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TOMUS XXI

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ПРЕДСТАВЛЕНИЕ ФУНКЦИЙ КЛАССОВ $L_p[0, 1]$, $0<p<1$, ОРТОГОНАЛЬНЫМИ РЯДАМИ

А. А. ТАЛАЛЯН (Ереван)

Посвящается акабемику Д. АЛЕКСИЧУ *к семидесятилетию со дня рождения*

Введение. Доказываются следующие теоремы.

Теорема 1. *Если $\{\varphi_n(x)\}$ полная в $L_2[0, 1]$ ортонормированная система, то для любой функции $f(x)\in L_p[0, 1]$, $0<p<1$, существует ряд*

$$\sum_{n=1}^{\infty} c_n\varphi_n(x), \tag{1}$$

который сходится к $f(x)$ в метрике $L_p[0, 1]$, т. е.

$$\lim_{n\to\infty}\int_0^1 \left| f(x)-\sum_{k=1}^{n} c_k\varphi_k(x)\right|^p dx = 0. \tag{2}$$

Теорема 2. *Если $\{\varphi_n(x)\}$ полная в $L_2[0, 1]$ ортонормированная система, то существует ряд (1), у которого не все коэффициенты C_n, $n=1, 2, \dots$ равны нулю и который сходится к нулю в метрике всех пространств $L_p[0, 1]$, $0<p<1$, т. е.*

$$\lim_{n\to\infty}\int_0^1 \left| \sum_{k=1}^{n} c_k\varphi_k(x)\right|^p dx = 0 \tag{3}$$

для всех p, $0<p<1$.

Из теоремы 2 следует, что сходящийся в метрике $L_p[0, 1]$, $p<1$, к функции $f(x)$ ряд (1) не единственный. Кроме того, теорема 2 окончательна в том смысле, что для $p=1$ она уже не верна.

§ 1. Доказательство основных лемм. Лемма 1. *Пусть $f(x)$, $f_1(x)$, $\dots$, $f_n(x)$ интегрируемы с квадратом на отрезке $\Delta\equiv[a, b]$* и *$1>\varepsilon>0$, $\sigma>0$ произвольные положительные числа.*

Тогда существует $\psi(x)\in L_2[a, b]$, обладающая следующими свойствами:

$$1.\quad \psi(x)=0 \quad \text{при} \quad x\in\Delta - e, \tag{1. 1}$$

где e — измеримое множество, такое, что

$$e\subset\Delta,\quad \mu(e)\leqq \varepsilon\cdot\mu(\Delta), \tag{1. 2}$$

$$2.\quad \int_\Delta \psi^2(x)\,dx \leqq \frac{1}{\varepsilon}\int_\Delta f^2(x)\,dx, \tag{1. 3}$$

$$3.\quad \left|\int_\Delta f(x)f_k(x)\,dx-\int_\Delta \psi(x)f_k(x)\,dx\right| < \sigma,\quad 1\leqq k\leqq n. \tag{1. 4}$$

Подробное доказательство этой леммы в более общем случае, когда $f(x)\in L_p(\Delta)$, $f_k(x)\in L_q(\Delta)$, $k=1, 2, \dots n$, где $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, приведено в работе [1] (см. [1], стр. 86 и [2], стр. 144). Для полноты изложения заметим, что лемма немедленно вытекает из следующего легко проверяемого утверждения.

Пусть $\Delta_i^{(N)}$, $i=1, 2, \dots N$ интервалы, полученные разбиением интервала Δ на N равных частей, и пусть

$$\delta_i^{(N)}\subset \Delta_i^{(N)}, \quad \mu(\delta_i^{(N)}) = \varepsilon\cdot\mu(\Delta_i^{(N)}). \tag{1.5}$$

Положим

$$\psi_N(x) = \begin{cases} \dfrac{1}{\varepsilon\mu(\Delta_i^{(N)})}\displaystyle\int_{\Delta_i^{(N)}} f(x)\,dx, & x\in\delta_i^{(N)},\ 1\leqq i\leqq N \\ 0 & x\,\overline{\in}\,\displaystyle\bigcup_{i=1}^{N}\delta_i^{(N)} \end{cases} \tag{1.6}$$

и

$$\varphi_N(x) = f(x)-\psi_N(x). \tag{1.7}$$

Нетрудно проверить, что последовательность $\{\varphi_N(x)\}$ слабо сходится к нулю, т. е. (см. [2], стр. 143)

$$\lim_{N\to\infty}\int_{\Delta} F(x)\varphi_N(x)\,dx = 0, \quad F(x)\in L_2(\Delta). \tag{1.8}$$

Для достаточно большого N функция $\psi(x)=\psi_N(x)$ будет удовлетворять требованиям леммы.

Лемма 2. *Пусть $\{\varphi_n(x)\}$ — полная в $L_2[0, 1]$ ортонормированная система, $\Phi(x)\in L_p[0, 1]$, $0<p<1$ и p_0 — фиксированное число $0<p_0<p$.*

Тогда для всякого натурального N и положительного $\eta>0$ существует полином по системе $\{\varphi_n(x)\}$ вида

$$H(x) = \sum_{k=N}^{m} a_k\varphi_k(x), \quad m>N, \tag{1.9}$$

который удовлетворяет условиям

$$\alpha)\quad \int_0^1 |H(x)-\Phi(x)|^q\,dx<\eta, \quad p_0\leqq q\leqq p \tag{1.10}$$

$$\beta)\quad \int_0^1 \left|\sum_{k=N}^{n} a_k\varphi_k(x)\right|^q dx\leqq \eta+\int_0^1 |\Phi(x)|^q\,dx \tag{1.11}$$

$$(N\leqq n\leqq m,\ p_0\leqq q\leqq p).$$

Доказательство. Пусть $f(x)\in L_2[0,1]$ и

$$\int_0^1 |f(x)-\Phi(x)|^q\,dx < \frac{\eta}{4},\quad p_0\leqq q\leqq p. \tag{1.12}$$

Обозначим

$$r=1-p. \tag{1.13}$$

Поскольку $r>0$, можно выбрать $0<\varepsilon<1$, удовлетворяющее неравенству

$$\varepsilon^r\left(\int_0^1 f^2(x)\,dx\right)^{q/2} < \frac{\eta}{4},\quad p_0\leqq q\leqq p. \tag{1.14}$$

Возьмем $\delta>0$ такое, чтобы для всякой функции $\varphi(x)\in L_2[0,1]$ из неравенства

$$\int_0^1 \varphi^2(x)\,dx<\delta, \tag{1.15}$$

вытекало неравенство

$$\int_0^1 |\varphi(x)|^q\,dx < \frac{\eta}{4},\quad p_0\leqq q\leqq p. \tag{1.16}$$

Далее отрезок $[0,1]$ разделим на конечное число непересекающихся интервалов $\Delta_1,\Delta_2,\dots,\Delta_\nu$ таким образом, чтобы

$$\frac{4}{\varepsilon}\int_{\Delta_i} f^2(x)\,dx<\delta,\quad 1\leqq i\leqq \nu. \tag{1.17}$$

Если N_i заранее заданное натуральное число и $\sigma>0$, то в силу леммы 1 существует $\psi_i(x)\in L_2[0,1]$, удовлетворяющая условиям:

$$\psi_i(x)=0 \quad\text{при}\quad x\in[0,1]-e_i,\quad e_i\subset\Delta_i,\quad \mu(e_i)\leqq\varepsilon\mu(\Delta_i); \tag{1.18}$$

$$\left|\int_{\Delta_i}(\psi_i(x)-f(x))\varphi_k(x)\,dx\right|\leqq\sigma,\quad 1\leqq k\leqq N_i-1; \tag{1.19}$$

$$\int_{\Delta_i}\psi_i^2(x)\,dx\leqq\frac{1}{\varepsilon}\int_{\Delta_i}f^2(x)\,dx\leqq\frac{\delta}{4}. \tag{1.20}$$

Положим

$$F_i(x)=\begin{cases} f(x)-\psi_i(x), & x\in\Delta_i\\ 0 & \text{при}\quad x\,\overline{\in}\,\Delta_i\end{cases} \tag{1.21}$$

$$a_k=\int_0^1 F_i(x)\varphi_k(x)\,dx,\qquad k=1,2,\dots. \tag{1.22}$$

В неравенствах (1. 19) число σ можно считать настолько малым, что

$$\int_0^1\left|\sum_{k=1}^{N_i-1}a_k\varphi_k(x)\right|^q dx\leqq\frac{\eta}{8\nu};\quad p_0\leqq q\leqq p. \tag{1.23}$$

С другой стороны, поскольку система $\{\varphi_k(x)\}$ полная, можно выбрать $m_i > N_i$ такое, что

$$\int_0^1 \left| \sum_{k=1}^{m_i} a_k \varphi_k(x) - F_i(x) \right|^q dx \leqq \frac{\eta}{8\nu}, \quad p_0 \leqq q \leqq p. \tag{1.24}$$

Из (1. 23) и (1. 24) вытекает

$$\int_0^1 \left| \sum_{k=N_i}^{m_i} a_k \varphi_k(x) - F_i(x) \right|^q dx \leqq \frac{\eta}{4\nu}, \quad p_0 \leqq q \leqq p. \tag{1.25}$$

Далее, из (1. 20), (1. 21) и (1. 22) получаем

$$\int_0^1 \left| \sum_{k=N_i}^{n} a_k \varphi_k(x) \right|^2 dx \leqq \int_0^1 F_i^2(x)\, dx \leqq \frac{4}{\varepsilon} \int_{\Delta_i} f^2(x)\, dx \leqq \delta \tag{1.26}$$

$$(N_i \leqq n \leqq m_i)$$

и согласно выбору числа δ (см. (1. 15), (1. 16)) будем иметь

$$\int_0^1 \left| \sum_{k=N_i}^{n} a_k \varphi_k(x) \right|^q dx \leqq \frac{\eta}{4}, \quad N_i \leqq n \leqq m_i,\ p_0 \leqq q \leqq p. \tag{1.27}$$

Таким образом, для каждого i, $1 \leqq i \leqq \nu$ определяется полином по системе $\{\varphi_n(x)\}$ вида

$$\sum_{k=N_i}^{m_i} a_k \varphi_k(x), \quad m_i > N_i, \quad 1 \leqq i \leqq \nu, \tag{1.28}$$

удовлетворяющий условиям (1.25) и (1. 27), при этом, очевидно, можно добиться того, чтобы было

$$N = N_1, \quad N_i = m_{i-1} + 1, \quad 1 < i \leqq \nu. \tag{1.29}$$

Докажем, что полином

$$H(x) = \sum_{k=N}^{m} a_k \varphi_k(x) = \sum_{i=1}^{\nu} \sum_{k=N_i}^{m_i} a_k \varphi_k(x) \quad (m_\nu = m) \tag{1.30}$$

удовлетворяет требованиям леммы 2.

Обозначим

$$Q_j(x) = \sum_{i=1}^{j} \psi_i(x), \quad 1 \leqq j \leqq \nu, \tag{1.31}$$

$$\Phi_j(x) = \sum_{i=1}^{j} F_i(x), \quad 1 \leqq j \leqq \nu, \tag{1.32}$$

$$A_j = \bigcup_{i=1}^{j} \Delta_i, \quad 1 \leqq j \leqq \nu, \tag{1.33}$$

$$B_j = \bigcup_{i=1}^{j} e_i, \quad 1 \leqq j \leqq \nu. \tag{1.34}$$

Из (1. 25) следует, что

$$(1.35)\qquad \int_0^1 \left|\sum_{k=N}^{m_j} a_k\varphi_k(x)-\Phi_j(x)\right|^q dx \leqq \sum_{i=1}^{j}\int_0^1\left|\sum_{k=N_i}^{m_i} a_k\varphi_k(x)-F_i(x)\right|^q dx < \frac{\eta}{4}$$

$$(1\leqq j\leqq \nu,\ p_0\leqq q\leqq p).$$

С другой стороны, поскольку (см. (1. 18), (1. 31))

$$(1.36)\qquad \int_{A_j}|f(x)-\Phi_j(x)|^q\,dx=\int_{B_j}|Q_j(x)|^q\,dx,\quad 1\leqq j\leqq\nu;$$

$$(1.37)\qquad \mu(B_j)\leqq\sum_{i=1}^{j}\varepsilon\mu(\Delta_i)\leqq\varepsilon,\quad 1\leqq j\leqq\nu;$$

применяя неравенство Гельдера, для $s=\frac{2}{q}$, $s'=\frac{s}{s-1}$ получим

$$(1.38)\qquad \int_{B_j}|Q_j(x)|^q\,dx\leqq\left(\int_{B_j}|Q_j(x)|^2\,dx\right)^{1/s}\cdot(\mu(B_j))^{1/s'}=$$

$$=\varepsilon^{1/s'}\cdot\left(\sum_{i=1}^{j}\int_{\Delta_i}\psi_i^2(x)\,dx\right)^{1/s}\leqq\varepsilon^{1/s'}\left(\frac{1}{\varepsilon}\cdot\int_{A_j}f^2(x)\,dx\right)^{1/s}.$$

Так как

$$(1.39)\qquad \frac{1}{s'}-\frac{1}{s}=1-\frac{2}{s}=1-q\geqq 1-p,\quad p_0\leqq q\leqq p.$$

из (1. 14), (1. 36) и (1. 38) получаем неравенство

$$(1.40)\qquad \int_{A_j}|f(x)-\Phi_j(x)|^q\,dx\leqq\frac{\eta}{4},\quad 1\leqq j\leqq\nu;\ p_0\leqq q\leqq p,$$

которое вместе с (1. 35), если положить $j=\nu$, влечет

$$(1.41)\qquad \int_0^1|H(x)-f(x)|^q\,dx\leqq\frac{\eta}{2},\quad p_0\leqq q\leqq p.$$

Остается проверить, что полином $H(x)$ удовлетворяет также условию β).

Поскольку $\Phi_j(x)=0$ при $x\bar{\in} A_j$, то из (1. 35) следует неравенство

$$(1.42)\qquad \int_0^1\left|\sum_{k=N}^{m_j}a_k\varphi_k(x)\right|^q dx\leqq\int_{A_j}|\Phi_j(x)|^q\,dx+\frac{\eta}{4},\quad 1\leqq j\leqq\nu;\ p_0\leqq q\leqq p$$

и, сравнивая (1. 42) с (1. 40), получим

$$(1.43)\qquad \int_0^1\left|\sum_{k=N}^{m_j}a_k\varphi_k(x)\right|^q dx\leqq\int_{A_j}|f(x)|^q\,dx+\frac{\eta}{2}\leqq\int_0^1|f(x)|^q\,dx+\frac{\eta}{2}$$

$$(1\leqq j\leqq\nu;\ p_0\leqq q\leqq p).$$

Пусть $N \leqq n \leqq m$ и число j в силу (1. 29) выбрано так, что

(1. 44) $$N_{j+1} \leqq n \leqq m_{j+1}, \quad 0 \leqq j \leqq \nu - 1.$$

Тогда

(1. 45)* $$\sum_{k=N}^{n} a_k \varphi_k(x) = \sum_{k=N}^{m_j} a_k \varphi_k(x) + \sum_{k=N_{j+1}}^{n} a_k \varphi_k(x)$$

и, учитывая (1. 43) и (1. 27), будем иметь

(1. 46) $$\int_0^1 \left| \sum_{k=N}^{n} a_k \varphi_k(x) \right|^q dx \leqq \int_0^1 |f(x)|^q \, dx + \frac{\eta}{2} + \frac{\eta}{4} \quad (N \leqq n \leqq m; \; p_0 \leqq q \leqq p).$$

Лемма 2 доказана.

§ 2. Доказательство теорем. Пусть $f(x) \in L_p[0, 1]$, $0 < p < 1$ и $\{\eta_i\}$ последовательность положительных чисел, где

(2. 1) $$\lim_{i \to \infty} \eta_i = 0$$

Предположим, что определен полином вида

(2. 2) $$G_i(x) = \sum_{k=1}^{m_i} a_k \varphi_k(x),$$

удовлетворяющий условию

(2. 3) $$\int_0^1 |f(x) - G_i(x)|^p \, dx < \eta_i.$$

Согласно лемме 2, в формулировке которой берется $N = m_i + 1$, $\eta = \eta_{i+1}$ и $\Phi(x) = f(x) - G_i(x)$, существует полином

(2. 4) $$H_i(x) = \sum_{k=m_i+1}^{m_{i+1}} a_k \varphi_k(x), \quad m_{i+1} > m_i,$$

удовлетворяющий условиям:

(2. 5) $$1^\circ \quad \int_0^1 |G_{i+1}(x) - f(x)|^p \, dx < \eta_{i+1},$$

где

(2. 6) $$G_{i+1}(x) = G_i(x) + H_i(x);$$

(2. 7) $$2^\circ \quad \int_0^1 \left| \sum_{k=m_i+1}^{n} a_k \varphi_k(x) \right|^p dx \leqq \int_0^1 |f(x) - G_i(x)|^p \, dx + \eta_{i+1} \leqq \eta_i + \eta_{i+1}$$

$$(m_i < n \leqq m_{i+1}).$$

* Первую сумму правой части (1. 45) при $j = 0$ считаем равным нулю.

Ясно, что по индукции определяются последовательности полиномов $\{H_i(x)\}$, $\{G_i(x)\}$, удовлетворяющих условиям (2. 5), (2. 6) и (2. 7) для всех $i=1, 2, \dots$.

Полученный ряд

$$\sum_{k=1}^{\infty} a_k \varphi_k(x) \equiv \sum_{i=1}^{\infty} \sum_{k=m_{i-1}+1}^{m_i} a_k \varphi_k(x), \quad m_0 = 0, \tag{2. 8}$$

сходится к $f(x)$ в метрике $L_p[0, 1]$, $0<p<1$.

В самом деле, если

$$m_i < n \leqq m_{i+1}, \tag{2. 9}$$

то

$$\int_0^1 \left| \sum_{k=1}^{n} a_k \varphi_k(x) - f(x) \right|^p dx = \int_0^1 |G_i(x) - f(x)|^p \, dx +$$

$$+ \int_0^1 \left| \sum_{k=m_i+1}^{n} a_k \varphi_k(x) \right|^p dx \leqq 2\eta_i + \eta_{i+1},$$

где $2\eta_i + \eta_{i+1} \to 0$ при $n \to \infty$.

Теперь докажем теорему 2. Возмьем последовательность $\{P_k\}$, где

$$0 < p_1 < \dots < p_k < p_{k+1} < \dots; \quad \lim_{k \to \infty} p_k = 1. \tag{2. 10}$$

Согласно лемме 2, в формулировке которой берется $\eta = \eta_1$, $N=2$, $p=p_1$ и $\Phi(x) = \varphi_1(x)$, существует полином

$$\tau_1(x) = \sum_{k=2}^{m_1} a_k \varphi_k(x), \tag{2. 11}$$

который удовлетворяет условию

$$\int_0^1 |\varphi_1(x) - \tau_1(x)|^{p_1} \, dx < \eta_1. \tag{2. 12}$$

Предположим, что определены полиномы

$$\tau_i(x) = \sum_{k=m_i+1}^{m_{i+1}} a_k \varphi_k(x), \quad 0 \leqq i \leqq j-1, \tag{2. 13}$$

где

$$1 = m_0 < m_1 < \dots < m_{j-1} < m_j \tag{2. 14}$$

и имеет место неравенство

$$\int_0^1 \left| \varphi_1(x) - \sum_{k=2}^{m_j} a_k \varphi_k(x) \right|^q dx < \eta_j, \quad p_1 \leqq q \leqq p_j. \tag{2. 15}$$

Согласно лемме 2, где положено $p_0=p_1$, $p=p_{j+1}$, $N=m_{j+1}$, $\eta=\eta_{j+1}$ и $\Phi(x)=$ $=\varphi_1(x)-\sum_{k=2}^{m_j} a_k\varphi_k(x)$, существует полином

$$\sum_{k=m_j+1}^{m_{j+1}} a_k\varphi_k(x), \tag{2.16}$$

который обладает свойствами:

$$\int_0^1 \left|\varphi_1(x)-\sum_{k=2}^{m_{j+1}} a_k\varphi_k(x)\right|^q dx \leqq \eta_{j+1}, \quad p_1\leqq q\leqq p_{j+1}, \tag{2.17}$$

$$\int_0^1 \left|\sum_{k=m_j+1}^{n} a_k\varphi_k(x)\right|^q dx \leqq \int_0^1 \left|\varphi_1(x)-\sum_{k=2}^{m_j} a_k\varphi_k(x)\right|^q dx+\eta_{j+1}, \tag{2.18}$$

$$(m_j<n\leqq m_{j+1},\ p_1\leqq q\leqq p_{j+1}).$$

Продолжая это построение, определяем ряд

$$\sum_{k=2}^{\infty} a_k\varphi_k(x) \tag{2.19}$$

и последовательность натуральных чисел $\{m_j\}$, для которых условия (2. 15) и (2. 18) выполняются при всех $j=2, 3, \ldots$. Из (2. 15) и (2.18) следует, что ряд (2. 19) сходится к $\varphi_1(x)$ в любой метрике L_q, $0<q<1$.

В самом деле, пусть $p_1\leqq q<1$ и натуральное число n выбрано настолько большим, что если $m_j<n\leqq m_{j+1}$, то $p_j\geqq q$. Тогда имеем

$$\int_0^1 \left|\varphi_1(x)-\sum_{k=2}^{n} a_k\varphi_k(x)\right|^q dx \leqq \int_0^1 \left|\varphi_1(x)-\sum_{k=2}^{m_j} a_k\varphi_k(x)\right|^q dx+$$

$$+\int_0^1 \left|\sum_{k=m_j+1}^{n} a_k\varphi_k(x)\right|^q dx \leqq \eta_j+\eta_j+\eta_{j+1},$$

и, в силу (2. 1), левая часть неравенства стремится к нулю при $n\to\infty$.

Ясно, что ряд $\sum_{k=1}^{\infty} c_k\varphi_k(x)$, где $c_1=1$ и $c_k=-a_k$, $k\geqq 2$, сходится к нулю одновременно во всех метриках $L_q[0, 1]$, $0<q<1$. Теорема 2 доказана.

Замечание. 1. Доказанные теоремы верны также для базисов пространства $L_p[0, 1]$, $p>1$. Соответствующие теоремы можно доказать применением леммы 1 работы [1] (см. стр. 86).

2. Из формулировки леммы 2 видно, что можно доказать следующую теорему.

Если $\{\varphi_n(x)\}$ — полная ортонормированная система, то из нее можно удалить бесконечную подсистему $\{\varphi_{n_k}(x)\}$ таким образом, чтобы оставшаяся система $\{\psi_n(x)\}=\{\varphi_n(x)\}-\{\varphi_{n_k}(x)\}$ также удовлетворяла требованиям теоремы 1, т. е. чтобы для любой функции $f(x)\in L_p[0,1]$, $0<p<1$ существовал ряд $\sum_{n=1}^{\infty} c_n\psi_n(x)$, сходящийся к $f(x)$ в метрике $L_p[0,1]$.

3. В формулировке теоремы 1 вместо пространства $L_p[0,1]$, $0<p<1$, можно взять любое пространство $L_\varphi[0,1]$ с метрикой $\varrho(f,g)=$ $=\int_0^1 \varphi(|f(x)-g(x)|)\,dx$, где $\varphi(y)$ любая непрерывная функция, удовлетворяющая условиям:

$$1.\quad \varphi(y)\uparrow;\ \varphi(0)=0,$$

$$2.\quad \frac{y}{\varphi(y)}\to+\infty \text{ при } y\to\infty,$$

$$3.\quad \varphi(a+b)\leqq\varphi(a)+\varphi(b),\ a\geqq 0,\ b\geqq 0.$$

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GÉNÉRALISATION D'UN THÉORÈME DE WIENER ET DE LÉVY

Par
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Dédié à GEORGES ALEXITS *à l'occasion de son 70-ième anniversaire*

§ 1. Introduction

1. 1. — Nous allons considérer certains ensembles des fonctions sommables et 2π-périodiques de la variable réelle t. Les ensembles des fonctions bornées, des fonctions continues, des fonctions continues et développables en séries de Fourier uniformément convergentes, ainsi que des fonctions ayant de séries de Fourier à r-ième ($r>0$) puissance absolument convergentes soient notés respectivement par $\mathcal{B}$, $\mathcal{C}$, $\mathcal{U}$ et $\mathcal{A}^r$. Donc, si la fonction $g\in\mathcal{A}^r$, on a

$$(1.1)\qquad g(t)\sim\sum_{\nu=-\infty}^{\infty} b_\nu e^{i\nu t} \quad\text{et}\quad \sum_{\nu=-\infty}^{\infty}|b_\nu|^r<\infty.$$

On écrira $\mathcal{A}$ au lieu $\mathcal{A}^1$. Il est évident que $\mathcal{A}^{r_1}\subset\mathcal{A}^{r_2}$ si $r_1<r_2$.

Le théorème de WIENER et de LÉVY en question s'énonce comme suit ([8], I, p. 245).

THÉORÈME A. — (i) *Si $f\in\mathcal{A}$, et si les valeurs (en général complexes) de $f(t)=z$ se trouvent sur une courbe C en chaque point delaquelle une fonction $F(z)$ à variable complexe est analytique (non nécessairement uniforme) telle que $F(z)$ revienne à sa détermination initiale après un parcours complet de C correspondant à une variation de 2π de t, alors $F(f)\in\mathcal{A}$.*

(ii) *Si, en particulier, $f(t)\neq 0$ et $f\in\mathcal{A}$, alors $f^{-1}\in\mathcal{A}$.* [1]

Nous avons constaté dans la note [1] que, pour certaines fonctions particulières, (ii) subsiste même si l'on y remplace $\mathcal{A}$ par $\mathcal{A}^r$. Nous allons montrer, entre autres, que (i) a aussi lieu, en substituant $\mathcal{A}^r$ à $\mathcal{A}$, pour *toute* fonction $f\in\mathcal{A}^r$, pourvu que $0<r<1$.

Il est également tout indiqué de soulever la question: que peut-on dire lorsque $r>1$? Nous n'avons résolu ce problème que dans quelques cas particuliers. Ce qui s'explique, en partie au moins, par là que $f\in\mathcal{A}^r$, $0<r\leqq 1$ est une restriction beaucoup plus forte que $f\in\mathcal{A}^r$, $r>1$. Quand $f\in\mathcal{A}^r$, $0<r\leqq 1$, il est permis et naturel d'admettre qu'en même temps $f\in\mathcal{C}$, ce qui assure que $f\in\mathcal{U}\subset\mathcal{C}\subset\mathcal{B}$. Cette remarque d'apparence triviale est motivée par le fait que deux fonctions sommables, ne différant que sur un ensemble de mesure nulle, ont la même série de Fourier. La démonstration du théorème A donnée dans [8] se sert aussi forcément de l'hypothèse tacite que $f\in\mathcal{C}$ qui ne découle ni de la condition $f\in\mathcal{A}$ ni de celle que les valeurs de $f(t)$ se trouvent sur une courbe C. Si $f\in\mathcal{C}\cap\mathcal{A}$, alors C est une courbe continue fermée

[1] On doit (i) à LÉVY [4] et (ii) à WIENER [6]; cité d'après [8]. — Nous avons modifié l'énoncé du théorème A donné dans [8], en tenant compte d'une remarque de INGHAM ([3], pp. 156—157).

ou un arc continu parcouru plusieurs fois quand t varie de 2π. Dans la démonstration de notre théorème 1 nous allons tirer parti des propriétés sus-mentionnées de f.

Par contre, si $r>1$, il existe $f_0 \in \mathscr{A}^r$ telle que $f_0 \notin \mathscr{B}$, et *a fortiori* $f_0 \notin \mathscr{C}$, $f_0 \notin \mathscr{U}$. Pour illustrer ce fait il suffit d'invoquer l'exemple de la fonction

$$f_0(t) = \sum_{\nu=1}^{\infty} \frac{\cos \nu t}{\nu}.$$

$f_0 \in \mathscr{A}^r$ pour tout $r>1$, mais $f_0(0)=f_0(2\pi)=\infty$. C'est pourduoi, quand $r>1$, la condition $f \in \mathscr{A}^r$ sera complétée par d'autres stipulations.

$f(t)$ étant, par hypothèse, sommable et 2π-périodique, elle est développable en série de Fourier, soit

$$f(t) \sim \sum_{\nu=-\infty}^{\infty} a_\nu e^{i\nu t}. \tag{1. 2}$$

Pour formuler la première condition additionnelle rappelons que

$$\omega_1(\delta) = \omega_1(\delta; f) = \sup_{0 \leqq h \leqq \delta} \frac{1}{2\pi} \int_0^{2\pi} |f(t+h) - f(t)| \, dt$$

est le module de continuité intégrale de f dans L; si $\omega_1(\delta) = O(\delta^\alpha)$ $(0<\alpha \leqq 1,\ \delta \to +0)$ nous disons avec ZYGMUND que f appartient à la classe de fonctions Λ_α^1. Lorsque $g \in \Lambda_\alpha^1$ (g étant définie sous (1. 1)), on a $b_\nu = O(|\nu|^{-\alpha})$ ([8], I, p. 46). Il s'ensuit que $\Lambda_\alpha^1 \subset \mathscr{A}^r$ pourvu que $r^{-1} < \alpha \leqq 1$. Nous admettrons, chaque fois que $r>1$, comme condition complémentaire principale: $f \in \Lambda_1^\alpha$, avec $r^{-1} < \alpha \leqq 1$.

L'approximation uniforme de f par les sommes partielles de la série (1. 2) est un des moyens de la démonstration du théorème A et aussi de notre théorème 1 dans lequel $0<r\leqq 1$. Quand $r>1$, on peut utiliser encore, dans la preuve du théorème correspondant, l'approximation uniforme de f par des polynomes trigonométriques, en supposant que $f \in \mathscr{C} \cap \mathscr{A}^r$. Or, cette hypothèse implique déjà que $f \in L^2 \equiv \mathscr{A}^2$ et que $F(f) \in L^2 \equiv \mathscr{A}^2$; il suffit donc d'examiner alors les cas $1<r<2$.

Toutefois, quand $r>1$, la restriction $f \in \mathscr{C}$ n'est pas inévitable, comme l'indiquent les fonctions satisfaisant aux conditions $f \in L^2 - \mathscr{C}$, $f(t) \neq 0$, car nous avons alors $f^{-1} \in L^2 \equiv \mathscr{A}^2$. En supposant que $f \notin \mathscr{C}$, nous admettrons d'abord, sous certaines conditions concernant $F(z)$, que $f \in \mathscr{B} \cap \Lambda_\alpha^1$. Cette dernière hypothèse signifie aussi que $f \in L^2$. On peut donc se borner encore à l'étude des cas $1<r<2$.

Néanmoins il ne faut pas non plus restreindre nos investigations aux $f \in \mathscr{B} \cap \Lambda_\alpha^1$. En imposant à $F(z)$ quelques restrictions nouvelles, il suffira de supposer que $f \in \Lambda_\alpha^1$; mais alors les cas $r \geqq 2$ deviennent également intéressants.

1. 2. — Nous allons établir les trois propositions ci-après.

THÉORÈME 1. — (i) *Si $f \in \mathscr{C} \cap \mathscr{A}^r$, $0<r\leqq 1$, et si les valeurs (en général complexes) de $f(t)=z$ se trouvent sur une courbe C en chaque point delaquelle une fonction $F(z)$ à variable complexe est analytique (non nécessairement uniforme) telle que $F(z)$ revienne à sa détermination initiale après un parcours complet de C correspondant à une variation de 2π de t, alors $F(f) \in \mathscr{A}^r$.*

(ii) *Si, en particulier, $f(t) \neq 0$ et $f \in \mathscr{C} \cap \mathscr{A}^r$, alors $f^{-1} \in \mathscr{A}^r$.*

THÉORÈME 2. — (i) *Si* $f \in \mathscr{C} \cap \Lambda^1_\alpha$, $r^{-1} < \alpha \leqq 1$, $1 < r < 2$, *et si les autres conditions du théorème* 1 *sont réalisées, alors* $F(f) \in \mathscr{A}^r$.

(ii) *Si, en particulier,* $f(t) \neq 0$ *et* $f \in \mathscr{C} \cap \Lambda^1_\alpha$, *alors* $f^{-1} \in \mathscr{A}^r$.

On peut améliorer le théorème 2 en considérant même des fonctions $f(t)$ sommables et non bornées, si l'on assujétit $F(z)$ à une condition qui dans sa forme générale est moins restrictive que l'analyticité, mais qui est réalisée même dans le cas particulier où $F(z)$ est analytique et *uniforme*.

Pour énoncer cette proposition plus générale, notons par $\Lambda_1(E)$ la classe de fonctions à variable complexe z qui, sur l'ensemble E du plan des z, satisfont à une condition de Lipschitz d'ordre 1. En d'autres termes $F \in \Lambda_1(E)$ si, pour chaque couple de points $z_1, z_2 \in E$, on a

$$|F(z_1) - F(z_2)| \leqq K|z_1 - z_2| \tag{1. 3}$$

où la constante K est indépendante de z_1 de z_2 et de $|z_1 - z_2|$.

THÉORÈME 3. — (i) *Si* $f \in \Lambda^1_\alpha$, $r^{-1} < \alpha \leqq 1$, *et si les valeurs de* $f(t) = z$ *forment un ensemble* E *sur lequel on peut définir une fonction* $F(z) \in \Lambda_1(E)$, *alors* $F(f) \in \Lambda^1_\alpha \subset \mathscr{A}^r$.

(ii) *Si* $f(t)$ *remplit les conditions de* (i), *et si* $F(z)$ *est analytique et uniforme dans un domaine* D *dont* E *est un sous-ensemble ayant une distance positive de la frontière de* D, *alors* $F(f) \in \Lambda^1_\alpha \subset \mathscr{A}^r$.

Nous appellerons F la fonction extérieure et f la fonction intérieure.

§ 2. Fonction extérieure analytique, et $0 < r \leqq 1$

2. 1. — Nous admettons que dans ce paragraphe $0 < r \leqq 1$.

Avant de passer à la démonstration du théorème 1, introduisons certaines notations, rappelons quelques relations connues et démontrons deux lemmes simples. Écrivons, pour toute fonction $g \in \mathscr{A}^r$ (définie par (1. 1)),

$$S_r(g) = \sum_{\nu=-\infty}^{\infty} |b_\nu|^r; \quad \mathrm{M}g = \max_t |g(t)|. \tag{2. 1}$$

Les relations suivantes sont bien connues: Si $g_i \in \mathscr{A}^r$ et c_i est constante $(i = 1, 2, \ldots)$, on a

$$S_r\left(\sum c_i g_i\right) \leqq \sum |c_i|^r S_r(g_i). \tag{2. 2}$$

Si $g_1, g_2 \in \mathscr{A}^r$, alors

$$S_r(g_1 g_2) \leqq S_r(g_1) S_r(g_2), \tag{2. 3}$$

d'où il vient pour $g \in \mathscr{A}^r$ et $n = 2, 3, \ldots$,

$$S_r(g^n) \leqq S_r^n(g). \tag{2. 4}$$

Soit C^k la classe de fonctions k fois continûment dérivables.

LEMME 1. — *Si* $g \in C^k$, *si* g *est* 2π-*périodique et* $kr > 1$, *alors*

$$S_r(g) \leqq (\mathrm{M}g)^r + c_r(\mathrm{M}g^{(k)})^r \quad \left(c_r = 2\sum_{\nu=1}^{\infty} \nu^{-kr}\right), \tag{2. 5}$$

c'est que $g \in \mathscr{A}^r$.

Il suffit évidemment de prendre $k=[r^{-1}]+1$ ($[x]$ désigne la partie entière de x); lorsque $r=1$, il faut avoir $k=2$ au moins. Le lemme 1 subsiste d'ailleurs même pour $r>1$ et $k=1$.

DÉMONSTRATION. — On a $|b_0| \leqq \mathrm{M}g$. Pour $\nu \neq 0$, on obtient, en intégrant k fois par parties, $|b_\nu| \leqq |\nu|^{-k}\mathrm{M}g^{(k)}$, d'où l'inégalité (2. 5).

LEMME 2. — *Si* $H(u) \in C^k$, $u=q(t) \in C^k$, *et si* $H[q(t)]$ *existe pour* $0 \leqq t \leqq 2\pi$, *alors*

$$\mathrm{M}\frac{d^k}{dt^k}H[q(t)] \leqq K_q \max_{l=0,1,\dots,k} \mathrm{M}H^{(l)}[q(t)] \tag{2. 6}$$

où K_q *ne dépend que de* k *et de* $q(t)$.

DÉMONSTRATION. — La règle de la dérivation des fonctions composées donne

$$\frac{d^k}{dt^k}H[q(t)] = \sum_{l=1}^{k} U_l(k)H^{(l)}[q(t)]q'(t)^{n_1(k,l)}q''(t)^{n_2(k,l)}\dots q^{(l)}(t)^{n_l(k,l)}.$$

Les $U_l(k)$ ainsi que les $n_1(k,l), \dots, n_l(k,l)$ sont d'entiers positifs indépendants de $H(t)$ et de $q(t)$. Par conséquent, en posant

$$K_q = \mathrm{M}\sum_{l=1}^{k} U_l(k)|q'(t)^{n_1(k,l)}\dots q^{(l)}(t)^{n_l(k,l)}|,$$

on obtient (2. 6).

2. 2. — DÉMONSTRATION DU THÉORÈME 1. — Soit D le domaine d'analyticité de $F(z)$. La courbe $\complement$ a, par hypothèse, une distance positive de la frontière de D. Il existe donc un nombre $\varrho>0$ tel que $F(z)$ soit analytique dans chaque cercle $|z-f(t)| \leqq 2\varrho$. D'autre part $f \in \mathscr{C} \cap \mathscr{A}^r \subset \mathscr{U}$, c'est-à-dire que la série de Fourier (1. 2) de $f(t)$ a une somme partielle $s_\mu(t)=p(t)$ telle que les deux inégalités suivantes soient simultanément vérifies:

$$\mathrm{M}(f-p) < \frac{\varrho}{2}; \quad S_r(f-p) = \sum_{|\nu|>\mu} |a_\nu|^r < \left(\frac{\varrho}{2}\right)^r. \tag{2. 7}$$

Ce qui permet d'écrire

$$F[f(t)] = F[f(t)-p(t)+p(t)] = \sum_{m=0}^{\infty} \frac{1}{m!}F^{(m)}[p(t)][f(t)-p(t)]^m. \tag{2. 8}$$

Appliquons à (2. 8) les formules (2. 2), (2. 3), (2. 4) et (2. 7), nous obtenons

$$S_r\{F[f(t)]\} \leqq \sum_{m=0}^{\infty} \frac{1}{(m!)^r} S_r\{F^{(m)}[p(t)]\}\left(\frac{\varrho}{2}\right)^{rm}. \tag{2. 9}$$

Nous allons évaluer $S_r\{F^{(m)}[p(t)]\}$. $F(z)$ étant analytique, $g_m(t)=F^{(m)}[p(t)]$ $(m=0, 1, 2, \dots)$ est indéfiniment dérivable et, par suite, vu (2. 5),

$$S_r(g_m) \leqq (\mathrm{M}g_m)^r + c_r(\mathrm{M}g_m^{(k)})^r, \quad (k=[r^{-1}]+1) \tag{2. 10}$$

où, d'après (2. 6),

$$\mathrm{M}g_m^{(k)} \leqq K_p \max_{l=0,1,\dots,k} \mathrm{M}F^{(m+l)}[p(t)] = K_p|F^{(m+\lambda_m)}[p(t_m)]| \quad (1 \leqq \lambda_m \leqq k). \tag{2. 11}$$

λ_m est un entier, $t_m \in [0, 2\pi]$ est un nombre convenablement choisi. Il en découle que

$$S_r(g_m) \leqq (1+c_r K_p^r) \max\{(\mathrm{M}g_m)^r, |F^{(m+\lambda_m)}[p(t_m)]|^r\} = A_m^r. \tag{2. 12}$$

On obtient finalement, en vertu des expressions (2. 9)—(2. 12),

$$S_r\{F[f(t)]\} \leqq \sum_{m=0}^{\infty} \left(\frac{A_m}{m!}\right)^r \left(\frac{\varrho}{2}\right)^{rm}. \tag{2. 13}$$

Pour établir le théorème 1, il suffit donc de prouver que

$$\overline{\lim_{m\to\infty}} \left(\frac{A_m}{m!}\right)^{r/m} < \left(\frac{2}{\varrho}\right)^r. \tag{2. 14}$$

Observons tout d'abord que, grâce à la première relation (2. 7), $F(z)$ est analytique dans chaque cercle $|z-p(t)| \leqq 3\varrho/2$. On en tire que

$$\mathrm{M} \max_{|z-p(t)|\leqq\varrho} |F(z)| = \mu < \infty.$$

Il s'ensuit, en appliquant la formule de Cauchy,

$$\left|\frac{F^{(m)}[p(t)]}{m!}\right|^{1/m} = \left|\frac{1}{2\pi} \int\limits_{|z-p(t)|=\varrho} \frac{F(z)\,dz}{[z-p(t)]^{m+1}}\right|^{1/m} \leqq \mu^{1/m}\varrho^{-1}. \tag{2. 15}$$

Remarquons ensuite que

$$\Lambda = \overline{\lim_{m\to\infty}} \left|\frac{F^{(m+\lambda_m)}[p(t_m)]}{m!}\right|^{1/m} =$$

$$= \overline{\lim_{m\to\infty}} \left|\frac{F^{(m+\lambda_m)}[p(t_m)]}{(m+\lambda_m)!}\right|^{\frac{1}{m+\lambda_m}\left(1+\frac{\lambda_m}{m}\right)} \{(m+1)\dots(m+\lambda_m)\}^{1/m} =$$

$$= \overline{\lim_{m\to\infty}} \left|\frac{F^{(m+\lambda_m)}[p(t_m)]}{(m+\lambda_m)!}\right|^{\frac{1}{m+\lambda_m}}.$$

Il résulte ainsi de (2. 15) que

$$\Lambda \leqq \frac{1}{\varrho} < \frac{2}{\varrho}. \tag{2. 16}$$

(2. 12), (2. 15) et (2. 16) vérifient (2. 14). Le second membre de (2. 13) est donc une série convergente. Ce qui prouve le théorème 1.

§ 3. Fonction extérieure analytique, fonction intérieure continue, et $1<r<2$

3. 1. — Nous supposons que dans ce paragraphe $1<r<2$. Les inégalités (2. 2)—(2. 4), si $r>1$, en général ne sont plus valables, même la définition de $S_r(g)$ doit être changée. Si $g\in\mathscr{B}$, la notation de $\mathrm{M}g$ garde son sens signalé sous (2. 1). Soit maintenant

$$S_r(g)=\left(\sum_{\nu=-\infty}^{\infty}|b_\nu|^r\right)^{1/r}. \tag{3. 1}$$

L'inégalité (2. 2) sera-t-ainsi remplacée par la suivante:

$$S_r\left(\sum c_i g_i\right)\leqq \sum |c_i|\, S_r(g_i). \tag{3. 2}$$

(C'est l'inégalité de Minkowski.)

Cependant (2. 3) et (2. 4) n'ont pas d'équivalents directs. Nous allons donc montrer deux lemmes qui peuvent jouer les rôles de (2. 3) et de (2. 4).

LEMME 3. — *Si* $r>1$, $g_1\in\mathscr{A}$, $g_2\in\mathscr{A}^r$, *alors*

$$S_r(g_1g_2)\leqq S_1(g_1)S_r(g_2), \tag{3. 3}$$

autrement dit $g_1g_2\in\mathscr{A}^r$.

DÉMONSTRATION. — (3. 3) n'est autre chose qu'un cas particulier de l'inégalité suivante plus générale de W. H. YOUNG ([2], p. 199, Theorem 277). *Si* $\lambda>0$, $\mu>0$, $\lambda+\mu<1$, *et si* $g_1\in\mathscr{A}^{1/(1-\lambda)}$, $g_2\in\mathscr{A}^{1/(1-\mu)}$, *alors* $g_1g_2\in\mathscr{A}^{1/(1-\lambda-\mu)}$ *et*

$$S_{1/(1-\lambda-\mu)}(g_1g_2)\leqq S_{1/(1-\lambda)}(g_1)S_{1/(1-\mu)}(g_2). \tag{3. 4}$$

(3. 4) se réduit à (3. 3) pour $\lambda=0$ et $1/(1-\mu)=r$; on peut effectuer ce changement sans qu'il soit nécessaire de modifier la démonstration faite pour établir (3. 4).

LEMME 4. — (a) *Si* $g\in\Lambda_\alpha^1$, $r^{-1}<\alpha\leqq 1$, *alors* $g\in\mathscr{A}^r$. — (b) *Si* $g\in\mathscr{B}\cap\Lambda_\alpha^1$, $r^{-1}<\alpha\leqq 1$, *alors* $g^n\in\mathscr{A}^r$ $(n=2,3,\ldots)$ *et*

$$S_r(g^n)\leqq An\mathrm{M}g^{n-1} \tag{3. 5}$$

où $A=A(g)$ *est une constante.*

DÉMONSTRATION. — (a) Comme nous l'avons déjà fait remarquer (a) est une conséquence de la relation $b_\nu=O(|\nu|^{-\alpha})$. (b) Posons

$$g^n(t)\sim\sum_{\nu=-\infty}^{\infty}b_{n\nu}e^{i\nu t}.$$

Nous avons ainsi, pour $\nu\neq 0$,

$$\int_0^{2\pi}g^n(t)e^{-i\nu t}\,dt=-\int_0^{2\pi}g^n\left(t+\frac{\pi}{\nu}\right)e^{-i\nu t}\,dt=\frac{1}{2}\int_0^{2\pi}\left[g^n(t)-g^n\left(t+\frac{\pi}{\nu}\right)\right]e^{-i\nu t}\,dt$$

et de là

$$|b_{n\nu}| \leqq \frac{1}{4\pi}\int_0^{2\pi}\left|g^n\left(t+\frac{\pi}{\nu}\right)-g^n(t)\right|dt \leqq$$

$$\leqq \frac{1}{4\pi}\int_0^{2\pi}\left|g\left(t+\frac{\pi}{\nu}\right)-g(t)\right|\sum_{j=1}^{n}\left|g^{n-j}\left(t+\frac{\pi}{\nu}\right)g^{j-1}(t)\right|dt \leqq$$

$$\leqq \frac{1}{2}\,\omega_1\left(\frac{\pi}{\nu};g\right)n\mathrm{M}g^{n-1} = O\left(|\nu|^{-\alpha}n\mathrm{M}g^{n-1}\right);$$

de plus $|b_{n0}| \leqq \mathrm{M}g^n$. Par conséquent

$$S_r(g^n) = O(n\mathrm{M}g^{n-1}).$$

C'est exactement (3. 5), donc $g^n \in \mathscr{A}^r$.

3. 2. — DÉMONSTRATION DU THÉORÈME 2. — La condition $f \in \mathscr{C}$ assure, selon le théorème d'approximation de Weierstrass, l'existence d'un polynome trigonométrique $p(t)$ vérifiant l'inégalité

$$\mathrm{M}(f-p) < \frac{\varrho}{2} \tag{3. 6}$$

où ϱ a le même sens que dans le § 2. Nous pouvons ainsi écrire à nouveau la série de Taylor (2. 8) de $F[f(t)]$. $g_m(t) = F^{(m)}[p(t)]$ étant indéfiniment dérivable, $g_m \in \mathscr{A}$, selon le lemme 1 (avec $r=1$ et $k=2$); et $(f-p)^m \in \mathscr{A}^r$, d'après le lemme 4. En tenant compte donc de (3. 1), (3. 2), (3. 3), (3. 5) et (3. 6), on trouve l'expression analogue à (2. 9)

$$S_r\{F[f(t)]\} \leqq A_0 + A\sum_{m=1}^{\infty}\frac{1}{m!}S_1\{F^{(m)}[p(t)]\}m\left(\frac{\varrho}{2}\right)^{m-1} =$$

$$= A_0 + A\sum_{m=1}^{\infty}\frac{A_m}{(m-1)!}\left(\frac{\varrho}{2}\right)^{m-1}$$

où A_m est défini comme dans (2. 12), avec $r=1$ et $k=2$. Les relations (2. 15), (2. 16) et un raisonnement pareil à celui du paragraphe précédent conduisent encore à la conclusion que $S_r\{F[f(t)]\} < \infty$.

§ 4. Fonction extérieure non nécessairement analytique, fonction intérieure non nécessairement bornée, et $r > 1$

DÉMONSTRATION DU THÉORÈME 3. — (i) Posons $F[f(t)] = G(t)$. Il est à voir avant tout que $G(t)$ possède un développement de Fourier. Dans ce but nous allons établir que $G(t)$ est sommable, en montrant d'abord que $G(t)$ est mesurable et ensuite que $|G(t)|$ est majorée par une fonction sommable. $f(t)$ étant sommable, la relation (1. 3) assure déjà que $G(t)$ soit mesurable.

Soit t_1 un point où $|f(t_1)|<\infty$ et, en vertu de (1. 3), $|G(t_1)|<\infty$. En appliquant encore l'inégalité (1. 3), on a

$$|G(t)| \leqq |G(t)-G(t_1)| + |G(t_1)| \leqq K\,|f(t)-f(t_1)| + |G(t_1)|.$$

Le dernier membre de cette inégalité étant sommable, $G(t)$ l'est aussi, et on peut écrire

$$G(t) \sim \sum_{\nu=-\infty}^{\infty} B_\nu e^{i\nu t}.$$

Nous avons ainsi, toujours d'après (1. 3), pour $\nu \neq 0$,

$$|B_\nu| \leqq \frac{1}{4\pi}\int_0^{2\pi}\left|G\left(t+\frac{\pi}{\nu}\right)-G(t)\right|dt \leqq \frac{K}{4\pi}\int_0^{2\pi}\left|f\left(t+\frac{\pi}{\nu}\right)-f(t)\right|dt \leqq$$

$$\leqq \frac{K}{2}\,\omega_1\left(\frac{\pi}{\nu};f\right) = O\big(|\nu|^{-\alpha}\big),$$

d'où il vient $F(f)\in \Lambda_\alpha^1 \subset \mathscr{A}^r$.

(ii) Il suffit de montrer, selon (i), que les conditions imposées à $F(z)$ entraînent que $F\in\Lambda_1(E)$.

Observons d'abord que, si $F(z)$ n'est pas uniforme dans D, il peut arriver qu'il existe de couples de points $\zeta_1, \zeta_2 \in E$ tels que $|\zeta_1-\zeta_2|$ soit arbitrairement petit, tandis que $|F(\zeta_1)-F(\zeta_2)|$ reste supérieur à une quantité positive. Pour ces couples de points l'inégalité (1. 3) ne pourrait donc pas subsister.

Ce n'est pas le cas si $F(z)$ est uniforme dans D. Le fait que 3ϱ, la distance de E de la frontière de D est positive, implique que l'ensemble $E'\subset D$ union des disques circulaires de centre en E et de rayon 2ϱ est à distance ϱ de la frontière de D. Il en découle que $\max\limits_{z\in E}|F(z)|=Q<\infty$, $\max\limits_{z\in E'}|F'(z)|=Q'<\infty$ et que $F(z)$ est analytique dans chaque cercle $|z-f(t)|\leqq 2\varrho$. Quand $|z_1-z_2|\leqq\varrho$, z_2 est un point intérieur du cercle $|z-z_1|\leqq 2\varrho$. Le segment $[z_1, z_2]$ est donc entièrement dans E' lorsque $z_1, z_2\in E$. Il en résulte que, dans ce cas,

$$|F(z_1)-F(z_2)| = \left|\int_{z_1}^{z_2} F'(z)\,dz\right| \leqq Q'|z_1-z_2|.$$

D'autre part si $|z_1-z_2|>\varrho$, on a

$$|F(z_1)-F(z_2)| \leqq 2Q < \frac{2Q}{\varrho}|z_1-z_2|.$$

On en conclut que $F\in\Lambda_1(E)$, avec $K=\max(Q', 2Q/\varrho)$ et, par suite, $F\in\Lambda_\alpha^1\subset\mathscr{A}^r$.

Remarques ajoutées à l'occasion de la correction des épreuves.

1. — Nous avons rendu compte des résultatspubli és dans cette note au Colloque sur la Théorie Constructive des Fonctions (tenu à Budapest du 24 août au 3 septembre 1969). Dans leurs interventions, suivies de notre conférence, les professeurs R. ASKEY et Z. CIESELSKI ont attiré notre attention aux travaux [5] et [7]. La

note [5] fait mention d'un résultat de MARCINKIEWICZ amélioré par ZYGMUND analogue au théorème 1, toutefois avec la différence que $f(t)=x$ y est à valeur réelle et, pour $0<r<1$, $F(x)$ appartient à une classe plus large de fonctions que celle des fonctions analytiques. Dans la note [7] le théorème 1 est établi au moyen d' unealgèbre de Banach généralisée.

2. — Une proposition analogue à celle de LÉVY a été prouvée par INGHAM [3]. Il remplace la fonction intérieure ayant une série de Fourier absolument convergente par la fonction $z=\varphi(s)$ $(s=\sigma+it)$ développable, pour $\sigma\geqq 0$, en série de Dirichlet, avec $\sigma=0$ comme abscisse de convergence absolue. La fonction extérieure $\Phi(z)$ est analytique et remplit des conditions déterminées. $\Phi[\varphi(s)]$ possède alors également, pour $\sigma\geqq 0$, un développement de Dirichlet, avec $\sigma=0$ comme abscisse de convergence absolue.

Une modification légère de la démonstration de INGHAM permet de montrer que $\varphi(it)\in\mathscr{A}^r$, $0<r<1$, entraîne $\Phi[\varphi(it)]\in\mathscr{A}^r$.

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INTRODUCTION À LA THÉORIE DES „FONCTIONS SPLINE"

Par

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Hommage au Professeur G. ALEXITS, *à l'occasion de son 70e anniversaire*

Considérons la différence divisée $[x_0, x_1, ..., x_n; f]$ de la fonction f sur les noeuds $x_0, x_1, ..., x_n$ où $x_0 < x_1 < ... < x_n$ et supposons que la fonction f soit de la classe $c\,[x_0, x_n]$. Nous avons donné [2], la représentation intégrale

$$[x_0, x_1, \dots, x_n; f] = \int_{x_0}^{x_n} \psi(s) f^{(n)}(s)\, ds, \tag{1}$$

où la fonction ψ a été obtenue par la solutions d'un problème aux limites et les conditions aux limites nous ont permis de démontrer que la fonction ψ est positive sur l'intervale (x_0, x_n) et qu'on a

$$\int_{x_0}^{x_n} \psi(s)\, ds = \frac{1}{n!}. \tag{2}$$

Considérons la différence d'ordre n, de la fonction f avec des pas différents $h_1, h_2, ..., h_n$ définie par la formule

$$\overset{n}{\underset{h_1, h_2, \dots, h_n}{\Delta}} f(x) = f(x+h_1+\dots+h_n) - \sum f(x+h_1+\dots+h_{n-1}) + \dots + (-1)^n f(x). \tag{3}$$

M. FRÉCHET [1] a donné une caractérisation fonctionnelle des polynômes par le théorème suivant:

THÉORÈME 1. *Les solutions continues de l'équation fonctionnelle*

$$\overset{n}{\underset{h_1, h_2, \dots, h_n}{\Delta}} f(x) = 0, \tag{4}$$

quels que soient x et les pas $h_1, h_2, ..., h_n$, sont des polynomes de degré $n-1$ au plus.

En supposant que $0 < h_1 \leqq h_2 \leqq ... \leqq h_n$ et que $f \in C^n[x, x+h_1+...+h_n]$ nous avons montré [3] qu'on a la représentation intégrale

$$\overset{n}{\underset{h_1, h_2, \dots, h_n}{\Delta}} f(x) = \int_{x}^{x+h_1+\dots+h_n} \theta(s) f^{(n)}(s)\, ds, \tag{5}$$

où la fonction θ s'obtient par la solution d'un problème aux limites. Les conditions aux limites nous ont permis de démontrer que la fonction θ est positive sur l'intervale $[[x, x+h_1+...+h_n]$ et que

$$\int_{x}^{x+h_1+\dots+h_n} \theta(s)\, ds = h_1 h_2 \dots h_n. \tag{6}$$

T. Popoviciu [6] a considéré la fonctionnelle

$$\sum_{i=0}^{n} A_i f(x+\alpha_i h), \tag{7}$$

où les coefficients A_i et α_i sont liés par les équations

$$\sum_{i=0}^{n} A_i \alpha_i^j = 0, \tag{8}$$

pour $j=0, 1, ..., m-1$, et

$$\sum_{i=0}^{n} A_i \alpha_i^m \neq 0. \tag{9}$$

T. Popoviciu a donné une caractérisation fonctionnelle des polynômes par le théorème suivant:

Théorème 2. *Dans les conditions* (8) *et* (9), *les solutions continues de l'équations fonctionnelle*

$$\sum_{i=0}^{n} A_i f(x+\alpha_i h) = 0, \tag{10}$$

quels que soient x et h, sont des polynômes de degré $m-1$ au plus.

Les exemples précédents nous conduisent à considérer en général la fonctionnelle $L[f]$, de la forme

$$L[f] = A_0 f(x_0) + A_1 f(x_1) + \ldots + A_n f(x_n), \tag{11}$$

où la fonction f est définie sur les noeuds $x_0 < x_1 < \ldots < x_n$. On suppose que les noeuds $x_0, x_1, ..., x_n$ et les coefficients $A_0, A_1, ..., A_n$ soient tels que l'on ait

$$L[1] = 0,\ L[x] = 0, \ldots, L[x^{m-1}] = 0,\ L[x^m] = 1 \tag{12}$$

avec $m < n+1$. Nous avons donc les relations:

$$\begin{aligned} A_0 + A_1 + \ldots + A_n &= 0, \\ A_0 x_0 + A_1 x_1 + \ldots + A_n x_n &= 0, \\ &\cdots\cdots\cdots \\ A_0 x_0^{m-1} + A_1 x_1^{m-1} + \ldots + A_n x_n^{m-1} &= 0, \end{aligned} \tag{13}$$

et

$$A_0 x_0^m + A_1 x_1^m + \ldots + A_n x_n^m = 1. \tag{14}$$

Dans ce travail nous montrerons surtout que, si $f \in C^m[x_0, x_n]$, on a la représentation intégrale

$$L[f] = \int_{x_0}^{x_n} \varphi(s) f^{(m)}(s)\, ds, \tag{15}$$

où la fonction φ s'obtient par la solution d'un problème aux limites.

Si les noeuds $x_0, x_1, ..., x_n$ sont donnés, on peut fixer les coefficients $A_m, A_{m+1}, ..., A_n$ et alors les coefficients $A_0, A_1, ..., A_{m-1}$ sont parfaitement déterminés par les équations (13). On peut aussi écrire les coefficients $A_m, A_{m+1}, ..., A_n$ sous la forme $HA'_m, HA'_{m+1}, ..., HA'_n$ où $A'_m, A'_{m+1}, ..., A'_n$ sont donnés et alors les coefficients $A_0, A_1, ..., A_{m-1}$ seront de la forme $HA'_0, HA'_1, ..., HA'_{m-1}$. Si $A_0 x_0^m + A_1 x_1^m + \ldots + A_n x_n^m \neq 0$, on peut déterminer le facteur H par la condition (14).

Pour déterminer les coefficients $A_0, A_1, \ldots, A_n$ on peut ajouter aux équations (13) d'autre équations entre ces coefficients qu'on peut prendre de différentes manières. De cette façon on obtient des classes de fonctionnelles de la forme (11).

Nous donnons l'exemple suivant:

Supposons que les noeuds $x_0, x_1, \ldots, x_n$ forment une progression arithmétique de raison h et qu'on envisage la fonctionnelle (11), lorsque entre les coefficients $A_0, A_1, \ldots, A_n$ on a les équations

$$A_0 = A_n = 1,\ A_1 = A_{n-1}, \ldots, A_{p-1} = A_{n-p+1}, \tag{16}$$
$$A_p = A_{p+1} = \ldots = A_{n-p}.$$

On suppose que les coefficients $A_1, A_2, \ldots, A_p$ soient donnés par les formules

$$\begin{aligned} A_1 &= \lambda_1 - C'_{2p}\lambda_0 \\ A_2 &= \lambda_2 - C'_{2p}\lambda_1 + C^2_{2p}\lambda_0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ A_p &= \lambda_p - C'_{2p}\lambda_{p-1} + \ldots + (-1)^p C^p_{2p}\lambda_0, \end{aligned} \tag{17}$$

où

$$\lambda_j = \frac{C^p_{j+p} C^p_{n-p-j}}{C^p_{n-p}} \tag{18}$$

pour $j = 0, 1, \ldots, p$.

Dans ces conditions, nous avons montré que

$$L[1] = 0,\ L[x] = 0, \ldots, L[x^{2p-1}] = 0 \tag{19}$$

et que si $f \in C^{2p}[x_0, x_n]$, nous avons la représentation intégrale

$$L[f] = \int_{x_0}^{x_n} \theta(s) f^{(2p)}(s)\, ds, \tag{20}$$

où la fonction θ est positive sur l'intervalle (x_0, x_n).

Les résultats de ce travail ont été communiqués au «Colloque sur les équations fonctionnelles», Bucarest—Mamaia [4] (11—17 sept. 1968).

1. Supposons que $f \in C^m[x_0, x_n]$ et cherchons une représentation intégrale de la fonctionnelle (11). Pour cela rattachons aux intervalles $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ les polynomes $\varphi_1, \varphi_2, \ldots, \varphi_n$ solutions des équations différentielles

$$\varphi_1^{(m)} = 0,\ \varphi_2^{(m)} = 0, \ldots, \varphi_n^{(m)} = 0. \tag{21}$$

Nous pouvons écrire alors les formules suivantes:

$$\begin{aligned} (\varphi_1^{(m-1)} f - \varphi_1^{(m-2)} f' + \ldots + (-1)^{m-1} \varphi_1 f^{(m-1)})_{x_0}^{x_1} &= (-1)^{m-1} \int_{x_0}^{x_1} \varphi_1 f^{(m)}\, ds, \\ (\varphi_2^{(m-1)} f - \varphi_2^{(m-2)} f' + \ldots + (-1)^{m-1} \varphi_2 f^{(m-1)})_{x_1}^{x_2} &= (-1)^{m-1} \int_{x_1}^{x_2} \varphi_2 f^{(m)}\, ds, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ (\varphi_n^{(m-1)} f - \varphi_n^{(m-2)} f' + \ldots + (-1)^{m-1} \varphi_n f^{(m-1)})_{x_{n-1}}^{x_n} &= (-1)^{m-1} \int_{x_{n-1}}^{x_n} \varphi_n f^{(m)}\, ds. \end{aligned} \tag{22}$$

Relativement à ces formules nous traiterons le

Problème aux limites: déterminer les polynomes $\varphi_1, \varphi_2, \ldots, \varphi_n$ qui vérifient les conditions aux limites

$$\varphi_1(x_0) = 0, \varphi_1'(x_0), \ldots, \varphi_1^{(m-2)}(x_0) = 0, \quad \varphi_1^{(m-1)}(x_0) = (-1)^m A_0$$

$$\varphi_1(x_1) = \varphi_2(x_1), \ \varphi_1'(x_1) = \varphi_2'(x_1), \ldots, \varphi_1^{(m-2)}(x_1) = \varphi_2^{(m-2)}(x_1),$$

$$\varphi_1^{(m-1)}(x_1) - \varphi_2^{(m-1)}(x_1) = (-1)^{m-1} A_1$$

....

$$\begin{aligned} (23) \quad & \varphi_{n-1}(x_{n-1}) = \varphi_n(x_{n-1}), \ \varphi_{n-1}'(x_{n-1}) = \varphi_n'(x_{n-1}), \ldots, \\ & \varphi_{n-1}^{(m-2)}(x_{n-1}) = \varphi_n^{(m-2)}(x_{n-1}), \\ & \varphi_{n-1}^{(m-1)}(x_{n-1}) - \varphi_n^{(m-1)}(x_{n-1}) = (-1)^{m-1} A_{n-1} \end{aligned}$$

$$\varphi_n(x_n) = 0, \ \varphi_n'(x_n) = 0, \ldots, \varphi_n^{(m-2)}(x_n) = 0, \ \varphi_n^{(m-1)}(x_n) = (-1)^{m-1} A_n.$$

Il est facile de résoudre ce problème. On vérifie sans difficulté que les polynomes

$$\varphi_1(s) = (-1)^m A_0 \frac{(s-x_0)^{m-1}}{(m-1)!},$$

$$(24) \quad \varphi_2(s) = (-1)^m \left[A_0 \frac{(s-x_0)^{m-1}}{(m-1)!} + A_1 \frac{(s-x_1)^{m-1}}{(m-1)!} \right],$$

....

$$\varphi_n(s) = (-1)^m \left[A_0 \frac{(s-x_0)^{m-1}}{(m-1)!} + A_1 \frac{(s-x_1)^{m-1}}{(m-1)!} + \ldots + A_{n-1} \frac{(s-x_{n-1})^{m-1}}{(m-1)!} \right]$$

vérifient les conditions aux limites (23) aux points $x_0, x_1, \ldots, x_{n-1}$. Il reste à montrer que les conditions aux limites au point x_n sont également vérifiées. En effet, ces conditions s'expriment par

$$(25) \quad \begin{array}{llll} A_0 & + A_1 & + \ldots + A_{n-1} + A_n & = 0, \\ A_0(x_n - x_0) & + A_1(x_n - x_1) & + \ldots + A_{n-1}(x_n - x_{n-1}) & = 0, \\ \ldots & \ldots & \ldots & \\ A_0(x_n - x_0)^{m-1} & + A_1(x_n - x_1)^{m-1} & + \ldots + A_{n-1}(x_n - x_{n-1})^{m-1} & = 0, \end{array}$$

et il est facile voir que ces conditions peuvent être écrites sous la forme:

$$(26) \quad \begin{array}{llll} A_0 & + A_1 & + \ldots + A_n & = 0, \\ A_0 x_0 & + A_1 x_1 & + \ldots + A_n x_n & = 0, \\ \ldots & \ldots & \ldots & \\ A_0 x_0^{m-1} & + A_1 x_1^{m-1} & + \ldots + A_n x_n^{m-1} & = 0 \end{array}$$

et ces conditions se confondent avec les équations (13), ce qui veut dire qu'elles sont satisfaites.

Ainsi le problème aux limites est parfaitement résolu. En remplaçant dans les formules (22) les polynomes $\varphi_1, \varphi_2, \dots, \varphi_n$ par les formules (24) et en ajoutant membre à membre toutes ces formules, on obtient, d'après les conditions aux limites (23), la représentation intégrale

$$L[f] = A_0 f(x_0) + A_1 f(x_1) + \dots + A_n f(x_n) = \int_{x_0}^{x_n} \varphi(s) f^{(m)}(s)\, ds, \tag{27}$$

où la fonction φ coïncide sur les intervales $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ avec es polynomes $\varphi_1, \varphi_2, \dots, \varphi_n$.

2. En employant la notation des „fonctions spline"

$$(s-x_i)_+^{m-1} = \begin{cases} (s-x_i)^{m-1} & \text{si} \quad s \geqq x_i \\ 0 & \text{si} \quad s \leqq x_i . \end{cases} \tag{28}$$

On peut écrire les formules (24) sous la forme

$$\varphi(s) = (-1)^m \left[A_0 \frac{(s-x_0)_+^{m-1}}{(m-1)!} + A_1 \frac{(s-x_1)_+^{m-1}}{(m-1)!} + \dots + A_{n-1} \frac{(s-x_{n-1})_+^{m-1}}{(m-1)!} \right] \tag{29}$$

ou encore, sous la forme

$$\varphi(s) = (-1)^m \left[A_0 \frac{(s-x_0)_+^{m-1}}{(m-1)!} + A_1 \frac{(s-x_1)_+^{m-1}}{(m-1)!} + \dots + A_n \frac{(s-x_n)_+^{m-1}}{(m-1)!} \right] \tag{30}$$

puisque le derniere terme est nul sur l'intervalle $[x_0, x_n]$.

La fonction φ de la représentation intégrale (27), jouit aussi de la propriété exprimée par la formule

$$\int_{x_0}^{x_n} \varphi(s)\, ds = \frac{1}{m!}. \tag{31}$$

En effet si nous remplaçons dans la formule (27), la fonction f par x^m et l'on tient compte de la formule (14), il en résultera la formule (31).

Nous avons étendu ainsi la propriété que la différence divisée $[x_0, x_1, \dots, x_n; f]$ peut être représentée par une intégrale définie de la forme (1), à des fonctionnelle de la forme (11) où entre les coefficients et les noeuds il y a les m équations (13), le nombre $n+1$ des noeuds étant plus grand que m. (Le cas $m=n$ correspond au cas de la différence divisée.)

3. I. J. SCHOENBERG [7] a défini la fonction spline d'ordre $m-1$ relativement aux noeuds $x_0, x_1, \dots, x_n$ où $x_0 < x_1 < \dots < x_n$ sur toute la droite, par la formule

$$\psi(x) = P_{m-1}(s) + \sum_{i=0}^{n} \mu_i (s-x_i)_+^{m-1}, \tag{32}$$

où $P_{m-1}(s)$ est un polynôme de degré $m-1$ au plus et $\mu_0, \mu_1, \ldots, \mu_n$ sont des coefficients constants.

Il résulte de cette définition que la fonction ψ est égale au polynôme $P_{m-1}(s)$ pour $s \leqq x_0$ et au polynôme

$$P_{m-1}(s) + \sum_{i=0}^{n} \mu_i (s-x_0)^{m-1} \tag{33}$$

pour $s \geqq x_n$.

Lorsque les noeuds $x_0, x_1, \ldots, x_n$ sont donnés la fonction spline dépend de $m+n+1$ paramètres, les coefficients de $P_{m-1}(s)$ et les coefficients μ_i.

On peut identifier la fonction φ de la représentation intégrale (27) avec une fonction spline. Mais pour réaliser cette identité, il faut d'abord prolonger la fonction φ donnée par la formule (30) à gauche de x_0 par $\varphi_0(s) = 0$ et à droite de x_n, comme dans le cas des fonctions spline, par

$$\varphi_{n+1}(s) = (-1)^m \sum_{i=0}^{n} A_i \frac{(s-x_i)^{m-1}}{(m-1)!}. \tag{34}$$

Les équations (13) montrent que nous avons $\varphi_{n+1}(s) = 0$.

En faisant ce prolongement on peut dire que *la fonction φ de la représentation* (27) *est la fonction spline* (32) *d'ordre $m-1$, relativement aux noeuds $x_0, x_1, \ldots, x_n$ avec le polynôme $P_{m-1}(s)$ identiquement nul et dont les coefficients μ_i sont égaux à*

$$\mu_i = (-1)^m \frac{A_i}{(m-1)!}. \tag{35}$$

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ON THE ENUMERATION OF SEARCH-CODES

By

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Dedicated to Professor G. ALEXITS *at the occasion of his 70th birthday*

§ 1. Definition of search-codes

Any finite sequence of nonnegative integers is called a *codeword.* If $a=(a_1, a_2, ..., a_s)$ is a codeword, we call s the *length* of the codeword a and put $|a|=s$. The empty sequence (of length 0) is considered as a codeword too. We denote by Z the set of all codewords. If $a=(a_1, a_2, ..., a_r)$ and $b=(b_1, b_2, ..., b_s)$ are any two codewords, we define the codeword ab as $ab=(a_1, a_2, ..., a_r, b_1, b_2, ..., b_s)$. Clearly $|ab| = |a|+|b|$, and Z is a semigroup with respect to the multiplication defined above. Denoting by e the empty codeword, e is the unit element of the semigroup Z. A codeword a is called a *prefix* of the codeword b if there exists a codeword c such that $b=ac$; in this case c is called a *suffix* of b. If $a=(a_1, ..., a_s)$, $a_1, ..., a_s$ are called the *letters* of a.

Any finite set $C=\{c_1, c_2, ..., c_n\}$ of different codewords $c_j=(c_{j,1}, c_{j,2}, ..., c_{j,l_j})$ is called a *code*. It is convenient to consider the empty set also as a code: we call it *the empty code*. The code consisting of the single codeword e, where e is the empty codeword is a nonempty code, which will be called *the trivial code*.

If C is a code and a any codeword (not necessarily in C) we denote by C_a the set of those codewords $b\in Z$ for which $ab\in C$. In other words C_a is obtained by taking all those codewords c of C of which a is a prefix and removing from these codewords the prefix a, i.e. preserving only the suffix of c following a. Evidently $C_e=C$, and $(C_a)_b=C_{ab}$. Of course for certain $a\in Z$ (in fact for all but a finite number of $a\in Z$) C_a is the empty code. If $a=c\in C$ then C_c is not empty, as it contains certainly the empty codeword e, but C_c may be the trivial code: if C_c is for every $c\in C$ the trivial code, C is called a *prefix code*. Thus a code C is a prefix code if none of its codewords is the prefix of an other codeword in C.

If C is a code, we shall denote by $N(C)$ the number of elements of C, and we shall call $N(C)$ the *size* of C.

We shall call a code C *well-branched*, if one of the following three cases takes place: 1. C is the empty code; 2. C is the trivial code; 3. C does not contain the empty codeword e and there exists an integer $b(C)$ — called *the branching number of* C — such that $b(C)\geqq 2$ and denoting by $\{k\}$ the codeword of length 1, consisting of the single letter k ($k=0, 1, ...$) the code $C_{\{k\}}$ is not empty if $k<b(C)$ and $C_{\{k\}}$ is empty if $k\geqq b(C)$. If C is the empty code we put $b(C)=0$ while if C is the trivial code we put $b(C)=1$; thus the branching number $b(C)$ is defined for every well-branched code.

We call a code C a *search code* if C_a is a well-branched code for every $a\in Z$. Clearly if C is a search code, then C is necessarily a prefix code. As a matter of fact if $c\in C$ the code C_c has to be a well branched code, and as it contains the empty codeword e, C_c has to be the trivial code: thus C is a prefix code. However, not every prefix code is a search-code: for instance the code consisting of a single codeword, which is different from the empty codeword e, is a prefix code, but not a search code.

If C is a search code, we call those $a \in Z$ for which $b(C_a) \geqq 2$, the *branching points* of C. We shall denote the total number of branching points of C by $B(C)$.

A search code C is called *regular of degree* q $(q \geqq 2)$, if each of its branching points has the branching number q, i.e. if for each $a \in Z$ such that $b(C_a) \geqq 2$, one has $b(C_a) = q$. The trivial code is by definition regular of every degree $q \geqq 2$.

The study of search codes is motivated by the connection of search-codes with *the mathematical theory of search* (see e.g. [1]). Let us consider a simple search process in which we want to identify an unknown element x of a finite set S. Suppose that we can not observe x directly, however we may choose some functions $f_1, f_2, \ldots, f_N$ from a given set F of nonnegative integer valued functions f defined on the set S, and observe the values $f_1(x), f_2(x), \ldots, f_N(x)$ of these functions at the unknown point x. A method for choosing the functions f_k, one after the other, from the set F (so that the choice of f_k may depend on the previously observed values $f_1(x), \ldots, f_{k-1}(x)$) so that after a finite number N of steps ($N = N(x)$ may depend on x) the unknown x is uniquely determined by the sequence $f_1(x), f_2(x), \ldots, f_N(x)$, is called *a strategy of search*. Clearly each strategy defines a code, the codewords of which are the sequences $(f_1(x), f_2(x), \ldots, f_{N(x)}(x))$ $(x \in S)$. A strategy is called *irreducible*, if none of the observations prescibed by the strategy is unnecessary. It is easy to see that the codes corresponding to irreducible strategies of search are just the search-codes defined above, provided that for every possible sequence $f_1(x), \ldots, f_{k-1}(x)$, if n is a possible value of $f_k(x)$, then so is every $m < n$, as well.

To every search-code there corresponds a rooted tree, defined as follows: the root of the tree corresponds to the empty code word, the branching points of the tree correspond to the branching points of the code, and the endpoints of the tree correspond to codewords belonging to the code. Two vertices of the tree are connected by an edge if the two corresponding codewords a and b are such that a is a prefix of b and $|b| - |a| = 1$.

In what follows we shall count search codes satisfying certain conditions. We shall use throughout the paper the method of generating functions. An other way to solve these problems is by means of the mentioned correspondence between codes and trees, making use of the methods of the theory of trees. However, this approach has the disadvantage, that to different codes there may correspond the same tree. In dealing with regular codes this does not cause any difficulty, but in the general case the counting of trees is somewhat more involved: as a matter of fact for some types of codes in the present paper, we get explicit formulae for the total number of codes, while for the number of the corresponding trees we were able to get explicit formulas only for the corresponding generating functions. In view of this we do not deal here at all with the counting of trees corresponding to the search-codes considered. However, we inted to return to these questions elsewhere.

§ 2. Enumeration of regular search-codes

Let C be a regular search code of degree $q \geqq 2$. First we prove the following

LEMMA. *If C is a regular search-code of degree q then the size $N(C)$ of the code C satisfies the following congruence relation:*

$$N(C) \equiv 1 \bmod (q-1). \tag{2. 1}$$

Conversely, if n is a natural number such that $n-1$ is divisible by $q-1$, then there exist regular search codes of degree q and size n.

PROOF. Let $E(C)$ denote the number of pairs (a, b) of code-words, such that both C_a and C_b are different from the empty code, further a is a prefix of b and $|b|-|a| = 1$. To determine $E(C)$ we may first fix a and count in how many ways b can be chosen, or conversely, fix b and consider in how many ways a can be chosen. Thus we obtain two expressions, and equating these we shall get (2. 1). Evidently a can be chosen as one of the branching points of C, i.e. in $B(C)$ ways, and if a is fixed, by definition b can be chosen in $b(C_a)=q$ ways; thus

$$E(C)=q\cdot B(C). \tag{2. 2}$$

On the other hand, b can be chosen in $B(C)+N(C)-1$ ways,* and to every b there corresponds a uniquely determined a, obtained by omitting the last letter of b. Thus

$$E(C) = B(C)+N(C)-1. \tag{2. 3}$$

Comparing (2. 2) and (2. 3) we get

$$N(C) = B(C)(q-1)+1 \tag{2. 4}$$

which proves (2. 1).

REMARK. $E(C)$ is equal to the number of edges of the tree corresponding to the code C.

The second statement of the lemma can be proved by induction as follows: If $n=1$, there exists a (unique) regular code of degree q and size $N(C)=1$, namely the trivial code. Suppose we have already constructed a regular search code of degree q and having size $N(C)=n$, where $n-1$ is divisible by $q-1$. Take any one of the codewords c belonging to the code C, omit c from C and add to C the q code-words obtained by adding to the end of c the suffix of length 1 consisting of the single letter k, where k is any one of the numbers $0, 1, \ldots, q-1$. In this way we obtain a code C' which is regular of degree q and has size $N(C') = n+q-1$. This proves Lemma 1.

We shall determine now the total number $C_q(n)$ of regular search codes of degree q and size n, where of course $n \equiv 1 \bmod (q-1)$. We shall prove the following

THEOREM 1. *For every natural number n of the form $n = k(q-1)+1$ $(k=0, 1, \ldots)$ one has*

$$C_q(n) = \frac{(kq)!}{k!\cdot n!}. \tag{2. 5}$$

* We can namely choose for b any codeword belonging to C and any branching point of C, except e.

PROOF. Clearly $C_q(1)=1$, and for every $n = k(q-1)+1$ with $k \geqq 1$ we have the recursion formula

(2. 6) $$C_q(n) = \sum_{n_1+n_2+\ldots+n_q=n} C_q(n_1)C_q(n_2)\ldots C_q(n_q),$$

where the summation has to be extended only over those values of n_j for which $C_q(n_j)$ is defined i.e. for $n_j \equiv 1 \bmod (q-1)$.

Let us put

(2. 7) $$G_q(y) = \sum_{n\equiv 1 \bmod (q-1)} C_q(n)x^n.$$

We get from (2. 6)

(2. 8) $$G_q(y) = y+(G_q(y))^q.$$

From (2. 8) one can deduce, by the Bürmann—Lagrange formula for the power series of the inverse function of a given function, that

(2. 9) $$G_q(y) = \sum_{k=0}^{\infty} \frac{(kq)!}{k!(k(q-1)+1)!} y^{k(q-1)+1},$$

i.e. that (2. 5) holds.

REMARK. The power series expansion (2. 9) can be interpreted as the power series of that real root x of the trinomial equation $x-x^q = y$, which reduces to 0 for $y=0$. It is easy to see that the power series (2. 9) is convergent for $|y| \leqq \frac{q-1}{q^{\frac{q}{q-1}}}$ and thus (2. 9) solves the equation $x-x^q = y$ for a given value of y if $|y| \leqq \frac{q-1}{q^{\frac{q}{q-1}}} = R_q$. Notice that R_q tends increasingly to 1 for $q \to +\infty$.

We want to add some remarks on the special case $q=2$, i.e. the case of *binary search codes*. If $q=2$, then by Lemma 1 $N(C) = B(C)+1$ i.e. there exist regular binary search codes of any size $n \geqq 1$, and we get from (2. 5) that their total number is

(2. 10) $$C_2(n) = \frac{\binom{2n-1}{n}}{(2n-1)} = \frac{(2n-3)!!\,2^{n-1}}{n!}.$$

This formula has been known already to A. CAYLEY (see [2] and [3]), who solved the corresponding problem for counting trees. Cayley pointed out also that $C_2(n)$ is equal to the number of possible interpretations of a „product" of n factors with respect to a nonassociative operation, i.e. $C_2(n)$ is equal to the possible bracketings of such a „product".

Thus Theorem 1 is a straightforward generalization of Cayley's classical results. It seems improbable that this generalization has not been made previously, but I did not find it in the literature.

Let us notice that for $q=2$ we get from (2. 8) the explicit formula

(2. 11) $$G_2(y) = \frac{1-\sqrt{1-4y}}{2}.$$

§ 3. Enumeration of all search codes of given size

Let $D(n, k)$ denote the total number of search codes of size n and having k branching points. Evidently $k \leqq n-1$, as for a given size the number of branching points is maximal if the code is binary. It is easy to see that $D(1, 0)=1$, $D(n, 0)=0$ if $n \geqq 2$, and for $n \geqq 2$ the following recursion formula holds:

$$D(n, k) = \sum_{l=2}^{n} \sum_{\substack{n_1+n_2+\ldots+n_l=n \\ k_1+k_2+\ldots+k_l=k-1}} D(n_1, k_1) D(n_2, k_2) \ldots D(n_l, k_l). \tag{3. 1}$$

Let us put

$$\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} D(n, k) x^n y^k = d(x, y). \tag{3. 2}$$

It follows that

$$d(x, y) = x + \frac{y\, d^2(x, y)}{1-d(x, y)}. \tag{3. 3}$$

Thus we get

$$d(x, y) = \frac{1+x-\sqrt{(1+x)^2-4x(y+1)}}{2(y+1)}. \tag{3. 4}$$

Let us denote now by $D(n)$ the total number of search codes of size n. It follows, putting

$$\delta(x) = \sum_{n=1}^{\infty} D(n) x^n, \tag{3. 5}$$

that

$$\delta(x) = d(x, 1) = \frac{1+x-\sqrt{1-6x+x^2}}{4}. \tag{3. 6}$$

From (3. 4) and (3. 6) we can get explicit formulae for $D(n, k)$ and $D(n)$, respectively. By some easy calculations we get

$$D(n, k) = \frac{1}{n} \binom{n-2}{k-1} \binom{n+k-1}{k} \quad (n \geqq 2,\ k \geqq 1) \tag{3. 7}$$

and thus

$$D(n) = \frac{1}{n} \sum_{k=1}^{n-1} \binom{n-2}{k-1} \binom{n+k-1}{k} \quad (n \geqq 2). \tag{3. 8}$$

From (3. 6) we obtain for $D(n)$ the asymptotic formula

$$D(n) \sim \frac{\sqrt{3-2\sqrt{2}}}{4\sqrt{\pi}} \frac{(3+2\sqrt{2})^n}{n^{3/2}}. \tag{3. 9}$$

The first values of $D(n)$ are $D(2)=1$, $D(3)=3$, $D(4)=11$, $D(5)=45$, and $D(6)=197$.

We consider now the following enumeration problem: let $D_r(n)$ denote the total number of search codes of size n consisting of codewords all having length

$\leq r$ $(r=1, 2, \ldots)$. By the same type of argument which led us to (3. 3) we obtain, putting

$$\delta_r(x) = \sum_{n=1}^{\infty} D_r(n) x^n \tag{3. 10}$$

that

$$\delta_r(x) = x + \frac{\delta_{r-1}^2(x)}{1-\delta_{r-1}(x)}. \tag{3. 11}$$

Thus we get successively

$$\delta_1(x) = \frac{x}{1-x},$$

$$\delta_2(x) = x + \frac{x^2}{(1-x)(1-2x)},$$

$$\delta_3(x) = x + \frac{x^2(1-2x+2x^2)^2}{(1-x)(1-2x)(1-4x+4x^2-2x^3)},$$

etc.

Evidently

$$\lim_{r\to+\infty} \delta_r(x) = \delta(x). \tag{3. 12}$$

One can determine similarly the generating functions of the total number of *regular* search codes of degree q and size n, consisting of codewords all having the length $\leq r$. If the number of such codes is denoted by $B_q(n, r)$, and we put

$$\beta_q(r, x) = \sum_{\substack{n=1 \\ n\equiv 1 \bmod (q-1)}}^{\infty} B_q(n, r) x^n \tag{3. 13}$$

we obtain $\beta_q(0, x) = x$ and

$$\beta_q(r, x) = x + [\beta_q(r-1, x)]^q \qquad (r=1, 2, \ldots). \tag{3. 14}$$

Especially we get for the binary case $\beta_2(0, x) = x$ and

$$\beta_2(r, x) = x + \beta_2^2(r-1, x) \quad \text{for} \quad r \geq 1. \tag{3. 15}$$

Thus we get successively the polynomials

$$\beta_2(1, x) = x + x^2,$$

$$\beta_2(2, x) = x + (x+x^2)^2,$$

$$\beta_2(3, x) = x + [x + (x+x^2)^2]^2,$$

etc.

Let $B_2(r)$ denote the total number of binary search codes, consisting of codewords all having length $\leq r$, irrespectively of the number of codewords in the code. As clearly

$$B_2(r) = \sum_{n=1}^{\infty} B_2(n, r) = \beta_2(r, 1), \tag{3. 16}$$

we get the recursion formula

$$B_2(0) = 1, \quad B_2(r+1) = 1 + B_2^2(r) \quad \text{for} \quad r \geqq 0 \tag{3.17}$$

from which one can calculate successively the values of $B(r)$. We have $B_2(1)=2$, $B_2(2)=5$, $B_2(3)=26$, $B_2(4)=677$, $B_2(5)=458330$. (Notice that the trivial code is counted here as the unique binary code consisting of a single codeword of length 0.) If $b(r)$ denotes the total number of binary search codes such that the length of the longest codeword of the code is equal to r, then $b_2(r) = B_2(r) - B_2(r-1)$; thus we have $b_2(1)=1$, $b_2(2)=3$, $b_2(3)=21$, $b_2(4)=651$ etc.

One can get from (3. 17) also asymptotic formulae for the sequences $B_2(r)$ resp. $b_2(r)$ for $r \to +\infty$.

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ÜBER DIE FUNKTIONALGLEICHUNG $f(a_0+a_1x+a_2y+a_3xy)+g(b_0+b_1x+b_2y+b_3xy)=h(x)+k(y)$

Von

I. FENYŐ (Rostock)

Herrn Professor G. ALEXITS *zum 70. Geburtstag gewidmet*

1. Auf einer Tagung über Funktionalgleichungen hat Herr. M. Hosszu folgende Frage gestellt: Es soll die allgemeinste meßbare Lösung der Funktionalgleichung

$$f(x+y-xy)+f(xy) = f(x)+f(y)$$

angegeben werden. Die Frage haben wir in einer früheren Arbeit beantwortet ([1]). Eine Verallgemeinerung der obigen Funktionalgleichung ist die im Titel stehende Gleichung

$$f(a_0+a_1x+a_2y+a_3xy)+g(b_0+b_1x+b_2y+b_3xy) = h(x)+k(y), \tag{1}$$

wobei f, g, h, und k unbekannte Funktionen, a_i, b_i gegebene Konstanten sind. Unser Ziel ist, die allgemeinste lokalintegrierbare, bzw. eine noch allgemeinere Lösung von (1) zu bestimmen.

Wir werden die betrachtete Funktionalgleichung in folgende Gestalt umschreiben:

$$f[r_0+(r_1x+r_2)(r_3y+r_4)]+g[s_0+(s_1x+s_2)(s_3y+s_4)] = h(x)+k(y). \tag{2}$$

Es wird angenommen, daß $r_1r_3\neq 0$ und $s_1s_3\neq 0$ ist.

Durch Einführung gewisser Transformationen für Distributionen werden wir die Gleichung (2) in eine Distributionenfunktionalgleichung mit vier unbekannten Distributionen überführen und werden ihre allgemeinste Lösung im Bereich der Distributionen aufsuchen.

2. Wir werden folgende Bezeichnungen einführen:

Δ_k ist der lineare Raum der Schwartzschen Grundfunktionen von k Variablen;

Δ_k' ist der lineare Raum der Distributionen über Δ_k;

$$D^r = \frac{d^r}{dt^r}; \quad D_1^r = \frac{\partial^r}{\partial x^r}; \quad D_2^r = \frac{\partial^r}{\partial y^r} \qquad (r = 1, 2, 3, \ldots);$$

D^0 ist der Identitätsoperator;

$\Delta_1(\alpha)$ ist der lineare Raum der Grundfunktionen von einer Veränderlichen, deren sämtliche Ableitungen im Punkt $x=\alpha$ verschwinden;

$\Gamma(\alpha)$ ist die Menge derjenigen Grundfunktionen einer Veränderlichen, deren Träger den Punkt α nicht enthält (d. h. zu jeder Funktion aus $\Gamma(\alpha)$ gibt es eine Umgebung von α in welcher die Funktion identisch verschwindet).

$\Delta_2^x(\alpha)$ (bzw. $\Delta_2^y(\beta)$) sei die Menge aller Grundfunktionen von zwei Veränderlichen, die entlang der Gerade $x=\alpha$ (bzw. an $y=\beta$) samt allen Ableitungen verschwinden;

C_k^∞ sei der lineare Raum aller unbeschränkt oft differenzierbarer Funktionen von k unabhängigen Veränderlichen;

I_1, I_2 seien fogende Operationen:

$$(I_1 f)(y) = \int_{-\infty}^{\infty} f(x, y)\,dx, \qquad (I_2 f)(x) = \int_{-\infty}^{\infty} f(x, y)\,dy.$$

3. Wir benötigen die Einführung einiger Transformationen für Distributionne. Für Funktionen einer, bzw. zwei Veränderlichen werden wir folgende Koordinatentransformationen betrachten:

$$T_1(p, q)f(t) = |p| f(px+q),$$

bzw.

$$T_2(p, q; r, s)g(x, y) = |pr|\, g(px+q, ry+s).$$

Offensichtlich ist

$$T_1^{-1}(p, q) = T_1\left(\frac{1}{p}, -\frac{q}{p}\right), \quad \text{bzw.} \quad T_2^{-1}(p, q; r, s) = T_2\left(\frac{1}{p}, -\frac{q}{p}; \frac{1}{r}, -\frac{s}{r}\right).$$

Sei nun V eine Distribution über Δ_1 bzw. Δ_2, dann definieren wir folgende Transformationen

$$T_1(p, q)V\cdot\varphi = V\cdot T_1^{-1}(p, q)\varphi \qquad (V\in\Delta_1',\ \varphi\in\Delta_1),$$

bzv.

$$T_2(p, q; r, s)V\cdot\varphi = V\cdot T_2^{-1}(p, q; r, s)\varphi \qquad (V\in\Delta_2',\ \varphi\in\Delta_2).$$

Falls V eine durch die Funktion f erzeugte reguläre Distribution ist, dann ist auch $T_1(p, q)f$ eine reguläre Distribution erzeugt durch $f(px+q)$. Dasselbe gilt auch für den Fall zweier Veränderlicher.

Man kann leicht folgende Differentiationsregeln beweisen:

$$DT_1(p, q)V = pT_1(p, q)(DV) \qquad (V\in\Delta_1')$$

(3)
$$D_1T_2(p, q; r, s)V = pT_2(p, q; r, s)(D_1V);$$

$$D_2T_2(p, q; r, s)V = qT_2(p, q; r, s)(D_2V) \qquad (V\in\Delta_2').$$

Es genügt, nur die erste der Formeln (3) nachzuweisen. Für eine beliebige Grundfunktion $\varphi\in\Delta_1$ gilt nämlich

$$DT_1(p, q)V\cdot\varphi = -T_1(p, q)V\cdot D\varphi = -V\cdot T_1^{-1}(p, q)D\varphi =$$
$$= -V\cdot pDT_1^{-1}(p, q)\varphi = pDV\cdot T_1^{-1}(p, q)\varphi = pT_1(p, q)DV\cdot\varphi,$$

womit die Behauptung bewiesen ist.

Es wird auch von folgender Regel Gebrauch gemacht: Wenn $a(x, y)\in C_2^\infty$ und $V\in\Delta_2'$, dann gilt

(4)
$$T_2(p, q; r, s)(aV) = T_2(p, q; r, s)aT_2(p, q; r, s)V.$$

(Die entsprechende Aussage gilt auch für den Fall einer Veränderlichen, was wir aber nicht benötigen werden.) Da im Beweis von (4) p, q, r, s festgehaltene Zahlen sind, werden wir für $T_2(p, q; r, s)$ kurz nur T_2 schreiben. Sei nun $\varphi\in\Delta_2$ beliebig, dann ist

$$T_2(aV)\cdot\varphi = aV\cdot T_2^{-1}\varphi = V\cdot aT_2^{-1}\varphi = V\cdot T_2^{-1}[(T_2a)\varphi] =$$
$$= T_2(V)\cdot(T_2a)\varphi = T_2(a)T_2(V)\cdot\varphi,$$

womit die Behauptung bewiesen ist.

4. Es wird eine Abbildung von $\Delta_2^x(0) \cup \Delta_2^y(0)$ in Δ_1, bezeichnet durch p, betrachtet ([1]):

$$p\varphi = p\varphi(x, y) = \int_{-\infty}^{\infty} \varphi\left(u, \frac{t}{u}\right) \frac{du}{|u|} = \chi(t) \quad \text{für} \quad \varphi(x, y) \in \Delta_2^x(0)$$

und

$$p\varphi = p\varphi(x, y) = \int_{-\infty}^{\infty} \varphi\left(\frac{t}{u}, u\right) \frac{du}{|u|} = \chi(t) \quad \text{für} \quad \varphi(x, y) \in \Delta_2^y(0).$$

Man kann beweisen, daß $\chi(t) = p\varphi(x, y) \in \Delta_1$ (vgl. [1]) ist. p ist offensichtlich additiv und homogen, weitere Eigenschaften dieser Abbildung (wie z. B. Stetigkeit) werden wir nicht gebrauchen.

Mit Hilfe dieser Abbildung werden wir folgende Distributionentransformation definieren ([1]):

$$P(V) \cdot \varphi = V \cdot p(\varphi) \qquad (V \in \Delta_1'; \ \varphi \in \Delta_2^x(0) \cup \Delta_2^y(0)).$$

$P(V)$ ist ein additives und homogenes Funktional, definiert über dem Raum $\Delta_2^x(0) \cup \Delta_2^y(0)$. Es wurde in [1] bewiesen, daß die Beziehungen

(5) $$D_1 P(V) = y P(DV); \qquad D_2 P(V) = x P(DV)$$

gelten. Falls $V = f(t)$, d. h. wenn V eine reguläre Distribution ist, dann ist auch $P(V)$ ein reguläres Funktional, das durch die Funktion $f(xy)$ erzeugt ist. Es gilt ferner: Wenn $a(t) \in C_1^\infty$, dann gilt

(6) $$P(aV) = P(a) P(V).$$

Den Beweis s. in [1].

Es wird auch folgender Zusammenhang von Nutzen sein ([1])

(7) $$D_1 D_2 P(V) = D_2 D_1 P(V) = P(DtDV) \qquad (V \in \Delta_1').$$

5. In den späteren Ausführungen werden wir folgenden Approxiomationssatz benötigen.

HILFSSATZ. *Sei* $R(t) \in C_1^\infty$ *und*

$$S(t) = a_0 + a_1 t + \cdots + a_\nu t^\nu$$

ein Polynom. Ferner sei $\chi(t) \in \Gamma(0)$ *so beschaffen, daß*

$$\Omega(t) = \frac{\chi(t)}{R(t) + C} \qquad (C = a_0 + a_1 + a_2 + \cdots + a_\nu)$$

auch eine Grundfunktion aus $\Gamma(0)$ *sei. Behauptung: Es gibt eine Familie von Grundfunktionen* $\varphi_\varepsilon(x, y) \in \Delta_2^x(0)$ *bzw.* $\psi_\varepsilon(x, y) \in \Delta_2^y(0)$, *für welche*

$$p[(R(x) + S(y)) \varphi_\varepsilon(x, y)] \to \chi(t) \quad \text{für} \quad \varepsilon \to 0$$

bzw.

$$p[(R(y) + S(x)) \psi_\varepsilon(x, y)] \to \chi(t) \quad \text{für} \quad \varepsilon \to 0$$

gilt. Der Grenzübergang ist dabei im Sinne des Konvergenzbegriffes, eingeführt im Raum Δ_1, *zu verstehen.*

BEWEIS. Wir werden den Beweis zuerst für eine solche Grundfunktion $\chi(t)$ beweisen, die zusätzlich zu den Bedingungen des Hilfssatzes auch noch an der Halbachse $(-\infty, 0)$ identisch verschwindet.

Jetzt sei vorläufig $\varphi(x,y) \in \Delta_2^x(0)$ so gewählt, daß $\varphi(x,y) \equiv 0$ für $x \leqq 0$ und y beliebig. Dann gilt

$$p[(R(x)+S(y))\varphi(x,y)] = \int_{-\infty}^{+\infty} R(u)\varphi\left(u, \frac{t}{u}\right)\frac{du}{|u|} + \int_{-\infty}^{+\infty} S\left(\frac{t}{u}\right)\varphi\left(u, \frac{t}{u}\right)\frac{du}{|u|} =$$

$$= \int_0^{\infty} R(u)\varphi\left(u, \frac{t}{u}\right)\frac{du}{u} + \int_0^{\infty} S\left(\frac{t}{u}\right)\varphi\left(u, \frac{t}{u}\right)\frac{du}{u} =$$

$$= \int_{-\infty}^{\infty} R(e^v)\varphi(e^v, e^{\tau-v})\,dv + \int_{-\infty}^{+\infty} \sum_{k=0}^{\nu} a_k e^{k\tau} e^{-k\tau}\varphi(e^v, e^{\tau-v})\,dv =$$

$$= \int_{-\infty}^{\infty} \tilde{R}(v)\tilde{\varphi}(v, \tau-v)\,dv + \sum_{k=0}^{\nu} a_k e^{k\tau} \int_{-\infty}^{\infty} e^{-kv}\tilde{\varphi}(v, \tau-v)\,dv,$$

wobei $v = \log u$, $\tau = \log t$, $\tilde{\varphi}(\xi, \eta) = \varphi(e^\xi, e^\eta)$, $\tilde{R}(v) = R(e^v)$ ist. Offensichtlich ist $R(v) \in C_1^\infty$, $\tilde{\varphi}(\xi, \eta) \in C_2^\infty$.

Wir bilden nun aus $\Omega(t)$ die Funktion $\tilde{\Omega}(\tau) = \Omega(e^\tau)$. Auch $\tilde{\Omega}(t)$ ist eine Grundfunktion, denn einerseits gilt $\tilde{\Omega}(\tau) \in C_1^\infty$, andererseits, falls der Träger von $\Omega(t)$ in $0 < a < b$ liegt, liegt der Träger von $\tilde{\Omega}(\tau)$ in $\log a < \tau < \log b$, also in einem beschränkten Intervall.

Zunächst wollen wir die Funktion

$$\varkappa_\varepsilon(t) = \begin{cases} \exp \dfrac{-\varepsilon^2}{\varepsilon^2 - \log^2 t} & \text{für} \quad e^{-\varepsilon} < t < e^{\varepsilon} \\ 0 & \text{sonst} \end{cases}$$

betrachten und bilden

$$\varphi_\varepsilon(x, y) = \Omega(x)\varkappa_\varepsilon(y).$$

Wegen der Eigenschaften von $\Omega(t)$ ist $\varphi_\varepsilon(x,y) \in \Delta_2^x(0)$ und $\varphi_\varepsilon(x,y) \equiv 0$ für $x \leqq 0$ und y beliebig. Somit gilt auf Grund eines wohlbekannten Satzes ([3] p. 142) über $\tilde{\varkappa}_\varepsilon(\xi) = \varkappa_\varepsilon(e^\xi)$:

$$p[(R(x)+S(y))\varphi_\varepsilon(x,y)] = \int_{-\infty}^{+\infty} \tilde{R}(v)\tilde{\Omega}(v)\tilde{\varkappa}_\varepsilon(\tau-v)\,dv +$$

$$+ \sum_{k=0}^{\nu} a_k e^{k\tau} \int_{-\infty}^{\infty} e^{-kv}\tilde{\Omega}(v)\varkappa_\varepsilon(\tau-v)\,dv \to \tilde{R}(\tau)\tilde{\Omega}(\tau) + \sum_{k=0}^{\nu} a_k e^{k\tau}e^{-k\tau}\tilde{\Omega}(\tau) =$$

$$= (\tilde{R}(\tau)+C)\tilde{\Omega}(\tau) = (R(t)+C)\Omega(t) = \chi(t),$$

wobei auch hier der Konvergenzbegriff in Δ_1 gemeint ist. Damit ist aber die Behauptung für den betrachteten Sonderfall bewiesen.

Für eine Funktion $\psi(t) \in \Gamma(0)$ die den Forderungen des Hilfssatzes genügt und zusätzlich noch identisch für $t \geqq 0$ verschwindet bleibt voriger Gedankengang ohne Änderung anwendbar, so daß der Hilfssatz auch für solche Grundfunktionen gültig ist. Da aber jede Grundfunktion aus $\Gamma(0)$ in die Gestalt $\chi(t)+\psi(t)$ zerlegbar ist (sogar auch eindeutig), ist die Behauptung völlig bewiesen.

Ähnlich verläuft der Beweis bezüglich der Behauptung über

$$p[(S(x)+R(y))\psi_\varepsilon(x,y)].$$

6. Mit Hilfe der oben definierten Transformationen werden wir eine weitere definieren, die auf dem Raum Δ_1' erklärt ist:

$$M(r_0, r_1, r_2, r_3, r_4) = T_2(r_1, r_2; r_3, r_4)PT_1(1, r_0).$$

(Wenn die Gefahr eines Irrtums nicht vorhanden ist, werden wir die Parameterwerte nicht ausschreiben.)

Da

$$M(V)\cdot\varphi = T_2\{P[T_1(V)]\}\cdot\varphi = P[T_1(V)]\cdot T_2^{-1}\varphi =$$
$$= P[T_1(V)]\cdot\frac{1}{|r_1 r_3|}\varphi\left(\frac{x-r_2}{r_1}, \frac{y-r_4}{r_3}\right)$$

ist, ist somit $M(V)$ auf dem Funktionenraum $\Delta_2^x\left(-\frac{r_2}{r_1}\right)\cup\Delta_2^y\left(-\frac{r_4}{r_3}\right)$ definiert und dort ein additives und homogenes Funktional.

Leicht zu beweisen sind folgende Zusammenhänge:

(8) $$D_1 M(V) = r_1(r_3 y + r_4)M(DV)$$
und $$(V\in\Delta_1').$$
$$D_2 M(V) = r_3(r_1 x + r_2)M(DV)$$

Es soll die Gültigkeit z. B. der ersten Formel nachgewiesen werden. Durch Anwendung von (3), (5), (4) und nochmals (3), ergibt sich

$$D_1 M(V) = r_1 T_2(D_1 P(T_1(V))) = r_1 T_2(yP(DT_1(V)) =$$
$$= r_1(r_3 y + r_4)T_2(P(T_1(DV))) = r_1(r_3 y + r_4)M(DV).$$

Nach (8), (6) und (3) ergibt sich

$$D_2 D_1 M(V) = r_1 D_2(r_3 y + r_4)M(DV) =$$
$$= r_1 r_3 M(DV) + r_1 r_3(r_1 x + r_2)(r_3 y + r_4)M(D^2 V) =$$
$$= r_1 r_3 T_2[PT_1(DV) + PtT_1(D^2 V)] = r_1 r_3 T_2 P[DT_1(V) + tD^2 T_1(V)] =$$
$$= r_1 r_3 T_2 P(DtDT_1(V)),$$

somit gilt

(9) $$D_1 D_2 M(V) = D_2 D_1 M(V) = r_1 r_3 P(DtDT_1(V)).$$

Wenn hier $V=f(t)$ ist, dann geht die Transformation M auf Grund der Bedeutung von T_1 und T_2 angewendet auf Funktionen in die Abbildung

$$f(t) \to f(r_0 + (r_1 x + r_2)(r_3 y + r_4))$$

über.

7. Zum Schluß soll noch eine weitere Distributionstransformation erwähnt werden ([2]) die $\Delta_1' \times \Delta_1'$ in Δ_2' abbildet. Seien U und V Distributionen aus Δ_1', dann ist

$$S(U, V)\varphi = U \cdot I_2\varphi + V \cdot I_2\varphi \qquad (U, V \in \Delta_1', \varphi \in \Delta_2).$$

Es gilt ([2])

$$D_1 D_2 S(U, V) = D_2 D_1 S(U, V) = 0, \tag{10}$$

und wenn $U = h(t)$, $V = k(t)$ ist, dann gilt

$$S(h, k) = h(x) + k(y).$$

8. Nun wollen wir anstatt der Funktionalgleichung (2) die folgende Distributionsfunktionalgleichung

$$M(r_0, r_1, r_2, r_3, r_4)(F) + M(s_0, s_1, s_2, s_3, s_4)(G) = S(H, K) \tag{11}$$

betrachten, wobei F, G, H, K unbekannte Distributionen (aus Δ_1') sind. Nach den früheren Bemerkungen bezüglich der Bedeutung von M und S angewendet auf Funktionen, geht (11) in die Gleichung (2) über falls F, G, H, K reguläre Distributionen sind.

Der Kürze halber wollen wir für $M(r_0, r_1, r_2, r_3, r_4)$ nur $M(r)$ schreiben, ähnliche Bedeutung soll auch $M(s)$ haben.

Wir nehmen vorläufig an, daß Distributionen F, G, H, K existieren, die der Gleichung (11) genügen. Dann wenden wir den Differentialoperator $D_1 D_2$ auf beiden Seiten von (11) an und erhalten nach Beachtung von (9) und (10) folgendes:

$$r_1 r_3 T_2(r) P\big(DtDT_1(1, r_0)(F)\big) + s_1 s_3 T_2(s) P\big(DtDT_1(1, s_0)(G)\big) = 0. \tag{12}$$

Durch Einführung folgender Bezeichnungen

$$r_1 r_3 DtDT_1(1, r_0)(F) = U; \qquad s_1 s_3 DtDT_1(1, s_0)(G) = V,$$

nimmt (12) die Gestalt

$$P(U) = T_2^{-1}(r) T_2(s) P(V)$$

an. Da aber, wie ein leichtes Rechnen zeigt

$$T_2^{-1}(r) T_2(s) = T_2\left(\frac{s_1}{r_1}, \frac{r_1 s_2 - r_2 s_1}{r_1}; \frac{s_3}{r_3}, \frac{r_3 s_4 - r_4 s_3}{r_3}\right) =$$

$$= T_2(\omega_1, \omega_2; \omega_3, \omega_4) = T_2(\omega)$$

gilt, ist nun

$$P(U) = -T_2(\omega) P(V).$$

Daraus folgt nach (5) und (3)

$$D_1 P(U) = yP(DU) = -\omega_1 T_2(\omega)\big(D_1 P(V)\big) = -\omega_1 T_2(\omega)\big(yP(DV)\big)$$

$$D_2 P(U) = xP(DU) = -\omega_2 T_2(\omega)\big(D_2 P(V)\big) = -\omega_2 T_2(\omega)\big(xP(DV)\big),$$

und wenn wir diese zwei Gleichungen kombinieren, ergibt sich durch Berücksichtigung von (4)

$$0 = \omega_2 y T_2(\omega)\big(xP(DV)\big) - \omega_1 x T_2(\omega)\big(yP(DV)\big) =$$

$$= T_2(\omega)\left[\omega_3 x\left(\frac{1}{\omega_3}y - \frac{\omega_4}{\omega_3}\right)P(DV) - \omega_1 y\left(\frac{1}{\omega_1}x - \frac{\omega_2}{\omega_1}\right)P(DV)\right] =$$

$$= T_2(\omega)[(\omega_1\omega_2 y - \omega_3\omega_4 x)P(DV)] = 0,$$

d. h.

(14) $$(\beta x + \alpha y)P(DV) = 0 \quad (\alpha = \omega_1\omega_2,\ \beta = -\omega_3\omega_4).$$

Um aus (14) auf die Gestalt von V schließen zu können, müssen wir zwei Fälle unterscheiden. Der erste ist derjenige, wo beide Größen α und β verschwinden, der zweite, wo mindestens eine dieser Zahlen von Null verschieden ist.

9. Der Fall $\alpha=\beta=0$ tritt genau dann ein, wenn $r_1 s_2 - r_2 s_1 = 0$ und $r_3 s_4 - r_4 s_3 = 0$ ist. In diesem Fall ist aber

$$T_2(\omega) = T_2(\omega_1, 0; \omega_3, 0),$$

deswegen gilt

$$T_2(\omega)P(V)\cdot\varphi = \frac{1}{|\omega_1\omega_3|}P(V)\cdot\varphi\left(\frac{1}{\omega_1}x, \frac{1}{\omega_3}y\right),$$

also ist T_2 eine Ähnlichkeitstransformation ([3], p. 20). Offensichtlich ist aber

$$\frac{1}{|\omega_1\omega_3|}P(V)\cdot\varphi\left(\frac{1}{\omega_1}x, \frac{1}{\omega_3}y\right) = \frac{1}{|\omega_1\omega_3|}V\cdot\int_{-\infty}^{\infty}\varphi\left(u, \frac{1}{\omega_1\omega_3}\frac{t}{u}\right)\frac{du}{|u|} =$$

$$= P(\alpha_{\frac{1}{\omega_1\omega_3}}V)\cdot\varphi$$

für $\varphi\in\Delta_2^x(0)$, und änliche Überlegungen führen zum selben Resultat wenn wir eine Funktion aus $\Delta_2^y(0)$ wählen. α_i bedeutet hier die Ähnlichkeitstransformation. Somit können wir anstatt (13) schreiben

$$P(U) = -P(\alpha_{\frac{1}{\omega_1\omega_3}}V)$$

oder

$$P(U + \alpha_{\frac{1}{\omega_1\omega_3}}V) = 0.$$

Wir setzen

(15) $$U + \alpha_{\frac{1}{\omega_1\omega_3}}V = Z$$

und werden aus

(16) $$P(Z) = 0$$

das Verschwinden von Z folgern.

10. Dazu betrachten wir eine beliebige Grundfunktion $\chi(t)$ aus $\Gamma(0)$. Nach unserem Hilfssatz wählen wir eine Familie von Grundfunktionen $\varphi_\varepsilon(x, y) \in \Delta_2^x(0) \cup \Delta_2^y(0)$ so daß

$$p(\varphi_\varepsilon(x, y)) \to \chi(t) \qquad (\varepsilon \to 0)$$

in Sinne der Konvergenz in Δ_1, gilt. Wegen der Stetigkeit von Z folgt

$$P(Z) \cdot \varphi_\varepsilon(x, y) = Z \cdot p(\varphi_\varepsilon(x, y)) \to Z \cdot \chi(t) \qquad (\varepsilon \to 0).$$

Andererseits aber ist laut (16) die linke Seite dieser Gleichung gleich 0, somit gilt

$$Z \cdot \chi(t) = 0$$

für jede Grundfunktion aus $\Gamma(0)$. Da die Umgebung des Ursprunges in welcher $\chi(t) \equiv 0$ gilt, beliebig klein sein kann, folgt, daß Z eine auf den Nullpunkt konzentrierte Distribution ist. Deswegen kann Z nur folgende Gestalt haben

$$(17) \qquad Z = \lambda_0 \delta + \lambda_1 \delta' + \cdots + \lambda_k \delta^{(k)} + \cdots$$

wobei $\lambda_0, \lambda_1, \cdots$ gewisse Konstanten sind, $\delta^{(k)}$ die k-te Ableitung der Diracschen Distribution bedeutet.

Wegen (15) besteht die folgende Beziehung:

$$(18) \qquad P(Z) = \lambda_0 P(\delta) + \lambda_1 P(\delta') + \cdots + \lambda_k P(\delta^{(k)}) + \cdots.$$

Jetzt betrachten wir die Funktion $y^k/k!$ in einer Umgebung des Nullpunktes und setzen sie fort zu eines Grundfunktion $b(y) \in \Delta_1$. Nachher sei $a(x) \in \Delta_1(0)$ so beschaffen daß

$$\int_{-\infty}^{\infty} \frac{a(u)}{|u|\, u^{k-1}}\, du \neq 0$$

gilt. Wir setzen $\varphi(x, y) = a(x) b(y)$, (dann ist selbstverstädlich $\varphi(x, y) \in \Delta_2^x(0)$) und nach (18)

$$0 = P(Z) \cdot \varphi = \lambda_k \int_{-\infty}^{\infty} \frac{a(u)}{|u|\, u^{k-1}}\, du,$$

woraus $\lambda_k = 0$ folgt $(k = 0, 1, 2, \cdots)$. Wegen der Darstellung (17) ist $Z = 0$.

11. Es gilt nun nach (15)

$$U + \alpha_{\frac{1}{\omega_1 \omega_3}} V = 0$$

und wegen der Definition von U und V gilt

$$r_1 r_3 Dt DT_1(1, r_0)(F) = -s_1 s_3 \alpha_{\frac{1}{\omega_1 \omega_3}} Dt DT_1(1, s_0)(G).$$

Für jede Distribution (aus Δ_1') Φ gelten aber die folgende Identitäten

$$\alpha_j D\Phi = \frac{1}{j} D\alpha_j \Phi; \qquad \alpha_j(f(t)\Phi) = \alpha_j(f(t)\Phi)\alpha_j \ (f \in C_1^\infty, \Phi \in \Delta_1')$$

deswegen können wir schreiben

$$r_1 r_3 Dt DT_1(1, r_0)(F) = -\frac{s_1 s_3}{r_1 r_3} D\alpha_{\frac{1}{\omega_1 \omega_3}} t DT_1(1, s_0)(G)$$

was unmittelbar zur Beziehung

$$DT_1(1, r_0)(F) = -D\alpha_{\frac{1}{\omega_1 \omega_3}} T_1(1, s_0)(G) + K_1/t$$

führt, wobei K_1 eine Konstante ist. Daher wird

$$T_1(1, r_0)(F) = -\alpha_{\frac{1}{\omega_1 \omega_3}} T_1(1, s_0)(G) + K_1 \log|t| + K_2$$

(K_2 = Konstante), oder anders geschrieben

$$T_1(1, r_0)(F) = -T_1(\omega_1 \omega_3, s_0)(G) + K_1 \log|t| + K_2.$$

Daraus folgt

$$F = -T_1^{-1}(1, r_0) T_1(\omega_1 \omega_3, s_0)(G) + K_1 T_1^{-1}(1, r_0)(\log|t|) + K_2$$

oder ausführlicher geschrieben

$$F = -T_1 \left[\frac{s_1 s_3}{r_1 r_3}, s_0 - \frac{s_1 s_3}{r_1 r_3}(r_0 + r_2 r_4)\right](G) + K_1 T_1^{-1}(1, r_0)(\log|t|) + K_2.$$

Ein leichtes Rechnen zeigt, daß

$$M(r) T_1 \left[\frac{s_1 s_3}{r_1 r_3}, s_0 - \frac{s_1 s_3}{r_1 r_3}(r_0 + r_2 r_4)\right] = M(s)$$

gilt, wenn wir die Zusammenhänge $r_1 s_2 - r_2 s_1 = 0$, $r_3 s_4 - r_4 s_3 = 0$ beachten. Somit ist

$$M(r)(F) + M(s)(G) = K_1 M(r) T_1^{-1}(1, r_0) \log|t| + K_2 =$$

$$= K_1 \log|(r_1 x + r_2)(r_3 y + r_4)| + K_2 = K_1 \log|r_1 x + r_2| + K_1 \log|r_3 y + r_4| + K_2.$$

Daraus können wir schon die allgemeinste Lösung im Bereich der Distributionen von (11) für den Fall $\omega_2 = \omega_4 = 0$ festlegen:

G = beliebige Distribution,

$$F = -T_1 \left[\frac{s_1 s_3}{r_1 r_3}, s_0 - \frac{s_1 s_3}{r_1 r_3}(r_0 + r_2 r_4)\right](G) + K_1 \log|t - r_0| + K_2$$

$$H = K_1 \log|r_1 t + r_2|$$

$$K = K_1 \log|r_3 t + r_4| + K_2.$$

12. Wir wollen jetzt zum Fall übergehen, wenn in (14) mindestens eine der Konstanten α und β nicht verschwindet. Wir zeigen nun daß $W = DV = 0$ ist.

Wir nehmen an daß z. B. $\beta \neq 0$ ist. Dann gilt für jede $\varphi(x, y) \in \Delta_2^x(0) \cup \Delta_2^y(0)$

$$(\beta x + \alpha y) P(W) \cdot \varphi = W \cdot p((\beta x + \alpha y)\varphi) = 0.$$

Wenn nun $\chi(t)\in\Gamma(0)$ eine solche Grundfunktion ist, die an der Stelle $t=-\frac{\alpha}{\beta}$ mindestens von erster Ordnung verschwindet (im Falle $\alpha=0$ entfällt diese Bedingung), dann folgt aus dem Hilfssatz $W\cdot\chi=0$. Daraus schließen wir genau wie im Punkt 10, daß der Träger von W höchstens aus der Menge $\{0\}\cup\left\{-\frac{\alpha}{\beta}\right\}$ besteht. (Im Fall $\alpha=0$ haben wir nicht weiter zu argumentieren, denn alles verläuft wörtlich wie im Punkt 10, deswegen dürfen wir annehmen, daß auch $\alpha\neq0$ ist.) Somit hat also W folgende Gestalt

$$W=\lambda_0\delta+\lambda_1\delta'+\cdots+\mu_0\delta_1+\mu_1\delta_1'+\cdots$$

wobei $T_1\left(1,\frac{\alpha}{\beta}\right)(\delta)=\delta_1$ ist. Wir wollen in einer hinreichend kleinen Umgebung von $-\frac{\alpha}{\beta}$ die Funktion $B(y)=\left(y+\frac{\alpha}{\beta}\right)^k\Big/k!$ betrachten und sie zu einer Grundfunktion $b(y)$ so fortsetzen, daß sie samt allen Ableitungen im Punkt $y=0$ verschwindet. Es soll auch eine weitere Grundfunktion $a(x)$ betrachtet werden, die folgende Eigenschaften besitzt:

1° $\{\operatorname{supp} a\times\operatorname{supp} b\}\cap\{\beta x+\alpha y=0\}=\emptyset$

2° $\displaystyle\int_{-\infty}^{\infty}\frac{a(u)}{|u|\,u^{k-1}}\,du\neq0.$

Aus 1° folgt, daß $a(x)\in\Gamma(0)$ ist, und deswegen existiert das Integral in 2°.)

Wenn wir jetzt

$$\varphi(x,y)=a(x)b(y)\quad(\in\Delta_2^x(0))$$

setzen, dann ist einerseits

$$P(W)\cdot\varphi=W\cdot p(\varphi)=\mu_k\int_{-\infty}^{\infty}\frac{a(u)}{|u|\,u^{k-1}}\,du.$$

Anderseits aber ist $P(W)\cdot\varphi=0$ für alle Grundfunktionen φ, die der Bedingung 1° genügen, erfüllt. Daraus folgt, daß $\mu_k=0$ ist.

Wenn wir von $B(y)=y^k/k!$ ausgehen, erhalten wir, daß auch $\lambda_k=0$ ist.

Damit ist aber nachgewiesen, daß $W=0$ ist.

13. Aus $W=DV=0$ folgt, daß $V=K_1$ (eine Konstante) ist. Aus der Definition von V ergibt sich

$$G=k_1(t-s_0)+k_2\log|t-s_0|+k_3.$$

Da die Rolle von U und V vertauschbar ist, hat F folgende Gestalt

$$F=l_1(t-r_0)+l_2\log|t-r_0|+l_3.$$

Somit ist

$$M(r)F+M(s)G=l_1(r_1x+r_2)(r_3y+r_4)+l_2\log|r_1x+r_2|+l_2\log|r_3y+r_4|+$$
$$+l_3+k_1(s_1x+s_2)(s_3y+s_4)+k_2\log|s_1x+s_2|+k_2\log|s_3y+s_4|+k_3.$$

Da laut Voraussetzung $r_1 r_3 \neq 0$ und $s_1 s_3 \neq 0$ ist, kann der eben gewonnene Ausdruck eine identisch verschwindende Ableitung nur dann besitzen, wenn $k_1 = -\frac{r_1 r_3}{s_1 s_3} l_1$ gilt.

14. Unsere Ergebnisse können wir nun im folgenden Satz zusammenfassen:

SATZ. *Die allgemeinste Lösung von* (11) *bzw.* (2) ((1)) *im Bereich der Distributionen (unter der Bedingung daß* $r_1 r_3 \neq 0$, $s_1 s_3 \neq 0$ *ist)*

a) *falls* $(r_1 s_2 - r_2 s_1)^2 + (r_3 s_4 - r_4 s_3)^2 \neq 0$ *gilt, sind Distributionen erzeugt durch die folgenden Funktionen*

$$(19)\qquad \begin{cases} F = l_1(t-r_0) + l_2 \log|t-r_0| + l_3, \\ G = -\dfrac{r_1 r_3}{s_1 s_3} l_1(t-s_0) + k_2 \log|t-s_0| + k_3, \\ H = \nu t + l_2 \log|r_1 t + r_2| + k_2 \log|s_1 t + s_2| + \mu, \\ K = \varrho t + l_2 \log|r_3 t + r_4| + k_2 \log|s_3 t + s_4| + \tau, \end{cases}$$

wobei l_i $(i=1,2,3)$, k_2, k_3, ν, μ, ϱ, τ *beliebige Konstanten sind.*

b) *falls aber* $(r_1 s_2 - r_2 s_1)^2 + (r_3 s_4 - r_4 s_3)^2 = 0$ *ist, dann ist die allgemeine Lösung von* (11) *durch folgendes System von Distributionen gegeben:*

$$(20)\qquad \begin{cases} F = -T_1\left[\dfrac{s_1 s_3}{r_1 r_3}, s_0 - \dfrac{s_1 s_3}{r_1 r_3}(r_0 + r_2 r_4)\right] G + \varkappa_1 \log|t-r_0| + \varkappa_2, \\ H = \varkappa_1 \log|r_1 t + r_2| + \varkappa_3, \\ K = \varkappa_1 \log|r_3 t + r_4| + \varkappa_2 - \varkappa_3, \end{cases}$$

wobei G eine beliebige Distribution ist (aus Δ_1'*) und* $\varkappa_i$ $(i=1,2,3)$ *beliebige Zahlenwerte sind.*

Der obige Satz gibt die allgemeinste Lösung von (11) an. Wenn wir daraus auf die allgemeine lokalintegrierbare Funktionenlösung von (2) (bzw. (1)) für alle x, y Werte schließen möchten, so können wir nach einer Bemerkung von H. SWIATAK [4] sagen, daß die allgemeinste lokalintegrierbare Lösung für alle x und y Werte im Fall a) Funktionen sind, die fast überall mit den in (19) angegebenen Funktionen übereinstimmen. Wenn wir nach der allgemeinsten stetigen Lösung für alle x und y Werte fragen, dann sind diese durch die folgenden Funktionen gegeben:

$$f(t) = \alpha, \qquad g(t) = \beta(t-s_0) + \gamma, \qquad h(t) = \nu t + \mu, \qquad k(t) = \varrho t + \tau,$$

wobei $\alpha, \beta, \gamma, \nu, \mu, \varrho, \tau$ Konstanten sind. (Wenn wir die Geraden $x = -\frac{r_2}{r_1}$, $y = -\frac{r_4}{r_3}$, $x = -\frac{s_2}{s_1}$, $y = -\frac{s_4}{s_3}$ von der (x, y)-Ebene streichen und das Restgebiet mit Q bezeichnen, so geben die Funktionen (19) die allgemeinsten stetigen Lösungen von (2) in Q an.)

Wenn wir im Fall b) für G eine durch die lokalintegrierbare Funktion $g(t)$ erzeugte reguläre Distribution einsetzen, so sind die allgemeinsten lokalintegrierbaren Lösungen von (2) für alle x und y Werte mit den in (20) angegebenen Funktionen fast überall gleich. Analoge Bemerkungen wie im Fall a) bezüglich der stetigen Lösungen gelten auch hier.

Als Schlußbemerkung sei erwähnt, daß bei der Hosszuschen Funktionalgleichung $r_0=1$, $r_1=-1$, $r_2=1$, $r_3=1$, $r_4=-1$; $s_0=0$, $s_1=1$, $s_2=0$, $s_3=1$, $s_4=0$, d. h. $\omega_2^2+\omega_4^2\neq 0$ ist, dabei muß noch $F=G=H=K$ gelten. Das aber kann für jeden Wert von t nur dann erfüllt sein, falls $l_2=k_2=0$ ist. Somit erhalten wir das schon in [1] bewiesene Resultat: Die allgemeine Lösung der Hosszuschen Funktionalgleichung im Bereich der Distributionen (und auch im Bereich der lokalintegrierbaren Funktionen im obigen Sinn) ist durch die Menge der Funktionen von der Gestalt $mt+n$ gegeben.

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POSITIVE ORTHONORMAL SYSTEMS

By

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To Professor G. ALEXITS *on his 70th birthday*

1. Introduction. In 1935, Hardy and Littlewood made the following observation:

(1. 1) *If $\Sigma a_\nu e^{i\nu x}$ and $\Sigma A_\nu e^{i\nu x}$ are trigonometric polynomials and $|a_\nu| \leqq A_\nu$ for all ν, then for all even integers p:*

(1. 2) $$\int_0^{2\pi} |\sum a_\nu e^{i\nu x}|^p \, dx \leqq \int_0^{2\pi} |\sum A_\nu e^{i\nu x}|^p \, dx.$$

This result was the starting point of the paper in which Hardy and Littlewood formulated the now famed "majorant problem" [7].

Our point of departure here is the following simple proof of (1. 1). If we set $p=2r$, the left hand side of (1. 2) can be expanded in the form

(1. 3) $$\sum_{\nu_1} \cdots \sum_{\nu_r} \sum_{\mu_1} \cdots \sum_{\mu_r} a_{\nu_1} \dots a_{\nu_r} \bar{a}_{\mu_1} \dots \bar{a}_{\mu_r} \int_0^{2\pi} e^{i\nu_1 x} \dots e^{i\nu_r x} e^{-i\mu_1 x} \dots e^{-i\mu_r x} \, dx.$$

Notice then that each of the integrals in (1. 3) is non negative, namely equal to 0 or 2π. Hence (1. 3) is majorized by

$$\sum_{\nu_1} \cdots \sum_{\nu_r} \sum_{\mu_1} \cdots \sum_{\mu_r} A_{\nu_1} \dots A_{\nu_r} A_{\mu_1} \dots A_{\mu_r} \int_0^{2\pi} e^{i\nu_1 x} \dots e^{i\nu_r x} e^{-i\mu_1 x} \dots e^{-i\mu_r x} \, dx,$$

but this expression collapses into the right hand side of (1. 2).

Observe that the only property of the system $\{e^{i\nu x}\}$ which we used here explicitly is the non negativity of the integrals in (1. 3). This suggests the following definition.

(1. 4) DEFINITION. *Let the real or complex valued functions $\varphi_1, \varphi_2, \dots$ belong to L_p of some measure space, where $p=2r$ is a positive even integer. We shall say that $\{\varphi_\nu\}$ is a "positive system for p" if for any choice of indices $\nu_1, \dots, \nu_r, \mu_1 \dots, \mu_r$ we have*

(1. 5) $$\int \varphi_{\nu_1} \cdots \varphi_{\nu_r} \bar{\varphi}_{\mu_1} \cdots \bar{\varphi}_{\mu_r} \geqq 0. *$$

This given we see that in fact the analogue of (1. 1) must hold for such a system as well.

More precisely

(1. 6) *If $\{\varphi_\nu\}$ is a positive system for p, $\Sigma a_\nu \varphi_\nu$ and $\Sigma A_\nu \varphi_\nu$ are finite linear combinations of the φ_ν's, and $|a_\nu| \leqq A_\nu$ for all ν, then we have*

(1. 7) $$\int |\sum a_\nu \varphi_\nu|^p \leqq \int |\sum A_\nu \varphi_\nu|^p.$$

* It is tacitly assumed that none of the φ_ν's are a. e. zero.

Positive systems are quite abundant in analysis and it is thus worthwhile to investigate the consequences of this inequality. Indeed, it may be good at this point to familiarize ourselves with some of the examples:

a) The exponential functions $\{e^{i\nu x}\}$ on $[0, 2\pi]$,
b) The Rademacher functions on $[0, 1]$,
c) The Paley—Walsh functions on $[0, 1]$
all form positive systems for any even $p \geqq 2$.

d) If the functions $X_1, X_2, \dots$ are independent p-integrable random variables and satisfy the condition

$$\mathbf{E}(\bar{X}_\nu^\alpha X_\nu^\beta) \geqq 0 \ \forall \ 0 \leqq \alpha \leqq \frac{p}{2}, \quad 0 \leqq \beta \leqq \frac{p}{2},^*$$

then $\{X_\nu\}$ is a positive system for p. Such a condition is satisfied for instance if each X_ν is uniformly distributed on the unit circle of the complex plane.

From given positive systems there are two basic methods of generating new ones.

(i) By grouping the functions of a given positive system into finite bunches and taking linear combinations of the functions within each bunch, with *positive coefficients*, we obtain further positive systems. By this method, the "cosine" system

$$1, \ \cos x, \ \cos 2x, \ \dots$$

can be obtained from the exponential system $\{e^{i\nu x}\}$, and is therefore also a positive system for any p.**

(ii) From two or more positive systems on a finite measure space, further positive systems can be obtained by a procedure which, in a certain sense, can be used to generate all such systems.

For simplicity we shall present it in the case that the functions are real and the underlying spaces have total measure one. It goes as follows. Starting from a system $\{\varphi_\nu\}$ of measurable functions on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, the Kolmogoroff consistency theorem gives us a measure μ, on the Borel field $\mathcal{B}$ generated by the cylinder sets of the space R_∞ of all real sequences $x=(x_1, x_2, \dots)$, such that $\forall n$

$$\mu\{x: x_1 \leqq \alpha_1, \dots, x_n \leqq \alpha_n\} = \mathbf{P}\{\varphi_1 \leqq \alpha_1, \dots, \varphi_n \leqq \alpha_n\}.$$

Then the coordinate functions $\{x_\nu\}$ form a positive system on $(R_\infty, \mathcal{B}, \mu)$ if and only if $\{\varphi_\nu\}$ is a positive system on $(\Omega, \mathcal{F}, \mathbf{P})$. Different systems $\{\varphi_\nu\}$ will in general yield different measures μ. We may thus consider the set K_p of all such probability measures on $(R_\infty, \mathcal{B})$ with respect to which the coordinate functions $x_1, x_2, \dots$ form a positive system for a given p. Clearly convex linear combinations of measures in K_p yield further measures in K_p and thus lead to further positive systems.

In other words, K_p is a convex subset of the linear space of measures on $(R_\infty, \mathcal{B})$. It would be very interesting if the extreme points of K_p could be characterized somehow.

* Here $\mathbf{E}(X_\nu)$ means the expectation of X_ν, i. e. $\mathbf{E}(X_\nu) = \int_\Omega X_\nu \, d\mathbf{P}$.

** The sine system $\{\sin \nu x\}$ is not positive for any $p > 2$ [5].

Given a system $\{\varphi_\nu\}$, on a space $(\Omega, \mathcal{F}, \mu)$, which is positive for some p, the first question that presents itself is whether or not the inequality in (1. 7) must hold for infinite sums as well. To be more precise, suppose that the system $\{\varphi_\nu\}$ is also orthonormal and suppose that for the functions f and F we have in the L_2 sense

$$f = \sum a_\nu \varphi_\nu \quad \text{and} \quad F = \sum A_\nu \varphi_\nu$$

with $|a_\nu| \leqq A_\nu$ for all ν. Then do we necessarily have

(1. 8) $$\int_\Omega |f|^p \, d\mu \leqq \int_\Omega |F|^p \, d\mu \,?$$

It turns out that this inequality for general p-systems is not as immediate a consequence of (1. 7) as one might think at first sight.

To see the significance of (1. 8), suppose that $\varphi_\nu = e^{i\nu x}$ and set

$$f = \sum a_\nu e^{i\nu x}, \qquad F = \sum A_\nu e^{i\nu x}$$

$$S_n(x, f) = \sum_{\nu=-n}^{n} a_\nu e^{i\nu x}, \qquad S_n(x, F) = \sum_{\nu=-n}^{n} A_\nu e^{i\nu x}.$$

Then by the Marcel Riesz theorem we have

$$\|S_n(x, F)\|_p \leqq c_p \|F\|_p.$$

By the positivity of $\{e^{i\nu x}\}$ we get

(1. 9) $$\|S_n(x, f)\|_p \leqq \|S_n(x, F)\|_p \leqq c_p \|F\|_p,$$

and (1. 8) is then obtained by letting n tend to infinity in this inequality and making a further use of Marcel Riesz's result.

One of the main efforts in this paper has been to show the validity of (1. 8). This will be obtained in the next section (theorem (2.13)), but only under the further assumptions that each of the functions φ_ν is bounded and that the underlying space has finite measure.

Given a system $\{\varphi_\nu\}$ which is positive for some p our considerations lead naturally to the introduction of the class X_p of formal series $\Sigma a_\nu \varphi_\nu$ whose "majorant" series $\Sigma |a_\nu| \varphi_\nu$ have partial sums which are uniformly bounded in the L_p norm.

We shall show that this class X_p turns out to be a Banach space under the norm

(1. 10) $$N_p\left(\sum a_\nu \varphi_\nu\right) = \sup_n \left\| \sum_{\nu=1}^{n} |a_\nu| \varphi_\nu \right\|_p.$$

For series in X_p we can show that

$$\left\| \sum_{\nu=1}^{n} a_\nu \varphi_\nu \right\|_p \uparrow \left\| \sum a_\nu \varphi_\nu \right\|_p, \quad \left\| \sum a_\nu \varphi_\nu \right\|_p \leqq \left\| \sum |a_\nu| \varphi_\nu \right\|_p.$$

The inequality (1. 9) shows then that, for the exponential system, X_p can be identified with the class of functions $f \sim \Sigma a_\nu e^{i\nu x}$ in L_p whose majorant function $\bar{f} \sim \Sigma |a_\nu| e^{i\nu x}$ also belongs to L_p. Our theorem (2. 13) in essence states that the same holds true for any bounded orthonormal system on a space of finite measure.

In the third section, we shall compare our norm (1. 10) with the Menchov—Paley "norm". We were brought to do so by the following considerations.

Note that the order in which we take the functions φ_ν is immaterial. Indeed, the rearrangement of any X_p series is an X_p series with respect to the rearranged system. The reason for this can be easily seen from (1. 6). If E is any subset of $\{1, 2, ..., n\}$, then

$$\left\| \sum_{\nu \in E} a_\nu \varphi_\nu \right\|_p \leqq \left\| \sum_{\nu=1}^{n} |a_\nu| \varphi_\nu \right\|_p,$$

thus our norm $N_p(\Sigma a_\nu \varphi_\nu)$ is an upper bound for the L_p norms of the partial sums of $\Sigma a_\nu \varphi_\nu$ in *any* rearrangement.

By an argument based on the Bohnemblust—Spitzer combinatorial lemma [8], similar to that used by the first author [3] for the case $p=2$, but considerably more intricate, the second author [5], has shown that any series in X_p can be rearranged so that the supremum S^* of the absolute values of the partial sums belongs to L_p (see theorem 3. 3).

The classical result of MENCHOV—PALEY ([9] VII. p. 189), on the other hand, states that a sufficient condition for

$$S^*(x, a) = \sup_n \left| \sum_{\nu=1}^{n} a_\nu \varphi_\nu(x) \right|$$

to be in $L_p\,(p>2)$' in case of a uniformly bounded orthonormal system of functions $\{\varphi_\nu\}$, is that

$$B_p^*(a) = \left[\sum_\nu (a_\nu^*)^p \nu^{p-2} \right]^{\frac{1}{p}} \tag{1. 11}$$

be finite, where $\{a_\nu^*\}$ is the non increasing rearrangement of $\{|a_\nu|\}$.

In short, the condition $B_p^*(a) < \infty$ implies $S^* \in L_p$ *without* rearrangement, whereas the condition $\Sigma a_\nu \varphi_\nu \in X_p$ implies $S^* \in L_p$ for *some* rearrangement. It is thus of interest to know whether there are series in X_p such that $B_p^* = \infty$, in other words, whether the condition $B_p^* < \infty$ is indeed more restrictive than the condition $\Sigma a_\nu \varphi_\nu \in X_p$.

For the cosine system this is indeed true and easy to show. The gap series

$$\sum_{k=2}^{\infty} \frac{1}{\sqrt{k} \log k} \cos 2^k x$$

is an example in point. In fact this is the Fourier series of a function in all L_p and thus is an X_p series relative to the cosine system, yet its B_p^* is infinite for all $p>2$.

In the third section we shall show by a combinatorial argument that this situation holds quite generally. Indeed such examples will be constructed for every uniformly bounded orthonormal system. (See theorem 3. 8.)

Finally, we would like to point out that the notion of positive system admits a "dual" notion. To see this, note that the map

$$f \sim \sum a_n \varphi_n \to \hat{f} \sim \sum |a_n| \varphi_n,$$

which sends a function into its "majorant" can be dualized. Indeed, we can formally decompose this map into the three steps

$$f \xrightarrow{\mathscr{F}} \{a_n\} \xrightarrow{A} \{|a_n|\} \xrightarrow{\mathscr{F}^{-1}} \hat{f}.$$

Where $\mathscr{F}$ and A denote respectively the operation of taking Fourier coefficients and the operation of taking absolute values.

The dual map $\{a_n\} \to \{\hat{a}_n\}$ is obtained by reversing these steps, namely

$$\{a_n\} \xrightarrow{\mathscr{F}^{-1}} f \xrightarrow{A} |f| \xrightarrow{\mathscr{F}} \{\hat{a}_n\}.$$

The inequality that is dual to

$$\int |f|^p \leqq \int |\hat{f}|^p$$

is then

$$\sum |a_n|^p \leqq \sum |\hat{a}_n|^p.$$

This inequality is in fact true for the exponential system, a fact which appears to have been discovered first by P. CIVIN [1].

In the last section of this paper we shall present a simple proof of this result along the same lines we have followed in the proof of (1. 1).

The interesting fact which is brought forth by this approach is that a dual notion to that of positivity for a given $p=2r$ is the positivity of the "multiple" Poisson kernel

$$P_\varrho(x_1, \dots, x_r, y_1, \dots, y_r) = \sum_\nu \varrho^\nu \varphi_\nu(x_1) \dots \varphi_\nu(x_r) \overline{\varphi_\nu(y_1)} \dots \overline{\varphi_\nu(y_r)}.$$

2. The spaces X_p. Let $\varphi_1, \varphi_2, \dots$ be a positive system for p. The objects under examination will be formal series $S = \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu$, or in other words, just sequences of complex coefficients $a_1, a_2, \dots$. These form a linear space under the usual operations of addition and scalar multiplication.

If $|a_\nu| \leqq A_\nu$, $\nu = 1, 2, \dots$, we say that the series $\Sigma A_\nu \varphi_\nu$ *majorizes* the series $\Sigma a_\nu \varphi_\nu$, and we write

$$\sum a_\nu \varphi_\nu \ll \sum A_\nu \varphi_\nu.$$

This is a notation used by HARDY and LITTLEWOOD in [7]. If $S = \Sigma a_\nu \Phi_\nu$, then the series $\hat{S} = \Sigma |a_\nu| \varphi_\nu$ is called the *least majorant* of S.

Our starting point is (1. 6), namely

(2. 1) *If* $S = \sum_{\nu=1}^{n} a_\nu \varphi_\nu \ll \sum_{\nu=1}^{n} A_\nu \varphi_\nu$ *then*

$$\int \Big| \sum_{\nu=1}^{n} a_\nu \varphi_\nu \Big|^p \leqq \int \Big| \sum_{\nu=1}^{n} A_\nu \varphi_\nu \Big|^p, \tag{2. 2}$$

and in particular

$$\|S\|_p \leqq \|\hat{S}\|_p. \tag{2. 3}$$

We also have the inequality

$$\sum_{\nu=1}^{n} |a_\nu|^p \int |\varphi_\nu|^p \leqq \int \Big| \sum_{\nu=1}^{n} |a_\nu| \varphi_\nu \Big|^p, \tag{2. 4}$$

since the expansion of the right hand side consists of non-negative terms, among which are the terms of the left hand sum.

Define a functional $N_p(S)$ on finite sums $S=\sum_{\nu=1}^{n} a_\nu \varphi_\nu$ by

$$N_p(S) = \|\hat{S}\|_p.$$

We will show that *N_p is a norm on the space of finite linear combinations of the φ_ν.*

First, (2. 4) shows that $N_p(S)=0$ implies $S=0$, i. e., all a_ν are 0. Obviously $N_p(cS)=|c|N_p(S)$. For any series S and T,

$$(S+T)\hat{} \ll \hat{S}+\hat{T}.$$

Hence for finite sums, $N_p(S+T) = \|(S+T)\hat{}\|_p \leqq \|\hat{S}+\hat{T}\|_p$ by (2. 1). Then $\|\hat{S}+\hat{T}\|_p \leqq \|\hat{S}\|_p+\|\hat{T}\|_p = N_p(S)+N_p(T)$. This proves the triangle inequality.

All that we have done so far can be extended to infinite series. If $S=\sum_{\nu=1}^{\infty} a_\nu\varphi_\nu$ let $S_n=\sum_{\nu=1}^{n} a_\nu\varphi_\nu$. Observe that $\hat{S}_n \ll \hat{S}_{n+1}$. By (2. 1), $N_p(S_n)\leqq N_p(S_{n+1})$; the norms of the partial sums are nondecreasing.

DEFINITION. *Let $\varphi_1, \varphi_2, \ldots$ be a positive system for p. We say that the series $S=\sum_{\nu=1}^{\infty} a_\nu\varphi_\nu$ belongs to the class X_p if the sequence of norms $N_p(S_n)$ is bounded, and hence tends to a finite limit, which we call $N_p(S)$.*

Evidently $S\in X_p$ if and only if $\hat{S}\in X_p$, and $N_p(S)=N_p(\hat{S})$. Two inequalities for series in X_p may be written down immediately.

(2. 5) *Let $T\in X_p$ and $S\ll\hat{T}$. Then $S\in X_p$ and $N_p(S)\leqq N_p(T)$.* This is true since $\|\hat{S}_n\|_p\leqq\|\hat{T}_n\|_p\leqq N_p(T)$, and $N_p(S)=\lim\|\hat{S}_n\|_p$.

(2. 6) *Let $S=\sum_{\nu=1}^{\infty} a_\nu\varphi_\nu\in X_p$. Then $\sum_{\nu=1}^{\infty}|a_\nu|^p\int|\varphi_\nu|^p\leqq N_p^p(S)$.* We merely let $n\to\infty$ in (2. 4).

(2. 7) THEOREM. *The class X_p is a linear space, N_p is a norm, and with this norm X_p is a Banach space.*

PROOF. Let $S, T\in X_p$. Since N_p is a norm on finite sums, $N_p(S_n+T_n) \leqq N_p(S_n)+ +N_p(T_n) \leqq N_p(S)+N_p(T)$. Hence $S+T\in X_p$, and $N_p(S+T)\leqq N_p(S)+N_p(T)$. If $S\in X_p$ and $N_p(S)=0$, then (2. 6) shows that all the coefficients of S are 0. Thus N_p is a norm on X_p.

We must prove that X_p is complete under this norm. Given a sequence $S^{(1)}$, $S^{(2)}, \ldots$ of series in X_p such that $N_p(S^{(j)}-S^{(k)})\to 0$ as $j, k\to\infty$, we may immediately identify the series S toward which $\{S^{(j)}\}$ converges in norm. Let $S^{(j)}=\sum_{\nu=1}^{\infty} a_{j\nu}\varphi_\nu$. By (2. 6), $|a_{j\nu}-a_{k\nu}|\,\|\varphi_\nu\|_p \leqq N_p(S^{(j)}-S^{(k)})$. Therefore, for each ν, the sequence $a_{1\nu}, a_{2\nu}, \ldots$ is a Cauchy sequence. Let $a_\nu=\lim_{j\to\infty} a_{j\nu}$, $S=\sum_{\nu=1}^{\infty} a_\nu\varphi_\nu$.

