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TOMUS XXII

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ASYMMETRISCHE REGULÄRE GRAPHEN. II

Von

G. BARON (Wien)

In [1] hat der Autor gemeinsam mit W. IMRICH asymmetrische reguläre spezielle Graphen (d.h. Graphen ohne Schlingen und Mehrfachkanten) untersucht und für gewisse Zahlenpaare (n, k) die Existenz resp. Nichtexistenz von n-punktigen asymmetrischen regulären Graphen vom Grad k nachgewiesen. Die Resultate sind in Tabelle 1 wiedergegeben. Die dort nicht aufscheinenden Paare (n, k) mit $k \geqq 3$ und n gerade für k ungerade stellen die durch diese Arbeit nicht entschiedenen Fälle dar.

Tabelle 1

Grad k	Ne	E
3	$n \leqq 11$	$12 \leqq n$
4	$n \leqq 9$	$10 \leqq n$
5	$n \leqq 9$	$10 \leqq n \leqq 12, \quad 20 \leqq n$
6	$n \leqq 10$	$11 \leqq n \leqq 12, \quad 21 \leqq n$
$7 \leqq k \leqq 13$, unger.	$n \leqq k+4$	$n = k+5, \quad 2k+10 \leqq n$
$k \geqq 15$, unger.	$n \leqq k+4$	$k+5 \leqq n \leqq 2k-6, \quad 2k+10 \leqq n$
$k \geqq 8$, ger.	$n \leqq k+3$	$k+4 \leqq n \leqq \max(k+5, 2k-7), \quad 2k+8 \leqq n$

Für die Paare (14, 5), (16, 5), (18, 5), (14, 8), (16, 10) und (18, 12) wies IZBICKI in [2] die Existenz nach.

In [1] wurden zwei Methoden angegeben, um aus den Graphen mit $(n, 3)$ und $(n, 4)$ weitere asymmetrische Graphen zu konstruieren.

1) Komplementbildung.

2) Anhängen eines asymmetrischen „Ventils“ an einen „Autoreifen“.

Wir wollen hier eine weitere Methode angeben, mit der wir nicht nur viele der nach [1] und [2] noch nicht entschiedenen Fälle im positiven Sinne (d.h. Existenz) entscheiden, sondern auch die in [2] entschiedenen Fälle mit einem Schlag mitentscheiden. Die Methode läßt sich wie folgt beschreiben.

3) Ersetzen eines Punktes durch einen vollständigen k-punktigen Graphen C_k.

Durch diese Methode wird aus einem Graphen $X(n, k)$ ein Graph $Y(n+k-1, k)$. Da wir diese Methode für $k \geqq 10$ nur für $n < 2k$ anwenden müssen, ist in diesen Graphen höchstens ein C_k vorhanden, der keine Punkte mit einem anderen C_k gemeinsam hat. Auch in den Fällen $5 \leqq k \leqq 9$ sieht man leicht, daß die Graphen, auf die wir die Methode 3 anwenden, diese Bedingung erfüllen. Ist in X ein C_k vorhanden, so wählen wir einen seiner Punkte, sonst einen beliebigen und ersetzen ihn durch einen C_k, wobei in Y von jedem Punkt vom C_k eine Kante nach „alten" Punkten läuft wie in X selbst. Da die „neuen" Punkte in Y die einzigen in einem „isolierten" C_k sind, bilden sie einen Block. Jeder Automorphismus von Y muß also die „alten" Punkte wieder in „alte" Punkte überführen, liefert also einen Automorphismus von X. Daher ist auch Y asymmetrisch.

Durch Anwendung der Methoden 1 und 3 erhält man aus den Paaren (10, 5) und (12, 5) die in [2] entschiedenen Paare. Aus den anderen Paaren von [1] erhält man für $k \geqq 21$ alle Paare außer (n, k) mit $2k \leqq n \leqq 2k+2$. Für $k \leqq 20$ bleiben noch weitere Lücken, die aber auf die Fälle (n, k) mit $2k \leqq n \leqq 2k+2$ und $k \geqq 8$ und (13, 6), (14, 6) und (15, 6) zurückgeführt werden können. Für die letzten drei Fälle geben wir die folgenden Graphen X_n, $13 \leqq n \leqq 15$ an. Wie in [1] seien die Graphen durch $V(X_n) = \{i \mid 1 \leqq i \leqq n\}$ und $E(X_n)$ beschrieben.

$$E(X_{13}) = \{(1, 4), (1, 6), (1, 8), (1, 9), (1, 10), (1, 11), (2, 4), (2, 6), (2, 7), (2, 9), (2, 11), (2, 13), (3, 6), (3, 7), (3, 8), (3, 9), (3, 10), (3, 11), (4, 6), (4, 7), (4, 10), (4, 11), (5, 7), (5, 8), (5, 9), (5, 10), (5, 11), (5, 13), (6, 12), (6, 13), (7, 8), (7, 12), (8, 12), (8, 13), (9, 12), (9, 13), (10, 12), (10, 13), (11, 12)\}$$

$$E(X_{14}) = \{(1, 6), (1, 8), (1, 9), (1, 10), (1, 11), (1, 13), (2, 4), (2, 6), (2, 7), (2, 9), (2, 11), (2, 13), (3, 6), (3, 7), (3, 8), (3, 9), (3, 10), (3, 14), (4, 6), (4, 7), (4, 10), (4, 11), (4, 14), (5, 7), (5, 8), (5, 9), (5, 10), (5, 11), (5, 13), (6, 12), (6, 14), (7, 8), (7, 14), (8, 12), (8, 13), (9, 12), (9, 13) (10, 12), (10, 13), (11, 12), (11, 14), (12, 14)\}$$

$$E(X_{15}) = \{(1, 2), (1, 7), (1, 9), (1, 10), (1, 11), (1, 12), (2, 3), (2, 5), (2, 10), (2, 11), (2, 15), (3, 4), (3, 6), (3, 10), (3, 14), (3, 15), (4, 8), (4, 9), (4, 10), (4, 13), (4, 14), (5, 6), (5, 7), (5, 8), (5, 11), (5, 13), (6, 7), (6, 9), (6, 12), (6, 15), (7, 8), (7, 11), (7, 14), (8, 9), (8, 13), (8, 15), (9, 12), (9, 14), (10, 12), (10, 13), (11, 12), (11, 15), (12, 13), (13, 14), (14, 15)\}.$$

Die Asymmetrie dieser Graphen kann man leicht durch Betrachtung der Nachbargraphen nachweisen.

Ohne Entscheidung bleiben noch folgende Gruppen übrig

I) $(2k, k)$ für $k \geqq 8$

II) $(2k+1, k)$ für $k \geqq 8$ gerade

III) $(2k+2, k)$ für $k \geqq 8$

IV) (22, 7), (22, 14), (35, 14), (35, 20)

V) (26, 9), (26, 16)

Die Fälle der Gruppe V lassen sich auf den Fall $(18, 9) \in \mathrm{I}$ zurückführen; analog Gruppe IV auf $(16, 8) \in \mathrm{I}$ und Gruppe III auf $(2k+2, k+1) \in \mathrm{I}$. Es bleiben also im Wesentlichen nur die Gruppen I und II zur Behandlung übrig.

Für diese Gruppen wollen wir Graphen angeben, deren Asymmetrie wieder durch Betrachtung der Nachbargraphen leicht nachgewiesen werden kann. Die Indizes sollen modulo k betrachtet werden.

Gruppe I: $X_{2k,k}$

$$V(X_{2k,k}) = \{P_i | 1 \leqq i \leqq k\} \cup \{Q_i | 1 \leqq i \leqq k\}$$

$$\begin{aligned} E(X_{2k,k}) = &\big(\{(P_i P_j) | 1 \leqq i < j \leqq k\} - \{(P_i P_{i+1}) | 1 \leqq i \leqq k\}\big) \cup \\ &\cup \big(\{(Q_i Q_j) | 1 \leqq i < j \leqq k\} - \{(Q_1 Q_2), (Q_2 Q_3), (Q_1 Q_3)\} - \\ &- \{(Q_i Q_{i+1}) | 4 < i \leqq k\} - \{(Q_k Q_4)\}\big) \cup \\ &\cup \{(P_i Q_i) | 1 \leqq i \leqq k\} \cup \{(P_i Q_{i+1}) | 1 \leqq i \leqq k\} \cup \\ &\cup \{(P_i Q_{i+3}) | 1 \leqq i \leqq k\} \end{aligned}$$

$X_{2k,k}$ besteht aus zwei Schichten: die P-Schichte ist das Komplement eines k-Ecks, die Q-Schichte das Komplement des aus einem Dreieck und einem $(k-3)$-Eck bestehenden Graphen. Die zwei Schichten sind durch $3k$ Kanten verbunden, welche nur Drehungen zulassen, die aber wegen der verschiedenen Struktur der Schichten verboten sind.

Gruppe II: $X_{2k+1,k}$

$$V(X_{2k+1,k}) = \{P_i | 1 \leqq i \leqq k\} \cup \{Q_i | 1 \leqq i \leqq k\} \cup \{R\}$$

$$\begin{aligned} E(X_{2k+1,k}) = &\{(P_i P_{i+1}) | 1 \leqq i \leqq k\} \cup \{(Q_i Q_{i+1}) | 1 \leqq i \leqq k, \ i \text{ unger.}\} \cup \\ &\cup \{(RQ_i) | 1 \leqq i \leqq k\} \cup \big(\{(P_i Q_j) | 1 \leqq i, j \leqq k, i \neq j\} - \\ &- \{(P_i Q_{i+1}) | 4 \leqq i \leqq k\} - \{(P_2 Q_3), (P_3 Q_2), (P_1 Q_4)\}\big) \end{aligned}$$

$X_{2k+1,k}$ besteht aus drei Schichten: die P-Schichte ist ein k-Eck, die Q-Schichte $k/2$ Einzelstrecken und die R-Schichte nur ein einzelner Punkt. Die R-Schichte ist vollständig mit der Q-Schichte verbunden. Die an einer Stelle unregelmäßige Verbindung zwischen P- und Q-Schichte macht den Graphen asymmetrisch.

Es sind nun alle Fälle erledigt und die Resultate können in Tabelle 2 zusammengefaßt werden, wobei für ungerade Werte von k wieder nur gerade Werte von n interessant sind.

Tabelle 2

Grad k	NE	E
3	$n \leqq 11$	$12 \leqq n$
4,5	$n \leqq 9$	$10 \leqq n$
6	$n \leqq 10$	$11 \leqq n$
$k \geqq 7$, unger.	$n \leqq k+4$	$k+5 \leqq n$
$k \geqq 8$, ger.	$n \leqq k+3$	$k+4 \leqq n$

BEMERKUNG: Alle Existenzbeweise wurden konstruktiv geführt. Dabei traten vier Serien und einige Einzelfälle von Graphen auf, welche durch Anwendung dreier Methoden in weitere Graphen transformiert wurden. Man definiert das Asymmetriemaß eines Graphen als die kleinste Zahl von Kantenänderungen (Wegnahme und/oder Hinzufügen), die notwendig sind, um einen symmetrischen Graphen (d.h. einen Graphen, der einen nicht trivialen Automorphismus zuläßt) zu erhalten. Da erstens die Ausgangsgraphen das Asymmetriemaß 2 haben; zweitens die Methode 1 das Asymmetriemaß nicht ändert und drittens die Methoden 2 und 3 Graphen vom Asymmetriemaß 2 erzeugen, haben alle in den Beweisen angegebenen asymmetrischen regulären Graphen das Asymmetriemaß 2, das ja für reguläre Graphen das kleinste von Null verschiedene ist.

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ON THE IMPOSSIBILITY TO CONSTRUCT CERTAIN CLASSES OF GRAPHS BY EXTENSIONS

By

F. GLIVJAK and J. PLESNÍK (Bratislava)

In this paper *graph* means a non-empty undirected connected graph, without loops and multiple edges. We allow infinite graphs. An edge of the graph is called *superfluous* if after deleting it the diameter of the graph does not increase. A β_r-graph is a graph of diameter r without superfluous edges. A δ_r-graph is a graph of diameter r without k-gons, for $k = 3, 4, \ldots, r+1$. The distance between the vertices x, y in the graph G will be denoted by $\varrho_G(x, y)$. The basic notions and notations not defined here are meant as in [1] and [2].

One can see that every δ_r-graph is a β_r-graph (see e.g. [2]). In the paper [3] it is shown that the vertex set of a β_r-graph can be a finite, countable or uncountable infinite set.

In the sequel let $\mathscr{A}$ be a certain class of graphs. Let $\mathscr{P}$ and $\mathscr{S}$ be a finite sets of graphs not necessarily disjoint.

A graph G will be called a *Q-extension* of the graph R if both graphs Q and R are vertex disjoint section graphs of the graph G, and G contains vertices of the graphs Q and R only. We say that a class $\mathscr{A}$ of graphs can be constructed from the set $\mathscr{P}$ by extensions $\mathscr{S}$ if for every graph $G \in \mathscr{A}$ either $G \in \mathscr{P}$ or there exists a sequence $H_0, H_1, \ldots, H_n, \ldots$ of graphs such that:

1) $H_0 \in \mathscr{P}$, $H_i \in \mathscr{A}$ for $i \geqq 1$;
2) H_{i+1} is a Q-extension of the graph H_i ($i \geqq 0$) for some graph $Q \in \mathscr{S}$;
3) there exists a natural number j such that $H_j = G$.

We shall prove that the class $\mathscr{A}$ can not be constructed from the set $\mathscr{P}$ by extension $\mathscr{S}$, whereby $\mathscr{A}$ is

1) the class of all finite δ_2-graphs
2) the class of all finite β_r-graphs, $r \geqq 2$
3) the class of all graphs of diameter r, $r \geqq 2$.

DEFINITION 1. Let $G = (U, H)$ and $Q = (V, E)$ be vertex disjoint graphs. Let $G_1 = (U_1, H_1)$ be a graph such that $U_1 = U \cup V$ and the graphs G and Q are section graphs of the graph G_1. Then we say that the graph G_1 is a *connection* of the graphs G, Q or G_1 is a *Q-extension* of the graph G. We also say that the graph G is a *Q-reduction* of the graph G_1.

DEFINITION 2. Let $G \in \mathscr{A}$ and let Q be a graph. Then we say:

a) The graph G is *($\mathscr{P}$, Q)-irreducible* if $G \notin P$ and it does not contain a section graph isomorphic to the graph Q or for every Q-reduction R of the graph G is $R \notin \mathscr{A}$, $R \notin \mathscr{P}$.

b) The graph G is *($\mathscr{P}$, $\mathscr{S}$)-irreducible* if it is ($\mathscr{P}$, Q)-irreducible for every $Q \in \mathscr{S}$.

DEFINITION 3. We say that a class $\mathscr{A}$ of graphs *can be constructed from the set $\mathscr{P}$ by extensions $\mathscr{S}$* if the set of $(\mathscr{P}, \mathscr{S})$-irreducible graphs of the class $\mathscr{A}$ is empty.

THEOREM 1. Let $\mathscr{P}$, $\mathscr{S}$ be finite sets of finite graphs. Then the class of all finite δ_2-graphs can not be constructed from the set of graphs $\mathscr{P}$ by extensions $\mathscr{S}$.

PROOF. Let $k, m \geqq 4$ natural numbers. First we give a construction of certain δ_2-graphs $T(k, m)$ and then we prove $(\mathscr{P}, \mathscr{S})$-irreducibility of these graphs. Such graphs have been studied in [4] too. The vertex set of the graph $T(k, m)$ is:

$$\{v_0, v_1, \dots, v_k\} \cup \{u_1, u_2, \dots, u_{k-2}\} \cup \bigcup_{i=1}^{k} X_i, \quad \text{where} \quad X_i = \{x_{i1}, x_{i2}, \dots, x_{im}\}$$

The sketch of this graph is in Fig. 1.

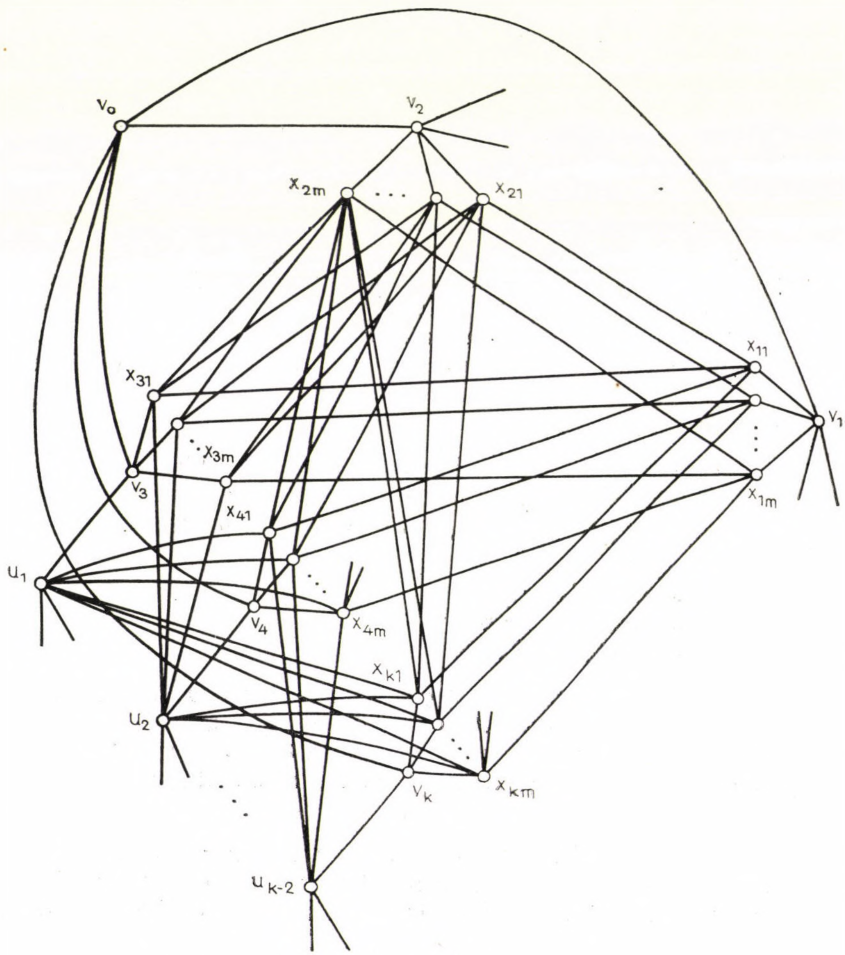

Fig. 1

Put

$$X_{ij} = X_i - \{x_{ij}\} \quad \text{for} \quad i=1, 2, \dots, k; \quad j=1, 2, \dots, m.$$

We determine the graph $T(k, m)$ by setting the neighbourhood to every vertex:

$$\Omega(v_0) = \{v_1, v_2, \dots, v_k\}$$

$$\Omega(v_j) = \{v_0, u_1, \dots, u_{k-2}\} \cup X_j \quad \text{for } j=1, 2.$$

$$\Omega(v_i) = \{v_0, u_{i-2}\} \cup X_i \quad \text{for } i=3, 4, \dots, k$$

$$\Omega(u_j) = \{v_1, v_2, v_{j+2}\} \cup \bigcup_{\substack{p=3 \\ p \neq j+2}}^{k} X_p \quad \text{for } j=1, 2 \dots, k-2$$

$$\Omega(x_{1i}) = \{v_1\} \cup \bigcup_{j=2}^{k} \{x_{ji}\} \quad \text{for } i=1, 2, \dots, m$$

$$\Omega(x_{2i}) = \{v_2, x_{1i}\} \cup \bigcup_{j=3}^{k} X_{ji} \quad \text{for } i=1, 2, \dots, m$$

$$\Omega(x_{ji}) = \{v_j, x_{1i}\} \cup X_{2i} \cup \bigcup_{\substack{p=1 \\ p \neq j-2}}^{k-2} \{u_p\} \quad \text{for } j=3, 4, \dots, k; \; i=1, 2, \dots, m$$

One can easily verify that the graph constructed above is a δ_2-graph.

Let N be a natural number greater than the number of vertices of every graph in the sets $\mathscr{P}$ and $\mathscr{S}$. In what follows let $k, m > N+2$. Now we assert that the graph $T(k, m)$ is a $(\mathscr{P}, \mathscr{S})$-irreducible δ_2-graph. In the opposite case the graph $T(k, m)$ could be considered as a connection of two graphs G and Q whereby either $Q \in \mathscr{S}$, $G \in \mathscr{P}$ or $Q \in \mathscr{S}$, G is a δ_2-graph.

The case $Q \in \mathscr{S}$, $G \in \mathscr{P}$ is not possible because the number of the vertices of the graph $T(k, m)$ is greater than $2N$. Now we prove that the second case is impossible too.

Let $G=(V, E)$, then from the preceding argument it follows that $|V| > k(m+2)-1-N$, i.e. at most $N-1$ vertices of the graph $T(k, m)$ do not belong to the graph G. So the graph G contains at least one vertex from every vertex set X_i of the graph $T(k, m)$. Let us suppose that $v_0 \notin V$. Then at most one vertex v_i, $i>2$, belongs to the graph G. In the opposite case we would have $\varrho_G(v_i, v_j)>2$, since the path (v_i, v_0, v_j) $(i, j>2, i \neq j)$, is the only path of length 2 between v_i and v_j. So at least $1+k-2 > N$ vertices of the graph $T(k, m)$ do not belong to the graph G, and this is impossible. Hence $v_0 \in V$, thus the graph G contains all vertices v_i, $i=1, 2, \dots, k$ because in the opposite case we would have $\varrho_G(v_0, x)>2$ (according to which is proved above, for every $i=1, 2, \dots, k$ at least one $x_{ij} \in V$ for a suitable j). The graph G contains all vertices u_i too, $i = 1, 2, \dots, k-2$. Since in the opposite case $\varrho_G(v_{i+2}, x_{ps})>2$ would hold for $x_{ps} \in V$ and $p \neq i+2$, $p \geqq 3$.

If there exists an $x_{1j} \notin V$ then the set $\{x_{2j}, x_{3j}, \dots, x_{kj}\}$ of N vertices does not belong to V (because in the opposite case we would have $\varrho_G(v_1, x_{ij})>2$ for $i=2, 3, \dots, k$ which is impossible). Hence $x_{1j} \in V$ for $j=1, 2, \dots, m$.

Also $x_{2j} \in V$ for $j=1, 2, \dots, m$, for otherwise $\varrho_G(v_2, x_{1j})>2$. And finally $x_{ij} \in V$ for $i=3, 4, \dots, k$; $j=1, 2, \dots, m$, because in the opposite case we would have $\varrho_G(v_i, x_{1j})=3$. So the graph G must coincide with the graph $T(k, m)$ and this is impossible. Hence the case $Q \in \mathcal{S}$, G is a δ_2-graph is impossible too. Thus the graph $T(k, m)$ is a $(\mathcal{P}, \mathcal{S})$-irreducible δ_2-graph for every $k, m > N+2$. This completes the proof.

COROLLARY 1. *Let $\mathcal{P}$, $\mathcal{S}$ be finite sets of finite graphs. Then the class of all finite* β_2-graphs *cannot be constructed from the set $\mathcal{P}$ by extensions $\mathcal{S}$.*

PROOF. *The graphs $T(k, m)$ are $(\mathcal{P}, \mathcal{S})$-irreducible β_2-graphs for $k, m > N+2$.*

THEOREM 2. *Let $\mathcal{P}$, $\mathcal{S}$ be finite sets of finite graphs and let $r \geqq 3$ be a natural number. Then the class of all finite β_r-graphs cannot be constructed from the set $\mathcal{P}$ by extensions $\mathcal{S}$.*

PROOF. Let $r \geqq 3$ be given. First we construct certain β_r-graphs $H(k, m)$ and then we prove $(\mathcal{P}, \mathcal{S})$-irreducibility of these graphs. Let $k, m \geqq 2$ be natural. The vertex set of the graph $H(k, m)$ is $U = \{w\} \cup \bigcup_{i=1}^{r-1} X_i \cup \bigcup_{i=1}^{k} Y_i$, where $X_i = \{x_{i1}, x_{i2}, \dots, x_{ik}\}$, $Y_i = \{y_{i1}, y_{i2}, \dots, y_{im}\}$. Obviously $|U| = k(r+m-1)+1$. Now we define the graph $H(k, m)$ by the neighbourhoods of the vertices:

$$\Omega(w) = X_1$$

$$\Omega(x_{1j}) = \{w, x_{2j}\}, \; j = 1, 2, \dots, k$$

$$\Omega(x_{ij}) = \{x_{i-1,j}, x_{i+1,j}\}, \; i = 2, 3, \dots, r-2; \; j = 1, 2, \dots, k$$

$$\Omega(x_{r-1,j}) = \{x_{r-2,j}\} \cup Y_j, \; j = 1, 2, \dots, k$$

$$\Omega(y_{ij}) = \{x_{r-1,i}, y_{1j}, y_{2j}, \dots, y_{i-1,j}, y_{i+1,j}, \dots, y_{kj}\}, \; i = 1, 2, \dots, k; \; j = 1, 2, \dots, m.$$

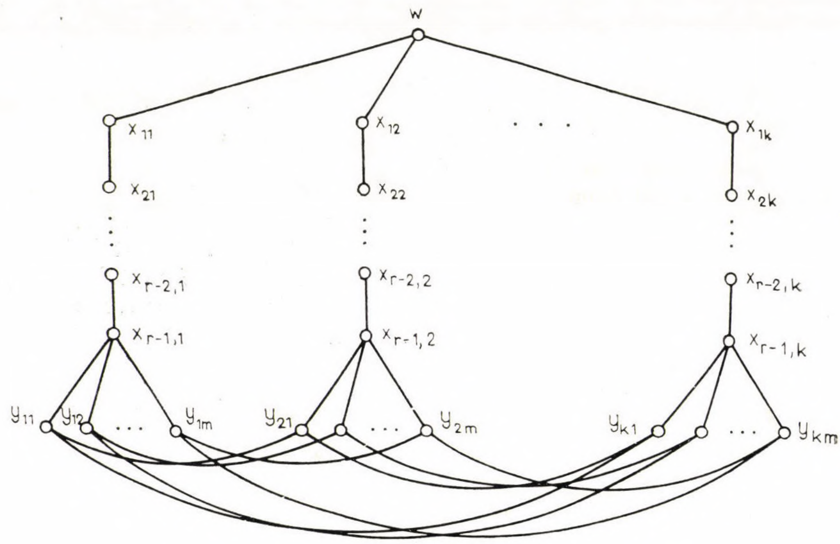

Fig. 2

The sketch of the graph $H(k, m)$ is shown in Fig. 2. One can easily verify that the graph $H(k, m)$ is a β_r-graph.

Let N be a natural number greater than the number of vertices of any graph of the sets $\mathscr{P}$, $\mathscr{S}$. Further we assume that $k, m > N$. Let the graph $H(k, m)$ be not $(\mathscr{P}, \mathscr{S})$-irreducible. Then $H(k, m)$ is a connection of two graphs G and Q, whereby either $Q \in \mathscr{S}$, $G \in \mathscr{P}$ or $Q \in \mathscr{S}$, G is a β_r-graph. The case $Q \in \mathscr{S}$, $G \in \mathscr{P}$ is impossible because the number of vertices of graph $H(k, m)$ is greater than $2N$.

In the second case the number of vertices of the graph $G = (V, E)$ is more than $k(r+m-1)+1-N$. If $w \notin V$ then at least $k-1$ vertices from the set X_1 do not belong to the set V (because the path (x_{1i}, w, x_{1j}), $i \neq j$, is the only path of length s, $s \leqq r$ in the graph $H(k, m)$). This is impossible, hence $w \notin V$.

Let $x_{ij} \notin V$ for some indices i, j. Then $\varrho_G(w, y_{jp}) > r$, for $p = 1, 2, \ldots, m$. Since $m > N$, so $x_{ij} \in V$ for all i, j. If $y_{ij} \notin V$ for some indices i, j then $\varrho_G(x_{1i}, y_{pj}) > r$ for $p = 1, 2, \ldots, k$; $p \neq i$, and hence at least $k > N$ vertices do not belong to the graph G which is impossible. Thus G coincides with the graph $H(k, m)$ which is a contradiction. Hence the graphs $H(k, m)$ are $(\mathscr{P}, \mathscr{S})$-irreducible for $k, m > N$. This completes the proof.

COROLLARY 2. *Let $\mathscr{P}$, $\mathscr{S}$ be finite sets of not necessarily finite graphs and let $r \geqq 2$. Then there exists an infinite number of $(\mathscr{P}, \mathscr{S})$-irreducible finite β_r-graphs.*

PROOF. Let $\mathscr{P}_1 \subset \mathscr{P}$, $\mathscr{S}_1 \subset \mathscr{S}$ be subsets containing all finite graphs of the sets $\mathscr{P}$, $\mathscr{S}$ respectively. Let N be a natural number greater than the number of vertices of any graph from the sets $\mathscr{P}_1$ and $\mathscr{S}_1$. If $r = 2$ then, by Theorem 1 and Corollary 1, the graphs $T(k, m)$ are $(\mathscr{P}_1, \mathscr{S}_1)$-irreducible for $k, m > N+2$. Since they are finite, therefore they are $(\mathscr{P}, \mathscr{S})$-irreducible too. The proof for $r > 2$ follows from Theorem 2 analogously.

COROLLARY 3. Let $\mathscr{P}$, $\mathscr{S}$ be finite sets of not necessarily finite graphs and let $r \geqq 2$. Then the set of all graphs of diameter r cannot be constructed from the set $\mathscr{P}$ by extensions $\mathscr{S}$.

PROOF. In the proofs of the Theorem 1 and Theorem 2 we have not used the superfluousness of edges of the graphs $T(k, m)$ and $H(k, m)$.

REMARK. The autors do not know whether the classes of the graphs studied above can be contructed from a finite set of graphs by a finite set of some operations.

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ON THE MEASURABLE SOLUTIONS OF A FUNCTIONAL EQUATION

By

Z. DARÓCZY (Debrecen)

1. Introduction. We denote by

$$\Delta_n = \left\{(p_1, p_2, \ldots, p_n)\colon p_i \geqq 0,\ i=1,2,\ldots,n;\ \sum_{i=1}^{n} p_i = 1\right\}$$

the set of all complete finite n-ary probability distributions ($n=1, 2, \ldots$). The following result was proved by T. W. CHAUNDY and J. B. MCLEOD [2]: *Let $F(x)$ be a real function defined in the interval* [0, 1]. *We suppose that $F(x)$ satisfies the functional equation*

$$\sum_{i=1}^{n}\sum_{k=1}^{m} F(p_i q_k) = \sum_{i=1}^{n} F(p_i) + \sum_{k=1}^{m} F(q_k) \tag{1}$$

for all $(p_1, p_2, \ldots, p_n) \in \Delta_n$ and $(q_1, q_2, \ldots, q_m) \in \Delta_m$ ($n, m = 1, 2, \ldots$). If $F(x)$ is continuous in [0, 1] *then we have*

$$F(x) = \begin{cases} cx \log x & \text{for} \quad x \in (0, 1] \\ 0 & \text{for} \quad x = 0, \end{cases} \tag{2}$$

where c is a constant.

J. ACZÉL and Z. DARÓCZY [1] proved that this result is true under the assumption of the functional equation (1) for $n=m$ ($n=1, 2, \ldots$), too. Recently the author [3] has proved, that the result of [2] is true under the assumption of the functional equation (1) for $n=2$ and $m=1, 2, \ldots$.

In this paper we give a generalization of the above theorems. In this generalization we suppose only the measurability of the function $F(x)$ in the open interval $(0, 1)$ under the assumption of the functional equation (1) for $n=2$, $m=3$ and $n=m=1$.

2. Algebraic results. If we take in (1) $n=m=1$, then we have $F(1)=0$. *We define $\mathfrak{M}$ as the set of all real functions $F(x)$ defined on* [0, 1], *satisfying the functional equation*

$$\sum_{i=1}^{2}\sum_{k=1}^{3} F(p_i q_k) = \sum_{i=1}^{2} F(p_i) + \sum_{k=1}^{3} F(q_k) \tag{3}$$

for all $(p_1, p_2) \in \Delta_2$ and $(q_1, q_2, q_3) \in \Delta_3$ and the condition $F(1)=0$. If $F(x) \in \mathfrak{M}$, then we have $F(0)=0$ (we take in (3) $(p_1, p_2)=(1, 0)$ and $(q_1, q_2, q_3)=(1, 0, 0)$).

LEMMA 1. *If* $F(x)\in\mathfrak{M}$, *then the function* $\varphi_x(q)$ *for all fix* $x\in[0,1]$ *defined by*

$$\varphi_x(q) = F(xq)+F((1-x)q)-F(q) \qquad (q\in[0,1]) \tag{4}$$

satisfies the functional equation

$$\varphi_x(u+v) = \varphi_x(u)+\varphi_x(v) \tag{5}$$

for all $(u,v)\in D$, *where*

$$D=\{(u,v):\ \ 0\leqq u<1,\ \ 0\leqq v<1,\ \ u+v\leqq 1\}.$$

PROOF. Let $x\in[0,1]$ and $(u,v)\in D$ be arbitrary. Setting in (3) $(p_1,p_2)=(x,1-x)$ and $(q_1,q_2,q_3)=(u,v,1-u-v)$, we find

$$\begin{aligned} &F(xu)+F((1-x)u)+F(xv)+F((1-x)v)+F(x(1-u-v))+\\ &+F((1-x)(1-u-v)) = F(x)+F(1-x)+F(u)+F(v)+F(1-u-v). \end{aligned} \tag{6}$$

We now take in (3) $(p_1,p_2)=(x,1-x)$ and $(q_1,q_2,q_3)=(u+v,1-u-v,0)$, then we have by $F(0)=0$

$$\begin{aligned} &F(x(u+v))+F((1-x)(u+v))+F(x(1-u-v))+F((1-x)(1-u-v)) =\\ &= F(x)+F(1-x)+F(u+v)+F(1-u-v). \end{aligned} \tag{7}$$

From (7) and (6) it follows

$$\begin{aligned} &F(x(u+v))+F((1-x)(u+v))-F(xu)-F((1-x)u)-\\ &-F(xv)-F((1-x)v) = F(u+v)-F(u)-F(v), \end{aligned}$$

which implies the equation (5) by (4).

LEMMA 2. *Let* $F(x)\in\mathfrak{M}$ *be arbitrary and we suppose that the function* $\varphi_x(q)$ *defined by* (4) *is*

$$\varphi_x(q)=\varphi_x(1)q \qquad (q\in[0,1]). \tag{8}$$

Then the function $f(x)$ *defined by*

$$f(x) = F(x)+F(1-x) \qquad (x\in[0,1]) \tag{9}$$

satisfies the functional equation

$$f(u)+(1-u)f\left(\frac{v}{1-u}\right)=f(v)+(1-v)f\left(\frac{u}{1-v}\right) \tag{10}$$

for all $(u,v)\in D$.

PROOF. From (4) we have by (8)

$$F(xq)+F((1-x)q)-F(q) = q[F(x)+F(1-x)] \tag{11}$$

for all $x, q \in [0, 1]$. If $(u, v) \in D$ is arbitrary, then it follows by (9) and (11)

$$f(u)+(1-u)f\left(\frac{v}{1-u}\right) = F(u)+F(1-u)+(1-u)\left[F\left(\frac{v}{1-u}\right)+F\left(\frac{1-u-v}{1-u}\right)\right] =$$

$$= F(u)+F(1-u)+F(v)+F(1-u-v)-F(1-u) = F(u)+F(v)+F(1-u-v),$$

that is, the right hand side is symmetric in u and v, which proves (10).

3. Measurable solutions. We begin with the following known result due to Z. DARÓCZY and L. LOSONCZI [4]:

LEMMA 3. *Let $\varphi(q)$ be a real function defined on* [0, 1] *and satisfy the functional equation*

$$\varphi(u+v) = \varphi(u)+\varphi(v) \tag{12}$$

for all $(u, v) \in D$. If $\varphi(q)$ is measurable in the interval (0, 1), *then we have $\varphi(q)=\varphi(1)q$ for all $q \in [0, 1]$.*

The fundamental lemma is the following result due to P. M. LEE [5]:

LEMMA 4. *Let $f(x)$ be a real function defined on* [0, 1] *and satisfying the functional equations* (10) *for all $(u, v) \in D$ and $f(u) = f(1-u)$ for all $u \in [0, 1]$. If $f(x)$ is measurable on* (0, 1), *then we have*

$$f(x) = \begin{cases} c[x \log x+(1-x)\log(1-x)] & \textit{for} \quad x \in (0, 1) \\ 0 & \textit{for} \quad x=0, 1, \end{cases} \tag{13}$$

where c is a constant.

The main result of this paper is the following

THEOREM. *If $F(x) \in \mathfrak{M}$ is measurable on the interval* (0, 1), *then $F(x)$ is the function* (2).

PROOF. If $F(x) \in \mathfrak{M}$ is measurable on (0, 1), then the function $\varphi_x(q)$ defined by (4) is measurable on (0, 1) and satisfies the functional equation (5), which implies by lemma 3 $\varphi_x(q)=\varphi_x(1)q$ for all $q \in [0, 1]$. From this assertion it follows by lemma 2 that the function $f(x)$ defined by (9) is measurable on (0, 1) and satisfies the functional equations (10) and $f(u) = f(1-u)$, which implies by lemma 4

$$f(x) = c[x \log x+(1-x)\log(1-x)] \qquad (x \in (0, 1)) \tag{14}$$

and $f(0)=f(1)=0$. Setting the function (14) in (11), we find

$$F(xq)+F((1-x)q)-F(q) = qc[x \log x+(1-x)\log(1-x)] \tag{15}$$

for all $x, q \in [0, 1]$, where we take $0 \log 0=0$ per definitionem. Let $(u, v) \in D$ be arbitrary, then there are $x, q \in [0, 1]$ with the property $u=xq$ and $v=(1-x)q$, such that we have by (15)

$$F(u)+F(v)-F(u+v) = (u+v)c\left[\frac{u}{u+v}\log\frac{u}{u+v}+\frac{v}{u+v}\log\frac{v}{u+v}\right] =$$

$$= cu \log u+cv \log v-c(u+v)\log(u+v)$$

for all $(u, v) \in D$. From this equation it follows with $\varphi(u) = F(u) - cu \log u$ $(u \in [0, 1])$ that $\varphi(u)$ satisfies the equation (12) for all $(u, v) \in D$. The function $\varphi(u)$ is measurable on (0, 1), this implies by lemma 3 $\varphi(u) = \varphi(1)u$; but by $F(1) = 0$ we have $\varphi(1) = 0$, that is $\varphi(u) = 0$ for all $u \in [0, 1]$. From this result we have $F(u) = cu \log u$, which proves the theorem.

REMARK. The result of the theorem is best possible in the sense that there are *nonmeasurable* functions $F(x) \in \mathfrak{M}$. For example the function

$$F(x) = \begin{cases} xA(\log x) & \text{for} \quad x \in (0, 1] \\ 0 & \text{for} \quad x = 0 \end{cases}$$

is in $\mathfrak{M}$, if $A(t)$ is a nonmeasurable solution of the Cauchy's functional equation $A(t_1 + t_2) = A(t_1) + A(t_2)$.

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ON DIVERGENCE AND ABSOLUTE CONVERGENCE OF SERIES ARISING FROM STRONGLY MULTIPLICATIVE ORTHOGONAL FUNCTIONS

By

F. MÓRICZ (Szeged)

1. ALEXITS [2] introduced the following definitions. The sequence $\{\varphi_n(x)\}$ $(n=1, 2, \ldots)$ of real measurable functions defined in the interval (0, 1) is called a *multiplicative system* if all their finite products are Lebesgue integrable with

$$\int_0^1 \varphi_{n_1}(x)\varphi_{n_2}(x) \ldots \varphi_{n_k}(x)\,dx = 0 \qquad (n_1<n_2<\ldots<n_k;\ k=1,2,\ldots).$$

The sequence $\{\varphi_n(x)\}$ is called an *equinormed strongly multiplicative system* (in abbreviation: ESMS) if all the finite products

$$\{\varphi_{n_1}(x)\varphi_{n_2}(x)\ldots\varphi_{n_k}(x)\} \qquad (1\leqq n_1<n_2<\cdots<n_k;\quad k=1,2,\ldots)$$

form an orthonormal system. In other words, $\{\varphi_n(x)\}$ is an ESMS if the following conditions are satisfied:

$$\int_0^1 \varphi_n(x)\,dx = 0, \qquad \int_0^1 \varphi_n^2(x)\,dx = 1 \qquad (n=1,2,\ldots);$$

$$\int_0^1 \varphi_{n_1}^{\alpha_1}(x)\varphi_{n_2}^{\alpha_2}(x)\ldots\varphi_{n_k}^{\alpha_k}(x)\,dx = \int_0^1 \varphi_{n_1}^{\alpha_1}(x)\,dx \int_0^1 \varphi_{n_2}^{\alpha_2}(x)\,dx \ldots \int_0^1 \varphi_{n_k}^{\alpha_k}(x)\,dx$$

$$(n_1<n_2<\cdots<n_k;\ k=2,3,\ldots),$$

where the exponents $\alpha_1, \alpha_2, \ldots, \alpha_k$ can be equal to 1 or 2.[1]

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[1] The notion of ESMS can be much more generally defined as follows (see in detail RÉVÉSZ [7]): Let $\{\Omega, \mathcal{S}, \mathbf{P}\}$ be a probability space, and let $\xi_1, \xi_2, \ldots$ be a sequence of random variables on Ω with $E(\xi_n)=0$, $E(\xi_n^2)=1$ $(n=1,2,\ldots)$ where $E(\xi)$ means the expectation of ξ, i.e. $E(\xi)=\int_\Omega \xi(\omega)\,d\mathbf{P}$. The sequence $\xi_1, \xi_2, \ldots$ is called an ESMS if

$$E(\xi_{n_1}^{\alpha_1}\xi_{n_2}^{\alpha_2}\ldots\xi_{n_k}^{\alpha_k}) = E(\xi_{n_1}^{\alpha_1})E(\xi_{n_2}^{\alpha_2})\ldots E(\xi_{n_k}^{\alpha_k}) \qquad (n_1<n_2<\cdots<n_k;\ k=2,3,\ldots),$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ can be equal to 1 or 2.

In this paper, we consider only the particular case: $\Omega=(0,1)$, $\mathcal{S}$ is the class of Lebesgue measurable subsets of (0, 1), and $\mathbf{P}$ is the common Lebesgue measure on it. Our results hold in the general case as well, but, of course, by the term "almost everywhere" we must mean "everywhere on Ω with the exception at most of a set of measure zero with respect to the probability measure $\mathbf{P}$ in question".

2. Evidently, a sequence of independent functions with mean 0 and square integral 1 forms an ESMS. Another example is a strongly lacunary sequence of trigonometric functions, i.e. a system $\{\sqrt{2}\sin 2\pi m_k x\}$, where $m_{k+1}/m_k \geqq 3$ $(k=1, 2, \dots)$.[2]

The behaviour of the series arising from the functions of an ESMS resembles, in many respects, that of series arising from independent functions. As to convergence properties, ALEXITS proved the following two theorems.

THEOREM A. (ALEXITS [2]; see also ALEXITS and TANDORI [3].) *If* $\{\varphi_n(x)\}$ *is a uniformly bounded* ESMS *then the relation*

$$\sum_{n=1}^{\infty} c_n^2 < \infty \tag{1}$$

implies the convergence of the orthogonal series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \tag{2}$$

almost everywhere.

In the sequel we shall use "a.e." instead of almost everywhere.

Before stating Theorem B in an explicit form we must sketch the definition of a permanent Toeplitz summation process. Let $T=(a_{ik})$ $(i, k=1, 2, \dots)$ be a doubly infinite matrix of numbers. We say that the series (2) with partial sums $s_k(x)$ is *T-summable* at a point $x\in(0, 1)$ if the *T-mean*

$$t_i(x) = \sum_{k=1}^{\infty} a_{ik} s_k(x)$$

tends to a limit for $i\to\infty$ (provided the series on the right converges for all i). A summation process determined by the matrix T (in abbreviation "*summation process*") is said to be *permanent* if $s_k\to s$ imlies $t_i\to s$. The necessary and sufficient conditions for the permanence of a summation process are known. (See, for example, ZYGMUND [8], p. 74.) In what follows we shall consider only permanent summation processes.

THEOREM B. (ALEXITS [1], p. 194; see also ZYGMUND [9].) *Let* $\{\varphi_n(x)\}$ *be a multiplicative system having the following properties:*

$$\begin{aligned} &\int_0^1 \varphi_n^2(x)\,dx = 1 \qquad (n=1, 2, \dots), \\ &\int_0^1 \varphi_m^2(x)\varphi_n^2(x)\,dx = 1 \qquad (m, n=1, 2, \dots;\ m\neq n), \end{aligned} \tag{3}$$

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[2] We note that the trigonometric series $\sum_{k=1}^{\infty}(a_k\cos m_k x + b_k \sin m_k x)$ is called lacunary if $m_{k+1}/m_k\geqq q>1$ $(k=1, 2, \dots)$.

and furthermore, for every set $\mathcal{E} \subset (0, 1)$ *of positive measure and for every sufficiently large n the relation*[3]

$$\int_{\mathcal{E}} \varphi_n^2(x)\,dx \geqq K|\mathcal{E}|$$

holds, where K denotes a positive constant depending only on $\mathcal{E}$. *If the series* (2) *is summable in a set of positive measure by a permanent summation process then its coefficients satisfy condition* (1).[4]

3. We shall show that in case $\{\varphi_n(x)\}$ is uniformly bounded this condition is a consequence of (3). More exacty, our result reads as follows:

THEOREM 1. *If* $\{\varphi_n(x)\}$ *is a uniformly bounded system satisfying* (3) *then for any set* $\mathcal{E} \subset (0, 1)$ *of positive measure we have*

$$\lim_{n\to\infty} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} \varphi_n^2(x)\,dx = 1. \tag{4}$$

Combining this result with Theorem B we obtain

THEOREM 2. *Let* $\{\varphi_n(x)\}$ *be a uniformly bounded multiplicative system satisfying* (3). *If the series* (2) *is summable on a set of positive measure by a permanent summation process then condition* (1) *is satisfied.*

Thus it may be stated that the series arising from the functions of a uniformly bounded multiplicative system satisfying (3) whose coefficients do not satisfy condition (1) are practically nonsummable. Comparing this result with Theorem A we may assert: *a series arising from the functions of a uniformly bounded* ESMS *is convergent or divergent a.e. according as* Σc_n^2 *is convergent or divergent.* This assertion remains valid even if the convergence is replaced by a permanent summation process. In probability theory this fact is called *the law of zero or unity.*

4. Now we are going to prove Theorem 1. We need an earlier result which is due to the author [5].

THEOREM C. *If* $\{\varphi_n(x)\}$ *is a uniformly bounded system satisfying* (3) *then the arithmetic mean* $\frac{1}{N}\sum_{n=1}^{N} \varphi_n^2(x)$ *converges in measure to* 1.[5]

PROOF OT THEOREM 1. If (4) were not satisfied for some set $\mathcal{E}_0$ of positive measure then there would exist a sequence $\{n_k\}$ of indices and a number $\alpha(\neq 1)$ such that

$$\lim_{k\to\infty} \frac{1}{|\mathcal{E}_0|} \int_{\mathcal{E}_0} \varphi_{n_k}^2(x)\,dx = \alpha. \tag{5}$$

[3] $|\mathcal{E}|$ denotes the Lebesgue measure of the set $\mathcal{E}$.

[4] Here we give the original theorems of ALEXITS with a slight modification.

[5] I.e. for every positive number ε the measure of the set at the points of which $\left|\frac{1}{N}\sum_{n=1}^{N} \varphi_n^2(x) - 1\right| \geqq \varepsilon$ holds tends to 0 as $N\to\infty$.

This is impossible. In fact, let us consider the subsystem $\{\varphi_{n_k}(x)\}$ which is also a uniformly bounded system satisfying (3). On account of Theorem C we find that the arithmetic mean $\frac{1}{N}\sum_{k=1}^{N}\varphi_{n_k}^2(x)$ also converges in measure to 1.

For any number $\varepsilon>0$, given in advance, consider the sets

$$\mathscr{E}_{1N}=\left\{x\in\mathscr{E}_0:\left|\frac{1}{N}\sum_{k=1}^{N}\varphi_{n_k}^2(x)-1\right|<\varepsilon\right\}\quad\text{and}\quad \mathscr{E}_{2N}=\mathscr{E}_0-\mathscr{E}_{1N},$$

and decompose the integral as follows:

$$\int_{\mathscr{E}_0}\sum_{k=1}^{N}\varphi_{n_k}^2(x)\,dx=\left\{\int_{\mathscr{E}_{1N}}+\int_{\mathscr{E}_{2N}}\right\}\sum_{k=1}^{N}\varphi_{n_k}^2(x)\,dx\qquad(N=1,2,\ldots).$$

Denoting by K a common bound of the system $\{\varphi_n(x)\}$ we obtain that

$$\left|\frac{1}{N}\int_{\mathscr{E}_0}\sum_{k=1}^{N}\varphi_{n_k}^2(x)\,dx-|\mathscr{E}_0|\right|\leqq\varepsilon|\mathscr{E}_{1N}|+(K^2+1)|\mathscr{E}_{2N}|.$$

Taking into consideration that here ε is an arbitrarily small positive number and that, by virtue of the convergence in measure of the arithmetic mean mentioned above, $|\mathscr{E}_{2N}|$ tends to 0 ($N\to\infty$), we conclude

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\frac{1}{|\mathscr{E}_0|}\int_{\mathscr{E}_0}\varphi_{n_k}^2(x)\,dx=1.$$

This contradicts (5) and proves Theorem 1.

5. We shall briefly deal with an important consequence of Theorem 1 which is very useful to study convergence properties of series arising from functions of ESMS.

THEOREM 3. *Let* $\{\varphi_n(x)\}$ *be a uniformly bounded* ESMS. *Suppose we are given a set* $\mathscr{E}\subset(0,1)$ *of positive measure and a number* $\lambda>1$. *Then there exists an integer* $\mu_0=\mu_0(\mathscr{E},\lambda)$ *such that for any finite sum* $S(x)=\sum_{n=\mu}^{\nu}c_n\varphi_n(x)$ *with* $\nu>\mu\geqq\mu_0$ *we have*

$$\lambda^{-1}|\mathscr{E}|\sum_{n=\mu}^{\nu}c_n^2\leqq\int_{\mathscr{E}}S^2(x)\,dx\leqq\lambda|\mathscr{E}|\sum_{n=\mu}^{\nu}c_n^2. \tag{6}$$

The inequalities hold also if $S(x)$ *is an infinite series with* $\sum_{n=\mu}^{\infty}c_n^2<\infty$ *and* $\mu\geqq\mu_0$.

We note that this theorem generalizes the known inequalities concerning lacunary trigonometric series and Rademacher's system. (See ZYGMUND [9], and see also ZYGMUND [8], pp. 203—204.)

PROOF OF THEOREM 3. A simple calculation shows that

(7) $$\int_{\mathscr{E}} S^2(x)\,dx = \sum_{n=\mu}^{\nu} c_n^2 \int_{\mathscr{E}} \varphi_n^2(x)\,dx + \sum_{\substack{m\neq n\\ \mu\leqq m,\, n\leqq \nu}} c_m c_n \int_{\mathscr{E}} \varphi_m(x)\varphi_n(x)\,dx.$$

By Schwarz's inequality the last sum does not exceed

(8) $$\Big\{\sum_{\mu\leqq m,\, n\leqq \nu} c_m^2 c_n^2\Big\}^{1/2} \Big\{ \sum_{\substack{m\neq n\\ \mu\leqq m,\, n\leqq \nu}} \Big[\int_{\mathscr{E}} \varphi_m(x)\varphi_n(x)\,dx\Big]^2\Big\}^{1/2} = \\ = \Big\{\sum_{n=\mu}^{\nu} c_n^2\Big\}\Big\{ \sum_{\substack{m\neq n\\ \mu\leqq m,\, n\leqq \nu}} \Big[\int_{\mathscr{E}} \varphi_m(x)\varphi_n(x)\,dx\Big]^2\Big\}^{1/2}.$$

Since the integrals $\int_{\mathscr{E}} \varphi_m(x)\varphi_n(x)\,dx$ are the coefficients in the expansion of the characteristic function of $\mathscr{E}$ in the functions of the orthonormal system $\{\varphi_m(x)\varphi_n(x)\}$ $(m\neq n)$, by applycation of Bessel's inequality we can make the right-hand side of (8) less than

$$\frac{\lambda-1}{\lambda(\lambda+1)}\,|\mathscr{E}| \sum_{n=\mu}^{\nu} c_n^2$$

if μ and ν are large enough, say $\nu>\mu\geqq\mu_1(\mathscr{E},\lambda)$. By virtue of Theorem 1 we find the estimate

(9) $$\frac{2}{1+\lambda} \leqq \frac{1}{|\mathscr{E}|}\int_{\mathscr{E}} \varphi_n^2(x)\,dx \leqq \frac{1+\lambda}{2}$$

for every sufficiently large n, say $n\geqq\mu_2(\mathscr{E},\lambda)$. Collecting results (7), (8) and (9), we obtain the desired assertion (6) with $\mu_0=\max\{\mu_1,\mu_2\}$.

If $S(x)$ is an infinite series with $\Sigma c_n^2<\infty$ we first apply (6) to the partial sums $s_\nu(x)$ of $S(x)$. Then making $\nu\to\infty$ and observing that

$$\int_{\mathscr{E}} s_\nu^2(x)\,dx \to \int_{\mathscr{E}} S^2(x)\,dx$$

we get the required result.

Thus we have finished the proof of Theorem 3.

6. We note that the proof of Theorem B (and also that of Theorem 2) remains valid if we merely assume that the T-means of the series (2) are bounded at every point of a set $\mathscr{E}$ with $|\mathscr{E}|>0$, where T is a permanent summation process. For some problems it is desirable to have a similar result for one-sided boundedness.

Our theorem, which is a simple generalization of that of ZYGMUND [10], reads as follows:

THEOREM 4. *Let* $\{\varphi_n(x)\}$ *be a uniformly bounded* ESMS. *If* Σc_n^2 *diverges then the set of points* x *at which*[6]

$$\lim_{i\to\infty} \frac{t_i^+(x)}{\left\{\sum_{k=1}^{\infty} c_k^2 A_{ik}^2\right\}^{1/2}} = 0$$

is of measure zero, where $A_{ik} = \sum_{l=k}^{\infty} a_{il}$ $(k=1, 2, \ldots)$.

The sum in curly brackets tends to $+\infty$ with i, since $A_{ik}\to 1$ for fixed k. (This is an immediate consequence of the conditions of permanence.) Hence Theorem 4 implies *if the T-means of* $\Sigma c_n\varphi_n(x)$ *are bounded above (or below) at every point of a set of positive measure, then the series* Σc_n^2 *converges.*

The proof of Theorem 4 follows closely that of a similar theorem concerning lacunary trigonometric series in the book of ZYGMUND [8], pp. 205—206. We only remark that the proof is based on Theorem 3 and on a result which originates also from the author [4], namely

THEOREM D. *If* $\{\varphi_n(x)\}$ *is a uniformly bounded* ESMS, *then for every positive number* p *we have the inequalities*

$$C_p\{\sum c_n^2\}^{1/2} \leqq \left\{\int_0^1 |S(x)|^p dx\right\}^{p/1} \leqq D_p\{\sum c_n^2\}^{1/2},$$

where $S(x)=\Sigma c_n\varphi_n(x)$ *means a finite sum or infinite series with* $\Sigma c_n^2<\infty$ *and* C_p, D_p *are positive constants depending only on* p.

7. The following definitions are due to ORLICZ [6]. (See also ALEXITS [1], pp. 326—329.) We say that an arbitrary orthonormal system $\{\varphi_n(x)\}$ possesses property C if the convergence $\sum_{n=1}^{\infty} c_n\varphi_n(x)$ in a set of positive measure implies the relation $\lim_{n\to\infty} c_n=0$; and we say that the system $\{\varphi_n(x)\}$ possesses property D if the absolute convergence of $\sum_{n=1}^{\infty} c_n\varphi_n(x)$ in a set of positive measure implies the relation

$$\sum_{n=1}^{\infty} |c_n| < \infty. \tag{10}$$

The following theorem was also proved by ORLICZ [6].

THEOREM E. *In order that an arbitrary system* $\{\varphi_n(x)\}$ *possess the property* C *or* D *it is necessary and sufficient that the condition*

$$\liminf_{n\to\infty} \int_{\mathcal{E}} |\varphi_n(x)|\,dx > 0 \tag{11}$$

be satisfied for all sets $\mathcal{E}$ *of positive measure.*

[6] Here $t_i^+(x) = \max\{0, t_i(x)\}$. We remark that the sum in curly brackets is finite if we suppose that the series defining $t_i(x)$ converges in a set of positive measure. This follows from Theorem 2.

By virtue of Theorem 1 we obtain that in the case of a uniformly bounded system satisfying (3) condition (11) is satisfied for every set $\mathcal{E}$ with $|\mathcal{E}|>0$. In fact, (4) implies that

$$\liminf_{n\to\infty} \int_{\mathcal{E}} |\varphi_n(x)|\,dx \geqq \frac{1}{K} \lim_{n\to\infty} \int_{\mathcal{E}} \varphi_n^2(x)\,dx = \frac{|\mathcal{E}|}{K} > 0,$$

where K denotes a common bound of the system $\{\varphi_n(x)\}$.

Also we have

THEOREM 5. *A uniformly bounded system satisfying* (3) *always possesses the properties C and D.*

It is clear from Theorem A that for uniformly bounded ESMS condition (1) implies also convergence a.e. of every rearrangement of the series (2).[7] We note the interesting phenomenon that in the case of a uniformly bounded ESMS the convergence a.e. in every rearrangement of the terms does not coincide with the absolute convergence a.e. For example, the sequence $c_n = 1/n$ $(n=1, 2, \ldots)$ fulfils condition (1) but not (10). Thus the corresponding series (2) with an arbitrary uniformly bounded ESMS is convergent a.e. in every rearrangement of its terms, but nevertheless, it fails to be absolutely convergent at the points of a set of positive measure.

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[7] Here we understand, of course, that the term "a.e." means that the set of measure zero of the points of divergence may vary with every rearrangement.

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DISTRIBUTION OF CROSSINGS IN RESTRICTED PATHS

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1. Introduction

For the one-dimensional unrestricted symmetric random walk the distribution of crossings through a specified point have been studied by CSÁKI [1], CSÁKI and VINCZE [2], FELLER [9], ORA ENGELBERG [6], JAIN [11], DWASS [4] and the author [12]. In these studies there are no barrier restrictions on the particle, e.g. its confinement between two specified positions, its touching or crossing one or both of two specified positions. We now investigate the distributions of the above-mentioned type, in case the particle reaches at the end of the walk some specified position (including the starting position).

Consider a sequence of identical and independent auxiliary random variables $\xi_1, \xi_2, \dots, \xi_n$, such that

$$\xi_i = \begin{cases} +1, & \text{if the } i\text{th step is to the right,} \\ -1, & \text{if the } i\text{th step is to the left} \end{cases}$$

and let

$$S_0=0, \; S_i = \xi_1+\xi_2+\dots+\xi_i,$$

then S_i denotes the position of the particle at the end of the ith step and the sequence of points $\{i, S_i\}$, $i=0, 1, \dots, n$, joined in succession represents the space-time graph of the symmetric random walk.

Let $R_1(>0)$ and $-R_2(R_2>0)$ be the extreme positions reached by the particle during the n steps. Given preassigned positive values a, b, KEMPERMAN [13] has obtained, for a particle starting from $i(>0)$ and arriving at $j(>0)$ on the nth step, the probability for each of the following contingencies.

(i) $R_1<a$, and the path does not touch the x-axis,

(ii) the particle returns r times to the x-axis and $R_1<a$,

(iii) additional to (ii), the path does not touch the line $y=-1$.

CSÁKI and VINCZE [3] determined the probability that a particle starting from the origin and returning to it after $2n$ steps crosses the x-axis l times and $R_i<a$, $i=1, 2$. JAIN [10] calculated the probabilities for a particle, starting from the origin, to arrive at m on the nth step, such that

(i) $R_1<a, \quad R_2<b$

(ii) $R_1\geqq a, \quad R_2\geqq b$

(iii) $R_1\geqq a, \quad R_2<b$

(iv) the particle crosses the x-axis l times from the $+$ve side and $R_1<a$.

We obtain here the generating functions of the numbers of the following types of paths collected into two sets:

SET I: Starting from the origin and returning to it at the nth step after crossing the origin l times such that

(i) $R_1 < a$, $R_2 < b$, i.e. in the presence of absorbing barriers at a and $-b$.
(ii) $R_1 \geqq a$, $R_2 < b$, i.e. in the presence of an absorbing barrier at $-b$ and reaching or crossing the height a,
(iii) $R_1 < a$, $R_2 \geqq b$, i.e. in the presence of an absorbing barrier at a and reaching or crossing the height $-b$.
(iv) $R_1 \geqq a$, $R_2 \geqq b$;

SET II: Starting from the origin and arriving at $m(>0)$ on the nth step after crossing the origin l times under the various conditions (i) to (iv) of Set I.

2. Symbols

For convenience of writing we introduce the following symbols:

C = intersection or crossing with the x-axis: a point (i, S_i) of the path for which either (i) $S_{i-1} = -1$, $S_i = 0$, $S_{i+1} = +1$; or (ii) $S_{i-1} = +1$, $S_i = 0$, $S_{i+1} = -1$.

S = Section: the segment of a path included between two consecutive C-points. In general, S is used to denote a path of unspecified length with ends on the x-axis and with $S_i \geqq 0$ or $S_i \leqq 0$ in-between.

$S^+(S^-)$: a section (S) with $S_i \geqq 0$ or $S_i \leqq 0$ in-between. Thus $S^+(a^-)$ is an S^+ with $R_1 < a$.

$A_{n,m}$: a path $\{S_0, S_1, \ldots, S_n\}$ with $S_n = m$, $0 \leqq |m| < n$, $A_{n,0} \equiv A_n$ (n even). Thus an S^+ and S^- with specified lengths, say $2i$ steps, are equivalent respectively to an A_{2i}^+ and A_{2i}^-.

$A_{n,m}^l$: an $A_{n,m}$ with exactly l C-points; $A_{n,0}^l \equiv A_n^l$ (n even).

The $+$ sign on the top right of a or b indicates $\geqq$ and the $-$ sign indicates $<$. Thus $A_{n,m}^l(a^+, -b^-)$ is an $A_{n,m}^l$ with $R_1 \geqq a$ and $R_2 < b$, $A_{n,m}^l(-b^-)$ is an $A_{n,m}^l$ with $R_2 < b$.

The $+$ sign (or $-$ sign) on the top right of A indicates that the path never goes below (or above) the x-axis. Thus $A_{n,m}^+(a^-)$ is a path from the origin to (n, m) with $S_i \geqq 0$, $i = 1, 2, \ldots, n$ and $R_1 < a$.

To simplify the notations we write $A_{n,0}^l(\cdots) \equiv A_n^l(\cdots)$.

$C_{n,m}$: an $A_{n,m}$ reaching the height m for the first time at the nth step.

$C_{n,m}^l$: a $C_{n,m}$ with lC.

$C_{n,m}^l(-b^-)$: a $C_{n,m}^l$ with $R_2 < b$,

$A(n, i, j, a)$: a path leading from i $(0 < i < a)$ to j $(0 < j < a)$ at the nth step and not reaching any of the lines $y = 0$ and $y = a$.

$(\cdots)$: number of all possible paths of the type $\ldots$.

$\alpha_\zeta \equiv \sum_{n=1}^{\infty} (\alpha_n)\zeta^n$: the generating function of the sequence $\{(\alpha_n)\}$. We denote the generating function of the sequence $\{(\alpha_{2n})\}$, namely $\sum_{n=1}^{\infty} (\alpha_{2n})\zeta^n$ by $\alpha_{\sqrt{\zeta}}$ and the g.f. of the sequence $\{(\alpha_{2n+1})\}$, namely $\sum_{n=1}^{\infty} (\alpha_{2n+1})\zeta^n$, by $\zeta^{-\frac{1}{2}} \alpha_{\sqrt{\zeta}}$.

3. Some important generating functions

It is convenient to use the following symbols:

(1a) $$\omega = \frac{1-\sqrt{1-4\zeta}}{1+\sqrt{1-4\zeta}}, \quad |\zeta| < \frac{1}{4}, \quad |\omega| < 1, \quad \sqrt{1-4\zeta} = \frac{1-\omega}{1+\omega}, \quad \zeta = \frac{\omega}{(1+\omega)^2};$$

$$\omega_1 = \left(\frac{1-\omega^{a-1}}{1-\omega^{a+1}}\right) \quad \text{and} \quad \omega_2 = \left(\frac{1-\omega^{b-1}}{1-\omega^{b+1}}\right).$$

LEMMA 1 (CSÁKI and VINCZE [3]). *For* $a = 1, 2, \ldots$

(1) $$A^{+}_{\sqrt{\zeta}}(a^-) = \omega\omega_1 \quad \text{and} \quad A^{-}_{\sqrt{\zeta}}(-b^-) = \omega\omega_2.$$

PROOF. From a result due to ELLIS [5]

(2) $$(A(n,i,j,a)) = \sum_{r=-\infty}^{\infty}\left[\binom{n}{\frac{1}{2}(n-i+j)+ra} - \binom{n}{\frac{1}{2}(n+i+j)+ra}\right].$$

Now consider paths of the type $A^{+}_{2n}(a^-)$ starting from and arriving after $2n$ steps at the origin without reaching any of the lines $y=-1$ and $y=a$. Such a path can, by shifting the origin to $(0, -1)$, be seen to be equivalent to an $A(2n, 1, 1, a+1)$; hence

$$(A^{+}_{2n}(a^-)) \equiv (A(2n, 1, 1, a+1)).$$

Its generating function is, on using (2).

(3) $$A^{+}_{\sqrt{\zeta}}(a^-) \equiv \sum_{n=1}^{\infty}(A^{+}_{2n}(a^-))\zeta^n = \sum_{n=1}^{\infty}\sum_{r=-\infty}^{\infty}\left[\binom{2n}{n+r(a+1)} - \binom{2n}{n+1+r(a+1)}\right]\zeta^n.$$

Omitting the evanescent terms, the first double sum

$$= \left(\sum_{r=0}^{\infty} + \sum_{-r=1}^{\infty}\right)\sum_{n=|r|(a+1)}^{\infty} - 1;$$

here the term -1 cancels the extraneous term corresponding to $n=0=r$ which (introduced in the first term) has been taken to be 1, like the term corresponding to $n=0=p$ in (4) below. Similarly the second double sum

$$= \sum_{r=0}^{\infty}\sum_{n=1+|r|(a+1)}^{\infty} + \sum_{-r=1}^{\infty}\sum_{n=-1+|r|(a+1)}^{\infty}.$$

Hence, putting $-r=s$ when $-r$ ranges from 1 to ∞

$$A^{+}_{\sqrt{\zeta}}(a^-) = \sum_{r=0}^{\infty}\sum_{n=r(a+1)}^{\infty}\binom{2n}{n+r(a+1)}\zeta^n - \sum_{r=0}^{\infty}\sum_{n=1+r(a+1)}^{\infty}\binom{2n}{n+1+r(a+1)}\zeta^n +$$

$$+\sum_{s=1}^{\infty}\sum_{n=s(a+1)}^{\infty}\binom{2n}{n+s(a+1)}\zeta^n - \sum_{s=1}^{\infty}\sum_{n=-1+s(a+1)}^{\infty}\binom{2n}{n+s(a+1)-1}\zeta^n - 1.$$

We know [7] that

$$(4) \qquad \sum_{n=p}^{\infty} \binom{2n}{n+p} \zeta^n = \frac{\zeta^{-p}}{\sqrt{1-4\zeta}} \left(\frac{2\zeta}{1+\sqrt{1-4\zeta}} \right)^{2p} = \frac{1+\omega}{1+\omega} \cdot \omega^p.$$

Hence

$$A^+_{\sqrt{\zeta}}(a^-) = \frac{1+\omega}{1-\omega^{a+1}} - \frac{1+\omega}{\omega} \cdot \frac{\omega^{a+1}}{1-\omega^{a+1}} - 1,$$

which leads to the first result in (1). The second result follows on observing that there is a one-to-one correspondence between paths of the types $A^+_{2n}(-b^-)$ and $A^-_{2n}(b^-)$.

LEMMA 2. *For* $a=1, 2, \ldots$

$$(5) \qquad A^+_{\sqrt{\zeta}}(a^+) = \omega(1-\omega_1) \quad \text{and} \quad A^-_{\sqrt{\zeta}}(-b^+) = \omega(1-\omega_2).$$

PROOF:

$$(6) \qquad \big(A^+_{2n}(a^+)\big) = (A^+_{2n}) - \big(A^+_{2n}(a^-)\big).$$

Since $(A^+_{2n}) = (C_{2n+1,1}) = \frac{1}{2n+1}\binom{2n+1}{n}$, a known formula [7] shows that

$$(7) \qquad A^+_{\sqrt{\zeta}} \equiv \sum_{n=1}^{\infty} \frac{1}{2n+1} \binom{2n+1}{n} \zeta^n = \sum_{n=1}^{\infty} \frac{1}{n+1} \binom{2n}{n} \zeta^n = \omega = A^-_{\sqrt{\zeta}}$$

and taking g.f. of (6), Lemma 1 immediately yields the first result in (5). The second one follows immediately on replacing ω_1 by ω_2 in the first result of (5), since

$$\big(A^-_{2n}(-b^+)\big) \equiv \big(A^+_{2n}(b^+)\big).$$

LEMMA 3. *For* $0<2m$,

$$(8) \qquad A^+_{\sqrt{\zeta},\,2m} = (1+\omega)\omega^m,$$

PROOF: On adding a -1 step at the beginning, an $A^+_{2n,2m}$ becomes a $C_{2n+1,\,2m+1}$ so that

$$(9) \qquad (A^+_{2n,\,2m}) = \frac{2m+1}{2n+1} \binom{2n+1}{n-m} = \binom{2n}{n+m} - \binom{2n}{n+m+1}.$$

Hence

$$A^+_{\sqrt{\zeta},\,2m} = \frac{1+\omega}{1-\omega}\omega^m - \frac{1+\omega}{1-\omega}\omega^{m+1}$$

by (4), which leads to (8).

LEMMA 4. *For* $0<2m<a$,

$$(10) \qquad A^+_{\sqrt{\zeta},\,2m}(a^-) = \frac{(1+\omega)(\omega^m - \omega^{-m+a})}{1-\omega^{a+1}}.$$

PROOF: Replacing n, i, j, a respectively by $2n$, 1, $2m+1$, $a+1$ in (2),

$$(A^{+}_{2n,2m}(a^{-})) = (A(2n, 1, 2m+1, a+1)). \tag{11}$$

Then from (2)

$$A^{+}_{\sqrt{\zeta},2m}(a^{-}) = \sum_{r=0}^{\infty} \sum_{n=m+r(a+1)}^{\infty} \binom{2n}{n+m+r(a+1)} \zeta^{n} +$$

$$+\sum_{s=1}^{\infty} \sum_{n=-m+s(a+1)}^{\infty} \binom{2n}{n-m+s(a+1)} \zeta^{n} - \sum_{r=0}^{\infty} \sum_{n=m+1+r(a+1)}^{\infty} \binom{2n}{n+m+1+r(a+1)} \zeta^{n} -$$

$$-\sum_{s=1}^{\infty} \sum_{n=-m-1+s(a+1)}^{\infty} \binom{2n}{n-m-1+s(a+1)} \zeta^{n}$$

leading to (10), by using (4).

LEMMA 5. *For* $0<2m<a$,

$$A^{+}_{\sqrt{\zeta},2m}(a^{+}) = (1+\omega)\frac{\omega^{a}(\omega^{-m}-\omega^{m+1})}{(1-\omega^{a+1})}. \tag{12}$$

PROOF: Since

$$(A^{+}_{2n,2m}(a^{+})) = (A^{+}_{2n,2m}) - (A^{+}_{2n,2m}(a^{-})), \tag{13}$$

hence by (8) and (10)

$$A^{+}_{\sqrt{\zeta},2m}(a^{+}) = A^{+}_{\sqrt{\zeta},2m} - A^{+}_{\sqrt{\zeta},2m}(a^{-})$$

which leads to (12).

Reflection of the paths in the x-axis shows that

$$A^{-}_{\sqrt{\zeta}}(-a^{-}) \equiv A^{+}_{\sqrt{\zeta}}(a^{-});\quad A^{-}_{\sqrt{\zeta}}(-a^{+}) \equiv A^{+}_{\sqrt{\zeta}}(a^{+});\quad A^{-}_{\sqrt{\zeta},-2m} \equiv A^{+}_{\sqrt{\zeta},2m};$$

$$A^{-}_{\sqrt{\zeta},-2m}(-a^{-}) \equiv A^{+}_{\sqrt{\zeta},2m}(a^{-});\quad A^{-}_{\sqrt{\zeta},-2m}(-a^{+}) \equiv A^{+}_{\sqrt{\zeta},2m}(a^{+}).$$

4. Restricted paths with even number of crossings

THEOREM 1. *For $a>0$, $b>0$, the g.f. of $\{(A^{2k}_{2n})\}$ having the characteristics mentioned within the round brackets is*

$$A^{2k}_{\sqrt{\zeta}}(a^{+}, -b^{+};\ S_1=+1, iS^{+}(a^{+}), jS^{-}(-b^{+})) \equiv \tag{14}$$

$$\equiv \binom{k+1}{i}\binom{k}{j}\omega^{2k+1}\omega_1^{k+1}\omega_2^{k}(\omega_1^{-1}-1)^{i}(\omega_2^{-1}-1)^{j}.$$

NOTE. If $i=0$, the $+$ sign on the top right of a should be replaced by the $-$ sign, since in that case the path does not reach or cross the line $y=a$. Similarly for the cases $j=0$ and $i=0=j$. Thus

(i) $$A^{2k}_{\sqrt{\zeta}}(a^+,\ -b^+;\ S_1=+1,\ \text{no}\ S^+(a^+),\ jS^-(-b^+))\equiv$$
$$\equiv A^{2k}_{\sqrt{\zeta}}(a^-,\ -b^+;\ S_1=+1,\ jS^-(-b^+)),$$

(ii) $$A^{2k}_{\sqrt{\zeta}}(a^+,\ -b^+;\ S_1=+1,\ iS^+(a^+),\ \text{no}\ S^-(-b^+))\equiv$$
$$\equiv A^{2k}_{\sqrt{\zeta}}(a^+,\ -b^-;\ S_1=+1,\ iS^+(a^+)),$$

(iii) $$A^{2k}_{\sqrt{\zeta}}(a^+,\ -b^+;\ S_1=+1,\ \text{no}\ S^+(a^+),\ \text{no}\ S^-(-b^+))\equiv$$
$$\equiv A^{2k}_{\sqrt{\zeta}}(a^-,\ -b^-;\ S_1=+1).$$

PROOF. An $A^{2k}_{2n}(a^+,\ -b^+)$ stipulated in (14) consists of the following: $(k+1)\ S^+$, of which i are $S^+(a^+)$ while $k+1-i$ are $S^+(a^-)$, and $k\ S^-$, of which j are $S^-(-b^+)$ while $k-j$ are $S^-(-b^-)$.

Let the $i\ S^+(a^+)$ be of lengths $2\alpha_s$ $(s=1,\dots,i)$ steps, the $(k+1-i)\ S^+(a^-)$ of lengths $2\alpha'_t$ $(t=1,\dots,k+1-i)$ steps, the $jS^-(-b^+)$ of lengths $2\beta_u$ $(u=1,\dots,j)$ steps and the $(k-j)S^-(-b^-)$ of lengths $2\beta'_v$ $(v=1,\dots,k-j)$ steps, so that

$$2\left[\sum_{s=1}^{i}\alpha_s+\sum_{t=1}^{k+1-i}\alpha'_t+\sum_{u=1}^{j}\beta_u+\sum_{v-1}^{k-j}\beta'_v\right]=2n.$$

Any i of the $(k+1)\ S^+$ can be chosen to be $S^+(a^+)$ and any j out of the kS^- can be chosen to be $S^-(-b^+)$. The two choices are possible in $\binom{k+1}{i}\binom{k}{j}$ ways. Further an S^+ of length 2β steps is equivalent to an $A^+_{2\beta}$ and an S^- to an $A^-_{2\beta}$. Hence the number of paths of the type indicated in (14) is

(15) $$\binom{k+1}{i}\binom{k}{j}\Sigma\prod_{s=1}^{i}(A^+_{2\alpha_s}(a^+))\prod_{t=1}^{k+1-i}(A^+_{2\alpha'_t}(a^-))\prod_{u=1}^{j}(A^-_{2\beta_u}(-b^+))\prod_{v=1}^{k-j}(A^-_{2\beta'_v}(-b^-))$$

where the summation Σ extends over values of α's and β's for which

$$\Sigma\alpha_s+\Sigma\alpha'_t+\Sigma\beta_u+\Sigma\beta'_v=n.$$

This sum is amenable to the Convolution Theorem on generating functions and the GF of (15) is

$$\binom{k+1}{i}\binom{k}{j}\{A^+_{\sqrt{\zeta}}(a^+)\}^i\{A^+_{\sqrt{\zeta}}(a^-)\}^{k+1-i}\{A^-_{\sqrt{\zeta}}(-b^+)\}^j\{A^-_{\sqrt{\zeta}}(-b^-)\}^{k-j},$$

which on using (1) and (5) leads to (14).

If instead of $+1$, $S_1=-1$, then a, b, i, j in (14) have to be replaced respectively by b, a, j, i and

(16) $$A^{2k}_{\sqrt{\zeta}}(a^+,\ -b^+;\ S_1=-1,\ iS^+(a^+),\ jS^-(-b^+))=$$
$$=\binom{k}{i}\binom{k+1}{j}\omega^{2k+1}\omega_1^k\omega_2^{k+1}(\omega_1^{-1}-1)^i(\omega_2^{-1}-1)^j.$$

Adding (14) and (16) we dispense with the restriction on S_1 and have

(17) $$A^{2k}_{\sqrt{\zeta}}(a^+,\ -b^+;\ iS^+(a^+),\ jS^-(-b^+)) =$$
$$= (k+1)\binom{k}{i}\binom{k}{j}\omega^{2k+1}\omega_1^k\omega_2^k(\omega_1^{-1}-1)^i(\omega_2^{-1}-1)^j\left(\frac{\omega_1}{k+1-i}+\frac{\omega_2}{k+1-j}\right).$$

DEDUCTIONS. (i) Summing (14) over $1 \leqq i \leqq k+1$ and $1 \leqq j \leqq k$ as also (16) over $1 \leqq i \leqq k$ and $1 \leqq j \leqq k+1$ and adding the two sums we dispense with the restriction on the numbers of S^+ and S^-, getting

(18) $$A^{2k}_{\sqrt{\zeta}}(a^+,\ -b^+) = \omega^{2k+1}\left[(1-\omega_1^{k+1})(1-\omega_2^k)+(1-\omega_1^k)(1-\omega_2^{k+1})\right].$$

(ii) For $i=j=0$, a path envisaged in (17) has no S^+ reaching or crossing a nor any S^- reaching or crossing $-b$, and (17) becomes

(19) $$A^{2k}_{\sqrt{\zeta}}(a^-,\ -b^-) = \omega^{2k+1}\omega_1^k\omega_2^k(\omega_1+\omega_2).$$

Taking $a=b$ in this we verify for an even number of crosses the result [3]. Further letting a, b tend to ∞, the barriers are decimated; and since $|\omega|<1$, $\omega_1\to 1$ and $\omega_2\to 1$, we have
$$A^{2k}_{\sqrt{\zeta}} = 2\omega^{2k+1},$$
so that by using the known formula [7]

(19a) $$\sum_{n=p}^{\infty}\frac{1}{n}\binom{2n}{n+p}\zeta^n = \frac{\omega^p}{p},\quad p=1,2,\ldots$$

we have
$$(A^{2k}_{2n}) = 2\,\frac{2k+1}{n}\binom{2n}{n+2k+1},$$
verifying the result [2].

Letting $b\to\infty$, (19) becomes
$$A^{2k}_{\sqrt{\zeta}}(a^-) = \omega^{2k+1}\omega_1^k(1+\omega_1)$$
and the coefficient of ζ^n in this gives the number of A^{2k}_{2n} with $R_1<a$, verifying for an even number of crosses the result [10].

(iii) Putting $i=0$ in (14) and (16) and summing over j from 1 to k in (14) and from 1 to $k+1$ in (16) by the binomial theorem,

(20) $$A^{2k}_{\sqrt{\zeta}}(a^-,\ -b^+) = \omega^{2k+1}\omega_1^k(1+\omega_1-\omega_1\omega_2^k-\omega_2^{k+1}).$$

(iv) Putting $j=0$ in (14) and (16) and summing over i from 1 to $k+1$ in (14) and 1 to k in (16) we get

(21) $$A^{2k}_{\sqrt{\zeta}}(a^+,\ -b^-) = \omega^{2k+1}\omega_2^k(1+\omega_2-\omega_2^{k+1}-\omega_2\omega_1^k).$$

5. Restristed paths with odd number of crossings

THEOREM 2

$$A_{\sqrt{\zeta}}^{2k-1}(a^+,\ -b^+;\ iS^+(a^+),\ jS^-(-b^+)) = \tag{22}$$

$$= 2\binom{k}{i}\binom{k}{j}\omega^{2k}\omega_1^k\omega_2^k(\omega_1^{-1}-1)^i(\omega_2^{-1}-1)^j.$$

NOTE. If i or/and j are zero, then the $+$ sign on the top right of a or/and b will be replaced by the $-$ sign.

PROOF. An $A_{2n}^{2k-1}(a^+,\ -b^+)$ of the type envisaged consists of
(i) kS^+ of which i are $S^+(a^+)$, while $k-i$ are $S^+(a^-)$, and (ii) kS^- of which j are $S^-(-b^+)$, while $k-j$ are $S^-(-b^-)$.

It may have either $S_1 = +1$ or -1; which accounts for the factor 2 in formula (22). Also i of the kS^+ can be chosen to be $S^+(a^+)$ and j of the kS^- can be chosen to be $S^-(-b^+)$. Hence

$$(A_{2n}^{2k-1}(a^+,\ -b^+;\ iS^+(a^+),\ jS^-(-b^+)) =$$

$$= 2\binom{k}{i}\binom{k}{j}\sum\prod_{s=1}^{i}(A_{2\alpha_s}^+(a^+))\prod_{t=1}^{k-i}(A_{2\alpha_t'}^+(a^-))\prod_{u=1}^{j}(A_{2\beta_u}^-(-b^+))\prod_{v=1}^{k-j}(A_{2\beta_v'}^-(-b^-)),$$

where the summation Σ is taken over values of α's and β's for which

$$\Sigma\alpha_s+\Sigma\alpha_t'+\Sigma\beta_u+\Sigma\beta_v' = n.$$

As in the derivation of (15), the generating function of the above expression is obtained by using the convolution theorem along with (1) and (5).

DEDUCTIONS. (i) Summing (22) over $1\leq i\leq k$ and $1\leq j\leq k$ we get

$$A_{\sqrt{\zeta}}^{2k-1}(a^+,\ -b^+) = 2\omega^{2k}(1-\omega_1^k)(1-\omega_2^k). \tag{23}$$

(ii) For $i=j=0$, (22) gives

$$A_{\sqrt{\zeta}}^{2k-1}(a^-,\ -b^-) = 2\omega^{2k}\omega_1^k\omega_2^k. \tag{24}$$

Putting $a=b$ in (24) we verify for an odd number of crossings the result [3]. On letting $b\to\infty$ in (24), we verify for an odd number of crossings the result [10].
(iia) The limits $a\to\infty$ and $b\to\infty$ lead to

$$A_{\sqrt{\zeta}}^{2k-1} = 2\omega^{2k}, \quad \text{so that} \quad (A_{2n}^{2k-1}) = 2\frac{2k}{n}\binom{2n}{n+2k}, \tag{25}$$

verifying for an odd number of crossings the result [2].
(iii) Putting $i=0$ in (22) and summing over $i\leq j\leq k$,

$$A_{\sqrt{\zeta}}^{2k-1}(a^-,\ -b^+) = 2\omega^{2k}\omega_1^k(1-\omega_2^k). \tag{26}$$

(iv) Putting $j=0$ in (22) and summing over $1\leq i\leq k$,

$$A_{\sqrt{\zeta}}^{2k-1}(a^+,\ -b^-) = 2\omega^{2k}\omega_2^k(1-\omega_1^k). \tag{27}$$

6. Restricted paths with final position specified

The generating functions of $(A^l_{2n,2m})$ $(0<2m<a)$ under the following conditions are now determined:

	a, b finite	a or/and $b\to\infty$
(i)	$R_1\geqq a,\quad R_2\geqq b$	
(ii)	$R_1<a,\quad R_2<b$	$R_1<a,\quad R_2<b$
(iii)	$R_1<a,\quad R_2\geqq b$	$R_2\geqq b$
(iv)	$R_1\geqq a,\quad R_2<b$	$R_1\geqq a.$

(A) Crossing the x-axis $2k$ times

THEOREM 3. For $0<2m<a$,

$$A^{2k}_{\sqrt{\zeta},2m}(a^+,\ -b^+)=(1+\omega)\omega^{2k+m}(1-\omega_2^k)\left[1-\frac{\omega_1^k(1-\omega^{a-2m})}{1-\omega^{a+1}}\right]. \tag{28}$$

PROOF. Let PQ be an $A^{2k}_{2n,2m}(a^+,\ -b^+)$ and $P(2i,0)$ its last intersection with the x-axis. S_1 must be $+1$ and considering the two possible types of segments OP and PQ, summing $(OP)(PQ)$ over $1\leqq i\leqq n-m$ in each case and adding

$$\left(A^{2k}_{2n,2m}(a^+,\ -b^+)\right)=\sum_i\left(A^{2k-1}_{2i}(a^+,\ -b^+;\ S_1=+1)\right)(A^+_{2n-2i,2m})+$$
$$+\sum_i\left(A^{2k-1}_{2i}(a^-,\ -b^+;\ S_1=+1)\right)\left(A^+_{2n-2i,2m}(a^+)\right),$$

so that the generating function

$$A^{2k}_{\sqrt{\zeta},2m}(a^+,\ -b^+)=A^{2k-1}_{\sqrt{\zeta}}(a^+,\ -b^+;\ S_1=+1)A^+_{\sqrt{\zeta},2m}+$$
$$+A^{2k-1}_{\sqrt{\zeta}}(a^-,\ -b^+;\ S_1=+1)A^+_{\sqrt{\zeta},2m}(a^+),$$

leading to (28) by using (8), (12), (23) and (26).

THEOREM 4. For $0<2m<a$,

$$A^{2k}_{\sqrt{\zeta},2m}(a^-,\ -b^-)=(1+\omega)\omega^{2k+m}\omega_1^k\omega_2^k\frac{1-\omega^{a-2m}}{1-\omega^{a+1}}. \tag{29}$$

PROOF. An $A^{2k}_{2n,2m}(a^-,\ -b^-)$ path OQ must have $S_1=+1$; let $P(2i,0)$ be its last intersection with the x-axis. Considering the nature of the two segments OP and PQ

$$A^{2k}_{\sqrt{\zeta},2m}(a^-,\ -b^-)=A^{2k-1}_{\sqrt{\zeta}}(a^-,\ -b^-;\ S_1=+1)A^+_{\sqrt{\zeta},2m}(a^-)$$

leading to (29) by (10) & (24).

DEDUCTIONS FROM THEOREM 4. (i) On letting a or/and $b\to\infty$, (29) yields

(30) $$A^{2k}_{\sqrt{\zeta},2m}(a^-) = (1+\omega)\omega^{2k+m}\omega_1^k \frac{1-\omega^{a-2m}}{1-\omega^{a+1}};$$

(31) $$A^{2k}_{\sqrt{\zeta},2m}(-b^-) = \omega^{2k+m}\omega_2^k(1+\omega);$$

(32) $$A^{2k}_{\sqrt{\zeta},2m} = \omega^{2k+m}(1+\omega),$$

wherein the coefficient of ζ^n shows that

(33) $$(A^{2k}_{2n,2m}) = \frac{4k+2m+1}{2n+1}\binom{2n+1}{n+2k+m+1}, \quad \text{by (19a)}$$

the result [12].

(ii) *First passage paths.* The paths $A^{2k}_{2n-2,2m-2}(2m^-, -b^-)$ remain below the line $y=2m$ and terminate at $y=2m-2$ on the $(2n-2)$th step. Since a path which is a first passage through $2m$ at the $2n$th step has to pass through $2m-2$ at the $(2n-2)$th step, the generating function of $(C^{2k}_{2n,2m}(-b^-))$ is

(34) $$C^{2k}_{\sqrt{\zeta},2m}(-b^-) = \zeta A^{2k}_{\sqrt{\zeta},2m-2}(2m^-, -b^-).$$

(iii) A path which is a first passage through $2m+1$ at the $(2n+1)$th step has to pass through $2m$ at the $2n$th step. Therefore (29) with $a=2m+1$ gives the generating function

(35) $$\zeta^{-1/2}C^{2k}_{\sqrt{\zeta},2m+1}(-b^-) = (1-\omega^2)\omega^{2k+m}\omega_2^k\frac{(1-\omega^{2m})^k}{(1-\omega^{2m+2})^{k+1}}.$$

(iv) On letting $b\to\infty$, (35) becomes

(36) $$\zeta^{-1/2}C^{2k}_{\sqrt{\zeta},2m+1} = (1-\omega^2)\omega^{2k+m}\frac{(1-\omega^{2m})^k}{(1-\omega^{2m+2})^{k+1}}.$$

THEOREM 5. For $0<2m<a$,

(37) $$A^{2k}_{\sqrt{\zeta},2m}(a^-, -b^+) = (1+\omega)\omega^{2k+m}\omega_1^k(1-\omega_2^k)\frac{1-\omega^{a-2m}}{1-\omega^{a+1}}.$$

PROOF: Let OQ be an $A^{2k}_{2n,2m}(a^-, -b^+)$, it must have $S_1=+1$. Let $P(2i, 0)$ be its last intersection with the x-axis; then considering the two segments OP and PQ,

$$A^{2k}_{\sqrt{\zeta}}(a^-, -b^+) = A^{2k-1}_{\sqrt{\zeta}}(a^-, -b^+;\ S_1=+1)A^{+}_{\sqrt{\zeta},2m}(a^-)$$

leading to (37) by (10) and (26).

DEDUCTIONS. (i) On letting $a\to\infty$, (37) gives

(38) $$A^{2k}_{\sqrt{\zeta},2m}(-b^+) = \omega^{2k+m}(1+\omega)(1-\omega_2^k).$$

(ii) As in the derivation of (34) we get from (37)

(39) $$C^{2k}_{\sqrt{\zeta},2m}(-b^+) \equiv \zeta A^{2k}_{\sqrt{\zeta},2m-2}(2m^-, -b^+)$$

(iii) Also as in the derivation of (35),

$$\zeta^{-1/2} C^{2k}_{\sqrt{\zeta},\,2m+1}(-b^{+}) \equiv A^{2k}_{\sqrt{\zeta},\,2m}(2m+1^{-},\ -b^{+}). \tag{40}$$

THEOREM 6. For $0<2m<a$,

$$A^{2k}_{\sqrt{\zeta},\,2m}(a^{+},\ -b^{-}) = (1+\omega)\omega^{2k+m}\omega_2^{k}\left[1-\frac{\omega_1^{k}(1-\omega^{a-2m})}{1-\omega^{a+1}}\right]. \tag{41}$$

PROOF. As in Theorem 3, considering the last intersection with the x-axis and invoking the convolution theorem,

$$A^{2k}_{\sqrt{\zeta},\,2m}(a^{+},\ -b^{-}) = A^{2k-1}_{\sqrt{\zeta}}(a^{+},\ -b^{-};\ S_1=+1)A^{+}_{\sqrt{\zeta},\,2m}+$$
$$+A^{2k-1}_{\sqrt{\zeta}}(a^{-},\ -b^{-};\ S_1=+1)A^{+}_{\sqrt{\zeta},\,2m}(a^{+}),$$

which leads to (41) by (8), (12), (24) and (27).

When $b\to\infty$, (41) becomes

$$A^{2k}_{\sqrt{\zeta},\,2m}(a^{+}) = \omega^{2k+m}(1+\omega)\left[1-\frac{\omega_1^{k}(1-\omega^{a-2m})}{1-\omega^{a+1}}\right]. \tag{42}$$

(B) Crossing the x-axis $(2k-1)$ times

THEOREM 7. For $0<2m<a$,

$$A^{2k-1}_{\sqrt{\zeta},\,2m}(a^{+},\ -b^{+}) = \omega^{2k+m-1}(1+\omega)(1-\omega_2^{k})\left[1-\frac{\omega_1^{k-1}(1-\omega^{a-2m})}{1-\omega^{a+1}}\right]. \tag{43}$$

PROOF. As in Theorem 3, considering the last intersection with the x-axis,

$$A^{2k-1}_{\sqrt{\zeta},\,2m}(a^{+},\ -b^{+}) = A^{2k-2}_{\sqrt{\zeta}}(a^{+},\ -b^{+};\ S_1=-1)A^{+}_{\sqrt{\zeta},\,2m}+$$
$$+A^{2k-2}_{\sqrt{\zeta}}(a^{-},\ -b^{+};\ S_1=-1)A^{+}_{\sqrt{\zeta},\,2m}(a^{+}). \tag{43a}$$

Of the four g.f. on the r.h.s. the second is given by (8) and fourth by (12). To evaluate the first, we replace k by $k-1$ in (16) and the binomial sums over $1\leqq i\leqq k-1$ and $1\leqq j\leqq k$, yield

$$A^{2k-2}_{\sqrt{\zeta}}(a^{+},\ -b^{+};\ S_1=-1) = \omega^{2k-1}(1-\omega_1^{k-1})(1-\omega_2^{k}).$$

Similarly putting $k-1$ for k and $i=0$ in (16) and summing over $1\leqq j\leqq k$

$$A^{2k-2}_{\sqrt{\zeta}}(a^{-},\ -b^{+};\ S_1=-1) = \omega^{2k-1}\omega_1^{k-1}\omega_2^{k}(\omega_2^{-k}-1).$$

Substituting these values in $(43a)$ and using (8) & (12), we prove (43).

THEOREM 8. *For* $0<2m<a$,

$$A^{2k-1}_{\sqrt{\zeta},\,2m}(a^{-},\ -b^{-}) = \omega^{2k+m-1}\omega_1^{k-1}\omega_2^{k}(1+\omega)\left(\frac{1-\omega^{a-2m}}{1-\omega^{a+1}}\right). \tag{44}$$

PROOF. Considering the last intersection of the x-axis and an $A^{2k-1}_{2n,2m}(a^-, -b^-)$ we have as in Theorem 7

$$A^{2k-1}_{\sqrt{\zeta},2m}(a^-, -b^-) = A^{2k-2}_{\sqrt{\zeta}}(a^-, -b^-; S_1 = -1)A^{+}_{\sqrt{\zeta},2m}(a^-),$$

and the theorem is proved on using (16) with $i=j=0$ and (10).

DEDUCTIONS. (i) On letting a or/and $b\to\infty$, (44) becomes

$$A^{2k-1}_{\sqrt{\zeta},2m}(a^-) = \omega^{2k+m-1}\omega_1^{k-1}(1+\omega)\frac{1-\omega^{a-2m}}{1-\omega^{a+1}}, \tag{45}$$

$$A^{2k-1}_{\sqrt{\zeta},2m}(-b^-) = \omega^{2k+m-1}\omega_2^{k}(1+\omega), \tag{46}$$

$$A^{2k-1}_{\sqrt{\zeta},2m} = \omega^{2k+m-1}(1+\omega). \tag{47}$$

The coefficient of ζ^n in this shows that

$$(A^{2k-1}_{2n,2m}) = \frac{4k+2m-1}{2n+1}\binom{2n+1}{n+2k+m}, \quad \text{by (19a)}$$

verifying for an odd number of crossings the result [12].

(ii) Further, as explained in the derivation of (34), we have

$$C^{2k-1}_{\sqrt{\zeta},2m}(-b^-) = \zeta A^{2k-1}_{\sqrt{\zeta},2m-2}(2m^-, -b^-). \tag{48}$$

(iii) On letting $b\to\infty$, this becomes

$$C^{2k-1}_{\sqrt{\zeta},2m} = \zeta A^{2k-1}_{\sqrt{\zeta},2m-2}(2m^-). \tag{49}$$

(iv) As in the derivation of (35), we have

$$\zeta^{-1/2}C^{2k-1}_{\sqrt{\zeta},2m+1}(-b^-) = A^{2k-1}_{\sqrt{\zeta},2m}(2m+1^-, -b^-). \tag{50}$$

(v) On letting $b\to\infty$, it becomes

$$\zeta^{-1/2}C^{2k-1}_{\sqrt{\zeta},2m+1} = A^{2k-1}_{\sqrt{\zeta},2m}(2m+1^-). \tag{51}$$

THEOREM 9. For $0<2m<a$,

$$A^{2k}_{\sqrt{\zeta},2m}(a^-, -b^+) = \omega^{2k+m-1}\omega_1^{k-1}(1+\omega)(1-\omega_2^k)\left(\frac{1-\omega^{a-2m}}{1-\omega^{a+1}}\right). \tag{52}$$

PROOF. Considering the last intersection of the x-axis and an $A^{2k-1}_{2n,2m}(a^-, -b^+)$,

$$A^{2k-1}_{\sqrt{\zeta},2m}(a^-, -b^+) = A^{2k-2}_{\sqrt{\zeta}}(a^-, -b^+; S_1 = -1)A^{+}_{\sqrt{\zeta},2m}(a^-).$$

Put $i=0$ and replace k by $k-1$ in (16), and sum over $1\leq j\leq k$, then

$$A^{2k-2}_{\sqrt{\zeta}}(a^-, -b^+; S_1 = -1) = \omega^{2k-1}\omega_1^{k-1}(1-\omega_2^k).$$

On using this and (10), the theorem is established.

DEDUCTIONS. (i) On letting $a \to \infty$, (52) becomes

$$A^{2k-1}_{\sqrt{\zeta}, 2m}(-b^+) = \omega^{2k+m-1}(1+\omega)(1-\omega_2^k). \tag{53}$$

(ii) As in the derivation of (34) we have

$$C^{2k-1}_{\sqrt{\zeta}, 2m}(-b^+) = \zeta A^{2k-1}_{\sqrt{\zeta}, 2m-2}(2m^-, \ -b^+). \tag{54}$$

(iii) As in the derivation of (35),

$$\zeta^{-1/2} C^{2k-1}_{\sqrt{\zeta}, 2m+1}(-b^+) = A^{2k-1}_{\sqrt{\zeta}, 2m}(2m+1^-, \ -b^+). \tag{55}$$

THEOREM 10. For $0 < 2m < a$,

$$A^{2k-1}_{\sqrt{\zeta}, 2m}(a^+, \ -b^-) = \omega^{2k+m-1}\,\omega_2^k(1+\omega)\left[1 - \frac{\omega_1^{k-1}(1-\omega^{a-2m})}{1-\omega^{a+1}}\right]. \tag{56}$$

PROOF. Considering the last point of intersection with the x-axis of an $A^{2k-1}_{2n, 2m}(a^+, -b^-)$, we have as in Theorem 3,

$$\begin{aligned} A^{2k-1}_{\sqrt{\zeta}, 2m}(a^+, \ -b^-) &= A^{2k-2}_{\sqrt{\zeta}}(a^+, \ -b^-; \ S_1 = -1) A^+_{\sqrt{\zeta}, 2m} + \\ &+ A^{2k-2}_{\sqrt{\zeta}}(a^-, \ -b^-; \ S_1 = -1) A^+_{\sqrt{\zeta}, 2m}(a^+). \end{aligned} \tag{57}$$

Of the four g.f. on the r.h.s. the second is given by (8) and the fourth by (12). To evaluate the first we put $j=0$ and replace k by $k-1$ in (16) and then sum over $1 \leqq i \leqq k-1$, by the binomial theorem. We thus have

$$A^{2k-2}_{\sqrt{\zeta}}(a^+, \ -b^-; \ S_1 = -1) = \omega^{2k-1}\,\omega_2^k(1-\omega_1^{k-1}).$$

Similarly putting $k-1$ for k and $i=j=0$ in (16),

$$A^{2k-2}_{\sqrt{\zeta}}(a^-, \ -b^-; \ S_1 = -1) = \omega^{2k-1}\,\omega_1^{k-1}\,\omega_2^k.$$

Substituting these values in (57) and using (8) & (12) the theorem is established.

When $b \to \infty$, (56) becomes

$$A^{2k-1}_{\sqrt{\zeta}, 2m}(a^+) = \omega^{2k+m-1}(1+\omega)\left[1 - \frac{\omega_1^{k-1}(1-\omega^{a-2m})}{1-\omega^{a+1}}\right]. \tag{58}$$

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PATHS CROSSING TWO AND THREE LINES

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1. Symbols

In this paper we investigate the joint distribution of crosses with two or three lines when the final position is specified. This investigation seems to be a new one. Only the generating functions of various types of paths have been determined.

In addition to the notations given in [2] we shall use the following:

$A^{+,l}_{n,m;h}$: $(h>0)$ an $A^l_{n,m;h}$ with $S_i\geqq 0$, $i=1,2,\dots,n$; $A^{+,l}_{n,0;h}\equiv A^{+,l}_{n;h}$ (n even).

$A^{l_1,l_2,l_3}_{n,m;h_1,h_2,h_3}$: an $A_{n,m}$ crossing l_i $(i=1,2,3)$ times the line $y=h_i$; $A^{l_1,l_2,l_3}_{n,0;h_1,h_2,h_3}\equiv A^{l_1,l_2,l_3}_{n;h_1,h_2,h_3}$ (n even).

S_u : an S above or below the line $y=u$; $S_0\equiv S$.

S^+_u; S^-_u : an S_u with $S_i\geqq u$; $S_i\leqq u$ in-between. S^+_u is thus a +ve section with respect to the line $y=u$; $S^+_0\equiv S^+$, $S^-_0\equiv S^-$.

$S^+_u(h^+)$: an S^+_u which touches or crosses the line $y=h(>u)$. $S^+_0(h^+)\equiv S^+(h^+)$.

$S^+_u(h^-)$: an S^+_u which does not reach the line $y=h(>u)$. $S^+_0(h^-)\equiv S^+(h^-)$.

$S^{+,l}_u(h^+)$: an $S^+_u(h^+)$ crossing l times the line $y=h-1$. Thus $S^{+,0}_u(h^+)=S^+_u(h^-)$; $S^{+,l}_0(h^+)\equiv S^{+,l}(h^+)$, $h>1$.

$S^-_u(h^+)$: an S^-_u which touches or crosses the line $y=h(<u)$; $S^-_0(h^+)\equiv S^-(h^+)$, $h<0$.

$S^-_u(h^-)$: an S^-_u which does not reach the line $y=h(<u)$; $S^-_0(h^-)\equiv S^-(h^-)$, $h<0$.

$S^{-,l}_u(h^+)$: an $S^-_u(h^+)$ crossing l times the line $y=h+1$; $S^{-,l}_0(h^+)\equiv S^{-,l}(h^+)$, $h<-1$. Thus an $S^{-,0}_u(h^+)$ is an $S^-_u(h^-)$.

2. Some important generating functions

We shall find it convenient to use the symbol:

$$\omega=\frac{1-\sqrt{1-4\zeta}}{1+\sqrt{1-4\zeta}},\quad |\zeta|<\frac{1}{4},\quad |\omega|<1; \tag{1}$$

so that

$$\sqrt{1-4\zeta}=\frac{1-\omega}{1+\omega},\qquad \zeta=\frac{\omega}{(1+\omega)^2}.$$

LEMMA 1. *For* $a>2m>0,\ b\geqq 0,$

$$A_{\sqrt{\zeta},2m}(a^-,\ -b^-)=\frac{1+\omega}{1-\omega}\cdot\frac{1-\omega^{a-2m}}{1-\omega^{a+b}}(1-\omega^b)\omega^m. \tag{2}$$

PROOF. We consider paths of the type $A_{2n,2m}(a^-,\ -b^-)$ starting from the origin and reaching on the $2n$th step the height $2m$ without reaching earlier any of the lines $y=-b$ and $y=a$. Such a path can, on shifting the origin to $(0,\ -b)$, be seen to be equivalent to an $A\,(2n,\ b,\ 2m+b,\ a+b)$; hence its generating function is, on using [2. 2], and writing for convenience $c=a+b$

$$A_{\sqrt{\zeta},2m}(a^-,\ -b^-)=\sum_{n=1}^{\infty}\sum_{r=-\infty}^{\infty}\left[\binom{2n}{n+m+rc}-\binom{2n}{n+m+b+rc}\right]\zeta^n$$

which on using [2.4] gives (2).

LEMMA 2. For $a>2m+1,\ m\geqq 0,\ b\geqq 0,$

$$\zeta^{-1/2}A_{\sqrt{\zeta},2m+1}(a^-,\ -b^-)=\frac{(1+\omega)^2}{1-\omega}\cdot\frac{\omega^m(1-\omega^b)(1-\omega^{a-2m-1})}{1-\omega^{a+b}}. \tag{3}$$

PROOF. As in Lemma 1, we have

$$\big(A_{2n+1,2m+1}(a^-,\ -b^-)\big)\equiv\big(A(2n+1,\ 2m+b+1,\ a+b)\big),$$

and on putting $c=a+b$ its generating function is given by

$$\zeta^{-1/2}A_{\sqrt{\zeta},2m+1}(a^-,\ -b^-)=\sum_{r=0}^{\infty}\sum_{n=m+rc}^{\infty}\binom{2n+1}{n+m+1+rc}\zeta^n+$$

$$+\sum_{s=1}^{\infty}\sum_{n=-m-1+sc}^{\infty}\binom{2n+1}{n+m+1-sc}\zeta^n-$$

$$-\sum_{r=0}^{\infty}\sum_{n=m+b+rc}^{\infty}\binom{2n+1}{n+m+b+1+rc}\zeta^n-\sum_{s=1}^{\infty}\sum_{n=-m-b-1+sc}^{\infty}\binom{2n+1}{n+m+b+1-sc}\zeta^n.$$

We know [1] that for $q=0,1,2,\ldots$.

$$\sum_{\alpha=0}^{\infty}\binom{2\alpha+q}{\alpha}\zeta^\alpha=\frac{(1+\omega)^{q+1}}{1-\omega}, \tag{4}$$

so that

$$\sum_{\alpha=p}^{\infty}\binom{2\alpha+1}{\alpha-p}\zeta^\alpha=\frac{(1+\omega)^2}{1-\omega}\omega^p. \tag{5}$$

Hence on using (5), (3) is established.

3. Determination of $GF\{(A_{2n;h}^{+,2l})\}$ and $GF\{(A_{2n;h,0}^{l_1,l})\}$

THEOREM 1. For $h>0$,

$$A_{\sqrt{\zeta};h}^{+,2l} = \omega^{h+2l-1}(1-\omega^2)^2 \frac{(1-\omega^h)^{l-1}}{(1-\omega^{h+2})^{l+1}}. \tag{6}$$

PROOF. Let OPQ be a path of the type $A_{2n;h}^{+,2l}$, P being the point, say (α, h), where the path crosses the line $y=h$ for the first time. Then the path consists of two segments (i) OP, a segment of the type $A_{\alpha,h}^{+}(h+1^-)$ and (ii) PQ, a segment of the type $A_{2n-\alpha,-h}^{2l-1}(-h-1^-)$ with $P(\alpha, h)$ as the origin. The second segment is equivalent to an $A_{2n-\alpha,h}^{2l-1}(h+1^-)$ which is further equivalent to a $C_{2n-\alpha+1,h+1}^{2l-1}$.
Hence

$$A_{\sqrt{\zeta};h}^{+,2l} = \sum_{n=1}^{\infty} \sum_{\alpha=h}^{2n-4l-h+2} (A_{\alpha,h}^{+}(h+1^-))(C_{2n-\alpha+1,h+1}^{2l-1})\zeta^n.$$

CASE (i) h even; then α will also be even. Let $h=2r$ and $\alpha=2i$, then

$$A_{\sqrt{\zeta};2r}^{+,2l} = \sum_{i=r}^{\infty} (A_{2i,2r}^{+}(2r+1^-))\zeta^i \sum_{n-i=2l+r-1}^{\infty} (C_{2n-2i+1,2r+1}^{2l-1})\zeta^{n-i}.$$

Putting $n-i=j$ and using [2.10] as also [2.51] we obtain a result which is in agreement with (6) for $h=2r$.

CASE (ii) h odd; then α will also be odd. Let $h=2r+1$ and $\alpha=2i+1$, then

$$A_{\sqrt{\zeta};2r+1}^{+,2l} = \sum_{i=r}^{\infty} (A_{2i+1,2r+1}^{+}(2r+2^-))\zeta^i \sum_{n-i=2l+r}^{\infty} (C_{2n-2i,2r+2}^{2l-1})\zeta^{n-i}.$$

Putting $n-i=j$ and using [2.49] and (3) we get the result agreeing with (6) for $h=2r+1$, in view of (1).

THEOREM 2. *For* $h>0$,

$$A_{\sqrt{\zeta};h,0}^{2l_1,2l} = \left(\frac{1-\omega^h}{1-\omega^{h+2}}\right)^{l_1+l} \omega^{2l_1+2l+1} \cdot \sum_{k>0} \binom{l_1-1}{k-1} \omega^{k(h-2)} \left(\frac{1-\omega^2}{1-\omega^h}\right)^{2k} \left\{\binom{l+1}{k}\left(\frac{1-\omega^h}{1-\omega^{h+2}}\right) + \binom{l}{k}\right\}, \tag{7}$$

and[1]

$$A_{\sqrt{\zeta};h,0}^{2l_1,2l-1} = 2\omega^{2l_1+2l}\left(\frac{1-\omega^h}{1-\omega^{h+2}}\right)^{l_1+l} \sum_{k>0} \binom{l_1-1}{k-1}\binom{l}{k} \omega^{k(h-2)} \left(\frac{1-\omega^2}{1-\omega^h}\right)^{2k}. \tag{8}$$

PROOF. PART I. An $A_{2n;h,0}^{2l_1,2l}$ envisaged in (7) must have either (i) $S_1=+1$, $h \geqq S_{2n-1}>0$, or (ii) $S_1=-1$, $S_{2n-1}<0$.

[1] The upper limit of summation in (7) is min $(l_1, l+1)$ and in (8) it is min (l_1, l). For greater values of k the summand vanishes; hence the upper limit need not be mentioned.

In case (i) the path consists of $(l+1)\ S^+$, lS^-, l_1, S_h^+ and $(l_1, -1)S_h^-$. Suppose k out of the $(l+1)S^+$ are $S^+(h+1^+)$ such that the ith one crosses the line $y=h$ $2l_{1i}\geqq 2$ $(i=1, 2, \dots, k)$ times and $\sum_{i=0}^{k} l_{1\iota} = l_1$. Then an $A_{2n;h,0}^{2l_1,2l}$ may be considered as consisting of three types of sections $S^{+,2l_{1i}}(h+1^+)$, $S^+(h+1^-)$ and S^-, their numbers being respectively k, $l+1-k$ and l. Suppose that taken in order, the $kS^{+,2l_{1i}}(h+1^+)$ are of $2\alpha_i$ steps $(i=1,2,\dots,k)$, the $l+1-kS^+(h+1^-)$ are of $2\beta_j$ steps $(j=1,\dots,l+1-k)$ and the lS^- are of $2\gamma_m$ steps $(m=1,\dots,l)$. Since these three types of sections are individually equivalent to $A_{2\alpha_i;h}^{+,2l_{1i}}$, $A_{2\beta_j}^{+}(h+1^-)$ and $A_{2\gamma_m}^{-}$ respectively; hence

$$\left(A_{2n;h,0}^{2l_1,2l}(S_1=+1, kS^+(h+1^+))\right)\Big/\binom{l+1}{k} =$$

$$= \Sigma_2\Sigma_1 \prod_{i=1}^{k}(A_{2\alpha_i;h}^{+,2l_{1i}}) \prod_{j=1}^{l+1-k}(A_{2\beta_j}^{+}(h+1^+)) \prod_{m=1}^{l}(A_{2\gamma m}^{-})$$

the summations Σ_1, Σ_2 being respectively over the ranges

$$\sum_{i=1}^{k} l_{1i}=l_1 \quad \text{and} \quad \Sigma\alpha_i+\Sigma\beta_j+\Sigma\gamma_m = n.$$

The convolution theorem gives for generating functions

$$A_{\sqrt{\zeta};h,0}^{2l_1,2l}(S_1=+1,\ kS^+(h+1^+))\Big/\binom{l+1}{k} =$$

$$= \Sigma_1 \prod_{i=1}^{k} A_{\sqrt{\zeta};h}^{+,2l_{1i}}\{A_{\sqrt{\zeta}}^{+}(h+1^-)\}^{l+1-k}\{A_{\sqrt{\zeta}}^{-}\}^{l}$$

Using (6), (2.1] and [2.7] this equals

$$\text{(9)}\qquad \binom{l_1-1}{k-1}\omega^{2l_1+2l+1+k(h-2)}\left(\frac{1-\omega^h}{1-\omega^{h+2}}\right)^{l_1+l+1}\left(\frac{1-\omega^2}{1-\omega^h}\right)^{2k},$$

the first factor being the result of Σ_1, i.e. the number of partitions of l_1 into k +ve integers (>0).

In case (ii) the path consists of lS^+, $(l+1)S^-$, $l_1\ S_h^+$ and $(l_1\ -1)S_h^-$. Suppose k out of the lS^+ are $S^+(h+1^+)$ such that the ith one crosses the line $y=h$ $2l_{1i}$ times $(i=1, 2, \dots, k)$ and $\Sigma l_{1\iota}=l_1$, then as above

$$\text{(10)}\qquad \begin{gathered} A_{\sqrt{\zeta};h,0}^{2l_1,2l}(S_1=-1,\ kS^+(h+1^+))\Big/\binom{l}{k} = \\ \binom{l_1-1}{k-1}\omega^{2l_1+2l+1+k(h-2)}\left(\frac{1-\omega^h}{1-\omega^{h+2}}\right)^{l_1+l}\left(\frac{1-\omega^2}{1-\omega^h}\right)^{2k}. \end{gathered}$$

Summing from (9) over $1 \leqq k \leqq \min(l_1, l+1)$ and from (10) over $1 \leqq k \leqq \min(l_1, l)$ and adding the two sums we prove (7).

PART II. For a path envisaged in (8) we note that it can begin with $S_1 = +1$ or $S_1 = -1$ and in each case the path would consist of $lS^+, lS^-, l_1 S_h^+$ and $(l_1 - 1)S_h^-$. Then as before

$$A_{\sqrt{\zeta};h,0}^{2l_1,2l-1} = 2\sum_{k>0}\binom{l_1-1}{k-1}\binom{l}{k} A_{\sqrt{\zeta};h,0}^{2l_1,2l-1}(S_1=+1,\ kS^+(h+1^+))$$

leading to (8).

4. Generating function of the sequence $\{(A_{2n,2m:2m,0}^{l_1,l})\}$

THEOREM 3. *For* $m>0$,

(11) $$A_{\sqrt{\zeta},2m;2m,0}^{2l_1,2l} = \omega^{2l_1+2l+1} v_1^{l_1+l+1} \sum_{k\geqq 0}\binom{l_1}{k}\binom{l}{k} v_2^{2k+1},$$

(12) $$A_{\sqrt{\zeta},2m;2m,0}^{2l_1,2l-1} = \omega^{2l_1+2l} v_1^{l_1+l} \sum_{k\geqq 0}\binom{l_1}{k}\binom{l-1}{k} v_2^{2k+1},$$

(13) $$A_{\sqrt{\zeta},2m;2m,0}^{2l_1-1,2l} = \omega^{2l_1+2l} v_1^{l_1+l} \sum_{k\geqq 0}\binom{l_1-1}{k}\binom{l}{k} v_2^{2k+1},$$

(14) $$A_{\sqrt{\zeta},2m;2m,0}^{2l_1-1,2l-1} = \omega^{2l_1+2l-1} v_1^{l_1+l-1} \sum_{k\geqq 0}\binom{l_1-1}{k}\binom{l-1}{k} v_2^{2k+1},$$

where $$v_1 = \left(\frac{1-\omega^{2m}}{1-\omega^{2m+2}}\right) \quad\text{and}\quad v_2 = \left(\frac{\omega^{m-1}-\omega^{m+1}}{1-\omega^{2m}}\right).$$

PROOF. (i) An $A_{2n,2m;2m,0}^{2l_1,2l}$ must have $S_1 = +1$ and $S_{2n-1} = 2m-1$; let $P(2i, 0)$ be the last point where it crosses the x-axis. Suppose the segment OP crosses $y=2m$, $2l_{11}$ (>0) times. Then the segment OP is an $A_{2i;2m,0}^{2l_{11},2l-1}$ $(+)$ and the remaining segment, from P to $Q(2n,2m)$ is an $A_{2n-2i,2m;2m}^{+,2l_1-2l_{11}}$ equivalent to an $A_{2n-2i,2m}^{2l_1-2l_{11}}(2m+1^-)$. Another possibility is that $l_{11}=0$, the segment OP does not cross the line $y=2m$. Then OP is an $A_{2i}^{2l-1}(2m+1^-; S_1=+1)$ and PQ is an $A_{2n-2i,2m;2m}^{+,2l_1}$ equivalent to an $A_{2n-2i,2m}^{2l_1}(2m+1^-)$. Hence

$$A_{\sqrt{\zeta},2m;2m,0}^{2l_1,2l} = \sum_{l_{11}=1}^{l_1} A_{\sqrt{\zeta};2m,0}^{2l_{11},2l-1}(S_1=+1) A_{\sqrt{\zeta},2m}^{2l_1-2l_{11}}(2m+1^-)+$$

$$+A_{\sqrt{\zeta}}^{2l-1}(2m+1^-;\ S_1=+1) A_{\sqrt{\zeta},2m}^{2l_1}(2m+1^-).$$

On letting $b\to\infty$ in [2.24] and then using this, [2.30], (8) and remembering that the upper limit for k is $\min(l_1, l)$ it becomes

$$=\omega^{2l_1+2l+m}(1-\omega^2)\frac{(1-\omega^{2m})^{l_1+l}}{(1-\omega^{2m+2})^{l_1+l+1}}\left[\sum_{k>0}\binom{l}{k}\left(\frac{\omega^{m-1}-\omega^{m+1}}{1-\omega^{2m}}\right)^{2k}\sum_{l_{11}=k}^{l_1}\binom{l_{11}-1}{k-1}+1\right]$$

and since

$$\sum_{l_{11}=k}^{l_1}\binom{l_{11}-1}{k-1}=\sum_{l_{11}=k}^{l_1}\left\{\binom{l_{11}}{k}-\binom{l_{11}-1}{k}\right\}=\binom{l_1}{k}, \tag{15}$$

this can be put in the form (11).

(ii) To prove (12). An $A^{2l_1, 2l-1}_{2n, 2m; 2m, 0}$ must have $S_1=-1$ and $S_{2n-1}=2m-1$. Considering the two segments made by the last point of intersection of the path with the x-axis we have, as above

$$A^{2l_1, 2l-1}_{\sqrt{\zeta}, 2m; 2m, 0}=\sum_{l_{11}=1}^{l_1}A^{2l_{11}, 2l-2}_{\sqrt{\zeta}; 2m, 0}(S_1=-1)A^{2l_1-2l_{11}}_{\sqrt{\zeta}, 2m}(2m+1^-)+$$

$$+A^{2l-2}_{\sqrt{\zeta}}(2m+1^-;\ S_1=-1)A^{2l_1}_{\sqrt{\zeta}, 2m}(2m+1^-),$$

[2.16] on putting $i=j=0$ and letting $b\to\infty$, so that $\omega_2\to 1$ gives

$$A^{2l-2}_{\sqrt{\zeta}}(2m+1^-;\ S_1=-1)=\omega^{2l-1}\left(\frac{1-\omega^{2m}}{1-\omega^{2m+2}}\right)^{l-1}. \tag{16}$$

On using this, [2.30], (10) and (15), (12) is established.

(iii) To prove (13). An $A^{2l_1-1, 2l}_{2n, 2m; 2m, 0}$ must have $S_1=+1$ and $S_{2n-1}=2m+1$; and proceeding as before, we have

$$A^{2l_1-1, 2l}_{\sqrt{\zeta}, 2m; 2m, 0}=\sum_{l_{11}=1}^{l_1}A^{2l_{11}, 2l-1}_{\sqrt{\zeta}; 2m, 0}(S_1=+1)A^{2l_1-2l_{11}-1}_{\sqrt{\zeta}, 2m}(2m+1)+$$

$$+A^{2l-1}_{\sqrt{\zeta}}(2m+1^-;\ S_1=+1)A^{2l_1-1}_{\sqrt{\zeta}, 2m}(2m+1^-)$$

Now a use of [2.24], [2.45], (8) and (15) leads to (13).

(iv) Finally to prove (14). An $A^{2l_1-1, 2l-1}_{2n, 2m; 2m, 0}$ must have $S_1=-1$ and $S_{2n-1}=2m+1$ and as before

$$A^{2l_1-1, 2l-1}_{\sqrt{\zeta}, 2m; 2m, 0}=\sum_{l_{11}=1}^{l_1}A^{2l_{11}, 2l-2}_{\sqrt{\zeta}; 2m, 0}(S_1=-1)A^{2l_1-2l_{11}-1}_{\sqrt{\zeta}, 2m}(2m+1^-)+$$

$$+A^{2l-2}_{\sqrt{\zeta}}(2m+1^-;\ S_1=-1)A^{2l_1-1}_{\sqrt{\zeta}, 2m}(2m+1^-),$$

On using [2.45], (10), (15) and (16), we establish (14).

THEOREM 4. *For* $m \geqq 0$.

(17) $$\zeta^{-1/2} A^{2l_1, 2l}_{\sqrt{\zeta}, 2m+1; 2m+1, 0} = \omega^{2l_1+2l+m}(1+\omega) v_3^{l_1+l+1} \sum_{k \geqq 0} \binom{l_1}{k}\binom{l}{k} v_4^{2k+1} \omega^{k(2m-1)},$$

(18) $$\zeta^{-1/2} A^{2l_1, 2l-1}_{\sqrt{\zeta}, 2m+1; 2m+1, 0} = \omega^{2l_1+2l+m-1}(1+\omega) v_3^{l_1+l} \sum_{k \geqq 0} \binom{l_1}{k}\binom{l-1}{k} v_4^{2k+1} \omega^{k(2m-1)},$$

(19) $$\zeta^{-1/2} A^{2l_1-1, 2l}_{\sqrt{\zeta}, 2m+1; 2m+1, 0} = \omega^{2l_1+2l+m-1}(1+\omega) v_3^{l_1+l} \sum_{k \geqq 0} \binom{l_1-1}{k}\binom{l}{k} v_4^{2k+1} \omega^{k(2m-1)},$$

(20) $$\zeta^{-1/2} A^{2l_1-1, 2l-1}_{\sqrt{\zeta}, 2m+1; 2m+1, 0} =$$
$$= \omega^{2l_1+2l+m-2}(1+\omega) v_3^{l_1+l-1} \sum_{k \geqq 0} \binom{l_1-1}{k}\binom{l-1}{k} v_4^{2k+1} \omega^{k(2m-1)},$$

where $$v_3 = \left(\frac{1-\omega^{2m+1}}{1-\omega^{2m+3}}\right) \quad \text{and} \quad v_4 = \left(\frac{1-\omega^2}{1-\omega^{2m+1}}\right).$$

PROOF. Let $P(2i, 0)$ be the last point where a path OPQ of the type envisaged in the above formulae intersects the x-axis. As in (11) to (14) the segment PQ is, for a suitable value of l, equivalent to an $A^l_{2n+1-2i, 2m+1}(2m+\bar{2})$ or to a $C^l_{2n+2-2i, 2m+2}$.

(i) For (17) we have

$$(A^{2l_1, 2l}_{2n+1, 2m+1; 2m+1, 0}) = \sum_{l_{11}=1}^{l_1} \sum_i (A^{2l_{11}, 2l-1}_{2i; 2m+1, 0}(S_1=+1))(A^{2l_1-2l_{11}}_{2n+1-2i, 2m+1}(2m+1^-)+$$
$$+\sum_i (A^{2l-1}_{2i}(2m+2^-;\ S_1=+1))(A^{2l_1}_{2n+1-2i, 2m+1}(2m+2^-).$$

Now $(A^{2l_1-2l_{11}}_{2n+1-2i, 2m+1}(2m+2^-)) \equiv (C^{2l_1-2l_{11}}_{2n+2-2i, 2m+2})$, so that

$$\zeta^{-1/2} A^{2l_1, 2l}_{\sqrt{\zeta}, 2m+1; 2m+1, 0} = \sum_{l_{11}=1}^{l_1} A^{2l_{11}, 2l-1}_{\sqrt{\zeta}; 2m+1, 0}(S_1=+1)\zeta^{-1} C^{2l_1-2l_{11}}_{\sqrt{\zeta}, 2m+2}+$$
$$+A^{2l-1}_{\sqrt{\zeta}}(2m+2^-;\ S_1=+1)\zeta^{-1/2} A^{2l_1}_{\sqrt{\zeta}, 2m+1}(2m+2^-).$$

Using (8), [2.24], [2.34] and (15) and noting that the summand is nonzero only for $k \leqq \min(l_1, l)$ we are led to (17).

(ii) To obtain (18) we have

$$\zeta^{-1/2} A^{2l_1, 2l-1}_{\sqrt{\zeta}, 2m+1; 2m+1, 0} = \sum_{l_{11}=1}^{l_1} A^{2l_{11}, 2l-2}_{\sqrt{\zeta}; 2m+1, 0}(S_1=-1)\zeta^{-1/2} A^{2l_1-2l_{11}}_{\sqrt{\zeta}, 2m+1}(2m+2^-)+$$
$$+A^{2l-2}_{\sqrt{\zeta}}(2m+2^-;\ S_1=-1)\zeta^{-1/2} A^{2l_1}_{\sqrt{\zeta}, 2m+1}(2m+2^-),$$

which on using [2.16], [2.34], (10) and (15) leads to (18).

(iii) To obtain (19) we have

$$\zeta^{-1/2} A^{2l_1-1,2l}_{\sqrt{\zeta},2m+1;2m+1,0} = \sum_{l_{11}=1}^{l_1} A^{2l_{11},2l-1}_{\sqrt{\zeta};2m+1,0}(S_1=+1)\zeta^{-1/2} A^{2l_1-2l_{11}-1}_{\sqrt{\zeta},2m+1}(2m+2^-)+$$

$$+A^{2l-1}_{\sqrt{\zeta}}(2m+2^-;\ S_1=+1)\zeta^{-1/2} A^{2l_1-1}_{\sqrt{\zeta},2m+1}(2m+2^-)$$

using [2·24], [2·49], (8) and (15), we establish (19).

(iv) Lastly to obtain (20), we have

$$\zeta^{-1/2} A^{2l_1-1,2l-1}_{\sqrt{\zeta},2m+1;2m+1,0} = \sum_{l_{11}=1}^{l_1} A^{2l_{11},2l-2}_{\sqrt{\zeta};2m+1,0}(S_1=-1)\zeta^{-1/2} A^{2l_1-2l_{11}-1}_{\sqrt{\zeta},2m+1}(2m+2^-)+$$

$$+A^{2l-2}_{\sqrt{\zeta}}(2m+2^-;\ S_1=-1)\zeta^{-1/2} A^{2l_1-1}_{\sqrt{\zeta},2m+1}(2m+2^-)$$

and this, on using [2.16], [2.49], (10) and (15) yields (20).

THEOREM 5. *For* $m>0,\ h>0,$

$$(21)\quad A^{2l}_{\sqrt{\zeta},2m;-h}(2m+1^-) = \omega^{2l-1+m+h}(1+\omega)(1-\omega^2)\frac{(1-\omega^{2m+1})(1-\omega^{2m+h})^{l-1}}{(1-\omega^{2m+h+2})^{l+1}},$$

$$(22)\quad \zeta^{-1/2} A^{2l}_{\sqrt{\zeta},2m+1;-h}(2m+2^-) = \omega^{2l-1+m+h}(1+\omega)^2(1-\omega^2)\frac{(1-\omega^{2m+2})(1-\omega^{2m+h+1})^{l-1}}{(1-\omega^{2m+h+3})^{l+1}}.$$

PROOF. To prove (21), let $P(\alpha,-h)$ be the first point where the line $y=-h$ is crossed by an $A^{2l}_{2n,2m;-h}(2m+1^-)$. Then the segment OP is an $A_{\alpha,-h}(2m+1^-,\ -h-1^-)$ equivalent to an $A_{\alpha,h}(h+1^-,\ -2m-1^-)$ and the segment from P to $Q(2n,2m)$ is seen, on taking the origin at $P(\alpha,-h)$, to be equivalent to an $A^{2l-1}_{2n-\alpha,2m+h}(2m+h+1^-)$. Thus

$$(23)\ (A^{2l}_{2n,2m;-h}(2m+1^-)) = \sum_{\alpha}(A_{\alpha,h}(h+1^-,\ -2m-1^-))(A^{2l-1}_{2n-\alpha,2m+h}(2m+h+1^-)).$$

CASE 1. h is even, $=2r$, say; then α has to be even and

$$(24)\quad \begin{aligned} A^{2l}_{\sqrt{\zeta},2m;-2r}(2m+1^-) &= A_{\sqrt{\zeta},2r}(2r+1^-,\ -2m-1^-)A^{2l-1}_{\sqrt{\zeta},2m+2r}(2m+2r+1^-) \\ &= \omega^{2l+m+2r-1}(1+\omega)(1-\omega^2)\frac{(1-\omega^{2m+1})(1-\omega^{2m+2r})^{l-1}}{(1-\omega^{2m+2r+2})^{l+1}}, \end{aligned}$$

by (2) and [2.45].

CASE 2. h is odd, $=2r+1$, say, then α has to be odd, $=2\beta+1$, say, and the last type of path in (23) is equivalent to a C-path. Thus (23) becomes

$$(A^{2l}_{2n,2m;-2r-1}(2m+1^-)) = \sum_{\beta}(A_{2\beta+1,2r+1}(2r+2^-,\ -2m-1^-))(C^{2l-1}_{2n-2\beta,2m+2r+2})$$

so that

$$(25)\quad A^{2l}_{\sqrt{\zeta},2m;-2r-1}(2m+1^-)=\zeta^{-1/2}A_{\sqrt{\zeta},2r+1}(2r+2^-,\ -2m-1^-)C^{2l-1}_{\sqrt{\zeta},2m+2r+2}$$

This, on using (3) and [2.49], and (24) are contained in (21).

To prove (22), let $P(\alpha,\ -h)$ be the first point where an $A^{2l}_{2n+1,2m+1;-h}(2m+2^-)$ crosses the line $y=-h$; then, as above

$$(26)\quad \begin{aligned}&(A^{2l}_{2n+1,2m+1;-h}(2m+2^-))=\\&=\sum_{\alpha}(A_{\alpha,h}(h+1^-,\ -2m-2^-))(A^{2l-1}_{2n+1-\alpha,2m+1+h}(2m+h+2^-))\end{aligned}$$

CASE 1. $h=2r$; then α is even and the last type of path is equivalent to a first passage path. Convolution theorem makes (26) yield

$$(27)\quad \zeta^{-1/2}A^{2l}_{\sqrt{\zeta},2m+1;-2r}(2m+2^-)=A_{\sqrt{\zeta},2r}(2r+1^-,\ -2m-2^-)\zeta^{-1}C^{2l-1}_{\sqrt{\zeta},2m+2r+2}.$$

CASE 2. $h=2r+1$; then α is odd and (26) yields

$$(28)\quad \begin{aligned}&\zeta^{-1/2}A^{2l}_{\sqrt{\zeta},2m+1;-2r-1}(2m+2^-)=\\&=\zeta^{-1/2}A_{\sqrt{\zeta},2r+1}(2r+2^-,\ -2m-2^-)A^{2l-1}_{\sqrt{\zeta},2m+2r+2}(2m+2r+3^-).\end{aligned}$$

(27), on using (2) and [2.49], and (28), on using (3) and [2.45], make up (22).

5. A_{2n} paths crossing two lines

Here we consider paths of the type $A^{2l_1,2l_2}_{2n;h_1,-h_2}$ under four different cases according to which line is crossed first and which one is crossed last. The ordinates of these lines will be written in order within square brackets. The four types are thus

(i) $A^{2l_1,2l_2}_{2n;h_1,-h_2}[h_1,h_1]$, (ii) $A^{2l_1,2l_2}_{2n;h_1,-h_2}[h_1,\ -h_2]$

(iii) $A^{2l_1,2l_2}_{2n;h_1,-h_2}[-h_2,h_1]$, (iv) $A^{2l_1,2l_2}_{2n;h_1,-h_2}[-h_2,\ -h_2]$.

THEOREM 6. *For* $h_1,h_2>0$,

$$(29)\quad \begin{aligned}&A^{2l_1,2l_2}_{\sqrt{\zeta};h_1,-h_2}[h_1,h_1]=\omega^{2l_1+2l_2+h_1-1}(1+\omega)^2(1-\omega^{h_2+1})^2\cdot\\&\cdot\frac{(1-\omega^{h_1+h_2})^{l_1+l_2-1}}{(1-\omega^{h_1+h_2+2})^{l_1+l_2+1}}\sum_{k>0}\binom{l_1-1}{k}\binom{l_2-1}{k-1}\left(\frac{1-\omega^2}{1-\omega^{h_1+h_2}}\right)^{2k}\omega^{k(h_1+h_2-2)}.\end{aligned}$$

PROOF. Let the first and the last intersections of an $A^{2l_1,2l_2}_{2n;h_1,-h_2}$ path $OPQR$ with the line $y=h_1$ be $P(\alpha,h_1)$ and $Q(2j+\alpha,h_1)$. Then OP is an $A_{\alpha,h_1}(h_1+1^-,\ -h_2-1^-)$, PQ referred to P as the origin, is an $A^{2l_1-2,2l_2}_{2j;0,-h_1-h_2}(S_1=+1)$ equivalent to an

$A^{2l_2,2l_1-2}_{2j;h_1+h_2,0}(S_1=-1)$, QR, referred to Q as origin is an $A_{2n-\alpha-2j,-h_1}(1^-,-h_1-h_2-1^-)$ equivalent to an $A_{2n-2j-\alpha,h_1}(h_1+1^-,\ -h_2-1^-)$. Hence

$$(30)\qquad (A^{2l_1,2l_2}_{2n;h_1,-h_2})=\sum_{\alpha,j}(A_{\alpha,h_1}(h_1+1^-,\ -h_2-1^-))(A^{2l_2,2l_1-2}_{2j;h_1+h_2,0}(S_1=-1))\cdot \cdot(A_{2n-\alpha-2j,h_1}(h_1+1^-,\ -h_2-1^-)).$$

CASE 1. $h_1=2r$; then α is even and (30) yields

$$(31)\qquad A^{2l_1,2l_2}_{\sqrt{\zeta};2r,-h_2}= =A_{\sqrt{\zeta},2r}(2r+1^-,\ -h_2-1^-)A^{2l_2,2l_1-2}_{\sqrt{\zeta};2r+h_2,0}(S_1=-1)A_{\sqrt{\zeta},2r}(2r+1^-,\ -h_2-1^-).$$

CASE 2. $h_1=2r+1$, then α is odd and (30) yields

$$(32)\qquad A^{2l_1,2l_2}_{\sqrt{\zeta};2r+1,-h_2}=\zeta^{-1/2}A_{\sqrt{\zeta},2r+1}(2r+2^-,\ -h_2-1^-)A^{2l_2,2l_1-2}_{\sqrt{\zeta};2r+h_2+1,0}(S_1=-1)\cdot \cdot\zeta\zeta^{-1/2}A_{\sqrt{\zeta},2r+1}(2r+2^-,\ -h_2-1^-)$$

which, on using (3) and (10), along with (31), on using (2) and (10), makes up (29).

THEOREM 7. *For* $h_1,h_2>0$

$$(33)\qquad A^{2l_1,2l_2}_{\sqrt{\zeta};h_1,-h_2}[h_1,\ -h_2]=\omega^{2l_1+2l_2+h_1-2}(1+\omega)^3(1-\omega)(1-\omega^{h_1+1})(1-\omega^{h_2+1})\cdot \cdot\frac{(1-\omega^{h_1+h_2})^{l_1+l_2-2}}{(1-\omega^{h_1+h_2+2})^{l_1+l_2+1}}\sum_{k\geq 0}\binom{l_1-1}{k}\binom{l_2-1}{k}\left(\frac{1-\omega^2}{1-\omega^{h_1+h_2}}\right)^{2k}\omega^{k(h_1+h_2-1)}.$$

PROOF. Let $OPQR$ an $A^{2l_1,2l_2}_{2n;h_1,-h_2}$ first cross $y=h_1$ at $P(\alpha,h_1)$; also let its last intersection be with $y=-h_2$ at $Q(\alpha+\beta,\ -h_2)$. Then OP is an $A_{\alpha,h_1}(h_1+1^-,\ -h_2-1^-)$, PQ referred to P as origin is an $A^{2l_1-1,2l_2-1}_{\beta,-h_1-h_2;0,-h_1-h_2}$ equivalent to an $A^{2l_2-1,2l_1-1}_{\beta,h_1+h_2;h_1+h_2,0}$ and QR referred to Q as origin is an $A_{2n-\alpha-\beta,h_2}(h_1+h_2+1^-,\ -1^-)$. Hence

$$(34)\qquad (A^{2l_1,2l_2}_{2n;h_1,-h_2})=\sum_{\alpha,\beta}(A_{\alpha,h_1}(h_1+1^-,\ -h_2-1^-))(A^{2l_2-1,2l_1-1}_{\beta,h_1+h_2;h_1+h_2,0})\cdot \cdot(A_{2n-\alpha-\beta,h_2}(h_1+h_2+1^-,\ -1^-)).$$

From the consideration that h_1, h_2, α, β may be even or odd, we have four contingencies:

(i) $h_1=2r,\quad h_2=2s,\quad \alpha=2i,\quad \beta=2j$

(ii) $h_1=2r,\quad h_2=2s+1,\quad \alpha=2i,\quad \beta=2j+1$

(iii) $h_1=2r+1\quad h_2=2s,\quad \alpha=2i+1,\quad \beta=2j+1$

(iv) $h_1=2r+1,\quad h_2=2s+1\quad \alpha=2i+1,\quad \beta=2j$

In case (i), (34) gives

(35) $A^{2l_1, 2l_2}_{\sqrt{\zeta}; 2r, -2s} =$

$$= A_{\sqrt{\zeta}, 2r}(2r+1^-, \ -2s-1^-) A^{2l_2-1, 2l_1-1}_{\sqrt{\zeta}, 2r+2s; 2r+2s, 0} A_{\sqrt{\zeta}, 2s}(2r+2s+1^-, \ -1^-),$$

which, on using (2) and (14), is in agreement with (33) for $h_1 = 2r$, $h_2 = 2s$.

CASE (ii). The g.f.'s are now related by

(36) $$A^{2l_1, 2l_2}_{\sqrt{\zeta}; 2r, -2s-1} = A_{\sqrt{\zeta}, 2r}(2r+1^-, \ -2s-2^-) \zeta^{-1/2} A^{2l_2-1, 2l_1-1}_{\sqrt{\zeta}, 2r+2s+1; 2r+2s+1, 0} \cdot \\ \cdot \zeta^{1/2} A_{\sqrt{\zeta}, 2s+1}(2r+2s+2^-, \ -1^-)$$

which again, on using (2), (3) and (20) agrees with (33) for $h_1 = 2r$, $h_2 = 2s+1$.

CASE (iii). The generating functions have now the relation

(37) $$A^{2l_1, 2l_2}_{\sqrt{\zeta}; 2r+1, -2s} = \zeta^{-1/2} A_{\sqrt{\zeta}, 2r+1}(2r+2^-, \ -2s-1^-) \zeta^{-1/2} A^{2l_2-1, 2l_1-1}_{\sqrt{\zeta}, 2r+2s+1; 2r+2s+1, 0} \cdot \\ \cdot \zeta A_{\sqrt{\zeta}, 2s}(2r+2s+2^-, \ -1^-),$$

and this, on using (2), (3) and (20) leads to (33) for $h_1 = 2r+1$, $h_2 = 2s$.

CASE (iv). The relation between the generating functions is now

(38) $$A^{2l_1, 2l_2}_{\sqrt{\zeta}; 2r+1, -2s-1} = \zeta^{-1/2} A_{\sqrt{\zeta}, 2r+1}(2r+2^-, \ -2s-2^-) A^{2l_2-1, 2l_1-1}_{\sqrt{\zeta}, 2r+2s+2; 2r+2s+2, 0} \cdot \\ \cdot \zeta^{1/2} A_{\sqrt{\zeta}, 2s+1}(2r+2s+3^-, \ -1^-).$$

Using (3) and (14), one is led to (33) for $h_1 = 2r+1$, $h_2 = 2s+1$. Thus Formulae (35), (36), (37) and (38) combine to give (33).

THEOREM 8. *For* $h_1, h_2 > 0$,

(39) $$A^{2l_1, 2l_2}_{\sqrt{\zeta}; h_1, -h_2}[-h_2, \ h_1] = \textit{r. h. expression in } (33)$$

This follows on observing that by reflection

$$A^{2l_1, 2l_2}_{2n; h_1, -h_2}[-h_2, \ h_1] \equiv A^{2l_2, 2l_1}_{2n; h_2, -h_1}[h_2, \ -h_1].$$

THEOREM 9. *For* $h_1, h_2 > 0$,

(40) $A^{2l_1, 2l_2}_{\sqrt{\zeta}; h_1, -h_2}[-h_2, \ -h_2] =$ *expressions* (29) *with the subscripts* 1 *and* 2 *interchanged.*

This follows on observing that by reflection

$$A^{2l_1, 2l_2}_{2n; h_1, -h_2}[-h_2, \ -h_2] \equiv A^{2l_2, 2l_1}_{2n; h_2, -h_1}[h_2, \ h_2].$$

Hence

(41) $$A^{2l_1, 2l_2}_{\sqrt{\zeta}; h_1, -h_2} = \textit{expressions } (29)+(33)+(33)+(40).$$

6. A_{2n} paths crossing three lines

In this section we consider only A_{2n} paths in the context of their crossing three lines and prove

THEOREM 10. *For* $h_i > 0$, $i = 1, 2$,

$$(42)\quad A^{2l_1, 2l, 2l_2}_{\sqrt{\zeta}; h_1, 0, -h_2}(S_1 = +1) = \omega^{2l_1+2l+2l_2+1} \frac{(1-\omega^{h_1})^{l_1+l+1}(1-\omega^{h_2})^{l+l_2}}{(1-\omega^{h_1+2})^{l_1+l+1}(1-\omega^{h_2+2})^{l+l_2}} \cdot$$

$$\cdot \sum_{i>0} \sum_{j>0} \binom{l_1-1}{i-1}\binom{l+1}{i}\binom{l}{j}\binom{l_2-1}{j-1} \omega^{i(h_1-2)+j(h_2-2)} \left(\frac{1-\omega^2}{1-\omega^{h_1}}\right)^{2i} \left(\frac{1-\omega^2}{1-\omega^{h_2}}\right)^{2j},$$

$$(43)\quad A^{2l_1, 2l, 2l_2}_{\sqrt{\zeta}; h_1, 0, -h_2}(S_1 = -1) = \textit{expression (42) with subscripts 1 and 2 interchanged.}$$

$$(44)\quad A^{2l_1, 2l, 2l_2}_{\sqrt{\zeta}; h_1, 0, -h_2} = \textit{expression (42)} + \textit{expression (43)}.$$

PROOF. An $A^{2l_1, 2l, 2l_2}_{2n; h_1, 0, -h_2}$ with $S_1 = +1$ has $S_{2n-1} > 0$.
It has

$$(l+1)S^+, \quad l_1 S^+_{h_1}, \quad (l_2-1) S^+_{-h_2},$$

$$lS^-, \quad (l_1-1) S^-_{h_1}, \quad l_2 S^-_{-h_2}.$$

(i) Suppose that out of the $(l+1)S^+$, i are $S^+(h_1+1^+)$, the kth one crossing $y = h_1$ $2l_{1k}$ times, so that $\sum_{k=1}^{i} l_{1k} = l_1$ and (ii) out of the lS^-, j are $S^-(-h_2, -1^+)$, the sth crossing $y = -h_2$ $2l_{2s}$ times such that $\sum_{s=1}^{j} l_{2s} = l_2$. Hence

$$(45)\quad A^{2l_1, 2l, 2l_2}_{\sqrt{\zeta}; h_1, 0, -h_2}(S_1 = +1, iS^+(h_1+1^+), jS^-(-h_2-1^+)) =$$

$$= \binom{l+1}{i}\binom{l}{j} \Sigma_1 \prod_{k=1}^{i} A^{+, 2l_{1k}}_{\sqrt{\zeta}; h_1} \Sigma_2 \prod_{s=1}^{j} A^{-, 2l_{2s}}_{\sqrt{\zeta}; -h_2} \{A^+_{\sqrt{\zeta}}(h_1+1^-)\}^{l+1-i} \{A^-_{\sqrt{\zeta}}(-h_2-1^-)\}^{l-j},$$

the summations Σ_1, Σ_2 being respectively over the ranges

$$\sum_{k=1}^{i} l_{1k} = l_1 \quad \text{and} \quad \sum_{s=1}^{j} l_{2s} = l_2,$$

since

$$A^{-, 2l_{2s}}_{\sqrt{\zeta}; -h_2} \equiv A^{+, 2l_{2s}}_{\sqrt{\zeta}; h_2} \quad \text{and} \quad A^-_{\sqrt{\zeta}}(-h_2-1^-) \equiv A^+_{\sqrt{\zeta}}(h_2+1^-),$$

(45), on using [2·1] and (6), and then summing it over i and j, leads to (42).

To establish (43) we observe that an $A^{2l_1, 2l, 2l_2}_{2n; h_1, 0, -h_2}$ with $S_1 = -1$ (and consequently $S_{2n-1} < 0$) becomes, on rotating through 180°, an $A^{2l_2, 2l, 2l_1}_{2n; h_2, 0, -h_1}$ with $S_1 = +1$ (and $S_{2n-1} > 0$). Hence (42) with the subscripts 1 and 2 interchanged leads to (43).

THEOREM 11. *For* $h_i > 0$, $i = 1, 2$,

$$A^{2l_1\, 2l-1,\, 2l_2}_{\sqrt{\zeta};\, h_1,\, 0,\, -h_2} = 2\omega^{2l_1+2l+2l_2} \frac{(1-\omega^{h_1})^{l_1+l}(1-\omega^{h_2})^{l+l_2}}{(1-\omega^{h_1+2})^{l_1+l}(1-\omega^{h_2+2})^{l+l_2}} \cdot \tag{46}$$

$$\cdot \sum_{i>0} \sum_{j>0} \binom{l}{i}\binom{l}{j}\binom{l_1-1}{i-1}\binom{l_2-1}{j-1} \omega^{i(h_1-2)+j(h_2-2)} \left(\frac{1-\omega^2}{1-\omega^{h_1}}\right)^{2i} \left(\frac{1-\omega^2}{1-\omega^{h_2}}\right)^{2j}.$$

PROOF. An $A^{2l_1,\, 2l-1,\, 2l_2}_{2n;\, h_1,\, 0,\, -h_2}$ consists of

$$lS^+, \quad (l_1-1)\, S^-_{h_1}, \quad (l_2-1)\, S^+_{-h_2},$$

$$lS^-, \quad l_1 S^+_{h_1}, \quad l_2 S^-_{-h_2},$$

since S_1 may be $+1$ as well as -1, (46) follows on using arguments as in Theorem 10.

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SIGNIERTE ZELLENZERLEGUNGEN. I

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Einleitung

In den letzten zwei Jahrzehnten befaßten sich mehrere Verfasser mit solchen Zerlegungen der Sphäre, in denen mit jeder Ecke p Kanten inzidieren (p-Kantzerlegung) und die Anzahl der Ecken eines jeden Flächenstücks ein Vielfaches der Zahl q ist (multi-q-gonale Fläche). Eine Eberhardsche Fragestellung ([4] S. 84, [9] S. 291) beantwortend, zeigte MOTZKIN [16], daß die Flächenanzahl von Dreikantzerlegungen mit lauter multitrigonalen Flächen stets eine gerade Zahl ist (s. auch [7] S. 733 sowie [8], [14]). GALLAGHER (s. [17]) verallgemeinert diese Behauptung, indem er Teilbarkeitseigenschaften der Kantenanzahlen von Dreikantzerlegungen mit lauter multi-q-gonalen Flächen bestimmt ($q=3, 4, 5$). Zur Behandlung der Teilbarkeitsprobleme formuliert MOTZKIN [17] einen Abbildungssatz bezüglich solcher Zerlegungen, die aus lauter multi-q-gonalen Flächen bestehen und in denen mit einer jeden Ecke eine durch p teilbare Anzahl von Kanten inzidieren. GRÜNBAUM ([9] Kapitel 13. 4) untersucht die Teilbarkeitseigenschaften der Flächenanzahlen auch bei solchen Dreikantzerlegungen, wo alle Flächen mit Ausnahme von zwei multi-q-gonal sind ($q=3, 4$). (S. auch [3], [15].) GRÖTZSCH [6], [7] und HAUSCHILD [10], [11] behandeln verschiedene Signierungs- und Färbungsprobleme bezüglich Drei- und Vierkantzerlegungen mit lauter multi-q-gonalen Flächen. In der vorliegenden Arbeit wollen wir diese Untersuchungen in mehrere Richtungen weiterführen. Es werden „signierte" Zerlegungen betrachtet, d.h. Zerlegungen, deren Ecken bzw. Flächen mit Signaturen versehen sind. Solche Zerlegungen wurden zuerst — in Zusammenhang mit der Vierfarbenvermutung — von HEAWOOD [12] untersucht. Von einer wohlbekannten Heawoodschen Behauptung ausgehend (s. § 1), formulieren wir in § 2 mehrere Abbildungssätze. Diese — dem Wesen nach sehr einfachen — Behauptungen ermöglichen uns durch Anwendung elementarer kombinatorisch-topologischer Methoden eine einheitliche und durchsichtige Behandlung der erwähnten Probleme. (Es sei bemerkt, daß unser Satz (2. 8) mit dem erwähnten Motzkinschen Abbildungssatz in engem Zusammenhang steht.) § 3 enthält die Beweise der Abbildungssätze und eine Anwendung des Satzes (2. 8) für Signierungs- und Färbungsprobleme. In § 4 wird der Begriff „Abbildungsgrad" erklärt und mittels dieses ein allgemeiner Teilbarkeitssatz bezüglich der Flächenanzahlen bewiesen (Satz (4. 4)). Es wird ferner an einem Spezialfall die Grundidee jenes Verfahren vorgeführt, das die Anwendung unserer Ergebnisse auf den erwähnten Grünbaumschen Problemenkreis ermöglicht.

In den Fortsetzungen dieser Arbeit wollen wir uns mit weiteren Verallgemeinerungen der Teilbarkeitssätze sowie mit einigen anderen Eigenschaften der signierten Zerlegungen von geschlossenen und berandeten Flächen befassen.

§ 1. Umformung einer Heawoodschen Behauptung

In [12] betrachtet HEAWOOD solche Eckensignierungen von Dreikantzerlegungen, wo jeder Ecke eine der Zahlen $+1$ und -1 als Signatur zugeordnet ist. Er fand, daß eben jenen Signierungen eine besondere Bedeutung zukommt, bei welchen für eine jede Fläche der Zerlegung die Summe der Signaturen ihrer Ecken durch 3 teilbar ist. Nennt man solche Signierungen kurz 3-regulär, so läßt sich die in der Einleitung erwähnte Heawoodsche Behauptung folgendermaßen formulieren:

(1. 1) *Die Flächen einer Dreikantzerlegung der Sphäre lassen sich dann und nur dann mit vier Farben zulässig färben, wenn die Zerlegung eine 3-reguläre Eckensignierung besitzt.* (Die Färbung der Flächen heißt zulässig, wenn je zwei Flächen mit gemeinsamer Kante stets verschiedene Farben bekommen.)

Wir haben nun gefunden, daß der Dual dieser Behauptung sich in sehr natürlicher Weise in einen Abbildungssatz verwandeln läßt. Dieser besagt nämlich folgendes:

(1. 2) *Die Ecken einer Triangulation (Dreieckzerlegung) der Sphäre lassen sich dann und nur dann mit vier Farben zulässig färben, wenn die Zerlegung eine 3-reguläre Flächensignierung besitzt.*

Dabei wird die Färbung der Ecken dann zulässig genannt, wenn je zwei mit einer Kante verbundene Ecken stets verschiedene Farben erhalten; eine 3-reguläre Flächensignierung bedeutet eine solche Signierung der Flächen mit den Zahlen $+1$ und -1, wo die Summe der Signaturen der mit einer jeden Ecke inzidenten Flächen durch 3 teilbar ist. Um aus (1. 2) einen Abbildungssatz zu erhalten, muß man nur die vier Farben als die vier Ecken eines Tetraeders T auffassen. Eine Färbung der Ecken der Triangulation K bedeutet dann eine Abbildung φ der Eckmenge von K in die Eckmenge von T. Haben je zwei Dreiecksflächen von K mit nicht leerem Durchschnitt entweder eine Ecke oder eine Kante gemeinsam, so kann man K als einen zweidimensionalen simplizialen Zellenkomplex auffassen (s. [1]). Wird auch die ebenfalls mit T bezeichnete Zerlegung der Oberfläche von T als ein solcher Komplex betrachtet, so bedeutet die Zulässigkeit der Färbung, daß φ eine *entartungsfrei* simpliziale Abbildung von K in T ist. (Durch eine entartungsfreie simpliziale Abbildung wird jedes Simplex von K auf ein gleichdimensionales Simplex von T abgebildet.) Durch eine solche Abbildung φ wird jedoch — vorausgesetzt, daß die Trägerflächen von K und T irgendwie orientiert sind — eine „natürliche" Flächensignierung für K induziert, nämlich diejenige, wo eine jede positiv orientierte Fläche X von K die Signatur $+1$ oder -1 erhält, je nachdem das Bild von X positiv oder negativ orientiert ist. Wie nun leicht ersichtlich, ist diese Signierung 3-regulär. Es besteht jedoch auch die folgende Umkehrung dieser Behauptung: Liegt eine 3-reguläre Flächensignierung σ für K vor, so gibt es eine solche entartungsfreie simpliziale Abbildung φ von K in T, daß die durch φ induzierte Signierung von K mit σ zusammenfällt. Um die indirekte Fassung zu vermeiden, führen wir den Begriff „signaturtreue Abbildung" ein. Es bezeichne K_σ die mit der Flächensignierung σ versehene Triangulation K. Eine entartungsfreie simpliziale Abbildung φ von K_σ in T heißt eine signaturtreue simpliziale Abbildung von K_σ in T, wenn das Bild einer jeden positiv orientierten Fläche X von K_σ positiv oder negativ orientiert ist, je nachdem ob die Signatur von X gleich $+1$ oder -1 ist. Die angedeutete Umformung von (1. 1) bzw. (1. 2) lautet dann wie folgt:

(1. 3) *Es sei K eine Triangulation der (orientierten) Sphäre und T die Triangulation der (orientierten) Tetraederoberfläche.*

a) Ist σ eine 3-reguläre Flächensignierung von K, so gibt es eine signaturtreue simpliziale Abbildung von K_σ in T.

b) Ist φ eine entartungsfreie simpliziale Abbildung von K in T, so ist die durch φ induzierte Flächensignierung von K 3-regulär.

(In der Tat sagt (1. 3) etwas mehr aus als (1. 2); es enthält jedoch nichts Neues da die bekannten Beweise von (1. 1) zugleich auch die Bestätigung für (1. 3) liefern.)

§ 2. Formulierung der Abbildungssätze

(2. 1) Zur Formulierung, sowie zu den späteren Untersuchungen benötigen wir mehrere Erklärungen und Bezeichnungen. Unsere Betrachtungen beziehen sich ausschließlich auf Zellenzerlegungen von orientierbaren Flächen. Die „Trägerflächen" können geschlossen oder berandet sein und sind stets als topologische Gebilde anzusehen. Die Oberfläche eines konvexen Polyeders wird also auch als zweidimensionale Sphäre, ein Vieleck auch als (abgeschlossene) Kreisscheibe betrachtet. Eine (zweidimensionale) Zelle der Zerlegung K, zusammen mit ihrem Rande, heißt eine *Fläche (Flächenstück)* von K. Es wird stets vorausgesetzt, daß jedes Flächenstück das topologische Bild eines sphärischen n-Ecks ($n \geqq 2$) ist. Die Bilder der Ecken und Seiten des n-Ecks sind die Ecken und Kanten des Flächenstücks, das selbst als ein n-Eck bezeichnet wird. (Schlingen werden also nicht zugelassen.) Der Durchschnitt zweier Flächen einer Zerlegung kann nur gemeinsame Ecken und Kanten der beiden Flächen enthalten. Enthält er eine gemeinsame Kante, so heißen die Flächen *benachbart.* Zwei Ecken, die durch eine Kante verbunden sind, heißen ebenfalls benachbart. Die am Rande der Trägerfläche liegenden Ecken und Kanten der Zerlegung K sind die *Randecken* und *Randkanten* von K.

Wir wollen voraussetzen, daß jede vorkommende Trägerfläche F mit einer — wenn nicht anderes gesagt wird — beliebig gewählten Orientierung versehen ist. Bezüglich dieser kann dann jedes Flächenstück einer Zerlegung von F entweder positiv oder negativ orientiert werden. Bezeichnet X ein orientiertes Flächenstück, so wird durch

$$X=[x_1 x_2 \dots x_p] \qquad (p \geqq 3)$$

ausgedrückt, daß $x_1, x_2, \dots, x_p$ die Ecken von X sind und die Reihenfolge $x_1, x_2, \dots, x_p$ der Orientierung von X entspricht. $-X$ bzw. $|X|$ bezeichnet dann dasselbe Flächenstück mit entgegengesetzter, bzw. ohne Orientierung. ($-X=[x_p x_{p-1} \dots x_1]$.) Die Menge der positiv orientierten Flächen der Zerlegung K wird mit $\mathscr{F}_+(K)$ bezeichnet.

Sind alle Flächen einer Zerlegung p-Ecke, so heißt sie *p-Eckzerlegung.* Eine Ecke, mit der genau q Kanten inzidieren, heißt *q-valent*; ist mit ihr eine durch q teilbare Anzahl von Kanten inzident, so heißt sie *multi-q-valent.* In unseren Untersuchungen werden nur p-Eckzerlegungen mit $p \geqq 3$ betrachtet. (Der Fall $p=2$ ist von geringen Interesse und seine Einbeziehung würde eine Erschwerung unserer Ausdruckweise verursachen.)

Wir bemerken, daß die betrachteten Zerlegungen auch als zweidimensionale Zellenkomplexe aufgefaßt werden können; eine Möglichkeit, von der gelegentlich Gebrauch gemacht wird.

(2. 2) Die Rolle der Triangulation der Tetraederoberfläche wird in unseren Verallgemeinerungen durch das reguläre Mosaik $\{p, q\}$ übernommen ($p \geqq 3$, $q \geqq 2$; s. [2], [5]). $\{p, q\}$ ist eine q-valente p-Eckzerlegung der Sphäre, der Euklidischen oder der hyperbolischen Ebene, je nachdem ob in der Relation $(p-2)(q-2) \lesseqqgtr 4$ das Zeichen $<$, $=$ oder $>$ besteht. Es kann nur in den beiden letzten Fällen vorkommen, daß die Trägerfläche einer von uns betrachteten Zerlegung weder geschlossen noch berandet ist. Entsprechend unserer vorherigen Voraussetzung soll auch die Trägerfläche des Mosaiks $\{p, q\}$ mit irgendeiner Orientierung versehen sein. Die Menge der positiv orientierten Flächen (Zellen) von $\{p, q\}$ wird mit $\mathscr{F}_+\{p, q\}$ bezeichnet. Es sei noch bemerkt, daß zufolge der Annahme $p \geqq 3$, zwei Ecken in $\{p, q\}$ höchstens durch eine Kante verbunden sind. Eine Kante wird daher durch ihre Endpunkte eindeutig bestimmt. Falls $q > 2$ ist, sind auch die Flächen von $\{p, q\}$ durch ihre Eckpunkte eindeutig bestimmt. Im Falle $q=2$ besteht $\{p, q\}$ aus zwei p-Eckflächen A und $\bar{A}$. Sind $A, \bar{A} \in \mathscr{F}_+\{p, q\}$, so können $A=[a_1 a_2 \dots a_p]$ und $\bar{A}=[a_p a_{p-1} \dots a_1]$ gesetzt werden.

(2. 3) Wenn nicht anderes gesagt wird, wollen wir im folgenden unter einer Signierung stets eine Signierung der Flächenstücke einer Zerlegung K verstehen, und zwar eine solche, wo als Signaturen nur die Zahlen $+1$ und -1 zugelassen sind (Heawoodsche Signierung). Die Signatur, die durch die Signierung σ der Fläche X zugeteilt ist, wird mit $\sigma(X)$, die signierte Zerlegung mit K_σ bezeichnet. Sind sämtliche vorkommenden Signaturen gleich $+1$, so heißt σ *vollpositiv*.

