)$$

and for $0 \leqq m < \infty$

$$P_{r,m+r}(z)|_{r=0}^{m} \in \mathfrak{MD}\{f(z)\}, \quad P_{m+r,r}(z)|_{r=0}^{\infty} \in \mathfrak{MD}\{f(z)\},$$

$$P_{r,2m-r}(z)|_{r=0}^{m} \in \mathfrak{MD}\{P_{m,m}(z)\}, \quad P_{r,2m-r+1}(z)|_{r=0}^{m+1} \in \mathfrak{MD}\{P_{m+1,m}(z)\},$$

$$P_{2m-r,r}(z)|_{r=0}^{m} \in \mathfrak{MD}\{P_{m,m}(z)\}, \quad P_{2m-r+1,r}(z)|_{r=0}^{m} \in \mathfrak{MD}\{P_{m+1,m}(z)\}.$$

The results of the above theorem for the case in which $\alpha = 0$, $\beta = \infty$ (clause i) then alone has meaning) have already been derived ([10]).

Again referring the values of $f(z)$ and the quotients $\{P_{i,j}(z)\}$ to a system of rectangular coordinates x', y', z', the results of Theorem 6 may be interpreted as follows. When $z \in (-\beta^{-1}, 0)$ all quotients $\{P_{i,j}(z)\}$ lie on an upturned convex hull as described above in connection with Theorem 5, where now the keel lies between the sequences $P_{m,m}(z)$ $(m = 0, 1, \ldots)$, $P_{m+1,m}(z)$ $(m = 0, 1, \ldots)$. In this case the plane $z' = f(z)$ lies above the hull, which has a pronounced prow (both sequences $P_{0,m}(z)$ $(m = 0, 1, \ldots)$, $P_{m,0}(z)$ $(m = 0, 1, \ldots)$ converge monotonically to $f(z)$). When

$z\in(-\infty, -\alpha^{-1})$ all quotients lie on a hull which is not upturned (i.e. it is concave viewed from the positive z'-direction), its keel lies between the sequences $P_{m,m}(z)$ $(m=0, 1, \ldots)$, $P_{m+1,m}(z)$ $(m=0, 1, \ldots)$, it has a receding prow, and the plane $z'=f(z)$ lies below the hull. When $z\in(0, \infty)$, two hulls are involved. The quotients $P_{2i,2j+1}(z)$, $P_{2i+1,2j}(z)$ $(i, j=0, 1, \ldots)$ are situated precisely as described in connection with Theorem 5. The quotients $\{P_{i',j'}(z)\}$ taken from the set $P_{2i,2j}(z)$, $P_{2i+1,2j+1}(z)$ $(i, j=0, 1, \ldots)$ for which $j'\geqq i'$ lie on a semi-hull (not upturned) with the sequence $P_{m,m}(z)$ $(m=0, 1, \ldots)$ lying on the keel. The quotients $\{P_{i',j'}(z)\}$ of this set with $j'<i'$ may lie on a simple completion of this hull. It is also possible for this continuation to be fractured, with a side of the hull extending to infinity in the positive z'-direction; the continuation of the surface then reappears from the negative z' direction and is convex when viewed from the positive z'-direction. If $z<\beta^{-1}$, fracture does not occur; both hulls have pronounced prows. If $z>\beta^{-1}$, fracture may occur, and the hull and semi-hull described in general have receding prows. In all cases, the plane $z'=f(z)$ lies between the complete and fractured hulls, or between the two complete hulls as is appropriate.

It is clear that of all the approximations to the numerical value of $f(z)$ to be obtained from the quotients considered in Theorems 5 and 6, the best belong, in the case of Theorem 5, to the diagonal sequence $P_{m+1,m}(z)$ $(m=0, 1, \ldots)$ and, in the case of Theorem 6, to the diagonal sequences $P_{m,m}(z)$ $(m=0, 1, \ldots)$, $P_{m+1,m}(z)$ $(m=0, 1, \ldots)$.

Theorem 5 may be used to derive a convergence result. When

$$-\infty < \alpha < 0 < \beta < \infty, \quad \gamma = \min(-\alpha^{-1}, \beta^{-1}), \quad \mathfrak{MS}\{\sigma\in\mathfrak{D}_\alpha^\beta\} \equiv \{t_\nu\}$$

and $f(z)=\mathcal{A}[\sigma : z]_\alpha^\beta$, then the series $\sum t_\nu z^\nu$ converges to $f(z)$ for $z\in(-\gamma, \gamma)$. A similar result holds for the series $\sum \tilde{t}_\nu z^\nu$. We derive

THEOREM 7. *Let* $-\infty<\alpha<0<\beta<\infty$, $\gamma=\min(-\alpha^{-1}, \beta^{-1})$, $\mathfrak{MS}\{\sigma\in\mathfrak{D}_\alpha^\beta\}\equiv\{t_\nu\}$, $\mathfrak{P}\{\sum t_\nu z^\nu\}\equiv\{P_{i,j}(z)\}$ *and* $f(z)=\mathcal{A}[\sigma : z]_\alpha^\beta$. *Then any infinite sequence of quotients* $P_{i,j}(z)$ *with distinct pairs of suffixes* i, j *taken from the set* $P_{2i,2j+1}(z)$, $P_{2i+1,2j}(z)$ $(i, j=0, 1, \ldots)$ *converges uniformly to* $f(z)$ *over any closed interval contained in* $(-\gamma, \gamma)$.

Theorem 6 may also be used to derive a convergence result similar to that of Theorem 7; but this result is a special case of a far more general theorem due to the author ([11]), and we therefore omit the details.

We conclude with three remarks. i) We have assumed that the function $\sigma(\varrho)$ occurring in expressions such as $\mathcal{ST}[\sigma : \lambda]_\alpha^\beta$, $\mathcal{A}[\sigma : z]_\alpha^\beta$ is given. If one of the limits α, β is infinite, it may occur that there are many functions $\sigma(\varrho)$ for which $\{t_\nu\}\equiv$ $\equiv\mathfrak{MS}\{\sigma\in\mathfrak{D}_\alpha^\beta\}$. In these circumstances we have correspondingly many functions $f(z)$ generating the same Padé table; in Theorems 5 and 6, $f(z)$ may be taken to be any one of these functions. When α and β are finite, this possibility does not arise, and $f(z)$ in Theorem 7 is a unique function. In the remaining theorems, $\sigma(\varrho)$ occurs explicitly. ii) If the function $\sigma(\varrho)$ described at the commencement of this note is merely a simple step function with a finite number of salti, $\mathcal{A}\{\mathcal{F}(\lambda)\}$ and $\mathcal{C}\{\mathcal{F}(\lambda)\}$ (if it exists) terminate; the Padé table generated by $\sum t_\nu z^\nu$ contains an infinite block of identical quotients. The results of Theorems 1 and 2 then hold for the convergents that are defined, and those of Theorems 3—6 hold for those

quotients not belonging to the infinite block. iii) We might have presented the theory in terms of the functions $S'(\lambda)=\int_{\alpha}^{\beta} d\sigma(\varrho)/(\lambda-\varrho)$ and $f'(z)=\int_{\alpha}^{\beta} d\sigma(\varrho)/(1-z\varrho)$. But these functions are simply related to those considered $(S'(\lambda)=-S(-\lambda), f'(z)=f(-z))$ and, for convenience, we chose to deal with the functions $S(\lambda)$ and $f(z)$.

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References

[1] O. PERRON, *Die Lehre von den Kettenbrüchen,* **2** Teubner (Stuttgart, 1957).
[2] H. S. WALL, *Analytic theory of continued fractions,* Van Nostrand (Princeton, 1948).
[3] R. NEVANLINNA, Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjes'sche Momentenproblem, *Ann. Acad. Sci. Fenn.,* Ser. A, **18** (1922).
[4] H. HAMBURGER, Über eine Erweiterung des Stieltjes'schen Momentenproblems, *Math. Ann.,* **81** (1920), pp. 235—319; **82** (1921), pp. 120—164, 168—187.
[5] T. J. STIELTJES, Recherches sur les fractions continues, *Ann. Fac. Sci. Toulouse,* **8** (1894), pp. 1—122; **9** (1895), pp. 1—47.
[6] H. PADÉ, Sur la représentation approchée d'une fonction par des fractions rationnelles, *Ann. Sci. Ecole Norm. Sup.,* **9** (1892) (Supplément), pp. 1—93.
[7] H. S. WALL, On the Padé approximants associated with a positive definite power series, *Trans. Amer. Math. Soc.,* **33** (1931), pp. 511—532.
[8] E. B. VAN VLECK, On an extension of the 1894 memoir of Stieltjes, *Trans. Amer. Math. Soc.,* **4** (1903), pp. 297—332.
[9] H. S. WALL, On the Padé approximants associated with the continued fraction and series of Stieltjes, *Trans. Amer. Math. Soc.,* **31** (1929), pp. 91—116.
[10] P. WYNN, Upon the Padé table derived from a Stieltjes series, *SIAM Jour. Numer. Anal.,* **5** (1968), pp. 805—834.
[11] P. WYNN, Upon a convergence result in the theory of the Padé table, *Trans. Amer. Math. Soc.,* **165** (1972), pp. 239—249.

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MODULES ARITHMÉTIQUES

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Introduction

Soit A un anneau commutatif et unitaire. Il est bien connu que le treillis $\mathscr{L}(A)$ des idéaux de A est un treillis modulaire. L. FUCHS a appelé anneaux arithmétiques les anneaux A pour lesquels $\mathscr{L}(A)$ est un treillis distributif. Ces anneaux ont été étudiés par de nombreux auteurs, notamment par I. S. COHEN, K. ASANO, C. U. JENSEN, R. B. WARFIELD, J.-P. LAFON. Les anneaux arithmétiques constituent des généralisations des anneaux de Prüfer et des anneaux *ZPI* généraux [6]: dire qu'un anneau intègre (resp. noethérien) A est un anneau arithmétique revient à dire que A est un anneau de Prüfer (resp. *ZPI* général).

Dans ce travail on se propose d'étudier des modules ayant le treillis distributif des sous-modules, modules appelés par nous modules arithmétiques.

A désignera toujours un anneau commutatif et unitaire et $\operatorname{Max}(A)$ l'ensemble des idéaux maximaux de A. Tous les modules sont supposés unitaires. Si M est un A-module, on désigne par $\mathscr{L}_A(M)$ le treillis des sous-A-modules de M. La terminologie et les notations non explicités sont celles de [8].

1. Quelques caractérisations et propriétés des modules arithmétiques

DÉFINITION 1.1. *Un A-module M est appelé module arithmétique si $\mathscr{L}_A(M)$ est un treillis distributif, c'est-á-dire si M satisfait à l'une des conditions équivalentes suivantes:*

(i) $P\cap(Q+R) = (P\cap Q)+(P\cap R)$ *quels que soient* $P, Q, R\in\mathscr{L}_A(M)$;

(ii) $P+(Q\cap R) = (P+Q)\cap(P+R)$ *quels que soient* $P, Q, R\in\mathscr{L}_A(M)$.

DÉFINITION 1.2. *Un A-module M est appelé module de Bézout si tout sous-module de type fini de M est monogène.*

PROPOSITION 1.3. *Soient A un anneau local et M un A-module. Les assertions suivantes sont équivalentes:*

(1) *M est un A-module de Bézout.*

(2) *Tout sous-module de M engendré par deux éléments est monogène.*

(3) *$\mathscr{L}_A(M)$ est un ensemble totalement ordonné par inclusion.*

(4) *L'ensemble des sous-modules monogènes de M est totalement ordonné par inclusion.*

(5) *M est un A-module arithmétique.*

PREUVE. Les implications (1)$\Leftrightarrow$(2), (3)$\Rightarrow$(4), (4)$\Rightarrow$(2) et (3)$\Rightarrow$(5) sont évidentes.

Prouvons que (4)$\Rightarrow$(3): soient $P, Q\in\mathscr{L}_A(M)$ et supposons que $P\nsubseteq Q$. Si $x\in P\setminus Q$, on a $Ay\subseteq Ax$ pour tout $y\in Q$, donc $Q\subseteq P$.

Pour l'implication (2)⇒(4), il suffit de prouver que si $x, y \in M$, alors $Ax+Ay$ est engendré par x ou par y. Mais $Ax+Ay=Az$, donc il existe des éléments $a, b, c, d \in A$ tels que

$$x = az, \quad y = bz, \quad z = cx+dy.$$

Par conséquent $(1-ca-db)z=0$. Si $z=0$, alors $Ax=Ay=0$; si $z\neq 0$, alors

$$1-ca-db \in \mathfrak{m},$$

où $\mathfrak{m}$ est l'idéal maximal de A. Il s'ensuit que $a \notin \mathfrak{m}$ ou $b \notin \mathfrak{m}$, c'est-á-dire que $Az=Ax$ ou $Az=Ay$.

Enfin il reste à prouver l'implication (5)⇒(4). Soient $x, y \in M$. Comme

$$Ax = Ax \cap (Ay + A(x-y)),$$

il en résulte que

$$Ax = (Ax \cap Ay) + (Ax \cap A(x-y)),$$

donc $x=u+a(x-y)$, où $u \in Ax \cap Ay$ et $a \in A$, d'où il vient

$$x(1-a) = u-ay.$$

Si $a \in \mathfrak{m}$, alors $x \in Ay$, donc $Ax \subseteqq Ay$; si $a \notin \mathfrak{m}$, alors $y \in Ax$, donc $Ay \subseteqq Ax$.

LEMME 1.4. *Soient S une partie multiplicative de A et M un A-module. Si M est un A-module arithmétique, alors $S^{-1}M$ est un $S^{-1}A$-module arithmétique.*

PREUVE. Soient $\varphi : M \to S^{-1}M$ le morphisme canonique et $P', Q', R' \in \mathscr{L}_{S^{-1}A}(S^{-1}M)$. Si $P=\varphi^{-1}(P')$, $Q=\varphi^{-1}(Q')$ et $R=\varphi^{-1}(R')$, alors $P, Q, R \in \mathscr{L}_A(M)$, d'où l'égalité:

$$P \cap (Q+R) = (P \cap Q) + (P \cap R).$$

Il en résulte que

$$S^{-1}P \cap (S^{-1}Q + S^{-1}R) = (S^{-1}P \cap S^{-1}Q) + (S^{-1}P \cap S^{-1}R).$$

Comme $S^{-1}P=P'$, $S^{-1}Q=Q'$ et $S^{-1}R=R'$, on obtient l'égalité:

$$P' \cap (Q'+R') = (P' \cap Q') + (P' \cap R'),$$

c'est-á-dire $S^{-1}M$ est un $S^{-1}A$-module arithmétique.

LEMME 1.5 ([8], chap. 2). *Soient M un A-module et $N \in \mathscr{L}_A(M)$. Alors $N=M$ si et seulement si $N_{\mathfrak{m}}=M_{\mathfrak{m}}$ pour tout $\mathfrak{m} \in \operatorname{Max}(A)$.*

THÉORÈME 1.6. *Soit M un A-module. Alors M est un A-module arithmétique si et seulement si, pour tout $\mathfrak{m} \in \operatorname{Max}(A)$, $M_{\mathfrak{m}}$ est un $A_{\mathfrak{m}}$-module arithmétique.*

PREUVE. Si M est un A-module arithmétique, alors, d'après 1.4, $M_{\mathfrak{m}}$ est un $A_{\mathfrak{m}}$-module arithmétique pour tout $\mathfrak{m} \in \operatorname{Max}(A)$.

Réciproquement, soient $P, Q, R \in \mathscr{L}_A(M)$. On a l'inclusion évidente:

$$(P \cap Q) + (P \cap R) \subseteqq P \cap (Q+R).$$

Mais, pour tout $\mathfrak{m} \in \operatorname{Max}(A)$, on a

$$(P_{\mathfrak{m}} \cap Q_{\mathfrak{m}}) + (P_{\mathfrak{m}} \cap R_{\mathfrak{m}}) = P_{\mathfrak{m}} \cap (Q_{\mathfrak{m}} + R_{\mathfrak{m}}),$$

donc, en vertu de 1.5,

$$(P \cap Q) + (P \cap R) = P \cap (Q + R).$$

LEMME 1.7. *Soient A un anneau local et M, M' deux A-modules arithmétiques non nuls. Alors $M \oplus M'$ n'est pas un A-module arithmétique.*

PREUVE. Supposons que $M \oplus M'$ est un A-module arithmétique. Alors, en vertu de 1.3, $\mathscr{L}_A(M \oplus M')$ serait un ensemble totalement ordonné. Soient $x \in M$, $x \neq 0$ et $x' \in M'$, $x' \neq 0$, donc

$$A(x, 0) \subseteq A(0, x') \quad \text{ou} \quad A(0, x') \subseteq A(x, 0).$$

Il en résulte que $x = 0$ ou $x' = 0$, d'où la contradiction.

PROPOSITION 1.8. *Soit $(M_i)_{i \in I}$ une famille de A-modules arithmétiques. Les assertions suivantes sont équivalentes:*

(1) $\bigoplus_{i \in I} M_i$ *est un A-module arithmétique.*

(2) $\operatorname{Supp}(M_i) \cap \operatorname{Supp}(M_j) = \varnothing$ *pour tout $i \neq j$ de I.*

PREUVE. (1)⇒(2): supposons qu'il existe $i \neq j$ de I tels que $\operatorname{Supp}(M_i) \cap \cap \operatorname{Supp}(M_j) \neq \varnothing$. Soit $\mathfrak{p} \in \operatorname{Supp}(M_i) \cap \operatorname{Supp}(M_j)$. Mais

$$\left(\bigoplus_{i \in I} M_i\right)_{\mathfrak{p}} \simeq \bigoplus_{i \in I} (M_i)_{\mathfrak{p}}.$$

Comme $(M_i)_{\mathfrak{p}} \neq 0$ et $(M_j)_{\mathfrak{p}} \neq 0$, on obtient, d'après 1.7, une contradiction.

(2)⇒(1) résulte aussitôt de 1.6.

THÉORÈME 1.9. *Les assertions suivantes sont équivalentes pour un A-module M:*

(1) *M est un A-module arithmétique.*

(2) *Pour tout $P, Q \in \mathscr{L}_A(M)$ avec $P \subseteq Q$ et Q de type fini, il existe un idéal $\mathfrak{a}$ de A tel que*

$$P = \mathfrak{a}Q.$$

(3) $(P+Q):R = (P:R) + (Q:R)$ *pour tout $P, Q, R \in \mathscr{L}_A(M)$ avec R de type fini.*

(4) $P:(Q \cap R) = (P:Q) + (P:R)$ *pour tout $P, Q, R \in \mathscr{L}_A(M)$ avec Q et R de type fini.*

Pour la preuve de ce théorème on utilise le résultat suivant bien connu: Si S est une partie multiplicative de A, M est un A-module et $P, Q \in \mathscr{L}_A(M)$ avec Q du type fini, alors

$$S^{-1}(P:Q) = S^{-1}P : S^{-1}Q.$$

Puis on procède comme dans [11], où ce théorème est établi dans le cas particulier $M = A$.

COROLLAIRE 1.10. *Tout A-module de Bézout est un A-module arithmétique.*

PREUVE. Soient $P, Q \in \mathscr{L}_A(M)$ avec $P \subseteq Q$ et Q du type fini. Alors $Q = Ax$, où $x \in Q$. Mais $P = (P:Ax)Ax$, donc, en vertu de 1.9, M est un A-module arithmétique.

THÉORÈME 1.11. *Soit M un A-module arithmétique tel que* $\mathrm{Supp}(M)\cap\mathrm{Max}(A)$, *soit un ensemble fini (en particulier cette condition est remplie si A est un anneau semi-local). Alors M est un A-module de Bézout.*

Nous omettons la preuve, elle étant similaire à celle du théorème 5 de [11], énoncé pour le cas $M=A$.

PROPOSITION 1.12. *Soient M un A-module arithmétique et N un sous-module $\mathfrak{p}$-primaire de M, c'est-á-dire* $\mathrm{Ass}_A(M/N)=\{\mathfrak{p}\}$ *(voir* [8], *chap.* 4*). Alors N est un sous-module irréductible de M.*

PREUVE. Soient $P, Q\in\mathscr{L}_A(M)$ tels que $N=P\cap Q$ et $\varphi:M\to M_\mathfrak{p}$ le morphisme canonique; donc $N_\mathfrak{p}=P_\mathfrak{p}\cap Q_\mathfrak{p}$, d'où selon 1.4 et 1.3, $N_\mathfrak{p}=P_\mathfrak{p}$ ou $N_\mathfrak{p}=Q_\mathfrak{p}$. Supposons que $N_\mathfrak{p}=P_\mathfrak{p}$. Comme N est $\mathfrak{p}$-primaire, il en résulte que $\varphi^{-1}(N_\mathfrak{p})=N$ [13]. Mais $P\subseteqq\varphi^{-1}(P_\mathfrak{p})=\varphi^{-1}(N_\mathfrak{p})=N$, donc $P\subseteqq N$, d'où l'on obtient $N=P$, c'est-à-dire que N est un sous-module irréductible de M.

LEMME 1.13. *Soient $\varphi:A\to B$ un épimorphisme plat d'anneaux, M un A-module et $f:M\to M\otimes_A B$ le morphisme canonique: $f(x)=x\otimes 1$ pour tout $x\in M$. Alors, pour tout $P'\in\mathscr{L}_B(M\otimes_A B)$, on a*

$$P'=f\big(f^{-1}(P')\big)B.$$

PREUVE. Le morphisme φ met en évidence deux foncteurs:

$$\mathrm{Mod}(A)\underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}}\mathrm{Mod}(B),$$

où $\varphi^*(M)=M\otimes_A B$ et $\varphi_*(N)=N_{[\varphi]}$.

Comme $B\otimes_A B\cong B$, on en déduit:

$$\varphi^*(M)/P'\simeq\big(\varphi^*(M)/P'\big)\otimes_B B\simeq\big(\varphi^*(M)/P'\big)\otimes_B(B\otimes_A B)\simeq\big(\varphi^*(M)/P'\big)\otimes_A B.$$

Soit $P=f^{-1}\big(\varphi_*(P')\big)\in\mathscr{L}_A(M)$. En tensoriant par B le monomorphisme canonique

$$M/P\hookrightarrow\varphi_*\big(\varphi^*(M)/P'\big),$$

on obtient un monomorphisme

$$(M/P)\otimes_A B\hookrightarrow\varphi^*(M)/P'$$

donc un monomorphisme

$$\varphi^*(M)/f(P)B\hookrightarrow\varphi^*(M)/P'.$$

Comme c'est évidemment une surjection, on en tire que $P'=f(P)B$.

THÉORÈME 1.14. *Soit $\varphi:A\to B$ un épimorphisme plat d'anneaux. Si M est un A-module arithmétique, alors $M\otimes_A B$ est un B-module arithmétique.*

PREUVE. Il suffit de montrer que $(M\otimes_A B)_\mathfrak{q}$ est un $B_\mathfrak{q}$-module arithmétique pour tout $\mathfrak{q}\in\mathrm{Max}(B)$. Soient $\mathfrak{q}\in\mathrm{Max}(B)$ et $\mathfrak{p}=\varphi^{-1}(\mathfrak{q})$. Comme $\varphi(A\setminus\mathfrak{p})\subseteqq B\setminus\mathfrak{q}$, il résulte que $B_\mathfrak{q}$ est un anneau de fractions de l'anneau $B_\mathfrak{p}=B\otimes_A A_\mathfrak{p}$. Il suffit donc, en vertu de 1.4, de prouver que $(M\otimes_A B)\otimes_B B_\mathfrak{p}\cong M_\mathfrak{p}\otimes_{A_\mathfrak{p}} B_\mathfrak{p}$ est un $B_\mathfrak{p}$-module arithmétique pour tout $\mathfrak{p}\in\mathrm{Spec}(A)$. Si $\mathfrak{p}\in\mathrm{Spec}(A)$, alors $\varphi_\mathfrak{p}:A_\mathfrak{p}\to B\otimes_A A_\mathfrak{p}=B_\mathfrak{p}$

est un épimorphisme plat d'anneaux. $M_\mathfrak{p}$ étant un $A_\mathfrak{p}$-module arithmétique, $\mathscr{L}_{A_\mathfrak{p}}(M_\mathfrak{p})$ est un ensemble totalement ordonné par inclusion. Il en résulte, en vertu de 1.13, que $\mathscr{L}_{B_\mathfrak{p}}(M_\mathfrak{p}\otimes_{A_\mathfrak{p}} B_\mathfrak{p})$ est un ensemble totalement ordonné, c'est-á-dire que $M_\mathfrak{p}\otimes_{A_\mathfrak{p}} B_\mathfrak{p}$ est un $B_\mathfrak{p}$-module arithmétique.

COROLLAIRE 1.15. *Si $\varphi: A\to B$ est un épimorphisme plat d'anneaux et si A est un anneau arithmétique, B l'est aussi.*

2. Produit tensoriel des modules arithmétiques

LEMME 2.1. *Soient I un ensemble ordonné filtrant croissant et $(M_\alpha, \varphi_{\beta\alpha})_{\alpha\in I}$ un système inductif de A-modules arithmétiques. Alors $\varinjlim_{\alpha\in I} M_\alpha$ est un A-module arithmétique.*

PREUVE. On peut supposer que A est un anneau local. Pour tout $\alpha\in I$, soit $\varphi_\alpha: M\to\varinjlim_{\alpha\in I} M_\alpha$ le morphisme canonique. Si $x, y\in\varinjlim_{\alpha\in I} M_\alpha$, il existe $\beta\in I$ et $x_\beta, y_\beta\in M_\beta$ tel que $x=\varphi_\beta(x_\beta)$ et $y=\varphi_\beta(y_\beta)$. Comme M_β est un A-module arithmétique, alors $Ax_\beta\subseteq Ay_\beta$ ou $Ay_\beta\subseteq Ax_\beta$, donc $\varphi_\beta(Ax_\beta)\subseteq\varphi_\beta(Ay_\beta)$ ou $\varphi_\beta(Ay_\beta)\subseteq\varphi_\beta(Ax_\beta)$, c'est-à-dire $Ax\subseteq Ay$ ou $Ay\subseteq Ax$. On en condu, en vertu de 1.3 et 1.6, que $\varinjlim_{\alpha\in I} M_\alpha$ est un A-module arithmétique.

REMARQUE. Une limite inductive quelconque de A-modules arithmétiques n'est pas nécessairement un A-module arithmétique, comme cela résulte de 1.7.

PROPOSITION 2.2. *Soient M et N deux A-modules arithmétiques. Alors $M\otimes_A N$ est un A-module arithmétique.*

PREUVE. Comme $(M\otimes_A N)_\mathfrak{m}\cong M_\mathfrak{m}\otimes_{A_\mathfrak{m}} N_\mathfrak{m}$ pour tout $\mathfrak{m}\in\mathrm{Max}(A)$, on peut supposer que A est un anneau local.

Soit N' un sous-module de type fini de N. Alors, en vertu de 1.3, N' est un A-module monogène. Si $N'=Ax$, la suite exacte canonique

$$A\to Ax\to 0$$

permet d'écrire la suite exacte

$$M\cong M\otimes_A A\to M\otimes_A Ax\to 0,$$

donc $M\otimes_A N'$ est un A-module arithmétique.

Comme $N=\varinjlim_{\alpha\in I} N_\alpha$, où I est un ensemble ordonné filtrant croissant et N_α sont des A-modules de type fini pour tout $\alpha\in I$, il en résulte que

$$M\otimes_A N\cong\varinjlim_{\alpha\in I}(M\otimes_A N_\alpha),$$

donc, en vertu de 2.1, $M\otimes_A N$ est un A-module arithmétique.

COROLLAIRE 2.3. *Soient A un anneau arithmétique et $\mathscr{A}(A)$ la classe des classes d'équivalence de A-modules arithmétiques pour la relation d'isomorphie. Si $[M]$ désigne l'image dans $\mathscr{A}(A)$ d'un A-module arithmétique M, alors $\mathscr{A}(A)$, muni de la "loi de composition":*

$$[M]+[N] = [M\otimes_A N],$$

est un "monoïde commutatif", admettant $[A]$ pour "élément neutre". Le "groupe" des éléments inversibles de $\mathscr{A}(A)$ est formé par les classes des A-modules projectifs de rang 1.

PREUVE. La première partie de la proposition résulte aussitôt de 2.2.

Soit maintenant $[M]\in\mathscr{A}(A)$ un élément inversible; il existe alors un A-module arithmétique N tel que $M\otimes_A N\cong A$. Pour le A-module M il existe une suite exacte:

$$A^{(J)} \to A^{(I)} \to M \to 0$$

d'où l'où déduit la suite exacte:

$$N^{(J)} \to N^{(I)} \to M\otimes_A N \to 0.$$

Comme $M\otimes_A N\cong A$, il existe un A-module N' tel que $N^{(I)}\cong A\oplus N'$, donc

$$A^{(I)} \cong (M\otimes_A N)^{(I)} \simeq M\otimes_A (N^{(I)}) \simeq M\oplus(M\otimes_A N')$$

et, par suite, M est un A-module projectif. On obtient de la même manière que N est un A-module projectif.

Si $\mathfrak{m}\in\mathrm{Max}\,(A)$, alors $M_{\mathfrak{m}}\otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}}\cong A_{\mathfrak{m}}$; comme $M_{\mathfrak{m}}$ et $N_{\mathfrak{m}}$ sont $A_{\mathfrak{m}}$-modules projectifs, ils sont, d'après un résultat classique de KAPLANSKY, $A_{\mathfrak{m}}$-modules libres. On a donc $\mathrm{rg}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})=\mathrm{rg}_{A_{\mathfrak{m}}}(N_{\mathfrak{m}})=1$, c'est-á-dire que $\mathrm{rg}_A(M)=\mathrm{rg}_A(N)=1$.

3. Modules de torsion arithmétiques

Rappelons brièvement quelques définitions et notations. Soient M un A-module et S un A-module simple. On dit que M est un A-module de torsion (resp. un A-module de torsion S-primaire) si tout A-module quotient non nul de M contient un sous-module simple (resp. un sous-module simple isomorphe à S) [9]. Si M est un A-module de torsion, on définit la série de LOEWY de M:

$$0 = L_0(M) \subset L_1(M) \subset \ldots \subset L_\alpha(M) \subset \ldots \subset L_{\lambda(M)}(M) = M$$

de la manière suivante: $L_0(M)=0$, $L_{\alpha+1}(M)$ est défini tel que $L_{\alpha+1}(M)/L_\alpha(M)$ soit le socle $S_A(M/L_\alpha(M))$ de $M/L_\alpha(M)$ et si α est un ordinal limite, alors $L_\alpha(M)=\bigcup_{\beta<\alpha} L_\beta(M)$. $\lambda(M)$ est le plus petit ordinal λ tel que $L_\lambda(M)=L_{\lambda+1}(M)$ et est appelé la longueur de LOEWY de M. On a nécessairement $M=L_{\lambda(M)}(M)$.

Si M est un A-module quelconque, on désigne par $E_A(M)$ son enveloppe injective et par $S_A(M)$ son socle.

LEMME 3.1. *Soient I un ensemble et $(S_i)_{i\in I}$ une famille de A-modules simples. Alors $\bigoplus_{i\in I} S_i$ est un A-module arithmétique si et seulement si pour tout $i\neq j$ de I, $S_i\ncong S_j$.*

PREUVE. On applique 1.8.

THÉORÈME 3.2. *Soient S un A-module simple et M un A-module de torsion S-primaire. Les assertions suivantes sont équivalentes:*

(1) *M est un A-module arithmétique.*

(2) *$L_{\alpha+1}(M)/L_\alpha(M)$ est un A-module simple pour tout $\alpha<\lambda(M)$.*

(3) *$\mathscr{L}_A(M)$ est un ensemble totalement ordonné par inclusion.*

(4) *$\mathscr{L}_A(M)$ est un ensemble bien ordonné par inclusion.*

(5) *$L_\alpha(M)$ où $0\leqq\alpha\leqq\lambda(M)$ sont les seuls sous-modules de M.*

PREUVE. On peut munir M d'une structure canonique de $A_{\mathfrak{m}}$-module, où $\mathfrak{m}=\mathrm{Ann}_A(S)$ [9]. Soit M^* le $A_{\mathfrak{m}}$-module ainsi obtenu. En outre, $\mathscr{L}_A(M)=\mathscr{L}_{A_{\mathfrak{m}}}(M^*)$, $\lambda(M)=\lambda(M^*)$ et $L_\alpha(M)=L_\alpha(M^*)$ pour tout $0\leqq\alpha\leqq\lambda(M)$. On peut donc supposer que A est un anneau local.

Les implications (1)$\Leftrightarrow$(3) résultent de 1.3, l'implication (1)$\Rightarrow$(2) découle de 3.1 et des implications (5)$\Rightarrow$(4)$\Rightarrow$(3) sont immédiates. Il reste à prouver l'implication (2)$\Rightarrow$(5). Soit P un sous-module de M tel que $P\neq M$ et $P\neq 0$. Il existe donc $\gamma\leqq\lambda(M)$ tel que $L_\gamma(M)\nsubseteqq P$. Soit $\alpha=\min\{\gamma|\gamma\leqq\lambda(M), L_\gamma(M)\nsubseteqq P\}$. Alors $0<\alpha\leqq$ $\leqq\lambda(M)$ et α n'est pas un ordinal limite, donc $\alpha=\beta+1$ et $L_\beta(M)\subseteqq P$. Si $L_\beta(M)\neq P$, alors $P/L_\beta(M)\neq 0$, donc

$$0 \neq S_A\big(P/L_\beta(M)\big) \subseteqq S_A\big(M/L_\beta(M)\big) = L_\alpha(M)/L_\beta(M).$$

Il en résulte que $S_A\big(P/L_\beta(M)\big)=L_\alpha(M)/L_\beta(M)$, car $L_\alpha(M)/L_\beta(M)$ est un A-module simple, donc $L_\alpha(M)/L_\beta(M)\subseteqq P/L_\beta(M)$, c'est-à-dire que $L_\alpha(M)\subseteqq P$, d'où la contradiction. C'est que $P=L_\beta(M)$.

LEMME 3.3. *Soit A un anneau parfait* [7] *local arithmétique. Alors A est un anneau uniséríel.*

PREUVE. Soit $\mathfrak{m}$ l'idéal maximal de A. Si $\mathfrak{m}\neq\mathfrak{m}^2$, soit $x\in\mathfrak{m}\setminus\mathfrak{m}^2$. Comme A est un anneau arithmétique, $\mathfrak{m}/\mathfrak{m}^2$ est un A-module simple, donc $\mathfrak{m}=Ax$. A étant un anneau parfait, A est un anneau semi-artinien [12], donc $\mathfrak{m}$ est le seul idéal premier de A. Par conséquent, le nilradical de A est $\mathfrak{m}$, c'est-à-dire x est nilpotent. Il résulte que A est un anneau primaire, donc artinien. En conclusion A est un anneau uniséríel.

Si $\mathfrak{m}=\mathfrak{m}^2$, alors tout A-module de torsion est semi-simple. A étant un anneau parfait, A est un anneau semiartinien, [12] donc A est un A-module de torsion, donc A est un corps, c'est-à-dire un anneau uniséríel.

PROPOSITION 3.4. *Soient S un A-module simple et M un A-module de torsion S-primaire arithmétique. On a alors:*

(1) *$\lambda(M)\leqq\omega$, où ω est le premier ordinal transfini.*

(2) *Les seuls sous-modules propres de M sont les $L_n(M)$ où $0\leqq n<\omega$.*

PREUVE. On peut supposer l'anneau A local. Si $\lambda(M)>\omega$, alors $L_\omega(M)\neq M$, donc il existe $x\in M$ tel que $x\notin L_\omega(M)$ et, par suite, $L_\omega(M)\subseteqq Ax$. Mais $A/\mathrm{Ann}_A(x)$ est un anneau parfait [4], local et arithmétique, donc, d'aprés 3.3, $A/\mathrm{Ann}_A(x)$ est un anneau uniséríel. En conclusion Ax est un A-module de longueur finie, donc $\lambda(Ax)=n_0<\omega$. Comme $Ax=L_{n_0}(Ax)=Ax\cap L_{n_0}(M)$, il résulte que $x\in L_{n_0}(M)\subseteqq$ $\subseteqq L_\omega(M)$, d'où la contradiction, donc $\lambda(M)\leqq\omega$. Pour prouver (2) il suffit d'appliquer 3.2.

COROLLAIRE 3.5. *Soit M un A-module de torsion arithmétique non nul tel que* $\mathrm{Ass}(M)$ *soit un ensemble fini* $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_k\}$. *Si* $(L_\alpha(M, S_i))_\alpha$ *désigne la* S_i*-série de* LOEWY *de* M *pour tout* $1 \leqq i \leqq k$, *où* $S_i = A/\mathfrak{m}_i$ $(\mathrm{Ass}(M) \subseteqq \mathrm{Max}(A)$, *d'après* [3]), *alors on a:*

(1) $\lambda(M) \leqq \omega$.

(2) *Pour tout sous-module P de M il existe la famille d'ordinaux* $(\alpha_i)_{1 \leqq i \leqq k}$, *où* $0 \leqq \alpha_i \leqq \omega$ *pour tout* $1 \leqq i \leqq k$, *tel que:*

$$P = \bigoplus_{i=1}^{k} L_{\alpha_i}(M, S_i).$$

PREUVE. (1) Comme $\mathrm{Ass}(M)$ est un ensemble fini, M est la somme directe de ses composantes S-primaires $\mathscr{T}_S(M)$ ([2] et [5]):

$$M = \bigoplus_{i=1}^{k} \mathscr{T}_{S_i}(M).$$

Mais $\lambda(M) = \sup_{1 \leqq i \leqq k} (\lambda(\mathscr{T}_{S_i}(M)))$, donc $\lambda(M) \leqq \omega$.

(2) On a

$$P = \bigoplus_{i=1}^{k} \mathscr{T}_{S_i}(P).$$

En outre

$$\mathscr{T}_{S_i}(P) = \mathscr{T}_{S_i}(M) \cap P \quad \text{et} \quad L_\alpha(M, S_i) = L_\alpha(\mathscr{T}_{S_i}(M))$$

pour tout $0 \leqq \alpha \leqq \omega$ et $1 \leqq i \leqq k$. Puis, on applique 3.4.

COROLLAIRE 3.6. *Soit M un A-module de torsion arithmétique. Les assertions suivantes sont équivalentes:*

(1) $\mathrm{Ass}(M)$ *est un ensemble fini.*

(2) M *est un A-module artinien.*

PREUVE. Le corollaire résulte aussitôt de 3.5.

PROPOSITION 3.7. *Soient A un anneau commutatif et* $\mathfrak{m} \in \mathrm{Max}(A)$. *Les assertions suivantes sont équivalentes:*

(1) $E_A(A/\mathfrak{m})$ *est un A-module de torsion arithmétique.*

(2) $A_\mathfrak{m}$ *est soit un anneau unisériel soit un anneau de valuation discrète.*

PREUVE. (1)$\Rightarrow$(2): comme $E_A(A/\mathfrak{m})$ est un A-module de torsion, il en résulte que $E_A(A/\mathfrak{m})$ est un A-module de torsion S-primaire, où $S = A/\mathfrak{m}$ d'après [2]. On peut donc munir $E_A(A/\mathfrak{m})$ d'une structure canonique de $A_\mathfrak{m}$-module, et $\mathscr{L}_A(E_A(A/\mathfrak{m})) = \mathscr{L}_{A_\mathfrak{m}}(E_A(A/\mathfrak{m}))$ [9]. Par ailleurs $E_A(A/\mathfrak{m}) \cong E_{A_\mathfrak{m}}(A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m})$, on peut donc supposer que A est un anneau local d'idéal maximal $\mathfrak{m}$. Soient $E = E_A(A/\mathfrak{m})$, $\mathscr{I}(A)$ le treillis des idéaux de A et

$$\varphi : \mathscr{I}(A) \to \mathscr{L}_A(E)$$

l'application définie par

$$\varphi(\mathfrak{a}) = \mathrm{Ann}_E(\mathfrak{a}) = \{x \in E \mid \mathfrak{a} \subseteqq \mathrm{Ann}_A(x)\}$$

pour tout $\mathfrak{a} \in \mathscr{I}(A)$.

A étant un anneau local, φ est une application injective. Comme, d'après 3.6, E est un A-module artinien, il vient aussitôt que A est un anneau noethérien. Il en résulte en outre que A est un anneau arithmétique, car E est un A-module arithmétique et φ est injective. En conclusion, A est un anneau arithmétique noethérien, donc un anneau ZPI [6], c'est-á-dire un anneau uniséríel ou un anneau de valuation discrète.

(2)⇒(1): $A_{\mathfrak{m}}$ étant un anneau noethérien, $E_A(A/\mathfrak{m})$ est un A-module de torsion, donc $\mathscr{L}_A(E_A(A/\mathfrak{m}))=\mathscr{L}_{A_{\mathfrak{m}}}(E_A(A/\mathfrak{m}))$ [9]. On peut donc supposer que A est un anneau local. Puis on applique [1] et 3.2.

COROLLAIRE 3.8. *Soient A un anneau, $\mathfrak{m}\in\mathrm{Max}\,(A)$, $S=A/\mathfrak{m}$ et M un A-module de torsion S-primaire arithmétique tel que $\lambda(M)=\omega$. Les assertions suivantes sont équivalentes:*

(1) *M est un A-module injectif.*

(2) *$M=E_A(S)$.*

(3) *$A_{\mathfrak{m}}$ est un anneau de valuation discrète.*

PREUVE. D'après 3.2, $S_A(M)=S$; comme M est un A-module de torsion, $S_A(M)\subseteqq M$ est une extension essentielle, donc

$$S\subseteqq M\subseteqq E_A(S).$$

Par conséquent (1)⇔(2).

La preuve de 3.7 montre qu'on peut supposer l'anneau A local. Prouvons l'implication (2)⇒(3): On peut tirer de 3.7 que A est soit un anneau uniséríel, soit un anneau de valuation discrète. Si A était un anneau uniséríel, on aurait $E_A(S)\cong A$, donc $\lambda(M)<\omega$ et, par suite, A est un anneau de valuation discrète. Enfin, si A est un anneau de valuation discrète, alors $E_A(S)$ est un A-module arithmétique d'après [1]. Comme $S\subseteqq M\subseteqq E_A(S)$, il découle de 3.4 que $M=E_A(S)$ ou $M=L_n(E_A(S))$ pour un nombre naturel n. Mais $\lambda(M)=\omega$, donc $M=E_A(S)$.

REMARQUE. D. T. GILL a prouvé dans [10] que si A est un anneau local d'idéal maximal $\mathfrak{m}$, alors $E_A(A/\mathfrak{m})$ est un A-module arithmétique si et seulement si A est un anneau arithmétique presque maximal (ou anneau de valuation presque maximal). Cette condition est encore équivalente à la condition suivante: tout A-module de type fini est une somme directe de A-modules monogènes.

UN EXEMPLE. Soit A un anneau local non noethérien d'idéal maximal $\mathfrak{m}$ principal tel que pour tout nombre naturel n, $\mathfrak{m}^n/\mathfrak{m}^{n+1}\cong A/\mathfrak{m}$. On peut considérer, par exemple, un corps commutatif K, un nombre naturel $s\geqq 2$ et le groupe totalement ordonné par l'ordre lexicographique $\mathbf{Z}^s$. On le sait ([8], chap. 6) qu'on peut construire une valuation canonique sur le corps $K(X_1, X_2, \ldots, X_s)$ dont le groupe des ordres soit $\mathbf{Z}^s$ et dont le corps résiduel soit K. Si A est l'anneau de cette valuation, l'idéal maximal $\mathfrak{m}$ de A est principal, car le monoïde des éléments strictement positifs de $\mathbf{Z}^s$ possède un plus petit élément, à savoir $(0, 0, \ldots, 1)$. Donc $\dim_{A/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})=1$ pour tout nombre naturel n.

Soit $E=E_A(A/\mathfrak{m})$. On a $\mathscr{T}(E)=\bigcup_{n=0}^{\infty} E_n$, où $\mathscr{T}(E)$ est le sous-module de torsion de E (c'est-à-dire le plus grand sous-module de E qui est un module de torsion) et

$E_n=\mathrm{Ann}_E(\mathfrak{m}^n)$ pour tout nombre naturel n. En effet, il est clair que $\bigcup_{n=0}^{\infty} E_n \subseteqq \mathscr{T}(E)$, car $E_n=L_n(E)$ pour tout $0\leqq n<\omega$. Soit maintenant $x\in\mathscr{T}(E)$, $x\neq 0$; alors $B=A/\mathrm{Ann}_A(x)$ est un anneau parfait ([4] ou [5]), local, d'idéal maximal principal $\mathfrak{n}=By$. Mais $\mathfrak{n}$ est le nilradical de B, donc y est nilpotent, autrement dit, il existe un nombre naturel k tel que $\mathfrak{m}^k\subseteqq\mathrm{Ann}_A(x)$. Par conséquent $x\in E_k$.

$\mathscr{T}(E)=\bigcup_{n=0}^{\infty} E_n$ est un A-module de torsion arithmétique, car, d'après [14], $L_n(\mathscr{T}(E))=L_n(E)\cap\mathscr{T}(E)=E_n$ et $E_{n+1}/E_n\cong\mathrm{Hom}_{A/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1}, A/\mathfrak{m})\cong A/\mathfrak{m}$. E n'est pas donc un A-module de torsion, en vertu de 3.8, $\lambda(\mathscr{T}(E))=\omega$, et $\mathscr{T}(E)$ est un A-module de torsion S-primaire, arithmétique et non injectif. En outre, E est, d'après [10], un A-module arithmétique si et seulement si A est un anneau de valuation presque maximal.

4. L'anneau d'endomorphismes d'un module arithmétique

LEMME 4.1. *Soient M et N deux A-modules arithmétiques. Si M est de type fini, alors $\mathrm{Hom}_A(M, N)$ est un A-module arithmétique.*

PREUVE. Pour tout $\mathfrak{m}\in\mathrm{Max}\,(A)$, le morphisme canonique

$$(\mathrm{Hom}_A(M, N))_{\mathfrak{m}} \to \mathrm{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}})$$

est injectif car M est un A-module de type fini ([8], chap. 2). Il suffit donc de prouver que $\mathrm{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}})$ est un $A_{\mathfrak{m}}$-module arithmétique, ce qui revient à montrer que $\mathscr{L}_{A_{\mathfrak{m}}}(\mathrm{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}))$ est un ensemble totalement ordonné par inclusion. On peut donc supposer l'anneau A local et dans ce cas le A-module arithmétique de type fini M est monogène: $M=Ax$ où $x\in M$.

Soient $f, g\in\mathrm{Hom}_A(Ax, N)$. Alors $Af(x)\subseteqq Ag(x)$ ou $Ag(x)\subseteqq Af(x)$, car N est un A-module arithmétique. Supposons que $Af(x)\subseteqq Ag(x)$. Il existe alors $a\in A$ tel que $f(x)=ag(x)$, donc $f(bx)=ag(bx)$ pour tout $b\in A$, c'est-à-dire $f=ag$. Par conséquent $\mathrm{Hom}_A(M, N)$ est un A-module arithmétique.

THÉORÈME 4.2. *Si M est un A-module arithmétique de type fini, l'anneau $\mathrm{Hom}_A(M, M)$ est un anneau arithmétique commutatif isomorphe avec l'anneau $A/\mathrm{Ann}_A(M)$.*

PREUVE. Comme tout idéal de l'anneau $\mathrm{Hom}_A(M, M)$ est un sous-A-module de $\mathrm{Hom}_A(M, M)$, il résulte de 4.1 que $\mathrm{Hom}_A(M, M)$ est un anneau arithmetique.

Soit

$$\alpha: A \to \mathrm{Hom}_A(M, M)$$

le morphisme canonique

$$\alpha(a) = \varphi_a \quad (a\in A)$$

où $\varphi_a(x)=ax$ pour tout $x\in M$.

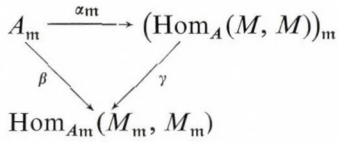
Nous allons montrer que α est surjectif, ce qui revient à prouver que

$$\alpha_{\mathfrak{m}}: A_{\mathfrak{m}} \to (\mathrm{Hom}_A(M, M))_{\mathfrak{m}}$$

est un épimorphisme de $A_{\mathfrak{m}}$-modules pour tout $\mathfrak{m}\in\mathrm{Max}\,(A)$. Alors il résultera que

$$A/\mathrm{Ann}_A(M) \simeq \mathrm{Hom}_A(M, M).$$

Considérons le diagramme commutatif de $A_{\mathfrak{m}}$-modules:



β et γ étant morphismes canoniques.

Pour prouver que $\alpha_{\mathfrak{m}}$ est surjectif il suffit de montrer que β est surjectif, car γ est injectif.

La question se ramène donc au cas suivant: A est un anneau local et M est un A-module arithmétique monogène. On va prouver que le morphisme canonique

$$\alpha: A \to \mathrm{Hom}_A(M, M)$$

est surjectif, où $M=Ax$. Soit $f\in\mathrm{Hom}_A(Ax, Ax)$; alors $f(x)=ax$ où $a\in A$, donc $f(bx)=$ $=a(bx)$ pour tout $b\in A$, c'est-à-dire $f=\alpha(a)$.

COROLLAIRE 4.3. *Si M est un A-module arithmétique de présentation finie, alors M est un facteur direct d'une somme directe de A-modules monogènes.*

PREUVE. On peut considérer M comme un module sur l'anneau arithmétique $A/\mathrm{Ann}_A(M)$. Le $A/\mathrm{Ann}_A(M)$-module M est de présentation finie. D'après un théorème de WARFIELD [15] on en déduit le résultat ci-après.

COROLLAIRE 4.4. *Si A est un anneau de Prüfer et I est un idéal non nul du type fini de A, alors $\mathrm{Hom}_A(I, I)$ est un anneau commutatif isomorphe avec l'anneau A.*

PROPOSITION 4.5. *Soient A un anneau local d'idéal maximal $\mathfrak{m}$ et M un A-module arithmétique non nul. Supposons que $\mathrm{Hom}_A(M, M)$ est un A-module arithmétique. On a alors:*

(1) *Tout $f\in\mathrm{Hom}_A(M, M)$ est soit une homothétie soit f est bijectif et f^{-1} est une homothétie.*

(2) *$\mathrm{Hom}_A(M, M)$ est un anneau commutatif.*

(3) *Si $\mathfrak{m}\in\mathrm{Ass}(M)$ (en particulier si M est un A-module de torsion S-primaire, où $S=A/\mathfrak{m}$) alors tout $f\in\mathrm{Hom}_A(M, M)$ est une homothétie, donc*

$$\mathrm{Hom}_A(M, M) \simeq A/\mathrm{Ann}_A(M).$$

PREUVE. (1) Soit $f\in\mathrm{Hom}_A(M, M)$. Comme $\mathrm{Hom}_A(M, M)$ est un A-module arithmétique, on a

$$Af \subseteqq A1_M \quad \text{ou} \quad A1_M \subseteqq Af.$$

Si $Af\subseteqq A1_M$, alors f est une homothétie. Si $A1_M\subseteqq Af$, il existe $a\in A$, $a\neq 0$ tel que $1_M=af$. Donc $1_M=\varphi_a\circ f=f\circ\varphi_a$, c'est-à-dire f est bijectif et f^{-1} est l'homothétie φ_a.

(2) Soient $f, g\in\mathrm{Hom}_A(M, M)$. Si f ou g est une homothétie alors $f\circ g=g\circ f$. Si f et g ne sont pas des homothéties, alors f et g sont bijectifs et en outre f^{-1} et g^{-1} sont des homothéties et, par suite, $f^{-1}\circ g^{-1}=g^{-1}\circ f^{-1}$, donc $f\circ g=g\circ f$.

(3) Soit $f \in \mathrm{Hom}_A(M, M)$. Si f n'est pas une homothétie, alors f est bijectif et f^{-1} est une homothétie φ_a où $a \in \mathfrak{m}$. Comme $\mathfrak{m} \in \mathrm{Ass}(M)$, il existe $x \in M$ tel que $\mathfrak{m} = \mathrm{Ann}_A(x)$. On a donc $f^{-1}(x) = ax = 0$ et, par suite, $x = 0$, d'où $\mathfrak{m} = A$, ce qui est absurde.

REMARQUE. Si A est un anneau local et M un A-module arithmétique quelconque, le A-module $\mathrm{Hom}_A(M, M)$ n'est pas en général un A-module arithmétique. En effet, on peut considérer un nombre premier $p > 0$, l'anneau local $\mathbf{Z}_{(p)}$ et le $\mathbf{Z}_{(p)}$-module $E_{\mathbf{Z}}(\mathbf{Z}/p\mathbf{Z})$ qui est isomorphe à $E_{\mathbf{Z}_{(p)}}(\mathbf{Z}_{(p)}/p\mathbf{Z}_{(p)})$. L'anneau

$$\mathbf{Z}_p = \mathrm{Hom}_{\mathbf{Z}_{(p)}}(E_{\mathbf{Z}}(\mathbf{Z}/p\mathbf{Z}), E_{\mathbf{Z}}(\mathbf{Z}/p\mathbf{Z}))$$

est l'anneau des entiers p-adiques. Mais $\mathbf{Z}_p$ n'est pas un $\mathbf{Z}_{(p)}$-module arithmétique, comme cela résulte de 4.5, (3).

CONJECTURE. *Si A est un anneau local et M un A-module arithmétique, alors* $\mathrm{Hom}_A(M, M)$ *est un anneau arithmétique.*

LEMME 4.6. *Soient A un anneau local et M, N deux A-modules arithmétiques. Supposons que N satisfait à la condition suivante:*

(*) $x \in N$, $x \neq 0$, $a \in A$ *et* $ax = 0$ *entraîne* $a = 0$.

Alors $\mathrm{Hom}_A(M, N)$ *est un A-module arithmétique.*

PREUVE. Soient $f, g \in \mathrm{Hom}_A(M, N)$. On va prouver qu'il existe $a \in A$ tel que $f = ag$ ou $g = af$. On peut donc supposer que $f \neq 0$ et $g \neq 0$. Il existe alors $x_0, x_1 \in M$ tel que $f(x_0) \neq 0$ et $g(x_1) \neq 0$. Comme M est un A-module arithmétique, il existe $c \in A$ tel que $x_0 = cx_1$ ou $x_1 = cx_0$. Supposons que $x_1 = cx_0$ il vient $f(x_0) \neq 0$ et $g(x_0) \neq 0$.

N étant un A-module arithmétique, $Af(x_0) \subseteqq Ag(x_0)$ ou $Ag(x_0) \subseteqq Af(x_0)$. Supposons que $Af(x_0) \subseteqq Ag(x_0)$. Il existe ainsi $a_0 \in A$ tel que

$$f(x_0) = a_0 g(x_0).$$

Soit $x \in Ax_0$; alors $x = dx_0$ où $d \in A$, donc

$$f(x) = f(dx_0) = da_0 g(x_0) = a_0 g(x).$$

Soit maintenant $y \in M \setminus Ax_0$ (si $M = Ax_0$, la preuve est terminée d'après 4.1). On voit que $Ax_0 \subset Ay$, donc $x_0 = by$, où $b \in \mathfrak{m}$, $\mathfrak{m}$ étant l'idéal maximal de A. Mais $Af(y) \subseteqq Ag(y)$ ou $Ag(y) \subseteqq Af(y)$. Admettons que $Ag(y) \subset Af(y)$, ce qui donne $g(y) = af(y)$, où $a \in \mathfrak{m}$. D'autre part

$$f(x_0) = a_0 g(x_0) = a_0 g(by) = a_0 baf(y) = a_0 af(by) = a_0 af(x_0),$$

donc

$$(1 - a_0 a) f(x_0) = 0.$$

Comme $f(x_0) \neq 0$, il suit que $1 - a_0 a \in \mathfrak{m}$, donc $a_0 a \notin \mathfrak{m}$, ce qui contredit au fait que $a \in \mathfrak{m}$ et, par suite, on a nécessairement

$$Af(y) \subseteqq Ag(y).$$

Nous avons montré ainsi que pour tout $x \in M$, il existe $a_x \in A$ tel que

$$f(x) = a_x g(x).$$

De plus, si $Ax \subseteqq Ax_0$, on peut choisir $a_x = a_0$.

Soit $x \in M$ tel que $x \notin Ax_0$; alors $Ax_0 \subset Ax$, donc $x_0 = \lambda x$, où $\lambda \in \mathfrak{m}$. Par suite

$$a_0 g(x_0) = f(x_0) = \lambda f(x) = \lambda a_x g(x) = a_x g(x_0),$$

donc

$$(a_0 - a_x) g(x_0) = 0.$$

On a, par hypothèse, $a_x = a_0$, et de lá

$$f = a_0 g,$$

ce qui achève la démonstration.

THÉORÈME 4.7. *Soient A un anneau local et M un A-module arithmétique qui satisfait à la condition suivante:*

$(*)$ $a \in A$, $x \in M$, $x \neq 0$ *et* $ax = 0$ *entraïne* $a = 0$.

Alors $\mathrm{Hom}_A(M, M)$ *est un anneau arithmétique commutatif.*

PREUVE. On applique 4.5 et 4.6.