We will show $S \in X_p$. Since $|N_p(S^{(j)}) - N_p(S^{(k)})| \leqq N_p(S^{(j)} - S^{(k)})$, the sequence of norms $\{N_p(S^{(j)})\}$ converges to a finite limit L. Write

$$N_p(S_n) \leqq N_p(S_n - S_n^{(j)}) + N_p(S_n^{(j)}),$$

and observe that for each fixed n, $N_p(S_n - S_n^{(j)}) \to 0$ as $j \to \infty$, simply because $a_{j\nu} \to a_\nu$. Hence $N_p(S_n) \leqq L$ for each n, and so $S \in X_p$.

The proof that $S^{(j)} \to S$ in X_p is similar: Write

$$N_p(S_n^{(j)} - S_n) \leqq N_p(S_n^{(j)} - S_n^{(k)}) + N_p(S_n^{(k)} - S_n) \leqq N_p(S^{(j)} - S^{(k)}) + N_p(S_n^{(k)} - S_n).$$

Let $k \to \infty$, then let $n \to \infty$. This gives

$$N_p(S^{(j)} - S) \leqq \limsup_{k \to \infty} N_p(S^{(j)} - S^{(k)}),$$

which tends to 0 as $j \to \infty$.

(2. 8) THEOREM. *The partial sums S_n of a series S in X_p*

(i) *converge to S in the norm of X_p*

(ii) *converge in L_p.*

PROOF. Let $m < n$. The two functions $\hat{S}_m$ and $\hat{S}_n - \hat{S}_m$ by themselves form a positive system, since they are linear combinations of the φ_ν with non-negative coefficients. The estimate (2. 4) with $n = 2$ applied to this new system gives

$$\int |\hat{S}_m|^p + \int |\hat{S}_n - \hat{S}_m|^p \leqq \int |\hat{S}_m + (\hat{S}_n - \hat{S}_m)|^p, \tag{2. 9}$$

$$N_p^p(S_n - S_m) \leqq N_p^p(S_n) - N_p^p(S_m). \tag{2. 10}$$

The $S_n - S_m$, $n > m$, are the partial sums of $S - S_m$. Therefore if we let $n \to \infty$ we obtain $N_p^p(S - S_m) \leqq N_p^p(S) - N_p^p(S_m)$, which tends to 0 as $m \to \infty$. This proves (i).

To prove (ii), we notice that according to (2. 3), $\|S_n - S_m\|_p \leqq N_p(S_n - S_m)$. Thus (i) shows that $\{S_n\}$ is an L_p Cauchy sequence.

COROLLARY. *The space X_p is the completion of the metric space M_p of finite linear combinations of the φ_ν, with the norm N_p.*

For, the last theorem shows that M_p is dense in the complete metric space X_p.

We may now extend the basic estimate (2. 1) to X_p series. If $S \in X_p$ we will allow S and $\hat{S}$ to stand also for the L_p limit functions of S_n and $\hat{S}_n$. Thus $N_p(S) = \|\hat{S}\|_p$.

(2. 11) *Let S, $T \in X_p$, $S \ll \hat{T}$. Then*

$$\|S\|_p \leqq \|\hat{S}\|_p \leqq \|\hat{T}\|_p. \tag{2. 12}$$

This is an obvious consequence of the last theorem and of (2. 1), applied to the partial sums of the series in question.

The reason we still distinguish X_p series from their L_p limit functions is that there is nothing so far to prevent a series with coefficients not all zero from converging to zero in L_p. However, this cannot happen if the coefficients are non-negative, as then the L_p norms of the partial sums are non-decreasing. If $\{\varphi_\nu\}$, besides being a positive system, is orthonormal on a space of finite measure, then this problem is eliminated, and we will identify L_2-convergent series with their limit functions. For example, any L_2 function $f=\Sigma a_\nu\varphi_\nu$ has a least majorant $\hat{f}=\Sigma|a_\nu|\varphi_\nu$ in L_2.

In this situation, we will look upon X_p as the set of functions f in the L_2 span of $\{\varphi_\nu\}$ such that the partial sums of the Fourier series of $\hat{f}$ are bounded in L_p (hence convergent). Then $X_p\subset L_p$. Suppose, however, we only know that $\hat{f}\in L_p$. Can we conclude that its partial sums are bounded in L_p, and hence $f\in X_p$?

Consider first the exponential system $\{e^{i\nu x}: \nu=0, \pm 1, \ldots\}$. We have seen that M. Riesz' theorem on partial sums answers "yes" to the previous question. However, this can be proved without using Riesz' theorem. Let $\hat{S}$ be the Fourier series of $\hat{f}$. Then $\hat{S}_n \ll 2\hat{\sigma}_{2n+1}-\hat{\sigma}_n$, where $\hat{\sigma}_n$ is the n^{th} Cesàro mean of $\hat{S}$. Since $\|\hat{\sigma}_n\|_p\leqq\|\hat{f}\|_p$, we see that the numbers $\|\hat{S}_n\|_p$ are bounded. Therefore $f\in X_p$. *Thus for the exponential system, X_p is precisely the class of L_2 functions whose least majorants are in L_p.*

We can extend this result as follows:

(2. 13) THEOREM. *Given an orthonormal positive system $\{\varphi_\nu\}$ for p on a space of finite measure such that each φ_ν is bounded. Then X_p is the set of functions in the L_2 span of $\{\varphi_\nu\}$ whose least majorants belong to L_p.*

This theorem is a corollary of a somewhat more general statement, namely

(2. 14) THEOREM. *Given a positive system for p consisting of bounded L_2 functions φ_ν. Let S be the series $\sum_{\nu=1}^{\infty} a_\nu\varphi_\nu$, where $a_\nu\geqq 0$. If $\{S_n\}$ converges in L_2 to a function f which also belongs to L_p, then $S\in X_p$.*

We need the following

(2. 15) LEMMA. *Under the hypotheses of* (2. 14), *the functions $f, \varphi_1, \varphi_2, \ldots$ are a positive system.*

To see how theorem (2. 14) follows from the lemma, replace f by $f-S_n$. The lemma implies that $f-S_n$ and S_n form a positive system. Hence as in (2. 9),

$$\int|S_n|^p+\int|f-S_n|^p\leqq\int|f|^p.$$

Then $\|S_n\|_p\leqq\|f\|_p$, which shows that $S\in X_p$.

PROOF OF (2. 15). Let $p=2r$. We must verify that

$$\int f^j\bar{f}^k\varphi_{\nu_1}\ldots\varphi_{\nu_l}\bar{\varphi}_{\mu_1}\ldots\bar{\varphi}_{\mu_m}\geqq 0 \tag{2. 16}$$

where $j+l=r$, $k+m=r$, $1\leqq j+k\leqq p-1$. An induction on $j+k$ will proceed in two stages. In the following, Ψ is a product of the φ_ν and $\bar{\varphi}_\nu$ such that there are $p/2$ unconjugated and $p/2$ conjugated factors in the integral in which Ψ appears.

First stage. We will prove (2. 16) for $1\leqq j+k\leqq r+1$.

Let $j+k=1$ (say $j=1$). Since $S_n\to f$ in L_2, and $\Psi\in L_2$, we have

$$0\leqq\int S_n\Psi\to\int f\Psi.$$

Hence (2. 16) holds for $j+k=1$.

Suppose $1 \leqq j+k \leqq r$, and that (2. 16) holds. We wish to prove (2. 16) with an extra factor f or $\bar{f}$, and less a factor φ_ν or $\bar{\varphi}_\nu$. We may assume that the extra factor is f. Since S_n is a linear combination of φ_ν with non-negative coefficients, the induction hypothesis implies

$$\int f^j \bar{f}^k S_n \Psi \geqq 0.$$

Now $S_n \to f$ in L^2, $f^j \bar{f}^k \in L_2$, and Ψ is bounded. Thus

$$\int f^j \bar{f}^k S_n \Psi \to \int f^j \bar{f}^k f \Psi.$$

The first stage induction is complete.

Let $q=r$ or $r+1$, whichever is even. For the second stage we need

$$\int |S_n - f|^q |\varphi_\nu|^{p-q} \to 0 \quad \text{as} \quad n \to \infty. \tag{2. 17}$$

Here is the proof of (2. 17): Let $\Psi_0 = |\varphi_\nu|^{p-q}$. Notice that the first stage result applied to $f - S_n$ shows that $\{f - S_n, \varphi_1, \varphi_2, \ldots\}$ is a positive system for q with respect to the measure $\Psi_0 \, d\mu$. (Here, μ is the measure we have been integrating with respect to all along.) Again, the argument which led to (2. 9) gives

$$\int |f|^q \Psi_0 \, d\mu \geqq \int |S_n|^q \Psi_0 \, d\mu + \int |f - S_n|^q \Psi_0 \, d\mu \geqq \int |S_n|^q \Psi_0 \, d\mu,$$

an inequality which shows that $\sum_{\nu=1}^{\infty} a_\nu \varphi_\nu$ is in X_q with respect to the system $\{\varphi_\nu\}$ and the measure $\Psi_0 \, d\mu$. By (2. 8), $\{S_n\}$ converges in $L_q(\Psi_0 \, d\mu)$. Since Ψ_0 is bounded, $S_n \to f$ in $L_2(\Psi_0 \, d\mu)$. Hence $S_n \to f$ in $L_q(\Psi_0 \, d\mu)$; in other words, (2. 17) holds.

Second stage. We know that (2. 16) holds for $j+k = q$. Suppose $q \leqq j+k \leqq p-2$, and that (2. 16) holds. Then

$$\left| \int f^j \bar{f}^k (S_n - f) \Psi \right| \leqq \left(\int |f|^{(j+k)q'} \right)^{1/q'} \left(\int |S_n - f|^q |\Psi|^q \right)^{1/q}, \tag{2. 18}$$

where $q' = q/(q-1)$. Now $q \geqq p/2$ implies

$$q' \leqq \frac{p}{p-2}, \qquad (j+k)q' \leqq p.$$

Hence the first integral on the right-side of (2. 18) is finite. We can throw away all of the factors of Ψ except one (say φ_ν) and write $|\Psi|^q \leqq \text{const}\, |\varphi_\nu|^q \leqq \text{const}\, |\varphi_\nu|^{p-q}$, since the φ_ν are bounded. Thus

$$\left| \int f^j \bar{f}^k (S_n - f) \Psi \right| \leqq \text{const} \left(\int |S_n - f|^q |\varphi_\nu|^{p-q} \right)^{1/q} \to 0 \quad \text{as} \quad n \to \infty.$$

Using the induction hypothesis we see that

$$\int f^j \bar{f}^k f \Psi \geqq 0.$$

This proves (2. 16) for $r+1 \leqq j+k \leqq p-1$, and completes the proof of Lemma (2. 15).

3. Theorems on rearrangements. In this section, A_p, $A(r, s, \ldots)$ denote constants, not necessarily the same at each occurrence, which depend only on the indicated parameters.

Suppose $S=\sum_{\nu=1}^{\infty} a_\nu\varphi_\nu$ is in X_p of the positive system $\{\varphi_\nu\}$. Consider any rearrangement $\sum_{\nu=1}^{\infty} a_{\sigma_\nu}\varphi_{\sigma_\nu}$ of S. The partial sums of this rearrangement are majorized by $\hat{S}$, hence are bounded in L_p by $\|\hat{S}\|_p$. In other words, any rearrangement of a series in X_p belongs to X_p of the rearranged system $\{\varphi_{\sigma_\nu}\}$. As far as this theory is concerned, the labeling of φ_ν is incidental; we might as well have defined $S\in X_p$ by requiring

$$\sup\left\|\sum_{\nu\in E}|a_\nu|\,\varphi_\nu\right\|_p<+\infty,$$

where the sup is taken over all finite subsets E of indices ν.

It is reasonable then, that there be theorems which bound the suprema of partial sums of rearranged X_p series. The following theorem on finite sums was proved by a combinatorial argument [5] based on the Bohnenblust—Spitzer combinatorial Lemma.

(3. 1) THEOREM. *Let $\varphi_1, \ldots, \varphi_n$ be a positive system for p. Let $S=\sum_{\nu=1}^{n} a_\nu\varphi_\nu$. Given a permutation $\sigma=(\sigma_1, \ldots, \sigma_n)$ of $\{1, \ldots, n\}$ let*

$$S^*(x,\sigma)=\max_{1\leq m\leq n}\left|\sum_{\nu=1}^{m} a_{\sigma_\nu}\varphi_{\sigma_\nu}(x)\right|.$$

Then

$$\frac{1}{n!}\sum_{\sigma}\int[S^*(x,\sigma)]^p\,dx\leq A_p\int|\hat{S}|^p, \tag{3. 2}$$

where the sum is taken over the n! permutations.

COROLLARY. *There is a σ such that $\|S^*(\cdot,\sigma)\|_p\leq A_p\|\hat{S}\|_p$.*

Theorem (3. 1) is probably the best result of this kind that can be obtained from the Bohnenblust—Spitzer Lemma. Using quite different methods, the first author has improved the right-side of (3. 2) to read $A_p\int|S|^p$ assuming $\{\varphi_\nu\}$ is an orthogonal system, not necessarily a positive system [4].

In [5], Theorem (3. 1) was extended to infinite X_p series for the trigonometric case. One version of the extended theorem used M. Riesz' theorem on partial sums. Replacing Riesz' theorem by (2. 11), we prove this version for general positive systems.

(3. 3) THEOREM. *If $S=\sum_{\nu=1}^{\infty} a_\nu\varphi_\nu$ is in X_p of the positive system $\{\varphi_\nu\}$ then there is a permutation $\sigma=(\sigma_1, \sigma_2, \ldots)$ of the positive integers such that*

$$S^*(x,\sigma)=\sup_{m}\left|\sum_{\nu=1}^{m} a_{\sigma_\nu}\varphi_{\sigma_\nu}(x)\right|$$

is in L_p and

$$\|S^*(\cdot,\sigma)\|_p\leq A_p\|\hat{S}\|_p. \tag{3. 4}$$

PROOF. Let $n_0=0$ and $n_1, n_2, \dots$ be an increasing sequence of positive integers: We will consider permutations σ which rearrange the integers of each block $\{\nu: n_k+1 \leqq \nu \leqq n_{k+1}\}$ separately. Theorem (3. 1) will thus be applied to the functions

$$\sum_{\nu=n_k+1}^{n_{k+1}} a_\nu \varphi_\nu .$$

Let $S_n(x, \sigma) = \sum_{\nu=1}^{n} a_{\sigma\nu} \varphi_{\sigma\nu}(x)$, $S_0 = 0$. Notice that $S_{n_k}(x) = S_{n_k}(x, \sigma)$. For $n_k+1 \leqq n \leqq n_{k+1}$ $(k \geqq 1)$ write

$$S_n(x, \sigma) = S(x) + \big(S_{n_k}(x) - S(x)\big) + \big(S_n(x, \sigma) - S_{n_k}(x, \sigma)\big).$$

Then

$$|S_n(x, \sigma)|^p \leqq 3^{p-1}\big(|S(x)|^p + |S_{n_k}(x) - S(x)|^p + \delta_k^*(x, \sigma)^p\big)$$

where

$$\delta_k^*(x, \sigma) = \max_{n_k+1 \leqq n \leqq n_{k+1}} |S_n(x, \sigma) - S_{n_k}(x)|, \qquad k \geqq 0.$$

For $1 \leqq n \leqq n_1$ simply write $|S_n(x, \sigma)| \leqq \delta_0^*(x, \sigma)$. Thus

$$S^*(x, \sigma)^p \leqq \delta_0^*(s, \sigma)^p + 3^{p-1}\Big(|S(x)|^p + \sum_{k=1}^{\infty} |S_{n_k}(x) - S(x)|^p + \sum_{k=1}^{\infty} \delta_k^*(x, \sigma)^p\Big).$$

Applying the corollary of (3. 1) to each δ_k^*, we see that there is a σ such that

$$\int \delta_k^*(x, \sigma)^p \, dx \leqq A_p \int |\hat{S}_{n_{k+1}} - \hat{S}_{n_k}|^p .$$

Hence

$$\int S^*(x, \sigma)^p \, dx \leqq \tag{3. 5}$$

$$\leqq A_p \int |\hat{S}_{n_1}|^p + 3^{p-1}\Big\{\int |S|^p + \sum_{k=1}^{\infty} \int |S_{n_k} - S|^p + A_p \sum_{k=1}^{\infty} \int |\hat{S}_{n_{k+1}} - \hat{S}_{n_k}|^p\Big\}.$$

Since $\hat{S}_n \to \hat{S}$ in L_p, the n_k may be chosen to increase so fast that

$$\sum_{k=1}^{\infty} \int |\hat{S}_{n_k} - \hat{S}|^p \leqq \int |\hat{S}|^p . \tag{3. 6}$$

But $\hat{S} - \hat{S}_{n_k}$ is a majorant of $S - S_{n_k}$ and of $\hat{S}_{n_{k+1}} - \hat{S}_{n_k}$. Therefore by (2. 11) the last two sums in (3.5) are also not more than $\int |\hat{S}|^p$. Since $\int |\hat{S}_{n_1}|^p \leqq \int |\hat{S}|^p$, $\int |S|^p \leqq \int |\hat{S}|^p$, we obtain (3. 4).

Having chosen the n_k so that (3. 6) holds, we could make the set of permutations which fix the corresponding blocks of indices into a probability space as did the first author in [3], and prove that (3. 4) holds when the left-side is integrated over σ.

The Menchov—Paley and Hardy—Littlewood theorems. The condition $S \in X_p$ guarantees that S may be rearranged so that $S^*(\cdot, \sigma) \in L_p$. There is another condition on the coefficients which implies $S^* \in L_p$ without rearrangement. Given a sequence $a=(a_1, a_2, \dots)$ tending to zero let $a_1^*, a_2^*, \dots$ be the nonincreasing rearrangement of the nonzero members of $|a_1|, |a_2|, \ldots,$ with repetitions if some of the $|a_\nu|$ are equal. Put

$$B_p^*(a) = \left[\sum_{\nu=1}^{\infty} (a_\nu^*)^p \nu^{p-2}\right]^{1/p}.$$

We remark that $p \geqq 2$ and $B_p^*(a) < \infty$ implies $\Sigma |a_\nu|^2 < \infty$.

(3. 6) THEOREM (MENCHOV—PALEY [9, v. II, p. 189]). *Let $\varphi_1, \varphi_2, \ldots$ be a uniformly bounded orthonormal system, $|\varphi_\nu| \leqq M$. Let $p > 2$. If $B_p^*(a) < \infty$ then $S^*(x) = \sup_n \left|\sum_{\nu=1}^{n} a_\nu \varphi_\nu(x)\right|$ is in L_p and*

$$\|S^*\|_p \leqq A_p M^{\frac{p-2}{p}} B_p^*(a).$$

(3. 7) COROLLARY. *In addition, let $\{\varphi_\nu\}$ be a positive system for p, p even. If $B_p^*(a) < \infty$ then $S = \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu$ is in X_p, and*

$$\|\hat{S}\|_p \leqq A_p M^{\frac{p-2}{p}} B_p^*(a).$$

The corollary does not require the full power of (3. 6), but only a weaker result, which is also due to PALEY, ([9] v. II, pp. 121—123), to the effect that if $B_p^*(a) < \infty$ then $\Sigma a_\nu \varphi_\nu$ converges in L_2 to an $f \in L_p$ and

$$\left(\int |f|^p\right)^{\frac{1}{p}} \leqq A_p M^{\frac{p-2}{2}} B_p^*(a).$$

We see from (3. 7) that $S \in X_p$ is an apparently weaker condition than $B_p^*(a) < \infty$, and Theorem (3. 3) gives an appropriately weaker result than the Menchov—Paley theorem. Hence we should suspect that there are series $S \in X_p$ such that $B_p^*(a) = \infty$.

These will be series to which the Menchov—Paley theorem does not apply but which can nevertheless, by theorem (3. 3), be rearranged so that the supremum of their partial sums is in L_p. In the introduction we mentioned such an example for the trigonometric case. The next theorem shows the existence of examples in more general circumstances.

(3. 8) THEOREM. *Let $\varphi_1, \varphi_2, \ldots$ be a uniformly bounded ($|\varphi_\nu| \leqq M$) orthonormal positive system for an even $p \geqq 4$ on a space of finite measure h. Then there is a series $S = \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu$ with $a_\nu > 0$ such that $S \in X_p$ but $B_p^*(a) = \infty$.*

PROOF. This will be carried out in two stages. First we obtain an estimate on the L_p norm of finite sums with nonnegative coefficients, averaged over all rearrangements of the *coefficients*. Next, such sums are pieced together to form the L_p-convergent series S.

Rearrangements of coefficients in finite sums. Given a sum $\sum_{\nu=1}^{n} b_\nu \varphi_\nu$ with $b_\nu \geqq 0$ $(\nu = 1, \ldots, n)$ we will establish an upper bound on the expression

$$I = \frac{1}{n!} \sum_\sigma \int \left|\sum_{\nu=1}^{n} b_{\sigma\nu} \varphi_\nu\right|^p, \tag{3. 9}$$

p even. Here $\sum_{\sigma}$ sums over all the $n!$ permutations. Expanding the p^{th} power $(p=2r)$ gives

$$I=\frac{1}{n!}\sum_{\sigma}\int\sum_{\nu_1\dots\nu_r}\sum_{\mu_1\dots\mu_r}\prod_{j=1}^{r} b_{\sigma_{\nu_j}}\varphi_{\nu_j}b_{\sigma_{\mu_j}}\bar{\varphi}_{\mu_j}.$$

We classify the p-tuples $(\nu_1, \dots, \nu_r, \mu_1, \dots, \mu_r)$, writing

$$(3.10)\qquad I=\sum_{s}\sum_{\alpha,\beta}k(\alpha,\beta)\sum_{N}\frac{1}{n!}\sum_{\sigma}b_{\sigma_{n_1}}^{\gamma_1}\dots b_{\sigma_{n_s}}^{\gamma_s}\int\varphi_{n_1}^{\alpha_1}\bar{\varphi}_{n_1}^{\beta_1}\dots\varphi_{n_s}^{\alpha_s}\bar{\varphi}_{n_s}^{\beta_s},$$

where $1\leqq s\leqq\min(p,n)$, $\alpha=(\alpha_1,\dots,\alpha_s)$, $\beta=(\beta_1,\dots,\beta_s)$, $\alpha_i\geqq 0$, $\beta_i\geqq 0$, $\gamma_i=\alpha_i+\beta_i$, $1\leqq\gamma_1\leqq\gamma_2\leqq\dots\leqq\gamma_s$, $\alpha_1+\dots+\alpha_s=\beta_1+\dots+\beta_s=r$, $N=(n_1,\dots,n_s)$ where the n_i are distinct, and $k(\alpha,\beta)$ is a positive combinatorial factor bounded above by a constant A_p.

The sum over σ in (3. 10) does not depend on the choice of the n_i; thus

$$(3.11)\qquad I=\sum_{s}\sum_{\alpha,\beta}k(\alpha,\beta)\frac{(n-s)!}{n!}\sum_{M}b_{m_1}^{\gamma_1}\dots b_{m_s}^{\gamma_s}\int\sum_{N}\varphi_{n_1}^{\alpha_1}\bar{\varphi}_{n_1}^{\beta_1}\dots\varphi_{n_s}^{\alpha_s}\bar{\varphi}_{n_s}^{\beta_s},$$

where $M=(m_1,\dots,m_s)$, the m_i distinct. Now since $b_\nu\geqq 0$ and the φ_ν form a positive system for p, we will remove the restriction that M and N consist of distinct indices. This adds only non-negative terms to the right-side of (3. 11). Thus

$$(3.12)\qquad I\leqq\sum_{s}\sum_{\alpha,\beta}k(\alpha,\beta)U(\alpha,\beta),$$

$$U(\alpha,\beta)=\frac{(n-s)!}{n!}\sum_{\nu=1}^{n}b_\nu^{\gamma_1}\dots\sum_{\nu=1}^{n}b_\nu^{\gamma_s}\int\sum_{\nu=1}^{n}\varphi_\nu^{\alpha_1}\bar{\varphi}_\nu^{\beta_1}\dots\sum_{\nu=1}^{n}\varphi_\nu^{\alpha_s}\bar{\varphi}_\nu^{\beta_s}.$$

We will estimate the non-negative quantities $U(\alpha,\beta)$ by functions of n and the b_ν. First observe

$$(3.13)\qquad \int\Big|\sum_{\nu=1}^{n}\varphi_\nu\Big|^l=\int\Big|\sum\varphi_\nu\Big|^{l-2}\Big|\sum\varphi_\nu\Big|^2\leqq(Mn)^{l-2}\int\Big|\sum\varphi_\nu\Big|^2=M^{l-2}n^{l-1}$$

if $l\geqq 2$. Also $(n-s)!/n!\leqq A_p n^{-s}$.

Case 1: $\gamma_1\geqq 2$. We simply estimate $|\varphi_\nu|\leqq M$ and thus

$$(1)\qquad U(\alpha.\beta)\leqq\frac{A_p}{n^s}\sum b_\nu^{\gamma_1}\dots b_\nu^{\gamma_s}M^{\gamma_1}n\dots M^{\gamma_s}nh=A_p hM^p\sum b_\nu^{\gamma_1}\dots\sum b_\nu^{\gamma_s}=U_1.$$

Case 2: $\gamma_1=\dots=\gamma_l=1$, $l\geqq 2$, and $\gamma_{l+1}\geqq 2$ if $l<s$. From (3. 13),

$$\int\sum\varphi_\nu^{\alpha_1}\bar{\varphi}_\nu^{\beta_1}\dots\sum\varphi_\nu^{\alpha_s}\bar{\varphi}_\nu^{\beta_s}\leqq M^{\gamma_{l+1}}n\dots M^{\gamma_s}n\int\Big|\sum\varphi_\nu\Big|^l\leqq M^{p-2}n^{s-1},$$

$$(2)\qquad U(\alpha,\beta)\leqq A_pM^{p-2}\frac{1}{n}\Big(\sum b_\nu\Big)^l\sum b_\nu^{\gamma_{l+1}}\dots\sum b_\nu^{\gamma_s}=U_2.$$

Case 3: $\gamma_1 = 1$, and $\gamma_2 \geqq 2$ if $s \geqq 2$. By Schwarz' Inequality,

$$\int \left(\sum \varphi_\nu^{\alpha_1} \bar{\varphi}_\nu^{\beta_1}\right)\left(\sum \varphi_\nu^{\alpha_2} \bar{\varphi}_\nu^{\beta_2} \dots \sum \varphi_\nu^{\alpha_s} \bar{\varphi}_\nu^{\beta_s}\right) \leqq \left[\int \left|\sum \varphi_\nu\right|^2 \int \left|\sum \varphi_\nu^{\alpha_2} \bar{\varphi}_\nu^{\beta_2} \dots \sum \varphi_\nu^{\alpha_s} \bar{\varphi}_\nu^{\beta_s}\right|^2\right]^{\frac{1}{2}} \leqq$$

$$\leqq [n(M^{\gamma_2} n \dots M^{\gamma_s} n)^2 h]^{\frac{1}{2}} = h^{\frac{1}{2}} M^{p-1} n^{s-\frac{1}{2}},$$

$$(3) \qquad U(\alpha, \beta) \leqq A_p h^{\frac{1}{2}} M^{p-1} n^{-\frac{1}{2}} \sum b_\nu \sum b_\nu^{\gamma_2} \dots \sum b_\nu^{\gamma_s} = U_3.$$

Estimates (1), (2), and (3) will be used to prove a later lemma.

Piecing together finite sums. Our series S will be obtained by modifying the series $\sum_{\nu=1}^{\infty} \nu^{-\lambda} \varphi_\nu$, $\lambda = 1 - 1/p$. Take integers n_k and a permutation σ as in the beginning of the proof of (3. 3). Write

$$\Phi_k(x, \sigma) = \sum_{\nu=n_k+1}^{n_{k+1}} \sigma_\nu^{-\lambda} \varphi_\nu(x), \qquad (k = 0, 1, 2, \dots).$$

$$B_k = \sum_{\nu=n_k+1}^{n_{k+1}} \nu^{-\lambda p} \nu^{p-2} = \sum_{\nu=n_k+1}^{n_{k+1}} \frac{1}{\nu}$$

Choose positive convergence factors $\varepsilon_0 \geqq \varepsilon_1 \geqq \varepsilon_2 \geqq \dots$ such that $\sum_0^\infty \varepsilon_k < \infty$. The factor ε_k will multiply the k^{th} block of $\Sigma \nu^{-\lambda} \varphi_\nu$. Make $\{n_k\}$ increase so fast that

$$(3.\ 14) \qquad \sum_{k=1}^{\infty} \varepsilon_k^p B_k = \infty.$$

Let $I(m, n)$ be the expression I of (3. 9) associated with $\sum_{\nu=m+1}^{n} \nu^{-\lambda} \varphi_\nu$. Here $n - m$ replaces n in (3. 9).

(3. 15) LEMMA. *$I(m, n) \leqq A(p, M, h)$ for all m and n.*

We will prove this later. Since $I(n_k, n_{k+1})$ is the average of $\int |\Phi_k(x, \sigma)|^p \, dx$ over all permutations of $(n_k, n_{k+1}]$, we can determine σ on each block $(n_k, n_{k+1}]$ such that

$$(3.\ 16) \qquad \int |\Phi_k(x, \sigma)|^p \, dx \leqq A(p, M, h)$$

for all k. Then $\sum_{k=0}^{\infty} \varepsilon_k \|\Phi_k(\cdot, \sigma)\|_p < \infty$. Hence $\sum_{k=0}^{\infty} \varepsilon_k \Phi_k(x, \sigma)$ converges in L_p. When this series is written out in terms of the φ_ν, it has positive coefficients and the partial sums are bounded in L_p. This is because a subsequence of them is bounded and the φ_ν are a positive system. The coefficients a_ν and their decreasing rearrangement a_ν^* are

$$a_\nu = \varepsilon_k \sigma_\nu^{-\lambda}, \qquad a_\nu^* = \varepsilon_k \nu^{-\lambda}, \qquad n_k + 1 \leqq \nu \leqq n_{k+1}.$$

Thus (3. 14) states that $B_p^*(a) = \infty$.

It remains to prove the lemma. In view of (3. 12) and estimates (1), (2), (3) we need only prove that the expressions U_1, U_2, U_3 associated with $\sum_{\nu=m+1}^{n} \nu^{-\lambda}\varphi_\nu$ are bounded by an $A(p, M, h)$. Evidently $\Sigma b_\nu^\gamma \leqq \sum_{m+1}^{n} \nu^{-3/2} < \sum_{1}^{\infty} \nu^{-3/2} = C$ if $\gamma \geqq 2$. Since $s \leqq p$,

$$U_1 \leqq A_p h M^p C^p \leqq A_p h M^p C^p.$$

To bound U_2 we observe that, since $l \leqq p$,

$$\frac{1}{n-m}\left(\sum b_\nu\right)^l = \frac{1}{n-m}\left(\sum_{m+1}^{n}\frac{1}{\nu^\gamma}\right)^l < \frac{1}{n-m}\left(\int_m^n x^{-\gamma}\,dx\right)^l = p^l n^{\frac{l}{p}-1}\frac{(1-a)^l}{1-a^p} =$$

$$= p^l n^{\frac{l}{p}-1}\frac{(1-a)^{l-1}}{1+a+\cdots+a^{p-1}} < p^l n^{\frac{l}{p}-1} \leqq p^p,$$

where $a = \left(\frac{m}{n}\right)^{1/p}$. Then there are factors Σb_ν^γ as before.

Similarly for U_3,

$$\frac{1}{(n-m)^{\frac{1}{2}}}\sum b_\nu < pn^{\frac{1}{p}-\frac{1}{2}}\frac{1-a}{(1-a^p)^{\frac{1}{2}}} = pn^{\frac{1}{p}-\frac{1}{2}}\left[\frac{1-a}{1+a+\cdots+a^{p-1}}\right]^{\frac{1}{2}} < pn^{\frac{1}{p}-\frac{1}{2}} \leqq p.$$

This completes the proof of (3. 8). A corollary of the proof is an estimate on how much smaller N_p can be made on finite sums than can B_p^*.

(3. 17) *Given the conditions on* $\{\varphi_\nu\}$ *of Theorem* (3. 8). *For each* n *there is a sum* $S = \sum_{\nu=1}^{n} a_\nu \varphi_\nu$ *with* $a_\nu > 0$ $(\nu = 1, \ldots, n)$ *such that*

(3. 18) $$(\log n)\int |S|^p \leqq A(p, M, h)[B_p^*(a)]^p.$$

This follows directly from (3. 16) if we set $n_1 = n$ and $S(x) = \Phi_0(x, \sigma)$. Here

$$[B_p^*(a)]^p = \sum_{\nu=1}^{n} \nu^{-1} \sim \log n.$$

A theorem of HARDY and LITTLEWOOD [6] states that for cosine series S with nonincreasing non-negative coefficients the expressions B_p^* and $\|S\|_p$ are equal within a factor A_p. This allows us to state (3. 17) in a different form for the case $\varphi_\nu(x) = \cos(\nu-1)x$.

(3. 19) *For each even* $p \geqq 4$ *and each* n *there is a trigonometric polynomial* $S(x) = \sum_{\nu=0}^{n} a_\nu \cos \nu x$ *with* $a_\nu > 0$ $(\nu = 0, \ldots, n)$ *such that*

(3. 20) $$(\log n)\int_0^{2\pi} |S|^p \leqq A_p \int_0^{2\pi} \left|\sum_{\nu=0}^{n} a_\nu^* \cos \nu x\right|^p dx,$$

where $a_0^*, a_1^*, \ldots, a_n^*$ *is the nonincreasing rearrangement of* $a_0, \ldots, a_n$.

It would be interesting to know whether it is possible to improve upon the factor $\log n$ in (3. 20). It is known [2] that the nonincreasing rearrangement gives the largest L_p norm. Perhaps the specific permutation of the a_ν^* which gives the smallest L_p norm may be determined.

4. The dual property. We return to the trigonometric system and present a result that is dual to (1. 1). At the beginning of section 2 we started with the natural ordering on the coefficients of series; here we start with the natural ordering on functions or, more generally, on measures. Let m and M be finite Borel measures on $[0, 2\pi]$, m complex, M positive. The notation $|m| \leqq M$ will mean that $|m|(E) \leqq M(E)$ for any Borel set E, where $|m|$ is the total variation measure of m.

(4. 1) THEOREM. *Let the measures m and M have Fourier coefficients*

$$a_\nu = \int_0^{2\pi} e^{-i\nu x} m(dx); \quad A_\nu = \int_0^{2\pi} e^{-i\nu x} M(dx) \qquad (\nu = 0, \pm 1, \ldots).$$

If $|m| \leqq M$, then $\Sigma |a_\nu|^p \leqq \Sigma |A_\nu|^p$, for all even p.

PROOF. Let $p = 2r$. If $0 < \varrho < 1$ then

$$\sum_{\nu=-\infty}^{\infty} |a_\nu|^p \varrho^{|\nu|} = \sum_\nu \varrho^{|\nu|} a_\nu^r \bar{a}_\nu^r =$$

$$= \sum_\nu \varrho^{|\nu|} \int \cdots \int \exp[-i\nu(x_1 + \cdots + x_r - y_1 - \cdots - y_r)] m(dx_1) \cdots m(dx_r) \bar{m}(dy_1) \cdots \bar{m}(dy_r) =$$

$$= \int \cdots \int P_\varrho(x_1 + \cdots + x_r - y_1 - \cdots - y_r) m(dx_1) \cdots m(dx_r) \bar{m}(dy_1) \cdots \bar{m}(dy_r),$$

where $P_\varrho(x)$ is the Poisson kernel. This expression is majorized by the corresponding expression with m replaced by M, since $P_\varrho(x) > 0$. The latter expression then collapses back into $\Sigma |A_\nu|^p \varrho^{|\nu|}$. Letting $\varrho \to 1$ gives the desired result.

The positivity of P_ϱ is dual to the positivity of $\{e^{i\nu x}\}$. Obviously the Fejér kernel would have done just as well, and in fact the same kind of thing can be done for Fourier transforms by using, say, the kernel $e^{-\frac{1}{2}x^2}$.

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THE GIBBS PHENOMENON AND LEBESGUE CONSTANTS FOR REGULAR $[J, f(x)]$ MEANS

By

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To Professor G. ALEXITS *on his 70th birthday*

Preface

1. Introduction. Professor G. Alexits has made many noteworthy contributions to the problem, important both within mathematics and for its applications, of determining when an orthogonal development generated by a function of specified type will reproduce, or fail to reproduce, the generating function. His well-known book [1] (which appeared not only in the English edition cited here, but previously in German and later in Russian versions) provides an illuminating discussion of a number of crucial problems in this field.

Here we restrict ourselves to the most classical orthogonal development, Fourier series, and consider two inter-related questions, namely the Lebesgue constants arising from the $[J, f(x)]$ summation of Fourier series (Part II) and the corresponding problems for the Gibbs phenomenon (Part I).

The general significance of the Lebesgue constants (Lebesgue functions in the non-periodic case) for summability problems is explained in Chapter III of [1], where background literature is cited. For our purposes, it suffices to recall that the unboundedness of the sequence of the Lebesgue constants (norms) arising from the application of a summation method to Fourier series implies the existence of two continuous functions, each of period 2π, such that (i) the Fourier series of one of them fails to be summable to the function at a prescribed point (the du Bois Reymond singularity), while (ii) the Fourier series of the other is summable everywhere to the function, but not uniformly in any neighbourhood of a prescribed point (the Lebesgue singularity).

Another non-uniformity problem prominent in the theory of Fourier series gives rise to the well-known Gibbs phenomenon. Chief attributes are summarized in the opening sections of O. SZÁSZ's discussion of its occurrence for Hausdorff summation [15] and will not be repeated here.

2. The $[J, f(x)]$ transform. The summation method which we apply to Fourier series is that introduced by A. JAKIMOVSKI [6] as the sequence-to-function analogue of the Hausdorff means. Its best known special cases are the Abel scale and Borel's exponential means.

Let $f(x)$ be differentiable infinitely often in $[0, \infty)$. With a sequence $\{s_n\}$, $n=0, 1, 2, \ldots$, associate the transform

$$t(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} f^{(n)}(x) s_n, \qquad x \geqq 0. \tag{2. 1}$$

If $t(x)$ converges for all large x, and if $\lim t(x) = s$, $x \to \infty$, then s is called the $[J, f(x)]$-limit of the sequence $\{s_n\}$.

When $f(x) = e^{-x}$, (2. 1) defines the Borel exponential mean. When $f(x) = (1+x)^{-\gamma-1}$, $\gamma > -1$, (2. 1) becomes the generalized Abel A_γ transform, $\gamma = 0$ giving the classic Abel (or Abel—Poisson) method.

JAKIMOVSKI [6] proved that the $[J, f(x)]$ transform is regular if and only if

$$f(x) = \int_0^\infty e^{-ux}\, d\alpha(u), \qquad x \geqq 0, \tag{2. 2}$$

where $\alpha(u)$ is of bounded variation in $[0, \infty)$ and

$$\alpha(0) = \alpha(0+) = 0, \qquad \alpha(\infty -) = 1. \tag{2. 3}$$

In this form, the Borel exponential mean arises on taking $\alpha(u) = 0$, $0 \leqq u < 1$; $\alpha(u) = 1$, $1 \leqq u < \infty$. The A_γ transform has an absolutely continuous weight function, namely

$$\alpha(u) = [\Gamma(\gamma+1)]^{-1} \int_0^u v^\gamma e^{-v}\, dv, \qquad \gamma > -1,$$

which becomes $\alpha(u) = 1 - e^{-u}$ in the classic Abel case $(\gamma = 0)$.

It can be shown [7] that a regular $[J, f(x)]$ method with weight function $\alpha(u)$ is *totally regular* if and only if $\alpha(u)$ is non-decreasing, $0 \leqq u < \infty$. (A method is totally regular, by definition, if and only if $t(x) \to s$ as $x \to \infty$, whenever $s_n \to s$ as $n \to \infty$, both for s finite and for s infinite.) The Borel and A_γ, $\gamma > -1$, methods are totally regular.

Part I. The Gibbs phenomenon

3. Introduction. The ordinary Gibbs phenomenon, as might be displayed by the Fourier series of a function at a jump discontinuity, can, as is well-known, be discussed in terms of the particular periodic function $\sum_1^\infty k^{-1} \sin kx$, at the origin (cf. vol. I of [18], p. 61).

Accordingly, we construct an integral representation for the Gibbs ratio of the regular $[J, f(x)]$ transform of that function (Theorem 1), analogous to the one found by O. SZÁSZ [15, Theorem 2] for Hausdorff means, then construct various regular $[J, f(x)]$ methods, chiefly with discontinuous $\alpha(u)$, which exhibit the Gibbs phenomenon (§ 5), and finally determine the behaviour of the Abel A_γ methods, $\gamma > -1$, with respect to the Gibbs phenomenon (§ 6). It turns out that the A_γ methods which exhibit the Gibbs phenomenon are precisely those for which $\gamma > 1$.

4. The Gibbs ratio. Hereafter we define

$$s_0(t) \equiv 0, \quad s_n(t) = \frac{2}{\pi} \sum_{k=1}^n \frac{\sin kt}{k}, \qquad n = 1, 2, \ldots, \tag{4. 1}$$

and denote by $J_x(t)$ the $[J, f(x)]$ transform (2. 1) of the sequence $\{s_n(t)\}$, $n=0, 1, \dots$. The function $\alpha(u)$ is determined by (2. 2) from $f(x)$. In this notation, we establish the following result, which generalizes that of [11] and includes an analogue of Theorem 2 of [15]. As usual,

$$\mathrm{Si}(y) = \int_0^y v^{-1} \sin v \, dv.$$

THEOREM 1. *Let $[J, f(x)]$ be a regular transform with corresponding weight function $\alpha(u)$, $0 \leqq u < \infty$. Then, for each T, $0 \leqq T \leqq \infty$,*

$$\lim_{x\to\infty} J_x(t_x) = \frac{2}{\pi} \int_0^\infty \int_0^T \frac{\sin uy}{y} \, dy \, d\alpha(u), \tag{4. 2}$$

where $xt_x \to T$ and $t_x \to 0+$ as $x \to \infty$.

COROLLARY 1. *For regular $[J, f(x)]$,*

$$\lim_{x\to\infty} J_x(t_x) \leqq \frac{2}{\pi} \mathrm{Si}(\pi) \int_0^\infty |d\alpha(u)|. \tag{4. 3}$$

If $[J, f(x)]$ is totally regular, this becomes

$$\lim_{x\to\infty} J_x(t_x) \leqq \frac{2}{\pi} \mathrm{Si}(\pi). \tag{4. 4}$$

Equality prevails in (4. 4) *if* [11] *and only if $[J, f(x)]$ is a Borel exponential method.*

The last few words of Corollary 1 warrant some explanation. As JAKIMOVSKI observes in Example 3, p. 141 of [6], there is no real difference between the methods $[J, f(x)]$ and $[J, f(cx)]$, c a positive constant. In the Borel case, this suggests that we identify with one another all the $[J, f(x)]$ methods for which $f(x)=e^{-cx}$, $c>0$, *i. e.*, the ones for which $\alpha(u)$ is the one-jump step-function, $\alpha(u)=0$, $0\leqq u<c$; $\alpha(u)=1$, $c\leqq u<\infty$. This we do, and it is in this sense that the corollary is intended.

PROOF OF THEOREM 1. In the course of proof there comes a point, namely (4. 9), after which it is necessary to separate the cases $T<\infty$ and $T=\infty$. But until then no such distinction is required.

Our starting point is the application to (2. 1) of the known representation (cf. vol. I of [18], p. 61)

$$s_n(t) = -\frac{1}{\pi} t + \frac{2}{\pi} \int_0^t \frac{\sin (n+\frac{1}{2})y}{2 \sin \frac{1}{2} y} \, dy, \qquad n = 0, 1, 2, \dots,$$

together with the consequence

$$f^{(n)}(x) = \int_0^\infty (-u)^n e^{-ux} \, d\alpha(u), \qquad n = 0, 1, 2, \dots,$$

of (2. 2).

Changing the order of integration, as permitted by the Lebesgue dominated convergence theorem, this gives

$$J_x(t) = \frac{2}{\pi}\int_0^\infty\int_0^t e^{-ux}\left[\sum_{n=0}^\infty \frac{\sin(n+\frac{1}{2})y}{2\sin\frac{1}{2}y}\,\frac{(ux)^n}{n!}\right] dy\, d\alpha(u) + o(1),$$

as $t\to 0+$. The bracketed infinite series can be summed in closed form, as in formula (6) of [11]. Using this, $J_x(t)$ can be written

$$J_x(t) = \frac{2}{\pi}\int_0^\infty\int_0^t \{\exp[ux(\cos y-1)]\}\frac{\sin(ux\sin y+\frac{1}{2}y)}{2\sin\frac{1}{2}y}\, dy\, d\alpha(u) + o(1) =$$

$$= \frac{1}{\pi}\int_0^\infty\int_0^t \{\exp[ux(\cos y-1)]\}\sin(ux\sin y)\cot\tfrac{1}{2}y\, dy\, d\alpha(u) +$$

$$+\frac{1}{\pi}\int_0^\infty\int_0^t \{\exp[ux(\cos y-1)]\}\cos(ux\sin y)\, dy\, d\alpha(u) + o(1) \equiv \frac{1}{\pi}I_1 + \frac{1}{\pi}I_2 + o(1).$$

Now,

$$(4.5)\qquad |I_2| \leqq \int_0^\infty\int_0^t dy\,|d\alpha(u)| = t\int_0^\infty |d\alpha(u)| = o(1), \qquad t\to 0+.$$

In I_1 we wish to replace $\cot\frac{1}{2}y$ by $2/y$. To do so, we recall from p. 444 of [15] that $|\cot y - y^{-1}| \leqq cy$ for $0<y<\frac{1}{2}\pi$, so that

$$\int_0^\infty\int_0^t \{\exp[ux(\cos y-1)]\}\,|\sin(ux\sin y)|\,|\cot\tfrac{1}{2}y - 2y^{-1}|\, dy\,|d\alpha(u)| \leqq$$

$$\leqq \frac{1}{2}ct^2\int_0^\infty |d\alpha(u)| = o(1), \quad \text{as} \quad t\to 0+.$$

Thus,

$$(4.6)\qquad I_1 = 2\int_0^\infty\int_0^t \{\exp[ux(\cos y-1)]\}y^{-1}\sin(ux\sin y)\, dy\, d\alpha(u) + o(1).$$

Next we wish to simplify $\sin(ux\sin y)$ to $\sin(uxy)$. Here we have

$$0 \leqq |\sin(uxy) - \sin(ux\sin y)| = 2|\sin\{\tfrac{1}{2}ux(y-\sin y)\}\cos\{\tfrac{1}{2}ux(y+\sin y)\}| \leqq$$

$$\leqq ux(y-\sin y) \leqq uxy^3$$

so that, taking $0 \leqq t \leqq \frac{1}{2}\pi$, we have $y \leqq \frac{1}{2}\pi \sin y$, for $0 \leqq y \leqq t$, and

$$\int_0^\infty \int_0^t \{\exp[ux(\cos y - 1)]\} y^{-1} |\sin(uxy) - \sin(ux \sin y)| \, dy \, |d\alpha(u)| \leqq$$

$$\leqq \int_0^\infty \int_0^t \{\exp[ux(\cos y - 1)]\} uxy^2 \, dy \, |d\alpha(u)| \leqq$$

$$\leqq \frac{1}{2}\pi t \int_0^\infty \int_0^t \{\exp[ux(\cos y - 1)]\} ux \sin y \, dy \, |d\alpha(u)| \leqq$$

$$\leqq \frac{1}{2}\pi t \int_0^\infty |d\alpha(u)| = o(1) \quad \text{as} \quad t \to 0+.$$

Hence, $\sin(ux \sin y)$ can be replaced by $\sin(uxy)$ in the expression for I_1 in (4. 6) with an error of $o(1)$ as $t \to 0+$. Doing so, and substituting $2y$ for y in the result, yields

$$I_1 = 2 \int_0^\infty \int_0^{t/2} \{\exp[-2ux \sin^2 y]\} y^{-1} \sin(2uxy) \, dy \, d\alpha(u) + o(1). \tag{4. 7}$$

Next, we simplify the exponential factor, replacing in it $\sin^2 y$ by y^2. The resulting error is dominated in absolute value by

$$2 \int_0^\infty \int_0^{t/2} \{1 - \exp[-2ux(y^2 - \sin^2 y)]\} \{\exp(-2ux \sin^2 y)\} y^{-1} \, dy \, |d\alpha(u)| \leqq$$

$$\leqq 2 \int_0^\infty \int_0^{t/2} 2ux(y^2 - \sin^2 y) \{\exp(-2ux \sin^2 y)\} y^{-1} \, dy \, |d\alpha(u)| \leqq$$

$$\leqq \frac{4}{3} \int_0^\infty \int_0^{t/2} uxy^3 \{\exp(-2ux \sin^2 y)\} \, dy \, |d\alpha(u)| \leqq$$

$$\leqq \frac{4}{3} \int_0^\infty \int_0^{t/2} uxy^3 \{\exp(-8\pi^{-2} uxy^2)\} \, dy \, |d\alpha(u)| \leqq$$

$$\leqq \frac{t^2}{3} \int_0^\infty \int_0^{t/2} uxy \{\exp(-8\pi^{-2} uxy^2)\} \, dy \, |d\alpha(u)| =$$

$$= \frac{\pi^2}{48} t^2 \int_0^\infty \{1 - \exp(-2\pi^{-2} t^2 ux)\} \, |d\alpha(u)| < \frac{\pi^2}{48} t^2 \int_0^\infty |d\alpha(u)| = o(1) \quad \text{as} \quad t \to 0+,$$

where we have used the standard inequality $1 - e^{-v} \leqq v$, $v \geqq 0$, for

$$v = 2ux(y^2 - \sin^2 y).$$

Thus, (4. 7) can be written as

$$I_1 = 2\int_0^\infty \int_0^{t/2} \{\exp[-2uxy^2]\} y^{-1} \sin(2uxy)\,dy\,d\alpha(u) + o(1), \tag{4. 8}$$

as $t \to 0+$.

The next step is to replace the exponential factor in I_1 by unity. More precisely, we must prove (i)

$$D_x(t_x) \equiv \int_0^\infty \int_0^{t_x/2} y^{-1}\{1-\exp[-2uxy^2]\} \sin(2uxy)\,dy\,d\alpha(u) = o(1), \tag{4. 9}$$

$$\text{as} \quad t_x \to 0+ \quad \text{and} \quad xt_x \to T, \quad x \to \infty,$$

for $0 \leqq T < \infty$ and from this infer (4. 2). Then (ii) for $T = \infty$, (4. 2) will be verified directly.

(i) *Proof of* (4. 9) *and* (4. 2) *for* $T < \infty$. Define, as in [9] and [11], $f_{ux}(y) = y^{-1}(1-\exp[-2uxy^2])$. Then

$$f'_{ux}(y) = \frac{df_{ux}(y)}{dy} = \frac{(4uxy^2+1)(\exp[-2uxy^2]) - 1}{y^2}$$

so that, from formulae (8), (9) and (10) of [11], with some constant $K > 0$,

$$|D_x(t_x)| \leqq K\int_0^\infty [t_x + (ux)^{-1}\int_0^{t_x/2} |f'_{ux}(y)|\,dy]\,|d\alpha(u)|. \tag{4. 10}$$

As shown in §6 of [9], the function $\psi(v) = (2v+1)e^{-v} - 1$ vanishes for one and only one positive value of v, say δ, and is positive for all $0 < v < \delta$.

Since $T < \infty$, there exists $A > 0$ such that $xt_x^2 \leqq A$ for $x \geqq 0$. We choose $u_0 > 0$ small enough that $A \leqq 2\delta u_0^{-1}$. Then, for $0 \leqq u \leqq u_0$ and $0 \leqq y \leqq \frac{1}{2} t_x$, we have $2uxy^2 \leqq \frac{1}{2} u_0 A \leqq \delta$, so that $f'_{ux}(y) \geqq 0$, for such u and y. Hence

$$\int_0^{t_x/2} |f'_{ux}(y)|\,dy = \int_0^{t_x/2} f'_{ux}(y)\,dy = f_{ux}\left(\frac{1}{2}t_x\right) = \tag{4. 11}$$

$$= \frac{2}{t_x}\left[1 - \exp\left(-\frac{1}{2}uxt_x^2\right)\right] \leqq uxt_x, \quad 0 \leqq u \leqq u_0.$$

This suggests rewriting inequality (4. 10) as

$$|D_x(t_x)| \leqq Kt_x\int_0^\infty |\,d\alpha(u)| + K\int_0^{u_0} (ux)^{-1}\int_0^{t_x/2} |f'_{ux}(y)|\,dy\,|d\alpha(u)| +$$

$$+K\int_{u_0}^\infty (ux)^{-1}\int_0^{t_x/2} |f'_{ux}(y)|\,dy\,|d\alpha(u)|.$$

The first term on the right clearly is $o(1)$. By (4. 11) the second term is $o(1)$.

The last term is dominated [9; (6. 7)] by

$$K \int_{u_0}^{\infty} (ux)^{-\frac{1}{2}} \, |d\alpha(u)|,$$

and this is also $o(1)$, as $x \to \infty$, since $u_0 > 0$.

This proves (4. 9) for $T < \infty$. In view of this and (4. 5), we may write

$$J_x(t_x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{t_x/2} \frac{\sin(2uxy)}{y} \, dy \, d\alpha(u) + o(1) = \frac{2}{\pi} \int_0^{\infty} \int_0^{xt_x} \frac{\sin uy}{y} \, dy \, d\alpha(u) + o(1).$$

The uniform boundedness of the inner integral now permits the passage to the limit $xt_x \to T$. This establishes (4. 2) for the case $0 \leqq T < \infty$.

(ii) *Proof of* (4. 2) *for* $T = \infty$. In this case, the result (4. 2) to be demonstrated can be expressed as

$$\text{(4. 12)} \qquad \lim_{x \to \infty} \int_0^{\infty} \int_0^{t_x/2} \{\exp(-2uxy^2)\} y^{-1} \sin(2uxy) \, dy \, d\alpha(u) = \frac{1}{2} \pi,$$

when $xt_x \to \infty$, and $t_x \to 0+$, since

$$\mathrm{Si}(\infty) = \frac{1}{2} \pi \quad \text{and} \quad \int_0^{\infty} d\alpha(u) = 1.$$

This is the analogue of Theorem 1b of O. SZÁSZ [15] and can be established using calculations similar to his. Corresponding to the definition (4. 2) on p. 450 of [15], we put

$$p_x(u, t) = \int_0^{t/2} \{\exp(-2uxy^2)\} y^{-1} \sin(2uxy) \, dy,$$

so that $p_x(0, t) = 0$, and employ also the familiar function

$$\mathrm{si}(y) = -\int_y^{\infty} v^{-1} \sin v \, dv = \mathrm{Si}(y) - \frac{1}{2} \pi.$$

First replacing $2uxy$ by y and then integrating by parts, we obtain, for $u > 0$,

$$p_x(u, t) = \int_0^{uxt} \left\{ \exp\left(-\frac{y^2}{2ux} \right) \right\} d\,\mathrm{si}(y) =$$

$$= \frac{1}{2} \pi + \{\mathrm{si}(uxt)\} \left\{ \exp\left(-\frac{1}{2} uxt^2 \right) \right\} + \frac{1}{ux} \int_0^{uxt} y \, \mathrm{si}(y) \left\{ \exp\left(-\frac{y^2}{2ux} \right) \right\} dy.$$

Hence,

$$\left| p_x(u, t) - \frac{1}{2}\pi \right| < |\mathrm{si}\,(uxt_x)| + \frac{1}{ux}\{\max_{y>0} |\mathrm{si}(y)|\} \int_0^{\omega} y \exp\left(-\frac{y^2}{2ux}\right) dy +$$

$$+ \frac{1}{ux}\{\max_{y \geqq \omega} |\mathrm{si}(y)|\} \int_{\omega}^{\infty} y \exp\left(-\frac{y^2}{2ux}\right) dy <$$

$$< |\mathrm{si}(uxt_x)| + \{\max_{y>0} |\mathrm{si}(y)|\} \left\{1 - \exp\left(-\frac{\omega^2}{2ux}\right)\right\} + \max_{y \geqq \omega} |\mathrm{si}(y)|,$$

where $\omega > 0$ and $u > 0$.

Now let $\omega = \omega_x \to \infty$ in such a way that $\omega_x^2 = o(x)$, and recall that $xt_x \to T = \infty$. Thus, $p_x(u, t_x) \to \frac{1}{2}\pi$ as $x \to \infty$, in $0 < u < \infty$. Moreover, the convergence is bounded, $0 < u < \infty$, since $|\mathrm{si}\,(y)| \leqq \mathrm{Si}\,(\pi)$ for all y. Hence, the bounded convergence theorem for Stieltjes integrals may be applied, showing that

$$\lim_{x \to \infty} \int_{0+}^{\infty} p_x(u, t_x)\, d\alpha(u) = \int_{0+}^{\infty} \frac{1}{2}\pi\, d\alpha(u) = \frac{1}{2}\pi.$$

But, $\alpha(0+) = \alpha(0) = 0$, and so

$$\int_0^{\infty} p_x(u, t_x)\, d\alpha(u) = \int_{0+}^{\infty} p_x(u, t_x)\, d\alpha(u) \to \frac{1}{2}\pi.$$

This proves (4. 12), and thus also (4. 2), for $T = \infty$.

The proof of Theorem 1 is now complete.

Corollary 1 follows readily. Inequalities (4. 3) and (4. 4) are immediate. In the final sentence, the "if" part is known [11].

To establish the "only if" part, we recall that the function $\mathrm{Si}\,(w)$ achieves its maximum when and only when $w = \pi$, and write the right member of (4. 2) as $(2/\pi) \int_0^{\infty} \mathrm{Si}\,(Tu)\, d\alpha(u)$. Moreover [7], $\alpha(u)$ is non-decreasing, $0 \leqq u < \infty$.

For T finite, we pick a, b, $0 \leqq a < b \leqq \infty$, with $\alpha(a) < \alpha(b)$, and π/T not in the closed interval $[a, b]$. If this is impossible, then $\alpha(u)$ is a one-jump step-function, *i. e.*, it generates a Borel exponential method, as required. If such a selection is possible, we have

$$\int_0^{\infty} \mathrm{Si}(Tu)\, d\alpha(u) = \int_0^a + \int_a^b + \int_b^{\infty} \leqq$$

$$\leqq [\mathrm{Si}(\pi)]\alpha(a) + \left[\max_{a \leqq u \leqq b} \mathrm{Si}(Tu)\right][\alpha(b) - \alpha(a)] + [\mathrm{Si}(\pi)][1 - \alpha(b)] <$$

$$< [\mathrm{Si}(\pi)]\alpha(a) + [\mathrm{Si}(\pi)][\alpha(b) - \alpha(a)] + [\mathrm{Si}(\pi)][1 - \alpha(b)] = \mathrm{Si}(\pi).$$

For $T = \infty$, (4. 2) equals $1 < (2/\pi)\,\mathrm{Si}\,(\pi)$.

This proves Corollary 1.

REMARKS. A similar argument shows that *the only totally regular Hausdorff methods whose Gibbs ratios equal* $(2/\pi)\,\mathrm{Si}\,(\pi)$ *are the Euler* (E, k) *methods*, $k \geqq 0$, *where* $(E, 0)$ *denotes convergence.* All others have smaller Gibbs ratios. SZÁSZ's Theorem 2 [15] replaces (4. 2) and the monotonicity of the relevant weight-function is inferred from W. A. HURWITZ's Theorem VI [4] instead of [7].

5. Some transforms displaying the Gibbs phenomenon. I. In this section it is shown that some of the $[J, f(x)]$ means analogous to the Hausdorff means considered by the IZUMIS [5] exhibit the Gibbs phenomenon. Chiefly, but not exclusively, these are methods with discontinuous $\alpha(u)$.

Nothing in this section casts any light on the Abel A_γ methods. These are discussed in detail in § 6.

For $T=\infty$, the limit (4. 2) becomes 1. For $T=0$, the corresponding limit is 0. Accordingly, we can (and do) restrict ourselves to $0<T<\infty$.

The limit (4. 2) can be rewritten as

$$G_\alpha(T) \equiv \frac{2}{\pi}\int_0^\infty\int_0^T \frac{\sin uy}{y}\,dy\,d\alpha(u) = \frac{2}{\pi}\int_0^\infty \frac{\sin Ty}{y}[1-\alpha(y)]\,dy. \tag{5. 1}$$

To verify this, we integrate by parts the first integral in (5. 1) considered as

$$\lim_{b\to\infty}\int_0^b h(u)\,d\alpha(u).$$

This shows both the existence of the remaining integral and the equation (5. 1). Thus, Theorem 1 can be expressed in the following form for $0<T<\infty$:

COROLLARY 2. *For regular* $[J, f(x)]$ *means,*

$$G_\alpha(T) = \lim_{x\to\infty} J_x(t_x) = \frac{2}{\pi}\int_0^\infty \frac{\sin Ty}{y}[1-\alpha(y)]\,dy,$$

where $0<T<\infty$, *and* $xt_x\to T$, $t_x \geqq 0$, *as* $x\to\infty$.

A regular $[J, f(x)]$ transform, therefore, will display the Gibbs phenomenon if and only if

$$F(T) \equiv \int_0^\infty y^{-1}(\sin Ty)\alpha(y)\,dy < 0, \tag{5. 2}$$

for some T, $0<T<\infty$, since $\int_0^\infty y^{-1}\sin Ty\,dy = \frac{1}{2}\pi$, $T>0$.

Thus, the problem of this section is to find conditions on $\alpha(u)$ which imply $F(T)<0$ for some positive T. This is analogous to the discussion for Hausdorff means, initiated by O. SZÁSZ [15] and continued by A. E. LIVINGSTON [8], D. J. NEWMAN [14], J. MANN [13] and, most recently, by M. and S. IZUMI [5].

Our next results correspond to Theorems 1, 2 and 3 of [5].

To formulate them, it is convenient to emphasize the role of the function $\alpha(u)$. We shall denote a $[J, f(x)]$ method also as $[J, \alpha(u)]$, when $\alpha(u)$ and $f(x)$ are connected by the relationship (2. 2). The following three theorems distinguish the behaviour of $\alpha(u)$ at $u=0$.

THEOREM 2. *If* $[J, \alpha(u)]$ *is regular, if* $\beta(u)=\alpha(u)/u$ *is of bounded variation in* $0\leqq u<\infty$, *if* $\beta(0)=\beta(0+)=0$, *and if* $\beta(u)$ *has at least one point of discontinuity, then*

$$\liminf_{A\to\infty}([mE_A]/A) > 0, \tag{5. 3}$$

where $E_A=\{T\,|\,0<T\leqq A,\ F(T)<0\}$, *and* mE_A *denotes the Lebesgue measure of* E_A.

Theorem 2 implies the (already known [11]) existence of the Gibbs phenomenon for Borel means.

THEOREM 3. *If* $[J, \alpha(u)]$ *is regular, if* $\beta(u)=\alpha(u)/u$ *is of bounded variation* $0\leqq u<\infty$, *and if* $\beta(0+)<0$, *then* (5. 3) *holds.*

Theorem 3, as distinguished from Theorems 2 and 4, does not require $\alpha(u)$ to have any discontinuities.

THEOREM 4. *If* $[J, \alpha(u)]$ *is regular, if* $\beta(u)=\alpha(u)/u$ *is of bounded variation in* $0\leqq u<\infty$, *with* $\beta(0+)>0$, *then* (5. 3) *holds, provided*

$$\beta(0+) < \mathcal{M}\Big(\Big|\beta(0+)+\sum_j d_j\cos x_j t\Big|\Big), \tag{5. 4}$$

where

$$\mathcal{M}(\varphi(t)) = \lim_{x\to\infty}\frac{1}{x}\int_0^x \varphi(t)\,dt,$$

$\{x_j\}$ *is the set (necessarily countable) of points of discontinuity of* $\alpha(u)$, *and* $d_j=\beta(x_j+)-\beta(x_j-)$.

Central to the proofs of these three theorems is a corollary of a result due to N. WIENER [17] (cf. Theorem (4. 19), p. 261, of [18]):

LEMMA 1. *If* $\beta(u)$ *is continuous and of bounded variation in* $0\leqq u<\infty$, *and if* $\beta(0)=\beta(\infty-)=0$, *then*

$$\int_0^A\left|\int_0^\infty T\beta(u)\sin Tu\,du\right|dT = o(A), \quad as \quad A\to\infty. \tag{5. 5}$$

PROOF. Integration by parts of the inner integral changes the left member of (5. 5) into a form covered by WIENER's theorem.

Theorems 2, 3 and 4 can be established now in the same way as were Theorems 1, 2, and 3 of [5]. The details are left to the reader.

6. Methods exhibiting the Gibbs phenomenon. II: the Abel scale. These differ from the methods considered in § 5 and are not covered by the theorems in that section. A sequence $\{s_n\}$ is said to be summable A_γ, $\gamma>-1$, to the value s, if

$$\lim_{y\to\infty}(1+y)^{-\gamma-1}\sum_{n=0}^{\infty}\binom{n+\gamma}{n}s_n\left(\frac{y}{1+y}\right)^n \tag{6. 1}$$

exists and equals s [2]. When $\gamma=0$, this defines the classic Abel method. Obviously,

$$\binom{n+\gamma}{n} > 0, \quad \gamma > -1. \tag{6.2}$$

The familiar integral representation for the gamma function [3, § 3.381, formula 4, p. 317, last line], makes it clear that the A_γ methods are of $[J, f(x)]$ type. Here $\Gamma(\gamma+1)\alpha'(u) = u^\gamma e^{-u}$, so that the method is totally regular, $\gamma > -1$.

Thus, Theorem 1 is applicable, showing that here

$$G_\gamma(T) \equiv \lim_{x\to\infty} J_x(t_x) = \frac{1}{\Gamma(\gamma+1)} \frac{2}{\pi} \int_0^\infty \left[\int_0^T \frac{\sin uy}{y}\, dy\right] u^\gamma e^{-u}\, du, \tag{6.3}$$

for $\gamma > -1$, when $t_x \to 0+$ and $xt_x \to T$, $0 \leqq T \leqq \infty$, as $x\to\infty$.

Defining the Gibbs ratio

$$G_\gamma \equiv \max G_\gamma(T), \quad 0 \leqq T \leqq \infty, \tag{6.4}$$

we note that an A_γ method exhibits the Gibbs phenomenon if and only if $G_\gamma > 1$.

THEOREM 5. *The Abel A_γ method exhibits the Gibbs phenomenon if and only if $\gamma>1$. More precisely,*

$$G_\gamma = 1 \quad \textit{if} \quad -1 < \gamma \leqq 1, \tag{6.5}$$

while

$$G_\gamma = \frac{2}{\pi} \int_0^{\pi/(\gamma+1)} \frac{\sin\{(\gamma+1)x\}}{\sin x} \cos^\gamma x\, dx > 1, \quad \gamma > 1. \tag{6.6}$$

Furthermore, G_γ increases (strictly) from 1 to $(2/\pi)\,\mathrm{Si}(\pi)$ as γ increases from 1 to ∞.

REMARKS. It is not surprising that the Gibbs ratio increases with γ, since BORWEIN [2, p. 320(1)] has shown that the A_γ method becomes weaker as the parameter γ increases. On the other hand, he has proved also that any A_γ method with $\gamma > -1$ is stronger than all the CESÀRO methods [2, Theorem 4]. Hence, an A_γ method with $\gamma>1$ illustrates the known fact that a summability method may still possess a Gibbs phenomenon even if a weaker method does not.

PROOF OF THEOREM 5. To determine the behaviour of G_γ, we interchange the order of integration in (6.3) to obtain another representation of $G_\gamma(T)$, namely,

$$G_\gamma(T) = \frac{1}{\Gamma(\gamma+1)} \frac{2}{\pi} \int_0^T \frac{1}{y} \int_0^\infty u^\gamma e^{-u} \sin yu\, du\, dy.$$

Using the known value of the inner integral [3, § 3.944, formula 5, p. 490, line 1], we can write

$$G_\gamma(T) = \frac{2}{\pi} \int_0^T \frac{1}{y} \frac{\sin\{(\gamma+1)\arctan y\}}{(1+y^2)^{(\gamma+1)/2}}\, dy.$$

With $v=\arctan y$, this becomes

$$G_\gamma(T) = \frac{2}{\pi} \int_0^{\arctan T} \frac{\sin\{(\gamma+1)v\}}{\sin v} \cos^\gamma v\, dv. \tag{6.7}$$

When $-1<\gamma\leqq 1$, the integrand is non-negative for $0\leqq v\leqq\frac{1}{2}\pi$, *i. e.*, for $0\leqq T\leqq\infty$. Hence $G_\gamma=G_\gamma(\infty)$ if $-1<\gamma\leqq 1$. But $G_\gamma(\infty)=1$, for $\gamma>-1$, as was established in the course of proving Theorem 1. (This evaluation of $G_\gamma(\infty)$ follows also from [3, § 3. 718, formula 3, p. 404, last line]; perhaps our Theorem 1 can be regarded as a new (and real-valued) proof of the appropriate special case of that formula.)

This proves assertion (6. 5).

To establish (6. 6) is quite easy; indeed, it is obvious from the graph of the integrand of $G_\gamma(\infty)$. The first arch, *i. e.*, from $v=0$ to $v=\pi/(\gamma+1)$, is positive. Subsequent arches (if any) alternate in sign and bound successively decreasing areas (taken as absolute values) since $\cos^\gamma v$, $\gamma>1$, and $1/\sin v$ decrease in $0\leqq v\leqq\frac{1}{2}\pi$.

Thus, $G_\gamma=G_\gamma(\tan\{\pi/(\gamma+1)\})>G_\gamma(\infty)=1$ when $\gamma>1$, as asserted in (6. 6).

The final sentence of Theorem 5 contains two assertions: one that

$$\lim_{\gamma\to\infty} G_\gamma=(2/\pi)\,\mathrm{Si}(\pi), \tag{6. 8}$$

the other, that

$$G_\mu<G_\lambda \quad \text{when} \quad 1\leqq\mu<\lambda. \tag{6. 9}$$

To prove (6. 8) we take $\gamma>1$ and write, in view of (6. 6),

$$G_\gamma=\frac{2}{\pi}\int_0^{\pi/(\gamma+1)}\frac{\sin\{(\gamma+1)v\}}{v}\cos^\gamma v\,dv+\frac{2}{\pi}\int_0^{\pi/(\gamma+1)}\left(\frac{1}{\sin v}-\frac{1}{v}\right)\sin\{(\gamma+1)v\}\cos^\gamma v\,dv=$$

$$=\frac{2}{\pi}\int_0^{\pi/(\gamma+1)}\frac{\sin\{(\gamma+1)v\}}{v}\cos^\gamma v\,dv+O\left(\frac{1}{\gamma}\right)=$$

$$=\frac{2}{\pi}\int_0^{\pi/(\gamma+1)}\frac{\sin\{(\gamma+1)v\}}{v}\,dv-\frac{2}{\pi}\int_0^{\pi/(\gamma+1)}\frac{1-\cos^\gamma v}{v}\sin\{(\gamma+1)v\}\,dv+O\left(\frac{1}{\gamma}\right)=$$

$$=\frac{2}{\pi}\int_0^{\pi/(\gamma+1)}\frac{\sin\{(\gamma+1)v\}}{v}\,dv+O\left(\frac{1}{\gamma}\right),\quad \gamma\to\infty,$$

since the functions $\left(\frac{1}{\sin v}-\frac{1}{v}\right)$ and $(1-\cos^\gamma v)/v$ are bounded in $0\leqq v\leqq\pi/(\gamma+1)$.

Putting $x=(\gamma+1)v$ now completes the proof of (6. 8).

The proof of (6. 9) is based on BORWEIN's Lemma 2 [2, p. 319], together with the well-known inequalities

$$2>s_n(x)=\frac{2}{\pi}\sum_{k=1}^{n}\frac{\sin kx}{k}>0,\quad 0<x<\pi,\qquad n=1,2,\dots. \tag{6. 10}$$

The lower bound was conjectured first by L. Fejér and proved by several authors, including Fejér, in a variety of ways. (References and interesting generalizations are given by P. TURÁN [16].)

Denoting, in analogy with [2], the A_γ transform of $s_n(x)$ by $\sigma_\gamma(y;x)$, where the A_γ limit arises on allowing y to become infinite, it follows immediately from (6. 2) and (6. 10) that

(6. 11) $$2>\sigma_\gamma(y;x)>0, \qquad \gamma>-1, \qquad 0<x<\pi.$$

Now we restrict ourselves to λ, μ with $1\leqq\mu<\lambda$, and use BORWEIN's formula [2, Lemma 2 (i) p. 319] to obtain

$$0<\sigma_\mu\left(y;\left(\tan\frac{\pi}{\mu+1}\right)\frac{1}{y}\right)=$$

$$=\frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)}y^{-\lambda}\int_0^y(y-t)^{\lambda-\mu-1}t^\mu\sigma_\lambda\left(t;\left(\tan\frac{\pi}{\mu+1}\right)\frac{1}{y}\right)dt=$$

$$=\frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)}y^{-\lambda}\int_0^{\omega(y)}+\frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)}y^{-\lambda}\int_{\omega(y)}^y,$$

where $\omega(y)=o(y)$ is a positive increasing function of y, with $y>\omega(y)\to\infty$.

Now,

$$y^{-\lambda}\left[\max_{0\leqq t\leqq\omega(y)}(y-t)^{\lambda-\mu-1}\right]\int_0^{\omega(y)}t^\mu\,dt<\frac{\omega(y)}{y}\left[\frac{\omega(y)}{y-\omega(y)}\right]^\mu=o(1),\quad y\to\infty,$$

so that the first term is $o(1)$ as $y\to\infty$, in view of the upper bound in (6. 11).

The lower bound in (6. 11), together with the case $n=0$ of Lemma 1 of [2], implies that

$$0<\frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)}y^{-\lambda}\int_{\omega(y)}^y(y-t)^{\lambda-\mu-1}t^\mu\sigma_\lambda\left(t;\left(\tan\frac{\pi}{\mu+1}\right)\frac{1}{y}\right)dt<$$

$$<\max_{\omega(y)\leqq t\leqq y}\sigma_\lambda\left(t;\left(\tan\frac{\pi}{\mu+1}\right)\frac{1}{y}\right)=\sigma_\lambda\left(\tau_y;\left(\tan\frac{\pi}{\mu+1}\right)\frac{1}{y}\right),$$

where τ_y is an appropriately chosen value of t, given y.

Now

$$\limsup_{y\to\infty}\sigma_\lambda\left(\tau_y;\left(\tan\frac{\pi}{\mu+1}\right)\frac{1}{y}\right)=G_\lambda(S),$$

for some S. But $G_\lambda(S)\leqq G_\lambda$ for all S. Hence

$$G_\mu=\limsup_{y\to\infty}\sigma_\mu\left(y;\left(\tan\frac{\pi}{\mu+1}\right)\frac{1}{y}\right)\leqq G_\lambda.$$

To complete the proof of (6. 9) it suffices now to show $G_\mu\neq G_\lambda$, $1\leqq\mu<\lambda$. To this end, we consider G_γ, $\gamma>1$, as expressed by (6. 6). Regarding γ now as a complex variable, we note that G_γ is analytic, $1<|\gamma|<\infty$. If $G_\mu=G_\lambda$ for a pair of real values, $1\leqq\mu<\lambda$, it would follow from the monotonicity already established that G_γ is constant in the open real interval (μ,λ) contained within the domain of analyticity of G_γ.

Thus, G_γ would have to be constant, $1<|\gamma|<\infty$. But $G_\gamma \to 1$, as $\gamma \to 1$, whereas, from (6. 8), $G_\gamma \to (2/\pi)\ \mathrm{Si}\,(\pi)>1$, as $\gamma \to \infty$, through real values.

This contradiction completes the proof of (6. 9) and, with it, of Theorem 5.

Part II. The Lebesgue constants

7. Introduction. We obtain here asymptotic estimates for the Lebesgue constants of regular $[J, f(x)]$ methods. The chief result is Theorem 6 (§ 8). The Borel means turn out to be extremal here, just as for the Gibbs phenomenon (§ 4, Corollary 1). The A_γ methods are also discussed (§ 11), not only because these methods are interesting in themselves, but also because their Lebesgue constants are bounded and the results obtained illustrate what can be expected of well-behaved sub-schemes of $[J, f(x)]$ methods having this property. As a by-product of this discussion we learn (Corollary 3) that the Lebesgue constants for Euler $(E, 1)$ summation of Fourier series form a non-decreasing sequence. The results and methods parallel those for regular Hausdorff means [12], with Borel exponential summation playing the role in the $[J, f(x)]$ family filled by the Euler methods in the Hausdorff scheme in Theorem 1 (6) of [12]. The A_γ methods display the same type of behaviour as that established for the CESÀRO and HÖLDER means in Theorems 3 and 4 of [12].

One distinction is apparent between the results for the Hausdorff means and those for the $[J, f(x)]$ means. It can be seen in the discrepancy between the magnitudes of the constant factors occurring in the respective principal terms of the Lebesgue constants (contrast Theorem 6 here with Theorem 1 of [12]). This arises because convergence is a (totally regular) Hausdorff method, but is not equivalent to any $[J, f(x)]$ mean [7].

The x-th Lebesgue constant for the $[J, \alpha(u)]$ method will be denoted by $L(x; \alpha)$. By definition

$$L(x;\alpha)=\frac{2}{\pi}\int_0^{\pi/2}|D_x^{(\alpha)}(t)|\,dt, \tag{7. 1}$$

where $D_x^{(\alpha)}(t)$ is the $[J, \alpha(u)]$ transform of the Dirichlet kernel

$$D_n(t)=\{\sin(2n+1)t\}/\sin t.$$

The similarity between the Dirichlet kernel and the function $s_n(t)$ defined in (4. 1) permits the construction of an explicit representation for $L(x; \alpha)$ along the lines of § 4. Doing so yields the formula

$$L(x;\alpha)=\frac{2}{\pi}\int_0^{\pi/2}\left|\int_0^{\infty}\{\exp[-2ux\sin^2 y]\}\frac{\sin(ux\sin 2y+y)}{\sin y}\,d\alpha(u)\right|dy. \tag{7. 2}$$

The simplifications introduced into the corresponding integrands for $J_x(t)$ in § 4 can be made here, even though we do not now have $t\to 0$. They lead, just as in the case of (4. 8), to the following expression:

$$L(x;\alpha)=\frac{2}{\pi}\int_0^{\pi/2}y^{-1}\left|\int_0^{\infty}\{\exp[-2uxy^2]\}\sin(2uxy)\,d\alpha(u)\right|dy+o(1),\quad x\to\infty. \tag{7. 3}$$

8. The main theorem and related results. In the notation of § 7, we can formulate now the principal result of this Part:

THEOREM 6. *Let* $[J, \alpha(u)]$ *be regular and suppose*

$$\int_0^\infty u|d\alpha(u)| < \infty. \tag{8. 1}$$

Then

$$L(x;\alpha) = B(\alpha)\log x + o(\log x), \quad x\to\infty, \tag{8. 2}$$

where

$$B(\alpha) = (1/\pi)\mathcal{M}\Big\{\Big|\sum_j d_j \sin x_j t\Big|\Big\}. \tag{8. 3}$$

Here x_j *is the j-th discontinuity (jump) of* $\alpha(u)$ *and the summation extends over all such (possibly countably infinite) values.* $\mathcal{M}\{\Phi(t)\}$ *represents, as usual, the mean value of the almost periodic function* $\varphi(t)$. *Furthermore,*

$$0 \leqq B(\alpha) \leqq \frac{2}{\pi^2}\int_0^\infty |d\alpha(u)|, \tag{8. 4}$$

and

$$B(\alpha)=0 \text{ if and only if } \alpha(u) \text{ is continuous.} \tag{8. 5}$$

If, in addition, $[J, \alpha(u)]$ *is totally regular, then*

$$0 \leqq B(\alpha) \leqq 2/\pi^2, \tag{8. 6}$$

with $B(\alpha)=2/\pi^2$ *if* [9] *and only if* $[J, \alpha(u)]$ *is a Borel exponential method.*

Regarding the last words of this Theorem, the reader should bear in mind the comments made in § 4 following Corollary 1 which dealt with the corresponding extremal situation for the Gibbs phenomenon.

This Theorem implies that *any regular* $[J, \alpha(u)]$ *method satisfying* (8. 1) *with* $\alpha(u)$ *having a discontinuity exhibits both the du Bois Reymond singularity and the Lebesgue singularity* (§1).

Plainly, (8. 5) does *not* imply the absence of these singularities from regular $[J, \alpha(u)]$ methods with everywhere continuous $\alpha(u)$, since the remainder term in (8. 2) could be unbounded.

Even an absolutely continuous $\alpha(u)$ could generate these singularities, since the remainder term in (8. 2) is "best possible" even for such functions. This follows from the next Theorem.

THEOREM 7. *Let* $\varepsilon(x)\downarrow 0$ *as* $x\to\infty$. *There exists a non-decreasing, absolutely continuous weight function* $\alpha(u)$ *for which* $L(x;\alpha)\neq o(\varepsilon(x)\log x)$, *as* $x\to\infty$.

9. Proof of Theorem 6. Corresponding to Lemma 1 of [12] we need the following result here.

LEMMA 2. *If* $[J, \alpha(u)]$ *is a regular method with* $\alpha(u)$ *satisfying* (8. 1), *then*

$$L(x;\alpha) = \frac{2}{\pi}\int_1^{x^{1/2}} y^{-1}\left|\int_0^\infty \sin uy\, d\alpha(u)\right| dy + o(\log x). \tag{9. 1}$$

Proof of Lemma 2. Let $1<A<\frac{\pi}{2}\sqrt{x}$ be fixed and decompose the integral in (7. 3) into $(2/\pi)\,L_1(x;\alpha)+(2/\pi)\,L_2(x;\alpha)$ where

$$L_1(x;\alpha)=\int_0^{A/x^{1/2}} y^{-1}\left|\int_0^\infty \{\exp[-2uxy^2]\}\sin(2uxy)\,d\alpha(u)\right|dy$$

and

$$L_2(x;\alpha)=\int_{A/x^{1/2}}^{\pi/2} y^{-1}\left|\int_0^\infty \{\exp[-2uxy^2]\}\sin(2uxy)\,d\alpha(u)\right|dy.$$

Now,

$$0\leqq L_2(x;\alpha)\leqq \int_{A/x^{1/2}}^{\pi/2} y^{-1}\int_0^\infty e^{-2uA^2}|d\alpha(u)|\,dy=\varphi(A)\log x+\theta(A),$$

where

$$\varphi(A)=o(1)\quad\text{and}\quad \theta(A)=o(1),\quad\text{as}\quad A\to\infty,$$

since $\alpha(0+)=\alpha(0)=0$.

Now,

$$L_1(x;\alpha)=\int_0^{A/x^{1/2}} y^{-1}\left|\int_0^\infty \sin(2uxy)\,d\alpha(u)\right|dy+R_1,$$

where

$$0\leqq R_1\leqq \int_0^{A/x^{1/2}} y^{-1}\int_0^\infty [1-\exp\{-2uxy^2\}]\,|d\alpha(u)|\,dy\leqq$$

$$\leqq \int_0^{A/x^{1/2}} y^{-1}\int_0^\infty 2uxy^2\,|d\alpha(u)|\,dy=x\int_0^\infty u\,|d\alpha(u)|\int_0^{A/x^{1/2}} 2y\,dy=$$

$$=A^2\int_0^\infty u\,|d\alpha(u)|=O(1),\quad\text{as}\quad x\to\infty.$$

Now,

$$\int_{1/(2x^{1/2})}^{A/x^{1/2}} y^{-1}\left|\int_0^\infty \sin(2uxy)\,d\alpha(u)\right|dy\leqq(\log 2A)\int_0^\infty|d\alpha(u)|=O(1),\quad\text{as}\quad x\to\infty,$$

and so (with $t=2xy$),

$$L_1(x;\alpha)=\int_0^{1/(2x^{1/2})} y^{-1}\left|\int_0^\infty \sin(2uxy)\,d\alpha(u)\right|dy+O(1)=$$

$$=\int_0^{x^{1/2}} t^{-1}\left|\int_0^\infty \sin(tu)\,d\alpha(u)\right|dt+O(1)=\int_1^{x^{1/2}}+\int_0^1+O(1).$$

But

$$0\leqq\int_0^1 t^{-1}\left|\int_0^\infty \sin(tu)\,d\alpha(u)\right|dt\leqq\int_0^1\int_0^\infty u\,|d\alpha(u)|\,dt=O(1).$$

Hence

$$L_1(x;\alpha) = \int_1^{x^{1/2}} t^{-1} \left| \int_0^\infty \sin tu \, d\alpha(u) \right| dt + O(1) \equiv L_1^*(x;\alpha) + O(1), \quad x \to \infty.$$

Thus,

$$\frac{1}{\log x} \left| L(x;\alpha) - \frac{2}{\pi} L_1^*(x;\alpha) \right| \leqq \varphi(A) + O\left(\frac{1}{\log x}\right), \quad \text{as} \quad x \to \infty,$$

and so,

$$\limsup_{x\to\infty} \frac{1}{\log x} \left| L(x;\alpha) - \frac{2}{\pi} L_1^*(x;\alpha) \right| \leqq \varphi(A).$$

Letting $A \to \infty$ completes the proof of the Lemma since $\varphi(A) \to 0$ as $A \to \infty$.

The final major preliminary step is to establish a needed extension of Lemma 2 of [12], namely,

LEMMA 3. *If $\alpha(u)$ is continuous and of bounded variation, $0 \leqq u < \infty$, then*

$$\int_0^x y^{-1} \left| \int_0^\infty \sin yu \, d\alpha(u) \right| dy = o(\log x), \quad x \to \infty. \tag{9. 2}$$

PROOF OF LEMMA 3. As for Lemma 1 (§ 5), proof will be done with the aid of WIENER's theorem ([17]; [18], Theorem (4. 19), p. 261). Let

$$s(y) = \left| \int_0^\infty \sin yu \, d\alpha(u) \right|$$

and define

$$t(x) = x^{-1} \int_1^x s(y)\, dy, \quad v(x) = \int_1^x y^{-1} s(y)\, dy.$$

The Wiener theorem tells us that $t(x) = o(1)$, $x \to \infty$. This will be shown to imply that $v(x) = o(\log x)$, the assertion of Lemma 3. (This inference, incidentally, does not depend on the present definition of $s(y)$.)

We can write

$$v(x) = \int_1^x y^{-1} t(y)\, dy + t(x),$$

since $s(x) = t(x) + xt'(x)$. For $\varepsilon > 0$, we choose N such that $|t(y)| < \varepsilon$ for $y \geqq N$ and so

$$|v(x)| < \int_1^N y^{-1} t(y)\, dy + \varepsilon \log\left(\frac{x}{N}\right) + t(x), \quad x > N.$$

Thus,

$$\limsup_{x\to\infty} \frac{|v(x)|}{\log x} \leqq \varepsilon.$$

and the Lemma is proved.

Theorem 6 follows readily. The argument provided in § 3 of [12] for the proof of Theorem 1 of [12] applies; only obvious rewordings are required. To prove (8. 6), we recall [7] that $\alpha(u)$ is non-decreasing when $[J, \alpha(u)]$ is totally regular. That the Borel exponential means provide the only cases of equality in (8. 6) follows from Lemma 3 of [12].

10. Proof of Theorem 7. Theorem 7 is an immediate consequence of Theorem 2 of [12]. We take the function $g(u)$ constructed in the proof of the latter theorem, $0 \leqq u \leqq 1$, and define $\alpha(u) = g(u)$ for $0 \leqq u \leqq 1$; $\alpha(u) = 1$, for $1 < u < \infty$. Corresponding to formula (23) of [12] we note that

$$L(x; \alpha) \geqq \frac{2}{\pi} \int_0^{\pi x^{1/2}} y^{-1} \left| \int_0^{\infty} \sin yu \, d\alpha(u) \right| dy + O(1),$$

which becomes, for the $\alpha(u)$ just defined,

$$L(x; \alpha) \geqq \frac{2}{\pi} \int_0^{\pi x^{1/2}} y^{-1} \left| \int_0^{1} \sin yu \, d\alpha(u) \right| dy + O(1), \tag{10. 1}$$

to which Theorem 2 of [12] can be applied directly.

11. The Lebesgue constants for A_γ summation. Changing our notation slightly, we denote by $L(x, \gamma)$ the x-th Lebesgue constant corresponding to the A_γ mean of Fourier series, $\gamma > -1$. We prove

THEOREM 8. *As $x \to \infty$, $\lim L(x, \gamma)$, say $L(\gamma)$, exists for each $\gamma > -1$, and*

$$L(\gamma) = \frac{2}{\pi} \int_0^{\pi/2} \frac{|\sin \{\gamma + 1) v\}|}{\sin v} \cos^\gamma v \, dv, \qquad -1 < \gamma < \infty, \tag{11. 1}$$

with

$$L(\gamma) = 1, \qquad -1 < \gamma \leqq 1. \tag{11. 2}$$

Moreover, $L(\gamma)$ is a non-decreasing function of γ, $\gamma > -1$, and

$$L(\gamma) = \frac{2}{\pi^2} \log \gamma + \lambda + O(\gamma^{-1/2}), \quad as \quad \gamma \to \infty, \tag{11. 3}$$

where

$$\lambda = -\frac{2}{\pi^2} C - \frac{2}{\pi^2} \log \frac{1}{2} \pi^2 + 2 \int_0^1 \log \Gamma(t) \cos \pi t \, dt, \tag{11. 4}$$

C being Euler's constant.

From Theorem 8 we obtain at once the following consequence in view of the identification of $L(\gamma)$ above with $L_E(\gamma)$ of [10].

COROLLARY 3. *The Lebesgue constants for Euler $(E, 1)$ summation of Fourier series form a non-decreasing sequence.*

PROOF OF THEOREM 8. The existence of $L(\gamma)$ and its representation (11.1) are established by an argument fashioned after the proofs of Theorems 3 and 4 of [12].

For our present $\alpha(u)$, formula (7. 3) becomes

$$\Gamma(\gamma+1)L(x,\gamma) = \frac{2}{\pi}\int_0^{\pi/2} y^{-1}\left|\int_0^\infty \{\exp[-2uxy^2]\}u^\gamma e^{-u}\sin(2uxy)\,du\right| dy + o(1) =$$

$$= \frac{2}{\pi}\left\{\int_0^{x^{\beta-1}} + \int_{x^{\beta-1}}^{\pi/2}\right\} + o(1) \equiv \frac{2}{\pi}[\Lambda_1(x)+\Lambda_2(x)] + o(1),$$

where $\frac{1}{2} < \beta < 1$.

Now,

$$0 \leqq \Lambda_2(x) \leqq \int_{x^{\beta-1}}^{\pi/2} y^{-1}\int_0^\infty \{\exp[-ux^{2\beta-1}]\}u^\gamma\,du\,dy =$$

$$= \Gamma(\gamma+1)(x^{2\beta-1})^{-\gamma-1}\int_{x^{\beta-1}}^{\pi/2} y^{-1}\,dy = o(1), \qquad x\to\infty.$$

In simplifying $\Lambda_1(x)$ we are led to the following iterated integral, which we transform by integrating the inner integral by parts, to obtain

$$\int_0^{x^{\beta-1}} y^{-1}\left|\int_0^\infty [1-\exp\{-2uxy^2\}]e^{-u}u^\gamma \sin(2uxy)\,du\right| dy =$$

$$= \int_0^{x^{\beta-1}} y^{-1}\left|\int_0^\infty [(\gamma u^{\gamma-1}-u^\gamma)e^{-u}(1-\exp[-2uxy^2]) + \right.$$

$$\left. + 2xy^2\{\exp[-2uxy^2]\}u^\gamma e^{-u}](2xy)^{-1}\cos(2uxy)\,du\right| dy.$$

Using again the inequality $1-e^{-v} \leqq v$, with $v=2uxy^2$, we have

$$0 < \frac{1}{2}x^{-1}\int_0^{x^{\beta-1}} y^{-2}\int_0^\infty (\gamma u^{\gamma-1}+u^\gamma)e^{-u}[1-\exp\{-2uxy^2\}]\,du\,dy \leqq$$

$$\leqq \int_0^{x^{\beta-1}}\int_0^\infty (\gamma u^\gamma + u^{\gamma+1})e^{-u}\,du\,dy = o(1), \qquad x\to\infty.$$

Moreover,

$$\int_0^{x^{\beta-1}}\int_0^\infty \{\exp[-2uxy^2]\}u^\gamma e^{-u}\,du\,dy \leqq \Gamma(\gamma+1)\int_0^{x^{\beta-1}} dy = o(1), \qquad x\to\infty.$$

Hence,

$$L(x,\gamma)=\frac{2}{\pi}[\Gamma(\gamma+1)]^{-1}\int\limits_0^{x^{\beta-1}} y^{-1}\left|\int\limits_0^{\infty} u^\gamma e^{-u}\sin(2uxy)\,du\right| dy+o(1)=$$

$$=\frac{2}{\pi}[\Gamma(\gamma+1)]^{-1}\int\limits_0^{2x^{\beta}} y^{-1}\left|\int\limits_0^{\infty} u^\gamma e^{-u}\sin yu\,du\right| dy+o(1)=$$

$$=\frac{2}{\pi}[\Gamma(\gamma+1)]^{-1}\int\limits_0^{\infty} y^{-1}\left|\int\limits_0^{\infty} u^\gamma e^{-u}\sin yu\,du\right| dy+o(1),$$

as $x\to\infty$, since the last integral exists.

Thus, $L(\gamma)$ exists, as asserted in Theorem 8. It can be written in the form (11. 1) by following the same steps as led to (6. 7).

Equation (11. 2) is an immediate consequence of (11. 1).

To show that $L(\gamma)$ is a non-decreasing function of γ, it suffices to employ the reasoning on p. 297 of [12], and take into account BORWEIN's Lemma 2(i) [2] which shows that the transform taking $\sigma_\lambda(t)$ to $\sigma_\mu(y)$, $\lambda>\mu>-1$, has a non-decreasing weight function.

Formulae (11. 3) and (11. 4) are already known [10], since formula (11. 1) coincides with the expression for $L_E(\gamma)$ studied in [10].

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SUR CERTAINS ENSEMBLES DE SALEM

Par
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Dédié au Professeur G. ALEXITS *pour son 70^e anniversaire*

On dit qu'un sous-ensemble de la droite réelle R est un *ensemble de translation* s'il est infini et si, pour tout $\varepsilon > 0$, il peut s'écrire comme somme algébrique $S+K$ où S est un ensemble fini de pas supérieur à ε (le pas de S est la plus petite distance de points distincts de S), et K est un compact de diamètre inférieur à ε. Cette définition équivaut à celle de [2], p. 18. Tout ensemble de translation est parfait, totalement discontinu, et il porte une mesure naturelle — la L-mesure —, positive et de masse totale 1, qui charge de masses égales deux portions de l'ensemble translatées l'une de l'autre.

SALEM a démontré le théorème suivant ([3], [2], chap. VIII).

Etant donné un nombre α strictement compris entre 0 *et* 1 *il existe un ensemble de translation E, de dimension de Hausdorff α, tel que la L-mesure correspondante $d\mu$ (qui est une mesure singulière) satisfasse $\hat{\mu}(u) = O(u^{-\beta/2})$ $(u \to \infty)$ pour tout $\beta < \alpha$, $\hat{\mu}(u)$ désignant la transformée de Fourier—Stieltjes de $d\mu$.*

La démonstration de Salem est probabiliste et non constructive. Le but de cet article est d'étendre le théorème au cas $\alpha = 1$, en donnant alors une construction explicite de l'ensemble E ou ce qui revient au même, de la mesure $d\mu$.

Soit n_j une suite strictement croissante d'entiers $\geqq 0$ $(j = 0, 1, 2, \ldots, ;\ n_0 = 0)$. Posons

$$p_j(u) = \prod_{n=n_j+1}^{n_{j+1}} \cos \frac{u}{2^n}.$$

Ainsi

$$\prod_{j=k}^{\infty} p_j(u) = S\left(\frac{u}{2^{n_k}}\right)$$

avec

$$S(u) = \prod_{n=1}^{\infty} \cos \frac{u}{2^n} = \frac{\sin u}{u}.$$

On a aussi

$$p_j(u) = 2^{-(n_{j+1}-n_j)} \sum_{m \text{ impair},\ |m| \leqq 2^{n_{j+1}-n_j}} \exp \frac{imu}{2^{n_{j+1}}}.$$

Selon une construction de RUDIN et SHAPIRO ([2], p. 134), il existe une constante absolue C et un choix des signes $+$ et $-$ tels que le polynôme trigonométrique

$$q_j(u) = 2^{-(n_{j+1}-n_j)} \sum_{m \text{ impair},\ |m| \leqq 2^{n_{j+1}-n_j}} \pm \exp \frac{imu}{2^{n_{j+1}}}$$

satisfasse $q_j(0) = 0$, et

$$|q_j(u)| \leqq C 2^{-\frac{1}{2}(n_{j+1}-n_j)}.$$

Posons

$$\hat{\mu}(u) = \prod_{j=0}^{\infty} (p_j(u) + q_j(u)).$$

Comme $p_j(u) + q_j(u)$ est la transformée de Fourier—Stieltjes d'une mesure positive $d\mu_j$, de masse totale 1, portée par la moitié des points de la forme $\frac{m}{2^{n_{j+1}}}$ (m impair $|m| \leqq 2^{n_{j+1}-n_j}$), $\hat{\mu}(u)$ est la transformée de Fourier—Stieltjes d'une mesure $d\mu$ qu, est la convolution de toutes les mesures $d\mu_j$. On vérifie facilement que $d\mu$ est li L-mesure d'un ensemble de translation, de mesure de Lebesgue nulle. D'autre paar

$$|\hat{\mu}(u)| \leqq \sum_{k=1}^{\infty} \prod_{j=0}^{k-1} |p_j(u) + q_j(u)|\, |q_k(u)| \prod_{j=k+1}^{\infty} |p_j(u)| \leqq C \sum_{k=1}^{\infty} 2^{-\frac{1}{2}(n_{k+1}-n_k)} \left| S\left(\frac{u}{2^{n_{k+1}}}\right)\right|.$$

Supposons $n_{k+1} \geqq 2n_k$. En décomposant la somme $\sum_{k=1}^{\infty}$ en $\sum_{k=1}^{\nu-1} + \sum_{k=\nu}^{\infty}$ et en tenant compte des inégalités $|S(u)| \leqq 1$ et $|S(u)| \leqq \frac{1}{u}$ $(u > 0)$, on obtient

$$|\hat{\mu}(u)| \leqq C' \left(\frac{1}{u} 2^{\frac{1}{2}(n_\nu + n_{\nu-1})} + 2^{\frac{1}{2}(n_\nu - n_{\nu+1})} \right),$$

où C' est une constante absolue, et ν un entier arbitraire. Choisissons ν de sorte qut

$$2^{n_\nu - n_{\nu-1}} \leqq u < 2^{n_{\nu+1} - n_\nu}$$

(ce qui est possible pour u assez grand). Alors

$$|\hat{\mu}(u)| \sqrt{u} \leqq C'(2^{n_{\nu-1}} + 1).$$

Soit $\omega(u)$ une fonction positive tendant vers l'infini quand $u \to \infty$. On peut choisir la suite n_j de façon que

$$u \geqq 2^{n_\nu - n_{\nu-1}} \Rightarrow \omega(u) > 2^{n_{\nu-1}}.$$

Alors $\sqrt{u}\, |\hat{\mu}(u)| \leqq 2\omega(u)$. Cette majoration, lorsque $\omega(u)$ croit moins vite que toute puissance de u, entraîne d'ailleurs que la dimension de Hausdorff du support de $d\mu$ est égale à 1 ([2], p. 34). Enonçons le résultat

Etant donné une fonction positive $\omega(u)$ de $u > 0$, tendant vers l'infini quand $u \to \infty$, il existe un ensemble de translation E, de mesure de Lebesgue nulle, tel que la L-mesure $d\mu$ ayant E pour support satisfasse $\hat{\mu}(u) = O\left(\frac{\omega(u)}{\sqrt{u}}\right)$ $(u \to \infty)$.

Si l'on renonce à l'exigence que E soit un ensemble de translation, on a de meilleures majorations possibles pour $\hat{\mu}(u)$ (voir [1]).

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UN EXEMPLE DANS LA THÉORIE DU PROLONGEMENT ANALYTIQUE D'UNE SÉRIE DE DIRICHLET

Par

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Dédié au professeur G. ALEXITS *pour son soixante-dixième anniversaire*

Rappelons d'abord le théorème suivant:[1]

THÉORÈME A. *Soit* $\{\lambda_n\}$ *une suite* $0<\lambda_1<\lambda_2<\cdots$, $\lambda_n\to\infty$, *avec*

(1) $$\liminf(\lambda_{n+1}-\lambda_n)=h>0,$$

(2) $$\limsup\frac{n}{\lambda_n}=D^{\bullet},$$

et supposons que la série

(3) $$f(s)=\sum a_n e^{-\lambda_n s}$$

admet une abscisse de convergence finie σ_c:

(4) $$-\infty<\sigma_c=\limsup\frac{\log|a_n|}{\lambda_n}<\infty.$$

Il existe alors pour $\alpha>0$, $\beta\geqq 0$ *une fonction continue* $A(\alpha,\beta)$, *avec* $A(\alpha,0)=0$, *telle que, quelle que soit la valeur réelle de* t_0, *la fonction* $f(s)$ *admet un point singulier situé sur le rectangle* ($s=\sigma+it$):

(5) $$\sigma_c-A(h,D^{\bullet})\leqq\sigma\leqq\sigma_c,\qquad |t-t_0|\leqq\pi D^{\bullet}.$$

On peut prendre pour $A(\alpha,\beta)$ *la fonction suivante:*

(6) $$A(\alpha,\beta)=\pi\beta-[3\log(\alpha\beta)-9]\beta,\quad \textit{pour}\quad \beta>0,$$
$$A(\alpha,0)=0.$$

Dans ce théorème on désigne évidemment par la même lettre la série de Dirichlet et son prolongement analytique direct. Par le prolongement direct de la série dans un domaine Δ contenant le demi-plan de convergence on entend une fonction qui est holomorphe dans Δ et qui est égale à la série dans le demi-plan de convergence.

Dans le théorème précédent on peut choisir pour $A(\alpha,\beta)$ ($\beta>0$) toute fonction de la forme

(7) $$A(\alpha,\beta)=\pi\beta+[A(a)-2a\log(\alpha\beta)]\beta=\pi\beta+L(a,\alpha,\beta),$$

$A(a)$ étant la fonction

(8) $$A(a)=a\{4+a+\log[(a^2-1)/a^3]+\log[(a+1)/(a-1)]/a\}.$$

[1] Pour la démonstration de ce théorème voir, par exemple, S. Mandelbrojt: Séries adhérentes. Régularisation des suites. Applications. Gauthier-Villars, 1952.

Dans $L(a, h, D^{\bullet})$, et par conséquent aussi dans l'expression que nous venons d'indiquer pour $A(h, D^{\bullet})$, intervient $\log(hD^{\bullet})$. Comme $hD^{\bullet} \leqq 1$, l'expression $-2aD^{\bullet}\log(hD^{\bullet})$ est positive, et pour h fixe (par exemple $h=1$) tend vers zéro, lorsque $D^{\bullet}$ tend vers zéro, comme $-2aD^{\bullet}\log D^{\bullet}$. C'est donc de la même manière que les expressions $L(a, h, D^{\bullet})$ et $A(h, D^{\bullet})$ tendent vers zéro lorsque $D^{\bullet}$ tend vers zéro, pour h fixe. Nous désirons démontrer que cette rapidité des expressions indiquées de tendre vers zéro est très caractéristique. On a même nécessairement, lorsque h est fixe:

$$\liminf_{D^{\bullet}\to 0} A(hD^{\bullet})/(-D^{\bullet}\log D^{\bullet}) = \liminf_{D^{\bullet}\to 0} L(a, h, D^{\bullet})/(-D^{\bullet}\log D^{\bullet}) \geqq 1.$$

Autrement dit, dans l'expression de $A(\alpha, \beta)$ intervenant *dans l'énoncé du théorème A le nombre* 3 *ne peut certainement pas être remplacé par un nombre inférieur à un.* Il est donc, à plus forte raison, impossible de prendre pour $A(h, D^{\bullet})$ (h fixe) une expression de la forme $cD^{\bullet}$, où c est indépendant de $D^{\bullet}$.

Nous pouvons, en effet, démontrer le théorème suivant:

THÉORÈME B. *Soit p un entier $p \geqq 2$, et soit $\{\mu_n\}$ une suite d'entiers positifs satisfaisant aux conditions*

$$\mu_{n+1}/\mu_n > p/(p-1), \quad \lim \Big(\sum_{k\leqq n} \mu_k\Big)\Big/\mu_{n+1} = 0. \tag{9}$$

Soit $\{b_n\}$ une suite telle que

$$\limsup |b_n|^{1/n} = 1. \tag{10}$$

La série

$$\sum b_n \left(\frac{e^{-ps} - e^{-(p-1)s}}{2}\right)^{\mu_n} \tag{11}$$

converge pour $\sigma > 0$, elle y représente une fonction holomorphe $f(s)$ qui est aussi représentée dans ce demi-plan par une série de la forme (3) *possédant les propriétés suivantes:*

1°) *les λ_n sont entiers, $h=1$ (h défini par* (1))

2°) $\sigma_c = 0$

3°) $D^{\bullet} = 1/p$ ($D^{\bullet}$ *défini par* (2))

4°) *le prolongement analytique $f(s)$ de la série est holomorphe sur la demi-bande $|t| \leqq \pi D^{\bullet}$, $\sigma \geqq -\mu D^{\bullet}$, où μ est la racine positive de*

$$D^{\bullet 2} e^{2\mu}(\mu^2 + \pi^2) = 4. \tag{12}$$

Lorsque p tend vers l'infini, la quantité μ ainsi définie satisfait à la relation

$$\mu = -\log D^{\bullet} - \log(-\log D^{\bullet}) + \log 2 + O(1), \qquad (D^{\bullet} = 1/p \to 0).$$

On voit ainsi que, quel que soit le choix de $A(\alpha, \beta)$ correspondant à l'énoncé du théorème A, on a lorsque $D^{\bullet}$ tend vers zéro:

$$A(1, D^{\bullet}) \geqq -[\log D^{\bullet} + \log(-\log D^{\bullet}) - \log 2 + O(1)]D^{\bullet} \sim -D^{\bullet}\log D^{\bullet}. \tag{13}$$

DÉMONSTRATION DU THÉORÈME B. 1°) et 2°) sont évidents: il suffit de raisonner sur la série

$$\sum b_n[(z^p - z^{p-1})/2]^{\mu_n} \tag{14}$$

en tenant compte de (10).

Pour voir 3°), remarquons qu'en désignant par $N(x)$ le nombre de λ_n non supérieurs à x, on a

$$N(p\mu_n) = \sum_{k \leqq n} (\mu_k + 1) = n + \sum_{k \leqq n} \mu_k,$$

donc

$$\frac{N(p\mu_n)}{p\mu_n} = \frac{n}{p\mu_n} + \frac{1}{p} + \sum_{k \leqq n-1} \mu_k (p\mu_n)^{-1}.$$

Et comme $\lim n/\mu_n = 0$, on a

$$D^* \geqq \limsup N(p\mu_n)/p\mu_n = \frac{1}{p}.$$

On a aussi pour $p\mu_{n-1} < \lambda_k \leqq p\mu_n$ $(n \geqq 2)$

$$\frac{k}{\lambda_k} \leqq \frac{N(p\mu_n)}{p\mu_n},$$

et, par conséquent:

$$D^* = \limsup \frac{k}{\lambda_k} = \frac{1}{p}.$$

Démontrons maintenant 4°). Remarquons d'abord que pour $0 < D < 1$ $\mu > 0: \cos \pi D > 1 - (\pi D^2)/2$, $1 - e^{-\mu D} < \mu D$; on peut donc écrire:

$$e^{2\mu}(1 + e^{-2\mu D} - 2e^{-\mu D} \cos \pi D) < e^{2\mu}[1 + e^{-2\mu D} - 2e^{-\mu D} + e^{-\mu D}(\pi D)^2] <$$
$$< e^{2\mu}[(1 - e^{-\mu D})^2 + (\pi D)^2] < e^{2\mu}(\mu^2 + \pi^2) D^2. \tag{15}$$

Si

$$s \in S = \{s \,|\, |t| \leqq \pi D^*, \ -\mu D^* \leqq \sigma \leqq 0\},$$

on a

$$|e^{-ps} - e^{-(p-1)s}| = e^{-(p-1)\sigma}|e^s - 1| \leqq e^{\mu(p-1)D^*}(e^{2\mu D^*} + 1 - 2e^{\mu D^*} \cos \pi D^*)^{\frac{1}{2}} =$$
$$= e^{\mu(1-D^*)}(e^{2\mu D^*} - 2e^{\mu D^*} \cos \pi D^* + 1)^{\frac{1}{2}},$$

c'est-à-dire, d'après (15),

$$|e^{-ps} - e^{-(p-1)s}|^2 \leqq e^{2\mu}(e^{-2\mu D^*} - 2e^{-\mu D^*} \cos \pi D^* + 1) < e^{2\mu}(\mu^2 + \pi^2) D^{*2},$$

et, si μ est une racine positive de (12), on a

$$\left| \frac{e^{-ps} - e^{-(p-1)s}}{2} \right| < 1.$$

La fonction est donc, d'après (10), holomorphe sur S.

Posons

$$\mu = -\log D^* - \log(-\log D^*) + \log 2 + c(D^*)$$

et démontrons que $c = c(D^{\bullet}) = O(1)$ lorsque $D^{\bullet} \to 0$. On a, en effet:

$$\frac{4}{D^{\bullet 2}} = e^{2\mu}(\mu^2 + \pi^2) =$$

$$= \frac{4e^{2c}}{D^{\bullet 2}(\log D^{\bullet})^2}[-\log D^{\bullet} - \log(-\log D^{\bullet}) + \log 2 + c]^2 \frac{\mu^2 + \pi^2}{\mu^2} \sim \frac{4e^{2c}}{D^{\bullet 2}},$$

ce qui prouve que $c(D^{\bullet}) = O(1)$. Le théorème est ainsi démontré.

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ON THE SATURATION THEOREM FOR THE CESARO MEANS OF FOURIER SERIES

By

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To Professor G. ALEXITS *on the occasion of his 70th birthday*

1. Let $f(x)$ be a real-valued, 2π-periodic integrable function on the real line. We denote its Fourier series by

$$f(x) \sim \frac{1}{2} a_0 + \sum_{k=1}^{\infty} A_k(x), \quad A_k(x) = a_k \cos kx + b_k \sin kx,$$

a_k and b_k are its Fourier coefficients. Its conjugate series is defined by

$$f^{\sim}(x) \sim \sum_{k=1}^{\infty} B_k(x), \quad B_k(x) = a_k \sin kx - b_k \cos kx.$$

In 1941 G. ALEXITS [1] proved a result in the theory of approximation which is known as the saturation theorem for the Fejér means of Fourier series of continuous functions: *Let*

$$\sigma_n(f; x) = \frac{1}{2} a_0 + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) A_k(x), \qquad n = 0, 1, 2, \dots,$$

be the Fejér means of a function f in $C_{2\pi}$.

(a) *If for* $f, g \in C_{2\pi}$

$$\lim_{n \to \infty} \|(n+1)\{\sigma_n(f; \cdot) - f(\cdot)\} - g(\cdot)\|_{C_{2\pi}} = 0,$$

then $f^{\sim} \in AC_{2\pi}$ *and* $f^{\sim\prime} = -g$, *and vice versa. If, in particular, g vanishes, then f is a constant function.*

(b) *Necessary and sufficient for an* $f \in C_{2\pi}$ *to satisfy*

$$\|\sigma_n(f; \cdot) - f(\cdot)\|_{C_{2\pi}} = O\left(\frac{1}{n+1}\right) \qquad (n \to \infty)$$

s that $f^{\sim} \in \mathrm{Lip}\,(1; C_{2\pi})$.

The so-called "o"-theorem, part (a) with $g = 0$, already goes back to S. Bernstein, 1931. In this formulation, part (a) is due to P. L. BUTZER—E. GŐRLICH [4]. Alexits' result, part (b), was at that time surprising, and might be considered as the beginning of what is now known as the saturation theory for summation processes of Fourier series. For the sake of completeness we have to mention that in 1947 J. Favard rediscovered and reformulated this phenomenon for general summation processes of Fourier series. The first systematic study of the saturation problem for polynomial processes is due to M. Zamansky, 1949; this was followed by the abstract saturation theorem of P. L. Butzer for semi-groups of operators on Banach spaces and its applications, 1956/57, as well as the two papers of G. Sunouchi—C. Watari on the subject, 1958/59. For detailed information we refer to P. L. Butzer—E. Gőrlich, *loc. cit.*

The above theorem has been generalized in two directions. First, the theorem remains true for the $L^p_{2\pi}$-spaces, $1 \leq p < \infty$. Secondly, the Fejér means can be replaced by the (C, α)-means, $\alpha > 0$. In this case, the order of approximation is

$$a_n^\alpha = \sum_{k=n+1}^{\infty} \frac{\alpha}{k(k+1)} \sim \frac{\alpha}{n} \qquad (n \to \infty).$$

(For $\alpha = 1$, Fejér's process, $a_n^1 = 1/(n+1)$.) See again the paper of P. L. Butzer—E. Gőrlich. Moreover, local versions of these saturation theorems, i. e., statements on given subintervals (a, b) in $[-\pi, \pi)$, are due to G. SUNOUCHI [6].

Recently, V. A. ANDRIENKO [2] published a pointwise "o"-theorem for the Fejér means. He proved the following: *Let $f \in L_{2\pi}$, be finite and such that $\sigma_n(f; x)$ converges to $f(x)$ as $n \to \infty$ for all x in some interval (a, b). If*

$$\lim_{n \to \infty} (n+1)\{\sigma_n(f; x) - f(x)\} = 0 \tag{1}$$

for all $x \in (a, b)$, then for almost all x, $f^{\sim}(x)$ is a constant in (a, b). If, in particular, f is continuous then the result remains true even if (1) *is violated in a denumerable set of points.*

Andrienko's result considerably weakens the "o"-theorem for the Fejér means, in particular, for continuous functions. His proof is based on Verblunsky's uniqueness theorem for Abel summable trigonometric series.

It is the aim of this paper to generalize the result of V. A. Andrienko for the (C, α)-means. Moreover, we replace the zero function on the right-hand side of (1) by a finitely-valued, integrable function $g(x)$, defined in (a, b), to get a characterization of $f^{\sim}(x)$ via $g(x)$ on this interval. Thus, from a certain point of view, this problem can be considered as the converse of the Voronovskaya-type problem for the (C, α)-means, once again see P. L. Butzer—E. Gőrlich, *loc. cit.*, for details. As in Andrienko's proof, we reduce our proof to Verblunsky's theorem, mentioned above. Our reduction, however, is different, and the method is of general interest in connection with the approximation theorems for the (C, α)-means of Fourier series.

2. For a function $f(x)$ in $L_{2\pi}$ the (C, α)-means of its Fourier series are defined by

$$[\sigma_n^\alpha f](x) = \frac{1}{2} a_0 + \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^\alpha A_k(x) \qquad (n = 0, 1, 2, \dots;\ \alpha > -1),$$

where $A_n^\alpha = \binom{n+\alpha}{n}$. Obviously, $[\sigma_n^1 f](x) = \sigma_n(f; x)$.

THEOREM 1. *Let $f \in L_{2\pi}$, and $\alpha > 0$. For all x for which $\lim_{n \to \infty} [\sigma_n^\alpha f](x) = c(x)$ exists finitely, we have that the sequence*

$$\frac{[\sigma_n^\alpha f](x) - c(x)}{a_n^\alpha}, \quad a_n^\alpha = \sum_{k=n+1}^{\infty} \frac{\alpha}{k(k+\alpha)},$$

converges as $n \to \infty$ if, and only if,

$$\lim_{n \to \infty} -[\sigma_n^\alpha f]^{\sim\prime}(x)$$

exists. Moreover, the limits are equal.

For the proof of the theorem we need the following lemma on numerically-valued series.

LEMMA. *Let $u_0+u_1+u_2+\cdots$ be a real-valued series, and let the summation process τ_n^α of Σu_k, given by*

$$\tau_n^\alpha = \frac{1}{a_n^\alpha} \sum_{k=n+1}^{\infty} \frac{\alpha}{k(k+\alpha)} \sigma_k^{\alpha-1} \qquad (n = 0, 1, 2, \dots; \ \alpha > 0),$$

be well-defined. The series Σu_k is (C, α)-summable to c if, and only if, τ_n^α converges as $n \to \infty$, its limit is also c.

PROOF. It is convenient to introduce the notations

$$s_n^\alpha = \sum_{k=0}^{n} A_{n-k}^\alpha u_k \quad \text{and} \quad t_n^\alpha = \sum_{k=n+1}^{\infty} \frac{\alpha}{k(k+\alpha)} \sigma_k^{\alpha-1}.$$

Suppose, $\sigma_n^\alpha \to c$ as $n \to \infty$. Taking into account that $s_n^\alpha = \sum_{k=0}^{n} s_k^{\alpha-1}$, we obtain by summation by parts

$$t_n^\alpha = \sum_{k=n+1}^{\infty} \frac{1}{kA_k^\alpha} s_k^{\alpha-1} = -\frac{s_n^\alpha}{(n+1)A_{n+1}^\alpha} + \sum_{k=n+1}^{\infty} \frac{1}{kA_k^{\alpha+1}} s_k^\alpha,$$

or

$$\tau_n^\alpha = -\frac{1}{(n+\alpha+1)a_n^\alpha} \sigma_n^\alpha + \frac{a_n^{\alpha+1}}{a_n^\alpha} \tau_n^{\alpha+1}. \tag{2}$$

Since

$$a_n^\alpha = -\frac{1}{n+\alpha+1} + a_n^{\alpha+1}, \quad a_n^\alpha \sim \frac{\alpha}{n} \quad (n \to \infty),$$

and since, trivially, $\sigma_n^\alpha \to c$ as $n \to \infty$ implies $\tau_n^{\alpha+1} \to c$ as $n \to \infty$, we immediately conclude from (2) that also $\tau_n^\alpha \to c$ as $n \to \infty$. Conversely, let $\tau_n^\alpha \to c$ as $n \to \infty$. With $s_0^{\alpha-1} = t_{-1}^\alpha = u_0$ and

$$s_n^{\alpha-1} = nA_n^\alpha \{t_{n-1}^\alpha - t_n^\alpha\}, \qquad n = 1, 2, \dots,$$

we obtain, again by partial summation,

$$s_n^\alpha = \sum_{k=0}^{n} s_k^{\alpha-1} = u_0 - nA_n^\alpha t_n^\alpha + (1+\alpha) \sum_{k=0}^{n-1} A_k^\alpha t_k^\alpha$$

or

$$\sigma_n^\alpha = \frac{u_0}{A_n^\alpha} - na_n^\alpha \tau_n^\alpha + \frac{(1+\alpha)}{A_n^\alpha} \sum_{k=0}^{n-1} A_k^\alpha a_k^\alpha \tau_k^\alpha. \tag{3}$$

By use of the identity

$$A_n^\alpha = 1 - nA_n^\alpha a_n^\alpha + (1+\alpha) \sum_{k=0}^{n-1} A_k^\alpha a_k^\alpha,$$

equation (3) implies that $\sigma_n^\alpha \to c$ as $n \to \infty$.

To prove Theorem 1, it is enough to remark that

$$(4) \qquad [\sigma_n^\alpha f](x)-c(x) = \sum_{k=n+1}^{\infty} \{[\sigma_{k-1}^\alpha f](x)-[\sigma_k^\alpha f](x)\} =$$

$$= -\sum_{k=n+1}^{\infty} \frac{\alpha}{k(k+\alpha)} [\sigma_k^{\alpha-1} f]^{\sim\prime}(x).$$

Replacing the series Σu_k in the lemma by $-\Sigma k A_k(x)$, and we have the assertion of the theorem.

Relations of the type (4) are not new (cf. [8, p. 269]), they occur quite regularly in connection with characterizations of function classes with respect to a given summation method. For a recent paper in this respect see P.L. BUTZER—S. PAWELKE [5].

The next theorem states the main result of the paper.

THEOREM 2. *Let $f\in L_{2\pi}$, $\alpha>0$, and let $f(x)$ be finite in some interval (a, b) and such that $\lim_{n\to\infty} [\sigma_n^\alpha f](x)=f(x)$ for all x in (a, b). If*

$$(5) \qquad \lim_{n\to\infty} \frac{[\sigma_n^\alpha f](x)-f(x)}{a_n^\alpha} = g(x)$$

exists finitely for all $x\in(a, b)$, with $g(x)$ integrable, then for almost all x in (a, b)

$$(6) \qquad f^{\sim}(x) = C - \int_a^x g(u)\,du, \quad C \text{ is some constant.*}$$

If, in particular, f is in $C_{2\pi}$, then (6) remains true even if (5) is violated in a denumerable set of points.

The theorem reduces to Andrienko's result in case $\alpha=1$ and $g(x)=0$ in (a, b). Its proof is a consequence of Theorem 1 and Verblunsky's theorem, which we shall formulate in the following specialized form: *Let $f\in L_{2\pi}$. If the limit*

$$(7) \qquad \lim_{r\to 1-} -f^{\sim\prime}(r;x) = g(x), \quad f(r;x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} r^k A_k(x) \quad (0 \leqq r < 1),$$

exists finitely for all x in some interval (a, b) with $g(x)$ integrable, then for almost all x in (a, b)

$$f^{\sim}(x) = C - \int_a^x g(u)\,du.$$

The assertion remains true, even if (7) does not hold in a denumerable set of points in (a, b), supposing that

$$(1-r)f^{\sim\prime}(r;x) = o(1) \quad (r\to 1^-)$$

at these points.

* If the conjugate function $f^{\sim}(x)$ is defined by $\lim_{r\to 1-} f^{\sim}(r;x)$, $f^{\sim}(r;x)=\Sigma r^k B_k(x)$ $(0\leqq r<1)$, whenever it exists, then the relation (6) holds for all x in (a, b).

Indeed, by Theorem 1 condition (5) implies that the series

$$-\sum_{1}^{\infty} kA_k(x)$$

is (C, α)-summable to $g(x)$ in (a, b) which in turn implies Abel summability, i. e., (7) holds true. Verblunsky's theorem then establishes the first part of the theorem.

If the function f belongs to $C_{2\pi}$, then it is known (cf. [3, p. 235] or [8, p. 263]) that

$$\|f^{\sim\prime}(r; \cdot)\|_{C_{2\pi}} = o[(1-r)^{-1}] \quad (r \to 1^-),$$

proving the second part of the theorem.

The given form of Verblunsky's theorem can be easily derived from Theorem 8. 1 in [8, p. 358]. A version quite similar to that above can be found in F. WOLF [7, Theorem 60].

As a final result we can conclude out of the saturation theorem, Theorem 1, and M. Riesz' theorem for (C, α)-summability (cf. [8, p. 94])

THEOREM 3. *Let $f \in L_{2\pi}$. If $f^{\sim}$ is in $AC_{2\pi}$, then for almost all x*

$$\lim_{n\to\infty} \frac{[\sigma_n^\alpha f](x) - f(x)}{a_n^\alpha} = -f^{\sim\prime}(x).$$

More generally, if

$$\|\sigma_n^\alpha f - f\|_{L_{2\pi}} = O(a_n^\alpha) \quad (n \to \infty),$$

then for almost all x the limit

$$\lim_{n\to\infty} \frac{[\sigma_n^\alpha f](x) - f(x)}{a_n^\alpha}$$

exists.

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ON A SPECIAL KIND OF POLYHEDRA

By

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Dedicated to G. Alexits *on his 70th anniversary*

1. In the Euclidean 3-space a closed non-empty set of points with connected interior is said to be a polyhedron if it is the union of a finite set of tetrahedra. A polihedron P is bounded by a finite number of polygons (defined analogously in the plane), the *faces* of P, no two of which form one polygon. The sides and vertices of these polygons are called *edges* and *vertices* of P, respectively. If P is homeomorphic to a ball, it is said to be *simple*.

We define the *support-set* of P as the set of those points of P which are contained in a support plane of P. Intuitively speaking, the support-set of P consists of those points which will get dirty by rolling P topsyturvy on a dirty plane. If C denotes the convex hull of P and $\bar{C}$ the boundary of C then $S = P \cap \bar{C}$ is the support-set of P.

Clearly, the support-set consists of some vertices, edges and faces of P. All vertices of C belong to S, since all these are vertices of P as well. Consequently, there are always vertices of P which belong to S. Erecting upon each face of a regular tetrahedron a sufficiently spiky tetrahedron we obtain a polyhedron whose support-set consists of some vertices of the polyhedron and contains none of its edges or faces.

In the following we shall consider polyhedra whose all vertices belong to their support-sets and study the question what else does necessarily belong to the support-set of such a polyhedron.

We show by an example that *there are simple polyhedra with all their vertices and none of their faces in the support-set of the polyhedron.* Let us consider an icosahedron I with two opposite vertices A and B. The 20 faces of I form 10 pairs of faces, each of which consists of a triangle with a vertex in A or B and an adjacent triangle disjoint from A and B. If we remove from I the tetrahedra defined as the convex hulls of the pairs of triangles we mentioned, we get a polyhedron of required property.

The next question which arises is whether a polyhedron exists with all the vertices but nothing else in its support-set. By other words: is there a polyhedron such that rolling it on a dirty plane each of its vertices but none of its edges and faces gets dirty? This is the problem we want to deal with in this paper and we shall answer it affirmatively by the following

Theorem. *There exists a simple polyhedron whose support-set is identical with the set of its vertices.*

2. First we construct an unbounded polyhedral body of required property inscribed into a cylinder. We choose two relatively prime integers $a>1$, $b>1$. We consider a cylinder and an infinite sequence of its equidistant parallel circles. We cut them orthogonally by equidistant generators the number of which is divisible by $a+b$. We number these generators by $1, 2, \ldots, n(a+b)$ considered as representatives of residue classes mod $n(a+b)$. We number the parallel circles by $\ldots, -2, -1, 0, 1, 2, \ldots$. The points of intersection of circle k and generators

$$ka,\ ka+(a+b),\ \ldots,\ ka+(n-1)(a+b)$$

define a regular n-gon inscribed in this circle. We construct the convex hulls of each pair of such n-gons inscribed in two adjacent circles k and $k+1$. The union of all these convex hulls is the polyhedric body P we wanted to construct.

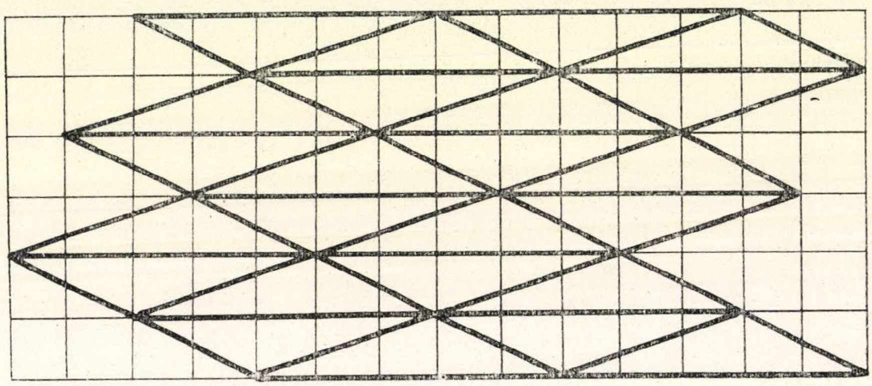

Fig. 1.

The surface of P consists of congruent triangles. Fig. 1 shows the planar map of a part of this surface in case $a=2$, $b=3$. In this map parallel circles and generators are represented by horizontal and vertical lines, respectively. Owing to $(a, b)=1$, P has an infinite sequence of vertices on each numbered generator. Consequently, the convex hull of P is an $n(a+b)$-sided regular prism, the boundary of which does not contain any edge of P, owing to $a>1$, $b>1$, while all the vertices of P are on this boundary. These are just the properties we had to prove.

3. We may easily get further examples of the same kind by taking only those vertices of the above example which are on a given segment of the cylinder, i. e. intercepted by two given generators. This segment should be broad enough, such that from each regular n-gon $m \geqq 3$ vertices should belong to the segment. The convex hull of these m vertices plays now the role of the regular n-gon. In what follows we consider the case where from each n-gon exactly 3 vertices are contained in the segment. Consequently, the polyhedral body we obtain is the union of octahedra, namely the convex hulls of two triangles inscribed in adjacent parallel circles. It is easily seen that the properties we are interested in and their proofs remain unaltered.

We start now from our last example and bend the cylinder until it becomes a torus. This bending transforms the generators and parallel circles of the cylinder into parallel and meridian circles of the torus, respectively. We consider the case when the vertices of the body we constructed inscribed into the cylinder are transported into a finite number of points of the "outer part" of the torus, and the transformation is such that neighbouring vertices of a generator are mapped into neighbouring vertices of the corresponding meridian circle. Consequently, these vertices define, if connected as before the bending, a polyhedron P whose support-set contains all its vertices. If the meridian circles were chosen dense enough, a segment of the torus intercepted by two meridian circles and containing a given number of meridian circles, is near enough to a cylinder and according to the case of our previous example none of the edges of P lays on the support-set of P. This completes the proof that the support-set of P is identical with the set of its vertices.

4. We answered our question affirmatively, but first by an unbounded polyhedral body, then by a polyhedron which is not simple. We finish by proving that the answer remains affirmative even if restricting to simple polyhedra.

We start from a polyhedron P we constructed inscribed in a torus. This is the union of octahedra. We remove one octahedron from P and obtain a simple polyhedron P_S.

Since we did not remove any vertex, the convex hulls of P and P_S are identical. Consequently, all vertices of P_S are on the boundary of its convex hull, while all edges of P_S run in the interior of the convex hull. This proves that P_S is of the required property and completes the proof of our theorem.

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DEPARTURE FROM INDEPENDENCE: THE STRONG LAW, STANDARD AND RANDOM-SUM CENTRAL LIMIT THEOREMS

By

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In memory of Professor ALFRÉD RÉNYI

1. Introduction. Most of the research in probability dealing with sequences of dependent random variables involves the specification of a fixed degree of departure from independence (e.g. absolute fairness, Markov dependence, m-dependence etc.). However, there seems to be no systematic investigation of exactly how far one can deviate from independence and still have the usual theorems holding.

One way of attacking the problem of allowable departure from independence is to use known results about absolutely fair sequences of random variables. Investigation of the difference between an arbitrary sequence and its related absolutely fair sequence will be shown to give information about the original sequence, and the permissible degree of non-independence.

Let $\{X_j\}$ be a sequence of random variables with zero mean on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and let us write

$$X_j = Y_j + W_j, \qquad j = 1, 2, \dots, \tag{1.1}$$

where

$$Y_j = X_j - \mathbf{E}(X_j | X_1, \dots, X_{j-1}),$$

and

$$W_j = \mathbf{E}(X_j | X_1, \dots, X_{j-1}),$$

with $X_0 = 0$. Then the sequence $\{Y_j\}$ has the property

$$\mathbf{E}(Y_j | X_1, \dots, X_{j-1}) = 0, \qquad j = 1, 2, \dots, \tag{1.2}$$

with probability 1, which in turn implies that the Y_j are absolutely fair ([6]); that is to say

$$\mathbf{E}(Y_j | Y_1, \dots, Y_{j-1}) = 0, \qquad j = 1, 2, \dots, \tag{1.3}$$

with probability 1.

It is known that the central limit theorem ([8], [5]) and the random-sum central limit theorem ([4]) hold for partial sums of absolutely fair random variables, under appropriate further restrictions, if one requires that the sequence $\{Y_j\}$ be like independent random variables in the sense that

$$\mathbf{E}(Y_j^2 | Y_1, \dots, Y_{j-1}) = 1, \qquad j = 1, 2, \dots, \tag{1.4}$$

with probability 1. In fact the right hand side can be replaced by σ_j^2 constant ([5], [4]), but to avoid complications we use the above form.

The purpose of this paper is to consider the strong law, the central limit theorem, the mixing of partial sums and the random-sum central limit theorem for the original sequence $\{X_j\}$.

First we approach these problems assuming a basic independence-like condition on the sequence $\{X_j\}$ which results in (1. 4) on the absolutely fair part $\{Y_j\}$. Proceeding from this viewpoint we prove various propositions under some relaxation of independence conditions involving only the sequence $\{W_j\}$.

The second approach starts out from a different point of view. Namely, the basic assumption is now that the normalized partial sums of the sequence $\{X_j\}$ are mixing. In this case it is shown that we need consider independence-like condiions involving only the sequence $\{W_j\}$.

2. Preliminaries. We first state some definitions and results that will be needed tin the sequel. All limits are taken as n approaches infinity.

LEMMA 1 (page 254 of [3]). *Let $\{u_n\}$ be a sequence of random variables which converge in distribution to the distribution function F, and suppose $\{v_n\}$ is a sequence of random variables which converge to zero in probability. Then $\{u_n+v_n\}$ converges to F in distribution.*

THEOREM A. *Let $\{Y_j\}$ be an absolutely fair sequence of random variables satisfying also the condition* (1. 4) *and Lindeberg's condition. Let $T_n=\sum_1^n Y_j$. Then*

$$\mathbf{P}(T_n/\sqrt{n}\leqq x)\to\Phi(x),$$

where Φ is the standard normal distribution function.

This result is due to P. LÉVY [8] and a proof may be also found on page 383 of [5].

DEFINITION 1. A sequence of random variables $\{\eta_n\}$ is called mixing with the distribution function F if for every event A in $\mathscr{A}$ with $\mathbf{P}(A)>0$ and for every x which is a point of continuity of F we have

$$\mathbf{P}(\eta_n\leqq x|A)\to F(x).$$

A sufficient condition for this property to hold is

$$\mathbf{P}(\eta_n\leqq x|\eta_k\leqq x)\to F(x),\quad\text{for}\quad k=1,2,\ldots.$$

For a proof of this as well as a general discussion of the concept of mixing we refer the reader to the elegant basic paper of RÉNYI [12] on this subject.

THEOREM B ([4]). *Let $\{Y_n\}$ and $\{T_n\}$ be as in Theorem A. Then $\{T_n/\sqrt{n}\}$ is mixing with the standard normal distribution function Φ.*

DEFINITION 2. Let $\{Z_j\}$ be a sequence of random variables and write $R_n=\sum_1^n Z_j$. We say the sequence $\{Z_j\}$ satisfies the central limit theorem with norming factors $\{b_n\}$ if

$$\mathbf{P}(R_n/b_n\leqq x)\to\Phi(x).$$

DEFINITION 3. Let $\{\nu_n\}$ be a sequence of positive integer valued random variables on $(\Omega,\mathscr{A},\mathbf{P})$. We say that the random-sum central limit theorem holds for $\{Z_j\}$ and $\{b_n\}$ if

$$\mathbf{P}(R_{\nu_n}/b_{\nu_n}\leqq x)\to\Phi(x).$$

Random-sum central limit theorems have been proved by many people (see e.g. [1], [2], [4], [10], [11], [13], [14] and [15] in the bibliography of this paper). Essentially the results obtained in the independent case may be stated as follows.

THEOREM C ([2], [10]). *If $\{X_j\}$ is a sequence of independent identically distributed random variables with mean* 0 *and variance* 1 *and if v_n/n converges in probability to a positive random variable v then the random sum central limit theorem holds for $\{X_j\}$ and $\{\sqrt{n}\}$.*

In [4] this result has been extended to the case when the random variables are absolutely fair as follows.

THEOREM D. *Let the sequence $\{Y_j\}$ be as in Theorem A and suppose v_n/n converges in probability to a positive random variable v. Then the random-sum central limit theorem holds for $\{Y_j\}$ and $\{\sqrt{n}\}$.*

3. Results obtained with control on the absolutely fair part. Throughout this section we assume that the assumption of (1. 4) holds. This is implied by the following condition, which involves only the sequence $\{X_j\}$,

$$\mathbf{E}\big(X_j^2-\mathbf{E}^2(X_j|X_1,\ldots,X_{j-1})|X_1,\ldots,X_{j-1}\big)=1, \qquad j=1,2,\ldots,$$

with probability 1. It should be noted that this condition is the only one assumed that involves second order moments. Despite this it will be seen that the classical norming factors $\{\sqrt{n}\}$ suffice.

PROPOSITION 1. *Let $\{X_j\}$ be a sequence of random variables with mean zero and write X_j as in* (1. 1). *Let $\{Y_j\}$ satisfy Lindeberg's condition. Then, if $\sum_1^n W_j/n$ converges to* 0 *in probability, $\{X_j\}$ satisfies the central limit theorem with norming factors $\{\sqrt{n}\}$.*

PROOF. Under the hypothesis (1. 4) and Lindeberg's condition Theorem A implies that the $\{Y_j\}$ sequence part of $\{X_j\}$ satisfies the central limit theorem with norming factors $\{\sqrt{n}\}$. This combined with Lemma 1 implies the result.

PROPOSITION 2. *Let $\{X_j\}$ be as in Proposition* 1 *and write $S_n=\sum_1^n X_j$. Then, if $\sum_1^n W_j/\sqrt{n}$ converges to* 0 *in probability, the sequence $S_n/\sqrt{n}$ is mixing with the distribution function Φ.*

PROOF. By Theorem B $T_n/\sqrt{n}$ is mixing with the distribution function Φ. Therefore, given any event A in $\mathscr{A}$ with $\mathbf{P}(A)>0$,

$$\mathbf{P}(T_n/\sqrt{n}\leqq x|A)\to\Phi(x).$$

Lemma 1 combined with our hypothesis on the sequence $\{W_j\}$ implies now

$$\mathbf{P}(S_n/\sqrt{n}\leqq x|A)=\mathbf{P}\left(\left(T_n+\sum_1^n W_j\right)\Big/\sqrt{n}\leqq x|A\right)\to\Phi(x),$$

and this is the statement of Proposition 2.

For our next theorem we need the following result which perhaps has some interest of its own.

LEMMA 2. *Let $\{Z_n\}$ be a sequence of random variables such that $Z_n \to 0$ with probability 1 and let $\{\nu_n\}$ be a sequence of positive integer valued random variables defined on the same probability space. If $\nu_n \to \infty$ in probability, then $Z_{\nu_n} \to 0$ in probability.*

PROOF. For every n, $a>0$ and $\varepsilon>0$ we have

$$\mathbf{P}(|Z_{\nu_n}| \geqq \varepsilon) =$$

$$= \mathbf{P}(|Z_{\nu_n}| \geqq \varepsilon, \nu_n \geqq a) + \mathbf{P}(|Z_{\nu_n}| \geqq \varepsilon, \nu_n < a) \leqq \mathbf{P}(|Z_{\nu_n}| \geqq \varepsilon, \nu_n \geqq a) + \mathbf{P}(\nu_n < a) =$$

$$= \sum_{k=a}^{\infty} \mathbf{P}(|Z_k| \geqq \varepsilon, \nu_n = k) + \mathbf{P}(\nu_n < a) \leqq$$

$$\leqq \sum_{k=a}^{\infty} \mathbf{P}(\sup_{a \leqq k} |Z_k| \geqq \varepsilon, \nu_n = k) + \mathbf{P}(\nu_n < a) \leqq \mathbf{P}(\sup_{a \leqq k} |Z_k| \geqq \varepsilon) + \mathbf{P}(\nu_n < a).$$

Now given any small positive number δ, choose the value of a in the first probability statement of the last line so large that it becomes less than or equal to $\delta/2$. This can be done for we assume that $Z_n \to 0$ with probability 1. For this value of a choose the value of n so large that the second probability statement of the last line above becomes less than $\delta/2$. This can be done for we assume $\nu_n \to \infty$ in probability. The proof of Lemma 2 is now complete.

The conclusion of this lemma does not necessarily hold if we only assume that $Z_n \to 0$ in probability. This is shown by the following counterexample.

Let $\Omega = [0, 1]$, $\mathscr{A}$ the Lebesgue measurable subsets of Ω and $\mathbf{P}$ the Lebesgue measure restricted to Ω. Write I_A for the indicator function of the set A. Define $Z_1 = 1$, $Z_2 = I_{[0,1/2]}$, $Z_3 = I_{[1/2,1]}$ and in general, if $a_m = [m(m-1)/2]+1$, define $Z_{a_m+j} = I_{[j/m, j+1/m]}$ for $j = 0, 1, \dots, m-1$; $m = 1, 2, \dots$. Let $\nu_{a_m+k} = \sum_{j=0}^{m-1} a_{m+j} I_{[j/m, j+1/m]}$ for $k = 0, 1, 2, \dots, m-1$; $m = 1, 2, \dots$. Then $Z_n \to 0$ in probability but not with probability 1 and $\nu_n \to \infty$ in probability (since $\nu_n \to \infty$ with probability 1). However $Z_{\nu_n} = 1$ with probability 1 for all n.

THEOREM 1. *Let $\{X_j\}$ and $\{S_n\}$ be as in Proposition 2 and let $\{\nu_n\}$ be a sequence of positive integer valued random variables defined on the same probability space. Assume that the following two conditions hold:*

(3. 1) *ν_n/n converges in probability to a positive random variable ν,*

(3. 2) *$\sum_1^n W_j/\sqrt{n}$ converges to 0 with probability 1.*

Then the random-sum central limit theorem holds for $\{X_j\}$ and $\{\sqrt{n}\}$; that is to say

$$\mathbf{P}\{S_{\nu_n}/\sqrt{\nu_n} \leqq x\} \to \Phi(x).$$

PROOF. Write X_j as in (1. 1). Given the conditions (1. 4) and (3. 1), the random-sum central limit theorem holds for the absolutely fair sequence $\{Y_j\}$ and $\{\sqrt{n}\}$ (Theorem D). A simple contradiction argument shows that condition (3. 1) implies $v_n \to \infty$ in probability and so conditions (3. 1) and (3. 2) combined with Lemma 2 imply that $\sum_1^{v_n} W_j/\sqrt{v_n} \to 0$ in probability. Theorem 1 now follows from Lemma 1.

The question arises if in fact the condition (3.2) can be replaced by the weaker condition used to obtain the non-random results of Propositions 1 and 2, where only convergence in probability was required. The above counterexample shows that (3. 2) cannot be replaced by $\sum_1^{n} W_j/\sqrt{n} \to 0$ in probability under the assumption $v_n \to \infty$ in probability. However, since in fact $v_n/n \to 1$ with probability 1 in the example, we have that (3. 2) cannot be weakened even under (3. 1). Since $\mathbf{E}(X_j|X_1, \ldots, X_{j-1})$ is equal to 0 in both the independent and absolutely fair cases, one would expect that the contribution of the sequence $\{W_j\}$ must be small in some sense, in order for the various results to hold. In this light our condition (3. 2) appears to be quite natural.

PROPOSITION 3. *Under the assumptions* (1. 4) *and* (3. 2) *the strong law of large numbers holds for the sequence* $\{X_j\}$ *of Theorem* 1.

PROOF. Condition (1. 4) implies that $\sum_1^{\infty} j^{-2}\mathbf{E}(Y_j^2) = \sum_1^{\infty} j^{-2} < \infty$. By the stability theorem (page 387 of [9]) we then have $n^{-1}\sum_1^{n} Y_j \to 0$ with probability 1. This combined with the hypothesis of (3. 2) implies that $n^{-1}\sum_1^{n} X_j \to 0$ with probability 1.

For our next proposition we need the following result which we quote from section 2 of [4].

LEMMA 3. *Let* $\{Z_n\}$ *be a sequence of random variables such that* $Z_n \to 0$ *with probability* 1 *and let* $\{v_n\}$ *be a sequence of positive integer valued random variables defined on the same probability space. If* $v_n \to \infty$ *with probability* 1, *then* $Z_{v_n} \to 0$ *with probability* 1.

PROPOSITION 4. *Under the assumptions* (1. 4), (3.2) *and* $v_n \to \infty$ *with probability* 1, *the random-sum strong law of large numbers holds for the sequence* $\{X_j\}$ *of Theorem* 1; *that is to say* $v_n^{-1}\sum_1^{v_n} X_j \to 0$ *with probability* 1.

This statement is an immediate corollary of Proposition 3 and Lemma 3.

4. Results obtained without assumptions on the absolutely fair part. There are certain special cases where one can prove the central limit theorem and that the normalized partial sums are mixing. For example it is known that the central limit theorem holds for certain random variables of the form $X_k(t) = f(2^k t)$, where f is a function defined on the unit interval and extended by periodicity to the whole real line (see e.g. [7]). Starting from this knowledge S. TAKAHASHI [14] has demonstrated for these functions that their normalized partial sums are in fact mixing with the distribution function Φ.