Die zur Ecke x gehörige *Signatursumme* $s_\sigma(x)$ bedeutet die Summe der Signaturen der mit x inzidenten Flächen. Ist x eine innere Ecke von K und σ die vollpositive Signierung, so fällt der Wert $s_\sigma(x)$ mit der Anzahl der mit x inzidenten Kanten (von K), d.h. mit der Valenz bzw. dem Grad von x, zusammen. Ist x eine innere Ecke von K und besteht

$$s_\sigma(x) \equiv 0 \pmod q,$$

so heißt die Signierung σ in der Ecke x *q-regulär* und x eine *q-reguläre Ecke* von K_σ. Ist σ in jeder inneren Ecke von K q-regulär, so heißt sie eine *q-reguläre Signierung* von K.

(2. 4) Es sei K eine p-Eckzerlegung und φ eine derartige Abbildung der Eckmenge von K in die Eckmenge des Mosaiks $\{p, q\}$, daß zu einer jeden orientierten Fläche $X=[x_1 x_2 \dots x_p]$ von K eine orientierte Fläche $A=[a_1 a_2 \dots a_p]$ von $\{p, q\}$ mit $\varphi(x_i)=a_i$ $(i=1, 2, \dots, p)$ existiert. Ist $q>2$, so ist nach (2. 2) A durch X eindeutig bestimmt. Es kann dann $A=\varphi(X)$ gesetzt werden. Für $q=2$ gibt es in $\{p, 2\}$ außer A noch eine zweite Fläche $A'=[a_1 a_2 \dots a_p]$. (Ist A positiv orientiert, so ist A' negativ orientiert, und umgekehrt.) In diesem Falle soll die Abbildung der Ecken durch eine Abbildung der Flächen ergänzt werden, indem man einem jeden X eine der Flächen A und A' als Bild $\varphi(X)$ zuordnet. Wir wollen die so erklärte Abbildung φ als eine *entartungsfreie polygonale* Abbildung von K in $\{p, q\}$ bezeichnen. Es sei

bemerkt: Bei einer entartungsfreien polygonalen Abbildung ordnet sich einer jeden Kante k von K eine eindeutig bestimmte Bildkante $\varphi(k)$ von $\{p, q\}$ zu.

Ist K eine p-Eckzerlegung, so wird durch eine entartungsfreie polygonale Abbildung φ von K in $\{p, q\}$ eine natürliche Signierung σ in K induziert, nämlich jene, bei der für jedes $X \in \mathscr{F}_+(K)$ die Signatur $\sigma(X)$ gleich $+1$ oder -1 ist, je nachdem ob $\varphi(X)$ positiv oder negativ orientiert ist.

Es bezeichne K_σ eine signierte p-Eckzerlegung. Unter einer *signaturtreuen polygonalen* Abbildung von K_σ in $\{p, q\}$ verstehen wir eine entartungsfreie polygonale Abbildung φ von K_σ in $\{p, q\}$, die folgende Eigenschaft besitzt: für jedes $X \in \mathscr{F}_+(K_\sigma)$ ist $\varphi(X)$ positiv oder negativ orientiert, je nachdem ob $\sigma(X)$ gleich $+1$ oder -1 ist (d.h. die durch φ induzierte Signierung fällt mit σ zusammen). Es sei hervorgehoben, daß (auch im Falle $q=2$) eine signaturtreue polygonale Abbildung durch die Angabe der Abbildung der Eckmenge von K_σ eindeutig bestimmt ist.

Wir sind nun in der Lage, die angekündigten Abbildungssätze formulieren zu können. Der erste von ihnen ist eine sehr naheliegende Verallgemeinerung von (1. 3). Mit Rücksicht auf die Anwendungen sollen die Verallgeimeinerungen der Behauptungen *a)* und *b)* von (1. 3) als gesonderte Sätze ausgesprochen werden:

(2. 5) SATZ. *Es seien p und q ganze Zahlen ($p \geqq 3$, $q \geqq 2$) und K sei eine p-Eckzerlegung der Sphäre. Ist σ eine q-reguläre Signierung von K, so gibt es eine signaturtreue polygonale Abbildung von K_σ in das Mosaik $\{p, q\}$.*

(2. 6) *Mit Benützung der Bezeichnungen von* (2. 5) *sei φ eine entartungsfreie polygonale Abbildung von K in $\{p, q\}$. Dann ist die durch φ induzierte Signierung von K q-regulär.*

Wie es sich leicht ergeben wird, bleibt die Behauptung (2. 6) auch dann bestehen, wenn die Sphäre durch eine beliebige andere Fläche ersetzt wird. Einfache Beispiele zeigen, daß das Gleiche für (2. 5) nicht gilt. Es bleibt jedoch auch die Behauptung von (2. 5) richtig, wenn man statt der Sphäre die ebenfalls einfach zusammenhängende Kreisscheibe nimmt. Wegen der Anwendungen soll diese Feststellung auch als ein selbstständiger Satz formuliert werden.

(2. 7) SATZ. *Eine q-regulär signierte p-Eckzerlegung der (abgeschlossenen) Kreisscheibe kann stets durch eine signaturtreue polygonale Abbildung in das Mosaik $\{p, q\}$ abgebildet werden ($p \geqq 3$, $q \geqq 2$).*

Wir wollen — ebenfalls in Anbetracht der Anwendungen — einen Spezialfall von (2. 5) gesondert formulieren. Sind alle Ecken einer p-Eckzerlegung K der Sphäre multi-q-valent (es muß dann $(p-2)(q-2) < 4$ gelten), so ist die vollpositive Signierung von K q-regulär. Wird nun K mit dieser Signierung versehen, so kann man darauf (2. 5) anwenden. Wir erhalten so folgende Behauptung:

(2. 8) SATZ. *Sind alle Ecken einer p-Eckzerlegung K der Sphäre multi-q-valent, so kann die Eckmenge von K derart in die Eckmenge des Mosaiks $\{p, q\}$ abgebildet werden, daß die Bilder der Ecken einer jeden positiv orientierten Fläche $X=[x_1 x_2 \dots x_p]$ von K der Reihe nach mit den Ecken einer (von X abhängigen) positiv orientierten Fläche $A=[a_1 a_2 \dots a_p]$ von $\{p, q\}$ zusammenfallen ($p \geqq 3$, $q \geqq 2$, $(p-2)(q-2) < 4$).*

Im Falle $p=3$, $q=3$ gibt (2. 8) den Dual derjenigen bekannten Heawoodschen Behauptung an, wonach die Vierfarbenvermutung für Dreikantzerlegungen mit lauter multitrigonalen Flächen richtig ist [12]. Satz (2. 8) steht mit einem Spezialfall des Theorems 1 von [17] in engem Zusammenhang.

§ 3. Die Beweise der Abbildungssätze. Anwendungen

(3. 1) Zu den Beweisen benötigen wir einige weitere Erklärungen und Bezeichnungen. Die in unseren Betrachtungen vorkommenden Kanten sind meistens gerichtet. Sie werden mit den Buchstaben k und $\varkappa$ (auch mit Indizes und anderen Zeichen versehen) bezeichnet. $|k|$ bedeutet die ungerichtete Kante, die geeignet orientiert die Kante k ergibt. Sind die Ecken x und y nur durch eine Kante verbunden, so wird diese mit $|xy|=|yx|$ bezeichnet. xy bedeutet dann die gerichtete Kante mit Anfangspunkt x und Endpunkt y. Dieselbe Bezeichnungsweise wird auch dann benützt, wenn die Ecken x und y eventuell durch mehrere Kanten verbunden sind, aus dem Text jedoch klar hervorgeht, welche von diesen unter $|xy|$ bzw. xy zu verstehen ist. Ist $X=[x_1 x_2 \dots x_p]$ ein orientiertes Flächenstück, so heißt die mit X inzidente Kante $x_1 x_2$ bzw. $x_2 x_1$ in X positiv bzw. negativ orientiert.

(3. 2) Es sei K_σ eine signierte Zerlegung und $|k|$ und $|k'|$ zwei Kanten K_σ, die beide die Ecke x enthalten. Wir wollen den „Winkelwert" $\langle |k| x |k'| \rangle$ (in K_σ) definieren. Dazu sei angenommen, daß x der gemeinsame Anfangspunkt der gerichteten Kanten k und k' ist. Ist $k=k'$, so sagen wir, daß $\langle |k| x |k'| \rangle$ (in K_σ) existiert und gleich Null ist. Nun sei $k \neq k'$. Dann gibt es (entsprechend dem positiven Umlauf um die Ecke x) in K_σ höchstens eine Folge

$$k_0 X_1 k_1 \dots k_{j-1} X_j k_j \qquad (j \geqq 1,\ k_0=k,\ k_j=k'), \tag{1}$$

die abwechselnd aus *verschiedenen* Kanten und positiv orientierten Flächen von K_σ besteht und wobei x der gemeinsame Anfangspunkt der Kanten ist, k_{i-1} und k_i mit X_i inzident sind und in X_i k_{i-1} positiv, k_i negativ orientiert ist $(i=1, \dots, j)$. Existiert diese Folge, so wird sie der (kxk')-*Winkelraum* (von K_σ) genannt. Wir sagen dann, daß der Winkelwert $\langle |k| x |k'| \rangle$ (in K_σ) existiert und setzen

$$\langle |k| x |k'| \rangle = \sum_{i=1}^{j} \sigma(X_i). \tag{2}$$

Bezeichnen jetzt k und k' beliebig gerichtete, mit x inzidente Kanten, so wird

$$\langle kxk' \rangle = \langle |k| x |k'| \rangle$$

angenommen. Sind ferner in K_σ die Kanten xy und xz einwandfrei erklärt, so wird mit $k=xy$ und $k'=xz$

$$\langle yxz \rangle = \langle kxk' \rangle \tag{3}$$

gesetzt. Sind auch weiterhin k und k' mit x inzident, so kann behauptet werden, daß einer der Werte $\langle kxk' \rangle$ und $\langle k'xk \rangle$ stets vorhanden ist, und zwar, wenn x eine Randecke und $k \neq k'$ ist, so nur der eine, wenn x eine innere Ecke ist, so beide. Gilt im zweiten Falle $k \neq k'$, so besteht

$$\langle kxk' \rangle + \langle k'xk \rangle = s_\sigma(x). \tag{4}$$

Ist $X \in \mathscr{F}_+(K_\sigma)$ und $X=[x_1 x_2 \dots x_p]$, so existieren stets die Werte $\langle x_{i+1} x_i x_{i-1} \rangle$ und es ist

$$\langle x_{i+1} x_i x_{i-1} \rangle = \sigma(X) \qquad (i=1, \dots, p;\ x_0=x_p,\ x_{p+1}=x_1). \tag{5}$$

Wir werden die oben erklärten Winkelwerte auch bei dem Mosaik $\{p, q\}$ benützen. Dazu wird $\{p, q\}$ als eine mit der vollpositiven Signierung versehene, signierte Zerlegung betrachtet. Zufolge der Voraussetzung $p \geqq 3$ (s. (2. 2)) ist hier die Bezeichnungsweise $\langle xyz \rangle$ ohne Beschränkung anwendbar. Sind ab und ac Kanten von $\{p, q\}$, so ist es klar, daß bei der nich negativen Drehung ($<2\pi$), die ab in ac überführt, ein Drehwinkel von der Größe $\frac{2\pi}{q} \langle bac \rangle$ beschrieben wird. Es folgt daraus: Sind b, c, d mit a benachbarte Ecken von $\{p, q\}$, so gelten die folgenden Relationen unbeschränkt:

(6) $$\langle cab \rangle \equiv -\langle bac \rangle \quad (\operatorname{mod} q),$$

(7) $$\langle bac \rangle + \langle cad \rangle \equiv \langle bad \rangle \quad (\operatorname{mod} q).$$

Die folgende Aussage behauptet eine Invarianzeigenschaft der Winkelwerte.

(3. 3) *Es sei K_σ eine signierte p-Eckzerlegung und φ eine signaturtreue polygonale Abbildung von K_σ in $\{p, q\}$ ($p \geqq 3, q \geqq 2$). Sind dann die Kanten k und k' mit der Ecke x inzident und existiert (in K_σ) der Wert $\langle kxk' \rangle$, so gilt mit $\varphi(x) = a$, $\varphi(k) = \varkappa$, $\varphi(k') = \varkappa'$ die Relation*

(8) $$\langle kxk' \rangle \equiv \langle \varkappa a \varkappa' \rangle \quad (\operatorname{mod} q).$$

Beweis. Es darf angenommen werden, daß x der gemeinsame Anfangspunkt der Kanten k und k' ist. Besteht $k = k'$, so ist (8) trivial. Es sei $k \neq k'$, $k_0 X_1 k_1 \dots k_{j-1} X_j k_j$ ($k_0 = k, k_j = k'$) der (kxk')-Winkelraum von K_σ und $\varphi(k_i) = \varkappa_i$, $\varphi(X_i) = A_i$ ($i = 1, \dots, j$; $\varkappa = \varkappa_0$). Dann sind $\varkappa_{i-1}$ und $\varkappa_i$ mit A_i inzident und in A_i ist $\varkappa_{i-1}$ positiv, $\varkappa_i$ negativ orientiert. Zufolge der Signaturtreue von φ ist A_i positiv oder negativ orientiert, je nachdem ob $\sigma(X_i)$ gleich $+1$ oder -1 ist. Nach (5) und (6) gilt daher $\langle \varkappa_{i-1} a \varkappa_i \rangle \equiv \sigma(X_i)$ (mod q), $i = 1, \dots, j$. Daraus ergib sich nach (7)

$$\langle kxk' \rangle \equiv \sum_{i=1}^{j} \langle \varkappa_{i-1} a \varkappa_i \rangle \equiv \langle \varkappa a \varkappa' \rangle \quad (\operatorname{mod} q).$$

Aus (4), (8) und (6) folgt unmittelbar

(3. 4) *Bestehen dieselben Voraussetzungen, wie in* (3. 3), *und ist y eine innere Ecke von K_σ, so ist die Signierung von K_σ in y q-regulär.*

Nun ergibt (3. 4) sofort die Richtigkeit der Behauptung (2. 6). Ist nämlich σ die durch φ induzierte Signierung von K, so ist φ eine signaturtreue polygonale Abbildung von K_σ in $\{p, q\}$, und daher ist σ in jeder inneren Ecke von K q-regulär. (Offensichtlich kann dabei die Trägerfläche von K eine beliebige (orientierbare) Fläche sein.)

Von den Sätzen (2. 5) und (2. 7) beweisen wir zuerst (2. 7). Satz (2. 5) wird sich dann als unmittelbare Folge von (2. 7) ergeben.

(3. 5) Der Beweis von (2. 7) ist ein einfacher Induktionsschluß bezüglich der Flächenanzahl f der Zerlegung. Die Behauptung des Satzes ist im Falle $f = 1$ trivial. Es sei angenommen, daß sie für alle Zerlegungen mit einer Flächenanzahl $< f$ ($f > 1$) richtig ist, und bezeichne K_σ eine q-regulär signierte p-Eckzerlegung der Kreisscheibe F mit der Flächenanzahl f. Bekanntlich (s. [19] S. 137) enthält K_σ eine solche

Fläche Y, daß die Herausschneidung von Y aus F eine Kreisscheibe F' zurückläßt. Die zu F' gehörigen Flächen von K_σ bilden eine q-regulär signierte p-Eckzerlegung K'_σ von F'. K'_σ besteht aus $f-1$ Flächen. Die Orientierung von F' soll der Orientierung von F entsprechen, d.h. jede orientierte Fläche von K'_σ soll auf F' und F dasselbe Orientierungsvorzeichen erhalten. Die Randecken und Randkanten von K'_σ bilden ein (topologisches) n-Eck V ($n \geqq 2$). Es seien $v_1, v_2, \dots, v_n$ die Ecken von V, wobei im Falle $n > 2$ vorausgesetzt ist, daß die angegebene Reihenfolge der Ecken dem negativen Umlaufsinn des Randes entspricht. Wir nehmen an, daß $Y = [y_1 y_2 \dots y_p]$ positiv orientiert ist. Die gemeinsamen Ecken und Kanten von Y und K'_σ bilden einen Kantenzug U. Es darf angenommen werden, daß $v_1, v_2, \dots, v_j$ eben die Ecken von U sind und $y_i = v_i$ ($i = 1, \dots, j$; $2 \leqq j \leqq p$) besteht. Laut der Induktionsannahme gibt es eine signaturtreue polygonale Abbildung φ von K'_σ in $\{p, q\}$. Es sei $\varphi(v_i) = a_i$ ($i = 1, \dots, j$).

Ist $j > 2$, so sind $v_2, \dots, v_{j-1}$ innere Ecken von K_σ. Da σ q-regulär ist, bestehen nach (4) für jene Winkelwerte (in K_σ), die zu den benachbarten Kanten von U gehören,

$$\langle v_{i-1} v_i v_{i+1} \rangle + \langle v_{i+1} v_i v_{i-1} \rangle = s_\sigma(v_i) \equiv 0 \quad (\text{mod}\, q), \ i = 2, \dots, j-1.$$

Laut (5) ist $\langle v_{i+1} v_i v_{i-1} \rangle = \langle y_{i+1} y_i y_{i-1} \rangle = \sigma(Y)$, woraus $\langle v_{i-1} v_i v_{i+1} \rangle \equiv -\sigma(Y)$ (mod q) folgt. Nach (3. 3) gilt daher ($\langle v_{i-1} v_i v_{i+1} \rangle$ existiert auch in K'_σ und fällt mit dem in K_σ vorhandenen Wert zusammen)

$$\langle a_{i-1} a_i a_{i+1} \rangle \equiv -\sigma(Y) \quad (\text{mod}\, q), \ i = 2, \dots, j-1.$$

Laut (5) und (6) existiert dann eine (eindeutig bestimmte) orientierte Fläche $B = [b_1 b_2 \dots b_p]$ von $\{p, q\}$ derart, daß $b_i = a_i$ ($i = 1, \dots, j$) besteht und B positiv oder negativ orientiert ist, je nachdem ob $\sigma(Y)$ gleich $+1$ oder -1 ist.

Im Fall $j = 2$ ist die Existenz einer solchen Fläche B offensichtlich.

Besteht $j = p$, so gibt φ selbst eine Abbildung der Eckmenge von K_σ in die Eckmenge von $\{p, q\}$ an. Ist $j < p$, so wird φ durch die Ansätze $\varphi(y_i) = b_i$ ($i = j+1, \dots, p$) zu einer solchen Abbildung erweitert. Zufolge der oben festgestellten Eigenschaften von B, stellt φ in beiden Fällen eine signaturtreue polygonale Abbildung von K_σ in $\{p, q\}$ dar. Damit ist der Beweis von (2. 7) beendet.

(3. 6) Zum Beweis von (2. 5) sei K_σ eine q-regulär signierte p-Eckzerlegung der Sphäre F. Wir schneiden F entlang einer Kante $|k|$ von K_σ auf. Dann entstehen eine Kreisscheibe $\dot{F}$ und eine q-regulär signierte Zerlegung $\dot{K}_\sigma$ von $\dot{F}$. (Die Orientierung von $\dot{F}$ soll der Orientierung von F entsprechen.) Nach (2. 7) existiert eine signaturtreue polygonale Abbildung φ von $\dot{K}_\sigma$ in $\{p, q\}$. Offensichtlich ist aber φ auch eine signaturtreue polygonale Abbildung von K_σ in $\{p, q\}$. Damit ist die Richtigkeit des Satzes (2. 5) gezeigt worden.

Es ist eine bemerkenswerte Tatsache, daß die Annahme, wonach σ auch in den Endpunkten der Kante $|k|$ q-regulär ist, in unserem Beweis gar nicht benutzt wurde. Wird diese Eigenschaft von σ nicht vorausgesetzt, so ergibt die Existenz von φ zusammen mit (3. 4), daß σ sie dennoch besitzt. Es gilt daher die folgende Behauptung:

(3. 7) Satz. *Es sei K eine p-Eckzerlegung der Sphäre. Dann existiert keine solche Signierung von K, die mit Ausnahme einer Ecke oder zweier benachbarter Ecken in allen übrigen Ecken q-regulär ist.*

Im Falle $p=3$, $q=3$ ergibt (3. 7) die duale Behauptung eines Heawoodschen Satzes [13].

Wir erwähnen noch einen anderen Spezialfall von (3. 7). Dieser steht mit dem Satz (2. 8) in Zusammenhang. Es sei K eine solche p-Eckzerlegung der Sphäre, in der jede Ecke — eventuell mit Ausnahme der benachbarten Ecken x_1 und x_2 — multi-q-valent ist. Die vollpositive Signierung von K ist dann in jeder Ecke (eventuell mit Ausnahme von x_1 und x_2) q-regulär. Laut (3. 7) muß dann σ auch in x_1 und x_2 q-regulär sein. Demzufolge gilt:

(3. 8) *Es existiert keine solche p-Eckzerlegung der Sphäre, in der jede Ecke mit Ausnahme einer oder zweier benachbarter multi-q-valent ist* ($p\geqq 3$, $q\geqq 2$).

Die Behauptung (3. 8) ist in einem allgemeinen Satz von MALKEVITCH [15] enthalten (s. auch [9] Kapitel 13. 4 und [3]). Es sei noch bemerkt, daß man im Falle $(p-2)(q-2)<4$ die Behauptung (3. 7) aus (3. 8) herleiten kann.

(3. 9) In der Arbeit [6] befaßt sich GRÖTZSCH mit Ecken- und Kantensignierungsproblemen solcher Dreikantzerlegungen, in denen jede Fläche multi-q-gonal ist. Unter anderen wird in [6] dem Wesen nach folgendes behauptet: Für die betrachteten Zerlegungen erweist sich das reguläre Mosaik $\{q, 3\}$ $(2\leqq q\leqq 5)$ gewissermassen als Prototyp, indem seine Eigenschaften sich auf diese Zerlegungen übertragen. Wir haben nun gefunden, daß der wahre Grund dieses Verhaltens durch den Satz (2. 8) angegeben wird. Es soll dies in Zusammenhang mit einem Spezialfall des Eckensignierungssatzes von [6] gezeigt werden. Der Dual dieses Spezialfalles lautet wie folgt:

(3. 10) (GRÖTZSCH). *Es sei K eine Triangulation der Sphäre, in der jede Ecke multi-q-valent ist* ($q=2, 4, 5$). *Dann lassen sich die Flächen von K mit den Zahlen* $1, 2, \ldots, q$ *derart signieren, daß die erhaltene Signierung* σ *folgende Eigenschaft besitzt: Ist x eine beliebige Ecke von K und sind* $X_1, X_2, \ldots, X_{\lambda q}$ *die mit x inzidenten Flächen in derjenigen zyklischen Reihenfolge, in welcher sie bei x nebeneinander Platz nehmen, so sind die Zahlen* $1, 2, \ldots, q$ *in der Folge* $\sigma(X_1), \sigma(X_2), \ldots, \sigma(X_{\lambda q})$ *derart gleichverteilt, daß bei der Durchlaufung dieser Folge eine (von x abhängige) Permutation von* $1, 2, \ldots, q$ *sich* λ*-mal periodisch wiederholt.*

Nun läßt sich für die Mosaiken $\{3, 2\}$, $\{3, 4\}$ und $\{3, 5\}$ je eine dem Satz (3. 10) entsprechende Flächensignierung leicht angeben (und bestätigen, daß für $\{3, 3\}$ keine solche existiert). Wir nehmen nach der Wahl dieser Signierungen eine Abbildung φ, welche, entsprechend der Forderungen des Satzes (2. 8), die Triangulation K von (3. 10) in $\{3, q\}$ abbildet. Dann teilen wir einer jeden Fläche X von K die Signatur von $\varphi(X)$ zu. Haben nun x und $X_1, \ldots, X_{\lambda q}$ dieselbe Bedeutung wie in (3. 10), so folgt aus der Eigenschaft von φ, wonach mit X auch $\varphi(X)$ positiv orientiert ist, daß $\varphi(X_{qi+1}), \varphi(X_{qi+2}), \ldots, \varphi(X_{qi+q})$ eben mit den zur Ecke $\varphi(x)$ inzidenten Flächen $A_1, A_2, \ldots, A_q$ von $\{3, q\}$ zusammenfallen $(i=0, 1, \ldots, \lambda-1)$. Dabei gibt $A_1, A_2, \ldots$ $\ldots, A_q$ die bei $\varphi(x)$ vorhandene zyklische Reihenfolge der Flächen an. Bezeichnen $j_1, j_2, \ldots, j_q$ die Signaturen von $A_1, A_2, \ldots, A_q$, so kann behauptet werden, daß in der Folge der Signaturen von $X_1, \ldots, X_{\lambda q}$ die Folge $j_1, j_2, \ldots, j_q$ sich λ-mal periodisch wiederholt.

Der gleiche Gedankengang läßt sich auch für solche p-Eckzerlegungen K der Sphäre anwenden, in denen jede Ecke multi-q-valent ist oder die Flächen mit einer

q-regulären Heawoodschen Signierung versehen sind. Dabei können für die Mosaiken $\{p, q\}$ neben Flächensignierungen auch Kanten- und Eckensignierungen betrachtet werden. Man erhält so Flächen-, Kanten- und Eckensignierungssätze für K (siehe z. B. einige Sätze bzw. Teilaussagen derselben in [6], [10], [11]).

§ 4. Der Abbildungsgrad. Teilbarkeitseigenschaften der Flächenanzahlen

Zur Behandlung der in der Einleitung erwähnten Teilbarkeitsprobleme benötigen wir den Begriff des Abbildungsgrades. Die nachfolgende Erklärung stimmt dem Wesen nach mit der üblichen, für simpliziale Abbildungen von Pseudomannigfaltigkeiten gegebenen Definition überein. (Siehe [1] Vol. 2, S. 86—87.) Sie beruht auf der folgenden Behauptung.

(4. 1) *Es sei K eine p-Eckzerlegung der geschlossenen Fläche F und φ eine entartungsfreie polygonale Abbildung von K in $\{p, q\}$. Ist $A \in \mathscr{F}_+\{p, q\}$, so soll v_A^+ bzw. v_A^- die Anzahl derjenigen $X \in \mathscr{F}_+(K)$ bezeichnen, für die $\varphi(X) = A$ bzw. $\varphi(X) = -A$ besteht. Dann ist die Zahl $v_A^+ - v_A^-$ für jedes $A \in \mathscr{F}_+\{p, q\}$ die gleiche.*

Beweis. Es sei aa' eine beliebige (gerichtete) Kante von $\{p, q\}$. Wir bezeichnen mit $\mathscr{K}_{aa'}$ die Menge derjenigen (gerichteten) Kanten k von K, für die $\varphi(k) = aa'$ besteht. Es sollen A und A' die mit aa' inzidenten Flächen von $\mathscr{F}_+\{p, q\}$, X_k und X_k' die mit k inzidenten Flächen von $\mathscr{F}_+(K)$ bezeichnen, wobei aa' in A positiv, in A' negativ, k in X_k positiv, in X_k' negativ orientiert ist. Zufolge der Entartungsfreiheit von φ ist mit einer Fläche von K höchstens eine Kante von $\mathscr{K}_{aa'}$ inzident. Daher besteht für die Menge

$$\mathscr{F}_{aa'} = \{X_k : k \in \mathscr{K}_{aa'}\}$$

die Relation

$$|\mathscr{F}_{aa'}| = |\mathscr{K}_{aa'}|. \tag{1}$$

Ist nun $k \in \mathscr{K}_{aa'}$, so ist entweder $\varphi(X_k) = A$ oder $\varphi(X_k) = -A'$. Umgekehrt, wenn für ein $X \in \mathscr{F}_+(K)$ entweder $\varphi(X) = A$ oder $\varphi(X) = -A'$ gilt, so ist mit X genau eine Kante von $\mathscr{K}_{aa'}$ inzident und diese ist in X positiv orientiert, d.h. es besteht $X = X_k$ mit einem $k \in \mathscr{K}_{aa'}$. Daraus folgt nach (1)

$$v_A^+ + v_{A'}^- = |\mathscr{K}_{aa'}|. \tag{2}$$

Gleichfalls besteht

$$v_{A'}^+ + v_A^- = |\mathscr{K}_{a'a}|. \tag{3}$$

Man kann jedoch aus $\mathscr{K}_{aa'}$ die Menge $\mathscr{K}_{a'a}$ dadurch erhalten, daß man die Richtungen seiner Kanten umkehrt. Dies ergibt $|\mathscr{K}_{a'a}| = |\mathscr{K}_{aa'}|$, woraus nach (2) und (3)

$$v_A^+ - v_A^- = v_{A'}^+ - v_{A'}^- \tag{4}$$

folgt. Die Relation (4) gilt für jedes aus benachbarten Flächen bestehendes Flächenpaar A, A' von $\{p, q\}$. Man kann aber in $\{p, q\}$ von einer Fläche ausgehend jede andere mittels solcher Schritte erreichen, bei denen man stets zu benachbarten Flächen übergeht. Demzufolge ist (4) auch für nicht benachbarte A und A' richtig. Damit ist die Behauptung (4. 1) bewiesen.

Es wird nun die in (4. 1) erklärte und von der Fläche A unabhängige Zahl $\nu_A^+ - \nu_A^-$ als der Abbildungsgrad d_φ der Abbildung φ bezeichnet:

(5) $$d_\varphi = \nu_A^+ - \nu_A^-.$$

Die Summen
$$\nu^+(K) = \Sigma \nu_A^+ \quad \text{und} \quad \nu^-(K) = \Sigma \nu_A^-,$$
wobei A alle Flächen von $\mathscr{F}_+\{p, q\}$ durchläuft, ergeben die Anzahlen derjenigen Flächen X von $\mathscr{F}_+(K)$, deren Bilder positiv, bzw. negativ orientiert sind. (Da K stets nur aus einer endlichen Anzahl von Flächen besteht, enthält jede Summe nur eine endliche Anzahl von Null verschiedener Glieder.) Werden nun die Gleichungen (5) für alle $A \in \mathscr{F}_+\{p, q\}$ addiert, so ergibt sich, falls die Anzahl $f_{p,q}$ der Flächen von $\{p, q\}$ endlich ist,

(6) $$f_{p,q} \cdot d_\varphi = \nu^+(K) - \nu^-(K).$$

Ist $f_{p,q} = \infty$, d.h. gilt $(p-2)(q-2) \geqq 4$, so muß ein $A \in \mathscr{F}_+\{p, q\}$ mit $\nu_A^+ = 0$ und $\nu_A^- = 0$ existieren. Es ist dann $d_\varphi = 0$, und daraus folgt $\nu^+(K) - \nu^-(K) = 0$.

(4. 2) Es sei jetzt angenommen, daß die p-Eckzerlegung K mit einer q-regulären Signierung σ versehen ist und φ eine signaturtreue polygonale Abbildung von K_σ in $\{p, q\}$ bedeutet. Dann geben $\nu^+(K)$ und $\nu^-(K)$ die Anzahlen der mit $+1$ bzw. -1 signierten Flächen von K_σ an. Wir wollen diese Anzahlen mit $\nu_+(K_\sigma)$ und $\nu_-(K_\sigma)$ bezeichnen und

(7) $$\delta(K_\sigma) = \nu_+(K_\sigma) - \nu_-(K_\sigma)$$

setzen. Nach dem Vorangehenden besteht dann

(8) $$\delta(K_\sigma) = d_\varphi \cdot f_{p,q},$$

wobei dies im Falle $f_{p,q} = \infty$ die Bedeutung $d_\varphi = 0$ und $\delta(K_\sigma) = 0$ hat.

(4. 3) Kehren wir nun zur Relation (6) zurück. Wird durch die Abbildung φ jedes $X \in \mathscr{F}_+(K)$ auf ein $A \in \mathscr{F}_+\{p, q\}$ abgebildet, d.h. besteht $\nu^-(K) = 0$, so ergibt $\nu^+(K)$ die Flächenanzahl f von K an. Laut (6) ist dann

(9) $$f = d_\varphi \cdot f_{p,q}.$$

f ist also ein Vielfaches von $f_{p,q}$. Nach (2. 8) kann man daher folgenden Satz aussprechen:

(4. 4) Satz. *Sind alle Ecken einer p-Eckzerlegung K der Sphäre multi-q-valent, so ist die Flächenanzahl von K ein ganzzahliges Vielfaches der Flächenanzahl des regulären Mosaiks $\{p, q\}$ $(p \geqq 3, q \geqq 2)$.*

Im Falle $p = 3$, $q = 3$ bzw. $p = 3$, $q = 3, 4, 5$ ist die Behauptung von (4. 2) dem in der Einleitung erwähnten Motzkinschen bzw. Gallagherschen Satze äquivalent.

(4. 5) Wir wollen im Falle $p = 3$, $q = 3$ eine Erweiterung von (4. 4) beweisen. Es soll nämlich gezeigt werden, daß die Flächenanzahl f einer Triangulation K der Sphäre F auch dann noch durch $f_{3,3} = 4$ teilbar ist, wenn alle Ecken, mit Ausnahme von zwei, multitrivalent sind. Diese Behauptung ist einer Abschwächung eines Grünbaumschen Satzes äquivalent. (Siehe [9], Theorem 13. 4. 4 (3) sowie

[3], [15]. Der Dual des Grünbaumschen Satzes bezieht sich auch auf solche Zerlegungen, in denen einige Ecken desselben Flächenstückes zusammenfallen können.)

Es seien nun x_1 und x_2 die Ausnahmeecken und U sei ein Weg (einfacher Kantenzug) von K, der x_1 mit x_2 verbindet. Wir schneiden F entlang U auf. Dann entsteht aus F eine Kreisscheibe $\dot{F}$, aus K eine solche p-Eckzerlegung $\dot{K}$ von $\dot{F}$, in der alle inneren Ecken multitrivalent sind. Der Weg U wird, zufolge der Aufschneidung, in die Wege $\hat{U}$ und $\check{U}$ von $\dot{K}$ zerspaltet. $\hat{U}$ und $\check{U}$ bilden zusammen den Rand von $\dot{F}$. Wir nehmen nachher drei gesonderte Exemplare von $\dot{F}$ (bzw. $\dot{K}$) und unterscheiden diese durch die oberen Indizes $i=1, 2, 3$. Dieselbe Bezeichnung wird auch für die Exemplare der Wege $\hat{U}$ und $\check{U}$ benutzt. Danach fügen wir $\dot{F}^i$ und $\dot{F}^{i+1}$ entlang der Wege $\hat{U}^i$ und $\check{U}^{i+1}$ zusammen ($i=1,2,3$; $\dot{F}^4=\dot{F}^1$, $\check{U}^4=\check{U}^1$). Es entsteht so eine geschlossene Fläche $\bar{F}$ und eine p-Eckzerlegung $\bar{K}$ von $\bar{F}$. ($\bar{K}$ ist ein Überlagerungskomplex von K.) Es ist leicht ersichtlich, daß $\bar{F}$ mit der Sphäre homeomorph ist und alle Ecken von $\bar{K}$ multitrivalent sind. Es kann daher (4. 4) auf $\bar{K}$ angewendet werden. Dies ergibt, daß die Flächenanzahl $\bar{f}$ von $\bar{K}$ durch 4 teilbar ist. Da ferner $\bar{f}=3f$ besteht, muß auch f durch 4 teilbar sein.

Für Wertepaare p, q, die von $p=3$, $q=3$ verschieden sind, können die Teilbarkeitseigenschaften von f nicht so einfach bestimmt werden. Mittels einer geeigneten Verfeinerung des oben benützten Gedankenganges kann jedoch auch dieses Problem sowie eine Verallgemeinerung derselben behandelt werden. Dies möchten wir in einer Fortsetzung unserer Arbeit mitteilen.

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ON CERTAIN LINEAR OPERATORS. I

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1. Introduction

The object of this paper is to prove some theorems, which will be the complements of the theorems of the paper [1]. The proof of some theorems will be a modification of the ideas of the above mentioned paper. In these cases we give only a sketch of the proof with reference to the corresponding parts of [1].

We begin with some notations.

Let us denote by $\omega(t)$ a function with the following properties.

(i) $\omega(t)>0$ for $t>0$, $\omega(T)\geqq\omega(t)$ if $T\geqq t$, $\omega(t)$ is a continuous function for $t\geqq 0$,

(ii) $\dfrac{t^m}{\omega(t)}$ is a monotonous increasing function for $t\geqq 0$,

(iii) $\lim\limits_{t=+0}\dfrac{t^m}{\omega(t)}=0$ ($m\geqq 1$, fixed integer).

Let $[a, b]$ be an arbitrary finite interval, $a\leqq x_{1n}<x_{2n}<\cdots<x_{pn}\leqq b$ $(n=1, 2, \dots)$, where

$$p=p(n) \quad\text{and}\quad \overline{\lim_{n\to\infty}}\, p(n)=\infty. \tag{1. 1}$$

Let us denote by $C_m^{[a,b]}(\omega)$ the class of all continuous functions on $[a, b]$ for which

$$\omega_m(f;t)\leqq a_m(f)\,\omega(t) \tag{1. 2}$$

holds. Here $\omega_m(f;t)$ is the modulus of smoothness of order m of $f(x)$, $a_m(f)$ depends only on $f(x)$, $\omega(t)$ is defined by (i), (ii) and (iii).

Let

$$l_{kn}(x) \qquad (n=1, 2, \dots;\ k=1, 2, \dots, p) \tag{1. 3}$$

be continuous functions on $[a, b]$,

$$L_n(f;x)=\sum_{k=1}^{p} f(x_{kn})\,l_{kn}(x), \tag{1. 4}$$

$$\lambda_n(x)=\sum_{k=1}^{p}\left|l_{kn}(x)\right|;\quad \lambda_n=\max_{a\leqq x\leqq b}\lambda_n(x), \tag{1. 5}$$

$$d_n=\min_{1\leqq k\leqq p-1}(x_{k+1,n}-x_{kn}). \tag{1. 6}$$

The main theorem proved in [1] reads as follows:

THEOREM 1. 1. *If* $\overline{\lim}_{n\to\infty} \lambda_n > 1$ *or* $\underline{\lim}_{n\to\infty} \lambda_n < 1$, *then there exists an* $f(x) \in C_m^{[a,b]}(\omega)$ *such that*[1]

$$\|L_{n_k}(f; x) - f(x)\| > \lambda_{n_k} \omega(d_{n_k}) \quad (k = 1, 2, \ldots). \tag{1.7}$$

(Here $0 < n_1 < n_2 < \ldots$ *are integers.)*

2. The next theorem shows that in a special case the conditions for $\omega(t)$ can be weakened. (See Theorem 2 in [2].) Namely we have the following

THEOREM 2. 1. *If* $\omega(t)$ *is defined only by* (i) *and* (ii), *further*

$$\overline{\lim_{n\to\infty}} \omega(d_n) \lambda_n = \infty \tag{2.1}$$

then there exists an $f(x) \in C_m^{[a,b]}(\omega)$ *such that*

$$\overline{\lim_{n\to\infty}} \|L_n(f; x)\| = \infty.$$

PROOF. For the sake of brevity let us denote $x_{kn}, l_{kn}(x), \ldots$ by $x_k, l_k(x), \ldots$, respectively. We shall refer to the formulae (1), (2), (3), ... of the paper [1] with the notations (1*), (2*), (3*), ..., respectively.

Let z_n be a point such that

$$\lambda_n = \lambda_n(z_n) = \sum_{k=1}^{p} |l_k(z_n)| \qquad (n = 1, 2, \ldots). \tag{2.2}$$

We define the continuous functions $g_n(x)$ as follows. Let

$$\begin{cases} g_n(x_k) = \operatorname{sign} l_k(z_n), \\ g_n(x_k) = 1 \quad \text{if} \quad l_k(z_n) = 0 \quad (k = 1, 2, \ldots, p), \end{cases} \tag{2.3}$$

furthermore

$$g_n(x) = \begin{cases} g_n(x_1) & \text{if} \quad a \leqq x < x_1 \\ g_n(x_p) & \text{if} \quad b \geqq x > x_p. \end{cases} \tag{2.4}$$

If $g_n(x_k) = g_n(x_{k+1})$, then let $g_n(x) = g_n(x_k)$ for $x \in [x_k, x_{k+1}]$.

On the other hand, if $g_n(x_k) \neq g_n(x_{k+1})$, let

$$g_n(x) = g_n(x_k)\left[1 - \frac{(2m-1)!!}{(2m-2)!!} \int_{-1}^{2\frac{x-x_k}{x_{k+1}-x_k}-1} (1-t^2)^{m-1}\, dt\right] \quad (x_k \leqq x \leqq x_{k+1}).$$

It is easy to see (see (17*)—(21*)) that

$$\omega_m(g_n; t) \leqq B_m d_n^{-m} t^m \qquad \left(0 < t \leqq \frac{b-a}{m}\right)^2, \tag{2.5}$$

$$\|g_n(x)\| \leqq 1 \qquad (n = 1, 2, \ldots), \tag{2.6}$$

$$\|L_n(g_N; x)\| \leqq \lambda_n \qquad (n, N = 1, 2, \ldots), \tag{2.7}$$

$$L_n(g_n; z_n) = \lambda_n \qquad (n = 1, 2, \ldots). \tag{2.8}$$

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[1] $\|g(x)\| = \max_{a \leqq x \leqq b} |g(x)|$.

[2] B_m depends only on m.

If there exists a fixed number N such that

$$\overline{\lim_{n\to\infty}} \|L_n(g_N; x)\| = \infty$$

then the theorem is proved; so we can suppose that

$$(2.9)\qquad \|L_n(g_N; x)\| \leqq C_N \qquad (n, N = 1, 2, \ldots).$$

Now we define the sequence $0 < n_1 < n_2 < \ldots$ of indices as follows.

$$(2.10)\qquad \omega(d_{n_k})\lambda_{n_k} \geqq k\cdot 2^k \qquad (k = 1, 2, \ldots),$$

$$(2.11)\qquad \omega(d_{n_k})\lambda_{n_k} \geqq 3\cdot 2^k \sum_{i=1}^{k-1} 2^{-i}\omega(d_{n_i})C_{n_i} \qquad (k = 2, 3, \ldots),$$

$$(2.12)\qquad \omega(d_{n_i}) \leqq \frac{1}{2}\omega(d_{n_{i-1}}) \qquad (i = 2, 3, \ldots).$$

These are possible, taking into account (2. 1), (1. 6) and (i). Let

$$(2.13)\qquad f(x) = \sum_{i=1}^{\infty} 2^{-i}\omega(d_{n_i})g_{n_i}(x).$$

Here the right hand series converges for all $a \leqq x \leqq b$ by virtue of (2. 12) and (2. 6). Evidently, $f(x)$ is a continuous function in $[a, b]$. At first we show that $f(x) \in C_m^{[a,b]}(\omega)$. Let t $\left(0 \leqq t \leqq \dfrac{b-a}{m}\right)$ be arbitrary. Then

$$(2.14)\qquad \omega_m(f; t) \leqq \sum_{i=1}^{\infty} 2^{-i}\omega(d_{n_i})\omega_m(g_{n_i}; t).$$

Let

$$(2.15)\qquad d_{n_{j+1}} < t \leqq d_{n_j}.$$

(If $d_{n_1} < t$, then $j = 0$).

By (2. 5) and (ii) we get

$$\sum_{i=1}^{j} 2^{-i}\omega(d_{n_i})\omega_m(g_{n_i}; t) \leqq B_m \sum_{i=1}^{j} 2^{-i}\omega(d_{n_i})d_{n_i}^{-m}t^m \leqq B_m \sum_{i=1}^{j} 2^{-i}\omega(t) = O[\omega(t)].$$

On the other hand, by virtue of (2. 6) and (i) we have

$$\sum_{i=j+1}^{\infty} 2^{-i}\omega(d_{n_i})\omega_m(g_{n_i}; t) \leqq 2^m \sum_{i=j+1}^{\infty} 2^{-i}\omega(d_{n_i}) \leqq 2^m \omega(d_{n_{j+1}}) \sum_{i=j+1}^{\infty} 2^{-i} = O[\omega(t)].$$

I.e., $f(x) \in C_m^{[a,b]}(\omega)$, as we stated.

Now we can easily prove our statement.

By (2. 13)

$$(2.16)\qquad L_{n_k}(f; z_{n_k}) = \sum_{i=1}^{\infty} 2^{-i}\omega(d_{n_i})L_{n_k}(g_{n_i}; z_{n_k}).$$

We know (see (2. 8)) that

$$(2.17)\qquad 2^{-k}\omega(d_{n_{kk}})L_{n_k}(g_{n_k}; z_{n_k}) = 2^{-k}\omega(d_{n_k})\lambda_{n_k}.$$

Using (2. 9) and (2. 11), we have

$$\text{(2. 18)} \qquad \sum_{i=1}^{k-1} 2^{-i}\omega(d_{n_i})\left|L_{n_k}(g_{n_i}; z_{n_k})\right| \leqq \sum_{i=1}^{k-1} 2^{-i}\omega(d_{n_i})C_{n_i} \leqq \frac{1}{3}2^{-k}\omega(d_{n_k})\lambda_{n_k}.$$

In virtue of (2. 7) and (2. 12) we obtain

$$\text{(2. 19)} \qquad \sum_{i=k+1}^{\infty} 2^{-i}\omega(d_{n_i})\left|L_{n_k}(g_{n_i}; z_{n_k})\right| \leqq \lambda_{n_k} \sum_{i=k+1}^{\infty} 2^{-i}\omega(d_{n_i}) \leqq \frac{1}{3}\lambda_{n_k}2^{-k}\omega(d_{n_k}).$$

I.e., we obtain by (2. 16)—(2. 19) and (2. 10)

$$L_{n_k}(f; z_{n_k}) \geqq \frac{1}{3}2^{-k}\omega(d_{n_k})\lambda_{n_k} \geqq \frac{k}{3} \qquad (k=1, 2, \ldots),$$

in accordance with our statement.

3. Now we turn to other generalizations. Although the results obtained here, elucidate from several points of view the behaviour of $L_n(f; x)$ in the whole interval $[a, b]$, they afford no lower estimations at a fixed point x in $[a, b]$. We may state the problem: whether a statement like Theorem 1. 1 is true if we investigate the behaviour of $L_n(f; x)$ at an arbitrary point x of $[a, b]$.

It can be shown by means of the method applied in the proof of Theorem 1. 1 that the following holds

THEOREM 3. 1. *Let x_0 be an arbitrary point in $[a, b]$. If $\overline{\lim}_{n\to\infty} \lambda_n(x_0) > 1$ or $\underline{\lim}_{n\to\infty} \lambda_n(x_0) < 1$, then there exists an $f(x) \in C_m^{[a,b]}(\omega)$ such that*

$$\text{(3. 1)} \qquad \left|L_{n_k}(f; x_0) - f(x_0)\right| > \lambda_{n_k}(x_0)\omega(d_{n_k}) \qquad (k=1, 2, \ldots).$$

(Here $0 < n_1 < n_2 < \ldots$ are integers.)

PROOF. The proof runs as in [1]. When

$$(\alpha) \quad \underline{\lim}_{n\to\infty} \lambda_n(x_0) = 1 - 2\varepsilon \qquad (\varepsilon > 0)$$

we can choose a sequence $0 < n_1 < n_2 < \ldots$ such that $\lambda_{n_k}(x_0) \leqq 1 - \varepsilon$ $(k=1, 2, \ldots)$, $d_{n_1} > d_{n_2} > \ldots$ and $\omega(d_{n_1}) < \varepsilon$. Then for $f(x) \equiv 1$ we obtain

$$f(x_0) - L_{n_k}(f; x_0) = 1 - \sum_{k=1}^{p} l_k(x_0) \geqq 1 - (1-\varepsilon) > \omega(d_{n_1}) \geqq \lambda_{n_k}\omega(d_{n_k}) \quad (k=1, 2, \ldots),$$

as we stated.

Let us consider the case

$$(\beta) \quad \overline{\lim}_{n\to\infty} \lambda_n(x_0) = \infty \quad \text{or} \quad \overline{\lim}_{n\to\infty} \lambda_n(x_0) = 1 + 2\varepsilon \qquad (\varepsilon > 0).$$

Let us define $g_n(x)$ as in Part 2, replacing z_n by x_0. We have (2. 5) and (2. 6), further

$$\text{(3. 2)} \qquad |L_n(g_N; x_0)| \leqq \lambda_n(x_0) \qquad (n, N=1, 2, \ldots),$$

$$\text{(3. 3)} \qquad L_n(g_n; x_0) = \lambda_n(x_0) \qquad (n=1, 2, \ldots).$$

If there exists a fixed number N and a sequence $0<n_1<n_2<\dots$ such that

$$|L_n(g_N;x_0)-g_N(x_0)|>\omega(d_n)\lambda_n(x_0) \qquad (n=n_1,n_2,\dots),$$

then the theorem is true; so we can suppose that

$$\text{(3. 4)} \qquad |L_n(g_N;x_0)-g_N(x_0)| \leqq \omega(d_n)\lambda_n(x_0) \quad \text{if} \quad n>M(N).$$

Now we define the sequence of indices $0<n_1<n_2<\dots$ as follows (see in [1]). Let

$$\text{(3. 5)} \qquad \lambda_{n_i}(x_0) \geqq 1+\varepsilon \qquad (i=1,2,\dots),$$

$$\text{(3. 6)} \qquad \omega(d_{n_1})\leqq q, \quad \omega(d_{n_{i+1}})\leqq q\omega(d_{n_i}) \quad (i=1,2,\dots), \text{ where } 0<q<1,$$

$$\text{(3. 7)} \qquad \sum_{i=1}^{j-1}\omega(d_{n_i})d_{n_i}^{-m} \leqq \omega(d_{n_j})d_{n_j}^{-m} \qquad (j=2,3,\dots),$$

$$\text{(3. 8)} \qquad n_{i+1}>M(n_i) \qquad (i=1,2,\dots).$$

Let

$$f(x) = Q\sum_{i=1}^{\infty}\omega(d_{n_i})g_{n_i}(x) \qquad (Q>1).$$

By our assumptions we have (cf. [1]) that[3]

$$\omega_m(f;t) = Q\sum_{i=1}^{\infty}\omega(d_{n_i})\omega_m(g_{n_i};t) = Q\left[\sum_{i=1}^{j}+\sum_{i=j+1}^{\infty}\right] \leqq Q\left(2B_m+\frac{2^m}{1-q}\right)\omega(t),$$

i.e., $f(x)\in C_m^{[a,b]}(\omega)$.

Now we can prove (3. 1).

Omitting the details, we obtain

$$L_{n_k}(f;x_0)-f(x_0) \geqq Q\Bigg[\omega(d_{n_k})L_{n_k}(g_{n_k};x_0)-\sum_{i=1}^{k-1}\omega(d_{n_i})\left|L_{n_k}(g_{n_i};x_0)-g_{n_i}(x_0)\right|-$$

$$-\sum_{i=k+1}^{\infty}\omega(d_{n_i})\left|L_{n_k}(g_{n_i};x_0)\right|-\sum_{i=k}^{\infty}\omega(d_{n_i})\left|g_{n_i}(x_0)\right|\Bigg] \geqq$$

$$\geqq Q\left[\omega(d_{n_k})\lambda_{n_k}(x_0)-\frac{q}{1-q}\omega(d_{n_k})\lambda_{n_k}(x_0)-\frac{q}{1-q}\omega(d_{n_k})\lambda_{n_k}(x_0)-\omega(d_{n_k})\frac{1}{1-q}\right] =$$

$$= \omega(d_{n_k})\left[Q\lambda_{n_k}(x_0)-2\lambda_{n_k}(x_0)\frac{Qq}{1-q}-\frac{Q}{1-q}\right].$$

As in [1], we can easily justify by (β) that

$$Q\lambda_{n_k}(x_0)-2\lambda_{n_k}(x_0)\frac{Qq}{1-q}-\frac{Q}{1-q}>\lambda_{n_k}(x_0)$$

with suitable Q and q.

So we perfectly proved our statement. —

When the order of magnitude of the distance of the neighbouring points x_{kn}

[3] j is defined by (2.15).

is equal $\left(\text{e.g. } O\left(\frac{1}{n}\right)\right)$, the expression d_n is perfectly characteristic of the whole interval $[a, b]$. But in many cases we wish to investigate such point-systems x_{kn} where the d_n characterizes only a subinterval of $[a, b]$ (e.g. choosing as fundamental points x_{kn} the zeros of the n-th Jacobi polynomials). This consideration gives the idea to improve Theorem 1. 1, in such direction.

Let us denote by $I_n = \{x_{j(n);n}; x_{l(n),n}\}$ $(1 \leqq j(n) < l(n) \leqq p)$ general intervals, further let $[I_n] = [x_{j(n),n}; x_{l(n),n}]$ and $(I_n) = (x_{j(n),n}; x_{l(n),n})$; $I_{n+1} \supseteqq I_n$ $(n = 1, 2, \dots)$

Let

$$d_n(I_n) = \min_{x_{kn},\, x_{k+1,n} \in [I_n]} (x_{k+1,n} - x_{kn}) \tag{3. 9}$$

such that

$$\lim_{n \to \infty} d_n(I_n) = 0 \tag{3. 10}$$

and

$$\lambda_n(I_n; x) = \sum_{\substack{k \\ x_{kn} \in (I_n)}} |l_{kn}(x)|. \tag{3. 11}$$

The following theorem holds.

Theorem 3. 2. *If $x_0 \in [a, b]$ and I_n are such that*

$$\overline{\lim_{n \to \infty}}\, \lambda_n(I_n; x_0) > 1 \quad \text{or} \quad \lim_{n \to \infty} \lambda_n(I_n; x_0) < 1$$

then there exists an $f(x) \in C_m^{[a,b]}(\omega)$ for which

$$|f(x_0) - L_{n_k}(f; x_0)| > \lambda_{n_k}(I_n; x_0)\, \omega(d_{n_k}(I_{n_k})) \qquad (k = 1, 2, \dots). \tag{3. 12}$$

(Here $0 < n_1 < n_2 < \dots$ are integers.)

Proof. The ideas are similar to that of [1]. The only essential difference is the definition of $g_n(x)$. Let

$$g_n(x_{kn}) = \operatorname{sign} l_{kn}(x_0) \quad \text{if} \quad x_{kn} \in (I_n). \tag{3. 13}$$

For $x \in [x_{kn}, x_{k+1,n}] \subset [I_n]$ we define $g_n(x)$ as in Theorem 2. 1.

Finally, let

$$g_n(x) = 0 \quad \text{if} \quad x \notin (I_n). \tag{3. 14}$$

Obviously

$$\omega_m(g_n; t) \leqq B_m d_n^{-m} t^m \left(0 < t \leqq \frac{b-a}{m}\right), \tag{3. 15}$$

$$|g_n(x)| \leqq 1, \tag{3. 16}$$

$$|L_n(g_N; x_0)| = \left| \sum_{k=1}^{p} g_N(x_{kn})\, l_{kn}(x_0) \right| \leqq \sum_{x_{kn} \in (I_n)} |l_{kn}(x_0)| = \lambda_n(I_n; x_0), \quad (N \geqq n), \tag{3. 17}$$

$$L_n(g_n; x_0) = \lambda_n(I_n; x_0). \tag{3. 18}$$

Repeating the proof of Theorem 3. 1 and replacing d_n and $\lambda_n(x_0)$ by $d_n(I_n)$ and $\lambda_n(I_n; x_0)$, respectively, we can easily prove (3. 12). —

4. Notes

a) As in [1], by means of the applied method we can prove the analogous trigonometric theorems.

Then $[a, b]$ is the interval $[0, 2\pi)$, $f(x)\in \tilde{C}_m(\omega)$ provided that, besides (1. 2), $f(x)$ is 2π-periodic, $l_{kn}(x)$ are continuous 2π-periodic functions and

$$d_n = \min_{1\leq k\leq p}(x_{k+1,n}-x_{kn}) \qquad (\text{Here } x_{p+1,n}\equiv x_{1n}+2\pi).$$

b) If $\overline{\lim\limits_{n\to\infty}}\,\lambda_n(x_0)\,\omega(d_n)=\infty$ or $\overline{\lim\limits_{n\to\infty}}\,\lambda_n(I_n;\,x_0)\,\omega\big(d_n(I_n)\big)=\infty$, then we may get rid of the condition (iii) in Theorem 3. 1 and Theorem 3. 2, respectively.

c) We can easily generalize the last theorem, if we consider the system of the distinct intervals

$$\bigcup_{k=1}^{s}\{x_{j(n,k),n};\, x_{l(n,k),n}\} \text{ instead of } I_n.$$

d) All theorems will hold, if we want to estimate the difference

$$f(x)-L_n^*(f;\,x)$$

where $L_n^*(f;\,x) = L_n(f;\,x)+F(x)$ and

$$F(x) = o\big[\lambda_n\omega(d_n)\big] \quad \text{or} \quad F(x_0) = o\big[\lambda_n(x_0)\,\omega(d_n)\big]$$

or $F(x_0) = o\big[\lambda_n(I_n;\,x_0)\,\omega\big(d_n(I_n)\big)\big]$, respectively.

e) With a little modification in the definition of $g_n(x)$ we can obtain the formulae

$$\lambda_n(\{a,\,b\};\,x_0)=\lambda_n(x_0),\; d_n(\{a,\,b\})=d_n$$

in addition to (3. 12). I.e., Theorem 3. 1 follows from Theorem 3. 2.

We remark that Theorem 3.1 isn't true if $\lim\limits_{n\to\infty}\lambda_n(x_0)=1$. We can be prove it if $L_n(f;\,x)=f(x_0)$.

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ISOTOPES OF SOME SPECIAL QUASIGROUP VARIETIES

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1. Introduction. Since every quasigroup is isotopic to a loop, it is of interest to see if one can characterize intrinsically, the classes of all quasigroups that are isotopic to some well known classes of loops. So far work on this problem has been restricted to studying conditions under which classes of quasigroups are related to groups in this way. EVANS in [5] showed that a quasigroup is isotopic to a group if and only if it satisfies a law:

$$((xP_1 \cdot yP_2)P_3 \cdot zP_4)P_5 = (xQ_1 \cdot (yQ_2 \cdot zQ_3)Q_4)Q_5$$

where P_i, Q_i, $i=(1, 2, 3, 4, 5)$ are permutations on the quasigroup. ACZÉL, BELOUSOV and HOSSZU in [2] give the general solution for the functional equation $(x1y)2z = x3(y4z)$, defined on a set Q which forms quasigroups under the operations $i=(1, 2, 3, 4)$, and show that the four quasigroups are isotopic to a group, unique up to automorphism. BELOUSOV in [3] showed that a quasigroup is isotopic to a group if and only if it satisfies a balanced identity. We will generalize, slightly, the notion of balanced identity and give a brief alternative proof, based upon the above result by EVANS.

We next apply the same technique to the quasigroup isotopes of M_1, M_2, and Moufang loops, obtaining identities which are necessary and sufficient conditions for quasigroups to be isotopic to these loops. It is no accident that our results are restricted to M_1, M_2, and Moufang loops. These are the best known of varieties of loops, other than groups which have the property that every loop isotope of a loop in the variety is also in the variety. It will be shown elsewhere that if a quasigroup variety is closed under isotopy, then it must be the class of all quasigroups isotopic to some loop in the subvariety of loops it contains.

2. Definitions. A *quasigroup* is a non-empty set, Q, which is closed with respect to three binary operations, multiplication $(\cdot)$, left division $(\backslash)$, right division $(/)$, and such that for all $x, y \in Q$,

1. $x \cdot (x \backslash y) = y$ 3. $(y / x) \cdot x = y$
2. $x \backslash (x \cdot y) = y$ 4. $(y \cdot x) / x = y$

A *loop* is a quasigroup which satisfies $x / x = y \backslash y$ for all x, y. Here we follow Evans [7].

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Let $(G, \cdot, \diagdown, \diagup)$ and $(H, \bigcirc, \obslash, \oslash)$ be quasigroups. $(G, \cdot, \diagdown, \diagup)$ is said to be *isotopic to* or an *isotope* of $(H, \bigcirc, \obslash, \oslash)$ if there exist one-to-one mappings α, β, γ of G upon H such that $x\alpha \bigcirc y\beta = (x \cdot y)\gamma$ [4].

Let F be the free quasigroup on $x_1, x_2, \ldots$ [7]. Let w be a word in F. The *exponent* $e_i(x_i)$ of x_i in w is defined as follows, [6].

(i) $e_i(x_i) = 1, \; e_i(x_j) = 0, \; i \neq j$

(ii) if w is $u \cdot v$, $e_i(w) = e_i(u) + e_i(v)$

if w is $u \diagdown v$, $e_i(w) = -e_i(u) + e_i(v)$

if w is $u \diagdown v$, $e_i(w) = e_i(u) - e_i(v)$

Let w_1 and w_2 be reduced words in F. $w_1 = w_2$ is a *generalized group identity*, if the following conditions hold.

(i) There exist distinguished generators x, y, z, occurring exactly once in both w_1 and w_2, such that w_1 contains a subword u which contains x and y, but not z; and w_2 contains a subword v which contains y and z, but not x.

(ii) The greatest common divisor, g, of $\{e_i(w_1 \diagdown w_2) | i = 1, 2, \ldots\}$ is not 1.

We note that every generalized group identity is satisfied by some non-trivial quasigroup since it is satisfied by every abelian group of exponent g.

Let Q be a quasigroup and P_i, Q_i, $(i = 1, 2, 3, 4, 5)$ be permutations of Q. The following laws will be called *generalized associative* laws.

A. $[(xP_1 \bigcirc_1 yP_2)P_3 \bigcirc_2 zP_4]P_5 = [uQ_1 \bigcirc_3 (vQ_2 \bigcirc_4 wQ_3)Q_4]Q_5$

B. $[(xP_1 \bigcirc_1 yP_2)P_3 \bigcirc_2 zP_4]P_5 = [(vQ_1 \bigcirc_3 wQ_2)Q_3 \bigcirc_4 uQ_4]Q_5$

C. $[zP_1 \bigcirc_1 (xP_2 \bigcirc_2 yP_3)P_4]P_5 = [uQ_1 \bigcirc_3 (vQ_2 \bigcirc_4 wQ_3)Q_4]Q_5$

where $\{\bigcirc_i | i = 1, 2, 3, 4\} = \{\cdot, \diagdown, \diagup\}$ and u, v, w is a permutation of x, y, z such that $\{v, w\} \neq \{x, y\}$. The following special case of A, where all operations occurring in the equation are taken to be the same, is due to EVANS, [5].

D. $[(xP_1 \bigcirc yP_2)P_3 \bigcirc zP_4]P_5 = [xQ_1 \bigcirc (yQ_2 \bigcirc zQ_3)Q_4]Q_5$

If Q is a quasigroup and c is a fixed element of Q, then we define the mappings R_c, L_c, T_c of Q into Q by

$$xR_c = x \cdot c, \qquad xL_c = c \cdot x, \qquad xT_c = c \diagup x$$

It is easily seen that R_c, L_c, and T_c are permutations of Q and that their inverses are given by

$$xR_c^{-1} = x \diagup c, \qquad xL_c^{-1} = c \diagdown x, \qquad xT_c^{-1} = x \diagdown c.$$

We will occasionally omit the multiplication symbol $(\cdot)$ in an expression in order to simplify notation.

4. Quasigroup isotopes of groups. In this section we will prove the following theorem.

THEOREM 1. *A necessary and sufficient condition that a quasigroup be isotopic to a group is that it satisfy a generalized group identity.*

The proof will be given as a sequence of lemmas which we now list in order to make the basic ideas clear.

LEMMA 1. *If a quasigroup Q, satisfies a generalized group identity,*

$$w_1(x, y, z, x_1, \dots, x_n) = w_2(x, y, z, x_1, \dots, x_n),$$

then Q satisfies a generalized associative law.

LEMMA 2. *If a quasigroup Q satisfies a generalized associative law, then Q satisfies Law D in some one of its operations.*

LEMMA 3. *If a quasigroup $(Q, \cdot, \diagdown, \diagup)$ satisfies Law D in any one of its operations ($\bigcirc$), then it is isotopic to a group with ($\cdot$) corresponding to the group operation.*

The preceding three lemmas show that if a quasigroup satisfies a generalized group identity, then it is isotopic to a group. The next lemma establishes the converse.

LEMMA 4. *If a quasigroup is isotopic to a group, then it satisfies the generalized group identity*

$$([(x\diagup u)\cdot(v\diagdown y)]\diagup u)\cdot(v\diagdown z) = (x\diagup u)\cdot(v\diagdown[(y\diagup u)\cdot(v\diagdown z)])$$

PROOF OF LEMMA 1. Let u_1 be the subword of w_1 of maximum length containing x, but not y and u_2 the subword of w_1 of maximum length containing y, but not x. Let u_3 be the subword of w_1 of maximum length containing both u_1 and u_2, but not z and u_4 be the subword of w of maximum length containing z, but not u_3. Now, one of $u_3 \bigcirc u_4$ or $u_4 \bigcirc u_3$ is a subword of w_1, for if not, then let u' be the subword of w_1 of minimum length containing u_3 and u_4 and let s and t be the major components of u', $u' = s \bigcirc t$. Clearly both u_3 and u_4 cannot be subwords of the same major component since u' has minimum length. Assume that u_3 and u_4 are subwords of s and t, respectively. Then, from the maximality of u_3 and u_4, it follows that $u_3 = s$ and $u_4 = t$. Hence $u_3 \bigcirc u_4$ is a subword. A similar argument holds in the other case.

Let a be a fixed element of Q and w_1', u_1', u_2', u_3', u_4' be the expressions obtained from w_1, u_1, u_2, u_3, u_4, by replacing x_i, $(i = 1, 2, \dots, n)$ by a. Then $u_1' = xP_1$, $u_2' = yP_2$, $u_3' = (xP_1 \bigcirc_1 yP_2)P_3$ or $(yP_2 \bigcirc_1 xP_1)P_3$, $u_4' = zP_4$, $w_1' = (u_3' \bigcirc_2 zP_4)P_5$ or $(zP_4 \bigcirc_2 u_3')P_5$ where P_i, $i = (1, 2, 3, 4, 5)$ are products of R_a, L_a, T_a and their inverses. Hence $w_1' = [(uP_1 \bigcirc_1 vP_2)P_3 \bigcirc_2 zP_4]P_5$ or $w_1' = [zP_4 \bigcirc_2 (uP_1 \bigcirc_1 vP_2)P_3]P_5$ where $\{u, v\} = \{x, y\}$. Similarly, we show that $w_2' = [(rQ_2 \bigcirc_4 sQ_3)Q_4 \bigcirc_3 xQ_1]Q_5$ or $w_2' = [xQ_1 \bigcirc_3 (rQ_2 \bigcirc_4 sQ_3)Q_4]Q_5$ where $\{r, s\} = \{y, z\}$.

Since $w_1(x, y, z, a, \dots, a) = w_2(x, y, z, a, \dots, a)$ holds in Q, we obtain one of the laws A, B, or C.

PROOF OF LEMMA 2. Let Q satisfy A and let a be a fixed element of Q. Substituting a for x, y, and z, in turn, and simplifying the resulting expressions, we get

$$\left.\begin{aligned} y\bigcirc_2 z &= (rD_1\bigcirc_4 sD_2)D_3 \\ x\bigcirc_2 z &= (mE_1\bigcirc_3 nE_2)E_3 \\ x\bigcirc_1 y &= (hF_1\bigcirc_3 kF_2)F_3 \end{aligned}\right\} \quad (1)$$

or

$$\left.\begin{aligned} y\bigcirc_2 z &= (rD'_1\bigcirc_3 sD'_2)D'_3 \\ x\bigcirc_2 z &= (mE'_1\bigcirc_4 nE'_2)E'_3 \\ x\bigcirc_1 y &= (hF'_1\bigcirc_3 kF'_2)F'_3 \end{aligned}\right\} \quad (2)$$

where $\{r, s\}=\{y, z\}$, $\{m, n\}=\{x, z\}$, $\{h, k\}=\{x, y\}$; $D_i, E_i, F_i, D'_i, E'_i, F'_i$ $i=(1, 2, 3)$ are permutations on Q. Using (1) or (2) to replace occurences of $\bigcirc_2$, $\bigcirc_3$, $\bigcirc_4$ in A by $\bigcirc_1$, we have

$$[(xP'_1\bigcirc_1 yP'_2)P'_3\bigcirc_1 zP'_4]P'_5 = [uQ'_1\bigcirc_1(vQ'_2\bigcirc_1 wQ'_3)Q'_4]Q'_5.$$

There are four possibilities for (u, v, w): (i) (x, y, z), (ii) (x, z, y). (iii) (y, z, x), (iv) (y, x, z). (3) is clearly Law D if (i) occurs. Let a be a fixed element of Q. If (ii), (iii) or (iv) occur, by substituting a for x, y, or z, respectively, in (3) we get a law $y\bigcirc_1 x = (xG_1\bigcirc_1 yG_2)G_3$ where G_1, G_2, G_3 are permutations on Q. Using this law, we can transform A, with some computation, into Law D.

An analogous argument holds if Q satisfies B or C.

PROOF OF LEMMA 3. $(Q, \bigcirc, \varnothing, \circledbslash)$ is isotopic to a group $(Q, *, \backslash *, /*)$ as a direct consequence of the result of EVANS [5]. Let the isotopy be given by $x\bigcirc y = (x\alpha * y\beta)\gamma$. If $\bigcirc$ is $/$, then $x\cdot y = (x\gamma^{-1}/* y\beta)\alpha^{-1} = (x\gamma^{-1} * y\beta I)\alpha^{-1}$ where I is the inverse operation in the group. If $\bigcirc$ is $\backslash$, then $x\cdot y = (x\alpha\backslash * y\gamma^{-1})\beta^{-1} = (x\alpha I * y\gamma^{-1})\beta^{-1}$. Thus, the quasigroup is isotopic to a group with the multiplication operation in the quasigroup corresponding to the group operation.

PROOF OF LEMMA 4. Let quasigroup $(Q, \cdot, \backslash, /)$ be isotopic to a group. Since every loop isotope of a group is a group, every loop isotope of $(Q, \cdot, \backslash, /)$ is a group [4]. Let $u, v\in Q$. Then $(Q, \bigcirc, \varnothing, \circledbslash)$ given by $x\bigcirc y = (x\backslash u)\cdot(v\backslash y)$ is a group and therefore satisfies $(x\bigcirc y)\bigcirc z = x\bigcirc(y\bigcirc z)$. Since this is true for all $u, v\in Q$, the quasigroup satisfies the identity

$$([x/u)(v\backslash y)]/u)(v\backslash z) = (x/u)(v\backslash[(y/u)(v\backslash z)])$$

for all $x, y, u, v\in Q$.

5. M_1, M_2, and Moufang Loops. The varieties of M_1, M_2 and Moufang loops are defined by identities:

$$M_1 \quad y(z\cdot yx) = (y\cdot zy)x$$

$$M_2 \quad (xy\cdot z)y = x(yz\cdot y)$$

$$\text{Moufang} \quad (xy\cdot z)y = x(y\cdot zy)$$

A loop is a Moufang loop if and only if it is both M_1 and M_2. Every loop isotope of a M_1, M_2, or Moufang loop is, respectively, a M_1, M_2, or Moufang loop [1].

Let w_1 and w_2 be reduced words in the free quasigroups on $x_1 x_2, \ldots$. $w_1 = w_2$ is a *generalized M_1 identity* if the following conditions hold.

(i) There exist three distinguished generators, x, y, z, such that x and z occur exactly once and y occurs exactly twice in both w_1 and w_2 with order of occurrence y, z, y, x.

(ii) $w_1 = y \cdot u$, $w_2 = v \cdot x$ where (1) yx is a subword of u and (2) there exist subwords, v_1, v_2, such that $v_1 = z \cdot t(y, x_1, \ldots, x_n)$ is a subword of v_2 and $y \cdot v_2$ is a subword of v.

(iii) The greatest common divisor, g, of $\{e_i(w_1/w_2) | i = 1, 2, \ldots\}$ is not 1.

The definition of a *generalized M_2 identity* is the same as that of a generalized M_1 identity with the following exceptions.

(a) in condition (i) the order of occurrence of x, y, and z is x, y, z, y.

(b) in condition (ii), part (1), xy is a subword of u and in part (2), there exist subwords v_1, v_2 such that $v_1 = t(y, x_1, \ldots, x_n) \cdot z$ is a subword of v_2 and $v_2 \cdot y$ is a subword of v.

THEOREM 2. *A necessary and sufficient condition that a quasigroup be isotopic to a M_1 (M_2) loop is that it satisfy a generalized M_1 (M_2) identity.*

The proofs of the two cases involved in this theorem are analogous due to the symmetry in the definitions of generalized M_1 and M_2 identities. For this reason we present only the proof for M_1 loops. This proof will be a direct consequence of the three lemmas which are stated below.

LEMMA 1. *If a quasigroup Q satisfies a generalized M_1 identity, then Q and every isotope of Q satisfies a law of the form* E: $y \cdot [zP_1 \cdot (yx)P_2]P_3 = [y \cdot (z \cdot yQ_1)Q_2]Q_3 \cdot x$ *where P_i, Q_i, $(i = 1, 2, 3)$ are permutations on Q.*

LEMMA 2. *If a loop L satisfies a law of the form* E, *then L is a M_1 loop.*

Since every quasigroup is isotopic to a loop, it is clear in view of these two lemmas that if a quasigroup satisfies a generalized M_1 identity, then it is isotopic to a M_1 loop. Thus, the sufficiency of the stated condition is established. The next lemma establishes it as a necessary condition.

LEMMA 3. *If a quasigroup is isotopic to a M_1 loop, then it satisfies the generalized M_1 identity,*

$$y \cdot (v \backslash [z \cdot (v \backslash yx)]) = ([y \cdot (v \backslash [z \cdot (v \backslash uy)])] / u) \cdot x$$

PROOF OF LEMMA 1. Suppose quasigroup Q satisfies the generalized M_1 identity, $w_1(x, y, z, x_1, \ldots, x_n) = w_2(x, y, z, x_1, \ldots, x_n)$. Let u_1 be the subword of u of maximum length containing yx, but not z, and u_2 the subword of u of maximum length containing z but not yx. Let $u_1', u_2', u', v_1', v_2', v', w_1', w_2'$ be the expressions resulting from substituting a fixed element $a \in Q$ for x_i, $(i = 1, 2, \ldots, n)$ in the words $u_1, u_2, u, v_1, v_2, v, w_1, w_2$. Then $u_1' = (yx)P_2$, $u_2' = zP_1$, $u' = [zP_1 \bigcirc (yx)P_2]P_3$, $v_1' = z \cdot t(y, a, a, \ldots, a) = z \cdot yQ_1$, $v_2' = (z \cdot yQ_1)Q_2$, $v' = [y \cdot (z \cdot yQ_1)Q_2]Q_3$, $w_1 = y \cdot u' = y \cdot (zP_1 \bigcirc (yx)P_2)P_3$, $w_2' = v' \cdot x = [y \cdot (z \cdot yQ_1)Q_2]Q_3 \cdot x$. Therefore, Q satisfies

$$y \cdot [zP_1 \bigcirc (yx)P_2]P_3 = [y \cdot (z \cdot yQ_1)Q_2]Q_3 \cdot x \tag{1}$$

where the P_i, Q_i, $(i = 1, 2, 3)$ are products of R_a, L_a, T_a and their inverses.

If we now set $y=a$ in (1), we find that $z \bigcirc x = (zA \cdot xB)C$ for some permutations A, B, C. Using this result and (1), we have

$$y \cdot [zP'_1 \cdot (yx)P'_2]P'_3 = [y \cdot (z \cdot yQ_1)Q_2]Q_3 \cdot x$$

Let an isotope of Q be given by $x \bigcirc y = (x\alpha \cdot y\beta)\gamma$. Let $P''_1 = \alpha P'_1 \alpha^{-1}$, $P''_2 = \gamma^{-1} P'_2 \beta^{-1}$, $P''_3 = \gamma^{-1} P'_3 \beta^{-1}$, $Q''_1 = \alpha Q_1 \beta^{-1}$, $Q''_2 = \gamma^{-1} Q_2 \beta^{-1}$, $Q''_3 = \gamma^{-1} Q_3 \alpha^{-1}$. Then computation reveals that the isotope satisfies the law

$$y \bigcirc [zP''_1 \bigcirc (y \bigcirc x)P''_2]P''_3 = [y \bigcirc (z \bigcirc yQ''_1)Q''_2]Q''_3 \bigcirc x$$

PROOF OF LEMMA 2. Suppose that L satisfies the law

$$y \cdot [zA \cdot (yx)B]C = [y \cdot (z \cdot yD)E]F \cdot x$$

where A, B, C, D, E, F are permutations. Then the permutation group of L satisfies

(2) $$AR_{(yx)B} CL_y = R_{yD} EL_y FR_x$$

Setting $x=1$ in (2) we have

(3) $$AR_{yB} CL_y = R_{yD} EL_y F$$

Combining (2) and (3), we get

(4) $$R_{(yx)B} CL_y = R_{yB} CL_y R_x$$

Let $y=1$ in (4). Then

(5) $$R_{xB} C = R_{1B} CR_x$$

Combining this with (4), we get

$$R_{yx} L_y = R_y L_y R_x$$

Therefore L satisfies $y(z \cdot yx) = (y \cdot zy)x$.

PROOF OF LEMMA 3. Since every loop isotope of a M_1 loop is a M_1 loop, it follows that every loop isotope of Q is M_1. Let $u, v \in Q$. Then the loop isotope $(Q, \bigcirc, \oslash, \circledbslash)$ given by $x \bigcirc y = (x/u) \cdot (v \backslash y)$ satisfies the M_1 identity, $y \bigcirc (z \bigcirc (y \bigcirc x)) = (y \bigcirc (z \bigcirc y)) \bigcirc x$. Since this is true for all $u, v \in Q$, Q must satisfy the identity

$$(y/u) \cdot (v \backslash [(z/u) \cdot (v \backslash [(y/u)(v \backslash x)])]) = ([(y/u) \cdot (v \backslash [(z/u) \cdot (v \backslash y)])]/u) \cdot (v/x)$$

But this implies that Q satisfies the generalized M identity,

$$y \cdot (v \backslash [z \cdot (v \backslash yx)]) = ([y \cdot (v \backslash [z \cdot (v \backslash yu)])]/u) \cdot x$$

Noting that a loop is Moufang if and only if it is both M_1 and M_2, we have the following result.

COROLLARY. *A necessary and sufficient condition that a quasigroup be isotopic to a Moufang loop is that it satisfy a generalized M_1 identity and a generalized M_2 identity.*

It is possible to find a single identity satisfied by the quasigroup isotopes of Moufang loops without using the above results for M_1 and M_2 loops. However, we must give up the advantage of having a large number of identities with which we can identify quasigroups that are isotopic to Moufang loops.

THEOREM 3. *A quasigroup is isotopic to a Moufang loop if and only if it satisfies the identity*

$$([(xy/u)(v\backslash zu)]/u)\cdot y = x\cdot(v\backslash[(vy/u)\cdot(v\backslash zy)])$$

PROOF. Let Q be a quasigroup isotopic to a Moufang loop. It is clear that every loop isotope of Q is Moufang. Let $u, v\in Q$ and let $L=(Q, \bigcirc, \oslash, \circledbslash)$ be the isotope of $(Q, \cdot, \backslash, /)$ given by $x\bigcirc y=(x/u)(v\backslash y)$. Then, since L is Moufang, it must satisfy $y\bigcirc(z\bigcirc(y\bigcirc x))=(y\bigcirc(z\bigcirc y))\bigcirc x$. Since u and v are arbitrary, Q must satisfy the identity

$$([(x/u)(v\backslash y)]/u)(v\backslash y) = (x/u)(v\backslash[(y/u)\cdot(v\backslash[(z/u)(v\backslash y)])]$$

Conversely, if Q is a quasigroup satisfying (1), then every loop isotope obviously satisfying $y\bigcirc(z\bigcirc(y\bigcirc x))=(y\bigcirc(z\bigcirc y))\bigcirc x$ and hence is Moufang. Thus a quasigroup is isotopic to a Moufang loop if and only if it satisfies (1). We conclude the proof by noting that (1) is equivalent to the identity

$$([(xy/u)(v\backslash zu)]/u)\cdot y = x\cdot(v\backslash[(vy/u)\cdot(v\backslash zy)])$$

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ЗАМЕЧАНИЯ О ПОГРЕШНОСТИ ТРИГОНОМЕТРИЧЕСКОГО (0—2)-ИНТЕРПОЛИРОВАНИЯ

О. КИШ (Будапешт)

Пусть

(1) $$x_{kn} = \frac{2\pi k}{n} \qquad (k=0, 1, \dots, n-1;\ n=1, 3, 5, 7, \dots).$$

В заметке [1] было доказано, что 1) существует единственный тригонометрический многочлен

$$R_n(x) = a_0 + \sum_{j=1}^{n-1} (a_j \cos jx + b_j \sin jx) + a_n \cos nx,$$

удовлетворяющий условиям

$$R_n(x_{kn}) = f(x_{kn}) \qquad (k=0, 1, \dots, n-1),$$

$$R_n''(x_{kn}) = 0 \qquad (k=0, 1, \dots, n-1),$$

где $f(x)$ любая определенная в точках x_{kn} функция; 2) если она 2π-периодична, непрерывна и удовлетворяет условию

(2) $$f(x+h) - 2f(x) + f(x-h) = o(h),$$

то последовательность (0—2) — интерполяционных многочленов $R_n(x)$ равномерно сходится к функции $f(x)$ на всей вещественной оси. В статье [3] доказано, что 3) условие (2) нельзя заменить условием

$$f(x+h) - 2f(x) + f(x-h) = O(h).$$

В настоящей заметке мы докажем следующие результаты:

Теорема 1. *Если модуль гладкости $\omega_2(f, t)$ функции $f(x)$ удовлетворяет условию*

(3) $$\omega_2(f, t) = O\big(\varphi(t)\big)$$

и функция $\varphi(t)$ удовлетворяет условию

(4) $$\sum_{j=1}^{k} 4^j \varphi(2^{-j}) = O\big(4^k \varphi(2^{-k})\big),$$

то

(5) $$\big|R_n(x) - f(x)\big| = O\left(n\varphi\left(\frac{1}{n}\right)\right).$$

(Заметим, что условие (4) выполняется, если

(6) $$\varphi(t)=t^{\alpha} \qquad (0<\alpha<2);$$

а условие (2) выполняется, если

$$\varphi(t)=o(t),$$

но тогда из (5) следует равномерная сходимость интерполяционного процесса, т. е. теорема 2) из [1] является следствием теоремы 1.)

Теорема 2. *Если $\varphi(0)=0$; $\varphi(t)>0$ при $t>0$; функция $\varphi(t)$ непрерывна и неубывает; функция $\varphi(t)t^{-2}$ невозрастает и*

(7) $$\lim_{t\to+0}\omega(t)t^{-2}=+\infty,$$

то существует 2π-периодическая непрерывная функция $f(x)$, удовлетворяющая условию (3), *для которой*

(8) $$|R_n(\pi)-f(\pi)|>\frac{n}{5}\varphi\left(\frac{1}{n}\right) \qquad (n=n_1, n_2, n_3, \ldots),$$

где n_k возрастающая последовательность натуральных чисел.

(Таким образом оценка (5) не может быть улучшена. Заметим, что условия теоремы 2 выполняются для функций (6); если $\varphi(t)=t$, то из теоремы 2 следует теорема 3) из [3].)

Доказательство теоремы 1. Обозначим через $U_n(x)$ тригонометрический многочлен $n-1$-ого порядка, наименее уклоняющийся от функций $f(x)$. Известно (см. [2], стр. 274), что

$$|f(x)-U_n(x)|=O\left(\omega_2\left(f,\frac{1}{n}\right)\right).$$

Поэтому

(9) $$|f(x)-U_n(x)|=O\left(\varphi\left(\frac{1}{n}\right)\right).$$

Пусть

$$2^k\leqq n<2^{k+1}, \qquad T_n(x)=U_{2^k}(x).$$

Очевидно

(10) $$|f(x)-T_n(x)|=O\left(\varphi\left(\frac{1}{n}\right)\right).$$

Так как

$$T_n(x)=U_1(x)+\sum_{j=1}^{k}[U_{2^j}(x)-U_{2^{j-1}}(x)],$$

то

$$T_n''(x)=\sum_{j=1}^{k}[U_{2^j}''(x)-U_{2^{j-1}}''(x)].$$

Ввиду (9)

$$|U_{2^j}(x)-U_{2^{j-1}}(x)|=O(\varphi(2^{-j}));$$

в силу неравенства С. Н. Бернштейна

$$\left|U''_{2^j}(x)-U''_{2^{j-1}}(x)\right| = O\big(4^j\varphi(2^{-j})\big),$$

поэтому

$$\left|T''_n(x)\right| = O\left(\sum_{j=1}^{k} 4^j\varphi(2^{-j})\right).$$

Отсюда и из (4) получаем:

$$\left|T''_n(x)\right| = O\big(4^k\varphi(2^{-k})\big) = O\left(n^2\varphi\left(\frac{1}{n}\right)\right). \tag{11}$$

В [1] доказано (стр. 272), что при $n=1, 3, 5, \ldots$

$$f(x)-R_n(x) = f(x)-T_n(x)+\sum_{k=0}^{n-1}\left[T_n(x_{kn})-f(x_{kn})\right]u_{kn}(x)+\sum_{k=0}^{n-1}T''_n(x_{kn})v_{kn}(x), \tag{12}$$

где $u_{kn}(x)$ и $v_{kn}(x)$ фундаментальные многочлены интерполирования, причем

$$\sum_{k=0}^{n-1}\left|u_{kn}(x)\right| = O(n), \tag{13}$$

$$\sum_{k=0}^{n-1}\left|v_{kn}(x)\right| = O\left(\frac{1}{n}\right). \tag{14}$$

Из (10)—(14) получаем доказываемое неравенство (5).

Доказательство теоремы 2. В заметке [1] доказано, что

$$R_n(x) = \sum_{k=0}^{n-1} f(x_{kn})u_{kn}(x) \qquad (n=1, 3, 5, \ldots), \tag{15}$$

где фундаментальные многочлены $u_{kn}(x)$ суть тригонометрические многочлены и поэтому непрерывные 2π-периодические функции, причем

$$\lambda_n(\pi) = \sum_{k=0}^{n-1}\left|u_{kn}(\pi)\right| \geqq \frac{n}{5} \qquad (n=1, 3, 5, \ldots) \tag{16}$$

и поэтому

$$\lim_{m\to\infty}\lambda_{2m-1}(\pi) = \infty. \tag{17}$$

Из (1) следует, что расстояние соседних узлов x_{kn} равно

$$d_n = \frac{2\pi}{n} \qquad (n=1, 3, 5, \ldots). \tag{18}$$

В статье [4] доказано, что если выполняются условия теоремы 2, функции $u_{kn}(x)$ 2π-периодичны и непрерывны и выполняется условие (17), то для любых линейных операторов вида (15) можно подобрать 2π-периодическую функцию $f(x)$, удовлетворяющую условию (3) и

$$|R_n(\pi)-f(\pi)| \geqq \lambda_n(\pi)\varphi(d_n) \qquad (n=n_1, n_2, n_3, \dots).$$

Отсюда, из (16), (18) и монотонности функции $\varphi(t)$ следует доказываемое неравенство (8).

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ÄQUIVALENZRELATION FÜR DIE CHARAKTERISIERUNG DES JACOBSONSCHEN RADIKALS

Von

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Für eine universale Algebra $\mathfrak{A}=(A, \Omega)$ im Sinne von G. BIRKHOFF und für eine Teilmenge B der Menge A der Strukturelemente bezeichne B_Ω die durch B erzeugte Teilstruktur. Enthält B nur ein enziges Element b, so wird B_Ω mit $(b)_\Omega$ bezeichnet. Durch die Gleichung $(a)_\Omega=(b)_\Omega$ wird in A eine Äquivalenzrelation definiert.

Ist nun insbesondere $\mathfrak{A}=(A, \Omega)$ ein assoziativer Ring, wobei Ω das System der Addition, Substraktion und aller Rechtsmultiplikationen bezeichnet, so stimmt $(a)_\Omega$ mit dem Hauptrechtsideal $(a)_r$ von A überein, und dann ist die durch die Gleichung $(a)_r=(b)_r$ definierte Äquivalenzrelation $\equiv$ auch eine Linkskongruenz der multiplikativen Halbgruppe des Ringes A, die mit der Hilfe von ganzen rationalen Zahlen und von Ringelementen auch explizit beschrieben werden kann.

Das Ziel dieser Note ist, das Jacobsonsche Radikal J eines Ringes A (vgl. JACOBSON [3]) mit der Hilfe der obigen Äquivalenzrelation zu charakterisieren. Demnach läßt sich J als ein einseitig antieinfaches Radikal ansehen (vgl. ANDRUNAKIEWITSCH [1]). Die im Beweis benützte Methode ist teilweise einem Beweis von SĄSIADA [5] und des Verfassers [6] ähnlich.

Es gilt nämlich der folgende

SATZ. *Das Jacobsonsche Radikal J eines Ringes A stimmt mit der Menge K derjenigen Elemente x von A überein, für welche die (im vorigen erklärte) Äquivalenzrelation $y \equiv y+yz$ mit jedem Element $y\in A$ und mit jedem Element z des Hauptrechtsideals $(x)_r$ gilt.*

BEWEIS. Bezeichnen $B^{-1}C$ für beliebige Teilmengen B und C des Ringes A die Menge

$$\{w;\, w\in A,\ Bw\subseteqq C\}$$

und Φ_r das Frattinische Rechtsideal (d.h. den Durchschnitt aller maximalen Rechtsideale) von A, so läßt sich nach HILLE [2] bzw. KERTÉSZ [4] $J=A^{-1}\Phi_r$ bestätigen.

Zuerst beweisen wir $K\subseteqq J$. Gilt nämlich $x\notin J$ für ein $x\in A$, so existieren wegen $x\notin A^{-1}\Phi_r$ ein Element $y\in A$ und ein maximales Rechtsideal R von A mit $yx\notin R$. Dann ergibt sich $A^2\nsubseteqq R$, $y\notin R$ und der A-Rechtsmodul A/R ist einfach. Folglich gibt es ein $u\in A$ mit $y+R = yxu+R$. Dann gilt wegen $z = -xu\in(x)_r$, und

$$(y+y(-xu))_r \subseteqq R \neq A = (y)_r+A$$

gewiß $x\notin K$.

Zweitens zeigen wir, daß auch $J\subseteqq K$ gilt. Ist nämlich $x\notin K$ für ein $x\in A$, so existieren Elemente $z\in(x)_r$ und $y\in A$, derart, daß $y+yz$ und y nicht äquivalent bezüglich $\equiv$ sind. Man erhält dann $(y)_r \neq (y+yz)_r$ und $(y)_r \supset (y+yz)_r$. Ist nun $\mathfrak{M}$ die Menge derjenigen Rechtsideale R von A, für die

$$(y+yz)_r \subseteqq R \quad \text{und} \quad y \notin R$$

gelten, so ist $\mathfrak{M}$ nichtleer und induktiv. Es sei M ein maximales Rechtsideal aus $\mathfrak{M}$ (welches nach dem Zornschen Lemma existiert). Der A-Rechtsmodul A/M ist dann subdirekt irreduzibel, denn $(y)_r + M/M$ liegt in jedem von Null verschiedenen A-Teilmodul von A/M. Wegen $y \notin M$ und $y+yz \in M$ erhält man $(yz)_r + M = (y)_r + M$, weiterhin

$$(yz)_r + M \subseteqq y(x)_r + M \neq M.$$

Da $(y)_r + M/M$ ein einfacher A-Rechtsmodul ist, der von dem Radikal J annihiliert wird, gilt $x \notin J$.

Hiernach gilt $J = K$, womit der Satz bewiesen ist.

Bemerkung (am 27. August 1971): Übungsaufgabe 5.24 des Lehrbuches „Vorlesungen über Artinsche Ringe" von A. Kertész führt ein Radikal $K^*(M)$ für jeden A-Rechtsmodul M über einem Ring A ein. Nach dem Verfasser (Rings, which are radical modules; *Proc. Japan Acad.*, im Erscheinen) ist A dann und nur dann ein Jacobsonscher Radikalring, wenn A, als ein A-Rechtsmodul A angesehen, ein Radikalmodul ist, d. h. $K^*(A) = A$ gilt. Andererseits hat der Verfasser gezeigt (Notes on modules, III; *Proc. Japan Acad.* 46:3 (1970) 354—357), dass $K^*(A)$ klein Radikal, im Sinne von S. A. Amitsur und A. G. Kurosch, des Ringes A ist.

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ON FOURIER SERIES WITH POSITIVE COEFFICIENTS

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Introduction

Recently R. P. BOAS [2] proved several interesting theorems concerning Fourier series with positive coefficients. We recall some of his theorems:

THEOREM A. *If φ_n are the Fourier sine or cosine coefficients of $\varphi(x)$ on the interval $0 \leqq x \leqq \pi$, $\varphi_n \geqq 0$ and $0 < \gamma < 1$, then the conditions*

$$\sum_{n=1}^{\infty} n^{\gamma-1} \varphi_n < \infty$$

and

$$\int_{a^+}^{\pi} (x-a)^{-\gamma} \varphi(x)\, dx < \infty \qquad 0 \leqq a < \pi$$ [1]

are equivalent. If $a=0$, then, for the sine series, γ can be allowed to equal to 1.

THEOREM B. *If $\varphi_n \geqq 0$, $p > 1$ and $1/p < \gamma < (p+1)/p$, then the condition*

$$|x-a|^{-\gamma} |\varphi(x)-\varphi(a)| \in L^p \ (\text{for any } a,\ 0 \leqq a < \pi)$$

is equivalent to either of the conditions

$$\sum_{n=1}^{\infty} n^{p\gamma-p-2} \left(\sum_{k=1}^{n} k \varphi_k \right)^p < \infty$$

and

$$\sum_{n=1}^{\infty} n^{p\gamma-2} \left(\sum_{k=n}^{\infty} \varphi_k \right)^p < \infty.$$

If $(1-p)/p < \gamma < 1/p$ then, using Theorem B, BOAS proved

THEOREM C. *If $p > 1$, $(1-p)/p < \gamma < 1/p$, $\varphi_n \geqq 0$ and $\varphi(x) x^{-\gamma} \in L^p$, then*

$$\sum_{n=1}^{\infty} n^{p\gamma-2} \left(\sum_{k=0}^{n} \varphi_k \right)^p < \infty,$$

or, equivalently,

$$\sum_{n=1}^{\infty} n^{p\gamma+p-2} \left(\sum_{k=n}^{\infty} k^{-1} \varphi_k \right)^p < \infty.$$

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[1] $\int_{a^+}^{b}$ means $\lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b}$.

Prior to this ASKEY and WAINGER [1] proved the following

THEOREM D. *If* $\varphi(x)\in L$, $p>1$, $(1-p)/p<\gamma<1/p$ *and*

$$\sum_{n=1}^{\infty} n^{p\gamma+p-2}\left(\sum_{k=n}^{\infty}|\varphi_k-\varphi_{k+1}|\right)^p<\infty, \tag{1}$$

then $\varphi(x)(\sin x)^{-\gamma}\in L^p$.

In the proofs of these theorems Hardy's inequalities play an important role. First we generalize these inequalities (Lemmas 1, 2, and 4). This will enable us to prove some generalizations of the above theorems.

We shall use the following notations throughout this paper:

$$g(x)\sim\sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n=\frac{2}{\pi}\int_0^{\pi} g(x)\sin nx dx;$$

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx, \qquad a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx,$$

and $\varphi(x)$ denotes either an $f(x)$ or a $g(x)$ and φ_n denote the Fourier coefficients of $\varphi(x)$.

We prove the following theorems.

THEOREM 1. *Let* $\varphi_n\geqq 0$ *and let* $\{\lambda_n\}$ *be a nondecreasing sequence of positive numbers satisfying the conditions*

$$\sum_{n=1}^{m}\lambda_n n^{-1}\leqq K\lambda_m\ ^2 \tag{2}$$

and

$$\sum_{n=1}^{m}\lambda_n^{-1}\leqq Km\lambda_m^{-1}, \tag{3}$$

and, in the case of the cosine series, in addition

$$n(n+1)(\lambda_{n+1}-\lambda_n)\uparrow \quad \textit{and} \quad \lambda_n n^{-1}\leqq K\sum_{k=n+1}^{\infty}(\lambda_k-\lambda_{k-1})k^{-1}. \tag{4}$$

Then we have

$$\sum_{n=1}^{\infty}\lambda_n n^{-1}\varphi_n<\infty \tag{5}$$

if and only if

$$\int_{a^+}^{a+\pi}\lambda(x-a)\varphi(x)dx<\infty, \qquad 0\leqq a<\pi, \tag{6}$$

where $\lambda(x)$ $(x>0)$ *is the continuous function for which* $\lambda(n^{-1})=\lambda_n$ *and which is linear on the intervals* $[(n+1)^{-1}, n^{-1}]$ *and equals to* λ_1 *for* $x\geqq 1$.

[2] K and K_i denote either absolute constants or constants depending on certain functions and numbers which it is not necessary to explain in detail, not necessarily the same at each occurrence.

More precisely, (5) is necessary for (6) with $a=0$, and sufficient for (6) with any a.

Note that for sine series and for $a=0$ condition (3) can be replaced by the weaker condition

$$\sum_{n=1}^{m} n\lambda_n^{-1} \leqq Km^2\lambda_m^{-1}. \tag{7}$$

Secondly we mention that for $a=0$ the statement (5)⇒(6) was proved by ROBERTSON [4]; actually it is easy to prove that (2) and (3) are equivalent to the conditions

$$\int_x^1 \lambda(t)t^{-1}dt \leqq K\lambda(x) \quad \text{and} \quad \int_0^x \lambda(t)\,dt \leqq Kx\lambda(x),$$

respectively.

Theorem 1 has e.g. the following

COROLLARY 1. *Let $0<\gamma<1$, let β be an arbitrary real number, and let $\varphi_n \geqq 0$. Then the conditions*

$$\int_{a+}^{a+\pi} (x-a)^{-\gamma}\left(\ln\left(\frac{1}{x-a}+1\right)\right)^{\beta} \varphi(x)\,dx < \infty \qquad (\textit{for any } a,\ 0 \leqq a < \pi)$$

and

$$\sum_{n=2}^{\infty} n^{\gamma-1}(\ln n)^{\beta}\varphi_n < \infty$$

are equivalent.

For $p>1$ we have

THEOREM 2. *Let $p>1$, $\varphi_n \geqq 0$, and let $\{\lambda_n\}$ be a non-decreasing sequence of positive numbers satisfying the conditions*

$$\sum_{n=1}^{m} \lambda_n^p n^{-2} \leqq K\lambda_m^p m^{-1} \tag{8}$$

and

$$\sum_{n=1}^{m} n^p \lambda_n^{-p} \leqq Km^{1+p}\lambda_m^{-p}. \tag{9}$$

Then the condition

$$\int_a^{a+\pi} \lambda^p(x-a)|\varphi(x)-\varphi(a)|^p\,dx < \infty, \qquad 0 \leqq a < \pi, \tag{10}$$

is equivalent to either of the conditions

$$\sum_{n=1}^{\infty} \lambda_n^p n^{-p-2}\left(\sum_{k=1}^{n} k\varphi_k\right)^p < \infty \tag{11}$$

and

$$\sum_{n=1}^{\infty} \lambda_n^p n^{-2}\left(\sum_{k=n}^{\infty} \varphi_k\right)^p < \infty, \tag{12}$$

where $\lambda(x)$ has the same meaning as in Theorem 1.

More precisely, (11) is necessary for (10) with $a=0$, and sufficient for (10) with any a.

From this theorem we can get the following corollaries. First, for instance, taking $\lambda_n = n^{2/p}[\log(n+1)]^\beta$, we get

COROLLARY 2. *Let $p>1$, β an arbitrary real number, and $\varphi_n \geqq 0$. Then the condition*

$$\int_a^{a+\pi} (x-a)^{-2}\left(\ln\left(\frac{1}{x-a}+1\right)\right)^{p\beta} |\varphi(x)-\varphi(a)|^p\,dx<\infty \tag{13}$$

is equivalent to either of the conditions

$$\sum_{n=2}^{\infty} (\ln n)^{p\beta} n^{-p}\left(\sum_{k=1}^{n} k\varphi_k\right)^p<\infty$$

and

$$\sum_{n=2}^{\infty} (\ln n)^{p\beta}\left(\sum_{k=n}^{\infty} \varphi_k\right)^p<\infty.$$

From Theorem 2 and Lemma 2 we obtain

COROLLARY 3. *Let $p>1$, $\varphi_n\downarrow 0$, and let $\{\lambda_n\}$ satisfy* (8) *and* (9). *Then* (10) *holds if and only if*

$$\sum_{n=1}^{\infty} \lambda_n^p n^{p-2}\varphi_n^p<\infty.$$

COROLLARY 4. *Let $p>1$, β be an arbitrary real number, and $\varphi_n\downarrow 0$. Then* (13) *holds if and only if*

$$\sum_{n=2}^{\infty} (\ln n)^{p\beta} n^p\varphi_n^p<\infty.$$

Applying Lemma 4 we easily get from Theorem 2 the following

THEOREM 3. *Suppose that $p>1$ and that $\{\lambda_n\}$ satisfies the conditions*

$$\sum_{n=1}^{m} n^{p-2}\lambda_n^p \leqq K\lambda_m^p m^{p-1} \tag{14}$$

and

$$\sum_{n=1}^{m} \lambda_n^{-p} \leqq Km\lambda_m^{-p}. \tag{15}$$

Then

$$\int_0^{\pi} \lambda^p(x)|\varphi(x)|^p\,dx<\infty \tag{16}$$

implies

$$\sum_{n=1}^{\infty} \lambda_n^p n^{-2}\left(\sum_{k=1}^{n} \varphi_k\right)^p<\infty, \tag{17}$$

or equivalently

$$\sum_{n=1}^{\infty} \lambda_n^p n^{p-2}\left(\sum_{k=n}^{\infty} k^{-1}\varphi_k\right)^p<\infty. \tag{18}$$

From the proof of Theorem 2 it will be seen that (10) is implied also by (11) and (12) with $|\varphi_n|$ instead of φ_n, with no assumption about the sign of φ_n.

In view of this we can deduce from Theorem 2

THEOREM 4. *Suppose that $\varphi(x)\in L$, $p>1$, and that*

$$\sum_{n=1}^{m} n^{p-2}\lambda_n^p \leqq K\lambda_m^p m^{p-1} \tag{19}$$

and

$$\sum_{n=1}^{m} \lambda_n^{-p} \leqq Km\lambda_m^{-p}. \tag{20}$$

Then

$$\sum_{n=1}^{\infty} \lambda_n^p n^{p-2}\left(\sum_{k=n}^{\infty}|\varphi_k-\varphi_{k+2}|\right)^p<\infty \tag{21}$$

implies

$$\int_0^{\pi} \lambda^p(x)\left(\frac{\sin x}{x}\right)^p|\varphi(x)|^p\,dx<\infty. \tag{22}$$

Corollaries similar to those which were deduced from Theorem 2 could also be obtained from Theorem 3 and 4.

§ 1. Lemmas

LEMMA 1. *Let $\{a_n\}$ and $\{\eta_n\}$ be sequences of non-negative numbers such that*

$$\sum_{k=n}^{\infty}\eta_k=\beta_n\eta_n.$$

Then for any $\beta\geqq 1$

$$\sum_{k=1}^{\infty}\eta_k\left(\sum_{n=1}^{k}a_n\right)^{\beta}\leqq \beta^{\beta}\sum_{k=1}^{\infty}\eta_k(\beta_k a_k)^{\beta}.$$

This lemma can be found in [3].

LEMMA 2. *Let $\beta\geqq 1$, and let $\{\varrho_n\}$ and $\{a_n\}$ be sequences of non-negative numbers such that*

$$\sum_{n=1}^{m}\varrho_n\leqq Km\varrho_m. \tag{1. 1}$$

Then

$$\sum_{n=1}^{\infty}\varrho_n\left(\sum_{k=n}^{\infty}a_k\right)^{\beta}\leqq K_1\sum_{n=1}^{\infty}\varrho_n(na_n)^{\beta}. \tag{1. 2}$$

PROOF. We suppose, as we may do without loss of generality, that $\varrho_1>0$ and the series on the right-hand side converges. If $\beta=1$, we obtain (1. 2) by interchanging the order of summation and using (1. 1).

Let $\beta>1$; $s_n=\sum_{k=n}^{\infty} a_k$ and $\bar\varrho_n=\min_{2^n<k\leqq 2^{n+1}}\varrho_k$. First we show that

$$\left(\sum_{n=1}^{m}\varrho_n\right)s_m^\beta\to 0. \tag{1.3}$$

By (1.1) we have

$$\sum_{n=1}^{m}2^n\bar\varrho_n\leqq N2^m\bar\varrho_m, \tag{1.4}$$

where N is an integer independent of m. Let $r_i=\sum_{n=(i-1)N+1}^{iN}2^n\bar\varrho_n$. Using (1.4) an easy calculus gives

$$\sum_{i=1}^{j}r_i\leqq r_{j+1}$$

and from this one can deduce the estimates

$$2^{n-1}r_m\leqq r_{m+n}\leqq N2^{(m+n)N}\bar\varrho_{(m+n)N} \tag{1.5}$$

for any positive integers n and m. Since

$$\sum_{n=m}^{\infty}\varrho_n(na_n)^\beta\to 0,$$

we have with $2^{\mu N}<m\leqq 2^{(\mu+1)N}$

$$\begin{aligned}\sum_{n=m}^{\infty}a_n &\leqq \left\{\sum_{n=m}^{\infty}\varrho_n n^\beta a_n^\beta\right\}^{\frac{1}{\beta}}\left\{\sum_{n=m}^{\infty}\left(n\varrho_n^{\frac{1}{\beta}}\right)^{\frac{\beta}{1-\beta}}\right\}^{\frac{\beta-1}{\beta}}=\\ &=o(1)\left\{\left(\sum_{n=m}^{2^{(\mu+3)N}}+\sum_{n=2^{(\mu+3)N}+1}^{\infty}\right)\left(n^\beta\varrho_n\right)^{\frac{1}{1-\beta}}\right\}^{\frac{\beta-1}{\beta}}.\end{aligned} \tag{1.6}$$

By (1.1)

$$\sum_{n=m}^{2^{(\mu+3)N}}(n^\beta\varrho_n)^{\frac{1}{1-\beta}}\leqq K_2\sum_{n=m}^{2^{(\mu+3)N}}\left(n^\beta n^{-1}\sum_{k=1}^{m}\varrho_k\right)^{\frac{1}{1-\beta}}\leqq K_3\left(\sum_{k=1}^{m}\varrho_k\right)^{\frac{1}{1-\beta}} \tag{1.7}$$

and by (1.1), (1.4), (1.5) and by the definition of r_i

$$\begin{aligned}\sum_{n=2^{(\mu+3)N}}^{\infty}(n^\beta\varrho_n)^{\frac{1}{1-\beta}} &\leqq \sum_{i=(\mu+3)N}^{\infty}2^i(2^{i\beta}\bar\varrho_i)^{\frac{1}{1-\beta}}=\\ &=\sum_{i=(\mu+3)N}^{\infty}(2^i\bar\varrho_i)^{\frac{1}{1-\beta}}\leqq\sum_{k=\mu+2}^{\infty}\sum_{i=kN+1}^{(k+1)N}(2^i\bar\varrho_i)^{\frac{1}{1-\beta}}\leqq\\ &\leqq K\sum_{k=\mu+2}^{\infty}(2^{kN}\bar\varrho_{kN})^{\frac{1}{1-\beta}}\leqq K_1(2^{(\mu+2)N}\bar\varrho_{(\mu+2)N})^{\frac{1}{1-\beta}}\leqq K_2\left(\sum_{k=1}^{m}\varrho_k\right)^{\frac{1}{1-\beta}}.\end{aligned} \tag{1.8}$$

Inserting (1. 7) and (1. 8) into (1. 6) gives (1. 3).

We are now in a position to prove (1. 2) easily. Indeed, we have by (1. 1) and (1. 3) for ν large enough

$$\sum_{n=1}^{\nu+1} \varrho_n s_n^\beta = \sum_{n=1}^{\nu} \left(\sum_{i=1}^{n} \varrho_i\right)(s_n^\beta - s_{n+1}^\beta) + \left(\sum_{i=1}^{\nu+1} \varrho_i\right) s_{\nu+1}^\beta \leqq K_1 \sum_{n=1}^{\nu} n\varrho_n (s_n^\beta - s_{n+1}^\beta) \leqq$$

$$\leqq K_2 \sum_{n=1}^{\nu} n\varrho_n a_n s_n^{\beta-1} \leqq K_2 \left\{\sum_{n=1}^{\nu} \varrho_n a_n^\beta n^\beta\right\}^{\frac{1}{\beta}} \left\{\sum_{n=1}^{\nu} s_n^\beta \varrho_n\right\}^{\frac{\beta-1}{\beta}}$$

and from this the statement (1. 2) follows obviously.

LEMMA 3. *Let $p \geqq 1$, $s > 0$ and let $\{a_n\}$ and $\{\alpha_n\}$ be sequences of non-negative numbers. Denote*

$$\sum\nolimits_1 = \sum_{n=1}^{\infty} \alpha_n^p n^{-2} \left(\sum_{k=n}^{\infty} a_k\right)^p$$

and

$$\sum\nolimits_2 = \sum_{n=1}^{\infty} \alpha_n^p n^{-2-pS} \left(\sum_{k=1}^{n} k^S a_k\right)^p.$$

If

$$(1.\ 9) \qquad \sum_{k=n}^{\infty} \alpha_k^p k^{-2-pS} \leqq K\alpha_n^p n^{-1-pS},$$

then there exists a $K_1 = K_1(\{\alpha_n\}, p, s)$ such that

$$(1.\ 10) \qquad \Sigma_2 \leqq K_1 \Sigma_1 .$$

If

$$(1.\ 11) \qquad \sum_{k=1}^{n} \alpha_k^p k^{-2} \leqq K\alpha_n^p n^{-1}$$

then there exists a $K_2 = K_2(\{\alpha_n\}, p, s)$ such that

$$(1.\ 12) \qquad \Sigma_1 \leqq K_2 \Sigma_2 .$$

We remark that for $p > 1$ and $\alpha_n^p = n^{2+c}$, with certain conditions on c, this lemma was proved by BOAS ([2], Lemma 5).

PROOF. First we prove (1. 10). We set $A_n = \sum_{k=n}^{\infty} a_k$. Then

$$\sum\nolimits_2 = \sum_{n=1}^{\infty} \alpha_n^p n^{-2-pS} \left(\sum_{k=1}^{n} k^S (A_k - A_{k+1})\right)^p \leqq K_3 \sum_{n=1}^{\infty} \alpha_n^p n^{-2-pS} \left(\sum_{k=1}^{n} k^{S-1} A_k\right)^p.$$

Hence, applying Lemma 1 with $\eta_n = \alpha_n^p n^{-2-ps}$ and $a_k = k^{s-1} A_k$, we obtain (1. 10).

We begin the proof of (1. 12) with proving the inequality

$$(1.\ 13) \qquad \sum\nolimits_3 = \sum_{k=3}^{\infty} \alpha_k^{\frac{p}{1-p}} k^{\frac{2-p}{p-1}} < \infty .$$

Let $\bar{\alpha}_n = \min_{2^n < k \leqq 2^{n+1}} \alpha_k^p k^{-2}$. Following the same lines as in the proof of Lemma 2, we obtain

$$2^n \sum_{k=(m-1)N+1}^{mN} 2^k \bar{\alpha}_k \leqq \sum_{k=(m+n)N+1}^{(m+n+1)N} 2^k \bar{\alpha}_k$$

for any m and n. Using this and the estimate

$$\sum\nolimits_3 = \sum_{m=1}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} \alpha_k^{\frac{p}{1-p}} k^{\frac{2-p}{p-1}} \leqq K_1 \sum_{m=1}^{\infty} (2^m \bar{\alpha}_m)^{\frac{1}{1-p}},$$

the same computation as in (1. 8) gives (1. 13).

Now we write

$$A_n = \sum_{k=1}^{n} k^S a_k.$$

Then

$$\sum_{k=n}^{\mu} a_k = \sum_{k=n}^{\mu} k^{-S}(A_k - A_{k-1}) \leqq K \sum_{k=n}^{\mu-1} k^{-S-1} A_k + A_\mu \mu^{-S}. \tag{1. 14}$$

Since

$$\sigma_{n,m} = \sum_{k=n}^{m} k^{-S-1} A_k = \sum_{k=n}^{m} A_k \alpha_k k^{-\frac{2}{p}} k^{-S} k^{\frac{2}{p}-1} \alpha_k^{-1};$$

for $p=1$

$$\sigma_{n,m} \leqq K \Sigma_2,$$

namely, by (1. 11), $\{n\alpha_n^{-1}\}$ is bounded; for $p>1$

$$\sigma_{n,m} \leqq K \left\{ \sum_{k=n}^{m} \alpha_k^p k^{-2-Sp} A_k^p \right\}^{\frac{1}{p}} \left\{ \sum_{k=n}^{m} \alpha_k^{\frac{p}{1-p}} k^{\frac{2-p}{p-1}} \right\}^{\frac{p-1}{p}},$$

and by (1. 13)

$$\sigma_{n,m} \leqq K \sum\nolimits_2^{\frac{1}{p}}.$$

Thus

$$\sum_{k=n}^{2n} k^{-S-1} A_k \to 0$$

which implies

$$n^{-S} A_n \to 0.$$

Therefore, by (1. 14), we have

$$\sum_{k=n}^{\infty} a_k \leqq K \sum_{k=n}^{\infty} k^{-S-1} A_k.$$

Applying Lemma 2 we obtain

$$\sum\nolimits_1 \leqq K \sum_{n=1}^{\infty} \alpha_n^p n^{-2} \left(\sum_{k=n}^{\infty} k^{-S-1} A_k \right)^p \leqq K \sum_{n=1}^{\infty} \alpha_n^p n^{-2} (n n^{-S-1} A_n)^p = K \sum\nolimits_2.$$

The proof is thus complete.

LEMMA 4. *Let* $r \geqq 1$ *and let* $p(x)$ $(x \geqq 0)$ *be a non-negative function and* $\lambda(x)$ $(x>0)$ *be a positive monotonic function having the properties:*

$$\lambda\left(\frac{1}{x}\right) = \lambda^{-1}(x) \tag{1. 15}$$

and

$$\sum_{k=1}^{n} k^{r-2}\lambda\left(\frac{1}{k}\right) \leqq K\lambda\left(\frac{1}{n}\right) n^{r-1}. \tag{1. 16}$$

If $p^r(x)\lambda(x)$ *is integrable on* $(0, \infty)$, *then* $\left(x^{-1}P(x)\right)^r\lambda(x)$ *is also integrable on* $(0, \infty)$ *and*

$$\int_0^\infty \left(\frac{P(x)}{x}\right)^r \lambda(x)\,dx \leqq K \int_0^\infty p^r(x)\lambda(x)\,dx, \tag{1. 17}$$

where

$$P(x) = \int_0^x p(t)\,dt.$$

We mention that Lemma 4 for $r>1$ can be deduced also from a theorem of SYSOEVA [5], but this deduction is almost the same long as to give an elementary direct proof for any $r \geqq 1$, therefore we choose the latter.

PROOF. We may assume $p(x) \not\equiv 0$. First we show that the estimate

$$\Lambda(x) = \int_x^\infty \lambda(t)t^{-r}\,dt \leqq K\lambda(x)x^{1-r} \tag{1. 18}$$

follows from (1. 16). If $\lambda(t)$ decreases then (1. 18) is obvious. Let $\lambda(t)$ increase, furthermore let

$$I_m = \int_{2^m}^\infty \lambda(t)t^{-r}\,dt \qquad (m=0, \pm 1, \pm 2, \ldots),$$

$$\varrho_i = 2^{i(r-1)}\lambda(2^{-i}) \quad \text{and} \quad r_i = \sum_{n=(i-1)N+1}^{iN} \varrho_n \qquad (i=1, 2, \ldots),$$

where N is an integer having, by (1. 16), the property:

$$\sum_{i=1}^{j} \varrho_i \leqq N\varrho_j \qquad (j=1, 2, \ldots). \tag{1. 19}$$

From (1. 19), by a straightforward calculation, we get

$$\sum_{i=1}^{j} r_i \leqq r_{j+1}$$

and

$$2^{n-1}r_m \leqq r_{m+n} \leqq N\varrho_{(n+m)N} \tag{1. 20}$$

for any positive integers m and n.

Since

$$I_m = \sum_{n=m+1}^{\infty} \int_{2^{n-1}}^{2^n} \lambda(t) t^{-r} dt \leqq K \sum_{n=m}^{\infty} \lambda(2^n) 2^{n(1-r)},$$

thus if $m \geqq 0$ and $\mu N \leqq m < (\mu+1)N$, then, by (1. 15), (1. 20) and $\lambda(t)\uparrow$, we obtain

$$\text{(1. 21)}\quad \begin{aligned} I_m &\leqq \sum_{i=\mu}^{\infty} \sum_{n=iN}^{(i+1)N} (\lambda(2^{-n})2^{n(r-1)})^{-1} \leqq K_1 \sum_{i=\mu}^{\infty} (\lambda(2^{-iN})2^{iN(r-1)})^{-1} \leqq \\ &\leqq K_2 (\lambda(2^{-\mu N})2^{\mu N(r-1)})^{-1} \leqq K_3 \lambda(2^m) 2^{m(1-r)}. \end{aligned}$$

If $m<0$, then by (1. 20)

$$\text{(1. 22)}\quad I_m \leqq \sum_{n=1}^{|m|} \lambda(2^{-n}) 2^{n(r-1)} + K \leqq K_1 \lambda(2^m) 2^{m(1-r)}.$$

If $2^m \leqq x < 2^{m+1}$, by (1. 21), (1. 22) and $\lambda(t)\uparrow$, we have

$$\Lambda(x) \leqq I_m \leqq K\lambda(x) x^{1-r},$$

which is the required (1. 18).

Now we can begin to prove (1. 17). If $r=1$, then by (1. 16) $\lambda^{-1}(t)$ increases, therefore for any $0 \leqq \xi < x$

$$\int_{\xi}^{x} p(t)\lambda(t)\lambda^{-1}(t)\, dt \leqq \lambda^{-1}(x) \int_{\xi}^{x} p(t)\lambda(t)\, dt,$$

that is

$$\text{(1. 23)}\quad P(x) = o(\lambda^{-1}(x)) \quad \text{as} \quad x \to 0 \quad \text{or} \quad x \to \infty.$$

If $r>1$, applying Hölder's inequality, we get

$$\text{(1. 24)}\quad \int_{\xi}^{x} p(t)\, dt \leqq \left\{ \int_{\xi}^{x} p^r(t)\lambda(t)\, dt \right\}^{\frac{1}{r}} \left\{ \int_{\xi}^{x} \lambda^{\frac{1}{1-r}}(t)\, dt \right\}^{\frac{r-1}{r}}.$$

Next we show that

$$\text{(1. 25)}\quad I(x) = \int_{0}^{x} \lambda^{\frac{1}{1-r}}(t)\, dt \leqq Kx\lambda^{\frac{1}{1-r}}(x).$$

If $\lambda^{\frac{1}{1-r}}(t)$ increases then (1. 25) is clear. If $\lambda^{\frac{1}{1-r}}(t)$ decreases and $2^{mN} \leqq x < 2^{(m+1)N}$ then

$$I(x) \leqq \int_{0}^{2^{(m+1)N}} \lambda^{\frac{1}{1-r}}(t)\, dt \leqq \sum_{n=-(m+1)N}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \lambda^{\frac{1}{1-r}}(t)\, dt \leqq$$

$$\leqq K \sum_{n=-(m+1)N}^{\infty} 2^{-n} \lambda^{\frac{1}{1-r}}(2^{-n}) = K \sum_{n=-(m+1)N}^{\infty} (2^{n(r-1)}\lambda(2^{-n}))^{\frac{1}{1-r}}.$$

If $m<0$, using (1. 20), we have

$$I(x) \leqq K \sum_{i=-(m+1)}^{\infty} \sum_{n=iN}^{(i+1)N} \left(2^{n(r-1)}\lambda(2^{-n})\right)^{\frac{1}{1-r}} \leqq K_1 \sum_{i=-(m+1)}^{\infty} \left(2^{iN(r-1)}\lambda(2^{-iN})\right)^{\frac{1}{1-r}} \leqq \tag{1. 26}$$

$$\leqq K_2\left(2^{-(m+1)N(r-1)}\lambda(2^{(m+1)N})\right)^{\frac{1}{1-r}} \leqq K_3 x\left(\lambda(x)\right)^{\frac{1}{1-r}}.$$

If $m \geqq 0$, by (1. 20) and $\left(\lambda(t)\right)^{\frac{1}{1-r}} \downarrow$, we get

$$I(x) \leqq \sum_{n=1}^{(m+1)N} \left(2^{-n(r-1)}\lambda(2^{-n})\right)^{\frac{1}{1-r}} + K \leqq K_1 \sum_{n=1}^{(m+1)N} \left(2^{n(r-1)}\lambda(2^{-n})\right)^{\frac{1}{r-1}} \leqq$$

$$\leqq K_1 \sum_{i=1}^{m+1} \sum_{n=(i-1)N+1}^{iN} \left(2^{n(r-1)}\lambda(2^{-n})\right)^{\frac{1}{r-1}} \leqq K_2 \sum_{i=1}^{m+1} \left(2^{iN(r-1)}\lambda(2^{-iN})\right)^{\frac{1}{r-1}} \leqq \tag{1. 27}$$

$$\leqq K_3\left(2^{(m+1)N(r-1)}\lambda(2^{-(m+1)N})\right)^{\frac{1}{r-1}} \leqq K_4 x \lambda^{\frac{1}{1-r}}(x).$$

Thus (1. 25) follows from (1. 26) and (1. 27).
From (1. 24) and (1. 25) we get

$$P(x) = o\left(x^{\frac{r-1}{r}} \lambda^{-\frac{1}{r}}(x)\right) \quad \text{as} \quad x \to 0 \quad \text{or} \quad x \to \infty. \tag{1. 28}$$

Let $0<a<b<\infty$. By partial integration we obtain

$$\int_a^b \left(\frac{P(x)}{x}\right)^r \lambda(x)\,dx \leqq \left[P^r(x)\Lambda(x)\right]_a^b + K_1 \int_a^b (P(x))^{r-1} p(x)\Lambda(x)\,dx. \tag{1. 29}$$

If $r=1$, in view of (1. 18) and (1. 23), (1. 29) implies the statement (1. 17). If $r>1$ we obtain, applying Hölder's inequality,

$$\int_a^b (P(x))^{r-1} p(x)\lambda(x)x^{1-r}\,dx \leqq \left\{\int_a^b p^r(x)\lambda(x)\,dx\right\}^{\frac{1}{r}} \left\{\int_a^b \left(\frac{P(x)}{x}\right)^r \lambda(x)\,dx\right\}^{\frac{r-1}{r}}.$$

From this and (1. 29), by (1. 18) and (1. 28), we obtain (1. 17), and this completes the proof.

§ 2. Proof of the theorems

(THEOREM 1) We first show that (6) implies (5) for sine series. Putting

$$G(y) = \int_{0+}^{y} \lambda(t)g(t)\,dt$$

and following the same lines as BOAS did ([2], Theorem 6) we get

$$\sum_{n=1}^{m} b_n \int_0^1 \lambda(t)\sin nt\,dt \leqq K_1 \tag{2. 1}$$

for any m.

In view of (2) we have

$$\sum_{n=1}^{m} \lambda_{2^n} \leqq N\lambda_{2^m}, \tag{2. 2}$$

where N is an integer independent of m. Let $\gamma_i = \sum_{n=(i-1)N+1}^{iN} \lambda_{2^n}$. Then, by (2. 2),

$$\sum_{i=1}^{j} \gamma_i \leqq \gamma_{j+1}.$$

Hence

$$2^{\nu-1}\gamma_\mu \leqq \gamma_{\mu+\nu} \tag{2. 3}$$

follows easily. By (2. 3) we obtain

$$2^\nu \lambda_{2^{\mu N}} \leqq \lambda_{2^{(\mu+\nu+2)N}}, \tag{2. 4}$$

and hence for any ν ($\geqq 1$) and for any positive t

$$2^\nu \lambda(t) \leqq \lambda(t \cdot 2^{-(\nu+4)N}). \tag{2. 5}$$

According to (2. 5) there exists a number $M(\geqq 2)$ such that for any positive t

$$\lambda\left(\frac{t}{M}\right) > 2\lambda(t).$$

Using this

$$\int_0^1 \lambda(t) \sin nt\, dt \geqq \int_0^{\frac{\pi}{2Mn}} \lambda(t) \sin nt\, dt,$$

which is graphically obvious. Since $\lambda(t)$ decreases

$$\int_0^{\frac{\pi}{2Mn}} \lambda(t) \sin nt\, dt \geqq \frac{K_2}{n} \lambda\left(\frac{1}{n}\right),$$

thus

$$\int_0^1 \lambda(t) \sin nt\, dt \geqq \frac{K_2}{n} \lambda\left(\frac{1}{n}\right).$$

Hence and from (2. 1) we obtain that

$$\sum_{n=1}^{\infty} \lambda_n n^{-1} b_n$$

converges.

To prove that (5) follows from (6) for cosine series we put

$$F(y) = \int_{0+}^{y} \lambda(t) f(t)\, dt \quad \text{and} \quad F_1(y) = \int_0^y f(t)\, dt.$$

By (6) $F(y)\to 0$ as $y\to 0^+$. Furthermore

$$(2.6)\qquad \begin{aligned} |F_1(y)| &= \left|\int_0^y f(t)\,dt\right| = \left|\int_0^y \lambda^{-1}(t)\,dF(t)\right| = \\ &= \lambda^{-1}(y)F(y) + \left|\int_0^y \frac{\lambda'(t)}{\lambda^2(t)}F(t)\,dt\right| = o\big(\lambda^{-1}(y)\big). \end{aligned}$$

Hence

$$\lambda(y)y^{-1}\int_0^y F_1(t)\,dt = o(1) \quad \text{as} \quad y\to 0^+.$$

If $0<\varepsilon<1$, we have

$$\begin{aligned} \int_\varepsilon^1 \lambda'(t)F_1(t)\,dt &= \int_\varepsilon^1 \left(\lambda'(t)\int_0^t f(u)\,du\right)dt = \\ &= \int_0^\varepsilon f(u)\,du\int_\varepsilon^1 \lambda'(t)\,dt + \int_\varepsilon^1 f(u)\,du \int_u^1 \lambda'(t)\,dt = \\ &= -F_1(\varepsilon)\big(\lambda(\varepsilon)-\lambda(1)\big) - \int_\varepsilon^1 \lambda(u)f(u)\,du + \int_\varepsilon^1 f(u)\lambda(1)\,du. \end{aligned}$$

If $\varepsilon\to 0$ we obtain

$$-\int_{0^+}^1 \lambda'(t)F_1(t)\,dt \leqq K.$$

Thus, if $0<y<1$

$$(2.7)\qquad \int_y^1 |\lambda'(t)|F_1(t)\,dt = \frac{a_0}{2}\int_y^1 |\lambda'(t)|t\,dt + \sum_{n=1}^\infty \frac{a_n}{n}\int_y^1 |\lambda'(t)|\sin nt\,dt \leqq K_1.$$

Similarly to the proof of (2. 4), and using (3) instead of (2), one can prove that

$$2^\nu \frac{2^{\mu N}}{\lambda_{2^{\mu N}}} \leqq K_2 \frac{2^{(\mu+\nu)N}}{\lambda_{2^{(\mu+\nu)N}}}$$

is valid for any $\nu, \mu(\geqq 1)$. Using this, (2) and (3) we get

$$(2.8)\qquad \sum_{k=m}^\infty \lambda_k k^{-2} \leqq K_3\lambda_m m^{-1}.$$

Let us define a function $\lambda^*(t)$ on $(0, 1]$ as follows:

$$\lambda^*(t) = (\lambda_{n+1}-\lambda_n)n(n+1) \quad \text{if} \quad t\in\left(\frac{1}{n+1}, \frac{1}{n}\right].$$

It is clear that

$$(2.9)\qquad \lambda^*(t) = |\lambda'(t)| \quad \text{if} \quad t\in\left(\frac{1}{n+1}, \frac{1}{n}\right]$$

and, by (4), $\lambda^*(t)$ decreases on (0, 1]. Thus, by (2. 8), we have

$$\lambda^*(x)x^2 \leqq K_4 \int_0^x \lambda^*(t)tdt \leqq K_5 \sum_{k=\left[\frac{1}{x}\right]}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \lambda^*(t)tdt \leqq K_6 \sum_{k=\left[\frac{1}{x}\right]}^{\infty} \lambda_k k^{-2} \leqq K_7 \lambda(x)x$$

i.e.

(2. 10) $$\lambda^*(x)x \leqq K_7 \lambda(x).$$

Hence and from (2. 8) and (2. 9) we get

$$\int_0^1 |\lambda'(t)|tdt \leqq K_8,$$

and thus, by (2. 7) and (2. 9),

(2. 11) $$\left|\sum_{n=1}^{\infty} n^{-1} a_n \int_y^1 \lambda^*(t)\sin ntdt\right| \leqq K_9.$$

By (3), (2. 6), (2. 9) and (2. 10) we also have

$$\lambda^*(y)\sum_{n=1}^{\infty} n^{-1} a_n \int_0^y \sin ntdt = \lambda^*(y)\int_0^y F_1(t)dt - \lambda^*(y)\int_0^y \frac{1}{2} a_0 tdt \leqq K.$$

Hence and from (2. 11) we obtain

$$\left|\lambda^*(y)\sum_{n=1}^{\infty} n^{-1} a_n \left\{\int_0^y \sin ntdt + (\lambda^*(y))^{-1}\int_y^1 \lambda^*(t)\sin ntdt\right\}\right| \leqq K_1.$$

Observing that the expression in braces is nonnegative, the second integral can be expressed, by the second mean-value theorem, in the form

$$\lambda^*(y)\int_y^u \sin ntdt \qquad y \leqq u < 1.$$

We have for any m

$$0 \leqq \lambda^*(y)\sum_{n=1}^{m} n^{-1} a_n \left\{\int_0^y \sin ntdt + (\lambda^*(y))^{-1}\int_y^1 \lambda^*(t)\sin ntdt\right\} \leqq K_1.$$

Hence if $y \to 0$ we get

(2. 12) $$\sum_{n=1}^{m} n^{-1} a_n \int_0^1 \lambda^*(t)\sin ntdt \leqq K_1.$$

By (2. 4) for any K there exists an integer $q = q(K)$ such that for any n

(2. 13) $$\lambda_{nq} > K\lambda_n.$$

Using (2. 10) and (2. 13) with $K=8K_7$ $(K_7\geqq 1)$, we have

$$\lambda^*\left(\frac{1}{n}\right)\frac{1}{n}\leqq K_7\lambda\left(\frac{1}{n}\right)=K_7\lambda_n<\frac{1}{8}\lambda_{nq}<\frac{1}{4}(\lambda_{nq}-\lambda_n)=\frac{1}{4}\int_{\frac{1}{nq}}^{\frac{1}{n}}\lambda^*(t)\,dt\leqq\frac{1}{4n}\lambda^*\left(\frac{1}{nq}\right),$$

therefore, for any positive t, there exists an integer $N(\geqq 1)$ such that

$$\lambda^*\left(\frac{t}{N}\right)>2\lambda^*(t).$$

In virtue of this

$$\int_0^1\lambda^*(t)\sin nt\,dt\geqq\int_0^{\frac{\pi}{2nN}}\lambda^*(t)\sin nt\,dt, \tag{2.14}$$

which is obvious graphically. By (4)

$$\int_0^{\frac{\pi}{2nN}}\lambda^*(t)\sin nt\,dt\geqq\frac{2}{\pi}n\int_0^{\frac{1}{8nN}}\lambda^*(t)\,t\,dt\geqq\frac{n}{2}\sum_{k=8nN}^{\infty}\frac{1}{k+1}(\lambda_{k+1}-\lambda_k)\geqq K_1\lambda_{8nN}\geqq K_1\lambda_n.$$

Hence, by (2. 12) and (2. 14), we obtain that

$$\sum_{n=1}^{\infty}\lambda_n n^{-1}a_n$$

converges.

This proves (5) for cosine series.

We now show that (5) implies (6). Suppose $\Sigma\lambda_n n^{-1}b_n<\infty$. Since, by (2. 8), for any $0\leqq a<\pi$ and $0<x<y<\pi-a$

$$\left|\int_{a+x}^{a+y}\lambda(t-a)\sin nt\,dt\right|=\left|\sin na\int_x^y\lambda(t)\cos nt\,dt+\cos na\int_x^y\lambda(t)\sin nt\,dt\right|\leqq$$

$$\leqq 2\int_0^{\frac{4}{n}}\lambda(t)\,dt\leqq 2\sum_{k=\frac{n}{4}}^{\infty}\int_{\frac{1}{k+1}}^{\frac{1}{k}}\lambda(t)\,dt\leqq K\sum_{k=\frac{n}{4}}^{\infty}\lambda_{k+1}\frac{1}{k^2}\leqq K_1\frac{\lambda_n}{n},$$

thus

$$\sum_{n=1}^{\infty}b_n\int_{a+x}^{a+y}\lambda(t-a)\sin nt\,dt \tag{2.15}$$

converges uniformly in x and y. Therefore (2. 15) tends to a limit as $x\to 0^+$. On the other hand, since $\Sigma b_n\sin nx$ is a Fourier series, (2. 15) is equal to

$$\int_{a+x}^{a+y}\lambda(t-a)g(t)\,dt.$$

Hereby we proved that

$$\int_{a^+}^{a+y} \lambda(t-a)g(t)\,dt < \infty.$$

For cosine series the same argument gives that $\Sigma \lambda_n n^{-1} a_n < \infty$ implies

$$\int_{a^+}^{a+y} \lambda(t-a)f(t)\,dt < \infty.$$

The proof is complete.

(THEOREM 2) First we prove that (10) with $a=0$ implies (11) and (12) for sine series. Since $\lambda(x)g(x) \in L$ we have

$$\int_x^y \lambda(t)g(t)\,dt = \sum_{n=1}^{\infty} b_n \int_x^y \lambda(t)\sin nt\,dt. \tag{2.16}$$

On the other hand, by (8) and (9), a standard computation gives that

$$\sum_{n=m}^{\infty} \frac{\lambda_n^p}{n^{p+2}} \leqq K \frac{\lambda_m^p}{m^{p+1}} \tag{2.17}$$

and thus we have

$$\sum_{n=m}^{\infty} \lambda_n \frac{1}{n^3} \leqq \left\{\sum_{n=m}^{\infty} \frac{\lambda_n^p}{n^{p+2}}\right\}^{\frac{1}{p}} \left\{\sum_{n=m}^{\infty} \frac{1}{n^2}\right\}^{\frac{p-1}{p}} \leqq K_1 \frac{\lambda_m}{m^2}.$$

Using this we obtain for any $0<x<y<\pi$

$$\left|\int_x^y \lambda(t)\sin nt\,dt\right| \leqq n \int_0^{\frac{4}{n}} \lambda(t)t\,dt \leqq K_2 n \sum_{k=\frac{n}{4}}^{\infty} \lambda_k \frac{1}{k^3} \leqq K_3 \frac{\lambda_n}{n}.$$

Consequently, the series on the right in (2.16) converges uniformly in x and y. Hence, if $x \to 0^+$, we obtain

$$\int_0^y \lambda(t)g(t)\,dt = \sum_{n=1}^{\infty} b_n \int_0^y \lambda(t)\sin nt\,dt. \tag{2.18}$$

Applying Hardy's inequality (see Lemma 4 if $r=p>1$, $\lambda(x)\equiv 1$ and $p(t)=|\lambda(t)g(t)|$) to the left-hand side of (2.18), we get

$$\int_0^{\pi} \left\{\frac{1}{y} \int_0^y \lambda(t)|g(t)|\,dt\right\}^p dy \leqq K_4 \int_0^{\pi} |\lambda(t)g(t)|^p\,dt.$$

Hence, by (10) and (2.18),

$$\int_0^{\pi} \left(\sum_{n=1}^{\infty} b_n \frac{1}{y} \int_0^y \gamma(t)\sin nt\,dt\right)^p dy < K_5$$

follows. Thus, by $\lambda(x)\downarrow$, we have

$$K_5 > \int_0^1 \left(\sum_{n=1}^{\frac{1}{x}} b_n \frac{1}{x} \int_0^x \lambda(t) \sin nt dt \right)^p dx \geqq K_6 \int_0^1 \left(\sum_{n=1}^{\frac{1}{x}} b_n n\lambda(x) x \right)^p dx \geqq$$

$$\geqq K_7 \sum_{k=1}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \left(\sum_{n=1}^{k} b_n n \lambda_k \frac{1}{k} \right)^p dx \geqq K_8 \sum_{k=1}^{\infty} k^{-2-p} \lambda_k^p \left(\sum_{n=1}^{k} b_n n \right)^p,$$

which gives (11). By (8) and (9) (see also (2. 17)) we can apply Lemma 3 with $\alpha_n = \lambda_n$ and $s=1$, and thus we obtain that (11) is equivalent to (12).

For cosine series therefore it is enough to prove that (10) with $a=0$ implies (12). First we show that

$$x^{-1}(f(x)-f(0)) \in L. \tag{2. 19}$$

By (10)

$$\int_0^{\pi} t^{-1} |f(t)-f(0)| dt \leqq \left\{ \int_0^{\pi} (t\lambda(t))^{\frac{p}{1-p}} dt \right\}^{\frac{p-1}{p}} \left\{ \int_0^{\pi} \lambda^p(t) |f(t)-f(0)|^p dt \right\}^{\frac{1}{p}} \leqq$$

$$\leqq K_1 \left\{ \sum_{k=1}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} (t\lambda(t))^{\frac{p}{1-p}} dt \right\}^{\frac{p-1}{p}} \leqq K_2 \left\{ \sum_{k=1}^{\infty} k^{\frac{2-p}{p-1}} \lambda_k^{\frac{p}{1-p}} \right\}^{\frac{p-1}{p}} = \tag{2. 20}$$

$$= K_3 \left\{ \sum_{n=1}^{\infty} (2^n \lambda_{2^n}^{-p})^{\frac{1}{p-1}} \right\}^{\frac{p-1}{p}}.$$

Similarly to the proof of (2. 4), using (8) instead of (2), it can be proved that

$$2^{\nu} \lambda_{2^{\mu N}}^p 2^{-\mu N} \leqq K_4 \lambda_{2^{(\mu+\nu)N}}^p 2^{-(\mu+\nu)N} \tag{2. 21}$$

holds for any $\nu, \mu (\geqq 1)$. Consequently

$$\sum_{n=1}^{\infty} (2^n \lambda_{2^n}^{-p})^{\frac{1}{p-1}} < K_5$$

and thus, by (2. 20), (2. 19) is proved.

According to (2. 19)

$$f(0)-f(x) = \sum_{n=1}^{\infty} a_n (1-\cos nx).$$

Hence

$$\int_0^x \lambda(t)(f(0)-f(t)) dt = \sum_{n=1}^{\infty} a_n \int_0^x \lambda(t)(1-\cos nt) dt,$$

and in view of (10), by Hardy's inequality, we obtain

$$\int_0^{\pi}\left(\sum_{n=1}^{\infty} a_n \frac{1}{x}\int_0^x \lambda(t)(1-\cos nt)\,dt\right)^p dx = \int_0^{\pi}\left(\frac{1}{x}\int_0^x \lambda(t)\big(f(0)-f(t)\big)\,dt\right)^p dx \leqq$$

$$\leqq K\int_0^{\pi} \lambda^p(t)\big(f(0)-f(t)\big)^p\,dt \leqq K_1.$$

Since the terms of the series are non-negative, we have

$$K_1 \geqq \sum_{k=2}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \left\{\sum_{n=1}^{k} a_n k \int_0^{\frac{1}{k+1}} \lambda(t)(1-\cos nt)\,dt\right\}^p dx \geqq$$

$$\geqq K_2 \sum_{k=2}^{\infty} k^{p-2}\left(\sum_{n=1}^{k} a_n n^2 \int_0^{\frac{1}{k+1}} \lambda(t)t^2\,dt\right)^p \geqq K_3 \sum_{k=2}^{\infty} k^{-2p-2}\lambda_k^p\left(\sum_{n=1}^{k} n^2 a_n\right)^p.$$

Using Lemma 3 with $\alpha_n=\lambda_n$ and $s=2$, we obtain (12).

Now we prove that (11) and (12) (as has been proved (11) is equivalent to (12)) imply (10) for any $0\leqq a<\pi$. We prove this only for sine series because the proof for cosine series runs similarly.

We have

$$g(x)-g(a) = 2\sum_{n=1}^{\infty} b_n \cos\frac{1}{2}n(x+a)\sin\frac{1}{2}n(x-a)$$

and

$$\int_a^{\pi+a}\left|\lambda(x-a)\sum_{k=1}^{N} b_k \cos\frac{1}{2}k(x+a)\sin\frac{1}{2}k(x-a)\right|^p dx \leqq$$

$$\leqq \int_0^{\pi}\left|\lambda(2t)\sum_{k=1}^{\left[\frac{1}{t}\right]} b_k \sin kt\right|^p dt + \int_0^{\pi}\left(\lambda(2t)\sum_{k=\left[\frac{1}{t}\right]+1}^{N} b_n|\sin kt|\right)^p dt \equiv I_1+I_2.$$

By (11)

$$I_1 \leqq K\sum_{n=2}^{\infty}\int_{\frac{1}{n}}^{\frac{1}{n-1}}\left(\lambda(t)t\sum_{k=1}^{\left[\frac{1}{t}\right]} kb_k\right)^p dt \leqq K_1\sum_{n=1}^{\infty}\lambda_n^p n^{-p-2}\left(\sum_{k=1}^{n} kb_k\right)^p \leqq K_2,$$

and by (12)

$$I_2 \leqq K_3\left(1+\sum_{n=2}^{\infty}\int_{\frac{1}{n}}^{\frac{1}{n-1}}\left(\lambda(t)\sum_{k=n}^{\infty} b_k\right)^p dt\right) \leqq K_4\left(1+\sum_{n=1}^{\infty}\lambda_n^p n^{-2}\left(\sum_{k=n}^{\infty} b_k\right)^p\right) \leqq K_5.$$

Summing up we get the required statement (10).

We have proved Theorem 2.

(THEOREM 3) The proof for sine series is similar to that of cosine series, therefore we carry out only the last one. By (14) and (16) $f(x)$ is integrable. Namely, by (14), for any $\nu, \mu (\geqq 1)$

$$2^{\nu} \lambda_{2^{\mu N}}^{p} 2^{\mu(p-1)N} \leqq K_1 \lambda_{2^{(\mu+\nu)N}}^{p} 2^{(\mu+\nu)(p-1)N}$$

holds (for detail and the meaning of N we refer to the proof of (2. 4)). Thus we have

$$\sum\nolimits_1 \equiv \sum_{n=1}^{\infty} (\lambda_{2^n}^{p} 2^{n(p-1)})^{\frac{1}{1-p}} < \infty$$

and by Hölder's inequality

$$\int_0^{\pi} |f(x)| dx \leqq \left\{ \int_0^{\pi} |f(x)|^p \lambda^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\pi} \lambda(x)^{\frac{p}{1-p}} dx \right\}^{\frac{p-1}{p}} \leqq$$

$$\leqq K_2 \left\{ \sum_{k=1}^{\infty} k^{-2} \lambda_k^{\frac{p}{1-p}} \right\}^{\frac{p-1}{p}} \leqq K_3 \left\{ \sum_{n=1}^{\infty} 2^{-n} \lambda_{2^n}^{\frac{p}{1-p}} \right\}^{\frac{p-1}{p}} \equiv K_3 \left\{ \sum\nolimits_1 \right\}^{\frac{p-1}{p}} < \infty .$$

Let us suppose for simplicity that $a_0 = 0$. Then we have

$$f_1(x) = \int_0^x f(t) dt = \sum_{n=1}^{\infty} n^{-1} a_n \sin nx.$$

An application of Lemma 4 gives that $\lambda(x) x^{-1} f_1(x) \in L^p$, therefore we can apply Theorem 2 with $f_1(x)$, $\{\lambda_n n\}$ and $a = 0$.

Hence, by (11) and (12), the statements (17) and (18) hold, which is the required proof.

(THEOREM 4) Here we detail only the sine version. For simplicity we suppose that $b_1 = b_2 = 0$. Since $g(x)$ is integrable we have

$$2g(x) \sin x \sim \sum_{n=1}^{\infty} (b_{n+2} - b_n) \cos (n+1) x.$$

Then, by (19), (20) and (21), we can apply Theorem 2 with $2g(x) \sin x$, $\{\lambda_n n\}$ and $a = 0$ (cf. the remark mentioned before the Theorem 4); thus we obtain the required (22).

The proof for cosine series is similar, therefore we omit it.

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A NOTE ON BOUNDED PRIMARY GROUPS

By

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Let p be a prime and let G be an abelian group which is metric in its p-adic topology. Denote by G^* the completion of G. It is well known that if H is a subgroup of G, then the map of H^* into G^* induced by the inclusion map of H into G is not necessarily monic. If, however, G is a bounded p-primary group, then G is complete as are all of its subgroups H and in this case the induced maps are obviously monic. Among the p-primary groups this property actually characterizes the bounded ones. More precisely, one has the following theorem.

THEOREM. *Let G be a p-primary metric group. Then G is bounded if and only if for all subgroups H of G the induced map of H^* into G^* is monic.*

PROOF. We must show that if G is not bounded then G contains a subgroup H for which the induced map of H^* into G^* is not monic. Let B be a basic subgroup of G. Suppose that B were bounded. Then, since B is pure in G, B would be a summand. Say $G = B \otimes S$. Since G/B is divisible, S is divisible. But G is metric and therefore $S=0$ which implies that $G=B$ is bounded. It follows that B is not bounded.

Choose cyclic summands $B_1, B_2, B_3, \ldots$ of B of orders $p^{n_1}, p^{n_2}, p^{n_3}, \ldots$ where $1<n_1<n_2\ldots$ and let u_k be a generator of B_k. Define elements x_k and y_k by

$$x_k=p^{n_k-1}u_k, \quad y_k = p^{n_{k+1}-n_k}u_{k+1}-u_k \qquad (k=1,2,3,\ldots).$$

Let H be the subgroup of G generated by the x_k's and y_k's. Since

$$p^{n_k-1}y_k = p^{n_{k+1}-1}u_{k+1}-p^{n_k-1}u_k = x_{k+1}-x_k$$

the sequence $\{x_k\}$ is Cauchy in H. Let x^* be the limit of the x_k's in H^*. In G one has $x_k=p^{n_k-1}u_k$ and therefore $\{x_k\}$ is a null sequence in G and hence in G^*. In order to show then that the induced map of H^* into G^* is not $1-1$ it is sufficient to show that $x^*\neq 0$.

Suppose $x^*=0$. Then for a sufficiently large j $x_j \in p^{n_1-1}H^*$. Therefore

$$x_1 = \sum_{i=1}^{j-1}(x_i-x_{i+1})+x_j = -\sum_{i=1}^{j-1} p^{n_i-1}y_i+x_j \in p^{n_1-1}H^*.$$

Since H is pure in H^*, $x_1 \in p^{n_1-1}H$. Let $x\in H$ be such that $p^{n_1-1}x=x_1$. Since each of the x_k's is of order p, we may assume that x is of the form

$$x = \sum_{k=1}^{s} a_k y_k.$$

Thus

$$p^{n_1-1}u_1 = x_1 = p^{n_1-1}x = p^{n_1-1}\sum_{k=1}^{s} a_k y_k = p^{n_1-1}\sum_{k=1}^{s} a_k(p^{n_{k+1}-n_k}u_{k+1}-u_k) =$$

$$= p^{n_1-1}\left(a_1u_1 + \sum_{k=1}^{s-1}(a_k p^{n_{k+1}-n_k} - a_{k+1})u_{k+1} + a_s p^{n_{s+1}-n_s}u_{s+1}\right).$$

Therefore

$$p^{n_1-1} \equiv a_1 p^{n_1-1} (p^{n_1})$$

$$p^{n_1-1}(a_k p^{n_{k+1}-n_k} - a_{k+1}) \equiv 0 \quad (p^{n_{k+1}}) \qquad (k=1, 2, \dots, s-1)$$

$$a_s p^{n_{s+1}-n_s+n_1-1} \equiv 0 \quad (p^{n_{s+1}})$$

The last equation implies that $a_s \equiv 0(p^{n_s-n_1+1})$. Suppose now that $a_{k+1} \equiv \equiv 0(p^{n_{k+1}-n_1+1})$. Then from the k^{th} equation one has

$$0 \equiv p^{n_1-1}(a_k p^{n_{k+1}-n_k} - a_{k+1}) \equiv a_k p^{n_{k+1}-n_k+n_1-1}(p^{n_{k+1}}).$$

Therefore $a_k \equiv 0(p^{n_k-n_1+1})$. For $k=1$ this gives $a_1 \equiv 0(p)$. But this contradicts the first equation. Therefore it must be the case that the equation $x_1 = p^{n_1-1}x$ is not solvable in H and thus $x^* \neq 0$.

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ON TWO POLYNOMIAL INEQUALITIES. I.

By

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A. Introduction. The aim of this paper is the proof and the application of the following results:

THEOREM 1. *Let ε be an arbitrary fixed number, $\pi_n(x)$ a polynomial of degree n at most satisfying*

(1) $$|\pi_n(x)| \leqq e^{x^2/2} \qquad [|x| \leqq (1+\varepsilon)\sqrt{2n}]$$

then we have

(2) $$|\pi_n(x)| \leqq c_1(\varepsilon) e^{x^2/2} \qquad (-\infty < x < +\infty)$$

and

(3) $$|\pi_n'(x)| \leqq c_2(\varepsilon) n^{1/2} e^{x^2/2} \qquad (-\infty < x < +\infty)$$

Here (and in what follows) $c_k(\varepsilon)$ $(k=1, 2, \ldots)$ denote positive numbers depending on ε at most.

In turn, we are going to show, that the value $(1+\varepsilon)\sqrt{2n}$ in (1) is optimal.

THEOREM 2. *For an arbitrary $\varepsilon > 0$ there exists a sequence $\{\varrho_n(\varepsilon; x)\}$ of polynomials in x so that the degree of $\varrho_n(\varepsilon; x)$ is n at most, we have*

(4) $$|\varrho_n(\varepsilon; x)| \leqq e^{x^2/2} \qquad [|x| \leqq (1-\varepsilon)\sqrt{2n};\ n=1, 2, \ldots]$$

but in spite of this there exist two real sequences $\{\zeta_n\}$ and $\{\zeta_n^*\}$ so that

(5) $$\lim_{n\to\infty} e^{-\zeta_n^2/2} |\varrho_n(\varepsilon; \zeta_n)| = \infty$$

and

(6) $$\lim_{n\to\infty} n^{-1/2} e^{-\zeta_n^{*2}/2} |\varrho_n'(\varepsilon; \zeta_n^*)| = \infty.$$

Further, we show that the right side of (3) gives us the correct order of magnitude:

THEOREM 3. *There exists a sequence of polynomials $\{\sigma_n(x)\}$, so that the degree of σ_n is n at most, we have*

$$\max |\sigma_n(x)| \leqq e^{x^2/2} \qquad (-\infty < x < +\infty;\ n=1, 2, \ldots)$$

and

$$\max_{|x| \leqq 1} |\sigma_n'(x)| \geqq A_0 n^{1/2}$$

with an absolute constant A_0.

In what follows, we denote also by A_i $(i=0, 1, 2, \ldots)$ absolute positive constans.

Theorem 1 sharpens a special case of a theorem of M. M. DZRBASIAN [1] who proved that from $|\pi_n(x)| \leqq e^{x^2/2}$ follows $\pi_n'(x) = O(n^{1/2})$ uniformly in every finite interval and gave analogous results for other weight-functions. The estimation of the derivative under the condition $|\pi_n(x)| \leqq e^x$ was treated by G. SZEGŐ [8]. The special case of Dzrbasian's result which we have in mind could be easily deduced from Szegő's result. In turn, by the transformation $x = t^2/2$ we obtain from Theorem 1:

THEOREM 1*. *Let $\varepsilon > 0$, $\pi_n(x)$ be a polynomial with degree n at most and*

(1*) $$|\pi_n(x)| \leqq e^x \qquad [0 \leqq x \leqq (1+\varepsilon)2n]$$

then we have

(2*) $$|\pi_n(x)| \leqq c_1^*(\varepsilon)e^x \qquad (0 \leqq x < +\infty)$$

and

(3*) $$|\pi_n'(x)| \leqq c_2^*(\varepsilon)x^{-1/2}n^{1/2}e^x \qquad (0 \leqq x < +\infty)$$

For large values of x this is sharper than Szegő's result. Nevertheless Theorem 1* does not give any estimate for $\pi_n'(0)$, but G. SZEGŐ [8] proved also directly that $\pi_n'(0) = O(n)$. We remark that this could be deduced by a further argument from (3*). According to a personal communication, Prof. P. TURÁN is able to deduce a result similar to Theorem 1 along the lines of G. SZEGŐ's paper quoted above.

The proof of these theorems is based on properties of Hermite's orthogonal polynomials $H_m(x)$. Concerning all the properties of $H_m(x)$ which we do not treat here in detail, we refer to the book of G. SZEGŐ [7].

We use the notation $z_m(x) = e^{-x^2/2}H_m(x)$; then

(7) $$z_m'' + (2m+1-x^2)z_m = 0$$

so that there are $m+1$ values $\xi_{0m} > \xi_{1m} > \cdots > \xi_{mm}$ for which $z_m(x)$ attains a local extremum with alternating sign and for which we have

(8) $$|\xi_{km}| \leqq \sqrt{2m+1} \qquad (k=0, 1, \ldots, m)$$

From (7) we conclude

(9) $$x\frac{d}{dx}\left[z_m^2 + (2m+1-x^2)^{-1}z_m'^2\right] \geqq 0$$

and

(10) $$x\frac{d}{dx}\left[(2m+1-x^2)z_m^2 + z_m'^2\right] \leqq 0.$$

From (9) and (10) it follows that for fixed m, $|z_m(\xi_{km})|$ is an increasing function of $|\xi_{km}|$ and that $\sqrt{2m+1-\xi_{km}^2}\,|z_m(\xi_{km})|$ is a decreasing function of $|\xi_{km}|$. Let now m be a number divisible by 4, so that $z_m(0)$ is a local extremum and $z_m(0) > 0$. We obtain

(11) $$H_m(0)e^{\xi_{km}^2/2} \leqq (-1)^k H_m(\xi_{km}) \leqq \sqrt{\frac{2m+1}{2m+1-\xi_{km}^2}}\,H_m(0)e^{\xi_{km}^2/2}.$$

B. Proof of the inequality (2). Without restricting our statement we can suppose that n is so large that there exists an m divisible by 4 and

(12) $$n<m<(1+\varepsilon/2)^2 n.$$

Then by (8) all the values ξ_{km} are between

$$-(1+\varepsilon/2)\sqrt{2n} \quad \text{and} \quad (1+\varepsilon/2)\sqrt{2n}.$$

Let $0<\eta<1$. We consider the polynomials

(13) $$\Phi_{nj}(x)=(1+\eta)\frac{H_m(x)}{H_m(0)}+(-1)^j\pi_n(x) \qquad (j=1,2).$$

As the ξ_{km} are ordered decreasingly, we have for every $0<\eta<1$ by (1) and (11)

(14) $$(-1)^k\Phi_{nj}(\xi_{km})>0 \qquad (k=0,1,\dots,m).$$

It follows that $\Phi_{n1}(x)$, as well as $\Phi_{n2}(x)$, has at least one zero in each of the intervals $(\xi_{rm},\xi_{r-1,m})$ $(r=1,2,\dots,m)$.

This makes m zeros and Φ_{nj} $(j=1,2)$ has the degree m. We conclude that these are all the zeros of Φ_{n1} and Φ_{n2}. In particular, we must have

$$\Phi_{nj}(x)>0 \qquad (j=1,2;\ |x|>\xi_{0m}=-\xi_{mm}),$$

i.e.

(15) $$|\pi_n(x)|\leqq(1+\eta)\frac{H_m(x)}{H_m(0)}<2\frac{H_m(x)}{H_m(0)} \qquad (|x|>\xi_{0m}).$$

By the second of Plancherel—Rotach's asymptotic formulas, we have for $(1+\varepsilon/4)\sqrt{2m+1}\leqq x\leqq 2\sqrt{2m+1}$

(16) $$\frac{H_m(x)}{H_m(0)}\leqq c_1(\varepsilon)e^{x^2/2}.$$

From the differential equation (7) we see that $e^{-x^2/2}H_m(x)$ is a positive decreasing function for $x\geqq\sqrt{2m+1}$, so that (16) holds for all $x>\sqrt{2m+1}\,(1+\varepsilon/4)$. Because $H_m(x)$ is even, (16) is true for all $|x|\geqq(1+\varepsilon/4)\sqrt{2m+1}$ and so by (12) (at least for sufficiently large n) for $|x|\geqq(1+\varepsilon)\sqrt{2n}$. From (15) and (16) we see that (2) holds for $|x|\geqq(1+\varepsilon)\sqrt{2n}$ and for $|x|<(1+\varepsilon)\sqrt{2n}$ it is true by our assumption (1); this proves (2).

C. Proof of the inequality (3). Let in what follows — in contrast to (12) — $m=4n$, $|\Theta_m|\leqq 10m^{-1/2}$ and

$$\chi_m(x+\Theta_m)=H_m(x+\Theta_m)/H_m(0).$$

LEMMA 1. *At the points* $\xi^*_{km}=\xi_{km}-\Theta_m$ $(k=0,1,\dots,m)$ *we have*

(17) $$(-1)^k\chi_m(\xi^*_{km}+\Theta_m)>A_1e^{\xi^{*2}_{km}/2}.$$

PROOF. We use (11) and (8).

$$(-1)^k \chi_m(\xi^*_{km}+\Theta_m) = (-1)^k \frac{H_m(\xi_{km})}{H_m(0)} \geqq e^{\xi^2_{km}/2} =$$

$$= e^{(\xi^*_{km}+\Theta_m)^2/2} \geqq e^{\xi^{*2}_{km}/2} e^{-10\frac{\sqrt{2m+1}}{\sqrt{m}}} > A_1 e^{\xi^{*2}_{km}/2}.$$

Q.E.D.

We turn now to the proof of (3). For $|x| \leqq \sqrt{m} = 2\sqrt{n}$ the factor of (7) is

$$2m+1-x^2 > 2m+1-m > m,$$

so that by Sturm's theorem each two of the consecutive zeros of $z_m(x)$ inside $[-2\sqrt{n}, 2\sqrt{n}]$ have a distance less than $\frac{\pi}{\sqrt{m}}$. Between each pair of consecutive zeros of $z_m(x)$ there is exactly one ξ_{km}, i.e. we have

$$\xi_{km}-\xi_{k+1,m} < \frac{2\pi}{\sqrt{m}}; \quad \xi_{km}, \xi_{k+1,m} \in [-2\sqrt{n}, 2\sqrt{n}].$$

We conclude from this and (11), (17) that for each $|x| \leqq 3/2 \cdot \sqrt{n}$ there are two numbers Θ'_m, Θ''_m so that

a) $$-\frac{10}{\sqrt{m}} < \Theta'_m \leqq 0 \leqq \Theta''_m < \frac{10}{\sqrt{m}};$$

b) $$|\chi_m(x+\Theta'_m)| > A_1 e^{x^2/2} \quad \text{and} \quad |\chi_m(x+\Theta''_m)| > A_1 e^{x^2/2};$$

c) $\chi_m(x+\Theta'_m)$ and $\chi(x+\Theta''_m)$ have opposite signs;

d) $\chi_m(x+\Theta_m)$ is monotone for $\Theta_m \in [\Theta'_m, \Theta''_m]$.

Let us now consider a fixed value $|x| \leqq 2\sqrt{n}$ and a fixed $0<\eta<1$. If $\pi_n(x)$ satisfies (1), we know that also (2) is satisfied. It follows that there exist two numbers $|\Theta^{(1)}_m| \leqq \frac{10}{\sqrt{m}}$ and $|\Theta^{(2)}_m| \leqq \frac{10}{\sqrt{m}}$ so that

(18a) $$\pi_n(x) = (1+\eta)A_1^{-1}c_1(\varepsilon)\chi_m(x+\Theta^{(1)}_m)$$

and

(18b) $$-\pi_n(x) = (1+\eta)A_1^{-1}c_1(\varepsilon)\chi_m(x+\Theta^{(2)}_m).$$

It follows from Lemma 1 and (2) that the polynomials of degree m

$$\psi_{ni}(t) = \pi_n(t)+(-1)^i(1+\eta)A_1^{-1}c_1(\varepsilon)\chi_m(t+\Theta^{(i)}_m) \qquad (i=1,2)$$

attain $m+1$ values with alternating sign at the points $\xi^{*(1)}_{km}$ resp. $\xi^{*(2)}_{km}$, $\xi^{*(i)}_{km} = \xi_{km}-\Theta^{(i)}_m$ $(k=0,1,2,\dots,m)$. We conclude that $\pi'_m(x)$ is between[1] $-(1+\eta)A_1^{-1}c_1(\varepsilon)\chi'_m(x+\Theta^{(2)}_m)$ and $(1+\eta)A_1^{-1}c_1(\varepsilon)\chi'_m(x+\Theta^{(1)}_m)$.

[1] Our argument is the same as in a well-known proof of Bernstein's inequality. See. I. G. NATANSON [6].