COROLLAIRE 4.8. *Soient A un anneau local intègre et M un A-module arithmétique sans torsion. Alors* $\mathrm{Hom}_A(M, M)$ *est un anneau arithmétique commutatif.*

COROLLAIRE 4.9. *Soient A un anneau de valuation et I un idéal de A. Alors* $\mathrm{Hom}_A(I, I)$ *est un anneau arithmétique commutatif.*

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Bibliographie

[1] T. ALBU, Modules injectifs et modules artiniens sur les anneaux *ZPI* généraux, *C. R. Acad. Sci. Paris,* **271**, Série A (1970), p. 209—212.
[2] T. ALBU, Modules de torsion à support fini, *C. R. Acad. Sci. Paris,* **273**, Série A (1971), p. 335—338.
[3] T. ALBU, Modules décomposables de Dickson, *C. R. Acad. Sci. Paris,* **273**, Série A (1971), p. 369—372.
[4] T. ALBU et C. NĂSTĂSESCU, Décomposition primaire dans les modules de torsion, *C. R. Acad. Sci. Paris,* **273**, Série A (1971), p. 696—699.
[5] T. ALBU et C. NĂSTĂSESCU, Décomposition primaire des modules, *J. Algebra,* 1972 (à paraître).
[6] K. ASANO, Über kommutative Ringe in denen jedes Ideal als Produkt von Primidealen darstellbar ist, *J. Math. Soc. Japan,* **3** (1951), p. 82—90.
[7] H. BASS, Finitistic dimension and a homological generalization of semi-primary rings, *Trans. Amer. Math. Soc.,* **95** (1960), p. 466—488.
[8] N. BOURBAKI, *Algèbre commutative,* chap. 1—7, Hermann (Paris, 1965).
[9] S. E. DICKSON, Decomposition of modules II: Rings without chain conditions, *Math. Z.,* **104** (1968), p. 349—357.
[10] D. T. GILL, Almost maximal valuation rings, *J. London Math. Soc.,* **(2), 4** (1971), p. 140—146.
[11] C. U. JENSEN, Arithmetical rings, *Acta Math. Acad. Sci. Hung.,* **17** (1966), p. 115—123.
[12] C. NĂSTĂSESCU et N. POPESCU, Anneaux semi-artiniens, *Bull. Soc. Math. France,* **96** (1968), p. 357—368.
[13] N. RADU, *Inele locale,* I, Editura Academiei (Bucureşti, 1968).
[14] A. ROSENBERG and D. ZELINSKY, Finiteness of the injective hull, *Math. Z.,* **70** (1959), p. 372—380.
[15] R. B. WARFIELD, Decomposability of finitely presented modules, *Proc. Amer. Math. Soc.,* **25** (1970), p. 167—172.

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Tomus 25 (3—4), (1974), pp. 313—319.

PATH CONNECTED GRAPHS

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1. Introduction

The graphs we discuss will be finite, undirected with no loops or multiple edges. A graph G with vertices V and edges E will be denoted by $G(V, E)$, or simply by G. If $|V|=n$ and $|E|=q$ we shall call $G=G(V, E)$ an (n, q) graph. Whenever vertices u and v are adjacent in G, we write $\{u, v\}\in E$ and refer to edge uv.

If there exists a path of length i connecting the vertices u and v of V, then we shall say that property $P_i(u, v)$ holds. Furthermore if $P_i(u, v)$ holds for all distinct pairs (u, v) in V, then we simply say property P_i holds. For subsequent use we also introduce the following notation.

$$d(v) = \text{degree of vertex } v$$

$$\delta(G) = \min \{d(v) | v\in V\}$$

$$\sigma(G) = \min \{d(u)+d(v) | u, v\in V, u \neq v\}$$

$$\bar{\sigma}(G) = \min \{d(u)+d(v) | \{u, v\}\notin E, u \neq v\}.$$

Observe that $\bar{\sigma}(G)$ is not defined when $G\cong K_n$. Although this is artificial we will define $\bar{\sigma}(G)=n+1$ when $G\cong K_n$ to take care of special cases. Notation not specifically introduced will be as in [2].

The purpose of this paper is to generalize a theorem due to OYSTEIN ORE given in [3]. In [3] ORE discusses sufficient degree and edge conditions for property P_{n-1} to hold. Such graphs he calls H-connected. A natural question arises. Under the degree and edge conditions given by ORE, for what i does P_i hold? For the edge conditions given this question is answered in [1] Corollary 2.2. The answer to the remainder of the question is the content of this paper. Specifically ORE shows (in Theorem 3.1 of [3]) when G is an (n, q) graph satisfying $\bar{\sigma}(G)\geqq n+1$ that G satisfies P_{n-1} (is H-connected). We shall show, under the same conditions, that P_i holds for not only $i=n-1$, but also for all i, $4\leqq i\leqq n-1$. Thus under the same conditions much more can be said concerning paths connecting pairs of points of G. Furthermore it will be shown that this result is the best possible, i.e., that under the conditions given P_i need not hold for $i=1, 2, 3$.

2. Generalization of Ore's theorem

Throughout this section the graph $G=G(V, E)$ will be assumed to be of order n with, until further notice, $\bar{\sigma}(G)\geqq n+1$. Also, since subgraphs of G frequently satisfy the same type of degree condition, we give the following convention. *A subgraph G' of G is said to satisfy condition* (OC), "*Ore's condition,*" *if* $\bar{\sigma}(G')\geqq|G'|+1$.

The proof of the generalized theorem is lengthy and depends on several intermediate results which we now state.

LEMMA 2.1. *For* $n \geq 4$, $\delta(G) \geq 3$.

PROOF. Let $u \in G$ such that $d(u) = \delta(G)$. If $d(u) \leq 2$ then there exists a $v \in G$ such that $\{u, v\} \notin E$. But $d(u) + d(v) \geq n+1$ so that $d(v) \geq n-1$, a contradiction.

LEMMA 2.2. *If* $\{u, v\} \notin E$, *then there exist distinct elements* $v_1, v_2, v_3 \in V$ *such that* $\{u, v_i\}, \{v, v_i\} \in E$ $(1 \leq i \leq 3)$.

PROOF. Since $\bar{\sigma}(G) \geq n+1$ and $\{u, v\} \notin E$, we have $n \geq 5$. Let $R = \{x \in V \mid \{x, u\} \in E\}$ and $S = \{x \in V \mid \{x, v\} \in E\}$. Now $R \cup S \subseteq V \setminus \{u, v\}$ and $|R| + |S| \geq n+1$. Hence $|R \cap S| \geq (n+1) - (n-2) = 3$.

LEMMA 2.3. *If* $\{u, v\} \notin E$ *or* $d(u) + d(v) \geq n+2$, *then* $P_2(u, v)$ *and* $P_3(u, v)$ *hold in* G.

PROOF. Either condition requires that $n \geq 5$ or $G \cong K_4$. If $G \cong K_4$, then the result is obvious, so take $n \geq 5$.

Let R and S be as defined in Lemma 2.2. In either case $|R \cap S| \geq 2$ so that $P_2(u, v)$ holds.

If $R \setminus \{v\} = S \setminus \{u\}$, then since $d(u) + d(v) \geq n+1$ for $\{u, v\} \notin E$ and $d(u) + d(v) \geq n+2$ otherwise, we have $|R \setminus \{v\}| \geq n/2 > 2$. For this case let $T = R \setminus \{v\} = S \setminus \{u\}$. If $w_1, w_2 \in T$ and $\{w_1, w_2\} \in E$ then surely $P_3(u, v)$ holds. If no pair in T is adjacent, then $d(w) \leq (n-1) - (n/2 - 1) \leq n/2$ for all $w \in T$. Therefore $d(w_1) + d(w_2) \leq n$ for $w_1, w_2 \in T$, $\{w_1, w_2\} \notin E$, a contradiction.

Thus we may assume $R \setminus \{v\} \neq S \setminus \{u\}$ and that there exists a $w \in R$, $w \notin S$, $w \neq u, v$. Then by Lemma 2.2 $P_2(w, v)$ holds with at least one such path not involving u. Hence $P_3(u, v)$ holds in G.

We next prove a theorem which gives that P_i holds for about one-half of the i's desired.

THEOREM 2.4 *Property* P_i *holds in* G *for* $n/2 < i \leq n-1$.

PROOF. Since $\bar{\sigma}(G) \geq n+1$, $G \cong K_n$ for $n \leq 4$ and then the result is trivial. Therefore we take $n \geq 5$.

Let $u = x_1$ and $v = x_n$ be fixed points of V. By Ore's theorem ([3] Theorem 3.1) there exists a path of length $n-1$ between u and v which we denote by $x_1 x_2 \ldots x_n$. Let $x = x_{n-i+1}$ and $y = x_i$ (with i fixed, $n/2 < i < n-1$) and assume that $P_i(u, v)$ fails in G.

If $\{u, x\} \in E$, then $u x_{n-i+1} x_{n-i+2} \ldots x_n$ is a path of length i. Likewise if $\{y, v\} \in E$, then $u x_2 x_3 \ldots x_i v$ is a path of length i. Hence $\{u, x\}, \{y, v\} \notin E$, and since $\bar{\sigma}(G) \geq n+1$, we have $d(x) + d(u) \geq n+1$ and $d(y) + d(v) \geq n+1$. If $\{u, x_j\}, \{x, x_{j+1}\} \in E$, for $n-i+1 \leq j \leq n-1$, then $u x_j x_{j-1} \ldots x x_{j+1} x_{j+2} \ldots v$ is a path and $P_i(u, v)$ holds. In the same way if $\{v, x_k\}, \{y, x_{k-1}\} \in E$, for $2 \leq k \leq i$, then $u x_2 x_3 \ldots x_{k-1} y x_{i-1} \ldots x_k v$ is a path and $P_i(u, v)$ holds. Therefore, these pairs of adjacencies also cannot occur.

Next let r and s be the number of vertices of $\{x_1, x_2, \ldots, x_{n-i+1}\}$ to which u and x are adjacent, and r' and s' be the number of vertices of $\{x_i, x_{i+1}, \ldots, x_n\}$ to which v and y are adjacent, respectively. By the comments of the previous paragraph there exists a one-to-one map between the points of $\{x_{n-i+1}, x_{n-i+2}, \ldots, x_{n-1}\}$ to which u is adjacent and a subset of the points $\{x_{n-i+2}, x_{n-i+3}, \ldots, x_n\}$ to which x

is not adjacent. Thus if there are k such points to which u is adjacent, we have that $r \geqq d(u)-k-1$ and $s \geqq d(x)-[(i-1)-k]$. Hence $r+s \geqq d(u)+d(x)-i \geqq n+1-i$. Similarly $r'+s' \geqq n+1-i$.

Note that $i > n/2$ gives $n-i+1 \leqq i$. If $\{u, x_j\}, \{y, x_{n+2-j}\} \in E$ $(2 \leqq j \leqq n-i+1)$ then $ux_j x_{j+1} \dots y x_{n+2-j} x_{n+3-j} \dots v$ is a path of length i from u to v, and if $\{v, x_k\}, \{x, x_{n-k}\} \in E$ $(i \leqq k \leqq n-1)$ then $ux_2 \dots x_{n-k} x x_{n-i+2} \dots x_k v$ is again a path of length i from u to v. Thus for every point in $\{x_2, x_3, \dots, x_{n-i+1}\}$ to which u is adjacent there exists a point in $\{x_{i+1}, x_{i+2}, \dots, x_n\}$ to which y is not adjacent. There are r such points. Likewise to each of the s points in $\{x_1, x_2, \dots, x_{n-i}\}$ to which x is adjacent, v is not adjacent to one in $\{x_i, x_{i+1}, \dots, x_{n-1}\}$. Furthermore these given correspondences are one-to-one. Hence $s' \leqq [(n-i+1)-1]-r$ and $r' \leqq [(n-i+1)-1]-s$ so that $r'+s' \leqq 2(n-i+1)-2-(r+s) \leqq 2(n-i+1)-2-(n-i+1) = n-i-1$, a contradiction.

Thus the original assumption that $P_i(u, v)$ fails for some i between $n/2$ and $n-1$ is false and the proof is complete.

To prove the generalization of Ore's theorem we need one additional lemma.

LEMMA 2.5. *If $x_1 x_2 \dots x_i x_{i+1}$ is a path of length i between $u = x_1$ and $v = x_{i+1}$ $(1 \leqq i < n-1)$ in G, then one of the following occurs:*

(i) *$P_{i+1}(u, v)$ holds in G,*

(ii) *the graph $G' = G \setminus \{x_1, x_2, \dots, x_{i+1}\}$ satisfies* (OC),

(iii) *there exist w_1 and w_2 in G distinct from u and v such that the graph $G'' = G \setminus \{w_1, w_2\}$ satisfies* (OC),

(iv) *$i = 2$ and $P_4(u, v)$ holds in G.*

PROOF. We can again assume $n \geqq 5$, for otherwise $G \cong K_n$ and the result is trivial. Let $L = \{x_1, x_2, \dots, x_{i+1}\}$. If $i > n/2$, then $P_{i+1}(u, v)$ holds by Theorem 2.4, so we take $i \leqq n/2$ for the remainder of the proof. If there exists a $w \in G \setminus L$ with $\{w, x_j\}, \{w, x_{j+1}\} \in E$ for some j, $1 \leqq j \leqq i$, then $x_1 x_2 \dots x_j w x_{j+1} \dots x_{i+1}$ and $P_{i+1}(u, v)$ holds and (i) is satisfied. Thus we assume that no such w in $G \setminus L$ exists.

Let w_1 and w_2 be distinct elements of $G' = G \setminus L$ with $\{w_1, w_2\} \notin E$. Thus $d(w_1) + d(w_2) \geqq n+1$. Letting d' denote the degree of points in G' we have, by the last assumption, that $d'(w_1)+d'(w_2) \geqq (n+1)-(i+1) = |G'|+1$, unless i is even and $\{w_1, x_j\}, \{w_2, x_j\} \in E$ for $j = 1, 3, 5, \dots, i+1$. If $d'(w_1)+d'(w_2) \geqq |G'|+1$ for all non-adjacent w_1 and w_2 in G', then G' satisfies (OC), and hence (ii). Therefore we further assume that i is even and $\{w_1, x_j\}, \{w_2, x_j\} \in E$ for $j = 1, 3, 5, \dots, i+1$.

If $i = 2$, then by Lemma 2.2, there exists a path $w_1 w w_2$ between w_1 and w_2 with $w \neq u, v$. Hence $u w_1 w w_2 v$ is a path, $P_4(u, v)$ holds, and (iv) is satisfied.

Thus take $i \geqq 4$. $d(w_1)+d(w_2) \geqq n+1$, and therefore by the above assumptions $d'(w_1)+d'(w_2) \geqq (n+1)-(i+2) = n-i-1$. Since $|G'| = n-i-1$, there must exist a $w \in G' \setminus \{w_1, w_2\}$ such that $\{w, w_1\}, \{w, w_2\} \in E$. If w_1 and w are not adjacent to any common vertex in G then the graph $G'' = G \setminus \{w, w_1\}$ satisfies (OC), and hence (iii). Hence we assume that there exists an $x \in G$ such that $\{x, w_1\}, \{x, w\} \in E$. If $x \in G'$ then $u w_1 x w w_2 x_5 \dots v$ is a path and $P_{i+1}(u, v)$ holds so (i) is satisfied. If $x = x_j$ (j odd), then by symmetry we can assume $j \geqq 3$. In this case $u x_2 \dots x_{j-2} w_1 w x_j \dots v$ is a path and again $P_{i+1}(u, v)$ holds with (i) satisfied. If $x = x_j$ (j even) then w_1 is adjacent to two consecutive x_j's in the path $x_1 x_2 \dots x_{i+1}$, a contradiction. This completes the proof of the lemma.

We now prove the generalization of Ore's theorem.

THEOREM 2.6. *G satisfies property P_i for $4 \leqq i \leqq n-1$.*

PROOF. By Theorem 2.4 the result is clear for $n \leqq 7$ so we take $n \geqq 8$.

Suppose the theorem is false with G a graph of minimal order for which it is not true. Note that $|G| \geqq 8$. Thus there exist points u and v in G and an integer j, $4 \leqq j \leqq n-1$, such that $P_j(u, v)$ fails. By Lemma 2.3 either $P_1(u, v)$ or $P_2(u, v)$ holds. If $P_4(u, v)$ holds then let i be the largest integer such that $P_l(u, v)$ holds for $4 \leqq l \leqq i$ and $P_{i+1}(u, v)$ fails. Otherwise let i be the largest integer ($i<4$) such that $P_i(u, v)$ holds. Note that by Theorem 2.4 $i<[n/2]$.

Let $u=x_1$, $v=x_{i+1}$ and $x_1 x_2 \ldots x_{i+1}$ be a path of length i between u and v and apply Lemma 2.5. From the choice of i it is clear that (i) and (iv) of the lemma cannot occur. If (iii) of Lemma 2.5 occurs, then by the minimality of $|G|$, $P_j(u, v)$ holds in G'' ($4 \leqq j \leqq n-3$). Since $n \geqq 8$ and thus $n-3 > n/2$, this says in particular that $P_i(u, v)$ holds for $4 \leqq i \leqq n/2$, a contradiction. Hence also (iii) of Lemma 2.5 does not occur. Thus we may assume that (ii) of Lemma 2.5 does hold with $i<[n/2]$ and $G'=G \setminus \{x_1, x_2, \ldots, x_{i+1}\}$ satisfying (OC). Since $|G| \geqq 8$ and $i<[n/2]$ we have $|G'|=n-i-1 \geqq 4$, $n/2$. Throughout the remainder of the proof d' will denote the degree of vertices in G'.

Let $R=\{x \in G' \mid \{x, u\} \in E$ and $S=\{x \in G' \mid \{x, v\} \in E\}$.

Case 1. There exists an $x \in R$ and a $y \in S$ with $x \neq y$.

For this case we assume further that $|G'| \geqq 5$, since $|G'|=4$ with G' satisfying (OC) gives $G' \cong K_{|G'|}$, from which P_l holds in G ($4 \leqq l \leqq n/2$).

By assumption (since $|G'|<|G|$) $P_l(x, y)$ holds in G' for $4 \leqq l \leqq |G'|-1$ and hence $P_l(u, v)$ holds in G for $6 \leqq l \leqq n/2 < |G'|+1$. Also if $\{x, y\} \notin E$ or either $d'(x)=n-i-2$ or $d'(y)=n-i-2$, then (since $d'(x) \geqq 3$ and $d'(y) \geqq 3$ by Lemma 2.1) $P_2(x, y)$ and $P_3(x, y)$ hold in G' (by Lemma 2.3) and thus $P_4(u, v)$ and $P_5(u, v)$ hold in G. Therefore we assume that no element of $R \cup S$ has degree $n-i-2$ and each element of S is adjacent to each element of R.

Hence pick a $z \in G'$ such that $\{x, z\} \notin E$. Then there exists a $z' \in G'$ such that $\{x, z'\}, \{z', z\} \in E$. Also $\{v, z\} \notin E$, so by Lemma 2.2 there exists a $y' \in G$ such that $\{v, y'\}, \{y', z\} \in G$ and $y' \neq u, x$, and z'. Therefore $uxz'zy'v$ is a path and $P_5(u, v)$ holds in G.

If there exists a $z \in G'$ such that $\{u, z\}, \{v, z\} \notin E$, then by 2.2 there exists x' and y' such that $ux'zy'v$ is a path and $P_4(u, v)$ holds in G. Hence we can assume that $R \cup S = G'$. This implies $R \not\subseteq S$ (or $S \not\subseteq R$) for otherwise either x or y would have degree $n-i-2$. Hence we assume $x \notin S$ and $y \notin R$. If there exists an $x' \in R$ with $\{x, x'\} \in E$ then $ux'xyv$ is a path and $P_4(u, v)$ holds in G. Thus we assume no element of $R(S)$ is adjacent to any other element of $R(S)$ from which $R \cap S = \emptyset$. This implies $d'(r)=|S|$ and $d'(s)=|R|$ for $r \in R$ and $s \in S$. Hence if $|R| \geqq |S|$, then $d'(r_1)+d'(r_2) \leqq |G'|$ for $r_1, r_2 \in R$, $r_1 \neq r_2$, a contradiction. But then $R=\{x\}$ with $d'(x)=m-i-2$, another contradiction.

We have established if there exists an $x \in R$ and a $y \in S$ with $x \neq y$, then $P_l(u, v)$ holds in G for $4 \leqq l \leqq n/2$. But this contradicts the fact that $P_{i+1}(u, v)$ fails, so that if the theorem is false this case does not occur.

Case 2. $R \neq \emptyset$ and $S=\emptyset$, or $R=S=\{x\}$.

Let $x \in R$. If $d'(x)=n-i-2$ then let y be an arbitrary element of $G' \setminus \{x\}$. Again as in the first case $d'(x)=n-i-2$ and $d'(y) \geqq 3$ so that by Lemma 2.3 $P_2(x, y)$ and $P_3(x, y)$ hold in G'. If $d'(x)<n-i-2$ then let y be such that $\{x, y\} \notin E$ and again by

2.3 $P_2(x, y)$ and $P_3(x, y)$ hold in G'. Hence $P_l(x, y)$ holds in G' for $2 \leqq l \leqq |G'|-1$. Since $S=\varnothing$ or $S=\{x\}$, we have that $\{v, y\} \notin E$ so that by 2.2 there exists a $v' \in G \setminus (G' \cup \{u\})$ with $\{v, v'\}, \{v', y\} \in E$. Therefore $P_l(u, v)$ holds in G for $5 \leqq l \leqq n/2 < |G'|+1$. To see that $P_4(u, v)$ holds pick a $y \in G'$ such that $\{x, y\} \in E$ and pick v' as above. Thus for this second case $P_l(u, v)$ holds for $4 \leqq l \leqq n/2$, which again contradicts that $P_{i+1}(u, v)$ fails.

Case 3. $R=\varnothing$ and $S=\varnothing$.

Let $x, y \in G'$. If x and y are chosen such that $\{x, y\} \notin E$, then by 2.3, $P_2(x, y)$ and $P_3(x, y)$ hold in G'. Thus using the assumption on G' we again have $P_l(x, y)$ holds in G' for $2 \leqq l \leqq |G'|-1$. Now $\{u, x\}, \{v, y\} \notin E$ (since $R=S=\varnothing$), so that there exists $x' \neq y'$ in $G \setminus (G' \cup \{u, v\})$ with $\{u, x'\}, \{x', x\}, \{v, y'\}, \{y', y\} \in E$. Therefore $P_l(u, v)$ holds in G for $6 \leqq l \leqq n/2 < |G'|+3$. If x and y are chosen in G' such that $x=y$, then $ux'xy'v$ is a path and $P_4(u, v)$ holds, while if x and y are chosen such that $\{x, y\} \in E$ then $xx'xyy'v$ is a path and $P_5(u, v)$ holds. Hence again $P_l(u, v)$ holds in G for $4 \leqq l \leqq n/2$, a contradiction.

Since $i < [n/2]$ and $G' = G \setminus \{x_1, x_2, \ldots, x_{i+1}\}$ satisfying (OC) always gives that $P_l(u, v)$ holds for $4 \leqq l \leqq n/2$, the original supposition that $P_{i+1}(u, v)$ fails is false and the theorem is established.

For the remainder of this section we drop the assumption that $\bar{\sigma}(G) \geqq |G|+1$. *We still assume that* $|G|=n$.

In [1] PLD-maximal graphs are discussed. The graph G is said to be PLD-maximal if property P_i holds for $2 \leqq i \leqq |G|-1$. It is easy, in light of the theorem, to find sufficient connectedness conditions for G to be PLD-maximal. This is done in the following corollary.

COROLLARY 2.7. *Let* $G=G(V, E)$ *be an* $[n/2+1]$*-connected graph. Then* G *is* PLD-*maximal. Furthermore the condition given is as sharp as possible.*

PROOF. Surely G $[n/2+1]$-connected implies that for $v \in V$, $d(v) \geqq [n/2+1]$. Thus $\bar{\sigma}(G) \geqq 2[n/2+1] \geqq n+1$ and G satisfies (OC). Hence, by the theorem, P_i holds for $4 \leqq i \leqq |G|-1$. Thus the result will follow by showing that P_2 and P_3 hold.

Let $u, v \in V$. Since $d(u)+d(v) \geqq n+1$ it follows that there exists an $x \in V$, such that $\{u, x\}, \{x, v\} \in E$ and $P_2(u, v)$ holds. Thus it remains to prove that P_3 holds.

Suppose that $P_3(u, v)$ fails for some $u, v \in V$. Let $A=\{x \in V \mid \{x, u\}, \{x, v\} \in E\}$, $B=\{x \in V \mid \{x, u\} \in E,\ \{x, v\} \notin E,\ x \neq v\}$, $C=\{x \in V \mid \{x, v\} \in E,\ \{x, u\} \notin E,\ x \neq u\}$, and $D=\{x \in V \mid \{x, u\}, \{x, v\} \notin E,\ x \neq u, v\}$. Observe that for $P_3(u, v)$ to fail the following must occur: (1) no two points of A are adjacent and (2) no point of A is adjacent to a point of B (respectively C). If $|A|=0$, then since the removal of $B \cup \{v\}$ (respectively $C \cup \{u\}$) makes G disconnected, the $[n/2+1]$-connectivity of G implies $|B|, |C| \geqq [n/2]$ from which $|G| \geqq |B|+|C|+2 \geqq 2[n/2]+2$, a contradiction. Therefore let $l=|A| \geqq 1$. But since A has none of its points adjacent, the removal of $D \cup \{u, v\}$ makes G disconnected. Hence $|D| \geqq [n/2]-1$ and $n=|G| \geqq |D|+|A|+2 \geqq ([n/2]-1)+l+2$, which in turn gives $n-[n/2] \geqq l+1$ so that $[n/2] \geqq l$. Also the removal of $A \cup B \cup \{v\}$ (respectively $A \cup C \cup \{u\}$) makes G disconnected so that $|B|, |C| \geqq [n/2]-l$. Thus $n = |G| = |A|+|B|+|C|+|D|+2 \geqq l+2([n/2]-l)+[n/2]-1+2 = 3[n/2]-l+1$ which implies $l \geqq [n/2]$. Hence $[n/2]=l$. Thus $|B|=|C|=0$, giving that G is at most $[n/2]$-connected, which is absurd. Therefore the supposition that P_3 fails is false, and the proof that G is PLD-maximal is complete.

The sharpness of the connectedness condition $[n/2+1]$ is shown by example in the next section.

As an additional corollary to Theorem 2.6 we determine, for the graph G, the minimum value of $\bar{\sigma}(G)$ such that G is PLD-maximal.

COROLLARY 2.8. *For* $n \geq 4$ *and* $\bar{\sigma}(G) \geq \left[\frac{3}{2}n-1\right]$ *we have that* G *is* PLD-*maximal. Furthermore the condition given cannot be weakened.*

PROOF. Surely for $n \geq 4$ $\bar{\sigma}(G) \geq [3n/2-1] \geq n+1$ so by the theorem property P_i holds for $4 \leq i \leq n-1$. Also for $u, v \in V$ with $\{u, v\} \notin E$ we have that $P_2(u, v)$ and $P_3(u, v)$ hold by Lemma 2.3. Thus for G to be PLD-maximal it suffices to show that $P_2(u, v)$ and $P_3(u, v)$ hold for $\{u, v\} \in E$.

Suppose that there exists $u, v \in V$, $\{u, v\} \in E$ such that $P_2(u, v)$ fails. Let $R=\{x \in V \mid \{x, u\} \in E, x \neq v\}$, $S=\{x \in V \mid \{x, v\} \in E, x \neq u\}$, and $T=\{x \in V \mid \{x, u\}, \{x, v\} \notin E\}$. Then, since $P_2(u, v)$ fails, R, S, and T are mutually disjoint with $n=|R|+|S|+|T|+2$. By Lemma 2.1 $\delta(G) \geq 3$ so $R, S \neq \varnothing$. Let $r \in R$ and $s \in S$. Then $d(r)+d(v) \leq (n-2)+ +|S|+1=(n-1)+|S|$ and $d(s)+d(u) \leq |R|+1+(n-2)=(n-1)+|R|$. Hence $[3n/2-1] \leq \bar{\sigma}(G) \leq |R|+(n-1)$, $|S|+n-1$. Therefore $3n-3 \leq 2[3n/2-1] \leq |R|+|S|+ +2n-2$ so that $|G|=n \leq |R|+|S|+1<|G|$, which is impossible. Therefore the supposition that $P_2(u, v)$ fails is false and P_2 holds.

Next suppose that there exists $u, v \in V$, $\{u, v\} \in E$ such that $P_3(u, v)$ fails. Let $R=\{x \in V \mid \{x, u\} \in E, \{x, v\} \notin E, x \neq v\}$, $S=\{x \in V \mid \{x, v\} \in E, \{x, u\} \notin E, x \neq u\}$, $T= =\{x \in V \mid \{x, u\}, \{x, v\} \in E\}$, and $L=\{x \in V \mid \{x, u\}, \{x, v\} \notin E\}$. Observe that R, S, T, L are mutually disjoint with $n=|R|+|S|+|T|+|L|+2$. Furthermore for $P_3(u, v)$ to fail each of following must hold: (1) no two points of T are adjacent, and (2) no points of R (respectively S) are adjacent to points of $T \cup S$ (respectively $T \cup R$). Now $d(u) \leq |T|+|R|+1$ and $d(v) \leq |T|+|S|+1$. Also for $r \in R$ and $s \in S$ we have $d(r) \leq |R|+|L|$, $d(s) \leq |S|+|L|$, and $\{r, v\}, \{s, u\} \notin E$. Hence if either S or R is non-empty, then $n+1 \leq \bar{\sigma}(G) \leq |R|+|S|+|L|+|T|+1=n-1$ which is absurd. Therefore $S=R=\varnothing$. But by Lemma 2.1, $\delta(G) \geq 3$ so that $d(u), d(v) \geq 3$ implying $|T| \geq 2$. Also since no two points of T are adjacent and $d(t) \geq 3$ for $t \in T$, we have $|L| \geq 1$. Let $t_1, t_2 \in T$, $l \in L$ and notice that $d(t_1)+d(t_2) \leq 2|L|+4$, $d(u)+d(l) \leq (|T|+1)+(n-3)$ so that $[3n/2-1] \leq \bar{\sigma}(G) \leq 2|L|+4$, $|T|+(n-2)$. But since $R=S=\varnothing$, $n=|L|+|T|+2$ and $2|L|+4=2n-2|T|$. Hence $9n-9 \leq 6[3n/2-1] \leq 4(|T|+n-2)+2(2n-2|T|)= =8n-8$ implying $n \leq 1$, a contradiction to the choice of n. Thus the supposition that $P_3(u, v)$ fails is false and P_3 holds.

An example in the next section shows that the condition given cannot be weakened.

3. Examples and observations

Within this section we shall show by example the sharpness of several of the results given in the previous section. The examples needed can all be formed by looking at the union and join operations on graphs. Define for graphs $G_1=G_1(V_1, E_1)$ and $G_2=G_2(V_2, E_2)$, the *union* of G_1 and G_2, written $G_1 \cup G_2$, as the graph with vertices $V_1 \cup V_2$ and edges $E_1 \cup E_2$, and the *join* of G_1 and G_2, written G_1+G_2, as the graph $G_1 \cup G_2$ together with all edges joining V_1 and V_2 (see [2]).

EXAMPLE 3.1. As in [1] define an (m, n, k) graph to be the graph $\overline{K}_n+(K_m \cup K_k)$. The vertices of an (m, n, k) graph have the following degrees:

$$d(v) = m+n-1, \quad v \in K_m$$

$$d(v) = m+k, \quad v \in \overline{K}_n$$

$$d(v) = k+n-1, \quad v \in K_k.$$

Clearly an $(m, n, 1)$ graph does not satisfy P_2 while an $(m, n, 2)$ graph does not satisfy P_3. Thus for N not too small it is easy to check that the following occurs:

$$\bar{\sigma}(G) = n+3, \quad \text{when} \quad G = (N, N-1, 2), \qquad n = |G| = 2N+1$$

$$\bar{\sigma}(G) = n+3, \quad \text{when} \quad G = (N+1, N-1, 1), \quad n = |G| = 2N+1$$

$$\bar{\sigma}(G) = n+2, \quad \text{when} \quad G = (N-1, N-1, 2), \quad n = 2N$$

$$\bar{\sigma}(G) = n+4, \quad \text{when} \quad G = (N+1, N-2, 1), \quad n = 2N.$$

These specific (m, n, k) graphs demonstrate that when G satisfies (OC), property P_i may fail for $i=1, 2, 3$.

The sharpness of Corollary 2.7 is apparent by inspecting the $(n-2-[n/2], [n/2], 2)$ graph for $n \geqq 4$. This graph of order n is $[n/2]$-connected, but P_3 fails.

EXAMPLE 3.2. For $n \geqq 4$ form the graph $G=\left(K_{\left[\frac{n-2}{2}\right]} \cup \{v\}\right)+\left(K_{n-\left[\frac{n-2}{2}\right]-2} \cup \{u\}\right)$. It is a simple computation to check that for this graph of order n, $\bar{\sigma}(G)=[3n/2-2]$ but P_2 fails. Thus the condition given in Corollary 2.8 is the best possible.

Several comments concerning the results of this paper are in order. The results of Theorem 2.6 are of special interest in light of Theorem 2.1 of [1]. To have properties P_2 and P_3 hold, we see from 2.1 of [1], that in addition to condition (OC), G has to satisfy $\sigma(G) \geqq n+1$ and, for n odd, $\bar{\sigma}(G) \geqq n+2$. Thus to have P_2 and P_3 hold places additional restriction on G. This is also consistent with what we proved in Corollaries 2.7 and 2.8. Forgetting about degree, edge, or connectedness conditions it would be nice to know how the P_i's of G relate, in particular, does P_{n-1} holding in G imply that other P_i's hold? If so, which ones?

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References

[1] R. J. FAUDREE, C. C. ROUSSEAU and R. H. SCHELP, Theory of path length distributions I, *Discrete Math.*, **6** (1973), pp. 35—53.
[2] F. HARARY, *Graph theory*, Addison-Wesley (Reading, Mass., 1969).
[3] O. ORE, Hamilton connected graphs, *J. Math. Pures Appl.*, **42** (1963), pp. 21—27.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (3—4), (1974), pp. 321—327.

TWO PARALLEL RELATIVE HOMOLOGICAL ALGEBRAS

By

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In recent years, relative homological algebra has played an increasingly important part in the theory of groups.[1] The most notable example of this is Nunke's discovery of the class of totally projective groups, which is the class of projectives for a certain relative homological algebra [3].

DEFINITION (BUCHSBAUM [1]). Let $\hat{R}$ be a class of short exact sequences of groups, and let A, B, and C be groups such that $A \subset B \subset C$. Then $\hat{R}$ is a relative homological algebra if the following six conditions are satisfied:

(i) If $E \in \hat{R}$, then any sequence equivalent to E is in $\hat{R}$.

(ii) All split short exact sequences are in $\hat{R}$.

(iii) If $0 \to A \to C \to C/A \to 0 \in \hat{R}$, then $0 \to A \to B \to B/A \to 0 \in \hat{R}$.

(iv) If $0 \to B \to C \to C/B \to 0 \in \hat{R}$, then $0 \to B/A \to C/A \to C/B \to 0 \in \hat{R}$.

(v) If $0 \to A \to B \to B/A \to 0 \in \hat{R}$ and $0 \to B \to C \to C/B \to 0 \in \hat{R}$, then $0 \to A \to C \to C/A \to \to 0 \in \hat{R}$.

(vi) If $0 \to A \to C \to C/A \to 0 \in \hat{R}$ and $0 \to B/A \to C/A \to C/B \to 0 \in \hat{R}$, then $0 \to B \to C \to \to C/B \to 0 \in \hat{R}$.

In this paper, two particular classes of short exact sequences are studied. Each class is shown to be a relative homological algebra, and a characterization is given for each. Here no assumptions are made regarding the groups considered and, in general, no distinction is made between the short exact sequence $0 \to A \to B \to C \to 0$ and the element it represents in the group Ext (C, A). However, if E is the sequence $0 \to A \to B \to B/A \to 0$, the notation nE, where $n \in Z$, always denotes the sequence $0 \to A \xrightarrow{j} H \xrightarrow{s} B/A \to 0$ where $H = \{(b, b_1 + A) \mid b + A = nb_1 + A\}$, $j(a) = (a, 0)$, and $s(b, b_1 + A) = b_1 + A$. nE does represent n times the element represented by E in Ext $(B/A, A)$. Also, the notation E_n denotes the sequence $0 \to A/A[n] \to A + nB/A[n] \to \to n(B/A) \to 0$, where the maps are induced by those of E. The torsion subgroup of Ext (C, A) is denoted by Text (C, A), the divisible part of Ext (C, A) by Dext (C, A), and the elements of infinite height by Pext (C, A).

In [4], C. WALKER proved that an exact sequence $E: 0 \to A \to B \to B/A \to 0$ represents an element of Text $(B/A, A)$ if and only if for some n the sequence E_n is split exact. The characterizations of the two relative homological algebras studied in this paper are similar to this characterization of Text.

DEFINITION. (a) A sequence $E: 0 \to A \to B \to B/A \to 0$ is quasi-pure if nE is pure for some $n \in Z$. The class of all such E is denoted $\hat{A}$.

[1] Throughout this paper, the word group always means Abelian group.

(b) A sequence $E: 0 \to A \to B \to B/A \to 0$ is quasi-torsion split if nE is torsion split for some $n \in Z$. The class of all such E is denoted $\hat{C}$.

It should be noted that the classes $\hat{A}$ and $\hat{C}$ could also be defined in the following way: For any groups A and B, let $\hat{A}(B, A)/\text{Pext}(B, A) = (\text{Ext}(B, A)/\text{Pext}(B, A))_t$, and $\hat{C}(B, A)/\text{Dext}(B, A) = (\text{Ext}(B, A)/\text{Dext}(B, A))_t$.

REMARK. Let $\text{Ext}(B, A) = \text{Dext}(B, A) \oplus R(B, A)$. Then $(\text{Ext}(B, A)/\text{Dext}(B, A)_t) = (\text{Dext}(B, A) \oplus R(B, A)/\text{Dext}(B, A))_t = \text{Dext}(B, A) \oplus R_t(B, A)/\text{Dext}(B, A)$, where $R_t(B, A)$ denotes the torsion subgroup of $R(B, A)$. Therefore, $\hat{C} = \text{Dext} + \text{Text}$.

To illustrate the parallel between the results given for both of these classes, a theorem is first proved concerning $\hat{A}$, and then the corresponding one for $\hat{C}$.

THEOREM 1 (a). *A sequence* $E: 0 \to A \to B \to B/A \to 0$ *is quasi-pure if and only if* E_n *is pure for some* n.

PROOF. If E is quasi-pure, then $nE: 0 \to A \xrightarrow{j} H \xrightarrow{s} B/A \to 0$ is pure, where $H = \{(b, b_1 + A) \mid b + A = nb_1 + A\}$, $j(a) = (a, 0)$, $s(b, b_1 + A) = b_1 + A$. Recall $E_n: 0 \to A/A[n] \to A + nB/A[n] \to n(B/A) \to 0$. Let $m \in Z$ and suppose $m(nb + A[n]) \in A/A[n]$. Then $mnb \in A$. $(nb, b + A) \in H$ and hence $mn(nb, b + A) \in mnH$. But $mn(nb, b + A) = (mn^2 b, mnb + A) = (a_1, 0) \in j(A)$ since $mnb \in A$. Therefore $mn(nb, b + H) \in mnH \cap j(A)$ and there exists an $a \in A$ such that $mn(nb, b + A) = mn(a, 0)$ since nE is pure. Hence $mn^2 b = mna$. Therefore $n(mnb - ma) = 0$ and $mnb - ma \in A[n]$. Thus $m(nb + A[n]) = m(a + A[n])$, that is, E_n is pure. Now suppose E_n is pure. $n^2 E: 0 \to A \to H \to B/A \to 0$ where $H = \{(b, b_1 + A) \mid b + A = n^2 b_1 + A\}$. From here on, A is identified with $j(A)$. Let $m \in Z$ and suppose $m(b, b_1 + a) = (a, 0)$, that is, $(a, 0) \in mH \cap A$. Thus $mb \in A$, $mb_1 \in A$, and $n^2 b_1 = b + a_1$ for some $a_1 \in A$. Consider $m(nb_1 + A[n])$. $nb_1 + A[n] \in A + nB/A[n]$, and since $mb_1 \in A$, $m(nb_1 + A[n]) \in A/A[n]$. Therefore, since E_n is pure, there exists an $a' \in A$ such that $m(nb_1 + A[n]) = m(a' + A[n])$. Hence $n(mnb_1 - ma') = 0$. But $n^2 b_1 = b + a_1$. Therefore $m(b + a_1) = mna'$, that is, $mb = m(na' - a_1)$. Thus $m(b, b_1 + A) = m(na' - a_1, 0)$, and hence $n^2 E$ is pure and E is quasi-pure.

THEOREM 1 (b). *A sequence* $E: 0 \to A \to B \to B/A \to 0$ *is quasi-torsion split if and only if* E_n *is torsion split for some* n.

PROOF. Suppose $nE: 0 \to A_t \xrightarrow{j} H_t \xrightarrow{s} B/A \to 0$ is torsion split, where

$$H = \{(b, b_1 + A) \mid b + A = nb_1 + A\}.$$

Then $0 \to A_t \xrightarrow{j} H_t \xrightarrow{s} (B/A)_t \to 0$ is split exact, and therefore there exists a homomorphism $\varphi: (B/A)_t \to H_t$ such that for each $b + A \in (B/A)_t$, $s\varphi(b + A) = b + A$. Notice that if $\varphi(b + A) = (b_1, b + A) \in H_t$, $b_1 + A = nb + A$ and hence $\varphi(b + A)$ can always be written $(nb + a, b + A)$ for an appropriate $a \in A$. Consider $E_n: 0 \to A/A[n] \to A + nB/A[n] \xrightarrow{\pi} \xrightarrow{\pi} n(B/A) \to 0$. Define $\psi: (n(B/A))_t \to (A + nB)_t/A[n]$ by $\psi(nb + A) = (nb + a) + A[n]$ where $\varphi(b + A) = (nb + a, b + A)$. Suppose $nb + A = nb_1 + A \in (n(B/A))_t$, $\varphi(b + A) = (nb + a, b + A)$, and $\varphi(b_1 + A) = (nb_1 + a_1, b_1 + A)$. Since φ is a homomorphism, $\varphi(nb + A) = n\varphi(b + A) = n(nb + a, b + A) = (n^2 b + na, nb + A)$, and similarly $\varphi(nb_1 + A) = (n^2 b_1 + na_1, nb_1 + A)$. But φ is well defined and hence $(n^2 b_1 + na_1, nb_1 + A) = (n^2 b + na, nb + A)$. Therefore $n^2 b_1 + na_1 = n^2 b + na$ and hence $nb_1 + a_1 - (nb + a) \in A[n]$ and ψ is well defined. That ψ is a homomorphism follows immediately from the fact that φ is a homomorphism. If $nb + A \in (n(B/A))_t$, then $\pi\psi(nb + A) = \pi((nb + a) + A[n]) = nb + A$ and hence the sequence $0 \to A_t/A[n] \to (A + nB)_t/A[n] \to (n(B/A))_t \to 0$ is split. It is

clearly exact since for any $nb+A\in(n(B/A))_t$, ψ defines an element in $(A+nB)_t/A[n]$ mapping onto $nb+A$ under π. Therefore E_n is torsion split. Now suppose E_n is torsion split, that is, the sequence $0\to A_t/A[n]\to(A+nB)_t/A[n]\xrightarrow{\pi}(n(B/A))_t\to 0$ is split exact. Then there exists a splitting map $\varphi:(n(B/A))_t\to(A+nB)_t/A[n]$ such that $\pi\varphi(nb+A)=nb+A$. Since $\pi((a+nb)+A[n])=n(b+A)$, $\varphi(nb+A)=(a+nb')+A[n]$ where $n(b'-b)\in A$. n^2E: $0\to A\to H\xrightarrow{s}B/A\to 0$ where $H=\{(b,b_1+A)\,|\,b+A=n^2b_1+A\}$. Consider $0\to A_t\to H_t\xrightarrow{s}(B/A)_t\to 0$ where $H_t=\{(b_t,b_1+A)\in H|$ for some $k\in Z$, $kb_t=0$, $kb_1\in A\}$. Define $\psi:(B/A)_t\to H_t$ as follows: Let $b+A\in(B/A)_t$. Then $n(b+A)\in(n(B/A))_t$. Suppose $\varphi(nb+A)=(a+nb')+A[n]$. Then define $\psi(b+A)=(na+n^2b',b+A)$. Since $n(b'-b)\in A$, $na+n^2b'+A=n^2b+A$ and hence $(na+n^2b',b+A)\in H$. This and the fact that $a+nb'\in(A+nB)_t$ and $b+A\in(B/A)_t$ imply that $(na+n^2b',b+a)\in H_t$. Suppose $b+A=b_1+A\in(B/A)_t$. Then $nb+A=nb_1+A\in(n(B/A))_t$. Let $\varphi(nb+A)=(a+nb')+A[n]$, $\varphi(nb_1+A)=(a_1+nb_1')+A[n]$. Then $n(a+nb')=n(a_1+nb_1')$ and hence $\psi(b+A)=(nb+n^2b',b+A)=(na_1+n^2b_1',b_1+A)=\psi(b_1+A)$. Thus ψ is well defined. It is clear that ψ is a homomorphism since φ is a homomorphism. Let $b+A\in(B/A)_t$. Then $s\psi(b+A)=s(na+n^2b',b+A)$ for appropriate a and b' as indicated above. Therefore $s\psi(b+A)=b+A$ and the sequence splits. It is clearly exact since given $b+A\in(B/A)_t$, ψ defines an element $x\in H_t$ such that $s(x)=b+A$. Therefore, if E_n is torsion split, n^2E is torsion split and hence E is quasi-torsion split.

The following pair of theorems shows that both $\hat{A}$ and $\hat{C}$ are relative homological algebras. Although Dext and Text are relative homological algebras, it is not immediate that $\hat{C}$ is a relative homological algebra.

THEOREM 2 (a). *$\hat{A}$ is a relative homological algebra.*

PROOF. BUTLER and HORROCKS [2] have shown that the first four conditions in the given definition of a relative homological algebra are satisfied by any class $\hat{F}$ of short exact sequences, such that $\hat{F}$ is a functor which is contravariant in the first variable and covariant in the second variable. Thus, only conditions (v) and (vi) remain to be verified for both $\hat{A}$ and $\hat{C}$.

(v) Let E': $0\to A\to B\to B/A\to 0$ and E'': $0\to B\to C\to C/B\to 0\in\hat{A}$. Then there is some integer n such that nE': $0\to A\to H'\to B/A\to 0$ and nE'': $0\to B\to H''\to C/B\to 0$ are both pure where $H'=\{(b,b_1+A)\,|\,b+A=nb_1+A\}$, $H''=\{(c,c_1+B)\,|\,c+B=nc_1+B\}$. Let E: $0\to A\to C\to C/A\to 0$ and n^2E: $0\to A\to H\to C/A\to 0$ where $H=\{(c,c_1+A)|c+A=n^2c_1+A\}$. Let $m\in Z$ and suppose $m(c,c_1+A)\in mH\cap A$. Then $c-n^2c_1\in A$, $mc_1\in A$, $mc\in A$. Clearly $(nc_1,c_1+B)\in H''$, and $m(nc_1,c_1+B)\in mH''\cap B$ since $nc_1\in A$ and hence B. Therefore, since nE'' is pure, there exists $b\in B$ such that $m(nc_1,c_1+B)=m(b,0)$, that is, $mnc_1=mb$. $(nb,b+A)\in H$, and since $mc_1\in A$, $m(nb,b+A)\in A$. Thus, since nE' is pure, there is an $a\in A$ such that $m(nb,b+A)=m(a,0)$, that is, $mnb=ma$. But $mnb=mn^2c_1$ and since $c-n^2c_1=a_1$ for some $a_1\in A$, $mn^2c_1=mc-ma_1$. Therefore $mc=ma+ma_1=m(a+a_1)$ and n^2E is pure and hence $E\in\hat{A}$.

(vi) Let $A\subset B$ and let E': $0\to A\to C\to C/A\to 0$ and E'': $0\to B/A\to C/A\to C/B\to 0\in\hat{A}$. Then there exists an $n\in Z$ such that nE': $0\to A\to H'\to C/A\to 0$ and nE'': $0\to B/A\to H''\to C/B\to 0$ are pure, where $H'=\{(c,c_1+A)|c+A=nc_1+A\}$ and $H''=\{(c+A,c_1+B)\,|\,c+B=nc_1+B\}$. Let E: $0\to B\to C\to C/B\to 0$, n^2E: $0\to B\to H\to C/B\to 0$, where $H=\{(c,c_1+B)\,|\,c+B=n^2c_1+B\}$. Suppose $m\in Z$ and $m(c,c_1+B)\in mH\cap B$. Then mc, mc_1, and $c-n^2c_1\in B$. Thus $m(nc_1+A,c_1+B)\in mH''\cap B/A$ and therefore,

since nE'' is pure, there exists $b\in B$ such that $m(nc_1+A, c_1+B)=m(b+A, 0)$, that is, $mnc_1-mb\in A$. Similarly $m\big(n(nc_1-b), (nc_1-b)+A\big)\in mH'\cap A$. Since nE' is pure, there exists an $a\in A$ such that $m\big(n(nc_1-b), (nc_1-b)+A\big)=m(a, 0)$. Therefore $m(n^2c_1-nb)=$ $=ma$. But $n^2c_1=c-b_1$ for some $b_1\in B$. Therefore $m\big((c-b_1)-nb\big)=ma$, that is, $mc=m(b_1+nb+a)$. Let $b_2=b_1+nb+a$. Then $m(c, c_1+B)=m(b_2, 0)$. Therefore n^2E is pure and $E\in\hat{A}$. Thus $\hat{A}$ is a relative homological algebra.

THEOREM 2 (b). *$\hat{C}$ is a relative homological algebra.*

PROOF. (v) Let $E'\colon 0\to A\to B\to B/A\to 0$ and $E''\colon 0\to B\to C\to C/B\to 0\in\hat{C}$. Then there exists an $n\in Z$ such that $nE'\colon 0\to A\to H'\to B/A\to 0$ and $nE''\colon 0\to B\to H''\to$ $\to C/B\to 0$ are torsion split, where $H'=\{(b, b_1+A)\,|\,b+A=nb_1+A\}$ and $H''=$ $=\{(c, c_1+B)\,|\,c+B=nc_1+B\}$. That is, $nE_t'\colon 0\to A_t\to H_t'\xrightarrow{\alpha}(B/A)_t\to 0$ and $nE_t''\colon 0\to B_t\to$ $\to H_t''\xrightarrow{\beta}(C/B)_t\to 0$ are split exact. Therefore, there exist homomorphisms $\varphi_1\colon (B/A)_t\to$ $\to H_t'$ and $\varphi_2\colon (C/B)_t\to H_t''$ such that $\alpha\varphi_1(b+A)=b+A$ for all $b+A\in(B/A)_t$ and $\beta\varphi_2(c+B)=c+B$ for all $c+B\in(C/B)_t$. Let $E\colon 0\to A\to C\to C/A\to 0$. Then $n^2E\colon 0\to A\to$ $H\to C/A\to 0$ where $H=\{(c, c_1+A)\,|\,c+A=n^2c_1+A\}$. Consider $n^2E_t\colon 0\to A_t\to H_t\xrightarrow{\gamma}$ $\xrightarrow{\gamma}(C/A)_t\to 0$. Define $\varphi\colon(C/A)_t\to H_t$ as follows: Let $c+A\in(C/A)_t$. Then $c+B\in(C/B)_t$. Let $\varphi_2(c+B)=(c_t, c+B)\in H_t''$. Then $c_t+B=nc+B$, that is, $c_t-nc\in B$. Since $c_t\in C_t$ and $c+A\in(C/A)_t$, there exists an $m\in Z$ such that $m(c_t-nc)\in A$. Therefore, $(c_t-nc)+$ $+A\in(B/A)_t$. Let $\varphi_1\big((c_t-nc)+A\big)=(b_t, c_t-nc)\in H_t'$. Thus $b_t+A=(nc_t-n^2c)+A$, that is, $n^2c+A=(nc_t-b_t)+A$. Note that $nc_t-b_t\in C_t$. Therefore $(nc_t-b_t, c+A)\in H_t$. Now define $\varphi(c+A)=(nc_t-b_t, c+A)$. If $c+A=c_1+A$, then $c+B=c_1+B$ and hence, since φ_2 is well defined, $\varphi_2(c+B)=\varphi_2(c_1+B)$, that is, c_t is unique. Also, since φ_1 is well defined, b_t is unique. Therefore φ is well defined. It is clearly a homomorphism and hence n^2E_t splits. n^2E_t is exact since the proof that n^2E_t splits shows that $\gamma(nc_t-b_t, c+A)=c+A$ for any $c+A\in(C/A)_t$ and appropriate c_t and b_t. Therefore n^2E is torsion split and $E\in\hat{C}$.

(vi) Let $A\subset B$ and suppose $E'\colon 0\to A\to C\to C/A\to 0$ and $E''\colon 0\to B/A\to C/A\to$ $\to C/B\to 0\in\hat{C}$. Then there exists an $n\in Z$ such that $nE'\colon 0\to A\to H'\to C/A\to 0$ and $nE''\colon 0\to B/A\to H''\to C/B\to 0$ are torsion split, where $H'=\{(c, c_1+A)\,|\,c+A=nc_1+A\}$ and $H''=\{(c+A, c_1+B)\,|\,c+B=nc_1+B\}$. That is, $nE_t'\colon 0\to A_t\to H_t'\xrightarrow{\alpha}(C/A)_t\to 0$ and $nE_t''\colon 0\to(B/A)_t\to H_t''\xrightarrow{\beta}(C/B)_t\to 0$ are split exact. Therefore, there exist homomorphisms $\varphi_1\colon(C/A)_t\to H_t'$ and $\varphi_2\colon(C/B)_t\to H_t''$ such that $\alpha\varphi_1(c+A)=c+A$ for all $c+A\in(C/A)_t$ and $\beta\varphi_2(c+B)=c+B$ for all $c+B\in(C/B)_t$. Let $E\colon 0\to B\to C\to C/B\to 0$. Then $n^2E\colon 0\to B\to H\to C/B$ as before. Consider $n^2E_t\colon 0\to B_t\to H_t\to(C/B)_t\to 0$. Define $\varphi\colon(C/B)_t\to H_t$ as follows: Let $c+B\in(C/B)_t$ and $\varphi_2(c+B)=(c_1+A, c+B)\in H_t''$. Therefore $c_1-nc\in B$ and $c_1+A\in(C/A)_t$. Let $\varphi_1(c_1+A)=(c_t, c_1+A)\in H_t''$. Then $c_t-nc_1\in A$ and $c_t-n^2c\in B$. Now define $\varphi(c+B)=(c_t, c+B)$. That this is well defined follows since φ_1 and φ_2 are well defined in much the same way that was given in the proof of (v) and is omitted. φ is clearly a homomorphism and hence n^2E_t is split. Since n^2E_t is exact as in (v), n^2E is torsion split and $E\in\hat{C}$. Therefore $\hat{C}$ is a relative homological algebra.

From the standpoint of group theory, perhaps the most important reason to consider various relative homological algebras is that the classes of groups which are $\hat{R}$-projective or $\hat{R}$-injective for a given relative homological algebra $\hat{R}$ might be interesting classes that have never been considered. Unfortunately, this is not the case with the two relative homological algebras studied here.

THEOREM 3 (a). *P is $\hat{A}$-projective if and only if P is free.*

PROOF. If P is free, it is clearly $\hat{A}$-projective since it is Ext-projective. Suppose P is $\hat{A}$-projective. Since $\mathrm{Ext} \subset \hat{A}$, direct sums of cyclic groups are the only possibilities for $\hat{A}$-projectives. But P must be torsion free. If not, then there exists some $n \in Z$ such that Z_n is a summand of P. Since summands of projectives are projective, Z_n would be $\hat{A}$-projective. Consider $E: 0 \to nZ \to Z \to Z/nZ \to 0$. Since $E_n: 0 \to nZ/nZ[n] \to \to (nZ+nZ)/nZ[n] \to n(Z/nZ) \to 0$ splits, E clearly is in A. But $f: Z_n \to Z/nZ$ by $f(1) = = 1+nZ$ is a well defined homomorphism and yet $\mathrm{Hom}(Z_n, Z) = 0$ since Z_n is torsion and Z is torsion-free. Therefore Z_n is not $\hat{A}$-projective and P is torsion-free. Hence P is free.

THEOREM 3 (b). *P is $\hat{C}$-projective if and only if P is free.*

PROOF. If P is free, P is $\hat{C}$-projective. Let P be $\hat{C}$-projective. Since $\mathrm{Text} \subset \hat{C}$ and $\mathrm{Dext} \subset \hat{C}$, the class of $\hat{C}$-projective must be contained in the intersection of the class of Text-projectives and the class of Dext-projectives. The Dext-projectives have the form $F \oplus T$ where F is free and T is torsion, while the Text-projectives have the form $F \oplus D$ where F is free and D is torsion-free divisible. Thus the class of $\hat{C}$-projectives must be the class of free groups.

THEOREM 4 (a). *I is $\hat{A}$-injective if and only if I is divisible.*

PROOF. If I is divisible, then I is Ext-injective and hence $\hat{A}$-injective. Suppose I is $\hat{A}$-injective. Since $\mathrm{Text} \subset \hat{A}$ and Text-injectives are the divisible groups, I is divisible.

THEOREM 4 (b). *I is $\hat{C}$-injective if and only if I is divisible.*

PROOF. The proof is the same as that of Theorem 4 (a).

To show that both $\hat{A}$ and $\hat{C}$ have neither enough projectives nor injectives, the following two lemmas, dual to one another, are proved. To prove them, it is convenient to use an equivalent definition of a relative homological algebra due to BUTLER and HORROCKS [2]. They define a relative homological algebra in terms of maps rather than sequences.