This suggests the following general problem. Given that the normalized partial sums of a sequence of random variables satisfy the central limit theorem and that they are also mixing with the distribution function Φ, under what conditions will the random-sum central limit theorem hold.

Our next result is a partial answer to this question. It is important to point out that here we no longer need the assumption (1. 4) which, in section 3, provided us with what we are assuming now: the central limit theorem and the mixing of the normalized partial sums.

THEOREM 2. *Let* $\{X_j\}$ *be a sequence of random variables with mean* 0 *and variance* 1, *and let* $\{\nu_n\}$ *be a sequence of positive integer valued random variables defined on the same probability space. Assume that the sequence* $\{X_j\}$ *satisfies the central limit theorem with norming factors* $\{\sqrt{n}\}$. *Assume further that the thus normalized partial sums are mixing with the distribution function* Φ. *If*

(4. 1) ν_n/n *converges in probability to a positive discrete random variable* ν,

$$\sum_1^n W_j/\sqrt{n} \to 0 \quad \text{with probability } 1, \tag{4. 2}$$

then the random-sum central limit theorem holds for $\{X_j\}$ *and* $\{\sqrt{n}\}$.

PROOF. Again write $X_j = Y_j + W_j$ as in (1. 1), and consider $\sum_1^n Y_j/\sqrt{n} = \left(\sum_1^n X_j - \sum_1^n W_j\right)/\sqrt{n}$. Using the argument of Proposition 2 we deduce that $\left\{\sum_1^n Y_j/\sqrt{n}\right\}$ is mixing with the distribution function Φ.

Next we prove that under the condition (4. 1) the random-sum central limit theorem holds for the absolutely fair sequence $\{Y_j\}$ and $\{\sqrt{n}\}$. To do this we make the following general observation deduced by analysing the proof of Theorem 2 of RÉNYI's paper [13].

OBSERVATION. *Let* $\{X_j\}$ *be an arbitrary sequence of random variables with mean* 0 *and variance* 1 *and suppose that the sequence* $\{X_j\}$ *satisfies the central limit theorem with norming factors* $\{\sqrt{n}\}$. *Assume also that the normalized partial sums are mixing with the distribution function* Φ. *If the condition* (4. 1) *holds and if the sequence* $\{X_j\}$ *also satisfies the following*

*Condition (*ANSCOMBE [1]*): Suppose that for given* $\varepsilon > 0$ *and* $\delta > 0$ *there exists a positive number* c *and an integer* n_0 *such that if* $n \geqq n_0$

$$\mathbf{P}\Big(\max_{|m-n| \leqq nc} |S_n - S_m| \geqq \varepsilon\sqrt{n}\Big) \leqq \delta,$$

then the random-sum central limit theorem holds for $\{X_j\}$ *and* $\{\sqrt{n}\}$.

Thus to show that the random-sum central limit theorem holds for the $\{Y_j\}$ and $\{\sqrt{n}\}$ we only have to verify the Anscombe condition. But this immediately follows from the Kolmogorov inequality for martingales and from the fact that the variance of Y_j is less than or equal to 1. The latter in turn follows from the orthogonality of Y_j and W_j.

We now complete the proof of the theorem. Because of hypothesis (4. 1) v_n approaches infinity in probability. This combined with condition (4. 2) allows us to use Lemma 2 on the sequence $\{\sum_1^n W_j/\sqrt{n}\}$, resulting in the convergence in probability of $\sum_1^{v_n} W_j/\sqrt{v_n}$ to 0. By Lemma 1 the proof of Theorem 2 is now complete

COMMENT 1. Originally the authors used the principle of the Observation' contained in the above proof, directly on the sequence $\{X_j\}$. The proof then boiled down to showing that the Anscombe condition was satisfied for the partial sums of the sequence $\{W_j\}$. But the simplest condition for this to be true appears to be $\sum_1^\infty \mathbf{E}(X_j|X_1, \ldots, X_{j-1}) < \infty$ with probability 1 which, of course, is a much stronger requirement than the present condition (4. 2) which is the same as the condition (3. 2) of Theorem 1.

COMMENT 2. The now classical result of ANSCOMBE [1], which in fact sparked the current flow of random-sum papers, actually is in the spirit of this section if one interprets his result in terms of partial sums of random variables. In this interpretation Anscombe's theorem can be formulated as follows.

THEOREM E. *Let $\{X_j\}$ satisfy the central limit theorem with norming factors $\{b_n\}$ and assume that Anscombe's above quoted condition holds (with $\sqrt{n}$ replaced by b_n now). Then, if v_n/n converges in probability to a constant then the random-sum central limit theorem holds for $\{X_j\}$ and $\{b_n\}$.*

In the light of this statement our Theorem 2 is in essence a generalization of his result. To further emphasize this, we give here the following more general form of Theorem 2.

THEOREM 3. *Let $\{X_j\}$ be a sequence of random variables with mean 0 and $\mathbf{E}(X_j^2) < \infty$ for all j. Let $\{v_n\}$ be a sequence of positive integer valued random variables defined on the same probability space as $\{X_j\}$. Assume that the sequence $\{X_j\}$ satisfies the central limit theorem with norming factors $\{b_n\}$. Assume further that the thus normalized partial sums are mixing with the distribution function Φ. We write X_j as in* (1. 1). *If the condition* (4. 1) *is true and we also have*

$$\sum_1^n W_j/b_n \to 0 \quad \textit{with probability } 1, \tag{4. 3}$$

$$\lim_{c\to 0} \limsup_{n\to\infty} \sum_{j=n+1}^{[n(1+c)]} \mathbf{E}(Y_j^2)/b_n^2 = 0, \tag{4. 4}$$

then the random-sum central limit theorem holds for $\{X_j\}$ and $\{b_n\}$.

The proof of this statement is similar to that of Theorem 2. Condition (4. 4) appears when we apply the principle of the Observation of the proof of Theorem 2 and use the Kolmogorov inequality for martingales to verify the Anscombe condition for the absolutely fair sequence $\{Y_j\}$.

At this stage we should perhaps point out that although the subsequent papers were inspired by the Anscombe result, they belong in spirit to the approach of the third section of this paper. That is to say they assume either independence ([11], [13], [2], [10] and [15]) or absolute fairness [4].

COMMENT 3. Naturally one should want to change condition (4. 1) so as to drop the condition of the discreteness of ν.

PROPOSITION 5. *Under the assumption that the variance of X_j is equal to* 1, *condition* (4. 2) *implies that the strong law of large numbers holds for the sequence $\{X_j\}$ of Theorem* 2.

PROOF. Similar to that of Proposition 3.

PROPOSITION 6. *Under the assumption that the variance of X_j is equal to* 1, *condition* (4. 2) *and $\nu_n \to \infty$ with probability* 1 *imply that the random-sum strong law of large numbers holds for the sequence $\{X_j\}$ of Theorem 2.*

This statement is an immediate corollary of Proposition 5 and Lemma 3.

5. Concluding remarks. The following "principle" has suggested itself to the authors while working on this paper. Whenever one wants to go from the central limit theorem to the random-sum central limit theorem one must impose conditions on the sequence $\{X_j\}$ which will also imply the strong law of large numbers for the sequence $\{X_j\}$.

The orthogonal decomposition (1. 1) of the sequence $\{X_j\}$ was utilized in section 3 to the extent that assuming (1. 4) for the sequence $\{Y_j\}$ we were able to avoid direct assumptions on the variances of the sequence $\{X_j\}$. Also, this orthogonal decomposition did away with covariance considerations which usually play a role in the non-independent case.

In section 4 the orthogonal decomposition (1. 1) made it possible to control the variances of the absolutely fair sequence $\{Y_j\}$ simply by assuming that the variance of X_j was equal to 1. In case of Theorem 3 the orthogonal decomposition (1. 1) enabled us to control the variances of the absolutely fair sequence $\{Y_j\}$ by assuming $\mathbf{E}(X_j^2) < \infty$ and then the condition (4. 4) helped us to prove the random-sum central limit theorem. Thus, again, covariance analysis did not play a role.

It would be interesting to know if results and techniques developed for orthogonal functions could be of more use in the area discussed here.

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APPENDIX. For not wanting to disturb the random-sum central limit theorem oriented line of thought of our paper we decided to attach here this appendix in order to point out that the technique of section 4 is applicable to prove random-sum limit theorems in general for arbitrary sequences of random variables.

We first restate here the Observation of the proof of our Theorem 2, using again the proof of Theorem 2 of RÉNYI's paper [13] as our guide line, as follows.

LEMMA 4. *Let* $\{X_j\}$ *be an arbitrary sequence of random variables with mean* 0. *Let* $\{\nu_n\}$ *be as in Theorem* 3. *Assume that the normed partial sum sequence* $\{S_n/d_n\}$ *of* $\{X_j\}$ *is mixing with the distribution function F in the sense of Definition* 1. *Then, if the condition* (4. 1) *holds and the sequence* $\{X_j\}$ *satisfies the Anscombe condition (with* $\sqrt{n}$ *replaced by* d_n*), the random-sum limit theorem*

$$\mathbf{P}(S_{\nu_n}/d_{\nu_n} \leqq x) \to F(x)$$

holds for every real x which is a point of continuity of F.

Using this lemma we can now prove

THEOREM 4. *Let* $\{X_j\}$ *be a sequence of random variables with mean* 0 *and* $\mathbf{E}(X_j^2) < \infty$ *for all j. Let* $\{\nu_n\}$ *be as in Theorem* 3. *Assume that the normed partial sum sequence* $\{S_n/d_n\}$ *of* $\{X_j\}$ *is mixing with the distribution function F in the sense of Definition* 1. *Write* X_j *as in* (1. 1). *If the condition* (4. 1) *is true and we also have*

$$\sum_1^n W_j/d_n \to 0 \quad \textit{with probability } 1, \tag{A. 1}$$

$$\lim_{c\to 0} \limsup_{n\to\infty} \sum_{j=1}^{[n(1+c)]} \mathbf{E}(Y_j^2)/d_n^2 = 0, \tag{A. 2}$$

then the following random-sum limit theorem holds

$$\mathbf{P}(S_{\nu_n}/d_{\nu_n} \leqq x) \to F(x), \tag{A. 3}$$

for every real x which is a point of continuity of F.

PROOF. Using the argument of Proposition 2 as we did in the proof of Theorem 2 we deduce that the normed partial sum sequence of $\{\sum_1^n Y_j/d_n\}$ of the absolutely fair part sequence $\{Y_j\}$ of $\{X_j\}$ is mixing with the distribution function F in the sense of Definition 1.

Next we prove that under the conditions (4. 1) and (A. 2) and (A. 3) type random-sum limit theorem is also true for the normed partial sum sequence $\{\sum_1^n Y_j/d_n\}$ of the absolutely fair sequence $\{Y_j\}$, by applying Lemma 4 and then Kolmogorov's inequality for martingales combined with the hypothesis (A. 2).

To complete the proof of the theorem we again note that condition (4. 1) implies that ν_n approaches infinity in probability. This combined with condition (A. 1) enables us to use Lemma 2 on the sequence $\{\sum_{1}^{n} W_j/d_n\}$, resulting in the convergence in probability of $\sum_{1}^{\nu_n} W_j/d_{\nu_n}$ to 0. By Lemma 1 the proof of Theorem 4 is complete.

This proof of course differs only notationally from that of Theorem 2.

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PARACOMPACT SUBSPACES

By

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1. Introduction. The definition of paracompactness for a topological space $(X, \mathcal{J})$ states that every open cover of the space have a locally finite open refinement. When applying the definition to a subspace S of $(X, \mathcal{J})$, the phrases 'open cover' and 'open refinement', of course, refer to the relative topology on S. Locally finite also is in reference to open sets in S. Immediate questions arise when these phrases are used in reference to the topology for the whole space X. That is, one can consider a subspace S of $(X, \mathcal{J})$ to be *strongly paracompact* if every cover of S by members of $\mathcal{J}$ has a locally finite in X refinement by members of $\mathcal{J}$. (In [1], such subspaces were called α-paracompact.)

It is clear that every strongly paracompact subspace of a topological space is paracompact. However, the two notions are not equivalent. This follows from the example of a completely regular, non-normal space due to Niemytzki. Let X be that subset of the plane $R \times R$ consisting of the points (x, y) for which $Y \geqq 0$. Let S be that subset of X consisting of the points $(x, 0)$ of R. The relative product topology $\mathcal{J}$ on X is enlarged to include also as neighbourhoods of the points in S the ε-spheres tangent to the points $(x, 0)$ together with $\{(x, 0)\}$ for all $\varepsilon > 0$. That is, the sets

$$K(x, \varepsilon) = \{(x, 0) \cup \{(u, v) \in S : (u-x)^2 + (v-\varepsilon)^2 < \varepsilon\}\}$$

are also included for all $\varepsilon > 0$. With this new topology S is a closed discrete subspace of X and hence is paracompact. However, S is not strongly paracompact. For the cover $(K(x, 1/2))_{x \in S}$ is an open in X cover of S that has no refinement and is not locally finite in X. For any $\mathcal{J}$ neighbourhood N of a point $(x, 0)$ in S will meet an infinite number of the sets $K(x, 1/2)$. As our main theorem will show this subspace S is not strongly paracompact since it is not P-embedded in X.

In particular, it will be shown that for normal Hausdorff spaces a subspace is strongly paracompact if and only if it is paracompact and P-embedded. A subset S is *P-embedded* in a topological space X if every continuous pseudometric on S extends to a continuous pseudometric on X. This concept was studied in [3] and [5]. Its relationship to paracompactness was studied in [4] and [6]. The notion of P-embedding generalizes the notion of C-embedding where extensions of continuous real valued functions are required.

The importance of P-embedding can be readily realized from the fact that it plays the same role in collectionwise normal spaces as C-embedding plays in normal spaces. In particular a space is *collectionwise normal if and only if every closed subset is P-embedded* (see [3]). Hence the notion of strongly paracompact, from the results in this paper, will have applications in such spaces.

The notion of strongly paracompact can also be generalized by making use of infinite cardinal numbers γ. Such techniques have already been applied to paracompactness and P-embedding. Hence we will relate this new notion of 'strongly γ-paracompact' to the notions of γ-paracompact and P^{γ}-embedded. Throughout this paper emphasis will be placed on the notions in the definitions of paracompactness that deal with the topologies of the subspace and of the space. Hence we will refer to, for example, the local finiteness of a cover as being either locally finite in S (the subspace) or locally finite in X. Also $(X, \mathcal{J})$ or X will mean a topological space with topology $\mathcal{J}$.

2. Definitions and basic results. Let S be a subset of the topological space $(X, \mathcal{J})$ and let γ be an infinite cardinal number. The relative topology on S will be denoted by $\mathcal{J}_S$. The subset S is *γ-paracompact* if every cover by members of $\mathcal{J}_S$ of cardinality at most γ has a locally finite in S refinement by members of $\mathcal{J}_S$. The subset is *strongly γ-paracompact* if every cover by members of $\mathcal{J}$ of cardinality at most γ has a locally finite in X refinement by members of $\mathcal{J}$. In the special case of $\gamma = \aleph_0$ then γ-paracompact is the usual notion of countably paracompact. A pseudometric d on S is *γ-separable* if there is a subset G of S whose cardinality is at most γ and such that G is dense in S relative to the pseudometric topology $\mathcal{J}_d$. The subset S is *P^{γ}-embedded* in X if every γ-separable continuous pseudometric on S has a continuous extension to X.

The subspace S is *paracompact* (respectively *strongly paracompact*) if every cover of S by members of $\mathcal{J}_S$ (respectively $\mathcal{J}$) has a locally finite in S (respectively in X) refinement by members $\mathcal{J}_S$ (respectively $\mathcal{J}$). It is *P-embedded* if every continuous pseudometric on S extends to a continuous pseudometric on X. This is equivalent to saying that S is paracompact (respectively strongly paracompact, respectively P-embedded) if and only if S is γ-paracompact (respectively strongly γ-paracompact, respectively P^{γ}-embedded) for all cardinal numbers γ. By an extension $\mathcal{U}^*$ of a cover $\mathcal{U}$ on S is meant a cover of X such that the trace of $\mathcal{U}^*$ on S is $\mathcal{U}$.

If $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ is a cover of a set X and if A is a subset of X, then by the star of A with respect to $\mathcal{U}$, written st$(A, \mathcal{U})$, we mean $\cup \{U_\alpha : \alpha \in I$ and $U_\alpha \cap A \neq 0\}$. We set $\mathcal{U}^* = (\mathrm{st}\,(U_\alpha, \mathcal{U}))_{\alpha \in I}$ and say that *$\mathcal{U}$ is a star-refinement* of a cover $\mathcal{V}$ in case $\mathcal{U}^*$ is a refinement of $\mathcal{V}$. A sequence $(\mathcal{U}_n)_{n \in \tilde{N}}$ of covers of X is said to be *normal* in case $\mathcal{U}_{n+1}$ is a star-refinement of $\mathcal{U}_n$ for each $n \in \tilde{N}$. A cover $\mathcal{U}$ of a topological space X is said to be *normal* in case there exists a normal sequence $(\mathcal{U}_n)_{n \in \tilde{N}}$ of open covers of X such that $\mathcal{U}_1$ is a refinement of $\mathcal{U}$.

The definitions of other terms used in this paper can be found in [3]. The following results will be needed in the main part of this paper.

THEOREM 2. 1 (see [3]). *If S is a subspace of X and if γ is an infinite cardinal number then the following statements are equivalent:*

(1) *S is P^{γ}-embedded in X.*

(2) *Every locally finite in S normal cover of S by members of $\mathcal{J}_S$ of power at most γ has a refinement by members of $\mathcal{J}_S$ that can be extended to a locally finite in X normal cover of X by members of $\mathcal{J}$.*

THEOREM 2. 2 (see for example [3, Theorem 4. 5]). *If X is a topological space then the following statements are equivalent:*

(1) *X is normal.*

(2) *Every locally finite open cover of X is normal.*

LEMMA 2. 3 *Suppose that $\mathcal{U}=(U_\alpha)_{\alpha\in I}$ is an open cover of a topological space X and that $\mathcal{V}=(V_\beta)_{\beta\in J}$ is a locally finite open cover of X that refines $\mathcal{U}$. Then there exists a locally finite open cover $\mathcal{W}=(W_\alpha)_{\alpha\in I}$ of X such that $W_\alpha\subset U_\alpha$ for each $\alpha\in I$.*

PROOF. Let $\pi:J\to I$ be a function such that $V_\beta\subset U_{\pi(\beta)}$ for each $\beta\in J$. For each $\alpha\in I$, let $W_\alpha=\bigcup_{\beta\in\pi^{-1}(\alpha)}V_\beta$. Then one easily verifies that $\mathcal{W}=(W_\alpha)_{\alpha\in I}$ is a locally finite open cover of X such that $W_\alpha\subset U_\alpha$ for each $\alpha\in I$.

3. Main results. THEOREM 3. 1. *If X is a topological space and if S is a normal γ-paracompact P^γ-embedded subset of X, then S is strongly γ-paracompact.*

PROOF. Let $\mathcal{U}=(U_\alpha)_{\alpha\in I}$ be a cover of S by members of $\mathcal{I}$ with cardinality at most γ. Since S is γ-paracompact, the trace $\{U_\alpha\cap S:\alpha\in I\}$ of $\mathcal{U}$ on S has a locally finite in S refinement $\mathcal{V}=(V_\beta)_{\beta\in J}$ by members of $\mathcal{I}_S$. Thus there exists a function $\lambda:J\to I$ such that $V_\beta\subset U_{\lambda(\beta)}\cap S$. By Lemma 2. 3, $\mathcal{V}$ can be assumed to have cardinality at most γ. Normality of S implies that $\mathcal{V}$ is normal in S by Theorem 2. 2. Moreover, since S is P^γ-embedded in X, Theorem 2. 1 states that $\mathcal{V}$ has a refinement that extends to a locally finite in X normal cover $(W_\delta)_{\delta\in K}$ of X by members of $\mathcal{I}$. Thus there exists a function $\pi:K\to J$ such that $W_\delta\cap S\subset V_{\pi(\delta)}$. But then the family

$$\{W_\delta\cap U_\alpha:\delta\in K \quad\text{and}\quad \alpha=\lambda(\pi(\delta))\}$$

is a locally finite family in X which is a cover of S and a refinement of $\mathcal{U}$ by members of the topology $\mathcal{I}$ on X. Hence S is strongly γ-paracompact.

THEOREM 3. 2. *If $(X,\mathcal{I})$ is a normal topological space and if S is a closed subset of X, then the following statements are equivalent:*

(1) *S is γ-paracompact and P^γ-embedded in X.*

(2) *S is strongly γ-paracompact.*

PROOF. Since S is normal, (2) follows from (1) by Theorem 3. 1. On the other hand, if S is strongly γ-paracompact then S is γ-paracompact. Theorem 2. 1 is now used to show that S is P^γ-embedded in X. In fact, let $\mathcal{U}=(U_\alpha)_{\alpha\in I}$ be a locally finite in S cover of S by members of $\mathcal{I}_S$ of cardinality at most γ. For each $\alpha\in I$, $U_\alpha\in\mathcal{I}_S$ so there exists a $V_\alpha\in\mathcal{I}$ such that $V_\alpha\cap S=U_\alpha$. Then $\mathcal{V}=(V_\alpha)_{\alpha\in I}$ is a cover of S by members of $\mathcal{I}$ with cardinality at most γ. By (2), there is a locally finite in X cover $\mathcal{W}^*$ of S by members of $\mathcal{I}$ such that $\mathcal{W}^*$ refines $\mathcal{V}$. Then the trace $\mathcal{W}$ of $\mathcal{W}^*$ on S is the refinement of $\mathcal{U}$ by members of $\mathcal{I}_S$ needed in 2. 1. It is clear that $\mathcal{W}$ is locally finite in S. The family $\mathcal{A}=\mathcal{W}^*\cup\{X\setminus S\}$ is a locally finite in X cover of X by members of $\mathcal{I}$ whose trace with S is $\mathcal{W}$. By 2. 2 and the fact that X is normal, $\mathcal{A}$ is normal. Hence $\mathcal{A}$ is the required extension of $\mathcal{W}$ and S is P^γ-embedded in X.

COROLLARY 3. 3. *If X is a normal Hausdorff topological space and if S is a subset of X, then the following statements are equivalent:*

(1) *S is strongly paracompact.*

(2) *S is paracompact and P-embedded in X.*

PROOF. In [1, Corrolary 4] it is shown that a strongly paracompact subset is closed. Thus 3. 2 shows that (1) *implies* (2). Since paracompact Hausdorff spaces are normal, Theorem 3. 1 shows that (2) *implies* (1). This completes the proof.

COROLLARY 3. 4. *If X is a topological space and if S is a paracompact Hausdorff P-embedded subset of X, then S is strongly paracompact in X.*

COROLLARY 3. 5. *If X is a topological space and if S is a normal Hausdorff countably paracompact $P^{\aleph_0}$-embedded subset of X, then S is strongly countably paracompact.*

In [2, Theorem 2. 19] it is shown that *a normal space X is γ-collectionwise normal if and only if every closed subset is P^{γ}-embedded in X.* In particular, it was shown that *X is normal if and only if every closed subset is $P^{\aleph_0}$-embedded in X.* Using these results it is now possible to obtain the following three collaries.

COROLLARY 3. 6. *If S is a closed subset of a normal γ-collectionwise normal space X, then S is γ-paracompact if and only if S is strongly γ-paracompact.*

COROLLARY 3. 7. *If S is a closed subset of a normal space X then S is countably paracompact if and only if S is strongly countably paracompact.*

COROLLARY 3. 8. *If S is a closed subset of a collectionwise normal space X, then S is paracompact if and only if S is strongly paracompact.*

Suppose F is a strongly γ-paracompact subset of S and S is a subset of a topological space X. It is now possible to give some sufficient conditions for F to be strongly γ-paracompact in X.

THEOREM 3. 9. *Let F be a strongly γ-paracompact subset of the closed subspace S of the topological space X. If there is an open set G in X such that $F \subset G \subset S$ then F is strongly γ-paracompact in X.*

PROOF. Let $\mathcal{U}^*$ be a cover of F by members of $\mathcal{I}$ with cardinality at most γ. The trace $\mathcal{U}$ of $\mathcal{U}^*$ on S is a cover of F by members of $\mathcal{I}_S$. By assumption $\mathcal{U}$ has a locally finite in S refinement $\mathcal{V}^*$ by members of $\mathcal{I}_S$. The trace $\mathcal{V}$ of $\mathcal{V}^*$ on G is a locally finite in S refinement of $\mathcal{U}^*$ by members of $\mathcal{I}_G$ and covers F. From $G \subset S$ and G open in X it follows that $\mathcal{V}$ is open in X. Clearly $\mathcal{V}$ is locally finite in X since S is closed.

COROLLARY 3. 10. *If F is a strongly paracompact subset of the closed subspace S of $(X, \mathcal{I})$ and if there is an open set G in X such that $F \subset G \subset S$ then F is strongly paracompact in X.*

COROLLARY 3. 11. *If F is a paracompact Hausdorff P-embedded subset of the closed subspace S of $(X, \mathcal{I})$ and if there is an open set G in X such that $F \subset G \subset S$ then F is strongly paracompact in X.*

PROOF. By Corollary 3. 4, F is a strongly paracompact subset of the closed subspace S. Hence by Corollary 3. 10 F is strongly paracompact in X.

COROLLARY 3. 12 (see [1]). *If S is a closed paracompact subset of X and if F is a closed Hausdorff subset of X that is contained in the interior of S, then F is strongly paracompact in X.*

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$\omega^n I$-BISIMPLE SEMIGROUPS

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The purpose of this paper is to determine the structure of $\omega^{n-1}I$-bisimple semigroups mod groups for all natural numbers n and to develop a homomorphism theory and a theory of congruences for these classes of semigroups. The ω^n-bisimple semigroups [10] and the $\omega^{n-1}I$-bisimple semigroups are the only classes of simple semigroups except completely simple semigroups whose structure has been determined mod groups.

Let S be a bisimple semigroup and let E_S denote the set of idempotents of S. As usual, E_S is partially ordered in the following fashion: if $e, f \in E_S$, $e \leqq f$ if and only if $ef = fe = e$. We then say that E_S is under or assumes its natural partial order. Let I denote the set of all integers, let I^0 denote the set of non-negative integers, and let n denote a natural number. If E_S, under its natural order, is order isomorphic to $(I^0)^n (I \times (I^0)^{n-1})$, under the reverse of the usual lexicographic order, S is called an ω-bisimple ($\omega^{n-1}I$-bisimple) semigroup. (See [10] for an explanation of notation.)

In Section 1, we first describe the homomorphisms of an ω^n-bisimple semigroup S onto an ω^n-bisimple semigroup S^* in terms of homomorphisms of the group of units of S onto the group of units S^*, and give necessary and sufficient conditions for S to be isomorphic to S^*. We utilize this isomorphism theorem, the structure theorem for ω^n-bisimple semigroups [10], and techniques first introduced in [9, 10] to determine the structure of $\omega^{n-1}I$-bisimple semigroups mod groups. (This result was announced in [10].) In addition, we describe the homomorphisms of an ω^n-bisimple semigroup S onto an ω^k-bisimple semigroup $(n > k) S^*$ in terms of homomorphisms of the group of units of S into the group of units of S^*. This result has applications to the theory of congruences of ω^n-bisimple semigroups [11].

In section 2, we describe the homomorphisms of an $\omega^{n-1}I$-bisimple semigroup S onto an $\omega^{n-1}I$-bisimple semigroup S^* in terms of the homomorphisms of a fixed subgroup G of S onto a fixed subgroup G^* of S^*. We also give necessary and sufficient conditions for two $\omega^{n-1}I$-bisimple semigroups to be isomorphic.

In section 3, we determine the congruence relations on an $\omega^{n-1}I$-bisimple semigroup S mod group congruences. We first show that each congruence ϱ on S is a group congruence (S/ϱ is a group), an idempotent separating congruence (each ϱ-class contains at most one idempotent), or an $\omega^{k-1}I$-bisimple congruence (S/ϱ is an $\omega^{k-1}I$-bisimple semigroup) for some $k \in \{1, 2, ..., n-1\}$. To determine the group congruences, we give an explicit determination of the maximal group homomorphic image H of S including the defining homomorphism. Clearly, the group congruences of S are in a 1—1 correspondence with the normal subgroups of H. We determine the idempotent separating congruences and the $\omega^{k-1}I$-bisimple congruences of S in terms of certain normal subgroups of a fixed subgroup of S.

The theory of extensions of $\omega^{n-1}I$-bisimple semigroups is given in [7, 8, and 13].

Let n be a natural number. We say $(I^0)^n$ $(I\times(I^0)^{n-1})$ is under the reverse of the usual lexicographic order if $(a_1, a_2, ..., a_n)<(b_1, b_2, ..., b_n)$ if and only if $a_1>b_1$, or $a_1=b_1$ and $a_2>b_2$, or $...a_i=b_i$ for $1\leqq i<k$ and $a_k>b_k$, or ..., or $a_i=b_i$ for $1\leqq i<n$ and $a_n>b_n$.

Let R, L, H, and D denote Green's relations [2], and let R_a denote the R-class containing the element a. Unless otherwise specified, we will use the definitions and terminology of [2].

1. Structure theory. Let C_1 be the bicyclic semigroup, i.e. $C_1 = I^0\times I^0$ under the multiplication $(i,j)(k,s) = (i+k-\min(j,k), j+s-\min(j,k))$. Let X be an arbitrary semigroup. We define $C_1\circ X$ to be $C_1\times X$ under the multiplication $((m,n),s)((p,q),t)=((m,n)(p,q),f(n,p))$ where $f(n,p)=s, t$, or st according to whether $n>p$, $p>n$, or $n=p$ and where juxtaposition denotes multiplication in C_1 and X. We define $C_2=C_1\circ C_1$, $C_3=C_1\circ C_2$, ..., $C_n=C_1\circ C_{n-1}$. We characterized C_n in [6] and we called C_n the $2n$-cyclic semigroup in [6]. Let $C_1^* = I\times I$, where I is the group of integers, under the multiplication $(i,j)(k,s) = (i+k-\min(j,k), j+s-\min(j,k))$. We called C_1^* the extended bicyclic semigroup in [9]. We define $C_*^1\circ X$ to be $C_1^*\times X$ under the multiplication $((m,n),s)((p,q),t)=((m,n)(p,q),g(n,p))$ where $g(n,p)=s,t$, or st according to whether $n>p$, $p>n$, or $n=p$. Let $C_n^* = C_1^*\circ C_{n-1}$ for $n\geqq 2$. We called C_n^* the extended $2n$-cyclic semigroup in [10]. We will show that S is an $\omega^{n-1}I$-bisimple semigroup if and only if $S\cong G\times C_n^*$, where G is a group, under a suitable multiplication.

Let us now give a brief indication of the structure theorems on which our development is based. The structure of an ω^n-bisimple semigroup S is described completely in terms of a group G (the group of units of S), endomorphisms $\gamma_1, \gamma_2, ..., \gamma_n$ of G, elements $t_1, t_2, ..., t_{\varphi(n)}$ $\left(\varphi(n) = \dfrac{n(n-1)}{2}\right)$ of G, and C_n. Thus, if S is an ω^n-bisimple semigroup, we will write $S=(G, C_n, \gamma_1, \gamma_2, ..., \gamma_n; t_1, t_2, ..., t_{\varphi(n)})$. In fact, $S\cong G\times C_n$ under a suitable multiplication and conversely [10, theorem 2. 3]. We also utilize a structure theorem for a class of right cancellative semigroups. Let P be a right cancellative semigroup with identity. The partially ordered system of principal left ideals of P ordered by inclusion is termed the ideal structure of P. If the ideal structure of P is order isomorphic to $(I^0)^n$ under the reverse of the usual lexicographic order, P is called an ω^n-right cancellative semigroup. The structure of an ω^n-right cancellative semigroup P is described completely in terms of a group G (the group of units of P), endomorphisms $\gamma_1, \gamma_2, ..., \gamma_n$ of G, elements $t_1, t_2, ..., t_{\varphi(n)}$ of G, and $(I^0)^n$. Thus, we write, $P=(G, (I^0)^n, \gamma_1, \gamma_2, ..., \gamma_n, t_1, t_2, ..., t_{\varphi(n)})$. In fact, P is an ω^n-right cancellative semigroup if and only if $P\cong G\times(I^0)^n$, where G is a group under a suitable multiplication [10, theorem 1. 4].

We first establish the homomorphism theorem for ω^n-bisimple semigroups. The theorem is utilized in developing the homomorphism theory of ω^n-bisimple semigroups (section 2), and the isomorphism part of the theorem is crucial in the determination of the structure of ω^nI-bisimple semigroups. The proof of the homomorphism theorem depends on [3, theorem 1. 1 and theorem 1. 2], [1, main theorem], [12, theorem 1] and a technique introduced in the proof of [10, theorem 2. 3].

THEOREM 1. 1. *Let* $S=(G, C_n, \gamma_1, \gamma_2, \ldots, \gamma_n, w_1, w_2, \ldots, w_{\varphi(n)})$ *and* $S^*=(G^*, C_n, \alpha_1, \alpha_2, \ldots, \alpha_n, t_1, t_2, \ldots, t_{\varphi(n)})$ *be* ω^n*-bisimple semigroups. Let* $z_1, z_2, \ldots, z_n$ *be elements of* G^* *and let* f *be a homomorphism of* G *onto* G^* *such that*

$$(1.1) \qquad f\alpha_k C_{z_k} = \gamma_k f \quad \textit{for} \quad 1 \leqq k \leqq n \quad (xC_{z_k} = z_k x z_k^{-1} \quad \textit{for} \quad x \in G^*)$$

and

$$(1.2) \quad (z_{k+s}\alpha_k)t_{\varphi(n-k)+s}C_{z_k} = w_{\varphi(n-k)+s}f \quad \textit{for} \quad 1 \leqq k < n \quad \textit{and} \quad 1 \leqq s \leqq n-k.$$

For each) $g, (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)) \in S$, *let*

$$(1.3) \qquad (g, (a_1, b_1), \ldots, (a_n, b_n))\Phi =$$

$$= \Bigg(\Bigg(\prod_{j=a_1-1}^{0} z_1^{-1}\alpha_1^j \alpha_2^{a_2} \ldots \alpha_n^{a_n}\Bigg) \ldots \Bigg(\prod_{j=a_{n-1}-1}^{0} z_{n-1}^{-1}\alpha_{n-1}^j \alpha_n^{a_n}\Bigg) \cdot$$

$$\cdot \Bigg(\prod_{j=a_n-1}^{0} z_n^{-1}\alpha_n^j\Bigg)(gf)\Bigg(\prod_{j=0}^{b_n-1} z_n\alpha_n^j\Bigg) \ldots \Bigg(\prod_{j=0}^{b_1-1} z_1\alpha_1^j\alpha_2^{b_2} \ldots \alpha_n^{b_n}\Bigg), (a_1, b_1), \ldots, (a_n, b_n)\Bigg),$$

if $a_i, b_i \geqq 1$ $(1 \leqq i \leqq n)$. *If* $a_k = 0 (b_k = 0)$ *for some* k, *we let the corresponding factor above be* e^*, *the identity of* G^*.

Then, Φ *is a homomorphism of* S *onto* S^* *and conversely every homomorphism of* S *onto* S^* *is obtained in this fashion. In addition,* S *is isomorphic to* S^* *if and only if* G *is isomorphic to* G^*.

PROOF. By [10, theorem 2. 3] and [10, theorem 1. 4], the right unit subsemigroups of S and S^* are $P=(G, (I^0)^n, \gamma_1, \gamma_2, \ldots, \gamma_n, w_1, w_2, \ldots, w_{\varphi(n)})$ and $P^*=(G^*, (I^0)^n, \alpha_1, \alpha_2, \ldots, \alpha_n, t_1, t_2, \ldots, t_{\varphi(n)})$ respectively. If we apply [1, main theorem] to $P=(G, (I^0)^n, \gamma_1, \gamma_2, \ldots, \gamma_n, w_1, w_2, \ldots, w_{\varphi(n)})$, we will denote the resulting semigroup by T. Let $e(e^*)$ denote the identity of $G(G^*)$. Since the group of units of P is $\{(g, 0, 0, \ldots, 0) : g \in G\}$ by [10, theorem 1. 4], it follows from [1, p. 548, equation 1. 2] that each element of T may be given the "normal representation" $((e, a_1, a_2, \ldots, a_n), (g, b_1, b_2, \ldots, b_n))$ (notation of [1, main theorem]). We showed in [10, proof of theorem 2. 3] that $((e, a_1, a_2, \ldots, a_n), (g, b_1, b_2, \ldots, b_n))\delta = (g, (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n))$ defines an isomorphism of T onto $S = (G, C_n, \gamma_1, \gamma_2, \ldots, \gamma_n, w_1, w_2, \ldots, w_{\varphi(n)})$. Let S^* denote the corresponding isomorphism of T^* onto S^*. Let Φ be a homorphism of S onto S^*. Hence, $M = \delta\Phi(\delta^*)^{-1}$ is a homomorphism of T onto T^*. Thus, M is given by [3, p. 1114, equation 1. 14] by virtue of [3, p. 1113, theorem 1. 1]. Since M maps T onto T^*, it follows that the k in [3, p. 1114, equation 1. 14] is an element of the group of units of P^* by virtue of [1, p. 548, equation 1. 2]. Hence, by [3, p. 1113, equation 1. 13], k may be taken to be the identity of P^*. (Actually, the homomorphism N in [3, equation 1. 14] is replaced by $NC_k - 1$ where $xC_k - 1 = k^{-1}xk$ for $x \in G^*$.) By [11, theorem 1. 1], M maps E_T isomorphically into E_{T^*}. If $(h, c_1, c_2, \ldots, c_n) \in P^*$, there exists $(e, a_1, a_2, \ldots, a_n)$, $(g, b_1, b_2, \ldots, b_n) \in P$ such that

$$[(e^*, 0, \ldots, 0), (h, c_1, \ldots, c_n)] = ((e, a_1, \ldots, a_n), (g, b_1, \ldots, b_n))M =$$

$$= ((e, a_1, \ldots, a_n)N, (g, b_1, \ldots, b_n)N)$$

(notation of [3, theorem 1. 1]) since M maps T onto T^*. Hence, utilizing [1, equation 1. 2 and equation 1. 3], we obtain

$$((e, a_1, \dots, a_n), (e, a_1, \dots, a_n))M = ((e, a_1, \dots, a_n), (g, b_1, \dots, b_n))M \cdot$$
$$\cdot ((g, b_1, \dots, b_n), (e, a_1, \dots, a_n))M = ((e^*, 0, \dots, 0), (e^*, 0, \dots, 0)).$$

Since $(e, 0, \dots, 0)$ is the identity of P by [10, theorem 1. 4], $((e,0,\dots,0),(e,0,\dots,0))$ is the identity of T by [3, p. 1113, equation 1. 6]. Hence,

$$((e, a_1, \dots, a_n), (e, a_1, \dots, a_n)) = ((e, 0, \dots, 0), (e, 0, \dots, 0)).$$

Thus, $a_i = 0$ for $1 \leqq i \leqq n$ by [1, equation 1. 2]. Hence, by [1, equation 1. 2], $(h, c_1, \dots, c_n) = (g, b_1, \dots, b_n)N$ and N is a homomorphism of P onto P^* by [3, theorem 1. 1]. Thus, N is given by [12, theorem 1] and (1. 1) and (1. 2) are valid by virtue of [12, theorem 1]. If we substitute N in the modified form ($k = e^*$) of [3, equation 1. 14] we obtain

$$(1.4)\quad ((e, a_1, \dots, a_n), (g, b_1, \dots, b_n))M = \Bigg[\Bigg(\Bigg(\prod_{j=0}^{a_n-1} z_n \alpha_n^j\Bigg)\Bigg(\prod_{j=0}^{a_{n-1}-1} z_{n-1}\alpha_{n-1}^j \alpha_n^{a_n}\Bigg) \dots$$
$$\dots \Bigg(\prod_{j=0}^{a_1-1} z_1 \alpha_1^j \alpha_2^{a_2} \dots \alpha_n^{a_n}\Bigg), a_1, \dots, a_n\Bigg), \Bigg((gf)\Bigg(\prod_{j=0}^{b_n-1} z_n \alpha_n^j\Bigg) \dots$$
$$\dots \Bigg(\prod_{j=0}^{b_1-1} z_1 \alpha_1^j \alpha_2^{b_2} \dots \alpha_n^{b_n}\Bigg), b_1, \dots, b_n\Bigg)\Bigg]$$

for $a_i, b_i \geqq 1$. If $a_i(b_i) = 0$, the corresponding factor is e^*.

If we apply [1, equation 1. 2] to the right hand side of (1. 4), we obtain

$$(1.5)\quad ((e, a_1, \dots, a_n), (g, b_1, \dots, b_n))M =$$
$$= \Bigg((e, a_1, \dots, a_n), \Bigg(\Bigg(\prod_{j=a_1-1}^{0} z_1^{-1} \alpha_1^j \alpha_2^{a_2} \dots \alpha_n^{a_n}\Bigg) \dots$$
$$\dots \Bigg(\prod_{j=a_n-1}^{0} z_n^{-1} \alpha_n^j\Bigg)(gf)\Bigg(\prod_{j=0}^{b_n-1} z_n \alpha_n^j\Bigg)\Bigg(\prod_{j=0}^{b_{n-1}-1} z_{n-1}\alpha_{n-1}^j \alpha_n^{b_n}\Bigg) \dots$$
$$\dots \Bigg(\prod_{j=0}^{b_1-1} z_1 \alpha_1^j \alpha_2^{b_2} \dots \alpha_n^{b_n}\Bigg), b_1, \dots, b_n\Bigg)\Bigg).$$

Thus, $\Phi = \delta^{-1} M \delta^*$ is given by (1. 3).

Conversely, suppose that the conditions of the theorem are satisfied. Hence, [12, theorem 1] defines a homomorphism N of P onto P^*. By [10, theorem 1. 4], $(g, a_1, \dots, a_n) \mathsf{L} (h, b_1, \dots, b_n)$ if and only if $a_i = b_i$ for $1 \leqq i \leqq n$. Hence, it is easy to see that N is a sl-homomorphism [3, p. 1113] of P onto P^*. Thus, repeating the above procedure, Φ or (1. 3) defines a homomorphism of S onto S^*.

Suppose that there exists an isomorphism f of G onto G^*. Thus, there exists an isomorphism N of P onto P^* by [12, theorem 1]. Hence, M is an isomorphism of T onto T^* by [3, theorem 1. 2]. Thus, Φ is an isomorphism of S onto S^*. Conversely, suppose that there exists an isomorphism Φ of S onto S^*. Thus, M is an isomorphism of T onto T^*. Thus, by [3, theorem 1. 2], there exists an isomorphism N of P onto P^*. Hence, by [12, theorem 1], there exists an isomorphism f of G onto G^*.

REMARK. In the case $n=1$, we obtain [9, theorem 2. 1] and in the case $n=2$ we obtain [10, theorem 3. 2].

We next provide a converse to [11, lemma 1. 1]. This theorem is utilized in determining the types of congruences that are admitted by an ω^n-bisimple semigroup and in giving a complete determination of the ω^k-bisimple congruences ($k<n$) of an ω^n-bisimple semigroup [11].

THEOREM 1. 2. *Let* $S=(G,\gamma_1,\gamma_2,\dots,\gamma_n,w_1,w_2,\dots,w_{\varphi(n)})$ *and* $S^*=(G^*,\alpha_1,\alpha_2,\dots$ $\dots,\alpha_k,t_1,t_2,\dots,t_{\varphi(k)})$ *be* ω^n *and* ω^k*-bisimple semigroups respectively with* $n>k$. *Let* Φ *be a homomorphism of* S *onto* S^*. *Then there exists a homomorphism* f *of* G *into* G^* *and elements* $z_1, z_2, \dots, z_n$ *of* G^* *such that*

$$z_{l+s}\alpha_l t_{\varphi(k-l)+s}C_{z_l} = w_{\varphi(n-l)+s}f \quad \text{if} \quad 1\leqq l<k, \quad 1\leqq s\leqq k-l. \tag{1. 6}$$

$$z_{l+s}\alpha_l C_{z_l} = w_{\varphi(n-l)+s}f \quad \text{if} \quad k-l+1\leqq s\leqq n-l, \quad 1\leqq l\leqq k. \tag{1. 7}$$

$$z_{l+s}C_{z_l} = w_{\varphi(n-l)+s}f \quad \text{if} \quad k<l<n, \quad 1\leqq s\leqq n-l. \tag{1. 8}$$

$$f\alpha_i C_{z_i} = \gamma_i f \quad \text{if} \quad 1\leqq i\leqq k. \tag{1. 9}$$

$$fC_{z_i} = \gamma_i f \quad \text{if} \quad 1+k\leqq i\leqq n. \tag{1. 10}$$

For each $(g,(a_1,b_1),(a_2,b_2),\dots,(a_n,b_n))\in S$,

$$(g,(a_1,b_1),(a_2,b_2),\dots,(a_n,b_n))\Phi =$$

$$=\Bigg(\Bigg(\prod_{a_1-1}^{j=0} z_1^{-1}\alpha_1^j\alpha_2^{a_2}\dots\alpha_k^{c_k}\Bigg)\dots\Bigg(\prod_{a_{k-1}-1}^{j=0} z_{k-1}^{-1}\alpha_{k-1}^j\alpha_k^{a_k}\Bigg)\Bigg(\prod_{a_k-1}^{j=0} z_k^{-1}\alpha_k^j\Bigg)\cdot$$

$$\cdot z_{k+1}^{-a_{k+1}}\dots z_n^{-a_n}gf\cdot z_n^{b_n}\dots z_{k+1}^{b_{k+1}}\Bigg(\prod_{j=0}^{b_k-1} z_k\alpha_k^j\Bigg)\Bigg(\prod_{j=0}^{b_{k-1}-1} z_{k-1}\alpha_{k-1}^j\alpha_k^{b_k}\Bigg)\dots$$

$$\dots\Bigg(\prod_{j=0}^{b_1-1} z_1\alpha_1^j\alpha_2^{b_2}\dots\alpha_k^{b_k}\Bigg),(a_1,b_1),(a_k,b_k)\Bigg)$$

if $a_1,b_1,\dots,a_k,b_k\geqq 1$. *If* $a_j(b_j)=0$ *for some* j, *the corresponding factor is* e^*, *the identity of* G^*.

Conversely, let $z_1,z_2,\dots,z_n$ *be elements of* G^* *and let* f *be a homomorphism* G *onto* G^* *such that* (1. 6)—(1. 10) *are valid. Then,* Φ *defines a homomorphism of* S *onto* S^*.

PROOF. The direct part of the theorem is precisely [11, lemma 1. 1]. The proof of the converse is similar to the proof of the converse of theorem 1. 1 except that we utilize [12, theorem 2] instead of [12, theorem 1].

Before proving the structure theorem, we will need the following result.

THEOREM 1. 3 (WARNE, [4], [5]). *Let* S *be a regular bisimple semigroup and let* $e\in E_S$. *Then,* eSe *is a regular bisimple semigroup with identity* e. *If* E_S *is linearly ordered,* $S=\bigcup(eSe: e\in E_S)$ *and* $eSe\subset fSf$ *if and only if* $e\leqq f$ *and each* eSe *is a bisimple inverse semigroup with identity* e *with* $E_{eSe}=\{f\in E_S\ f\leqq e\}$.

We will sketch the proof of the following theorem at points where it closely parallels the proof of [10, theorem 3. 3].

THEOREM 1. 4. *S is an ω^{n-1} I-bisimple semigroup if and only if $S \cong G \times C_n^*$ where G is a group and C_n^* is the extended $2n$-cyclic semigroup under the multiplication*

$$(X, (a_1, b_1), \dots, (a_n, b_n))(Y, (c_1, d_1), \dots, (c_n, d_n)) =$$
$$= (t, ((a_1, b_1), \dots, (a_n, b_n))((c_1, d_1), \dots, (c_n, d_n)))$$

where

$$t = X\left(\left(f^{-1}_{b_1-c_1, c_1}\left(\left(\left(\prod_{k=n-2}^{k=0} z^{-c_{n-k}}_{\varphi(n)-k}\right)(Y\beta_1)\left(\prod_{k=0}^{n-2} z^{d_{n-k}}_{\varphi(n)-k}\right)\right)\beta_1^{b_1-c_1-1}\right)f_{b_1-c_1, d_1}\right)\prod_{i=2}^{n}\beta_i^{b_i}\right)\cdot$$
$$\cdot\left(\left(f^{-1}_{c_1-b_1, a_1}\left(\left(\left(\prod_{k=n-2}^{k=0} z^{-a_{n-k}}_{\varphi(n)-k}\right)(X\beta_1)\left(\prod_{k=0}^{k=n-2} z^{b_{n-k}}_{\varphi(n)-k}\right)\right)\beta_1^{c_1-b_1-1}\right)f_{c_1-b_1, b_1}\right)\prod_{i=2}^{n}\beta_i^{c_i}Y\right)\cdot$$
$$\cdot\left(X\left(\left(\left(\prod_{k=n-l-1}^{k=0} z^{-c_{n-k}}_{\varphi(n-l+1)-k}\right)(Y\beta_l)\left(\prod_{k=0}^{n-l-1} z^{d_{n-k}}_{\varphi(n-l+1)-k}\right)\right)\beta_l^{b_l-c_l-1}\prod_{i=l+1}^{n}\beta_i^{b_i}\right)\right)\cdot$$
$$\cdot\left(\left(\left(\prod_{k=n-l-1}^{k=0} z^{-a_{n-k}}_{\varphi(n-l+1)-k}\right)(X\beta_1)\left(\prod_{k=0}^{n-l-1} z^{b_{n-k}}_{\varphi(n-l+1)-k}\right)\right)\beta_l^{c_l-b_l-1}\right)\prod_{i=l+1}^{n}\beta_i^{c_i}Y\right)$$

($X\beta_n^{c_n-s}Y\beta_n^{b_n-s}$ where $s = \min(b_n, c_n)$) if $b_1 > c_1$ ($c_1 > b_1$) ($b_l > c_l$ where $l \in \{2, \dots, n-l\}$ and $b_i = c_i$ for $1 \leqq i \leqq l-1$) ($c_l > b_l$ where $l \in \{2, \dots, n-l\}$ and $b_i = c_i$ for $1 \leqq i \leqq l-1$) (otherwise). Juxtaposition denotes multiplication in G and C_n^* and $\varphi(x) = x(x-1)/2$ for $x \in I^0$ and $x \geqq 1$. $\{\beta_i : 1 \leqq i \leqq n\}$ is a collection of endomorphisms of G and $\{z_i : 1 \leqq i \leqq \varphi(n)\}$ is a collection of elements of G such that

(A) $$\beta_{k+l}\beta_l = \beta_l C_{z_{\varphi(n-l)+k}}$$

where $1 \leqq l \leqq n-1$ and $1 \leqq k \leqq n-l$ and

(B) $$z_{\varphi(n-l+1)-i} C_{z_{\varphi(n-l)+k}} = z_{\varphi(n-k-l+1)-i}\beta_l$$

where $1 \leqq l \leqq n-2$, $1 \leqq k \leqq n-l-1$, and $0 \leqq i \leqq n-k-l-1$. For $s \in I^0$, $t \in I$,

$$f_{0,t} = e, \text{ the identity of } G, \text{ while if } s > 0$$
$$f_{s,t} = m_{t+1}\beta_1^{s-1} m_{t+2}\beta_1^{s-2} \dots m_{t+(s-1)}\beta_1 m_{t+s}$$

where $\{m_t : t \in I\}$ is a collection of elements of G with $m_t = e$, if $t > 0$.

PROOF. Let $S \cong G \times C_n^*$, where G is a group, under the multiplication given in the statement of the theorem. For each $i \in I$, let $S_i = \{g, (a_1 + i, b_1 + i), (a_2, b_2), \dots$ $\dots, (a_n, b_n) : g \in G,\ a_i, b_i \in I^0$ for $1 \leqq i \leqq n\}$. Then, $S = \bigcup(S_i : i \in I, i \leqq 0)$ and $S_i \subset S_j$ if and only if $i \geqq j$. By [10, theorem 2. 3], S_0 is an ω^n-bisimple semigroup. For $i \in I$, $i \leqq 0$, and $(X, (a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)) \in S_0$, let

$$(X, (a_1, b_1), (a_2, b_2), \dots, (a_n, b_n))\Phi_i =$$
$$= \left(\left(f^{-1}_{a_1, i}\prod_{i=2}^{n}\beta_i^{a_i}\right)X\left(f_{b_1, i}\prod_{i=2}^{n}\beta_i^{b_i}\right), (a_1 + i, b_1 + i), (a_2, b_2), \dots, (a_n, b_n)\right).$$

Clearly, Φ_i is a 1—1 mapping of S_0 onto S_i. We shall show that Φ_i is an isomorphism. To do this, we first note that

$$f_{s,t}\beta_1^c = f_{s+c,t}f_{c,s+t}^{-1} \quad \text{for} \quad s, c \in I^0 \quad \text{and} \quad t \in I \tag{1. 11}$$

and, by [10, (3) of theorem 2. 3],

$$\left(g \prod_{i=l+1}^{n} \beta_i^{c_i}\right)\beta_1 = \left(\prod_{k=0}^{n-l-1} z_{\varphi(n-l+1)-k}^{c_{n-k}}\right)(g\beta_1)\left(\prod_{k=n-l-1}^{0} z_{\varphi(n-l+1)-k}^{-c_{n-k}}\right) \tag{1. 12}$$

where $c_i \in I^0$ for $1 \leqq i \leqq n$, $k \in I^0$, and $1 \leqq l \leqq n-1$.

Let us first suppose that $b_1 > c_1$. Hence, by (1. 11) and (1. 12),

$$(X, (a_1, b_1), \dots, (a_n, b_n))\Phi_i(Y, (c_1, d_1), \dots, (c_n, d_n))\Phi_i =$$

$$= \left(\left(f_{a_1,i}^{-1} \prod_{i=2}^{n} \beta_i^{a_i}\right) X \left(f_{b_1,i} \prod_{i=2}^{n} \beta_i^{b_i}\right),\ (a_1+i, b_1+i), \dots, (a_n, b_n)\right) \cdot$$

$$\cdot \left(\left(f_{c_1,i}^{-1} \prod_{i=2}^{n} \beta_i^{c_i}\right) Y \left(f_{d_1,i} \prod_{i=2}^{n} \beta_i^{d_i}\right),\ (c_1+i, d_1+i), \dots, (c_n, d_n)\right) =$$

$$= \left(\left(\left(f_{a_1,i}^{-1} \prod_{i=2}^{n} \beta_i^{a_i}\right) X \left(f_{b_1,i} \prod_{i=2}^{n} \beta_i^{b_i}\right)\right)\left(\left(f_{b_1-c_1,c_1+i}^{-1} \cdot\right.\right.\right.$$

$$\cdot \left(\left(\left(\prod_{k=n-2}^{k=0} z_{\varphi(n)-k}^{-c_{n-k}}\right)\left(\left(\left(f_{c_1,i}^{-1} \prod_{i=2}^{n} \beta_i^{c_i}\right) Y \left(f_{d_1,i} \prod_{i=2}^{n} \beta_i^{d_i}\right)\right)\beta_1\right)\cdot\right.\right.$$

$$\cdot \left.\left.\left(\prod_{k=0}^{n-2} z_{\varphi(n)-k}^{d_{n-k}}\right)\right)\right)\beta_1^{b_1-c_1-1}\right) f_{b_1-c_1,d_1+i}\Bigg) \prod_{i=2}^{n} \beta_i^{b_i}\Bigg),$$

$$(a_1+i, b_1-c_1+d_1+i), (a_2, b_2), \dots, (a_n, b_n)\Bigg) =$$

$$= \left(\left(f_{a_1,i}^{-1} \prod_{i=2}^{n} \beta_i^{a_i}\right) X \left(\left(\left(\left(\left(\prod_{k=n-2}^{k=0} z_{\varphi(n)-k}^{-c_{n-k}}\right)(Y\beta_1)\left(\prod_{k=0}^{n-2} z_{\varphi(n)-k}^{d_{n-k}}\right)\right)\right)\cdot\right.\right.\right.$$

$$\left.\left.\cdot \beta_1^{b_1-c_1-1}\right)(f_{d_1,i}\beta_1^{b_1-c_1})f_{b_1-c_1,d_1+i}\right) \prod_{i=2}^{n} \beta_i^{b_i}\Bigg),$$

$$(a_1+i, b_1-c_1+d_1+i), (a_2, b_2), \dots, (a_n, b_n)\Bigg) =$$

$$= \left(\left(f_{a_1,i}^{-1} \prod_{i=2}^{n} \beta_i^{a_i}\right) X \left(\left(\left(\left(\left(\prod_{k=n-2}^{k=0} z_{\varphi(n)-k}^{-c_{n-k}}\right)(Y\beta_1)\left(\prod_{k=0}^{n-2} z_{\varphi(n)-k}^{d_{n-k}}\right)\right)\right)\beta_1^{b_1-c_1-1}\right)\cdot\right.\right.$$

$$\left.\left.\cdot f_{d_1+(b_1-c),i}\right) \prod_{i=2}^{n} \beta_i^{b_i}\right),\ (a_1+i, b_1+d_1-c_1+i), (a_2, b_2), \dots, (a_n, b_n)\Bigg) =$$

$$= \left(\left(X\left(\left(\left(\prod_{k=n-2}^{k=0} z_{\varphi(n)-k}^{-c_{n-k}}\right)(Y\beta_1)\left(\prod_{k=0}^{n-2} z_{\varphi(n)-k}^{d_{n-k}}\right)\right)\right)\beta_1^{b_1-c_1-1} \prod_{i=2}^{n} \beta_i^{b_i}\right),\right.$$

$$\left.(a_1, b_1-c_1+d_1), (a_2, b_2), \dots, (a_n, b_n)\right)\Bigg)\Phi_i =$$

$$= ((X, (a_1, b_1), \dots, (a_n, b_n))(Y_1(c_1, d_1), \dots, (c_n, d_n)))\Phi_i.$$

The case $c_1 > b_1$ is similar. Next, we suppose that $b_s = c_s$ for $1 \leqq s \leqq l-1$ and $b_l > c_l$ where $2 \leqq l \leqq n-1$. Thus, utilizing (1. 12),

$$(X, (a_1, b_1), \dots, (a_n, b_n))\Phi_i(Y, (c_1, d_1), \dots, (c_n, d_n))\Phi_i =$$

$$= \left(\left(f_{a_1,i}^{-1} \prod_{i=2}^{n} \beta_i^{a_i}\right) X \left(f_{b_1,i} \prod_{i=2}^{n} \beta_i^{b_i}\right), (a_1+i, b_1+i), \dots, (a_n, b_n)\right) \cdot$$

$$\cdot \left(\left(f_{c_1,i}^{-1} \prod_{i=2}^{n} \beta_i^{c_i}\right) Y \left(f_{d_1,i} \prod_{i=2}^{n} \beta_i^{d_i}\right), (c_1+i, d_1+i), \dots, (c_n, d_n)\right) =$$

$$= \left(\left(f_{a_1,i}^{-1} \prod_{i=2}^{n} \beta_i^{a_i}\right) X \left(f_{b_1,i} \prod_{i=2}^{n} \beta_i^{b_i}\right) \left(\left(\left(\prod_{k=n-l-1}^{0} z_{\varphi(n-l+1)-k}^{-c_{n-k}}\right) \cdot\right.\right.\right.$$

$$\cdot \left(\left(\left(f_{c_1,i}^{-1} \prod_{i=2}^{n} \beta_i^{c_i}\right) Y \left(f_{d_1,i} \prod_{i=2}^{n} \beta_i^{d_i}\right)\right) \beta_1\right) \left(\prod_{k=0}^{n-l-1} z_{\varphi(n-l+1)-k}^{d_{n-k}}\right)\right) \cdot$$

$$\cdot \beta_l^{b_l - c_l - 1} \left.\prod_{i=l+1}^{n} \beta_i^{b_i}\right), (a_1+i, d_1+i), (a_2, d_2), \dots$$

$$\dots, (a_{l-1}, d_{l-1}), (a_l, b_l + d_l - c_l), (a_{l+1}, b_{l+1}), \dots, (a_n, b_n)\Big) =$$

$$= \left(\left(f_{a_1,i}^{-1} \prod_{i=2}^{n} \beta_i^{a_i}\right) X \left(f_{b_1,i} \prod_{i=2}^{n} \beta_i^{b_i}\right) \left(\left(\left(\left(f_{c_1,i}^{-1} \prod_{i=2}^{l} \beta_i^{c_i}\right) \beta_l\right) \left(\prod_{k=n-l-1}^{k=0} z_{\varphi(n-l+1)-k}^{-c_{n-k}}\right) \cdot\right.\right.\right.$$

$$\cdot (Y\beta_l) \left(\prod_{k=0}^{n-l-1} z_{\varphi(n-l+1)-k}^{d_{n-k}}\right) \left(\left(f_{d_1,i} \prod_{i=2}^{l} \beta_i^{d_i}\right) \beta_1\right)\right) \beta_l^{b_l - c_l - 1} \left.\prod_{i=l+1}^{n} \beta_i^{b_i}\right),$$

$$(a_1+i, d_1+i), (a_2, d_2), \dots, (a_{l-1}, d_{l-1}), (a_l, b_l + d_l - c_l), (a_{l+1}, b_{l+1}), \dots, (a_n, b_n)\Big) =$$

$$= \left(X \left(\left(\left(\prod_{k=n-l-1}^{0} z_{\varphi(n-l+1)-k}^{-c_{n-k}}\right) (Y\beta_l) \prod_{k=0}^{n-l-1} z_{\varphi(n-l+1)-k}^{d_{n-k}}\right) \beta_l^{b_l - c_l - 1} \prod_{i=l+1}^{n} \beta_i^{b_i}\right),\right.$$

$$(a_1, d_1), \dots, (a_{l-1}, d_{l-1}), (a_l, b_l + d_l - c_l), (a_{l+1}, b_{l+1}), \dots, (a_n, b_n)\Big) \Phi_i =$$

$$= ((X, (a_1, b_1), \dots, (a_n, b_n))(Y, c_1, d_1), \dots, (c_n, d_n)))\Phi_i.$$

The case $b_s = c_s$ for $1 \leqq s \leqq l-1$ where $2 \leqq l \leqq n-1$ and $c_l > b_l$ is similar. The other cases are proved in a straightforward manner.

By methods of the proof of [10, theorem 3. 3], S is an $\omega^{n-1}I$-bisimple semigroup.

Let S^* be an $\omega^{n-1}I$-bisimple semigroup and let $E_{S^*} = (e_{(k_1, k_2, \dots, k_n)}: k_1 \in I,$ $k_i \in I^0$ for $2 \leqq i \leqq n)$ under the reverse of the usual lexicographic order. If we let $S_i^* = e_{(i,0,\dots,0)} S^* e_{(i,0,\dots,0)}$, $S^* = \cup(S_i^*: i \in I, i \leqq 0)$ and $S_i^* \subset S_j^*$ if and only if $i \geqq j$ by theorem 1. 3. By theorem 1. 3, each S_i^* is an ω^n-bisimple semigroup with $E_{S_i^*} = \{e_{(k_1,k_2,\dots,k_n)}: k_1 \in I,\ k_1 \geqq i \text{ and } k_i \in I^0 \text{ for } 2 \leqq i \leqq n\}$. Thus, by [10, theorem 2. 3], we may write $S_i^* = (G_i, C_n, \theta_{i,1}, \theta_{i,2}, \dots, \theta_{i,n}, t_{i,1}, t_{i,2}, \dots, t_{i,\varphi(n)})_i$, say. Let $G = G_0, \gamma_{0,j} = \theta_{0,j}$ for $1 \leqq j \leqq n$, and $w_{0,j} = t_{0,j}$ for $1 \leqq j \leqq \varphi(n)$. Thus, we may write $S_0^* = (G_0, C_n, \theta_{0,1}, \theta_{0,2}, \dots, \theta_{0,n}, t_{0,1}, t_{0,2}, \dots, t_{0,\varphi(n)})_0 = [G, C_n, \gamma_{0,1}, \gamma_{0,2}, \dots, \gamma_{0,n}, w_{0,1}, w_{0,2}, \dots, w_{0,\varphi(n)}]_0 = S_0'$. Suppose that $S_{i+1}^* = [G, C_n, \gamma_{i+1,1}, \dots, \gamma_{i+1,n}, w_{i+1,1} \dots w_{i+1,\varphi(n)}]_{i+1} = S_{i+1}'$. Utilizing [10, theorem 2. 3], $[g, (0,0), \dots, (0,0)]_{i+1} = (gf_i, (l,l), (0,0), \dots, (0,0))_i$, where f_i is an isomorphism of G onto G_i. Suppose that $[e, (0,0), \dots, (0,1), (0,0), \dots, (0,0)]_{i+1} = (y_{i,k}, (l,l), (0,0), \dots, (0,1), (0,0), \dots, (0,0))_i$ where $(0,1)$ is the k^{th} 2-tuple $(k>1)$ and $y_{i,k} \in G_i$. For $g \in G$, let $g\gamma_{i,1} = gf_i\theta_{i,1}f_i^{-1}, g\gamma_{i,j} = gf_i\theta_{i,j}f_i^{-1}C_{y_{i,j}f_i^{-1}}$ for $j>1$, and $w_{i,\varphi(n-k)+s} = (y_{i,k+s}f_i^{-1})\gamma_{i,k}(t_{i,\varphi(n-k)+s}f_i^{-1})C_{y_{i,k}f_i^{-1}}$ for $1 \leqq k < n$ and $1 \leqq s \leqq n-k$. Hence, by theorem 1. 1, we may set

$$(g, (a_1, b_1), \dots, (a_n, b_n))_i =$$

$$= \Bigg[\Bigg(\prod_{j=a_2-1}^{0} y_{i,2}f_i^{-1}\gamma_{i,2}^{j}\gamma_{i,3}^{a_3}\cdots\gamma_{i,n}^{a_n}\Bigg)\cdots\Bigg(\prod_{j=a_{n-1}-1}^{0} y_{i,n-1}f_i^{-1}\gamma_{i,n-1}^{j}\gamma_{i,n}^{a_n}\Bigg)\cdot$$

$$\cdot\Bigg(\prod_{j=a_n-1}^{0} y_{i,n}f_i^{-1}\gamma_n^{j}\Bigg)(gf_i^{-1})\Bigg(\prod_{j=0}^{b_n-1} y_{i,n}^{-1}f_i^{-1}\gamma_{i,n}^{j}\Bigg)\cdots$$

$$\cdots\Bigg(\prod_{j=0}^{b_2-1} y_{i,2}^{-1}f_i^{-1}\gamma_{i,2}^{j}\gamma_{i,3}^{b_3}\cdots\gamma_{i,n}^{b_n}\Bigg), (a_1, b_1), \dots, (a_n, b_n)\Bigg]_i$$

if $a_i, b_i \geqq 1$ $(1 \leqq i \leqq n)$. If $a_k = 0$ $(b_k = 0)$ for some k, we let the corresponding factor be e, the identity of G. Thus, we may let $S^* = \bigcup(S_i' : i \in I, i \leqq 0)$ where $S_i = [G, \gamma_{i,1}, \dots, \gamma_{i,n}, w_{i,1}, \dots, w_{\varphi(n)}]_i$ and

$$[g, (0,0), \dots, (0,0)]_{i+1} = [g, (1,1), (0,0), \dots, (0,0)]_i \tag{1. 13}$$

$$[e, (0,0), \dots, (0,1), (0,0), \dots, (0,0)]_{i+1} = [e, (1,1), (0,0), \dots, (0,1), (0,0), \dots, \dots, (0,0)]_i \tag{1. 14}$$

where $(0, 1)$ is the k^{th} 2-tuple $(k>1)$.

Clearly,

$$[e, (0,1), (0,0), \dots, (0,0)]_{i+1} = [x_i, (1,2), (0,0), \dots, (0,0)]_i \tag{1. 15}$$

for some $x_i \in G$.

Hence, utilizing (1. 13), (1. 14), (1. 15), and [10, theorem 2. 3], we obtain

$$[g, (a_1, b_1), \dots, (a_n, b_n)]_{i+1} = \tag{1. 16}$$

$$= ([e, (1,0), (0,0), \dots, (0,0)]_{i+1})^{a_1}([e, (0,0), (1,0), (0,0), \dots, (0,0)]_{i+1})^{a_2}\cdots$$

$$\cdots([e, (0,0), \dots, (1,0)]_{i+1})^{a_n}[g, (0,0), \dots, (0,0)]_{i+1}\cdot$$

$$\cdot([e, (0,0), \dots, (0,1)]_{i+1})^{b_n}\cdots([e, (0,1), (0,0), \dots, (0,0)]_{i+1})^{b_1} =$$

$$= [(x_i^{-1}\gamma_{i,1}^{a_1-1}\cdots x_i^{-1}\gamma_{i,1}x_i^{-1})\gamma_{i,2}^{a_2}\cdots\gamma_{i,n}^{a_n}g\cdot$$

$$\cdot(x_i\cdot x_i\gamma_{i,1}\cdots x_i\gamma_{i,1}^{b_1-1})\gamma_{i,2}^{b_2}\cdots\gamma_{i,n}^{b_n}, (a_1+1, b_1+1), (a_2, b_2), \dots, (a_n, b_n)]_i$$

for $a_1 \geqq 1$, $b_1 \geqq 1$. If $a_1 = 0$ $(b_1 = 0)$, the corresponding factor is e.

As in the proof of [10, theorem 3. 3], we obtain

(1. 17) $$g\gamma_{i,1} = m_{i+1}^{-1} g\gamma_{0,1} m_{i+1} \quad \text{for} \quad i \leqq 0$$

and

(1. 18) $$m_{i+1} = x_{-1}x_{-2}\dots x_i \quad \text{for} \quad i \leqq -1 \quad \text{and} \quad m_{i+1}=e \quad \text{for} \quad i \geqq 0.$$

(Actually, we just substitute "$[h, (a_1, b_1), (a_2, b_2), (0, 0), \dots, (0, 0)]_i$" for "$[h, (a_1, b_1), (a_2, b_2)]_i$", "$\gamma_{i,1}$" for "$\gamma_i$", "(1. 16)" for "(3. 4)", and "[10, theorem 2. 3]" for "(3. 3)" in the corresponding part of the proof of [10, theorem 3. 3]).

In the remainder of the proof, we will use the multiplication in [10, theorem 2. 3] without explicit mention. By (1. 16),

$$[e, (0, 0), \dots, (1, 1), (0, 0), \dots, (0, 0)]_{i+1}[g, (0, 0), \dots, (0, 0)]_{i+1} =$$
$$= [g\gamma_{i+1,s}, (0, 0), \dots, (1, 1), (0, 0), \dots, (0, 0)]_{i+1} =$$
$$= [g\gamma_{i+1,s}, (1, 1), (0, 0), \dots, (1, 1), (0, 0), \dots, (0, 0)]_{i+1} =$$
$$= [e, (1, 1), (0, 0), \dots, (1, 1), (0, 0), \dots, (0, 0)]_i \cdot [g, (1, 1), (0, 0), \dots, (0, 0)]_i =$$
$$= [g\gamma_{i,s}, (1, 1), (0, 0), \dots, (1, 1), (0, 0), \dots, (0, 0)]_i$$

where (1, 1) occurs as the first and s^{th} 2-tuple with $s>l$. Hence, $g\gamma_{i+1,s}=g\gamma_{i,s}$ for $s>1$. Therefore,

(1. 19) $$\gamma_{i,s}=\gamma_{0,s} \quad \text{for all} \quad i \leqq 0 \quad \text{and} \quad s>1.$$

By (1. 16),

$$[e, (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_{i+1}[e, (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_{i+1} =$$
$$= [w_{i+1,\varphi(n-l)+s}, (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_{i+1} =$$
$$= [e, (1, 1), (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_i \cdot$$
$$\cdot [e, (1, 1), (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_i =$$
$$= [w_{i,\varphi(n-l)+s}, (1, 1), (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_i =$$
$$= [w_{i+1,\varphi(n-l)+s}, (1, 1), (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_i$$

where on the left hand side (0, 1) occurs as the l^{th} 2-tuple and the $(l+s)^{\text{th}}$ 2-tuple with $l>1$ and $1 \leqq s \leqq n-l$ (reading from the left to right). Hence, $w_{i+1,\varphi(n-l)+s} = w_{i,\varphi(n-l)+s}$ for $l>1$ and $1 \leqq s \leqq n-l$. Thus,

(1. 20) $$w_{i,\varphi(n-l)+s} = w_{0,\varphi(n-l)+s} \quad \text{for} \quad l>1 \quad \text{and} \quad 1 \leqq s \leqq n-l.$$

Similarly,

$$[e, (0, 1), (0, 0), \dots, (0, 0)]_{i+1}[e, (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_{i+1} =$$
$$= [w_{i+1,\varphi(n-1)+s}, (0, 1), (0, 0), \dots, (0, 0)]_{i+1} =$$
$$= [w_{i+1,\varphi(n-1)+s}x_i, (1, 2), (0, 0), \dots, (0, 0)]_i =$$
$$= [x_i, (1, 2), (0, 0), \dots, (0, 0)]_i[e, (1, 1), (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_i =$$
$$= [x_i w_{i,\varphi(n-1)+s}, (1, 2), (0, 0), \dots, (0, 0)]_i$$

where, in the right hand brackets on the left hand side, (0, 1) occurs as the $(s+1)^{\text{st}}$ 2-tuple with $1 \leqq s \leqq n-1$. Hence, $w_{i,\varphi(n-1)+s} = x_i^{-1} w_{i+1,\varphi(n-1)+s} x_i$. Thus,

(1.21) $$w_{i,\varphi(n-1)+s} = m_{i+1}^{-1} w_{0,\varphi(n-1)+s} m_{i+1}.$$

Let $\beta_l = \gamma_{0,l}$ for $1 \leqq l \leqq n$, let $z_s = w_{0,s}$ for $1 \leqq s \leqq \varphi(n)$, and let $(m_t : t \in I)$ be defined by (1.18). With β_l, z_s, and m_t so defined, let $S = G \times C_n^*$ under the multiplication given in the statement of the theorem. We will use this multiplication without explicit mention. It is easily seen that

$$S_i = (e, (i, i), (0, 0), \dots, (0, 0))\, S\, (e, (i, i), (0, 0), \dots, (0, 0)) =$$
$$((X, (a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)) : a_1, b_1 \in I \text{ with } a_1, b_1 \geqq i, a_s, b_s \in I^0 \text{ for } 2 \leqq s \leqq n))$$

where e is the identity of G, and that each S_i is an ω^n-bisimple semigroup with $E_{S_i} = ((e, (a_1, a_1), (a_2, a_2), \dots, (a_n, a_n)) : a_1 \in I$ with $a_1 \geqq i$ and $a_s \in I^0$ for $2 \leqq s \leqq n)$. By a routine calculation, every element of S_i may be uniquely expressed in the form $x = r_{i,1}^{-a_1} r_{i,2}^{-a_2} \dots r_{i,n}^{-a_n} g r_{i,n}^{b_n} \dots r_{i,s}^{b_s} r_{i,1}^{b_1}$ where $a_s, b_s \in I^0$ for $1 \leqq s \leqq n$, $r_{i,1} = (e, (i, i+1), (0, 0), \dots, (0, 0))$, $r_{i,s} = (e, (i, i), (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0))$ where (0, 1) is the s^{th} 2-tuple with $s > 1$, and $g = (g, (i, i), (0, 0), \dots, (0, 0))$ where $g \in G$. By a straightforward calculation, $r_{i,s} g = g\beta_{i,s} r_{i,s}$ for $1 \leqq s \leqq n$ where $\beta_{i,s}$ is an endomorphism of G. Again by a straightforward calculation, $r_{i,l} r_{i,k+l} = z_{i,\varphi(n-l)+k} r_{i,l}$ where $1 \leqq l \leqq n-1$, $1 \leqq k \leqq n-l$, and $z_{i,\varphi(n-l)+k} \in G$. Thus, as in the second proof of [10, theorem 2.3], $(r_{i,1}^{-a_1} r_{i,2}^{-a_2} \dots r_{i,n}^{-a_n} g r_{i,n}^{b_n} \dots r_{i,1}^{b_1})\varphi_i = \langle g, (a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\rangle_i$ defines an isomorphism of S_i onto $\langle G, C_n, \beta_{i,1}, \beta_{i,2}, \dots, \beta_{i,n}, z_{i,1}, z_{i,2}, \dots, z_{i,\varphi(n)}\rangle_i$. Thus, we lose no generality in assuming that $\langle g, (a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\rangle_i = r_{i,1}^{-a_1} r_{i,2}^{-a_2} \dots r_{i,n}^{-a_n} g r_{i,n}^{b_n} \dots r_{i,1}^{b_1}$. Since $r_{i,s} h = h\beta_{i,s} r_{i,s}$ for $s > 1$ and $h \in G$, $\beta_s = \beta_{i,s}$ for all $i \leqq 0$ and $s > 1$. Thus, since $\beta_s = \gamma_{0,s}$, $\beta_{i,s} = \gamma_{i,s}$ for $i \leqq 0$ and $s > 1$ by virtue of (1.19). Since $r_{i,1} h = h\beta_{i,1} r_{i,1}$ for $h \in G$, $h\beta_{i,1} = m_{i+1}^{-1} h\beta_1 m_{i+1}$. Thus, since $\beta_1 = \gamma_{0,1}$, $\beta_{i,1} = \gamma_{i,1}$ for $i \leqq 0$ by (1.17). Since $r_{i,l} r_{i,l+s} = z_{i,\varphi(n-l)+s} r_{i,l}$ for $i \leqq 0$, $2 \leqq l \leqq n-1$, and $1 \leqq s \leqq n-l$, $z_{i,\varphi(n-l)+s} = z_{\varphi(n-l)+s}$ for $i \leqq 0$, $2 \leqq l \leqq n-1$, and $1 \leqq s \leqq n-l$. Thus, since $z_{\varphi(n-l)+s} = w_{0,\varphi(n-l)+s}$, $z_{i,\varphi(n-l)+s} = w_{i,\varphi(n-l)+s}$ for $i \leqq 0$, $2 \leqq l \leqq n-1$, and $1 \leqq s \leqq n-l$ by (1.20). Since $r_{i,1} r_{i,s+1} = z_{i,\varphi(n-1)+s} r_{i,1}$ for $i \leqq 0$ and $1 \leqq s \leqq n-1$, $z_{i,\varphi(n-1)+s} = m_{i+1}^{-1} z_{\varphi(n-1)+s} m_{i+1}$. Thus, since $z_s = w_{0,s}$, $z_{i,\varphi(n-1)+s} = w_{i,\varphi(n-1)+s}$ for $i \leqq 0$ and $1 \leqq s \leqq n-1$ by (1.21). Hence $z_{i,s} = w_{i,s}$ for $1 \leqq s \leqq \varphi(n)$ and $i \leqq 0$. Thus, $[g, (a_1, b_1), \dots, (a_n, b_n)]_i \Phi_i = \langle g, (a_1, b_1), \dots, (a_n, b_n)\rangle_i$ defines an isomorphism of $S_i' = [G, \gamma_{i,1}, \dots, \gamma_{i,n}, w_{i,1}, \dots, w_{i,\varphi(n)}]_i$ onto $S_i = \langle G, C_n, \beta_{i,1}, \dots, \beta_{i,n}, z_{i,1}, \dots, z_{i,\varphi(n)}\rangle_i$. Since $r_{i+1,1}^{-0} r_{i+1,2}^{-0} \dots r_{i+1,n}^{-0} g r_{i+1,n}^{0} \dots r_{i+1,2}^{0} r_{i+1,1}^{0} = r_{i,1}^{-1} r_{i,2}^{-0} \dots r_{i,n}^{-0} g r_{i,n}^{0} \dots r_{i,2}^{0} r_{i,1}^{1}$,

(1.22) $$\langle g, (0, 0), \dots, (0, 0)\rangle_{i+1} = \langle g, (1, 1), (0, 0), \dots, (0, 0)\rangle_i.$$

Since $r_{i+1,1} = r_{i,1}^{-1} f_{1,i+1} f_{1,i} r_{i,1}^2$ and (1.18) is valid,

(1.23) $$\langle e, (0, 1), (0, 0), \dots, (0, 0)\rangle_{i+1} = \langle x_i, (1, 2), (0, 0), \dots, (0, 0)\rangle_i.$$

Since $r_{i+1,k} = r_{i,1}^{-1} e r_{i,k} r_{i,1}$ for $k > 1$,

(1.24) $$\langle e, (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)\rangle_{i+1} = \\ = \langle e, (1, 1), (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)\rangle_i$$

where (0, 1) is the k^{th} 2-tuple with $k > 1$.

Utilizing (1. 22)—(1. 24), as we utilized (1. 13)—(1. 15) above, we obtain

$$\langle g, (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\rangle_{i+1} = \tag{1. 25}$$

$$= \langle (x_i^{-1} \gamma_{i,1}^{a_1 - 1} \ldots x_i^{-1} \gamma_{i,1} x_i^{-1}) \gamma_{i,2}^{a_2} \ldots \gamma_{i,n}^{a_n} g \cdot$$

$$\cdot ((x_i \cdot x_i \gamma_{i,1}, \ldots, x_i \gamma_{i,1}^{b_1 - 1}) \gamma_{i,2}^{b_2} \ldots \gamma_{i,n}^{b_n}), (a_1+1, b_1+1), (a_2, b_2), \ldots, (a_n, b_n)\rangle_i$$

for $a_1, b_1 \geqq 1$. If $a_1 = 0$ ($b_1 = 0$), the corresponding factor is e. Let

$$x\Phi = x\Phi_i \quad \text{if} \quad x \in S_i'.$$

By virtue of (1. 16) and (1. 25), Φ is an isomorphism of S^* onto S.

Since $r_{0,l} r_{0,l+k} = z_{0,\varphi(n-l)+k} r_{0,l}$ for $1 \leqq l \leqq n-1$ and $1 \leqq k \leqq n-1$, and $r_{0,s} g = g\beta_{0,s} r_{0,s}$ for $1 \leqq s \leqq n$, $\beta_{0,s} = \beta_s$ for $1 \leqq s \leqq n$, and $z_{0,s} = z_s$ for $1 \leqq s \leqq \varphi(n)$, (A) and (B) may be established as in the second proof of [10, theorem 2. 3].

REMARK. Let S be an $\omega^{n-1} I$-bisimple semigroup. In view of theorem 1. 4, we will write $S = (G, C_n^*, \alpha_1, \alpha_2, \ldots, \alpha_n, w_1, w_2, \ldots, w_{\varphi(n)}, u_i)$ where G is the structure group of S; where C_n^* is the extended $2n$-cyclic semigroup; where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are endomorphisms of G (the structure endomorphisms of S); where $w_1, w_2, \ldots, w_{\varphi(n)}$ are elements of G (the distinguished elements of G); and where $\{u_i : i \in I\}$ with $u_i = e$ $i > 0$ is a sequence of elements of G (the distinguished sequence of G).

REMARK. Utilizing (2. 1) of theorem 2. 1, it is easy to construct an $\omega^n I$-bisimple semigroup with a non-trivial distinguished sequence.

2. The homomorphism theory. In this section, we give necessary and sufficient conditions for two $\omega^n I$-bisimple semigroups to be isomorphic. We also describe the homomorphisms of an $\omega^n I$-bisimple semigroup S onto an $\omega^n I$-bisimple semigroup S^* in terms of homomorphisms of the structure group G of S onto the structure group G^* of S^*. In addition to the structural properties of $\omega^n I$-bisimple semigroups, we will utilize theorem 1. 1 in our development.

If G is a group and $x \in G$, $xC_y = yxy^{-1}$ where $y \in G$.

THEOREM 2. 1. *Let* $S = (G, C_n^*, \alpha_1, \alpha_2, \ldots, \alpha_n, \; w_1, w_2, \ldots, w_{\varphi(n)}, u_i)$ *and* $S^* = (G^*, C_n^*, \beta_1, \beta_2, \ldots, \beta_n, \; r_1, r_2, \ldots, r_{\varphi(n)}, v_i)$ *be* $\omega^{n-1} I$*-bisimple semigroups. Then* S *is isomorphic to* S^* *if and only if there exist* n *sequences* t_s $(1 \leqq s \leqq n)$*, where*

$$t_s = \{z_{i,s} \colon i \in I, \; i \leqq 0\},$$

of elements of G^**, a sequence* $\{f_i : i \in I, \; i \leqq 0\}$ *of isomorphisms of* G *onto* G^**, and* $a \in I$ *such that for all* $i \in I$, $i \leqq 0$,

$$z_{i+1,1} v_{i+a+2}^{-1} v_{i+a+1} = (u_{i+2}^{-1} u_{i+1}) f_i C_{z_{i,1}^{-1}} (z_{i,1} \beta_1) C_{v_{i+a+1}^{-1}}, \tag{2. 1}$$

$$f_i = f_{i+1} C_{z_{i,1}}, \tag{2. 2}$$

$$z_{i,1}^{-1} z_{i,k} (z_{i,1} \beta_{i+a,k}) = z_{i+1,k} \quad \textit{for} \quad k > 1, \tag{2. 3}$$

$$\alpha_1 C_{u_{i+1}^{-1}} f_i = f_i \beta_1 C_{z_{i,1} v_{i+a+1}^{-1}}, \tag{2. 4}$$

$$\alpha_k f_i = f_i \beta_k C_{z_{i,k}} \quad \textit{for} \quad k > 1, \tag{2. 5}$$

$$w_{\varphi(n-1)+s} C_{u_{i+1}^{-1}} f_i = z_{i,1+s} \beta_1 C_{v_{i+1+a}^{-1}} r_{\varphi(n-1)+s} C_{z_{i,1} v_{i+1+a}^{-1}} \quad \textit{for} \quad 1 \leqq s \leqq n-1, \tag{2. 6}$$

$$w_{\varphi(n-k)+s} f_i = (z_{i,k+s} \beta_k) r_{\varphi(n-k)+s} C_{z_{i,k}} \quad \textit{for} \quad 2 \leqq k < n \quad \textit{and} \quad 1 \leqq s \leqq n-k. \tag{2. 7}$$

PROOF. As in the proof of theorem 1. 4, $S=\bigcup(S_i : i\in I,\ i\leqq 0)$ where

$$S_i = (G, C_n, \alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,n}, w_{i,1}, w_{i,2}, \dots, w_{i,\varphi(n)}),$$

$$\alpha_{i,1} = \alpha_1 C_{u_{i+1}^{-1}}, \tag{2. 8}$$

$$\alpha_{i,s}=\alpha_s \quad \text{for} \quad s>1, \tag{2. 9}$$

$$w_{i,\varphi(n-1)+s} = w_{\varphi(n-1)+s} C_{u_{i+1}^{-1}} \quad \text{for} \quad 1\leqq s\leqq n-1, \tag{2. 10}$$

$$w_{i,\varphi(n-l)+s} = w_{\varphi(n-l)+s} \quad \text{for} \quad 2\leqq l\leqq n-1 \quad \text{and} \quad 1\leqq s\leqq n-l, \tag{2. 11}$$

$$(g, (a_1, b_1), (a_2, b_2), \dots, (a_n, b_n))_{i+1} = \tag{2. 12}$$

$$= ((s_i^{-1}\alpha_{i,1}^{a_1-1}\dots s_i^{-1}\alpha_{i,1}s_i^{-1})\alpha_{i,2}^{a_2}\dots\alpha_{i,n}^{a_n} g\,\cdot$$

$$\cdot((s_i\cdot s_i\alpha_{i,1}\dots s_i\alpha_{i,1}^{b_1-1})\alpha_{i,2}^{b_2}\dots\alpha_{i,n}^{b_n}),\ (a_1+1, b_1+1), (a_2, b_2), \dots, (a_n, b_n))_i$$

where if $a_1=0$ $(b_1=0)$ the corresponding factor is e, the identity of G and

$$s_i = u_{i+2}^{-1}u_{i+1}. \tag{2. 13}$$

Similarly, $S^*=\bigcup(S_i^* : i\in I,\ i\leqq 0)$ where $S_i^*=(G^*, C_n^*, \beta_{i,1}, \dots, \beta_{i,n}, r_{i,1}, \dots, r_{i,\varphi(n)})_i$

$$\beta_{i,1} = \beta_1 C_{v_{i+1}^{-1}}, \tag{2. 14}$$

$$\beta_{i,s}=\beta_s \quad \text{for} \quad s>1, \tag{2. 15}$$

$$r_{i,\varphi(n-1)+s} = r_{\varphi(n-1)+s} C_{v_{i+1}^{-1}} \quad \text{for} \quad 1\leqq s\leqq n-1, \tag{2. 16}$$

$$r_{i,\varphi(n-l)+s} = r_{\varphi(n-l)+s} \quad \text{if} \quad 2\leqq l\leqq n-1 \quad \text{and} \quad 1\leqq s\leqq n-l, \tag{2. 17}$$

$$[g, (a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)]_{i+1} = \tag{2. 18}$$

$$= [(t_i^{-1}\beta_{i,1}^{a_1-1}\dots t_i^{-1}\beta_{i,1}t_i^{-1})\beta_{i,2}^{a_2}\dots\beta_{i,n}^{a_n} g\,\cdot$$

$$\cdot((t_i\cdot t_i\beta_{i,1}\dots t_i\beta_{i,1}^{b_1-1})\beta_{i,2}^{b_2}\dots\beta_{i,n}^{b_n}),\ (a_1+1, b_1+1), (a_2, b_2), \dots, (a_n, b_n)]_i$$

where if $a_1=0$ $(b_1=0)$, the corresponding factor is e^*, the identity of G^* and

$$t_i = v_{i+2}^{-1}v_{i+1}. \tag{2. 19}$$

First, suppose that θ is an isomorphism of S onto S^*. Clearly, we may assume that $(e, (0, 0), \dots, (0, 0))_0\theta=[e^*, (0, 0), \dots, (0, 0)]_a$ for some $a\in I$. Thus, θ induces an isomorphism θ_0 of $S_0=(e, (0, 0), \dots, (0, 0))_0\, S(e, (0, 0), \dots, (0, 0))_0$ onto $[e^*, (0, 0), \dots, (0, 0)]_a\, S^*[e^*, (0, 0), \dots, (0, 0)]_a=S_a^*$. Hence, θ induces an isomorphism θ_i of S_i onto S_{i+a}^* for each $i\in I$ with $i\leqq 0$. Thus, by virtue of theorem 1. 1, for each i there exists an isomorphism f_i of G onto G^* and $z_{i,1}, z_{i,2}, \dots, z_{i,n}\in G^*$

such that

$$\alpha_{i,k} f_i = f_i \beta_{i+a,k} C_{z_{i,k}} \quad \text{for} \quad 1 \leqq k \leqq n, \tag{2.20}$$

$$w_{i,\varphi(n-k)+s} f_i = (z_{i,k+s} \beta_{i+a,k}) r_{i+a,\varphi(n-k)+s} C_{z_{i,k}} \tag{2.21}$$

for $1 \leqq k \leqq n$ and $1 \leqq s \leqq n-k$, and

$$(g, (a_1; b_1), \ldots, (a_n, b_n))_i \theta_i = \Bigg[\Bigg(\prod_{j=a_1-1}^{0} z_{i,1}^{-1} \beta_{i+a,1}^{j} \beta_{i+a,2}^{a_2} \cdots \beta_{i+a,n}^{a_n}\Bigg) \cdots \tag{2.22}$$

$$\cdots \Bigg(\prod_{j=a_{n-1}-1}^{0} z_{i,n-1}^{-1} \beta_{i+a,n-1}^{j} \beta_{i+a,n}^{a_n}\Bigg) \Bigg(\prod_{j=a_n-1}^{0} z_{i,n}^{-1} \beta_{i+a,n}^{j}\Bigg) (gf_i) \cdot$$

$$\cdot \Bigg(\prod_{j=0}^{b_n-1} z_{i,n} \beta_{i+a,n}^{j}\Bigg) \cdots \Bigg(\prod_{j=0}^{b_1-1} z_{i,1} \beta_{i+a,1}^{j} \beta_{i+a,2}^{b_2} \cdots \beta_{i+a,n}^{b_n}\Bigg), (a_1, b_1), \ldots, (a_n, b_n)\Bigg]_{i+a}$$

for $a_1, b_1 \geqq 1$. If $a_i(b_i) = 0$, the corresponding factor is e^*.

Combining (2. 8), (2. 14), and (2. 20), we obtain (2. 4) while combining (2. 9), (2. 15), and (2. 20), we obtain (2. 5). If we combine (2. 10), (2. 16), (2. 14), and (2. 21), we obtain (2. 6). If we combine (2. 11), (2. 17), (2. 15), and (2. 21), we obtain (2. 7).

If $x \in S_{i+1} \subset S_i$, $x\theta = x\theta_{i+1} = x\theta_i$. Thus, since $(e, (0, 1), (0, 0), \ldots, (0, 0))_{i+1} = (s_i, (1, 2), (0, 0), \ldots, (0, 0))_i$ by (2. 12), we may, utilizing (2. 22), (2. 18), (2. 19), (2. 13), and (2. 14), proceed essentially as in the proof of [9, theorem 2. 2] to obtain (2. 1).

Since $(e, (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0))_{i+1} = (e(1, 1), (0, 0), \ldots, (0, 1), (0, 0), \ldots \ldots, (0, 0))_i$ where (0, 1) is the k^{th} 2-tuple with $k > 1$, by (2. 12),

$$[z_{i+1,k}, (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0)]_{i+a+1} = \tag{2.23}$$

$$= [z_{i,1}^{-1} z_{i,k} (z_{i,1} \beta_{i+a,k}), (1, 1), (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0)]_{i+a}$$

by virtue of (2. 22). However, by (2. 18),

$$[z_{i+1,k}, (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0)]_{i+a+1} = \tag{2.24}$$

$$= [z_{i+1,k}, (1, 1), (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0)]_{i+a}.$$

Combining (2. 23) and (2. 24), we obtain (2. 3).

By (2. 12), $(g, (0, 0), (0, 0), \ldots, (0, 0))_{i+1} = (g, (1, 1), (0, 0), \ldots, (0, 0))_i$. Thus, utilizing (2. 22) and (2. 18), we may proceed as in the proof of [9, theorem 2. 2] to obtain (2. 2).

Let us now assume the conditions of the theorem are valid. By (2. 8), (2. 14), (2. 4), (2. 9), (2. 15), and (2. 5), (1. 1) (with the obvious modifications) is valid. By (2. 10), (2. 14), (2. 16), (2. 11), (2. 15), (2. 6), (2. 7), and (2. 17), (1. 2) is valid. Hence, (2. 22) defines an isomorphism of S_i onto S_{i+a}^*.

Utilizing (2. 22), (2. 18), (2. 19), (2. 1), (2. 13), (2. 14), and (2. 22), we may again proceed as in the proof of [9, theorem 2. 2] to show that

$$(e, (0, 1), (0, 0), \ldots, (0, 0))_{i+1} \theta_{i+1} = (s_i, (1, 2), (0, 0), \ldots, (0, 0))_i \theta_i.$$

Thus, by [10, theorem 2. 3],

$$(2.25)\qquad (e, (0, n), (0, 0), \dots, (0, 0))_{i+1}\theta_{i+1} = \\ = (s_i \cdot s_i\alpha_{i,1} \dots s_i\alpha_{i,1}^{n-1}, (1, n+1), (0, 0), \dots, (0, 0))_i\theta_i$$

for $n \geqq 1$. Taking inverses, we obtain, for $n \geqq 1$,

$$(2.26)\qquad (e, (n, 0), (0, 0), \dots, (0, 0))_{i+1}\theta_{i+1} = \\ = (s_i^{-1}\alpha_{i,1}^{n-1} \dots s_i^{-1}\alpha_{i,1}s_i^{-1}, (n+1, 0), (0, 0), \dots, (0, 0))_i\theta_i.$$

Utilizing (2. 22), (2. 18), (2. 2), and (2. 22) and proceeding as in the proof of [9, theorem 2. 2], we obtain

$$(2.27)\qquad (g, (0, 0), \dots, (0, 0))_{i+1}\theta_{i+1} = (g, (1, 1), (0, 0), \dots, (0, 0))_i\theta_i.$$

By (2. 22), (2. 3), (2. 18), and (2. 22),

$$(e, (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0))_{i+1}\theta_{i+1} = \\ = [z_{i+1,k}, (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_{i+a+1} = \\ = [z_{i,1}^{-1}z_{i,k}(z_{i,1}\beta_{i+a,k}), (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_{i+a+1} = \\ = [z_{i,1}^{-1}z_{i,k}(z_{i,1}\beta_{i+a,k}), (1, 1), (0, 0), \dots, (0, 1), (0, 0), \dots, (0, 0)]_{i+a} = \\ = (e, (1, 1), (0, 0), \dots, (0, 1)\ (0, 0), \dots, (0, 0))_i\theta_i$$

where (0, 1) is the k^{th} 2-tuple with $k > 1$. Thus, by [10, theorem 2. 3],

$$(2.28)\qquad (e, (0, 0), \dots, (0, n), (0, 0), \dots, (0, 0))_{i+1}\theta_{i+1} = \\ = (e, (1, 1), (0, 0), \dots, (0, n), (0, 0), \dots, (0, 0))_i\theta_i$$

where $(0, n)$ is the k^{th} 2-tuple with $k > 1$. Taking inverses, we obtain

$$(2.29)\qquad (e, (0, 0), \dots, (n, 0), (0, 0), \dots, (0, 0))_{i+1}\theta_{i+1} = \\ = (e, (1, 1), (0, 0), \dots, (n, 0), (0, 0), \dots, (0, 0))_i\theta_i.$$

Hence, by [10, theorem 2. 3], (2. 25), (2. 26), (2. 27), (2. 28), (2. 29), and [10, theorem 2. 3],

$$(2.30)\qquad (g, (a_1, b_1), \dots, (a_n, b_n))_{i+1}\theta_{i+1} = \\ = (e, (a_1, 0), \dots, (a_n, 0))_{i+1}\theta_{i+1}(g, (0, 0), \dots, (0, 0))_{i+1}\theta_{i+1} \cdot \\ \cdot (e, (0, b_1), \dots, (0, b_n))_{i+1}\theta_{i+1} = \\ = (e, (a_1, 0), (0, 0), \dots, (0, 0))_{i+1}\theta_{i+1}, (e, (0, 0), (a_2, 0), (0, 0), \dots, (0, 0))_{i+1}\theta_{i+1} \dots \\ \dots (e, (0, 0), \dots, (0, a_n))_{i+1}\theta_{i+1}(g, (0, 0), \dots, (0, 0))_{i+1}\theta_{i+1} \cdot \\ \cdot (e, (0, 0), \dots, (0, b_n))_{i+1}\theta_{i+1} \dots (e, (0, b_1), (0, 0), \dots, (0, 0))_{i+1}\theta_{i+1} = \\ = s(_i^{-1})\alpha_{i,1}^{a_1-1} \dots s_i^{-1}\alpha_{i,1}s_i^{-1})\alpha_{i,2}^{a_2} \dots \alpha_{i,n}^{a_n}g \cdot \\ \cdot ((s_i \cdot s_i\alpha_{i,1} \dots s_{i,1}\alpha_{i,1}^{b_1-1})\alpha_{i,2}^{b_2} \dots \alpha_{i,n}^{b_n}), (a_1+1, b_1+1), (a_2, b_2), \dots, (a_n, b_n))_i\theta_i$$

for $a_1 \geqq 1$ and $b_1 \geqq 1$. If $a_1 = 0$ $(b_1 = 0)$, the corresponding factor is e.

Let us define

$$x\theta = x\theta_i \quad \text{if} \quad x \in S_i.$$

Hence, by (2. 12) and (2. 30), θ defines an isomorphism of S onto S^*.

Let G be a group, let $\beta_1, \beta_2, \ldots, \beta_n$ be a collection of endomorphisms of G, let $\{m_i : i \in I\}$ be a collection of elements of G such that $m_i = e$ for $i > 0$, and let $a_1, a_2, \ldots, a_n$ be non-negative integers. We define

$$(2.31) \qquad t(m_i, \beta, a_1, \ldots, a_n) = $$

$$= \begin{cases} (m_{a_1+i}^{-1} \cdot m_{a_1+i-1}^{-1}\beta_1 \ldots m_{i+3}^{-1}\beta_1^{a_1-3} m_{i+2}^{-1}\beta_1^{a_1-2} m_{i+1}\beta_1^{a_1-2} \ldots \\ \qquad \ldots m_{i+1}\beta_1 m_{i+1})\beta_2^{a_2} \ldots \beta_n^{a_n} \quad \text{if} \quad a_1 \geqq 2; \\ e, \quad \text{otherwise.} \end{cases}$$

Let N denote the natural numbers.

THEOREM 2. 2. *Let* $S = (G, C_n^*, \alpha_1, \ldots, \alpha_n, w_1, \ldots, w_{\varphi(n)}, u_i)$ *and* $S^* = (G^*, C_n^*, \beta_1, \beta_2, \ldots, \beta_n, r_1, \ldots, r_{\varphi(n)}, v_i)$ *be* $\omega^{n-1}I$*-bisimple semigroups. Let* $t_s (l \leqq s \leqq n)$, *where*

$$t_s = \{z_{is} : i \in I,\ i \leqq 0\},$$

be n sequences of elements of G^*, *let* $\{f_i : i \in I,\ i \leqq 0\}$ *be a sequence of homomorphisms of G onto* G^*, *and let* $a \in I$ *such that for each* $i \in I \setminus N$, (2. 1)—(2. 7) *of theorem* 2. 1 *are valid.*

For each $(g, (a_1, b_1), \ldots, (a_n, b_n)) \in (e, (i, i), (0, 0), \ldots, (0, 0))S(e, (i, i), (0, 0), \ldots, (0, 0))$, *define*

$$(2.32) \qquad (g, (a_1, b_1), \ldots, (a_n, b_n))\theta = \Bigg(t(v_{i+a}, \beta, a_1 - i, a_2, \ldots, a_n) \cdot$$

$$\cdot \left(\prod_{j=a_1-i-1}^{0} (v_{i+a+1}^{-1} \cdot v_{i+a+1}^{-1}\beta_1 \ldots v_{i+a+1}^{-1}\beta_1^{j-1}(z_{i,1}^{-1}\beta_1^j) \cdot \right.$$

$$\left. \cdot w_{i+a+1}\beta_1^{j-1} \ldots v_{i+a+1}\beta_1 v_{i+a+1})\beta_2^{a_2} \ldots \beta_n^{a_n} \right) \ldots \left(\prod_{j=a_{n-1}^{-1}}^{0} z_{i,n-1}^{-1}\beta_{n-1}^j \beta_n^{a_n} \right) \cdot$$

$$\cdot \left(\prod_{j=a_n-1}^{0} z_{i,n}^{-1}\beta_n^j \right) (((t(u_i, \alpha, a_1 - i, a_2, \ldots, a_n))^{-1} g t(u_i, \alpha, b_1 - i, b_2, \ldots, b_n))f_i) \cdot$$

$$\cdot \left(\prod_{j=0}^{b_n-1} z_{i,n}\beta_n^j \right) \ldots \left(\prod_{j=0}^{b-i-1} (v_{i+a+1}^{-1} \cdot v_{i+a+1}^{-1}\beta_1 \ldots v_{i+a+1}^{-1}\beta_1^{j-1}(z_{i,1}\beta_1^j) \cdot \right.$$

$$\left. \cdot v_{i+a+1}\beta_1^{j-1} \ldots v_{i+a+1}\beta_1 v_{i+a+1})\beta_2^{a_2} \ldots \beta_n^{a_n} \right) \cdot$$

$$\cdot (t(v_{i+a}, \beta, b_1 - i, b_2, \ldots, b_n))^{-1}, (a_1 + a, b_1 + a), (a_2, b_2), \ldots, (a_n, b_n) \Bigg)$$

for $a_1, b_1 > i$, $a_s, b_s > 0$ *for* $2 \leqq s \leqq n$, *and* $j > 0$. *If* $a_1(b_1) = i\ (a_s(b_s) = 0)$, *the corresponding factor becomes* e^*, *the identity of* G^*. *If* $j = 0$, *the corresponding factor becomes* $z_{i,1}^{-1}\ (z_{i,1})$.

Then, θ is a homo morphism of S onto S^* and conversely every such homomorphism is obtained in this fashi on.

PROOF. Let θ be a homomorphism of S onto S^*. Thus, by [2, Vol. 2, p. 57, lemma 7. 34], θ maps E_S onto E_S^*. By theorem 3. 1, θ maps E_S isomorphically onto E_{S^*}. We may assume that $(e, (0, 0), \dots, (0, 0))_0\theta = [e, (0, 0), \dots, (0, 0)]_a$ for some $a \in I$. Hence, $(e, (0, 0), \dots, (0, 0))_i\theta = [e, (0, 0), \dots, (0, 0)]_{i+a}$. We may proceed essentially as in the proof of theore m 2. 1 to show that $x\theta = x\theta_i$ for $x \in S_i$ (notation of the proof of theorem 2. 1) where θ_i is given by (2. 22) with appropriate modifications. Conversely, proceeding as in the proof of theorem 2. 1, we may show that θ so defined is a homomorphism of S onto S^*. We now express θ in a more convenient form. From the proof of theorem 1. 4, it follows that

$$(2.33)\quad \big(t(u_i,\alpha,a_1,\dots,a_n)g\big(t(u_i,\alpha,b_1,\dots,b_n)\big)^{-1},(a_1+i,b_1+i),(a_2,b_2),\dots,(a_n,b_n)\big)= = (g,\ (a_1,b_1),(a_2,b_2),\dots,(a_n,b_n))_i$$

$$(2.34)\quad (g,(a_1+i,b_1+i),(a_2,b_2),\dots,(a_n,b_n)) = = \big(\big(t(u_i,\alpha,a_1,\dots,a_n)\big)^{-1}gt(u_i,\alpha,b_1,\dots,b_n),(a_1,b_1),(a_2,b_2),\dots,(a_n,b_n)\big)_i.$$

From (2. 14), we obtain that

$$(2.35)\quad g\beta^n_{i,1} = (v_{i+1}^{-1}\cdot v_{i+1}^{-1}\beta_1\dots v_{i+1}^{-1}\beta_1^{n-1}g\beta_1^n v_{i+1}\beta_1^{n-1}\dots v_{i+1}\beta_1 v_{i+1} \quad \text{for} \quad n \geqq 1.$$

If we apply (2. 34), (2. 35), (2. 15), and (2. 33) to (2. 22), we obtain (2. 32).

We established theo rem 2. 3 for the case $n=1$ (I-bisimple semigroup) in [9]. However, in that paper, θ was stated in terms of the ω-bisimple representations of $(e,(i,i))S(e,(i,i))$ and $(e,(i+a,i+a))S^*(e,(i+a),(i+a))$. We now give θ in a form independent of these rep resentations.

THEOREM 2. 3 [WARNE, 9]. *Let $S=(G, C_1^*, \alpha, u_i)$ and $S^*=(G^*, C_1^*, \beta, v_i)$ be I-bisimple semigroups. Let $\{z_i : i \in I,\ i \leqq 0\}$ be a sequence of elements of G^*, let $\{f_i : i \in I,\ i \leqq 0\}$ be a sequence of ho momorphisms of G onto G^*, and let a be an element of I such that*

$$(2.36)\quad z_{i+1}v_{i+a+2}^{-1}v_{i+a+1} = \big((u_{i+2}^{-1}u_{i+1})f_iC_{z_i}-1\big)(z_i\beta C_{v_{i+a+1}^{-1}})$$

$$(2.37)\quad f_i = f_{i+1}C_{z_i}$$

$$(2.38)\quad \alpha C_{u_{i+1}^{-1}}f_i = f_i\beta C_{z_iv_{i+a+1}^{-1}}.$$

For each $(g,(a_1,b_1)) \in (e,(i,i))S(e,(i,i))$, define

$$(2.39)\quad (g,(a_1,b_1))\theta = \Bigg(t(v_{i+a},\beta,a_1-i)\cdot \cdot\Bigg(\prod_{j=a_1-i-1}^{0}(v_{i+a+1}^{-1}\cdot v_{i+a+1}^{-1}\beta\dots v_{i+a+1}^{-1}\beta^{j-1}z_i^{-1}\beta^j v_{i+a+1}\beta^{j-1}\dots v_{i+a+1}\beta v_{i+a+1})\Bigg)\cdot \cdot\Big(\Big(\big(t(u_i,\beta,a_1-i)\big)^{-1}gt(u_1,\alpha,b_1-i)\Big)f_i\Big)\cdot \cdot\Bigg(\prod_{j=0}^{b_1-i-1}(v_{i+a+1}^{-1}\cdot v_{i+a+1}^{-1}\beta\dots v_{i+a+1}^{-1}\beta^{j-1}(z_i\beta^j)v_{i+a+1}\beta^{j-1}\dots v_{i+a+1}\beta v_{i+a+1})\Bigg)\cdot \cdot t(v_{i+a},\beta,b_1-i),(a_1+a,b_1+a)\Bigg)$$

for $a_1, b_1 > i$ *and* $j > 0$. *If* $a_1 = 0$ $(b_1 = 0)$, *the corresponding factor becomes* e^*, *the identity of* G^*. *If* $j = 0$, *the corresponding factor becomes* z_i^{-1} (z_i).

Then, θ *is a homomorphism of* S *onto* S^* *and conversely every such homomorphism is obtained in this fashion.*

3. The congruences. In this section, we will determine the congruence relations on an $\omega^{n-1}I$-bisimple semigroup $S = (G, C_n^*, \gamma_1, \gamma_2, \dots, \gamma_n, w_1, w_2, \dots, w_{\varphi(n)})$. We first show that each congruence on S is a group congruence, an idempotent separating congruence, or an $\omega^{k-1}I$-bisimple congruence for some $k \in \{1, 2, \dots, n-1\}$. To determine the group congruences of S, we give an explicit determination of the maximal group homomorphic image of S in terms of G. We next determine the idempotent separating congruences of S in terms of the $\gamma_1 - \gamma_2 - \dots - \gamma_n$ invariant subgroups of G. Finally, the $\omega^{k-1}I$-bisimple congruences of S are determined in terms of $\gamma_1 - \gamma_2 - \dots - \gamma_n$ invariant subgroups of G that obey certain conditions.

We first determine the nature of the congruence relations admitted by an $\omega^{n-1}I$-bisimple semigroup. Theorem 1. 3 and [11, theorem 1. 1 and corollary 1. 1] are important tools in this determination.

THEOREM 3. 1. *If* S *is an* $\omega^{n-1}I$*-bisimple semigroup, each congruence on* S *is a group congruence, an idempotent separating congruence, or an* $\omega^{k-1}I$*-bisimple congruence for some* $k \in \{1, 2, \dots, n-1\}$.

PROOF. We will denote E_S by $\{e_{(a_1, a_2, \dots, a_n)} : a_1 \in I$ and $a_s \in I^0$ for $2 \leqq s \leqq n\}$. Let ϱ be a congruence relation on S. If $S_i = e_{(i,0,\dots,0)} S e_{(i,0,\dots,0)}$, $S = \bigcup \{S_i : i \in I$ and $i \leqq 0\}$ by theorem 1. 3. Clearly, $\varrho | S_i \times S_i$ is a congruence relation ϱ_i on S_i. Since S_i is an ω^n-bisimple semigroup by theorem 1. 3, ϱ_i is a group congruence, an idempotent separating congruence, or an ω^k-bisimple congruence for some $k \in \{1, 2, \dots, n-1\}$ by [11, theorem 1. 1].

If $e_{(i+a_1, a_2, \dots, a_n)} \in S_i$ $(a_s \in I^0$ for $1 \leqq s \leqq n$ and $i \in I$, $i \leqq 0)$, $e_{(i+a_1, a_2, \dots, a_n)}$ becomes $(e, (a_1, a_1), \dots, (a_n, a_n))_i$ in the ω^n-bisimple representation of S_i. Let us suppose that ϱ_0 is an ω^k-bisimple congruence for some $k \in \{1, 2, \dots, n-1\}$. Assume that ϱ_{i+1} is an ω^k-bisimple congruence. Suppose that ϱ_i is an ω^s-bisimple congruence where $s \in \{1, 2, \dots, n-1\}$ and $s \neq k$. Let $e_{(a_1, a_2, \dots, a_n)}$; $e_{(b_1, b_2, \dots, b_n)} \in S_{i+1} \subseteqq S_i$. Thus, by [11, corollary 1. 1], $e_{(a_1, a_2, \dots, a_n)} \varrho_i e_{(b_1, b_2, \dots, b_n)}$ if and only if $a_r = b_r$ for $1 \leqq r \leqq s$. Hence $e_{(a_1, a_2, \dots, a_n)} \varrho_{i+1} e_{(b_1, b_2, \dots, b_n)}$ if and only if $a_r = b_r$ for $1 \leqq r \leqq s$ and we have a contradiction by virtue of [11, corollary 1. 1]. Let $e_{(a_1, a_2, \dots, a_n)}, e_{(b_1, b_2, \dots, b_n)} \in S_{i+1} \subseteqq S_i$ and suppose that $a_1 \neq b_1$. If ϱ_i is a group congruence, $e_{(a_1, a_2, \dots, a_n)} \varrho_i e_{(b_1, b_2, \dots, b_n)}$ and hence $e_{(a_1, a_2, \dots, a_n)} \varrho_{i+1} e_{(b_1, b_2, \dots, b_n)}$ and we again contradict [11, corollary 1. 1]. Next, suppose that ϱ_i is an idempotent separating congruence. By [11, corollary 1. 1], there exist distinct idempotents $e, f \in S_{i+1} \subseteqq S_i$ such that $e \varrho_{i+1} f$. Thus, $e \varrho_i f$ and we have a contradiction. Hence, if ϱ_0 is an ω^k-bisimple congruence, $e_{(a_1, a_2, \dots, a_n)} \varrho e_{(b_1, b_2, \dots, b_n)}$ if and only if $a_r = b_r$ for $1 \leqq r \leqq k$. Thus, ϱ is an $\omega^{k-1}I$-bisimple congruence by [2, Vol. 2 p. 62, theorem 7. 48] and [3, p. 1111]. The other cases are handled similarly.

REMARK. In the case $n = 1$, we obtain [9, theorem 4. 2].

We next determine the maximal group homomorphic image of an $\omega^{n-1}I$-bisimple semigroup. The basic tools are the construction of an $\omega^{n-1}I$-bisimple semigroup as an "inverse limit" of ω^n-bisimple semigroups, [10, theorem 2. 3], [11, theorem 2. 2

and theorem 2. 3] and theorem 1. 4. The most difficult part of the proof is the determination of the defining homomorphism.

If σ is an equivalence relation on a set X, we let x_σ denote the equivalence class containing the element x of X.

THEOREM 3. 2. *Let* $S=(G, C_n^*, \alpha_1, \alpha_2, \dots, \alpha_n, w_1, w_2, \dots, w_{\varphi(n)}, u_i)$ *be an* $\omega^{n-1}I$-*bisimple semigroup and let* e *be the identity of* G. *If* $N=\{g\in G \mid g\alpha_1^n=e \text{ for some } n\in I^0\}$, N *is a normal subgroup of* G. *If* $(xN)\theta=(x\alpha_1)N$, $x\in G$, θ *is an endomorphism of* G/N. *Let* $g\to\bar{g}$ *denote the natural homomorphism of* G *onto* G/N. *Let us define a relation* σ *on* $G/N\times(I^0)^2$ *by the rule*

$$((\bar{g}, a, b), (\bar{h}, c, d))\in\sigma$$

if and only if there exists $x, y\in I^0$ *such that* $x+a=y+c$, $x+b=y+d$, *and* $\bar{g}\theta^x=\bar{h}\theta^y$. *Then,* σ *is an equivalence relation on* $G/N\times(I^0)^2$. *Furthermore, the rule*

$$(\bar{g}, a, b)_\sigma(\bar{h}, c, d)_\sigma=(\bar{g}\theta^c\bar{h}\theta^b, a+c, b+d)_\sigma$$

defines a binary operation on $G/N\times(I^0)^2/\sigma=H$ *whereby* H *becomes a group which is the maximal group homomorphic image of* S. *The canonical homomorphism of* S *onto* H *is given by*

$$(g, (a_1, b_1), (a_2, b_2), \dots, (a_n, b_n))\Phi=$$

$$=\begin{cases} \big((y_{i,1}^{-1}\theta^{a_1-i-1}\dots y_{i,1}^{-1}\theta y_{i,1}^{-1})(y_{i,2}^{-a_2}\dots y_{i,n}^{-a_n}) \\ \big((\overline{t(u_i, \alpha, a_1-i, a_2, \dots, a_n)})^{-1}\bar{g}\,\overline{t(u_i, \alpha, b_1-i, \dots, b_n)}\big)\cdot \\ \cdot(y_{i,n}^{b_n}\dots y_{i,2}^{b_2})(y_{i,1}\cdot y_{i,1}\theta\dots y_{i,1}\theta^{b_1-i-1}), a_1-i, b_1-i\big)_\sigma \\ \text{for } i\leqq-1. \text{ If } a_1=i\ (b_1=i), \text{ the corresponding factor is } \bar{e}; \\ \left(\left(\prod_{s=2}^{n} y_{0,s}^{-a_s}\right)(\bar{g}\theta)\left(\prod_{s=n}^{2} y_{0,s}^{b_s}\right), a_1+1, b_1+1\right)_\sigma \quad \text{for} \quad i=0 \end{cases}$$

where $(g, (a_1, b_1), (a_2, b_2), \dots, (a_n, b_n))\in(e, (i, i), (0, 0), \dots, (0, 0))S(e, (i, i), (0, 0), \dots$ $\dots, (0, 0))$ *and where*

$$y_{i,s}=(\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\dots\bar{u}_{i+1}^{-1}\theta^{-(i+1)})(\bar{w}_{\varphi(n-1)+(s-1)}\theta^{-(i+1)})(\bar{u}_{i+1}\theta^{-(i+1)}\dots$$
$$\dots\bar{u}_{-1}\theta\bar{u}_0) \quad \text{for} \quad 1<s\leqq n \quad \text{and} \quad i\leqq-1,$$

$$y_{0,s}=\bar{w}_{\varphi(n-1)+(s-1)} \quad \text{for} \quad 1<s\leqq n,$$

$$y_{0,1}=\bar{e},$$

$$y_{-1,1}=\bar{u}_0^{-1},$$

$$y_{i,1}=\bar{u}_0^{-1}(\bar{u}_{-1}^{-1}\theta)\dots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+2}\theta^{-(i+1)}\bar{u}_{i+3}\theta^{-(i+2)}\dots\bar{u}_0\theta \quad \text{for} \quad i\leqq-2,$$

$$\bar{g}\delta_0=\bar{g}, \text{ and}$$

$$\bar{g}\delta_i=\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\dots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{g}\theta^{-i}\bar{u}_{i+1}\theta^{-(i+1)}\dots\bar{u}_{-1}\theta\bar{u}_0 \quad \text{for} \quad i\leqq-1.$$

PROOF. As in the proof of theorem 2. 1, $S=\bigcup(S_i : i\in I,\ i\leqq 0)$ where $S_i=(G, C_n, \alpha_{i,1}, \ldots, \alpha_{i,n}, w_{i,1}, \ldots, w_{i,\varphi(n)})_i$ and (2. 8)—(2. 13) are valid. Since $S_0=(G, C_n, \alpha_{0,1}, \ldots, \alpha_{0,n}, w_{0,1}, \ldots, w_{0,\varphi(n)})_0$ is an ω^n-bisimple semigroup, it is only necessary to exhibit a homomorphism of S onto H whereby H becomes the maximal homomorphic image of S by virtue of (2. 8) and [11, theorem 2. 3]. We use the multiplication given in H without explicit mention in most cases. We first utilize [11, theorem 2. 2] to determine a homomorphism Φ_i of S_i into H for each $i\leqq 0$. In the notation of [11, theorem 2. 2], let $z_{0,s}=(y_{0,s}, 1, 1)_\sigma$ for $1<s\leqq n$; $z_{i,s}=(y_{i,s}, 0, 0)_\sigma$ for $i<0$ and $1<s\leqq n$; $z_{i,1}=(y_{i,1}, 0, 1)_\sigma$ for $i\leqq 0$; and $gf_i=(\bar{g}\delta_i, 0, 0)_\sigma$. We first verify [11, (1) of theorem 2. 2]. Utilizing (2. 10), it is easy to see that $z_{i,1}z_{i,1+(s-1)}z_{i,1}^{-1}=w_{i,\varphi(n-1)+(s-1)}f_i$ for $i\leqq -1$ and $1<s\leqq n$. Again utilizing (2. 10), we obtain for $1<r<n$, $1\leqq k<n$, and $i\leqq -1$

$$z_{i,r}z_{i,r+k}z_{i,r}^{-1}=\big(\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\ldots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\cdot \tag{3. 1}$$

$$\cdot\big((\bar{u}_{i+1}\bar{w}_{i,\varphi(n-1)+(r-1)}\bar{w}_{i,\varphi(n-1)+(r+k-1)}\bar{w}_{i,\varphi(n-1)+(r-1)}^{-1}\bar{u}_{i+1}^{-1})\theta^{-(i+1)}\big)\cdot$$

$$\cdot\bar{u}_{i+1}\theta^{-(i+1)}\ldots\bar{u}_{-1}\theta\bar{u}_0, 0, 0\big)_\sigma.$$

Using [10, (2) of theorem 2. 3] and the fact that $\varphi(x)+x=\varphi(x+1)$ for $x\in N$, we obtain, for $i\leqq 0$,

$$w_{i,\varphi(n-1)+(r-1)}w_{i,\varphi(n-1)+(r+k-1)}w_{i,\varphi(n-1)+(r-1)}^{-1}=w_{i,\varphi(n-r)+k}\alpha_{i,1}. \tag{3. 2}$$

Thus, by virtue of (3. 1), (3. 2), and (2. 8), we have

$$z_{i,r}z_{i,r+k}z_{i,r}^{-1}=\big(\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\ldots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\big(\overline{(u_{i+1}w_{i,\varphi(n-r)+k}\alpha_{i,1}u_{i+1}^{-1})}\theta^{-(i+1)}\big)\cdot$$

$$\cdot\bar{u}_{i+1}\theta^{-(i+1)}\ldots\bar{u}_{-1}\theta\bar{u}_0, 0, 0\big)_\sigma=$$

$$=(\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\ldots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{w}_{i,\varphi(n-r)+k}\theta^{-i}\bar{u}_{i+1}\theta^{-(i+1)}\ldots\bar{u}_0, 0, 0)_\sigma=w_{i,\varphi(n-r)+k}f_i.$$

By a straightforward calculation, $z_{0,1}z_{0,1+(s-1)}z_{0,1}^{-1}=w_{0,\varphi(n-1)+(s-1)}f_0$ where $s>l$. Utilizing (3. 2), (2. 8), and (2. 11), it is easy to see that $z_{0,r}z_{0,r+k}z_{0,r}^{-1}=w_{0,\varphi(n-r)+k}f_0$ where $1<r\leqq n-1$ and $1\leqq k\leqq n-1$.

We next verify [11, (2) of theorem 2. 2]. If $i<0$ and $s>1$, we easily see that

$$z_{i,s}gf_iz_{i,s}^{-1}=\big(\bar{u}_0^{-1}\cdot\bar{u}_{-1}\theta\ldots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\cdot \tag{3. 3}$$

$$\cdot(\bar{w}_{\varphi(n-1)+(s-1)}\bar{g}\theta\bar{w}_{\varphi(n-1)+(s-1)}^{-1})\theta^{-(i+1)}\bar{u}_{i+1}\theta^{-(i+1)}\ldots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma.$$

By [10, (3) of theorem 2. 3], (2. 10), and (2. 8), for $i\leqq 0$ and $s>1$,

$$g\alpha_{i,s}\alpha_{i,1}=w_{i,\varphi(n-1)+(s-1)}g\alpha_{i,1}w_{i,\varphi(n-1)+(s-1)}^{-1}=$$

$$=u_{i+1}^{-1}w_{\varphi(n-1)+(s-1)}u_{i+1}g\alpha_{i,1}u_{i+1}^{-1}w_{\varphi(n-1)+(s-1)}^{-1}u_{i+1}=$$

$$=u_{i+1}^{-1}w_{\varphi(n-1)+(s-1)}g\alpha_1w_{\varphi(n-1)+(s-1)}^{-1}u_{i+1}.$$

Hence, by (2. 8), for $s>1$ and $i\leqq 0$,

$$g\alpha_{i,s}\alpha_1 = w_{\varphi(n-1)+(s-1)}g\alpha_1 w^{-1}_{\varphi(n-1)+(s-1)}$$

or

$$\overline{g\alpha_{i,s}}\theta = \overline{w}_{\varphi(n-1)+(s-1)}\bar{g}\theta\overline{w}^{-1}_{\varphi(n-1)+(s-1)}.$$

Thus, for $s>1$ and $i<0$, $z_{i,s}\, gf_i z_{i,s}^{-1} = g\alpha_{i,s}f_i$ by virtue of (3. 3). In the case $s=1$ and $i\leqq -2$, we proceed exactly as in the proof of [9, theorem 3. 6 (equations 3. 12, 3. 13, and 3. 14)]. Utilizing (2. 8), the case $s=1$ and $i=-1$, and the case $s=1$ and $i=0$ are easily verified. The case, $s>1$ and $i=0$, is a consequence of (2. 8), (2. 10) and [10, (3) of theorem 2. 3]. For $(g,(a_1,b_1),\dots,(a_n,b_n))_i\in S_i$, let

$$(g,(a_1,b_1),\dots,(a_n,b_n))_i\Phi_i =$$

$$=\begin{cases}((y_{i,1}^{-1}\theta^{a_1-1}\dots y_{i,1}^{-1}\theta y_{i,1}^{-1})(y_{i,2}^{-a_2}\dots y_{i,n}^{-a_n})(\bar{g}\delta_i)\\ (y_{i,n}^{b_n}\dots y_{i,2}^{b_2})(y_{i,1}\cdot y_{i,1}\theta\dots y_{i,1}\theta^{b_1-1}),a_1,b_1)_\sigma\\ \text{for } i\leqq -1. \text{ If } a_1=0\ (b_1=0), \text{ the corresponding factor is } \bar{e};\\ \left(\left(\prod_{s=2}^{n} y_{0,s}^{-a_s}\right)(\bar{g}\theta)\left(\prod_{s=n}^{2} y_{0,s}^{b_s}\right),a_1+1,b_1+1\right)_\sigma \quad \text{for}\quad i=0,\end{cases}$$

where $y_{i,s}(i\leqq 0$ and $1\leqq s\leqq n)$, $\delta_i(1\leqq i\leqq n)$, and θ are defined in the statement of the theorem. Hence, by [11, theorem 2. 2], Φ_i defines a homomorphism of S_i into H for all $i\leqq 0$.

Let us define $x\Phi=x\Phi_i$ if $x\in S_i$. We will show that Φ is a homomorphism of S onto H.

As in the proof of [9, theorem 3. 6], we may show that

$$(g,(1,1),(0,0),\dots,(0,0))_i\Phi_i = (g,(0,0),\dots,(0,0))_{i+1}\Phi_{i+1}$$

and

$$(s_i,(1,2),(0,0),\dots,(0,0))_i\Phi_i = (e,(0,1),(0,0),\dots,(0,0))_{i+1}\Phi_{i+1}$$

where $s_i = u_{i+2}^{-1}u_{i+1}$.

It is easily seen that

$$(e,(0,0),\dots,(0,1),(0,0),\dots,(0,0))_{i+1}\Phi_{i+1} =$$

$$=(e,(1,1),(0,0),\dots,(0,1),(0,0),\dots,(0,0))_i\Phi_i$$

where (0, 1) is the k^{th} 2-tuple with $k>1$.

Thus, utilizing [10, theorem 2. 3 (the multiplication given there)], we obtain $(e,(0,n),(0,0),\dots,(0,0))_{i+1}\Phi_{i+1} = (s_i\cdot s_i\alpha_{i,1}\dots s_i\alpha_{i,1}^{n-1},(1,n+1),(0,0),\dots,(0,0))_i\Phi_i$, where $n\geqq 1$ and

$$(e,(0,0),\dots,(0,n),(0,0),\dots,(0,0))_{i+1}\Phi_{i+1} =$$

$$=(e,(1,1),(0,0),\dots,(0,n),(0,0),\dots,(0,0))_i\Phi_i.$$

Hence, again utilizing [10, theorem 2. 3], it is easy to see that

$$(g, (a_1, b_1), \ldots, (a_n, b_n))_{i+1} \Phi_{i+1} =$$

$$= (e, (a_1, 0), \ldots, (a_n, 0))_{i+1} \Phi_{i+1} (g, (0, 0), \ldots, (0, 0))_{i+1} \Phi_{i+1} \cdot$$

$$\cdot (e, (0, b_1), \ldots, (0, b_n))_{n+1} \Phi_{i+1} =$$

$$= (e, (a_1, 0), (0, 0), \ldots, (0, 0))_{i+1} \Phi_{i+1} (e, (0, 0), (a_2, 0), \ldots, (0, 0))_{i+1} \Phi_{i+1} \cdots$$

$$\cdots (e, (0, 0), \ldots, (a_n, 0))_{i+1} \Phi_{i+1} (g, (0, 0), \ldots, (0, 0))_{i+1} \Phi_{i+1} \cdot$$

$$\cdot (e, (0, 0), \ldots, (0, b_n))_{i+1} \Phi_{i+1} \cdots (e, (0, b_1), \ldots, (0, 0))_{i+1} \Phi_{i+1} =$$

$$= ((s_i^{-1} \alpha_{i,1}^{a_1 - 1} \ldots s_i^{-1} \alpha_{i,1} s_i^{-1}) \alpha_{i,2}^{a_2} \ldots \alpha_{i,n}^{a_n} g ((s_i \cdot s_i \alpha_{i,1} \ldots s_i \alpha_{i,1}^{b_1 - 1}) \cdot$$

$$\cdot \alpha_{i,2}^{b_2} \ldots \alpha_{i,n}^{b_n}), (a_1 + 1, b_1 + 1), (a_2, b_2), \ldots, (a_n, b_n))_i \Phi_i$$

where if $a_1 = 0$ ($b_1 = 0$), the corresponding factor is e, the identity of G. Thus, if $x \in S_{i+1} \subseteqq S_i$, $x\Phi_{i+1} = x\Phi_i$ by virtue of (2. 12). Hence, Φ is a homomorphism of S onto H.

We will now show that H is the maximal group homomorphic image of S under the homomorphism Φ. Let G^* be an arbitrary group and let ϱ be a homomorphism of S onto G^*. We will denote $\varrho | S_i$ by ϱ_i. Thus, ϱ_i is a homomorphism of S_i into G^*. Since H is the maximal group homomorphic image of S_0 under the homomorphism Φ_0 by [11, theorem 2. 3], there exists a homomorphism γ of H onto the subgroup $S_0 \varrho_0$ of G^* such that $(g, (a_1, b_1), \ldots, (a_n, b_n)_0 \Phi_0 \gamma = (g, (a_1, b_1), \ldots, (a_n, b_n))_0 \varrho_0$ for all $(g, (a_1, b_1), \ldots, (a_n, b_n))_0 \in S_0$. Next, suppose that

$$(g, (a_1, b_1), \ldots, (a_n, b_n))_{i+1} \Phi_{i+1} \gamma = (g, (a_1, b_1), \ldots, (a_n, b_n))_{i+1} \varrho_{i+1}$$

where γ is a homomorphism of H onto $S_{i+1} \varrho_{i+1}$. By virtue of [11, theorem 2. 2], there exist, for each $i \in I$, $i \leqq 0$, $v_{i,1}, v_{i,2}, \ldots$, and $v_{i,n} \in G^*$ and a homomorphism η_i of G into G^* such that

$$v_{i,r} v_{i,r+k} v_{i,r}^{-1} = w_{i,\varphi(n-r)+k} \eta_i$$

for $1 \leqq r \leqq n-1$ and $1 \leqq k \leqq n-r$, and

$$v_{i,s} g\eta_i v_{i,s}^{-1} = g\alpha_{i,s} \eta_i$$

for $1 \leqq s \leqq n$.

Furthermore,

$$(g, (a_1, b_1), \ldots, (a_n, b_n))_i \varrho_i = \left(\prod_{s=1}^{n} v_{i,s}^{-a_s} \right) (g\eta_i) \left(\prod_{s=n}^{1} v_{i,s}^{b_s} \right)$$

for $(g, (a_1, b_1), \ldots, (a_n, b_n))_i \in S_i$.

Thus, as in the proof of [9, theorem 3. 6], we may show that for $i < 0$, $v_{i,1} = (s_i^{-1} \eta_{i+1}) v_{i+1,1}$ and, for $i \leqq 0$, $(g, (0,0), \ldots, (0,0))_i \varrho_i = (g, (0, 0), \ldots, (0, 0))_i \Phi_i \gamma$ and $(e, (0, 1), (0, 0), \ldots, (0, 0))_i \varrho_i = (e, (0, 1), (0, 0), \ldots, (0, 0)_i \Phi_i \gamma$.

Since

$(e, (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0))_{i+1} = (e, (1, 1), (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0))_i$ by (2. 12), $(e, (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0))_{i+1}\varrho_{i+1} = (e, (1, 1), (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0))_i\varrho_i$ where (0, 1) is the k^{th} two-tuple with $k > 1$. Thus,

$$v_{i+1,k} = v_{i,1}^{-1}v_{i,k}v_{i,1} = v_{i+1,1}^{-1}(s_i\eta_{i+1})v_{i,k}(s_i^{-1}\eta_{i+1})v_{i+1,1}.$$

Hence

$$v_{i,k} = (s_i^{-1}\eta_{i+1})(v_{i+1,1}v_{i+1,1+(k-1)}v_{i+1,1}^{-1})(s_i\eta_{i+1}) =$$
$$= (s_i^{-1}\eta_{i+1})w_{i+1,\varphi(n-1)+(k-1)}\eta_{i+1}(s_i\eta_{i+1}) = (s_i^{-1}w_{i+1,\varphi(n-1)+(k-1)}s_i)\eta_{i+1}.$$

Thus, utilizing (2. 13) and (2. 10), we obtain

$$(e, (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0))_i\varrho_i = v_{i,k} = (s_i^{-1}w_{i+1,\varphi(n-1)+(k-1)}s_i)\eta_{i+1} =$$
$$= (s_i^{-1}w_{i+1,\varphi(n-1)+(k-1)}s_i, (0, 0), \ldots, (0, 0))_{i+1}\varrho_{i+1} =$$
$$= (s_i^{-1}w_{i+1,\varphi(n-1)+(k-1)}s_i, (0, 0), \ldots, (0, 0))_{i+1}\Phi_{i+1}\gamma =$$
$$= (u_{i+1}^{-1}(u_{i+2}w_{i+1,\varphi(n-1)+(k-1)}u_{i+2}^{-1})u_{i+1}, (0, 0), \ldots, (0, 0))_{i+1}\Phi_{i+1}\gamma =$$
$$= (u_{i+1}^{-1}w_{\varphi(n-1)+(k-1)}u_{i+1}(0, 0), \ldots, (0, 0))_{i+1}\Phi_{i+1}\gamma.$$

However,

$$(e, (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0))_i\Phi_i = (y_{i,k}, 0, 0)_\sigma \text{ (in the case } i < -1) =$$
$$= (\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\ldots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{w}_{\varphi(n-1)+(k-1)}\theta^{-(i+1)}\bar{u}_{i+1}\theta^{-(i+1)}\ldots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma =$$
$$= (\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\ldots\bar{u}_{i+2}^{-1}\theta^{-(i+2)}((\bar{u}_{i+1}^{-1}\bar{w}_{\varphi(n-1)+(k-1)}\bar{u}_{i+1})\theta^{-(i+1)})\cdot$$
$$\cdot\bar{u}_{i+2}\theta^{-(i+2)}\ldots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma = ((\bar{u}_{i+1}^{-1}\bar{w}_{\varphi(n-1)+(k-1)}\bar{u}_{i+1})\delta_{i+1}, 0, 0)_\sigma =$$
$$= (u_{i+1}^{-1}w_{\varphi(n-1)+(k-1)}u_{i+1}, (0, 0), \ldots, (0, 0))_{i+1}\Phi_{i+1}$$

and the case $i = -1$ is handled similarly.
Hence,

$$(e, (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0))_i\varrho_i = (e, (0, 0), \ldots, (0, 1), (0, 0), \ldots, (0, 0))_i\Phi_i\gamma$$

where (0, 1) is the k^{th} two-tuple with $k > 1$.

Thus, by the usual methods, $(g, (a_1, b_1), \ldots, (a_n, b_n))_i\varrho_i = (g, (a_1, b_1), \ldots, \ldots, (a_n, b_n))_i\Phi_i\gamma$ for all $(g, (a_1, b_1), \ldots, (a_n, b_n))_i \in S_i$. Thus, if $x \in S$, $x\varrho = x\Phi\gamma$. Hence, H is the maximal group homomorphic image of S under the homomorphism Φ. Utilizing, (2. 34) and (2. 31), we obtain the expression for Φ given in the statement of the theorem.

In the case $n = 1$, we obtain [9, theorem 3. 6]. However, we obtain an improved form of this result.

THEOREM 3. 3 (WARNE, [9]). *Let $S = (G, C_1^*, \alpha, u_i)$ be an I-bisimple semigroup and let e be the identity of G. If $N = \{g \in G \mid g\alpha^n = e$ for some $n \in I^0\}$, N is a normal subgroup of G. If $(xN)\theta = (x\alpha)N$, $x \in G$, θ is an endomorphism of G/N. Let $g \to \bar{g}$*

denote the natural homomorphism of G onto G/N. Let us define a relation σ on $G/N\times(I^0)^2$ by the rule

$$((\bar{g}, a, b), (\bar{h}, c, d))\in\sigma$$

if and only if there exists $x, y\in I^0$ such that $x+a=y+c$, $x+b = y+d$, and $\bar{g}\theta^x=\bar{h}\theta^y$. Then σ is an equivalence relation on $G/N\times(I^0)^2$. Furthermore, the rule

$$(\bar{g}, a, b)_\sigma(\bar{h}, c, d)_\sigma = (\bar{g}\theta^c\bar{h}\theta^b, a+c, b+d)_\sigma$$

defines a binary operation on $G/N\times(I^0)^2/\sigma = H$ whereby H becomes a group which is the maximal group homomorphic image of S. The canonical homomorphism of S onto H is given by

$$(g, a_1, b_1)\Phi =$$

$$=\begin{cases}\big((y_{i,1}^{-1}\theta^{a_1-i-1}\dots y_{i,1}^{-1}\theta y_{i,1}^{-1})\big(\big(\big(\overline{t(u_i,\alpha,a_1-1)}\big)^{-1}\,\bar{g}\,\overline{t(u_i,\alpha,b_1-1)}\big)\delta_i\big)\cdot\\ \cdot(y_{i,1}\cdot y_{i,1}\theta\dots y_{i,1}\theta^{b_1-1-i}),\ a_1-i,\ b_1-i\big)_\sigma\\ \text{for } i\leqq -1. \text{ If } a_1=i\ (b_1=i), \text{ the corresponding factor is } \bar{e};\\ (\bar{g}, a_1, b_1)_\sigma \text{ for } i=0\end{cases}$$

where $(g, (a_1, b_1))\in(e, i, i)S(e, i, i)$ and where

$$y_{0,1}=\bar{e},$$

$$y_{-1,1}=\bar{u}_0^{-1},$$

$$y_{i,1}=\bar{u}_0^{-1}(\bar{u}_{-1}^{-1}\theta)\dots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+2}\theta^{-(i+1)}\bar{u}_{i+3}\theta^{-(i+2)}\dots\bar{u}_0\theta \text{ for } i\leqq -2,$$

$$\bar{g}\delta_0=\bar{g}, \quad \text{and}$$

$$\bar{g}\delta_i=\bar{u}_0^{-1}(\bar{u}_{-1}^{-1}\theta)\dots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{g}\theta^{-i}\bar{u}_{i+1}\theta^{-(i+1)}\dots\bar{u}_{-1}\theta\bar{u}_0 \text{ for } i\leqq -1.$$

We next determine the idempotent separating congruences of an ω^{n-1} I-bisimple semigroup. The basic tools are the "inverse limit technique", theorem 1. 4, and [11, theorem 3. 1].

An invariant subgroup V of a group G is called $\gamma_1-\gamma_2-\dots-\gamma_n$ invariant if $V\gamma_i\subseteqq V$ for $i=1, 2, \dots, n$, where the $\gamma_i (1\leqq i\leqq n)$ are endomorphisms of G.

THEOREM 3. 4. *Let $S=(G, C_n^*, \alpha_1, \dots, \alpha_n, w_1, \dots, w_{\varphi(n)}, u_i)$ be an ω^{n-1} I-bisimple semigroup. There exists a one-to-one correspondence between the idempotent separating congruences on S and the $\alpha_1-\alpha_2-\dots-\alpha_n$ invariant subgroups of G. If ϱ^V is the congruence corresponding to the $\alpha_1-\alpha_2-\dots-\alpha_n$ invariant subgroup V, $\varrho^V(g, (a_1, b_1), \dots, (a_n, b_n))=\{(vg, (a_1, b_1), \dots, (a_n, b_n)), v\in V\}$, i.e. $(g, (a_1, b_1), \dots, (a_n, b_n))\,\varrho^V(h, (c_1, d_1), \dots, (c_n, d_n))$ if and only if $a_i=c_i$ and $b_i=d_i$ for $1\leqq i\leqq n$ and $Vg=Vh$. If V_1, V_2 are $\alpha_1-\alpha_2-\dots-\alpha_n$ invariant subgroups of G, $V_1\subseteqq V_2$ if and only if $\varrho^{V_1}\subseteqq\varrho^{V_2}$.*

PROOF. As in the proof of theorem 2. 1, $S = \cup\{S_i : i\in I, i\leqq 0\}$ where $S_i=(G, C_n, \alpha_{i,1}, \dots, \alpha_{i,n}, w_{i,1}, \dots, w_{i,\varphi(n)})$ and (2. 8)—(2. 13) are valid. If ϱ is an idempotent separating congruence of S, let V_i be the $\alpha_{i,1}-\alpha_{i,2}-\dots-\alpha_{i,n}$ invariant subgroup of G corresponding to $\varrho_i=\varrho\mid S_i$ in [11, theorem 3. 1]. If we let $\varrho\theta=V_0$, θ defines a single valued mapping of the set of idempotent separating congruences of S

into the set of $\alpha_1-\alpha_2-\cdots-\alpha_n$ invariant subgroups of G. We will proceed to show that θ is "one-to-one" and "onto". We note that a subgroup N of G is $\alpha_1-\alpha_2-\cdots-\alpha_n$ invariant if and only if it is $\alpha_{i,1}-\alpha_{i,2}-\cdots-\alpha_{i,n}$ invariant ($i\in I$, $i\leqq 0$) by (2. 8) and (2. 9). Let $y\in V_{i+1}$. Hence, $(y, (0,0), \ldots, (0,0))_{i+1}\varrho_{i+1}(e, (0,0), \ldots$ $\ldots, (0,0))_{i+1}$ by [11, theorem 3. 1]. Thus, by (2. 12), $(y, (1,1), (0,0), \ldots, (0,0))_i\varrho_i$ $(e, (1,1), (0,0), \ldots, (0,0))_i$ and $y\in V_i$ by [11, theorem 3. 1]. In a similar manner, $V_i\subseteqq V_{i+1}$. Hence, $V_i=V_{i+1}$ and $V_i=V_0$ for all $i\in I$, $i\leqq 0$. Now, suppose that $\varrho\theta=\tau 0$. Hence, $V_0=X_0$ where X_i denotes the $\alpha_1-\alpha_2-\cdots-\alpha_n$ invariant subgroup associated with $\tau_i=\tau\mid S_i$. Suppose $x\varrho y$. If $x, y\in S_i$, we may write $x=(g, (a_1, b_1), \ldots$ $\ldots, (a_n, b_n))_i$ and $y=(h, (c_1, d_1), \ldots, (c_n, d_n))_i$. Hence, $x\varrho_i y$ and, thus, $a_i=c_i$ for $1\leqq i\leqq n$, $b_i=d_i$ for $1\leqq i\leqq n$, and $V_ig=V_ih$ by [11, theorem 3. 1]. Thus, $X_ig=X_ih$ and $x\tau y$ by [11, theorem 3. 1]. Similarly, $\tau\subseteqq\varrho$ and, hence, $\tau=\varrho$ and θ is one-to one. Let V be an $\alpha_1-\alpha_2-\cdots-\alpha_n$ invariant subgroup of G. Define $(g, (a_1, b_1), \ldots, (a_n, b_n))_i\varrho_i$ $(h, (c_1, d_1), \ldots, (c_n, d_n))_i$ if and only if $Vg=Vh$, $a_i=c_i$ for $1\leqq i\leqq n$, and $b_i=d_i$ for $1\leqq i\leqq n$. Thus, by [11, theorem 3. 1], ϱ_i is an idempotent separating congruence on S_i. Define ϱ as follows: for $x, y\in S_i$, $x\varrho y$ if and only if $x\varrho_i y$. Utilizing theorem 1. 3, it is easy to see that ϱ is an idempotent separating congruence on S. Clearly, $\varrho\theta=V$, and, hence, θ is an "onto" mapping. If $\varrho\theta=V$, we will denote ϱ by ϱ^V. Utilizing [11, theorem 3. 1], we see that $V_1\subseteqq V_2$ if and only if $\varrho^{V_1}\subseteqq\varrho^{V_2}$. The remainder of the theorem may be established by utilizing [11, theorem 3. 1] and (2. 34).