Making use of the relation $\chi_m'(t) = \sqrt{2m}\chi_{m-1}(t)$, we have, using the Plancherel—Rotach asymptotic formulas,

$$(19)\qquad \begin{aligned} |\pi_n'(x)| &\leqq 2A_1^{-1}c_1(\varepsilon)\sqrt{2m}\chi_{m-1}(x+\Theta_m^{(1)}) \leqq \\ &\leqq c_3(\varepsilon)\sqrt{m}\,e^{\left(|x|+\frac{10}{\sqrt{m}}\right)^2/2} \leqq c_4(\varepsilon)\sqrt{n}\,e^{x^2/2} \qquad (|x| \leqq 3/2\cdot\sqrt{n}). \end{aligned}$$

Using the first part of theorem 1, which was proved in part *B*, we obtain (3), Q.E.D.

D. Proof of the counterexamples. The example formulated in Theorem 2 is furnished simply by the Hermite polynomials. We know from the first Plansherel—Rotach asymptotic formula that for an r divisible by 4

$$(20)\qquad |H_r(x)| \leqq c_5(\varepsilon)H_r(0)e^{x^2/2} \qquad (|x| \leqq (1-\varepsilon)\sqrt{2(r+3)}),$$

but

$$(21)\qquad \sup e^{-x^2/2}|H_r(x)| > A_2 r^{1/4-1/12}H_r(0).$$

For a fixed $\varepsilon>0$ and an arbitrary n we choose $n-3 \leqq r \leqq n$ and r divisible by 4,

$$\varrho_n(\varepsilon; x) = \frac{H_r(x)}{c_5(\varepsilon)H_r(0)}.$$

We see from (20), (21) and $H_r'(x)/H_r(0) = \sqrt{2r}\,H_{r-1}(x)/H_{r-1}(0)$ that $\varrho_n(\varepsilon; x)$ has the properties required in Theorem 2.

Now we turn to the proof of Theorem 3. Let n be an even number and

$$(22)\qquad K_n(x) = \sum_{k=0}^{n}\frac{H_k(0)H_k(x)}{2^k k!} = \frac{1}{2^{n+1}n!}\,\frac{H_n(0)H_{n+1}(x)}{x}.$$

Using the known inequality (see e.g. G. FREUD—S. KNAPOWSKI [3])

$$\sum_{k=0}^{n}\frac{H_k^2(x)}{2^k k!} \leqq A_3\sqrt{n}\,e^{x^2}$$

we have by Cauchy's inequality

$$(23)\qquad |K_n(x)| \leqq A_4\sqrt{n}\,e^{x^2/2}.$$

Let $0 < \zeta_n < \dfrac{A_5}{\sqrt{n}}$ be the least positive zero of $H_{n+1}'(x)$. Then we have (e.g. by the Plancherel—Rotach formula)

$$(24)\qquad |H_{n+1}(\zeta_n)| \geqq A_6\sqrt{2^{n+1}(n+1)!}\,n^{-1/4}$$

so that

$$(25)\qquad \begin{aligned} |K_n'(\zeta_n)| &= \frac{|H_n(0)|}{2^{n+1}n!}\,\frac{|H_{n+1}(\zeta_n)|}{\zeta_n^2} = \frac{|H_n(0)|}{\sqrt{2^n n!}}\,\frac{|H_{n+1}(\zeta_n)|}{\sqrt{2^{n+1}(n+1)!}}\sqrt{(n+1)/2}\,\frac{1}{\zeta_n^2} \geqq \\ &\geqq \frac{A_7}{\sqrt{n+1}}\sqrt{2(n+1)}A_5^{-2}n \geqq A_8 n. \end{aligned}$$

We see from (23) and (25) that the polynomials $\sigma_n(x)=A_4^{-1}n^{-1/2}K_{2[n/2]}(x)$ $(n=1,2,\ldots)$ have the properties required in Theorem 3, Q.E.D.

E. On differentiated sequences of approximating polynomials. For every function $g(x)$ continuous on the real axis we define

$$\mathcal{E}_n(g) = \inf_{\pi_n} \sup_{-\infty<x<+\infty} e^{-x^2/2}|g(x)-\pi_n(x)| \tag{26}$$

where $\pi_n(x)$ runs through the polynomials of degree n at most. We are going to prove

THEOREM 4. *Let $f(x)$ be continuously differentiable for $-\infty<x<+\infty$, let*

$$\lim_{|x|\to\infty} f(x)x^{2r}e^{-x^2/2} = 0 \qquad (r=0,1,\ldots) \tag{27}$$

and for each n let there exist a polynomial $p_n(x)$ with degree n at most, for which we have

$$e^{-x^2/2}|f(x)-p_n(x)| \leqq \varepsilon_n \qquad (n=1,2,\ldots), \tag{28}$$

then

$$e^{-x^2/2}|f'(x)-p_n'(x)| \leqq A_9\{[\varepsilon_n+\mathcal{E}_n(f)]\sqrt{n}+\mathcal{E}_n(f')\} \tag{29}$$

holds.

This theorem is an analogue of a previous result of the author (see J. CZIPSZER—G. FREUD [2]). One could simplify (29) after having proved

$$\mathcal{E}_n(f) \leqq A_{10}n^{-1/2}\mathcal{E}_n(f'),$$

but this problem seems to be open[2].

As an application of theorem 4 let us consider the Lagrangean interpolatory polynomials $L_n(f;x)$ of $f(x)$ over the zeros of $H_n(x)$. If the function admits a modulus of continuity $\omega_r(f;\delta)$ of order r in respect to the whole real axis, then we have

$$e^{-x^2/2}|f(x)-L_n(f;x)| = O(\log n)\omega_r\left(f;\frac{1}{\sqrt{n}}\right);$$

further we have

$$\mathcal{E}_n(f) = O(1)\omega_r\left(f;\frac{1}{\sqrt{n}}\right)$$

(see G. FREUD [4] and [5]). Let $f(x)$ be differentiable and let $\omega_k(f;\delta)$ $(k\leqq r-1)$ exist in respect to the whole real axis. We observe that $\omega_{k+1}(f;\delta)\leqq\delta\omega_k(f';\delta)$ and conclude from Theorem 4 that

$$|f'(x)-L_n'(f;x)| = O(\log n)\omega_k\left(f';\frac{1}{\sqrt{n}}\right).$$

[2] (Note added in proof.) The indicated inequality was proved by the author in *Доклады Ак. Наук СССР* **191** (1970) pp. 293—294. A generalization and refinement of theorem 4 by the author will be published in *Studia Sci. Math. Hung.*

PROOF OF THEOREM 4. Let

$$f(x)\sim\sum_{k=0}^{\infty}a_k(f)H_k(x) \tag{30}$$

be the orthogonal expansion in Hermite-polynomials of $f(x)$ and let $s_n(f;x)$ be the partial sums of degree n of (30). We consider the de la Vallée-Poussin means

$$V_n(f;x)=\frac{1}{n}\sum_{\nu=1}^{n}s_{n+\nu}(f;x). \tag{31}$$

We observe that as a consequence of (27) and the well-known relation $H_n'(x)=$ $=2nH_{n-1}(x)$ we have the orthogonal expansion

$$f'(x)\sim\sum_{k=1}^{\infty}a_k(f)H_k'(x)=\sum_{k=0}^{\infty}2(k+1)a_{k+1}(f)H_k(x). \tag{32}$$

From (32) and (31) we get

$$V_n'(f;x)=\frac{1}{n}\sum_{\nu=0}^{n-1}s_{n+\nu}(f';x). \tag{33}$$

By a theorem of G. FREUD—S. KNAPOWSKI [3] we have

$$e^{-x^2/2}|f(x)-V_n(f;x)|\leqq A_{11}\mathscr{E}_{n+1}(f)\leqq A_{11}\mathscr{E}_n(f) \tag{34}$$

and from (33) we conclude

$$e^{-x^2/2}|f'(x)-V_n'(f;x)|\leqq A_{11}\mathscr{E}_n(f'). \tag{35}$$

From (28) and (34) follows

$$e^{-x^2/2}|p_n(x)-V_n(f;x)|\leqq \varepsilon_n+A_{11}\mathscr{E}_n(f). \tag{36}$$

$p_n-V_n(f)$ is a polynomial of degree $2n$ at most so that by Theorem 1 we get

$$e^{-x^2/2}|p_n'(x)-V_n'(f;x)|\leqq A_{12}\sqrt{n}\,[\varepsilon_n+A_{11}\mathscr{E}_n(f)]. \tag{37}$$

From (35) and (37) we obtain (29), Q.E.D.

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ON THE CONVERGENCE OF THE TRIGONOMETRIC (O, M) INTERPOLATION

By
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1. Introduction

Before proceeding to the main object of this paper we give a short survey of the results we shall use in the sequel.

We introduce the following definitions (see in [4]). Let us suppose that some function $\omega_m(t)$ has the following properties.

(i) $\omega_m(t)>0$ for $t>0$, $\omega_m(0)=0$, $\omega_m(T)\geqq\omega_m(t)$, if $T\geqq t$,

$\omega_m(t)$ is continuous for $t>0$,

(ii) $\dfrac{t^m}{\omega_m(t)}$ is monotone increasing for $t\geqq 0$,

(iii) $\lim\limits_{t=+0}\dfrac{t^m}{\omega_m(t)}=0$

(m is a fixed integer, $m\geqq 1$).[1]

Let us denote by $\tilde{C}(\omega_m)$ the class of all 2π-periodic continuous functions for which

$$\omega_m(f;t)\leqq a_m(f)\omega_m(t). \tag{1.1}$$

Here $\omega_m(f;t)$ is the modulus of smoothness of order m of $f(x)$; $a_m(f)>0$ depends only on $f(x)$; $\omega_m(t)$ is defined by (i), (ii) and (iii).

Let

$$0\leqq x_{0n}<x_{1n}<\cdots<x_{n-1,n}<2\pi \qquad (n=1,2,\ldots) \tag{1.2}$$

be an infinite point-system,

$$l_{kn}(x) \qquad (k=0,1,\ldots,n-1;\ n=1,2,\ldots) \tag{1.3}$$

2π-periodic continuous functions,

$$L_n(f;x)=\sum_{k=0}^{n-1}f(x_{kn})l_{kn}(x), \tag{1.4}$$

$$\lambda_n(x)=\sum_{k=0}^{n-1}\left|l_{kn}(x)\right|,\qquad \lambda_n=\max_{0\leqq x<2\pi}\lambda_n(x). \tag{1.5}$$

[1] (i), (ii) and (iii) are fulfilled if, e.g., $\omega_m(t)=t^\alpha$ $(0<\alpha<m)$.

Then the following theorem holds (cf. in [4]):

THEOREM 1.1. If $-\infty < x_0 < \infty$ and $\lim_{n\to\infty} \lambda_n(x_0) \neq 1$[2], then there exist an $f(x) \in \tilde{C}(\omega_m)$ and integers $0 < n_1 < n_2 < \dots$ such that

$$(1.6)\qquad |f(x_0) - L_{n_k}(f; x_0)| > \lambda_{n_k}(x_0)\omega_m(d_{n_k}) \qquad (k=1, 2, \dots),$$

where

$$(1.7)\qquad d_n = \min(x_{k+1,n} - x_{kn}) \qquad (x_{nn} = x_{0n} + 2\pi;\ n=1, 2, \dots).$$

NOTE. The theorem will be true if

$$L_n(f; x_0) = \sum_{k=0}^{n-1} f(x_{kn}) l_{kn}(x_0) + o(1)\lambda_n(x_0)\omega_m(d_n) \qquad (n=1, 2, \dots)$$

(cf. [4], Note d).

2. On the trigonometric (O, M) -interpolation

2. 1. Let

$$0 \leqq x_{0n} < x_{1n} < \cdots < x_{n-1,n} < 2\pi \qquad (n=1, 2, \dots)$$

be an infinite point-system. As in [2] A. SHARMA and A. K. VARMA (following P. TURÁN and O. KIS; see [1]), by trigonometric $(0, M)$ interpolating polynomials we mean those trigonometric polynomials $R_n(x)$ of order $\leqq n$ whose values and M-th derivatives at the x_{kn}'s are prescribed:

$$R_n(x_{kn}) = \alpha_{kn}, \qquad R_n^{(M)}(x_{kn}) = \beta_{kn} \qquad (k=0, 1, \dots, n-1;\ n=1, 2, \dots).$$

In [2] SHARMA and VARMA give a generalization for the theorem of [1] as follows

THEOREM 2. 1. (SHARMA—VARMA). *Let the x_{kn}'s be such that*

$$(2.1)\qquad x_{kn} = \frac{2k\pi}{n} \qquad (k=0, 1, \dots, n-1;\ n=1, 3, 5, \dots).$$

If we suppose that M is even ($M=2, 4, 6, \dots$) and n is odd ($n=1, 3, 5, \dots$), then there exist uniquely determined trigonometric $(0, M)$-interpolating polynomials $R_n(x)$ of order $\leqq n$.

Further, if $f(x)$ is a 2π-periodic continuous function and satisfies the conditions

$$(2.2)\qquad f(x+h) - 2f(x) + f(x-h) = o(h),$$

and

$$\beta_{kn} = o(n^{M-1}) \qquad (k=0, 1, \dots, n-1;\ n=1, 3, 5, \dots)$$

[2] It may occur that the limit does not exist.

then[3]

$$R_n(x) = \sum_{k=0}^{n-1} [f(x_{kn})F_n(x-x_{kn}) + \beta_{kn} G_n(x-x_{kn})] \tag{2.3}$$

converges uniformly to $f(x)$ on the whole real axis. Even if every β_{kn} is zero, the condition (2. 2) can not be replaced by the Lipschitz condition of order $\alpha < 1$.

The problem of presenting a generally exact estimation for the order of the difference $f(x) - R_n(x)$ was raised by O. KIS (cf. [3]). He has given the answer for the (0, 2) case. Theorem 2. 3 is a generalization of this result.

First of all we prove the following

LEMMA 2. 2. *If $f(x)$ is a 2π-periodic continuous function and $\omega_m(f; t) = O[\omega(t)]$ where $\omega(t)$ satisfies the condition*[4]

$$\sum_{j=1}^{k} (2^j)^s \omega\left(\frac{1}{2^j}\right) = O(1) 2^{ks} \omega\left(\frac{1}{2^k}\right) \tag{A_s}$$

for an arbitrary integer $S \geqq 1$, then there exists a trigonometric polynomial $T_n(x)$ of order $\leqq n$ such that

$$|f(x) - T_n(x)| = O\left[\omega\left(\frac{1}{n}\right)\right], \tag{2.4}$$

$$|T_n^{(s)}(x)| = O\left[n^s \omega\left(\frac{1}{n}\right)\right] \qquad (n = 1, 2, \ldots;\ s = 1, 2, \ldots). \tag{2.5}$$

PROOF. As it is well known (see [5], p. 274), there are trigonometric polynomials $U_n(x)$ of order $\leqq n$ such that

$$|f(x) - U_n(x)| = O\left[\omega\left(\frac{1}{n}\right)\right] \qquad (n = 1, 2, \ldots). \tag{2.6}$$

Let us choose a k such that

$$2^k \leqq n < 2^{k+1}.$$

Further let

$$T_n(x) = U_{2^k}(x) = \sum_{j=1}^{k} [U_{2^j}(x) - U_{2^{j-1}}(x)] + U_1(x).$$

[3] Here $F_n(x)$ and $G_n(x)$ are trigonometric polynomials of order $n-1$ and n, respectively, (depending on M as well) such that

$$F_n(x_{in}) = \begin{cases} 0 & \text{if } i \neq 0 \\ 1 & \text{if } i = 0, \end{cases} \qquad G_n(x_{in}) = 0 \quad (i = 0, 1, \ldots, n-1)$$

$$F_n^{(M)}(x_{in}) = 0 \quad (i = 0, 1, \ldots, n-1), \quad G_n^{(M)}(x_{in}) = \begin{cases} 0 & \text{if } i \neq 0 \\ 1 & \text{if } i = 0. \end{cases}$$

[4] This condition is fulfilled e.g. for $\omega_m(f; t) = \omega(t) = t^\alpha\ (0 < \alpha < S)$. If $m \geqq 2$ and $1 < \alpha < S$ ($S = 2, 3, \ldots$), then $f(x)$ satisfies (2.2) too.

The validity of (2. 4) is obvious, furthermore by (2. 6)

$$T_n^{(s)}(x) = \sum_{j=1}^{k} [U_{2^j}(x) - U_{2^{j-1}}(x)]^{(s)} + U_1^{(s)}(x) = O(1) \sum_{j=1}^{k} (2^j)^s \omega\left(\frac{1}{2^j}\right).$$

But owing to (A_s) and the relation for k, we get (2. 5). Q.E.D.

By this lemma we can prove the following

THEOREM 2. 3. *Let $f(x)$ be a 2π-periodic continuous function, $\omega_m(f; t) = O[\omega_m(t)]$ where $\omega_m(t)$ satisfies (A_M). If we suppose that M is even $(M=2, 4, 6, \ldots)$ and n is odd $(n=1, 3, 5, \ldots)$, then there exist the uniquely determined trigonometric (O, M)-interpolating polynomials*

$$R_n(x) = R_n(f; x) = \sum_{k=0}^{n-1} f(x_{kn}) F_n(x - x_{kn}) \qquad (n=1, 3, \ldots)$$

of order $\leqq n$ for the nodes (2. 1) *such that*[5]

$$\|f(x) - R_n(f; x)\| = O\left[n\omega_m\left(\frac{1}{n}\right)\right]. \tag{2. 7}$$

On the other hand, if (i), (ii) *and* (iii) *hold with $\omega_m(t)$, then there exists an $f^*(x) \in \tilde{C}(\omega_m)$ such that*

$$\left|f^*(\pi) - R_n(f^*; \pi)\right| \geqq cn\omega_m\left(\frac{1}{n}\right) \qquad (n=n_1, n_2, \ldots), \tag{2. 8}$$

where $0 < n_1 < n_2 < \ldots$ are odd and $c > 0$.

PROOF. For sake of brevity sometimes we omit the double indices. We shall use the following estimations (see [2]):

$$\begin{cases} \sum_{k=0}^{n-1} \left|F_n(x - x_k)\right| = O(n), \quad \sum_{k=0}^{n-1} \left|F_n(\pi - x_k)\right| \geqq c_1 n \qquad (c_1 > 0), \\ \sum_{k=0}^{n-1} \left|G_n(x - x_k)\right| = O(n^{-M+1}). \end{cases} \tag{2. 9}$$

Using the unique determination, we have for an arbitrary trigonometric polynomial $T_{n-1}(x)$ of order $\leqq n-1$

$$T_{n-1}(x) \equiv \sum_{k=0}^{n-1} [T_{n-1}(x_k) F_n(x - x_k) + T_{n-1}^{(M)}(x) G_n(x - x_k)]$$

$$(n=1, 3, \ldots; \ M=2, 4, \ldots).$$

---

[5] The unique existence of $R_n(x)$ is already known from [2] $\left(\|g(x)\| = \max_{0 \leqq x < 2\pi} |g(x)|\right)$.

Let $T_{n-1}(x)$ be the trigonometric polynomial, defined by Lemma 2. 2. Then

$$f(x)-R_n(f;\,x)\equiv f(x)-\sum_{k=0}^{n-1} f(x_k)F_n(x-x_k)=f(x)-T_{n-1}(x)+$$

$$+\sum_{k=0}^{n-1}\left[T_{n-1}(x_k)-f(x_k)\right]F_n(x-x_k)+{}^{\prime}\sum_{k=0}^{n-1} T_{n-1}^{(M)}(x_k)G_n(x-x_k).$$

Here by (2. 4)

$$|f(x)-T_{n-1}(x)|=O\left[\omega_m\left(\frac{1}{n}\right)\right], \tag{2. 10}$$

further, in virtue of (2. 9) and (2. 4)

$$\sum_{k=0}^{n-1}|T_{n-1}(x_k)-f(x_k)||F_n(x-x_k)|=O(n)\omega_m\left(\frac{1}{n}\right). \tag{2. 11}$$

Finally, from (A_M) and (2. 9)

$$\sum_{k=0}^{n-1}|T_{n-1}^{(M)}(x_k)||G_n(x-x_k)|=O(n)\omega_m\left(\frac{1}{n}\right). \tag{2. 12}$$

The estimations (2. 10)—(2. 12) give (2. 7). To obtain (2. 8), we pay attention to

$$\sum_{k=0}^{n-1}|F_n(x-x_k)|\geqq c_1 n.$$

Then we can apply our Theorem 1. 1, with "the cast" $l_k(x)=F_n(x-x_k)$, $L_n(f;x)=$ $=R_n(f;x)$ and $x_0=\pi$. Using that $d_n=\dfrac{2\pi}{n}$, we obviously obtain (2. 8). Q.E.D.

NOTES. 1. If $\omega_m(f;t)=O[\omega_m(t)]$ and we suppose (A_M) for $\omega_m(t)$ then the condition $\omega_m(t)=o(t)$ is sufficient for the uniform convergence of $R_n(f;x)$ to $f(x)$. Even if $m=2$, then (2. 2) is true for $s=2,3,\dots$, i.e. for an arbitrary even M. On the other hand, the condition $\omega_m\left(\frac{1}{n}\right)=O\left(\frac{1}{n}\right)$ is generally not sufficient. I.e., if (A_M), (i), (ii) and (iii) hold with $\omega_m(t)$, the condition $\omega_m\left(\frac{1}{n}\right)=o\left(\frac{1}{n}\right)$ is necessary and sufficient for the uniform convergence of $R_n(f;x)$ to an arbitrary $f(x)\in\tilde{C}(\omega_m)$.

2. If $f(x)$ is a "good function", e.g. $\omega_m(f;t)=\omega_m(t)=t^m$, then for $M>m$ (so $\omega_m(f;t)$ satisfies (A_M)) the rapidity of the convergence is at least $O(n^{-m+1})$.

3. Let

$$R_n(x)=\sum_{k=0}^{n-1}\left[f(x_{kn})F_n(x-x_{kn})+\beta_{kn}G_n(x-x_{kn})\right].$$

By (2. 9) we can easily prove (2. 7) for

$$\beta_{kn} = O(n^M)\omega_m\left(\frac{1}{n}\right).$$

(For a convergent procedure $\omega_m\left(\frac{1}{n}\right)=o(n)$, i.e. $\beta_{kn}=o(n^{M-1})$). On the other hand (cf. Note) we can prove (2. 8) only in the case

$$\beta_{kn} = o(n^M)\omega_m\left(\frac{1}{n}\right).$$

2. 2. Now we intend to investigate the cases when M is odd. From [2] the following result is known.

THEOREM 2. 4. (SHARMA—VARMA). *If* $x_{kn} = \frac{2k\pi}{n}$, *M is odd and n is arbitrary, then there exist uniquely determined trigonometric* (O, M)*-interpolating polynomials* $R_n(x)$ *of order* $\leqq n$.

Further, if $f(x)$ *is a* 2π*-periodic continuous function and* $\beta_{kn}=o\left(n^M\cdot(\log n)^{-1}\right)$ *then* $R_n(x)$ *defined by* (2. 3) *converges uniformly to* $f(x)$ *on the whole real axis.*

As before, we intend to estimate the order of difference $f(x)-R_n(x)$. The following theorem holds.

THEOREM 2. 5. *Let* $f(x)$ *be a* 2π*-periodic continuous function. If we suppose that* M *is odd* $(M=1, 3, \ldots)$, *then there exist uniquely determined trigonometric* (O, M)*-interpolating polynomials*[6]

$$R_n(x) \equiv R_n(f; x) = \sum_{k=0}^{n-1} f(x_{kn})F_n(x-x_{kn}) \qquad (n=1,2,\ldots) \tag{2. 13}$$

of order $\leqq n$ *for the nodes* (2. 1) *such that*

$$\begin{cases} |f(x)-R_n(f;x)| = O(1)\omega_1\left(\frac{\log n}{n}\right) & (M=1,3,\ldots) \\ |f(x)-R_n(f;x)| = O(\log n)\omega_m\left(\frac{1}{n}\right) & (m>1,\ M=1,3,\ldots) \end{cases} \tag{2. 14}$$

uniformly on the whole real axis. We supposed $\omega_m(f;t)=O[\omega_m(t)]$ *and* $\omega_m(t)$ *satisfies* (A_M) *if* $m>1$.

PROOF. I. $m=1$. We need the following formulae from [2] (p. 348—349).

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n-1} c_{n,j}jt_j(x)+\frac{t_n(x)}{(n-1)^M+1}$$

[6] The unique existence of $R_n(x)$ is already known from [2].

where

$$t_j(x) = \frac{1}{j}\left(\frac{\sin j\frac{x}{2}}{\sin\frac{x}{2}}\right)^2$$

and

$$c_{n,j} = O\left(\frac{1}{n^2}\right).$$

I.e., we obtain the estimation

$$(2.15)\qquad |F_n(x)| = O(1)\left[\frac{1}{n^3}\sum_{j=1}^{n-1}\left(\frac{\sin j\frac{x}{2}}{\sin\frac{x}{2}}\right)^2 + \frac{1}{n^{M+1}}\left(\frac{\sin n\frac{x}{2}}{\sin\frac{x}{2}}\right)^2\right].$$

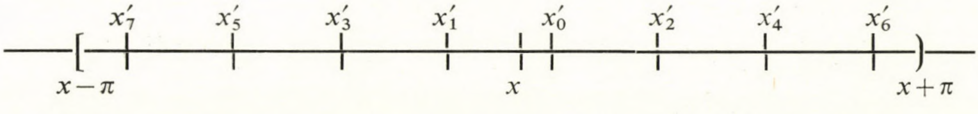
We shall use the following well-known estimations.

$$(2.16)\qquad \omega(f;\lambda t) \leqq (\lambda+1)\omega(f;t) \qquad (\lambda\geqq 0),$$

$$(2.17)\qquad \sin x \geqq \frac{2}{\pi}x \qquad \left(0\leqq x\leqq\frac{\pi}{2}\right),$$

$$(2.18)\qquad |\sin kx| \leqq |k\sin x|.$$

Evidently, it is sufficient to prove (2.14) for the interval $I=[x-\pi, x+\pi)$. Then the new nodes of (O, M) interpolation will be the points of I having the form $\frac{2l\pi}{n}$. Let us count these fundamental points as follows ($n=7$).



Then, denoting x_k' repeatedly by x_k, we have

$$(2.19)\qquad |x-x_k|\leqq\pi,\qquad c_1\frac{k}{n}\leqq|x-x_k|\leqq c_2\frac{k}{n}\qquad (c_1, c_2>0,\ k\neq 0).$$

We know that

$$(2.20)\qquad \sum_{k=0}^{n-1}F_n(x-x_k)\equiv 1.$$

(see note (3)).

If we denote $\omega(f;t)$ by $\omega(t)$, then we have by (2. 20); (2. 16), (2. 17); (2. 18), (2. 19); (2. 18), (2. 19) and for the time being undetermined function $h_n>0$

$$|f(x)-R_n(x)| = \left|\sum_{k=0}^{n-1}[f(x)-f(x_k)]F_n(x-x_k)\right| \leqq \sum_{k=0}^{n-1}\omega(|x-x_k|)\,|F_n(x-x_k)| =$$

$$= O(1)\sum_{k=0}^{n-1}\omega(h_n^{-1})[h_n|x-x_k|+1]\left[\frac{1}{n^3}\sum_{j=1}^{n-1}\left(\frac{\sin j\frac{x-x_k}{2}}{\sin\frac{x-x_k}{2}}\right)^2+\frac{1}{n^{M+1}}\left(\frac{\sin n\frac{x-x_k}{2}}{\sin\frac{x-x_k}{2}}\right)^2\right] =$$

$$= O(1)\omega(h_n^{-1})\left[\frac{h_n}{n^3}\sum_{k=0}^{n-1}|x-x_k|\sum_{j=1}^{n-1}\left(\frac{\sin j\frac{x-x_k}{2}}{\sin\frac{x-x_k}{2}}\right)^2+\right.$$

$$+\frac{h_n}{n^{M+1}}\sum_{k=0}^{n-1}|x-x_k|\left(\frac{\sin n\frac{x-x_k}{2}}{\sin\frac{x-x_k}{2}}\right)^2+\frac{1}{n^3}\sum_{k=0}^{n-1}\sum_{j=1}^{n-1}\left(\frac{\sin j\frac{x-x_k}{2}}{\sin\frac{x-x_k}{2}}\right)^2+$$

$$\left.+\frac{1}{n^{M+1}}\sum_{k=0}^{n-1}\left(\frac{\sin n\frac{x-x_k}{2}}{\sin\frac{x-x_k}{2}}\right)^2\right] =$$

$$= O(1)\omega(h_n^{-1})\left\{\frac{h_n}{n^3}\left[\sum_{k=1}^{n-1}|x-x_k|\sum_{j=1}^{n-1}\frac{1}{|x-x_k|^2}+|x-x_0|\sum_{j=1}^{n-1}\frac{j}{|x-x_0|}\right]+\right.$$

$$+\frac{h_n}{n^{M+1}}\left[\sum_{k=1}^{n-1}|x-x_k|\frac{1}{|x-x_k|^2}+|x-x_0|\frac{n}{|x-x_0|}\right]+$$

$$\left.+\frac{1}{n^3}\left[\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}\frac{1}{|x-x_k|^2}+\sum_{j=1}^{n-1}j^2\right]+\frac{1}{n^{M+1}}\left[\sum_{k=1}^{n-1}\frac{1}{|x-x_k|^2}+n^2\right]\right\} =$$

$$= O(1)\omega(h_n^{-1})\left[\frac{h_n}{n^3}(n^2\log n+n^2)+\frac{h_n}{n^{M+1}}(n\log n+n)+\frac{1}{n^3}(n^3+n^3)+\right.$$

$$\left.+\frac{1}{n^{M+1}}(n^2+n^2)\right] = O(1)\omega(h_n^{-1})\left(\frac{h_n\log n}{n}+\frac{h_n}{n^M}\log n+\frac{2}{n^{M-1}}+2\right).$$

But $M\geqq 1$, so for $h_n=\dfrac{n}{\log n}$ the procedure converges and

$$|f(x)-R_n(x)| = O(1)\omega\left(f;\,\frac{\log n}{n}\right) = O(1)\omega_1\left(\frac{\log n}{n}\right) \qquad (n=2,3,\ldots).$$

II. $m>1$. As in THEOREM 2. 3 we have

$$f(x)-R_n(f;x)=$$

$$=f(x)-T_{n-1}(x)+\sum_{k=0}^{n-1}\left[T_{n-1}(x_k)-f(x_k)\right]F_n(x-x_k)+\sum_{k=0}^{n-1}T_{n-1}^{(M)}c_{-n}(x-x_k).$$

By (2. 4); (2. 4),

$$\sum_{k=0}^{n-1}\left|F_n(x-x_k)\right|=O(1),\quad \sum_{k=0}^{n-1}\left|G_n(x-x_k)\right|=O\left(\frac{\log n}{n^M}\right), \tag{2. 21}$$

(see [2], p. 347 and 350.) and (2. 5) we have

$$f(x)-R_n(x)=O(1)\left[\omega_m\left(\frac{1}{n}\right)+\omega_m\left(\frac{1}{n}\right)+n^M\omega_m\left(\frac{1}{n}\right)\frac{\log n}{n^M}\right]=O(\log n)\omega_m\left(\frac{1}{n}\right).$$

NOTES. 1. For $M=m=1$ E. MOLDOVAN proved a similar theorem in the case of the Hermite—Fejér polynomials. (The theorem and the proof can be seen e.g. in [6].)

2. Obviously, it is worth to use the second formula of (2. 14) when $f(x)$ is a "good function", e.g. $\omega_m(f;t)=t^\alpha$ $(\alpha\geqq 1)$. Let $f(x)$ be a good function such that $\omega_m(f;t)=t^m$. Then for $M>m$ we can say that (A_M) is valid, so

$$f(x)-R_n(f;x)=O\left(\frac{\log n}{n^m}\right).$$

I.e., if M is large enough, then the convergence of $R_n(f;x)$ is very rapid even as compared with the trigonometric polynomials of best approximation. With other words, the lacunary interpolation process could be very useful for good functions. On the other hand, if $\omega_m(f;t)=\omega_m(t)=t^m$ but M is not large enough, then by the method of Lemma 2. 2 we obtain $(\omega_m(f;t)=\omega_m(t)=\omega(t)=t^m)$

$$T_n^{(M)}(x)=O(\log n)\quad \text{for}\quad M=m,$$

$$T_n^{(M)}(x)=O(1)\quad \text{for}\quad 1\leqq M<m.$$

Then the order of the convergence is

$$\frac{\log^2 n}{n^M}>\frac{\log n}{n^m}\equiv \log n\omega_m\left(\frac{1}{n}\right)\quad \text{for}\quad M=m,$$

$$\frac{\log n}{n^M}>\frac{\log n}{n^m}\equiv \log n\omega_m\left(\frac{1}{n}\right)\quad \text{for}\quad M<m.$$

That is, if $M\leqq m$, we obtain a worse estimation for good functions.

We can make the above mentioned remarks (with a little modification in the analogous formulae) for Theorem 2. 3 also.

3. Using the estimation (2. 21) for $G_n(x)$ we can easily prove (2. 14) for

$$\beta_{kn} = O\left(\frac{n^M}{\log n}\right)\omega\left(f;\ \frac{\log n}{n}\right)$$

or

$$\beta_{kn} = O(n^M)\omega_m\left(\frac{1}{n}\right).$$

4. We remark the following fact. For odd M there exist trigonometric (O, M)-interpolating polynomials $R_n(x)$ of order $\leqq n$ having the form (2. 3) such that

$$f(x) - R_n(x) = O(1)\omega_m\left(f;\ \frac{1}{n}\right) \qquad (m=1, 2, \dots;\ n=1, 2, \dots).$$

For, let $U_n(x)$ the trigonometric polynomials of best approximation of order $\leqq n$ for $f(x)$. Then (see Theorem 2. 3)

$$f(x) - R_n(x) = f(x) - U_n(x) + \sum_{k=0}^{n-1}\left[U_n(x_k) - f(x_k)\right]F_n(x - x_k) +$$

$$+ \sum_{k=0}^{n-1}\left[U_n^{(M)}(x_k) - \beta_{kn}\right]G_n(x - x_k).$$

If

$$\left|\beta_{kn} - U_n^{(M)}(x_k)\right| = O\left[\frac{n^M}{\log n}\omega_m\left(f;\ \frac{1}{n}\right)\right]$$

then by

$$\sum_{k=0}^{n-1}\left|F_n(x - x_k)\right| = O(1)$$

we obtain our statement.

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NOTES ON THE CONVERGENCE OF (0, 2) AND (0, 1, 3) INTERPOLATIONS

By

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1. Introduction

Before proceeding to the consideration of our object, we wish to have a short survey of the resource we shall use in the sequel.

At first let us consider the following definitions (s. [7]).

Let us suppose that some function $\omega_m(t)$ satisfies the following properties.

(i) $\omega_m(t)>0$ for $t>0$, $\omega_m(0)=0$, $\omega_m(T)\geqq\omega_m(t)$ if $T\geqq t$,

$$\omega_m(t) \text{ is continuous for } t>0,$$

(ii) $\dfrac{t^m}{\omega_m(t)}$ is monotone increasing for $t\geqq 0$,

(iii) $\displaystyle\lim_{t=+0}\frac{t^m}{\omega_m(t)}=0$

(m is a fixed integer, $m\geqq 1$).[1]

Let us denote by $C^{[-1,1]}(\omega_m)$ the class of all continuous functions $f(x)$ on the interval $[-1, 1]$ for which

$$\omega_m(f;t)\leqq a_m(f)\omega_m(t). \tag{1. 1}$$

Here $\omega_m(f;t)$ is the modulus of smoothness of order m of $f(x)$; $a_m(f)>0$ depends only on $f(x)$; $\omega_m(t)$ is defined by (i), (ii) and (iii).

Let

$$-1\leqq x_{nn}<x_{n-1,n}<\cdots<x_{2n}<x_{1n}\leqq 1 \qquad (n=1,2,\ldots) \tag{1. 2}$$

be an infinite point-system,

$$l_{kn}(x) \qquad (k=1,2,\ldots,n;\ n=1,2,\ldots) \tag{1. 3}$$

continuous functions on the interval $[-1, 1]$,

$$L_n(f;x)=\sum_{k=1}^{n}f(x_{kn})l_{kn}(x) \qquad (n=1,2,\ldots). \tag{1. 4}$$

Let us denote by $I_n=\{x_{j(n),n};x_{l(n),n}\}$ $(1\leqq j(n)<l(n)\leqq n)$ general intervals, further let $[I_n]=[x_{j(n),n};x_{l(n),n}]$ and $(I_n)=(x_{j(n),n};x_{l(n),n})$, $I_{n+1}\subseteqq I_n$ $(n=1,2,\ldots)$.

[1] E.g. if $\omega_m(t)=t^\alpha$ $(0<\alpha<m)$, (i), (ii) and (iii) are fulfilled.

If

(1. 5) $$d_n(I_n) = \min_{x_{k+1,n};\, x_{k,n} \in [I_n]} (x_{k,n} - x_{k+1,n})$$

such that

$$d_n(I_n) \to 0 \quad \text{if} \quad n \to \infty$$

and

(1. 6) $$\lambda_n(I_n; x) = \sum_{\substack{k \\ x_{kn} \in (I_n)}} |l_{kn}(x)|$$

then the following theorem holds (cf. [7]).

THEOREM 1. 1. *If* $-1 \leqq x_0 \leqq 1$ *and* I_n *are such that*[2]

(1. 7) $$\lim_{n \to \infty} \lambda_n(I_n; x_0) \neq 1$$

then there exist an $f(x) \in C^{[-1,1]}(\omega_m)$ *and integers* $0 < n_1 < n_2 < \dots$ *such that*

(1. 8) $$|f(x_0) - L_n(f; x_0)| > \lambda_n(I_n; x_0)\,\omega_m(d_n(I_n)) \qquad (n = n_1, n_2, \dots).$$

NOTE. The theorem will be true if

$$L_n(f; x_0) = \sum_{k=1}^{n} f(x_{kn}) l_{kn}(x_0) + o(1)\lambda_n(I_n; x_0)\,\omega_m(d_n(I_n)).$$

(cf. [7], Note *d.*)

2. On the (0, 2) interpolation

Let

(2. 1) $$-1 \leqq x_{nn} < x_{n-1,n} < \dots < x_{2n} < x_{1n} \leqq 1 \qquad (n = 1, 2, \dots)$$

be an infinite point-system. As in [1], by (0, 2)-interpolating polynomials we mean those polynomials $R_n(x)$ of degree $\leqq 2n-1$ whose values and second derivatives at the x_{kn}'s are prescribed:

(2. 2) $$R_n(x_{kn}) = \alpha_{kn}, \quad R_n''(x_{kn}) = \beta_{kn} \quad (k = 1, 2, \dots, n;\ n = 1, 2, \dots).$$

In [2] and [3] P. TURÁN and his collaborators proved the following theorems: Let us choose as fundamental points x_{kn} of the (0, 2) interpolation the zeros x_{kn} ($k = 1, 2, \dots, n$; $n = 1, 2, \dots$) of the polynomials

(2. 3) $$\Pi_n(x) = (1 - x^2) P_{n-1}(x)$$

where $P_n(x)$ stands for the n^{th} Legendre polynomial with the normalization $P_n(1) = 1$.

If $n = 2k$, then to prescribed values α_{kn} and β_{kn} there is a uniquely determined polynomial[3]

(2. 4) $$R_n(x) = \sum_{k=1}^{n} \alpha_{kn} r_k(x) + \sum_{k=1}^{n} \beta_{kn} \varrho_k(x)$$

[2] It may occur that the limit does not exist.

where the polynomials $r_k(x)$ and $\varrho_k(x)$, the so called fundamental functions of the first and second kind of degree $\leqq 2n-1$, are uniquely determined by the conditions

$$r_k(x_j) = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}, \quad r_k''(x_j) = 0 \quad (j, k = 1, 2, \dots, n),$$

and

$$\varrho_k(x_j) = 0, \quad \varrho_k''(x_j) = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} \quad (j, k = 1, 2, \dots, n),$$

respectively. Further, owing to the above mentioned uniqueness theorem, we have

$$(2.5) \qquad P_{2n-1}(x) = \sum_{k=1}^{n} P_{2n-1}(x_k) r_k(x) + \sum_{k=1}^{n} P_{2n-1}''(x_k) \varrho_k(x)$$

for an arbitrary polynomial $P_{2n-1}(x)$ of degree $\leqq 2n-1$.

In [6], G. Freud improved the convergence theorem from [3] as follows:

Theorem 2. 1 (G. Freud). *Let $f(x)$ be a continuous function on the interval $[-1, 1]$ which satisfies the condition*

$$(2.6) \qquad f(x+h) - 2f(x) + f(x-h) = o(h) \qquad (-1 \leqq x-h < x+h \leqq 1).$$

Let

$$\beta_{1n} = o(n^2), \quad \beta_{kn} = \frac{o(n)}{\sqrt{1-x_{kn}^2}} \quad (k = 2, 3, \dots, n-1), \quad \beta_{nn} = o(n^2).$$

If we choose as fundamental points x_{kn} the zeros x_{kn} of the polynomials $\Pi_n(x)$, then for $n = 4, 6, 8, \dots$ the sequence of the uniquely determined (0, 2)-interpolating polynomials

$$R_n(x) = \sum_{k=1}^{n} [f(x_{kn}) r_{kn}(x) + \beta_{kn} \varrho_{kn}(x)] \qquad (n = 4, 6, 8, \dots)$$

uniformly converges to $f(x)$ in $[-1, 1]$. Even if all β_{kn} are zeros, the condition (2. 6) cannot be replaced by a Lipschitz condition of order $\alpha < 1$.

The object of this part is to give another and improved form of Theorem 2. 1. Namely, we shall present a generally exact estimation for the order of the difference $f(x) - R_n(x)$. We follow the method of the papers [6] and [8].

Definition. Let $\omega_k(f; t)$ be the continuity modulus of second order of the continuous function $f(x)$. We say that $f(x)$ or $\omega_2(f; t)$ satisfies the condition (A_2)

[3] Here and in many cases we denote $\alpha_{kn}, \beta_{kn}, x_{kn}, r_{kn}(x)$ and $\varrho_{kn}(x)$ by $\alpha_k, \beta_k, x_k, r_n(x)$ and $\varrho_k(x)$, respectively.

if $\omega_2(f;t)=O[\omega_2(t)]$, where $\omega_2(t)$ is a non-negative increasing function, $\omega_2(ct)\leqq \leqq K_c\omega_2(t)$, further

$$(A_2)\qquad \sum_{j=1}^{k}\frac{\omega_2\left[\frac{\sqrt{1-x^2}}{2^j}+\frac{1}{(2^j)^2}\right]}{\left[\frac{\sqrt{1-x^2}}{2^j}+\frac{1}{(2^j)^2}\right]^2}=O(1)\frac{\omega_2\left[\frac{\sqrt{1-x^2}}{2^j}+\frac{1}{(2^j)^2}\right]}{\left[\frac{\sqrt{1-x^2}}{2^j}+\frac{1}{(2^j)^2}\right]^2}$$

uniformly for $x\in[-1,1]$.[4]

Now we prove the following

LEMMA 2. 2. *If $f(x)$ is a continuous function in $[-1,1]$ for which (A_2) is valid, then there is a polynomial $P_n(x)$ of degree $\leqq n$ such that*

$$(2.7)\qquad |f(x)-P_n(x)|\leqq c\omega_2\left(f;\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)=O\left[\omega_2\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right],$$

$$(2.8)\qquad |P_n''(x)|=O(1)\frac{\omega_2\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)}{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^2}\qquad(-1\leqq x\leqq 1,\ n=1,2,\ldots)$$

PROOF. As well known (s. [9], p. 281), there are polynomials $Q_n(f;x)\equiv Q_n(x)$ of degree $\leqq n$ such that

$$(2.9)\qquad |f(x)-Q_n(x)|=O\left[\omega_2\left(f;\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right].$$

Let us choose k such that

$$2^k\leqq n<2^{k+1}.$$

Further, let

$$P_n(x)=Q_{2^k}(x)=\sum_{j=1}^{k}[Q_{2^j}(x)-Q_{2^{j-1}}(x)]+Q_1(x).$$

The validity of (2. 7) is obvious, furthermore[5]

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[4] These are valid if e.g. $\omega_2(t)=t^\alpha$ $(0<\alpha<2)$; if $1<\alpha<2$, then (2.6) is valid, too.

[5] We use the following theorem (see [9], p. 234). If $S_p(x)$ is a polynomial of degree $\leqq p$ and

$$|S_p(x)|\leqq c_1\varphi\left(\frac{\sqrt{1-x^2}}{p}+\frac{1}{p^2}\right)$$

where $\varphi(t)\leqq\varphi(T)$ for $t\leqq T$ and $\varphi_1(t_1+t_2)\leqq M[\varphi(t_1)+\varphi(t_2)]$ then

$$|S_p^{(l)}(x)|\leqq c_l\frac{\varphi\left(\frac{\sqrt{1-x^2}}{p}+\frac{1}{p^2}\right)}{\left(\frac{\sqrt{1-x^2}}{p}+\frac{1}{p^2}\right)^l}\qquad(-1\leqq x\leqq 1;\ c_l \text{ does not depend on } x \text{ and } n).$$

By $|S_p(x)|=|Q_{2^j}(x)-Q_{2^{j-1}}(x)|\leqq\omega_2\left(f;\frac{\sqrt{1-x^2}}{2^j}+\frac{1}{2^{2j}}\right)$ and $\omega_2(f;t_1+t_2)\leqq 2^2[\omega_2(f;t_1)+ +\omega_2(f;t_2)]$ we obtain (2. 10).

(2. 10) $$|P_n''(x)| =$$

$$= \left|\sum_{j=1}^{k}[Q_{2^j}(x)-Q_{2^{j-1}}(x)]''\right| = \sum_{j=1}^{k}\left(\frac{\sqrt{1-x^2}}{2^j}+\frac{1}{2^{2j}}\right)^{-2} O\left[\omega_2\left(f;\ \frac{\sqrt{1-x^2}}{2^j}+\frac{1}{2^{2j}}\right)\right] =$$

$$= \sum_{j=1}^{k}\left(\frac{\sqrt{1-x^2}}{2^j}+\frac{1}{2^{2j}}\right) O\left[\omega_2\left(\frac{\sqrt{1-x^2}}{2^j}+\frac{1}{2^{2j}}\right)\right].$$

But owing to (A_2) and the relation for k, we get (2. 8). Q.E.D.

Now we can prove the following

THEOREM 2. 3. *If $f(x)$ is a continuous function on the interval* $[-1, 1]$ *satisfying the condition (A_2), then*

(2. 11) $$\|f(x)-R_n(f;x)\|_{[-1,1]} = O\left[n\omega_2\left(\frac{1}{n}\right)\right] \quad \text{if} \quad n=4,6,8,\ldots$$

where

$$R_n(f;x) = \sum_{k=1}^{n} f(x_{kn})r_{kn}(x) \qquad (n=4,6,8,\ldots)$$

are the uniquely determined (0, 2)*-interpolating polynomials for the zeros x_{kn} of the polynomials $\Pi_n(x)$.*[6]

On the other hand, if (i), (ii) *and* (iii) *hold with $\omega_2(t)$ then there exists an $f^*(x)\in C^{[-1,1]}(\omega_2)$ such that*

(2. 12) $$|f^*(0)-R_n(f^*;0)| \geqq cn\omega_2\left(\frac{1}{n}\right) \qquad (n=n_1, n_2, \ldots)$$

where $N_0<n_1<n_2$ are even and $c>0$. $\left(\|g(x)\| = \max_{-1\leqq x\leqq 1}|g(x)|\right)$.

PROOF. For the sake of brevity let us suppose throughout this paper that $n=2k\geqq 4$.

We shall use the following estimations (see e.g. [3] or [6]):

(2. 13)

$$\begin{cases} \varrho_{1n}(x) = O(n^{-7/2}), \qquad \varrho_{nn}(x) = O(n^{-7/2}), \\ \varrho_{kn}(x) = O(n^{-2})l_{kn}(x)(1-x_{kn}^2)k^{1/2}+O(n^{-7/2})k^{3/2} \qquad \left(2\leqq k\leqq \frac{n}{2}\right), \\ \varrho_{kn}(x) = O(n^{-2})l_{kn}(x)(1-x_{kn}^2)(n-k)^{1/2}+O(n^{-7/2})(n-k)^{3/2} \quad \left(\frac{n}{2}<k\leqq n-1\right), \\ r_{1n}(x) = O(n), \qquad r_{nn}(x) = O(n), \\ r_{kn}(x) = O(n^{1/2})k^{-1/2} \qquad \left(2\leqq k\leqq \frac{n}{2}\right), \\ r_{kn}(x) = O(n^{1/2})(n-k)^{-1/2} \qquad \left(\frac{n}{2}<k\leqq n-1\right). \end{cases}$$

[6] The unique existence of $R_n(f, x)$ is already known from [3].

where $l_{kn}(x)$ is the k^{th} fundamental polynomial of the Lagrange-interpolation,

$$\tag{2.14}\begin{cases} c_1\dfrac{k}{n} \leqq \sqrt{1-x_{kn}^2} \leqq c_2\dfrac{k}{n} & \left(2 \leqq k \leqq \dfrac{n}{2}\right), \\ c_1\dfrac{n-k}{n} \leqq \sqrt{1-x_{kn}^2} \leqq c_2\dfrac{n-k}{n} & \left(\dfrac{n}{2} < k \leqq n-1\right), \\ \dfrac{c_3}{n} \leqq \vartheta_{k+1,n}-\vartheta_{kn} \leqq \dfrac{c_4}{n}, \end{cases}$$

where c_1, c_2, c_3 and c_4 are fixed positive numbers and $x_{kn}=\cos\vartheta_{kn}$ $(0 \leqq \vartheta_{kn} \leqq \pi)$,

$$\tag{2.15} l_{kn}^2(x) \leqq \sum_{k=1}^{n} l_{kn}^2(x) \leqq 1 \qquad (x \in [-1,1]).$$

Now we can prove our theorem. Let us consider the estimation (2. 11). Let $P_n(x)$ be the polynomial defined by Lemma 2. 2. Then with (2. 5)

$$\tag{2.16}\begin{aligned} f(x)-R_n(f;x) \equiv f(x)-\sum_{k=1}^{n} f(x_k)r_k(x) = f(x)-P_n(x)+ \\ +\sum_{k=1}^{n}[P_n(x_k)-f(x_k)]r_k(x)+\sum_{k=1}^{n}P_n''(x_k)\varrho_k(x). \end{aligned}$$

Here

$$\tag{2.17} |f(x)-P_n(x)| = O\left[\omega_2\left(\frac{1}{n}\right)\right]$$

by (2. 7), further, using the estimations (2. 13) and (2. 7) we have

$$\tag{2.18}\begin{aligned} \left|\sum_{k=1}^{n/2}[P_n(x_k)-f(x_k)]r_k(x)\right| = O(n)\omega_2\left(\frac{1}{n^2}\right)+\left|\sum_{k=2}^{n/2}[P_n(x_k)-f(x_k)]r_k(x)\right| = \\ = O(n)\omega_2\left(\frac{1}{n^2}\right)+O(\sqrt{n})\sum_{k=2}^{n/2}\omega_2\left(\frac{k+1}{n^2}\right)\frac{1}{\sqrt{k}} = O(n)\omega_2\left(\frac{1}{n}\right). \end{aligned}$$

Let us estimate the remaining part. From (2. 13), (2. 8); (2. 8), (2. 13) and (2. 15) we have

$$\tag{2.19}\begin{aligned} \left|\sum_{k=1}^{n/2}P_n''(x_k)\varrho_k(x)\right| = O(n^{-7/2})n^4\omega_2\left(\frac{1}{n^2}\right)+O(1)\sum_{k=2}^{n/2}\omega_2\left(\frac{k+1}{n^2}\right)\frac{n^4}{(k+1)^2}\cdot \\ \cdot\left[n^{-2}l_k(x)\frac{k^2}{n^2}k^{1/2}+n^{-7/2}k^{3/2}\right] = O(\sqrt{n})\omega_2\left(\frac{1}{n^2}\right)+O\left[\omega_2\left(\frac{1}{n}\right)\right]\sum_{k=2}^{n/2}\left[l_k(x)\sqrt{k}+\frac{\sqrt{n}}{\sqrt{k}}\right] = \\ = O(\sqrt{n})\omega_2\left(\frac{1}{n^2}\right)+O\left[\omega_2\left(\frac{1}{n}\right)\right]\left\{\left[\sum_{k=2}^{n/2}l_k^2(x)\right]^{1/2}\left[\sum_{k=2}^{n/2}k\right]^{1/2}+\sqrt{n}\sum_{k=2}^{n/2}\frac{1}{\sqrt{k}}\right\} = O(n)\omega_2\left(\frac{1}{n}\right). \end{aligned}$$

Similarly

$$(2.20)\qquad \left|\sum_{k>n/2+1}^{n} [P_n(x_k)-f(x_k)]r_k(x)+P_n''(x_k)\varrho_k(x)\right| = O(n)\omega_2\left(\frac{1}{n}\right).$$

By using the formulae (2. 16)—(2. 20) we have (2. 11).

To complete our proof we have to justify (2. 12). For this reason we remark that

$$(2.22)\qquad \sum_{k=\left[\frac{n}{8}\right]}^{\left[\frac{n}{4}\right]} |r_k(0)| > c_5 n \qquad (c_5>0).$$

if n is large enough, $n>N_0$ (s. [3]). Let $n>N_0$; then we can apply our Theorem 1. 1, with "the cast" $l_k(x)=r_k(x)$, $I_n=\{x_{\left[\frac{n}{4}\right]+1}, x_{\left[\frac{n}{8}\right]-1}\}$ and $x_0 = 0$. Using — for these I_n — the well-known relation $d_n(I_n) \geqq \frac{c_6}{n}$ $(c_6 > 0)$ (we could easily justify it by (2. 14), too), we obviously obtain (2. 12). Q.E.D.

NOTES. 1. If we suppose (A_2) for $f(x)$, then the condition $\omega_2(t)=o(t)$ is sufficient to the uniform convergence of $R_n(f; x)$ to $f(x)$. This is equivalent to the condition (2. 6). On the other hand, the condition $\omega_2(t)=O(t)$ is generally not sufficient. I.e., if (A_2), (i), (ii) and (iii) are valid, the condition $\omega_2(t)=o(t)$ is necessary and sufficient to the uniform convergence of $R_n(f; x)$ to an arbitrary $f(x)\in C^{[-1,1]}(\omega_2)$.

2. The formula (2. 11) — as we can easily prove — will be true if $\beta_{kn}\neq 0$, but

$$|\beta_{kn}| = O(1)\frac{\omega_2\left(\frac{k+1}{n^2}\right)}{\frac{(k+1)^2}{n^4}}.$$

On the other hand, we can prove (2. 12) only in the case

$$|\beta_{kn}| = o(1)\frac{\omega_2\left(\frac{k+1}{n^2}\right)}{\left(\frac{k+1}{n^2}\right)^2}$$

(see the Note in Part 1).

3. On the (0, 1, 3) interpolation

R. B. SAXENA and A. SHARMA — extending the program of TURÁN — introduced the definition of (0, 1, 3) interpolation in [4].

We seek a polynomial $R_n(x)$ of degree $\leqq 3n-1$, where the values of the function, its first and third derivatives are prescribed at the n points x_{kn} where

$$-1 \leqq x_{nn} < x_{n-1,n} < \cdots < x_{1n} \leqq 1,$$

i.e.,

$$R_n(x_{kn}) = \alpha_{kn}, \qquad R_n'(x_{kn}) = \beta_{kn}, \qquad R_n'''(x_{kn}) = \gamma_{kn}, \tag{3.1}$$

say. SAXENA and SHARMA proved the following (s. [4] and [5]).

If we choose the fundamental points x_{kn} as in the previous part (see (2. 3)) and if $n=2k$, then to prescribed values α_{kn}, β_{kn} and γ_{kn} there is a uniquely determined polynomial[7]

$$R_n(x) = \sum_{k=1}^{n} \left[\alpha_{kn} u_{kn}(x) + \beta_{kn} v_{kn}(x) + \gamma_{kn} w_{kn}(x)\right]$$

of degree $\leqq (3n-1)$ such that (3. 1) is valid. Further, we have for an arbitrary polynomial $P_{3n-1}(x)$ of degree $\leqq (3n-1)$

$$P_{3n-1}(x) = \sum_{k=1}^{n} \left[P_{3n-1}(x_{kn}) u_{kn}(x) + P_{3n-1}'(x_{kn}) v_{kn}(x) + P_{3n-1}'''(x_{kn}) w_{kn}(x)\right]. \tag{3.2}$$

The following theorem holds:

THEOREM 3. 1. (SAXENA and SHARMA). *Let $f(x)$ have continuous derivative of order 2 in $[-1, 1]$ with the continuity module $\omega(f''; t)$ of $f''(x)$ such that*

$$\int_0 \frac{\omega(f''; u)}{u}\, du \tag{3.3}$$

exists. Supposing that for arbitrary small $\varepsilon > 0$ we have

$$|\gamma_{kn}| \leqq \varepsilon n^2 \qquad (n = 2l > n_0(\varepsilon),\ k = 1, 2, \ldots n),$$

the sequence

$$R_n(x) = \sum_{k=1}^{n} \left[f(x_{kn}) u_{kn}(x) + f'(x_{kn}) v_{kn}(x) + \beta_{kn} w_{kn}(x)\right] \tag{3.4}$$

converges to $f(x)$ uniformly in $[-1, 1]$. (x_{kn} are the roots of (2. 3).)

Our object is to give an improvement of the above mentioned theorem. Namely, we prove the following

[7] Here the polynomials $u_{kn}(x)$, $v_{kn}(x)$ and $w_{kn}(x)$ as fundamental functions, are of degree $\leqq (3n-1)$ and they play a similar role and have similar properties as $r_{kn}(x)$ and $\varrho_{kn}(x)$.

THEOREM 3. 2. *Let $f(x)$ have continuous derivative of order 1 with the continuity module $\omega_2(f'; t)$ of order 2, for which (A_2) is valid, then*[8]

$$\|f(x)-R_n(f;x)\|_{[-1,1]} = O\left[n\omega_2\left(\frac{1}{n}\right)\right] \quad \text{if} \quad n=4,6,8,\ldots, \tag{3. 5}$$

where

$$R_n(f;x) = \sum_{k=1}^{n} f(x_{kn})u_{kn}(x) + \sum_{k=1}^{n} f'(x_{kn})v_{kn}(x)$$

are the uniquely determined (0, 1, 3) *interpolating polynomials for the zeros x_{kn} of the polynomials $\Pi_n(x)$.*[9]

PROOF. Let $n=2l$, $n\geqq 4$.

We shall apply the following estimations (s. [5] (6. 1)—(6. 3), Lemma 7. 1, Lemma 7. 2, (8. 1)—(8. 3)).

(3. 6)

$$\begin{cases} u_1(x) = O(n), \qquad u_n(x) = O(n) \\ u_k(x) = O(1)[n^{1/2}l_k(x)k^{-1}+nk^{-5/2}+k^{1/2}l_k^2(x)+l_k^3(x)] \qquad \left(2\leqq k\leqq \frac{n}{2}\right), \\ u_k(x) = O(1)[n^{1/2}l_k(x)(n-k)^{-1}+n(n-k)^{-5/2}+(n-k)^{1/2}l_k^2(x)+l_k^3(x)] \\ \qquad\qquad \left(\frac{n}{2}+1\leqq k\leqq n-1\right), \\ v_1(x) = O(n^{-1/2}), \qquad v_n(x) = O(n^{-1/2}), \\ v_k(x) = O(1)[n^{-1}+n^{-3/2}l_k(x)k] \qquad \left(2\leqq k\leqq \frac{n}{2}\right), \\ v_k(x) = O(1)[n^{-1}+n^{-3/2}l_k(x)(n-k)] \qquad \left(\frac{n}{2}+1\leqq k\leqq n-1\right), \\ w_1(x) = O(n^{-5}), \qquad w_n(x) = O(n^{-5}), \\ w_k(x) = O(1)[n^{-7/2}l_k(x)(1-x_k^2)k+n^{-5}k^2] \qquad \left(2\leqq k\leqq \frac{n}{2}\right), \\ w_k(x) = O(1)[n^{-7/2}l_k(x)(1-x_k^2)(n-k)+n^{-5}(n-k)^2] \qquad \left(\frac{n}{2}+1\leqq k\leqq n-1\right). \end{cases}$$

[8] Here $\omega_2\left(f'; \frac{1}{n}\right) = o\left[\omega_2\left(\frac{1}{n}\right)\right]$.

[9] The unique existence of $R_n(f; x)$ is already known from [4].

By the Lemma 2. 2 there exist polynomials $P_n(f'; x) \equiv P_n(x)$ such that

$$(3.7)\quad |f'(x)-P_n(x)| = O(1)\omega_2\left(f'; \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right) = O(1)\omega_2\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)$$

$$(-1 \leqq x \leqq 1),$$

$$(3.8)\quad |P_n''(x)| = O(1)\frac{\omega_2\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)}{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^2} \qquad (-1 \leqq x \leqq 1).$$

If

$$Q_{n+1}(x) = \int_{-1}^{x} P_n(t)\,dt$$

then

$$|f(x)-f(-1)-Q_{n+1}(x)| = \left|\int_{-1}^{x}[f'(t)-P_n(t)]\,dt\right| = O(1)\omega_2\left(f'; \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right).$$
(3. 9)

Now we obtain by the above defined polynomials $P_n(x)$ and $Q_{n+1}(x)$ (see (3. 2))

$$f(x)-R_n(f;x) \equiv f(x)-\sum_{k=1}^{n} f(x_k)u_k(x)-\sum_{k=1}^{n} f'(x_k)v_k(x) = f(x)-f(-1)-Q_{n+1}(x)-$$
$$-\sum_{k=1}^{n}[f(x_k)-f(-1)-Q_{n+1}(x_k)]u_k(x)-$$
$$-\sum_{k=1}^{n}[f'(x_k)-P_n(x_k)]v_k(x)+\sum_{k=1}^{n} P_n''(x_k)w_k(x).$$

At first we intend to prove (3. 5). By (3. 9)

$$(3.10)\quad |f(x)-f(-1)-Q_{n+1}(x)| = O\left[\omega_2\left(f'; \frac{1}{n}\right)\right],$$

further, by the formulae (3. 9), (3. 6) and (2. 15)

$$(3.11)\quad \left|\sum_{k=1}^{n/2}[f(x_k)-f(-1)-Q_{n+1}(x_k)]u_k(x)\right| = O(n)\omega_2\left(\frac{1}{n^2}\right)+$$
$$+O(1)\omega_2\left(\frac{1}{n}\right)\left[\sqrt{n}\sum_{k=2}^{n/2}\frac{l_k(x)}{k}+n\sum_{k=2}^{n/2}\frac{1}{k^{5/2}}+\sum_{k=2}^{n/2} l_k^2(x)k^{1/2}+\sum_{k=2}^{n/2} l_k^3(x)\right] =$$
$$= O(n)\omega_2\left(\frac{1}{n^2}\right)+O(1)\omega_2\left(\frac{1}{n}\right)\left\{\sqrt{n}\left[\sum_{k=2}^{n/2} l_k^2(x)\right]^{1/2}\left[\sum_{k=2}^{n/2} k^{-2}\right]^{1/2}+\frac{1}{\sqrt{n}}+\right.$$
$$\left.+\left[\sum_{k=2}^{n/2} l_k^4(x)\sum_{k=2}^{n/2} k\right]^{1/2}+1\right\} = O(n)\omega_2\left(\frac{1}{n^2}\right)+O(1)\omega_2\left(\frac{1}{n}\right)\left(1+\frac{1}{\sqrt{n}}+n+1\right) = O(n)\omega_2\left(\frac{1}{n}\right).$$

Moreover, from (3. 7), (3. 6) and (2. 15) we have

$$(3.12)\qquad \left|\sum_{k=1}^{n/2}[f'(x_k)-P_n(x_k)]v_k(x)\right| = O\left(\frac{1}{\sqrt{n}}\right)\omega_2\left(\frac{1}{n^2}\right)+$$

$$+O(1)\omega_2\left(\frac{1}{n}\right)\left\{\sum_{k=2}^{n/2}n^{-1}+n^{-3/2}\left[\sum_{k=2}^{n/2}l_k^2(x)\sum_{k=2}^{n/2}k^2\right]^{1/2}\right\} = O\left(\frac{1}{\sqrt{n}}\right)\omega_2\left(\frac{1}{n^2}\right)+$$

$$+O(1)\omega_2\left(\frac{1}{n}\right)(1+1) = \underline{O(1)\omega_2\left(\frac{1}{n}\right)}.$$

Finally, by (3. 8), (2. 14), (3. 6) and (2. 15) we obtain

$$(3.13)\qquad \left|\sum_{k=1}^{n/2}P_n''(x_k)w_k(x)\right| = O(n^{-5})n^4\omega_2\left(\frac{1}{n^2}\right)+O(1)\omega_2\left(\frac{1}{n}\right)\left\{\sum_{k=2}^{n/2}\left[\frac{n^4 l_k(x)k^2 k}{(k+1)^2 n^{7/2} n^2}+\right.\right.$$

$$\left.\left.+\frac{n^4k^2}{(k+1)^2n^5}\right]\right\} = O\left(\frac{1}{n}\right)\omega_2\left(\frac{1}{n^2}\right)+O(1)\omega_2\left(\frac{1}{n}\right)\left\{n^{-3/2}\left[\sum_{k=2}^{n/2}l_k^2(x)\sum_{k=2}^{n/2}k^2\right]^{1/2}+\sum_{k=2}^{n/2}\frac{1}{n}\right\} =$$

$$= O\left(\frac{1}{n}\right)\omega_2\left(\frac{1}{n^2}\right)+O(1)\omega_2\left(\frac{1}{n}\right)(1+1) = \underline{O(1)\omega_2\left(\frac{1}{n}\right)}.$$

We obtain analogous estimations for the parts $\sum_{k=\frac{n}{2}+1}^{n}$. I.e., from (3. 10)—(3. 13) we have

$$|f(x)-R_n(f;x)| = O(n)\omega_2\left(\frac{1}{n}\right),$$

as was stated.

NOTES. 1. If $\omega_2\left(f';\frac{1}{n}\right)$ satisfies the condition (A_2), then the restriction $\omega_2\left(f';\frac{1}{n}\right)=o(n)$ is sufficient to the uniform convergence of $R_n(f;x)$ to $f(x)$. On the other hand, at present *I* have no estimation like (2. 22), so I do not know whether the above mentioned restriction is generally also necessary.

2. As in the previous part, by (3. 5) we can choose

$$|\gamma_{kn}| = O(1)\frac{\omega_2\left(\frac{k+1}{n^2}\right)}{\left(\frac{k+1}{n^2}\right)^2}$$

instead of $\gamma_{kn}=0$.

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DER MÜNTZSCHE SATZ BEIM ÜBERGANG VOM UNENDLICHEN ZUM ENDLICHEN INTERVALL

Von

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1. Einleitung

Sei $w(t)>0$ auf $[0,\infty)$, $L_w^p(0,\infty)=\left\{f\colon \|f\|_{p,w}=\left(\int\limits_0^\infty |f|^p w\,dt\right)^{1/p}<\infty\right\}$ für $1\leqq p<\infty$. Grundlegend sind die folgenden Definitionen:

Definition 1. Eine Folge $f_n\in L_w^p(0,\infty)$ heißt vollständig in $L_w^p(0,\infty)$, wenn zu beliebigem $f\in L_w^p(0,\infty)$ und beliebigem $\varepsilon>0$ eine Linearkombination $\Sigma c_i f_i$ existiert mit $\|f-\Sigma c_i f_i\|_{p,w}<\varepsilon$.

Definition 2. Eine Folge $f_n\in L_w^p(0,\infty)$ heißt abgeschlossen in $L_w^p(0,\infty)$, wenn aus $\int\limits_0^\infty f_n g w\,dt=0$ $(n=1,2,\dots)$ $g\in L_w^q(0,\infty)$, $\frac{1}{p}+\frac{1}{q}=1$ folgt, daß $g=0$ f. ü.

Bekanntlich sind, mit diesen Definitionen, die Eigenschaften Vollständigkeit und Abgeschlossenheit äquivalent.

Ist $t^\lambda\in L_w^p(0,\infty)$ für $\lambda\geqq 0$ und $1\leqq p<\infty$, so lautet das Müntzsche Problem: Welches sind notwendige und hinreichende Bedingungen für eine Folge $\Lambda=\{\lambda_i\}_1^\infty$, so daß $\{t^{\lambda_i}\}_1^\infty$, abgeschlossen (vollständig) ist in $L_w^p(0,\infty)$[1]?

Wir betrachten hier die Belegungsfunktionen $w_\alpha(t)=\exp(-t^\alpha)$ für $\alpha>0$. Um auch das Intervall $[0,1]$ in unsere Betrachtungen gleich einzuschließen, setzen wir

$$w_\infty(t)=\lim_{\alpha\to\infty} w_\alpha(t)=\begin{cases}1 & 0\leqq t<1\\ \dfrac{1}{e} & t=1\\ 0 & t>1\end{cases}$$

Es ist dann Abgeschlossenheit (Vollständigkeit) in $L_{w_\infty}^p(0,\infty)$ äquivalent mit Abgeschlossenheit (Vollständigkeit) in $L^p(0,1)$ $(1\leqq p<\infty)$, d.h. für $\alpha=\infty$ liegt das klassische Müntzsche Problem vor.

Offenbar ist $t^\lambda\in L_{w_\alpha}^p(0,\infty)$ für $\lambda\geqq 0$, $1\leqq p<\infty$, $0<\alpha\leqq\infty$. Wir setzen im folgenden voraus, dass $0<\lambda_1<\dots<\lambda_n\uparrow\infty$. Ferner sei

$$\Lambda(t)=\sum_{\lambda_i\leqq t}1,\qquad \lambda(t)=\begin{cases}\dfrac{2}{\lambda_1} & 0\leqq t\leqq\lambda_1\\ 2\displaystyle\sum_{\lambda_i<t}\frac{1}{\lambda_i} & \lambda_1<t\end{cases}$$

[1] Wir beschränken uns in dieser Arbeit auf die Behandlung der L_w^p-Räume mit $1\leqq p<\infty$. Für eine Ausdehnung auf gewisse abgeschlossene Unterräume von L^∞ vergleiche man etwa Fuchs [7], Schwartz [9].

Das erste Resultat ist der klassische Satz von MÜNTZ:

SATZ 1. $\{t^{\lambda_i}\}_1^\infty$ ist genau dann abgeschlossen in $L^p_{w_\infty}(0, \infty)$ $(1 \leqq p < \infty)$ wenn $\sum_1^\infty \frac{1}{\lambda_i} = \infty$.

Das erste Resultat für das unendliche Intervall stammt von FUCHS [6][2]:

SATZ 2. Existiert $k > 0$, so daß $\frac{\Lambda(t)}{t} \geqq \frac{1}{2} - \frac{k}{t}$ $(t \geqq t_0)$, so ist $\{t^{\lambda_i}\}_1^\infty$ abgeschlossen in $L^p_{w_1}(0, \infty)$ $(1 \leqq p < \infty)$.

In derselben Arbeit zeigt FUCHS auch die Äquivalenz von Abgeschlossenheit in $L^2_{w_1}(0, \infty)$ mit einer Aussage über Differenzengleichungen:

SATZ 3. Sei λ_i natürlich für $i = 1, 2, \ldots$. $\{t^{\lambda_i}\}_1^\infty$ ist genau dann abgeschlossen in $L^2_{w_1}(0, \infty)$, wenn die Folge $\Lambda = \{\lambda_i\}_1^\infty$ die folgende Eigenschaft hat:

(D) Aus $\Delta^{\lambda_i} S_0 = 0$, $S_n = O(n^k)$ folgt $S_n = P(n)$, wo $P(t)$ ein Polynom ist.

Ein vereinfachter Beweis dieses Satzes wurde von BOAS—POLLARD [4] gegeben. Danach ist jedes Kriterium für (D) auch ein Kriterium für Abgeschlossenheit in $L^2_{w_1}(0, \infty)$. AGNEW [1] zeigte, daß $\Lambda = \{2i\}_1^\infty$ und $\Lambda = \{2i-1\}_1^\infty$ die Eigenschaft (D) besitzen; der Fall $\Lambda = \{2i\}_1^\infty$ wurde später unabhängig nochmals von POLLARD [8] untersucht. Alle diese Ergebnisse sind enthalten in Satz 2. Nicht enthalten, sondern unvergleichbar mit Satz 2, ist das folgende Resultat von ENDL [5]:

SATZ 4. Sei λ_i natürlich für $i = 1, 2, \ldots$. Ist

$$D(\Lambda) = \varliminf_{x \to \infty} \frac{2}{\pi} \int_0^\infty \frac{\Lambda(t)}{t} \frac{x}{x^2 + t^2}\, dt > \frac{1}{2}$$

so besitzt Λ die Eigenschaft (D).

SPENCER [10] zeigt nämlich die Existenz von Folgen mit[3]

$$\varliminf \frac{\Lambda(t)}{t} < \frac{1}{2} < D(\Lambda).$$

Satz 2 wurde verschärft von Boas [3][4]:

SATZ 5. Existiert $0 < \delta(t)$ mit $\int_1^\infty \frac{\delta(t)}{t}\, dt < \infty$, und gilt

$$\frac{\Lambda(t)}{t} \geqq \frac{1}{2} - \delta(t)$$

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[2] Zuerst für $p = 2$, mit einer Bemerkung in FUCHS [7], S. 91—92 für die übrigen p.

[3] Man vergleiche auch ENDL [5], S. 286.

[4] Die Bezeichnung „complete" (S. 61) in dieser Arbeit entspricht weder unserer Definition von Vollst. noch von Abgeschlossenheit.

so ist $\{t^{\lambda_i}\}_1^\infty$ abgeschlossen in $L^p_{w_1}(0, \infty)$ $(1 \leqq p < \infty)$. Auch dieser Satz ist unvergleichbar mit Satz 4.

Die Sätze 2—5 beziehen sich alle auf $w_1(t) = e^{-t}$ und sind hinreichende Kriterien in Form von Dichteaussagen. Die in diesen Kriterien auftretende Konstante 1/2 ist scharf (Fuchs, [6]). Betrachten wir für den Moment nur natürliche Zahlen λ_i, so braucht man also zur Approximation auf $(0, \infty)$ mit der Belegungsfunktion $w_1(t)$, grob gesagt, die Hälfte der Potenzen, während man für $w_\infty(t)$, den Müntzschen Fall, noch mit Teilmengen der natürlichen Zahlen von der Dichte 0 auskommt. Die Klärung der Rolle dieser Konstanten 1/2 war Ausgangpunkt dieser Arbeit.

Es ist anschaulich klar, daß für wachsendes α der Einfluß der großen t-Werte immer kleiner wird, da der Beitrag zur Norm wegen der stärkeren Konvergenz von $w_\alpha(t)$ gegen 0 immer kleiner wird. Man wird also erwarten, daß die „Anzahl" der zur Abgeschlossenheit benötigten Potenzen von den asymptotischen Eigenschaften der Belegungsfunktion abhängt. In dieser Arbeit wird die Frage untersucht, ob es nicht ein einheitliches Kriterium für unsere Familie von Belegungsfunktionen $w_\alpha(t)$ $(0 < \alpha \leqq \infty)$ gibt, so daß sich der klassische Fall des Intervalls (0, 1) als Grenzfall des Intervalls $(0, \infty)$ mit der Belegungsfunktion $w_\alpha(t)$ für $\alpha \to \infty$ ergibt.

SPENCER [10] zeigt, im Anschluß an die Arbeit von BOAS [3]:

SATZ 6. Sei $0 < \alpha < \infty$. Existiert $0 < \delta(t)$ mit $\int\limits_1^\infty \frac{\delta(t)}{t}\,dt < \infty$ und gilt

$$\frac{\Lambda(t)}{t} \geqq \frac{1}{2\alpha} - \delta(t)$$

so ist $\{t^{\lambda_i}\}_1^\infty$ abgeschlossen in $L^p_{w_\alpha}(0, \infty)$ $(1 \leqq p < \infty)$.

Für $\alpha \to \infty$ folgt hieraus nur, daß $\{t^{\lambda_i}\}_1^\infty$ bezüglich $L^p_{w_\infty}(0, \infty)$ abgeschlossen ist für Folgen Λ von beliebig kleiner Dichte. Dies ist natürlich viel schwächer als der Satz von MÜNTZ.

Wir werden zeigen, daß sich für $p = 2$, unter der zusätzlichen Bedingung $\lambda_{i+1} - \lambda_i \geqq c > 0$, das gesuchte einheitliche Kriterium ergibt. Wir stützen uns dazu auf das tiefliegende Resultat von Fuchs [7]:

SATZ 7. Sei $\lambda_{i+1} - \lambda_i \geqq c > 0$; dann ist $\{t^{\lambda_i}\}_1^\infty$ genau dann abgeschlossen in $L^2_{w_1}(0, \infty)$, wenn

$$\int\limits_1^\infty \frac{e^{\lambda(r)}}{r^2}\,dr = \infty.$$

Zur Formulierung des gemeinsamen Kriteriums benutzen wir die aus der Funktionalanalysis üblichen Definitionen für die L^p-Räume. Wir definieren, für eine beliebige, auf [0, 1] meßbare Funktion f:

$$\|f\|_{L^\alpha[0,1]} = \begin{cases} \left(\int\limits_0^1 |f(r)|^\alpha\,dr\right)^{\frac{1}{\alpha}} & 0 < \alpha < \infty \\ \sup_0\limits_{[0,1]} |f(r)| & \alpha = \infty \end{cases}$$

Bekanntlich ist für jede solche Funktion f[5]

$$\lim_{\alpha\to\infty} \|f\|_{L^\alpha[0,1]} = \|f\|_{L^\infty[0,1]} \tag{1.1}$$

Mit diesen Definitionen können wir jetzt zeigen, daß die Funktion $e^{\lambda\left(\frac{1}{r}\right)}$ „verantwortlich" für Abgeschlossenheit in $L^2_{w_\alpha}$ ist $(0<\alpha\leqq\infty)$.

Wir zeigen:

SATZ 8. Sei $\lambda_{i+1}-\lambda_i \geqq c>0$, $0<\alpha\leqq\infty$. Dann ist $\{t^{\lambda_i}\}_1^\infty$ genau dann abgeschlossen in $L^2_{w_\alpha}(0,\infty)$, wenn

$$\|e^{\lambda\left(\frac{1}{r}\right)}\|_{L^\alpha[0,1]} = \infty$$

Aus der Definition von $\lambda(t)$ folgt sofort, daß für $\alpha=\infty$, den Müntz'schen Fall, der Satz 8 richtig ist:

$$\|e^{\lambda\left(\frac{1}{t}\right)}\|_{L^\infty[0,1]} = \sup_{[0,1]}{}_0\, e^{\lambda\left(\frac{1}{t}\right)} = \infty \Leftrightarrow \sum_1^\infty \frac{1}{\lambda_i} = \infty$$

Es bleibt also nur noch die Richtigkeit für $0<\alpha<\infty$ zu zeigen. Wegen (1. 1) ist das Kriterium für $\alpha=\infty$ tatsächlich die Form, in welche formal das Kriterium für $0<\alpha<\infty$ übergeht, wenn $\alpha\to\infty$. Wir haben damit also über die Belegungsfunktionen $w_\alpha(t)=e^{-t^\alpha}$ den „stetigen Anschluß" vom unendlichen Intervall zum endlichen Müntzschen Fall erzielt.

2. Abgeschlossenheit in $L^2_{w_\alpha}(0,\infty)$ $(0<\alpha<\infty)$.

SATZ 9. $\{t^{\lambda_i}\}_1^\infty$ ist genau dann abgeschlossen in $L^2_{w_\alpha}(0,\infty)$, wenn

$$\int_1^\infty \frac{e^{\alpha\lambda(r)}}{r^2}\,dr = \infty.$$

BEWEIS: Wir zeigen, daß das System von Bedingungen

$$\text{(I)}\qquad (2,1) = \int_0^\infty u^{\lambda_n} g(u) e^{-u^\alpha}\,du = 0 \qquad n=1,2,3,\dots \qquad g\in L^2_{w_\alpha}(0,\infty)$$

äquivalent ist zu

$$\text{(II)}\qquad (2,2) = \int_0^\infty t^{\frac{\lambda_n}{\alpha}} f(t) e^{-t}\,dt = 0 \qquad n=1,2,3,\dots \qquad f\in L^2_{w_1}(0,\infty)$$

wobei $f(t)=g\left(t^{\frac{1}{\alpha}}\right)\frac{1}{\alpha}t^{\frac{1}{\alpha}-1}$ bzw. $g(u)=f(u^\alpha)\alpha u^{\alpha-1}$ ist. Wir gehen dazu von (I)

[5] TITCHMARSH, *Trig. Series*, 2nd Edition, S. 18.

aus und substituieren $u = t^{1/\alpha}$ in (2, 1):

$$(2,1) = \int_0^\infty t^{\frac{\lambda_n}{\alpha}} g\left(t^{\frac{1}{\alpha}}\right) e^{-t} \frac{1}{\alpha} t^{\frac{1}{\alpha}-1}\, dt = \int_0^\infty t^{\frac{\lambda_n}{\alpha}} f(t) e^{-t}\, dt$$

wobei $f(t) = g\left(t^{\frac{1}{\alpha}}\right) \frac{1}{\alpha} t^{\frac{1}{\alpha}-1} \in L^2_{w_1}(0,\infty)$ ist. Denn, in der Zerlegung

$$f(t) = g\left(t^{\frac{1}{\alpha}}\right) \frac{1}{\sqrt{\alpha}} t^{\frac{1}{2\alpha}-\frac{1}{2}} \cdot \frac{1}{\sqrt{\alpha}} t^{\frac{1}{2\alpha}-\frac{1}{2}}$$

sind beide Faktoren aus $L^2_{w_1}(0,\infty)$ und damit auch das Produkt:

$$\int_0^\infty \left[g\left(t^{\frac{1}{\alpha}}\right) \frac{1}{\sqrt{\alpha}} t^{\frac{1}{2\alpha}-\frac{1}{2}} \right]^2 e^{-t}\, dt = \int_0^\infty g^2(u) e^{-u^\alpha}\, du < \infty$$

$$\int_0^\infty \left[\frac{1}{\sqrt{\alpha}} t^{\frac{1}{2\alpha}-\frac{1}{2}} \right]^2 e^{-t}\, dt = \int_0^\infty e^{-u^\alpha}\, du < \infty .$$

Gehen wir umgekehrt von (II) aus und substituieren $t = u^\alpha$ in (2, 2)

$$(2,2) = \int_0^\infty u^{\lambda_n} f(u^\alpha) e^{-u^\alpha} \alpha \cdot u^{\alpha-1}\, du = \int_0^\infty u^{\lambda_n} g(u) e^{-u^\alpha}\, du$$

so ist wieder $g(u) = f(u^\alpha) \alpha u^{\alpha-1} \in L^2_{w_\alpha}(0,\infty)$, denn mit der Zerlegung

$$g(u) = f(u^\alpha) \sqrt{\alpha}\, u^{\frac{\alpha}{2}-1} \cdot \sqrt{\alpha}\, u^{\frac{\alpha}{2}-\frac{1}{2}}$$

ist wieder jeder Faktor aus $L^2_{w_\alpha}(0,\infty)$. Es folgt, daß $\{t^{\lambda_n}\}_1^\infty$ genau dann in $L^2_{w_\alpha}(0,\infty)$ abgeschlossen ist, wenn $\{t^{\frac{\lambda_n}{\alpha}}\}_1^\infty$ in $L^2_{w_1}(0,\infty)$ abgeschlossen ist. Nun ist aber $\{t^{\frac{\lambda_n}{\alpha}}\}_1^\infty$ genau dann in $L^2_{w_1}(0,\infty)$ abgeschlossen, wenn

$$\int_1^\infty \exp\left\{ 2 \sum_{\frac{\lambda_n}{\alpha} < r} \frac{\alpha}{\lambda_n} \right\} \frac{dr}{r^2} = \infty$$

Die Substitution $\alpha r = \varrho$ liefert

$$\alpha \int_\alpha^\infty \exp\left\{ 2\alpha \sum_{\lambda_n < \varrho} \frac{1}{\lambda_n} \right\} \frac{d\varrho}{\varrho^2} = \infty$$

oder, in äquivalenter Form,

$$\int_1^\infty \exp\{\alpha \lambda(\varrho)\} \frac{d\varrho}{\varrho^2} = \infty$$

BEMERKUNG: Die Bedingung von Satz 9 ist auch hinreichend für Abgeschlossenheit in $L^p_{w_\alpha}(0,\infty)$ $(1 \leq p < \infty)$[6]

[6] Man beachte Fußnote 2.

3. Beweis des gemeinsamen Kriteriums für $0<\alpha<\infty$

Wir formen zuerst das Kriterium für die Räume $L^2_{w_\alpha}(0, \infty)$ um. Die Bedingung $\int\limits_1^\infty \exp\{\alpha\lambda(r)\}\frac{dr}{r^2}=\infty$ ist, wie die Substitution $r=\frac{1}{\varrho}$ zeigt, äquivalent zu

$$\int\limits_0^1 \left(e^{\lambda\left(\frac{1}{\varrho}\right)}\right)^\alpha d\varrho=\infty$$

oder

$$\left(\int\limits_0^1 \left(e^{\lambda\left(\frac{1}{\varrho}\right)}\right)^\alpha d\varrho\right)^{\frac{1}{\alpha}}=\|e^{\lambda\left(\frac{1}{\varrho}\right)}\|_{L^\alpha(0,1)}=\infty .$$

4. α-Vollständigkeit in $L^p(0, \infty)$

Es ist klar, daß jede Folge $\{t^{\lambda_n}\}_1^\infty$, die vollständig in $L^p_{w_\alpha}(0, \infty)$ $(0<\alpha<\infty)$ für ein festes α ist, auch vollständig in $L^p_{w_\infty}(0, \infty)$ ist. Die Umkehrung gilt nicht. Aus Vollständigkeit in $L^p_{w_\infty}(0, \infty)$ braucht nicht Vollständigkeit in $L^p_{w_\alpha}(0, \infty)$, auch für noch so großes α, zu folgen. Wohl aber folgt Vollständigkeit in $L^p(0, \infty)$, wenn wir für die Approximationsnorm α variabel zulassen. Wir definieren dazu (man beachte, daß $L^p(0, \infty)\subset L^p_{w_\alpha}(0, \infty)$ $(0<\alpha\leqq\infty)$):

Definition 3. Sei $0<\alpha<\infty$. $\{t^{\lambda_n}\}_1^\infty$ heißt α-vollständig in $L^p(0, \infty)$, wenn zu jedem $f\in L^p(0, \infty)$ und $\varepsilon>0$ ein α existiert, so dass für eine passende Linearkombination $\Sigma c_i t^{\lambda_i}$ gilt:

$$\|f-\Sigma c_i t^{\lambda_i}\|_{p, w_\alpha}<\varepsilon$$

Es gilt dann:

Satz 10. $\{t^{\lambda_n}\}_1^\infty$ ist genau dann α-vollständig in $L^p(0, \infty)$, wenn vollständig in $L^p_{w_\infty}(0, \infty)$.

Beweis: 1. Sei $\{t^{\lambda_n}\}_1^\infty$ vollständig in $L^p_{w_\infty}(0, \infty)$. Dann existiert zu $f\in L^p(0, \infty)$ und $\varepsilon>0$ eine Linearkombination $\Sigma c_i t^{\lambda_i}$ mit

$$\int\limits_0^1 |f-\Sigma c_i t^{\lambda_i}|^p dt<\frac{\varepsilon^p}{2}.$$

Für α groß genug wird

$$\int\limits_1^\infty |f-\Sigma c_i t^{\lambda_i}|^p e^{-t^\alpha} dt<\frac{\varepsilon^p}{2}$$

Es folgt

$$\int\limits_0^\infty |f-\Sigma c_i t^{\lambda_i}|^p e^{-t^\alpha} dt<\int\limits_0^1 |f-\Sigma c_i t^{\lambda_i}|^p dt+\int\limits_1^\infty |f-\Sigma c_i t^{\lambda_i}|e^{-t^\alpha} dt<e^p.$$

2. Sei $\{t^{\lambda_n}\}_1^\infty$ α-vollständig. Dann existiert zu $f \in L^p(0, \infty)$ und $\varepsilon > 0$ ein α und eine Linearkombination $\Sigma c_i t^{\lambda_i}$ mit

$$\int_0^\infty |f - \Sigma c_i t^{\lambda_i}|^p e^{-t^\alpha} dt < \frac{\varepsilon^p}{e}.$$

Hieraus folgt:

$$\int_0^1 |f - \Sigma c_i t^{\lambda_i}|^p dt < \int_0^1 |f - \Sigma c_i t^{\lambda_i}|^p \cdot e e^{-t^\alpha} dt < \varepsilon^p$$

Wir geben zum Schluß ein Kriterium für α-Vollständigkeit, das nochmals zeigt, wie der klassische Müntzsche Fall als Grenzfall der Approximation in $L^p_{w_\alpha}(0, \infty)$-Räumen für $\alpha \to \infty$ auftritt:

SATZ 11. $\{t^{\lambda_n}\}_1^\infty$ ist genau dann α-vollständig in $L^p(0, \infty)$, wenn

$$\lim_{\alpha \to \infty} \|e^{\lambda\left(\frac{1}{r}\right)}\|_\alpha = \infty$$

BEWEIS: 1. Sei $\{t^{\lambda_n}\}_1^\infty$ α-vollständig in $L^p(0, \infty)$ und $K > 0$. Dann existiert nach Müntz $\varepsilon > 0$ mit

$$e^{\lambda\left(\frac{1}{\varepsilon}+0\right)} > K$$

Da

$$\lim_{\alpha \to \infty} \left(\int_\varepsilon^1 \left(e^{\lambda\left(\frac{1}{r}\right)}\right)^\alpha dr \right)^{\frac{1}{\alpha}} = \sup_{[\varepsilon, 1]} e^{\lambda\left(\frac{1}{r}\right)} = e^{\lambda\left(\frac{1}{\varepsilon}+0\right)}$$

gibt es ein α mit

$$\|e^{\lambda\left(\frac{1}{r}\right)}\|_\alpha \geqq \left(\int_\varepsilon^1 \left(e^{\lambda\left(\frac{1}{r}\right)}\right)^\alpha dr \right)^{\frac{1}{\alpha}} > K$$

Hieraus folgt

$$\lim_{\alpha \to \infty} \|e^{\lambda\left(\frac{1}{r}\right)}\|_\alpha = \infty$$

2. Es sei die Bedingung von Satz 10 erfüllt. Wäre $\sum_1^\infty \frac{1}{\lambda_i} = M < \infty$, so hätte man

$$\|e^{\lambda\left(\frac{1}{r}\right)}\|_\alpha = \left(\int_0^1 \left(e^{\lambda\left(\frac{1}{r}\right)}\right)^\alpha dr \right)^{\frac{1}{\alpha}} \leqq e^M$$

im Widerspruch zur Voraussetzung.

BEMERKUNG: Gelänge es, Satz 10 direkt zu beweisen, so hätte man einen neuen Beweis des Satzes von MÜNTZ.

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NOTE ON FIRST ISOMORPHISM THEOREM IN LATTICES

By
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G. GRÄTZER and E. T. SCHMIDT prove in [1] (cf. page 78, theorem 23) that "An ideal I of a modular lattice of locally finite length with zero satisfies the first isomorphism theorem if and only if it is neutral". Their proof makes use of the concept of weak projectivity and is rather long. In what follows we give a very short proof of the same theorem without making use of the notion of weak projectivity. We also give an example of a modular lattice L and an ideal I of L such that I satisfies the first isomorphism theorem and despite this I is not a homomorphism kernel and hence not a neutral ideal in theorem 2. This solves problem 20 of [1]. Further we show that any ideal of a finite weakly modular lattice satisfying the first isomorphism theorem need not be a homomorphism kernel with the help of an example. This gives a partial solution to problem 19 of [1].

THEOREM 1. *An ideal N of a modular lattice L of locally finite length with zero satisfies the first isomorphism theorem if and only if it is neutral.*

PROOF. If N is neutral, then it satisfies the first isomorphism theorem is well known.

Conversely let L be a modular lattice of locally finite length and let if possible N be a nonneutral ideal of L satisfying the first isomorphism theorem. Now as L is modular, $I(L)$ the lattice of ideals of L is modular. The nonneutrality of the ideal N implies the existence of principal ideals $X=(x)$ and $Y=(y)$ of L such that $(N+X)(N+Y) \gneq N+XY$; as $I(L)$ being modular any one of the distributive laws implies all the others. Consider $A=(N+X)(NX+Y)$ and $B=(N+Y)(NY+X)$. Now $A \neq B$. For if $A=B$ then $A=AB$ implies $N+A=N+AB$; i.e., $(N+X)(N+Y)=N+XY$, a contradiction. Also $N+A=N+B=(N+X)(N+Y)$ and $NA=NB=NX+NY$. Further $NA \lneq N$, as otherwise $NX+NY=N(X+Y)$ would imply $N+XY = (N+X)(N+Y)$ (as $I(L)$ is modular).

Now the interval $(N, N+A)$ is isomorphic to the interval (NA, A); as they are transposes of each other in the modular lattice $I(L)$. Further L being a lattice of locally finite length; every interval of L is a lattice of finite length, and in particular the interval $L_1=(0, x+y)$. Therefore $I(L_1)$ is isomorphic to L_1 and hence (NA, A) being an interval of $I(L_1)$ is of finite length. Hence $(N, N+A)$ being isomorphic to (NA, A), is of finite length. By hypothesis N satisfies the first isomorphism theorem and hence lattice $N+A/N$ is isomorphic to the lattice A/NA. Note that the lattice $N+A/N$ is not isomorphic to the interval $(N, N+A)$, as the interval $(N+XY, (N+X)(N+Y))$ is annulled by the congruence generated by N. Thus A/NA is not isomorphic to the interval (NA, A); which means NA is nonneutral in A.

Put $NA = N_1$. N_1 is nonneutral in A. Replacing N by N_1 and L by A and repeating the argument, we can get ideals A_1, B_1 $(A_1 \neq B_1)$ both contained in A such that $N_1 + A_1 = N_1 + B_1$ and $N_1 A_1 = N_1 B_1$ with $N_1 A_1 \lneqq N_1$.

Now $N + A_1 = (N + N_1) + A_1 = N + (N_1 + A_1) = N + (N_1 + B_1) = (N + N_1) + B_1 = N + B_1$, and $NA_1 = N(AA_1) = (NA)A_1 = N_1 A_1 = N_1 B_1 = (NA)B_1 = N(AB_1) = NB_1$. N satisfies the first isomorphism theorem and by a similar argument $N_2 = N_1 A_1$ is nonneutral in A_1 and $N_2 \lneqq N_1$. Proceeding thus we will arrive at an infinite sequence of nonneutral ideals $N_1 \gneqq N_2 \gneqq N_3 \ldots$, all contained in the lattice consisting of the interval $(0, x+y)$ which is of finite length. A contradiction. Therefore no nonneutral ideal of L can satisfy the first isomorphism theorem.

THEOREM 2. *Let L be any modular lattice with zero. An ideal I of L satisfying the first isomorphism theorem need not necessarily be a homomorphism kernel.*

PROOF: By an example.

Let $S = (a_1, a_2, a_3, \ldots, a_n, \ldots)$ be an infinite set. Consider the distributive lattice D — a sublattice of the lattice of all subsets of S whose join irreducible elements are $\varnothing$ (the null set), a_1, a_2, $a_1 + a_3$, $a_1 + a_4$, $a_2 + a_5$, $a_2 + a_6$; $a_1 + a_3 + a_7$, $a_1 + a_3 + a_8$, $a_1 + a_4 + a_9$, $a_1 + a_4 + a_{10}$, $a_2 + a_5 + a_{11}$, $a_2 + a_5 + a_{12}$, $a_2 + a_6 + a_{13}$, $a_2 + a_6 + a_{14}$; $\ldots$. That is D has 2^0 join irreducible elements of dimension 0, 2^1 join irreducible elements of dimension *I*, 2^2 join irreducible elements of dimension 2, $\ldots$, 2^n join irreducible elements of dimension n such that each join irreducible element of D is covered by two join irreducible elements of D. Take the ordinal sum $M \oplus D$; where M is the five element nondistributive modular lattice $(0, x, y, n, \varnothing)$. Insert elements x_1, x_2 and y_1, y_2 such that we get the lattice $K_{\varnothing}$, as shown in figure 1.

Next, consider the elements a_1 and a_2 of $K_{\varnothing}$; i.e., the two join irreducible ele-

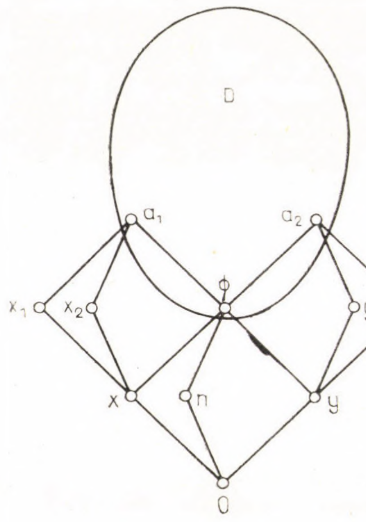

Fig. 1

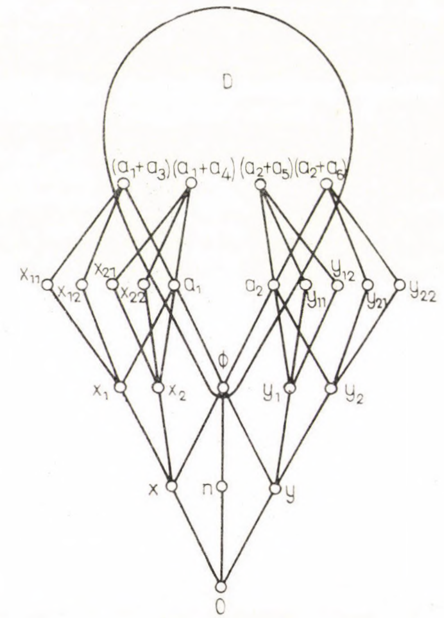

Fig. 2

ments of dimension I of D. a_1 has 2 join irreducible elements covering it namely a_1+a_3, a_1+a_4 and covers two meet irreducible elements x_1, x_2. Similarly a_2 has two join irreducible elements a_2+a_5, a_2+a_6 covering it and covers two meet irreducible elements y_1, y_2. Insert elements x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22} such that we get the lattice K_1 as shown in figure 2.

Next, consider the 2^2 join irreducible elements of dimension 2 of D in K_1. Each is covered by two join irreducible elements and covers two meet irreducible elements of K_1. Again insert 4 elements for each join irreducible element of dimension 2 of

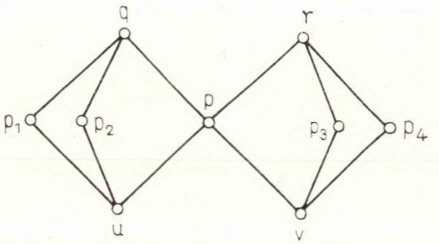

Fig. 3

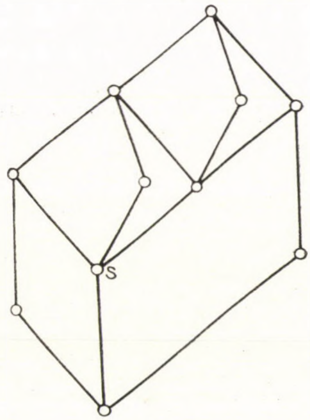

Fig. 4

D and obtain lattice sums and products by completing the diagram as shown in figure 3; where p is a join irreducible element u, v are the meet irreducible elements of K_1 covered by p and q, r are the join irreducible elements covering p. Next, carry out a similar construction for the 2^3 join irreducible elements of dimension 3 of D. Repeat the process step by step for all join irreducible elements of D. Let K be the lattice so obtained. Add an unit element I to K and let L' be the lattice so obtained. It is easy to see that the lattice L' is modular, as it has no sublattice isomorphic to the five element nonmodular lattice.

Let L be the dual of L'. Then L is a modular lattice with zero. The ideal S generated by n is not a homomorphism kernel, nevertheless it satisfies the first isomorphism theorem as $S+I/S$ and I/SI are one element lattices, for any ideal I of L.

COROLLARY 1. An ideal L of a modular lattice satisfying the first isomorphism theorem need not necessarily be neutral.

Proof follows from the fact that any neutral ideal of a lattice is a homomorphism kernel.

THEOREM 3. *An ideal satisfying the first isomorphism theorem of a finite weakly modular lattice need not necessarily be a homomorphism kernel.*

PROOF: By an example.

Consider the lattice L as shown in figure 4. L is a simple lattice and hence is weakly modular. L being simple, no proper ideal of L is a homomorphism kernel; nevertheless the ideal I generated by s satisfies the first isomorphism theorem.

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ON THE CONVERGENCE OF HERMITE—FEJÉR INTERPOLATION

By

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1. For a continuous function on the interval $[-1, 1]$ we define the well-known uniquely determined Hermite—Fejér interpolating polynomials of degree $\leqq 2n-1$ as follows.

$$(1.1)\qquad \begin{cases} H_n(f;\, x) = \sum_{k=1}^{n} f(x_{kn}) A_k^{(n)}(x) \quad \text{or} \\ \overline{H}_n(f;\, x) = H_n(f;\, x) + \sum_{k=1}^{n} \beta_{kn} B_k^{(n)}(x), \end{cases}$$

where

$$(1.2)\qquad \begin{cases} A_k^{(n)}(x) = \left[\dfrac{T_n(x)}{n(x-x_{kn})}\right]^2 (1-x x_{kn}) \qquad (k=1, 2, \dots, n), \\ B_k^{(n)}(x) = \left[\dfrac{T_n(x)}{n(x-x_{kn})}\right]^2 (1-x_{kn}^2)(x-x_{kn}) \qquad (k=1, 2, \dots, n), \\ T_n(x) = \cos n(\arccos x), \qquad x_{kn} = \cos\dfrac{2k-1}{2n}\pi \qquad (k=1, 2, \dots, n), \\ \beta_{kn} \text{ are prescribed.} \end{cases}$$

It is well known (see e.g. [1]) that

$$(1.3)\qquad A_k^{(n)}(x) \geqq 0 \quad \text{for} \quad k=1, 2, \dots, n \quad \text{and every} \quad x \in [-1, 1],$$

$$(1.4)\qquad \sum_{k=1}^{n} A_k^{(n)}(x) \equiv 1,$$

$$(1.5)\qquad H_n(f;\, x_{kn}) = \overline{H}_n(f;\, x_{kn}) = f(x_{kn}) \qquad (k=1, 2, \dots, n),$$

$$(1.6)\qquad H_n'(f;\, x_{kn}) = 0, \qquad \overline{H}_n'(f;\, x_{kn}) = \beta_{kn} \qquad (k=1, 2, \dots, n).$$

A classical result of FEJÉR states that $H_n(f;\, x)$ converges uniformly to $f(x)$ on $[-1, 1]$. A similar statement is valid for $\overline{H}_n(f;\, x)$ if

$$\beta_{kn} = O\left(\frac{1}{\sqrt{1-x_{kn}^2}} \frac{n}{\ln n}\right).$$

As to the rapidity of convergence, E. MOLDOVAN published (see [2]) the estimate[1]

(1. 7) $$\|f(x)-H_n(f;x)\|_{[-1,1]} = O(1)\omega\left(f; \frac{\log n}{n}\right),$$

where $\omega(f;t)$ is the modulus of continuity of $f(x)$ in $[-1, 1]$. In [3] O. SHISHA and B. MOND have given a generalization to the above mentioned results for functions of several variables. The purpose of this part is to give an improvement of the estimation (1. 7).

Namely, the following theorem holds:

THEOREM 1. 1. *If $f(x)$ is a continuous function in $[-1, 1]$ and $\omega(f;t)=O[\omega_1(t)]$, where $\omega_1(t)$ is a modulus of continuity, then*[2]

(1. 8) $$|f(x)-H_n(f;x)| = O(1)\left[\sum_{i=1}^{n}\frac{1}{i^2}\omega_1\left(\frac{i\sqrt{1-x^2}}{n}\right)+\sum_{i=1}^{n}\frac{1}{i^2}\omega_1\left(\frac{i^2}{n^2}\right)\right].$$

Before the proof we remark that our theorem is indeed an improvement of (1. 7). In fact, from (1. 8) we have

$$|f(x)-H_n(f;x)| = O(1)\omega_1\left(\frac{\log n}{n}\right)\sum_{i=1}^{n}\frac{1}{i^2}\left(\frac{i\sqrt{1-x^2}}{\log n}+1+\frac{i^2}{n\log n}+1\right) =$$

$$= O(1)\omega_1\left(\frac{\log n}{n}\right).$$

Further, if $\omega(f;t)=\omega_1(t)=t^\alpha$ $(0<\alpha<1)$, we obtain by (1. 8)

$$|f(x)-H_n(f;x)| = O(1)\left[\frac{\sqrt{(1-x^2)^\alpha}}{n^\alpha}\sum_{k=1}^{n}\frac{1}{i^{2-\alpha}}+\frac{1}{n^{2\alpha}}\sum_{i=1}^{n}\frac{1}{i^{2-2\alpha}}\right].$$

I.e.,

(1. 9) $$|f(x)-H_n(f;x)| = O(1)\left(\frac{\sqrt{(1-x^2)^\alpha}}{n^\alpha}+\frac{1}{n}\right) \qquad (0<\alpha<1;\ -1\leqq x\leqq 1),$$

(1. 10) $$|f(x)-H_n(f;x)| = O(1)\left(\sqrt{1-x^2}\,\frac{\log n}{n}+\frac{1}{n}\right) \qquad (\alpha=1;\ -1\leqq x\leqq 1).$$

PROOF. We need the following estimations. It is easy to verify (see e.g. in [4]) that if

(1. 11) $$\frac{j-1}{n}\pi \leqq \vartheta \leqq \frac{j}{n}\pi, \qquad x=\cos\vartheta,$$

[1] $\|g(x)\|_{[a,b]} = \max_{a\leqq x\leqq b} |g(x)|.$

[2] After lecture of these results G. FREUD has mentioned the unpublished statements of R. BOJANIČ having the form

$$|f(x)-H_n(f;x)| = O\left(\frac{1}{n}\right)\sum_{k=1}^{n}\omega\left(f;\frac{1}{k}\right).$$

then[3]

(1. 12) $$|f(x)-f(x_k)| = \begin{cases} O(1)\left[\omega\left(\frac{\sin\vartheta}{n}\right)+\omega\left(\frac{1}{n^2}\right)\right] & \text{if } k=j, \\ O(1)\left[\omega\left(\frac{i\sin\vartheta}{n}\right)+\omega\left(\frac{i^2}{n^2}\right)\right] & \begin{cases}\text{if } j<k=j+i\leqq n \\ \text{or } 1\leqq k=j-i<j.\end{cases}\end{cases}$$

Further, we have by a simple computation

(1. 13) $$(1-\cos\vartheta\cos\vartheta_k)\frac{\cos^2 n\vartheta}{(\cos\vartheta-\cos\vartheta_k)^2}\leqq$$

$$\leqq(1-\cos\vartheta\cos\vartheta_k+\sin\vartheta\sin\vartheta_k)\frac{\cos^2 n\vartheta}{(\cos\vartheta-\cos\vartheta_k)^2}=2\frac{\sin^2 n\frac{\vartheta+\vartheta_k}{2}\sin^2 n\frac{\vartheta-\vartheta_k}{2}}{\sin^2\frac{\vartheta-\vartheta_k}{2}}.$$

It is well known that

(1. 14) $$\frac{1}{\sin\frac{|\vartheta-\vartheta_k|}{2}}=O\left(\frac{n}{i}\right)\qquad (k\neq j).$$

Then we obtain, in virtue of (1. 1); (1. 4); (1. 3); (1. 11), (1. 12), (1. 2), (1. 13); (1. 14) and (1. 11), that

$$|f(x)-H_n(f;x)| = \left|f(x)-\sum_{k=1}^n f(x_k)A_k^{(n)}(x)\right| = \left|\sum_{k=1}^n [f(x)-f(x_k)]A_k^{(n)}(x)\right|\leqq$$

$$\leqq\sum_{k=1}^n|f(x)-f(x_k)|A_k^{(n)}(x) = O(1)\left[\omega\left(\frac{\sin\vartheta}{n}\right)+\omega\left(\frac{1}{n^2}\right)\right]A_j^{(n)}(x)+$$

$$+O(1)\sum_{k\neq j}\left[\omega\left(\frac{i\sin\vartheta}{n}\right)+\omega\left(\frac{i^2}{n^2}\right)\right]\left(\frac{\sin n\frac{\vartheta-\vartheta_k}{2}}{n\sin\frac{\vartheta-\vartheta_k}{2}}\right)^2=$$

$$=O(1)\left\{\omega\left(\frac{\sin\vartheta}{n}\right)+\omega\left(\frac{1}{n^2}\right)+\frac{1}{n^2}\sum_{k\neq j}\left[\omega\left(\frac{i\sin\vartheta}{n}\right)+\omega\left(\frac{i^2}{n^2}\right)\right]\frac{n^2}{i^2}\right\}=$$

$$=O(1)\left[\sum_{i=1}^n\frac{1}{i^2}\omega_1\left(\frac{i\sqrt{1-x^2}}{n}\right)+\sum_{i=1}^n\frac{1}{i^2}\omega_1\left(\frac{i^2}{n^2}\right)\right],$$

as we stated.

2. Now the question arises whether the order of our estimations is best possible. Evidently this is true for $f(x)\in\mathrm{Lip}\,\alpha$ $(0<\alpha<1)$ when $-1<x<1$. On the other hand, we prove the following statements.

[3] with $\omega(t)=\omega(f;t)$ and $x_{kn}=x_k$ (or $\cos\vartheta_{kn}=\cos\vartheta_k$).

THEOREM 2. 1. *There exist functions* $f(x)\in \mathrm{Lip}\,1$, $g(x)\in \mathrm{Lip}\,1$ and $h(x)\in \mathrm{Lip}\,\alpha$ $(0<\alpha<1)$ *such that*[4]

$$|f(0)-H_n(f;0)| \geqq c\frac{\log n}{n} \qquad (n=2,4,6,\ldots), \tag{2.1}$$

$$|g(1)-H_n(g;1)| = \frac{1}{n} \qquad (n=1,2,3,\ldots), \tag{2.2}$$

$$|h(1)-H_n(h;1)| > \frac{c}{n} \qquad (n=4,5,6,\ldots). \tag{2.3}$$

PROOF. The proofs are very similar. At first let

$$f(x)=|x|, \qquad x=\cos\vartheta=0, \qquad \vartheta=\frac{\pi}{2}, \qquad n=2,4,6,\ldots. \tag{2.4}$$

Then by (1. 1), (1. 2) and (2. 4)

$$H_n(f;0)-f(0)=\sum_{k=1}^{n}[f(x_k)-f(0)]\,A_k^{(n)}(0) \geqq \frac{1}{n^2}\sum_{k=1}^{\frac{n}{2}}\cos\vartheta_k\frac{\cos^2 n\vartheta}{\cos^2\vartheta_k}=$$

$$=\frac{1}{n^2}\sum_{k=1}^{\frac{n}{2}}\frac{1}{\cos\vartheta_k-\cos\vartheta}=\frac{1}{2n^2}\sum_{k=1}^{\frac{n}{2}}\frac{1}{\sin\frac{\vartheta+\vartheta_k}{2}\,\sin\frac{\vartheta-\vartheta_k}{2}}=$$

$$=\frac{1}{2n^2}\sum_{k=1}^{\frac{n}{2}}\frac{1}{\sin\left[\frac{\pi}{4}\left(\frac{2k-1}{n}+1\right)\right]\sin\left[\frac{\pi}{4}\left(1-\frac{2k-1}{n}\right)\right]}>\frac{1}{n^2}\sum_{k=1}^{\frac{n}{2}}\frac{1}{\sin\frac{\pi}{2}\cdot\frac{\pi}{4}\cdot\frac{n-2k+1}{n}}>\frac{c\log n}{n}.$$

Now let

$$g(x)=|x|, \qquad x=\cos\vartheta=1, \qquad \vartheta=0, \qquad n=1,2,3,\ldots. \tag{2.5}$$

Then with (1. 2) and (2. 5) we obtain that

$$g(1)-H_n(g;1)=\frac{1}{n^2}\sum_{k=1}^{n}(\cos\vartheta-\cos\vartheta_k)(1-\cos\vartheta\cos\vartheta_k)\frac{\cos^2 n\vartheta}{(\cos\vartheta-\cos\vartheta_k)^2}=$$

$$=\frac{1}{n^2}\sum_{k=1}^{n}(\cos\vartheta-\cos\vartheta_k)^2\frac{\cos^2 n\vartheta}{(\cos\vartheta-\cos\vartheta_k)^2}=\frac{1}{n}.$$

At last let

$$h(x)=|x|^\alpha \quad (0<\alpha<1), \qquad x=\cos\vartheta=1, \qquad \vartheta=0, \qquad n=4,5,6,\ldots. \tag{2.6}$$

[4] We denote by $c, c_1, \ldots$ suitable positive constants.

By the Lagrange mean-value theorem for $x \geqq \frac{1}{2}$ and $0 < k \leqq \left[\frac{n}{4}\right]$ we have

$$|x|^\alpha - |x_k|^\alpha = \alpha t_k^{\alpha-1}(x-x_k) \geqq c(x-x_k) \qquad \left(x \geqq \frac{1}{2}, \quad 0<k\leqq\left[\frac{n}{4}\right]\right). \tag{2.7}$$

So as above, we get by (2. 6) and (2. 7)

$$h(1)-H_n(h;1) \geqq \frac{1}{n^2}\sum_{k=1}^{\left[\frac{n}{4}\right]}(\cos^\alpha \vartheta - \cos^\alpha \vartheta_k)(1-\cos\vartheta\cos\vartheta_k)\frac{\cos^2 n\vartheta}{(\cos\vartheta-\cos\vartheta_k)^2} \geqq$$

$$\geqq \frac{c}{n^2}\sum_{k=1}^{\left[\frac{n}{4}\right]}(\cos\vartheta-\cos\vartheta_k)^2\frac{\cos^2 n\vartheta}{(\cos\vartheta-\cos\vartheta_k)^2} = \frac{c}{n}.$$

Q.E.D.

3. Notes.

a) We can prove similar results for $f(x)-\overline{H}_n(f;x)$ if β_{kn} are suitable expressions.

b) We can obtain analogous results for functions f of several variables.

c) The statements (2. 2) and (2. 3) seem to be new. As to (2. 1) I think it is known, but I have never seen it in a paper.

4. In the following parts we intend to investigate the analogous question for the trigonometric Hermite—Fejér interpolation and the so called trigonometric $(0, M)$ interpolation. We have the following (see [5] and [6]).

For a 2π-periodic continuous function and $M=1, 3, 5, \ldots$[5] there exist the uniquely determined trigonometric $(0, M)$ interpolating polynomials

$$\begin{cases} R_n(f;x) = \sum_{k=0}^{n-1} f(x_{kn})F_n(x-x_{kn}) \quad \text{or} \\ \overline{R}_n(f;x) = R_n(f;x)+\sum_{k=0}^{n-1}\beta_{kn}G_n(x-x_{kn}) \end{cases} \tag{4.1}$$

of order $\leqq n$ where

$$\begin{cases} x_{kn} = \dfrac{2k\pi}{n} \qquad (k=0,1,\ldots,n-1;\ n=1,2,3,\ldots), \\ F_n(x_{kn}) = G_n^{(M)}(x_{kn}) = \begin{cases} 0 & \text{if } k=1,2,\ldots,n-1 \\ 1 & \text{if } k=0, \end{cases} \\ F_n^{(M)}(x_{kn}) = G_n(x_{kn}) = 0 \qquad (k=0,1,\ldots,n-1). \end{cases} \tag{4.2}$$

[5] The case $M=1$ is due to D. JACKSON (see D. JACKSON, *Theory of Approximation*, Amer. Math. Soc. Colloqu. Publ. vol. XI, 1930.)

$F_n(x-x_{kn})$ and $G_n(x-x_{kn})$ are trigonometric polynomials of order $n-1$ and n, respectively, depending on M as well,

$$(4.3)\quad F_n(x) = O(1)\left[\frac{1}{n^3}\sum_{j=1}^{n-1}\left(\frac{\sin j\frac{x}{2}}{\sin\frac{x}{2}}\right)^2 + \frac{1}{n^{M+1}}\left(\frac{\sin n\frac{x}{2}}{n\sin\frac{x}{2}}\right)^2\right] \qquad (M=1,3,\dots),$$

$$(4.4)\quad \begin{cases} \sum_{k=0}^{n-1} F_n(x-x_{kn}) \equiv 1, \\ \sum_{k=0}^{n-1} |G_n(x-x_{kn})| = O\left(\frac{\log n}{n^M}\right), \end{cases} \qquad (M=1,3,\dots),$$

$$(4.5)\quad \begin{cases} R_n(f;x_{kn}) = \bar{R}_n(f;x_{kn}) = f(x_{kn}) & (k=0,1,\dots,n-1), \\ R_n^{(M)}(f;x_{kn}) = 0, \quad \bar{R}_n^{(M)}(f;x_{kn}) = \beta_{kn} & (k=0,1,\dots,n-1;\ M=1,3,\dots). \end{cases}$$

In [6] we proved that

$$(4.6)\quad |f(x)-R_n(f;x)| = O\left[\omega\left(f;\frac{\log n}{n}\right)\right] \qquad (n=2,3,\dots;\ M=1,3,\dots).$$

Now we intend to improve this estimation. Namely the following theorem holds.

THEOREM 4.1. *If $f(x)$ is a 2π-periodic continuous function such that $\omega(f;t) = O[\omega_1(t)]$*[6] *then*

$$(4.7)\quad |f(x)-R_n(f;x)| = O(1)\sum_{i=1}^{n}\frac{1}{i^2}\omega_1\left(\frac{i}{n}\right) \qquad (n=1,2,\dots;\ M=1,3,\dots).$$

Obviously from (4.7) we have

$$|f(x)-R_n(f;x)| = O(1)\omega_1\left(\frac{\log n}{n}\right)\left[\sum_{i=1}^{n}\left(\frac{1}{i^2}\frac{i}{\log n}+1\right)\right] = O(1)\omega_1\left(\frac{\log n}{n}\right)$$
$$(n=2,3,\dots),$$

furthermore for $\omega(f;t)=\omega_1(t)=t^\alpha$

$$|f(x)-R_n(f;x)| = O\left(\frac{1}{n^\alpha}\right) \qquad (0<\alpha<1),$$

$$|f(x)-R_n(f;x)| = O\left(\frac{\log n}{n}\right) \qquad (\alpha=1,\ n=2,3,\dots)$$

holds.

PROOF. The proof runs similarly as in Part 1. By the 2π-periodicity we may suppose (see (4.2)) that $|x-x_k| \leqq \pi$ $(k=0,1,\dots,n-1)$. Let

$$(4.8)\quad \frac{2(j-1)}{n}\pi + \frac{\pi}{n} \leqq x \leqq \frac{2j\pi}{n}.$$

[6] $\omega_1(t)$ is a modulus of continuity.

Then we can easily verify that

$$(4.9)\qquad c_1\frac{i}{n}<|x_k-x|<c_2\frac{i}{n}\quad \text{if}\quad j<k=j+i\leq n\quad \text{or}\quad 1\leq k=j-i<j,$$

$$(4.10)\qquad \frac{1}{\sin\frac{|x-x_k|}{2}}=O\left(\frac{n}{i}\right)\qquad (k\neq j).$$

We shall also use the estimation

$$(4.11)\qquad |\sin lx|\leq |l\sin x|$$

Then, in virtue of (4.4); (4.9); (4.3), (4.8); (4.9), (4.10), (4.8) and (4.11) we obtain

$$|f(x)-R_n(f;x)|=\left|\sum_{k=0}^{n-1}[f(x)-f(x_k)]F_n(x-x_k)\right|\leq\sum_{k=0}^{n-1}\omega(|x-x_k|)|F_n(x-x_k)|=$$

$$=O(1)\left\{\sum_{k\neq j}\omega\left(\frac{i}{n}\right)\left[\frac{1}{n^3}\sum_{l=1}^{n-1}\left(\frac{\sin l\frac{x-x_k}{2}}{\sin\frac{x-x_k}{2}}\right)^2+\frac{1}{n^{M+1}}\left(\frac{\sin n\frac{x-x_k}{2}}{\sin\frac{x-x_k}{2}}\right)^2\right]+\right.$$

$$\left.+\omega\left(\frac{1}{n}\right)\left[\frac{1}{n^3}\sum_{l=1}^{n-1}\left(\frac{\sin l\frac{x-x_j}{2}}{\sin\frac{x-x_j}{2}}\right)^2+\frac{1}{n^{M+1}}\left(\frac{\sin n\frac{x-x_j}{2}}{\sin\frac{x-x_j}{2}}\right)^2\right]\right\}=$$

$$=O(1)\left[\sum_{k\neq j}\omega\left(\frac{1}{n}\right)\left(\frac{1}{n^3}\sum_{l=1}^{n-1}\frac{n^2}{i^2}+\frac{1}{n^{M+1}}\frac{n^2}{i^2}\right)+\omega\left(\frac{1}{n}\right)\left(\frac{1}{n^3}\sum_{l=1}^{n-1}n^2+\frac{1}{n^{M+1}}n^2\right)\right]=$$

$$=O(1)\left[\sum_{k\neq j}\omega\left(\frac{1}{n}\left(\frac{1}{i^2}+\frac{1}{n^{M-1}}\frac{1}{i^2}\right)+\omega\left(\frac{1}{n}\right)\left(1+\frac{1}{n^{M-1}}\right)\right]=O(1)\sum_{i=1}^{n}\frac{1}{i^2}\omega_1\left(\frac{i}{n}\right).$$

Q.E.D.

5. To show that generally our estimation cannot be improved we prove the following theorem.

THEOREM 5.1. *There exist a function* $f(x)\in\text{Lip }1$, *an odd* M *and points* x_n^* *such that*

$$(5.1)\qquad |f(x_n^*)-R_n(f;x_n^*)|\geq\frac{c\log n}{n}\qquad (n=N,N+1,\dots;\ M=1).$$

PROOF. Indeed, let $f(x)=|x|$ for $-\pi\leq x<\pi$ and 2π-periodic elsewhere, further

$$(5.2)\qquad x_n^*=\frac{\pi}{n}\qquad (n=N,N+1,\dots).$$

It is well known that for $M=1$

$$F(x)=\frac{1}{n^2}\left(\frac{\sin n\frac{x}{2}}{\sin\frac{x}{2}}\right)^2. \tag{5.3}$$

So by (5.2), (5.3); (4.4); (4.10) and (4.10).

$$R_n(f;x_n^*)-f(x_n^*)\geqq\frac{1}{n^2}\sum_{k=0}^{\left[\frac{n}{2}\right]}\left(\frac{2k\pi}{n}-\frac{\pi}{n}\right)\left[\frac{\sin n\left(\frac{2k\pi}{2n}-\frac{\pi}{2n}\right)}{\sin\frac{2k\pi-\pi}{2n}}\right]^2\geqq$$

$$\geqq\frac{1}{n^2}\left(-\frac{\pi}{n}\right)\left(\frac{2n}{\pi}\right)^2+\frac{1}{n^2}\sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{2k-1}{n}\pi\left[\frac{\sin(2k-1)\frac{\pi}{2}}{\sin(2k-1)\frac{\pi}{2n}}\right]^2\geqq$$

$$\geqq-\frac{4}{\pi n}+\frac{1}{n^2}\sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{2k-1}{n}\pi\frac{4n^2}{(2k-1)^2\pi^2}\geqq c\frac{\log n}{n},$$

as we stated. Q.E.D.

6. Notes. The notes *a)* and *b)* of Part 3 can be applied to the results of Part 5 as well.

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DETERMINING GROUPS FROM ENDOMORPHISM RINGS FOR ABELIAN GROUPS MODULO BOUNDED GROUPS[1]

By

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Let $\hat{A}$ be the category of Abelian groups, $\hat{B}$ the SERRE class of bounded Abelian groups, and let $\hat{A}/\hat{B}$ denote the quotient category defined by GROTHENDIECK.

It is shown in this paper that BAER's theorem that a p-group is determined by its endomorphism ring holds for divisible p-groups in the quotient category $\hat{A}/\hat{B}$. It is also shown that, when considering the theorem for arbitrary p-groups in $\hat{A}/\hat{B}$, it suffices to consider only reduced p-groups. Finally, it is shown that the theorem holds for bounded p-groups in the related quotient category $\hat{A}/\hat{F}$, where $\hat{F}$ is the class of finite Abelian groups.

Let $\hat{A}$ be the category of groups[2] and $\hat{B}$ the SERRE class of bounded groups. Then $\hat{A}/\hat{B}$ is the quotient category as defined by GROTHENDIECK [3]. The objects of $\hat{A}/\hat{B}$ are the objects of $\hat{A}$. $\mathrm{Hom}_{\hat{A}/\hat{B}}(G, H) = \varinjlim_{(G', H')\in D} \mathrm{Hom}(G', H/H')$, where $D = \{(G', H') | G' \subset G,\ H' \subset H;\ G/G',\ H' \in \hat{B}\}$. D is directed by $(G', H') \leqq (G'', H'')$ if and only if $G'' \subset G'$ and $H' \subset H''$.

One of the classical theorems in Abelian group theory is REINHOLD BAER's theorem [1] that a p-group is determined by its endomorphism ring. That is, if G and H are p-groups such that $E(G) \simeq E(H)$, then $G \simeq H$, where for any group K, $E(K)$ denotes the endomorphism ring of K. It seems reasonable that this theorem should be true in the category $\hat{A}/\hat{B}$. Making this more plausible is the fact that E. WALKER showed in [4] that the category $\hat{A}/\hat{B}$ is equivalent to the category $\hat{A}_Q$ whose objects are those of $\hat{A}$ and whose maps are given by $Q \otimes \mathrm{Hom}(A, B) = \mathrm{Hom}_{\hat{A}_Q}(A, B)$, where Q means the field of rational numbers. It is proved in this paper that Baer's theorem is true in $\hat{A}/\hat{B}$ for divisible p-groups. However, even the case where the groups in question are direct sums of cyclic p-groups seems to be difficult.

All groups considered are p-groups. The work is done in the category $\hat{A}_Q$ rather than $\hat{A}/\hat{B}$ since computations seem easier there. Note that every element in $Q \otimes E(A)$ can be written in the form $1/n \otimes f$ with $n > 0$, and composition of maps is given by $(1/n \otimes f)(1/m \otimes g) = 1/mn \otimes fg$.

LEMMA 1. *Let $\hat{C}$ be an Abelian category, G an object in $\hat{C}$. Let ϱ be an idempotent in $E_{\hat{C}}(G)$. Then $\varrho E_{\hat{C}}(G) \varrho \simeq E_{\hat{C}}(\varrho G)$.*

PROOF. Since $\hat{C}$ is Abelian, and ϱ is an idempotent in $E_{\hat{C}}(G)$, there exist maps $\pi \in \hat{C}(G, \varrho G)$ and $i \in \hat{C}(\varrho G, G)$ such that $\pi i = 1_{\varrho G}$, $i\pi = \varrho$, $\pi\varrho = \pi$, and $\varrho i = i$. Define

[1] The results of this paper were part of a doctoral dissertation completed in May, 1969, under Professor ELBERT A. WALKER at New Mexico State University.

[2] Throughout this paper, the word group always means Abelian group.

$\varphi: \varrho E_{\hat{C}}(G)\varrho \to E_{\hat{C}}(\varrho G)$ by $\varphi(\varrho\alpha\varrho) = \pi\alpha i$. Suppose $\varrho\alpha\varrho = 0$. Then $\pi\alpha i = \pi\varrho\alpha\varrho i = 0$ and hence φ is well defined. $\varphi(\varrho\alpha\varrho) + \varphi(\varrho\alpha'\varrho) = \pi\alpha i + \pi\alpha' i = \pi(\alpha+\alpha')i = \varphi(\varrho(\alpha+\alpha')\varrho)$. Also, $(\varphi(\varrho\alpha\varrho))(\varphi(\varrho\alpha'\varrho)) = \pi\alpha i\pi\alpha' i = \pi\alpha\varrho\alpha' i = \varphi(\varrho\alpha\varrho\alpha'\varrho)$. Therefore, φ is a homomorphism. Suppose $\pi\alpha i = 0$. Then $i\pi\alpha i\pi = 0$, that is, $\varrho\alpha\varrho = 0$. Therefore, φ is $1-1$. Now suppose $\beta \in E_{\hat{C}}(\varrho G)$. Then $\varrho i\beta\pi\varrho \in \varrho E_{\hat{C}}(G)\varrho$, and $\varphi(\varrho i\beta\pi\varrho) = \pi i\beta\pi i = \beta$. Therefore, φ is an isomorphism.

THEOREM 2. *Let G be a divisible p-group and H an arbitrary p-group. If $Q\otimes E(G) \simeq Q\otimes E(H)$, then $G \simeq H$ in $\hat{A}/\hat{B}$.*

PROOF. Let $G = \sum_{i\in I} Z(p^\infty)_i$. Suppose $Q\otimes E(G) \simeq Q\otimes E(H)$, where H is an arbitrary p-group. For each $i\in I$, let $\pi_i: G\to Z(p^\infty)_i$ be the usual projection, and let $1\otimes\pi_i$ correspond to $1/n_i\otimes\varrho_i \in Q\otimes E(H)$ under the given isomorphism. It is well known that $E(Z(p^\infty)) \simeq P$, the p-adic integers, and hence $Q\otimes E(Z(p^\infty))$ is the p-adic number field. By Lemma 1, $Q\otimes E(\pi_i G) \simeq Q\otimes E((1\otimes\pi_i)G) \simeq (1\otimes\pi_i)(Q\otimes E(G))(1\otimes\pi_i) \simeq (1/n_i\otimes\varrho_i)(Q\otimes E(H))(1/n_i\otimes\varrho_i) \simeq Q\otimes E((1/n_i\otimes\varrho_i)H) \simeq Q\otimes E(\varrho_i H)$. Since $\pi_i G = Z(p^\infty)_i$, $Q\otimes E(\varrho_i H)$ is isomorphic to the p-adic number field. In particular, $Q\otimes E(\varrho_i H)$ has no divisors of zero. Thus $\varrho_i H$ cannot be decomposed into two unbounded summands. For suppose $\varrho_i H = A\oplus B$, where A and B are both unbounded. Let ϱ_A and ϱ_B denote the usual projections of $\varrho_i H$ onto A and B respectively. Then $1\otimes\varrho_A\varrho_B = 0$, yet $1\otimes\varrho_A \neq 0$ and $1\otimes\varrho_B \neq 0$; that is, they are divisors of zero. Now suppose $\varrho_i H$ is unbounded and reduced. Let $\alpha: \varrho_i H\to B$ be an epimorphism, where B is a basic subgroup of $\varrho_i H$. Since B is unbounded, write $B = B_1\oplus B_2$, where B_1 and B_2 are both unbounded. Let $\alpha_1: B\to B_1$ be the projection of B onto B_1 with kernel B_2. Let $\alpha_2: B_1\to \operatorname{Ker}(\alpha_1\alpha)$ be such that $\operatorname{Im}\alpha_2$ is unbounded, and let γ denote the composition $\alpha_2\alpha_1\alpha$. Then $1\otimes\gamma\neq 0$, but $(1\otimes\gamma)(1\otimes\gamma) = 0$. Therefore $\varrho_i H = Z(p^\infty)\oplus B_i$, where B_i is bounded. Now suppose $H = D\oplus R$, where D is divisible and R is reduced. Let $\varrho: H\to R$ be the usual projection, and suppose R is unbounded. Let $1\otimes\varrho$ correspond to $1/n\otimes\alpha\in Q\otimes E(G)$. Then $1/n\otimes\alpha \neq 0$, and hence, since G is divisible, $(1/n\otimes\alpha)(1\otimes\pi_i)\neq 0$ for some i. Therefore $(1\otimes\varrho)(1/n_i\otimes\varrho_i)\neq 0$. But $\operatorname{Im}\varrho_i = Z(p^\infty)\oplus B_i$, and since $\operatorname{Im}\varrho$ is reduced, $\varrho\varrho_i H$ is bounded. Hence R is bounded. Therefore, without loss of generality, it can be assumed that H is divisible and $\operatorname{Im}\varrho_i = Z(p^\infty)$. Suppose $|G|\geqq 2^{\aleph_0}$. Then $r(G) = |G|$. Since $E(G)$ and $E(H)$ are torsion-free, $|E(G)| = |Q\otimes E(G)| = |Q\otimes E(H)| = |E(H)|$. Now $|E(G)| = 2^{|G|}$, and $|E(H)| = 2^{|H|}$. Hence $|G| = |H|$ and $r(G) = r(H)$. (Note that the generalized continuum hypothesis is assumed here.) Suppose $r(G) = r < \aleph_0$. Then $G = \sum_{i=1}^{r} Z(p^\infty)_i$. Let π_i be the projection $G\to Z(p^\infty)_i$ as before, and let $1\otimes\pi_i$ correspond to $1/n_i\otimes\varrho_i\in Q\otimes E(H)$. Then $(1/n_i\otimes\varrho_i)^2 = 1/n_i\otimes\varrho_i$, so that $\operatorname{Im}(\varrho_i^2 - n_i\varrho_i)$ is bounded. Since H is divisible and the $1/n_i\otimes\varrho_i$ are mutually orthogonal, $\varrho_i^2 = n_i\varrho_i$, $\varrho_i\varrho_j = 0$ if $i\neq j$, and $\varrho_1 H$ is divisible. It follows that $r(H)\geqq r$. By symmetry, $r(G) = r(H)$ and the theorem is proved.

THEOREM 3. *Let G and H be p-groups, $G = D\oplus R$, $H = D'\oplus R$, where D and D' are the maximal divisible subgroups of G and H respectively. If $Q\otimes E(R) \simeq Q\otimes E(R')$ implies $R\simeq R'$ in $\hat{A}/\hat{B}$, then $Q\otimes E(G)\simeq Q\otimes E(H)$ implies $G\simeq H$ in $\hat{A}/\hat{B}$.*

PROOF. Let $\pi_1: G\to D$ and $\pi_2: G\to R$ be the usual projections, and let $\pi_i\otimes 1$ correspond to $1/n_i\otimes\varrho_i$ under the given isomorphism $Q\otimes E(G)\simeq Q\otimes E(H)$. Then

$Q\otimes E(\pi_i G) \simeq Q\otimes E(\varrho_i H)$, and since $\pi_1 G = D$ is divisible, $\varrho_1(H) = D_1 \oplus B$, where $D_1 \simeq D$ and B is bounded by Theorem 2. It follows readily that $\varrho_2 H$ is reduced and that $D_1 = D'$. Since $(1/n_1 \otimes \varrho_1) + (1/n_2 \otimes \varrho_2) = 1\otimes 1$, $1/n_1 n_2 \otimes (n_2 \varrho_1 + n_1 \varrho_2 - 1) = 0$, so $mn_2 \varrho_1 + mn_1 \varrho_2 = m$ for some $m > 0$. Furthermore, $\varrho_2 H \cap \varrho_1 H$ is bounded since $1/n_1 \otimes \varrho_1$ and $1/n_2 \otimes \varrho_2$ are mutually orthogonal idempotents. Thus $H = D' \oplus S$ where $S \simeq \varrho_2 H$ in $\hat{A}/\hat{B}$. The theorem follows.

In view of the apparent difficulty of proving Baer's theorem for direct sums of cyclics in $\hat{A}/\hat{B}$, it is of interest that the theorem is true for bounded groups in a related quotient category.

Let $\hat{F}$ be the Serre class of finite groups. Form the quotient category $\hat{A}/\hat{F}$ in the same way $\hat{A}/\hat{B}$ is formed. Then $G \simeq H$ in $\hat{A}/\hat{F}$ if and only if there exist subgroups S and A of G, and T and B of H such that $S/A \simeq T/B$ and G/S, A, H/T, $B \in \hat{F}$. Here again only p-groups are considered.

Let A be a p-group and let $F(A) = \{f\in E(A) \mid \operatorname{Im} f \in F\}$. $F(A)$ is clearly an ideal in $E(A)$ and thus it makes sense to consider the quotient ring $E(A)/F(A)$.

LEMMA 4. *Let $p^{\omega}A = 0$. Then $E(A)/F(A) \simeq E_{\hat{A}/\hat{F}}(A)$.*

PROOF. BERTHOLF [2] has shown that the elements of $E_{\hat{A}/\hat{F}}(A)$ can be represented by elements of Hom $(A, A/F)$ where $F\in \hat{F}$. Since A has no elements of infinite height, F can be imbedded in a finite summand of A and thus elements of $E_{\hat{A}/\hat{F}}(A)$ can be represented by elements of Hom $(A, A) = E(A)$. That is, $E(A) \overset{\varphi}{\to} E_{\hat{A}/\hat{F}}(A) \to 0$ is exact, where if $f\in E(A)$, $\varphi(f)$ is the element of $E_{\hat{A}/\hat{F}}(A)$ represented by f. φ is clearly a ring homomorphism by the definition of $E_{\hat{A}/\hat{F}}(A)$. $\varphi(f) = 0$ if and only if $\operatorname{Im} f \in F$ [4]. Therefore, Ker $\varphi = F(A)$ and $E(A)/F(A) \simeq E_{\hat{A}/\hat{F}}(A)$

LEMMA 5. *Let G be any group and $\pi + F(G)$ any idempotent in $E(G)/F(G)$.* Then $(\pi + F(G))(E(G)/F(G))(\pi + F(G) \simeq E(\pi G)/F(\pi G)$.

PROOF. Consider $\Phi : (\pi + F(G))(E(G)/F(G))(\pi + F(G)) \to E(\pi G)/F(\pi G)$ by $\Phi((\pi + F(G))(\alpha + F(G))(\pi + F(G))) = \pi\alpha\pi + F(\pi G)$. Since $\pi + F(G)$ is an idempotent, there exists an $f\in F(G)$ such that $\pi^2 = \pi + f$. Suppose

$$(\pi + F(G))(\alpha + F(G))(\pi + F(G)) = (\pi + F(G))(\alpha' + F(G))(\pi + F(G)).$$

Then $\pi\alpha\pi + F(G) = \pi\alpha'\pi + F(G)$, that is, there exists a $g\in F(G)$ such that $\pi\alpha\pi = \pi\alpha'\pi + g$. $\pi\alpha\pi(\pi G) = \pi\alpha\pi^2(G) = \pi\alpha(\pi + f)(G) = \pi\alpha\pi(G) = \pi\alpha f(G) = \pi\alpha'\pi(G) + g(G) + \pi\alpha f(G)$. Therefore $g(G) \in \pi G$, that is, $g \in F(\pi G)$ and Φ is well defined since $\pi\alpha\pi + F(\pi G) = \pi\alpha'\pi + F(\pi G)$. Φ is a ring homomorphism since its action is essentially multiplication. Suppose $\Phi((\pi + F(G))(\alpha + F(G))(\pi + F(G))) = \pi\alpha\pi + F(\pi G) = 0$. Then $\pi\alpha\pi \in F(\pi G)$. Then $\pi\alpha\pi(\pi G)$ is finite, and therefore $(\pi\alpha\pi + \pi\alpha f)(G)$ is finite. But $\pi\alpha f(G)$ is finite, and therefore $\pi\alpha\pi(G)$ is finite, that is, $\pi\alpha\pi + F(G) = 0$, and Φ is $1-1$. Now suppose $\alpha + F(\pi G) \in E(\pi G)/F(\pi G)$. Then $\alpha\pi \in E(G)$ and

$$\Phi((\pi + F(G))(\alpha\pi + F(G))(\pi + F(G))) = \pi\alpha\pi^2 + F(\pi G) = \alpha + F(\pi G).$$

It must be shown that $(\pi\alpha\pi^2 - \alpha) \in F(\pi G)$. $\pi\alpha\pi^2 = \pi\alpha\pi + \pi\alpha f$, and therefore $\pi\alpha\pi^2 + F(\pi G) = \pi\alpha\pi + F(\pi G)$. $\pi\alpha\pi + F(\pi G) = \pi\alpha + F(\pi G)$ by the same reasoning. Notice $\pi^2\alpha + F(\pi G) = \pi\alpha + F(\pi G)$ since $\pi^2\alpha = (\pi + f)\alpha = \pi\alpha + f\alpha$ and hence $f\alpha \in f(\pi G)$. Suppose $\pi\alpha + F(\pi G) \neq \alpha + F(\pi G)$. Then $H = \operatorname{Im}(\pi\alpha - \alpha) \subset \pi G$ is infinite. Let $K \subset G$ be such that $\pi(K) = H$. Then $\pi(H) = \pi^2(K) = \pi(K) + f(K)$, that is, $\pi(H)$

is infinite since $\pi(K)$ is infinite. But this contradicts the fact that $\pi^2\alpha+F(\pi G) = \pi\alpha+F(\pi G)$. Therefore $\pi\alpha+F(\pi G) = \alpha+F(\pi G)$, Φ is onto, and the isomorphism is established.

THEOREM 6. Let G be a bounded p-group, H any p-group such that $p^\omega H=0$. If $E_{\hat{A}/\hat{F}}(G)=E_{\hat{A}/\hat{F}}(H)$, then $G\simeq H$ in $\hat{A}/\hat{F}$.

PROOF. Let $G = \sum_{i=1}^{n} G_i$ where each G_i is a direct sum of cyclic groups of order p^i. By Lemma 4 and the hypothesis, it can be assumed that there exists an isomorphism $\varphi: E(G)/F(G) \to E(H)/F(H)$. Since $p^n G=0$, $p^n+F(G) = 0$ and hence $\varphi(p^n+F(G)) = p^n+F(H) = 0$, which implies that $p^n H$ is finite and hence H is bounded. Therefore, it can be assumed that $H = \sum_{i=1}^{n} H_i$ where each H_i is a direct sum of cyclic groups of order p^i. Let $\pi_i: G\to G_i$ be the usual projections and $\varphi(\pi_i+F(G)) = \pi_i'+F(H)$.

By Lemma 5,

$$E(G_i)/F(G_i) = E(\pi_i G)/F(\pi_i G) \simeq (\pi_i+F(G))(E(G)/F(G))(\pi_i+F(G)) \simeq$$

$$(\pi_i'+F(G))(E(H)/F(H))(\pi_i'+F(H)) \simeq E(\pi_i' H)/F(\pi_i' H).$$

As before, since $p^i G_i = 0$, $p^i+F(G_i) = 0$ and hence $p^i+F(\pi_i' H) = 0$ and hence $p^i\pi_i' H$ is finite and it can be assumed that $\pi_i' H = \sum_{j=1}^{i} K_j$, where K_j is a direct sum of cyclic groups of order p^j for all j. But $E(G_i)/F(G_i)$ has no non-trivial idempotents of additive order less than p^i, and therefore neither does $E(\pi_i' H)/F(\pi_i' H)$, which means that each K_j is finite for $j<i$. Thus it can be assumed that $\pi_i' H=K_i$, that is, $E(\pi_i' H)/F(\pi_i' H) \simeq E(K_i)/F(K_i)$. Thus $E(G_i)/F(G_i) \simeq E(K_i)/F(K_i)$.

Assuming G_i is infinite, $2^{|G_i|} = |E(G_i)/F(G_i)| = |E(K_i)/F(K_i)| = 2^{|K_i|}$. Therefore, again using the generalized continuum hypothesis, $|G_i|=|K_i|$. Therefore, $|G_i|\geqq|H_i|$, and symmetrically, $|H_i|\geqq|G_i|$. Therefore $G_i\simeq H_i$ in $\hat{A}/\hat{F}$ and hence $G\simeq H$ in $\hat{A}/\hat{F}$.

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СИММЕТРИЧЕСКАЯ ПОЛУГРУППА ПРЕОБРАЗОВАНИЙ ПОКРЫВАЕТСЯ СВОИМИ ИНВЕРСНЫМИ ПОДПОЛУГРУППАМИ

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Полугруппа называется квазиинверсной, если каждый её элемент принадлежит некоторой её инверсной подполугруппе. Исследуются некоторые свойства квазиинверсных полугрупп, ставятся нерешённые проблемы. Основной результат: полугруппы всех полных и всех частичных преобразований произвольного множества квазиинверсны.

Всюду далее $\mathcal{T}(A)$ обозначает полугруппу всех полных преобразований множества A, а $\mathcal{F}(A)$ обозначает полугруппу всех частичных преобразований множества A. Обе полугруппы рассматриваются относительно операции суперпозиции преобразований. Хорошо известно, что эти полугруппы регулярны, но не инверсны (за исключением случая, когда A содержит менее двух различных элементов). Однако эти полугруппы содержат довольно богатый набор инврсных подполугрупп, в частности, они содержат изоморфную копию любой инверсной полугруппы, порядок которой не превосходит мощности множества A. На VI Всесоюзном коллоквиуме по общей алгебре в г. Минске (1964 г.) была поставлена проблема: будут ли полугруппы $\mathcal{T}(A)$ и $\mathcal{F}(A)$ полностью покрываться своими инверсными подполугруппами. Основное содержание настоящей заметки заключается в положительном ответе на эту проблему.

Предварительно напомним основные определения. Элемент v полугруппы G называется *обобщённо обратным* для элемента u, если

$$uvu = u, \qquad vuv = v \tag{1}$$

Полугруппа, для каждого элемента которой существует обобщённо обратный, называется *регулярной*. Полугруппа, для каждого элемента которой существует единственный обобщённо обратный, называется *инверсной* (или *обобщённой группой*).

Будем говорить, что полугруппа G покрывается своими инверсными подполугруппами, если каждый элемент полугруппы G содержится в некоторой инверсной подполугруппе этой полугруппы.

Элемент v полугруппы G называется *квазиобратным* для элемента u этой полугруппы, если выполняются соотношения (1) и подполугруппа полугруппы G, порождённая элементами u и v, инверсна. Полугруппа, для каждого элемента которой существует квазиобратный, будет называться *квазиинверсной*. Кольцо, мультипликативная полугруппа которого квазиинверсна, назовём *сильно регулярным*.

Ясно, что квазиинверсные полугруппы регулярны, а инверсные полугруппы квазиинверсны. Обратное неверно: некоммутативная идемпотентная полугруппа квазиинверсна, но не инверсна. То, что класс квазирегулярных полугрупп уже класса регулярных полугрупп, показывает

Пример 1. Рассмотрим пятиэлементную полугруппу Γ с элементами $\{a, b, c, d, e\}$ и с таблицей умножения

	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	b	c
d	a	a	a	d	e
e	a	d	e	d	e

Эта полугруппа регулярна, но не инверсна (см. [1], стр. 3, последняя строка пятого столбца). Легко видеть, что единственным обобщённо обратным для элемента b будет элемент e. Поэтому всякая регулярная подполугруппа нашей полугруппы, содержащая элемент b, будет также содержать элементы e, $b^2=a$, $be=c$ и $eb=d$, т. е. будет несобственной. Поэтому b не принадлежит никакой инверсной подполугруппе полугруппы Г. Ясно, что Γ не является квазиинверсной.

Замечание. Среди полугрупп небольшого конечного порядка квазиинверсные полугруппы встречаются достаточно часто, как показывает следующая таблица, составленная на основе таблиц Кэли всех полугрупп, порядок которых не превышает 6 (изоморфные и антиизоморфные полугруппы не различаются) [1]:

Порядок полугруппы	1	2	3	4	5	6
Общее число полугрупп	1	4	18	126	1160	15 973
В том числе регулярных	1	3	9	42	206	1352
Инверсных	1	2	5	16	52	208
Квазиинверсных	1	3	9	42	205	1343

Поскольку полугруппа Γ, приведённая в примере 1, антиизоморфна сама себе, получаем, что квазиинверсных, но не инверсных полугрупп порядка не выше 4 не существует; всякая квазиинверсная, но не инверсная полугруппа порядка 5 изоморфна полугруппе Γ. Для удобства приведём координаты всех девяти квазиинверсных, но не инверсных полугрупп согласно списку [1] (указывая номера страницы, строки и столбца, где помещена таблица Кэли соответствующей полугруппы): 48. 4. 2; 48. 4. 3; 48. 4. 4; 48. 4. 5; 48. 4. 11; 50. 5. 7; 50. 6. 10; 59. 12. 11; 64. 8. 11.

Ясно, что *полугруппа тогда и только тогда покрывается своими инверсными подполугруппами, когда она квазиинверсна.*

Аналогом квазиинверсных полугрупп являются так называемые *вполне регулярные полугруппы*, т. е. полугруппы, покрываемые своими подгруппами. Вполне регулярные полугруппы хорошо изучены [2]. Очевидно, что вполне регулярные полугруппы квазиинверсны. Обратное, однако, неверно. Все регулярные полугруппы не выше четвёртого порядка вполне регулярны. Всякая квазиинверсная, но не вполне регулярная полугруппа пятого порядка изоморфна инверсной полугруппе 3. 14. 4 из [1]. Всякая квазиинверсная, но не вполне регулярная полугруппа шестого порядка изоморфна либо одной из инверсных полугрупп 48. 4. 9; 50. 5. 6; 64. 8. 10; 67. 14. 15, либо изоморфна или антиизоморфна квазиинверсной, но не инверсной полугруппе 48. 4. 1 из [1].

Вполне регулярные полугруппы допускают разбиение, классами которого являются подгруппы. Квазиинверсные полугруппы, порядок которых не выше 6, допускают разбиение, классы которого суть инверсные подполугруппы. Однако, как покажет пример 2, в общем случае это неверно.

Лемма. *Элемент v полугруппы G будет квазиобратным для элемента u этой полугруппы тогда и только тогда, когда выполняются соотношения* (1) *и при всех натуральных m и n элементы $u^m v^m$ и $v^n u^n$ коммутируют при умножении, т. е.*

$$u^m v^{m+n} u^n = v^n u^{m+n} v^m \tag{2}$$

Необходимость. Если v — квазиобратный для u, то элементы вида $u^m v^m$ и $v^n u^n$ будут идемпотентами инверсной подполугруппы, порождённой элементами u и v, а потому коммутируют.

Достаточность. Пусть выполняются соотношения (1) и (2). Элементы $u^m v^m$ и $v^n u^n$ принадлежат подполугруппе, порождённой u и v, поэтому без ущерба для общности можно считать, что u и v порождают всю полугруппу G. Покажем теперь индукцией по n, что v^n будет обобщённо обратным для u^n. При $n=1$ это очевидно. Пусть это верно для $n=m-1$. Тогда $u^m v^m u^m = uu^{m-1}v^{m-1}vuu^{m-1} = uvuu^{m-1}v^{m-1}u^{m-1} = uu^{m-1} = u^m$.
Аналогично проверяем, что $v^m u^m v^m = v^m$.

Пользуясь этим и условиями (1) и (2) доказываем лемму 1. 2 и опирающуюся на неё теорему 1. 3 работы Л. М. Глускина [3], где указанные утверждения доказаны в предположении, что идемпотенты G коммутируют при умножении. Однако доказательство Л. М. Глускина без всяких изменений проходит при более слабых предположениях, сделанных нами. Поэтому мы опустим здесь эти доказательства и используем далее утверждение, что всякий элемент полугруппы G имеет вид $v^k u^l v^m$, где $l \geqq k \geqq 0$, $l \geqq m > 0$ и $l > 0$.

Покажем теперь, что все идемпотенты G имеют вид $u^m v^{m+n} u^n$, где $m \geqq 0$, $n \geqq 0$, $m+n > 0$. Пусть элемент $x = v^k u^l v^m$ есть идемпотент. Если $l = m+k$, то $x = u^m v^{m+k} u^k$. Остаётся рассмотреть два случая: когда $l > k+m$ и когда $l < k+m$. Пусть $l > k+m$. Тогда $x = v^k u^l v^m = x^2 = v^k u^l v^{m+k} u^l v^m = v^k u^{l-m-k} u^{m+k} v^{m+k} u^{m+k} u^{l-m-k} v^m = v^k u^{l-m-k} u^{m+k} u^{l-m-k} v^m = v^k u^{2l-m-k} v^m$, т. е. $v^k u^l v^m = v^k u^{2l-m-k} v^m$. Здесь $k \leqq l$, $m \leqq l$, $k \leqq 2l-m-k$ и $m \leqq 2l-m-k$. Умножая последнее равенство слева на u^k и справа на u^m, получаем после очевидного преобразования, что $u^l = u^{2l-m-k}$. Введём обозначения: $a = l-k$ и $b = l-m$. Тогда $u^l = u^{a+b}$, откуда, используя то, что $a+b \geqq l$, получаем: $v^{a+b} = v^l u^l v^{a+b} u^l v^l = v^l u^{a+b} v^{a+b} u^{a+b} v^l = v^l u^{a+b} v^l = v^l u^l v^l = v^l$. Поэтому $x = v^k u^l v^m = u^{l-k} v^l u^{l-m} = u^a v^{a+b} u^b$. Если же $l < k+m$, то, согласно лемме 1. 2 работы [3], $x = u^a v^l u^b$ и $l > a+b$. Этот случай отличается от предыдущего лишь тем, что здесь u и v поменялись местами. Поэтому в этом случае $x = v^c u^{c+d} v^d = u^c v^{c+d} u^d$.

Наконец, используя лемму 1. 2 работы [3], легко проверяем, что если $m \geqq n$, то $u^m v^m u^n v^n = u^n v^n u^m v^m = u^m v^m$ и $v^m u^m v^n u^n = v^n u^n v^m u^m = v^m u^m$. Пользуясь этим, соотношениями (2) и только что установленным видом идемпотентов полугруппы G, получаем, что идемпотенты G коммутируют при умножении. Согласно утверждению 1. 3. 1 работы [3], полугруппа G инверсна, что и требовалось доказать.

Доказанная лемма может рассматриваться как некоторое усиление вышеупомянутых результатов Л. М. Глускина из [3]. Заметим, что вид идемпотентных элементов, найденный нами в ходе доказательства леммы, позволяет весьма просто описать полурешётку главных левых (или правых) идеалов элементарной инверсной полугруппы в смысле Л. М. Глускина [3], если известен тип этой полугруппы (ибо такая полурешётка изоморфна полурешётке идемпотентов инверсной полугруппы).

Теорема 1. *Симметрическая полугруппа $\mathcal{T}(A)$ полных преобразований любого множества A квазиинверсна.*

Доказательство. Если множество A пусто или одноэлементно, то $\mathcal{T}(A)$ есть одноэлементная группа, поэтому без ущерба для общности можно предположить, что A содержит более одного элемента.

Пусть $\varphi \in \mathcal{T}(A)$. Достаточно показать, что найдётся преобразование $\psi \in \mathcal{T}(A)$, являющееся квазиобратным для φ. Отысканию такого преобразования ψ посвящена вся оставшаяся часть доказательства.

Напомним, что если n — натуральное число, то, по определению, $\overset{1}{\varphi} = \varphi$ и $\overset{n+1}{\varphi} = \varphi \circ \overset{n}{\varphi}$. Кроме того, положим $\overset{0}{\varphi} = \Delta_A$, где Δ_A есть тождественное бинарное отношение на множестве A. Как обычно, $pr_1\tau$ обозначает первую проекцию (т. е. область определения), а $pr_2\tau$ — вторую проекцию (т. е. область значений) частичного преобразования τ. Ясно, что $pr_2 \overset{n+1}{\tau} \subset pr_2 \overset{n}{\tau}$. Назовём *глубиной* элемента $a \in A$ и обозначим через $d(a)$ наибольшее из таких чисел n, для которых $a \in pr_2 \overset{n}{\varphi}$. Если наибольшего из таких чисел не существует, полагаем $d(a) = \infty$. Таким образом, $0 \leqq d(a) \leqq \infty$. Если $a \in pr_2 \overset{m}{\varphi}$, то $\overset{n}{\varphi}(a) \in pr_2 \overset{m+n}{\varphi}$. Поэтому для всех n выполняется формула $d(\overset{n}{\varphi}(a)) \geqq d(a) + n$, которую мы будем использовать далее без специальных оговорок. Если $d(\overset{n}{\varphi}(a)) = n$ для всех n, то назовём элемент a *специальным*. Легко видеть, что в этом случае $d(a) = 0$.

Пусть $d(a) > 0$. В множестве $\overset{-1}{\varphi}\langle a \rangle$ всех прообразов элемента a относительно φ выберем элемент $\pi(a)$ наибольшей возможной глубины. В результате получим частичное преобразование π множества A, существование которого гарантируется аксиомой выбора. Ясно, что $pr_1 \pi = pr_2 \varphi$.

Если множество специальных элементов непусто, то зафиксируем какой-либо из них, скажем, a_0. Тогда из $a \in \overset{-1}{\varphi}\langle \overset{n}{\varphi}(a_0) \rangle$ следует, что $n = d(\overset{n}{\varphi}(a_0)) = d(\varphi(a)) \geqq d(a) + 1$, т. е. $d(a) \leqq n - 1$. Поскольку $\overset{n-1}{\varphi}(a_0) \in \overset{-1}{\varphi}\langle \overset{n}{\varphi}(a_0) \rangle$, получаем, что $\overset{n-1}{\varphi}(a_0)$ имеет среди элементов $\overset{-1}{\varphi}\langle \overset{n}{\varphi}(a_0) \rangle$ наибольшую возможную глубину. Выберем π так, чтобы $\pi(\overset{n}{\varphi}(a_0)) = \overset{n-1}{\varphi}(a_0)$ для всех n. Как мы только что показали, это не противоречит ранее наложенным на π условиям.

Индукцией по n докажем формулу

(3) $$a \in pr_1 \overset{n}{\pi} \to d(\overset{n}{\pi}(a)) = d(a) - n.$$

Пусть $n=1$ и $a \in pr_1 \pi$. Тогда $d(a) > 0$. Если $\pi(a) \in pr_2 \overset{n}{\varphi}$, то $a = \varphi(\pi(a)) \in \in \varphi(pr_2 \overset{n}{\varphi}) = pr_2 \overset{n+1}{\varphi}$, откуда следует, что $d(\pi(a)) \leqq d(a) - 1$. Ясно, что $d(a) - 1 \geqq 0$. Поскольку $a \in pr_2 \overset{d(a)}{\varphi}$, существует элемент $a_1 \in pr_2 \overset{d(a)-1}{\varphi}$, такой, что $\varphi(a_1) = a$. Ясно, что $a_1 \in \overset{-1}{\varphi}\langle a\rangle$ и $d(a_1) \geqq d(a) - 1$. Вспоминая, что элемент $\pi(a)$ имеет максимальную глубину среди элементов $\overset{-1}{\varphi}\langle a\rangle$, получаем, что $d(\pi(a)) \geqq d(a_1)$, откуда $d(\pi(a)) = d(a) - 1$. Поэтому формула (3) справедлива при $n=1$.

Допустим теперь, что эта формула справедлива при $n = m-1$ и пусть $a \in pr_1 \overset{m}{\pi}$. Тогда $\pi(a) \in pr_1 \overset{m-1}{\pi}$, откуда, по формуле (3), $d(\overset{m}{\pi}(a)) = d(\overset{m-1}{\pi}(\pi(a))) = = d(\pi(a)) - (m-1) = d(a) - 1 - (m-1) = d(a) - m$. Формула (3) доказана.

Пользуясь этой формулой, докажем индукцией по n формулу

(4) $$n \leqq d(a) \to a \in pz_1 \overset{n}{\pi}.$$

Для $n=0$ эта формула следует из определения $\overset{0}{\pi}$. Пусть формула справедлива для $n = m-1$ и пусть $m \leqq d(a)$. Тогда $m-1 \leqq d(a) - 1 = d(\pi(a))$, откуда $\pi(a) \in pr_1 \overset{m-1}{\pi}$, т. е. $a \in pr_2 \overset{m}{\pi}$. Формула (4) доказана.

Пусть a не является специальным элементом. Тогда $d(\overset{n}{\varphi}(a)) > n$ для некоторого n. Наименьшее из таких n назовём *высотой* a и обозначим через $h(a)$. Для удобства высоту специальных элементов положим равной ∞.

Определим теперь преобразование ψ множества A. Если элемент a — специальный, положим $\psi(a) = a_0$. В противном случае положим $\psi(a) = \overset{h(a)+1}{\pi}(\overset{h(a)}{\varphi}(a))$. Существование правой части этого равенства следует из формулы (4).

Докажем индукцией по n формулу

(5) $$n \leqq d(a) \to \overset{n}{\psi}(a) = \overset{n}{\pi}(a).$$

При $n=0$ она очевидна. Пусть она верна при $n = m-1$ и пусть $m \leqq d(a)$. По предположению индукции и формуле (3), $\overset{m-1}{\psi}(a) = \overset{m-1}{\pi}(a)$, $d(\overset{m-1}{\psi}(a)) = = d(\overset{m-1}{\pi}(a)) = d(a) - (m-1) > 0$. Отсюда следует, что $h(\overset{m-1}{\psi}(a)) = 0$, т. е. $\overset{m}{\psi}(a) = \psi(\overset{m-1}{\psi}(a)) = \pi(\overset{m-1}{\psi}(a)) = \overset{m}{\pi}(a)$. Формула (5) доказана. Учитывая, что, по определению π, $\overset{n}{\varphi} \circ \overset{n}{\pi}(a)$ равно либо a либо не определено, получаем в качестве следствия из формулы (5) формулу

(6) $$n \leqq d(a) \to \overset{n}{\psi} \circ \overset{n}{\varphi}(a) = a.$$

Пусть $h_n(a)$ обозначает наименьшее из таких чисел p, для которых $d(\overset{p}{\varphi}(a)) > n+p-1$. Если таких p не существует, положим $h_n(a)=\infty$. Ясно, что $h_1(a) = h(a)$. Индукцией по n докажем формулу

(7) $$h_n(a) = p \neq \infty \to \overset{n}{\psi}(a) = \overset{p+n}{\pi} \circ \overset{p}{\varphi}(a)$$

При $n=1$ она превращается в определение $\psi(a)$ для не специального элемента a. Пусть эта формула верна для $n = m-1$ и пусть $h_m(a) = p \neq \infty$. Легко видеть, что $h_{m-1}(a) \leqq h_m(a)$, поэтому $h_{m-1}(a) = q \neq \infty$. Пользуясь предположением индукции, получаем: $\overset{m}{\psi}(a) = \psi(\overset{m-1}{\psi}(a)) = \psi \circ \overset{m-1+q}{\pi} \circ \overset{q}{\varphi}$.