LEMMA 5. *Let $\hat{R}$ be a relative homological algebra with enough projectives.*[2] *Let $\hat{R}'$ be the class of all sequences for which all of the $\hat{R}$-projectives are projective. Then $\hat{R}' = \hat{R}$.*

PROOF. Suppose $E: 0 \to A \to B \xrightarrow{\alpha} C \to 0 \in \hat{R}'$. Since $\hat{R}$ has enough projectives, there exists an $\hat{R}$-projective P and an epimorphism $\psi: P \to C$. But since P is $\hat{R}$-projective, P is $\hat{R}'$-projective, and hence there exists a homomorphism $\beta: P \to B$ such that $\alpha\beta = \psi$. This insures that $E \in \hat{R}$ by Butler and Horrocks' definition of a relative homological algebra. Clearly $\hat{R} \subset \hat{R}'$, and therefore $\hat{R} = \hat{R}'$.

LEMMA 6. *Let $\hat{R}$ be a relative homological algebra with enough injectives. Let $\hat{R}'$ be the class of all sequences for which all of the $\hat{R}$-injectives are injective. Then $\hat{R}' = \hat{R}$.*

[2] $\hat{R}$ has enough projectives if for each C there is an exact sequence $0 \to A \to P \to C \to 0$ with P $\hat{R}$-projective. Enough injectives are defined dually.

PROOF. The proof of this lemma is dual to the proof of Lemma 5.

THEOREM 7 (a). *$\hat{A}$ has neither enough projectives nor enough injectives.*

PROOF. If $\hat{A}$ had enough projectives, then $\hat{A}=\text{Ext}$ by Lemma 5. If $\hat{A}$ had enough injectives, then $\hat{A}=\text{Ext}$ by Lemma 6. Thus, $\hat{A}$ has neither enough projectives nor enough injectives.

THEOREM 7 (b). *$\hat{C}$ has neither enough projectives nor enough injectives.*

PROOF. The proof here is the same as for Theorem 7 (a).

The final two pairs of theorems of this paper are analogs of Theorem 2 and Theorem 3, respectively, in [4].

THEOREM 8 (a). *Let $E: 0\to A\to B\to B/A\to 0\in\text{Ext}(B/A, A)$ and $n\in Z$. If $E': 0\to A\to A+nB\to n(B/A)\to 0$ is pure, then nE is pure.*

PROOF. As before, $nE: 0\to A\to H\to B/A\to 0$, where $H=\{(b, b_1+A)\,|\,b+A=nb_1+A\}$. Let $m\in Z$ and suppose $m(b, b_1+A)\in mH\cap A$. Then $mb\in A$, $mb_1\in A$, and $nb_1=b+a_1$ for an appropriate $a_1\in A$. $mnb_1\in(A+nB)\cap A$. By hypothesis, E' is pure, so there exists an $a\in A$ such that $mnb_1=ma$. But $nb_1=b+a_1$, and therefore $m(b+a_1)=$ $=m(a)$, that is $mb=m(a-a_1)$. Therefore $m(b, b_1+A)=m(a-a_1, 0)$, that is, nE is pure.

THEOREM 8 (b). *Let $E: 0\to A\to B\to B/A\to 0\in\text{Ext}(B/A, A)$ and $n\in Z$. If $E': 0\to A\to A+nB\to n(B/A)\to 0$ is torsion split, then nE is torsion split.*

PROOF. Suppose E' is torsion split. Then $E'_t: 0\to A_t\to(A+nB)_t\xrightarrow{\eta}\big(n(B/A)\big)_t\to 0$ is split exact. Therefore, there exists a homomorphism $\varphi: \big(n(B/A)\big)_t\to(A+nB)_t$ such that $\eta\varphi(nb+A)=nb+A$ for each $nb+A\in\big(n(B/A)\big)_t$. $nE: 0\to A\to H\to B/A\to 0$, where $H=\{(b, b_1+A)\,|\,b+A=nb_1+A\}$. Consider $nE_t: 0\to A_t\to H_t\xrightarrow{f}(B/A)_t\to 0$. Define $\psi: (B/A)_t\to H_t$ by $\psi(b+A)=\big(\varphi(nb+A), b+A\big)$. $\big(\varphi(nb+A), b+A\big)\in H_t$ since if $\varphi(nb+A)=nb_1+a_1$, $\eta(nb_1+a_1)=nb_1+A=nb+A$ since φ is a splitting map for E'_t. If $b+A=b_1+A\in(B/A)_t$, then $nb+A=nb_1+A$ and hence $\varphi(nb+A)=\varphi(nb_1+A)$. Therefore $\psi(b+A)=\psi(b_1+A)$ and ψ is well defined. ψ is a homomorphism since φ is a homomorphism. By definition of the map $f: H_t\to(B/A)_t$, $f\psi(b+A)=$ $=f\big(\varphi(nb+A), b+A\big)=b+A$. Thus nE_t is split exact and nE is torsion split.

THEOREM 9 (a). *Let $n\in Z$. If $A[n]=0$ or $(B/A)[n]=0$, then $E: 0\to A\to B\to B/A\to 0$ is such that nE is pure if and only if $E': 0\to A\to A+nB\to n(B/A)\to 0$ is pure.*

PROOF. Suppose $nE: 0\to A\to H\to B/A\to 0$, where $H=\{(b, b_1+A)\,|\,b+A=nb_1+A\}$, is pure. If $A[n]=0$, then $E'=E_n$, and by Theorem 1 (a), E' is pure. Suppose nE is pure and $(B/A)[n]=0$. Let $m\in Z$ and $m(nb+a)\in m(A+nB)\cap A$. Then $mnb\in A$ and hence $mb+A\in(B/A)[n]$. But $(B/A)[n]=0$ and therefore $mb\in A$. Therefore $m(nb+a, b+A)\in mH\cap A$. Since nE is pure, there exists an $a'\in A$ such that $m(nb+a, b+A)=m(a', 0)$, that is, $m(nb+a)=ma'$. Therefore E' is pure. The converse is contained in Theorem 8 (a).

THEOREM 9 (b). *Let $n\in Z$. If $A[n]=0$ or $(B/A)[n]=0$, then $E: 0\to A\to B\to B/A\to 0$ is such that nE is torsion split if and only if $E': 0\to A\to A+nB\to n(B/A)\to 0$ is torsion split.*

PROOF. Suppose $nE: 0 \to A \to H \to B/A \to 0$, where $H = \{(b, b_1 + A) \mid b + A = nb_1 + A\}$, is torsion split. That is, $nE_t: 0 \to A_t \to H_t \xrightarrow{f} (B/A)_t \to 0$ is split exact. If $A[n] = 0$, then as above $E' = E_n$ and by Theorem 1 (b), E' is torsion split. Suppose nE is torsion split and $(B/A)[n] = 0$. Let $E'_t: 0 \to A_t \to (A + nB)_t \xrightarrow{\alpha} (n(B/A))_t \to 0$. Since nE is torsion split there exists a homomorphism $\varphi: (B/A)_t \to H_t$ such that for each $b + A \in (B/A)_t$, $f\varphi(b + A) = b + A$. Define $\psi: (n(B/A))_t \to (A + nB)_t$ by $\psi(nb + A) = nb + a$, where $\varphi(b + A) = (nb + a, b + A)$. Suppose $nb + A = nb_1 + A \in (n(B/A))_t$. Then $n(b - b_1) \in A$, that is, $(b - b_1) + A \in (B/A)[n]$. But $(B/A)[n] = 0$, and hence $b - b_1 \in A$, that is, $b + A = b_1 + A$. Therefore $\varphi(b + A) = \varphi(b_1 + A)$ and ψ is well defined. ψ is a homomorphism since φ is is a homomorphism and $\alpha\psi(nb + A) = \alpha(nb + a) = nb + A$ for all $nb + A \in (n(B/A))_t$. Therefore E'_t is split exact and E' is torsion split. The converse is contained in Theorem 8 (b).

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References

[1] D. BUCHSBAUM, A note on homology in categories, *Ann. of Math.*, **69** (1958), pp. 66—74.
[2] M. C. R. BUTLER and G. HORROCKS, Classes of extensions and resolutions, *Phil. Trans. Royal Soc. London*, Ser A, **254** (1961), pp. 155—222.
[3] R. J. NUNKE, Homology and direct sums of countable Abelian groups, *Math. Zeitschr.*, **101** (1967), pp. 182—212.
[4] C. P. WALKER, Properties of Ext and quasi-splitting of Abelian groups, *Acta Math. Acad. Sci. Hung.*, **15** (1964), pp. 157—160.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (3—4), (1974), pp. 329—333.

ОБОБЩЕННАЯ ОДНОЛИНЕЙНАЯ СИСТЕМА МАССОВОГО ОБСЛУЖИВАНИЯ С НЕУЧЕТОМ ЗАТРАЧЕННОЙ ДО ПРЕРЫВАНИЯ РАБОТЫ

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Продолжим начатое в [2] исследование приоритетных систем массового обслуживания, функционирование которых описывается кусочно-линейным марковским процессом. Будем рассматривать систему массового обслуживания, в которой для требований второго типа затраченная на отрезке (ν, μ) работа в случае поступления требования первого типа не учитывается, а после окончания обслуживания требования первого типа с вероятностью 1 вернемся к началу отрезка (ν, μ) и все обслуживание начинается заново. При этом не ограничимся случаем одного прерывания, а считаем возможным любое число прерываний.

Как известно по [1] время пребывания в нулевом состоянии для требований первого типа имеет показательное распределение, так распределение длительности промежутка от момента начала цикла порядка ν до поступления требования первого типа также является показательным. Пусть необходимая для перехода из состояния ν в состояние μ работа равна x, а функция распределения необходимой для этого работы будет $H_\nu(x)$. Вероятность того, что будет именно n прерываний равносильна вероятности события, в n случаях обслуживание прерывается перед выполнением x единиц работы, а после этого в течение времени x/α_ν поступления требования первого типа не будет. Вышеупомянутая вероятность равна

$$[1-\exp(-\lambda_0 x/\alpha_\nu)]^n \exp(-\lambda_0 x/\alpha_\nu). \tag{1}$$

В дальнейшем будем следовать по схеме, описанной в [1]. Там сначала была определена $l_{\nu\mu}$-вероятность перехода из состояния ν в состояние μ, $|\nu|-1 \leqq |\mu| \leqq |\nu|$. Вычислим $l_{\nu\mu}$ при условии, что было i прерываний. Вероятность условия определяется по (1), а вероятность после выполнения x единиц работы находится в состоянии μ равна

$$\sum_{|\nu|=|\mu'|} p_{\mu'\mu}\left[1-e^{-\frac{\lambda_0}{\alpha_\nu}x}\right]^i e^{-\frac{\lambda_0}{\alpha_\nu}x} l_{\nu\mu'}(x).$$

$l_{\nu\mu i}$ при таком условии получим очевидно интегрированием по x.

В дальнейшем будем рассматривать один приближенный метод. Если нас удовлетворяет знание характеристик с некоторой вероятностью, тогда можно пользоваться следующей схемой. В практических приложениях можем считать, что достаточно ограничиться для числа прерываний некоторым n, происшест-

вие больше чем n прерываний можно считать практически невозможным. Выбираем n из условия

$$\int_0^\infty \sum_{i=0}^{n} [1-\exp(-\lambda_0 x/\alpha_v)]^i \exp(-\lambda_0 x/\alpha_v)\, dH_v(x) \geqq p$$

где p—некоторый наперед заданный уровень вероятности.

В состоянии v' мы можем находиться до момента прерывания, между прерываниями и в течение выполнения x единиц работы, когда уже прерывания не будет. Определим $l_{v\mu} t^{v'}_{v\mu}$ при условии, что было именно i прерываний.

Имеет место формула:

$$t^{v'}_{v\mu i} l_{v\mu} = \sum_{|\mu'|=|v|} p_{\mu'\mu} \int_0^\infty \left\{ \int_0^x \frac{\frac{\lambda_0}{\alpha_v} \exp\left(-\frac{\lambda_0}{\alpha_v} u\right)}{1-\exp\left(-\frac{\lambda_0}{\alpha_v} x\right)} \cdot i \cdot \int_0^u \left[\frac{1}{\alpha_{v'}} \cdot l_{vv'}(y) \times \right.\right.$$

$$\times \Big(1 - \sum_{\substack{\varepsilon=\mu' \\ |\varepsilon|=|v|-1}} l_{v'\varepsilon}(u-y)\Big)^- + \sum_{|v_1|=|v_2|=|v|} \frac{L_{v_1 v_2} T^{v'}_{v_1 v_2}}{\alpha_{v_1}} l_{vv_1}(y) \cdot \Big(1 - \sum_{\substack{\varepsilon=\mu' \\ |\varepsilon|=|v|-1}} l_{v_2\varepsilon}(u-y)\Big) \Big] \times$$

$$\times dy \cdot du + \int_0^\infty \left[\frac{1}{\alpha_{v'}} l_{vv'}(y) l_{v'\mu'}(x-y) + \right.$$

$$\left.\left. + \sum_{|v_1|=|v_2|=|v|} \frac{L_{v_1 v_2} T^{v'}_{v_1 v_2}}{\alpha_{v_1}} l_{vv_1}(y) l_{v_2\mu'}(x-y) \right] dy \right\} \cdot \left(1 - e^{-\frac{\lambda_0}{\alpha_v} x}\right)^i e^{-\frac{\lambda_0}{\alpha_v} x}\, dH_v(x).$$

В состоянии v' мы можем находиться с одной стороны в течечие времени обслуживания, исходящего из состояния v и закончившегося поступлением требования первого типа, т. е. прерыванием в обслуживании. Такая картина повторяется i раз. С другой стороны в $i+1$-ом цикле прерывания не будет, так имеет место формула, приведенная в [1]. А нужная нам величина равна сумме двух этих величин.

В этой формуле u случайная величина, имеющая усеченное показательное распределение с правой точкой усечения x:

$$F(u) = \frac{1 - e^{-\frac{\lambda_0}{\alpha_v} u}}{1 - e^{-\frac{\lambda_0}{\alpha_v} x}} \qquad (0 \leqq u < x).$$

$F(u)$—функция распределения затраченной работы для обслуживания требования второго типа на отрезке (v, μ) до момента поступления требования первого типа. Очевидно после выполнения $u (u < x)$ единиц работы мы будем находиться в некоторый промежуточный момент при обслуживании, нам

нужно установить, в каком состоянии мы можем быть при этом. С одной стороны очевидно, что мы не можем быть в состоянии μ', в таком случае наш цикл кончился бы после выполнения меньше чем необходимых по условию x единиц работы. С другой стороны в силу данной в [1] интерпретации кусочно-линейных процессов в некоторый промежуточный момент количество необходимых операций для возврата в некоторое состояние μ (исходным состоянием является ν и выполняется соотношение $|\nu|=|\mu|$), будет некоторым неотрицательным числом. Так в такой момент мы можем находитсья лишь в таком состоянии ε, для которого $|\varepsilon|\geqq|\nu|$. Система дифференциальных уравнений для $l_{\nu\mu}(x)$, данная в [1] определяет $l_{\nu\mu}(x)$ для таких μ, для которых $|\nu|-1\leqq|\mu|\leqq|\nu|$. Таким образом если имело бы место $|\varepsilon|=|\nu|-1$, это означало бы окончание нашего цикла, что по условию не имеет места: для вероятности перехода из состояния ν' (или ν_1) в состояние ε, для которого выполняются вышесказанные требования, будет справедлива формула:

$$1-\sum_{\substack{\varepsilon=\mu'\\|\varepsilon|=|\nu|-1}} l_{\nu'\varepsilon}(u-y)$$

где условия суммирования рассматриваются независимо друг от друга. Наконец

$$\left(1-e^{-\frac{\lambda_0}{\alpha_\nu}x}\right)^i e^{-\frac{\lambda_0}{\alpha_\nu}x}$$

вероятность осуществления именно i прерываний.

Дальше мы можем следовать по схеме, приведенной в [1], отличие будет лишь в последней формуле. Здесь мы должны рассмотреть в отдельности случаи $0, 1, \ldots, n$ прерываний. $l_{\nu\mu}$, $t^{\nu'}_{\nu\mu i}$, $\tilde{L}_{\mu'\mu}$ для нас известны, а последняя формула имеет вид:

$$\tilde{L}_{\nu\mu}\tilde{T}^{\nu'}_{\nu\mu}=l_{\nu\mu}t^{\nu'}_{\nu\mu n}+\sum_{|\mu'|=|\nu|}\sum_{i+j=n} l_{\nu\mu'}L_{\mu'\mu}(t^{\nu'}_{\nu\mu i}+T^{\nu'}_{\mu'\mu j}).$$

Здесь мы должны последовательно перебрать все значения числа прерываний от нуля до n.

Такой подход может оказаться в некоторой степени неудобным. Так в последнем случае нам необходимо решать n систем уравнений, с другой стороны в случае медленной сходимости

$$\int_0^\infty \sum_{i=0}^{n}\left(1-e^{-\frac{\lambda_0}{\alpha_\nu}x}\right)^i e^{-\frac{\lambda_0}{\alpha_\nu}x}\,dH_\nu(x)$$

n может оказаться слишком большим, так в случае ограничения малым n не можем достаточной полнотой охарактеризовать нашу систему. Ниже определим нужные нам характеристики при поступлении любого числа требований первого типа.

Для $l_{\nu\mu}$ остаются в силе уже приведенные рассуждения, мы сразу же начнем с определения $l_{\nu\mu}\, t^{\nu'}_{\nu\mu}$.

$$t^{\nu'}_{\nu\mu} l_{\nu\mu} = \sum_{|\mu'|=|\nu|} p_{\mu'\mu} \int_0^\infty \sum_{i=0}^{\infty} \left\{ \int_0^x \frac{\frac{\lambda_0}{\alpha_\nu} \exp\left(-\frac{\lambda_0}{\alpha_\nu} u\right)}{1-\exp\left(-\frac{\lambda_0}{\alpha_\nu} x\right)} \cdot i \cdot \int_0^u \left[\frac{1}{\alpha_{\nu'}} \cdot l_{\nu\nu'}(y) \times \right.\right.$$

$$\times \Big(1 - \sum_{\substack{\varepsilon=\mu' \\ |\varepsilon|=|\nu|-1}} l_{\nu'\varepsilon}(u-y)\Big) + \sum_{|\nu_1|=|\nu_2|=|\nu|} \frac{L_{\nu_1\nu_2} T^{\nu'}_{\nu_1\nu_2}}{\alpha_{\nu_1}} l_{\nu\nu_1}(y) \times$$

$$\left.\left. \times \Big(1 - \sum_{\substack{\varepsilon=\mu' \\ |\varepsilon|=|\nu|-1}} l_{\nu'\varepsilon}(u-y)\Big) \right] dy \cdot du + \int_0^\infty \left[\frac{1}{\alpha_{\nu'}} l_{\nu\nu'}(y) l_{\nu'\mu'}(x-y) + \right.\right.$$

$$\left.\left. + \sum_{|\nu_1|=|\nu_2|=|\nu|} \frac{L_{\nu_1\nu_2} T^{\nu'}_{\nu_1\nu_2}}{\alpha_{\nu_1}} l_{\nu\nu_1}(y) l_{\nu_2\mu'}(x-y) \right] dy \right\} \cdot (1 - e^{-\frac{\lambda_0}{\alpha_\nu} x})^i e^{-\frac{\lambda_0}{\alpha_\nu} x} \, dH_\nu(x) =$$

$$= \sum_{|\mu'|=|\nu|} p_{\mu'\mu} \int_0^\infty \left\{ e^{\frac{\lambda_0}{\alpha_\nu} x} \int_0^x \frac{\lambda_0}{\alpha_\nu} \exp\left(-\frac{\lambda_0}{\alpha_\nu} u\right) \cdot \int_0^u \left[\frac{1}{\alpha_{\nu'}} l_{\nu\nu'}(y) \times \right.\right.$$

$$\times \Big(1 - \sum_{\substack{\varepsilon=\mu' \\ |\varepsilon|=|\nu|-1}} l_{\nu'\varepsilon}(u-y)\Big) + \sum_{|\nu_1|=|\nu_2|=|\nu|} \frac{L_{\nu_1\nu_2} T^{\nu'}_{\nu_1\nu_2}}{\alpha_{\nu_1}} l_{\nu\nu_1}(y) \Big(1 -$$

$$\left.\left. - \sum_{\substack{\varepsilon=\mu' \\ |\varepsilon|=|\nu|-1}} l_{\nu'_2\varepsilon}(u-y)\Big) \right] dy\, du + \int_0^\infty \left[\frac{1}{\alpha_{\nu'}} l_{\nu\nu'}(y) l_{\nu'\mu'}(x-y) + \right.\right.$$

$$\left.\left. + \sum_{|\nu_1|=|\nu_2|=|\nu|} \frac{L_{\nu_1\nu_2} T^{\nu'}_{\nu_1\nu_2}}{\alpha_{\nu_1}} l_{\nu\nu_1}(y) l_{\nu_2\mu'}(x-y) \right] dy \right\} dH_\nu(x).$$

В дальнейшем вычисление характеристик осуществляется в полном соответствии с схемой, описанной в [1]. Таким образом на поставленный в [1] вопрос и в случае описанной приоритетной системы можем дать положительный ответ.

На первый взгляд может оказаться весьма непривычным, что мы все время исходим из необходимой для обслуживания в случае непрерывания работы, когда ясно, что в силу прерываний и неучета затраченной на обслуживание работы до прерывания, необходимая для обслуживания работа будет равна или больше x. Даже можем различать работу, необходимую для обслуживания требования первого типа и для требований второго типа. При наличии n прерываний работа, затраченная на обслуживание требования второго

типа очевидно складывается из n усеченно — показательно распределенных случайных величин с функцией распределения

$$F(u) = \frac{1 - \exp\left(-\frac{\lambda_0}{\alpha_\nu} u\right)}{1 - \exp\left(-\frac{\lambda_0}{\alpha_\nu} x\right)} \quad (0 \leqq u < x).$$

и из одной случайной величины с функцией распределения $H_\nu(x)$. Эта последняя величина означает работу, которую нужно выполнить при непоступлении требования первого типа. Свертки этих распределений задают нам функции распределения в случаях осуществления именно n прерываний и взяв эти свертки с весами

$$\left[1 - \exp\left(-\frac{\lambda_0}{\alpha_\nu} x\right)\right]^n \exp\left(-\frac{\lambda_0}{\alpha_\nu} x\right)$$

получим функцию распределения работы, затраченной вообще на обслуживание требований второго типа. Для исследования вышеописанной системы массового обслуживания для нас существенным явилось то обстоятельство, что на обслуживание требования второго типа (для перехода из состояния ν в состояние μ без прерывания) необходима была работа, равная x единицам и в случае прерывания сразу же восстановились первоначальные условия. Отсюда следует, что вероятность перехода не зависит от того сколько у нас было прерываний, а лишь от необходимой для полного обслуживания работы, когда уже прерывания не будет. Данная для $t'_{\nu\mu i}\, l_{\nu\mu}$ формула уже содержит в себе индикатор осуществления именно i прерываний.

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Литература

[1] Б. В. Гнеденко, И. Н. Коваленко, *Введение в теорию массового обслуживания*, изд. «Наука», 1966.

[2] Л. Лакатош, Обобщение теоремы об аналитическом виде характеристик обобщенных однолинейных систем массового обслуживания для одного типа приоритетных систем, *Acta Math. Acad. Sci Hung.*, **25** (1974), стр. 55—60.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (3—4), (1974), pp. 335—340.

APPLICATIONS OF PRODUCT COLOURING

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Introduction

It will be shown that under some (rather restrictive) conditions the n-colourings of the (weak) direct product (cardinal product, Kronecker product, conjunction) are only those trivially induced by n-colourings of the factors. This result will be used to show that there are uniquely n-colourable graphs without short odd cycles and to prove the following theorem: A graph G can be embedded into a line-critical n-chromatic graph if and only if the contraction of any edge of G results in an $(n-1)$-colourable graph.

1. Notation and terminology

The graphs considered in this paper have neither loops nor multiple edges. Given a collection G_α, $\alpha \in A$, of graphs we define their *direct product,* denoted by $\prod_{\alpha \in A} G_\alpha$, as the graph whose vertex set is the cartesian product of the $V(G_\alpha)$ with two vertices adjacent if and only if they are adjacent in every component. Denote a vertex of $\prod_{\alpha \in A} G_\alpha$ by $\langle a_\alpha \rangle$ and a product $\prod_{\alpha \in A} G_\alpha$ where $G_\alpha \cong G_\beta \cong G$ for every $\alpha, \beta \in A$ by $\prod_A G$. Let $\varkappa(G)$ denote the chromatic number of G. If γ is an n-colouring of G then $\gamma(v)$ will denote the colour of v (which will always be an integer between 1 and n).

2. Colourings of products

Let γ be an n-colouring of G_{α_0} for some $\alpha_0 \in A$, then γ induces an n-colouring of $\prod_{\alpha \in A} G_\alpha$ in the natural way, giving each point of the colour the colour of its α_0-component. Here the question arises

(a) Are these all the n-colourings of the product? Also, this observation yields

$$\chi\left(\prod_{\alpha \in A} G_\alpha\right) \leqq \min_{\alpha \in A} \chi(G_\alpha).$$

(b) Does here the equality hold if A is finite?

Question (b) is, as far as we know, unanswered. The answer to question (a) is negative in general. However, we are going to give positive answer to (a) in some special cases.

THEOREM 1. *Suppose G is a connected graph such that any two n-colourings of G differ in at least two points. Then each n-colouring of $K_n \times G$ is induced by an n-colouring of one of the factors.*

PROOF. We distinguish two cases:

Case 1: There is an $x \in V(G)$ such that $\beta(1, x), \ldots, \beta(n, x)$ are different. Let y be adjacent to x in G. Then $\beta(i, y) \neq \beta(j, x)$ for $i \neq j$, whence $\beta(i, y) = \beta(i, x)$. As G is connected we conclude that $\beta(i, x)$ is independent of x, i.e. it is induced by an n-colouration of K_n.

Case 2: For all $x \in V(G)$, there are $i \neq j$ such that $\beta(i, x) = \beta(j, x)$. Denote this colour by $\gamma'(x)$. Fix a point $x \in V(G)$ and define

$$\gamma_k(y) = \begin{cases} \gamma'(y) & \text{if} \quad y \neq x \\ \beta(k, x) & \text{if} \quad y = x. \end{cases}$$

Now γ_k is an n-colouring of G. For let u, v be adjacent vertices of G, then we may assume $u \neq x$. Then $\gamma_k(u) = \gamma'(u) = \beta(i, u) = \beta(j, u)$ for some $i \neq j$. Moreover, $\gamma_k(v) = \beta(m, v)$ for some m, no matter if $v = x$ or not. We may assume $i \neq m$. Then (i, u) and (m, v) are adjacent in the product and so

$$\gamma_k(u) = \beta(i, u) \neq \beta(m, v) = \gamma_k(v).$$

The n-colourings $\gamma_1, \ldots, \gamma_n$ differ only in x; therefore, by our assumption on G, they are identical. Hence $\beta(i, x)$ is the name for every i, which proves that β is induced by the n-colouring γ_k of G.

COROLLARY 1. *If $\chi(G) > n$ and G is connected then $K_n \times G$ is uniquely n-colourable.*

Now if G has no odd cycles shorter than a given s, then, as it is easily checked, neither does $K_n \times G$. So using the fact that such graphs G exist for any n and s (which is a consequence of a theorem of ERDŐS [1], but also follows by much simpler constructions, see e.g., ERDŐS—HAJNAL [2]) we get

COROLLARY 2. *For all $n \geq 3$, there is a uniquely n-colourable graph without odd cycles shorter than a given number s.*

This result improves the results of HARARY, HEDETNIEMI and ROBINSON [3] and NEŠETŘIL [4].

It is easy to see that $(K_n)^k = \prod_{\{1, \ldots, k\}} K_n$ is connected for $n \geq 3$, so by induction on k, we have the following useful corollary to Theorem 1.

COROLLARY 3. *$(K_n)^k$ has exactly k n-colourings.*

One can give the following formulation to this result: Given k switches, each having n positions, a traffic signal has n lights of different colours, and, depending on the position of the switches, always one of these lights is on. We know that if we change the position of *all* switches, the colour of the traffic signal changes. Then the signal depends on one of the switches only.

The n^{k-1} points of $(K_n)^k$ with one component fixed are independent. This is the most number of independent points of $(K_n)^k$ and these are the only maximum independent sets.

THEOREM 2. *The independence number of* $(K_n)^k$ *is* n^{k-1} *and furthermore the only maximum independent sets are those obtained by fixing one component.*

PROOF. We illustrate the proof of the theorem with the case $n=10$ and $k=6$. Consider any odometer from a '67 Buick. Any fixed position of it represents a point of $(K_{10})^6$, two of which are adjacent if they have different digits on each place. Fixing the relative positions of the dials only, we get 10 pairwise adjacent points. By fixing the sixth digit and looking at the 10^5 different positions of the other digits we obtain a partitioning of the points of $(K_{10})^6$ into 10^5 sets each of which induces a complete subgraph. Thus the independence number must be 10^5. Furthermore, given any maximum independent set S we can obtain a 10-colouring of the graph whose colour classes are $C_i=\{(a_\alpha+i):(a_\alpha)\in S\}$ where the addition is modulo n (with 0 replaced by n). Thus from Corollary 3 of Theorem 1 the points of S must have one component fixed.

There are at least two other proofs of Corollary 3. One of them is to prove Theorem 2 directly, which implies Corollary 3 immediately. The other proof generalizes to the infinite powers of K_n; we will give this generalization in section 4.

3. Forbidden subgraphs

This section is concerned with describing the forbidden subgraphs of critically n-chromatic graphs. For a graph G and an edge e of G we will denote by Ge the contraction of e in G. In a line critically n-chromatic graph G any edge when contracted gives a graph which is $n-1$ colourable. Therefore if a graph G contains an edge e such that $\chi(Ge)\geqq n$ then G is a forbidden subgraph of line critically n-chromatic graphs. This gives the easy direction of the following theorem.

THEOREM 3. *Let* $n\geqq 4$. *If* G *contains an edge* e *such that* $\chi(Ge)\geqq n$ *then* G *is a forbidden subgraph for a graph to be line critically* n*-chromatic. Otherwise* G *can be extended to a line critical* n*-chromatic graph. Furthermore any graph* G *with* $\chi(G)<n$ *can be extended to a point critical* n*-chromatic graph.*

PROOF. If for any edge e in G we have $\chi(Ge)<n$, then for each edge e_i, $1\leqq i\leqq$ $\leqq|E(G)|=q$, there is an $(n-1)$-colouring α_i of $G-e_i$ which colours the endpoints of e_i the same colour. By the corollary of theorem 1 the graph $(K_{n-1})^q$ has exactly $q\,(n-1)$-colourings. Denote them by β_i, $1\leqq i\leqq q$, where β_i colours any point with k in the i^{th} component colour k. We construct the graph K by taking the union of G and $(K_{n-1})^q$ and adding any edge between a point of G and a point of $(K_{n-1})^q$ which are not coloured the same in any of the colourings $\alpha_i\cup\beta_i$, $1\leqq i\leqq q$.

We claim now that for a fixed colouring β_i of $(K_{n-1})^q$ in K and any point v of G in K and any colour k different from $\alpha_i(v)$, there is a point of $(K_{n-1})^q$ in K adjacent to v with colour k. This is easy to see since the point with k in the i^{th} component and $\alpha_j(v)+1$ (addition modulo $n-1$ with 0 replaced by $n-1$) in the j^{th} component for $j\neq i$, is coloured k by β_i and is not coloured $\alpha_j(v)$ by β_j for any $j\neq i$.

Now it is easy to see that K is n-chromatic. For if there was an $(n-1)$-colouring

γ of K, this would coincide with one of the β_i's $(1 \leqq i \leqq q)$ on $(K_{n-1})^q$ by Corollyra 3 and thus, it would coincide with α_i on G by the previous claim.

Moreover, any edge of G is critical in K for $K-e_i$ has the $(n-1)$-colouring $\alpha_i \cup \beta_i$.

Hence, considering a line-critical n-chromatic subgraph of K, this contains G. This proves the theorem.

The construction also yields a

COROLLARY. *If G is a subgraph of a line-critical n-chromatic graph then it is an induced subgraph of a line-critical n-chromatic graph.*

A simpler construction can be given if $\varkappa(G) \leqq n-2$. Take a complete $(n-2)$-partite graph whose colour classes all contain $2m+1$ points. For a large enough value of m, G will be a subgraph of such a graph. Denote it by H and its colour classes by H_i, $1 \leqq i \leqq n-2$. We construct a graph K by taking the union of H and $n-2$ cycles C_i, $1 \leqq i \leqq n-2$, each of length $2m+1$ and we join each C_i with H_j, for $i \neq j$, with $2m+1$ independent edges. It is easy to verify that K is line critically n-chromatic.

Finally if $\chi(G-v) < n$ for all $v \in V(G)$ then G can be extended to a point critical n-chromatic graph. The construction is similar to the first construction in this proof. We have an $(n-1)$-colouring, α_i, of $G-v_i$ for each $1 \leqq i \leqq |V(G)| = p$. Let β_i, $1 \leqq i \leqq p$, be the $(n-1)$-colourings of $(K_{n-1})^p$. Define K to be the union of G and $(K_{n-1})^p$ together with any edge from a point of G to a point of $(K_{n-1})^p$ which are not coloured the same in any of the colourings $\alpha_i \cup \beta_i$. It is easy to see that for any $(n-1)$-colouring, β_i, of $(K_{n-1})^p$ in K the point v_i of G in K is forced to be coloured the n^{th} colour. Thus $\chi(G) = n$ and clearly $\alpha_i \cup \beta_i$ is an $n-1$ colouring of $G-v_i$ for all i. Hence G is a subgraph of a point critical subgraph of K.

4. The colourings of $\prod_A K_n$

As $\prod_A K_n$ contains a complete n-graph (the diagonal), its chromatic number is n. However, in the case when A is infinite, it always has n-colourings other than those induced by n-colourings of the factors. This was observed by L. PÓSA.

A $0-1$ *measure* on the set A is a function μ associating 0 or 1 with the subsets of A, and satisfying

(i) $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$ if $A_1 \cap A_2 = \varnothing$.

(ii) $\mu(A) = 1$.

The measure is trivial if there is a point $a \in A$ such that $\mu(\{a\}) = 1$ (i.e. $\mu(X) = 1$ iff $a \in X$). On a finite A, all measures are trivial but if A is infinite, there are non-trivial measures.

Let μ be a measure on A, and β an n-colouration of G_α $(\alpha \in A)$. Then define an n-colouration of $\prod_{\alpha \in A} G_\alpha$ as follows. Consider a point (a_α) and let $S_i = \{\alpha : \beta_\alpha(a_\alpha) = i\}$. As $\{S_1, \ldots, S_n\}$ is a finite partition of A, exactly one S_k has measure 1. Colour then (a_α) by k. This is a legitimate colouring as two sets of measure 1 have to intersect. We call this colouring induced by the β_α's and μ.

THEOREM 4.* *Each n-colouring of* $\prod_A K_n$ $(n \geqq 3)$ *is induced by n-colourings of the factors and a measure* μ *of* A.

PROOF. We use induction on n. Suppose first $n=3$. Let γ be an n-colouring of $\prod_A K_3$.

Let X be a subset of A and let $P_i(X)=\{(a_\alpha): a_\alpha=i$ for all $\alpha\in X$ and $a_\alpha\neq i$ for all $\alpha\notin X\}$, $1\leqq i\leqq 3$. Define $\mu(X)=1$ if and only if the points of $P_i(X)$ are coloured the same for at least two values of i.

(1) If $\mu(X)=1$ then $\mu(A-X)=0$.

Since $\mu(X)=1$ we know the points of $P_i(X)$ are coloured the same for say $i=1$ and 2. It is easy to see that this also implies that the points of $P_3(X)$ are coloured the same and that all three sets of points are coloured different. Hence the points $P\in P_1(X)$ with $a_\alpha=3$ for $\alpha\notin X$ and $Q\in P_2(X)$ with $a_\alpha=3$ for $\alpha\notin X$ are points of $P_3(A-X)$ which are coloured different colours. We similarly find points of $P_2(A-X)$ with different colours. Thus $\mu(A-X)=0$.

(2) If $\mu(X)=0$ then $\mu(A-X)=1$.

Since $\mu(X)=0$ there are points $P, Q\in P_1(X)$ and $P', Q'\in P_2(X)$ which receive different colours. Since every point of $P_1(A-X)$ is adjacent to P and Q they must all be coloured the same colour. Likewise every point of $P_2(A-X)$ is adjacent to P' and Q', and so must be coloured the same colour. Therefore $\mu(A-X)=1$.

(3) If $\mu(X)=1$ and $X\subseteq X'$, then $\mu(X')=1$.

Suppose $\mu(X')=0$. Then there are points $P, Q\in P_1(X')$ which are coloured different. Define a point $S=(a_\alpha)$ by $a_\alpha=2$ if $\alpha\in X$, $a_\alpha=3$ if $\alpha\in X'-X$ and $a_\alpha=1$ if $\alpha\in A-X'$; similarly let $T=(a'_\alpha)$ be defined by $a_\alpha=3$ if $\alpha\in X$, $a_\alpha=2$ if $\alpha\in X'-X$ and $a_\alpha=1$ if $\alpha\in A-X'$. Then both S, T are joined to P, Q and thus have the third colour. As $\mu(X)=1$, all points of $P_2(X)\cup P_3(X)$ have this third colour and this is a contradiction as $P_2(X)\cup P_3(X)$ is not an independent set. Furthermore, by the same type argument, we can see that the points of $P_1(X)\cup P_1(X')$ must all be coloured the same.

(4) If $\mu(X_1)=\mu(X_2)=1$ then $\mu(X_1\cap X_2)=1$.

This follows by a similar argument.

From (2), (3) and (4) we see that μ is a $0-1$ measure on A and it is clear from the definition of μ and the last statement in the proof of (3) that it induces the given colouring. The general statement now follows by induction.

Suppose $\prod_A K_n$ is n-coloured $(n\geqq 4)$ and any $(n-1)$-colouring of $\prod_A K_{n-1}$ is induced by some $0-1$ measure. Now consider the points (a_α) of $\prod_A K_n$ for which $a_\alpha<n$ for all $\alpha\in A$. They induce $\prod_A K_{n-1}$ and are coloured with only $n-1$ colours since the point (a_α) with $a_\alpha=n$ for all α is adjacent to all of these points. Hence there is a measure μ which induces this colouring of $\prod_A K_{n-1}$. Furthermore μ induces the colouring of the whole graph. For let $P\in P_n(X)$ where $\mu(X)=1$. Clearly P is adjacent to some point of $\prod_A K_{n-1}\cap P_i(X)$ for each $i<n$ and so must be coloured with the only remaining colour. Therefore the points of $P_n(X)$ are all coloured the

* Theorem 4 was proved independently by L. LEMPERT.

same. Likewise we can argue that the points of $P_i(X)$ are coloured the same for any i and so we have that μ induces the colouring of $\coprod_A K_n$.

If A is finite it has only trivial $0-1$ measures and hence corollary 3 follows from theorem 4.

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References

[1] P. ERDŐS, Graph theory and probability II, *Canad. J. Math.*, **13** (1961), pp. 346—352.
[2] P. ERDŐS, A. HAJNAL, On chromatic graphs (Hungarian, with English summary), *Mat. Lapok*, **18** (1967), pp. 1—4.
[3] F. HARARY, S. T. HEDETNIEMI and R. W. ROBINSON, Uniquely colorable graphs, *J. Comb. Theory*, **6** (1969), pp. 271—278.
[4] J. NEŠETŘIL, On uniquely colorable graphs without short cycles, to appear in *Cas. Pest. Math.*

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (3—4), (1974), pp. 341—361.

ЛОКАЛЬНЫЕ ТЕОРЕМЫ О СХОДИМОСТИ ИНТЕРПОЛЯЦИОННОГО ПРОЦЕССА ЛАГРАНЖА С УЗЛАМИ В КОРНЯХ МНОГОЧЛЕНОВ ЭРМИТА

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§ 1. Введение

Исследованию точечной сходимости интерполяционного процесса Лагранжа посвящены сотни статей и десятки книг. Однако, все эти работы почти без исключения занимаются лишь тем случаем, когда узлы интерполирования содержатся в некотором фиксированном конечном интервале. По видимому первая работа в которой исследована проблема сходимости интерполяционного процесса Лагранжа на бесконечном интервале была работа Г. Фройда [5], где было доказано следующее утверждение.[1]

Пусть суммируемая на всей оси функция w удовлетворяет условиям

$$w(x) \cong c_1 e^{-c_2 x^2} \quad (-\infty < x < \infty)$$

и

$$\int_{-\infty}^{\infty} e^{c_3 x^2} w(x)\,dx < \infty$$

(c_1, c_2 и c_3 положительные константы). Если функция f равномерно непрерывна и ограничена на всей оси, далее

$$\lim_{\delta \to 0} \delta^{-\frac{1}{2}} \sup_{-\infty < x < \infty} |f(x+\delta) - f(x)| = 0,$$

то тогда для интерполяционного процесса Лагранжа $L_n(w, x, f)$, основанного на корнях многочленов, ортогональных с весом w, справедливо соотношение

$$\lim_{n \to \infty} \sup_{-\infty < x < \infty} e^{-\frac{c_2 x^2}{2}} |f(x) - L_n(w, x f)| = 0.$$

Немного позднее Г. Фройд в работе [6] рассматривал интерполяционный процесс Лагранжа с узлами в корнях многочленов Эрмита, и установил, что функции Лебега этого процесса имеют порядок $\log n$. При помощи этой оценки он нашел аналог признака Дини—Липшица, при дополнительном пред-

[1] Проблема сходимости в среднем была изучена уже раньше в таботах Й. Шохата [20], Й. Шохата — Й. Тамаркина [21], Й. Балажа — П. Турана [1], [2] и Г. Фройда [4]. В работах [19] и [20] Шохат сформулировал несколько утверждений о точечной сходимости процесса Лагранжа с узлами в корнях многочленов Эрмита и Лагерра, однако при доказательстве этих утверждений он допустил одну ошибку (см. [19], формулу (62) и [20], формулу (74)), следовательно его доказательства не проходят.

положении, что индуцирующая функция равномерно непрерывна и ограничена на всей оси.

Отправляясь от идей Г. Фройда [6] мы доказали в нашей работе [11] аналогичные утверждения для интерполяционного процесса Лагранжа с узлами в корнях многочленов Лагерра, а О. Киш в работе [8] для процесса Лагранжа с узлами в корнях многочленов Сонина—Маркова.[2]

Рассматривая затем процесс Лагранжа с узлами в корнях многочленов Эрмита, мы показали, в работе [12], что если индуцирующая функция f равномерно непрерывна и ограничена на всей вещественной оси и имеет ограниченное изменение на некотором конечном интервале, то внутри этого интервала процесс Лагранжа равномерно сходится[3] к функции f. Аналогичные утверждения для узлов Лагерра были установлены нами в работе [13].

Затем в работе [15] мы исследовали процесс Лагранжа с узлами в корнях многочленов Эрмита и Лагерра, и доказали теоремы о сходимости при постоянно присутствующем предположении, что индуцирующая функция f, кроме некоторых локальных свойств, должна быть равномерно непрерывной на всем интервале интерполирования. Мы доказали например, что если функция f на некотором интервале (a, b) удовлетворяет одностороннему условию Дини—Липшица:[4]

$$f(x+\delta)-f(x) \geqq -\frac{v(\delta)}{|\log \delta|} \quad (a < x < x+\delta < b),$$

где $v(\delta)$ такая неубывающая функция, для которой

$$\lim_{\delta \to +0} v(\delta) = 0,$$

то интерполяционный процесс Лагранжа с узлами в корнях многочленов Эрмита или Лагерра, индуцированный при помощи функции f, сходится равномерно к функции f внутри (a, b). Для функций, имеющих ограниченное Φ-изменение, мы также доказали теоремы о сходимости и нашли порядок скорости сходимости.

В настоящей работе мы собираемся сделать новый шаг, который в некотором смысле завершает исследования по теории интерполяционного процесса Лагранжа с узлами в корнях многочленов Эрмита, а именно, мы покажем, что если индуцирующая функция f интегрируема в смысле Римана на любом конечном интервале и не очень быстро растет к бесконечности, когда независимая переменная стремится к бесконечности, то сходимость процесса Лагранжа, индуцированного при помощи f, в некоторой точке x_0 зависит лишь от поведения функции f в окресности точки x_0.

[2] Многочленами Лагерра называются многочлены, которые ортогональны на положительной полуоси с весом $x^{\alpha}e^{-x}(\alpha > -1)$, а многочленами Сонина—Маркова называются те многочлены, которые ортогональны на всей вещественной оси с весом $|x|^{\alpha}e^{-x^2}(\alpha > -1)$.

[3] Значит, равномерная сходимость имеет место на любом отрезке, строго принадлежащем к данному интервалу.

[4] Понятие одностороннего условия Дини-Липшица было введено нами в работе [14].

При помощи этого принципа локализации мы получим несколько признаков для сходимости процесса Лагранжа, часть из которых хорошо известна из теории тригонометрических рядов Фурье, и перенос которых в теорию Лагранжева интерполирования на бесконечном интервале до сих пор казался безнадежным.

§ 2. Основные результаты

В настоящей работе мы будем рассматривать функции из класса R, который состоит из тех функций f, определенных на всей вещественной оси, которые на любом конечном отрезке интегрируемы в смысле Римана и при больших значениях абсолютной величины аргумента x растут не быстрее, чем $e^{\frac{\varepsilon x^2}{2}}$, где $\varepsilon=\varepsilon(f)$ некоторое положительное число, строго меньшее, чем 1. Так как функции, которые интегрируемы в смысле Римана на некотором промежутке, всегда ограничены на этом же промежутке, то мы в праве считать, что для всех f из R мы имеем

$$(1)\quad |f(x)| \leqq Ae^{\frac{\varepsilon x^2}{2}} \quad (-\infty < x < \infty, \quad f \in R, \quad A = A(f) > 0,\ 0 < \varepsilon = \varepsilon(f) < 1).$$

Обозначим через

$$h_n(x) = (-1)^n \frac{1}{\pi^{\frac{1}{4}} \sqrt{2^n n!}} e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad (n = 0, 1, 2, \ldots)$$

многочлены Эрмита, ортонормированные на вещественной оси с весом e^{-x^2}, то есть

$$\int_{-\infty}^{\infty} h_n(x) h_m(x) e^{-x^2} dx = \begin{cases} 0, & \text{при} \quad n \neq m \\ 1, & \text{при} \quad n = m \end{cases} \quad (n, m = 0, 1, 2, \ldots).$$

Пусть

$$(2)\qquad x_1 \equiv x_{1n} > x_2 > \ldots > x_n \quad (n = 1, 2, \ldots)$$

корни n-го многочлена Эрмита $h_n(x)$.

Далее, обозначим через $L_n(x, f)$ $(n=1, 2, \ldots)$ интерполяционные многочлены Лагранжа степени не выше $(n-1)$, совпадающие с заданной функцией f из класса R в узлах (2). Мы имеем таким образом

$$L_n(x, f) = \sum_{k=1}^{n} f(x_k) l_k(x) \quad (f \in R, \quad n = 1, 2, \ldots),$$

где

$$l_k(x) \equiv l_{kn}(x) = \frac{h_n(x)}{h_n'(x_{kn})(x - x_{kn})} \quad (k = 1, 2, \ldots, n)$$

фундаментальные многочлены Лагранжева интерполирования по узлам (2).

Наш основной результат, который уместно назвать принципом локализации для Лагранжева интерполирования по узлам Эрмита, гласит следующим образом:

Теорема 1. *Если* $f\in R$ *и* $f(x)\equiv 0$ *на отрезке* $[a-\delta, b+\delta]$ $(\delta>0)$, *то тогда*

$$\max_{a\leqq x\leqq b} e^{-\frac{x^2}{2}}|L_n(x,f)| = o(1) \quad (n=1,2,\ldots), \tag{3}$$

причем константа в «о»-члене зависит лишь от f *и* δ.

Естественно, что для класса R выражение $o(1)$ в (3) может как угодно медленно стремиться к 0 при $n\to\infty$. Однако для класса V, состоящего из тех функций f из R, для которых $\overset{\infty}{\underset{-\infty}{\operatorname{var}}} f(x)e^{-\frac{\varepsilon x^2}{2}}<\infty$ при некотором ε $(0<\varepsilon<1)$, мы сможем доказать следующее утверждение:

Теорема 2. *Если* $f\in V$ *и* $f(x)\equiv 0$ *на отрезке* $[a-\delta, b+\delta]$ $(\delta>0)$, *то тогда*

$$\max_{a\leqq x\leqq b} e^{-\frac{x^2}{2}}|L_n(x,f)| = O(1)\frac{1}{\sqrt{n}} \quad (n=1,2,\ldots), \tag{4}$$

где константа, входящий в «О»-член независит от n.

Возможно, что в соотношении (4) величина $O(1)$ также независит от отрезка $[a, b]$, однако в настоящее время мы не в состоянии этого показать.

При помощи выше сформулированного принципа локализации мы докажем следующие признаки для сходимости $L_n(x,f)$ к $f(x)$ в некоторой индивидуальной точке x_0.

Теорема 3. *Пусть* $f\in R$. *Если в некоторой* $\delta_1(>0)$ *окресности точки* x_0 *мы имеем*

$$|f(x_0\pm t)-f(x_0)|\leqq \mu(t) \quad (0\leqq t\leqq \delta_1) \tag{5}$$

где μ *такая неубывающая функция от* t, *что*[5]

$$\int_0^{\delta_1}\frac{\mu(t)}{t}\,dt<\infty, \tag{6}$$

то многочлены $L_n(x,f)$ *в точке* x_0 *сходятся к значению* $f(x_0)$ *при* $n\to\infty$.

Таким образом признак Дини—Натансона,[6] который хорошо известен из теории тригонометрического интерполирования по равноотстоящим узлам, переносится в теорию Лагранжева интерполирования по узлам Эрмита.

[5] Здесь имеется в виду несобственный интеграл Римана.
[6] См. работу И. П. Натансона [9].

Следствие 4. *Если* $f\in R$ *и в точке* x_0

$$|f(x_0+t)-f(x_0)| = O\left(\frac{1}{|\log|t||^{\alpha}}\right) \quad (|t|\leqq\delta_1,\ \alpha>1),$$

то

$$L_n(x_0, f) \xrightarrow[n\to\infty]{} f(x_0).$$

Самым глубоким признаком для сходимости — на наш взгляд — является

Теорема 5. *Пусть* $f\in R$. *Если функция* f *на отрезке* $[a-\delta, b+\delta]$ $(\delta>0)$ *удовлетворяет одностороннему условию Дини—Липшица*[7]

$$f(x+t)-f(x) \geqq -\frac{v(t)}{|\log t|} \quad (a-\delta<x<x+t<b+\delta) \tag{7}$$

$(v(t)\downarrow 0$ *при* $t\to+0)$, *то* $L_n(x,f)$ *сходится при* $n\to\infty$ *к значению* $f(x)$ *на отрезке* $[a, b]$ *во всех точках непрерывности функции* f, *и сходимость равномерна на всех отрезках* $[a_1, b_1]\subseteqq[a, b]$, *где* f *непрерывна.*

Следствие 6. *Если* $f\in R$ *и на отрезке* $[a-\delta, b+\delta]$ $(\delta>0)$ *она непрерывна с модулем непрерывности* $\omega(t)$, *далее*

$$\omega(t) = o\left(\frac{1}{|\log t|}\right) \quad (t\to+0),$$

то $L_n(x, f)$ *разномерно на* $[a, b]$ *сходится к значению* $f(x)$ *при* $n\to\infty$.

Следствие 7. *Если* $f\in R$ *и* $\operatorname{var}_{a-\delta}^{b+\delta} f<\infty$ $(\delta>0)$, *то* $L_n(x,f)\xrightarrow[n\to\infty]{} f(x)$ *во всех точках* x *из* $[a, b]$, *где функция* f *непрерывна, далее сходимость равномерно на всех отрезках* $[a_1, b_1]\subseteqq[a, b]$, *где* f *непрерывна.*

Пусть Φ некоторая непрерывная, определенная на неотрицательной полуоси, которая обладает свойствами $\Phi(0)=0$, $\Phi(\xi_1)\leqq\Phi(\xi_2)$ $(0\leqq\xi_1<\xi_2)$.

Мы будем говорить, что функция f, определенная на отрезке $[a, b]$, имеет ограниченное Φ-изменение на $[a, b]$, то есть $f\in V_{\Phi}[a, b]$, если

$$v_{\Phi}(f, a, b) = \sup_{\tau}\sum_{i}\Phi(|f(b_i)-f(a_i)|)<\infty,$$

где τ произвольная конечная система попарно непересекающихся интервалов (a_i, b_i), принадлежащих к (a, b) (см. Л. Юнг [23]).

Теорема 8. *Пусть* $f\in R$. *Если* $f\in V_{\Phi}[a-\delta, b+\delta]$ $(\delta>0)$ *и*

$$\int_0^1 |\log\Phi(\xi)|\,d\xi<\infty, \tag{8}$$

то $L_n(x,f)$ *сходится к значению* $f(x)$ *при* $n\to\infty$ *во всех точках* x *из* $[a, b]$, *где функция* f *непрерывна, далее слодимость равномерна на всех отрезках* $[a_1, b_1]\subseteqq[a, b]$, *где* f *непрерывна.*

[7] Отметим, что из (7) не вытекает непрерывность функции f на отрезке $[a, b]$.

Теорема 9. *Пусть* $[a, b]$ *некоторый отрезок,* $\delta > 0$ *и* χ *характеристическая функция отрезка* $[a-\delta, b+\delta]$, *значит* $\chi(t)=1$ *на* $[a-\delta, b+\delta]$ *и* $\chi(t)=0$ *вне* $[a-\delta, b+\delta]$. *Если функция* f *непрерывна на* $[a-\delta, b+\delta]$ *с модулем непрерывности* $\omega(t)$, *далее* $(1-\chi)f \in V$, *то тогда*

$$\max_{a \leqq x \leqq b} |f(x) - L_n(x, f)| = O(1) \log n \omega\left(\frac{1}{\sqrt{n}}\right) \quad (n = 1, 2, \ldots), \tag{9}$$

где константа, входящий в «O»-член независит от n.

§ 3. Доказательства

Обозначим через $\lambda_n(x)$ $(n=1, 2, \ldots)$ функции Кристоффеля веса e^{-x^2}, то есть

$$\lambda_n(x) = \left[\sum_{k=0}^{n-1} h_n^2(x)\right]^{-1} \quad (n = 1, 2, \ldots), \tag{10}$$

и пусть $\lambda_k \equiv \lambda_{kn} = \lambda_n(x_{kn})$ $(k=1, 2, \ldots, n)$ соответствующие числа Кристоффеля, о которых известно, что

$$\begin{aligned} &\lambda_k < \lambda_j \quad (0 \leqq x_j < x_k) \\ &\lambda_k > \lambda_j \quad (x_j < x_k \leqq 0) \end{aligned} \tag{11}$$

(см. Г. Сеге [22], (15.3.9)), далее[8]

$$\lambda_n(x)^{-1} \leqq c_4 n^{\frac{1}{2}} e^{x^2} \quad (-\infty < x < \infty) \tag{12}$$

и

$$\lambda_k \leqq c_5 n^{-\frac{1}{2}} e^{-x_k^2} \quad (|x_k| \leqq \sqrt{n}) \tag{13}$$

(см. Г. Фройд [6]). Как показали Й. Балаж и П. Туран в работе [1], справедливы соотношения

$$\sum_{k=1}^{n} \lambda_k e^{\varrho x_k^2} \leqq \int_{-\infty}^{\infty} e^{(\varrho-1)x^2} dx \quad (0 < \varrho < 1, \quad n = 1, 2, \ldots) \tag{14}$$

и

$$\sum_{k=1}^{n} \lambda_k x_k^2 e^{\varrho x_k^2} \leqq \int_{-\infty}^{\infty} x^2 e^{(\varrho-1)x^2} dx \quad (0 < \varrho < 1, \quad n = 1, 2, \ldots). \tag{15}$$

Между $l_k(x)$ и $\lambda_n(x)$ существуют определенные связи, а именно

$$\lambda_n(x)^{-1} = \sum_{k=1}^{n} \frac{l_k^2(x)}{\lambda_k} \quad (n = 1, 2, \ldots) \tag{16}$$

[8] c_4 и в дальнейшем c_5, c_6, ... означают некоторые положительные константы.

(см. например Г. Фройд [7], (1.4.7)) и

$$l_k(x) = \frac{(-1)^{k+1}}{\sqrt{2}} \frac{\sqrt{\lambda_k}}{x - x_k} h_n(x) \quad (k = 1, 2, \ldots, n), \tag{17}$$

которое непосредственно следует из формул (15.3.6) и (5.5.1) книги Г. Сеге [22]. Далее, в силу асимптотической формулы Планшереля—Роташа [17] мы имеем

$$|h_n(x)| \leqq c_6 n^{-\frac{1}{4}} e^{\frac{x^2}{2}} \quad (|x| \leqq \sqrt{n}), \tag{18}$$

а согласно одному неравенству Х. Крамера

$$|h_n(x)| \leqq c_7 e^{\frac{x^2}{2}} \quad (-\infty < x < \infty) \tag{19}$$

(см. Г. Бейтмен—А. Эрдейи [3], (10.18.19)). Отметим еще, что для корней x_k многочлена $h_n(x)$ мы имеем

$$c_8 \frac{1}{\sqrt{n}} \leqq x_k - x_{k+1} \leqq c_9 \frac{1}{\sqrt{n}} \qquad (|x_k| \leqq \sqrt{n}) \tag{20}$$

(см. Г. Фройд [6], формулы (10) и (11)), и

$$|x_k| \leqq \sqrt{2n+1} \quad (k = 1, 2, \ldots, n) \tag{21}$$

(см. Г. Сеге [22], (6.32.6)).