REMARK. In the case $n=1$, we obtain [9, theorem 4. 4].

REMARK. If S is an $\omega^{n-1}I$-bisimple semigroup, $\mathscr{H}$ is the maximal idempotent separating congruence of S. This statement is a consequence of theorem 1. 4 and theorem 3. 4.

We next completely determine the $\omega^{k-1}I$-bisimple congruences ($1\leqq k\leqq n-1$) of an $\omega^{n-1}I$-bisimple semigroup. Important tools are "the inverse limit technique", [11, theorem 4. 1], and theorem 1. 4.

THEOREM 3. 5. *Let $S=(G, C_n^*, \beta_1, \beta_2, \ldots, \beta_n, w_1, w_2, \ldots, w_{\varphi(n)}, m_i)$ be an $\omega^{n-1}I$-bisimple semigroup and let k be a positive integer less than n. Let N be a $\beta_1-\beta_2-\cdots-\beta_n$ invariant subgroup of G such that $h\beta_n\in N$ implies that $h\in N$.*

First, suppose there exists $x\in G$ and a positive integer μ such that

(3. 4) $$x\beta_{k+1}\in xN,$$

(3. 5) $$(x^qh)N=(h\beta_{k+1}^{\mu q}x^q)N \quad \textit{for} \quad q\in I^0 \quad \textit{and} \quad h\in G,$$

and

(3. 6) $$x\beta_s\in w^{\mu}_{\varphi(n-s)+(k-s+1)}N \quad \textit{for} \quad 1\leqq s\leqq k.$$

Let

$$(g, (a_1, b_1), \ldots, (a_n, b_n))\varrho(h, (c_1, d_1), \ldots, (c_n, d_n))$$

if and only if

(3. 7) $$a_s=c_s \quad \textit{and} \quad b_s=d_s \quad \textit{for} \quad 1\leqq s\leqq k$$

and there exists $i \in I$ *with* $i \leqq 0$ *such that* $a_1, b_1, c_1, d_1 \geqq i$ *and*

$$(3.8)\qquad \left\{\left(\prod_{s=1}^{n-k-1} w_{\varphi(n-k-1)+s}^{-a_{k+s+1}}\right)\cdot\right.$$

$$\cdot\left(\left(\left(t(m_i, \beta, a_1-i, a_2, \ldots, a_n)\right)^{-1} g t(m_i, \beta, b_1-i, b_2, \ldots, b_n)\right)\beta_{k+1}\right)\cdot$$

$$\cdot\left.\left(\prod_{s=n-k-1}^{1} w_{\varphi(n-k-1)+s}^{b_{k+s+1}}\right)\right\} \beta_{k+1}^{d_{k+1}-\min(b_{k+1}, d_{k+1})} \left\{\left(\prod_{s=1}^{n-k-1} w_{\varphi(n-k-1)+s}^{-d_{k+s+1}}\right)\cdot\right.$$

$$\cdot\left(\left(\left(t(m_i, \beta, d_1-i, d_2, \ldots, d_n)\right)^{-1} h^{-1} t(m_i, \beta, c_1-i, \ldots, c_n)\right)\beta_{k+1}\right)\cdot$$

$$\cdot\left.\left(\prod_{s=n-k-1}^{1} w_{\varphi(n-k-1)+s}^{c_{k+s+1}}\right)\right\} \beta_{k+1}^{b_{k+1}-\min(b_{k+1}, d_{k+1})} \in N x^r$$

where $\mu r = (a_{k+1}-b_{k+1})+(d_{k+1}-c_{k+1})$ *for some integer* r.

Then, ϱ *is an* ω^{k-1} *I-bisimple congruence on* S. *The collection of* ϱ*-classes containing idempotents (*ϱ *is completely and uniquely determined by this collection) is* $\{N_{(a_1, a_2, \ldots, a_k)} : (a_1, a_2, \ldots, a_k) \in I \times (I^0)^{k-1}\}$ *where*

$$(3.9)\quad N_{(a_1, a_2, \ldots, a_k)} = \{(g, (a_1, a_1), \ldots, (a_k, a_k), (a_{k+1}, b_{k+1}), \ldots, (a_n, b_n)) : a_s, b_s \in I^0$$

for $k+1 \leqq s \leqq n$, $a_{k+1}-b_{k+1} = \mu r$ *for some integer* r, $g \in G$, *and*

$$\left(\left(t(m_i, \beta, a_1-i, a_2, \ldots, a_n)\right)^{-1} g t(m_i, \beta, a_1-i, a_2, \ldots, a_k, b_{k+1}, \ldots, b_n)\right)\beta_{k+1} \in$$

$$\in N\left(\prod_{s=n-k-1}^{1} w_{\varphi(n-k-1)+s}^{a_{k+s+1}}\right) x^r \left(\prod_{s=1}^{n-k-1} w_{\varphi(n-k-1)+s}^{-b_{k+s+1}}\right)$$

where $i \in I$, $i \leqq 0$, *and* $a_1 \geqq i\}$.

Secondly, let

$$(g, (a_1, b_1), \ldots, (a_n, b_n))\,\varrho\,(h, (c_1, d_1), \ldots, (c_n, d_n))$$

if and only if

$$(3.10)\qquad a_s = c_s \quad \text{and} \quad b_s = d_s \quad \text{for} \quad 1 \leqq s \leqq k,$$

$$(3.11)\qquad (a_{k+1}-b_{k+1})+(d_{k+1}-c_{k+1}) = 0,$$

and there exists $i \in I$, $i \leqq 0$ *such that* $a_1, b_1, c_1, d_1 \geqq i$ *and*

$$(3.12)\qquad \left\{\left(\prod_{s=1}^{n-k-1} w_{\varphi(n-k-1)+s}^{-a_{k+s+1}}\right)\cdot\right.$$

$$\cdot\left(\left(\left(t(m_i, \beta, a_1-i, a_2, \ldots, a_n)\right)^{-1} g t(m_i, \beta, b_1-i, b_2, \ldots, b_n)\right)\beta_{k+1}\right)\cdot$$

$$\cdot\left.\left(\prod_{s=n-k-1}^{1} w_{\varphi(n-k-1)+s}^{b_{k+s+1}}\right)\right\} \beta_{k+1}^{d_{k+1}-\min(b_{k+1}, d_{k+1})} \left\{\left(\prod_{s=1}^{n-k-1} w_{\varphi(n-k-1)+s}^{-d_{k+s+1}}\right)\cdot\right.$$

$$\cdot\left(\left(\left(t(m_i, \beta, d_1-i, d_2, \ldots, d_n\right)^{-1} h^{-1} t(m_i, \beta, c_1-i, c_2, \ldots, c_n)\right)\beta_{k+1}\right)\right)\cdot$$

$$\cdot\left.\left(\prod_{s=n-k-1}^{1} w_{\varphi(n-k-1)+s}^{c_{k+s+1}}\right)\right\} \beta_{k+1}^{b_{k+1}-\min(b_{k+1}, d_{k+1})} \in N.$$

Then, ϱ is an $\omega^{k-1}I$-bisimple congruence on S. The collection of ϱ-classes containing idempotents (ϱ is uniquely and completely determined by this collection) is $\{N_{(a_1,a_2,\dots,a_k)}:(a_1,a_2,\dots,a_k)\in I\times(I^0)^{k-1}$ where

$$(3.13)\qquad N_{(a_1,a_2,\dots,a_k)}=$$

$$=\{(g,(a_1,a_1),\dots,(a_k,a_k),(a_{k+1},a_{k+1}),(a_{k+2},b_{k+2}),\dots,(a_n,b_n)):$$

$$a_p,b_p\in I^0 \quad \textit{for} \quad k+1\leqq p\leqq n, g\in G, \quad \textit{and} \quad ((t(m_i,\beta,a_1-i,a_2,\dots,a_n))^{-1}\cdot$$

$$\cdot gt(m_i,\beta,a_1-i,a_2,\dots,a_{k+1},b_{k+2},\dots,b_n))\beta_{k+1}\in N\cdot$$

$$\cdot\left(\prod_{s=n-k-1}^{1} w_{\varphi(n-k-1)+s}^{a_{k+s+1}}\right)\left(\prod_{s=1}^{s=n-k-1} w_{\varphi(n-k-1)+s}^{-b_{k+s+1}}\right)$$

where $i\in I$, $i\leqq 0$, and $a_1\geqq i\}$.

In both the above cases, if $n=k+1$, any factor involving "Π" is replaced by e, the identity of G. Conversely, every $\omega^{k-1}I$-bisimple congruence of S is obtained in one of the above fashions.

PROOF. Let ϱ be an $\omega^{k-1}I$-bisimple congruence on S. Thus, $\varrho_i=\varrho\,|\,S_i$ is an ω^k-bisimple congruence of S by theorem 1. 3. Hence, by [11, theorem 4. 1], we have the following two possibilities:

Case I: $(g,(a_1,b_1),\dots,(a_n,b_n))_i\,\varrho_i\,(h,(c_1,d_1),\dots,(c_n,d_n))_i$ if and only if

$$(3.14)\qquad a_s=c_s \quad \text{and} \quad b_s=d_s \quad \text{for} \quad 1\leqq s\leqq k$$

and

$$(3.15)\qquad \left\{\left(\prod_{s=1}^{n-k-1} w_{i,\varphi(n-k-1)+s}^{-a_{k+s+1}}\right)(g\beta_{i,\,k+1})\left(\prod_{s=n-k-1}^{1} w_{i,\varphi(n-k-1)+s}^{b_{k+s+1}}\right)\right\}\beta_{i,k+1}^{d_{k+1}-\min(b_{k+1},d_{k+1})}\cdot$$

$$\cdot\left\{\left(\prod_{s=1}^{n-k-1} w_{i,\varphi(n-k-1)+s}^{-d_{k+s+1}}\right)(h^{-1}\beta_{i,k+1})\left(\prod_{s=n-k-1}^{s=1} w_{i,\varphi(n-k-1)+s}^{c_{k+s+1}}\right)\right\}\beta_{i,k+1}^{b_{k+1}-\min(b_{k+1},d_{k+1})}\in N_i x_i^r$$

where $(a_{k+1}-b_{k+1})+(d_{k+1}-c_{k+1})=\mu_i r$ for some integer r. Here, N_i is a $\beta_{i,1}-\beta_{i,2}-\dots-\beta_{i,n}$ invariant subgroup of G such that $h\beta_{i,n}\in N_i$ implies that $h\in N_i$, μ_i is a positive integer, and x_i is an element of G such that

$$(3.16)\qquad x_i\beta_{i,k+1}\in x_iN_i,$$

$$(3.17)\qquad (x_i^q h)N_i=(h\beta_{i,k+1}^{\mu_i q}x_i^q)N_i \quad \text{where} \quad q\in I^0 \quad \text{and} \quad h\in G,$$

and

$$(3.18)\qquad x_i\beta_{i,s}\in w_{i,\varphi(n-s)+(k-s+1)}^{\mu_i}N_i \quad \text{for} \quad 1\leqq s\leqq k.$$

Case II: $(g,(a_1,b_1),\dots,(a_n,b_n))_i\,\varrho_i^*\,(h,(c_1,d_1),\dots,(c_n,d_n))_i$ if and only if

$$(3.19)\qquad a_s=c_s \quad \text{and} \quad b_s=d_s \quad \text{for} \quad 1\leqq s\leqq k,$$

$$(3.20)\qquad (a_{k+1}-b_{k+1})+(d_{k+1}-c_{k+1})=0,$$

and

$$(3.21)\quad \left\{\left(\prod_{s=1}^{n-k-1} w_{i,\varphi(n-k-1)+s}^{-a_{k+s+1}}\right)(g\beta_{i,k+1})\left(\prod_{s=n-k-1}^{1} w_{i,\varphi(n-k-1)+s}^{b_{k+s+1}}\right)\right\}\beta_{i,k+1}^{d_{k+1}-\min(b_{k+1},d_{k+1})}\cdot$$

$$\cdot\left\{\left(\prod_{s=1}^{n-k-1} w_{i,\varphi(n-k-1)+s}^{-d_{k+s+1}}\right)(h^{-1}\beta_{i,k+1})\left(\prod_{s=n-k-1}^{1} w_{i,\varphi(n-k-1)+s}^{c_{k+s+1}}\right)\right\}\beta_{i,k+1}^{b_{k+1}-\min(b_{k+1},d_{k+1})}\in N_i,$$

where N_i is a $\beta_{i,1}-\beta_{i,2}-\cdots-\beta_{i,n}$ invariant subgroup of G such that $h\beta_{i,n}\in N_i$ implies that $h\in N_i$.

Suppose that ϱ_{i+1} is a congruence of Case I. Choose a_{k+1} and c_{k+1} such that $\mu_{i+1}=a_{k+1}-c_{k+1}$. By (3. 16), $x_{i+1}\beta_{i+1,k+1}\in x_{i+1}N_{i+1}$. Hence,

$$(x_{i+1},(0,0),\dots,(a_{k+1},0),(0,0),\dots,(0,0))_{i+1}\varrho_{i+1}\cdot$$
$$\cdot(e,(0,0),\dots,(c_{k+1},0),(0,0),\dots,(0,0))_{i+1}$$

by (3. 14) and (3. 15). Thus, by (2. 12),

$$(x_{i+1},(1,1),(0,0),\dots,(a_{k+1},0),(0,0),\dots,(0,0))_i\varrho_i\cdot$$
$$\cdot(e,(1,1),(0,0),\dots,(c_{k+1},0),(0,0),\dots,(0,0))_i.$$

Hence, since $a_{k+1}\neq c_{k+1}$, ϱ_i is not of Case II. Hence, if ϱ_{i+1} is of Case I, ϱ_i is of Case I. Similarly, if ϱ_{i+1} is of Case II, ϱ_i is of Case II.

Case A: ϱ_0 is of Case I. We first show that $N_{i+1}=N_i$ for all $i\in I$ with $i\leqq 0$. Let $g\in N_{i+1}$. Thus, $(g,(0,0),\dots,(0,0))_{i+1}\varrho_{i+1}(e,(0,0),\dots,(0,0))_{i+1}$. Hence, $(g,(1,1),(0,0),\dots,(0,0))_i\varrho_i(e,(1,1),(0,0),\dots,(0,0))_i$ and $g\beta_{i,k+1}\in N_i$. Hence, $g\beta_{i,k+1}^{\mu_i}\in N_i$ since N_i is $\beta_{i,k+1}$-invariant. Thus, it follows from (3. 17) that $g\in N_i$. Similarly, $N_i\subseteq N_{i+1}$. If we let $N=N_0$, $N=N_i$ for all $i\in I$ with $i\leqq 0$. We next show that $\mu_{i+1}=\mu_i$ for $i\leqq 0$. Suppose that $\mu_{i+1}=a_{k+1}-c_{k+1}$ where $a_{k+1},c_{k+1}\in I^0$. Thus, as above, $(x_{i+1},(1,1),(0,0),\dots,(a_{k+1},0),(0,0),\dots,(0,0))_i\varrho_i(e,(1,1),(0,0),\dots,\dots,(c_{k+1},0),(0,0),\dots,(0,0))_i$. Hence, there exists a positive integer s such that $s\mu_i=\mu_{i+1}$. In a similar manner, $\mu_i\geqq\mu_{i+1}$. If we let $\mu=\mu_0$, $\mu_i=\mu$ for all $i\in I$ with $i\leqq 0$. By what we have proved above, $x_{i+1}\beta_{i,k+1}\in Nx_i$. It then follows from (3. 16) and (3. 17) that $Nx_i=Nx_{i+1}$ for $i\leqq 0$. If we denote x_0 by x, then $Nx_i=Nx$ for all $i\in I$ with $i\leqq 0$. If we let $i=0$ in (3. 16)—(3. 18) and apply (2. 8)—(2. 11), we obtain (3. 4)—(3. 6). By (2. 8) and (2. 9), N is a $\beta_1-\beta_2-\beta_n$ invariant subgroup of G such that $h\beta_n\in N$ implies that $h\in N$.

Suppose $(g,(a_1,b_1),\dots,(a_n,b_n))\varrho(h,(c_1,d_1),\dots,(c_n,d_n))$. Hence, by theorem 1. 3 and theorem 1. 4 there exists $i\in I$ with $i\leqq 0$, and a_1', b_1', c_1', and $d_1'\in I^0$ such that $a_1=a_1'+i$, $b_1=b_1'+i$, $c_1=c_1'+i$, and $d_1=d_1'+i$. Hence, (3. 7) and (3. 8) follow from (2. 34), (3. 14), (3. 15), (2. 9), and (2. 11). Conversely, assume (3. 7) and (3. 8) and let $a_1'=a_1-i$, $b_1'=b_1-i$, $c_1'=c_1-i$, and $d_1'=d_1-i$. Thus, by (2. 9), (2. 11), (3. 14), (3. 15), and (2. 34), $(g,(a_1,b_1),\dots,(a_n,b_n))\varrho(h,(c_1,d_1),\dots\dots,(c_n,d_n))$.

Let us now determine the ϱ-classes containing idempotents. By (3. 7) and (3. 8), $(e,(a_1,a_1),\dots,(a_n,a_n))\varrho(e,(c_1,c_1),\dots,(c_n,c_n))$ if and only if $a_s=c_s$ for $1\leqq s\leqq k$. Thus, if T is the ϱ-class containing the idempotent $(e,(a_1,a_1),\dots,(a_n,a_n))$ and E_T denotes the set of idempotents of T, then

$$E_T=\{(e,(a_1,a_1),\dots,(a_k,a_k),(b_{k+1},b_{k+1}),\dots,(b_n,b_n)):b_s\in I^0\ \text{ for }\ k+1\leqq s\leqq n\}.$$

Hence, we may write $T=N_{(a_1,a_2,\dots,a_k)}$, and (3. 9) is an easy consequence of (3. 7) and (3. 8). By [2, Vol. 2, theorem 7. 48], ϱ is uniquely determined by

$$\{N_{(a_1,a_2,\dots,a_k)}: (a_1, a_2, \dots, a_k)\in I\times(I^0)^{k-1}\}.$$

Case B: ϱ_0 is of Case II. Since $g\beta_{i,k+1}\in N_i (i\in I,\ i\leqq 0)$ implies $g\in N_i$ by the proof of [11, theorem 4. 1], we may show as in Case A, that $N_{i+1}=N_i$ for all $i\in I$, $i\leqq 0$. Thus, if we let $N=N_0$, $N=N_i$ for all $i\in I$, $i\leqq 0$. Suppose that $(g, (a_1, b_1), \dots, \dots, (a_n, b_n))\varrho(h, (c_1, d_1), \dots, (c_n, d_n))$. Thus, by methods used in Case A, (3. 10)—(3. 12) follow from (2. 34), (3. 19)—(3. 21), (2. 9), and (2. 11). If we assume (3. 10)—(3. 12), $(g, (a_1, b_1), \dots, (a_n, b_n))\varrho(h, (c_1, d_1), \dots, (c_n, d_n))$ by virtue of (2. 9), (2. 11), (3. 19)—(3. 21), and (2. 34). We obtain (3. 13) as a consequence of (3. 10)—(3. 12).

Let us now establish the converse. Let N be a $\beta_1-\beta_2-\dots-\beta_n$ invariant subgroup of G such that $h\beta_n\in N$ implies that $h\in N$ and suppose there exists $x\in G$ and a positive integer μ such that (3. 4)—(3. 6) are valid. Thus, by employing (2. 8)—(2. 11), (3. 4)—(3. 6), (3. 7), (3. 8), (2. 34), and [11, theorem 4. 1], it is easily seen $\varrho_i=\varrho|S_i$, where ϱ is defined by (3. 7) and (3. 8), is a ω^k-bisimple congruence for all $i\in I$, $i\leqq 0$. Hence, ϱ is a congruence on S. By [3, p. 1111], S/ϱ is a bisimple semigroup. Thus, by [2, Vol. 2, p. 57, lemma 7. 34] and (3. 13), ϱ is an $\omega^{k-1}I$-bisimple congruence on S. The other case is handled in a similar manner.

REMARK. If $k+1=n$, (3. 8) may be replaced by

$$((t(m_i, \beta, a_1-i, a_2, \dots, a_n))^{-1}gt(m_i, \beta, b_1-i, b_2, \dots, b_n))\beta_n^{d_n-\min(b_n,d_n)}\cdot$$
$$\cdot((t(m_i, \beta, d_1-i, d_2, \dots, d_n))^{-1}h^{-1}t(m_i, \beta, c_1-i, c_2, \dots, c_n))\beta_n^{b_n-\min(b_n,d_n)}\in Nx^r.$$

This statement follows since $h\beta_n\in N$ implies $h\in N$ and (3. 4) is valid. If $k+1=n$, (3. 12) may be replaced by

$$((t(m_i, \beta, a_1-i, a_2, \dots, a_n))^{-1}gt(m_i, \beta, b_1-i, b_2, \dots, b_n))\beta_n^{d_n-\min(b_n,d_n)}\cdot$$
$$\cdot((t(m_i, \beta, d_1-i, d_2, \dots, d_n))^{-1}h^{-1}t(m_i, \beta, c_1-i, c_2, \dots, c_n))\beta_n^{b_n-\min(b_n,d_n)}\in N.$$

This follows since $h\beta_n\in N$ implies $h\in N$.

We state theorem 4. 1 in the case $n=2$ and $k=1$ as this represents the simplest possible case.

THEOREM 3. 6. *Let $S=(G, C_2^*, \beta_1, \beta_2, w_1, m_i)$ be an ωI-bisimple semigroup and let N be a $\beta_1-\beta_2$ invariant subgroup of G such that $h\beta_2\in N$ implies that $h\in N$.*

First, suppose there exists $x\in G$ and a positive integer q such that

(3. 22) $$x\beta_2\in xN,$$

(3. 23) $$(x^q h)N=(h\beta_2^{\mu q}x^q)N \quad \textit{for} \quad q\in I^0 \quad \textit{and} \quad h\in G, \quad \textit{and}$$

(3. 24) $$x\beta_1\in w_1^{\mu}N.$$

Let $(g, (a_1, b_1), (a_2, b_2))\varrho(h, (c_1, d_1), (c_2, d_2))$ if and only if

(3. 25) $$a_1=c_1 \quad \textit{and} \quad b_1=d_1$$

and there exists $i\in I$ with $i\leqq 0$ such that $a_1, b_1, c_1, d_1\geqq i$ and

(3. 26) $$((t(m_i, \beta, a_1-i, a_2))^{-1}gt(m_i, \beta, b_1-i, b_2))\beta_2^{d_2-\min(b_2,d_2)}\cdot$$
$$\cdot((t(m_i, \beta, d_1-i, d_2))^{-1}h^{-1}t(m_i, \beta, c_1-i, c_2))\beta_2^{b_2-\min(d_2,b_2)}\in Nx^r$$

where $\mu r=(a_2-b_2)+(d_2-c_2)$ for some $r\in I$.

Then, ϱ is an I-bisimple congruence on S. The collection of ϱ-classes containing idempotents (ϱ is completely and uniquely determined by this collection) is $\{N_{a_1}: a_1 \in I\}$ where

$$(3.27)\quad N_{a_1} = \{(g, (a_1, a_1), (a_2, b_2)): a_2, b_2 \in I^0, a_2 - b_2 = \mu r \text{ for some } r \in I, g \in G, \text{ and } (t(m_i, \beta, a_1 - i, a_2))^{-1} gt(m_i, \beta, a_1 - i, b_2) \in Nx^r$$

where $i \in I$, $i \leqq 0$, and $a_1 \geqq i\}$.

Secondly, let

$$(g, (a_1, b_1), (a_2, b_2))\varrho(h, (c_1, d_1), (c_2, d_2)) \text{ if and only if}$$

$$(3.28)\qquad a_1 = c_1 \text{ and } b_1 = d_1,$$

$$(3.29)\qquad (a_2 - b_2) + (d_2 - c_2) = 0,$$

and there exists $i \in I$ with $i \leqq 0$ such that $a_1, b_1, c_1, d_1 \geqq i$ and

$$(3.30)\qquad ((t(m_i, \beta, a_1 - i, a_2))^{-1} gt(m_i, \beta, b_1 - i, b_2))\beta_2^{d_2 - \min(b_2, d_2)} \cdot ((t(m_i, \beta, d_1 - i, d_2))^{-1} h^{-1} t(m_i, \beta, c_1 - i, c_2))\beta_2^{b_2 - \min(b_2, d_2)} \in N.$$

Then, ϱ is an I-bisimple congruence on S. The collection of ϱ-classes containing idempotents (ϱ is uniquely and completely determined by this collection) is $\{N_{a_1}: a_1 \in I\}$ where

$$(3.31)\qquad N_{a_1} = \{(g, (a_1, a_1), (a_2, a_2)): a_2 \in I^0, g \in G \text{ and } ((t(m_i, \beta, a_1 - i, a_2))^{-1} gt(m_i, \beta, a_1 - i, a_2)) \in N \text{ where } i \in I,\ i \leqq 0, \text{ and } a_1 \geqq i\}.$$

Conversely, every I-bisimple congruence of S is obtained in one of the above fashions.

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ON AN APPLICATION OF THE LARGE SIEVE: SHIFTED PRIME NUMBERS, WHICH HAVE NO PRIME DIVISORS FROM A GIVEN ARITHMETICAL PROGRESSION

By

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I. Introduction

In 1957 C. HOOLEY proved [1] on the base of the extended Riemann-conjecture the conjecture of Hardy and Littlewood stating that all sufficiently large integer can be represented as the sum of two squares and a prime. This assertion was proved in 1960 by YU. V. LINNIK [2] without using any conjectures. The proof of due to Linnik is a very ingenious but very complicated one. By new improvements of the large sieve due to E. BOMBIERI [3] the conjucture of Hardy and Littlewood can be proved by the simpler method of Hooley, too.

Using the method of Linnik or Hooley—Bombieri with unimportant modifications one can prove: if n is a large positive integer, then n can be represented as the sum of a prime and such a square-free summand which is a sum of two squares. In other words, the following assertion holds: if n is a large number, then it can be represented as a sum of a prime and such a number which has no prime divisors in the arithmetical progression $-1 \pmod 4$ (see [4]).

The following question seems to be interesting. Let D be an integer greater than 2 and l be coprime to D. We ask, whether the following assertion holds or not: all sufficiently large integer can be represented as the sum of a prime and such a summand which has no prime divisors in the arithmetical progression $l \pmod D$.

Similarly we can ask the following: whether for a non-zero fixed integer there can be found infinitely many (or at least one) representations of it as difference of a prime and such a number which has no prime factors in the arithmetical progression $l \pmod D$, or not.

In this paper we shall prove that for infinitely many pairs of l and D the answer is yes for both problems. The basic idea of the solution is very simple.

Let χ be a primitive character mod D, D a prime power. Using the large sieve and the method of Hooley for the estimation of Σ_B (see [1]) we prove the following asymptotical relation:

$$(1.1)\qquad \sum_{p\leqq x} r(p+a)|\mu(p+a)| = A(D,\chi,a)\,\mathrm{li}\,x + O\big(\mathrm{li}\,x\cdot(\log x)^{-\delta}\big),$$

where $r(n) = \sum_{d|n}\chi(d) = \prod_{p^\alpha\|n}\big(1+\chi(p)+\cdots+\chi(p^\alpha)\big)$, $\mu(n)$ is the Moebius-function, p runs through the primes, δ is a positive constant and $A(D,\chi,a)$ is a suitable complex number (see (2.2.4)). Furthermore, we shall show that $A(D,\chi,a)\neq 0$, except for the case aD is odd and $\chi(2)=-1$.

If l and D is such a pair of numbers that there exists a character χ satisfying $\chi(l)=-1$, then in the left hand side of (1.1) only those primes p occur with non-zero weight, for which $p-a$ has no prime factors in the arithmetical progression $l \pmod D$. Since $A(D, \chi, a) \neq 0$ disregarding the natural exceptions, from (1. 1) we obtain a positive answer for the second question.

For the solution of the first question we prove an asymptotical formula for the sum

$$\sum_{p<n} r(n-p)\,|\mu(n-p)|.$$

II. Asymptotical relations

2.1. Notation. a is a fixed non-zero integer. $p, p_1, \ldots, q, q_1, \ldots, p^*$ denote prime numbers. x, y, z are real numbers not less than 1. $d, k, l, m, n, r, s, t, \mu, \nu, u, v$ are natural numbers.

$[l, m]$ and (l, m) denote the least common multiple and the highest common factor of l and m, respectively.

We make use of Vinogradov's notation $A \ll B$, as an equivalent for $A = O(B)$.

Let $\pi(m, k, l)$ denote the number of primes not exceeding m which belong to the arithmetical progression $l \pmod k$. $\mu(n)$ denotes the Moebius-function, $\varphi(n)$ the Euler's function. For negative a we set $\varphi(a) = \varphi(|a|)$. $d(n)$ denotes the number of divisors of n. $\omega(n)$ is the number of different prime factors of n, $\Omega(n)$ is the total number of prime factors of n (counted according their multiplicity).

The letters $A, A_1, \ldots; c, c_1, \ldots$ denote suitable positive constants, ε denotes a sufficiently small positive number.

Let $x_1 = \log x$, $x_2 = \log x_1$.

Let $D = p^{*\beta}$ be a prime power, $D \geqq 3$. Let $\chi(n)$ be an arbitrary primitive character mod D. Let

$$r(n) = \sum_{d|n} \chi(d) = \prod_{p^\alpha \| n} \left(1 + \chi(p) + \cdots + \chi(p^\alpha)\right). \tag{2. 1. 1}$$

2.2. Formulation of theorems. Let

$$T(x) = \sum_{q \leqq x} r(q+a)\,|\mu(q+a)|, \tag{2. 2. 1}$$

$$K(x) = \sum_{q<n} r(n-q)\,|\mu(n-q)|. \tag{2. 2. 2}$$

THEOREM 1. *For any fixed integer $a \neq 0$, we have*

$$T(x) = A(D, \chi, a)\operatorname{li} x + O\left(\operatorname{li} x \cdot (\log x)^{-\delta}\right), \tag{2. 2. 3}$$

where δ is a positive constant,

$$A(D, \chi, a) = C(\chi) L(1, \chi) E(a, \chi) \prod_{p \nmid a} \left(1 - \frac{E(p, \chi) g(p, \chi)}{\varphi(p^2)}\right) - \tag{2. 2. 4}$$

$$- \varepsilon(D) \frac{\chi(a)}{\varphi(D)} C(\bar\chi) L(1, \bar\chi) E(a, \bar\chi) \prod_{p \nmid aD} \left(1 - \frac{E(p, \bar\chi) g(p, \bar\chi)}{\varphi(p^2)}\right)$$

and

(2. 2. 5)—(2. 2. 6) $$L(1,\chi)=\prod_p\left(1-\frac{\chi(p)}{p}\right)^{-1};\quad E(m,\chi)=\prod_{p|m}\frac{(p-1)(p-\chi(p))}{p^2-p+\chi(p)};$$

(2. 2. 7) $$C(\chi)=\prod_p\left(1+\frac{\chi(p)}{p(p-1)}\right),$$

where $g(n,\chi)$ *is a multiplicative function defined by the relation*

(2. 2. 8) $$g(p^a,\chi)=g(p,\chi)=1+\chi(p)+\chi^2(p)+\frac{\chi^3(p)}{p-\chi(p)}.$$

Further $\varepsilon(D)=1$, *when* $\beta=1$ *and* $=0$ *when* $\beta>1$.

We have $A(D,\chi,a)\neq 0$, *except for the case: a, D are odd and* $\chi(2)=-1$, *when* $A(D,\chi,a)=0$.

THEOREM 2. *For any n, we have*

(2. 2. 9) $$K(x)=A(D,\chi,n)\operatorname{li}n+O(\operatorname{li}n\cdot(\log n)^{-\delta})$$

where $\delta>0$. *Furthermore,*

$$|A(D,\varkappa,n)|\gg(\log\log n)^{-1},$$

except for the cas e n, D are odd and $\chi(2)=-1$.

We give a detailed proof only for Theorem 1. The proof of Theorem 2 runs in a similar but more complicated line. For the proof of Theorem 1 we need some lemmas. Many of these occur in the paper of C. Hooley.

2.3. Lemmata. Lemma 1 is due to BOMBIERI [3]. This is the principal lemma for our investigati ons.

LEMMA 1. *Let*

$$\mathscr{E}(x,k)=\max_{1\leq z\leq x}\max_{(l,k)=1}\left|\pi(z,k,l)-\frac{\operatorname{li}z}{\varphi(k)}\right|.$$

Then for any positive constant A the inequality

$$\sum_{k\leq Y}\mathscr{E}(x,k)\ll x\cdot x_1^{-A}$$

holds where $Y=x^{1/2}x_1^{-B}$ *and* $B\geq 4A+40$.

LEMMA 2. *For any m, and any non-principal character* χ mod D *we have*

$$\sum_{\substack{l\leq y\\(l,m)=1}}\frac{\chi(l)}{l}=L(1,\chi)\prod_{p|m}\left(1-\frac{\chi(p)}{p}\right)+O\left(\frac{1}{y}d(m;y)\right)+O(\sigma_{-1}(m;y)),$$

where

$$L(1,\chi)=\prod_p\left(1-\frac{\chi(p)}{p}\right)^{-1};$$

$$d(m;y)=\sum_{\substack{d|m\\d\leq y}}1;\quad \sigma_{-1}(m;y)=\sum_{\substack{d|m\\d>y}}\frac{1}{d}.$$

PROOF. We have

$$\sum_{\substack{l \le y \\ (l,m)=1}} \frac{\chi(l)}{l} = \sum_{l \le y} \frac{\chi(l)}{l} \sum_{d|(l,m)} \mu(d) = \sum_{\substack{d|m \\ d \le y}} \mu(d) \sum_{\substack{l=dt \\ l \le y}} \frac{\chi(l)}{l} =$$

$$= \sum_{d|m} \frac{\mu(d)\chi(d)}{d} \sum_{t \le \frac{y}{d}} \frac{\chi(t)}{t} = \sum_{\substack{d|m \\ d \le y}} \frac{\mu(d)\chi(d)}{d} \left\{ L(1,\chi) + O\left(\frac{d}{y}\right) \right\} =$$

$$= L(1,\chi) \sum_{d|m} \frac{\mu(d)\chi(d)}{d} + O\left(\frac{1}{y} \sum_{\substack{d|m \\ d \le y}} 1\right) + O\left(\sum_{\substack{d|m \\ d > y}} \frac{1}{d}\right) =$$

$$= L(1,\chi) \prod_{p|m} \left(1 - \frac{\chi(p)}{p}\right) + O\left(\frac{1}{y} d(m; y)\right) + O(\sigma_{-1}(m; y)).$$

LEMMA 3. *For any m and any non-principal character $\chi(\mathrm{mod}\ D)$ we have*

$$\sum_{\substack{l \le y \\ (l,m)=1}} \frac{\chi(l)}{\varphi(l)} = C(x) L(1,\chi) E(m,\chi) + O\left(\frac{\log 2y}{y} d(m)\right),$$

where $C(\chi)$ and $E(m,\chi)$ are defined by (2. 2. 6), (2. 2. 7).

Also, there exist positive constants A_1 and A_2 such that

$$\frac{A_1}{\log\log 10\, m} < |E(m,\chi)| < A_2 \log\log 10\, m.$$

The proof goes in the same way as that of Lemma 3 in [1] and so can be omitted.

LEMMA 4. *Let $D = p^{*\beta} (\ge 3)$, χ a primitive character* mod D, $(a, v) = 1$. *Let*

$$L(v) = \sum_{\substack{l (\mathrm{mod}\ D) \\ (lv-a, D)=1}} \chi(l).$$

Then

$$L(v) = \begin{cases} -\bar{\chi}(v)\chi(a), & \text{when } \beta = 1, \\ 0, & \text{when } \beta > 1. \end{cases}$$

PROOF.

$$L(v) = \sum_{l (\mathrm{mod}\ D)} \chi(l) - \sum_{\substack{l (\mathrm{mod}\ D) \\ lv - a \equiv 0 (\mathrm{mod}\ p^*)}} \chi(l) = \sum\nolimits_1 - \sum\nolimits_2,$$

say. It is obvious that $\Sigma_1 = 0$. If $(v, D) > 1$ or $(a, D) > 1$ then the sum Σ_2 contains either all or neither of the l in the reduced residue system mod D, and so $\Sigma_2 = 0$. If $\beta = 1$ and $(av, D) = 1$ then $\Sigma_2 = \bar{\chi}(v)\chi(a)$.

Let now $\beta > 1$, $(av, D) = 1$. Suppose first that $p^* \ge 3$. Let $\nu(n)$ denote the index of n relative to a fixed primitive root g, i.e. the exponent ν for which $g^\nu \equiv n (\mathrm{mod}\ D)$. Let $\omega = \exp(2\pi i/\varphi(D))$. Then our primitive character $\chi(n)$ can be represented in the form $\chi(n) = \omega^{m\nu(n)}$ where m is a suitable integer coprime to p^*. Let $l = l_1$ be a particular solution of the congruence $lv-a\equiv 0(\mathrm{mod}\, p^*)$. Then l is a solution of this congruence if and only if $v(l)\equiv v(l_1)(\mathrm{mod}\,(p-1))$. Consequently

$$\sum\nolimits_2=\omega^{v(l_1)m}\sum_{t=0}^{p^{*\beta-1}-1}\omega^{t(p-1)m}$$

holds. Since $p^*\nmid m$, the last sum is zero.

If $p^*=2$, $(av,p^*)=1$ then the condition $(lv-a,D)=1$ is equivalent to $(l,D)=1$, and so $\Sigma_2=0$.

LEMMA 5. *For any m, and non-principal character $\chi(\mathrm{mod}\, D)$ we have*

$$\sum_{\substack{l\leq y\\(l,m)=1}}\frac{\chi(l)\,l}{\varphi(l)}=O\big(d(m)\log 2y\big).$$

PROOF. Let l' denote a general square-free number. Then

$$\frac{l}{\varphi(l)}=\prod_{p|l}\left(1-\frac{1}{p}\right)^{-1}=\prod_{p|l}\left(1+\frac{1}{p-1}\right)=\sum_{l'|l}\frac{1}{\varphi(l')}.$$

So

$$\sum_{\substack{l\leq y\\(l,m)=1}}\frac{\chi(l)\,l}{\varphi(l)}=\sum_{\substack{l\leq y\\(l,m)=1}}\chi(l)\sum_{l=l'r}\frac{1}{\varphi(l')}=\sum_{\substack{l'\leq y\\(l',m)=1}}\frac{\chi(l')}{\varphi(l')}\sum_{\substack{r\leq\frac{y}{l'}\\(r,m)=1}}\chi(r).$$

Proving that inner sum is majorized by $d(m)$ we obtain immediately the statement of the lemma.

We take

$$\sum_{\substack{r\leq\frac{y}{l'}\\(r,m)=1}}\chi(r)=\sum_{r\leq\frac{y}{l'}}\chi(l)\sum_{\delta|(r,m)}\mu(\delta)=$$

$$=\sum_{\delta|m}\mu(\delta)\sum_{r=\delta s\leq\frac{y}{l'}}\chi(r)=\sum_{\delta|m}\mu(\delta)\chi(\delta)\cdot\sum_{s\leq\frac{y}{l'\delta}}\chi(s)\ll d(m).$$

Let $y=y(x)=x^{1/x_2^2}$, $P=\prod_{p\leq y}p$. For any positive integer t let

$$t^{(1)}=\prod_{\substack{p|t\\p\leq y}}p^\alpha,\quad t^{(2)}=\prod_{\substack{p|t\\p>y}}p^\alpha,$$

where

$$t=\prod p^\alpha.$$

We define the function $f(n)$ by the equation

$$f(n)=g(n)+h(n)$$

where

$$g(n)=\begin{cases}1 \text{ if } n \text{ is a prime not exceeding } y,\\ 0 \text{ otherwise,}\end{cases}$$

and

$$h(n)=\begin{cases}1 & \text{if } n \text{ is a prime to } P,\\ 0 & \text{otherwise.}\end{cases}$$

The following lemma is due to C. HOOLEY (see [1], Lemma 4).

LEMMA 6. *Let* $Y\leqq x$ *and* $k\ll x^{\theta}$ *where* θ *is a positive absolute constant less than* 1. *Then*

$$\sum_{\substack{v\leqq Y\\ v\equiv l \pmod k}} f(v)=\begin{cases}\dfrac{1}{\varphi(k)}B(x)Y+O\left(\dfrac{x}{k\cdot\log^5 x}\right) & \text{if}\quad (l^{(1)},k)=1\\[2ex] O\left(\dfrac{x}{k(\log x)^5}\right) & \text{if}\quad (l^{(1)},k)>1,\end{cases}$$

where $B(x)$ *depends only on* x *and satisfies*

$$B(x)\ll x_2^2\cdot x_1^{-1}.$$

LEMMA 7. *Let* χ *be a non-principal character* mod D, a *a fixed non-zero integer* $1\leqq k\leqq x$, $(k,a)=1$. *Then*

$$\sum_{\substack{l\geqq y\\ (l,a)=1}}\frac{\chi(l)}{\varphi(kl)}\ll x_2\frac{\log 2y}{yk}+\frac{x_2^2}{yk}.$$

PROOF. The proof is similar to that of Lemma 8 in [1].
We use the relation

$$\frac{1}{\varphi(kl)}=\frac{1}{l\varphi(k)}\prod_{\substack{p\mid l\\ p\nmid k}}\left(1-\frac{1}{p}\right)^{-1}.$$

From this we have

$$\frac{1}{\varphi(kl)}=\frac{1}{\varphi(k)l}\sum_{\substack{q\mid l\\ (q,k)=1}}\frac{1}{\varphi(q)}$$

where q denotes a general square-free number. Therefore

$$\varphi(k)\sum_{\substack{l\geqq y\\ (l,a)=1}}\frac{\chi(l)}{\varphi(kl)}=\sum_{\substack{qt\geqq y\\ (q,ak)=1\\ (t,a)=1}}\frac{\chi(qt)}{qt\varphi(q)}=$$

$$=\sum_{\substack{q\leqq y\\ (q,ak)=1}}\frac{\chi(q)}{q\varphi(q)}\sum_{\substack{t\geqq \frac{y}{q}\\ (t,a)=1}}\frac{\chi(t)}{t}+\sum_{(t,a)=1}\frac{\chi(t)}{t}\sum_{\substack{q>y\\ (q,ak)=1}}\frac{\chi(q)}{q\varphi(q)},$$

and this, by Lemma 2, is

$$O\left(\sum_{\substack{q\le y\\(q,ak)=1}}\frac{1}{q\varphi(q)}\frac{q}{y}d\left(a;\frac{y}{q}\right)\right)+O\left(\sum_{q\le y}\frac{1}{q\varphi(q)}\sigma_{-1}\left(a,\frac{y}{q}\right)\right)+O\left(\frac{x_2}{y}\right)=$$

$$=O\left(\frac{d(a)}{y}\sum_{q\le y}\frac{1}{\varphi(q)}\right)+O\left(\sigma_{-1}(a)\sum_{\frac{y}{|a|}\le q\le y}\frac{1}{q\varphi(q)}\right)+O\left(\frac{x_2}{y}\right)\ll$$

$$\ll\frac{\log 2y}{y}+\sum_{\frac{y}{|a|}\le q}\frac{x_2}{q^2}+\frac{x_2}{y}\ll\frac{\log 2y}{y}+\frac{x_2}{y}.$$

Since $k/\varphi(k)\le x_2$, the lemma follows.

LEMMA 8. *Let χ be a non-principal character* mod D, *a a non-zero fixed integer, ξ a positive integer not exceeding x, $(a,\xi)=1$. Then*

$$\sum_{\substack{r\le Y\\(r,a)=1}}\frac{\chi(r)r}{\varphi(\xi r)}\ll\frac{d(\xi)}{\varphi(\xi)}S(Y,\xi)\log 2Y,$$

where $S(Y,\xi)$ denotes the number of those integers in the interval $[1, Y]$ *all prime divisors of which divides ξ.*

Furthermore we have

$$S(Y,\xi)\ll(c\log 2Y)^{\omega(\xi)},$$

where c is a suitable positive constant.

PROOF. For a general r let $r=r_1r_2$, where r_1 contains only those prime factors of r which divides ξ, and $(r_2,\xi)=1$.

We have

$$L\stackrel{\text{def}}{=}\sum_{\substack{r\le Y\\(r,a)=1}}\frac{\chi(r)r}{\varphi(\xi r)}=\sum_{r_1\le Y}\frac{\chi(r_1)r_1}{\varphi(\xi)r_1}\sum_{\substack{r_2\le\frac{Y}{r_1}\\(r_2,\xi a)=1}}\frac{\chi(r_2)r_2}{\varphi(r_2)}.$$

Estimating the inner sum by Lemma 5 we have

$$L\ll\frac{d(\xi a)}{\varphi(\xi)}\sum_{r_1\le Y}\log\left(\frac{2Y}{r_1}\right)\ll\frac{d(\xi)}{\varphi(\xi)}S(Y,\xi)\log 2Y.$$

For the proof of the second part of the lemma let

$$\xi=p_1^{\alpha_1}\cdots p_r^{\alpha_r}\qquad(\omega(\xi)=r).$$

Then

$$S(Y,\xi)\le\sum_{p_1^{\beta_1}\le Y}1\sum_{p_2^{\beta_2}\le Y}1\ldots\sum_{p_r^{\beta_r}\le Y}1=\prod_{i=1}^{r}\left(\left[\frac{\log Y}{\log p_i}\right]+1\right)\le(c\log 2Y)^r.$$

The following lemma is the well-known Brun—Titchmarsh inequality (see [5]).

LEMMA 9. *For any $k \leqq x^\theta$ $(\theta < 1)$ and any l coprime to k we have*

$$\pi(x, k, l) \leqq C_\theta \frac{\operatorname{li} x}{\varphi(k)},$$

where C_θ is a positive constant depending only on θ.

LEMMA 10. *Let a be a non-zero fixed integer,*

$$D(x, k) = \sum_{\substack{q \leqq x \\ q+a \equiv l (\mathrm{mod}\, k)}} d(q+a).$$

We have

$$D(x, k) \ll \begin{cases} \dfrac{d(k)}{\varphi(k)} x, & \text{when } k \leqq x^{1/2} \\ \dfrac{d(k)}{k} x x_1, & \text{when } k \geqq x^{1/2}. \end{cases}$$

PROOF. Using the fact, that $d(mn) \leqq d(m) d(n)$ for all m, n, we have

$$D(x, k) \leqq 2 \sum_{\substack{q+a \equiv 0 (\mathrm{mod}\, k) \\ q \leqq x}} d(k) \sum_{\substack{u \leqq 2\sqrt{x/k} \\ q+a = kuv}} 1.$$

Let $k \leqq x^{1/2}$. Changing the order of summation in the above sum and applying Lemma 9 we obtain

$$D(x, k) \leqq 2d(k) \sum_{u \leqq 2(x/k)^{1/2}} \pi(x, ku, -a) \ll d(k) \operatorname{li} x \sum_{u \leqq 2x^{1/2}} \frac{1}{\varphi(ku)}.$$

Since $\varphi(ku) \geqq \varphi(k) \varphi(u)$ and

$$\sum_{u \leqq y} \frac{1}{\varphi(u)} \ll \log 2y$$

we have

$$D(x, k) \ll \frac{d(k)}{\varphi(k)} x.$$

For $k \geqq x^{1/2}$ we use the trivial estimation

$$D(x, k) \leqq \sum_{kn \leqq x+a} d(kn) \leqq d(k) \sum_{n \leqq \frac{x+a}{k}} d(n) \ll \frac{d(k)}{k} x x_1.$$

2.4. Decomposition of the sum (2.2.1). Since $|\mu(n)| = \sum_{\delta^2 | n} \mu(\delta)$, we have

(2. 4. 1) $$T(x) = \sum_{\delta^2 \leqq x+a} \mu(\delta) T(x, \delta^2),$$

where

(2. 4. 2) $$T(x, \delta^2) = \sum_{\substack{q \leqq x \\ q+a \equiv 0 (\mathrm{mod}\, \delta^2)}} r(q+a).$$

Taking into account that $|r(n)| \leqq d(n)$ we have

$$T(x, \delta^2) \leqq D(x, \delta^2).$$

So, using Lemma 9 we obtain

$$\left|\sum\right| \stackrel{\text{def}}{=} \Big|\sum_{\delta \leqq x_1^2} \mu(\delta) T(x, \delta^2)\Big| \ll \sum_{\delta \leqq x_1^2} D(x, \delta^2) \ll$$

$$\ll x \sum_{\delta \leqq x_1^2} \frac{d(\delta^2)}{\varphi(\delta^2)} + x x_1 \sum_{\delta \geqq x^{1/4}} \frac{d(\delta^2)}{\delta^2} = x \sum\nolimits_1 + x x_1 \sum\nolimits_2 .$$

Further by elementary calculations we obtain

$$\sum\nolimits_1 \ll x_1^{-3/2}, \quad \sum\nolimits_2 \ll x^{-1/5}.$$

Hence

$$\left|\sum\right| \ll x_1^{-1/2} \operatorname{li} x.$$

Furthermore, in the case $(a, \delta) > 1$ the sum $T(x, \delta^2)$ contains at most one summand, so $T(x, \delta^2) \ll x^{\varepsilon}$.

Consequently we have

$$T(x) = \sum_{\substack{\delta \leqq x_1^2 \\ (\delta, a) = 1}} \mu(\delta) T(x, \delta^2) + O\left(\frac{\operatorname{li} x}{\sqrt{x_1}}\right). \tag{2.4.3}$$

Let now

$$z_1 = \sqrt{x} \cdot x_1^{-100}; \quad z_2 = \sqrt{x} \cdot x_1^{100}; \quad z_1' = \frac{x+a}{z_2}. \tag{2.4.4}$$

Using the fact that $r(n) = \sum_{d|n} \chi(d)$, we have

$$T(x, \delta^2) = \sum_{\substack{q \leqq x \\ q+a \equiv 0(\delta^2) \\ q+a = uv}} \chi(u) = \sum_{u \leqq z_1} + \sum_{z_1 < u < z_2} + \sum_{u \geqq z_2} = \sum\nolimits_A^{(\delta)} + \sum\nolimits_B^{(\delta)} + \sum\nolimits_C^{(\delta)}. \tag{2.4.5}$$

Let

$$\sum\nolimits_A = \sum_{\substack{\delta \leqq x_1^2 \\ (\delta, a) = 1}} \mu(\delta) \sum\nolimits_A^{(\delta)}; \quad \sum\nolimits_C = \sum_{\substack{\delta \leqq x_1^2 \\ (\delta, a) = 1}} \mu(\delta) \sum\nolimits_C^{(\delta)}, \tag{2.4.6}$$

$$\sum\nolimits_B = \sum_{\substack{\delta \leqq x_1^2 \\ (\delta, a) = 1}} \mu(\delta) \sum\nolimits_B^{(\delta)}. \tag{2.4.7}$$

By (2. 4. 3) we have

$$T(x) = \sum\nolimits_A + \sum\nolimits_B + \sum\nolimits_C + O\left(\frac{\operatorname{li} x}{\sqrt{x_1}}\right). \tag{2.4.8}$$

2. 5. Estimation of $\Sigma_A^{(\delta)}$ and $\Sigma_C^{(\delta)}$. In this section we assume, that $(a, \delta) = 1$, $\delta \leqq x_1^2$.

A) *Estimation of* $\Sigma_A^{(\delta)}$. By (2. 4. 6) we have

$$\sum\nolimits_A^{(\delta)} = \sum_{u \leqq z_1} \chi(u) \pi(x, [u, \delta^2], -a).$$

From Lemma 1 it follows, that

$$\sum_A^{(\delta)} = S_\delta(\chi)\,\mathrm{li}\,x + O(x\cdot x_1^{-4}), \tag{2.5.1}$$

where

$$S_\delta(\chi) = \sum_{\substack{u\le z_1\\(u,a)=1}} \chi(u)/\varphi([u,\delta^2]). \tag{2.5.2}$$

For a general u let $u=u_1u_2$, where u_1 contains exactly those prime factors of u which divide δ and $(u_2,\delta)=1$. Using this notation we obtain

$$S_\delta(\chi) = \sum_{\substack{u_1\le z_1\\(u_1,a)=1}} \frac{\chi(u_1)}{\varphi([\delta^2,u_1])} \sum_{\substack{u_2\le \frac{z_1}{u_1}\\(u_2,a\delta)=1}} \frac{\chi(u_2)}{\varphi(u_2)}.$$

Hence, by Lemma 3

$$S_\delta(\chi) = C(\chi)L(1,\chi)E(\delta a,\chi)\sum_{u_1\le z_1}\frac{\chi(u_1)}{\varphi([\delta^2,u_1])} + \tag{2.5.3}$$

$$+O\left(d(\delta)\frac{x_1}{z_1}\sum_{u_1\le z_1}\frac{u_1}{\varphi([\delta^2,u_1])}\right).$$

Since $\frac{u_1}{\varphi([\delta^2,u_1])}\le 1$, the remainder term is majorized by

$$d(\delta)\frac{x_1}{z_1}S(z_1,\delta)$$

and by Lemma 8

$$\ll \frac{x_1}{z_1}(c\log x)^{\omega(\delta)}d(\delta)\ll x^{-1/4},$$

say. Furthermore

$$\sum_{\substack{u_1\le z_1\\(u_1,a)=1}}\frac{\chi(u_1)}{\varphi([\delta^2,u_1])}\ll\frac{1}{z_1^{1/2}}\sum_{u_1}\frac{u_1^{1/2}}{\varphi([\delta^2,u_1])} = \tag{2.5.4}$$

$$=\frac{1}{z_1^{1/2}}\prod_{p|\delta}\left(\frac{p^{1/2}}{\varphi(p^2)}+\frac{p}{\varphi(p^2)}+\frac{p^{3/2}}{\varphi(p^3)}+\cdots\right)\ll\frac{\sigma_{-1/2}(\delta)}{z_1^{1/2}\varphi(\delta)}\ll x^{-1/5},$$

say. Furthermore we have

$$\sum_{u_1}\frac{\chi(u_1)}{\varphi([\delta^2,u_1])} = \frac{1}{\varphi(\delta^2)}\prod_{p|\delta}\left(1+\chi(p)+\chi^2(p)+\frac{\chi^3(p)}{p-\chi(p)}\right) = \frac{g(\delta,\chi)}{\varphi(\delta^2)} \tag{2.5.5}$$

(see (2.2.8)).

From (2.5.3), (2.5.4), (2.5.5) it follows that

$$S_\delta(\chi) = C(\chi)L(1,\chi)E(a,\chi)E(\delta,\chi)g(\delta,\chi)\frac{1}{\varphi(\delta^2)}+O(x^{-1/6}). \tag{2.5.6}$$

B) *Estimation of* $\Sigma_C^{(\delta)}$. In the sum $\Sigma_C^{(\delta)}$ (see (2. 4. 6)) the condition $u \geqq z_2$ implies $v < \dfrac{x+a}{z} = z_1'$. Therefore

(2. 5. 7) $$\sum\nolimits_C^{(\delta)} = \sum_{v<z_1'} \sum\nolimits_v,$$

where

(2. 5. 8) $$\sum\nolimits_v = \sum_{\substack{q+a\equiv 0(\delta^2)\\ q+a=uv\\ z_2v-a<q<x}} \chi(u).$$

The summand in Σ_v depends only on the value of u taken only mod D. Consequently

(2. 5. 9) $$\sum\nolimits_v = \sum_{l(\mathrm{mod}\, D)} \chi(l) \sum_{\substack{q+a\equiv 0(\delta^2)\\ q+a\equiv lv(Dv)\\ z_2v-a<q<x}} 1.$$

The condition $(\delta^2, Dv)|lv$ is necessary and sufficient to the compatibility of the congruence-system in the variable x

(2. 5. 10) $$x+a \equiv 0(\delta^2), \qquad x+a \equiv lv(Dv),$$

This condition for $(l, D)=1$ is equivalent to the condition $(\mathrm{A}) \equiv \{(\delta^2, Dv)|v\}$.

Assuming (A), (2. 5. 10) has a unique solution $l^*(\mathrm{mod}\,[\delta^2, Dv])$. In the arithmetical progression $l^*(\mathrm{mod}\,[\delta^2, Dv])$ there exist at least two primes if and only if $(lv-a, Dv\delta^2)=1$. If this condition is satisfied for some l coprime to D then $(v, a)=1$. Let (B) denote the condition $(lv-a, D\delta^2)=1$.

Hence we obtain

(2. 5. 11) $$\sum\nolimits_v \ll x^\varepsilon$$

if $(a, v)>1$, or (A) is not satisfied.

Supposing that $(a, v)=1$ and (A) holds we have

(2. 5. 12) $$\sum\nolimits_v = \sum_{\substack{(l,D)=1\\ (\mathrm{B})}} \chi(l)\{\pi(x, [\delta^2, Dv], l^*) - \pi(z_2v-a, [\delta^2, Dv], l^*)\} + O(x^\varepsilon) =$$

$$= \sum\nolimits_v^{(1)} - \sum\nolimits_v^{(2)} + O(x^\varepsilon)$$

where

(2. 5. 13) $$\sum\nolimits_v^{(1)} = \sum_{\substack{l(D)\\ (\mathrm{B})}} \chi(l)\pi(x, [\delta^2, Dv], l^*); \quad \sum\nolimits_v^{(2)} = \sum_{\substack{l(D)\\ (\mathrm{B})}} \pi(z_2v-a, [\delta^2, Dv], l^*).$$

Using the notation of Lemma 1 and Lemma 4 we obtain

(2. 5. 14) $$\sum\nolimits_v^{(1)} = \frac{\mathrm{li}\, x}{\varphi([\delta^2, Dv])} L(v) + O(\mathscr{E}(x, [\delta^2, Dv]))$$

and

(2. 5. 15) $$\sum\nolimits_v^{(2)} = \frac{\mathrm{li}\,(z_2v-a)}{\varphi([\delta^2, Dv])} L(v) + O(\mathscr{E}(x, [\delta^2, Dv])).$$

Let $D=p^{*\beta}$, $\beta>1$. By Lemma 4 we obtain $L(v)=0$, for all v, and so by Lemma 1 we have

$$\sum\nolimits_C^{(\delta)} = \sum_{v<z_1'} \{\sum\nolimits_v^{(1)} - \sum\nolimits_v^{(2)} + O(x^{\varepsilon})\} \ll \frac{x}{x_1^4}. \tag{2.5.16}$$

Now we suppose, that $D=p^*$. In this case $L(v) = -\bar{\chi}(v)\chi(a)$ (see Lemma 4). Using Lemma 1 we have

$$S_1 \overset{\text{def}}{=} \sum_{v<z_1'} \sum\nolimits_v^{(1)} = -\chi(a)\,\mathrm{li}\,x \sum_{\substack{(v,a)=1\\ v\leqq z_1'\\ (\mathrm{A})}} \frac{\bar{\chi}(v)}{\varphi([\delta^2, Dv])} + O\left(\frac{x}{x_1^4}\right) = \tag{2.5.17}$$

$$= -\chi(a)\,\mathrm{li}\,x\, U_\delta(\bar{\chi}) + O\left(\frac{x}{x_1^4}\right).$$

From the condition (A) it follows that if $p^*|\delta$, then $p^*|v$. Consequently $\bar{\chi}(v)=0$. So we have

$$U_\delta(\bar{\chi}) = O \quad \text{if} \quad p^*|\delta. \tag{2.5.18}$$

If $(\delta, D)=1$ and $(D, v)=1$, then $\varphi([\delta^2, Dv]) = \varphi(D)\varphi([\delta^2, v])$. Hence

$$U_\delta(\bar{\chi}) = \frac{1}{\varphi(D)} \sum_{\substack{v\leqq z_1'\\ (v,a)=1}} \frac{\bar{\chi}(v)}{\varphi([v, \delta^2])}. \tag{2.5.19}$$

By (2. 5. 2) we have

$$U_\delta(\bar{\chi}) = \frac{1}{\varphi(D)} S_\delta(\bar{\chi}) + O(x^{-1/3}). \tag{2.5.20}$$

So by (2. 5. 6)

$$U_\delta(\bar{\chi}) = \frac{1}{\varphi(D)} C(\bar{\chi}) L(1, \bar{\chi}) E(a\delta, \bar{\chi}) g(\delta, \bar{\chi}) \frac{1}{\varphi(\delta^2)} + O(x^{-1/6}) \tag{2.5.21}$$

follows, if $(p^*, \delta)=1$.

Let

$$S_2 \overset{\text{def}}{=} \sum_{v<z_1'} \sum\nolimits_v^{(2)}. \tag{2.5.22}$$

Using Lemma 1 and Lemma 4 we have

$$S_2 = -\chi(a) \sum_{\substack{(v,a)=1\\ v\leqq z_1'\\ (\mathrm{A})}} \frac{\mathrm{li}\, z_2 v \cdot \bar{\chi}(v)}{\varphi([\delta^2, Dv])}. \tag{2.5.23}$$

Let $W=\dfrac{z_1}{x_1^4}$, and let

$$S_2 = -\chi(a)\{\sum\nolimits_K + \sum\nolimits_L\}, \tag{2.5.24}$$

where $v\leqq W$ in $\sum_K$, and $W<v\leqq z_1'$ in $\sum_L$.

Since $\operatorname{li} z_2 v \ll \frac{z_2 v}{x_1}$, we have

(2. 5. 25) $$\sum_K \ll \frac{z_2}{x_1} \sum_{v<W} \frac{v}{\varphi(v)} \ll \frac{z_2 W x_2}{x_1} \ll \frac{x}{x_1^4}.$$

Further in the sum $\sum_L$ the relation $\operatorname{li} z_2 v = \frac{z_2 v}{x_1} + O\left(\frac{z_2 v}{x_1^2}\right)$ holds. Consequently

(2. 5. 26) $$\sum_L = \sum_M + O\left(\sum_N\right),$$

where

(2. 5. 27) $$\sum_M = \frac{1}{x_1} \sum_{\substack{W \leqq v \leqq z_1' \\ (v,a)=1 \\ (A)}} \frac{\bar\chi(v) z_2 v}{\varphi([\delta^2, Dv])},$$

(2. 5. 28) $$\sum_N = \frac{1}{x_1^2} \sum_{W \leqq v \leqq z_1'} \frac{z_2 v}{\varphi([\delta^2, Dv])}.$$

We have

(2. 5. 29) $$\sum_N \ll \frac{x}{\delta x_1^{3/2}}$$

by trivial estimation. In the case $(\delta, p^*)>1$ $\Sigma_M=0$ from (A).

Let now $(\delta, p^*)=1$. For a general v we write $v=v_1 v_2$, where all prime divisors of v_1 are divisors of δ, and $(v_2, \delta)=1$. We have

$$\sum_M = \frac{z_2}{x_1 \varphi(D)} \sum_{\substack{W \leqq v \leqq z_1 \\ (v,a)=1}} \frac{\bar\chi(v_1) v_1}{\varphi([\delta^2, v_1])} \sum_{\substack{\frac{W}{v_1} \leqq v_2 \leqq \frac{z_1'}{v_1} \\ (v_2, \delta a)=1}} \frac{\bar\chi(v_2) v_2}{\varphi(v_2)}.$$

Using now Lemma 5 for the inner sum, hence

$$\sum_M \ll z_2 \sum_{v_1 \leqq z_1} \frac{v_1 d(\delta)}{\varphi([\delta^2, v_1])} \ll \frac{z_2 d(\delta)}{\delta} \sum_{v_1 \leqq z_1} \frac{v_1}{\varphi([\delta_1, v_1])}$$

follows. Using Lemma 8 and $\frac{v_1}{\varphi([\delta, v_1])} \leqq x_2$ we have

(2. 5. 30) $$\sum_M \ll \frac{z_2 d(\delta)}{\delta} S(z_1, \delta) \ll x^{0,8},$$

say. Consequently, by (2. 5. 25), (2. 5. 29)

(2. 5. 31) $$S_2 \ll \frac{x}{d x_1^{3/2}}$$

holds.

Since

$$\sum\nolimits_C^{(\delta)} = S_1 - S_2 + \sum_{v<z_1'} O(x^\varepsilon),$$

finally we have

$$\text{(2. 5. 32)} \qquad \sum\nolimits_C^{(\delta)} = -\varrho(D,\delta)\,\mathrm{li}\,x\,\frac{\chi(a)}{\varphi(D)}\,\frac{C(\bar\chi)L(1,\bar\chi)E(a\delta,\bar\chi)g(\delta,\bar\chi)}{\varphi(\delta^2)} + O\left(\frac{x}{\delta x_1^{3/2}}\right),$$

where

$$\text{(2. 5. 33)} \qquad \varrho(D,\delta) = \begin{cases} 1 & \text{when } D = p^*,\ (\delta, D) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

2. 6. Estimation of Σ_A and Σ_C. A) *The sum* Σ_A. Using (2. 4. 6), (2. 5. 1), (2. 5. 2), (2. 5. 6) we have

$$\text{(2. 6. 1)} \qquad \sum\nolimits_A = \mathrm{li}\,x\cdot C(\chi)L(1,\chi)E(a,\chi) \sum_{\substack{\delta \le x_1^2 \\ (\delta,a)=1}} \frac{\mu(\delta)E(\delta,\chi)g(\delta,\chi)}{\varphi(\delta^2)} + O\left(\frac{x}{x_1^2}\right).$$

From (2. 2. 6), (2. 2. 7) it follows $|g(p^\alpha,\chi)| \le 4$, $|E(p^\alpha,\chi)| \le 3$. So

$$\text{(2. 6. 2)} \qquad F \overset{\text{def}}{=} \sum_{\substack{\delta \ge x_1^2 \\ (\delta,a)=1}} \frac{\mu(\delta)E(\delta,\chi)g(\delta,\chi)}{\varphi(\delta^2)} \ll \sum_{\delta \ge x_1^2} \frac{(12)^{\omega(\delta)}}{\varphi(\delta^2)} \ll \frac{1}{x_1^{3/2}},$$

say. Furthermore

$$\text{(2. 6. 3)} \qquad C \overset{\text{def}}{=} \sum_{(\delta,a)=1} \frac{\mu(\delta)E(\delta,\chi)g(\delta,\chi)}{\varphi(\delta^2)} = \prod_{p\nmid a}\left(1 - \frac{E(p,\chi)g(p,\chi)}{\varphi(p^2)}\right).$$

Since the sum in the right hand side of (2. 6. 1) is equal to $F-G$, we have

$$\text{(2. 6. 4)} \qquad \sum\nolimits_A = \mathrm{li}\,x\cdot C(\chi)L(1,\chi)E(a,\chi) \prod_{p\nmid a}\left(1 - \frac{E(p,\chi)g(p,\chi)}{\varphi(p^2)}\right) + O(x\cdot x_1^{-3/2}).$$

B) *The sum* $\sum_C$. Using (2. 4. 6), (2. 5. 32) we have

$$\text{(2. 6. 5)} \qquad \sum\nolimits_C = -\varepsilon(D)\frac{\chi(a)}{\varphi(D)} C(\bar\chi)L(1,\bar\chi)E(a,\bar\chi) \sum_{\substack{\delta \le x_1^2 \\ (\delta,a)=1 \\ (\delta,D)=1}} \frac{\mu(\delta)E(\delta,\bar\chi)g(\delta,\bar\chi)}{\varphi(\delta^2)} + $$

$$+ O\left(\frac{x}{x_1^{3/2}} \sum_{\delta \le x_1^2} \frac{1}{\delta}\right).$$

Hence, extending the sum in the right hand side to infinity with error $O(x_1^{-3/2})$ we have

$$\text{(2. 6. 6)} \qquad \sum\nolimits_C = -\varepsilon(D)\,\mathrm{li}\,x\,\frac{\chi(a)}{\varphi(D)} C(\bar\chi)L(1,\bar\chi)E(a,\bar\chi) \prod_{p\nmid aD}\left(1 - \frac{E(p,\bar\chi)g(p,\bar\chi)}{\varphi(p^2)}\right) +$$

$$+ O(\mathrm{li}\,x\cdot x_1^{-0,4}).$$

From (2. 6. 4) and (2. 6. 5) it follows

(2. 6. 7) $$\sum_A + \sum_C = A(D, \chi, a)\,\mathrm{li}\,x + O(\mathrm{li}\,x \cdot x_1^{-0{,}4}).$$

So, for the proof of (2. 2. 3) it is enough to show that

$$\sum_B \ll \mathrm{li}\,x \cdot x_1^{-\delta}, \quad \delta > 0.$$

2. 7. Estimation of Σ_B. We use the method of Hooley.

Let

$$D(n) = \sum_{\substack{l|n \\ z_1 < l < z_2}} 1, \quad F(n) = \sum_{\substack{l|n \\ z_1 < l < z_2}} \chi(l).$$

We have

$$\left|\sum_B\right| = \left|\sum_{\substack{\delta \leqq x_1^2 \\ (\delta, a)=1}} \mu(\delta) \sum_{\substack{q \leqq x \\ q+a \equiv 0(\delta^2)}} F(q+a)\right| \leqq \sum_{q \leqq x} |F(q+a)| \left|\sum_{\substack{\delta^2 | q+a \\ (\delta, a)=1 \\ \delta \leqq x_1^2}} \mu(\delta)\right|.$$

Since the inner sum is majorized by

$$\left|\sum_{\substack{\delta^2 | q+a \\ (\delta, a)=1}} \mu(\delta)\right| + \sum_{\substack{\delta^2 | q+a \\ \delta > x_1^2}} 1 \ll 1 + \sum_{\substack{\delta^2 | q+a \\ \delta > x_1^2}} 1,$$

we have

(2. 7. 1) $$\sum_B \leqq \sum_{q \leqq x} |F(q+a)| + \sum_{q \leqq x} |F(q+a)| \{ \sum_{\substack{\delta^2 | q+a \\ \delta > x_1^2}} 1\} = \sum_B' + \sum_B''.$$

Using the inequality $|F(q+a)| \leqq d(q+a)$, we have by Lemma 9

(2. 7. 2) $$\sum_B'' \ll \sum_{\delta > x_1^2} \sum_{q+a \equiv 0(\delta^2)} |F(q+a)| \ll \sum_{\delta > x_1^2} D(x, \delta^2) \ll \frac{x}{x_1^{3/2}}.$$

Now we investigate Σ_B'. By the Cauchy—Schwarz inequality,

(2. 7. 3) $$\sum_B' \ll \Big(\sum_{D(q+a) \neq 0} 1\Big)^{1/2} \Big(\sum_{q \leqq x} |F(q+a)|^2\Big)^{1/2} = \sum_D^{1/2} \cdot \sum_E^{1/2},$$

say. Using the method of Hooley without any significant change, we obtain

(2. 7. 4) $$\sum_D \ll \mathrm{li}\,x \cdot (\log x)^{-\gamma} (\log \log x)^c,$$

where $\gamma = -\frac{1}{2} e \log 2 (>0)$, and c is a positive constant.

Estimation of Σ_E. We have, in the notation of Lemma 6,

$$\sum_E = \sum_{q \leqq x} |F^2(q+a)| \leqq \sum_{v \leqq x} |F^2(v+a)| f(v) = \sum_{\substack{l_1' m_1 = l_2' m_2 = v+a \\ z_1 < l_1', l_2' < z_2}} \chi(l_1')\bar{\chi}(l_2') f(v).$$

For given l_1', l_2', the values of $v+a$ in the above sum are equivalent to zero $\mathrm{mod}\,[l_1', l_2']$. Furthermore, if $(l_1', l_2') = d$, then $l_1' = dl_1$, $l_2' = dl_2$ and $[l_1', l_2'] = dl_1 l_2$, where $(l_1, l_2) = 1$. We now define (L_i), (H), (K) by $(\mathrm{L}_i) \equiv \left\{\frac{z_1}{d} < l_i < \frac{z_2}{d}\right\}$;

(H) $\equiv \{(l_1, l_2) = 1\}$; (K) $\equiv \{(dl_1 l_2, a) = 1\}$. Hence

$$(2.7.5)\qquad \sum_E \leqq \sum_{\substack{l_1 l_2 dm = \nu + a \\ (L_1), (L_2), (H)}} |\chi(d)|\chi(l_1)\bar\chi(l_2) f(\nu) = \sum_{d \geqq x^{1/8}} + \sum_{d \leqq x^{1/8}} = \sum_1 + \sum_2,$$

say. In $\sum_1$, we have $l_1 l_2 d < x^{7/8} x_1^{100}$. Hence, using Lemma 6

$$(2.7.6)\qquad \sum_1 = \sum_{(L_1), (L_2), (H)} |\chi(d)|\chi(l_1)\bar\chi(l_2) \sum_{\substack{\nu \equiv -a \pmod{dl_1 l_2} \\ \nu \leqq x}} f(\nu) =$$

$$= B(x)x \sum_{\substack{(K), (L_1), (L_2), (H) \\ d \geqq x^{1/8}}} \frac{|\chi(d)|\chi(l_1)\bar\chi(l_2)}{\varphi(dl_1 l_2)} + O\left(\frac{x}{x_1^5} \sum \frac{1}{dl_1 l_2}\right) =$$

$$= B(x)x\{\sum_3 + \sum_4\} + O\left(\frac{x}{x_1^2}\right),$$

where $x^{1/8} \leqq d \leqq z_1$ in $\sum_3$, $z_1 \leqq d \leqq z_2$ in $\sum_4$.

Now

$$\sum_3 = \sum_{\substack{(L_1) \\ (dl_1, a) = 1}} |\chi(d)|\chi(l_1) \sum_{\substack{(L_2) \\ (l_2, l_1 a) = 1}} \frac{\bar\chi(l_2)}{\varphi(dl_1 l_2)}.$$

Substituting the inner sum by Lemma 7, and then deleting the conditions $(dl_1, a) = 1$ and $d \geqq x^{1/8}$ from the outer summation, we obtain

$$\sum_3 \ll x_2^2 \sum_{\substack{d \leqq z_1 \\ z_1 \leqq dl_1 \leqq z_2}} \frac{d}{z_1} \frac{d(l_1)}{l_1 d} + x_2^2 \sum_{d \leqq z_1} \frac{1}{z_1 l_1} \ll \frac{x_2^2}{z_1} \sum_{\substack{d \leqq z_1 \\ z_1 \leqq dl_1 \leqq z_2}} \frac{d(l_1)}{l_1}.$$

Since

$$\sum_{\substack{d \leqq z_1 \\ z_1 \leqq dl_1 \leqq z_2}} \frac{d(l_1)}{l_1} \leqq \sum_{l_1 \leqq x_1^{200}} \frac{d(l_1)}{l_1} z_1 + \sum_{l_1 \geqq x_1^{200}} \frac{d(l_1) z_2}{l_1^2} \ll z_1 x_2^4 + z_2 \sum_{l_1 \geqq x_1^{200}} \frac{d(l_1)}{l_1^2} \ll z_1 \cdot x_2^7,$$

we have

$$(2.7.7)\qquad \sum_3 \ll x_2^9.$$

Also

$$(2.7.8)\qquad \sum_{\substack{z_1 \leqq d \leqq z_2 \\ l_1, l_2 \leqq x_1^{200}}} \frac{1}{\varphi(dl_1 l_2)} \ll x_2 \sum_{\substack{z_1 \leqq d \leqq z_2 \\ l_1, l_2 \leqq x_1^{200}}} \frac{1}{dl_1 l_2} \ll x_2^4.$$

We deduce from (2.7.6), (2.7.7) and (2.7.8)

$$(2.7.9)\qquad \sum_1 \ll \operatorname{li} x \cdot x_2^{11}.$$

To estimate Σ_2 we use the fact that for given l_1, l_2

$$\sum_{\substack{rt = l_1 \\ st = l_2}} \mu(t) = \begin{cases} 1, & \text{if } (l_1, l_2) = 1, \\ 0, & \text{if } (l_1, l_2) > 1. \end{cases}$$

We define (R), (S), (D), (DT) by $(\mathrm{R}) \equiv \left\{\frac{z_1}{dt} < r < \frac{z_2}{dt}\right\}$, $(\mathrm{S}) \equiv \left\{\frac{z_1}{dt} < s < \frac{z_2}{dt}\right\}$, $(\mathrm{D}) \equiv \{d < x^{1/8}\}$, $(\mathrm{DT}) \equiv \{d < x^{1/8},\ t < x^{1/8}\}$. We thus have

$$\sum\nolimits_2 = \sum_{\substack{rst^2\,dm = v+a \\ (\mathrm{R}),(\mathrm{S}),(\mathrm{D})}} \mu(t)|\chi(td)|\chi(r)\bar\chi(s)f(v) = \sum_{t \leqq x^{1/8}} \sum_{t > x^{1/8}} = \sum\nolimits_5 + \sum\nolimits_6,$$

say. Now, in $\sum_5$ the conditions of summation imply

$$rt^2\,dm \leqq \frac{x+a}{s} \leqq \frac{(x+a)\,dt}{std} \leqq \frac{x+a}{z_1}\,dt < z_2 x^{1/4}.$$

So

$$\sum\nolimits_5 = \sum_{\substack{rt^2\,dm < x^{1/4} z_2 \\ (\mathrm{R}),(\mathrm{DT})}} \mu(t)|\chi(td)|\chi(r) \sum_{\substack{rst^2\,dm = v+a \\ (\mathrm{S})}} \bar\chi(s)f(v). \tag{2.7.10}$$

Let $\lambda = rt^2\,dm$, $y_1 = \min(z_1 rtm, x)$; $y_2 = \min(z_2 rtm, x)$. Then for the inner sum we have

$$\left(\sum\nolimits_\lambda \overset{\mathrm{def}}{=}\right) \sum_{\substack{\lambda s = v+a \\ (\mathrm{S})}} \bar\chi(s)f(v) = \sum_{\substack{y_1 \leqq v \leqq y_2 \\ \lambda s = v+a}} \bar\chi(s)f(v) = \sum_{l(D)} \bar\chi(l) \sum_{\substack{s \equiv l(D) \\ y_1 \leqq v \leqq y_2 \\ \lambda s = v+a}} f(v) = \tag{2.7.11}$$

$$= \sum_{l \,(\mathrm{mod}\, D)} \bar\chi(l) \sum_{\substack{y_1 \leqq v \leqq y_2 \\ v+a \equiv l\lambda \,(\mathrm{mod}\, D\lambda)}} f(v).$$

Using Lemma 6 we have

$$\sum\nolimits_\lambda = B(x)(y_2 - y_1) \sum_{(l\lambda - a,\, D\lambda) = 1} \bar\chi(l) + O\left(\frac{x}{\lambda x_1^5}\right). \tag{2.7.12}$$

In the case $D = p^{*\beta}$, $\beta > 1$ by Lemma 4 the sum in (2.7.12) is equal to zero for all λ, consequently $\sum_\lambda \ll \frac{x}{\lambda x_1^5}$. So

$$\left(\sum\nolimits_0 \overset{\mathrm{def}}{=}\right) \sum_{r,d,m,t^2 \leqq x} \frac{x}{rdmt^2 x_1^5} \ll \frac{x}{x_1^2}. \tag{2.7.13}$$

Suppose now $D = p^*$. Using Lemma 4 we have

$$\sum_{(l\lambda - a,\, D\lambda) = 1} \bar\chi(l) = \begin{cases} -\bar\chi(x)\chi(\lambda), & \text{if } (a, \lambda) = 1, \\ 0, & \text{if } (a, \lambda) > 1 \end{cases} \tag{2.7.14}$$

and so

$$\sum\nolimits_\lambda = -\bar\chi(a)B(x)(y_2 - y_1)\chi(\lambda) + O\left(\frac{x}{\lambda x_1^5}\right). \tag{2.7.15}$$

Substituting (2. 7. 15) in (2. 7. 10), and taking into account (2. 7. 13), we have

$$\sum_5 = -\bar{\chi}(a)B(x) \sum_{\substack{rt^2 dm < x^{1/4} z_2 \\ (\mathrm{R}), (\mathrm{DT}) \\ (\lambda, a)=1}} \frac{\mu(t)|\chi(td)|\chi^2(rt)\chi(d)\chi(m)(y_2-y_1)}{\varphi(D\lambda)} + \tag{2. 7. 16}$$

$$+O\left(\sum_0\right) = -\bar{\chi}(a)B(x)\sum_7 + O\left(\frac{x}{x_1^2}\right),$$

say. Taking into account that

$$y_2 - y_1 = \begin{cases} 0, & \text{if} \quad rtm \geqq z_2, \\ x - z_1 rtm, & \text{if} \quad z_1 \leqq rtm \leqq z_2, \\ (z_2 - z_1)rtm, & \text{if} \quad rtm \leqq z_1, \end{cases}$$

we have

$$\sum_7 = \frac{z_2 - z_1}{\varphi(D)} \sum_8 + \frac{x}{\varphi(D)} \sum_9 - \frac{z_1}{\varphi(D)} \sum_{10}, \tag{2. 7. 17}$$

where

$$\sum_8 = \sum_{\substack{rtm \leqq z_1 \\ (\mathrm{R}), (\mathrm{DT}) \\ (\lambda, a)=1}} \frac{\mu(t)\chi^2(rt)\chi(d)\chi(m)rtm}{\varphi(rt^2 dm)}, \tag{2. 7. 18}$$

$$\sum_9 = \sum_{\substack{z_1 < rtm < z_2 \\ (\mathrm{R}), (\mathrm{DT}) \\ (\lambda, a)=1}} \frac{\mu(t)\chi^2(rt)\chi(d)\chi(m)}{\varphi(rt^2 dm)}, \tag{2. 7. 19}$$

and

$$\sum_{10} = \sum_{\substack{z_1 \leqq rtm < z_2 \\ (\mathrm{R}), (\mathrm{DT}) \\ (\lambda, a)=1}} \frac{\mu(t)\chi^2(rt)\chi(d)\chi(m)rtm}{\varphi(rt^2 dm)}. \tag{2. 7. 20}$$

We write

$$\sum_8 = \sum_{\substack{rtm \leqq z_1 \\ (\mathrm{R}) \\ (rt^2 m, a)=1}} \mu(t)\chi(r^2 t^2 m) rtm \sum_{\substack{\frac{z_1}{dt} \leqq d \leqq \frac{z_2}{rt} \\ (d, a)=1}} \frac{\chi(d)}{\varphi(rt^2 md)},$$

whence by Lemma 7 we obtain

$$\sum_8 \ll x_2 \sum_{rtm \leqq z_1} \frac{rtm}{rt^2 m} \frac{\log \frac{z_1}{rt}}{\frac{z_1}{rt}} + x_2^2 \sum_{rtm \leqq z_1} \frac{rtm}{rt^2 m \cdot \frac{z_1}{rt}} \ll \tag{2. 7. 21}$$

$$\ll \frac{x_2^2}{z_1} \sum_{rtm \leqq z_1} r \log \frac{z_1}{rt} \ll \frac{x_2^2}{z_1} \sum_{rt \leqq z_1} \left(\log \frac{z_1}{rt}\right) \frac{z_1}{t} \ll x_2^2 \sum_{n \leqq z_1} \log \frac{z_1}{n} \sigma_{-1}(n) \ll$$

$$\ll x_2^2 x_1 \sum_{n \leqq z_1 x_1^{-1}} \sigma_{-1}(n) + x_2^3 \sum_{n \leqq z_1} \sigma_{-1}(n) \ll x_2^3 z_1.$$

Similarly, we have

(2. 7. 22) $$\sum\nolimits_{10} \ll x_2^3 z_2.$$

Furthermore,

$$\sum\nolimits_9 \leqq \sum_{\substack{rt\leqq z_1\\ z_1\leqq rtm\leqq z_2\\ (rtm,a)=1}} \left| \sum_{\substack{z_1\leqq rtd\leqq z_2\\ (d,a)=1}} \frac{\chi(d)}{\varphi(rt^2md)} \right| + O\left(\sum_{\substack{z_1\leqq rt\leqq z_2\\ z_1\leqq rtm\leqq z_2\\ (\mathrm{R})}} \frac{1}{\varphi(rt^2dm)} \right) = \sum\nolimits_{11} + O\left(\sum\nolimits_{12}\right).$$

Using Lemma 7 for the estimation of the inner sum in Σ_{11}, we have

$$\sum\nolimits_{11} \ll x_2^2 \sum_{\substack{rt\leqq z_1\\ z_1\leqq rtm\leqq z_2}} \frac{1}{rt^2m} \frac{\log 2\frac{z_1}{rt}}{\frac{z_1}{rt}} \ll \frac{x_2^2}{z_1} \sum_{\substack{rt\leqq z_1\\ m\leqq\frac{z_2}{rt}}} \frac{\log 2\frac{z_1}{rt}}{tm} \ll$$

$$\ll \frac{x_2^2}{z_1} \sum_{rt\leqq z_1} \frac{\left(\log 2\frac{z_2}{rt}\right)^2}{t} \ll \frac{x_2^2}{z_1} \sum_{n\leqq z_1} \left(\log 2\frac{z_2}{n}\right)^2 \sigma_{-1}(n) \ll$$

$$\ll \frac{x_2^2}{z_1} \sum_{n\leqq z_1\cdot x_1^{-2}} x_1^2 \sigma_{-1}(n) + \frac{x_2^4}{z_1} \sum_{n\leqq z_1} \sigma_{-1}(n) \ll x_2^3.$$

Also, we have

$$\sum\nolimits_{12} \ll x_2 \sum_{\substack{z_1\leqq rt\leqq z_2\\ z_1\leqq rtm\leqq z_2\\ (\mathrm{R})}} \frac{1}{rt^2dm} \ll x_2 \sum_{z_1\leqq rt\leqq z_2} \frac{1}{rt^2} \left\{ \sum_{m\leqq\frac{z_2}{rt}} \frac{1}{m} \right\} \left\{ \sum_{d\leqq\frac{z_2}{rt}} \frac{1}{d} \right\} \ll$$

$$\ll x_2^3 \sum_{z_1\leqq rt\leqq z_2} \frac{1}{rt^2} = x_2^3 \sum_{z_1\leqq n\leqq z_2} \frac{\sigma_{-1}(n)}{n} \ll x_2^4.$$

Thus

(2. 7. 23) $$\sum\nolimits_9 \ll x_2^4.$$

Substituting the inequalities (2. 7. 21)—(2. 7. 23) into (2. 7. 17) we obtain

$$\sum\nolimits_7 \ll x_2^4$$

whence by $B(x) \ll \frac{x_2^2}{x_1}$ in Lemma 6

(2. 7. 24) $$\sum\nolimits_5 \ll \mathrm{li}\, x \cdot x_2^6$$

follows.

Finally, we estimate the sum Σ_6.

$$\sum\nolimits_6 \ll \sum_{\substack{rst^2dm\leqq x+a\\ t\geqq x^{1/8}}} 1 \ll \sum_{t\geqq x^{1/8}} \sum_{\mu\leqq\frac{x+a}{t^2}} d_4(\mu) \ll x\log^3 x \sum_{t\geqq x^{1/8}} \frac{1}{t^2} \ll x^{7/8} x_1^3 \ll \frac{x}{x_1^2}.$$

Combining our results, we obtain

(2. 7. 25) $$\sum_2 \ll \operatorname{li} x \cdot x_2^6$$

and so from (2. 7. 9) and (2. 7. 5)

$$\sum_E \ll \operatorname{li} x \cdot x_2^{11}.$$

Hence, by (2. 7. 3), (2. 7. 4)

$$\sum_B' \ll \operatorname{li} x \cdot x_1^{-\gamma/2} x_2^{\frac{c+11}{2}}.$$

So from (2. 7. 1) and (2. 7. 2) we deduce

(2. 7. 26) $$\sum_B \ll \operatorname{li} x \cdot x_1^{-\gamma/2} \cdot x_2^{c_1},$$

where $\gamma = 1 - \frac{1}{2} e \log 2 \ (>0)$, $c_1 > 0$ constant.

2. 8. Completion of the proof of Theorem 1. The fulfilment of the relation (2. 2. 3) now immediately follows from (2. 4. 8), (2. 6. 7), (2. 7. 26).

Now we investigate (2. 2. 4). We have

(2. 8. 1) $$A(D, \chi, a) = B(\chi, a) - \varepsilon(D) \frac{\chi(a)}{\varphi(D)} \left(1 - \frac{1}{\varphi(p^{*2})}\right)^{-1} \overline{B(\chi, a)},$$

where

(2. 8. 2) $$B(\chi, a) = C(\chi) L(1, \chi) E(a, \chi) \prod_{p \nmid a} \left(1 - \frac{E(p, \chi) g(p, \chi)}{\varphi(p^2)}\right).$$

Since $\left|\varepsilon(D) \frac{\chi(a)}{\varphi(D)} \left(1 - \frac{1}{\varphi(p^{*2})}\right)^{-1}\right| \neq 1$, so $A(D, \chi, a)$ is zero if and only if $B(\chi, a) = 0$.

From (2. 2. 7) it follows $C(\chi) \neq 0$, since the product is absolutely convergent and $\left|\frac{\chi(p)}{p(p-1)}\right| \leqq \frac{1}{2} < 1$. It is a well-known fact, that $L(1, \chi) \neq 0$. Further, from (2. 2. 6) $E(a, \chi) \neq 0$.

Since

$$\left|\frac{E(p, \chi) g(p, \chi)}{\varphi(p^2)}\right| < 1, \quad \text{if} \quad p \geqq 3,$$

and

$$\frac{E(2, \chi) g(2, \chi)}{\varphi(4)} = 1$$

if and only if $\chi(2) = -1$, so $B(\chi, a) \neq 0$, except for the case $\chi(2) = -1$, $2 \nmid aD$.

So the proof of Theorem 1 is completed.

III. Applications

3. 1. Lemmata. LEMMA 11. *Let $t(n)$ be a multiplicative function satisfying $t(p^\alpha) \ll \alpha^m$ for some fixed constant $m>0$, uniformly for all prime powers. Furthermore, let $t(p^\alpha) \geqq t(p^{\alpha-1})$, $\alpha=1, 2, \ldots$ hold. Then we have the inequality*

$$\sum_{p\leqq x} t(p+a) \ll \operatorname{li} x \exp\left(\sum_{p\leqq x} \frac{t(p)-1}{p}\right) \quad (a\neq 0, \text{ constant}). \tag{3. 1. 1}$$

This assertion is a special case of a more general theorem due to BARBAN [6]. Using the same method we can prove the following

LEMMA 12. *With the conditions of the previous lemma we have*

$$\sum_{p<n} t(n-p) \ll \operatorname{li} n \exp\left(\sum_{p<n} \frac{t(p)-1}{p}\right). \tag{3. 1. 2}$$

Let now χ be a character mod D and $r(n) = \prod_{p^\alpha \| n} \left(1+\chi(p)+\cdots+\chi(p^\alpha)\right)$. Let $t(n)$ be a multiplicative function defined by the relations:

$$t(p^\alpha) = t(p) = \max\left(1, |\chi(p)+1|^2\right).$$

This function satisfies the conditions stated in Lemmas 11, 12. Furthermore $|r(n)|^2 \leqq t(n)$ for all square-free n. Consequently from Lemma 11 and 12 immediately follows

LEMMA 13.

$$\sum_{p<x} |r^2(p+a)|\,|\mu(p+a)| \ll \operatorname{li} x \cdot x_1^K, \tag{3. 1. 3}$$

$$\sum_{p<n} |r^2(n-p)|\,|\mu(n-p)| \ll \operatorname{li} n \cdot (\log n)^K, \tag{3. 1. 4}$$

where

$$K = \frac{1}{\varphi(D)} \sum_{(l,D)=1} \max\left(1, |1+\chi(l)|^2\right). \tag{3. 1. 5}$$

3. 2. *Let $\mathscr{A}(D, l)$ denote the set of those square-free numbers, which have no prime factors in the arithmetical progression $l \pmod D$. Let $g(x, D, l, a)$ and $h(n, D, l)$ denote the number of solutions of the equations*

$$p-Q = a, \quad p\leqq x, \tag{3. 2. 1}$$

$$p+Q = n, \tag{3. 2. 2}$$

where p runs over the primes and Q over the elements of $\mathscr{A}(D, l)$, respectively.

From Theorem 1 and 2 we deduce the following assertions.

THEOREM 3. *Let $(l, D)=1$, and χ be such a character* mod D, *that* $\chi(l) = -1$. *Then*

$$g(x, D, l, a) \gg x/(\log x)^{2+K}, \tag{3. 2. 3}$$

except for the case D, a are odd and $\chi(2) = -1$.

THEOREM 4. *Let* $(l, D)=1$, *and* χ *be a character* mod D *such that* $\chi(l) = -1$. *Then*

$$h(n, D, l) \gg x/(\log x)^{2+K} \cdot (\log\log x), \tag{3.2.4}$$

except for the case n, D *are odd and* $\chi(2) = -1$.

PROOF. It is easy to see, that for any character χ (mod D) satisfying the condition $\chi(l) = -1$ there is a prime power divisor D^* of D and a primitive character χ^*(mod D^*) such that $\chi^*(l) = -1$. Furthermore, we have $\mathscr{A}(D^*, l) \subset \mathscr{A}(D, l)$ and so

$$g(x, D, l, a) \geqq g(x, D^*, l, a), \quad h(n, D, l) \geqq h(n, D^*, l).$$

So it is enough to prove our theorems only for prime-power D, and primitive character. In this case we use Theorems 1 and 2 as follows.

Since $r(n)=0$ for all such square-free integers, which do not belong to the set $\mathscr{A}(D, l)$, we have

$$|T(x)| \leqq \sum_{p<x} |r(p+a)|\,|\mu(p+a)| \leqq \tag{3.2.5}$$

$$\leqq (g(x, D, l, a))^{1/2} \{\sum_{q \leqq x} r^2(q+a)\,|\mu(q+a)|\}^{1/2}$$

and similarly

$$|K(n)| \leqq h^{1/2}(n, D, l) \{\sum_{p<n} |r^2(n-p)|\,|\mu(n-p)|\}^{1/2}. \tag{3.2.6}$$

Using Lemma 13 and Theorems 1, 2 we deduce the inequalities (3. 2. 3), (3. 2. 4).

3. 3. *Let* D *be an odd prime number, and let* $u(x, D, a)$ *and* $v(n, D)$ *be the number of solutions of the equations* $p-Q = a$, $p+Q = n$, *respectively, where* p *runs over the primes, and* Q *over those square-free integers, all prime divisors of which are quadratical residues* mod D. Applying Theorems 1, 2 choosing $\chi(n) = \left(\frac{n}{D}\right) \left(\left(\frac{n}{D}\right)\right.$ is the Legendre-symbol$\left.\right)$ and Lemma 13, we deduce

THEOREM 5.

$$u(x, D, a) \gg x/(\log x)^{4,5} \tag{3.3.1}$$

$$v(n, D) \gg x/(\log x)^{4,5} \cdot (\log\log x), \tag{3.3.2}$$

except 2 *is a quadratical non-residue* mod D *and* aD *is odd*, nD *is odd, respectively*.

3. 4. Remarks. 1. We can improve the inequalities in (3. 2. 3), (3. 2. 4), (3. 3. 1), (3. 3. 2) applying the sieve method of A. Selberg. But we cannot deduce the exact order of the number of solutions of the above equations.

2. Since $\chi(1)=1$ for all character, the above method is not applicable to the lower-estimation of $g(x, D, 1, a)$. I cannot prove the still weaker assertion: $\lim_{x\to\infty} g(x, D, 1, a)=\infty$.

3. If D is an odd prime, then for all $l \neq 1$, $(l, D)=1$ there is a character χ mod D such that $\chi(l) = -1$ if and only if D is a Gauss-prime, i.e. $D = 2^{2^k}+1$.

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ON THE DUALITY OF RADICAL AND SEMI-SIMPLE OBJECTS IN CATEGORIES

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§ 1.

A general theory of radicals and semi-simple objects in categories were studied in the papers of LIVŠIC [6], ŠULGEIFER [9], [10], RJABUHIN [7] and DICKSON [4], respectively.* In this note we lay stress on the duality between the concept of radical-ideals and that of semi-simple normal factorobjects. For this aim radical classes and semi-simple classes are defined axiomatically. In the categories of rings and groups, respectively, a radical class R defines a semi-simple class R^* which consists of all objects having zero R-radical. Moreover, a semi-simple class determines a radical class which consists of all objects having zero semi-simple images. Under certain (rather natural) conditions, we shall prove that a semi-simple class determines a radical class, however, we could not prove that to a radical class there belongs a semi-simple class (defined in the previous manner). The proof of the analogous statement for rings (cf. ANDERSON—DIVINSKY—SULIŃSKI [2]) makes strongly use of the operations defined on the ring. One could conjecture that generally a radical class does not determine a semi-simple class, further, that radical classes and semi-simple classes are dual, however, not equivalent classes (for the considered category is not selfdual).

Supposing a one-to-one correspondence between radical and semi-simple classes, we prove an intersection representation of radical ideals which were defined as a union of certain ideals. At last, applying Theorem 1 and 1* of [11] we obtain structure theorems for objects belonging to a hereditary radical class and semi-simple class, respectively.

§ 2.

In this paper we adopt the notions and notations of the preceding paper [11], *and we assume that the considered categories satisfy all of the axioms* (C_1)—(C_{10}). In addition, we need also categories in which every epimorphism is a normal one. For such categories the so-called Isomorphism Theorems are valid. To formulate them we remark that in such a category for any map $\alpha: a \to b$ and for any ideal (m, μ) of b there exists a complete counterimage (d, δ) of (m, μ) by α; the complete

* *Added in proof (5 September 1968).* In August 1968 there appeared RJABUHIN's paper "Radicals in categories (Russian), *Mat. Issl. (Kishinev)*, **2** (1967), pp. 107—165" where, among others, similar investigations are made to those of § 3.

counterimage (d, δ) means such an ideal of a for which

$$\begin{array}{ccccc} k & \longrightarrow & d & \xrightarrow{\nu} & m \\ & \searrow^{\varkappa} & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\mu} \\ & & a & \xrightarrow{\alpha} & b \end{array}$$

is a commutative diagram, where ν is a (normal) epimorphism and $(k, \varkappa) = \text{Ker}\,\alpha$ (cf. Šulgeifer [9] or Suliński [8]). A sequence $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ is called exact, if the normal image of α is just the kernel of β. By an exact diagram we understand a diagram consisting of exact rows and columns.

First Isomorphism Theorem. *Let $(k, \varkappa)$ and (m, μ) be ideals of an object a and b, respectively, and let*

$$0 \longrightarrow k \xrightarrow{\varkappa} a \xrightarrow{\alpha} b \longrightarrow 0$$

be an exact sequence. Denote by (d, δ) the complete counterimage of (m, μ) by the epimorphism α. Then there are maps β and γ such that

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & k & \to & d & \to & m & \to & 0 \\ & & & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\mu} & & \\ 0 & \to & k & \to & a & \to & b & \to & 0 \\ & & & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \\ & & 0 & \to & c & \to & c & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

is an exact commutative diagram.

Second Isomorphism Theorem. *Let $(k, \varkappa)$, (d_1, δ_1) and (d_2, δ_2) be ideals of an object $a \in C$ such that*

$$(k, \varkappa) = (d_1, \delta_1) \cap (d_2, \delta_2),$$

$$(a, \varepsilon_a) = (d_1, \delta_1) \cup (d_2, \delta_2)$$

hold. If

$$0 \to k \to d_1 \to b_1 \to 0$$

$$0 \to d_2 \to a \to b_2 \to 0$$

are exact sequences, then the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & k & \to & d_1 & \to & b_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & d_2 & \to & a & \to & b_2 & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

is exact and commutative (i.e. b_1 and b_2 are equivalent objects).

For these theorems we refer to [8], [12] or [13].

§ 3.

Let us consider a class R of objects of a category C satisfying

(a) *If* $a \in R$ *and* $\alpha: a \to b$ *is a normal epimorphism, then* $b \in R$;

(b) *For each object* $a \in C$, *the union of all ideals* $(k, \varkappa)$ *with* $k \in R$, *belongs to* R; *this union will be called the* R*-radical of* a *and will be denoted by* R-rad a;

(c) *If* $\alpha: a \to b$ *is a normal epimorphism with* Ker $\alpha = R$-rad a, *then* R-rad $b = (0, \omega)$ *holds.*

Such a class R will be called a *radical-class, the objects belonging to R are called R-radical objects.*

An *R-ideal* of an object a shall mean an ideal $(k, \varkappa)$ with $k \in R$. According to (b) R-rad a is the union of all R-ideals of a. Since $\omega: a \to 0$ is a normal epimorphism, so (a) implies $0 \in R$.

The dual class of R leads to the notion of semi-simple class. Let S be a class of objects of satisfying

(a*) *If* $a \in S$ *and* $\alpha: b \to a$ *is a normal monomorphism, then* $b \in S$;

(b*) *For each object* $a \in C$ *the union of all normal factorobjects* (λ, l) *with* $l \in S$, *belongs to* S; *this union will be called the* S*-semi-simple image of* a, *and will be denoted by* S-ses a.

(c*) *If* $\alpha: b \to a$ *is a normal monomorphism with Coker* $\alpha = S$-ses a, *then* S-ses $b = (\omega, 0)$ *holds.*

We call such a class S a *semi-simple class*, and the objects belonging to S are the *S-semi-simple objects*. By an *S-normal factorobject* we understand a normal factorobject (λ, l) with $l \in S$.

Let R be a radical class, and consider the class R^* consisting of all objects $a \in C$ whose R-radical is a zero object. Similarly, for a semi-simple class S, let S^* denote the class of all objects $a \in C$, whose S-semi-simple image is a zero object. Obviously both of $R \cap R^*$ and $S \cap S^*$ consist only from the zero objects.

THEOREM 1. *Assume that in the category C the product of two normal epimorphism is a normal one.* If S is a semi-simple class of objects of C, then the class* $S^* = \{a \in C \mid S\text{-ses } a = (\omega, 0)\}$ *forms a radical class.*

PROOF. Let a be an arbitrary element of S^*, and $\alpha: a \to b$ a (normal) epimorphism. Suppose $b \notin S^*$, i.e. S-ses $b = (\lambda, l) \neq (\omega, 0)$. Now $(\alpha\lambda, l)$ is an S-normal factorobject of a, and therefore we obtain the contradiction S-ses $a \neq (\omega, 0)$. Hence the class S^* satisfies condition (a).

Let a be an arbitrary element of C and consider all ideals $(k_i, \varkappa_i)$, $i \in I$ of a with $k_i \in S^*$. Denote the union $\bigcup_{i \in I} (k_i, \varkappa_i)$ by $(k, \varkappa)$. We shall show $k \in S^*$. Assume $k \notin S^*$. This implies S-ses $k = (\lambda, l) \neq (\omega, 0)$, and so Ker $\lambda = (d, \delta)$ differs from (k, ε_k). Thus for $\delta_0 = \delta\varkappa$ we have $(d, \delta_0) < \bigcup_{i \in I} (k_i, \varkappa_i) = (k, \varkappa)$, therefore there exists an

* This condition is satisfied, for instance, if every map has a normal image.

index $j_0 \in I$ with $(k_{j_0}, \varkappa_{j_0}) \not\leqq (d, \delta)$. Making use of the Second Isomorphism Theorem for

$$(r, \varrho) = (k_j, \varkappa_j) \cap (d, \delta_0),$$
$$(s, \sigma) = (k_j, \varkappa_j) \cup (d, \delta_0)$$

we obtain an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & r & \to & k_j & \to & b & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & d & \to & s & \to & b & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

According to (a) (proved already for S^*), from $k_j \in S^*$ it follows $b \in S^*$, so the First Isomorphism Theorem yields the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & d & \to & s & \to & b & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & d & \xrightarrow{\delta} & k & \xrightarrow{\lambda} & l & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & c & \to & c & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $l \in S$ holds, so by condition (a^*) $b \in S$ follows. Thus $b \in S \cap S^*$ is valid and so $(d, \delta_0) = (s, \sigma)$, further $(k_j, \varkappa_j) \leqq (d, \delta_0)$ follows which is a contradiction. Hence the class S^* fulfills condition (b).

At last we are going to prove the validity of condition (c) for the class S^*. Again, let a denote an arbitrary element of C and consider the union $(k, \varkappa) = \bigcup_{i \in I} (k_i, \varkappa_i)$ of all ideals of a with $k_i \in S^*$. We have to prove that for Coker $\varkappa = (\lambda, l)$ the object l has no non-zero ideal (d, δ) with $d \in S^*$. In the contrary, assume that there exists an ideal $(d, \delta) \neq (0, \omega)$ of l with $d \in S^*$. Let (c, γ) denote the complete counterimage of (d, δ) by $\lambda : a \to l$. Obviously $(k, \varkappa) < (c, \gamma)$ holds, and so we get $c \notin S^*$, i.e. S-ses $c = (\sigma, s) \neq (\omega, 0)$. Consider Ker $\sigma = (r, \varrho)$ and the ideal $(k, \varkappa_1)$ of c $(\varkappa_1 \gamma = \varkappa)$, moreover, the ideals

$$(*) \qquad \begin{aligned} (k, \varkappa_1) \cap (r, \varrho) &= (q, \vartheta) \\ (k, \varkappa_1) \cup (r, \varrho) &= (t, \tau). \end{aligned}$$

Let (m, μ) denote the image of (t, τ) by σ. Now we have the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & r & \to & t & \to & m & \to & 0 \\
 & & & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \mu} & & \\
0 & \to & r & \xrightarrow{\varrho} & c & \xrightarrow{\sigma} & s & \to & 0
\end{array}$$

and by (C_9) of [11] (m, μ) is an ideal of s. Hence from (a*) and $s \in S$ it follows $m \in S$. On the other hand the Second Isomorphism Theorem yields that

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & q & \to & k & \to & m & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & r & \to & t & \to & m & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

is an exact commutative diagram. Thus making use of (a) for the (normal) epimorphism ν, the relation $k \in S^*$ implies $m \in S^*$. Hence we obtain $m \in S \cap S^* = 0$ and $(q, \vartheta) = (k, \varkappa_1)$. Hence $(\ast)$ yields $(k, \varkappa_1) \leqq (r, \varrho)$, so by the First Isomorphism Theorem we get the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & k & \to & r & \to & m & \to & 0 \\
 & & & & \downarrow{\scriptstyle\varrho} & & \downarrow & & \\
0 & \to & k & \overset{\varkappa_1}{\to} & c & \to & d & \to & 0 \\
 & & & & \downarrow{\scriptstyle\sigma} & & \downarrow & & \\
 & & 0 & \to & s & \to & s & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $d \in S^*$, so condition (a) implies also $s \in S^*$. Thus we have established $s \in S \cap S^* = 0$, in contradiction to the definition of s. Thus the theorem is proved.

REMARK. If C_R denotes the category of (associative) rings, then for any radical class R *the class* $R^* = \{a \in C_R | R\text{-rad } a = (0, \omega)\}$ *is a semi-simple class*. In ANDERSON—DIVINSKY—SULIŃSKI [2] it is proved that for any radical class R, the class R^* has property (a*) in the category of associative rings and alternative rings, as well.

In the proof of this statement in [2], the operations defined on the ring, play an important rôle, so it seems to be rather difficult to obtain a dual statement* of that of Theorem 1 (of course, with the assumption of Theorem 1, and without the assumption that the product of two normal monomorphisms is a normal one). In the first part of AMITSUR [1] it is shown that any general radical R-rad A of a ring A coincides with the intersection of all ideals I_i for which R-rad $(A/I_i) = 0$. This means that (b*) is satisfied. (c*) follows almost trivially from the definition of R and R^*.

Let us mention that the same holds also for the category C_G of groups (cf. KUROŠ [5]).

* Recently E. P. ARMENDARIZ and W. G. LEAVITT have shown that in the category of all rings not every class R^* satisfies property (a*) (*Proc. Amer. Math. Soc.*, **18** (1967), pp. 1114—1117).

§ 4.

In what follows we omit the assumption that the product of two normal epimorphism is a normal one, but we suppose that

(C_{11}) *R^* is a semi-simple class and S^* is a radical class for any radical class R and semi-simple class S.*

THEOREM 2. *If R is an arbitrary radical class then $R^{**}=R$.*

PROOF. First, suppose $a\in R$. According to (b*), R^*-ses $a=(\lambda, l)$ is a normal factorobject with $l\in R^*$, i.e. R-rad $l=(0,\omega)$. On the other hand for the normal epimorphism λ condition (a) implies $l\in R$. Hence we have $(0,\omega)=R$-rad $l=(l_1\ \varepsilon_l)$. Thus R^*-ses $a=(\omega, 0)$, i.e. $a\in R^{**}$.

Conversely, let $a\notin R$, then R-rad $a=(k,\varkappa)$ is a proper ideal of a. Put $(\lambda, l)=$ $=\operatorname{Coker}\varkappa$. $(k,\varkappa)<(a,\varepsilon_a)$ implies $(\lambda, l)>(\omega, 0)$. By (c) we have R-rad $l=(\omega, 0)$ i.e. $l\in R^*$. Thus (b*) implies R^*-ses $a\geqq(\lambda, l)>(\omega, 0)$. Hence $a\notin R^{**}$.

THEOREM 2*. *If S is an arbitrary semi-simple class, then $S=S^{**}$.*

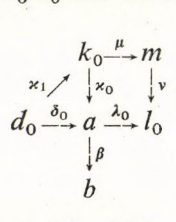
By definition, the R-radical $(k_0, \varkappa_0)$ of an object $a\in C$ is the union $\bigcup_{k\in R}(k,\varkappa)$ of all R-ideals of a. The following theorem gives an intersection representation of the R-radical. To formulate this, we shall call an ideal (d,δ) of an object $a\in C$ an *R^*-ideal,* if $\operatorname{Coker}\delta=(\lambda, l)$ is an R^*-normal factorobject (i.e. $l\in R^*$). Moreover, denote the R-radical and R^*-semi-simple image of a by $(k_0,\varkappa_0)$ and (λ_0, l_0), respectively. By Proposition 2 of [11] we obtain that $(d_0,\delta_0)=\operatorname{Ker}\lambda_0$ is the intersection of all R^*-ideals of a.

THEOREM 3. *The intersection of all R^*-ideals of $a\in C$ is equivalent to R-rad a, i.e. $(d_0,\delta_0)=(k_0,\varkappa_0)$.*

PROOF. Consider $\operatorname{Coker}\varkappa_0=(\beta, b)$. By condition (c) R-rad $b=(0,\omega)$, and so $b\in R^*$ holds. Therefore $(k_0,\varkappa_0)$ is an R^*-ideal and this implies

$$(d_0,\delta_0)\leqq(k_0,\varkappa_0).$$

According to (C_8) the map $\varkappa_0\lambda_0$ has an image, so we get the commutative diagram



where (m,ν) is the image of $\varkappa_0\lambda_0$ and by (C_9) ν is a normal monomorphism, and $(\beta, b)=\operatorname{Coker}\varkappa_0$. Since ν is a normal monomorphism, so (a*) implies $m\in R^*$. On the other hand, (C_9) and $k_0\in R$ and (a) imply $(\mu, m)=(\omega, 0)$. Since $(d_0,\delta_0)=$ $=\operatorname{Ker}\lambda_0$ and $\varkappa_0\lambda_0=\mu\nu=\omega$, so there exists a map $\varkappa_2: k_0\to d_0$ such that $\varkappa_2\delta_0=\varkappa_0$, and $\varkappa_2$ has to be a monomorphism. Therefore $(k_0,\varkappa_0)\leqq(d_0,\delta_0)$ is valid. Thus the theorem is proved.

Again, let $a \in C$ be an object with S-ses $a=(\lambda_0, l_0)$ and S^*-rad $a=(k_0, \varkappa_0)$. An S^*-normal factorobject (β', b') of $a \in C$ will mean a normal factorobject, whose kernel Ker $\beta'=(k, \varkappa)$ is an S^*-ideal i.e. $k \in S^*$. Denote by (β_0, b_0) the intersection of all S^*-normal factorobjects of a. By Proposition 2 of [11] $(\beta_0, b_0)=$Coker $\varkappa_0$ holds. Now the dual statement of that of Theorem 3 establishes the following

THEOREM 3*. *For the intersection* (β_0, b_0) *of all* S^*-*normal factorobjects* $(\beta_0\, b_0)= =(\lambda_0, l_0)$ *is valid.*

Let us mention that in view of Proposition 2 of [11], Theorems 3 and 3* are equivalent statements.

We say that the semi-simple class S is *hereditary* if it satisfies

(d*) *For any object* $a \in S$ *and normal epimorphism* $\alpha: a \to b$ *it follows* $b \in S$.

Hereditary semi-simplicity is sometimes called strongly semi-simplicity (cf. ANDRUNAKIEVIČ [3]). For such a class S Theorem 1 of [11] imples immediately

THEOREM 4. *Let* S *be a hereditary semi-simple class. Any object* $a \in S$ *whose ideal-lattice* L_a *is compactly generated, can be subdirectly embedded in a direct product of* S-*semi-simple objects, moreover, any direct factor is subdirectly irreducible if and only if condition* (I) *of* [11] *is fulfilled.*

Dualizing, a radical class R is said to be *hereditary*, if

(d) *For any object* $a \in R$ *and normal monomorphism* $\alpha: b \to a$ *it follows* $b \in R$.

For hereditary radicals we obtain

THEOREM 4*. *Let* R *be a hereditary radical class. Any* R-*radical object* a *whose ideal-lattice* L_a *is co-compactly generated, is a transfree image of a free product of* R-*radical objects* a_i, *moreover, any free factor* a_i *is transfreely irreducible, if and only if condition* (I*) *of* [11] *is fulfilled.*

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ÜBER DIE INDUZIERBARKEIT DES AFFIN-ZUSAMMENHÄNGENDEN RAUMES DER BIVEKTOREN

Von

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§ 1. Das Objekt der linearen Übertragung für Bivektoren

Ist ein affin-zusammenhängender Raum L_n mit einem Objekt $\Gamma_{\lambda\mu}^{\;\;\nu}$ der (nicht notwendig symmetrischen) linearen Übertragung gegeben, so kann man durch gewisse Anzahl der Postulate (Forderungen) in eindeutiger Weise die Objekte der linearen Übertragung für Tensoren aus dem Objekt $\Gamma_{\lambda\mu}^{\;\;\nu}$ ableiten, anders gesagt, man kann auch die kovariante Ableitung der Tensoren höherer Valenz definieren.

Insbesondere für Tensoren zweiter Ordnung $a^{\lambda\mu}$ folgt die Formel für kovariante Ableitung [1]:

(1. 1)
$$\nabla_\sigma a^{\lambda\mu} = \partial_\sigma a^{\lambda\mu} + \gamma_{\sigma\alpha\beta}^{\;\;\;\lambda\mu} a^{\alpha\beta},$$

wo

(1. 2)
$$\gamma_{\sigma\alpha\beta}^{\;\;\;\lambda\mu} \overset{\text{df}}{=} A_\alpha^\lambda \Gamma_{\sigma\beta}^{\;\;\mu} + A_\beta^\mu \Gamma_{\sigma\alpha}^{\;\;\lambda}.$$

Es ist bekannt [1], daß man von den a priori gegebenen $\gamma_{\sigma\alpha\beta}^{\;\;\;\lambda\mu}$ die Eigenschaft verlangen kann, daß die rechte Seite von (1. 1) für jeden Tensor $a^{\lambda\mu}$ einen Tensor dritter Ordnung darstellt, woraus sich die folgende Transformationsregel für γ [1], [2], [3] ergibt:

(1. 3)
$$\gamma_{\sigma'\alpha'\beta'}^{\;\;\;\;\;\;\lambda'\mu'} = A_{\sigma'\alpha'\beta'\lambda\;\mu}^{\sigma\;\alpha\;\beta\;\lambda'\mu'} \gamma_{\sigma\alpha\beta}^{\;\;\;\lambda\mu} + A_{\alpha'\mu}^{\lambda'\mu'} \partial_{\sigma'} A_{\beta'}^{\mu} + A_{\beta'\lambda}^{\mu'\lambda'} \partial_{\sigma'} A_{\alpha'}^{\lambda}.$$

Man kann auch die Frage stellen [3], ob für ein gegebenes Objekt γ ein Objekt Γ von der Eigenschaft (1. 2) existiert. Wenn dies der Fall ist, dann sagen wir, daß das Objekt γ durch das Objekt Γ induziert wird. L. TAMÁSSY hat [3] die notwendigen und hinreichenden Bedigungen für die Induzierbarkeit des Objektes γ durch das Objekt Γ gefunden. Diese Bedingungen gelten jedoch nicht für den Fall, wenn das Objekt γ ein Objekt der parallelen Übertragung der Bivektoren ist. Denn nehmen wir einen Bivektor $w^{\lambda\mu} = w^{[\lambda\mu]}$ als Spezialfall eines Tensores zweiter Ordnung, so ergibt sich für Bivektoren $w^{\lambda\mu}$ die folgende Formel für kovariante Ableitung [4]:

(1. 4)
$$\nabla_\sigma w^{\lambda\mu} = \partial_\sigma w^{\lambda\mu} + \Gamma_{\sigma\alpha\beta}^{\;\;\;\lambda\mu} w^{\alpha\beta},$$

wo

(1. 5)
$$\Gamma_{\sigma\alpha\beta}^{\;\;\;\lambda\mu} \overset{\text{df}}{=} 2\Gamma_{\sigma[\alpha}^{\;\;\;[\lambda} A_{\beta]}^{\mu]}.$$

Man kann von den $\Gamma_{\sigma\alpha\beta}^{\;\;\;\lambda\mu}$, die den folgenden zusätzlichen Bedingungen genügen

(1. 6)
$$\Gamma_{\sigma\alpha\beta}^{\;\;\;\mu\lambda} = -\Gamma_{\sigma\alpha\beta}^{\;\;\;\lambda\mu}, \quad \Gamma_{\sigma\beta\alpha}^{\;\;\;\lambda\mu} = -\Gamma_{\sigma\alpha\beta}^{\;\;\;\lambda\mu},$$

die Eigenschaft verlangen, daß die rechte Seite von (1.4) für jeden Bivektor $w^{\lambda\mu}$ einen Tensor dritter Ordnung darstellt, wobei $\nabla_\sigma w^{\lambda\mu} d\xi^\sigma$ zugleich ein Feld von Bivektoren ist. Dann ergibt sich aus (1. 4) für $\Gamma_{\sigma\alpha\beta}^{\;\;\;\lambda\mu}$ die folgende Transformationsregel [4]:

(1. 7)
$$\Gamma_{\sigma'\alpha'\beta'}^{\;\;\;\;\;\;\lambda'\mu'} = A_{\sigma'}^{\sigma} A_{[\alpha'\beta']}^{\alpha\;\beta} A_{\lambda\;\mu}^{[\lambda'\mu']} \Gamma_{\sigma\alpha\beta}^{\;\;\;\lambda\mu} + A_{\varkappa\pi}^{[\lambda'\mu']} \partial_{\sigma'} A_{[\alpha'\beta']}^{\varkappa\pi}.$$

Das Ergebnis von L. TAMÁSSY [3] über die notwendigen und hinreichenden Bedingungen der Induzierbarkeit der Konnexion $\Gamma_{\sigma\alpha\beta}^{\lambda\mu}$ kann in unserem Fall nicht verwendet werden, da unsere Konnexion $\Gamma_{\sigma\alpha\beta}^{\lambda\mu}$ vorasussetzungsgemäß noch die Schiefsymmetrieeigenschaft (1. 6) erfüllt was bei Tamássy nicht zutrifft. Folglich haben wir die Beziehungen

$$(1.8) \qquad \gamma_{1\,1\,t}^{\,1\,1} = \ldots = \gamma_{nnt}^{nn} = 0$$

welche aus (9a) in [3] nicht abzuleiten sind.

In der folgenden Mitteilung möchten wir die Ergebnisse von L. TAMÁSSY [3] ergänzen und zwar durch Angabe der notwendigen und hinreichenden Bedingungen für die Induzierbarkeit des mit der Eigenschaft (1. 6) versehenen Objektes $\Gamma_{\sigma\alpha\beta}^{\lambda\mu}$ durch das Objekt $\Gamma_{\sigma\lambda}^{\mu}$, d.h. durch Angabe der notwendigen und hinreichenden Bedigungen für die Lösbarkeit des Gleichungssystems (1. 5), in welchem die gegebenen Komponenten $\Gamma_{\sigma\alpha\beta}^{\lambda\mu}$ die Eigenschaft (1. 6) haben und $\Gamma_{\sigma\lambda}^{\mu}$ die Unbekannten sind. Für die zwei- und drei-dimensionale Räume wurde dieses Problem in der Arbeit [4] gelöst, für $n=4$ in der Arbeit [5] und schließlich allgemein für $n \geqq 5$ in der Arbeit [6]. Hier mögen alle diese Ergebnisse zusammengestellt werden ohne Angabe von Beweisen, die sich in [4], [5], [6] befinden.

§ 2. Induzierbarkeit des Objektes der linearen Übertragung der Bivektoren

SATZ 1 ([4]). *Das Objekt der linearen Übertragung der Bivektoren* $\Gamma_{\sigma\alpha\beta}^{\lambda\mu}$ *ist:*

I) *für* $n=2$ *immer induzierbar und zwar durch eine Menge der Objekte* $\Gamma_{\sigma\lambda}{}^{\mu}$, *die ein gemeinsames Objekt* Γ_σ *haben, wo*

$$(2.1) \qquad \Gamma_\sigma \overset{\text{df}}{=} \Gamma_{\sigma\alpha}^{\alpha}$$

und die z. B. durch folgende Formeln definiert sind:

$$(2.2) \qquad \Gamma_{\sigma 1}^{1},\, \Gamma_{\sigma 1}^{2},\, \Gamma_{\sigma 2}^{1} \text{ beliebig}, \quad \Gamma_{\sigma 2}^{2} = 2\Gamma_{\sigma 12}^{12} - \Gamma_{\sigma 1}^{1};$$

II) *für* $n=3$ *immer induzierbar und zwar durch das Objekt* $\Gamma_{\sigma\lambda}^{\mu}$, *das durch folgende Formel definiert ist:*

$$(2.3) \qquad \Gamma_{\sigma\lambda}^{\mu} = \tfrac{1}{2}\overset{2}{\Gamma}_\sigma A_\lambda^\mu + (-1)^{1+\lambda+\mu}\cdot 2\Gamma_{\sigma\alpha\beta}^{\varrho\tau},$$

wo die Indizes λ, ϱ, τ *und* μ, α, β *in folgender Beziehung zueinander stehen:*

$$(2.4) \qquad \begin{cases} \lambda = 1 & \textit{wenn} \quad \varrho = 2, \quad \tau = 3 \\ \lambda = 2 & \textit{wenn} \quad \varrho = 1, \quad \tau = 3 \\ \lambda = 3 & \textit{wenn} \quad \varrho = 1, \quad \tau = 2, \end{cases} \qquad \begin{cases} \mu = 1 & \textit{wenn} \quad \alpha = 2, \quad \beta = 3 \\ \mu = 2 & \textit{wenn} \quad \alpha = 1, \quad \beta = 3 \\ \mu = 3 & \textit{wenn} \quad \alpha = 1, \quad \beta = 2 \end{cases}$$

und $\overset{2}{\Gamma}_\sigma$ *folgendermaßen definiert ist:*

$$(2.5) \qquad \overset{2}{\Gamma}_\sigma \overset{\text{df}}{=} \Gamma_{\sigma\alpha\beta}^{\alpha\beta}.$$

SATZ 2. *Für die Induzierbarkeit des Objektes der parallelen Übertragung der Bivektoren im* 4-*dimensionalen Raum sind folgende Beziehungen notwendig und hinreichend* [5]:

$$(2.6) \qquad \Gamma_{\sigma\alpha\beta}^{\lambda\mu} = 0 \quad (\alpha \neq \lambda, \mu;\ \beta \neq \lambda, \mu;\ \sigma, \alpha, \beta, \lambda, \mu = 1, 2, 3, 4),$$

$$\Gamma_{\sigma\alpha\varrho}^{\alpha\tau} = \Gamma_{\sigma\beta\varrho}^{\beta\tau} \quad (\varrho \neq \tau, \alpha, \beta;\ \tau \neq \alpha, \beta;\ \sigma, \alpha, \beta, \varrho, \tau = 1, 2, 3, 4;\ n.\,s.\ \alpha, \beta^{*}), \tag{2.7}$$

$$\Gamma_{\sigma\alpha\beta}^{\alpha\beta} = \Gamma_{\sigma\varrho\tau}^{\varrho\tau} \quad (\alpha \neq \beta, \varrho, \tau;\ \beta \neq \varrho, \tau;\ \alpha, \beta, \varrho, \tau, \sigma = 1, 2, 3, 4;\ n.\,s.\ \alpha, \beta, \varrho, \tau). \tag{2.8}$$

SATZ 3 ([6]). *Für die Induzierbarkeit des Objektes der parallelen Übertragung der Bivektoren im n-dimensionalen Raum (wenn $n \geqq 5$) sind folgende Beziehungen notwendig und hinreichend:*

$$\Gamma_{\sigma\alpha\beta}^{\lambda\mu} = 0 \quad (\alpha \neq \lambda, \mu;\ \beta \neq \lambda, \mu;\ \sigma, \alpha, \beta, \lambda, \mu = 1, 2, \ldots, n), \tag{2.9}$$

$$\Gamma_{\sigma 1\varrho}^{1\tau} = \Gamma_{\sigma 2\varrho}^{2\tau} = \cdots = \Gamma_{\sigma\varrho-1\varrho}^{\varrho-1\tau} = \Gamma_{\sigma\varrho+1\varrho}^{\varrho+1\tau} = \cdots = \Gamma_{\sigma\tau-1\varrho}^{\tau-1\tau} = \Gamma_{\sigma\tau+1\varrho}^{\tau+1\tau} = \cdots = \Gamma_{\sigma n\varrho}^{n\tau}, \tag{2.10}$$

$$\Gamma_{\sigma 1\tau}^{1\varrho} = \Gamma_{\sigma 2\tau}^{2\varrho} = \cdots = \Gamma_{\sigma\varrho-1\tau}^{\varrho-1\varrho} = \Gamma_{\sigma\varrho+1\tau}^{\varrho+1\varrho} = \cdots = \Gamma_{\sigma\tau-1\tau}^{\tau-1\varrho} = \Gamma_{\sigma\tau+1\tau}^{\tau+1\varrho} = \cdots = \Gamma_{\sigma n\tau}^{n\varrho}$$

$$(\varrho < \tau,\ \varrho = 1, 2, \ldots, n-1;\ \tau = 2, 3, \ldots, n;\ n.\,s.\ \varrho-1, \varrho+1, \tau-1, \tau+1);$$

$$\sum_{E=1}^{m} \overset{E}{\underset{p}{C}} \cdot \overset{2}{\Gamma}{}_{\sigma E}^{E} = 0 \quad \left(m = \binom{n}{2},\ p = n+1, n+2, \ldots, \binom{n}{2}\right), \tag{2.11}$$

wo

$$\overset{2}{\Gamma}{}_{\sigma E}^{E} \overset{\text{df}}{=} \Gamma_{\sigma\alpha\lambda}^{\alpha\lambda} \quad (n.\,s.\ \alpha, \lambda), \tag{2.12}$$

(2.13)

$\overset{1}{\underset{p}{C}} = 1$ *wenn* $p = 2n-2, 2n-1, \ldots, \binom{n}{2}$

$\overset{2}{\underset{p}{C}} = 1$ *wenn* $p \leqq 2n-3$ *oder* $p = 3n-5, 3n-4, \ldots, \binom{n}{2}$

$\overset{i}{\underset{p}{C}} = -1$ *wenn* p *lakunär nur die Werte* $p = i + k \cdot \left(n - \frac{k+3}{3}\right)$ $(k = 1, 2, \ldots, i-1)$ *hat, oder wenn* p *alle aufeinanderfolgende Werte von* N_1 *bis* N_2 *annimmt, wo*

$$N_1 = i \cdot \left(n - \frac{i+1}{2}\right) + 1,\quad N_2 = (i+1)\left(n - \frac{i+2}{2}\right)$$

$$(i = 3, 4, \ldots, n-2)$$

$\overset{n-1}{\underset{p}{C}} = -1$ *wenn* $p = n - 1 + k \cdot \left(n - \frac{k+3}{2}\right)$ $(k = 1, 2, \ldots, n-2)$

$\overset{n}{\underset{p}{C}} = -1$ *wenn* $p = n+1, n+2, \ldots, \binom{n}{2}$

$\overset{q}{\underset{p}{C}} = \delta_p^q$ *wenn* $p, q = n+1, n+2, \ldots, \binom{n}{2}$; δ_p^q *bezeichnet das Kroneckersche Symbol*

$\overset{E}{\underset{p}{C}} = 0$ *wenn* $E = 1, 2, \ldots, m$, *aber* p *nicht die oben angegebenen Werte durchläuft,*

* „n. s. α, β" bedeutet: nicht summieren über α und β.

$$
(2.14) \qquad E \overset{\mathrm{df}}{=} \begin{cases} 1 & \text{wenn} \quad \alpha = 1, \quad \lambda = 2 \\ 2 & \text{wenn} \quad \alpha = 1, \quad \lambda = 3 \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ n-1 & \text{wenn} \quad \alpha = 1, \quad \lambda = n \\ n & \text{wenn} \quad \alpha = 2, \quad \lambda = 3 \\ n+1 & \text{wenn} \quad \alpha = 2, \quad \lambda = 4 \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ m-2 & \text{wenn} \quad \alpha = n-2, \quad \lambda = n-1 \\ m-1 & \text{wenn} \quad \alpha = n-2, \quad \lambda = n \\ m & \text{wenn} \quad \alpha = n-1, \quad \lambda = n, \qquad m = \binom{n}{2}. \end{cases}
$$

§ 3. Existenz der Objektes der paralellen Übertragung von Dichten

Man kann sich weiter die Frage stellen, wann das gegebene Objekt $\Gamma_{\sigma\alpha\beta}^{\lambda\mu}$ ein solches Objekt induziert, welches für die Bestimmung der parallelen Übertragung von Dichten ausreichend ist und zwar aus dem Objekt $\Gamma_{\sigma\lambda}^{\mu}$ durch Faltung in (2. 1) definiert ist.

KOROLLAR 1. Für die Existenz eines Objektes Γ_σ, welches für die Bestimmung der parallelen Übertragung von Dichten ausreichend ist, sind im vierdimensionalen Raum die Beziehungen (2. 8) notwendige und hinreichende Bedingungen, wobei

$$
(3.1) \qquad \Gamma_\sigma = 2(\Gamma_{\sigma 12}^{12} + \Gamma_{\sigma 34}^{34}).
$$

KOROLLAR 2. Für die Existenz eines Objektes Γ_σ, welches für die Bestimmung der parallelen Übertragung von Dichten ausreichend ist, bilden im n-dimensionalen Raum ($n \geqq 5$) die Beziehungen (2. 11) eine notwendige und hinreichende Bedingung.

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Zusatz bei der Korrektur (13. März 1970.): Es soll der Text mit den Arbeiten von E. Bompiani, A. Cossu, P. Mastrogiacomo, C. di Comite vergleichen werden, die teilweise vor 1961, teilweise nach 1961 publiziert worden sind. Vergleiche eine demnächst erscheinende Note in *Rendiconti della Academia dei Lincei,* wo die ergänzende Literatur zitiert ist.

KRAKÓW,
LOBZOWSKA 61,
POLSKA

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A PARTITION THEOREM FOR COLLECTIONS OF UNIVERSAL SUBCONTINUA

By
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This paper is concerned with conditions under which a collection of universal subcontinua may be partitioned into finitely many subcollections, each of which has the finite intersection property.

Throughout X will denote a Hausdorff topological space. A continuum $A \subset X$ will be called a *universal subcontinuum* of (USC) X if $A \cap B$ is a continuum for each continuum $B \subset X$. (Ordinarily it is assumed that X is a continuum, but this assumption is unnecessary for our purposes.) WALLACE, [3], studied USC under the title *semi-chains*. We are concerned with the following topological-combinatorial problem.

If α is a collection of universal subcontinua, if $n \geqq 2$ is a positive integer, and if of each n distinct elements of α at least two have a non-empty intersection, does there exist an integer $m \leqq n-1$ and m subcollections $\alpha_1, \dots, \alpha_m$ of α, each of which has the finite intersection property, such that $\alpha = \alpha_1 \cup \dots \cup \alpha_m$?

We have not been able to answer this question in general, but in this paper we will answer it affirmatively for $n=3$.

The importance of the present problem lies in the fact that if α is a collection of USC which has the finite intersection property then the intersection over α is a non-empty USC. The theorem presented here is of the type which has traditionally played a role in the study of the action of certain types of mappings on Hausdorff continua. It applies, for example, to collections of (i) compact nodal sets in a connected space, (ii) subcontinua of a hereditarily unicoherent continuum, and (iii) maximal cyclic elements in a peano continuum, [4]. It also applies to compact segments of the real line, where it overlaps a theorem of DEBRUNNER and HADWIGER [1]; however, this special case can be treated by simpler methods than those used here.

The following lemma is essentially lemma 1 of [2]. However, the proof is given here for the sake of completeness.

LEMMA. *Let α be a collection of universal subcontinua of X. If $A, B \in \alpha$ implies $A \cap B \neq \emptyset$, then α has the finite intersection property.*

PROOF. Assume that for some integer $n \geqq 2$, $A_1, \dots, A_n \in \alpha$ implies $A_1 \cap \dots \cap A_n \neq \emptyset$. Let $A_1, \dots, A_{n+1} \in \alpha$. Then by the inductive hypothesis $K = (A_1 \cap \dots \cap A_{n-1}) \cup A_n$ is a continuum, and since A_{n+1} is a USC,

$$A_{n+1} \cap K = (A_1 \cap \dots \cap A_{n-1} \cap A_{n+1}) \cup (A_n \cap A_{n+1})$$

is a continuum which is the union of two non-empty closed sets. Thus

$$A_1 \cap \dots \cap A_{n+1} \neq \emptyset,$$

and α has the finite intersection property.

THEOREM. *Let α be a non-empty collection of non-empty universal subcontinua of X such that of any three distinct elements of α, at least two have a non-empty intersection. Then either α has the finite intersection property or else there are two subcollections α_1 and α_2 of α, each of which has the finite intersection property such that $\alpha = \alpha_1 \cup \alpha_2$.*

PROOF. Let β be the set of all elements $A \in \alpha$ such that A is disjoint from some $B \in \alpha$. If $\beta = \emptyset$, then by the Lemma we are finished. If $\beta \neq \emptyset$, let γ consist of all pairs of non-empty subcollections (β_1, β_2) of β such that

(1) β_1 and β_2 have the finite intersection property.

(2) $A \in \beta_1$ $\{B \in \beta_2\}$ implies the existence of $B \in \beta_2$ $\{A \in \beta_1\}$ such that $A \cap B = \emptyset$.

Since $\beta \neq \emptyset$, also $\gamma \neq \emptyset$. If (β_1, β_2), $(\beta_1', \beta_2') \in \gamma$, define $(\beta_1, \beta_2) \leqq (\beta_1', \beta_2')$ if and only if $\beta_1 \subset \beta_1'$ and $\beta_2 \subset \beta_2'$. Then $\leqq$ is a partial order on γ, and any subset of γ which is totally ordered by $\leqq$ has a least upper bound in γ. Thus, by the maximal principle, γ has a maximal element (β_1, β_2). Choose a fixed $A_1 \in \beta_1$ and $B_1 \in \beta_2$ with $A_1 \cap B_1 = \emptyset$.

We will show that $\beta = \beta_1 \cup \beta_2$. Let $A \in \beta$. Suppose first that there is an $A_2 \in \beta_1$ and $B_2 \in \beta_2$ such that $A \cap A_2 = \emptyset = A \cap B_2$. By the assumption on α, we have $A_2 \cap B_2 \neq \emptyset$; furthermore, we may assume without loss of generality that $A \cap A_1 \neq \emptyset$. Because $K = A_1 \cup A_2 \cup B_2 \cup B_1$ is a continuum and $K \cap A = (A_1 \cap A) \cup (B_1 \cap A)$, the disjointness of A_1 and B_1 implies that $B_1 \cap A = \emptyset$. Consideration of the triple (A, A_2, B_1) shows that $A_2 \cap B_1 \neq \emptyset$. Choose $B_3 \in \beta_2$ such that $A_2 \cap B_3 = \emptyset$. Consideration of the triple (A, A_2, B_3) shows that $A \cap B_3 \neq \emptyset$. Then if $L = A_2 \cup A_1 \cup A \cup B_3$, we find that $L \cap B_1 = (A_2 \cap B_1) \cup (B_3 \cap B_1)$. The continuum $L \cap B_1$ is thus exhibited as the union of two disjoint non-empty closed subsets, and this is absurd. We have shown that $A \in \beta$ implies that A intersects every member of β_1 or every member of β_2.

Let $A \in \beta - (\beta_1 \cup \beta_2)$. Since A intersects every member of β_1 or every member of β_2 and (β_1, β_2) is maximal, A must intersect every member of β_1 and every member of β_2. Now choose $B \in \alpha$ such that $A \cap B = \emptyset$. Then, of course, $B \in \beta$, hence $B \in \beta \setminus (\beta_1 \cup \beta_2)$ as well. But then $B \cap (A_1 \cup A \cup B_1) = (B \cap A_1) \cup (B \cap B_1)$ is a continuum, and this is absurd, because B has to meet every element of $\beta_1 \cup \beta_2$, the same way as A does. This contradiction shows that $\beta = \beta_1 \cup \beta_2$.

Finally, it is easy to see that if $A \in \alpha - \beta$, then A intersects every member of α. Thus if $\alpha_1 = \beta_1 \cup (\alpha - \beta)$ and $\alpha_2 = \beta_2$, then α_1 and α_2 have the finite intersection property and $\alpha = \alpha_1 \cup \alpha_2$.

In attempting to give a general solution to the problem posed above, information of the following type may be useful.

PROPOSITION. *Let α and β be collections of universal subcontinua of X, let α and β have the finite intersection property, and let $A \in \alpha$ imply the existence of $B \in \beta$ for which $A \cap B = \emptyset$. Then if $A_0, \ldots, A_n \in \alpha$, there is a $B \in \beta$ for which $B \cap A_i = \emptyset$, $i = 0, \ldots, n$.*

PROOF. Assume the proposition is true for the integer $n-1$, where $n \geqq 1$, and let $A_0, \ldots, A_n \in \alpha$. For each integer k for which $0 \leqq k \leqq n$, we may use the inductive hypothesis to choose $B_k \in \beta$ such that

$$A_j \cap B_k = \emptyset, \qquad j = k, k+1, \ldots, k+n-1 \pmod{n+1}$$

If either $A_n \cap B_0 = \emptyset$ or $A_{n-1} \cap B_n = \emptyset$, we are finished. Hence assume that $A_n \cap B_0 \neq \emptyset$ and $A_{n-1} \cap B_n \neq \emptyset$. Let

$$K = A_n \cup B_0 \cup \cdots \cup B_n.$$

Then $A_{n-1} \cap K = (A_{n-1} \cap A_n) \cup (A_{n-1} \cap B_n)$ is a continuum which is the union of two non-empty disjoint closed subsets. This contradiction proves the proposition.

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ON QUASIMODULAR MAXIMAL LEFT IDEALS OF PRIMITIVE RINGS GENERATED BY THEIR MINIMAL LEFT IDEALS

By

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Let R be an associative ring. A left ideal L of R is modular (cf. JACOBSON [3]) if there exists an element $a \in R$ such that $x - xa \in L$ for all $x \in R$. A left ideal L of R is quasimodular (cf. SZÁSZ [1]) if $x \notin L$ implies that there exists $y \in R$ such that $xy \notin L$. The quasi-modular ideals of an associative ring were studied in [1] and we shall follow the terminology from that paper.

A quasi-modular maximal left ideal of a ring R which is not modular is called a distinguished left ideal. A ring R is called a left Ω-ring if R has a distinguished left ideal.

An example of a left Ω-ring was given in [1]. We shall here give new examples and we also answer some questions raised in [1]. The main part of this paper consists of a discussion about reduced socle rings which are introduced later. We refer to [3] for a general discussion about the socle of a ring.

DEFINITION 1. A ring R whose Jacobson radical is zero and where R coincides with its own socle and has no non trivial two-sided ideals is called a reduced socle ring.

DEFINITION 2. An idempotent e of a ring R is a minimal idempotent if the left principal ideal Re is a minimal left ideal of R.

If R is a reduced socle ring we know that e is a minimal idempotent if and only if eR is a minimal right ideal. There exists minimal left ideals L_a such that $R = \Sigma \oplus L_a$, i. e. R is the direct sum of these L_a. Each minimal left ideal of R is of the form Re where e is a minimal idempotent and they are all isomorphic as left R-modules.

DEFINITION 3. A reduced socle ring R is denumarable generated to the left if there exists a denumarable sequence $\{Re_n\}_1^\infty$ of minimal left ideals such that $R = \Sigma \oplus Re_n$.

THEOREM 1. *Let R be a reduced socle ring which is denumarable generated to the left. Then there exists a denumarable family $\{e_n\}_1^\infty$ of minimal orthogonal idempotents such that $R = \Sigma \oplus Re_n$.*

PROOF. Let us first recall that an idempotent e of R is a minimal idempotent if and only if $r(e) = \{x \in R, ex = 0\}$ is a maximal right ideal of R. The following two lemmas are fundamental for the construction which follows.

LEMMA 1. *Let e and f be two minimal idempotents of R such that $Re \cap Rf = 0$. Then $r(e) \neq r(f)$.*

PROOF. Suppose that $J=r(e)=r(f)$. Let $L = Re+Rf$. If $x \in L$ is not zero we see that $r(x)$ contains the maximal right ideal J and hence $r(x)=J$. It follows that if $x \in L$ is such that $xe=0$ then $x=0$. If x and y are two non zero element of L we can find $r \in R$ such that $y-rx$ belongs to the maximal left ideal $l(e)$, where $l(e)=\{x \in R, xe=0\}$. It follows that $y=rx$ and hence L is a minimal left ideal of R, a contradiction.

LEMMA 2. *Let e and f be two minimal idempotents of R such that $Re \cap Rf=0$. Then there exists a minimal idempotent g such that $Re+Rf = Re+Rg$ and $eg=ge=0$.*

PROOF. Since $r(e)$ is different from $r(f)$ we can find $x \in R$ such that $ex=e$ and $fx=0$. Namely, by the maximality of $r(e)$ and $r(f)$ one has $R = r(e)+r(f)$ and thus $e = (1-e)r_1+(1-f)r_2$ with $r_i \in R$, and we put $x = (1-f)r_2$. Let us put $f_1 = f-xf$. Clearly f_1 is an idempotent and since $ff_1=f$ we have $Re+Rf_1 = Re+Rf$. Finally we put $g = f_1-f_1e$ and then g is the desired minimal idempotent.

We now complete the proof. Let $R = \Sigma \oplus Re_n$ where $\{e_n\}_1^\infty$ is a family of minimal idempotents. Suppose we have found $f_1, f_2, \dots f_n$ such that $\{f_i\}_1^n$ is an orthogonal family of minimal idempotents and $Re_1+\dots+Re_n = Rf_1+\dots+Rf_n$. Now lemma 2 is applied to e_1 and e_{n+1}. Hence we can find a minimal idempotent g_1 such that $e_1g_1=g_1e_1=0$ and $Re_1+Re_{n+1} = Re_1+Rg_1$. Since e_{n+1} does not belong to Re_1+Re_2 if $n>1$ we see that $Re_2 \cap Rg_1 = 0$ and hence we can find a minimal idempotent g_2 such that $Re_2+Rg_1 = Re_2+Rg_2$ and $e_2g_2=g_2e_2=0$. But from the way g_2 is constructed in lemma 2 we see that $g_2e_1=0$ and using $g_2-e_1g_2$ we may assume that $e_1g_2=0$. Inductively we find g_k such that $Re_k+Rg_{k-1} = Re_k+Rg_k$ and g_k is orthogonal to $e_1, \dots, e_k$. Finally g_n gives the desired minimal idempotent. Let us put $f_{n+1}=g_n$. It is now clear that $\{f_n\}_1^\infty$ is an orthogonal family of minimal idempotents and $R = \Sigma \oplus Rf_n$.

THEOREM 2. *A reduced socle ring R is a regular ring.*

PROOF. It is almost trivial, that every reduced socle ring satisfies the minimum condition for principal left ideals. Moreover by Theorem (Satz) 2. 6. 1 and 2. 6. 3 of SZÁSZ [2] every semi-simple ring with minimum condition for principal left ideals is regular in the sense of Neumann.

We shall now try to find distinguished left ideals of a reduced socle ring.