Рассмотрим два случая:

Случай 1. Пусть $p=q$. Пользуясь формулой (3), получим: $d(\overset{m-1}{\psi}(a)) = d(\overset{m-1+q}{\pi} \circ \overset{q}{\varphi}(a)) = d(\overset{q}{\varphi}(a)) - (m-1+q) > 0$. По формуле (5), $\overset{m}{\psi}(a) = \pi(\overset{m-1}{\psi}(a)) = \overset{m+q}{\pi} \circ \overset{q}{\varphi}(a) = \overset{m+p}{\pi} \circ \overset{p}{\varphi}(a)$.

Случай 2. Пусть $p>q$. Тогда $d(\overset{q}{\varphi}(a)) \leqq m+q-1$, т. е. $d(\overset{q}{\varphi}(a)) = m+q-1$. Отсюда $d(\overset{p+m-1}{\varphi} \circ \overset{m-1}{\psi}(a)) = d(\overset{p+m-1}{\varphi} \circ \overset{m-1+q}{\pi} \circ \overset{q}{\varphi}(a)) = d(\overset{p-q}{\varphi} \circ \overset{q}{\varphi}(a)) = d(\overset{p}{\varphi}(a)) > p+m-1$. Пользуясь формулой (3) и определением числа p, без труда проверяем, что если $s < p+m-1$, то $d(\overset{s}{\varphi}(\overset{m-1}{\psi}(a))) \leqq s$. Поэтому $h(\overset{m-1}{\psi}(a)) = p+m-1$, откуда $\overset{m}{\psi}(a) = \overset{m+p}{\pi} \circ \overset{m+p-1}{\varphi}(\overset{m-1}{\psi}(a)) = \overset{m+p}{\pi} \circ \overset{m+p-1}{\varphi} \circ \overset{m-1+q}{\pi} \circ \overset{q}{\varphi}(a) = \overset{m+p}{\pi} \circ \overset{p}{\varphi}(a)$.

При доказательстве формулы (7) мы видели, что $h_m(a) < \infty \to h(\overset{m-1}{\psi}(a)) < \infty$, т. е. что из левой части формулы

(8) $$h_{n+1}(a) < \infty \leftrightarrow h(\overset{n}{\psi}(a)) < \infty$$

следует правая часть. Покажем индукцией по n, что из правой части следует левая. Для $n = 0$ это очевидно. Пусть это верно для $n = m-1$ и пусть $h(\overset{m}{\psi}(a)) = p < \infty$. Если $h(\overset{m-1}{\psi}(a)) = \infty$, то $\overset{m}{\psi}(a) = \psi(\overset{m-1}{\psi}(a)) = a_0$ и $h(\overset{m}{\psi}(a)) = \infty$, поэтому $h(\overset{m-1}{\psi}(a)) < \infty$. По предположению индукции, $h_m(a) = q < \infty$. По формуле (7), $\overset{m}{\psi}(a) = \overset{q+m}{\pi} \circ \overset{q}{\varphi}(a)$. Если $s = \max\{q+m, p\}$, то $\overset{s}{\varphi} \circ \overset{q+m}{\pi} \circ \overset{q}{\varphi}(a) = \overset{s-m}{\varphi}(a)$ и $s < d(\overset{s}{\varphi}(\overset{m}{\psi}(a))) = d(\overset{s-m}{\varphi}(a))$, откуда $h_{m+1}(a) \leqq s-m$. Формула (8) доказана.

Докажем теперь формулу

(9) $$h(\overset{n}{\psi}(a)) < \infty \leftrightarrow h(\overset{m+n}{\psi} \circ \overset{m}{\varphi}(a)) < \infty.$$

Пусть $h(\overset{n}{\psi}(a)) < \infty$. По формуле (8), $h_{n+1}(a) = p < \infty$. Применяя формулу (5) и определение p, получаем для $q = \max\{p, m\}$, что $d(\overset{q}{\varphi}(\overset{m}{\psi} \circ \overset{m}{\varphi}(a))) = d(\overset{q}{\varphi} \circ \overset{m}{\pi} \circ \overset{m}{\varphi}(a)) = d(\overset{q-m}{\varphi} \circ \overset{m}{\varphi}(a)) = d(\overset{q}{\varphi}(a)) = d(\overset{q-p}{\varphi}(\overset{p}{\varphi}(a))) \geqq q-p+d(\overset{p}{\varphi}(a)) >$

$>q-p+p+n = q+n$, откуда следует, что $h_{n+1}(\overset{m}{\psi}\circ\overset{m}{\varphi}(a))\leqq q$. По формуле (8), $h(\overset{m+n}{\psi}\circ\overset{m}{\varphi}(a)) < \infty$.

Пусть $h(\overset{m+n}{\psi}\circ\overset{m}{\varphi}(a)) < \infty$, тогда, по формуле (8), $h_{n+1}(\overset{m}{\psi}\circ\overset{m}{\varphi}(a)) = p < \infty$. Применяя формулу (5), получим при $q = \max\{p, m\}$, что $q+n < d(\overset{q}{\varphi}(\overset{m}{\psi}\circ\overset{m}{\varphi}(a)))= = d(\overset{q}{\varphi}\circ\overset{m}{\pi}\circ\overset{m}{\varphi}(a)) = d(\overset{q}{\varphi}(a))$, откуда следует, что $h_{n+1}(a) < q$. По формуле (8), $h(\overset{n}{\psi}(a))<\infty$. Формула (9) доказана.

Учитывая, что $d(\varphi(a))>0$, получаем, по формуле (6), что $\varphi\circ\psi\circ\varphi(a) = = \varphi(a)$, т. е. $\varphi\circ\psi\circ\varphi = \varphi$. Если a — специальный элемент, то $\psi\circ\varphi\circ\psi(a) = = \psi\circ\varphi(a_0) = a_0 = \psi(a)$. В противном случае, $\psi\circ\varphi\circ\psi(a) = \pi\circ\varphi\circ\overset{n+1}{\pi}\circ\overset{n}{\varphi}(a) = = \pi\circ\overset{n}{\pi}\circ\varphi(a) = \psi(a)$, где $n=h(a)$. Итак, $\psi\circ\varphi\circ\psi = \psi$. Поэтому преобразование ψ является обобщённо обратным для φ.

Пусть m и n — какие-либо натуральные числа. Введём обозначения $a_1 = \overset{n}{\varphi}\circ\overset{m+n}{\psi}\circ\overset{m}{\varphi}(a)$ и $a_2 = \overset{m}{\psi}\circ\overset{m+n}{\varphi}\circ\overset{n}{\psi}(a)$. Если $\overset{m+n-1}{\psi}\circ\overset{m}{\varphi}(a)$ — специальный элемент, то, по формуле (9), $\overset{n-1}{\psi}(a)$ — также специальный элемент. В этом случае $a_1 = \overset{n}{\varphi}(a_0)$ и $a_2 = \overset{m}{\psi}\circ\overset{m+n}{\varphi}(a_0) = \overset{n}{\varphi}(a_0) = a_1$. Пусть теперь $h(\overset{m+n-1}{\psi}\circ\overset{m}{\varphi}(a)) < < \infty$. По формуле (9), $h(\overset{n-1}{\psi}(a))<\infty$. По формуле (8), $h_n(\overset{m}{\psi}\circ\overset{m}{\varphi}(a)) = p<\infty$ и $h_n(a) = q < \infty$. Применяя формулу (7), получаем: $a_1 = \overset{n}{\varphi}\circ\overset{p+n}{\pi}\circ\overset{p}{\varphi}(\overset{m}{\psi}\circ\overset{m}{\varphi}(a)) = = \overset{p}{\pi}\circ\overset{p}{\varphi}\circ\overset{m}{\pi}\circ\overset{m}{\varphi}(a)$. Пусть $p\leqq m$. Тогда $a_1 = \overset{m}{\pi}\circ\overset{m}{\varphi}(a)$ и $p+n-1 < d(\overset{p}{\varphi}(\overset{m}{\psi}\circ\overset{m}{\varphi}(a)))= = d(\overset{m-p}{\pi}\circ\overset{m}{\varphi}(a)) = d(\overset{m}{\varphi}(a))-(m-p)$, откуда $m+n-1 < d(\overset{m}{\varphi}(a))$. Значит, $q\leqq m$. Поэтому $a_2 = \overset{m}{\psi}\circ\overset{m+n}{\varphi}\circ\overset{n}{\psi}(a) = \overset{m}{\pi}\circ\overset{m+n}{\varphi}\circ\overset{q+n}{\pi}\circ\overset{q}{\varphi}(a) = \overset{m}{\pi}\circ\overset{m-q}{\varphi}\circ\overset{q}{\pi}(a) = \overset{m}{\pi}\circ\overset{m}{\varphi}(a) = a_1$. Пусть теперь $p > m$. Тогда $a_1 = \overset{p}{\pi}\circ\overset{p}{\varphi}(a)$. С другой стороны, $p+n-1 < < d(\overset{p}{\varphi}(\overset{m}{\psi}\circ\overset{m}{\varphi}(a))) = d(\overset{p}{\varphi}\circ\overset{m}{\pi}\circ\overset{m}{\varphi}(a)) = d(\overset{p}{\varphi}(a))$, откуда следует, что $q\leqq p$. Если $q\leqq m$, то $m+n-1 < d(\overset{m}{\varphi}(a)) = d(\overset{m-q}{\pi}\circ\overset{m}{\varphi}(a))+(m-q) = d(\overset{q}{\varphi}(\overset{m}{\psi}\circ\overset{m}{\varphi}(a)))+(m-q)$, откуда $q+n-1 < d(\overset{q}{\varphi}(\overset{m}{\psi}\circ\overset{m}{\varphi}(a)))$ и $p\leqq q$. Поэтому $p = q$. Если же $q > m$, то $q+n-1 < d(\overset{q}{\varphi}(a)) = d(\overset{q}{\varphi}(\overset{m}{\psi}\circ\overset{m}{\varphi}(a)))$ и $p\leqq q$. Значит, $p=q$ и $a_2 = \overset{m}{\pi}\circ\overset{m+n}{\varphi}\circ\overset{q+n}{\pi}\circ\overset{q}{\varphi}(a) = = \overset{m}{\pi}\circ\overset{q-m}{\pi}\circ\overset{q}{\varphi}(a) = \overset{q}{\pi}\circ\overset{q}{\varphi}(a) = a_1$. Итак, $a_1 = a_2$. Поскольку это верно для всех $a\in A$, получаем, что $\overset{n}{\varphi}\circ\overset{m+n}{\psi}\circ\overset{m}{\varphi} = \overset{m}{\psi}\circ\overset{m+n}{\varphi}\circ\overset{n}{\psi}$. Таким образом, φ и ψ удовлетворяют (1) и (2) и, по лемме, ψ является квазиобратным для φ. Теорема 1 доказана.

Пример 2. Покажем, что полугруппа $\mathcal{T}(\{1, 2, 3\})$ не допускает разбиения, классы которого суть инверсные подполугруппы. Для этого рассмотрим следующие элементы нашей полугруппы, которые мы записываем как подстановки: $\alpha_i = \begin{pmatrix}1&2&3\\ i&i&i\end{pmatrix}$ для $i=1,2,3$; $\beta = \begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}$; $\gamma = \begin{pmatrix}1&2&3\\1&3&1\end{pmatrix}$; $\delta = \begin{pmatrix}1&2&3\\3&3&1\end{pmatrix}$;

$\varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}$. Легко проверить, что $\beta\circ\beta = \gamma\circ\gamma = \alpha_1$, $\delta\circ\delta = \gamma\circ\beta$, $\varepsilon\circ\varepsilon = \beta\circ\gamma$, $\varepsilon\circ\delta = \alpha_2$ и $\delta\circ\varepsilon = \alpha_3$ (напомним, что произведение преобразований мы записываем справа налево). Отсюда видно, что если наша полугруппа допускает разбиение в инверсные подполугруппы, то в компоненту, содержащую α_1, попадут также преобразования α_2 и α_3. Преобразования α_i идемпотентны, но не коммутируют при умножении. Поэтому компонента разбиения нашей полугруппы не может быть инверсной подполугруппой.

Этот пример был сообщён автору Л. М. Глускиным.

Теорема 2. *Симметрическая полугруппа $\mathscr{F}(A)$ частичных преобразований любого множества A квазиинверсна.*

Доказательство. Добавим к множеству A элемент ω, ранее не принадлежавший A, и рассмотрим подполугруппу симметрической полугруппы $\mathscr{T}(A\cup\{\omega\})$, состоящую из всех преобразований, оставляющих элемент ω неподвижным. Из доказательства теоремы 1 видно, что если преобразование φ имеет неподвижный элемент, то преобразование ψ, служащее для φ квазиобратным, всегда можно выбрать так, что ψ будет иметь тот же неподвижный элемент, что и φ. Поэтому наша подполугруппа квазиинверсна. Покажем, что она изоморфна полугруппе $\mathscr{F}(A)$. Для этого всякому преобразованию φ множества $A\cup\{\omega\}$, для которого $\varphi(\omega) = \omega$, поставим в соответствие преобразование $\bar\varphi\in\mathscr{F}(A)$, совпадающее с φ всюду, где $\Delta_A\circ\varphi$ определено. В работе [4] показано, что соответствие $\varphi\to\bar\varphi$ является изоморфизмом. Теорема 2 доказана.

Разумеется, не всякий элемент полугруппы $\mathscr{T}(A)$ или $\mathscr{F}(A)$ принадлежит некоторой подгруппе этой полугруппы. Нетрудно показать, что элемент φ какой-либо из этих полугрупп принадлежит подгруппе тогда и только тогда, когда $pr_2\varphi\subset pr_1\varphi$, $pr_2\overset{2}{\varphi} = pr_2\varphi$ и для любых $a_1, a_2\in pr_1\overset{2}{\varphi}$ из $\overset{2}{\varphi}(a_1) = \overset{2}{\varphi}(a_2)$ следует $\varphi(a_1) = \varphi(a_2)$.

В заключение отметим несколько нерешённых вопросов.

1. Привести пример регулярного, но не сильно регулярного кольца. Порядок такого кольца должен быть больше 6. Будет ли кольцо всех линейных преобразований векторного пространства над телом сильно регулярным? По-видимому, да. Нам удалось это доказать лишь для конечномерных векторных пространств над полями [5]. Вообще, будет ли полное матричное кольцо над сильно регулярным кольцом сильно регулярным?

2. Описать все квазиобратные для произвольного элемента полугруппы $\mathscr{T}(A)$ или $\mathscr{F}(A)$. Какие элементы обладают единственным квазиобратным? Если φ — преобразование, то можно показать, что на $pr_2\varphi$ любой квазиобратный для φ совпадёт с π, построенным при доказательстве теоремы 1.

3. Всякая инверсная подполугруппа полугруппы $\mathscr{T}(A)$ или $\mathscr{F}(A)$ включается в максимальную инверсную подполугруппу. В частности, инверсная полугруппа $\mathscr{K}(A)$ всех взаимно однозначных частичных преобразований множества A максимальна в $\mathscr{F}(A)$. Описать максимальные инверсные подполугруппы $\mathscr{T}(A)$ и $\mathscr{F}(A)$.

4. Полугруппа $\mathscr{P}(A)$ всех бинарных отношений на множестве A не регулярна. Её обобщённо обратимые элементы описаны в [6]. Описать её квазиобратимые элементы.

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ОЦЕНКА ОТКЛОНЕНИЯ ЧАСТНЫХ СУММ РЯДА ФУРЬЕ

О. КИШ (Будапешт)

Теорема. *Пусть n есть любое натуральное число, $s_n(x)$ частная сумма ряда Фурье непрерывной 2π-периодической функции f и ω ее модуль непрерывности. Если*

$$\lambda_n = \frac{2}{\pi}\int_0^{\pi/2} \frac{|\sin(2n+1)x|}{\sin x}\,dx, \tag{1}$$

то для всех вещественных чисел x

$$|f(x)-s_n(x)| \leqq \left(\frac{1}{2}\lambda_n+3\right)\omega\left(\frac{2\pi}{2n+1}\right). \tag{2}$$

Доказательство. Очевидно

$$f(x)-s_n(x) = f(x)-\frac{1}{2}\left[s_n(x)+s_n\left(x+\frac{2\pi}{2n+1}\right)\right]+\frac{1}{2}\left[s_n\left(x+\frac{2\pi}{2n+1}\right)-s_n(x)\right].$$

Поэтому

$$|f(x)-s_n(x)| \leqq \left|f(x)-\frac{1}{2}\left[s_n(x)+s_n\left(x+\frac{2\pi}{2n+1}\right)\right]\right|+\frac{1}{2}\left|s_n\left(x+\frac{2\pi}{2n+1}\right)-s_n(x)\right|. \tag{3}$$

Обозначим через $E_n(f)$ наилучшее приближение функции f тригонометрическими многочленами n-ого порядка и пусть

$$A = 2\int_0^{\pi} \frac{\sin x}{x}\,dx.$$

С. Н. Бернштейн доказал, (см. [1], стр. 525, формула (10)), что

$$\left|f(x)-\frac{1}{2}\left[s_n(x)+s_n\left(x+\frac{2\pi}{2n+1}\right)\right]\right| \leqq \left(\frac{A}{\pi}+1\right)E_n(f)+\frac{1}{2}\,\omega\left(\frac{2\pi}{2n+1}\right).$$

Здесь (см. [1], стр. 524, формула (7)) $\dfrac{A}{\pi}<\dfrac{4}{3}$ и поэтому

$$\left|f(x)-\frac{1}{2}\left[s_n(x)+s_n\left(x+\frac{2\pi}{2n+1}\right)\right]\right| \leqq \frac{7}{3}E_n(f)+\frac{1}{2}\,\omega\left(\frac{2\pi}{2n+1}\right).$$

Н. П. Корнейчук доказал (см. [2]), что

$$E_n(f) \leqq \omega\left(\frac{\pi}{n+1}\right). \tag{4}$$

Так как модуль непрерывности ω неубывает и $\frac{\pi}{n+1} < \frac{2\pi}{2n+1}$, то отсюда следует неравенство

$$E_n(f) \leqq \omega\left(\frac{2\pi}{2n+1}\right)$$

и поэтому

$$\left|f(x) - \frac{1}{2}\left[s_n(x) + s_n\left(x + \frac{2\pi}{2n+1}\right)\right]\right| \leqq \frac{17}{6}\,\omega\left(\frac{2\pi}{2n+1}\right) \leqq 3\omega\left(\frac{2\pi}{2n+1}\right). \tag{5}$$

По формуле Дирихле (см., например, [3], стр. 191)

$$s_n(x) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(x+2t)\,\frac{\sin(2n+1)t}{\sin t}\,dt. \tag{6}$$

Следовательно

$$s_n\left(x + \frac{2\pi}{2n+1}\right) - s_n(x) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left[f\left(x + \frac{2\pi}{2n+1} + 2t\right) - f(x+2t)\right]\frac{\sin(2n+1)t}{\sin t}\,dt,$$

$$\left|s_n\left(x + \frac{2\pi}{2n+1}\right) - s_n(x)\right| \leqq \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left|f\left(x + \frac{2\pi}{2n+1} + 2t\right) - f(x+2t)\right|\cdot\left|\frac{\sin(2n+1)t}{\sin t}\right|dt.$$

По определению модуля непрерывности здесь

$$\left|f\left(x + \frac{2\pi}{2n+1} + 2t\right) - f(x+2t)\right| \leqq \omega\left(\frac{2\pi}{2n+1}\right).$$

Отсюда и из (1) получаем:

$$\begin{aligned}\left|s_n\left(x + \frac{2\pi}{2n+1}\right) - s_n(x)\right| &\leqq \omega\left(\frac{2\pi}{2n+1}\right)\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left|\frac{\sin(2n+1)t}{\sin t}\right|dt = \\ &= \omega\left(\frac{2\pi}{2n+1}\right)\frac{2}{\pi}\int_{0}^{\pi/2}\frac{|\sin(2n+1)t|}{\sin t}\,dt = \lambda_n\omega\left(\frac{2\pi}{2n+1}\right).\end{aligned} \tag{7}$$

Резюмируя неравенства (3), (5) и (7), получаем доказываемое утверждение (2).

Замечания. 1. Обычно неравенство

$$|f(x) - s_n(x)| = O\left(\lambda_n\omega\left(\frac{1}{n}\right)\right) \tag{8}$$

доказывается следующим образом: по теореме Лебега

$$|f(x)-s_n(x)| \leqq (1+\lambda_n)E_n(f); \tag{9}$$

в силу теоремы Джексона (см., например, [3], стр. 117)

$$E_n(f) = O\left(\omega\left(\frac{1}{n}\right)\right); \tag{10}$$

из (9) и (10) следует (8). Если заменить (10) наиболее точным неравенстрвом такого вида (4), то вместо (8) получаем:

$$|f(x)-s_n(x)| \leqq (1+\lambda_n)\,\omega\left(\frac{\pi}{n+1}\right).$$

Для больших n эта оценка почти вдвое хуже, чем неравенство (2).

2. Если модуль непрерывности ω выпуклая функция, то имеет место более точная чем (2) оценка

$$|f(x)-s_n(x)| \leqq \left(\frac{1}{2}\lambda_n+\frac{1}{2}\right)\omega\left(\frac{\pi}{n+1}\right).$$

Это следует из неравенства (9) и формулы

$$E_n(f) \leqq \frac{1}{2}\,\omega\left(\frac{\pi}{n+1}\right),$$

которую Н. П. Корнейчук доказал в статье [4] для выпуклых модулей непрерывности ω.

3. Для постоянных Лебега λ_n выполняется равенство

$$\lambda_n = \frac{4}{\pi^2}\ln n + O(1)$$

(см., например, [5], стр. 115).

4. Коэффициент 1/2 в формуле (2) не может быть уменьшен, так как для любого положительного числа ε существует (зависящая от n) непрерывная 2π-периодическая функция f, для которой

$$s_n(0)-f(0) > \left(\frac{1}{2}\lambda_n+\frac{1}{2}-\varepsilon\right)\omega\left(\frac{2\pi}{2n+1}\right). \tag{11}$$

Такую функцию можно построить следующим образом. Рассмотрим разрывную функцию

$$f(x) = \operatorname{sign}\frac{\sin\frac{2n+1}{2}x}{\sin\frac{x}{2}} \qquad (x \neq 2k\pi;\ k=0,\ \pm 1,\ \pm 2,\ \dots).$$

Для нее из формул (6) и (1) получаем:

$$s_n(0) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(2t) \frac{\sin(2n+1)t}{\sin t} dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left| \frac{\sin(2n+1)t}{\sin t} \right| dt = \lambda_n. \tag{12}$$

Доопределим эту функцию, полагая

$$f(2k\pi) = -1 \quad (k = 0, \pm 1, \pm 2, \dots), \tag{13}$$

и заменим полученную функцию ее линейным интерполяционным многочленом в достаточно малой окрестности каждой ее точки разрыва. Таким образом мы получим 2π-периодическую непрерывную функцию, для которой вместо равенства (12) выполняется неравенство

$$s_n(0) > \lambda_n - 2\varepsilon.$$

Отсюда и из (13) получаем:

$$s_n(0) - f(0) > \lambda_n + 1 - 2\varepsilon. \tag{14}$$

Так как наша функция колеблется между -1 и 1, то

$$\omega\left(\frac{2\pi}{2n+1}\right) \leqq 2$$

(легко видеть, что имеет место точное равенство). Отсюда и из (14) следует доказываемое неравенство

$$s_n(0) - f(0) > (\lambda_n + 1 - 2\varepsilon) \frac{1}{2} \omega\left(\frac{2\pi}{2n+1}\right) = \left(\frac{1}{2}\lambda_n + \frac{1}{2} - \varepsilon\right) \omega\left(\frac{2\pi}{2n+1}\right).$$

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ON APPROXIMATION THEOREMS FOR UNIFORM SPACES

By

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0. Let $E \neq \emptyset$ be an arbitrary set and Φ a family of real-valued functions defined on E. Denote by $\mathcal{U}_\Phi$ the coarsest uniformity with respect to which all functions $f \in \Phi$ are uniformly continuous. $\mathcal{U}_\Phi$ is induced by the family of all pseudometrics

$$\sigma_f(x, y) = |f(x) - f(y)| \qquad (f \in \Phi).$$

There exist some approximation theorems of the following type:

If Φ fulfills suitable conditions, then each function of $C(\mathcal{U}_\Phi)$ is the limit of a uniformly convergent sequence taken from Φ.

Here $C(\mathcal{U})$ denotes, for a uniformity $\mathcal{U}$, the family of all functions which are uniformly continuous with respect to $\mathcal{U}$.

Conditions which can be used in the above theorem are as follows:

Φ consists of bounded functions, contains all constants, and

(1) (W. MAAK [4]) *is a ring*, i.e. $f, g \in \Phi$ implies $f-g \in \Phi$, $fg \in \Phi$;

(2) (G. NÖBELING and H. BAUER [6]) *is a vector lattice*, i.e. $f, g \in \Phi$, $\alpha \in R$ imply $\max(f, g) \in \Phi$, $\min(f, g) \in \Phi$, $f+g \in \Phi$, $\alpha f \in \Phi$;

(3) (Á. CSÁSZÁR and J. CZIPSZER [3]) *is a subtractive lattice*, i.e. $f, g \in \Phi$ implies $f-g \in \Phi$, $\max(f, g) \in \Phi$, $\min(f, g) \in \Phi$;

(4) (Á. CSÁSZÁR and J. CZIPSZER [3]) *is an affine lattice*, i.e. $f, g \in \Phi$, $\alpha \in R$ imply $\max(f, g) \in \Phi$, $\min(f, g) \in \Phi$, $\alpha f \in \Phi$, $f+\alpha \in \Phi$.

Clearly both (3) and (4) are generalizations of (2).

One of the aims of this paper is to prove a further generalization of (4) (see Theorem 1 below).

Another type of approximation theorems states that functions of $C(\mathcal{U}_\Phi)$ may be uniformly approximated by functions belonging to Φ only under suitable restrictions. One of them is the following:

(5) (G. NÖBELING and H. BAUER [6]) *If Φ is a vector lattice consisting of bounded functions and such that $f \in \Phi$, $\alpha > 0$ imply* $\min(f, \alpha) \in \Phi$, *then each function $h \in C(\mathcal{U}_\Phi)$ with the property that $\varepsilon > 0$ implies the existence of $\delta > 0$ and of functions $f_1, \dots, f_m \in \Phi$ with*

$$|f_i(x)| < \delta \qquad (i = 1, \dots, m) \Rightarrow |h(x)| < \varepsilon,$$

is limit of a uniformly convergent sequence taken from Φ.

Our second purpose is to show that the condition on Φ of being a vector lattice may be replaced here by that of being a subtractive lattice (Theorem 2).

We also present two kinds of applications of the above mentioned results. One of them is the proof of approximation theorems of the Stone—Weierstrass—Nakano type (Theorems 4 to 6), another furnishes new characterizations of the function classes having the form $C^*(\mathscr{U})$, i.e. coinciding with the family of all bounded functions uniformly continuous with respect to a uniformity $\mathscr{U}$ (Theorem 7).

Finally we use our results to a characterization of function classes having the form $C^*(\mathscr{T})$ and $C(\mathscr{T})$, i.e. coinciding with the totality of bounded continuous or of all continuous functions with respect to a topology $\mathscr{T}$ (Theorems 8 and 9).

1. THEOREM 1. *Let $\Phi \neq \emptyset$ be a family of bounded real functions defined on E which is a semi-affine lattice i.e. such that*

$$f, g \in \Phi \Rightarrow \max(f, g) \in \Phi, \quad \min(f, g) \in \Phi; \tag{1. 1}$$

$$f \in \Phi, \alpha \in R \Rightarrow f + \alpha \in \Phi, \tag{1. 2}$$

$$f \in \Phi, \gamma \in \Gamma \Rightarrow \gamma f \in \Phi, \tag{1. 3}$$

where $\Gamma \subset R$ is a set of real numbers containing 0 *and unbounded both from above and from below. Then $h \in C(\mathscr{U}_\Phi)$ implies that h is limit of a uniformly convergent sequence taken from Φ.*

THEOREM 2. *Let $\Phi \neq \emptyset$ be a family of bounded real functions defined on E which is a subtractive lattice i.e. satisfies* (1. 1) *and*

$$f, g \in \Phi \Rightarrow f - g \in \Phi. \tag{1. 4}$$

Moreover, suppose

$$f \in \Phi, \quad \alpha > 0 \Rightarrow \min(f, \alpha) \in \Phi. \tag{1. 5}$$

Assume that $h \in C(\mathscr{U}_\Phi)$ has the property that $\varepsilon > 0$ implies the existence of $\delta > 0$ and $f_1, \dots, f_m \in \Phi$ such that

$$|f_i(x)| < \delta \ (i = 1, \dots, m) \Rightarrow |h(x)| < \varepsilon.$$

Then h is limit of a uniformly convergent sequence taken from Φ.

We prove Theorem 1 and Theorem 2 simultaneously, referring the respective hypotheses by (I) and (II). The following proof is, in fact, the same as that of (5) given in [6] with some suitable modifications.

It is clear that $f \in \Phi$ implies $nf \in \Phi$ for every integer n in case (II). Hence (1. 3) holds in both cases. The condition $h \in C(\mathscr{U}_\Phi)$ can be formulated by saying that, to $\varepsilon > 0$, there correspond a $\delta > 0$ and functions $f_1, \dots, f_m \in \Phi$ such that

$$|f_i(x) - f_i(y)| < \delta \qquad (i = 1, \dots, m) \quad \text{implies} \quad |h(x) - h(y)| < \varepsilon.$$

The functions $f_i \in \Phi$ being bounded, there is a finite cover of E such that the oscillation of each f_i is less than δ on each of the members of this cover; therefore $h \in C(\mathscr{U}_\Phi)$ is bounded.

We may assume $h \geqq 0$, in case (I) in consequence of (1. 2), in case (II) by (1. 4), because, together with h, both $\max(h, 0)$ and $-\min(h, 0)$ clearly satisfy the conditions (II).

For a given $\varepsilon>0$ and $k=1, 2, \dots$ put

$$A_k = \{x:h(x) \leqq (k-1)\varepsilon\}, \tag{1.6}$$

$$B_k = \{x:h(x) \geqq k\varepsilon\}. \tag{1.7}$$

There is a k_0 with $B_k=\emptyset$ for $k\geqq k_0$. Let us select a $\delta>0$ and functions $f_1, \dots, f_m \in \Phi$ such that

$$|f_i(x)-f_i(y)|<\delta \qquad (i=1, \dots, m) \Rightarrow |h(x)-h(y)|<\varepsilon, \tag{1.8}$$

and in case (II) also

$$|f_i(x)|<\delta \qquad (i=1, \dots, m) \Rightarrow |h(x)|<\varepsilon. \tag{1.9}$$

Choose $\gamma\in\Gamma$, $\gamma<0$, and put

$$f_{m+i}=\gamma f_i \qquad (i=1, \dots, m), \tag{1.10}$$

$$\delta'=\min(\delta, |\gamma|\delta). \tag{1.11}$$

Then $f_i\in\Phi$ $(i=1, \dots, 2m)$. Moreover,

$$f_i(y)-f_i(x) < \delta' \qquad (i=1, \dots, 2m)$$

implies

$$f_i(y)-f_i(x) < \delta' \leqq \delta, \quad \gamma f_i(y)-\gamma f_i(x) < \delta' \qquad (i=1, \dots, m),$$

the latter implying

$$f_i(x)-f_i(y)< -\frac{\delta'}{\gamma} = \frac{\delta'}{|\gamma|} \leqq \delta.$$

Thus by (1.8)

$$f_i(y)-f_i(x) < \delta' \qquad (i=1, \dots, 2m) \Rightarrow |h(y)-h(x)| < \varepsilon. \tag{1.12}$$

A similar argument relying on (1.9), (1.10) and (1.11) shows in case (II) that

$$f_i(x)<\delta' \qquad (i=1, \dots, 2m) \Rightarrow h(x)<\varepsilon. \tag{1.13}$$

Let us now fix a k between 1 and k_0, and choose a $q\in\Gamma$ with $q>0$ and $\eta = \frac{k\varepsilon}{q} < \frac{\delta'}{2}$. Let $p>0$ denote an integer such that

$$-p\eta<f_i(x)<p\eta \qquad (x\in E,\ i=1, \dots, 2m),$$

and put, for integers $-p\leqq j_i\leqq p$ $(i=1, \dots, 2m)$,

$$C_{j_1, \dots, j_{2m}} = \{x\colon (j_i-1)\eta \leqq f_i(x) \leqq j_i\eta \ (i=1, \dots, 2m)\}. \tag{1.14}$$

Then

$$E = \bigcup_{j_1=-p}^{p} \dots \bigcup_{j_{2m}=-p}^{p} C_{j_1, \dots, j_{2m}}. \tag{1.15}$$

If now $x\in A_k$, $y\in B_k$, then by (1.6) and (1.7)

$$h(y)-h(x) \geqq \varepsilon,$$

hence by (1. 12) there is an index i with $1 \leqq i \leqq 2m$ and $f_i(y) - f_i(x) \geqq \delta' > 2\eta$, and if $x \in C_{j_1, \dots, j_{2m}}$ then

$$(j_i - 1)\eta \leqq f_i(x) \leqq j_i \eta, \qquad f_i(y) > (j_i + 1)\eta.$$

In other words, to $x \in A_k$ and $y \in B_k$ there correspond two indices i and j with

$$\text{(1. 16)} \qquad 1 \leqq i \leqq 2m, \quad -p \leqq j \leqq p, \quad f_i(x) \leqq j\eta, \quad f_i(y) > (j+1)\eta.$$

Let us now put in case (II)

$$f' = \min(\delta', \max(f_1, \dots, f_{2m})).$$

Then $f' \in \Phi$ by (1. 1) and (1. 5). Moreover, (1. 10) clearly implies

$$\text{(1. 17)} \qquad 0 \leqq f'(x) \leqq \delta' \qquad (x \in E),$$

and by (1. 7) and (1. 13)

$$\text{(1. 18)} \qquad f'(x) = \delta' \qquad (x \in B_k).$$

Choose $r \in \Gamma$ such that $r\delta' > p\eta$ and put

$$\text{(1. 19)} \qquad f_i' = f_i + rf'.$$

Then $f_i' \in \Phi$ by (1. 3) and (1. 4), and to $x \in A_k$, $y \in B_k$ there correspond in account of (1. 16), (1. 17) and (1. 18) two indices $1 \leqq i \leqq 2m$, $-p \leqq j \leqq p$ with

$$\text{(1. 20)} \qquad f_i'(x) \leqq j\eta + r\delta' = \alpha_j, \quad f_i'(y) > \alpha_j + \eta.$$

Putting

$$\text{(1. 21)} \qquad g_{ij} = q(f_i' - \min(f_i', \alpha_j))$$

we have $g_{ij} \in \Phi$ by (1. 5), (1. 4) and (1. 3), taking account of $\alpha_j > 0$. Moreover, $x \in A_k$ and $y \in B_k$ imply

$$\text{(1. 22)} \qquad g_{ij}(x) \leqq 0, \quad g_{ij}(y) > q\eta = k\varepsilon$$

for suitable indices $1 \leqq i \leqq 2m$, $-p \leqq j \leqq p$.

In case (I) we put quite simply

$$\text{(1. 23)} \qquad g_{ij} = q(f_i - j\eta).$$

Then again $g_{ij} \in \Phi$ by (1. 2) and (1. 3), and (1. 22) is valid for $x \in A_k$, $y \in B_k$ and suitable i, j by (1. 16).

For $x \in A_k$, $y \in B_k$, denote by $i(x, y)$ and $j(x, y)$ a pair of indices satisfying (1. 22). Define

$$\text{(1. 24)} \qquad g_k = \min_{x \in A_k} \max_{y \in B_k} g_{i(x,y),\, j(x,y)}.$$

As both $i(x, y)$ and $j(x, y)$ range over a finite set of values, we get $g_k \in \Phi$ by (1. 1), and (1. 22) implies

$$\text{(1. 25)} \qquad g_k(x) \leqq 0 \quad \text{for} \quad x \in A_k, \quad g_k(y) > k\varepsilon \quad \text{for} \quad y \in B_k.$$

In fact, if $x_0 \in A_k$, then $g_{i(x_0,y),j(x_0,y)}(x_0) \leqq 0$ for every $y \in B_k$, whence $g_k(x_0) \leqq 0$. For $y_0 \in B_k$, one has $g_{i(x,y_0),j(x,y_0)}(y_0) > k\varepsilon$ for every $x \in A_k$, hence

$$\max_{y \in B_k} g_{i(x,y),j(x,y)}(y_0) > k\varepsilon \qquad (x \in A_k),$$

and $g_k(y_0) > k\varepsilon$.

Put

$$h_k = \max(0, \min(g_k, k\varepsilon)). \tag{1. 26}$$

Then $h_k \in \Phi$ by (1. 1), (1. 2) and (1. 3) in case (I), by (1. 1), (1. 5) and (1. 4) in case (II). In account of (1. 25), we get

$$0 \leqq h_k(x) \leqq k\varepsilon \quad (x \in E), \qquad h_k(x) = 0 \quad (x \in A_k), \tag{1. 27}$$
$$h_k(y) = k\varepsilon \quad (y \in B_k).$$

We define finally

$$h' = \max(h_1, \ldots, h_{k_0}).$$

Then $h' \in \Phi$ by (1. 1), and if $x \in E$, there is a k_1 with $1 \leqq k_1 \leqq k_0$ and

$$(k_1 - 1)\varepsilon \leqq h(x) < k_1 \varepsilon.$$

Therefore, from (1. 6), (1. 7) and (1. 27), we get

$$x \in B_k, \quad h_k(x) = k\varepsilon \quad \text{for} \quad k = 1, \ldots, k_1 - 1,$$
$$0 \leqq h_{k_1}(x) \leqq k_1 \varepsilon,$$
$$x \in A_k, \quad h_k(x) = 0 \quad \text{for} \quad k = k_1 + 1, \ldots, k_0.$$

Thus $(k_1 - 1)\varepsilon \leqq h'(x) \leqq k_1 \varepsilon$, and

$$|h(x) - h'(x)| < \varepsilon \quad (x \in E).$$

By this, we have proved Theorems 1 and 2. Let us note that the second somewhat cumbersome condition on h may be replaced, in Theorem 2, by a simpler one under suitable hypotheses. This is the content of

THEOREM 3. *Let $\Phi \neq \emptyset$ be a subtractive lattice of bounded real functions defined on E, satisfying* (1. 5), *and suppose that*

$$Z(\Phi) = \bigcap_{f \in \Phi} Z(f) \neq \emptyset$$

where

$$Z(f) = \{x : f(x) = 0\}.$$

Then each function $h \in C(\mathcal{U}_\Phi)$ vanishing on $Z(\Phi)$ is limit of a uniformly convergent sequence taken from Φ.

In fact, if $x_0 \in Z(\Phi)$, and, for a given $\varepsilon > 0$, $f_1, \ldots, f_m \in \Phi$ and $\delta > 0$ are chosen in such a manner that

$$|f_i(x) - f_i(y)| < \delta \qquad (i = 1, \ldots, m) \Rightarrow |h(x) - h(y)| < \varepsilon,$$

then, in particular, $|f_i(x)|<\delta$ $(i=1, \ldots, m)$, i.e.

$$|f_i(x)-f_i(x_0)| < \delta \qquad (i=1, \ldots, m)$$

implies

$$|h(x)-h(x_0)| < \varepsilon,$$

i.e. $|h(x)|<\varepsilon$.

We also note that Theorem 2 implies (3). In fact, if Φ contains all constants then (1. 5) is a consequence of (1. 1); on the other hand, if h is an arbitrary real function and $\varepsilon>0$ is given, then we put $f=1$, $\delta=\frac{1}{2}$, and obtain $f\in\Phi$ and

$$|f(x)|<\delta \Rightarrow |h(x)|<\varepsilon.$$

1. In order to get approximation theorems of the Stone—Weierstrass and Nakano type, we first establish the following

LEMMA. *Let Φ be an arbitrary family of continuous real functions defined on a compact topological space E, and suppose that Φ separates the points of E, i.e. if $x, y\in E$, $x\neq y$ then $f(x)\neq f(y)$ for at least one $f\in\Phi$. Then E is Hausdorff and $\mathcal{U}_\Phi$ coindices with the (unique) uniformity which induces the topology of E.*

PROOF. Clearly E is Hausdorff so that its topology $\mathcal{T}$ is induced by a unique uniformity $\mathcal{U}$. As every $f\in\Phi$ is $\mathcal{T}$-continuous and $\mathcal{T}$ is compact, f is $\mathcal{U}$-uniformly continuous too, hence $\mathcal{U}_\Phi$ is, by definition, coarser than $\mathcal{U}$.

On the orher hand, let U be a symmetrical surrounding from $\mathcal{U}$; U also is a neighbourhood of the diagonal of $E\times E$ (with respect to the product topology). Given $x, y\in E$, denote by V_x and W_y neighbourhoods of x and y such that

for $x=y$: $V_x\times W_y\subset U$,

for $x\neq y$: there is an $f_{xy}\in\Phi$ with

$$|f_{xy}(u)-f_{xy}(x)|<\frac{1}{3}|f_{xy}(x)-f_{xy}(y)|=\varepsilon_{xy} \qquad (u\in V_x), \tag{2. 1}$$

$$|f_{xy}(v)-f_{xy}(y)|<\varepsilon_{xy} \qquad (v\in w_y). \tag{2. 2}$$

Such V_x and W_y exist, if $x\neq y$, since Φ separates the points of E and the elements of Φ are continuous. Now

$$E\times E=\bigcup_{i=1}^{m}(V_{x_i}\times W_{y_i})$$

by the compactness of $E\times E$. Put

$$\varepsilon=\min(\varepsilon_{x_1y_1}, \ldots, \varepsilon_{x_my_m}).$$

If $x, y\in E$ are so chosen that

$$|f_{x_iy_i}(x)-f_{x_iy_i}(y)|<\varepsilon \qquad (i=1, \ldots, m) \tag{2. 3}$$

and i_0 is an index with $(x, y) \in V_{x_{i_0}} \times W_{y_{i_0}}$ then $x_{i_0} = y_{i_0}$. In fact, $x_{i_0} \neq y_{i_0}$ would imply by (2. 1) and (2. 2)

$$|f_{x_{i_0} y_{i_0}}(x) - f_{x_{i_0} y_{i_0}}(x_{i_0})| < \varepsilon_{x_{i_0} y_{i_0}},$$

$$|f_{x_{i_0} y_{i_0}}(y) - f_{x_{i_0} y_{i_0}}(y_{i_0})| < \varepsilon_{x_{i_0} y_{i_0}}$$

whence

$$|f_{x_{i_0} y_{i_0}}(x) - f_{x_{i_0} y_{i_0}}(y)| > \varepsilon_{x_{i_0} y_{i_0}} \geqq \varepsilon.$$

Therefore (2. 3) implies $(x, y) \in U$. Thus $\mathcal{U}$ is coarser than $\mathcal{U}_\Phi$.

From this lemma and Theorems 1 and 2, we easily get

THEOREM 4. *Let be a compact topological space, and Φ either a semi-affine lattice or a subtractive lattice composed of continuous functions, containing all constants, and separating the points of E. Then every continuous function on E is limit of a uniformly convergent family taken from Φ.*

PROOF. Clearly, the functions $f \in \Phi$ are bounded and Φ satisfies (1. 5). Also, if h is continuous, then it is $\mathcal{U}_\Phi$-uniformly continuous by the lemma and by the compactness of E. Thus the assertion is obtained from Theorem 1 or Theorem 2; in the case of the latter, we must take into account the note at the end of section 1.

Another approximation theorem is obtained from the Lemma and Theorem 3:

THEOREM 5. *Let E be a compact topological space, and Φ a subtractive lattice of continuous functions on E, satisfying* (1. 5), *and separating the points of E. Assume $Z(\Phi) \neq \emptyset$. Then every continuous function on E which vanishes on $Z(\Phi)$ is limit of a uniformly convergent sequence taken from Φ.*

The following theorem of the Nakano type is a generalization of Theorem IV on p. 72 in [6] and is obtained from Theorem 5 by applying it to the one-point compactification of E:

THEOREM 6. *Let E be a locally compact topological space, and Φ a subtractive lattice of continuous functions on E, satisfying* (1. 5), *and separating the points of E. Assume that the functions from Φ vanish at infinity, i.e. that $f \in \Phi$, $\varepsilon > 0$ imply the existence of a compact set $K \subset E$ with $|f(x)| < \varepsilon$ for $x \in E - K$. Suppose that Φ separates the points of E from infinity, i.e. that $x \in E$ implies the existence of $f \in \Phi$ with $f(x) \neq 0$. Then each continuous function vanishing at infinity is limit of a uniformly convergent sequence taken from Φ.*

3. Let $\mathcal{U}$ be a uniformity on a set E, and $C^*(\mathcal{U})$ the family of all bounded $\mathcal{U}$-uniformly continuous real functions. Then the family $\Phi = C^*(\mathcal{U})$ clearly satisfies all conditions (1. 1) to (1. 4); moreover, Φ contains all constants, and fulfills

(3. 1) *If f is limit of a uniformly convergent sequence taken from Φ then $f \in \Phi$.*

Conversely, if Φ is either a semi-affine lattice or a subtractive lattice of bounded real functions, containing all constants, and satisfying (3. 1), then $\Phi = C^*(\mathcal{U}_\Phi)$, since $\Phi \subset C^*(\mathcal{U}_\Phi)$ by definition and $C^*(\mathcal{U}_\Phi) \subset \Phi$ by Theorem 1 or Theorem 2 (in the latter case, we again use it in the form (3)).

Hence we have proved:

THEOREM 7. *A family Φ of bounded real functions defined on a set $E \neq \emptyset$ can be written in the form $\Phi = C^*(\mathcal{U})$ with a suitable uniformity $\mathcal{U}$ on E if and only if it is either a semi-affine lattice or a subtractive lattice, contains all constants and satisfies* (3. 1).

This theorem furnishes two kinds of generalizations of Satz 6 in [6]. Thus we can characterize the families of the form $\Phi = C^*(\mathcal{U})$, in several manners, by the condition that they are composed of bounded functions and contain all constants, by (3. 1), and by a further condition $(*)$; the latter may consist of the assumption that Φ is a semi-affine lattice, or a subtractive lattice, or, in account of Satz 6 in [6], also of the condition that Φ is a ring. We shall denote by $(*)$ one of the three conditions listed here.

4. We turn to the question of establishing a connection between the characterization of families of the form $C^*(\mathcal{U})$ and of those of the form $C^*(\mathcal{T})$ where $\mathcal{T}$ is a topology and $C^*(\mathcal{T})$ denotes the collection of all bounded functions continuous with respect to $\mathcal{T}$. It is clear that the conditions $(*)$ mentioned above are necessary in order that we have $\Phi = C^*(\mathcal{T})$ with a suitable topology $\mathcal{T}$. Instead of (3. 1) another condition is necessary as well:

(4. 1) *If $f = \sup \Phi_1 = \inf \Phi_2$ where $\emptyset \neq \Phi_1 \subset \Phi$, $\emptyset \neq \Phi_2 \subset \Phi$, then $f \in \Phi$.*

In fact, if $\Phi = C^*(\mathcal{T})$, then $f = \sup \Phi_1$ $(f = \inf \Phi_2)$ implies that f is bounded from below (above) and semi-continuous from below (above).

The following theorem shows that (4. 1), together with $(*)$, characterizes the families of the form $C^*(\mathcal{T})$; similar characterizations were given in [1] and [5].

THEOREM 8. *A family Φ of bounded real functions defined on a set $E \neq \emptyset$ can be written in the form $\Phi = C^*(\mathcal{T})$ with a suitable topology $\mathcal{T}$ if and only if it contains all constants and satisfies* (4. 1) *and one of the conditions* $(*)$.

PROOF. The necessity of these conditions is proved already. Suppose they are fulfilled. Then Φ satisfies (3. 1). In fact, if $f_n \in \Phi$ and $f_n \to f$ uniformly, then there exist numbers $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ and $|f_n - f| \leqq \varepsilon_n$; clearly

$$f = \sup \{f_n - \varepsilon_n : n = 1, 2, \ldots\} = \inf \{f_n + \varepsilon_n : n = 1, 2, \ldots\},$$

hence $f \in \Phi$ by (4. 1) (each of the conditions $(*)$ implies that $f_n - \varepsilon_n \in \Phi$ and $f_n + \varepsilon_n \in \Phi$ since Φ contains all constants). Therefore there is a uniformity $\mathcal{U}$ such that $\Phi = C^*(\mathcal{U})$. Denoting by $\mathcal{T}$ the topology induced by $\mathcal{U}$, we thus have $\Phi \subset C^*(\mathcal{T})$.

On the other hand, assume $h \in C^*(\mathcal{T})$, $|h| \leqq M$. Given $x \in E$, $\varepsilon > 0$, let V be a neighbourhood of x such that

$$|h(t) - h(x)| < \varepsilon \quad \text{for} \quad t \in V.$$

Then there exists a function $f_{x,\varepsilon} \in C^*(\mathcal{U}) = \Phi$ with

$$0 \leqq f_{x,\varepsilon}(t) \leqq 1 \qquad (t \in E),$$

$$f_{x,\varepsilon}(x) = 0, \qquad f_{x,\varepsilon}(t) = 1 \qquad (t \in E - V).$$

Put

$$g'_{x,\varepsilon}(t) = h(x) - \varepsilon - 2Mf_{x,\varepsilon}(t),$$

$$g''_{x,\varepsilon}(t) = h(x) + \varepsilon + 2Mf_{x,\varepsilon}(t).$$

Then

$$g'_{x,\varepsilon} \in \Phi, \qquad g''_{x,\varepsilon} \in \Phi, \qquad g'_{x,\varepsilon} \leqq h, \qquad g''_{x,\varepsilon} \geqq h,$$

$$g'_{x,\varepsilon}(x) = h(x) - \varepsilon, \qquad g''_{x,\varepsilon}(x) = h(x) + \varepsilon.$$

Hence

$$h = \sup \{g'_{x,\varepsilon} : x \in E,\ \varepsilon > 0\} = \inf \{g''_{x,\varepsilon} : x \in E,\ \varepsilon > 0\}$$

and $h \in \Phi$ by (4. 1). Thus $C^*(\mathcal{T}) \subset \Phi$.

Let us finally turn to the characterization of families of the form $C(\mathcal{T})$ where $C(\mathcal{T})$ denotes the collection of all continuous functions with respect to the topology $\mathcal{T}$. Clearly, in order to have $\Phi = C(\mathcal{T})$, Φ has to contain all constants, fulfill $(*)$, (4. 1), and also the following condition:

(4. 2) *If* $f^{(s)} = \max\big(-s, \min(s, f)\big)$ *for every* $s > 0$, *then* $f \in \Phi$.

In fact, if $x \in E$, $\varepsilon > 0$ and $s > 0$ is chosen so that $f(x) + \varepsilon < s$, $f(x) - \varepsilon > -s$, then

$$|f^{(s)}(t) - f^{(s)}(x)| < \varepsilon$$

implies $|f(t) - f(x)| < \varepsilon$.

Conversely, we have

THEOREM 9. *A family Φ of real functions defined on a set $E \neq \emptyset$ can be written in the form $\Phi = C(\mathcal{T})$ with a suitable topology $\mathcal{T}$ if and only if it contains all constants and satisfies* (4. 1), (4. 2), *and one of the conditions* $(*)$.

PROOF. The necessity was proved already. Supposing these conditions to be satisfied, denote by Φ^* the family of all bounded functions belonging to Φ. Clearly Φ^* contains all constants and satisfies (4. 1) and $(*)$ as well. Hence $\Phi^* = C^*(\mathcal{T})$ with a suitable topology $\mathcal{T}$, by Theorem 8.

If $f \in \Phi$ then clearly $f^{(s)} \in \Phi$ for $s > 0$, so that $f^{(s)} \in \Phi^* = C^*(\mathcal{T}) \subset C(\mathcal{T})$ and $f \in C(\mathcal{T})$. Conversely, if $f \in C(\mathcal{T})$, then $f^{(s)} \in C^*(\mathcal{T}) = \Phi^* \subset \Phi$ for $s > 0$, whence $f \in \Phi$ by (4. 2). Therefore $\Phi = C(\mathcal{T})$.

We note that the question of characterizing the families of the form $C(\mathcal{U})$, with a suitable uniformity $\mathcal{U}$, is the subject of another paper [2].

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SUBHOMOGENE MITTELWERTE

Von
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1. Einleitung

Es sei I ein offenes Intervall, R die Menge der reellen Zahlen, R_+ die Menge der positiven Zahlen und E eine beliebige Menge. Wir führen die folgenden Funktionenklassen ein:

$$(1)\qquad \begin{aligned} Q(I) &= \{f | f: I \to R_+, \quad f \text{ stetig auf } I\}, \\ D(I) &= \{\varphi | \varphi: I \to R, \quad \varphi \text{ differenzierbar}, \quad \varphi'(x) \neq 0 \quad (x \in I)\}, \\ D_0(I) &= \{h | h: I \to I, \quad h \in D(I)\}, \\ D_E(I) &= \{k | k: E \times I \to I, \quad k(t, \cdot) \in D(I) \text{ für alle feste } t \in E\}. \end{aligned}$$

Liegt φ in $D(I)$, so folgt aus dem Darbouxschen Satz, daß φ streng monoton ist.

Definition. Es seien $\varphi \in D(I), f \in Q(I)$. Die Größe

$$(2)\qquad M_\varphi(x_i)_f = \varphi^{-1}\left(\frac{\sum_{i=1}^{n} f(x_i)\varphi(x_i)}{\sum_{i=1}^{n} f(x_i)}\right)$$

wird der *Mittelwert* der Zahlen $x_i \in I$ $(i = 1, 2, \dots, n)$ bezüglich der *Abbildungsfunktion* φ und der *Gewichtsfunktion* f genannt. Hier ist φ^{-1} die inverse Funktion von φ.

Der Mittelwert (2) wird *subhomogener Mittelwert bezüglich der Funktion* $k \in D_E(I)$ genannt, falls die Ungleichung

$$(3)\qquad M_\varphi\big(k(t, x_i)\big)_f \leqq k\big(t, M_\varphi(x_i)_f\big) \qquad (t \in E, \ x_i \in I, \ i = 1, 2, \dots, n; \ n = 2, 3, \dots)$$

gilt.

In dieser Arbeit werden die Ungleichung (3) und verwandte Ungleichungen untersucht. Im Abschnitt 2 beschäftigen wir uns mit den Ungleichungen:

$$(4)\qquad F(x_1 + a_1 t, \ x_2 + a_2 t, \dots, x_n + a_n t) \leqq F(x_1, x_2, \dots, x_n) + a_0 t,$$

$$(5)\qquad F(t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_n} x_n) \leqq t^{a_0} F(x_1, x_2, \dots, x_n),$$

wo $x_i \in (-\infty, \infty)$ bzw. $(0, \infty)$ $i = 1, 2, \dots, n$; $t \in (0, \infty)$ bzw. $(1, \infty)$; $a_0, a_1, \dots, a_n$ Konstanten sind. Eine Funktion, welche der Ungleichung (4) bzw. (5) genügt, kann auch *subhomogene Funktion* (bezüglich der Funktionen $x + a_i t$ bzw. $t^{a_i} x$) genannt werden. Entsprechend können die Begriffe *superhomogene* und *homogene Funktion* bzw. *Mittelwert* eingeführt werden.

Im Abschnitt 3 geben wir notwendige und hinreichende Bedingungen für die Subhomogenität von $M_\varphi(x_i)_f$.

Die Arbeit schließt mit der Untersuchung der „Potenzmittelwerte“

$$M_a(x_i)_p = \left(\frac{\sum_{i=1}^n x_i^{a+p}}{\sum_{i=1}^n x_i^p} \right)^{\frac{1}{a}} \qquad (a \neq 0,\ x_i \in (0, \infty),\ i = 1, 2, \dots, n), \tag{6}$$

$$M_0(x_i)_p = \lim_{a \to 0} M_a(x_i)_p = \exp \left(\frac{\sum_{i=1}^n x_i^p \ln x_i}{\sum_{i=1}^n x_i^p} \right) \qquad \bigl(x_i \in (0, \infty),\ i = 1, 2, \dots, n\bigr). \tag{7}$$

2. Subhomogene Funktionen

SATZ 1. *Es sei $F(x_1, x_2, \dots, x_n)$ eine auf R^n definierte stetig differenzierbare reelle Funktion, $a_0, a_1, \dots, a_n$ seien Konstanten. Die Ungleichung*

$$F(x_1 + a_1 t, x_2 + a_2 t, \dots, x_n + a_n t) \leqq F(x_1, x_2, \dots, x_n) + a_0 t \tag{4}$$

$$(x_i \in R,\ i = 1, 2, \dots, n;\ t \in R_+)$$

gilt dann und nur dann, falls die Ungleichung

$$a_1 \frac{\partial F}{\partial x_1} + a_2 \frac{\partial F}{\partial x_2} + \dots + a_n \frac{\partial F}{\partial x_n} \leqq a_0 \qquad (x_i \in R,\ i = 1, 2, \dots, n) \tag{8}$$

gilt.

BEWEIS. *Notwendigkeit.* Mit Hilfe der Taylorschen Formel erhalten wir aus (4)

$$\sum_{i=1}^n a_i \frac{\partial F}{\partial x_i} \bigg|_{(\xi_1, \xi_2, \dots, \xi_n)} \leqq a_0, \tag{9}$$

wo $\xi_i = x_i + \vartheta a_i t$ $(i = 1, 2, \dots, n)$, $0 < \vartheta < 1$ gilt. Mit $t \to 0+0$ ergibt sich (8).

Hinlänglichkeit. Wegen (8) gilt (9), also gilt auch (4).

SATZ 2. *Es sei $F(x_1, x_2, \dots, x_n)$ eine für $x_i \in R_+$ $(i = 1, 2, \dots, n)$ erklärte, positive, stetig differenzierbare, reelle Funktion; $a_0, a_1, \dots, a_n$ seien Konstanten. Die Ungleichung*

$$F(t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_n} x_n) \leqq t^{a_0} F(x_1, x_2, \dots, x_n) \tag{5}$$

$$\bigl(x_i \in R_+,\ i = 1, 2, \dots, n;\ t \in (1, \infty)\bigr)$$

gilt dann und nur dann, falls die Ungleichung

(10) $$a_1 x_1 \frac{\partial F}{\partial x_1} + a_2 x_2 \frac{\partial F}{\partial x_2} + \cdots + a_n x_n \frac{\partial F}{\partial x_n} \leqq a_0 F(x_1, x_2, \ldots, x_n)$$

$$(x_i \in R,\ i = 1, 2, \ldots, n)$$

gilt.

BEWEIS. Substituieren wir in (5) $t = e^s$, $x_i = e^{y_i}$, $\ln F(e^{y_1}, e^{y_2}, \ldots, e^{y_n}) = g(y_1, y_2, \ldots, y_n)$, dann erhalten wir

(11) $$g(y_1 + a_1 s, y_2 + a_2 s, \ldots, y_n + a_n s) \leqq g(y_1, y_2, \ldots, y_n) + a_0 s$$

$$(y_i \in R,\ i = 1, 2, \ldots, n;\ s \in R_+).$$

Durch die Anwendung des Satzes 1 ergibt sich unsere Behauptung.

BEMERKUNGEN 1. Der Satz 1 bleibt richtig, falls F eine für $x_i \in R_+$ $(i = 1, 2, \ldots, n)$ definierte Funktion und $a_0, a_1, \ldots, a_n$ nichtnegative Konstanten sind.

2. Steht $\geqq$ bzw. $=$ statt $\leqq$ in (4), (8) und in (5), (10), so gelten auch die Sätze 1, 2.

3. Ist in (4) $t \in (-\infty, 0)$ (dementsprechend in (5) $t \in (0, 1)$), so wird sich die Ungleichung (8) (bzw. (7)) umkehren. Gilt (4) für $t \in (-\infty, \infty)$ (bzw. (5) für $t \in (0, \infty)$), so muß in (4), (8) (bzw. in (5), (10)) das Gleichheitszeichen stehen.

Der Satz 2 ist eine Verallgemeinerung der Resultate von V. Alaci [2] und M. Ghermanescu [5].

3. Subhomogene Mittelwerte

SATZ 3. *Es seien $f, g \in Q(I)$, $\varphi, \psi \in D(I)$, $h \in D_0(I)$. Die Ungleichung*

(11) $$M_\varphi(h(x_i))_f \leqq h(M_\psi(x_i)_g) \qquad (x_i \in I;\ i = 1, 2, \ldots, n;\ n = 2, 3, \ldots)$$

gilt dann und nur dann, falls die Ungleichung

(12) $$\frac{\varphi(h(u)) - \varphi(h(\vartheta))}{\varphi'(h(\vartheta))} \frac{f(h(u))}{f(h(\vartheta))} \leqq \frac{\psi(u) - \psi(\vartheta)}{\psi'(\vartheta)} \frac{g(u)}{g(\vartheta)} h'(\vartheta) \qquad (u, \vartheta \in I)$$

erfüllt ist.

BEWEIS. Es sei z. B. $h'(t) > 0$, dann erhalten wir aus (11)

(13) $$M_\Phi(x_i)_F \leqq M_\psi(x_i)_g,$$

wo $\Phi(t) = \varphi(h(t)) \in D(I)$, $F(t) = f(h(t)) \in Q(I)$ gilt. Nach einem Satz von Z. DARÓCZY—L. LOSONCZI [4] besteht (13) genau dann, wenn

(14) $$\frac{\Phi(u) - \Phi(\vartheta)}{\Phi'(\vartheta)} \frac{F(u)}{F(\vartheta)} \leqq \frac{\psi(u) - \psi(\vartheta)}{\psi'(\vartheta)} \frac{g(u)}{g(\vartheta)} \qquad (u, \vartheta \in I).$$

gilt. Wegen $h'(t) > 0$ ist (14) gleichbedeutend mit (12). Im Falle $h'(t) < 0$ ist der Beweis ähnlich.

FOLGERUNG. *Es sei* $f \in Q(I)$, $\varphi \in D(I)$, $k \in D_E(I)$. *Der Mittelwert* $M_\varphi(x_i)_f$ *ist dann und nur dann subhomogen bezüglich der Funktion* $k(t, x)$, *d.h. es gilt die Ungleichung*

$$(3) \quad M_\varphi\big(k(t, x_i)\big)_f \leqq k\big(t, M_\varphi(x_i)_f\big) \qquad (t \in E,\ x_i \in I,\ i=1,2,\dots,n;\ n=2,3,\dots)$$

genau dann, falls

$$(15) \quad \frac{\varphi\big(k(t,u)\big)-\varphi\big(k(t,\vartheta)\big)}{\varphi'\big(k(t,\vartheta)\big)} \frac{f\big(k(t,u)\big)}{f\big(k(t,\vartheta)\big)} \leqq \frac{\varphi(u)-\varphi(\vartheta)}{\varphi'(\vartheta)} \frac{f(u)}{f(\vartheta)} \frac{\partial}{\partial \vartheta} k(t,\vartheta) \quad (t \in E;\ u, \vartheta \in I)$$

gilt.

BEWEIS. Führen wir — t momentan fix gehalten — die Bezeichnung $h_t(x) = k(t, x)$ ein, so wird aus (3)

$$M_\varphi\big(h_t(x_i)\big)_f \leqq h_t\big(M_\varphi(x_i)_f\big).$$

Auf Grund des Satzes 3 erhalten wir, daß (3) für ein festes $t \in E$ genau dann gilt, falls (15) für dieses t erfüllt ist. Daraus folgt unsere Behauptung.

Im Spezialfall $I = E = (0, \infty)$, $k(t, x) = t + x$ können die subhomogenen Mittelwerte einfacher charakterisiert werden als oben. Es gilt nämlich der

SATZ 4. *Es seien* $I = E = (0, \infty)$, $\varphi \in D(I)$ *zweimal,* $f \in Q(I)$ *einmal stetig differenzierbare Funktionen. Die Ungleichung*

$$(16) \quad M_\varphi(t + x_i)_f \leqq t + M_\varphi(x_i)_f \qquad \big(t, x_i \in (0, \infty),\ i=1,2,\dots,n;\ n=2,3,\dots\big)$$

gilt dann und nur dann, wenn die Ungleichung

$$(17) \quad \frac{\varphi'(u)-\varphi'(\vartheta)}{\varphi'(\vartheta)} + \left(\frac{\varphi(u)-\varphi(\vartheta)}{\varphi'(\vartheta)}\right)\left(\frac{f'(u)}{f(u)} - \frac{f'(\vartheta)}{f(\vartheta)} - \frac{\varphi''(\vartheta)}{\varphi'(\vartheta)}\right) \leqq 0 \qquad \big(u, \vartheta \in (0, \infty)\big)$$

erfüllt ist.

BEWEIS. Jetzt kann die Bedingung (15) in der Form

$$(18) \qquad G(u+t, \vartheta+t) \leqq G(u, \vartheta)$$

geschrieben werden, wobei

$$G(u, \vartheta) = \frac{\varphi(u)-\varphi(\vartheta)}{\varphi'(\vartheta)} \frac{f(u)}{f(\vartheta)}$$

gesetzt wird. Nach der Bemerkung 1 gilt die Ungleichung (18) genau dann, falls

$$(19) \qquad \frac{\partial G}{\partial u} + \frac{\partial G}{\partial \vartheta} \leqq 0 \qquad (u, \vartheta \in I)$$

ist. Wegen

$$\frac{\partial G}{\partial u} + \frac{\partial G}{\partial \vartheta} = \frac{f(u)}{f(\vartheta)} \left[\frac{\varphi'(u)-\varphi'(\vartheta)}{\varphi'(\vartheta)} + \frac{\varphi(u)-\varphi(\vartheta)}{\varphi'(\vartheta)} \left(\frac{f'(u)}{f(u)} - \frac{f'(\vartheta)}{f(\vartheta)} - \frac{\varphi''(\vartheta)}{\varphi'(\vartheta)}\right)\right],$$

erhalten wir (17) aus (19).

4. Potenzmittelwerte

In diesem Abschnitt wird die Subhomogenität der Mittelwerte $M_a(x_i)_p$ bezüglich der Funktion $k(t, x) = t+x$ untersucht. Bis jetzt wurde diese Frage nur im Spezialfall $a=1$ (Siehe E. F. Beckenbach [3]) behandelt.

SATZ 5. *Der Mittelwert $M_a(x_i)_p$ ist dann und nur dann subhomogen bezüglich der Funktion $k(t, x) = t+x$, d.h. es gilt die Ungleichung*

$$M_a(t+x_i)_p \leqq t+M_a(x_i)_p \qquad (t, x_i \in (0, \infty),\ i=1, 2, \ldots, n;\ n=2, 3, \ldots) \tag{20}$$

dann und nur dann, falls im Falle $a \geqq 0$ die Bedingung

$$p \geqq \max\{1-a, 0\}, \tag{21}$$

und im Falle $a<0$ die Bedingung

$$p \geqq \max\{1, -a\} \tag{21'}$$

erfüllt ist. Die umgekehrte Ungleichung

$$M_a(t+x_i)_p \geqq t+M_a(x_i)_p \qquad (t, x_i \in (0, \infty),\ i=1, 2, \ldots, n;\ n=2, 3, \ldots) \tag{22}$$

gilt genau dann, falls im Falle $a \geqq 0$ die Bedingung

$$p \leqq \min\{1-a, 0\}, \tag{23}$$

und im Falle $a<0$ die Bedingung

$$p \leqq \min\{1, -a\} \tag{23'}$$

erfüllt ist.

Zum Beweis brauchen wir einige elementare Ungleichungen.

LEMMA. *Es sei $J=(0, 1)\cup(1, \infty)$. Liegt a in $(0, 1)$, so gelten für $t \in J$ die Ungleichungen*

$$a-1 < \frac{1}{t-1} - \frac{a}{t^a-1}, \tag{24}$$

$$\frac{1}{t-1} - \frac{a}{t^a-1} < 0. \tag{25}$$

Liegt a in $(1, \infty)$, so gelten für $t \in J$ die Ungleichungen

$$\frac{1}{t-1} - \frac{a}{t^a-1} < a-1, \tag{26}$$

$$0 < \frac{1}{t-1} - \frac{a}{t^a-1}. \tag{27}$$

BEWEIS. Es besteht die Gleichheit

$$\frac{1}{t-1}-\frac{a}{t^a-1} = a-1+\frac{t}{(t-1)(t^a-1)}H(t) \qquad (t\in J),$$

wo $H(t) = t^a(1-a)+at^{a-1}-1$ ist. Wegen

$$H'(t) = a(a-1)t^{a-2}(1-t)$$

ist die Funktion $H(t)$ für $0<a<1$ streng monoton abnehmend in $(0, 1)$ und streng monoton wachsend in $(1, \infty)$. Auf Grund $H(0+0) = +\infty$, $H(1)=0$, $\frac{t}{(t-1)(t^a-1)}>0$ $(t\in J)$ wird

$$\frac{t}{(t-1)(t^a-1)}H(t)>0 \qquad (t\in J),$$

woraus (24) folgt.

Bei $1<a$ wird $H(t)$ in $(0, 1)$ streng monoton wachsend, in $(1, \infty)$ streng monoton abnehmend. Weil $H(0) = -1$, $H(1)=0$ gilt, so erhalten wir

$$\frac{t}{(t-1)(t^a-1)}H(t)<0 \qquad (t\in J),$$

woraus (26) folgt.

(25), (27) ergeben sich aus den bekannten (siehe [7]) Ungleichungen

$$t^a-1-a(t-1) < 0 \quad \text{für} \quad 0<a<1, \quad t\in J,$$

$$t^a-1-a(t-1) > 0 \quad \text{für} \quad a>1, \quad t\in J.$$

BEWEIS DES SATZES 5. **1.** $a>0$. Im Beweis des Satzes 4. haben wir gesehen, daß die Ungleichung (20) dann und nur dann gilt, falls

$$\text{(28)} \qquad \frac{\partial G}{\partial u}+\frac{\partial G}{\partial \vartheta} \leqq 0 \quad \text{für} \quad u, \vartheta\in(0, \infty),$$

ist, wo jetzt

$$G(u, \vartheta) = \frac{u^a-\vartheta^a}{a\vartheta^{a-1}}\frac{u^p}{\vartheta^p}$$

gesetzt wurde. $G(u, \vartheta)$ kann in der Form $G(u, \vartheta) = \vartheta l\left(\frac{u}{\vartheta}\right)$ mit $l(t) = \frac{t^{a+p}-t^p}{a}$ geschrieben werden. Es ist leicht zu sehen, daß (28) genau dann gilt, falls

$$\text{(29)} \qquad l'(t)(1-t)+l(t) \leqq 0 \qquad t\in(0, \infty)$$

erfüllt ist. Aus (29) erhalten wir

$$p \geqq 1-a+\frac{1}{t-1}-\frac{a}{t^a-1} \qquad (t\in J).$$

Diese Ungleichung gilt genau dann, falls

$$\text{(30)} \qquad p \geqq 1-a+S,$$

wo $S = \sup_{t\in J}\left(\frac{1}{t-1}-\frac{a}{t^a-1}\right)$. Für $0<a<1$ ist (wegen (25) und $\lim_{t\to\infty}\left(\frac{1}{t-1}-\frac{a}{t^a-1}\right)=0$) $S=0$; für $a=1$ ist offenbar $S=0$; während für $a<1$ (wegen (26) und $\lim_{t\to\infty}\left(\frac{1}{t-1}-\frac{a}{t^a-1}\right)=a-1$) $S=a-1$ ist. In jedem Falle erhalten wir (21) aus (30).

2. $a=0$. Genauso wie bei $a>0$ ergibt sich, daß die Ungleichung

$$p \geqq 1+\frac{1}{t-1}-\frac{1}{\ln t} \qquad (t\in J) \tag{31}$$

mit (20) äquivalent ist. Auf Grund von (25) und der Limesrelationen

$$\lim_{a\to 0}\left(\frac{1}{t-1}-\frac{a}{t^a-1}\right)=\frac{1}{t-1}-\frac{1}{\ln t}, \qquad \lim_{t\to\infty}\left(\frac{1}{t-1}-\frac{1}{\ln t}\right)=0$$

sieht man, daß (31) dann und nur dann gilt, falls $p\geqq 1$, d.h. falls (20) erfüllt ist.

3. $a<0$. Wegen der Identität

$$M_a(x_i)_p \equiv M_{-a}(x_i)_{a+p}$$

ist die Bedingung

$$a+p \geqq \max\{1-(-a), 0\},$$

d.h.

$$a \geqq \max\{1, -a\}$$

notwendig und hinreichend für (20).

Damit haben wir den ersten Teil des Satzes 5 bewiesen. Die andere Behauptung (bezüglich (22)) kann mit Hilfe der Ungleichungen (24), (27) ähnlich bewiesen werden.

Die Ergebnisse der Sätze 3, 4, 5 können für den Fall von Integralmittelwerten verallgemeinert werden. Wir beschäftigen uns hier nur mit der Verallgemeinerung des Satzes 5.

Es sei (X, S, μ) ein normierter Maßraum, in welchem zu jeder natürlichen Zahl n eine Folge $E_{1n}, E_{2n}, \dots, E_{nn}$ meßbarer Mengen existiert, welche die Bedingungen $E_{in}\cap E_{jn}=\emptyset$ $(i\neq j;\ i,j=1,2,\dots,n)$; $X=\bigcup_{i=1}^{n} E_{in}$; $\mu(E_{1n})=\mu(E_{2n})=\dots=\mu(E_{nn})$ erfüllt. Es seien a, p reelle Zahlen. Wir führen die folgende Bezeichnungen ein:

$$F_{ap}=\left\{ x(t) \;\middle|\; \begin{array}{l} x(t)\colon X\to(0,\infty), \quad x(t) \text{ messbar,} \\ x(t)^p, \begin{array}{l} x(t)^{a+p} \quad (a\neq 0) \\ x(t)^p \ln x(t) \quad (a=0) \end{array} \text{ sind integrierbar auf } X \end{array} \right\},$$

$$I_a(x(t))_p=\left(\frac{\int_X x(t)^{a+p}\,d\mu}{\int_X x(t)^p\,d\mu}\right)^{\frac{1}{a}} \qquad (a\neq 0,\ x(t)\in F_{ap}),$$

$$I_0(x(t))_p = \exp\left(\frac{\int_X x(t)^p \ln x(t)\,d\mu}{\int_X x(t)^p\,d\mu}\right) \qquad (x(t)\in F_{ap}).$$

SATZ 6. *Die Ungleichung*

$$(32) \qquad I_a(x(t)+y)_p \leqq I_a(x(t))_p + y \qquad (x(t)\in F_{ap},\ y>0)$$

gilt dann und nur dann, falls die Bedingungen (21) *bzw.* (21)′ *erfüllt sind. Die umgekehrte Ungleichung*

$$(33) \qquad I_a(x(t)+y)_p \geqq I_a(x(t))_p + y \qquad (x(t)\in F_{ap},\ y>0)$$

gilt genau dann, falls die Bedingungen (23) *bzw.* (23)′ *gelten.*

BEWEIS. *Notwendigkeit.* Setzen wir in (32) $x(t) = \sum_{i=1}^{n} x_i \chi_{E_{in}}(t)$ wo $\chi_{E_{in}}(t)$ die charakteristische Funktion von E_{in} ist und $E_{in}\cap E_{jn} = \emptyset$ $(i\neq j;\ i,j=1,2,\dots,n)$; $X = \bigcup_{i=1}^{n} E_{in}$; $\mu(E_{1n})=\mu(E_{2n})=\dots=\mu(E_{nn})$ sind. Wir erhalten

$$M_a(x_i+y)_p \leqq M_a(x_i)_p + y \qquad (x_i, y\in(0,\infty);\ i=1,2,\dots,n;\ n=2,3,\dots).$$

Diese Ungleichung ist genau (20) und auf Grund des Satzes 5 folgt die Notwendigkeit der Bedingungen (21), (21)′.

Hinlänglichkeit. Gilt (21), (21)′ so gilt (im Falle $a\neq 0$) wegen der Folgerung des Satzes 3 die Ungleichung

$$\frac{(u+y)^a-(\vartheta+y)^a}{a(\vartheta+y)^{a-1}}\,\frac{(u+y)^p}{(\vartheta+y)^p} \leqq \frac{u^a-\vartheta^a}{a\vartheta^{a-1}}\,\frac{u^p}{\vartheta^p} \qquad (u,\vartheta,y\in(0,\infty)).$$

Setzen wir hier $u=x(t)\in F_{ap}$, $\vartheta = I_a(x(t))_p$ und integrieren wir die Ungleichung auf X. Dann wird

$$(34) \qquad \frac{\int_X (x(t)+y)^{a+p}\,d\mu - (I_a(x(t))_p+y)^a \int_X (x(t)+y)^p\,d\mu}{a(I_a(x(t))_p+y)^{a+p-1}} \leqq 0.$$

Aus (34) erhalten wir die Ungleichung

$$I_a(x(t)+y) \leqq I_a(x(t))+y,$$

was zu beweisen war. Im Falle $a=0$ (bzw. (33)) verläuft der Beweis ähnlich.

BEMERKUNG BEI DER KORREKTUR: Die Sätze 1, 2 bleiben gültig auch dann, wenn die Funktion F nur (total) differenzierbar ist.

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ON DISCRETE BOREL SPACES

By

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Axiom of choice has been assumed throughout this note. Unless otherwise stated to the contrary, a set always means a non-empty set. *CH* means the continuum hypothesis. For any set X, C_X denotes the class of all subsets of X and a symbol like $C_X \times C_Y$ denotes the σ-algebra on $X \times Y$ generated by rectangles. The author has proved in [1] the following:

THEOREM 1: *Assume CH.* $C_X \times C_X = C_{X \times X}$ *iff* Card $X \leqq c$.

It is a pleasure to acknowledge the fact that this theorem has also been observed by Profs P. ERDŐS and S. M. ULAM — unpublished. Prof. JAN MYCIELSKI has informed the present author that the same was recently and independently observed by C. FERENS of Wroclaw and ROY. O. DAVIES of London — again unpublished. However the proof in all cases is the same.

In this note we extend the above theorem in two different directions.

THEOREM 2: *Assume CH.* $C_X \times C_Y = C_{X \times Y}$ *iff either both X and Y have* Card $\leqq c$, *or one of them is countable. More generally, a finite product of discrete Borel spaces is discrete iff either all factor spaces have* Card $\leqq c$ *or all but one spaces are countable.*

PROOF: Since the proof of the second sentence is similar to that of the first, we prove only the first sentence. Even here the „if" part is trivial by using theorem 1. To prove the „only if" part, suppose if possible both X and Y be uncountable and hence have Card $\geqq c$ by *CH*; and that one of them has Card $> c$ — to be more specific let Card $X > c$. Fix a cardinal α with

$$c < \alpha \leqq \text{Card } X; \quad \alpha \leqq 2^c.$$

Choose α-many distinct subsets of Y, say $(S_x;\, x \in X_0)$ indexed by a subset X_0 of X. Then the set

$$S = \bigcup_{x \in X_0} [(x) \times S_x]$$

can not belong to $C_X \times C_Y$. This proves the theorem.

Let $(X_\alpha,\, \alpha \in \Sigma)$ be a collection of sets and X be their cartesian product. A question raised by Dr. J. K. GHOSH is answered by the following theorem:

THEOREM 3: *Assume CH.* $C_X = \bigtimes_{\alpha \in \Sigma} C_{X_\alpha}$ *iff there is a finite subset F of* Σ *such that*

i) $\alpha \notin F$ *implies* X_α *is a singleton, and*

ii) *either* Card $X_\alpha \leqq c$ *for all* $\alpha \in F$ *or at most one* X_α *is uncountable.*

Here $\bigtimes_{\alpha \in \Sigma} C_{X_\alpha}$ means the σ-algebra on X generated by the finite dimensional cylinder sets. In fact, one can take it, for our theorem, to mean the σ-algebra generated by all boxes, i.e. sets of the form $\bigtimes_{\alpha \in \Sigma} B_\alpha$, where B_α is contained in X_α.

PROOF: „If" part is trivial from theorem 2. The „only if" part follows from the same theorem if we can exhibit an F satisfying i). For this observe that if infinitely many coordinate spaces are not singletons then X contains a subspace Y isomorphic to the countable product of two point spaces, and consequently can not be discrete. This proves the theorem.

Let a non-empty class B of subsets of a set Z be called an N-algebra if it is closed under complementation and unions of N-many sets; N being an infinite cardinal. A symbol like $C_X \bigtimes_N C_Y$ denotes the smallest N-algebra on $X \times Y$ generated by the rectangles. We immediately have,

THEOREM 4: Card $X \leqq N_{\alpha+1}$ *implies* $C_X \bigtimes_{N_\alpha} C_X = C_{X \times X}$.

Here N_α is an infinite cardinal number and as usual $N_{\alpha+1}$ is the smallest cardinal greater than N_α.

For $\alpha = 0$, (that is for $N_\alpha =$ the first infinite cardinal number) this is theorem 2 of [1]. The proof of this is more or less similar to that theorem and is hence omitted.

Analogues of theorems 1, 2, and 3 for N-algebras are left to the imagination of the reader.

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A FUNCTIONAL EQUATION ON A VECTOR SPACE

By

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In this paper, the cosine functional equation

$$f(x+y)+f(x-y) = 2f(x)f(y) \tag{1}$$

where f is a complex-valued function on a real vector space V is considered. It is known that [4], [6], if a complex-valued function $f(\neq 0)$ on an arbitrary group G satisfies (1) for all $x, y \in G$ and $f(x+y+z) = f(x+z+y)$ for all $x, y, z \in G$, then there is a homomorphism $g: G \to K$ (multiplication group of non-zero complex numbers) such that

$$f(x) = \frac{g(x)+g(x)^{-1}}{2}, \quad \text{for all } x \in G. \tag{2}$$

Further, if G is a locally compact topological group and f satisfying (1) on G is Haar measurable, then f is continuous. Here, we determine the solutions of (1) under the assumption that i) f is continuous along rays in V, ii) f is continuous in V. When $V = R^n$ (n positive integer $\geqq 1$), R real numbers, the condition of measurability of a solution of (1) can be weakened to the measurability on a measurable set of positive Lebesgue measure to yield the continuity of the function, thus taking the form cosh $(\alpha_1 x_1 + \cdots + \alpha_n x_n)$, α_i's complex constants, $(x_1, \ldots, x_n) \in R^n$, [3]. For similar results on a sine functional equation see [1].

From [4], it follows that if $f: V \to C$ (complex numbers) is a solution of (1) such that $\alpha = f(x_0) \neq \pm 1$ for some x_0 and β is a square root of $\alpha^2 - 1$, the corresponding homomorphism $g: V \to K$ satisfying (2) is given by

$$g(x) = \frac{1}{\beta}[f(x+x_0)+(\beta-\alpha)f(x)], \text{ for all } x \text{ in } V. \tag{3}$$

Definitions. A function $h: V \to C$ is said to be linear provided $h(\alpha x + \beta y) = \alpha h(x) + \beta h(y)$ for $x, y \in V$ and α, β real. A function $h: V \to C$ is said to be continuous along rays if for every $x \in V$, the mapping $r \to h(rx)$ is a continuous mapping of R into C. We prove the following theorems.

Theorem 1. *Let $f: V \to C$ satisfying (1) be continuous along rays. Then if f is not identically zero, there exists a linear function $L: V \to C$ such that*

$$f(x) = \cosh L(x) \quad \text{for all} \quad x \in V,$$

and conversely.

PROOF. Now there is a $g: V \to C$ satisfying (2) such that

$$g(x+y) = g(x)g(y) \quad \text{for all} \quad x, y \in V. \tag{4}$$

Now if $f(x) = \pm 1$ for all $x \in V$, then it follows from [4] that $g = f$ and so g is continuous along rays. So, let us assume that there is a y_0 such that $f(y_0) \neq \pm 1$. Let $x_0 \in V$. If $f(x_0) \neq \pm 1$, then by (3)

$$g(rx_0) = \frac{1}{\beta}\left[f((r+1)x_0) + (\beta - \alpha)f(rx_0)\right]$$

for all $r \in R$ where $\alpha = f(x_0)$ and $\beta^2 = \alpha^2 - 1$. As f is continuous along the ray determined by x_0, so is g. If $f(x_0) = \pm 1$, then $f(x+x_0) = f(x-x_0) = f(x)f(x_0) = \pm f(x)$, for all $x \in V$ (see [4]). Hence $f(y_0 - x_0) = \pm f(y_0) \neq \pm 1$. Thus $g(rx_0) = g[r(x_0 - y_0)]g(ry_0)$ for all $r \in R$ and hence g is continuous along rays. Hence, by (Lemma 1 of [1]), there is a linear functional $L: V \to C$ such that, $g(x) = \exp(L(x))$ and

$$f(x) = \frac{\exp(L(x)) + \exp(-L(x))}{2} = \cosh L(x)$$

for all $x \in V$. This completes the proof of this theorem.

THEOREM 2. Let V be a real linear topological space and $f: V \to C$ be a continuous solution of (1). Then for f not trivial, there is a continuous linear functional $L: V \to C$ such that $f(x) = \cosh L(x)$.

PROOF. The proof follows easily from theorem 1 and the fact (Lemma 2 of [1]) that when g satisfying (4) is continuous, the corresponding linear functional $L: V \to C$ is also continuous and $g(x) = \exp(L(x))$.

THEOREM 3. Let $f: V \to R$ be continuous along rays and be a solution of (1). Then there is a linear functional $L: V \to R$ such that either $f(x) = \cos L(x)$ or $f(x) = \cosh L(x)$, provided $f \not\equiv 0$.

PROOF. By theorem 1, there is a linear functional L such that $f(x) = \cosh L(x)$. Hence for $r \in R$, we have $f(rx) = \cosh rL(x)$. As f is real, we have $\cosh rL(x) = \cosh r\overline{L(x)}$, for all $r \in R$. Differentiating this equality twice with respect to r and putting $r = 0$ we get $L(x)^2 = \overline{L(x)}^2$. Thus $L(x)^2$ is real. Hence L is either real or purely imaginary. When L is real, we get $f(x) = \cosh L(x)$ and when L is purely imaginary we get $f(x) = \cos A(x)$, where $L = iA$ and A is a real linear functional on V. The proof of this theorem is thus complete.

Finally, we prove the following theorem, where μ denotes the Lebesque measure on R^n and for $A \subset R^n$ and $x \in R^n$ $A + x = \{a + x : a \in A\}$.

THEOREM 4. Let $f: R^n \to C$ be a solution of (1) such that f is measurable on a measurable subset A of positive Lebesgue measure. Then f is continuous and for $f \not\equiv 0$, $f(x) = \cosh(\alpha_1 x_1 + \dots + \alpha_n x_n)$ where $x = (x_1, \dots, x_n) \in R^n$, α_i's complex numbers.

PROOF. Without loss of generality, we can assume that $f(x) \not\equiv \pm 1$ for all $x \in R^n$. Otherwise, the corresponding homomorphism g satisfying (2) and (4) is such that

$g=f$ and hence g is measurable on A and hence g is continuous [2, p. 346]. Thus f is continuous.

Now, there exists no neighbourhood U of 0 such that $f(y) = \pm 1$ for all $y \in U$. If not, for every $x \in R^n$, there exists an integer m such that $\frac{x}{m} \in U$. Hence $f\left(\frac{x}{m}\right) = \pm 1$. By (2), $g\left(\frac{x}{m}\right) = \pm 1$. As g satisfies (4), $g(x) = \pm 1$, yielding $f(x) = \pm 1$, contradicting the assumption. Therefore, every neighborhood U of 0 contains a point x_0 such that $f(x_0) \neq \pm 1$.

It can be easily proved (analogous to the proof of Lemma 1 of [5]) that there is a neighborhood U of 0 such that $\mu(A \cap A+u) > 0$ for all $u \in U$. Hence, there is a point $x_0 \in U$ with the property that $\mu(A \cap A+x_0) > 0$, $f(x_0) \neq \pm 1$ and

$$f(x) = \frac{1}{\beta}\left[f(x+x_0)+(\beta-\alpha)f(x)\right] \text{ for all } x \in R^n.$$

As f is measurable on A, the restriction of g to $A \cap A+x_0$ is measurable and as before by [2], g is continuous. Thus the corresponding linear functional $L: R^n \to C$ is continuous and hence $L(x) = \alpha_1 x_1 + \dots + \alpha_n x_n$, where $x=(x_1, \dots, x_n) \in R^n$, α_i's complex constants. Hence $f(x) = \cosh(\alpha_1 x_1 + \dots + \alpha_n x_n)$. The proof is thus complete.

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DUALITY IN MODULES OVER PRINCIPAL IDEAL DOMAINS

By

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1. Introduction

In [2] the dual $D(E)$ of a left R-module E over a ring R with identity element $1 \neq O$ is defined as $\text{Hom}_R (E, Q)$, where Q is an injective cogenerator in the category of left R-modules. It is shown there that by taking annihilators of submodules of E in $D(E)$, one establishes an anti-monomorphism from the lattice $L(E)$ of submodules of E into the lattice $L(D(E))$ of a subgroups of $D(E)$. To obtain conditions on E under which this anti-monomorphism is an anti-isomorphism, one has to restrict the ring R or the module Q or both. In [1, pp. 94—97], BOURBAKI takes R to be a principal ideal domain which is not a field and Q the R-module K/R where K is the quotient field of R. It is then shown that if E is an R-module of finite length, then the natural homomorphism $c_E: E \to D^2(E)$, given by $c_E(x)(f) = f(x)$ for $x \in E$, $f \in D(E)$, is an isomorphism; moreover, the mapping which assigns to every submodule M of E its annihilator M° in $D(E)$ is an anti-isomorphism of the lattice $L(E)$ of the submodules of E onto the lattice $L(D(E))$ of submodules of $D(E)$. The purpose of this note is to prove the converse of these results in case R is not a valuation ring, that is, in case R has at least two non-associated prime elements; and to show that the condition that $c_E: E \to D^2(E)$ is an isomorphism is equivalent to the condition that $L(E)$ and $L(D(E))$ are anti-isomorphic.

2. Definitions and preliminary theorems

All modules that are considered are over a principal ideal domain R having at least two non-associated prime elements. K denotes the quotient field of R. The *dual* $D(E)$ of a module E is the module $\text{Hom}_R (E, K/R)$. If M is a submodule of E, the *annihilator* of M in $D(E)$ is the submodule M° of $D(E)$ consisting of all $f \in D(E)$ such that $f|M = O$. If M is a submodule of $D(E)$, its *annihilator* in E is the submodule M° of E consisting of all $x \in E$ such that $f(x) = O$ for all $f \in M$. A submodule M of E or of $D(E)$ is said to be *closed* if the annihilator $M^{\circ\circ}$ of M° is equal to M.

The following two properties of annihilators are consequences of the definitions and the injectivity of K/R.

I. If M is a submodule of a module E, then $D(E/M) \cong M^\circ$ and $D(M) \cong \cong D(E)/M^\circ$.

II. If M and N are submodules of E, then $(M+N)^\circ = M^\circ \cap N^\circ$ and $(M \cap N)^\circ = = M^\circ + N^\circ$.

The identification of $D(E/M)$ with M° and $D(M)$ with $D(E)/M^\circ$ is as follows: if $f \in M^\circ$ and $M + x \in E/M$, then $f(M+x) = f(x)$; if $M^\circ + f \in D(E)/M^\circ$ and $x \in M$, then $(M^\circ + f)(x) = f(x)$. This identification is independent of the choice of the representatives.

The fact that K/R is an injective cogenerator implies:

III. Every submodule of a module E is closed.

Note that a submodule N of $D(E)$ is closed if and only if there exists a submodule M of E such that $M^\circ = N$. Hence II and III imply that the assignment $M \to M^\circ$ is an anti-isomorphism from the lattice $L(E)$ of the submodules of E onto the lattice $\bar{L}(D(E))$ of the closed submodules of $D(E)$. A module E is said to be *dualizable* if $\bar{L}(D(E)) = L(D(E))$; that is if and only if $M^{\circ\circ} = M$ for every submodule M of $D(E)$. The theorem we want to prove is that a module E is dualizable if and only if it is of finite length.

If π is a prime element of R, then the submodule consisting of all $x \in K/R$ such that $\pi^n x = O$ for some integer $n \geqq 0$ will be denoted by U_π.

We recall a few facts about principal ideal domains and their modules which will be used without explicit reference.

(1) Every principal ideal domain of finite length is a field.

(2) If A is an ideal of a principal ideal domain R which is not a field, then R/A is of finite length if and only if $A \neq O$.

(3) Let M_π be a primary component of a torsion module E, and for every $x \in M_\pi$ let $n(x)$ be the smallest positive integer such that $\pi^{n(x)} x = O$. If $\{n(x) | x \in M_\pi\}$ is bounded, then M_π is isomorphic to a direct sum of cyclic modules of finite length. If $\{n(x) | x \in M_\pi\}$ is not bounded, then U_π is an epimorphic image of M_π.

3. THEOREM. *A module E is dualizable if and only if it is of finite length.*

The proof of the theorem will be preceded by four lemmas.

LEMMA 1. *If for a module E, the natural homomorphism $c_E: E \to D^2(E)$ is an isomorphism, then E is dualizable.*

PROOF. Let M be a submodule of $D(E)$, M' its annihilator in $D^2(E)$ and $N = c_E^{-1}(M')$. If $f \in M$, then $f(x) = c_E(x)(f) = O$ for every $x \in N$ and hence $f \in N^\circ$ and therefore $M \subseteqq N^\circ$. Let $f \in N^\circ$ and $f \notin M$. Since K/R is an injective cogenerator, there exists $h \in D^2(E)$ such that $h \in M'$ and $h(f) \neq O$. There exists $x \in N$ such that $c_E(x) = h$. Then $h(f) = c_E(x)(f) = f(x) = O$ which is a contradiction. Hence $M = N^\circ$ and therefore $M^{\circ\circ} = M$.

LEMMA 2. *A cyclic module is dualizable if and only if it has finite length.*

PROOF. Let E be a cyclic module of finite length. Then $E \cong D(E)$ and $c_E: E \to D^2(E)$ is an isomorphism [1, p. 95]. By lemma 1, E is dualizable.

Conversely, suppose that E is a cyclic module of infinite length. E is isomorphic to a quotient module R/A. Since the principal ideal domain R is a unique factorization domain, R/A is of infinite length if and only if $A = 0$. Hence $E \cong R$ and $D(E) \cong$ $\cong K/R$. If $O \neq M$ is a submodule of E, then it follows from what we just proved that E/M is a cyclic module of finite length. Hence $E/M \cong D(E/M)$. But by property I of § 2, M° is then of finite length for every submodule $M \neq O$ of E. On the other hand, since R has at least two non-associated prime elements, K/R has proper submodules which are not of finite length, namely U_π where π is a prime element of R. Thus E is not dualizable.

LEMMA 3. *U_π is not dualizable for any prime element π of R.*

PROOF. Let F be the cyclic submodule of $D(U_\pi)$ generated by the inclusion homomorphism $f: U_\pi \to K/R$. Then $F^\circ = O = D(U_\pi)^\circ$. Since $F \neq D(U_\pi)$, U_π is not dualizable.

LEMMA 4. *Every submodule and every quotient module of a dualizable module is dualizable.*

PROOF. Let F be a submodule of a dualizable module E and M a submodule of $D(F)$. Denote by M' the annihilator of M in F. We have to show that $M'' = M$. It is clear that $M \subseteq M''$. Let $f \in M''$ and N/F° the image of M under the isomorphism $D(F) \cong D(E)/F^\circ$. Then $M' = N^\circ \cap F$ and there exists $f' \in D(E)$ such that $f'|F = f$. Hence $f'|N^\circ \cap F = f'|M' = O$, that is $f' \in (N^\circ \cap F)^\circ$. By property II of § 2 and $N^{\circ\circ} = N$, we have $f' \in N + F^\circ = N$ and this implies $f'|F \in M$, that is $f \in M$. Hence $M'' = M$.

If E/F is a quotient module of E and M a submodule of $D(E/F)$ let N be the image of M under the isomorphism $D(E/F) \cong F^\circ$. Then $M' = N^\circ/F$. If $f \in M''$, then there exists $f' \in F^\circ$ such that $f(F+x) = f'(x)$ for every coset $F+x$ in E/F. In particular, $f(F+x) = f'(x) = O$ for all $x \in N^\circ$. Hence $f' \in N^{\circ\circ} = N$ and therefore $M'' = M$.

PROOF OF THE THEOREM. If E is of finite length, then $c_E: E \to D^2(E)$ is an isomorphism [1, p. 95]. By lemma 1, E is dualizable.

To prove the necessity of the condition, we show that if E is of infinite length then E is not dualizable. If E is not a torsion module, then it contains a submodule isomorphic to R. By lemmas 2 and 4, E is not dualizable. Suppose that E is torsion and let $E = \oplus M_\pi$ be a direct sum decomposition of E into its primary components. If the primary components are infinite in number, then $F = \oplus D(M_\pi)$ is a proper submodule of $D(E) = \Pi D(M_\pi)$ and $F^\circ = O = D(E)^\circ$ and hence E is not dualizable. Suppose now that the primary components are finite in number, then at least one component M_π is of infinite length. For every element $x \in M_\pi$ there exists a least integer $n(x) > 0$ such that $\pi^{n(x)} x = O$. If the set $\{n(x) | x \in M_\pi\}$ is bounded, then M_π is a direct sum of an infinite number of cyclic modules of finite length. Using the same argument as above, one shows that M_π and hence E is not dualizable. If the set $\{n(x) | x \in M_\pi\}$ is not bounded, then U_π is an epimorphic image of M_π. By lemmas 3 and 4, M_π and consequently E is not dualizable. Thus if E is dualizable, E must be of finite length.

COROLLARY 1. *$c_E: E \to D^2(E)$ is an isomorphism if and only if E is dualizable.*

COROLLARY 2. *If M is a submodule of a module E of finite length, then there exists a submodule N of E such that E/M is isomorphic to N and E/N is isomorphic to M.*

PROOF. Since E is of finite length, there exists an isomorphism $\tau: D(E) \to E$. Let $N = \tau(M^\circ)$. Then N has the desired properties.

REMARK 1. The hypothesis that R should have at least two non-associated prime elements cannot be relaxed. For if R is a valuation ring, then the ring R is dualizable and R is of infinite length.

REMARK 2. If R is the ring of integers Z, then the Z-modules of finite length are the finite abelian groups, and K/R is the group Q/Z of rationals mod 1. Thus an abelian group is dualizable if and only if it is finite.

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ON THE DISTRIBUTION OF PRIME NUMBERS WHICH ARE OF THE FORM "x^2+y^2+1" II

By

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1. In the previous paper [5] we have studied the number $I(N)$ of prime numbers p, not exceeding N, which are representable in the form x^2+y^2+1. Combining the HOOLEY—LINNIK theorem [2] [4]

$$\sum_{\substack{p-1=n^2+m^2\leqq N\\ p,n,m}} 1 = (1+o(1))\frac{N}{\log N}\prod_{p\equiv 1(\mathrm{mod}\ 4)}\left(1+\frac{1}{p(p-1)}\right)\prod_{p\equiv -1(\mathrm{mod}\ 4)}\left(1-\frac{1}{p(p-1)}\right)$$

with the argument of BRUN, we have obtained the estimate

$$I(N)\geqq c_0 N(\log N)^{-2}$$

with an absolute constant c_0 (hereafter $c_1, c_2, \dots$ denote absolute constants). And we have conjectured the asymptotic-formula

$$I(N)=(1+o(1))\mathfrak{S}N(\log N)^{-\frac{3}{2}},$$

where the constant $\mathfrak{S}$ has a rather complicated structure.

The purpose of this note is to prove the upper estimate of $I(N)$ which seems to be best possible. The result obtained is as follows;

THEOREM. *The inequality*

$$I(N)\leqq c_1 N(\log N)^{-\frac{3}{2}}$$

holds.

2. Now let Ω be the set of all integers whose prime divisors are all congruent to -1 mod 4, and Ω' be the set of all integers whose prime divisors are 2 or congruent to $+1$ mod 4.

Then we have

$$I(N)=\sum_{p-1=g^2h\leqq N,\ g\in\Omega,\ h\in\Omega'} 1=\sum_{\substack{p-1=g^2h\leqq N\\ g\in\Omega,\ g\leqq N^{\frac{1}{10}},\ h\in\Omega'}} 1+\sum_{\substack{p-1=g^2h\leqq N\\ g\in\Omega\ g>N^{\frac{1}{10}},\ h\in\Omega'}} 1=\sum\nolimits_1+\sum\nolimits_2.$$

The sum Σ_2 is easily estimated as follows;

$$\sum\nolimits_2=O\Big\{\sum_{\substack{p-1=g^2h\leqq N\\ g>N^{\frac{1}{10}}}} 1\Big\}=O\Big\{\sum_{N^{\frac{1}{10}}<g\leqq\sqrt{N}}\ \sum_{\substack{p\equiv 1\,(\mathrm{mod}\ g^2)\\ p\leqq N}} 1\Big\}=O\Big\{\sum_{N^{\frac{1}{10}}<g\leqq\sqrt{N}}\frac{N}{g^2}\Big\}=O(N^{\frac{9}{10}}).$$

(1)

3. Let us consider the sum Σ_1. We have

$$\sum\nolimits_1 = \sum_{g\in\Omega,\, g\leqq N^{\frac{1}{10}}} \sum_{\substack{p\equiv 1 \,(\mathrm{mod}\, g^2),\ \frac{p-1}{g^2}\in\Omega' \\ p\leqq N}} 1 = \sum_{g\in\Omega,\, g\leqq N^{\frac{1}{10}}} P_g(N), \text{ say.}$$

To $P_g(N)$ we will apply the sieve of A. SELBERG [6].

Now let λ_n be a sequence such that

$$\lambda_n \text{ is } \begin{cases} 1 \text{ if } n=1, \\ \text{arbitrary if } 1<n\leqq N^{\frac{1}{10}},\ n\in\Omega,\ (n,g)=1, \\ 0 \text{ otherwise.} \end{cases}$$

Then obviously we have

$$P_g(N) \leqq \sum_{\substack{p\equiv 1(\mathrm{mod}\, g^2) \\ p\leqq N}} \Big\{ \sum_{n \mid \frac{p-1}{g^2}} \lambda_n \Big\}^2.$$

And by an easy modification of SELBERG's argument, we obtain

$$P_g(N) \leqq \frac{\mathrm{li}\, N}{\varphi(g^2)} \left\{ \sum_{\substack{n\leqq N^{\frac{1}{10}} \\ n\in\Omega,\ (n,g)=1}} \frac{\mu^2(n)}{\varphi(n)\prod_{p\mid n}\left(1-\frac{1}{\varphi(p)}\right)} \right\}^{-1} + \tag{2}$$

$$+ O\left\{ \sum_{n_1, n_2\leqq N^{\frac{1}{10}}} |\lambda_{n_1}, \lambda_{n_2}| \left| \pi(N; g^2[n_1, n_2], 1) - \frac{\mathrm{li}\, N}{\varphi(g^2[n_1, {}_2])} \right| \right\},$$

where

$$\lambda_n = \mu(n) \prod_{p\mid n}\left(1-\frac{1}{\varphi(p)}\right)^{-1} \left\{ \sum_{\substack{m\leqq N^{\frac{1}{10}}/n \\ (n,m)=1,\ m\in\Omega,\ (n,g)=1}} \frac{\mu^2(m)}{\varphi(m)\prod_{p\mid m}\left(1-\frac{1}{\varphi(p)}\right)} \right\} \times \tag{3}$$

$$\times \left\{ \sum_{n\leqq N^{\frac{1}{10}},\ n\in\Omega,\ (n,g)=1} \frac{\mu^2(n)}{\varphi(n)\prod_{p\mid n}\left(1-\frac{1}{\varphi(p)}\right)} \right\}^{-1} = O(\log N)$$

and $[n_1, n_2]$ denotes the least common multiple of n_1 and n_2, and $\pi(N; k, l)$ is the number of prime numbers, not exceeding N, in the arithmetic progression $\equiv l$ modulo k.

4. Since the number of solutions of the equation

$$g^2[n_1, n_2] = k$$

is less than $\tau_4(k)$, that is, the number of representations of k as the product of four factors, we have, noting the equality (3),

$$\sum_{g \leqq N^{\frac{1}{10}}} \sum_{n_1, 2 \leqq N^{\frac{1}{10}}} |\lambda_{n_1} \lambda_{n_2}| \left| \pi(N; g^2[n_1, n_2], 1) - \frac{\operatorname{li} N}{\varphi(g^2[n_1, n_2])} \right| \leqq$$

$$\leqq c_2 (\log N)^2 \sum_{k \leqq N^{\frac{2}{5}}} \tau_4(k) \left| \pi(N; k, 1) - \frac{\operatorname{li} N}{\varphi(k)} \right| =$$

$$= c_2 (\log N)^2 \Bigg\{ \sum_{\substack{k \leqq N^{\frac{2}{5}} \\ \tau_4(k) \leqq (\log N)^{20}}} + \sum_{\substack{k \leqq N^{\frac{2}{5}} \\ \tau_4(k) > (\log N)^{20}}} \Bigg\} \tau_4(k) \left| \pi(N; k, 1) - \frac{\operatorname{li} N}{\varphi(k)} \right| =$$

$$= c_2 (\log N)^2 \{ \textstyle\sum_3 + \sum_4 \}, \text{ say.}$$

Now by the mean value theorem of Bombieri [1], it is easy to see that the inequality

$$\textstyle\sum_3 \leqq c_3 N (\log N)^{-20} \tag{4}$$

holds. Also by the theorem of Brun—Titchmarsh, we have

$$\sum_4 \leqq c_4 \sum_{k \leqq N^{\frac{2}{5}}} \frac{\tau_4(k)}{(\log N)^{20}} \cdot \frac{\tau_4(k)}{\varphi(k)} \cdot \frac{N}{\log N} \leqq c_4 N (\log N)^{-20} \sum_{k \leqq N^{\frac{2}{5}}} \frac{\tau_4^2(k)}{k} \leqq c_5 N (\log N)^{-4} \tag{5}$$

Hence we find from (4) and (5)

$$\sum_{g \leqq N^{\frac{1}{10}}} \sum_{n_1, n_2 \leqq N^{\frac{1}{10}}} |\lambda_{n_1} \lambda_{n_2}| \left| \pi(N; g^2[n_1, n_2], 1) - \frac{\operatorname{li} N}{\varphi(g^2[n_1, n_2])} \right| = O\big(N (\log N)^{-2}\big). \tag{6}$$

5. In order to calculate the sum of the main-term of the right side of (2), let us consider the function $(\operatorname{Re}(s) > 0)$

$$U_g(s) = \sum_{n \in \Omega, (n, g) = 1} \frac{\mu^2(n)}{n^s \varphi(n) \prod_{p|n} \left(1 - \frac{1}{\varphi(p)}\right)} =$$

$$= \prod_{p|g} \left(1 + \frac{1}{p^s(p-2)}\right)^{-1} \prod_{p \equiv -1 \pmod 4} \left(1 + \frac{2}{(p^{s+1}+1)(p-2)}\right) \prod_{p \equiv 1 \pmod 4} \left(1 + \frac{1}{p^{s+1}}\right).$$

Defining $L(s)$ by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \qquad (\operatorname{Re}(s) > 1),$$

we have

$$\left\{ \prod_{p \equiv -1 \pmod 4} \left(1 + \frac{1}{p^{s+1}}\right) \right\}^2 = \left(1 - \frac{1}{2^{s+1}}\right) \prod_{p \equiv -1 \pmod 4} \left(1 - \frac{1}{p^{2(s+1)}}\right) \frac{\zeta(s+1)}{L(s+1)},$$

where $\zeta(s)$ is the Riemann zeta-function.

Hence we have

$$\{U_g(s)\}^2 = \prod_{p|g}\left(1+\frac{1}{p^s(p-2)}\right)^{-2}\psi(s)\,\frac{\zeta(s+1)}{L(s+1)},$$

where $\psi(s)$ is a Dirichlet series which converges absolutely for $\mathrm{Re}\,(s) > -\frac{1}{2}$.

Therefore as in [3] we have

$$(7)\qquad \sum_{n\leqq N^{\frac{1}{10}},\, n\in\Omega,\, (n,g)=1} \frac{\mu^2(n)}{\varphi(n)\prod_{p|n}\left(1-\frac{1}{\varphi(p)}\right)} = (1+o(1))B\prod_{p|g}\left(1+\frac{1}{p-2}\right)^{-1}(\log N)^{\frac{1}{2}},$$

where B is an absolute constant which has a complicated expression.

6. Finally, collecting the results (1), (6) and (7), we obtain

$$I(N) \leqq c_6 N(\log N)^{-\frac{3}{2}} \sum_{g\leqq N^{\frac{1}{10}},\, g\in\Omega} \frac{1}{\varphi(g^2)}\prod_{p|g}\left(1+\frac{1}{p-2}\right) + O(N(\log N)^{-2}) \leqq$$

$$\leqq c_1 N(\log N)^{-\frac{3}{2}}.$$

This completes the proof of the theorem.

REMARK: Very similarily we can prove the following: Let $Q(\sqrt{D})$ be the quadratic number field with the discriminante D. Then the number $I_D(N)$ of prime numbers, not exceeding N, which are of the form

$$1+N(\mathfrak{A})$$

where $N(\mathfrak{A})$ is the norm of the integral ideal $\mathfrak{A}$ of $Q(\sqrt{D})$, satisfies the inequality

$$c_1(D)N(\log N)^{-2} \leqq I_D(N) \leqq c_2(D)N(\log N)^{-\frac{3}{2}}.$$

Here $c_1(D)$ and $c_2(D)$ are constants depending only on D.

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A COMMUTATIVITY THEOREM FOR PRIMARY RINGS

By

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In a recent paper [4], JOHNSEN, OUTCULT, and YAQUB have proved that a ring R with unity element which satisfies the identity $(xy)^2 = x^2y^2$ for all x, y in R is commutative. They also have shown by an example that, for any integer $k > 2$, there exists a noncummutative ring R with unity element satisfying the identity $(xy)^k = x^k y^k$ for all $x, y \in R$. These results are analogous to that in group theory. A well-known result in group theory says that a group which satisfies the identity $(xy)^k = x^k y^k$ for three consecutive integers k is necessarily abelian [2, p. 31]. The question therefore naturally arises: Whether the ring-theoretic analogue of this group-theoretic result is also valid? This note initiates an investigation of this problem. We shall give an affirmative answer of this question for primary rings.

Let R be a ring with unity element and J be its Jacobson radical. R is called primary if R/J is a simple ring (not necessarily artinian). R is called completely primary if R/J is a division ring. Clearly every completely primary ring is primary.

THEOREM. *If R is a primary ring which satisfies the identities*

$$(A) \qquad (xy)^k = x^k y^k, \quad k = n, n+1, n+2,$$

where n is a non-negative integer, then R is commutative.

We begin with

LEMMA 1. *Let R be a ring with unity element* 1. *If R satisfies the identities* (A), *then the Jacobson radical J of R is commutative.*

PROOF. Set $S = \{1 - x\colon x \in J\}$. It can be readily verified that S forms a multiplicative group having 1 as the unity element. Since S satisfies the identities (A), S is abelian. For any $x, y \in J$, $(1-x)(1-y) = (1-y)(1-x)$, and hence $xy = yx$.

LEMMA 2. *Let R be a ring with Jacobson radical J. If R/J is commutative, then, for $x \in R$, $x^n \in J$ implies $x \in J$.*

PROOF. For any $r \in R$, since $(xr)^n \equiv x^n r^n \equiv 0 \pmod J$, $(xr)^n$ is right quasi-regular. Consequently, xr is right quasi-regular and $x \in J$.

LEMMA 3. *Let R be a completely primary ring with Jacobson radical J. If $x \in R$ and $x \notin J$, then there exists $u \in R$ such that $ux = xu = 1$.*

PROOF. Since R/J by assumption is a division ring, there exists $y \in R$ such that $yx \equiv 1 \pmod J$, so $yx = 1 - s$ for some $s \in J$. By the quasi-regularity of s, yx is invertible. Hence there exists $u \in R$ such that $ux = 1$. Similarly $xv = 1$ for some $v \in R$.

It remains to show that $u=v$. Indeed, $u = u(1-v)+uv = u(x-1)v+uv = (1-u)v+uv = v$.

Lemma 4. (Herstein.) *If R is a ring in which $(xy)^k=x^ky^k$ for all $x, y\in R$ and a fixed integer $k>1$, then every commutator in R is nilpotent and the nilpotent elements of T form an ideal of R.*

Proof (see [1]).

Proof of the theorem. We assume that R is a primary ring and R satisfies the identities (A). By Lemma 4, R/J is commutative and hence it is a subdirect sum of fields. Since R/J is simple, R/J is a field, i.e. R is completely primary. Now let x and y be arbitrary elements in R. We shall show that $xy=yx$.

Case I. If $x, y\in J$, $xy=yx$ by Lemma 1.

Case II. If $x, y\notin J$, from the first two identities of (A),

$$x^{n+1}y^{n+1}=(xy)^{n+1}=(xy)^n(xy)=x^ny^nxy.$$

By Lemma 2, $x^n\notin J$. By Lemma 3, there exist $u, v\in R$ such that $x^nu=ux^n=yv=vy=1$. Thus we have

$$\text{(B)} \qquad xy^n=ux^nxy^nyv=ux^{n+1}y^{n+1}v=ux^ny^nxyv=y^nx.$$

Likewise, from the last two identities of (A), we obtain

$$\text{(C)} \qquad xy^{n+1}=y^{n+1}x.$$

From (B) and (C), it follows that

$$y^{n+1}x=xy^{n+1}=xy^ny=y^nxy.$$

Multiplying the equation by v^n from left, we obtain $yx=xy$.

Case III. If $x\in J$, $y\notin J$, then $1-x\notin J$. By the result of Case II, $(1-x)y = y(1-x)$, so

$$xy=yx.$$

Therefore, $xy=yx$ for all $x, y\in R$ and the theorem is now established.

We note that the hypothesis of the existence of the unity element in our theorem is essential.

Example 1. Let R be the ring of two by two matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, where a, b are real numbers. It is easy to see that R has no unity element, R/J is isomorphic to the field of real numbers and R satisfies the identies (A) for all $n\geqq 1$. However, R is not commutative.

We further note that the conclusion of the theorem does not follow if we assume identity $(xy)^k=x^ky^k$ for just two consecutive integers.

Example 2. Let S be a non-commutative ring of characteristic 3 such that $S^3=0$. Let R be the ring consisting of elements (a, n), where $a\in S$, $n\in Z_3$, the ring of integers mod 3. The addition and multiplication in R are defined by

$$(a, n)+(b, m) = (a+b, n+m)$$

and $(a, n)(b, m) = (ab+nb+ma, nm)$. Then the Jacobson radical J of R consists of all elements $(a, 0)$ where $a \in S$, and $R/J \cong Z_3$, so R is a completely primary ring. It can be verified easily that $(xy)^3 = x^3y^3$ and $(xy)^4 = x^4y^4$ for all $x, y \in R$. However, R is not commutative.

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ÜBER DIE ABSOLUTE SUMMIERBARKEIT DER ORTHOGONALREIHEN

Von

K. TANDORI (Szeged)

Herrn Professor PAUL TURÁN *zum 60. Geburtstag gewidmet*

1. In vorigen Arbeiten ist es gelungen, notwendige und hinreichende Bedingungen für die Konvergenz, unbedingte Konvergenz, Summierbarkeit und für die sehr starke Summierbarkeit der allgemeinen Orthogonalreihen anzugeben [2], [4]; die entsprechenden Bedingungen erfordern die Endlichkeit des Supremums gewisser Integralen. In dieser Arbeit werden wir eine ähnliche Betrachtung für die absolute Summierbarkeit vorführen.

2. Es sei $T=\|t_{i,k}\|_{i,k=0}^{\infty}$ eine Toeplitzsche Matrix, für die die Regularitätsbedingungen

$$1^0 \quad \lim_{i\to\infty} t_{i,k}=0 \qquad (k=0,1,\ldots),$$

$$2^0 \quad \lim_{i\to\infty} \sum_{k=0}^{\infty} t_{i,k} = 1,$$

$$3^0 \quad \sum_{k=0}^{\infty} |t_{i,k}| \leqq K \qquad (i=0,1,\ldots)$$

erfüllt werden.

Es sei weiterhin $\{a_n\}_{n=0}^{\infty}$ eine reelle Koeffizientenfolge und $\{\varphi_n(x)\}_{n=0}^{\infty}$ ein im Intervall (0, 1) orthonormiertes Funktionensystem (kurz ON-System). (Im folgenden — wenn anderes nicht erwähnt wird — nehmen wir an, daß das Orthogonalitätsintervall das Intervall (0, 1) ist.) Die k-te Partialsumme der Reihe

$$\sum_{n=0}^{\infty} a_n\varphi_n(x) \tag{1}$$

bezeichnen wir mit $s_k(x)$. Wir setzen

$$t_i(x) = \sum_{k=0}^{\infty} t_{i,k}s_k(x) \qquad (i=0,1,\ldots), \tag{2}$$

und $t_{-1}(x)\equiv 0$. Die Reihe (1) nennen wir absolut T-summierbar, oder kurz $|T|$-summierbar im Punkt x_0, wenn die Reihen (2) für jedes i im Punkt x_0 konvergieren und

$$\sum_{i=0}^{\infty} |t_i(x_0)-t_{i-1}(x_0)| < \infty$$

besteht.

Wir wollen die Menge $M(|T|)$ derjenigen Koeffizientenfolgen $\{a_n\}$ charakterisieren, für die die Reihe (1) für jedes ON-System $\{\varphi_n(x)\}$ fast überall $|T|$-summierbar ist. Da aus der $|T|$-Summierbarkeit der Reihe (1) in einem Punkt x die Existenz von $\lim_{i\to\infty} t_i(x)$, d.h. die T-Summierbarkeit im Punkt x folgt, und nach einem Satz von A. ZYGMUND [6] aus der T-Summierbarkeit der Rademacherschen Reihe $\Sigma a_n r_n(x)$ in einer Menge vom positiven Maß $\{a_n\}\in l^2$ (d.h. $\Sigma a_n^2<\infty$) sich ergibt, gilt $M(|T|)\subseteqq l^2$. So können wir im folgenden $\{a_n\}\in l^2$ annehmen.

Für eine angegebene Koeffizientenfolge $\{a_n\}$ setzen wir

$$\|\{a_n\};|T|\| = \sup_{\{\varphi_n\}} \int_0^1 \left(\sum_{i=0}^{\infty} |t_i(x)-t_{i-1}(x)|\right) dx = \lim_{N\to\infty} \sup_{\{\varphi_n\}} \int_0^1 \left(\sum_{i=0}^{N} |t_i(x)-t_{i-1}(x)|\right) dx,$$

wobei das Supremum für jedes ON-System gebildet ist. (Im folgenden bedeutet $\sup\limits_{\{\varphi_n\}}$ immer, daß das Supremum für jedes ON-System gebildet werden soll.)

Das Hauptresultat dieser Arbeit lautet folgenderweise:

SATZ. *$\{a_n\}\in M(|T|)$ gilt dann und nur dann, wenn $\|\{a_n\};|T|\|<\infty$ ist.*

Nach dem Satz von B. LEVI folgt aus $\|\{a_n\};|T|\|<\infty$, daß die Reihe (1) für jedes ON-System $\{\varphi_n(x)\}$ fast überall $|T|$-summierbar ist. Nur die entgegengesetzte Behauptung soll es beweisen. Wir werden genauer folgendes zeigen:

Gilt $\|\{a_n\};|T|\|=\infty$, dann gibt es ein ON-System $\{\Phi_n(x)\}$ derart, daß die Reihe

$$\Sigma a_n \Phi_n(x) \tag{3}$$

fast überall nicht $|T|$-summierbar ist.

3. BEMERKUNG. Es ist leicht zu beweisen, daß *$M(|T|)$ mit der Norm $\|\{a_n\};|T|\|$ und mit den gewöhnlichen vektoriellen Operationen ein Banachraum ist.*

$M(|T|)$ ist mit den gewöhnlichen vektoriellen Operationen offensichtlich ein linearer Raum. Nach unserem Satz gilt $\|\{a_n\};|T|\|<\infty$ für $\{a_n\}\in M(|T|)$; $\|\{a_n\};|T|\|\geqq 0$, $\|\{aa_n\};|T|\| = |a|\,\|\{a_n\};|T|\|$ und $\|\{a_n+b_n\};|T|\| \leqq \|\{a_n\};|T|\|+\|\{b_n\};|T|\|$ ($\{a_n\}$, $\{b_n\}\in(|T|)$, a beliebige reelle Zahl) sind auch offensichtlich. Nach der Definition von $\|\{a_n\};|T|\|$ gilt weiterhin

$$\int_0^1 |t_i^*(x)|dx \leqq \sup_{\{\varphi_n\}} \int_0^1 |t_i(x)|dx \leqq \|\{a_n\};|T|\| \tag{4}$$

für jedes i, wobei

$$t_i^*(x) = \sum_{k=0}^{\infty} t_{i,k}\big(a_0 r_0(x)+\ldots+a_k r_k(x)\big) = \sum_{n=0}^{\infty} T_{i,n} a_n r_n(x) \tag{5}$$

und $T_{i,n} = \sum\limits_{k=n}^{\infty} t_{i,k}$, $T_{-1,n}=0$ sind. (Die Reihe auf der rechten Seite konvergiert in der Metrik von L^2 zu der Summe auf der linken Seite.) Nach einem bekannten Satz (s. z. B. [7], S. 213.) gilt aber

$$\int_0^1 \left|\sum_{n=0}^{\infty} T_{i,n} a_n r_n(x)\right|^2 dx \geqq A\sqrt{\sum_{n=0}^{\infty} T_{i,n}^2 a_n^2} \tag{6}$$

mit einer positiven Konstante A. Aus (4), (5) und (6), auf Grund von 1° und 2° folgt aber

$$\|\{a_n\};|T|\| \geqq A\left(\sum_{n=0}^{\infty} a_n^2\right)^{1/2}. \tag{7}$$

Daraus folgt, daß $\|\{a_n\};|T|\|=0$ dann und nur dann besteht, wenn $a_n=0$ $(n=0, 1, \ldots)$ gilt. Auf Grund von (7) mit einer in [4] angewandten Methode kann man die Vollständigkeit von $M(|T|)$ zeigen.

BEMERKUNG II. Für gewisse spezielle Summationsverfahren ist dieser Satz schon bekannt. Mit $\sigma_i(x)$ bezeichnen wir das i-te $(C, 1)$-Mittel der Reihe (1). In einer vorigen Arbeit [5] wurde gezeigt, daß

$$\int_0^1\left(\sum_{i=0}^{\infty}|\sigma_i(x)-\sigma_{i-1}(x)|\right)dx \leqq K_1\left(|a_0|+|a_1|+\sum_{m=0}^{\infty}A_m\right)$$

für jedes ON-System $\{\varphi_n(x)\}$ besteht, wobei K_1 eine positive Konstante und $A_m^2 = a_{2^m+1}^2+\cdots+a_{2^{m+1}}^2$ $(m=0, 1, \ldots)$ sind. Weiterhin gibt es ein ON-System $\{\Phi_n(x)\}$ derart, daß für die $(C, 1)$-Mittel $\sigma_i^*(x)$ der Reihe $\Sigma a_n\Phi_n(x)$

$$\sum_{i=0}^{\infty}|\sigma_i^*(x)-\sigma_{i-1}^*(x)| \geqq K_2\left(|a_0|+|a_1|+\sum_{m=0}^{\infty}A_m\right)$$

mit einer positiven Konstante K_2 fast überall besteht. Also gilt

$$K_2\left(|a_0|+|a_1|+\sum_{m=0}^{\infty}A_m\right) \leqq \|\{a_n\};|C, 1|\| \leqq K_1\left(|a_0|+|a_1|+\sum_{m=0}^{\infty}A_m\right). \tag{8}$$

Nach dem Satz der erwähnten Arbeit ist aber die Reihe (1) für jedes ON-System fast überall dann und nur dann $|C, 1|$-summierbar, wenn $\sum_{m=0}^{\infty} A_m<\infty$ ist. (Siehe noch die Arbeit von P. BILLARD [1].) Aus (8) und aus diesem Satz ergibt sich unsere Behauptung für die $|C, 1|$-Summation; weiterhin gibt (8) eine Abschätzung für die Norm $\|\{a_n\};|C, 1|\|$. Die Norm für die Rieszsche absolute Summation kann man mit Hilfe der Resultaten von F. MÓRICZ [3] ähnlicherweise abschätzen.

4. Zum Beweis unserer Behauptung nehmen wir an, daß $\|\{a_n\};|T|\|=\infty$ ist. Die Summe (2) schreiben wir in der Form

$$t_i(\{\varphi_n\};x) = t_i(x) = \sum_{k=0}^{\infty} T_{i,k}a_k\varphi_k(x),$$

weiterhin setzen wir

$$t_i(\{\varphi_n\};M, N;x) = t_i(M, N;x) = \sum_{k=M}^{N} T_{i,k}a_k\varphi_k(x) \qquad (i=0, 1, \ldots),$$

$$t_{-1}(M, N;x)\equiv 0 \quad (M\leqq N).$$

Wir werden zwei Fälle unterscheiden.

a) Erstens nehmen wir an, daß für jedes N

$$\sup_{\{\varphi_n\}} \int_0^1 \left(\sum_{i=0}^{\infty} |t_i(N, N; x) - t_{i-1}(N, N; x)| \right) dx < \infty$$

besteht. Dann gilt offensichtlich

$$\sum_{i=0}^{\infty} |T_{i,N} - T_{i-1,N}||a_N| < \infty \qquad (N=0, 1, \ldots). \tag{9}$$

In diesem Falle besteht weiterhin

$$\lim_{N\to\infty} \sup_{\{\varphi_m\}} \int_0^1 \left(\sum_{i=i_0}^{\infty} |t_i(M, N; x) - t_{i-1}(M, N; x)| \right) dx = \infty . \tag{10}$$

für jedes i_0 und jedes M.

Durch Rekursion werden wir drei Indexfolgen $(0=)i_1 < i_1' < \cdots < i_r < i_r' < \cdots$, $(0=)N_0 < \cdots < N_r < \cdots$, eine Folge von meßbaren Mengen $E_r(\subseteqq(0, 1))$ $(r=1, 2, \ldots)$ und ein ON-System von Treppenfunktionen $f_n(x)$ $(n=0, 1, \ldots)$ derart definieren, daß die folgenden Bedingungen erfüllt sind.

1) für jedes r gilt

$$\sum_{i=i_r}^{i_r'} \left(\sum_{k=0}^{N_{r-1}} |T_{i,k} - T_{i-1,k}||a_k||f_k(x)| \right) < \frac{1}{4} \qquad (x \in (0, 1)); \tag{11}$$

2) für jedes r gilt

$$\sup_{\{\varphi_n\}} \int_0^1 \left(\sum_{i=i_s}^{i_r'} |t_i(M, N; x) - t_{i-1}(M, N; x)| \right) dx < \frac{3}{\sqrt{2}\, 4^2 4^r} \tag{12}$$

$$(s=1, \ldots, r-1;\ N_r \leqq M \leqq N);$$

3) für jedes r ist

$$\sum_{i=i_r}^{i_r'} \left| t_i(\{f_n\}, N_{r-1}+1, N_r; x) - t_{i-1}(\{f_m\}; N_{r-1}+1, N_r; x) \right| > 1 \qquad (x \in E_r) \tag{13}$$

mit

$$m(E_r) > 1 - \frac{1}{2 \cdot 4^r}. \tag{14}$$

Erstens wählen wir Indizes i_1' und N_1 derart, dass

$$\sup_{(\varphi_n)} \int_0^1 \left(\sum_{i=0}^{i_r'} |t_i(0, N_1; x) - t_{i-1}(0, N_1; x)| \right) dx > \sqrt{2} \tag{15}$$

und (12) für $r=1$ bestehen; auf Grund von $\{a_n\} \in l^2$, (9) und $\|\{a_n\}; |T|\| = \infty$ ist es möglich. Mit einer in der Arbeit [4] angewandten Methode kann man zeigen, daß es ein ON-System von Treppenfunktionen $\psi_n(x)$ $(n=0, \ldots, N_1)$ gibt, für welches

(16) $$\int_0^1 \left(\sum_{i=0}^{i_1'} \left| t_i(\{\psi_n\}; 0, N_1; x) - t_{i-1}(\{\psi_n\}; 0, N_1; x) \right| \right) dr > \sqrt{2}$$

ist. Da die Funktionen $\psi_n(x)$ $(n=0, \ldots, N_1)$ Treppenfunktionen sind (d.h. für jede Funktion $\psi_n(x)$ gibt es eine Zerlegung von (0, 1) auf endlichviele Intervalle derart, daß $\psi_n(x)$ in jedem Teilintervall konstant ist), gibt es paarweise disjunkte Intervalle $I_1, \ldots, I_s$ mit $\bigcup_{\sigma=1}^{s} I_\sigma = (0, 1)$ derart, daß

$$\sum_{i=0}^{i_1'} \left| t_i(\{\psi_n\}; 0, N_1; x) - t_i(\{\psi_n\}; 0, N_1; x) \right|$$

in jedem I_σ einen konstanten Wert w_σ aufnimmt. Aus (15) folgt:

$$\sum_{\sigma=1}^{s} w_\sigma m(I_\sigma) > \sqrt{2},$$

und so gilt auch

(17) $$\sum_{\sigma=1}^{s} w_\sigma^2 m(I_\sigma) > 2.$$

Es seien $\sigma_1 < \cdots < \sigma_p$ diejenige Indizes σ zwischen 1 und s, für die $w_\sigma \geqq 1$ ist, und $\sigma_1' < \cdots < \sigma_q'$ die übrigen Indizes zwischen 1 und s. Wir setzen

$$\alpha_l = \sum_{t=1}^{l-1} w_{\sigma_t}^2 \operatorname{mes}(I_{\sigma_t}) \qquad (l=1, \ldots, p+1),$$

$$\alpha_{p+1+l} = \alpha_{p+1} + \frac{1}{2\cdot 4} \sum_{t=1}^{l} m(I_{\sigma_t'}) \qquad (l=1, \ldots, q).$$

Es sei

$$\chi_n(x) = \begin{cases} \dfrac{1}{w_{\sigma_l}} \psi_n\left(\dfrac{x-\alpha_l}{w_{\sigma_l}^2 m(I_{\sigma_l})} \right) & (x \in (\alpha_l, \alpha_{l+1});\ l=1, \ldots, p), \\ \sqrt{2}\, 2\psi_n(2\cdot 4(x-\alpha_l)) & (x \in (\alpha_l, \alpha_{l+1});\ l=p+1, \ldots, p+q) \end{cases}$$

$(n=0, \ldots, N_1)$. Diese Treppenfunktionen bilden offensichtlich ein ON-System im Intervall $(0, \alpha_{p+q})$; weiterhin folgt aus (17) $\alpha_{p+q} > 1$, und aus der Definition von α_l ergibt sich $\alpha_{p+q} - \alpha_{p+1} \leqq \frac{1}{2\cdot 4}$ Wir setzen endlich

$$f_n(x) = \sqrt{\alpha_{p+q}}\, \chi_n(\alpha_{p+q} x) \qquad (n=0, \ldots, N_1).$$

Die Treppenfunktionen $f_n(x)$ $(n=0, \ldots, N_1)$ bilden ein ON-System im Intervall

(0, 1), weiterhin gibt es offensichtlich eine meßbare Menge $E_1(\subseteq(0,1))$ mit $m(E_1) > 1-\frac{1}{2\cdot 4}$ derart, daß

$$\sum_{i=0}^{i_1'} \left|t_i(\{f_n\};0,N_1;x)-t_{i-1}(\{f_n\};0,N_1;x)\right|>1 \qquad (x\in E_1)$$

gilt.

Es sei $r_0(>1)$ eine natürliche Zahl, und wir nehmen an, daß die Indizes $(0=)i_1<<i_1'<\cdots<i_{r_0-1}<i_{r_0-1}'$, $(0=)N_0<\cdots<N_{r_0-1}$, die Treppenfunktionen $f_n(x)$ $(n=0,\ldots,N_{r_0-1})$ und die meßbaren Mengen $E_1,\ldots,E_{r_0-1}$ schon definiert sind derart, dass diese Funktionen ein ON-System im (0, 1) bilden und (11), (12), (13), (14) für $r\leqq r_0-1$ erfüllt sind.

Da $\{a_n\}\in l^2$, $\|\{a_n\};|T|\|=\infty$ sind, (9) erfüllt wird, und die Funktionen $f_n(x)$ $(n=0,\ldots,N_{r_0-1})$ beschränkt sind, kann man Indizes $(i_{r_0-1}'<)i_{r_0}<i_{r_0}'$, $(N_{r_0-1}<)N_{r_0}$ derart angeben, daß (11), (12) für $r=r_0$ und

$$\text{(18)}\qquad \sup_{\{\varphi_n\}}\int_0^1\left(\sum_{i=i_{r_0}}^{i_{r_0}'}|t_i(N_{r_0-1}+1,N_{r_0};x)-t_{i-1}(N_{r_0-1}+1,N_{r_0};x)|\right)dx>\sqrt{2}$$

bestehen. Mit der erwähnten Methode ergibt sich aus (18) ein ON-System von Treppenfunktionen $\psi_n(x)$ $(n=N_{r_0-1}+1,\ldots,N_{r_0})$, für welches

$$\text{(19)}\qquad \int_0^1\left(\sum_{i=i_{r_0}}^{i_{r_0}'}|t_i(\{\psi_n\};N_{r_0-1}+1,N_{r_0};x)-t_{i-1}(\{\psi_n\};N_{r_0-1}+1,N_{r_0};x)|\right)dx>\sqrt{2}$$

ist. Da die Funktionen $\psi_n(x)$ $(n=N_{r_0-1}+1,\ldots,N_{r_0})$ Treppenfunktionen sind, gibt es paarweise disjunkte Intervalle $\bar I_1,\ldots,\bar I_{\bar s}$ mit $\bigcup_{\sigma=1}^{\bar s}\bar I_\sigma=(0,1)$ derart, daß

$$\sum_{i=i_{r_0}}^{i_{r_0}'}|t_i(\{\psi_n\};N_{r_0-1}+1,N_{r_0};x)-t_{i-1}(\{\psi_n\};N_{r_0-1}+1,N_{r_0};x)|$$

in jedem $\bar I_\sigma$ einen konstanten Wert $\bar\omega_\sigma$ aufnimmt. Aus (19) folgt:

$$\sum_{\sigma=1}^{\bar s}\bar w_\sigma m(\bar I_\sigma)>\sqrt{2}$$

und so gilt

$$\text{(20)}\qquad \sum_{\sigma=1}^{\bar s}\bar w_\sigma^2 m(\bar I_\sigma)>2.$$

Es seien $\bar\sigma_1<\cdots<\bar\sigma_{\bar p}$ diejenige Indizes σ zwischen 1 und $\bar s$, für die $\bar\omega_\sigma\geqq 1$ ist, und $\bar\sigma_1'<\cdots<\bar\sigma_{\bar q}'$ die übrigen Indizes zwischen 1 und $\bar s$. Wir setzen

$$\bar\alpha_l=\sum_{t=1}^{l-1}\bar w_{\bar\sigma_t}^2 m(\bar I_{\bar\sigma_t})\qquad (l=1,\ldots,\bar p+1),$$

$$\bar\alpha_{p+1+l}=\bar\alpha_{p+1}+\frac{1}{2\cdot 4^{r_0}}\sum_{t=1}^{l}m(\bar I_{\bar\sigma_t'})\qquad (l=1,\ldots,\bar q).$$

Es sei

$$\chi_n(x) = \begin{cases} \dfrac{1}{\bar{w}_{\bar{\sigma}_l}} \psi_n\left(\dfrac{x-\bar{\alpha}_l}{\bar{w}^2_{\bar{\sigma}_l} m(\bar{I}_{\bar{\sigma}_l})}\right) & (x \in (\bar{\alpha}_l, \bar{\alpha}_{l+1});\ l = 1, \dots, \bar{p}), \\ \sqrt{2}\, 2^{r_0} \psi_n(2 \cdot 4^{r_0}(x-\alpha_l)) & (x \in (\bar{\alpha}_l, \bar{\alpha}_{l+1});\ l = \bar{p}+1, \dots, \bar{p}+\bar{q}) \end{cases}$$

$(n := N_{r_0-1}+1, \dots, N_{r_0})$. Diese Treppenfunktionen bilden offensichtlich ein ON-System im $(0, \bar{\alpha}_{\bar{p}+\bar{q}})$; weiterhin folgt aus (20) $\bar{\alpha}_{\bar{p}+\bar{q}} > 1$, und aus der Definition von $\bar{\alpha}_l$ ergibt sich $\bar{\alpha}_{\bar{p}+\bar{q}} - \bar{\alpha}_{\bar{p}+1} \leqq \frac{1}{2 \cdot 4^{r_0}}$. Wir setzen

$$\bar{f}_n(x) = \sqrt{\bar{\alpha}_{\bar{p}+\bar{q}}}\, \chi_n(\bar{\alpha}_{\bar{p}+\bar{q}} x) \qquad (n = N_{r_0-1}+1, \dots, N_{r_0}).$$

Diese Treppenfunktionen bilden ein ON-System im (0, 1), weiterhin gibt es offensichtlich eine meßbare Menge $\bar{E}_{r_0}(\subseteqq (0,1))$ derart, daß

$$m(\bar{E}_{r_0}) \geqq 1 - \frac{1}{2 \cdot 4^{r_0}}, \tag{21}$$

$$\sum_{i=i_{r_0}}^{i'_{r_0}} \left| t_i(\{\bar{f}_n\}; N_{r_0-1}+1, N_{r_0}; x) - t_{i-1}(\{\bar{f}_n\}; N_{r_0-1}+1, N_{r_0}; x) \right| > 1 \qquad (x \in \bar{E}_{r_0}) \tag{22}$$

bestehen.

Da nach der Annahme die Funktionen $f_n(x)$ $(n = 0, \dots, N_{r_0-1})$ Treppenfunktionen sind, gibt es eine Zerlegung von (0, 1) auf endlichviele, paarweise disjunkte Intervalle $J_1, \dots, J_\varrho$ derart, daß jede Funktion $f_n(x)$ $(n = 0, \dots, N_{r_0-1})$ in jedem J_r konstant is.

Für ein endliches Intervall $I = (a, b)$, für eine Funktion $f(x)$ und für eine Menge $H(\subseteqq (0,1))$ setzen wir

$$f(I; x) = \begin{cases} f\left(\dfrac{x-a}{b-a}\right) & (x \in (a, b)), \\ 0 & \text{sonst}, \end{cases}$$

und mit $H(I)$ bezeichnen wir diejenige Untermenge von (a, b), die aus H mit der linearer Transformation $y = (b-a)x + a$ entsteht.

Wir setzen endlich

$$f_n(x) = \sum_{r=1}^{\varrho} \bar{f}_n(J'_r; x) - \sum_{r=1}^{\varrho} \bar{f}_n(J''_r; x) \qquad (n = N_{r_0-1}+1, \dots, N_{r_0}),$$

$$E_{r_0} = \left(\bigcup_{r=1}^{\varrho} \bar{E}_{r_0}(J'_r)\right) \cup \left(\bigcup_{r=1}^{\varrho} \bar{E}_{r_0}(J''_r)\right),$$

wobei J'_r, J''_r die zwei Hälften von J_r bezeichnen. Offensichtlich bilden die Treppenfunktionen $f_n(x)$ $(n = 0, \dots, N_{r_0})$ ein ON-System in (0, 1), nach (21) und (22) werden auch (13) und (14) für $r = r_0$ erfüllt.

Durch vollständiger Induktion erhalten wir also die Indexfolgen $\{i_r\}$, $\{i'_r\}$, $\{N_r\}$, die Mengenfolge $\{E_r\}$ und das ON-System $\{f_n(x)\}$, für die (11), (12), (13) und (14) bei jedem r erfüllt werden.

Endlich werden wir ein ON-System $\{\Phi_n(x)\}$ in (0, 1) und eine Folge von meßbaren Mengen $F_r(\subseteq(0,1))$ definieren. Wir setzen

$$\Phi_n(x) = \sqrt{2} f_n\left(\left(0, \frac{1}{2}\right); x\right) \qquad (n = N_{2\varrho}+1, \dots, N_{2\varrho+1};\ \varrho=0, 1, \dots),$$

$$\Phi_n(x) = \sqrt{2} f_n\left(\left(\frac{1}{2}, 0\right); x\right) \qquad (n = N_{2\varrho+1}+1, \dots, N_{2\varrho+2};\ \varrho=0, 1, \dots),$$

$$F_{2\varrho+1} = E_{2\varrho+1}\left(\left(0, \frac{1}{2}\right)\right) \qquad (\varrho=0, 1, \dots),$$

$$F_{2\varrho} = E_{2\varrho}\left(\left(\frac{1}{2}, 1\right)\right) \qquad (\varrho=1, 2, \dots).$$

Auf Grund von (11), (13) und (14) erhalten wir, daß

$$\text{(23)} \qquad \sum_{i=i_r}^{i'_r}\left(\sum_{k=0}^{N_{r-1}} |T_{i,k}-T_{i-1,k}||a_k||\Phi_k(x)|\right) < \frac{\sqrt{2}}{4} \qquad (x\in(0,1)),$$

$$\text{(24)} \qquad \sum_{i=i_r}^{i'_r} \left|t_i(\{\Phi_n\}; N_{r-1}+1, N_r; x) - t_{i-1}(\{\Phi_n\}, N_{r-1}+1, N_r; x)\right| > \sqrt{2} \qquad (x\in F_r)$$

und

$$\text{(25)} \qquad m(F_r) > \frac{1}{2}\left(1-\frac{1}{2\cdot 4^r}\right)$$

für jedes r erfüllt werden, weiterhin gilt für jedes r

$$\text{(26)} \qquad \sum_{i=i_r}^{i'_r} \left|t_i(\{\Phi_n\}, N_r+1, N_{r+1}; x) - t_{i-1}(\{\Phi_n\}, N_r+1, N_{r+1}; x)\right| = 0 \qquad (x\in F_r).$$

Für ein r setzen wir

$$\text{(27)} \qquad \sum_{i=i_r}^{i'_r} \left|t_i(\{\Phi_n\}; x) - t_{i-1}(\{\Phi_n\}; x)\right| \geqq \sum_{i=i_r}^{i'_r} \left|t_i(\{\Phi_n\}, N_{r-1}+1, N_r; x) - \right.$$
$$\left. - t_{i-1}(\{\Phi_n\}, N_{r-1}+1, N_r; x)\right| - \sum_{i=i_r}^{i'_r} \left|t_i(\{\Phi_n\}, 0, N_{r-1}; x) - t_{i-1}(\{\Phi_n\}, 0, N_{r-1}; x)\right| -$$
$$- \sum_{l=r}^{\infty} \sum_{i=i_r}^{i'_r} \left|t_i(\{\Phi_n\}, N_l+1, N_{l+1}; x) - t_{i-1}(\{\Phi_n\}, N_l+1, N_{l+1}; x)\right| =$$
$$= S_1(x) - S_2(x) - S_3(x).$$

Auf Grund von (24) ist

$$\text{(28)} \qquad S_1(x) > \sqrt{2} \qquad (x\in F_r).$$

Nach (23) gilt

(29) $$S_2(x) < \frac{\sqrt{2}}{4} \qquad (x \in (0, 1)).$$

Endlich, nach (11) besteht

$$\int_0^1 \left(\sum_{l=r+1}^{\infty} \sum_{i=i_r}^{i'_r} \left| t_i(\{\Phi_n\}, N_l+1, N_{l+1}; x) - t_{i-1}(\{\Phi_n\}, N_l+1, N_{l+1}; x) \right| \right) dx <$$

$$< \frac{3}{\sqrt{2} \cdot 4^2} \sum_{l=r+1}^{\infty} \frac{1}{4^l} = \frac{1}{\sqrt{2} \cdot 4 \cdot 4^r}.$$

So gibt es eine meßbare Menge $G_r (\subseteqq (0, 1))$ mit

(30) $$m(G_r) < \frac{1}{2 \cdot 4^r}$$

derart, daß

(31) $$\sum_{l=r+1}^{\infty} \sum_{i=i_r}^{i'_r} \left| t_i(\{\Phi_n\}, N_l+1, N_{l+1}) - t_{i-1}(\{\Phi_n\}, N_l+1, N_{l+1}; x) \right| < \frac{\sqrt{2}}{4}$$
$$(x \in (0, 1) \setminus G_r).$$

Es sei $H_r = E_r \setminus G_r$. Aus (25) und (30) folgt

(32) $$m(H_r) > \frac{1}{2} - \frac{1}{4^r}$$

für jedes r. Weiterhin, aus (26), (27), (28), (29) und (31) erhalten wir, daß

(33) $$\sum_{i=i_r}^{i'_r} \left| t_i(\{\Phi_n\}; x) - t_{i-1}(\{\Phi_n\}; x) \right| > \frac{\sqrt{2}}{2}$$

im Falle $x \in H_r$ für jedes r besteht.

Aus der Definitionen von F_r, H_r und (32) ergibt sich $m(\overline{\lim}_{\varrho \to \infty} H_{2\varrho}) = \frac{1}{2}$, $m(\overline{\lim}_{\varrho \to \infty} H_{2\varrho+1}) = \frac{1}{2}$ durch einfache Rechnung. Da $\overline{\lim}_{\varrho \to \infty} H_{2\varrho} \subseteqq \left(\frac{1}{2}, 1\right)$, $\overline{\lim}_{\varrho \to \infty} H_{2\varrho+1} \subseteqq \subseteqq \left(0, \frac{1}{2}\right)$ gelten, folgt, daß (33) fast überall für unendlich vieles r besteht, und deshalb ist fast überall

$$\sum_{i=0}^{\infty} \left| t_i(\{\Phi_n\}; x) - t_{i-1}(\{\Phi_n\}; x) \right| = \infty.$$

Damit haben wir unsere Behauptung im Fall *a*) bewiesen.

b) Wir nehmen an, daß es derartige Indizes N gibt, für die

$$\sup_{\{\varphi_n\}} \int_0^1 \left(\sum_{i=0}^{\infty} \left| t_i(N, N; x) - t_{i-1}(N, N; x) \right| \right) = \infty$$

besteht. Dann gilt offensichtlich

(34) $$\sum_{i=0}^{\infty} |T_{i,N}-T_{i-1,N}||a_N| = \infty .$$

Es seien $0 \leqq N_1 < \cdots < N_s < \cdots$ diejenigen Indizes, für die (34) besteht; die anderen Indizes bezeichnen wir mit $M_1 < \cdots < M_\sigma < \cdots$.

Wir sollen nun drei Fälle unterscheiden.

$b_1)$ Gilt (34) für jedes N, dann teilen wir das Intervall (0, 1) in paarweise disjunkte Intervalle $I_0, I_1, \ldots,$ und setzen wir

$$\Phi_n(x) = \begin{cases} \dfrac{1}{\sqrt{m(I_n)}} & (x \in I_n), \\ \text{sonst} \end{cases}$$

$(n=0, 1, \ldots)$. Offensichtlich bilden diese Funktionen ein ON-System. Ist $x \in (0, 1)$, dann gibt es einen Index N mit $x \in I_N$ und gilt

$$\sum_{i=0}^{\infty} \left|t_i(\{\Phi_n\};\, x) - t_{i-1}(\{\Phi_n\};\, x)\right| = \frac{1}{\sqrt{m(I_N)}} \sum_{i=0}^{\infty} |T_{i-1,N}-T_{i,N}||a_N| = \infty .$$

Die Reihe (3) ist also überall nicht $|T|$-summierbar.

Nehmen wir an, daß (34) nicht für jedes N besteht. Wir definieren eine Koeffizientenfolge $\{b_n\}_{n=0}^{\infty}$ folgenderweise: es sei $b_n = a_n$ für $n = N_s$ $(s = 1, 2, \ldots)$ und $b_n = 0$ sonst; offensichtlich ist $\{b_n\} \in l^2$. Für ein ON-System $\{\varphi_n(x)\}$ bilden wir die Summen

$$t_i(\{b_n\}, \{\varphi_n\};\, x) = \sum_{n=0}^{\infty} t_{i,n}\big(b_0\varphi_0(x) + \cdots + b_n\varphi_n(x)\big) = \sum_{n=0}^{\infty} T_{i,n} b_n \varphi_n(x)$$

$$(i=0, 1, \ldots), \quad t_{-1}(\{b_n\}, \{\varphi_n\};\, x) \equiv 0.$$

$b_2)$ Ist $\|\{b_n\};\, |T|\| < \infty$, dann teilen wir das Intervall (0, 1) in paarweise disjunkte Intervalle $I_1, I_2, \ldots$ und setzen wir

$$\Phi_{N_s}(x) = \begin{cases} \dfrac{1}{\sqrt{m(I_s)}} & (x \in I_s), \\ 0 & \text{sonst} \end{cases}$$

$(s = 1, 2, \ldots)$; für die Indizes M_σ definieren wir die Funktionen $\Phi_{M_\sigma}(x)$ $(\sigma = 1, 2, \ldots)$ derart, daß die Funktionen $\Phi_n(x)$ $(n=0, 1, \ldots)$ ein ON-System bilden. (Eine solche Konstruktion kann man leicht durchführen, z. B. setzen wir

$$\Phi_{M_\sigma}(x) = \sum_s r_\sigma(I_s;\, x) \qquad (\sigma = 1, 2, \ldots),$$

wobei $r_\sigma(x) = \operatorname{sign} \sin 2^\sigma \pi x$ die σ-te Rademachersche Funktion bezeichnet.) Nach der Voraussetzung gilt

(35) $$\sum_{i=0}^{\infty} \left|t_i(\{b_n\}, \{\Phi_n\};\, x) - t_{i-1}(\{b_n\}, \{\Phi_n\};\, x)\right| < \infty$$

fast überall. Ist $x \in (0, 1)$, dann gibt es einen Index s mit $x \in I_s$. Auf Grund der Definitionen von $\{\Phi_n\}$ und $\{b_n\}$ gilt offensichtlich

$$\sum_{i=0}^{\infty} \left|t_i(\{\Phi_n\}; x) - t_{i-1}(\{\Phi_n\}; x)\right| \geqq \frac{1}{\sqrt{m(I_s)}} \sum_{i=0}^{\infty} |T_{i,N_s} - T_{i-1,N_s}||a_{N_s}| -$$

$$- \sum_{i=0}^{\infty} \left|t_i(\{b_n\}, \{\Phi_n\}; x) - t_{i-1}(\{b_n\}, \{\Phi_n\}; x)\right|.$$

Da nach der Definition von N_s (34) besteht, erhalten wir daraus und aus (35), daß

$$\sum_{i=0}^{\infty} \left|t_i(\{\Phi_n\}; x) - t_{i-1}(\{\Phi_n\}; x)\right| = \infty \tag{36}$$

in I_s fast überall gilt. In diesem Falle haben wir also gezeigt, daß die Reihe (3) fast überall nicht $|T|$-summierbar ist.

$b_3)$ Endlich nehmen wir $\|\{b_n\}; |T|\| = \infty$ an. Dann teilen wir das Intervall $(0, 1/2)$ in paarweise disjunkte Intervalle $J_1, J_2, \ldots$, und setzen wir

$$\Phi_{N_s}(x) = \begin{cases} \dfrac{1}{\sqrt{m(J_s)}} & (x \in J_s), \\ 0 & \text{sonst} \end{cases}$$

$(s = 1, 2, \ldots)$. Weiterhin, unter Anwendung der Konstruktion im Falle $a)$ bilden wir das entsprechende ON-System $\{\omega_n(x)\}$, für welches also die Reihe

$$\sum_{n=0}^{\infty} b_n \omega_n(x) = \sum_{\sigma=1}^{\infty} b_{M_\sigma} \omega_{M_\sigma}(x)$$

fast überall nicht $|T|$-summierbar ist. Wir setzen

$$\Phi_{M_\sigma}(x) = \sqrt{2}\, \omega_{M_\sigma}\left(2\left(x - \frac{1}{2}\right)\right) \qquad (\sigma = 1, 2, \ldots).$$

Offensichtlich bilden die Funktionen $\Phi_n(x)$ $(n = 0, 1, \ldots)$ ein ON-System. Ist $x \in (0, 1/2)$, dann gibt es einen Index s mit $x \in J_s$, und nach der obigen Definitionen gilt

$$\sum_{i=0}^{\infty} \left|t_i(\{\Phi_n\}; x) - t_{i-1}(\{\Phi_n\}; x)\right| = \frac{1}{\sqrt{m(J_s)}} \sum_{i=0}^{\infty} |T_{i,N_s} - T_{i-1,N_s}||a_{N_s}| = \infty.$$

Ist aber $x \in (1/2, 1)$, dann gilt

$$\sum_{i=0}^{\infty} \left|t_i(\{\Phi_n\}; x) - t_{i-1}(\{\Phi_n\}; x)\right| =$$

$$= \sqrt{2} \sum_{i=0}^{\infty} \left| t_i\left(\{b_n\}, \{\omega_n\}; 2\left(x - \frac{1}{2}\right)\right) - t_{i-1}\left(\{b_n\}, \{\omega_n\}; 2\left(x - \frac{1}{2}\right)\right)\right|,$$

und die Summe dieser letzten Reihe ist aber nach der Definition von $\{\omega_n(x)\}$ fast überall in (1/2, 1) gleich Unendlich. Die Reihe (3) ist also auch in diesem Falle fast überall nicht summierbar.

Damit haben wir unsere Behauptung vollständig bewiesen.

(Eingegangen am 23. Oktober 1969)

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ON FINITE DIRICHLET SERIES

By

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To Professor P. TURÁN *on his* 60^{th} *birthday*

A *finite Dirichlet series* (abreviated by FDS in the sequel) is a function of the form

$$f(s) = \sum_{i=1}^{n} a_i i^s \qquad (a_i \text{ integer})$$

(we shall not use analitical properties of these; it is enough to consider them for integral s). Such a function is an analogue of a polynomial and several properties of polynomials are valid for such functions too. We are going to list such properties here — from a number-theoretical point of view.

Obviously, sum, difference and product of FDSs are FDSs too. Thus, the FDSs form a ring. As MCKENZIE mentioned[1], unique prime factorization holds in this ring. To show this, let p_i denote the i^{th} prime number. Every FDS can be written uniquely in the following form:

$$f(s) = \sum_{1 \leqq i_1 < \cdots < i_\nu} a_{i_1 \ldots, i_\nu} p_{i_1}^S p_{i_2}^S \cdots p_{i_\nu}^S .$$

From this we can deduce that the ring of FDSs is isomorphic with the ring of polynomials with infinitely many variables (over the ring of integers). It is well known that prime factorization is unique in this latter ring.

An FDS is called *primitive*, if its coefficients are coprime. The following analogue of GAUSS' lemma can be proved just in the same way as for polynomials:

The product of primitive FDSs is primitive.

The main purpose of this paper is to investigate some connections between the properties of FDSs and the number-theoretical properties of their values for different values of s. If $f(s)$ is an FDS and s_0 is a non-negative integer then $f(s_0)$ is an integer.

THEOREM 1. *Assume that* $f(s) = \sum_{i=1}^{n} a_i i^s$, $g(s) = \sum_{i=1}^{m} b_i i^s$ $(a_n, b_m \neq 0)$ *are FDSs such that*

$$f(s) \mid g(s)$$

for every $s \geqq 0$. *Then* $n \mid m$ *and*

$$h(s) = \left(\frac{m}{n} - 1\right)! \, \frac{g(s)}{f(s)}$$

is an FDS.

[1] Oral communication

PROOF: Put

$$u(s) = -\sum_{i=1}^{n-1} \frac{a_i}{a_n}\left(\frac{i}{n}\right)^s$$

and let p be a sufficiently large but fixed integer. Then

$$\frac{g(s)}{f(s)} = \frac{g(s)}{a_n \cdot n^s} \cdot \frac{1}{1-u(s)} = \left\{\sum_{j=0}^{p} (u(s))^j \frac{g(s)}{a_n \cdot n^s}\right\} + \frac{(u(s))^{p+1} g(s)}{a_n \cdot n^s(1-u(s))} = \left\{\sum_{i=1}^{N} \beta_i \gamma_i^s\right\} + R(s)$$

where N is some positive integer, β_i and γ_i are rationals $(1 \leqq i \leqq N)$, $\gamma_i > 0$ and for $R(s)$ we have

$$|R(s)| \leqq \left(\sum_{i=1}^{n-1} \left|\frac{a_i}{a_n}\right|\right)^{p+1} \left(\sum_{i=1}^{m} |b_i|\right) \frac{1}{|a_n|} \cdot \frac{1}{|1-u(s)|} \cdot \left[\left(\frac{n-1}{n}\right)^{p+1} \frac{m}{n}\right]^s.$$

This tends to 0 if $s \to 0$ and p is large enough.

Now let $\sum_{j=0}^{N} c_j x^j$ be a polinomial with integral coefficients and roots $\gamma_1, \ldots, \gamma_N$. Then

$$\sum_{j=0}^{N} c_j \sum_{i=1}^{N} \beta_i \gamma_i^{s+j} = \sum_{i=1}^{N} \beta_i \gamma_i^s \sum_{j=0}^{N} c_j \gamma_i^j = 0.$$

and hence

$$\sum_{j=0}^{N} c_j \frac{g(s+j)}{f(s+j)} = \sum_{j=0}^{N} c_j R(s+j).$$

The right-hand side tends to 0, while the left-hand side is always an integer. This implies that both vanish for $s \geqq s_0$.

We use now the following well-known theorem: if $\eta(s)$ is a function defined on the set of integers $s \geqq s_0$ and it satisfies the linear difference-equation

$$\sum_{j=0}^{M} d_j \eta(s+j) = 0$$

where the roots $\varrho_1, \ldots, \varrho_M$ of the polynomial $\sum_{j=0}^{M} d_j x^j$ are different then

$$\eta(s) = \sum_{i=1}^{M} \alpha_i \varrho_i^s$$

with some coefficients α_i. The application of this theorem yields

$$g(s) = f(s) \sum_{i=1}^{N} \alpha_i \gamma_i^s \tag{1}$$

with some coefficients α_i, $s \geqq s_0$. But it is easy to see that if (1) holds for every $s \geqq s_0$, then it holds formally, i.e. it holds for every $s \geqq 0$.

We have still to show that γ_i and $\left(\frac{m}{n}-1\right)!\ \alpha_i$ are integers. Let $\varphi_i(x)=\sum_{j=0}^{N-1} d_j x^j$ be a polynomial with integral coefficients and with roots γ_ν $(\nu\neq i)$. Then

$$\sum_{j=0}^{N-1} d_j \frac{g(s+j)}{f(s+j)} = \alpha_i \gamma_i^s \varphi_i(\gamma_i) \tag{2}$$

is an integer for every $s\geqq 0$. Hence γ_i is an integer, $n|m$ and

$$\frac{g(s)}{f(s)} = \sum_{i=1}^{m/n} \alpha_i' i^s$$

Put

$$\psi_i(x) = \prod_{\substack{\nu=1\\ \nu\neq i}}^{m/n} (x-\nu) = \sum_{j=0}^{m/n} e_j x^j.$$

then

$$\sum_{j=0}^{m/n} e_j \frac{g(s+j)}{f(s+j)} = (i-1)!\left(\frac{m}{n}-i\right)!\,\alpha_i'\cdot i^s.$$

Hence $(i-1)!\left(\frac{m}{n}-i\right)!\ \alpha_i'$ is an integer, consequently $\left(\frac{m}{n}-1\right)!\ \alpha_i' = \binom{\frac{m}{n}-1}{i}(i-1)!$ $\left(\frac{m}{n}-i\right)!\ \alpha_i'$ is an integer too, which proves the theorem.

Combining the analogue of GAUSS' lemma and theorem 1, we obtain

COROLLARY: *If $f(s)$, $g(s)$ are FDSs and $f(s)$ is primitive, furthermore $f(s)|g(s)$ for every $s\geqq 0$, then $\frac{g(s)}{f(s)}$ is an FDS.*

If $f(s)$ is not primitive, then the factor $\left(\frac{m}{n}-1\right)!$ cannot be omitted or replaced by a smaller one in general. Put

$$f(s)=(N-1)!\,h(s), \qquad g(s)=\left\{\sum_{i=1}^{N}(-1)^i\binom{N-1}{i-1}\cdot i^s\right\}h(s)$$

where $h(s)$ is primitive, then one can check easily that $f(s)|g(s)$ for every $s\geqq 0$, on the other hand $g(s)$ is primitive, and thus $K\frac{f(s)}{g(s)}$ has integral coefficients only if $(N-1)!|K$.

We prove another theorem which is an analogue of the theorem of BRISSE [1] and JENTZSCH [2] for polynomials.