Лемма 10. *Пусть* $0 < \varepsilon < 1$ *и* $c > 0$. *Тогда*

$$\sup_{-\infty < x < \infty} e^{-\frac{x^2}{2}} \sum_{|x_k| > c\sqrt{n}} e^{\frac{\varepsilon x_k^2}{2}} |l_k(x)| = O(e^{-c_{10} n}) \quad (n = 1, 2, \ldots). \tag{22}$$

Доказательство. Подберем число ε_1 такое, что $\varepsilon_1 > 0$ и $\varepsilon + \varepsilon_1 < 1$. Тогда

$$e^{-\frac{x^2}{2}} \sum_{|x_k| > c\sqrt{n}} e^{\frac{\varepsilon x_k^2}{2}} |l_k(x)| \leqq e^{-\frac{x^2}{2}} e^{-\frac{\varepsilon_1 c^2 n}{2}} \sum_{k=1}^{n} e^{\frac{\varepsilon + \varepsilon_1}{2} x_k^2} |l_k(x)| \leqq$$

$$\leqq e^{-\frac{\varepsilon_1 c^2 n}{2}} \left[e^{-x^2} \sum_{k=1}^{n} \frac{l_k^2(x)}{\lambda_k} \sum_{k=1}^{n} e^{(\varepsilon + \varepsilon_1) x_k^2} \lambda_k \right]^{\frac{1}{2}},$$

значит в силу соотношений (12), (14) и (16) мы имеем

$$e^{-\frac{x^2}{2}} \sum_{|x_k| > c\sqrt{n}} e^{\frac{\varepsilon x_k^2}{2}} |l_k(x)| \leqq \sqrt{c_4} e^{-\frac{\varepsilon_1 c^2 n}{2}} n^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{(\varepsilon + \varepsilon_1 - 1) x^2} dx \quad (-\infty < x < \infty),$$

откуда вытекает неравенство (22).

Лемма 11. *Если* $0<\varepsilon<1$ и $0<c<1$, *то тогда равномерно по всем таким* x, *для которых* $|x|\geqq\sqrt{n}$, *мы имеем*

$$e^{-\frac{x^2}{2}}\sum_{|x_k|<c\sqrt{n}}e^{\frac{\varepsilon x_k^2}{2}}|l_k(x)|=O\left(\frac{1}{n^{\frac{1}{4}}}\right)\quad(|x|\geqq\sqrt{n},\ n=1,2,\ldots). \tag{23}$$

Доказательство. В силу (12), (17) и (19)

$$e^{-\frac{x^2}{2}}\sum_{|x_k|<c\sqrt{n}}e^{\frac{\varepsilon x_k^2}{2}}|l_k(x)|\leqq\frac{c_7\sqrt{c_4}}{\sqrt{2}(1-c)}\frac{1}{n^{\frac{1}{4}}}\sum_{|x_k|<c\sqrt{n}}\lambda_k e^{\frac{\varepsilon+1}{2}x_k^2}\leqq$$

$$\leqq\frac{c_7\sqrt{c_4}}{\sqrt{2}(1-c)}\frac{1}{n^{\frac{1}{4}}}\sum_{k=1}^{n}\lambda_k e^{\frac{\varepsilon+1}{2}x_k^2},$$

значит (23) следует из неравенства (14).

Лемма 12. *Пусть* $0<\varepsilon<1$ *и* $\delta>0$. *Тогда для всех, как угодно малых* $\eta>0$ *существует число* T *такое, что*[9]

$$e^{-\frac{x^2}{2}}\sum_{\substack{|x-x_k|>\delta\\ T\leqq|x_k|\leqq\sqrt{n}}}e^{\frac{\varepsilon x_k^2}{2}}|l_k(x)|<\eta\quad(|x|\leqq\sqrt{n},\ n=1,2,\ldots). \tag{24}$$

Доказательсво. Мы имеем в силу соотношений (13), (17) и (18)

$$e^{-\frac{x^2}{2}}\sum_{\substack{|x-x_k|>\delta\\ T\leqq|x_k|\leqq\sqrt{n}}}e^{\frac{\varepsilon x_k^2}{2}}|l_k(x)|\leqq\frac{c_6\sqrt{c_5}}{\sqrt{2}\delta}\frac{1}{\sqrt{n}}\sum_{T\leqq|x_k|\leqq\sqrt{n}}e^{\frac{\varepsilon-1}{2}x_k^2}\quad(|x|\leqq\sqrt{n}).$$

Но функция $t^2e^{\frac{\varepsilon-1}{2}t^2}$ имеет на всей оси конечный максимум M, значит

$$e^{-\frac{x^2}{2}}\sum_{\substack{|x-x_k|>\delta\\ T\leqq|x_k|\leqq\sqrt{n}}}e^{\frac{\varepsilon x_k^2}{2}}|l_k(x)|\leqq\frac{c_6\sqrt{c_5}M}{\sqrt{2}\delta}\frac{1}{\sqrt{n}}\sum_{T\leqq|x_k|\leqq\sqrt{n}}\frac{1}{x_k^2}\quad(|x|\leqq\sqrt{n}),$$

следовательно, имея в виду сначала левую, затем правую часть в неравенствах (20) и симметричность относительно точки 0 расположение нулей многочлена $h_n(x)$,

$$e^{-\frac{x^2}{2}}\sum_{\substack{|x-x_k|>\delta\\ T\leqq|x_k|\leqq\sqrt{n}}}e^{\frac{\varepsilon x_k^2}{2}}|l_k(x)|\leqq\frac{c_6\sqrt{2c_5}M}{\delta c_8}\sum_{T\leqq x_k\leqq\sqrt{n}}\frac{x_k-x_{k+1}}{x_k^2}\leqq$$

$$\leqq\frac{c_6\sqrt{2c_5}M}{\delta c_8}\int_{T-c_9\frac{1}{\sqrt{n}}}^{\infty}\frac{1}{t^2}\,dt=\frac{c_6\sqrt{2c_5}M}{\delta c_8}\frac{1}{T-c_9\frac{1}{\sqrt{n}}}\quad(|x|\leqq\sqrt{n}),$$

что может быть сколь угодно малым, если только T достаточно большое.

[9] Значит, суммирование распространяется на те индексы k, для которых все условия, сформулированные под знаком суммы, выполняются.

Лемма 13. *Если* $\delta > 0$ *и* $0 < \varepsilon < 1$, *то равномерно по всем таким* x, *для которых* $|x| \leqq \sqrt{n}$, *мы имеем*

$$(25)\quad e^{-\frac{x^2}{2}} \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} |e^{\frac{\varepsilon x_k^2}{2}} l_k(x) + e^{\frac{\varepsilon x_{k+1}^2}{2}} l_{k+1}(x)| = O\left(\frac{1}{\sqrt{n}}\right) \quad (|x| \leqq \sqrt{n}, \quad n = 1, 2, \ldots).$$

Доказательство. Ясно, что

$$(26)\quad e^{-\frac{x^2}{2}} \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} |e^{\frac{\varepsilon x_k^2}{2}} l_k(x) + e^{\frac{\varepsilon x_{k+1}^2}{2}} l_{k+1}(x)| \leqq e^{-\frac{x^2}{2}} \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} \left|e^{\frac{\varepsilon x_k^2}{2}} - e^{\frac{\varepsilon x_{k+1}^2}{2}}\right| |l_k(x)| +$$

$$+ e^{-\frac{x^2}{2}} \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} e^{\frac{\varepsilon x_{k+1}^2}{2}} |l_k(x) + l_{k+1}(x)| = e^{-\frac{x^2}{2}} \Sigma_1 + e^{-\frac{x^2}{2}} \Sigma_2.$$

Рассмотрим сначала выражение $e^{-\frac{x^2}{2}} \Sigma_1$. Из теоремы Лагранжа следует, что

$$e^{-\frac{x^2}{2}} \Sigma_1 \leqq e^{-\frac{x^2}{2}} \varepsilon \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} |x_k - x_{k+1}| |\theta_k| e^{\frac{\varepsilon\theta_k^2}{2}} |l_k(x)|$$

где $x_{k+1} \leqq \theta_k \leqq x_k$. Так как $|x_k| \leqq \sqrt{n}$, то в силу правой части неравенств (20) мы имеем

$$|\theta_k| \leqq |x_k| + \frac{c_9}{\sqrt{n}} \leqq x_k^2 + 1 + c_9,$$

и следовательно

$$\theta_k^2 \leqq x_k^2 + \frac{2|x_k| c_9}{\sqrt{n}} + \frac{c_9^2}{n} \leqq x_k^2 + c_9(2 + c_9).$$

Таким образом — применяя (20) — мы получим

$$e^{-\frac{x^2}{2}} \Sigma_1 \leqq e^{-\frac{x^2}{2}} c_9 \varepsilon e^{\frac{\varepsilon c_9 (2+c_9)}{2}} \frac{1}{\sqrt{n}} \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} (x_k^2 + 1 + c_9) e^{\frac{\varepsilon x_k^2}{2}} |l_k(x)|,$$

значит, согласно (12), (17) и (18), мы имеем, что при $|x| \leqq \sqrt{n}$ справедливо неравенство

$$e^{-\frac{x^2}{2}} \Sigma_1 \leqq \frac{\sqrt{c_4} c_9 \varepsilon c_7 e^{\frac{\varepsilon c_9 (2+c_9)}{2}}}{\sqrt{2}\,\delta} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e^{\frac{\varepsilon+1}{2} x_k^2} (1 + c_9 + x_k^2) \lambda_k,$$

что вместе с неравенствами (14) и (15) показывает, что

$$(27)\qquad e^{-\frac{x^2}{2}} \Sigma_1 = O\left(\frac{1}{\sqrt{n}}\right) \quad (|x| \leqq \sqrt{n}).$$

Переходим теперь к оценке $e^{-\frac{x^2}{2}}\Sigma_2$ в (26). В силу (17)

$$e^{-\frac{x^2}{2}}\Sigma_2 = \frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} e^{\frac{\varepsilon x_{k+1}^2}{2}} \left| \frac{\sqrt{\lambda_k}}{x-x_k} - \frac{\sqrt{\lambda_{k+1}}}{x-x_{k+1}} \right| \leqq$$

$$\leqq \frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} e^{\frac{\varepsilon x_{k+1}^2}{2}} \sqrt{\lambda_k} \left| \frac{1}{x-x_k} - \frac{1}{x-x_{k+1}} \right| +$$

(28)

$$+ \frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} e^{\frac{\varepsilon x_{k+1}^2}{2}} \left|\sqrt{\lambda_k} - \sqrt{\lambda_{k+1}}\right| \cdot \frac{1}{|x-x_{k+1}|} =$$

$$= \frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \Sigma_{21} + \frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \Sigma_{22}.$$

Мы имеем в силу неравенств (13) и (18), что

$$\frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \Sigma_{21} \leqq \frac{c_6\sqrt{c_5}}{\sqrt{2}} \cdot \frac{1}{\sqrt{n}} \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} \frac{|x_k - x_{k+1}|}{|(x-x_k)(x-x_{k+1})|} \quad (|x| \leqq \sqrt{n}),$$

откуда легко следует, что

$$\frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \Sigma_{21} = O\left(\frac{1}{\sqrt{n}}\right) \left\{ \int_{-\sqrt{n}}^{x-\delta} + \int_{x+\delta}^{\sqrt{n}} \right\} \frac{1}{(x-t)^2}\, dt =$$

(29)

$$= O\left(\frac{1}{\sqrt{n}}\right) \int_{|t|\geqq\delta} \frac{1}{t^2}\, dt = O\left(\frac{1}{\sqrt{n}}\right) \quad (|x| \leqq \sqrt{n}).$$

Для оценки выражения $\frac{1}{\sqrt{2}} e^{-\frac{x^2}{2}} |h_n(x)| \Sigma_{22}$ нам понадобится свойство (11). Из него следует, что

$$\frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \Sigma_{22} = \frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \sum_{\substack{|x-x_k|>\delta \\ 0\leqq x_k\leqq\sqrt{n}}} e^{\frac{\varepsilon x_{k+1}^2}{2}} (\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k}) \frac{1}{|x-x_{k+1}|} +$$

$$+ \frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \sum_{\substack{|x-x_k|>\delta \\ -\sqrt{n}\leqq x_k<0}} e^{\frac{\varepsilon x_{k+1}^2}{2}} (\sqrt{\lambda_k} - \sqrt{\lambda_{k+1}}) \frac{1}{|x-x_{k+1}|} =$$

$$= \frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \Sigma_{221} + \frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \Sigma_{222}.$$

Для простоты рассмотрим $\frac{1}{\sqrt{2}} e^{-\frac{x^2}{2}} |h_n(x)| \Sigma_{221}$. Мы имеем в силу (13), что каждое слагаемое в Σ_{221} имеет порядок $O\left(\frac{1}{n^{\frac{1}{4}}}\right)$. Отсюда, при помощи суммирования по частям, мы получим, что

$$\frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \Sigma_{221} \leqq \frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \left\{ \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} \sqrt{\lambda_k} \left| \frac{e^{\frac{\varepsilon x_{k+1}^2}{2}}}{|x-x_{k+1}|} - \frac{e^{\frac{\varepsilon x_k^2}{2}}}{|x-x_k|} \right| + O\left(\frac{1}{n^{\frac{1}{4}}}\right) \right\} \leqq$$

$$\leqq \frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \left\{ \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} \sqrt{\lambda_k} e^{\frac{\varepsilon x_{k+1}^2}{2}} \left| \frac{1}{|x-x_k|} - \frac{1}{|x-x_{k+1}|} \right| + \right. \tag{30}$$

$$\left. + \sum_{\substack{|x-x_k|>\delta \\ |x_k|\leqq\sqrt{n}}} \sqrt{\lambda_k} \left| e^{\frac{\varepsilon x_{k+1}^2}{2}} - e^{\frac{\varepsilon x_k^2}{2}} \right| \frac{1}{|x-x_k|} + O\left(\frac{1}{n^{\frac{1}{4}}}\right) \right\}.$$

Первая сумма в правой части (30) оценивается в точности также, как и Σ_{21}, а вторая же, как и Σ_1. Таким образом, используя неравенство (18), мы получим, что $\frac{1}{\sqrt{2}} e^{-\frac{x^2}{2}} |h_n(x)| \sum_{221} = O\left(\frac{1}{\sqrt{n}}\right)$ при $|x| \leqq \sqrt{n}$, следовательно

$$\frac{e^{-\frac{x^2}{2}}|h_n(x)|}{\sqrt{2}} \Sigma_{22} = O\left(\frac{1}{\sqrt{n}}\right) \quad (|x| \leqq \sqrt{n}). \tag{31}$$

Соотношение (25) следует теперь из (26), (27), (28), (29) и (31).

Доказательство Теоремы 1. Пусть, значит, $f \in R$ и $f(x) \equiv 0$ на отрезке $[a-\delta, b-\delta]$ $(\delta > 0)$, далее $a \leqq x \leqq b$. Мы будем различать два случая, в зависимости от расположения точки x.

а) $|x| \geqq \sqrt{n}$. В этом случае мы имеем в силу (1), что

$$|L_n(x,f)| \leqq A \sum_{|x_k| \leqq \frac{1}{2}\sqrt{n}} e^{\frac{\varepsilon x_k^2}{2}} |l_k(x)| + A \sum_{|x_k| > \frac{1}{2}\sqrt{n}} e^{\frac{\varepsilon x_k^2}{2}} |l_k(x)|,$$

значит, согласно леммам 10 и 11,

$$e^{-\frac{x^2}{2}} |L_n(x,f)| = o(1) \quad (|x| \geqq \sqrt{n}).$$

б) $|x| \leqq \sqrt{n}$. Тогда из неравенства (1) и Леммы 10 следует, что

$$L_n(x,f) = \sum_{|x_k| \leqq \sqrt{n}} f(x_k) l_k(x) + O(e^{-c_{10} n}) e^{\frac{x^2}{2}}. \tag{32}$$

Далее ясно, что

$$\sum_{|x_k|\leqq\sqrt{n}} f(x_k)l_k(x) = \frac{1}{2}\sum_{|x_k|\leqq\sqrt{n}}\left[f(x_k)e^{-\frac{\varepsilon x_k^2}{2}} - f(x_{k-1})e^{-\frac{\varepsilon x_{k-1}^2}{2}}\right]e^{\frac{\varepsilon x_k^2}{2}}l_k(x) + \tag{33}$$

$$+\sum_{|x_k|\leqq\sqrt{n}}\left[f(x_k)e^{-\frac{\varepsilon x_k^2}{2}} + f(x_{k-1})e^{-\frac{\varepsilon x_{k-1}^2}{2}}\right]e^{\frac{\varepsilon x_k^2}{2}}l_k(x) = \frac{1}{2}\Sigma^1 + \frac{1}{2}\Sigma^2.$$

Пусть η как угодно малое, но фиксированное, положительное число. В силу предположений о функции f и Леммы 12 существует число T такое, что

$$|\Sigma^1| \leqq \Big|\sum_{|x_k|\leqq T}\left[f(x_k)e^{-\frac{\varepsilon x_k^2}{2}} - f(x_{k-1})e^{-\frac{\varepsilon x_{k-1}^2}{2}}\right]e^{\frac{\varepsilon x_k^2}{2}}l_k(x)\Big| + \eta e^{\frac{x^2}{2}} \quad (|x|\leqq\sqrt{n}).$$

Напомним, что здесь суммирование распространяется фактически только на те индексы k, для которых $|x-x_k|>\frac{\delta}{2}$. Значит, если мы применим соотношения (13), (17) и (18), то мы получим, что

$$|\Sigma^1| \leqq \frac{c_{11}}{\delta}e^{\frac{x^2}{2}}\frac{1}{\sqrt{n}}\sum_{|x_k|\leqq T}\left|f(x_k)e^{-\frac{\varepsilon x_k^2}{2}} - f(x_{k-1})e^{-\frac{\varepsilon x_{k-1}^2}{2}}\right|e^{\frac{(\varepsilon-1)}{2}x_k^2} + \eta e^{\frac{x^2}{2}} \quad (|x|\leqq\sqrt{n}),$$

где константа c_{11} зависит от c_5 и c_6, значит в силу левой части неравенств (20)

$$|\Sigma^1| \leqq \frac{c_{11}}{c_8\delta}e^{\frac{x^2}{2}}\sum_{|x_k|\leqq T}\left|f(x_k)e^{-\frac{\varepsilon x_k^2}{2}} - f(x_{k-1})e^{-\frac{\varepsilon x_{k-1}^2}{2}}\right|(x_{k-1}-x_k) + \eta e^{\frac{x^2}{2}} \quad (|x|\leqq\sqrt{n}).$$

Но на отрезке $[-T, T]$ функция $f(t)e^{-\frac{\varepsilon t^2}{2}}$ также интегрируема в смысле Римана, ибо $f\in R$. Отсюда — в силу определения интеграла Римана — мы получим, что если n достаточно большое, то

$$|\Sigma^1| \leqq 2\eta e^{\frac{x^2}{2}} \quad (|x|\leqq\sqrt{n}). \tag{34}$$

Рассмотрим теперь Σ^2 в (33). Так как $f(x)\equiv 0$ на отрезке $[a-\delta, b+\delta]$, а точка x содержится в отрезке $[a, b]$, то в силу (13), (17) и (18) каждое отдельно взятое слагаемое в Σ^2 имеет порядок $O\left(\frac{1}{\sqrt{n}}\right)e^{\frac{x^2}{2}}$ при $|x|\leqq\sqrt{n}$. Отсюда

$$|\Sigma^2| \leqq \sum_{|x_k|\leqq\sqrt{n}}\left|f(x_k)e^{-\frac{\varepsilon x_k^2}{2}}\right|\left|e^{\frac{\varepsilon x_k^2}{2}}l_k(x) + e^{\frac{\varepsilon x_{k+1}^2}{2}}l_{k+1}(x)\right| + O\left(\frac{1}{\sqrt{n}}\right)e^{\frac{x^2}{2}} \quad (|x|\leqq\sqrt{n}), \tag{35}$$

и из Леммы 13 немедленно вытекает, что

$$|\Sigma^2| = O\left(\frac{1}{\sqrt{n}}\right)e^{\frac{x^2}{2}} \quad (|x|\leqq\sqrt{n}), \tag{36}$$

ибо в (35) суммирование распространяется опять же в силу локальных свойств функции f лишь на те индексы k, для которых $|x-x_k|>\delta$. Оценки (32), (33), (34) и (36) показывают, что и в случае б)

$$e^{-\frac{x^2}{2}}|L_n(x,f)| = o(1) \quad (|x| \leqq \sqrt{n}).$$

Теорема 1 полностью установлена.

Доказательство Теоремы 2. Ясно, что $V \subset R$, значит в силу (32), (33) и (36) мы имеем

$$L_n(x,f) = \frac{1}{2} \sum_{|x_k| \leqq \sqrt{n}} \left[f(x_k)e^{-\frac{\varepsilon x_k^2}{2}} - f(x_{k-1})e^{-\frac{\varepsilon x_{k-1}^2}{2}}\right] e^{\frac{\varepsilon x_k^2}{2}} l_k(x) + O\left(\frac{1}{\sqrt{n}}\right) e^{\frac{x^2}{2}}$$

$$(a \leqq x \leqq b).$$

Применяя соотношения (13), (17) и (18) мы и получим, что

$$|L_n(x,f)| \leqq \frac{c_{12}}{\delta} e^{\frac{x^2}{2}} \frac{1}{\sqrt{n}} \sum_{k=2}^{n} \left|f(x_k)e^{-\frac{\varepsilon x_k^2}{2}} - f(x_{k-1})e^{-\frac{\varepsilon x_{k-1}^2}{2}}\right| + O\left(\frac{1}{\sqrt{n}}\right) e^{\frac{x^2}{2}}$$

$$(a \leqq x \leqq b; \quad n > n_0(a,b)),$$

ибо $f \equiv 0$ на отрезке $[a-\delta, b+\delta]$. Теорема 2 следует теперь из того, что $\operatorname{var}_{-\infty}^{\infty} f(t)e^{-\frac{\varepsilon t^2}{2}} < \infty$.

Лемма 14. *Пусть точка x_0 фиксирована. Если узел x_k таков, что $x_0 \leqq x_k \leqq x_0+1$, то тогда*

$$|l_k(x_0)| \leqq c(x_0) \frac{x_{k-1}-x_k}{x_{k-1}-x_0+\frac{1}{\sqrt{n}}}, \tag{37}$$

если только n достаточно большое ($n > n_0(x_0)$), где константа $c(x_0)$ конечна при всех фиксированных значениях x_0. Аналогичное неравенство имеет место и в случае $x_0-1 \leqq x_k \leqq x_0$.

Доказательство. Пусть $n > 2x_0^2$. Если $x_0 \leqq x_k \leqq x_0+\frac{1}{\sqrt{n}}$, то тогда в силу (20) $x_0 < x_{k-1} < x_0+(1+c_9)\frac{1}{\sqrt{n}}$, значит

$$|l_k(x_0)| \leqq \frac{\left[\lambda_k \sum_{j=1}^{n} \frac{l_j^2(x_0)}{\lambda_j}\right]^{\frac{1}{2}}}{(x_{k-1}-x_k)\sqrt{n}} (2+c_9) \frac{x_{k-1}-x_k}{x_{k-1}-x_0+\frac{1}{\sqrt{n}}}.$$

Отсюда, после применения соотношений (12), (13), (16) и (20), мы получим, что

$$|l_k(x_0)| \leqq \frac{\sqrt{c_4 c_5}(2+c_9)}{c_8} e^{\frac{x_0^2-x_k^2}{2}} \cdot \frac{x_{k-1}-x_k}{x_{k-1}-x_0+\frac{1}{\sqrt{n}}}.$$

Если же $x_0+\frac{1}{\sqrt{n}} \leqq x_k \leqq x_0+1$, то тогда в силу (13), (17) и (18)

$$|l_k(x_0)| \leqq \frac{\sqrt{c_5} c_6}{\sqrt{2}} e^{\frac{x_0^2-x_k^2}{2}} \frac{2}{(x_{k-1}-x_k)\sqrt{n}} \cdot \frac{x_{k-1}-x_k}{x_{k-1}-x_0+\frac{1}{\sqrt{n}}} \cdot \frac{x_{k-1}-x_0}{x_k-x_0},$$

значит, согласно неравенствам (20), мы имеем

$$|l_k(x_0)| \leqq \frac{\sqrt{2c_5} c_6}{c_8} e^{\frac{x_0^2-x_k^2}{2}} (1+c_9) \frac{x_{k-1}-x_k}{x_{k-1}-x_0+\frac{1}{\sqrt{n}}}.$$

Таким образом неравенство (37) установлено со значением $c(x_0)=c_{13}e^{|x_0|}$.

Доказательство Теоремы 3. Рассмотрим функцию g которая определяется равенством $g(x)=f(x)-f(x_0)$. Ясно, что

$$L_n(x_0, f)-f(x_0) = L_n(x_0, g).$$

Далее, так как $f \in R$, то и $g \in R$, значит по Теореме 1[10]

$$L_n(x_0, g) = \sum_{x_0-\delta \leqq x_k < x_0} g(x_k) l_k(x_0) + \sum_{x_0 \leqq x_k \leqq x_0+\delta} g(x_k) l_k(x_0) + o(1) \tag{38}$$

$$(n = 1, 2, \ldots),$$

где δ $(0<\delta<\delta_1)$ (см. формулу (5)) как угодно малое, но фиксированное число. В дальнейшем мы будем считать число n достаточно большим, а именно $n>n_0(x_0)$ (см. Лемму 14). Для простоты вычислений мы будем оценивать лишь вторую сумму, стоящую в правой части (38). Мы имеем согласно (5)

$$|g(x)| \leqq \mu(x-x_0) \quad (x_0 \leqq x \leqq x_0+\delta_1),$$

[10] Для этого нужно лишь представить функцию g в виде суммы двух функций, первая из которых равна g на отрезке $[x_0-\delta, x_0+\delta]$ и исчезает вне $[x_0-\delta, x_0+\delta]$, а вторая равна тождественно нулю на $[x_0-\delta, x_0+\delta]$ и равна g вне $[x_0-\delta, x_0+\delta]$.

значит по неравенствам (20), Лемме 14 и в силу монотонности функции

$$\Big|\sum_{x_0\leqq x_k\leqq x_0+\delta} g(x_k)\,l_k(x_0)\Big| \leqq c(x_0)\sum_{x_0\leqq x_k\leqq x_0+\delta}\mu(x_k-x_0)\,\frac{x_{k-1}-x_k}{x_{k-1}-x_0+\dfrac{1}{\sqrt{n}}}\leqq$$

$$\leqq c(x_0)\sum_{x_0\leqq x_k\leqq x_0+\delta}\int_{x_k}^{x_{k-1}}\frac{\mu(t-x_0)}{t-x_0+\dfrac{1}{\sqrt{n}}}\,dt \leqq c(x_0)\int_{x_0}^{x_0+\delta+\frac{c_9}{\sqrt{n}}}\frac{\mu(t-x_0)}{t-x_0+\dfrac{1}{\sqrt{n}}}\,dt =$$

$$= c(x_0)\int_0^{\delta+\frac{c_9}{\sqrt{n}}}\frac{\mu(t)}{t+\dfrac{1}{\sqrt{n}}}\,dt \leqq c(x_0)\,\mu\left(\frac{1}{\sqrt{n}}\right)+c(x_0)\int_0^{\delta+\frac{c_9}{\sqrt{n}}}\frac{\mu(t)}{t}\,dt,$$

что будет в силу (6) сколь угодно малым, если только δ достаточно малое, а n достаточно большое. Таким образом Теорема 3 установлена.

Следствие 4 немедленно вытекает из Теоремы 3.

Лемма 15. *Имеют место оценки*

$$\sup_{-\infty<x<\infty} e^{-\frac{x^2}{2}}\sum_{k=1}^{n}|l_k(x)+l_{k+1}(x)| \leqq c_{14} \quad (n=1,2,\ldots;\quad l_{n+1}(x)\equiv 0), \tag{39}$$

$$\sup_{-\infty<x<\infty} e^{-\frac{x^2}{2}}\sum_{k=1}^{n}|l_k(x)| \leqq c_{15}\log n \quad (n=1,2,\ldots) \tag{40}$$

и

$$|l_k(x)| \leqq c_{16}e^{|x|} \quad \left(|x-x_k|\leqq 1,\quad |x|\leqq\frac{1}{2}\sqrt{n},\quad n=1,2,\ldots\right). \tag{41}$$

Доказательство. (39) доказано в нашей работе [12] (см. еще [15]), а (40) в работе Г. Фройда [6]. Ясно, далее, что

$$l_k^2(x) \leqq \lambda_k\sum_{j=1}^{n}\frac{l_j^2(x)}{\lambda_j},$$

значит в силу (12), (13) и (16)

$$l_k^2(x) \leqq c_4c_5e^{x^2-x_k^2} \quad (|x_k|\leqq\sqrt{n}),$$

откуда вытекает (41).

Доказательство Теоремы 5. Пусть точка x_0 есть точка непрерывности функции f. Пусть $\eta>0$ как угодно малое, фуксированное число. Возьмем число $\sigma>0$ такое, что $|g(x)|<\eta$ $(|x-x_0|<2\sigma)$, где функция g, как и при доказательстве

Теоремы 3, определяется равенством $g(x)=f(x)-f(x_0)$. Так же, как и при доказательстве Теоремы 3, мы получим из Теоремы 1, что

$$L_n(x_0,f)-f(x_0)=\sum_{|x_0-x_k|<\sigma} g(x_k)\,l_k(x_0)+o(1). \tag{42}$$

Ясно далее, что

$$\begin{aligned}\sum_{|x_0-x_k|<\sigma} g(x_k)\,l_k(x_0) &= \frac{1}{2}\sum_{|x_0-x_k|<\sigma}[g(x_k)-g(x_{k-1})]\,l_k(x_0)+\\ &+\frac{1}{2}\sum_{|x_0-x_k|<\sigma}[g(x_k)+g(x_{k-1})]\,l_k(x_0)=\Sigma'+\Sigma''.\end{aligned} \tag{43}$$

Без ограничения общности мы можем считать, что $\sigma<\delta$ (см. формулу (7)). Но тогда в силу (7)

$$g(x_{k-1})-g(x_k)+\frac{v(x_{k-1}-x_k)}{|\log(x_{k-1}-x_k)|}\geqq 0 \quad (|x_0-x_k|<\sigma),$$

следовательно

$$\begin{aligned}|\Sigma'| &\leqq \sum_{|x_0-x_k|<\sigma}\left[g(x_{k-1})-g(x_k)+\frac{v(x_{k-1}-x_k)}{|\log(x_{k-1}-x_k)|}\right]|l_k(x_0)|+\\ &+\max_{|x_0-x_k|<\sigma}\left\{\frac{v(x_{k-1}-x_k)}{|\log(x_{k-1}-x_k)|}\right\}\cdot\sum_{k=1}^{n}|l_k(x_0)|\leqq\\ &\leqq\Big|\sum_{|x_0-x_k|<\sigma}[g(x_{k-1})-g(x_k)]\,|l_k(x_0)|\Big|+2\max_{|x_0-x_k|<\sigma}\left\{\frac{v(x_{k-1}-x_k)}{|\log(x_{k-1}-x_k)|}\right\}.\end{aligned}$$

Здесь второе слагаемое в силу (20) и (40) имеет порядок $O\left[v\left(\frac{c_9}{\sqrt{n}}\right)\right]$.[11] Далее, в сумме

$$\sum_{|x_0-x_k|<\sigma}[g(x_{k-1})-g(x_k)]\,|l_k(x_0)|$$

каждое слагаемое по абсолютной величине не превосходит $2c_{16}e^{|x_0|}\eta$ согласно (41) и выбору числа σ. Но тогда[12]

$$\Big|\sum_{|x_0-x_k|<\sigma}[g(x_{k-1})-g(x_k)]\,|l_k(x_0)|\Big|\leqq\sum_{|x_0-x_k|<\sigma}|g(x_k)|\,\big|l_k(x_0)+l_{k+1}(x_0)\big|+c_{17}e^{|x_0|}\eta,$$

значит в силу (39) и выбора числа σ

$$\Big|\sum_{|x_0-x_k|<\sigma}[g(x_{k-1})-g(x_k)]\,|l_k(x_0)|\Big|\leqq[c_{14}e^{\frac{x_0^2}{2}}+c_{17}e^{|x_0|}]\eta,$$

что вместе с η может стать как угодно малым. Таким образом при всех положительных η_1

$$|\Sigma'|<\eta_1, \tag{44}$$

[11] Напомним, что $a\leqq x_0\leqq b$.

[12] Сначала мы суммируем по частям, затем воспользуемся неравенством $|\,|l_k(x_0)|-|l_{k+1}(x_0)|\,|\leqq|l_k(x_0)+l_{k+1}(x_0)|$.

если только $\sigma > 0$ достаточно малое. Оценка выражения Σ'' в (43) совсем просто. Опять же воспользуемся тем, что отдельно взятые слагаемые в Σ'' по абсолютной величине не превосходят $2\, c_{16} e^{|x_0|}\eta$, значит

$$|\Sigma''| \leqq \frac{1}{2} \sum_{|x_0 - x_k| < \sigma} |g(x_k)|\,|l_k(x_0) + l_{k+1}(x_0)| + c_{18} e^{|x_0|}\,\eta,$$

следовательно, применяя (39), мы получим, что

$$|\Sigma''| \leqq \left[\frac{c_{14}}{2}\, e^{\frac{x_0^2}{2}} + c_{18} e^{|x_0|}\right] \eta,$$

то есть при всех $\eta_2 > 0$

$$|\Sigma''| < \eta_2 \tag{45}$$

если $\sigma > 0$ достаточно малое. Первая часть Теоремы 5 следует теперь из соотношений (42)—(45). Утверждение о равномерной сходимости на отрезке $[a_1, b_1] \subseteq [a, b]$, где f непрерывна, следует из того, что все наши оценки могут бчть равномерными на отрезке $[a_1, b_1]$, если f там непрерывна.

Следствие 6 непосредственно вытекает из Теоремы 5.

Следствие 7, которое является частным случаем Теоремы 8, также может быть выведено из Теоремы 5, что и свидетельствует об общности Теоремы 5. Для этого достаточно представить функцию f в виде $f_1 - f_2$, где

$$f_1(x) = \begin{cases} 0, & \text{при} \quad x \leqq a - \delta \\ \underset{a-\delta}{\overset{x}{\operatorname{var}}}\, f, & \text{при} \quad a - \delta < x < b + \delta \\ \underset{a-\delta}{\overset{b+\delta}{\operatorname{var}}}\, f, & \text{при} \quad x \geqq b + \delta \end{cases}$$

и соответственно

$$f_2(x) = \begin{cases} -f(x), & \text{при} \quad x \leqq a - \delta \\ -f(x) + \underset{a-\delta}{\overset{x}{\operatorname{var}}}\, f, & \text{при} \quad a - \delta < x < b + \delta \\ -f(x) + \underset{a-\delta}{\overset{b+\delta}{\operatorname{var}}}\, f, & \text{при} \quad x \geqq b + \delta, \end{cases}$$

и заметить что обе эти функции удовлетворяют условиям Теоремы 5. Действительно, обе монотонны на отрезке $[a - \delta, b + \delta]$, следовательно, на этом отрезке удовлетворяют одностороннему условию Дини—Липшица. Из конструкции ясно, что $f_1, f_2 \in R$. Далее, все точки непрерывности функции f являются точками непрерывности функций f_1 и f_2.[13]

[13] Относительно этого последнего см. например И. П. Натансон [10], гл. VIII, §5.

Лемма 16. *Пусть функция Φ удовлетворяет условию* (8). *Если функция f из класса $V_{\Phi}[a-\delta, b+\delta]$ ограничена на отрезке $[a-\delta, b+\delta]$, то тогда $f\in V_{\Phi_1}[a-\delta, b+\delta]$ также, где*

$$\Phi_1(\xi)=\frac{1}{\sup\limits_{a-\delta\leqq x\leqq b+\delta} f(x)-\inf\limits_{a-\delta\leqq x\leqq b+\delta} f(x)}\int_0^{\xi}\Phi(t)\,dt \quad (\xi\geqq 0),$$

при этом сопряженная с Φ_1 в смысле Юнга функция Ψ_1[14] *такова, что*

$$\sum_{k=1}^{\infty}\Psi_1\left(\frac{1}{k}\right)<\infty. \tag{46}$$

Это утверждение принадлежит К. И. Осколкову [16].

Доказательство Теоремы 8. Мы имеем в силу Леммы 16 — в обозначениях доказательства Теоремы 5 —, что $g\in V_{\Phi_1}[a-\delta, b+\delta]$. Пересмотря доказательство Теоремы 5 мы увидим, что наша задача состоит в следующем: нам нужно показать, что выражение $|\Sigma'|$ (см. (43)) будет как угодно малым, если только σ достаточно мало. Из (13), (17), (18), (20) и (41), немедленно следует, что

$$|\Sigma'|\leqq c_{19}\sum_{|k|=0}^{[c_{20}\sigma\sqrt{n}]}|g(x_{k+k_0})-g(x_{k+k_0-1})|\frac{1}{|k|+1},$$

где k_0 индекс корня x_k, ближащего к точке x_0, а константа c_{20} такова, что $|x_0-x_{k+k_0}|\leqq\sigma$, если $|k|\leqq[c_{20}\sigma\sqrt{n}]$, то есть[15]

$$\begin{aligned}|\Sigma'|&\leqq c_{19}\sum_{|k|=0}^{\left[\frac{1}{\eta}\right]}|g(x_{k+k_0})-g(x_{k+k_0-1})|\frac{1}{|k|+1}+\\&+c_{19}\sum_{|k|=\left[\frac{1}{\eta}\right]}^{[c_{20}\sigma\sqrt{n}]}|g(x_{k+k_0})-g(x_{k+k_0-1})|+\frac{1}{|k|+1}=\Sigma'_*+\Sigma'_{**}.\end{aligned} \tag{47}$$

Ясно, что в силу выбора числа σ,

$$\Sigma'_*\leqq c_{19}\eta\sum_{|k|=0}^{\left[\frac{1}{\eta}\right]}\frac{1}{|k|+1}\leqq c_{20}\eta|\log\eta|. \tag{48}$$

Для оценки Σ'_{**} мы возьмем последовательность $\varrho(k)>0$ $(k=1, 2, \ldots)$, обладающую свойствами:[16] $\varrho(k+1)\leqq\varrho(k)$, $\varrho(k)\xrightarrow[k\to\infty]{}0$ и $\sum\limits_{k=1}^{\infty}\Psi_1\left\{\frac{1}{k\varrho(k)}\right\}<\infty$. Такая последовательность в силу (46) существует. Но тогда

$$\Sigma'_{**}\leqq 2c_{19}\varrho\left(\left[\frac{1}{\eta}\right]\right)\sum_{|k|=\left[\frac{1}{\eta}\right]}^{[c_{20}\sigma\sqrt{n}]}|g(x_{k+k_0})-g(x_{k+k_0-1})|\frac{1}{|k|\varrho(|k|)},$$

[14] В силу монотонности Φ функция Φ_1 выпукла, значит Ψ_1 существует (см. А. Зигмунд [24], том I, стр. 25).

[15] Напомним, что σ и η фиксированы, а n считаем достаточно большим.

[16] Идея построения такой последовательности $\varrho(k)$ принадлежит Р. Салему [18].

следовательно

$$\Sigma'_{**} \leqq 2c_{19}\varrho\left(\left[\frac{1}{\eta}\right]\right)\left\{\sum_{|x_0-x_k|<\sigma}\Phi_1(|g(x_k)-g(x_{k-1})|)+2\sum_{k=\left[\frac{1}{\eta}\right]}^{\infty}\Psi_1\left(\frac{1}{k\varrho(k)}\right)\right\},$$

ибо[17]

$$uv \leqq \Phi_1(u)+\Psi_1(v) \quad (u, v \geqq 0).$$

Отсюда мы получим, что

$$\Sigma'_{**} \leqq c_{21}\varrho\left(\left[\frac{1}{\eta}\right]\right), \tag{49}$$

так как $g\in V_{\Phi_1}[a-\delta, b+\delta]$. Из неравенств (47)—(49) следует, что

$$|\Sigma'| = c_{20}\eta|\log\eta| + c_{21}\varrho\left(\left[\frac{1}{\eta}\right]\right),$$

если только $\sigma=\sigma(\eta, f, x_0)$ достаточно мало. Но это и означает, что $|\Sigma'|$ будет сколь угодно малым при малых значения σ. Отмечая, что все наши оценки равномерны на отрезках непрерывности функции f, мы и заканчиваем доказательство Теоремы 8.

Лемма 17. *Пусть функция f равномерно непрерывна и ограничена на всей вещественной оси. Тогда*

$$\max_{a\leqq x\leqq b}|f(x)-L_n(x, f)| \leqq c_{22}\log n\,\omega\left(f; \frac{1}{\sqrt{n}}\right) \quad (n = 1, 2, \ldots), \tag{50}$$

если отрезок $[a, b]$ фиксирован.

Лемма 17 доказана в работе Г. Фройда [6].

Доказательство Теоремы 9. Пусть $f=f_1+f_2$, где

$$f_1(x) = \begin{cases} f(a-\delta), & \text{при} \quad x \leqq a-\delta \\ f(x), & \text{при} \quad a-\delta < x < b+\delta \\ f(b+\delta), & \text{при} \quad x \geqq b+\delta \end{cases}$$

и

$$f_2(x) = \begin{cases} f(x)-f(a-\delta), & \text{при} \quad x \leqq a-\delta \\ 0, & \text{при} \quad a-\delta < x < b+\delta \\ f(x)-f(b+\delta), & \text{при} \quad x \geqq b+\delta. \end{cases}$$

Мы имеем в силу Леммы 17

$$\max_{a\leqq x\leqq b}|f_1(x)-L_n(x, f_1)| \leqq c_{22}\log n\,\omega\left(f_1; \frac{1}{\sqrt{n}}\right) = c_{22}\log n\,\omega\left(\frac{1}{\sqrt{n}}\right). \tag{51}$$

[17] Это и есть неравенство Юнга (см. например А. Зигмунд [24], I (9.1)).

Далее, по Теореме 2

$$\max_{a\leqq x\leqq b}|L_n(x,f_2)| = O(1)\frac{1}{\sqrt{n}} \quad O(1)\omega\left(\frac{1}{\sqrt{n}}\right), \tag{52}$$

ибо каждый модуль непрерывности $\omega(t)$ убывает при $t\to 0$ не быстрее, чем t. Соотношения (51) и (52) вместе показывают справедливость Теоремы 9.

§ 4. Заключение

Мы думаем, что наши результаты остаются в силе и для интерпольяционного процесса Лагранжа с узлами в корнях многочленов Лагерра или Сонина—Маркова.[18] При изучении этих процессов однако появляется новая трудность, а именно, что весовые функции этих многочленов обратятся в 0 или ∞ в точке 0. Мы надеемся, что нам скоро удается преодолеть эти неприятности, и сможем перенести наши теоремы и на этот случай.

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Цитированная литература

[1] J. BALÁZS—P. TURÁN, Notes on interpolation. VIII, *Acta Math. Acad. Sci. Hung.*, **12** (1961), стр. 469—474.
[2] J. BALÁZS—P. TURÁN, Notes on interpolation. IX, *Acta Math. Acad. Sci. Hung.*, **16** (1965), стр. 215—220.
[3] H. BATEMAN—A. ERDÉLYI. *Higher Transcendental Functions,* vol. 2 (New York, 1953).
[4] G. FREUD, Über die Konvergenz im Mittel von Lagrangeschen Interpolationspolynomfolgen, *Acta Math. Acad. Sci. Hung.*, **13** (1962), стр. 257—268.
[5] G. FREUD, Lagrange-féle interpolációs eljárások konvergenciája végtelen intervallumon, *Matematikai Lapok*, **18** (1967) стр. 289—292.
[6] G. FREUD, Lagrangesche Interpolation über die Nullstellen der Hermiteschen Orthogonalpolynome, *Studia Sci. Math. Hung.*, **4** (1968), стр. 179—190.
[7] G. FREUD, *Orthogonal Polynomials,* Akadémiai Kiadó (Budapest, 1971).
[8] О. Киш, О Лагранжевом интерполировании с узлами в корнях многочленов Сонина—Маркова, *Acta Math. Acad. Sci. Hung.*, **23** (1972), стр. 389—417.
[9] I. P. NATANSON, On the convergence of trigonometrical interpolation at equidistant knots, *Annals of Math.*, **45** (1944), стр. 457—471.
[10] И. П. Натансон, *Теория функций вещественной переменной,* ГИТТЛ (Москва, 1950).
[11] G. P. NÉVAI, A Laguerre-polinomok gyökein alapuló Lagrange-féle interpolációról, *Matematikai Lapok*, **22** (1971), стр. 149—164.
[12] Г. П. Неваи, Об интерполяционном процессе Лагранжа с узлами в корнях многочленов Эрмита, *Acta Math. Acad. Sci. Hung.*, **24** (1973), стр. 209—213.
[13] Г. П. Неваи, О сходимости Лагранжева интерполирования по узлам Лагерра, *Publicationes Math. Debrecen*, **20** (1973), стр. 235—239.

[18] См. сноску [2].

[14] Г. П. Неваи, О признаке Дини—Липшица, *Acta Math. Acad. Sci. Hung.*, **24** (1973), стр. 349—351.
[15] Г. П. Неваи, Замечания об интерполировании, *Acta Math. Acad. Sci. Hung.*, **25** (1974), стр. 123—144.
[16] К. И. Осколков, Обобщенная вариация, индикатриса Банаха и равномерная сходимость рядов Фурье, *Математические Заметки*, **12** (1972), стр. 313—324.
[17] M. Plancherel—W. Rotach, Sur les valeurs asymptotiques des polynomes d'Hermite, *Commentarii Math. Helvetici*, **1** (1929), стр. 227—254.
[18] R. Salem, Sur un test général pour la convergence uniforme des séries de Fourier, *Comptes Rendus de l'Acad. des Sci. de Paris*, **207** (1938), стр. 662—664.
[19] J. Shohat, On interpolation, *Annals of Math.*, **34** (1933), стр. 130—146.
[20] J. Shohat, Application of orthogonal Tchebycheff polynomials to Lagrangean interpolation and to the general theory of polynomials, *Annali di Mat.*, **18** (1938), стр. 201—238.
[21] J. Shohat—J. Tamarkin, *The Problem of Moments*, Amer. Math. Soc. (New York, 1943)
[22] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ. vol. XXIII, rev. edition (Providence, Rhode Island, 1967).
[23] L. C. Young, Sur une généralisation de la notion de variation de puissance p-iéme bornée au sens de M. Wiener, et sur la convergence des séries de Fourier, *Comptes Rendus de l'Acad. des Sci. de Paris*, **204** (1937), стр. 470—472.
[24] A. Zygmund, *Trigonometric Series*, vol. I—II, University Press (Cambridge, 1959).

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Tomus 25 (3—4), (1974), pp. 363—386.

ИССЛЕДОВАНИЕ ОДНОГО ИНТЕРПОЛЯЦИОННОГО ПРОЦЕССА. I

О. КИШ (Будапешт)

Введем следующие *обозначения:* $g(t)$ определенная на множестве вещественных чисел, вещественная, 2π-периодическая, непрерывная функция; n натуральное, k неотрицательное целое число; $a_i (0 \leqq i \leqq n-1)$ и $b_i (1 \leqq i \leqq n)$ вещественные числа; $S_{nk}(g, t)$ тригонометрический многочлен вида

$$a_0 + \sum_{i=1}^{n-1} (a_i \cos it + b_i \sin it) + b_n \sin nt,$$

удовлетворяющий условию

$$S_{nk}\left(g, \frac{2i-1}{2n}\pi\right) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} g\left(\frac{2i-2j+k-1}{2n}\pi\right) \quad (1 \leqq i \leqq 2n). \tag{1}$$

(Известно, что такой тригонометрический многочлен существует и единственен.)

Ниже мы двумя способами оценим погрешность $S_{nk}(g, t) - g(t)$. Аналогичным образом можно оценить отклонение от $g(t)$ тригонометрического многочлена не выше n-того порядка $T_{nk}(g, t)$, удовлетворяющего условиям

$$T_{nk}\left(g, \frac{2i-1}{2n+1}\pi\right) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} g\left(\frac{2i-2j+k-1}{2n+1}\pi\right) \quad (1 \leqq i \leqq 2n+1).$$

Поведение тригонометрических интерполяционных многочленов $S_{n0}(g, t)$ и $T_{n0}(g, t)$ хорошо изучено. Известно, что они могут не сходиться к $g(t)$, если $n \to \infty$. Случай $k=1$ и $k=2$ впервые изучал С. Н. Бернштейн в работах [1] и [2], заметив, что в этом случае интерполяционный процесс всегда равномерно сходится. Равномерную сходимость последовательности $T_{n2}(g, t)$ к $g(t)$ доказал также A. C. Offord в статье [8]. В работе [3] А. Ф. Харшиладзе доказал перавенство

$$|T_{n1}(g, t) - g(t)| \leqq (1 + 2\pi + 4\pi^2) E_n(g) + \omega\left(g, \frac{\pi}{2n+1}\right),$$

где $E_n(g)$ наилучшее приближение функции $g(t)$ тригонометрическими многочленами не выше n-того порядка, а $\omega(g, h)$ модуль непрерывности функции

$g(t)$. Этот результат доказан также в книге И. П. Натансона [4]. В статье [7] получено неравенство

$$|T_{n2}(g, t)-g(t)| \leqq \frac{13}{3\pi}\omega\left(g, \frac{2\pi}{2n+1}\right).$$

Введем следующие *обозначения:* $f(x)$ определенная на отрезке $[-1, +1]$, вещественная, непрерывная функция и, если

$$g(t) = f(\cos t),$$

то

$$P_{nk}(f, x) = S_{nk}(g, \arccos x).$$

Эти алгебраические многочлены не выше $n-1$-ой степени очевидно удовлетворяют условию

$$P_{nk}\left(f, \cos\frac{2i-1}{2n}\pi\right) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} f\left(\cos\frac{2i-2j+k-1}{2n}\pi\right) \quad (1 \leqq i \leqq n).$$

Поведение интерполяционных многочленов Лагранжа $P_{n0}(f, x)$ хорошо изучено. Известно, что они могут не сходиться к $f(x)$, если $n \to \infty$. В статье [9] G. Grünwald доказал, что последовательность $P_{n1}(f, x)$ всегда равномерно сходится к $f(x)$ при $n \to \infty$. В работе [10] T. M. Mills и A. K. Varma доказали неравенства

$$|P_{n1}(f, x)-f(x)| \leqq c_1\omega\left(f, \frac{\sqrt{1-x^2}}{n}\right) + c_2\omega\left(f, \frac{1}{n^2}\right),$$

$$|P_{n1}(f, x)-f(x)| \leqq c_3\omega_2\left(f, \frac{1}{n}\right),$$

где c_1, c_2 и c_3 положительные числа, а $\omega_2(f, h)$ модуль гладкости функции $f(x)$. В статье [1] С. Н. Бернштейн ввел интерполяционные многочлены $P_{n2}(f, x)$, заметив, что они также равномерно сходятся к $f(x)$ при $n \to \infty$. В работе [5] при четных k Д. Л. Берман ввел многочлены не выше $n-1$-ой степени $Q_{nk}(f, x)$, удовлетворяющие условию

$$Q_{nk}\left(f, \cos\frac{2i-1}{2n}\pi\right) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} f(x_{i-j+k/2}) \quad (1 \leqq i \leqq n),$$

где

$$x_{i-j+k,2} = \begin{cases} \cos\dfrac{2i-2j+k-1}{2n}\pi, & \text{если} \quad 1 \leqq i-j+\dfrac{k}{2} \leqq n, \\ \cos\dfrac{2i-1}{2n}\pi, & \text{если} \quad i-j+\dfrac{k}{2} < 1 \quad \text{или} \quad i-j+\dfrac{k}{2} > n. \end{cases}$$

В статье [6] Д. Л. Берман доказал неравенство

$$|Q_{n4}(f, x)-f(x)| \leqq \left(12\sqrt{2}+\frac{5}{2}\pi^2\cos n \arccos x\right)\omega\left(f, \frac{\pi}{n}\right).$$

Ниже мы оценим погрешность $P_{nk}(f, x)-f(x)$. Аналогичным образом можно при нечетных k оценить отклонение от $f(x)$ многочленов не выше $n-1$-ой степени $R_{nk}(f, x)$, удовлетворяющих условию

$$R_{nk}\left(f, \cos\frac{2i-1}{2n}\pi\right) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} f\left(\cos\frac{2i-2j+k}{2n}\pi\right) \quad (1 \leqq i \leqq n).$$

Интерполяционные многочлены $R_{n1}(f, x)$ также ввел С. Н. Бернштейн в статье [1], заметив, что и они всегда равномерно сходятся к $f(x)$ при $n\to\infty$.

Введем следующие *обозначения:*

$$d_n(t) = \frac{1}{2n}\sin nt \operatorname{ctg}\frac{t}{2}; \tag{2}$$

если k четное число, то

$$s_i(t) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} d_n\left(t-\frac{2i+2j-k-1}{2n}\pi\right) \quad (i = 0, \pm1, \pm2, \ldots), \tag{3}$$

а если k нечетное число, то

$$s_i(t) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} d_n\left(t-\frac{2i+2j-k}{2n}\pi\right) \quad (i = 0, \pm1, \pm2, \ldots). \tag{4}$$

Нижеследующие вспомогательные предложения понадобятся нам в дальнейшем.

Лемма 1. *Если k четное число, то*

$$S_{nk}(g, t) = \sum_{i=1}^{2n} g\left(\frac{2i-1}{2n}\pi\right) s_i(t), \tag{5}$$

а если k нечетное число, то

$$S_{nk}(g, t) = \sum_{i=1}^{2n} g\left(\frac{i\pi}{n}\right) s_i(t). \tag{6}$$

Доказательство. Очевидно

$$S_{nk}(g, t) = \sum_{i=1}^{2n} d_n\left(t-\frac{2i-1}{2n}\pi\right)\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} g\left(\frac{2i-2j+k-1}{2n}\pi\right). \tag{7}$$

Если k четно, то запишем эту формулу в виде

$$S_{nk}(g, t) = \sum_{i=1}^{2n} g\left(\frac{2i-1}{2n}\pi\right)\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} d_n\left(t-\frac{2i+2j-k-1}{2n}\pi\right), \tag{8}$$

а если k нечетно, то запишем (7) в виде

$$S_{nk}(g, t) = \sum_{i=1}^{2n} g\left(\frac{i\pi}{n}\right)\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} d_n\left(t-\frac{2i+2j-k}{2n}\pi\right). \tag{9}$$

Из (8)—(9) и (3)—(4) получаем (5)—(6).

Лемма 2. *Если* $u(t)$ *тригонометрический многочлен не выше* $n-1$*-ого порядка, то*

$$S_{nk}(u, t) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} u\left(t + \frac{k-2j}{2n}\pi\right). \tag{10}$$

Доказательство. Известно, что

$$u(t) = \sum_{i=1}^{2n} u\left(\frac{2i-1}{2n}\pi\right) d_n\left(t - \frac{2i-1}{2n}\pi\right).$$

Поэтому

$$\frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} u\left(t + \frac{k-2j}{2n}\pi\right) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \sum_{i=1}^{2n} u\left(\frac{2i-1}{2n}\pi\right) d_n\left(t - \frac{2i+2j-k-1}{2n}\pi\right).$$

Отсюда и из (8) получаем (10) для четных k. Если k нечетно, то (10) следует из (9) и из равенств

$$u(t) = \sum_{i=1}^{2n} u\left(\frac{i\pi}{n}\right) d_n\left(t - \frac{i\pi}{n}\right),$$

$$\frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} u\left(t + \frac{k-2j}{2n}\pi\right) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \sum_{i=1}^{2n} u\left(\frac{i\pi}{n}\right) d_n\left(t - \frac{2i+2j-k}{2n}\pi\right).$$

Обозначим через m целое число.

Лемма 3. *Если* i *четное число, то* i*-тая производная функции* $\operatorname{ctg} x$ *неотрицательна на отрезке* $\left(m\pi, m\pi + \frac{\pi}{2}\right]$ *и неположительна на отрезке* $\left[m\pi + \frac{\pi}{2}, m\pi + \pi\right)$, *если* i *нечетное число то эта производная отрицательна на отрезке* $(m\pi, m\pi + \pi)$.

Доказательство. Утверждение леммы следует из тождества

$$\frac{d^i \operatorname{ctg} x}{dx^i} = (-1)^i \sum_{k=0}^{\left[\frac{i+1}{2}\right]} a_k \operatorname{ctg}^{i-2k+1} x \quad (i = 0, 1, 2, \ldots),$$

где a_k положительные числа и $[y]$ обозначает целую часть числа y. Это тождество можно доказать методом математической индукции: оно верно при $i=0$ и из него следует аналогичное равенство для $i+1$ вместо i:

$$\frac{d^{i+1} \operatorname{ctg} x}{dx^{i+1}} = (-1)^{i+1} \sum_{k=0}^{\left[\frac{i+1}{2}\right]} (i-2k+1) a_k \operatorname{ctg}^{i-2k} x (1 + \operatorname{ctg}^2 x) =$$

$$= (-1)^{i+1} \sum_{k=0}^{\left[\frac{i+2}{2}\right]} b_k \operatorname{ctg}^{i-2k+2} x,$$

где

$$b_0 = (i+1) a_0, \quad b_k = (i-2k+1) a_k + (i-2k+3) a_{k-1} \quad \left(1 \leqq k \leqq \left[\frac{i+1}{2}\right]\right)$$

и, если i четно, то

$$b_{i/2+1} = a_{i/2}.$$

Лемма 4. *Пусть*

$$w_{ik} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-2j)^i \quad (i \geqq 0,\ k > 0).$$

Если $i < k$ *или* $k < i$ *и* $i-k$ *нечетно, то* $w_{ik}=0$, *а если* $k \leqq i$ *и* $i-k$ *четно, то* $w_{ik} > 0$.