PROPOSITION 1. *Let L be a maximal left ideal of a reduced socle ring R. Then L is modular if and only if there exists $x \in R$ such that $L=l(x)$.*

PROOF. Let us put $R = L+J$ where J is a minimal left ideal of R such that $L \cap J = 0$. If L is modular we have an element $a \in R$ such that $x-xa \in L$ fo ' $x \in R$. If we put $a = m+j$ with $m \in L$ and $j \in J$ we see that $x-xj \in L$ for all $x \in R$ and since $L \cap J = 0$ we see that $xj=0$ for all $x \in L$. Hence $L=l(j)$ in this case . Suppose conversely that $L=l(x)$ for some $x \in R$. We put $x = m+j$ as above and since $L \cap J = 0$ we see that $Lj=0$. If $j=0$ we choose $r \in R$ such that $j_1=mr$ is a minimal idempotent and since $Mj_1=0$ we see that j_1 cannot belong to M and $R = L+Rj_1$. Hence we may assume that $j \neq 0$ and so $L=l(j)$. Since jR is not contained in the Jacobson radical of R we can find $r \in R$ such that the left ideal $M_{jr} = \{x-xjr, x \in R\} \neq R$. Since $Ljr=0$ we see that $M_{jr}=L$ and hence L is modular.

PROPOSITION 2. *Let R be a reduced socle ring. If L is a maximal left ideal of R there exists $x \in R$ such that $y \notin L$ implies that $yx \notin L$.*

PROOF. We put $R = L + Re$ where e is a minimal idempotent such that $L \cap Re = 0$ and now e can be used.

PROPOSITION 3. *Let R be a reduced socle ring such that every maximal left ideal of R is modular. Then we can find a family $\{e_a\}$ of minimal orthogonal idempotents of R such that $R = \Sigma \oplus Re_a$.*

PROOF. Let us put $R = \Sigma_A \oplus Re_a$ where A is a well ordered set and e_a are minimal idempotents. Let a_0 be the first element of A and put $L_0 = \Sigma \oplus Re_a$ where the sum is taken over all a with $a > a_0$. By assumption L_0 is modular and hence $L_0 = l(r)$ for some $r \in R$. Since R has no proper two sided ideals we can choose r such that r is a minimal idempotent of Re_{a_0}. Let us put $f_{a_0} = r$. We have $e_a f_{a_0} = 0$ for all $a > a_0$. Now we let $L_1 = Rf_{a_0} + \Sigma \oplus Re_a$ where the sum is taken over all a with $a > a_1$. We can now find a minimal idempotent $f_{a_1} \in Re_{a_1}$ such that $L_1 f_{a_1} = 0$. Hence $e_a f_{a_1} = 0$ for all $a > a_1$ and since $f_{a_1} \in Re_{a_1}$ we also have $f_{a_1} f_{a_0} = f_{a_0} f_{a_1} = 0$. Inductively we obtain $f_a \in Re_a$ such that $\{f_a\}_A$ is an orthogonal family of minimal idempotents. Since $Rf_a = Re_a$ for all a we have the result.

PROPOSITION 4. *Let R be a reduced socle ring such that $R = \Sigma \oplus e_a R = \Sigma \oplus Re_a$ where $\{e_a\}_A$ is an infinite orthogonal family of minimal idempotents. Then R is a left Ω-ring.*

PROOF. Let $a_0 \in A$ be fixed and for each $a \in A$ we choose $r_a \in R$ such that $r_a e_0 = = e_a r_a e_0$ is different from zero. Here we have put $e_0 = e_{a_0}$. Let us put $x_a = e_a + r_a e_0$ for all $a \neq a_0$ and $L = \Sigma \oplus Rx_a$ where the sum is taken over all $a \in A$ with $a \neq a_0$. Since $R = L + Re_0$ it is clear that L is R or a maximal left ideal of R. If $L = R$ we get $e_0 = r_1 x_{a_1} + \dots + r_k x_{a_k}$. If we multiply with e_0 to the right we get $e_0 = r_1 r_{a_1} e_0 + \dots \dots + r_k r_{a_k} e_0$. If we multiply with e_{a_i} to the right we get $r_i e_{a_i} = 0$ and hence $r_i r_{a_i} = 0$ for each i, a contradiction. Suppose now that L is modular. Hence there exists a non zero element $x = e_0 r_0 + e_{a_1} r_1 + \dots + e_{a_k} r_k$ such that $Lx = 0$. If we choose $a \notin \{a_0, a_1, \dots, a_k\}$ we see that $x_a x = 0$ implies that $r_a e_0 r_0 = 0$ and hence $r_0 = 0$ since $e_0 R$ is a minimal right ideal. Now $x_{a_i} x = 0$ gives $e_{a_i} r_i = 0$ for all i. Hence $x = 0$, a contradiction.

THEOREM 3. *Let R be a reduced socle ring without a unit element. Then R is a left or a right Ω-ring.*

PROOF. Suppose that R is neither a left nor a right Ω-ring. Using proposition 3 we can find a family $\{e_a\}_A$ of minimal orthogonal idempotents of R such that $R = \Sigma \oplus Re_a$. This family must be infinite since R has no unit element. Now it is not clear that $R = \Sigma \oplus e_a R$ but since R is not a right Ω-ring we can find a family $\{f_b\}$ of minimal idempotents such that $f_b e_a = 0$ for all a and b and $\{f_b\}$ is an orthogonal family of idempotents and $R = \Sigma \oplus Re_a \Sigma \oplus Rf_b$. This can be proved with the same methods as in the proof of proposition 4. Using the idempotents $g_b = f_b - e_{a_0} f_b$ where a_0 is some fixed element of A we see that we may assume that $e_{a_0} f_b = 0$ also holds for all b. Now we choose r_a such that $e_a r_a = e_0 r_a e_a$ for all a different from zero. Here $e_0 = e_{a_0}$. In the same way we choose q_b such that $e_0 q_b = e_0 q_b f_b$ for all b. We put $x_a = e_a + e_0 r_a$ for all $a \neq a_0$ and $x_b = f_b + e_0 q_b$ for all b. Let J be the right ideal generated by these x_a and x_b. Using the same methods as in proposition 4 we can prove that J is a distinguished right ideal. This gives a contradiction.

We now study some reduced socle rings and the relationship between left and right distinguished ideals in these. Let V be an infinite dimensional vector-space over some field K. Let R_V be the ring of all linear transformations on V having a finite dimensional range. Clearly R_V is a reduced socle ring. Let $\{e_a\}$ be a basis of V and choose $g_a \in R_V$ such that $g_a(e_b) = D_{ab} e_a$ for all a and b. Here D_{ab} is the Kronecker delta-function. If $g \in R_V$ is such that $g(V) \subset Ke_1 + \dots + Ke_n$ we see that $g = g_1 g + \dots + g_n g$. Hence $\{g_a\}$ is an orthogonal family of minimal idempotents of R such that $R = \Sigma \oplus g_a R$. If J is a maximal right ideal of R_V it is easy to prove that $V_0 = \{g(x), x \in V \text{ and } g \in J\}$ is a proper subspace of V. Hence we can find a non zero $f \in R_V$ such that $fJ = 0$ and then proposition 1 shows that J is modular. We now prove that the situation to the left is quite different.

PROPOSITION 5. *Let R_V be as above. There does not exist a family $\{g_a\}$ of minimal orthogonal idempotents such that $R = \Sigma \oplus Rg_a$.*

PROOF. Suppose that $\{g_a\}$ is such a family. Choose $x_a \in V$ such that $g_a(x_a) = x_a$ for all a. Clearly $\{x_a\}$ is a linearly independent subset of V. Choose a fixed a_0 and let $y_0 = x_{a_0}$ while $y_a = x_a + y_0$ for all $a \neq a_0$. We can find $g \in R_V$ such that $g(y_0) = y_0$ while $g(y_a) = 0$ for all $a \neq a_0$. It is now obvious that g cannot belong to $\oplus Rg_a$, a contradiction.

THEOREM 4. *There exists a left Ω-ring R which has denumarable many modular maximal left ideals and non-denumarable many distinguished left ideals.*

PROOF. Let V be a denumarable dimensional vector space over the prime field with two elements. Let R_V be as before. If L is a modular maximal left ideal of R_V there exists an element $f \in R_V$ such that $L = l(f)$. It follows that $L = \{g \in R_V, g(x_0) = 0\}$ for some $x_0 \in V$. Hence there are only denumarable many modular maximal left ideals of R_V. Theorem 1 and proposition 5 show that R_V is not denumarable generated to the left. It follows that R_V has non-denumarable many distinguished left ideals.

Let us now consider an infinite dimensional vector space V and let $\{e_a\}$ be a basis of V. Let R_0 be the ring of all linear transformations on V such that $g \in R_0$ if and only if $g(e_a) \neq 0$ for only finitely many a. It is easily seen that R_0 is a reduced socle ring and $R_0 = \Sigma \oplus g_a R = \Sigma \oplus Rg_a$ where g_a are the projections associated with the basis e_a, i.e. $g_a(e_b) = D_{ab} e_b$ where D_{ab} is the Kronecker delta-function. Let us fix a_0 and choose $r_a \in R_0$ such that $r_a g_0 = g_a r_a g_0$ are different from zero for all $a \neq a_0$. We put $x_a = g_a + r_a g_0$ for all $a \neq a_0$. As before $g_0 = g_{a_0}$ here. Let L be the left ideal generated by these x_a. As in proposition 4 we see that L is a distinguished left ideal of R_0. Now it is easily seen that the ring generated by L and e_0 gives R. Hence we have found an answer to problem 1 in SZÁSZ [1].

We finally remark that everything we have proved for reduced socle rings in this paper also works for rings whose Jacobson radical is zero and where the rings coincide with their own socles. For such rings can be decomposed into homogenous components and each component is a reduced socle ring.

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A HÀJEK—RÉNYI EXTENSION OF LÉVY'S INEQUALITY AND SOME APPLICATIONS

By

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1. Introduction and summary. In 1955 [1] HÀJEK and RÉNYI gave an interesting generalization of the famous inequality of Kolmogorov and applied it to give a quick proof of the strong law of large numbers. Subsequently, their inequality was generalized by CHOW [2] to semi-martingales even as DOOB [4, p. 314] had already generalized Kolmogorov's inequality.

Let us for comparative purposes set down these well known results.

Suppose $X_1, \ldots, X_n$ are independent random variables with $\mathbf{E}(X_i)=0$ and $\mathbf{E}(X_i^2)<\infty$, $1 \leqq i \leqq n$, defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Let $S_k = \sum_{i=1}^{k} X_i$, $M_n = \max_{1 \leqq k \leqq n} |S_k|$.

Kolmogorov's inequality:

$$\mathbf{P}(M_n \geqq \varepsilon) \leqq \varepsilon^{-2}\mathbf{E}(S_n^2). \tag{1. 1}$$

Hàjek—Rényi inequality:

Let $U_n = c_n^2 S_n^2 + \sum_{i=1}^{n-1} (c_i^2 - c_{i+1}^2) S_i^2$, where $0 \leqq c_n \leqq c_{n-1} \leqq \cdots \leqq c_1$ are constants. Then,

$$\mathbf{P}\big(\max_{1 \leqq k \leqq n} c_k |S_k| \geqq \varepsilon\big) \leqq \varepsilon^{-2}\mathbf{E}(U_n). \tag{1. 2}$$

Let $Y_1, \ldots, Y_n$ be a nonnegative semi-martingale (expectation increasing) with respect to some sequence of sigma fields $\mathcal{A}_1 \subset \mathcal{A}_2 \cdots \subset \mathcal{A}_n \subset \mathcal{A}$ on Ω. Let $M_n^* = \max_{1 \leqq k \leqq n} Y_k$. Then,

Doob's inequality:

$$\mathbf{P}(M_n^* \geqq \varepsilon) \leqq \varepsilon^{-1} \int_{[M_n \geqq \varepsilon]} Y_n \, d\mathbf{P} \leqq \varepsilon^{-1}\mathbf{E}(Y_n). \tag{1. 3}$$

Finally if the c_i and Y_i are as above and we define $U_n^* = c_n Y_n + \sum_{i=1}^{n-1} (c_i - c_{i+1}) Y_i$, then, *Chow's inequality:*

(1. 4)
$$\mathbf{P}\big(\max_{1\le k\le n} c_k Y_k \geqq \varepsilon\big) \leqq$$
$$\leqq \varepsilon^{-1}\left\{\sum_{i=1}^{n-1}(c_i-c_{i+1})\mathbf{E}(Y_i)+c_n \int\limits_{[\max_{1\le k\le n} Y_k\geqq\varepsilon]} Y_n\,d\mathbf{P}\right\} \leqq \varepsilon^{-1}\mathbf{E}(U_n^*).$$

Another widely used inequality of the theory of sums of independent random variables which is of a somewhat different nature than the preceding is that of Lévy. (LOÈVE [8], p. 247.)

Lévy's inequality:

Suppose that the X_i as well as being independent are also symmetrically distributed about 0. Then,

(1. 5)
$$\mathbf{P}(M_n \geqq \varepsilon) \leqq 2\mathbf{P}(|S_n| \geqq \varepsilon).$$

In section 2 we shall prove the following generalization of (1. 5) analogous to, though necessarily more restrictive than (1. 4).

THEOREM 1. *Let the X_i be as in the statement of Lévy's inequality and $c_1, \ldots, c_n$ be as above. Let g be a nonnegative convex function on R. Define* $G_n = \sum_{i=1}^{n-1}(c_i-c_{i+1})g(S_i) + c_n g(S_n)$. *Then,*

(1. 5)
$$\mathbf{P}\big(\max_{1\le k\le n} c_k g(S_k) \geqq \varepsilon\big) \leqq 2\mathbf{P}(G_n \geqq \varepsilon).$$

Recently, ([3], [5], [6], and [12]) considerable attention has been focussed on optimal stopping problems involving pay offs of the form $c_n|S_n|^\alpha$. A great deal of simplification of some of the arguments involved in this work is achieved if one knows that,

(1. 6)
$$\mathbf{E}\big(\sup_n c_n |S_n|^\alpha\big) < \infty.$$

In [11] TEICHER gave some conditions under which (1. 6) holds. For the special case $c_n = n^{-\alpha}$, $1 \leqq \alpha \leqq 2$ a very satisfactory result along these lines had been obtained by MARCINKIEWICZ and ZYGMUND [10], theorem 7. In section 3, using theorem 1 and an approach due DOOB [4], p. 337, we prove a theorem encompassing the results of Marcinkiewicz and Zygmund and in part generalizing those of Teicher.

2. The inequality. We prove theorem 1. The argument hinges on,

LEMMA 2. 1. *Let the X_i, c_i, and g be as in the statement of theorem* 1. *Put $c_{n+1}=0$. Then, for any $0 \leqq r \leqq n-1$, and any real x,*

(2. 1)
$$\mathbf{P}\left(\sum_{j=1}^{n-r} g(S_j+x)(c_{r+j}-c_{r+j+1}) - c_{r+1}g(x) < 0\right) \leqq \frac{1}{2}.$$

PROOF. It is well known that a convex function on the line is continuous and possesses right and left hand derivatives at every point. We denote these derivatives by g_1' and g_{-1}' respectively. Note that

(2. 2)
$$g_1'(x) \geqq g_{-1}'(x)$$

for all x. Furthermore if $\operatorname{sgn} x = +1$ or -1 as $x \geqq 0$ or $x<0$ we have,

$$g(x+a)-g(x) \geqq [g'_{\operatorname{sgn} a}(x)]a. \tag{2. 3}$$

These facts may for instance be found in [8] (Theorems 111, 112, p. 91, 94.) Then,

$$\sum_{j=1}^{n-r} g(S_j+x)(c_{r+j}-c_{r+j+1})-c_{r+1}g(x) = \tag{2. 4}$$

$$= \sum_{j=1}^{n-r} [g(S_j+x)-g(x)](c_{r+j}-c_{r+j+1}) \geqq \sum_{j=1}^{n-r} g'_{\operatorname{sgn} S_j}(x)(c_{r+j}-c_{r+j+1})S_j.$$

But, by (2. 2) for all x, y,

$$[g'_{\operatorname{sgn} y}(y)]y \geqq [g'_{-\operatorname{sgn} y}(x)]y. \tag{2. 5}$$

From (2. 4) we see that,

$$\mathbf{P}\left(\sum_{j=1}^{n-r} g(S_j+x)(c_{r+j}-c_{r+j+1})-c_{r+1}g(x) < 0\right) \leqq \tag{2. 6}$$

$$\leqq \mathbf{P}\left(\sum_{j=1}^{n-r} g'_{\operatorname{sgn} S_j}(x)(c_{r+j}-c_{r+j+1})S_j < 0\right).$$

From the symmetry of the X_i the last expression equals

$$\mathbf{P}\left(\sum_{j=1}^{n-r} S_j(c_{r+j}-c_{r+j+1})g'_{\operatorname{sgn}(-S_j)}(x) > 0\right).$$

Finally by (2. 5),

$$\mathbf{P}\left(\sum_{j=1}^{n-r} S_j(c_{r+j}-c_{r+j+1})g'_{\operatorname{sgn}(-S_j)}(x) > 0\right) \leqq \tag{2. 7}$$

$$\leqq \mathbf{P}\left(\sum_{j=1}^{n-r} S_j(c_{r+j}-c_{r+j+1})g'_{\operatorname{sgn} S_j}(x) > 0\right).$$

Combining (2. 6) and (2. 7) we get the lemma.

We now proceed with the proof of theorem 1.

Define more generally

$$G_j = \sum_{k=1}^{j-1} g(S_k)(c_k-c_{k+1})+c_j g(S_j). \tag{2. 8}$$

In view of our assumptions it is clear that,

$$\mathbf{P}\left(\max_{1\leqq k\leqq n} c_k g(S_k) \geqq \varepsilon\right) \leqq \mathbf{P}\left(\max_{1\leqq j\leqq n} G_j \geqq \varepsilon\right). \tag{2. 9}$$

Let t be the smallest $k \leqq n$ such that $G_k \geqq \varepsilon$. If no such k exists let $t=n$. Then,

$$\text{(2. 10)} \qquad \mathbf{P}\big(\max_{1 \leqq j \leqq n} G_j \geqq \varepsilon\big) = \mathbf{P}(G_t \geqq \varepsilon) =$$

$$= \sum_{k=1}^{n} \{\mathbf{P}(t=k, G_n - G_t \geqq 0) + \mathbf{P}(t=k, G_n - G_t < 0)\} \leqq$$

$$\leqq \mathbf{P}(G_n \geqq \varepsilon) + \sum_{k=1}^{n} \mathbf{P}(t=k, G_n - G_k < 0).$$

But,

$$\text{(2. 11)} \qquad \mathbf{P}(t=k, G_n - G_k < 0) = \int_{[t=k]} \mathbf{P}(G_n - G_k < 0 | X_1, \ldots, X_k)\, d\mathbf{P},$$

where $\mathbf{P}[\cdot | X_1, \ldots, X_k]$ denotes as usual conditional probability.

But,

$$\text{(2. 12)} \qquad \mathbf{P}(G_n - G_k < 0 | X_1, \ldots, X_k) =$$

$$= \mathbf{P}\left(\sum_{j=k+1}^{n} (c_j - c_{j+1}) g(S_j) - c_{k+1} g(S_k) < 0 | X_1, \ldots, X_k\right).$$

Using the independence of the X_i's and lemma 2. 1 we see that,

$$\text{(2. 13)} \qquad \operatorname{ess\,sup} \mathbf{P}(G_n - G_k < 0 | X_1, \ldots, X_k) \leqq \frac{1}{2}.$$

Substituting in (2. 10) and (2. 11) we get

$$\text{(2. 14)} \qquad \mathbf{P}(\max_{1 \leqq j \leqq n} G_j \geqq \varepsilon) \leqq 2\mathbf{P}(G_n \geqq \varepsilon).$$

In view of (2. 9) the theorem is proved.

It may be worth noting that the inequality as well as its proof go over verbatim to the case where the X_i are sign invariant random elements (see [13]).

3. Maximal theorems for sums of independent random variables. Let $X_1, X_2, \ldots,$ be a sequence of independent random variables with $\mathbf{E}(X_i)=0$ for all i, $\{c_k\}$, $k \geqq 1$, a monotone decreasing sequence of positive constants. Define S_k as before and suppose that g is a nonnegative convex function such that for some $K < \infty$ and all x, y,

$$\text{(3. 1)} \qquad g(x+y) \leqq K\big(g(x) + g(y)\big),$$

and,

$$\text{(3. 2)} \qquad g(x) = g(-x).$$

The following theorem then holds.

THEOREM 2. *Under the above conditions,*

$$\text{(3. 3)} \qquad \mathbf{E}\{\sup_n c_n g(S_n)\} \leqq 4K\left[\sum_{n=1}^{\infty} (c_n - c_{n+1}) \mathbf{E}\big(g(S_n)\big) + \limsup_n c_n \mathbf{E}\big(g(S_n)\big)\right].$$

If the X_i are symmetric about 0 *the constant* $4K$ *may be replaced by* 2 *and conditions* (3. 1) *and* (3. 2) *may be dropped.*

PROOF. If the X_i are symmetric about 0 let F_n be the distribution function of $\max_{1 \leq k \leq n} c_k g(S_k)$ and F_n^* be the distribution function of G_n. From theorem 1 we have,

$$\text{(3. 4)} \qquad \mathbf{E}(\sup_{1 \leq k \leq n} c_k g(S_k)) = \int_0^\infty (1-F_n(t))\,dt \leq 2\int_0^\infty (1-F_n^*(t))\,dt =$$

$$= 2\mathbf{E}(G_n) = 2\left\{\sum_{k=1}^n (c_k - c_{k+1})\mathbf{E}(g(S_k)) + c_n \mathbf{E}(g(S_n))\right\}.$$

The symmetric case follows. In general let $X_1', X_2', \ldots$ be a sequence of random variables independent of each other and the X_i such that X_i and X_i' are identically distributed. As usual let $X_i^{(s)} = X_i - X_i'$, $S_i^{(s)} = \sum_{j=1}^i X_j^{(s)}$ be the symmetrized random variables and symmetrized sums. Then we have shown that,

$$\text{(3. 5)} \quad \mathbf{E}(\sup_n c_n g(S_n^{(s)})) \leq 2\left\{\sum_{n=1}^\infty (c_n - c_{n+1})\mathbf{E}[g(S_n^{(s)})] + \limsup_n c_n \mathbf{E}[g(S_n^{(s)})]\right\} \leq$$

$$\leq 4K\left\{\sum_{n=1}^\infty (c_n - c_{n+1})\mathbf{E}(g(S_n)) + \limsup_n c_n \mathbf{E}[g(S_n)]\right\}$$

by (3. 1), (3. 2) and the construction of the symmetrized sums. Finally,

$$\text{(3. 6)} \qquad \mathbf{E}[\sup_n c_n g(S_n^{(s)})] = \mathbf{E}\{\mathbf{E}(\sup_n c_n g(S_n^{(s)}) | X_1, X_2, \ldots)\} \geq$$

$$\geq \mathbf{E}\{\sup_n c_n \mathbf{E}(g(S_n^{(s)}) | X_1, X_2, \ldots)\} \geq \mathbf{E}\{\sup_n c_n g(\mathbf{E}(S_n^{(s)} | X_1, \ldots, X_n, \ldots))\} =$$

$$= \mathbf{E}\{\sup_n c_n g(S_n)\}.$$

The theorem is proved.

If we specialize to $g(t) = |t|^\alpha$, $\alpha \geq 1$, we can take $K = \max(2^{\alpha-1}, 1)$ and obtain

$$\text{(3. 7)} \quad \mathbf{E}(\sup_n c_n |S_n|^\alpha) \leq \max(2^{\alpha+1}, 4) \sum_{n=1}^\infty (c_n - c_{n+1})\mathbf{E}|S_n|^\alpha + \limsup_n c_n \mathbf{E}|S_n|^\alpha.$$

If $1 < \alpha < \infty$, MARCINKIEWICZ and ZYGMUND [10] showed that under our standard assumptions on the X_i,

$$\text{(3. 8)} \qquad \mathbf{E}|S_n|^\alpha \leq B_\alpha \mathbf{E}\left(\sum_{i=1}^n X_i^2\right)^{\frac{\alpha}{2}}$$

where B_α depends on α only. Applying a generalization of the c_α inequality, [9], p. 155 we have for $\alpha \geq 1$,

$$\text{(3. 9)} \qquad \mathbf{E}\left(\sum_{i=1}^n X_i^2\right)^{\frac{\alpha}{2}} \leq \max(1, n^{\frac{\alpha}{2}-1}) \sum_{i=1}^n \mathbf{E}|X_i|^\alpha.$$

Combining (3. 8) and (3. 9) we get if $\alpha \geqq 1$,

$$\mathbf{E}|S_n|^\alpha \leqq B_\alpha \max(1, n^{\frac{\alpha}{2}-1}) \sum_{i=1}^{n} \mathbf{E}|X_i|^\alpha. \tag{3. 10}$$

(Note that the inequality of v. BAHR—ESSEN [1] shows that in (3. 10) for $1 \leqq \alpha \leqq 2$, B_α may be taken equal to 1.)

We can now prove,

THEOREM 3. *Suppose the X_i are independent.*
If $1 \leqq \alpha \leqq 2$, and $c_n \downarrow 0$, then,

$$\mathbf{E}(\sup_n c_n |S_n|^\alpha) \leqq C_\alpha \sum_{n=1}^{\infty} c_n \mathbf{E}|X_n|^\alpha \tag{3. 11}$$

where C_α is a constant depending on α only.
If $\alpha \geqq 2$, $c_n \downarrow 0$ and $n^{\alpha/2-1} c_n \to 0$, then,

$$\mathbf{E}(\sup_n c_n |S_n|^\alpha) \leqq \tag{3. 12}$$

$$\leqq C_\alpha \sum_{n=1}^{\infty} n^{\frac{\alpha}{2}-1} c_n \mathbf{E}|X_n|^\alpha + \sum_{n=1}^{\infty} \left[\sum_{j=n+1}^{\infty} c_j [j^{\frac{\alpha}{2}-1} - (j-1)^{\frac{\alpha}{2}-1}] \right] \mathbf{E}|X_n|^\alpha$$

where C_α depends on α only.

PROOF. Substituting (3. 10) in (3. 7) we get for $1 \leqq \alpha \leqq 2$,

$$\mathbf{E}(\sup_n c_n |S_n|^\alpha) \leqq \tag{3. 13}$$

$$\leqq B_\alpha 2^{\alpha+1} \left\{ \sum_{n=1}^{\infty} (c_n - c_{n+1}) \left[\sum_{j=1}^{n} \mathbf{E}|X_j|^\alpha \right] + \limsup_n c_n \sum_{j=1}^{n} \mathbf{E}|X_j|^\alpha \right\} =$$

$$= B^\alpha 2^{\alpha+1} \left\{ \sum_{n=1}^{\infty} c_n \mathbf{E}|X_n|^\alpha + \limsup_n c_n \sum_{i=1}^{n} \mathbf{E}|X_i|^\alpha \right\}.$$

If the series on the right of (3. 13) diverges the inequality is trivial. On the other hand if it converges the Kronecker lemma [9], p. 238 shows that $\limsup_n c_n \sum_{i=1}^{n} \mathbf{E}|X_i|^\alpha = 0$ and (3. 11) follows. By the v. Bahr—Esseen inequality we may take $C_\alpha = 2^{\alpha+1}$.

By the same argument for $\alpha \geqq 2$

$$\mathbf{E}(\sup_n c_n |S_n|^\alpha) \leqq \tag{3. 14}$$

$$\leqq B_\alpha \left\{ \sum_{n=1}^{\infty} \left\{ \sum_{j=n}^{\infty} (c_j - c_{j+1}) j^{\frac{\alpha}{2}-1} \right\} \mathbf{E}|X_n|^\alpha + \limsup_n c_n n^{\frac{\alpha}{2}-1} \sum_{j=1}^{n} \mathbf{E}|X_j|^\alpha \right\}.$$

But

$$\sum_{j=n}^{\infty} (c_j - c_{j+1}) j^{\frac{\alpha}{2}-1} = c_n n^{\frac{\alpha}{2}-1} + \sum_{j=n+1}^{\infty} c_j (j^{\frac{\alpha}{2}-1} - (j-1)^{\frac{\alpha}{2}-1}). \tag{3. 15}$$

The conclusion (3. 12) again follows from the Kronecker lemma.

If $c_n = n^{-\alpha}$ and $1 \leqq \alpha \leqq 2$ our result is equivalent to theorem 7 of MARCINKIEWICZ and ZYGMUND [10].

If the X_i are identically distributed and $c_n \downarrow 0$ our argument shows that,

$$\mathbf{E}\left(\sup_n c_n |S_n|^\alpha\right) < \infty \tag{3. 16}$$

if,

$$\mathbf{E}|X_1|^\alpha < \infty, \tag{3. 17}$$

$$c_n = O\left(n^{-\frac{\alpha}{2}}\right) \tag{3. 18}$$

and

$$\sum_{n=1}^{\infty} c_n n^{\frac{\alpha}{2}-1} < \infty. \tag{3. 19}$$

TEICHER [11] has obtained the same conclusion for $\alpha = 2$ if (3. 17) is replaced by

$$\mathbf{E}(X_1^2 \log^+ X_1) < \infty, \tag{3. 20}$$

(3. 18) holds, and (3. 19) is replaced by,

$$\sum_{n=1}^{\infty} c_n^2 n < \infty. \tag{3. 21}$$

Of course (3. 20) is stronger than (3. 17) while (3. 21) is weaker than (3. 19). If $\alpha > 2$ and the X_i are identically distributed, Teicher's result is strictly stronger than ours since he needs only (3. 17), (3. 18) and

$$\sum_{n=1}^{\infty} n^{k-1} c_n^{\frac{2k}{\alpha}} < \infty \tag{3. 22}$$

where k is the greatest integer $\leqq \alpha$, which is always implied by (3. 19). On the other hand, his approach, which relies on the Wiener dominated ergodic theorem, does not generalize to the non-identically distributed case. A combination of the two methods may readily be shown to yield results stronger than either those of Teicher or those given here, if $\alpha > 2$, and $\mathbf{E}|X_i|^\alpha$ is independent of i. We shall not pursue this.

For the case of greatest current interest ([6])

$$c_n = n^{-\left(\frac{\alpha}{2}+\varepsilon\right)}$$

where $\varepsilon > 0$ and $\alpha \geqq 2$ our criteria are adequate.

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Note added in proof (24 February 1969). Professor R. Pyke has pointed out to me that the inequality (1. 5) generalizes to processes $\{x(t)\}$ with a. s. continuous sample functions and symmetric independent increments in a fashion analogous to BIRNBAUM and MARSHALL'S (*Ann. Math. Statist.*, **32** (1961)) generalization of the Kolmogorov inequality to continuous parameter martingales. It is also possible to generalize (1. 5) to variables $\{X_i\}$ taking their values in a linear topological space if the function g (now defined on the space in which the X_i take their values) is continuous.

An examination of the proof will show that all that is really needed is to apply the supporting hyperplane theorem to the set $\{(x, z): z \geqq g(x)\}$ in order to obtain an inequality of the form (2. 3),

$$g(x+y)-g(x) \geqq l(x)(y)$$

where $l(x)(\cdot)$ is the linear functional defining the supporting hyperplane. The proof then proceeds as before.

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ÜBER DIE METHODE DER RICHTENDEN FUNKTIONALE VON M. G. KREIN

Von

H. LANGER (Dresden)

In den letzten Jahren erschien eine Reihe von Arbeiten, in denen Ergebnisse der Theorie gewöhnlicher Differential- und Differenzengleichungen sowie Entwicklungssätze für positiv definite Kerne auf entsprechende Differential- oder Differenzengleichungen mit operatorwertigen Koeffizienten bzw. auf operatorwertige Kerne übertragen wurden (siehe z. B. [7], [6], [3], [4], [2]). Im Zusammenhang damit entsteht die Frage, ob auch die von M. G. Krein (siehe [5]) stammende allgemeine Methode der Entwicklung eines semidefiniten Skalarproduktes nach den richtenden Funktionalen eines symmetrischen Operators eine entsprechende Verallgemeinerung gestattet.

Wir zeigen in der vorliegenden Note, daß dies möglich ist (Satz 1) und betrachten anschließend drei Beispiele für die Anwendung des erhaltenen allgemeinen Entwicklungssatzes. Das erste Beispiel liefert einen Satz von J. M. BERESANSKIJ ([3], VII, Satz 2. 4) über die Existenz einer Spektralfunktion für symmetrische Differenzenoperatoren zweiter Ordnung mit operatorwertigen Koeffizienten, im zweiten Beispiel erhalten wir die Entwicklung eines operatorwertigen Kernes nach den Eigenfunktionen eines gewöhnlichen Differentialoperators (als Spezialfall ergibt sich daraus eine von M. L. GORBAČUK [4] angegebene Verallgemeinerung des Satzes von Bochner—Krein), und im dritten Beispiel zeigen wir die Existenz einer Spektralfunktion für eine kanonische Differentialgleichung im Hilbertraum. An die Stelle des endlichen Systems richtender Funktionale bei M. G. Krein tritt in allen diesen Beispielen ein „unendliches" System, das wir als sog. richtende Abbildung mit Werten in einem Hilbertraum auffassen.

1. Es sei H ein Hilbertraum mit dem Skalarprodukt (x, y); $\mathbf{V}$ bezeichne die Menge aller symmetrischen Operatorfunktionen $\mathbf{F}|\lambda \to F_\lambda$ von $\mathcal{R}$ in $[H]$* mit den Eigenschaften:

1) $F_0 = 0$;
2) F_λ ist nichtfallend bezüglich λ;
3) F_λ ist linksseitig stetig bezüglich λ.

Ein $\mathbf{F} \in \mathbf{V}$ definiert bekanntlich eine additive Mengenfunktion $\mathbf{F}|\Delta \to F(\Delta)$ auf dem von allen beschränkten Teilintervallen von $\mathcal{R}$ erzeugten Ring.

Im Folgenden sei Δ ein beschränktes Intervall der reellen Achse, $\bar{\Delta}$ dessen Abschließung und $\mathcal{Z}$ eine Zerlegung von Δ in endlich viele Teilintervalle Δ^σ, $\sigma = 1, 2, \ldots, r$; $\lim\limits_{\mathcal{Z}}$ bezeichne den Grenzwert des rechts daneben stehenden Ausdrucks, wenn $\mathcal{Z}$ den Filter aller solchen Zerlegungen durchläuft.

* $\mathcal{R}$ — Menge der reellen Zahlen; $[H]$ — Menge der beschränkten linearen Operatoren in H; $[H \to H_1]$ — Menge der beschränkten linearen Operatoren von H in H_1.

LEMMA 1. *Ist* $\mathbf{x}|\lambda \to x(\lambda)$ *eine auf* $\bar{\Delta}$ *holomorphe Funktion mit Werten in* H, φ *eine auf* $\bar{\Delta}$ *stetige komplexwertige Funktion, dann existiert der Grenzwert*

$$\lim_{\mathscr{Z}} \sum_{\sigma} \varphi(\lambda_\sigma) F(\Delta^\sigma) x(\lambda_\sigma) \quad (\lambda_\sigma \in \Delta^\sigma) \tag{1}$$

in der Normtopologie und ist unabhängig von der speziellen Wahl der Punkte λ_σ; *er werde mit* $\int_\Delta \varphi(\lambda) F(d\lambda) x(\lambda)$ *bezeichnet. Gestattet die Funktion* $\mathbf{x}$ *auf der Abschließung von* Δ *die absolut konvergente Darstellung* $x(\lambda) = \sum_{\nu=0}^{\infty} x_\nu (\lambda - \lambda_0)^\nu$ $(x_\nu \in H)$, *so gilt*

$$\int_\Delta \varphi(\lambda) F(d\lambda) x(\lambda) = \sum_{\nu=0}^{\infty} \Big(\int_\Delta \varphi(\lambda)(\lambda-\lambda_0)^\nu F(d\lambda)\Big) x_\nu;$$

die Reihe auf der rechten Seite konvergiert in der Normtopologie.

BEWEIS. Ohne Einschränkung der Allgemeinheit nehmen wir $0 \leqq F_\lambda \leqq I$ $(\lambda \in \Delta)$ an und setzen zunächst voraus, daß die Funktion $\mathbf{x}$ auf $\bar{\Delta}$ eine absolut konvergente Darstellung $x(\lambda) = \sum_{\nu=0}^{\infty} x_\nu (\lambda - \lambda_0)^\nu$ gestattet. Dann gilt

$$\lim_{\mathscr{Z}} \sum_{\sigma} \varphi(\lambda_\sigma) F(\Delta^\sigma) x(\lambda_\sigma) = \lim_{\mathscr{Z}} \sum_{\sigma} \varphi(\lambda_\sigma) F(\Delta^\sigma) \sum_{\nu=0}^{\infty} x_\nu (\lambda_\sigma - \lambda_0)^\nu =$$

$$= \lim_{\mathscr{Z}} \sum_{\nu=0}^{\infty} \Big(\sum_{\sigma} \varphi(\lambda_\sigma) F(\Delta^\sigma)(\lambda_\sigma - \lambda_0)^\nu\Big) x_\nu,$$

und es bleibt zu zeigen, daß dieser Limes existiert und gleich

$$\sum_{\nu=0}^{\infty} \lim_{\mathscr{Z}} \sum_{\sigma} \varphi(\lambda_\sigma)(\lambda_\sigma - \lambda_0)^\nu F(\Delta^\sigma) x_\nu = \sum_{\nu=0}^{\infty} \int_\Delta \varphi(\lambda)(\lambda-\lambda_0)^\nu F(d\lambda) x_\nu$$

ist. Wir setzen

$$a_{\nu;\mathscr{Z}} = \sum_{\sigma} \varphi(\lambda_\sigma)(\lambda_\sigma - \lambda_0)^\nu F(\Delta^\sigma) x_\nu; \quad a_\nu = \int_\Delta \varphi(\lambda)(\lambda - \lambda_0)^\nu F(d\lambda) x_\nu$$

und zeigen, daß zu jedem $\varepsilon > 0$ ein $N = N(\varepsilon)$ existiert, so daß $\sum_{\nu=N}^{\infty} \|a_{\nu;\mathscr{Z}}\| \leqq \varepsilon$ für alle Zerlegungen $\mathscr{Z}$ gilt. Dazu betrachten wir eine orthogonale Erweiterung $\tilde{\mathbf{F}}$ von $\mathbf{F}$ in einem Oberraum $\tilde{H}$ von H und bezeichnen die orthogonale Projektion von $\tilde{H}$ auf H mit P. Dann ergibt sich mit $\tau = \sup_{\lambda \in \Delta} |\lambda - \lambda_0|$:

$$\sum_{\nu=N}^{\infty} \|a_{\nu;\mathscr{Z}}\| = \sum_{\nu=N}^{\infty} \Big\|P \sum_{\sigma} (\lambda_\sigma - \lambda_0)^\nu \varphi(\lambda_\sigma) \tilde{F}(\Delta^\sigma) x_\nu\Big\| \leqq \sup_{\lambda \in \Delta} |\varphi(\lambda)| \sum_{\nu=N}^{\infty} \tau^\nu \|x_\nu\| \leqq \varepsilon$$

für hinreichend großes N und alle Zerlegungen $\mathscr{Z}$. Ist $\mathbf{x}$ eine beliebige holomorphe Funktion auf $\bar{\Delta}$, so gibt es eine Zerlegung $\mathscr{Z}$ von Δ, so daß für jedes Teilintervall Δ^σ dieser Zerlegung eine auf der Abschließung von Δ^σ absolut konvergente Darstellung $x(\lambda) = \sum_{\nu=0}^{\infty} x_\nu^{(\sigma)} (\lambda - \lambda^{(\sigma)})^\nu$ $(\lambda^{(\sigma)}, \lambda \in \Delta^\sigma)$ besteht. Daraus folgt leicht die Behauptung.

Für eine nur stetige Funktion **x** existiert der Grenzwert (1) bekanntlich i. a. nicht.

Die Beweise der folgenden beiden Lemmata verlaufen analog dem Beweis von Lemma 1 und seien dem Leser überlassen.

LEMMA 1'. *Unter den Voraussetzungen von Lemma* 1 *sei* **y** *eine auf* $\bar{\Delta}$ *holomorphe Funktion mit Werten in H. Dann existiert der Grenzwert*

$$\lim_{\mathfrak{T}} \sum_{\sigma} \varphi(\lambda_\sigma)\big(F(\Delta^\sigma)x(\lambda_\sigma), y(\lambda_\sigma)\big) \quad (\lambda_\sigma \in \Delta^\sigma)$$

und ist unabhängig von der speziellen Wahl der Punkte λ_σ; *er werde mit* $\int_\Delta \varphi(\lambda)\big(F(d\lambda)x(\lambda), y(\lambda)\big)$ *bezeichnet. Gestatten die Funktionen* **x** *und* **y** *auf der Abschließung von* Δ *absolut konvergente Darstellungen* $x(\lambda) = \sum_{\nu=0}^{\infty} x_\nu(\lambda-\lambda_0)^\nu$ *bzw.* $y(\lambda) = \sum_{\mu=0}^{\infty} y_\mu(\lambda-\lambda_0')^\mu$, *so gilt*

$$\int_\Delta \varphi(\lambda)\big(F(d\lambda)x(\lambda), y(\lambda)\big) = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} \int_\Delta (\lambda-\lambda_0)^\nu(\lambda-\lambda_0')^\mu \varphi(\lambda)\big(F(d\lambda)x_\nu, y_\mu\big);$$

die Reihe auf der rechten Seite konvergiert absolut.

LEMMA 1''. *Es seien* **A** *und* **B** *auf* $\bar{\Delta}$ *holomorphe Funktionen mit Werten in* $[H_1 \to H]$ *bzw.* $[H \to H_2]$ $(H_1, H_2$ — *Hilberträume*), φ *sei eine auf* $\bar{\Delta}$ *stetige komplexwertige Funktion. Dann existiert der Grenzwert*

$$\lim_{\mathfrak{T}} \sum_{\sigma} \varphi(\lambda_\sigma)B(\lambda_\sigma)F(\Delta^\sigma)A(\lambda_\sigma) \quad (\lambda_\sigma \in \Delta^\sigma)$$

in der Normtopologie von $[H_1 \to H_2]$ *und ist unabhängig von der speziellen Wahl der Punkte* λ_σ; *er werde mit* $\int_\Delta \varphi(\lambda)B(\lambda)F(d\lambda)A(\lambda)$ *bezeichnet. Gestatten die Funktionen* **A** *und* **B** *auf der Abschließung von* $\bar{\Delta}$ *absolut konvergente Darstellungen* $A(\lambda) = \sum_{\nu=0}^{\infty} A_\nu(\lambda-\lambda_0)^\nu$ *bzw.* $B(\lambda) = \sum_{\mu=0}^{\infty} B_\mu(\lambda-\lambda_0')^\mu\lambda$, *so gilt*

$$\int_\Delta \varphi(\lambda)B(\lambda)F(d\lambda)A(\lambda) = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} \int_\Delta (\lambda-\lambda_0)^\nu(\lambda-\lambda_0')^\mu B_\mu F(d\lambda)A_\nu =$$

$$= \sum_{\nu=0}^{\infty}\Big(\int_\Delta (\lambda-\lambda_0)^\nu \varphi(\lambda)B(\lambda)F(d\lambda)\Big)A_\nu = \sum_{\mu=0}^{\infty} B_\mu \int_\Delta (\lambda-\lambda_0')^\mu \varphi(\lambda)F(d\lambda)A(\lambda);$$

die Reihen konvergieren in der Normtopologie von $[H_1 \to H_2]$.

LEMMA 2. *Unter den Voraussetzungen von Lemma* 1 *seien* ϱ *und* $\varrho+\gamma$ $(\gamma \neq 0)$ *Punkte aus dem Inneren von* Δ *und*

$$\Delta_\varrho^\gamma = \begin{cases} (\varrho, \varrho+\gamma] & \textit{falls} \quad \gamma > 0, \\ [\varrho+\gamma, \varrho) & \textit{falls} \quad \gamma < 0. \end{cases}$$

Dann gilt

$$\lim_{\gamma\to 0}\frac{1}{\gamma}\int_{\Delta_\varrho^\gamma} F(d\lambda)(x(\lambda)-x(\varrho))=0$$

in der Normtopologie.

BEWEIS. Es sei wieder $0\leqq F_\lambda\leqq I$ $(\lambda\in\bar{\Delta})$. Wir wählen $\gamma_0>0$ so klein, daß $\mathbf{x}$ auf $[\varrho-\gamma_0,\ \varrho+\gamma_0]$ die Darstellung $x(\lambda)=\sum_{\nu=0}^{\infty}x_\nu(\lambda-\varrho)^\nu$ gestattet. Dann ergibt sich mit Lemma 1 für $|\gamma|<\gamma_0$

$$\left\|\frac{1}{\gamma}\int_{\Delta_\varrho^\gamma}F(d\lambda)(x(\lambda)-x(\varrho))\right\|\leqq\sum_{\nu=1}^{\infty}\left\|\int_{\Delta_\varrho^\gamma}\frac{(\lambda-\varrho)^\nu}{\gamma}F(d\lambda)x_\nu\right\|\leqq$$

$$\leqq\sum_{\nu=1}^{\infty}\left\|\int_{\Delta_\varrho^\gamma}\frac{(\lambda-\varrho)^\nu}{\gamma}\tilde{F}(d\lambda)\right\|\|\tilde{F}(\Delta_\varrho^\gamma)x_\nu\|\leqq\sum_{\nu=1}^{\infty}|\gamma|^{\nu-1}\|\tilde{F}(\Delta_\varrho^\gamma)x_\nu\|,$$

woraus leicht die Behauptung folgt.

Schließlich beweist man mit den üblichen Überlegungen noch das

LEMMA 3. *Es seien* $\mathbf{A}$ *und* $\mathbf{B}$ *auf* $\bar{\Delta}$ *holomorphe Funktionen mit Werten in* $[H_1\to H]$ *bzw.* $[H_2\to H_1]$. *Für ein beliebiges Teilintervall* Δ' *von* Δ *setzen wir*

$$G(\Delta')=\int_{\Delta'}A^*(\lambda)F(d\lambda)A(\lambda).$$

Dann wird durch $G_\lambda=G(\Delta\cap(-\infty,\lambda))$ $(\lambda\in\Delta)$ *auf* Δ *eine symmetrische nichtfallende Operatorfunktion mit Werten in* $[H_1]$ *definiert und es gilt*

$$\int_\Delta B^*(\lambda)G(d\lambda)B(\lambda)=\int_\Delta B^*(\lambda)A^*(\lambda)F(d\lambda)A(\lambda)B(\lambda).$$

Es sei wieder $\mathbf{F}\in\mathbf{V}$. Wir betrachten die lineare Hülle $\mathscr{L}$ aller Funktionen mit Werten in H der Gestalt $\chi_\Delta x$ ($x\in H$, χ_Δ-charakteristische Funktion des beschränkten Intervalls Δ) und setzen für $\mathbf{x},\mathbf{y}\in\mathscr{L}$

$$(\mathbf{x},\mathbf{y})_F=\int_{-\infty}^{\infty}(F(d\lambda)x(\lambda),y(\lambda)). \tag{2}$$

Durch Restklassenbildung nach $\mathscr{L}_0=\{\mathbf{x}\in\mathscr{L}:\ (\mathbf{x},\mathbf{x})_F=0\}$ und Vervollständigung des entstehenden Quotientenraumes bezüglich der vom Skalarprodukt (2) erzeugten Norm entsteht ein Hilbertraum, den wir mit $L_F^2(H)$ bezeichnen. Man sieht leicht, daß z. B. jede Funtion der Gestalt $\chi_\Delta\mathbf{x}$ mit einer auf $\bar{\Delta}$ holomorphen Funktion $\mathbf{x}$ mit Werten in H zu $L_F^2(H)$ gehört.

2. Es sei $\mathscr{H}$ ein linearer Raum mit einem positiv semidefiniten Skalarprodukt $[f,g]$ $(f,g\in\mathscr{H})$. Wir setzen $\mathscr{H}_0=\{f:[f,f]=0,\ f\in\mathscr{H}\}$, bilden den Faktorraum

$\mathscr{H}/\mathscr{H}_0$ (Elemente $\hat{f}, \hat{g}$ etc.) mit dem Skalarprodukt $[\hat{f}, \hat{g}]=[f, g]$ und vervollständigen diesen bezüglich der Norm $[\hat{f}, \hat{f}]^{\frac{1}{2}} = [f, f]^{\frac{1}{2}}$ $(f \in \hat{f},\, g \in \hat{g})$. Der entstehende Raum $\hat{\mathscr{H}}$ ist ein Hilbertraum; seine Elemente bezeichnen wir ebenfalls mit $\hat{f}, \hat{g}$ etc.; $\mathscr{H}/\mathscr{H}_0$ wird in natürlicher Weise mit einem Teil von $\hat{\mathscr{H}}$ identifiziert. Eine lineare Mannigfaltigkeit $\partial \subset \mathscr{H}$ heiße quasidicht in $\mathscr{H}$ bezüglich des Skalarproduktes $[f, g]$, wenn zu jedem $f \in \mathscr{H}$ eine Folge $(f_n) \subset \partial$ existiert mit $[f_n - f, f_n - f] \to 0$ $(n \to \infty)$. Genau dann ist $\partial \subset \mathscr{H}$ quasidicht in $\mathscr{H}$, wenn das Bild $\hat{\partial}$ von ∂ bezüglich des natürlichen Homomorphismus von $\mathscr{H}$ auf $\mathscr{H}/\mathscr{H}_0$ dicht ist in $\hat{\mathscr{H}}$. Eine Funktion **f**, erklärt auf einem Gebiet der komplexen Ebene mit Werten in $\mathscr{H}$, nennen wir holomorph, wenn $\hat{\mathbf{f}}|\lambda \to \hat{f}(\lambda) = \widehat{f(\lambda)}$ eine holomorphe Funktion (mit Werten in $\hat{\mathscr{H}}$) ist. Schließlich werde ein linearer Operator A, definiert auf einer linearen Mannigfaltigkeit $\partial(A) \subset \mathscr{H}$ mit Werten in $\mathscr{H}$, symmetrisch genannt, wenn

$$[Af, g] = [f, Ag] \qquad (f, g \in \partial(A))$$

gilt. Bekanntlich induziert ein symmetrischer Operator A in $\mathscr{H}$ mit einem quasidichten Definitionsbereich $\partial(A)$ einen symmetrischen Operator $\hat{A}$ in $\hat{\mathscr{H}}$, der sich stets zu einem selbstadjungierten Operator $\tilde{A}$ in einem Oberraum $\tilde{\mathscr{H}} \supset \hat{\mathscr{H}}$ erweitern läßt.

DEFINITION 1. Es sei A ein linearer Operator, definiert auf einer linearen Mannigfaltigkeit $\partial(A) \subset \mathscr{H}$ mit Werten in $\mathscr{H}$, H ein Hilbertraum mit dem Skalarprodukt (x, y) $(x, y \in H)$. Eine Abbildung $\Phi|(\lambda; f) \to \Phi_\lambda f$ von $\mathscr{R} \times \mathscr{H}$ in H heiße *richtend* für Operator A, wenn Folgendes gilt;

1) Für jedes feste $\lambda \in \mathscr{R}$ ist $\Phi_\lambda f$ linear in f;

2) für jedes feste $f \in \mathscr{H}$ ist $\Phi_\lambda f$ auf $\mathscr{R}$ holomorph in λ;

3) für $\lambda \in \mathscr{R}$ und $f \in \mathscr{H}$ hat die Gleichung $(A - \lambda I)g = f$ genau dann eine Lösung $g \in \partial(A)$, wenn $\Phi_\lambda f = 0$ gilt;

4) zu jedem kompakten Intervall Δ der reellen Achse gibt es eine Abbildung $\Psi^{(\Delta)}|(\lambda; x) \to \Psi_\lambda^{(\Delta)} x$ von $\Delta \times H$ in $\mathscr{H}$ mit den folgenden Eigenschaften:

(α) $\Psi_\lambda^{(\Delta)} x$ ist für jedes feste $\lambda \in \Delta$ linear in x und für jedes feste $x \in H$ holomorph in λ;

(β) $\Phi_\lambda \Psi_\lambda^{(\Delta)} x = x$ $(x \in H,\ \lambda \in \Delta)$;

(γ) der durch $\hat{\Psi}_\lambda^{(\Delta)} x = \widehat{\Psi_\lambda^{(\Delta)} x}$ definierte Operator $\hat{\Psi}_\lambda^{(\Delta)}$ von H in $\hat{\mathscr{H}}$ ist stetig.

(δ)* für beliebiges $\lambda_0 \in \mathscr{R}$ und $x \in H$ gibt es Elemente $f_\nu \in \mathscr{H}$ $(\nu = 0, 1, 2, \dots)$, so daß

$$\hat{\Psi}_\lambda^{(\Delta)} x = \sum_{\nu=0}^{\infty} (\lambda - \lambda_0)^\nu \hat{f}_\nu$$

gilt für alle hinreichend nahe bei λ_0 gelegenen λ.

Wir bemerken, daß der Operator $\hat{\Psi}_\lambda^{(\Delta)}$ nach einem bekannten Satz von N. Dunford auf Δ sogar bezüglich der Normtopologie von $[H \to \hat{\mathscr{H}}]$ holomorph ist in λ; daraus folgt, daß für eine auf Δ holomorphe Funktion $\mathbf{x}$ mit Werten in H auch $\lambda \to \hat{\Psi}_\lambda^{(\Delta)} x(\lambda)$ holomorph ist auf Δ.

* Bedingung 4δ) wird nur beim Beweis von Satz 3 benötigt.

Wie in [5] ergibt sich aus 1) und 3) unmittelbar

$$\Phi_\lambda(Ag) = \lambda\Phi_\lambda g \quad (\lambda\in\mathcal{R},\ g\in\partial(A)),$$

denn mit $f = Ag-\lambda g$ gilt $0 = \Phi_\lambda(f) = \Phi_\lambda(Ag)-\lambda\Phi_\lambda(g)$.

SATZ 1. *Es sei $\mathcal{H}$ ein linearer Raum mit einem positiv semidefiniten Skalarprodukt $[f,g]$ $(f,g\in\mathcal{H})$, A ein symmetrischer Operator in $\mathcal{H}$, definiert auf einem quasidichten Teil $\partial(A)\subset\mathcal{H}$. Der Operator A habe eine richtende Abbildung Φ mit Werten in einem Hilbertraum H. Dann existiert ein $\mathbf{F}\in\mathbf{V}$, so daß für $f,g\in\mathcal{H}$ gilt:*

$$(3)\qquad [f,g] = \int_{-\infty}^{\infty}(F(d\lambda)\Phi_\lambda f, \Phi_\lambda g) \quad \left(=\lim_{N,N'\to\infty}\int_{-N}^{N'}(F(d\lambda)\Phi_\lambda f,\Phi_\lambda g)\right).$$

BEWEIS. Wir betrachten den selbstadjungierten Operator $\tilde{A}$ in $\tilde{\mathcal{H}}$ (siehe die Bemerkungen vor Definition 1), $\tilde{\mathbf{E}}$ sei seine linksseitig stetige orthogonale Spektralfunktion, Δ_ϱ^γ $(\gamma\neq 0)$ habe die gleiche Bedeutung wie in Lemma 2.

Für ein Element $f\in\mathcal{H}$ mit $\Phi_\varrho f=0$ gilt

$$(4)\qquad \frac{1}{|\gamma|}\|\tilde{E}(\Delta_\varrho^\gamma)\hat{f}\| \to 0 \quad \text{für} \quad \gamma\to 0,$$

wenn $\|\tilde{h}\|$ die Norm des Elementes $\tilde{h}\in\tilde{\mathcal{H}}$ bezeichnet. In der Tat: Gemäß 3) gibt es ein $g\in\partial(A)$ mit $Ag-\varrho g = f$, also ist $\hat{A}\hat{g}-\varrho\hat{g} = \hat{f}$ und

$$\left[\tilde{E}(\Delta_\varrho^\gamma)\frac{\hat{f}}{\gamma},\frac{\hat{f}}{\gamma}\right] = \frac{1}{\gamma^2}\int_{\Delta_\varrho^\gamma}|\lambda-\varrho|^2[\tilde{E}(d\lambda)\hat{g},\hat{g}] \leqq [\tilde{E}(\Delta_\varrho^\gamma)\hat{g},\hat{g}] \to 0$$

für $\gamma\to 0$.

Wir wählen ein kompaktes Intervall Δ_0, das den Nullpunkt in seinem Inneren Δ_0^i enthält, und betrachten eine auf Δ_0 definierte Funktion $\mathbf{f}$ mit Werten in $\mathcal{H}$ und den folgenden Eigenschaften:

(i) $\mathbf{f}$ ist holomorph auf Δ_0;

(ii) Die Abbildung $\lambda\to\Phi_\lambda f(\lambda)$ ist holomorph auf Δ_0.

Für $\varrho\in\Delta_0^i$ gilt dann mit $f_\varrho = f(\varrho)-\Psi_\varrho^{(\Delta_0)}\Phi_\varrho f(\varrho)$ wegen 4ß)

$$\Phi_\varrho f_\varrho = \Phi_\varrho f(\varrho)-\Phi_\varrho\Psi_\varrho^{(\Delta_0)}\Phi_\varrho f(\varrho) = 0,$$

also besteht gemäß (4) die Beziehung

$$(5)\qquad \frac{1}{|\gamma|}\|\tilde{E}(\Delta_\varrho^\gamma)\hat{f}_\varrho\| \to 0 \quad \text{für} \quad \gamma\to 0.$$

Wir setzen jetzt

$$\varphi(\varrho) = \int_{\Delta_0^\varrho}\tilde{E}(d\lambda)\hat{f}(\lambda) - \int_{\Delta_0^\varrho}\tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0)}\Phi_\lambda f(\lambda).$$

Dann ergibt sich mit $\hat{g}_\varrho = \hat{\Psi}_\varrho^{(\Delta_0)} \Phi_\varrho f(\varrho)$:

$$\frac{\varphi(\varrho+\gamma)-\varphi(\varrho)}{\gamma} = \frac{1}{\gamma}\Big(\int_{\Delta_\varrho^\gamma} \tilde{E}(d\lambda)\hat{f}(\lambda) - \int_{\Delta_\varrho^\gamma} \tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0)}\Phi_\lambda f(\lambda)\Big) =$$

$$= \frac{1}{\gamma}\int_{\Delta_\varrho^\gamma} \tilde{E}(d\lambda)(\hat{f}(\lambda)-\hat{f}(\varrho)) + \frac{1}{\gamma}\int_{\Delta_\varrho^\gamma} \tilde{E}(d\lambda)(\hat{f}(\varrho)-\hat{\Psi}_\varrho^{(\Delta_0)}\Phi_\varrho f(\varrho)) +$$

$$+\frac{1}{\gamma}\int_{\Delta_\varrho^\gamma} \tilde{E}(d\lambda)(\hat{\Psi}_\varrho^{(\Delta_0)}\Phi_\varrho f(\varrho) - \hat{\Psi}_\lambda^{(\Delta_0)}\Phi_\lambda f(\lambda)) =$$

$$= \frac{1}{\gamma}\int_{\Delta_\varrho^\gamma} \tilde{E}(d\lambda)(\hat{f}(\lambda)-\hat{f}(\varrho)) + \frac{1}{\gamma}\tilde{E}(\Delta_\varrho^\gamma)\hat{f}_\varrho + \frac{1}{\gamma}\int_{\Delta_\varrho^\gamma} \tilde{E}(d\lambda)(\hat{g}_\varrho - \hat{g}_\lambda).$$

Für $\gamma \to 0$ streben alle Summanden der rechten Seite gegen 0: Der erste und dritte Summand auf Grund von Lemma 2, der zweite Summand wegen (5). Also ist $\varphi(\varrho)=0$ für alle $\varrho \in \Delta_0^i$, d. h., es gilt für ein beliebiges Intervall $\Delta \subset \Delta_0^i$

$$\int_\Delta \tilde{E}(d\lambda)\hat{f}(\lambda) = \int_\Delta \tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0)}\Phi_\lambda f(\lambda), \tag{6}$$

also inbesondere für $f(\lambda)=f \quad (\lambda \in \Delta_0)$

$$\tilde{E}(\Delta)\hat{f} = \int_\Delta \tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0)}\Phi_\lambda f.$$

Daraus folgt für $f, g \in \mathscr{H}$

$$[\tilde{E}(\Delta)\hat{f}, \hat{g}] = [\tilde{E}(\Delta)\hat{f}, \tilde{E}(\Delta)\hat{g}] = \Big[\int_\Delta \tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0)}\Phi_\lambda f, \int_\Delta \tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0)}\Phi_\lambda g\Big] =$$

$$= \int_\Delta [\tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0)}\Phi_\lambda f, \hat{\Psi}_\lambda^{(\Delta_0)}\Phi_\lambda g] = \int_\Delta (\hat{\Psi}_\lambda^{(\Delta_0)*}\tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0)}\Phi_\lambda f, \Phi_\lambda g),$$

wobei wir $\hat{\Psi}_\lambda^{(\Delta_0)}$ als Operator von H in $\tilde{\mathscr{H}}$ auffassen. Für beliebiges $\lambda \in \Delta_0^i$ definieren wir Operatoren F_λ in H durch

$$F_\lambda = \operatorname{sgn}\lambda \int_{\tilde{\Delta}_0^\lambda} \hat{\Psi}_\tau^{(\Delta_0)*}\tilde{E}(d\tau)\hat{\Psi}_\tau^{(\Delta_0)} \qquad \left(\tilde{\Delta}_0^\lambda = \begin{cases}[0,\lambda) & \text{für } \lambda > 0\\ [\lambda, 0) & \text{für } \lambda < 0\end{cases}\right), \tag{7}$$

$$F_0 = 0.$$

Gemäß Lemma 3 gilt dann

$$[\tilde{E}(\Delta)\hat{f}, \hat{g}] = \int_\Delta (F(d\lambda)\Phi_\lambda f, \Phi_\lambda g). \tag{8}$$

Wir zeigen, daß F_λ unabhängig ist von der speziellen Wahl des Intervalls Δ_0 und der Abbildung $\Psi^{(\Delta_0)}$. Sei $\mathbf{F}'$ die mit Hilfe eines Intervalls $\Delta_0' \supset \Delta_0$ und einer Abbildung $\Psi^{(\Delta_0')}$ entsprechend gebildete Operatorfunktion. Dann gilt für $x, y \in H$

$$(F'(\Delta)x, y) = \int_\Delta (\hat{\Psi}_\lambda^{(\Delta_0')*}\tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0')}x, y) = \Big[\int_\Delta \tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0')}x, \int_\Delta \tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0')}y\Big]. \tag{9}$$

Die Abbildung $\lambda \to \Psi_\lambda^{(\Delta'_0)} x$ hat die Eigenschaften (i) und (ii). Folglich gilt gemäß (6) und 4ß)

$$\int_\Delta \tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta'_0)} x = \int_\Delta \tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0)} \Phi_\lambda \hat{\Psi}_\lambda^{(\Delta'_0)} x = \int_\Delta \tilde{E}(d\lambda)\hat{\Psi}_\lambda^{(\Delta_0)} x$$

sowie eine entsprechende Beziehung mit y an Stelle von x, also ist wegen (9)

$$\big(F'(\Delta)x, y\big) = \big(F(\Delta)x, y\big) \quad (x, y \in H).$$

Aus (8) ergibt sich jetzt

$$[f, g] = [\hat{f}, \hat{g}] = \lim_{N, N' \to \infty} \int_{-N}^{N'} \big(F(d\lambda)\Phi_\lambda f, \Phi_\lambda g\big) = \int_{-\infty}^{\infty} \big(F(d\lambda)\Phi_\lambda f, \Phi_\lambda g\big),$$

womit Satz 1 bewiesen ist.

Wir sagen im Folgenden kurz, die durch (7) definierte Funktion $\mathbf{F} \in \mathbf{V}$ werde von dem selbstadjungierten Operator $\tilde{A}$ erzeugt.

Man zeigt jetzt ohne Schwierigkeit (vgl. [5]), daß eine eineindeutige Beziehung zwischen den verallgemeinerten Spektralfunktionen $\hat{\mathbf{E}}$ (oder den verallgemeinerten Resolventen $\hat{\mathbf{R}}$) von $\hat{A}$ in $\hat{\mathscr{H}}$ einerseits und den Funktionen $\mathbf{F} \in \mathbf{V}$ aus Satz 1 andererseits besteht, die durch folgende Gleichungen gegeben ist ($f, g \in \mathscr{H}$):

$$[\hat{E}(\Delta)\hat{f}, \hat{g}] = \int_\Delta \big(F(d\lambda)\Phi_\lambda f, \Phi_\lambda g\big); \tag{10}$$

$$[\hat{R}_z \hat{f}, \hat{g}] = \int_{-\infty}^{\infty} \frac{1}{\lambda - z}\big(F(d\lambda)\Phi_\lambda f, \Phi_\lambda g\big) \quad (\operatorname{Im} z \neq 0).$$

Damit folgt aus [1], S. 112, Satz 2 der

SATZ 2. *Die Funktion* $\mathbf{F} \in \mathbf{V}$ *aus Satz* 1 *ist genau dann eindeuting bestimmt, wenn der Operator* $\hat{A}$ *in* $\hat{\mathscr{H}}$ *maximal symmetrisch ist.*

Es seien jetzt die Voraussetzungen von Satz 1 erfüllt. Wir betrachten den Raum $L_F^2(H)$ für die dort erhaltene Funktion $\mathbf{F} \in \mathbf{V}$. Beziehung (3) besagt dann, daß die Funktionen $\Phi_\lambda f\, (f \in \mathscr{H})$ zu $L_F^2(H)$ gehören und $\hat{\mathscr{H}}$ der Abschließung $\bar{L}_0$ der linearen Hülle L_0 aller dieser Funktionen in $L_F^2(H)$ isomorph ist. Bei dieser Isomorphie $f \to \Phi_\lambda f$ geht $\hat{A}$ über in eine gewisse Einschränkung A_0 des Operators der Multiplikation mit der unabhängigen Veränderlichen in $L_F^2(H)$ und die verallgemeinerte Spektralfunktion $\hat{\mathbf{E}}$ in eine verallgemeinerte Spektralfunktion $\hat{\mathbf{E}}_0$ in $\bar{L}_0$.

SATZ 3. *Unter den Voraussetzungen von Satz* 1 *liegt die lineare Hülle* L_0 *der Funktionen* $\Phi_\lambda f\, (f \in \mathscr{H})$ *genau dann dicht in* $L_F^2(H)$, *wenn die zu* $\mathbf{F}$ *gemäß* (10) *gehörige verallgemeinerte Spektralfunktion* $\hat{\mathbf{E}}$ *in* $\hat{\mathscr{H}}$ *sogar orthogonal ist, d. h., wenn* $\hat{A}$ *selbstadjungierte Erweiterungen in* $\hat{\mathscr{H}}$ *besitzt und* $\mathbf{F}$ *durch eine solche selbstadjungierte Erweiterung erzeugt wird.*

BEWEIS. Die Spektralfunktion $\hat{\mathbf{E}}$ in $\hat{\mathscr{H}}$ ist genau dann orthogonal, wenn $\hat{\mathbf{E}}_0$ in $\bar{L}_0$ orthogonal ist.

Mit $\mathbf{E}$ bezeichnen wir die in $L_F^2(H)$ durch die Gleichung

(11) $$(E(\Delta)\mathbf{x}, \mathbf{y})_F = \int_\Delta (F(d\lambda)x(\lambda), y(\lambda)) \quad (\mathbf{x}, \mathbf{y} \in \mathscr{L})$$

definierte orthogonale Spektralfunktion. Für $\mathbf{x}, \mathbf{y} \in L_0$, $x(\lambda) = \Phi_\lambda f$, $y(\lambda) = \Phi_\lambda g$ gilt wegen (10) andererseits

$$(\hat{E}_0(\Delta)x, y)_F = \int_\Delta (F(d\lambda)x(\lambda), y(\lambda)),$$

woraus sich mit (11) die Beziehung $\hat{\mathbf{E}}_0 = P_0\mathbf{E}$ ergibt, wenn P_0 die orthogonale Projektion von $L_F^2(H)$ auf $\bar{L}_0$ bezeichnet.

Gilt $\bar{L}_0 = L_F^2(H)$, dann ist P_0 die identische Abbildung und $\hat{\mathbf{E}}_0 = \mathbf{E}$.

Es sei jetzt umgekehrt $\hat{\mathbf{E}}_0$ eine orthogonale Spektralfunktion. Für $\mathbf{x}, \mathbf{y} \in L_0$ und ein beliebiges beschränktes Intervall Δ gilt dann mit $\Delta' = \mathscr{R} \setminus \Delta$ und $\mathbf{x}_\Delta = \chi_\Delta \mathbf{x}$:

(12) $$\|\hat{\mathbf{E}}_0(\Delta)\mathbf{x} - \mathbf{y}\|_F^2 = (\hat{E}_0(\Delta)(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y})_F + (\hat{E}_0(\Delta')\mathbf{y}, \mathbf{y})_F =$$
$$= \int_\Delta (F(d\lambda)(x(\lambda)-y(\lambda)), x(\lambda)-y(\lambda)) + \int_{\Delta'} (F(d\lambda)y(\lambda), y(\lambda)) =$$
$$= \int_{-\infty}^{\infty} (F(d\lambda)(x_\Delta(\lambda)-y(\lambda)), x_\Delta(\lambda)-y(\lambda)).$$

Aus $\mathbf{x} \in L_0$ folgt $\hat{\mathbf{E}}_0(\Delta)\mathbf{x} \in \bar{L}_0$, also gibt es zu vorgegebenem $\varepsilon > 0$ ein $\mathbf{y} \in L_0$ mit $\|\hat{\mathbf{E}}_0(\Delta)\mathbf{x} - \mathbf{y}\|_F \leqq \varepsilon$, d. h. wegen (12), es ist $\mathbf{x}_\Delta \in \bar{L}_0$. Für eine Zerlegung $\mathscr{Z}$ von Δ und jede natürliche Zahl n gilt dann auch

$$\sum_{\sigma=1}^{r} \lambda_\sigma^n \mathbf{x}_{\Delta^\sigma} \in \overline{L_0} \quad (\lambda_\sigma \in \Delta^\sigma),$$

woraus leicht $p\mathbf{x}_\Delta \in \bar{L}_0$ folgt für ein beliebiges Polynom p.

Es sei jetzt $\mathbf{f}$ eine auf $\mathscr{R}$ definierte Funktion mit Werten in $\mathscr{H}$ und den folgenden Eigenschaften:

(i) $\hat{\mathbf{f}}$ ist holomorph auf $\mathscr{R}$;

(ii) die Abbildung $\lambda \to \Phi_\lambda f(\lambda)$ ist holomorph auf $\mathscr{R}$;

(iii) für beliebiges $\lambda_0 \in \mathscr{R}$ gibt es Elemente $f_\nu \in \mathscr{H}$ $(\nu = 0, 1, 2, \ldots)$, so daß

(13) $$\hat{f}(\lambda) = \sum_{\nu=0}^{\infty} (\lambda - \lambda_0)\hat{f}_\nu$$

gilt für alle hinreichend nahe bei λ_0 gelegenen λ. Wir zeigen, daß dann auch $(\Phi_\lambda f(\lambda))_\Delta$ zu $\bar{L}_0$ gehört. Auf der Abschließung von Δ bestehe eine Darstellung (13). Mit $f^{(N)}(\lambda) = \sum_{\nu=0}^{N} (\lambda - \lambda_0)\hat{f}_\nu$ folgt dann bei Beachtung von (8)

$$\int_\Delta (F(dt)(\Phi_\lambda f^{(N)}(\lambda) - \Phi_\lambda f(\lambda)), \Phi_\lambda f^{(N)}(\lambda) - \Phi_\lambda f(\lambda)) =$$
$$= \int_\Delta [\tilde{E}(d\lambda)(\hat{f}^{(N)}(\lambda) - \hat{f}(\lambda)), \hat{f}^{(N)}(\lambda) - \hat{f}(\lambda)] \to 0 \quad \text{für} \quad N \to \infty.$$

Da nach den vorangehenden Überlegungen aber $(\Phi_\lambda f^{(N)}(\lambda))_\Delta$ zu $\bar{L}_0$ gehört, ergibt sich $(\Phi_\lambda f(\lambda))_\Delta \in \bar{L}_0$. Setzen wir in dieser Beziehung speziell $f(\lambda)=\Psi_\lambda x$ für $x \in H$, so folgt $(\Phi_\lambda f(\lambda))_\Delta = \chi_\Delta\, x \in \bar{L}_0$, also muß notwendig $\bar{L}_0 = L_F^2(H)$ sein.

Der Satz 3 ist anscheinend neu auch für den Fall eines endlichen Systems von mehr als einem richtenden Funktional im Sinne von M. G. KREIN [5].

3. Als erste Anwendung (vgl. [3], VII, § 2) betrachten wir einen Hilbertraum H und im Raume $\mathscr{H}$ aller finiten Folgen $f=(u_j)$, $g=(v_j)$ von Elementen aus H mit dem Skalarprodukt $[f, g] = \sum_{j=0}^{\infty} (u_j, v_j)$ den Operator

$$A = \begin{pmatrix} B_0 & A_0 & 0 & 0 & \dots \\ A_0 & B_1 & A_1 & 0 & \dots \\ 0 & A_1 & B_2 & A_2 & \dots \\ 0 & 0 & A_2 & B_3 & \dots \\ . & . & . & . & \dots \end{pmatrix};$$

dabei seien die A_j und B_j beschränkte selbstadjungierte und die A_j außerdem beschränkt invertierbare Operatoren in H $(j=0, 1, 2, \dots)$. Wir definieren die zugehörigen Polynome erster Art durch die Gleichungen

$$P_0(\lambda)=I,\quad P_1(\lambda) = \overset{-1}{A}_0(\lambda I - B_0),\quad P_{k+1}(\lambda) = \overset{-1}{A}_k(\lambda I - B_k)P_k(\lambda) - \overset{-1}{A}_k A_{k-1} P_{k-1}(\lambda)$$
$$(k=1, 2, \dots).$$

Die Abbildung

$$\Phi \,|\, f \to \Phi_\lambda f = \sum_{j=0}^{\infty} P_j^*(\lambda) x_j, \quad f = (x_j) \in \mathscr{H},$$

von $\mathscr{H}$ in H ist richtend für den Operator A. In der Tat: Die Bedingungen 1) und 2) von Definition 1 sind offensichtlich erfüllt, und der durch die Gleichung

$$\Psi_\lambda^{(\Delta)} x = (x_j) \quad \text{mit} \quad x_0 = x,\ x_j = 0 \quad (j \geqq 1)$$

definierte Operator von H in $\mathscr{H}$ hat die Eigenschaften 4α)—4δ). Wir betrachten die Gleichung

(14) $$(A - \lambda I)g = f$$

im Raume $\mathscr{H}$. Es sei $f=(u_j)$, $g=(v_j)$. Gilt $v_j=0$ für $j>j_0$, so folgt $u_j=0$ für $j > j_0+1$. Ist umgekehrt $u_j=0$ für $j > j_0+1$, so ergibt sich aus $g \in H$ und der Invertierbarkeit von A_j leicht $v_j=0$ für $j>j_0$. Deshalb bleibt an Stelle von (14) nur die Gleichung

(15) $$A^{(0)} g^{(0)} - \lambda g^{(0)} = f^{(0)}$$

mit $f^{(0)} = (u_j)_{j=0,1,\dots,j_0+1}$, $g^{(0)} = (v_j)_{j=0,1,\dots,j_0+1}$,

$$A^{(0)} = \begin{pmatrix} B_0 & A_0 & 0 & . & . & . & 0 & 0 \\ A_0 & B_1 & A_1 & . & . & . & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & . & . & . & B_{j_0} & 0 \\ 0 & 0 & 0 & . & . & . & A_{j_0} & 0 \end{pmatrix}$$

zu betrachten.

Man sieht leicht, daß der Wertebereich von $A^{(0)}-\lambda I^{(0)}$ abgeschlossen ist. Deshalb hat (15) genau dann eine Lösung $g^{(0)}$, wenn $f^{(0)}$ auf allen Lösungen der adjungierten homogenen Gleichung orthogonal ist, d. h., falls $(f^{(0)}, h^{(0)})=0$ für alle Lösungen $h^{(0)}$ der Gleichung $(A^{(0)*}-\lambda I^{(0)})h^{(0)} = 0$ gilt. Diese Lösungen $h^{(0)}$ haben aber die Gestalt

$$h^{(0)} = (w_j)_{j=0,1,\ldots,j_0} \quad \text{mit} \quad w_j = P_j(\lambda)w_0, \quad w_0 \in H \text{ beliebig.}$$

Daraus folgt, daß die Gleichung (14) genau dann eine Lösung in $\mathscr{H}$ hat, falls $\Phi_\lambda f=0$ gilt, also genügt Φ auch der Bedingung 3) von Definition 1.

Aus Satz 1 ergibt sich jetzt die Existenz einer Operatorfunktion $\mathbf{F}\in\mathbf{V}$ in H, so daß für $f, g\in\mathscr{H}$ gilt:

$$\sum_{j=0}^{\infty}(u_j, v_j) = \int_{-\infty}^{\infty}\left(F(dt)\sum_{j=0}^{\infty}P_j^*(t)u_j, \sum_{k=0}^{\infty}P_k^*(t)v_k\right) \quad (f=(u_j), g=(v_j)).$$

Diese Beziehung ist aber äquivalent der folgenden:

$$\int_{-\infty}^{\infty}P_k(t)F(dt)P_j^*(t) = \delta_{jk}I.$$

Dabei ist $\mathbf{F}\in\mathbf{V}$ genau dann eindeutig bestimmt, wenn der Operator A in der Vervollständigung $l^2(H)$ von $\mathscr{H}$ maximal symmetrisch ist; die lineare Hülle von $P_j^*(\lambda)x$ $(j=0, 1, 2, \ldots;\ x\in H)$ ist genau dann dicht in $L_F^2(H)$, wenn A selbstadjungierte Erweiterungen in $l^2(H)$ besitzt und $\mathbf{F}$ durch eine solche selbstadjungierte Erweiterung erzeugt wird.

4. Es seien $\mathfrak{H}$ ein separabler Hilbertraum mit dem Skalarprodukt (u, v), $C_0(\mathfrak{H}; (a, b))$, $-\infty\leqq a<0<b\leqq\infty$, der Raum der stetigen Funktionen, definiert auf (a, b) mit Werten in $\mathfrak{H}$, die bei $t=a$ und $t=b$ identisch verschwinden, $\mathbf{A}_k$ $(k=0, 1, \ldots, r)$ auf (a, b) definierte stetige Funktionen mit Werten in $[\mathfrak{H}]$, die k-mal stetig differenzierbar sind und für die $A_r(s)$ $(s\in(a, b))$ beschränkt invertierbar ist.

Wir betrachten den Differentialoperator A:

$$(Au)(t) = \sum_{k=0}^{r}A_k(t)\frac{d^k u(t)}{dt^k}, \tag{16}$$

definiert auf der Menge $C_0^{(r)}(\mathfrak{H}; (a, b))$ aller r-mal stetig differenzierbaren Funktionen $\mathbf{u}\in C_0(\mathfrak{H}; (a, b))$. Mit Y_j bezeichnen wir die Lösung der zum adjungierten Differentialausdruck von (16) gehörigen Differentialgleichung in $[\mathfrak{H}]$:

$$\sum_{k=0}^{r}(-1)^k\frac{d^k}{dt^k}(Y(t;\lambda)A_k(t))-\lambda Y(t;\lambda) = 0$$

mit der Anfangsbedingung $Y_j^{(l)}(0;\lambda)=\delta_{jl}I$ $(j, l = 0, 1, \ldots r-1)$.

Es sei Weiter $\mathbf{K}|(s, t)\to K(s, t)$ ein auf $(a, b)\times(a, b)$ definierter Kern, dessen Werte beschränkte lineare Operatoren in $\mathfrak{H}$ sind und der in der schwachen Operatorentopologie auf $(a, b)\times(a, b)$ stetig von (s, t) abhängt. Außerdem sei $\mathbf{K}$ positiv semidefinit, d.h., es gelte

$$\int_a^b\int_a^b(K(s, t)u(s), u(t))\,ds\,dt \geqq 0 \quad (\mathbf{u}\in C_0(\mathfrak{H}; (a, b))).$$

Den Raum $C_0(\mathfrak{H};(a,b))$, versehen mit dem positiv semidefiniten Skalarprodukt

$$(17)\qquad [\mathbf{u},\mathbf{v}]=\int_a^b\int_a^b (K(s,t)u(s),v(t))\,ds\,dt \quad (\mathbf{u},\mathbf{v}\in C_0(\mathfrak{H};(a,b))),$$

bezeichnen wir mit $\mathscr{H}$. Man sieht leicht, daß $C_0^{(r)}(\mathfrak{H};(a,b))$ bezüglich des Skalarproduktes (17) in $\mathscr{H}$ quasidicht ist.

Zwischen dem Differentialoperator (16) und dem Kern $\mathbf{K}$ bestehe jetzt die Beziehung

$$[A\mathbf{u},\mathbf{v}]=[\mathbf{u},A\mathbf{v}] \quad (\mathbf{u},\mathbf{v}\in C_0^{(r)}(\mathfrak{H};(a,b))),$$

d.h., es gelte im Sinne verallgemeinerter Ableitungen

$$\sum_{k=0}^{r}(-1)^k\frac{d^k}{ds^k}(K(s,t)A_k(s))=\sum_{k=0}^{r}(-1)^k\frac{d^k}{dt^k}(A_k^*(t)K(s,t)).$$

Der Ausdruck

$$\Phi_\lambda\mathbf{f}=\begin{pmatrix}\int_a^b Y_0(t;\lambda)f(t)\,dt\\ \int_a^b Y_1(t;\lambda)f(t)\,dt\\ \vdots\\ \int_a^b Y_{r-1}(t;\lambda)f(t)\,dt\end{pmatrix},\quad \mathbf{f}\in C_0(\mathfrak{H};(a,b))$$

definiert eine richtende Abbildung Φ für den Operator A in $\mathscr{H}$; die Werte von Φ betrachten wir als Elemente des Raumes

$$H=\mathfrak{H}_1\oplus\mathfrak{H}_2\oplus\cdots\oplus\mathfrak{H}_r,\quad \mathfrak{H}_1=\mathfrak{H}_2=\cdots=\mathfrak{H}_r=\mathfrak{H}.$$

Offensichtlich hat Φ die Eigenschaften 1) und 2) aus Definition 1. Von Eigenschaft 3) kann man sich z. B. durch Zurückführung der Differentialgleichung

$$\sum_{k=0}^{r}A_k(t)\frac{d^k u(t)}{dt^k}-\lambda u(t)=f(t)$$

auf eine Differentialgleichung erster Ordnung im Raume $C_0(H;(a,b))$ überzeugen. Die Existenz der in Bedingung 4) geforderten Abbildung $\Psi^{(\Delta)}$ ergibt sich folgendermaßen. Wir wählen eine δ-Folge (δ_n) nichtnegativer, r-mal differenzierbarer Funktionen, deren Träger sich auf den Nullpunkt zusammenziehen. Dann gilt in $[\mathfrak{H}]$

$$\int_a^b Y_j^{(l)}(t;\lambda)\delta_n(t-t_0)\,dt\to\delta_{jl}I\quad(n\to\infty),$$

gleichmäßig für $\lambda\in\Delta$ (Δ—beliebiges beschränktes Intervall). Das folgt leicht aus der Tatsache, daß $Y_j^{(l)}(t;\lambda)$ auf $(a,b)\times\mathscr{R}$ stetig von $(t;\lambda)$ abhängt. Wir setzen $A_{j,l}^{(n)}(\lambda) = (-1)^l \int_a^b Y_j^{(l)}(t;\lambda)\delta_n(t-t_0)\,dt$ und bilden die Matrix

$$\mathscr{A}_\Delta^{(n)}(\lambda) = \left(A_{jl}^{(n)}(\lambda)\right).$$

Für hinreichend großes n ist $\mathscr{A}_\Delta^{(n)}(\lambda)$ $(\lambda\in\Delta)$ sicher invertierbar; wir wählen ein solches $n=n_0$ und setzen $\mathscr{A}_\Delta(\lambda)=\mathscr{A}_\Delta^{(n_0)}(\lambda)$. Für $x\in H$, $x=\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_r\end{pmatrix}$, hat dann

$$\Psi_\lambda^{(\Delta)}x = \left(\delta_{n_0}(t-t_0),\ \delta'_{n_0}(t-t_0),\ \dots,\ \delta_{n_0}^{(r-1)}(t-t_0)\right)\overset{-1}{\mathscr{A}_\Delta}(\lambda)x$$

die Eigenschaften (4α)—(4δ): Für (α), (γ) und (δ) ist das offensichtlich, (β) folgt leicht aus der Beziehung

$$\int_a^b \begin{pmatrix} Y_0(t;\lambda)\\ Y_1(t;\lambda)\\ \vdots\\ Y_{r-1}(t;\lambda)\end{pmatrix} \left(\delta_{n_0}(t-t_0),\ \delta'_{n_0}(t-t_0),\ \dots,\ \delta_{n_0}^{(r-1)}(t-t_0)\right)dt = \mathscr{A}_\Delta(\lambda).$$

Satz 1 sichert jetzt die Existenz einer linksseitig stetigen nichtabnehmenden Matrixfunktion $\mathbf{F}=(\mathbf{F}_{jl})_{j,l=0,1,\dots,r-1}$ mit $F(0)=0$; $F_{jl}(\lambda)\in[\mathfrak{H}]$ $(\lambda\in\mathscr{R})$, so daß für $\mathbf{u},\mathbf{v}\in C_0(\mathfrak{H};(a,b))$ gilt:

$$\int_a^b\int_a^b \left(K(\sigma;\tau)u(\sigma),\,v(\tau)\right)d\sigma\,d\tau =$$

$$= \int_{-\infty}^{\infty} \sum_{j,l=0}^{r-1} \left(F_{jl}(d\lambda)\int_a^b Y_j(\sigma;\lambda)u(\sigma)\,d\sigma,\ \int_a^b Y_l(\tau;\lambda)v(\tau)\,d\tau\right).$$

Setzen wir für $\mathbf{u}$ und $\mathbf{v}$ speziell $u_n(\sigma) = \delta_n(\sigma-s)u$ mit der oben eingeführten δ-Folge (δ_n), $u\in\mathfrak{H}$ und beachten die gleichmäßig auf jedem kompakten λ-Intervall Δ gültige Beziehung

$$\int_a^b Y_j(\sigma;\lambda)\delta_n(\sigma-s)\,d\sigma \to Y_j(s;\lambda) \qquad (l=0,1,\dots,r-1),$$

so ergibt sich

$$\int_\Delta \sum_{j,l=0}^{r-1}\left(F_{jl}(d\lambda)Y_j(s;\lambda)u,\ Y_l(s;\lambda)u\right) =$$

$$= \lim_{n\to\infty}\int_\Delta \sum_{j,l=0}^{r-1}\left(F_{jl}(d\lambda)\int_a^b Y_j(\sigma;\lambda)u_n(\sigma)\,d\sigma,\ \int_a^b Y_l(\tau;\lambda)u_n(\tau)\,d\tau\right) \leqq$$

$$\leqq \lim_{n\to\infty}\int_a^b\int_a^b \left(K(\sigma,\tau)u_n(\sigma),\,u_n(\tau)\right)d\sigma\,d\tau = \left(K(s,s)u,\,u\right)$$

und aus der Schwarzschen Ungleichung für $u, v \in \mathfrak{H}$

$$\left(\int_\Delta \left| \sum_{j,l=0}^{r-1} (F_{jl}(d\lambda) Y_j(s;\lambda)u, Y_l(t;\lambda)v) \right|\right)^2 \leqq$$

$$\leqq \int_\Delta \sum_{j,l=0}^{r-1} (F_{jl}(d\lambda)Y_j(s;\lambda)u, Y_l(s;\lambda)u) \int_\Delta \sum_{j,l=0}^{r-1} (F_{jl}(d\lambda)Y_j(t;\lambda)v, Y_l(t;\lambda)v),$$

woraus leicht die Darstellung

$$K(s,t) = \int_{-\infty}^{\infty} \sum_{j,l=0}^{r-1} Y_l^*(t;\lambda) F_{jl}(d\lambda) Y_j(s;\lambda)$$

im Sinne der schwachen Operatorentopologie folgt.

5. Mit zwei orthogonalen Projektionen $P_+, P_- \in [\mathfrak{H}]$, $P_+ + P_- = I$, bilden wir jetzt den Operator $J = P_+ - P_-$. $\mathfrak{N}_0$ sei ein abgeschlossener Teilraum von $\mathfrak{H}$ mit der Eigenschaft $(Jx, x) = 0$ $(x \in \mathfrak{N}_0)$. Diesen Teilraum stellen wir als Wertebereich eines Operators $N_0 \in [\mathfrak{H}]$ mit den Eigenschaften $N_0^2 = N_0$, $(JN_0)^2 = JN_0$ dar* und setzen noch $F^* = I - JN_0$. Weiter seien **B** und **M** zwei auf $[0, \infty)$ definierte stetige Funktionen mit Werten in $[\mathfrak{H}]$ und $B(t) = B^*(t)$, $M(t) \geqq 0$ $(t \in [0, \infty))$. Wir versehen den Raum $C_0 = C_0(\mathfrak{H}; [0, \infty))$ mit dem Skalarprodukt

$$[\mathbf{f}, \mathbf{g}] = \int_0^\infty (M(t)f(t), g(t))\, dt \quad (\mathbf{f}, \mathbf{g} \in C_0). \tag{18}$$

Ist $[\mathbf{f}, \mathbf{f}] = 0$, so schreiben wir kurz $\mathbf{f} = 0$ (M). Der Faktorraum nach dem Teilraum aller **f** mit dieser Eigenschaft werde mit $\mathscr{H}$, seine Elemente wieder mit **f**, **g** etc. bezeichnet.

Weiter sei **Z** die stetige Lösung der Integralgleichung

$$Z(t;\lambda) = I - iJ \int_0^t (B(\tau) - \lambda M(\tau)) Z(\tau;\lambda)\, d\tau,$$

d. h., es gelte

$$-iJ \frac{dZ(t;\lambda)}{dt} + B(t)Z(t;\lambda) - \lambda M(t) Z(t;\lambda) = 0, \quad Z(0;\lambda) = I.$$

Dann folgt bekanntlich

$$J\overset{-1}{Z}(t;\lambda)J = Z^*(t;\bar\lambda) \quad (t \in [0, \infty)). \tag{19}$$

Schließlich setzen wir noch $Z(t;0) = Z(t)$ und überlassen es dem Leser, sich von

* Ist z. B. $\dim P_-\mathfrak{H} \leqq \dim P_+\mathfrak{H}$, so lassen sich die Elemente von $\mathfrak{N}_0$ in der Form $Px_+ + KPx_+$ $(x_+ \in P_+\mathfrak{H})$ mit einer orthogonalen Projektion P in $P_+\mathfrak{H}$ und einem partiell isometrischen Operator K von $P_+\mathfrak{H}$ in $P_-\mathfrak{H}$ darstellen. Dann kann $N_0 x = PP_+x + KPP_+x$ gewählt werden.

der Gültigkeit der Beziehung

$$Z^*(t;\bar\lambda) = Z^*(t) - i\lambda \int_0^t Z^*(\tau;\bar\lambda) M(\tau) Z(\tau)\, d\tau \overset{-1}{Z}(t) J \tag{20}$$

zu überzeugen.

Es seien weiter die folgenden Voraussetzungen erfüllt:

1) Die Menge aller $\mathbf{f} \in C_0$ der Gestalt

$$f(t) = -iZ(t) \int_t^\infty \overset{-1}{Z}(\tau) J M(\tau) \varphi(\tau)\, d\tau \tag{21}$$

mit $\boldsymbol{\varphi} \in C_0$, $\int_0^\infty \overset{-1}{Z}(\tau) J M(\tau) \varphi(\tau)\, d\tau = 0$ ist quasidicht in C_0 bezüglich des Skalarproduktes (18).

2) $F^* M(0) F$ bildet den Wertebereich von F^* auf sich ab.

Die Voraussetzung 1) besagt mit anderen Worten, daß die Menge aller Funktionen $\mathbf{f} \in C_0^{(1)}(\mathfrak{H}; [0, \infty))$, für die ein $\boldsymbol{\varphi} \in C_0$ existiert mit

$$-iJ \frac{df(t)}{dt} + B(t) f(t) = M(t) \varphi(t), \quad f(0) \in \mathfrak{N}_0,$$

in C_0 quasidicht ist bezüglich des Skalarproduktes (18).

Lemma 4. *Unter Voraussetzung* 1) *folgt für* $\boldsymbol{\varphi}_0 \in C_0$, $\varphi_0 \in \mathfrak{H}$ *aus*

$$Z(t)\varphi_0 + Z(t) \int_t^\infty \overset{-1}{Z}(\tau) J M(\tau) \varphi_0(\tau)\, d\tau = 0 \ (M)$$

stets $Z(t)\varphi_0 = 0 \ (M)$, $\boldsymbol{\varphi}_0 = 0 \ (M)$.

Beweis. Nach Voraussetzung gilt für jedes $\boldsymbol{\varphi} \in C_0$ mit der Eigenschaft $\int_0^\infty \overset{-1}{Z}(\tau) J M(\tau) \varphi(\tau)\, d\tau = 0$:

$$\int_0^\infty \left(Z(t)\varphi_0 + Z(t) \int_t^\infty \overset{-1}{Z}(\tau) J M(\tau) \varphi_0(\tau)\, d\tau,\ M(t)\varphi(t) \right) dt = 0.$$

Daraus folgt durch partielle Integration bei Beachtung von (19)

$$0 = \int_0^\infty (Z(t)\varphi_0, M(t)\varphi(t))\, dt + \int_0^\infty \left(\overset{-1}{Z}(t) J M(t) \varphi_0(t),\ \int_0^t Z^*(\tau) M(\tau) \varphi(\tau)\, d\tau \right) dt =$$

$$= \int_0^\infty \left(M(t)\varphi_0(t),\ Z(t) \int_t^\infty \overset{-1}{Z}(\tau) J M(\tau) \varphi(\tau)\, d\tau \right) dt,$$

also auf Grund von Voraussetzung 1) $\boldsymbol{\varphi}_0 = 0 \ (M)$. Die letzte Gleichung zieht aber

$$Z(t) \int_t^\infty \overset{-1}{Z}(\tau) J M(\tau) \varphi_0(\tau)\, d\tau = 0 \quad (t \in [0, \infty))$$

und somit auch $Z(t)\varphi_0 = 0 \ (M)$ nach sich.

Auf der Menge ∂ aller $\mathbf{f}\in\mathscr{H}$, die einen Repräsentanten der Form (21) mit $\boldsymbol{\varphi}\in C_0$, $\int\limits_0^\infty \overset{-1}{Z}(\tau)JM(\tau)\varphi(\tau)\,d\tau\in\mathfrak{N}_0$ haben, definieren wir jetzt einen Operator A durch die Gleichung $A\mathbf{f}=\boldsymbol{\varphi}$. Auf Grund von Lemma 4 ist diese Definition korrekt; ∂ liegt quasidicht in $\mathscr{H}$ und A ist symmetrisch: Mit $\mathbf{g}\in\partial$, $A\mathbf{g}=\boldsymbol{\gamma}$ gilt nämlich

$$(A\mathbf{f},\mathbf{g})=(\boldsymbol{\varphi},\mathbf{g})=\int\limits_0^\infty\left(M(t)\varphi(t),\,-iZ(t)\int\limits_t^\infty\overset{-1}{Z}(\tau)JM(\tau)\gamma(\tau)\,d\tau\right)dt=$$

$$=i\int\limits_0^\infty\left(\int\limits_0^t Z^*(\tau)M(\tau)\varphi(\tau)\,d\tau,\,\overset{-1}{Z}(t)JM(t)\gamma(t)\right)dt=$$

$$=-i\int\limits_0^\infty\left(Z(t)\int\limits_t^\infty\overset{-1}{Z}(\tau)JM(\tau)\varphi(\tau)\,d\tau,\,M(t)\gamma(t)\right)dt+$$

$$+i\left(\int\limits_0^\infty J\overset{-1}{Z}(\tau)JM(\tau)\varphi(\tau)\,d\tau,\,\int\limits_0^\infty\overset{-1}{Z}(t)JM(t)\gamma(t)\,dt\right).$$

Der zweite Summand auf der rechten Seite verschwindet wegen $f(0), g(0)\in\mathfrak{N}_0$, woraus unmittelbar die Behauptung folgt.

Lemma 5. *Die durch die Gleichung*

$$\Phi_\lambda\mathbf{f}=F^*\int\limits_0^\infty Z^*(t;\lambda)M(t)f(t)\,dt$$

auf $\mathscr{H}$ definierte Abbildung mit Werten in $H=F^*\mathfrak{H}$ ist richtend für den Operator A.

Beweis. Man sieht leicht, daß $\mathbf{f}=0\,(M)$ auch $\Phi_\lambda\mathbf{f}=0$ nach sich zieht, deshalb können wir Φ_λ als Abbildung von $\mathscr{H}$ in H auffassen. Die Bedingungen 1) und 2) von Definition 1 sind offensichtlich erfüllt. Es sei weiter $A\mathbf{g}-\lambda\mathbf{g}=\mathbf{f}$ $(\mathbf{g}\in\partial,\ A\mathbf{g}=\boldsymbol{\gamma})$, d.h.

$$\gamma(t)+\lambda iZ(t)\int\limits_t^\infty\overset{-1}{Z}(\tau)JM(\tau)\gamma(\tau)\,d\tau=f(t).$$

Dann folgt bei Beachtung von (20) und $g(0)\in\mathfrak{N}_0$:

$$\Phi_\lambda\mathbf{f}=F^*\int\limits_0^\infty\left(Z^*(t;\bar\lambda)+\lambda i\int\limits_0^t Z^*(\tau;\bar\lambda)M(\tau)Z(\tau)\,d\tau\overset{-1}{Z}(t)J\right)M(t)\gamma(t)\,dt=$$

$$=F^*\int\limits_0^\infty Z^*(t)M(t)\gamma(t)\,dt=iF^*Jg(0)=i(I-JN_0)JN_0g(0)=0.$$

Ist umgekehrt $\Phi_\lambda\mathbf{f}=0$, so betrachten wir

$$g(t)=-iZ(t;\lambda)J\int\limits_t^\infty Z^*(\tau;\bar\lambda)M(\tau)f(\tau)\,d\tau.$$

Dann gilt

$$g(0) = -iJ \int_0^\infty Z^*(\tau, \bar{\lambda}) M(\tau) f(\tau)\, d\tau = -iN_0 \int_0^\infty Z^*(\tau; \bar{\lambda}) M(\tau) f(\tau)\, d\tau \in \mathfrak{N}_0,$$

und eine einfache Rechnung ergibt

$$-iJ \frac{dg(t)}{dt} + B(t) g(t) - \lambda M(t) g(t) = M(t) f(t),$$

also ist auch

$$g(t) = -iZ(t) \int_t^\infty \overset{-1}{Z}(\tau) J M(\tau) \big(\lambda g(\tau) + f(\tau)\big)\, d\tau,$$

d.h. $\mathbf{g} \in \partial$, $A\mathbf{g} - \lambda \mathbf{g} = \mathbf{f}$.

Um zu zeigen, daß Φ auch der Bedingung 4) genügt, setzen wir

$$F^* Z^*(t; \lambda) M(t) Z(t; \lambda) F - F^* M(0) F = C(t; \lambda);$$

dann gilt $C(0; \lambda) = 0$, und $C(t; \lambda)$ ist stetig in (t, λ), d.h., zu jedem kompakten Intervall Δ gibt es ein $t_\Delta > 0$, so daß

$$\|C(t; \lambda)\| \leqq \frac{1}{2\|(F^* M(0) F)^{-1}\|} \qquad (0 \leqq t \leqq t_\Delta;\ \lambda \in \Delta)$$

ausfällt. Mit einer δ-Folge nichtnegativer stetiger Funktionen δ_n mit $\delta_n(0) = 0$, deren Träger in $\left[0, \frac{1}{n}\right]$ liegen, definieren wir Operatoren $A_n(\lambda)$:

$$A_n(\lambda) x = F^* \int_0^\infty Z^*(t; \bar{\lambda}) M(t) Z(t; \lambda) F \delta_n(t) x =$$

$$= F^* \left(F^* M(0) F + \int_0^\infty C(t; \lambda) \delta_n(t)\, dt \right) F x \quad (x \in H).$$

Dann ergibt sich

$$\big(A_n(\lambda) x, x\big) \geqq \frac{\|Fx\|^2}{2\|(F^* M(0) F)^{-1}\|} \geqq \delta \|x\|^2 \quad (x \in H),$$

also hat $A_n(\lambda)$ für hinreichend großes n und alle $\lambda \in \Delta$ eine beschränkte Inverse als Abbildung von H in sich. Wir wählen ein solches $n = n_0$. Dann ist $A_{n_0}^{-1}(\lambda)$ wieder holomorph in λ. Die durch die Gleichung

$$(\Psi_\lambda^{(\Delta)} x)(t) = Z(t; \lambda) F \delta_{n_0}(t) \overset{-1}{A}_{n_0}(\lambda) x$$

definierte Abbildung $\Psi^{(\Delta)}$ von $\Delta \times H$ in $\mathscr{H}$ hat alle in Bedingung 4) geforderten Eigenschaften. Damit ist das Lemma bewiesen.

Aus Satz 1 folgt jetzt die Existenz einer Funktion $\mathbf{F} \in \mathbf{V}$, so daß für $\mathbf{f}, \mathbf{g} \in C_0$ gilt:

$$\int_0^\infty \big(M(t) f(t), g(t)\big)\, dt = \int_{-\infty}^\infty \big(F(d\lambda) \Phi_\lambda f, \Phi_\lambda g\big) \tag{22}$$

mit

$$\Phi_\lambda \mathbf{f} = F^* \int_0^\infty Z^*(t;\lambda) M(t) f(t)\, dt.$$

Schließlich bemerken wir, daß eine Darstellung (22) i. a. nicht möglich ist, falls die Voraussetzung 1) nicht erfüllt ist. Das zeigt folgendes Beispiel:

$$\mathfrak{H} = R^2, \quad B(t) = 0, \quad M(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (0 \leqq t < \infty)$$

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathfrak{N}_0 = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : f_2 = 0 \right\}.$$

(Eingegangen am 5. August 1968.)

Anmerkung bei der Korrektur (16. März 1970.): Die vorliegende Arbeit entstand im wesentlichen 1967 während meines Aufenthaltes am Department of Mathematics der Universität Toronto, Kanada. F. S. Rofe-Beketov und M. G. Krein machten mich freundlicherweise darauf aufmerksam, daß der erste der genannten Herren bereits 1960 ein unendlches System richtender Funktionale betrachtete im Zusammenhang mit unendlichen Systemen von Differentialgleichungen (siehe F. S. Rofe-Beketov, Entwicklung nach Eigenfunktionen unendlicher Systeme von Differentialgleichungen (russisch). Funktionalanalysis und ihre Anwendung — *Arbeiten der V. Allunions-Konferenz über Funktionalanalysis und ihre Anwendung* (Baku 1961)). Er formulierte jedoch nicht die allgemeinen Aussagen der obigen Sätze 1—3.

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A SHORT PROOF OF STONE'S THEOREM

By

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Let b be a nonzero element of a Boolean ring B. The subset $\{b\}$ of B has the following obvious properties: (1) it contains b as an element; (2) it is closed under multiplication; (3) it does not contain 0 as an element. From Zorn's lemma it follows readily: (4) for every nonzero element b of B, there is a maximal (w. r. t. $\subset$) subset of B, called an *ultrafilter* of B, which satisfies (1) to (3). But then (2) and (3) imply: (5) if M is an ultrafilter of B and $c \in (B-M)$ then there is $m \in M$ such that $cm=0$. Because otherwise, $M \cup \{c\} \cup \{cm | m \in M\}$ would contradict the maximality of M.

LEMMA. *Let M be an ultrafilter of B. Then for every element a and c of B*

(6) $((a \in M) \wedge (c \in M)) \leftrightarrow ((ac) \in M)$, (7) $((a \in M) \wedge (c \in M)) \rightarrow ((a+c) \notin M)$

(8) $((a \in M) \wedge (c \notin M)) \rightarrow ((a+c) \in M)$, (9) $((a \notin M) \wedge (c \notin M)) \rightarrow ((a+c) \notin M)$.

PROOF. By (2), if $a \in M$ and $c \in M$ then $(ac) \in M$. On the other hand, if $(ac) \in M$, and, say, $c \notin M$ then by (5) there is $m \in M$ such that $cm=0$ which in view of (2) implies $acm=0 \in M$, contradicting (3). Thus, (6) is proved. To prove (7), assume the contrary that $(a+c) \in M$. By (6) the hypothesis of (7) implies $(ac) \in M$. But then (2) implies $(a+c)ac = 0 \in M$, contradicting (3). Next, assume the hypothesis of (8). By (5) there is $m \in M$ such that $cm=0$. But then (2) implies $(am+cm) = am = (a+c)m \in M$, which in view of (6) implies $(a+c) \in M$. Thus, (8) is proved. Finally, assume the hypothesis of (9). By (5) there is $n \in M$ and $m \in M$ such that $an = cm = 0 = (a+c)nm$. However, (2) implies $(nm) \in M$. Hence $(a+c) \notin M$, since otherwise it would contradict (3). Thus, (9) is proved.

THEOREM. *Every Boolean ring B is isomorphic to a set D of dyadic sequences where in D addition and multiplication are performed coordinatewise* mod 2.

PROOF. Let $(\ldots, M_i, \ldots)_{i \in J}$ be a wellordering of all ultrafilters of B. For every $x \in B$, let $\theta(x)$ be the dyadic sequence $(\ldots, x_i, \ldots)_{i \in J}$ where $x_i = 1$, if $x \in M_i$ and $x_i = 0$, if $x \notin M_i$. Let D be the set of all dyadic sequences thus obtained. But then (6) to (9) imply that θ is a homomorphism from B onto D where in D addition and multiplication are performed coordinatewise mod 2. However, (1) and the definition of θ imply that $\theta(x)$ is the zero sequence if and only if $x=0$. Thus, θ is an isomorphism, as desired.

NOTE. If in the above $\theta(x)$ had been defined as the set of all the ultrafilters of B each containing x as an element, then clearly, θ would have been an isomorphism from B into the ring of sets of the set of all $\theta(x)$'s.

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ON THE AVERAGE ORDER OF MAGNITUDE OF DIRICHLET SERIES

By

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Let

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad (s = \sigma + it)$$

be an ordinary Dirichlet series, with bounded coefficients, say. The series converges absolutely in the half-plane $\sigma = \operatorname{Re} s > 1$ and represents a regular function there. Suppose that this function can be continued analytically into a larger half-plane $\sigma \geqq \alpha$ $(0 < \alpha < 1)$. We are interested in the order and the average order of magnitude of $F(s)$ (to be defined below) as $t = \operatorname{Im} s$ tends to infinity. The exact situation is not even known for the most special function of the above type, the Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is well-known how important this knowledge would be in number theory. In the present paper we show that this special case would have implication also for general $F(s)$. For the sake of simplicity we assume the so far unproved Lindelöf conjecture

$$\zeta(\sigma + it) = O(|t|^{\varepsilon}) \quad (\sigma \geqq \tfrac{1}{2}),$$

although in certain cases similar results undoubtedly follow from the already known estimations of $\zeta(s)$.

Before turning to the point, let us mention that theorems of this kind (i.e. where the ζ-function plays a role in the case of a general function) were proved first in [1] for coefficient sums, in [2] for boundary behaviour and in [3] for zeros of Dirichlet series, that part of the method being always the same as here.

Provided $F(s)$ is at most a power of t in magnitude, uniformly for $\sigma \geqq \alpha$, let us define the "average order" $\mu_p(\sigma)$ as the infimum of those values ν for which

$$\left(\frac{1}{2T} \int_{-T}^{T} |F(\sigma + it)|^p \, dt\right)^{1/p} = O(T^{\nu}).$$

Since $F(s)$ is bounded for $\sigma \geqq 1 + \delta$, trivially $\mu_p(\sigma) = 0$ $(\sigma \geqq 1)$. Carlson's theorem (see e. g. [4]) states that this is so even a little to the left of 1:

$$\mu_p(\sigma) \leqq \max\left(0, \mu \frac{1-\sigma}{1-\alpha} - \frac{1}{p} \cdot \frac{\sigma - \alpha}{1-\alpha}\right) \quad (\sigma \geqq \alpha)$$

for $\sigma \geqq \frac{1}{2}$, if $\mu_p(\alpha) \leqq \mu$ and p is an even integer. (The linear function is the equation of the straight line joining (α, μ) and $\left(1, -\frac{1}{p}\right)$.)

Instead of the inequality $\mu_p(\alpha) \leqq \mu$ for the average, we impose the stronger requirement

$$F(\sigma+it) = O(|t|^{\mu}) \quad (\sigma \geqq \alpha).$$

Carlson's method does not give more even under this condition, we prove, however, the following

THEOREM. *Provided that Lindelöf's conjecture concerning $\zeta(s)$ is true, we have for the above $F(s)$*

$$\mu_p(\sigma) \leqq \max\left(0, \mu\frac{1-\sigma}{1-\alpha} - \frac{1}{p}, \mu\frac{1-\sigma}{1-\alpha} - \frac{1}{2}\cdot\frac{\sigma-\alpha}{1-\alpha}\right) \quad (\sigma \geqq \alpha)$$

at least if $\sigma \geqq 3/4$.

This 3/4 appeared also in [3] and seems to be a limit of the method, although here it could give something more complicated even for $\sigma < 3/4$. Anyway, if $\sigma \geqq 3/4$, our estimation and in particular the segment where $\mu_p(\sigma) = 0$, is better than in Carlson's theorem for any value $p > 2$.

PROOF. First we need a representation for $F(s)$ in the strip $\alpha < \sigma < 1$, the case of $\sigma \geqq 1$ being trivial. We, too, choose the most convenient way, namely approximation by Abel means. We think σ fixed and denote by $c, c_1, \ldots$ positive constants which may depend on σ and naturally on $F(s)$.

As is well known,

$$e^{-y} = \frac{1}{2\pi i}\int\limits_{(\gamma)} \frac{\Gamma(w)}{y^w}\,dw \quad (y>0),$$

integrated over any vertical line $u = \operatorname{Re} w = \gamma (>1)$. Replacing y by ny, multiplying by $\frac{a_n}{n^s}$ and summing over n

$$P(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{-ny} = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \frac{1}{2\pi i}\int\limits_{(\gamma)} \frac{\Gamma(w)}{n^w y^w}\,dw = \frac{1}{2\pi i}\int\limits_{(\gamma)} \frac{\Gamma(w)}{y^w} F(s+w)\,dw.$$

The exchange of summation and integration is legitimate by the criterium $\Sigma\int|\;| < \infty$ since $\Gamma(w)$ vanishes exponentially at $u \pm i\infty$, while $|y^w| = y^u$ is bounded and the series of $F(s+w)$ converges absolutely. For the same reason, since $F(w)$ is only a power of $v = \operatorname{Im} w$ for $u \geqq \alpha$, i.e. $F(w+s)$ for $u \geqq \alpha - \sigma$, we way push the path of integration to the left up to the vertical line $(\alpha - \sigma)$. The integral over the latter becomes

$$\left|\int\limits_{(\alpha-\sigma)} \frac{\Gamma(w)}{y^w} F(s+w)\,dw\right| \leqq y^{\sigma-\alpha} \int\limits_{-\infty}^{\infty} |\Gamma(\alpha-\sigma+iv)|\,|F(\alpha+i[t+v])|\,dv \leqq$$

$$\leqq c_1 y^{\sigma-\alpha} \int\limits_{-\infty}^{\infty} e^{-c|v|}(|t|^{\mu}+|v|^{\mu})\,dv \leqq c_2 y^{\sigma-\alpha}|t|^{\mu} \quad (|t| \geqq 1).$$

In view of $0<\alpha<\sigma<1$, the abscissa of integration $\alpha-\sigma$ lies between -1 and 0, therefore a residue $F(s)$ appears because of the pole of $\Gamma(w)$ at $w=0$. Hence by the residue theorem

$$|P(s)| = |F(s)|+O\left(y^{\sigma-\alpha}|t|^{\mu}\right). \tag{1}$$

Next we estimate the level sets of $F(s)$ through this sum $P(s)$. Let Ω denote the set of those values t in $|t|\leqq T$ for which

$$|F(\sigma+it)| \geqq M.$$

If y is defined so as to make the above remainder term, the half of M, say,

$$c_2 y^{\sigma-\alpha}T^{\mu} = \frac{M}{2}, \quad y = c_3\left(\frac{M}{T^{\mu}}\right)^{\frac{1}{\sigma-\alpha}}, \tag{2}$$

then for $t\in\Omega$

$$|P(\sigma+it)| \geqq \frac{M}{2}.$$

Suppose that we have N numbers $t_k\in\Omega$ $(k=1, \dots, N)$. A lower restriction will be imposed on $|t_i-t_j|$ $(i\neq j)$ later, and under this restriction we seek a bound for N.

We may write

$$|P(\sigma+it_k)| \geqq \frac{M}{2}$$

in the form

$$\varrho_k P(\sigma+it_k) \geqq \frac{M}{2}$$

with a ϱ_k of modulus 1. Summation over $k=1, \dots, N$ gives

$$N\frac{M}{2} \leqq \sum_{k=1}^{N} \varrho_k P(\sigma+it_k) = \sum_{k=1}^{N} \varrho_k \sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma+it_k}} e^{-ny} = \tag{3}$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma}} e^{-ny} \sum_{k=1}^{N} \varrho_k n^{-it_k} \leqq \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} e^{-ny} \left|\sum_{k=1}^{N} \varrho_k n^{-it_k}\right|.$$

By assumption a_n can be replaced by a c_4. To the remaining series we apply Schwartz's inequality grouping the terms like

$$\frac{e^{-ny/2}}{n^{1/2}} \cdot \frac{e^{-ny/2}}{n^{\sigma-1/2}} \left|\sum_{k=1}^{N} \varrho_k n^{-it_k}\right|:$$

$$\sum_{n=1}^{\infty} \frac{e^{-ny}}{n^{\sigma}} \left|\sum_{k=1}^{N} \varrho_k n^{-it_k}\right| \leqq \sqrt{\sum_{n=1}^{\infty} \frac{e^{-ny}}{n} \cdot \sum_{n=1}^{\infty} \frac{e^{-ny}}{n^{2\sigma-1}} \left|\sum_{k=1}^{N} \varrho_k n^{-it_k}\right|^2}.$$

The first sum under the square root is exactly

$$\log\frac{1}{1-e^{-y}} \leqq \max\left(1+\log\frac{1}{y}, c_5\right).$$

It can be seen from the definition (2) of y that $\frac{1}{y}$ does not exceed a power of T (M will namely never fall below 1), $\log\frac{1}{y}=O(T^{\varepsilon})$. Let us denote such quantities $O(T^{\varepsilon})$ (which could of course be neglected as well as constants) in the sequel by A_i.

Let us carry out the squaring in the inner sum

$$\left|\sum_{k=1}^{N}\varrho_k n^{-it_k}\right|^2=\sum_{i,j=1}^{N}\varrho_i\bar{\varrho}_j n^{-i(t_i-t_j)}.$$

After another change in the order of summation, the second sum under the square root takes the form

$$\sum_{i,j=1}^{N}\varrho_i\bar{\varrho}_j\sum_{n=1}^{\infty}\frac{e^{-ny}}{n^{2\sigma-1+i(t_i-t_j)}}.$$

The estimation of the N terms $i=j$ is trivial:

$$N\sum_{n=1}^{\infty}\frac{e^{-ny}}{n^{2\sigma-1}}\leqq N\int_0^{\infty}\frac{e^{-xy}}{x^{2\sigma-1}}dx=Ny^{2\sigma-2}\int_0^{\infty}\frac{e^{-x}}{x^{2\sigma-1}}dx=c_6Ny^{2\sigma-2}$$

as $\frac{1}{2}<\sigma<1$. The remaining terms cannot be estimated trivially, and here we use our presumed knowledge on the ζ-function. Each term is namely a $P(s^*)$ corresponding to the ζ-function ($a_n\equiv 1$) where $s^*=2\sigma-1+i(t_i-t_j)$. Let us apply (1) for this case. Putting $\alpha=\frac{1}{2}$, the Lindelöf conjecture means $\mu=\varepsilon$, hence

$$\left|\sum_{n=1}^{\infty}\frac{e^{-ny}}{n^{s^*}}\right|=|\zeta(s^*)|+A_1y^{2\sigma-1-\frac{1}{2}}.$$

Here, by $\sigma\geqq\frac{3}{4}$, $\operatorname{Re}s^*=2\sigma-1\geqq\frac{1}{2}$ and the first term is $O(T^{\varepsilon})$. Also, the exponent of y is non-negative and since y will never be greater than 1, the second term is $O(T^{\varepsilon})$ as well.

The argument, however, is incorrect in so far as (1) was proved under the assumption that $F(w)$ is regular, but $\zeta(w)$ has a pole at $w=1$. The only difference is the emergence in the course of contour integration of another residue from $\frac{\Gamma(w)}{y^w}\zeta(s^*+w)$ at $s^*+w=1$:

$$\left|\frac{\Gamma(1-s^*)}{y^{1-s^*}}\right|\leqq\frac{e^{-c|t_i-t_j|}}{y^{2-2\sigma}}.$$

We have already remarked that $1/y$ never exceeds a fixed power of T. Therefore, to make this term $O(T^{\varepsilon})$, let us stipulate e.g. $|t_i-t_j|\geqq\log^2 T\,(i\neq j)$. In this case the proof of

$$\left|\sum_{n=1}^{\infty}\frac{e^{-ny}}{n^{2\sigma-1+i(t_i-t_j)}}\right|\leqq A_2\quad(i\neq j)$$

is complete and the contribution of these at most N^2 terms is less than $N^2 A_2$. Our original inequality (3) becomes

$$N\frac{M}{2} \leqq \sqrt{c_4 A_3 (c_6 N y^{2\sigma-2} + N^2 A_2)}, \quad N^2 M^2 \leqq A_4 N y^{2\sigma-2} + N^2 A_4.$$

If the coefficient of N^2 on the right is less than on the left, e.g.

$$M^2 \geqq 2A_4,$$

then

$$\frac{1}{2} N^2 M^2 \leqq A_4 N y^{2\sigma-2},$$

$$N \leqq 2A_4 \frac{y^{2\sigma-2}}{M^2} = A_5 T^{\mu \frac{2-2\sigma}{\sigma-\alpha}} M^{-\frac{2-2\alpha}{\sigma-\alpha}}.$$

Summing up, this is an upper bound for the number of elements in any sequence $t_k \in \Omega$ (i.e. $|t_k| \leqq T$ and $|F(\sigma+it_k)| \geqq M$) with $|t_i - t_j| \geqq \log^2 T\,(i \neq j)$.

Let t_1 be the smallest number of the closed set Ω, t_2 the smallest $t \in \Omega$ with $t_2 \geqq t_1 + \log^2 T$, t_3 the smallest with $t \geqq t_2 + \log^2 T$ and so on. In this way we get a finite number of points from Ω the mutual distance of which is at least $\log^2 T$. Therefore their number N cannot exceed the above bound. But the N intervals $[t_k, t_k + \log^2 T]$ cover the whole of Ω so that for its measure

$$|\Omega| \leqq N \log^2 T \leqq A_5 T^{\mu \frac{2-2\sigma}{\sigma-\alpha}} M^{-\frac{2-2\alpha}{\sigma-\alpha}} \log^2 T = A_6 T^{\mu \frac{2-2\sigma}{\sigma-\alpha}} M^{-\frac{2-2\alpha}{\sigma-\alpha}}.$$

The only condition was that $M^2 \geqq 2A_4$, $M \geqq A_7$.

Finally, applying this bound, we estimate the integral of $|F(\sigma+it)|^p$ in, so to speak, Lebesgue's conception, i.e. dividing the range of integration $[-T, T]$ according to the magnitude of $|F(\sigma+it)|$.

Our estimation for $|\Omega|$ is not valid if $M < A_7$. Also it is not worth while to use it, if it does not give more than the trivial $2T$, i.e. if

$$T^{\mu \frac{2-2\sigma}{\sigma-\alpha}} M^{-\frac{2-2\alpha}{\sigma-\alpha}} \geqq T, \quad M \leqq T^{\mu \frac{1-\sigma}{1-\alpha} - \frac{1}{2} \cdot \frac{\sigma-\alpha}{1-\alpha}}$$

say. Let M_0 denote the greater one of this threshold and A_7. For the measure of the set of t values defined by $|F(\sigma+it)| < M_0$ ($|t| \leqq T$) we know, therefore, but the trivial $2T$ and the contribution of this set is

$$\text{(4)} \qquad \int_{|F| < M_0} |F(\sigma+it)|^p \, dt \leqq 2TM_0^p \leqq A_8 \left(T + T^{p\left(\mu \frac{1-\sigma}{1-\alpha} - \frac{1}{2} \cdot \frac{\sigma-\alpha}{1-\alpha}\right)+1}\right).$$

Let now $M = M_h = M_0 2^h$ $(h = 0, 1, \ldots)$ and let us consider the set $\{t : |t| \leqq T,$ $M_h \leqq |F(\sigma+it)| < M_{h+1}\}$. It is contained in $\{t : |t| \leqq T, |F(\sigma+it)| \geqq M_h\}$, i.e. in our Ω with $M = M_h$ and in this case our inequality for $|\Omega|$ gives already a non-trivial bound

$$A_6 T^{\mu \frac{2-2\sigma}{\sigma-\alpha}} M_h^{-\frac{2-2\alpha}{\sigma-\alpha}}.$$

On the other hand, the integrand is less than $M_{h+1}^p = 2^p M_h^p$ and the contribution is bounded by

$$\int\limits_{M_h \leqq |F| < M_{h+1}} |F(\sigma+it)|^p \, dt \leqq A_6 T^{\mu \frac{2-2\sigma}{\sigma-\alpha}} M_h^{-\frac{2-2\alpha}{\sigma-\alpha}} 2^p M_h^p = A_9 T^{\mu \frac{2-2\sigma}{\sigma-\alpha}} M_h^{p-\frac{2-2\alpha}{\sigma-\alpha}}.$$

There is no need to deal with $M_h \leqq |F(\sigma+it)| < M_{h+1}$ if M_h is larger than the maximum of $|F(\sigma+it)|$ over $|t| \leqq T$. Since $F(\alpha+it) = O(|t|^\mu)$ and $F(1+\varepsilon+it) = O(1)$, by the convexity property of the order of magnitude this maximum is less than $A_{10} T^{\mu \frac{1-\sigma}{1-\alpha}}$, consequently the last M_h we have to consider is also below this bound. This last M_h makes our estimate for $\int\limits_{M_h \leqq |F| < M_{h+1}}$ maximal in case the exponent of M_h in it is non-negative:

$$\int\limits_{M_h \leqq |F| < M_{h+1}} |F(\sigma+it)|^p \, dt \leqq A_9 T^{\mu \frac{2-2\sigma}{\sigma-\alpha}} \left(A_{10} T^{\mu \frac{1-\sigma}{1-\alpha}}\right)^{p-\frac{2-2\alpha}{\sigma-\alpha}} = A_{11} T^{p\mu \frac{1-\sigma}{1-\alpha}}.$$

There are at most $O(\log T)$ sets $M_h \leqq |F| < M_{h+1}$ (M_h being less than a power of T and increasing by the powers of 2) and to get an upper bound for their total contribution, we have only to multiply the last one by $O(\log T)$.

If, on the other hand, the exponent $p - \dfrac{2-2\alpha}{\sigma-\alpha}$ of M_h is negative, then the maximum is attained at M_0. The corresponding quantity (or an even larger one) occours already in (4), possibly without A_g and this $O(\log T)$.

Summing up, together with (4) we have found the following estimate for the total integral:

$$\int\limits_{-T}^{T} |F(\sigma+it)|^p \, dt \leqq A_{12} \left(T + T^{p\left(\mu \frac{1-\sigma}{1-\alpha} - \frac{1}{2} \cdot \frac{\sigma-\alpha}{1-\alpha}\right)+1} + T^{p\mu \frac{1-\sigma}{1-\alpha}}\right),$$

$$\left(\frac{1}{2T} \int\limits_{-T}^{T} |F(\sigma+it)|^p \, dt\right)^{1/p} \leqq O(T^\varepsilon) \max \left(1, T^{\mu \frac{1-\sigma}{1-\alpha} - \frac{1}{2} \cdot \frac{\sigma-\alpha}{1-\alpha}}, T^{\mu \frac{1-\sigma}{1-\alpha} - \frac{1}{p}}\right)$$

and by the definition of $\mu_p(\sigma)$

$$\mu_p(\sigma) \leqq \max \left(0, \mu \frac{1-\sigma}{1-\alpha} - \frac{1}{2} \cdot \frac{\sigma-\alpha}{1-\alpha}, \mu \frac{1-\sigma}{1-\alpha} - \frac{1}{p}\right).$$

Q.e.d.

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FRACTIONAL DIMENSION AND THE BOREL–CANTELLI LEMMA

By

R. KAUFMAN (Urbana)

Let (r_n) be a decreasing sequence of positive numbers and $\sum_{n=1}^{\infty} r_n < \infty$. For every sequence (I_n) of open intervals of lengths r_n, the set $\limsup I_n \equiv \bigcap_{k>1} \bigcup_{n>k} I_n$ has Lebesgue measure 0. This conclusion can not be much improved: let h be a real function on $(0, \infty)$ and

$$h' > 0, \quad h'' \leqq 0, \quad h(0+) = 0.$$

$$\sum_{n=1}^{\infty} h(r_n) = \infty. \tag{2}$$

THEOREM 1. *The sequence of intervals* (I_n) *can be so chosen that* $\limsup I_n$ *has positive h-measure: for some Borel probability measure* μ *concentrated on this set,*

$$\mu([a, a+b]) \leqq C\, h(b), \quad 0 < b \leqq 1, \quad -\infty < a < \infty.$$

PROOF. We assume that $h(+\infty) = \infty$ (as can easily be achieved) and choose integers $v(k)$ and $k(n)$ $(1 \leqq k, n < \infty)$ as the smallest meeting the requirements

$$h(2^{-v(k)}) \leqq 2^{-k} < h(2^{1-v(k)}), \quad 2^{-v(k(n))} < \frac{1}{2} r_n.$$

By the concavity (1) of h,

$$2^{-k-1} < \frac{1}{2} h(2^{1-v(k)}) \leqq h(2^{-v(k)}),$$

and therefore $v(k+1) \geqq v(k)+1$. Also, if $k(n) > 1$,

$$2^{1-k(n)} \geqq h(2^{-v(k(n)-1)}) \geqq h\left(\frac{1}{2} r_n\right) \geqq \frac{1}{2} h(r_n)$$

and $\sum_{n=1}^{\infty} 2^{-k(n)} = \infty$.

Denote by $X_{k,n}$ $(1 \leqq k < \infty, 0 \leqq n < \infty)$ an array of mutually independent random variables with identical laws

$$\mathbf{P}(X=1) = \mathbf{P}(X=0) = \frac{1}{2}, \quad \text{and} \quad Z_n = \sum_{k=1}^{\infty} 2^{-v(k)} X_{k,n} \quad (0 \leqq n < \infty).$$

Then

$$(X_{k,n}=X_{k,0},\ 1\leqq k\leqq k(n))\subseteq\left(|Z_n-Z_0|<\frac{1}{2}r_n\right)$$

because $\sum_{k(n)<k} 2^{-v(k)}\leqq 2^{-v(k(n))}<\frac{1}{2}r_n$. Denoting by $\mathbf{P}^*$ conditional probability relative to the variable Z_0,

$$\sum_{n=1}^{\infty}\mathbf{P}^*\left(|Z_n-Z_0|<\frac{1}{2}r_n\right)\geqq\sum_{n=1}^{\infty}2^{-k(n)}=\infty.$$

By the Borel—Cantelli Lemma

$$\mathbf{P}^*\left(|Z_n-Z_0|<\frac{1}{2}r_n \text{ infinitely often}\right)=1$$

and $\mathbf{P}\left(|Z_n-Z_0|<\frac{1}{2}r_n \text{ i.o.}\right)=1$. Thus by defining

$$I_n=\left(Z_n-\frac{1}{2}r_n,\ Z_n+\frac{1}{2}r_n\right)$$

we obtain $\mathbf{P}(Z_0\in\limsup I_n)=1$.

Because Z_0 is independent of the intervals I_n $(n\geqq 1)$ the last formula can be stated $\mathbf{P}(\mu(\limsup I_n)=1)=1$, where μ is the distribution function of Z_0.

Finally, let b be small enough to solve the inequalities $2^{-v(k+1)}<b\leqq 2^{-v(k)}$ with $k\geqq 1$. Then

$$\mathbf{P}(a\leqq Z_0\leqq a+b)\leqq\mathbf{P}(a\leqq Z_0\leqq a+2^{-v(k)})\leqq 2^{-k}<$$
$$<h(2^{1-v(k)})\leqq 4h(2^{-v(k+1)})\leqq 4h(b),\quad\text{q.e.d.}$$

The same argument gives a curious fact. If v is a singular probability measure on $(-\infty,\infty)$, then by a classical theorem [2; IV 9]

$$\lim_{r\to 0^+} r^{-1}v(x-r,x+r)=\infty\quad\text{for } v\text{-almost all}\quad x.$$

Hence for certain sets E_N and numbers $s_N\to 0$

$$v(E_N)\geqq 1-N^{-2},\quad v(x-s_N,x+s_N)\geqq N^2 s_N,\quad x\in E_N.$$

The sequence (r_n) is formed from the sequence (s_N) by writing $2s_N$ in succession $[(N^2 s_N)^{-1}]$ times, $N=1,2,3,\ldots$. Then $\sum_{n=1}^{\infty} r_n\leqq 2\sum_{N=1}^{\infty}N^{-2}<\infty$, while

$$\sum_{n=1}^{\infty}v(x-r_n,x+r_n)=\infty\quad\text{for } v\text{-almost all}\quad x.$$

By the same reasoning as before,

THEOREM 2. *For some sequence of intervals* (I_n) *of lengths* $|I_n|$,

$$\sum_{n=1}^{\infty} |I_n| < \infty, \quad \nu(\limsup I_n) = 1, \text{ almost surely.}$$

In the next theorem the intervals I_n are distributed according to a uniform law.

THEOREM 3. *Let* (Y_n) *be a sequence of independent random variables, distributed uniformly upon* $[-1, 1]$, *and let* I_n *have center* Y_n *and length* n^{-s}, $1 < s < \infty$.

Then it is almost sure that $\limsup I_n$ *has Hausdorff dimension* s^{-1} *(obviously best possible).*

PROOF. Let (N_j) be an increasing sequence of integers such that

$$N_j \log N_j = o(N_{j+1} - N_j), \quad \log N_{j+1} = \big(1 + o(1)\big) \log N_j \qquad (j \to \infty).$$

Let E_j denote the event that each of the intervals $\big(kN_j^{-1}, (k+1)N_j^{-1}\big)$ — where $k = -N_j, 1-N_j, \ldots, N_j - 1$ — contain some number Y_n with $N_j \leqq n < N_{j+1}$. The probability that one fixed interval contain none of these variables is $(1-2N_j^{-1})^{N_{j+1}-N_j}$. Hence $\mathbf{P}(E_j) \to 1$ and $\mathbf{P}(\limsup E_j) = 1$.

For each instance of the event E_j, and each even integer k in $[-N, N-1]$, choose some Y_n in $\big(kN_j^{-1}, (k+1)N_j^{-1}\big)$ and let V_j comprise the intervals of length N_{j+1}^{-s} and centers Y_n. Thus $V_j \subseteqq \bigcup_n (I_n | N_j \leqq n < N_{j+1})$.

In every infinite set J of natural numbers there is an infinite subset J' such that $\bigcap_{J'} V_j$ has Hausdorff dimension s^{-1}. Suppose indeed that $j_1 < \cdots < j_r$ have been chosen, and consider those intervals of V_{j_r} contained entirely in $V_{j_1} \cap \cdots \cap V_{j_r}$. If j_{r+1} is sufficiently large, each of these contains $\geqq \frac{1}{3} N_{j_r+1}^{-s} N_{j_{r+1}}$ intervals of $V_{j_{r+1}}$. The distances between the intervals of V_j are ultimately $> d_j^{s-1}$, d_j being their common length. By a theorem of EGGLESTON [1, p. 55] $\bigcap_{r=1}^{\infty} V_{j_r}$ has dimension s^{-1}, q.e.d.

The last result exhibits a property of our random sets not common to all sets of fractional dimension.

THEOREM 4. *It is almost sure that the set of (additive) differences of* $\limsup I_n$ *has Hausdorff dimension* $\geqq \min(1, 2s^{-1}) > s^{-1}$.

In the proof (N_j) is a sequence of integers like that in Theorem 3 and

$$M_j = \big[\tfrac{1}{2}(N_j + N_{j+1})\big].$$

LEMMA. *Let* J_1 *and* J_2 *be intervals in* $[-1, 1]$, *and let* E_j' *be the following event:*

(i) *Each number in* J_1 *is within* N_j^{-1} *of some* Y_n *in* J_1, *with* $N_j \leqq n < M_j$; *the same for* J_2, *with values* Y_m, $M_j \leqq m < N_{j+1}$.

(ii) *Each number in the difference* $J_1 - J_2$ *is within* N_j^{-2} *of a difference* $Y_n - Y_m$, $Y_n \in J_1$, $Y_m \in J_2$. *Then* $\mathbf{P}(E_j') = 1$.

PROOF. Statement (i) is clear, as $M_j - N_j + 1 \geqq \frac{1}{2}(N_{j+1} - N_j)$. To prove (ii), we consider the probability of the event described, conditional upon the variables

Y_n, $N_j \leqq n < M_j$. Let J_3 be an interval of length N_j^{-2} in $J_1 - J_2$, and $M_j \leqq m < N_{j+1}$. Then the event

$$Y_n - Y_m \in J_3 \quad \text{with} \quad Y_n \in J_1,\ Y_m \in J_2$$

has conditional probability $\geqq \frac{1}{2} \operatorname{mes} \bigcup \{(Y_n - J_3) \cap J_2\}$, where the summation is extended to all values of $Y_n \in J_1$. If the first statement in (i) is assumed, the measure is $\geqq \varepsilon N_j^{-1}$ (with ε dependent upon J_1 and J_2). But the variables $Y_n - Y_m$ are conditionally independent for different m, so we find a conditional probability $\geqq 1 - (1 - \varepsilon N_j^{-1})^{N_{j+1} - M_j}$. Now the power in this expression is $o(N_j^{-r})$ for $r = 1, 2, 3, \ldots$. Taking $r = 2$, we obtain our lemma.

To complete the proof, we construct the sets V_j as before, using values Y_n with $N_j \leqq n < M_j$, and W_j, using $M_j \leqq m < N_{j+1}$. It is necessary to consider a difference set, say

$$(V_{j_1} \cap \cdots \cap V_{j_r}) - (W_{j_1} \cap \cdots \cap W_{j_r})$$

and the corresponding set with $r+1$ intersections. To choose j_{r+1} we apply the lemma to *every* pair of intervals, chosen one from $V_{j_1} \cap \cdots \cap V_{j_r}$ and one from $W_{j_1} \cap \cdots \cap W_{j_r}$. We can then apply Eggleston's theorem to the sequence of differences; the exponent -2 in the lemma thus gives the improvement claimed. Finally, observe that the sequence of differences does indeed narrow down to a subset of the difference set of lim sup I_n. This completes the proof.

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STRUCTURES OF CONTINUOUS FUNCTIONS. I

By
S. MRÓWKA (Buffalo)

§ 1. Introduction

There are numerous theorems in the literature concerning representation of certain maps (functionals) defined on sets of continuous functions. As an example we shall quote the following two.

KAKUTANI—RIESZ THEOREM. *If X is a Hausdorff compact space and $C(X)$ is the set of all real-valued continuous functions defined on X, then every linear positive functional φ on $C(X)$ (i.e., a real-valued map φ on $C(X)$ satisfying: $\varphi(f+g) = \varphi(f)+\varphi(g)$ and $\varphi(f)\geqq 0$ for $f\geqq 0$) admits the integral representation*

$$\varphi(f) = \int f\,d\mu$$

where μ is a Baire measure in X.

MAZUR THEOREM. *If X is a separable metric space and F is a subring of $C(X)$ such that F contains all constant functions on X, F is closed under inversion (i.e., if $f\in F$ and $f(p)\neq 0$ for every $p\in X$, then $1/f\in F$), and F satisfies the following condition*
(1) *if $f_1, f_2, \ldots$ are members of F such that $0\leqq f_n(p)\leqq 1$ for every $p\in X$ and every n, then there exists a sequence of positive numers α_n such that $\sum_n \alpha_n < +\infty$ and $\sum \alpha_n \cdot f_n \in F$,*
then every linear multiplicative functional φ on F is either identically equal to 0 or admits the following trivial representation

$$\varphi(f) = f(p_0)$$

where p_0 is a fixed point of X.

The purpose of this paper is to provide a general framework for the discussion of such representation theorems. These theorems exhibit the following pattern (precise definitions of the terms involved will be given in the next section): we consider a topological space E on which certain algebraic operations and/or relations are defined — we will refer to such an E as a topological algebraic structure. Given an arbitrary space X we denote by $C(X, E)$ the set of all continuous functions f with $f\colon X\to E$. Every operation (relation) in E gives rise to a "pointwisely defined" operation (relation, respectively in $C(X, E)$); $C(X, E)$ becomes therefore an algebraic structure. Let F be a substructure of $C(X, E)$; we are concerned with representation of homomorphisms (functionals) of F; i.e., maps φ of F that preserve the given operations or relations. Note that in such a general setting, although we deal with continuous functions only, we do not exclude the case of a discrete X and in this case F is simply a substructure of the direct product of copies of E. In fact, the thechnique of structures of continuous functions is applicable to problems which — in their original formulation — involve no topology (see Sections 6 and 7 of the present paper). Finally, note that there are two essentially different types of representation: in Mazur's theorem the representation formula involves only one point of the space; in the Kakutani—Riesz theorem the value of $\varphi(f)$ depends upon values of f on a whole set of points.

The present paper is the first in the series "Structures of continuous functions". Three papers of this series, III, IV, V, ([11], [17], [18]) have been already published. (In III, E is the set of integers considered as a ring and as a lattice; in IV, E is the lattice of the real numbers and V contains one result concerning the case of arbitrary E.) The second paper of this series was intended as a summary of results on E-compact spaces; however, in view of a rapid development in this area its publication was continuously delayed. A partial summary of related results will be published outside this series [13].

§ 2. Structures of continuous functions

The purpose of this section is to provide necessary definitions. By an *algebraic structure* we mean a triplet

(1) $$\{E; \{o_0, \dots, o_\xi, \dots\}_{\xi<\alpha}; \quad \{\varrho_0, \dots, \varrho_\eta, \dots\}_{\eta<\beta}\},$$

where E is a set, o_ξ are operations on E, and ϱ_η are relations on E. We do not assume that these operations and relations are finitary. Whenever no confusion seems possible the structure (1) will be denoted simply by E. The type of the structure (1) is the pair of transfinite sequences

(2) $$\{\nu_0, \dots, \nu_\xi, \dots\}_{\xi<\alpha}; \quad \{\mu_0, \dots, \mu_\eta, \dots\}_{\eta<\beta}$$

such that o_ξ is a ν_ξ-ary operation and ϱ_η is an μ_η-ary relation. For structures of the same type it is possible to define the concept of a homomorphism and that of an isomorphism. Let E and E_1 be structures of the same type; let o_ξ and $\bar{o}_\xi$ (ϱ_η and $\bar{\varrho}_\eta$) be the corresponding operations (relations) in E and E_1, respectively. For simplicity of notation we will assume at this moment that these operations and relations are binary. A map $\varphi\colon E\to E_1$ is called a *homomorphism* provided that

(3) $$\varphi(x_1 o_\xi x_2) = \varphi(x_1)\bar{o}_\xi\varphi(x_2) \quad \text{for every} \quad x_1, x_2, \in E \quad \text{and for every} \quad \xi<\alpha$$

and

(4) $$x_1\varrho_\eta x_2 \quad \text{implies} \quad \varphi(x_1)\bar{\varrho}_\eta\varphi(x_2) \quad \text{for every} \quad x_1, x_2\in E \quad \text{and for every} \quad \eta<\beta.$$

φ is called an *isomorphism* provided that φ is one-to-one, φ satisfies (3), and φ satisfies (4) with "implies" replaced by "if and only if". Note that according to the above definitions a one-to-one homomorphism need not to be an isomorphism.

A *substructure* E_0 of (1) is a subset of E whose operations and relations are those of (1) restricted to E_0 and which is closed under all of the operations of (1).

A *topological algebraic structure* is a structure (1) in which E is a Hausdorff topological space and such that all the operations o_ξ are continuous (relative to the product topology in the corresponding power of E). In general, we will not make any topological assumptions on the relations of (1), however, it is sometimes useful to assume that they are closed (in the respective powers of E) or that they are E-compact.

If E is a topological algebraic structure and X is an arbitrary topological space, then by $C(X, E)$ we shall denote the algebraic structure consisting of all continuous functions $f: X \to E$; operations and relations in $C(X, E)$ are the pointwisely defined counterparts of the operations and relations in E. That is, if o_ξ is an operation and ϱ_η is a relation in E (assumed, for simplicity of notation, to be binary), then the pointwisely defined counterparts $o_\xi^{(X)}$ and $\varrho_\eta^{(X)}$ in $C(X, E)$ of o_ξ and ϱ_η, respectively, are defined as follows:

(4) $$h = fo_\xi^{(X)}g \quad \text{if, and only if,} \quad h(p) = f(p)o_\xi g(p) \quad \text{for every} \quad p \in X;$$

and

(5) $$f\varrho_\eta^{(X)}g \quad \text{if, and only if,} \quad f(p)\varrho_\eta g(p) \quad \text{for every} \quad p \in X.$$

The superscript X in $o_\xi^{(X)}$ and $\varrho_\eta^{(X)}$ will be omitted whenever possible. Note that the structure $C(X, E)$ is of the same type as E. Furthermore, the assumption that the operations o_ξ are continuous implies that $C(X, E)$ is closed with respect to the operations $o_\xi^{(X)}$.

Throughout the rest of the paper we shall use the following notations: E will be a topological algebraic structure, E_1 will be an algebraic structure of the same type as E; F will be a substructure of $C(X, E)$ and φ will be a homomorphism of F into E_1 (note that F and E_1 are of the same type).

To conclude this section observe that in the Kakutani—Riesz Theorem we have $E = E_1 =$ the ordered group of the reals (i.e., $E = E_1 = \{\mathscr{R}; +; \leqq\}$) and in the Mazur theorem, $E = E_1 =$ the ring of the reals (i.e., $E = E_1 = \{\mathscr{R}; +; \cdot\}$) (where $+, \cdot, \leqq$ denote, respectively, the addition, the multiplication, and the "less than or equal to" relation in the set $\mathscr{R}$ of the reals).

§ 3. Supports and weak supports

Our main tool in dealing with the representation problem will be the concept of a support and that of a weak support.[1] The algebraic structure will not enter into the considerations of this section (so one may consider E as a plain topological space and E_1 as a plain set; thus φ is an arbitrary map with $\varphi: F \to E_1$).

A closed set $A \subset X$ is called a *support* of φ provided that for every $f, g \in F$ the equality $f|A = g|A$ implies $\varphi(f) = \varphi(g)$. A is called a *weak support* of φ provided that for every open set $U \subset X$ with $A \subset U$ and for every $f, g \in F$, the equality $f|U = g|U$ implies $\varphi(f) = \varphi(g)$.

Obviously, a support of φ is a weak support of φ (the converse is not necessarily true, see Examples 4. 3 and 4. 4). The concept of a weak support admits a natural and useful generalization: if εX is an extension of X (i.e., εX is a Hausdorff superspace of X in which X is dense), then a closed subset A of εX is called a weak support prof φ ovided that for every open subset U of εX with $A \subset U$ the equality $f|U \cap X = g|U \cap X$ implies $\varphi(f) = \varphi(g)$ (for every $f, g \in F$). An analogous generalization of the concept of a support, is, of course, superfluous.

[1] These concepts were introduced in [18].

Note that the empty set is a support of φ iff φ is a constant map. Any superset of a support (a weak support) of φ is again a support (a weak support) of φ, in particular, the whole space X is always a support of φ. We shall therefore be interested in the existence of a smallest support[2] or a smallest weak support.[2]

Note that the existence of a one-point support completely solves the representation problem; indeed we have the following.

3. 1. *Suppose that F contains all constant functions from $C(X, E)$ and let p_0 be a point from X. $\{p_0\}$ is a support of φ if, and only if, φ can be represented in the form*

$$\varphi(f)=\alpha(f(p_0)) \quad \textit{for every} \quad f\in F,$$

where α is a fixed homomorphism of E into E_1.