THEOREM 2. *Let $f(s)=\sum_{i=1}^{n} a_i\cdot i^s$ be an FDS and assume that $f(s)$ is the k^{th} power of an integer for every $s\geqq 0$. Then $f(s)$ is the k^{th} power of an FDS.*

PROOF: Put

$$m = \sqrt[k]{n}, \quad b = \sqrt[k]{a_n}, \quad \varepsilon_\mu = e^{\frac{2\pi i \mu}{k}}, \quad u(s) = \sum_{i=1}^{n-1} \frac{a_i}{a_n} \cdot \left(\frac{i}{n}\right)^s.$$

Now $u(s) \to 0$ if $s \to \infty$, hence $|u(s)| < \frac{1}{2}$ for $s \geqq s_0$ and the Newton formula is valid:

$$\sqrt[k]{f(s)} = b \cdot m^s \sqrt[k]{1+u(s)} = \sum_{j=0}^{\infty} \binom{\frac{1}{k}}{j} u(s)^j \cdot b \cdot m^s.$$

Let p be a sufficiently large fixed integer and put

$$R_1(s) = \sum_{j=p+1}^{\infty} \binom{\frac{1}{k}}{j} u(s)^j b \cdot m^s.$$

Then

$$|R_1(s)| \leqq \sum_{j=p+1}^{\infty} |u(s)|^j b \cdot m^s \leqq 2 \cdot b \cdot \left(\sum \left|\frac{a_i}{a_n}\right|\right)^{p+1} \left[\left(\frac{n-1}{m}\right)^{p+1} m\right]^s$$

since

$$\left|\binom{\frac{1}{k}}{j}\right| \leqq 1.$$

Thus, if p is sufficiently large, $R_1(s) \to 0$ for $s \to \infty$. On the other hand,

$$\sum_{j=0}^{r} \binom{\frac{1}{k}}{j} u(s)^j b \cdot m^s = \sum_{i=1}^{N} b \cdot \alpha_i (m\gamma_i)^s + \sum_{i=1}^{N'} b \cdot \beta_i (m\delta_i)^s = Q(s) + R_2(s),$$

where $m\gamma_i \geqq 1$, but $m\delta_i < 1$, and α_i, β_i, γ_i, δ_i are rationals. Put

$$R(s) = R_1(s) + R_2(s)$$

and

$$\varphi(x) = A \prod_{i=1}^{N} (x^k - \gamma_i^k n) = \sum_{j=0}^{kN} c_j x^j$$

where A is chosen so that the c_j-s are integers. Then

$$\sum_{j=0}^{kN} c_j \sqrt[k]{f(s+j)} = \sum_{j=0}^{kN} c_j R(s+j) \to 0 \qquad (s \to \infty).$$

Hence

$$\sum_{j=0}^{kN} c_j R(s+j) = 0 \qquad (s \geqq s_1).$$

As in the proof of Theorem 1, this implies

$$R(s+j) = \sum_{\nu=1}^{N} \sum_{\mu=1}^{k} \varrho_{\nu\mu}(m_\nu \varepsilon_\mu)^s.$$

Put for a given pair μ, ν

$$\psi(x) = \frac{\varphi(x)}{x - m\gamma_\nu \varepsilon_\mu} = \sum_{i=0}^{kN-1} d_i x^i.$$

Then

$$\sum_{i=0}^{kN-1} d_i R(s+i) = \varrho_{\nu\mu} \psi(m\gamma_\nu \varepsilon_\mu)[m\gamma_\nu \varepsilon_\mu]^s$$

which can tend to 0 only if $\varrho_{\gamma\mu} = 0$. This shows that for $s \geqq s_1$ $R(s) = 0$, i.e.

$$\sqrt[k]{f(s)} = \sum_{i=1}^{N} b_i \alpha_i (m\gamma_i)^s \qquad (s \geqq s_1)$$

Putting here $s' = ks_1$, $s'' = ks_1 + 1$:

$$\sqrt[k]{f(s')} = bn^{s_1} \sum_{i=1}^{N} \alpha_i \gamma_i^{s'}, \qquad \sqrt[k]{f(s'')} = bmn^{s_1} \sum_{i=1}^{N} \alpha_i \gamma_i^{s''},$$

from which it follows that b and m are rationals, consequently integers. From this we can deduce similarly to the proof of Theorem 1 that, choosing the indeces appropriately,

$$\sqrt[k]{f(s)} = \sum_{i=1}^{N} b\alpha_i i^s.$$

The analogue of GAUSS' lemma gives now that $b\alpha_i$ is an integer. This proves Theorem 2.

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HERMITE—FEJÉR INTERPOLATION BASED ON THE ROOTS OF HERMITE POLYNOMIALS

By
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1. In this paper we intend to continue our investigation begun in [2] (see, further, [3] and [4]).

For a continuous function $f(x)$ on the interval $(-\infty, \infty)$ we define the uniquely determined Hermite—Fejér interpolating polynomials of degree $\leqq 2n-1$ as follows.

$$\text{(1.1)}\qquad \begin{cases} S_n(f;x) = \sum_{k=-m}^{m}{}^* f(x_{kn})h_{kn}(x) \\ \bar{S}_n(f;x) = S_n(f;x) + \sum_{k=-m}^{m}{}^* \beta_{kn}\chi_{kn}(x),^1 \end{cases}$$

where $H_n(x)$ is the n-th Hermite polynomial defined by the well-known relation

$$e^{-x^2}H_n(x) = (-1)^n \frac{d^n}{dx^n}(e^{-x^2}),$$

x_{kn} are the roots of $H_n(x)$ of the property

$$\text{(1.2)}\qquad -\infty < x_{-m,n} < x_{-m+1,n} < \cdots < x_{-1,n} < 0 < x_{1,n} < x_{2,n} < \cdots < x_{m,n} \text{ for } n = 2m,$$

$$-\infty < x_{-m,n} < x_{-m+1,n} < \cdots < x_{-1,n} < x_{0n} \equiv 0 < x_{1,n} < x_{2,n} < \cdots < x_{m,n} \text{ for } n = 2m+1,$$

$$x_{ln} = -x_{ln} \quad \left(l = 1, 2, \ldots, \left[\frac{n}{2}\right]\right),$$

$$\text{(1.3)}\qquad l_{kn}(x) = \frac{H_n(x)}{H_n'(x_{kn})(x-x_{kn})},$$

$$\text{(1.4)}\qquad h_{kn}(x) = \left[1 - \frac{H_n''(x_{kn})}{H_n'(x_{kn})}(x-x_{kn})\right] l_{kn}^2(x) \equiv v_{kn}(x)\, l_{kn}^2(x),$$

$$\text{(1.5)}\qquad \chi_{kn}(x) = (x-x_{kn})\, l_{kn}^2(x),$$

β_{kn} are prescribed (see [1], 14. 1). It is well known (see [1], 14. 1) that

$$\text{(1.6)}\qquad \sum_{k=-m}^{m}{}^* h_{kn}(x) \equiv 1 \quad (n = 1, 2, \cdots; x \in (-\infty, \infty)),$$

$$\text{(1.7)}\qquad S_n(f; x_{kn}) = \bar{S}_n(f; x_{kn}) = f(x_{kn}) \text{ for } \{k\}_{k=-m}^{*m},$$

$$\text{(1.8)}\qquad S_n'(f; x_{kn}) = 0, \quad \bar{S}_n'(f; x_{kn}) = \beta_{kn} \text{ for } \{k\}_{k=-m}^{*m}.$$

[1] Throughout this paper the sign* means that we omit the term with $k=0$ if n is even.

2. The purpose of this paper is to give an estimation for the difference $S_n(f; x) - f(x)$. For this aim let ε be an arbitrarily small but fixed positive number. Denote by $\omega_\Delta(f; t)$ the modulus of continuity of $f(x)$ on the interval $[-\Delta-\varepsilon, \Delta+\varepsilon]$, further let $\omega_\Delta(t)$ be a modulus of continuity on $[-\Delta-\varepsilon, \Delta+\varepsilon]$. Then we have

THEOREM 2. 1. *Let $f(x)$ be a continuous function on $(-\infty, \infty)$, $|f(x)| = O(x^{2s})$ when $x \to \pm\infty$ ($s \geqq 0$ is a fixed integer). Suppose $\omega_\Delta(f; t) = O[\omega_\Delta(t)]$. Then for any number x of the interval $[-\Delta, \Delta]$ we have the relation*

$$|f(x) - S_n(f; x)| = O(1) \sum_{i=1}^{[\sqrt{n}]} \omega_\Delta\left(\frac{i}{\sqrt{n}}\right)\frac{1}{i^2}, \tag{2. 1}$$

where the sign O depends only on Δ and $f(x)$. Before proving this we investigate some special cases.

2. 1. First we prove that $S_n(f; x) \to f(x)$ uniformly for $x \in [-\Delta, \Delta]$. Indeed, from (2. 1) we have

$$|f(x) - S_n(f; x)| = O(1)\omega_\Delta\left(\frac{\log n}{\sqrt{n}}\right) \sum_{i=1}^{[\sqrt{n}]} \left(\frac{1}{i \log n} + \frac{1}{i^2}\right) =$$

$$= O(1)\omega_\Delta\left(\frac{\log n}{\sqrt{n}}\right)\left(\frac{\log \sqrt{n}}{\log n} + \sum_{i=1}^{[\sqrt{n}]} \frac{1}{i^2}\right) = O(1)\omega_\Delta\left(\frac{\log n}{\sqrt{n}}\right) \quad (-\Delta \leqq x \leqq \Delta).$$

If $f \in \operatorname{Lip} \varrho$ $(0 < \varrho \leqq 1)$ then we obtain

$$|f(x) - S_n(f; x)| = O(1) n^{-\varrho/2} \sum_{i=1}^{[\sqrt{n}]} i^{\varrho-2}. \tag{2. 2}$$

I.e.

$$|f(x) - S_n(f; x)| = \begin{cases} O(n^{-\varrho/2}) & \text{for } 0 < \varrho < 1, \ -\Delta \leqq x \leqq \Delta, \\ O\left(\dfrac{\log n}{n}\right) & \text{for } \varrho = 1, \ -\Delta \leqq x \leqq \Delta \end{cases}$$

2. 2. PROOF OF THEOREM 2. 1. We shall use some relations from [1]. First we mention the most important formulae

$$\begin{cases} H_{2m}(x) = (-1)^m 2^{2m} m! L_m^{(-1/2)}(y), \\ H_{2m+1}(x) = (-1)^m 2^{2m+1} m! x L_m^{(1/2)}(y) \quad \text{for } y = x^2. \end{cases} \tag{2. 3}$$

Here $L_m^{(\alpha)}(y)$ is the m-th Laguerre polynomial defined by the relation

$$e^{-y} y^\alpha L_m^{(\alpha)}(y) = \frac{1}{m!} \frac{d^m}{dy^m}(e^{-y} y^{m+\alpha}) \quad (\alpha > -1, y \geqq 0)$$

([1], (5. 6. 1) and (5. 1. 5)). By (2. 3) we have

$$\begin{cases} x_{kn} = \sqrt{y_{km}^{(-1/2)}}, & x_{-kn} = -\sqrt{y_{km}^{(-1/2)}} \quad \text{for } n = 2m, \ k = 1, 2 \ldots, m, \\ x_{kn} = \sqrt{y_{km}^{(1/2)}}, & x_{-kn} = -\sqrt{y_{km}^{(1/2)}} \quad \text{for } n = 2m+1, \ k = 1, 2, \ldots, m, \end{cases} \tag{2. 4}$$

where $y_{kn}^{(\alpha)}$ are the k-th roots of $L_m^{(\alpha)}(y)$.[2] Further, we have the following relations[3]

$$(2.5)\qquad v_{kn}(x) = 1-2x_k x+2x_k^2 \quad \left(n = 1,2,\cdots;\ k=(0),\ \pm1,\ \pm2,\ \dots,\pm\left[\frac{n}{2}\right]\right),$$

$$(2.6)\qquad \sqrt{y_{km}^{(\alpha)}} = \frac{1}{2\sqrt{m}}\,[k\pi+O(1)] \qquad (0<y_{km}^{(\alpha)}\leqq\Omega;\ m=1,2,\dots),$$

$$(2.7)\qquad H_n'(x)=2nH_{n-1}(x),$$

$$(2.8)\qquad |L_m'^{(\alpha)}(y_k^{(\alpha)})|\sim k^{-\alpha-3/2}m^{\alpha+1} \qquad (0<y_k^{(\alpha)}\leqq\Omega,\ m=1,2,\dots),$$

$$(2.9)\qquad |L_m^{(\alpha)}(y)| = \begin{cases} y^{-\alpha/2-1/4}\,O(m^{\alpha/2-1/4}) & \text{for } cm^{-1}\leqq y\leqq\Omega,\\ O(m^\alpha) & \text{for } 0\leqq y\leqq cm^{-1}\end{cases}$$

Here the notation $z_n\sim w_n$ means that $c_1 \leqq |z_n|/|w_n| \leqq c_2$ $(n\geqq N)$ where $0<c_1\leqq \leqq c_2<\infty$, $|w_n|\neq 0$; c and Ω are arbitrary fixed positive numbers; in (2.6), (2.8) and (2.9) O, c_1 and c_2 depend only on α and Ω (see [1], (14.5.6), (8.9.10), (5.5.10), (8.9.11) and (7.6.8)).

Now we estimate the difference $f(x)-f(x_k)$. Denote by x_{jn} the root nearest x (clearly $j=j(n)$; if $|x_{ln}-x| = |x_{l+1,n}-x|$ then let $j=l$). Then by (2.4) and (2.6) we have

$$(2.10)\qquad |x_k-x| = \left|\frac{k\pi-j\pi+O(1)}{2\sqrt{m}}\right| = O(1)\frac{i}{\sqrt{n}} \text{ if } k\neq j \quad \begin{matrix} k=j+i \text{ for } k>j,\\ k=j-i \text{ for } k<j,\end{matrix}$$

for $-\Delta \leqq x,\ x_k\leqq\Delta$. So

$$(2.11)\qquad \begin{cases} |f(x)-f(x_k)| = O(1)\,\omega_\Delta\left(\dfrac{i}{\sqrt{n}}\right) & (k\neq j,\ k=j\pm i,\ -\Delta\leqq x,\ x_k\leqq\Delta),\\ |f(x)-f(x_j)| = O(1)\,\omega_\Delta\left(\dfrac{1}{\sqrt{n}}\right).^4 \end{cases}$$

By (2.4) and (2.6) we can conclude the following estimations.

$$(2.12)\qquad \begin{cases} |j(n)|=O(1) \text{ if } x=0,\\ |j(n)| = O(\sqrt{n}),\quad |k|=O(\sqrt{n}) \text{ for } |x|\leqq\Omega_1\end{cases}$$

where $\Omega_1>0$ is a fixed number.[5] Now we estimate the Lagrange fundamental polynomials $l_{kn}(x)$ for $0\leqq|x|,\ |x_k|\leqq\sqrt{\Omega}\equiv\Omega_1$.

---

[2] It is well known that $0<y_{1m}^{(\alpha)}<y_{2m}^{(\alpha)}<\dots<y_{mm}^{(\alpha)}<\infty$ (s. [1]).

[3] Sometimes we omit the unnecessary indices.

[4] This relation is obvious from $|x-x_j|<|x_{j+1}-x_{j-1}| = O\left(\dfrac{1}{\sqrt{n}}\right)$

[5] Indeed if $|x_k| = \dfrac{1}{2\sqrt{m}}[k\pi+O(1)]\leqq\Omega_1$ then $|k| = O(\sqrt{n})$.

By (1. 3) and (2. 3) we have for $n=2m$, $y=x^2$

$$|l_{k,2m}(x)| = \left| \frac{L_m^{(-1/2)}(y)}{2x_k L_m'^{(-1/2)}(y_k)(x-x_k)} \right| = \tag{2. 13}$$

Using (2. 9), (2. 6), (2. 8) and (2. 10) we have

$$= O(1) \frac{n^{-1/2}}{|kn^{-1/2}||k^{-1}\cdot n^{1/2}|(i\cdot n^{-1/2})} = O\left(\frac{1}{i}\right) \text{ for } k\neq j,\ 0\leqq |x|,\ |x_k|\leqq \Omega_1.$$

Further for $k=j$, $x\neq x_j$ we have by (2. 7), (2. 8) and (2. 9)

$$|l_{j,2m}(x)| = \left| \frac{H_{2m}(x)-H_{2m}(x_j)}{x-x_j}\cdot\frac{1}{H'_{2m}(x_j)} \right| = \left| \frac{H'_{2m}(x^*)}{H'_{2m}(x_j)} \right| = \tag{2. 14}$$

$$= \left| \frac{4mH_{2m-1}(x^*)}{H'_{2m}(x_j)} \right| = \left| \frac{4m2^{2m-1}(m-1)!x^* L_{m-1}^{(1/2)}(y^*)}{2^{2m}m!2x_j L_m'^{(-1/2)}(y_j)} \right| =$$

$$= \begin{cases} O(1)\dfrac{x^*(y^*)^{-1/2}}{|jn^{-1/2}||j^{-1}n^{1/2}|} = O(1) \text{ for } cm^{-1}\leqq y^* \equiv (x^*)^2 \leqq \Omega, \\ O(1)\dfrac{n^{-1/2}n^{1/2}}{|jn^{-1/2}||j^{-1}n^{1/2}|} = O(1) \text{ for } 0\leqq y^*\equiv (x^*)^2\leqq cm^{-1}. \end{cases}$$

Let us consider the case $n=2m+1$. Using (1. 3) and (2. 3) we obtain for $k\neq j$, 0

$$|l_{k,2m+1}(x)| = \left| \frac{xL_m^{(1/2)}(y)}{2x_k^2 L_m'^{(1/2)}(y_k)(x-x_k)} \right| = \tag{2. 15}$$

As before we get

$$= \begin{cases} O(1)\dfrac{xy^{-1/2}}{(k^2n^{-1})(k^{-2}n^{3/2})(in^{-1/2})} = O\left(\dfrac{1}{i}\right) \text{ for } k\neq 0, j;\ cm^{-1}\leqq y\leqq \Omega, \\ O(1)\dfrac{n^{-1/2}n^{1/2}}{i} = O\left(\dfrac{1}{i}\right) \text{ for } k\neq 0, j;\ 0\leqq y\leqq cm^{-1}. \end{cases}$$

If $k=j\neq 0$, we have for $0\leqq |x|\leqq \Omega_1$

$$|l_{j,2m+1}(x)| = \left| \frac{H'_{2m+1}(x^*)}{H_{2m+1}(x_j)} \right| = \left| \frac{2(2m+1)H_{2m}(x^*)}{H_{2m+1}(x_j)} \right| = \tag{2. 16}$$

$$= O(1)\left| \frac{nL_m^{(-1/2)}(y^*)}{x_j L_m^{(1/2)}(y_j)} \right| = O(1)\frac{n^{1/2}}{|jn^{-1/2}||j^{-2}n^{3/2}|} = O(1)\frac{|j|}{\sqrt{n}} = O(1).$$

If $k=0$ but $j\neq 0$ then $x\sim |j|n^{-1/2}\left(y\sim \dfrac{j^2}{h}\right)$ i.e. $y\geqq cm^{-1}$. For $k=j=0$ we have

$0 \leqq y \leqq cm^{-1}$. So we obtain by (2. 3), (2. 7) and (2. 9)

$$(2.17) \qquad |l_{0,2m+1}(x)| = \left| \frac{H_{2m+1}(x)}{H'_{2m+1}(0)x} \right| = \left| \frac{H_{2m+1}(x)}{2(2m+1)H_{2m}(0)x} \right| =$$

$$= \frac{|xL_m^{(1/2)}(y)|}{|(2m+1)L_m^{(-1/2)}(0)x|} = \begin{cases} O(1)\dfrac{y^{-1/2}}{nn^{-1/2}} = O\left(\dfrac{1}{j}\right) \equiv O\left(\dfrac{1}{i}\right) & \text{for } j \neq 0,\ |x| \leqq \Omega_1, \\ O(1)\dfrac{n^{1/2}}{nn^{-1/2}} = O(1) & \text{for } k=j=0. \end{cases}$$

Thus from formulae (2. 13)—(2. 17) we can conclude the remarkable relations

$$(2.18) \qquad \begin{cases} |l_{kn}(x)| = O\left(\dfrac{1}{i}\right) & \text{for } k \neq j \ \ -\Omega_1 \leqq x, x_{kn} \leqq \Omega_1,\ n=1,2,\ldots, \\ |l_{jn}(x)| = O(1) & \text{for } \ -\Omega_1 \leqq x \leqq \Omega_1,\ n=1,2,\ldots. \end{cases}$$

By (2. 5) we have

$$(2.19) \qquad |v_{kn}(x)| = O(1) \quad \text{for} \quad -\Omega_1 \leqq x, x_{kn} \leqq \Omega_1,\ n=1,2,\ldots.$$

2. 3. To estimate the difference $f(x) - S_n(f; x)$ for the $x_{kn}s$ "far" we have to use the Gauss—Jacobi quadrature formula. If $P_{2n-1}(x)$ is a polynomial of degree $\leqq 2n-1$ then we have

$$\int_{-\infty}^{\infty} P_{2n-1}(x)e^{-x^2}dx = \sum_{k=-m}^{m}{}^{*} \lambda_{nk}P_{2n-1}(x_{kn}) \quad (n=1,2,\ldots),$$

where $n=2m$ or $n = 2m+1$, x_{2n} defined by (1. 2),

$$\lambda_{kn} = \sqrt{\pi}\, 2^{n+1} n! \{H_n'(x_{kn})\}^{-2} \qquad (\{k\}_{k=-m}^{*m},\ n=1,2,\ldots).$$

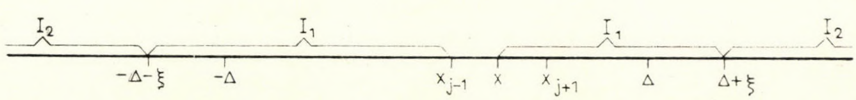
(s. [1], 3. 4. and (15. 3. 6.)). Further we have

$$\int_{-\infty}^{\infty} x^{2r}e^{-x^2}dx = \sqrt{\pi}\,\frac{\Gamma(2r+1)}{4^r\Gamma(r+1)} \qquad (r=0,1,2,\ldots)$$

(s. [5], 247. §). By these formulae we get

$$(2.20) \qquad \sum_{k=-m}^{m}{}^{*} x_{kn}^{2r}\{H_n'(x_{kn})\}^{-2} = \frac{\Gamma(2r+1)}{4^r\Gamma(r+1)}\,\frac{1}{2^{n+1}n!} \qquad (2r \leqq 2n-1).$$

2. 4. Now we can easily prove our main statement. Let $0 < \xi \leqq \varepsilon$ a fixed number. We split up the interval $(-\infty, \infty)$ as follows



By (1. 1) and (1. 6) we get

$$|f(x)-S_n(f;x)| = \left|\sum_{k=-m}^{m}{}^{*}[f(x)-f(x_k)]h_k(x)\right| = O(1)\big[|f(x)-f(x_j)|\,|h_j(x)|+$$

$$+\sum_{x_k\in I_1}|f(x)-f(x_k)|\,|h_k(x)|+\sum_{x_k\in I_2}|f(x)-f(x_k)|\,|h_k(x)|\big].$$

Using (2. 11), (2. 18) and (2. 19) we have

$$|f(x)-f(x_j)|\,|h_j(x)| = O(1)\omega_\Delta\left(\frac{1}{\sqrt{n}}\right).$$

Further by (2. 12), (2. 11), (2. 18) and (2. 19), we get

$$\sum_{x_k\in I_1}|f(x)-f(x_k)|\,|h_k(x)| = O(1)\sum_{i=1}^{[\sqrt{n}]}\omega_\Delta\left(\frac{i}{\sqrt{n}}\right)\frac{1}{i^2}.$$

Finally, by $|f(x)-f(x_k)| = O(x_k^{2s})$ (if $|x_k|$ is "large"), (2. 5), (2. 9) and (2. 20), we obtain (if $n\geqq N$).

$$\sum_{x_k\in I_2}|f(x)-f(x_k)|\,|h_k(x)| = O(1)\sum_{x_k\in I_2}x_k^{2s}|v_k(x)|\left[\frac{H_n(x)}{H_n'(x_k)(x-x_k)}\right]^2 =$$

$$= O(1)\sum_{x_k\in I_2}x_k^{2s+2}\{H_n'(x_k)\}^{-2}\xi^{-2}O(n) = O(1)\frac{\Gamma(2s+3)}{4^{s+1}\Gamma(s+2)}\frac{n}{2^{n+1}n!} = O\left(\frac{1}{n}\right).$$

By these estimations we have

$$|f(x)-S_n(f;x)| = O(1)\sum_{i=1}^{[\sqrt{n}]}\omega_\Delta\left(\frac{i}{\sqrt{n}}\right)\frac{1}{i^2},$$

as we stated.

3. Note. We can obtain similar results for $\bar{S}_n(f;x)$ as well.

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ON THE LARGE SIEVE WITH PRIMES

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1. Introduction

In paper [9] I proved the following theorem.

Let a_n $(M < n \leqq M+N)$ be any complex numbers, let

$$S(\alpha) = \sum_{M<n\leqq M+N} a_n e(n\alpha) \qquad \left(e(\beta)=e^{2\pi i\beta}\right),$$

$$Z = \sum_{M<n\leqq M+N} |a_n|^2.$$

Let p denote primes. Then, for $\varepsilon>0$, $A>0$, $Q\geqq Q_0(\varepsilon, A)$, and $N\leqq Q(\ln Q)^A$, we have

$$\sum_{p\leqq Q}\sum_{b=1}^{p-1}\left|S\left(\frac{b}{p}\right)\right|^2 \leqq \frac{Q^2 Z}{(\ln Q)^{1-\varepsilon}}. \tag{1.1}$$

It is one aim of this paper to sharpen (1. 1) as follows.

THEOREM 1. *Let $Q\geqq 10$, $0<\delta<1$, $N\leqq Q^{1+\delta}$. Then there is an absolute constant C so that*

$$\sum_{p\leqq Q}\sum_{b=1}^{p-1}\left|S\left(\frac{b}{p}\right)\right|^2 \leqq \frac{C}{1-\delta}\,\frac{Q^2\ln\ln Q\, Z}{\ln Q}$$

holds.

The usual upper bound for the sum on the left side of (1. 1) — without any restrictions on N and Q — is

$$\ll (Q^2+N)Z \tag{1.2}$$

(see [3]). One sees immediately that Theorem 1 is better than (1. 2) if

$$Q\geqq N^{1/2}(\ln N)^{C'} \qquad (C'>0, \text{ suff. large}).$$

The point $Q=N^{1/2}$, which is the most interesting one in applications, cannot be reached by the method used below.

As an application, I show an asymptotic formula which improves an inequality of [9].

THEOREM 2. *Let $s(p, b)$ be the least squarefree number $\equiv b \bmod p$. Then, for $0<\alpha<1$, we have*

$$\sum_{p \leq Q} \sum_{b=1}^{p-1} (s(p, b))^\alpha = (D(\alpha)+o(1)) \Pi(Q) Q^{1+\alpha}.$$

$$\big(D(\alpha)>0 \quad Q \to \infty, \quad \Pi(Q) \text{ is the number of primes } \leq Q\big).$$

Finally, I wish to thank Prof. HALBERSTAM, Prof. RICHERT, and H. SIEBERT for helpful advice.

2. Notation

Q will denote sufficiently large positive numbers,
p will be used for primes,
$C_1, C_2, \ldots$ will denote absolute constants.
As usual, $f \ll g$ means $|f| \leq Cg$. If nothing different is mentioned, the constants implied by the $\ll$ — and the 0 — symbol will be absolute. $[x]$ denotes the integral part of x.

3. Proof of Theorem 1.

As in [9], I use an upper estimation for the number of FAREY fractions of order Q and prime denominator in small intervals.

LEMMA 1. *Let* $0<\delta \leq 1-\dfrac{4 \ln \ln Q}{\ln Q}$, $\Delta = Q^{-1-\delta}$, *and, for real* α,

$$I(\alpha) = [\alpha - \Delta, \alpha + \Delta], \qquad P(\alpha) = \sum_{\substack{p \leq Q,\ (b, p)=1 \\ \frac{b}{p} \in I(\alpha)}} 1 .$$

Then we have

$$P(\alpha) \leq \frac{C_1}{1-\delta} \frac{Q^2 \ln \ln Q}{\ln Q} \Delta.$$

With the help of the Lemma Theorem 1 can easily be proved. In the case $1-\dfrac{4 \ln \ln Q}{\ln Q}<\delta<1$ the theorem is not better than (1. 2), so there is nothing to be shown. In the case $0<\delta \leq 1-\dfrac{4 \ln \ln Q}{\ln Q}$ we use a general inequality due to DAVENPORT and HALBERSTAM (see [3], in the original paper [1] there is a factor 2. 2 instead of 2).

Write

$$\|x\| = \min (x-[x], [x]+1-x).$$

Let $x_1, \ldots, x_R$ *be any real numbers with*

$$\|x_r - x_s\| \geq \eta \quad \text{if} \quad r \neq s \quad \left(0<\eta \leq \frac{1}{2}\right).$$

Then

$$\sum_{r=1}^{R} |S(x_r)|^2 \leqq 2\max(N, \eta^{-1})Z. \tag{3. 1}$$

Because of Lemma 1 the set $\left\{\frac{b}{p} \,|\, p \leqq Q;\ b=1, \dots, p-1\right\}$ can be split up into at most

$$\frac{C_2}{1-\delta}\frac{Q^2 \ln\ln Q}{\ln Q}\Delta \tag{3. 2}$$

classes K_i, so that for every i

$$\left\|\frac{b_1}{p_1}-\frac{b_2}{p_2}\right\| \geqq \Delta \quad \text{if} \quad \frac{b_1}{p_1} \neq \frac{b_2}{p_2} \quad \text{and} \quad \frac{b_1}{p_1}, \frac{b_2}{p_2} \in K_i.$$

For fixed i, (3. 1) gives

$$\sum_{\frac{b}{p}\in K_i} \left|S\left(\frac{b}{p}\right)\right|^2 \leqq 2\max(N, \Delta^{-1})Z = 2\Delta^{-1}Z.$$

Summing over i and using (3. 2), one gets Theorem 1.

PROOF OF LEMMA 1. Let

$$\tau = Q^{\frac{1+\delta}{2}}. \tag{3. 3}$$

Then α can be written

$$\alpha = \frac{a}{n}+z \quad \text{where} \quad n \leqq \tau, \quad (a, n)=1, \quad |z| \leqq \frac{1}{n\tau} \tag{3. 4}$$

(see [4], Theorem 36). For $n \leqq \tau$ we have

$$\frac{1}{n\tau} \geqq \frac{1}{\tau^2} = \Delta. \tag{3. 5}$$

We first note that we can restrict ourselves to the case

$$z \geqq \Delta. \tag{3. 6}$$

A. For $\alpha \in \left[\frac{a}{n}-\Delta, \frac{a}{n}+\Delta\right]$ $I(\alpha)$ is a subset of $I\left(\frac{a}{n}-\Delta\right) \cup I\left(\frac{a}{n}+\Delta\right)$. Because of (3. 5) the numbers $\frac{a}{n}-\Delta$ and $\frac{a}{n}+\Delta$ lie in the interval $\left[\frac{a}{n}-\frac{1}{n\tau}, \frac{a}{n}+\frac{1}{n\tau}\right]$. So we can assume $|z| \geqq \Delta$.

B. As $P(\alpha)=P(-\alpha)$ we choose z positive.
Write

$$\Delta_1 = \frac{\Delta}{z} \qquad (0<\Delta_1 \leqq 1), \tag{3.7}$$

$$j = \left[\frac{1}{\Delta_1}\right]+1, \qquad \Delta_2=j^{-1}. \tag{3.8}$$

Then, obviously,

$$\frac{\Delta_1}{2} \leqq \Delta_2 \leqq \Delta_1. \tag{3.9}$$

Put

$$y_i=i\Delta_2 Q, \qquad Y_i=(y_i, y_{i+1}] \qquad (i=0, 1, \dots) \tag{3.10}$$

and

$$P(\alpha) = \sum_{i=0}^{j-1} \sum_{p\in Y_i} \sum_{\substack{(b,p)=1\\ \frac{b}{p}\in I(\alpha)}} 1 = \sum_{i=0}^{j-1} P_i(\alpha), \text{ say.} \tag{3.11}$$

Now, for $0 \leqq i \leqq j-1$, and

$$p\in Y_i, \qquad (b,p)=1, \qquad \frac{b}{p}\in I(\alpha) \tag{3.12}$$

we have $p(\alpha-\Delta) \leqq b \leqq p(\alpha+\Delta)$ or, by (3. 4) and (3. 6), $y_i n(z-\Delta) \leqq bn-ap \leqq$ $\leqq y_{i+1}n(z+\Delta)$. Write

$$W_i = [w_i, w_i'] = [y_i nz - Qn\Delta, y_{i+1}nz+Qn\Delta]. \tag{3.13}$$

Thus, (3. 12) implies $bn-ap = k, k\in W_i$. Let a' (mod n) be defined by the congruence $aa'\equiv 1 \bmod n$, then

$$P_i(\alpha) \leqq \sum_{k\in W_i} \sum_{\substack{p\in Y_i\\ p\equiv -a'k \bmod n}} 1.$$

In the case $(k,n)=1$ the inner sum can be estimated by the BRUN—TITCHMARSH Theorem (see [5])

$$P_i(\alpha) \leqq \sum_{\substack{k\in W_i\\ (k,n)=1}} \frac{3(y_{i+1}-y_i)}{\varphi(n)\ln\frac{y_{i+1}-y_i}{n}} + \sum_{\substack{k\in W_i\\ (k,n)>1}} \sum_{\substack{p\in Y_i\\ p\mid n}} 1 \tag{3.14}$$

($\varphi(n)$ is Euler's function). (3. 10), (3. 9), (3. 7), (3. 4), and (3. 3) give

$$\frac{y_{i+1}-y_i}{n} = \frac{Q\Delta_2}{n} \geqq \frac{Q\Delta_1}{2n} = \frac{Q\Delta}{2nz} \geqq \frac{Q\Delta n\tau}{2n} = \frac{1}{2}Q^{\frac{1-\delta}{2}},$$

that is

$$\ln\frac{y_{i+1}-y_i}{n} \gg (1-\delta)\ln Q. \tag{3.15}$$

From (3. 13), (3. 9), and (3. 7) follows

$$w_i' - w_i = Q\Delta_2 nz + 2Q\Delta n \leqq 3Q\Delta n. \tag{3. 16}$$

Hence, by (3. 14), (3. 15), (3. 16), and by summation over i,

$$P(\alpha) \ll \frac{Q\Delta_2}{(1-\delta)\varphi(n)\ln Q} \sum_{i=0}^{j-1} \sum_{\substack{k\in W_i\\(k,n)=1}} 1 + Q\Delta n \sum_{p|n} 1. \tag{3. 17}$$

(3. 13), (3. 7), (3. 9), and (3. 10) imply $w_i' \leqq nz(y_{i+1} + 2Q\Delta_2) = nzy_{i+3}$, and, similarly, $w_i \geqq nzy_{i-2}$, $nzy_{j+2} \leqq nzQ(1+2\Delta_1) \leqq 3nzQ$, $nzy_{i-2} \geqq -2nzQ$. It follows that the intervals W_i $(0 \leqq i \leqq j-1)$ cover the interval $V=[-2nzQ, 3nzQ]$ at most six times. Hence, with (3. 17),

$$P(\alpha) \ll \frac{Q\Delta_2}{(1-\delta)\varphi(n)\ln Q} \sum_{\substack{k\in V\\(k,n)=1}} 1 + Q\ln Q\,\Delta n \ll \frac{Q^2\Delta_2 zn}{(1-\delta)\ln Q\,\varphi(n)} + Q\ln Q\,\Delta n.$$

In the first term we use [4], Theorem 328. Because of (3. 3), (3. 4), and the assumption on δ, the second term is

$$\leqq Q^{2-\frac{1-\delta}{2}} \ln Q\,\Delta \leqq Q^{2-\frac{2\ln Q}{\ln\ln Q}} \ln Q\,\Delta = \frac{Q^2\Delta}{\ln Q} \leqq \frac{1}{1-\delta}\,\frac{Q^2 \ln\ln Q\,\Delta}{\ln Q}.$$

This proves Lemma 1.

If one studies the sum $\sum_{n\in V,\ (k,n)=1} 1$ more carefully one can replace the factor $\ln\ln Q$ by $\ln\ln(Q^\delta)$. For example, this is $\ll \ln\ln\ln Q$ if $N \leqq Q(\ln Q)^{C_3}$.

4. Some Lemmas

LEMMA 2. *Let* $0<\alpha<1$, $0\leqq k\leqq(\ln Q)^{0\cdot 2}$, $N\leqq Q(\ln Q)^{0\cdot 2}$, $\frac{Q}{\ln Q}<p\leqq Q$. *Then*

$$\text{(i)}\qquad \sum_{\substack{n\leqq N\\ \mu^2(n-p)=\mu^2(n-2p)=\cdots=\mu^2(n-kp)=0}} \mu^2(n) = E(k)N + O\big(Q\exp(-\ln^{0\cdot 2} Q)\big).$$

($E(k)$ *is a positive constant* $\leqq 1$, *which depends on* k *only*, $\mu(n)$ *is Möbius' function.*)

$$\text{(ii)}\qquad \sum_{\substack{n\leqq N\\ \mu^2(n-p)=\cdots=\mu^2(n-kp)=0}} n^\alpha \mu^2(n) = \frac{E(k)}{1+\alpha} N^{1+\alpha} + O\big(Q^{1+\alpha}\exp(-\ln^{0\cdot 2} Q)\big).$$

Part (i) — which is far from being best possible — can be proved by the method of MIRSKY's papers [6] and [7]. The Q-dependence of k does not lead to essentially new difficulties. (ii) follows from (i) by partial summation.

LEMMA 3. *For every integer* $k\geqq 2$ *the inequality*

$$E(k) \leqq C_4 \frac{\ln k}{k^2} \quad \text{holds.}$$

PROOF. Although the numbers $E(k)$ can be given explicitely, it seems rather difficult to derive a good upper bound from that representation. We use the large sieve instead.

Let k be sufficiently large and write

$$k = 4l = (\ln Q)^{0.2}. \tag{4. 1}$$

Consider the sum

$$S_1 = \sum_{\frac{Q}{2}<p\leqq Q} \sum_{\substack{b=1 \\ s(p,b)\in(lQ,\,2lQ]}}^{p-1} 1. \tag{4. 2}$$

Clearly

$$S_1 = \sum_{\frac{Q}{2}<p\leqq Q} \sum_{\substack{lQ<n\leqq 2lQ \\ \mu^2(n-p)=\dots=\mu^2\left(n-\left[\frac{n}{p}\right]p\right)=0}} \mu^2(n) \geqq \sum_{\frac{Q}{2}<p\leqq Q} \sum_{\substack{lQ<n\leqq 2lQ \\ \mu^2(n-p)=\dots=\mu^2(n-4lp)=0}} \mu^2(n).$$

Applying Lemma 2, (i) to the inner sum this gives

$$S_1 \geqq \left(\Pi(Q)-\Pi\left(\frac{Q}{2}\right)\right)\left(lQE(4l)+O(Q\exp(-(\ln Q)^{0.2}))\right)$$

whence, by (4. 1),

$$E(4l) \ll \frac{S_1 \ln Q}{lQ^2} + O(l^{-2}). \tag{4. 3}$$

Let

$$M(N) = \sum_{n\leqq N} \mu^2(N), \qquad M(N,p,b) = \sum_{\substack{n\leqq N \\ n\equiv b \bmod p}} \mu^2(n).$$

If, for $\frac{Q}{2} < p \leqq Q$, $s(p,b) > lQ^2$, then $M(lQ,p,b)=0$, which implies

$$\left(M(lQ,p,b)-\frac{M(lQ)}{p}\right)^2 = \frac{M^2(lQ)}{p^2} \gg \frac{(lQ)^2}{Q^2} = l^2.$$

Hence, using the equality

$$p\sum_{b=0}^{p-1}\left(M(lQ,p,b)-\frac{M(lQ)}{p}\right)^2 = \sum_{b=1}^{p-1}\left|S_{lQ}\left(\frac{b}{p}\right)\right|^2,$$

where

$$S_{lQ}\left(\frac{b}{p}\right) = \sum_{n\leqq lQ} \mu^2(n) e\left(n\frac{b}{p}\right),$$

we get

$$S_1 \ll l^{-2} \sum_{\frac{Q}{2}<p\leqq Q} \sum_{b=1}^{p-1} \left(M(lQ,p,b)-\frac{M(lQ)}{p}\right)^2 \ll \frac{1}{Ql^2} \sum_{p\leqq Q} \sum_{b=1}^{p-1} \left|S_{lQ}\left(\frac{b}{p}\right)\right|^2.$$

The last sum can be estimated by Theorem 1 $\left(\text{put } \delta=\frac{1}{2},\ Z=M(lQ)\right)$.

$$S_1 \ll \frac{1}{Ql^2}\frac{Q^2 \ln\ln Q}{\ln Q} M(lQ) \ll \frac{Q^2 \ln\ln Q}{l \ln Q}.$$

Together with (4. 3) and (4. 1) this gives

$$E(k)=E(4l)\ll \frac{\ln\ln Q}{l^2}+\frac{1}{l^2}\ll \frac{\ln k}{k^2},$$

which is the desired inequality.

LEMMA 4. *For* $0<\alpha<1$ *we have*

$$\sum_{\frac{Q}{2}<p\leqq Q}\ \sum_{\substack{b=1\\ s(p,b)>Q(\ln Q)^{0.15}}}^{p-1} s^{\alpha}(p,b) \ll \Pi(Q)Q^{1+\alpha}(\ln Q)^{-0.1(1-\alpha)}.$$

(The $\ll$ — constant may depend on α).

PROOF. Write $N_0=Q(\ln Q)^{0.15}$, $N_{i+1}=2N_i$, $N_{i_0}<Q^{7/4}\leqq N_{i_0+1}$. Because of PRACHAR's Theorem

$$s(p,b)\leqq p^{7/4} \quad \text{for} \quad p\geqq p_0 \quad (\text{see [8]})$$

the sum to be estimated is

$$\leqq \sum_{0\leqq i\leqq i_0} N_{i+1}^{\alpha} \sum_{\substack{\frac{Q}{2}<p\leqq Q\\ s(p,b)\in(N_i,N_{i+1}]}} \sum_{b=1}^{p-1} 1.$$

The inner sum can be estimated in the same way as the sum S_1 in the proof of Lemma 3. The rest is easy calculation.

5. Proof of Theorem 2.

Let

$$\frac{1}{2}>\varepsilon>\frac{1}{\ln Q}, \tag{5. 1}$$

and

$$S_2=\sum_{Q(1-\varepsilon)<p\leqq Q}\ \sum_{\substack{b=1\\ s(p,b)\leqq Q[(\ln Q)^{0.2}]}}^{p-1} s^{\alpha}(p,b)=$$

$$=\sum_{0\leqq k<[(\ln Q)^{0.2}]}\ \sum_{Q(1-\varepsilon)<p\leqq Q}\ \sum_{\substack{b=1\\ s(p,b)\in(kQ),(k+1)Q]}}^{p-1} s^{\alpha}(p,b) \tag{5. 2}$$

$$=\sum_k S_{2,k} \text{ say.}$$

For fixed $k<[(\ln Q)^{0.2}]$ the inequalities

$$kQ < n \leqq (k+1-(k+1)\varepsilon)Q \quad \text{and} \quad Q(1-\varepsilon) < p \leqq Q$$

imply $0 < n-kp \leqq p$. Hence

$$S_{2,k} = \sum_{Q(1-\varepsilon)<p\leqq Q} \sum_{b=1}^{p-1} \sum_{\substack{kQ<n\leqq (k+1)(1-\varepsilon)Q \\ n=s(p,b)}} n^\alpha \mu^2(n) +$$

$$+ \sum_{Q(1-\varepsilon)<p\leqq Q} \sum_{\substack{b=1 \\ (k+1)(1-\varepsilon)Q<s(p,b)\leqq (k+1)Q}}^{p-1} s^\alpha(p,b) =$$

$$= \sum_{Q(1-\varepsilon)<p\leqq Q} \sum_{\substack{kQ<n\leqq (k+1)(1-\varepsilon)Q \\ \mu^2(n-p)=\cdots=\mu^2(n-kp)=0}} n^\alpha \mu^2(n) +$$

$$+ O\Big(((k+1)Q)^\alpha \varepsilon Q(k+1)\big(\Pi(Q)-\Pi(Q(1-\varepsilon))\big)\Big).$$

To the first sum we apply Lemma 2, (ii).

$$S_{2,k} = \sum_{Q(1-\varepsilon)<p\leqq Q} \left\{\frac{E(k)}{1-\alpha}\big(((k+1)(1-\varepsilon))^{1+\alpha}-k^{1+\alpha}\big)Q^{1+\alpha}+\right.$$

(5. 3) $$\left.+O\Big((Q(k+1))^{1+\alpha}\exp(-\ln^{0.2} Q)\Big)\right\} + O\big(\varepsilon^2((k+1)Q)^{1+\alpha}\Pi(Q)\big) =$$

$$= \big(\Pi(Q)-\Pi(Q(1-\varepsilon))\big)\frac{E(k)}{1+\alpha}\big((k+1)^{1+\alpha}-k^{1+\alpha}\big)Q^{1+\alpha} + O\big(\varepsilon^2((k+1)Q)^{1+\alpha}\Pi(Q)\big).$$

Because of Lemma 3 we have

$$\sum_{k\geqq [\ln^{0.2} Q]} E(k)\big((k+1)^{1+\alpha}-k^{1+\alpha}\big) \ll \sum_{k\geqq \ln^{0.2} Q} \frac{k^\alpha \ln k}{k^2} \ll \frac{\ln\ln Q}{(\ln Q)^{0.2(1-\alpha)}}.$$

Hence, by (5. 3) and summation over k,

$$S_2 = \big(\Pi(Q)-\Pi(Q(1-\varepsilon))\big)Q^{1+\alpha}D(\alpha)\big\{1+O\big(\varepsilon(\ln Q)^{0.2(2+\alpha)}\big)+$$
$$+O\big(\ln\ln Q(\ln Q)^{0.2(\alpha-1)}\big)\big\}$$

where

(5. 4) $$D(\alpha) = \frac{1}{1+\alpha}\sum_{k=0}^{\infty} E(k)\big((k+1)^{1+\alpha}-k^{1+\alpha}\big).$$

Splitting up the interval $\left(\frac{Q}{2}, Q\right)$ into intervals I_ν of length L_ν $\left(\frac{\varepsilon}{2} \leqq L_\nu \leqq \varepsilon\right)$ and treating the sums $S_2(I_\nu)$ in the same manner as S_2 we arrive at

$$\sum_{\frac{Q}{2}<p\leqq Q} \sum_{b=1}^{p-1} s^\alpha(p,b) = \left(\Pi(Q)-\Pi\left(\frac{Q}{2}\right)\right)Q^{1+\alpha}D(\alpha)\cdot$$

$$\cdot\big\{1+O\big(\varepsilon(\ln Q)^{0.2(2+\alpha)}\big)+O\big((\ln Q)^{0.1(\alpha-1)}\big)\big\} + O\left(\sum_{\frac{Q}{2}<p\leqq Q} \sum_{\substack{b=1 \\ s(p,b)>Q(\ln Q)^{0.15}}}^{p-1} s^\alpha(p,b)\right).$$

If we now put $\varepsilon=(\ln Q)^{0.1(\alpha-1)-0.2(2+\alpha)}$ and use Lemma 4 we get an error term of order $(\ln Q)^{0.1(\alpha-1)}$. Finally, we sum over intervals of the type $\left(\frac{Q'}{2}, Q'\right]$, and the theorem is proved.

6. In a similar way one proves the following theorem about the distribution of $s(p, b)$.

THEOREM 3. *There is a distribution function $F(t)$ with the properties*

(i) $$F(t)>1-\frac{C_5 \ln t}{t} \qquad (t \geqq 2)$$

and

(ii) $$\lim_{Q\to\infty}\left\{\frac{1}{\Pi(Q)Q}\sum_{p\leqq Q}\sum_{\substack{b=1\\ s(p,b)<tQ}}^{p-1} 1\right\} = F(t) \text{ for every } t.$$

The convergence is uniform in every finite interval.

7. More interesting are the corresponding statements concerning $q(p, b)$, the least prime congruent p mod b (see ERDŐS [2]). Similarly to Theorem 4 of [9] one can show

$$\sum_{p\leqq Q}\sum_{\substack{b=1\\ q(p,b)\notin(\varepsilon Q\ln Q,\, \varepsilon^{-1}Q\ln Q]}}^{p-1} q^{\alpha}(p, b) = \eta(\varepsilon)\Pi(Q)Q(Q\ln Q)^{\alpha}$$

where $\eta(\varepsilon)\to 0$ if $\varepsilon\to 0$ and $0<\alpha<\alpha_0$ (α_0 is a sufficiently small positive constant).

In order to estimate the sum in which

$$\varepsilon Q \ln Q < q(p, b) \leqq \varepsilon^{-1} Q \ln Q$$

one needs an average result of type Lemma 2 for primes. I hope to return to this question in a further paper.

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A CHARACTERIZATION OF SEMIGROUPS WITH IDENTITY ELEMENT

By

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Since semigroups having identity elements play an important role in many areas of mathematics, it is important to characterize this class of semigroups among all semigroups.

By F. SZÁSZ [2] is given the following interesting characterization of semigroups with an identity[1]:

THEOREM. *The following assertions concerning a semigroup S are equivalent:*

(I) *S is a semigroup with an identity;*

(II) *S has the following properties:*

(α) *there esists at least one left cancellable element f in S such that* $Sf \subseteq fS$;

(β) *there exists at least one right cancellable element e in S such that* $eS \subseteq Se$;

(γ) *no homomorphic image of* S° *has any non-zero left annihilator element;*

(δ) *no homomorphic image of* S° *has any non-zero right annihilator element.*

In his paper [2] SZÁSZ remarks that for a ring R the following conditions are equivalent (see [3]):

(A) R is a ring with identity,

(B) R has properties (α), (γ);

but, conditions (A) and (B) for a semigroup S instead of a ring R are probably not equivalent.

F. SZÁSZ [2] proposes to investigate this problem. Consider the semigroup S defined by the table

	e	a	b
e	e	a	b
a	b	a	b
b	b	a	b.

[1] We use the terminology of CLIFFORD-PRESTON's book [1]. Let S be a semigroup. An element e of S is called a *left (right) identity* element of S if $ea=a$ $(ae=a)$ for all a in S. An element e of S called a two-sided identity (or simply *identity*) element of S if it is both a left and a right identity element of S. An element z of S is called the *zero* element of S if $za=az=z$ for every a in S.

Let 0 be a symbol different from any element of S. Extend the given binary operation in S to one in $S\cup 0$ by defining $00=0$ and $0a=a0=0$ for every a in S. Denote by S° the following semigroup:

$$S^0 = \begin{cases} S \text{ if } S \text{ has a zero element and } |S|>1 \\ S\cup 0 \text{ otherwise.} \end{cases}$$

An element a of S° is called a *left (right) annihilator* of S if for every element x of S°, $ax=0$ $(xa=0)$.

An element a of S is said to be *left (right) cancellable* if for any x and y in S, $ax=ay$ $(xa=ya)$ implies $x=y$.

Let I be an ideal of S. Define $a\varrho b$ $(a, b\in S)$ to mean that either $a=b$ or else both a and b belong to I. We call ϱ the Rees congruence modulo I. The equivalence classes of S mod ϱ are I itself and every one-element set $\{a\}$ with $a\in S/I$. We shall write S/I instead of S/ϱ and we call S/I the *Rees factor semigroup* modulo I.

The element e of S is a left cancellable element, for which $Se \subseteqq eS$ holds; moreover, since S is a band, it satisfies (γ), but S has no identity.

Although conditions (A) and (B) for a semigroup S instead of a ring R are not equivalent, one can easily prove some similar statements. For this reason we use the following

LEMMA. *Let S be a semigroup having property (δ), and let $a \in S$ such that $Sa \subseteqq aS$. Then $a \in aS$.*

PROOF. Let $I = S \cup SaS \subseteqq aS$ and let φ be the natural homomorphism of S° onto the Rees factor semigroup of S modulo I.

Since $a\varphi$ is a right annihilator of S°/I, by (δ) $a \in I \subseteqq aS$. Q.E.D.

We can state the following theorems:

THEOREM 1. *The following assertions concerning a semigroup S are equivalent:*

(I) *S is a semigroup with a left identity element;*

(II) *S has properties (α), (δ).*

PROOF. It is clear that a semigroup S with a left identity element has properties (α), (δ).

Let S be a semigroup having properties (α), (δ). Let f be a left cancellable element of S for which $Sf \subseteqq fS$. By our Lemma there exists an element e of S such that $fe = f$, thus, for every $x \in S$, $fex = fx$, which implies $ex = x$.

THEOREM 2. *The following assertions for a semigroup S are equivalent:*

(I) *S is a semigroup with an identity;*

(II) *S has properties (β)′, (δ), where:*

(β)′ *there exists at least one right cancellable element e in S such that $eS = Se$.*

PROOF. It is clear that a semigroup S with an identity has properties (β)′, (δ).

Let S be a semigroup having properties (β)′, (δ). Let e be a right cancellable element of S for which $Se = eS$. By the Lemma we have $e \in eS$, and by $eS = Se$ also $e = e'e$ with an element e' in S. Since for every $x \in S$ $xe'e = xe$ implies $xe' = x$, thus e' is a right identity element of S.

Since $Se = eS$ implies that for every $x \in S$ there exists an $y \in S$ such that $xe = ey$, we have $e'xe = e'ey = ey = xe$, and this implies $e'x = x$. This means that e' is the identity of S.

One can prove the left-right dual of the Lemma and of Theorems 1 and 2.

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ON CERTAIN LINEAR OPERATORS. II

By

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1. The aim of this paper is to continue the investigations of [1]. In the mentioned paper we proved (among others) the following theorem.

Let us denote by $\omega_m(t)$ a function such that

(i) $\omega_m(t)>0$ for $t>0$, $\omega_m(0)=0$, $\omega_m(T)\geqq\omega_m(t)$ for $T\geqq t$, $\omega_m(t)$ is a continuous function for $t\geqq 0$,

(ii) $\dfrac{t^m}{\omega_m(t)}$ is a monotone increasing function for $t\geqq 0$ ($m\geqq 1$ a fixed integer number).

Let $[a, b]$ be an arbitrary finite interval,

$$\text{(1. 1)}\qquad \begin{cases} a \leqq x_{1n}<x_{2n}<\cdots<x_{pn}\leqq b & (n=1,2,3,\ldots), \text{ where} \\ p=p(n) \quad \text{and} \quad \overline{\lim_{n\to\infty}}\, p(n)=\infty, \end{cases}$$

$C^{[a,b]}(\omega_m)$ the class of all continuous functions on $[a, b]$ for which

$$\text{(1. 2)}\qquad \omega_m(f; t)\leqq a_m(f)\omega_m(t)$$

holds. Here $\omega_m(f; t)$ is the modulus of smoothness of order m of $f(x)$, $a_m(f)$ depends only on $f(x)$. Let

$$\text{(1. 3)}\qquad l_{kn}(x) \qquad (n=1,2,\ldots;\ k=1,2,\ldots,p)$$

be continuous functions on $[a, b]$,

$$\text{(1. 4)}\qquad L_n(f; x)=\sum_{k=1}^{p} f(x_{kn})\, l_{kn}(x),$$

$$\text{(1. 5)}\qquad \lambda_n(x)=\sum_{k=1}^{p}|l_{kn}(x)|, \qquad \lambda_n=\max_{a\leqq x\leqq b}\lambda_n(x),$$

$$\text{(1. 6)}\qquad d_n=\min_{1\leqq k\leqq p-1}(x_{k+1,n}-x_{kn}).$$

In the mentioned paper we proved (among others) the following

THEOREM 1. 1. *Let x_0 be an arbitrary point of $[a, b]$. If $\overline{\lim_{n\to\infty}}\,\lambda_n(x_0)>1$ or $\underline{\lim_{n\to\infty}}\,\lambda_n(x_0)<1$, further*

$$\text{(1. 7)}\qquad \lim_{t\to+0}\frac{t^m}{\omega_m(t)}=0$$

then there exists an $f(x)\in C^{[a,b]}(\omega_m)$ *such that*

$$|L_{n_k}(f;x_0)-f(x_0)|>\lambda_{n_k}(x_0)\omega_m(d_{n_k}) \qquad (k=1,2,\dots). \tag{1.8}$$

(Here $0<n_1<n_2<\cdots$ are integers; s. Theorem 3. 1 from [1]).

2. It is natural to ask what is the situation if $\omega_m(t)$ is best possible, i.e., $\omega_m(t)=t^m$ (for $\omega_m(t)=t^m$ (1. 7) is not valid). Considering that our proof does not ensure the relation (1. 8) for the class $C^{[a,b]}(t^m)$ we could think that there exists an $x_0\in[a,b]$, finite positive numbers $c(f)$ depending on $f(x)$ and a sequence $\varepsilon_n\searrow 0$ such that[1]

$$|L_n(f;x_0)-f(x_0)| < c(f)\varepsilon_n\lambda_n(x_0)\,d_n^m \qquad (n=1,2,\dots) \tag{2.1}$$

for every $f(x)\in C^{[a,b]}(t^m)$. But in this paper we prove that it is not true. Namely, the following theorem is valid.

THEOREM 2. 1. *For every* $x_0\in[a,b]$ *and positive sequence* $\varepsilon_n\searrow 0$ (*supposing that* $\overline{\lim}_{n\to\infty}\lambda_n(x_0)>1$ *or* $\underline{\lim}_{n\to\infty}\lambda_n(x_0)<1$) *there exists an* $f(x)\in C^{[a,b]}(t^m)$ *such that*

$$\overline{\lim_{n\to\infty}}\frac{|L_n(f;x_0)-f(x_0)|}{\varepsilon_n\cdot\lambda_n(x_0)\,d_n^m}=\infty. \tag{2.2}$$

I.e., using Theorem 1. 1, we can state (with the conditions of Theorem 2. 1) that for every $\omega_m(t)$ there exists an $f(x)\in C^{[a,b]}(\omega_m)$ such that

$$\overline{\lim_{n\to\infty}}\frac{|L_n(f;x_0)-f(x_0)|}{\varepsilon_n\lambda_n(x_0)\omega_m(d_n)}=\infty. \tag{2.2$^+$}$$

PROOF. The proof is similar to the proof of Theorem 3. 1 in [1].

If

$$\overline{\lim_{n\to\infty}}\lambda_n(x_0)=1-2\delta \qquad \left(0<\delta\leq\frac{1}{2}\right), \tag{α}$$

we can choose the sequence $0<n_1<n_2<\cdots$ such that $\lambda_{n_k}(x_0)\leqq 1-\delta$ $(k=1,2,\dots)$, $\varepsilon_{n_1}\leqq 1$, $d_{n_1}>d_{n_2}>d_{n_3}>\cdots$ and $d_{n_1}^m<\delta$. Then for $f(x)\equiv 1$ we obtain[2]

$$f(x_0)-L_{n_k}(f;x_0)=1-\sum_{k=1}^{p}l_k(x_0)\geqq 1-(1-\delta)>d_{n_k}^m>\lambda_{n_k}(x_0)d_{n_k}^m$$

whence

$$\frac{f(x_0)-L_{n_k}(f;x_0)}{\varepsilon_{n_k}\lambda_{n_k}(x_0)\,d_{n_k}^m}>\frac{1}{\varepsilon_{n_k}} \qquad (k=1,2,3,\dots)$$

in accordance with (2. 2). (We can see that if $\underline{\lim}_{n\to\infty}\lambda_n(x_0)<1$ the sharper statement (1. 8) is valid for $\omega_m(t)=t^m$ as well.)

[1] The notation $\varepsilon_n\searrow 0$ means that $\varepsilon_1\geqq\varepsilon_2\geqq\cdots\geqq\varepsilon_n\geqq\cdots>0$ and $\lim_{n\to\infty}\varepsilon_n=0$.

[2] Sometimes we omit the unnecessary indices.

Let us consider the case

$$(\beta) \qquad \overline{\lim_{n\to\infty}}\, \lambda_n(x_0) = \infty \quad \text{or} \quad \overline{\lim_{n\to\infty}}\, \lambda_n(x_0) = 1+2\delta \qquad (\delta>0).$$

First we remark that is enough to prove the relation

$$(2.\,2^{++}) \qquad \frac{|L_{n_k}(f;\, x_0)-f(x_0)|}{\varepsilon'_{n_k}\lambda_{n_k}(x_0)\, d^m_{n_k}} > 1 \qquad (k=1, 2, \dots)$$

for an arbitrary sequence $\varepsilon'_n \searrow 0$. For, if $(2.\,2^{++})$ is true, we have

$$\frac{|L_{n_k}(f;\, x_0)-f(x_0)|}{\varepsilon'^2_{n_k}\lambda_{n_k}(x_0)\, d^m_{n_k}} > \frac{1}{\varepsilon'_{n_k}} \qquad (k=1, 2, \dots).$$

Substituting $\varepsilon'^2_n = \varepsilon_n$ and using that $\varepsilon^{-1}_{n_2} \to \infty$ we get (2. 2).

Now let us prove $(2.\,2^{++})$. Define the continuous functions $g_n(x)$ as follows. Let

$$(2.\,3) \qquad g_n(x_k) = \begin{cases} \operatorname{sign} l_k(x_0) \\ 1 \quad \text{if} \quad l_k(x_0)=0 \end{cases} \qquad (k=1, 2, \dots, p),$$

furthermore

$$(2.\,4) \qquad g_n(x) = \begin{cases} g_n(x_1) & \text{for} \quad a \leqq x \leqq x_1 \\ g_n(x_p) & \text{for} \quad x_p \leqq x \leqq b. \end{cases}$$

If $g_n(x_k)=g_n(x_{k+1})$, then let $g_n(x)=g_n(x_k)$ for $x\in[x_k, x_{k+1}]$. On the other hand, if $g_n(x_k)\neq g_n(x_{k+1})$, let

$$g_n(x) = g_n(x_k)\left[1-\frac{(2m-1)!!}{(2m-2)!!}\int\limits_1^{2\frac{x-x_k}{x_{k+1}-x_k}-1} (1-t^2)^{m-1}\,dt\right] \qquad (x_k \leqq x \leqq x_{k+1}).$$

We can prove (s. [2]) that

$$(2.\,5) \qquad \omega_m(g_n;\, t) \leqq B_m d_n^{-m} t^m \qquad \left(0<t\leqq \frac{b-a}{m}\right),$$

i.e.

$$g_n(x)\in C^{[a,b]}(t^m),$$

$$(2.\,6) \qquad |g_n(x)| \leqq 1 \qquad (n=1, 2, 3, \dots),$$

$$(2.\,7) \qquad |L_n(g_N;\, x)| \leqq \lambda_n(x_0) \qquad (n, N=1, 2, 3, \dots),$$

$$(2.\,8) \qquad L_n(g_n;\, x_0) = \lambda_n(x_0) \qquad (n=1, 2, 3, \dots).$$

If there exist a fixed number N and a sequence $0<n_1<n_2<\cdots$ such that

$$|L_n(g_N;\, x_0)-g_N(x_0)| > \lambda_n(x_0)\, d^m_n \qquad (n=n_1, n_2, \dots)$$

then the theorem is true; so we can suppose

$$(2.\,9) \qquad |L_n(g_N;\, x_0)-g_N(x_0)| \leqq \lambda_n(x_0)\, d^m_n \quad \text{if} \quad n>M(N).$$

Now we define the sequence $0<n_1<n_2<\cdots$ of incides as follows

(2. 10) $$\lambda_{n_k}(x_0) \geqq 1+\delta \qquad (k=1, 2, \dots),$$

(2. 11) $$\varepsilon_{n_k} \leqq \frac{1}{k^2} \qquad (k=1, 2, \dots),$$

(2. 12) $$\varepsilon_{n_1} d_{n_1}^m \leqq q, \qquad \varepsilon_i d_i^m \leqq q\varepsilon_{i-1} d_{i-1}^m \qquad (i=n_1, n_2, \dots), \quad \text{where} \quad 0<q<1,$$

(2. 13) $$n_{k+1} > M(n_k) \qquad (k=1, 2, 3, \dots).$$

Let

$$f(x) = Q \sum_{i=1}^{\infty} \varepsilon_{n_i} d_{n_i}^m g_{n_i}(x) \qquad (Q>1).$$

We have to verify that $f(x) \in C^{[a, b]}(t^m)$. By (2. 5) and (2. 11) we get

$$\omega_m(f; t) \leqq Q \sum_{i=1}^{\infty} \varepsilon_{n_i} d_{n_i}^m \omega_m(g_{n_i}; t) \leqq QB_m \sum_{i=1}^{\infty} \varepsilon_{n_i} d_{n_i}^m d_{n_i}^{-m} t^m = QB_m t^m \sum_{i=1}^{\infty} \varepsilon_{n_i} = O(t^m),$$

as it was stated.

Now we can prove (2. 2^{++}). As in [2], we have (without details)

$$L_{n_k}(f; x_0) - f(x_0) \geqq Q \Bigg[\varepsilon_{n_k} d_{n_k}^m L_{n_k}(g_{n_k}; x_0) - \sum_{i=1}^{k-1} \varepsilon_{n_i} d_{n_i}^m |L_{n_k}(g_{n_i}; x_0) - g_{n_i}(x_0)| -$$

$$- \sum_{i=k+1}^{\infty} \varepsilon_{n_i} d_{n_i}^m |L_{n_k}(g_{n_i}; x_0)| - \sum_{i=k}^{\infty} \varepsilon_{n_i} d_{n_i}^m |g_{n_i}(x_0)| \Bigg] \geqq$$

$$\geqq Q \left[\varepsilon_{n_k} d_{n_k}^m \lambda_{n_k}(x_0) - \frac{q}{1-q} \varepsilon_{n_k} d_{n_k}^m \lambda_{n_k}(x_0) - \frac{q}{1-q} \varepsilon_{n_k} d_{n_k}^m \lambda_{n_k}(x_0) - \varepsilon_{n_k} d_{n_k}^m \frac{1}{1-q} \right] =$$

$$= \varepsilon_{n_k} d_{n_k}^m \left[Q\lambda_{n_k}(x_0) - 2\lambda_{n_k}(x_0) \frac{Qq}{1-q} - \frac{Q}{1-q} \right] \qquad (k=1, 2, \dots).$$

As in [2], by (β) we can prove that

$$Q\lambda_{n_k}(x_0) - 2\lambda_{n_k}(x_0) \frac{Qq}{1-q} - \frac{Q}{1-q} > \lambda_{n_k}(x_0) \qquad (k=1, 2, \dots)$$

for a suitable $Q>1$ and $0<q<1$.

Thus we have proved (2. 2^{++}), i.e. (2. 1). Q.E.D.

3. In the paper [1] we have seen that in a certain sense the condition $\lim_{t=+0} t^m/\omega_m(t) = 0$ was replaceable by the condition $\overline{\lim}_{n\to\infty} \omega_m(d_n)\lambda_n(x_0) = \infty$. Namely we proved the following

THEOREM 3. 1. *If*

(3. 1) $$\overline{\lim_{n\to\infty}} \omega_m(d_n)\lambda_n(x_0) = \infty$$

then there exists an $f(x)\in C^{[a,b]}(\omega_m)$ *such that*

$$\overline{\lim_{n\to\infty}}\,|L_n(f;\,x_0)|=\infty \tag{3.2}$$

(s. [1], Theorem 2. 1. and 4. b.).

Now by our theorem proved above we give a very simple proof of this statement. Indeed, from (3. 1) the existence of a sequence $\varepsilon_n\searrow 0$ such that $\overline{\lim\limits_{n\to\infty}}\,\varepsilon_n\omega_m(d_n)\lambda_n(x_0)=\infty$ follows, thus (2. 2^+) gives the statement.

4. NOTE. Theorems and notes analogous to the theorems and notes of [1] are true in our case as well.

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ON THE CONVERGENCE OF MULTIPLICATIVELY ORTHOGONAL SERIES

By

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To P. TURÁN *on his 60th birthday*

1. Introduction. The classical three-series theorem of KOLMOGOROV states essentially that if $\{\varphi_n(x)\}$ is a system of stochastically independent functions of mean-value zero and L^2-norm 1, then $\Sigma c_n^2 < \infty$ implies $\Sigma c_n \varphi_n < \infty$ a.e. Further, if $E_n = \{x: |c_n \varphi_n(x)| \geqq K\}$ for an arbitrary $K > 0$ and $\sum_{n=0}^{\infty} |E_n| < \infty$ then in order that $\Sigma c_n \varphi_n(x)$ converges a.e., the condition $\Sigma c_n^2 < \infty$ is also necessary.

Let $\mu(x)$ be a positive, bounded, non-decreasing function defined in $[a, b]$. Then, in consequence of the independence, in the three series theorem

$$\int_a^b \varphi_{n_1}^{r_1}(x) \varphi_{n_2}^{r_2}(x) \dots \varphi_{n_m}^{r_m}(x) \, d\mu(x) = 0 \tag{1}$$

is assumed for every finite product of φ's and every positive integer $r_1, r_2, \dots, r_m$ with at least one $r_k = 1$.

For bounded systems $\{\varphi_n(x)\}$ condition (1) can be considerably weakened. Indeed, TANDORI and one of the authors proved (see e.g. [1], p. 189) that, for uniformly bounded $\{\varphi_n(x)\}$, it is sufficient to suppose $r_k = 1$ or 2 $(k = 1, 2, \dots, m)$ with at least one $r_k = 1$, provided that

$$\int_a^b \varphi_{n_1}^2(x) \varphi_{n_2}^2(x) \dots \varphi_{n_m}^2(x) \, d\mu(x) = C$$

for every finite product of φ's. Recently it was proved [2] that the last restricting condition of equinorming also can be dropped without altering the statement.

The object of this note is to bring these improvements to a certain final stage by showing that the assertion of the first author [2] remains valid even if $\{\varphi_n(x)\}$ is only a bounded *multiplicatively orthogonal system* (MS), that is

$$\int_a^b \varphi_{\nu_1}(x) \dots \varphi_{\nu_m}(x) \, d\mu(x) = 0 \qquad (0 \leqq \nu_1 < \nu_2 < \dots < \nu_m),$$

without any other condition. More precisely, we prove the following

THEOREM 1. *Suppose $\mu(x)$ is a positive, bounded, non-decreasing function on $[a, b]$ and $\{\varphi_n(x)\}$ a* MS *uniformly bounded in $[a, b]$. Then $\sum_{n=1}^{\infty} c_n^2 < \infty$ implies the convergence of the series $\sum_{n=1}^{\infty} c_n \varphi_n(x)$ μ-a.e. in $[a, b]$.*

If we also suppose that

$$\int_a^b \varphi_k^2 \varphi_l \varphi_m d\mu(x) = 0, \tag{1.1}$$

k, l, m being different from each other and

$$\int_E \varphi_n^2(x) d\mu(x) \geqq K|E|_\mu \qquad (K>0) \tag{1.2}$$

for every set E with $|E|_\mu > 0$ (where $|E|_\mu$ denotes the μ-measure of E), *then the T-summability of* $\sum_{n=1}^{\infty} c_n \varphi_n(x)$ *in a set E with* $|E|_\mu > 0$ *implies* $\sum_1^\infty c_n^2 < \infty$ *where T is any regular method of summability.*

Theorem 1 can be applied to the unconditional convergence a.e. of some types of series whose convergence properties are of interest in themselves. By unconditional convergence a.e. of the series $\Sigma a_n f_n(x)$ we mean, as usual, that this series is convergent a.e. after an arbitrary reordering of its terms. (The set of divergence may depend on the reordering.) As a consequence of Theorem 1 we get

THEOREM 2. *Let $F(x)$ be a bounded, 2-periodic, odd measurable function in* $(-1, 1)$, *then* $\sum_{n=0}^{\infty} c_n^2 < \infty$ *implies the unconditional convergence a.e. of* $\sum_{n=0}^{\infty} c_n F(2^n x)$.

The known Rademacher system $\{r_n(x)\}$ defined in $[-1, 1]$ corresponds to $F(x) = 2^{-\frac{1}{2}}$ in $(0, 1)$ and $-2^{-\frac{1}{2}}$ in $(-1, 0)$. By Theorem 2, we can take for $F(x)$ any bounded measurable function which is antisymmetric about the middle point of a finite interval (a, b).

Another application of Theorem 1 concerns lacunary series of orthogonal polynomials. Every distribution $d\mu(x)$ determines under a given norming a unique system $\{p_n(x)\}$ of polynomials such that $p_n(x)$ should have the exact degree n. For lacunary series of these general orthogonal polynomials, we get the following

THEOREM 3. *If the system of orthogonal polynomials determined in* $[a, b]$ *by the distribution* $d\mu(x)$ *is so normed that* $\{p_n(x)\}$ *is uniformly bounded, then the lacunary series* $\Sigma c_k p_{n_k}(x)$ *with* $n_{k+1}/n_k \geqq q > 1$ *is unconditionally convergent a.e. in* $[a, b]$, *provided* $\Sigma c_k^2 < \infty$.

Other applications in probability theory concerning the strong law of large numbers are evident and are brought out in § 6.

2. A Lemma. Given a MS, we form the product system $\{\psi_n(x)\}$ by putting

$$\begin{cases} \psi_n(x) = \varphi_{\nu_1+1}(x) \dots \varphi_{\nu_m+1}(x) & \text{for} \quad n = 2^{\nu_1} + \dots + 2^{\nu_m}, \\ \psi_0(x) \equiv 1. \end{cases} \tag{2.1}$$

Then multiplicative orthogonality of $\{\varphi_n\}$ is equivalent to

$$\int_a^b \psi_n(x) d\mu(x) = 0 \qquad (n = 1, 2, \dots).$$

We now formulate the following

LEMMA. *Let* $\{\varphi_n(x)\}$ *be a* MS *on* $[a, b]$ *and* $\{\psi_n(x)\}$ *the product system* (2. 1). *If*

(2. 2) $$\|\psi_n\|^2 = \int_a^b \psi_n^2 d\mu(x) > 0$$

and $|\varphi_n(x)| \leqq 1$ *on* $[a, b]$, *then* $\sum_{n=1}^{\infty} c_n^2 < \infty$ *implies the convergence a.e. of* $\sum_{n=1}^{\infty} c_n \varphi_n(x)$.

PROOF. We extend the distribution $d\mu(x)$ to the interval $[a, c]$ with $c > b$ by putting

$$\mu^*(x) = \begin{cases} \mu(x), & a \leqq x \leqq b, \\ \mu(x) + x - b, & b \leqq x \leqq c. \end{cases}$$

By using the method of MENCHOFF [3] we extend the first 2^n functions $\psi_k(x)$ to 2^n functions $\chi_k(x)$ orthogonal on an interval (a, c) with $c > b$. Divide the interval (b, c) into $2^n(2^n - 1)$ equal parts $\{J_{k,l}\}_{k \neq l}$ with $|J_{k,l}| = J$ and $k, l = 0, 1, \ldots, 2^n - 1$. Set

$$I_{kl} = \int_a^b \psi_k \psi_l d\mu(x).$$

Define the functions $\chi_k^{(l)}(x)$ $(k \neq l)$ as follows:

$$\chi_k^{(l)}(x) = \begin{cases} \sqrt{\dfrac{1}{2J} |I_{kl}|}, & x \in J_{kl}, \\ -\sqrt{\dfrac{1}{2J} |I_{kl}|} \operatorname{sgn} I_{kl}, & x \in J_{lk}, \\ 0 \text{ otherwise.} \end{cases}$$

Put

$$\chi_k(x) = \begin{cases} \psi_k(x), & x \in [a, b], \\ \sum_{\substack{l=0 \\ l \neq k}}^{2^n - 1} \chi_k^{(l)}(x), & x \in (b, c). \end{cases}$$

Because $\chi_k^{(i)}(x)\chi_l^{(j)}(x) = 0$ except when $i = l$ and $j = k$, we obtain

$$\int_a^c \chi_k(x)\chi_l(x) d\mu^*(x) = I_{kl} + \int_b^c \sum_{i \neq k} \chi_k^{(i)}(x) \sum_{j \neq l} \chi_l^{(j)}(x) dx =$$

$$= I_{kl} + \Big(\int_{J_{kl}} + \int_{J_{lk}}\Big) \chi_k^{(l)}(x)\chi_l^{(k)}(x) dx = I_{kl} - |I_{kl}| \operatorname{sgn} I_{kl} = 0.$$

Thus $\chi_0(x), \chi_1(x), \ldots, \chi_{2^n - 1}(x)$ form an orthogonal system on (a, c) with the distribution $d\mu^*(x)$.

Set $a_0=0$ and, for $k\geqq 1$,

$$a_k=\begin{cases} c_{\nu+1} & \text{if } k=2^\nu, \\ 0 & \text{if } k\neq 2^\nu; \end{cases}$$

$$S_n(x)=\sum_{k=0}^{n} a_k\chi_k(x), \qquad s_n(x)=\sum_{k=1}^{n} c_k\varphi_k(x).$$

Then for $a\leqq x\leqq b$

$$S_{2^n-1}(x)=s_n(x).$$

Denote by $n(x)$ the least index ν for which

$$s_{n(x)}(x)=\max_{1\leqq\nu\leqq n} s_\nu(x).$$

Then $\{s_{n(x)}(x)\}$ is a non-decreasing sequence. From (2. 2) we have

$$\|\chi_n\|^2=\int_a^c \chi_n^2(x)\,d\mu^*(x)\geqq\|\psi_n\|^2>0,$$

so that putting

$$\sigma_{2^n-1}(x)=\sum_{k=1}^{2^n-1}\frac{a_k\chi_k(x)}{\|\chi_k\|},$$

we see that the function $\sigma_{2^n-1}(x)$ is well defined. Then

$$\left|\int_a^b s_{n(x)}(x)\,d\mu(x)\right|=\left|\int_a^b S_{2^{n(x)}-1}(x)\,d\mu(x)\right|=$$

$$=\left|\int_a^b\int_a^c \sigma_{2^n-1}(t)\sum_{k=0}^{2^{n(x)}-1}\frac{\chi_k(t)\chi_k(x)}{\|\chi_k\|}\,d\mu^*(t)\,d\mu(x)\right|\leqq$$

$$\leqq\left\{\int_a^c\sigma_{2^n-1}^2(t)\,d\mu^*(t)\int_a^c\left(\int_a^b\sum_{k=0}^{2^{n(x)}-1}\frac{\chi_k(t)\chi_k(x)}{\|\chi_k\|}\,d\mu(x)\right)^2 d\mu^*(t)\right\}^{1/2}.$$

Now

$$\int_a^c\sigma_{2^n-1}^2(t)\,d\mu^*(t)=\sum_{k=0}^{2^n-1}a_k^2=\sum_{k=1}^{n}c_k^2\leqq\sum_{k=1}^{\infty}c_k^2=A^2.$$

Setting $n(x,y)=\min\,(n(x),n(y))$, we get by a well-known method of estimation (see e.g. [1], p. 173) due to Kolmogorov-Seliverstov and Plessner:

$$\int_a^c\left(\int_a^b\sum_{k=0}^{2^{n(x)}-1}\frac{\chi_k(t)\chi_k(x)}{\|\chi_k\|}\,d\mu(x)\right)^2 d\mu^*(t)=$$

$$=\int_a^c\int_a^b\int_a^b\sum_{k=0}^{2^{n(x)}-1}\frac{\chi_k(t)\chi_k(x)}{\|\chi_k\|}\sum_{k=0}^{2^{n(y)}-1}\frac{\chi_k(t)\chi_k(y)}{\|\chi_k\|}\,d\mu(x)\,d\mu(y)\,d\mu^*(t)=$$

$$=\int_a^b\int_a^b\sum_{k=0}^{2^{n(x,y)}-1}\chi_k(x)\chi_k(y)\,d\mu(x)\,d\mu(y).$$

Hence

$$\left|\int_a^b s_{n(x)}(x)\,d\mu(x)\right| \leqq 2A\left\{\int_a^b\int_a^b \left|\sum_{k=0}^{2^{n(x)}-1} \chi_k(x)\chi_k(y)\right| d\mu(x)\,d\mu(y)\right\}^{1/2}. \tag{2. 3}$$

Since for $x\in[a, b]$, $\chi_n(x)=\psi_n(x)$ and $|\varphi_k(x)|\leqq 1$, we have

$$\sum_{k=0}^{2^{n(x)}-1} \chi_k(x)\chi_k(y) = \sum_{k=0}^{2^{n(x)}-1} \psi_k(x)\psi_k(y) = \prod_{k=1}^{n(x)} (1+\varphi_k(x)\varphi_k(y)) \geqq 0.$$

We may therefore omit the absolute value sign in the integrand on the right of (2. 3) so that by the multiplicative orthogonality of $\{\varphi_n(x)\}$ we get

$$\begin{aligned}\left|\int_a^b s_{n(x)}(x)\,d\mu(x)\right| &\leqq 2A\left\{\int_a^b\int_a^b \sum_{k=0}^{2^{n(x)}-1} \psi_k(x)\psi_k(y)\,d\mu(x)\,d\mu(y)\right\}^{1/2} = \\ &= 2A\left\{\int_a^b\int_a^b \psi_0(x)\psi_0(y)\,d\mu(x)\,d\mu(y)\right\}^{1/2} = 2A[\mu(b)-\mu(a)].\end{aligned} \tag{2. 4}$$

Since $\{s_{n(x)}(x)\}$ is a non-decreasing sequence, we have $\lim_{n\to\infty} s_{n(x)}(x)=s(x)$ a.e. Hence from (2. 4) it follows by Beppo Levi's theorem that $\int_a^b s(x)\,d\mu(x) \leqq$ $\leqq 2A[\mu(b)-\mu(a)]$ and consequently $s(x)$ is finite a.e. on $[a, b]$. Reasoning similarly with $\{-s_{n(x)}(x)\}$ we get finally that the sequence $\{|s_n(x)|\}$ is bounded on $[a, b]$ by an a.e. finite function.

Now the convergence of Σc_n^2 implies $\Sigma c_n^2\mu_n^2<\infty$ with some $\mu_n^2\uparrow\infty$. Then denoting by $s_n(\mu, x)$ the sum $\sum_{k=1}^{n} c_k\mu_k\varphi_k(x)$, we have

$$\left|\sum_{k=n}^{n+p} c_k\varphi_k(x)\right| \leqq \sum_{k=n}^{n+p-1} (\mu_k^{-1}-\mu_{k+1}^{-1})|s_k(\mu, x)| + \frac{|s_n(\mu, x)|}{\mu_n} + \frac{|s_{n+p}(\mu, x)|}{\mu_{n+p}} = o_x(1) \quad \text{a.e.}$$

This completes the proof of the Lemma.

3. Proof of Theorem 1. To prove the first part of Theorem 1, suppose $|\varphi_n(x)|\leqq$ $\leqq M$ $(n=1, 2, \ldots;\ a\leqq x\leqq b)$. We first extend the function $\mu(x)$ to $\bar\mu(x)$ over $[a, b+1]$ as follows:

$$\bar\mu(x) = \begin{cases} \mu(x), & a\leqq x\leqq b,\\ \mu(x)+x-b, & b\leqq x\leqq b+1.\end{cases}$$

Consider the functions

$$\Phi_n(x) = \begin{cases} \varphi_n(x)/M, & a\leqq x\leqq b,\\ \bar r_n(x), & b<x\leqq b+1,\end{cases}$$

where $\bar r_n(x)$ denotes the n^{th} Rademacher function over $[b, b+1]$. Then $|\Phi_n(x)|\leqq 1$ and $\{\Phi_n(x)\}$ is MS on $[a, b+1]$. Also

$$\|\Phi_n\|^2 \geqq \int_b^{b+1} \bar r_n^2(x)\,dx = 1.$$

Thus all the conditions of the Lemma are satisfied by $\{\Phi_n(x)\}$ on $[a, b+1]$. Hence $\Sigma c_n^2 < \infty$ implies the convergence of $\Sigma c_n \Phi_n(x)$ a.e. on $[a, b+1]$ and a fortiori the convergence of $\Sigma c_n \varphi_n(x)$ a.e. on $[a, b]$. This proves the first part of Theorem 1.

As to the second part, our proof follows the train of ideas of a similar theorem based on stronger conditions on $\{\varphi_n(x)\}$ (see [1], pp 194—195). We consider a regular T-method and let

$$t_n(x) = \sum_{k=0}^{\infty} \alpha_{nk} s_k(x).$$

Then

$$\sum_{k=0}^{\infty} |\alpha_{nk}| < B, \quad \lim_{n\to\infty} \alpha_{nk} = 0, \quad k = 0, 1, \dots; \quad \text{and} \quad \lim_{n\to\infty} \sum_{k=0}^{\infty} \alpha_{nk} = 1. \tag{3. 1}$$

It is easily seen that

$$t_n(x) = \sum_{k=0}^{\infty} c_k R_{nk} \varphi_k(x), \qquad R_{nk} = \sum_{l=k+1}^{\infty} \alpha_{nl}$$

and from (3. 1) it follows that

$$|R_{nk}| < B \quad \text{and} \quad \lim_{n\to\infty} R_{nk} = 1 \quad \text{for fixed } k. \tag{3. 2}$$

Suppose $\{t_n(x)\}$ converges on a set of positive μ-measure. Then for a fixed integer N, the sequence

$$t_n^{(N)}(x) = \sum_{k=N}^{\infty} c_k \varphi_k R_{nk}$$

converges uniformly on E with $|E|_\mu > 0$ and hence there exists $C > 0$ such that $|t_n^{(N)}(x)| \leqq C$, $x \in E$. Thus it follows from (1. 2), that

$$C^2 |E|_\mu \geqq K |E|_\mu \sum_{k=N}^{\infty} c_k^2 R_{nk}^2 + S_n \tag{3. 3}$$

where

$$S_n = 2 \sum_{k=N}^{\infty} c_k R_{nk} \sum_{l=k+1}^{\infty} c_l R_{nl} \int_E \varphi_k(x) \varphi_l(x) d\mu(x).$$

If $\Gamma(x)$ is the characteristic function of the set E and

$$\Gamma(x) \sim \sum_{k=0}^{\infty} \sum_{\substack{l=0 \\ l \neq k}}^{\infty} \gamma_{kl} \frac{\varphi_k(x)\varphi_l(x)}{\|\varphi_k \varphi_l\|}$$

denotes the expansion of $\Gamma(x)$ in the orthonormal system $\left\{\frac{\varphi_k(x)\varphi_l(x)}{\|\varphi_k \varphi_l\|}\right\}$, then

$$\left(\int_E \varphi_k \varphi_l d\mu(x)\right)^2 = \|\varphi_k \varphi_l\|^2 \gamma_{kl}^2,$$

so that

$$|S_n| \leqq 2\left\{\sum_{k=N}^{\infty}\sum_{l=k+1}^{\infty} c_k^2 c_l^2 R_{nk}^2 R_{nl}^2\right\}^{1/2}\left\{\sum_{k=N}^{\infty}\sum_{l=k+1}^{\infty}\left(\int_E \varphi_k\varphi_l d\mu(x)\right)^2\right\}^{1/2} \leqq$$

$$\leqq 2\sum_{j=N}^{\infty} c_j^2 R_{nj}^2 \left\{\sum_{k=N}^{\infty}\sum_{l=k+1}^{\infty}\gamma_{kl}^2\|\varphi_k\varphi_l\|^2\right\}^{1/2}.$$

Since $|\varphi_\nu(x)|\leqq M$, using Bessel's inequality on the last double sum on the right, we get

$$\sum_{k=0}^{\infty}\sum_{l=k+1}^{\infty}\gamma_{kl}^2\|\varphi_k\varphi_l\|^2 \leqq M^2|E|_\mu.$$

For N sufficiently large, we then have

$$\sum_{k=N}^{\infty}\sum_{l=k+1}^{\infty}\gamma_{kl}^2\|\varphi_k\varphi_l\|^2 \leqq \frac{K^2}{9}|E|_\mu^2.$$

Hence from (3. 3) we obtain

$$\sum_{k=N}^{\infty} c_k^2 R_{nk}^2 \leqq \frac{3C^2}{K}.$$

From the second relation in (3. 2) we get for every ν,

$$\lim_{n\to\infty}\sum_{k=N}^{N+\nu} c_k^2 R_{nk}^2 = \sum_{k=N}^{N+\nu} c_k^2 \leqq \frac{3C^2}{K}$$

so that

$$\sum_{k=N}^{\infty} c_k^2 \leqq \frac{3C^2}{K}.$$

This completes the proof of Theorem 1.

If $\{\varphi_n(x)\}$ is not uniformly bounded, but satisfies $|\varphi_n(x)|\leqq M_n$, from theorem 1 we get at once the following

COROLLARY 1. *If $\{\varphi_n(x)\}$ is* MS *such that $|\varphi_n(x)|\leqq M_n$, the convergence of $\Sigma c_n^2 M_n^2$ implies the convergence a.e. of the series $\Sigma c_n\varphi_n(x)$.*

We have to apply theorem 1 on the series $\Sigma\bar{c}_n\bar{\varphi}_n(x)$ with $\bar{c}_n = c_n M_n$ and $\bar{\varphi}_n(x) = \varphi_n(x)/M_n$ and Corollary 1 follows.

4. Proof of Theorems 2 and 3. The proof of theorem 2 depends on the observation that $\{F(2^n x)\}$ is a MS. In other words, we can see easily that

(4. 1) $$\int_{-1}^{1} F(2^{\nu_1}x)\dots F(2^{\nu_m}x)\,dx = 0 \qquad (0\leqq \nu_1<\nu_2<\dots<\nu_m).$$

Indeed, since $F(2^{\nu_1}x)$ is periodic with period $2.\,2^{-\nu_1}$ and $F(x)$ is odd, $F(2^{\nu_1}x)$ is antisymmetric about the point $2k.\,2^{-\nu_1}$ in the interval $I_k = [(2k-1)2^{-\nu_1}, (2k+1)2^{-\nu_1}]$. Also the period of the product-function $F(2^{\nu_2}x)\dots F(2^{\nu_m}x)$ is $2.\,2^{-\nu_2}\leqq 2^{-\nu_1}$, so that it is symmetric in I_k. Then the integrand in (4. 1) is an antisymmetric function in each of the intervals I_k; hence by the periodicity of the integrand

$$\int_{-1}^{1} = \int_{-1-2^{-\nu_1}}^{1-2^{-\nu_1}} = \sum_{k=-2^{\nu_1-1}}^{2^{\nu_1-1}-1} \int_{I_k} F(2^{\nu_1}x) \dots F(2^{\nu_m}x)\,dx = 0.$$

Theorem 2 is now a direct consequence of Theorem 1.

Passing over to the proof of theorem 3, assume first $\{\nu_j\}$ is such that $\nu_{j+1}/\nu_j > 2$. Then the product-polynomial $p_{\nu_1}(x)p_{\nu_2}(x)\dots p_{\nu_{k-1}}(x)$ is of degree $\sum_{j=1}^{k-1} \nu_j \leqq \leqq \nu_{k-1}\left(1+\frac{1}{2}+\dots+\frac{1}{2^{k-2}}\right) < 2\nu_{k-1} < \nu_k$, so that $p_{\nu_k}(x)$ is orthogonal to this product-polynomial:

$$\int_a^b p_{\nu_1}(x) \dots p_{\nu_{k-1}}(x) p_{\nu_k}(x)\,d\mu(x) = 0.$$

Thus the polynomials $\{p_{\nu_j}(x)\}$ form a bounded MS, hence by Theorem 1 the series $\Sigma a_n p_{\nu_n}(x)$ converges a.e., whenever $\Sigma a_n^2 < \infty$. But in theorem 3 $\{p_{n_k}(x)\}$ does not form a MS, since we supposed only $n_{k+1}/n_k \geqq q > 1$. However, in this case it is possible to decompose the sequence $\{p_{n_k}(x)\}$ as a finite union of sequences, each of which is a MS. For if s is the smallest integer such that $q^s > 2$, choose a number $r > q^s > 2$. Let $\{n_k^{(r)}\}$ be a subsequence of $\{n_k\}$ such that $n_{k+1}^{(r)}/n_k^{(r)} > r$. Since there are only a finite number of subsequences $\{n_k^{(j)}\}$, $j = 1, 2, \dots, s$ of $\{n_k\}$ such that $r \geqq n_{k+1}^{(j)}/n_k^{(j)} > 2$, we have

$$\sum_{k=1}^{\infty} c_k p_{n_k}(x) = \sum_{j=1}^{s} \sum_{k=1}^{\infty} c_k^{(j)} p_{n_k}^{(j)}(x)$$

where $\{c_k^{(j)}\}$ is a subsequence of $\{c_k\}$ and every $\{p_{n_k}^{(j)}(x)\}$ is a MS. Theorem 1 then applies to each sequence $\{p_{n_k}^{(j)}(x)\}$ so that $\Sigma c_k^2 < \infty$ implies $\Sigma c_k^{(j)} p_{n_k}^{(j)}(x)$ is convergent a.e. Hence $\Sigma c_k p_{n_k}(x)$ is also convergent a.e.

As a matter of fact, in the proof of theorems 2 and 3 we did not use any other properties than the multiplicative orthogonality and the boundedness of the systems $\{F(2^n x)\}$ and $\{p_{n_k}^{(j)}(x)\}$ respectively. But these properties do not depend on the arrangement of the terms. Hence instead of convergence a.e. we can assert also the unconditional convergence a.e. of the respective series. With this remark the proof of theorems 2 and 3 is complete.

5. The strong law of large numbers. Given a sequence $\{\xi_n\}$ of random variables defined on a probability space Ω, the number

$$\mathbf{E}(\xi_n) = \int_{\Omega} \xi_n\,d\mu$$

is the expectance of ξ_n. We shall say that $\{\xi_n\}$ is a *stochastically multiplicative* system, if

$$\mathbf{E}\left(\prod_{k=1}^{m} \xi_{\nu_k}\right) = \prod_{k=1}^{m} \mathbf{E}(\xi_{\nu_k}) \tag{5. 1}$$

for every collection of indices $\nu_1 < \nu_2 < \cdots < \nu_m$. Put

$$\zeta_n = \xi_n - \mathbf{E}(\xi_n),$$

then $\{\zeta_n\}$ is a MS, provided that $\{\xi_n\}$ is stochastically multiplicative. Indeed, Ω plays the role of the interval $[a, b]$ and by (5. 1) we have

$$\int_\Omega \zeta_{\nu_1} \zeta_{\nu_2} \dots \zeta_{\nu_m} d\mu = 0$$

for every collection of indices $\nu_1 < \nu_2 < \cdots < \nu_m$. Thus we can apply our previous results to the so- called strong law of great numbers.

THEOREM 4. *Let $\{\xi_n\}$ be a stochastically multiplicative system such that $|\xi_n| \leqq M_n$ where $\{M_n\}$ is not decreasing. Choosing two sequences of positive numbers $\{p_n\}$ and $\{q_n\}$ satisfying the conditions*

$$\sum_{n=1}^{\infty} \frac{p_n^2}{q_n^2} < \infty, \qquad q_n < q_{n+1} \to \infty,$$

we have

$$\mathbf{P}\left(\frac{\sum_{k=1}^{n} p_k \zeta_k}{q_n M_n} \to 0 \right) = 1,$$

where $\mathbf{P}(A)$ denotes the probability of the event A.

Indeed, by $|\xi_n| \leqq M_n$ we have $|\zeta_n| \leqq 2M_n$ and $\{\zeta_n\}$ is a MS. Apply Corollary 1 to series $\Sigma c_n \varphi_n(x)$ with $c_n = \dfrac{p_n}{q_n M_n}$ and $\varphi_n(x) = \zeta_n$. Then $\Sigma c_n^2 < \infty$, hence $\Sigma c_n \varphi_n(x)$ converges a.e., or in other terms the series

$$\sum_{n=1}^{\infty} \frac{p_n}{q_n M_n} \zeta_n$$

converges with probability 1. But $q_n M_n \nearrow \infty$, hence it follows by a well- known lemma of Kronecker that

$$\mathbf{P}\left(\sum_{k=1}^{n} p_k \zeta_k = o(q_n M_n) \right) = 1,$$

as we have asserted.

COROLLARY 2. *If $\{\xi_n\}$ is stochastically multiplicative and uniformly bounded, then the weighted arithmetic means of $\{\zeta_n\}$ satisfy the relation*

$$\mathbf{P}\left(\frac{\sum_{k=1}^{n} p_k \zeta_k}{P_n} \to 0 \right) = 1$$

with $P_n = \sum_{k=1}^{n} p_k$, provided that

$$\sum \frac{p_n^2}{P_n^2} < \infty \qquad (P_n \to \infty).$$

Put $M_n = M$ and $q_n = P_n$, then the conditions of theorem 4 are satisfied and Corollary 2 follows.

COROLLARY 3. *Let* $\{\lambda_n\}$ *be a non-decreasing sequence of positive numbers such that* $\Sigma n^{-1}\lambda_n^{-1} < \infty$. *If* $\{\xi_n\}$ *is stochastically multiplicative and* $|\xi_n| = O(n^{1/2}\lambda_n^{-1/2})$, *then*

$$\mathbf{P}\left(\frac{\zeta_1 + \zeta_2 + \cdots + \zeta_n}{n} \to 0\right) = 1.$$

Put $p_n = 1, \quad q_n = n^{1/2}\lambda_n^{1/2}, \quad M_n = n^{1/2}\lambda_n^{-1/2}$. Then

$$\sum_{n=1}^{\infty} \frac{p_n^2}{q_n^2} = \sum_{n=1}^{\infty} \frac{1}{n\lambda_n} < \infty,$$

hence we can apply theorem 4 by which our statement follows.

REMARK. Noticing the many points of similarity between the convergence properties of series whose terms form a uniformly bounded stochastically multiplicative system and those of series of independent random variables, we are tempted to ask whether some form of the law of iterated logarithm and of the central limit theorem is true for bounded stochastically multiplicative systems. We are not able to decide whether this is true or false.

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GYÖRGY HAJÓS
(1912–1972)

Two years passed since the death of A. RÉNYI and another mournfull loss hit the Redaction of the Acta Mathematica Academiae Scientiarum Hungaricae, the Academy and the whole Hungarian mathematical life. GYÖRGY HAJÓS, editor of this journal, member of the Hungarian Academy of Sciences, corresponding member of alien Academies, professor of the Eötvös Loránd University, president of the Bolyai János Mathematical Society passed away on March 17, 1972 after a long illness which paralysed his exceptional creative power since years.

GYÖRGY HAJÓS was born on February 21, 1912 in Budapest. He studied at the University of Budapest where he took his doctor's degree in 1935. In the same year he became assistant professor at the Technical University of Budapest. In 1949 he accepted a full professorship at the Eötvös Loránd University in Budapest. HAJÓS was elected corresponding member of the Hungarian Academy of Sciences in 1948 and became member in 1953. He was a highly estimated scientist awarded two times with the Kossuth prize (1948 and 1962).

The scientific work of HAJÓS culminated in his famous proof (1941) of Minkowski's conjecture (1896) concerning the lattice covering of the n-dimensional space with cubes. The proof rests on his theorem of the factorization of Abelian groups the importance of which was later recognized in the international mathematical literature. RÉDEI called it the second fundamental theorem of the theory of Abelian groups. Besides this excellent achievement, HAJÓS' scientific work covers many fields of mathematics as geometry, theory of graphs, determinants, numerical and graphical methods, geometry of lattices, any of his notes being a model of precision and concision.

HAJÓS was also an excellent teacher, His lectures had a country-wide fame. He was able to present the matter very clearly, very interesting and, at the same time, with highest logical precision. His book, "Introduction to Geometry" (1960) was published four times in Hungarian and translated into German (1970).

HAJÓS loved his country profoundly. After many visits at foreign universities and congresses, he enjoyed to return to his pupils and tried to enrich and to improve the Hungarian mathematical life. He left a great lack behind him and we shall always cherish his memory.

The Editorial Board

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ON POLYTOPES FIXED BY THEIR VERTICES

By
P. MANI (Bern)

A convex body P is fixed by a subset $M \subset \hat{P}$ if every body P', which is the result of a small translation on P, contains a point of M in its interior. Of course each body is fixed by its whole boundary $\hat{P}$, and so we may look for fixing subsets which are minimal with respect to various conditions. B. GRÜNBAUM in (1) has derived upper bounds for the cardinality of smallest fixing sets. We want to answer a similar question in the case where P is a polytope and M a subset of its vertices.

Let E^d be the d-dimensional Euclidean space. Given a set $X \subset E^d$ we denote its interior by X, its boundary by $\hat{X}$ and its closure by $\bar{X}$. By conv X we understand the convex hull of X, by relint X and relbd X the interior and the boundary of X in the topology of its affine hull aff X. If $P \subset E^d$ is a polytope we denote by $\mathfrak{P}$ the complex consisting of all faces F of P and by $\hat{\mathfrak{P}} = \mathfrak{P} - \{P\}$ its boundary complex. For any collection $\mathfrak{S}$ of polytopes in E^d we set $|\mathfrak{S}| = \bigcup_{X \in \mathfrak{S}} X$, $\mathfrak{S}^k = \{X \in \mathfrak{S} | \dim X \leqq k\}$, $\Delta^k(\mathfrak{S}) = \mathfrak{S}^k - \mathfrak{S}^{k-1}$ $(-1 \leqq k \leqq d)$. If $\mathfrak{P}$ is the complex of a polytope $P \subset E^d$ we will occasionally write $\Delta^k(P)$ instead of $\Delta^k(\mathfrak{P})$. Given a vertex $p \in \Delta^0(\mathfrak{P})$, we denote its star in $\hat{\mathfrak{P}}$ by st $(p, \hat{\mathfrak{P}})$ or, if no confusion may arise, by st$p$. link $(p, \hat{\mathfrak{P}})$, or simply link p stands for the link of p in $\hat{\mathfrak{P}}$. Notice that for a simplicial d-polytope P each complex link p $(p \in \Delta^0(P))$ is isomorphic to the boundary complex of a simplicial $(d-1)$-polytope. In this case we have for the edge-valence $v(p)$ of p in $\mathfrak{P}$ the relation $v(p) = \text{card}\, \Delta^0(\text{link}\, p)$. By $A(P, p)$ $(p \in \Delta^0(P))$ we denote the interior angle of P at the vertex p. $A(P, p)$ is a subset of the unit sphere S^{d-1} in E^d. A d-polytope $P \subset E^d$ is said to be hindered (fixed) by its vertices if for each direction $t \in S^{d-1}$ there is a number $\varepsilon > 0$ such that every polytope $P + \delta t$ with $0 < \delta < \varepsilon$ contains a vertex $p \in \Delta^0(P)$ (contains a vertex $p \in \Delta^0(P)$ in its interior). Equivalently, $P \subset E^d$ is hindered by its vertices if and only if $\bigcup_{p \in \Delta^0(P)} A(P, p) = S^{d-1}$. H. HADWIGER, in studying the existence of minima of convex functions defined on E^d, raised the following question. Denote by $\mathfrak{F}_d(\mathfrak{H}_d)$ the set of d-polytopes which are fixed (hindered) by their vertices, and define $f(d) = \min \{\text{card}\, \Delta^0(P) | P \in \mathfrak{F}_d\}$, $h(d) = \min \{\text{card}\, \Delta^0(P) | P \in \mathfrak{H}_d\}$. Which are the values $f(d)$ and $h(d)$? Direct computations show $f(2) = 5$, $h(2) = 4$, and in general we have $d+1 \leqq h(d) \leqq f(d) \leqq 2d+2$, where the upper bound is given by a Kleetope over a simplex S, whose new vertices are sufficiently close to the centroids of the facets of S. Our aim is to prove that, for each $d \geqq 3$, the equality $h(d) = f(d) = 2d+2$ holds. We begin with two elementary lemmas about the boundary complexes of simplicial polytopes.

LEMMA 1. *Let P be a simplicial d-polytope and $p, q \in \Delta^0(P)$ two verices which are not connected by an edge $K \in \Delta^1(P)$. If $\Delta^0(\text{link } p) = \Delta^0(\text{link } q)$, and if $L = \Delta^0(\text{link } p) = \Delta^0(\text{link } q)$ has cardinality* card $L = d$, *then* $\hat{\mathfrak{P}} = \text{st } p \cup \text{st } q$.

PROOF. For each facet $F \in \Delta^{d-1}(\text{st } p)$ we have card $(\Delta^0(F) \cap L) = d-1$. Now L contains exactly d subsets X with card $X = d-1$. Since card $\Delta^{d-1}(\text{st } p) \geqq d$ we immediately conclude $\Delta^{d-1}(\text{st } p) = \{\text{conv}(X \cup \{p\}) | X \subset L, \text{card } X = d-1\}$, and further $|\text{link } p| = \text{relbd conv } L$. Similarly $|\text{link } q| = \text{relbd conv } L$, and therefore $|\text{st } p \cup \text{st } q|$ is a piecewise linear $(d-1)$-sphere contained in $|\hat{\mathfrak{P}}|$. This is only possible if $|\text{st } p \cup \text{st } q| = |\hat{\mathfrak{P}}|$.

LEMMA 2. *Let P be a simplicial d-polytope with $d \geqq 3$. Assume that $\Delta^0(P)$ contains two vertices p, q which are not connected by an edge $K \in \Delta^1(P)$ and for which $\Delta^0(\text{link } p) \neq \Delta^0(\text{link } q)$. Then there is a facet $F \in \Delta^{d-1}(P)$ and a triangle $D \in \Delta^2(P)$ disjoint to F.*

PROOF. We can suppose that there is a point $x \in \Delta^0(\text{link } q)$ not contained in link p. Let $L \in \Delta^1(\text{link } q)$ be an edge issuing from x and denote by $y \in \Delta^0(\text{link } q)$ its other endpoint. There exists a facet $G \in \Delta^{d-2}(\text{link } p)$ which does not contain y. The facet $F = \text{conv}(G \cup \{p\}) \in \Delta^{d-1}(P)$ and the triangle $D = \text{conv}(L \cup \{q\}) \in \Delta^2(P)$ are obviously disjoint.

The next statement is the main tool for our proof. We feel that there should be for it a much shorter argument than ours.

PROPOSITION 1. *Assuming $d \geqq 3$ let P be a simplicial d-polytope with $d+i$ vertices. Then*

(I) *if $i \leqq 3$ there are two facets $F_1, F_2 \in \Delta^{d-1}(P)$ such that $\Delta^0(F_1) \cup \Delta^0(F_2) = \Delta^0(P)$.*

(II) *if $i \geqq 4$ there are three facets F_1, F_2, F_3 in $\Delta^{d-1}(P)$ such that* card $\left(\bigcup_{k=1}^{3} \Delta^0(F_k)\right) \geqq d+4$.

PROOF. We proceed by induction on the dimension d and notice that for $d=3$ both statements are easily derived from the fact that there are two disjoint triangles in the boundary complex of each simplicial 3-polytope with more than five vertices.

For $d \geqq 4$ we first prove (I), and it is enough to consider all d-polytopes with $d+3$ vertices. If there is a point $p \in \Delta^0(P)$ whose valence $v(p)$ is $d+2$, we find, by the inductive hypothesis, two facets $G_1, G_2 \in \Delta^{d-2}(\text{link } p)$ such that $\Delta^0(G_1) \cup \Delta^0(G_2) = \Delta^0(\text{link } p)$. The facets $F_i = \text{conv}(G_i \cup \{p\})$ $(1 \leqq i \leqq 2)$ of P have the required properties. If there is a vertex $p \in \Delta^0(P)$ with $v(p) = d$ we choose $q \in \Delta^0(P) - \Delta^0(\text{st } p)$. We can not have $\Delta^0(\text{link } q) = \Delta^0(\text{link } p)$ since otherwise lemma 1 would imply $\hat{\mathfrak{P}} = \text{st } p \cup \text{st } q$, but the last complex has only $d+2$ vertices. Therefore, by lemma 2, there exist in $\hat{\mathfrak{P}}$ a facet F_1 and a triangle D disjoint to F_1. Choosing any facet $F_2 \supset D$ we have card $(\Delta^0(F_1) \cup \Delta^0(F_2)) \geqq \text{card}(\Delta^0(F_1) \cup \Delta^0(D)) = d+3$, and so $\Delta^0(F_1) \cup \Delta^0(F_2) = \Delta^0(P)$. It remains to show that there is no simplicial d-polytope P with $d+3$ vertices all of which have valence $d+1$ in $\hat{\mathfrak{P}}$. Otherwise we choose a pair of points $p, q \in \Delta^0(P)$ not connected by an edge $L \in \Delta^1(P)$. It follows that $\Delta^0(\text{link } p) = \Delta^0(\text{link } q)$, and this set contains $d+1$ vertices of P. There must be a vertex $x \in \text{link } p$

whose valence in link p is d, for otherwise link p would be isomorphic to the boundary complex of a $(d-1)$-dimensional simplex, and consequently could not contain more than d vertices. Since x is connected to p and q as well, we find $v(x) = d+2$, a contradiction.

In order to prove (II) we choose $p \in \Delta^0(P)$. If there is a vertex $q \in \Delta^0(P) - \Delta^0(\mathrm{st}\, p)$ such that $\Delta^0(\mathrm{link}\, q) \neq \Delta^0(\mathrm{link}\, p)$ we find, by lemma 2, a facet $F \in \Delta^{d-1}(P)$ and a triangle $D \in \Delta^2(P)$ disjoint to F. Let r be a point in $\Delta^0(P) - \Delta^0(F \cup D)$. Any set of facets F_i $(1 \leqq i \leqq 3)$ with $F_1 = F$, $F_2 \supset D$ and $F_3 \supset \{r\}$ fulfils the conditions of (II).

So from now on we can assume $\Delta^0(\mathrm{link}\, q) = \Delta^0(\mathrm{link}\, p)$ for each $q \in \Delta^0(P) - \Delta^0(\mathrm{st}\, p)$. Let us set $L = \Delta^0(\mathrm{link}\, p)$. If every vertex $q \neq p$ is in L there are, by the inductive assumption, facets G_i $(1 \leqq i \leqq 3)$ in $\Delta^{d-2}(\mathrm{link}\, p)$ such that their union contains at least $(d-1)+4$ vertices. The facets $F_i = \mathrm{conv}\,(G_i \cup \{p\})$ of P have the required properties. If there is exactly one point, say q, in $\Delta^0(P) - \Delta^0(\mathrm{st}\, p)$, we have $\mathrm{card}\, L = d+i-2 = (d-1)+(i-1)$. In the case $i=4$ there exist, by (I), two facets G_1, G_2 in $\Delta^{d-2}(\mathrm{link}\, p)$ such that $\mathrm{card}\,(\Delta^0(G_1) \cup \Delta^0(G_2)) = d+i-2$. Setting $F_k = \mathrm{conv}\,(G_k \cup \{p\})$ $(1 \leqq k \leqq 2)$ and choosing any facet F_3 which contains q, we find $\mathrm{card}\left(\bigcup_{k=1}^{3} \Delta^0(F_k)\right) = d+4$. If $i \geqq 5$, the inductive assumption provides us with three facets $G_k \in \Delta^{d-2}(\mathrm{link}\, p)$ such that $\mathrm{card}\left(\bigcup_{k=1}^{3} \Delta^0(G_k)\right) \geqq (d-1)+4$, and the facets $F_k = \mathrm{conv}\,(G_k \cup \{p\})$ are as required. Let us now suppose that there are at least two points, say q and r, in $\Delta^0(P) - \Delta^0(\mathrm{st}\, p)$. By our assumption about the links, no two of the points p, q, r are connected by an edge in $\mathfrak{P}$. If $\mathrm{card}\, L \geqq d+1$ we find a facet $G \in \Delta^{d-2}(\mathrm{link}\, p)$ and two points x, y in $L - \Delta^0(G)$. We set $F_1 = \mathrm{conv}\,(G \cup \{p\})$ and choose facets F_2, containing the edge $\mathrm{conv}\,\{x, q\} \in \Delta^1(P)$, and F_3, containing $\mathrm{conv}\,\{y, r\}$. It follows that $\mathrm{card}\left(\bigcup_{k=1}^{3} \Delta^0(F_k)\right) \geqq d+4$. The case $\mathrm{card}\, L = d$ is excluded by lemma 1.

It would easily follow from the above proposition that $f(d) = 2d+2$. For our purpose we need a few more remarks. The next lemma states a well known property of convex polytopes.

LEMMA 3. *Let $P \subset E^d$ be a d-polytope and X a face of P. If $E \subset E^d$ is a flat which is not parallel to any flat contained in* aff X, *then there is a facet $F \in \Delta^{d-1}(P)$ which contains X and for which* $\dim\,(\mathrm{aff}\, F \cap E) = \dim E - 1$.

LEMMA 4. *Let P be a nonsimplicial d-polytope with $d \geqq 3$ and* $\mathrm{card}\, \Delta^0(P) \geqq d+3$. *There are two facets $F_1, F_2 \in \Delta^{d-1}(P)$ such that* $\mathrm{card}\,(\Delta^0(F_1) \cup \Delta^0(F_2)) \geqq d+3$ *and* $\mathrm{aff}\, F_1 \cap \mathrm{aff}\, F_2 \neq \emptyset$.

PROOF. We choose a facet $F_1 \in \Delta^{d-1}(P)$ with $\mathrm{card}\, \Delta^0(F_1) \geqq d+1$, and a point $p \in \Delta^0(P) - \Delta^0(F_1)$. If $\mathrm{card}\, \Delta^0(F_1) \geqq d+2$ we apply lemma 3 to $X = \{p\}$ and $E = \mathrm{aff}\, F_1$, and find a facet $F_2 \supset \{p\}$ such that $\mathrm{aff}\, F_2 \cap \mathrm{aff}\, F_1 \neq \emptyset$. F_1 and F_2 have the required properties. So let us assume $\mathrm{card}\, \Delta^0(F_1) = d+1$. By $V(p) \subset \Delta^0(P)$ we denote the set of vertices which are connected to p by an edge of P. If there is a vertex $v \in V(p) - \Delta^0(F_1)$, let L be the edge which has p and v as its endpoints. Now we proceed as above, with $X = L$ and $E = \mathrm{aff}\, F_1$ in lemma 3. Assuming on the other hand $V(p) \subset \Delta^0(F_1)$ let x be a point of link p and $S = \mathrm{conv}\,\{x, p\}$. There is a facet $G \in \Delta^{d-1}(\mathrm{st}\, p)$

and a subset $Y \subset \Delta^0(G) \cap V(p)$ such that conv Y contains a point $y \in S$. $H = \operatorname{aff} G \cap \cap \operatorname{aff} F_1$ supports F_1 in aff F_1, and we have $Y \subset H \cap F_1$, therefore $y \in \operatorname{relbd} F_1$. This implies $y = x$ and furthermore $|\operatorname{link} p| \subset \operatorname{relbd} F_1$. Since $|\operatorname{link} p|$ and relbd F_1 are both polyhedral $(d-2)$-spheres it follows that $|\operatorname{link} p| = \operatorname{relbd} F_1$ and further $P = \operatorname{conv}(F_1 \cup \{p\})$. This however is not possible, because $\Delta^0(F_1) \cup \{p\}$ contains only $d+2$ vertices of p.

Now we are ready to prove our main result.

PROPOSITION 2. *For $d \geqq 3$, no d-polytope with less than $2d+2$ vertices is hindered by its vertices.*

PROOF. Let P be a d-polytope with card $\Delta^0(P) < 2d+1$. By adding an appropriate number of pyramidal caps we obtain a polytope P' with card $(\Delta^0(P')) = = 2d+1$. If P is hindered by its vertices then each P' obtained in this way also is. So we can restrict our attention to the d-polytopes with $2d+1$ vertices. First we show that for each such polytope P there is a set $\mathfrak{F} \subset \Delta^{d-1}(P)$ of facets such that

$$(*) \qquad \begin{gathered} \operatorname{card} \mathfrak{F} \leqq d-1, \quad \dim\Big(\bigcap_{F \in \mathfrak{F}} \operatorname{aff} F\Big) = d - \operatorname{card} \mathfrak{F}, \\ \operatorname{card}\Big(\bigcup_{F \in \mathfrak{F}} \Delta^0(F)\Big) \geqq d + \operatorname{card} \mathfrak{F} + 1. \end{gathered}$$

We distinguish three cases. If P is not simplicial, we set $\mathfrak{F} = \{F_1, F_2\}$, where F_1 and F_2 are the facets mentioned in lemma 4. If P is a simplicial three-polytope we find, for example by direct examination of the five possible combinatorial types, three facets F_i in $\Delta^2(P)$ such that $F_1 \cap (F_2 \cup F_3) = \emptyset$. Since $\Delta^0(P) = 7$, this means that $F_2 \cap F_3$ is an edge $K \in \Delta^1(P)$. By applying lemma 3 to K and the flat aff F_1, we find an index $i \in \{2, 3\}$ such that $\operatorname{aff} F_1 \cap \operatorname{aff} F_i \neq \emptyset$. The set $\mathfrak{F} = \{F_1, F_i\}$ has the required properties. If P is a simplicial d-polytope with $d \geqq 4$, there are, by proposition 1, (II), three facets F_i in $\Delta^{d-1}(P)$ such that card $\left(\bigcup_{i=1}^{3} \Delta^0(F_i)\right) \geqq d+4$. If $\bigcap_{i=1}^{3} \Delta^0(F_i) \neq \emptyset$, the set $\mathfrak{F} = \{F_1, F_2, F_3\}$ has the required properties. Otherwise we have for each choice of distinct indices $\Delta^0(F_i \cap F_j) \cap \Delta^0(F_i \cap F_k) = \emptyset$, which means that there are indices i, j such that card $\Delta^0(F_i \cap F_j) \leqq (1/2)$ card $\Delta^0(F_i) = d/2$, or, in other words, card $(\Delta^0(F_i) - \Delta^0(F_j)) \geqq d/2$. For $d \geqq 5$ it follows that card $(\Delta^0(F_i) \cup \cup \Delta^0(F_j)) \geqq d+3$. It is easy to see that also for $d=4$ there must be two indices i, j with card $(\Delta^0(F_i) \cup \Delta^0(F_j)) \geqq d+3$, since otherwise it would follow that for all i, j card $(\Delta^0(F_i) \cap \Delta^0(F_j)) = 2$. This is only possible for configurations with card $\left(\bigcup_{i=1}^{3} \Delta^0(F_i)\right) = 6$, which contradicts the choice of our facets F_i. So we find always two facets F_1, F_2 with card $(\Delta^0(F_1) \cup \Delta^0(F_2)) \geqq d+3$. If $\operatorname{aff} F_1 \cap \operatorname{aff} F_2 \neq \emptyset$ we can set $\mathfrak{F} = \{F_1, F_2\}$; otherwise denote by $G \in \Delta^{d-2}(P)$ any $(d-2)$-dimensional face of F_2. By applying lemma 3 to G and aff F_1, we find a facet $F_0 \supset G$ such that $\operatorname{aff} F_0 \cap \operatorname{aff} F_1 \neq \emptyset$. $\mathfrak{F} = \{F_0, F_1\}$ fulfils the conditions mentioned under (*). Now let $\mathfrak{F}_0$ be a maximal set of facets satisfying (*). We want to show card $\Big(\bigcup_{F \in \mathfrak{F}_0} \Delta^0(F)\Big) \geqq 2d$. Otherwise we obviously have card $\mathfrak{F}_0 \leqq d-2$ and, with $E_0 = \bigcap_{F \in \mathfrak{F}_0} \operatorname{aff} F$, $\dim E_0 \geqq 2$.

Let p be a vertex in $\Delta^0(P)-\bigcup_{F\in\mathfrak{F}_0}\Delta^0(F)$. By lemma 3, applied to $\{p\}$ and E_0, we find a facet G containing p such that $\dim(\operatorname{aff} G\cap E_0)=\dim E_0-1$. The set $\mathfrak{F}_0\cup\{G\}$ fulfils $(*)$, contradicting the maximality of $\mathfrak{F}_0$. So we have found for each d-polytope with $2d+1$ vertices a set $\mathfrak{F}_0\subset\Delta^{d-1}(P)$ such that, with $E_0=\bigcap_{F\in\mathfrak{F}_0}\operatorname{aff} F$, $\dim E_0\geqq 1$, $\operatorname{card}\left(\Delta^0(P)-\bigcup_{F\in\mathfrak{F}_0}\Delta^0(F)\right)\leqq 1$. We can assume without restriction that E_0 contains the origin of E^d, such that each $\operatorname{aff} F$ $(F\in\mathfrak{F}_0)$ is a subspace. Set $H(F)=\operatorname{aff} F$ and denote the two closed halfspaces, which are determined by $H(F)$, by $H^+(F)$ and $H^-(F)$, where $H^+(F)$ is the halfspace containing P. Set $H^+=\bigcap_{F\in\mathfrak{F}_0}H^+(F)$ and $H^-=\bigcap_{F\in\mathfrak{F}_0}H^-(F)$. Since $H^-=-H^+$, we easily derive that H^- is a convex cone with nonempty interior H^-, whose lineality space is a subspace $E_0\subset E^d$ of dimension $\geqq 1$. Therefore, considering that $\Delta^0(P)-\bigcup_{F\in\mathfrak{F}_0}\Delta^0(F)$ contains at most one point, we can choose a unit vector $u\in H^-$ such that, if $\Delta^0(P)-\bigcup_{F\in\mathfrak{F}_0}\Delta^0(F)$ consists of the point p_0, we have $u\notin A(P,p_0)$. It immediately follows that u is not contained in any angle $A(P,p)$ $\left(p\in\Delta^0(P)\right)$, for let p be a vertex different from p_0. This means that p is contained in a facet $F\in\mathfrak{F}_0$, and since u lies in the halfspace $H^-(F)$, it cannot belong to the interior angle $A(P,p)$. In other words, P is not hindered by its vertices.

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AN EXTREMAL GRAPH PROBLEM

By

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Throughout this paper graphs are supposed not to contain loops and multiple edges. G^n denotes a graph of n vertices but only if n is an upper index. $e(G)$ denotes the number of edges, $v(G)$ denotes the number of vertices, $\chi(G)$ denotes the chromatic number of G. $G_1 \times \cdots \times G_d$ or $\bigtimes_{i=1}^{d} G_i$ denotes the product of the G_i's, i.e. the graph obtained from the graphs $G_1, \ldots, G_d$ by joining any two vertices belonging to different G_i's. Here the graphs $G_1, \ldots, G_d$ are supposed to be vertex-independent. $K_d(r_1, \ldots, r_d)$ denotes the complete d-chromatic graph with r_i vertices of the i^{th} colour, i.e. $K_d(r_1, \ldots, r_d) = \bigtimes_{i=1}^{d} G_i$ where $e(G_i)=0$, $v(G_i)=r_i$. If E is any set, $|E|$ denotes the number of its elements.

Introduction

P. TURÁN proved in 1941 [1] that if $K^n = \bigtimes_{i=1}^{p-1} G^{n_i}$ where $n_i = \left[\frac{n}{p-1}\right]$ or $n_i = \left[\frac{n}{p-1}\right]+1$, and $e(G^{n_i})=0$ then K^n does not contain a complete p-graph and if G^n is an arbitrary other graph not containing a complete p-graph, then $e(G^n) < e(K^n)$.

This is the source of the following problems:

PROBLEM 1. Let $G_1, \ldots, G_l$ be given graphs. What is the maximum number of edges a graph can have if it does not contain any G_j as a subgraph?

Putting

$$(1) \qquad f(n; G_1, \ldots, G_l) = \max\{e(G^n) : G_i \nsubseteq G^n, \quad i = 1, \ldots, l\}$$

the problem can be rephrased:

Determine the function $f(n; G_1, \ldots, G_l)$ for given graphs $G_1, \ldots, G_l$.

PROBLEM 2. The graphs attaining the maximum in (1) are called extremal graphs. Determine the structure of the extremal graphs for given $G_1, \ldots, G_l$ and n.

The answer for these problems is fairly similar to the answer for TURÁN's original problem:

I. We have proved [2] that

$$f(n; G_1, \dots, G_l) = \binom{n}{2}\left(1-\frac{1}{d}+o(1)\right) \tag{2}$$

$$\text{where} \quad d+1 = \min_{1\leq i\leq l} \chi(G_i). \tag{3}$$

(2) and (3) express that $f(n; G_1, \dots, G_l)$ depends very loosely on the structure of the graphs $G_1, \dots, G_l$, its order of magnitude is already determined by the minimal chromatic number.

II. Later we proved independently [3], [4] that the structure of the extremal graphs is also fairly independent of the G_i's. Our most interesting results connected with Problem 2 can be summarized as follows:

Let $G_1, \dots, G_l$ be given graphs, K^n be an extremal graph for $G_1, \dots, G_l$ and n be large enough. Then there exists an integer $r>0$ (depending on some colouring properties of G_i's) such that

A) K^n can be obtained from a graph-product $\bigtimes_{i=1}^{d} N_i$ by omitting $O\left(n^{2-\frac{1}{r}}\right)$ edges from and adding $O\left(n^{2-\frac{1}{r}}\right)$ new edges to it. Here

$$d+1 = \min \chi(G_i).$$

B) The components of the product are of almost equal size:

$$n_i = v(N_i) = \frac{n}{d}+O\left(n^{1-\frac{1}{r}}\right)$$

C) Each vertex $x\in K^n$ has valency greater than $\frac{n}{d}(d-1)-c_1 n^{1-\frac{1}{r}}$ where c_1 is a suitable constant.

D) Let $\varepsilon>0$ be fixed. There is a constant K_ε such that the number of vertices of N_i joined to at least εn_i vertices of N_i is less than K_ε.

These assertions have asymptotic character. They illustrate that the extremal graphs are very similar to that one in TURÁN's original theorem. They are the best possible in a certain way. The theorem we prove in this paper has "exact character" but the graphs G_i are more special.

Here we have to remark, that this theorem is the first one, which describes the structure of rather complicated extremal graphs fairly well.

THEOREM. *Let $r_1=1$, 2 or 3, $r_1\leqq r_2\leqq\dots\leqq r_{d+1}$ be given integers. If n is large enough, then each extremal graph K^n for $K_{d+1}(r_1, \dots, r_{d+1})$ is a graph product:*

$$K^n = \bigtimes_{i=1}^{d} N_i$$

where

1) $n_i = v(N_i) = \frac{n}{d}+o(n)$;

2) *N_1 is an extremal graph for $K_2(r_1, r_2)$;*

3) *$N_2, \dots, N_d$ are extremal graphs for $K_2(1, r_2)$.*

Conversely, if $\hat{N}_1, \ldots, \hat{N}_d$ *are given graphs such that*

4) *there exists an extremal graph* $\overset{d}{\underset{i=1}{\times}} N_i$ *satisfying* 1), 2), 3) *such that* $v(\hat{N}_i) = v(N_i)$;

5) $\hat{N}_1$ *is an extremal graph for* $K_2(r_1, r_2)$;

6) $\hat{N}_i$ *is an extremal graph for* $K_2(1, r_2)$, $K_2(2, 2)$, $K_3(1, 1, 1)$ $(i \neq 1)$,

then $\hat{K}^n = \overset{d}{\underset{i=1}{\times}} \hat{N}_i$ *is an extremal graph for* $K_{d+1}(r_1, \ldots, r_{d+1})$.

REMARK 1. Our theorem does not characterize the extremal graphs for $K_{d+1}(r_1, \ldots, r_{d+1})$ completely. First of all, we do not know the extremal graphs for $K_2(r_1, r_2)$ sufficiently well. Further, just because of this lack of knowledge about the extremal graphs we do not know the exact values of n_i for given n. The extremal graphs are those among the described ones which have the maximum number of edges. As far as we know this can occur for many different choices of the n_i.

REMARK 2. For $r_1 = 1$ [4] proves the statement. We shall prove it only for $r_1 = 3$. The case $r_1 = 2$ can be treated similarly.

REMARK 3.

$$f(n; K_2(r_1 - 1, r_2)) = o(f(n; K_2(r_1, r_2))) \quad \text{if} \quad r_1 \leqq r_2 \tag{4}$$

probably always holds, but we do not know it for $r_1 \geqq 4$. This is why we can prove the theorem only for $r_1 < 4$. (4) can be proved for $r_1 = 2$ as follows: T. KŐVÁRI, V. T. SÓS and P. TURÁN [5] and independently P. ERDŐS (unpublished) proved that

$$f(n; K_2(p, q)) = O\left(n^{2-\frac{1}{p}}\right) \quad \text{if} \quad p \leqq q. \tag{5}$$

P. ERDŐS, A. RÉNYI and V. T. SÓS proved for $p = 2$, BROWN for $p = 2, 3$ that (5) can not be improved [6], [7]:

$$f(n; K_2(2, 2)) = \frac{1}{2} n^{3/2} + o(n^{3/2}) \tag{5a}$$

and

$$\underline{\lim} f(n; K_2(3, 3))/n^{5/3} > 0 \quad \text{if} \quad n \to \infty. \tag{5b}$$

Now, (5a), (5b) and (5) imply (4) if $r_1 = 3$.

Trivially (5b) gives a lower estimation for $f(n; K_2(4, 4))$. We do not know any better lower estimation for it.

REMARK 4. In a forthcoming paper M. SIMONOVITS is going to prove some generalizations, based on Remark 3.

Proofs

First we prove two lemmas.

LEMMA 1. *Let G_1 be a graph not containing $K_2(r_1, r_2)$, let G_i $(i=2, \dots, d)$ be graphs not containing $K_2(1, r_2)$, $K_2(2, 2)$, $K_3(1, 1, 1)$, where $r_1 \leqq r_2 \leqq \dots \leqq r_{d+1}$ are given positive integers. Then $\bigtimes_{i=1}^{d} G_i$ does not contain $K_{d+1}(r_1, \dots, r_{d+1})$.*

PROOF. It is sufficient to consider only the case $r_2 = r_3 = \dots = r_{d+1}$. We prove, that if G_d does not contain any of $K_2(r_1, r_2)$, $K_2(2, 2)$, $K_3(1, 1, 1)$ and G does not contain any $K_d(r_1, r_2, r_2, \dots, r_2)$, then $G \times G_d$ does neither contain any $K_{d+1}(r_1, r_2, r_2, \dots, r_2)$. From this the lemma follows immediately by mathematical induction.

First we remark, that $K_{d+1}(r_1, r_2, \dots, r_2)$ has the following property: If we omit some vertices $x_1, x_2, \dots, x_\lambda$ from it and either all these vertices belong to the same class or $x_2, x_3, \dots, x_\lambda$ belong to the same class and $\lambda < r_2$, then the remaining graph contains a $K_d(r_1, r_2, \dots, r_2)$. This assertion is trivial if all the vertices belong to the same class. In the other case let us denote by $U_1, \dots, U_{d+1}$ the classes of $K_{d+1}(r_1, r_2, \dots, r_2)$ and suppose that $x_1 \in U_j$, $x_2, \dots, x_\lambda \in U_k$. Let V be the empty set if $U_k = \{x_2, \dots, x_\lambda\}$ and a set containing exactly one vertex of $U_k - \{x_2, \dots, x_\lambda\}$ otherwise. Then one can easily show that the classes U_i $(i \neq j, k)$ and $U_j \cup V - \{x_1\}$ span a graph containing $K_d(r_1, r_2, \dots, r_2)$.

Let us consider now $G \times G_d$ and suppose that it contains a $K_{d+1}(r_1, r_2, \dots, r_2)$ the classes of which are $U_1, U_2, \dots, U_{d+1}$. We show, that either G_d contains only vertices of one U_j or it contains one vertex from a U_j and at most $r_2 - 1$ vertices belonging to another U_k.

Indeed, if there were $x, y, z \in G_d$ belonging to different U_j's then they would determine a $K_3(1, 1, 1) \subseteqq G_d$ contradicting our assumptions. Thus $G_d \cap U_j$ is empty for all but at most two values of j. If there existed $u_1, u_2 \in U_j \cap G_d$, $v_1, v_2 \in U_k \cap G_d$ then they would determine a $K_2(2, 2) \subseteqq G_d$ contradicting our assumptions. Thus, $G_d \cap K_{d+1}(r_1, r_2, \dots, r_2)$ contains vertices, belonging to the same U_j or a vertex $x \in U_k$ and at most $r_2 - 1$ other vertices belonging to the same U_j indeed. ($|U_j \cap G_d| < r_2$ since G_d does not contain a $K_2(1, r_2)$.) Because of this there is a $K_d(r_1, \dots, r_2)$ determined by the other vertices of $K_{d+1}(r_1, \dots, r_2)$ which is contained by $G \times G_d - G_d = G$. This contradiction proves Lemma 1.

LEMMA 2. *Let G^ν, r be given, $r \geqq 3$. There exists a constant $c_{\delta, r} > 0$ depending only on δ and r such that if G^ν is a graph not containing $K_2(3, r)$ and $x \in G^\nu$ is a vertex of valency greater than $\delta\nu$ in it then*

$$e(G^\nu) \leqq f(\nu; K_2(3, r)) - c_{\delta, r} \cdot \nu^{5/3} \tag{6}$$

PROOF. Let C be a subclass of vertices of G^ν consisting of $\approx \delta\nu$ vertices, each of which is joined to x. Then for no $p_1, \dots, p_r \in C$, $u, v \in G^\nu - \{x\}$ the set of these vertices determines a $K_2(2, r)$ the first class of which is $\{u, v\}$; otherwise $\{x, u, v\}$ and $\{p_1, \dots, p_r\}$ would determine a $K_2(3, r) \subseteqq G^\nu$. Therefore the graph determined by the edges both endpoints of which belong to C, does not contain a $K_2(2, r)$. Similarly the bipartite graph, determined by the edges one endpoint of which belongs to C, the other to $G^\nu - C - \{x\}$, does neither contain a $K_2(2, r)$ the second class of which is in C. Therefore the number of these edges is $O(\nu^{3/2})$. (The proof in [5] also gives this.) The remaining edges of G^v have both their endpoints in $G^v - C$, thus the number of these edges is at most $f((1-\delta)v; K_2(3, r))$. Thus

$$e(G^v) \leqq f((1-\delta)v; K_2(3, r)) + O(v^{3/2}).$$

Since the disjoint union of two extremal graphs for $K_2(3, r)$ does not contain a $K_2(3, r)$ either,

(7) $$f(v_1+v_2; K_2(3, r)) \geqq f(v_1; K_2(3, r)+f(v_2; K_2(3, r)).$$

Thus

(8) $$e(G^v) \leqq f(v; K_2(3, r)) + O(n^{3/2}) - f(\delta v; K_2(3, r)).$$

Since $f((\delta v; K_2(3, r)) \geqq c_r(\delta v)^{5/3}$, (8) implies (6).

PROOF OF THEOREM. Let K^n be an extremal graph for $K_{d+1}(r_1, \ldots, r_{d+1})$ and colour it by d colours so that the number of edges, having endpoints of the same colour be minimal. Then there exist an integer r and graphs $N_1, \ldots, N_d$ so that A), B), C), D) hold (see Introduction and [4], [3]). We shall use them only in the following weaker form:

α) C_i denotes the class of vertices of N_i, $|C_i| = n_i = \frac{n}{d} + o(n)$.

β) All the vertices have valency greater than $\frac{n}{d}(d-1) - o(n)$.

γ) Let $\varepsilon > 0$ be a small constant (fixed only later). Let us denote the class of vertices of C_i, joined to at most εn vertices of the same C_i by C_i'. Then there exists a constant K_ε depending only on ε and $r_1, \ldots, r_{d+1}$ such that $|C_i - C_i'| < K_\varepsilon$. The vertices of $C_i - C_i'$ will be called exceptional vertices, and γ) expresses that their number is bounded. Clearly, if $x \in C_i'$, then x is joined to at most εn vertices of C_i but if $n > n_0(\varepsilon)$ it is joined to at least $|C_j| - 2\varepsilon n$ vertices of C_j because of α) and β) $(i \neq j)$.

I. Let $E = \sum_{1 \leqq i < j \leqq d} n_i n_j$. Trivially, E is the number of pairs of vertices in K^n belonging to different classes.

Lemma 1 implies that

(9) $$f(n; K_{d+1}(r_1, \ldots, r_{d+1})) = e(K^n) \geqq E + f(n_1; K_2(3, r_2)) + \sum_{i=2}^{d} f(n_i; K_2(1, r_2)).$$

Indeed, if G^{n_1} is an extremal graph for $K_2(r_1, r_2)$, $G^{n_1}, \ldots, G^{n_d}$ are extremal graphs for $\{K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1)\}$, then $G^n = \bigtimes_{i=1}^{d} G^{n_i}$ does not contain a $K_{d+1}(r_1, \ldots, r_{d+1})$, thus $e(K^n) \geqq e(G^n)$. It is easy to see that the extremal graphs for $\{K_2(1, r_2)\ K_2(2, 2), K_3(1, 1, 1)\}$ are also extremal graphs for $K_2(1, r_2)$, if n is large enough. If $n_i(r_2-1)$ is even, the extremal graphs for $K_2(1, r_2)$ are regular graphs of degree $r_2 - 1$. If $n_i(r_2-1)$ is odd, such graphs do not exist, the extremal graphs have $n_i - 1$ vertices of valency $r_2 - 1$ and one vertex of valency $r_2 - 2$. If n_i is large enough, among these graphs there exist graphs not containing either $K_2(2, 2)$ or $K_3(1, 1, 1)$. This and

$$f(n; K_2(1, r_2)) \geqq f(n; K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1))$$

prove that

$$f(n; K_2(1, r_2)) = f(n; K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1))$$

for large values of n_i. This implies, that each extremal graph for $\{K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1)\}$ is also an extremal graphs for $K_2(1, r_2)$. Therefore, the right hand side of (9) equals to $e(G^n) \leqq e(K^n)$. Thus (9) holds.

II. First we remark, that C_i' does not contain a $K_2(3, r_2)$; for if it contained a $K_2(3, r_2)$, we could find a $K_{d-1}(r_3, \dots, r_{d+1})$ in the graph spanned by the other classes so that $K_2(3, r_2) \times K_{d-1}(r_3, \dots, r_{d+1}) = K_{d+1}(3; r_2, \dots, r_{d+1})$ would be contained by K^n.

Now we prove that if C_1' contains a $K_2(2, r_2)$, then for $i \geqq 2$, C_i' does not contain a $K_2(1, r_2)$. Let us denote by B_j $(j=2, \dots, d)$ the class of vertices of C_j' $(j=2, \dots, d)$ joined to all vertices of the fixed $K_2(2, r_2) \subseteqq C_1'$. If there were a $u \in B_j$ and $v_1, \dots, v_{r_3} \in B_j$ joined to u $(j \geqq 2)$, then these r_3+1 vertices and the fixed $K_2(2, r_2) \subseteqq C_1'$ and $r_4, \dots, r_{d+1}$ suitable vertices of $B_3, \dots, B_d$ (if $d \geqq 3$) would determine a $K_{d+1}(3, r_2, \dots, r_{d+1})$ in K^n if ε is small enough. (The expression "suitable" means: the other vertices must determine a $K_{d-3}(r_4, \dots, r_{d+1})$ each vertex of which is joined to each vertex of the fixed $K_2(2, r_2)$ and to $u, v_1, \dots, v_{r_3}$.) Therefore the set $\{u, v_1, \dots, v_{r_3}\}$ can not exist. Thus B_j contains $O(n)$ edges. Let us consider the j^{th} class, $j \geqq 2$. The number of edges in $C_j' - B_j$ is $O(m_j^{5/3})$, where

$$m_j = |C_j' - B_j| \leqq (2+r_2) \cdot 2\varepsilon n.$$

The remaining edges of K^n in C_j' join $C_j' - B_j$ to B_j. Their number is $O(nm_j^{2/3})$.[1]

Let us divide B_j into classes of $\approx m_j$ vertices. Each of these classes together with $C_j' - B_j$ determines a graph of $\approx 2m_j$ vertices, not containing $K_2(3, r_2)$. Therefore each of them has $O(m_j^{5/3})$ edges and their number is $\approx \frac{n}{dm_j}$. Thus C_j' contains

$$O(n) + O(m_j^{5/3}) + O(nm_j^{2/3}) = \varepsilon^{2/3} \cdot O(n^{5/3})$$

edges and the same bound holds for C_j. Thus $C_2, \dots, C_d$ contain $\varepsilon^{2/3} \cdot O(n^{5/3})$ edges.

Let us suppose now that C_2' contains a $K_2(1, r_2)$ and let A_1 denote the set of vertices of C_1' joined to this $K_2(1, r_2)$. Clearly, A_1 does not contain any $K_2(2, r_3)$, otherwise $C_2' \cup A_1$ would contain a $K_2(2, r_3) \times K_2(1, r_2) \supseteqq K_3(3, r_2, r_3)$ and taking suitable vertices from the other classes we could complete this $K_3(3, r_2, r_3)$ into a $K_{d+1}(3, r_2, r_3, \dots, r_{d+1}) \subseteqq K^n$. Therefore, the method used above gives that C_1' contains only $\varepsilon^{2/3} O(n^{5/3})$ edges. The same bound is valid for C_1, thus

$$e(K^n) \leqq E + \varepsilon^{2/3} O(n^{5/3}). \tag{10}$$

Now we fix ε so that (10) should contradict (9). Thus C_2' does not contain $K_2(1, r_2)$ and generally, C_j' $(j \geqq 2)$ also does not contain it.

In general it could happen that C_1' did not contain $K_2(2, r_2)$. But if no C_j' contained a $K_2(2, r_2)$, then

$$e(K^n) \leqq E + d \cdot O(n^{3/2}) + O(n)$$

would hold contradicting (9). Thus we may assume that C_1' does contain a $K_2(2, r_2)$ and $C_2', \dots, C_d'$ do not contain any $K_2(1, r_2)$.

[1] This can also be derived directly from the proof of [5].

III. Now we show that if n is sufficiently large, then there exist no exceptional vertices: $C_i'=C_i$. Actually we prove that if $\varepsilon' = \frac{1}{2}r_{d+1}\cdot d\cdot\varepsilon$ and n is sufficiently large, then K^n contains no vertices joined to at least $\varepsilon' n$ vertices of each class. Since ε is an arbitrarily small positive number, this gives that the maximal valency in N_i is $o(n)$. This, of course, implies that $C_i'=C_i$ for $n>n_0$.

Let us suppose that $x\in K^n$ is joined to at least $\varepsilon' n$ vertices of each class. Then the graph G^* spanned by x and C_1' can not contain a $K_2(3, r_2)$. Indeed, since C_1' does not contain a $K_2(3, r_2)$, if G^* does, then x must be a vertex of this $K_2(3, r_2)$. Since each non-exceptional vertex is joined to all the vertices of the other classes but at most εn, we may select succesively $r_3, \dots, r_{d+1}$ vertices of $C_2', \dots, C_d'$ so that the selected vertices span a $K_{d-1}(r_3, \dots, r_{d+1})$ and are joined to each vertex of the fixed $K_2(3, r_2)$. Thus K^n contains a $K_{d+1}(3, r_2, \dots, r_{d+1})$. This contradiction proves that G^* can not contain any $K_2(3, r_2)$. Thus C_1' (and C_1 as well) contain $f(n_1; K_2(3, r_2))-cn^{5/3}$ edges (Lemma 2) where $c>0$. Since C_i' $(i\geqq 2)$ does not contain any $K_2(1, r_2)$,

$$e(K^n) \leqq E+f(n_1; K_2(3, r_2))-cn^{5/3}+O(n). \tag{12}$$

But (12) contradicts (9). This proves that K^n has no exceptional vertices: $C_i'=C_i$. Thus C_1 does not contain $K_2(3, r_2)$, $C_2, \dots, C_d$ do not contain $K_2(1, r_2)$ and consequently

$$e(K^n) \leqq E+f(n_1; K_2(3, r_2))+\sum_{i=2}^{d} f(n_i, K_2(1, r_2)). \tag{13}$$

(13) and (9) proves that

$$e(K^n) = E+f(n_1, K(3, r_2))+\sum_{i=2}^{d} f(n_i, K_2(1, r_2)). \tag{14}$$

Since C_1 does not contain $K_2(3, r_2)$, the graph spanned by it must be an extremal graph for $K_2(3, r_2)$, otherwise the "equal" of (14) would be "definitely less". Similarly, the graphs spanned by $C_2, \dots, C_d$ are extremal graphs for $K_2(1, r_2)$ and if they are denoted by $N_1, N_2, \dots, N_d$, then $K^n = \bigtimes_{i=1}^{d} N_i$, i.e. every two vertices are joined, if they belong to different C_i's.

The second part of the Theorem is trivial now: If $\hat{N}_i$ satisfies our conditions, $\hat{K}^n = \bigtimes_{i=1}^{d} \hat{N}_i$ has the same number of edges as K^n and according to Lemma 1 it does not contain a $K_{d+1}(r_1, \dots, r_{d+1})$. Therefore it is an extremal graph for it. This completes our proof.

REMARK 5. An easy discussion shows that if $r_1\geqq 2$, $r_2\geqq 3$, $\{K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1)\}$ can be replaced by $\{K_2(1, r_2), K_2(2, 2)\}$ but it cannot be replaced by $K_2(1, r_2)$.

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PACKING CONVEX SETS INTO A SIMILAR SET

By

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I. Introduction. This paper deals with a subject which arises out of a famous problem.[1] The problem, which is an old chestnut and makes the rounds every so often, goes as follows: Let squares of perimeter a and b be enclosed in a square of perimeter c so that their interiors do not overlap. Then one must show that $a+b \leqq c$.

The problem, which is of very modest difficulty, becomes more interesting when one realizes that the same conclusion holds trivially if the squares are replaced by circles of perimeter (or radius or diameter) a, b, and c. There is a property here (the property is described in detail below) which is shared by the square and the circle, and certain other figures but not all. The property can be described as follows:

Let K be a convex figure, and let K_0, K_1, and K_2 be figures similar to K, with K_1 and K_2 contained in K_0, and K_1 and K_2 having non-overlapping interiors, i.e., K_1 and K_2 are packed into K_0. Then K is called *tight* if for every such configuration the perimeters of K_1 and K_2 add up to no more than the perimeter of K_0. Our opening comments, then, in this language, were to the effect that the square and the circle are tight figures. This problem is a genetic problem in the sense of FEJES TÓTH [2, preface]. The solution which this problem generates is indeed interesting in as much as it consists of two much studied and beautiful classes of convex sets. We shall prove, *inter alia*, that a figure is tight exactly if it is either a regular polygon or a curve of constant width.

Looking at the same problem a little differently, we can define a number $\hat{K}$ for each convex figure K as the supremum of $\dfrac{\pi(K_1)+\pi(K_2)}{\pi(K_0)}$ for K_0, K_1, and K_2 where K_1 and K_2 are packed into K_0, and π is the symbol for perimeter. We show that $1 \leqq \hat{K} \leqq \sqrt{2}$ for all convex K. The case $\hat{K}=1$ was already mentioned. The case $\hat{K}=\sqrt{2}$ occurs iff K is either an isosceles right triangle or a parallelogram whose sides are in the ratio $\sqrt{2}$.

We can also ask the same questions on the assumption that K_0, K_1, and K_2 are not only similar, but that the similarity is positive. In that case, the same answer still applies for $\hat{K}=1$; for $\hat{K}=\sqrt{2}$, K is either an isosceles right triangle or a *rectangle* whose sides are in the ratio $\sqrt{2}$.

In the course of developing this material, we have come upon an interesting condition similar to the condition of constant width, which we call the condition

[1] The authors first heard of this problem from NEWMAN. The first author from D. J. NEWMAN and the second from MORRIS NEWMAN when he sent this problem in to the Mathematical Talent Search at the University of Wisconsin.

of constant minimum width, and we use this condition to show a characterization (possibly new) of curves of constant width. Our work in this paper is restricted to the plane.

These problems give rise to similar problems in higher dimensions and also to the problem of packing more than two similar sets in a similar set. However the generalizations of the tight case appear much harder. The upper bound and its cases of equality may often be easier.

II. Definitions, notation, and preliminary remarks. Let K be any closed bounded convex plane set. The perimeter of K, we denote by $\pi(K)$, the area of K by $A(K)$, the boundary of K by $\partial(K)$, and the interior of K by K^0. We say that K_1 and K_2 are *packed* in K_0 if $K_1 \cup K_2 \subset K_0$ and $K_1^0 \cap K_2^0 = \Box$. We use the symbol $\cong$ to denote congruence, and $\cong^+$ to denote positive congruence, with $\sim$ and $\sim^+$ used in the same way for similarity. $\hat{K}$ is defined by

$$\hat{K} = \sup \left\{ \frac{\pi(K_1)+\pi(K_2)}{\pi(K_0)} \,|K \sim K_0 \sim K_1 \sim K_2| \text{ and } K_1 \text{ and } K_2 \text{ are packed in } K_0 \right\}.$$

$\hat{K}^+$ is defined likewise, using $\sim^+$ instead of $\sim$.

A *diameter* of a convex set K is a chord of maximal length, and that length, denoted by $\mathbf{d}(K)$ is called *the diameter* of K. This ambiguous usage rarely causes any difficulty. We recall that the lines perpendicular to a diameter at its endpoints are support lines ([3], p. 9).

One configuration with which we will often use concerns the case where K_1 and K_2 are packed in K_0 in a special way.

Let $\overline{P_1 P_2}$ be a chord of K_0 such that P_1 and P_2 have parallel support lines l_1 and l_2. Let λ be chosen so that $0<\lambda<1$. Let K_1 be the set obtained by a dilation of K_0 with ratio λ taking P_1 as the origin. Let K_2 be the set obtained by a dilation of ratio $1-\lambda$ with origin P_2. Then K_1 and K_2 are packed in K_0 since they are sepatared by their common support line l where $l \| l_1$ and l divides $\overline{P_1 P_2}$ in ratio $\lambda:(1-\lambda)$. We say in this case that K_1 and K_2 are *arrayed* along the chord $\overline{P_1 P_2}$ with parameter λ. If K_1 and K_2 are packed in K_0 so that $\pi(K_1)+\pi(K_2) = \hat{K}\pi(K_0)$, then K_1 and K_2 are called a *maximal pair* in K_0. We normalize our units so that $\pi(K_0)=1$.

If $\partial(K)$ contains a line segment, then that line segment is called a *side* of K. If, at some point D of ∂K, there is more than one line of support, then D is an *angle* (or *angle point*) of K. If P is an angle point, and also the endpoint of two sides, then P is a *vertex* of K.

1. Lemma. *For any closed, bounded, convex figure K, we have $1 \leqq \hat{K} \leqq \sqrt{2}$.*

Proof. To prove the first half of the lemma, let K_0 be any figure similar to K and let $\overline{AB}$ be a diameter of K_0. Let K_1 and K_2 be arrayed along the diameter $\overline{AB}$ with parameter λ, for some λ. Then $\pi(K_1)=\lambda$, $\pi(K_2) = (1-\lambda)$, and $\pi(K_1)+\pi(K_2) = 1$. Thus, the set of ratios over which we maximize to find $\hat{K}$ includes 1, assuring that $1 \leqq \hat{K}$.

On the other hand, we know that there exists a constant a with the property that $A(K')=a(\pi(K'))^2$, for every figure K' similar to K. If K_1 and K_2 are packed in K_0, then $A(K_1)+A(K_2) \leqq A(K_0)$, so that $1 = (\pi(K_0))^2 \geqq (\pi(K_1))^2+(\pi(K_2))^2$. If we let $b_1=\pi(K_1)$, $b_2=\pi(K_2)$, then we wish to maximize b_1+b_2, given that

$b_1^2+b_2^2 \leqq 1$. It is well-known that this maximum is $\sqrt{2}$, and is achieved exactly when $b_1 = b_2 = \frac{1}{2}\sqrt{2}$. Q.E.D.

We note that in order for $\hat{K}=\sqrt{2}$, it is necessary that $A(K_1) = A(K_2) = \frac{1}{2} A(K_0)$.

2. LEMMA. *For every closed, bounded, convex K, there always exists a maximal pair with neither set degenerate.*

PROOF. We take two cases, according as $\hat{K}=1$ or $\hat{K}>1$. In the former case, any pair arrayed along any diameter with parameter λ, $0<\lambda<1$ is a maximal pair. In the latter, we observe that the possible configurations of K_1 and K_2 for a given K_0 form an eight dimensional set,[2] and the instances where K_1 and K_2 are packed in K_0 form a closed, bounded subset. The function $\pi(K_1)+\pi(K_2)$ is continuous on this set and thus achieves its maximum. We note that since the perimeter of the smaller of K_1, K_2 must exceed $(\hat{K}-1)\pi(K_0)$, the maximum actually represents a configuration in which neither is reduced to a point. Q.E.D.

III. The condition $\hat{K}=\sqrt{2}$. In our study of the extreme values which $\hat{K}$ can assume, we begin with consideration of which figures K can have $\sqrt{2}$ for that value. In this case, we see from Lemma 1 that $K_1 \cong K_2$ and that $K_1 \cup K_2 = K_0$. We see easily that $K_1 \cap K_2$ is a closed, non-empty subset of K_0. Since K_1 and K_2 are convex, $K_1 \cap K_2$ is a subset of a line. Since K_0 is convex, $K_1 \cap K_2$ is not a single point. Since K_0 is bounded, $K_1 \cap K_2$ is a line segment, which we will call $\overline{AB}$.

Each of A and B may or may not be an angle of K_0, but each must be an angle of K_1 and of K_2 as well. Thus, K has at least two angles. Our next lemma deals with how many angles K can have. We recall that for a convex set the sum of the absolute change of outer normal at all boundary points is 2π radians.

3. LEMMA. *K has no more than four angles.*

PROOF. Suppose that K has five or more angles. We know from the convexity of K that it may have many angles of nearly π radians but can have only finitely many small angles, since the sum of the angles supplementary to the angles of K can total no more than 2π radians. Let γ be the fifth largest angle in K, and assume that K has k angles of size γ or smaller. Note that $5 \leqq k \leqq \frac{2\pi}{\pi-\gamma}$. Of these $2k$ angles of K_1 and K_2, $2k-4$ of them do not occur on $\overline{AB}$, and thus must occur on $\partial(K_0)$. It follows that K_0 must have at least $2k-4$ angles of size γ or smaller; in fact K_0 has $2k-4$, $2k-3$, or $2k-2$ such angles. Since K_0 also has k such angles, $2k-4 \leqq k$. Thus $k \leqq 4$. Q.E.D.

4. COROLLARY. *If K has four angles, then K_0 has no angles at A or at B. If K has three angles, then K_0 has an angle at exactly one of A, B. If K has two angles, then K_0 has angles at A and at B.*

[2] Four dimensions for each of K_1 and K_2: the location (two dimensions). size, and orientation of the figure. This is really a special case of the Blaschke Selection Theorem ([1] pp. 64 ff).

5. Corollary. *K has at least three angles.*

Proof. Suppose K has only two angles. Let α be the smaller angle in K. Then α occurs at A or at B in K_1, assume without loss of generality it is at A, and the angle at B in K_1 is at least as large. Thus, both angles of K_0 are larger than α, and K_0 cannot be similar to K_1, contrary to hypothesis. Q.E.D.

We must now take up the cases in which K has three or four angles, and we will call these Case 3 and Case 4 respectively. We will study Case 4 first.

6. Lemma. *In Case 4, K is a quaerilateral.*

Proof. We wish to show that the sum of the angles of K is 2π rad. It will then follow that each of the arcs of $\partial(K)$ joining these angles is straight ([3], p. 15). Let the angles of K_1 and K_2 which lie at A and B be denoted as the *selected angles*; the others are *rejected angles*. Then the sum of the selected angles according to Corollary 4 is 2π rad., while the sum of the rejected angles is Σ, the sum of the angles of K_0. Thus, the sum of the selected and rejected angles of K_1 and K_2 is $2\pi+\Sigma$. On the other hand, the sum of the selected and rejected angles is the sum of all the angles of K_1 and K_2 and is thus 2Σ, since $K_1 \sim K_2 \sim K_0$. It follows that $2\Sigma = 2\pi+\Sigma$, or that $\Sigma=2\pi$. Q.E.D.

7. Lemma. *In Case 4, every angle in K appears in K at least twice.*

Proof. Suppose not, and suppose, that the angle of K_0 occuring once is at C. Without loss of generality C is a vertex of K_1. Let the vertices of K_1 be denoted as $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and those of K_2 as $\beta_1, \beta_2, \beta_3, \beta_4$, as shown in figure 1. Let Φ be a similarity of K_1 with K_0. Then clearly $\Phi(C)=C$, since we have assumed that this angle appears only once. If we had $\Phi(E)=E$, then Φ would be an isometry and $\Phi(K_1)=K_0$ which is impossible since $\hat{K}>1$. Therefore, $\Phi(E)=D$, $\Phi(A)=E$, and $\Phi(B)=F$. Thus $\alpha_2=\alpha_3$, $\alpha_3=\beta_2$, and $\alpha_4=\beta_4$. Thus, $\overline{AB}\|\overline{DF}$. Therefore $\overline{AC}\|\overline{BE}$ or $\overline{AB}\|\overline{CE}$, since $K_1 \cong K_2$. Applying Φ, $\overline{AB}\|\overline{CE}$ implies $\overline{EF}\|\overline{CD}$, and $\overline{AC}\|\overline{EB}$ implies $\overline{CE}\|\overline{DF}$, while $\overline{DF}\|\overline{AB}$ by the discussion above. Thus, in either case K_1 is a parallelogram. This contradicts our hypothesis that the angle occurs only once. Q.E.D.

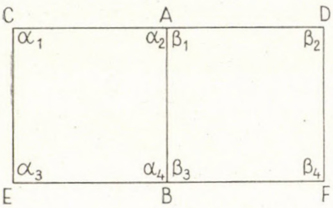

Fig. 1

8. Theorem. *In Case 4, K is a parallelogram.*

Proof. If the smallest and largest angles are equal, then K is a rectangle. If they are different, then there are exactly two of each, so that K is a parallelogram or an isosceles trapezoid. However, it is not possible to assemble two congruent isosceles trapezoids to make an isosceles trapezoid unless they are rectangles. Thus K is a parallelogram. Q.E.D.

9. Corollary. *The sides of K are in the ratio $\sqrt{2}$.*

Proof. Clear.

10. COROLLARY. *If, in the original statement of the problem, we had resticted ourselves to positive similarities only, then the parallelogram mentioned in Theorem 8 would have been a rectangle.*

PROOF. The additional restriction does not change Theorem 8 or Corollary 9, but the final configuration could not then be constructed for skew parallelograms. Q.E.D.

We now turn to Case 3. In Case 3, we have six angles between K_1 and K_2, two at A and two at B. Thus, there are two other angles in K_0. It follows that exactly one of A and B is an angle of K_0. Let B be the angle. Label the angles of K_1 with

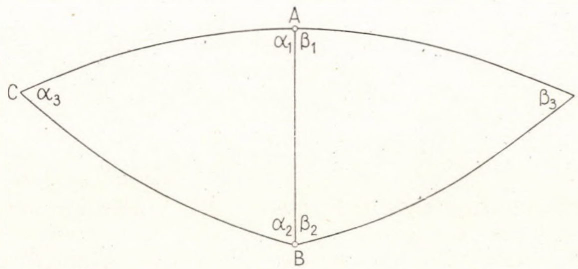

Fig. 2

α's, those of K_2 with β's. Let α_1 and β_1 be at A, α_2 and β_2 at B (see figure 2). The angle of K_0 at B is greater than one of the angles of K_1, and is thus not the smallest of the angles of K_0. Thus, either α_3 or β_3 is the smallest, and without loss of generality we take it to be α_3 and call the angle C. Label the angle at β_3 as D. Let Φ be a similarity of K_1 with K_0. Let us look at $\Phi(C)$. We know that $\alpha_2+\beta_2 > \alpha_2 \geqq \alpha_3$, since the smallest angle of K_0 is also the smallest in K_1. Therefore, $\Phi(C) \neq B$.

If $\Phi(C)=C$, then $\Phi(B)=D$, since $\Phi(B)=B$ would imply $\alpha_2 = \alpha_2+\beta_2$. Thus $\alpha_2=\beta_3$ and $\Phi(A)=B$, so that $\alpha_1 = \alpha_2+\beta_2$. Calculating, we have

$$\beta_1+\beta_2+\beta_3 = \beta_1+\beta_2+\alpha_2 = \beta_1+\alpha_1 = \pi.$$

Thus, K_2 is a triangle. We have $\alpha_1 = \alpha_2+\beta_2 > \alpha_2 = \beta_3$. Thus $\alpha_1 \neq \beta_2$, $\alpha_1 \neq \beta_3$, so that $\alpha_1=\beta_1$. Thus, K_1 and K_2 are congruent right triangles and have equal hypotenuses. It follows that K_0 is isosceles if $\Phi(C)=C$.

The other possibility is that $\Phi(C)=D$. Then the smallest two angles in K_0 are equal. Since the same must be true for K_1 and K_2, and neither α_2 nor β_2 can be the biggest angle in its figure, we have $\alpha_3=\alpha_2$, $\beta_3=\beta_2$. Therefore, α_1 is the biggest angle in K_1 and β_1 is the biggest in K_2. Thus $\alpha_1=\beta_1=\pi/2$. Since $\alpha_1 = \alpha_2+\beta_2$, $\alpha_2=\alpha_3=\beta_3=\beta_2=\pi/4$. We have immediately that K_1 and K_2 are isosceles right triangles.

Thus, in either case we have the following:

11. THEOREM. *In Case 3, K is an isosceles right triangle.*

PROOF. Given above.