Доказательство проводится методом математической индукции. Для $k=1$ утверждение леммы верно: $w_{i1} = 1-(-1)^i$. Предположим, что утверждение леммы верно до некоторого k включительно и покажем, что тогда оно верно и для $k+1$ вместо k: очевидно $w_{0k+1}=0$, а если $i>0$, то

$$w_{ik+1} = \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (k+1-2j)^i = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k+1-2j)^i +$$

$$+ \sum_{j=1}^{k+1} (-1)^j \binom{k}{j-1} (k+1-2j)^i = \sum_{j=0}^{k} (-1)^j \binom{k}{j} [(k-2j+1)^i - (k-2j-1)^i] =$$

$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{l=0}^{i} \binom{i}{l} (k-2j)^{i-l} [1-(-1)^l] =$$

$$= 2 \sum_{l=1}^{\left[\frac{i+1}{2}\right]} \binom{i}{2l-1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-2j)^{i-2l+1} = 2 \sum_{l=1}^{\left[\frac{i+1}{2}\right]} \binom{i}{2l-1} w_{i-2l+1k};$$

отсюда действительно следует утверждение леммы с $k+1$ вместо k.

Лемма 5. *Пусть* $0 < 2kh < \pi$. *Если* k *четное число, то функция*

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \operatorname{ctg}(x-2jh+kh) \tag{11}$$

неотрицательна на отрезке $\left[m\pi+kh, m\pi+\frac{\pi}{2}\right]$ *и неположительна на отрезке* $\left[m\pi+\frac{\pi}{2}, m\pi+\pi-kh\right)$, *а если* k *нечетное число, то функция* (11) *отрицательна на отрезке* $(m\pi+kh, m\pi+\pi-kh)$.

Доказательство. На отрезке $(m\pi+kh, m\pi+\pi-kh)$

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \operatorname{ctg}(x-2jh+kh) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{d^i \operatorname{ctg} x}{dx^i} (k-2j)^i h^i =$$

$$= \sum_{i=0}^{\infty} \frac{h^i}{i!} \frac{d^i \operatorname{ctg} x}{dx^i} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-2j)^i = \sum_{i=0}^{\infty} w_{ik} \frac{h^i}{i!} \frac{d^i \operatorname{ctg} x}{dx^i}.$$

Отсюда и из лемм 3 и 4 следует утверждение леммы 5.

Обозначим через l то целое число, для которого выполняется условие

$$\frac{2l-1}{2n}\pi \leqq t < \frac{2l+1}{2n}\pi. \tag{12}$$

Лемма 6. *Если* $2n \leqq k$, *то*

$$s_i(t) \geqq 0 \quad (i = 0, \pm 1, \pm 2, \ldots). \tag{13}$$

Если $2n > k$ *и* k *четное число, то*

$$s_i(t) \geqq 0 \quad \left(2mn+l-\frac{k}{2} \leqq i \leqq 2mn+l+\frac{k}{2}+1\right), \tag{14}$$

$$(-1)^{i+l+\frac{k}{2}} s_i(t) \leqq 0 \quad \left(2mn+l+\frac{k}{2} < i \leqq 2mn+l+n\right), \tag{15}$$

$$(-1)^{i+l+\frac{k}{2}} s_i(t) \geqq 0 \quad \left(2mn+l+n < i \leqq 2mn+2n+l-\frac{k}{2}\right). \tag{16}$$

Если $2n > k$ *и* k *нечетное число, то*

$$s_i(t) \geqq 0 \quad \left(2mn+l-\frac{k+1}{2} \leqq i \leqq 2mn+l+\frac{k+1}{2}\right), \tag{17}$$

$$(-1)^{i+l+\frac{k+1}{2}} s_i(t) \geqq 0 \quad \left(2mn+l+\frac{k+1}{2} \leqq i \leqq 2mn+2n+l-\frac{k+1}{2}\right). \tag{18}$$

Доказательство. Если k четное число, то ввиду (3) и (2)

$$s_i(t) = \frac{1}{n2^{k+1}} \sum_{j=0}^{k} \binom{k}{j} \sin n\left(t-\frac{2i+2j-k-1}{2n}\pi\right) \operatorname{ctg}\left(\frac{t}{2}-\frac{2i+2j-k-1}{4n}\pi\right).$$

Поэтому

$$s_i(t) = \frac{1}{n2^{k+1}} \sin n\left(t-\frac{2l-1}{2n}\pi\right)(-1)^{i+l+\frac{k}{2}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \operatorname{ctg}(x-2jh+kh),$$

где

$$x = \frac{t}{2}-\frac{2i-1}{4n}\pi, \quad h = \frac{\pi}{4n}.$$

Если $2n > k$, то условие $0 < 2kh < \pi$ леммы 5 выполняется. Если

$$2mn+l+\frac{k}{2} < i \leqq 2mn+l+n,$$

то ввиду (12)

$$x < \frac{2l+1}{4n}\pi - \frac{4mn+2l+k+1}{4n}\pi = -m\pi - kh,$$

$$x \geqq \frac{2l-1}{4n}\pi - \frac{4mn+2l+2n-1}{4n}\pi = -m\pi - \frac{\pi}{2},$$

поэтому в силу леммы 5 функция (11) неположительна и следовательно имеет место (15). Если

$$2mn+l+n < i \leqq 2mn+2n-\frac{k}{2}+l,$$

то

$$x < \frac{2l+1}{4n}\pi - \frac{4mn+2l+2n+1}{4n}\pi = -m\pi-\frac{\pi}{2},$$

$$x > \frac{2l-1}{4n}\pi - \frac{4mn+4n+2l-k-1}{4n}\pi = -m\pi-\pi,$$

в силу леммы 5 функция (11) неотрицательна и следовательно имеет место (16). Для нечетных k неравенство (18) доказывается аналогичным образом.

(13), (14) и (17) доказывается методом математической индукции. Если $k=0$, то (14) следует из (16) и (15). Предположим, что (13) и (14) выполняется для некоторого четного k. Обозначая $s_i(t)$ через $s_{ik}(t)$ получаем из (4) и (3):

$$s_{ik+1}(t) = \frac{1}{2^{k+1}}\sum_{j=0}^{k+1}\binom{k+1}{j} d_n\left(t-\frac{2i+2j-k-1}{2n}\pi\right) =$$

$$= \frac{1}{2^{k+1}}\sum_{j=0}^{k}\binom{k}{j} d_n\left(t-\frac{2i+2j-k-1}{2n}\pi\right)+$$

$$+\frac{1}{2^{k+1}}\sum_{j=1}^{k+1}\binom{k}{j-1} d_n\left(t-\frac{2i+2j-k-1}{2n}\pi\right) = \frac{1}{2}s_{ik}(t)+\frac{1}{2}s_{i+1k}(t).$$

Если $2n \leqq k$, то отсюда и из (13) следует (13) с $k+1$ вместо k, а если $2n > k$, то из (14) следует (17) с $k+1$ вместо k. Аналогичным образом можно доказать (13) и (14) для $k+1$ вместо k, если (13) и (17) имеет место для нечетного k.

Введем следующее *обозначение:*

$$r(t) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} g\left(t+\frac{k-2j}{2n}\pi\right)-g(t). \tag{19}$$

Теорема 1.

$$|S_{nk}(g,t)-g(t)| \leqq \left[1+\sum_{i=1}^{2n}|s_i(t)|\right]E_{n-1}(g)+|r(t)|. \tag{20}$$

Доказательство. Обозначим через $u(t)$ тригонометрический многочлен не выше $n-1$-ого порядка, наименее уклоняющийся от функции $g(t)$. Очевидно

$$S_{nk}(g,t)-g(t) = S_{nk}(g,t)-S_{nk}(u,t)+S_{nk}(u,t)-\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} g\left(t+\frac{k-2j}{2n}\pi\right)+r(t).$$

Если k четное число, то из формулы (5) и неравенства

$$|g(t)-u(t)| \leqq E_{n-1}(g)$$

получаем:

$$|S_{nk}(g, t)-S_{nk}(u, t)| = \left|\sum_{i=1}^{2n}\left[g\left(\frac{2i-1}{2n}\pi\right)-u\left(\frac{2i-1}{2n}\pi\right)\right]s_i(t)\right| \leqq E_{n-1}(g)\sum_{i=1}^{2n}|s_i(t)|,$$

а если k нечетное число, то ввиду (6)

$$|S_{nk}(g, t)-S_{nk}(u, t)| = \left|\sum_{i=1}^{2n}\left[g\left(\frac{i\pi}{n}\right)-u\left(\frac{i\pi}{n}\right)\right]s_i(t)\right| \leqq E_{n-1}(g)\sum_{i=1}^{2n}|s_i(t)|.$$

Из формулы (10) следует:

$$\left|S_{nk}(u, t)-\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}g\left(t+\frac{k-2j}{2n}\pi\right)\right| =$$

$$= \left|\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left[u\left(t+\frac{k-2j}{2n}\pi\right)-g\left(t+\frac{k-2j}{2n}\pi\right)\right]\right| \leqq E_{n-1}(g).$$

Резюмируя сказанное, приходим к (20).

Замечания. 1. В силу (1) и (19) в узлах интерполирования $\frac{2i-1}{2n}\pi$ имеет место более точное чем (20) соотношение

$$S_{nk}\left(g, \frac{2i-1}{2n}\pi\right)-g\left(\frac{2i-1}{2n}\pi\right) = r\left(\frac{2i-1}{2n}\pi\right) \quad (i = 0, \pm 1, \pm 2, \ldots).$$

2. В формуле (20) выражение $E_{n-1}(g)$ можно заменить на $\omega\left(g, \frac{\pi}{n}\right)$ или $c\omega_2\left(g, \frac{\pi}{n}\right)$, где c положительное число, так как по теореме Н. П. Корнейчука

$$E_{n-1}(g) \leqq \omega\left(g, \frac{\pi}{n}\right) \tag{21}$$

и известно, что

$$E_{n-1}(g) \leqq c\omega_2\left(g, \frac{\pi}{n}\right).$$

3. Покажем, что существуют зависящие от t и n функции $g(x)$, для которых

$$S_{nk}(g, t)-g(t) = \left(1+\sum_{i=1}^{2n}|s_i(t)|\right)E_{n-1}(g),$$

если $t \neq \frac{2i-1}{2n}\pi$ при четных k и $t \neq \frac{i\pi}{n}$ при нечетных k $(i=0, \pm 1, \pm 2, \ldots)$.

Если k четно, то определим непрерывную 2π-периодическую функцию $g(x)$ так, чтобы выполнялись следующие условия: $|g(x)| \leqq 1$,

$$g\left(\frac{2i-1}{2n}\pi\right) = \operatorname{sign} s_i(t) \quad (i = 0, \pm 1, \pm 2, \ldots),$$

$g(t) = -1$, в $2n$ последовательных точках отрезка $[0, 2\pi)$ функция $g(x)$ попеременно принимает значения $+1$ и -1. Тогда ввиду (5)

$$S_{nk}(g, t) = \sum_{i=1}^{2n} |s_i(t)|,$$

по теореме П. Л. Чебышева $E_{k-1}(g)=1$ и наше утверждение доказано. Случай нечетных k рассматривается аналогичным образом.

Заметим еще, что для функции $g(x)$

$$S_{nk}(g, t) - g(t) = \frac{1}{2}\left[1 + \sum_{i=1}^{2n} |s_i(t)|\right] \omega\left(g, \frac{\pi}{n}\right),$$

так как $\omega\left(g, \frac{\pi}{n}\right) = 2$.

Оценим выражение $r(t)$, фигурирующее в (20).

Теорема 2. *Если k четное число, то*

$$|r(t)| \leqq \frac{k}{2^{k+1}} \binom{k}{k/2} \omega\left(g, \frac{\pi}{n}\right), \tag{22}$$

$$|r(t)| \leqq \frac{k}{8} \omega_2\left(g, \frac{\pi}{2n}\right). \tag{23}$$

Если k нечетное число, то

$$|r(t)| \leqq \frac{k+1}{2^{k+1}} \binom{k+1}{\frac{k+1}{2}} \omega\left(g, \frac{\pi}{2n}\right), \tag{24}$$

$$|r(t)| \leqq \frac{k}{2} \omega_2\left(g, \frac{\pi}{2n}\right). \tag{25}$$

Доказательство. Запишем (19) в виде

$$r(t) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \left[g\left(t + \frac{k-2j}{2n}\pi\right) - g(t)\right]. \tag{26}$$

Если k четное число, то здесь

$$\left|g\left(t + \frac{k-2j}{2n}\pi\right) - g(t)\right| \leqq \left|\frac{k}{2} - j\right| \omega\left(g, \frac{\pi}{n}\right).$$

Отсюда, из (26) и из равенства

$$\sum_{j=0}^{k} \binom{k}{j} \left|\frac{k}{2} - j\right| = \sum_{j=0}^{k/2} \binom{k}{j} (k-2j) = k \sum_{j=0}^{k/2} \binom{k}{j} - 2k \sum_{j=1}^{k/2} \binom{k-1}{j-1} =$$

$$= \frac{k}{2}\left[2^k + \binom{k}{k/2}\right] - k2^{k-1} = \frac{k}{2}\binom{k}{k/2}$$

получаем (22). Неравенство (24) доказывается аналогичным образом:

$$\left|g\left(t+\frac{k-2j}{2n}\pi\right)-g(t)\right| \leqq |k-2j|\omega\left(g, \frac{\pi}{2n}\right)$$

и, если k нечетное число, то

$$\sum_{j=0}^{k}\binom{k}{j}|k-2j| = 2\sum_{j=0}^{\frac{k-1}{2}}\binom{k}{j}(k-2j) = 2k\sum_{j=0}^{\frac{k-1}{2}}\binom{k}{j}-4k\sum_{j=1}^{\frac{k-1}{2}}\binom{k-1}{j-1} =$$

$$= k2^k - k\left[2^k - 2\binom{k-1}{\frac{k-1}{2}}\right] = 2k\binom{k-1}{\frac{k-1}{2}} = \frac{k+1}{2}\binom{k+1}{\frac{k+1}{2}};$$

отсюда и из (26) следует (24).

Чтобы доказать (23), заметим, что ввиду (19)

$$r(t) = \frac{1}{2^{k+1}}\sum_{j=0}^{k}\binom{k}{j}\left[g\left(t+\frac{k-2j}{2n}\pi\right)-2g(t)+g\left(t-\frac{k-2j}{2n}\pi\right)\right]$$

и, если k четное число, то здесь

$$\left|g\left(t+\frac{k-2j}{2n}\pi\right)-2g(t)+g\left(t-\frac{k-2j}{2n}\pi\right)\right| \leqq \left(\frac{k}{2}-j\right)^2\omega_2\left(g, \frac{\pi}{n}\right). \tag{27}$$

Отсюда и из формулы

$$\sum_{j=0}^{k}\binom{k}{j}\left(\frac{k}{2}-j\right)^2 = \sum_{j=0}^{k}\binom{k}{j}\left[\frac{k^2}{4}-j(k-1)+j(j-1)\right] =$$

$$= \frac{k^2}{4}\sum_{j=0}^{k}\binom{k}{j}-k(k-1)\sum_{j=1}^{k}\binom{k-1}{j-1}+k(k-1)\sum_{j=2}^{k}\binom{k-2}{j-2} =$$

$$= k^2 2^{k-2} - k(k-1)2^{k-1} + k(k-1)2^{k-2} = k2^{k-2}$$

получаем (23). Соотношение (25) доказывается аналогичным образом, только вместо (27) используется неравенство

$$\left|g\left(t+\frac{k-2j}{2n}\pi\right)-2g(t)+g\left(t-\frac{k-2j}{2n}\pi\right)\right| \leqq (k-2j)^2\omega_2\left(g, \frac{\pi}{2n}\right).$$

Замечания. 1. Из формулы Стирлинга следует асимптотическое равенство

$$\frac{k}{2^{k+1}}\binom{k}{k/2} \sim \sqrt{\frac{k}{2\pi}}. \tag{28}$$

2. Пусть $2n \geqq k$. Если $g(x) = |x-t|$ на отрезке $[t-\pi, t+\pi]$, то в соотношениях (22) и (24) имеет место равенство, а если $g(x) = (x-t)^2$, то равенство имеет место в формулах (23) и (25).

Фигурирующая в неравенстве (20) функция Лебега $\sum_{i=1}^{2n} |s_i(t)|$ может быть преобразована следующим образом:

Теорема 3. *На отрезке* (12)

$$\sum_{i=1}^{2n} |s_i(t)| = \frac{1}{2^k} \sum_{i=1}^{2n} b_i d_n\left(t - \frac{2l+2i-1}{2n}\pi\right). \tag{29}$$

Если k *положительное четное число и* $2n \geqq 3k$, *то*

$$b_i = \sum_{j=0}^{k-i} \binom{k}{j} [1+(-1)^{i+j}] \quad (1 \leqq i \leqq k), \tag{30}$$

$$b_i = 0 \quad \left(k < i \leqq n - \frac{k}{2}\right), \tag{31}$$

$$b_i = 2 \sum_{j=n+\frac{k}{2}-i+1}^{k} (-1)^{i+j} \binom{k}{j} \quad \left(n - \frac{k}{2} < i \leqq n\right), \tag{32}$$

$$b_i = b_{2n+1-i} \quad (n < i \leqq 2n). \tag{33}$$

Если k *нечетное число и* $n \geqq k$, *то*

$$b_i = \sum_{j=0}^{k-i} \binom{k}{j} [1-(-1)^{i+j}] \quad (1 \leqq i \leqq k), \tag{34}$$

$$b_i = 0 \quad (k < i \leqq n) \tag{35}$$

и имеет место (33).

Доказательство. Из (2)—(4) следует:

$$\sum_{i=1}^{2n} |s_i(t)| = \sum_{i=1}^{2n} |s_{l+i}(t)|. \tag{36}$$

Пусть k положительное четное и m целое число,

$$a_i = 1 \quad \left(2mn - \frac{k}{2} < i \leqq 2mn + \frac{k}{2}\right),$$

$$a_i = (-1)^{i+\frac{k}{2}+1} \quad \left(2mn + \frac{k}{2} < i \leqq 2mn + n\right),$$

$$a_i = (-1)^{i+\frac{k}{2}} \quad \left(2mn + n < i \leqq 2mn + 2n - \frac{k}{2}\right).$$

Ввиду (36) и (13)—(16)

$$\sum_{i=1}^{2n} |s_i(t)| = \sum_{i=1}^{2n} a_i s_{l+i}(t). \tag{37}$$

Из (3) получаем:

$$s_{l+i}(t) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} d_n\left(t - \frac{2l+2i+2j-k-1}{2n}\pi\right).$$

Отсюда и из (37) следует (29), где

$$b_i = \sum_{j=0}^{k}\binom{k}{j} a_{i-j+\frac{k}{2}} = \sum_{j=0}^{k}\binom{k}{j} a_{i+j-\frac{k}{2}}.$$

Пусть сначала $1 \leqq i \leqq k \leqq n$. Тогда

$$b_i = \sum_{j=0}^{k-i}\binom{k}{j} + \sum_{j=k-i+1}^{k}\binom{k}{j}(-1)^{i+j+1}.$$

Отсюда и из равенства

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j = 0 \quad (k \geqq 1) \tag{38}$$

получаем (30). Полагая $2n \geqq 3k$ и $k < i \leqq n - \frac{k}{2}$, видим, что

$$b_i = \sum_{j=0}^{k}\binom{k}{j}(-1)^{i+j+1} = 0$$

и формула (31) доказана. Если $n - \frac{k}{2} < i \leqq n + \frac{k}{2}$, то

$$b_i = \sum_{j=0}^{n+\frac{k}{2}-i}\binom{k}{j}(-1)^{i+j+1} + \sum_{j=n+\frac{k}{2}-i+1}^{k}\binom{k}{j}(-1)^{i+j}$$

и поэтому имеет место (32) и соотношение

$$b_i = \sum_{j=i+\frac{k}{2}-n}^{k}\binom{k}{j}(-1)^{i+j+1} + \sum_{j=0}^{i+\frac{k}{2}-n-1}\binom{k}{j}(-1)^{i+j} = b_{2n+1-i}, \tag{39}$$

а если $n + \frac{k}{2} < i \leqq 2n - k$, то

$$b_i = \sum_{j=0}^{k}\binom{k}{j}(-1)^{i+j} = 0 = b_{2n+1-i}. \tag{40}$$

Пусть наконец $2n - k < i \leqq 2n$, тогда

$$\begin{aligned} b_i &= \sum_{j=0}^{2n-i}\binom{k}{j}(-1)^{i+j} + \sum_{j=2n-i+1}^{k}\binom{k}{j} = \\ &= \sum_{j=k-i-2n}^{k}\binom{k}{j}(-1)^{i+j} + \sum_{j=0}^{k+i-2n-1}\binom{k}{j} = b_{2n+1-i}. \end{aligned} \tag{41}$$

Из (39)—(41) получаем (33).

Для четных k теорема доказана. При нечетных k можно рассуждать аналогичным образом.

Замечание. Если $2n \leqq k+2$, то

$$\sum_{i=1}^{2n} |s_i(t)| = 1,$$

так как ввиду леммы 6

$$\sum_{i=1}^{2n} |s_i(t)| = \sum_{i=1}^{2n} s_i(t)$$

и в силу (5), (6) и (10)

$$\sum_{i=1}^{2n} s_i(t) = S_{nk}(1, t) = 1. \tag{42}$$

Заметим еще, что из (42) следует неравенство

$$\sum_{i=1}^{2n} |s_i(t)| \geqq 1. \tag{43}$$

Укажем некоторые следствия теорем 1—3.

1. Если $k \geqq 1$ фиксированно и $n \to \infty$, то $S_{nk}(g, t)$ равномерно сходится к $g(t)$, так как ввиду (2) $|d_n(t)| \leqq 1$ и поэтому и в силу (29)—(35) фигурирующая в неравенстве (20) функция Лебега $\sum_{i=1}^{2n} |s_i(t)|$ ограничена, кроме того $E_{n-1}(g) \to \infty$ и, на оснований (22) и (24), $r(t) \to 0$ при $n \to \infty$.

2. Если $g(t) = f(\cos t)$, то из (20) следует:

$$|P_{nk}(f, x) - f(x)| \leqq \left[1 + \sum_{i=1}^{2n} |s_i(\arccos x)|\right] E_{n-1}(f) + |r(\arccos x)|,$$

где $E_{n-1}(f)$ наилучшее приближение функции $f(x)$ алгебраическими многочленами не выше $n-1$-ой степени (очевидно $E_{n-1}(g) = E_{n-1}(f)$). В неравенствах (22) и (24) можно заменить g на f, так как

$$\omega(g, h) \leqq \omega(f, h).$$

3. Если $k = 1$, то ввиду (33)—(35) равенство (29) можно записать в виде

$$\frac{1}{2} \sum_{i=1}^{2n} |s_i(t)| = d_n\left(t - \frac{2l-1}{2n}\pi\right) + d_n\left(t - \frac{2l+1}{2n}\pi\right).$$

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$$d_n\left(t - \frac{2l-1}{2n}\pi\right) + d_n\left(t - \frac{2l+1}{2n}\pi\right) < \frac{4}{\pi}.$$

Поэтому и в силу (20), (24) и (21)

$$|S_{n1}(g,t)-g(t)| \leqq \left(1+\frac{4}{\pi}\right)E_{n-1}(g)+\omega\left(g,\frac{\pi}{2n}\right) \leqq$$

(44)

$$\leqq \left(1+\frac{4}{\pi}\right)\omega\left(g,\frac{\pi}{n}\right)+\omega\left(g,\frac{\pi}{2n}\right) \leqq \left(3+\frac{8}{\pi}\right)\omega\left(g,\frac{\pi}{2n}\right).$$

4. Если $k=2$, то ввиду (30)—(33)

$$b_1 = b_{2n} = 4, \quad b_2 = b_{2n-1} = 2, \quad b_n = b_{n+1} = 2(-1)^n,$$

а остальные коэффициенты b_i равны нулю. Можно показать, что

$$\frac{1}{4}\sum_{i=1}^{2n} b_i d_n\left(t-\frac{2l+2i-1}{2n}\pi\right) < \frac{10}{3\pi}.$$

Поэтому и в силу (20) и (22)

$$|S_{n2}(g,t)-g(t)| \leqq \left(1+\frac{10}{3\pi}\right)E_{n-1}(g)+\frac{1}{2}\omega\left(g,\frac{\pi}{n}\right) \leqq \left(\frac{3}{2}+\frac{10}{3\pi}\right)\omega\left(g,\frac{\pi}{n}\right). \tag{45}$$

Замечание. Если $k\geqq 3$, то из теорем 1—3 получаются более плохие оценки, чем (44) и (45), так как ввиду (28) коэффициенты правых частей неравенств (22)—(25) стремятся к бесконечности с возрастанием k.

Теорема 4. *Если $k\geqq 2$ четное число и $2n\geqq 3k$, то*

$$|S_{nk}(g,t)-g(t)| \leqq \frac{1}{2^k}\omega\left(g,\frac{\pi}{n}\right)\sum_{i=1}^{2n} b_i d_n\left(t-\frac{2l+2i-1}{2n}\pi\right), \tag{46}$$

где

$$b_i = \sum_{j=0}^{\frac{k}{2}-i}\binom{k}{j}\left\{1+[1-(-1)^{i+j}]\left(\frac{k}{2}-i-j\right)\right\}+\sum_{j=\frac{k}{2}-i+1}^{k-i}\binom{k}{j}[1+(-1)^{i+j}]\left(i+j-\frac{k}{2}\right)$$

(47)

$$\left(1 \leqq i \leqq \frac{k}{2}\right),$$

$$b_i = \sum_{j=0}^{k-i}\binom{k}{j}[1+(-1)^{i+j}]\left(i+j-\frac{k}{2}\right) \quad \left(\frac{k}{2} < i \leqq k\right), \tag{48}$$

$$b_i = 0 \quad \left(k < i \leqq n-\frac{k}{2}\right), \tag{49}$$

$$b_i = (2n+1)\sum_{j=n+\frac{k}{2}-i+1}^{k}\binom{k}{j}(-1)^{i+j} \quad \left(n-\frac{k}{2} < i \leqq n\right), \tag{50}$$

$$b_i = b_{2n+1-i} \quad (n < i \leqq 2n), \tag{51}$$

Если $k \geqq 3$ нечетное число и $2n > 3k$, то

$$|S_{nk}(g, t) - g(t)| \leqq \frac{1}{2^k}\,\omega\left(g, \frac{\pi}{2n}\right) \sum_{i=1}^{2n} b_i d_n\left(t - \frac{2l+2i-1}{2n}\,\pi\right), \tag{52}$$

где

$$b_i = \sum_{j=0}^{\frac{k-1}{2}-i} \binom{k}{j} \{2 + [1 + (-1)^{i+j}](k - 2i - 2j)\} + \sum_{j=\frac{k+1}{2}-i}^{k-i} \binom{k}{j} [1 - (-1)^{i+j}](2i + 2j - k) \tag{53}$$

$$\left(1 \leqq i \leqq \frac{k-1}{2}\right),$$

$$b_i = \sum_{j=0}^{k-i} \binom{k}{j} [1 - (-1)^{i+j}](2i + 2j - k) \quad \left(\frac{k+1}{2} \leqq i \leqq k\right), \tag{54}$$

$$b_i = 0 \quad \left(k < i \leqq n - \frac{k-1}{2}\right), \tag{55}$$

$$b_i = 2 \sum_{j=n-i+\frac{k+3}{2}}^{k} (-1)^{i+j} \binom{k}{j} (2n - 2i - 2j + k + 1) \quad \left(n - \frac{k-3}{2} \leqq i \leqq n\right) \tag{56}$$

и имеет место (51).

Доказательство. Если k четное число, то формулу (5) очевидно можно записать в виде

$$S_{nk}(g, t) = \sum_{i=1}^{2n} g\left(\frac{2l+2i-1}{2n}\,\pi\right) s_{l+i}(t),$$

а формулу (42) в виде

$$\sum_{i=1}^{2n} s_{l+i}(t) = 1.$$

Поэтому

$$S_{nk}(g, t) - g(t) = \sum_{i=1}^{2n} \left[g\left(\frac{2l+2i-1}{2n}\,\pi\right) - g(t)\right] s_{l+i}(t). \tag{57}$$

Если $1 \leqq i \leqq n$, то ввиду (12) здесь

$$\left|g\left(\frac{2l+2i-1}{2n}\,\pi\right) - g(t)\right| \leqq i\omega\left(g, \frac{\pi}{n}\right),$$

а если $n < i \leqq 2n$, то

$$\left|g\left(\frac{2l+2i-1}{2n}\,\pi\right) - g(t)\right| \leqq (2n + 1 - i)\,\omega\left(g, \frac{\pi}{n}\right).$$

Следовательно

$$|S_{nk}(g, t) - g(t)| \leqq \omega\left(g, \frac{\pi}{n}\right)\left[\sum_{i=1}^{n} i|s_{l+i}(t)| + \sum_{i=n+1}^{2n} (2n + 1 - i)|s_{l+i}(t)|\right].$$

Так как $2n>k$, то отсюда получаем:

$$|S_{nk}(g,t)-g(t)| \leqq \omega\left(g,\frac{\pi}{n}\right)\sum_{i=1}^{2n} a_i s_{l+i}(t), \tag{58}$$

где ввиду (14)—(16)

$$a_i = i \quad \left(1 \leqq i \leqq \frac{k}{2}\right),$$

$$a_i = i(-1)^{i+\frac{k}{2}+1} \quad \left(\frac{k}{2} < i \leqq n\right),$$

$$a_i = (2n+1-i)(-1)^{i+\frac{k}{2}} \quad \left(n < i \leqq 2n-\frac{k}{2}\right),$$

$$a_i = 2n+1-i \quad \left(2n-\frac{k}{2} < i \leqq 2n\right).$$

Пусть

$$a_i = a_{i+2n} \quad (i<1),$$

$$a_i = a_{i-2n} \quad (i>2n).$$

Тогда в силу (58) и (3) имеет место (46), где

$$b_i = \sum_{j=0}^{k}\binom{k}{j} a_{i-j+\frac{k}{2}} = \sum_{j=0}^{k}\binom{k}{j} a_{i+j-\frac{k}{2}}.$$

Пусть $1 \leqq i \leqq \frac{k}{2}$. Тогда, так как $n \geqq k$,

$$b_i = \sum_{j=0}^{\frac{k}{2}-i}\binom{k}{j}\left(\frac{k}{2}-i-j+1\right) + \sum_{j=\frac{k}{2}-i+1}^{k-i}\binom{k}{j}\left(i+j-\frac{k}{2}\right) +$$

$$+ \sum_{j=k-i+1}^{k}\binom{k}{j}\left(i+j-\frac{k}{2}\right)(-1)^{i+j+1}.$$

Принимая во внимание тождество (38) и

$$\sum_{j=0}^{k}\binom{k}{j} j(-1)^j = k\sum_{j=1}^{k}\binom{k-1}{j-1}(-1)^j = 0 \quad (k \geqq 2),$$

получаем (47). Пусть теперь $\frac{k}{2} < i \leqq k$. Тогда при $2n \geqq 3k$

$$b_i = \sum_{j=0}^{k-i}\binom{k}{j}\left(i+j-\frac{k}{2}\right) + \sum_{j=k-i+1}^{k}\binom{k}{j}\left(i+j-\frac{k}{2}\right)(-1)^{i+j+1}$$

и поэтому имеет место (48). Если $k<i\leqq n-\frac{k}{2}$, то получаем (49):

$$b_i=\sum_{j=0}^{k}\binom{k}{j}\left(i+j-\frac{k}{2}\right)(-1)^{i+j+1}=0.$$

При $n-\frac{k}{2}<i\leqq n+\frac{k}{2}$ имеем:

$$b_i=\sum_{j=0}^{n-i+\frac{k}{2}}\binom{k}{j}\left(i+j-\frac{k}{2}\right)(-1)^{i+j+1}+\sum_{j=n-i+\frac{k}{2}+1}^{k}\binom{k}{j}\left(2n+1-i-j+\frac{k}{2}\right)(-1)^{i+j}.$$

Отсюда получаем (50) и равенство

$$b_i=\sum_{j=i+\frac{k}{2}-n}^{k}\binom{k}{j}\left(i-j+\frac{k}{2}\right)(-1)^{i+j+1}+$$

(59)

$$+\sum_{j=0}^{i+\frac{k}{2}-n-1}\binom{k}{j}\left(2n+1-i+j-\frac{k}{2}\right)(-1)^{i+j}=b_{2n+1-i}.$$

Если $n+\frac{k}{2}<i\leqq 2n-k$, то

$$b_i=\sum_{j=0}^{k}\binom{k}{j}\left(2n+1-i-j+\frac{k}{2}\right)(-1)^{i+j}=0=b_{2n+1-i}, \tag{60}$$

а если $2n-k<i<2n-\frac{k}{2}$, то

$$b_i=\sum_{j=0}^{2n-i}\binom{k}{j}\left(2n+1-i-j+\frac{k}{2}\right)(-1)^{i+j}+\sum_{j=2n-i+1}^{k}\binom{k}{j}\left(2n+1-i-j+\frac{k}{2}\right)=$$

(61)

$$=\sum_{j=i+k-2n}^{k}\binom{k}{j}\left(2n+1-i+j-\frac{k}{2}\right)(-1)^{i+j}+\sum_{j=0}^{i+k-2n+1}\binom{k}{j}\left(2n+1-i+j-\frac{k}{2}\right)=$$

$$=b_{2n+1-i}.$$

Пусть, наконец, $2n-\frac{k}{2}<i\leqq 2n$. Тогда

$$b_i=\sum_{j=0}^{2n-i}\binom{k}{j}\left(2n+1-i-j+\frac{k}{2}\right)(-1)^{i+j}+\sum_{j=2n-i+1}^{2n-i+\frac{k}{2}}\binom{k}{j}\left(2n+1-i-j+\frac{k}{2}\right)+$$

(62)

$$+\sum_{j=2n-i+\frac{k}{2}+1}^{k}\binom{k}{j}\left(i+j-\frac{k}{2}-2n\right)=\sum_{j=k+i-2n}^{k}\binom{k}{j}\left(2n+1-i+j-\frac{k}{2}\right)(-1)^{i+j}+$$

$$+\sum_{j=i+\frac{k}{2}-2n}^{i+k-2n-1}\binom{k}{j}\left(2n+1-i+j-\frac{k}{2}\right)+\sum_{j=0}^{i+\frac{k}{2}-2n-1}\binom{k}{j}\left(-2n+i-j+\frac{k}{2}\right)=b_{2n+1-i}.$$

Из (59)—(62) получаем (51) и теорема доказана для четных k. Аналогичным образом можно доказать теорему и для нечетных k.

Получим два *следствия* теоремы 4.

1. Если $k=2$ и $n\geqq 3$, то из (47)—(51) получаем:

$$b_1 = b_{2n} = 5, \quad b_2 = b_{2n-1} = 2, \quad b_n = b_{n+1} = (2n+1)(-1)^n,$$

а остальные коэффициенты b_i исчезают. Можно доказать, что

$$\frac{1}{4}\sum_{i=1}^{2n} b_i d_n\left(t - \frac{2l+2i-1}{2n}\pi\right) < \frac{13}{3\pi}.$$

Отсюда и из (46) получаем:

$$|S_{n2}(g, t) - g(t)| \leqq \frac{13}{3\pi}\,\omega\left(g, \frac{\pi}{n}\right).$$

Это неравенство лучше, чем (45) и уже было опубликовано в статье [7]. Во второй части работы мы покажем, что число $\frac{13}{3\pi}$ здесь можно заменить на $\frac{5}{4}$ и нельзя заменить на меньшее число.

2. Если $k=3$ и $n\geqq 4$, то из (53)—(56) и (51) следует:

$$b_1 = b_{2n} = 20, \quad b_2 = b_{2n-1} = 18, \quad b_3 = b_{2n-2} = 6, \quad b_n = b_{n+1} = 4(-1)^n,$$

а остальные коэффициенты b_i исчезают. Можно доказать, что

$$\sum_{i=1}^{2n} b_i d_n\left(t - \frac{2l+2i-1}{2n}\pi\right) \leqq 20.$$

Поэтому и ввиду (52)

$$|S_{n3}(g, t) - g(t)| \leqq \frac{5}{2}\,\omega\left(g, \frac{\pi}{2n}\right).$$

3. Если $g(t)=f(\cos t)$, то неравенство (46) можно записать в виде

$$|P_{nk}(f, x) - f(x)| \leqq \frac{1}{2^k}\,\omega\left(g, \frac{\pi}{n}\right)\sum_{i=1}^{2n} b_i d_n\left(\arccos x - \frac{2l+2i-1}{2n}\pi\right). \tag{63}$$

Здесь

$$\omega\left(g, \frac{\pi}{n}\right) \leqq \omega\left(f, \frac{\pi}{n}\right).$$

Аналогичным образом можно преобразовать и формулу (52).

Имеет место и более точный результат:

Теорема 5. *Если k четное число, $k\geqq 4$ и $2n\geqq 3k$, то*

$$|P_{nk}(f, x) - f(x)| \leqq \frac{1}{2^k}\,\omega\left(f, \frac{\pi}{n}\sqrt{1-x^2}\right)\sum_{i=1}^{2n} b_i d_n\left(\arccos x - \frac{2l+2i-1}{2n}\pi\right) + \tag{64}$$

$$+ \frac{1}{2^k}\,\omega\left(f, \frac{\pi^2}{2n^2}|x|\right)\sum_{i=1}^{2n} c_i d_n\left(\arccos x - \frac{2l+2i-1}{2n}\pi\right),$$

где числа b_i *определяются формулами* (47)—(51) *и*

$$c_i = \sum_{j=0}^{\frac{k}{2}-i} \binom{k}{j}\left[\left(i+j-\frac{k}{2}-1\right)^2+(-1)^{i+j}\left(i+j-\frac{k}{2}\right)^2\right]+$$
$$+\sum_{j=\frac{k}{2}-i+1}^{k-i}\binom{k}{j}[1+(-1)^{i+j}]\left(i+j-\frac{k}{2}\right)^2 \quad \left(1 \leqq i \leqq \frac{k}{2}\right), \tag{65}$$

$$c_i = \sum_{j=0}^{k-i}\binom{k}{j}[1+(-1)^{i+j}]\left(i+j-\frac{k}{2}\right)^2 \quad \left(\frac{k}{2} < i \leqq k\right), \tag{66}$$

$$c_i = 0 \quad \left(k < i \leqq n-\frac{k}{2}\right), \tag{67}$$

$$c_i = \sum_{j=n-i+\frac{k}{2}+1}^{k}\binom{k}{j}\left[\left(2n+1-i-j+\frac{k}{2}\right)^2+\left(i+j-\frac{k}{2}\right)^2\right](-1)^{i+j} \quad \left(n-\frac{k}{2} < i \leqq n\right), \tag{68}$$

$$c_i = c_{2n+1-i} \quad (n < i \leqq 2n). \tag{69}$$

Если k *нечетное число,* $k \geqq 3$ *и* $2n > 3k$, *то*

$$|P_{nk}(f, x)-f(x)| \leqq \frac{1}{2^k}\,\omega\left(f, \frac{\pi}{2n}\sqrt{1-x^2}\right)\sum_{i=1}^{2n} b_i d_n\left(\arccos x-\frac{2l+2i-1}{2n}\pi\right)+$$
$$+\frac{1}{2^k}\,\omega\left(f, \frac{\pi^2}{8n^2}|x|\right)\sum_{i=1}^{2n} c_i d_n\left(\arccos x-\frac{2l+2i-1}{2n}\pi\right), \tag{70}$$

где числа b_i *определяются формулами* (53)—(56) *и* (51),

$$c_i = \sum_{j=0}^{\frac{k-1}{2}-i}\binom{k}{j}[(2i+2j-k-2)^2-(-1)^{i+j}(2i+2j-k)^2]+$$
$$+\sum_{j=\frac{k+1}{2}-i}^{k-i}\binom{k}{j}[1-(-1)^{i+j}](2i+2j-k)^2 \quad \left(1 \leqq i \leqq \frac{k-1}{2}\right), \tag{71}$$

$$c_i = \sum_{j=0}^{k-i}\binom{k}{j}[1-(-1)^{i+j}](2i+2j-k)^2 \quad \left(\frac{k+1}{2} \leqq i \leqq k\right), \tag{72}$$

$$c_i = 0 \quad \left(k < i \leqq n-\frac{k-1}{2}\right), \tag{73}$$

$$c_i = \sum_{j=n-i+\frac{k+3}{2}}^{k}\binom{k}{j}[(4n-2i-2j+k+2)^2-(2i+2j-k)^2](-1)^{i+j} \tag{74}$$
$$\left(n-\frac{k-3}{2} \leqq i \leqq n\right)$$

и имеет место (69).

Доказательство. Пусть k четное число. Если $g(t)=f(\cos t)$, то в формуле (57)

$$\left|g\left(\frac{2l+2i-1}{2n}\pi\right)-g(t)\right| = \left|f\left(\cos\frac{2l+2i-1}{2n}\pi\right)-f(x)\right| \leqq$$

$$\leqq \omega\left(f, \left|\cos t-\cos\frac{2l+2i-1}{2n}\pi\right|\right).$$

Здесь

$$\cos t-\cos\frac{2l+2i-1}{2n}\pi = \cos t-\cos\left(\frac{2l+2i-1}{2n}\pi-t+t\right) =$$

$$=\left[1-\cos\left(\frac{2l+2i-1}{2n}\pi-t\right)\right]\cos t+\sin\left(\frac{2l+2i-1}{2n}\pi-t\right)\sin t =$$

$$=2\sin^2\left(\frac{2l+2i-1}{4n}\pi-\frac{t}{2}\right)\cos t+\sin\left(\frac{2l+2i-1}{2n}\pi-t\right)\sin t.$$

Если $1\leqq i\leqq n$, то ввиду (12) получаем отсюда:

$$\left|\cos t-\cos\frac{2l+2i-1}{2n}\pi\right| \leqq \frac{i^2\pi^2}{2n^2}|\cos t|+\frac{i\pi}{n}|\sin t|,$$

а если $n<i\leqq 2n$, то

$$\left|\cos t-\cos\frac{2l+2i-1}{2n}\pi\right| \leqq \frac{(2n-i+1)^2}{2n^2}\pi^2|\cos t|+\frac{2n+1-i}{n}\pi|\sin t|.$$

Поэтому при $1\leqq i\leqq n$

$$\left|g\left(\frac{2l+2i-1}{2n}\pi\right)-g(t)\right| \leqq i\omega\left(f, \frac{\pi}{n}|\sin t|\right)+i^2\omega\left(f, \frac{\pi^2}{2n^2}|\cos t|\right),$$

а если $n<i\leqq 2n$, то

$$\left|g\left(\frac{2l+2i-1}{2n}\pi\right)-g(t)\right| \leqq (2n+1-i)\omega\left(f, \frac{\pi}{n}|\sin t|\right)+(2n+1-i)^2\omega\left(f, \frac{\pi^2}{2n^2}|\cos t|\right).$$

Отсюда и из (57) следует:

$$|S_{nk}(g,t)-g(t)| \leqq \omega\left(f, \frac{\pi}{n}|\sin t|\right)\left[\sum_{i=1}^{n} i|s_{l+i}(t)|+\sum_{i=n+1}^{2n}(2n+1-i)|s_{l+i}(t)|\right]+$$

$$+\omega\left(f, \frac{\pi^2}{2n^2}|\cos t|\right)\left[\sum_{i=1}^{n} i^2|s_{l+i}(t)|+\sum_{i=n+1}^{2n}(2n+1-i)^2|s_{l+i}(t)|\right]. \tag{75}$$

Доказывая теорему 4, мы получили равенство

$$\sum_{i=1}^{n} i|s_{l+i}(t)|+\sum_{i=n+1}^{2n}(2n+1-i)|s_{l+i}(t)| = \frac{1}{2^k}\sum_{i=1}^{2n} b_i d_n\left(t-\frac{2l+2i-1}{2n}\pi\right), \tag{76}$$

где числа b_i определяются формулами (47)—(51). Покажем, что имеет место аналогичное равенство

$$(77)\quad \sum_{i=1}^{n} i^2|s_{l+i}(t)| + \sum_{i=n+1}^{2n} (2n+1-i)^2|s_{l+i}(t)| = \frac{1}{2^k}\sum_{i=1}^{2n} c_i d_n\left(t - \frac{2l+2i-1}{2n}\pi\right),$$

где числа c_i определяются формулами (65)—(69).

Пусть

$$a_i = i^2 \quad \left(1 \leqq i \leqq \frac{k}{2}\right),$$

$$a_i = i^2(-1)^{i+\frac{k}{2}+1} \quad \left(\frac{k}{2} < i \leqq n\right),$$

$$a_i = (2n+1-i)^2(-1)^{i+k} \quad \left(n < i \leqq 2n - \frac{k}{2}\right),$$

$$a_i = (2n+1-i)^2 \quad \left(2n - \frac{k}{2} < i \leqq 2n\right),$$

$$a_i = a_{i+2n} \quad (i < 1),$$

$$a_i = a_{i-2n} \quad (i > 2n).$$

Тогда ввиду (14) и (16)

$$\sum_{i=1}^{2n} i^2|s_{l+i}(t)| + \sum_{i=n+1}^{2n} (2n+1-i)^2|s_{l+i}(t)| = \sum_{i=1}^{2n} a_i s_{l+i}(t) = \sum_{i=1}^{2n} c_i d_n\left(t - \frac{2l+2i-1}{2n}\pi\right),$$

где

$$c_i = \sum_{j=0}^{k}\binom{k}{j} a_{i-j+\frac{k}{2}} = \sum_{j=0}^{k}\binom{k}{j} a_{i+j-\frac{k}{2}}.$$

Пусть, сначала, $1 \leqq i \leqq \frac{k}{2}$. Тогда

$$c_i = \sum_{j=0}^{\frac{k}{2}-i}\binom{k}{j}\left(i+j-\frac{k}{2}-1\right)^2 + \sum_{j=\frac{k}{2}-i+1}^{k-i}\binom{k}{j}\left(i+j-\frac{k}{2}\right)^2 +$$

$$+ \sum_{j=k-i+1}^{k}\binom{k}{j}\left(i+j-\frac{k}{2}\right)^2(-1)^{i+j+1}.$$

Отсюда и из тождества

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^m = 0 \quad (m = 0, 1, \ldots, k-1)$$

получаем формулу (65). Если $\frac{k}{2} < i \leqq k$, то

$$c_i = \sum_{j=0}^{k-i}\binom{k}{j}\left(i+j-\frac{k}{2}\right)^2 + \sum_{j=k-i+1}^{k}\binom{k}{j}\left(i+j-\frac{k}{2}\right)^2(-1)^{i+j+1}$$

и следовательно имеет место (66). Полагая $k<i\leqq n-\frac{k}{2}$, докажем (67):

$$c_i=\sum_{j=0}^{k}\binom{k}{j}\left(i+j-\frac{k}{2}\right)^2(-1)^{i+j+1}=0.$$

Пусть теперь $n-\frac{k}{2}<i\leqq n+\frac{k}{2}$. Тогда

$$\begin{aligned}c_i=&\sum_{j=0}^{n-i+\frac{k}{2}}\binom{k}{j}\left(i+j-\frac{k}{2}\right)^2(-1)^{i+j+1}+\\&+\sum_{j=n-i+\frac{k}{2}+1}^{k}\binom{k}{j}\left(2n+1-i-j+\frac{k}{2}\right)^2(-1)^{i+j}.\end{aligned}\tag{78}$$

Отсюда следует (68) и кроме того

$$\begin{aligned}c_i=&\sum_{j=i+\frac{k}{2}-n}^{k}\binom{k}{j}\left(i-j+\frac{k}{2}\right)^2(-1)^{i+j+1}+\\&+\sum_{j=0}^{i+\frac{k}{2}-n-1}\binom{k}{j}\left(2n+1-i+j-\frac{k}{2}\right)^2(-1)^{i+j}=c_{2n+1-i}.\end{aligned}\tag{79}$$

Если $n+\frac{k}{2}<i\leqq 2n-k$, то

$$c_i=\sum_{j=0}^{k}\binom{k}{j}\left(2n+1-i+j-\frac{k}{2}\right)^2(-1)^{i+j}=0=c_{2n+1-i}.\tag{80}$$

Полагая $2n-k<i\leqq 2n-\frac{k}{2}$, получаем:

$$\begin{aligned}c_i=&\sum_{j=0}^{2n-i}\binom{k}{j}\left(2n+1-i-j+\frac{k}{2}\right)^2(-1)^{i+j}+\\&+\sum_{j=2n-i+1}^{k}\binom{k}{j}\left(2n+1-i-j+\frac{k}{2}\right)^2=c_{2n+1-i}.\end{aligned}\tag{81}$$

Если, наконец, $2n-\frac{k}{2}<i\leqq 2n$, то

$$\begin{aligned}c_i=&\sum_{j=0}^{2n-i}\binom{k}{j}\left(2n+1-i-j+\frac{k}{2}\right)^2(-1)^{i+j}+\sum_{j=2n-i+1}^{2n-i+\frac{k}{2}}\binom{k}{j}\left(2n+1-i-j+\frac{k}{2}\right)^2+\\&+\sum_{j=2n-i+\frac{k}{2}+1}^{k}\binom{k}{j}\left(i+j-\frac{k}{2}-2n\right)^2=c_{2n+1-i}.\end{aligned}\tag{82}$$

Из (79)—(82) получаем (69).

Полагая $t=\arccos x$, получаем (64) из (75)—(77). Аналогичным обазом можно доказать (69)—(74).

Укажем два *следствия* теоремы 5.

1. Если $k=3$, то ввиду (71), (72) и (74)

$$c_1 = 64, \quad c_2 = 54, \quad c_3 = 18, \quad c_n = (-1)^n 8(2n+1).$$

Мы уже упоминали, что на отрезке (12)

$$\sum_{i=1}^{2n} b_i d_n\left(t - \frac{2l+2i-1}{2n}\pi\right) \leqq 20.$$

Можно показать, что

$$\sum_{i=1}^{2n} c_i d_n\left(t - \frac{2l+2i-1}{2n}\pi\right) \leqq 64.$$

Поэтому

$$|P_{n3}(f,x)-f(x)| \leqq \frac{5}{2}\omega\left(f, \frac{\pi}{2n}\sqrt{1-x^2}\right) + 8\omega\left(f, \frac{\pi^2}{8n^2}|x|\right).$$

2. Пусть $k=4$. Тогда из (47)—(50) и (65)—(68) следует:

$$b_1 = 23, \quad b_2 = 25, \quad b_3 = 16, \quad b_4 = 4,$$

$$b_{n-1} = (-1)^{n+1}(2n+1), \quad b_n = (-1)^{n+1}3(2n+1),$$

$$c_1 = 39, \quad c_2 = 49, \quad c_3 = 32, \quad c_4 = 8,$$

$$c_{n-1} = (-1)^{n+1}(2n^2+2n+1), \quad c_n = (-1)^{n+1}(6n^2+6n-1).$$

Можно доказать, что

$$\sum_{i=1}^{2n} b_i d_n\left(t - \frac{2l+2i-1}{2n}\pi\right) \leqq 23,$$

$$\sum_{i=1}^{2n} c_i d_n\left(t - \frac{2l+2i-1}{2n}\pi\right) \leqq 39.$$

Поэтому

$$|P_{n4}(f,x)-f(x)| \leqq \frac{23}{16}\omega\left(f, \frac{\pi}{n}\sqrt{1-x^2}\right) + \frac{39}{16}\omega\left(f, \frac{\pi^2}{2n^2}|x|\right).$$

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Цитированная литература

[1] С. Н. Бернштейн, Об одном видоизменении интерполяционной формулы Лагранжа, *Зап. Харк. матем. тов.*, **5** (1931), стр. 49—57.
[2] С. Н. Бернштейн, О тригонометрическом интерполировании по способу наименьших квадратов, *ДАН* **4** (1934), стр. 1—8.
[3] А. Ф. Харшиладзе, Об одной теореме С. Н. Бернштейна из теории интерполяций, *Труды института точн. механ. и оптики*, **1**:3 (1941), стр. 207—212.
[4] И. П. Натансон, *Конструктивная теория функций*. Москва, гос. изд. техн.—теор. лит., 1949.
[5] Д. Л. Берман, Сходимость некоторых интерполяционных операций, *ДАН* **64** (1949), стр. 5—8.
[6] Д. Л. Берман, Исследование сходимости интерполяционных процессов, *ДАН* **102** (1955), стр. 867—869.
[7] О. Киш, О некоторых интерполяционных процессах С. Н. Бернштейна, *Acta Math. Acad. Sci. Hung.*, **24** (1973), стр. 353—361.
[8] A. C. Offord, Approximation to functions by trigonometric polynomials, *Duke Math. Journal*, **6** (1940), стр. 505—510.
[9] G. Grünwald, On a convergence theorem for the Lagrange interpolation polynomials, *Bull. Amer. Math. Soc.*, **47** (1941), стр. 271—275.
[10] T. M. Mills, A. K. Varma, New estimates for G. Grünwald's convergence theorem concerning Lagrange interpolation and a new proof of A. F. Timan theorem, preprint, 1973.

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AN INTEGRAL INEQUALITY SATISFIED BY BESSEL FUNCTIONS

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1. Let $j_{\alpha 1}, j_{\alpha 2}, \ldots$ denote the positive roots of the Bessel function $J_\alpha(x)$ in increasing order. R. ASKEY [1] conjectured that if $\beta(\alpha)$ is the root of the transcendental equation

$$\int_0^{j_{\alpha 2}} \xi^{-\beta(\alpha)} J_\alpha(\xi)\, d\xi = 0,$$

where $-1<\alpha<1/2$ and $\beta>\beta(\alpha)$ then

$$\int_0^y \xi^{-\beta} J_\alpha(\xi)\, d\xi > 0 \quad (y>0).$$

In the present paper we shall give a proof of this inequality in the case $-1/2<\alpha<1/2$.[1]

Actually we shall prove somewhat more, namely: *if $C_\alpha(x)$ is any real solution of Bessel's differential equation of order α ($|\alpha|<1/2$), $c_{\alpha k}$ is its k'th positive zero, $C_\alpha(x)>0$ if $0<x<c_{\alpha 1}$, further $\beta(\alpha)$ is defined by*

$$\int_0^{c_{\alpha 2}} \xi^{-\beta(\alpha)} C_\alpha(\xi)\, d\xi = 0, \tag{1}$$

then, if $\beta>\beta(\alpha)$ and $y>0$, one has

$$k(y) = \int_0^y \xi^{-\beta} C_\alpha(\xi)\, d\xi > 0. \tag{2}$$

We shall use the following theorem [2]:

Let the differentiable function $f(x)$ be monotonic in $x_0<x<X_0$ (where X_0 is finite or infinite). Let further $y(x)$ be that particular solution of the differential equation $y''+f(x)y=0$, whose three consecutive zeros in $x_0\leqq x\leqq X_0$ are x_1, x_2 and x_3. Then, if $f(x)$ is strictly decreasing in (x_0, X_0),

$$|y(x_2-u)| < |y(x_2+u)| \quad (0<u\leqq x_2-x_1)$$

[1] The validity of this inequality for $-1<\alpha\leqq -1/2$ was proved in the meantime by R. Askey and J. Steinig (oral communication).

holds and if $f(x)$ strictly increases in (x_0, X_0), we have

$$|y(x_2-u)| > |y(x_2+u)| \quad (0 < u \leqq x_3-x_2).$$

A corollary of this theorem is the following.

Let $g(x)$ be a positive and monotonic function defined in (x_0, X_0). Then if $g(x)$ is nondecreasing and $f(x)$ decreasing in (x_0, X_0) we have

$$\int_{x_1}^{x_2} g(x)|y(x)|dx < \int_{x_2}^{x_3} g(x)|y(x)|dx.^2 \tag{3}$$

If, however, $g(x)$ is nonincreasing and $f(x)$ increasing, the reverse inequality holds.[3]

2. Turning now to the proof of (2) we start from the differential equation

$$\frac{d^2\eta}{dx^2}+q(x)\eta = 0, \quad q(x) = \gamma^2 x^{2\gamma-2}+\frac{1-4\alpha^2\gamma^2}{4x^2}, \tag{4}$$

a solution of which is

$$\eta = \sqrt{x}\,C_\alpha(x^\gamma). \tag{5}$$

We assume

$$|\alpha| < \frac{1}{2}, \quad 1 < \gamma < \frac{1}{2|\alpha|}. \tag{6}$$

By these assumptions the first term of $q(x)$ is increasing and the second one decreasing for positive x's, thus $q(x)$ has a unique minimum in $(0, \infty)$. Denoting the place of the minimum by $x_\alpha(\gamma)$ a simple calculation leads to

$$[x_\alpha(\gamma)]^\gamma = \frac{\sqrt{1-4\alpha^2\gamma^2}}{2\gamma\sqrt{\gamma-1}}. \tag{7}$$

[2] If $y(x)$ has but two zeros in (x_0, X_0), say x_2 and x_3, then of course

$$\int_{x_0}^{x_2} g(x)|y(x)|dx < \int_{x_2}^{x_3} g(x)|y(x)|dx.$$

[3] Moreover, in this case, if $y(x)>0$ in the open interval (x_1, x_2), one has

$$\varphi(X) = \int_{x_1}^{X} g(x)y(x)dx > 0 \quad (x_1 < X < X_0).$$

Indeed, since all the zeros $x_1, x_2, \ldots$ of $y(x)$ are simple ones, $\operatorname{sgn} y(x)=(-1)^n$ in (x_{n-1}, x_n) and the local minima of the function $\varphi(X)$ are at the places x_{2k+1} $(k=1, 2, \ldots)$. These minima are positive, since

$$\varphi(x_{2k+1}) = \sum_{i=1}^{k}\Big[\int_{x_{2i-1}}^{x_{2i}} g(x)|y(x)|dx - \int_{x_{2i}}^{x_{2i+1}} g(x)|y(x)|dx\Big]$$

and by our assumption the quantities in the brackets are positive ones.