§ 4. The compact case

In this section X will be assumed to be a Hausdorff space. We shall give a few sufficient conditions for the existence of smallest supports and weak supports in the case of a compact X as well as discuss a few counter-examples.

Let $\mathfrak{C}$ be a multiplicative [3] base for closed subsets of X.[4] We shall say that φ *has the property Π relative to* $\mathfrak{C}$ (in symbols: $\Pi(\varphi, \mathfrak{C})$ holds) provided that the intersection of two supports of φ from C is a support of φ.

4. 1. THEOREM. *Let X be compact. If φ has the property Π relative to $\mathfrak{C}$, then the intersection of all supports of φ from $\mathfrak{C}$ is the smallest weak support of φ.*

PROOF. Let

$$\mathfrak{Z}_\varphi=\{A\colon A\in\mathfrak{C} \quad \text{and } A \text{ is a support of } \varphi\},\ Z_\varphi=\bigcap\mathfrak{Z}_\varphi.$$

It is easy to see that the class $\mathfrak{Z}_\varphi$ is multiplicative. Let U be an open subset of X with $Z_\varphi\subset U$ and let $f|U=g|U$, $f, g\in F$. Since $\mathfrak{Z}_\varphi$ is multiplicative (and X is compact), there is an $A\in\mathfrak{Z}_\varphi$ with $A\subset U$. We have $f|A=g|A$, therefore $\varphi(f)=\varphi(g)$. Thus Z_φ is a weak support of φ.

Now, Z_φ is the smallest weak support of φ. Indeed, assume that Z is a weak support of φ and assume that $Z_\varphi\not\subset Z$. Let $p_0\in Z_\varphi\setminus Z$. There is an open set G such that $Z\subset G$ and $p_0\notin\bar{G}$. Since $\mathfrak{C}$ is a base for closed sets, there is an $A\in\mathfrak{C}$ such that $\bar{G}\subset A$ and $p_0\notin A$. Since $Z\subset \operatorname{Int} A$, A is a support of φ. Thus $A\in\mathfrak{Z}_\varphi$, hence $Z_\varphi\subset A$, contrary to the fact that $p_0\in Z_\varphi$ and $p_0\notin A$.

We shall now give sufficient conditions for $\Pi(\varphi, \mathfrak{C})$.

We say that F *has the property* (K) *relative to* $\mathfrak{C}$ (in symbols: $K(F, C)$ holds) provided that the following condition is satisfied

for every $A, B\in\mathfrak{C}$ and for every $f, g\in F$ with $f|A\cap B=g|A\cap B$ there exists an $h\in F$ such that $f|A=h|A$ and $g|B=h|B$.

[2] A smallest support (weak support) of φ is a support (weak support) which is contained in every support (weak support) of φ.

[3] A class $\mathfrak{C}$ of sets is said to be *multiplicative* provided that $A, B\in\mathfrak{C}$ implies $A\cap B\in\mathfrak{C}$.

[4] A base for closed sets is a class of closed sets such that every closed set is an intersection of some members of this class.

4. 2. THEOREM. *If F has the property (K) relative to $\mathfrak{C}$, then φ has the property Π relative to $\mathfrak{C}$.*

PROOF. Let $A, B \in \mathfrak{C}$ and let A and B be supports of φ. If $f|C = g|C$, where $f, g \in F$ and $C = A \cap B$, then there is an $h \in F$ with $f|A = h|A$ and $g|B = h|B$. Since A and B are supports of φ, we have $\varphi(f) = \varphi(h)$ and $\varphi(g) = \varphi(h)$, hence $\varphi(f) = \varphi(g)$. Thus $\mathfrak{C}$ is a support of φ.

Note that $F = C(X, E)$ has the property (K) relative to an additive[5] (and multiplicative) class $\mathfrak{C}$ of closed subsets of X whenever X has the following extension property: for every $A \in \mathfrak{C}$ every continuous function $f: A \to E$ admits a continuous extension $f: X \to E$ (e.g., $F = C(X, E)$ has the property (K) relative to the class of all closed subsets of X whenever X is normal and E is an absolute (metric retract). Indeed, in this case we define $h_0(p) = f(p)$ for $p \in A$ and $h_0(p) = g(p)$ for $p \in B$ (so that $h_0: A \cup B \to E$) and then take a continuous extension h of h_0 with $h: X \to E$. A particular case of the above is: if X is 0-dimensional, then $F = C(X, E)$ has always the property K relative to the class $\mathfrak{C}$ of all open-closed subsets of X. Consequently, *if X is a 0-dimensional compact space, then every map φ on $C(X, E)$ has a smallest weak support.*

We shall now consider a few examples.

4. 3. EXAMPLE. Let $E = E_1$ be the lattice of the real numbers (i.e., $E = E_1 = \langle \mathscr{R}; \vee, \wedge \rangle$, where $\vee$, and $\wedge$ stand for maximum and minimum, respectively) and let $F = C(X, E)$ (i.e., F is the lattice of real-valued continuous functions on X). If X is compact, then every (lattice-) homomorphism $\varphi: C(X, E) \to E$ has a one-point weak support (consequently, φ has a smallest weak support). If X is infinite (completely regular, but not necessarily compact), then there is a homomorphism φ without a one-point support (for these results see [17]). Such a φ has the property Π relative to the class of all closed subsets of X (obvious by 4. 2) and this implies that φ does not have a smallest support.

4. 4. EXAMPLE. Let X be the closed unit interval, let $E = \mathscr{R}$ and let F consist of all continuously differentiable functions on X. Let $\mathfrak{C}$ be the class of all finite unions of intervals of the form $[a, b]$ where $0 \leqq a < b \leqq 1$, where a is rational, b is irrational or $b = 1$. $\mathfrak{C}$ is an additive and multiplicative base for closed subsets of X. F has the property K relative to $\mathfrak{C}$ (on the other hand, F has the property K relative to neither the class of all closed subsets of X nor the class of all finite unions of arbitrary closed subintervals of X). Thus (by 4. 2 and 4. 1), every map $\varphi: F \to E_1$, where E_1 is an arbitrary set, has a smallest weak support. The map φ, defined by $\varphi(f) = f'(x_0)$, (x_0 — a fixed point of X) has $\{x_0\}$ as its smallest weak support; φ does not have a smallest support.

4. 5. EXAMPLE. Let $X = [0, 1]$, let $E = E_1$ be the ring of the reals $\mathscr{R}$, and let F consist of all polynomials in $C(X, E)$. Define $\varphi(f) = f(2)$ for every $f \in F$. F is a subring of $C(X, E)$ and F contains all constant functions in $C(X, E)$. φ is a (ring-) homomorphism of F. Every infinite closed subset of X is a support of φ whereas no finite subset of X is. If $\mathfrak{C}$ is an arbitrary multiplicative base for closed subsets of X, then φ does not have the property Π relative to $\mathfrak{C}$ ($\mathfrak{C}$ contains infinite disjoint

[5] A class $\mathfrak{C}$ of sets is said to be *additive* provided that $A, B \in \mathfrak{C}$ implies $A \cup B \in \mathfrak{C}$.

sets). Thus, F does not have the property K (relative to any such $\mathfrak{C}$). Every one-point set is a weak support of φ; consequently φ does not have a smallest weak support.

4. 6. EXAMPLE. Let $\mathscr{R}_\alpha$, where α is an ordinal, denote the ordered product $[0, 1) \times S(\alpha)$[6] (ordered according to second coordinates). Elements of $\mathscr{R}_\alpha$ of the form $(0, \xi)$ will be denoted by ξ. $\mathscr{R}_\alpha$ will be considered as a lattice. $\mathscr{R}_\alpha$ is connected relative to its order topology.

Let $X=[0, 1]$ and let α be a fixed ordinal with $\alpha > \Omega$. Let $F=C(X, \mathscr{R}_\alpha)$. From the connectedness of X it is easy to infer that

(i) if $f \in C(X, \mathscr{R}_\alpha)$ and $f(x_0) < \Omega$ for some $x_0 \in X$, then $f(x) < \Omega$ for every $x \in X$.

Let $F_1 = \{f \in F\colon f(x) < \Omega$ for every $x \in X\}$, $F_2 = \{f \in F\colon f(x) \geqq \Omega$ for every $x \in X\}$. Clearly, $F_1 \cap F_2 = \emptyset$ and from (i) infer that $F = F_1 \cup F_2$. Define $\varphi(f) = f(0)$ for $f \in F_1$ and $\varphi(f) = f(1)$ for $f \in F_2$. φ is a (lattice-) homomorphism of $C(X, \mathscr{R}_\alpha)$ (into the chain $\mathscr{R}_\alpha$). The set $A = \{0, 1\}$ is the smallest support of φ (as well as its smallest weak support). It follows that the intersection of any two supports of φ is again a support of φ. On the other hand, $F = C(X, \mathscr{R}_\alpha)$ does not have the property K relative to any multiplicative base $\mathfrak{C}$ for closed subsets of X.

We have seen that a smallest weak support need not exist. But if X is compact, then φ always has a minimal weak support (i.e., a weak support that does not contain properly another weak support). This follows immediately from the KURATOWSKI lemma ([5], statement (41), p. 88): repeating the proof of 4. 1 we can show that the intersection of a chain of weak supports is again a weak support.

We shall now turn to the existence of smallest supports. In some cases it is possible to prove that weak supports of φ are, in fact, its supports. This is, for instance, the case when φ is continuous (in a certain sense) and if functions from F that agree on a weak support A of φ can be approximated by functions that agree on neighborhoods of A. A formal statement to this effect can be formulated as follows.

Let A be a closed subset of X and let D be a directed set. Suppose that we can define a convergence $\underset{(1)}{\to}$ for D-nets (i.e., nets with D as the set of indices) of elements of F such that

(1) *for every $f, g \in F$ with $f|A = g|A$ there exist nets $\{f_n\colon n \in D\}$ and $\{g_n\colon n \in D\}$ of functions for F and a net $\{U_n\colon n \in D\}$ of open subsets of X such that $f_n \underset{(1)}{\to} f$, $g_n \underset{(1)}{\to} g$, $A \subset U_n$, and $f_n|U_n = g_n|U_n$.*

We have

4. 7. *Let A be a weak support of φ and suppose that convergences $\underset{(1)}{\to}$ and $\underset{(2)}{\to}$ of D-nets in F and E_1, respectively, are defined. If $\underset{(1)}{\to}$ satisfies condition* (1) *and φ is continuous relative to these convergences, then A is a support of φ.*

Let us mention some cases when a convergence satisfying (1) can be defined.

4. 8. *If E is a normed linear space, then the uniform convergence of sequences in $C(X, E)$ satisfies condition* (1) *(relative to any closed subset of X).*

[6] $S(\alpha)$ denotes the set of all ordinals $\xi < \alpha$.

PROOF. In a normed linear space closed spheres are retracts of the whole space; consequently, we can define a sequence of continuous functions $r_n\colon E\to E$, $n=1, 2, \ldots$, such that $\|r_n(e)\| \leqq \frac{1}{n}$ and $r_n(e)=e$ for $\|e\| \leqq \frac{1}{n}$. Let A be a closed subset of X and let $f, g\in C(X, E)$, $f|A=g|A$. Set $f_n(x) = f(x)+r_n(g(x)-f(x))$ and $g_n(x) = g(x)$ for $n=1, 2, \ldots$. Clearly, $f_n\to f$ and $g_n\to g$ uniformly on X; furthermore $f_n|U_n = g_n|U_n$, where $U_n = \left\{x\in X\colon \|f(x)-g(x)\| < \frac{1}{n}\right\}$; U_n is an open subset of X containing A.

Clearly, 4. 8 can be generalized to other types of linear topological spaces which have bases of (closed) neighborhoods that are retracts of the whole space. If such a space does not satisfy the first axiom of countability, then one has to consider convergence of uncountable nets.

4. 9. *Let E be a topological abelian group (written additively) having a base $\mathfrak{G}$ of neighborhoods of the zero-element* 0 *with* card $\mathfrak{G}\leqq m$. *Let X be 0-dimensional compact. The uniform convergence of nets of cardinality $\leqq m$ in $C(X, E)$ satisfies condition* (1) *(relative to any closed subset of X).*

PROOF. Consider $\mathfrak{G}$ as a directed set; G precedes G_1 iff $G\supset G_1$. Let $f, g\in C(X, E)$, $f|A=g|A$, where A is a closed subset of X. For every $G\in\mathfrak{G}$ there exists a closed-open subset U_G of X such that $A\subset U_G$ and $f(p)-g(p)\in G$ for $p\in U_G$. Define $f_G(p)=g(p)$ for $p\in U_G$, $f_G(p)=f(p)$ for $p\in X\setminus U_G$ and $g_G=g$.

A trivial case in which weak supports are is given by the following.

4. 10. *If either E or X is discrete, then every weak support of a $\varphi\colon F\to E_1$ is a support of φ.*

PROOF. If E is discrete, then the diagonal of $E\times E$ is open in $E\times E$; consequently, if two functions agree on a subset A of X, then they agree on an open superset of A. The case of a discrete X is obvious.

§ 5. Compact case: one-point weak supports

We shall now consider the following question: for what structures E and E_1 is it true that all homomorphisms $\varphi\colon C(X, E)\to E_1$, where X is an arbitrary Hausdorff compact space, have one-point weak supports? We conjecture (see [18]) that this question can be decided by examining finite spaces. A partial success, concerning only 0-dimensional compact spaces, has been obtained in [18]. We shall quote this result.[7]

We say that a topological algebraic structure E is an *s-algebra* provided that among the operations of E there is a binary operation o satisfying the following condition

(s) *for every compact subset C of E there exist elements 0_C and 1_C such that $0_C o e = 0_C o e'$ for every $e, e'\in C$ and $1_C o e=e$ for every $e\in C$.*

[7] This result has been announced in [16].

Examples of s-algebras: every topological ring E with unit element is an s-algebra; one takes o to be the multiplication, 0_C and 1_C to be the zero element and the unit element of E, respectively. Every ordered set considered as a lattice with the order topology is an s-algebra; one takes, for instance, o to be the maximum $(\vee)$ and $0_C = \sup C$, $1_C = \inf C$ for every compact subset C of E.

5. 1. THEOREM. *Let E be an s-algebra and let E_1 be an algebraic structure of the same type as E.*[8] *If every homomorphism $\varphi\colon C(\mathscr{D}_2, E) \to E_1$, where $\mathscr{D}_2$ is the two-point discrete space, has a one-point support, then every homomorphism $\varphi\colon C(X, E) \to E_1$, where X is an arbitrary Hausdorff 0-dimensional compact space, has a one-point weak support.*

As it was pointed out in [18] the above theorem fails if "weak support" is replaced by "support" in its conclusion. The theorem also fails if "0-dimensional" is removed from its assumption. Consider the chain $\mathscr{R}_\alpha$ $(\alpha > \Omega)$ described in Example 4. 6 and let $E = E_1 = \mathscr{R}_\alpha$. It is easy to see that the assumptions of Theorem 5. 1 are satisfied, but its conclusion fails for X = the closed interval $[0, 1]$. But note also that $C(X, E)$, $X = [0, 1]$, does not have the property (K) (sec. 4); it appears that assumptions of this type would enable us to extend Theorem 5. 1 to arbitrary Hausdorff compact spaces.

§ 6. *E*-compact spaces

In the absence of compactness of X the study of supports become more difficult. In particular, it may happen that all functions in $C(X, E)$ can be continuously extended over some extension εX of X (in fact, $C(X, E)$ may turn out to be isomorphic to $C(\varepsilon X, E)$) and homomorphisms of $C(X, E)$ may have very simple supports in εX which, however, are not contained in X. To eliminate such difficulties one needs to assume that X coincides with some of its extensions; an exact formulation of this assumption is that X is E-compact (see statement 6. 3 below). An exposition of the various facts concerning E-compact spaces and the related concept of E-completely regular spaces can be found in [4], [2], [19], [12], [13]; the purpose of the present section is to state in a concise form some information that is relevant to our discussion. Only 6. 4 is proved since its proof cannot be found in the quoted literature.

A space X is said to be E-completely regular (E-compact) provided that, for some cardinal $\mathfrak{m}$, X is homeomorphic to a subspace (a closed subspace, respectively) of some topological power $E^{\mathfrak{m}}$ of E.

6. 1. *Every structure of continuous functions $C(X, E)$, where X is an arbitrary space is isomorphic to the structure $C(X', E)$, where X' is an E-completely regular space. In fact, there is a continuous map Φ of X onto X' such that the map $\overleftarrow{\Phi}$ defined by $\overleftarrow{\Phi}(g) = g\Phi$ for every $g \in C(X', E)$ is an isomorphism of $C(X', E)$ onto $C(X, E)$.*

From now on all spaces will be assumed to be Hausdorff. An *extension* of X is a pair $(X, \varepsilon X)$, where εX is a superspace of X in which X is dense. We will usually

[8] E_1 has therefore at least one binary operation, but we do not assume that E_1 is an s-algebra.

denote $(X, \varepsilon X)$ simple by εX. Two extensions εX and $\varepsilon_1 X$ are said to be equal in the sense of extensions (in symbols: $\varepsilon X \underset{\text{ext}}{=} \varepsilon_1 X$) provided that there exists a homeomorphism h of εX onto $\varepsilon_1 X$ such that $h(p)=p$ for every $p \in X$.

6. 2. *For every E-completely regular space X there exists an (unique up to* $\underset{\text{ext}}{=}$ *) E-compact extension* $\beta_E X$ *of X such that every continuous function* $f \in C(X, Y)$ *where Y is an arbitrary E-compact space, admits a continuous extension* $f^* \in C(\beta_E X, Y)$.

6. 3. *Assume that X is E-completely regular. X is E-compact if, and only if,* $\beta_E X = X$.

According to 6. 2 we can define a map Ψ of $C(X, E)$ onto $C(\beta_E X, E)$ by setting $\Psi(f)$ = the continuous extension f^* of f with $f^* \in C(\beta_E X, E)$; X being dense in $\beta_E X$ implies the uniqueness of f^*. In most cases, Ψ turns out to be an isomorphism.

6. 4. *Let E be a topological algebraic structure such that all the relations of E are E-compact. Let X be E-completely regular. The map* Ψ *defined by*

$\Psi(f)$ *= the continuous extension* $f^* \in C(\beta_E X, E)$ *of* $f \in C(X, E)$*, is an isomorphism of* $C(X, E)$ *onto* $C(\beta_E X, E)$.

PROOF. That Ψ preserves the operations follows easily from the continuity of the operations. Let ϱ be a relation in E; assume for simplicity of notations that ϱ is binary. Let $f, g \in C(X, E)$, let f^* and g^* be continuous extensions of f and g, respectively, with $f^*, g^* \in C(\beta_E X, E)$. We have to show that $f \varrho^{(X)} g$ iff $f^* \varrho^{(\beta_E X)} g^*$. The "if" part is obvious. Assume $f \varrho^{(X)} g$. Define a map h of X into $E \times E$ setting $h(p) = (f(p), g(p))$ for every $p \in X$. The assumption $f \varrho^{(X)} g$ implies that, in fact, $h \in C(X, \varrho)$. Consequently, h admits a continuous extension h^* with $h^* \in C(\beta_E X, \varrho)$. In other words, $h^*(p) \in \varrho$ for every $p \in \beta_E X$. But $h_1(p) = (f^*(p), g^*(p))$ is a continuous map of $\beta_E X$ into $E \times E$ which agrees with h^* on a dense subset of $\beta_E X$, hence $h_1(p) = h^*(p)$ for every $p \in \beta_E X$. This implies that $h_1(p) = (f^*(p), g^*(p)) \in \varrho$ (i.e., $f^*(p) \varrho g^*(p)$) for every $p \in \beta_E X$; i.e., $f^* \varrho^{(\beta_E X)} g^*$.

Recall that every subspace of a finite power $\mathcal{R}^n$ of the reals $\mathcal{R}$ is $\mathcal{R}$-compact; in other words, every finitary relation in $\mathcal{R}$ is $\mathcal{R}$-compact. Consequently, as a particular case of 6. 4 we obtain:

For every completely regular space X the structures $C(X, \mathcal{R})$ *and* $C(\beta_{\mathcal{R}} X, \mathcal{R})$ *are isomorphic relative to all pointwisely defined operations and all finitary pointwisely defined relations.*

NOTE. If εX is an arbitrary extension of X, then Ψ is defined only on the substructure $F_{\varepsilon X}$ of $C(X, E)$ consisting of all those functions f in $C(X, E)$ that admit an extension belonging to $C(\varepsilon X, E)$. Again, the continuity of operations implies that Ψ preserves them; however, in this case Ψ need not preserve E-compact relations. For instance, let $E = \mathcal{R}$, X = the open interval $(0, 1)$, εX = the closed interval $[0, 1]$. $F_{\varepsilon X}$ consists of all uniformly continuous functions on X. Ψ does not preserve the relation $<$; in fact, there are functions $f \in F_{\varepsilon X}$ such that $f(x) > 0$ for every $x \in X$, but it is not true that $f^*(x) > 0$ for every $x \in \varepsilon X$. On the other hand, an argument similar to that used in the proof of 6. 4 shows that in case of an arbitrary extension εX of X, Ψ preserves all relations that are closed in the respective powers of E.

The class of all E-completely regular (E-compact) spaces will be denoted by

$\mathfrak{C}(E)$ ($\mathfrak{K}(E)$, respectively). Note that $\mathfrak{K}(E) \subset \mathfrak{C}(E)$ and $\mathfrak{K}(E) = \mathfrak{K}(E_1)$ implies $\mathfrak{C}(E) = \mathfrak{C}(E_1)$.

A space E is called *admissible* if there is a compact space E^* with $\mathfrak{C}(E) = \mathfrak{C}(E^*)$. If E is admissible, then there exists a compact superspace E_1 of E with $\mathfrak{C}(E) = \mathfrak{C}(E_1)$ (for instance, $E_1 = \beta_{E^*}E$).

6. 5. *Let E be an admissible space and let E_1 be a compact superspace of E with $\mathfrak{C}(E) = \mathfrak{C}(E_1)$. An E-completely regular space X is E-compact if, and only if, the following condition is satisfied*

for every $p_0 \in \beta_{E_1}X \setminus X$ there is a continuous function $f: \beta_{E_1}X \to E_1$ such that $f[X] \subset E$ and $f(p_0) \notin E$.

Note that the extension $\beta_E X$ depends only upon the class of compactness of E; in other words,

6. 6. *If $\mathfrak{K}(E) = \mathfrak{K}(E_1)$, then for every E-completely regular X we have $\beta_E X \underset{\text{ext}}{=} \beta_{E_1} X$.*

Let us now discuss a few examples. If $E = \mathscr{I}$ (= the unit interval [0, 1]) or if E is the space of the reals $\mathscr{R}$, then $\mathfrak{C}(E)$ is the class of all (Hausdorff) completely regular spaces. $\mathfrak{C}(\mathscr{D})$, where $\mathscr{D}$ is a two-point discrete space, is the class of all (Hausdorff) 0-dimensional spaces; in fact, *$\mathfrak{C}(E) = \mathfrak{C}(\mathscr{D})$ iff E is a 0-dimensional space containing more than one point.* $\mathfrak{K}(\mathscr{I})$ is the class of all compact spaces. $\mathfrak{K}(\mathfrak{D})$ is the class of all 0-dimensional compact spaces; in fact *$\mathfrak{K}(E) = \mathfrak{K}(\mathscr{D})$ iff E is a 0-dimensional compact space containing more than one point.* In the next section we shall frequently refer to the class $\mathfrak{K}(\mathscr{N})$[9] where $\mathscr{N}$ is the space of non-negative integers (= the discrete space of cardinality $\aleph_0$). A discrete space is $\mathscr{N}$-compact iff its cardinality is non-measurable in the Ulam sense. We have *$\mathfrak{K}(E) = \mathfrak{K}(\mathscr{N})$ iff E is $\mathscr{N}$-compact and E contains a closed copy of $\mathscr{N}$.* Every $\mathscr{N}$-compact space is 0-dimensional; every Lindelöf 0-dimensional space is $\mathscr{N}$-compact. In particular, for every 0-dimensional non-compact subspace E of the reals $\mathscr{R}$ we have $\mathfrak{K}(E) = \mathfrak{K}(\mathscr{N})$.

§ 7. Non-compact case: $F = C(X, E)$

The purpose of the present and the next section is to show how the previously obtained results can be applied to the case of an arbitrary space X. No general theorems will be proved in these two sections; however, a general procedure will be described in rough terms and then illustrated by a few theorems concerning particular structures E and E_1. In this section we shall discuss the case when F is the whole structure $C(X, E)$; the case of substructures of $C(X, E)$ will be discussed in the next section.

We shall assume that E is admissible; let E_1 be a compact superspace of E. We denote by $C^*(X, E)$ the set of all functions f from $C(X, E)$ such that $f[X]$ is contained in a compact subset of E. (If $E =$ the space of integers, then $C^*(X, E)$ consists of all bounded functions in $C(X, E)$; however, if E = the space of rational numbers, then $C^*(X, E)$ does not contain all bounded functions.) By 6. 2 every $f \in C^*(X, E)$ admits an extension $f^* \in C(\beta_{E_1}X, E)$; in most cases $C^*(X, E)$ and $C(\beta_{E_1}X, E)$ are isomorphic.

[9] This case was first mentioned in [4].

A homomorphism $\varphi: C(X, E) \to E_1$ induces a homomorphism $\varphi^*: C(\beta_{E_1}X; E) \to \to E$; φ^* is defined by $\varphi^*(g) = \varphi(g|X)$ for every $g \in C(\beta_{E_1}X; E)$. Now, $\beta_{E_1}X$ is a compact space; suppose that we are able to prove that a set $A \subset \beta_{E_1}X$ is the smallest support or the smallest weak support of φ^*. Assuming that X is E-compact, we will try to prove that $A \subset X$; here we appeal to statement 6. 5. If $A \subset X$ is proved, then A is a support of φ (or weak support) restricted to $C^*(X, E)$; the last step is to show that A is a support of the whole φ. On the other hand, if X is not E-compact, then we will try to get a negative result: to show an existence of a φ which does not have such supports as those which exists in the case of a compact or E-compact X.

We shall now illustrate the above procedure.

To start with we shall reprove a theorem due essentially to BIAŁYNICKI—BIRULA and ŻELAZKO [1] (see also [7]).

7. 1a. THEOREM. *Let B an algebra over a field K, having the unit element e (both B and K are assumed to carry the discrete topology). If X is K-compact, then every homomorphism $\varphi: C(X, B) \to K$ has a one point support.*

PROOF. Assume first that X is a two-point space, $X = \{p_1, p_2\}$. If there is a homomorphism $\varphi: C(X, B) \to K$ such that none of the points p_i is a support of φ, then there are four functions f_i, g_i, $i = 1, 2$, such that $f_i(p_i) = g_i(p_i)$ and $\varphi(f_i) \neq \neq \varphi(g_i)$ for $i = 1, 2$. The function $f = (f_1 - g_1)(f_2 - g_2)$ is identically equal to 0, but $\varphi(f) = (\varphi(f_1) - \varphi(g_1))(\varphi(f_2) - \varphi(g_2)) \neq 0$ which is impossible. Thus, the conclusion of the theorem is satisfied for a two point space X; consequently, by Theorem 5. 1, if X is a 0-dimensional compact space, then every $\varphi: C(X, B) \to K$ has a one-point weak support. But B is discrete, hence by 4. 10, φ has a one-point support.

If K is finite, then the theorem is shown; in fact, in this case being K-compact is equivalent to X being 0-dimensional Hausdorff compact. Assume therefore that K is infinite and let X be a K-compact space. We shall assume that K is contained in B. Let e be the unit element of B; e is also the unit element of K; let $C_0(X, K)$ denote the set of all constant functions $f: X \to K$. For every $k \in K$ we denote by $f^{(k)}$ the constant function on X whose value is k. We can assume that

$$(1) \qquad \varphi(f^{(k)}) = k \quad \text{for every} \quad k \in K;$$

indeed, φ restricted to $C_0(X, K)$ induces in a natural way an endomorphism of K, say α; this endomorphism does not vanish identically $(\varphi(f^{(e)}) \neq 0$; for otherwise $\varphi(f) = 0$ for every $f \in C(X, B))$, hence α is one-to-one; compose φ with α^{-1}. Clearly, if $\alpha^{-1} \circ \varphi$ has a one-point support then φ has also.

Let K_1 be the one-point compactification of K; K_1 is a compact superspace of K with $\mathfrak{C}(K_1) = \mathfrak{C}(K) = \mathfrak{C}(B)$. $C^*(X, B)$ consists of all functions in $C(X, B)$ having finitely many values. Each function $f \in C^*(X, B)$ admits a continuous extension $f^* \in C(\beta_{K_1}X, B)$. Let us set $\varphi^*(g) = \varphi(g|X)$ for every $g \in (\beta_{K_1}X, B)$ and, by the first part of the proof, φ^* has a one-point support $\{p_0\}$ in $\beta_{K_1}X$. We shall prove that $p_0 \in X$.

Assume that $p_0 \in \beta_{K_1}X \setminus X$. There is continuous function $g_0: \beta_{K_1}X \to K_1$ such that $g_0[X] \subset K$ and $g_0(p_0) = \infty$ (where ∞ is the ideal point of the one point compactification K_1 of K). Let $f_0 = g_0|X$; clearly $f_0 \in C(X, B)$. Let $k_0 = \varphi(f_0)$. There is a neighborhood U of p_0 such that $g_0(p) \neq k_0$ for every $p \in U$. Let $A = \{p \in \beta_{K_1}X: g_0(p) = k_0\}$; we have $A \cap U = \emptyset$. Take a $k_1 \in K$ with $k_1 \neq k_0$ and set $g_1(p) = k_1$ for $p \in A$ and $g_1(p) = k_0$ for $p \in \beta_{K_1}X \setminus A$. $g_1 \in C(\beta_{K_1}X, B)$ and from (1) we infer that $\varphi^*(g_1) = k_0$. Setting $f_1 = g_1 | X$, we have $\varphi(f_1) = k_0$, consequently, $\varphi(f_0 - f_1) = 0$; but $(f_0 - f_1)(p) \in K$ and $(f_0 - f_1)(p) \neq 0$ for every $p \in X$; therefore $f_0 - f_1$ has an inverse in $C(X, B)$. This contradicts the fact that $\varphi(f_0 - f_1) = 0$; hence $p_0 \in X$.

It follows from the above that $\{p_0\}$ is a support of φ restricted to $C^*(X, B)$. Let f_1 and f_2 be two arbitrary functions in $C(X, B)$ with $f_1(p_0) = f_2(p_0)$. Let $A = \{p \in X: f_1(p) = f_2(p)\}$; A is a closed-open subset of X. Set $f_3(p) = e$ for $p \in A$, $f_3(p) = 0$ for $p \in X \setminus A$. Then $f_3 \in C^*(X, B)$, therefore $\varphi(f_3) = \varphi(f^{(e)}) = e$. On the other hand, the function $(f_1 - f_2)f_3$ is identically equal to 0, therefore $\varphi(f_1 - f_2) \cdot \varphi(f_3) = \varphi((f_1 - f_2) \cdot f_3) = 0$; therefore $\varphi(f_1 - f_2) = 0$; thus $\varphi(f_1) = \varphi(f_2)$. Consequently, $\{p_0\}$ is a support of φ.

The following is the converse of 7. 1.a.

7. 1. b. THEOREM. *Let K be a field with the discrete topology. If X is not K-compact, then there exists a homomorphism $\varphi: C(X, K) \to K$ which does not have a one-point support in X.*

PROOF. Every function $f \in C(X, K)$ admits an extension $f^* \in C(\beta_K X, K)$; take a point $p_0 \in \beta_K X \setminus X$ (note that $\beta_K X \neq X$) and let $\varphi(f) = f^*(p_0)$.

Theorem 7. 1.a and 7. 1.b such be compared with the results of [1] (or with a more general version of these results given in [7]). If K is finite, then (as it was already observed) X is K-compact iff X is compact; hence, in this case, a discrete X is K-compact iff X is finite. If $\aleph_0 \leqq \text{card } K < \aleph_I$, where $\aleph_I$ is the first measurable cardinal (in the Ulam sense), then X is K-compact iff X is N-compact; hence, in this case, a discrete X is K-compact iff card $X < \aleph_I$. In general, setting $m = \text{card } K$, we have that a discrete X is K-compact iff card $X < \aleph(m)$. $\aleph(m)$ is used here in the sense of [7].

Theorem 7. 1.a is not the best one. The proof shows that this theorem remains valid if K is integral domain satisfying the condition

(2) *for every space X and every non-constant homomorphism $\varphi: C(X, K) \to K$, if $f \in C(X, K)$ and $f(p) \neq 0$ for every $p \in X$, then $\varphi(f) \neq 0$.*

It has been shown in [11] that the ring of integers satisfies (2) (see [11], § 5, (v)). Consequently, Theorem 7. 1.a is true if K is the ring of integers. (The last statement is more general than Theorem 2 in [11].)

REMARK 1. Condition (2) obviously implies the following one

(3) *for every non-constant endomorphism α of K we have $\alpha(k) \neq 0$ for every $k \in K$, $k \neq 0$.*

We do not know if (3) implies (2). It is easy to see that (3) is equivalent to

(3a) *every endomorphism α of K can be extended to an endomorphism $\tilde{\alpha}$ of $\tilde{K}$, where $\tilde{K}$ is the field of quotients of K.*

Similarly, (2) is equivalent to

(2a) *for every space X, every homomorphism $\varphi: C(X, K) \to K$ can be extended to a homomorphism $\tilde{\varphi}: C(X, \tilde{K}) \to \tilde{K}$, where $\tilde{K}$ is the field of quotients of K.*

We shall now discuss the case where $E = E_1$ is an ordered subgroup of the reals $\mathscr{R}$. In other words, we shall discuss maps φ of $C(X, E)$ into E. That preserve $+$ and $\leqq$. If $E = \mathscr{R}$, then such maps coincide with integrals (in the case of compact, or more generally, $\mathscr{R}$-compact, X); consequently, they need not to have finite supports. In contrast to this we shall show

7. 2. a. THEOREM. *If E is a proper ordered subgroup of the additive group of the reals $\mathscr{R}$ and X is an $\mathscr{N}$-compact space, then every homomorphism $\varphi: C(X, E) \to E$ has a finite support.*

One can assume without the loss of generality that E contains the number 1. This assumption will be kept throughout the following discussion.

The above theorem for the case of a $\mathscr{D}$-compact X has been announced in [10]. We shall start with the proof of this particular case. We need the following:

7. 3. LEMMA. *Let E be a subgroup of the additive group of the reals. Assume that there is a sequence $\alpha_1, \alpha_2, \ldots$ of positive numbers such that*

(a) $\sum_n \alpha_n < +\infty$

and

(b) *for every sequence $x_n \in E$ with $x_n \to 0$ we have $\sum_n \alpha_n x_n \in E$.*

Then $E = \mathscr{R}$.

PROOF. Let $x \in E$, $x \neq 0$. Then $\alpha_n x \in E$ for $n = 1, 2, \ldots$; hence E contains a sequence convergent to 0, therefore E is dense in $\mathscr{R}$.

Let c be an arbitrary real. By induction one can define a sequence $x_1, x_2, \ldots$ of elements of E such that

$$(4) \qquad |\alpha_1 x_1 + \cdots + \alpha_n x_n - c| < \min\left\{\frac{1}{2n}\alpha_n, \frac{1}{2(n+1)}\alpha_{n+1}\right\}.$$

Clearly, $\sum_n \alpha_n x_n = c$. It remains to show that $x_n \to 0$. We shall show that $|x_{n+1}| < \frac{1}{n+1}$ for $n = 1, 2, \ldots$. Let $c_n = \alpha_1 x_1 + \cdots + \alpha_n x_n - c$; we have

$$|c_n| < \min\left\{\frac{1}{2n}\alpha_n, \frac{1}{2(n+1)}\alpha_{n+1}\right\}.$$

Now

$$|x_{n+1}| = \frac{1}{\alpha_{n+1}}|\alpha_{n+1}x_{n+1}| = \frac{1}{\alpha_{n+1}}\left(|\alpha_{n+1}x_{n+1}| - |c_n| + |c_n|\right) \leqq$$

$$\leqq \frac{1}{\alpha_{n+1}}\left(|c_n + \alpha_{n+1}x_{n+1}| + |c_n|\right) = \frac{1}{\alpha_{n+1}}\left(|c_{n+1}| + |c_n|\right) \leqq$$

$$\leqq \frac{1}{\alpha_{n+1}}\left(\frac{1}{2(n+1)}\alpha_{n+1} + \frac{1}{2(n+1)}\alpha_{n+1}\right) = \frac{1}{\alpha_{n+1}} \cdot \frac{\alpha_{n+1}}{n+1} = \frac{1}{n+1}.$$

Proof of Theorem 7. 2.a *for a $\mathscr{D}$-compact X*. Let X be $\mathscr{D}$-compact (i.e., 0-dimensional and compact) and let $\varphi\colon C(X, E)\to E$ be a given homomorphism. By remarks after Theorem 4. 2, φ has a smallest weak support A. Let $\underset{(1)}{\to}$ be the uniform convergence of sequences in $C(X, E)$; by 4. 9, $\underset{(1)}{\to}$ satisfies condition (1) of §4. Clearly, φ is continuous relative to $\underset{(1)}{\to}$ and the usual convergence in E; consequently, A is the smallest support of φ. It remains to show that A is finite.

Assume A is infinite. There is a sequence $U_1, U_2, \dots$ of mutually disjoint closed-open subsets of X with $U_n \cap A \neq \emptyset$ for $n=1, 2, \dots$. Set $f_n(p)=1$ for $p\in U_n$ and $f_n(p)=0$ for $p\in X\setminus U_n$. Let $\alpha_n=\varphi(f_n)$. We have $\alpha_n\in E$ and $\alpha_n>0$ (if $\alpha_n=0$, then $X\setminus U_n$ would be a support of φ). On the other hand, $\varphi(f_1+\dots+f_n)\leqq\varphi(g)$, where g is the function identically equal to 1; therefore the series $\sum\limits_n \alpha_n$ is convergent. Let $x_1, x_2, \dots$ be an arbitrary sequence of elements of E with $x_n\to 0$. The function f, defined by $f(p)=x_n$ for $p\in U_n$ and $f(p)=0$ for $p\in X\setminus\bigcup\{U_n\colon n=1, 2, \dots\}$, belongs to $C(X, E)$; moreover, $f=\sum\limits_n x_n\cdot f_n$, the convergence of the series being uniform. It follows that $\sum\limits_n \alpha_n\cdot x_n=\sum\limits_n x_n\cdot\varphi(f_n)=\sum\limits_n \varphi(x_n\cdot f_n)=\varphi(f)\in E$; consequently, by Lemma 7. 3, $E=\mathscr{R}$, contrary to the assumption.

To complete the proof we need still two lemmas.

7. 4. Lemma. *For every $f\in C(X, E)$ there is a sequence $g_1, g_2, \dots$ of functions from $C(X, E)$ such that each g_n has only finitely many values and the set of functions $nf-g_n, n=1, 2, \dots$, is bounded in $C(X, E)$ (i.e., there is an $h\in C(X, E)$ such that $|nf-g_n|\leqq h$ for every n).*

Proof. Select a sequence of numbers $0<a_1<a_2<\dots$ such that $a_n\notin E$ and $a_n\to\infty$. For every n select a $b_n\in E$ with $a_n^2<b_n$. Let $A_1=\{p\in X\colon |f(p)|<a_1\}$ and $A_n=\{p\in X\colon a_{n-1}<|f(p)|<a_n\}$ for $n=2, 3, \dots$. The sets A_n are closed and open and $\bigcup\limits_n A_n=X$. Define $h(p)=b_n+2$ for $p\in A_n$. Clearly, $h\in C(X, E)$ and

$$f^2(p)+2<h(p) \quad \text{for every} \quad p\in X.$$

Now, for a given n select $\alpha_0<\alpha_1<\dots<\alpha_s$ so that

$$\alpha_0\leqq -n^2<n^2\leqq\alpha_s, \quad 1<\alpha_{i+1}-\alpha_i<2, \quad \alpha_i\setminus E.$$

Since $\alpha_{i+1}-\alpha_i>1$ (and $1\in E$), we can find $\beta_i\in E$ with $\alpha_i<\beta_i<\alpha_{i+1}$ for $i=0, 1, \dots, s-1$. Set $B_i=\{p\in X\colon \alpha_i<nf(p)<\alpha_{i+1}\}$ for $i=0, 1, \dots, s-1$. B_i are closed and open; the set $B=\bigcup\{B_i\colon i=0, \dots, s-1\}$ is also closed and open. Set

$$g_n(p)=\beta_i \quad \text{for} \quad p\in B_i, \quad g_n(p)=0 \quad \text{for} \quad p\in X\setminus B.$$

We then have

$$|nf(p)-g_n(p)|\leqq h(p) \quad \text{for every} \quad p\in X.$$

Indeed, if $p\in B_i$ (for some i), then

$$|nf(p)-g_n(p)|\leqq\alpha_{i+1}-\alpha_i<2\leqq h(p);$$

on the other hand, if $p\in X\setminus B$, then $|nf(p)|\geqq n^2$, hence $|f(p)|\geqq n$, therefore $|nf(p)\leqq \leqq f^2(p)\leqq h(p)$.

We shall now consider additive maps of $C(X, E)$ into E that are bounded (i.e., they carry bounded sets of functions in $C(X, E)$ into bounded sets of numbers).

Every homomorphism of $C(X, E)$ into E is an additive bounded map; the difference of two additive bounded maps is again an additive bounded map.

7. 5. LEMMA. *Let $C^{**}(X, E)$ be the set of all functions in $C(X, E)$ that have only finitely many values. If two additive bounded maps of $C(X, E)$ into E agree on $C^{**}(X, E)$, then they agree everywhere on $C(X, E)$.*

PROOF. It suffices to show that if an additive bounded map ψ of $C(X, E)$ into E vanishes on $C^{**}(X, E)$, then ψ vanishes everywhere. Let f be an arbitrary function in $C(X, E)$. By Lemma 7. 4; there exists a sequence $g_1, g_2, \ldots$ of functions from $C^{**}(X, E)$ such that the set $nf-g_n$, $n=1, 2, \ldots$, is bounded. Consequently, the set of numbers $\psi(nf-g_n)$, $n = 1, 2, \ldots$, is bounded. But $\psi(g_n) = 0$, hence $\psi(nf-g_n) = n\psi(f)$; therefore $\psi(f)=0$.

Proof of Theorem 7. 2a *for the general case.* Recall the material of §6 and the remarks at the beginning of the present section. Let E_1 be a 0-dimensional compact superspace of E. We have $\beta_{E_1}X \underset{\text{ext}}{=} \beta_{\mathcal{D}}X$. Let φ be a homomorphism of $C(X, E)$ into E; we can assume that φ does not vanish identically. Since $C^*(X, E)$ is isomorphic to $C(\beta_{\mathcal{D}}X, E)$ (and the theorem is true in the compact case), we infer that φ restricted to $C^*(X, E)$ has a finite support A contained in $\beta_{\mathcal{D}}X$. Let $A=\{p_1, \ldots, p_k\}$. It is clear that we have

$$\varphi(f) = \alpha_1 f^*(p_1)+\cdots+\alpha_k f^*(p_k) \quad \text{for every} \quad f\in C^*(X, E) \tag{5}$$

where $\alpha_1, \ldots, \alpha_k$ are fixed numers and f^* denotes the continuous extension of f over $\beta_{\mathcal{D}}X$. We can assume that all α_i are positive.

We shall prove that $A\subset X$. Indeed, assume that $p_{i_0}\in\beta_{\mathcal{D}}X\setminus X$. Since X is $\mathcal{N}$-compact, there is a continuous function $f_0^*\colon \beta_{\mathcal{D}}X\to\mathcal{N}^*$ ($\mathcal{N}^*=\mathcal{N}\cup\{\infty\}$ is the one-point compactification of $\mathcal{N}$) such that $f_0^*(p_{i_0})=\infty$ and $f_0^*(p)\in\mathcal{N}$ for every $p\in X$; see 6. 5. Clearly, it can be assumed that $f_0^*(p_i)=0$ for $i\neq i_0$. Let $f_0=f_0^*|X$; we have (in view of the assumption $1\in E$) $f_0\in C(X, E)$. Let $f_0^{(n)}=f_0\wedge n$ for $n=1, 2, \ldots$. Clearly, $f_0^{(n)}\in C^*(X, E)$, hence, from (5) we infer that $\varphi(f_0^{(n)})=\alpha_{i_0}\cdot n$. But $0\leqq f_0^{(n)}\leqq f_0$, therefore $0\leqq\varphi(f_0^{(n)})\leqq\varphi(f_0)$ for $n=1, 2, \ldots$; and this implies that, contrary to the assumption, $\alpha_{i_0}=0$. Thus $A\subset X$.

Knowing that $A\subset X$ we can rewrite (5) as follows

$$\varphi(f) = \alpha_1 f(p_1)+\cdots+\alpha_k(p_k) \quad \text{for every} \quad f\in C^*(X, E). \tag{6}$$

It suffices to show that (6) holds for every $f\in C^*(X, E)$. This, however, follows immediately from Lemma 7. 5. Indeed, the left-hand side of (6) defines a homomorphism of $C(X, E)$ which agrees with φ on $C^{**}(X, E)$ (in fact, on $C^*(X, E)$). Therefore the left-hand side of (6) agrees with φ everywhere on $C(X, E)$.

Theorem 7. 2a is shown.

The converse of Theorem 7. 2a is obvious.

7. 2b. THEOREM. *If X is not E-compact, then there exists a homomorphism $\varphi\colon C(X, E)\to E$ without a finite support.*

PROOF. It suffices to set

$$\varphi(f)=f^*(p_0) \quad \text{for every} \quad f\in C(X, E),$$

where p_0 is a fixed point of $\beta_E X \setminus X$ and f^* denotes the continuous extension of f with $f^*: \beta_E X \to E$. It is clear that no compact subset of X is a support of φ.

As a still another example of the above procedure one could mention a generalization of a result of Turowicz due to R. C. Moore. TUROWICZ [20] considers multiplicative functionals $\varphi: C(X, \mathscr{R}) \to \mathscr{R}$ that are continuous with respect to the uniform convergence and proves that if X is compact, then every such functional has a countable support — in fact, Turowicz obtains a representation formula for such functionals.[10] R. C. MOORE [6] proves that every such functional has a countable compact support in X (and hence is representable in Turowicz's form) iff X is $\mathscr{R}$-compact.[11]

In [2] BLEFKO proves a result[12] related to Theorems 7.1a and 7.1b and Theorem 2 in [11].

7. 6. THEOREM (BLEFKO). *Let $\mathscr{P}$ be the ring of rationals with the standard topology. Every homomorphism $\varphi: C(X, \mathscr{P}) \to \mathscr{P}$ has a one-point support in X if, and only if, X is $\mathscr{N}$-compact.*

The above seems to be the only result concerning a non-locally compact structure.

§ 8. Non-compact case: $F \subset C(X, E)$

When dealing with substructures F of $C(X, E)$ it can always be assumed that F separates points and closed sets of X. A formal statement to this effect is as follows. Let $f_1, \ldots, f_k$ be functions from X into E. We denote by $\langle f_1, \ldots, f_k \rangle$ the map of X into the product E^k whose value at a point $p \in X$, $\langle f_1, \ldots f_k \rangle(p)$, is the point $(f_1(p), \ldots, f_k(p))$ of E^k. A class F of continuous functions from X into E is called an *E-separating* class for X provided that for every closed set $A \subset X$ and every point $p \in X \setminus A$ there is a finite number of functions $f_1, \ldots, f_k$ from F such that $\langle f_1, \ldots, f_k \rangle(p) \notin \mathrm{cl}\, \langle f_1, \ldots, f_k \rangle[A]$, where cl denotes the closure in E^k. The following statement is a generalization of 6. 1.

8. 1. *Let $F \subset C(X, E)$. There exists an E-completely regular space X' and a continuous map Φ of X onto X' such that every $f \in F$ can be (uniquely) written in the form $f = g \circ \Phi$; furthermore, the class F' of all those $g \in C(X', E)$ for which $g \circ \Phi \in F$ is an E-separating class for X'.*

Thus, if we let (as in 6. 1) $\overleftarrow{\Phi}(g) = g \circ \Phi$ for every $g \in F'$, then $\overleftarrow{\Phi}$ is a one-to-one map of F' into F and obviously $\overleftarrow{\Phi}$ is an isomorphism relative to pointwisely defined operations and relations. In other words, F is isomorphic to an E-separating structure.

In the preceding section when studying the whole structure $C(X, E)$ we used certain relation between X and one of the maximal compactifications (statement 6. 5).

[10] Turowicz has formulated his result only for the case of a compact metric X. However, in [3], BOURGIN shows that the same procedure can be applied in case of arbitrary compact (Hausdorff) spaces.

[11] This result has been announced in [15].

[12] This result has been announced in [14].

Sometimes this procedure can be applied also to substructures of $C(X, E)$. For some substructures F of $C(X, E)$ it is possible to assign a compactification cX of X such that all homomorphisms of F have support of a certain type in X iff certain relation holds between X and cX. This procedure was applied in [8] to substructures of $C(X, \mathcal{R})$, where $\mathcal{R}$ is the ring of the reals; let us briefly recall the known facts.

If X is compact, then all homomorphisms of the ring $C(X, \mathcal{R})$ into $\mathcal{R}$ have one-point support. A subset P of a space X is said to be Q-closed in X provided that for every $p_0 \in X \setminus P$ there is a continuous function $f: X \to [0, 1]$ such that $f(p_0) = 0$ and $f(p) > 0$ for every $p \in P$. For an arbitrary (completely regular) space X all homomorphisms of the ring $C(X, \mathcal{R})$ into $\mathcal{R}$ have one-point supports in X iff X is Q-closed in βX. Consider now subrings F of $C(X, \mathcal{R})$ such that (a) F contains all constant functions on X, (b) F is inverse closed (i.e., if $f \in F$ and $f(p) \neq 0$ for every $p \in X$, then $1/f \in F$), and (c) F is closed with respect to uniform convergence. It was shown in [8] (Theorem 2) that to each subring F satisfying the above conditions it is possible to assign a compactification cX of X such that all homomorphisms of F have one-point supports in X iff X is Q-closed in cX. The compactification cX can be defined, for instance, as the smallest compactification such that all bounded functions in F can be continuously extended over cX.[13] It was shown in [9] that similar theorems hold true for some linear sublattices of $C(X, \mathcal{R})$.

In this section we shall give still another illustration of the above procedure. We shall obtain results paralleling those of [8] but concerning some subrings of $C(X, \mathcal{Z})$, where $\mathcal{Z}$ is the ring of integers (homomorphisms of the whole ring $C(X, \mathcal{Z})$ have been studied in [11]). These results, in turn, will be applied to obtain a characterization of the class of strongly non-measurable cardinals in the Ulam sense (see [12]).

8. 2a. THEOREM. *Let F be a subring of $C(X, \mathcal{Z})$ satisfying the following conditions:*

(a) *F contains all constant functions.*

(b) *$f \in F$ iff all truncations of f belong to F;*[14]

(c) *F is closed under composition with functions $\alpha: \mathcal{Z} \to \mathcal{Z}$ (i.e., for every $f \in F$ and for every $\alpha: \mathcal{Z} \to \mathcal{Z}$, the composition $\alpha \circ f$ belongs to F);*

(d) *F is $\mathcal{Z}$-separating.*

Let cX be the smallest compactification such that every function f in F admits a continuous extension $f^*: cX \to \mathcal{Z} \cup \{\pm\infty\}$. If X is Q-closed in cX, then every homomorphism $\varphi: F \to \mathcal{Z}$ has a one-point support in X.

The proof of this theorem is almost identical with that of Theorem 2 in [11]; let us only discuss the necessary changes. The compactification cX is 0-dimensional;[15]

[13] cX can also be defined as the smallest compactification such that every function f in F admits a continuous extension $f^*: cX \to \mathcal{R} \cup \{\pm\infty\}$, where $\mathcal{R} \cup \{\pm\infty\}$ is the (unique) two-point compactification of $\mathcal{R}$.

[14] The i-th truncation of f is defined by $f^{(i)} = -i \vee (f \wedge i)$.

[15] It is useful to formulate a general statement concerning such compactifications.

Let E be a compact space, let X be an E-completely regular space, and let F be an E-separating class for X.

(a) *There exists the smallest compactification cX of X having the property*

(i) *evevy $f \in F$ admits a continuous extension $f^*: cX \to E$.*

(b) *This compactification cX is E-completely regular.*

hence if $p_0 \in cX \setminus X$, then there is a continuous function $g: cX \to [0, 1]$ such that $g(p_0)=0$ and $g(p)>0$ for every $p \in X$. Using 0-dimensionality of cX we can modify g so that its values on X are of the form $1/n$. Taking the reciprocal of g we obtain a continuous function $f^*: cX \to \mathscr{Z} \cup \{\pm\infty\}$ with $f^*(p_0)=\infty$ and $0<f(p)<+\infty$ for every $p \in X$. It is now clear that the considerations of [11] can be applied if we shall show that F contains all functions f from $C(X, \mathscr{Z})$ that admit continuous extensions $f^*: cX \to \mathscr{Z} \cup \{\pm\infty\}$. This will be accomplished in the following two lemmas.

8. 3. LEMMA. *Let X be a compact space and let F be a subring of $C(X, \mathscr{Z})$ that satisfies* (a) *and* (c) *of Theorem* 8. 2a. *If F distinguishes points of X (i.e., if for every $p, q \in X$ with $p \neq q$ there is an $f \in F$ with $f(p) \neq f(q)$), then $F = C(X, \mathscr{Z})$.*

PROOF. A straightforward compactness argument shows that for each pair of disjoint closed subsets A and B of X there is an $f \in F$ with $f(p)=0$ for $p \in A$ and $f(p)=1$ for $p \in B$. Let g be an arbitrary function from $C(X, \mathscr{Z})$; let $k_1, \ldots, k_n$ be all the values of g. Let $A_i = g^{-1}[k_i]$. There are functions $f_1, \ldots, f_n \in F$ such that $f_i(p)=0$ for $p \in \bigcup\{A_j: j<i\}$ and $f_i(p)=1$ for $p \in \bigcup\{A_j: j \geqq i\}$. Let $f = f_1 + \cdots + f_n$. We have $f \in F$ and $f(p)=j$ for $p \in A_j$. It suffices to compose f with a function $\alpha: \mathscr{Z} \to \mathscr{Z}$ such that $\alpha(j)=k_j$ for $j=1, 2, \ldots, n$.

8. 4. LEMMA. *Under the notations and the assumptions of Theorem* 8. 2a, *F contains all functions f on X that admit continuous extensions $f^*: cX \to \mathscr{Z} \cup \{\pm\infty\}$.*

PROOF. Let F^* be the set of all bounded functions in F. It follows directly from condition (d) that the class of all continuous extensions of members of F^* over cX distinguishes points of cX (use also footnote[15]). Consequently, by the preceding lemma, F^* contains all bounded function from $C(X, \mathscr{Z})$ that admit continuous extensions over cX. The lemma now follows directly from condition (b).

Note that in the converse of Theorem 8. 2a we can relax the condition on F.

8. 2b. THEOREM. *Let F be an arbitrary subring of $C(X, \mathscr{Z})$ that is $\mathscr{Z}$-separating and let cX be defined as in* 8. 2a. *If X is not Q-closed in cX, then F admits a homomorphism $\varphi: F \to \mathscr{Z}$ which does not have a one-point support in X.*

PROOF. There is a point $p_0 \in cX \setminus X$ such that for no continuous function $g: cX \to [0, 1]$ it is true that $g(p_0)=0$ and $g(p)>0$ for every $p \in X$. It is clear that for every continuous extension $f^*: cX \to \mathscr{Z} \cup \{\pm\infty\}$ of an $f \in F$ we have $f^*(p_0) \in \mathscr{Z}$. Consequently, the formula $\varphi(f)=f^*(p_0)$ for every $f \in F$ defines a homomorphism of F into $\mathscr{Z}$. Clearly, φ does not have a compact support in X.

We are now ready to give the characterization of the class $\mathscr{Z}$ of strongly non-measurable cardinals (see [12]).

(c) *This compactification cX can also be characterized as the compactification having property* (i) *and the following one*
(ii) *for every $p, q \in cX \setminus X$, if $p \neq q$, then there is an $f \in F$ such that $f^*(p) \neq f^*(q)$, where f^* is the continuous extension of f with $f^*: cX \to E$.*
(Note that the implication in (ii) holds for every $p, q \in cX$).
Verification of the above statement is routine.
In the proof of Theorem 8. 2a we apply this statement with $E = \mathscr{Z} \cup \{\pm\infty\}$.

8. 5. THEOREM. *Let* $\mathfrak{m}$ *be a cardinal satisfying* $\mathfrak{m}^{\aleph_0}=\mathfrak{m}$ *and let* $X_{\mathfrak{m}}$ *be a discrete space of cardinality* $\mathfrak{m}$. *The following are equivalent*

(a) $\mathfrak{m}\in\mathcal{M}$;

(b) *there is a subring* F *of* $C(X_{\mathfrak{m}}, \mathscr{Z})$ *such that* F *is* $\mathscr{Z}$*-separating, every homomorphism* $\varphi: F\to\mathscr{Z}$ *has a one-point support in* $X_{\mathfrak{m}}$, *and card* $F=\mathfrak{m}$.

PROOF. Let $\mathfrak{m}\in\mathcal{M}$. By Theorems 4. 1 and 5. 1 in [12], there is a class H of continuous functions $h: \beta X_{\mathfrak{m}}\to[0, 1]$ such that $h(p)>0$ for every $p\in X_{\mathfrak{m}}$ and every $h\in H$ and for every $p\in\beta X_{\mathfrak{m}}\setminus X_{\mathfrak{m}}$ there is an $h\in H$ with $h(p)=0$; furthermore, card $H=\mathfrak{m}$. Using 0-dimensionality of $\beta X_{\mathfrak{m}}$ we can assume that all the functions h in H have values of the form $1/n$ on $X_{\mathfrak{m}}$. Let F_0 be the class of the reciprocals of the restrictions of members of H to $X_{\mathfrak{m}}$; let F_1 be an arbitrary $\mathscr{Z}$-separating class for $X_{\mathfrak{m}}$ with card $F_1=\mathfrak{m}$. Let F be the smallest subring of $C(X_{\mathfrak{m}}, Z)$ containing $F_0\cup F_1$ and satisfying conditions (a), (b), and (c) of Theorem 8. 2a. From $\mathfrak{m}^{\aleph_0}=\mathfrak{m}$ we infer that card $F\leqq\mathfrak{m}$. It is easy to see that $X_{\mathfrak{m}}$ is Q-closed in the corresponding compactification $cX_{\mathfrak{m}}$ of $X_{\mathfrak{m}}$. Consequently, the conclusion follows directly from Theorem 8. 2a.

Conversely, assume that (b) is satisfied. Let $cX_{\mathfrak{m}}$ be the compactification corresponding to F. By Theorem 8. 2b, $X_{\mathfrak{m}}$ is Q-closed in $cX_{\mathfrak{m}}$. From card $F=\mathfrak{m}$ we infer that $cX_{\mathfrak{m}}$ has a base of cardinality $\mathfrak{m}$; in fact, the class of all continuous extensions $f^*: cX_{\mathfrak{m}}\to Z\cup\{\pm\infty\}$ of functions $f\in F$ is a $\mathscr{Z}\cup\{\pm\infty\}$-separating class for $cX_{\mathfrak{m}}$. Consequently, by Theorem 5. 1 in [12], $\mathfrak{m}\in\mathcal{M}$.

It is easy to see that if the cardinal $\mathfrak{m}$ in the above theorem is of the form $\mathfrak{m}=2^{\mathfrak{n}}$, then we can find a ring F satisfying (b) which is closed relative to any system of $\mathfrak{m}$ operations each having $\leqq\mathfrak{n}$ arguments.

Theorem 8. 5 was announcend in [12]. As it was pointed out in [12], a similar theorem can be proved for subrings of $C(X_{\mathfrak{m}}, \mathscr{R})$ (where $\mathscr{R}$ is the ring of the reals). In general, with the aid of the class **M** one can prove for various structures E the existence of substructures F of $C(X_{\mathfrak{m}}, E)$ (i.e., of direct products of copies of E) such that F has essentially the same homomorphisms into E as $C(X_{\mathfrak{m}}, E)$ but F is not isomorphic to any $C(X, E)$. Furthermore, for sufficiently large Ulam non-measurable cardinals, F can assumed to be closed relative to large systems of operations of huge numbers of arguments. This indicates the impossibility of axiomatic description of direct products of E by means of formulas (of possibily infinite length) involving only elements and homomorphisms of $C(X_{\mathfrak{m}}, E)$, provided that the number of these formulas and their length is Ulam non-measurable. More remarks on this subject will be published later.

§ 9. Concluding remarks

In Section 7 we used the substructure $C^*(X, E)$ to reduce the study of supports to the compact case. Sometimes a different procedure is possible. If E admits a compact superstructure E^*, then $C(X, E)$ is isomorphic to a substructure of $C(\beta_{E^*}X, E^*)$. The same is true for substructures of $C(X, E)$. We can therefore use $C(\beta_{E^*}X, E)$ to reduce the study of supports to the compact case. This procedure can be used, for instance, when $C(X, E)$ is considered as a lattice of continuous functions with values in a chain E; indeed every chain E can be extended to a compact chain. In fact, this procedure has been used implicitly by several authors in the study of homomorphisms of lattices of continuous functions. The author plans to publish a paper containing further applications; it will be shown that results similar to those discussed in the preceding section can also be obtained for some sublattices of $C(X, \mathscr{R})$.

Representation theorems for homomorphisms frequently lead to the so-called "homeomorphism theorems". The first such theorem is due to Banach: if X and Y are compact metric spaces and $C(X, \mathscr{R})$ and $C(Y, \mathscr{R})$ are isomorphic as Banach spaces, then X and Y are homeomorphic. We shall say that a structure E is (topologically) *determining*, provided that for every E-compact spaces X and Y the isomorphism of $C(X, E)$ and $C(Y, E)$ implies the homeomorphism of X and Y. It follows from 6. 4 that *if the relations of E are E-compact, then the class of all E-compact spaces is a maximal class of spaces in which the above implication may hold.* There is a group of theorems asserting that the various structures on the set $\mathscr{R}$ of the reals are determining. Perhaps the best known is the one in which $\mathscr{R}$ is considered as a ring; at the same time, this is the weakest theorem in this direction. In fact, if Φ is a ring-isomorphism between $C(X, \mathscr{R})$ and $C(Y, \mathscr{R})$, then Φ is an isomorphism relative to all pointwisely defined operations and relations. The strongest out of presently known theorems is the one where $\mathscr{R}$ is considered as a lattice. It would be interesting to see whether this is, in fact, the strongest possible theorem in this direction. The question can be formulated as follows. Suppose that $E=\{\mathscr{R}; \{0_0, ..., 0_\xi, ...\}_{\xi<\alpha}; \{\varrho_0, ..., \varrho_\eta\}_{\eta<\beta}\}$ is a determining structure on the reals $\mathscr{R}$. Is it true that every isomorphism between $C(X, E)$ and $C(Y, E)$ is, in fact, a lattice-isomorphism?

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$\mathfrak{R}$-SPACES

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In [6] we have introduced definitions of spaces determined by, respectively, countable compact subsets, countable closed sets, and countable sets (these definitions will be reproduced below). Some information on these definitions can be found in [5]. In [1] and [2] FRANKLIN has studied spaces determined by countable compact spaces in more detail; in particular, he has obtained a complete characterization of such spaces by showing that a Hausdorff space is determined by countable compact subsets iff it is a quotient of a discrete union[1] of spaces each of which is homomorphic to a convergent sequence with the limit (i.e., each of which is the one-point compactification of a countable infinite discrete space). This result has been further generalized by HERRLICH [4].

The purpose of this paper is to consider a general schema of definitions (including in particular, the first two of the above mentioned) and to show that various of the previously obtained theorems hold true in this general setting.

§ 1. $\mathfrak{R}$-spaces

Let $\mathfrak{R}$ be a topological class of topological spaces (i.e., $\mathfrak{R}$ is a class of topological spaces such that $X \in \mathfrak{R}$ and $X = Y$ imply $Y \in \mathfrak{R}$). We shall always assume that $\mathfrak{R}$ is non-empty. We say that a space X is *locally in* $\mathfrak{R}$ provided that every point of X has a neighborhood whose closure is in $\mathfrak{R}$. $\mathfrak{R}|X$ denotes the class of closed subspaces of X which are in $\mathfrak{R}$.

1. 1. DEFINITION. A space X is said to be an $\mathfrak{R}$-*space* provided that the following condition is satisfied:

for every $F \subset X$, if $A \cap F$ is closed for every $A \in \mathfrak{R}|X$, then F is closed.

1. 2. ILLUSTRATION. Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be, respectively, the classes of all Hausdorff compact countable spaces and of all Hausdorff countable spaces. Spaces determined by countable compact spaces coincide with Hausdorff $\mathfrak{R}_1$-spaces and spaces determined by countable closed subspaces coincide with Hausdorff $\mathfrak{R}_2$-spaces. It was pointed out in [5] that Hausdorff $\mathfrak{R}_1$-spaces coincide with Hausdorff $\mathfrak{R}_1'$-spaces,

[1] The discrete union $X = \cup_d \{X_\xi : \xi \in \Xi\}$ of a collection $\{X_\xi : \xi \in \Xi\}$ of spaces is defined as follows: we take a collection $\{X_\xi^* : \xi \in \Xi\}$ of mutually disjoint copies of spaces X_ξ; the set of points of X is the set-theoretic union of the sets of points of the spaces X_ξ^* and an $F \subset X$ is closed in X iff $F \cap X_\xi^*$ is closed in X_ξ^* for every $\xi \in \Xi$.

where $\mathfrak{N}_1'$ consists of all spaces homeomorphic to a convergent sequence (with the limit). (This result is a particular case of statement 1. 16 below; indeed, every countable Hausdorff compact space is 1-st countable; hence it is an $\mathfrak{N}_1'$-space.) It is clear that, within T_1-spaces, $\mathfrak{N}_1'$-spaces can be defined as spaces in which every sequentially closed set is closed. Such spaces are also called spaces *determined by sequences* or *sequential* spaces.

1. 3. *The property of being an $\mathfrak{N}$-space is local* (i.e., if every point of X has a neighborhood whose closure is an $\mathfrak{N}$-space, then X is an $\mathfrak{N}$-space).

PROOF. Assume that F is not closed; let $p_0 \in \bar{F} \setminus F$. There is a neighbourhood U of p_0 such that $\bar{U}$ is an $\mathfrak{N}$-space. $\bar{U} \cap F$ is not closed in $\bar{U}$; consequently, there is an $A \in \mathfrak{N}|\bar{U}$ such that $A \cap \bar{U} \cap F$ is not closed in $\bar{U}$. It is clear that $A \cap F$ is not closed in X and $A \in \mathfrak{N}|X$. Thus X is an $\mathfrak{N}$-space.

The above statement has two immediate corollaries.

1. 4. COROLLARY. *Every space which is locally in $\mathfrak{N}$ is an $\mathfrak{N}$-space.*

Thus, the property of being an $\mathfrak{N}$-space is a form of localization of the property of being a member of $\mathfrak{N}$. We shall see (Theorem 1. 12) that in various cases $\mathfrak{N}$-spaces can be characterized as quotients of spaces that are locally in $\mathfrak{N}$.

1. 5. COROLLARY. *A discrete union of $\mathfrak{N}$-spaces is an $\mathfrak{N}$-space.*

PROOF. It is obvious that a discrete union of $\mathfrak{N}$-spaces is locally an $\mathfrak{N}$-space.

1. 6. *If every member of $\mathfrak{N}'$ is an $\mathfrak{N}$-space, then every $\mathfrak{N}'$-space is an $\mathfrak{N}$-space.*

1. 7. DEFINITION. *A map $\varphi: X \underset{\text{onto}}{\to} Y$ is called an $\mathfrak{N}$-quotient provided that φ is a quotient map and $\varphi[A] \in \mathfrak{N}|Y$ for every $A \in \mathfrak{N}|X$.*

1. 8. *The image of an $\mathfrak{N}$-space under an $\mathfrak{N}$-quotient map is an $\mathfrak{N}$-space.*

PROOF. Let $\varphi: X \underset{\text{onto}}{\to} Y$, X is an $\mathfrak{N}$-space, φ is an $\mathfrak{N}$-quotient. Let $F \subset Y$ and assume that $F \cap A$ is closed for every $A \in \mathfrak{N}|Y$. Let $F_1 = \varphi^{-1}[F]$, let $B \in \mathfrak{N}|X$. $\varphi[B] \in \mathfrak{N}|Y$, therefore $\varphi[B] \cap F$ is closed. Consequently, $\varphi^{-1}[\varphi[B] \cap F]$ is closed. It is easy to see that $B \cap F_1 = B \cap \varphi^{-1}[\varphi[B] \cap F]$; hence $B \cap F_1$ is closed. It follows that F_1 is closed, therefore F is closed. Thus Y is an $\mathfrak{N}$-space.

1. 9. COROLLARY. *If $\mathfrak{N}$ is closed under continuous maps, then the property of being an $\mathfrak{N}$-space is closed under continuous closed maps.*

PROOF. If $\mathfrak{N}$ is closed under continuous maps, then every continuous closed map is an $\mathfrak{N}$-quotient.

1. 10. COROLLARY. *If $\mathfrak{N}$ consists of compact spaces and $\mathfrak{N}$ is closed under continuous maps onto Hausdorff spaces, then the property of being an $\mathfrak{N}$-space is closed under quotient maps onto Hausdorff spaces.*

PROOF. Under present assumptions every quotient map onto a Hausdorff space is an R-quotient.

1. 11. *Let X be an $\mathfrak{N}$-space and let $\mathfrak{K}$ be a subclass of $\mathfrak{N}|X$ such that every member of $\mathfrak{N}|X$ is contained in some member of $\mathfrak{K}$ and $\bigcup \mathfrak{K} = X$. Then X is a quotient of the discrete union of members of $\mathfrak{K}$.*

PROOF. It is easy to check that the map which maps every point of the discrete union of $\mathfrak{K}$ onto the same point of X is quotient. An exact formalization of this proof is routine.

Note that, in general, we need not to have $\bigcup(\mathfrak{R}|X)=X$, even if X is an $\mathfrak{R}$-space. But if X is a T_1-space and $\mathfrak{R}$ contains one-point spaces, then obviously $\bigcup(\mathfrak{R}|X)=X$.

1. 12. THEOREM. *Assume that every member of $\mathfrak{R}$ is Hausdorff compact and that a continuous Hausdorff image of every member of $\mathfrak{R}$ is an $\mathfrak{R}$-space. The following conditions on a Hausdorff space X are equivalent:*

(a) *X is an $\mathfrak{R}$-space;*

(b) *X is a quotient of a discrete union of some members of $\mathfrak{R}$;*

(c) *X is a quotient of a Hausdorff space which is locally in $\mathfrak{R}$.*

PROOF. (a)$\Rightarrow$(b). X is an $\mathfrak{R}'$-space, where $\mathfrak{R}'=\mathfrak{R}\cup$ {one-point spaces}. By 1. 11, X is a quotient of discrete union of $\mathfrak{R}'|X$. But every one-point space is a quotient of a space in $\mathfrak{R}$. Thus (b) holds.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a). Let $\varphi\colon Y \xrightarrow[\text{onto}]{} X$, where Y is an $\mathfrak{R}$-space and φ is quotient. Let $\mathfrak{R}^*$ be the class of all continuous Hausdorff images of members of $\mathfrak{R}$. Y is an $\mathfrak{R}^*$-space, hence, by 1. 10, X is an $\mathfrak{R}^*$-space. By 1. 6, X is an $\mathfrak{R}$-space.

The verification of the following two statements presents no difficulty.

1. 13. *If $\mathfrak{R}$ is closed-hereditary, then every closed subspace of an $\mathfrak{R}$-space is again an $\mathfrak{R}$-space.*

1. 14. *If $\mathfrak{R}$ is closed-hereditary, then every open subspace of a regular $\mathfrak{R}$-space is an $\mathfrak{R}$-space.*

We shall now give a few illustrations of Theorem 1. 12.

If $\mathfrak{R}=$ the class of spaces homeomorphic to a convergent sequence, then Theorem 1. 12 gives the result of FRANKLIN [1] on spaces determined by sequences.

From now on all spaces will be assumed to be Hausdorff.

Let $\mathfrak{R}_w$ be the class of well-ordered compact spaces. (An ordered [a well-ordered] space is a space which is homeomorphic to an ordered [a well-ordered] set with its order topology.) We shall prove

1. 15. *A continuous image of a member of $\mathfrak{R}_w$ is an $\mathfrak{R}_w$-space.* (Note that a continuous image of a well-ordered compact space need not to be well-ordered.)

PROOF. Let $\varphi\colon X \xrightarrow[\text{onto}]{} Y$, where X is well-ordered compact and φ is continuous. Let F be a non-closed subset of Y; let $F_1=\varphi^{-1}[F]$. Let x_0 be the first element of X such that $F_1^*=F_1\cap[0,x_0]$[2] is not closed; let $\varphi_0=\varphi|F_1^*$. We have

(i) if $A\subset F^*$ and $\sup A<x_0$, then $\sup\varphi_0^{-1}[\varphi_0[A]]<x_0$.

Indeed, $\bar{A}\subset F^*$ and $\bar{A}$ is compact, hence $\varphi_0^{-1}[\varphi_0[\bar{A}]]=\varphi^{-1}[\varphi[\bar{A}]]\cap[0,x_0]$ is a closed subset of X that is contained in F_1^* and that contains $\varphi_0^{-1}[\varphi_0[A]]$. It follows that $\varphi_0^{-1}[\varphi_0[\bar{A}]]\subset F_1^*$, therefore $\sup\varphi_0^{-1}[\varphi_0[A]]<x_0$.

We infer from (i) that it is possible to define by transfinite induction an increasing transfinite sequence $\alpha_0,\dots,\alpha_\beta,\dots,\beta<\lambda$, of elements of F_1^* such that

(1) if β is non-limit, then $\alpha_\beta>\sup\varphi^{-1}[\varphi]\{\alpha_{\beta'}\colon\beta'<\beta\}]]$;

[2] 0 is the first element of X and $[0,x_0]$ is the closed interval from 0 to x_0.

(2) if β is limit, then $\alpha_\beta = \sup \{\alpha_{\beta'}: \beta' < \beta\}$;

(3) $\sup \{\alpha_\beta: \beta < \lambda\} = x_0$.

Now we let $X_0 = \{\alpha_\beta: \beta < \lambda\} \cup \{x_0\}$. X_0 is closed in X and φ is one-to-one on X_0. Consequently, $\varphi[X_0]$ is a well-ordered space, but $F \cap \varphi[X_0]$ is not closed. Thus Y is an $\mathfrak{R}_w$-space.

We have also

1. 16. *The discrete union $X = \bigcup_d \{X_\xi: \xi \in \Xi\}$ of ordered spaces can be ordered (with preservation of topology) in such a way that the ordering of X is an extension of each of the orderings of X_ξ's and each X_ξ is an interval of X.*

(An interval of an ordered set X is a subset I of X with the property: if $a, b \in I$ and $a < c < b$, then $c \in I$.)

PROOF. Apply an order of the type $(\omega^* + \omega) \cdot \alpha$ to the set Ξ (each such order type has the discrete topology) and define the order in X according to the definition of the sum of order types. (If Ξ is finite, then the proposition is trivial.)

The following is obvious

1. 17. *A locally compact ordered space is an $\mathfrak{R}_w$-space.*

Combining 1.15—1.17 with Theorem 1.12 we obtain immediately the following:

1. 18. THEOREM. *The following conditions on X are equivalent:*

(a) *X is an $\mathfrak{R}_w$-space (i.e., X is determined by well-ordered compact spaces);*

(b) *X is a quotient of a discrete union of well-ordered compact spaces;*

(c) *X is a quotient of an ordered locally well-ordered space;*

(d) *X is a quotient of a locally well-ordered space;*

(e) *X is a quotient of an ordered locally compact space.*

Another case of interest: Let $\mathfrak{R}_D =$ the class of all dyadic compacts. The class of $\mathfrak{R}_D$-spaces (i.e., the class of spaces determined by dyadic compacts) is, of course, wider than the class of all spaces determined by sequences. Barring the existence of weakly inaccessible cardinals we infer easily from the result of MAZUR [8] (see also remarks on the Mazur result in [7] that every $\mathfrak{R}_D$-space X has the following property (S): *every sequentially continuous function with values in a completely regular space is continuous.*

Note that if in the above property we replace "completely regular" by "Hausdorff", then spaces with this property coincide with spaces determined by sequences.

§ 2. k-spaces. Quotients of locally compact spaces

In this section we shall make a few remarks on k-spaces. For simplicity, all spaces will be assumed to be Hausdorff.

Since k-spaces coincide with $\mathfrak{R}$-spaces, where $\mathfrak{R}$ is the class of all compact spaces, all the results of the previous section apply to k-spaces. In particular, from 1. 12 we obtain:

2. 1. *The following are equivalent*

(a) *X is a k-space;*

(b) *X is a quotient of a discrete union of compact spaces;*
(c) *X is a quotient of a locally compact space.*

Since all k-spaces can be obtained as quotients of locally compact spaces, it is interesting to see how local compactness fails under quotient maps. Note that in a locally compact space every compact set has a neighbourhood with a compact closure. The following is obvious.

2. 2. *If* $\varphi: X \xrightarrow[\text{onto}]{} Y$, *$\varphi$ is a continuous open (closed, respectively) map, and* $\varphi^{-1}(q_0)$ $(q_0 \in Y)$ *has a neighbourhood with a compact closure, then Y is locally compact at* q_0.

The above statement fails for quotient maps. Let X be the subspace of the reals consisting of all numbers of the form $n+\frac{1}{m}$ for $n=0, 1, \dots;$ $m=1, 2, \dots$. Define $\varphi(n)=\frac{1}{n}$ for $n=1, 2, \dots$ and $\varphi(x)=x$ for every other $x \in X$. Y is the range of φ with topology defined so as to make φ a quotient. X is locally compact; Y fails to be locally compact at 0; while $\varphi^{-1}(0)$ contains only one point (in fact, $\varphi^{-1}(q)$ has at most two points for every $q \in Y$). Note that Y is normal[3] and X is the discrete union of countably many convergent sequences. Consequently

2. 3. *There is a countable normal space Y which is the image of the discrete union of countably many convergent sequences under a quotient map φ such that $\varphi^{-1}(q)$ contains at most two points for every $q \in Y$ and such that Y fails to be locally compact at a point q_0 such that $\varphi^{-1}(q_0)$ has only one point.*

The above constructed space Y is lacking only one nice thing: the first axiom of countability at 0. This, however, cannot be remedied. Indeed, we have:

2. 4. THEOREM. *If* $\varphi: X \xrightarrow[\text{onto}]{} Y$ *is quotient, Y is first countable and regular at a* $q_0 \in Y$, *and* $\varphi^{-1}(q_0)$ *has a compact neighborhood, then Y is locally compact at* q_0.

To prove 2. 4 we need a lemma.

2. 5. LEMMA. *If A_n and B_n are closed, $B_n \subset A_n$ and $A_{n+1} \subset A_n$, then* $\bigcup\limits_n B_n \cup \bigcap\limits_n A_n$ *is closed. Consequently, if* $\bigcap\limits_n A_n = \emptyset$, *then* $\bigcup\limits_n B_n$ *is closed.*

PROOF. We have

$$\bigcup_n B_n \cup \bigcap_n A_n = \bigcap_n (B_1 \cup \dots \cup B_n \cup A_n).$$

Proof of Theorem 2. 4. Assume Y is not locally compact at q_0; let U be a neighbourhood of $\varphi^{-1}(q_0)$ with $\overline{U}$ compact; let $V_1, V_2, \dots$ be a local base of q_0. We have $V_n \not\subset \varphi[U]$; let $p_n \in V_n \cap (Y \setminus \varphi[U])$; let $F=\{p_1, p_2, \dots\}$, $B_n=\varphi^{-1}(p_n)$,

[3] The space Y provides us with an example showing that "to be determined by sequences" is not a hereditary property. Indeed, Y, as a quotient of a 1st countable space, is determined by sequences. If we remove from Y all points of the form $\frac{1}{n}$; $n=1, 2, \dots$, then the resulting space is not determined by sequences.

$A_n = \varphi^{-1}[\bar{V}_n] \cap (X \setminus U)$. A_n and B_n satisfy the assumption of 2. 5, consequently, $\bigcup B_n$ is closed. But $\varphi^{-1}[F] = \bigcup_n B_n$; therefore F is closed. On the other hand, $q_0 \in \bar{F} \setminus F$. Contradiction.

§ 3. Concluding remarks

Spaces determined by countable sets were defined in [6] as spaces satisfying the condition

if $\bar{A} \subset F$ for every countable $A \subset F$, then F is closed.

This definition does not fit into the schema considered in this paper. We plan to publish a paper discussing other schemas of definitions as well as relations between these schemas. In particular, HERRLICH's results [4] can be obtained as particular cases of results concerning one of such schemas.

In [6] we asked if every Hausdorff compact space which is determined by countable sets is determined by countable closed sets. This problem still remains open; however, FRANKLIN [3] has shown that the asnwer is positive for Hausdorff compact spaces of cardinality $< 2^{\aleph_1}$.

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ISOMORPHISM PROBLEM FOR A SPECIAL CLASS OF GRAPHS

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1. The following research problem was proposed by A. ÁDÁM [1]:

PROBLEM. Let $0<k_1<k_2<\dots<k_m<n$ be given integers, and let $G_n(k_1, \dots, k_m)$ be the directed graph with vertices $P_1, \dots, P_n$ in which there is an edge from P_i to P_j if and only if there is some t such that $k_t \equiv j-i \pmod n$. A sufficient condition for two such graphs $G_n(k_1, \dots, k_m)$ and $G'_n(k'_1, \dots, k'_m)$ to be isomorphic is that there exists a number r, $0<r<n$, relatively prime to n and a permutation α of $\{1, \dots, n\}$ such that $k'_t \equiv rk_{\alpha(t)} \pmod n$ for $1 \leqq t \leqq n$. It is conjectured that this sufficient condition is also necessary.

We shall prove that the conjecture is true when $n=p$ is a prime. We shall also establish a necessary condition for isomorphism of two graphs G_n and G'_n when $n=p^a$ is a power of a prime.

Given a finite set S of nonnegative integers we associate to S the polynomial

$$S(x) = \sum_{s\in S} x^s.$$

As usual $|S|$ denotes the cardinality of S. The set $Z_n=\{0, 1, \dots, n-1\}$ is a ring with respect to addition and multiplication mod n. We denote by $z_1 \circ z_2$ the product of $z_1, z_2 \in Z_n$ in this ring. Let $N=\{1, 2, \dots, n-1\} \subset Z_n$. The multiplicative group M of units of Z_n operates on the set of subsets of N. The action of $a\in M$ on $K\subset N$ is determined by

$$K \overset{a}{\rightarrow} a\circ K = \{a\circ k \mid k\in K\}.$$

In particular, we have $|K|=|a\circ K|$. We shall write $K\sim K'$ if there exists $a\in M$ such that $K'=a\circ K$.

Let $K=\{k_1, \dots, k_m\}$ and $K'=\{k'_1, \dots, k'_m\}$. The above conjecture can be stated as follows: If $G_n(K)\cong G_n(K')$ then $K\sim K'$. In the sequel we shall use the converse which is obviously true: If $K\sim K'$ then $G_n(K)\cong G_n(K')$.

2. Let $a_{ij}=1$ or $a_{ij}=0$ $(i, j\in Z_n)$ according to whether there is an edge in $G_n(K)$ from P_i to P_j or not. The matrix $A=(a_{ij})$ is cyclic. The row $i=0$ of A has ones for $j\in K$ and zeros elsewhere. Let the matrix A' correspond to the graph $G_n(K')$. Since we assume that $G_n(K)\cong G_n(K')$ there exists a permutation matrix P such that

$$A' = P^{-1}AP. \tag{1}$$

Conversely, if P is a permutation matrix (1) implies that the graphs are isomorphic. From (1) we conclude that A and A' have the same eigenvalues, i.e., (cf. [3, p. 66])

$$K(\varepsilon^k) = K'(\varepsilon^{\pi(k)}), \quad k\in N \tag{2}$$

where ε is a primitive n-th root of unity and π is a permutation of N.

From now on let n be a power of a prime $n=p^a$. If $S\subset N$ then S^t will denote the set of all $s\in S$ which are divisible by p^t but not divisible by p^{t+1}.

LEMMA 1. *If $G_n(K)\cong G_n(K')$, $n=p^a$ then there exist $K_1\sim K$, $K_1'\sim K'$ and a permutation τ of N such that $\tau(k)=k$ for $k\in N^0$ and $K_1(\varepsilon^k)=K_1'(\varepsilon^{\tau(k)})$, $k\in N$.*

PROOF. Since $p^{a-1}-1$ numbers in N are divisible by p and

$$p^a-1>p(p^{a-1}-1)$$

we conclude that there exists at least one $k_0\in N^0$ such that also $\pi(k_0)\in N^0$. Putting $K_1=k_0\circ K$ and $K_1'=\pi(k_0)\circ K'$ we have $G_n(K)\cong G_n(K_1)$ and $G_n(K')\cong G_n(K_1')$. Since $G_n(K_1)\cong G_n(K_1')$ there exists a permutation σ of N such that

$$K_1(\varepsilon^k)=K_1'(\varepsilon^{\sigma(k)}),\quad k\in N.$$

Using the definition of K_1 and K_1' we conclude that

$$K_1(x)=\overline{K(x^{k_0})},\quad K_1'(x)=\overline{K'(x^{\pi(k_0)})}$$

where the bars indicate that the exponents have to be reduced mod n. This implies that

$$K_1(\varepsilon)=K_1'(\varepsilon),$$

and consequently

$$K_1(\varepsilon^k)=K_1'(\varepsilon^k),\quad k\in N^0.$$

Therefore we can replace σ by τ which satisfies $\tau(k)=k$ for $k\in N^0$. Q.E.D.

THEOREM 1. *If $n=p$ is a prime then the conjecture is true.*

PROOF. Let $G_n(K)\cong G_n(K')$. By Lemma 1 we have

$$K_1(\varepsilon^k)=K_1'(\varepsilon^k),\quad k\in N$$

where $K\sim K_1$ and $K'\sim K_1'$. If $K_1(x)-K_1'(x)\neq 0$ then it is a polynomial with integral coefficients of degree $\leqq p-1$, with no constant term. Such a polynomial is not a multiple of the minimal polynomial of ε which is

$$1+x+\cdots+x^{p-1}.$$

Therefore we must have $K_1(x)=K_1'(x)$, i.e., $K_1=K_1'$ and consequently $K\sim K'$. Q.E.D.

REMARK. It is possible that the matrices A and A' have the same eigenvalues and that $G_n(K)$ and $G_n(K')$ are not isomorphic. Example: $n=25$, $K=\{1, 2, 4, 5, 7, 12, 17, 22\}$, $K'=\{1, 3, 4, 5, 8, 13, 18, 23\}$.

LEMMA 2. *If $G_n(K)\cong G_n(K')$, $n=p^a$ then there exist $L\sim K$, $L'\sim K'$ and a permutation ω of N such that $\omega(N^t)=N^t$ for $t=0, 1, \ldots, a-1$ and*

$$L(\varepsilon^k)=L'(\varepsilon^{\omega(k)}),\quad k\in N.$$

PROOF. By Lemma 1 we may assume that (2) holds with $\pi(N^0)=N^0$. We shall not insist on $\pi(k)=k$ for $k\in N^0$. As in Lemma 1 there exists at least one $k_1\in N^1$

such that also $\pi(k_1) \in N^1$. Let $s, s' \in M = N^0$ be such that $s \circ k_1 = p$ and $s' \circ \pi(k_1) = p$. For $K_1 = s \circ K$ and $K_1' = s' \circ K'$ we have

$$K_1(\varepsilon^k) = K_1'(\varepsilon^{\sigma(k)}), \quad k \in N$$

where σ is a permutation of N such that $\sigma(N^0) = N^0$ and moreover $\sigma(p) = p$. Since

$$K_1(\varepsilon^p) = K_1'(\varepsilon^p)$$

implies that

$$K_1(\varepsilon^k) = K_1'(e^k), \quad k \in N^1$$

we can replace σ by τ which satisfies

$$\tau(N^0) = N^0, \ \tau(N^1) = N^1.$$

Repeating this process several times we shall get L, L' and ω with required properties. Q.E.D.

Note that the minimal polynomial of ε is (cf. [2, p. 206]).

$$\varphi(x) = 1 + x^{p^{a-1}} + x^{2p^{a-1}} + \dots + x^{p^a - p^{a-1}}.$$

THEOREM 2. *If $G_n(K) \cong G_n(K')$ and $n = p^a$ is a power of a prime then*

$$|K^t| = |(K')^t|, \qquad t = 0, 1, \dots, a-1.$$

PROOF. Note that $L \sim K$ implies that $|L^t| = |K^t|$ for all $t = 0, 1, \dots, a-1$. By Lemmas 1 and 2 we can assume that (2) holds and that $\pi(1) = 1$,

$$\pi(N^t) = N^t, \qquad t = 0, 1, \dots, a-1.$$

From $K(\varepsilon) - K'(\varepsilon) = 0$ we conclude that $\varphi(x)$ divides $K(x) - K'(x)$. Since $K(0) - K'(0) = 0$ this implies that $K(x) - K'(x)$ does not contain the terms x^v, $v \in N^{a-1}$. We infer that $K^{a-1} = (K')^{a-1}$. By induction assume that

$$|K^t| = |(K')^t|, \qquad t = s+1, \dots, a-1. \tag{3}$$

We have

$$\pi(p^{a-s-1}) = up^{a-s-1}, \quad (u, p) = 1.$$

Equation (2) gives

$$K(\varepsilon^{p^{a-s-1}}) - K'(\varepsilon^{up^{a-s-1}}) = 0. \tag{4}$$

Let us define

$$f(x) = \overline{K(x^{p^{a-s-1}})} - \overline{K'(x^{up^{a-s-1}})}$$

where the bars indicate that the exponents have to be reduced mod n.

The constant term of $\overline{K(x^{p^{a-s-1}})}$, and $\overline{K'(x^{up^{a-s-1}})}$ is $\sum_{t=s+1}^{a-1} |K^t|$, and $\sum_{t=s+1}^{a-1} |(K')^t|$, respectively. By (3) they are equal. Hence $f(0) = 0$. It is clear that the polynomials

$$K(x^{p^{a-s-1}}) \quad \text{and} \quad \overline{K(x^{p^{a-s-1}})}$$

have the same value for $x = \varepsilon$. Similarly,

$$K'(x^{up^{a-s-1}}) \quad \text{and} \quad \overline{K'(x^{up^{a-s-1}})}$$

have the same value for $x=\varepsilon$. Therefore (4) implies that $f(\varepsilon)=0$. Hence, we know that

$$f(x) = \sum_{v=1}^{n-1} a_v x^v, \quad n = p^a.$$

Since ε is a root of $f(x)$ the minimal polynomial of ε must divide $f(x)$. So,

$$f(x) = \varphi(x) g(x),$$
$$\varphi(x) = 1 + x^{p^{a-1}} + x^{2p^{a-1}} + \cdots + x^{p^a - p^{a-1}},$$
$$g(x) = \sum_{u=1}^{p^{a-1}-1} b_u x^u.$$

Clearly, the product $\varphi(x)g(x)$ does not contain the terms $a_v x^v$, $v \in N^{a-1}$.

Now, the sum of coefficients of $x^{p^{a-1}}, x^{2p^{a-1}}, \ldots, x^{p^a - p^{a-1}}$ in $\overline{K(x^{p^{a-s-1}})}$ and $\overline{K'(x^{up^{a-s-1}})}$ is equal to $|K^s|$ and $|(K')^s|$, respectively. Since $f(x)$ has no nonzero coefficient a_v, $v \in N^{a-1}$ it follows that the sum of coefficients of $x^{p^{a-1}}, x^{2p^{a-1}}, \ldots$ $\ldots, x^{p^a - p^{a-1}}$ in $\overline{K(x^{p^{a-s-1}})}$ and $\overline{K'(x^{up^{a-s-1}})}$ are the same, i.e., $|K^s| = |(K')^s|$.

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ЛОКАЛЬНЫЕ ТЕОРЕМЫ С ОСТАТОЧНЫМ ЧЛЕНОМ ДЛЯ ОДНОГО КЛАССА АРИФМЕТИЧЕСКИХ ФУНКЦИЙ

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Пусть n — натуральное число, q_n — произведение тех множителей канонического разложения n на простые, степень которых $\geqq 2$ (для $n=1$ и для бесквадратных n положим $q_n=1$). Будем говорить, что комплекснозначная арифметическая функция $f(n)$ принадлежит классу R, если для всех n

$$f(n)=f(q_n). \tag{1}$$

В настоящей работе изучается распределение значений функций класса R. Локальная теорема для функции такого типа впервые была получена А. Реньи [1], рассматривавшим функцию $f(n) = \Omega(n)-\omega(n)$, где $\omega(n)$ означает число различных простых делителей n, $\Omega(n)$ — число простых делителей n с учетом кратности. Он доказал, что для любого целого $k\geqq 0$ существует предел

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n\leqq N\\ \Omega(n)-\omega(n)=k}} 1 = \lambda_k, \tag{2}$$

причем λ_k порождаются степенным рядом

$$\sum_{k=0}^{\infty}\lambda_k z^k = \prod_p\left(1+\frac{1}{p-z}\right)\left(1-\frac{1}{p}\right). \tag{3}$$

Легко видеть, что класс R содержит, в частности, все аддитивные функции, равные нулю в простых, и все мультипликативные функции, равные единице в простых.

И. П. Кубилюс [2] получил следующее обобщение теоремы Реньи: если $f(n)$ — целозначная аддитивная функция, равная нулю в простых, то для любого целого k существует асимптотическая плотность

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n\leqq N\\ f(n)=k}} 1 = \lambda_k(f), \tag{4}$$

$\lambda_k(f)$ порождаются рядом Фурье

$$\sum_{k=-\infty}^{\infty}\lambda_k(f)e^{itk} = \prod_p\left(1+\sum_{r=2}^{\infty}\frac{e^{itf(p^r)}-e^{itf(p^{r-1})}}{p^r}\right). \tag{5}$$

Впоследствие многие авторы [3—7] занимались уточнением теорем Реньи и Кубилюса. И. Катаи [7] показал, что

$$\sum_{\substack{n \leqq N \\ \Omega(n)-\omega(n)=k}} 1 = \lambda_k N + O\left(\sqrt{N} (\log \log N)^{k-1}\right). \tag{6}$$

В работе [6] доказано, что если $f(n)$ — целозначная аддитивная функция, равная нулю в простых, то равномерно по k

$$\sum_{\substack{n \leqq N \\ f(n)=k}} 1 = \lambda_k(f) N + O\left(\sqrt{N} \log^2 N\right). \tag{7}$$

В настоящей работе доказывается локальная теорема для функций класса R с оценкой остаточного члена, которая даже в общем случае точнее, чем (6) и (7).

Теорема 1. *Пусть $f(n)$ — функция класса R, E — произвольное множество комплексных чисел. Тогда при $N \to \infty$*

$$\sum_{\substack{n \leqq N \\ f(n) \in E}} 1 = \lambda_E(f) N + O\left(\sqrt{N}\right), \tag{8}$$

где

$$\lambda_E(f) = \frac{6}{\pi^2} \sum_{\substack{n=1 \\ q_n=n \\ f(n) \in E}}^{\infty} \frac{1}{n \prod\limits_{p/n} \left(1 + \frac{1}{p}\right)}, \tag{9}$$

константа в символе O — абсолютная.

Для случая $f(n) = \Omega(n) - \omega(n)$ этот результат допускает дальнейшее уточнение.

Теорема 2. *Пусть $k \geqq 1$ — целое. Тогда*

$$\sum_{\substack{n \leqq N \\ \Omega(n)-\omega(n)=k}} 1 = \lambda_k N + \frac{\sqrt{N}}{\log N} \left\{ P_0(\log \log N) + O\left(\exp\left(-a \frac{(\log \log N)^{3/5}}{(\log \log \log N)^{1/5}} \right) \right) \right\}, \tag{10}$$

где $P_0(u)$ — полином степени $k-1$, $a > 0$ — константа.

Оценка остаточного члена связана с современными сведениями о нулях ζ — функции Римана. Более точный результат получается, если принять квазиримaнову гипотезу для $\zeta(s)$.

Теорема 3. *Из квазиримановой гипотезы вытекает асимптотическое разложение*

$$\sum_{\substack{n \leqq N \\ \Omega(n)-\omega(n)=k}} 1 \sim \lambda_k N + \frac{\sqrt{N}}{\log N} \sum_{\nu=1}^{\infty} \frac{P_\nu(\log \log N)}{(\log N)^\nu}, \tag{11}$$

где $P_\nu(u)$ — полиномы степени $\leqq k-1$ при $\nu \geqq 1$.

Доказательство теоремы I

Отметим прежде всего, что если $f(n)$ — функция класса R, а F — произвольная функция, определённая на множестве значений $f(n)$, то $F(f(n))$ принадлежит классу R. Так как

$$\sum_{\substack{n \leqq N \\ f(n) \in E}} 1 = \sum_{n \leqq N} F_E(f(n)),$$

где

$$F_E(x) = \begin{cases} 1, & \text{если } x \in E. \\ 0, & \text{если } x \bar{\in} E, \end{cases}$$

то всё сводится к подсчёту среднего значения для некоторых функций класса R, принимающих только значения 0 и 1. Мы докажем здесь, что если $f(n)$ — ограниченная функция класса R, то

$$\sum_{n \leqq N} f(n) = A_f N + O(M_f \sqrt{N}), \tag{12}$$

с абсолютной константой в символе O, где

$$A_f = \frac{6}{\pi^2} \sum_{\substack{n=1 \\ q_n = n}}^{\infty} \frac{f(n)}{n \prod_{p/n} \left(1 + \frac{1}{p}\right)}, \tag{13}$$

$$M_f = \sup_n |f(n)|. \tag{14}$$

Отсюда, конечно, будет следовать утверждение теоремы 1.

Для доказательства (12) заметим, что каждое натуральное число n однозначно представимо в виде $n = k \cdot l$, где $(k, l) = 1$, k бесквадратно, $q_l = l = q_n$. Поэтому

$$\sum_{n \leqq N} f(n) = \sum_{n \leqq N} f(q_n) = \sum_{\substack{kl \leqq N \\ (k,l)=1 \\ q_l = l}} \mu^2(k) f(l) = \sum_{\substack{l \leqq N \\ q_l = l}} f(l) \sum_{\substack{k \leqq \frac{N}{l} \\ (k,l)=1}} \mu^2(k).$$

Пусть $\alpha(n)$ означает произведение различных простых делителей n при $n > 1$, $\alpha(1) = 1$. Легко проверить тождество

$$\sum_{\substack{k \leqq x \\ (k,l)=1}} \mu^2(k) = \sum_{\substack{m \leqq x \\ \alpha(m)/\alpha(l)}} (-1)^{\Omega(m)} \sum_{k \leqq \frac{x}{m}} \mu^2(k),$$

из которого следует:

$$\sum_{n \leqq N} f(n) = \sum_{\substack{l \leqq N \\ q_l = l}} f(l) \sum_{\substack{m \leqq \frac{N}{l} \\ \alpha(m)/\alpha(l)}} (-1)^{\Omega(m)} \sum_{k \leqq \frac{N}{ml}} \mu^2(k) = \sum_{\substack{ml \leqq N \\ q_l = l \\ \alpha(m)/\alpha(l)}} (-1)^{\Omega(m)} f(l) \sum_{k \leqq \frac{N}{ml}} \mu^2(k). \tag{15}$$

Следуя И. Катаи [7], положим

$$\sum_{k \leqq x} \mu^2(k) = \frac{6}{\pi^2} x + \sqrt{x}\, \varepsilon(x).$$

Подставив в (15), получаем:

$$(16)\qquad \sum_{n\le N} f(n) = \frac{6}{\pi^2} N \sum_{\substack{ml\le N\\ q_l=l\\ \alpha(m)/\alpha(l)}} \frac{(-1)^{\Omega(m)} f(l)}{ml} + \sqrt{N} \sum_{\substack{ml\le N\\ q_l=l\\ \alpha(m)/\alpha(l)}} \frac{(-1)^{\Omega(m)} f(l)}{\sqrt{ml}} \varepsilon\left(\frac{N}{ml}\right) =$$

$$= A_f N - \frac{6}{\pi^2} N \sum_{\substack{ml> N\\ q_l=l\\ \alpha(m)/\alpha(l)}} \frac{(-1)^{\Omega(m)} f(l)}{ml} + \sqrt{N} \sum_{\substack{ml\le N\\ q_l=l\\ \alpha(m)/\alpha(l)}} \frac{(-1)^{\Omega(m)} f(l)}{\sqrt{ml}} \varepsilon\left(\frac{N}{ml}\right),$$

где

$$A_f = \frac{6}{\pi^2} \sum_{\substack{l=1\\ q_l=l}}^{\infty} \frac{f(l)}{l} \sum_{\substack{m=1\\ \alpha(m)/\alpha(l)}}^{\infty} \frac{(-1)^{\Omega(m)}}{m} = \frac{6}{\pi^2} \sum_{\substack{l=1\\ q_l=l}}^{\infty} \frac{f(l)}{l \prod_{p/l} \left(1+\frac{1}{p}\right)}.$$

Рассмотрим сумму

$$(17)\qquad g(k) = \sum_{\substack{ml=k\\ q_l=l\\ \alpha(m)/\alpha(l)}} (-1)^{\Omega(m)} f(l).$$

Если $q_k = k$, то

$$(18)\qquad g(k) = \sum_{\substack{l/k\\ q_l=l\\ \alpha(l)=\alpha(k)}} (-1)^{\Omega\left(\frac{k}{l}\right)} f(l) = (-1)^{\Omega(k)} \sum_{\substack{l/k\\ l\equiv O(\alpha^2(k))}} (-1)^{\Omega(l)} f(l) =$$

$$= (-1)^{\Omega(k)} \sum_{d/\frac{k}{\alpha^2(k)}} (-1)^{\Omega(d)} f\big(d\alpha^2(k)\big);$$

если же $q_k \ne k$, то $g(k)=0$, так как условия $ml=k$, $q_l=l$, $\alpha(m)/\alpha(l)$ ни для каких m и l не могут быть выполнены одновременно, и сумма (17) пуста. Из (16) теперь следует:

$$(19)\qquad \sum_{n\le N} f(n) = A_f N - \frac{6}{\pi^2} N \sum_{k>N} \frac{g(k)}{k} + \sqrt{N} \sum_{k\le N} \frac{g(k)}{\sqrt{k}} \varepsilon\left(\frac{N}{k}\right).$$

Так как из (18)

$$|g(k)| \le M_f \tau\left(\frac{k}{\alpha^2(k)}\right),$$

то

$$(20)\qquad \sum_{n\le N} f(n) - A_f N \ll$$

$$\ll M_f \left\{ N \sum_{\substack{k>N\\ q_k=k}} \frac{1}{k} \tau\left(\frac{k}{\alpha^2(k)}\right) + \sqrt{N} \sum_{\substack{k\le N\\ q_k=k}} \frac{1}{\sqrt{k}} \tau\left(\frac{k}{\alpha^2(k)}\right) \varepsilon\left(\frac{N}{k}\right) \right\}.$$

Применив известные оценки сумм мультипликативных функций ([3], теорема 2.1.1) к функции

$$h(k)=\begin{cases}\dfrac{1}{\sqrt{k}}\tau\left(\dfrac{k}{\alpha^2(k)}\right), & \text{если}\quad q_k=k,\\ 0, & \text{если}\quad q_k\neq k,\end{cases}$$

найдём:

$$\sum_{k\leqq x}h(k)=C_1\log x+O(1),$$

где

$$C_1=\frac{1}{2}\prod_p\left(1+\frac{1}{(\sqrt{p}-1)^2}\right)\left(1-\frac{1}{p}\right).$$

Отсюда, так как

$$\varepsilon(x)\ll e^{-\sqrt{\log x}},$$

(см., например, [8]), то

$$\text{(21)}\quad \sum_{k\leqq N}h(k)\varepsilon\left(\frac{N}{k}\right)\ll\sum_{k\leqq N}h(k)e^{-\sqrt{\log\frac{N}{k}}}=e^{-\sqrt{\log N}}\sum_{k\leqq N}h(k)-\sum_{k\leqq N}h(k)\int\limits_{\frac{N}{k}}^{N}de^{-\sqrt{\log u}}=$$

$$=e^{-\sqrt{\log N}}(C_1\log N+O(1))-\int\limits_1^N\sum_{\frac{N}{u}<n\leqq N}h(k)\,de^{-\sqrt{\log u}}\ll$$

$$\ll-\int\limits_1^N\log u\,de^{-\sqrt{\log u}}=O(1);$$

$$\text{(22)}\quad \sum_{k>N}\frac{h(k)}{\sqrt{k}}=\frac{1}{2}\sum_{k>N}h(k)\int\limits_k^\infty\frac{du}{u\sqrt{u}}=\frac{1}{2}\int\limits_N^\infty\sum_{N<k\leqq u}h(k)\frac{du}{u\sqrt{u}}\ll$$

$$\ll\int\limits_N^\infty\left\{\log\frac{u}{N}+O(1)\right\}\frac{du}{u\sqrt{u}}\ll\frac{1}{\sqrt{N}}.$$

Учитывая (20)—(22), получаем:

$$\sum_{n\leqq N}f(n)-A_fN\ll M_f\left\{N\sum_{k>N}\frac{h(k)}{\sqrt{k}}+\sqrt{N}\sum_{k\leqq N}h(k)\varepsilon\left(\frac{N}{k}\right)\right\}\ll M_f\sqrt{N}.$$

Тем самым формула (12), (а, значит, и теорема 1) доказана.

Интересно отметить, что оценка остатка в (12), вообще говоря, неулучшаема. Пусть, например,

$$f(n)=q_n^{\beta i},$$

где β вещественно. Ясно, что $f(n)$ ограничена и принадлежит классу R. Сумма $\sum_{n\leqq N}q_n^{\beta i}$ может быть подсчитана с помощью производящего ряда Дирихле

$$F(s)=\sum_{n=1}^\infty\frac{q_n^{\beta i}}{n^s}.$$

Нетрудно проверить, что

$$F(s) = \frac{\zeta(s)}{\zeta(2s)}\zeta(2s-2\beta i)H(s),$$

где $H(s)$ регулярна, ограничена и не обращается в нуль в области $\operatorname{Re} s > \frac{1}{3}+\varepsilon$ для любого $\varepsilon>0$. Применив контурное интегрирование и используя известные оценки $\zeta(s)$ в критической полосе (см., например, [9],) найдём:

$$\sum_{n\leqq N} q_n^{\beta i} = C_2 N + \frac{\zeta\left(\frac{1}{2}+\beta i\right) H\left(\frac{1}{2}+\beta i\right)}{(1+2\beta i)\zeta(1+2\beta i)} N^{\frac{1}{2}+\beta i} + O\left(\sqrt{N} e^{-\log N)^{\alpha}}\right), \quad \alpha > 0.$$

Выбрав $\beta \neq 0$, так, что $\zeta\left(\frac{1}{2}+\beta i\right) \neq 0$, получим:

$$\sum_{n\leqq N} q_n^{\beta i} - C_2 N \neq o(\sqrt{N}).$$

Остаточный член в теореме Реньи

Пусть k — целое неотрицательное. Полагая в (19)

$$f(n) = \begin{cases} 1, & \text{если } \Omega(n)-\omega(n) = k, \\ 0, & \text{если } \Omega(n)-\omega(n) \neq k, \end{cases}$$

найдём:

$$\sum_{\substack{n\leqq N \\ \Omega(n)-\omega(n)=k}} 1 = \lambda_k N - \frac{6}{\pi^2} N \sum_{n>N} \frac{g_k(n)}{n} + \sqrt{N} \sum_{n\leqq N} \frac{g_k(n)}{\sqrt{n}} \varepsilon\left(\frac{N}{n}\right), \tag{23}$$

где $g_k(n)=0$, если $q_n \neq n$, и

$$g_k(n) = (-1)^{\Omega(n)} \sum_{\substack{d / \frac{n}{\alpha^2(n)} \\ \Omega(d\alpha^2(n)) - \omega(d\alpha^2(n))=k}} (-1)^{\Omega(d)} =$$

$$= (-1)^{\Omega(n)} \sum_{\substack{d / \frac{n}{\alpha^2(n)} \\ \Omega(d)+\omega(n)=k}} (-1)^{\Omega(d)} = (-1)^{\Omega(n)-\omega(n)+k} \sum_{\substack{d / \frac{n}{\alpha^2(n)} \\ \Omega(d)+\omega(n)=k}} 1,$$

если $q_n = n$.

Лемма. *Имеют место асимптотические разложения*

$$\sum_{\substack{n\leqq N \\ q_n=n}} \frac{(-1)^{\Omega(n)-\omega(n)}}{\sqrt{n}} \sum_{\substack{d / \frac{n}{\alpha^2(n)} \\ \Omega(d)+\omega(n)=k}} 1 \sim \sum_{\nu=0}^{\infty} \frac{Q_\nu(\log\log N)}{(\log N)^\nu}, \tag{24}$$

$$\sum_{\substack{n\leqq N \\ q_n=n}} \frac{1}{\sqrt{n}} \sum_{\substack{d / \frac{n}{\alpha^2(n)} \\ \Omega(d)+\omega(n)=k}} 1 \sim \sum_{\nu=0}^{\infty} \frac{\bar{Q}_\nu(\log\log N)}{(\log N)^\nu}, \tag{25}$$

где $Q_\nu(u)$, $\bar{Q}_\nu(u)$ — полиномы степени $\leqq k$.

Мы приведём доказательство только (24); (25) доказывается совершенно наалогично. Пусть

$$S_N(z, w) = \sum_{\substack{n \leqq N \\ q_n = n}} \frac{(-1)^{\Omega(n)-\omega(n)}}{\sqrt{n}} z^{\omega(n)} \sum_{d / \frac{n}{\alpha^2(n)}} w^{\Omega(d)}. \tag{26}$$

Рассмотрим мультипликативную функцию

$$\xi(n) = \begin{cases} \dfrac{(-1)^{\Omega(n)-\omega(n)}}{\sqrt{n}} z^{\omega(n)} \displaystyle\sum_{d / \frac{n}{\alpha^2(n)}} w^{\Omega(d)}, & \text{если} \quad q_n = n, \\ 0, & \text{если} \quad q_n \neq n. \end{cases}$$

Применив к ней теорему 2.1.1 работы [3], получаем асимптотическое разложение для $S_N(z, w)$:

$$S_N(z, w) = \sum_{n \leqq N} \xi(n) \sim \sum_{\nu=0}^{\infty} C_\nu(z, w) (\log N)^{-z-\nu}; \tag{27}$$

коэффициенты $C_\nu(z, w)$ регулярны по каждой переменной при $|z| < \delta$, $|w| < \delta$, где δ — достаточно малое положительное число.

Пусть a и b — целые неотрицательные числа. Умножая обе части (27) на $\dfrac{1}{2\pi i w^{a+1}}$ и интегрируя по окружности $|w| = \dfrac{\delta}{2}$, находим:

$$\frac{1}{2\pi i} \int_{|w|=\frac{\delta}{2}} \frac{S_N(z, w)}{w^{a+1}} dw = \tag{28}$$

$$= \sum_{\substack{n \leqq N \\ q_n = n}} \frac{(-1)^{\Omega(n)-\omega(n)}}{\sqrt{n}} z^{\omega(n)} \sum_{\substack{d / \frac{n}{\alpha^2(n)} \\ \Omega(d) = a}} 1 \sim \sum_{\nu=0}^{\infty} D_\nu(z) (\log N)^{-z-\nu},$$

где

$$D_\nu(z) = \frac{1}{2\pi i} \int_{|w|=\frac{\delta}{2}} \frac{C_\nu(z, w)}{w^{a+1}} dw.$$

Аналогично, умножив (28) на $\dfrac{1}{2\pi i z^{b+1}}$ и интегрируя по окружности $|z| = \dfrac{\delta}{2}$, получаем:

$$\sum_{\substack{n \leqq N \\ q_n = n \\ \omega(n) = b}} \frac{(-1)^{\Omega(n)-\omega(n)}}{\sqrt{n}} \sum_{\substack{d / \frac{n}{\alpha^2(n)} \\ \Omega(d) = a}} 1 \sim \sum_{\nu=0}^{\infty} \frac{R_{b,\nu}(\log\log N)}{(\log N)^\nu}, \tag{29}$$

$R_{b,\nu}(u)$ — полиномы степени $\leqq b$.

Из (29) вытекает:

$$\sum_{\substack{n\leqq N\\ q_n=n}}\frac{(-1)^{\Omega(n)-\omega(n)}}{\sqrt{n}}\sum_{\substack{d/\frac{n}{\alpha^2(n)}\\ \Omega(d)+\omega(n)=k}}1=$$

$$=\sum_{\substack{a+b=k\\ a,b\geqq 0}}\sum_{\substack{n\leqq N\\ q_n=n\\ \omega(n)=b}}\frac{(-1)^{\Omega(n)-\omega(n)}}{\sqrt{n}}\sum_{\substack{d/\frac{n}{\alpha^2(n)}\\ \Omega(d)=a}}1\sim\sum_{\nu=0}^{\infty}\frac{Q_\nu(\log\log N)}{(\log N)^\nu},$$

где

$$Q_\nu(u)=\sum_{\substack{a+b=k\\ a,b\geqq 0}}R_{b,\nu}(u).$$

Лемма доказана. Из неё следует, что

$$G_k(N)=\sum_{n\leqq N}\frac{g_k(n)}{\sqrt{n}}\sim(-1)^k\sum_{\nu=0}^{\infty}\frac{Q_\nu(\log\log N)}{(\log N)^\nu}. \tag{30}$$

Отсюда

$$\sum_{n>N}\frac{g_k(n)}{n}=\frac{1}{2}\sum_{n>N}\frac{g_k(n)}{\sqrt{n}}\int_n^\infty\frac{du}{u\sqrt{u}}=\frac{1}{2}\int_N^\infty\sum_{N<n\leqq u}\frac{g_k(n)}{\sqrt{n}}\frac{du}{u\sqrt{u}}=$$

$$=\int_N^\infty\frac{1}{\sqrt{u}}\,d\sum_{n\leqq u}\frac{g_k(n)}{\sqrt{n}}=\int_N^\infty\frac{1}{\sqrt{u}}\,dG_k(u).$$

Дифференцируя асимптотическое разложение (30) (возможность дифференцирования нетрудно обосновать), после простых преобразований получим:

$$\sum_{n>N}\frac{g_k(n)}{n}\sim\frac{1}{\sqrt{N}}\sum_{\nu=1}^{\infty}\frac{\bar P_\nu(\log\log N)}{(\log N)^\nu}, \tag{31}$$

где $\bar P_1(u)$ — полином степени $\leqq k-1$, а при $\nu\geqq 2$ $\bar P_\nu(u)$ — полиномы степени $\leqq k$.

Рассмотрим теперь сумму

$$\sum_{n\leqq N}\frac{g_k(n)}{\sqrt{n}}\,\varepsilon\left(\frac{N}{n}\right). \tag{32}$$

Пусть $\varrho(x)>0$ — убывающая дифференцируемая при $x\geqq x_0$ функция, такая, что $|\varepsilon(x)|\leqq\varrho(x)$ при $x\geqq x_0$. Выберем y с условием $x_0\leqq y\leqq\sqrt{N}$ и разобьём сумму (32) на три части, которые будем подсчитывать по-разному.

$$\sum_{n\leqq N}=\sum_{n\leqq\sqrt{N}}+\sum_{\sqrt{N}<n\leqq\frac{N}{y}}+\sum_{\frac{N}{y}<n\leqq N}.$$

Имеем:

(33)
$$\sum_{n\leq\sqrt{N}} \frac{g_k(n)}{\sqrt{n}}\,\varepsilon\left(\frac{N}{n}\right) \ll \sum_{n\leq\sqrt{N}} \frac{|g_k(n)|}{\sqrt{n}}\,\varrho\left(\frac{N}{n}\right) \leq$$
$$\leq \varrho\left(\sqrt{N}\right) \sum_{n\leq\sqrt{N}} \frac{|g_k(n)|}{\sqrt{n}} \ll \varrho\left(\sqrt{N}\right)(\log\log N)^k.$$

Для второго интервала используем асимптотическое разложение (25).

$$\sum_{\sqrt{N}<n\leq\frac{N}{y}} \frac{g_k(n)}{\sqrt{n}}\,\varepsilon\left(\frac{N}{n}\right) \ll \sum_{\sqrt{N}<n\leq\frac{N}{y}} \frac{|g_k(n)|}{\sqrt{n}}\,\varrho\left(\frac{N}{n}\right) \ll$$
$$\ll \varrho\left(\sqrt{N}\right)(\log\log N)^k - \sum_{\sqrt{N}<n\leq\frac{N}{y}} \frac{|g_k(n)|}{n} \int_{\frac{N}{n}}^{\sqrt{N}} \varrho'(u)\,du =$$
$$= \varrho\left(\sqrt{N}\right)(\log\log N)^k - \int_{y}^{\sqrt{N}} \sum_{\frac{N}{u}<n\leq\frac{N}{y}} \frac{|g_k(n)|}{\sqrt{n}}\,\varrho'(u)\,du.$$

Из (25) вытекает, что при $y\leq u\leq\sqrt{N}$

$$\sum_{\frac{N}{u}<n\leq\frac{N}{y}} \frac{|g_k(n)|}{\sqrt{n}} \ll \frac{(\log\log N)^{k-1}}{\log N}\log u,$$

откуда

(34)
$$\sum_{\sqrt{N}<n\leq\frac{N}{y}} \frac{g_k(n)}{\sqrt{n}}\,\varepsilon\left(\frac{N}{n}\right) \ll \varrho\left(\sqrt{N}\right)(\log\log N)^k + \frac{(\log\log N)^{k-1}}{\log N}\int_{y}^{\infty} |\varrho'(u)|\log u\,du.$$

Наконец, для третьей суммы имеем:

$$\sum_{\frac{N}{y}<n\leq N} \frac{g_k(n)}{\sqrt{n}}\,\varepsilon\left(\frac{N}{n}\right) = \varepsilon(y)\sum_{\frac{N}{y}<n\leq N} \frac{g_k(n)}{\sqrt{n}} + \sum_{\frac{N}{y}<n\leq N} \frac{g_k(n)}{\sqrt{n}} \int_{\frac{N}{n}}^{y} d\varepsilon(u) =$$
$$= \int_{1}^{y} \sum_{\frac{N}{u}<n\leq N} \frac{g_k(n)}{\sqrt{n}}\,d\varepsilon(u) + O\left(\varrho(y)\sum_{\frac{N}{y}<n\leq N} \frac{|g_k(n)|}{\sqrt{n}}\right) =$$
$$= \int_{1}^{y} \sum_{\frac{N}{u}<n\leq N} \frac{g_k(n)}{\sqrt{n}}\,d\varepsilon(u) + O\left(\varrho(y)\log y\cdot\frac{(\log\log N)^{k-1}}{\log N}\right).$$

Используя асимптотическое разложение (24), получаем: если $1\leqq u\leqq y\leqq\sqrt{N}$, то

$$\sum_{\frac{N}{u}<n\leqq N}\frac{g_k(n)}{\sqrt{n}}\sim\sum_{\nu=1}^{\infty}\frac{T_\nu(\log\log N,\log u)}{(\log N)^\nu},$$

где $T_\nu(x,y)$ — полиномы, степень которых по x не превышает $k-1$. Поэтому, если $\int\limits_1^\infty \log^A u\,d\varepsilon(u)<\infty$ для всех $A>0$, то

$$\sum_{\frac{N}{y}<n\leqq N}\frac{g_k(n)}{\sqrt{n}}\varepsilon\left(\frac{N}{n}\right)=\sum_{\nu=1}^{A}\frac{\tilde{Q}_\nu(\log\log N)}{\log^\nu N}+$$

$$+O\left(\frac{(\log\log N)^{k-1}}{\log N}\left|\int\limits_y^\infty\log^B u\,d\varepsilon(u)\right|\right)+O\left(\frac{\log\log N)^{k-1}}{\log^A N}\log^B y\int\limits_1^y|d\varepsilon(u)|\right).$$

Так как

$$\varepsilon(u)=\frac{1}{\sqrt{u}}\sum_{n\leqq u}\mu^2(n)-\frac{6}{\pi^2}\sqrt{u},$$

то

$$\int\limits_1^y|d\varepsilon(u)|\ll\sqrt{y},$$

и

(35)
$$\sum_{\frac{N}{y}<n\leqq N}\frac{g_k(n)}{\sqrt{n}}\varepsilon\left(\frac{N}{n}\right)=\sum_{\nu=1}^{A}\frac{\tilde{Q}_\nu(\log\log N)}{(\log N)^\nu}+$$

$$+O\left(\frac{(\log\log N)^{k-1}}{\log N}\left|\int\limits_y^\infty\log^B u\,d\varepsilon(u)\right|\right)+O\left(\frac{\sqrt{y}\log^B y(\log\log N)^{k-1}}{\log^A N}\right).$$

Собирая оценки (33)—(35), находим:

(36)
$$\sum_{n\leqq N}\frac{g_k(n)}{\sqrt{n}}\varepsilon\left(\frac{N}{n}\right)-\sum_{\nu=1}^{A}\frac{\tilde{Q}_\nu(\log\log N)}{(\log N)^\nu}\ll\varrho(\sqrt{N})(\log\log N)^k+$$

$$+\frac{(\log\log N)^{k-1}}{\log N}\left\{\left|\int\limits_y^\infty\varrho'(u)\log^B u\,du\right|+\left|\int\limits_y^\infty\log^B u\,d\varepsilon(u)\right|+\frac{\sqrt{y}\log^B y}{(\log N)^{A-1}}\right\},$$

где A — сколь угодно большая константа, $B=B(A)$.

Современные сведения о нулях $\zeta(s)$ дают оценку (см. [8])

$$\varepsilon(x)\ll\exp\left\{-a\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right\},\quad a>0.$$

Выбирая в (35) $y = \log N$, получаем:

$$\sum_{n\leq N} \frac{g_k(n)}{\sqrt{n}} \varepsilon\left(\frac{N}{n}\right) = \frac{1}{\log N}\left\{\tilde{Q}_1(\log\log N) + O\left(\exp\left\{-a\frac{(\log\log N)^{3/5}}{(\log\log\log N)^{1/5}}\right\}\right)\right\} \tag{37}$$

(все слагаемые, кроме первого, суммы в левой части (36) идут в остаток).

Утверждениє теоремы 2 следует теперь из равенств (23), (31) и (37).

Если принять квазириманову гипотезу, то

$$\varepsilon(x) \ll x^{-\alpha}$$

с некоторым $\alpha > 0$. Тогда из (36)

$$\sum_{n\leq N} \frac{g_k(n)}{\sqrt{n}} \varepsilon\left(\frac{N}{n}\right) - \sum_{\nu=1}^{A} \frac{\tilde{Q}_\nu(\log\log N)}{(\log N)^\nu} \ll$$

$$\ll N^{-\beta} + \frac{(\log\log N)^{k-1}}{\log N}\left\{y^{-\beta} + \frac{\sqrt{y}\log^B y}{(\log N)^{A-1}}\right\}, \quad \beta > 0.$$

Полагая здесь $y = \log^A N$ и учитывая, что A можно взять сколь угодно большим, получаем:

$$\sum_{n\leq N} \frac{g_k(n)}{\sqrt{n}} \varepsilon\left(\frac{N}{n}\right) \sim \sum_{\nu=1}^{\infty} \frac{\tilde{Q}_\nu(\log\log N)}{\log^\nu N}. \tag{38}$$

Из (23), (31) и (38) вытекает утверждение теоремы 3.

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Примечание к корректуре. В последиее время автором получено безусловное доказательство асимптотического разложенпя (11) остаточного члена в теореме Реньи, не зависящее от каких-либо гипотез.

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STRUCTURES ELEMENTARILY EQUIVALENT RELATIVE TO INFINITARY LANGUAGES TO MODELS OF HIGHER POWER

By

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Introduction

The main result of this paper gives an explicit axiomatization by means of a set of sentences of $L_{\omega_1,\omega}$ of the class of countable structures which are ω_1, ω-elementarily equivalent to some uncountable structure (see 1. 4. Theorem, in partic ular the equivalence of 1. 4. 1 and 1. 4. 5).

We also prove that a countable discrete linear ordering* is characterizable up to isomorphism by a sentence of $L_{\omega_1,\omega}$ (1. 6 Theorem).

In § 2 we prove that if an $L_{\varkappa^+,\omega}$ sentence has arbitrarily large models, it also has a model of power $\varkappa$ which is ∞, ω-elementarily equivalent to arbitrarily large models. Here our method is based on order-indiscernible elements. This result was obtained by the author in the spring of 1967 as a positive answer to a question asked by D. W. Kueker and J. Malitz. After completing this paper the author learned that the result had been known to C. C. Chang.

§ 0. Notations

Our set-theoretical and logical conventions are those generally applied in model-theoretical literature. In the following we list the notations only which may need explanation.

μ, ν, ϱ range over ordinals, $\varkappa, \lambda$ over infinite cardinals. card A is the cardinality of A. R_ν is the set of sets of rank less than ν, $\beth_\nu =_{df} \text{card } R_\nu$.

For arbitrary finite sequences x and y, $\lambda(x)$ denotes the length of x and $x^\frown y$ denotes the concatenation of x and y. We write $x{*}n$ for $x^\frown\langle n\rangle$, $x \lhd y$ means that x is a proper initial segment of y.

Similarity types are sets of finitary predicate and operation symbols; the latter include the individual constants. Similarity types are denoted by τ, τ'. The language $L_{\varkappa,\omega}(\tau)$, associated with $\varkappa$ and τ, has as logical symbols the variables v_n for $n<\omega$, the equality symbol $\approx$, the usual finitary connectives and quantifiers ($\neg, \wedge, \vee, \rightarrow, \forall v_n, \exists v_n$) and the infinitary connectives $\bigwedge, \bigvee$. The latter are applied for sets of power $<\varkappa$ of formulas to obtain a new formula. We also write $\bigwedge_{i\in I} \varphi_i$ for $\bigwedge\{\varphi_i : i \in I\}$ and similarly with $\bigvee$ for $\bigwedge$. We assume that every formula has only finitely many free variables. $\mathscr{F}_n^{\varkappa,\omega}(\tau)$ denotes the set of formulas of $L_{\varkappa,\omega}(\tau)$ with at most the free variables $v_0, \ldots, v_{n-1}$, $\mathscr{F}_n^{\infty,\omega}(\tau) =_{df} \bigcup_{\varkappa} \mathscr{F}_n^{\varkappa,\omega}(\tau)$,

* A linear ordering is discrete if it does not contain a dense subordering.

$\mathscr{F}^{\varkappa,\omega}(\tau)=_{df}\bigcup_{n<\omega}\mathscr{F}_n^{\varkappa,\omega}(\tau)$, $\mathscr{F}^{\infty,\omega}(\tau)=_{df}\bigcup_{n<\omega}\mathscr{F}_n^{\infty,\omega}(\tau)$. $\mathscr{T}_n(\tau)$ is the set of terms, built up from operation symbols in τ only, and having at most the variables $v_0, \dots, v_{n-1}$. $\mathscr{T}(\tau)=_{df}\bigcup_{n<\omega}\mathscr{T}_n(\tau)$.

If φ is a formula, t is a term, $\varphi(t/v_n)$ denotes the result of substituting t for the free occurrences of v_n in φ, after renaming the bound variables of φ if necessary.

Let $\mathfrak{A}$ be a structure of (similarity) type τ, $|\mathfrak{A}|=A$ its universe, $\varphi\in\mathscr{F}^{\infty,\omega}(\tau)$. If α is a function such that every free variable of φ belongs to the domain of α and the range of α is induded in A, $\models_{\mathfrak{A}}\varphi[\alpha]$ means that φ is true in $\mathfrak{A}$ when the free variable v is interpreted as $\alpha(v)$. We write $\models_{\mathfrak{A}}\varphi[a/v_n, \alpha]$ for $\models_{\mathfrak{A}}\varphi[\{(v_n, a)\}\cup\alpha]$, $\models_{\mathfrak{A}}\varphi[a/v_n]$ for $\models_{\mathfrak{A}}\varphi[a/v_n, 0]$, $\models_{\mathfrak{A}}\varphi[x]$ for $\models_{\mathfrak{A}}\varphi[\{(v_i, x(i)): i<n\}]$ if $x\in A^n$ and $\models_{\mathfrak{A}}\varphi[a_i: i<n]$ for $\models_{\mathfrak{A}}\varphi[\langle a_i: i<n\rangle]$. For a term $t\in\mathscr{T}_n(\tau)$ and $x\in A^n$, $t^{\mathfrak{A}}[x]$ is the denotation of t in $\mathfrak{A}$ if v_i denotes $x(i)$ for $i<n$.

We apply the convention that the universe of $\mathfrak{A}, \mathfrak{A}_i, \mathfrak{A}', \mathfrak{B}, \mathfrak{B}_i, \mathfrak{B}', \dots$ is $A, A_i, A', B, B_i, B', \dots$, resp. $\mathfrak{A}$ is called countable, or of power $\varkappa$, if A is countable, or of power $\varkappa$, resp. $h: \mathfrak{A}\simeq\mathfrak{B}$ abbreviates that h is an isomorphism on $\mathfrak{A}$ onto $\mathfrak{B}$, $\mathfrak{A}\simeq\mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. If $a_0, \dots, a_{n-1}\in A$, we use the notation $(\mathfrak{A}, a_0, \dots, a_{n-1})$ to denote the expansion of $\mathfrak{A}$ with the distinguished elements $a_0, \dots, a_{n-1}$ named by n new individual constants.

Let $\mathfrak{A}, \mathfrak{B}$ be structures of similarity type τ. $\mathfrak{A}$ and $\mathfrak{B}$ are *$\varkappa, \omega$-elementarily equivalent* (in symbols: $\mathfrak{A}\equiv^{\varkappa,\omega}\mathfrak{B}$) if they satisfy the same sentences of $L_{\varkappa,\omega}(\tau)$. $\mathfrak{A}$ is a *$\varkappa, \omega$-elementary substructure* of $\mathfrak{B}$ (in symbols: $\mathfrak{A}\prec^{\varkappa,\omega}\mathfrak{B}$) if $A\subseteq B$ and $(\mathfrak{A}, a_0, \dots, a_{n-1})\equiv^{\varkappa,\omega}(\mathfrak{B}, a_0, \dots, a_{n-1})$ for all $n<\omega$, and any $a_0, \dots, a_{n-1}\in A$. We write $\mathfrak{A}\equiv^{\infty,\omega}\mathfrak{B}$, or $\mathfrak{A}\prec^{\infty,\omega}\mathfrak{B}$ if $\mathfrak{A}\equiv^{\varkappa,\omega}\mathfrak{B}$ for all $\varkappa$, or $\mathfrak{A}\prec^{\varkappa,\omega}\mathfrak{B}$ for all $\varkappa$, resp.

In case $\varkappa=\omega$, $L_{\omega,\omega}(\tau)$ reduces to ordinary first order logic. In this case we omit the superscript ω, ω and write $\mathscr{F}(\tau)$ for $\mathscr{F}^{\omega,\omega}(\tau)$, e.t.c.

A *theory* is a pair (T, τ) such that $T\subseteq\mathscr{F}_0(\tau)$, and a *model of the theory* (T, τ) has the similarity type τ (in contrast to arbitrary models of T).

For a structure $\mathfrak{B}$ of type τ and any $\Sigma\subseteq\mathscr{F}_1(\tau)$ we say that $\mathfrak{B}$ *omits* Σ if there is no $b\in B$ such that $\models_{\mathfrak{B}}\varphi[b/v_0]$ for all $\varphi\in\Sigma$.

Suppose that $X\subseteq A$ and $\mathfrak{A}$ is a structure of type τ. We say that $\mathfrak{A}$ is *generated* by X if

$$A = \{t^{\mathfrak{A}}[x]: t\in\mathscr{T}_n(\tau), x\in X^n, n<\omega\}.$$

Following J. Silver, we call a theory (T, τ) *tidy*, if for every $n<\omega$ and $\varphi\in\mathscr{F}_{n+1}(\tau)$ there is a term $t\in\mathscr{T}_n(\tau)$ such that

$$T\models\exists v_n\varphi\rightarrow\varphi(t/v_n).$$

0. 1. LEMMA. *For every theory (T, τ) there is a tidy theory (T', τ') such that $\tau\subseteq\tau'$, card $\tau'\leqq$ max (card τ, ω) and $\mathfrak{B}$ is a model of (T, τ) iff it is a reduct to τ of a model of (T', τ').*

§ 1. Characterization of countable structures elementarily equivalent in $L_{\omega_1,\omega}$ to some uncountable structure

In this section we prove our result concerning the characterization of countable structures which are ω_1, ω-elementarily equivalent to some uncountable structure.

The similarity type τ is fixed throughout the section; it is not necessarily countable. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{A}'$, ... denote structures of type τ. In what follows, we suppress all reference to τ, thus $\mathscr{F}^{\omega_1,\omega}$ is written for $\mathscr{F}^{\omega_1,\omega}(\tau)$, e.t.c.

In the next lemma we collect some well-known results we need.

1. 1. Lemma. 1. 1. 1. *If* $\mathfrak{A}_\mu \prec^{\infty,\omega} \mathfrak{A}_\nu$ *for* $\mu<\nu<\varrho$, *then we have* $\mathfrak{A}_\mu \prec^{\infty,\omega} \bigcup_{\nu<\varrho} \mathfrak{A}_\nu$ *for* $\mu<\varrho$. *The same is true with* ∞ *replaced by* $\varkappa$.

1. 1. 2. *If* $\mathfrak{A}, \mathfrak{B}$ *are countable and* $\mathfrak{A} \equiv^{\omega_1,\omega} \mathfrak{B}$, *then* $\mathfrak{A} \cong \mathfrak{B}$.

1. 1. 3. *If* $\mathfrak{A}$ *is countable and* $a \in A$, *then there is* $\varphi \in \mathscr{F}_1^{\omega_1,\omega}$ *such that*

$$\models_{\mathfrak{A}} \varphi[b/v_0] \quad \text{iff} \quad (\mathfrak{A}, b) \simeq (\mathfrak{A}, a)$$

for any $b \in A$.

1. 1. 4. (Scott's Isomorphism Theorem [8].) *Suppose* τ *is countable. If* $\mathfrak{A}$ *is countable, then there is a sentence* $\varphi \in \mathscr{F}_0^{\omega_1,\omega}$ *such that for any countable* $\mathfrak{B}$, $\models_{\mathfrak{B}} \varphi$ *iff* $\mathfrak{B} \cong \mathfrak{A}$. *Such a* φ *is called a Scott sentence for* $\mathfrak{A}$.

1. 1. 5. *A Scott sentence* φ *(for any structure) is complete, i.e. for every* $\psi \in \mathscr{F}_0^{\omega_1,\omega}$ *either* $\varphi \models \psi$ *or* $\varphi \models \neg\psi$.

Proofs. 1. 1. 1 is a generalization of Tarski's Union Theorem [9]; also its proof is essentially the same. 1. 1. 2 and 1. 1. 3 are rather easily proved (the second proof uses 1. 1. 2). For details see [2]. 1. 1. 5 is a consequence of the Downward Löwenheim—Skolem theorem for $L_{\omega_1,\omega}$ (see [8]).

For the sake of brevity we call a nonempty set S of finite sequences of natural numbers a *wellfounded* (w.f.) *tree*, if the following are satisfied (s, s_1, s_2 range over finite sequences of natural numbers):

(1) for any s_1, s_2, $s_1 \lhd s_2$ and $s_2 \in S$ imply $s_1 \in S$,

(2) if $s^*n \in S$ for some $n \in \omega$, $s^*m \in S$ for all $m \in \omega$,

(3) there is no infinite sequence $\langle s_n : n<\omega \rangle$ of elements s_n of S such that $s_n \lhd s_{n+1}$ for all $n<\omega$.

Suppose that S is a w.f. tree. s is an *endpoint* of S if for all $n<\omega$, $s^*n \notin S$. It is wellknown that a property P can be proved to be possessed by all points of S "by induction on $s \in S$", i.e. by showing that (i) every endpoint of S has P and (ii) if for all $n \in \omega$ s^*n has P then s has P. There is a corresponding scheme of definition "by induction on $s \in S$".

Let S be again a w.f. tree and $\Phi = \langle \varphi_s : s \in S \rangle$ a function with domain S such that for each $s \in S$ $\varphi_s \in \mathscr{F}^{\omega_1,\omega}_{\lambda(s)+1}$, moreover $\varphi_0 = v_0 \approx v_0$. Let us call Φ a *w.f. formula-tree*. We define by induction on $s \in S$ a formula $\psi_s = \psi_s^{(\Phi)}$ such that

(4) if s is an endpoint of S,

$$\psi_s = \exists! v_0 \varphi_s,$$

(5) if s is not an endpoint of S,

$$\psi_s = \exists v_{\lambda(s)+1}[\forall v_0[\varphi_s \to \bigvee_{n<\omega} \varphi_{s*n}] \wedge \bigwedge_{n<\omega} \psi_{s*n}].$$

Put $\psi^{(\Phi)}$ to be ψ_0.

REMARK. The sentences $\psi^{(\Phi)}$ were obtained by trying to express in $L_{\omega_1,\omega}$ in natural ways that a model is countable. The simplest way is to say that every element is definable in terms of some finitely many fixed elements. (Sentences thus obtained appear in [11] for another purpose.) Then one may divide the structure into countably many definable parts and ensure the countability of each part separately. The $\psi^{(\Phi)}$ are obtained essentially by a transfinite iteration of the last procedure.

1. 2. PROPOSITION. *For any w.f. formula-tree Φ, the sentence $\psi^{(\Phi)}$ has only countable models.*

PROOF. Let $\Phi = \langle \varphi_s : s \in S \rangle$, $\psi = \psi^{(\Phi)}$ and S' the set of elements of S which are not endpoints of S.

Suppose $\models_{\mathfrak{A}} \psi$. Using the definition of ψ, we can easily define by induction on $\lambda(s) \in \omega$ elements $a_s \in A$ such that, if we put

$$\alpha_s =_{df} \{(v_{i+1}, a_{s \restriction i}) : i \leqq \lambda(s)\},$$

$$\alpha'_s =_{df} \{(v_{i+1}, a_{s \restriction i}) : i < \lambda(s)\} = \alpha_{s \restriction \lambda(s)-1},$$

we have

(7) for $s \in S'$

$$\models_{\mathfrak{A}} \forall v_0[\varphi_s \to \bigvee_{n<\omega} \varphi_{s*n}][\alpha_s],$$

$$\models_{\mathfrak{A}} \bigwedge_{n<\omega} \psi_{s*n}[\alpha_s]$$

and

(8) for $s \in S - S'$

$$\models_{\mathfrak{A}} \exists!\, v_0 \varphi_s[\alpha'_s].$$

Now we show by induction on $s \in S$ that the set $A_s =_{df} \{a : \models_{\mathfrak{A}} \varphi_s[a/v_0, \alpha'_s]\}$ is countable. Indeed, by (8) A_s has exactly one element if $s \in S - S'$. By (7) we have $A_s \subseteq \bigcup_{n<\omega} A_{s*n}$ for $s \in S'$, and since by the induction hypothesis A_{s*n} is countable for each $n < \omega$, so is A_s. Our induction is complete.

Obviously $A = A_0$, hence A is countable, q.e.d.

Let $\mathfrak{A}$ be a countable structure, $A = \{a_n : n < \omega\}$. Let a and $c_0, \ldots, c_{k-1}$ be elements of A. We call *a undetermined by $\langle c_0, \ldots, c_{k-1} \rangle$ (in $\mathfrak{A}$)*, if there exists a sequence $\langle b_n : n < \omega \rangle$ such that $b_0 = a$, $b_n \neq b_{n+1}$ and

$$(\mathfrak{A}, c_0, \ldots, c_{k-1}, a_0, \ldots, a_{n-1}, b_n) \cong (\mathfrak{A}, c_0, \ldots, c_{k-1}, a_0, \ldots, a_{n-1}, b_{n+1})$$

for all $n < \omega$.

It is rather easy to see that this notion is independent from the choice of the enumeration $\langle a_n : n < \omega \rangle$ of A. We apply this notion and the next simple lemma in the proofs of both 1. 4 and 1. 6.

1. 3 LEMMA. *Suppose a is undetermined by $\langle c_0, ..., c_{k-1}\rangle$ in $\mathfrak{A}$.*

1. 3. 1. *If h is an automorphism of $\mathfrak{A}$, then $h(a)$ is undetermined by $\langle h(c_0), ..., ..., h(c_{k-1})\rangle$.*

1. 3. 2. *If x is a sequence such that the range of x is included in $\{c_0, ..., c_{k-1}\}$, then a is undetermined by x.*

1. 3. 3. *For any $c\in A$ there is an a' such that*

$$(\mathfrak{A}, c_0, ..., c_{k-1}, a) \cong (\mathfrak{A}, c_0, ..., c_{k-1}, a')$$

and a' is undetermined by $\langle c_0, ..., c_{k-1}, c\rangle$.

1. 3. 4. *$a \neq c_i$ for $i<k$.*

PROOF. 1. 3. 1 follows from the remark on the independence from $\langle a_n : n<\omega\rangle$. 1. 3. 2 is obvious. To see 1. 3. 3, note that $c=a_m$ for some $m<\omega$. Having the sequence $\langle b_n : n<\omega\rangle$ satisfying the requirements in the definition of "a is undetermined by $\langle c_0, ..., c_{k-1}\rangle$", we put $a'=b_{m+1}$ and $b'_n=b_{m+1+n}$ for $n<\omega$. It is easy to check that $\langle b'_n : n<\omega\rangle$ is a sequence with the properties required for a' being undetermined by $\langle c_0, ..., c_{k-1}, c\rangle$. The isomorphism in 1. 3. 3 is a consequence of $a=b_0$, $a'=b_{m+1}$. Finally, to show 1. 3. 4, note that

$$(\mathfrak{A}, c_0, ..., c_{k-1}, a) \cong (\mathfrak{A}, c_0, ..., c_{k-1}, b_1)$$

and $b_1 \neq a$. $a=c_i$ would hence imply that $b_1=c_i=a$, which is a contradiction.

1. 4 THEOREM. *Let $\mathfrak{A}$ be a countable structure. The following conditions are equivalent:*

1. 4. 1 *There is an uncountable structure $\mathfrak{B}$ such that $\mathfrak{B} \equiv^{\omega_1, \omega} \mathfrak{A}$.*

1. 4. 2 *There is a structure $\mathfrak{B}$ of power ω_1 such that $\mathfrak{A} \prec^{\infty, \omega} \mathfrak{B}$.*

1. 4. 3 *There is a structure $\mathfrak{A}'$ such that $A\subset A'$, $A\neq A'$ and for all $n<\omega$ and any elements $a_0, ..., a_{n-1}\in A$ we have*

$$(\mathfrak{A}, a_0, ..., a_{n-1}) \cong (\mathfrak{A}', a_0, ..., a_{n-1}).$$

1. 4. 4 *For any sequence $\langle a_n : n<\omega\rangle$ of elements a_n of A, there is a sequence $\langle b_n : n<\omega\rangle$ such that for all $n<\omega$, $b_n \neq b_{n+1}$ and*

$$(\mathfrak{A}, a_0, ..., a_{n-1}, b_n) \cong (\mathfrak{A}, a_0, ..., a_{n-1}, b_{n+1}).$$

1. 4. 5 *For every w.f. formula-tree Φ we have $\models_{\mathfrak{A}} \neg \psi^{(\Phi)}$.*

1. 4. 6 *For any $\psi \in \mathscr{F}_0^{\omega_1, \omega}$, if $\models_{\mathfrak{A}} \psi$, then ψ has at least one uncountable model.*

PROOF. The implication 1. 4. 1 $\Rightarrow$ 1. 4. 6 is obvious. 1. 4. 6 $\Rightarrow$ 1. 4. 5 follows from 1. 2.

(*ad* 1. 4. 5 $\Rightarrow$ 1. 4. 4). Let $\langle a_n : n<\omega\rangle$ be a fixed sequence of elements of A and suppose $\mathfrak{A}$ does not satisfy 1. 4. 4 with the given $\langle a_n : n<\omega\rangle$. By induction on the length $\lambda(s)\in\omega$ of the finite sequence s of natural numbers we define $A_s \subseteq A$ as follows. Let $A_0=A$. For any $n<\omega$ and any given $a\in A$, let us call the set $\{b : (\mathfrak{A}, a_0, ..., a_{n-1}, b) \cong (\mathfrak{A}, a_0, ..., a_{n-1}, a)\}$ *a class modulo* $(a_0, ..., a_{n-1})$. Supposing that A_s has been defined, let $\langle A_{s*n} : n<\omega\rangle$ be an enumeration (not necessarily without repetition) of those classes modulo $(a_0, ..., a_{\lambda(s)+1})$ which are included in A_s. Since A_s is a class modulo $(a_0, ..., a_{\lambda(s)})$, or else it is A, we have that $A_s = \bigcup_{n<\omega} A_{s*n}$.