IV. The condition $\hat{K}=1$: curves of constant width. Curves of constant width K have $\hat{K}=1$. In fact, a much stronger theorem holds:

12. THEOREM. *Let K_1 and K_2 be packed in K_0, where all three are curves of constant width, not necessarily similar. Then $\pi(K_1)+\pi(K_2) \leqq \pi(K_0)$.*

PROOF. Let l be any line dividing the convex set K_1 from K_2. Let l_1 and l_2 be support lines of K_0 parallel to l, so chosen that K_1 lies between l_1 and l, K_2 lies between l_2 and l. Clearly, l lies between l_1 and l_2. We set d_1 as the distance between l_1 and l, d_2 as the distance between l_2 and l, and d_0 as the distance between l_1 and l_2. Let $\mathbf{w}(K)$ denote the width of K, then we have $\mathbf{w}(K_1) \leqq d_1$ and $\mathbf{w}(K_2) \leqq d_2$, so that $\mathbf{w}(K_1)+\mathbf{w}(K_2) \leqq d_1+d_2 = d_0 = \mathbf{w}(K_0)$. By BARBIER's Theorem, [3, p. 74] we know that for any curve of constant width, K, $\pi(K) = \pi\mathbf{w}(K)$. Thus $\pi(K_1)+\pi(K_2) \leqq$ $\leqq \pi(K_0)$. Q.E.D.

13. COROLLARY. *If K is a curve of constant width, then $\hat{K}=1$, regardless of whether the similarities are restricted to be positive.*

V. The condition $\hat{K}=1$: regular polygons. All regular polygons K have $\hat{K}=1$. The proof will be broken into three cases: the case of the equilateral triangle, the case of other regular odd-gons, and the case of regular even-gons.

The general method we will follow will be the same in all cases of polygons. We assume that K_1 and K_2 are packed in K_0, all of which are regular n-gons. We choose l, a line separating K_1 from K_2 and let l_1 and l_2 be supporting lines for K_0 parallel to l, with K_1 between l_1 and l, K_2 between l_2 and l. Let $P_1 \in l_1 \cap K_0$, $P_2 \in l_2 \cap K_0$ be vertices of K_0. Consider the regular n-gon K_1' of the same size as K_1, lying in K_0, and with P_1 as one of its vertices. We wish to show that K_1' lies between l_1 and l, and similarly for K_2' defined between l and l_2. If we could show this, then we would see that K_1' and K_2' would occupy non-overlapping portions of the diagonal $\overline{P_1P_2}$ in K_0. It would follow that

$$\mathbf{d}(K_1')+\mathbf{d}(K_2') \leqq \mathbf{d}(K_0),$$

so that

$$\pi(K_1')+\pi(K_2') \leqq \pi(K_0),$$

and

$$\pi(K_1)+\pi(K_2) \leqq \pi(K_0).$$

To show this result, we will consider the triangle consisting of the two sides of K_0 at P_1, extended, together with the appropriate section of l. The easier case where l is parallel to one of the sides is left to the reader. We want to prove that if this triangle will accomodate any regular n-gon, then it will accomodate a regular n-gon of the same size in "standard position", *i.e.* with a vertex at P_1. Rephrasing the previous assertion, we want to prove

14. LEMMA. *Let $n \geqq 3$, and let Δ be a triangle containing an angle P of $\pi - \frac{2\pi}{n}$ rad. Let l be the side of Δ opposite P, and assume that a regular n-gon K is contained in Δ. Let K' be another regular n-gon of the same size as K, inscribed in Δ in standard position, and let l' be a support line for K' opposite P and parallel to l. Then l does not separate P from l'.*

If the lemma is true for regular n-gon K^* contained in Δ where K^* is larger than K then the lemma is true for K. Thus we may suppose that K has vertices on all three edges of Δ.

15. LEMMA. *Lemma* 14 *holds for* $n=3$.

PROOF. Let the vertices of Δ be P, P', and P''. Let Q be the vertex of K on l, and let h be the altitude of K, an equilateral triangle. The width of K in any direction is at least h, thus, the perpendicular distance from P'' to $\overline{PP'}$ is at least h, as is the distance from P' to $\overline{PP''}$. Since the angle at P is $\pi/3$, it follows that $\overline{PP'}$ is at least as long as the side of K, and $\overline{PP''}$ is also. Thus, K' lies inside Δ, and the lemma is proved.

16. LEMMA. *Lemma* 14 *holds when n is an odd number greater than* 3.

PROOF. Let us imagine the following process. We *start* with the polygon K', and we rotate K' about its center through an angle $\alpha < \frac{2\pi}{n}$ to give us a polygon K'' which is oriented parallel to K. We then translate K'' so that it is inscribed in

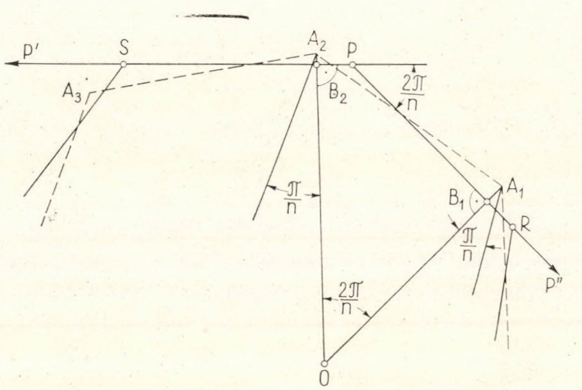

Fig. 3

the angle at P, at which time it will coincide with K. We will show K'' must intersect l', so that l cannot separate P from l'.

In the vicinity of P, the polygons K' and K'' look like figure 3, where P, R, and S are vertices of K' and A_1, A_2, and A_3 are vertices of K''. In order to inscribe K'' in the triangle Δ, it will be necessary to translate the figure K'' until A_1 and A_2 touch PR and PS respectively. Let $\overline{A_1B_1} \perp \overline{PR}$ and $\overline{A_2B_2} \perp \overline{PS}$ be extended to meet at O. Then the angle at O is determined only by the sides $\overline{PR}$ and $\overline{PS}$, and must be $\frac{2\pi}{n}$. By reasons of symmetry, $\overline{A_1B_1} = \overline{A_2B_2}$, and if we draw $\overline{A_1C_1}$ and $\overline{A_2C_2}$ making an angle of $\frac{\pi}{n}$ with $\overline{A_1B_1}$ and $\overline{A_2B_2}$ respectively, then $\overline{A_1C_1} \| \overline{A_2C_2}$ and if C_1 and C_2 are chosen as the points where these lines cross $\overline{PR}$ and $\overline{PS}$, res-

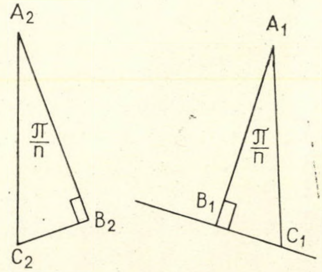

Fig. 4

pectively, then $\overline{A_1C_1} = \overline{A_2C_2}$ (see figure 4). It follows that the translation necessary to inscribe K'' in $\sphericalangle P'PP''$ is exactly the vector $\overrightarrow{A_1C_1}$. Since this vector makes an angle of $\frac{\pi}{2} - \frac{\pi}{n}$ to PR, it is in the direction of the bisector of the angle $\sphericalangle P'PP'' = \pi - \frac{2\pi}{n}$ and the magnitude is $\overline{A_1B_1} \sec \frac{\pi}{n}$. Please note that we have not yet made use of the fact that n is odd, and so we will use this argument for the next lemma as well.

We now turn our attention to what is happening at the other side of K'', where K' meets l'. We assume that K' meets l' at A^*, as shown in figure 5.

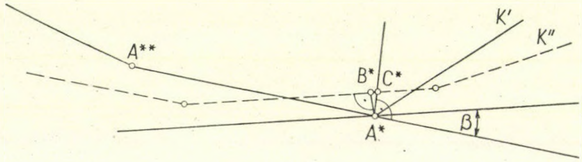

Fig. 5

The segment $\overline{A^*A^{**}}$ is the edge of K' opposite P, whence the bisector of $\sphericalangle P'PP''$ is perpendicular to $\overline{A^*A^{**}}$. Let C^* be the point on the edge of K'' such that $A^*C^* \perp \perp A^*A^{**}$. Let B^* be the foot of the perpendicular from A^* to this same edge of K''. Thus $\overline{A_1C_1} \| \overline{A^*C^*}$ and the angle $B^*A^*C^*$ is the angular displacement of K' to make K'' (see figure 6), i.e. $\sphericalangle B^*A^*C^* = \alpha$. It follows that $\overline{A^*C^*} < \overline{A^*D^*} = \overline{A_1C_1}$ if $\alpha < \frac{\pi}{n}$, and $\overline{A^*C^*} = \overline{A_1C_1}$ if $\alpha = \frac{\pi}{n}$. Thus, for $\alpha < \frac{\pi}{n}$, A^* is an interior point of K. When $\alpha = \frac{\pi}{n}$, the image of $\overline{A^*A^{**}}$ in K'' is parallel to $\overline{PS}$. Thus, the corresponding points in K lie at a distance h from $\overline{PS}$, where h is the minimum width of K. For values of α between $\frac{\pi}{n}$ and $\frac{2\pi}{n}$, the image of A^{**} lies even further from $\overline{PR}$, and somewhat closer to $\overline{PS}$, though not as close as A^* (see figure 7). Thus, for all such values

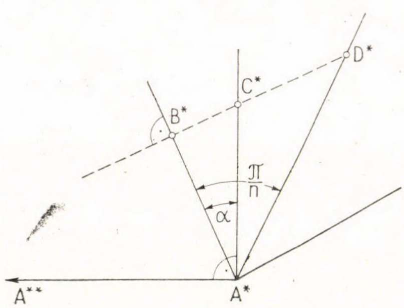

Fig. 6

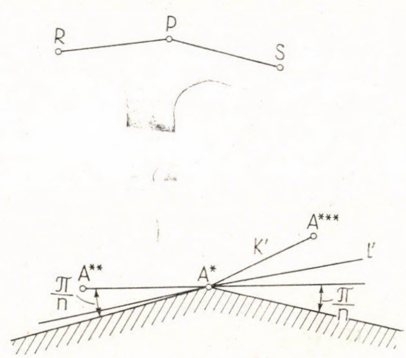

Fig. 7

of α, the image of A^{**} lies in the shaded area. It follows that l' intersects K, and l cannot separate l' from P. In the case $\alpha \geqq \frac{\pi}{n}$, l' separates P from l unless $\alpha = \frac{\pi}{n}$, in which case $l' = l$. Thus, the lemma is proven.

17. *Lemma* 14 *holds if n is even.*

PROOF. Let P, Q, R, S, A^*, and α be defined as in the proof of Lemma 16. Since n is even, l' passes through A^* the vertex of K' opposite P. Thus, there is symmetry enough so that we can assume that $\alpha \leqq \frac{\pi}{n}$, for if $\frac{\pi}{n} < \alpha < \frac{2\pi}{n}$, we can use instead the rotation through an angle $\alpha' = \frac{2\pi}{n} - \alpha$ in the other direction. The analysis of the magnitude and direction of the translation which takes K'' into K is the same as in the previous lemma. Now let us look at the situation around A^*. We let $\overline{A^*B^*}$ be the perpendicular to the rotated side, and C^* the intersection of the rotated side with the angle bisector at A^*. We wish to know that $\overline{A^*C^*} \leqq \overline{A_1C_1}$. The fact that $\alpha \leqq \frac{\pi}{n}$ implies that B^* lies to the right of C^* in figure 8. Let A' be the rotated posi-

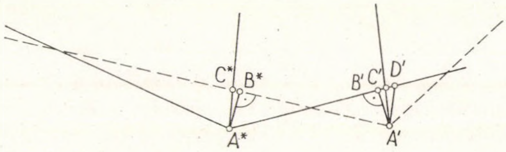

Fig. 8

tion of A^*, and let B' and C' be chosen symmetrically with B^* and C^*. Now if $\overline{A'D'}$ is chosen parallel to $\overline{A^*C^*}$, then $\overline{A'D'} = \overline{A_1C_1}$, since $\triangle A_1B_1C_1 \cong \triangle A'B'D'$. Since D' lies to the right of C', $\overline{A'D'} > \overline{A'C'} = A^*C^*$. Thus, A^* lies inside K, and l cannot divide P from l'. Q.E.D.

18. THEOREM. *If K is a regular polygon, then $\hat{K} = 1$.*

PROOF. Let K_1 and K_2 be packed in K_0, all these being regular n-gons. Let l be a line separating K_1 from K_2. Let the support lines to K_0 parallel to l pass through P_1 and P_2 respectively, and let K' be a regular n-gon of the same size as K_1, lying inside K_0 and having a vertex at P_1. Let K'' be chosen at P_2 with the same relation to K_2. Then by lemmas 15, 16, and 17, which together prove lemma 14, neither K' nor K'' intersects l, except possibly at points of $\partial(K')$ or $\partial(K'')$. The line $\overline{P_1P_2}$ is a diameter of K_0, and the intersections of these lines with K' and K'' are diameters of K' and K'' respectively. Since no point l is an interior point of K' or K'', we have

$$\mathbf{d}(K') + \mathbf{d}(K'') \leqq \mathbf{d}(K_0).$$

By similarity,

$$\pi(K') + \pi(K'') \leqq \pi(K_0),$$

and

$$\pi(K_1)+\pi(K_2) \leqq \pi(K_0),$$

since

$$K_1 \cong K', \quad K_2 \cong K''.$$

Q.E.D.

VI. Some examples when $K \neq 1$. Now we come to the hard part, showing that *only* curves of constant width and regular polygons are tight. We will start with the assumption that the figure K is tight, and is not a curve of constant width. To make our argument as general as possible, we will take an apparently weaker condition of tightness, namely that $\hat{K}^+=1$, and show the results in that case.

It will be instructive, at first, to consider some examples of figures which are not tight, for these examples will be clues to the techniques we will use later. In figure 9, we see that a rectangle twice as long as it is wide is not tight, for the perimeters are: $\pi(K_1)=3$, $\pi(K_2)=4\frac{1}{2}$, $\pi(K_0)=6$. In figure 10, we see the same technique applied

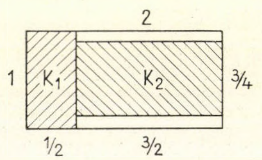

Fig. 9

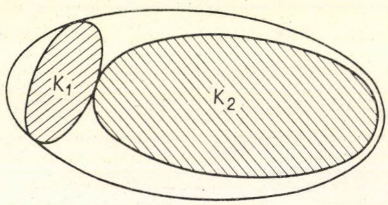

Fig. 10

to an ellipse, which is not tight. What we shall learn later is that every tight figure satisfies the "condition of constant minimum width", which will assure us that a rectangle is tight only if it is a square, and an ellipse is tight only if it is a circle.

In figure 11 we see a rhombus K_0, with two similar rhombi packed into it so that $\pi(K_1)+\pi(K_2) = \pi(K_0)$. If K_1 is rotated a small amount counterclockwise about any point in the shaded region, it will come out of contact with K_2 and remain inside K, thus it can be expanded. Such a motion is possible for every rhombus which is not a square, even though the rhombus satisfies the condition of constant

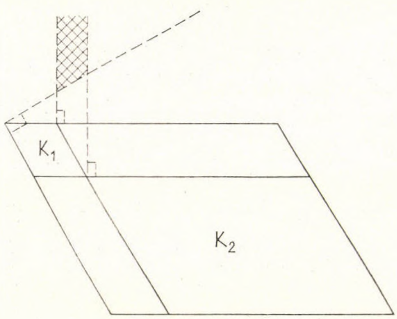

Fig. 11

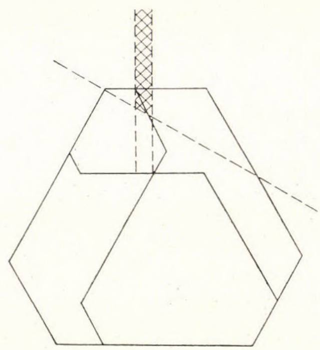

Fig. 12

minimum width. Another figure which satisfies the condition of constant minimum width is the truncated equilateral triangle shown in figure 12, with parallel opposite sides and the alternate sides equal. Unless it is a regular hexagon, we can array K_1 and K_2 along the diagonal $\overline{PQ}$ and then rotate K_1 about any point in the shaded region in a counterclockwise direction a little. K_1 will come out of contact with K_2 and remain inside K, and we will be able to expand it. These last two figures violate another principle, which we call the "Non-Rotation Condition", which is necessary for tightness in figures.

We consider one final example. Let K_0 be constructed (figure 13) from a quarter disc ABC with right angle at B, by replacing the part of the circular arc joining

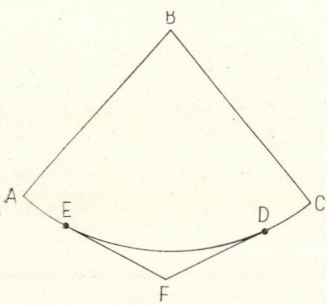

Fig. 13

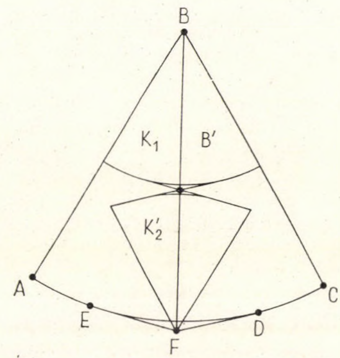

Fig. 14

two points D and E by the two tangent segments from D and E, which meet at some point, say F. Consider the figures K_1 and K_2 arrayed along the chord $\overline{BF}$ with K_1 at B and K_2 at F. Let B' be the point of BF where K_1 meets K_2. Rotate K_2 about the midpoint of $\overline{B'F}$ through 180°, to get K_2', as in figure 14. Since the angle at F is made of two tangents with less than $\frac{\pi}{2}$ rad of arc between them the angle at F is obtuse; hence we can rotate K_2' a little about F and it comes out of contact with K_1 and thus can be expanded. It follows that this figure is not tight.

In each of the next three sections we derive a necessary condition, arising from the above examples, for the condition $\hat{K}^+=1$.

VII. The minimum width condition. 19. DEFINITION. Let K be a convex figure, with $P\in\partial(K)$. The *minimum width at P* is the minimum of all widths of K measured to a support line passing through P, and is denoted by $\mathbf{w}(P)$. *K satisfies the condition of constant minimum width* (hereinafter called the *minimum width condition* or MWC) if $\mathbf{w}(P)$ is the same for all $P\in\partial(K)$. We define $\mathbf{m}(K)=\min\mathbf{w}(P)$.

The following theorem is well-known ([1] p. 77); we include a proof for completeness.

20. THEOREM. *Let K be a convex figure, and let l_1 and l_2 be support lines in a direction of minimal width. Then there exist points P_1 and P_2 in $l_1 \cap K$ and $l_2 \cap K$ respectively such that $\overline{P_1 P_2}$ is perpendicular to l_1.*

PROOF. If the theorem were false, then the perpendicular projection of $K \cap l_1$ on l_2 would not intersect $K \cap l_2$. Since both these sets are closed and convex, there is a point Q_2 in neither set which separates them in l_2. Let Q_1 be the projection of Q_2 on l_1. Then $\overline{Q_1 Q_2} = \mathbf{m}(K)$. For α_1 small enough, $0 < \alpha < \alpha_1$ assures that a line passing through Q_1 and making an angle α with l_1 will not intersect K. Similarly there is an α_2 for Q_2. Let $0 < \alpha < \min(\alpha_1, \alpha_2)$, and let l' and l'' be the parallel lines through Q_1 and Q_2 making angle α with l_1 and l_2. Then K lies between l' and l'', while

$$d(l', l'') < \overline{Q_1 Q_2} = d(l_1, l_2) = \mathbf{m}(K).$$

But this is a contradiction. Q. E. D.

21. LEMMA. *Let $P_1 \in \partial(K)$ and let l_1 and l_2 be chosen so that l_1 and l_2 are parallel support lines to K, $P_1 \in l_1$, and $d(l_1, l_2) = \mathbf{w}(P_1)$. Let $P_2 \in l_2 \cap K$. Then of the four rays into which P_1 and P_2 divide l_1 and l_2, at least one is tangent to K and makes a non-obtuse angle with $\overline{P_1 P_2}$.*

PROOF. If none of the rays which make non-obtuse angles were tangent to K, then we could find an angle α small enough so that the rays emanating from P_1 and P_2 and making an angle α with l_1 or l_2 would not intersect K except at P_1 or P_2. One of these rays makes a smaller angle with $\overline{P_1 P_2}$ than any angle made with it by l_1 or l_2. The lines l' and l'' through P_1 and P_2 parallel to that ray are support lines, and the distance between these lines is less than the distance between l_1 and l_2. Thus $\mathbf{w}(P_1) \leqq d(l', l'') < d(l_1, l_2) = \mathbf{w}(P_1)$, a contradiction. Thus, one of the rays is tangent. Q.E.D.

22. THEOREM. *If K is tight, then K satisfies the minimum width condition.*

PROOF. We will prove the contrapositive. Assume that $\mathbf{w}(P_1) > \mathbf{w}(P)$, where $P, P_1 \in \partial(K)$. We will show that $\hat{K}^+ > 1$.

Let P_2 and parallel support lines l_1 and l_2 be chosen so that $P_1 \in l_1 \cap \partial(K)$ $P_2 \in l_2 \cap \partial(K)$, and $\mathbf{w}(P_1) = d(l_1, l_2)$. Let r be the one of the four rays which is tangent, as in Lemma 21. We can assume that r is at P_1; there is no difference if it is at P_2.

Before giving the formal proof, we will sketch the idea. Let K_1 and K_2 be arrayed along the chord $\overline{P_1 P_2}$ with parameter λ where λ is very small and K_1, the smaller one of K_1 and K_2, is near P_1. Then we observe that $K \setminus K_2$, the set-theoretic difference of K and K_2 contains a piece in the vicinity of P_1 which looks like a long strip with parallel sides whose width is $\lambda \mathbf{w}(P_1)$. Since $\lambda \mathbf{w}(P) < \lambda \mathbf{w}(P_1)$, we could remove K_1 and replace it with K', a congruent figure re-oriented so as to fit into the "almost strip" without touching either $\partial(K)$ or K_2. Let K'' be an expanded copy of K', just a little larger than K' and still not touching K_2 or $\partial(K)$, though lying within K. Then

$$\pi(K'') + \pi(K_2) > \pi(K') + \pi(K_2) = \pi(K_1) + \pi(K_2) = \pi(K) = 1$$

so that $\hat{K} > 1$. That would prove the theorem.

We now set out to give the real proof. Let α be the angle which the ray r makes with $\overline{P_1P_2}$ and let $ABCD$ be a parallelogram about K where $\overline{AB}$ and $\overline{CD}$ lie on parallel support lines with width $w(P)$ and the angle at D is α. Let $\eta>0$ be chosen so that $\mathbf{w}(P)+\eta<\mathbf{w}(P_1)$, and let $\overline{AD}$ be extended to E so that E lies at a distance $\mathbf{w}(P)+\eta$ from the line through $\overline{AB}$ (figure 15). Let $\sphericalangle AEC=\beta$. We note that $\beta<\alpha$. Now let a secant be drawn (figure 16) from P_1 making an angle β with $\overline{P_1P_2}$ and

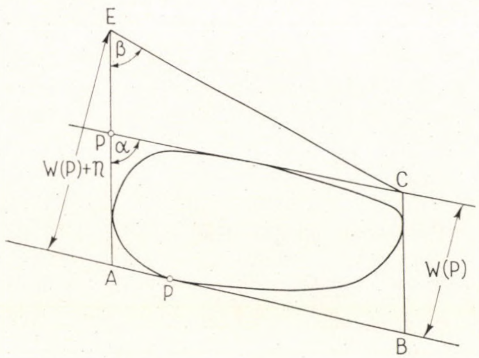

Fig. 15

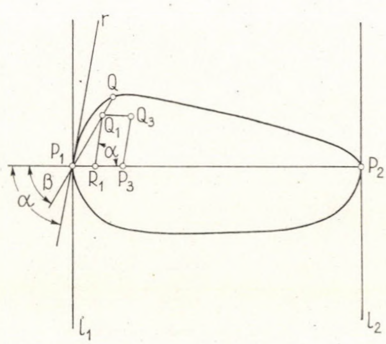

Fig. 16

meeting $\partial(K)$ at Q. Choose Q_1 on $\overline{P_1Q}$ so close to P_1 that the following construction remains in K: Draw a line through Q_1 parallel to r, and meeting $\overline{P_1P_2}$ at R_1 (see figure 16). On $\overline{Q_1R_1}$ construct a parallelogram $P_3Q_3Q_1R_1$ similar to $ABCD$ with $\overline{R_1P_3}$ corresponding to $\overline{DA}$. By our choice of Q_1 we have guaranteed that the parallelogram is in K. Now let K_1 and K_2 be arrayed along $\overline{P_1P_2}$ with parameter $\lambda=\dfrac{\overline{P_1P_3}}{\overline{P_1P_2}}$. Then $\pi(K_1)+\pi(K_2)=\pi(K)$. Let K' be a figure similar to K and contained in parallelogram $P_3Q_3Q_1R_1$ with constant of similarity $\mu=\dfrac{\overline{P_3R_1}}{\overline{AD}}$. Let h be the perpendicular distance from P_1 to $\overline{P_3Q_3}$. Since the figure $P_1Q_1Q_3P_3$ is similar to $ECBA$ with proportionately μ we see that $h=\mu(\mathbf{w}(P)+\eta)<\mu\mathbf{w}(P_1)$.

At the same time, $h=\lambda\mathbf{w}(P_1)$, since r and its parallels at P_3 and P_1 are in support lines. It follows that $\mu>\lambda$, and $\pi(K')>\pi(K_1)$. However, K' cannot intersect K_2, and lies within K. Thus, K' and K_2 are packed in K and

$$\pi(K')+\pi(K_2)>\pi(K_1)+\pi(K_2)=\pi(K)=1,$$

from which we have immediately $\hat{K}^+>1$. Q.E.D.

23. Corollary. *If, for every point $P\in\partial(K)$, there is only one width at P, then K is tight if and only if K is a figure of constant width.*

24. COROLLARY. *If K has no angles, then K is tight if and only if K is a figure of constant width.*

25. THEOREM. *If K is strictly convex (has no sides) and satisfies the MWC, then K is a figure of constant width.*

PROOF. Since K has no side, every support line intersects K in just one point, and at every point which is not an angle, we have a unique line of support. If K is not a figure of constant width, let l and l' be parallel support lines with $d(l, l')=h>\mathbf{m}(K)$. Denote $l\cap K=P$, $l'\cap K=P'$. Let r and r' be tangent rays to K at P and P' respectively and on opposite sides of $\overline{PP'}$. We can assume without loss of generality that the angle α made by r with $\overline{PP'}$ is not less than the angle α' made by r', so that the lines l_0 and l_0' passing through P and P' with angle α to $\overline{PP'}$ are support lines.

At most countably many points of $\partial(K)$ are angles, so that we can choose a sequence $P_n \to P$ "monotonically" along $\partial(K)$ on the side of r with each P_n a point with only one support line, l_n, through it. Let P_n' be the intersection of K with l_n' the opposite support line. Then the distance between l_n and l_n' is the unique width at P_n, for each n. Thus, by Theorem 20, $\overline{P_n P_n'} \perp l_n$ and

$$d(P_n, P_n')=d(l_n, l_n')=\mathbf{w}(P_n)=\mathbf{m}(K),$$

since K must satisfy the MWC. We know that the limiting position of the lines l_n must be the line l_0, and that the limiting position of the l_n' must be l_0'. The sequence P_n' converges, say to P'', and if $P'\neq P''$, then the line l_0' contains both points, and $\partial(K)$ contains $\overline{P'P''}$, contrary to hypothesis. Thus, $P_n'\to P'$. This gives us

$$d(P_n, P_n')\to d(P, P')\geqq d(l, l')=h>\mathbf{m}(K),$$

which is impossible, since $d(P_n, P_n')=\mathbf{m}(K)$ for all n. Thus we have a contradiction, and the theorem is proved. Q.E.D.

VIII. The non-rotation condition. 26. DEFINITION. Let P_1 and P_2 be opposite points. We say P_1 and P_2 satisfy the *Rotation Condition* if and only if there exist parallel support lines l_1 and l_2, through P_1 and P_2 respectively, such that there is a point R interior to the following three half planes H_1, H_2, H defined below.

H_2 is the half plane containing P_1 and bounded by the line m_2 with $P_2\in m_2$ and $m_2\perp l_2$.

Let r_1 and r be the tangent rays at P_1 labeled so that $r\subseteq H_2$. (See figure 17.) Let P be the point of $K\cap r_1$ farthest from P_1.

H is the half plane not containing r and bounded by the line m with $P_1\in m$ and $m\perp r$.

H_1 is the half plane containing the infinite part of r_1 and bounded by the line m_1 with $P\in m_1$ and $m_1\perp r_1$.

If it should happen that P_1 or P_2 is interior to a side of K then we can replace it by the endpoint of the side which is more distant from the opposite point. This increases the intersection $H_1\cap H_2\cap H$. We therefore suppose that P_1 and P_2 are always so chosen.

27. DEFINITION. K is said to satisfy the *Non-Rotation Condition*, NRC, if and only if no pair of opposite points satisfy the rotation condition.

28. THEOREM. *If K is tight, then K satisfies the non-rotation condition.*

PROOF. We suppose that P_1 and P_2 satisfy the rotation condition and show that K is not tight. Let R be a point in $H_1 \cap H_2 \cap H$.

Let Q_1 and Q_3 be the points of K_0 corresponding to P_1 and P_2, respectively. Array K_1 and K_2 along the chord $\overline{Q_1 Q_3}$ with parameter λ, $0<\lambda<1$. Let Q_2 be the

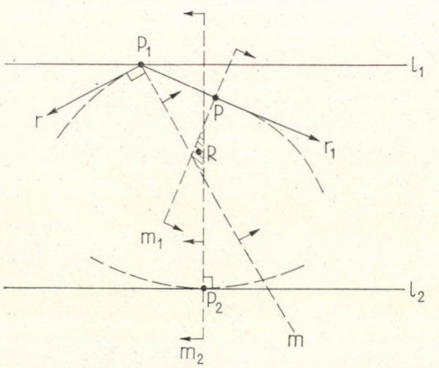

Fig. 17

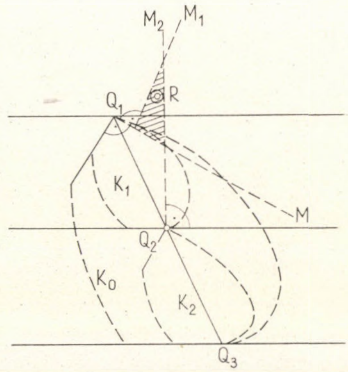

Fig. 18

point of K_1 corresponding to P_2. With Q_1 and Q_2 of K_1 replacing P_1 and P_2 of K find a point R_1 for K_1 corresponding to the point R for K. We note that the only points K_1 can possibly have in common with K_0 are on the tangent rays at Q_1 and the only point it has in common with K_2 is Q_2. Thus a small rotation about R_1 will bring K_1 out of contact with both K_2 and ∂K_0, allowing it to be expanded (see figure 18). Q.E.D.

IX. The opposite angle condition. 29. DEFINITION. The convex set K is said to satisfy the *opposite angle conditions*, OAC, if and only if

1. For any diameter $\overline{PQ}$ the tangent angles at P and Q are equal. Furthermore if Q is a vertex then either the tangent rays at P and Q are parallel or there is another diameter $\overline{PR}$ at P.

2. For opposite points P and Q which are such that all the points of ∂K in some neighbourhood of Q are closer to P than Q is, it cannot happen that the two angles at Q which $\overline{PQ}$ makes with the tangent rays at Q are both bigger than both the angles which $\overline{PQ}$ makes with the tangent rays at P.

30. THEOREM. *If K is tight, then K satisfies OAC.*

PROOF. We suppose K does not satisfy OAC and show it is not tight. We first suppose K violates OAC 2. Let P_0 and Q_0 be the points of K_0 corresponding to the points P and Q of K. Array K_1 and K_2 along $\overline{P_0 Q_0}$ with parameter λ, $0<\lambda<1$

with K_1 at P_0 and K_2 at Q_0. Let P_2 and Q_2 be the points of K_2 corresponding to P and Q. Since OAC 2 is violated, the angles at Q_2 are greater than those at P_2, thus we may rotate K_2 about the midpoint of $\overline{P_2Q_2}$ through 180° to obtain K_2' where P_2' is at Q_2 and the angle at P_2' is inside the angle of K_0 at $Q_0=Q_2$ (see figure 19). Thus if λ is chosen so that K_2 is small, K_2' will be inside K. Since Q_2' is the farthest point of K_2' from P_2' in a neighbourhood of Q_2', any small rotation of K_2' will bring it out of contact with K_1. Such a rotation is possible since the angle at Q_0 is larger than at P_2'. Thus after rotating slightly K_2' can be expanded and K is not tight. Thus a tight figure satisfies OAC 2.

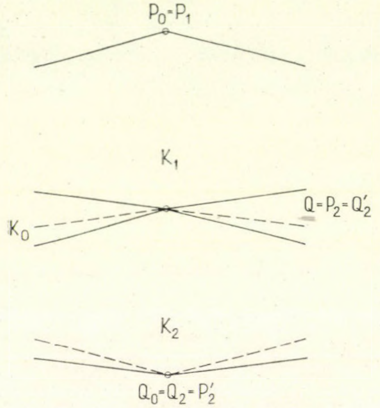

Fig. 19

The proof that a tight figure satisfies OAC 1 proceeds with two lemmas.

31. LEMMA. *If K satisfies MWC then the number of diameters of K is finite unless K has constant width.*

PROOF. Suppose K does not have constant width and that there are infinitely many diameters with length $d>h$, where h is the minimal width. Then we could find a sequence of diameters $\overline{P_iQ_i}$ with $P_i\to P_0$ and $Q_i\to Q_0$ so that $\overline{P_0Q_0}$ is also a diameter. Since the lines perpendicular to a diameter at its endpoints are support lines of K, it follows that the boundary points between Q_n and Q_0 are opposite those between P_n and P_0. But, if Q_n and P_n are close enough to Q_0 and P_0, then dist $(\overline{Q_nQ_0}, \overline{P_nP_0}) > d-\varepsilon > h$ for ε small enough and n large enough. Since K contains the segments $\overline{Q_nQ_0}$ and $\overline{P_nP_0}$ its boundary arcs $\widehat{Q_nQ_0}$ and $\widehat{P_nP_0}$ also satisfy dist $(\widehat{Q_nQ_0}, \widehat{P_nP_0})>h$. But this says that the width at a point interior to Q_nQ_0 is greater than h. This contradicts MWC. Q.E.D.

32. LEMMA. *If K is tight then the tangent angles at the ends of a diameter are equal.*

PROOF. Let $\overline{PQ}$ be a diameter. Suppose the angle at Q is larger than that at P. As in the proof of OAC 2 we array K_1 and K_2 along $\overline{P_0Q_0}$ with K_2 at Q. We now rotate K_2 through whatever angle is required and translate it to position K_2' so that the angle at P_2' fits inside the angle at Q_0. Again if K_2' is small enough it follows that K_2' is inside K_0. Since the angle at Q_0 is bigger than that at P_2' and since there are only a finite number of diameters we may rotate K_2' to a position where the chord of K_2' lying on P_0Q_0 is not a diameter. Hence K_2' may be expanded, contradicting the hypothesis. Q.E.D.

We now complete the proof that a tight figure satisfies OAC 1.

If the angle at Q_0 is a vertex then we can repeat the above argument but the position of K_2' is completely determined. Thus the chord of K_2' lying on $\overline{P_0Q_0}$ must be a diameter of K_2'. Unless the angles at P_0 and Q_0 are parallel this diameter will be different from $\overline{P_2'Q_2'}$. Thus if K is tight OAC 1 holds. Q.E.D.

X. $\hat{K}^+ = 1$, regular evengons. For the remainder of this paper we assume $\hat{K}^+ = 1$; in fact, the careful reader will note that all we use is MWC, NRC and OAC. If K is not of constant width then we choose two points P_1 and P_2 so that $\overline{P_1 P_2}$ is a diameter. Let l_1 be the support line through P_1 in a direction of minimal width, and let l_2 be the opposite support line parallel to l_1.

33. LEMMA. *If l_1 is chosen properly then $P_2 \in l_2$.*

PROOF. If there is more than one choice for l_1, choose l_1 so that dist $(l_2 \cap K, P_2)$ is as small as possible. Let Q_2 be the point of $l_2 \cap K$ closest to P_2. If $P_2 = Q_2$ we are done. If $P_2 \neq Q_2$, choose $Q_3 \in \partial K$ between P_2 and Q_2 so that Q_3 has a unique support line, l_3. The support line l parallel to l_3 must pass through P_1 since Q_3 is between P_2 and Q_2. Since this is the only support line at Q_3, dist $(l, l_3) = h$ by MWC. This contradicts the choice of l_1. Thus $P_2 = Q_2$. Q.E.D.

Since dist $(l_1, l_2) = h$ by Theorem 20 there are points $Q_1 \in l_1 \cap K$ and $Q_2 \in l_2 \cap K$ with $\overline{Q_1 Q_2} \perp l_1$. If Q_1 and Q_2 can be chosen so that neither $Q_1 = P_1$ nor $Q_2 = P_2$ then K has parallel sides at P_1 and P_2. If not, then without loss of generality $P_1 = Q_1$ and $\overline{P_1 Q_2} \perp l_2$, $h = \text{dist}(P_1, Q_2) < \text{dist}(P_1, P_2) = d$, and $\overline{P_2 Q_2}$ is part of an edge of K. We now consider two exhaustive, exclusive cases.

Case I. For some choice of diametrically opposed points, P_1 and P_2, we get parallel sides.

Case II. For no choice of diametrically opposed points do we get parallel sides.

34. THEOREM. *In Case I, K is a regular evengon.*

PROOF. We begin with a lemma.

35. LEMMA. *If K has a side $\overline{AB}$ and an opposite parallel tangent ray, then the ray contains a side $\overline{CD}$, and $\overline{AB}$ and $\overline{CD}$ are equal in length and perpendicularly opposite each other.*

PROOF. Let $\overline{CD}$ be the longest edge of K in the ray, possibly $C = D$. If the two parallel edges, $\overline{AB}$ and $\overline{CD}$, were not as described in the lemma then by proper labeling of the endpoints we may suppose that B does not project onto $\overline{CD}$ and that D is farther from B than C. It follows that NRC is violated with $P_1 = D$, $P_2 = B$ and $P = C$. Q.E.D.

Let $\overline{P_1 Q_1}$ and $\overline{P_2 Q_2}$ be the parallel edges. By Lemma 35 they are parallel and opposite, hence, dist $(Q_1, Q_2) = \text{dist}(P_1, P_2) = d$ so that $\overline{Q_1 Q_2}$ is also a diameter. The ray from P_i through Q_i is a tangent ray to K. Let r_1 and r_2 be the other tangent rays at P_1 and P_2. Since the tangent angles at P_1 and P_2 are equal by Lemma 32, it follows that $r_1 \| r_2$. If neither r_1 nor r_2 contains a side of K then pick R_1 on ∂K near P_1 on the side of P_1 as r_1. If R_1 is sufficiently close to P_1 all the points opposite R_1 will be close to P_2. Again if R_1 is sufficiently close to P_1 all its opposite points will be further away than $h < d = \text{dist}(P_1, P_2)$. This violates MWC, hence one of r_1 and r_2 contains a side. Again by Lemma 35, r_1 and r_2 must contain equal opposite sides. Let R_1 and R_2 be the ends of these sides. By MWC, dist $(\overline{P_1 R_1}, \overline{P_2 R_2}) = \text{dist}(\overline{P_1 Q_1}, \overline{P_2 Q_2}) = h$. Also $\overline{P_1 P_2}$ is a common diagonal to the rectangles

$P_1Q_1Q_2P_2$ and $P_1R_1R_2P_2$. Thus these two rectangles are congruent with $\overline{P_1R_1}$ corresponding to $\overline{P_1Q_1}$. Also the angle at P_1 is $2\left(\arcsin\frac{h}{d}\right)$ and length $\overline{P_1Q_1}$ is $\sqrt{d^2-h^2}$. Of course $\overline{Q_1Q_2}$ is a diameter just as $\overline{P_1P_2}$ is, so it must also have two pairs of parallel sides of length $\sqrt{d^2-h^2}$ meeting at angle $2\left(\arcsin\frac{h}{d}\right)$. We iterate this process and keep creating new pairs of sides of length $\sqrt{d^2-h^2}$ meeting at angles of $2\left(\arcsin\frac{h}{d}\right)$. Since the total perimeter is bounded we must eventually repeat. Thus the entire boundary of K is composed of parallel pairs of sides of length $\sqrt{d^2-h^2}$ meeting at equal angles; i.e., K is a regular evengon. Q.E.D.

XI. $\hat{K}^+=1$, **regular oddgons.** Case I having been disposed of in Theorem 34 we now dispose of Case II.

36. THEOREM. *In Case II, K is a regular oddgon.*

PROOF. From the discussion at the beginning of Section X we know that in this case we can find diametrically opposed points P_1 and P_2 such that K has a side containing $\overline{P_2R_2}$ where $\overline{P_1R_2}\perp\overline{P_2R_2}$ and $\operatorname{dist}(P_1,R_2)=h$. Let r_2' be the tangent ray from P_2 containing $\overline{P_2R_2}$ and let r_2 be the other tangent ray at P_2. Let r_1 be the tangent ray at P_1 on the opposite side of $\overline{P_1P_2}$ from r_2 and let r_1' be the other tangent ray at P_1. Let the angles α_1, α_2, β_1, and β_2 be as in figure 20. Since the perpendiculars at the ends of a diameter are support lines all these angles are non-obtuse. Also we know from the way Case II arose that there is a support line through P_1 parallel to r_2'. Thus $\alpha_1\leqq\alpha_2$. Furthermore $r_1'\parallel r_2'$ implies that there are parallel edges by Lemma 35, but this cannot happen in Case II, thus $\alpha_1<\alpha_2$. By Lemma 32 the tangent angles at P_1 and P_2 are equal so

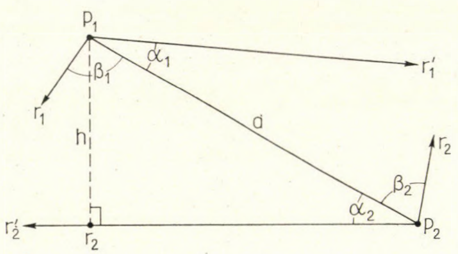

Fig. 20

$$\alpha_1+\beta_1=\alpha_2+\beta_2;\ \text{whence}\ \beta_1>\beta_2.$$

If r_1 does not contain an edge, choose R_1 close enough to P_1 so that for every point T between R_1 and P_1 both $\operatorname{dist}(T,P_2)>h$ and P_2 is the only point opposite T. This can be done since $\operatorname{dist}(P_1,P_2)=d>h$ and $\beta_1>\beta_2$. If we choose a support line, l, which touches ∂K strictly between R_1 and P_1, then only P_2 is opposite $l\cap\partial K$ and since all these points are too far from P_2, Theorem 22 yields a contradiction. Thus r_1 contains an edge and by Theorem 22 there is a point R_1 on this edge with $\overline{P_2R_1}\perp\overline{P_1R_1}$ and $\operatorname{dist}(R_1,P_2)=h$. At this point we have a symmetric situation between $\overline{P_1R_1}$ and $\overline{P_2R_2}$.

If r_1' does not contain an edge, then NRC is violated (with l_2 containing r_2', $l_1\parallel l_2$, $P_1\in l_1$, $r=r_1$, $r_1=r_1'$ and consequently $P=P_1$). By the above mentioned symmetry r_2 contains an edge also.

Since the angles at the ends of the diameter are not parallel and since P_1 and P_2 are both vertices it follows from OAC 1 that there are other diameters $\overline{P_0P_1}$ and $\overline{P_2P_3}$. Replacing $\overline{P_1P_2}$ first by $\overline{P_0P_1}$ then by $\overline{P_2P_3}$ in the above argument we find points P_{-1} and P_4 so that $\overline{P_{-1}P_0}$ and $\overline{P_3P_4}$ are diameters. Iterating we get a sequence of vertices ... $P_{-2}, P_{-1}, P_0, P_1, P_2$ ... with equal angles and $\overline{P_iP_{i+1}}$ a diameter. Since there can only be a finite number of such angles this sequence must repeat. We note that the angle at a vertex at the end of a diameter, is divided into two parts by the diameter, one of which is always $\arcsin\frac{h}{d}$. Thus opposite any of the points P_i there are precisely two other points P_{i-1} and P_{i+1} such that $\overline{P_iP_{i-1}}$ and $\overline{P_iP_{i+1}}$ are diameters. Since P_{i-1} and P_{i+1} are both opposite P_i the entire boundary between them is opposite P_i. Furthermore, since the angles at P_{i-1} and P_{i+1} are the same as the angle at P_i and the sides of the angles are not parallel, no points except those between P_{i-1} and P_{i+1} are opposite P_i. We will eventually show that the boundary between P_{i-1} and P_{i+1} is a side of length $2\sqrt{d^2-h^2}$.

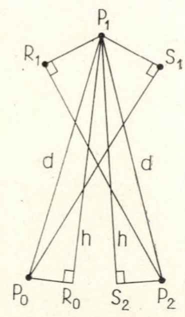

Fig. 21

We know from Theorem 20 that on the edge leaving P_1 which is opposite P_2 there is a point R_1 such that $\overline{P_2R_1} \perp \overline{P_1R_1}$ and length $\overline{P_2R_1}$ is h. Similarly on the side of the vertex P_1 which is opposite P_0 there is a point S_1 with $\overline{P_0S_1} \perp \overline{P_1S_1}$ and length $P_0S_1=h$. (See figure 21.) In general we get points R_{i-1} and S_{i+1} on the sides at P_{i-1} and P_{i+1} such that $\overline{P_iR_{i-1}} \perp \overline{P_{i-1}R_{i-1}}$ and $\overline{P_iS_{i+1}} \perp \overline{P_{i+1}S_{i+1}}$ and length $\overline{P_iR_{i-1}}$ = length $\overline{P_iS_{i+1}}=h$. Also we note that length $\overline{P_iR_i}$ = length $\overline{P_iS_i}=$ $=\sqrt{d^2-h^2}$. We next show that the sides at P_i do not end at R_i and S_i. It is sufficient to show that the side through P_1S_1 does not end at S_1. Suppose it does. We consider two cases. If the angle at P_i is acute then there can be only three such angles and K must look like figure 22. This figure violates NRC with l_2 taken to be the support line through $\overline{P_2S_2}$ and $P=S_1$. For in this case, both m_2 and m_1 are support lines of K, symmetric with respect to P_0, while P_0S_1 extended is m. Since $\overline{P_1P_2}$

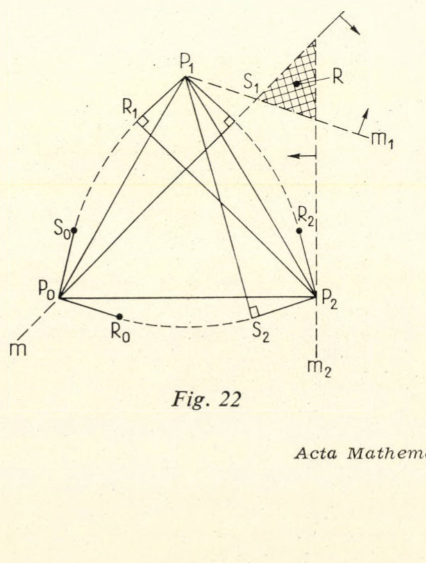

Fig. 22

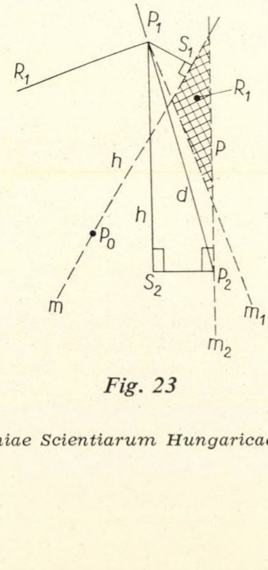

Fig. 23

is not a side we see that $S_1 \neq R_2$; hence m passes above $m_1 \cap m_2$ and there is a point R in the three open half-planes. If the angles are non-acute, then the line m_1 goes inside K and below S_1 while S_1 is on the same side of m_2 as P_1 since dist $(P_1, m_2) = \sqrt{d^2-h^2}$ = length $\overline{P_1 S_1}$, but $\overline{P_1 S_1}$ is not perpendicular to m_1. Thus we can again find a desired point R. (See figure 23.) In fact a point on ∂K just on the other side of S_1 from P_1 is the desired region. Thus in any case S_1 is not an end point of the side through $P_1 S_1$. Symmetry yields that neither R_i nor S_i can be end points of any sides of K. We now wish to show that $R_{i+1} = S_{i-1}$. For suppose $R_0 \neq S_2$, and consider the endpoint T of the side containing $P_2 S_2$, since $\overline{P_1 S_2} \perp P_2 S_2$ it follows that dist $(T, P_1) > h$. If we choose any point $Q \in \partial K$ close to T but between T and R_0, then P_1 is the only point opposite Q and hence by MWC and Theorem 20 there is a side of K containing Q and tangent to the circle, C, of radius h about P_1. By picking another point Q' we get another side of K which is tangent to C. Since K is convex the sides must be the same. Thus T is the endpoint of another side, say $\overline{TU}$, which is tangent to C. (See figure 24.) Since P_0 and P_2 are the only

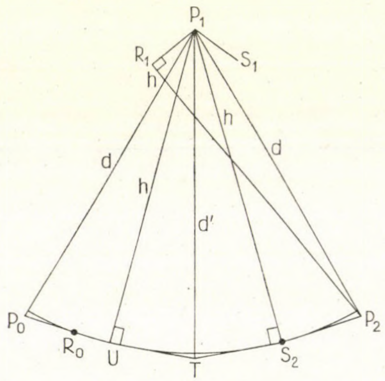

Fig. 24

diametrically opposed points to P_1, it follows that dist $(P_1, T) = d' < d$. Thus $\sphericalangle UTP_1 = \sphericalangle S_2TP_1 = \arcsin \frac{h}{d'} > \arcsin \frac{h}{d} = \sphericalangle P_0 P_1 S_1 = \sphericalangle P_2 P_1 R_1$. Thus both the angles $\sphericalangle UTP_1$ and $\sphericalangle S_2TP_1$ are bigger than both the angles $\sphericalangle TP_1 R_1$ and $\sphericalangle TP_1 S_1$. This violates OAC 2, a contradiction. Thus $R_0 = S_2$ and $\overline{P_0 P_2}$ is a side of K of length $2\sqrt{d^2-h^2}$. By symmetry $\overline{P_i P_{i+2}}$ is a side of K of length $2\sqrt{d^2-h^2}$. Thus the boundary of K consists of non-parallel sides of length $2\sqrt{d^2-h^2}$ meeting at equal angles. Thus K is a regular oddgon. Q.E.D.

XII. Conclusions. We come now to the main theorem of the paper:

37. THEOREM. *Let K be a convex, bounded figure. Then K is tight if and only if K is either a curve of constant width or a regular polygon.*

PROOF. Combination of Theorems 25, 31, and 33.

In the process of showing this theorem, we have proved another.

38. THEOREM. *A strictly convex figure K is a curve of constant width if and only if K satisfies the minimum width condition and has no side.*

The condition $\hat{K}=\sqrt{2}$ is satisfied exactly for isosceles right triangles and parallelograms whose sides are in the ratio $\sqrt{2}$. For $\hat{K}^+=\sqrt{2}$, only rectangles of this ratio and isosceles right triangles will suffice.

Thus we have found the necessary and sufficient conditions for the extrema of this packing problem.

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ON SEMIRINGS WHICH ARE EMBEDDABLE INTO A SEMIRING WITH IDENTITY

By
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We shall denote by $\mathcal{I}$ the class of all semirings which are embeddable into a semiring with identity. In an earlier paper [3], we proved the following:

THEOREM 1. *A semiring R is in $\mathcal{I}$ if and only if every translation of R is an additive endomorphism of R.*

The purpose of this paper is to investigate further properties of $\mathcal{I}$. This leads us to an explicit description of the greatest homomorphic image of a semiring which is embeddable into a semiring with identity. We wish to express our gratitude to Professors L. FUCHS and P. A. GRILLET for their helpful suggestions.

A *left (right) translation* of a semiring R is defined as a finite (pointwise) sum of the identity mapping of R or of inner left (right) translations of R. We denote by $\Lambda_R(\bar{P}_R)$ the set of such mappings. The reader is referred to [4] for the basic properties of $\bar{\Lambda}_R$ and $\bar{P}_R$.

In view of Theorem 1, it is easy to see that $\mathcal{I}$ is a variety (defined by a countable set of identities). This allows us to give properties of $\mathcal{I}$ as consequences of more general results. For instance, given any semiring R, we may assert the existence of a semiring R' and a homomorphism f of R into R' with a universal property relative to elements of $\mathcal{I}$. Observe that, since $\mathcal{I}$ is obviously hereditary, this statement is equivalent to the existence of a greatest homomorphic image of R which is in $\mathcal{I}$. We shall give an explicit description of such an R'.

This depends on the next result, where we recall that, for any semiring R, $\bar{\Lambda}_R(\bar{P}_R)$ is a near-semiring under (pointwise) addition and the composition of mappings as multiplication.

PROPOSITION 2. *Let f be a homomorphism of a semiring R onto a semiring R'. Then there exists a unique homomorphism $f_{\bar{\Lambda}}$ of $\bar{\Lambda}_R$ onto $\bar{\Lambda}_{R'}$ such that*

$$f_{\bar{\Lambda}}(\lambda)\circ f = f\circ\lambda \tag{1}$$

holds for all $\lambda\in\bar{\Lambda}_R$.

PROOF. Observe first that, if it exists, such homomorphism $f_{\bar{\Lambda}}$ is unique, since f is onto. To show the existence of $f_{\bar{\Lambda}}$, let now $\lambda\in\bar{\Lambda}_R$. Then we can write $\lambda=\sum_{i=1}^{n}\lambda_i$, where either $\lambda_i=\varepsilon_R$ is the identity mapping of R or $\lambda_i=\lambda_{x_i}$ (with $x_i\in R$) is an inner translation of the multiplicative semigroup of R. Then, for all $x\in R$,

$$f(\lambda(x)) = f\left(\sum_{i=1}^{n}\lambda_i(x)\right) = \sum_{i=1}^{n} f(\lambda_i(x)) = \sum_{i=1}^{n}\lambda_i'(f(x)) = \lambda'(f(x)),$$

with $\lambda' = \sum_{i=1}^{n} \lambda_i'$, where $\lambda_i' = \varepsilon_{R'}$ (identity mapping of R') in case $\lambda_i = \varepsilon_R$ and $\lambda_i' = \lambda_{f(x_i)}$ in case $\lambda_i = \lambda_{x_i}$. Therefore, for all $\lambda \in \bar{\Lambda}_R$, there exists $\lambda' \in \bar{\Lambda}_{R'}$ such that $\lambda' \circ f = f \circ \lambda$; since f is onto, λ' is unique. Hence $f_{\bar{\Lambda}}(\lambda) = \lambda'$ defines a mapping of $\bar{\Lambda}_R$ into $\bar{\Lambda}_{R'}$ such that (1) holds. Clearly, since f is onto, so is $f_{\bar{\Lambda}}$. Finally, using (1), we have:

$$f_{\bar{\Lambda}}(\lambda+\lambda')\circ f = f\circ(\lambda+\lambda') = f\circ\lambda + f\circ\lambda' = f_{\bar{\Lambda}}(\lambda)\circ f + f_{\bar{\Lambda}}(\lambda')\circ f = \big(f_{\bar{\Lambda}}(\lambda)+f_{\bar{\Lambda}}(\lambda')\big)\circ f;$$

$$f(\lambda\circ\lambda')\circ f = f\circ\lambda\circ\lambda' = f_{\bar{\Lambda}}(\lambda)\circ f\circ\lambda' = f_{\bar{\Lambda}}(\lambda)\circ f_{\bar{\Lambda}}(\lambda')\circ f,$$

for all $\lambda, \lambda' \in \bar{\Lambda}_R$ which shows that $f_{\bar{\Lambda}}$ is a homomorphism.

We now consider the smallest additive congruence $\mathscr{C}_R$ on a semiring R containing the set $\mathscr{A}_R$ of all pairs having one of the following forms:

$$(2) \qquad \big(\lambda(x+y),\quad \lambda(x)+\lambda(y)\big),$$

$$(3) \qquad \big(\varrho(x+y),\quad \varrho(x)+\varrho(y)\big),$$

where $x, y \in R$, $\lambda \in \bar{\Lambda}_R$ and $\varrho \in \bar{P}_R$. Obviously Theorem 1 simply means that $R \in \mathscr{I}$ if and only if $\mathscr{C}_R$ is the identity. From this remark and a few technical lemmas will follow our main result which is the following:

THEOREM 3. *$\mathscr{C}_R$ is a congruence and $R' = R/\mathscr{C}_R$ is the greatest homomorphic image of R which is embeddable into a semiring with identity. Furthermore R and R' have same universal semiring with identity.*

LEMMA 4. *$\mathscr{C}_R$ is a congruence of R.*

PROOF. Let us first show that $\mathscr{A}_R$ admits the multiplication of R. We simply consider pairs in $\mathscr{A}_R$ of the form (2), the proof would be dual for pairs of the form (3).

Let therefore $\big(\lambda(x+y), \lambda(x)+\lambda(y)\big) \in \mathscr{A}_R$ for some $x, y \in R$ and $\lambda \in \bar{\Lambda}_R$, and let $u \in R$. Then, by 1. 3 of [4], we know that λ is a left translation of the multiplicative semigroup of R, whence

$$\big(\lambda(x+y)\big)u = \lambda\big((x+y)u\big) = \lambda(xu+yu)$$

and

$$\big(\lambda(x)+\lambda(y)\big)u = \lambda(x)u + \lambda(y)u = \lambda(xu)+(yu);$$

thus $\big(\lambda(x+y)u, \big(\lambda(x)+\lambda(y)\big)u\big) \in \mathscr{A}_R$. Also, by 1. 7 of [4], $\lambda_u\lambda = \lambda_{\varrho(u)}$ for some $\varrho \in \bar{P}_R$; therefore

$$u\big(\lambda(x+y)\big) = \varrho(u)(x+y) = \varrho(u)x+\varrho(u)y = u\lambda(x)+u\lambda(y) = u\big(\lambda(x)+\lambda(y)\big),$$

so that $\big(u\big(\lambda(x+y)\big), u\big(\lambda(x)+\lambda(y)\big)\big) \in \mathscr{A}_R$ trivially. Thus $\mathscr{A}_R$ admits the multiplication of R. It follows easily that the smallest additive congruence $\mathscr{C}_R$ containing $\mathscr{A}_R$ has the same property which completes the proof.

We denote by f the canonical projection of R onto $R' = R/\mathscr{C}_R$. By lemma 4, R' is a semiring under the induced operations.

LEMMA 5. *$R' \in \mathscr{I}$; furthermore, for any homomorphism g of R onto a semiring $R'' \in \mathscr{I}$, there exists a unique homomorphism g' of R' onto R'' such that $g' \circ f = g$.*

PROOF. To prove that $R' \in \mathcal{I}$, we shall check that any translation of R' is an additive homomorphism of R'. If $\lambda' \in \bar{\Lambda}_{R'}$, then, by proposition 2, there exists $\lambda \in \bar{\Lambda}_R$ such that $\lambda' \circ f = f \circ \lambda$. Then, for all $x' = f(x)$, $y' = f(y)$ in R',

$$\lambda'(x'+y') = \lambda'(f(x)+f(y)) = \lambda'(f(x+y)) = f(\lambda(x+y))$$

and similarly $\lambda'(x')+\lambda'(y') = f(\lambda(x)+\lambda(y))$; hence, in view of (2), $\lambda'(x')+\lambda'(y') = \lambda'(x'+y')$. Thus λ' is an additive homomorphism of R'. Dually any right translation is an additive homomorphism of R' too.

Let now g be a homomorphism of R onto a semiring $R'' \in \mathcal{I}$. Clearly all we need to show is that the congruence $\ker g$ induced by g contains $\mathcal{C}_R$. If first $x, y \in R$, $\lambda \in \bar{\Lambda}_R$, then $g(\lambda(x+y)) = \lambda''(g(x)+g(y))$ and

$$g(\lambda(x)+\lambda(y)) = g(\lambda(x))+g(\lambda(y)) = \lambda''(g(x))+\lambda''(g(y))$$

for some $\lambda'' \in \bar{\Lambda}_{R''}$ by proposition 2. Also, since $R'' \in \mathcal{I}$, λ'' is an additive homomorphism; thus we get $(\lambda(x+y), \lambda(x)+\lambda(y)) \in \ker g$; dualy $\ker g$ contains all pairs of the form (3), which shows that $\ker g \supseteqq \mathcal{A}_R$. Clearly then $\ker g \supseteqq \mathcal{C}_R$ so that g factorizes uniquely through R'.

LEMMA 6. *R and R' have same universal semiring with identity.*

PROOF. Let R^1 be the universal semiring with identity of R' and denote by φ' the canonical injection of R' into R^1. We want to prove that R^1 is also the universal semiring with identity of R, with the obvious canonical homomorphism $\varphi' \circ f$.

Let g be any homomorphism of R to a semiring R'' with identity. By lemma 5, there exists a unique homomorphism g' of R' into R'' such that $g' \circ f = g$. Also, by the universal property of R^1, there exists a unique homomorphism g_1 of R^1 into R'' such that $g_1 \circ \varphi' = g'$. Thus g_1 is a homomorphism of R^1 into R'' such that $g_1 \circ (\varphi' \circ f) = g$ and is then unique with this property.

Thus R^1 is the universal semiring with identity of R.

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ON APPROXIMATE DERIVATES AND THEIR PROPERTIES

By

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1. Introduction

The notion of derivatives is generalized in many directions. DENJOY in his memoir [5] considered the approximate derivatives. BURKILL and HASLAM-JONES [8] obtained several results by considering approximate derivates (for definition see also [12]) and BURKILL [1] proved that if for an approximately continuous function f the approximate lower derivate is non-negative then f is non-decreasing. SUNOUCHI and UTAGAWA [14] obtained the same result by considering measurable functions. KHINTCHINE [9] proved that if for a monotone function f the approximate derivative f'_{ap} exists then the ordinary derivative f' also exists and $f'=f'_{\mathrm{ap}}$. GOFFMAN and NEUGEBAUER [6] obtained that the approximate derivative possesses Darboux property and the Mean value property and that for an approximately differentiable function, the set of non-differentiability is non-dense (see also [15] and [16]). MARCUS [11] proved that the approximate derivatives satisfy Denjoy property, i.e., the inverse image of an open interval under an approximate derivative is either void or of positive measure. Just as ZAHORSKI [18] refined the Denjoy property for ordinary derivatives proved by DENJOY [4], SELIVANOV [13] and CLARKSON [3], WEIL [17] refined this result of Marcus by showing that for every open interval $(a, b), x \in f'^{-1}_{\mathrm{ap}}((a, b))$ and $\{I_n\}$ a sequence of closed intervals with the following properties:

(i) $x \notin \bigcup I_n$,

(ii) every neigbourhood of x contains all but a finite number of the intervals I_n, and

(iii) $$m(f'^{-1}_{\mathrm{ap}}((a, b)) \cap I_n) = 0 \quad \text{for every} \quad n,$$

imply

$$\lim_{n \to \infty} \frac{m(I_n)}{d(x, I_n)} = 0$$

where

$$d(x, I_n) = \inf \{|x-y| : y \in I_n\}.$$

In the present paper we considered these results with the approximate derivatives f'_{ap} replaced by the approximate upper and lower derivates.

2. Definitions and notations

Throughout the paper f will denote a real function defined on the real line, D^+f etc. will denote, as usual, the Dini derivatives of f and m will denote the measure of a set in Lebesgue sense. For convenience we state the following known definitions:

I. The right upper approximate derivate of f at a point x is the lower bound of the numbers y such that the set

$$\left\{\xi : \frac{f(\xi)-f(x)}{\xi - x} > y,\ \xi > x\right\}$$

has the point x as a point of dispersion. This will be denoted by $AD^+f(x)$. The definitions of AD_+f, AD^-f and AD_-f are analogous.

II. A measurable set E is said to be d-open iff each point of E is a point of density of E. The collection of all such d-open sets is called the density topology. The complement of a d-open set is called d-closed (see [7]).

III. The function AD^+f has Darboux property iff for any two reals α, β with $\alpha<\beta$ and a number η lying between $AD^+f(\alpha)$ and $AD^+f(\beta)$ there is ξ, $\alpha<\xi<\beta$ such that $AD^+f(\xi)=\eta$.

AD^+f has mean value property iff for $\alpha<\beta$ there is ξ, $\alpha<\xi<\beta$ such that $f(\beta)-f(\alpha)=(\beta-\alpha)AD^+f(\xi)$.

For approximate limit and approximate continuity we follow the standard definitions as given in SAKS [12]. The function f is approximately upper semi continuous at ξ iff $f(\xi)$ is not less than the approximate upper limit of f at ξ.

IV. A sequence $\{I_n\}$ of closed intervals is said to converge to a point x iff x is not in the union of the intervals I_n and every neighbourhood of x contains all but a finite number of the intervals I_n.

3. Main results

LEMMA. *If f is approximately upper semi-continuous then the set*

$$F = \{x : f(x) \geqq \lambda\}$$

is d-closed for every λ, $-\infty<\lambda<\infty$.

PROOF. Since approximately upper semi-continuous functions are upper semi-continuous with respect to the density topology [7], the result follows.

THEOREM 1. *Let f be approximately upper semi-continuous and let K be a real number $0<K<1$ such that for every x, the right upper density of the set*

$$S(x) = \{\xi : f(\xi) \geqq f(x)\}$$

at x is not less than K. Then f is non-decreasing.

PROOF. For any two reals a, b, $a<b$ we shall show that $f(a)\leqq f(b)$. Choose K_1, $0<K_1<K$. Let X be a set such that

(i) $$X\subset S(a)\cap[a,b]=\{\xi:f(\xi)\geqq f(a)\}\cap[a,b]$$

(ii) $$x',x''\in X \quad\text{and}\quad x'<x'' \quad\text{imply}\quad \frac{m(S(a)\cap(x',x''))}{x''-x'}\geqq K_1.$$

Let $\mathcal{R}$ be the collection of all such sets X. We shall show that the collection $\mathcal{R}$ is non-empty. Since the right upper density of the set $S(a)$ at $x=a$ is not less than K, there is a sequence of intervals $\{I_n\}$ such that a is the left hand endpoint of I_n for each n, $m(I_n)\to 0$ as $n\to\infty$ and $\lim\limits_{n\to\infty}\dfrac{m(S(a)\cap I_n)}{m(I_n)}\geqq K$. Let us write $I_n=[a,b_n]$. We may suppose that $I_n\subset[a,b]$ for all n. From the sequence $\{I_n\}$ we construct another sequence of intervals $\{J_n\}$ such that the right hand endpoint of J_n belongs to $S(a)\cap[a,b]$ for each n. If the right hand endpoint of I_n is a member of $S(a)$ then we take $I_n=J_n$; otherwise we consider the following cases:

I. b_n is the limit point of $S(a)$ from the left. Since $b_n\notin S(a)$ and $S(a)$ is d-closed, b_n is a point of dispersion for the set $S(a)$. So there is a point $c_n\in S(a)$ such that $a<c_n<d_n$ and $\dfrac{m(S(a)\cap(c_n,b_n))}{b_n-c_n}<\dfrac{1}{n}$. In this case we take $J_n=[a,c_n]$.

II. b_n is not a limit point of $S(a)$ from the left. Then there is d_n, $a<d_n<b_n$ such that $(d_n,b_n)\cap S(a)=0$ and either $d_n\in S(a)$, or d_n is a limit point of $S(a)$ from the left. If $d_n\in S(a)$ we take $J_n=[a,d_n]$; otherwise we choose a point c_n as in Case I such that $c_n\in S(a)$, $a<c_n<d_n$ and $\dfrac{m(S(a)\cap(c_n,d_n))}{d_n-c_n}<\dfrac{1}{n}$, i.e. $\dfrac{m(S(a)\cap(c_n,b_n))}{b_n-c_n}<\dfrac{1}{n}$. In this case we take $J_n=[a,c_n]$. Now

$$\frac{m(S(a)\cap I_n)}{m(I_n)}=\frac{m(S(a)\cap J_n)}{m(I_n)}+\frac{m(S(a)\cap(c_n,b_n))}{m(I_n)}\leqq$$

$$\leqq\frac{m(S(a)\cap J_n)}{m(J_n)}+\frac{m(S(a)\cap(c_n,b_n))}{b_n-c_n}<\frac{m(S(a)\cap J_n)}{m(J_n)}+\frac{1}{n},$$

i.e.

$$\lim_{n\to\infty}\frac{m(S(a)\cap I_n)}{m(I_n)}\leqq\liminf_{n\to\infty}\frac{m(S(a)\cap J_n)}{m(J_n)}.$$

Thus

$$\liminf_{n\to\infty}\frac{m(S(a)\cap J_n)}{m(J_n)}\geqq K.$$

This proves that the collection $\mathcal{R}$ is non-empty. Let $\mathcal{R}$ be partially ordered by set inclusion. It can be verified that every linearly ordered subset of $\mathcal{R}$ has an upper bound in $\mathcal{R}$. Hence by Zorn's lemma $\mathcal{R}$ has a maximal element, say Z.

Let $\sup Z=\beta$. We shall prove that $\beta\in Z$. If $x\in Z$, $x<\beta$, then there is a sequence $\{x_n\}\subset Z$ such that $x<x_n\leqq x_{n+1}$ and $\lim\limits_{n\to\infty}x_n=\beta$. So

$$K_1\leqq\lim_{n\to\infty}\frac{m(S(a)\cap(x,x_n))}{x_n-x}=\frac{m(S(a)\cap(x,\beta))}{\beta-x}.$$

So β is a point of positive upper density of $S(a)$. Since f is approximately upper semi-continuous, by the lemma $\beta\in S(a)$. Since Z is maximal, from the above relation we conclude that $\beta\in Z$.

We now show that $b=\beta$. If possible, let $\beta<b$. Then since the right upper density of the set $S(\beta)=\{\xi:f(\xi)\geqq f(\beta)\}$ at $x=\beta$ is not less than K, there exists γ, $\beta<\gamma<b$, $\gamma\in S(\beta)$ such that

$$\frac{m\big(S(\beta)\cap(\beta,\gamma)\big)}{\gamma-\beta}\geqq K_1.$$

Since $S(\beta)\subset S(a)$, we conclude that $\dfrac{m\big(S(a)\cap(\beta,\gamma)\big)}{\gamma-\beta}\geqq K_1$. Also if α is any other member of Z, then $m\big(S(a)\cap(\alpha,\beta)\big)\geqq K_1(\beta-\alpha)$ and $m\big(S(a)\cap(\beta,\gamma)\big)\geqq K_1(\gamma-\beta)$ and hence $m\big(S(a)\cap(\alpha,\gamma)\big)\geqq K_1(\gamma-\alpha)$ i.e. $\dfrac{m\big(S(a)\cap(\alpha,\gamma)\big)}{\gamma-\alpha}\geqq K_1$. This shows that $Z\cup\{\gamma\}\in\mathscr{R}$. But since $\gamma\notin Z$ and Z is maximal, this is a contradiction. Thus $\beta=b$. So $b\in S(a)$ i.e., $f(a)\leqq f(b)$.

COROLLARY. *If f is approximately upper semi-continuous and if $AD_+f(x)\geqq 0$ for all x, then f is non-decreasing.*

THEOREM 2. *For a monotone function f,* (i) $AD_+f=D_+f$, (ii) $AD^+f=D^+f$, (iii) $AD_-f=D_-f$, (iv) $AD^-f=D^-f$.

PROOF. We shall prove (i) for non-decreasing function as well as non-increasing function. The other cases are analogous.

Let us first suppose that f is non-decreasing. If possible, suppose that $AD_+f\neq\neq D_+f$. Let

$$K_1=D_+f(\xi)<AD_+f(\xi)=K.$$

Choose ε, $0<\varepsilon<\dfrac{K-K_1}{2}$. Then since $K_1<K-2\varepsilon$, there is a sequence $\{h_n\}$ such that $h_n\to 0+$ as $n\to\infty$ and

$$\frac{f(\xi+h_n)-f(\xi)}{h_n}<K-2\varepsilon\quad\text{for all}\quad n. \tag{1}$$

From (1) we conclude that $\dfrac{f(\xi+h_n)-f(\xi)}{K-\varepsilon}<h_n$. For a given n we denote by J_n and I_n the intervals $[\xi,\xi+h_n]$ and $\left[\xi+\dfrac{f(\xi+h_n)-f(\xi)}{K-\varepsilon},\xi+h_n\right]$ respectively. Then

$$m(J_n)=h_n\quad\text{and}\quad m(I_n)=h_n-\frac{f(\xi+h_n)-f(\xi)}{K-\varepsilon}.$$

So from (1) we get

$$\frac{m(I_n)}{m(J_n)}=1-\frac{f(\xi+h_n)-f(\xi)}{h_n(K-\varepsilon)}>1-\frac{K-2\varepsilon}{K-\varepsilon}. \tag{2}$$

Since $AD_+ f(\xi)=K$, there is N such that the set

$$E=\left\{x: \frac{f(x)-f(\xi)}{x-\xi}-K>-\varepsilon,\ x>\xi\right\}$$

satisfies the inequality

$$\frac{m(E\cap J_n)}{m(J_n)}>1-\frac{\varepsilon}{K-\varepsilon} \quad \text{for all} \quad n\geqq N. \tag{3}$$

Now if $x\in I_n$ we have

$$f(x)\leqq f(\xi+h_n)\leqq f(\xi)+(K-\varepsilon)(x-\xi).$$

So

$$\frac{f(x)-f(\xi)}{x-\xi}\leqq K-\varepsilon$$

and hence

$$E\cap I_n=0.$$

So from (2) we get

$$\frac{m(E\cap J_n)}{m(J_n)}=\frac{m(E\cap(J_n-I_n))}{m(J_n)}\leqq\frac{m(J_n)-m(I_n)}{m(J_n)}=1-\frac{m(I_n)}{m(J_n)}<\frac{K-2\varepsilon}{K-\varepsilon}. \tag{4}$$

Since (3) and (4) are contradictory for $n\geqq N$, the result follows.

Next let f be non-increasing. If possible, suppose that $AD_+ f\neq D_+ f$. Let

$$K_1=D_+ f(\xi)<AD_+ f(\xi)=K.$$

Choose ε, $0<\varepsilon<\frac{K-K_1}{2}$. Then since $K_1<K-2\varepsilon$, there is a sequence $\{h_n\}$ such that $h_n\to 0+$ as $n\to\infty$ and

$$\frac{f(\xi+h_n)-f(\xi)}{h_n}<K-2\varepsilon \quad \text{for all} \quad n. \tag{5}$$

From (5) we conclude that $\frac{f(\xi+h_n)-f(\xi)}{K-\varepsilon}>h_n$. For a given n we denote by J_n and I_n the intervals $\left[\xi,\ \xi+\frac{f(\xi+h_n)-f(\xi)}{K-\varepsilon}\right]$ and $\left[\xi+h_n,\ \xi+\frac{f(\xi+h_n)-f(\xi)}{K-\varepsilon}\right]$ respectively. Then

$$m(J_n)=\frac{f(\xi+h_n)-f(\xi)}{K-\varepsilon}$$

and

$$m(I_n)=\frac{f(\xi+h_n)-f(\xi)}{K-\varepsilon}-h_n.$$

So from (5) we get

$$\frac{m(I_n)}{m(J_n)}=1-\frac{h_n}{\frac{f(\xi+h_n)-f(\xi)}{K-\varepsilon}}=1-\frac{K-\varepsilon}{\frac{f(\xi+h_n)-f(\xi)}{h_n}}>1-\frac{K-\varepsilon}{K-2\varepsilon}. \tag{6}$$

Since $AD_+ f(\xi)=K$, there is N such that the set

$$E=\left\{x:\frac{f(x)-f(x)}{x-\xi}-K>-\varepsilon,\ x>\xi\right\}$$

satisfies the inequality

$$\frac{m(E\cap J_n)}{m(J_n)}>1+\frac{\varepsilon}{K-2\varepsilon}\quad\text{for all}\quad n\geqq N. \tag{7}$$

Now if $x\in I_n$ we have

$$f(x)\leqq f(\xi+h_n)\leqq f(\xi)+(K-\varepsilon)(x-\xi).$$

So

$$\frac{f(x)-f(\xi)}{x-\xi}\leqq K-\varepsilon$$

and hence

$$E\cap I_n=0.$$

So from (6) we get

$$\frac{m(E\cap J_n)}{m(J_n)}=\frac{m\big(E\cap(J_n-I_n)\big)}{m(J_n)}\leqq\frac{m(J_n)-m(I_n)}{m(J_n)}=1-\frac{m(I_n)}{m(J_n)}<\frac{K-\varepsilon}{K-2\varepsilon}. \tag{8}$$

Since (7) and (8) are contradictory for $n\geqq N$, the result follows.

COROLLARY. *Let f be approximately continuous. If there exists a bounded derivative Φ' such that any one of*

(i) $AD_+ f(x)\geqq\Phi'(x)$ (ii) $AD_- f(x)\geqq\Phi'(x)$

(iii) $AD^- f(x)\leqq\Phi'(x)$ (iv) $AD^+ f(x)\leqq\Phi'(x)$

is true for all x, then f is continuous and the approximate derivates of f are respectively equal to its corresponding Dini derivatives.

PROOF. If $AD_+ f\geqq\Phi'$ then the function $g=f-\Phi$ is approximately continuous and $AD_+ g\geqq 0$ and hence g is non-decreasing. So g is continuous and from Theorem 2, the approximate derivates of g are equal to its corresponding Dini derivates and hence f is continuous and the approximate derivates of f are equal to its corresponding Dini derivates.

THEOREM 3. *Let f be approximately continuous and let $AD^- f=AD_+ f$. If $AD_+ f$ is of Baire class* 1, *then $AD_+ f$ has Darboux property.*

PROOF. Let α and β be any two real numbers such that $\alpha<\beta$. Suppose $AD_+ f(\alpha)<0$ and $AD_+ f(\beta)>0$. If possible, let there be no point $\xi\in(\alpha,\beta)$ for which $AD_+ f(\xi)=0$. Let

$$E^+=\{x:x\in[\alpha,\beta],\ AD_+ f(x)>0\}$$

$$E^-=\{x:x\in[\alpha,\beta],\ AD_+ f(x)<0\}.$$

Then $[\alpha,\beta]=E^+\cup E^-$.

We now prove that every component of E^+ and E^- is either a single point or a closed interval.

To prove this we assume that Q is a component of E^+ which is not a single point. Then Q is an interval. Let c, d be the endpoints of Q. Now by the corollary of Theorem 1, f is non-decreasing in (c, d). Since f is approximately continuous, f is non-decreasing in $[c, d]$. Hence $AD_+ f(c) \geqq 0$ and $AD^- f(d) \geqq 0$ i.e. $AD_+ f(d) \geqq 0$. Since $[\alpha, \beta] = E^+ \cup E^-$, we have $AD_+ f(c) > 0$ and $AD_+ f(d) > 0$. Hence $Q = [c, d]$. Similarly if Q is a component of E^- then Q is either a single point or a closed interval.

Let $\{Q^+\}$ and $\{Q^-\}$ be the collection of components of E^+ and E^- respectively, which are intervals and let $\{Q\} = \{Q^+\} \cup \{Q^-\}$. Then two distinct elements of Q are disjoint. Hence the set $P = [\alpha, \beta] - \bigcup_{Q \in \{Q\}} Q^0$ is perfect and $AD_+ f$ has no point of continuity in P relative to P. This contradicts that $AD_+ f$ is of Baire class 1. This completes the theorem.

COROLLARY. *Let f be approximately continuous and let $AD_+ f = AD^- f$. If $AD_+ f$ is of Baire class* 1, *then $AD_+ f$ has mean value property.*

PROOF. Let $\alpha < \beta$. We shall show that if $f(\alpha) = f(\beta)$, then there is ξ, $\alpha < \xi < \beta$, such that $AD_+ f(\xi) = 0$. To complete the proof we have only to consider the function g where $g(x) = f(x) - \frac{f(\alpha) - f(\beta)}{\alpha - \beta} x$. If f is constant on $[\alpha, \beta]$ the corollary is proved. So we suppose that f is not constant on $[\alpha, \beta]$. We conclude that there are two points $\eta', \eta'' \in (\alpha, \beta)$ such that $AD_+ f(\eta') < 0$ and $AD^- f(\eta'') > 0$. For, if $AD_+ f(x) \geqq 0$ for all $x \in (\alpha, \beta)$ then f would be non-decreasing on (α, β) and since $f(\alpha) = f(\beta)$, f would be constant on $[\alpha, \beta]$. So, by Theorem 3, there is ξ between η' and η'' such that $AD_+ f(\xi) = 0$.

THEOREM 4. *Let f be approximately continuous and let $AD^- f = AD_+ f$. Let $AD^- f$ be of Baire class* 1 *and let $E = \{x : AD^- f(x) = D^- f(x) = D_+ f(x) = AD_+ f(x)\}$. Then for every interval I, $E \cap I$ contains an interval.*

PROOF. Suppose there is an interval I such that $E \cap I$ contains no interval. Let $E^+ = \{x : x \in I;\ AD^- f(x) \geqq 0\}$ and $E^- = \{x : x \in I;\ AD^- f(x) < 0\}$. Then $I = E^+ \cup E^-$. Then by the corollary of Theorem 2, every component of E^+ and E^- is a single point and so E^+ and E^- are dense in I. Let $\alpha > 0$ and let $E_\alpha^- = \{x : x \in I;\ AD^- f(x) \leqq -\alpha\}$. Then E_α^- is dense in I. For, considering $g(x) = f(x) + \alpha x$ we see that the set $A = \{x : x \in I;\ AD^- g(x) = D^- g(x) = D_+ g(x) = AD_+ g(x)\}$ contains no interval and consequently since g satisfies the hypothesis of f, the set $E_g^- = \{x : x \in I;\ AD^- g(x) \leqq 0\}$ is dense in I. Since $AD^- g(x) = AD^- f(x) + \alpha$, we conclude that $E_g^- = E_\alpha^-$. Now since E^+ and E^- are dense in I, $AD^- f$ has no point of continuity in I which contradicts the fact that $AD^- f$ is of Baire class 1.

THEOREM 5. *Let f be approximately continuous, $AD^- f = AD_+ f$ and $AD^- f$ be finite except an enumerable set. If $AD^- f$ is of Baire class* 1, *then for any two reals α, β $(\alpha < \beta)$ the set*

$$E(\alpha, \beta) = \{x : \alpha < AD^- f(x) < \beta\}$$

is either void or of positive measure in every interval.

PROOF. Let $[a, b]$ be any interval. Let

$$E = E(\alpha, \beta) \cap [a, b],$$

$$E_\alpha = \{x : x \in [a, b],\ AD^- f(x) \leqq \alpha\},$$

$$E_\beta = \{x : x \in [a, b],\ AD^- f(x) \geqq \beta\}.$$

Then $[a, b] = E \cup E_\alpha \cup E_\beta$, the sets E, E_α and E_β being mutually disjoint.

If possible, let us suppose that E is not void but $mE=0$. We shall now show that $E \subset E'_\alpha \cap E'_\beta$. Let $x_0 \in E$ but $x_0 \notin E'_\alpha$. Then there is a neighbourhood V of x_0 such that $AD^- f(x) > \alpha$ for all $x \in V$. Then by considering the function $g(x) = f(x) - \alpha x$ it is seen that g is approximately continuous and $AD_+ g(x) > 0$ for all $x \in V$. Hence by the corollary of Theorem 2, g is continuous on V and $D^- g = AD^- g = AD_+ g = D_+ g$ on V and hence f is continuous and $D^- f = AD^- f = AD_+ f = D_+ f$ on V. By Theorem 1 of [10] we have $m(E \cap V) > 0$ which is a contradiction since $mE=0$. Thus $E \subset E'_\alpha$. Similarly $E \subset E'_\beta$. Hence $E \subset E'_\alpha \cap E'_\beta$. Since the set $E'_\alpha \cap E'_\beta$ is closed we conclude that $\bar{E} \subset E'_\alpha \cap E'_\beta$.

Let $\xi \in \bar{E}$. Then since $\bar{E} \subset E'_\alpha \cap E'_\beta$, we have for any interval J containing ξ

$$\sup_{x \in J} AD^- f(x) \geqq \beta, \quad \inf_{x \in J} AD^- f(x) \leqq \alpha.$$

Since by Theorem 3, $AD^- f(x)$ possesses Darboux property, it follows that

$$\sup_{x \in J \cap E} AD^- f(x) = \beta \quad \inf_{x \in J \cap E} AD^- f(x) = \alpha$$

So,

$$\sup_{x \in J \cap E} AD^- f(x) \geqq \beta \quad \inf_{x \in J \cap E} AD^- f(x) \leqq \alpha.$$

So the saltus of $AD^- f(x)$ at each point of $\bar{E}$ relative to $\bar{E}$ is at least $\beta - \alpha$. So the function $AD^- f(x)$ considered on $\bar{E}$ is everywhere discontinuous on $\bar{E}$. Now since the function $AD^- f(x)$ belongs to the Baire class 1, the discontinuities of $AD^- f(x)$ considered over a closed subset must form a set of the first category relative to the subset. Since $\bar{E}$ is a set of the second category on itself, it provides a contradiction. This proves the theorem.

COROLLARY. *Let f be approximately continuous, $AD^- f = AD_+ f$ and $AD^- f$ be finite except an enumerable set. If $AD^- f$ is of Baire class* 1 *and if $AD_+ f \geqq 0$ almost everywhere, then f is continuous and non-decreasing.*

PROOF. Since the set $\{x : AD_+ f(x) < 0\}$ is of measure zero, the set $E(-1, 0) = \{x : -1 < AD_+ f(x) < 0\}$ is also of measure zero and hence by Theorem 5, the set $E(-1, 0)$ is void. Since $AD_+ f$ satisfies Darboux property, the set $\{x : AD_+ f(x) \leqq -1\}$ is also void. Hence $AD_+ f(x) \geqq 0$ for all x and the conclusion follows from the corollary of Theorem 1.

REMARK. Another result in this direction can be found in [2].

The following theorem generalizes a theorem of WEIL [17].

THEOREM 6. *Let f be approximately continuous, $AD^- f = AD_+ f$ and let $AD^- f$ be finite except an enumerable set and of Baire class 1. For any two reals α, β ($\alpha < \beta$) denote*

$$E(\alpha, \beta) = \{x : \alpha < AD^- f(x) < \beta\}.$$

If the set $E(\alpha, \beta)$ contains a point x_0 where $f'_{ap}(x_0)$ exists, then $E(\alpha, \beta)$ has the following property:

For every sequence $\{I_n\}$ of closed intervals converging to x_0 with

$$m(E(\alpha, \beta) \cap I_n) = 0$$

for every n, we have

$$\lim_{n\to\infty} \frac{m(I_n)}{d(x_0, I_n)} = 0$$

where

$$d(x_0, I_n) = \inf \{|x_0 - y| : y \in I_n\}.$$

The proof of the theorem is based on the following statement:

STATEMENT. Let g be approximately continuous, $g(0)=0$, $g'_{ap}(0)$ exist and $g'_{ap}(0) > 0$ and let $\{I_n = [a_n, b_n]\}$ be a sequence of closed intervals with positive endpoints converging to 0 such that for every n, $x \in I_n$ implies $AD^- g(x) \leqq 0$. Then

$$\lim_{n\to\infty} \frac{m(I_n)}{d(0, I_n)} = 0.$$

PROOF OF THE STATEMENT. Let $g'_{ap}(0) > \varepsilon > 0$ and let

$$E = \left\{x : g'_{ap}(0) - \varepsilon < \frac{g(x)}{x} < g'_{ap}(0) + \varepsilon\right\}.$$

Then 0 is a point of density of the set E. Let n be any positive integer. We shall now show that

$$\frac{m(E \cap I_n)}{b_n} < \frac{2\varepsilon}{g'_{ap}(0)}. \tag{1}$$

Since $AD^- g(x) \leqq 0$ for all $x \in I_n$, g is non-increasing on I_n and hence g is continuous on I_n.

Let $A = E \cap I_n$. If A is empty, then $E \cap I_n$ is empty and so

$$\frac{m(E \cap I_n)}{b_n} = 0 < \frac{2\varepsilon}{g'_{ap}(0)}.$$

If A is non-empty, let $x_2 = \sup A$. Since g is continuous on I_n, $\frac{g(x)}{x}$ is also continuous on I_n. Therefore

$$\frac{g(x_2)}{x_2} \geqq g'_{ap}(0) - \varepsilon.$$

Let $x_1 = \dfrac{g'_{ap}(0)-\varepsilon}{g'_{ap}(0)+\varepsilon}x_2$. Since $0<\varepsilon<g'_{ap}(0)$, $0<x_1<x_2$. If $x_1 \leqq a_n$, then $A\subset[x_1, x_2]$. If $x_1>a_n$, then since g is non-increasing on I_n, $x\in I_n$ and $x<x_1$ imply

$$g(x) \geqq g(x_1) \geqq g(x_2)$$

and hence

$$\frac{g(x)}{x} \geqq \frac{g(x_2)}{x_1} = \frac{g'_{ap}(0)+\varepsilon}{g'_{ap}(0)-\varepsilon}\cdot\frac{g(x_2)}{x_2} \geqq g'_{ap}(0)+\varepsilon$$

showing that x cannot be a point of E and consequently $x\notin A$. Thus $A\subset[x_1, x_2]$. Since $E\cap I_n = A$,

$$m(E\cap I_n) \leqq x_2-x_1 = x_2-\frac{g'_{ap}(0)-\varepsilon}{g'_{ap}(0)+\varepsilon}x_2 =$$

$$= x_2\cdot\frac{2\varepsilon}{g'_{ap}(0)+\varepsilon} < x_2\cdot\frac{2\varepsilon}{g'_{ap}(0)} \leqq b_n\cdot\frac{2\varepsilon}{g'_{ap}(0)}.$$

Hence

$$\frac{m(E\cap I_n)}{b_n} < \frac{2\varepsilon}{g'_{ap}(0)}.$$

Thus in any case (1) is true. Also since 0 is a point of density of E, we have

$$\lim_{n\to\infty}\frac{m(E\cap[0, b_n])}{b_n} = 1 \tag{2}$$

Now from (1)

$$1 \geqq \frac{a_n}{b_n} \geqq \frac{m(E\cap[0, a_n])}{b_n} = \frac{m(E\cap[0, b_n])}{b_n} - \frac{m(E\cap I_n)}{b_n} > \frac{m(E\cap[0, b_n])}{b_n} - \frac{2\varepsilon}{g'_{ap}(0)}.$$

Hence from (2)

$$1 \geqq \limsup_{n\to\infty}\frac{a_n}{b_n} \geqq \liminf_{n\to\infty}\frac{a_n}{b_n} \geqq 1-\frac{2\varepsilon}{g'_{ap}(0)}.$$

Since $\varepsilon>0$ is arbitrary, $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = 1.$

So

$$\lim_{n\to\infty}\frac{b_n-a_n}{a_n} = 0.$$

PROOF OF THE THEOREM. We may assume $x_0=0$, $f(x_0)=0$. Suppose $\{I_n\}$ is a sequence of closed intervals converging to 0 such that $m(E(\alpha, \beta)\cap I_n) = 0$ for every n. So, by Theorem 5, $E(\alpha, \beta)\cap I_n$ is empty for every n. So, for each n, $x\in I_n$ implies either $AD^-f(x)\geqq\beta$ or $AD^-f(x)\leqq\alpha$. Since by Theorem 3, $AD^-f(x)$ possesses Darboux property it follows that for each n, either $AD^-f(x)\geqq\beta$ for all $x\in I_n$ or

$AD^-f(x) \leqq \alpha$ for all $x \in I_n$. Let N denote the set of positive integers. Let

$N_1 = \{n \colon I_n$ has positive endpoints and $x \in I_n$ implies $AD^-f(x) \geqq \beta\}$
$N_2 = \{n \colon I_n$ has positive endpoints and $x \in I_n$ implies $AD^-f(x) \leqq \alpha\}$
$N_3 = \{n \colon I_n$ has negative endpoints and $x \in I_n$ implies $AD^-f(x) \geqq \beta\}$
$N_4 = \{n \colon I_n$ has negative endpoints and $x \in I_n$ implies $AD^-f(x) \leqq \alpha\}$.

Since $N = \bigcup_{i=1}^{4} N_i$, to prove

$$\lim_{n\to\infty} \frac{m(I_n)}{d(0, I_n)} = 0$$

it suffices to show that

$$\lim_{\substack{n\to\infty \\ n\in N_i}} \frac{m(I_n)}{d(0, I_n)} = 0, \qquad i = 1, 2, 3, 4.$$

Let us consider the subset $N_1 \subset N$. Consider the transformation $g(x) = \beta x - f(x)$. Then $g(0)=0$ and $g'_{\mathrm{ap}}(0)>0$. Let $n \in N_1$. Then $AD_+ f(x) \geqq \beta$ on I_n and hence by the corollary of Theorem 1, the function $f(x)-\beta x$ is non-decreasing on I_n and so g is non-increasing on I_n. So $AD^-g(x) \leqq 0$ for all $x \in I_n$. Hence applying the above statement we conclude that

$$\lim_{\substack{n\to\infty \\ n\in N_i}} \frac{m(I_n)}{d(0, I_n)} = 0.$$

Considering the transformations $g(x) = f(x) - \alpha x$, $g(x) = \beta x + f(-x)$ and $g(x) = -\alpha x - f(-x)$ for the sets N_2, N_3 and N_4 respectively we get

$$\lim_{\substack{n\to\infty \\ n\in N_i}} \frac{m(I_n)}{d(0, I_n)} = 0, \qquad i = 2, 3, 4.$$

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IDEAUX A DROITE MAXIMAUX D'UN ANNEAU

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Nous montrons dans cet article que les idéaux à droite maximaux d'un anneau peuvent être répartis en trois classes disjointes et qu'on peut leur associer trois types d'anneaux: anneaux simples de carré nul, anneaux primitifs et anneaux pseudo-primitifs. La structure des anneaux pseudo-primitifs s'obtient à partir de celle des anneaux primitifs.

Soit un anneau R et soit A un idéal à droite de R. Posons $A:R=\{x|x\in R,\ Rx\subseteq A\}$ et $[A:R]=(A:R)\cap A$. Il est immédiat que $A:R$ et $[A:R]$ sont des idéaux de R et que $[A:R]\subseteq A$.

Un idéal à droite A de R est dit:

(1) *modulaire* [1] s'il existe $e\in R$ tel que $ex-x\in A$ pour tout $x\in R$.

(2) *quasi-modulaire* ([3], [4]) si $A:R\subseteq A$. Tout idéal à droite modulaire est quasi-modulaire, mais l'inverse est faux.

(3) *pseudo-modulaire* si $A:R\nsubseteq A$ et $A:R\neq R$.

Si A est un idéal à droite maximal de R, on voit facilement qu'on a les trois possibilités suivantes qui s'excluent mutuellement:

(1) $A:R=R$, c'est-à-dire $R^2\subseteq A$

(2) A est quasi-modulaire

(3) A est pseudo-modulaire.

Un idéal P de R sera dit *g-primitif à droite*, s'il existe un idéal à droite maximal A de R tel que $[A:R]=P$. Un anneau sera dit *g-primitif à droite*, si l'idéal (0) est g-primitif à droite. Suivant la propriété de l'idéal à droite maximal A tel que $[A:R]=$ $=(0)$, un anneau g-primitif à droite R est de l'un des trois types suivants:

(1) $A:R=R$. Alors $(0)=[A:R]=(A:R)\cap A=A$ et R est un anneau simple de carré nul.

(2) A est quasi-modulaire. Dans ce cas, STEINFELD [2] a montré que R est un anneau primitif.

(3) A est pseudo-modulaire. Dans ce cas, R sera dit un anneau *pseudo-primitif à droite*. Nous allons montrer que la structure d'un tel anneau s'obtient à partir de celle d'un anneau primitif.

EXEMPLE D'ANNEAU PSEUDO-PRIMITIF À DROITE. Soit A un anneau primitif (à droite) et soit V un A-module à droite fidèle et irréductible. Soit $P=V\times A$ et définissons dans P une addition et une multiplication comme suit:

$$(v, a)+(w, b) = (v+w, a+b)$$

$$(v, a)(w, b)=(vb, ab).$$

On vérifie que P est un anneau pseudo-primitif à droite, car $\bar{A}=\{(0, a)|a\in A\}$ est un idéal à droite maximal pseudo-modulaire tel que $[\bar{A}:A]=(0)$.

Nous allons montrer que tout anneau pseudo-primitif à droite a une telle structure.

THEOREME. *Soit R un anneau pseudo-primitif à droite et soit A un idéal à droite maximal et pseudo-modulaire tel que $[A:R]=(0)$. Posons $V=A:R$. On a les propriétés suivantes:*

(1) $V\neq(0)$, $V\cap A = (0)$ *et* $R = V\oplus A$

(2) $V=\{x|x\in R, Rx=(0)\}$

(3) *V est un idéal à droite minimal de R*

(4) *pour tout $v\in V$, $v\neq 0$, on a $vA=V$.*

(5) *V est un A-module irréductible et fidèle et A est un anneau primitif (à droite)*

(6) *l'anneau R est isomorphe à l'anneau $P = V\times A$, les opérations de P étant celles définies dans l'exemple précédent*

(7) *l'idéal* (0) *est l'intersection des idéaux à droite maximaux de R.*

PREUVE. (1) $V\neq(0)$ par définition. Si $a\in V\cap A$, alors $Ra\subseteq A$ et $a\in[A:R]=(0)$. Comme A est un idéal à droite maximal et comme V est un idéal $\nsubseteq A$, on a $R = V\oplus A$.

(2) Soit $x\in R$. Alors $x = v+a$, $v\in V$, $a\in A$. Si $w\in V$, on a $xw = vw+aw$. Comme V est un idéal et A un idéal à droite on a $aw\in V\cap A$ et donc $aw=0$. D'autre part $Rw\subseteq A$; d'où $vw\in V\cap A$ et $vw=0$. Par conséquent $xw=0$ et $RV=(0)$.

Si $Rx=(0)$, alors $Rx\subseteq A$ et $x\in A:R=V$.

(3) Cela découle immédiatement de (1) et du fait que A est un idéal à droite maximal.

(4) vA est un idéal à droite de R et $vA\subseteq V$. Comme V est minimal, on a soit $vA=V$, soit $vA=(0)$. Supposons que $vA=(0)$. Alors $vR=(0)$, car $R = V\oplus A$ et $RV=(0)$. Soit $X=\{x|x\in R, xR=(0)\}$; X est un idéal de R et, puisque $V\cap X\neq(0)$ on a $V\subseteq X$. D'où $VR=(0)$ et $R^2 = (V\oplus A)R \subseteq AR \subseteq A$. Par conséquent $R=A:R$, contre le fait que A est pseudo-modulaire.

(5) V est un A-module irréductible d'après (4). Soit $T=\{x|x\in A, Vx=0\}$. On a $Rx = (V\oplus A)x \subseteq Ax \subseteq A$ et $x\in(A:R)\cap A = [A:R] = (0)$. Donc V est fidèle et A est un anneau primitif (à droite).

(6) D'après (1), tout élément x de R a une décomposition unique de la forme $x = v+a$, $v\in V$, $a\in A$. Définissons une application f de R dans $P = V\times A$ par $f(x)=(v, a)$. On vérifie facilement que $f(x)$ est une application bijective de R sur P telle que

$$f(x+y) = f(x)+f(y), \quad f(xy)=f(x)f(y).$$

Par conséquent, l'anneau R est isomorphe à l'anneau P.

(7) A étant un anneau primitif, l'ensemble $\{M_i|i\in I\}$ des idéaux à droite maximaux de A n'est pas vide et $\bigcap_{i\in I} M_i = (0)$. Il est immédiat que $V_i = V\oplus M_i$ est un

idéal à droite maximal de R pour tout $i \in I$ et que $V = \bigcap_{i \in I} V_i$. Comme $A \cap V = (0)$ et comme A est un idéal à droite maximal de R, on voit que (0) est l'intersection des idéaux à droite maximaux de R.

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PSEUDOGEODÄTISCHE LINIEN AUF FLÄCHEN

Von

U. SIMON (Berlin)

Herrn Professor Strubecker zum 65. Geburtstag gewidmet

In einer Reihe von Arbeiten ([7]—[10]) hat W. WUNDERLICH pseudogeodätische Linien auf speziellen Flächen untersucht: dabei heißt eine Flächenkurve *Pseudogeodätische*, wenn der Winkel φ zwischen der Tangentialebene der Fläche und der Schmiegebene der Kurve längs der Kurve konstant ist. Spezialfälle sind die Geodätischen $\left(\varphi = \frac{\pi}{2}\right)$ und die Asymptotenlinien ($\varphi=0$).

W. VOGLER hat kürzlich die Frage gestellt,[1] ob sich die Pseudogeodätischen ebenso zur Charakterisierung gewisser Flächenklassen heranziehen lassen wie die Geodätischen (etwa analog zur lokalen Kennzeichnung der Kugel unter den Flächen nichtverschwindender Gaußscher Krümmung durch die Eigenschaft, daß jede Schattengrenze Geodätische ist; vgl. z. B. [1], S. 193, [3], [5]).

In der vorliegenden Note wird die Problemstellung von Herrn VOGLER aufgegriffen. Wir werden u. a. das folgende Ergebnis beweisen:

SATZ 1. *Auf dem Flächenstück $F \in C^2$ im euklidischen Raum E_3 sei die Gaußsche Krümmung ungleich Null bis auf eine nirgends dichte Punktmenge. Ist jede Schattengrenze gleichzeitig eine Pseudogeodätische, so liegt F auf einer Sphäre (und die Schattengrenzen sind dann notwendig Geodätische).*

Eine ähnliche Aussage liefert Satz 2 in § 2. Den Beweisen in § 2 stellen wir einfache Eigenschaften von Pseudogeodätischen auf Flächen in § 1 voran. Als methodisches Hilfsmittel verwenden wir bekannte Ergebnisse der Streifentheorie. Dadurch wird gleichzeitig auf den engen Zusammenhang mit Verbiegungsfragen hingewiesen, doch wollen wir darauf hier nicht näher eingehen.

§ 1. Pseudogeodätische Kurven und Streifen

Es sei $x=x(s)$ eine Raumkurve mit der Bogenlänge s als Parameter und mit dem begleitenden Dreibein[2] $\{\dot{x}, \mathfrak{h}, \mathfrak{b}\}$. $\mathfrak{S}=\{x(s), \xi(s)\}$ sei der Streifen mit der Trägerkurve x und dem begleitenden Streifendreibein $\{\dot{x}, \eta, \xi\}$. Die Kurvenkrümmung k sei im folgenden bis auf eine nirgends dichte Menge auf der Kurve ungleich Null. Zwischen den Invarianten k ($k \geqq 0$) und $\varkappa$ (Windung) der Kurve und den Invarianten

[1] Auf der Tagung der Österr. Math. Gesellschaft in Linz 1968, [6].
[2] Der Punkt · bezeichnet hier und im folgenden die Ableitung nach der Bogenlänge s.

a (geodätische Windung), b (Normalkrümmung) und c (geodätische Krümmung) des Streifens bestehen in Punkten mit $k>0$ die Relationen ([2], S. 150)

$$c = k \cdot \cos \varphi \tag{1. 1a}$$

$$b = -k \cdot \sin \varphi \tag{1. 1b}$$

$$\varkappa = a + \dot{\varphi}; \tag{1. 1c}$$

dabei ist $\varphi = \sphericalangle (\mathfrak{b}, \xi) = \sphericalangle (\mathfrak{h}, \eta)$. Wir definieren: der Streifen $\mathfrak{S}$ heißt pseudogeodätisch, falls längs des Streifens $\varphi = \text{const}$ gilt. Für die Untersuchung der wesentlichen geometrischen Eigenschaften der pseudogeodätischen Streifen wollen wir uns o. B. d. A. auf $0 \leqq \varphi \leqq \frac{\pi}{2}$ beschränken. Aus (1. 1) erhalten wir sofort die folgenden einfachen Ergebnisse:

LEMMA 1. 1. *Kennzeichnend für einen pseudogeodätischen Streifen ist die Relation*

$$\varkappa = a. \tag{1. 2}$$

LEMMA 1. 2. *Längs des Streifens $\mathfrak{S}$ sei $0 < \varphi \leqq \frac{\pi}{2}$. Der Streifen ist pseudogeodätisch genau dann, wenn gilt*

$$c = \alpha \cdot b, \qquad \alpha = \text{const} \leqq 0; \tag{1. 3}$$

dabei ist $\alpha = -\text{ctg}\, \varphi$.

Unter den Voraussetzungen von Lemma 1. 2. ist mit $k \neq 0$ auch $b \neq 0$.

LEMMA 1. 3. *Die Trägerkurve x des pseudogeodätischen Streifens $\mathfrak{S}$ mit $0 < \varphi \leqq \frac{\pi}{2}$ ist eben genau dann, wenn $a = 0$ gilt.*

Der Beweis ist wegen (1. 1c) klar.

Übrigens ist längs eines ebenen pseudogeodätischen Streifens $\left(0 < \varphi \leqq \frac{\pi}{2}\right)$ der Vektor $e = \eta + \alpha \xi$ konstant, was man leicht mit Hilfe der Ableitungsgleichungen der Streifentheorie ([2], S. 150) verifiziert; e spielt eine Rolle bei der Verbiegung von Mützen mit ebenen pseudogeodätischen Rändern.

Um Pseudogeodätische auf beliebigen Flächen untersuchen zu können, wollen wir die Differentialgleichung einer Pseudogeodätischen angeben. Sei M eine zweidimensionale differenzierbare Mannigfaltigkeit der Klasse C^r, $r \geqq 2$, und sei

$$x: M \to E_3$$

eine C^r-Immersion in den dreidimensionalen euklidischen Raum. Es seien (u^1, u^2) lokale Parameter auf M, $x = x(u^i)$ sei der „Ortsvektor" der Fläche $F = x(M)$ bezüglich eines Nullpunktes und $\xi = \xi(u^i)$ sei der Normalenvektor.[3] Eine Kurve $x(s) = x(u^i(s))$ auf $x(M)$ ist genau dann pseudogeodätisch, wenn der Streifen $\mathfrak{S} =$

[3] Für die Grundformeln der Flächentheorie verweisen wir auf den Nachtrag in [2], S. 149 ff. Wir übernehmen die dortige Bezeichnungsweise.

$=\{x(u^i(s)), \xi(u^i(s))\}$ pseudogeodätisch ist. Die Streifeninvarianten sind durch (vgl. [2], S. 150)

$$(1.4) \qquad a = B_{ik}\dot{u}^i n^k, \quad b = -B_{ik}\dot{u}^i\dot{u}^k, \quad c = g_{ik}n^i\frac{D\dot{u}^k}{ds}$$

mit den Grundtensoren der Fläche $x(M)$ verbunden. Dabei sind (g_{ik}) bzw. (B_{ik}) die Tensoren der ersten bzw. zweiten Grundform der Fläche, ε_{ik} ist der Diskriminantentensor; $n_k=\varepsilon_{ik}\dot{u}^i$ ist der „innere Normalenvektor" der Kurve auf der Fläche; $\frac{D}{ds}$ bezeichnet die kovariante Ableitung längs $x=x(s)$.

LEMMA 1. 4. *Eine Pseudogeodätische $x=x(s)\in C^2$ auf $x(M)$ mit $0<\varphi\leqq\frac{\pi}{2}$ genügt oder Differentialgleichung*

$$(1.5) \qquad \ddot{u}^k+\Gamma_{ij}{}^k\dot{u}^i\dot{u}^j = \alpha\cdot b\cdot n^k, \qquad \alpha = \text{const} \leqq 0.$$

($\Gamma_{ij}{}^k$ sind die Christoffelsymbole.)

BEWEIS. Für eine beliebige Flächenkurve gilt wegen (1. 4)

$$\ddot{u}^k+\Gamma_{ij}{}^k\dot{u}^i\dot{u}^j = \frac{D\dot{u}^k}{ds} = c\cdot n^k.$$

Mit Lemma 1. 2 folgt (1. 5).

BEMERKUNG. Offenbar ist jede Lösung von (1. 5) auch Pseudogeodätische. Eine Pseudogeodätische mit $\varphi=0$ ist eine Asymptotenlinie, d.h. Lösung der Differentialgleichung $b=0$.

§ 2. Lokale Kennzeichnungen der Kugel durch Pseudogeodätische

Wir beweisen Satz 1 aus der Einleitung. Um zu zeigen, daß $x(M)$ auf einer Sphäre liegt, genügt es zu zeigen, daß die Nabelpunkte (das sind die Nullstellen der auf $x(M)$ stetigen Funktion (H^2-K), $H=\frac{k_1+k_2}{2}$) auf $x(M)$ dicht liegen, d.h. jede nichtleere, offene Teilmenge $V\subset x(M)$ muß Nabelpunkte enthalten. Sei also V nichtleer und offen; wir nehmen an, V enthalte keine Nabelpunkte. Wegen der Voraussetzungen existiert eine offene nichtleere Teilmenge $U\subset V$ auf der für die Gaußsche Krümmung K gilt: $K\neq 0$. Sei $p\in U$ und sei $G: u^i=u^i(s)$ eine beliebige Schattengrenze durch p bezüglich einer gewissen Lichtrichtung; G sei aber nicht Asymptotenlinie (wegen $K\neq 0$ gibt es höchstens zwei Asymptotenlinien durch p). In U genügt G der Differentialgleichung[4].

$$(2.1) \qquad \ddot{u}^k+\Gamma_{ij}{}^k\dot{u}^i\dot{u}^j = -B^{r}_{i.\|j}\tilde{B}_{r.}{}^k\dot{u}^i\dot{u}^j.$$

[4] Vgl. [5], § 3. Mit dem Doppelstrich bezeichnen wir im folgenden die kovariante Ableitung bezüglich der Metrik.

Dabei sei $\tilde{B}^{jk}$ der zu B_{ij} inverse Tensor, d.h. es sei $\tilde{B}^{jk}B_{ij}=\delta_{i.}^{k}$. Wir nehmen an, G sei gleichzeitig Pseudogeodätische; G muß also (1. 5) und (2. 1) genügen, es muß daher gelten

(2. 2a) $$\alpha\cdot b\cdot n^k = -B_{i.\|j}^{r}\tilde{B}_{r.}^{k}\dot{u}^i\dot{u}^j$$

bzw.

(2. 2b) $$\alpha\cdot b\cdot \dot{u}^s = -\varepsilon_{k.}^{s}B_{ir\|j}\tilde{B}^{rk}\dot{u}^i\dot{u}^j.$$

Aus (2. 2a) bzw. (2. 2b) erhalten wir

(2. 3a) $$0=\alpha\cdot b\cdot n_s\dot{u}^s = -B_{i.\|j}^{r}\tilde{B}_{rk}\dot{u}^i\dot{u}^j\dot{u}^k$$

bzw. entsprechend

(2. 3b) $$0=\alpha\cdot b(\dot{u}^s\dot{u}^q-\dot{u}^q\dot{u}^s) = -B_{ir\|j}\tilde{B}^{rp}(\varepsilon_{p.}^{s}\delta_{k.}^{q}-\varepsilon_{p.}^{q}\delta_{k.}^{s})\dot{u}^i\dot{u}^j\dot{u}^k.$$

Nach Voraussetzung müssen die Gleichungen (2. 3) von fast allen Schattengrenzen durch p erfüllt werden, d.h. die Gleichungen (2. 3) gelten für fast alle Richtungen in p. Die kubischen Formen (2. 3) können aber nur dann für fast alle Richtungen verschwinden, wenn ihre Koeffizienten verschwinden, d.h. es muß gelten

(2. 4a) $$B_{ir\|j}\tilde{B}_{.k}^{r}+B_{jr\|k}\tilde{B}_{.i}^{r}+B_{kr\|i}\tilde{B}_{.j}^{r}=0$$

bzw.

(2. 4b) $$\tilde{B}^{rp}\{B_{ir\|j}(\varepsilon_{p.}^{s}\delta_{k.}^{q}-\varepsilon_{p.}^{q}\delta_{k.}^{s})+B_{jr\|k}(\varepsilon_{p.}^{s}\delta_{i.}^{q}-\varepsilon_{p.}^{q}\delta_{i.}^{s})+B_{kr\|i}(\varepsilon_{p.}^{s}\delta_{j.}^{q}-\varepsilon_{p.}^{q}\delta_{j.}^{s})\}=0.$$

Da p beliebiger Punkt in U war, gelten die Differentialgleichungen (2. 4a) und (2. 4b) in ganz U. Für die folgenden Umformungen notieren wir einige Beziehungen: es sei H die mittlere Krümmung und

$$\bar{B}^{ir}=\varepsilon^{ij}\varepsilon^{rs}B_{js}$$

der zu B_{ik} *adjungierte* Tensor. Es gilt in U (vgl. [2], S. 149) $\bar{B}^{ir}=K\cdot\tilde{B}^{ir}$, also $2K=\bar{B}^{ik}B_{ik}$ und $K_{\|j}=\bar{B}^{ik}B_{ik\|j}=K\cdot\tilde{B}^{ik}B_{ik\|j}$. Dann erhalten wir die Relationen (2. 5a) bzw. (2. 5b), indem wir (2. 4a) mit $K\cdot g^{ij}$ überschieben bzw. (2. 4b) mit $K\cdot g^{ij}$ überschieben und über die Indices (k, q) verjüngen (nach Voraussetzung war $K\neq 0$ in U):

(2. 5a) $$H_{\|r}\bar{B}_{.j}^{r}=K_{\|j};$$

(2. 5b) $$\varepsilon^{rj}\{3B_{j.}^{s}H_{\|r}-B_{j.}^{k}B_{r.\|k}^{s}\}=0;$$

da ε^{rj} schiefsymmetrisch ist, muß $A_{jr.}^{s}=3B_{j.}^{s}H_{\|r}-B_{j.}^{k}B_{r.\|k}^{s}$ symmetrisch in (j, r) sein, d.h. es ist trivialerweise auch

(2. 5c) $$g^{js}(A_{jrs}-A_{rjs})=0;$$

unter Berücksichtigung von

$$2B_{ik\|r}B^{ik}=(B_{ik}B^{ik})_{\|r}=2(2H^2-K)_{\|r}$$

folgt aus (2. 5c)

(2. 5d) $$2H\cdot H_{\|j}+K_{\|j}-B_{j.}^{k}H_{\|k}=0.$$

Setzt man (2. 5a) in (2. 5d) ein, so erhält man schließlich

(2. 6) $$2HH_{\|j}+H_{\|k}\bar{B}_{.j}^{k}-H_{\|k}B_{.j}^{k}=0.$$

Um die Differentialgleichung (2. 6) in U zu lösen, führen wir Krümmungslinienparameter in U ein (nach Annahme enthält U keine Nabelpunkte); es gilt dann mit k_1, k_2 als Hauptkrümmungen

$$(g_{ij}) = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \quad (B_{ij}) = \begin{pmatrix} k_1 \cdot g_{11} & 0 \\ 0 & k_2 \cdot g_{22} \end{pmatrix}, \quad (B^i_{.j}) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix},$$

$$(\bar{B}^i_{.j}) = \begin{pmatrix} k_2 & 0 \\ 0 & k_1 \end{pmatrix}.$$

(2. 6) erhält die Gestalt

$$k_2 \cdot H_{\|1} = 0, \quad k_1 \cdot H_{\|2} = 0. \tag{2.6a}$$

Wegen $K \neq 0$ in U folgt H=const und wegen (2. 5a) damit auch K=const. $\neq 0$ in U; also liegt U — entgegen der Annahme über $U \subset V$ — auf einer Sphäre, denn die einzigen Flächenstücke mit konstantem H und K sind bekanntlich die Ebene, die Sphäre und der Zylinder. Damit ist Satz 1 bewiesen.

Analog zum Vorhergehenden wird man vermuten, daß unter der Voraussetzung $K \neq 0$ die Sphäre dadurch gekennzeichnet wird, daß jede ebene Kurve eine Pseudogeodätische ist. Es zeigt sich, daß man sogar nur bestimmte ebene Kurven zu betrachten braucht. Es gilt:

SATZ 2. *Auf $x(M)$ sei $K \neq 0$ bis auf eine nirgends dichte Punktmenge. Gibt es einen Punkt $p_0 \in E_3$, $p_0 \notin x(M)$, mit der Eigenschaft, daß jede Ebene e durch p_0 die Fläche $x(M)$ in einer Pseudogeodätischen schneidet, so liegt $x(M)$ auf einer Sphäre (und die ebenen Schnitte sind alle geodätisch).*

BEWEIS. Es sei x_0 der Ortsvektor von p_0; für einen beliebigen Punkt $p \in x(M)$ mit dem Ortsvektor x sei

$$P = \langle x - x_0, \xi \rangle, \quad 2\varrho = \langle x - x_0, x - x_0 \rangle. \tag{2.7}$$

Wir wollen zuerst zeigen, daß die Punkte mit $P \neq 0$ dicht auf $x(M)$ liegen. Sei U eine beliebige, nichtleere, offene Teilmenge von $x(M)$. Nach Voraussetzung existiert ein Punkt $p \in U$ mit $K = K(p) \neq 0$. V sei eine offene Umgebung von p, in der $K \neq 0$ gilt; o. B. d. A. sei $V \subset U$. Wir zeigen, daß es in V Punkte mit $P \neq 0$ gibt. Ist nämlich $P=0$ für alle Punkte in V, so muß grad P in p verschwinden und es gilt wegen der Weingartenschen Ableitungsgleichungen $0 = P_i = \langle x - x_0, \xi_i \rangle = -B_{i.}^{k} \langle x - x_0, x_k \rangle$, woraus wegen $K \neq 0$ (d.h. $\| B_{i.}^{k} \| \neq 0$) in p folgt

$$\langle x - x_0, x_k \rangle = 0.$$

Andererseits ist in p

$$P = \langle x - x_0, \xi \rangle = 0;$$

es muß also $x = x_0$, d.h. $p = p_0$ gelten. Das ist aber ein Widerspruch zu $p \in x(M)$, $p_0 \notin x(M)$. U enthält also Punkte mit $P \neq 0$, so daß die Punkte mit $P \neq 0$ dicht auf $x(M)$ liegen. — Sei nun $q \in x(M)$ ein beliebiger Punkt mit $K(q) \neq 0$, $P(q) \neq 0$; sei $\bar{e} = e \cap x(M)$ ein ebener Schnitt durch q; o. B. d. A. sei $\bar{e}$ nicht Asymptotenlinie. Nach [5] läßt sich $\bar{e}$ als Autoparallele des symmetrischen Zusammenhanges $\Pi = \Pi(p_0)$ in einer offenen Umgebung $\bar{U}$ von q auffassen. Dabei hat Π die Komponenten

$$\Pi_{ij}{}^{k} = \Gamma_{ij}{}^{k} + P^{-1} \cdot \varrho^{k} B_{ij}; \tag{2.8}$$

da $\bar{e}$ auch Pseudogeodätische und nicht Asymptotenlinie ist, ergeben (1. 5) und (2. 8) mit einer geeigneten Funktion $\pi=\pi(s)$

$$\alpha\cdot b\cdot n^k = (\ddot{u}^k+\Pi_{ij}{}^k\dot{u}^i\dot{u}^j)-P^{-1}\cdot\varrho^k B_{ij}\dot{u}^i\dot{u}^j = \Pi(s)\cdot\dot{u}^k+P^{-1}\cdot\varrho^k\cdot b(s).$$

Aus

$$\alpha\cdot b = \alpha\cdot b\varepsilon_{ik}\dot{u}^i n^k = P^{-1}\cdot b\varepsilon_{ik}\dot{u}^i\varrho^k$$

erhalten wir wegen $b\neq 0$ für $\bar{e}$ die Differentialgleichung

$$-\dot{u}^2\varrho^1+\dot{u}^1\varrho^2 = \alpha\cdot P; \tag{2.9}$$

dabei haben wir die lokalen Parameter (u^i) so transformiert, daß in q gilt: $(g_{ij})=(\delta_{ij})$. Die Differentialgleichung (2. 9) soll für jeden ebenen Schnitt $\bar{e}$ durch q, d.h. für alle Richtungen in q gelten. Das ist aber dann und nur dann möglich, wenn $\alpha=0=\pi$ und $\varrho_1=0=\varrho_2$ gilt. Da q in $\bar{U}$ beliebig war und $K\neq 0$ ist, ist $\varrho=$const. in $\bar{U}$. Die Punkte mit $K\neq 0$, $P\neq 0$ liegen dicht auf $x(M)$, also ist $\varrho=$const. auf $x(M)$, d.h. $x(M)$ liegt auf einer Sphäre. Aus (2. 8) folgt $\Pi=\Gamma$, d.h. jeder Schnitt $\bar{e}$ is geodätisch.

In [4] hatten wir eine einfache Methode von Laugwitz zur Lösung entsprechender Probleme für Geodätische angewendet. Wegen der Struktur der Differentialgleichung (1. 5) — die Pseudogeodätischen lassen sich für $\varphi\neq\frac{\pi}{2}$ i. A. nicht als Autoparallelen eines symmetrischen Zusammenhangs darstellen — führt dieses Verfahren hier jedoch nicht zum Ziel.

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COMPLETIONS OF TOPOLOGICAL ABELIAN p-GROUPS

By

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In the study of an Abelian group a useful method is the consideration of its completions with respect to various topologies, e.g. the p-adic topology. Within a certain class of topologies, defined in terms of families of preradicals and including the p-adic topology, it is shown that even in the non-Hausdorff case completions do exist (although the uniqueness is lost). It is shown that every Abelian p-group of countable length is embeddable as an isotype subgroup of a fully complete Abelian group of the same length. Examples of the non-uniqueness of completions are also given.

The question of the existence of completions of non-Hausdorff topological Abelian p-groups was raised by Tom Head at the symposium on the theory of Abelian groups, held at Montpellier University in June 1967.

Let G be a topological Abelian group. By a completion of G we shall mean a pair (H, φ) where H is a complete topological Abelian group and φ is a topological isomorphism of G onto a dense subspace of H. We shall identify G with $\varphi(G)$, and we shall only consider those completions H of G for which the closure of the identity in H is contained in G. All groups referred to will be Abelian groups and in general the notation and terminology will be that of [8], [5], or [1].

Let $\mathscr{S}$ be a Serre class of Abelian groups (a non-empty class closed under subgroups, homomorphic images and extensions) and let $G\in\mathscr{S}$. A family of preradicals $\{r_\gamma\}_{\gamma\in\Gamma}$ will be said to be an *admissible family of preradicals* for G with respect to $\mathscr{S}$ if

i) $\{r_\gamma(G)\}_{\gamma\in\Gamma}$ is a filter base on G,

ii) $r_\gamma(K)/\bigcap_{\gamma\in\Gamma} r_\gamma(K) = r_\gamma\left(K/\bigcap_{\gamma\in\Gamma} r_\gamma(K)\right)$ for all $K\in\mathscr{S}$, and

iii) if $\mathscr{T}$ is the topology induced on G by taking $\{r_\gamma(G)\}_{\gamma\in\Gamma}$ as a base for the fundamental system of neighbourhoods of zero and $\alpha: G\to C$ is the natural homomorphism from G into the Hausdorff completion of $(G, \mathscr{T})$, then $r_\gamma(C)\cap\alpha(G) = r_\gamma(\alpha(G))$ for all $\gamma\in\Gamma$ and $\{r_\gamma(C)\}_{\gamma\in\Gamma}$ is a base for the neighbourhood system of zero in C.

The topologies we will consider will all arise from admissible families of preradicals; however, when considering the completion of an Abelian group G with respect to such a topology, we shall only show that condition ii) holds for those groups necessary to the proof of Theorem 1.

THEOREM 1. *Let G be an Abelian group, let $\{r_\gamma\}_{\gamma\in\Gamma}$ be an admissible family of preradicals for G, and let $\mathscr{T}$ be the topology induced on G by taking $\{r_\gamma(G)\}_{\gamma\in\Gamma}$ as a base for the fundamental system of neighbourhoods of zero. Then there exists a complete* *topological Abelian group* $(H, \mathcal{T}')$ *and a topological isomorphism* β *from* G *into* H *such that*

1) $\beta(G)$ *is a dense subspace of* H,
2) $\{r_\gamma(H)\}_{\gamma\in\Gamma}$ *is a base for the neighbourhood system of* H *at* 0,
3) $\beta(G)\cap r_\gamma(H) = r_\gamma(\beta G)$, *and*
4) $\bigcap_{\gamma\in\Gamma} r_\gamma(H) = \beta\left(\bigcap_{\gamma\in\Gamma} r_\gamma(G)\right)$.

PROOF. Let $G_0 = G/\bigcap_{\gamma\in\Gamma} r_\gamma(G)$ with the quotient topology $\mathcal{T}_0$. Let $\alpha: G\to C$ be the natural (topological) homomorphism from G into its Hausdorff completion C and let $\lambda: G_0\to C$ be the natural topological isomorphism from G_0 onto $\alpha(G)$. Consider the short exact sequence

$$0\to G_0 \xrightarrow{\lambda} C\to C/G_0\to 0.$$

This gives rise to the exact sequence

$$\mathrm{Ext}\left(C, \cap r_\gamma(G)\right)\xrightarrow{\lambda^*}\mathrm{Ext}\left(G_0, \cap r_\gamma(G)\right)\to 0.$$

Since λ^* is onto there exists an extension H of $\bigcap_{\gamma\in\Gamma} r_\gamma(G)$ by C and an embedding $\beta: G\to H$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & \cap r_\gamma(G) & \xrightarrow{\xi} & H & \xrightarrow{\sigma} & C & \to & 0 \\ & & \| & & \uparrow\beta & & \uparrow\lambda & & \\ 0 & \to & \cap r_\gamma(G) & \to & G & \to & G_0 & \to & 0 \end{array}$$

is commutative. Note that $r_\gamma(H)\cap\beta G = r_\gamma(\beta G)$ since r_γ is a preradical and $r_\gamma(C)\cap \cap\alpha(G) = \alpha(r_\gamma(G))$. Thus condition 3) is satisfied. Suppose $x\in \bigcap_{\gamma\in\Gamma} r_\gamma(H)$. Then $\sigma(x)\in \bigcap_{\gamma\in\Gamma} r_\gamma(C) = 0$ and hence $\bigcap_{\gamma\in\Gamma} r_\gamma H = \beta\left(\bigcap_{\gamma\in\Gamma} r_\gamma(G)\right)$. Hence condition 4) is satisfied. Thus we need only topologize H by taking $\{r_\gamma(H)\}_{\gamma\in\Gamma}$ as a base for the fundamental system of neighbourhoods of 0, and the construction is complete.

Note that any completion of G in which the closure of the identity in G is the same as the closure of the identity in the completion is such an extension. That two such extensions need not be isomorphic is clear from Example 1 of [3] where, in this case, we take G to be countable.

Note also that obtaining the completion was a purely algebraic matter. The topology was only put on as an after thought.

We will need the following terminology. Let G be a reduced Abelian group, β an ordinal $\geqq 1$, and let $\mathcal{T}^\beta$ be the topology induced on G by taking the chain of subgroups $\{p^\alpha G\}_{\alpha<\omega\beta}$ as a fundamental system of neighbourhoods of 0. We shall refer to this topology as the $p^{\omega\beta}$*-topology*. In case $\beta=1$ this is just the p-adic topology. Let $\mathcal{T}_n$ be the relative topology induced on $G[p^n]$ by $\mathcal{T}^\beta$. Let $\mathcal{T}_\beta$ be the inductive limit topology on G obtained by taking the inductive limit of the topological groups $(G[p^n], \mathcal{T}_n)$ in the category of topological Abelian groups (see [2]). We shall refer to $\mathcal{T}_\beta$ as the $\omega\beta$*-topology* on G. In case $\beta=1$, this is the large topology. As indicated in [2], the completion of a p-group of length $\omega\beta$ with respect to the $\omega\beta$-topology is just the torsion subgroup of the completion with respect to the $p^{\omega\beta}$-topology.

COROLLARY 2. *Let G be an Abelian group and let λ be an accessible ordinal. Let $\mathcal{T}^{\lambda}$ be the $p^{\omega\lambda}$-topology on G. Then G can be embedded (topological isomorphism) as a dense, $p^{\omega\lambda}$-pure subgroup of a complete topological Abelian group.*

PROOF. Theorem 2.2 of [9] establishes the conditions in Theorem 1, and Theorem 2.3 of [9] and Theorem 2.9 of [11] give the $p^{\omega\lambda}$-purity.

With the following lemma it will be clear that completions also exist for the large topology. Recall from [12] that a subgroup L of a reduced p-group G is said to be large if L is fully invariant and together with any basic subgroup of G generates G. We shall use freely the characterization of large subgroups given in [12].

LEMMA 3. *Let G be a reduced Abelian p-group, let $\varphi: G \to G/p^{\omega}G$ be the natural map, and let $\mathcal{L}$ and $\mathcal{L}'$ be the lattice of large subgroups of G and $G/p^{\omega}G$, respectively. Then $\Psi: \mathcal{L} \to \mathcal{L}': L \to \varphi(L)$ is a one-to-one correspondence between $\mathcal{L}$ and $\mathcal{L}'$. Also L and $\varphi(L)$ are determined by the same finite U-sequence.*

PROOF. Let $x \in G$. Then $U_G(x) = U_{G/p^{\omega}G}(\varphi(x))$ (a U-sequence of x is an Ulm sequence of x where no distinction is made between transfinite heights). Clearly $U_{G/p^{\omega}G}(\varphi(x)) \geqq U_G(x)$ ($\geqq$ the component wise partial ordering). The equality follows since φ preserves the height of elements of finite heights. Thus by Theorem 2.7 in [12], $\varphi(G\{\underset{\sim}{n}\}) = (G/G^1)\{\underset{\sim}{n}\}$ where $G\{\underset{\sim}{n}\}$ is the large subgroup of G determined by the finite U-sequence $\underset{\sim}{n}$.

COROLLARY 4. *Let G be a reduced Abelian p-group and let $\mathcal{T}$ be the large topology on G, i.e., the topology induced by taking as a base for the fundamental system of neighbourhoods of zero the large subgroup of G. Then G can be embedded (topological isomorphism) as a dense, pure subgroup of a complete topological Abelian p-group.*

The above lemma and corollary can be generalized for the $\omega\beta$-topologies.

Recall that a generalized primary group (written g.p. group) is an Abelian group in which $qG = G$ for all primes $q \neq p$. Following [6], we say that a reduced g.p. group G is fully complete if G is complete in the $p^{\omega\beta}$-topology for all $\beta \geqq 1$. We will say that a reduced p-group G is fully torsion complete if G is complete in the $\omega\beta$-topology for all ordinals $\beta \geqq 1$.

THEOREM 5. *Every reduced g.p. group of countable length α is embeddable as an isotype subgroup of a fully complete g.p. group of length α.*

PROOF. Let G be a g.p. group of countable Ulm type α. Let $\mathcal{S}$ be the class of all directed families $D_\gamma = \{H_\beta, \sigma_{\beta\eta}\}_{\beta \leqq \eta < \gamma}$, $\gamma \leqq \alpha + 1$ having the following properties:

1) $H_0 = G$,
2) H_β is complete in the $p^{\omega\beta}$-topology for $\beta \geqq 1$,
3) $\sigma_{\beta\eta}: H_\beta \to H_\eta$ is a monomorphism for all $\beta \leqq \eta < \gamma$,
4) $\sigma_{\beta\eta}(H_\beta)$ is isotype in H_η,
5) $\sigma_{\beta,\beta+1}(H_\beta)$ is dense in $H_{\beta+1}$ with respect to the $p^{\omega(\beta+1)}$-topology for all β such that $\beta + 1 < \gamma$,
6) $p^{\omega\eta}(\sigma_{\beta\eta} H_\beta) = p^{\omega\eta}(H_\eta)$ for all $\beta \leqq \eta \leqq \alpha + 1$, and
7) if $\beta < \gamma$ is a limit ordinal, then H_β is the completion of $\varinjlim_{\alpha \lesssim \beta} H_\alpha$ in the $p^{\omega\beta}$-topology.

Partially order $\mathcal{S}$ as follows. If $D_\gamma=\{H_\beta, \sigma_{\beta\eta}\}_{\beta\leqq\eta<\gamma}$ and $D_\delta=\{C_\beta, \xi_{\beta\eta}\}_{\beta\leqq\eta<\delta}$ are elements of $\mathcal{S}$, then $D_\delta\leqq D_\gamma$ if $\delta\leqq\gamma$ and there exists a family of isomorphisms $\lambda_{\delta\gamma}=\{\lambda_\beta: C_\beta\twoheadrightarrow H_\beta\}_{\beta<\delta}$ such that λ_0 is the identity map on G, and for all $\beta\leqq\eta<\delta$, the diagram

$$\begin{array}{ccc} & \sigma_{\beta\eta} & \\ H_\beta & \rightarrow & H_\eta \\ \lambda_\beta \uparrow & & \uparrow \lambda_\eta \\ C_\beta & \rightarrow & C_\eta \\ & \delta_{\beta\eta} & \end{array}$$

commutes.

Note that if $\lambda'_{\delta\gamma}=\{\lambda'_\beta: C_\beta\twoheadrightarrow H_\beta\}_{\beta<\delta}$ is another such family of isomorphisms then $\lambda'_\beta=\lambda_\beta$ for all $\beta<\delta$. To see this note first that $\lambda_0=\lambda'_0$ since both are the identity map on G. Also C_β/G is divisible for all $0<\beta<\delta$ since $C_{\beta+1}/C_\beta$ is divisible (see [9]) and if β is a limit ordinal C_β/C_η is divisible for $\eta<\beta$ since C_β is the completion of the direct limit of the C_η, $\eta<\beta$. Next note that H_β is reduced by condition 6. Thus $(\lambda_\beta-\lambda'_\beta)C_\beta$ is divisible since $H_0\subset\ker(\lambda_\beta-\lambda'_\beta)$ and hence $\lambda_\beta=\lambda'_\beta$ since H_β is reduced. Let $\{D_\beta\}_{\beta\in\Gamma}$ be a chain in $\mathcal{S}$. We may assume $\Gamma=\{\beta|\beta<\gamma$ for some ordinal $\gamma\leqq\alpha+1\}$. Also $\{D_\beta, \lambda_{\beta\eta}\}_{\beta\leqq\eta<\gamma}$ is a directed family and the direct limit D_γ exists and is an upper bound for the chain $\{D_\beta\}_{\beta\in\Gamma}$. Thus by Zorn's Lemma there is a maximal element D in $\mathcal{S}$. By Corollary 2 we have that D satisfies the desired properties.

THEOREM 6. *Every reduced p-group of countable length α is embeddable as an isotype subgroup of a fully torsion complete p-group of length α.*

The proof is essentially the same as that of Theorem 5, replacing the $p^{\omega\beta}$-topology by the $\omega\beta$-topology. Note that one just obtains the torsion subgroup of the fully complete g.p. group of Theorem 5.

Let G be a fully torsion complete p-group of countable length α. By Theorem 5, G can be embedded as an isotype subgroup of a fully complete g.p. group K of length α. Since G is the torsion subgroup of K and K is cotorsion (see [6] or [10]) we have that K is the cotorsion completion of G and is uniquely determined by G. Hence for any reduced p-group H of countable length α, these generalized completions of H are cotorsion. They are the cotorsion completion of H (and hence unique up to isomorphism) if and only if $H/p^\beta H$ is fully torsion complete for all $\beta<\alpha$.

We end this paper with the following example. Let G be an Abelian p-group with countable basic subgroup B such that $G/p^\omega G\cong p^\omega G\cong\bar{B}$ a closed p-group (the completion of B with respect to the large topology). Let H be a completion of G with respect to the p-adic topology. Let K be the completion of H with respect to the p^{ω^2}-topology. Since K is cotorsion it is determined up to isomorphism by G, its torsion subgroup. Hence K does not depend on which completion H we choose. We shall show that even in this case there are 2^c (c is the cardinality of the continuum) non-isomorphic completions H of G with respect to the p-adic topology.

First note that if F is the subgroup generated by a maximal linearly independent set of torsion free elements of $p^\omega K$ and T is a subgroup of $p^\omega G$ maximal with respect to the property of being disjoint from F, then T is the torsion subgroup of $p^\omega K$.

Next let $\{x\}_{x\in\Gamma}$ be a maximal linearly independent set of torsion free elements of H. Note that $|\Gamma|=c$. Let $y\in F-\{0\}$. Let $S=\{\{z_x|x\in\Gamma\}|z_x\in\{x+y, x\}\}$. Then $|\mathcal{S}|=2^c$. For each $A\in\mathcal{S}$, let H_A be a subgroup of K containing A and G and maximal with respect to the property of being disjoint from F. Note that H_A is pure in K and hence $(H_A+p^\omega K)/p^\omega K \cong H_A/H_A \cap p^\omega K = H_A/p^\omega H_A = H_A/(p^\omega K)_t$ ($(p^\omega K)_t$ is the torsion subgroup of $p^\omega K$). Also the natural homomorphism $\varphi: K \to K/p^\omega K$ takes H_A onto $K/p^\omega K$. This follows from the maximality of H_A. Thus H_A is a completion of G with respect to the p-adic topology for each $A \in \mathcal{S}$. Finally $|\mathrm{Hom}\,(H, K)| \leqq \leqq |\mathrm{Hom}\,(B, K)| = c$. Hence there are 2^c non-isomorphic groups H_A. The techniques used in this example are borrowed from [7].

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CONTROLLABLY PERIODIC PERTURBATIONS OF AUTONOMOUS SYSTEMS

By

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1. This paper is dealing with the behaviour of a periodic solution of a system of differential equations under perturbation. The case when the unperturbed system is autonomous and the perturbation periodic and non-autonomous will be considered. However, it will be assumed that the period of the perturbation is controllable. The results are stated and proved for D-periodic (derivo-periodic) solutions of cylindrical systems first (cf. [1]). Since the latter are, in a certain sense, generalizations of periodic solutions of arbitrary systems, the corresponding results for the ordinary case follow readily. The generalization involved is justified by the fact that D-periodic solutions of cylindrical systems occur frequently in applications (cf. e.g. [1], [2], [3]). It is to be noted here that already in [4] Vol. I., p. 80, H. POINCARÉ has mentioned a case which is basically the one called here a D-periodic solution of a cylindrical system.

2. Consider the system

$$\frac{dx}{dt} = f(x) + \mu g\left(\frac{t}{\tau}, x, \mu, \tau\right) \tag{1}$$

where x, f, g are n-dimensional vectors, t, μ, τ real scalars and the function on the right hand side

$$F\left(\frac{t}{\tau}, x, \mu, \tau\right) = f(x) + \mu g\left(\frac{t}{\tau}, x, \mu, \tau\right)$$

is in the C^1 class if x is in an open and connected region Ω of the n-dimensional space, $-\infty < t < +\infty$, $|\mu| < \alpha$, $|\tau - \tau_0| < \beta$, where $\alpha > 0$, $\tau_0 > 0$ and $0 < \beta < \tau_0$. We assume that the function g and, as a consequence, F is periodic in t with (least) period τ i.e.

$$g(s+1, x, \mu, \tau) \equiv g(s, x, \mu, \tau). \tag{2}$$

We assume furthermore that the function F is also *periodic in the vector x with "vector period"* $a \cdot \tau$, i.e.

$$F\left(\frac{t}{\tau}, x + a \cdot \tau, \mu, \tau\right) \equiv F\left(\frac{t}{\tau}, x, \mu, \tau\right) \tag{3}$$

for all t, $|\tau - \tau_0| < \beta$, $|\mu| < \alpha$ and $x \in \Omega$, where a is some vector which in general may depend on μ and τ (cf. [5] p. 57). However, we shall assume that

$$a \cdot \tau = a^0 \cdot \tau_0, \tag{4}$$

where a^0 is a constant vector. In case $a=0$, (3) does not mean any restriction on F. If $a\neq 0$, Ω is supposed to have the following property: from $x\in\Omega$ follows $x+at\in\Omega$, $-\infty<t<+\infty$, i.e. Ω is a "cylinder with axis parallel to a" in the n-dimensional space.

REMARK. As it is easy to see, condition (3) is slightly milder, than the assumption that $F\left(\frac{t}{\tau}, x, \mu, \tau\right)$ is *cylindrical* in some components of x (cf. [1], [6]), i.e. F is periodic in x_k with period $a_k\tau$, for $k=m+1, m+2, \dots, n$, $0\leqq m\leqq n-1$.

We note here that Theorems 2 and 3 of [1] remain valid if the periodicity assumption in some coordinates of x of the respective functions there is replaced by periodicity in the vector argument with appropriate vector period. In [7], which has been noticed by the author recently, a theorem analogous to Theorem 3 of [1] had earlier been proved on *asymptotical orbital stability* of a D-periodic solution.

DEFINITION 1. *We say that the continuously differentiable vector function* $\varphi(t)$ *is D-periodic with period* τ, *if its derivative is periodic with (least) period* τ (cf. [1]).

It is easy to see that $\varphi(t)$ is D-periodic if and only if it is the sum of a periodic and a linear function of t, i.e.

$$\varphi(t) = v(t) + a\cdot t,$$

where the vector function $v(t)$ is periodic:

$$v(t+\tau)\equiv v(t)$$

and a is a constant vector which will be called the *coefficient vector*.

LEMMA 1. *A* $\varphi(t)$ *solution of system* (1) *defined and with path contained in* Ω *at least for* $t_0\leqq t\leqq t_0+\tau$, *for some* t_0, *is D-periodic with period* τ *and coefficient vector* a *if and only if*

$$\varphi(t_0+\tau) = \varphi(t_0) + a\cdot\tau. \tag{5}$$

PROOF. Condition (5) is obviously necessary, we have to prove that it is sufficient as well. Since F is periodic in the vector x with vector period $a\tau$, if $\varphi(t)$ is a solution of (1), then $\varphi(t)+a\tau$ is a solution too. Furthermore, from the periodicity of F in t with period τ follows that $\varphi(t+\tau)$ is also a solution defined for $t_0-\tau\leqq t\leqq t_0$, at least. But from condition (5) follows that

$$[\varphi(t+\tau)]_{t=t_0} = \varphi(t_0+\tau) = \varphi(t_0)+a\cdot\tau = [\varphi(t)+a\cdot\tau]_{t=t_0}$$

and this means because of the uniqueness of the solutions that

$$\varphi(t+\tau)\equiv\varphi(t)+a\cdot\tau.$$

The latter identity has the immediate consequence that $\varphi(t)$ is D-periodic with period τ and coefficient vector a and this was to be proved.

Irrespective of whether F is periodic in the vector x or not, i.e. $a \neq 0$ or $a=0$, there holds obviously the following

CONSEQUENCE. *A $\varphi(t)$ solution of* (1), *defined and with path contained in Ω at least for $t_0 \leqq t \leqq t_0+\tau$, for some t_0, is periodic with period τ, if and only if*

$$\varphi(t_0+\tau) = \varphi(t_0). \tag{6}$$

3. Alongside with system (1) we shall consider the unperturbed autonomous system

$$\frac{dx}{dt} = f(x) \tag{7}$$

which we get to, if $\mu=0$ is substituted into $F\left(\frac{t}{\tau}, x, \mu, \tau\right)$. It will be assumed that (7) has a non-constant D-periodic solution $p(t)$ with period $\tau_0 > 0$, coefficient vector a^0 and with path contained in Ω for $-\infty < t < +\infty$. (In the special case when $a=a^0=0$, i.e. $f(x)$ is non-periodic in the vector x, $p(t)$ is periodic in the ordinary sense).

The first variational system of (7) corresponding to the solution $p(t)$, i.e.

$$\frac{dy}{dt} = \left.\frac{\partial f}{\partial x}\right|_{x=p(t)} \cdot y \tag{8}$$

is a linear system with periodic coefficients. We shall make use of the following simple

LEMMA 2. *Let $x=Az$ be a regular linear coordinate transformation, i.e. A a constant regular square matrix, and consider the transformed system*

$$\frac{dz}{dt} = A^{-1}f(Az) \tag{9}$$

and its solution $q(t)=A^{-1}p(t)$; the characteristic multipliers of the first variational system of (9) *corresponding to $q(t)$ are equal to the characteristic multipliers of* (8).

The proof is simple and will, therefore, be omitted.

4. The first problem to be answered is: given the unperturbed autonomous system (7) possessing a D-periodic solution; under what conditions will the perturbed system (1) have a D-periodic solution too. The attention is drawn to the fact that the period of the non-autonomous periodic perturbation $\mu g\left(\frac{t}{\tau}, x, \mu, \tau\right)$ occurs in g as a parameter (besides the "small parameter" μ) and can be chosen appropriately if necessary. We express this fact by saying that the *period of the perturbation is controllable.*

We introduce the following notations

$$\begin{aligned} p^0 &= p(0) = (p_1^0, p_2^0, \ldots, p_n^0), \\ \dot{p}^0 &= \dot{p}(0) = (\dot{p}_1^0, \dot{p}_2^0, \ldots, \dot{p}_n^0), \end{aligned} \tag{11}$$

where $\dot{p}^0 \neq 0$ will be assumed without loss of generality, and denote by

$$x = x(t; t_0, x^0, \mu, \tau) \tag{12}$$

the solution of the perturbed system (1) for which

$$x(t_0; t_0, x^0, \mu, \tau) = x^0$$

holds. If $|t_0|$, $\|x^0-p^0\|$, $|\mu|$ and $|\tau-\tau_0|$ are sufficiently small the path corresponding to (12) will intersect the hyperplane in the n-dimensional space of x which passes through the point p^0 and is orthogonal to the vector $\dot{p}^0$. This assertion follows from the fact that on a finite interval (12) is a continuous function of its arguments. As a consequence, all solutions of (1) for small enough $|\mu|$ and $|\tau-\tau_0|$ with initial values (t_0, x^0) sufficiently close to $(0, p^0)$ can be characterized with the parameter value $t=\vartheta$ at which this intersection takes place and the point p^0+h of intersection. Thus we consider the solutions in the following form

$$x = x(t; \vartheta, p^0+h, \mu, \tau), \tag{13}$$

where

$$x(\vartheta; \vartheta, p^0+h, \mu, \tau) = p^0+h$$

and the scalar product

$$(h, \dot{p}^0) = 0. \tag{14}$$

We are going to prove the following

THEOREM 1. *If 1 is a simple characteristic multiplier of the system* (8), *then to all sufficiently small values of $|\mu|$ and $|\vartheta|$ there belongs a unique period $\tau=\tau(\mu, \vartheta)$ and a unique $h=h(\mu, \vartheta)$ such that,*

$$\varphi(t, \mu, \vartheta) = x\big(t; \vartheta, p^0+h(\mu, \vartheta), \mu, \tau(\mu, \vartheta)\big) \tag{15}$$

is a D-periodic solution with period $\tau(\mu, \vartheta)$ and coefficient vector a of the perturbed system (1), *where also $\tau=\tau(\mu, \vartheta)$; the vector a is uniquely determined by condition* (4), *the functions $\tau(\mu, \vartheta)$ and $h(\mu, \vartheta)$ are in the C^1 class, $\tau(0,0)=\tau_0$, $h(0,0)=0$ and $\varphi(t, 0, 0)=p(t)$.*

REMARK 1. This theorem is a simple generalization of a similar theorem, due essentially to H. POINCARÉ, concerning periodic solutions and autonomous perturbations. The proof of the present theorem follows the outlines of that of the quoted one as given in [8] pp. 352—353.

REMARK 2. As it is well known, the periodic function $\dot{p}(t)$ is a solution of (8) and, as a consequence, 1 is a characteristic multiplier.

PROOF. Without loss of generality we may assume that the hyperplane passing through the the point p^0 and orthogonal to the vector $\dot{p}^0$ is the plane $x_1=p_1^0$, i.e.

$$\dot{p}^0 = (\dot{p}_1^0, 0, \dots, 0), \qquad \dot{p}_1^0 \neq 0$$

and, as a consequence,

$$h = (0, h_2, h_3, \dots, h_n).$$

If this were not the case, this could be achieved by a simple orthogonal transformation of the coordinate system which by Lemma 2 does not alter the characteristic multipliers.

By a well-known theorem (cf. e.g. [8] p. 58.), for sufficiently small $|\vartheta|$, $\|h\|$, $|\mu|$ and $|\tau-\tau_0|$ (13) is defined on the same interval as $p(t)$. Thus, according to Lemma

1, it is a D-periodic solution with period τ and coefficient vector a if and only if

(16) $$x(\vartheta+\tau;\vartheta,p^0+h,\mu,\tau)=x(\vartheta;\vartheta,p^0+h,\mu,\tau)+a\cdot\tau=p^0+h+a\cdot\tau,$$

i.e.

(17) $$z=x(\vartheta+\tau;\vartheta,p^0+h,\mu,\tau)-p^0-h-a\cdot\tau=0.$$

For fixed μ and ϑ, (17) is a system of equations in the unknowns τ, $h=(0,h_2,h_3,\dots,h_n)$ and $a=(a_1,a_2,\dots,a_n)$, to which we have to attach system (4). Thus, we have $2n$ equations for the $2n$ unknowns. If $\mu=\vartheta=0$ this system has a solution, namely $\tau=\tau_0$, $h=0$, $a=a^0$, since then (1) is reduced to (7) and $\varphi(t,0,0)=p(t)$. Substituting (4) into (17), we have

(18) $$z=x(\vartheta+\tau;\vartheta,p^0+h,\mu,\tau)-p^0-h-a^0\tau_0=0.$$

If the latter system has a solution τ, h and τ is close enough to $\tau_0>0$, then $\tau\neq 0$ and (4) can be solved uniquely for a. But z is continuously differentiable with respect to τ and the coordinates of h and thus, if at the point $P:\mu=\vartheta=0$, $\tau=\tau_0$, $h=0$ the Jacobian of the system (18) does not vanish, τ and h can be expressed from it as single valued uniquely determined and continuously differentiable functions of μ and ϑ in a neighbourhood of $\mu=0$, $\vartheta=0$.

Since the components of the vector on the left hand side of (18) are $z_1=x_1-c_1$, $z_i=x_i-c_i-h_i$, $(i=2,3,\dots,n)$, where c_1, c_i are constants and x does not depend on its last argument τ, if $\mu=0$,

$$\frac{\partial z_i(P)}{\partial\tau}=\dot{x}_i(\tau_0;0,p^0,0,\tau_0)=\dot{p}_i(\tau_0)=\dot{p}_i^0=\delta_{i1}\dot{p}_i^0,$$

$$\frac{\partial z_i(P)}{\partial h_k}=\frac{\partial x_i(P)}{\partial h_k}-\delta_{ik}\qquad(i=1,2,\dots,n;\ k=2,\dots,n)$$

where δ_{ik} is the Kronecker symbol. Thus

$$J=\left.\frac{\partial(z_1,z_2,\dots,z_n)}{\partial(\tau,h_2,\dots,h_n)}\right|_P=\left|\begin{matrix}\dot{p}_1^0 & \dfrac{\partial x_1}{\partial h_2} & \dots & \dfrac{\partial x_1}{\partial h_n}\\ 0 & \dfrac{\partial x_2}{\partial h_2}-1 & \dots & \dfrac{\partial x_2}{\partial h_n}\\ \dots & \dots & \dots & \dots\\ 0 & \dfrac{\partial x_n}{\partial h_2} & \dots & \dfrac{\partial x_n}{\partial h_n}-1\end{matrix}\right|_P,$$

or

(19) $$J=\dot{p}_1^0\left|\begin{matrix}\dfrac{\partial x_2}{\partial h_2}-1 & \dots & \dfrac{\partial x_2}{\partial h_n}\\ \dots & \dots & \dots\\ \dfrac{\partial x_n}{\partial h_2} & \dots & \dfrac{\partial x_n}{\partial h_n}-1\end{matrix}\right|_P,$$

where the usbscript P means that all elements are to be taken at P. Let us consider now the function

$$x=x(t;0,x^0,0,\tau)$$

which is the solution of (7) with the initial value

$$x^0 = x(0; 0, x^0, 0, \tau), \quad x^0 = (x_1^0, \ldots, x_n^0).$$

As it is well known the matrix

$$\frac{\partial x}{\partial x^0}\bigg|_{x^0=p^0} = \left[\frac{\partial x_i}{\partial x_k^0}\right]_{x_k^0=p_k^0} \tag{20}$$

is a matrix solution of (8) and

$$\frac{\partial x}{\partial x^0}\bigg|_{x^0=p^0,\, t=0} = U, \tag{21}$$

where U is the unit matrix. Hence, the characteristic multipliers of (8) are given by the roots of the equation

$$\det\left(\frac{\partial x}{\partial x^0}\bigg|_{x^0=p^0,\, t=\tau_0} - \lambda U\right) = 0. \tag{22}$$

We denote the first column of the matrix (20) by

$$\psi(t) = \frac{\partial x}{\partial x_1^0}\bigg|_{x^0=p^0};$$

this is a solution of (8) and by (21)

$$\psi(0) = (1, 0, 0, \ldots, 0) = \frac{1}{\dot{p}_1^0}(\dot{p}_1^0, 0, \ldots, 0) = \frac{1}{\dot{p}_1^0}\dot{p}^0.$$

Since (8) is a homogeneous linear system, we have

$$\psi(t) \equiv \frac{1}{\dot{p}_1^0}\dot{p}(t),$$

thus $\psi(t)$ is periodic with period τ_0,

$$\psi(t+\tau_0) = \psi(t)$$

and in particular $\psi(\tau_0)=\psi(0)$. Hence, the left hand side of (22) looks as follows:

$$\det\left(\frac{\partial x}{\partial x^0}\bigg|_{x^0=p^0,\, t=\tau_0} - \lambda U\right) = \begin{vmatrix} 1-\lambda & \frac{\partial x_1}{\partial x_2^0} & \cdots & \frac{\partial x_1}{\partial x_n^0} \\ 0 & \frac{\partial x_2}{\partial x_2^0} - \lambda & \cdots & \frac{\partial x_2}{\partial x_n^0} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{\partial x_n}{\partial x_2^0} & \cdots & \frac{\partial x_n}{\partial x_n^0} - \lambda \end{vmatrix}_{x^0=p^0,\, t=\tau_0} =$$

$$= (1-\lambda)\begin{vmatrix} \frac{\partial x_2}{\partial x_2^0} - \lambda & \cdots & \frac{\partial x_2}{\partial x_n^0} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial x_2^0} & \cdots & \frac{\partial x_n}{\partial x_n^0} - \lambda \end{vmatrix}_{x^0=p^0,\, t=\tau_0}.$$

Since 1 is a simple characteristic multiplier by assumption,

$$\left|\begin{matrix} \frac{\partial x_2}{\partial x_2^0}-1 & \cdots & \frac{\partial x_2}{\partial x_n^0} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial x_2^0} & \cdots & \frac{\partial x_n}{\partial x_n^0}-1 \end{matrix}\right|_{x^0=p^0,\, t=\tau_0} \neq 0.$$

Hence

$$J = \dot{p}_1^0 \left|\begin{matrix} \frac{\partial x_2}{\partial h_2}-1 & \cdots & \frac{\partial x_2}{\partial h_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial h_2} & \cdots & \frac{\partial x_n}{\partial h_n}-1 \end{matrix}\right|_P = \dot{p}_1^0 \left|\begin{matrix} \frac{\partial x_2}{\partial x_2^0}-1 & \cdots & \frac{\partial x_2}{\partial x_n^0} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial x_2^0} & \cdots & \frac{\partial x_n}{\partial x_n^0}-1 \end{matrix}\right|_{x^0=p^0,\, t=\tau_0} \neq 0.$$

Thus, there exist $\alpha_1>0$, $\alpha_2>0$ such that if $|\mu|<\alpha_1$, $|\vartheta|<\alpha_2$ then τ and h can be expressed from (18) as single valued continuously differentiable functions of μ and ϑ:

$$\tau=\tau(\mu,\vartheta),\quad h=h(\mu,\vartheta),\quad |\mu|<\alpha_1,\quad |\vartheta|<\alpha_2, \tag{23}$$

$\tau(0,0)=\tau_0$, $h(0,0)=0$ and then a can be determined uniquely from

$$a\cdot\tau(\mu,\vartheta) = a^0\cdot\tau_0.$$

These values of τ, h and a satisfy (16), i.e. for arbitrarily given values of μ and ϑ such that $|\mu|<\alpha_1$, $|\vartheta|<\alpha_2$, the function

$$\varphi(t;\mu,\vartheta) = x(t;\vartheta,p^0+h(\mu,\vartheta),\mu,\tau(\mu,\vartheta))$$

is a uniquely determined D-periodic solution with period $\tau(\mu,\vartheta)$ and coefficient vector

$$a = a^0\cdot\frac{\tau_0}{\tau(\mu,\vartheta)}$$

of the system

$$\frac{dx}{dt} = f(x)+\mu g\left(\frac{t}{\tau(\mu,\vartheta)}, x, \mu, \tau(\mu,\vartheta)\right) \tag{24}$$

and this was to be proved.

In the special case $a=a^0=0$, i.e. when the condition (3) on the periodicity of the right hand side of (1) in the vector x is dropped and, at the same time the solution $p(t)$ of (7) is periodic in the ordinary sense with period τ_0, we have the following

THEOREM 1′. *If* 1 *is a simple characteristic multiplier of system* (8), *then to all sufficiently small values of* $|\mu|$ *and* $|\vartheta|$ *there belongs a unique period* $\tau=\tau(\mu,\vartheta)$ *and a unique* $h=h(\mu,\vartheta)$ *such that*

$$\varphi(t,\mu,\vartheta) = x(t;\vartheta,p^0+h(\mu,\vartheta),\mu,\tau(\mu,\vartheta)) \tag{25}$$

is a periodic solution with period $\tau(\mu,\vartheta)$ *of the perturbed system* (1), *where also* $\tau=\tau(\mu,\vartheta)$; *the functions* $\tau(\mu,\vartheta)$ *and* $h(\mu,\vartheta)$ *are continuously differentiable*, $\tau(0,0)=\tau_0$, $h(0,0)=0$ *and* $\varphi(t,0,0)=p(t)$.

5. We are going to study now the question of stability of the D-periodic solution of the perturbed system (1). We assume from now on that $F\left(\frac{t}{\tau}, x, \mu, \tau\right)$ is in the C^2 class in the region specified above.

In what follows we will need a stability theorem, due originally to LYAPUNOV, in a modified form for the case of a D-periodic solution of a system periodic in t and in the vector argument.

Let the right hand side of the system

$$\frac{dx}{dt} = F(t, x) \tag{26}$$

be twice continuously differentiable, periodic in t with period τ and periodic in the vector argument x with vector period $a\tau$. Let (26) have a D-periodic solution $\varphi(t)$ with period τ and coefficient vector a. Consider the first variational system of (26) corresponding to $\varphi(t)$:

$$\frac{dy}{dt} = \left[\frac{\partial F(t, x)}{\partial x}\right]_{x=\varphi(t)} \cdot y \tag{27}$$

which is obviously a linear system with τ-periodic coefficients.

THEOREM 2. *If all the characteristic multipliers of* (27) *are in modulus less than* 1, *then* $\varphi(t)$ *is an asymptotically stable solution of* (26).

The standard proof for the original theorem (i.e. for the case of a periodic solution) applies to this case without alteration, therefore it will be omitted (see e.g. [8] p. 321).

Returning to the system (1) we assume that the conditions of Theorem 1 hold and, as a consequence, the existence of the uniquely determined D-periodic solution (15) of the system (24) is established for sufficiently small values of $|\mu|$, $|\vartheta|$. Consider the first variational system of (24) corresponding to the solution (15):

$$\frac{dy}{dt} = \left[\frac{\partial f(x)}{\partial x} + \mu \frac{\partial g\left(\frac{t}{\tau(\mu, \vartheta)}, x, \mu, \tau(\mu, \vartheta)\right)}{\partial x}\right]_{x=\varphi(t;\, \mu,\, \vartheta)} \cdot y. \tag{28}$$

Let us denote by $Y(t; \mu, \vartheta)$ the fundamental matrix solution of (28) for which $Y(0; \mu, \vartheta) = U$, (U is the unit matrix) holds. The "characteristic matrix" $C(\mu, \vartheta)$ of (28) corresponding to Y is then

$$C(\mu, \vartheta) = Y\big(\tau(\mu, \vartheta); \mu, \vartheta\big). \tag{29}$$

If $\mu = \vartheta = 0$, then $\tau(0, 0) = \tau_0$, (15) is reduced to $p(t)$ and the system (28) to (8). Thus

$$C(0, 0) = Y(\tau_0; 0, 0). \tag{30}$$

By assumptions of Theorem 1 one is a simple characteristic multiplier of the system (8), i.e. a simple eigenvalue of matrix (30). Let us denote by $\lambda_k(\mu, \vartheta)$ $(k = 1, 2, \dots, n)$, the eigenvalues of (29) as functions of μ and ϑ and let

$$\lambda_n(0, 0) = 1.$$

Since $C(\mu, \vartheta)$ is continuously differentiable in a neighbourhood of $\mu=\vartheta=0$ and the roots of a polynomial are continuous functions of the coefficients, it follows that $\lambda_n(\mu, \vartheta)$, a continuous function of μ and ϑ, is a simple eigenvalue of (29). It is easy to prove that a simple root of a polynomial is not only continuous but also a continuously differentiable function of the coefficients in a neighbourhood of the actual coefficients. Thus, we get that $\lambda_n(\mu, \vartheta)$ is a continuously differentiable function of μ and ϑ in a neighbourhood of $\mu=\vartheta=0$. Since in the same neighbourhood λ_n assumes values in a neighbourhood of 1 and no other eigenvalue may fall in the latter neighbourhood, λ_n is real valued.