Now we dispose of the value of the parameter γ and choose it so that the relation

$$\frac{\sqrt{1-4\alpha^2\gamma^2}}{2\gamma\sqrt{\gamma-1}} = c_{\alpha 2} \tag{8}$$

should hold.

This can be always done. Indeed, $[x_\alpha(\gamma)]^\gamma$ is a continuous function of γ in the open interval $\left(1, (2|\alpha|)^{-1}\right)$,

$$\lim_{\gamma\to 1}[x_\alpha(\gamma)]^\gamma = \infty, \quad \lim_{\gamma\to(2|\alpha|)^{-1}}[x_\alpha(\gamma)]^\gamma = 0,$$

hence for an appropriate choice of γ (8) holds.

Having thus fixed the quantity γ we see that $q(x)$ decreases in the interval $(0, c_{\alpha 2}^{1/\gamma})$ and increases in the interval $(c_{\alpha 2}^{1/\gamma}, \infty)$. Hence, if $\delta \geqq 0$ we have by footnote [2]

$$\int_0^{c_{\alpha 2}^{1/\gamma}} x^{\frac{1}{2}+\delta} C_\alpha(x^\gamma)\,dx < 0. \tag{9}$$

On the other hand, if $\varepsilon \geqq 0$ and $y > c_{\alpha 2}$ we have by footnote [3]

$$h(y) = \int_{c_{\alpha 2}^{1/\gamma}}^{y^{1/\gamma}} x^{\frac{1}{2}-\varepsilon} C_\alpha(x^\gamma)\,dx > 0. \tag{10}$$

Introducing now the new variable $\xi = x^\gamma$ we transform the integrals (9) and (10) into

$$\int_0^{c_{\alpha 2}} \xi^{\frac{3}{2\gamma}-1+\delta'} C_\alpha(\xi)\,d\xi < 0 \quad (\delta' \geqq 0) \tag{11}$$

and

$$\int_{c_{\alpha 2}}^{y} \xi^{\frac{3}{2\gamma}-1-\varepsilon'} C_\alpha(\xi)\,d\xi > 0 \quad (\varepsilon' \geqq 0,\ y > c_{\alpha 2}). \tag{12}$$

Comparing the definition (1) of $\beta(\alpha)$ with (11) we see that

$$\frac{3}{2\gamma} - 1 > -\beta(\alpha) > -\beta \tag{13}$$

and by (12) and (13)

$$\int_{c_{\alpha 2}}^{y} \xi^{-\beta} C_\alpha(\xi)\,d\xi > 0. \tag{14}$$

Now we show that the function $k(y)$ defined by (2) is positive if $y > 0$. This is obvious if $0 < y \leqq c_{\alpha 1}$. If $c_{\alpha 1} < y < c_{\alpha 2}$, we have $C_\alpha(y) < 0$, hence

$$k(y) \geqq \int_0^{c_{\alpha 2}} \xi^{-\beta} C_\alpha(\xi)\,d\xi \quad (c_{\alpha 1} < y \leqq c_{\alpha 2}).$$

Moreover in $0<\xi<c_{\alpha 2}$ we have

$$\xi^{-\beta}C_\alpha(\xi) > c_{\alpha 1}^{\beta(\alpha)-\beta}\xi^{-\beta(\alpha)}C_\alpha(\xi)$$

with the exception of $\xi=c_{\alpha 1}$, thus by (1)

$$k(y) > c_{\alpha 1}^{\beta(\alpha)-\beta}\int_0^{c_{\alpha 2}} \xi^{-\beta(\alpha)}C_\alpha(\xi)d\xi = 0 \quad (c_{\alpha 1} < y \leqq c_{\alpha 2}). \tag{15}$$

Finally if $y>c_{\alpha 2}$, then by (15) and (14)

$$k(y) = \int_0^{c_{\alpha 2}} \xi^{-\beta}C_\alpha(\xi)d\xi + \int_{c_{\alpha 2}}^{y} \xi^{-\beta}C_\alpha(\xi)d\xi > 0.$$

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References

[1] R. ASKEY, Positive Jacobi polynomial sums, *Tôhoku Math. J.*, 2nd series, **24** (1972), pp. 109—119.

[2] E. MAKAI, On a monotonic property of certain Sturm—Liouville functions, *Acta Math. Acad. Sci. Hung.*, **3** (1952), pp. 165—172.

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ON THE DIVERGENCE OF INTERPOLATION PROCESSES ON SETS HAVING POSITIVE MEASURE

By
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1. Results

1.1. Let us consider the interpolation process $L_n(f;\vartheta)=\sum_{k=1}^{2n} f(\vartheta_{kn})l_{kn}(\vartheta)$ for 2π-periodic continuous functions $f(\vartheta)$ (i.e. $f\in\tilde{C}$) on the $[0, 2\pi)$ with nodes and fundamental polynomials

$$\vartheta_{kn}=\frac{2k-1}{2n}\pi, \quad l_{kn}(\vartheta)=\frac{(-1)^k}{2n}\cos nx \operatorname{ctg}\frac{\vartheta-\vartheta_{kn}}{2} \quad (k=1, 2, \ldots, 2n), \tag{1.1}$$

respectively. As M. RIESZ showed in [1][1] for any trigonometric polynomial $T_{n-1}(\vartheta)$ of degree $\leqq n-1$ we have

$$T_{n-1}(\vartheta)=L_n(T_{n-1};\vartheta)=\sum_{k=1}^{2n} T_{n-1}(\vartheta_{kn})l_{kn}(\vartheta) \tag{1.2}$$

from where we can obtain

$$|L_n(f;\vartheta)-f(\vartheta)|\leqq c_m[\lambda_n(\vartheta)+1]\omega_m\left(f;\frac{1}{n}\right) \quad (n=1, 2, \ldots) \tag{1.3}$$

for any fixed ϑ where $\omega_m(f;t)$ is the m-th modulus of smoothness of $f(\vartheta)$,

$$\lambda_n(\vartheta)=\sum_{k=1}^{2n}|l_{kn}(\vartheta)|, \quad \lambda_n=\max_{0\leqq x<2\pi}\lambda_n(\vartheta). \tag{1.4}$$

We now can prove that $\lambda_n\sim\log n$ $(n=2, 3, 4, \ldots)$,[2] so we have

$$|L_n(f;\vartheta)-f(\vartheta)|=O(\log n)\omega_m\left(f;\frac{1}{n}\right) \quad (n=2, 3, 4, \ldots).$$

In this paper we wish to investigate the divergence-behaviour of L_n for (1.1) and some other systems of nodes.

Using the very surprising FABER's theorem we can obtain an $f(\vartheta)\in\tilde{C}$ such that $L_n(f;\vartheta)$ is not uniformly bounded, and so, in particular, does not converge uniformly ([4]).

Further, with the methods used by ERDŐS in [5] and [6] we can prove that there exists an $f\in\tilde{C}$ such that if $\vartheta_0=\frac{p}{q}\pi$, where p and q are odd then $\lim_{n\to\infty}|L_n(f;\vartheta_0)|=\infty$.

[1] See further [2], 4.2.2 and [3], 1.81 and 1.82.
[2] $f_n\sim g_n$ means that $f_n=O(g_n)$ further $g_n=O(f_n)$.

As in their famous papers G. GRÜNWALD [7] and J. MARCINKIEWICZ [8] have done, we can show that there is an $f\in\tilde{C}$ such that $L_n(f;\vartheta)$ diverges everywhere.

1.2. Let us denote by $\omega_m(t)$ a function such that

(i) $\omega_m(t)>0$ for $t>0$, $\omega_m(0)=0$, $\omega_m(T)\geqq\omega_m(t)$ for $T\geqq t$, $\omega_m(t)$ is continuous for $t\geqq 0$,

(ii) $\dfrac{t^m}{\omega_m(t)}$ is monotone increasing for $t\geqq 0$ ($m\geqq 1$ fixed integer).

Let us denote by $\tilde{C}(\omega_m)$ the functions $f(\vartheta)$ for which $f(\vartheta)\in\tilde{C}$ further $\omega_m(f;t)\leqq$ $\leqq a_m(f)\omega_m(t)$. Then we have the following estimation, which is a special case of a theorem proved by O. KIS in [9]:

If $\lim\limits_{t=+0}\left(t^m/\omega_m(t)\right)=0$ then there exists an $f(\vartheta)\in\tilde{C}(\omega_m)$ such that

$$\overline{\lim_{n\to\infty}}\,\|L_n(f;\vartheta)-f(\vartheta)\|_{[0,2\pi)}>\omega_m\left(\frac{1}{n}\right)\log n\quad(n=2,3,4,\ldots).^{3} \tag{1.5}$$

As a pointwise theorem, I succeeded in proving a theorem (see [10], Theorem 1.1, Theorem 2.1 and 2.8.3) as follows.

For any fixed $\vartheta^*\in[0,2\pi)$ there are $f(\vartheta)\in\tilde{C}(\omega_m)$ and $\{n_i\}_{i=1}^{\infty}$ such that

$$L_n(f;\vartheta^*)-f(\vartheta^*)>\omega_m\left(\frac{1}{n}\right)\log n\quad(n=n_1,n_2,n_3,\ldots), \tag{1.6}$$

further for any $\vartheta\neq\vartheta^*$ (where $0\leqq\vartheta<2\pi$) we have

$$|L_N(f;\vartheta)-f(\vartheta)|\leqq c(\vartheta)\,\omega_m\left(\frac{1}{N}\right)\quad(N\geqq n_0(\vartheta)). \tag{1.7}$$

1.3. Our aim is to generalize the above mentioned GRÜNWALD's and MARCINKIEWICZ's theorem considering first of all the ideas of [7], [9] and [10]. We prove

THEOREM 1.1. *If* $\lim\limits_{t=+0}\left(t^m/\omega_m(t)\right)=0$ *then there is a function* $h(\vartheta)\in\tilde{C}(\omega_m)$ *such that for any* $-\infty<\vartheta^*<\infty$ *there is a subsequence* $\{n_k\}_{k=1}^{\infty}$ *depending on* ϑ^* *for which*

$$|L_{n_k}(h;\vartheta^*)-h(\vartheta^*)|>\omega_m\left(\frac{1}{n_k}\right)\log k\quad(k=1,2,3,\ldots), \tag{1.8}$$

from where we have that for any fixed ϑ^*

$$\overline{\lim_{n\to\infty}}\,\frac{|L_n(h;\vartheta^*)-h(\vartheta^*)|}{\omega_m\left(\dfrac{1}{n}\right)}=\infty.$$

Comparing this result with (1.6) we can say that we lost the exact order of the divergence but we succeeded in proving the theorem for the whole interval by the only function $h(\vartheta)$.

[3] $\|g_n(\vartheta)\|_{[0,2\pi)}=\max\limits_{0\leqq x<2\pi}|g(\vartheta)|$.

Let us consider a special case. If $\omega_m(t)=t^\varrho$ $(0<\varrho<m)$ then we have for a suitable $h(\vartheta)\in\tilde{C}(t^\varrho)$

$$|L_{n_k}(h;\vartheta^*)-h(\vartheta^*)|>\frac{\log k}{n_k^\varrho}\quad(-\infty<\vartheta^*<\infty,\ \omega_m(t)=t^\varrho,\ k=1,2,3,\ldots). \tag{1.9}$$

By (1.3)

$$|L_n(h;\vartheta)-h(\vartheta)|\leqq c\frac{\log n}{n^\varrho}\quad(n=2,3,4,\ldots)$$

so these estimations do not "touch" each other.

But if the $\omega_m(t)$ is "bad", e.g.

$$\frac{\omega_m(t^{1+r})}{\omega_m(t)}\geqq c(r)>0\quad(r\geqq 0\ \text{real}) \tag{1.10}$$

then the following result is more useful.

THEOREM 1.2. *If* $\lim\limits_{t=+0}(t^m/\omega_m(t))=0$ *and* $\varepsilon>0$ *then there is a function* $h(\vartheta)\in\tilde{C}(\omega_m)$ *such that for any* $-\infty<\vartheta^*<\infty$ *there is a subsequence* $\{n_k\}_{k=1}^\infty$ *depending on* ϑ^* *and* ε *for which*

$$|L_{n_k}(h;\vartheta^*)-h(\vartheta^*)|>\omega_m\left(\frac{1}{n_k^{1+\varepsilon}}\right)\log n_k\quad(k=1,2,3,\ldots,\varepsilon>0). \tag{1.11}$$

Moreover, if

$$\omega_m(t)=\begin{cases}|\log t|^{s-1} & \text{if}\quad 0\leqq t\leqq e^{s-1}\\ \omega(e^{s-1}) & \text{if}\quad t>e^{s-1}\end{cases}\qquad(0\leqq s<1),$$

then by (1.3) and (1.11)

$$\overline{\lim_{n\to\infty}}\frac{|L_n(h;\vartheta^*)-h(\vartheta^*)|}{(\log n)^s}\sim 1\quad(-\infty<\vartheta^*<\infty;\ 0\leqq s<1), \tag{1.12}$$

i.e. our estimations "touch" each other.

1.4. Now let us consider the Lagrange interpolatory process

$$L_n(f;x)=\sum_{k=1}^n f(x_{kn})l_{kn}(x)$$

where (with $x=\cos\vartheta$)

$$x_{kn}=\cos\frac{2k-1}{2n}\pi=\cos\vartheta_{kn},\quad l_{kn}(x)=(-1)^{k-1}\frac{\sin\vartheta_{kn}}{n}\frac{\cos n\vartheta}{\cos\vartheta-\cos\vartheta_{kn}}$$
$$(k=1,2,\ldots,n) \tag{1.13}$$

where the nodes (1.3) are the roots of the Chebyshev polynomials of the first kind of degree n, $f(x)$ is continuous on $[-1,1]$ (i.e. $f\in C^{[-1,1]}$). Of course we can apply the FABER's theorem ([4]), even the mentioned ERDŐS' and GRÜNWALD—MARCINKIEWICZ's theorems were proved for this process $\left(\text{with } x_0=\cos\frac{p}{q}\pi\right)$.

The theorems analogous to (1.5) and (1.6) can be proved for $L_n(f; x)$ by supposing in (1.6) that $x^* \neq \pm 1$. The case $x^* = \pm 1$ demands distinct investigations (see [9] and [10]).

1.5. In this part we assert the following theorems.

THEOREM 1.3. *If* $\lim_{t=+0} (t^m/\omega_m(t)) = 0$ *further* $[a, b] \subseteqq [-1, 1]$ *then there is a function* $h(x) \in C^{[-1,1]}(\omega_m)$ *such that for any* $x^* \in [a, b]$ *there is a subsequence* $\{n_k\}$ *depending on* x^* *and* $[a, b]$ *for which*

$$|L_{n_k}(h; x^*) - h(x^*)| > \begin{cases} \omega_m\left(\frac{1}{n_k}\right)\log k & ([a, b] \subset (-1, 1);\ k = 1, 2, 3, \ldots), \\ \omega_m\left(\frac{1}{n_k}\right) & ([a, b] \equiv [-1, 1];\ k = 1, 2, 3, \ldots). \end{cases} \tag{1.14}$$

Moreover, if $\omega_m(t)$ is "bad", then we have

THEOREM 1.4. *If* $\lim_{t=+0} (t^m/\omega_m(t)) = 0$ *and* $\varepsilon > 0$ *then there is a function* $h(x) \in C^{[-1,1]}(\omega_m)$ *such that for any* $x^* \in [-1, 1]$ *there is a subsequence* $\{n_k\}$ *depending on* x^* *and* ε *for which*

$$|L_{n_k}(h; x^*) - h(x^*)| > \omega_m\left(\frac{1}{n_k^{2(1+\varepsilon)}}\right)\log n_k \quad (k = 1, 2, 3, \ldots). \tag{1.15}$$

1.6. We can prove similar theorems for another nodes. E.g. for

$$\vartheta_{kn} = \frac{2k\pi}{2n+1} \quad (k = 0, 1, \ldots, 2n;\ \vartheta^* \neq 0); \quad \vartheta_{kn} = \frac{k\pi}{n} \quad (k = 0, 1, \ldots, 2n-1;\ \vartheta^* \neq 0);$$

$$x_{kn} = \cos\frac{k\pi}{n} \quad (k = 0, 1, \ldots, n;\ x^* \neq \pm 1), \text{ etc.}$$

2. Proofs

2.1. PROOF OF THEOREM 1.1. Of course we suppose that $0 \leqq \vartheta^* < 2\pi$. Moreover, at first let

$$0 < \vartheta^* < 2\pi. \tag{2.1}$$

By the restrictions for $\omega_m(t)$ we can choose a matrix NP

$$\begin{array}{llll} n_{11} & 2n_{11} & p_1 & 2p_1 \\ n_{21} & 2n_{21}\ n_{22}\ 2n_{22} & p_2 & 2p_2\ p_3\ 2p_3 \end{array} \quad \text{or} \tag{2.2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$n_{k1}\ 2n_{k1}\ n_{k2}\ 2n_{k2} \ldots n_{kk}\ 2n_{kk} \qquad p_{s+1}\ 2p_{s+1}\ p_{s+2}\ 2p_{s+2} \ldots p_{s+k}\ 2p_{s+k}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where $0<p_1<p_2<\dots$ ($n_{11}=p_1$, $n_{21}=p_2$, $n_{22}=p_3$, etc.), further with $0<q<1$

$$\omega_m\left(\frac{1}{p_1}\right)\leqq q, \quad \omega_m\left(\frac{1}{p_{i+1}}\right)\leqq q\omega_m\left(\frac{1}{p_i}\right) \quad (i=1, 2, 3, \dots), \tag{2.3}$$

$$\sum_{i=1}^{j-1}\omega_m\left(\frac{1}{p_i}\right)p_i^m\leqq\omega_m\left(\frac{1}{p_j}\right)p_j^m \quad (j=2, 3, 4, \dots), \tag{2.4}$$

$$\log 2n_{ki}\,\omega_m\left(\frac{1}{n_{k,i+1}}\right)\leqq q\log k\,\omega_m\left(\frac{1}{n_{ki}}\right) \tag{2.5}$$

$$(k=1, 2, 3, \dots;\ i=1, 2, \dots, k;\ n_{k,k+1}\equiv n_{k+1,1})$$

(see further (2.17)).

Now we define the functions $g_{n_{ki}}(\vartheta)$ ($k=1, 2, 3, \dots$; $i=1, 2, \dots, k$). First of all we divide the interval $[0, 2\pi)$ into $(k+1)$ equal parts

$$I_{k1}=\left[0, \frac{2\pi}{k+1}\right),\ I_{k2}=\left[\frac{2\pi}{k+1}, 2\frac{2\pi}{k+1}\right), \dots,\ I_{k,k+1}=\left[k\frac{2\pi}{k+1}, 2\pi\right).$$

Using that $\left(\bigcup_k \vartheta_{kn}\right)\cap\left(\bigcup_j \vartheta_{j,2n}\right)=\varnothing$, with the notation

$$I_{k1}\cup I_{k2}\cup\dots\cup I_{ki}=\Delta_{ki}$$

we can define $g_{n_{ki}}(\vartheta)$ as follows.

$$\begin{cases} g_{n_{ki}}(\vartheta_{j,n_{ki}})=\begin{cases}(-1)^j & \text{if} \quad \vartheta_{j,n_{ki}}\in\Delta_{ki}\\ 0 & \text{if} \quad \vartheta_{j,n_{ki}}\notin\Delta_{ki}\end{cases} & (j=1, 2, \dots, n_{ki})\\ g_{n_{ki}}(\vartheta_{l,2n_{ki}})=\begin{cases}(-1)^l & \text{if} \quad \vartheta_{l,2n_{ki}}\in\Delta_{k_i}\\ 0 & \text{if} \quad \vartheta_{l,2n_{ki}}\notin\Delta_{ki}\end{cases} & (l=1, 2, \dots, 2n_{ki}).\end{cases} \tag{2.6}$$

Further let

$$g_{p_i}(0)=g_{p_i}(2\pi)=0 \quad (i=1, 2, 3, \dots). \tag{2.7}$$

If we denote by $d_{n_{ki}}$ the minimal distance of the points where $g_{n_{ki}}(\vartheta)=0$ or ±1 (see (2.6) and (2.7)), with a simple computation we get

$$d_{n_{ki}}\sim\frac{1}{n_{ki}} \quad (k=1, 2, 3, \dots;\ i=1, 2, \dots, k). \tag{2.8}$$

Moreover, with $\vartheta_0=0$ and $\vartheta_n=2\pi$ in the subintervals let

$$g_{n_{ki}}(\vartheta)=g_{n_{ki}}(\vartheta_j)\left[1-\frac{(2m+1)!!}{(2m)!!}\int_{-1}^{2\frac{\vartheta-\vartheta_j}{\vartheta_{j+1}-\vartheta_j}-1}(1-t^2)^m dt\right] \tag{2.9}$$

$$(\vartheta_j\leqq\vartheta\leqq\vartheta_{j+1},\ \vartheta_j\in[0, 2\pi))$$

and 2π-periodic elsewhere.

By (2.8) and (2.9) we can verify (see [9])

$$\omega(g_{n_{rs}}^{(i)}; t)\leqq B_{m+1}d_{rs}^{-(i+1)}t \quad (i=0, 1, \dots, m)$$

from where by [2], 3.3.1, (1)

$$\omega_{i+1}(g_{n_{rs}};\, t) \leqq B_{m+1} d_{rs}^{-(i+1)} t^{i+1} \quad (i = 0, 1, \ldots, m) \tag{2.10}$$

i.e. if $i+1=m$ we get $g_{n_{rs}} \in \tilde{C}(\omega_m)$.

As we can see $|g_{p_i}(\vartheta)| \leqq 1$, and hence

$$|L_{p_j}(g_{p_i};\, \vartheta)| \leqq c_5 \log p_j \quad (i, j = 1, 2, 3, \ldots). \tag{2.11}$$

Now we prove two lemmas.

LEMMA 2.1 (see [7]). *For any* ϑ

$$\max(|\cos\vartheta|, |\cos 2\vartheta|) \geqq \frac{1}{2}. \tag{2.12}$$

Indeed, if $|\cos\vartheta| < \frac{1}{2}$ then

$$|\cos 2\vartheta| = |1 - 2\cos^2\vartheta| > 1 - 2\frac{1}{4} = \frac{1}{2}.$$

LEMMA 2.2 (see [7], (8) and (9)). *If* $\vartheta^* \in I_{k,\, i+1}$ $(i \geqq 1)$ *then*

$$\begin{cases} |L_{n_{ki}}(g_{n_{ki}};\, \vartheta^*)| \geqq c_4 |\cos n_{ki}\vartheta^*| \log k & (k \geqq k_0(\vartheta^*)) \\ |L_{2n_{ki}}(g_{n_{ki}};\, \vartheta^*)| \geqq c_4 |\cos 2n_{ki}\vartheta^*| \log k & (k \geqq k_0(\vartheta^*)) \end{cases} \tag{2.13}$$

where c_4 *does not depend on* ϑ^*.

Indeed, using (2.6) and (1.1) we have

$$|L_{2n_{ki}}(g_{n_{ki}};\, \vartheta^*)| = \left| \sum_{j=1}^{4n_{ki}} g_{n_{ki}}(\vartheta_j) l_j(\vartheta^*) \right| = \left| \frac{\cos 2n_{ki}\vartheta^*}{4n_{ki}} \sum_{\vartheta_j \in \Delta_{ki}} \frac{\cos\dfrac{\vartheta^* - \vartheta_j}{2}}{\sin\dfrac{\vartheta^* - \vartheta_j}{2}} \right| \geqq$$

$$\geqq \frac{1}{\sqrt{2}\pi} \frac{|\cos 2n_{ki}\vartheta^*|}{4n_{ki}} \sum_{\substack{\vartheta^* - \vartheta_j \leqq \frac{\pi}{2} \\ \vartheta_j \in \Delta_{ki}}} \frac{1}{\vartheta - \vartheta_j} \geqq c \frac{|\cos 2n_{ki}\vartheta^*|}{4n_{ki}} 4n_{ki} \left[\log\left(n_{ki} \frac{\vartheta^*}{2\pi} \right) - \log\frac{n_{ki}}{k+1} \right] \geqq$$

$$\geqq c_4 |\cos 2n_{ki}\vartheta^*| \log k.$$

if $k \geqq k_0(\vartheta^*)$. The estimation for n_{ki} is similar.

Now let us consider the function

$$f(\vartheta) = Q \sum_{i=1}^{\infty} \omega_m\left(\frac{1}{p_i}\right) g_{p_i}(\vartheta) \equiv Q \sum_{r=1}^{\infty} \sum_{s=1}^{r} \omega_m\left(\frac{1}{n_{rs}}\right) g_{n_{rs}}(\vartheta). \tag{2.4}$$

At first we prove that $f(\vartheta) \in \tilde{C}(\omega_m)$. Indeed, if

$$\frac{1}{p_{j+1}} < t \leqq \frac{1}{p_j}$$

we have, using (2.10), $|g_{p_i}(\vartheta)| \leqq 1$, (2.4), (2.3) and (ii)

$$\omega_m(f;t) \leqq Q \sum_{i=1}^{\infty} \omega_m\left(\frac{1}{p_i}\right) \omega_m(g_{p_i};t) = Q\left(\sum_{i=1}^{j} + \sum_{i=j+1}^{\infty}\right) \leqq$$

$$\leqq Q\left[B_m t^m \sum_{i=1}^{j} \omega_m\left(\frac{1}{p_i}\right) p_i^m + 2^m \sum_{i=j+1}^{\infty} \omega_m\left(\frac{1}{p_i}\right)\right] \leqq Q 2 B_m t^m \omega_m\left(\frac{1}{p_j}\right) p_j^m +$$

$$+ Q 2^m \frac{\omega_m(t)}{1-q} \leqq 2Q\left(B_m + \frac{2^{m-1}}{1-q}\right) \omega_m(t).$$

Now we prove (1.8) for any $0 < \vartheta^* < 2\pi$. Let us consider for every k the number i such that $\vartheta^* \in I_{k,i+1}$. By (2.1) we have that $i \geqq 1$ if $k \geqq k(\vartheta^*)$. Then we choose from n_{ki} and $2n_{ki}$ the very value v for which $|\cos v\vartheta^*| > \frac{1}{2}$. This is possible by (2.12). So we have by (2.13)

$$|L_{2n_{ki}}(g_{n_{ki}};\vartheta^*)| \geqq \frac{c_4}{2} \log k \quad (\text{if } v = 2n_{ki}). \tag{2.15}$$

Now we prove that for any ϑ

$$|L_{2p_R}(g_{p_r};\vartheta) - g_{p_r}(\vartheta)| \leqq \omega_m\left(\frac{1}{p_R}\right) \quad \text{if} \quad R > r \tag{2.16}$$

supposing that $\{p_i\}$ is a suitable lacunary sequence.

Indeed, let $T_N(\vartheta)$ be the best approximating trigonometric polynomial to $g_{p_r}(\vartheta)$ on $[0, 2\pi)$ of degree $\leqq N$. Then by (2.10)

$$|g_{p_r}(\vartheta) - T_N(\vartheta)| \leqq c_{m+1} \omega_{m+1}\left(g_{p_r}; \frac{1}{N}\right) \leqq h_{m+1} \frac{p_r^{m+1}}{N^{m+1}}.$$

If $N = p_k - 1$, then we have

$$|L_{2p_R}(g_{p_r};\vartheta) - T_N(\vartheta) + T_N(\vartheta) - g_{p_r}(\vartheta)| \leqq e_{m+1} \log 2p_R \frac{p_r^{m+1}}{(p_R-1)^{m+1}}.$$

If we can choose $\{p_i\}$ such that

$$e_{m+1} \log 2p_R \frac{p_r^{m+1}}{(p_R-1)^{m+1}} < \frac{1}{p_R^m} \quad (R > r) \tag{2.17}$$

then, continuing the previous estimate, we get

$$\leqq \frac{1}{p_R^m} < \omega_m\left(\frac{1}{p_R}\right).$$

We can argue in the same way for L_{p_R}, so (2.16) is true. And now (if $n_{ki} = p_j$) we can write

$$|L_{2p_j}(f;\vartheta^*) - f(\vartheta^*)| = \left|Q \sum_{i=1}^{\infty} [L_{2p_j}(g_{p_i};\vartheta^*) - g_{p_i}(\vartheta^*)] \omega_m\left(\frac{1}{p_i}\right)\right| \geqq$$

$$\geqq Q|L_{2p_j}(g_{p_j};\vartheta^*)| \omega_m\left(\frac{1}{p_j}\right) - Q \sum_{i=1}^{j-1} |L_{2p_j}(g_{p_i};\vartheta^*) - g_{p_i}(\vartheta^*)| \omega_m\left(\frac{1}{p_i}\right) -$$

$$- Q \sum_{i=j+1}^{\infty} |L_{2p_j}(g_{p_i};\vartheta^*)| \omega_m\left(\frac{1}{p_i}\right) - Q \sum_{i=j}^{\infty} |g_{p_i}(\vartheta^*)| \omega_m\left(\frac{1}{p_i}\right).$$

Using (2.16), $i<j$ and (2.3) we get

$$\sum_{i=1}^{j-1} |L_{2p_j}(g_{p_i}; \vartheta^*) - g_{p_i}(\vartheta^*)| \omega_m\left(\frac{1}{p_i}\right) \leqq \omega_m\left(\frac{1}{p_j}\right) \frac{q}{1-q}. \tag{2.18}$$

By (2.3) and (2.5) we have

$$\sum_{i=j+1}^{\infty} |L_{2p_j}(g_{p_i}; \vartheta^*)| \omega_m\left(\frac{1}{p_i}\right) \leqq \log 2p_j \sum_{i=j+1}^{\infty} \omega_m\left(\frac{1}{p_i}\right) \leqq \tag{2.19}$$

$$\leqq \log 2p_j \omega_m\left(\frac{1}{p_{j+1}}\right) \frac{1}{1-q} \leqq \frac{q}{1-q} \log k \omega_m\left(\frac{1}{p_j}\right).$$

At last by (2.3) and $|g_{p_i}(\vartheta)| \leqq 1$ we have

$$\sum_{i=j}^{\infty} |g_{p_i}(\vartheta^*)| \omega_m\left(\frac{1}{p_i}\right) \leqq \omega_m\left(\frac{1}{p_j}\right) \frac{1}{1-q}. \tag{2.20}$$

Summarizing (2.15), (2.18), (2.19) and (2.20) we get

$$|L_{2n_{ki}}(f; \vartheta^*) - f(\vartheta^*)| \geqq \omega_m\left(\frac{1}{n_{ki}}\right)\left[\frac{c_4 Q}{2} \log k - \frac{Qq}{1-q} \log k - Q \frac{1+q}{1-q}\right].$$

But if Q is large enough we have that

$$[\ldots] > \log k$$

so we proved our assertion for $0<\vartheta^*<2\pi$. Let us investigate the case $\vartheta^*=0$. Certainly

$$|L_n(f; 0) - f(0)| \leqq C_m(\log n + 1) \omega_m\left(f; \frac{1}{n}\right) \leqq b_m(f) \log n \omega_m\left(\frac{1}{n}\right) \quad (n \geqq 2).$$

But if we consider the function $f_1(\vartheta)$ for which (1.6) and (1.7) are valid then the function

$$h(\vartheta) = f(\vartheta) + 2b_m f_1(\vartheta)$$

fulfills (1.8).

2.2. PROOF OF THEOREM 1.2. Let the matrix NP be as follows:

$$\begin{array}{l} n_1 \;\; 2n_1 \;\; 3n_1 \;\; 2\cdot 3n_1 \ldots 3^{L_1} n_1 \;\; 2\cdot 3^{L_1} n_1 \\ n_2 \;\; 2n_2 \;\; 3n_2 \;\; 2\cdot 3n_2 \ldots 3^{L_2} n_2 \;\; 2\cdot 3^{L_2} n_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ n_k \;\; 2n_k \;\; 3n_k \;\; 2\cdot 3n_k \ldots 3^{L_k} n_k \;\; 2\cdot 3^{L_k} n_k \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array} \tag{2.21}$$

where $L_k = [\overset{3}{\log} n_k^{\varepsilon}] + 1$. For $g_{n_k}(\vartheta)$ we have

$$g_{n_k}(\vartheta_{j,3^{s-1}n_k}) = \begin{cases} (-1)^{j+s+1} & \text{if} \quad \vartheta_{j,3^{s-1}n_k} \in \Delta_{3^{L_k+1},s} \\ 0 & \text{if} \quad \vartheta_{j,3^{s-1}n_k} \notin \Delta_{3^{L_k+1},s} \end{cases}$$

(2.22)

$$g_{n_k}(\vartheta_{j,2\cdot 3^{s-1}n_k}) = \begin{cases} (-1)^{j+s+1} & \text{if} \quad \vartheta_{j,2\cdot 3^{s-1}n_k} \in \Delta_{3^{L_k+1},s} \\ 0 & \text{if} \quad \vartheta_{j,2\cdot 3^{s-1}n_k} \notin \Delta_{3^{L_k+1},s} \end{cases} \quad (s = 1, 2, \ldots, L_k + 1),$$

further we use (2.7) and (2.9).

We can see by (1.1) that the definition of $g_{n_k}(\vartheta)$ has meaning. Now for $g_{n_k}(\vartheta)$

$$d_{n_k} \sim \frac{1}{n_k^{1+\varepsilon}}. \tag{2.23}$$

Let

$$f(\vartheta) = Q \sum_{i=1}^{\infty} \omega_m\left(\frac{1}{n_i^{1+\varepsilon}}\right) g_{n_i}(\vartheta) \tag{2.24}$$

where we use for $\{n_i\}$ (2.3), (2.4) and (2.27).

As in Lemma 2 we have for $\vartheta^* \in \Delta_{3^{L_k+1},s+1}$ $(s \geqq 1,\ 0 < \vartheta^* < 2\pi)$

$$\begin{cases} |L_{3^s n_k}(g_{n_k}; \vartheta^*)| \geqq c_4(\varepsilon)|\cos 3^s n_k \vartheta| \log n_k & (k \geqq k_0(\vartheta^*)) \\ |L_{2\cdot 3^s n_k}(g_{n_k}; \vartheta^*)| \geqq c_4(\varepsilon)|\cos 2 \cdot 3^s n_k \vartheta| \log n_k & (k \geqq k_0(\vartheta^*)) \end{cases} \tag{2.25}$$

where $c_4(\varepsilon)$ does not depend on ϑ^*.

Further if we replace m by $2m$ in (2.9) we have for any ϑ

$$|L_{p_R}(q_{p_r}; \vartheta) - g_{p_r}(\vartheta)| \leqq \omega_m\left(\frac{1}{p_R}\right) \quad \text{for} \quad R > r \tag{2.26}$$

if $\{p_i\}$ is a suitable lacunary sequence. For this end see the proof of (2.16).

Finally, we remark that

$$\log 3^s n_k \sim \log n_k \quad (s = 1, 2, \ldots L_k). \tag{2.27}$$

Using these new estimations and the former ones we can prove (1.11).

2.3. PROOF OF THEOREM 1.3. This is analogous to 2.1, if we replace 0 and 2π by a and b, respectively. We have (2.8) also because of $[a, b] \subset (-1, 1)$. If $[a, b] \equiv [-1, 1]$ we define $g_n(x)$ on a certain $[a_1, b_1] \subset (-1, 1)$ as above, further $g_n(x) = 0$ for $x \notin [a_1, b_1]$. The only difference is that now in (2.13) we have to replace $\log k$ by c_6 if $x \notin [a_1, b_1]$.

2.4. PROOF OF THEOREM 1.4. The only difference is that now $d_n \sim n^{-2(1+\varepsilon)}$. (See 2.2.)

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References

[1] M. RIESZ, Formule d'interpolation pour la dérivée d'un polynome trigonometrique, *Comptes Rendus,* **158** (1914), p. 1152.
[2] A. F. TIMAN, *Theory of Approximation of Functions of Real Variable,* Moscow, 1960 (Russian).
[3] A. H. TURECKI, *Theory of Interpolation through Problems,* Minsk, 1968 (Russian).
[4] G. FABER, Über die interpolatorische Darstellung stetiger Funktionen, *Jahr. D. Math. Verein.,* **23** (1914), pp. 194—210.
[5] P. ERDŐS, On divergence properties of the Lagrange interpolation parabolas, *Ann. of Math.,* **42** (1941), pp. 309—315.
[6] P. ERDŐS, Corrections to two of my papers, *ibid.,* **44** (1943), pp. 647—651.
[7] G. GRÜNWALD, Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome stetiger Funktionen, *ibid.,* **37** (1936), pp. 908—918.
[8] J. MARCINKIEWICZ, Sur la divergence des polynomes d'interpolation, *Acta Litt. Ac. Szeged,* **8** (1937), pp. 131—135.
[9] O. KIS, Notes on the order of interpolation errors, *Acta Math. Acad. Sci. Hung.,* **20** (1969), pp. 339—346 (Russian).
[10] P. O. H. VÉRTESI, On certain linear operators. VIII, *ibid.,* **25** (1974), pp. 171—187.

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Tomus 25 (3—4), (1974), pp. 401—412.

ON THE ORDER OF CONVERGENCE OF A FINITE ELEMENT SCHEME

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The finite element method is nothing else than the Rayleigh—Ritz method using special trial functions. The simplest idea goes back to COURANT who suggested triangulating the given region and using functions which are linear on each triangle as trial functions for solving elliptic boundary value problems of the second order (see [1]). The order of convergence of this method has been thoroughly discussed in [2]—[9]. ZLÁMAL has suggested in [10] using trial functions which are piecewise cubic polynomials (see also [11]—[13]). A great advantage in the use of Zlámal's method lies in the fact that it gives directly the first partial derivatives of the solution of the boundary value problem. In the present paper we shall discuss the order of convergence of Zlámal's finite element scheme.

1. Let R be a bounded open plane region whose boundary C consists of a finite number of piecewise-analytic simple closed curves. Denote by A_μ $(\mu=1, 2, \ldots, \nu)$ the corners of C, i.e. those points on C, where distinct analytic curves meet.

We consider the boundary value problem

$$Lu(x, y) = g(x, y), \quad (x, y)\in R, \quad u(x, y) = 0, \quad (x, y)\in C, \tag{1}$$

where

$$Lu \equiv \frac{\partial}{\partial x}\left[a(x, y)\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial x}\left[b(x, y)\frac{\partial u}{\partial y}\right]+\frac{\partial}{\partial y}\left[b(x, y)\frac{\partial u}{\partial x}\right]+$$
$$+\frac{\partial}{\partial y}\left[c(x, y)\frac{\partial u}{\partial y}\right]-f(x, y)u.$$

Let the coefficients $a(x, y)$, $b(x, y)$, $c(x, y)$, $f(x, y)$ and the right-hand side $g(x, y)$ be analytic in an open region G containing the closure of R in its interior. Suppose that at all points of R

$$a\xi^2+2b\xi\eta+c\eta^2 \geqq \alpha(\xi^2+\eta^2) \quad (\alpha = \text{const.} > 0) \tag{2}$$

for all real ξ, η. Moreover, we assume that $f(x, y)\geqq 0$.

Let Ω be an open region in the plane of R. Denote by $W_2^{(k)}(\Omega)$ the Hilbert space of all functions which, together with their generalized derivatives up to the k-th order, belong to $L_2(\Omega)$. The norm is given by

$$\|v\|^2_{k,\Omega} = \sum_{j=0}^{k} |v|^2_{j,\Omega}, \quad \text{where} \quad |v|^2_{j,\Omega} = \sum_{|i|=j} \|D^i v\|^2_{L_2(\Omega)}.$$

Here we use the notation $i = (i_1, i_2)$, $|i| = i_1+i_2$, $D^i v \equiv \dfrac{\partial^{|i|} v}{\partial x^{i_1}\partial y^{i_2}}$.

It is well-known that under the above-mentioned assumptions the solution $u(x, y)$ of the problem (1) minimizes the functional

$$F(v) = \iint_R \left[a\left(\frac{\partial v}{\partial x}\right)^2 + 2b\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + c\left(\frac{\partial v}{\partial y}\right)^2 + fv^2 + 2gv \right] dx\, dy \tag{3}$$

in the space $\overset{\circ}{W}{}_2^{(1)}(R)$ (see e.g. [14]). Here $\overset{\circ}{W}{}_2^{(1)}(R)$ is the space of functions which we get by completing in the norm $\| \ \|_{1,R}$ the set of differentiable functions with compact support in R.

LEMMA 1. *$u(x, y)$ is an analytic function of x as well as of y in R.*

For a proof, see [15], p. 213.

LEMMA 2. *$u(x, y)$ is analytic on C, excluding the corners.*

For a proof of a more general result, see [16].

LEMMA 3. *Let $A_\mu = (x_{A_\mu}, y_{A_\mu})$ be a corner of C with interior angle $\pi\alpha_\mu$ $(0 < \alpha_\mu < 2)$. Let*

$$x^* = k_{A_\mu}x + l_{A_\mu}y, \quad y^* = m_{A_\mu}x + n_{A_\mu}y \tag{4}$$

be a linear transformation which transforms the operator L into the normal form at the point A_μ. Let $r_{A_\mu} = [(x-x_{A_\mu})^2 + (y-y_{A_\mu})^2]^{\frac{1}{2}}$. If the transformation (4) transforms the angle $\pi\alpha_\mu$ into an angle $\pi\alpha_\mu^*$ $(0 < \alpha_\mu^* < 2)$, then for $\alpha_\mu^* \neq \frac{1}{m}$ (m an integer)

$$u(x, y) = u_1(x, y) + O\left(r_{A_\mu}^{\frac{1}{\alpha_\mu^*}}\right), \tag{5}$$

where $u_1(x, y)$ and its partial derivatives of all orders remain bounded when $(x, y) \to A_\mu$ in R, while

$$u(x, y) = u_1(x, y) + O\left(r_{A_\mu}^{\frac{1}{\alpha_\mu^*}} |\log r_{A_\mu}|\right), \tag{6}$$

when $\alpha_\mu^* = \frac{1}{m}$ $(m = 1, 2, \ldots)$. These relations may be indefinitely formally differentiated.

This lemma follows from the results of KONDRAT'EV (see [17]).

2. Let us first consider the case when the boundary C consists of a finite number of simple closed polygons. We triangulate R, i.e. we cover R by a finite number of arbitrary triangles such that any two triangles are either disjoint or have a common vertex or a common side. To every triangulation we associate two parameters: h, ϑ. h is the largest side and ϑ is the smallest angle of all triangles of the given triangulation. In the sequel we assume that

$$h < c_1\bar{h}, \quad \vartheta \geqq \vartheta_0 > 0, \tag{7}$$

where $\bar{h}$ is the smallest side of all triangles of the given triangulation; c_1 and ϑ_0 do not depend on h.

We denote by M_h the set of all triangles of the given triangulation of R. Let $P_i=(x_i, y_i)$ $(i=1, 2, 3)$ be the vertices of the triangle $T\in M_h$ and let $P_0=(x_0, y_0)$ be the centre of gravity of T. Let $p_i=p(P_i)$ $(i=0, 1, 2, 3)$, $q_i=q(P_i)$, $r_i=r(P_i)$ $(i=1, 2, 3)$ be any real-valued functions defined at the points P_i. It is well-known that there exists exactly one cubic polynomial $p(x, y)$ which satisfies the conditions $p(P_i)=p_i$ $(i=0, 1, 2, 3)$, $\frac{\partial p(P_i)}{\partial x}=q_i$, $\frac{\partial p(P_i)}{\partial y}=r_i$ $(i=1, 2, 3)$ (see e.g. [7]). Denote by $H_3(R)$ the set of all functions which are equal on each triangle $T\in M_h$ to the corresponding cubic polynomial $p(x, y)$ and denote by $\mathring{H}_3(R)$ the set of functions from $H_3(R)$ which vanish on C. It is easy to show that $H_3(R)\subset W_2^{(1)}(R)$ and $\mathring{H}_3(R)\subset \mathring{W}_2^{(1)}(R)$. The solution $u(x, y)$ of the problem (1) is approximated by the function $u_h(x, y)$ which minimizes the functional (3) in the class $\mathring{H}_3(R)$. The existence and uniqueness of $u_h(x, y)$ follows immediately from the condition (2) and the non-negativity of the function $f(x, y)$.

LEMMA 4. *Let $T\in M_h$, $v(x, y)\in W_2^{(k)}(T)$ $(k=3, 4)$ and let $p(x, y)$ be the cubic polynomial determined by the conditions*

$$p(P_i)=v(P_i)\ (i=0, 1, 2, 3),\quad \frac{\partial p(P_i)}{\partial x}=\frac{\partial v(P_i)}{\partial x},\quad \frac{\partial p(P_i)}{\partial y}=\frac{\partial v(P_i)}{\partial y}\quad (i=1, 2, 3).$$

Then

$$\|v-p\|_{1,T}\leqq \frac{c_2}{\sin\vartheta}h^{k-1}|v|_{k,T}, \tag{8}$$

where c_2 is a positive constant which does not depend on the triangle T and the function $v(x, y)$.

This lemma follows from the results of BRAMBLE and ZLÁMAL (see [7], p. 818).

LEMMA 5. *Let $T\in M_h$ and let $p(x, y)$ be the cubic polynomial determined by the conditions*

$$p(P_i)=p_i\quad (i=0, 1, 2, 3),\quad \frac{\partial p(P_i)}{\partial x}=q_i,\quad \frac{\partial p(P_i)}{\partial y}=r_i\quad (i=1, 2, 3),$$

where p_i, q_i, r_i are any real parameters. Then

$$|p|_{1,T}\leqq c_3(p_{\max}+hq_{\max}+hr_{\max}) \tag{9}$$

and

$$\|p\|_{L_2(T)}\leqq c_4h(p_{\max}+hq_{\max}+hr_{\max}), \tag{10}$$

where

$$p_{\max}=\max_{i=0,1,2,3}|p_i|,\quad q_{\max}=\max_{i=1,2,3}|q_i|,\quad r_{\max}=\max_{i=1,2,3}|r_i|,$$

c_3 and c_4 are positive constants which do not depend on the triangle T and the parameters p_i, q_i, r_i.

PROOF. Let us consider the integrals

$$\Phi(p) = |p|^2_{1,T} = \iint_T \left[\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2\right] dx\, dy \tag{11}$$

and

$$\Psi(p) = \|p\|^2_{L_2(T)} = \iint_T p^2 dx\, dy. \tag{12}$$

By means of the linear transformation

$$x = x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta, \qquad y = y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta \tag{13}$$

we get

$$\Phi(p) = \iint_{T_1} \left[\alpha\left(\frac{\partial s}{\partial \xi}\right)^2 + 2\beta \frac{\partial s}{\partial \xi}\frac{\partial s}{\partial \eta} + \gamma\left(\frac{\partial s}{\partial \eta}\right)^2\right] d\xi\, d\eta \tag{14}$$

and

$$\Psi(p) = \iint_{T_1} s^2 |J|\, d\xi\, d\eta. \tag{15}$$

Here T_1 is the triangle in the (ξ, η)-plane with vertices (0, 0), (1, 0), (0, 1), J is the Jacobian of (13), $s = s(\xi, \eta) = a_1 + a_2\xi + \ldots + a_{10}\eta^3$ is the cubic polynomial determined by the ten conditions

$$\begin{aligned} s(0, 0) &= p_1, \\ \frac{\partial s(0, 0)}{\partial \xi} &= (x_2 - x_1)q_1 + (y_2 - y_1)r_1, \\ \frac{\partial s(0, 0)}{\partial \eta} &= (x_3 - x_1)q_1 + (y_3 - y_1)r_1, \\ &\vdots \\ s\left(\frac{1}{3}, \frac{1}{3}\right) &= p_0 \end{aligned} \tag{16}$$

and the coefficients α, β, γ are given by

$$\alpha = \frac{(x_3 - x_1)^2 + (y_3 - y_1)^2}{|J|}, \quad \gamma = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{|J|},$$

$$\beta = -\frac{(x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1)}{|J|}.$$

Let $a^* = (a_1, a_2, \ldots, a_{10})$ and let z be the vector whose components are the right-hand sides of the equations (16). Then

$$a = D^{-1}z, \tag{17}$$

where $D = (d_{jk})$ is a non-singular matrix whose elements do not depend on the triangle T and the parameters p_i, q_i, r_i. Since $\dfrac{\partial s}{\partial \xi}$ can be written in the form $\dfrac{\partial s}{\partial \xi} = l^* a$, where

$l^* = (0, 1, 0, 2\xi, \eta, 0, 3\xi^2, 2\xi\eta, \eta^2, 0)$, we have

$$\iint_{T_1} \alpha \left(\frac{\partial s}{\partial \xi}\right)^2 d\xi\, d\eta = \alpha a^* L a, \tag{18}$$

where $L = (l_{jk})$, $l_{jk} = \iint_{T_1} l_j l_k\, d\xi\, d\eta$. Substituting (17) into (18) we find that

$$\iint_{T_1} \alpha \left(\frac{\partial s}{\partial \xi}\right)^2 d\xi\, d\eta = \alpha z^* A z, \quad A = (D^{-1})^* L D^{-1}. \tag{19}$$

Since $\dfrac{\partial s}{\partial \eta} = m^* a$, where $m^* = (0, 0, 1, 0, \xi, 2\eta, 0, \xi^2, 2\xi\eta, 3\eta^2)$ we have

$$\iint_{T_1} \gamma \left(\frac{\partial s}{\partial \eta}\right)^2 d\xi\, d\eta = \gamma z^* C z, \quad C = (D^{-1})^* M D^{-1} \tag{20}$$

where $M = (m_{jk})$, $m_{jk} = \iint_{T_1} m_j m_k\, d\xi\, d\eta$. In a similar way, we find that

$$\iint_{T_1} 2\beta \frac{\partial s}{\partial \xi} \frac{\partial s}{\partial \eta}\, d\xi\, d\eta = \beta z^* (B^* + B) z, \quad B = (D^{-1})^* N D^{-1}, \tag{21}$$

where $N = (n_{jk})$, $n_{jk} = \iint_{T_1} l_j m_k\, d\xi\, d\eta$. It follows from the conditions (7) that $|J| = \frac{1}{2}\operatorname{mes} T > c_5 h^2$ and, consequently, $\max\{|\alpha|, |\beta|, |\gamma|\} < c_6$, where c_5 and c_6 are positive constants which do not depend on the triangle T and the parameters p_i, q_i, r_i. On the other hand, the definition of the vector z implies that

$$\max_{j=1,2,\ldots,10} |z_j| = O(p_{\max} + h q_{\max} + h r_{\max}). \tag{22}$$

Substituting (22) into (19), (20), (21) and, using the boundedness of the coefficients α, β, γ, we get the inequality (9).

Substituting (22) into (17) we find that

$$\max_{j=1,2,\ldots,10} |a_j| = O(p_{\max} + h q_{\max} + h r_{\max}).$$

and, consequently,

$$\max_{(\xi,\eta)\in T_1} |s(\xi, \eta)| = O(p_{\max} + h q_{\max} + h r_{\max}). \tag{23}$$

Substituting (23) into (15) and using the trivial estimate $J = O(h^2)$ we get (10). This completes the proof of Lemma 5.

THEOREM 1. *Assume that the boundary C consists of a finite number of simple closed polygons and consider a triangulation of R such that the conditions* (7) *are valid. Then*

$$\|u - u_h\|_{1,R} < c_7 h^{\beta^*} \tag{24}$$

and

$$\max_{(x,y)\in R} |u(x,y)-u_h(x,y)| < c_8 h^{\beta^*} |\log h|^{\frac{1}{2}}. \tag{25}$$

Here

$$\beta^* = \begin{cases} \dfrac{1}{\max\limits_{\mu=1,2,\ldots,\nu} \alpha_\mu^*}, & \text{if} \quad \max\limits_{\mu=1,2,\ldots,\nu} \alpha_\mu^* > \dfrac{1}{3} \quad \text{and} \quad \max\limits_{\mu=1,2,\ldots,\nu} \alpha_\mu^* \neq \dfrac{1}{2}, \\ 2-\varepsilon, & \text{if} \quad \max\limits_{\mu=1,2,\ldots,\nu} \alpha_\mu^* = \dfrac{1}{2}, \\ 3-\varepsilon, & \text{if} \quad \max\limits_{\mu=1,2,\ldots,\nu} \alpha_\mu^* = \dfrac{1}{3}, \\ 3, & \text{if} \quad \max\limits_{\mu=1,2,\ldots,\nu} \alpha_\mu^* < \dfrac{1}{3}, \end{cases}$$

c_7 and c_8 are positive constants which do not depend on the triangulation of R, ε is any positive real number.

PROOF. Let $P_i=(x_i, y_i)$ $(i=1, 2, 3)$ be the vertices of the triangle $T\in M_h$ and let $P_0=(x_0, y_0)$ be the centre of gravity of T. The functions $p_i=p(P_i)$ $(i=0, 1, 2, 3)$, $q_i=q(P_i)$, $r_i=r(P_i)$ $(i=1, 2, 3)$ are defined as follows. Let $p_0=u(P_0)$. If the vertex P_i is not a corner of C, then we set $p_i=u(P_i)$, $q_i=\dfrac{\partial u(P_i)}{\partial x}$, $r_i=\dfrac{\partial u(P_i)}{\partial y}$. If $P_i=A_\mu$, where A_μ is a corner of C with interior angle $\pi\alpha_\mu$ $(0<\alpha_\mu<2)$, then we set $p_i=q_i=r_i=0$. (If $0<\alpha_\mu<1$, then the two definitions are equivalent. In this case from Lemma 3 it follows that the partial derivatives $\dfrac{\partial u(A_\mu)}{\partial x}$ and $\dfrac{\partial u(A_\mu)}{\partial y}$ exist and, since the derivatives of u along the sides of C vanish, we have

$$\frac{\partial u(A_\mu)}{\partial x}\cos\omega_1+\frac{\partial u(A_\mu)}{\partial y}\sin\omega_1=0, \quad \frac{\partial u(A_\mu)}{\partial x}\cos\omega_2+\frac{\partial u(A_\mu)}{\partial y}\sin\omega_2=0, \tag{26}$$

where ω_1 and ω_2 are the angles which are formed between the x-axis and the corresponding sides of C. (26) implies that $\dfrac{\partial u(A_\mu)}{\partial x}=\dfrac{\partial u(A_\mu)}{\partial y}=0$.)

Let $p(x, y)$ be the cubic polynomial determined by the ten conditions $p(P_i)=p_i$ $(i=0, 1, 2, 3)$, $\dfrac{\partial p(P_i)}{\partial x}=q_i$, $\dfrac{\partial p(P_i)}{\partial y}=r_i$ $(i=1, 2, 3)$ and let $\varrho_h u$ be the function which is equal on each triangle $T\in M_h$ to the corresponding polynomial $p(x, y)$. It is easy to show that $\varrho_h u\in \mathring{H}_3(R)$. Let the functional $D(v)$ be defined by

$$D(v)=\iint_R \left[a\left(\frac{\partial v}{\partial x}\right)^2+2b\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+c\left(\frac{\partial v}{\partial y}\right)^2+fv^2\right]dx\,dy.$$

Then, using a standard technique (see e.g. [10], p. 396), we can easily prove that

$$D(u-u_h) \leqq D(u-\varrho_h u).$$

Hence, using the Friedrichs inequality we obtain

$$\|u-u_h\|_{1,R} = O(\|u-\varrho_h u\|_{1,R}). \tag{27}$$

Denote by $r(A_\mu, T)$ the distance from the corner A_μ to the triangle $T \in M_h$. If $r(A_\mu, T) > r_1$ for $\mu = 1, 2, \ldots, \nu$, where r_1 is a fixed small positive real number, then, using Lemmas 1, 2 and 4 we obtain

$$\|u-\varrho_h u\|^2_{1,T} = O(h^6 \operatorname{mes} T).$$

Hence it follows that

$$\sum_{T \in M_h,\, r(A_\mu, T) > r_1,\ \mu=1,2,\ldots,\nu} \|u-\varrho_h u\|^2_{1,T} = O(h^6). \tag{28}$$

Let A_μ be a corner of C with interior angle $\pi\alpha_\mu$ $(0<\alpha_\mu<2)$. If $\frac{1}{3}<\alpha^*_\mu<2$, and $\alpha^*_\mu \neq \frac{1}{2}$, then, using Lemmas 3 and 4, we get

$$\sum_{T \in M_h,\, h \leqq r(A_\mu, T) \leqq r_1} \|u-\varrho_h u\|^2_{1,T} = O\Big(h^6 \sum_{T \in M_h,\, h \leqq r(A_\mu, T) \leqq r_1} (\operatorname{mes} T)[r(A_\mu, T)]^{\frac{2}{\alpha^*_\mu}-8}\Big) =$$

$$= O\Big(h^6 \Big\{ \iint_{h \leqq r(A_\mu, P) \leqq r_1} [r(A_\mu, P)]^{\frac{2}{\alpha^*_\mu}-8}\, dx\, dy \Big\}\Big) = O\Big(h^6 \int_h^{r_1} r^{\frac{2}{\alpha^*_\mu}-7}\, dr\Big) = O\Big(h^{\frac{2}{\alpha^*_\mu}}\Big), \tag{29}$$

where $r(A_\mu, P)$ is the distance between the points A_μ and $P=(x, y)$. If $\alpha^*_\mu = \frac{1}{2}$ or $\alpha^*_\mu = \frac{1}{3}$, then in a similar way we get

$$\sum_{T \in M_h,\, h \leqq r(A_\mu, T) \leqq r_1} \|u-\varrho_h u\|^2_{1,T} = O\Big(h^{\frac{2}{\alpha^*_\mu}-\varepsilon}\Big). \tag{30}$$

If $0<\alpha^*_\mu<\frac{1}{3}$, then by Lemma 3 we have $u(x, y) \in W_2^{(4)}(R_{A_\mu})$, where $R_{A_\mu} = \{P \in R,\ 0<r(A_\mu, P)<r_1\}$. From Lemma 4 it follows then that

$$\sum_{T \in M_h,\, h \leqq r(A_\mu, T) \leqq r_1} \|u-\varrho_h u\|^2_{1,T} = O(h^6). \tag{31}$$

(29), (30) and (31) imply that

$$\sum_{T \in M_h,\, h \leqq r(A_\mu, T) \leqq r_1} \|u-\varrho_h u\|^2_{1,T} = O(h^{2\beta^*}) \tag{32}$$

for $\mu = 1, 2, \ldots, \nu$.