Let S be the set of the finite sequences s of natural numbers such that $A_{s \upharpoonright \lambda(s)-1}$ contains at least two different elements. Conditions (1) and (2) in the definition of w.f. tree are obviously satisfied by S. Condition (4) is easily seen to be equivalent to the assumption that $\mathfrak{A}$ does not satisfy 1. 4. 4 with the given $\langle a_n : n < \omega \rangle$. As a conclusion, S is a w.f. tree.

By 1. 1. 3, for every $s \in S$ there is a formula $\varphi_s \in \mathscr{F}^{\omega_1;\omega}_{n+1}$ (where $n = \lambda(s)$) such that

$$A_s = \{a : \models_{\mathfrak{A}} \varphi_s[a/v_0, \{(v_{i+1}, a_i) : i < n\}]\}.$$

Let $\Phi = \langle \varphi_s : s \in S \rangle$. By definition, Φ is a w.f. formula-tree. Using that $A_s = \bigcup_{n<\omega} A_{s*n}$ and that for every endpoint s of S A_s has exactly one element, a straightforward induction on $s \in S$ shows that

$$\models_{\mathfrak{A}} \psi_s^{(\Phi)}[\{(v_{i+1}, a_i) : i < \lambda(s)\}]$$

and in particular

$$\models_{\mathfrak{A}} \psi^{(\Phi)}.$$

We have obtained that $\mathfrak{A}$ does not satisfy 1. 4. 5, q.e.d.

(*ad* 1. 4. 4$\Rightarrow$1. 4. 3). Suppose that $\mathfrak{A}$ satisfies 1. 4. 4 and let $A = \{a_n : n < \omega\}$. Using the terminology introduced before 1. 3, the supposition means that there is an element $a \in A$ undetermined by the empty sequence (in $\mathfrak{A}$). By induction on k we define a sequence $\langle c_k : k < \omega \rangle$ such that for all $k < \omega$

(9) a is undetermined by $\langle c_0, \ldots, c_{k-1} \rangle$ and

(10) $(\mathfrak{A}, a_0, \ldots, a_{k-1}) \cong (\mathfrak{A}, c_0, \ldots, c_{k-1})$.

Suppose $c_0, \ldots, c_{k-1}$ have been defined such that (9) and (10) hold. By (10) let g be an automorphism of $\mathfrak{A}$ such that $g(a_i) = c_i$ for $i < k$ and let $c = g(a_k)$. By 1. 3. 3 there exists an automorphism h of $\mathfrak{A}$ such that $h(c_i) = c_i$ for $i < k$ and $a' =_{df} h(a)$ is undetermined by $\langle c_0, \ldots, c_{k-1}, c \rangle$. Put $c_k =_{df} h^{-1}(c)$. Applying 1. 3. 1 with h^{-1} for h, we obtain that a is undetermined by $\langle c_0, \ldots, c_{k-1}, c_k \rangle$. On the other hand, $h^{-1}g : (\mathfrak{A}, a_0, \ldots, a_{k-1}, a_k) \cong (\mathfrak{A}, c_0, \ldots, c_{k-1}, c_k)$. Hence (9) and (10) are fulfilled with $k+1$ for k, consequently our induction is complete.

It follows from (10) and $A = \{a_n : n < \omega\}$ that the mapping f defined by

$$f(a_n) = c_n$$

for $n < \omega$ is an isomorphism of $\mathfrak{A}$ into itself. Let $\mathfrak{C} \subset \mathfrak{A}$ be the image of $\mathfrak{A}$ by f. By (9) and 1. 3. 4 we have that $a \notin C = \{c_n : n < \omega\}$, hence $C \neq A$.

We show that the structure $\mathfrak{C}$ satisfies (iii), $\mathfrak{C}$ standing for $\mathfrak{A}$ in (iii). Since $\mathfrak{C} \cong \mathfrak{A}$, it will follow that $\mathfrak{A}$ itself satisfies (iii). To this end consider any finitely many elements, say $c_0, \ldots, c_{k-1}$, of C. By (10)

$$g : (\mathfrak{A}, a_0, \ldots, a_{k-1}) \cong (\mathfrak{A}, c_0, \ldots, c_{k-1})$$

for some g. The mapping gf^{-1} is an isomorphism of $\mathfrak{C}$ onto $\mathfrak{A}$ leaving $c_0, \ldots, c_{k-1}$ fixed, which shows that $\mathfrak{C}$ indeed satisfies (iii).

Our proof of 1. 4. 4$\Rightarrow$1.4.3 is complete.

(*ad* 1. 4.3$\Rightarrow$1. 4. 2). By transfinite induction on $\nu<\omega_1$ we define the countable structure $\mathfrak{A}_\nu$ such that

$$\mathfrak{A}_0=\mathfrak{A}$$

and for all $\mu<\nu<\omega_1$

$$\mathfrak{A}_\mu \prec^{\infty,\omega} \mathfrak{A}_\nu \tag{11}$$

and

$$\mathfrak{A}_{\mu+1}\not\cong\mathfrak{A}_\mu. \tag{12}$$

Suppose $0<\nu<\omega_1$ and that $\mathfrak{A}_\mu$ is defined for all $\mu<\nu$.

If ν is a limit ordinal, put $\mathfrak{A}_\nu=_{df}\bigcup_{\mu<\nu}\mathfrak{A}_\mu$, $\mathfrak{A}_\nu$ is obviously countable and by 1. 1. 1 $\mathfrak{A}_\mu\prec^{\infty,\omega}\mathfrak{A}_\nu$ for $\mu<\nu$.

If $\nu=\mu+1$, by $\mathfrak{A}_0=\mathfrak{A}\prec^{\infty,\omega}\mathfrak{A}_\mu$ it follows that $\mathfrak{A}\cong\mathfrak{A}_\mu$ (see 1. 1. 2). Let $\mathfrak{A}_{\mu+1}$ be a structure related to $\mathfrak{A}_\mu$ as $\mathfrak{A}'$ is related to $\mathfrak{A}$ in 1. 4. 3. Then $\mathfrak{A}_{\mu+1}\not\cong\mathfrak{A}_\mu$. Since $(\mathfrak{A}_\mu, a_0, \dots, a_{n-1})\cong(\mathfrak{A}_{\mu+1}, a_0, \dots, a_{n-1})$ trivially implies $(\mathfrak{A}_\mu, a_0, \dots, a_{n-1})\equiv$ $\equiv^{\infty,\omega}(\mathfrak{A}_{\mu+1}, a_0, \dots, a_{n-1})$, it follows that $\mathfrak{A}_\mu\prec^{\infty,\omega}\mathfrak{A}_{\mu+1}$.

This completes the definition of $\langle\mathfrak{A}_\mu\colon \nu<\omega_1\rangle$ with the required properties.

Finally put $\mathfrak{B}=_{df}\bigcup_{\nu<\omega_1}\mathfrak{A}_\nu$. From (12) card $B=\omega_1$ and by (11) and 1. 1. 1 $\mathfrak{A}\prec^{\infty,\omega}\mathfrak{B}$, q.e.d.

The implication 1. 4. 2$\Rightarrow$1. 4. 1 is obvious.

This completes the proof of 1. 4.

REMARKS. (a) The implication 1. 4. 3$\Rightarrow$1. 4. 2 is implicitly given by KUEKER [4] with the same proof (see the proof of Corollary 3. 3. 3 in [4]). The equivalence of 1. 4. 1—1. 4. 3 was found by the author in the spring of 1967 before he had known Kueker's work.

(b) The equivalence of conditions 1. 4. 1—1. 4. 4 and 1. 4. 6 can be proved by using Scott's Isomorphism Theorem (1. 1. 4) instead of referring to the formulas in 1. 4. 5.

First we derive 1. 4. 1$\Rightarrow$1. 4. 3 for τ countable. Suppose $\mathfrak{B}\equiv^{\omega_1,\omega}\mathfrak{A}$, $\mathfrak{B}$ is uncountable. Let $A=\{a_n\colon n<\omega\}$, and by 1. 1. 4 let $\varphi_n\in\mathscr{F}_n^{\omega_1,\omega}$ for each $n<\omega$ such that for any countable $\mathfrak{A}'$ and $b_i\in A'$, $\models_{\mathfrak{A}'}\varphi_n[b_i\colon i<n]$ iff $(\mathfrak{A}', b_0, \dots, b_{n-1})\cong$ $\cong(\mathfrak{A}, a_0, \dots, a_{n-1})$.

Let Σ be the smallest set of formulas such that (i) $\varphi_n\in\Sigma$ for $n<\omega$, (ii) every subformula of any member of Σ is in Σ, (iii) the negation of any member of Σ is in Σ. Σ is obviously countable.

Repeating the proof in [9] of the Downward—Löwenheim—Skolem theorem, we construct a countable A' such that "$\mathfrak{A}'\prec_\Sigma\mathfrak{B}$", i.e.

$$\models_{\mathfrak{A}'}\varphi[b_i\colon i<n] \quad \text{iff} \quad \models_{\mathfrak{B}}\varphi[b_i\colon i<n]$$

for any $n<\omega$, $\varphi\in\Sigma\cap\mathscr{F}_n^{\omega_1,\omega}$ and $b_i\in A'$ $(i<n)$. Adding an element from $B-A'\neq 0$ to A', we then construct $\mathfrak{A}''\prec_\Sigma\mathfrak{B}$ such that $A'\subset A''$, $A'\neq A''$. It follows $\mathfrak{A}'\prec_\Sigma\mathfrak{A}''$. Since $\varphi_0\in\Sigma$, we have $\mathfrak{A}'\cong\mathfrak{A}$, and in general by $\varphi_n\in\Sigma$ we have

$$(\mathfrak{A}', a_0', \dots, a_{n-1}')\cong(\mathfrak{A}'', a_0', \dots, a_{n-1}')$$

for arbitrary $a_0', \dots, a_{n-1}'\in A'$, which proves that $\mathfrak{A}'$, hence $\mathfrak{A}$ too, satisfies (iii).

It is not difficult to show directly that if $\mathfrak{A}$ does not satisfy (iii), the reduct of $\mathfrak{A}$ to some countable similarity type does not satisfy (iii) either. It follows that 1. 4. 1$\Rightarrow$1. 4. 3 holds generally.

As to the implication 1. 4. 3$\Rightarrow$1. 4. 4, suppose $\mathfrak{A}$ and $\mathfrak{A}'$ are as in 1. 4. 3. Taking any isomorphism h of $\mathfrak{A}$ onto $\mathfrak{A}'$ and any $c \in A' - A$, we easily check that the element $h^{-1}(c) \in A$ is undetermined by the empty sequence in $\mathfrak{A}$.

Finally, the implication 1. 4. 6$\Rightarrow$1. 4. 1 is immediate from 1. 1. 4 and 1. 1. 5 in case τ is countable. Hence also 1. 4. 6$\Rightarrow$1. 4. 3 holds if τ is countable. By the remark in the last but one paragraph 1. 4. 6$\Rightarrow$1. 4. 3 follows generally, q.e.d.

Suppose τ is countable. According to [10], a countable structure $\mathfrak{A}$ is called *saturated* if for any $n < \omega$, $a_0, \ldots, a_{n-1} \in A$ and $\Sigma \subset \mathscr{F}_{n+1}$, if for all finite subsets Σ' of Σ we have $\models_{\mathfrak{A}} \exists v_n \bigwedge \Sigma'[a_i: i < n]$, then for some $a_n \in A$, $\models_{\mathfrak{A}} \bigwedge \Sigma[a_i: i \leqq n]$. Using Theorem 4. 4 and 4. 6 of [10], one easily sees that 1. 4. 3 is satisfied by any saturated $\mathfrak{A}$. Hence we have

1. 5. COROLLARY. *If $\mathfrak{A}$ is a countable saturated structure of a countable similarity type, then there exists a structure $\mathfrak{B}$ of power ω_1 such that $\mathfrak{A} \prec^{\infty, \omega} \mathfrak{B}$.*

We apply the equivalence of 1. 4. 4 and 1. 4. 6 together with Scott's theorem to prove the next result on linear orderings. We call a (linear) ordering $(A, <)$ *dense* if it has at least two elements and if $a_1 < a_2$, then there is $b \in A$ such that $a_1 < b < a_2$. Cantor's well-known theorem says that a countable dense ordering has one of the order-types $\eta, 1+\eta, \eta+1, 1+\eta+1$ where η is the order-type of the set of the rational numbers ordered in the natural way. Following P. Erdős and A. Hajnal, we call an ordering *discrete* if it does not have a dense subordering. Finally, let us call a (countable) structure $\mathfrak{A}$ *characterizable* in $L_{\omega_1, \omega}$ if there is $\varphi \in \mathscr{F}_0^{\omega_1, \omega}$ such that $\mathfrak{B}$ is a model of φ iff $\mathfrak{B} \cong \mathfrak{A}$.

1. 6. THEOREM. *Every countable discrete ordering is characterizable in $L_{\omega_1, \omega}$.*

PROOF. Let $(A, <)$ be a countable ordering and suppose that $(A, <)$ is not characterizable in $L_{\omega_1, \omega}$. We show that $(A, <)$ is not discrete.

By 1. 1. 4, let φ be a Scott sentence for $\mathfrak{A} = (A, <)$. By assumption φ must have an uncountable model. Hence by 1. 1. 5 every sentence true in $\mathfrak{A}$ has an uncountable model, i.e. 1. 4. 6 is satisfied. By 1. 4, 1. 4. 4 is satisfied, too.

In what follows we use the notion "undetermined by" introduced before 1. 3. Note that there is an element in A undetermined by the empty sequence (in $\mathfrak{A}$).

For the sake of convenience let us call a finite subset X of A *a good set* if for any $a, b \in X$ such that $a < b$ and such that there is no $c \in X$ with $a < c < b$ (briefly: *b follows immediately a in X*) *either* b is undetermined by a (i.e. by $\langle a \rangle$) *or* a is undetermined by b.

1. 7. LEMMA. *If X is a good set, if $a, b \in X$ and if $a < b$, then there is a good set Y such that $X \subseteqq Y$ and for some $c \in Y$ we have $a < c < b$.*

PROOF OF 1. 7. Suppose X, a, b are given as in the lemma. If there is $c \in X$ such that $a < c < b$, we may take $Y =_{df} X$. Hence we may and do suppose that b follows immediately a in X. Since X is good, either b is undetermined by a, or conversely. By symmetry, it is enough to handle the first case. By 1. 3. 3 there is $d \in A$ undetermined by $\langle a, b\rangle$ such that for some automorphism h of $(A, <)$ $h(a)=a$ and $h(b)=d$. Since $a<b$, we have $h(a)<h(b)$, i.e. $a<d$. By 1. 3. 4 $d\neq b$, hence either $a<d<b$ (Case 1) or $b<d$ (Case 2).

In Case 1 let $Y=X\cup\{d\}$. Using also 1. 3. 2, we find that d is undetermined both by a and b. Since if y follows x immediately in Y then either y follows x immediately in X, or else $(x, y)=(a, d)$ or $(x, y)=(d, b)$, it follows that Y is a good set satisfying the requirements of the lemma.

In Case 2 let $c=h^{-1}(b)$. From $a<b<d$ we obtain $h^{-1}(a)<h^{-1}(b)<h^{-1}(d)$ i.e. $a<c<b$. Since b is undetermined by a and d is undetermined by b, we conclude by 1. 3. 1 that $c=h^{-1}(b)$ is undetermined by $a=h^{-1}(a)$ and $b=h^{-1}(d)$ is undetermined by $c=h^{-1}(b)$. Hence as before $Y=_{df}X\cup\{c\}$ satisfies the requirements of the lemma, which completes the proof.

Turning now to the proof of 1. 6, let a be an arbitrary element of A and $b\in A$ an element undetermined by a. Such b exists by supposition and 1. 3. 3. By 1. 3. 4, $b\neq a$. Put $X_0=\{a, b\}$; X_0 is a good set. Proceeding by induction, suppose $X_0, \dots, X_n$ are good sets such that $X_0\subseteq\dots\subseteq X_n$ and let $X_{n+1}\supseteq X_n$ be a good set such that for any $a, b\in X_n$ with $a<b$ there is $c\in X_{n+1}$ with $a<c<b$. Such an X_{n+1} is obtained by repeated application of 1. 7. Having defined the sequence $\langle X_n: n<\omega\rangle$, we put $B=\bigcup_{n<\omega} X_n$. It is obvious from the construction that the ordering $(B, <\restriction B)$ is dense, which completes the proof of 1. 6.

§ 2. Sentences having arbitrarily large models

It is to be noted that there are sentences of $L_{\omega_1,\omega}$ which (a) have uncountable models but (b) do not have any countable model ω_1, ω-elementarily equivalent to some uncountable model. Such sentences are given by Scott [8]. Scott defines for every ν, $1\leqq\nu<\omega_1$ a sentence φ_ν of $L_{\omega_1,\omega}$ with $\approx$ and the binary predicate E such that $\mathfrak{A}=(A, E^A)$ satisfies φ_ν iff $\mathfrak{A}$ is isomorphic to a transitive substructure of $(R_\nu, \in\restriction R_\nu)$. So it follows by a well-known theorem of Mostowski [7] that no model of φ_ν has any nontrivial automorphism. This in turn implies by the equivalence of 1. 4. 1 and 1. 4. 4 (1. 4 Theorem) (or also, it is seen directly) that φ_ν satisfies (b).

Incidentally, this shows that 1. 2 is not a consequence of 1. 4.

On the other hand, for any $\nu<\omega_1$, (c) φ_ν does not have arbitrarily large models. A corollary to the main result of this section (2. 8 Theorem) is that the property (c) of φ_ν is a consequence of (b).

For a linearly ordered set, or simply an *ordering* $(X, <)$, $X^{n,<}$ denotes the set of sequences $\langle x_i: i<n\rangle\in X^n$ such that $x_i<x_{i+1}$ for $i<n-1$. If also $(Y, <')$ is an ordering and $x\in X^n$, $y\in Y^n$, we write $(x)_<\cong(y)_{<'}$ in case $x(i)<x(j)$ iff $y(i)<y(j)$ for any $i, j<n$.

2. 1. Definition. Suppose $\mathfrak{B}$ is a structure of type τ', $(X, <)$ is an ordering and $X\subset B$.

2. 1. 1 X is called a *set of* $<$*-atomic-indiscernibles for* $\mathfrak{B}$ if for every $n<\omega$, every atomic $\varphi\in\mathscr{F}_n(\tau')$ and any $x, x'\in X^{n,<}$ we have:

$$\models_{\mathfrak{B}}\varphi[x] \quad \text{iff} \quad \models_{\mathfrak{B}}\varphi[x'].$$

X is a *set of* $<-\infty, \omega$*-indiscernibles for* $\mathfrak{B}$ if the above holds with any $\varphi\in\mathscr{F}_n^{\infty,\omega}(\tau')$.

2. 1. 2. We denote the set of all atomic formulas $\varphi \in \mathscr{F}(\tau')$ such that $\models_{\mathfrak{B}} \varphi[x]$ for some $n < \omega$ and $x \in X^{n,<}$ by $P(\mathfrak{B}, X, <)$.

2. 1. 3 $\mathfrak{B}$ is called an $(X, <)$-*structure* if X is a set of $<-\omega$, ω-indiscernibles for $\mathfrak{B}$ and $\mathfrak{B}$ is generated by X.

The following strengthened version of the basic result of MORLEY [6] was used by J. Malitz to prove the part of 2. 8 Theorem obtained by omitting 2. 8. 2 and 2. 8. 5.

2. 2 LEMMA. *Suppose that* $\operatorname{card} \tau' \leqq \varkappa$, $\Sigma \subseteqq \mathscr{F}_1(\tau)$ *and* (T, τ') *is a tidy theory. Suppose moreover that* (T, τ') *has a model omitting* Σ *and of power* $\geqq \beth_\nu$ *for each* $\nu < (2^\varkappa)^+$ *(resp. for each* $\nu < \omega_1$ *if* $\varkappa = \omega$*). Then for every linearly ordered set* $(X, <)$ *there is a model* $\mathfrak{B}$ *of* (T, τ') *such that* $\mathfrak{B}$ *is an* $(X, <)$*-structure and* $\mathfrak{B}$ *omits* Σ.

2. 2 is an easy consequence of the following two statements, (A) and (B). (A) is Lemma 4. 1 in MORLEY [6] and (B) is a version of the main theorem of EHRENFEUCHT—MOSTOWSKI [3].

(A) Suppose the hypotheses of 2. 2. Then there is a function $\langle \varphi_t : t \in \mathscr{T}(\tau') \rangle$ such that $\varphi_t \in \Sigma$ for $t \in \mathfrak{T}(\tau')$ and for every linearly ordered set $(X, <)$ there is a model $\mathfrak{B}$ of $(\mathscr{T}, \tau')$ with the properties that $\mathfrak{B}$ is generated by X and $\models_{\mathfrak{B}} \neg \varphi_t(t/v_0)[x]$ for all $n < \omega$, $t \in \mathscr{T}_n(\tau')$ and $x \in X^{n,<}$.

(B) Suppose that the binary predicate symbol E is in τ'' and that the theory (T, τ'') has at least one model $\mathfrak{B}$ such that $E^{\mathfrak{B}}$ is an infinite linear ordering of the field of $E^{\mathfrak{B}}$. Then for every linear ordering $(X, <)$ there is a model $\mathfrak{B}$ of (T, τ'') such that X is a set of $<-\omega$, ω-indiscernibles for $\mathfrak{B}$ and $< \subseteqq E^{\mathfrak{B}}$.

The next lemma is due to LOPEZ—ESCOBAR [5] (see Lemma 3. 2 in [5]) who used it together with MORLEY's results [6] to estimate from above the Hanf number of $L_{\varkappa^+,\omega}$ (2. 8 without 2. 8. 2—2. 8. 5).

2. 3. LEMMA. *Suppose* $\psi \in \mathscr{F}_0^{\varkappa^+,\omega}(\tau)$, $\operatorname{card} \tau \leqq \varkappa$. *There is a theory* (T, τ') *and a set* $\Sigma \subseteqq \mathscr{F}_1(\tau')$ *such that* $\tau \subseteqq \tau'$, $\operatorname{card} \tau' \leqq \varkappa$ *and for any structure* $\mathfrak{A}$ *of type* τ *of power at least* $\varkappa$, $\models_{\mathfrak{A}} \psi$ *holds iff there is an expansion* $\mathfrak{A}'$ *of* $\mathfrak{A}$ *which is a model of* (T, τ') *and which omits* Σ.

Our improvement of the results of Lopez—Escobar's and Malitz's mentioned previously rests on the following simple observation, 2. 4. Lemma, which is a special case of a theorem of CHANG (see [1]; 2. 4. and 2. 8. were found by the author in the spring of 1967, independently of Chang).

2. 4. LEMMA. *Suppose that* $(X_j, <_j)$ *is a dense (linear) ordering without first and last element and* $\mathfrak{B}_j$ *is an* $(X_j, <_j)$*-structure of type* τ' *for* $j = 1, 2$. *Suppose moreover that* $P(\mathfrak{B}_1, X_1, <_1) = P(\mathfrak{B}_2, X_2, <_2)$. *Then*

$$\models_{\mathfrak{B}_1} \varphi[x] \quad \textit{iff} \quad \models_{\mathfrak{B}_2} \varphi[y]$$

for any $n < \omega$, $\varphi \in \mathscr{F}_n^{\infty,\omega}(\tau')$, $x \in X_1^{n,<_1}$, $y \in X_2^{n,<_2}$. *In particular*, $\mathfrak{B}_1 \equiv^{\infty,\omega} \mathfrak{B}_2$.

If in addition $(X_1, <_1) \subset (X_2, <_2)$, *then* $\mathfrak{B}_1 \prec^{\infty,\omega} \mathfrak{B}_2$.

PROOF. By induction on the formula $\varphi \in \mathscr{F}^{\infty,\omega}(\tau')$ we prove that

(1) $$\models_{\mathfrak{B}_1} \varphi\left[t_i^{\mathfrak{B}_1}[x_i] : i < n\right]$$

if and only if

(2) $$\models_{\mathfrak{B}_2} \varphi\left[t_i^{\mathfrak{B}_2}[y_i]: i < n\right]$$

whenever $n < \omega$, $\varphi \in \mathscr{F}_n^{\infty, \omega}(\tau')$, $t_i \in \mathscr{T}_{m_i}(\tau')$, $x_i \in X_1^{m_i}$, $y_i \in X_2^{m_i}$ (for $i < n$) and

(3) $$(x_0{}^\frown x_1{}^\frown \dots {}^\frown x_{n-1})_{<_1} \cong (y_0{}^\frown y_1{}^\frown \dots {}^\frown y_{n-1})_{<_2}.$$

The statement to be proved is an easy consequence of the hypotheses if φ is an atomic formula. The induction steps corresponding to the (finitary or infinitary) propositional connectives are trivial.

What remains is to consider a formula φ of form $\varphi = \exists v \psi$. We may assume that $\psi \in \mathscr{F}_{n+1}^{\infty, \omega}(\tau')$ and $v = v_n$. Assume (1). Using that $\mathfrak{B}_1$ is generated by X_1, we may choose $m_n < \omega$, $t_n \in \mathscr{T}_{m_n}(\tau')$ and $x_n \in X_1^{m_n}$ such that

$$\models_{\mathfrak{B}_1} \psi\left[t_i^{\mathfrak{B}_1}[x_i]: i < n+1\right].$$

Applying (3) and the fact that $(X_2, <_2)$ is dense without first and last element, we have $y_n \in X_2^{m_n}$ such that

$$(x_0{}^\frown x_1{}^\frown \dots {}^\frown x_{n-1}{}^\frown x_n)_{<_1} \cong (y_0{}^\frown y_1{}^\frown \dots {}^\frown y_{n-1}{}^\frown y_n)_{<_2}.$$

Hence by the induction hypothesis applied for ψ, we obtain

$$\models_{\mathfrak{B}_2} \psi\left[t_i^{\mathfrak{B}_2}[y_i]: i < n+1\right]$$

and a fortiori

$$\models_{\mathfrak{B}_2} \exists v_n \psi[t_i^{\mathfrak{B}_2}[y_i]: i < n],$$

i.e. (2) indeed holds. By symmetry, we also have (2)$\Rightarrow$(1).

Thus our induction, and hence also the proof of the lemma, is complete.

2. 5 COROLLARY. *Suppose that $(X, <)$ is a dense ordering without first and last element and $\mathfrak{B}$ is an $(X, <)$-structure. Then X is a set of $< -\infty$, ω-indiscernibles for $\mathfrak{B}$.*

PROOF. Apply 2.4 with $(X_1, <_1) = (X_2, <_2) = (X, <)$ and $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}$.

We state the next two easily proved lemmas without proof. The first one can be verified by a straightforward "canonical" construction, or else by using the Compactness Theorem. The second one is Lemma 5. 4 in [3].

2. 6 LEMMA. *Suppose that $(X_1, <_1)$, $(X_2, <_2)$ are infinite orderings and $\mathfrak{B}_1$ is an $(X_1, <_1)$-structure. Then there exists an $(X_2, <_2)$-structure $\mathfrak{B}_2$ such that*

$$P(\mathfrak{B}_2, X_2, <_2) = P(\mathfrak{B}_1, X_1, <_1)$$

($\mathfrak{B}_2$ is uniquely determined up to isomorphisms leaving the elements of X_2 fixed). If in addition $(X_1, <_1) \subset (X_2, <_2)$, then $\mathfrak{B}_2$ can be chosen so that also $\mathfrak{B}_1 \subset \mathfrak{B}_2$.

2. 7. LEMMA. *If $\mathfrak{B}$ is an $(X, <)$-structure, then any automorphism of $(X, <)$ can be (uniquely) extended to an automorphism of $\mathfrak{B}$.*

2. 8 THEOREM. *Let $\varkappa$ be an infinite cardinal,* $\operatorname{card} \tau \leqq \varkappa$. *Let ψ be a sentence of $L_{\varkappa^+, \omega}(\tau)$ and suppose that for each $\nu < (2^\varkappa)^+$ (in case $\varkappa = \omega$, for each $\nu < \omega_1$) ψ has a model of power at least $\beth_\nu$. For each infinite cardinal $\lambda \geqq \varkappa$, let $(X_\lambda, <_\lambda)$*

be a densely ordered set of power λ without first and last element. Then there are structures $\mathfrak{A}_\lambda$ of type τ and of power λ such that for any $\lambda \geqq \varkappa$, $X_\lambda \subseteq A_\lambda$ and

2. 8. 1. *$\mathfrak{A}_\lambda$ is a model of φ,*

2. 8. 2. $\mathfrak{A}_\lambda \equiv^{\infty, \omega} \mathfrak{A}_\varkappa$,

2. 8. 3. *every automorphism of $(X_\lambda, <_\lambda)$ can be extended to an automorphism of $\mathfrak{A}_\lambda$, and*

2. 8. 4. *X_λ is a set of $<_\lambda - \infty$, ω-indiscernibles for $\mathfrak{A}_\lambda$.*

If in addition $(X_{\lambda_1}, <_{\lambda_1}) \subset (X_{\lambda_2}, <_{\lambda_2})$ for $\varkappa \leqq \lambda_1 < \lambda_2$, then $\mathfrak{A}_\lambda$ can be chosen so that also

2. 8. 5. *$\mathfrak{A}_{\lambda_1} \prec^{\infty, \omega} \mathfrak{A}_{\lambda_2}$ for $\varkappa \leqq \lambda_1 < \lambda_2$.*

REMARK. As was mentioned before, 2. 8 with 2. 8. 2—2. 8. 5 omitted is due to LOPEZ—ESCOBAR [5] and with 2. 8. 2, 2. 8. 5 omitted, to J. MALITZ (unpublished).
For related applications of indiscernibles, see [1] and [2].

PROOF. Suppose the hypotheses of the theorem. Applying 2. 3, we have a theory (T, τ') and $\Sigma \subseteq F_1(\tau')$ such that $\tau \subseteq \tau'$, card $\tau' \leqq \varkappa$ and for any structure $\mathfrak{A}$ of type τ and of power at least $\varkappa$, $\models_{\mathfrak{A}} \psi$ holds iff there is an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ which is a model of (T, τ') and which omits Σ. By 0. 1 Lemma, we may assume that (T, τ') is tidy. It is clear that (T, τ') and Σ satisfy the hypotheses of 2. 2. By 2. 2 let $\mathfrak{B}_\varkappa$ be a model of (T, τ') such that $\mathfrak{B}_\varkappa$ is an $(X_\varkappa, <_\varkappa)$-structure and $\mathfrak{B}_\varkappa$ omits Σ. For an arbitrary $\lambda > \varkappa$, let by 2. 6 $\mathfrak{B}_\lambda$ be an $(X_\lambda, <_\lambda)$-structure such that

$$P(\mathfrak{B}_\lambda, X_\lambda, <_\lambda) = P(\mathfrak{B}_\varkappa, X_\varkappa, <_\varkappa). \tag{4}$$

Put finally $\mathfrak{A}_\lambda =_{df} \mathfrak{B}_\lambda \restriction \tau$ for $\lambda \geqq \varkappa$. We show that the $\mathfrak{A}_\lambda$ satisfy the requirements.

By card $\tau' \leqq \varkappa$, card $X_\lambda = \lambda$ and the fact that $\mathfrak{B}_\lambda$ is generated by X_λ, we have card $\mathfrak{A}_\lambda = \lambda$ for $\lambda \geqq \varkappa$.

Since $\mathfrak{B}_\varkappa$ is a model of (T, τ') and omits Σ, $\mathfrak{A}_\varkappa$ is a model of φ.

Using the facts that $\mathfrak{B}_\lambda$ is an $(X_\lambda, <_\lambda)$-structure, (4) holds and $(X_\lambda, <_\lambda)$ is dense without first and last element for $\lambda \geqq \varkappa$, we obtain by 2. 4, 2. 5 and 2. 7 that 2. 8. 2, 2. 8. 4 and 2. 8. 3 are true, respectively. The structures $\mathfrak{A}_\lambda$ with the additional property 2. 8. 5 can be obtained by using the additions to 2. 4 and 2. 6.

The proof is complete.

2. 9. COROLLARY. *If a sentence of $L_{\varkappa^+, \omega}$ has arbitrarily large models, it has a model of power $\varkappa$ which is ∞, ω-elementarily equivalent to some structure of power λ for any $\lambda \geqq \varkappa$.*

The special case of 2. 9 when the sentence in question is a conjunction of finitary sentences was obtained independently by D. KUEKER [4] using different methods.

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DIE PAIRSCHEN[1] FREIEN BINOÏDEN ALS SPEZIALFÄLLE DER ANGEORDNETEN FREIEN HOLOMORPHEN MENGEN

von

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I. J. 1959 habe ich [2] den allgemeinen Begriff der angeordneten freien holomorphen Mengen (Mengen mit „zahlenartig" aufbaubaren Elementen) eingeführt, und die Theorie der rekursiven Funktionen für solche abstrakte Mengen als Definitionsbereiche verallgemeinert; ferner als wichtigsten Spezialfall den Fall der „Wortemengen" („freie Monoïden", d. h. Mengen der endlichen Folgen aus Elementen je einer gegebenen Menge) über „Alphabete" beliebiger Mächtigkeit ausgearbeitet. Bis zur letzten Zeit wurden unter den Spezialfällen des allgemeinen Begriffes nur die Wortmengen vielseitig untersucht und angewandt. Nun wurde in [1] ein in wichtigen Anwendungen, insbesondere in der mathematischen Grammatik nützlicher Begriff, der Begriff der „freien Binoïden" eingeführt, von welchen in vorliegender Arbeit gezeigt wird, daß — und wie — auch sie als angeordnete freie holomorphe Mengen definiert werden können, und wie sich die allgemeine Theorie der rekursiven Funktionen für solche Definitionsbereiche gestaltet.

In **I** gebe ich die notwendigen Kenntnisse über die angeordneten freien holomorphen Mengen an, teils schon an den hier betrachteten Spezialfall angepaßt, mit Hervorhebung einiger Fragen, die in verschiedenen Spezialfällen verschieden beantwortet werden können. Diese Fragen werden für den Fall der freien Binoïden in **III** beantwortet, wo auch verschiedene, in [1] eigeführte Begriffe auf Grund der allgemeinen Theorie als primitiv-rekursiv definiert werden (ich verwende aber zumeist von den dortigen abweichende Bezeichnungen). In **II** gebe ich die Definition aus [1] einer freien Binoïde $\hat{V}$ über eine Menge V an. Diese ist die Menge solcher Graphen, durch welche z. B. die „Tiefstrukturen" der zu einer Sprache gehörigen Texte veranschaulicht werden. In einer früheren Arbeit [3] habe ich für die sog. KANTOROWITSCH-schen Formel-Graphen eine geeignete „linearisierte" Form angegeben; für die hier betrachteten Graphen gebe ich eine analoge Form an: für die Graphen in [3] war es günstig, sie als Mengen gewisser geordneter Paare zu betrachten; für die als Binoïdenelemente auftretenden Graphen eignet es sich,

[1] C. PAIR—A. QUERE, *Definition et Etude des Bilangages Reguliers* (Université de Nancy, 1968). — Zitiert als [1].

[2] Siehe: R. PÉTER, Über die Verallgemeinerung der Theorie der rekursiven Funktionen für abstrakte Mengen geeigneter Struktur als Definitionsbereiche, *diese Acta*, Teil I **12** (1961), S. 271—314 (mit Angabe in der ersten Fußnote der Geschichte dieses Themenkreises); Teil II **13** (1962), S. 1—24 (mit einigen Berichtigungen zum Teil I). — Zitiert als [2a]. Berichtigungen dazu sind auch in der in Fußnote[3] zitierten Arbeit und im „Appendix" folgenden Buches zu finden: R. PÉTER, *Recursive Functions* (Budapest—New York—London, 1967). — Zitiert als [2b]. Eine Berichtigung zu [2b] werde ich in vorliegender Arbeit angeben.

[3] R. PÉTER, Über die Primitiv-Rekursivität einiger den Aufbau von Formeln charakterisierender Wortfunktionen, *diese Acta*, **14** (1963), S. 149—172. — Zitiert als [3].

diese als Mengen gewisser geordneter Tripel exakt (und auch ohne Kenntnis der — hier übrigens auch durch Berufung auf die Anschauung unterstützten — graphentheoretischen Begriffe eindeutig verständlich) zu definieren.

I

Die Eigenschaft einer Menge „auf Art der natürlichen Zahlen aufbaubar zu sein" bedeutet nach [2a] exakt das Folgende:

Sei H eine beliebige nicht leere Menge, H_0 eine nicht leere Teilmenge von H (alle Elemente von H_0 werden die Rolle von 0 spielen) und F eine nicht leere Teilmenge der auf H definierten und nur Werte aus H annehmenden Funktionen beliebig vieler Variablen (alle Elemente von F werden die Rolle der Nachfolgerfunktion spielen). Ferner sei H_n für $n=1, 2, \ldots$ folgenderweise angegeben: sind die Mengen $H_0, H_1, \ldots, H_n$ bereits definiert, dann sei H_{n+1} die Menge jener Elemente von H, welche als Werte der Funktionen aus F angenommen werden, falls für ihre Argumente Elemente aus $H_0 \cup H_1 \cup \ldots \cup H_n$ gesetzt werden, und dabei mindestens für ein Argument ein Element aus H_n. Wird nun H durch die Union sämtlicher Teilmengen $H_0, H_1, H_2, \ldots$ erschöpft, dann wird H eine *holomorphe Menge* genannt. Und zwar eine *freie holomorphe Menge,* falls die Mengen H_n paarweise disjunkt sind, und für jedes nicht zu H_0 gehörige Element von H eindeutig bestimmt ist, aus welcher zu F gehörigen Funktion und durch Einsetzen welcher Elemente für deren Argumente es entsteht. Im weiteren wird H immer eine freie holomorphe Menge bezeichnen, und auch die Bezeichnungen $F, H_0, H_1, H_2, \ldots$ werden beibehalten. $h \in H_i$ wird auch so ausgedrückt, daß die Ordnung $o(h)$ von h gleich i ist.

Es ist auch die Einführung des Begriffes der *Vorgänger* je eines Elementes von H notwendig; es empfiehlt sich aber, dies nicht allgemein festzulegen. In verschiedenen Anwendungen kann verschiedenartig angegeben werden, welche Teilmengen von H als Vorgängermengen je einem Element von H zugeordnet werden; die Relationen „y ist ein Vorgänger bzw. echter (d. h. von x verschiedener) Vorgänger von x" — durch $y \preceq x$ bzw. $y \prec x$ bezeichnet — haben aber immer die folgenden Forderungen zu erfüllen:

(1) $x \preceq x$.

(2) Ist x der Form $f(y_1, \ldots, y_n)$, wo $f \in F$ ist, so gehören die „unmittelbaren Konstitutienten" $y_1, \ldots, y_n$ von x zu den Vorgängern von x.

(3) Aus $x \prec y$ und $y \prec z$ folgt $x \prec z$.

(4) Ein echter Vorgänger von x ist immer kleinerer Ordnung als x.

Die angedeuteten Berichtigungen meiner früheren Arbeiten beziehen sich zumeist auf den ebenfalls notwendigen Begriff der *unmittelbaren Vorgänger* der Elemente von H (in manchen Anwendungen ist es nämlich nicht zweckmäßig. diese mit den unmittelbaren Konstituenten zu identifizieren). Hier bringe ich eine Berichtigung zum „Appendix" meines Buches [2b]: Die dort gegebene Definition für diesen Begriff ist nicht immer brauchbar. Es empfiehlt sich auch das nicht ein für allemal festzulegen, welche Teilmenge der Vorgängermenge je eines Elementes x als die Menge der unmittelbaren Vorgänger von x gelten soll, nur die folgenden Forderungen dafür festzulegen:

(5) Es gebe für jedes festgewählte $f \in F$ eine von den Argumenten unabhängige

obere Schranke für die Anzahl der unmittelbaren Vorgänger von Elementen der Struktur $x=f(...)$.

(6) Jeder echte Vorgänger eines x sei Vorgänger eines unmittelbaren Vorgängers von x.

Eine freie holomorphe Menge H mit einem den Forderungen (1)—(6) genügenden Vorgängerbegriff wird eine (partiell) *angeordnete freie holomorphe Menge* genannt. Von nun an sei durch H immer eine solche Menge bezeichnet.

Es kann nun auf Grund einer beliebigen angeordneten freien holomorphen Menge H als Definitionsbereich eine allgemeine Theorie der rekursiven Funktionen aufgebaut werden.

In vorliegender Arbeit wird H_0 aus einem einzigen (durch $\wedge$ bezeichneten) „leeren Objekt" bestehen, und F nur die zweistelligen Funktionen

$$f_{a_i}(x_1, x_2)$$

enthalten, wobei a_i die Elemente einer gegebenen Menge $V \subset H$ durchläuft. Das Element $\wedge$ besitzt keine echten Vorgänger. Für ein von $\wedge$ verschiedenes Element x gelten als echte Vorgänger seine unmittelbare Konstituenten und deren Vorgänger. Für diesen Spezialfall lautet das in [2a] angegebene Schema der primitiven Rekursion:

$$f(\wedge, u_1, ..., u_r) = g(u_1, ..., u_r)$$

und für $a_i \in V$

$$f(f_{a_i}(x_1, x_2), u_1, ..., u_r) = g_{a_i}(x_1, x_2, u_1, ..., u_r, f(x_1, u_1, ..., u_r), f(x_2, u_1, ..., u_r)).$$

Allgemein ist eine solche Definition nicht konstruktiv; es können aber daraus durch zu den betrachteten Anwendungen passende Einschränkungen konstruktive Definitionen geschöpft werden. Im Fall der in [1] betrachteten freien Binoïden ergibt sich die Konstruktivität der Definition ohne weiteres daraus, daß die Menge V (die als „Vokabular" einer Sprache behandelt wird) immer endlich ist.

Es werden die Elemente von H_0 und F allgemein als Ausgangsfunktionen aufgenommen, zu welchen eventuell noch weitere Ausgangsfunktionen anzuschließen sind. (In **III** werde ich zeigen, daß in unserem Spezialfall nur die charakteristische Funktion der Gleichheit zu den Ausgangsfunktionen hinzuzunehmen ist.) Eine Funktion wird (in den Ausgangsfunktionen) primitiv-rekursiv genannt, wenn sie von den Ausgangsfunktionen ausgehend durch endlich viele Substitutionen und primitive Rekursionen erhalten werden kann. Dabei darf jede Funktion so betrachtet werden, als eine Funktion ihrer Argumente und beliebig (endlich) vieler „hinzugenommenen" Argumente, von denen sie nicht tatsächlich abhängt.

Eine Beziehung B heißt primitiv-rekursiv, falls ihre charakteristische Funktion (welche in unserem Fall den Wert $\wedge$ oder a_0 annimmt — wo a_0 ein festgewähltes Element von V ist — je nachdem an der betrachteten Stelle B besteht oder nicht) primitiv-rekursiv ist.

Allgemein kann je ein bestimmtes Element $h_0, h_1, h_2, ...$ der Mengen $H_0, H_1, H_2, ...$ angegeben werden; diese können dann mit den natürlichen Zahlen 0, 1, 2, ... identifiziert werden. In unserem Fall können diese — mit dem bereits festgewählten Element a_0 von V — wie folgt gewählt werden:

$$h_0 = \wedge, \quad h_1 = f_{a_0}(h_0, h_0), \quad h_2 = f_{a_0}(h_1, h_1), \ldots$$

Man sieht, daß hier

$$h_m \leqq h_n \quad \text{gleichbedeutend mit} \quad m \leqq n$$

ist. Merken wir auch, daß für jede natürliche Zahl n (die mit h_n identifiziert wurde)

$$o(n) = o(h_n) = n = h_n,$$

insbesondere

$$o(n) = n$$

gilt.

Die charakteristischen Funktionen $\mathrm{sg}(x)$ und $\overline{\mathrm{sg}}(x)$ der Beziehungen $x = \wedge$ bzw. $x \neq \wedge$ sind primitiv-rekursiv. Diese benutzend erhält man nach [2a], daß die Primitiv-Rekursivität der Beziehungen B, bzw. B_1 und B_2 die Primitiv-Rekursivität der Negation $\bar{B}$, bzw. der Disjunktion $B_1 \vee B_2$ (d. h. „oder" im Sinne „vel") und der Konjunktion $B_1 \,\&\, B_2$ nach sich zieht. Auch dies spielte eine Rolle im Beweis der Primitiv-Rekursivität der durch Fallunterscheidung wie folgt aus primitiv-rekursiven Funktionen $g_1, \ldots, g_{n+1}$ und Beziehungen $B_1, \ldots, B_n$ definierten Funktion f, wobei an jeder Stelle höchstens eine dieser Beziehungen gilt:

$$f(u_1, \ldots, u_r) = \begin{cases} g_1(u_1, \ldots, u_r), & \text{falls} \quad B_1(u_1, \ldots, u_r) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ g_n(u_1, \ldots, u_r), & \text{falls} \quad B_n(u_1, \ldots, u_r) \\ g_{n+1}(u_1, \ldots, u_r) & \text{sonst.} \end{cases}$$

In meinen allgemeinen Untersuchungen habe ich zur primitiv-rekursiven Definition der Ordnung $o(x)$ eine Funktion x^{-1} benutzt, welche einen bestimmten unmittelbaren Vorgänger nächstkleinerer Ordnung von x angibt; und gefordert, daß x^{-1} als primitiv-rekursiv gelten, notwendigerfalls zu den Ausgangsfunktionen hinzugenommen werden soll. In **III** werde ich dafür die charakteristische Funktion der Beziehung $y \leqq x$ als primitiv-rekursive Funktion definieren, dann daraus eine primitiv-rekursive Definition von $o(x)$ erhalten; daraus erhielte man leicht auch für die (nicht mehr gebrauchte) Funktion x^{-1} eine primitiv-rekursive Definition.

Mit Verwendung von $o(x)$ wurde der — im folgenden als **Satz in I** zitierte — Satz erhalten, daß jede zahlentheoretische primitiv-rekursive Funktion zu einer primitiv-rekursiven Funktion in H erweitert werden kann. Genauer: Es gibt zu jeder primitiv-rekursiven zahlentheoretischen Funktion $\varphi(n_1, \ldots, n_r)$ eine primitiv-rekursive Funktion $f(z_1, \ldots, z_r)$ in H, für welche

$$o\big(f(z_1, \ldots, z_r)\big) = \varphi\big(o(z_1), \ldots, o(z_r)\big)$$

gilt. Daraus folgt, daß die in H primitiv-rekursive Funktion $g(z_1, \ldots, z_r) = o\big(f(z_1, \ldots, z_r)\big)$ und die zahlentheoretische primitiv-rekursive Funktion φ für natürliche Zahlen als Argumente übereinstimmen.

Um auch nicht-primitive Rekursionen (z. B. die Wertverlaufsrekursion, die simultane und die eingeschachtelte Rekursion) untersuchen zu können, wurde in [2a] eine Abbildung der Menge der aus Elementen von H gebildeten endlichen

Folgen auf eine Teilmenge von H gebraucht. Das Element von H, welches dabei der Elementenfolge $u_0, u_1, \ldots, u_n$ entspricht, wurde mit

$$c_n(u_0, u_1, \ldots, u_n)$$

bezeichnet (im zahlentheoretischen Fall kann dieses Element z. B.

$$p_0^{\mu_0} p_1^{\mu_1} \ldots p_n^{\mu_n}$$

sein, wo $p_0, p_1, \ldots$ die wachsende Primzahlenfolge ist). Es wurden Funktionen $\operatorname{long}(x)$ und $k_i(x)$ eingeführt, für welche, falls mit gewissen $u_0, u_1, \ldots, u_n \in H$

$$x = c_n(u_0, u_1, \ldots, u_n)$$

ist,

$$\operatorname{long}(x) = n$$

und für $i = 1, 2, \ldots, \operatorname{long}(x)$

$$k_i(x) = u_i$$

besteht, und es wurde gefordert, daß nach geeigneter Festsetzung der zu verwendenden Abbildung je eine solche Ausdehnung der Definitionen von $\operatorname{long}(x)$ und von $k_i(x)$ als Funktion der beiden Variablen x und i auf die ganze Menge H als Definitionsbereich, wobei immer

$$k_i(x) \leqq x$$

gilt, als primitiv-rekursiv gelten (notwendigerfalls zu den Ausgangsfunktionen hinzugenommen werden) soll.

In Bemerkung (1) (S. 299) des I. Teils von [2a] wird eine geeignete Abbildung und die primitiv-rekursive Definition der gewünschten Ausdehnungen der dazu gehörigen Funktionen $\operatorname{long}(x)$ und $k_i(x)$ für den Fall angegeben, wobei F wenigstens eine mindestens zweistellige Funktion enthält. Das eignet sich auch für unsere Zwecke; unser F enthält ja lauter zweistellige Funktionen. Aus diesen hat man eine zu fixieren; diese kann $f_{a_0}(x, y)$ sein. Damit werden den Folgen

$$\begin{array}{l} u_0 \\ u_0, u_1 \\ u_0, u_1, u_2 \\ \cdots\cdots \end{array}$$

der Reihe nach die Elemente

$$\text{(F)} \quad \begin{array}{l} c_0(u_0) = f_{a_0}(u_0, h_0) \\ c_1(u_0, u_1) = f_{a_0}(f_{a_0}(u_0, h_0), f_{a_0}(u_1, h_1)) \\ c_2(u_0, u_1, u_2) = f_{a_0}(f_{a_0}(f_{a_0}(u_0, h_0), f_{a_0}(u_1, h_1)), f_{a_0}(u_2, h_2)) \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

zugeordnet. Die Verwendung der $h_0, h_1, \ldots$ (wodurch die Bestimmung von $\operatorname{long}(x)$ sehr einfach wird) fordert eine geringe Änderung der Funktionen $c(x)$ und $p(x, y)$, durch welche in der allgemeinen Behandlung die Art der Entstehung von

$$c_{n+1}(u_0, \ldots, u_n, u_{n+1})$$

aus

$$c_n(u_0, \ldots, u_n) \quad \text{und} \quad u_{n+1}$$

nachgebildet wird, und deren Primitiv-Rekursivität (notwendigerfalls Hinzunahme zu den Ausgangsfunktionen) verlangt wird; deshalb komme ich darauf in **III** zurück.

Endlich ergab sich noch die Notwendigkeit der Primitiv-Rekursivität (wenn es anders nicht geht, die Hinzunahme zu den Ausgangsfunktionen) von vier weiteren Funktionen, die ich für unseren Fall spezialisiert angebe:

Zur Bildung der „Wertverlaufsfunktion" einer Funktion wird eine Folge (etwa mit Wiederholungen)

$$\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_s$$

aus den Vorgängern je eines $x \in H$ gebildet. Als eine zweckmäßige Anordnung wählen wir hier jene, wobei für $x = \wedge$ die Folge aus dem einzigen Glied $\wedge$ besteht, und für ein x der Form $f_{a_i}(x_1, x_2)$ erst die Vorgänger von x_1, dann die Vorgänger von x_2, und als letztes Glied x aufgezählt werden. Man sieht, daß hierbei die Zahl s nicht von a_i abhängig ist.

Sei i ein bestimmter jener Indizes 1 oder 2, für welchen $\bar{x}_j$ bei einem $j < s$ ein Vorgänger von x_i ist; und sei l ein bestimmter der Indizes, mit welchen $\bar{x}_j$ in der festgesetzten Reihenfolge der Vorgänger von x_i vorkommt. Diese i und l sollen als Funktionen von x und j derart auf die ganze Menge H ausgedehnt definiert werden, daß sie überall nur Zahlenwerte annehmen, und für jedes $x = f_{a_i}(x_1, x_2)$ nur von x_1, x_2 und j abhängig sind. Die ersten drei der vier in Frage stehenden Funktionen sind das obige s und diese beiden Funktionen:

$$s(x), \quad i(x, j), \quad l(x, j).$$

Die vierte in Frage stehende Funktion v ist zur Behandlung der „eingeschachtelten Rekursionen" notwendig. Ihre Bedeutung ist, daß $k_i(x)$ (das immer ein Vorgänger von x ist) als Vorgänger mit dem Index v von x in der festgelegten Reihenfolge der Vorgänger von x vorkommt. Ein solches v soll als Funktion von x und i auf die ganze Menge H ausgedehnt primitiv-rekursiv sein.

In vielen Definitionen wurden auch die Konstanten $a_i \in V$ und die Identitätsfunktion $f(x) = x$ verwendet. Natürlich müssen auch diese als primitiv-rekursiv gelten (notwendigerfalls zu den Ausgangsfunktionen hinzugenommen werden).

II

Eine freie Binoïde $\hat{V}$ über eine Menge V ist eine Graphen-Menge, deren Elemente „Tiefstrukturen" von Texten veranschaulichen, welche aus Sätzen von Sprachen über das gegebene Vokabular V bestehen. (Für die Weiteren ist die Kenntnis der exakten Definitionen der in dieser Erklärung gebrauchten Begriffe der mathematischen Grammatik nicht notwendig.) Da es üblich ist, gewisse Graphen mit Namen wie „Baum", „Setzbaum" (d. h. Baum mit einem ausgezeichneten „Wurzelpunkt"), „Wald", sogar „Kaktus" zu benennen, wäre die passende Benennung für ein Element von $\hat{V}$ eigentlich „Spiegelbild im Strom einer herbstlichen Allee" (wofür ich kurz nur „Allee" sagen werde). Unser Graph wäre nämlich ein Wald, doch mit solchen Bäumen als Komponenten, die in eine endliche Folge geordnet sind („Allee"). Ferner sind diese Komponenten recht bunt: sowohl Punkten- als auch Kantengefärbt („herbstlich"). Endlich sind die Komponenten orientierte Setzbäume; doch in den Abbildungen (da es sich um „Tiefstrukturen" handelt) immer vom Wurzelpunkt nach unten orientiert („Spiegelbild"). Wie jeder Graph, bleibt auch dieser invariant gegenüber stetigen Deformationen (Verzerrungen des Bildes im „Strom").

In diesem romantischen Bild wurde schon halbwegs die exakte Definition des Begriffes „Allee" angegeben: eine Allee über V ist mit den Fachausdrücken der Graphentheorie ausgedrückt eine endliche Folge von vom Wurzelpunkt ausgehend orientierten Setzbäumen — wobei mit Ausnahme des Wurzelpunktes, in den keine Kanten einlaufen, in jeden Knotenpunkt (kurz: „Punkt") genau eine Kante einläuft — deren Punkte mit Elementen einer gegebenen Menge V „gefärbt" sind (unter „Färbung mit einem z" die Zuordnung von z zu einem Punkt oder zu einer Kante verstanden), und deren aus je einem Punkt herauslaufenden Kanten untereinander geordnet sind. Diese Teilordnung kann damit angedeutet werden, daß die aus einem Punkt herauslaufenden Kanten mit 1, 2, 3, ... „gefärbt" werden. Nun ist $\hat{V}$ die Menge sämtlicher Alleen über V. (Im weiteren bedeutet „Allee" immer „Allee über V".)

Da in je einen Punkt eines Setzbaumes ein einziger Weg aus dem Wurzelpunkt führt, und dieser durch die Folge der an ihm entlang auftauchenden Farben-Zahlen (kurz: durch die zum Punkt gehörige „Farbenfolge") angegeben werden kann, ist ein Punkt P einer Allee durch ein geordnetes Tripel vollkommen charakterisiert, dessen erstes Glied die Ordnungszahl in der Komponentenfolge der P enthaltenden Komponente ist, das zweite Glied die zu P gehörige Farbenfolge, und das dritte Glied die (durch die ersten beiden eindeutig bestimmte) Farbe von P. Den Anfangs- bzw. Endpunkt einer Kante k charakterisierende Tripel T_1 und T_2 besitzen dann dieselbe Zahl als erstes Glied; und als zweites Glied besitzt T_2 die Fortsetzung durch die „Farbe" von k der in T_1 als zweites Glied auftretenden Farbenfolge.

Diese Darstellung bietet auch eine natürliche Anordnung der Punkte einer Allee: der durch das Tripel T_1 charakterisierte Punkt liegt dann „vor" dem durch das Tripel T_2 charakterisierten Punkt, wobei als zweites Glied in T_1 die Farbenfolge $(m_1, m_2, \ldots, m_r)$ und in T_2 die Farbenfolge $(n_1, n_2, \ldots, n_s)$ enthalten ist, wenn entweder das erste Glied von T_1 kleiner als das erste Glied von T_2 ist, oder ihre ersten Glieder übereinstimmen, doch für das kleinste i, wofür $m_i \neq n_i$ gilt, m_i entweder „leer" (d. h. gar nicht vorhanden) oder kleiner als n_i ist.

Nun schlage ich vor, als eine exakte (auch ohne Kenntnis der graphentheoretischen Begriffe eindeutig verständliche) Definition einer Allee über V eine endliche Menge geordneter Tripel zu betrachten, welche entweder leer (mit der Bezeichnung Λ) oder wie folgt beschaffen ist:

1. Die ersten Glieder der Tripel sind positive ganze Zahlen, wobei eine Zahl nur dann als erstes Glied einer Tripel vorkommen kann, wenn auch alle kleineren Zahlen als erste Glieder von Tripeln vorkommen. (Die Tripel mit gleichem ersten Glied bilden je eine Komponente der Allee. Das größte vorkommende erste Glied ist die Anzahl der Komponenten.)

2. Die zweiten Glieder der Tripel sind „Farbenfolgen", d. h. endliche Folgen von positiven ganzen Zahlen (auch die mit $\varnothing$ bezeichnete „leere" Folge zugelassen, woraus durch Anschließung einer Folge diese Folge selbst entsteht), wobei eine nicht leere Farbenfolge

$$(n_1, \ldots, n_r, n_{r+1})$$

nur dann als zweites Glied einer Tripel mit dem ersten Glied m vorkommen kann, wenn unter den Tripeln mit dem ersten Glied m als zweites Glied einerseits auch die Folge

$$(n_1, \ldots, n_r)$$

vorkommt (daraus ergibt sich durch Wiederholung, daß neben jedem vorkommenden ersten Glied auch $\emptyset$ als zweites Glied vorkommen muß; das zweite Glied $\emptyset$ ist für die Wurzelpunkte der Komponenten charakteristisch); andererseits für $n_{r+1} > 1$ auch die Folgen

$$(n_1, \ldots, n_r, 1), (n_1, \ldots. n_r, 2), \ldots, (n_1, \ldots, n_r, n_{r+1}-1)$$

vorkommen (in diese führen die „vorherigen" Kanten aus dem gemeinsamen Punkt mit der Farbenfolge $(n_1, \ldots, n_r)$ der m-ten Komponente).

3. Die dritten Glieder der Tripel sind durch die ersten beiden Glieder eindeutig bestimmte Elemente von V.

In [1] werden zur Erzeugung der Alleen von der leeren Allee $\wedge$ ausgehend zwei Operationen eingeführt. Diese werden durch $\rightarrow$ bzw. $\uparrow$ bezeichnet, was darauf hinweist, daß die Alleen durch die erste Operation „nach rechts", durch die zweite „nach oben" erweitert werden. Durch $x \rightarrow y$ wird nämlich das assoziative Nacheinandersetzen *(„Konkatenation")* der Alleen x und y bezeichnet; durch $a_i \uparrow x$, mit $a_i \in V$, die *„Verwurzeligung"* der Allee x, d. h. die Aufnahme von einem mit a_i gefärbten Wurzelpunkt, und von daraus in die Wurzelpunkte der Komponenten von x (in der Reihenfolge dieser Komponenten geordnet) führenden Kanten.

Für als Tripelmengen definierte Alleen über V bedeutet $x \rightarrow y$ die Union von x mit y, nachdem für nicht leeres x und y erst y derart modifiziert wird, daß darin zum ersten Glied jedes Tripels m addiert wird, wobei m die größte Zahl ist, die in einem zu x gehörigen Tripel als erstes Glied vorkommt; und die Bedeutung von $a_i \uparrow x$ ist (in Betracht gezogen, daß durch diese Operation ein zusammenhängender, eine einzige Komponente enthaltender Graph entsteht): es wird zu x die Tripel $(1, \emptyset, a_i)$ hinzugenommen, und — für nicht leeres x — auch in jedem Tripel von x das erste Glied — nachdem es als Anfang der im Tripel vorkommenden Farbenfolge hinzugefügt wird — durch 1 ersetzt.

In [1] wird als *Grundeigenschaft* der Alleen hervorgehoben, daß jede nicht leere Allee r über V mit eindeutig bestimmten

$$a_i \in V, \quad r_1 \in \hat{V}, \quad r_2 \in \hat{V}$$

als

(D) $$r = r_1 \rightarrow (a_i \uparrow r_2)$$

dargestellt werden kann. Für als Tripelmenge definiertes r erhält man a_i, r_1 und r_2 (in Betracht gezogen, daß in dieser Darstellung $a_i \uparrow r_2$ das letzte Glied der Komponentenfolge von r, und a_i der Wurzelpunkt dieser letzten Komponente ist) folgenderweise: Ist m die größte Zahl, die als erstes Glied einer zu r gehörigen Tripel vorkommt, so ist a_i das dritte Glied jenes (in r sicher enthaltenen) Tripels, dessen erstes Gied m und zweites Glied $\emptyset$ ist. Die Tripel, deren erste Glieder kleiner als m sind, bilden r_1. Endlich entstehen die zu r_2 gehörigen Tripel aus den Tripeln von r mit m als erstes Glied, indem dasjenige mit dem zweiten Glied $\emptyset$ weggelassen wird, und in jedem anderen das erste Glied der in ihm vorkommenden Farbenfolge abgetrennt und an Stelle des ersten Gliedes des Tripels gesetzt wird.

Die Allee r besteht dann und nur dann aus einem einzigen Setzbaum, wenn r_1 leer ist. Für $r_1=r_2=\wedge$ besteht r aus dem einzigen Wurzelpunkt $(1, \emptyset, a_i)$; diese Allee kann mit der Farbe ihres einzigen Punktes, mit a_i identifiziert werden (wonach V zu einer Teilmenge von $\hat{V}$ wird).

Da $a_i \uparrow r_2$ in der Darstellung (D) einer Allee r zusammenhängend, und so eine Komponente von r ist, und zwar das letzte Glied der Komponentenfolge von r, zeigt (D) die Erzeugung je einer Allee, von der leeren $\wedge$ ausgehend, durch das Anschließen immer neuer Komponenten ans Ende der bereits vorhandenen Komponentenfolge. Wären die Komponenten (die zusammenhängenden Alleen) als weiter unauftrennbare Objekte behandelt, so hätten wir es hier mit einer freien Monoïde über die Menge der zusammenhängenden Alleen zu tun. Doch die Verwendung der mit $\uparrow$ bezeichneten „Verwurzeligung" in der letzten Komponente $a_i \uparrow r_2$ von r zeigt, daß es sich hier auch um die Weitergliederung der zusammenhängenden Alleen, um die Aufdeckung ihrer „Tiefstruktur" handelt. So kommt man zum Begriff der Binoïden. Eine Binoïde über eine Menge V ist eine Menge B mit einer Regel der „inneren" (auf je zwei Elemente von B verwendbaren), assoziativen Aneinandersetzung von Elementen, mit einem dazu gehörigen Einheitselement (das ist für die Binoïde $\hat{V}$ das Element $\wedge$); und mit einer Regel für das Anschließen eines „äußeren" (d.h. zu V gehörigen) Elementes an ein Element von B. In [1] wird $\hat{V}$ in dem Sinne eine „freie" Binoïde über V genannt, daß sie in eine beliebig gegebene Binoïde B über V in genau einer Weise homomorph abgebildet werden kann.

Von nun an wird die Frage angegriffen, ob und wie die Binoïde $\hat{V}$ über V als ein Spezialfall der angeordneten freien holomorphen Mengen betrachtet werden kann.

So sicher nicht, daß neben $H_0=\{\wedge\}$ die Funktionen $x\rightarrow y$ und $a_i \uparrow x$ für alle $a_i \in V$ als Elemente von F aufgenommen werden; da dann wegen der Assoziativität von $x\rightarrow y$ der Aufbau einer aus mehr als zwei Komponenten bestehenden Allee (als das einzige Element $\wedge$ von H_0 durch endlichmalige Anwendung von in F enthaltenen Funktionen) keineswegs eindeutig wäre. Wird aber, mit der Bezeichnung

$$f_{a_i}(x, y)=x\rightarrow(a_i \uparrow y),$$

F als die Menge aller Funktionen $f_{a_i}(x, y)$ gebildet, wobei $a_i \in V$ ist, so zeigt die eindeutige Darstellbarkeit (D) jeder Allee, daß die durch diese Wahl von F (neben $H_0=\{\wedge\}$) erzeugte holomorphe Menge aus den Elementen von $\hat{V}$ besteht, ferner eine *freie* holomorphe Menge ist, da jedes seiner von $\wedge$ verschiedenen Elemente der Form

$$f_{a_i}(x_1, x_2)$$

ist, mit eindeutig bestimmten a_i und Argumente. Nun empfielt es sich, die Vorgänger und unmittelbaren Vorgänger der Elemente von $\hat{V}$ so zu definieren, wie es in **II** angedeutet wurde (wonach $\wedge$ keine echte Vorgänger, ein Element $x=f_{a_i}(x_1, x_2)$ als unmittelbare Vorgänger x_1 und x_2, als echte Vorgänger sämtliche Vorgänger von x_1 und x_2 besitzen wird); und damit erhalten wir $\hat{V}$ als eine angeordnete freie holomorphe Menge. Im weiteren wird mit $\hat{V}$ immer dies bezeichnet (wobei V eine gegebene Menge ist).

III

Nun kommt es darauf an, für eine gegebene (im in [1] behandelten Fall endliche) Menge V die Primitiv-Rekursivität in $\hat{V}$ jener Funktionen nachzuweisen, von welchen dies in **I** gefordert wurde. Dieser Begriff wird infolge der Bisherigen folgenderweise festgestellt:

Das Schema der primitiven Rekursion lautet:

$$f(\wedge, u_1, \ldots, u_r) = g(u_1, \ldots, u_r)$$

(S) und für alle $a_i \in V$

$$f(f_{a_i}(x_1, x_2), u_1, \ldots, u_r) = g_{a_i}(x_1, x_2, u_1, \ldots, u_r, \quad f(x_1, u_1, \ldots, u_r), f(x_2, u_1, \ldots, u_r)).$$

Eine Funktion heißt primitiv-rekursiv (worunter hier immer „primitiv-rekursiv in $\hat{V}$" verstanden wird), falls sie von den Funktionen

$$\wedge, \quad f_{a_i}(x_1, x_2) \quad \text{für} \quad a_i \in V$$

und von der charakteristischen Funktion der Beziehung

$$x = y$$

ausgehend durch eine endliche Kette von Substitutionen und primitiven Rekursionen entsteht.

Bereits in **I** wurde auf die Primitiv-Rekursivität mehrerer Funktionen hingezeigt; diese werden in den Folgenden benutzt.

Die durch $\mathrm{uv}_1(x)$ bzw. $\mathrm{uv}_2(x)$ bezeichneten unmittelbaren Vorgänger von x (dessen Wert für $x = \wedge$ belanglos ist) können durch folgende primitive Rekursionen definiert werden:

$$\mathrm{uv}_1(\wedge) = \wedge \qquad \mathrm{uv}_2(\wedge) = \wedge$$

$$\text{und für} \quad a_i \in V$$

$$\mathrm{uv}_1(f_{a_i}(x_1, x_2)) = x_1 \qquad \mathrm{uv}_2(f_{a_i}(x_1, x_2)) = x_2.$$

Bereits in diesen Definitionen wurde für die Funktionen g_{a_i} des Schemas (S) die Identitätsfunktion $\mathrm{id}(x) = x$ benutzt (als Funktion mehrerer Variablen betrachtet, was aber zugelassen wurde). Diese ist infolge der Definition

$$\mathrm{id}(\wedge) = \wedge$$

$$\text{und für} \quad a_i \in V$$

$$\mathrm{id}(f_{a_i}(x_1, x_2)) = f_{a_i}(x_1, x_2)$$

primitiv-rekursiv.

Da in **II** durch Identifizierung der aus je einem einzigen Punkt bestehenden Alleen mit der Farbe des betreffenden Punktes erreicht wurde, daß auch die Elemente von V zu $\hat{V}$ gehören, können auch diese als (konstante) primitiv-rekursive Funktionen in $\hat{V}$ erhalten werden: in Betracht gezogen — was auch in den weiteren Definitionen immer vor Augen zu halten ist —, daß f_{a_i} für ein beliebiges $a_i \in V$ als

$$f_{a_i}(x_1, x_2) = x_1 \rightarrow (a_i \uparrow x_2)$$

eingeführt wurde, erhält man a_i durch die Substitution

$$a_i = f_{a_i}(\wedge, \wedge).$$

Die charakteristische Funktion $\operatorname{vorg}(x, y)$ der Relation $y \preceq x$ ergibt sich durch Fallunterscheidung, die — wie darauf, und auch auf die Primitiv-Rekursivität der logischen Verknüpfungen, in **I** hingezeigt wurde — ihre Primitiv-Rekursivität sichert:

$$\operatorname{vorg}(\wedge, y) = \operatorname{sg}(y)$$

und für $a_i \in V$

$$\operatorname{vorg}(f_{a_i}(x_1, x_2), y) = \begin{cases} \wedge, & \text{falls } f_{a_i}(x_1, x_2) = y \vee \operatorname{vorg}(x_1, y) = \wedge \vee \\ & \vee \operatorname{vorg}(x_2, y) = \wedge \\ a_0 & \text{sonst.} \end{cases}$$

Damit ist auch die Relation $x \prec y$ primitiv-rekursiv, da sie (die Negation der Gleichheit wie üblich bezeichnet) mit

$$x \preceq y \,\&\, x \neq y$$

gleichbedeutend ist.

Damit kann auch die Ordnung $o(x)$ von x primitiv-rekursiv definiert werden. In **I** wurde der Begriff der Ordnung auseinandergesetzt, wonach der Wert von $o(x)$ immer eine natürliche Zahl ist; dabei wurden die natürlichen Zahlen

$$0, 1, 2, \ldots$$

der Reihe nacht mit

$$\wedge, \quad h_1 = f_{a_0}(\wedge, \wedge), \quad h_2 = f_{a_0}(f_{a_0}(\wedge, \wedge), f_{a_0}(\wedge, \wedge)), \ldots$$

identifiziert. Nun läßt sich $o(x)$ wie folgt definieren:

$$o(\wedge) = \wedge$$

und für $a_i \in V$

$$o(f_{a_i}(x_1, x_2)) = \begin{cases} f_{a_0}(o(x_1), o(x_1)), & \text{falls } o(x_2) \preceq o(x_1) \\ f_{a_0}(o(x_2), o(x_2)) & \text{sonst.} \end{cases}$$

Mit $o(x)$ ist auch die damit identische Funktion $h_{o(x)}$ primitiv-rekursiv.

Die in [1] eingeführten Grundoperationen $a_i \uparrow x$ und $y \rightarrow x$ können nun wie folgt primitiv-rekursiv definiert werden:

$$a_i \uparrow x = f_{a_i}(\wedge, x)$$

ferner

$$y \rightarrow \wedge = y$$

und für $a_i \in V$

$$y \rightarrow f_{a_i}(x_1, x_2) = f_{a_i}(y \rightarrow x_1, x_2).$$

Mit diesen lassen sich auch weitere, in [1] rekursiv definierte Funktionen durch unsere primitive Rekursion definieren. Solche sind: die „Wurzelpunktenfolge", die „Blätterfolge", und die charakteristische Funktion der Beziehung „eine Nachkommenschaft eines bestimmten Vorfahrs sein".

Die Wurzelpunktenfolge $\mathrm{wpf}(x)$ einer Allee x bedeutet die Konkatenation der Wurzelpunkte der Komponenten von x, in der Reihenfolge der Komponenten. Dafür ergibt sich die folgende Definition:

$$\mathrm{wpf}(\wedge) = \wedge$$

$$\text{und für } a_i \in V$$

$$\mathrm{wpf}(f_{a_i}(x_1, x_2)) = \mathrm{wpf}(x_1) \rightarrow a_i.$$

In [1] wird auch der Begriff des i-ten Wurzelpunktes verwendet. Dieser ist die i-te Komponente der durch die Wurzelpunktenfolge dargestellten Allee. Es kann allgemein eine primitiv-rekursive Funktion $\mathrm{komp}(x, i)$ derart definiert werden, daß sie für natürliche Zahlenwerte von i mit der i-ten Komponente von x übereinstimmt (an anderen Stellen ist ihr Wert belanglos).

Dazu definiere ich erst die Komponentenanzahl $\mathrm{kompz}(x)$ von x als primitiv-rekursive Funktion durch

$$\mathrm{kompz}(\wedge) = \wedge$$

$$\text{und für } a_i \in V$$

$$\mathrm{kompz}(f_{a_i}(x_1, x_2)) = o(\mathrm{kompz}(x_1)) + 1.$$

Erklärung zur rechten Seite der letzten Gleichung: der **Satz in I** ergibt auf die primitiv-rekursive zahlentheoretische Funktion $n_1 + n_2$ angewandt, daß es eine in $\hat{V}$ primitiv-rekursive Funktion $f(z_1, z_2)$ mit

$$o(f(z_1, z_2)) = o(z_1) + o(z_2)$$

gibt; daraus ergibt sich (da $o(1) = 1$ ist) die hier verwendete Funktion durch Substitution.

Damit erhält man die folgende Definition für $\mathrm{komp}(x, i)$:

$$\mathrm{komp}(\wedge, i) = \wedge$$

$$\text{und für } a_i \in V$$

$$\mathrm{komp}\,(f_{a_i}(x_1, x_2), i) = \begin{cases} \mathrm{komp}(x_1, i), & \text{falls } o(i) < o\,(\mathrm{kompz}(x_1)) \\ a_i \uparrow x_2 & \text{sonst.} \end{cases}$$

Erklärung zur rechten Seite der letzten Gleichung: die zahlentheoretische Beziehung $n_1 < n_2$ ist mit dem Verschwinden der zahlentheoretischen Funktion

$$(n_1 + 1) \dot{-} n_2$$

gleichbedeutend (wo mit $\dot{-}$ die sog. „arithmetische Differenz" bezeichnet wird). Wird der **Satz in I** auf die primitiv-rekursive zahlentheoretische Funktion

$$(n_1 + n_3) \dot{-} n_2$$

angewandt, so erhält man eine in $\hat{V}$ primitiv-rekursive Funktion $f(z_1, z_2, z_3)$ mit

$$o(f(z_1, z_2, z_3)) = (o(z_1) + o(z_3)) \dot{-} o(z_2).$$

Die Beziehung

$$o(i) < o\,(\text{kompz}(x))$$

kann dann mit

$$f(i, \text{kompz}(x), 1) = \Lambda$$

vertreten werden. (Im folgenden werde ich den **Satz in I** ohne weitere Erklärungen anwenden.)

Für natürliche Zahlen als i ist nun der i-te Wurzelpunkt von x:

$$\text{komp}(\text{wpf}\,(x), i).$$

Mit der Vorstellung, daß die Endpunkte der Bäume unserer Alleen „Blätter" sind (in einem allein aus seinem Wurzelpunkt bestehenden Baum diesen Wurzelpunkt als Endpunkt betrachtend), bedeutet die Blätterfolge bf(x) die Konkatenation der „Farben" dieser Blätter von x, und zwar nach der angegebenen Ordnung der Punkte fortschreitend in Betracht gezogen, Dafür ergibt sich die Definition:

$$\text{bf}(\Lambda) = \Lambda$$

und für $a_i \in V$

$$\text{bf}(f_{a_i}(x_1, x_2)) = \begin{cases} \text{bf}(x_1) \to a_i, & \text{falls} \quad x_2 = \Lambda \\ \text{bf}(x_1) \to \text{bf}(x_2) & \text{sonst.} \end{cases}$$

Das i-te Blatt von x ist dann $\text{komp}(\text{bf}\,(x), i)$.

Unter der Nachkommenschaft eines Punktes P in einer Allee x wird die Konkatenation der „Farben" jener Punkte verstanden, nach welchen aus P Kanten führen, in der Reihenfolge dieser Kanten in Betracht gezogen. Ist a_j die Farbe von P, so wird die Nachkommenschaft von P „eine Nachkommenschaft des Vorfahrs a_j" genannt. Die charakteristische Funktion $\text{nk}_{a_j}(x, y)$ der Beziehung: „y ist eine Nachkommenschaft des Vorfahrs a_j in der Allee x", wobei a_j ein fixiertes Element von V ist, läßt sich wie folgt definieren:

$$\text{nk}_{a_j}(\Lambda, y) = a_0$$

und für $a_i \in V$

$$\text{nk}_{a_j}(f_{a_i}(x_1, x_2), y) = \begin{cases} \Lambda, & \text{falls} \quad \text{nk}_{a_j}(x_1, y) = \Lambda \vee \text{nk}_{a_j}(x_2, y) = \Lambda \vee \\ & \qquad \vee (a_i = a_j \;\&\; y = \text{wpf}(x_2)) \\ a_0 & \text{sonst.} \end{cases}$$

In [1] wird auch der Begriff des Reversierten einer Allee x verwendet, worunter jene Allee zu verstehen ist, welche durch Umkehrung der Reihefolge der Komponenten von x entsteht. Mit der Bezeichnung rev(x) kann dies durch folgende primitive Rekursion definiert werden:

$$\text{rev}(\Lambda) = \Lambda$$

und für $a_i \in V$

$$\text{rev}(f_{a_i}(x_1, x_2)) = (a_i \uparrow x_2) \to \text{rev}(x_1).$$

Nach diesen Mustern lassen sich gewiß auch ferner die in der Untersuchung der freien Binoïden einzuführenden Begriffe definieren.

Es kann aber dabei auch die Verwendung von gewissen anderen Rekursionsarten notwendig werden. Ich bin noch schuldig mit der primitiv-rekursiven Definition der in **I** eingeführten Hilfsfunktionen

$$p(x, y),\quad s(x),\quad i(x, j),\quad l(x, j),\quad v(x, i),$$

welche die Auflösung anderer Rekursionsarten auf primitive Rekursionen und Substitutionen ermöglichen.

In **I** wurde mit (F) bezeichnet eine Zuordnung gewisser Elemente von $\hat{V}$ zu den endlichen Elementenfolgen von $\hat{V}$ vorgeschlagen, wobei die der Folge $u_0, u_1, \ldots, u_n$ zugeordnete Zahl mit $c_n(u_0, u_1, \ldots, u_n)$ bezeichnet wurde. Die Rolle von $p(x, y)$ ist die Art der Entstehung von $c_{n+1}(u_0, \ldots, u_n, u_{n+1})$ aus $c_n(u_0, \ldots, u_n)$ und u_{n+1} nachzubilden. Aus (F) sieht man, daß

$$\text{(C)} \qquad c_{n+1}(u_0, \ldots, u_n, u_{n+1}) = f_{a_0}(c_n(u_0, \ldots, u_n), f_{a_0}(u_{n+1}, h_{n+1}))$$

besteht. Nach [2a] ist die Funktion $\operatorname{long}(x)$, welche für jedes x, das einer endlichen Folge zugeordnet ist, den größten der darin vorkommenden Gliederindizes, und sonst $\wedge$ als Wert annimmt (also immer eine natürliche Zahl ist, so daß immer

$$\operatorname{long}(x) = o(\operatorname{long}(x))$$

besteht), primitiv-rekursiv in unserem Fall. Wird nun mit Verwendung des **Satzes in I** $p(x, y)$ durch Substitutionen primitiv-rekursiver Funktionen wie folgt definiert:

$$p(x, y) = f_{a_0}(x, f_{a_0}(y, h_{\operatorname{long}(x)+1}))$$

und für $c(x)$ die Funktion $\operatorname{id}(x) = x$ gewählt, so besitzen diese primitiv-rekursiven Funktionen die in [2a] geforderte Eigenschaft, daß wegen

$$p(c_n(u_0, \ldots, u_n), c(u_{n+1})) = f_{a_0}(c_n(u_0, \ldots, u_n), f_{a_0}(u_{n+1}, h_{n+1}))$$

infolge (C) immer

$$c_{n+1}(u_0, \ldots, u_n, u_{n+1}) = p(c_n(u_0, \ldots, u_n), c(u_{n+1}))$$

besteht.

Die weiteren zu untersuchenden Hilfsfunktionen beziehen sich auf eine die Vorgänger eines $x \in \hat{V}$ (mit Wiederholungen) enthaltende Folge

$$\wedge = \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_s = x.$$

welche für $x = \wedge$ allein aus $\bar{x}_0 = \wedge$ besteht, und für beliebiges $x = f_{a_i}(x_1, x_2)$ (mit $a_i \in V$) erst die Vorgänger von x_1 und nachher, bis auf $\bar{x}_s = x$, die Vorgänger von x_2 in ihrer Reihenfolge enthält. So ergibt sich hier $s = s(x)$ mit Verwendung des **Satzes in I** durch die primitive Rekursion:

$$s(\wedge) = 1$$

und für $a_i \in V$

$$s(f_{a_i}(x_1, x_2)) = s(x_1) + s(x_2) + 2$$

(da alle Werte von s natürliche Zahlen sind, stimmt $s(z)$ für jedes z mit $o(s(z))$ überein; das Analoge gilt auch für die folgenden Definitionen). Dabei sieht man, daß der Wert von $s(x)$ für jedes x nur von den unmittelbaren Vorgängern von x abhängig ist (d. h. für Stellen mit denselben unmittelbaren Vorgängern stimmen die Werte von s überein).

Die Funktion $i(x, j)$ gibt für $x \neq \wedge, j < s(x)$ einen der Indizes 1, 2 an, für welchen $\bar{x}_j$ ein Vorgänger des damit versehenen unmittelbaren Vorgängers von x ist; falls $x = \wedge$ oder $j \geqq s(x)$ oder j keine natürliche Zahl ist, ist der Wert von $i(x, j)$ belanglos. Dafür bietet sich die folgende primitiv-rekursive Definition:

$$i(\wedge, j) = \wedge$$

und für $a_r \in V$

$$i(f_{a_r}(x_1, x_2), j) = \begin{cases} 1, & \text{falls} \quad o(j) \leqq s(x_1) \\ 2 & \text{sonst.} \end{cases}$$

Man sieht, daß der Wert von $i(x, j)$ (der immer eine natürliche Zahl ist) an jeder Stelle nur von j und von den unmittelbaren Vorgängern von x (sogar nur vom ersten) abhängig ist.

Die Funktion $l(x, j)$ gibt für $x \neq \wedge, j < s(x)$ einen Index an, mit welchem $\bar{x}_j$ in der festgesetzten Reihenfolge der Vorgänger von $x_{i(x,j)}$ auftritt; auch die Werte dieser Funktion sind belanglos für $x = \wedge$ oder $j \geqq s(x)$ oder für j-Werte, die keine natürlichen Zahlen sind. Es bietet sich dafür die primitiv-rekursive Definition:

$$l(\wedge, j) = \wedge$$

und für $a_r \in V$

$$l(f_{a_r}(x_1, x_2), j) = \begin{cases} o(j), & \text{falls} \quad i(f_{a_r}(x_1, x_2), j) = 1 \\ o(j) \dot{-} (s(x_1) + 1) & \text{sonst.} \end{cases}$$

Auch der Wert von $l(x, j)$ (der ebenfalls immer eine natürliche Zahl ist) hängt an jeder Stelle nur von j und vom ersten unmittelbaren Vorgänger von x ab.

Endlich gibt die Funktion $v(x, i)$ einen Index an, mit welchem $k_i(x)$, das immer ein Vorgänger von x ist, in der festgesetzten Reihenfolge der Vorgänger von x auftritt. Hier erinnere ich daran, daß die in [2a] als primitiv-rekursiv definierte zweistellige Funktion $k_i(x)$ für solche x, die keiner Elementenfolge zugeordnet sind, den Wert x annimmt, ferner für jedes solche x, das einer Elementenfolge zugeordnet ist, für $o(i) > \text{long}(x)$ den Wert $\wedge$, für $o(i) \leqq \text{long}(x)$ aber das $o(i)$-te Glied der betreffenden Elementenfolge als Wert annimmt. Die charakteristische Funktion $\text{ef}(x)$ der Beziehung: „x ist einer Elementenfolge zugeordnet" (in [2a] durch $\text{ch}(x)$ bezeichnet) hat sich in [2a] auch als primitiv-rekursiv erwiesen.

Ist x einer aus dem einzigen Glied u_0 bestehenden Folge zugeordnet, so ist

$$x = c_0(u_0) = f_{a_0}(u_0, h_0)$$

und $k_i(x)$ verschwindet nur für $i = 0$ nicht. Für diesen Fall kommt $k_0(x) = u_0$, als der erste unmittelbare Vorgänger von x, mit dem Index $s(u_0)$ in der zu x gehörigen Vorgängerfolge vor.

Ist x einer Elementenfolge $u_0, \ldots, u_n, u_{n+1}$ zugeordnet, so gilt nach (C)

$$x = c_{n+1}(u_0, \ldots, u_n, u_{n+1}) = f_{a_0}(c_n(u_0, \ldots, u_n), f_{a_0}(u_{n+1}, h_{n+1})),$$

wonach x als ersten bzw. zweiten unmittelbaren Vorgänger

$$x_1 = c_n(u_0, \ldots, u_n) \quad \text{bzw.} \quad x_2 = f_{a_0}(u_{n+1}, h_{n+1})$$

enthält. So ist $k_i(x) = k_i(x_1)$ für $i = 0, 1, 2, \ldots, n$ ein ebensovielter Vorgänger von x als von x_1; und $k_{n+1}(x) = u_{n+1}$ ist der erste unmittelbare Vorgänger von x_2:

$$k_{n+1}(x) = \mathrm{uv}_1(x_2),$$

so stimmt dies mit dem

$$s(x_1) + s(\mathrm{uv}_1(x_2)) + 1\text{-}$$

ten Glied der zu x gehörigen Vorgängerfolge überein. So bietet sich als primitiv-rekursive Definition für $v(x, i)$:

$$v(\wedge, i) = 0$$

und für $a_j \in V$

$$v(f_{a_j}(x_1, x_2), i) = \begin{cases} f_{a_j}(x_1, x_2), \text{ falls } \mathrm{ef}(f_{a_j}(x_1, x_2)) = a_0 \\ s(x_1), \text{ falls } \mathrm{ef}(f_{a_j}(x_1, x_2)) = \wedge \ \& \\ \qquad \& \ \mathrm{long}(f_{a_j}(x_1, x_2)) = o(i) = 0 \\ v(x_1, i), \text{ falls } \mathrm{ef}(f_{a_j}(x_1, x_2)) \wedge \ \& \\ \qquad \& \ o(i) < \mathrm{long}(f_{a_j}(x_1, x_2)) \\ s(x_1) + s(\mathrm{uv}_1(x_2)) + 1, \text{ falls } \mathrm{ef}(f_{a_j}(x_1, x_2) = \wedge \ \& \\ \qquad \& \ o(i) > 0 \ \& \ o(i) = \mathrm{long}(f_{a_j}(x_1, x_2)) \\ \wedge \text{ sonst.} \end{cases}$$

Damit sind nun alle Hilfsmittel beisammen, mit welchen die in [2a] entwickelte Theorie auf den Fall der freien Binoïden angewandt werden kann, ohne mehr als die charakteristische Funktion der Beziehung $x = y$ zu den Ausgangsfunktionen $\wedge$ und $f_{a_i}(x_1, x_2)$ für $a_i \in V$ hinzuzunehmen.

Prüft man nach, wie viel von den speziellen Eigenschaften von $\hat{V}$ dabei wesentlich benutzt wurde, so erkennt man leicht, daß das für $\hat{V}$ Ausgesagte für jede angeordnete freie holomorphe Menge H gilt, zu welchem ein aus solchen Funktionen f_{a_i} bestehendes F gehört, wobei a_i eine Indexmenge I durchläuft, deren Elemente als primitiv-rekursive Konstanten von H betrachtet werden können; ferner wenigstens eine zu F gehörige Funktion mindestens 2-stellig, aber für die Variablenanzahl der zu F gehörigen Funktionen eine obere Schranke t vorhanden ist; endlich die Elemente von H_0 keine echten Vorgänger besitzen, und ein Element $f_{a_i}(x_1, \ldots, x_r)$ von H als unmittelbare Vorgänger seine unmittelbaren Konstituenten $x_1, \ldots, x_r$ und als echte Vorgänger $x_1, \ldots, x_r$ und deren echte Vorgänger enthält. Jede Funktion $f_{a_i}(x_1, \ldots, x_r)$ kann dabei als eine t-stellige Funktion $f_{a_i}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_t)$ betrachtet werden, wenn z. B.

$$f_{a_i}(x_1, \ldots, x_r) = f_{a_i}(x_1, \ldots, x_r, x_r, \ldots, x_r)$$

gesetzt wird. Aus einem mindestens 2-stelligen Element f_{a_0} von F erhält man durch eventuelle weitere Einsetzungen die zur Theorie festzusetzende 2-stellige Funktion

$$f_{a_0}(x_1, x_2) = f_{a_0}(x_1, x_2, x_2, \ldots, x_2)$$

wie dies in [2a] eingeführt und angewendet wurde. Die Rolle von $\wedge$ übernimmt ein festgewähltes Element h_0 von H_0. An Stelle des Schemas (S) tritt das folgende allgemeinere Schema der primitiven Rekursion:

$$f(h, u_1, \ldots, u_r) = g_h(u_1, \ldots, u_r) \quad \text{für} \quad h \in H_0$$

$$\text{und für} \quad a_i \in I$$

$$f(f_{a_i}(x_1, \ldots, x_t), u_1, \ldots, u_r) =$$

$$= g'_{a_i}(x_1, \ldots, x_t, u_1, \ldots, u_r, f(x_1, u_1, \ldots, u_r), \ldots, f(x_t, u_1, \ldots, u_r)),$$

woraus durch zu den Anwendungen passende Einschränkungen konstruktive Definitionen geschöpft werden können.

Insbesondere folgt nach den Ergebnissen von [2a] aus den bisherigen Betrachtungen bezüglich $\hat{V}$ und aus leicht durchführbaren analogen Betrachtungen bezüglich der eben beschriebenen Verallgemeinerungen davon, daß in diesen Mengen verschiedenartige Rekursionen auf primitive Rekursionen und Substitutionen aufgelöst werden können: die „Wertverlaufsrekursion", wobei zur Definition eines Funktionswertes Funktionswerte an echten, aber nicht notwendig unmittelbaren Vorgängern verwendet werden; die simultan-rekursive Definition mehrerer Funktionen; Rekursionen, wobei für die Parameter Einsetzungen erfolgen, sogar Einsetzungen von Werten der zu definierenden Funktion (an Vorgänger-Stellen). Die Spezialisierung der in [2a] angegebenen Definition für in H allgemein- und partiell-rekursiven Funktionen und der Herstellung ihrer expliziten Form geht ohne weiteres, da diesbezüglich keine Alternativen für die Spezialfälle übrigblieben.

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ON A LAW OF ITERATED LOGARITHM FOR STRONGLY MULTIPLICATIVE SYSTEMS

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Introduction

1. Preliminaries. Let $\{X_n\}$ be a sequence of uniformly bounded random variables on some probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ with $\mathbf{E}(X_n)=0$ and $\mathbf{E}(X_n^2)=1$. We quote a definition of ALEXITS [1] as it is given by in RÉVÉSZ [8].

DEFINITION 1. A uniformly bounded sequence $\{X_n\}$ of random variables is called an equinormed strongly multiplicative system (ESMS) if

$$\mathbf{E}(X_i) = 0, \quad \mathbf{E}(X_i^2) = 1, \qquad i = 1, 2, \dots, \tag{1.1}$$
$$\mathbf{E}(X_{i_1}^{r_1} X_{i_2}^{r_2} \dots X_{i_k}^{r_k}) = \mathbf{E}(X_{i_1}^{r_1})\mathbf{E}(X_{i_2}^{r_2}) \dots \mathbf{E}(X_{i_k}^{r_k}),$$

where $i_1 < i_2 < \dots < i_k$, $k=1, 2, \dots$, and $r_1, r_2, \dots, r_k$ can be equal to 1 or 2.

Obviously, a sequence $\{X_n\}$ of independent uniformly bounded random variables with zero mean and variance 1 is an ESMS. Another example (ALEXITS [1], RÉVÉSZ [8]) is the sequence $\{\sqrt{2} \sin n_k x\}$ on the interval $[0, 2\pi]$ if $n_{k+1}/n_{k \geqq 3}$ and if the probability measure $\mathbf{P}$ on the Borel measurable subsets of the interval $[0, 2\pi]$ is defined by $\mathbf{P}(A) = \lambda(A)/2\pi$, where λ is the common Lebesgue measure. Also, an absolutely fair (for this terminology see e.g. FELLER's book [4] or (3. 2) of this paper) bounded sequence of random variables $\{X_n\}$ satisfying also the conditions

$$\mathbf{E}(X_1^2) = 1, \quad \mathbf{E}(X_n^2 | X_1, \dots, X_{n-1}) = 1, \quad n \geqq 2,$$

with probability 1, is an ESMS.

ALEXITS [1] proved that an ESMS has the property of independent random variables that $\sum_1^\infty c_i X_i$ is convergent if and only if $\sum_1^\infty c_i^2 < \infty$. RÉVÉSZ [8] proved that the central limit theorem and a law of iterated logarithm also hold for an ESMS, and we quote these results here, writing $S_n = X_1 + \dots + X_n$.

THEOREM A. *If* $\{X_n\}$ *is an* ESMS *then*

$$\lim_{n \to \infty} \mathbf{P}\left(\frac{S_n}{\sqrt{n}}\right) = \Phi(x),$$

where Φ *is the unit normal distribution function.*

THEOREM B. *If* $\{X_n\}$ *is an* ESMS *then*

$$\mathbf{P}\left(\limsup_{n \to \infty} \frac{S_n}{(n \log \log n)^{\frac{1}{2}}} \leqq 6\right) = 1.$$

We will show in section 2 of this paper (Theorem 1) that, under some further restrictions on the random variables involved, Theorem B (Theorem 2 of [8]) can be improved. In section 3 a generalization of Theorem 1 is proved (Theorem 2).

2. A law of iterated logarithm. We will first need some further definitions.

DEFINITION 2. We say that a finite number $\mu(X)$ is a median of X, if

$$\mathbf{P}\{X \geqq \mu(X)\} \geqq \frac{1}{2} \leqq \mathbf{P}\{X \leqq \mu(X)\}.$$

DEFINITION 3. A random variable $\mu(X|\mathfrak{D})$ is said to be a conditional median of given $\mathfrak{D}$, where $\mathfrak{D}$ is a sub σ-field of events of the σ-field of events $\mathfrak{A}$ of $(\Omega, \mathfrak{A}, \mathbf{P})$, if

$$\mathbf{P}^{\mathfrak{D}}\{X-\mu(X|\mathfrak{D}) \geqq 0\} \geqq \frac{1}{2} \leqq \mathbf{P}^{\mathfrak{D}}\{X-\mu(X|\mathfrak{D}) \leqq 0\}$$

with probability 1 (here the notation of Loève's book [6] is used).

First we remark that for the median $\mu(S_k - S_n)$ of $(S_k - S_n)$, where S_n is defined in terms of an ESMS, we have

$$|\mu(S_k - S_n)| \leqq (2\mathbf{E}\{(S_n - S_k)^2\})^{\frac{1}{2}} \leqq (2\mathbf{E}(S_n^2))^{\frac{1}{2}} \leqq 2n^{\frac{1}{2}}.$$

The proof of this statement is identical with that for mutually independent random variables which can be found in LOÈVE's book [6], page 244. Throughout this section we assume that the conditional median of $S_k - S_n$ given the sub σ-field of events generated by $S_1, S_2, \ldots, S_k$, where these partial sums are defined in terms of an ESMS, has the same bound as the unconditional median of $S_k - S_n$, that is

$$|\mu(S_k - S_n|S_1, \ldots, S_k)| \leqq 2n^{\frac{1}{2}}. \tag{2.1}$$

The following law of iterated logarithm is going to be proved here.

THEOREM 1. *Let* $\{X_n\}$ *be an* ESMS *with* $|X_n| \leqq K$ *for all* n. *If the assumption of* (2. 1) *holds then*

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{|S_n|}{(2n \log\log n)^{\frac{1}{2}}} \leqq 1\right) = 1. \tag{2.2}$$

It is likely that the constant 1 of Theorem 1 is the best possible but I could not prove this.

The proof of Theorem 1 is based on the following lemmas.

LEMMA 1. *Let* X *be an arbitrary random variable with* $\mathbf{E}(X) = M$ *and* $\mathbf{E}((X-M)^2) = \sigma^2$. *Define* $\mathbf{E}(e^{\varepsilon(X-M)}) = \mathfrak{M}(\varepsilon)$, *where* $\varepsilon > 0$. *Then, with* $t > 0$,

$$\mathbf{P}(X \geqq M + (t + \log \mathfrak{M}(\varepsilon))/\varepsilon) \leqq e^{-t}. \tag{2.3}$$

This lemma follows immediately from Markov's inequality (see e.g. page 313 of RÉNYI's book [7]).

LEMMA 2. *Let* $\{X_n\}$ *be an* ESMS *with* $|X_n| \leqq K$ *for all* n. *Define* $\mathfrak{M}(\varepsilon) = \mathbf{E}(e^{\varepsilon S_n})$, $\varepsilon > 0$. *Then*

$$\log \mathfrak{M}(\varepsilon) \leqq (n\varepsilon^2/2)(1 + \varepsilon K e^{\varepsilon K}/3). \tag{2. 4}$$

This lemma is proved on page 322 and 323 of RÉNYI's book [7] for mutually independent random variables. Its proof for an ESMS is similar. To indicate this we consider

$$e^{\varepsilon X_k} = \sum_{j=0}^{\infty} \frac{\varepsilon^j X_k^j}{j!} \leqq 1 + \varepsilon X_k + \frac{\varepsilon^2 X_k^2}{2} + \sum_{j=3}^{\infty} \frac{\varepsilon^j K^{j-2} X_k^2}{j!} \leqq$$

$$\leqq 1 + \varepsilon X_k + \varepsilon^2 X_k^2 \left(\frac{1}{2} + \frac{\varepsilon K e^{\varepsilon K}}{6} \right).$$

Therefore,

$$\mathbf{E}\left(e^{\varepsilon \sum_{k=1}^{n} X_k}\right) \leqq \mathbf{E}\left\{ \prod_{k=1}^{n} \left(1 + \varepsilon X_k + \varepsilon^2 X_k^2 \left(\frac{1}{2} + \frac{\varepsilon K e^{\varepsilon K}}{6} \right) \right) \right\} = \prod_{k=1}^{n} \left(1 + \varepsilon^2 \left(\frac{1}{2} + \frac{\varepsilon K e^{\varepsilon K}}{6} \right) \right),$$

where the last inequality follows from the assumptions of (1. 1). Since $1 + x < e^x$, if $x > 0$, we have

$$\mathbf{E}\left(e^{\varepsilon \sum_{k=1}^{n} X_k}\right) \leqq e^{n \frac{\varepsilon^2}{2} \left(1 + \frac{\varepsilon K e^{\varepsilon K}}{3} \right)}.$$

Taking the logarithm of both sides of the last inequality, the statement of Lemma 2 follows.

LEMMA 3. (A generalization of Bernstein's inequality for an ESMS). *Let* $\{X_n\}$ *be as in Lemma* 2. *If* $0 < \lambda < \sqrt{n}/K$, *then*

$$\mathbf{P}(|S_n| \geqq \lambda \sqrt{n}) \leqq 2 \exp\left(-\lambda^2/2(1 + \lambda K/2\sqrt{n})^2\right). \tag{2. 5}$$

Lemma 3 follows from Lemmas 1 and 2 and its proof for mutually independent random variables can be found on pages 323 and 324 of RÉNYI's book [7]. Its proof for an ESMS is exactly the same (we only have to use Lemma 2 of this section instead of the lemma of page 322 of [7] at the appropriate place).

Now we formulate the extended Lévy inequality (see e.g. page 385, [6]) as it is needed in this paper.

LEMMA 4. *If* $X_1, X_2, \ldots X_n$ *are arbitrary random variables and* $S_k = X_1 + \cdots + X_k$, *then for every* x

$$\mathbf{P}\left(\max_{1 \leqq k \leqq n} (S_k - \mu(S_k - S_n | S_1, \ldots, S_k)) \geqq x\right) \leqq 2\mathbf{P}(S_n \geqq x). \tag{2. 6}$$

Lemma 4 and the assumption of (2. 1) imply the following statement.

LEMMA 5. *Let* $\{X_n\}$ *be an* ESMS. *If the assumption of* (2. 1) *holds, then for every* x

$$\mathbf{P}\left(\max_{1 \leqq k \leqq n} S_k \geqq x\right) \leqq 2\mathbf{P}(S_n \geqq x - 2n^{\frac{1}{2}}). \tag{2. 7}$$

This inequality follows from the fact that Lemma 4 remains valid if $\mu(S_k - S_n | S_1, \ldots, S_k)$ is replaced by $-2n^{\frac{1}{2}}$ on assuming the condition (2. 1). Changing now x into $x - 2n^{\frac{1}{2}}$ in this modified (2. 6), we get the inequality of (2. 7).

Using now the Borel—Cantelli Lemma, the inequality of (2. 7) and Lemma 3, the proof of Theorem 1 can be carried out exactly the same way as that of Theorem 1 of page 337 of RÉNYI's book [7].

3. **A generalization of Theorem 1.** In this section the boundedness assumption of an ESMS is going to be dropped and we are going to work with random variables which satisfy the conditions of the following definition.

DEFINITION 4. A sequence $\{X_n\}$ of random variables will be called a normed strongly multiplicative system (NSMS) if

$$\mathbf{E}(X_i) = 0, \quad \mathbf{E}(X_i^2) = \sigma_i^2, \qquad i = 1, 2, \ldots,$$
$$(3.\ 1) \qquad \mathbf{E}(X_{i_1}^{r_1} X_{i_2}^{r_2} \ldots X_{i_k}^{r_k}) = \mathbf{E}(X_{i_1}^{r_1})\mathbf{E}(X_{i_2}^{r_2}) \ldots \mathbf{E}(X_{i_k}^{r_k}),$$

where $i_1 < i_2 < \cdots < i_k$, $k = 1, 2, \ldots$, and $r_1, r_2, \ldots, r_k$ can be equal to 1 or 2.

The difference between this definition and Definition 1 is that here we do not require the sequence $\{X_n\}$ to be uniformly bounded and equinormed. As an example we again mention that a sequence $\{X_n\}$ of independent random variables with zero mean and variance σ_n^2, $n = 1, 2, \ldots$, is a NSMS. Also, a sequence $\{X_n\}$ of absolutely fair random variables, that is to say for which we have

$$(3.\ 2) \qquad \mathbf{E}(X_1) = 0, \quad \mathbf{E}(X_n | X_1, \ldots, X_{n-1}) = 0, \qquad n \geqq 2,$$

with probability 1, is a NSMS if it also satisfies the conditions

$$(3.\ 3) \qquad \mathbf{E}(X_1^2) = \sigma_1^2, \ \mathbf{E}(X_n^2 | X_1, \ldots, X_{n-1}) = \sigma_n^2, \qquad n \geqq 2.$$

with probability 1, where $\sigma_1, \sigma_2, \ldots$ are non-negative constants. We will again write $S_n = X_1 + \cdots + X_n$ and introduce the notation $\hat{\sigma}_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$ for use in this section.

The $\{S_n\}$ process of an absolutely fair sequence $\{X_n\}$ is a martingale with probability 1. Conversely, if $\{S_n\}$ is a martingale process then $\{X_n\}$ is absolutely fair. The condition (3. 2) lies between zero correlation and independence of the X_n's. For details on these statements we refer the reader to DOOB's book [3], pp. 91—92.

Condition (3. 3) is also less restrictive than the customary assumption of mutual independence of random variables. Independent random variables with zero expectations and finite variances satisfy (3. 3). Also, (3. 2) and (3. 3) together imply that the process $\{S_j^2 - \hat{\sigma}_j^2, \mathfrak{F}_j, j \leqq n\}$ is a martingale, where $\mathfrak{F}_j$ is the Borel field of ω sets, $(\omega \in \Omega)$, determined by conditions on $X_1, \ldots, X_j$. Conversely, if (3. 2) is true and if $\{S_j^2 - \hat{\sigma}_j^2, \mathfrak{F}_j, j \leqq n\}$ is a martingale process, then (3. 3) is true. Again for details we refer to DOOB's book [3], pp. 382—384.

Condition (3. 3) was probably first used by P. LÉVY to show that the central limit theorem is applicable to martingales much as it is to sums of mutually independent random variables ([3], p. 384). In [2] it was shown that the random-sum central limit theorem is also applicable to martingales under the assumptions of (3. 3). LÉVY has also shown (Theorem 70, p. 260 of [5]) that the law of iterated logarithm remains true for sequences of random variables satisfying (3. 2), (3. 3) and some further conditions.

The conditions of (3. 1) are weaker than those of (3. 2) and (3. 3) and it would be interesting to know what properties of martingale partial sums of random variables remain valid for a NSMS. In this section we prove the following law of iterated logarithm for a NSMS.

THEOREM 2. *Let* $\{X_n\}$ *be a* NSMS. *Assume also that the conditional median* $\mu(S_k - S_n | S_1, \ldots, S_k)$ *of the random variable* $S_k - S_n$, *which is defined in terms of a* NSMS, *satisfies the condition*

$$(3.4) \qquad |\mu(S_k - S_n | S_1, \ldots, S_k)| \leqq (2\hat{\sigma}_n^2)^{\frac{1}{2}}.$$

If $\hat{\sigma}_n^2 \to +\infty$ *and* $|X_n|/\hat{\sigma}_n = o([\log\log]^{-\frac{1}{2}} \hat{\sigma}_n^2)$, $t_n = (2 \log\log \hat{\sigma}_n^2),^{\frac{1}{2}}$ *then*

$$(3.5) \qquad \mathbf{P}\left(\limsup_{n\to\infty} \frac{|S_n|}{\hat{\sigma}_n t_n} \leqq 1\right) = 1.$$

About the condition (3. 4) we remark that it is true for the median $\mu(S_k - S_n)$ of $S_k - S_n$ defined in terms of a NSMS and we assume here that the conditional median of $S_k - S_n$ given the sub σ-field of events generated by $S_1, S_2, \ldots, S_k$ has the same bound.

If we replace the NSMS of Theorem 2 by an absolutely fair sequence of random variables which also satisfy the conditions of (3. 3) then the statement of Theorem 2 remains true and it becomes a special case of Theorem 70, page 260 of LÉVY's book [5].

The proof of Theorem 2 is based on the following lemma.

LEMMA 6. *Let* $\{X_n\}$ *be a* NSMS. *Let also* $c = \max_{1 \leqq k \leqq n} |X_k|/\hat{\sigma}_n$ *and* $\varepsilon > 0$. *We have*

(3. 6) *If* $\varepsilon c \leqq 1$, *then* $\mathbf{P}(S_n > \varepsilon\hat{\sigma}_n) < \exp\left[-\frac{\varepsilon^2}{2}\left(1 - \frac{\varepsilon c}{2}\right)\right]$ *and, if* $\varepsilon c \geqq 1$, *then*

$$\mathbf{P}(S_n > \varepsilon\hat{\sigma}_n) < \exp\left(-\frac{\varepsilon}{4c}\right).$$

(3. 7) *Given* $\gamma > 0$, *if* $c = c(\gamma)$ *is sufficiently small and* $\varepsilon = \varepsilon(\gamma)$ *is sufficiently large, then* $\mathbf{P}\{S_n > \varepsilon\hat{\sigma}_n\} > \exp\left[-\frac{\varepsilon^2}{2}(1+\gamma)\right]$.

These exponential bounds are verified for mutually independent random variables on pages 255—257 of LOÈVE's book [6]. Its proof for a NSMS is similar. To see this we let $t > 0$, $|X_k| \leqq c < \infty$. Then

$$e^{tX_k} = \sum_{j=0}^{\infty} \frac{t^j X_k^j}{j!} < 1 + tX_k + \frac{t^2 X_k^2}{2} + \sum_{j=3}^{\infty} \frac{t^j c^{j-2} X_k^2}{j!}.$$

Since $e^{t(1-t)} < 1 + t < e^t$, it follows that, for $tc \leqq 1$,

$$e^{tX_k} < 1 + tX_k + \frac{t^2 X_k^2}{2}\left(1 + \frac{tc}{3} + \frac{t^2 c^2}{3 \cdot 4} + \ldots\right) < 1 + tX_k + \frac{t^2 X_k^2}{2}\left(1 + \frac{tc}{2}\right)$$

and so

$$\mathbf{E}(e^{tX_k}) < 1+\frac{t^2\sigma_k^2}{2}\left(1+\frac{tc}{2}\right) < \exp\left[\frac{t^2\sigma_k^2}{2}\left(1+\frac{tc}{2}\right)\right],$$

and

$$e^{tX_k} > 1+tX_k+\frac{t^2X_k^2}{2}\left(1-\frac{tc}{3}-\frac{t^2c^2}{3\cdot 4}-\cdots\right) > 1+tX_k+\frac{t^2X_k^2}{2}\left(1-\frac{tc}{2}\right)$$

and so

$$\mathbf{E}(e^{tX_k}) > 1+\frac{t^2\sigma_k^2}{2}\left(1-\frac{tc}{2}\right) > \exp\left[\frac{t^2\sigma_k^2}{2}\left(1-\frac{tc}{2}\right)\right].$$

Thus we obtain

$$\prod_{k=1}^{n}\left\{1+\frac{tX_k}{\hat{\sigma}_n}+\frac{t^2X_k^2}{2\hat{\sigma}_n^2}\left(1-\frac{tc}{2}\right)\right\} < e^{\frac{tS_n}{\hat{\sigma}_n}} < \prod_{k=1}^{n}\left\{1+\frac{tX_k}{\hat{\sigma}_n}+\frac{t^2X_k^2}{2\hat{\sigma}_n^2}\left(1+\frac{tc}{2}\right)\right\}$$

and, taking expected values on both sides, it follows from the NSMS property (3. 1) of $\{X_n\}$ that we have

$$\exp\left[\frac{t^2}{2}(1-tc)\right] < \mathbf{E}\left(e^{\frac{tS_n}{\hat{\sigma}_n}}\right) < \exp\left[\frac{t^2}{2}\left(1+\frac{tc}{2}\right)\right], \quad tc \leqq 1. \tag{3. 8}$$

From Markov's inequality and (3. 8) it follows now

$$\mathbf{P}\left\{\frac{S_n}{\hat{\sigma}_n} > \varepsilon\right\} \leqq e^{-t\varepsilon}\mathbf{E}\left(e^{\frac{tS_n}{\hat{\sigma}_n}}\right) < \exp\left[-t\varepsilon+\frac{t^2}{2}\left(1+\frac{tc}{2}\right)\right], \tag{3. 9}$$

and replacing t by ε or $1/c$ according as $\varepsilon c \leqq 1$ or $\geqq 1$, inequalities (3. 6) follow. The proof of inequality (3. 7) is the same as that of inequality (ii) of page 254. [6].

For the proof of Theorem 2 we will only need the first inequality of (3. 6) and the following version of Lemma 5.

LEMMA 7. *Let* $\{X_n\}$ *be a* NSMS. *Assume* (3. 4). *Then, for every* x,

$$\mathbf{P}(\max_{1\leqq k\leqq n} S_k \geqq x) \leqq 2\mathbf{P}(S_n \geqq x-(2\hat{\sigma}_n^2)^{\frac{1}{2}}). \tag{3. 10}$$

The proof of Lemma 7 is identical to that of Lemma 5.

Using now the Borel—Cantelli Lemma, (3. 10) and the first upper exponential bound of (3. 6), the proof of Theorem 2 can be carried out exactly the same way as that of the first part of the law of iterated logarithm on page 260 of LOÈVE's book [6] under the heading Proof, 1° on pages 261 and 262.

Just as in case of Theorem 1 it is again likely that the constant 1 of Theorem 2 is the best possible but I could not prove this. For example, when one tries to prove that $S_n > (1-\delta)\hat{\sigma}_n t_n$ for infinitely many n with probability 1, Loève's Proof. 2° on page 262, [6], breaks down in our case at the point where the events A_k are defined, for these events A_k are not independent any more when they are defined in terms of a NSMS.

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RELATIONSHIP BETWEEN NOETHER LATTICES AND x-SYSTEMS

By

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Noether lattices and x-systems were introduced primarily as frameworks for abstract ideal theory. R. P. DILWORTH [2] and E. W. JOHNSON [3] extended some results of ideal theory in Noetherian rings to Noether lattices, while K. E. AUBERT [1] generalized some classical results of ideal theory to x-systems. The purpose of this paper is to give conditions under which the above two notions are equivalent.

A closure operator defined on the subsets ($A \to A_x$) of a commutative semigroup S (with an identity element 1) is called an *x-operation* on S if $AB_x \subseteq B_x \cap (AB)_x$, for all subsets A, B of S. The collection $X(S)$ of x-closed subsets of S is called an *x-system* in S [1]. If $aB_x = (aB)_x$ (hence $\{a\}_x = aS$), for every $a \in S$ and $B \subseteq S$, then $X(S)$ is called an *r-system* in S. An element M of a multiplicative lattice L (as defined in [2]) is said to be *principal* if $(B \wedge (C:M))M = BM \wedge C$ and $(B \vee CM):M = (B:M) \vee C$, for all $B, C \in L$ [2]. A modular multiplicative lattice L which satisfies the ascending chain condition is called a *Noether lattice* in case each element of L is a join of principal elements [2]. For terminology concerning x-systems and Noether lattices see [1] and [2], respectively.

Let S be a commutative semigroup with an identity element 1. An x-system $X(S)$ in S is said to be *additive* in case $nmS \subseteq A_x \cup_x mB_x$ implies that there exist $pS \subseteq B_x$ and $qS \subseteq nS \cup_x pS$ such that $qmS \subseteq A_x$, $nS \subseteq qS \cup_x pS$ and $pS \subseteq (nmS \cup_x qmS): mS$. The collection of ideals of a commutative ring (with identity) is an example of an additive x-system. A Noether lattice L is said to be *additive* in case there exists a multiplicatively closed family $\mathcal{J}$ of principal elements of L ($I \in \mathcal{J}$) such that every element of L is a finite join of elements belonging to $\mathcal{J}$, and if $M \leqq H \vee G$, where $H, G \in L$ and $M \in \mathcal{J}$, then there exist elements $P, N \in \mathcal{J}$ such that $N \leqq H$, $P \leqq G$, $M \leqq N \vee P$, $N \leqq M \vee P$, and $P \leqq N \vee M$. An additive Noether lattice is said to satisfy *condition* (M) in case $M \leqq NG$, where $G \in L$ and $M, N \in \mathcal{J}$, implies that there exists an element $P \in \mathcal{J}$ such that $P \leqq G$ and $M = NP$. The lattice of ideals of a Noetherian ring (with identity) is an example of an additive Noether lattice which satisfies condition (M).

The purpose of this paper is to prove the following two theorems:

THEOREM A. *Let S be a commutative semigroup with identity and let $X(S)$ be an additive r-system in S which satisfies the ascending chain condition. Then $X(S)$ is an additive Noether lattice which satisfies condition* (M).

THEOREM B. *Let L be an additive Noether lattice which satisfies condition* (M). *Then there exists a commutative semigroup $\mathcal{J}$ with an identity and there exists an additive r-system $X(\mathcal{J})$ in $\mathcal{J}$ (which satisfies the ascending chain condition) such that L and $X(\mathcal{J})$ are isomorphic as multiplicative Noether lattices.*

PROOF OF THEOREM A. Let S be a commutative semigroup with an identity and let $X(S)$ be an additive r-system which satisfies the ascending chain condition. AUBERT [1] proved that any x-system in S is a lattice, with respect to set intersection and x-union, in which the x-product is commutative, associative and distributive over finite x-unions. To satisfy the definition of a multiplicative lattice (as defined in [2]) it remains to show that the x-product is distributive over arbitrary x-unions. This distributive property will be established for arbitrary subsets of S. Let $A \subseteq S$ and let $\mathscr{K}$ be an arbitrary family of subsets of S and let $C = A \circ (\cup_x \{B | B \in \mathscr{K}\})$. Clearly C is an upper bound of the $A \circ B$, $B \in \mathscr{K}$. To show that C is the least upper bound of the $A \circ B$, $B \in \mathscr{K}$ let D be an element in $X(S)$ such that $A \circ B \subseteq D$, for every element B in $\mathscr{K}$. Hence $B \subseteq D:A$, for all $B \in \mathscr{K}$. Since $D:A$ belongs to $X(S)$ ([1]) we conclude that $\cup_x \{B | B \in \mathscr{K}\} \subseteq D:A$. Therefore $C \subseteq D$, and so $A \circ (\cup_x \{B | B \in \mathscr{K}\}) = \cup_x \{A \circ B | B \in \mathscr{K}\}$. Therefore $X(S)$ is a multiplicative lattice.

To prove that $X(S)$ is modular let $A_x, B_x, C_x \in X(S)$ such that $A_x \subseteq C_x$. It will now be shown that $C_x \cap (A_x \cup_x B_x) \subseteq A_x \cup_x (C_x \cap B_x)$, since the opposite inclusion is clear. If $nS \subseteq C_x \cap (A_x \cup_x B_x)$, then $nS \subseteq A_x \cup_x B_x$. Setting $m=1$ in the additive condition we obtain that there exist $qS \subseteq A_x$ and $pS \subseteq B_x$ such that $nS \subseteq qS \cup_x pS$ and $pS \subseteq nS \cup_x qS \subseteq C_x$. Hence $nS \subseteq qS \cup_x pS \subseteq A_x \cup_x (B_x \cap C_x)$. Since the ascending chain condition holds in $X(S)$ each x-ideal is finitely generated by x-ideals of the form kS. We shall now show that, for each $k \in S$, kS is a principal element in $X(S)$. If $m \in kA_x \cap B_x$, then $m = nk \in B_x$, where $n \in A_x \cap (B_x : k)$, hence $kA_x \cap B_x \subseteq (A_x \cap (B_x : k))k$. Consequently $kS \circ A_x \cap B_x \subseteq (A_x \cap (B_x : kS)) \circ kS$. Since the opposite inclusion is clear we conclude that kS is meet principal in $X(S)$. To show that kS is join principal it suffices to show that, for all $A_x, B_x \in X(S)$, $(A_x \cup_x kB_x):k \subseteq A_x : k \cup_x B_x$. Now let $n \in (A_x \cup_x kB_x):k$, so $nkS \subseteq A_x \cup_x kB_x$. The additive condition for $X(S)$ implies that there exist $pS \subseteq B_x$ and $qS \subseteq nS \cup_x pS$ such that $qkS \subseteq A_x$ (hence $qS \subseteq A_x : k$) and $nS \subseteq qS \cup_x pS$, consequently $n \in A_x : k \cup_x B_x$. This completes the proof that $X(S)$ is a Noether lattice. Since the collection $\mathscr{J} = \{kS | k \in S\}$ is multiplicatively closed, $X(S)$ is an additive Noether lattice. To see that $X(S)$ satisfies condition (M) let $kS \subseteq qS \circ B_x$. But $qS \circ B_x = qB_x$ ([1]), implies that $kS \subseteq qB_x$. Therefore there exists an element $p \in B_x$ such that $kS = qSpS = qS \circ pS$, q.e.d.

PROOF OF THEOREM B. Let $\mathscr{J}$ denote a multiplicatively closed family of principal elements of L which satisfies the additive condition for L and condition (M). Define the product of two elements of $\mathscr{J}$ to be their product in L. Hence $\mathscr{J}$ is a commutative semigroup with the identity element I. For $A, B \subseteq \mathscr{J}$, let AB denote the complex product, that is $AB = \{MN | M \in A \text{ and } N \in B\}$. For $A \subseteq \mathscr{J}$, define $A_x = \left\{ M \in \mathscr{J} \mid \text{there exist } M_i \in A,\ 1 \leqq i \leqq n, \text{ such that } M \leqq \bigvee_{i=1}^{n} M_i \right\}$. It will now be shown that $X(\mathscr{J}) = \{A \subseteq \mathscr{J} | A_x = A\}$ is an r-system. Clearly the mapping $A \to A_x$ is a closure operation for which $AB_x \subseteq B_x$, for all $A \subseteq \mathscr{J}$ an $B_x \in X(\mathscr{J})$. Now take $N \in AB_x$. Then $N = PM$, where $P \in A$ and $M \in B_x$, so there exist $M_i \in B$, $1 \leqq i \leqq n$, such that $PM \leqq \bigvee_{i=1}^{n} PM_i$. Since for each i, $1 \leqq i \leqq n$, $PM_i \in AB$, it follows that $N \in (AB)_x$, and so $AB_x \subseteq (AB)_x$. Further if $M \in (NB)_x$, for $N \in \mathscr{J}$, then there exist elements $M_i \in B$, $1 \leqq i \leqq n$, such that $M \leqq N \left(\bigvee_{i=1}^{n} M_i \right)$. Therefore, by condition (M),

$M=NP$, for some element $P\in\mathscr{J}$, such that $P\leqq\bigvee_{i=1}^{n} M_i$. Since $P\in B_x$, $M\in NB_x$. Hence $X(\mathscr{J})$ is an r-system in $\mathscr{J}$. From the definition of A_x it is clear that $X(\mathscr{J})$ is of finite character, hence the chain-closure condition holds for all the x-ideals. This together with the fact that L is Noether implies that the ascending chain condition holds in $X(\mathscr{J})$.

It will now be shown that $X(\mathscr{J})$ and L are isomorphic as multiplicative lattices. Define $\theta: X(\mathscr{J})\to L$ by $\theta(A_x)=\vee\{M|M\in A_x\}$. Clearly $A_x\subseteq B_x$ implies $\theta(A_x)\leqq\theta(B_x)$. If $\theta(A_x)\leqq\theta(B_x)$, then, for $M\in A_x$, $M\leqq\theta(A_x)\leqq\theta(B_x)=\bigvee_{i=1}^{n} M_i$, where $M_i\in B_x$, hence $M\in(B_x)_x=B_x$. This proves that $A_x\subseteq B_x$ if and only if $\theta(A_x)\leqq\theta(B_x)$. θ maps $X(\mathscr{J})$ onto L since each $H\in L$ is the image of $\{M\in\mathscr{J}|M\leqq H\}_x$. Since $M\notin B_x$ implies $M\not\leqq\theta(B_x)$, θ is a one-to-one function. Consequently θ is a lattice isomorphism. To show that θ preserves multiplication, let $A_x, B_x\in X(\mathscr{J})$ and set $A_x=\bigcup_{x\, i=1}^{m} M_i\mathscr{J}$ and $B_x=\bigcup_{x\, j=1}^{n} N_j\mathscr{J}$. Then $A_x\circ B_x=\bigcup_{x\,(i,j)}(M_i\mathscr{J})\circ(N_j\mathscr{J})$. Since θ is a lattice isomorphism, $\theta(A_x\circ B_x)=\bigvee_{(i,j)}\theta[(M_i\mathscr{J})\circ(N_j\mathscr{J})]=\bigvee_{(i,j)} M_iN_j=\theta(A_x)\,\theta(B_x)$. This completes the proof that $X(\mathscr{J})$ an L are isomorphic as multiplicative lattices.

It remains to show that $X(\mathscr{J})$ is an additive r-system. Let $NM\mathscr{J}\subseteq A_x\cup_x MB_x$. By definition of the x-operation $NM\leqq H\vee MG$, where $H=\bigvee_{i=1}^{n} N_i$, $N_i\in A_x$ and $G=\bigvee_{i=1}^{p} P_i$, $P_i\in B_x$, hence $N\leqq(H\vee MG):M$. Since M is a principal element in L, $N\leqq H:M\vee G$. The additivity of L now implies that there exist elements $P, Q\in\mathscr{J}$ such that $Q\leqq H:M$, $P\leqq G$, $N\leqq Q\vee P$, $Q\leqq P\vee N$, and $P\leqq N\vee Q$. Since $P\leqq G$, $P\mathscr{J}\subseteq B_x$ and similarly $QM\mathscr{J}\subseteq A_x$. The remaining three inclusions in the additive condition (for x-systems) hold by definition of the above x-operation and the above lattice inequalities.

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CONCAVE MAJORANTS OF POSITIVE FUNCTIONS

By

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1. Introduction

We consider positive measurable functions $f=f(t)$ on the positive half-axis $R_+^1=(0,\infty)$. Let $F=F(t)$ be the least concave majorant of f, i.e. the least concave function such that $f(t)\leqq F(t)$ for all t. We shall find conditions on f under which we have

$$F(t)\leqq C\int_0^\infty k\left(\frac{t}{\tau}\right)f(\tau)\frac{d\tau}{\tau} \tag{1}$$

where k is a given positive function and C a constant depending on f. Estimates of the type (1) are of interest because they can be used to show that F satisfies certain growth conditions if corresponding growth conditions are imposed on f. In this sense our results contain some previous results of BEURLING [1] and GOSSELIN [3] (cf. [4]).

1. Estimation of the least concave majorant

A function f is *concave* if and only if

$$(1-\lambda)f(s)+\lambda f(t)\leqq f\big((1-\lambda)s+\lambda t\big),\quad 0<\lambda<1.$$

It is easily seen that if f is concave (and positive!) $f(t)$ is increasing but $\dfrac{f(t)}{t}$ is decreasing. This can be summarized in the inequality

$$f(s)\leqq\max\left(1,\frac{s}{t}\right)f(t). \tag{2}$$

It is interesting to note that to some extent the converse holds true.

LEMMA 1 (cf. [5]). *Suppose that f satisfies the inequality*

$$f(s)\leqq C\max\left(1,\frac{s}{t}\right)f(t) \tag{3}$$

for some constant C. Then f is equivalent to its least concave majorant F.

(Two functions f and F are said to be equivalent if we have

$$C_1 \leqq \frac{f(t)}{F(t)} \leqq C_2$$

for some constants C_1 and C_2, $0<C_1\leqq C_2<\infty$.)

PROOF. We have clearly

$$F(t)=\sup \Sigma\lambda_i f(t_i), \quad \Sigma\lambda_i=1, \quad \Sigma\lambda_i t_i=t, \quad \lambda_i\geqq 0.$$

But by (3) we have

$$\Sigma\lambda_i f(t_i) \leqq C\Sigma\lambda_i \max\left(1, \frac{t_i}{t}\right) f(t) \leqq C\left(\Sigma\lambda_i+\frac{1}{t}\Sigma\lambda_i t_i\right) f(t) \leqq 2Cf(t).$$

Hence

$$F(t)\leqq 2Cf(t).$$

Since also

$$f(t)\leqq F(t)$$

the proof is complete.

REMARK 1. In the case of the inequality (2) (i.e. $C=1$ in inequality (3)) the above proof gives

$$\frac{1}{2} \leqq \frac{f(t)}{F(t)} \leqq 1.$$

Here the bounds are the best possible, as is seen by the example $f(t)=\max(1, t)$ in which case $F(t) = 1+t$. Hence (2) does not imply concavity. (Note also that if f is differentiable then (2) is equivalent to $0\leqq tf'(t)\leqq f(t)$.)

COROLLARY 1. *Let f be any function. Then its least concave majorant F is equivalent to the function*

$$F^*(t) = \sup \min\left(1, \frac{t}{s}\right) f(s).$$

PROOF. Immediate.

Let us call a function f *pseudo-concave* if it satisfies inequality (3) for some C. It is then readily seen that F^* is the least pseudo-concave majorant of f.

From Corollary 1 we see that (1) is equivalent to

$$\min\left(1, \frac{t}{s}\right) f(s) \leqq C\int_0^\infty k\left(\frac{t}{\tau}\right) f(\tau)\frac{d\tau}{\tau}, \quad s\in(0,\infty). \tag{4}$$

We can use (4) to get some information on k. Assuming e.g. that (4) holds for the characteristic function of an interval (α, β), with $\alpha<\beta$, we obtain from (4)

$$\min\left(1, \frac{t}{s}\right) \leqq C\int_\alpha^\beta k\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \quad \text{if} \quad s\in(\alpha, \beta)$$

or, with a new constant C,

$$\min(1, t) \leqq C\int_\alpha^\beta k\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau}.$$

Assuming moreover that

$$\int_{\alpha}^{\beta} k\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \leqq Ck(t)$$

we obtain

$$\min(1, t) \leqq C\,k(t).$$

We shall now show that conversely, (1) holds with $k(t)=\min(1,t)$ for a sufficiently large class of functions (including in particular characteristic functions of intervals). We denote by $\mathscr{F}_0(a, b)$, where $0<a<b<\infty$, the class of functions f such that for some constant C (depending on f) holds the inequality

$$f(t) \leqq C\int_{at}^{bt} f(\tau)\frac{d\tau}{\tau}. \tag{5}$$

It follows from (3) that every pseudo-concave function and thus in particular every concave function is of class $\mathscr{F}_0(a, b)$ for all a, b. As another example we cite *subadditive* functions. A function f is said to be subadditive if and only if we have

$$f(s+t) \leqq f(s)+f(t). \tag{6}$$

(Note that concave functions are subadditive and that conversely *increasing* subadditive functions are pseudo-concave.) Indeed more generally we may consider functions satisfying

$$f(s+t) \leqq C\big(f(s)+f(t)\big) \tag{7}$$

for some constant $C \geqq 1$. Write (7) as

$$f(t) \leqq C\big(f(\tau)+f(t-\tau)\big), \qquad 0<\tau\leqq t/2$$

and integrate over $\left(\left(\frac{1}{2}-\varepsilon\right)t, \frac{t}{2}\right)$, with $0<\varepsilon<\frac{1}{2}$. It follows that

$$f(t)\int_{\left(\frac{1}{2}-\varepsilon\right)t}^{\frac{t}{2}} d\tau \leqq C\left(\int_{\left(\frac{1}{2}-\varepsilon\right)t}^{\frac{t}{2}} f(\tau)\,d\tau + \int_{\left(\frac{1}{2}-\varepsilon\right)t}^{\frac{t}{2}} f(t-\tau)\,d\tau\right)$$

or

$$f(t) \leqq C\frac{1}{\varepsilon t}\int_{\left(\frac{1}{2}-\varepsilon\right)t}^{\left(\frac{1}{2}+\varepsilon\right)t} f(\tau)\,d\tau$$

which implies (5) with $a=\frac{1}{2}-\varepsilon$, $b=\frac{1}{2}+\varepsilon$. Thus in particular subadditive functions are of class $\mathscr{F}_0(a, b)$ whenever $a<\frac{1}{2}<b$ (cf. [3]).

THEOREM 1. *Let f be of class $\mathscr{F}_0(a, b)$ for some a, b with $a<b$. Then* (1) *holds with $k(t)=k_0(t)=\min(1, t)$.*

PROOF. By the inequality

$$\min\left(1, \frac{t}{s}\right) \leqq \max\left(1, \frac{\tau}{s}\right) \min\left(1, \frac{t}{\tau}\right)$$

we get from (5)

$$\min\left(1, \frac{t}{s}\right) f(s) \leqq C \int_{as}^{bs} \max\left(1, \frac{\tau}{s}\right) \min\left(1, \frac{t}{\tau}\right) f(\tau) \frac{d\tau}{\tau} \leqq C \int_{0}^{\infty} \min\left(1, \frac{t}{\tau}\right) f(\tau) \frac{d\tau}{\tau}.$$

Thus (4), and hence (1), follows with $k(t)=k_0(t)$.

As we have already told above this is about the best result for the class $\mathscr{F}_0(a, b)$. To push further we have to consider more restricted classes of functions. Let us say that f is of class $\mathscr{F}_1(a, b)$ if we have

$$f(s) \leqq C \int_{at}^{bt} f(\tau) \frac{d\tau}{\tau} \quad \text{if} \quad s \leqq t \tag{8}$$

and of class $\mathscr{F}_2(a, b)$ if

$$\frac{t}{s} f(s) \leqq C \int_{at}^{bt} f(\tau) \frac{d\tau}{\tau} \quad \text{if} \quad s \geqq t. \tag{9}$$

Let us say that f is of class $\mathscr{F}_3(a, b)$ if it is simultaneously of class $\mathscr{F}_1(a, b)$ and of class $\mathscr{F}_2(a, b)$. (Note that f is of class $\mathscr{F}_1(a, b)$ if it is of class $\mathscr{F}_0(a, b)$ and if $f(t)$ is increasing, and of class $\mathscr{F}_0(a, b)$ and if $\dfrac{f(t)}{t}$ is decreasing.)

THEOREM 2. (i) *Let f be of class $\mathscr{F}_1(a, b)$. Then* (1) *holds with*

$$k(t) = k_1(t) = \begin{cases} t & \text{if} \quad t \leqq a^{-1} \\ 0 & \text{if} \quad t > a^{-1} \end{cases}$$

(ii) *Let f be of class $\mathscr{F}_2(a, b)$. Then* (1) *holds with*

$$k(t) = k_2(t) = \begin{cases} 0 & \text{if} \quad t < b^{-1} \\ 1 & \text{if} \quad t \geqq b^{-1} \end{cases}$$

PROOF. We prove only (i) since (ii) can be proven by a similar (dual) argument. From the proof of Theorem 1 we get

$$\frac{t}{s} f(s) \leqq C \int_{at}^{\infty} \frac{t}{\tau} f(\tau) \frac{d\tau}{\tau} \quad \text{if} \quad s \geqq t.$$

But (8) implies

$$f(s) \leqq C \int_{at}^{bt} f(\tau) \frac{d\tau}{\tau} \leqq C \int_{at}^{\infty} \frac{t}{\tau} f(\tau) \frac{d\tau}{\tau} \quad \text{if} \quad s \leqq t.$$

Thus (4) follows with $k(t)=k_1(t)$.

THEOREM 3. *Let f be of class $\mathcal{F}_3(a, b)$. Then holds* (1) *with*

$$k(t) = k_3(t) = \begin{cases} 1 & \text{if } \; b^{-1} \leqq t \leqq a^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Immediate.

Let us apply our results to the case of subadditive functions. We know already that a subadditive function is of class $\mathcal{F}_0(a, b)$, $a<\frac{1}{2}<b$. Thus we can apply Theorem 1. Let us show that a subadditive function is of class $\mathcal{F}_2(a, b)$, $a<\frac{1}{2}$, $1>b$. Indeed by iteration of inequality (6) is obtained

$$f(s) \leqq 2^n f\left(\frac{s}{2^n}\right)$$

for any integer $n \geqq 0$. Set $\sigma = \dfrac{s}{2^n}$. Choose n such that

$$t \leqq \sigma \leqq 2t.$$

Then we get

$$f(s) \leqq C\frac{s}{t}\int\limits_{\sigma\left(\frac{1}{2}-\varepsilon\right)}^{\sigma\left(\frac{1}{2}+\varepsilon\right)} f(\tau)\frac{d\tau}{\tau} \leqq C\frac{s}{t}\int\limits_{t\left(\frac{1}{2}-\varepsilon\right)}^{t(1+2\varepsilon)} f(\tau)\frac{d\tau}{\tau}, \quad s \geqq t.$$

In other words f is of class $\mathcal{F}_2(a, b)$, $a<\frac{1}{2}$, $1>b$. Thus we may apply theorem 2, part (ii).

Let us finally impose the following inequality of f:

$$f(t) \leqq C\big(f(t+s)+f(s)\big). \tag{10}$$

GOSSELIN [3] calls a function *subadditive-even* if it is subadditive and satisfies (10). Let us show that a function which satisfies (10) is of class $\mathcal{F}_1(a, b)$, $b-a>1$. Indeed writing (10) as

$$f(s) \leqq C\big(f(s+\tau)+f(\tau)\big)$$

and integrating over $\big(at, (a+\varepsilon)t\big)$ we obtain

$$f(s)\int\limits_{at}^{(a+\varepsilon)t} d\tau \leqq C\int\limits_{at}^{(a+\varepsilon)t} f(s+\tau)\,d\tau + \int\limits_{at}^{(a+\varepsilon)t} f(\tau)\,d\tau$$

from which follows if $s \leqq t$:

$$f(s) \leqq \frac{C}{\varepsilon t}\left(\int\limits_{at}^{(1+a+\varepsilon)t} f(\tau)\,d\tau + \int\limits_{at}^{(a+\varepsilon)t} f(\tau)\,d\tau\right).$$

Thus f is of class $\mathcal{F}_1(a, a+1+\varepsilon)$. Therefore Theorem 2, part (ii) applies to such functions. In particular Theorem 3 applies to subadditive-even functions.

2. Growth conditions

THEOREM 4. *Assume that* f *satisfies*

$$\int_0^\infty \varphi(t) f(t) \frac{dt}{t} < \infty$$

and that (1) *holds. Then* F *satisfies*

$$\int_0^\infty \Phi(t) F(t) \frac{dt}{t} < \infty$$

where Φ *and* φ *are related by*

$$\varphi(t) = \int_0^\infty k\left(\frac{\tau}{t}\right) \Phi(\tau) \frac{d\tau}{\tau}.$$

PROOF. Immediate.

We consider some special cases.

Let us first consider the special case $\Phi(t)=t^{-\theta}$. Then $\varphi(t)=ct^{-\theta}$ with

$$c = c_\theta = \int_0^\infty k(\tau)\tau^{-\theta} \frac{d\tau}{\tau}.$$

Of course the result is of interest only if $c<\infty$. We see easily that $c<\infty$ in the following ranges of θ:

$$\begin{aligned} 0<\theta<1 &\quad \text{if} \quad k=k_0, \\ -\infty<\theta<1 &\quad \text{if} \quad k=k_1, \\ 0<\theta<\infty &\quad \text{if} \quad k=k_2, \\ -\infty<\theta<\infty &\quad \text{if} \quad k=k_3. \end{aligned}$$

Thus e.g. for functions f of class $\mathcal{F}_3(a, b)$ holds

$$\int_0^\infty t^{-\theta} f(t) \frac{dt}{t} < \infty \tag{11}$$

if and only if

$$\int_0^\infty t^{-\theta} F(t) \frac{dt}{t} < \infty, \tag{12}$$

this for any θ, $-\infty<\theta<\infty$. For subadditive-even functions this result was established by GOSSELIN [3], Theorem 1.

Note also that since F is concave condition (12) implies $F=0$, if $\theta<0$ or $\theta>1$. Therefore, for the same values of θ, condition (11) implies $f=0$ (cf. GOSSELIN [3], Theorem 2.)

Next let us take

$$\varphi(t) = \begin{cases} t^{-\theta} & \text{if} \quad t \geqq 1 \\ 0 & \text{if} \quad t < 1. \end{cases}$$

If $k=k_3$ we then get

$$\varphi(t)=\int_{\max(1,b^{-1}t)}^{a^{-1}t}\tau^{-\theta}\frac{d\tau}{\tau}$$

and in particular

$$\varphi(t)=\begin{cases}ct^{-\theta} & \text{if} \quad t\geqq b\\ 0 & \text{if} \quad t\leqq a\end{cases}$$

with $c<\infty$ for all θ. Thus if f is of class $\mathscr{F}_3(a,b)$ holds

$$\int_1^\infty t^{-\theta}f(t)\frac{dt}{t}<\infty \tag{13}$$

if and only if

$$\int_1^\infty f^{-\theta}F(t)\frac{dt}{t}<\infty. \tag{14}$$

For subadditive functions which are restrictions of symmetric subadditive functions on the whole axis $R^1=(-\infty,\infty)$ — such functions are subadditive-even — the special case $\theta=1$ is contained in a lemma by Beurling, which was reproduced as theorem 1. 2. 7 in BJÖRCK [2]. (Actually there the corresponding result in R^n is needed but the transition to the case R^1_+ causes no difficulty.) Note that (14) is trivially fulfilled if $\theta>1$ (since $F(t)=O(t)$, $t\to\infty$; cf. (2)).

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QUASI-NOETHERIAN x-SYSTEMS

By

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A closure operator defined on the subsets $(A \to A_x)$ of a semigroup S with an identity is called an *x-operator* if $AB_x \subseteq B_x \cap (AB)_x$, for all subsets A, B of S. The collection $X(S)$ of x-ideals (x-closed subsets of S) is called an *x-system* [1]. An x-ideal A_x is *quasi-noetherian* if, for every x-ideal B_x, $(A_x: B_x^n) \cap (A_x \cup_x B_x^n) = A_x$, for sufficiently large n. This condition, for ideals in a Noetherian ring, was studied in [2]. If every x-ideal is quasi-noetherian, then the x-system and the x-operator are said to be *quasi-noetherian*. In this paper we will study quasi-noetherian x-systems.

We adopt the notation and terminology of [1] and we assume throughout that S is a commutative semigroup with an identity.

In general, x-systems (including those which satisfy the ascending chain condition) may contain irreducible x-ideals which are not primary [1]. Thus an additional condition on the x-operator is needed to establish the primary decomposition theory in x-systems. The following lemma provides a criterion for determining when an irreducible x-ideal is primary.

LEMMA 1. *In an x-system, an irreducible x-ideal is primary if and only if it is quasi-noetherian.*

PROOF. Suppose that A_x is irreducible and quasi-noetherian and that $C_x \circ B_x \subseteq A_x$ but $C_x \not\subseteq A_x$. Consequently, $A_x \neq A_x: B_x^m$ for all m, and since $A_x = (A_x: B_x^m) \cap \cap (A_x \cup_x B_x^m)$ for sufficiently large m, it follows that $A_x = A_x \cup_x B_x^m$ for some m.

Now, assume A_x is an irreducible x-ideal which is primary and let B_x be arbitrary. If A_x contains some power of B_x, then $A_x \cup_x B_x^m = A_x$, for sufficiently large m, and the desired identity holds. Hence, assume A_x does not contain a power of B_x. Since $[(A_x: B_x^n) \cap (A_x \cup_x B_x^n)] \circ B_x^n \subseteq A_x$, it follows that $(A_x: B_x^n) \cap (A_x \cup_x B_x^n) = A_x$, for all n, q.e.d.

THEOREM 2. *In an x-system $X(S)$ which satisfies the ascending chain condition, the following are equivalent:*

a) *Every irreducible x-ideal is primary.*

b) *Every irreducible x-ideal is quasi-noetherian.*

c) *$X(S)$ is quasi-noetherian.*

d) *Every x-ideal is a finite intersection of primary x-ideals.*

PROOF. From Lemma 1 and the fact that every x-ideal is a finite intersection of irreducible x-ideals, it follows that conditions a), b), and d) are equivalent. We need only to show that b) implies c). Let A_x be any x-ideal and write A_x as a finite intersection, $A_x = \bigcap_{i=1}^{n} (Q_i)_x$, of irreducible x-ideals. Let B_x be arbitrary. For each $i=1, \ldots, n$, there exists an integer m_i such that $((Q_i)_x : B_x^k) \cap ((Q_i)_x \cup_x B_x^k) = (Q_i)_x$, for every $k \geqq m_i$. For any integer k which is greater than the maximum of the m_i we have

$$(A_x : B_x^k) \cap (A_x \cup_x B_x^k) =$$

$$= \left[\bigcap_{i=1}^{n} ((Q_i)_x : B_x^k)\right] \cap \left[\left(\bigcap_{i=1}^{n} (Q_i)_x\right) \cup_x B_x^k\right] = \bigcap_{i=1}^{n} (Q_i)_x = A_x, \quad \text{q.e.d.}$$

LEMMA 3. *If $X(S)$ is quasi-noetherian, then, for all $A_x, B_x, A_x \cap (B^n)_x \subseteqq A_x \circ B_x$ for sufficiently large n.*

PROOF. The Lemma follows since $A_x \cap (B^n)_x \subseteqq (A_x \circ B_x : B_x^n) \cap (A_x \circ B_x \cup_x B_x^n) \subseteqq \subseteqq A_x \circ B_x$ for sufficiently large n, q.e.d.

An x-system is said to be modular if $A_x \supseteqq B_x$ implies $A_x \cap (B_x \cup_x C_x) = = B_x \cup_x (A_x \cap C_x)$. The following theorem shows that the converse of Lemma 3 holds in modular x-systems which satisfy the ascending chain condition.