We are able to prove now

THEOREM 3. *Under the assumptions of Theorem* 1, *if the remaining* $n-1$ *characteristic multipliers of the system* (8) *are in modulus less than one, there exist* $\varrho_1>0$, $\varrho_2>0$ *such that, if* μ *and* ϑ *satisfy the conditions*

$$|\mu|<\varrho_1, \quad |\vartheta|<\varrho_2 \tag{31}$$

and

$$\mu\left[\frac{\partial\lambda_n(\mu, \vartheta)}{\partial\mu}\right]_{\mu=\vartheta=0}<0, \tag{32}$$

then the D-periodic solution (15) *of the perturbed system* (24) *is asymptotically stable* (*cf.* [9] *Theorem* 3).

PROOF. Since $n-1$ of the eigenvalues of the matrix (30) are in modulus less than one and $C(\mu, \vartheta)$ is a continuous function of μ and ϑ, there can be found $\beta_1>0$, $\beta_2>0$ and $0<q<1$ such that for $|\mu|<\beta_1$, $|\vartheta|<\beta_2$ the $n-1$ corresponding roots of $C(\mu, \vartheta)$ are still in modulus less than q.

For $\mu=0$, $\varphi(t; 0, \vartheta) = p(t-\vartheta)$ for all ϑ, as a consequence, $\tau(0, \vartheta)\equiv\tau_0$, $h(0, \vartheta)=0$ and (28) is reduced to

$$\frac{dy}{dt}=\left[\frac{\partial f(x)}{\partial x}\right]_{x=p(t-\vartheta)}\cdot y. \tag{33}$$

A fundamental matrix solution $\tilde{Y}_\vartheta(t)$ of (33) can be got from the fundamental solution $Y(t; 0, 0)$ of (8) by simply writing

$$\tilde{Y}_\vartheta(t) = Y(t-\vartheta; 0, 0).$$

The corresponding "characteristic matrix" of the system (33) is then

$$\tilde{C}_\vartheta = \tilde{Y}_\vartheta^{-1}(t)\tilde{Y}_\vartheta(t+\tau_0) = Y^{-1}(t-\vartheta; 0, 0)\cdot Y(t+\tau_0-\vartheta; 0, 0) =$$
$$= Y^{-1}(0; 0, 0)\cdot Y(\tau_0; 0, 0) = C(0, 0).$$

Hence, we have that for $\mu=0$ the characteristic multipliers of the system (28) do not depend on ϑ. In particular,

$$\left[\frac{\partial\lambda_n(0, \vartheta)}{\partial\vartheta}\right]_{\vartheta=0}=0. \tag{34}$$

$\lambda_n(\mu, \vartheta)$ is locally decreasing in the point $\mu=\vartheta=0$ at directions in the μ, ϑ plane forming an obtuse angle with the vector $\text{grad}_{(0,0)}\lambda_n(\mu, \vartheta)$. Because of (34) these directions are characterized by inequality (32). Since $\lambda_n(0,0)=1$, there exist $\gamma_1>0$, $\gamma_2>0$ such that for μ, ϑ satisfying (32) and $|\mu|<\gamma_1$, $|\vartheta|<\gamma_2$; $q<\lambda_n(\mu,\vartheta)<1$. Let us introduce the notations

$$\varrho_i=\min(\alpha_i,\beta_i,\gamma_i) \qquad i=1,2.$$

We have that, if μ, ϑ satisfy conditions (31), (32), then all the characteristic multipliers of system (28) are in modulus less than one and this, by Theorem 2, yields the proof.

In the special case $a=a^0=0$, i.e. when the condition (3) on the periodicity of the right hand side of (1) in the vector x is dropped and, at the same time, the solution $p(t)$ of (7) is periodic in the ordinary sense, we have the following

THEOREM 3′. *Under the assumptions of Theorem* 1′ *if the remaining* $n-1$ *characteristic multipliers of system* (8) *are in modulus less than one, there exist* $\varrho_1>0$, $\varrho_2>0$ *such that if* μ *and* ϑ *satisfy the conditions* (31) *and* (32), *then the periodic solution* (25) *of the perturbed system* (24) *is asymptotically stable.*

REMARK. Condition (32) is satisfied either by positive or by negative μ's depending on the sign of $\left[\frac{\partial\lambda_n(\mu,\vartheta)}{\partial\mu}\right]_{\mu=\vartheta=0}$. In case

$$\left[\frac{\partial\lambda_n(\mu,\vartheta)}{\partial\mu}\right]_{\mu=\vartheta=0}=0$$

no μ exist satisfying (32). In the latter case we have asymptotical stability or unstability for all sufficiently small $|\mu|$ and $|\vartheta|$ according to that, whether the point $\mu=\vartheta=0$ is a maximum or a minimum point of the function $\lambda_n(\mu,\vartheta)$ respectively. Naturally, critical cases may occur too.

6. EXAMPLE. Consider the perturbed system

$$\begin{aligned}\dot{x}_1&=f_1(x_1,x_2)-\mu\left(x_2+\cos\frac{2\pi}{\tau}t\right)\\ \dot{x}_2&=f_2(x_1,x_2)+\mu\left(x_1+\sin\frac{2\pi}{\tau}t\right),\end{aligned}\tag{35}$$

where

$$f_1(x_1,x_2)=\begin{cases}x_2-\frac{1}{3}x_1^3+\frac{1}{3}(1-x_2^2)^{3/2}, & \text{if}\quad 0\leqq x_1\leqq 1,\ |x_2|\leqq 1\\ x_2-\frac{1}{3}x_1^3-\frac{1}{3}(1-x_2^2)^{3/2}, & \text{if}\quad -1\leqq x_1<0,\ |x_2|\leqq 1\end{cases}$$

$$f_2(x_1,x_2)=\begin{cases}-x_1-\frac{1}{3}x_2^3+\frac{1}{3}(1-x_1^2)^{3/2}, & \text{if}\quad |x_1|\leqq 1,\quad 0\leqq x_2\leqq 1\\ -x_1-\frac{1}{3}x_2^3-\frac{1}{3}(1-x_1^2)^{3/2}, & \text{if}\quad |x_1|\leqq 1,\ -1\leqq x_2<0.\end{cases}$$

The corresponding unperturbed system is

(36) $$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2).$$

A periodic solution of (36) with period 2π is $p(t) = (\sin t, \cos t)$; $p(0) = (0, 1)$, $\dot{p}(0) = (1, 0)$. The first variational system of (36) corresponding to $p(t)$ is

$$\dot{y}_1 = -\sin^2 t \cdot y_1 + (1 - \sin t \cos t) \cdot y_2$$
$$\dot{y}_2 = -(1 + \sin t \cos t) \cdot y_1 - \cos^2 t \cdot y_2.$$

It is easy to find a fundamental matrix solution and through it the characteristic multipliers, they are: $\lambda_1 = e^{-2\pi}$, $\lambda_2 = 1$. The perturbed system (35) has the following periodic solution:

$$\varphi(t, \mu) = \left(\sin \frac{2\pi}{\tau} t, \cos \frac{2\pi}{\tau} t\right),$$

provided that $|\mu| < \frac{1}{2}$, $\tau = 2\pi(1-2\mu)^{-1}$ and this value of τ is substituted into (35) too. "ϑ" has been kept equal to zero for sake of simplicity. The characteristic multipliers of the first variational system of (35) corresponding to $\varphi(t, \mu)$ can be determined with some difficulty, they are

$$\lambda_1 = \exp\{\pi(2\mu - 1)^{-1}[1 + (1 - 8\mu^2)^{1/2}]\},$$
$$\lambda_2 = \exp\{\pi(2\mu - 1)^{-1}[1 - (1 - 8\mu^2)^{1/2}]\}.$$

$$\frac{d\lambda_2(0)}{d\mu} = 0.$$

However, $\mu = 0$ is a maximum point of the function $\lambda_2(\mu)$ and $|\lambda_i(\mu)| < 1$, if $0 < |\mu| < \frac{1}{2}$ $(i = 1, 2)$. Nevertheless, no conclusion of asymptotic stability can be deduced from here, since the right hand side of (35) is not continuous on the segments $\{(x_1, x_2): x_1 = 0, |x_2| < 1\}$, and $\{(x_1, x_2): |x_1| < 1, x_2 = 0\}$.

7. In case the right hand side of system (1) is analytic, theorems similar to Theorem 1 and 3, respectively, hold with the additional result that the solution $\varphi(t, \mu, \vartheta)$ and the functions $\tau(\mu, \vartheta)$, $h(\mu, \vartheta)$ and $\lambda_n(\mu, \vartheta)$ are analytic in (μ, ϑ) in a neighbourhood of $\mu = 0$, $\vartheta = 0$. Thus Poincaré's method can be applied to the actual determination of the D-periodic solution of the perturbed system and its period. $\varphi(t, \mu, \vartheta)$ and $\tau(\mu, \vartheta)$ can be expanded by powers of μ and the coefficients of the respective expansions determined successively. See [10].

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MINIMAL SCRAMBLING SETS OF SIMPLE ORDERS

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Let n, k be positive integers, $2 \leqq k \leqq n$. Let $[1, n]$ denote the n-set of integers x, $1 \leqq x \leqq n$. Let R_i denote a simple order on $[1, n]$. In this paper we look for minimal sets $S = \{R_1, \ldots, R_N\}$ that "scramble" the elements of $[1, n]$ up in various ways.

DEFINITION. $S = \{R_1, \ldots, R_N\}$ will be called k-suitable (for n) if for every k-subset of the n elements, say

$$a_1, \ldots, a_k$$

and any distinguished element of the set, say a_k, there is some simple order $R_i \in S$ such that $a_j R_i a_k$ for every $1 \leqq j < k$, that is, a_k is the "largest" element of the k-set under R_i. The cardinality of the smallest k-suitable S will be denoted by $N(n, k)$.

DEFINITION. $S = \{R_1, \ldots, R_N\}$ will be called *k^*-suitable* if for every ordered k-set of the n elements, say

$$(a_1, \ldots, a_k)$$

there is some $R_i \in S$ such that, for $s, t \in [1, k]$,

$$a_s R_i a_t \quad \Leftrightarrow \quad s \leqq t.$$

That is, the R_i give all $k!$ permutations of every k-set. The cardinality of the minimal k^*-suitable S will be denoted by $N^*(n, k)$.

The notion of k-suitability is due to DUSHNIK [1], who found a simple formula for $N(n, k)$ when $2[\sqrt{n}] - 1 \leqq k \leqq n$. In this paper we consider k fixed and $n \to +\infty$. If R is a simple order let R^c denote the converse order $[aRb \Leftrightarrow bR^c a]$. We note $N(n, 2) = N^*(n, 2) = 2$ as $\{R, R^c\}$ forms a 2^*-suitable set for any R. We also note that $N \leqq N^*$, and N and N^* are monotonically increasing in n and k.

THEOREM 1. *For $k \geqq 3$, fixed, and $n \to \infty$*

$$\log_2 n \leqq N^*(n, k) \leqq \frac{k}{\log\left[\dfrac{k!}{k!-1}\right]} \log n.$$

PROOF. Let $R_1, \ldots, R_s$ be 3^*-suitable on $[1, n]$ and let y be the largest element in R_s. For $A \subseteq \{1, \ldots, s-1\}$ set

$$T_A = \{x : x \neq y, xR_i y \Leftrightarrow i \in A\}.$$

Then the T_A are disjoint with union $[1, n] - \{y\}$ of cardinality $n-1$. If any $|T_A| \geqq 2$, say $x, z \in T_A$ then y is never between x and z, a contradiction. Thus all $|T_A| \leqq 1$ so $n-1 \leqq 2^{s-1}$,

$$N^*(n, k) \geqq N^*(n, 3) \geqq 1 + \log_2 (n-1) \geqq \log_2 n.$$

For the upper bound we use probabilistic techniques. There are $n!^s$ different $(R_1, \ldots, R_s)$ on $[1, n]$. There are $\binom{n}{k} k! < n^k$ ordered $(x_1, \ldots, x_k)$ $(x_i \in [1, n]$, distinct$)$ that might not appear in any of the R's. There are $n!(1-(1/k!))R$'s that do not contain $x_1, \ldots, x_k$ in that specific order. Thus there are less than $n^k[n!(1-(1/k!))]^s$ $(R_1, \ldots, R_s)$ that do not contain every $(x_1, \ldots, x_k)$.

$$n^k[n!(1-(1/k!))]^s < n!^s$$

when

$$\frac{k}{\log\left[\frac{k!}{k!-1}\right]} \log n < s.$$

In that case, some $(R_1, \ldots, R_s)$ must contain every $(x_1, \ldots, x_n)$. Thus

$$N^*(n, k) \leqq \frac{k}{\log\left[\frac{k!}{k!-1}\right]} \log n. \qquad \text{Q.E.D.}$$

It is possible to improve the bound on $N^*(n, 3)$ by constructive techniques. For $k \geqq 4$, we have not been able to show $N^*(n, k) \leqq C_k \log n$ by constructive means.

THEOREM 2. *For* $n, k \geqq 3$

$$N(n, k) \geqq N(n, 3) \geqq \log_2 \log_2 n.$$

PROOF. We use the well known theorem of ERDŐS and SZEKERES [4] that if n^2+1 elements are ordered in two different ways some $(n+1)$-set is monotone under both orders. By a simple induction, if $2^{2^s}+1$ elements are ordered in $(s+1)$ ways some triple (x, y, z) is monotone under all orders. So if $n > 2^{2^{s-1}}+1$ and $R_1, \ldots, R_s$ are simple orders on $[1, n]$ we can find x, y, z so that, for $1 \leqq i \leqq s$, $xR_i yR_i z$ or $zR_i yR_i x$. We never have $x, zR_i y$ so $R_1, \ldots, R_s$ is not 3-suitable. Thus

$$N(n, 3) \geqq 2 + [\log_2 \log_2 (n-1)].$$

We now derive an upper bound for $N(n, k)$. This bound is due to A. Hajnal. We first require a lemma which is perhaps of independent interest.

DEFINITION. Call a family $\mathfrak{J} = \{S_1, \ldots, S_r\}$ of sets t-scrambling if for all $1 \leqq i_1 < < i_2 < \cdots < i_t \leqq n$ and all $A \subseteqq [1, t]$ there exists x,

$$(^*) \qquad x \in S_{i_j} \quad \text{if and only if } j \in A.$$

Intuitively, any t sets yield a Venn diagram with all 2^t components nonempty. Set $M(n, t)$ = the maximal cardinality $|\mathfrak{J}|$ of a t-scrambling family of subsets of $\{1, \ldots, n\}$.

LEMMA. $$M(2n, 2) = \binom{2n-1}{n-1}, \quad M(2n+1, 2) = \binom{2n}{n-1},$$

$$M(n, t) \geqq \left[\frac{1}{2}[(1-2^{-t})^{-1/t}]^n\right].$$

PROOF. If $\mathfrak{J}$ is 2-scrambling on $[1, 2n]$, $\mathfrak{J} \cup \{S : S^c \in \mathfrak{J}\}$ is a family of noncomparable sets; so by Sperner's Lemma [5]

$$2|\mathfrak{J}| \leqq \binom{2n}{n}, \qquad |\mathfrak{J}| \leqq \frac{1}{2}\binom{2n}{n} = \binom{2n-1}{n}.$$

As $\mathfrak{J} = \{S : S \subseteqq [1, 2n],\ 1 \in S,\ |S| = n\}$ is 2-scrambling, $M(2n, 2) = \binom{2n-1}{n}$.

If $\mathfrak{J}$ is 2-scrambling on $[1, 2n+1]$, $\mathfrak{J}' = \{S : S \in \mathfrak{J}$ or $S^c \in \mathfrak{J},\ |S| \leqq n\}$ is a family of noncomparable sets. By a theorem of ERDŐS, KO, and RADO [3] this implies

$$|\mathfrak{J}| = |\mathfrak{J}'| \leqq \binom{2n+1}{n-1}.$$

As $\mathfrak{J} = \{S : S \subseteqq [1, 2n+1] : 1 \in S,\ |S| = n\}$ is 2-scrambling, $M(2n+1, 2) = \binom{2n}{n-1}$.

There are 2^{ns} ordered s-tuples $(S_1, \dots, S_s)$ of subsets of $[1, n]$. For each $i_1, \dots, i_t$, A at most $2^{ns}(1-2^{-t})^n$ of these s-tuples have no x, $1 \leqq x \leqq n$ satisfying (*). As there are $< \binom{s}{t} 2^t < (2s)^t$ such $i_1, \dots, i_t$, A the number of s-tuples that are not t-scrambling

$$< (2s)^t (1-2^{-t})^n 2^{ns} < 2^{ns}$$

for

$$s < \frac{1}{2}[(1-2^{-t})^{-1/t}]^n.$$

Thus for such s there exists a t-scrambling $\mathfrak{J}$ on $[1, n]$, $|\mathfrak{J}| = s$.

Now let n, k be given and fix a $(k-1)$-scrambling family $\{S_1, \dots, S_m\}$ of subsets of $[1, n]$ where $m = M(n, k-1)$. Set $r = 2^{M(n,k-1)}$. Let $Q_1, \dots, Q_r$ be the subsets of $\{1, \dots, M(n, k-1)\}$. For $1 \leqq i \leqq n$ define an order R_i on the Q's by $Q_s R_i Q_t$ if, setting j = minimal integer in $Q_s \triangle Q_t$, either $i \in S_j$, $j \notin Q_s$, and $j \in Q_t$ or $i \notin S_j$, $j \in Q_s$, and $j \notin Q_t$. I claim that $\{R_1, \dots, R_n\}$ is a k-suitable family of simple orders on $\{Q_1, \dots, Q_r\}$. Clearly the R_i are simple orders. Now fix distinct $Q_{a_1}, \dots, Q_{a_k}$. For $1 \leqq j \leqq k-1$ set d_j = the minimal element of $Q_{a_j} \triangle Q_{a_k}$. Set $A = \{j : 1 \leqq j \leqq k-1,\ d_j \in Q_{a_k}\}$. As $\{S_1, \dots, S_m\}$ is a $(k-1)$-scrambling family there exists x, $1 \leqq x \leqq n$, so that

$$x \in S_{d_j} \text{ if and only if } j \in A.$$

Then $Q_{a_j} R_x Q_{a_k}$ for $1 \leqq j \leqq k-1$. Therefore

$$N(2^{M(n,k-1)}, k) \leqq n.$$

Combining this result with the lemma and the monotonicity of N yields

THEOREM 3. $N(n, 3) < \log_2 \log_2 n + \frac{1}{2} \log_2 \log_2 \log_2 n + \log_2 (\sqrt{2}\pi) + o(1)$.

For $k \geqq 4$, fixed, and n approaching infinity

$$N(n, k) < (\log_2 \log_2 n)\left(-t/\log_2(1-2^{-t})\right) + o(1).$$

M. A. MULLIN (personal correspondence) has asked about the infinite case. For k finite we now find $N(n, k)$ and $N^*(n, k)$ for n an infinite cardinal.

LEMMA. *Let k, l be integres, t an infinite cardinal, $|T| = t$. Then thern exists a family F of functions $f: T \to [1, k]$ sucht that $|F| = 2^t$ and such that for all distinct $f_1, \ldots, f_k \in F$ and any $v_1, \ldots, v_k \in [1, l]$ there exists $x \in T$*

$$f_i(x) = v_i \qquad 1 \leqq i \leqq k.$$

PROOF. As $t = t^k l^{2^k}$ let T be the set of ordered $(k+1)$-tuples $(x_1, \ldots, x_k, g)$ where $1 \leqq x_i < t$ and $g: [0, 2^k) \to [1, l]$. For $S \subseteq [1, t]$ set

$$f_s((x_1, \ldots, x_k, g)) = g\Big(\sum_{x_i \in S} 2^{i-1}\Big).$$

Then $F = \{f_s\}$ is the desired family. (We suppress the proof.)

THEOREM 4. *For k finite, $k \geqq 3$, and n infinite*

$$N(n, k) = \min\{m : 2^{2^m} \geqq n\}.$$

PROOF. Suppose $2^{2^m} < n$ and let $R_1, \ldots$ be a family of m simple orders of $[1, n]$. Define a colouration of the complete graph on $[1, n]$ by colouring $\{i, j\}$ with $A_{ij} = \{k : iR_k j\}$. As the colours are subsets of $[1, m]$ there are at most 2^m colours used. By a theorem of ERDŐS, HAJNAL, and RADO [2] this implies that there exists a monochromatic triangle. That is, there exist x, y, z such that $A_{xy} = A_{xz} = A_{yz}$. Therefore x, y, z may be permuted in only two ways and so $R_1, \ldots$ cannot be 3-suitable.

We find a k-suitable family of m simple orders on 2^{2^m} points by the same method as Theorem 3. The above lemma with $l = 2$ replaces its finitistic analogue.

THEOREM 5. *For k finite, $k \geqq 3$, and n infinite*

$$N^*(n, k) = \min\{m : 2^m \geqq n\}.$$

PROOF. The proof used in Theorem 1 shows that $2^{N^*(n,k)} \geqq n$. For any k, m we find a family F of 2^m functions $f_\alpha: [1, m] \to [1, k]$ satisfying the lemma for $l = k$. We induce simple orders R_α by setting $xR_\alpha y$ if $f_\alpha(x) < f_\alpha(y)$. If $f_\alpha(x) = f_\alpha(y)$ we order x and y in any consistent manner. Then the R_α form a k^*-suitable family of m simple orders on 2^m points.

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ON THE NORMAL PRODUCT OF FC-NILPOTENT AND FC-HYPERCENTRAL GROUPS

By

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I. Introduction and notation. Let $F_0 = E$, the unit subgroup and let $F_1(G)$ be the set of elements of a group G which posses a finite number of conjugates. For all ordinals α we define inductively $F_{\alpha+1}(G)$ to be the complete inverse image of $F_1(G/F_\alpha(G))$. If α is a limit ordinal, then we define $F_\alpha(G) = \cup \{F_\beta(G) : \beta < \alpha\}$. For all ordinals α, $F_\alpha(G)$ is a characteristic subgroup of G. A group G is called FC-nilpotent of class n if there exists an integer n such that $F_{n-1}(G) \neq G$ and $F_n(G) = G$, and G is called FC-hypercentral of class α if there exists an ordinal α such that $F_\beta(G) \neq G$ for $\beta < \alpha$ and $F_\alpha(G) = G$.

We need the following facts about FC-nilpotent groups and FC-hypercentral groups: Subgroups, direct sums, and homomorphic images of FC-nilpotent groups (FC-hypercentral groups) are FC-nilpotent groups (FC-hypercentral groups). For further information see [1]. The following theorem of Fitting is well known (see [5: 7. 4. 1]): If A and B are normal nilpotent subgroups of a group G, then AB is also a normal nilpotent subgroup of G. Further, if A is of class a and B is of class b, then AB is of class $a+b-1$. P. Hall has shown [3: p. 334] that the normal product of two ZA-groups is a ZA group. Similar theorems for FC-nilpotent and FC-hypercentral groups do not seem to have appeared in the literature.

In this paper we shall prove that the normal product of two FC-nilpotent groups is FC-nilpotent and that the normal product of two FC-hypercentral groups is FC-hypercentral. Bounds will be given on the class of the product.

II. LEMMA 1. *Let L and M be normal subgroups of the group G. Suppose $L \subseteq M$ and $L \subseteq F_\gamma(G)$ for some ordinal γ. Then $M \subseteq F_{\gamma+1}(G)$ if $M/L \subseteq F_1(G/L)$.*

PROOF. If $M/L \subseteq F_1(G/L)$, then for $m \in M$, the index of the centralizer of mL in G/L is finite. Now $G/F_\gamma(G)$ is a homomorphic image of G/L, so the centralizer of $mF_\gamma(G)$ has finite index in $G/F_\gamma(G)$. Therefore $M \subseteq F_{\gamma+1}(G)$.

LEMMA 2. *Let H and K be normal subgroups of a group G. For any pair of non-negative integers (i, j) define a subgroup by $G(i, j) = F_i(H) \cap F_j(K)$. Then $G(i, j) \subseteq$ $\subseteq F_{i+j-1}(HK)$.*

PROOF. Let $F(k)$ denote $F_k(HK)$. Since $F_i(H)$ is a characteristic subgroup of H and H is a normal subgroup of G, $F_i(H)$ is a normal subgroup of G. A similar remark shows that $F_j(K)$ is normal in G, so that $G(i, j)$ is a normal subgroup of G.

We have that $G(0, 0) = G(1, 0) = G(0, 1) = E$. Let $s = i+j$. We shall induct on s. If $s = 1$, the result is clear. Assume the statement to be true for $s = 1, 2, \ldots, t$ and suppose that $i+j = t+1$. If i is 0, then $G(0, j) = F_0(H) \cap F_j(K) = E$. Similarly if $j = 0$, then $G(i, 0) = E$. Thus we may assume that $i \neq 0 \neq j$.

We have by the induction assumption that both $G(i-1,j)$ and $G(i,j-1)$ are normal subgroups of $F(i+j-2)$. Let L denote their product, and let $x \in G(i,j)$. Let S be a nonempty subset of G. By $\mathrm{Con}(x, S)$ we shall mean the class whose members are of the form x^s where $s \in S$. Then $\mathrm{Con}(x, H)$ has only a finite number of members modulo $G(i-1,j)$ and so it has only a finite number of members modulo L. The same observation using $G(i,j-1)$ for $G(i-1,j)$ shows that $\mathrm{Con}(x, K)$ has only a finite number of members modulo L. Thus $\mathrm{Con}(\mathrm{Con}(x, H), K)$ has only a finite number of members modulo L. Thus $G(i,j)/L \subseteqq F_1(HK/L)$. By Lemma 1, $G(i,j) \subseteqq F(i+j-1)$, and the proof is completed.

THEOREM 1. *If H and K are normal subgroups of a group G, and if H and K are FC-nilpotent of class n and m, respectively, with $n \geqq m$, then HK is FC-nilpotent of class at most $2n+m-1$.*

PROOF. Lemma 2 shows that $H \cap K = G(n, m) \subseteqq F(n+m-1)$. As the class of $HK/(H \cap K)$ is less than or equal to n, the class of HK is less than or equal to $n+(n+m-1) = 2n+m-1$ by [2: p. 506].

III. We shall state without proof this expanded version of a result of HOELZER [4: p. 10]:

THEOREM 2. *If H is a non-trivial normal subgroup of an FC-hypercentral group G, then $H \cap F_1(G) \neq E$.*

LEMMA 3. *If H and K are normal subgroups of a group G, and if H and K are FC-hypercentral groups and $HK \neq E$, then $F_1(HK) \neq E$.*

PROOF. If $H \cap K = E$, then $HK = H+K$, and $F_1(H+K) = F_1(H)+F_1(K) \neq E$. If $H \cap K \neq E$, then $H \cap K$ is a non-trivial normal subgroup of H. By Theorem 2, $L = (H \cap K) \cap F_1(H) \neq E$. Note that L is a normal subgroup of G as $F_1(H)$ is a characteristic subgroup of H and H is normal in G. Now $L \cap F_1(K) \neq E$ as L is a non-trivial normal subgroup of K. But

$$L \cap F_1(K) = [(H \cap K) \cap F_1(H)] \cap F_1(K) = F_1(H) \cap F_1(K) = M,$$

and M is a normal subgroup of G. Let $x \in M \setminus E$. Consider the set $\{x^{hk} : h \in H$ and $k \in K\}$. This is a subset of M. As h ranges over H, x^h takes on a finite number of values, say $x_1, \ldots, x_n$, all of which lie in M. As $x_i \in F_1(K)$, x_i^k takes on a finite number of values as k ranges over K for $1 \leqq i \leqq n$. Thus $x \in F_1(HK)$, and so $F_1(HK) \neq E$.

THEOREM 3. *Let H and K be non-trivial normal subgroups of a group G such that $G = HK$. If H and K are FC-hypercentral, then G is FC-hypercentral.*

PROOF. Deny. By Lemma 3, $F_1(HK) \neq E$. Suppose that there exists an ordinal α such that $F_\alpha(G) = F_{\alpha+1}(G) \neq G$. Then $\bar{G} = G/F_\alpha(G) = [HF_\alpha(G)]/F_\alpha(G) \cdot [KF_\alpha(G)]/F_\alpha(G)$. Now $\bar{G}$ is the product of two normal FC-hypercentral groups and $\bar{G} \neq E$. By Lemma 2, $F_1(\bar{G}) \neq E$. Therefore $F_{\alpha+1}(G)$, which is the complete inverse image of $F_1(\bar{G})$, is strictly greater than $F_\alpha(G)$. This is a contradiction.

IV. In this section we shall indicate how to find the bound of the class of an FC-hypercentral group which is the product of two normal FC-hypercentral groups.

If α is a limit ordinal we define, $G(\alpha, \beta) = \bigcup \{G(\mu, \beta): \mu < \alpha\}$, $G(\alpha) = G(\alpha, \alpha) = \bigcup \{G(\mu, \mu): \mu < \alpha\}$, and $F(\alpha) = \bigcup \{F(\mu): \mu < \alpha\}$. The following lemma is proved by an argument similar to the one used in Lemma 2.

LEMMA 4. *If* $G(\alpha, \beta+1) \subseteqq F(\gamma)$ *and* $G(\alpha+1, \beta) \subseteqq F(\gamma)$, *then* $G(\alpha+1, \beta+1) \subseteqq F(\gamma+1)$.

The following proposition is central in the process of determining the bound.

PROPOSITION 1. *Let* α *be an infinite ordinal and assume that* $G(\alpha) \subseteqq F(\gamma)$. *Then if* $1 \leqq n, m < \omega$, $G(\alpha+n, \alpha+m)$ *is contained in* $F(\gamma+\alpha+(n+m-1))$. *Furthermore, it follows that* $G(\alpha+\omega)$ *is contained in* $F(\gamma+\alpha+\omega)$.

PROOF. The first assertion will follow if we can show that for all n, $1 \leqq n < \omega$, $G(\alpha+n, \alpha) \subseteqq F(\gamma+\alpha)$ and that $G(\alpha, \alpha+n) \subseteqq F(\gamma+\alpha)$. For then the bound $\gamma+\alpha+(n+m-1)$ can be derived by induction on $n+m$, exactly as in the finite case.

We wish first to establish by induction that for $1 \leqq n < \omega$ and $1 \leqq \mu < \alpha$,

$$G(\alpha+n, \mu) \subseteqq F(\gamma+\mu+(n-1)). \tag{1}$$

Assume that $1 \leqq k < \omega$, $1 \leqq \mu < \xi$, and that (1) is true

(i) whenever $1 \leqq n < k$ for any $\mu < \alpha$, and

(ii) for $n = k$ and $1 \leqq \mu < \xi$.

We will establish (1) for (k, ξ) which will complete the induction.

Case I. $\xi > 1$. If ξ is a limit ordinal we have by (ii), for $\mu < \xi$, $G(\alpha+k, \mu) \subseteqq F(\gamma+\mu+(k-1))$. Hence

$$G(\alpha+k, \xi) = \bigcup \{G(\alpha+k, \mu): \mu < \xi\} \subseteqq \bigcup \{F(\gamma+\mu+(k-1): \mu < \xi\} =$$
$$= F(\gamma+\xi) \subseteqq F(\gamma+\xi+(k-1)).$$

If ξ is not a limit ordinal, then by (ii), $G(\alpha+k, \xi-1) \subseteqq F(\gamma+(\xi-1)+(k-1))$. If $k=1$, we have $G(\alpha+k-1, \xi) = G(\alpha, \xi) \subseteqq F(\gamma) \subseteqq F(\gamma+(\xi-1)+(k-1))$. If $k \geqq 1$, we have by (i), $G(\alpha+k-1, \xi) \subseteqq F(\gamma+\xi+(k-2)) = F(\gamma+(\xi-1)+(k-1))$. In any even an application of Lemma 4 gives $G(\alpha+k, \xi) \subseteqq F(\gamma+\xi+(k-1))$.

Case II. $\xi = 1$. If $k = 1$, we have $G(\alpha+1, 1) \subseteqq F(\gamma+1) = F(\gamma+1+(k-1))$, because $G(\alpha, 1) F(\gamma)$ and $G(\alpha+1, 0) = E \subseteqq F(\gamma)$. If $k > 1$, then by (i) $G(\alpha+(k-1), 1) \subseteqq F(\gamma+1+(k-2))$. Since $G(\alpha+k, 0) = E \subseteqq F(\gamma+1+(k-2))$, we have by Lemma 4, $Gk, 1)\ (\alpha+ \subseteqq F(\gamma+1+(k-1))$.

Thus (1) is established. It follows that for $1 \leqq n < \omega$ we have

$$G(\alpha+n, \alpha) = \bigcup \{G(\alpha+n, \mu): \mu < \alpha\} \subseteqq \bigcup \{F(\gamma+\mu+(n-1): \mu < \alpha\} = F(\gamma+\alpha).$$

This proves the first assertion of the proposition. The second follows easily since

$$G(\alpha+\omega) = \bigcup \{G(\alpha+n, \alpha+n): n < \omega\} \subseteqq \bigcup \{F(\gamma+\alpha(2n-1)): n < \omega\} = F(\gamma+\alpha+\omega).$$

The second part of the proposition gives the primary recursion property needed to derive a bound in closed form. The derivation proceeds as follows.

(I) Using the primary recursion property for each power ω^λ of ω where λ is any ordinal greater than zero, a number $\sigma(\lambda)$ can be computed such that $G(\omega^\lambda) \subseteqq F(\sigma(\lambda))$. This will solve the problem for computing a bound when the lengths of H and K are not greater than some power of ω and the number will be: $\lambda=1$: $\sigma(\lambda)=\omega$; $1<\lambda<\omega$: $\sigma(\lambda)=\omega^{2(\lambda-1)}$; $\lambda=\omega^\alpha c$ where $\alpha\geqq 1$, $1\leqq c<\omega$: $\sigma(\lambda)==\omega^{\omega^\alpha(2c-1)}$; otherwise: $\sigma(\lambda)=\omega^{\lambda 2}$.

(II) For any $\lambda\geqq 1$ and any $\mu\geqq\omega^\lambda$, μ a limit ordinal, a number $\sigma(\mu+\omega^\lambda)$ can be computed such that $G(\mu+\omega^\lambda)\subseteqq F(\sigma(\mu+\omega^\lambda))$. This number will be as follows:

$$\lambda=1 \quad : \quad \sigma(\mu+\omega)=\sigma(\mu)+\mu+\omega \quad \text{where} \quad G(\mu)\subseteqq F(\sigma(\mu))$$

$$1<\lambda<\omega: \quad \sigma(\mu+\omega^\lambda)=\sigma(\mu)+\mu\omega^{\lambda-1} \quad \text{where} \quad G(\mu)\subseteqq F(\sigma(\mu))$$

$$\lambda\geqq\omega \quad : \quad \sigma(\mu+\omega^\lambda)=\sigma(\mu)+\mu\omega^\lambda \quad \text{where} \quad G(\mu)\subseteqq F(\sigma(\mu)).$$

There are two cases of special interest. If α is greater than or equal to the FC-hypercentral classes of H and K and $\alpha=\omega^{\omega^\mu}$ where $\mu\geqq 0$, then the FC-hypercentral class of HK is at most $\alpha 2$. If β is greater than or equal to the FC-hypercentral class of H and K, and $\omega^\beta=\beta$, then the FC-hypercentral class of HK is at most $\beta 2$.

It is not known if these bounds (in the finite and in the transfinite cases) are the best possible.

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ON A THEOREM OF S. DANCS AND P. TURÁN

By

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We consider functions of the form

$$F(z) = \sum_{k=1}^{m} p_k(z)\, e^{\omega_k z} \tag{1}$$

where the ω_k are distinct complex numbers and the $p_k(z)$ polynomials not all trivial of some bounded degree. Then S. DANCS and P. TURÁN have shown that the number of zeros (counted according to multiplicity) of such a function in *any* square of side length S in the complex plane whose sides are parallel to the axes does not exceed

$$SM\left(4+\frac{\varrho}{10m}\right)+9\varrho^2 \log 2m\varrho+5\varrho m+\varrho(2m+\varrho)\log\left(1+\frac{5m^2\varrho}{S\Delta}\right) \tag{2}$$

where the degree of the $p_k(z) \leqq \varrho-1$, $n \geqq 2$ and

$$M = \max_{k,\, l} |\omega_k-\omega_l|, \quad \Delta = \min_{h \neq l} |\omega_h-\omega_l|.$$

By a similar method to that employed by DANCS and TURÁN, but from a different formula as our starting point, we prove a similar result to (2) avoiding a difficulty in the original proof. Briefly, our method of proof is as follows:

By an application of the Cauchy inequality to our starting result, we find a point z_0 in a chosen square of side S so that we have a lower bound for $|F(z_0)|$. This indeed is the critical step in the proof. We then apply an inequality of JENSEN to the effect that the number of zeros (counted according to multiplicity) of $F(z)$ in a circle of radius r about z_0 cannot exceed

$$\max_{|z-z_0|<er} \log\left|\frac{F(z)}{F(z_0)}\right|. \tag{3}$$

Choosing r so that the abovementioned circle covers the chosen square we obtain the required upper bound for the number of zeros of $F(z)$ in the square. Moreover this upper bound will prove to be independent of the coefficients of the polynomials $p_k(z)$ in $F(z)$. We easily show that this implies that our upper bound is one for the number of zeros of $F(z)$ in *any* square of side S in the complex plane.

Our starting point is Lemma 1, which is Theorem 4 of [2]. We refer the reader to that paper for the proof. Throughout we employ the following notation:

Let $\omega_1, \omega_2, \dots, \omega_m$ be distinct complex numbers and let $p_1(z), p_2(z), \dots, p_m(z)$ be polynomials (not all trivial) of the form

$$p_k(z) = a_{k1} + a_{k2}z + \dots + a_{k\varrho_k} z^{\varrho_k - 1} \qquad (k = 1, 2, \dots, m)$$

and

$$\varrho_1 + \varrho_2 + \dots + \varrho_m = \sigma.$$

We write

$$F(z) = \sum_{k=1}^{m} p_k(z) e^{\omega_k z}.$$

Further let

$$\min_{\substack{j=1,2,\dots,m \\ h \neq j}} |\omega_h - \omega_j| = \delta_h, \qquad \max_{j=1,2,\dots,m} |\omega_j| = \Omega$$

and write

(4)
$$\max_{h=1,2,\dots,m} \varrho_h = \varrho, \qquad \min_{\substack{h=1,2,\dots,m \\ t=1,2,\dots,\varrho_h}} \delta_h^{\sigma-t} = \delta^{\sigma-1}.$$

LEMMA 1. *There is an integer μ such that $1 \leqq \mu \leqq \sigma$ and*

$$\left| \left\{ \frac{d^{\mu-1}}{dz^{\mu-1}} \left(\sum_{k=1}^{m} p_k(z) e^{\omega_k z} \right) \right\}_{z=0} \right| \geqq |a_{ht}| \frac{\delta_h^{\sigma-t}}{\sigma^{\varrho_h - t}(1+\Omega)^{\sigma-1}}$$

for $h=1, 2, \dots, m$, $t=1, 2, \dots, \varrho_h$.

LEMMA 2. *Let $R \geqq 2\sigma/e$. Then there is a point z_0 such that $|z_0| = R$ and*

$$|F(z_0)| \geqq |a_{ht}| \frac{\delta_h^{\sigma-t}}{\sigma^{\varrho_h - t}(1+\Omega)^{\sigma-1}} \geqq |a_{ht}| \frac{\delta^{\sigma-1}}{\sigma^{\varrho-1}(1+\Omega)^{\sigma-1}}$$

$$(h = 1, 2, \dots, m;\ t = 1, 2, \dots, \varrho_h).$$

PROOF. Choosing μ by Lemma 1 we have by Cauchy's Lemma

(5)
$$F^{(\mu-1)}(0) = \frac{(\mu-1)!}{2\pi i} \oint_{|z|=R} \frac{F(z)\,dz}{z^\mu}.$$

Taking absolute values in (5) and applying Lemma 1 we obtain

$$\frac{(\mu-1)!}{R^{\mu-1}} \max_{|z|=R} |F(z)| \geqq |F^{(\mu-1)}(0)| \geqq |a_{ht}| \frac{\delta_h^{\sigma-t}}{\sigma^{\varrho_h - t}(1+\Omega)^{\sigma-1}}.$$

Then let z_0 be a point such that $|z_0| = R$ and

$$\max_{|z|=R} |F(z)| = |F(z_0)|.$$

Observing that

$$\min_{\mu=1,2,\dots,\sigma} \frac{R^{\mu-1}}{(\mu-1)!} \geqq 1$$

and noting the definitions (4) we obtain the lemma.

LEMMA 3. *Let z_1 be any point such that $|z_1| \leqq 9R$. Then*

$$\left| \frac{F(z_1)}{F(z_0)} \right| \leqq (9R)^{\varrho-1} e^{9R\Omega} \sigma^{\varrho}(1+\Omega)^{\sigma-1} \delta^{-(\sigma-1)}.$$

PROOF. By Lemma 2 we have

$$|F(z_0)|\,|z_1^{t-1} e^{\omega_h z_1}| \geqq |a_{ht} z_1^{t-1} e^{\omega_h z_1}| \frac{\delta^{\sigma-1}}{\sigma^{\varrho-1}(1+\Omega)^{\sigma-1}}. \tag{6}$$

Summing the inequality (6) over $h=1, 2, \ldots, m$; $t=1, 2, \ldots, \varrho_h$ we obtain

$$\sigma|F(z_0)|(9R)^{\varrho-1} e^{9R\Omega} \geqq \sum_{h=1}^{m} \sum_{t=1}^{\varrho_h} |F(z_0)| \max_{h,t} |a_{ht} z_1^{t-1} e^{\omega_h z_1}| \geqq$$

$$\geqq \sum_{h=1}^{m} \sum_{t=1}^{\varrho_h} |a_{ht} z_1^{t-1} e^{\omega_h z_1}| \frac{\delta^{\sigma-1}}{\sigma^{\varrho-1}(1+\Omega)^{\sigma-1}} \geqq |F(z_1)| \frac{\delta^{\sigma-1}}{\sigma^{\varrho-1}(1+\Omega)^{\sigma-1}}$$

whence the lemma follows.

LEMMA 4. *Let N be the number of zeros (counted according to multiplicity) of $F(z)$ in the square of side $2R$, centre origin (i.e. with vertices at $z=\pm R \pm iR$). Then*

$$e^N \leqq (9R)^{\varrho-1} e^{9R\Omega} \sigma^{\varrho}(1+\Omega)^{\sigma-1} \delta^{-(\sigma-1)}.$$

PROOF. Since $|z_0|=R$, z_0 lies in or on the chosen square and a circle of radius $2R\sqrt{2}$ about z_0 covers the square. Hence by Jensen's inequality described above (3)

$$N \leqq \max_{|z-z_0| \leqq 2eR\sqrt{2}} \log \left| \frac{F(z)}{F(z_0)} \right|. \tag{7}$$

Then if z_1 is a point at which $|F(z)|$ attains its maximum on the disc $|z-z_0| \leqq 2eR\sqrt{2}$ we have

$$|z_1| \leqq |z_0| + 2eR\sqrt{2} \leqq R + 2eR\sqrt{2} \leqq 9R$$

whence since by (7)

$$e^N \leqq \left| \frac{F(z_1)}{F(z_0)} \right|$$

we may apply Lemma 3 to obtain the assertion.

LEMMA 5. *Let $K>0$ and let N' be the number of zeros (counted according to multiplicity) of $F(z)$ in the square of side $2R/K$, centre origin (i.e. with vertices at $z=\pm R/K \pm iR/K$). Then*

$$e^{N'} \leqq (9R)^{\varrho-1} e^{9R\Omega/K} \sigma^{\varrho}(K+\Omega)^{\sigma-1} \delta^{-(\sigma-1)}.$$

PROOF. Consider the function

$$F(z/K) = G(z) = \sum_{k=1}^{m} p_k(z/K) e^{z\omega_k/K}.$$

It is immediate that $G(z)$ has exactly N' zeros (counted according to multiplicity) in the square of side $2R$, centre origin (i.e. with vertices at $z=\pm R\pm iR$). Hence Lemma 4 applies to $G(z)$, and gives

$$e^{N'} \leqq (9R)^{\varrho-1} e^{9R(\Omega/K)} \sigma^{\varrho} (1+\Omega/K)^{\sigma-1} (\delta/K)^{-(\sigma-1)}$$

whence the assertion. (Note that this trick works because the upper bound in Lemma 4 does not depend on the coefficients of the polynomials $p_k(z)$, only roughly on their degrees).

LEMMA 6. *The quantity Ω in Lemma 5 may be replaced by $M/\sqrt{3}$ where*

(8) $$M = \max_{h,\, j=1,2,\dots,m} |\omega_h - \omega_j|.$$

PROOF: This result depends only on the well-known result that given arbitrarily many points $\omega_1, \dots, \omega_m$ in the plane such that (8) is the case, there exists a point α such that

$$\max_{h=1,2,\dots,m} |\alpha-\omega_h| \leqq \frac{M}{\sqrt{3}}.$$

Since the functions $F(z)$ and $e^{-\alpha z}F(z)$ have precisely the same zeros, we may without loss of generality apply our results to the function

$$e^{-\alpha z}F(z) = \sum_{k=1}^{m} p_k(z) e^{(\omega_k-\alpha)z}$$

to obtain the assertion of the lemma.

THEOREM. *Let $\omega_1, \omega_2, \dots, \omega_m$ be distinct complex numbers and let $p_1(z), p_2(z), \dots$ $\dots, p_m(z)$ be polynomials (not all trivial) of degree at most $\varrho_1-1, \varrho_2-1, \dots, \varrho_m-1$ respectively. Further let*

$$\varrho_1+\varrho_2+\dots+\varrho_m = \sigma, \quad \max_{k=1,2,\dots,m} \varrho_k = \varrho,$$

$$M = \max_{h,\, k=1,2,\dots,m} |\omega_h-\omega_k|,$$

$$\delta = \min_{\substack{h,\, j=1,2,\dots,m\,(h\neq j)\\ t=1,2,\dots,\varrho_h}} |\omega_h-\omega_j|^{\frac{\sigma-t}{\sigma-1}} \left(= \min_{\substack{h,\, j=1,2,\dots,m\\ h\neq j}} |\omega_h-\omega_j|, \quad \text{if} \quad \delta<1\right).$$

Then the number of zeros (counted according to multiplicity) of the function

$$F(z) = \sum_{k=1}^{m} p_k(z) e^{\omega_k z}$$

in any square of side S in the complex plane, does not exceed

$$\frac{3\sqrt{3}}{2} SM + (\sigma-1)\log\left(\frac{\sigma}{\delta S e} + \frac{M}{\delta\sqrt{3}}\right) + (2\varrho-1)\log\sigma + (\varrho-1)\log\frac{18}{e}.$$

PROOF. In lemma 5 (as amended by lemma 6) put $S=\dfrac{2R}{K}$, thus choosing $K=2R/S$. Further put $R=\dfrac{2\sigma}{e}$ (in accordance with lemma 2; but this choice is not optimal — we could in lemma 2 in any case have taken $R = \{(\sigma-1)!\}^{1/\sigma-1})$. Then if N is the number of zeros (counted according to multiplicity) of $F(z)$ in the square of side S, centre origin (i.e. with vertices at $z=\pm S/2 \pm iS/2$) we have

$$(9) \qquad N \leqq \frac{3\sqrt{3}}{2} SM + (\sigma-1)\log\left(\frac{\sigma}{\delta Se}+\frac{M}{\delta\sqrt{3}}\right) + (2\varrho-1)\log\sigma + (\varrho-1)\log\frac{18}{e}.$$

Thus we need only show that our result applied to *any* square of side S in the complex plane.

Since the result (9) depends only on the absolute values of the mutual differences of ω's and (loosely) on the degrees of the polynomials $p(z)$ it is clear that the right hand side of (9) is also an upper bound for the number of zeros (counted according to multiplicity) of the function

$$(10) \quad F(\theta w+\beta) = H(w) = \sum_{k=1}^{m} p_k(\theta w+\beta)e^{\omega_k(\theta w+\beta)} = \sum_{k=1}^{m} \{e^{\omega_k\beta}p_k(\theta w+\beta)\}e^{(\omega_k\theta)w}$$

in the square of side S, centre origin (i.e. with vertices $w=\pm S/2\pm iS/2$) where $|\theta|=1$ and β is any complex number.

Then by writing $z=\theta w+\beta$ in (10) we see that the right hand side of (9) is an upper bound for the number of zeros (counted according to multiplicity) of the function

$$F(z) = \sum_{k=1}^{m} p_k(z)e^{\omega_k z}$$

in the square of side S, centre β (with vertices $z=(\pm S/2\pm iS/2)\theta+\beta$) and we have the assertion of the theorem.

COROLLARY. *If $a_1, a_2, \ldots, a_m$ be complex constants (not all zero) then the number of zeros (counted according to multiplicity) of the function*

$$\sum_{k=1}^{m} a_k e^{\omega_k z}$$

in any square of side S in the complex plane, does not exceed

$$\frac{3\sqrt{3}}{2} SM + (m-1)\log\left(\frac{4m}{\delta Se}+\frac{M}{\delta\sqrt{3}}\right) + \log m$$

where

$$M = \max_{h,k=1,2,\ldots,m} |\omega_h-\omega_k|, \quad \delta = \min_{\substack{h,j=1,2,\ldots,m\\ h\neq j}} |\omega_h-\omega_j|.$$

We mention the particular result of the corollary for comparison with the corresponding result of Turán mentioned in [1], where the upper bound is

$$5SM + m\log\left(2+\frac{2m^3}{\delta S}\right) + \log 2m.$$

We need hardly remark on the startling aspects of the form of the result. We should observe however that our method of proof will provide similar results for more general functions for which we have an analogue of lemma 1. Indeed only in the final step when we generalized the result to any square (and in lemma 6) did we otherwise make use of properties specific to the exponential function.

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ENUMERATION OF TRANSITIVE, STEP-TYPE RELATIONS

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1. Introduction

Let $S=\{s_1, s_2, \ldots, s_n\}$ be a finite set and let α be a relation on S. It will be convenient to identify α with its adjacency matrix $A=(a_{ij})$, where $a_{ij}=1$ if $(s_i, s_j)\in\alpha$ and $a_{ij}=0$ if $(s_i, s_j)\notin\alpha$.

DEFINITION 1. A relation α is called *step-type* iff for each i, j

$$a_{ij}=1 \quad \text{implies} \quad \begin{array}{ll} a_{pj}=1 & \text{for each} \quad p>i, \\ a_{iq}=1 & \text{for each} \quad q<j. \end{array}$$

By noting the one-to-one correspondence between step-type relations on S and their row sum vectors (which are isotone), it is readily seen that the number of such relations is $\binom{2n}{n}$. An *irreflexive* ($a_{ii}=0$ for each i) step-type relation is necessarily transitive; in fact, it corresponds to a decision pattern [5], as will be shown in Section 3. An arbitrary step-type relation, however, need not be transitive; for example

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

In this paper we construct a generating function which enumerates all transitive, step-type relations. Related topological questions are considered in the final section.

2. Reflexive relations

If A is a reflexive, step-type relation, then $a_{ij}=1$ for all $i\geqq j$.

LEMMA 1. *If A is a reflexive, transitive, step-type relation, then there is an integer v, $1\leqq v\leqq n$, such that $a_{ij}=0$ for each (i,j) satisfying $i\leqq v$ and $j>v$.*

PROOF. The case $v=n$ corresponds uniquely to the *unit* relation $J = S\times S$. Assume that $A\neq J$ and note that for some $v<n$, $a_{v,v+1}=0$. Since A is step-type, $a_{ij}=0$ for each (i,j) such that $i\leqq v$ and $j\geqq v+1$.

THEOREM 1. *If $g_1(n)$ denotes the number of reflexive, transitive, step-type relations on S, then for $n\geqq 2$*

$$g_1(n)=2^{n-1}.$$

PROOF. Consider the supra-diagonal $(n-1)$-tuple $(a_{12}, a_{23}, \ldots, a_{n-1,n})$. Since A is of step-type there is a rectangular block of 0's in A above and to the right of each 0 entry in the $(n-1)$-tuple. By transitivity, each of the remaining entries in A is 1. Thus there is a one-to-one correspondence between these $(n-1)$-tuples and the required relations.

Since $g_1(1)=1$, it follows that the generating function is

$$G_1(x) = \sum_{n=1} g_1(n)x^n = \sum_{n=1} 2^{n-1}x^n,$$

$$G_1(x) = \frac{x}{1-2x}. \tag{1}$$

As in most problems of this sort, we are interested also in equivalence classes of relations, where A and B are considered *equivalent* iff there is a permutation matrix P with transpose P' such that [1]

$$B=P'AP.$$

Step-type relations are canonical representatives of their equivalence classes, hence (1) enumerates a subset of the non-homeomorphic topologies on S [4]. The interesting question is one of identification, and Theorem 2, below, provides an easily applied test.

DEFINITION 2. A relation with column sum vector $(c_1, c_2, \ldots, c_n)$ is called *excessive* iff for each i, $1 \leqq i \leqq n$, there are at least i coordinates c_j such that

$$c_j \geqq n-i+1.$$

Any relation is equivalent to one having antitone column sum vector, and the property of being excessive is an invariant. If $(c_1', c_2', \ldots, c_n')$ is the column sum vector in antitone form, then the condition in Definition 2 is equivalent to

$$c_j' \geqq n-j+1 \quad \text{for each} \quad j.$$

THEOREM 2. *Among the transitive relations A on S the following statements are equivalent:*

(1) *A is excessive,*

(2) *A is reflexive and is equivalent to a step-type relation.*

PROOF. That (2) $\rightarrow$ (1) follows from the remarks preceding Theorem 2. The proof that (1) $\rightarrow$ (2) is by mathematical induction on the columns of A. We may assume that the column sum vector $(c_1, c_2, \ldots, c_n)$ of A is antitone, hence $c_j \geqq n-j+1$ for each j. Thus $c_1=n$, and $a_{i1}=1$ for each i. Now assume that for each $j \leqq k$ $a_{jj}=1$ and the step-type condition of Definition 1 holds. Note that $c_{k+1} \geqq n-k$. If $a_{i,k+1}=0$ for each $i \leqq k$, then $a_{i,k+1}=1$ for each $i \geqq k+1$. If $a_{i,k+1}=1$ for some $i \leqq k$, then for each $p \geqq i$ $a_{pi}=1$, and by transitivity $a_{p,k+1}=1$. The column sum vector is antitone, hence in either case both required conditions hold in the first $k+1$ columns of A, and the theorem is proved.

3. Decision patterns

The major task in this section is the enumeration of irreflexive, step-type relations on S, and in the following section this result is combined with (1) to solve the general problem.

Observe first that there is a one-to-one correspondence between the set, $\mathcal{I}_n$, of irreflexive step-type relations on S (for $n\geqq 2$) and the set of $(n-1)$-tuples $(c_1, c_2, \ldots$ $\ldots, c_{n-1})$ satisfying the properties

(a) $0\leqq c_j\leqq n-1$ for each j,

(b) the coordinates are antitone,

(c) $c_j \leqq n-j$ for each j.

Note that each such $(n-1)$-tuple represents the first $n-1$ coordinates of the column sum vector

$$(c_1, c_2, \ldots, c_{n-1}, 0)$$

associated with some member of $\mathcal{I}_n$. It will be more convenient to modify the problem in a way suggested by replacing $(c_1, c_2, \ldots, c_{n-1})$ with $(n-c_1, n-c_2, \ldots, n-c_{n-1})$.

Let n and k denote positive integers and let $g(n, k)$ denote the number of k-tuples

$$(d_1, d_2, \ldots, d_k)$$

subject to the conditions

(I) $d_i\in\{1, 2, \ldots, n\}$ for each i,

(II) coordinates are isotone,

(III) $d_i\geqq i$ for each i.

In a typical k-tuple subject to (I), (II) and (III), d_k can be replaced by n or by one of the integers $n-1, \ldots, k$. It is easy to see then that the following recursion formulas hold

$$g(n, k) = g(n, k-1)+g(n-1, k), \qquad 2\leqq k\leqq n-1,$$
$$g(n, n) = g(n, n-1),$$

together with the boundary conditions

$$g(n, 1)=n$$
$$g(n, k) \text{ is undefined for } k>n.$$

This problem is equivalent to the *ballot number* problem (see, for example, [3; p. 130 and p. 152]). Thus

$$g(n, k) = \binom{n+k}{k}-\binom{n+k}{k-1}=\frac{n+1-k}{n+1}\binom{n+k}{k}.$$

If $g_0(n)$ denotes the number of irreflexive, step-type relations on S, then

$$g_0(n) = g(n, n-1) = g(n, n) = \frac{1}{n+1}\binom{2n}{n}, \tag{2}$$

and the generating function is [3; p. 101]

$$G_0(x) = \sum_{n=0}\frac{1}{n+1}\binom{2n}{n}x^n = \frac{1-\sqrt{1-4x}}{2x}, \tag{3}$$

with $g_0(0)=1$ by convention.

In 1957, WINE and FREUND published a short study of decision patterns, in which the enumerating formula was shown to be (2) above [5]. Although not noted in that paper, decision patterns can be taken to represent a certain subset of the transitive relations. Because of the interest in the problem of enumerating transitive relations, we digress briefly to translate the concept of decision pattern into the more familiar language of relation matrices, and to outline a proof that the family of irreflexive, step-type relations can be interpreted as the set of decision patterns.

Suppose now that the elements of S are real numbers, and that there is a decision procedure which for each ordered pair (s_i, s_j) asserts one and only one of the following:

s_i is significantly greater than s_j	$(s_i \gg s_j)$,
s_i is significantly less than s_j	$(s_i \ll s_j)$,
s_i, s_j are not significantly different	$(s_i \doteq s_j)$.

To guard against "circularity" we impose the conditions

(i) $s_i \gg s_j$ iff $s_j \ll s_i$,

(ii) $s_i \gg s_j \doteq s_k \gg s_m \to s_i \gg s_m$,

(iii) $s_i \gg s_j \gg s_k \doteq s_m \to s_i \gg s_m$.

DEFINITION 3. Let A be a relation on S such that $a_{ij}=1$ iff $s_i \gg s_j$. Then A is called a *decision relation*, and the equivalence class containing A is called a *decision pattern.*

We note that in a decision relation A, $a_{ij}=a_{ji}=0$ corresponds to the case $s_i \doteq s_j$. Proofs of the following lemmas are straightforward.

LEMMA 2. *If A is a decision relation, then*

(a) *A is anti-symmetric,*

(b) *A is irreflexive,*

(c) *A is transitive,*

(d) $s_m \doteq s_i \gg s_j \gg s_k \to s_m \gg s_k$.

LEMMA 3. *Any irreflexive, step-type relation is a decision relation.*

LEMMA 4. *Any decision relation is equivalent to an irreflexive, step-type relation.*

4. General case

Let $f(n)$ denote the number of transitive, step-type relations on S, and let the desired generating function be

$$F(x) = \sum_{n=0} f(n)x^n,$$

where $f(0)=1$ by convention. Let

$g_0(n)$ be the number of irreflexive, step-type relations on S,

$g_1(n)$ be the number of reflexive, transitive, step-type relations on S,

$h_0(n)$ be the number of transitive, step-type relations on S with leading diagonal entry 0,

$h_1(n)$ be the number of transitive, step-type relations on S with leading diagonal entry 1.

Let the respective generating functions be

$$G_0(x) = \sum_{n=0} g_0(n)x^n, \qquad g_0(0) = 1,$$

$$G_1(x) = \sum_{n=0} g_1(n)x^n, \qquad g_1(0) = 0,$$

$$H_0(x) = \sum_{n=0} h_0(n)x^n, \qquad h_0(0) = 1,$$

$$H_1(x) = \sum_{n=0} h_1(n)x^n, \qquad h_1(0) = 1.$$

The following recursion formulas are now (independently) evident:

$$f(n) = h_0(n)+h_1(n) \quad \text{if} \quad n \geqq 1, \quad f(0) = 1$$

$$f(n) = g_0(0)h_1(n)+g_0(1)h_1(n-1)+\dots+g_0(n-1)h_1(1)+g_0(n)h_1(0), \qquad n \geqq 0$$

$$= \sum_{k=0}^{n} g_0(k)h_1(n-k),$$

$$h_1(n) = g_1(1)h_0(n-1)+g_1(2)h_0(n-2)+\dots+g_1(n-1)h_0(1)+g_1(n)h_0(0), \qquad n \geqq 1$$

$$= \sum_{k=0}^{n} g_1(k)h_0(n-k), \qquad n \geqq 0.$$

Hence

(4) $$F(x) = H_0(x)+H_1(x)-1,$$

(5) $$F(x) = G_0(x)H_1(x),$$

(6) $$H_1(x) = 1+G_1(x)H_0(x).$$

For a recent discussion of operations on formal power series, see [2]. From (4), (5) and (6),

(7) $$F(x) = \frac{G_0(x)\big(1+G_1(x)\big)}{1+G_1(x)-G_0(x)G_1(x)}.$$

From (1) and (3),

$$F(x) = \frac{\dfrac{1-\sqrt{1-4x}}{2x}\left(1+\dfrac{x}{1-2x}\right)}{1+\dfrac{x}{1-2x}-\left(\dfrac{1-\sqrt{1-4x}}{2x}\right)\left(\dfrac{x}{1-2x}\right)},$$

(8) $$F(x) = \frac{(1-4x+3x^2)-(1-2x+x^2)\sqrt{1-4x}}{2x^3}.$$

Recalling that
$$\sqrt{1-4x} = 1 - \sum_{n=1}^{\infty} \binom{2}{n} \binom{2n-2}{n-1} x^n,$$
the first few terms of the enumerating series are

(9) $$F(x) = 1+2x+6x^2+19x^3+62x^4+207x^5+704x^6+2431x^7+\cdots.$$

5. Remarks

A decision relation α is irreflexive, but the inclusion of (s_i, s_i) in α does not adversely affect its transitivity. If $\mathscr{I}_n$ again denotes the set of irreflexive, step-type relations on S, then for each member of $\mathscr{I}_n$ its row sum vector $(r_1, r_2, \dots, r_n)$ is isotone and its column sum vector $(c_1, c_2, \dots, c_n)$ is antitone. Let $V=((r_1, c_1), (r_2, c_2), \dots \dots, (r_n, c_n))$ and let $\mathscr{V}$ denote the set of all such n-tuples of ordered pairs. Then
$$|\mathscr{V}| = |\mathscr{I}_n| = \frac{1}{n+1}\binom{2n}{n}.$$
For each $V \in \mathscr{V}$, suppose that there are v different ordered pairs in V which appear in multiplicities $\mu_1, \mu_2, \dots, \mu_v$. (Note that $1 \leqq v \leqq n$, $1 \leqq \mu_i \leqq n$, and $\Sigma \mu_i = n$.) Then the number of non-equivalent transitive relations A on S which, except for reflexive pairs (s_i, s_i), are decision relations is

(10) $$\sum_{V \in \mathscr{V}} \prod_{i=1}^{v} (1+\mu_i).$$

As a final comment, we note that the topologies on a finite set S can be identified with the reflexive, transitive relations on S, the T_0 topologies corresponding to those which are also antisymmetric.

DEFINITION 4. A topology $\mathscr{T}$ on S is called *nested* iff the members of its minimal base [4] are linearly ordered under inclusion.

It follows immediately that a nested topology is identical with its minimal base $\mathscr{B}$, hence $|\mathscr{T}|=|\mathscr{B}|$. There is a one-to-one correspondence between the non-homeomorphic nested topologies and the reflexive, transitive, step-type relations, thus (1) is also the generating function for non-homeomorphic nested topologies. Exactly one of these is T_0, that for which $|\mathscr{T}| = n+1$.

There does not appear to be a useful characterization of topologies related to decision patterns. If we agree that a topology with matrix T is of *decision type* if $T-I$ is a decision relation, then the number of non-homeomorphic topologies on S which are either nested or of decision type is

(11) $$\frac{1}{n+1}\binom{2n}{n} + 2^{n-1} - 1.$$

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CONCERNING NIL GROUPS FOR NEAR-RINGS

By

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The question concerning "trivial" multiplications for near-rings is raised in [2] by BERMAN and SILVERMAN and interpretations of this question are answered both by CLAY in [3] and MALONE in [6].

For rings, in [5] nil abelian groups are defined to be abelian groups over which the only associative distributive multiplications definable are the "trivial" zero multiplications. Sums of the quasi-cyclic groups $(Z(p^\infty)$, p a prime$)$ are shown to be nil abelian groups in [5] by using only the two distributive laws. If, for sums of quasi-cyclic groups, one only assumes one distributive law and the associative law for multiplication to yield near-rings, the number of examples is greatly increased; some of these examples might properly be called "trivial" while others are of questionable classification as to triviality.

In this paper we wish to give many examples, including all near-rings on the quasi-cyclic groups, and we wish to give compatibility equations for defining new near-rings on group direct sums of near-rings.

We also hope to indicate the apparent lack of correspondence for near-rings to the concept of nil groups for rings.

1. Preliminaries

A *near-ring* is a triple $(N, +, \cdot)$ where $(N, +)$ is a group, $(N, \cdot)$ a semi-group, and $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$, for all $a, b, c \in N$. A normal subgroup $(I, +)$ of $(N, +)$ is a left ideal if $NI \subset I$ and is an *ideal* if $(I, +, \cdot)$ is the kernel of a near-ring homomorphism, [4].

Throughout let $(G, +)$ denote an arbitrary group and $E_G = E = \mathrm{Hom}\,((G, +), (G, +))$ with Θ and I the zero and identity homomorphisms, respectively; if X and Y are non-void sets, let Y^X denote the set of all functions with domain X and range contained in Y, [3].

We state two theorems from [3]:

1. 1. THEOREM. *If $f \in E^G$, then f defines a multiplication $*$ on $(G, +)$ by $a*b = f_a(b)$ where f defines the mapping $a \to f_a$. Conversely, if $*$ is a multiplication on G, then $*$ defines an element $f \in E^G$ by $a \to f_a$ where $f_a(b) = a*b$. These correspondences between E^G and multiplications on $(G, +)$ are inverse one-to-one correspondences. That is, if $f \in E^G$ defines the multiplication $*$, and $*$ defines $g \in E^G$, then $f=g$; and if $*$ defines $f \in E^G$ and f defines $*'$ in G, then $* = *'$.*

1. 8. THEOREM. *Let $f \in E^G$ and let $*$ be the corresponding multiplication. Let Θ be the zero in E and let I be the identity in E. If $f(0)=\Theta$ and $f(x) \in \{\Theta, I\}$ for all $x \in G$, then $*$ is associative. If $f(0)=I$, then $*$ is associative if and only if $f(x)=I$ for all $x \in G$.*

For any group G, the multiplications provided by 1. 8. Theorem should be among those considered " trivial" for near-rings, [6].

2. All near-rings on $Z(p^\infty)$

First we make some observations concerning the general theory.

REMARK 2. 1. Let G be a group and let $f \in E^G$ define a near-ring on G as in 1. 1 Theorem. For each $\alpha \in f(G)$ define $G_\alpha = \{a \in G : f(a) = \alpha\}$. For $a \in G_\alpha$ define $a^2 = \alpha(a)$, $a^i = \alpha^{i-1}(a)$, $i > 1$. Note that if $a \in G$, then $a^i \in G_{\alpha^i}$, for $i \geqq 1$, since $a*(a*(\dots \dots(a*b)\dots) = \alpha^i(b) = a^i * b$, for all $b \in G$. Hence, $a^{n+1}=0$ implies that $f(0) = f(a^{n+1}) = \alpha^{n+1}$ and $a^{n+1}=a$ implies that $\alpha = f(a) = f(a^{n+1}) = \alpha^{n+1}$. For the latter, α an automorphism implies that $\alpha^n = I$. From $a*(b*c) = (a*b)*c$, we have that if $a \in G_\alpha$, $b \in G_\beta$, then $\alpha(b) \in G_{\alpha \circ \beta}$ and if, in addition, β is an automorphism and $\alpha(b) = b$, then $\alpha = I$, since $\alpha \circ \beta = f(\alpha(b)) = f(b) = \beta$.

Next we wish to motivate and prove theorem 2. 2.

Let $G = Z(p^\infty) = \bigcup_{n=1}^{\infty} H_n$ where $H_n = \left\langle \frac{1}{p^n} \right\rangle$. Suppose $f \in E^G$ defines a near-ring on G. Since $0*(0*a) = (0*0)*a = 0*a$ for all $a \in G$, $f(0)$ must act as the identity on the range of $f(0)$. But every non-zero endomorphism of $Z(p^\infty)$ is onto $Z(p^\infty)$ so that $f(0)=\Theta$ or $f(0)=I$. If $f(0)=I$, $f(x)=I$ for all $x \in G$, by 1. 8 theorem. So henceforth in this section we suppose that $f(0)=\Theta$.

Suppose for some $a \in G$, $f(a)=\alpha$, α an endomorphism of G with $\ker(\alpha) \neq \{0\}$. Then, $\bigcup_{i=1}^{\infty} \ker(\alpha^i) = G$, so that for some $n \geqq 1$, $\alpha^n(a) = 0$. But then $\Theta = f(0) = f(a^{n+1}) = \alpha^{n+1}$ and $\alpha = \Theta$, again since every non-zero endomorphism of $Z(p^\infty)$ is onto $Z(p^\infty)$. Thus the only non-zero endomorphisms that may occur in $f(G)$ are automorphisms.

Suppose $f(H_i) = \{\Theta\}$ for $1 \leqq i < n$ but $f(H_n) \neq \{\Theta\}$. Let $f\left(\frac{i}{p^n}\right) = \alpha_i$ where if p divides i, $\alpha_i = \Theta$, and, for some fixed m, $(m, p) = 1$, $\alpha_m \neq \Theta$. Since H_n is a finite group, there must be a $k \geqq 1$ such that $\alpha_m^k|_{H_n} = I_{H_n}$. Hence, for such a k we must have $\alpha_m = f\left(\frac{m}{p^n}\right) = f\left(\left(\frac{m}{p^n}\right)^{k+1}\right) = \alpha_m^{k+1}$ and $\alpha_m^k = I$. Thus, there is a t_0 $\left(\text{e.g. } \frac{t_0}{p^n} = \left(\frac{m}{p^n}\right)^k\right)$ with $f\left(\frac{t_0}{p^n}\right) = I$. For any $\alpha_s \neq \Theta$, $\alpha_s^{-1} = \alpha_s^{k_s}$, for some $0 < k_s$, k_s minimal.

Now, if $f(a) = \beta \neq \Theta$, then for some s, $\beta\left(\frac{s}{p^n}\right) = \frac{t_0}{p^n}$ and, since $a*\left(\frac{s}{p^n}*c\right) = f\left(\beta\left(\frac{s}{p^n}\right)\right)(c) = c = \beta \circ \alpha_s(c)$, for all c, we have that $\beta \circ \alpha_s = I$ or that $\beta = \alpha_s^{k_s}$:

(1) $f(a)=\beta$ and $\beta\left(\frac{s}{p^n}\right)=\frac{t_0}{p^n}$ if and only if $\beta=\alpha_s^{k_s}$. This gives that *the* $\Theta\neq\alpha_s$ *determined by this fixed* H_n *give all of* $f(G)$ *and form a group of automorphisms under composition.*

Also, by Remark 2.1 and the above results all α_s have the property that if $\alpha_s^l(c)=c$ and $f(c)\neq\Theta$, then $\alpha_s^l=I$.

Now we determine $f(H_{n+k})$ as follows:

Assign $f\left(\frac{1}{p^{n+k}}\right)=\alpha_{t_1}\in f(H_n)$ arbitrarily and for $\alpha_s\in f(H_n)$, $f\left(\alpha_s\left(\frac{1}{p^{n+k}}\right)\right)=$ $=\alpha_s\circ\alpha_{t_1}$. If $a_1\in H_{n+k}-\left(H_{n+k-1}\cup\left\{\alpha_s\left(\frac{1}{p^{n+k}}\right):\alpha_s\in f(H_n)\right\}\right)$, assign $f(a_1)=\alpha_{t_{a_1}}\in f(H_n)$ arbitrarily and $f(\alpha_s(a_1))=\alpha_s\circ\alpha_{t_{a_1}}$. Having assigned $\frac{1}{p^{n+k}}$, $a_1, a_2, \dots, a_{r-1}$, if

$$a_r\in H_{n+k}-\left(H_{n+k-1}\cup\left\{\alpha_s\left(\frac{1}{p^{n+k}}\right),\quad a_s(a_i):1\leqq i<r,\quad \alpha_s\in f(H_n)\right\}\right),$$

assign $f(a_r)=\alpha_{t_{a_r}}\in f(H_n)$ arbitrarily and $f(\alpha_s(a_r))=\alpha_s\circ\alpha_{t_{a_r}}$. Then f defines a near-ring on $Z(p^\infty)$.

We summarize:

THEOREM 2. 2. *Let* $f\in E^G$ *define a non-zero near-ring on* $G=Z(p^\infty)$. *There are two cases:*

(i) *If* $f(0)=I$, *then* $f(x)=I$ *for all* $x\in G$.

(ii) *Suppose* $f(0)\neq I$. *Then* $f(0)=\Theta$ *and there is an* n *such that for* $1\leqq i<n$, $f(H_i)=\{\Theta\}$, $f(H_n)=f(G)$ *where* $(f(G)-\{\Theta\})$ *is a group of automorphisms under composition. If for* $\frac{i}{p^n}\in H_n$ *we write* $f\left(\frac{i}{p^n}\right)=\alpha_i$, *we have for* $(i,p)\geqq p$, $\alpha_i=\Theta$, *for* $(i,p)=1$, $\alpha_i=\Theta$ *or* α_i *is an automorphism. For some* m *with* $(m,p)=1$, α_m *is an automorphism and for any automorphism* $\alpha_s\in f(G)$, *if* $\alpha_s^l(c)=c$ *where* $f(c)\neq\Theta$, *then* $\alpha_s^l=I$. *Furthermore, the assignment of* $f(a)=\alpha_s$ *for any* $a\in H_{n+k}$, $k\geqq 1$, *is arbitrary except for the stipulation that* $f(a)=\alpha_s$ *gives that* $f(\alpha_t(a))=\alpha_t\circ\alpha_s$ *for all* $\alpha_t\in f(G)$.

EXAMPLE 2. 3. $Z(5^\infty)$. Let α be the automorphism associated with the p-adic integer $u=2+5+2\cdot 5^2+5^3+2\cdot 5^4+5^5+\cdots$, [5, p. 211]. Then $u^2=-1$ and $u^4=1$, [1, p. 70]. Let f define a near-ring on $Z(p^\infty)$ by

$$f\left(\frac{1}{5}\right)=\alpha,\quad f\left(\alpha\left(\frac{1}{5}\right)\right)=f\left(\frac{2}{5}\right)=\alpha^2,\quad f\left(\alpha^2\left(\frac{1}{5}\right)\right)=f\left(\frac{4}{5}\right)=\alpha^3,$$

$$f\left(\alpha^3\left(\frac{1}{5}\right)\right)=f\left(\frac{3}{5}\right)=\alpha^4=I,$$

and

$$f\left(\alpha^4\left(\frac{1}{5}\right)\right)=f\left(\frac{1}{5}\right)=\alpha^5=\alpha;$$

for all other $b\in Z(p^\infty)$, $f(b)=\Theta$.

3. The existence of other „non-trivial" near-rings

EXAMPLE 3. 1. If G is a non-divisible torsion abelian group, then non-trivial rings may be defined on G, [5].

EXAMPLE 3. 2. Let G be any group with an automorphism α of G of prime order p. Take k maximal such that for some $a \in G$, and $1 \leqq i \leqq k$, $\alpha^i(a) \neq a$. Let $f \in E^G$ be defined by $f(0) = \Theta$, $f(a^i) = \alpha^{ik}$, and $f(b) = \Theta$ for $b \notin \{a, a^2, a^3, \ldots\}$, where a satisfies the condition on $\alpha, \alpha^2, \ldots, \alpha^k$. Then f defines a near-ring on G:

$$a^i * (a^j * x) = \alpha^{k(i+j)}(x)$$

and

$$(a^i * a^j) * x = \alpha^{ki}\left(\alpha^{k(j-1)}(a)\right) * (x) = \alpha^{k(i+j-1)}(a) * x = \alpha^{k(i+j)}(x).$$

(G satisfies the conditions above, e.g., if there is a non-central $a \in G$ of finite order.)

EXAMPLE 3. 3. Let G be any group with an endomorphism α such that for some $a \in G$, $\alpha^i(a) \neq a$ for $1 \leqq i$. Let $f \in E^G$ be defined by $f(0) = \Theta$, $f(a^i) = \alpha^i$, $1 \leqq i$, and $f(b) = \Theta$ for $b \notin \{a, a^2, a^3, \ldots\}$. Again, f defines a near-ring on G.

(G satisfies the above conditions, e.g., if G is a free group.)

4. „Trivial" near-rings

Throughout, let $f \in E^G$ as in 1. 1 Theorem.

First we list four examples to show the manner in which some "trivial" near-rings may be constructed.

EXAMPLE 4. 1. $f(x) = I$, for all $x \in G$, [3].

EXAMPLE 4. 2. $G = Z \cup C$ where $0 \in Z$ and $Z \cap C = \varnothing$.

$$f(z) = \Theta, \quad \text{for all} \quad z \in Z,$$

and

$$f(c) = I, \quad \text{for all} \quad c \in C, \quad [3].$$

EXAMPLE 4. 3. G an abelian group with $G = Z \cup C \cup D$ where $0 \in Z$ and these sets are pairwise disjoint with $x \in C$ implying that $-x \in D$ and $x \in D$ implying that $-x \in C$.

$$f(z) = \Theta, \quad \text{for all} \quad z \in Z,$$

$$f(c) = I \quad \text{for all} \quad c \in C,$$

and

$$f(d) = -I, \quad \text{for all} \quad d \in D.$$

That f defines an associative multiplication is straight forward.

For all following comments concerning $A \oplus B$, the notation $(f+g)(a+b)$ will be used instead of $f(a)\,\mathrm{op}_A + g(b)\,\mathrm{op}_B$.

EXAMPLE 4. 4. $G = A \oplus B$, $\Theta = \Theta_A + \Theta_B$, and $I = I_A + I_B$. For all $a \in A$, let $B = Z_a \cup C_a$ with $f_a \in E(B)^B$ defined as in Example 4. 2. Define $f \in E^G$:

$$f(a+b) = I_A + f_a(b) \quad \text{with} \quad f_a(0) = \Theta_B \quad \text{for all} \quad a \in A, \quad b \in B.$$

For this example we have

$$(a_1+b_1) * [(a_2+b_2) * (a_3+b_3)] = a_3 + f_{a_1}(b_1)[f_{a_2}(b_2)[b_3]] \tag{2}$$

and

$$[(a_1+b_1) * (a_2+b_2)] * (a_3+b_3) = a_3 + f_{a_2}(f_{a_1}(b_1)[b_2])[b_3]. \tag{3}$$

Associativity follows from (2) and (3) using the cases $f_{a_1}(b_1) = \Theta_B$ and $f_{a_1}(b_1) = I_B$.

B is an ideal in G under any assignment above. A is an ideal in G if and only if $f(a+b) = f(b)$, for all $a \in A$, $b \in B$.

(Obviously in this example we could interchange the roles of A and B.)

REMARK 4. 5. We could give further examples for near-rings on $A \oplus B$ using various combinations of Examples 4. 1, 4. 2, 4. 3 applied to A and to B, as was done for Example 4. 4; and in so doing list all near-rings definable on $A \oplus B$ in terms of $f \in E^G$ with $f(G) \subseteq \{\Theta, I, -I, I_A + \Theta_B, -I_A + \Theta_B, \Theta_A + I_B, \Theta_A - I_B\}$. These near-rings should also be included in the list of "trivial" near-rings. Instead we give

LEMMA 4. 6. *Let $G = A \oplus B$ and, for each $a \in A$, $f_a \in E(B)^B$, and, for each $b \in B$, $f_b \in E(A)^A$ define near-rings on B and A, respectively. Define $f \in E^G$: $f(a+b) = f_b(a) + f_a(b)$.*

(i) *Suppose $f_b(a) = I_A$ for all $a \in A$, $b \in B$. Then f defines a near-ring on G if and only if*

$$\lambda \circ f_a(b) = f_a(\lambda(b)), \quad \textit{for all} \quad \lambda \in \bigcup_{a \in A} f_a(B). \tag{4}$$

(ii) *Suppose $f_b(a) = f_0(a)$, $f_a(b) = f_0(b)$, $f_a(0) = \Theta_B$, and $f_b(0) = \Theta_A$ for all $a \in A$, $b \in B$. Then f defines a near-ring on G if and only if*

$$\lambda \circ f_0(a) = f_0(\lambda(b)), \quad \textit{for all} \quad \lambda \in f_0(A), \tag{5}$$

and

$$\lambda \circ f_0(b) = f_0(\lambda(b)), \quad \textit{for all} \quad \lambda \in f_0(B). \tag{6}$$

(iii) *Suppose $f_0(b) = \Theta_B$, $f_b(0) = \Theta_A$, $f_b(a) = f_0(a)$, and $f_a(0) = \Theta_B$, for all $a \in A$ and $b \in B$. Then f defines a near-ring on G if and only if*

$$\lambda \circ f_0(a) = f_0(\lambda(a)), \quad \textit{for all} \quad \lambda \in f_0(A), \tag{7}$$

and

$$f_{a_1}(b_1) \circ f_{a_2}(b_2) = f_{f_0(a_1)(a_2)}(f_{a_1}(b_1)(b_2)), \quad \textit{for all} \quad a_1, a_2 \in A, b_1, b_2, \in B. \tag{8}$$

PROOF. The proof follows from cases using

$$(a_1+b_1) * [(a_2+b_2) * (a_3+b_3)] = f_{b_1}(a_1) \circ f_{b_2}(a_2)[a_3] + f_{a_1}(b_1) \circ f_{a_2}(b_2)[b_3] \tag{9}$$

and

$$\begin{aligned} &[(a_1+b_1) * (a_2+b_2)] * (a_3+b_3) = \\ &= f_{f_{a_1}(b_1)(b_2)}(f_{b_1}(a_1)(a_2))[a_3] + f_{f_{b_1}(a_1)(a_2)}(f_{a_1}(b_1)(b_2))[b_3]. \end{aligned} \tag{10}$$

Certainly the results for direct sums of two groups may be extended to direct sums of any number of groups.

From the above results one sees the large number of near-rings possible for various groups over which only trivial rings are definable. However, it may be possible that there exist groups, e.g. torsion-free groups, mixed abelian groups, and non-abelian groups not satisfying the conditions of Examples 3.2 or 3.3, for which the only multiplications are "trivial" in the sense of Section 4.

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ZWEI EXTREMALAUFGABEN FÜR KONVEXE BEREICHE

Von

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Unter einem m-Eck verstehen wir ein ebenes konvexes Polygon mit höchstens m Ecken. E. SAS [4] und L. FEJES TÓTH [2, S. 36—38] haben den folgenden Satz bewiesen: *Ist K ein ebener konvexer Bereich vom Flächeninhalt $F(K)$ und bezeichnet $F_m(K)$ (für $m\in\{3, 4, \dots\}$) das Maximum der Flächeninhalte aller dem Bereich einbeschriebenen m-Ecke, so gilt*

$$F_m(K) \geqq \frac{m}{2\pi}\sin\frac{2\pi}{m}F(K);$$

Gleichheit besteht hier genau dann, wenn K eine Ellipse ist. Der Beweis der Ungleichung rührt von Sas, der Unitätsbeweis von FEJES TÓTH her. FEJES TÓTH weist darauf hin, daß für das analoge Problem bezüglich der umbeschriebenen m-Ecke minimalen Flächeninhalts die Ellipsen nicht extremal sind (der Extremalbereich hängt von m ab). Es wird sodann bemerkt [2, S. 39], daß die den Umfang betreffenden analogen Probleme weder für ein- noch für umbeschriebene m-Ecke gelöst seien. Wie wir sehen werden, läßt sich aber der Grundgedanke des SAS—FEJES TÓTHschen Beweises (in einer auf naheliegende Weise abgewandelten Form) auch bei der Behandlung dieser beiden Fragen verwenden. Wir bezeichnen mit $L_m(K)$ das Minimum der Umfänge aller dem Bereich K umbeschriebenen m-Ecke, mit $l_m(K)$ das Maximum der Umfänge aller K einbeschriebenen m-Ecke, mit $L(K)$ den Umfang von K selbst, und beweisen:

SATZ. *Für jeden ebenen konvexen Bereich K und für $m=3, 4, \dots$ gilt*

$$L_m(K) \leqq \frac{m}{\pi}\operatorname{tg}\frac{\pi}{m}L(K), \tag{1}$$

$$l_m(K) \geqq \frac{m}{\pi}\sin\frac{\pi}{m}L(K). \tag{2}$$

Ist K ein Kreis, so gilt in beiden Fällen das Gleichheitszeichen. In (1) *besteht das Gleichheitszeichen nur für Kreise; in* (2) *steht es jedenfalls dann nur für Kreise, wenn $m\leqq 21$ und m ungerade oder $m\leqq 42$ und m gerade ist.*

Die Ungleichung (1) ist bereits in [5] aus einem allgemeineren Satz hergeleitet worden; die dort verwendete Methode gestattete es jedoch nicht, die Kreise als einzige Extremalbereiche festzustellen. Ob für (2) bei beliebigem m die Kreise die einzigen Extremalbereiche sind, haben wir leider nicht entscheiden können.

Beweis des Satzes. Mit $H(\alpha)$ bezeichnen wir die (K enthaltende) Stützhalbebene an K, deren äußere Normalenrichtung mit einer festen Richtung den Winkel α bildet. $G(\alpha)$ sei die zugehörige Stützgerade und $p(\alpha)$ deren Abstand von dem etwa im Innern von K gewählten Koordinatenursprung. Wir setzen $\tau = \pi/m$. Dann ist

$$P_m(\alpha) = \bigcap_{j=0}^{m-1} H(\alpha + 2j\tau)$$

ein dem Bereich K umbeschriebenes Polygon mit höchstens m Ecken. Für seinen Umfang liefert eine elementargeometrische Betrachtung (oder siehe Meissner [3, S. 313])

$$L(P_m(\alpha)) = 2 \operatorname{tg} \tau \sum_{j=0}^{m-1} p(\alpha + 2j\tau). \tag{3}$$

Sei x_j ein Punkt der Berührmenge $G(\alpha+2j\tau) \cap K$, $j = 0, 1, \ldots, m-1$. Die konvexe Hülle $P_m^*(\alpha)$ der Punkte $x_0, \ldots, x_{m-1}$ ist ein dem Bereich K einbeschriebenes konvexes Polygon mit höchstens m Ecken (es ist nicht eindeutig bestimmt, falls der Rand von K Strecken enthält; im übrigen kann es auch in eine Strecke ausarten). Für den Umfang von $P_m^*(\alpha)$ gilt (vgl. Fejes Tóth [2, S. 40])

$$L(P_m^*(\alpha)) \geqq \cos \tau L(P_m(\alpha)). \tag{4}$$

Gleichheit gilt hier nur, wenn für $j = 0, 1, \ldots, m-1$ die Punkte x_j und x_{j+1} (wobei $x_m = x_0$ gesetzt sei) gleichen Abstand vom Schnittpunkt der beiden Geraden $G(\alpha+2j\tau)$ und $G(\alpha+2(j+1)\tau)$ haben.

Wegen

$$\int_0^{2\pi} p(\alpha)\, d\alpha = L(K)$$

erhält man aus (3) für den Mittelwert von $L(P_m(\alpha))$

$$\frac{1}{2\pi} \int_0^{2\pi} L(P_m(\alpha))\, d\alpha = \frac{1}{\tau} \operatorname{tg} \tau L(K).$$

Es gibt daher einen Winkel α_0 mit

$$L(P_m(\alpha_0)) \leqq \frac{1}{\tau} \operatorname{tg} \tau L(K). \tag{5}$$

(Vgl. a. Meissner [3, S. 313]. Im Fall $m = 3$ ergibt sich speziell, daß K ein umbeschriebenes gleichseitiges Dreieck vom Umfang $\leqq (3\sqrt{3}/\pi) L(K)$ besitzt. Dies ist auf die gleiche Weise von Eggleston [1], Theorem 6 (ii), gezeigt worden.) Ferner gibt es einen Winkel α_1 mit

$$L(P_m(\alpha_1)) \geqq \frac{1}{\tau} \operatorname{tg} \tau L(K),$$

also mit

$$L(P_m^*(\alpha_1)) \geqq \frac{1}{\tau} \sin \tau L(K). \tag{6}$$

Daraus folgen die beiden Ungleichungen (1) und (2).

Gilt nun in (1) das Gleichheitszeichen, so gibt es keinen Winkel α_0, für den in (5) Ungleichheit besteht, es ist also

$$(P_m(\alpha)) = \frac{1}{\tau} \operatorname{tg} \tau L(K) = L_m(K), \qquad 0 \leqq \alpha \leqq 2\pi.$$

Jedes der Polygone $P_m(\alpha)$ hat somit kleinsten Umfang unter den K umbeschriebenen m-Ecken. Daraus folgt (FEJES TÓTH [2, S. 6]): Der Kreis, der die auf der Geraden $G(\alpha)$ gelegene (als nicht entartet angenommene) Seite des Polygons $P_m(\alpha)$ und die Geraden $G(\alpha \pm 2\tau)$ berührt und nicht in $H(\alpha)$ liegt, berührt auch K. Der Mittelpunkt jeder Seite von $P_m(\alpha)$ ist also ein Punkt von K. Da dies für alle $\alpha \in [0, 2\pi]$ gilt, folgert man leicht, daß der Rand von K keine Strecken enthält. Die Stützfunktion p von K ist daher differenzierbar.

Sei $S(\alpha)$ der Schnittpunkt der Stützgeraden $G(\alpha - 2\tau)$ und $G(\alpha)$, sei $a(\alpha)$ der Abstand des Berührpunktes $B(\alpha)$ der Geraden $G(\alpha)$ vom Punkt $S(\alpha)$, und sei $b(\alpha)$ der Abstand des Punktes $B(\alpha)$ von $S(\alpha + 2\tau)$. Eine einfache Rechnung ergibt (siehe MEISSNER [3, Kap. II])

$$(7) \qquad \begin{cases} a(\alpha)\sin 2\tau = p(\alpha - 2\tau) - p(\alpha)\cos 2\tau + p'(\alpha)\sin 2\tau, \\ b(\alpha)\sin 2\tau = p(\alpha + 2\tau) - p(\alpha)\cos 2\tau - p'(\alpha)\sin 2\tau. \end{cases}$$

Nun hatte sich aber gezeigt, daß der Berührpunkt jeder Seite jedes Polygons $P_m(\alpha)$ der Mittelpunkt dieser Seite ist; es ist also $a(\alpha) = b(\alpha)$ und daher

$$(8) \qquad p'(\alpha) = \frac{1}{2} \operatorname{cosec} 2\tau [p(\alpha + 2\tau) - p(\alpha - 2\tau)], \qquad 0 \leqq \alpha \leqq 2\pi.$$

Die Stützfunktion des Extremalbereichs K genügt also einer linearen Differenzen-Differentialgleichung. Überraschenderweise ist es dieselbe Gleichung, auf die FEJES TÓTH [2, S. 37] bei dem eingangs erwähnten Inhaltsproblem geführt worden ist. Die einzigen mit 2π periodischen Lösungen der Gleichung (8) sind von der Form

$$(9) \qquad p(\alpha) = a_0 + a_1 \cos \alpha + b_1 \sin \alpha$$

mit Konstanten a_0, a_1, b_1, wie FEJES TÓTH (loc. cit.) gezeigt hat. Durch (9) wird aber die Stützfunktion eines Kreises gegeben.

Gilt in (2) das Gleichheitszeichen, so muß für $\alpha \in [0, 2\pi]$ in (4) das Gleichheitszeichen bestehen. Daraus folgt (FEJES TÓTH [2, S. 40]): Jede Ecke S des Polygons $P_m(\alpha)$ hat von den (als Ecken des einbeschriebenen Polygons $P_m^*(\alpha)$ gewählten) Berührpunkten der beiden durch S gehenden Seiten gleichen Abstand. Daraus folgert man wieder, daß die Berührpunkte eindeutig bestimmt sind und die Funktion p also differenzierbar ist. Die Beziehung $b(\alpha) = a(\alpha + 2\tau)$ bedeutet nach (7)

$$(10) \qquad p'(\alpha + 2\tau) + p'(\alpha) = \operatorname{cotg} \tau [p(\alpha + 2\tau) - p(\alpha)]$$

für $0 \leqq \alpha \leqq 2\pi$. Auf eine eng damit zusammenhängende Funktionalgleichung ist auch schon MEISSNER [3, S. 322] bei Behandlung eines anderen Problems geführt worden.

Um Aussagen über die mit 2π periodischen Lösungen p der Gleichung (10) machen zu können, betrachten wir die Fourier-Koeffizienten von p, also

$$a_k = \frac{1}{\pi}\int_0^{2\pi} p(\alpha)\cos k\alpha\, d\alpha, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} p(\alpha)\sin k\alpha\, d\alpha, \quad k = 0, 1, 2, \ldots.$$

Multipliziert man die Gleichung (10) mit $\cos k\alpha$ bzw. $\sin k\alpha$ und integriert von 0 bis 2π, so findet man nach partieller Integration und elementaren Umformungen

$$a_k(\operatorname{cotg}\tau \operatorname{tg} k\tau - k) + b_k(k \operatorname{cotg} k\tau - \operatorname{cotg}\tau) = 0,$$

$$-a_k(k \operatorname{cotg} k\tau - \operatorname{cotg}\tau) + b_k(\operatorname{cotg}\tau \operatorname{tg} k\tau - k) = 0.$$

Die Determinante dieses Gleichungssystems verschwindet genau dann, wenn $\operatorname{tg} k\tau = k \operatorname{tg}\tau$ ist. Nehmen wir nun also an, die Gleichung (10) habe eine Lösung p, die nicht von der Form (9) ist, so gibt es eine Zahl $k \geqq 2$ mit $a_k^2 + b_k^2 \neq 0$, also mit

$$k \operatorname{tg}\frac{\pi}{m} = \operatorname{tg} k\frac{\pi}{m}. \tag{11}$$

Um zu zeigen, daß die Kreise die einzigen Extremalbereiche für die Ungleichung (2) sind, müßte also gezeigt werden, daß die Gleichung (11) keine ganzzahligen Lösungen k, m mit $k \geqq 2$, $m \geqq 3$ besitzt. Wir können nur zeigen, daß es keine derartige Lösung mit ungeradem $m \leqq 21$ oder mit geradem $m \leqq 42$ gibt (die Abschätzungen ließen sich mit etwas mehr Aufwand geringfügig verbessern).

Wir nehmen also an, es gelte (11) mit ganzzahligen $k \geqq 2$, $m \geqq 3$. Für $0 < x < \pi/2$ ist $k \operatorname{tg} x < \operatorname{tg} kx$; wegen $\operatorname{tg}\pi/m > 0$ muß $\operatorname{tg} k\pi/m > 0$ sein, also ist

$$k\pi/m \in \big(n\pi/2, (n+1)\pi/2\big) \tag{12}$$

mit einer geraden Zahl $n \geqq 2$. Insbesondere ist

$$k > m. \tag{13}$$

Wir zeigen

$$\left\{\begin{matrix} k\pi/m \neq (n+1)\pi/2 - \pi/m \\ k\pi/m \neq (n+1)\pi/2 - 2\pi/m \end{matrix}\right\} \quad \text{für gerade} \quad m \tag{14}$$

$$k\pi/m \neq (n+1)\pi/2 - \pi/2m. \tag{15}$$

Beweis. Wäre $k\pi/m = (n+1)\pi/2 - \pi/m$, so wäre

$$k \operatorname{tg}\pi/m = \operatorname{tg}\big((n+1)\pi/2 - \pi/m\big) = \operatorname{cotg}\pi/m,$$

also

$$\operatorname{tg}^2 \pi/m = 1/k. \tag{16}$$

Aus $\operatorname{tg} m\pi/m = 0$ folgt

$$\binom{m}{1} - \binom{m}{3}\operatorname{tg}^2 \pi/m + \binom{m}{5}\operatorname{tg}^4 \pi/m - \cdots = 0. \tag{17}$$

Einsetzen von (16) ergibt, wenn m gerade ist,

$$\binom{m}{1}k^{(m-2)/2}-\binom{m}{3}k^{(m-4)/2}+\dots\pm\binom{m}{m-3}k\mp\binom{m}{m-1}=0.$$

Es ist also k ein Teiler von m, was (13) widerspricht.

Wäre $k\pi/m=(n+1)\pi/2-2\pi/m$, so wäre k tg $\pi/m=$ cotg $2\pi/m$, woraus $\mathrm{tg}^2\,\pi/m = 1/(2k+1)$ folgt. Einsetzen in (17) führt bei geradem m analog wie oben auf einen Widerspruch.

Wäre $k\pi/m=(n+1)\pi/2-\pi/2m$, so ergäbe sich analog $\mathrm{tg}^2\,\pi/2m=1/(2k+1)$, woraus nun folgt, daß $2k+1$ ein Teiler von $2m$ ist. Das ist wieder nicht möglich. Die Ungleichungen (14) und (15) sind damit bewiesen.

Nun sei m gerade. Wegen der Periodizität der tg-Funktion und wegen (12) ist tg $k\pi/m$ einer der Werte tg $j\pi/m$, $j=1,\dots,m/2-1$. Wegen (14) kommen davon $j=m/2-1$ und $j=m/2-2$ nicht in Frage. Es ist also

$$\mathrm{tg}\,k\pi/m \leqq \mathrm{tg}\,(m/2-3)\pi/m=\mathrm{cotg}\,3\pi/m$$

und daher

(18) $$k\leqq \mathrm{cotg}\,\pi/m\;\mathrm{cotg}\,3\pi/m\leqq m^2/3\pi^2.$$

Für ungerades m findet man unter Berücksichtigung von (15) analog

(19) $$k\leqq \mathrm{cotg}\,\pi/m\;\mathrm{cotg}\,3\pi/2m\leqq 2m^2/3\pi^2.$$

Nun ist

$$\mathrm{tg}\,k\pi/m=k\;\mathrm{tg}\,\pi/m\geqq k\pi/m,$$

und wegen (13) ist $k\pi/m>\pi$. Da tg $x\leqq x$ für $\pi\leqq x\leqq 10\pi/7$ ist, muß also $k\pi/m\geqq 10\pi/7$ sein. Zusammen mit (18) ergibt das für gerades m

$$m\geqq(10/7)\cdot 3\pi^2>42$$

und mit (19) für ungerades m

$$m\geqq(10/7)\cdot 3\pi^2/2>21.$$

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NOTWENDIGE UND HINREICHENDE BEDINGUNGEN FÜR DIE DISKRETHEIT DES SPEKTRUMS STURM—LIOUVILLESCHER OPERATOREN

Von

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Betrachtet wird der Sturm—Liouvillesche Differentialausdruck

$$L = -\frac{d}{dx} p(x) \frac{d}{dx} + q(x), \qquad 0 < x < a \leqq \infty, \tag{1}$$

dessen Koeffizienten $p(x)$ und $q(x)$ stets folgende Voraussetzungen erfüllen sollen:

1. $p(x)$ ist im (offenen) Intervall $(0, a)$ positiv und stetig differenzierbar.
2. $q(x)$ ist in $(0, a)$ nicht negativ und lokal quadratisch integrierbar.

Legt man den Hilbertraum $L^2(0, a)$ mit seinem Skalarprodukt

$$(f, g) = \int_0^a f(x)\overline{g(x)}\,dx$$

zugrunde, so wird durch L der symmetrische Differentialoperator L_0 mit dem Definitionsbereich[1]

$$D(L_0) = C_0^2(0, a), \quad L_0 u = Lu, \quad u \in D(L_0), \tag{2}$$

erklärt. Zahlreiche Autoren haben Untersuchungen darüber angestellt, wie das Verhalten der Koeffizienten $p(x)$ und $q(x)$ das Spektrum selbstadjungierter Erweiterungen von L beeinflußt. Für den Spezialfall $p(x) \equiv 1$, $q(x) \geqq q_0$, $a = \infty$ hat MOLČANOV ([6]) eine notwendige und hinreichende Bedingung dafür angegeben, daß das Spektrum einer selbstadjungierten Erweiterung $\tilde{L}_0$ von L_0 rein diskret ist. Diese Bedingung wurde von I. BRINK ([1]) und R. S. ISMAGILOV ([4]) auf den Fall verallgemeinert, daß $q(x)$ nicht beschränkt nach unten ist. Das Kriterium von A. M. MOLČANOV wird in der vorliegenden Arbeit auf den Fall (1) für gewisse Funktionen $p(x)$ verallgemeinert. Außerdem ergibt sich ein Resultat über die Lokalisierung des wesentlichen Spektrums,[2] wobei es sich als wesentlich erweist, wie sich die Mittelwerte von $\frac{1}{p(x)}$ und $q(x)$ an den gegebenenfalls singulären Endpunkten[3] des Intervalls $(0, a)$ verhalten.

[1] $C_0^2(0, a)$ ist wie üblich die Menge der in $(0, a)$ zweimal stetig differenzierbaren (komplexwertigen) Funktionen mit kompaktem Träger.

[2] Ein Punkt des Spektrums gehört genau dann zum wesentlichen Spektrum, wenn er kein isolierter Eigenwert ist.

[3] Zur Definition des regulären und singulären Endpunktes sei auf [8], § 15, verwiesen.

Wir setzen im folgenden voraus, daß der Endpunkt 0 regulär und der Endpunkt a singulär ist. Aufgrund der Zerlegungsmethode ([8]; § 24, 1) entscheidet dann allein das Verhalten der Koeffizienten $p(x)$ und $q(x)$ am Endpunkt a über die Natur des wesentlichen Spektrums, das für alle selbstadjungierten Erweiterungen von L_0 das gleiche ist ([7]; § 19, 4). Durch Abschätzung der quadratischen Form

$$(3) \qquad (L_0 u, u) = \int_0^a (p(x)|u'|^2 + q(x)|u|^2)dx, \qquad u \in C_0^2(0, a),$$

für die Funktionen $u(x)$, deren Träger an den singulären Endpunkt a heranrücken, nach unten gelingt es, untere Schranken für das wesentliche Spektrum einer selbstadjungierten Erweiterung $\tilde{L}_0$ zu finden. Zur Abschätzung der quadratischen Form (3) beweisen wir folgenden

HILFSSATZ. *Bezeichnet* (ω) *das Intervall* (α, β), $0 < \alpha < \beta < a$, $\omega = \beta - \alpha$ *und*

$$(4) \qquad \mu_\omega = \frac{1}{\omega} \int_{(\omega)} q(x)\,dx$$

den Mittelwert der positiven Funktion $q(x)$ *auf* (ω), *so gilt die Ungleichung*[4]

$$(5) \qquad \int_{(\omega)} (p(x)|u'|^2 + q(x)|u|^2)dx \geqq \frac{\mu_\omega}{1 + \omega\mu_\omega \int_{(\omega)} \frac{dx}{p(x)}} \|u\|_{(\omega)}^2.$$

BEWEIS. Die reelle Funktion $\varphi(x) \in C^2[\alpha, \beta]$ besitze die Nullstelle $x_0 \in [\alpha, \beta]$. Mit Hilfe der Schwarzschen Ungleichung folgt dann

$$\varphi^2(x) = \left(\int_{x_0}^{x} \varphi'(t)\,dt\right)^2 \leqq \left|\int_{x_0}^{x} \frac{dt}{p(t)}\right| \cdot \left|\int_{x_0}^{x} p(t)[\varphi'(t)]^2\,dt\right| \leqq$$

$$\leqq \left(\int_{(\omega)} \frac{dx}{p(x)}\right)\left(\int_{(\omega)} p(x)[\varphi'(x)]^2\,dx\right),$$

woraus sich durch Integration

$$(6) \qquad \|\varphi\|_{(\omega)}^2 \leqq \omega \left(\int_{(\omega)} \frac{dx}{p(x)}\right)\left(\int_{(\omega)} p(x)[\varphi'(x)]^2\,dx\right)$$

ergibt. Bezeichnet $u_0 = \varphi_0 + i\psi_0$ denjenigen Wert der komplexwertigen Funktion

$$u(x) = \varphi(x) + i\psi(x) \in C^2[\alpha, \beta],$$

[4] $\|\cdot\|_{(\omega)}$ bezeichnet die Norm im Hilbertraum $L^2(\alpha, \beta)$.

für welchen $|u(x)|$ auf $[\alpha, \beta]$ minimal ist, so folgen nach (6) die Ungleichungen

$$\int_{(\omega)} (\varphi(x)-\varphi_0)^2 dx \leqq \omega \left(\int_{(\omega)} \frac{dx}{p(x)} \right) \left(\int_{(\omega)} p(x)[\varphi'(x)]^2 dx \right)$$

und

$$\int_{(\omega)} (\psi(x)-\psi_0)^2 dx \leqq \omega \left(\int_{(\omega)} \frac{dx}{p(x)} \right) \left(\int_{(\omega)} p(x)[\psi'(x)]^2 dx \right).$$

Durch Addition erhält man daraus

$$\int_{(\omega)} |u(x)-u_0|^2 dx \leqq \omega \left(\int_{(\omega)} \frac{dx}{p(x)} \right) \left(\int_{(\omega)} p(x)|u'|^2 dx \right). \tag{7}$$

In der folgenden Abschätzung wird die Ungleichung (7) benutzt:

$$\int_{(\omega)} (p(x)|u'|^2 + q(x)|u|^2) dx \geqq \frac{1}{\omega \int_{(\omega)} \frac{dx}{p(x)}} \|u-u_0\|^2_{(\omega)} + \mu_\omega \|u_0\|^2_{(\omega)} =$$

$$= \frac{\mu_\omega}{1+\delta^2} \left[\frac{\delta^2}{\omega\mu_\omega \int_{(\omega)} \frac{dx}{p(x)}} (1+\delta^{-2}) \|u-u_0\|^2_{(\omega)} + (1+\delta^2) \|u_0\|^2_{(\omega)} \right].$$

Setzt man

$$\delta^2 = \omega\mu_\omega \int_{(\omega)} \frac{dx}{p(x)}$$

und benutzt die Ungleichung

$$(1+\delta^{-2}) \|u-u_0\|^2_{(\omega)} + (1+\delta^2) \|u_0\|^2_{(\omega)} \geqq \|u\|^2_{(\omega)},$$

so ergibt sich die Behauptung (5).

Um die Abschätzung (5) lokal anwenden zu können, zerlegen wir das Intervall $(0, a)$ durch Teilpunkte $x_1=0, x_2, \dots, x_n, \dots; x_n<x_{n+1}$, in abzählbar viele Intervalle $(\omega_n)=(x_n, x_{n+1})$, deren Längen $\omega_n = x_{n+1}-x_n$ durch ein fixiertes $\varrho>0$ aus

$$\varrho = \omega_n \int_{x_n}^{x_n+\omega_n} \frac{dx}{p(x)} \tag{8}$$

eindeutig bestimmt werden. Ist der singuläre Endpunkt a endlich, so ergeben sich wegen der Divergenz von

$$\int_{a-\varepsilon}^{a} \frac{dx}{p(x)}, \qquad 0 < \varepsilon < a,$$

aus (8) unendlich viele Intervalle (ω_n). Unendlich viele Intervalle entstehen offenbar auch im Falle $a=\infty$. Die quadratische Form $(L_0 u, u)$ kann in beiden Fällen mittels Ungleichung (5) wie folgt abgeschätzt werden; es wird $\mu_{\omega_n}=\mu_n$ gesetzt:

$$(9)\quad u\in C_0^2(0,a),\quad (L_0 u, u) = \sum_{n=1}^{\infty} \int_{(\omega_n)} (p(x)|u'|^2+q(x)|u|^2)\,dx \geqq \sum_{n=1}^{\infty} \frac{\mu_n}{1+\varrho\mu_n} \|u\|^2_{(\omega_n)}.$$

Diese Abschätzung kann dazu verwendet werden, eine untere Schranke für das wesentliche Spektrum $\sigma(\tilde{L}_0)$ einer selbstadjungierten Erweiterung $\tilde{L}_0$ von L_0 zu bekommen. Es gilt folgender

SATZ 1. *Es bezeichne*

$$(10)\qquad \mu(\varrho,\xi) = \frac{1}{\omega(\varrho,\xi)} \int_{(\omega(\varrho,\xi))} q(x)\,dx$$

den Mittelwert der Funktion $q(x)$ *auf dem durch*

$$(11)\qquad \varrho = \omega \int_{\xi}^{\xi+\omega} \frac{dx}{p(x)}, \qquad 0 \leqq \xi < a,$$

bestimmten Intervall $(\omega(\varrho,\xi)) = (\xi, \xi+\omega)$ *der Länge* $\omega(\varrho,\xi)$. *Wenn dann für jedes fixierte* $\varrho>0$

$$(12)\qquad \varliminf_{\xi\to a} \mu(\varrho,\xi) \geqq \varkappa$$

ist, so gilt

$$(13)\qquad \sigma(\tilde{L}_0) \cap (-\infty, \varkappa) = \emptyset.$$

BEWEIS. Es kann $\varkappa>0$ vorausgesetzt werden. Es sei $\varkappa<\infty$. Ein $\varepsilon(>0)$ wird beliebig gewählt und $\varrho = \dfrac{\varepsilon}{\varkappa(\varkappa-\varepsilon)}$, $\varkappa-\varepsilon>0$, gesetzt. Wegen (12) sind in (9) alle $\mu_n > \varkappa-\varepsilon$ für hinreichend große n $(n\geqq N(\varepsilon))$. Dann gilt für diese n

$$(14)\qquad \frac{\mu_n}{1+\varrho\mu_n} = \frac{1}{\dfrac{1}{\mu_n}+\varrho} \geqq \frac{\varkappa-\varepsilon}{1+\dfrac{\varepsilon}{\varkappa}} > \varkappa-2\varepsilon.$$

Die Vereinigung der (ω_n) mit $n\geqq N(\varepsilon)$ überdeckt ein Intervall (c,a), $0<c<a$. Aus (9) folgt dann wegen (14)

$$(L_c u, u)_{(c,a)} \geqq (\varkappa-2\varepsilon)\|u\|^2_{(c,a)}, \quad D(L_c) = C_0^2(c,a), \quad L_c u = Lu,$$

woraus sich für das wesentliche Spektrum einer selbstadjungierten Erweiterung $\tilde{L}_c$ von L_c

$$\sigma(\tilde{L}_c) \cap (-\infty, \varkappa-2\varepsilon) = \emptyset$$

ergibt. Aufgrund der Zerlegungsmethode ist $\sigma(\tilde{L}_0)=\sigma(\tilde{L}_c)$, was

$$\sigma(\tilde{L}_0) \cap (-\infty, \varkappa-2\varepsilon) = \emptyset$$

bedeutet. Da ε beliebig gewählt war, folgt die Behauptung des Satzes 1.

SATZ 2. *Für die durch ein beliebiges* $\varrho>0$ *bestimmte Familie von Intervallen* $(\omega(\varrho, \xi))$, $0\leqq\xi<a$, *(vergl. Satz 1) sei das Produkt der Mittelwerte von* $p(x)$ *und* $\frac{1}{p(x)}$ *gleichmäßig beschränkt,*

$$(15)\qquad \left(\frac{1}{\omega(\varrho,\xi)}\int\limits_{(\omega(\varrho,\xi))} p(x)\,dx\right)\left(\frac{1}{\omega(\varrho,\xi)}\int\limits_{(\omega(\varrho,\xi))}\frac{dx}{p(x)}\right)\leqq M(\varrho).$$

Das Spektrum $\sigma(\tilde{L}_0)$ *einer selbstadjungierten Erweiterung* $\tilde{L}_0$ *von* L_0 *ist genau dann rein diskret, wenn für jedes fixierte* $\varrho>0$

$$(16)\qquad \lim_{\xi\to a}\frac{1}{\omega(\varrho,\xi)}\int\limits_{(\omega(\varrho,\xi))} q(x)\,dx=\infty$$

gilt.

BEWEIS. Daß die Bedingung (16) für die Diskretheit des Spektrums hinreichend ist, folgt unmittelbar aus Satz 1. Um zu zeigen, daß die Bedingung notwendig ist, nehmen wir an, daß sie nicht erfüllt ist. Danach gibt es ein $\varrho^*>0$, so daß für ein gewisses $K<\infty$ eine Folge von Punkten $\xi_1, \xi_2, \dots, \xi_\nu\to a$, derart existiert, daß

$$(17)\qquad \frac{1}{\omega(\varrho^*,\xi_\nu)}\int\limits_{(\omega(\varrho^*,\xi_\nu))} q(x)\,dx\leqq K,\qquad \nu=1,2,\dots,$$

gilt. Die Intervalle $(\omega(\varrho^*, \xi_\nu))$ darf man sich paarweise disjunkt vorstellen. Es wird im folgenden in Anwendung eines bekannten Satzes von RELLICH (vergl. [8]; § 24, Satz 11) eine in $L^2(0, a)$ nicht kompakte Folge von Funktionen $v_\nu(x)$ konstruiert, die in der Norm $(L_0 v_\nu, v_\nu)^{\frac{1}{2}}$ beschränkt ist. $\varphi(x)$ sei eine reelle Funktion aus $C_0^2(0, 1)$ mit der Norm Eins. Wir setzen

$$(18)\qquad \omega(\varrho^*,\xi_\nu)=\omega_\nu,\quad v_\nu(x)=\frac{1}{\sqrt{\omega_\nu}}\varphi\left(\frac{x-\xi_\nu}{\omega_\nu}\right),\quad \xi_\nu\leqq x\leqq\xi_\nu+\omega_\nu,$$

und dehnen den Definitionsbereich von $v_\nu(x)$ jeweils durch die Vorschrift

$$v_\nu(x)=0,\quad x\in(0,a)\setminus(\omega_\nu),\qquad \nu=1,2,3,\dots,$$

auf das gesamte Intervall $(0, a)$ aus. Dann ist

$$\int\limits_0^a v_\nu^2(x)\,dx=\frac{1}{\omega_\nu}\int\limits_{(\omega_\nu)}\varphi^2\left(\frac{x-\xi_\nu}{\omega_\nu}\right)dx=\int\limits_0^1\varphi^2(t)\,dt=1,$$

$$v_\nu^2(x)\leqq\frac{1}{\omega_\nu}\cdot\max_{t\in[0,1]}\varphi^2(t)=\frac{c_1}{\omega_\nu}\quad\text{und}\quad[v_\nu'(x)]^2\leqq\frac{1}{\omega_\nu^3}\cdot\max_{t\in[0,1]}[\omega'(t)]^2=\frac{c_2}{\omega_\nu^3},$$

und es folgt aus (15) und (17)

$$(L_0 v_\nu, v_\nu) = \int\limits_{(\omega_\nu)} p(x)[v'_\nu(x)]^2 dx + \int\limits_{(\omega_\nu)} q(x) v_\nu^2(x) dx \leqq$$

$$\leqq \frac{c_2}{\omega_\nu^3} \int\limits_{(\omega_\nu)} p(x) dx + \frac{c_1}{\omega_\nu} \int\limits_{(\omega_\nu)} q(x) dx \leqq \frac{c_2 M(\varrho^*)}{\varrho^*} + c_1 K, \qquad \nu = 1, 2, 3, \ldots.$$

Satz 2 ist damit bewiesen.

Zum Schluß geben wir zwei einfache Kriterien für die Funktion $p(x)$ an, so daß (15) erfüllt ist.

I. $a=\infty$. Es existieren positive Konstanten c_1 und c_2, so daß für hinreichend große x, $x \geqq x_0$,

(20) $$c_1 x^\sigma \leqq p(x) \leqq c_2 x^\sigma, \qquad -\infty < \sigma \leqq 2,$$

gilt.

Im Falle $\sigma=0$ folgt sofort

$$\left(\frac{1}{\omega}\int\limits_{\xi}^{\xi+\omega} p(x)dx\right)\left(\frac{1}{\omega}\int\limits_{\xi}^{\xi+\omega} \frac{dx}{p(x)}\right) \leqq c_2 \cdot \frac{1}{c_1} = \text{const}, \qquad \xi \geqq x_0.$$

Wenn σ durch $0<\sigma\leqq 2$ eingeschränkt ist, kann wie folgt abgeschätzt werden, $\xi \geqq x_0$:

$$\varrho = \omega \int\limits_{\xi}^{\xi+\omega} \frac{dx}{p(x)} \leqq \frac{\omega}{c_1} \int\limits_{\xi}^{\xi+\omega} x^{-\sigma} dx \leqq \frac{\omega^2}{c_1} \xi^{-\sigma}.$$

Aus $\xi^\sigma \leqq \dfrac{\omega^2}{c_1 \varrho}$ folgt dann, daß für hinreichend große ξ, $\xi \geqq x_1(\varrho) > x_0$, die Abschätzung $\omega(\varrho, \xi) \geqq 1$ richtig ist, so daß für solche ξ folgendes gilt:

$$\int\limits_{\xi}^{\xi+\omega} p(x)dx \leqq c_2 \int\limits_{\xi}^{\xi+\omega} x^\sigma dx \leqq c_2 \omega(\xi+\omega)^\sigma \leqq c_3 \omega(\xi^\sigma + \omega^\sigma) \leqq$$

$$\leqq c_4 \frac{\omega^3}{\varrho} + c_3 \omega^{1+\sigma} \leqq \left(c_3 + \frac{c_4}{\varrho}\right) \omega^3.$$

Somit ergibt sich für hinreichend große ξ, $\xi \geqq x_1(\varrho)$,

$$\left(\frac{1}{\omega}\int\limits_{\xi}^{\xi+\omega} p(x)dx\right)\left(\frac{1}{\omega}\int\limits_{\xi}^{\xi+\omega} \frac{dx}{p(x)}\right) \leqq \left(c_3 + \frac{c_4}{\varrho}\right)\omega^2 \cdot \frac{\varrho}{\omega^2} = c_3 \varrho + c_4; \qquad c_3, c_4 > 0.$$

Das ist mit $M(\varrho) = c_3 \varrho + c_4$ die Ungleichung (15) für hinreichend große ξ, was für den Satz 2 ausreicht.

Wenn σ negativ ist, kann ähnlich geschlossen werden.

In (20) ist $\sigma=2$ die natürliche obere Grenze: Für $p(x)=x^{2+\varepsilon}$, $x\geqq x_0$, ist für ein beliebiges $\varepsilon>0$ die Bedingung (16) nicht mehr notwendig für die Diskretheit des Spektrums. Das Spektrum ist auch im Falle $q(x)\equiv 0$ diskret. (Vergl. [3], S. 183.)

Der Satz von MOLČANOV entspricht dem Spezialfall $\sigma=0$ und $c_1=c_2=1$.

II. $a\leqq\infty$. Es existiert eine Konstante C, so daß für hinreichend große x, $x\geqq x_0$, die Abschätzung

$$|p'(x)|\leqq C\sqrt{p(x)} \tag{21}$$

gilt.

Dann ist wieder die Bedingung (15) erfüllt: Wir verfolgen die Funktion $p(x)$, wenn sie der Bedingung (21) genügt, von einem Punkt (ξ, η), $\xi\geqq x_0$, $\eta>0$, aus für wachsende x. Sie liegt offenbar zwischen den Lösungen der Differentialgleichungen

$$p'(x)=C\sqrt{p(x)} \quad \text{und} \quad p'(x)=-C\sqrt{p(x)} \tag{22}$$

die durch den Punkt (ξ, η) gehen. Diese Lösungen sind die Parabelbögen

$$y_1=\frac{C^2}{4}\left(x-\xi+\frac{2}{C}\sqrt{\eta}\right)^2 \quad \text{und} \quad y_2=\frac{C^2}{4}\left(x-\xi-\frac{2}{C}\sqrt{\eta}\right)^2.$$

Es wird $\omega=\omega(\varrho, \xi)$ abgeschätzt:

$$\varrho=\omega\int_{\xi}^{\xi+\omega}\frac{dx}{p(x)}\leqq\omega\int_{\xi}^{\xi+\omega}\frac{4}{C^2\left(x-\xi-\frac{2}{C}\sqrt{\eta}\right)^2}dx=\frac{2\omega^2}{\sqrt{\eta}\,(2\sqrt{\eta}-C\omega)}.$$

Daraus folgt

$$\omega\geqq\sqrt{\eta}\left(\sqrt{\varrho+\frac{\varrho^2C^2}{16}}-\frac{\varrho C}{4}\right)=A(\varrho, C)\cdot\sqrt{p(\xi)}, \tag{23}$$

und es ergibt sich

$$\int_{\xi}^{\xi+\omega}p(x)\,dx\leqq\frac{C^2}{4}\int_{\xi}^{\xi+\omega}\left(x-\xi+\frac{2}{C}\sqrt{\eta}\right)^2dx=\frac{C^2}{12}\omega^3+\frac{C}{2}\omega^2\sqrt{\eta}+\omega\eta\leqq B\omega^3,$$

$$B=B(C,\varrho)=\frac{C^2}{12}+\frac{C}{2A}+\frac{1}{A}, \qquad A=A(\varrho, C)=\sqrt{\varrho+\frac{C^2\varrho^2}{16}}-\frac{\varrho C}{4}.$$

Die Bedingung (15) ist für $p(x)$ also mit $M(\varrho)=\varrho B(\varrho)$ erfüllt.

In konkreten Fällen kann das Kriterium (21) oft mit Erfolg angewendet werden, z. B. wenn $p(x)=(1-x)^\sigma$ ist $(a=1)$: (21) ist für $\sigma\geqq 2$ zu erfüllen, und das Spektrum von $\tilde{L}_0$ ist also nach Satz 2 genau dann rein diskret, wenn (16) erfüllt ist. Ist aber $\sigma<2$, so ist Bedingung (16) nicht mehr notwendig. In diesem Fall ist das Spektrum bereits diskret, wenn $q(x)\equiv 0$ ist (vergl. [5], S. 200).

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A PROOF OF THE COMPLETE NORMALITY OF CHAINS

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A linearly ordered set or chain has a natural topology defined in terms of its order relation. This is sometimes called the *intrinsic* topology of the chain. If the chain has more than one element, this topology is obtained by taking as a subbase for the open sets the *open rays* $(a, +\infty)$ and $(-\infty, b)$ consisting of the elements x such that $a<x$ and $x<b$ respectively. The open rays and their intersections, the *open intervals* $(a, b)=[x: a<x<b]$ form a base for the open sets.

It is natural to ask what separation properties hold for this topology with any chain. GARRETT BIRKHOFF has shown ([3], p. 39, theorem 6) that every chain is a normal Hausdorff space in its intrinsic topology. S. A. GAAL has shown ([5], p. 93, theorem 3) that every conditionally complete chain is completely normal in its intrinsic topology. N. BOURBAKI ([4], Chapter IX, p. 68) has suggested that the complete normality of chains be proved first for complete chains (and GAAL has given such a proof). It is then suggested that the result for chains in general be obtained by imbedding the chain in its completion by cuts, and using the fact that the intrinsic topology of a chain is equivalent to its relative topology as a subspace of the completion, together with the fact that complete normality is a hereditary property.[1] It is possible to make the proof in this way, and it is also possible to prove directly that every chain is completely normal without relying on other theorems.

It is easily verified that a chain is a Hausdorff space in its intrinsic topology. In fact, if $a<b$ are two elements of the chain, and there is an element c between them, then the open rays $(-\infty, c)$ and $(c, +\infty)$ contain a and b respectively and are disjoint. If no such element c exists, then the rays $(-\infty, b)$ and $(a, +\infty)$ are disjoint and separate a and b.

A topological space is said to be *completely normal* if any pair A, B of separated sets of the space lie in disjoint open sets. The sets A and B are said to be *separated* if neither set contains a point of the closure of the other. Complete normality of a space is known to be equivalent to the condition every subspace be a normal space in its relative topology.

THEOREM 1. *Every chain is a completely normal Hausdorff space in its intrinsic topology.*

PROOF. We have shown that a chain C is a Hausdorff space. Let A and B be two separated sets of C and let S be the complementary set $S = C-(A\cup B)$. For

[1] It is well known that in general, the instrinsic topology of a subchain of a chain C is distinct from its relative topology as a subspace of C. In [1] we have given a necessary and sufficient condition that the two topologies of a subchain be equivalent.

each element $a \in A$ and $b \in B$ we wish to select an open neighbourhood I_a and I_b of a and b respectively so that the union of the sets I_a is disjoint from the union of the sets I_b. Usually these neighbourhoods will be open intervals $I_a = (a_l, a_r)$ and $I_b = (b_l, b_r)$. We shall give rules for selecting the left and right endpoints a_l, b_l, a_r, b_r of these open intervals so that every set I_a is disjoint from every set I_b.

An element $a \in A$ or $b \in B$ which happens to be the smallest element of the chain C (if there is such a smallest element), is not in any open interval. Such an element will be assigned a neighbourhood which is an open ray of the form $(-\infty, a_r)$ or $(-\infty, b_r)$, and the rules for selecting the right endpoint a_r or b_r of such a ray will be the same as for the right endpoint of open intervals. Likewise the rules for selecting the single left endpoints of open rays assigned as neighbourhoods of the largest element of the chain, if there is one, are the same as those for selecting the left endpoints of open intervals. We now state these selection rules.

(1) Select I_a and I_b always so that $I_a \cap B = \emptyset = I_b \cap A$.
(2) If possible, select $a_l, a_r \in A$ and $b_l, b_r \in B$.
(3) If for $a \in A$ there is a $b \in B$ such that $a < b$ and $(a, b) = \emptyset$, then b is unique and no a_r satisfies (1) and (2); in this case choose $a_r = b$ and $b_l = a$.
(4) If for $a \in A$ there is a $b \in B$ such that $a < b$ and $\emptyset \neq (a, b) \subset S$, then b is unique and no a_r satisfies (1) and (2); in this case select any element $s \in S$ from (a, b) and take $a_r = s = b_l$.
(5) If $a \in A$ and rules (2), (3), and (4) do not determine the choice of a_r, then select a_r at random to satisfy rule (1). In this case $a_r \in S$ and $(a, a_r) \subset S$.
(6) For the other endpoints a_l, b_l, and b_r, use rules similar to (3), (4), and (5), with the proper interchanges of left and right, and of sets A and B.

Since A and B are separated sets, it is always possible to follow rule (1), since elements of A have neighbourhoods disjoint from B, and conversely. We now show that if the selection rules (1)—(6) are followed, then every neighbourhood I_a is disjoint from every neighbourhood I_b.

Suppose $a < b$ and the neighbourhoods $I_a = (a_l, a_r)$ and $I_b = (b_l, b_r)$ have an element c in common. Then $c \in S$, since I_a is disjoint from B and I_b from A. The elements a, b, c, a_r, b_l must be in the order $a < b_l < c < a_r < b$. Then $a_r \in I_b$, hence $a_r \in S$, since I_b contains no points of A. It follows that a_r was selected using rules (4) or (5), since in the cases of rules (2) and (3) a_r is in A or B. Hence $(a, a_r) \subset S$. Similarly $(b_l, b) \subset S$. It follows that $(a, b) \subset S$, since $(a, b) = (a, a_r) \cup (b_l, b)$. But then selection rule (4) would have applied; the intervals I_a and I_b would have a common endpoint $a_r = b_l = s$, and they would be disjoint.

A similar argument shows that any pair of neighbourhoods I_a and I_b with $b < a$ are disjoint. Finally the special cases where a or b are the smallest or largest elements of the chain C and hence have neighbourhoods which are open rays instead of open intervals are handled in the same way.

It follows that the union of all the neighbourhoods I_a of points of A is an open set disjoint from the open union of the neighbourhoods I_b of points of B. Since the separated sets A and B lie in disjoint open sets, the chain C is completely normal in its intrinsic topology. Hence Theorem 1 has been proved.

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ON COSINE SERIES WITH POSITIVE COEFFICIENTS

By

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Dedicated to Professor P. TURÁN *on his 60-th birthday*

Introduction

R. ASKEY [2] proved the following

THEOREM A. *Let $\{a_n\}$ be a monotone decreasing sequence and the Fourier cosine coefficients of $f(x)$ and let*

$$0<\alpha<2, \quad 1<p<\infty, \quad \frac{1}{p}+\frac{1}{q}=1, \quad 1\leqq r\leqq\infty.$$

Then

$$\|f\|_{\alpha,p,r}=\left(\int_0^1\left(\int_0^{2\pi}\left|\frac{f(x+t)+f(x-t)-2f(x)}{t^\alpha}\right|^p dx\right)^{r/p}\frac{dt}{t}\right)^{1/r}<\infty$$

if and only if

$$\left(\sum_{n=1}^{\infty} n^{r\left(\alpha+\frac{1}{q}\right)-1} a_n^r\right)^{1/r}<\infty.$$

We remark that this theorem includes a number of results on integrated Lipschitz conditions which have been proved by various authors ([5], [6], [10]) previously.

Recently M. and S. IZUMI [4] generalized Theorem A. Their theorem states:

THEOREM B. *Let $0<\alpha<2$, $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $1\leqq r\leqq\infty$. Let $\{a_n\}$ be a positive sequence and the Fourier cosine coefficients of $f(x)$. Then*

$$\left\{\sum_{n=1}^{\infty} n^{r\left(\alpha-\frac{1}{p}\right)-1}\left(\sum_{m=n/2}^{n} a_m\right)^r\right\}^{1/r}\leqq K\|f\|_{\alpha,p,r}.^{1}$$

If $\sum_{n=1}^{\infty}|\Delta a_n|<\infty$, then

$$\|f\|_{\alpha,p,r}\leqq K\left(\sum_{n=1}^{\infty} n^{r\left(\alpha+\frac{1}{q}\right)-1}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^r\right)^{1/r}.$$

[1] $\sum_{k=a}^{b}$, where a and b are not integers, means a sum over all integers between a and b. K and K_i denote either absolute constants or constants depending on certain functions and numbers which are not necessary to explain in detail, not necessarily the same at each occurences.

It is clear that this theorem reduces to Theorem A if the sequence $\{a_n\}$ is monotone decreasing.

First we generalize this theorem as follows:

THEOREM 1. *Let* $1<p<\infty$, $1\leqq r<\infty$ *and* $\lambda(x)$ $(x\geqq 1)$ *a positive monotone function with* $K_1\lambda(2^n)\leqq\lambda(2^{n+1})\leqq K_2\lambda(2^n)$, *where* $K_2\geqq K_1>0$. *Let* $\{a_n\}$ *be a positive sequence and the Fourier cosine coefficients of* $f(x)$. *Then*

$$(1)\qquad \sum_{n=1}^{\infty}\lambda(n)n^{-r}\left(\sum_{k=n/2}^{n}a_k\right)^r\leqq$$

$$\leqq K\int_0^1\frac{\lambda\left(\frac{1}{t}\right)}{t^{2+\frac{r}{p}-r}}\left(\int_0^\pi|f(x+t)+f(x-t)-2f(x)|^p\,dx\right)^{r/p}dt=I(f,\lambda,r,p).$$

If $\sum_{k=1}^{\infty}|\Delta a_k|<\infty$ *and* $\lambda(x)$ *satisfies the conditions*

$$(2)\qquad \sum_{n=1}^{m}\lambda(n)n^{\frac{r}{p}-r}\leqq K\lambda(m)m^{\frac{r}{p}-r+1}$$

and

$$(3)\qquad \sum_{n=m}^{\infty}\lambda(n)n^{r\left(\frac{1}{p}-3\right)}\leqq K\lambda(m)m^{1+r\left(\frac{1}{p}-3\right)},$$

then

$$(4)\qquad I(f,\lambda,r,p)\leqq K\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^r.$$

Supposing the monotonicity of the sequence $\{a_n\}$ we get the following

COROLLARY. *If* $1<p<\infty$, $1\leqq r<\infty$ *and* $\lambda(x)$ *satisfies* (2) *and* (3) *and, further, if the sequence* $\{a_n\}$ *is monotone decreasing then*

$$(5)\qquad \sum_{n=1}^{\infty}\lambda(n)a_n^r<\infty$$

if and only if

$$(6)\qquad \int_0^1\frac{\lambda\left(\frac{1}{t}\right)}{t^{2-\frac{r}{q}}}\left(\int_0^\pi|f(x+t)+f(x-t)-2f(x)|^p\,dx\right)^{r/p}dt<\infty.$$

If $0<\alpha<2$ and $\lambda(t)=t^{r\left(\alpha+\frac{1}{q}\right)-1}$ then this corollary reduces to Theorem A.

Using this corollary and some known results we prove the following general equivalent theorem which includes the following classical result of HARDY and LITTLEWOOD (see e.g. [3] p. 35).

If $\{a_n\}$ *is monotonically decreasing to zero and* $1<r<\infty$ *then* $f(x)\in L^r(0,\pi)$ *if and only if* $\Sigma n^{r-2}a_n^r<\infty$.

We use the following notations:

$$\omega_p(\delta, f) = \sup_{0 \leq h \leq \delta} \left\{ \int_0^\pi |f(x+h) - f(x)|^p \, dx \right\}^{1/p},$$

and $E_n^{(p)}(f)$ denotes the best approximation of $f(x)$ in $L^p(0, \pi)$.

THEOREM 2. *Let $1<p<r$ and let $\{\varphi_n\}$ be a non-negative non-decreasing sequence of numbers satisfying $\varphi_{n^2} \leq K\varphi_n$ for all n. Define*

$$\Phi(x) = \sum_{n=1}^{x} n^{\frac{r}{p}-2} \varphi_n \quad \text{and} \quad \varphi(x) = \varphi_n \quad \text{if} \quad x \in (n-1, n].$$

Let $\{a_n\}$ be a monotone decreasing sequence and the Fourier cosine coefficients of $f(x)$. Then the statements

$$\Sigma_1 = \sum_{n=1}^{\infty} \varphi_n n^{r-2} a_n^r < \infty, \tag{7}$$

$$\Sigma_2 = \sum_{n=1}^{\infty} \varphi_n n^{-rs+\frac{r}{p}-2} \left(\sum_{k=1}^{n} k^{(s+1)p-2} a_k^p \right)^{r/p} < \infty, \quad \text{for any} \quad s > \frac{1}{p} - \frac{1}{r}, \tag{8}$$

$$\Sigma_3 = \sum_{n=1}^{\infty} \varphi_n n^{\frac{r}{p}-2} \left(\sum_{k=n}^{\infty} k^{p-2} a_k^p \right)^{r/p} < \infty, \tag{9}$$

$$\Sigma_4 = \sum_{n=1}^{\infty} \varphi_n n^{\frac{r}{p}-2} \omega_p \left(\frac{1}{n}, f \right)^r < \infty, \tag{10}$$

$$\Sigma_5 = \sum_{n=1}^{\infty} \varphi_n n^{\frac{r}{p}-2} (E_n^{(p)}(f))^r < \infty, \tag{11}$$

$$I_1 = \int_0^\pi |f(x)|^{r-\frac{r}{p}+1} \Phi(|f(x)|) \, dx < \infty, \tag{12}$$

$$I_2 = \int_0^\pi |f(x)|^r \varphi(|f(x)|) \, dx < \infty, \tag{13}$$

$$I_3 = \int_0^\pi |f(x)|^r \varphi\left(\frac{1}{x}\right) dx < \infty \tag{14}$$

and

$$I_4 = \int_0^\pi \varphi\left(\frac{1}{t}\right) t^{-\frac{r}{p}} \left(\int_0^\pi |f(x+t) + f(x-t) - 2f(x)|^p \, dx \right)^{r/p} dt < \infty \tag{15}$$

are equivalent.

§ 1. Lemmas

LEMMA 1 ([9], Theorem 1). *If $p \geqq 1$ and $\alpha_n \geqq 0$, then for any sequence $\{\lambda_n\}$ of positive numbers we have*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{\nu=1}^{n} \alpha_\nu \right)^p \leqq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \left(\sum_{\nu=n}^{\infty} \lambda_\nu \right)^p \alpha_n^p \tag{1.1}$$

and

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{\nu=n}^{\infty} \alpha_\nu \right)^p \leqq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \left(\sum_{\nu=1}^{n} \lambda_\nu \right)^p \alpha_n^p. \tag{1.2}$$

LEMMA 2 ([8], Lemma 3). *Let $\beta \geqq 1$, $\gamma > 0$ and let $\{\alpha_n\}$ and $\{\lambda_n\}$ be sequences of non-negative numbers. Denote*

$$\sigma_1 = \sum_{n=1}^{\infty} \lambda_n^\beta n^{-2} \left(\sum_{k=n}^{\infty} \alpha_k \right)^\beta$$

and

$$\sigma_2 = \sum_{n=1}^{\infty} \lambda_n^\beta n^{-2-\beta\gamma} \left(\sum_{k=1}^{n} k^\gamma \alpha_k \right)^\beta.$$

If

$$\sum_{k=n}^{\infty} \lambda_k^\beta k^{-2-\beta\gamma} \leqq K \lambda_n^\beta n^{-1-\beta\gamma}, \tag{1.3}$$

then there exists a $K_1 = K_1(\{\lambda_n\}, p, \gamma)$ such that

$$\sigma_2 \leqq K_1 \sigma_1. \tag{1.4}$$

If

$$\sum_{k=1}^{n} \lambda_k^\beta k^{-2} \leqq K \lambda_n^\beta n^{-1}, \tag{1.5}$$

then there exists a $K_2 = K_2(\{\lambda_n\}, p, \gamma)$ such that

$$\sigma_1 \leqq K_2 \sigma_2. \tag{1.6}$$

LEMMA 3 ([8], Theorem 3). *Suppose that $p > 1$ and that $\{\lambda_n\}$ satisfies the conditions*

$$\sum_{n=1}^{m} n^{p-2} \lambda_n^p \leqq K \lambda_m^p m^{p-1} \tag{1.7}$$

and

$$\sum_{n=1}^{m} \lambda_n^{-p} \leqq K \lambda_m^{-p} m. \tag{1.8}$$

Then

$$\int_0^\pi \lambda^p(x) |f(x)|^p \, dx < \infty \tag{1.9}$$

implies

$$\sum_{n=1}^{\infty} \lambda_n^p n^{-2} \left(\sum_{k=1}^{n} a_k \right)^p < \infty, \tag{1.10}$$

where a_n *are the Fourier cosine coefficients of* $f(x)$ *and* $\lambda(x)$ $(x>0)$ *is a continuous function linear on the intervals* $[(n+1)^{-1}, n^{-1}]$, $\lambda(n^{-1})=\lambda_n$ *and* $\lambda(x)=\lambda_1$ *for* $x\geqq 1$.

LEMMA 4 ([7], Theorem 1). *Let* $\{\varphi_k\}$ *be a non-negative non-decreasing sequence of numbers with* $\sum_{k=m}^{\infty} \varphi_k k^{-2} \leqq K\varphi_m m^{-1}$, $\nu>p\geqq 1$ *and* $f(x)\in L^p(0,\pi)$. *Define*

$$\Phi(x) = \sum_{k=1}^{x} k^{\frac{\nu}{p}-2} \varphi_k .$$

Then

$$\sum_{n=1}^{\infty} n^{\frac{\nu}{p}-2} \varphi_n \omega_p^{\nu}\left(\frac{1}{n}, f\right) < \infty$$

implies

$$\int_0^{\pi} |f(x)|^{\nu-\frac{\nu}{p}+1} \Phi(|f(x)|)\, dx < \infty.^2$$

LEMMA 5 ([1], Theorem 2). *Let* $\{a_n\}$ *be a sequence which is monotonically decreasing to zero and such that for a fixed* p $(1<p<\infty)$

$$\sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty.$$

If $f(x)$ *is the sum of the series* $\sum_{n=1}^{\infty} a_n \cos nx$, *then*

$$\omega_p\left(\frac{1}{n}, f\right) \leqq K\frac{1}{n}\left\{\sum_{\nu=1}^{n} \nu^{2p-2} a_\nu^p\right\}^{1/p} + K\left\{\sum_{\nu=n}^{\infty} \nu^{p-2} a_\nu^p\right\}^{1/p}. \tag{1.11}$$

LEMMA 6 ([4], p. 864). *If* $\{a_n\}$ *is the sequence of Fourier cosine of* $f(x)$ *satisfying the condition*

$$\sum_{n=1}^{\infty} |\Delta a_k| < \infty$$

then for $p\geqq 1$

$$\int_0^{\pi} |f(x+t)+f(x-t)-2f(x)|^p\, dx \leqq Kt^{2p} \sum_{n=0}^{1/t} n^{3p-2}\left(\sum_{k=n}^{\infty} |\Delta a_k|\right)^p + $$

$$+K \sum_{n=1/t}^{\infty} n^{p-2}\left(\sum_{k=n}^{\infty} |\Delta a_k|\right)^p.$$

LEMMA 7 ([11], p. 338). *If* $p\geqq 1$ *and* $f(x)\in L^p(0,\pi)$ *then we have*

$$E_n^{(p)}(f) \leqq K\omega_p\left(\frac{1}{n}, f\right).$$

[2] We remark that in [7] this theorem was proved for (0,1) instead of $(0, \pi)$ but this has no importance.

LEMMA 8 ([11], p. 344). *If* $p \geqq 1$ *and* $f(x) \in L^p(0, \pi)$ *then we have*

$$\omega_p\left(\frac{1}{n}, f\right) \leqq Kn^{-1} \sum_{\nu=0}^{n} E_\nu^{(p)}(f).$$

LEMMA 9 ([12], Lemma 12). *Let* $A(x)$ *be a non-negative non-decreasing function on* $[0, \infty)$ *such that* $A(x^2) \leqq KA(x)$ *for any* $x \in [0, \infty)$ *and let* $B(x)$ *be a non-negative function on* $(0, \pi]$. *Then*

$$\int_0^\pi B(x) A\big(B(x)\big)\, dx < \infty$$

implies

$$\int_0^\pi B(x) A\left(\frac{1}{x}\right) dx < \infty.$$

§ 2. Proof of the theorems

AD THEOREM 1. First we prove (1). By the Parseval formula, with $\Phi_x(t) = f(x+t)+f(x-t)-2f(x)$, we have

$$I_{m,n} \equiv -\frac{1}{8\pi} \int_0^\pi \Phi_x(t) \sum_{\nu=m}^{n} \cos \nu x\, dx = \sum_{\nu=m}^{n} a_\nu \sin^2 \frac{1}{2} \nu t.$$

Hence we get with $q = \dfrac{p}{p-1}$

$$I_{m,n} \leqq K\left\{\int_0^\pi |\Phi_x(t)|^p\, dx\right\}^{1/p} \left\{\int_0^\pi \left| \sum_{\nu=m}^{n} \cos \nu x \right|^q dx\right\}^{1/q} \leqq \tag{2. 1}$$

$$\leqq K\left\{\int_0^\pi |\Phi_x(t)|^p\, dx\right\}^{1/p} \left\{\int_0^{1/n} n^q\, dx + \int_{1/n}^\pi x^{-q}\, dx\right\}^{1/q} \leqq K\left\{\int_0^\pi |\Phi_x(t)|^p\, dx\right\}^{1/p} n^{1/p}.$$

Putting $m = \left[\dfrac{n}{2}\right]$, we obtain by (2. 1)

$$\left(\sum_{\nu=\frac{n}{2}}^{n} a_\nu\right)^r \leqq Kn^{2+\frac{r}{p}} \int_{1/(n+1)}^{1/n} \left\{\int_0^\pi |\Phi_x(t)|^p\, dx\right\}^{\frac{r}{p}} dt$$

and hence

$$\sum_{n=1}^{\infty} \lambda(n) n^{-r} \left(\sum_{\nu=\frac{n}{2}}^{n} a_\nu\right)^r \leqq K \int_0^\pi \lambda\left(\frac{1}{t}\right) t^{-2-\frac{r}{p}+r} \left(\int_0^\pi |\Phi_x(t)|^p\, dx\right)^{\frac{r}{p}} dt,$$

which is the required inequality.

Now we prove the inequality (4). By Lemma 6 we obtain

$$\sum_{n=2}^{\infty}\int_{1/n}^{1/(n-1)}\lambda\left(\frac{1}{t}\right)t^{r-2-\frac{r}{p}}\left(\int_0^{\pi}|\Phi_x(t)|^p\,dx\right)^{\frac{r}{p}}dt\leqq$$

$$\leqq K\sum_{n=1}^{\infty}\lambda(n)n^{\frac{r}{p}-r-2r}\left(\sum_{\nu=1}^{n}\nu^{3p-2}\left(\sum_{k=\nu}^{\infty}|\Delta a_k|\right)^p\right)^{\frac{r}{p}}+$$

$$+K\sum_{n=1}^{\infty}\lambda(n)n^{\frac{r}{p}-r}\left(\sum_{\nu=n}^{\infty}\nu^{p-2}\left(\sum_{k=\nu}^{\infty}|\Delta a_k|\right)^p\right)^{\frac{r}{p}}\equiv\Sigma_1+\Sigma_2.$$

If $r\leqq p$, then by (3) we have

$$\Sigma_1\leqq K\sum_{n=1}^{\infty}\lambda(n)n^{\frac{r}{p}-3r}\left\{\sum_{m=0}^{\log n}\sum_{\nu=2^m}^{2^{m+1}}\nu^{3p-2}\left(\sum_{k=2^m}^{\infty}|\Delta a_k|\right)^p\right\}^{\frac{r}{p}}\leqq$$

$$\leqq K\sum_{n=1}^{\infty}\lambda(n)n^{\frac{r}{p}-3r}\sum_{m=0}^{\log n}2^{m\left(3r-\frac{r}{p}\right)}\left(\sum_{k=2^m}^{\infty}|\Delta a_k|\right)^r\leqq$$

$$\leqq K\sum_{m=0}^{\infty}2^{m\left(3r-\frac{r}{p}\right)}\left(\sum_{k=2^m}^{\infty}|\Delta a_k|\right)^r\sum_{n=2^m}^{\infty}\lambda(n)n^{\frac{r}{p}-3r}\leqq$$

$$\leqq K\sum_{m=0}^{\infty}2^m\lambda(2^m)\left(\sum_{k=2^m}^{\infty}|\Delta a_k|\right)^r\leqq K\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^r;$$

and by (2) we obtain

$$\Sigma_2\leqq K\sum_{n=1}^{\infty}\lambda(n)n^{\frac{r}{p}-r}\left(\sum_{m=\log n}^{\infty}2^{m(p-1)}\left(\sum_{k=2^m}^{\infty}|\Delta a_k|\right)^p\right)^{\frac{r}{p}}\leqq$$

$$\leqq K\sum_{n=1}^{\infty}\lambda(n)n^{\frac{r}{p}-r}\sum_{m=\log n}^{\infty}2^{mr\left(1-\frac{1}{p}\right)}\left(\sum_{k=2^m}^{\infty}|\Delta a_k|\right)^r\leqq$$

$$\leqq K\sum_{m=1}^{\infty}2^{mr\left(1-\frac{1}{p}\right)}\left(\sum_{k=2^m}^{\infty}|\Delta a_k|\right)^r\sum_{n=1}^{2^m}\lambda(n)n^{\frac{r}{p}-r}\leqq$$

$$\leqq K\sum_{m=1}^{\infty}2^m\lambda(2^m)\left(\sum_{k=2^m}^{\infty}|\Delta a_k|\right)^r\leqq K\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^r.$$

If $p<r$ we use Lemma 1 to estimate the sums Σ_1 and Σ_2. By (3) and (1. 1) with $\lambda_n=\lambda(n)n^{\frac{r}{p}-3r}$ and $\alpha_n=n^{3p-2}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p$ we obtain

$$\Sigma_1\leqq K\sum_{n=1}^{\infty}\left(\lambda(n)n^{\frac{r}{p}-3r}\right)^{1-\frac{r}{p}}\left(\lambda(n)n^{\frac{r}{p}-3r+1}\right)^{\frac{r}{p}}\cdot\left(n^{3p-2}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p\right)^{\frac{r}{p}}\leqq$$

$$\leqq K\sum_{n=1}^{\infty}\lambda(n)\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^r.$$

By (2) and (1. 2) with $\lambda_n = \lambda(n) n^{\frac{r}{p}-r}$ and $\alpha_n = n^{p-2}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p$ we get

$$\Sigma_2 \leqq K \sum_{n=1}^{\infty}\left(\lambda(n) n^{\frac{r}{p}-r}\right)^{1-\frac{r}{p}}\left(\lambda(n) n^{\frac{r}{p}-r+1}\right)^{\frac{r}{p}} \cdot \left(n^{p-2}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p\right)^{\frac{r}{p}} \leqq$$

$$\leqq K \sum_{n=1}^{\infty} \lambda(n)\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^r.$$

The proof is thus complete.

AD THEOREM 2. The scheme of the proof can be illustrated in the following way:

$$(7)\Rightarrow(8)\Rightarrow(9)\Rightarrow(8), \quad (9)\Rightarrow(10)\Rightarrow(11)\Rightarrow(10)\Rightarrow(12)\Rightarrow(13)\Rightarrow(14)\Rightarrow(7)\Rightarrow(15)\Rightarrow(7).$$

The implication (7)$\Rightarrow$(8) can be proved by using the inequality (1. 1) of Lemma 1 with $\lambda_n = \varphi_n n^{-rs+\frac{r}{p}-2}$, $\alpha_n = n^{(s+1)p-2}$ and with exponent $\frac{p}{r}$. First we remark that the condition

$$\varphi_{n^2} \leqq K\varphi_n \tag{2. 2}$$

and $s > \frac{1}{p} - \frac{1}{r}$ imply the estimation

$$\sum_{n=m}^{\infty} \varphi_n n^{r\left(\frac{1}{p}-s\right)-2} \leqq K\varphi_m m^{r\left(\frac{1}{p}-s\right)-1}.$$

Thus we have

$$\Sigma_2 \leqq K \sum_{n=1}^{\infty}\left(\varphi_n n^{r\left(\frac{1}{p}-s\right)-2}\right)^{1-\frac{r}{p}}\left(\varphi_n n^{r\left(\frac{1}{p}-s\right)-1}\right)^{\frac{r}{p}} \cdot \left(n^{(s+1)p-2} a_n^p\right)^{\frac{r}{p}} \leqq$$

$$\leqq K \sum_{n=1}^{\infty} \varphi_n n^{r-2} a_n^r \leqq K\Sigma_1,$$

which gives the conclusion.

To prove (8)$\Rightarrow$(9) we use the inequality (1. 6) of Lemma 2 with $\lambda_n = \varphi_n^{p/r} n$, $\alpha_n = n^{p-2} a_n^p$, $\gamma = sp$ and $\beta = \frac{r}{p}$, since in this case, by (2. 2), the inequality (1. 5) is satisfied. Hence

$$\Sigma_3 \leqq K \sum_{n=1}^{\infty} (\varphi_n^{p/r} n)^{r/p} n^{-2-rs}\left(\sum_{k=1}^{n} k^{(s+1)p-2} a_k^p\right)^{r/p} = K\Sigma_2.$$

The statement (9)$\Rightarrow$(8) can be proved similarly using the inequality (1. 4) of Lemma 2 with the same λ_n, α_n, γ and β as before. Then, by (2. 2) and $s > \frac{1}{p} - \frac{1}{r}$,

(1. 3) is satisfied, thus (1. 4) gives

$$\Sigma_2 \leqq K \sum_{n=1}^{\infty} (\varphi_n^{p/r} n)^{r/p} n^{-2} \left(\sum_{k=n}^{\infty} k^{p-2} a_k^p \right)^{r/p} = K\Sigma_3 .$$

In order to prove that (9) implies (10) we apply Lemma 5. It is clear that (8) with $s=1$ follows from (9), that is we have

$$\sum_{n=1}^{\infty} \varphi_n n^{\frac{r}{p}-r-2} \left(\sum_{k=1}^{n} k^{2p-2} a_k^p \right)^{\frac{r}{p}} < \infty .$$

Hence and from (9) and (1. 11) we obviously obtain (10).

The implication (10)⇒(11) is trivial by Lemma 7.

The implication (11)⇒(10) can be proved by Lemma 8 and Lemma 1. Namely

$$\Sigma_4 \leqq K \sum_{n=1}^{\infty} \varphi_n n^{\frac{r}{p}-2-r} \left(\sum_{\nu=0}^{n} E_\nu^{(p)}(f) \right)^r \leqq$$

$$\leqq K \sum_{n=1}^{\infty} \left(\varphi_n n^{\frac{r}{p}-2-r}\right)^{1-r} \left(\varphi_n n^{\frac{r}{p}-1-r}\right)^r E_n^{(p)}(f)^r \leqq K \sum_{n=1}^{\infty} \varphi_n n^{\frac{r}{p}-2} \left(E_n^{(p)}(f)\right)^r \leqq K\Sigma_5 .$$

The implication (10)⇒(12) is also obvious by Lemma 4 with $\nu=r$.

To prove (12)⇒(13) we give the following inequality:

$$x^{\frac{r}{p}-1} \varphi(x) \leqq K \sum_{n=\frac{x}{2}}^{x} n^{\frac{r}{p}-2} \varphi(n) \leqq K\Phi(x) \qquad (x \geqq 1).$$

Hence the required inequality $I_2 \leqq KI_1$ plainly follows.

The implication (13)⇒(14) can be proved by using Lemma 9. Namely it is clear by (2. 2) that

$$\varphi\left(|f(x)|^r\right) \leqq K\varphi\left(|f(x)|\right)$$

and thus, by (13), we have

$$\int_0^\pi |f(x)|^r \varphi\left(|f(x)|^r\right) dx \leqq KI_2 .$$

Hence, by Lemma 9, we obtain $I_3 \leqq KI_2$.

In order to prove the implication (14)⇒(7) we use Lemma 3 with $p=r$ and $\lambda(x) = \varphi\left(\frac{1}{x}\right)^{\frac{1}{r}}$. Thus we get

$$\sum_{n=1}^{\infty} \varphi_n n^{-2} \left(\sum_{k=1}^{n} a_k \right)^r < \infty$$

which implies $\Sigma_1 < \infty$.

The implications (7)⇒(15)⇒(7) can easily be proved by the Corollary given above, putting $\lambda(x)=x^{r-2}\varphi(x)$.

Collecting our results we have completed our proof.

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ÜBER DIREKTE ZERLEGUNGEN VON ERWEITERUNGSRINGEN

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1. Einleitung

Es bezeichne A einen kommutativen Ring mit Einselement 1 und A sei direkte Summe einer endlichen Familie $(B_i)_{1 \leqq i \leqq n}$ von Unterringen. Bekanntlich gibt es dann (vgl. RÉDEI [3], § 44; KERTÉSZ [1], § 47) orthogonale Idempotente $e_1 \in B_1$, $e_2 \in B_2, \ldots, e_n \in B_n$ derart, daß $1 = e_1 + e_2 + \cdots + e_n$ ist und für jedes B_i $(1 \leqq i \leqq n)$ $B_i = e_i A$ gilt, also

(1) $$A = e_1 A \oplus e_2 A \oplus \cdots \oplus e_n A.$$

Ist umgekehrt e_i ein idempotentes Element aus A, dann ist A direkte Summe von $\mathfrak{a} = e_i A$ und dem Annulatorideal $\mathfrak{b}$ von e_i.

Sei nun R ein kommutativer Ring mit Einselement, der A enthält. Es ist klar, daß jede direkte Zerlegung (1) von A eine direkte Zerlegung von R induziert: $R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_n R$. In vorliegender Arbeit wollen wir untersuchen, unter welchen Bedingungen auch jede direkte Zerlegung von R eine direkte Zerlegung von A induziert, das heißt, mit jeder direkten Zerlegung

(2) $$R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_n R$$

von R auch folgt, daß

(3) $$e_1 A + e_2 A + \cdots + e_n A$$

eine direkte Zerlegung von A ist. Wir nennen (3) die Einschränkung der direkten Zerlegung (2) auf A.

Im allgemeinen scheint es schwierig zu sein, diese Frage zu beantworten. Wir beweisen im folgenden für den Spezialfall, daß R der Polynomring $A[x_1, x_2, \ldots, x_n]$ in n Unbestimmten über A ist, daß die Einschränkung einer direkten Zerlegung von $A[x_1, x_2, \ldots, x_n]$ auf A stets eine direkte Zerlegung von A ist. Weiters geben wir eine notwendige und hinreichende Bedingung dafür an, daß die Einschränkung jeder direkten Zerlegung des vollen Quotientenringes von A auf A eine direkte Zerlegung von A ist und zeigen, daß jeder artinsche Ring dieser Bedingung genügt.

Allgemein gilt folgender

HILFSSATZ 1. *Ist R kommutativer Ring mit Einselement der A enthält, so ist die Einschränkung* (3) *einer direkten Zerlegung* (2) *von R auf A dann und nur dann eine direkte Zerlegung von A, wenn alle $e_i \in A$ $(i = 1, 2, \ldots, n)$.*

BEWEIS. Seien die $e_i \in A$ $(i = 1, 2, \ldots, n)$, dann sind die $e_i A$ Unterringe von A und jedes Element aus R und damit erst recht jedes Element $a \in A$ besitzt eine eindeutige Darstellung $a = e_1 a + e_2 a + \cdots + e_n a$. Daher gilt $A = e_1 A \oplus e_2 A \oplus \cdots \oplus e_n A$.

Die Umkehrung ist unmittelbar einzusehen.

BEMERKUNG 1. Ist R Integritätsbereich, so wird das Problem trivial. R und A sind dann direkt unzerlegbar.

Die direkten Zerlegungen eines Ringes R stehen auf Grund ihrer Verbindung mit den Idempotenten aus R in engem Zusammenhang mit der Existenz von gewissen Systemen von Abbildungen von R in sich. Siehe dazu [2].

2. Der Polynomring über A

Sei $A[x_1, x_2, ..., x_n]$ der Polynomring in n Unbestimmten über A. Es gilt folgender

SATZ 1. *Ist $A[x_1, x_2, ..., x_n]$ direkte Summe einer endlichen Familie von Unterringen, so ist die Einschränkung dieser direkten Zerlegung auf den Grundring A stets eine direkte Zerlegung von A.*

BEWEIS. Wir schreiben für den Polynomring $A[x_1, x_2, ..., x_n]$ zur Abkürzung $A[x_r]$. Es sei $A[x_r] = B_1[x_r] \oplus B_2[x_r] \oplus \cdots \oplus B_n[x_r]$ eine direkte Zerlegung von $A[x_r]$. Wie wir wissen, gilt dann $B_i[x_r] = e_i A[x_r]$ $(i=1, 2, ..., n)$ mit geeignet gewählten

$$e_i = e_i e_i \in B_i[x_r]. \tag{4}$$

Denken wir uns die e_i $(i=1, 2, ..., n)$ als Summen von Formen g_k^i $(0 \leqq k \leqq m)$ vom Grad k geschrieben:

$$e_i = g_0^i + g_s^i + g_{s+1}^i + \cdots + g_m^i \quad \text{mit} \quad g_s^i \neq 0. \tag{5}$$

Durch Einsetzen von (5) in (4) und Gradvergleich ergibt sich sofort $g_0^i = g_0^i g_0^i$ und

$$g_s^i = 2 g_0^i g_s^i. \tag{6}$$

Daraus folgt $g_0^i g_s^i = 2 g_0^i g_s^i$, also $g_0^i g_s^i = 0$ und daher wegen (6) auch $g_s^i = 0$. Damit ist (5) unmöglich und es folgt $e_i = g_0^i \in A$ $(i=1, 2, ..., n)$. Somit sind also die Voraussetzungen von Hilfssatz 1 erfüllt und die Einschränkung von $A[x_r] = B_1[x_r] \oplus \oplus B_2[x_r] \oplus \cdots \oplus B_n[x_r]$ auf A ist eine direkte Zerlegung von A und es gilt $A = B_1 \oplus \oplus B_2 \oplus \cdots \oplus B_n$, wobei $B_i = e_i A$ $(i=1, 2, ..., n)$.

3. Der volle Quotientenring von A

Es sei A wiederum kommutativer Ring mit Einselement und A_s bezeichne den vollen Quotientenring von A. Dieser wird bekanntlich (vgl. etwa [3], § 47) durch die Gesamtheit der Elemente ab^{-1} gebildet, wobei a ein beliebiges, b ein reguläres Element aus A ist. Gleichheit, Summe und Produkt von solchen Elementen sind in der üblichen Art definiert. Sei nun

$$A_s = e_1 A_s \oplus e_2 A_s \oplus \cdots \oplus e_n A_s \tag{7}$$

eine direkte Zerlegung von A_s. Nach Hilfssatz 1 ist die Einschränkung von (7) auf A genau dann eine direkte Zerlegung von A, wenn alle e_i $(i=1, 2, ..., n)$ aus A sind. Wir beweisen dazu

HILFSSATZ 2. *Ein idempotentes Element* $\frac{f}{g} \in A_s$ *ist dann und nur dann aus* A, *wenn es eine Darstellung* $f=ha$ *besitzt, wobei* h *reguläres Element aus* A *und* a *Idempotente aus* A *ist.*

BEWEIS. Die Bedingung ist notwendig, denn ist $\frac{f}{g} = a \in A$, so folgt aus $\frac{f}{g} = \frac{f^2}{g^2}$, daß a Idempotente ist und es gilt $f=ga$.

Umgekehrt sei $\frac{f}{g}$ Idempotente aus A mit

$$f=ha, \tag{8}$$

wobei h reguläres und a idempotentes Element aus A ist. Es folgt dann aus $\frac{f}{g} = \frac{f^2}{g^2}$

$$fg=f^2 \tag{9}$$

und, wenn wir (8) in (9) einsetzen, $hag=h^2 a$. Da h invertierbar in A_s ist, gilt

$$a\frac{g}{h} = a. \tag{10}$$

Wir setzen nun (10) in (8) ein und bekommen $f = ha\frac{g}{h} = ag$, also folgt $\frac{f}{g} = a \in A$.