Let us now consider a triangle $T \in M_h$ such that $r(A_\mu, T) \leqq h$. In the sequel we shall prove that if $\frac{1}{2} < \alpha_\mu^* < 2$, then

$$\|u\|_{1,T} = O\left(h^{\frac{1}{\alpha_\mu^*}}\right) \tag{33}$$

and

$$p_i = O\left(h^{\frac{1}{\alpha_\mu^*}}\right), \quad q_i = O\left(h^{\frac{1}{\alpha_\mu^*}-1}\right), \quad r_i = O\left(h^{\frac{1}{\alpha_\mu^*}-1}\right) \tag{34}$$

for $i=0, 1, 2, 3$ and $i=1, 2, 3$, respectively.

Let us first consider the case when $1 < \alpha_\mu^* < 2$. In this case the relations (34) follow directly from Lemma 3 and the conditions (7). On the other hand, using Lemma 3 we get

$$|u|_{1,T}^2 = O\Big(\iint\limits_{0<r(A_\mu, T)<h} [r(A_\mu, P)]^{\frac{2}{\alpha_\mu^*}-2}\, dx\, dy\Big) = O\Big(\int_0^h r^{\frac{2}{\alpha_\mu^*}-1}\, dr\Big) = O\left(h^{\frac{2}{\alpha_\mu^*}}\right) \tag{35}$$

and

$$\|u\|_{L_2(T)}^2 = O\left(h^{\frac{2}{\alpha_\mu^*}+2}\right). \tag{36}$$

The estimate (33) follows directly from (35) and (36).

Let us now consider the case when $\frac{1}{2} < \alpha_\mu^* < 1$. We have already proved that if $0 < \alpha_\mu^* < 1$ then $\frac{\partial u(A_\mu)}{\partial x} = \frac{\partial u(A_\mu)}{\partial y} = 0$. Hence, using Lemma 3, we obtain that

$$\frac{\partial u_1(A_\mu)}{\partial x} = \frac{\partial u_1(A_\mu)}{\partial y} = 0. \tag{37}$$

By Taylor's theorem we have

$$\max_{(x,y)\in T} |u_1(x, y)| = O(h^2) \tag{38}$$

and

$$\max_{(x,y)\in T} \left|\frac{\partial u_1(x, y)}{\partial x}\right| = O(h), \quad \max_{(x,y)\in T} \left|\frac{\partial u_1(x, y)}{\partial y}\right| = O(h). \tag{39}$$

(5), (38) and (39) imply that if $\frac{1}{2} < \alpha_\mu^* < 1$, then the relations (34) hold. On the other hand, using (5), (38) and (39) we have

$$|u|_{1,T}^2 = O\Big(\iint\limits_{0<r(A_\mu, P)<h} \left\{[r(A_\mu, P)]^{\frac{2}{\alpha_\mu^*}-2} + h^2\right\} dx\, dy\Big) = \tag{40}$$

$$= O\Big(\int_0^h r^{\frac{2}{\alpha_\mu^*}-1}\, dr + h^4\Big) = O\left(h^{\frac{2}{\alpha_\mu^*}}\right)$$

and

$$\|u\|_{L_2(T)}^2 = O\left(h^{\frac{2}{\alpha_\mu^*}+2} + h^6\right) = O\left(h^{\frac{2}{\alpha_\mu^*}+2}\right). \tag{41}$$

From (40) and (41) it follows that if $\frac{1}{2}<\alpha_\mu^*<1$, then the relation (33) holds.

Using Lemma 5 we have from (34) that

$$\|p\|_{1,T}^2 = O\left(h^{\frac{2}{\alpha_\mu^*}}\right). \tag{42}$$

(33) and (42) imply that if $\frac{1}{2}<\alpha_\mu^*<2$, then

$$\|u-\varrho_h u\|_{1,T}^2 = O\left(h^{\frac{2}{\alpha_\mu^*}}\right). \tag{43}$$

If $\alpha_\mu^*=\frac{1}{2}$, then in a similar way we get

$$\|u-\varrho_h u\|_{1,T}^2 = O(h^{4-\varepsilon}). \tag{44}$$

Let us now consider the case when $\frac{1}{3}<\alpha_\mu^*<\frac{1}{2}$. In this case from Lemma 3 it follows that $u(x, y)\in W_2^{(3)}(T)$ and, using Lemmas 3 and 4 we obtain that

$$\|u-\varrho_h u\|_{1,T}^2 = O(h^4|u|_{3,T}^2) = O\left(h^4\iint_I [r(A_\mu, P)]^{\frac{2}{\alpha_\mu^*}-6}\,dx\,dy\right) = \tag{45}$$

$$= O\left(h^4\int_0^h r^{\frac{2}{\alpha_\mu^*}-5}\,dr\right) = O\left(h^{\frac{2}{\alpha_\mu^*}}\right).$$

If $\alpha_\mu^*=\frac{1}{3}$, then in a similar way we get

$$\|u-\varrho_h u\|_{1,T}^2 = O(h^{6-\varepsilon}). \tag{46}$$

Finally, if $0<\alpha_\mu^*<\frac{1}{3}$, then from Lemma 3 it follows that $u(x, y)\in W_2^{(4)}(T)$ and, using Lemma 4, we get

$$\|u-\varrho_h u\|_{1,T}^2 = O(h^6). \tag{47}$$

(43), (44), (45), (46) and (47) imply that

$$\sum_{T\in M_h, r(A_\mu, T)\leqq h} \|u-\varrho_h u\|_{1,T}^2 = O(h^{2\beta^*}) \tag{48}$$

for $\mu=1, 2, \ldots, \nu$.

Summing (32) and (48) over all corners of C and using (28) we find that

$$\|u-\varrho_h u\|_{1,R} = O(h^{\beta^*}). \tag{49}$$

Substituting (49) into (27) we get the inequality (24). The inequality (25) follows directly from (24) and a theorem of V. P. IL'IN (see [18], p. 101). This completes the proof of Theorem 1.

3. Let us now consider the general case when the boundary C consists of a finite number of piecewise-analytic simple closed curves.

If t is a positive integer, then we denote by $C^{(t)}(\bar{R})$ the set of functions which have continuous partial derivatives of all orders up to t in the closed region $\bar{R}=R\cup C$. Let $\Theta=\Theta(x, y)$ be a real function such that $\Theta(x, y)\in C^{(l)}(\bar{R})$ $(l=4, 5)$ and

$$\Theta(x, y) > 0, \quad (x, y)\in R, \quad \Theta(x, y) = 0, \quad (x, y)\in C, \tag{50}$$

$$\left|\frac{\partial\Theta(x, y)}{\partial x}\right| + \left|\frac{\partial\Theta(x, y)}{\partial y}\right| \neq 0, \quad (x, y)\in C.$$

In [19] and [20] some specific cases are indicated in which one can explicitely construct such a function $\Theta(x, y)$.

Let S be a polygon which contains $\bar{R}$ in its interior and consider a triangulation of S such that the conditions (7) are valid. We denote by $\mathring{I}_3(R)$ the set of functions $w(x, y)$ of the form $w(x, y)=\Theta(x, y)v(x, y)$ for all $v(x, y)\in H_3(S)$. Then clearly $\mathring{I}_3(R)\subset\mathring{W}_2^{(1)}(R)$. Following KANTOROVICH and KRYLOV (see [19]) we approximate the solution $u(x, y)$ of the problem (1) by the function $u_{h,\Theta}(x, y)$ which minimizes the functional (3) in the class $\mathring{I}_3(R)$. The existence and uniqueness of $u_{h,\Theta}(x, y)$ follows immediately from the condition (2) and the non-negativity of the function $f(x, y)$.

If t is an integer, $t\geqq 2$, then we denote by $B^{(t)}(R)$ the set of all functions $\varphi(x, y)\in$ $\in C^{(t-1)}(\bar{R})$ such that $D^s\varphi$ is absolutely continuous in R for all $|s|\leqq t-1$ and $D^s\varphi\in$ $\in L_2(R)$ for all $|s|=t$.

THEOREM 2. *Let R be a bounded open plane region whose boundary C consists of a finite number of piecewise-analytic simple closed curves. Assume that $\Theta(x, y)\in$ $\in C^{(k+1)}(\bar{R})$, $u(x, y)\in B^{(k+1)}(R)$* $(k=3, 4)$ and $\Theta(x, y)$ satisfies the conditions* (50). *Then*

$$\|u-u_{h,\Theta}\|_{1,R} < c_9 h^{k-1}\|u\|_{k+1,R} \tag{51}$$

and

$$\max_{(x,y)\in R}|u(x, y)-u_{h,\Theta}(x, y)| < c_{10}h^{k-1}|\log h|^{\frac{1}{2}}\|u\|_{k+1,R} \tag{52}$$

where c_9 and c_{10} are positive constants which depend only on the coefficients of the operator L and on the region R.

PROOF. As in the proof of Theorem 1, we have

$$\|u-u_{h,\Theta}\|_{1,R} = O(\|u-w\|_{1,R}) \tag{53}$$

for all $w\in\mathring{I}_3(R)$. We set $z(x, y)=u(x, y)/\Theta(x, y)$ for all $(x, y)\in R$. Then, using a theorem of SCHULTZ (see [21], p. 529), we obtain that

$$\|z\|_{k,R} \leqq c_{11}\|u\|_{k+1,R} \tag{54}$$

where c_{11} is a positive constant which depends only on the region R. If $v(x, y)\in H_3(S)$ and $w(x, y)=\Theta(x, y)v(x, y)$, then

$$\|u-w\|_{1,R} = \|\Theta(z-v)\|_{1,R}$$

* This condition is, in general, not satisfied if the boundary C has corners (see [17]).

and, consequently

$$\|u-w\|_{1,R} \leqq c_{12}\|z-v\|_{1,R} \tag{55}$$

where c_{12} depends only on the region R. By the Calderón extension theorem (see [22], p. 74) there exists a function $\hat{z}(x, y)\in W_2^{(k)}(S)$ such that $\hat{z}(x, y)=z(x, y)$ for all $(x, y)\in R$ and

$$\|\hat{z}\|_{k,S} \leqq c_{13}\|z\|_{k,R} \tag{56}$$

where c_{13} depends only on the region R. From Lemma 4 it follows that there exists a function $v(x, y)\in H_3(S)$ such that

$$\|\hat{z}-v\|_{1,S} \leqq c_{14}h^{k-1}\|\hat{z}\|_{k,S} \tag{57}$$

where c_{14} does not depend on the triangulation of S. Substituting (54) and (56) into (57) we find that

$$\|z-v\|_{1,R} \leqq \|\hat{z}-v\|_{1,S} \leqq c_{15}h^{k-1}\|u\|_{k+1,R}, \tag{58}$$

where c_{15} depends only on the region R. Substituting (58) into (55) we have

$$\|u-w\|_{1,R} \leqq c_{16}h^{k-1}\|u\|_{k+1,R} \tag{59}$$

for $k=3, 4$. Finally, substituting (59) into (53) we get the inequality (51). The inequality (52) follows directly from (51) and a theorem of V. P. IL'IN (see [18], p. 101). This completes the proof of Theorem 2.

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References

[1] R. COURANT, Variational methods for the solution of problems of equilibrium and vibrations, *Bull. Amer. Math. Soc.*, **49** (1943), pp. 1—23.

[2] JU. K. DEM'JANOVIČ, Approximation and convergence of the method of nets in elliptic problems (Russian), *Dokl. Akad. Nauk SSSR*, **170** (1966), pp. 27—30.

[3] JU. K. DEM'JANOVIČ, Uniform convergence of variation-grid schemes (Russian), *Dokl. Akad. Nauk SSSR*, **185** (1969), pp. 264—266.

[4] L. A. OGANESJAN and L. A. RUHOVEC, Variational difference schemes for second order linear equations in a two-dimensional region with a piecewise-smooth boundary (Russian), *Z. Vyčisl. Mat. Fiz.*, **8** (1968), pp. 97—114.

[5] L. A. OGANESJAN and L. A. RUHOVEC, Investigation of the convergence rate of variational difference schemes for second order elliptic equations in a two-dimensional region with a smooth boundary (Russian), *Z. Vyčisl. Mat. i Mat. Fiz.*, **9** (1969), pp. 1102—1120.

[6] J. NITSCHE, Linear Spline-Funktionen und die Methoden von Ritz für elliptische Randwertsprobleme, *Arch. Rational Mech. Anal.*, **36** (1970), pp. 348—355.

[7] J. H. BRAMBLE and M. ZLÁMAL, Triangular elements in the finite element method, *Math. Comp.*, **24** (1970), pp. 809—819.

[8] L. VEIDINGER, On the order of convergence of the Rayleigh—Ritz method with piecewise linear trial functions, *Acta Math. Acad. Sci. Hung.*, **23** (1972), pp. 507—517.

[9] I. BABUŠKA and M. B. ROSENZWEIG, A finite element scheme for domains with corners, *Numer. Math.*, **20** (1972), pp. 1—21.
[10] M. ZLÁMAL, On the finite element method, *Numer. Math.*, **12** (1968), pp. 394—409.
[11] M. ZLÁMAL, On some finite element procedures for solving second order boundary value problems, *Numer. Math.*, **14** (1969), pp. 42—48.
[12] M. ZLÁMAL, A finite element procedure of the second order of accuracy, *Numer. Math.*, **14** (1970), pp. 394—402.
[13] G. STRANG, Approximation in the finite element method, *Numer. Math.*, **19** (1972), pp. 81—98.
[14] V. A. IL'IN and I. A. ŠIŠMAREV, The connection between generalized and classical solutions of the Dirichlet problem (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.*, **24** (1960), pp. 521—530.
[15] C. MIRANDA, *Partial differential equations of elliptic type,* Springer (Berlin—Heidelberg—New York, 1970).
[16] C. D. MORREY and L. NIRENBERG, On the analyticity of linear elliptic systems of partial differential equations, *Comm. Pure Appl. Math.*, **10** (1957), pp. 271—290.
[17] V. A. KONDRAT'EV, Boundary value problems for elliptic equations in domains with conical or angular points (Russian), *Trudy Moskov. Mat. Obšč.*, **16** (1967), pp. 209—292.
[18] V. P. IL'IN, Some inequalities in function spaces and their application to the investigation of the convergence of variational processes (Russian), *Trudy Mat. Inst. Steklov,* **53** (1959), pp. 64—127.
[19] L. V. KANTOROVICH and V. I. KRYLOV, *Approximate methods in higher analysis,* Interscience (New York, 1958).
[20] V. L. RVAČEV and L. I. ŠKLJAROV, On the application of the Bubnov—Galerkin method to the solution of boundary value problems for regions of complicated form (Russian), *Differencial'nye Uravnenija,* **1** (1965), pp. 1537—1543.
[21] M. H. SCHULTZ, Rayleigh—Ritz—Galerkin methods for multidimensional problems, *SIAM J. Numer. Anal.*, **6** (1969), pp. 524—538.
[22] C. B. MORREY, *Multiple integrals in the calculus of variations,* Springer (Berlin—Heidelberg—New York, 1966).

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A REMARK ON A PAPER BY ABIAN

By
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A recent paper [1] by A. ABIAN has as its main result that the theory of atomless Boolean algebras is not categorical in certain uncountable powers. This is a weak special case of the following known result (see [3]).

THEOREM. *If a theory in the lower predicate calculus has a model with a definable binary relation which linearly orders any infinite subset of that model then it is not categorical in any uncountable power K.*

In particular, no extension of the theory of Boolean algebras can be categorical in any uncountable power.

For a wide class of cardinals K the above theorem was proved earlier by EHRENFEUCHT [2]. See also footnote 18 of [3].

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References

[1] A. ABIAN, Nonisomorphic atomless Boolean algebras, *Acta Math. Acad. Sci. Hung.*, **24** (1973), pp. 139—141.
[2] A. EHRENFEUCHT, On theories categorical in power, *Fund. Math.*, **44** (1957), pp. 241—248.
[3] M. MORLEY, Categoricity in power, *Trans. AMS*, **114** (1965), pp. 514—538.

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ON SUMMABILITY OF FOURIER TRANSFORMS

By
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Let f be a Lebesgue integrable function over the real axis. Let us denote by

$$F(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt} dx \tag{1}$$

the Fourier transform of the function f and by

$$f_\lambda(x) = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} F(t) e^{itx} dt \tag{2}$$

the symmetric "partial" inverse Fourier transform of the function F of order λ.

Considering the Bernstein—Rogosinski means

$$B_\lambda(x, f) = \frac{1}{2}\left[f_\lambda(x) + f_\lambda\left(x + \frac{\pi}{\lambda}\right)\right]$$

I. P. NATANSON [1] has proved the following statement:

If $f \in L(-\infty, \infty) \cap \mathrm{Lip}_\alpha(-\infty, \infty)$ $(0 < \alpha \leqq 1)$ *then we have uniformly in* $x \in (-\infty, \infty)$

$$f(x) - B_\lambda(x, f) = \begin{cases} O(1)\dfrac{1}{\lambda^\alpha}, & \text{if } 0 < \alpha < 1 \\ O(1)\dfrac{\log \lambda}{\lambda}, & \text{if } \alpha = 1 \end{cases} \tag{3}$$

$(\lambda > 1)$. Furthermore he asked whether in case $\alpha = 1$ "O" might be replaced by "o".

In this note we are going to show that in the estimation (3) even the logarithmic factor may be dropped. More precisely we shall prove the following theorem.

THEOREM. *If* $f \in L(-\infty, \infty)$ *and the modulus of continuity* $\omega(f, \delta)$ *of the function* f *is such that* $\omega(f, \delta) \to 0$ $(\delta \to 0)$, *then*

$$f(x) - B_\lambda(x, f) = O(1)\omega\left(f, \frac{1}{\lambda}\right)$$

$(\lambda \geqq 1)$, *where the factor* $O(1)$ *may depend on* f, *but it is independent of* x *and* λ.

COROLLARY. *Let* $f \in L(-\infty, \infty)$ *and* $\omega(f, \delta) \to 0$ $(\delta \to 0)$. *Then we have uniformly in* $x \in (-\infty, \infty)$

$$f_\lambda(x) - f(x) = \frac{1}{2}\left[f_\lambda(x) - f_\lambda\left(x + \frac{\pi}{\lambda}\right)\right] + O(1)\omega\left(f, \frac{1}{\lambda}\right) \quad (\lambda \geqq 1)$$

and consequently $f_\lambda(x)$ *converges to* $f(x)$ *in case* $\lambda \to +\infty$ *uniformly in* $x \in (-\infty, \infty)$ *iff* $\omega\left(f_\lambda, \frac{1}{\lambda}\right) \to 0$ $(\lambda \to +\infty)$. *Furthermore the following estimate is fulfilled:*

$$f(x) - f_\lambda(x) = O(1)\left[\omega\left(f_\lambda, \frac{1}{\lambda}\right) + \omega\left(f, \frac{1}{\lambda}\right)\right] \quad (\lambda \geqq 1, \ -\infty < x < \infty).$$

To prove our theorem we need the following

LEMMA. *If* $f \in L(-\infty, \infty)$ *and*

$$\omega(f, \delta) \to 0 \quad (\delta \to 0) \tag{4}$$

then

$$\lim_{t \to \infty} f(t) = 0. \tag{5}$$

PROOF.[1] Let us suppose that (5) does not hold. Then there exist a number $\varepsilon > 0$ and a sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $|f(x_n)| > \varepsilon$ $(n = 1, 2, \ldots)$ and $x_n \to \infty$ $(n \to \infty)$. We obtain from (4) that

$$|f(x_n)| > \omega(f, \delta) \quad (n = 1, 2, \ldots)$$

for every fixed δ such that $0 < \delta < \delta_1(\varepsilon)$. We can suppose without any restriction that $x_{n+1} > x_n + \delta$. We have

$$|f(t) - f(x_n)| < \omega\left(f, \frac{\delta}{2}\right) \quad \left(|t - x_n| < \frac{\delta}{2}\right)$$

and so

$$|f(t)| > \omega(f, \delta) - \omega\left(f, \frac{\delta}{2}\right) \quad \left(|t - x_n| < \frac{\delta}{2}\right).$$

Hence we obtain

$$\int_{x_n - \frac{\delta}{2}}^{x_n + \frac{\delta}{2}} |f(t)|\,dt > \delta\left[\omega(f, \delta) - \omega\left(f, \frac{\delta}{2}\right)\right] \quad (n = 1, 2, \ldots).$$

It is easy to see that $\delta\left[\omega(f, \delta) - \omega\left(f, \frac{\delta}{2}\right)\right] > 0$ for some $0 < \delta < \delta_1(\varepsilon)$ unless $f \equiv 0$. It means that

$$\int_{-\infty}^{\infty} |f(t)|\,dt = +\infty$$

unless $f \equiv 0$ and we have got a contradiction.

Now we turn to the

[1] The basic idea of this proof is due to I. P. NATANSON [1].

PROOF OF THE THEOREM. Let us choose a point x and suppose it to be fixed. Let f^x be a 2π periodic continuous function which is defined on the interval $\left[x-\frac{\pi}{2}, x+\frac{3\pi}{2}\right]$ as follows:

$$f^x(t) = \begin{cases} f(t), & \text{if} \quad x-\frac{\pi}{2} \leqq t \leqq x+\frac{\pi}{2} \\ f(2x+\pi-t), & \text{if} \quad x+\frac{\pi}{2} \leqq t \leqq x+\frac{3}{2}\pi \end{cases} \tag{6}$$

and let

$$f = f^x + {}^x f. \tag{7}$$

First we show that

$$f^x(x) - B_\lambda(x, f^x) = O(1)\omega\left(f, \frac{1}{\lambda}\right) \quad (\lambda \geqq 1). \tag{8}$$

Let us denote by $S_{[\lambda]}(t, f^x)$ the $[\lambda]$-th partial sum of the trigonometric Fourier series of the function f^x, and by a_k and b_k $(k=1, 2, \ldots)$ the Fourier coefficients of f^x. It is well known that[2]

$$f^x_\lambda(t) = S_{[\lambda]}(t, f^x) - \frac{[1-\lambda+[\lambda]]}{2}(a_{[\lambda]}\cos[\lambda]t + b_{[\lambda]}\sin[\lambda]t)$$

since f^x has period 2π. Thus we have

$$|f^x(x) - B_\lambda(x, f^x)| \leqq \tag{9}$$

$$\leqq \left|f^x(x) - \frac{1}{2}\left[S_{[\lambda]}(x, f^x) + S_{[\lambda]}\left(x+\frac{\pi}{\lambda}, f^x\right)\right]\right| + \frac{1}{2}|a_{[\lambda]}| + \frac{1}{2}|b_{[\lambda]}|.$$

It is easy to see that the first expression on the right side of (9) is of order $\omega\left(f^x, \frac{1}{\lambda}\right)$. Indeed, denoting by $H_{[\lambda]}$ the trigonometric polynomial of best approximation in the uniform metric to f^x of order $[\lambda]$, we obviously have

$$\left|f^x(x) - \frac{1}{2}\left[S_{[\lambda]}(x, f^x) + S_{[\lambda]}\left(x+\frac{\pi}{\lambda}, f^x\right)\right]\right| \leqq$$

$$\leqq \frac{1}{2}\left\{\left|f^x(x) - f^x\left(x+\frac{\pi}{\lambda}\right)\right| + |H_{[\lambda]}(x) - f^x(x)| + \left|H_{[\lambda]}\left(x+\frac{\pi}{\lambda}\right) - f^x\left(x+\frac{\pi}{\lambda}\right)\right| + \right.$$

$$\left. + \left|S_{[\lambda]}(x, f^x - H_{[\lambda]}) + S_{[\lambda]}\left(x+\frac{\pi}{\lambda}, f^x - H_{[\lambda]}\right)\right|\right\}$$

and using the theorems of Rogosinski and Jackson[3] we obtain

$$\left|f^x(x) - \frac{1}{2}\left[S_{[\lambda]}(x, f^x) + S_{[\lambda]}\left(x+\frac{\pi}{\lambda}, f^x\right)\right]\right| = O(1)\omega\left(f^x, \frac{1}{\lambda}\right). \tag{10}$$

[2] See e.g. A. ZYGMUND [2], II. (7.2).
[3] See e.g. A. ZYGMUND [2], III.12 and III.13.

Further we have[4]

$$|a_{[\lambda]}|+|b_{[\lambda]}| \leqq 2\omega\left(f^x, \frac{\pi}{[\lambda]}\right). \tag{11}$$

Remarking the obvious inequality $\omega(f^x, \delta)\leqq\omega(f, \delta)$ $(0<\delta<\pi)$ we obtain (8) from (9), (10) and (11).

Now we estimate the expression $B_\lambda(x, {}^xf)$. We get from (1) and (2) that

$${}^xf_\lambda(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} {}^xf(u+t)\frac{\sin\lambda u}{u}\,du$$

and consequently

$$B_\lambda(t, {}^xf) = \frac{1}{2\pi}\int_{-\infty}^{\infty} {}^xf(u+t)\sin\lambda u\left[\frac{1}{u}-\frac{1}{u-\frac{\pi}{\lambda}}\right]du.$$

We obtain from the lemma and formulas (6) and (7) that $|{}^xf(u)|=O(1)$ $(-\infty<u<\infty)$ and ${}^xf(u+x)\equiv 0$ if $|u|\leqq\frac{\pi}{2}$. Hence

$$B_\lambda(x, {}^xf) = O(1)\frac{1}{\lambda}\int_{|u|\geqq\frac{\pi}{2}} \frac{1}{|u|\cdot\left|u-\frac{\pi}{\lambda}\right|}du \quad (\lambda\geqq 1)$$

and so we have

$$B_\lambda(x, {}^xf) = O(1)\cdot\frac{1}{\lambda} = O(1)\omega\left(f, \frac{1}{\lambda}\right) \quad (\lambda\geqq 1), \tag{12}$$

since every modulus of continuity $\omega(f, \delta)$ tends to zero if $\delta\to 0$ not faster than δ. The theorem follows now from (6), (7), (8) and (12).

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References

[1] I. P. NATANSON, On a summation method of Fourier integrals, *Mat. Sbornik*, 7 (**49**) (1940), pp. 313—320.
[2] A. ZYGMUND, *Trigonometric series*, vol. I (Cambridge, University Press, 1959).

[4] See e.g. A. ZYGMUND [2], II(4.1).

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WEITERE BEMERKUNGEN ÜBER DAS MAXIMUM VON TEILSUMMEN ORTHOGONALER REIHEN

Von

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1. Es sei $1 \leqq K \leqq \infty$. Für eine Zahlenfolge $\{a_n\}_1^\infty$ setzen wir

$$\|\{a_n\}_1^\infty; K\| = \sup_{|\varphi_n| \leqq K} \left\{ \int_0^1 \sup_{\substack{i,j \\ 1 \leqq i \leqq j < \infty}} (a_i \varphi_i(x) + \ldots + a_j \varphi_j(x))^2 dx \right\}^{1/2},$$

wobei $\sup_{|\varphi_n| \leqq K}$ bedeutet, daß Supremum für jedes im Grundintervall (0, 1) orthonormierte System $\{\varphi_n(x)\}_1^\infty$ mit

$$|\varphi_n(x)| \leqq K \quad (0 \leqq x \leqq 1, n = 1, 2, \ldots)$$

gebildet wird. (Im Falle $K = \infty$ soll man das Supremum für jedes orthonormierte System $\{\varphi_n(x)\}$ in (0, 1) bilden.)

Nach der Definition gilt die Ungleichung

$$\|\{a_n\}_1^\infty; K_1\| \leqq \|\{a_n\}_1^\infty; K_2\| \leqq \|\{a_n\}_1^\infty; \infty\| \quad (1 \leqq K_1 \leqq K_2)$$

für jede Folge $\{a_n\}_1^\infty$.

In dieser Note werden wir die Abhängigkeit der Norm $\|\{a_n\}_1^\infty; K\|$ von K betrachten.

2. In einer vorherigen Arbeit [1] haben wir das Problem aufgeworfen, ob auch die entgegengesetzte Ungleichung

$$\|\{a_n\}_1^\infty; \infty\| \leqq C(K) \|\{a_n\}_1^\infty; K\| \quad (K > 1)$$

mit eine nur von K abhängigen positiven Konstante $C(K)$ für jede Folge $\{a_n\}_1^\infty$ richtig ist. Dieses Problem ist noch ungelöst. (Die obige Ungleichung wurde nur für gewisse Folgen $\{a_n\}_1^\infty$ bewiesen s. [1], [2].) In dieser Note werden wir bezüglich dieses Problems die folgende Behauptung beweisen:

SATZ I. *Ist* $\|\{a_n\}_1^\infty; \infty\| < \infty$, *dann gibt es für jedes* $\varepsilon (> 0)$ *eine, von der Folge* $\{a_n\}_1^\infty$ *abhängige positive Konstante* $K_0 = K_0(\{a_n\}) (> 1)$ *derart, daß*

$$\|\{a_n\}_1^\infty; \infty\| \leqq \|\{a_n\}_1^\infty; K_0\| + \varepsilon$$

besteht.

Um diese Behauptung zu beweisen brauchen wir einen Hilfssatz.

Für eine Folge $\{a_n\}_1^\infty$ und für eine natürliche Zahl N setzen wir

$$\{a_n\}_1^N = \{a_1, \ldots, a_N, 0, \ldots\}.$$

HILFSSATZ I. *Es sei* $\{a_n\}_1^\infty$ *eine Folge und* N *eine natürliche Zahl. Für jedes* $\varepsilon>0$ *gibt es ein in* $(0, 1)$ *orthonormiertes, von* ε *und von der Folge* $\{a_n\}_1^N$ *abhängiges System von Treppenfunktionen*[1] $\psi_1(x), \ldots, \psi_N(x)$ *derart, daß*

$$\|\{a\}_1^N; \infty\|^2 - \varepsilon \leqq \int_0^1 \max_{1\leqq i\leqq j\leqq N} (a_i\psi_i(x) + \ldots + a_j\psi_j(x))^2\,dx$$

gilt.

Dieser Hilfssatz ist bekannt (s. [3]).

BEWEIS DES SATZES I. Nach der Definition gilt

$$\|\{a_n\}_1^N; K\| \nearrow \|\{a_n\}_1^\infty; K\| \quad (N \to \infty) \tag{1}$$

für jede Folge $\{a_n\}_1^\infty$ und für jedes K $(1 \leqq K \leqq \infty)$. So gibt es eine natürliche Zahl N_0 mit

$$\|\{a_n\}_1^{N_0}; \infty\| \geqq \|\{a_n\}_1^\infty; \infty\| - \varepsilon/2. \tag{2}$$

Weiter erhalten wir durch Anwendung des Hilfssatzes I, daß es ein in $(0, 1)$ orthonormiertes System von Treppenfunktionen $\psi_1(x), \ldots, \psi_N(x)$ derart existiert, daß

$$\|\{a_n\}_1^{N_0}; \infty\| - \varepsilon/2 \leqq \left\{\int_0^1 \max_{1\leqq i\leqq j\leqq N_0} (a_i\psi_i(x) + \ldots + a_j\psi_j(x))^2\,dx\right\}^{1/2} \tag{3}$$

gilt. Da die $\psi_n(x)$ $(n=1, \ldots, N_0)$ Treppenfunktionen sind, gibt es eine Konstante $K_0(>1)$ mit

$$|\psi_n(x)| \leqq K_0 \quad (0 \leqq x \leqq 1,\ n = 1, \ldots, N_0).$$

So folgt aus (2) und (3)

$$\|\{a_n\}_1^\infty; \infty\| \leqq \|\{a_n\}_1^{N_0}; K_0\| + \varepsilon \leqq \|\{a_n\}_1^\infty; K_0\| + \varepsilon.$$

3. Im folgenden werden wir die Stetigkeitsverhältnisse der Norm $\|\{a_n\}_1^\infty; K\|$ als Funktion von K diskutieren. Wir werden den folgenden Satz zeigen:

SATZ II. *Für jede Folge* $\{a_n\}_1^\infty$ *ist* $\|\{a_n\}_1^\infty; K\|$ *als Funktion von* K *in* $(1, \infty]$ *stetig.*

Zum Beweis des Satzes II benötigen wir noch weitere Hilfssätze.

HILFSSATZ II. *Es sei* $\{a_n\}_1^\infty$ *eine Folge,* N *eine natürliche Zahl, und* $K>1$. *Für jedes* $\varepsilon>0$ *gibt es ein in* $(0, 1)$ *orthonormiertes System von Treppenfunktionen* $\psi_1(x), \ldots, \psi_N(x)$ *mit*

$$|\psi_n(x)| \leqq K \quad (0 \leqq x \leqq 1,\ n = 1, \ldots, N) \tag{4}$$

und

$$\|\{a_n\}_1^N; K\|^2 - \varepsilon \leqq \int_0^1 \max_{1\leqq i\leqq j\leqq N} (a_i\psi_i(x) + \ldots + a_j\psi_j(x))^2\,dx. \tag{5}$$

[1] Eine Funktion in $(0, 1)$ wird Treppenfunktion genant, wenn eine Einteilung von $(0, 1)$ in endlich viele, paarweise disjunkte Teilintervalle derart existiert, daß die Funktion in jedem Teilintervalle mit einer Konstante gleich ist.

Dieser Hilfssatz ist die Verschärfung des Hilfssatzes I.

Auf Grund der Definition der Norm $\|\{a_n\}_1^N; K\|$ gibt es ein in $(0, 1)$ orthonormiertes System $\{\bar{\varphi}_n(x)\}_1^N$ mit

$$|\bar{\varphi}_n(x)| \leqq K \quad (0 \leqq x \leqq 1,\ n = 1, \ldots, N) \tag{6}$$

und

$$\int_0^1 \max_{1 \leqq i \leqq j \leqq N} \left(a_i \bar{\varphi}_i(x) + \ldots + a_j \bar{\varphi}_j(x)\right)^2 dx \geqq \|\{a_n\}_1^N; K\|^2 - \varepsilon/3. \tag{7}$$

Es seien $\varphi_n(x)$ $(n=1, \ldots, N)$ Treppenfunktionen mit

$$\int_0^1 \left(\bar{\varphi}_n(x) - \varphi_n(x)\right)^2 dx < \eta \quad (i = 1, \ldots, N)$$

und

$$|\varphi_n(a)| \leqq |\bar{\varphi}_n(x)| \quad (0 \leqq x \leqq 1,\ n = 1, \ldots, N). \tag{8}$$

Wir setzen

$$\alpha_{ij} = \int_0^1 \varphi_i(x)\varphi_j(x)dx \quad (1 \leqq i, j \leqq N).$$

Es sei a $(0<a<1)$ so gewählt, daß die Ungleichungen

$$\left(1 - (1-a^2)^2(1-a)\right)/a < K^2, \tag{9}$$

$$(1-a)(1-a^2) \int_0^1 \max_{1 \leqq i \leqq j \leqq N} \left(a_i \bar{\varphi}_i(x) + \ldots + a_j \bar{\varphi}_j(x)\right)^2 dx \geqq \tag{10}$$

$$\geqq \int_0^1 \max_{1 \leqq i \leqq j \leqq N} \left(a_i \bar{\varphi}_i(x) + \ldots + a_j \bar{\varphi}_j(x)\right)^2 dx - \frac{\varepsilon}{3}$$

erfüllt werden.

Teilen wir das Intervall $(1-a^2, 1)$ in $N(N-1)$ paarweise disjunkte Teilintervalle I_{ij} $(1 \leqq i, j \leqq N, i \neq j)$ von gleicher Länge $|I_{ij}| = \dfrac{a^2}{N(N-1)}$. Ist η genügend klein $(\eta < \eta_1)$, dann gelten die Abschätzungen

$$\int_0^1 \max_{1 \leqq i \leqq j \leqq N} \left(a_i \varphi_i(x) + \ldots + a_j \varphi_j(x)\right)^2 dx > \tag{11}$$

$$> \int_0^1 \max_{1 \leqq i \leqq j \leqq N} \left(a_i \bar{\varphi}_i(x) + \ldots + a_j \bar{\varphi}_j(x)\right)^2 dx - \frac{\varepsilon}{3},$$

$$\frac{N}{2a^2} \max_{\substack{1 \leqq i, j \leqq N \\ i \neq j}} |\alpha_{ij}| \leqq K^2, \tag{12}$$

$$\alpha_{nn} \geqq 1 - a^2 \quad (n = 1, \ldots, N). \tag{13}$$

Wir setzen

$$\overline{\psi}_n(x) = \begin{cases} \varphi_n\left(\dfrac{x}{1-a^2}\right) & (0 \leqq x \leqq 1-a^2), \\ \sqrt{\dfrac{N}{2a^2}|\alpha_{ni}|} & (x \in I_{ni};\ 1 \leqq i, j \leqq N,\ i \neq j), \\ -\sqrt{\dfrac{N}{2a^2}|\alpha_{ni}|}\ \operatorname{sign}\alpha_{ni} & (x \in I_{in};\ 1 \leqq i, j \leqq N,\ i \neq j), \\ 0 & \text{sonst.} \end{cases} \qquad (n = 1, \ldots, N)$$

Offensichtlich bilden die Treppenfunktionen $\overline{\psi}_n(x)$ (x) $(n=1, \ldots, N)$ ein orthonormiertes System in (0, 1), weiterhin gelten auf Grund von (6), (8), (12), (13) und der Definition der Funktionen $\overline{\psi}_n(x)$ die folgenden Beziehungen:

$$|\overline{\psi}_n(x)| \leqq K \quad (0 \leqq x \leqq 1 \ \ n = 1, \ldots, N), \tag{14}$$

$$\int_0^1 \max_{1 \leqq i \leqq j \leqq N} \left(a_i\overline{\psi}_i(x) + \ldots + a_j\overline{\psi}_j(x)\right)^2 dx \geqq \tag{15}$$

$$\geqq \int_0^{1-a^2} \max_{1 \leqq i \leqq j \leqq N} \left(a_i\overline{\psi}_i(x) + \ldots + a_j\overline{\psi}_j(x)\right)^2 dx =$$

$$= (1-a^2) \int_0^1 \max_{1 \leqq i \leqq j \leqq N} \left(a_i\varphi_i(x) + \ldots + a_j\varphi_j(x)\right)^2 dx,$$

$$\int_0^1 \overline{\psi}_n^2(x)\,dx \geqq \int_0^{1-a^2} \overline{\psi}_n^2(x)\,dx = (1-a^2)\alpha_{nn} \geqq (1-a^2)^2 \quad (n = 1, \ldots, N). \tag{16}$$

Es sei endlich

$$\psi_n(x) = \begin{cases} \overline{\psi}_n\left(\dfrac{x}{1-a}\right) & (0 \leqq x \leqq 1-a), \\ m_n r_n\left(\dfrac{x-1+a}{a}\right) & (1-a < x \leqq 1) \end{cases} \qquad (n = 1, \ldots, N),$$

wobei m_n derart bestimmt ist, daß die Funktion $\psi_n(x)$ normiert ist, d. h.

$$(1-a)\int_0^1 \overline{\psi}_n^2(x)\,dx + m_n^2 a = 1 \quad (n = 1, \ldots, N) \tag{17}$$

gilt. Auf Grund von (9), (16) und (17) folgt:

$$m_n \leqq \sqrt{\frac{1-(1-a^2)^2(1-a)}{a}} \leqq K \quad (n = 1, \ldots, N).$$

Daraus und aus (14) erhalten wir (4). Offensichtlich bilden die Treppenfunktionen $\psi_n(x)$ $(n=1, \ldots, N)$ ein orthonormiertes System in (0, 1). Weiterhin gilt

$$\int_0^1 \max_{1\leq i\leq j\leq N} \left(a_i\psi_i(x)+\ldots+a_j\psi_j(x)\right)^2 dx \geqq \tag{18}$$

$$\geqq \int_0^{1-a} \max_{1\leq i\leq j\leq N} \left(a_i\psi_i(x)+\ldots+a_j\psi_j(x)\right)^2 dx =$$

$$= (1-a)\int_0^1 \max_{1\leq i\leq j\leq N} \left(a_i\overline{\psi}_i(x)+\ldots+a_j\overline{\psi}_j(x)\right)^2 dx.$$

Endlich erhalten wir (5) auf Grund von (9), (10), (11), (15) und (18).

Damit haben wir den Hilfssatz II bewiesen.

HILFSSATZ III. *Es sei* $\|\{a_n\}_1^\infty; K_0\|<\infty$ *für ein* K_0 $(1<K_0<\infty)$. *Dann gilt*

$$\|\{a_n\}_1^\infty; K\| \leqq \|\{a_n\}_1^\infty; K_0\| \leqq \sqrt{\frac{K_0^2-1}{K^2-1}}\,\|\{a_n\}_1^\infty; K\| \quad (1<K<K_0).$$

Auf Grund der Definition von $\|\{a_n\}_1^\infty; K\|$ ist die erste Ungleichung klar. Es soll nun die zweite beweisen. Ist $\|\{a_n\}_1^\infty; K'\|=0$ für ein $K'(\geqq 1)$, dann gilt nach der Definition der Norm $\|\{a_n\}_1^\infty; K'\|$ offensichtlich $a_n=0$ $(n=1, 2, \ldots)$, und somit ist $\|\{a_n\}_1^\infty; K\|=0$ für jedes $K(\geqq 1)$; in diesem Falle ist aber die Behauptung richtig. Wir dürfen also $\|\{a_n\}_1^\infty; K\|>0$ $(1\leqq K\leqq\infty)$ voraussetzen.

Auf Grund von (1) wählen wir einen Index N mit

$$\|\{a_n\}_1^\infty; K_0\| - \|\{a_n\}_1^N; K_0\| < \frac{\varepsilon}{3}. \tag{19}$$

Auf grund des Hilfssatzen II gibt es ein in (0, 1) orthonormiertes System von Treppenfunktionen $\psi_1(x), \ldots, \psi_N(x)$ mit

$$|\psi_n(x)| \leqq K_0 \quad (0\leqq x\leqq 1,\ n=1, \ldots, N) \tag{20}$$

und

$$\|\{a_n\}_1^N; K_0\| - \left\{\int_0^1 \max_{1\leq i\leq j\leq N} \left(a_i\psi_i(x)+\ldots+a_j\psi_j(x)\right)^2 dx\right\}^{1/2} \leqq \frac{\varepsilon}{3}. \tag{21}$$

Es sei

$$a = \frac{K_0^2-K^2}{K^2(K_0^2-1)} \quad (<1). \tag{22}$$

Wir setzen

$$\chi_n(x) = \begin{cases} \dfrac{K}{K_0}\,\psi_n\left(\dfrac{x}{1-a}\right) & (0\leqq x\leqq 1-a), \\ \varrho r_n\left(\dfrac{x-1+a}{a}\right) & (1-a<x\leqq 1) \end{cases} \qquad (n=1, \ldots, N),$$

wobei ϱ derart bestimmt ist, daß die Funktion $\chi_n(x)$ normiert ist, d. h.

$$\frac{K^2}{K_0^2}(1-a)+\varrho^2 a = 1 \tag{23}$$

gilt. Die Treppenfunktionen $\chi_n(x)$ $(n=1, \ldots, N)$ bilden offensichtlich ein orthonormiertes System in (0, 1), ferner gilt nach (22) und (23)

$$\varrho = \sqrt{\frac{1-\frac{K^2}{K_0^2}(1-a)}{a}} = K.$$

Daraus und aus (20) erhalten wir

$$|\chi_n(x)| \leqq K \quad (0 \leqq x \leqq 1,\ n = 1, \ldots, N). \tag{24}$$

Es gilt weiterhin

$$\int_0^1 \max_{1\leqq i\leqq j\leqq N} (a_i\chi_i(x)+\ldots+a_j\chi_j(x))^2 dx \geqq \tag{25}$$

$$\geqq \int_0^{1-a} \max_{1\leqq i\leqq j\leqq N} (a_i\chi_i(x)+\ldots+a_j\chi_j(x))^2 dx =$$

$$= \frac{K^2}{K_0^2}(1-a)\int_0^1 \max_{1\leqq i\leqq j\leqq N} (a_i\psi_j(x)+\ldots+a_j\psi_j(x))^2 dx.$$

Aus (19), (21), (24) und (28) bekommen wir

$$\|\{a_n\}_1^\infty;\ K_0\| \leqq \|\{a_n\}_1^N;\ K_0\| + \frac{\varepsilon}{3} \leqq$$

$$\leqq \left\{\int_0^1 \max_{1\leqq i\leqq j\leqq N} (a_i\psi_i(x)+\ldots+a_j\psi_j(x))^2 dx\right\}^{1/2} + \frac{2\varepsilon}{3} \leqq$$

$$\leqq \frac{K_0}{K}\frac{1}{\sqrt{1-a}}\|\{a_n\}_1^N;\ K\| + \frac{2\varepsilon}{3} \leqq \frac{K_0}{K}\frac{1}{\sqrt{1-a}}\|\{a_n\}_1^\infty;\ K\| + \frac{2\varepsilon}{3} =$$

$$= \sqrt{\frac{K_0^2-1}{K^2-1}}\|\{a_n\}_1^\infty;\ K\| + \frac{2\varepsilon}{3}.$$

Da $\varepsilon(>0)$ beliebig ist, ergibt sich mithin die Behauptung des Hilfssatzes III.

Der Satz II folgt unmittelbar aus dem Satz I und dem Hilfssatz III.

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Schriftenverzeichnis

[1] K. TANDORI, Über das Maximum der Summen orthogonaler Funktionen, *Acta Sci. Math. Szeged,* **32** (1971), S. 317—326.

[2] K. TANDORI, Über die unbedingte Konvergenz der Orthogonalreihen, *Acta Sci. Math. Szeged,* **32** (1971), S. 11—40.

[3] K. TANDORI, Über die Konvergenz der Orthogonalreihen, *Acta Sci. Math. Szeged,* **24** (1963), S. 139—151.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 25 (3—4), (1974), pp. 425—433.

A LAW OF THE ITERATED LOGARITHM FOR WEAKLY MULTIPLICATIVE SYSTEMS AND ITS APPLICATIONS

By
P. RÉVÉSZ (Budapest)

Introduction

The sequence $\varphi_1, \varphi_2, \ldots$ of random variables is called a multiplicative system (MS) if

$$\int \varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_k} = 0 \quad (i_1 < i_2 < \ldots < i_k;\ k = 1, 2, \ldots). \tag{1}$$

This definition was introduced by G. ALEXITS ([9], [10]).

In [1] the following was proved:

THEOREM A. *Let $\varphi_1, \varphi_2, \ldots$ be a uniformly bounded* MS *and let $a_1, a_2, \ldots$ be a sequence of real numbers for which*

$$a_n = o\left(\frac{A(n)}{\sqrt{\log\log A(n)}}\right) \quad \text{and} \quad A(n) \to \infty \tag{2}$$

where $A^2(n) = \sum_{k=1}^{n} a_k^2$. Further let $M_k = M_k(\theta)$ defined by

$$A^2(M_k - 1) < \theta^k \leqq A^2(M_k)$$

and suppose that

$$\varliminf_{k\to\infty} \frac{T^2(M_k)}{A^2(M_k)} > 0 \quad a.e. \tag{3}$$

for any $\theta > 1$ where $T^2(n) = \sum_{k=1}^{n} a_k^2\varphi_k^2$. Then

$$\varlimsup_{n\to\infty} \frac{S(n)}{\sqrt{2T^2(n)\log\log A(n)}} \leqq 1 \quad a.e. \tag{4}$$

where $S(n) = \sum_{k=1}^{n} a_k\varphi_k$.

Because of some applications (see § 3) it is natural to ask how can we replace condition (1) by a weaker one. A concept of weak multiplicativity was introduced and studied by G. ALEXITS ([11], [12]). In this paper a similar concept (see (5) and (5*)) will be defined and two laws of iterated logarithm will be proved.

THEOREM 1. *Let $\varphi_1, \varphi_2, \ldots$ be a sequence of uniformly bounded random variables ($|\varphi_i| \leqq K_1$; $i = 1, 2, \ldots$) for which there exists a positive constant K_2 such that*

$$\left|\int \prod_{i=n}^{m} (1 + b_i\varphi_i)\right| \leqq K_2 \quad (1 \leqq n < m < \infty) \tag{5}$$

where $b_1, b_2, \ldots$ *is any sequence of real numbers for which*

$$|b_i| \leqq 1 \quad (i = 1, 2, \ldots).$$

Further let $a_1, a_2, \ldots$ *be a sequence of real numbers for which* (2) *holds and finally suppose the validity of* (3).

Then we have (4).

THEOREM 2. *Let* $\varphi_1, \varphi_2, \ldots$ *be a sequence of uniformly bounded random variables* ($|\varphi_i| \leqq K_1$; $i=1, 2, \ldots$) *for which there exists a positive constant* K_2^* *such that*

$$(5^*) \qquad \left| \int \prod_{i=n}^{m} (1+b_i\varphi_i) \right| \leqq K_2^* \quad (1 \leqq n < m < \infty)$$

provided that $|b_i| \leqq 3\sqrt{\frac{\log m}{m}}$. *Further suppose that*

$$(3^*) \qquad \varlimsup_{n\to\infty} \frac{\tilde{T}^2(\theta^n)}{\theta^n} > 0 \ \text{a.e.} \quad (\text{if } \theta > 1)$$

where $\tilde{T}^2(n) = \varphi_1^2 + \varphi_2^2 + \ldots + \varphi_n^2$. *Then*

$$\varlimsup_{n\to\infty} \frac{\tilde{S}(n)}{\sqrt{2\tilde{T}^2(n)\log\log n}} \leqq 1$$

where $\tilde{S}(n) = \varphi_1 + \varphi_2 + \ldots + \varphi_n$.

Clearly a common generalization of Theorems 1 and 2 would be interesting. The importance of this will be seen in the Applications (§ 3).

The proofs of Theorems 1 and 2 are very similar to each other and they are quite similar to the proof of Theorem A. This latter one was prepared in a detailed form in [1], therefore we take the liberty only to give the proof of Theorem 1 (and not that of 2) in a non-very detailed form.

§ 1. Inequalities

THEOREM B.[1] *Let* $\varphi_1, \varphi_2, \ldots, \varphi_n$ *be a sequence of uniformly bounded random variables* ($|\varphi_i| \leqq K_1$; $i=1, 2, \ldots, n$; $K_1 > 1$) *for which* (8) *holds and let* $a_1, a_2, \ldots, a_n$ *be a sequence of real numbers, further let* λ *be a positive number. Then*

$$\int e^{\lambda S} \leqq K_2 \exp\left\{\frac{\lambda^2 A^2 K_1^2}{2}\right\}$$

where $S = \sum_{k=1}^{n} a_k\varphi_k$ *and* $A^2 = \sum_{k=1}^{k} a_k^2$.

[1] This theorem is essentially due to AZUMA [2].

PROOF. Since e^z is a convex function, for $|x| \leqq 1$, $a \neq 0$ we have

$$e^{ax} \leqq e^{|a|} \frac{|a|+ax}{2|a|} + e^{-|a|} \frac{|a|-ax}{2|a|} = \cosh(|a|) + \frac{ax}{|a|} \sinh(|a|).$$

Hence (making use of the inequality $|\operatorname{tgh} x| \leqq 1$)

$$\int e^{\lambda S} = \int \prod_{k=1}^{n} \exp\left(a_k \lambda K_1 \frac{\varphi_k}{K_1}\right) \leqq \int \prod_{k=1}^{n} \left[\cosh(|a_k|\lambda K_1) + \frac{a_k}{|a_k|} \frac{\varphi_k}{K_1} \sinh(|a_k|\lambda K_1)\right] =$$

$$= \prod_{k=1}^{n} \cosh(|a_k|\lambda K_1) \int \prod_{k=1}^{n} \left(1 + \frac{a_k}{|a_k|} \frac{\varphi_k}{K_1} \operatorname{tgh}(|a_k|\lambda K_1)\right) \leqq$$

$$\leqq K_2 \prod_{k=1}^{n} \cosh(|a_k|\lambda K_1) = K_2 \prod_{k=1}^{n} \sum_{m=0}^{\infty} \frac{(\lambda K_1 |a_k|)^{2m}}{(2m)!} \leqq$$

$$\leqq K_2 \prod_{k=1}^{n} \sum_{m=0}^{\infty} \frac{(\lambda K_1 |a_k|)^{2m}}{2^m m!} = K_2 \prod_{k=1}^{n} \exp\left(\frac{\lambda^2 K_1^2 a_k^2}{2}\right) = K_2 \exp\left(\frac{\lambda^2 K_1^2 A^2}{2}\right)$$

i.e. Theorem B is proved.

THEOREM C (see [1], Theorem 2). *Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be a sequence of uniformly bounded random variables ($|\varphi_i| \leqq K_1$; $i=1, 2, \ldots, n$) and let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers. Then*

$$\exp\left(\lambda S - \frac{\lambda^2 T^2}{2}\left(1 + 2\lambda K_1 \max_{1 \leqq k \leqq n} |a_k|\right)\right) \leqq \prod_{k=1}^{n} (1 + \lambda a_k \varphi_k)$$

where $S = \sum_{k=1}^{n} a_k \varphi_k$, $T^2 = \sum_{k=1}^{n} a_k^2 \varphi_k^2$ *and λ is any positive number for which*

$$\lambda K_1 \max_{1 \leqq k \leqq n} |a_k| \leqq 1/2.$$

THEOREM 3. *Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be a sequence of uniformly bounded random variables ($|\varphi_i| \leqq K_1$; $i=1, 2, \ldots, n$; $K_1 \geqq 1$) obeying condition (5) and let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers, further let y be a positive number. Then*

$$\mathbf{P}\{|S| \geqq y K_1 A \sqrt{2}\} \leqq 2K_2 e^{-y^2}$$

where $S = \sum_{k=1}^{n} a_k \varphi_k$ *and* $A^2 = \sum_{k=1}^{n} a_k^2$.

PROOF. Let

$$\lambda = \frac{\sqrt{2} y}{K_1 A}$$

then by Theorem B we have

$$\int e^{\lambda |S|} \leqq \int e^{\lambda S} + \int e^{-\lambda S} \leqq 2K_2 \exp\left(\frac{\lambda^2 A^2 K_1^2}{2}\right)$$

and by Markov inequality

$$\mathbf{P}\{|S| \geqq yK_1A\sqrt{2}\} = \mathbf{P}\{e^{\lambda|S|} \geqq \exp(\lambda yK_1A\sqrt{2})\} \leqq$$

$$\leqq 2K_2 \exp\left(\frac{\lambda^2A^2K_1^2}{2} - \lambda yK_1A\sqrt{2}\right) = 2K_2 \exp(y^2 - 2y^2) = 2K_2e^{-y^2}$$

which proves our Theorem 3.

THEOREM 4. *Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be a sequence of uniformly bounded random variables obeying condition* (5) *and let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers. Then*

$$\mathbf{P}\{\max_{1\leqq m\leqq n} |S(m)| \geqq K_3\sqrt{A^2 \log\log A}\} \leqq K_4 \exp(-2\log\log A)$$

where $S(m) = \sum_{k=1}^{m} a_k\varphi_k$, $A^2 = \sum_{k=1}^{n} a_k^2$. *$K_3$ and K_4 are suitable positive constants.*

The proof of this theorem is step by step the same as that of Theorem 4 of [1]. Hence the proof is omitted.

§ 2. The proof of Theorem 1

The proofs of the following four lemmas are exactly the same as the proofs of the corresponding lemmas of [1].

LEMMA 1. *If* (2) *and* (5) *hold then we have*

$$\mathbf{P}\left\{\frac{S(M_k)}{\sqrt{2A^2(M_k)\log\log A(M_k)}} \geqq \frac{T^2(M_k)}{2CA^2(M_k)}\left(1 + 2\lambda K_1 \max_{1\leqq j\leqq M_k} |a_j|\right) + (1+\varepsilon)\frac{C}{2}\right\} = O\left(\frac{1}{k^{1+\varepsilon}}\right)$$

for any $C>0$ where

$$\lambda = \lambda(C) = \sqrt{\frac{2\log\log A(M_k)}{C^2A^2(M_k)}}.$$

LEMMA 2. *If* (3) *holds then for any $\varrho>0$ one can find a set F, a positive number $\varkappa$ and an integer k_0 such that*

$$\mathbf{P}(F) \leqq \varrho$$

and

$$\sqrt{\frac{T^2(M_k)}{A^2(M_k)}} \geqq \varkappa \quad \text{if} \quad k \geqq k_0 \quad \text{and} \quad x \notin F.$$

LEMMA 3. *Define the event $\mathfrak{A}_k$ by*

$$\mathfrak{A}_k = \left\{\frac{S(M_k)}{\sqrt{2T^2(M_k)\log\log A(M_k)}} \geqq 1+\delta\right\} \quad (\delta > 0).$$

Then (under the conditions of Theorem 1*) only finitely many $\mathfrak{A}_k$ can occur with probability* 1.

LEMMA 4. *Set*

$$F_k = \left\{ \max_{M_k \leqq N < M_{k+1}} |S(N) - S(M_k)| \geqq \varepsilon K_2 \sqrt{A^2(M_k) \log\log A(M_k)} \right\}$$

and suppose that conditions (2) *and* (5) *hold. Then among the events* F_k *only finitely many occur with probability* 1 *(where* ε *is any positive number).*

PROOF OF THEOREM 1. Theorem 1 is a trivial consequence of Lemmas 2, 3 and 4 and the inequality:

$$\left| \frac{S(N)}{\sqrt{2T^2(N) \log\log A(N)}} \right| \leqq \left| \frac{S(N) - S(M_k)}{\sqrt{2A^2(M_k) \log\log A(M_k)}} \right| \sqrt{\frac{A^2(M_k)}{T^2(M_k)}} +$$

$$+ \left| \frac{S(M_k)}{\sqrt{2T^2(M_k) \log\log A(M_k)}} \right|.$$

§ 3. Applications

1. *Subsequences of normed sequences*

M. WEISS ([3]) proved the following

THEOREM D. *Let* $\{\varphi_n\}$ *be a uniformly bounded orthonormal system of real-valued functions on the interval* [0, 1]. *Then there exists a subsequence* $\{\varphi_{n_k}\}$ *and a real valued function* $f(x)$, $\int_0^1 f^2(x)\,dx = 1$, $0 \leqq f(x) \leqq B$ *where* B *is the uniform bound of* $\{\varphi_n\}$, *such that for an arbitrary sequence* $\{a_k\}$ *of real numbers satisfying* (2) *we have*

$$\overline{\lim_{N \to \infty}} \frac{S_N(x)}{\sqrt{2A^2(N) \log\log A(N)}} = f(x) \quad a.e.$$

where $S_N(x) = \sum_{k=1}^{N} a_k \varphi_{n_k}(x)$.

In [4] we formulated a theorem (5.15) showing that the condition of orthogonality in Theorem D can be omitted (in some sense). Now we show that the upper part of Theorem D and Theorem 5.1.5 of [4] is a trivial consequence of Theorem 1, i.e. we prove our

THEOREM 5. *Let* $\{\varphi_n\}$ *be a sequence of uniformly bounded random variables. Then there exists a subsequence* $\{\varphi_{n_k}\}$, *a bounded random variable* μ *and a non-negative, bounded random variable* σ *such that for an arbitrary sequence* $\{a_k\}$ *of real numbers satisfying* (2) *we have*

$$\overline{\lim_{N \to \infty}} \frac{\sum_{k=1}^{N} a_k (\varphi_{n_k} - \mu)}{\sqrt{2A^2(N) \log\log A(N)}} \leqq \sigma \quad a.e. \tag{6}$$

PROOF. Since $\{\varphi_n\}$ is weakly compact in the space L^2 one can find a subsequence $\{\varphi_{n_k}\}$ which converges weakly to a bounded random variable μ, i.e.

$$\int \varphi_{n_k} g \to \int \mu g \quad (k \to \infty)$$

for every $g \in L^2$. Set

$$\varphi_{n_k} - \mu = \psi_k .$$

Similarly $\{\psi_k^2\}$ is also weakly compact i.e. there exists a subsequence $\{\psi_{n_k}^2\}$ which converges weakly to a bounded non-negative random variable σ^2. (This sequence $\{n_k\}$ is not certainly the same as the sequence of indices of φ. The meaning of $\{n_k\}$ will be understood in the sequel in a similar way.) Set

$$\mathfrak{A} = \mathfrak{A}_\varepsilon = \{\sigma^2 \geqq \varepsilon\}, \quad \mathfrak{B} = \{\sigma^2 = 0\}, \quad \psi_{n_k} = \chi_k .$$

We will prove that for every $\varepsilon > 0$ there exists a subsequence $\{\chi_{n_k}\}$ such that

$$\overline{\lim_{k\to\infty}} \frac{a_1\chi_{n_1}(x) + a_2\chi_{n_2}(x) + \ldots + a_k x_{n_k}(x)}{\sqrt{2A^2(k)\log\log A(k)}} \leqq \sigma \quad \text{if} \quad x \in \mathfrak{A}_\varepsilon \tag{7}$$

and

$$\overline{\lim_{h\to\infty}} \frac{a_1\chi_{n_1}(x) + a_2\chi_{n_2}(x) + \ldots + a_k\chi_{n_k}(x)}{\sqrt{2A^2(k)\log\log A(k)}} \leqq 0 \quad \text{if} \quad x \in \mathfrak{B} \tag{8}$$

provided that $\{a_k\}$ satisfies (2).

Making use of the diagonal method (7) and (8) imply (6). Now we turn to the

PROOF OF (7). Set

$$\frac{\chi_k}{\sigma} = \vartheta_k .$$

Clearly ϑ_k and ϑ_k^2 converge weakly to 0 and 1 on the set $\mathfrak{A}_\varepsilon$, respectively, i.e.

$$\int_{\mathfrak{A}_\varepsilon} \vartheta_k g \to 0 \quad (k \to \infty) \quad \text{and} \quad \int_{\mathfrak{A}_\varepsilon} \vartheta_k^2 g \to \int_{\mathfrak{A}_\varepsilon} g \quad (k \to \infty)$$

for any $g \in L^2$.

Now we construct the subsequence $\{\vartheta_{n_k}\}$ as follows. Let n_1 be the smallest integer for which

$$\left|\int_{\mathfrak{A}_\varepsilon} \vartheta_{n_1}\right| \leqq \frac{1}{4} \quad \text{and} \quad \left|\int_{\mathfrak{A}_\varepsilon} (\vartheta_{n_1}^2 - 1)\right| \leqq \frac{1}{4} .$$

Suppose that the integers $n_1 < n_2 < \ldots < n_k$ have already been chosen, then choose $n_{k+1} > n_k$ as the smallest integer for which

$$\left|\int_{\mathfrak{A}_\varepsilon} \vartheta_{n_{i_1}} \vartheta_{n_{i_2}} \ldots \vartheta_{n_{i_l}} \vartheta_{n_{k+1}}\right| \leqq \frac{1}{2^{2(k+1)}} \quad (i_1 < i_2 < \ldots < i_l \leqq k;\ l = 0, 1, 2, \ldots, k)$$

and

$$\left|\int_{\mathfrak{A}_\varepsilon} (\vartheta_{n_{i_1}}^2 - 1)(\vartheta_{n_{i_2}}^2 - 1) \ldots (\vartheta_{n_{i_l}}^2 - 1)(\vartheta_{n_{k+1}}^2 - 1)\right| \leqq \frac{1}{2^{2(k+1)}}$$

$$(i_1 < i_2 < \ldots < i_l \leqq k;\ l = 0, 1, 2, \ldots, k).$$

Clearly this sequence $\{\vartheta_{n_k}\}$ (and its any subsequence) is obeying conditions (3) and (5) i.e. by Theorem 1 we have (7).

Before the proof of (8) we formulate the very simple

LEMMA 5. *Let*

$$\varepsilon_n \leqq \frac{1}{n^2 \exp(e^n)}$$

and $\{a_n\}$ *be a sequence of real numbers obeying condition* (2). *Then*

$$\frac{a_1\varepsilon_1 + a_2\varepsilon_2 + \ldots + a_n\varepsilon_n}{\sqrt{2A^2(n)\log\log A(n)}} \to 0.$$

PROOF. By (2), if n is big enough then

$$A^2(n) \leqq \frac{nA^2(n)}{\log\log A(n)}$$

i.e.

$$|a_n| \leqq A(n) \leqq e^{e^n}.$$

Hence the series $\sum_{k=1}^{\infty} a_k\varepsilon_k$ is convergent what proves the Lemma.

PROOF OF (8). Clearly

$$\int_{\mathfrak{B}} \chi_{n_k}^2 \to 0.$$

Hence for any $\varepsilon>0$ one can find a set F_ε $(\mathbf{P}(F_\varepsilon)>1-\varepsilon)$ and a subsequence $\{\chi_{n_{k_j}}\}$ such that $\chi_{n_{k_j}} \to 0$ uniformly on F_ε and moreover

$$|\chi_{n_{k_j}}| \leqq \frac{1}{j^2 e^{ej}} \quad \text{if} \quad x \in F_\varepsilon.$$

Now one can get (8) by Lemma 5 and making use of the diagonal method.

2. *Martingales*

Let $\xi_0=0, \xi_1, \xi_2, \ldots$ be a martingale and let

$$\eta_i = \xi_i - \xi_{i-1} \quad (i = 1, 2, \ldots).$$

Then clearly

$$\mathbf{E}(\eta_{i_1}\eta_{i_2}\ldots\eta_{i_k}) = 0 \quad (1 \leqq i_1 < i_2 < \ldots < i_k;\ k = 1, 2, \ldots)$$

provided that the expectation exists, i.e. $\eta_1, \eta_2, \ldots$ constitutes a MS if the expectation of $\eta_{i_1}\eta_{i_2}\ldots\eta_{i_k}$ exists.

W. F. STOUT ([5], Theorem 1) proved the following

THEOREM E.[2] *Let* $\xi_0=0, \xi_1, \xi_2, \ldots$ *be a martingale and set*

$$\eta_i = \xi_i - \xi_{i-1} \quad (i = 1, 2, \ldots).$$

[2] The original form of Stout's theorem is somewhat more general.

Suppose that

$$S_n^2 = \sum_{i=1}^{n} \mathbf{E}(\eta_i^2 | \xi_{i-1}, \xi_{i-2}, \ldots, \xi_0) \to +\infty \quad a.e.$$

and

$$|\eta_n| \leqq K_n \frac{S_n}{\sqrt{\log\log S_n}} \quad a.e.$$

where K_n is a random variable going to 0 a.e. and being measurable with respect to[3] $B(\xi_1, \xi_2, \ldots, \xi_{n-1})$. *Then*

$$\overline{\lim_{n\to\infty}} \frac{\xi_n}{\sqrt{2S_n^2 \log\log S_n}} \leqq 1.$$

Now we can get an analogue of Theorem E.

THEOREM 6. *Let $\xi_0=0, \xi_1, \xi_2, \ldots$ be a martingale, set*

$$\eta_i = \xi_i - \xi_{i-1} \quad (i = 1, 2, \ldots), \quad T^2(n) = \eta_1^2 + \eta_2^2 + \ldots + \eta_n^2$$

and suppose that there exists a sequence $\{a_n\}$ of real numbers such that $|\eta_n| \leqq a_n$ and

$$\max_{1\leqq k\leqq n} a_k = o\left(\frac{A(n)}{\sqrt{\log\log A(n)}}\right) \quad \left(A^2(n) = \sum_{k=1}^{n} a_k^2\right).$$

Further assume that

$$\underline{\lim_{n\to\infty}} \frac{T^2(n)}{A^2(n)} > 0 \quad a.e. \tag{3**}$$

Then we have

$$\overline{\lim_{n\to\infty}} \frac{\xi_n}{\sqrt{2T_n^2 \log\log A(n)}} \leqq 1.$$

PROOF. This Theorem is a straight consequence of Theorem A.

REMARK. In some sense Theorem 6 is weaker than Theorem E, especially condition (3**) seems to be too strong.

3. *Subsequences of Walsh functions*

Let $W_n(x)$ be the n-th Walsh function. Several authors ([6], [7], [8]) have proved laws of the iterated logarithm for lacunary Walsh functions. The best result — in some sense — is due to A. FÖLDES:

THEOREM F ([8]). *Let $W_{n_1}, W_{n_2}, \ldots$ be a subsequence of the Walsh functions for which*

$$\frac{n_{k+1}}{n_k} \geqq 1 + \frac{1}{k^\alpha} \quad (\alpha < 1/2). \tag{9}$$

Then

$$\overline{\lim_{k\to\infty}} \frac{\sum_{j=1}^{k} w_{n_j}}{\sqrt{2k \log\log k}} \leqq 1 \quad a.e.$$

[3] $B(\xi_1, \xi_2, \ldots, \xi_k)$ is the smallest σ-algebra with respect to which the random variables $\xi_1, \xi_2, \ldots, \xi_k$ are measurable.

This theorem can be considered as a consequence of Theorem 2, just we have to check that

$$\left|\int_0^1 \prod_{j=k}^{k+l} (1+b_j w_{n_j})\right| \leqq K_2^* \tag{10}$$

provided that (9) holds and $|b_j| \leqq 3\sqrt{\frac{\log l}{l}}$. The proof of (10) is not very simple, but quite elementary using only combinatorial tools (see [8]).

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References

[1] P. RÉVÉSZ, A new law of the iterated logarithm for multiplicative systems, *Acta Sci. Math. Szeged,* **34** (1973), pp. 349—358.
[2] K. AZUMA, Weighted sums of certain dependent random variables, *Tohoku Math. Journ.,* **19** (1967), pp. 357—367.
[3] M. WEISS, On the law of iterated logarithm, *Trans. Amer. Math. Soc.,* **92** (1959), pp. 531—553.
[4] P. RÉVÉSZ, *The laws of large numbers,* Akadémiai Kiadó (Budapest, 1967).
[5] W. F. STOUT, A martingale analogue of Kolmogorov's law of the iterated logarithm, *Z. Wahrscheinlichkeitstheorie,* **15** (1970), pp. 279—290.
[6] В. Ф. Гапошкин, Лакунарные ряды и независимые функции, *Успехи Матем. Наук,* **21/6** (1966), pp. 3—82.
[7] P. RÉVÉSZ—M. WSCHEBOR, On the statistical properties of the Walsh functions, *Publ. Math. Inst. Hung. Acad. Sci.,* **9** (1964), pp. 343—353.
[8] A. FÖLDES, Further statistical properties of the Walsh functions, *Studia Sci. Math. Hung.,* **7** (1972), pp. 147—153.
[9] G. ALEXITS, *Convergence problems of orthogonal series,* Pergamon Press (Oxford, 1961).
[10] G. ALEXITS—K. TANDORI, Über das Konvergenzhalten einer Klasse von Orthogonalreihen, *Annales Univ. Sci. Budapest, Sectio Math.,* **3** (1961), pp. 15—19.
[11] G. ALEXITS, Stochastische Unabhängigkeit und Orthogonalität, *Mitt. Math. Giessen,* **92** (1971), pp. 1—10.
12] G. ALEXITS, On the convergence of function series, *Acta Sci. Math. Szeged,* **34** (1973), pp. 1—9.

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Tomus 25 (3—4), (1974), pp. 435—442.

SEPARATION AND APPROXIMATION THEOREMS ON DERIVATIVES

By
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1. Introduction

In this paper, we consider the following problems:

(i) (the problem of uniform approximation) characterize the sets $H \subset [0, 1]$ having the property that, for every Baire 1 function $f(x)$ and for every $\varepsilon > 0$, there exists a derivative $g(x)$ such that

$$|f(x) - g(x)| < \varepsilon \quad (x \in H);$$

(ii) (the problem of separation) characterize the pairs H_1, H_2 ($H_1 \subset [0, 1]$, $H_2 \subset [0, 1]$) for which there exists a derivative $g(x)$ with $g|_{H_1} = 0$, $g|_{H_2} = 1$.

These two problems are closely related to each other. We are going to prove here a separation theorem (certainly not the best possible) by the aid of which we give a full solution of the approximation problem.*

As for (i) two instances were proved in [1]:

(ia) if H is a nowhere dense closed set, then the uniform approximation property holds on H ([1], Corollary 4.9);

(ib) if H is a set of measure zero, then, for every Baire 1 function $f(x)$, there exists a derivative $g(x)$ with $f(x) = g(x)$ $(x \in H)$ ([1], theorem 4.14).

As for (ii) an important special case was proved by Z. Zahorski (as a matter of fact, it concerns to approximately continuous functions, see [3], Lemma 12):

(iia) if H_1, H_2 are disjoint G_δ and D-closed sets, then there exists an approximately continuous function $f(x)$ with $f|_{H_1} = 0$, $f|_{H_2} = 1$, and $0 < f(x) < 1$ elsewhere.

2. Notations and lemmas

We apply here the same notations as in [1] and we need some results proved in [1]. For the sake of the reader's convenience we repeat here the notations and list the referred results as lemmas.

— Let $\mathscr{B}^1$ and $\mathscr{D}$ denote the family of Baire 1 functions and derivatives, respectively (all these functions are defined on [0, 1]).

— If $\mathscr{F}$ is any family of functions and $H \subset [0, 1]$, then $\mathscr{F}_H$ denotes the family of restricted $\mathscr{F}$-functions, i.e. $\mathscr{F}_H = \{f|_H : f \in \mathscr{F}\}$.

— $\lambda(H)$ denotes the Lebesgue outer measure of H.

— The set H is called D-open, if every $x \in H$ is of inner density 1 in H.

* The complete solution of the separation problem has recently been found by M. LACZKOVICH ([2]).

— A measurable set H is said to be metrically dense in an interval $[a, b]$ if, for every subinterval $I \subset [a, b]$, $\lambda(I \cap H) > 0$.

— The closure and the interior of any set H is denoted by $\operatorname{cl} H$ and $\operatorname{int} H$, respectively.

LEMMA 1 (see [1], Lemma 4.3). *For any subset* $\emptyset \neq E \subset (-\infty, +\infty)$ *we define*

$$h(x, E) = [\inf \{|x-y| : y \in E\}]^2.$$

Then $h(x, E)$ *is continuous and for* $x \in \operatorname{cl} E$ *even the derivative* $h'(x, E)$ *exists and*

$$h(x, E) = h'(x, E) = 0 \quad (x \in \operatorname{cl} E).$$

If further $F(x)$ *is differentiable then, for any* $G(x)$ *with* $F(x) \leqq G(x) \leqq F(x) + h(x, E)$, *the relations*

$$G'(x) = F'(x) \quad (x \in \operatorname{cl} E), \quad G(x) = F(x) \quad (x \in \operatorname{cl} E)$$

hold (the proof is trivial).

LEMMA 2 (see [1], Corollary 4.7). *Let* $E_1, \ldots, E_n, \ldots$ *and* $E = \bigcup E_n$ *be nowhere dense and closed sets,* $E_i \cap E_j = \emptyset$ $(i \neq j)$. *Let* $f(x)$ *be defined on* E *and suppose that* $f|_{E_n}$ *is constant for* $n = 1, 2, \ldots$. *Then* $f \in \mathscr{D}_E$.

LEMMA 3. *If* $H \subset (a, b)$; $(a, b) \setminus H$ *is metrically dense in* (a, b), $\varphi(x) < \psi(x)$ *are continuous functions in* (a, b), *then there exist differentiable functions* $F(x)$ *and* $G(x)$ *such that*

(i) $F'(x) = 0$ $(x \in H)$ *and* $\varphi(x) < F(x) < \psi(x)$ $(a < x < b)$,

(ii) $G'(x) = 1$ $(x \in H)$ *and* $\varphi(x) < G(x) < \psi(x)$ $(a < x < b)$.

PROOF. Assertion (i) is Corollary 4.12 in [1]. In order to prove (ii) we put $\varphi_1(x) = \varphi(x) - x$, $\psi_1(x) = \psi(x) - x$ and applying (i) we find $F(x)$ such that $F'(x) = 0$ $(x \in H)$ and

$$\varphi_1(x) < F(x) < \psi_1(x) \quad (a < x < b).$$

Hence $G(x) = F(x) + x$ satisfies the required conditions.

LEMMA 4. *Let* H_1, H_2 *be disjoint* G_δ *sets and denote*

$$E = [\operatorname{cl} H_1 \setminus \operatorname{int} (\operatorname{cl} H_1)] \cup [\operatorname{cl} H_2 \setminus \operatorname{int} (\operatorname{cl} H_2)]. \tag{1}$$

Then

(a) *E is a nowhere dense closed set;*

(b) *if* (a, b) *is any interval contiguous to* E *then*

(b1) $(a, b) \cap H_i \neq \emptyset$ *implies that* $(a, b) \cap H_i$ *is everywhere dense in* (a, b), $i = 1, 2$,

(b2) $(a, b) \cap H_1 \neq \emptyset$ *implies* $(a, b) \cap H_2 = \emptyset$.

PROOF. (a) is trivial. For (b1) we observe that $(a, b) \cap \operatorname{cl} H_i$ is open. In fact, $x \in (a, b) \cap \operatorname{cl} H_i$ implies $x \notin E$, hence $x \in \operatorname{int} (\operatorname{cl} H_i)$ by the definition of E. But $(a, b) \cap \operatorname{cl} H_i$ is also closed (relatively) in (a, b), thus by the connectedness of (a, b) either $\operatorname{cl} H_i \cap (a, b) = \emptyset$ or $\operatorname{cl} H_i \cap (a, b) = (a, b)$ holds which proves (b1). Finally if $H_1 \cap (a, b) \neq \emptyset$, $H_2 \cap (a, b) \neq \emptyset$, then by (b1) both sets are everywhere dense in (a, b). Since H_1, H_2 are G_δ sets, $(a, b) \cap H_1 \cap H_2$ is everywhere dense in (a, b) as well, and this contradicts the assumption $H_1 \cap H_2 = \emptyset$.

3. The separation theorem

THEOREM 1. *Suppose that H_1 and H_2 are disjoint G_δ subsets in* $[0, 1]$ *such that both* $[0, 1]\setminus H_1$ *and* $[0, 1]\setminus H_2$ *are metrically dense in* $[0, 1]$. *Then there exists a derivative $g(x)$ such that $g|_{H_1}=0$, $g|_{H_2}=1$.*

REMARK. Theorem 1 fails to hold if the derivative separating H_1 and H_2 is is required to be a bounded function (see the counterexample in [1], section 5.1).

PROOF. Let E denote the set defined by formula (1). We consider the G_δ sets

$$H_1^* = E \cap H_1, \quad H_2^* = E \cap H_2$$

and choose disjoint G_δ sets $H_1^{**} \supset H_1^*$, $H_2^{**} \supset H_1^*$ such that

$$H_1^{**} \cup H_2^{**} = E.$$

(This choice of H_i^{**} is based upon the following well known theorem (e.g. [4]): if $\Phi = \bigcup_{i=1}^{n} \Phi_i$ $(n \leqq +\infty)$ is a union of F_σ sets then there exist F_σ sets $\Psi_i \subset \Phi_i$ such that the system $\{\Psi_i\}$ is pairwise disjoint and $\cup \Psi_i = \cup \Phi_i$.) By

$$H_1^{**} = E \setminus H_2^{**}, \quad H_2^{**} = E \setminus H_1^{**}$$

these two sets are also of type F_σ, thus

$$H_1^{**} = \bigcup_{k=0}^{\infty} E_{2k+1}, \quad H_2^{**} = \bigcup_{k=1}^{\infty} E_{2k},$$

where $\{E_n\}$ is a system of pairwise disjoint nowhere dense closed sets, and $E = \bigcup_{n=1}^{\infty} E_n$.

For $x \in E$ we define $\tilde{f}(x)$ by

$$\tilde{f}(x) = \begin{cases} 0, & x \in E_{2k+1}, \quad k = 0, 1, \ldots \\ 1, & x \in E_{2k}, \quad k = 1, 2, \ldots . \end{cases}$$

Referring to Lemma 2 we have $\tilde{f} \in \mathscr{D}_E$. Let f denote a function $f \in \mathscr{D}$ with $f_E| = \tilde{f}$ and let F be a primitive of f. Let $\{(a_k, b_k)\}$ denote the system of intervals contiguous to E (we can suppose $(a_k, b_k) \subset [0, 1]$, and though $a_k = 0$ or $b_k = 1$ may occur, it does not make any trouble). We define the functions $G_k(x)$ for $x \in (a_k, b_k)$, $k = 1, 2$, as follows:

— If $(a_k, b_k) \cap (H_1 \cup H_2) = \varnothing$ then $G_k(x) = F(x)$,

— If $(a_k, b_k) \cap H_1 \neq \varnothing$ then by Lemma 4, (b2) we have $(a_k, b_k) \cap H_2 = \varnothing$ and we apply Lemma 3 (i) for the interval $(a, b) = (a_k, b_k)$ and for the functions

$$\varphi(x) := F(x), \quad \psi(x) := F(x) + h(x, E).$$

This step is justified by the assumption that $[0, 1]\setminus H_1$ is metrically dense in $[0, 1]$, i.e. $(a_k, b_k)\setminus H_1$ is metrically dense in (a_k, b_k). Let now $G_k(x)$ denote the function provided by Lemma 3, (i).

— If $(a_k, b_k) \cap H_2 \neq \varnothing$ then $(a_k, b_k) \cap H_1 = \varnothing$, and now we apply Lemma 3, (ii) analogously to the preceding case.

Finally

$$G(x) = \begin{cases} F(x), & x \in E \\ G_k(x), & x \in (a_k, b_k), \quad k = 1, \ldots \end{cases}$$

is the definition of the primitive of our separating derivative.

In fact, by $F(x) \leqq G(x) \leqq F(x) + h(x, E)$ $(x \in [0, 1])$ Lemma 1 assures the existence of $g(x) = G'(x)$ for $x \in E$, and $g(x) = G'(x) = G_k'(x)$ otherwise. If $x \in H_i$ then for $x \in E$ we have $x \in H_i^*$, thus

$$g(x) = G'(x) = F'(x) = \begin{cases} 0, & i = 1 \\ 1, & i = 2, \end{cases}$$

for $x \notin E$ we have $x \in (a_k, b_k)$ and

$$g(x) = G'(x) = G_k'(x) = \begin{cases} 0, & i = 1 \\ 1, & i = 2 \end{cases}$$

by the construction of $G_k(x)$. Hence $g(x) = G'(x)$ separates H_1 and H_2, and the theorem is proved.

4. The approximation theorem

Preparing our main result we state and prove some lemmas. Actually, these are the steps of the proof of Theorem 2. First we prove that the problem can be reduced to the case of measurable sets (Lemma 5), then Lemmas 6 and 7 yield the result for the special cases when the Baire 1 function to be approximated is bounded and locally bounded, respectively.

LEMMA 5. *Suppose that $\mathscr{D}_H$ is uniformly dense in $\mathscr{B}_H^1$ where $H \subset [0, 1]$ and let H^* denote any measurable hull of H. Then $[0, 1] \setminus H^*$ is metrically dense in* $[0, 1]$.

PROOF. The proof is indirect. Suppose that there exists an interval $I = (a, b) \subset [0, 1]$ with $\lambda(I \setminus H^*) = 0$. We define $f \in \mathscr{B}^1$ by

$$f(x) = \begin{cases} 0, & \text{if} \quad x < c = \dfrac{1}{2}(a+b) \\ 1, & \text{if} \quad x \geqq c. \end{cases}$$

By our assumption, there exists $g \in \mathscr{D}$ such that, for every $x \in H$,

$$|f(x) - g(x)| < \frac{1}{4}. \tag{2}$$

Since $f - g \in \mathscr{B}^1$, the set

$$M = \left\{ x \in [0, 1], \ |f(x) - g(x)| < \frac{1}{4} \right\}$$

is measurable and $M \supset H$, hence, apart from a set of measure zero, we have $M \supset H^*$, thus $\lambda(I \setminus M) = 0$. Thus the estimation $|f(x) - g(x)| < \dfrac{1}{4}$ holds for almost every $x \in I$. In particular, $g(x) < \dfrac{1}{4}$ for a.e. $x \in (a, c)$ and $g(x) > \dfrac{3}{4}$ for a.e. $x \in (c, b)$. According

to a known theorem of DENJOY [5], for any $g\in\mathscr{D}$ the set $\{x: \alpha<g(x)<\beta\}$ is either empty or has positive measure. Applying Denjoy's theorem for the intervals (a, c) and (c, b) we obtain

$$g(x) \leqq \frac{1}{4} \quad (x\in(a, c)), \quad g(x) \geqq \frac{3}{4} \quad (x\in(c, b)).$$

Hence $g(x)$ cannot possess Darboux property on (a, b), a contradiction with $g\in\mathscr{D}$.

LEMMA 6. *Let* $[a, b]\subset[0, 1]$, $H\subset[a, b]$ *a* G_δ *set and suppose that* $[a, b]\setminus H$ *is metrically dense in* $[a, b]$. *Let* $f\in\mathscr{B}^1$, $|f|\leqq 1$. *Then, for every* $\varepsilon>0$, *there exists a function* $G(x)$ *differentiable in* $[a, b]$, *such that* $|f(x)-G'(x)|<\varepsilon$ $(x\in H)$. *If further* $a\notin H$, $b\notin H$, *then* $G(x)$ *can be chosen to satisfy* $G'(a)=G'(b)=0$.

PROOF. Taking an integer m with $\varepsilon>\frac{1}{m}$, we consider the sets

$$H_k = \left\{x:(k-1)\frac{1}{m} < f(x) < (k+1)\frac{1}{m}\right\}, \quad -m \leqq k \leqq m.$$

By $f\in\mathscr{B}^1$, these sets are of type F_σ and, by our assumption $|f|\leqq 1$, $\bigcup_{k=-m}^{m} H_k=[0, 1]$. Recalling the theorem mentioned in the proof of Theorem 1 we can choose F_σ sets $E_k\subset H_k$ such that $[0, 1]=\bigcup_{k=-m}^{m} E_k$ and $E_k\cap E_i=\varnothing$ for $k\neq i$. Since $E_k=[0, 1]\setminus\bigcup_{j\neq k} E_j$, these sets are also of type G_δ. Taking a fixed k we apply Theorem 1 for the disjoint G_δ sets

$$H_1' = H\cap\left(\bigcup_{j\neq k} E_j\right), \quad H_2' = H\cap E_k,$$

both having metrically dense complements. If $a, b\notin H$ then we replace H_1' by $H_1'\cup\{a, b\}$. By Theorem 1 we obtain a derivative $g_k\in\mathscr{D}$ such that

$$g_k(x) = \begin{cases} 1, & \text{if} \quad x\in H\cap E_k \\ 0, & \text{if} \quad x\in H\cap\left(\bigcup_{j\neq k} E_j\right) \end{cases}$$

and moreover $g_k(a)=g_k(b)=0$ holds if $a, b\notin H$. Now put

$$g(x) = \sum_{k=-m}^{m} \frac{k}{m} g_k(x).$$

Then $g\in\mathscr{D}$ and for $x\in H$ there exists a unique k with $x\in H\cap E_k$, thus

$$|g(x)-f(x)| = \left|k\cdot\frac{1}{m}-f(x)\right| < \frac{1}{m} < \varepsilon,$$

since, for $x\in E_k\subset H_k$

$$(k-1)\frac{1}{m} < f(x) < (k+1)\frac{1}{m}$$

holds. We denote by $G(x)$ the primitive of $g(x)$ restricted to $[a, b]$ and the lemma is proved.

LEMMA 7. *Let H be a G_δ set, $H\subset(a, b)\subset[0, 1]$, and suppose that $(a, b)\setminus H$ is metrically dense in (a, b). Let $f\in\mathscr{B}^1$ and let f be locally bounded in (a, b)* (i.e. *f is bounded in every closed subinterval $[c, d]\subset(a, b)$*). *Let $\varphi(x)<\psi(x)$ be continuous functions in (a, b). Then, for every $\varepsilon>0$, there exists a function $G(x)$ differentiable in (a, b) such that*

$$|f(x)-G'(x)| < \varepsilon \quad (x\in H)$$

and

$$\varphi(x) < G(x) < \psi(x).$$

PROOF. Since $(a, b)\setminus H$ is metrically dense, a fortiori it is everywhere dense in (a, b). Thus we can take a sequence $a<a_k<b$, $-\infty<k<+\infty$ such that $a_k<a_{k+1}$, $a_k\notin H$, $k=0, \pm1, \dots$, and $\lim_{k\to-\infty} a_k=a$, $\lim_{k\to+\infty} a_k=b$. In particular,

$$\bigcup_{k=-\infty}^{+\infty} [a_k, a_{k+1}] = (a, b)$$

is a non-overlapping decomposition of (a, b).

Now applying Lemma 6 for each of the intervals $[a_k, a_{k+1}]$, we obtain the functions $G_k(x)$ such that $G_k'(x)$ exists for $x\in[a_k, a_{k+1}]$,

$$|G_k'(x)-f(x)| < \varepsilon \quad (x\in H\cap[a_k, a_{k+1}])$$

and

$$G_k'(a_k) = G_k'(a_{k+1}) = 0. \tag{3}$$

By (3) the function

$$\gamma(x) = G_k'(x) \quad \text{if} \quad x\in[a_k, a_{k+1}], \quad -\infty < k < +\infty$$

is defined unambiguously in (a, b). It is obvious that $\gamma(x)$ is a derivative in (a, b) and by the construction of $G_k(x)$ we have

$$|\gamma(x)-f(x)| < \varepsilon \quad (x\in H).$$

Let $\Gamma(x)$ denote a primitive of $\gamma(x)$ and applying Lemma 3 (i) for the continuous functions

$$\varphi_1(x) := \varphi(x)-\Gamma(x), \quad \psi_1(x) := \psi(x)-\Gamma(x)$$

we obtain the differentiable function $F(x)$ with the properties

$$F'(x) = 0 \quad (x\in H)$$

and

$$\varphi_1(x) < F(x) < \psi_1(x) \quad (x\in(a, b)).$$

Then the differentiable function $G(x)=F(x)+\Gamma(x)$ satisfies

$$\varphi(x) < G(x) < \psi(x) \quad (x\in(a, b))$$

and by $G'(x)=\Gamma'(x)=\gamma(x)$ $(x\in H)$ the estimation

$$|G'(x)-f(x)| < \varepsilon \quad (x\in H)$$

holds as well.

THEOREM 2. *Let $H\subset[0,1]$ be an arbitrary subset. The family $\mathscr{D}_H$ is everywhere dense in $\mathscr{B}_H^1$ with respect to the uniform convergence if and only if $[0,1]\setminus H^*$ is metrically dense in $[0,1]$, where H^* denotes any measurable hull of H.*

PROOF. We know by Lemma 5 that the condition imposed on $[0,1]\setminus H^*$ is necessary. Turning to the proof of the sufficiency, if H^* is any measurable hull of H, then we can take a G_δ set H^{**} such that $H^{**}\supset H^*$ and $\lambda(H^{**})=\lambda(H^*)$. In particular, also $[0,1]\setminus H^{**}$ is metrically dense in $[0,1]$. If we prove that $\mathscr{D}_{H^{**}}$ is uniformly dense in $\mathscr{B}_{H^{**}}^1$ then this obviously implies that $\mathscr{D}_H$ is uniformly dense in $\mathscr{B}_H^1$. Thus, there is no loss of generality in assuming that H itself is G_δ and $[0,1]\setminus H$ is metrically dense.

Let $f\in\mathscr{B}^1$ be arbitrary and denote

$$U=\{x: f \text{ is locally bounded at } x\}.$$

U is obviously open and since the continuity points of f form an everywhere dense set, U is an everywhere dense open set. Hence $V=[0,1]\setminus U$ is nowhere dense and closed. As we mentioned under (ia) in the introduction, the uniform approximation property holds on V, i.e. for $\varepsilon>0$ there exists $g_0\in\mathscr{D}$ such that

$$|g_0(x)-f(x)|<\varepsilon \quad (x\in V).$$

Let G_0 denote a primitive of g_0. We decompose U into the intervals contiguous to V (we can assume $0,1\in V$):

$$U=\bigcup_{k=1}^{\infty}(a_k,b_k).$$

Now taking a k fixed we put

$$H=H\cap(a_k,b_k),\quad (a,b)=(a_k,b_k),$$

$$\varphi=G_0|_{(a,b)},\quad \psi(x)=[G_0(x)+h(x,V)]|_{(a,b)}$$

and apply Lemma 7. By this we obtain a differentiable function $G_k(x)$ on (a_k,b_k) and finally we define $G(x)$ by

$$G(x)=\begin{cases}G_0(x), & \text{if } x\in V,\\ G_k(x), & \text{if } x\in(a_k,b_k),\ k=1,\ldots.\end{cases}$$

Then by Lemma 1 $G'(x)$ exists everywhere in $[0,1]$ and in particular, for $x\in V$, we have $G'(x)=G_0'(x)$. Hence $|f(x)-G'(x)|<\varepsilon$ is fulfilled for $x\in H\cap V$. If $x\in H\cap(a_k,b_k)$, then

$$|f(x)-G'(x)|=|f(x)-G_k'(x)|<\varepsilon$$

holds by Lemma 7. Putting $g(x)=G'(x)$ we conclude $g\in\mathscr{D}$ and $|g(x)-f(x)|<\varepsilon$ $(x\in H)$ and the theorem is proved.

REMARK. The following problem remains open:

Characterize the sets $H\subset[0,1]$ for which $\mathscr{D}_H$ is uniformly closed. We know that

(i) if H is an interval or $H=I\setminus H_1$, where I is an interval and $\lambda(H_1)=0$, then $\mathscr{D}_H$ is closed (trivial).

(ii) if $\lambda(H)=0$ then $\mathscr{D}_H$ is closed (by Theorem 4.19 in [1], $\mathscr{D}_H=\mathscr{B}_H^1$, and $\mathscr{B}_H^1$ is always closed).

(iii) if $\lambda(H)>0$ and $[0, 1]\setminus H$ is metrically dense then $\mathscr{D}_H$ is *not* uniformly closed (by Theorem 2 above $\mathscr{D}_H$ is uniformly dense in $\mathscr{B}_H^1$ and by Theorem 4.19 in [1] $\mathscr{D}_H\neq\mathscr{B}_H^1$).

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References

[1] G. PETRUSKA and M. LACZKOVICH, Baire 1 functions, approximately continuous functions and derivatives, *Acta Math. Acad. Sci. Hung.*, **25** (1974), pp. 189—212.
[2] M. LACZKOVICH, Separation of sets by derivatives, to appear in *Acta Math. Acad. Sci. Hung.*
[3] Z. ZAHORSKI, Sur la première dérivée, *Trans. Amer. Math. Soc.*, **69** (1950), pp. 1—54.
[4] I. P. NATANSON, *Theorie der Funktionen einer reellen Veränderlichen,* Akademie Verlag (Berlin, 1961).
[5] A. DENJOY, Sur une propriété des fonctions dérivées, *Enseignement Math.*, **18** (1916), pp. 320—328.

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ON THE OPTIMALITY OF A SIMPLE PREFIX CODE

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§. 1

n symbols are given with probabilities $p_1 \geqq p_2 \geqq p_n \geqq 0$ $\left(\sum_{i=1}^{n} p_i = 1\right)$. A *binary code* of the symbols is a set of $0-1$ sequences $f(i)=e_1(i)\dots e_{l_i}(i)$ where $e_s(i)=0$ or 1 for every $i=1, \dots, n$ and $1 \leqq s \leqq l_i$. ($f(i)$ is a codeword, l_i is the length of $f(i)$.)

The code is *prefix* if there are no two different i and j such that $l_j \geqq l_i$ and

$$e_1(i) = e_1(j), \dots, e_{l_i}(i) = e_{l_i}(j).$$

In other words, a code is prefix if no codeword is a segment of an other one. In what follows "code" means prefix code.

The average code length is defined by

$$L = \sum_{i=1}^{n} p_i l_i.$$

HUFFMAN [1] gave an algorithm to find the shortest prefix code of given probabilities, that is to find a code for which L is minimal, see [2] too. However, we may determine the minimal value of L and the optimal code without going through the algorithm only in special cases.

The simplest special case is: $p_1 = \dots = p_n = \frac{1}{n}$.

The optimal average code length for this case was given by SANDELIUS [3].

The exercise 3.15 in GALLAGER's book [2] states that in the same case the optimal code has at most two different lengths, which are consecutive integers. From this statement the average code length is easily determinable.

SOBEL [4] has a result for probabilities forming an arithmetical progression with $p_1 < 2p_n$.

These results follow from the theorem of KATONA and LEE [5], [6];

If $p_n + p_{n+1} > p_1$ then the optimal code has at most two different lengths, which are consecutive integers and

$$L(p_1 \dots p_n) = [\log n] + \sum_{i=k+1}^{n} p_i \tag{1}$$

where

$$k = 2^{[\log n]+1} - n. \tag{2}$$

[1] $\log n$ means $\log_2 n$, $[x]$ is the integer part of x.

As for every optimal code

$$l_1 \leqq l_2 \leqq \ldots \leqq l_n \tag{3}$$

holds [1], [2], the first statement is equivalent to

$$l_n - l_1 \leqq 1.$$

Now we introduce some notations. If $f = e_1 \ldots e_l$ is a $0-1$ sequence, then f^*, f^- and fe are defined by

$$f^* = e_1 \ldots e_{l-1} \tag{4}$$

$$f^- = e_1 \ldots e_{l-1}(1-e_l) \tag{5}$$

$$fe = e_1 \ldots e_l e \quad \text{where} \quad e = 0 \quad \text{or} \quad 1. \tag{6}$$

In the next definition we use that if k is given by (2) then $n-k$ is even and

$$k + \frac{n-k}{2} = 2^{[\log n]}.$$

DEFINITION. *An $f(1)\ldots f(n)$ prefix binary code is standard if the sequences $f(1), f(2), \ldots, f(k)$, $f^*(k+1) = f^*(k+2)$, $f^*(k+3) = f^*(k+4)$, $\ldots$, $f^*(n-1) = f^*(n)$ are just the $2^{[\log n]}$ $0-1$ sequences of length* $[\log n]$.

So the standard code consists of k words having length $[\log n]$ and $n-k$ words having length $[\log n]+1$. For any given n we can construct the standard code trivially. If the symbols have probabilities $p_1 \geqq p_2 \ldots \geqq p_n$, the average code length of the standard code is given by (1).

From the proof of the previous theorem of KATONA and LEE [5], [6] we obtain:

(7) *If $p_n + p_{n-1} \geqq p_1$ then the standard code is optimal.*[2]

(If $p_n + p_{n-1} > p_1$ then the values l_i of every optimal code are the same, if $p_n + p_{n-1} = p_1$ then some other optimal codes can exist.)

The aim of §. 2 is to give a necessary and sufficient condition of the optimality of the standard code.

In §. 3 we generalize the result for non-binary coding.

In §. 4 we give a simple algorithm to construct the best binary code having at most three lengths which are consecutive integers. We give a necessary condition, which has to be satisfied by the probabilities if there exists an optimal code which has only three consecutive lengths.

The general form of the KATONA—LEE theorem ([5]—[6]) is:

If $p_n + tp_{n-1} > p_1$ then for every optimal code

$$l_n - l_1 \leqq t.$$

Using this with $t=2$, by a slight modification we get a sufficient condition:

If $p_n + 2p_{n-1} \geqq p_1$ then our algorithm gives an optimal code.

[2] This follows from the Huffman-algorithm too by changing a little the section 1 in the proof of the necessity part of Theorem 1 of this paper (§. 2).

§. 2

THEOREM 1. *In the binary case the standard code is optimal if and only if*

1. $p_n+p_{n-1}\geqq p_{k+1}$,
2. *the sum of any two of the numbers* $p_k, p_{k-1}, p_n+p_{n-1}, p_{n-2}+p_{n-3}$ *is at least* p_1.

Here, according to (2), $k=2^{[\log n]+1}-n$ and per definitionem $p_0=p_{-1}=1$ and $p_{n+1}=0$.

Before the proof we introduce two transformations which make from certain prefix binary codes an other one. We use the notations (4), (5), (6).

Reduction. If $f(i)$ and $f(j)$ are two codewords which have maximal length and $f^-(i)=f(j)$ then let us omit $f(i)$ and $f(j)$ and take $f^*(i)=f^*(j)$ with probability p_i+p_j. In this way we obtain a prefix code. We know from the proof of the Huffman-algorithm that the new code is optimal if and only if the original code was also optimal.

Transfer at the words of length r. Let $f(t), f(t+1), \ldots, f(t+s)$ be the codewords of length r in the given code. Assume, that for their probabilities $p_t\geqq p_{t+1}\geqq\ldots\geqq p_{t+s}$ holds.

We assume that $f^*(t)=f^*(t+1)$ and $s\geqq 2$. (In the proof we can assume this.) Let us define the new code $g(i)$ by:

$g(i)=f(i)$ if $i<t$ or $i>t+s$

$g(t)=f^*(t)=f^*(t+1)$

$g(i)=f(i+1)$ if $t+1\leqq i\leqq t+s-2$

$g(t+s-1)=f(t+s)0$

$g(t+s)\quad=f(t+s)1.$

Trivially, the new code is prefix, too. The average code length decreases with

$$p_t+p_{t+s-1}-p_{t+s}. \tag{8}$$

The proof of necessity part of Theorem 1. 1. Assume, that the standard code $f(1)\ldots f(n)$ is optimal. Then using a "reduction" with $i=n-1, j=n$ we get an optimal code including a word of length $[\log n]$ with probability p_n+p_{n-1}. As $f(k+1)$ has length $[\log n]+1$ and its probability is $p_k, p_n+p_{n-1}\geqq p_{k+1}$ follows from (3).

2. Starting with the standard code, first we reduce $f(n)$ and $f(n-1)$, then $f(n-2)$ and $f(n-3)$. Thus we get an optimal code. Let us array the words in order of decreasing probabilities. The first word has length $[\log n]$ and its probability is at least p_1. The probabilities of the two smallest of the numbers are $p_k, p_{k-1}, p_n+p_{n-1}, p_{n-2}+p_{n-3}$. As the code is optimal, a transfer at words of length $[\log n]$ can not decrease the average code length. Thus, (8), in the present case, can not be positive, which means that the condition (2) in Theorem 1 holds.

Our proof is complete only if $2\leqq k\leqq n-4$. This inequality does not hold if $n=2^r$, $n=2^r+1$, $n=2^r-1$. In this cases the statement could be proved similarly.

The proof of sufficiency. Starting with the standard code, let us make reductions successively with $f(n)$ and $f(n-1)$, then $f(n-2)$ and $f(n-3)$ etc. and finally with $f(k+2)$ and $f(k+1)$. From the condition (1) of Theorem 1 it follows that the prob-

abilities which we reduce are always the smallest ones. So it follows from the Huffman-algorithms that if this new code is optimal, then the original standard code is optimal, too.

The new code consists of $2^{[\log n]}$ words which have length $[\log n]$ — this is also a standard code. In consequence of (7) it is enough to prove that the sum of the two smallest probabilities is at least the greatest probability.

Trivially, the two least probabilities are just the two least from the numbers p_k, p_{k-1}, p_n+p_{n-1}, $p_{n-2}+p_{n-3}$. The greatest probability can be

a) $p_{k+1}+p_{k+2}$,

b) p_1.

a) From condition (1) and the decreasing array of the probabilities we get that all of the numbers p_k, p_{k-1}, p_n+p_{n-1}, $p_{n-2}+p_{n-3}$ are at least p_{k+1}. Thus the sum of the two smallest is $\geqq 2p_{k+1} \geqq p_{k+1}+p_{k+2}$.

b) What we need in this case is just condition (2) of Theorem 1.

§. 3

If we have d letters for coding, the codewords are $f(i)=e_1(i)\dots e_{l_i}(i)$, where $e_s(i)$ is one of the d letters ($i=1, \dots, n$, $1 \leqq s \leqq l_i$).

Theorem 1 and its proof could be generalized for $d>2$.

Let us define k by

$$k = \left[\frac{d^{[\log_d n]+1}-n}{d-1}\right] \tag{9}$$

and b by

$$n = m(d-1)+b, \quad 2 \leqq b \leqq d. \tag{10}$$

(b plays role in the d-letters Huffman-algorithm too [2].)

It is easy to see that $\dfrac{n-k-b}{d}$ is integer and

$$k+\frac{n-k-b}{d}+1 = d^{[\log_d n]}.$$

DEFINITION. A prefix code $f(1)\dots f(n)$ of d letters is standard if the sequences

$$f(1), \dots, f(k), \quad f^*(k+1)=\dots=f^*(k+d), \quad f^*(k+d+1)=\dots=f^*(k+2d), \dots$$

$$\dots, f^*(n-b-d+1)=\dots=f^*(n-b), \quad f^*(n-b+1)=\dots=f^*(n)$$

are just the $d^{[\log_d n]}$ sequences of length $[\log_d n]$ forming from the d letters.

So the standard code consists of k words of length $[\log_d n]$ and $n-k$ words of length $[\log_d n]+1$. The average code length is:

$$L^d(p_1 \dots p_n) = [\log_d n] + \sum_{i=k+1}^{n} p_i.$$

THEOREM 1*. *The standard code is optimal if and only if*

1. $p_n+p_{n-1}+\dots p_{n-b+1} \geqq p_{k+1}$,

2. *the sum of any d of the numbers* p_k, p_{k-1}, ..., p_{k-d+1}, $p_n+p_{n-1}+\ldots+p_{n-b+1}$, $p_{n-b}+p_{n-b-1}+\ldots+p_{n-b-d+1}$, $p_{n-b-d}+\ldots+p_{n-b-2d+1}$, ..., $p_{n-b-(d-2)d}+\ldots+p_{n-b-(d-1)d+1}$ *is at least* p_1.

Here k and b are given by (9) and (10) and by definition $p_i=1$ if $i\leqq 0$ and $p_i=0$ if $i\geqq n+1$.

§. 4

In this paragraph we deal with the best binary code which has at most three lengths. Let these lengths be $h-1$, h, $h+1$.

Trivially, in the best code $h=[\log n]$ (type A) or $h=[\log n]+1$ (type B). If $h<[\log n]$ the code could not be prefix, if $h>[\log n]+1$ the standard code is better.

Let x, y and z be the number of the words of length $h-1$, h, $h+1$ in the same order in a best code of type A. Then

$$x+y+z=n, \tag{11}$$

$$2x+y+\frac{z}{2}=2^{[\log n]}. \tag{12}$$

(From the optimality among the codes of type A follows that z is even and (12) holds.)

If we make a transfer at the words of length $[\log n]$ x times starting with the standard code, we get a code of type A for which (11) and (12) hold. A transfer reduces the number of words of length $[\log n]$ with 3.

As $x\leqq\frac{2^{[\log n]+1}-n}{3}=\frac{k}{3}$ follows from (11) and (12), we can make x transfers.

So, the best code of type A *is obtainable from the standard code by some transfers at the words of length* $\log n$.

At the first transfer the code length decreases with $p_1-p_k-p_{k-1}$, at the second with $p_2-p_{k-2}-p_{k-3}$ etc. As $p_1\geqq p_2\geqq\ldots\geqq p_n$, the code length at the beginning decreases, then begins to grow. So, *we obtain the best code of type* A *by making the transfers written above until the code length decreases.*

If $p_k+p_{k-1}\geqq p_1$ the standard code is the best code of type A. If the code length decreases at every step, we finish the transfers when the number of the words of length $\log n$ is already less than 3.

We can find the best code of type B in the same way, making the transfers at the words of length $[\log n]+1$. The best code having at most three lengths which are consecutive integers is the better from the best code of types A and B.

If for the probabilities p_1, ..., p_n there exists an optimal code for which $l_n-l_1\leqq 2$ then our procedure gives an optimal code. This is the situation if $p_n+2p_{n-1}\geqq p_1$ (see [5]).

Now we prove, that if $p_n+p_{n-1}<p_{k+1}$ then the best code of type A could not be optimal.

Let $f(t)$ denote the first word which has length $[\log n]+1$. Then $t\leqq k+1$ so $p_t\geqq p_{k+1}>p_n+p_{n-1}$. From (8) follows that a transfer at the words of length $[\log n]+1$ decreases the average code length. This proves the statement.

Similarly, if $p_k+p_{k-1}<p_1$ then the best code of type **B** could not be optimal, because a transfer at the words of length $[\log n]$ decreases the average code length.

So we obtained the following necessary condition:

THEOREM 2. *If for the probabilities* $p_1, \ldots, p_n$ *there exists an optimal code with* $l_n-l_1\leqq 2$ *then one of the inequalities* $p_k+p_{k-1}\geqq p_1$ *and* $p_n-p_{k-1}\geqq p_{k+1}$ *is true.*

A sufficient condition is: $p_n+2p_{n-1}\geqq p_1$ ([5]). Here $k=2^{[\log n]+1}-n$ and by definition $p_0=1$, $p_{n+1}=0$.

From Theorem 2 follows that if there exists an optimal code for which $l_n-l_1\leqq 2$ and we want to obtain it by the procedure written above, we do not have to choose between the best codes of types A and **B**. In fact, if $p_k+p_{k-1}\geqq p_1$ then the best code of type A is the standard code so the optimal code is of type **B**. Similarly, if $p_n+p_{n-1}\geqq \geqq p_{k+1}$ then the optimal code is of type A.

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References

[1] D. A. HUFFMAN, A method for the construction of minimum redundancy codes, *Proc. I.R.E* 40 1098 (1952).
[2] R. D. GALLAGER, *Information theory and reliable communication,* Wiley (New York, 1968).
[3] M. SANDELIUS, On an optimal search procedure, *Amer. Math. Monthly,* **68** (1961), pp. 133—134.
[4] M. SOBEL, Binomial and hypergeometric group-testing, *Studia Sci. Math. Hung.,* **3** (1968), pp. 19—42.
[5] G. O. H. KATONA—M. A. LEE, Some remarks on the construction of optimal codes, *Acta Math. Acad. Sci. Hung.,* **24** (1973), pp. 439—442.
[6] G. O. H. KATONA, Combinatorial search problems, *A survey of combinatorial theory*. Edited by J. N. Srivastava. North-Holland Publishing Company, (1973), pp. 285—308.

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SUPPLEMENT TO MY PAPER "ON CERTAIN LINEAR OPERATORS. VIII"

By

P. O. H. VÉRTESI (Budapest)

Here I should like to publish the Remark 2.8.3 which was unfortunately omitted from the above mentioned paper (see *Acta Math. Acad. Sci. Hung.*, **25** (1974), pp. 171—187). I use the notations applied there.

2.8.3. It is an interesting and sometimes very useful supplement to Theorems 2.1—2.4 the following:

We can choose the function $f(x)\in\tilde{C}(\omega_m)$ such that besides the assertions of the above mentioned theorems we have for any $x\neq x^*$

$$|L_N(f;x)-f(x)| \leqq c_1(x)\omega_m(\delta_N) \quad (N\geqq n_0(x)).$$

Indeed, if e.g. $x<x^*$, then using our separating relation (2.11) we have for a fixed $\alpha=\alpha(x)$ that

$$f(x) = Q\omega_m(\delta_{n_\alpha})g_{n_\alpha}(x).$$

Let $n_\alpha < n_k \leqq N < n_{k+1}$. Then we have

$$L_N(f;x)-f(x) = Q\sum_{i=1}^{\infty}\omega_m(\delta_{n_i})[L_N(g_{n_i};x)-g_{n_i}(x)] = Q\left(\sum_{i=1}^{k-1}+\sum_{i=k}^{\infty}\right).$$

But if we replace m by a suitable s $(s>m)$ at the definition of $g_n(x)$, we have with a fix $r\geqq m$

$$|L_N(g_n;x)-g_n(x)| \leqq \delta_N^r$$

if $N=N(r)$ is large enough. Namely (see [6])

$$\omega(g_n^{(i)};t) \leqq B_s\delta_n^{-i-1}t \quad (i=0,1,\ldots,s-1)$$

from where by [13], 3.3. (1)

$$\omega_{i+1}(g_n;t) \leqq B_s\delta_n^{-i-1}t^{i+1} \quad (i=0,1,\ldots,s-1).$$

If $P_{N-1}(g_n;x)$ is the best approximating polynomial for $g_n(x)$ in $[-1,1]$ of order $\leqq N-1$, we have

$$|g_n(x)-P_{N-1}(g_n;x)| \leqq c_s\omega_s\left(g_n;\frac{1}{N}\right) \leqq B_s'(\delta_n N)^{-s}$$

applying the relation $i=s-1$. So

$$|L_N(g_n;x)-g_n(x)| = |L_N(g_n;x)-P_{N-1}(x)+P_{N-1}(x)-g_n(x)| \leqq 2B_s'\frac{\log N}{(\delta_n N)^s}.$$

If $p^{-\gamma-1} \leqq \delta_p < p^{-\gamma}$ $(p = 1, 2, 3, \ldots)$, then let $s = r(\gamma+1)+1$. Thus

$$2B'_s \frac{\log N}{(\delta_n N)^{r(\gamma+1)+1}} < \frac{n^{(\gamma+1)[r(\gamma+1)+1]} \log N}{N^{r(\gamma+1)+1}} < \frac{1}{N^{r(\gamma+1)}} \leqq \delta_N^r \,^1$$

if N is large enough. So we have for a suitable lacunary $\{n_i\}$

$$\left| Q \sum_{i=1}^{k-1} \omega_m(\delta_{n_i})[L_N(g_{n_i}; x) - g_{n_i}(x)] \right| \leqq Q\omega_r(\delta_N) \sum_{i=1}^{k-1} \omega_m(\delta_{n_i}) \leqq \frac{Qq}{1-q} \omega_r(\delta_N).$$

Finally

$$\left| Q \sum_{i=k}^{\infty} \omega_m(\delta_{n_i})[L_N(g_{n_i}; x) - g_{n_i}(x)] \right| = \left| Q \sum_{i=k}^{\infty} \omega_m(\delta_{n_i}) L_N(g_{n_i}; x) \right| =$$

$$= \left| Q \sum_{i=k}^{\infty} \omega_m(\delta_{n_i}) \sum_{j=1}^{N} g_{n_i}(x_{j,N}) l_{j_N}(x) \right| \leqq Q \sum_{i=k}^{\infty} \omega_m(\delta_{n_i}) \sum_{x_{j,N} \in \Delta_{n_i}} |l_{j_N}(x)| \leqq$$

$$\leqq \frac{Q}{c(x)} \sum_{i=k}^{\infty} \omega_m(\delta_{n_i}) \leqq \frac{Q}{c(x)} \frac{1}{1-q} \omega_m(\delta_N).$$

(Here we used that $x - x_{j,N} \geqq c(x)$ if $x_{j_N} \in \Delta_{n_i}$ $(i > \alpha)$ further that the number of such x_{j_N}'s is $\sim \Delta_{n_i} N(2\pi)^{-1} < N$; if $x = \pm 1$, in Theorem 2.4 we have to use (2.15.)

So

$$|L_N(f; x)| - f(x)| \leqq \left[\frac{Q}{1-q} + \frac{Q}{c(x)(1-q)} \right] \omega_m(\delta_N) = c_1(x)\omega_m(\delta_N) \quad (N \geqq n_0(x))$$

as it was stated. The case $x > x^*$ is similar.

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[1] In Theorem 2.4 $\lambda_N \asymp \log N$, so we have to use another s.

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