Aus Hilfssatz 1 und Hilfssatz 2 folgt unmittelbar

SATZ 2. *Die Einschränkung einer direkten Zerlegung* $A_s = e_1 A_s \oplus e_2 A_s \oplus \cdots \oplus e_n A_s$ *auf* A *ist dann und nur dann eine direkte Zerlegung von* A, *wenn es für jedes idempotente Element* $e_i = \frac{f_i}{g_i}$ $(i=1, 2, \dots, n)$ *eine Darstellung* $f_i=h_i a_i$ *gibt, wobei* h_i *reguläres und* a_i *idempotentes Element aus* A *ist.*

Leider müssen wir das Problem der Charakterisierung aller Ringe A_s, in denen die Bedingung von Hilfssatz 2 für alle Idempotenten zutrifft, offen lassen. Ersichtlich sind das genau jene Ringe A_s, bei denen die Einschränkung jeder direkten Zerlegung auf A eine direkte Zerlegung von A ist.

Es gilt jedoch folgender hinreichender

HILFSSATZ 3. *Sei* $\frac{f}{g}$ *Idempotente aus* A_s. *Bricht die absteigende Kette von Hauptidealen* $fA \supseteqq f^2 A \supseteqq \cdots$ *nach endlich vielen Schritten ab, so gilt* $\frac{f}{g} \in A$.

BEWEIS. Da die Idealkette $fA \supseteqq f^2 A \supseteqq \cdots$ nach endlich vielen Schritten abbrechen soll, muß für ein natürliches n gelten

$$f^n A = f^{n+1} A. \tag{11}$$

Aus (11) folgt

$$f^n = af^{n+1} \quad \text{mit} \quad a \in A. \tag{12}$$

Wegen $\frac{f^2}{g^2} = \frac{f}{g}$ gilt

$$f^2 = fg. \tag{13}$$

Durch k-maliges Anwenden von (13) ergibt sich

(14) $$f^{k+1} = fg^k.$$

Formen wir (12) mittels (14) um, so erhalten wir $fg^{n-1} = afg^n$, woraus $\frac{f}{g} = af \in A$ folgt.

BEMERKUNG 2. Es gibt einfache Beispiele dafür, daß die Bedingung von Hilfssatz 3 nicht notwendig ist. Ist jedoch A artinscher Ring, dann sind die Voraussetzungen von Hilfssatz 3 erfüllt, denn es gilt dann sogar, daß jede fallende Idealkette aus A abbricht.

Als Folgerung von Bemerkung 2, Hilfssatz 3 und Hilfssatz 1 erhalten wir

SATZ 3. *Ist A artinscher Ring, so ist die Einschränkung jeder direkten Zerlegung von A_s auf A eine direkte Zerlegung von A.*

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THE NUMBER OF CUT VERTICES AND CUT ARCS IN A STRONG DIRECTED GRAPH

By

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In this paper we determine the ranges of the number of cut vertices and the number of cut arcs in a strong graph on n vertices with m arcs. Also it is proved that a strong complete graph on $n(>3)$ vertices has at most $n-2$ cut vertices, without using the existence of a Hamiltonian circuit, thus solving a problem posed by KORVIN

0. Introduction and definitions

In [5], one of the authors determined the ranges of the number of cut vertices and the number of cut edges in an undirected (connected) graph on n vertices with m edges. In this paper we solve the corresponding problem for strong directed graphs, and characterize some of the extremal graphs. This generalizes some of the results of GUPTA [2]. Also it is proved that a strong complete graph on $n>3$ vertices has at most $n-2$ cut vertices, without using Camion's theorem. This solves a problem raised by KORVIN at the International Symposium on Graph Theory held in Rome [4].

For notation and terminology, BERGE [1] is generally followed. However we use the following definitions.

All the graphs considered are directed graphs with neither loops nor multiple arcs and are strong unless otherwise mentioned.

An arc (x, y) of G is symmetric if $(y, x) \in G$. A symmetric graph whose associated undirected graph is a tree is called a symmetric tree. G^s denotes the partial graph of G consisting of all symmetric arcs of G.

A vertex (arc) of a strong graph G is called a cut vertex (cut arc) if its deletion makes the graph not strong.

$n(G)$, $m(G)$, $r(G)$, $s(G)$ denote the numbers of vertices, arcs, cut vertices and cut arcs of G respectively. Also $R(G)$ denotes the set of cut vertices of G.

$d_G^+(x)$, $d_G^-(x)$ denote the outdegree and the indegree of x in G respectively. The degree of x is the sum of the indegree and the outdegree of x.

The converse of G has the same vertices as G and (x, y) is an arc in the converse if and only if $(y, x) \in G$.

1. The number of cut vertices in a graph

In this section, by a maximal graph on n vertices with r cut vertices, we mean a graph such that the addition of any arc decreases the number of cut vertices. An extremal graph is a graph with n vertices, r cut vertices and with the maximum number of arcs.

LEMMA 1. 1. *If G is maximal and there are two vertex disjoint paths from x to y, then $(x, y) \in G$.*

PROOF. If $(x, y) \notin G$, add the arc (x, y) to G. If this converts some cut vertex u of G into a non-cut vertex, then u belongs to every path from x to y in G, a contradiction.

LEMMA 1. 2. *Let G be a maximal graph and x a non-cut vertex of G. Then $R(G-x) \subseteqq R(G)$. Further if $y \in R(G)-R(G-x)$, then either $d^+(x)=1$ and $(x, y) \in G$ or $d^-(x)=1$ and $(y, x) \in G$.*

PROOF. Suppose some non cut vertex z of G is a cut vertex of $G-x$. Then there exist vertices u, v such that every path from u to v in $G-x$ passes through z. Since z is a non-cut vertex of G, there is a path ν from u to v in $G-z$. Let $\mu=[u, u_1, \ldots$ $\ldots, u_{k-1}, z, u_{k+1}, \ldots, v]$ be a path from u to v in $G-x$. If μ and ν are vertex disjoint, $(u, v) \in G$ since G is maximal. Even otherwise $(u_i, u_j) \in G$ for some $i \leqq k-1$ and some $j \geqq k+1$. Thus there is a path from u to v not passing through z in $G-x$, a contradiction.

Now let $y \in R(G)-R(G-x)$. We consider two cases.

Case (i). Every path from x to some vertex v in G passes through y. Now if u is any vertex other than y and $(x, u) \in G$, then since $G_1 = G-\{x, y\}$ is strong, there is a path from u to v in G_1. Thus there is a path from x to v, not passing through y, in G. This contradiction shows that $d^+(x)=1$ and $(x, y) \in G$.

Case (ii). Every path from some vertex v to x in G passes through y. As in case (i), it follows that $d^-(x)=1$ and $(y, x) \in G$. This completes the proof of the lemma.

LEMMA 1. 3. *Let G be a maximal graph and x a non-cut vertex of G such that $R(G) = R(G-x)$. If the degree of x in G is at least $2n-2-k$, then $r(G) \leqq k$.*

PROOF. If $k = n-1$, the lemma is trivial. So let $k \leqq n-2$. Evidently now there exists a path $[x, y, z]$ in G such that $(x, z) \notin G$. Since G is maximal, it follows that y is a cut vertex of G and is a non-cut vertex of the graph G^* obtained from G by adding the arc (x, z). If G^* is not maximal let H be a maximal graph containing G^* such that $R(G^*) = R(H) = R(G)-y$. By lemma 1. 2, $R(H-x) \subseteqq R(H)$. Let now u be a non-cut vertex of $H-x$. Then $H-\{x, u\}$ is strong. So if u is a cut vertex of H, then either $d_H^+(x)=1$ or $d_H^-(x)=1$. Now $d_H^+(x)=1$ is not possible, so $d_H^-(x)= =1=d_G^-(x)$. Since $k \leqq n-2$, by hypothesis, $d_G^+(x) \geqq n-1$, a contradiction since $(x, z) \notin G$. Thus u is a non-cut vertex of H and $R(H-x) = R(H)$. Now H satisfies the hypothesis of the lemma with k replaced by $k-1$ and the proof is completed by using induction on k.

LEMMA 1. 4. *Let G be a maximal graph, x a non-cut vertex of G and let $r(G) = = n-1$. Then the degree of x is at most $n-1$.*

PROOF. This lemma follows from lemma 1. 3 if $R(G-x) = R(G)$. If $r(G) = = r(G-x)+2$, then by lemma 1. 2, $d^+(x)=d^-(x)=1$. So we may take that $R(G-x) = R(G)-y$ where $y \in R(G)$. Now we prove the lemma by induction on n. The lemma is trivial for $n=3$, so assume the result for $n-1$ and let G be a graph with n vertices and satisfying the hypothesis of the lemma. Without loss of generality we may take $d^+(x)=1$ and $(x, y) \in G$. If possible let the degree of $x \geqq n$. Then evidently

$d^-(x) = n-1$. Since G is maximal and $G-x$ is strong, it follows by lemma 1. 1 that $d^-(y) = n-1$.

First we show that $G-x$ is maximal. Otherwise add an arc α to $G-x$ such that the resulting graph has the same cut vertices as $G-x$. Let H be the graph $G+\alpha$. Now $R(H) \subseteq R(G)$. If possible let $u \in R(G)-R(H)$. Since $d_H^+(x)=1$, $y \in R(H)$ and $u \neq y$. Suppose now that u is a cut vertex of $H-x$ and v_1, v_2 are two vertices of $H-x$ such that every path from v_1 to v_2 in $H-x$ passes through u. Since $u \notin R(H)$, there is a path μ from v_1 to v_2 not containing u in H. Evidently now $x \in \mu$. Now the vertex succeeding x on μ is y and if z is the vertex preceding x on μ, then $(z, y) \in H$ since $d_H^-(y) = n-1$. Thus there is a path from v_1 to v_2 in $H-x$ and not containing u. This contradiction shows that u is a non-cut vertex of $H-x$. By the definition of α, u is a non-cut vertex of $G-x$ and so of G. This contradiction shows that $R(H) = R(G)$. But then G is not maximal. This finally proves that $G-x$ is maximal.

Since $r(G) = n-1$, $r(G-x) = n-2$ and y is a non-cut vertex of $G-x$. Hence by induction hypothesis, the degree of y in $G-x$ is at most $n-2$. This is a contradiction since $G-x$ is strong and the indegree of y in $G-x$ is $n-2$. Thus our supposition that the degree of x in $G \geqq n$ leads to a contradiction and the lemma is proved.

THEOREM 1. 5. *The maximum number of arcs in a strong graph with $n \geqq 4$ vertices and r cut vertices is*

$$A(n, r) = \begin{cases} \binom{n-1}{2} + 3 & \text{if } r = n \\ \binom{n}{2} + 1 & \text{if } r = n-1 \\ \binom{n}{2} + \binom{n-r}{2} + r & \text{if } r \leqq n-2 \end{cases}$$

Further the extremal graphs with n vertices and r cut vertices are $G_i(n)$, $i=1, 2, \dots, 7$ described below.

$G_i(n)$ has the vertices $1, 2, \dots, n$.

In $G_1(n)$, (i, j) is an arc if either $1 \leqq i < j \leqq n-2$ or $i = j+1$ or $i=1$ and $j=n$ or $i = n-1$ and $j=n$.

In $G_2(n)$, (i, j) is an arc if either $1 \leqq i < j \leqq n-4$ or $i = j+1$ or $1 \leqq i \leqq n-3$ and $j = n-2$ or $i = n-1$ and $1 \leqq j \leqq n-4$ or $i = n-1$ and $j=n$ or $i=1$ and $j=n$.

$G_3(n)$ is the converse of $G_2(n)$.

In $G_4(n)$, (i, j) is an arc if either $1 \leqq i < j \leqq n-1$ or $i = j+1$ or $i=1$ and $j=n$.

In $G_5(n)$, (i, j) is an arc if either $1 \leqq i < j \leqq n-2$ or $i = j+1$ or $i = n-1$ or $i=1$ and $j=n$.

$G_6(n)$ is the converse of $G_5(n)$.

In $G_7(n, r)$, (i, j) is an arc if either $1 \leqq i < j \leqq n$ or $i = j+1$ or $k \leqq j < i \leqq k+n-r-1$, where k is a fixed integer such that $1 \leqq k \leqq r+1$.

It may be noted that $G_i(n)$ has n or $n-1$ cut vertices according as $1 \leqq i \leqq 3$ or $4 \leqq i \leqq 6$ and $G_7(n, r)$ has r cut vertices.

PROOF. We prove the theorem by induction on n. The theorem is easily verified when $n=4$ using the list of digraphs on 4 vertices given in [3]. So assume the result for $n-1$ and let G be an extremal graph on $n \geqq 5$ vertices with r cut vertices.

The graphs described in the statement of the theorem have $A(n, r)$ arcs where n is the number of vertices and r is the number of cut vertices. Thus G has at least $A(n, r)$ arcs. We now prove that G has at most $A(n, r)$ arcs and is one of the graphs described in the theorem.

First let $r=n$. Let x be a vertex of G. Then there exist vertices y and z such that (y, x) and (x, z) are arcs of G, and every path in G from y to z passes through x. Let H be the graph obtained from G by adding the arc (y, z) and dropping the vertex x. Since G is maximal, H is strong. Evidently any vertex of G other than x, y, z is a cut vertex of H also.

If $r(H) = n-1$, then consider the graph $G' = G+(y, z)$. If G^* is a maximal graph containing G' with $R(G^*)=R(G')$, then by lemma 1. 4, the degree of x in G^* and hence in G is at most $n-1$. Now by induction hypothesis, H has at most $\binom{n-2}{2}+3$ arcs. Thus G has at most $\binom{n-2}{2}+3+(n-1)-1 = A(n, n)$ arcs. Now it also follows that H is an extremal graph and the degree of x in G is $n-1$. We consider the case $H = G_2(n-1)$, the other two cases can be disposed similarly. Now the only cut arcs of $G_2(n-1)$ are $(1, n-1)$ and $(i+1, i)$ where $i \neq n-3$. Thus (y, z) is $(1, n-1)$ or $(i+1, i)$ with $i \neq n-3$. If $i = n-2$, then x is not adjacent to any of the vertices $1, 2, \dots, n-3$, for otherwise $n-2$ or $n-1$ will be a non-cut vertex of G. This is a contradiction since the degree of x in G is $n-1$. The case $i = n-4$ is similar. If (y, z) is $(1, n-1)$ or $(i+1, i)$ with $1 \leqq i \leqq n-5$, then the symmetric path $[1, 2, \dots, n-5]$ extends in length by unity with the vertex x. Thus $G=G_2(n)$. This proves that if $r(H) = n-1$, then $G=G_i(n)$ for some $i \leqq 3$.

Now let $r(H) \leqq n-2$. Then we may take that y is a non-cut vertex of H, for, the case z is a non-cut vertex of H is similar. Evidently then $G-\{x, y\}$ is strong. If $d_G^-(x) \geqq 2$, then y is a non-cut vertex of G. If $d_G^+(y) \geqq 2$, then there is a path in G from y to z and not using x. Thus $d^-(x)=d^+(y)=1$. Now let G_0 be the graph obtained from G by amalgamating the vertices x and y. Let x_0 denote the vertex of G_0 obtained by amalgamating x and y of G. If any vertex u of G_0 other than x_0 is a non-cut vertex of G_0, then u is a non-cut vertex of G also (since (y, x) is the only arc incident into x and is the only arc incident out from y). This contradiction shows that $r(G_0) = n-2$. We now show that G_0 is maximal. If G_0 is not maximal, let the addition of an arc (a, b) not affect the cut vertices of G_0. Then consider the graph $G+(a, b)$, $G+(a, y)$ or $G+(x, b)$ according as $a \neq x_0$ and $b \neq x_0$, $b=x_0$ or $a=x_0$. This graph has the same cut vertices as G, a contradiction. Thus G_0 is maximal and by lemma 1. 4, the degree of x_0 in G_0 is at most $n-2$.

If now $d_{G_0}^-(x_0)=1$ and $d_{G_0}^+(x_0)=1$, then $G-\{x, y\}$ has at least $n-4$ cut vertices, hence by induction hypothesis, $G-\{x, y\}$ has at most $\binom{n-2}{2}+n-3$ arcs. Thus G has at most $\binom{n-2}{2}+n-3+4 = A(n, n)$ arcs. It also follows that $G-\{x, y\}$ has exactly $n-4$ cut vertices and is extremal and $G=G_1(n)$.

If $d_{G_0}^-(x_0) \geqq 2$ or $d_{G_0}^+(x_0) \geqq 2$, then $G-\{x, y\}$ has at least $n-3$ cut vertices, hence by induction hypothesis, $G-\{x, y\}$ has at most $\binom{n-2}{2}+1$ arcs. Thus G has at most $\binom{n-2}{2}+1+(n-2)+2 = A(n, n)$ arcs. It also follows that $G-\{x, y\}$ has

$n-3$ cut vertices and is extremal. To be specific, let $d^-_{G_0}(x_0) \geqq 2$. If $G-\{x, y\}$ is $G_5(n-2)$, then it follows that $d^+_{G_0}(x_0)=1$ and $(x, n-4) \in G$. If $(n-2, y) \in G$, then $n-3$ is a non-cut vertex of G. So $(i, y) \in G$ for $1 \leqq i \leqq n-3$. Thus $G=G_2(n)$. If $G-\{x, y\} = G_4(n-2)$, then again $d^+_{G_0}(x_0)=1$ and $(x, n-2) \in G$. If $n \geqq 6$, then $(i, y) \in G$ for some i with $2 \leqq i \leqq n-3$, hence 1 is a non-cut vertex of G. Thus $n=5$ and $G=G_2(5)$. The case $G-\{x, y\} = G_6(n-2)$ is similar to the case $G-\{x, y\} = G_5(n-2)$. When $d^+_{G_0}(x_0) \geqq 2$ the proof is similar. Thus we have proved that when $r=n$, $G=G_i(n)$ for some $i \leqq 3$.

Next let $r \leqq n-1$. Then G has a non-cut vertex x. By lemma 1. 2, $r(G) \geqq r(G-x) \geqq r(G)-2$. Thus we have three cases.

Case (i). $r(G-x) = r(G)$. Then by lemma 1. 3, there are at most $2n-2-r$ arcs of G incident with x. By induction hypothesis, $G-x$ has at most $A(n-1, r)$ arcs. So G has at most $A(n, r)$ arcs. It also follows that $r \leqq n-3$, $G-x$ is extremal and the degree of x in G is $2n-2-r$. By induction hypothesis, $G-x$ is $G_7(n-1, r)$. Now observe that i is a cut vertex of $G_7(n-1, r)$ if and only if $2 \leqq i \leqq k$ or $k+n-r-2 \leqq i \leqq n-2$. Also if (x, i), (x, j) are symmetric arcs of G, then all vertices λ with $i<\lambda<j$ are non-cut vertices of $G-x$. Now since the degree of x in G is $2n-2-r$, there are at least $n-1-r$ symmetric arcs incident with x.

If $r = n-3$, evidently there are exactly 2 symmetric arcs (x, i_0) and (x, i_0+1) incident with x and since G is extremal, $(j, x) \in G$ for $1 \leqq j \leqq i_0-1$ and $(x, j) \in G$ for $i_0+2 \leqq j \leqq n-1$. Thus $G=G_7(n, r)$.

If $r \leqq n-4$, then there is no symmetric arc (x, i) whenever $i<k$ or $i > k+n-r-2$. Thus (x, i) is a symmetric arc for $k \leqq i \leqq k+n-r-2$, $(i, x) \in G$ for $1 \leqq i \leqq k-1$, and $(x, i) \in G$ for $k+n-r-1 \leqq i \leqq n-1$. Thus $G=G_7(n, r)$.

Case (ii). $r(G-x) = r(G)-1$. Let $y \in R(G)-R(G-x)$. Then by induction hypothesis, $G-x$ has at most $A(n-1, r-1)$ arcs. By lemma 1. 2, we may take, without loss of generality, that $d^+(x)=1$ and $(x, y) \in G$. So there are at most n arcs of G incident with x.

If now $r \leqq n-2$, then G has at most $A(n, r)$ arcs. It also follows that the degree of x is n and $G-x$ is extremal. So $G-x = G_7(n-1, r-1)$ and evidently either $y = n-1$ or $k=r<y$. Thus $G=G_7(n, r)$.

If $r = n-1$, then by lemma 1. 4, the degree of x in G is at most $n-1$. By induction hypothesis, $G-x$ has at most $A(n-1, n-2)$ arcs. So G has at most $A(n, n-1)$ arcs and $G-x$ is extremal. If now $G-x = G_4(n-1)$, then the vertex 1 of $G-x$ is a non-cut vertex of G, a contradiction. If $G-x = G_5(n-1)$, then $(n-1, x) \notin G$, hence $(i, x) \in G$ for $i \leqq n-2$. Thus $G=G_5(n)$. If $G-x = G_6(n-1)$, then $(n-2, x) \notin G$, hence $(i, x) \in G$ for all $i \neq n-2$. But then 1 is a non-cut vertex of G, a contradiction This completes case (ii).

Case (iii). $r(G-x) = r(G)-2$. Let $y, z \in R(G)-R(G-x)$. Then by induction hypothesis, $G-x$ has at least $A(n-1, r-2)$ arcs. Also by $G-x$ has at most $A(n-1, r-2)$ arcs. Also by lemma 1. 2, $d^+(x)=d^-(x)=1$. Since G has at least $A(n, r)$ arcs, it follows that $r = n-1$ and $G-x$ is extremal. Thus $G-x$ is $G_7(n-1, n-3)$ and $G=G_4(n)$. This completes the proof of the theorem.

THEOREM 1. 6. *The maximum number of cut vertices in a strong graph with n vertices and m arcs is $r=r(n, m)$ where*

$$r(n, m)=\max \{q : m \leqq A(n, q)\}.$$

PROOF. By theorem 1. 5 it follows that the number of cut vertices in a graph with n vertices and m arcs is at most $r(n, m)$. To show that this bound is attained, consider the following graph. Take a graph on n-vertices, with $r=r(n, m)$ cut vertices and with $A(n, r)$ arcs. By theorem 1. 5, it has a Hamiltonian circuit and the deletion of $A(n, r)-m$ arcs not belonging to the Hamiltonian circuit gives the required graph. This completes the proof of the theorem.

Now it is not difficult to prove that the range of r, the number of cut vertices in a graph on n vertices with m arcs is

$$0 \leqq r \leqq r(n, m) \quad \text{if} \quad m \geqq 2n,$$

$$2n-m \leqq r \leqq r(n, m) \quad \text{if} \quad n \leqq m \leqq 2n-1 \quad \text{and} \quad m \neq 2n-2,$$

$$1 \leqq r \leqq r(n, m) \quad \text{if} \quad m = 2n-2.$$

2. The number of cut vertices in a strong complete graph

In [4], KORVIN has asked for a proof of the fact that any strong complete graph on $n>3$ vertices has at most $n-2$ cut vertices without using Camion's theorem. In this section we give a simple proof of the result using induction on n.

The result is easily verified for $n=4$, so assume it for $n-1$ and let G be a strong complete graph with n vertices where $n \geqq 5$. We may assume that G is maximal, i.e., the addition of any arc to G converts some cut vertex into a non-cut vertex. We consider two cases.

Case (i). G has a non-cut vertex x. Then by induction hypothesis, $G-x$ has two non-cut vertices y, z. Since $d_G^+(x)>1$ or $d_G^-(x)>1$, it follows that one of y, z is a non-cut vertex of G. Thus G has at least two non-cut vertices.

Case (ii). All vertices of G are cut vertices. Let x be any vertex of G. Then there exist vertices y and z such that every path from y to z in G passes through x and (y, x), $(x, z) \in G$. Let $G_1 = G+(y, z)$. Since G is maximal, x is a non-cut vertex of G_1. By induction hypothesis G_1-x has at least two non-cut vertices. Evidently, y, z are non-cut vertices of G_1-x and as in case (i) it follows that one of y, z is a non-cut vertex of G, a contradiction. Thus case (ii) does not occur and the proof is complete.

It may be remarked that the above result and a simple induction on the number of vertices in a strong complete graph, yield a new proof of Camion's theorem.

3. The number of cut arcs in a graph

In this section, by a maximal graph on n vertices with s cut arcs we mean a graph such that the addition of any arc decreases the number of cut arcs. An extremal graph is a graph with n vertices, s cut arcs and with the maximum number of arcs.

It is convenient to note down the following facts which will be used repeatedly.

If G is a strong graph and the addition of a new arc (x, y) converts some cut arc (u, v) into a non-cut arc then (u, v) belongs to every path from x to y in G. Further in G, there are paths from u to x and y to v not including the arc (u, v).

If G is a strong graph and the removal of a non-cut arc (x, y) converts some non-cut arc (u, v) into a cut arc, then (u, v) belongs to every path from x to y in $G-(x, y)$ and (x, y) belongs to every path from u to v in $G-(u, v)$.

An arc (u, v) is a cut arc of G if and only if (v, u) is a cut arc of the converse of G.

LEMMA 3. 1. *If there are two arc disjoint paths from a vertex x to another vertex y in a maximal graph G, then $(x, y) \in G$.*

PROOF. This lemma is an immediate consequence of the first observation made above.

LEMMA 3. 2. *If G is a maximal graph there is no circuit of length greater than 2 without chords.*

PROOF. If possible let $C=[x_1, x_2, \dots, x_k, x_{k+1}=x_1]$ be a circuit without chords and $k \geqq 3$. Then add the arc (x_2, x_1). Since G is maximal this converts some cut arc (u, v) of G into a non-cut arc. Then $(u, v) \in C$ and there exist paths in G from u to x_2 and x_1 to v not using the arc (u, v). Hence by lemma 3. 1, C has a chord, a contradiction.

LEMMA 3. 3. *There exists an extremal graph containing a symmetric tree on n or $n-1$ vertices according as $s \neq 2n-3$ or $s = 2n-3$.*

PROOF. We start with any extremal graph G with n vertices and s cut arcs. If G does not contain a symmetric tree as stated in the lemma, we describe a procedure by which we can increase the maximum size of a symmetric tree contained in G. Let T be a maximal symmetric tree contained in G.

Suppose there is a circuit containing exactly one vertex u of T. Then we can choose the circuit such that it has no chord incident with u. Let this circuit be $C=[u, x_1, \dots, x_k, u]$.

First let there be a chain $v=[x_j, y_1, \dots, y_p, v]$, with $v \neq (u, x_1)$ and (x_k, u), connecting some vertex of C and a vertex v of T with all intermediate vertices outside $C \cup T$. If v is not a path, we may assume without loss of generality that $(y_{p-1}, y_p) \in G$ and $(v, y_p) \in G$. We call vertices like y_p distinguished vertices of v. Now consider a path μ from y_p to u. Let z be the first vertex on this path belonging to $C \cup T$. If $z \in C-u$, then we have the path $(v, y_p)+\mu[y_p, z]$ instead of the chain v. If $z \in T$, we have the chain $v[x_j, y_p]+\mu[y_p, z]$ and this has fewer distinguished vertices than v. Proceeding further we see that we can take v to be a path from x_j to v. Then by lemma 3. 1, $(x_j, u) \in G$, hence $j=k$. Now add the new arc (x_1, u) to G. If any cut arc (x_i, x_{i+1}), $1 \leqq i \leqq k-1$, converts into a non-cut arc, then $(u, x_{i+1}) \in G$, a contradiction. Also (x_k, u) is a non-cut arc of G, so the new graph has the same cut arcs as G, a contradiction.

If there is no chain connecting a vertex of $C-u$ and a vertex of $T-u$, then u is a cut vertex of the undirected graph H associated with G. Let L be the piece of H with respect to u containing the vertices of C. Then by lemma 3. 2, L contains a symmetric arc. Now turn the piece L (in G) at u such that a symmetric arc becomes incident with u. Then the size of the tree T is increased.

Thus we may take that there is no circuit containing exactly one vertex of the tree T. Let $\mu=[u_1, u_2, \dots, u_p]$ be the chain in T connecting vertices u_1 and u_p of T

and $\nu=[u_1, x_1, x_2, \ldots, x_k, u_p]$ a path from u_1 to u_p with all intermediate vertices outside T. We can choose the paths μ and ν such that the only arcs of the type (u_i, x_j) or (x_j, u_i) are (u_1, x_1) and (x_k, u_p).

Now consider the graph $G_1 = G+(u_p, x_k)$. If some cut arc of G belonging to μ becomes a non-cut arc in G_1, we get a contradiction. If one of the arcs (x_i, x_{i+1}), $1 \leqq i \leqq k-1$ is a cut arc in G and a non-cut arc in G_1, then $(x_i, u_p) \in G$, a contradiction. Since G is extremal, it follows that (u_1, x_1) is a cut arc of G and a non-cut arc of G_1.

Now let $k+1$ = length of $\nu \geqq 3$. Then $(x_k, x_1) \in G$. Now construct a graph G^* from G by deleting the arc (u_1, x_1) and adding the new arc (u_1, x_k). Evidently G^* is strong and (u_1, x_k) is a cut arc of G^*. If some cut arc (x, y) of G becomes a non-cut arc in G^*, then $(x, y)=(x_i, x_{i+1})$ for some $i \leqq k-1$. Also then there is a path not using (x_i, x_{i+1}) from x_i to u_1, hence by lemma 3. 1, $(x_i, u_p) \in G$, a contradiction. Thus no cut arc of G becomes a non-cut arc in G^*. Since $s(G^*) \geqq s(G)$ and $m(G^*)= =m(G)$, G^* is extremal. Next let a non-cut arc (x, y) of G become a cut arc in G^*. Then since $[u_1, x_k, x_1]$ is a path from u_1 to x_1 in $G+(u_1, x_k)$, it follows that $(x, y)= (x_k, x_1)$. Now construct the graph G^{**} from G^* by adding the new arc (u_p, x_k). It can be seen that this does not convert any cut arc of G^* belonging to μ to a non-cut arc in G^{**}. Also (u_1, x_k) is a non-cut arc in G^{**}. Thus

$$s(G) = s(G^*)-1 = s(G^{**}).$$

This is a contradiction since $m(G^{**}) = m(G)+1$. Thus $s(G^*)=s(G)$. In the graph G^* we have the path $[u_1, x_k, u_p]$ from u_1 to u_p. Thus we may choose the path ν in G with length 2. Let $\nu=[u_1, x_1, u_p]$.

If now (u_1, x_1) is a non-cut arc of G, then $G+(u_p, x_1)$ has the same cut arcs as G, a contradiction. Thus we may assume that (u_1, x_1) and (x_1, u_p) are cut arcs of G. If $p-1$ = the length of $\mu \geqq 2$, add the new arc (u_p, x_1) and delete the arc (u_1, x_1). This does not change the number of cut arcs in the graph since $(u_1, u_p) \in G$. Thus the size of the tree can be increased when $p \geqq 3$. So let $p=2$.

If there is a chain, other than (u_1, x_1) and (x_1, u_2), connecting x_1 and a vertex of T and with all intermediate vertices outside T, then as before it can be shown that there is a path with the same properties. This contradicts the assumption that (u_1, x_1) and (x_1, u_2) are cut arcs. Thus either $d^+(x_1)=d^-(x_1)=1$ or x_1 is a cut vertex of the undirected graph H associated with G. In the latter case, a piece of H with respect to x_1 which does not include any vertex of T contains a symmetric arc and this piece can be transferred (in G) to a vertex of T in such a way that the size of the tree increases. Thus we may take $d^+(x_1)=d^-(x_1)=1$. If now there is any path connecting two vertices of T with all intermediate vertices outside T, we first reduce its length to 2 and then make the indegree and the outdegree of the middle vertex unity. Thus we take that if x is any vertex outside T, then there are exactly two arcs (u_1, x) and (x, u_2) incident with x and u_1, u_2 are adjacent in T.

Let now q be the number of vertices outside T. Let J be the subgraph of G generated by the vertices of T. Then

$$n(J) = n-q, \quad m(J) = m-2q.$$

Also let $s(J) = s(G)-2q+\beta$. Clearly $\beta \leqq q$. Now consider an extremal graph J_1 on $q+1$ vertices with $2q-\beta$ cut arcs.

To show the existence of such an extremal graph in the required cases, it is enough to exhibit a graph (not necessarily extremal) on $q+1$ vertices with exactly $2q-\beta$ cut arcs and having at least $2q$ arcs, where $0 \leqq \beta \leqq q$. For this consider the graph $H_3(n, s)$ described at the end of the proof of theorem 3. 4 with $n = q+1$ and $s = 2q-\beta$. Now attaching J_1 suitably to J, we get a graph with the same number of cut arcs as G and the size of the tree is increased.

If $q=\beta=1$ then let x be the vertex outside T and let (u_1, x), $(x, u_2) \in G$. Since $\beta=1$, (u_1, u_2) is a cut arc of J. If (u_2, u_1) is a non-cut arc of J, then replace the arc (u_1, x) by (x, u_1) and add the new arc (u_2, x). The resulting graph has the same number of cut arcs as G, thus G is not extremal. Thus (u_2, u_1) is a cut arc of J, then each of the vertices u_1, u_2 is either a pendant vertex[1] of T or a cut vertex of the undirected graph H associated with T. If J has an arc (u, v) not belonging to T, we may assume that u, v belong to that piece of H with respect to u_1 which does not contain u_2. Then this piece can be turned around at u_1 (in G) such that the arc (u, v) becomes incident into u_1. Then (u, u_2) is an arc of, and $\beta=0$ for, the new graph. Thus we can increase the size of T unless $q=1$ and $T=J$, i.e., $s = 2n-3$. This completes the proof of the lemma.

It is easy to deduce the following

COROLLARY. *The maximum number of cut arcs in a strong graph on n vertices is* $2n-2$.

This result was proved earlier by GUPTA [2].

THEOREM 3. 4. *The maximum number of arcs in a strong graph with n vertices with s cut arcs is*

$$B(n,s) = \begin{cases} \binom{n-s-1}{2} + \binom{n}{2} + n - 1 & \text{if } 0 \leqq s \leqq n-1 \\ \binom{2n-s-1}{2} + s & \text{if } n \leqq s \leqq 2n-2. \end{cases}$$

PROOF. If $s = 2n-3$, there is no graph with n vertices, s cut arcs and with a spanning symmetric tree. Now by lemma 3. 3, there is an extremal graph, with $n-1$ vertices generating a symmetric tree and with the indegree and the outdegree of nth vertex being unity. Thus the theorem is proved when $s = 2n-3$.

So let $s \neq 2n-3$. Then we prove the theorem by induction on n. The theorem is trivial for $n=2$ and assume the result for $n-1$. By lemma 3. 3 there is an extremal graph G with n vertices, s cut arcs and with a spanning symmetric tree T. Let (x, y) be a pendant edge of T and x a pendant vertex. We have four cases.

Case (i). Both (x, y) and (y, x) are cut arcs of G. Then $d^+(x)=d^-(x)=1$. Also by induction hypothesis, $G-x$ has at most $B(n-1, s-2)$ arcs and so G has at most $B(n-1, s-2)+2 \leqq B(n, s)$ arcs.

Case (ii). (x, y) is a cut arc of G and (y, x) is not a cut arc of G. Then $d^+(x)=1$. Also $G-x$ has at least $s-1$ cut arcs and so by induction hypothesis has at most $B(n-1, s-1)$ arcs. If $s \leqq n-1$, then there are at most n arcs incident with x in G. If $s \geqq n$, then there are at least $s-n+1$ edges (u, v) of $T-x$ such that both (u, v)

[1] A vertex u of a tree T is a pendant vertex of T if the degree of u in T is unity.

and (v, u) are cut arcs of G. If now $[x, \ldots, u, v]$ is a chain in T, then $(v, x) \notin G$. Thus we see that if $s \geqq n$, there are at most $2n-s-1$ arcs of G incident with x. Thus G has at most $B(n, s)$ arcs.

Case (iii). (x, y) is a non-cut arc of G and (y, x) is a cut arc of G. This is similar to case (ii).

Case (iv). Both (x, y) and (y, x) are non-cut arcs of G. Then $G-x$ has at least s cut arcs and so has at most $B(n-1, s)$ arcs. Now let $[x, \ldots, u, v]$ be a chain of T. If (u, v) is a cut arc of G, then $(x, v) \notin G$. If (v, u) is a cut arc of G, then $(v, x) \notin G$. Thus there are at most $2n-s-2$ arcs of G incident with x. So G has at most $B(n-1, s)+$ $+2n-s-2 \leqq B(n, s)$ arcs.

To show that the bound $B(n, s)$ is attained, consider the following graphs $H_i(n, s)$ defined on vertices $1, 2, \ldots, n$.

In $H_1(n, s)$, (i, j) is an arc if either $i<j$ or $i=j+1$ or $k \leqq j < i \leqq k+n-s-1$ where k is a fixed integer such that $1 \leqq k \leqq s+1$. Here $s < n-2$.

In $H_2(n)$, (i, j) is an arc if either $i<j$ or $i=j+1$ and $i \neq k+1, k+2$ or $i=k+2$ and $j=k$ or $i=l$ and $j=l-1$ where k, l are fixed integers such that $1 \leqq k \leqq n-2$ and $k+1 \leqq l \leqq k+2$. Here $s=n-2$.

$H_3(n, s)$ is the graph obtained by attaching symmetric trees with a total of $2k$ arcs at some of the following graph H. H has vertices $1, 2, \ldots, n-k$ where $0 \leqq k \leqq$ $\leqq s+1-n$ and $k \leqq n-3$ if $s=2n-3$. In H, (i, j) is an arc if either $1 \leqq i<j \leqq$ $\leqq 2n-s-1$ or $i=j+1 \leqq 2n-s-1$. Further for each i with $2n-s \leqq i \leqq n-k$, $d_H^+(i)=d_H^-(i)=1$ and if $(i_1, i), (i, i_2) \in H$, then $1 \leqq i_1 < i_2 \leqq 2n-s-1$. Here $s \geqq n-1$.

This completes the proof of the theorem.

It may be remarked that theorem 3. 4 can be proved without using lemma 3. 3 if it can be proved that there is an extremal graph on n vertices with s cut arcs and with a non-cut vertex. Also if every extremal graph has a non-cut vertex then it can be proved that $H_1(n, s)$, $H_2(n)$, $H_3(n, s)$ are the only extremal graphs.

COROLLARY. *Any strong graph G on n vertices with $2n-2$ cut arcs is a symmetric tree.*

PROOF. By the theorem 3. 4, G has at most $2n-2$ arcs, thus G is extremal. By lemma 3. 2, G has no circuit of length greater than 2 and so G is a symmetric tree.

This was also proved by GUPTA [2].

THEOREM 3. 5. *The maximum number of cut arcs in a strong graph on n vertices with m arcs is $s=s(n, m)$ where*

$$s(n, m) = \max\{q : q \leqq m \leqq B(n, q)\}.$$

PROOF. By theorem 3. 4, the number of cut arcs in any graph on n vertices with m arcs is at most $s(n, m)$. To show that the bound is attained, consider the following graph. If $m < 2n-2$, consider a circuit on $2n-m$ vertices with an attached symmetric path of length $m-n$. If $m \geqq 2n-2$, take a graph with a spanning symmetric tree on n vertices, with $s=s(n, m)$ cut arcs and with $B(n, s)$ arcs and delete $B(n, s)-m$ arcs not belonging to the spanning symmetric tree. This completes the proof of the theorem.

Now it is not difficult to prove that the range of s, the number of cut arcs in a graph on n vertices with m arcs is

$$0 \leqq s \leqq s(n, m) \quad \text{if} \quad m \geqq 2n,$$

$$2n-m+1 \leqq s \leqq s(n, m) \quad \text{if} \quad n+1 \leqq m \leqq 2n-1,$$

$$s = n \quad \text{if} \quad m = n.$$

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A SHR-F_n ADMITTING AN AFFINE MOTION

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The existence of an affine motion has been studied in detail by K. TAKANO in a series of his papers [1][1] to [4] for non-Riemannian spaces of recurrent curvature. Recently this discussion has been extended to Finsler spaces of recurrent curvature by R. S. SINHA [5] and the present authors [7]. Following [2] for the special consideration of the covariant derivative of affine motion-generating vector we have similarly defined a special infinitesimal transformation. The existence of an affine motion in a recurrent Finsler space admitting the said transformation has been discussed in the present paper. The notation used here mainly depends upon [6] and [7].

1. Introduction

Let F_n be an n-dimensional Finsler space endowed with the metric function F^2 of class at least C^5 in its all $2n$ arguments. For a set of symmetric connection parameters G^i_{jk} the covariant derivative (of Berwald) $\mathscr{B}_k X^i$ for a vector $\mathfrak{X}$ is given by

$$\mathscr{B}_k X^i = \partial_k X^i - (\dot{\partial}_j x^i) G^j_k + X^j G^i_{jk}, \quad \partial_k \equiv \partial/\partial x^k, \quad \dot{\partial}_k \equiv \partial/\partial \dot{x}^k. \tag{1.1}$$

The functions G^j_k are positively homogeneous of degree one in $\dot{x}^i$s and are connected with the connection parameters by $G^j_k = G^j_{kh}\dot{x}^h$. Also the entities $G^i_{jkh} \overset{\text{def}}{=} \dot{\partial}_j G^i_{kh}$ constitute a tensor-field symmetric in its all covariant indices and satisfy

$$G^i_{jkh}\dot{x}^j = 0. \tag{1.2}$$

In what follows we shall use also the relation $\mathscr{B}_k \dot{x}^i = 0$.

Given a vector $\mathfrak{X}$ the operators $\dot{\partial}_j$ and $\mathscr{B}_k$ commute according to

$$(\dot{\partial}_j \mathscr{B}_k - \mathscr{B}_k \dot{\partial}_j) X^i = G^i_{jkh} X^h, \tag{1.3}$$

and

$$2\mathscr{B}_{[j}\mathscr{B}_{k]} X^i = H^i_{jkh} X^h - H^r_{jk} \dot{\partial}_r X^i, \text{ [3]} \tag{1.4}$$

[1] Numbers in square brackets refer to the references given at the end.

[2] All the geometric entities will be considered functions of the line-element $(x^i, \dot{x}^i)$ unless stated otherwise. The indices $i, j, k, \ldots$ vary over the positive integers from 1 to n.

[3] Square brackets denote the skew-symmetric part with respect to the indices enclosed within them.

where the functions H^i_{jkh} constitute a tensor $\mathfrak{H}$ called the curvature tensor (of Berwald). The functions H^i_{jk} are connected with the curvature tensor by

$$H^i_{jkh}\dot{x}^h = H^i_{jk}. \tag{1.5}$$

A space characterized by

$$\mathscr{B}_l H^i_{jkh} = \lambda_l H^i_{jkh}, \tag{1.6}$$

for some non-null vector-field λ_l is called a recurrent Finsler space and has been denoted by HR-F_n [7]. It has been seen [6] that a HR-F_n also admits the equation

$$\mathscr{B}_l H^i_{jk} = \lambda_l H^i_{jk}. \tag{1.7}$$

An infinitesimal point transformation

$$\bar{x}^i = x^i + \varepsilon v^i \tag{1.8}$$

generated by the vector-field v^i (independent of the directional arguments) defines the Lie derivative

$$\mathscr{L}G^i_{kh} = \mathscr{B}_k\mathscr{B}_h v^i + v^j H^i_{jkh} + G^i_{jkh}\mathscr{B}_r v^j \dot{x}^r. \tag{1.9}$$

The vanishing of the Lie derivative of G^i_{kh} is a necessary and sufficient condition that the transformation (1. 8) be an affine motion.

2. Recurrent vector-field v^i

Analogous to (1. 6) we define the recurrence property of the vector-field v^i generating the infinitesimal transformation (1. 8). The vector-field v^i will be called recurrent if there exists some non-null vector-field μ_h such that

$$\mathscr{B}_h v^i = \mu_h v^i. \tag{2.1}$$

Accordingly the second order covariant derivative of this vector is given by

$$\mathscr{B}_k\mathscr{B}_h v^i = \mu_{kh} v^i, \tag{2.2}$$

where we have put

$$\mu_{kh} \overset{\text{def}}{=} \mathscr{B}_k \mu_h + \mu_k \mu_h. \tag{2.3}$$

Also the Lie derivative of G^i_{kh} and H^i_{jkh} in a HR-F_n admitting (2. 1) reduce to

$$\mathscr{L}G^i_{kh} = \mu_{kh} v^i + (H^i_{jkh} + G^i_{jkh}\mu_r \dot{x}^r) v^j, \tag{2.4}$$

and

$$\mathscr{L}H^i_{jkh} = v^r \lambda_r H^i_{jkh} - \mu_r H^r_{jkh} v^i - 2\mu_{[j} H^i_{k]rh} v^r + H^i_{jkr}\mu_h v^r + (\dot{\partial}_r H^i_{jkh})\mu_s v^r \dot{x}^s. \tag{2.5}$$

Now onwards we shall consider the infinitesimal transformation (1. 8) of a special type when (2. 1) holds:

$$\bar{x}^i = x^i + \varepsilon v^i, \quad \mathscr{B}_h v^i = \mu_h v^i. \tag{2.6}$$

DEFINITION 2. 1. A HR-F_n admitting a special type of infinitesimal transformation given by (2. 6) will be called a *special* HR-F_n and will be denoted by SHR-F_n.

If we assume that (2. 6) defines an affine motion in a SHR-F_n we have, from (2. 4),

$$\mu_{kh} v^i + (H^i_{jkh} + G^i_{jkh}\mu_r \dot{x}^r) v^j = 0. \tag{2. 7}$$

In view of (1. 2) and (1. 5) the above equation, when transvected by $\dot{x}^h$, yields

$$\mu_{kh} v^i \dot{x}^h + H^i_{jk} v^j = 0. \tag{2. 8}$$

In consequence of (1. 7) and (2. 1) the covariant differentiation of this relation determines

$$(\mathscr{B}_m \mu_{kh} + \mu_m \mu_{kh}) v^i \dot{x}^h + (\lambda_m + \mu_m) H^i_{jk} v^j = 0. \tag{2. 9}$$

On the other hand when (2. 8) is transvected by v^k and the fact that (being H^i_{jk} skew-symmetric in j, k) $H^i_{jk} v^j v^k$ vanishes identically is used there follows the equation

$$v^i \mu_{kh} v^k \dot{x}^h = 0,$$

and so the equation

$$\mu_{kh} v^k \dot{x}^h = 0$$

as v^i is an arbitrary vector-field. The covariant derivative of the last relation, in consequence of (2. 1), determines

$$(\mathscr{B}_m \mu_{kh} + \mu_m \mu_{kh}) v^k \dot{x}^h = 0. \tag{2. 10}$$

Contracting (2. 9) for i and k, and applying (2. 10) it therefore follows that the vector-fields λ_m and μ_m are connected by

$$(\lambda_m + \mu_m) H^i_{ji} v^j = 0,$$

which gives rise to

THEOREM 2. 1. *If a SHR-F_n admits an affine motion there holds either of the following conditions*

$$\text{(i)} \quad \mu_m = -\lambda_m, \qquad \text{(ii)} \quad H^i_{ji} v^j = 0. \tag{2. 11}$$

3. Implications of the affine motion in a SHR-F_n

It has been seen above that the conditions given by (2. 11) hold necessarily in a SHR-F_n which admits an affine motion.[4] In the present section we consider the further implications of the conditions given by (2. 11). We begin with the condition expressed by (2. 11) (i). Assuming (2. 11) (i) we thus find, in an ASHR-F_n,

$$\mathscr{B}_h v^i = -\lambda_h v^i, \tag{3. 1}$$

$$\mathscr{B}_k \mathscr{B}_h v^i = (-\mathscr{B}_k \lambda_h + \lambda_k \lambda_h) v^i, \tag{3. 2}$$

[4] Henceforward the space SHR-F_n admitting an affine motion will be briefly denoted by ASHR-F_n.

and

(3.3) $$\mathcal{L}H^i_{jkh} = v^r\mathcal{B}_r H^i_{jkh} + \lambda_r H^r_{jkh} v^i + 2\lambda_{[j} H^i_{k]rh} v^r - H^i_{jkr}\lambda_h v^r - (\dot\partial_r H^i_{jkh})\lambda_s v^r \dot x^s.$$

Arranging the terms by means of (1. 7) the above relation reduces to

$$\mathcal{L}H^i_{jkh} = v^r(\mathcal{B}_r H^i_{jkh} + \mathcal{B}_j H^i_{krh} + \mathcal{B}_k H^i_{rjh}) + \lambda_r H^r_{jkh} v^i - H^i_{jkr}\lambda_h v^r - (\dot\partial_r H^i_{jkh})\lambda_s v^r \dot x^s.$$

Multiplying it by $\dot x^h$, using the contraction given by (1. 5) and, finally, applying the Bianchi identity for the tensor-field H^i_{jk} we derive

(3.4) $$\mathcal{L}H^i_{jk} = \lambda_r H^r_{jk} v^i - H^i_{jkr} v^r \lambda_h \dot x^h.$$

Also we know that the Lie derivative of the curvature tensor $\mathfrak{H}$ (and so that of the tensor-field H^i_{jk}) vanishes when the space admits an affine motion. Thus for an ASHR-F_n we obtain, from (3. 4), that the recurrence vector-field λ_i satisfies

(3.5) $$\lambda_r H^r_{jk} v^i = H^i_{jkr} v^r \lambda_h \dot x^h.$$

This establishes

THEOREM 3. 1. *If an ASHR-F_n satisfies* (2. 11) (i) *it also satisfies* (3. 5).

Applying the commutation formula (1. 4) for the tensor-field $\mathcal{B}_h v^i$ we obtain

$$2\mathcal{B}_{[j}\mathcal{B}_{k]}\mathcal{B}_h v^i = H^i_{jkr}\mathcal{B}_h v^r - H^r_{jkh}\mathcal{B}_r v^i - H^r_{jk}\dot\partial_r \mathcal{B}_h v^i.$$

Interchanging the processes of covariant and partial differentiation by means of (1. 3) and noting the fact that v^i is independent of $\dot x^i$'s the above identity further reduces to

$$2\mathcal{B}_{[j}\mathcal{B}_{k]}\mathcal{B}_h v^i = H^i_{jkr}\mathcal{B}_h v^r - H^r_{jkh}\mathcal{B}_r v^i - H^r_{jk} G^i_{hrs} v^s.$$

When transvected with $\dot x^h$ this identity, in consequence of (3. 5), for an ASHR-F_n reduces to

(3.6) $$2\mathcal{B}_{[j}\mathcal{B}_{k]}\mathcal{B}_h v^i \dot x^h = 0.$$

Thus we have

THEOREM 3. 2. *The processes of covariant differentiation commute for the vector-field $\dot x^h \mathcal{B}_h v^i$ in an ASHR-F_n.*

From (1. 6), (1. 7) and (3. 1) we can easily verify that the tensor-fields $v^j H^i_{jkh}$ and $v^j H^i_{jk}$ are covariant constants in an ASHR-F_n. Thus we have

(3.7) $$\mathcal{B}_l(v^j H^i_{jkh}) = 0 = \mathcal{B}_l(v^j H^i_{jk}).$$

Consequently the covariant differentiation of (2. 8) yields

(3.8) $$\dot x^h \mathcal{B}_j \mathcal{B}_k \mathcal{B}_h v^i = 0.$$

4. Integrability conditions of (3. 6)

To deduce the integrability conditions of (3. 6) let us consider a scalar point function $\sigma(x^j)$ satisfying the differential equation

(4. 1) $$\mathscr{B}_k(\sigma \dot{x}^h \mathscr{B}_h v^i) = 0$$

and so obviously to

(4. 2) $$2\mathscr{B}_{[j}\mathscr{B}_{k]}(\sigma \dot{x}^h \mathscr{B}_h v^i) = 0$$

also. Carrying out the covariant differentiation as indicated therein and noting the fact that $2\mathscr{B}_{[j}\mathscr{B}_{k]}\sigma$ vanishes the equation (4. 2) simplifies to

$$2\sigma(\mathscr{B}_{[j}\mathscr{B}_{k]}\dot{x}^h \mathscr{B}_h v^i) = 0,$$

from which, for σ being an arbitrary function, there follows the equation (3. 6). Thus (4. 1) represents the integrability condition of (3. 6).

For an ASHR-F_n, where (3. 1) holds, equation (4. 1) reduces to

$$\dot{x}^h(\lambda_h \mathscr{B}_k \sigma + \sigma \mathscr{B}_k \lambda_h - \sigma \lambda_k \lambda_h) v^i = 0.$$

Putting

(4. 3) $$\lambda = \dot{x}^h \lambda_h,$$

and

(4. 4) $$\sigma_k = (\mathscr{B}_k \sigma)/\sigma = \mathscr{B}_k \log \sigma$$

we get, from the above equation,

(4. 5) $$\mathscr{B}_k \lambda + \lambda(\sigma_k - \lambda_k) = 0,$$

as v^i is a non-null vector-field. Thus we may conclude

THEOREM 4. 1. *In an ASHR-F_n there exists a scalar point function σ satisfying* (4. 1) *and it is connected with the recurrence vector-field λ_k by* (4. 5).

Now we consider a particular case of the integrability conditions of (3. 6). It is easy to see that the equation

(4. 6) $$\mathscr{B}_k(\dot{x}^h \mathscr{B}_h v^i) = 0$$

also satisfies the differential equation (3. 6). Thus, proceeding with (4. 4) in the way similar to above we obtain

(4. 7a) $$\mathscr{B}_k \lambda = \lambda \lambda_k,$$

or

(4. 7b) $$\lambda_k = \mathscr{B}_k(\log \lambda).$$

The last relation expresses the gradient property of the vector-field λ_k. Conversely, when (4. 7a) holds it may be seen from (3. 2) that the derivative $\mathscr{B}_k \mathscr{B}_h v^i$, when transvected with $\dot{x}^h$, vanishes identically. Thus the equation (4. 7a) gives rise to the integrability condition (4. 6) of (3. 6).

Concluding these remarks we may state

THEOREM 4. 2. *In an ASHR-F_n the equation* (4. 6) *will represent the integrability condition of* (3. 6) *if and only if the recurrence vector-field λ_k is a gradient one.*

The condition expressed by (4. 6) also expresses the fact that the vector-field λv^i is invariant under the covariant differentiation.

5. Some characteristics of the function σ

The function $\sigma(x^j)$ has been introduced in the equation (4. 1). Currently we discuss some of its properties. Carrying out the covariant differentiation as indicated in (4. 1) we get

$$\dot{x}^h\{(\mathscr{B}_k\sigma)\mathscr{B}_h v^i + \sigma\mathscr{B}_k\mathscr{B}_h v^i\} = 0. \tag{5. 1}$$

Eliminating the second order covariant derivative of v^i with the help of (2. 2) and (2. 8) the above equation, for an ASHR-F_n, reduces to

$$(\mathscr{B}_k\sigma)\dot{x}^h\mathscr{B}_h v^i - \sigma H^i_{jk}v^j = 0.$$

Transvecting by v^k and simplifying by means of (3. 1) and (4. 3) the last relation further reduces to

$$(v^k\mathscr{B}_k\sigma)\lambda v^i = 0. \tag{5. 2}$$

For λv^i being non-null it then follows that $v^k\mathscr{B}_k\sigma$ vanishes identically. The vanishing of this expression which is simply the Lie derivative of the scalar function σ characterizes an important property of σ. Thus we have proved

THEOREM 5. 1. *The function σ is a Lie-invariant in an ASHR-F_n.*

The equation (5. 1) satisfied by σ may be rewritten when we make use of (4. 4):

$$\dot{x}^h(\sigma_k\mathscr{B}_h v^i + \mathscr{B}_k\mathscr{B}_h v^i) = 0. \tag{5. 3}$$

Forming its covariant differentiation and simplifying by means of (3. 8) there follows

$$\dot{x}^h\{(\mathscr{B}_j\sigma_k)\mathscr{B}_h v^i + \sigma_k\mathscr{B}_j\mathscr{B}_h v^i\} = 0.$$

Eliminating the second order covariant derivative of v^i from the above two equations we derive

$$\mathscr{B}_j\sigma_k = \sigma_j\sigma_k, \tag{5. 4}$$

as $\dot{x}^h\mathscr{B}_h v^i \equiv \lambda v^i$ is a non-null vector-field. This relation characterizes another important property of the function σ.

THEOREM 5. 2. *In an ASHR-F_n the scalar function σ satisfies* (5. 4).

The derivative $\mathscr{B}_j\sigma_k$ being symmetric in j, k we may derive, from (5. 4), by differentiating it covariantly and observing (5. 4) itself

$$\mathscr{B}_h\mathscr{B}_j\sigma_k = 2\sigma_j\sigma_k\sigma_h. \tag{5. 5}$$

Thus the second order covariant derivative of σ_k is also symmetric and so its skew-symmetric part vanishes. Making an application of the commutation formula (1. 4) we may derive, from (5. 5),

$$\sigma_r H^r_{hjk} - (\dot{\partial}_r \sigma_k) H^r_{hj} = 0.$$

For $\log \sigma$ being a scalar point function the derivative $\dot{\partial}_r \sigma_k$ may be seen to vanish from (1. 3) and (4. 4). Consequently the above relation reduces to

$$\sigma_r H^r_{hjk} = 0. \tag{5. 6}$$

Thus we may conclude

THEOREM 5. 3. *The relation* (5. 6) *represents the integrability condition of the equation* (5. 4).

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ON ISOTONE AND HOMOMORPHIC MAPS OF ORDERED UNARY ALGEBRAS

By

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1. Introduction

Professor Miroslav NOVOTNÝ gave me the following

PROBLEM. Find necessary and sufficient conditions under which the system of all isotone maps of a set A into a set B is identical with the system of all homomorphic maps of A into B; an ordering and a unary operation is defined on each of the sets A, B.

This work solves the above-mentioned problem and, in fact, in several ways. The required necessary and sufficient conditions are always given by pairs of theorems A 1 and A 2, B 1 and B 2, C 1 and C 2; the first pair holds under the restrictive assumption card $B > 1$, whereas the remaining two pairs hold quite generally.

Theorem D shows then that, for single-valued determination whether the above-mentioned systems of maps are identical (or not), it is sufficient to observe only those maps where the image of A is at most a two-element set.

The work is concluded with a remark on the connection between this problem and the topology.

The author is greatly indebted to Professor NOVOTNÝ for his valuable advice and careful revision of the manuscript.

2. Fundamental notions, symbols and simple relations

2. 1. Throughout this work denote: A, B sets, f and g unary (partial) operations on A and B respectively, H the set of all homomorphic maps $(A, f) \to (B, g)$, I the set of all isotone maps $(A, \leqq) \to (B, \preceq)$, Z the set of all maps $A \to B$. Let further

$$Z_{\mathfrak{k}} := \{\varphi \in Z \mid \text{card } \varphi A = \mathfrak{k}\}, \quad H_{\mathfrak{k}} := H \cap Z_{\mathfrak{k}}, \quad I_{\mathfrak{k}} := I \cap Z_{\mathfrak{k}},$$

$$Z_{\mathfrak{k}}^* := \{\varphi \in Z \mid \text{card } \varphi A \leqq \mathfrak{k}\}, \quad H_{\mathfrak{k}}^* := H \cap Z_{\mathfrak{k}}^*, \quad I_{\mathfrak{k}}^* := I \cap Z_{\mathfrak{k}}^*$$

for every cardinal number $\mathfrak{k}$.

2. 2. The current symbols are used in this work as follow: (A, f) means the algebra with the carrier-set A and with the operation f, $(A, \leqq)$ means the set A ordered by the relation $\leqq$, $A \to B$ means the map of A into B, id_A means the identical map $A \to A$, $\mathrm{dom} f := \{a \in A \mid fa \text{ is defined}\}$.

(A, f) is called a *discrete algebra* iff $\mathrm{dom}\, f = \emptyset$. $a_1 \| a_2$ means that the elements a_1, a_2 are not comparable (with respect to the corresponding ordering).

$(A, \leqq)$ is called an *antichain* iff $a_1 \leqq a_2$ implies $a_1 = a_2$ for each $a_1, a_2 \in A$.

If $fa = a$ for each $a \in A$, the operation f will be identified with id_A.

The symbols introduced with respect to A, f, $\leqq$ will be used analogically with respect to B, g, $\preceq$, too.

2. 3. It is evident that $Z_{\mathfrak{k}}^* = \bigcup Z_{\mathfrak{m}} (\mathfrak{m} \leqq \mathfrak{k})$ and therefore $H_{\mathfrak{k}}^* = \bigcup H_{\mathfrak{m}} (\mathfrak{m} \leqq \mathfrak{k})$, $I_{\mathfrak{k}}^* = \bigcup I_{\mathfrak{m}} (\mathfrak{m} \leqq \mathfrak{k})$.

It is evident that always $H_0 = I_0 = Z_0$ and $Z_0 = \emptyset$ for $A \neq \emptyset$, whereas $Z_0 = Z = \{o\}$ for $A = \emptyset$, where o means the empty map.

It is evident that always $H_1 \subseteqq I_1 = Z_1$ and in accordance with the foregoing also $H_1^* \subseteqq I_1^* = Z_1^*$.

If M, N denote some two of H, I, Z and ϱ means equality or inclusion, and if $\mathfrak{k} > \mathfrak{m} \geqq \mathfrak{n} \geqq \mathfrak{p}$ are cardinal numbers, then evidently

$$M \varrho N \Rightarrow \bigcup M_{\mathfrak{m}} (\mathfrak{m} < \mathfrak{k}) \varrho \bigcup N_{\mathfrak{m}} (\mathfrak{m} < \mathfrak{k}) \Rightarrow M_{\mathfrak{m}}^* \varrho N_{\mathfrak{m}}^* \Rightarrow M_{\mathfrak{n}}^* \varrho N_{\mathfrak{n}}^* \Rightarrow M_{\mathfrak{p}} \varrho N_{\mathfrak{p}}$$

hold.

2. 4. The classes of the decomposition of A according to the equivalence formed as the transitive symmetrical hull of the relation $\leqq$ will be called $\leqq$*-components*.

The classes of the decomposition of A according to the equivalence formed as the reflexive transitive symmetrical hull of the relation ω: $a_1 \omega a_2 \Leftrightarrow f a_1 = a_2$ for $a_1, a_2 \in A$ will be called *f-components*.

It should be remarked that in fact the f-components are the classes of the decomposition R_f introduced in [3], Def. 3 (but where f is understood as a map of the set M into itself and the set M corresponds to our $\operatorname{dom} f$).

It is evident that the elements $a_1, a_2 \in A$ belong to the same $\leqq$-component (eventually f-component) if and only if there exist a suitable natural number p and a sequence $t_0 = a_1, t_1, t_2, \ldots, t_p = a_2$ of elements of A such that $t_i \leqq t_{i+1}$ or $t_{i+1} \leqq t_i$ (eventually $f t_i = t_{i+1}$ or $f t_{i+1} = t_i$) hold for every natural number i, $0 \leqq i < p$.

3. Trivial cases

3. 1. LEMMA. *Let* $A = \emptyset$ *or* $B = \emptyset$. *Then* $I = H$.

PROOF. If $A = \emptyset$, then $Z = \{o\}$; o is (in trivial manner) isotone and homomorphic. If $B = \emptyset$ and $A \neq \emptyset$, then $Z = \emptyset = I = H$. (See 2. 3.)

3. 2. LEMMA. *Let* card $A = 1$ *or* card $B = 1$. *Then every map* $A \to B$ *is isotone, therefore* $H \subseteqq I$.

PROOF. If $A = \emptyset$ or $B = \emptyset$, then this assertion holds according to 3. 1. If $A = \{a\}$ and $B \neq \emptyset$, then an arbitrary map $A \to B$ maps a and also the whole A on some element of B, consequently it is isotone.

If $B = \{b\}$ and $A \neq \emptyset$, then there exists one and only one map $A \to B$, which maps each element of A on b, consequently it is isotone.

3. 3. LEMMA. *Let* card $A = 1$ *and* $B \neq \emptyset$, *or* card $B = 1$ *and* $A \neq \emptyset$. *Then the following statements are equivalent:*

(i) $I = H$.

(ii) $g = \mathrm{id}_B$ *or* (A, f) *is discrete.*

(iii) *Every map* $A \to B$ *is isotone and also homomorphic.*

PROOF. (i)$\Rightarrow$(ii): If (A,f) is discrete, then (ii) is satisfied.

Let now (A,f) fail to be discrete.

If $A=\{a\}$, then $fa=a$. Let $b\in B$ be an arbitrary element. The map $\varphi:A\to B$, such that $\varphi a=b$, is evidently isotone and in accordance with (i) homomorphic, too. It is $\varphi a=\varphi fa=b$. Consequently gb is defined and $gb=b$.

If $B=\{b\}$, then for an arbitrary $\alpha\in\operatorname{dom}f(\neq\emptyset)$ and for $\varphi\in Z$ it holds $\varphi\alpha=\varphi f\alpha=b$. Evidently φ is homomorphic, consequently gb is defined and $gb=b$.

Therefore $g=\mathrm{id}_B$.

(ii)$\Rightarrow$(iii): Let φ be an arbitrary map $A\to B$. If (A,f) is discrete, then every map $A\to B$ is homomorphic. Let now (A,f) fail to be discrete, consequently in accordance with (ii) $g=\mathrm{id}_B$. If $A=\{a\}$, then $fa=a$ and $\varphi a=\varphi fa$. Consequently $g\varphi a=\mathrm{id}_B\varphi a=\varphi a=\varphi fa$. If $B=\{b\}$, then for each $\alpha\in\operatorname{dom}f(\neq\emptyset)$ it is $\varphi\alpha=b=\varphi f\alpha$ and also $gb=b$. Therefore in each case $\varphi\in H$ and, with the help of 3. 2, $\varphi\in I$.

(iii)$\Rightarrow$(i) quite evidently.

3. 4. REMARK. If (A,f) is discrete, then every map $A\to B$ is evidently homomorphic; if $(A,\leqq)$ is an antichain, then every map $A\to B$ is evidently isotone; therefore if the two foregoing cases occur at the same time, then evidently $I=H=Z$.

If $g=\mathrm{id}_B$, then $H_1=Z_1$ and consequently (by 2. 3) $I_1=H_1$ is quite evident.

4. Further lemmas

4. 1. LEMMA. *The following conditions are equivalent:*

(i) $I_1\subseteqq H_1$.

(ii) $I_1=H_1$.

(iii) *Either* (A,f) *is a discrete algebra or* $g=\mathrm{id}_B$.

PROOF. (i)$\Rightarrow$(ii) directly with the help of 2. 3.

(ii)$\Rightarrow$(iii): If (A,f) is discrete, then (iii) holds; thus hereafter let (A,f) fail to be discrete.

If $B=\emptyset$, then $g=\mathrm{id}_B$ (trivially), and (iii) holds; thus let hereafter $B\neq\emptyset$.

Let $b\in B$ be arbitrary. Let the map $\varphi:A\to B$ be defined in this way: $\varphi a=b$ for each $a\in A$. Evidently $\varphi\in I_1$, consequently (by (ii)) $\varphi\in H_1$. Hence $g\varphi x$ is defined and $\varphi fx=g\varphi x$ for each $x\in\operatorname{dom}f\neq\emptyset$. Furthermore $\varphi fx=\varphi x=b$, consequently $gb=b$. Hence $g=\mathrm{id}_B$; (iii) holds again.

(iii)$\Rightarrow$(i) evidently: If (A,f) is discrete, then (by 3. 4) $H=Z$, therefore (by 2. 3) $H_1=Z_1$ and (i) holds. If $g=\mathrm{id}_B$, then (by 3. 4) (ii) and consequently (i) holds.

4. 2. LEMMA. *Let* $I_1\subseteqq H_1$, $I_2\supseteqq H_2$, $\operatorname{card}B>1$. *Then either* $(A,\leqq)$ *is an antichain or* $g=\mathrm{id}_B$.

PROOF. Assume that, on the contrary, $(A,\leqq)$ fails to be an antichain and $g\neq\mathrm{id}_B$. Then there exist $a_1,a_2\in A$, $b_1,b_2\in B$ such that $a_1<a_2$, $b_1\not\leqq\not\geqq b_2$.

Let us define the map $\varphi:A\to B$:

$$\varphi a=\begin{cases} b_1 & \text{for } a\leqq a_1\\ b_2 & \text{for } a\not\leqq a_1.\end{cases}$$

Evidently $\varphi \notin I_2$. By 4. 1 (A, f) is discrete, consequently $\varphi \in H_2$ (by 3. 4). Hence $I_2 \not\supseteq H_2$, which a contradiction of the premise.

4. 3. LEMMA. *Let $I_2^* \subseteq H_2^*$. Then either $(B, \trianglelefteq)$ is an antichain or $f \subseteq \mathrm{id}_A$.*

PROOF. Assume that, on the contrary, $(B, \trianglelefteq)$ fails to be an antichain and $f \not\subseteq \mathrm{id}_A$; then there exist elements $b_1, b_2 \in B$, $x \in A$ such that $b_1 \lhd b_2$, fx is defined, and $fx \neq x$. Hence, by 4. 1, $g = \mathrm{id}_B$.

Let $fx < x$. Put for each $a \in A$

$$\varphi a = \begin{cases} b_1 & \text{if} \quad a \leqq fx \\ b_2 & \text{if} \quad a \not\leqq fx. \end{cases}$$

It is easily seen that $\varphi \in I_2$, consequently $\varphi \in H_2$: hence $\varphi fx = g\varphi x$. Moreover $\varphi fx = b_1$, whereas $g\varphi x = \mathrm{id}_B \varphi x = \varphi x = b_2$, from where $b_1 = b_2$, which is a contradiction.

Let $fx \not< x$. Put for each $a \in A$

$$\psi a = \begin{cases} b_1 & \text{if} \quad a \leqq x \\ b_2 & \text{if} \quad a \not\leqq x. \end{cases}$$

It is easily seen that $\psi \in I_2$, consequently $\psi \in H_2$, hence $\psi fx = g\psi x$. Moreover $\psi fx = b_2$, whereas $g\psi x = b_1$, from where $b_2 = b_1$, which is a contradiction again. Therefore the assertion holds.

4. 4. LEMMA. *Let $I_2^* = H_2^*$. Then at least one of the ordered sets $(A, \leqq)$, $(B, \trianglelefteq)$ is an antichain.*

PROOF. Assume that, on the contrary, $(A, \leqq)$ and $(B, \trianglelefteq)$ fail to be antichains. Then there exist elements $a_1, a_2 \in A$, $b_1, b_2 \in B$ such that $a_1 < a_2$, $b_1 \lhd b_2$.

Put $\varphi a_1 = b_2$, $\varphi a = b_1$ for each $a \in A - \{a_1\}$. It is easily seen that $\varphi \in Z_2$, but $\varphi \notin I_2 = H_2$, hence $H_2 \neq Z_2$ (so that, by 2. 3, $H \neq Z$). Hence evidently the algebra (A, f) fails to be discrete (by 3. 4). Hence by 4. 1 $g = \mathrm{id}_B$. Then, however, $f \not\subseteq \mathrm{id}_A$, for otherwise $H = Z$ would occur. This is a contradiction, for the premises imply, by 4. 3, $f \subseteq \mathrm{id}_A$.

5. Theorems

5. 1. THEOREM A 1. *Let $g \neq \mathrm{id}_B$,* card $B > 1$. *Then the following statements are equivalent:*

(i) $I = H$.

(ii) *(A, f) is a discrete algebra and $(A, \leqq)$ is an antichain.*

(iii) *Every map $A \to B$ is isotone and also homomorphic.*

(iv) *$I_1 \subseteq H_1$ and $I_2 \supseteq H_2$.*

PROOF. (ii)$\Rightarrow$(iii) quite evidently (by 3. 4). (iii)$\Rightarrow$(i)$\Rightarrow$(iv) quite trivially (by 2. 3). (iv)$\Rightarrow$(ii) directly by 4. 1 and 4. 2.

5. 2. THEOREM A 2. *Let* $g=\mathrm{id}_B$, card $B>1$. *Then the following statements are equivalent:*

(i) $I=H$.

(ii) *The system of all f-components is identical with the system of all* $\leqq$*-components. At least one of the ordered sets* $(A, \leqq)$, $(B, \preceq)$ *is an antichain.*

(iii) $I_2=H_2$.

PROOF. (i)$\Rightarrow$(iii) quite trivially (by 2. 3).

(iii)$\Rightarrow$(ii): Assume, on the contrary, that the first part of (ii) fails to hold; then there exist two elements $a_1, a_2 \in A$ such that one of the following cases occurs:

(α) a_1, a_2 belong to the same f-component but they do not belong to the same $\leqq$-component;

(β) a_1, a_2 do not belong to the same f-component but they belong to the same $\leqq$-component.

Let be $b_1, b_2 \in B$ such that $b_1 \neq b_2$.

Let (α) be the case. Let K denote that $\leqq$-component which includes a_1. Put

$$\varphi a = \begin{cases} b_1 & \text{for } a \in K \\ b_2 & \text{for } a \in A -- K (\neq \emptyset). \end{cases}$$

Evidently $\varphi \in I_2$. By 2. 4, there exists a sequence $t_0=a_1, t_1, \ldots, t_p=a_2$ of elements of A (where p is a natural number) such that for every $i=1, 2, \ldots, p$ either $ft_{i-1}=t_i$ or $ft_i=t_{i-1}$.

Let i_0 $(0 \leqq i_0 \leqq p)$ be the least index such that $\varphi t_{i_0} \neq b_1$ (evidently such an i_0 exists and $i_0 \geqq 1$). We have $b_1 = \varphi t_{i_0-1} \neq \varphi t_{i_0} = b_2$. But $g=\mathrm{id}_B$ so that $g\varphi t_{i_0-1} = \varphi t_{i_0-1} \neq \neq \varphi t_{i_0} = g\varphi t_{i_0}$, considering that either $ft_{i_0-1}=t_{i_0}$ or $ft_{i_0}=t_{i_0-1}$. Hence $\varphi \notin H \supseteqq H_2$. But this is a contradiction of (iii).

Let (β) be the case. Let S denote that f-component which includes a_1. Put

$$\psi_1 a = \begin{cases} b_1 \\ b_2 \end{cases} \qquad \psi_2 a = \begin{cases} b_2 & \text{for } a \in S \\ b_1 & \text{for } a \in A - S (\neq \emptyset). \end{cases}$$

Evidently $\psi_1, \psi_2 \in H_2$. Since a_1, a_2 belong to the same $\leqq$-component, there exists (by 2. 4) a sequence $v_0=a_1, v_1, \ldots, v_r=a_2$ of elements of A such that for every $j=1, 2, \ldots, r$ either $v_{j-1}<v_j$ or $v_{j-1}>v_j$.

Let j_0 $(0<j_0 \leqq r)$ be the least index such that $v_{j_0} \notin S$. It can be easily derived:

$$\begin{array}{llll} \text{if} \quad v_{j_0-1}<v_{j_0} & \text{and} \quad b_1 \prec b_2, & \text{then} \quad \psi_2 \notin I \supseteqq I_2; \\ \text{if} \quad v_{j_0-1}<v_0 & \text{and} \quad b_1 \not\prec b_2, & \text{then} \quad \psi_1 \notin I \supseteqq I_2; \\ \text{if} \quad v_{j_0-1}>v_0 & \text{and} \quad b_1 \prec b_2, & \text{then} \quad \psi_1 \notin I \supseteqq I_2; \\ \text{if} \quad v_{j_0-1}>v_0 & \text{and} \quad b_1 \not\prec b_2, & \text{then} \quad \psi_2 \notin I \supseteqq I_2. \end{array}$$

In all cases a contradiction of (iii) occurs. Therefore the first part of (ii) holds true.

Now, assume on the contrary that the second part of (ii) fails to hold; then there exist four elements $a_1, a_2 \in A$, $b_1, b_2 \in B$ such that $a_1<a_2$, $b_1 \prec b_2$. Put for each $a \in A$

$$\chi a = \begin{cases} b_1 & \text{if} \quad a \leqq a_1 \\ b_2 & \text{if} \quad a \not\leqq a_1. \end{cases}$$

Evidently $\chi \in I_2$. By the first part of (ii) which has been proved, the elements a_1, a_2 are included in the same f-component. Analogously as for φ it can be proved that $\chi \notin H \supseteq H_2$, which is a contradiction of (iii) again. Therefore (ii) is true.

(ii)$\Rightarrow$(i): Let $\varphi \in H$ be arbitrary. As $g = \mathrm{id}_B$, the image of every f-component under the map φ is a singleton and by the first part of (ii) also the image of every $\leqq$-component under the map φ is a singleton. Hence $\varphi \in I$. Therefore $H \subseteq I$.

Let $\varphi \in I$ be arbitrary. If $(A, \leqq)$ is an antichain then $I = Z$. By the first part of (ii) $f \subseteq \mathrm{id}_A$ thus $H = Z$. Hence $I = H$.

If $(B, \trianglelefteq)$ is an antichain, then the image of every $\leqq$-component under the map φ is a singleton and by the first part of (ii) also the image of every f-component under the map φ is a singleton. Hence $\varphi \in H$. Therefore $I \subseteq H$.

Thus (i) is true.

5. 3. THEOREM B 1. *Let $(B, \trianglelefteq)$ fail to be an antichain. Then the following statements are equivalent:*

(i) $I = H$.

(ii) *$(A, \leqq)$ is an antichain;*
if (A, f) is not discrete, then $f \subseteq \mathrm{id}_A$ and $g = \mathrm{id}_B$.

(iii) *Every map $A \to B$ is isotone and also homomorphic.*

(iv) $I_2^* = H_2^*$.

PROOF. (iii)$\Rightarrow$(i) quite evidently, as well as (i)$\Rightarrow$(iv) (by 2. 3).

(iv)$\Rightarrow$(ii): By 4. 4, $(A, \leqq)$ is an antichain. If (A, f) is not discrete, then by 4. 1, $g = \mathrm{id}_B$; by 4. 3, $f \subseteq \mathrm{id}_A$.

(ii)$\Rightarrow$(iii): From the premise that $(A, \leqq)$ is an antichain follows (by 3. 4) that $I = Z$. If (A, f) is discrete, then (by 3. 4) $H = Z$. If (A, f) is not discrete, then (by (ii)) $f \subseteq \mathrm{id}_A$ and $g = \mathrm{id}_B$, and consequently $H = Z$ again.

5. 4. THEOREM B 2. *Let $(B, \trianglelefteq)$ be an antichain. Then the following statements are equivalent:*

(i) $I = H$.

(ii) *If* card $B > 1$, *then the system of all f-components is identical with the system of all $\leqq$-components.*
If (A, f) is not discrete, then $g = \mathrm{id}_B$.

(iii) $I_2^* = H_2^*$.

PROOF. (i)$\Rightarrow$(iii) quite trivially (by 2. 3).

(iii)$\Rightarrow$(ii): When card $A = 0$, (ii) holds (by 2. 3, trivially); thus let hereafter card $A > 0$. If (A, f) is not discrete, then by 4. 1 it is $g = \mathrm{id}_B$; the second part of (ii) holds.

Assume hereafter that card $B > 1$.

Let (A, f) fail to be discrete. By the second part of (ii) already proved, $g = \mathrm{id}_B$ and by 5. 2, the first part of (ii) holds. Let (A, f) be discrete. Then by 3. 4 $H = Z$, hence (by (iii) and 2. 3) $I_2 = Z_2$, which is possible in the case card $B > 1$ only when $(A, \leqq)$ is an antichain; otherwise elements $a_1, a_2 \in A$, $b_1, b_2 \in B$ would exist such that $a_1 < a_2$, $b_1 \neq b_2$ (therefore in accordance with the premise of the theorem it would be $b_1 \| b_2$) and putting $\varphi a_1 = b_1$, $\varphi a = b_2$ for $a \in A - \{a_1\}$ we would have $\varphi \in Z_2$, $\varphi \notin I_2$, which would be a contradiction. Consequently every $\leqq$-component is a singleton and so is every f-component, too. Therefore the first part of (ii) holds again.

(ii)$\Rightarrow$(i): If card $B \leqq 1$, then (i) is satisfied by 3. 1 and 3. 3. Thus hereafter let card $B > 1$. If $g = \mathrm{id}_B$, then (i) holds by 5. 2. If $g \neq \mathrm{id}_B$, then (by (ii)) (A, f) is discrete and $(A, \leqq)$ is an antichain. By 5. 1, (i) holds again.

5. 5. THEOREM C 1. *Let $(A, \leqq)$ fail to be an antichain. Then the following statements are equivalent:*

(i) $I = H$.

(ii) (α) *$(B, \trianglelefteq)$ is an antichain.*

(β) *If* card $B > 1$, *then the system of all f-components is identical with the system of all $\leqq$-components.*

(γ) *If (A, f) fails to be discrete or* card $B > 1$, *then* $g = \mathrm{id}_B$.

(iii) (α) *and* (β) *from* (ii) *hold.*

(γ') *If (A, f) fails to be discrete, then* $g = \mathrm{id}_B$.

(iv) $I_2^* = H_2^*$.

PROOF. (i)$\Rightarrow$(iv) quite trivially (by 2. 3).

(iv)$\Rightarrow$(ii): By 4. 4, (α) holds. By 5. 4, (α) implies (β). By 4. 1 and 4. 2, (γ) holds.

(ii)$\Rightarrow$(iii) quite trivially.

(iii)$\Rightarrow$(i) by 5. 4.

5. 6. REMARK ON THEOREMS B 2 AND C 1. If we take in account the premises of these two teorems (i.e. $(B, \trianglelefteq)$ is an antichain and $(A, \leqq)$ is not) and if we add the premise card $B > 1$ (card $A > 1$ follows from the foregoing premises), we obtain the premises of theorem 3 from [2].

Given these premises $((\gamma) \equiv (\gamma'))$, $(\beta) \wedge (\gamma)$ ($\equiv$ (ii) from B 2) implies card $\varphi K = 1$ for every homomorphic map φ of an arbitrary $\leqq$-component $K \subseteq A$ into B, which is essentially contained in the part (B) of theorem 3 of [2].

Further (γ) implies $g = \mathrm{id}_B$; analogously in (B) $a' \cdot a'$ is defined and $a'^2 = a'$ for each $a' \in G'$ (G' corresponds to our B).

The remaining condition from (B) for the product ab has no analogy in a unary algebra when $a \neq b$, whereas, when $a = b$, it corresponds to the condition: a, fa are included in the same $\leqq$-component.

Therefore under the premises taken in account the parts of theorems B 2 and C 1 are very similar to theorem 3 from [2].

5. 7. THEOREM C 2. *Let $(A, \leqq)$ be an antichain. Then the following statements are equivalent:*

(i) $I = H$.

(ii) *If* card $B > 1$, *then* $f \subseteq \mathrm{id}_A$. *If (A, f) fails to be discrete, then* $g = \mathrm{id}_B$.

(iii) *Every map $A \to B$ is isotone and also homomorphic.*

(iv) $I_2^* \subseteq H_2^*$.

PROOF. (iii)$\Rightarrow$(i)$\Rightarrow$(iv) quite trivially (by 2. 1 and 2. 3).

(iv)$\Rightarrow$(ii): The premise card $B > 1$ and $f \not\subseteq \mathrm{id}_A$ leads to a contradiction of (iv): Then there exist $\alpha \in \mathrm{dom}\, f (\neq \emptyset)$ such that $f\alpha \neq \alpha$. Let $b_1 \in B$ be arbitrary. If gb_1 is not defined, let us choose $b_2 \in B$ arbitrarily; if gb_1 is defined, let us choose $b_2 \in B$ so that $b_2 \neq gb_1$ (such a b_2 exists for card $B > 1$). Define the map $\varphi: A \to B: \varphi\alpha = b_1$, $\varphi a = b_2$ for $a \in A - \{\alpha\}$. Evidently (by 3. 4 and 2. 3) $\varphi \notin H_2^*$, although $\varphi \in I_2^*$.

The second part of (ii) follows from (iv) directly by 4. 1.

(ii)$\Rightarrow$(iii): For card $A=0$ or card $B=0$ the statement (iii) is trivially satisfied by 2. 3.

For card $B=1\wedge$card $A>0$ (iii) is satisfied by 3. 3. Thus hereafter let card $B>1$ so that $f\subseteqq \mathrm{id}_A$.

If (A,f) is discrete, (iii) is obtained directly by 3. 4. Thus hereafter let (A,f) fail to be discrete so that $g=\mathrm{id}_B$. $f\subseteqq \mathrm{id}_A$ and $g=\mathrm{id}_B$ imply $H=Z$; the premise of the theorem implies $I=Z$.

5. 8. REMARK ON THEOREM C 2. The equivalence of the conditions (i) and (ii) of theorem C 2 is an analogy to theorem 2 from [2] (and in certain sense it is a "consequence" of theorem 2 — when the binary operation "becomes" the unary one; it is a "consequence" generalized by omitting the premises card $B>1$ and card $A\geqq 1$).

5. 9. THEOREM D. *The following statements are equivalent:*

(i) $I=H$.

(ii) $I_2^*=H_2^*$.

This theorem is direct consequence of theorems B 1 and B 2 (or C 1 and C 2).

6. Remarks on analogous problems in topology

In [4] and [5] analogous problems are studied — our H corresponds to the set of all continuous maps $(A,u)\to(B,v)$, where u and v are topologies defined for A and B, respectively. Some results of the present work are very similar to some results of the works [4] and [5].

The author dealt (on recommendation of Professor NOVOTNÝ) with the question, whether the results of his work could be derived from the results of the works [4] and [5], and that with the help of topologizing the sets A and B in such a manner that the system of all continuous maps $A\to B$ (with regard to the topologies established) becomes identical with the system H.

In the following let the topology be understood in the sense of ČECH's paper [6].

It was found that the above-mentioned topologization is generally not possible, which follows from the following theorem.

THEOREM. *Let u and v be topologies for A and B respectively, S the set of all continuous maps $(A,u)\to(B,v)$. Then*

$$S=H\Rightarrow g=\mathrm{id}_B\vee \mathrm{dom}\, f=\emptyset.$$

PROOF. Let, on the contrary, $g\neq \mathrm{id}_B$ and $\mathrm{dom}\, f\neq\emptyset$. Then there exists a $\beta\in B$ such that $g\beta=\beta$ does not hold. Put $\varphi a=\beta$ $(a\in A)$. Evidently $\varphi\notin H$, since for $\alpha\in\in\mathrm{dom}\, f(\neq\emptyset)$ it is $\varphi\alpha=\varphi f\alpha=\beta$, but with respect to the foregoing $g\varphi\alpha=\varphi f\alpha$ does not hold. Now it will be shown that $\varphi\in S$ which is a contradiction of the premise $S=H$: Let $a\in A$, U an arbitrary neighbourhood of the point φa. $\varphi^{-1}U\supseteqq\varphi^{-1}\varphi a=\varphi^{-1}\beta=A$. By [6] 10. 1. 1 $\varphi\in S$.

REMARK. The premise $S=H$ of the foregoing theorem can be replaced by the less strict premise $S_1 \subseteqq H_1$, where $S_1 := \{\varphi \in S \mid \text{card } \varphi A = 1\}$. This follows from the foregoing proof.

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m-ADIC SPACES[1]

By

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Dyadic spaces were defined by P. S. ALEXANDROFF [1] as Hausdorff continuous images of topological powers of a two element discrete space. It turns out that various properties of dyadic spaces hold for a much wider class of spaces. For each cardinal m we denote by A_m the one point compactification of a discrete space of cardinality m (for finite m, A_m is just a discrete space with m elements). A topological space is said to be *m-adic* provided that it is a Hausdorff continuous image of some topological power of A_m. This concept was introduced by MRÓWKA [9].

This paper is devoted to a further study of m-adic spaces. The first section is a survey of product-theoretic theorems in a form suitable for application to m-adic spaces. The second section contains various theorems about m-adic spaces. The main results include the following. Every collection of mutually disjoint nonempty open subsets of an m-adic space has cardinality at most m. If $\aleph_0 \leqq m \leqq n$, then an m-adic space has pseudo-character n if and only if it has weight n. And, a dyadic space X of weakly accessible weight is metrizable if and only if the diagonal of $X \times X$ is a sequential G_δ-set. Two types of results are a consequence of a theorem of MAZUR [8].

§ 1. Product-theoretic theorems

The Tihonov product of a collection $\{X_\alpha : \alpha \in A\}$ of spaces will be denoted by X. A *support* of a function f defined on $S \subset X$ is defined to be a set $B \subset A$ for which $f(x) = f(y)$ for every $x, y \in S$ with $x|B = y|B$ ($x|B$ denotes the restriction of x to B). A *support* of a subset F of X is defined to be a set $B \subset A$ for which $x \in F$, $y \in X$, and $x|B = y|B$ imply that $y \in F$ for every $x \in F$, $y \in X$. Clearly, a superset of a support (of a function or a set) is also a support.

The *pseudo-character* of a space Y is defined to be the smallest cardinal q such that for every $y \in Y$ there is a collection $\mathcal{O}_y$ of open subsets of Y for which $\{y\} = \bigcap \mathcal{O}_y$ and card $\mathcal{O}_y \leqq q$. The *character* (local weight) of a space at a point p is defined to be the smallest infinite cardinal m such that there is a neighbourhood base of p having cardinality at most m. The *character* of a space is defined to be the supremum of the characters at its points. The *weight* of a space is defined to be the smallest infinite cardinal m such that the space has a base of cardinality not exceeding m. The *density character* of a space is defined to be the smallest cardinal m such that the space has a dense subset of cardinality m.

[1] This paper is a part of the author's doctoral dissertation directed by Professor S. Mrówka and submitted to The Pennsylvania State University in December, 1969.

Throughout the remainder of this paper, the cardinal m will be assumed infinite and all spaces will be assumed Hausdorff.

1. 1. *Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces each having density character at most m. Then the cardinality of every family of mutually disjoint nonempty open subsets of the product X does not exceed m.*

This is a generalization of a result of MARCZEWSKI [7]. In [5, p. 80] 1. 1 is proved for $m = \aleph_0$. The proof for arbitrary m is analogous.

1. 2. *Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces each having density character at most m. Then every canonical closed subset (i.e., a set that is the closure of its interior) of the product X has a support of cardinality not exceeding m.*

This is a generalization of a result of ROSS and STONE [10]. The proof is analogous to that in [10].

1. 3. COROLLARY. *Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces each having weight not exceeding m. For each pair U, V of disjoint open subsets of the product X there are disjoint sets U', V', each of which can be represented as a union of a collection of cardinality at most m of elementary neighbourhoods, and such that $U \subset U'$ and $V \subset V'$.*

This is a generalization of a theorem of BOCKSTEIN [3]. Note, however, that it follows directly from 1. 2.

The following example shows that 1. 2 and 1. 3 cannot be strengthened under present hypotheses even if the spaces are compact. Let a^* denote the ideal point of A_m and let D denote the discrete space $\{0, 1\}$. Let $X = A_m \times \Pi\{D_a : a \in A_m \setminus \{a^*\}\}$ where $D_a = D$ for every $a \in A_m \setminus \{a^*\}$.

For every $a \in A_m \setminus \{a^*\}$ let π_a be the projection of X onto D_a and let π be the projection of X onto A_m. Let $U_a = \pi^{-1}(a) \cap \pi_a^{-1}(0)$, $V_a = \pi^{-1}(a) \cap \pi_a^{-1}(1)$, $U = \bigcup\{U_a : a \in A_m \setminus \{a^*\}\}$, and $V = \bigcup\{V_a : a \in A_m \setminus \{a^*\}\}$. Then U and V are disjoint open subsets of X and a routine check reveals that every pair U', V' of sets as in 1. 3 cannot be represented as collections of cardinalities less than m of elementary neighbourhoods. Also, every support of $\overline{U}$ has cardinality m.

1. 4. *Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces each having density character not exceeding m. Let f be a continuous closed map of the product X onto a space Y. Then every canonical closed subset C of Y is the image of a canonical closed subset F of X having a support B of cardinality at most m. Furthermore, if X is compact. then $F|B$ is a canonical closed subset of $X|B$.*

PROOF. Let $U \subset Y$ be open with $\overline{U} = C$. By, 1. 2, $\overline{f^{-1}[U]}$ has a support B with card $B \leqq m$. Let $F = \overline{f^{-1}[U]}$. Then $f[F] \subset \overline{U}$ by the continuity of f and $\overline{U} \subset f[F]$ since f is a closed map. Thus $f[F] = \overline{U}$. It is easy to show that $F = (F|B) \times (X|A \setminus B)$.

Now suppose that X is also compact. By the continuity of π_B, $F|B \subset \overline{\pi_B[f^{-1}[U]]}$; and, the reverse inclusion holds since π_B is a closed map. Consequently, $F|B$ is a canonical closed set in $X|B$.

1. 5. THEOREM. *Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces each having density character not exceeding m and let f be a continuous map from the product X into a space Y having pseudo-character q. Then f has a support of cardinality at most* max $\{m, q\}$.

PROOF. First we shall show that for every $x \in X$ there is a set $B_x \subset A$ with card $B_x \leqq \leqq \max(q, \aleph_0\}$ and such that if $y \in X$ and $y|B_x = x|B_x$, then $f(y) = f(x)$. To this end let $S_x = f^{-1}(f(x))$. Then S_x is a G^q_δ-set (i.e., there is a collection $\mathcal{O}$ of open sets such that $S_x = \bigcap \mathcal{O}$ and card $\mathcal{O} \leqq q$). Let $\mathcal{O}$ be such a collection. For every $G \in \mathcal{O}$ choose an elementary neighbourhood $U(G)$ such that $x \in U(G) \subset G$. Let B_x be the set of indices α such that α is a distinguished index for $U(G)$ for some $G \in \mathcal{O}$. Clearly, card $B_x \leqq \leqq \max\{q, \aleph_0\}$. Now suppose that $y \in X$ and $f(y) \neq f(x)$. Then $y \notin S_x$. Consequently, there is an $\alpha \in B_x$ such that $y(\alpha) \neq x(\alpha)$. Thus $y|B_x \neq x|B_x$.

Let $z \in X$ be arbitrary. By the above there is a set $B_z \subset A$ of cardinality at most $\max\{q, \aleph_0\}$ such that $f(y) = f(z)$ for every $y \in X$ with $y|B_z = z|B_z$.

Let $n = \max\{m, q\}$ and $B_0 = B_z$. Since the density character of $X|B_0$ is at most n, there is a set $D_0 \subset X$ with card $D_0 \leqq n$ and such that $D_0|B_0$ is dense in $X|B_0$ and $D_0|A \setminus B_0 = \{z\}|A \setminus B_0$.

As above, for every $x \in D_0$ there is a set $B_x \subset A$ with card $B_x \leqq n$ and such that $f(y) = f(x)$ for every $y \in X$ with $y|B_x = x|B_x$. Let $B_1 = \bigcup\{B_x : x \in D_0\}$. Then card $B_1 \leqq n$. There is a set $D_1 \subset X$ with card $D_1 \leqq n$ and such that $D_1|B_1$ is dense in $X|B_1$ and $D_1|A \setminus B_1 = \{z\}|A \setminus B_1$.

Suppose that B_ξ and D_ξ have been defined for every $\xi < \eta < \omega(n)$ so that card $B_\xi \leqq n$, $D_\xi \subset X$, card $D_\xi \leqq n$, $D_\xi|B_\xi$ is dense in $X|B_\xi$, and $D_\xi|A \setminus B_\xi = \{z\}|A \setminus B_\xi$.

Let $B^\eta = \bigcup\{B_\xi : \xi < \eta\}$. Then card $(B^\eta) \leqq n$. There is a set $D_\eta \subset X$ such that card $D_\eta \leqq n$, $D_\eta|B^\eta$ is dense in $X|B^\eta$, and $D_\eta|A \setminus B^\eta = \{z\}|A \setminus B^\eta$.

Again as above, for every $x \in D_\eta$ there is a set $B_x \subset A$ with card $B_x \leqq n$ and such that $f(y) = f(x)$ for every $y \in X$ with $y|B_x = x|B_x$. Let $B_\eta = \bigcup\{B_x : x \in D_\eta\}$. Then card $B_\eta \leqq n$. The induction is complete.

Let $B = \bigcup\{B_\xi : \xi < \omega(n)\}$ and $D = \bigcup\{D_\xi : \xi < \omega(n)\}$. Then card $B \leqq n$, card $D \leqq n$, $D|B$ is dense in $X|B$, and $D|A \setminus B = \{z\}|A \setminus B$.

We now show that $f(x) = f(x_0)$ for every $x \in X$, $x_0 \in D$ such that $x|B = x_0|B$. Since $x_0 \in D$, $x_0 = (x_0|B) \cup (z|A \setminus B)$. Since $D|B$ is dense in $X|B$, there is a net $\{x_n\}$ in D converging to $x|B$. Let $x_n^* = (x_n|B) \cup (x|A \setminus B)$ for every n in the directed set. Then the net $\{x_n^*\}$ converges to x. However, the net $\{x_n\}$ converges to x_0. For every n, $x_n|B = x_n^*|B$ and $x_n \in D$. Thus $f(x_n) = f(x_n^*)$ for every n. By the continuity of f, the nets $\{f(x_n)\}$ and $\{f(x_n^*)\}$ converge to $f(x_0)$ and $f(x)$ respectively. And since limits are unique in Y, $f(x) = f(x_0)$.

To conclude the proof it suffices to show that B is a support of f. To this end let $x, y \in X$ with $x|B = y|B$. Then there is a net $\{x_n\}$ in D such that the net $\{x_n|B\}$ converges to $x|B$. Let $x_n^* = (x_n|B) \cup (x|A \setminus B)$ and $y_n = (x_n|B) \cup (y|A \setminus B)$ for every n in the directed set. Then the nets $\{x_n^*\}$ and $\{y_n\}$ converge to x and y respectively. Since f is continuous, the nets $\{f(x_n^*)\}$ and $\{f(y_n)\}$ converge to $f(x)$ and $f(y)$ respectively. For every n, $x_n \in D$ and $x_n^*|B = y_n|B = x_n|B$. Thus $f(x_n^*) = f(y_n)$ for every n. Finally, since limits are unique in Y, $f(x) = f(y)$.

For the *p-topology* of the product (i.e., elementary neighbourhoods are of the form $\bigcap\{\pi_{\alpha_\xi}^{-1}[U_\xi] : \xi \in \Xi\}$ where card $\Xi < p$ and U_ξ is an open set in X_{α_ξ} for every $\xi \in \Xi$), we have the following

1.5'. THEOREM. *Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces each having density character not exceeding m. Let f be a continuous function from the product X (with the p-topology) into a space Y having pseudo-character q. Then there is a support of f of cardinality not exceeding*

$$\ldots\varphi(n\varphi(\ldots n\varphi(n\varphi(n))\ldots))\ldots$$

where φ is taken for every $\xi<\omega(n)$, $\varphi(\gamma)$ denotes the supremum, over all sets $A_0\subset A$ with card $A_0=\gamma$, *of the density characters of $X|A_0$, and $n=$*max $\{m, p, q\}$. *In particular, if $\varphi(n)\leqq n$ (e.g., if $p=\aleph_0$ or if $m^{pq}=m$), then there is a support of f of cardinality not exceeding n.*

REMARKS. Theorem 1. 5 is a generalization of a theorem of GLEASON (cited in [6]). In [6], Theorem 1. 5 is proved for m and q countable. The proof of 1. 5′ is a modification of the proof of 1. 5.

In 1. 5′, an upper bound for the cardinality of a support of f is

$$\ldots(\exp(\exp\ldots(\exp(\exp(n)))\ldots))\ldots$$

where exp is taken for every ordinal $\xi<\omega(n)$ and exp (β) denotes 2^β for every cardinal β.

Theorem 1. 5 cannot be strengthened under present hypotheses even if q is finite (see [5, p. 120] for an example).

1. 6. COROLLARY. *Let $\{X_\alpha:\alpha\in A\}$ be a collection of spaces each having weight not exceeding m. For every continuous map f from the product X into a space Y having pseudo-character q there is a subspace X_f of X of weight not exceeding* max $\{m, q\}$ *and such that $f[X_f]=f[X]$. Moreover, if X is compact, then $f[X]$ has weight not exceeding* max $\{m, q\}$.

PROOF. Since the density character of a space never exceeds its weight, by 1. 5, f has a support $B\subset A$ such that card $B\leqq$max $\{m, q\}$. That is, there is a continuous map g from $X|B$ into Y such that $f=g\circ\pi_B$.

Let $x\in X$ be a fixed element. We define $X_f\subset X$ so that $X_f|B=X|B$ and $X_f|A\setminus B=\{x\}|A\setminus B$. Then $f[X_f]=f[X]$. Also, the weight of X_f does not exceed $m\cdot$card $B=$ max $\{m, q\}$.

If X is compact, then X_f is also compact. Thus $f[X]$ has weight at most max $\{m, q\}$ since continuous maps do not increase weights for compact spaces.

1. 7. COROLLARY. *Let $\{X_\alpha:\alpha\in A\}$ be a collection of spaces each having density character not exceeding m. Let F be a subset of the product X. Then F is a zero-set if and only if $F=(F|B)\times(X|A\setminus B)$ for some set $B\subset A$ such that* card $B\leqq m$ *and $F|B$ is a zero-set in $X|B$.*

PROOF. Suppose that F is a zero-set in X. Then there is a continuous function f from X into I (the interval [0, 1]) such that $F=f^{-1}(0)$. By 1. 5, f has a support $B\subset A$ with card $B\leqq m$. That is, there is a continuous function g from $X|B$ into I such that $f=g\circ\pi_B$. Consequently, $F|B=g^{-1}(0)$ and hence $F|B$ is a zero-set in $X|B$. Clearly, $F=(F|B)\times(X|A\setminus B)$.

Conversely, suppose that $F=D\times(X|A\setminus B)$, card $B\leqq m$, and D is a zero-set in $X|B$. There is a continuous function g from $X|B$ into I such that $D=g^{-1}(0)$. It follows that $f=g\circ\pi_B$ is a continuous function with $F=f^{-1}(0)$.

Let $\{X_\alpha:\alpha\in A\}$ be a collection of spaces and x^* be a fixed element of the product X. For every $x\in X$ and every $B\subset A$ we denote by x_B the element $(x|B)\cup(x^*|A\setminus B)$ in X. A set $S\subset X$ is said to be *invariant under projections* provided that $x_B\in S$ for $x\in S$ and every $B\subset A$.

We shall denote by m_0 the smallest cardinal that is weakly inaccessible from $\aleph_0$ (i.e., m_0 is the smallest uncountable cardinal such that (a) if $\aleph_\xi<m_0$, then

$\aleph_{\xi+1} < m_0$ and (b) if $m_\xi < m_0$ for every $\xi \in \Xi$ and card $\Xi < m_0$, then $\Sigma\{m_\xi : \xi \in \Xi\} < m_0$).

The following is a corollary of a result of MAZUR [8].

1. 8. *Let X be a second-countable space and Y a space such that the diagonal Δ of $Y \times Y$ is a sequential G_δ-set (i.e., Δ is a countable intersection of sequentially open sets). Let $S_\alpha \subset X^m$ be invariant under projections for every $\alpha \in A$ and let $S = \Pi\{S_\alpha : \alpha \in A\}$. If* max $\{m$, card $A\} < m_0$, *then every sequentially continuous function from S into Y has a countable support and is continuous.*

PROOF. It suffices to embed S into X^n $(n < m_0)$ so that S is invariant under proections. Then the conclusion follows from [8].

§ 2. m-adic spaces

We recall that X is m-adic provided that it is a continuous image of some topological power of A_m. Note that if $2 \leqq m \leqq \aleph_0$, then X is m-adic if and only if it is dyadic. Hence no generality is lost by assuming m to be infinite. Note further that an m-adic space is n-adic for every $n \geqq m$.

2. 1. THEOREM. *Every collection of mutually disjoint nonempty open subsets of an m-adic space has cardinality at most m.*

PROOF. Suppose that f is a continuous map of A_m^p onto X for some cardinal p and $\mathcal{O}$ is a collection of mutually disjoint nonempty open subsets of X with card $\mathcal{O} > m$. Then $f^{-1}[\mathcal{O}] = \{f^{-1}[G] : G \in \mathcal{O}\}$ is a collection of mutually disjoint nonempty open subsets of A_m^p and card $(f^{-1}[\mathcal{O}]) > m$. However, this is a contradiction of 1. 1 since the density character of A_m is m.

2. 2. COROLLARY. *A_n is never m-adic for every pair m, n of cardinals with $m < n$.*

The following result of R. ENGELKING (see [9]) is similar to Theorem 2. 1.

2. 3. (ENGELKING). *Let $n \leqq m$. An m-adic space X is n-adic if nd only if every collection of mutually disjoint nonempty open subsets of X has cardinality at most n.*

Another similar result is the following unpublished result of MRÓWKA.

2. 4. (MRÓWKA). *Every m-adic space X having density character n is n-adic.*

PROOF. If $n \geqq m$, then it is obvious that X is n-adic. Thus let $n < m$ and D be a dense subset of X of cardinality n. Since X is m-adic, there is a continuous map f of A_m^p onto X for some cardinal p.

Let Ξ be a set of cardinality p so that A_m^p is the product of copies of A_m indexed by Ξ. Let $S \subset A_m^p$ so that card $S = n$ and $f[S] = D$. Then card $(\pi_\xi[S]) \leqq n$ for every $\xi \in \Xi$. Consequently, for every $\xi \in \Xi$, $\pi_\xi[S]$ is homeomorphic to a subspace of A_n.

Regarding A_n^p as a subspace of A_m^p, we have that D is a continuous image of $S \subset A_n^p$. Since $f[A_n^p]$ is compact and hence closed, and $D \subset f[A_n^p]$, $X = f[A_n^p]$. Thus X is n-adic.

2. 5. THEOREM. *Every m-adic space X having weight n is a continuous image of A_m^n.*

PROOF. Suppose that f is a continuous map of A_m^p onto X for some cardinal p. If $p \leqq n$, then $f \circ \pi$ is a continuous map of A_m^n onto X where π is the projection of A_m^n onto A_m^p. Therefore, we may assume that $n < p$.

We regard X as a subspace of I^n where I denotes the interval $[0, 1]$. Let Ξ be an index set of cardinality n and $\mathscr{F} = \{\pi_\xi \circ f : \xi \in \Xi\}$ where π_ξ is the projection of I^Ξ onto $I_\xi = I$ for every $\xi \in \Xi$. By a classical theorem (every real-valued continuous function on a product of compact spaces has a countable support), each member of $\mathscr{F}$ has a countable support. Consequently, f has a support of cardinality n and hence there is a continuous map of A_m^n onto X.

2. 6. THEOREM. *Let $m \leqq n$. An m-adic space has pseudo-character n if and only if it has weight n.*

PROOF. The other case being obvious, we suppose that f is a continuous map of A_m^p onto X for some cardinal p and that X has pseudo-character n. Since A_m^p is compact and A_m has weight m, the conclusion follows from Corollary 1. 6.

REMARKS. The space $\mathscr{A}_7$ of [2, p. 76] is compact, separable, first-countable, but has uncountable weight. By Theorem 2. 6, it is not $\aleph_0$-adic; and, by 2. 4, it is not m-adic (for any m). In particular, this answers affirmatively a question of MRÓWKA [9] as to whether there exists a first-countable compact space which is not m-adic.

2. 7. THEOREM. *Let $m, n < m_0$. If X is an m-adic space having weight n and the diagonal Δ of $X \times X$ is a sequential G_δ-set, then $n \leqq m$.*

PROOF. By Theorem 2. 5, there is a continuous map f of A_m^n onto X. Let Ξ be a set of cardinality n so that A_m^n is the product of copies of A_m indexed by Ξ. It is easy to show that A_m can be embedded in $\{0, 1\}^m$ so that it is invariant under projections. By 1. 8, f has a countable support $\Gamma \subset \Xi$. Consequently, there is a continuous map g of $A_m^n | \Gamma$ onto X such that $f = g \circ \pi_\Gamma$. Thus the weight of X is not greater than m, the weight of $A_m^n | \Gamma$.

A consequence of Theorem 2. 7 is the following characterization of metrizability in dyadic spaces.

2. 8. COROLLARY. *Let X be a dyadic space having weight less than m_0. Then the diagonal Δ of $X \times X$ is a sequential G_δ-set if and only if X is metrizable.*

PROOF. If X is metrizable, then Δ is a sequential G_δ-set since every open set is sequentially open.

Conversely, suppose that Δ is a sequential G_δ-set. Clearly, X is $\aleph_0$-adic. Thus by Theorem 2. 7, X is second-countable and hence metrizable.

REMARKS. Corollary 2. 8 is not true for compact spaces in general. For instance, βN, the Stone—Čech compactification of the space of positive integers, is obviously not metrizable; however, $\Delta \subset \beta N \times \beta N$ is a sequential G_δ-set. It is known that a compact space X such that the diagonal of $X \times X$ is a G_δ-set is metrizable. We do not know, however, if every dyadic space with sequential G_δ-set points is metrizable.

EFIMOV [4] proved that a dyadic space is metrizable if and only if it is Fréchet. In fact, the following stronger result holds. The proof is analogous to that in [4].

2. 9. THEOREM. *A dyadic space is metrizable if and only if it is sequential.*

(Recall that a space is Fréchet provided that the sequential closure of every set coincides with its closure and is sequential provided that every sequentially closed set is closed.)

We conclude with two results on the structure of closed sets in m-adic spaces.

2. 10. *Every canonical closed set F in an m-adic space X is a continuous image of $C \times A_m^n$ where C is a canonical closed subset of A_m^n and n does not exceed the weight of X.*

PROOF. By Theorem 2. 5, there is a continuous map f of A_m^p onto X where p is the weight of X. Clearly, f is a closed map. The conclusion now follows from 1. 4 with $n = \text{card}\,(A \setminus B)$. Of course, if $B = A$, we use the convention that A_m^0 is a singleton.

2. 11. *Every closed G_δ-set F in an m-adic space X is a continuous image of $C \times A_m^n$ where C is a closed G_δ-set in A_m^n and n does not exceed the weight of X. Furthermore, if the weight of X and m are each less than m_0, then C is a closed G_δ-set in $A_m^{\aleph_0}$.*

PROOF. By theorem 2. 5, there is a continuous map f of A_m^p onto X where p is the weight of X. Clearly, $f^{-1}[F]$ is a zero-set. Consequently, by Corollary 1. 7, $f^{-1}[F] = ((f^{-1}[F])|B) \times A_m^n$ where card $B \leqq m$, $n \leqq p$, and $(f^{-1}[F])|B$ is a closed G_δ-set in A_m^m.

For the second part, there is a continuous map g from A_m^p into I such that $f^{-1}[F] = g^{-1}(0)$. By 1. 8 (see e.g., the proof of 2. 7), g has a countable support. Consequently, the set B above is countable and the conclusion follows.

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ON THE NILSTUFE OF THE DIRECT SUM OF TWO GROUPS

By
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1. All groups in this paper will be assumed to be abelian groups with addition as the group operation. An associative operation on a group G satisfying both distributive laws will be called a multiplication on G. The multiplication $gh=0$ for all $g, h \in G$ is called the zero multiplication, and can be defined on every group G. If G allows no other multiplication, then G is called a nil-group. The notion of a nil-group was introduced by SZELE [2], and studied further in [1]. SZELE proved [1], [2] that a torsion group is nil iff it is divisible and that there are no nil mixed groups. In [3] SZELE generalized the concept of a nil-group as follows. Let n be a positive integer. If there exists a multiplication on G under which $G^n \neq 0$, but $G^{n+1}=0$ under every multiplication, then G is said to have nilstufe n, denoted by $\nu(G)=n$. Clearly the nil-groups are the groups G with $\nu(G)=1$. If $\nu(G)>n$ for every positive integer n, then we will denote $\nu(C)=\infty$.

The original intent of the research which led to this paper was to consider a group $G = G_1 \oplus G_2$, $\nu(G_1)=\nu(G_2)=1$, and to see what could be said about $\nu(G)$. Szele's theorem assures us that neither G_1 nor G_2 can be mixed. We further have that if G_1 and G_2 are both torsion groups then they are both divisible and G is therefore a divisible, torsion group, and hence nil by Szele's theorem. In theorem 1 we consider the case where G_1 is a torsion group, and G_2 is torsion free. However, we consider there the more general situation, $\nu(G_1)=n$, $\nu(G_2)=m$, n, m arbitrary positive integers. In § 3 we study the case where both G_1 and G_2 are torsion free and of rank 1. If either G_1 or G_2 is not nil then its nilstufe is ∞ ([1], p. 270). Therefore $\nu(G_1)=\nu(G_2)=1$ is the most general case to consider when both G_1 and G_2 are torsion free groups of rank 1.

The case G_1 and G_2 torsion free and of arbitrary rank remains an open question.

2. THEOREM 1. *Let $G = G_1 \oplus G_2$, G_1 torsion, $\nu(G_1)=n$, G_2 torsion free, $\nu(G_2)=m$, then $\nu(G) \leqq (n+1)(m+1)-1$.*

PROOF. Consider the cartesian product $G_i \times G_j$, $i=1$ or 2, $j=1$ or 2. Every multiplication on G is a bilinear mapping of $G_i \times G_j$ into G and therefore factors through $G_i \otimes G_j$. If either $i=1$ or $j=1$ then $G_i \otimes G_j$ is a torsion group, and $g_i g_j \in G_1$, $g_i \in G_i$, $g_j \in G_j$. We therefore have:

(1) $$G_1^2 \subset G_1$$

(2) $$G_1 G_2 \subset G_1,$$

and

(3) $$G_2 G_1 \subset G_1$$

under every multiplication on G.

Consider the restriction of a multiplication on G onto G_1. By (1) it is a multiplication on G_1, and therefore;

(4) $$G_1^{n+1}=0 \text{ under every multiplication on } G.$$

Let X_G be a multiplication on G, $g, h \in G_2$, and $gX_G h = g_1+g_2$, $g_1 \in G_1$, $g_2 \in G_2$. Put $gX_{G_2}h=g_2$. X_{G_2} so defined is a multiplication on G_2. Therefore:

(5) $$G_2^{m+1} \subset G_1.$$

Let

$$l = (m+1)(n+1), \quad g_i \in G, \qquad i=1, \dots, l.$$

Under every multiplication on G

$$\prod_{i=1}^{l} g_i = \left(\prod_{i=1}^{m+1} g_i\right)\left(\prod_{i=m+2}^{2m+2} g_i\right)\cdots\left(\prod_{i=n(m+1)+1}^{l} g_i\right) = 0.$$

COROLLARY. *Let G_1 be a nil torsion group, G_2 a nil torsion free group, and $G = G_1 \oplus G_2$. Then $\nu(G)=2$ or 3. If $pG_2=G_2$ for all primes p for which G_1 has a non-trivial p-component, then $\nu(G)=2$.*

PROOF. By theorem 1, $\nu(G) \leqq 3$. G is a mixed group and therefore by (2), $\nu(G)>1$.

If $pG_2=G_2$ for every prime p for which G_1 has a non-zero p-component then $G_1 \otimes G_2 = 0$ ([1], p. 255, theorem 65. 4). Therefore (2) and (3) in the proof of theorem 1 become

(2^1) $$G_1 G_2=0,$$

(3^1) $$G_2 G_1=0.$$

The restriction of a multiplication on G to G_1 is a multiplication on G_1 by (1) but G_1 is nil. We therefore have:

(1^1) $$G_1^2=0.$$

Let $g, h, j \in G$, $g = g_1+g_2$, $h = h_1+h_2$, $j = j_1+j_2$, $g_1, h_1, j_1 \in G_1$, $g_2, h_2, j_2 \in G_2$.

By (1^1), (2^1), (3^1) and (5) $gh=k_1 \in G_1$ therefore, $ghj = k_1(j_1+j_2) = 0$ by (1^1) and (2^1) q.e.d.

3. Let G_1 and G_2 be torsion free groups. Let $r(G_i)$ and $T(G_i)$ denote the rank and type of G_i, $i=1, 2$ respectively. (If $r(G_i)=1$ then it makes sense to consider $T(G_i)$, [1], p. 148). $H_p(x)$ will denote the height of an element at a prime p.

THEOREM 2. *Let $G = G_1 \oplus G_2$, G_1 and G_2 torsion free groups, $r(G_i)=\nu(G_i)=1$, $i=1, 2$. Then $\nu(G) \leqq 3$. If $T(G_1)=T(G_2)$ then G is nil.*

PROOF. Let φ_1 be the projection of G onto G_1, and φ_2 be the projection of G onto G_2. If X_G is a multiplication on G, then $\varphi_i \circ X_G$ is a multiplication on G_i, $i=1, 2$. G_1 and G_2 are both nil, therefore:

(6) $$G_1^2 \subset G_2,$$

and

(7) $$G_2^2 \subset G_1$$

under every multiplication on G.

Let $g_1 \in G_1$, $g_2 \in G_2$, $g_1 \neq 0$, $g_2 \neq 0$, and let $g_1 g_2 = \bar{g}_1 + \bar{g}_2$, $g_1 \in G_1$, $g_2 \in G_2$. Suppose $T(G_1) \neq T(G_2)$. Without loss of generality, we may assume the existence of infinitely many primes $\{p_i\}$ such that

(8) $$H_{p_i}(g_1) < H_{p_i}(g_2) \quad \text{for all} \quad i.$$

For all primes p, $H_p(g_1 g_2) \geqq H_p(g_2)$ and $H_p(g_1 g_2) = \min\big(H_p(\bar{g}_1), H_p(\bar{g}_2)\big)$.

For almost all i, $H_{p_i}(g_1) = H_{p_i}(\bar{g}_1)$. Therefore

(9) $$G_1 G_2 \subset G_2$$

and similarly

(10) $$G_2 G_1 \subset G_2.$$

Let $g_2' \in G_2$, $g_2' \neq 0$. If $g_2 g_2' \neq 0$, then $H_{p_i}(g_2 g_2') > H_{p_i}(g_1)$ for all i and by (7), $H_{p_i}(g_2 g_2') = H_{p_i}(g_1)$ for almost all i. Therefore

(11) $$G_2^2 = 0.$$

Let $g, h, j, k \in G$. (6), (9), (10) and (11) imply that $gh \in G_2$ and $jk \in G_2$. Therefore $ghjk \in G_2^2 = 0$, so that $G^4 = 0$, or $\nu(G) \leqq 3$.

Suppose $T(G_1) = T(G_2)$. Let $g_1 \in G_1$, $g_1 \neq 0$. If $H_p(g_1) = 0$ or ∞ for all but finitely many primes, then $\nu(G_1) = \infty$ ([1], p. 270). Therefore there exist infinitely many primes $\{p_i\}$ such that $0 < H_{p_i}(g_1) < \infty$.

Let $\bar{g}_1 \in G_1$, $\bar{g}_1 \neq 0$. If $g_1 \bar{g}_1 \neq 0$, then $H_{p_i}(g_1 \bar{g}_1) > H_{p_i}(g_1)$ for all but finitely many i. Therefore $T(g_1 \bar{g}_1) \neq T(g_1)$. However $g_1 \bar{g}_1 \in G_2$ by (6) so that $T(g_1 \bar{g}_2) = T(G_2) = T(G_1) = T(g_1)$ a contradiction. Therefore:

(12) $$G_1^2 = 0,$$

and similarly

(13) $$G_2^2 = 0.$$

Let $g_1 \in G_1$, $g_2 \in G_2$, $g_1 \neq 0$, $g_2 \neq 0$. $H_p(g_1) = H_p(g_2)$ for all but finitely many primes p, so that $H_{p_i}(g_1) = H_{p_i}(g_2)$ for almost all i, the p_i are as above, $0 < H_{p_i}(g_1) < \infty$. Therefore, if $g_1 g_2 \neq 0$, then

$$H_{p_i}(g_1 g_2) > H_{p_i}(g_1) \quad \text{for almost all } i, \text{ so that} \quad T(g_1 g_2) \neq T(g_1).$$

However, $T(g_1 g_2) = T(G_1) \cap T(G_2)$ ([1], p. 147) $= T(G_1)$, a contradiction. Therefore

(14) $$G_1 G_2 = 0,$$

and similarly

$$G_2 G_1 = 0. \tag{15}$$

(12), (13), (14), and (15) imply that $G^2=0$, or $\nu(G)=1$.

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ÜBER FINSLERRÄUME VON ZWEIFACH REKURRENTER KRÜMMUNG

Von

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§ 1. Einleitung

Die Finslerräume, für die eine Relation von der Form

$$\nabla_m R_{h\ jk}^{*i} = \varkappa_m R_{h\ jk}^{*i} \tag{1. 1}$$

besteht, wo $R_{h\ jk}^{*i}$ jetzt und im folgenden einen der Krümmungstensoren des Raumes, $\varkappa_m(x, \dot{x})$ ein kovariantes Vektorfeld, bzw. ∇_m eine der kovarianten Ableitungen bedeutet, werden Finslerräume von rekurrenter Krümmung genannt (vgl. [4] § 3 und [6]). Wenn statt (1. 1) eine Relation von der Form:

$$\nabla_p \nabla_m R_{h\ jk}^{*i} = T_{mp} R_{h\ jk}^{*i} \tag{1. 2}$$

besteht, wo T_{mp} ein rein-kovariantes Tensorfeld zweiter Stufe bezeichnet, so werden wir den Raum *einen Finslerraum von zweifach rekurrenter Krümmung* nennen, während die durch (1. 1) charakterisierten Räume *Finslerräume von einfach rekurrenter Krümmung* genannt werden. Im folgenden wollen wir die Grundeigenschaften der Finslerräume von zweifach rekurrenter Krümmung bestimmen.

Vor allem bemerken wir, daß ein Finslerraum von einfach rekurrenter Krümmung immer auch ein solcher von zweifach rekurrenter Krümmung ist. Das folgt aus (1. 1) unmittelbar nach einer kovarianten Ableitung beider Seiten nach ∇_p unter Verwendung von (1. 1) selbst. Es wird dann

$$T_{mp} = \nabla_p \varkappa_m + \varkappa_m \varkappa_p. \tag{1. 3}$$

Aus (1. 2) folgt aber selbstverständlich im allgemeinen die Relation (1. 1) nicht; wir werden aber hinreichende Bedingungen angeben, damit ein Finslerraum von zweifach rekurrenter Krümmung gleichzeitig ein Finslerraum von einfach rekurrenter Krümmung sei, ferner werden wir für zwei bestimmte kovariante Ableitungen beweisen, daß der Finslerraum von zweifach rekurrenter Krümmung auch von einfach rekurrenter Krümmung sei. *Die Sätze aber, die wir in Bezug auf die Räume von zweifach rekurrenter Krümmung ableiten werden, sind unabhängig davon, daß der Raum auch ein Finslerraum von einfach rekurrenter Krümmung ist, oder nicht*; wir verweisen in erster Reihe auf unsere Sätze 3, 4, 4a, 5 und 6.

Das Vektorfeld $\varkappa_m$ in (1. 1) ist bekanntlich immer ein Gradientvektorfeld (vgl. Satz I. von [4]), aber auch T_{mp} in (1. 2) hat eine bestimmte Form, wie wir das zeigen werden, undzwar eben die Form (1. 3), wo $\varkappa_m$ nach unserem Satz 3 ein Gradientvektorfeld ist. (In diesem Falle bedeutet natürlich $\varkappa_m$ nicht unbedingt das in (1. 1) vorhandene Vektorfeld, da (1. 1) überhaupt nicht gültig sein braucht.)

In gewissen Fällen kann $R_i^{*j}{}_{kl}$ durch Tensoren zweiter Stufe ausgedrückt werden. Diese Type sind gewissermaßen analog zu denen von Herrn R. N. SEN als einfach (simple) bezeichnete Type (vgl. [6] § 5, S. 297).

Bezüglich der Zerlegbarkeit dieser Räume sind ganz ähnliche Sätze gültig, wie im Falle der einfach rekurrenter Finslerräume.

Endlich werden wir im letzten Paragraphen die möglichen Verallgemeinerungen unserer Finslerräume von rekurrenter Krümmung besprechen.

§ 2. Grundgrößen der Finslerräume

Ein Finslerraum F_n ist eine Mannigfaltigkeit der Linienelemente $(x^i, \dot{x}^i)$ $(i=1, 2, \dots, n)$, in dem eine Metrik durch eine in den $\dot{x}^i$ von erster Dimension positiv homogene Grundfunktion $F(x^i, \dot{x}^i)$ definiert ist. Ist die Punktmannigfaltigkeit von F_n n-dimensional, so ist die Linienelementmannigfaltigkeit $(2n-1)$-dimensional, da bei den $\dot{x}^i$ nur ihre Verhältnisse in Betracht kommen.

Die aus $F(x, \dot{x})$[1] ableitbaren Grundgrößen der Finslerräume sind die folgenden:

1) Der in i, k symmetrische metrische Grundtensor

$$g_{ik} \overset{\text{def}}{=} \frac{1}{2} \frac{\partial^2 F^2}{\partial \dot{x}^i \partial \dot{x}^k}.$$

2) Der Einheitsvektor l^i der die Richtung seines Stützelementes hat:

$$l^i \overset{\text{def}}{=} \frac{\dot{x}^i}{F}, \quad l_i = \frac{\partial F}{\partial \dot{x}^i}.$$

3) Der Torsionstensor A_{ijk} der in allen ihren Indexen symmetrisch ist. Es gilt:

$$A_{ijk} \overset{\text{def}}{=} \frac{1}{4} F \frac{\partial^3 F^2}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k}, \quad A_i \overset{\text{def}}{=} A_i{}^k{}_k.$$

4) Die Cartanschen kovarianten Ableitungen sind für einen gemischten Tensor $X^i{}_k$ durch die folgenden Formeln bestimmt:

$$X^i{}_k\|_j = F \frac{\partial X^i{}_k}{\partial \dot{x}^j}, \tag{2.1}$$

$$X^i{}_{k;j} = X^i{}_k\|_j + A_r{}^i{}_j X^r{}_k - A_k{}^r{}_j X^i{}_r, \tag{2.2}$$

$$X^i{}_k|_j = \frac{\partial X^i{}_k}{\partial x^j} - X^i{}_k\|_r \Gamma_0^{*r}{}_k + X^r{}_k \Gamma_r^{*i}{}_j - X^i{}_r \Gamma_k^{*r}{}_j, \tag{2.3}$$

wo der Index „0" die Überschiebung mit l^i und die $\Gamma_k^{*r}{}_j$ die Cartanschen Übertragungsparameter bedeuten. Bezüglich ihre explizite Form verweisen wir auf [2].

[1] Jetzt, und im folgenden, werden wir statt $(x^i, \dot{x}^i)$ kurz $(x, \dot{x})$ schreiben.

5) Die Berwaldsche kovariante Ableitung ist durch

$$X^i{}_{k(j)} = \frac{\partial X^i{}_k}{\partial x^j} - X^i{}_k\|_r G_0{}^r{}_j + X^r{}_k G_r{}^i{}_k - X^i{}_r G_k{}^r{}_j \tag{2. 4}$$

festgelegt, wo die Berwaldschen Übertragungsparameter $G_j{}^i{}_k$ mit den Cartanschen Übertragungsparametern $\Gamma_j^{*i}{}_k$ durch die Formeln

$$G_j{}^i{}_k = \Gamma_j^{*i}{}_k + A_j{}^i{}_k|_0 \tag{2. 5}$$

zusammenhängen (vgl. [5], Kap. III. § 3, S. 76—82).

Wir bemerken schon jetzt, daß die Cartanschen kovarianten Ableitungen (2. 2) und (2. 3) metrisch sind, d.h. es gilt:

$$g_{ij;k} = 0, \quad g_{ij}|_k = 0, \tag{2. 6}$$

ferner es ist auch

$$l^i|_k = 0, \quad l_i|_k = 0, \tag{2. 7}$$

und ebenso auf Grund von (2. 5):

$$l^i{}_{(k)} = 0, \quad l_{i(k)} = 0; \tag{2. 8}$$

hingegen ist

$$g_{ij(k)} = -2A_{ijk}|_0, \tag{2. 9}$$

und

$$l^i{}_{;k} = l^i\|_k = \delta^i_k - l^i l_k. \tag{2.10}$$

6) Die fundamentalen Krümmungstensoren des Raumes F_n sind die folgenden: der Hauptkrümmungstensor

$$\hat{R}_i{}^j{}_{kl} \stackrel{\text{def}}{=} \frac{\partial \Gamma_i^{*j}{}_k}{\partial x^l} - \Gamma_i^{*j}{}_k\|_t \Gamma_0^{*t}{}_l + \Gamma_i^{*t}{}_k \Gamma_t^{*j}{}_l - (k/l), \tag{2. 11}$$

wo (k/l) den vorigen Ausdruck mit vertauschten Indexen k und l bedeutet; der vollständige Krümmungstensor:

$$R_i{}^j{}_{kl} \stackrel{\text{def}}{=} \hat{R}_i{}^j{}_{kl} + A_i{}^j{}_t \hat{R}_0{}^t{}_{kl}, \tag{2. 12}$$

der Berwaldsche Krümmungstensor:

$$K_i{}^j{}_{kl} = \frac{\partial G_i{}^j{}_k}{\partial x^l} - G_i{}^j{}_k\|_t G_0{}^t{}_l + G_i{}^t{}_k G_t{}^j{}_l - (k/l); \tag{2. 13}$$

ferner die beiden weiteren Krümmungstensoren von Cartan

$$P_i{}^j{}_{kl} \stackrel{\text{def}}{=} \Gamma_i^{*j}{}_k\|_l + A_i{}^j{}_t A_l{}^t{}_k|_0 - A_i{}^j{}_l|_k, \tag{2. 14}$$

und

$$S_i{}^j{}_{kl} \stackrel{\text{def}}{=} A_i{}^t{}_l A_t{}^j{}_k - A_i{}^t{}_k A_t{}^j{}_l. \tag{2. 15}$$

(Vgl. [5], Kap. IV. insb. S. 99—101, die Formel (6. 7)).

Bezüglich der Eigenschaften dieser Krümmungstensoren bemerken wir, daß sie mit der Ausnahme von $P_i{}^j{}_{kl}$, alle in k, l schiefsymmetrisch sind. Wenn wir durch Überschiebung mit g_{jh} den Index j herunterziehen, so werden R_{ihkl}, P_{ihkl} und S_{ihkl} in den Indexen i, h auch schiefsymmetrisch sein (vgl. [5], Kap. IV. § 2).

Die Bianchischen Identitäten dieser Krümmungsgrößen sind die folgenden:

$$\hat{R}_i{}^j{}_{kl}|_m + \Gamma^{*}_i{}^j{}_k\|_t \hat{R}_0{}^t{}_{lm} + \{zykl.\}_{klm} = 0, \tag{2.16}$$

[2]

$$R_i{}^j{}_{kl}|_m + P_i{}^j{}_{kt} R_0{}^t{}_{lm} + \{zykl.\}_{klm} = 0, \tag{2.17}$$

$$K_i{}^j{}_{kl(m)} + G_i{}^j{}_k\|_t K_0{}^t{}_{lm} + \{zykl.\}_{klm} = 0. \tag{2.18}$$

Wir wollen noch darauf hinweisen, daß mit der Methode von H. RUND (vgl. [5], Kap. IV. § 3) auch für den Krümmungstensor $S_i{}^j{}_{kl}$ bezüglich der kovarianten Ableitung (2. 2) leicht eine „Bianchische Identität" abgeleitet werden könnte, doch werden wir in unserem letzten Paragraphen zeigen, daß für den Krümmungstensor $S_i{}^j{}_{kl}$ bezüglich der kovarianten Ableitung (2. 2) und auch bezüglich (2. 1) kein Finslerraum von rekurrenter Krümmung existieren kann.

§ 3. F_n-Räume von zweifach rekurrenter Krümmung die auch einfach rekurrent sind

Nehmen wir an, daß die Relation (1. 2) für eine der Krümmungstensoren (2. 11)—(2. 15) gilt, wo ∇_m eine der kovarianten Ableitungen (2. 3), oder (2. 4) bedeutet. Unter diesen Bedingungen gilt der folgende:

SATZ 1. *Gelten in einem F_n-Raum die Relationen* (1. 2) *und* (1. 3), *in dem $\varkappa_m$ ein Vektorfeld bedeutet, hat ferner das Differentialgleichungssystem:*

$$\nabla_p H_i{}^j{}_{klm} + \varkappa_m H_i{}^j{}_{klp} = 0 \tag{3.1}$$

außer $H_i{}^j{}_{klm}=0$ keine weitere Lösung, so gilt auch die Relation (1. 1).[3]

BEWEIS. Die Relation

$$\nabla_m R^{*}_i{}^j{}_{kl} = \varkappa_m R^{*}_i{}^j{}_{kl} + H_i{}^j{}_{klm} \tag{3.2}$$

ist eine Identität, falls

$$H_i{}^j{}_{klm} \equiv \nabla_m R^{*}_i{}^j{}_{kl} - \varkappa_m R^{*}_i{}^j{}_{kl}$$

besteht. Der Satz 1 ist somit mit $H_i{}^j{}_{klm}=0$ äquivalent. Bilden wir nun die kovariante Ableitung ∇_p auf beiden Seiten von (3. 2), beachten wir dann die Relation (3. 2) selbst, so bekommen wir:

$$\nabla_p \nabla_m R^{*}_i{}^j{}_{kl} = (\nabla_p \varkappa_m + \varkappa_m \varkappa_p) R^{*}_i{}^j{}_{kl} + \nabla_p H_i{}^j{}_{klm} + \varkappa_m H_i{}^j{}_{klp}.$$

Beachten wir jetzt (1. 3), (1. 2) und unsere Annahme über (3. 1), so folgt aus der letzten Relation, daß $H_i{}^j{}_{klm}=0$, d.h. (1. 1) gültig ist, w. z. b. w.

[2] $\{zykl.\}_{klm}$ bedeutet die zyklische Permutation auf die Indizes k, l, m.

[3] Für die Punkträume haben wir schon diesen Satz in einer noch nicht veröffentlichten Arbeit bewiesen. Vollständigkeitshalber geben wir auch hier den Beweis.

Der Satz 1 gibt hinreichende Bedingungen dafür, daß ein Finslerraum F_n von zweifach rekurrenter Krümmung auch ein Finslerraum von einfach rekurrenter Krümmung sei. Bezüglich der kovarianten Ableitungen (2. 1) und (2. 2) gilt aber der wichtige

SATZ 2. *Ist ein Finslerraum F_n ein Raum von zweifach rekurrenter Krümmung bezüglich einer der kovarianten Ableitungen* (2. 1) *und* (2. 2), *d.h. gilt die Relation* (1. 2), *in dem ∇_m eine der beiden genannten kovarianten Ableitungen bezeichnet, so ist der Finslerraum F_n auch von einfach rekurrenter Krümmung.*

BEWEIS. Wir werden den Satz nur bezüglich der kovarianten Ableitung (2. 1) beweisen, da der Beweis, bezüglich der kovarianten Ableitung (2. 2) vollständig analog geführt werden kann. Es gilt also:

$$R^{*i}_{h\ jk}\|_m\|_p = T_{mp} R^{*i}_{h\ jk}.$$

Eine Überschiebung mit l^m gibt wegen (2. 10) und wegen der Identität

$$R^{*i}_{h\ jk}\|_m l^m = R^{*i}_{h\ jk;m} l^m = 0,$$

die Formel:

$$-R^{*i}_{h\ jk}\|_m l^m\|_p \equiv -R^{*i}_{h\ jk}\|_p = T_{0p} R^{*i}_{h\ jk},$$

woraus

$$R^{*i}_{h\ jk}\|_p = -T_{0p} R^{*i}_{h\ jk}$$

folgt, und das beweist schon den Satz, da $(-T_{0p})$ ein kovarianter Vektor ist. Im Falle $T_{0p}=0$ wäre auf Grund unserer letzten Formel $R^{*i}_{h\ jk}\|_m = 0$, und somit auch $R^{*i}_{h\ jk}\|_m\|_p = 0$.

BEMERKUNG. Bei den kovarianten Ableitungen (2. 1) und (2. 2) ist immer $T_{m0}=0$, da

$$M^{\cdot\cdot}_{\cdots}\|_0 = M^{\cdot\cdot}_{\cdots;0} = 0$$

für jeden beliebigen Tensor $M^{\cdot\cdot}_{\cdots}$ besteht.

Wir wollen jetzt zeigen, daß die Bedingung (1. 3) in manchen Fällen eine Folgerung von (1. 2) ist. Es gilt:

SATZ 3. *Bedeutet in* (1. 2) *die kovariante Ableitung ∇_m eine der kovarianten Ableitungen* (2. 3), *oder* (2. 4), *ist ferner $R^{*k}_{0\ 0k} \neq 0$, so hat T_{mp} die Form* (1. 3), *wo das Vektorfeld $\varkappa_m$:*

(3. 3a) $$\varkappa_m = \nabla_m \log |R^{*k}_{0\ 0k}|$$

ist. Wenn ∇_m eine der kovarianten Ableitungen (2. 2), *oder* (2. 3) *bedeutet, und gilt noch $R^{*ik}_{\ \ ik} \neq 0$, so hat T_{mp} die Form* (1. 3) *mit*

(3. 3b) $$\varkappa_m = \nabla_m \log |R^{*ik}_{\ \ ik}|.$$

BEWEIS. Aus (1. 2) folgt durch Verjüngung über i und k die Relation:

(3. 4) $$\nabla_p \nabla_m R^{*k}_{h\ jk} = T_{mp} R^{*k}_{h\ jk}.$$

Bedeutet nun ∇_m eine der kovarianten Ableitungen (2. 3) oder (2. 4), so ist nach

(2. 7) bzw. (2. 8) $\nabla_m l^i = 0$; somit wird aus (3. 4) nach einer Überschiebung mit l^h und dann mit l^j:

$$\nabla_p \nabla_m R^{*k}_{0\ 0k} = T_{mp} R^{*k}_{0\ 0k}. \tag{3. 5}$$

Drücken wir jetzt aus dieser Formel T_{mp} aus, so wird:

$$T_{mp} = \frac{1}{R} \nabla_p \nabla_m R, \quad R \overset{\text{def}}{=} |R^{*k}_{0\ 0k}|. \tag{3. 6}$$

Da R ein Skalar ist, bekommt man durch unmittelbare Berechnung:

$$\nabla_p \nabla_m \log R = \frac{1}{R} \nabla_p \nabla_m R - \nabla_m \log R \, \nabla_p \log R.$$

Benützen wir die Bezeichnung (3. 3a), so wird aus der letzten Gleichung:

$$\frac{1}{R} \nabla_p \nabla_m R = \nabla_p \varkappa_m + \varkappa_m \varkappa_p,$$

und (3. 6) beweist somit die erste Hälfte des Satzes.

Bezüglich der zweiten Hälfte des Satzes beachten wir, daß wenn ∇_m eine der kovarianten Ableitungen (2. 2), oder (2. 3) bedeutet, dann besteht auf Grund von (2. 6): $\nabla_m g^{hj} = 0$. Da die Relation (3. 4) offenbar auch jetzt gültig ist, bekommt man nach einer Überschiebung mit g^{hj}

$$\nabla_p \nabla_m R^{*jk}{}_{jk} = T_{mp} R^{*jk}{}_{jk} \tag{3. 7}$$

und diese Gleichung ist von derselben Form, wie (3. 5), nur steht jetzt statt $R^{*k}_{0\ 0k}$ der Skalar $R^{*ik}{}_{ik}$. Aus der Formel (3. 7) gelangt man ebenso, wie im vorigen Fall zur Formel (3. 3b), womit auch die zweite Hälfte des Satzes 3 bewiesen ist.

Die Formel (1. 3) ist also für T_{mp} aus (1. 2) im allgemeinen — abgesehen vom Fall der Nulltensoren — ableitbar, wenn eine der Grundgrößen l^i bzw. g_{ik} bezüglich ∇_m verschwindende kovariante Ableitung hat. Für die durch die Formel (2. 1) bestimmte kovariante Ableitung ist (1. 3) aus (1. 2) im allgemeinen nicht ableitbar.

§ 4. Folgerungen der Bianchischen Identitäten

Die verschiedenen Type der Bianchischen Identitäten sind durch die Gleichungen (2. 16)—(2. 18) angegeben. Die kovarianten Ableitungen, die in diesen Identitäten vorkommen, haben die Eigenschaft, daß sie auf den Einheitsvektor l^i verwendet den Nulltensor geben. Ferner ist nach (2. 5)

$$G_i{}^j{}_k\|_t l^i = \Gamma^{*j}_{i\ k}\|_t l^i + A_i{}^j{}_k|_0\|_t l^i = \Gamma^{*j}_{i\ k}\|_t l^i + A_0{}^j{}_k|_0\|_t - A_i{}^j{}_k|_0(\delta^i_t - l^i l_t)$$

und nach den Eigenschaften des Torsionstensors gilt:

$$G_i{}^j{}_k\|_t l^i = \Gamma^{*j}_{i\ k}\|_t l^i - A_t{}^j{}_k|_0. \tag{4. 1}$$

Beachten wir jetzt, daß die Bewaldschen Übertragungsparameter die Form:

$$G_i{}^j{}_k \overset{\text{def}}{=} \frac{\partial^2 G^j}{\partial \dot{x}^i \, \partial \dot{x}^k}$$

haben, und noch homogen von nullter Ordnung sind, so folgt

(4.2a) $$G_i{}^j{}_k \|_t l^i = 0,$$

und im Hinblick auf (4.1):

(4.2b) $$\Gamma^{*j}_{i\ k} \|_t l^i = A_t{}^j{}_k|_0 = A_k{}^j{}_t|_0 .$$

Aus den Identitäten (2.18) bzw. (2.17) folgt nun nach einer Überschiebung mit l^i:

(4.3) $$K_0{}^j{}_{kl(m)} + \{zykl.\}_{klm} = 0$$

bzw.

(4.4) $$R_0{}^j{}_{kl}|_m + A_k{}^j{}_t|_0 R_0{}^t{}_{lm} + \{zykl.\}_{klm} = 0$$

da auf Grund der Formel (2.14) die Relationen:

(4.5) $$P_0{}^j{}_{kt} = \Gamma^{*j}_{i\ k} \|_t l^i = A_k{}^j{}_t|_0$$

folgen. Auf Grund von (4.2b) und (2.16) sind die Identitäten (4.4) auch bezüglich des Hauptkrümmungstensors (2.11) gültig, es besteht sogar nach (2.12)

$$R_0{}^j{}_{kl} \equiv \hat{R}_0{}^j{}_{kl} .$$

Wir untersuchen erstens die Folgerungen der Identitäten (4.3), wenn der Finslerraum bezüglich des Berwaldschen Krümmungstensors $K_i{}^j{}_{kl}$, und bezüglich der Berwaldschen kovarianten Ableitung (2.4) von zweifach rekurrenter Krümmung ist, d.h. wenn

(4.6) $$K_i{}^j{}_{kl(m)(p)} = T_{mp} K_i{}^j{}_{kl}$$

besteht. Differenziert man (4.3) kovariant nach x^p, beachtet man ferner (4.6), so wird:

(4.7) $$T_{mp} K_0{}^j{}_{kl} + T_{kp} K_0{}^j{}_{lm} + T_{lp} K_0{}^j{}_{mk} = 0.$$

Auf Grund der Relation

(4.8) $$K_0{}^j{}_{kl} = \hat{R}_0{}^j{}_{kl} = R_0{}^j{}_{kl}$$

(vgl. [5], Kap. IV. Formel (6.9d)) besteht (4.7) auch für den Hauptkrümmungstensor und vollständigen Krümmungstensor des Raumes.

Eine Überschiebung der Identität (4.7) mit l^k und dann nach einer Verjüngung bezüglich j, l gibt unter Beachtung der schiefen Symmetrie des Krümmungstensors in den beiden letzten Indexen, die Identität:

(4.9) $$T_{jp}(\delta^j_m K^* - l^j K_0{}^l{}_{ml} - K_0{}^j{}_{0m}) = 0,$$

wo $K^* \equiv K_0{}^l{}_{0l}$ den Krümmungsskalar des Finslerraumes bedeutet. Wir bemerken hier, daß das mittlere Glied in (4. 9) nicht vorkommt, falls $T_{0p}=0$ ist.

Wir beweisen den folgenden

SATZ 4. *Ist* Det $(T_{mp})\neq 0$ und $n>2$, *so ist* $K_i{}^j{}_{kl}=0$. *Ist* Det $(T_{mp})=0$, und $T_{0p}\neq 0$, *so ist der Krümmungstensor* $K_0{}^j{}_{kl}$ *durch* T_{mp} *und durch einen gemischten Tensor zweiter Stufe:* $Q^j{}_k$ *in algebraischer Form ausdrückbar.*

BEMERKUNG. Nach der ersten Behauptung des Satzes ist im Wesentlichen in einem Finslerraum von rekurrenter Krümmung Det $(T_{mp})=0$, da widrigenfalls der Krümmungstensor Null wäre, und der Definitionsformel (1. 2) wäre somit in trivialer Weise erfüllt.

BEWEIS DES SATZES 4. Wäre Det $(T_{mp})\neq 0$, so könnte das homogen lineare Gleichungssystem (4. 9), in dem als Unbekannten die in Klammern stehenden Glieder betrachtet werden können, nur die triviale Lösung haben, d.h. es wäre:

$$\delta^j_m K^* - l^j K_0{}^l{}_{ml} - K_0{}^j{}_{0m} = 0. \tag{4.10}$$

Eine Verjüngung über j, m ergibt aus dieser Gleichung

$$K^*(n-2) = 0,$$

woraus $K^*=0$ folgt.

Der vollständige Krümmungstensor R_{ijkl} ist in i, j schiefsymmetrisch. Auf Grund von (4. 8) folgt somit

$$K_0{}^0{}_{kl} = \hat{R}_0{}^0{}_{kl} = R_0{}^0{}_{kl} = 0. \tag{4.11}$$

Eine Überschiebung von (4. 10) mit l_j gibt dann wegen $K^*=0$

$$K_0{}^l{}_{ml} = 0,$$

woraus dann nach (4. 10)

$$K_0{}^j{}_{0m} \equiv F^{-2} K_i{}^j{}_{km} \dot{x}^i \dot{x}^k = 0 \tag{4.12}$$

folgt. Offenbar erhalten wir diese Relation in vollständig ähnlicher Weise auch dann, falls in (4. 9) $T_{jp} l^j \equiv T_{0p}=0$ ist, nur das mittlere Glied in (4. 10) wird fehlen.

Für die Berwaldschen Krümmungstensoren gelten nun die folgenden Relationen:

$$K_0{}^i{}_{jk} \equiv R_0{}^i{}_{jk} = \frac{1}{3F}\left\{\frac{\partial(F^2 K_0{}^i{}_{0k})}{\partial \dot{x}^j} - \frac{\partial(F^2 K_0{}^i{}_{0j})}{\partial \dot{x}^k}\right\}, \tag{4.13}$$

$$K_h{}^i{}_{jk} \stackrel{\text{def}}{=} \frac{\partial}{\partial \dot{x}^h}(F K_0{}^i{}_{jk}), \tag{4.14}$$

(vgl. [5], Kap. IV. § 6.). Aus diesen Formeln folgt wegen (4. 12) unmittelbar die erste Hälfte des Satzes.

Nehmen wir nun an, daß Det $(T_{mp})=0$ ist. Auf Grund der Identität (4. 9) ist in diesem Falle

$$\delta^j_m K^* - l^j K_0{}^l{}_{ml} - K_0{}^j{}_{0m} \equiv Q^j{}_m, \tag{4.15}$$

wo $Q^j{}_m$ einen gemischten Tensor bedeutet, für den $Q^j{}_m \neq 0$ bestehen muß, sonst wäre (4. 10) gültig, woraus, wie wir gezeigt haben, $K_h{}^i{}_{jk}=0$ folgt.

Nach einer Überschiebung von (4. 15) mit l_j folgt wegen (4. 11):

$$K_0{}^l{}_{ml} \equiv -K_0{}^l{}_{lm} = l_m K^* - Q^0{}_m.$$

Beachten wir ferner, daß nach einer Verjüngung von (4. 15) bezüglich j und m die Formel

$$K^* = \frac{1}{n-2} Q^j{}_j$$

folgt, so sieht man, daß aus (4. 15) für $K_0{}^j{}_{0m}$ die Formel:

$$(4.16) \qquad K_0{}^j{}_{0m} = \frac{1}{n-2}(\delta^j_m - l^j l_m) Q^t{}_t + l^j Q^0{}_m - Q^j{}_m$$

folgt. Aus (4. 7) folgt nach einer Kontraktion mit l^k:

$$(4.17) \qquad T_{mp} K_0{}^j{}_{0l} + T_{0p} K_0{}^j{}_{lm} - T_{lp} K_0{}^j{}_{0m} = 0 \quad \text{(nicht summieren über } p),$$

woraus wegen der Annahme: $T_{0p} \neq 0$ das Bestehen der Formel

$$K_0{}^j{}_{lm} = (T_{0p})^{-1}(T_{lp} K_0{}^j{}_{0m} - T_{mp} K_0{}^j{}_{0l})$$

gefolgert werden kann, und das drückt die zweite Hälfte des Satzes aus.

Wir wollen hier darauf hinweisen, daß mit Hilfe der partiellen Ableitungen (d.h. nicht in algebraischer Form), aus (4. 13) und (4. 14) im Hinblick auf (4. 16) die Krümmungstensoren $K_0{}^i{}_{jk}$ und auch $K_h{}^i{}_{jk}$ allein durch den Tensor $Q^j{}_m$ ausgedrückt werden können.

Aus (4. 17) folgt der folgende:

SATZ 4a. *Sind die Bedingungen* Det $(T_{mp}) = 0$ *und* $T_{00} \neq 0$ *gültig, so ist der Krümmungstensor* $K_0{}^j{}_{km}$ *durch den Vektor* T_{m0} *und* $Q^j{}_m$ *in algebraischer Form ausdrückbar.*

BEWEIS. Aus $T_{00} \neq 0$ folgt offenbar, daß auch $T_{0p} \neq 0$ gelten muß. Die Bedingungen des Satzes 4 bestehen, und die Formel (4. 17) ist also ableitbar. Aus (4. 17) folgt nun nach Überschiebung mit l^p:

$$K_0{}^j{}_{lm} = \frac{1}{T_{00}}(T_{l0} K_0{}^j{}_{0m} - T_{m0} K_0{}^j{}_{0l}),$$

und das beweist den Satz 4a.

BEMERKUNG. Der Satz 4a ist sehr ähnlich zum zweiten Teile des Satzes 4. Der Unterschied besteht darin, daß die Rolle von T_{mp} im Falle $T_{00} \neq 0$ durch den Vektor T_{m0} übernommen wird. —

Ist nun der Finslerraum bezüglich der *Cartanschen kovarianten Ableitung* (2. 3) von zweifach rekurrenter Krümmung, so wollen wir die Identitäten (4. 4) analysieren. Überschieben wir (4. 4) mit l^k, so wird:

$$R_0{}^j{}_{0l}|_m + R_0{}^j{}_{lm}|_0 - R_0{}^j{}_{0m}|_l - A_l{}^j{}_t|_0 R_0{}^t{}_{0m} + A_m{}^j{}_t|_0 R_0{}^t{}_{0l} = 0.$$

Differenzieren wir jetzt diese Identität kovariant nach x^p, beachten wir ferner, daß nach unserer Annahme

$$R_i{}^j{}_{kl}|_m|_p = T_{mp} R_i{}^j{}_{kl}$$

gilt, so bekommt man:

$$T_{mp}R_0{}^j{}_{0l}+T_{0p}R_0{}^j{}_{lm}-T_{lp}R_0{}^j{}_{0m}=(A_l{}^j{}_t|_0 R_0{}^t{}_{0m}-A_m{}^j{}_t|_0 R_0{}^t{}_{0l})|_p.$$

Eine Verjüngung bezüglich j und l gibt:

(4.18)
$$T_{ip}(\delta^i_m R+l^i R_0{}^l{}_{lm}-R_0{}^i{}_{0m})=H_{mp}$$

mit

$$H_{mp}\overset{\text{def}}{=}(A_t|_0 R_0{}^t{}_{0m})|_p-(A_m{}^j{}_t|_0 R_0{}^t{}_{0j})|_p,$$

$$R\overset{\text{def}}{=}R_0{}^j{}_{0j}(=K^*).$$

Aus (4. 18) folgt auf Grund der Theorie der linearen Gleichungssysteme unmittelbar der

SATZ 5. *Gelten die Relationen*

$$\operatorname{Det}(\delta^i_m-R_0{}^i{}_{0m})\neq 0,\quad \operatorname{Det}(\delta^i_m-l^i R_0{}^l{}_{lm}-R_0{}^i{}_{0m})\neq 0,$$

so ist im Falle $H_{mp}=0$ *auch* $T_{mp}=0$ *und im Falle* $H_{mp}\neq 0$ *ist* T_{mp} *durch einmalige kovariante Ableitung von* $R_0{}^i{}_{0k}$ *bestimmbar.*

Ist $T_{0p}=0$, so reduzieren sich die Determinantenbedingungen auf das erstere, da aus (4. 18) das Glied $l^i R_0{}^l{}_{0m}$ ausfällt. Ist $T_{0p}\neq 0$, so ist offenbar nur die zweite Bedingung nötig.

§ 5. Zerlegbare Räume von zweifach rekurrenter Krümmung

Die Definition der Zerlegbarkeit der differentialgeometrischen Räume ist, wie folgt:

DEFINITION. *Wenn die Komponenten der Grundgrößen* (g_{ij}, *oder* $\Gamma_i{}^j{}_k$) *eines Raumes in zwei Teile zerfallen, undzwar so, daß diejenigen Komponenten, die durch die Indizes* $1, \ldots, r$ *gekennzeichnet sind, nur von* $(x^1, \ldots, x^r, \dot{x}^1, \ldots, \dot{x}^r)$ *abhängen, ferner diejenigen, die durch die Indizes* $r+1, \ldots, n$ *gekennzeichnet sind, nur von* $(x^{r+1}, \ldots, x^n, \dot{x}^{r+1}, \ldots, \dot{x}^n)$ *abhängen, und die übrigen, sogenannten gemischten Komponenten verschwinden, so wird der Raum (metrisch- bzw. affin-) zerlegbar genannt.*

Im folgenden wollen wir annehmen, daß unsere Finslerräume von zweifach rekurrenter Krümmung affin-zerlegbar sind. Es sollen $\alpha, \beta, \gamma, \delta, \varepsilon$ die Zahlen $1, 2, \ldots, r$ und $\mu, \varrho, \sigma, \tau$ die Zahlen $r+1, \ldots, n$ bedeuten. Es gilt der folgende

SATZ 6. *Ist ein Finslerraum bezüglich der Berwaldschen kovarianten Ableitung* (2. 4) *affinzerlegbar und von rekurrenter Krümmung, mit* $T_{ij}\neq 0$, *so ist eine der Komponenträume eben, d.h. es verschwindet der zugehörige Berwaldsche Krümmungstensor.*

BEWEIS. Wählen wir in der Identität (4. 7) für die Indizes j, k, l die Buchstaben α, β, γ und für m, p die Buchstaben μ, ϱ, so bekommt man aus (4. 7):

$$T_{\mu\varrho}K_0{}^\alpha{}_{\beta\gamma}=0,$$

da die gemischten Komponenten verschwinden. Wenn wir jetzt für die Indizes j, k, l die Buchstaben ϱ, σ, τ und für m, p die Buchstaben α, β wählen, so gibt (4. 7):

$$T_{\alpha\beta} K_0{}^{\varrho}{}_{\sigma\tau} = 0.$$

Aus den beiden letzten Gleichungen folgt wegen $T_{ij} \neq 0$, daß entweder $K_0{}^{\alpha}{}_{\beta\gamma}$, oder $K_0{}^{\varrho}{}_{\sigma\tau}$ verschwinden muß. Wenn z. B. $K_0{}^{\alpha}{}_{\beta\gamma} = 0$ ist, so folgt aus (4. 14) unmittelbar, daß auch $K_{\delta}{}^{\alpha}{}_{\beta\gamma} = 0$ ist, und das beweist den Satz.

BEMERKUNG. In der Formel von $K_h{}^i{}_{jk}$ kommt die Grundfunktion F nicht vor. Aus (4. 14) folgt nämlich:

$$K_h{}^i{}_{jk} = \frac{\partial}{\partial \dot{x}^h} (K_r{}^i{}_{jk} \dot{x}^r).$$

(Vgl. [1] oder [5], Kapitel IV. § 6, insb. die Formel (6. 6)).

Für den Hauptkrümmungstensor bzw. vollständigen Krümmungstensor kann eine analoge Beweisführung nicht durchgeführt werden, wie das aus (2. 16), (2. 17) bzw. (4. 4) gefolgert werden kann. Wir wollen aber darauf hinweisen, daß wenn der Finslerraum statt (1. 1) im Sinne der Relation:

$$\hat{R}_i{}^j{}_{kl}|_m + \Gamma^{*}{}_i{}^j{}_k\|_t \hat{R}_0{}^t{}_{lm} = \varkappa_m \hat{R}_i{}^j{}_{kl}$$

bzw.

$$R_i{}^j{}_{kl}|_m + R_i{}^j{}_{kt} R_0{}^t{}_{lm} = \varkappa_m R_i{}^j{}_{kl}$$

von rekurrenter Krümmung wäre, so könnte man einen dem Satze 6 entsprechenden Satz auch für den Haupt- bzw. vollständigen Krümmungstensor formulieren. Das kann aus (2. 16) bzw. (2. 17) leicht bewiesen werden, sogar müßte man die genannten Identitäten nicht mit l^i überschieben.

§ 6. Schlußbemerkungen

Wir wollen in diesem Paragraphen das Problem untersuchen, ob in welcher Weise unsere Resultate bezüglich der Finslerräume von rekurrenter Krümmung in allgemeineren Räume übertragbar sind. Der Begriff der einfach bzw. zweifach rekurrenten Krümmung kann z. B. auch in affinzusammenhängenden Linienelementräumen L_n (vgl. [7]) formuliert werden. Die Rolle l^i nimmt in diesen Räumen $\dot{x}^i$ über, ein metrischer Grundtensor existiert aber nicht. Diejenigen Sätze, bei denen in den Beweisen Überschiebungen mit g^{ik} durchgeführt werden, können somit nicht auf L_n-Räume formuliert werden, wie das z. B. bei dem zweiten Teil des Satzes 3, insbesondere bei der Formel (3. 3b) der Fall ist.

Hingegen können solche Type der Räume von rekurrenter Krümmung existieren, die im Finslerschen Fall nicht vorhanden sind. Es gilt nämlich der folgende

SATZ 7. *Es existieren außer* $S_i{}^j{}_{kl} = 0$ *keine Finslerräume von einfach und zweifach rekurrenter Krümmung bezüglich des Krümmungstensors* $S_i{}^j{}_{kl}$, *wenn die kovariante Ableitung die durch* (2. 1), *oder* (2. 2) *bestimmte kovariante Ableitung ist.*

Vor dem Beweis des Satzes wollen wir bemerken, daß auf Grund unseres Satzes 2, im Falle $T_{0p} \neq 0$, aus der Bedingung, daß der Raum von zweifach rekurrenter

Krümmung sei, die Tatsache folgt, daß unser Raum auch von einfach rekurrenter Krümmung ist. Doch müssen wir auch den Fall der Räume von zweifach rekurrenter Krümmung untersuchen, da auch $T_{0p}=0$ möglich sein kann, und in diesem Falle kann der Satz 2 nicht verwendet werden.

BEWEIS DES SATZES 7. Wir werden den Beweis nur für die kovariante Ableitung (2. 2) durchführen, da für die kovariante Ableitung (2. 1) der Beweis auf Grund der Identität (2. 10) vollständig analog geführt werden kann.

Auf Grund von (2. 15) ist

$$S_0{}^j{}_{kl}=0. \tag{6. 1}$$

Besteht nun

$$S_i{}^j{}_{kl;m}=\varkappa_m S_i{}^j{}_{kl}, \tag{6. 2}$$

so wird auf Grund der Identität (2. 10) und

$$S_i{}^j{}_{kl;m}l^i \equiv S_0{}^j{}_{kl;m}-S_i{}^j{}_{kl}l^i_{;m} \equiv -S_m{}^j{}_{kl} \tag{6. 3}$$

nach einer Überschiebung von (6. 2) mit l^i:

$$S_m{}^j{}_{kl}=0,$$

und das beweist schon die Tatsache, daß (6. 2) in den Finslerraum nur für den Nulltensor existieren kann.

Wir zeigen nun, daß Finslerräume von zweifach rekurrenter Krümmung bezüglich $S_i{}^j{}_{kl}$ und bezüglich der kovarianten Ableitungen (2. 1) und (2. 2) auch nicht existieren, selbstverständlich außer $S_i{}^j{}_{kl}=0$. Wäre nämlich

$$S_i{}^j{}_{kl;m;p}=T_{mp}S_i{}^j{}_{kl}, \tag{6. 4}$$

so wäre nach einer Überschiebung mit l^i auf Grund von (6. 1) und (2. 10):

$$(S_i{}^j{}_{kl;m}l^i)_{;p}-S_p{}^j{}_{kl;m}+S_i{}^j{}_{kl;m}l^il_p = 0.$$

In Hinsicht auf die Identität (6. 3) bekommt man aus dieser Gleichung:

$$S_m{}^j{}_{kl;p}+S_p{}^j{}_{kl;m}+S_m{}^j{}_{kl}l_p = 0.$$

Wegen

$$S_p{}^j{}_{kl;m}l^m \equiv S_p{}^j{}_{kl}\|_m l^m=0,$$

wird nach einer Überschiebung mit l^m, wieder in Hinsicht auf die Identität (6. 3) bzw. (6. 1)

$$S_p{}^j{}_{kl}=0,$$

womit der Satz 7 vollständig bewiesen ist.

(Eingegangen am 11. August 1970.)

ERDÉSZETI- ÉS FAIPARI EGYETEM,
ERDŐMÉRNÖKI KAR,
MATEMATIKAI TANSZÉK,
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