THEOREM 4. *Let $X(S)$ be a modular x-system which satisfies the ascending chain condition. Then each of the following is equivalent to the conditions in Theorem 2:*

1) *If A_x is an irreducible x-ideal, then for each x-ideal B_x, $(A_x : B_x^m) \cap (B^m)_x \subseteqq A_x$, for sufficiently large m.*

2) *For all x-ideals $A_x, B_x, A_x \cap (B^m)_x \subseteqq A_x \circ B_x$, for sufficiently large m.*

3) *For all x-ideals $A_x, B_x, (A_x : B_x^m) \cap (B^m)_x \subseteqq A_x$, for sufficiently large m.*

PROOF. By modularity $(A_x : B_x^n) \cap (A_x \cup_x B_x^n) = A_x \cup_x [(A_x : B_x^n) \cap (B^n)_x]$. Hence, condition 3) above is equivalent to condition c) of Theorem 2. Now, assume condition 3). For all $A_x, B_x, (A_x \circ B_x : B_x^m) \cap (B^m)_x \subseteqq A_x \circ B_x$, for sufficiently large m. Since $[A_x \cap (B^m)_x] \circ (B^m)_x \subseteqq A_x \circ B_x$, it follows that $A_x \cap (B^m)_x \subseteqq (A_x \circ B_x : B_x) \cap (B^m)_x \subseteqq \subseteqq A_x \circ B_x$, and so condition 3) implies condition 2). For the converse, choose an integer k such that $A_x : B_x^k = A_x : B_x^{k+1}$. Applying condition 2) to $A_x : B_x^k$ and $(B^k)_x$ we conclude that $(A_x : B_x^k) \cap (B^{km})_x \subseteqq (A_x : B_x^k) \circ (B^k)_x \subseteqq A_x$, for sufficiently large m. Thus, by our choice of k, $(A_x : B_x^n) \cap (B^n)_x \subseteqq A_x$, for sufficiently large n. The argument which was used to prove that b) and c) (of Theorem 2) are equivalent can be adapted to show that conditions 1) and 3) are equivalent, q.e.d.

An x-operator (hence the x-system) is *intersection continuous* if $(B_x \cap aC_x)_x = = B_x \cap ((a)_x \circ C_x)$ for every $a \in S$ and for all $B_x, C_x \in X(S)$. With this definition we have the following:

PROPOSITION 5. *A modular intersection continuous x-system which satisfies the ascending chain condition is quasi-noetherian.*

PROOF. Let A_x be an x-ideal which is not primary and choose elements $a, b \in S$ such that $ab \in A_x$, $a \notin A_x$, and $b^n \notin A_x$ for all n. Choose k such that $A_x : b^k = A_x : b^{k+1}$ and let $B_x = (A_x : b^k) \cap (A_x \cup_x (b^k)_x)$. To complete the proof we need only to show that $B_x \subseteqq A_x$, since both components in the definition of B_x strictly contain A_x. By modularity, $B_x = A_x \cup_x (A_x : b^k) \cap (b^k)_x$. By our choice of k, $(A_x : b^k) \cap b^k S \subseteqq A_x$, hence $(A_x : b^k) \cap (b^k)_x = [(A_x : b^k) \cap b^k S]_x \subseteqq A_x$, since the x-system is intersection continuous. Consequently, A_x is not irreducible, therefore the x-system is quasi-noetherian, q.e.d.

We will now investigate some intersection properties in quasi-noetherian x-systems. In particular we will consider the Krull Intersection Theorem and the Artin—Rees Lemma.

LEMMA 6. *In a quasi-noetherian x-system,* $A_x \subseteqq \bigcap_{n=1}^{\infty} (B^n)_x$ *if and only if* $A_x \circ B_x = A_x$.

PROOF. If $A_x \subseteqq \bigcap_{n=1}^{\infty} (B^n)_x$, then $A_x = A_x \cap (B^n)_x \subseteqq A_x \circ B_x \subseteqq A_x$ for sufficiently large n (Lemma 3). Hence $A_x = A_x \circ B_x$, q.e.d.

The "only if" part of Lemma 6 implies the Krull Intersection Theorem in local rings. However, consideration of the semigroup $S = \{0, a, 1\}$ with $a^2 = a$, and the system $X(S) = \{\{0\}, \{0, a\}, \{0, a, 1\}\}$, shows that the Intersection Theorem need to hold in a quasi-noetherian x-system. The main problem in this example is that $\{0, a\} \circ \{0, a\} = \{0, a\}$. This observation leads to the following Corollary which is immediate from Lemma 6.

COROLLARY 7. *Let* $X(S)$ *be a quasi-noetherian x-system and let* 0 *denote the least element in* $X(S)$. *For each* $A_x \neq S$, $\bigcap_{n=1}^{\infty} (A^n)_x = 0$ *if and only if* $B_x \circ C_x = B_x$ *implies* $C_x = S$ *or* $B_x = 0$ *(for all x-ideals* B_x *and* C_x*).*

In the following two theorems we consider the more general forms of the Krull Intersection Theorem. We will first consider the form of the Intersection Theorem which holds in local rings [3], i.e., if $C_x \neq S$, then $\bigcap_{n=1}^{\infty} (A_x \cup_x C_x^n) = A_x$, for all A_x.

THEOREM 8. *Let* $X(S)$ *be a modular quasi-noetherian x-system. For* $C_x \neq S$, $\bigcap_{n=1}^{\infty} (A_x \cup_x C_x^n) = A_x$ *(for all* A_x*) if and only if* $B_x \subseteqq G_x \cup_x B_x \circ D_x$ *and* $D_x \neq S$ *imply* $B_x \subseteqq G_x$ *(for all* B_x, G_x, *and* D_x*).*

PROOF. Let $B_x \subseteqq G_x \cup_x B_x \circ D_x$ and $D_x \neq S$. Consequently, $B_x \subseteqq G_x \cup_x B_x \circ D_x^n \subseteqq \subseteqq G_x \cup_x D_x^n$ for all n, hence $B_x \subseteqq \bigcap_{n=1}^{\infty} (G_x \cup_x D_x^n) = G_x$. For the converse, take $C_x \neq S$ and for an arbitrary A_x let $\bigcap_{n=1}^{\infty} (A_x \cup_x C_x^n) = M_x$. Choose m such that $M_x \cap (C^m)_x \subseteqq \subseteqq M_x \circ C_x$ (Lemma 3). By modularity, $M_x = M_x \cap (A_x \cup_x (C^m)_x) = A_x \cup_x (M_x \cap (C^m)_x) \subseteqq \subseteqq A_x \cup_x M_x \circ C_x$, hence $M_x \subseteqq A_x$, q.e.d.

Let $A_x = \bigcap_{i=1}^{\infty} (Q_i)_x$ be a primary decomposition of A_x, let $(P_1)_x, \ldots, (P_n)_x$ denote the associated primes, and let C_x be an arbitrary x-ideal. The collection of $(P_i)_x$ with the property $(P_i)_x \cup_x C_x \neq S$ is an isolated set, hence $A_x[C_x] = = \bigcap \{(Q_i)_x | (P_i)_x \cup_x C_x \neq S\}$ is an isolated component of A. We will now consider the form of the Intersection Theorem which holds in Noetherian rings [3], i.e., if $C_x \neq S$, then $\bigcap_{n=1}^{\infty} (A_x \cup_x C_x^n) = A_x[C_x]$, for all A_x.

THEOREM 9. *Let $X(S)$ be a modular x-system in which each x-ideal has a primary decomposition. For $C_x \neq S$, $\bigcap_{n=1}^{\infty} (A_x \cup_x C_x^n) = A_x[C_x]$ (for all A_x) if and only if $B_x \subseteq G_x \cup_x B_x \circ D_x$ and $D_x \neq S$ imply $B_x \subseteq G_x[D_x]$ (for all x-ideals B_x, D_x, and G_x).*

PROOF. For x-ideals B_x, G_x, and D_x ($D_x \neq S$), $B_x \subseteq G_x \cup_x B_x \circ D_x$ implies $B_x \subseteq \bigcap_{n=1}^{\infty} (G_x \cup_x D_x^n) = G_x[D_x]$. For the converse, let A_x be an arbitrary x-ideal, let $A_x = \bigcap_{i=1}^{k} (Q_i)_x$ be a primary decomposition of A_x, let $C_x \neq S$, and let $\bigcup_{n=1}^{\infty} (A_x \cup_x C_x^n) = M_x$. We will show that $M_x = A_x[C_x]$. Let $M_x \circ C_x = \bigcap_{i=1}^{n} (R_i)_x$ be a primary decomposition of $M_x \circ C_x$. For each i, $M_x \subseteq (R_i)_x$ or $(C^t)_x \subseteq (R_i)_x$ for some positive integer t. Therefore there exists a positive integer m such that $(C^m)_x \subseteq (R_i)_x$ for $i = 1, \ldots, q$ and $M_x \subseteq (R_i)_x$ for $i = q+1, \ldots, n$, hence $M_x \cap (C^m)_x \subseteq M_x \circ C_x$. Now, by modularity we have $M_x = M_x \cap (A_x \cup_x (C^m)_x) = A_x \cup_x (M_x \cap (C^m)_x \subseteq A_x \cup_x M_x \circ C_x$. Therefore, $M_x \subseteq A_x[C_x]$.)

For the opposite inclusion, let $A_x\langle C_x\rangle = \cap \{(Q_j)_x | (P_j)_x \cup_x C_x = S\}$ and observe that $(P_j)_x \cup_x C_x = S$ implies $(\Pi\{(Q_j)_x | (P_j)_x \cup_x C_x = S\}) \cup_x C_x = S$. Consequently, $A_x\langle C_x\rangle \cup_x C_x = S$. From this it follows that $A_x \cup_x (C^s)_x = (A_x[C_x] \cap A_x\langle C_x\rangle) \cup_x (C^s)_x = A_x[C_x] \cup_x (C^s)_x$ for all positive integers s. Therefore, $A_x[C_x] \subseteq M_x$, q.e.d.

An x-operator (hence the x-system) is *residuation continuous* if $(aB_x)_x : a = (aB_x : a)_x$ for all $a \in S$ and $B_x \in X(S)$. We now note the following more special result which assumes cancellation of non-zero elements in the semigroup.

COROLLARY 10. *Let S satisfy the cancellation law for non-zero elements. In a quasi-noetherian and residuation continuous x-system, $\bigcap_{n=1}^{\infty} (B^n)_x = 0$, for every x-ideal $B_x \neq S$.*

PROOF. If $a \in A_x = \bigcap_{n=1}^{\infty} (B^n)_x$, then $(a)_x \circ B_x = (aB_x)_x = (a)_x$. Hence, $S = (a)_x : a = (aB_x)_x : a = (aB_x : a)_x$, so if a is not a zero element in S, then $S = B_x$. It follows that $A_x = 0$, q.e.d.

In Proposition 5 we proved that a modular intersection continuous x-system in which the ascending chain condition holds is quasi-noetherian. We will now obtain the Artin—Rees Lemma in such an x-system.

THEOREM 11. *If $X(S)$ is a modular intersection continuous x-system in which the ascending chain condition holds, then, given $A_x, B_x \in X(S)$, there exists an integer k such that $A_x \cap (B^n)_x = (A_x \cap (B^k)_x) \circ (B^{n-k})_x$ for all $n \geqq k$.*

PROOF. Let A_x be an x-ideal different from S and let L be the collection of all sequences $\{(B_i)_x\}_{i \geqq 0}$ of x-ideals satisfying $(A^i)_x \supseteq (B_i)_x \supseteq (B_{i+1})_x \supseteq A_x \circ (B_i)_x$ for every $i \geqq 0$. L is a partially ordered set with respect to the following relation: For $B, C \in L$, $B \leqq C$ if and only if $(B_i)_x \subseteq (C_i)_x$ for all $i \geqq 0$. Let $\mathcal{J}$ be the collection of all sequences $\{(B_i)_x\}_{i \geqq 0}$ in L for which there fails to exist an integer n such that $(B_{n+i})_x = (B_n)_x \circ (A^i)_x$ for all $i \geqq 0$. Since $\{A_x \cap (B^i)_x\}_{i \geqq 0}$ belongs to L for every x-ideal B_x, it is sufficient to show that $\mathcal{J} = \emptyset$. Hence assume $\mathcal{J} \neq \emptyset$, let $\mathcal{C}$ be any chain of elements of $\mathcal{J}$ and observe that $T=\{\bigcup_{x\, C\in\mathscr{C}}(C_i)_x\}_{i\geqq 0}$ belongs to L. If T is not in $\mathcal{J}$, then there exists an integer n such that $(T_i)_x=(T_n)_x\circ(A^{i-n})_x$ for all $i\geqq n$. Since $X(S)$ satisfies the ascending chain condition, for each integer j, $0\leqq j\leqq n$, there exists an element of the chain whose j^{th} component is $(T_j)_x$. For D, the maximum of these elements $D=T$. It is clear that $(D_j)_x=(T_j)_x$ for each j, $0\leqq j\leqq n$. For $j>n$, $(T_j)_x\supseteqq(D_j)_x\supseteqq(D_n)_x\circ(A^{j-n})_x=(T_n)_x\circ(A^{j-n})_x=(T_j)_x$. This contradiction proves that T is in $\mathcal{J}$, and hence $\mathcal{J}$ has a maximal element B.

Since $\{(A^i)_x\}_{i\geqq 0}$ is not in $\mathcal{J}$, there exists an element $a\in A_x$ such that $(B_1)_x\subsetneqq(B_1)_x\cup_x(aS)_x\subseteqq A_x$. The sequence $E=\{(B_i)_x\cup_x(aS)_x\circ(A^{i-1})_x\}_{i\geqq 0}$ (where $A^{-1}=S$) is in L, but not in $\mathcal{J}$, since $B\not\geqq E$. Hence there exists an integer k such that $(E_n)_x\circ A_x=(E_{n+1})_x$ for all $n\geqq k$ and so $(B_{n+1})_x\subseteqq(B_n)_x\circ A_x\cup_x(aS)_x\circ(A^n)_x$ for all $n\geqq k$. Then since $X(S)$ is modular and intersection continuous, we have $(B_{n+1})_x=(B_{n+1})_x\cap[(B_n)_x\circ A_x\cup_x(aS)_x\circ(A^n)_x]=(B_n)_x\circ A_x\cup_x[(B_{n+1})_x\cap((aS)_x\circ(A^n)_x)]=(B_n)_x\circ A_x\cup_x[(A^n)_x\cap((B_{n+1})_x:(aS)_x)]\circ(aS)_x$, for all $n\geqq k$.

Now the sequence $G=\{((B_{i+1})_x:(aS)_x)\cap(A^i)_x\}$ is in L. Observe that $B\not\geqq G$. Otherwise, the above equalities imply $(B_{n+1})_x=(B_n)_x\circ A_x\cup_x(B_n)_x\circ(aS)_x=(B_n)_x\circ A_x$ for all $n\geqq k$ (but $B\in\mathcal{J}$). Therefore there exists an integer $m(\geqq k)$ such that $(G_{n+1})_x=(G_n)_x\circ A_x$ for all $n\geqq m$. Consequently, for all $n\geqq m$, $(B_{n+1})_x=(B_n)_x\circ A_x\cup_x[(A^n)_x\cap((B_{n+1})_x:(aS)_x)]\circ(aS)_x=(B_n)_x\circ A_x\cup_x[(A^{n-1})_x\cap((B_n)_x:(aS)_x)]\circ A_x\circ(aS)_x=(B_n)_x\circ A_x$. This contradicts the fact that $B\in\mathcal{J}$. Thus $\mathcal{J}$ is empty, q.e.d

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ON FINITE Δ-SYSTEMS OF ERDŐS AND RADO

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1. Introduction

The main purpose of this paper is to give some new estimations for a problem stated in [1] and to show its connection with Ramsey's theorem.

First, we have to repeat necessary definitions from [1]:

"A system Σ_1: Y_ν $(\nu \in N)$ of sets Y_ν, where ν ranges over the index set N, is said to contain the system Σ_0: X_μ $(\mu \in M)$ if, for every μ_0 of M, the set X_{μ_0} occurs in Σ_1 at least as often as in Σ_0, i.e., if

$$|\{\nu\colon \nu \in N;\ Y_\nu = X_{\mu_0}\}| \geqq |\{\mu\colon \mu \in M;\ X_\mu = X_{\mu_0}\}|.$$

If Σ_1 contains Σ_0 and, at the same time, Σ_0 contains Σ_1, then we do not distinguish between the systems Σ_0 and Σ_1.

The system Σ_0 is called an (a, b)-system if it consists of a (not necessarily distinct) sets of cardinal b, i.e., if $|M| = a$ and $|X_\mu| = b$ for $\mu \in M$. The system Σ_0 is called a Δ-system if it has the property that the intersections of any two of its sets (not necessarily distinct sets but sets having distinct indices μ) have the same value, i.e. if for

$$\mu_0, \mu_1, \mu_2, \mu_3 \in M; \quad \mu_0 \neq \mu_1; \quad \mu_2 \neq \mu_3$$

we always have $X_{\mu_0} X_{\mu_1} = X_{\mu_2} X_{\mu_3}$. More specifically, Σ_0 is a Δ (a)-system with kernel K if $|M| = a$ and $X_{\mu_0} X_{\mu_1} = K$ whenever $\mu_0, \mu_1 \in M$; $\mu_0 \neq \mu_1$.

. . . Expressions such as

$$(>a,\ \leqq b)\text{-system}, \quad \Delta(>a)\text{-system}$$

have their obvious meaning."

We shall concern ourselves only with a, b finite $(1 \leqq a,\ b < \aleph_0)$.

Erdős and Rado proved that given any a, b there exists a least number $f(a, b)$ such that every $(>f,\ \leqq b)$-system contains a $\Delta(>a)$-system. In fact, they proved

$$a^{b+1} < f(a, b) \leqq b!\,a^{b+1}\left(1 - \frac{1}{2!\,a} - \frac{2}{3!\,a^2} - \dots - \frac{b-1}{b!\,a^{b-1}}\right)$$

and conjectured $f(a, b) \leqq c^b a^{b+1}$ for some absolute positive constant c.

We shall show

$$a^{b+1}\left(1 + \frac{1}{a}\right)^{\left[\frac{b}{2}\right]} \leqq f(a, b) \text{ for } a \text{ even}$$

$$a^{b+1}\left(1 + \frac{1}{2a} - \frac{1}{2a^2}\right)^{\left[\frac{b}{2}\right]} \leqq f(a, b) \text{ for } a \text{ odd}$$

(Theorem 3). For $a \geqq b^2-b+1$, $b \geqq 3$, we shall prove

$$f(a, b) \leqq \frac{(b+1)!}{2b} a^{b+1}\left(1-\frac{1}{3a}-\frac{1}{3a^2}\right)$$

(Theorem 5).

We shall also show that the existence of $f(a, b)$ is a consequence of Ramsey's theorem.

Erdős and Rado denoted by $\Phi(a, b)$ the least number provided that every $(>\Phi, \leqq b)$-system $\Sigma: X_\mu(\mu \in M)$ which satisfies $X_\mu \neq X_\nu$ for $\mu \neq \nu$ contains a $\Delta(>a)$-system. It is easy to see that

$$f(a, b)=a\Phi(a, b).$$

Hence, we shall confine ourselves only with systems of distinct sets.

One may "polarize" the notion of $\Phi(a, b)$ in the following way:

Define $\pi(a_0, a_1, \ldots, a_{b+1}, b)$ as the least number such that every $(>\pi, \leqq b)$-system contains a $\Delta(>a_j)$-system with kernel K, $|K|=j$ for some j. Especially, put $\pi(n, a, b)=\pi(n, a, a, \ldots, a, b)$.

Evidently, $\pi(n, a, 1)=\pi(n, 1, b)=n$.

2. Results

THEOREM 1.

(1)
$$\pi(n, a, 2) \geqq na+\left[\frac{n}{\left[\frac{a+1}{2}\right]}\right]\cdot\left[\frac{a}{2}\right]$$

where $[x]$ *is the greatest integer not exceeding* x. *Especially,* $\Phi(a, 2) \geqq a^2+a$ *for* a *even,* $\Phi(a, 2) \geqq a^2+\frac{a-1}{2}$ *for* a *odd.*

THEOREM 2. *If* $n=1, 2$ *or* $a=2$ *then in* (1) *the equality holds.*

THEOREM 3. *If* $n \leqq a$ *then*

$$\pi(n, a, b) \geqq \Phi(a, 2)^{\frac{b}{2}-1}\cdot\pi(n, a, 2) \quad \text{for } b \text{ even}$$

$$\pi(n, a, b) \geqq \Phi(a, 2)^{\frac{b-1}{2}}\cdot n \quad \text{for } b \text{ odd.}$$

If $n = ka+q$, $k \geqq 0$, $1 \leqq q \leqq a$ *then*

$$\pi(n, a, b) \geqq k\cdot\Phi(a, b)+\pi(q, a, b).$$

Especially,

(2)
$$\begin{cases} \Phi(a, b) \geqq a^b\left(1+\dfrac{1}{a}\right)^{\left[\frac{b}{2}\right]} & \text{for } a \text{ even} \\ \Phi(a, b) \geqq a^b\left(1+\dfrac{1}{2a}-\dfrac{1}{2a^2}\right)^{\left[\frac{b}{2}\right]} & \text{for } a \text{ odd.} \end{cases}$$

The following theorem shows that (2) is not best possible.

THEOREM 4. $\pi(1, 2, 3) = 10$.

As regards the upper bound, we have the two following theorems:

THEOREM 5. *Assume* $a \geqq b(b-1)+1$. *Then*

$$\pi(n, a, b) \leqq n \cdot \frac{3}{2} a \left(1 - \frac{1}{3a}\right) \qquad \text{for } b = 2$$

$$\pi(n, a, b) \leqq n \cdot \frac{(b+1)!}{2^b} a^{b-1} \left(1 - \frac{1}{3a} - \frac{1}{3a^2}\right) \text{ for } b \geqq 3.$$

THEOREM 6. *If* $b \geqq 2$ *then*

$$\pi(n, a, b) \leqq \frac{3}{4} b!\, n a^{b-1} \left(1 - \frac{2}{3a}\right)^{b-2}.$$

Finally, we shall present the result concerning Ramsey's theorem. Following [2], by

(3) $$a \to (b_0, b_1, \ldots, b_{k-1})^r$$

we mean the following assertion:

If X is any set, $|X| \geqq a$ and all unordered r-tuples of distinct elements of X are divided into k classes, then there exists an index j and a set Y, $Y \subset X$, $|Y| \geqq b_j$ such that all r-tuples of its elements belong to the j-th class.

A finite version of Ramsey's theorem asserts that given any positive integers $b_0, b_1, \ldots, b_{k-1}, r$ there exists a positive integer

$$R_r(b_0, b_1, \ldots, b_{k-1})$$

such that $a \geqq R$ implies (3).

THEOREM 7. *Given any positive integer b, there exist the least integers*

$$p_1(b), p_2(b), \ldots, p_{b-1}(b)$$

such that $a_i > p_i(b)$ *for all* $i = 1, 2, \ldots, b-1$ *implies*

$$\pi(a_0, a_1, \ldots, a_{b-1}, b) < R_2(a_0, a_1, \ldots, a_{b-1}).$$

In fact, we have

$$p_k(b) \leqq \binom{b}{k}(b-k)+1.$$

Remark that the last theorem has no quantitative importance as the numbers R are too great.

3. Lemmas

LEMMA 1. $\pi(n_1+n_2, a, b) \geqq \pi(n_1, a, b) + \pi(n_2, a, b)$.

PROOF. Take the systems Σ_1, Σ_2 respective to $\pi(n_1, a, b)$, $\pi(n_2, a, b)$ provided, that $X \in \Sigma_1$, $Y \in \Sigma_2 \Rightarrow X \cap Y = \emptyset$ and consider a system

$$\Sigma_1 \oplus \Sigma_2 = \{Z;\ Z \in \Sigma_1 \text{ or } Z \in \Sigma_2\}.$$

LEMMA 2. $\pi(n, a, b_1+b_2) \geqq \pi(n_1, a_1, b_1)\cdot\pi(n_2, a_2, b_2)$ *where* $n=\min(n_1, n_2)$, $a=\max(n_1, n_2, a_1, a_2)$. *Especially, if* $n \leqq a$ *then*

$$\pi(n, a, b_1+b_2) \geqq \pi(n, a, b_1)\cdot \Phi(a, b_2).$$

PROOF. Take the systems Σ_1, Σ_2 respective to $\pi(n_1, a_1, b_1)$, $\pi(n_2, a_2, b_2)$ provided that $X\in\Sigma_1,\ Y\in\Sigma_2 \Rightarrow X\cap Y=\emptyset$ and consider a system

$$\Sigma_1\otimes\Sigma_2=\{X\cup Y;\ X\in\Sigma_1,\ Y\in\Sigma_2\}.$$

It does not contain any $\Delta(>n)$-system with empty kernel as both "projections" of such a system are $\Delta(>n)$-systems with empty kernels; it does not contain any $\Delta(>a)$-system with non-empty kernel as at least one "projection" of such a system is a $\Delta(>a)$-system (the second one may be fixed).

LEMMA 3. $\pi(n, a, b) \leqq \pi(n-1, a, b)+1+\sum_{j=1}^{b-1}\binom{b}{j}\pi(a-1, a, j)$.

PROOF. Take a system Σ respective to $\pi(n, a, b)$, choose $B\in\Sigma$ and define $\mathfrak{A}_j$, $j=0, 1, \dots, b$ by

$$X\in\mathfrak{A}_j \Leftrightarrow X\in\Sigma \quad\text{and}\quad |X\cap B|=j.$$

Evidently, $|\Sigma|=\sum_{j=0}^{b}|\mathfrak{A}_j|$,

$$|\mathfrak{A}_0| \leqq \pi(n-1, a, b),\quad |\mathfrak{A}_j| \leqq \binom{b}{j}\pi(a-1, a, b-j)\ \text{ for } j=1, \dots, b-1,\quad |\mathfrak{A}_b|=1.$$

LEMMA 4. $\pi(1, a, b) \leqq 1+\sum_{j=1}^{b-1}\binom{b}{j}\pi(a-1, a, j)$.

PROOF. $|\mathfrak{A}_0|=0$ here.

LEMMA 5. *If* $a \geqq b(b-1)+1$ *then*

$$\pi(1, a, b) \leqq 1+\sum_{j=1}^{b-2}\binom{b}{j}\pi(a-1, a, j)+\pi(a-1, a, b-1).$$

PROOF. By the preceding arguments, it suffices to show that

$$|\mathfrak{A}_1| \leqq \pi(a-1, a, b-1). \tag{4}$$

Choose $B=\{n_1, n_2, \dots, n_b\}\in\Sigma$ and define $\mathfrak{A}(n_i)$ by

$$x\in\mathfrak{A}(n_i) \Leftrightarrow X = Y-B;\ Y\in\Sigma,\ Y\cap B=\{n_i\}.$$

Clearly,
$$|\mathfrak{A}_1| = \sum_{j=1}^{n}|\mathfrak{A}(n_j)|.$$

If there is at most one index j such that $\mathfrak{A}(n_j)\neq\emptyset$ then (4) is evident. If

$$|\mathfrak{A}(n_j)| \leqq \pi(b-1, a, b-1)$$

for all j then (4) is implied by $a \geqq b(b-1)+1$ and Lemma 1 as

$$b\pi(b-1, a, b-1) \leqq \pi(b(b-1), a, b-1) \leqq \pi(a-1, a, b-1).$$

Finally, assume

$$|\mathfrak{A}(n_i)| > \pi(b-1, a, b-1), \ \mathfrak{A}(n_j) \neq \emptyset$$

for a couple of distinct indices i, j. Then $\mathfrak{A}(n_i)$ either contains a $\Delta(>a)$-system, which is a contradiction, or a $\Delta(\geqq b)$-system with empty kernel. At the same time, $|Y| = b-1$ for any $Y \in \mathfrak{A}(n_j)$. Hence, given any $Y \in \mathfrak{A}(n_j)$ there is an $X \in \mathfrak{A}(n_i)$ such that $X \cap Y = \emptyset$ in the second case. But this is a contradiction, too, because

$$(X \cup \{n_i\}) \cap (Y \cup \{n_j\}) = \emptyset$$

and Σ is not allowed to contain two independent sets.

To estimate $|\mathfrak{A}_2|$ better, we shall need the following lemma. Starting from it, we shall use now and then usual graph-theoretical notions and omit their definitions.

LEMMA 6. *Let G be a graph with v vertices and e edges. Denote by d the number of the edges incident with any edge in G. If $v \geqq 4$ then*

(5) $$d \leqq 2 \quad \textit{for} \quad e = v$$

(6) $$d \leqq 1 \quad \textit{for} \quad e > v$$

(7) $$d = 0 \quad \textit{for} \quad e \geqq 2v-2.$$

PROOF. (5), (6): Assume $d \geqq 1$. Then there exists an edge $e_0 = (v_1, v_2)$ incident with any edge in G. Denote by (v_i) the degree of a vertex v_i.

If $e > v$ then $(v_1)+(v_2) = e+1 \geqq v+2$ but $(v_1) \leqq v-1$, $(v_2) \leqq v-1$. Hence, $(v_1) \geqq 3$, $(v_2) \geqq 3$. Similarly, if $e = v$ then $(v_1) \geqq 2$, $(v_2) \geqq 2$ and the assumptions $e = v \geqq 4$ $(v_1) = (v_2) = 2$ leads to a contradiction, hence $(v_1) \geqq 3$ or $(v_2) \geqq 3$. Finally, it is easy to see that $(v_1) \geqq 3$, $(v_2) \geqq 3$ implies $a = 1$ and $(v_1) \geqq 3$, $(v_2) \geqq 2$ implies $d \leqq 2$; the respective graphs are on the following figure:

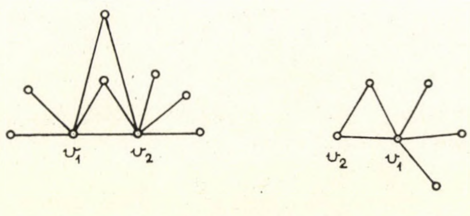

Fig. 1

(7): No full subgraph of G with $v-2$ vertices is empty. Now, we are able to prove the following lemma.

LEMMA 7. *If $a \geqq b(b-1)+1$, $b \geqq 4$ then*

$$\pi(1, a, b) \leqq 1 + \sum_{j=1}^{b-3} \binom{b}{j} \pi(a-1, a, j) + (b-1)\pi(a-1, a, b-2) + \pi(a-1, a, b-1).$$

PROOF. By preceding arguments, it suffices to show

$$|\mathfrak{A}_2| \leqq (b-1)\pi(a-1, a, b-2).$$

Consider a graph $[B, \{X \cap B;\ X \in \mathfrak{A}_2\}]$. If e is any edge of G then a system

$$\mathfrak{A}(e) = \{X - B;\ X \cap B = e\}$$

contains no $\Delta(>a)$-system with non-empty kernel and no $\Delta(\geqq a)$-system with empty kernel. Moreover, if there exists an edge in G disjoint with e then we conclude, similarly as in Lemma 5, that $\mathfrak{A}(e)$ contains no $\Delta(>b-2)$-system with empty kernel. Thus, using the notation of Lemma 6, one has

$$|\mathfrak{A}_2| \leqq d\cdot\pi(a-1, a, b-2) + (e-d)\cdot\pi(b-2, a, b-2) =$$
$$= e\cdot\pi(b-2, a, b-2) + d\big(\pi(a-1, a, b-2) - \pi(b-2, a, b-2)\big).$$

For $b \geqq 4$, one has by Lemma 6

$$|\mathfrak{A}_2| \leqq \operatorname{Max}\left\{\begin{matrix} (b-1)\,\pi(a-1, a, b-2) \\ 2\pi(a-1, a, b-2) + (b-2)\pi(b-2, a, b-2) \\ \pi(a-1, a, b-2) + 2(b-2)\pi(b-2, a, b-2) \\ \binom{b}{2}\,\pi(b-2, a, b-2) \end{matrix}\right\}.$$

By Lemma 1, we have $\pi(a-1, a, b-2) \geqq (b-2)\pi(b-2, a, b-2)$ as $a-1 \geqq b(b-1) > (b-2)^2$. Moreover, $b-1 \geqq 3$ and $(b-1)(b-2) \geqq \binom{b}{2}$ for $b \geqq 4$. Hence, the above maximum is equal to $(b-1)\pi(a-1, a, b-2)$.

LEMMA 8. $\pi(n, a, b) \leqq \dfrac{n}{2}\left(1 + \pi(1, a, b) + \sum\limits_{j=1}^{b-1} \binom{b}{j}\pi(a-1, a, j)\right)$.

PROOF. Let $\Sigma = \{X_\nu;\ \nu \in N\}$ be a system respective to $\pi(n, a, b)$ and $\Sigma_0 = \{X_\nu;\ \nu \in N_0\}$ a maximal Δ-system with empty kernel contained in Σ. Clearly, $|N_0| \leqq n$. One may divide Σ into (not necessarily disjoint) classes $\mathfrak{B}_\nu$, $\nu \in N_0$ by

$$X \in \mathfrak{B}_\nu \Leftrightarrow X \cap X_\nu \neq \emptyset$$

and define

$$\mathfrak{B}_\nu^* = \mathfrak{B}_\nu - \bigcup_{\mu \neq \nu} \mathfrak{B}_\mu.$$

Evidently, any set contained in $\mathfrak{B}_\nu - \mathfrak{B}_\nu^*$ meets at least two sets in Σ_0. Hence we have

$$|N| \leqq \sum_{\nu \in N_0}\left(|\mathfrak{B}_\nu^*| + \frac{1}{2}|\mathfrak{B}_\nu - \mathfrak{B}_\nu^*|\right) = \frac{1}{2}\sum_{\nu \in N_0}\left(|\mathfrak{B}_\nu^*| + |\mathfrak{B}_\nu|\right),$$

$$|\mathfrak{B}_\nu^*| \leqq \pi(1, a, b), \quad |\mathfrak{B}_\nu| \leqq 1 + \sum_{j=1}^{b}\binom{b}{j}\pi(a-1, a, j),$$

and comparing these inequalities one gets the desired result.

LEMMA 9. *Given any positive integer b and an integer k, $0<k<b$ there exists a least number $p_k(b)$, with the following property:*
"If Σ is a $(>p, b)$-system provided that

(8) $$A, B\in\Sigma, \quad A\neq B \Rightarrow |A\cap B| = k$$
then Σ is a Δ-system."
In fact,

$$p_k(b) \leqq \binom{b}{k}(b-k)+1.$$

PROOF. Let Σ be an (a, b)-system fulfilling (8) but not a Δ-system. It suffices to show that

(9) $$a \leqq \binom{b}{k}(b-k)+1.$$

Choose $B\in\Sigma$ and define $\mathfrak{A}(B^*)$ for any $B^*\subset B, |B^*|=k$ by

$$X\in\mathfrak{A}(B^*) \Leftrightarrow X\in\Sigma \quad \text{and} \quad X\cap B=B^*.$$

To prove (9), it suffices to show that

$$|\mathfrak{A}(B^*)| \leqq b-k$$

for all B^*. Given any B^* there is $C\in\Sigma$ such that $C\notin\mathfrak{A}(B^*)$. If $X\in\mathfrak{A}(B^*)$ then

$$C\cap X = (C\cap B^*)\cup(C\cap(X-B)), \quad |C\cap X| = k, \quad |C\cap B^*|<k.$$

Moreover, if $X, Y\in\mathfrak{A}(B^*)$ and $X\neq Y$ then

$$(X-B)\cap(Y-B) = \varnothing.$$

Thus, the sets $C\cap(X-B)$ where $X\in\mathfrak{A}(B^*)$ are non-empty and mutually disjoint subsets of $C-B$.
But $|C-B| = b-k$ and (10) follows.

4. Proofs

PROOF OF THEOREM 1. The case $b=2$ is a case of graphs and the problem is to find a graph with maximal number of edges containing neither $>n$ independent edges nor a vertex of degree $>a$.
Evidently, a graph with $2n+1$ vertices does not contain $n+1$ independent edges.
Given a even, consider a complete graph with $a+1$ vertices: it has $\binom{a+1}{2} = a\left[\frac{a+1}{2}\right]+\left[\frac{a}{2}\right]$ edges, all vertices of degree a and does not contain $\frac{a}{2}+1 = \left[\frac{a+1}{2}\right]+1$ independent edges.
Given a odd, consider a graph with $a+1$ vertices of degree a and one vertex of degree $a-1$ (such a graph exists, indeed — see [3]) it has $\frac{1}{2}\left(a(a+1)+(a-1)\right) =$

$=a\left[\frac{a+1}{2}\right]+\left[\frac{a}{2}\right]$ edges, no vertex of degree $>a$ and does not contain $\frac{a+1}{2}+1$ independent edges.

Thus, we have proved

$$\pi\left(\left[\frac{a+1}{2}\right], a, 2\right) \geqq a\left[\frac{a+1}{2}\right]+\left[\frac{a}{2}\right].$$

A star of degree a shows that

$$\pi(1, a, 2) \geqq a.$$

Now, by Lemma 1,

$$\pi(n, a, 2) \geqq \left[\frac{n}{\left[\frac{a+1}{2}\right]}\right]\left(a\left[\frac{a+1}{2}\right]+\left[\frac{a}{2}\right]\right)+\left(n-\left[\frac{a+1}{2}\right]\left[\frac{n}{\left[\frac{a+1}{2}\right]}\right]\right)a =$$

$$= na+\left[\frac{a}{\left[\frac{a+1}{2}\right]}\right]\left[\frac{a}{2}\right].$$

PROOF OF THEOREM 2. $n=1$: By Lemmas 4, 5, $\pi(1, 2, 2) \leqq 3$ and $\pi(1, a, 2) \leqq a$ for $a \geqq 3$.

$a=2$: By Lemmas 4, 3, one has $\pi(1, 2, 2) \leqq 3$ $\pi(n, 2, 2) \leqq 3+\pi(n-1, 2, 2)$.

$n=2$: We shall proceed by an exhaustion; it may be helpful to divide all connected graphs containing at most 2 independent edges into the following five sections (evidently, these graphs contain no open path of length 5):

α. G is a tree;
β. the longest circuit of G is a triangle; G contains no open path of length 4;
γ. the longest circuit of G is a triangle; G contains an open path of length 4;
δ. the longest circuit of G is a 4-gon;
ε. the longest circuit of G is a 5-gon.

Denote by E a set of edges of G and by a the maximal degree of its vertices.

The case $a=2$ was investigated already, thus, one may suppose $a \geqq 3$. By simple reflections (the respective graphs are on Fig. 2)

α. $|E| \leqq 2a$;

β. $|E| \leqq a+1$;

γ. $|E| \leqq \max(a+3, 6, 2a-1)$;

δ. $|E| \leqq \max(a+3, 2a)$;

ε. $|E| \leqq \min\left(\left[\frac{5a}{2}\right], 10\right)$.

Moreover, $|E| \leqq 2\pi(1, a, 2)=2a$ in the disconnected case. Hence, $\pi(3, 2, 2) \leqq 7$, $\pi(4, 2, 2) \leqq 10$, $\pi(a, 2, 2) \leqq 2a$ for $a \geqq 5$.

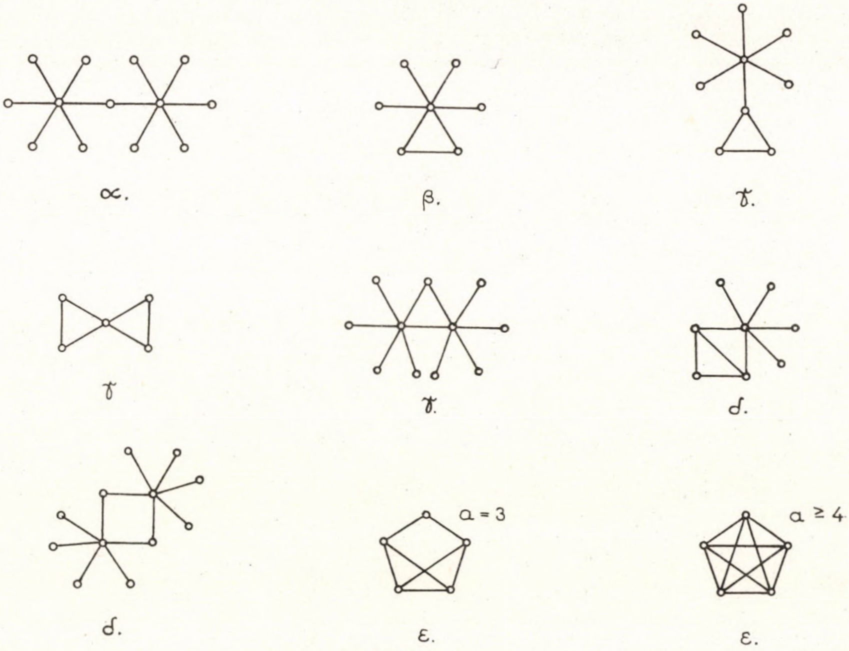

Fig. 2

PROOF OF THEOREM 3. It is a simple consequence of Theorem 1 and Lemmas 1, 2.

PROOF OF THEOREM 4. The following (10, 3)-system contains neither a $\Delta(>1)$-system with empty kernel nor a $\Delta(>2)$-system with non-empty kernel:

$$123,\ 145,\ 156,\ 256,\ 264,\ 364,\ 345,\ 124,\ 235,\ 316$$

where $abc=\{a, b, c\}$. Hence, $\pi(1, 2, 3)\geqq 10$. To prove $\pi(1, 2, 3)\leqq 10$ it suffices to prove.

$$|\mathfrak{A}_1|\leqq 6,$$

by Lemmas 3, 4. Consider the system $G=\{X-B;\ X\in\mathfrak{A}_1\}$ as a multigraph (i.e. a graph where multiple edges are allowed) with the following properties:

(α) any edge in G is either single or double;
(β) G does not contain two independent edges;
(γ) any single edge is coloured by one colour;
(δ) any double edge is coloured by two distinct colours;
(ε) there are at most three colours in the whole G;
(ϑ) no "monochromatic graph" contains a vertex of degree >2.

Clearly, the three "monochromatic graphs" are the systems $\mathfrak{A}(n_i)$ where $B=\{n_1, n_2, n_3\}$ and $\mathfrak{A}(n_i)=\{X-B;\ X\in\Sigma,\ X\cap B=\{n_i\}\}$.

If no $\mathfrak{A}(n_i)$ contains a triangle then $|\mathfrak{A}(n_i)| \leqq 2$, which implies $|\mathfrak{A}_1| \leqq 6$. If there is an $\mathfrak{A}(n_i)$ containing a triangle then G has precisely three vertices and $|\mathfrak{A}_1| \leqq 6$, too.

PROOF OF THEOREM 5. By Lemmas 7, 5 one gets

$$\pi(n, a, b) \leqq n\left(\frac{b+1}{2}\pi(a-1, a, b-1)+1+\sum_{j=1}^{b-2}\binom{b}{j}\pi(a-1, a, j)\right)$$

which gives

$$\pi(n, a, 2) \leqq n(\tfrac{3}{2}a-\tfrac{1}{2}), \quad \pi(n, a, 3) \leqq n(3a^2-a-1).$$

For $b \geqq 4$ one has, by Lemmas 7 and 6,

$$\pi(n, a, b) \leqq n\left(\frac{b+1}{2}\pi(a-1, a, b-1)+\right. \tag{13}$$

$$\left.+\frac{(b-1)(b+2)}{4}\pi(a-1, a, b-2)+1+\sum_{j=1}^{b-3}\binom{b}{j}\pi(a-1, a, j)\right).$$

For large b, it would be rather difficult to estimate $\pi(n, a, b)$ by (13), directly. However, one may deduce an explicit estimation loosing some sharpness from it.

Define

$$g(1) = 1, \quad g(2) = \frac{3}{2}a-\frac{1}{2}, \quad g(3) = 3a^2-a-1,$$

$$g(b) = \frac{b+1}{2}ag(b-1) \quad \text{for} \quad b \geqq 4. \tag{14}$$

By induction, we shall prove that

$$\pi(n, a, b) \leqq n \cdot g(b).$$

If $b=1, 2, 3$ then this inequality follows from (11), (12). Now, take a $\beta \geqq 4$ and suppose (14) for all $b<\beta$. (13) implies

$$\frac{\pi(n, a, \beta)}{n} \leqq 1+(a-1)\left(\frac{\beta+1}{2}g(\beta-1)+\frac{(\beta-1)(\beta+2)}{4}g(\beta-2)+\sum_{j=1}^{\beta-3}\binom{\beta}{j}g(j)\right) =$$

$$= g(\beta)+\sum_{j=1}^{\beta-2}R_j$$

where

$$R_1 = -\binom{\beta}{1}g(1)+1,$$

$$R_j = -\binom{\beta}{j}g(j)+a\binom{\beta}{j-1}g(j-1) \quad \text{for} \quad j = 2, \dots, \beta-3,$$

$$R_{\beta-2} = -\frac{\beta+1}{2}g(\beta-1)+(a-1)\frac{(\beta-1)(\beta+2)}{4}g(\beta-2)+a\binom{\beta}{\beta-3}g(\beta-3).$$

It suffices to show $R_0 \leqq 0$ for all $j=1, 2, \dots, \beta-2$.

1. This fact is trivial if $j=1$.
2. Consider an integer j, $2 \leqq j \leqq \beta-3$. We have

$$\binom{\beta}{j} = \binom{\beta}{j-1} \cdot \frac{\beta-j+1}{j} \geqq \binom{\beta}{j-1} \cdot \frac{4}{j},$$

and

$$R_j \leqq \binom{\beta}{j-1}\left(-\frac{4}{j}\, g(j)+ag(j-1)\right).$$

2. 1. If $j=2$ then

$$-\frac{4}{j}\, g(j)+ag(j-1) = \left(-2a+\frac{2}{3}\right)+a = -a+\frac{2}{3} < 0$$

hence, $R_j < 0$.

2. 2. If $j=3$ then

$$-\frac{4}{j}\, g(j)+ag(j-1) = -\left(4a^2-\frac{4}{3}\, a-\frac{4}{3}\right)+a\left(\frac{3}{2}\, a-\frac{1}{2}\right) = \frac{1}{6}(-15a^2+5a+8) < 0$$

hence, $R_j < 0$.

2. 3. If $j \geqq 4$ then

$$-\frac{4}{j}\, g(j)+ag(j-1) = ag(j-1)\left(-\frac{4}{j}\cdot\frac{j+1}{2}+1\right) = ag(j-1)\left(-1-\frac{2}{j}\right) < 0$$

hence, $R_j < 0$.

3. 1. If $\beta=4$ then

$$R_{\beta-2} = -\frac{5}{2}\, g(3)+(a-1)\,\frac{9}{2}\, g(2)+a\binom{4}{1} g(1) =$$

$$= -\frac{5}{2}(3a^2-a-1)+\frac{9}{2}(a-1)\left(\frac{3}{2}\, a-\frac{1}{2}\right)+4a = \frac{1}{4}(-3a^2-10a+19) < 0$$

(as $a \geqq \beta(\beta-1)+1 = 13$) hence, $R_{\beta-2} < 0$.

3. 2. If $\beta \geqq 5$ then

$$-\frac{\beta+1}{2}\, g(\beta-1)+a\,\frac{(\beta-1)(\beta+2)}{4}\, g(\beta-2) =$$

$$= \left(-\frac{\beta+1}{2}\cdot\frac{\beta}{2}+\frac{(\beta-1)(\beta+2)}{4}\right) ag(\beta-2) = -\frac{a}{2}\, g(\beta-2) \leqq -\frac{\beta(\beta+1)+1}{2}\, g(\beta-2)$$

and

$$R_{\beta-2} \leqq \left(-\frac{\beta(\beta-1)+1}{2}-\frac{(\beta-1)(\beta+2)}{4}\right) g(\beta-2)+a\binom{\beta}{3} g(\beta-3) =$$

$$= -\frac{\beta(3\beta-1)}{4}\, g(\beta-2)+a\binom{\beta}{3} g(\beta-3).$$

3.2.1. If $\beta = 5$ then

$$R_{\beta-2} \leqq -\frac{35}{2} g(3)+a\binom{5}{3} g(2) =$$

$$= -\frac{35}{2}(3a^2-a-1)+10a\left(\frac{3}{2}a-\frac{1}{2}\right) = \frac{1}{2}(-75a^2-45a+35) < 0.$$

3.2.2. If $\beta \geqq 6$ then

$$R_{\beta-2} \leqq \left(-\frac{\beta(3\beta-1)}{4}a\frac{\beta-1}{2}+a\frac{\beta(\beta-1)(\beta-2)}{6}\right) g(\beta-3) =$$

$$= -\frac{5}{4}\binom{\beta+1}{3} ag(\beta-3) < 0.$$

Proof of Theorem 6. Define $k(1)=l(1)=1$, $k(2)=l(2)=\frac{3}{2}a$,

$$k(b) = ab\left(1-\frac{2}{3a}\right)k(b-1) = \frac{3}{4}b!\,a^{b-1}\left(1-\frac{2}{3a}\right)^{b-2} \quad \text{for} \quad b \geqq 3,$$

$$l(b) = 1+(a-1)\sum_{j=1}^{b-1}\binom{b}{j} l(j) \quad \text{for} \quad b \geqq 3.$$

It suffices to prove

(15) $$\pi(n, a, b) \leqq nl(b) \leqq nk(b)$$

for any b. We shall proceed by induction. Evidently, (15) holds for $b=1, 2$ (in the second case, see Theorems 2, 5); take $\beta \geqq 3$ and assume (15) for any $b<\beta$. Then by Lemmas 3, 4

$$\pi(n, a, \beta) \leqq n\left(1+(a-1)\sum_{j=1}^{\beta-1} l(j)\right) = nl(\beta).$$

Remark that

$$l(\beta)-l(\beta-1) = (a-1)\left(\beta l(\beta-1)+\sum_{j=1}^{\beta-2}\binom{\beta-1}{j-1} l(j)\right)$$

$$l(\beta-1)-1 = (a-1)\sum_{j=1}^{\beta-2}\binom{\beta}{j} l(j) =$$

$$= (a-1)\sum_{j=1}^{\beta-2}\frac{\beta}{j}\binom{\beta-1}{j-1} l(j) \geqq \frac{3}{\beta-2}(a-1)\sum_{j=1}^{\beta-2}\binom{\beta-1}{j-1} l(j)$$

as $j \leqq \beta-2$, $\beta \geqq 3$ in the very last inequality.

Hence,

$$l(\beta) \leqq l(\beta-1)+(a-1)\beta l(\beta-1)+\frac{\beta-2}{3}(l(\beta-1)-1) \leqq$$

$$\leqq a\beta\left(1-\frac{2}{3a}\right) l(\beta-1) \leqq k(\beta).$$

PROOF OF THEOREM 7. Let Σ be an (a, b)-system where $a \geqq R_2(a_0, a_1, \ldots, a_{b-1})$. Divide all unordered couples of distinct sets of Σ into b classes $\mathfrak{C}_0, \mathfrak{C}_1, \ldots, \mathfrak{C}_{b-1}$ by

$$(X, Y) \in \mathfrak{C}_j \Leftrightarrow |X \cap Y| = j.$$

By Ramsey's theorem, there is an index i and an (a_i, b)-system Σ' contained in Σ such that

$$A, B \in \Sigma', \ A \neq B \Rightarrow |A \cap B| = i.$$

If $i=0$ then Σ' is evidently a Δ-system. If $i>0$ then $a_i > p_i(b)$ and Σ' is a Δ-system, by Lemma 9.

5. Remarks

1. The following theorem does not concern Δ-systems, directly. However, concerning independent edges and degrees of vertices, it may be of some interest.

THEOREM 8. *Let G be a graph, V the set of its vertices and, for any $v \in V$, $d(v)$ denotes the degree of v. If there is $\{v_0, v_1, \ldots, v_{n-1}\} \subset V$ such that*

$$d(v_i) \geqq n+i \tag{16}$$

then G contains n independent edges.

Moreover, if one replaces the numbers $n+i$ in (16) by any numbers $\delta(i)$ such that $\delta(j) < n+j$ for at least one j, then the above implication fails.

PROOF. We shall find a set $\{x_0, x_1, \ldots, x_{n-1}\} \subset V$ such that $\{v_0, v_1, \ldots, v_{n-1}\} \cap \cap \{x_0, x_1, \ldots, x_{n-1}\} = \emptyset$ and $(v_i, x_i) \in E$ (the set of edges of 6) for all i. First, $d(v_0) = n$ and $|\{v_1, \ldots, v_{n-1}\}| = n-1$. Thus, there is a vertex x_0 such that $\{x\}_0 \cap \cap \{v_0, \ldots, v_{n-1}\} = \emptyset$ and $(v_0, x_0) \in E$. Further, assume that there is a set $\{x_0, x_1, \ldots, x_k\}$ such that $\{x_0, x_1, \ldots, x_k\} \cap \{v_0, v_1, \ldots, v_{n-1}\} = \emptyset$ and $(v_i, x_i) \in E$ for all $i \leqq k$. Then

$$|\{x_0, x_1, \ldots, x_k, v_0, v_1, \ldots v_{n-1}\}| = n+k+1 \quad \text{and} \quad d(v_{k+1}) \geqq n+k+1.$$

Thus, there is a vertex x_{k+1} such that $(v_{k+1}, x_{k+1}) \in E$ and

$$\{x_{k+1}\} \cap \{x_0, x_1, \ldots, x_k, v_0, v_1, \ldots v_{n-1}\} = \emptyset.$$

Now, we shall prove the second part.

Given any positive integers $\delta(i)$, $i=0, 1, \ldots, n-1$ such that $\delta(0) \leqq \delta(1) \leqq \ldots \ldots \leqq \delta(n-1)$ and $\delta(j) < n+j$ for certain j there is a graph G with the following properties:

(i) G does not contain n independent edges,

(ii) there is a set $\{v_0, \ldots, v_{n-1}\}$ of vertices of G such that $d(v_i) \geqq \delta(v_i)$ for all i.

We shall construct G in the following way:

Take a complete graph of $n+j$ vertices and choose a $(n-j-1)$-point subset of its vertices, say $V^* = \{v_{j+1}, v_{j+2}, \ldots, v_{n-1}\}$. Join each $v_i \in V^*$ by edges with $\delta(i) - (n+j-1)$ new vertices.

Indeed, this graph, although connected, "consists" of $n-j-1$ stars and a complete graph of $2j+1$ vertices. The last "component" contains at most j independent edges and each of preceding ones at most one.

Moreover, $d(v_i) = \delta(i)$ for $i > j$, $d(v_i) = n+j-1 \geqq \delta(i)$ for $i \leqq j$.

2. Finally, we shall give some better estimations for $p_k(b)$.

THEOREM 9. $p_2(4) \leqq 7$, $p_2(5) \leqq 13$, $p_2(b) \leqq \binom{b}{2}\left[\frac{b-2}{2}\right]+1$ *for* $b \geqq 6$.

PROOF. The system $E=\{X\cap B;\ X\in\Sigma,\ X\neq B\}$ may be considered as a set of edges of a graph. If $e\in E$ then $|\mathfrak{A}(e)| \leqq b-2$ where

$$\mathfrak{A}(e)=\{X\in\Sigma;\ X\cap B=e\}.$$

Moreover, if there is an $e'\in E$ such that $e\cap e'=\emptyset$, then by a similar reflection, $|\mathfrak{A}(e)| \leqq \left[\frac{b-2}{2}\right]$.

Hence, using the notation of Lemma 6,

$$|\Sigma| \leqq d(b-2)+(e-d)\left[\frac{b-2}{2}\right]+1.$$

If $b\geqq 4$, we have by Lemma 6,

$$|\Sigma| \leqq \operatorname{Max}\left\{\begin{matrix} (b-1)(b-2)+1 \\ 2(b-2)+(b-2)\left[\frac{b-2}{2}\right]+1 \\ (b-2)+2(b-2)\left[\frac{b-2}{2}\right]+1 \\ \binom{b}{2}\left[\frac{b-2}{2}\right]+1 \end{matrix}\right\} =$$

$$= \operatorname{Max}\left\{(b-1)(b-2)+1,\ \binom{b}{2}\left[\frac{b-2}{2}\right]+1\right\}.$$

THEOREM 10. $p_1(3)=7$, $p_1(4)=13$, $p_1(5)=21$, $p_1(6)=31$, $p_1(8)=57$, $p_1(9)=73$, $p_1(10)=91$, $p_1(12)=133$, $p_2(4)=7$, $p_2(5)=13$, $p_k(k+1) = k+2$ *for every positive integer* k.

PROOF. By Theorems 7, 9, it suffices to prove $p_k(b)\geqq n$ for all the above numbers.

Remember that a (v, k)-system Σ is called a (v, k, λ)-configuration if

(i) $\left|\bigcup_{X\in\Sigma} X\right| \leqq v$,

(ii) $X, Y\in\Sigma,\ X\neq Y \Rightarrow |X\cap Y|=\lambda$,

(iii) integers v, k, λ fulfill the condition $0<\lambda<k<v-1$. Clearly, an existence of a (v, k, λ)-configuration implies $p_k(k)\geqq v$ as the configuration is not a Δ-system, by (i). Denote by $[d_1, \dots d_k]_v$ (d_i, v integers, $0\leqq d_1<d_2<\dots<d_k<v$) a system

$$\{\{d_1+e, \dots, d_k+e\};\ e\in\mathbf{Z}_v\}$$

where $\mathbf{Z}_v$ is a ring of integers modulo v. Then $[d_1, d_2, \dots d_k]_v$ is a (v, k, λ)-configuration if and only if any integer $a \not\equiv 0 \pmod v$ may be expressed in precisely λ ways in the form

$$d_i - d_j \equiv a \pmod v.$$

Remember that then $\{d_1, d_2, \dots, d_k\}$ is called a perfect difference set; it is easy to see that

$$\lambda = \frac{v-1}{k(k-1)}.$$

There exist the following (v, k, λ)-configurations (see [4]):

$[0, 1, 3]_7$

$[0, 1, 3, 9]_{13}$

$[0, 1, 4, 14, 16]_{21}$

$[0, 1, 3, 8, 12, 18]_{31}$

$[0, 1, 3, 13, 32, 36, 43, 52]_{57}$

$[0, 1, 3, 7, 15, 31, 36, 54, 63]_{73}$

$[0, 1, 3, 9, 27, 49, 56, 61, 77, 81]_{91}$

$[0, 1, 3, 12, 20, 34, 38, 81, 88, 94, 104, 109]_{133}$

$[0, 1, 2, 4]_7$.

To prove $p_2(5) \geqq 13$ consider the following system:

12345	12ABC	13ADG	14AEK	15CEG
	12DEF	13BEH	14DHC	15FHA
	12GHK	13CFK	14GBF	15KBD.

Finally, to show $p_k(k+1) \geqq k+2$, consider a system of all $(k+1)$-point subsets of a $(k+2)$-point set.

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ON AN INEQUALITY OF P. TURÁN

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Both the result and the proof of the following paper have been improved by remarks of Mr. A. A. Balkema. I want to express my thanks for his valuable suggestions.

1. In his paper "A remark on linear differential equations", [7], P. TURÁN stresses the importance of certain inequalities in the theory of linear differential equations with constant coefficients. As a contribution he proves the following theorem:

Let

(1. 1) $$y^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_0 y = 0, \quad y=y(z),$$

a_ν complex numbers, be a differential equation, let further

(1. 2) $$\varphi(\omega) = \omega^n+a_{n-1}\omega^{n-1}+\ldots+a_0$$

be the corresponding characteristic polynomial.

Let Λ be a real number such that all zeros $\omega_1, \omega_2, \ldots, \omega_n$ of the polynomial (1. 2) are in the halfplane

(1. 3) $$\operatorname{Re} z \geqq \Lambda.$$

Then for fixed α, β and δ with

(1. 4) $$\alpha>\beta, \quad \delta>0$$

the inequality

(1. 5) $$\int_\alpha^{\alpha+\delta} |y(t)|^2\,dt \geqq \left(\frac{\delta}{2e(2\alpha-2\beta+\delta)}\right)^{n^2} e^{-2(\alpha-\beta+\delta)|\Lambda|} \int_\beta^{\beta+\frac{1}{2}\delta} |y(t)|^2\,dt$$

holds for all solutions $y(t)$ of (1. 1).

Turán conjectures that the exponent n^2 on the right-hand side can be replaced by cn, where c is a positive constant.

As a special case of a more general theorem we shall deduce the inequality

(1. 6) $$\int_\alpha^{\alpha+\delta} |y(t)|^2\,dt \geqq \left(\frac{\delta}{2e(\alpha-\beta+\delta)}\right)^{2n} e^{-2(\alpha-\beta+\delta)|\Lambda|} \int_\beta^{\beta+\delta} |y(t)|^2\,dt$$

for all solutions $y(t)$ of (1. 1).

2. We shall prove the following

THEOREM 1. *Let*

$$y^{(n)}+a_{n-1}y^{(n-1)}+\dots+a_0y=0 \tag{2. 1}$$

be a homogeneous linear differential equation with constant coefficients, and let Λ be a real number such that the zeros of the characteristic polynomial lie in the halfplane

$$\operatorname{Re} z \geqq \Lambda. \tag{2. 2}$$

Let $\alpha, \beta, \delta_1, \delta_2, p$ be real numbers, $\delta_1, \delta_2, p>0$, and such that

$$\alpha \geqq \beta+\delta_2. \tag{2. 3}$$

Put

$$\delta_0=\begin{cases}\delta_1 & \text{if } \Lambda \leqq 0,\\ -\delta_2 & \text{otherwise.}\end{cases} \tag{2. 4}$$

Then we have for every solution $y(t)$ of (2. 1)

$$\int_{\alpha}^{\alpha+\delta_1}|y(t)|^p\,dt \geqq \frac{(6n)^{p/2}\delta_1}{2\delta_1+n\delta_2}\left(\frac{\delta_1}{2e(\alpha-\beta+\delta_1)}\right)^{np} e^{p\Lambda(\alpha-\beta+\delta_0)}\int_{\beta}^{\beta+\delta_2}|y(t)|^p\,dt. \tag{2. 5}$$

If we take $p=2$ and $\delta_1=\delta_2=\delta>0$ we obtain for $\beta \leqq \alpha-\delta$

$$\int_{\beta}^{\beta+\delta}|y(t)|^2\,dt \leqq \frac{1}{2}\left(\frac{2e(\alpha-\beta+\delta)}{\delta}\right)^{2n} e^{2|\Lambda|(\alpha-\beta+\delta)}\int_{\alpha}^{\alpha+\delta}|y(t)|^2\,dt. \tag{2. 6}$$

Hence (1. 6) is true for $\beta \leqq \alpha-\delta$.

If $\alpha-\delta<\beta \leqq \alpha$ we have $[\beta, \beta+\delta]\subset[\alpha-\delta, \alpha]\cup[\alpha, \alpha+\delta]$. One easily sees that (2. 6) also holds for $\beta=\alpha$ since $n\geqq 1$. Applying inequality (2. 6) with $\beta=\alpha-\delta$ and $\beta=\alpha$ respectively and adding, we find that (1. 6) holds also in the case $\alpha-\delta<$ $<\beta \leqq \alpha$. Hence (1. 6) holds for arbitrary α, β and δ with $\alpha \geqq \beta$, $\delta>0$.

Suppose that the two reals Λ_1 and Λ_2 are such that

$$\Lambda_1 \leqq \operatorname{Re}\omega_\varkappa \leqq \Lambda_2, \qquad \varkappa=1, 2, \dots, n.$$

Then our theorem enables us to find a non-trivial upper and lower bound for the quotient

$$\int_{\alpha}^{\alpha+\delta_1}|y(t)|^p\,dt \Big/ \int_{\beta}^{\beta+\delta_2}|y(t)|^p\,dt \tag{2. 7}$$

only depending on $\alpha, \beta, \delta_1, \delta_2, p, \Lambda_1$ and Λ_2, even if the condition (2. 3) is not fulfilled.

In order to find an upper bound we define three intervals $[\gamma_i, \gamma_i^*]$, $i=1, 2, 3$, by

$$[\gamma_1, \gamma_1^*]=(-\infty, \beta]\cap[\alpha, \alpha+\delta_1],$$

$$[\gamma_2, \gamma_2^*]=[\beta, \beta+\delta_2]\cap[\alpha, \alpha+\delta_1],$$

and

$$[\gamma_3, \gamma_3^*]=[\beta+\delta_2, \infty)\cap[\alpha, \alpha+\delta_1].$$

The union of these intervals is equal to $[\alpha, \alpha+\delta_1]$. Some of them may be empty. First we apply our theorem to find an upper bound for

$$\int_{\gamma_1}^{\gamma_1^*} |y(t)|^p \, dt \Big/ \int_{\beta}^{\beta+\delta_2} |y(t)|^p \, dt.$$

For the integral over the second segment, $[\gamma_2, \gamma_2^*]$, we have a trivial estimation, viz.

$$\int_{\gamma_2}^{\gamma_2^*} |y(t)|^p \, dt \Big/ \int_{\beta}^{\beta+\delta_2} |y(t)|^p \, dt \leqq 1.$$

We apply our theorem once more in order to find an upper estimate for

$$\int_{-\gamma_3^*}^{-\gamma_3} |y(-t)|^p \, dt \Big/ \int_{-(\beta+\delta_2)}^{-\beta} |y(-t)|^p \, dt.$$

Since this quotient is equal to

$$\int_{\gamma_3}^{\gamma_3^*} |y(t)|^p \, dt \Big/ \int_{\beta}^{\beta+\delta_2} |y(t)|^p \, dt,$$

we find an upper bound for (2. 7) by adding the three upper bounds.

In a similar way one finds a lower bound for (2. 7) by computing an upper bound for the inverse quotient

$$\int_{\beta}^{\beta+\delta_2} |y(t)|^p \, dt \Big/ \int_{\alpha}^{\alpha+\delta_1} |y(t)|^p \, dt.$$

3. We shall base the proof of theorem 1 upon the following generalization of Turán's first main theorem (see [4] p. 104 or [5] theorem 2. 1):

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be k complex numbers. Let $P_1, P_2, \ldots, P_k$ be k polynomials of respective degrees $\varrho_1-1, \varrho_2-1, \ldots, \varrho_k-1$. Put $n=\sum_{\varkappa=1}^{k} \varrho_k$. Then for every integer $m \geqq 0$ there is an integer v with $m+1 \leqq v \leqq m+n$ such that

$$(3.\ 1) \qquad \left|\sum_{\varkappa=1}^{k} P_\varkappa(v)\alpha_\varkappa^v\right| \geqq \left\{\sum_{h=1}^{n} \binom{m+h-1}{h-1} 2^{h-1}\right\}^{-1} \left|\sum_{\varkappa=1}^{k} P_\varkappa(0)\right| \min_{j=1,\ldots,k} |\alpha_j|^v.$$

Theorem 1 may also be proved by using Turán's first main theorem itself, [1], [3], [6] p. 38, instead of this generalization. One then has to take an appropriate limit at the end of the proof. Compare [7].

As a first step we prove the following

LEMMA. *Let A, p and t_1 be real numbers, $p>0$, and m a non-negative number. Let $y(t)$ be an arbitrary solution of* (2. 1) *and let the zeros of the corresponding characteristic polynomial* (1. 2) *lie in the halfplane* $\operatorname{Re} z \geqq A$. *Then*

$$(3.\ 2) \quad |y(t_1)|^p \leqq \frac{1}{(6n)^{p/2}} \left(\frac{2e(m+n)}{n}\right)^{np} \sum_{\mu=m+1}^{m+n} |y(t_1+\mu)|^p \max_{v=m+1,\ldots,m+n} e^{-2Av}.$$

PROOF. We have

$$\sum_{h=1}^{n} 2^{h-1}\binom{m+h-1}{h-1} \leqq \binom{m+n-1}{n-1}\sum_{h=1}^{n} 2^{h-1} \leqq \binom{m+n}{n} 2^n \leqq \frac{(2(m+n))^n}{n!}.$$

Since

$$n! > \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{([2], Ch. VII, Appendix)}$$

it follows that

$$\sum_{h=1}^{n} 2^{h-1}\binom{m+h-1}{h-1} \leqq \frac{1}{\sqrt{2\pi n}}\left(\frac{2e(m+n)}{n}\right)^n. \tag{3. 3}$$

If $y(z)$ is a solution of (1. 1) then so is $y(z+t_1)$. Hence we can write $y(z+t_1)$ as a general exponential polynomial

$$\sum_{\varkappa=1}^{k} P_\varkappa(z) e^{\omega_\varkappa z}.$$

Here $\omega_1, \omega_2, \ldots, \omega_k$ denote the distinct zeros of the characteristic polynomial (1.2). If their respective multiplicities are $\varrho_1, \varrho_2, \ldots, \varrho_k$, then $P_\varkappa$ is a polynomial of degree $\leqq \varrho_x^{-1}$, $\varkappa = 1, 2 \ldots, k$. Further $\sum_{\varkappa=1}^{k} \varrho_\varkappa = n$. The $\omega_\varkappa$'s are in the halfplane (2. 2). Hence

$$|e^{\omega_\varkappa}| \geqq e^{A}, \qquad \varkappa = 1, 2, \ldots, k. \tag{3.4}$$

From the above mentioned generalization of Turán's first main theorem, using also (3. 3) and (3. 4), we see that there exists an integer v with $m+1 \leqq v \leqq m+n$ such that

$$|y(t_1+v)| = \left|\sum_{\varkappa=1}^{k} P_\varkappa(v) e^{\omega_\varkappa v}\right| \geqq \left\{\sum_{h=1}^{n} 2^{h-1}\binom{m+h-1}{h-1}\right\}^{-1} \left|\sum_{\varkappa=1}^{k} P_\varkappa(0)\right| \min_{j=1,\ldots,k} |e^{\omega_j}|^v \geqq$$

$$\geqq \sqrt{2\pi n}\left(\frac{n}{2e(m+n)}\right)^n |y(t_1)| e^{Av}.$$

Hence, using the monotony of the functions $x^{-p/2}$ and e^{-pAx} for $x > 0$,

$$|y(t_1)|^p \leqq \frac{1}{(2\pi n)^{p/2}}\left(\frac{2e(m+n)}{n}\right)^{np} \max_{v=m+1,\ldots,m+n} \left(|y(t_1+v)|^p e^{-pAv}\right) \leqq$$

$$\leqq \frac{1}{(6n)^{p/2}}\left(\frac{2e(m+n)}{n}\right)^{np} \sum_{\mu=m+1}^{m+n} |y(t_1+\mu)|^p \max_{v=m+1,m+n} e^{-pAv},$$

as asserted.

4. PROOF OF THEOREM 1. Put

$$C(m) = \frac{1}{(6n)^{p/2}}\left(\frac{2e(m+n)}{n}\right)^{np} \max_{v=m+1,m+n} e^{-pAv}. \tag{4. 1}$$

Integrating (3. 2) from $-m-1$ to $-m$ we get

$$(4.\ 2)\qquad \int_{-m-1}^{-m} |y(t)|^p\,dt \leqq C(m) \sum_{\mu=m+1}^{m+n} \int_{-m-1}^{-m} |y(t+\mu)|^p\,dt = C(m)\int_0^n |y(t)|^p\,dt$$

for every non-negative integer m.

Let ξ_1 and ξ_2 be real numbers such that

$$(4.\ 3)\qquad \xi_1 < \xi_2 \leqq 0.$$

Define the non-positive integers η_1, η_2 by

$$(4.\ 4)\qquad \eta_1 = [\xi_1], \quad \eta_2 = -[-\xi_2].$$

Hence we have by (4. 2)

$$(4.\ 5)\qquad \int_{\xi_1}^{\xi_2} |y(t)|^p\,dt \leqq \int_{\eta_1}^{\eta_2} |y(t)|^p\,dt = \sum_{j=\eta_1+1}^{\eta_2} \int_{j-1}^{j} |y(t)|^p\,dt \leqq$$

$$\leqq \left(\int_0^n |y(t)|^p\,dt\right) \sum_{j=\eta_1+1}^{\eta_2} C(-j).$$

The definitions (4. 4) imply

$$(4.\ 6)\qquad \eta_1 \geqq \xi_1 - 1, \quad \eta_2 \leqq \xi_2 + 1, \quad \eta_2 - \eta_1 \leqq \xi_2 - \xi_1 + 2.$$

Using these estimates we deduce from (4. 1)

$$\sum_{j=\eta_1+1}^{\eta_2} C(-j) \leqq (\eta_2-\eta_1) \max_{j=\eta_1+1,\dots,\eta_2} C(-j) \leqq$$

$$\leqq \frac{\xi_2-\xi_1+2}{(6n)^{p/2}} \left(\frac{2e(-\eta_1-1+n)}{n}\right)^{np} \max_{j=\eta_1+1,\dots,\eta_2} \max_{\nu=-j+1,\dots,-j+n} e^{-pA\nu} \leqq$$

$$\leqq \frac{\xi_2-\xi_1+2}{(6n)^{p/2}} \left(\frac{2e(-\xi_1+n)}{n}\right)^{np} \max_{j=-\eta_2+1,\dots,-\eta_1-1+n} e^{-pAj} \leqq$$

$$\leqq \frac{\xi_2-\xi_1+2}{(6n)^{p/2}} \left(\frac{2e(-\xi_1+n)}{n}\right)^{np} \max_{j=-\xi_2,\dots,-\xi_1+n} e^{-pAj}.$$

Substituting this estimate in the inequality (4. 5) we get

$$(4.\ 7)\qquad \int_{\xi_1}^{\xi_2} |y(t)|^p\,dt \leqq \frac{\xi_2-\xi_1+2}{(6n)^{p/2}} \left(\frac{2e(-\xi_1+n)}{n}\right)^{np} \max_{j=-\xi_2,\dots,-\xi_1+n} e^{-pAj} \int_0^n |y(t)|^p\,dt.$$

5. The function $y_1(t) = y\left(\frac{\delta_1 t}{n}\right)$ is a solution of the differential equation

$$(5.\ 1)\qquad y^{(n)} + a_{n-1}\frac{\delta_1}{n} y^{(n-1)} + a_{n-2}\left(\frac{\delta_1}{n}\right)^2 y^{(n-2)} + \dots + a_0\left(\frac{\delta_1}{n}\right)^n y = 0.$$

The zeros of the corresponding characteristic polynomial are in the halfplane

$$\operatorname{Re} z \geqq \frac{\delta_1 \Lambda}{n}.$$

Applying (4. 7) to $y_1(t)$ we find

(5. 2)
$$\int_{\frac{\delta_1 \xi_1}{n}}^{\frac{\delta_1 \xi_2}{n}} |y(t)|^p\, dt = \int_{\xi_1}^{\xi_2} |y_1(t)|^p\, dt \leqq$$

$$\leqq \frac{\xi_2 - \xi_1 + 2}{(6n)^{p/2}} \left(\frac{2e(-\xi_1 + n)}{n} \right)^{np} \max_{j = -\xi_2, -\xi_1 + n} e^{-\frac{\delta_1 \Lambda p j}{n}} \int_0^{\delta_1} |y(t)|^p\, dt.$$

Now we take

$$\xi_1 = \frac{n}{\delta_1}(\beta - \alpha), \quad \xi_2 = \frac{n}{\delta_1}(\beta - \alpha + \delta_2);$$

in view of (2. 3) we have indeed (4. 3).

Hence by (5. 2)

$$\int_{\beta-\alpha}^{\beta-\alpha+\delta_2} |y(t)|^p\, dt \leqq \frac{n\delta_2 + 2\delta_1}{(6n)^{p/2}\delta_1} \left(\frac{2e(\alpha - \beta + \delta_1)}{\delta_1} \right)^{np} \max_{j = \alpha - \beta - \delta_2, \alpha - \beta + \delta_1} e^{-p\Lambda j} \int_0^{\delta_1} |y(t)|^p\, dt.$$

Remembering the definition of δ_0 in (2. 4) we find

$$\max_{j = \alpha - \beta - \delta_2, \alpha - \beta + \delta_1} e^{-p\Lambda j} = e^{-\Lambda(\alpha - \beta + \delta_0)}.$$

The theorem now follows by a simple translation:

$$\int_{\beta}^{\beta+\delta_2} |y(t)|^p\, dt \leqq \frac{n\delta_2 + 2\delta_1}{(6n)^{p/2}\delta_1} \left(\frac{2e(\alpha - \beta + \delta_1)}{\delta_1} \right)^{np} e^{-p\Lambda(\alpha - \beta + \delta_0)} \int_{\alpha}^{\alpha+\delta_1} |y(t)|^p\, dt.$$

NOTE. Mr. A. A. BALKEMA proved that the factor

$$\left(\frac{\delta}{2e(\alpha - \beta + \delta)} \right)^{np}$$

in the right-hand side of (2. 5) can be replaced by the factor

$$\frac{1}{e^p} \left(\frac{\delta}{2e(\alpha - \beta + \delta)} \right)^{(n-1)p}$$

and that this new exponent $(n-1)p$ is best possible. His proof runs by an improvement of inequality (3. 3). (Compare [5], formula (1. 4. 2).)

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A REMARK ON MENGER'S THEOREM

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Introduction

Let us consider a finite, undirected graph $\mathcal{G}$ without loops and multiple edges; let M be the set of its vertices. If $\mathcal{P}$ is a path in $\mathcal{G}$ and $S \subseteqq M$, then we say that $\mathcal{P}$ is independent of S if S contains no inner point of $\mathcal{P}$.

We fix two vertices a, b of $\mathcal{G}$ such that they are not joined. A set $S \subseteqq M$ will be said to separate a and b if no (a, b)-path [1] is independent of S. Since a and b are not joined, there exist sets separating them; let k denote the minimum cardinality of these sets. By a well-known theorem of K. MENGER, k is the maximum number of pairwise independent (a, b)-paths in $\mathcal{G}$. The interesting part of this theorem is the existence of k independent (a, b)-paths.

Let now $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_k$ be independent (a, b)-paths. We consider every $\mathcal{K}_i$ to be a chain (i.e. a lattice) with zero element a and unity b. If S is a set separating a and b and $|S| = k$, then S contains just one element of $\mathcal{K}_i$ for every $1 \leqq i \leqq k$, thus S may be considered to be an element of the lattice $\mathcal{K}_1 \times \mathcal{K}_2 \times \dots \times \mathcal{K}_k$. If $\mathcal{L}$ denotes the system of all separating sets of cardinality k, then it can be verified by rather simple arguments that it is a sublattice of $\mathcal{K}_1 \times \mathcal{K}_2 \times \dots \times \mathcal{K}_k$.

In this paper we shall follow a converse way: we define the lattice $\mathcal{L}$ and prove its properties directly (i.e. without using Menger's theorem) and then deduce Menger's theorem from these results.

The beginning of our argument is similar to HALIN's proof given in [1], but we investigate an other lattice and the end of our proof is more algebraic.

§ 1. Let us recall that $\mathcal{L}$ is the system of all sets separating a and b of the (minimal) cardinality k. First we introduce a notation: if $S \subset M$, $x \in M - S$ then let S_x denote the set of those vertices of S which are not separated from x by S, i.e. which are accessible from x on a path independent of S.

We state some simple properties of the sets of type S_x:

(A) *If S separates x and y then so does S_x.*

(B) *If $S_x \subseteqq T \subseteqq S$ then $S_x = T_x$.* Really, the vertices of S_x are not separated from x by T (not even by S), hence $S_x \subseteqq T_x$. Conversely, the vertices of T_x are not separated from x by T, hence not by S_x, consequently not by S (see (A)), hence they belong to S_x.

Put now $S = P \cup Q$, $T = P_x \cup Q_x$, then trivially $S_x \subseteqq T \subseteqq S$ and thus from (B) we can deduce

(C) $(P \cup Q)_x = (P_x \cup Q_x)_x$.

[1] If $a, b \in M$ ($U, V \subseteqq M$) then (a, b)-path ((U, V)-path) means a path joining a and b (a vertex of U to a vertex of V, resp.).

Finally, the following statement is clear:

(D) *If* $P, Q \subseteq M$, $x, y \in M-P-Q$, $R \subseteq P \cap Q$ *and* R *separates* x *and* y *then* R *has a common vertex with every* (P_x, Q_y)*-path.*

Now we state our fundamental

LEMMA. *Introduce the following operations on* $\mathscr{L}$: *if* $S, T \in \mathscr{L}$ *then let*

$$S \wedge T = (S \cup T)_a, \quad S \vee T = (S \cup T)_b.$$

Then $\mathscr{L}$ *is a lattice.*

In knowledge of Menger's theorem we could state that this lattice is distributive, since it is a sublattice of the direct product of k chains. This fact could be proved directly too; but we shall not need it.

PROOF OF THE LEMMA. First of all we have to show, that the defined operations are operations on $\mathscr{L}$, i.e. if $S, T \in \mathscr{L}$ then $S \wedge T \in \mathscr{L}$, $S \vee T \in \mathscr{L}$.

$S \wedge T$, $S \vee T$ are separating sets by (A), therefore we have only to prove

$$|S \wedge T| = |S \vee T| = k. \tag{1}$$

It is obvious that

$$|S \wedge T| \geqq k, \quad |S \vee T| \geqq k. \tag{2}$$

Put $P = Q = S \cup T$, $x = a$, $y = b$, $R = S$ in (D), then

$$(S \wedge T) \cap (S \vee T) \subseteq S.$$

By symmetry

$$(S \wedge T) \cap (S \vee T) \subseteq S \cap T. \tag{3}$$

Using the trivial "dual" of (3)

$$(S \wedge T) \cup (S \vee T) \subseteq S \cup T, \tag{4}$$

we get by (2), (3) and (4)

$$2k \leqq |S \wedge T| + |S \vee T| = |(S \wedge T) \cap (S \vee T)| + |(S \wedge T) \cup (S \vee T)| \leqq$$
$$\leqq |S \cup T| + |S \cap T| = |S| + |T| = 2k.$$

and this shows that in (2) the equality must hold; this proves (1). By the way we have obtained that the equality must hold in (3) and (4) too, i.e.

(E) $(S \wedge T) \cap (S \vee T) = S \cap T$,

(F) $(S \wedge T) \cup (S \vee T) = S \cup T$.

From (E) and (F) we can deduce other formulas showing set-theoretical connections of our operations; what we shall need is the following:

(EF) $(P \wedge Q) - P = Q - (P \vee Q)$.

We have still to verify that the axioms of lattices hold. The symmetry in definitions of the operations enables us to prove only one of the dual axioms.

I. $T \wedge T = T$,

II. $S \wedge T = T \wedge S$ are trivial.

III. Instead of the associativity we prove somewhat more:

$$(S\wedge T)\wedge U=(S\cup T\cup U)_a.$$

This equality can be obtained from (C) by putting $P=T\cup S$, $Q=U$, $x=a$.

IV. To show

$$T\vee(S\wedge T)=T \tag{5}$$

let us note that (E) and (F) imply trivially that

$$P\wedge Q=P$$

is equivalent to

$$P\vee Q=Q.$$

Thus (5) is equivalent to

$$T\wedge(S\wedge T)=S\wedge T$$

which follows trivially from I, II and III.

We have finished the proof of the lemma. We close this section with a property of the lattice $\mathscr{L}$:

(G) *If* $S, T, U\in\mathscr{L}$, $S\leqq T\leqq U$ [2], *then* $S\cap U\subseteq T$.

This follows from (D) putting $P=S\cup T$, $Q=U\cup T$, $x=a$, $y=b$, $R=T$.

§2. We formulate some equivalents of Menger's theorem. One of these is the equivalence of the following two statements:

(i) *There exist k independent* (a, b)*-paths in* $\mathscr{G}$.

(ii) *Any set separating a and b has cardinality* $\geqq k$.

Since (i) trivially implies (ii), we have to prove that (ii) implies (i) too. Obviously it is enough to show this under a further supposition:

(iii) *Omitting any edge of* $\mathscr{G}$ *condition* (ii) *does not hold.*

If we choose a vertex of every (a, b)-path, we obtain a set separating a and b. Hence the number of all (a, b)-paths is surely $\geqq k$. We have only to show that if (ii) and (iii) hold, then any two (a, b)-paths are independent. This follows from the statement, that

(iv) *Every vertex of* $\mathscr{G}$ *except a and b is of valency* $\leqq 2$.

From (ii), (iii) and (iv) one could deduce easily that $\mathscr{G}$ consists of k independent (a, b)-paths.

PROOF OF (iv). Let c be any vertex of $\mathscr{G}$ different from a and b. By (iii), the edges incident to c can be divided into two classes $\mathscr{E}=\{E_1, \dots, E_m\}$, $\mathscr{F}=\{F_1, \dots, F_n\}$ as follows: omitting any E_i $(1\leqq i\leqq m)$ there exists a set K_i of cardinality $k-1$ separating a from both of b and c, and omitting any F_i there exists a set L_i of cardinality $k-1$ separating b from both of a and c.

It is enough to show that $m\leqq 1$, since then similarly $n\leqq 1$ and the valency of c is $n+m\leqq 2$. If a and c are joined then the edge joining them is obviously the

[2] $V\leqq W$ means, as usual, $V\wedge W=V$.

only element of $\mathscr{E}$, hence we may suppose that the endpoints x_i, c of E_i differ from a for any $1 \leqq i \leqq m$.

Obviously $c \notin K_i$, $x_i \notin K_i$, moreover $T_i = K_i \cup \{x_i\}$ and $S_i = K_i \cup \{c\}$ are separating sets in $\mathscr{G}$, and hence elements of the lattice $\mathscr{L}$. Since

$$T_i \wedge S_i \subset K_i \cup \{x_i\} \cup \{c\}$$

and by the definition of K_i

$$c \notin T_i \wedge S_i = (K_i \cup \{x_i\} \cup \{c\})_a$$

we have

$$T_i \wedge S_i = T_i \quad \text{i.e.} \quad T_i \leqq S_i.$$

Furthermore, we show that S_i "covers" T_i, i.e.

(H) *If* $T_i \leqq X \leqq S_i$ *then* $X = S_i$ *or* $X = T_i$.

Really, the application of (D) with $P = T_i \cup X$, $Q = S_i \cup X$, $R = X$, $x = a$, $y = b$ gives that X contains all elements of $S_i \cap T_i = K_i$ and one point of the edge E_i.

Consider now the sets

$$S = S_1 \vee \ldots \vee S_m, \quad T = T_1 \vee \ldots \vee T_m, \quad U_i = S_i \wedge T, \quad V_i = S_i \vee T.$$

We are going to calculate U_i and V_i. Since $T_i \leqq U_i \leqq S_i$, (H) implies that either $U_i = S_i$ or $U_i = T_i$. But $U_i = S_i$ would imply $S_i \leqq T \leqq S$ and hence by (G) $c \in T$, which contradicts (F). Thus

$$U_i = S_i \wedge T = T_i. \tag{6}$$

Similarly, it is clear that $T_j \leqq V_i \wedge S_j \leqq S_j$ $(1 \leqq i, j \leqq m)$ hence either $V_i \wedge S_j = T_j$ or $V_i \wedge S_j = S_j$. But (F) implies $c \in V_i$ (since $c \notin U_i$) and hence by (E) $c \in V_i \wedge S_j$, thus

$$V_i \wedge S_j = S_j, \quad \text{i.e.} \quad V_i \geqq S_j.$$

Since this holds for any $1 \leqq j \leqq m$, we have $V_i \geqq S$. On the other hand $V_i = S_i \vee T \leqq S$, hence

$$V_i = S_i \vee T = S. \tag{7}$$

Substitute now $P = S_i$, $Q = T$ in (EF), then by (6) and (7)

$$T - S = T - (S_i \vee T) = (S_i \wedge T) - S_i = T_i - S_i = \{x_i\}.$$

Now $T - S$ does not depend on i; this is possible only if $m = 1$.

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SOME RESULTS AND PROBLEMS ON CERTAIN POLARIZED PARTITIONS

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§ 1. Introduction. Notation

1. 1. A short list of general notations

$\alpha, \beta, \gamma, \delta$ denote cardinals. $\xi, \zeta, \eta, \mu, \nu, \sigma, \varrho$ denote ordinals. $|A|$ is the cardinality of the set A. α^+ is smallest cardinal greater than α. ω_ξ is the sequence of infinite cardinals $\omega_0=\omega$. i, j, r, s, l, k denote integers (cardinals $<\omega$). α is a strong limit cardinal if $2^\beta<\alpha$ for every $\beta<\alpha$. For $\alpha\geqq\omega$ $\mathrm{cf}(\alpha)$ is the least cardinal cofinal with α

$$[A]^\alpha=\{X: X\subset A\wedge|X|=\alpha\},\ [A]^{<\alpha}=\{X: X\subset A\wedge|X|<\alpha\}.$$

For the convenience of the reader we recall the definition of some of the partition symbols defined in earlier papers [1], [2], [3].

DEFINITION 1. 1. 1. *The ordinary partition symbol.* $\alpha\to(\beta_\nu)^\delta_{\nu<\gamma}$ denotes that the following statement is true.

Whenever $[\alpha]^\delta=\bigcup_{\nu<\gamma} I_\nu$ then there are $A\subset\alpha$, $\nu<\gamma$ such that $|A|=\beta_\nu$, $[A]^\delta\subset I_\nu$.

Here and for all other symbols to be defined $\alpha\nrightarrow(\beta_\nu)^\delta_{\nu<\gamma}$ denotes the negations of the corresponding statements. $\alpha\to(\beta)^\delta_\gamma$ denotes $\alpha\to(\beta_\nu)^\delta_{\nu<\gamma}$ where $\beta_\nu=\beta$ for $\nu<\gamma$.

We use some other self explanatory abbreviations which are defined in detail in [2].

Note that the ordinary partition symbol and some of the other symbols can be defined for types instead of ordinals in a natural way. If α; β_ν, $\nu<\gamma$ are types, $\alpha\to(\beta_\nu)^\delta_\gamma$ means the following:

Whenever $A, \prec$ is an ordered set, $\mathrm{tp}(\prec)=\alpha$ and $[A]^\delta=\bigcup_{\nu<\gamma} I_\nu$, then there are $A'\subset A$, $\nu<\gamma$ such that $\mathrm{tp}\,A'(\prec)=\beta_\nu$ and $[A']^\delta\subset I_\nu$.

Since we do not investigate these problems here, we will give all the definitions for cardinals.

DEFINITION 1. 1. 2. *The polarized partition symbol.* Let $r, s<\omega$; $r=r_0+\dots+r_{s-1}$. Let α_i, $\beta_{i,\nu}$ be cardinals for $i<s$, $\nu<\gamma$.

$\begin{pmatrix}\alpha_0\\ \cdots\\ \alpha_{s-1}\end{pmatrix}\to\begin{pmatrix}\beta_{0,\nu}\\ \cdots\\ \beta_{s-1,\nu}\end{pmatrix}^{r_0,\dots,r_{s-1}}_{\nu<\gamma}$ means that the following statement is true.

Whenever

$$[\alpha_0]^{r_0}\times\dots\times[\alpha_{s-1}]^{r_{s-1}}=\bigcup_{\nu<\gamma} I_\nu$$

then there exist sets $A_i\subset\alpha_i$, $i<s$ and $\nu<\gamma$ such that

$$[A_0]^{r_0}\times\dots\times[A_{s-1}]^{r_{s-1}}\subset I_\nu \quad\text{and}\quad |A_i|=\beta_{i,\nu} \quad\text{for}\quad i<s.$$

DEFINITION 1. 1. 3. $\alpha \to (\beta)_\gamma^{<\omega}$ means that the following statement is true: Whenever

$$[\alpha]^r = \bigcup_{\nu<\gamma} I_\nu^r \quad \text{for} \quad r < \omega$$

then there are $A \subset \alpha$ and $f \in {}^\omega\gamma$ such that $|A| = \beta$ and $[A]^r \subset I_{f(r)}$ for $r < \omega$.

DEFINITION 1. 1. 4. $\alpha \to [\beta_\nu]_{\nu<\gamma}^\delta$ means that the following statement is true. Whenever $[\alpha]^\delta = \bigcup_{\nu<\gamma} I_\nu$ then there are $A \subset \alpha$, $\nu_0 < \gamma$ such that

$$|A| = \beta_{\nu_0} \quad \text{and} \quad A \subset \bigcup_{\nu<\gamma, \nu \neq \nu_0} I_\nu.$$

If $\beta_\nu = \beta$ for $\nu < \gamma$, we write $\alpha \to [\beta]_\gamma^\delta$.

DEFINITION 1. 1. 5. $\alpha \to [\beta]_{\gamma_0, \gamma_1}^\delta$ means that the following statement is true: Whenever $[\alpha]^\delta = \bigcup_{\nu<\gamma_0} I_\nu$ then there are $A \subset \alpha$ and $C \subset \gamma_0$ such that $|A| = \beta$, $|C| \leqq \gamma_1$ and $A \subset \bigcup_{\nu \in C} I_\nu$. The symbols defined in 1. 1. 4, 1. 1. 5 are the "square bracket" symbols corresponding to the ordinary partition symbol. Quite similarly two square bracket symbols

$$\begin{pmatrix}\alpha_0\\ \cdots \\ \alpha_{s-1}\end{pmatrix} \to \begin{bmatrix}\beta_{0,\nu}\\ \cdots \\ \beta_{s-1,\nu}\end{bmatrix}^{r_0,\ldots,r_{s-1}}, \quad \begin{pmatrix}\alpha_0\\ \cdots \\ \alpha_{s-1}\end{pmatrix} \to \begin{bmatrix}\beta_0\\ \cdots \\ \beta_{s-1}\end{bmatrix}^{r_0,\ldots,r_{s-1}}_{\gamma_0,\gamma_1}$$

can be defined corresponding to the polarized partition symbol.

The symbols defined in 1. 1, 1. 2, 1. 3 were defined in [2], where we gave a detailed discussion of the ordinary partition symbol and the special case. $s=2$, $r=2$, $r_0=r_1=1$, $\gamma=2$ of the polarized partition symbol.

The aim of the present is to consider the special cases

$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_{0,\nu}\\ \beta_{1,\nu}\end{pmatrix}^{1,r}, \quad 2 \leqq r < \omega$$

of the polarized partition symbol, mainly in case $r=2$, $\gamma=2$ and some related problems.

1. 2 A new notation for the main problems considered

1. 2. 1. $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_{0,\nu}\\ \beta_{1,\nu}\end{pmatrix}^{1,r}_{\nu<\gamma}$ is obviously equivalent to the following statement:

Let $[\alpha_1]^r = \bigcup_{\nu<\gamma} I_\nu^\xi$ for $\xi < \alpha_0$. Then there are $A_0 \subset \alpha_0$, $A_1 \subset \alpha_1$, $\nu < \gamma$ such that $|A_0| = \beta_0, \nu$, $|A_1| = \beta_{1,\nu}$ and

$$[A_1]^r \subset \bigcap_{\xi \in A_0} I_\nu^\xi.$$

If $\mathscr{B}_\nu = \{[A]^r : A \subset \alpha_1 \wedge |A| = \beta_{1,\nu}\}$ then 1. 2. 1 can be expressed as follows: There are $A_0 \subset \alpha$, $\nu < \gamma$ such that $|A_0| = \beta_{0,\nu}$ and there is an $X \in \mathscr{B}_\nu$ such that

$$X \subset \bigcap_{\xi \in A_0} I_\nu^\xi.$$

We will consider the more general problem when $\mathscr{B}_\nu$ can be more general classes.

DEFINITION 1. 2. 2. Let $\alpha_0, \alpha_1; \beta_\nu, \nu<\gamma$ be cardinals $r<\omega$, and let $\mathscr{B}_\nu, \nu<\gamma$ be a sequence, where $\mathscr{B}_\nu \subset \mathscr{P}([\alpha_1]^r)$. Then $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_\nu\\ \mathscr{B}_\nu\end{pmatrix}^{1,r}_{\nu<\gamma}$ means that the following statement is true. Whenever

$$[\alpha_1]^r = \bigcup_{\nu<\gamma} I_\nu^\xi \quad \text{for} \quad \xi < \alpha_0$$

then there are $A_0 \subset \alpha_0$, $\nu<\gamma$ such that $|A_0| = \beta_\nu$ and there is $X \in \mathscr{B}_\nu$ with

$$X \subset \bigcap_{\xi \in A_0} I_\nu^\xi.$$

We will write

$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_0 & \beta_1\\ \mathscr{B}_0, & \mathscr{B}_1\end{pmatrix}^{1,r} \quad \text{for} \quad \begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_\nu\\ \mathscr{B}_\nu\end{pmatrix}^{1,r}_{\nu<2}.$$

Note that an $X \subset [\alpha_1]^2$ can be considered as a graph $\langle \alpha, X \rangle$ whose vertices and edges are the elements of α and the elements of X, respectively. We will sometimes use graph terminology for expressing certain properties of such classes.

We will sometimes use the $\vee$ (or) sign in the symbol; e.g.

$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_0' & & \beta_0'' & \beta_1' & & \beta_1''\\ \mathscr{B}_0' & \vee & \mathscr{B}_0'', & \mathscr{B}_1' & \vee & \mathscr{B}_1''\end{pmatrix}^{1,r}$$

has the following self explanatory meaning.

Whenever

$$[\alpha_1]^r = \bigcup_{\nu<2} I_\nu^\xi \quad \text{for} \quad \xi < \alpha_0$$

then there are $A_0 \subset \alpha$, $\nu<2$ such that either $|A_0| = \beta_\nu'$ and there is an $X \in \mathscr{B}_\nu'$ for which

$$X \subset \bigcap_{\xi \in A_0} I_\nu^\xi,$$

or $|A_0| = \beta_\nu''$ and there is an $X \in \mathscr{B}_\nu''$ for which

$$X \subset \bigcap_{\xi \in A_0} I_\nu^\xi.$$

1. 3 About the results

Though we have defined above a general symbol which can be used to express the results and problems we are going to state, it will be clear to everyone familiar with the subject that a systematical discussion of all the problems involved is hopeless presently (and perhaps would not even be worth while). We came across the special cases of these problems when working on ordinary partition problems. Some of the results are ten years old, some are new and give the solution of several porblems stated in our paper [3].

We will consider different instances in different chapters and we give short summaries there.

Though most of the results will only interest those who know the basic results un partition relations in detail, there will be some simple unsolved problems which seem to be fundamental. Trying to clear these problems up we will prove some obviously not final partial results too.

We mention that Theorems 4. 1, 4. 3, 6. 1, 6. 3 give solution of Problems 61, 59, 60 stated in [3], respectively.

§ 2. A positive result for measurable α

DEFINITION 2. 1. $\mathscr{B}_{\alpha,\gamma,\delta}$ will denote the class of complete γ, δ even graphs with set of vertices α i.e.

$$\mathscr{B}_{\alpha,\gamma,\delta}=\{X\subset[\alpha]^2:\exists C, D(C\subset\alpha\wedge D\subset\alpha\wedge C\cap D=\bigcirc\wedge|C|= =\gamma\wedge|D|=\delta\wedge(\{\xi,\eta\}\in X\Leftrightarrow\zeta\in C\wedge\eta\in D))\}.$$

If $X\in\mathscr{B}_{\alpha,\gamma,\delta}$ we write $X=[C, D]$.

2. 2. *Let* $\alpha\geqq\omega$ *and* $\mathscr{B}^0_\alpha=\{X\subset[\alpha]^2:\langle\alpha, X\rangle$ *contains an odd circuit*$\}$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\nrightarrow\begin{pmatrix}1 & \alpha\\ \mathscr{B}^0_\alpha, & \mathscr{B}_{\alpha,1,\alpha}\end{pmatrix}^{1,2}.$$

PROOF. Put $I^\xi_0=\{\{\zeta,\eta\}\in[\alpha]^2:\zeta\leqq\xi<\eta\}$ $I^\xi_1=[\alpha]^2-I^\xi_0$ for $\xi<\alpha_0$. Then $I^\xi_0\notin\mathscr{B}^0_\alpha$. If $X\in\mathscr{B}_{\alpha,1,\alpha}$ i.e. $X=[\{\xi_0\}, D]$, $|D|=\alpha$ then $X\cap I^\xi_0\neq 0$ for $\xi\geqq\xi_0$.

Note that if $\mathscr{B}^1_\alpha=\{X\subset[\alpha]^2: X$ contains α edges$\}$ then one can prove e.g.

2. 3. $$\begin{pmatrix}\omega\\ \omega\end{pmatrix}\rightarrow\begin{pmatrix}\omega & \omega\\ k, & \mathscr{B}^1_\omega\end{pmatrix}^{1,2} \quad \textit{for} \quad k<\omega,$$

which shows that in 2. 2 $\mathscr{B}_{\alpha,1,\alpha}(\subset\mathscr{B}^1_\alpha)$ can not be replaced by $\mathscr{B}^1_\alpha$. We omit the routine proof.

2. 4. *Let* $\alpha\geqq\omega$. *Put* $\mathscr{B}^2_\alpha=\{X\subset[\alpha]^2: X\neq\bigcirc\}$. *We have*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\nrightarrow\begin{pmatrix}1 & \alpha\\ \mathscr{B}^1_\alpha, & \mathscr{B}^2_\alpha\end{pmatrix}^{1,2}.$$

PROOF. For $\xi<\alpha$ put $I^\xi_0=[\xi]^2$, $I^\xi_1=[\alpha]^2-I^\xi_0$.

Then $|I^\xi_0|<\alpha$, hence $I^0_\xi\notin B^1_\alpha$ and if $X\neq 0$ then there is $\xi_0<\alpha$ such that $X\cap I^\xi_0\neq 0$ for $\xi\geqq\xi_0$.

The above negative results suggest the formulation of the following property:

2. 5. $\mathbf{P}(\alpha)\Leftrightarrow$ for every $\beta,\gamma<\alpha$, $\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\rightarrow\begin{pmatrix}1 & 1 & \alpha\\ \beta\vee & \mathscr{B}_{\alpha,\gamma,\alpha}, & \alpha\end{pmatrix}^{1,2}$.

Our main result in this chapter is

THEOREM 1. 1. *Let* $\alpha\geqq 0$ *be* 0, 1-*measurable. Then for every* $\beta,\gamma<\alpha$

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\rightarrow\begin{pmatrix}\alpha & 1 & \alpha\\ \beta\vee & \mathscr{B}_{\alpha,\gamma,\alpha}, & \alpha\end{pmatrix}^{1,2}.$$

As a corollary of this, $\mathbf{P}(\alpha)$ holds for measurable α. Before proving the theorem we mention

PROBLEM 1. a) Can one prove Theorem 1. 1 under some weaker hypothesis than the measurability of α? (E.g. $\alpha \to (\alpha)_2^2$?)

b) Is $\mathbf{P}(\alpha)$ true for the first strongly inaccessible cardinal?

Note that we will prove that $\mathbf{P}(\alpha)$ is false for cardinals not strongly inaccessible.

PROOF OF THEOREM 1. 1. By the definitions 1. 2. 1, 1. 2. 2 it is obviously sufficient to prove the following statement.

Let I_ξ, $\xi<\alpha$ be a sequence of type α, where $I_\xi \subset [\alpha]^2$. Assume that the following conditions (1), (2) hold:

(1) If $A, B \subset \alpha$, $A \cap B = 0$, $|A| = \gamma$, $|B| = \alpha$ then $[A, B] \not\subset I_\xi$ for $\xi < \alpha$.

(2) If $A, B \subset \alpha$, $|A| = \beta$, $|B| = \alpha$ then $[A]^2 \not\subset \bigcup_{\xi \in B} I_\xi$.

Then there are $C, D \subset \alpha$, $|C| = |D| = \alpha$ such that

$$[C]^2 \cap I_\xi = 0 \quad \text{for} \quad \xi < \alpha.$$

Let μ denote a non-trivial α-complete 0, 1-valued measure on α. For each $P \in [\alpha]^2$ put

$$N(P) = \{\xi \in \alpha : P \in I_\xi\}. \tag{3}$$

Put

$$I = \{P \in [\alpha]^2 : \mu(N(P)) = 1\}. \tag{4}$$

α being strongly inaccessible we have $\alpha \to (\beta, \alpha)^2$ (see [1]). Applying this for the partition $[\alpha]^2 = I \cup ([\alpha]^2 - I)$ and using the assumption (2) we obtain that

(5) There is an $A_0 \subset \alpha$, $|A_0| = \alpha$, $[A_0]^2 \cap \mathrm{I} = 0$ i.e.

$$\mu(N(P)) = 0 \quad \text{for each} \quad P \in [A_0]^2.$$

Let now μ' be a non trivial α-complete 0, 1-valued measure on A_0.

Put

(6) $U_\xi(x) = \{y \in A_0 : \{x, y\} \in I_\xi\}$ for every $x \in A_0$, $\xi < \alpha$

$M(x) = \{\xi < \alpha : \mu'(U_\xi(x)) = 1\}$ for every $x \in A_0$

$T = \{x \in A_0 : \mu(M(x)) = 1\}$

$T_\xi = \{x \in A_0 : \xi \in M(x)\}$ for every $\xi < \alpha$.

We prove

(7) $|T_\xi| < \gamma$ for every $\xi < \alpha$. In fact if $T' \subset T_\xi$ $|T'| = \gamma$ then $\mu'\left(\bigcap_{x \in T'} U_\xi(x)\right) = 1$ and $\left[T', \bigcap_{x \in T'} U_\xi(x) - T'\right] \subset I_\xi$ which contradicts the assumption (1).

It follows that

(8) $|T| < \gamma$, for if $T' \subset T$, $|T'| = \gamma$ then there is a $\xi \in \bigcap_{x \in T'} M(x)$ and $T' \subset T_\xi$ for this ξ.

Put $A_1 = A_0 - T$. By (5) and (6) and (8)

(9) $\mu'(A_1)=1$, $\mu(N(P))=0$ for every $P\in[A_1]^2$, $\mu(M(x))=0$ for every $x\in A_1$.

We define the sequences $\{x_\varrho\}_{\varrho<\alpha}\subset A_1$, $\{\xi_\varrho\}_{\varrho<\alpha}\subset\alpha$ by induction on ϱ as follows. Assume that $\varrho<\alpha$, X_σ and ξ_τ are already defined and $\xi_\tau\notin M(X_\sigma)$ for $\tau,\sigma<\varrho$. Then by (8) and (9)

$$\mu'\Big(\bigcup_{\tau<\varrho}\bigcup_{\sigma<\varrho}U_{\xi_\tau}(x_\sigma)\Big)=0$$

hence by (7) there is an $X_\varrho\in A_1$, $X_\varrho\neq X_\sigma$ for $\sigma<\varrho$ such that

(10) $$x_\varrho\notin\bigcup_{\tau<\varrho}\bigcup_{\sigma<\varrho}U_{\xi_\tau}(x_\sigma)\cup\bigcup_{\tau<\varrho}T_{\xi_\tau}.$$

By (9) the set

$$\bigcup_{\tau<\sigma\leqq\varrho}N(\{x_\tau,x_\sigma\})\cup\bigcup_{\sigma\leqq\varrho}M(x_\sigma)$$

has μ measure 0, hence there is a $\xi_\varrho<\alpha$, $\xi_\varrho\neq\xi_\tau$ for $\tau<\varrho$ such that

(11) $$\xi_\varrho\notin\bigcup_{\tau<\sigma\leqq\varrho}N(\{x_\tau,x_\sigma\})\cup\bigcup_{\sigma\leqq\varrho}M(x_\sigma).$$

By (10) and (11) $\xi_\tau\notin M(x_\sigma)$ holds for $\tau,\sigma,\ \leqq\varrho+1$ as well. Thus the sequences are defined. Put $C=\{x_\varrho\}_{\varrho<\alpha}$, $D=\{\xi_\varrho\}_{\varrho<\alpha}$. By the definition $|C|=|D|=\alpha$. Let $\tau<\sigma<\alpha$, $\varrho<\alpha$. We prove $\{x_\tau,x_\sigma\}\notin I_{\xi_\varrho}$. We distinguish two cases: (i) $\varrho<\sigma$, (ii) $\sigma\leqq\varrho$. By (10) $x_\sigma\notin U_{\xi_\varrho}(x_\tau)$ if (i) holds. By (11) $\xi_\varrho\notin N(\{x_\tau,x_\sigma\})$ it (ii) holds.

Hence by (3) and (6) $\{x_\tau,x_\sigma\}\notin I_{\xi_\varrho}$ in both cases. This proves that $[C]^2\cap I_\xi=0$ for every $\xi\in D$.

Now we prove a number of negative results which show that $\mathbf{P}(\alpha)$ is false for not strongly inaccessible cardinals.

2. 6. *Assume* $\mathrm{cf}(\alpha)<\alpha$ *and* $\alpha\geqq\omega$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\nrightarrow\begin{pmatrix}1 & & 1 & \alpha\\ \mathrm{cf}\,(\alpha) & \vee & \mathscr{B}^1_\alpha, & \alpha\end{pmatrix}^{1,2}.$$

We give the (trivial) proof in § 7 (Theorem 7. 1) where we discuss singular cardinals.

2. 7. $$\begin{pmatrix}\beta^+\\ \beta^+\end{pmatrix}\nrightarrow\begin{pmatrix}1 & & 1 & \beta^+\\ \omega & \vee & \mathscr{B}^1_{\beta^+}, & \beta^+\end{pmatrix}^{1,2},\quad \textit{for}\quad \beta\geqq\omega.$$

PROOF. For each $\xi<\beta^+$ let $\prec_\xi$ be a well ordering of ξ such that $\mathrm{tp}\,\xi(\prec_\xi)\leqq\beta$. Put

$$I_0^\xi=\{\{\zeta,\eta\}\in[\xi]^2:\zeta<\eta\wedge\eta\prec_\xi\zeta\},\quad I_1^\xi=[\beta^+]^2-I_0^\xi.$$

Obviously I_0^ξ does not contain a complete ω graph, and $I_0^\xi\notin\mathscr{B}^1_{\beta^+}$. On the other hand if $X\subset\alpha$, $\mathrm{tp}X(<)=\beta+1$ then there is a $\xi_0<\beta$, $X\subset\xi_0$ and $X\cap I_0^\xi\neq\bigcirc$ for $\xi\geqq\xi_0$.

With a similar idea one gets

2. 8. *Assume* $\alpha\geqq\omega$, $\beta\nrightarrow(\gamma,\gamma)^2$ *for every* $\beta<\alpha$, $\mathrm{cf}\,(\alpha)\neq\mathrm{cf}\,(\gamma)$ *then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\nrightarrow\begin{pmatrix}1 & & 1 & \alpha\\ \gamma & \vee & \mathscr{B}^1_\alpha, & \gamma\end{pmatrix}^{1,2}.$$

PROOF. For each $\xi<\alpha$, let $I_0^\xi \subset [\xi]^2$ be such that the partition $[\xi]^2 = I_0^\xi \vee ([\xi]^2 - I_0^\xi)$ establishes the negative partition relation $|\xi| \nrightarrow (\gamma, \gamma)^2$. Put $I_1^\xi = [\alpha]^2 - I_0^\xi$. Obviously $I_0^\xi \notin B_\alpha^1$ and I_0^ξ does not contain a complete γ-graph. Assume $X \subset \alpha$, $|X| = \gamma$. Then by the assumption $\mathrm{cf}(\gamma) \neq \mathrm{cf}(\alpha)$, there are $Y \subset X$, $|Y| = \gamma$ and $\xi_0 < \alpha$ such that $Y \subset \xi_0$. But then by the construction $I_0^\xi \cap [Y]^2$ for $\xi \geqq \xi_0$.

2. 9. COROLLARY. *Assume $\alpha = (2^\beta)^+ \geqq \omega$. Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix} \nrightarrow \begin{pmatrix}1 & & 1 & \alpha\\ \beta^+ & \vee & \mathscr{B}_\alpha^1, & \beta^+\end{pmatrix}^{1,2}.$$

PROOF by 2. 8 considering that $2^\beta \nrightarrow (\beta^+)_2^2$.

2. 10. COROLLARY. $\mathbf{P}(\alpha)$ *is false if $\alpha \geqq \omega$ is not strongly inaccessible.*

PROOF. By 2. 6 and 2. 7 we assume that α is regular, $2^\beta \geqq \alpha$ for some $\beta < \alpha$, and $\beta^+ < \alpha$. Considering $2^\beta \nrightarrow (\beta^+)_2^2$ the statement follows from 2. 8.

Without using G.C.H. we could not prove stronger negative results. Assuming G.C.H., much stronger negative results will be proved in § 3.

It is obvious that many quantitative questions can be asked here; we point out one.

PROBLEM 2. Can one prove without assuming C.H. that

$$\begin{pmatrix}\omega_1\\ \omega_1\end{pmatrix} \nrightarrow \begin{pmatrix}1 & & 1 & \omega_1\\ 3 & \vee & \mathscr{B}_{\omega_1}^1, & \omega_1\end{pmatrix}^{1,2} ?$$

This should be compared with 2. 7 and Theorems 3. 1 and 3. 2.

§ 3. Stronger counter examples for $P(\alpha^+)$, assuming G.C.H.

DEFINITON 3. 1. Let $\mathscr{B}_\alpha^3 = \{X \subset [\alpha]^2;\ \langle \alpha, X\rangle$ contains a circuit$\}$,

$$\mathscr{B}_{\alpha^+}^4 = \{X \subset [\alpha^+]^2 \colon |X| = \alpha\}.$$

THEOREM 3. 1. *Assume G.C.H., $\alpha \geqq \omega$. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & & 1 & \alpha^+\\ \mathscr{B}_{\alpha^+}^1 & \vee & \mathscr{B}_{\alpha^+}^3, & \mathscr{B}_{\alpha^+}^4\end{pmatrix}^{1,2}.$$

PROOF. First we prove

(1) Assume $\{Y_\mu\}_{\mu<\alpha}$ is a sequence of type α of elements of $\mathscr{B}_{\alpha^+}^4$. Then there is a set $I \subset [\alpha^+]^2$, $|I| = \alpha$, such that the graph $\langle \alpha^+, I\rangle$ does not contain a circuit, and $I \cap Y_\mu \neq 0$ for $\mu < \alpha$.

First we define a sequence $\{P_\mu\}_{\mu<\alpha}$ of elements of $[\alpha^+]^2$ by transfinite induction on μ. Assume P_ν is defined for every $\nu < \mu$ for some $\mu < \alpha$. Then $\left|\bigcup_{\nu<\mu} P_\nu\right| < \alpha$, hence $Y_\mu - [\bigcup_{\nu<\mu} P_\nu]^2 \neq 0$ and let P_μ be an element of it. Put $I = \{P_\mu\}_{\mu<\alpha}$. Then I obviously satisfies the requirements of (1).

By G.C.H., there exists a well-ordering $\{X_\xi\}_{\xi<\alpha^+} = \mathscr{B}^4_{\alpha^+}$ of type α^+ of $\mathscr{B}^4_{\alpha^+}$. By (1) for each $\xi<\alpha^+$ there exists an $I_0^\xi \notin \mathscr{B}^1_{\alpha^+} \cup \mathscr{B}^3_{\alpha^+}$ such that $I_0^\xi \cap X_\zeta \neq 0$ for each $\zeta<\xi$. Put $I_1^\xi = [\alpha^+]^2 - I_0^\xi$ for $\xi<\alpha^+$.

If $X \in \mathscr{B}^4_{\alpha^+}$, then $X = X_\zeta$ for some $\zeta<\alpha^+$ and $X \cap I_0^\xi \neq 0$ for $\zeta<\xi<\alpha^+$.

DEFINITION 3. 2. Let $\mathscr{B}^5_\alpha = \{X \subset [\alpha]^2 : P \cap Q \neq$ for some $P \neq Q \in X\}$. $\mathscr{B}^6_\alpha = \{X \subset [\alpha]^2 : X$ consists of α disjoint edges$\}$. Our next theorem is incomparable with Theorem 3. 1 since $\mathscr{B}^1_{\alpha^+} \cup \mathscr{B}^3_{\alpha^+} \subset \mathscr{B}^1_{\alpha^+} \cup \mathscr{B}^5_{\alpha^+}$ but $\mathscr{B}^6_{\alpha^+} \subset \mathscr{B}^4_{\alpha^+}$.

THEOREM 3. 2. *Assume G.C.H., $\alpha \geqq \omega$. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & & 1 & & \alpha^+\\ \mathscr{B}^1_{\alpha^+} & \vee & \mathscr{B}^5_{\alpha^+} & , & \mathscr{B}^6_{\alpha^+}\end{pmatrix}^{1,2}.$$

PROOF. First we prove

(1) If $\{Y_\mu\}_{\mu<\alpha}$ is a sequence of type α of elements of $\mathscr{B}^6_\alpha$ then there is an $I \subset [\alpha^+]^2$, $|I| \leqq \alpha$, which consists of disjoint pairs such that $Y_\mu \cap I \neq 0$ for every $\mu<\alpha$.

To prove this we define a sequence $\{P_\mu\}_{\mu<\alpha} \subset [\alpha^+]^2$ by transfinite induction on μ. Assume P_ν is defined for every $\nu<\mu$ for some $\mu<\alpha$. Then $\left|\bigcup_{\nu<\mu} P_\nu\right| < \alpha$. Considering $Y_\mu \in \mathscr{B}^6_{\alpha^+}$ there is a $P_\mu \in Y_\mu$ such that $P_\mu \cap \bigcup_{\nu<\mu} P_\nu = 0$, $I = \{P_\mu\}_{\mu<\alpha}$ satisfies the requirements of (1). By G.C.H., there exists a well-ordering $\{X_\xi\}_{\xi<\alpha^+} = \mathscr{B}^6_{\alpha^+}$ of type α^+ of $\mathscr{B}^6_{\alpha^+}$. Applying (1) for $\{X_\zeta\}_{\zeta<\xi}$ for each $\xi<\alpha^+$ we obtain that there exists an $I_0^\xi \subset [\alpha^+]^2$, $|I_0^\xi| = \alpha$ $I_0^\xi \notin \mathscr{B}^1_{\alpha^+} \cup \mathscr{B}^5_{\alpha^+}$ such that

$$I_0^\xi \cap X_\zeta \neq 0 \quad \text{for every} \quad \zeta<\xi.$$

Put $I_1^\xi = [\alpha^+]^2 - I_0^\xi$ to $\xi<\alpha^+$. If $X \in \mathscr{B}^6_{\alpha^+}$ then $X = X_\zeta$ for some $\zeta<\alpha^+$ and

$$I_0^\xi \cap X \neq 0 \quad \text{for every} \quad \xi>\zeta.$$

Our next theorem shows that Theorems 3. 1 and 3. 2 do not have a common generalisation.

THEOREM 3. 3. *Assume $\alpha \geqq \omega$, $\gamma^+<\alpha$. Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix} \rightarrow \begin{pmatrix}1 & & \alpha\\ \mathscr{B}^5_\alpha & , & \mathscr{B}_{\alpha,\gamma,\alpha}\end{pmatrix}^{1,2}.$$

PROOF. Let $[\alpha]^2 = I_0^\xi \cup I_1^\xi$ for $\xi<\alpha$.

Assume $I_0^\xi \notin \mathscr{B}^5_\alpha$ for every $\xi<\alpha$. Let $C \subset \alpha$, $|C| = \gamma$.

Put $B = \alpha - C$. For each $\xi<\alpha$ let

(1) $B_\xi = \{y \in B$: there is an $x \in C$ such that $\{x, y\} \in I_0^\xi\}$.

Considering $I_0^\xi \notin \mathscr{B}^5_\alpha$, we have $|B_\xi| \leqq \gamma$ for every $\xi<\alpha$.

By a theorem of G. FODOR [7] then there exist $D \subset B$, $A \subset \alpha$ such that $|D| = |A| = \alpha$ and $D \cap B_\xi = 0$ for $\xi \in A$.

Then $[C, D] \in \mathscr{B}_{\alpha,\gamma,\alpha}$ and by (1) $[C, D] \subset I_1^\xi$ for every $\xi \in A$.

As to the counterexamples in Theorem 3. 1, 3. 2, it is obvious that assuming G.C.H., neither of the classes $\mathscr{B}^4_{\alpha^+}$, $\mathscr{B}^6_{\alpha^+}$ can be replaced by a class containing graphs with fewer than α edges since then α^+ graphs coincide on a set of power γ (where $\gamma<\alpha$) and the problems are reduced to ordinary partition problems.

§ 4. Further counterexamples (assuming G.C.H).

In this § we consider problems of the type

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \alpha \\ \mathscr{B}_0, \mathscr{B}_1\end{pmatrix}^{1,2}$$

where the complement of B_0 consists of very small graphs and the graphs in $\mathscr{B}_1$ have α^+ edges.

Though most of the results are negative and technically complicated to prove, they are surprisingly sharp. That is why we think it is worth to give them in detail.

DEFINITION 4. 1. $\mathscr{B}^7_\alpha = \{X \subset [\alpha]^2 : \langle \alpha, X\rangle$ contains an infinite path$\}$.

THEOREM 4. 1. *Assume G.C.H., $\alpha \geqq \omega$. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & 1 & \alpha \\ \mathscr{B}^3_{\alpha^+} & \vee\ \mathscr{B}^7_{\alpha^+}, & \mathscr{B}^1_{\alpha^+}\end{pmatrix}^{1,2}$$

i.e. in a set of power α^+ we can define α^+ forests not containing infinite paths so that given α^+ edges all but less than α of the forests have an edge among the given edges.

Note that $\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \alpha \\ \mathscr{B}^3_{\alpha^+}, & \mathscr{B}^1_{\alpha^+}\end{pmatrix}^{1,2}$ could technically much simpler be proved.

PROOF. We will define a sequence $I_\xi \subset [\alpha^+]^2$, $\xi < \alpha^+$ with the intention that the partitions

(1) $$I_\xi = I_0^\xi,\ I_1^\xi = [\alpha^+]^2 - I_0^\xi,\ [\alpha^+]^2 = I_0^\xi \cup I_1^\xi$$

should establish the required counterexample.

For each $\varrho < \alpha^+$ we will define a function β_ϱ and its domain $D_\varrho \subset \varrho$, $\beta_\varrho \in {}^{D_\varrho}\varrho$ and we will put

(2) $$I_\xi = \{\{\beta_\varrho(\xi), \varrho\} : \xi \in D_\varrho \wedge \varrho < \alpha^+\} \quad \text{for every} \quad \xi < \alpha^+.$$

We will define β_ϱ, D_ϱ and a one-to-one mapping φ_ϱ of D_ϱ onto an ordinal $\leqq \alpha$ by transfinite induction on ϱ.

By G.C.H., there exists a well-ordering $\{R_\nu\}_{\nu < \alpha^+} = [\alpha^+]^\alpha$ of type α^+ of $[\alpha^+]^\alpha$.

Assume $\varrho < \alpha^+$ and β_σ, D_σ, φ_σ are already defined for every $\sigma < \varrho$.

We want β_ϱ, D_ϱ, and φ_ϱ to satisfy the following conditions (3) and (4)

(3) For every $\sigma < \varrho$ and for every $\nu < \varrho$ $R_\nu \subset \varrho$ there is a $\xi \in R_\nu$ such that

$$\beta_\varrho(\xi) = \sigma.$$

(4) If $\xi \in D_\sigma \cap D_\varrho$, $\sigma < \varrho$ and $\beta_\varrho(\xi) = \sigma$ then

$$\varphi_\sigma(\xi) > \varphi_\varrho(\xi).$$

We define $\varphi_\varrho^{-1}(\mu)$ by transfinite induction on μ. Let $\mathscr{H} = \{R_\nu : \nu < \varrho \wedge R_\nu \subset \varrho\}$.

If $\mathscr{H} = 0$ put $D_\varrho = \varphi_\varrho = \beta_\varrho = 0$. If $\mathscr{H} \neq 0$, $\varphi_\varrho^{-1}(\mu)$ will be defined for every $\mu < \alpha$.

Let $\{\langle P_\mu, \sigma_\mu\rangle\}_{\mu<\alpha}$ be a sequence containing all elements of $\mathscr{H}\times\varrho$ (may be with repetitions) and suppose that $\varphi_\varrho^{-1}(\tau)$ is defined for every $\tau<\mu$ for some $\mu<\alpha$.

(5) If $|P_\mu \cap D_{\sigma_\mu}|<\alpha$ then let $\varphi_\varrho^{-1}(\mu)$ be an element of $P_\mu - D_{\sigma_\mu} - \{\varphi_\varrho^{-1}(\tau)\colon \tau<\mu\}$.

(6) If $|P_\mu \cap D_{\sigma_\mu}|=\alpha$ then φ_{σ_μ} being one-to-one $P_\mu \cap D_{\sigma_\mu} - \{\varphi_\varrho^{-1}(\tau)\colon \tau<\mu\}$ has an element ξ such that $\varphi_{\sigma_\mu}(\xi)>\mu$. Put $\varphi_\varrho^{-1}(\mu)=\xi$ for this ξ. Thus $\varphi_\varrho^{-1}(\mu)$ is defined for every $\mu<\alpha$. Put

(7) $\mathscr{R}(\varphi_\varrho^{-1}(\mu))=D_\varrho$.

Then if $\mathscr{H}\neq 0$, φ_ϱ is a one-to-one mapping of D_ϱ onto α.
Put

(8) $\beta_\varrho(\xi)=\sigma$ if $\xi\in D_\varrho$, $\xi=\varphi_\varrho^{-1}(\mu)$, $\sigma_\mu=\sigma$.

This defines $\beta_\varrho(\xi)$ for $\xi\in D_\varrho$.

Assume $\sigma<\varrho$, $\nu<\varrho$, $R_\nu\subset\varrho$. Then $R_\nu\in\mathscr{H}$, hence there is $\mu<\alpha$ such that $P_\mu=R_\nu$, $\sigma_\mu=\sigma$. Put $\xi=\varphi_\varrho^{-1}(\mu)$. Then by (7) and (8) $\xi\in D_\varrho$, $\beta_\varrho(\xi)=\sigma$. By (5) and (6) $\xi\in R_\nu=P_\mu$. Thus (3) is satisfied. Assume $\xi\in D_\sigma\cap D_\varrho$, $\beta_\varrho(\xi)=\sigma$. Then by (5) and (6) $\xi=\varphi_\varrho^{-1}(\mu)$, $\sigma=\sigma_\mu$, $\varphi_{\sigma_\mu}(\xi)>\mu=\varphi_\varrho(\xi)$. Thus β_ϱ, D_ϱ and φ_ϱ satisfy (4) as well.

It remains to show that the I_ξ defined by (2) and the I_0^ξ, I_0^ξ, defined by (1) satisfy the requirements of our theorem.

By (2) for every pair $\xi<\alpha^+$, $\varrho<\alpha^+$ there is at most one $\sigma<\varrho$ for which $\{\sigma,\varrho\}\in I_\xi$. This means that the I_ξ are forests, i.e. $I_0^\xi\notin\mathscr{B}^3_{\alpha^+}$ for $\xi<\alpha^+$. Using the above property of the I_ξ, if it contains an infinite path, then there is an increasing sequence $\{\varrho_n\}_{n<\omega}$ of type ω of ordinals $<\alpha^+$ such that $\{\varrho_n, \varrho_{n+1}\}\in I_\xi$ for every $n<\omega$. Then by (2) $\xi\in D_{\varrho_{n+1}}$, $\beta_{\varrho_{n+1}}(\xi)=\varrho_n$ for $n<\omega$. Hence by (4) $\varphi_{\varrho_n}(\xi)>\varphi_{\varrho_{n+1}}(\xi)$ for $n<\omega$, a contradiction. It follows by (1) that $I_0^\xi\notin\mathscr{B}^7_{\alpha^+}$.

Let now $X\in B^1_{\alpha^+}$ i.e. $X\subset[\alpha^+]^2$, $|X|=\alpha^+$. Put $T=\{\varrho<\alpha^+\colon \exists\sigma(\sigma<\varrho\wedge\{\sigma,\varrho\}\in X)\}$. Then $|T|=\alpha^+$. Let $C\subset\alpha^+$, $|C|=\alpha$. Then there is a $\nu<\alpha^+$ such that $C=R_\nu$. There is a $\varrho<\alpha^+$ such that $\nu<\varrho$, $R_\nu\subset\varrho$ and $\varrho\in T$. There is $\sigma<\varrho$ such that $\{\sigma,\varrho\}\in X$. By (3) there is a $\xi\in D_\varrho$ such that $\xi\in C$, $\beta_\varrho(\xi)=\sigma$. By (2) that means $\{\sigma,\varrho\}\in I_\xi$, hence

$$X\cap I_\xi\neq 0 \quad \text{for} \quad \xi\in C.$$

By (1) that means $C\subset\alpha^+$, $|C|=\alpha^+$, $X\in B^1_{\alpha^+}$ imply

$$X\not\subset\bigcap_{\xi\in C} I_1^\xi$$

This proves the theorem.

Our next theorem shows that the forests defined in Theorem 4. 1 can not be edge disjoint.

THEOREM 4. 2. *Let $\alpha\geqq\omega$, $\beta<\alpha$. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix}\to\begin{pmatrix}2 & \alpha^+\\ \mathscr{B}^2_{\alpha^+}, & \mathscr{B}_{\alpha^+,\beta,\alpha^+}\end{pmatrix}^{1,2}.$$

PROOF. Let $[\alpha^+]^2=I_0^\xi\cup I_1^\xi$ for $\xi<\alpha^+$ be arbitrary. Put briefly $I_0^\xi=I_\xi$. We assume that the I_ξ are disjoint.

Let $B \subset \alpha^+$, $|B| = \beta$ be arbitrary. For each $\zeta \in \alpha^+ - B$ put $V_\zeta = \{\xi < \alpha^+ : I_\xi \cap [B, \{\zeta\}] \neq 0\}$. By the assumption $|V_\zeta| \leqq \beta$ for each $\zeta \in \alpha^+ - B$. By the result of G. FODOR [7] already mentioned there are $C \subset \alpha^+$, $D \subset \alpha^+ - B$, $|C| = |D| = \alpha^+$ such that $V_\zeta \cap C = 0$ for every $\zeta \in D$. Put $X = [B, D]$. Then $X \in \mathscr{B}_{\alpha^+, \beta, \alpha^+}$ and $X \cap I_\xi = 0$ for $\xi \in C$ i.e.

$$X \subset \bigcap_{\xi \in C} I_1^\xi.$$

However one can prove a theorem corresponding to Theorem 4. 1 for edge disjoint forests as well.

THEOREM 4. 3. *Assume G.C.H., $\alpha \geqq \omega$. Then*

$$\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} 1 & 2 & \alpha^+ \\ \mathscr{B}^3_{\alpha^+} \vee \mathscr{B}^2_{\alpha^+}, & \mathscr{B}_{\alpha^+, \alpha, \alpha^+} \end{pmatrix}^{1,2}.$$

We postpone the proof to p. 384, where we are going to state two more general Theorems.

DEFINITION 4. 2. Let $\mathscr{B}^8_{\alpha,k} = \{X \subset [\alpha]^2 : \langle \alpha, X \rangle$ contains a path of length $k\}$ for $1 \leqq k < \omega$.

Note that $\mathscr{B}^8_{\alpha,1} = \mathscr{B}^2_\alpha$, $\mathscr{B}^8_{\alpha,2} = \mathscr{B}^5_\alpha$. Forests not contained in $\mathscr{B}^8_{\alpha,3}$, are usually called stars.

We will briefly write $\mathscr{B}^8_\alpha$ for $\mathscr{B}^8_{\alpha,3}$. Obviously $\mathscr{B}^8_\alpha \subset \mathscr{B}^3_\alpha$.

We turn back to the problem considered in Theorem 4. 1.

A very strong negative result holds still if we assume that the forests defined in Theorem 4. 1 are even smaller.

THEOREM 4. 4. *Assume G.C.H., $\alpha \geqq \omega$. Then*

a) $$\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} 1 & \alpha \\ \mathscr{B}^8_{\alpha^+}, & \mathscr{B}_{\alpha^+, \omega, \alpha^+} \end{pmatrix}^{1,2} \quad \textit{if} \quad \mathrm{cf}(\alpha) > \omega,$$

b) $$\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} 1 & \alpha \\ \mathscr{B}^8_{\alpha^+}, & \mathscr{B}_{\alpha^+, \delta, \alpha^+} \end{pmatrix}^{1,2} \quad \textit{where} \quad \delta = \min[\alpha^+, \omega_2] \quad \textit{if} \quad \mathrm{cf}(\alpha) = \omega.$$

We mention

PROBLEM 3. Assume G.C.H. Let $\alpha = \omega_\omega$. Does

$$\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} 1 & \alpha \\ \mathscr{B}^8_{\alpha^+}, & \mathscr{B}_{\alpha^+, \omega_1, \alpha^+} \end{pmatrix}^{1,2} \quad \text{hold?}$$

PROOF OF THEOREM 4. 4. Theorems of [2] say that if G.C.H. is assumed then the following relations hold:

$$\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} \alpha^+ & \omega & 1 \\ \alpha &, \alpha & \vee \alpha^+ \end{pmatrix}^{1,1} \quad \text{for} \quad \mathrm{cf}(\alpha) > \omega.$$

$$\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} \alpha^+ & \omega_2 & 1 \\ \alpha &, \alpha & \vee \alpha^+ \end{pmatrix}^{1,1} \quad \text{for} \quad \mathrm{cf}(\alpha) = \omega_1, \quad \alpha > \omega_1.$$

$$\begin{pmatrix} \omega_1 \\ \omega_1 \end{pmatrix} \nrightarrow \begin{pmatrix} \omega_1 & \omega_1 & 1 \\ \omega &, \omega & \vee \omega_1 \end{pmatrix}^{1,1}.$$

Note that we do not know if ω_2 can be replaced by ω_1 in the second negative relation (see [2]). This explains why we have Problem 3 unsolved.

By definitions the above results mean the following:

(1) There exists a sequence $\{S_\varrho\}_{\varrho<\alpha^+}$ of subsets of α^+, $|S_\varrho|=\alpha$ for $\varrho<\alpha$ satisfying the following conditions:

(a) If $E\subset\alpha^+$, $|E|=\alpha^+$ then $|\alpha^+ - \bigcup_{\varrho\in E} S_\varrho| < \alpha$

(b) If $F\subset\alpha$ then $|\bigcap_{\varrho\in F} S_\varrho| < \alpha$ provided one of the following conditions holds.

(i) $|F|=\omega \wedge \mathrm{cf}(\alpha)>\omega$, (ii) $|F|=\omega_2\wedge \mathrm{cf}(\alpha)=\omega_1\wedge\alpha>\omega$,
(iii) $|F|=\omega_1\wedge\alpha=\omega$.

Let $\{R_\nu\}_{\nu<\alpha^+}=[\alpha^+]^\alpha$ be a well-ordering of type α^+ of $[\alpha^+]^\alpha$.

We will define a sequence I_ξ, $\xi<\alpha^+$, $I_\xi\subset[\alpha^+]^2$ with the intention that

(2) $$I_0^\xi = I_\xi,\quad I_1^\xi = [\alpha^+]^2 - I_\xi$$

should establish the required counterexample.

Similarly as in the proof of Theorem 4. 1 we will define a function β_ϱ for every $\varrho<\alpha^+ (\beta_\varrho\in {}^{D_\varrho}\varrho,\ D_\varrho\subset S_\varrho)$ and we will put

(3) $$I_\xi = \big\{\{\beta_\varrho(\xi),\varrho\}\colon \text{for } \varrho<\alpha^+,\ \xi\in D_\varrho\big\}\quad \text{for every}\quad \xi<\alpha^+.$$

We want β_ϱ to satisfy the following condition:

(4) For each $\sigma<\varrho$ and for each $\nu<\varrho$ for which $|R_\nu\cap S_\varrho - S_\sigma| = \alpha$ there is a $\xi\in(R_\nu\cap S_\varrho)-S_\sigma$ such that $\sigma=\beta_\sigma(\xi)$. To do this we need the following lemma. Let H be a set, H_σ $\sigma<\varrho$, $|\varrho|\leqq\alpha$ be a sequence of subsets of H, and let $\mathscr{H}_\sigma\subset[H_\sigma]^\alpha$, $|\mathscr{H}_\sigma|\leqq\alpha$ for $\sigma<\varrho$. Then there exists a sequence $T_\sigma\subset H_\sigma$ $\sigma<\varrho$ such that the T_σ are disjoint and each T_σ meets each element of $\mathscr{H}_\sigma$. This is an easy generalization of a well-known theorem of F. BERNSTEIN. The proof is left to the reader.

Put $H=S_\varrho$, $H_\sigma = S_\varrho - S_\sigma$

$$\mathscr{H}_\sigma = \{(R_\nu\cap S_\varrho)-S_\sigma\colon \nu<\varrho\wedge|(R_\nu\cap S_\varrho)-S_\sigma| = \alpha\}\qquad\text{for}\quad \sigma<\varrho.$$

We obtain the existence of T_σ and we put $\beta_\varrho(\xi)=\sigma$ for $\xi\in T_\sigma$ (hence $D_\varrho=D(\beta_\varrho)=\bigcup_{\varrho<\sigma} T_\sigma$). Then the β_ϱ satisfy (4). The $I_\xi\notin\mathscr{B}^3_{\alpha^+}$ since by (3) for each ϱ there is at most one $\sigma<\varrho$ for $\{\sigma,\varrho\}\in I_\xi$. Using this property it is easy to see that if an I_ξ contained a path of length 3 then there were $\tau<\sigma<\varrho$ such that both $\{\tau,\sigma\}$ and $\{\sigma,\varrho\}$ would belong to I_ξ. By (3) then $\beta_\varrho(\xi)=\sigma$, $\beta_\sigma(\xi)=\tau$ hence by (4) $\xi\in(S_\sigma-S_\tau)\cap(S_\varrho-S_\sigma)$ a contradiction. Thus $I_\xi\notin\mathscr{B}^8_{\alpha^+}$. Hence the I_ξ are stars.

Let now $X=[F,E]\in\mathscr{B}_{\alpha^+,\delta,\alpha^+}$ where $\delta=\omega$, $\delta=\omega_2$, $\delta=\omega_1$, if $\mathrm{cf}(\alpha)>\omega$ $\mathrm{cf}(\alpha)=\omega\wedge\alpha>\omega_1$, $\alpha=\omega_1$ respectively, and let $C\subset\alpha^+$, $|C|=\alpha$. Then there is a $\nu<\alpha^+$ such that $C=R_\nu$. By the assumption (1),b we have

(5) $$|R_\nu\cap\bigcap_{\sigma\in F'} S_\sigma| < \alpha$$

for every $F'\subset F$, $|F'|=\delta$. Using $\mathrm{cf}(\delta)\neq\mathrm{cf}(\alpha)$ and the theorem: $\begin{pmatrix}\delta\\ \alpha\end{pmatrix}\to\begin{pmatrix}\delta & 1\\ \alpha, & \alpha\end{pmatrix}^{1,1}$

for cf $(\delta)=$ cf (α) of [2] it results from (5) that there is a $\sigma\in F$ for which $|R_\nu - S_\sigma|=\alpha$. Applying $\begin{pmatrix}\alpha^+\\ \alpha\end{pmatrix}\to\begin{pmatrix}\alpha^+ & 1\\ \alpha & ,\alpha\end{pmatrix}^{1,1}$ again it follows that there is a $\varrho\in E$, $\varrho>\max[\nu,\sigma]$ such that

$$|(R_\nu\cap S_\varrho)-S_\sigma| = \alpha.$$

By (4) there is a $\xi\in R_\nu$ such that $\sigma=\beta_\varrho(\xi)$. By (3) that means $\{\sigma,\varrho\}\in[F,E]\cap I_\xi$. By (2) this means

$$X\not\subset\bigcap_{\xi\in C} I^1_\xi.$$

Our next theorem shows that except for the case stated in Problem 3, Theorem 4. 4 is best possible of its kind.

DEFINITION 4. 3. Let $\mathscr{B}^9_{\alpha,k}=\{X\subset[\alpha]^2$: there are $\varrho_0<\cdots<\varrho_k<\alpha^+$ such that $\{\varrho_i,\varrho_{i+1}\}\in X$ for $i<k\}$ for $1\leqq k<\omega$, i.e $\langle\alpha, X\rangle$ contains an increasing path of length k. Obviously $\mathscr{B}^9_{\alpha,k}\subset\mathscr{B}^8_{\alpha,k}$ and we have

THEOREM 4. 5. *Assume G.C.H.,* $\alpha\geqq\omega$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix}\to\begin{pmatrix}1 & \alpha\\ \mathscr{B}^9_{\alpha,k} & ,\mathscr{B}_{\alpha^+,\delta,\alpha^+}\end{pmatrix}^{1,2}\quad \textit{for every}\quad k<\omega$$

and for $\delta<\omega$ *or* $\delta=\omega$ *if* cf $(\alpha)>\omega$ *or* cf $(\alpha)=\omega$ *respectively.*

PROOF. We prove the statement by induction on k. For $k=1$ it is trivial. Assume $k>1$ and the statement is true for $k-1$.

Let $[\alpha^+]^2=I^\xi_0\cup I^\xi_1$ for $\xi<\alpha$. Put briefly $I^\xi_0=I_\xi$ for $\xi<\alpha$. Put

(1) $T_\xi=\{\zeta<\alpha^+:\zeta$ is the greatest point of an increasing path of length $k-1$ contained in $I_\xi\}$. Theorems of [2] say that

$$\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix}\to\begin{pmatrix}\alpha & \alpha\\ \alpha^+ & ,\delta\end{pmatrix}^{1,1}\quad\text{for}\quad\delta<\omega$$

and

$$\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix}\to\begin{pmatrix}\alpha & \alpha\\ \alpha^+ & ,\delta\end{pmatrix}^{1,1}\quad\text{for}\quad\delta\leqq\omega\quad\text{if}\quad\mathrm{cf}(\alpha)=\omega.$$

It results that one of the following conditions hold:

(2) There are $C\subset\alpha$, $D\subset\alpha^+$, $|C|=\alpha$, $|D|=\alpha^+$ such that $D\cap T_\xi=0$ for every $\xi\in C$.

(3) There are $C\subset\alpha$, $D\subset\alpha^+$, $|C|=\alpha$ such that

$$D\subset\bigcap_{\xi\in D}T_\xi$$

and $|D|=\delta$ or $|D|=\omega$ if cf$(\alpha)\neq\omega$ or cf$(\alpha)=\omega$ respectively. If (2) holds then $\bar I^\xi_0=[D]^2\cap I_\xi$, $I^\xi_1=[D]^2-\bar I^\xi_0$, $\xi\in C$ are α 2-partitions of a set of power α^+, and by (1) $\bar I^\xi_0\notin\mathscr{B}^9_{\alpha,k-1}$; hence the result follows from the induction hypothesis.

Assume (3) holds. Let E be an arbitrary subset of α^+ such that $D<E$, $|E|=\alpha^+$. Then again by (1) $[D,E]\cap I_\xi=0$ for every $\xi\in C$; hence $[D,E]\subset\bigcap_{\xi\in C}I'_\xi$. Note

that in case $\alpha=\omega$ G.C.H. is not used since $\binom{\omega}{\omega_1} \to \binom{\omega \quad \omega}{\omega_1, \omega}^{1,1}$ can be proved without any hypothesis.

DEFINITION 4. 4. $\mathscr{B}_\alpha^{10} = \bigcap_{1 \leqq k < \omega} \mathscr{B}_{\alpha,k}^8$.

Note that $\mathscr{B}_\alpha^7$ is a proper subset of $\mathscr{B}_\alpha^{10}$. Assuming G.C.H. Theorem 4. 5 implies trivially for $\alpha \geqq \omega$ that

$$\binom{\alpha^+}{\alpha^+} \to \binom{1 \quad \alpha}{\mathscr{B}_\alpha^{10}, \mathscr{B}_{\alpha^+,\delta,\alpha^+}}^{1,2} \quad \text{for} \quad \delta < \omega$$

and

$$\binom{\alpha}{\alpha^+} \to \binom{1 \quad \alpha}{\mathscr{B}_\alpha^{10}, \mathscr{B}_{\alpha^+,\delta,\alpha^+}}^{1,2} \quad \text{for} \quad \delta < \omega \quad \text{if} \quad \mathrm{cf}(\alpha) \neq \omega.$$

However the following improvement of Theorem 4. 1 is still possible.

THEOREM 4. 6. *Assume G.C.H.* $\mathrm{cf}(\alpha)=\omega$. *Then*

$$\binom{\alpha}{\alpha^+} \not\to \binom{1 \qquad 1 \qquad \alpha}{\mathscr{B}_{\alpha^+}^3 \vee \mathscr{B}_\alpha^{10}, \mathscr{B}_{\alpha^+}^1}^{1,2}.$$

PROOF. We will define a sequence $I_\zeta = I_0^\zeta \subset [\alpha^+]^2$, $I_1^\zeta = [\alpha^+]^2 - I_0^\zeta$, $\zeta < \alpha$ with the intention that the partitions

(1) $[\alpha^+]^2 = I_0^\zeta \cup I_1^\zeta$, $\zeta < \alpha$

should establish the required counterexample. For each $\zeta < \alpha^+$ we will define a function β_ϱ and its domain $D_\varrho \subset \alpha$, $\beta_\varrho \in {}^{D_\varrho}\varrho$ and we will put

(2) $I_\zeta = \{\{\beta_\varrho(\zeta), \varrho\} : \zeta \in D_\varrho, \varrho < \alpha^+\}$ for every $\zeta < \alpha$.

Similarly as in the proof of Theorem 4. 1, we will define β_ϱ, D_ϱ and a one-to-one mapping φ_ϱ of D_ϱ onto an ordinal $\leqq \alpha$, by transfinite induction on ϱ.

By G.C.H. there exists a well-ordering $\{R_\nu\}_{\nu<\alpha^+} = [\alpha]^\alpha$ of type α^+ of $[\alpha]^\alpha$. Considering that $\mathrm{cf}(\alpha)=\omega$ we may assume that

(3) $\alpha = \bigcup_{k<\omega} A_k$ where $|A_k| < \alpha$ and the A_k are disjoint, $|A_0| < \ldots < |A_k| < \ldots$.

Put $k(\zeta) = k$ if $\zeta \in A_k$.

Assume $\varrho < \alpha^+$ and β_σ, D_σ, φ_σ are defined for every $\sigma < \varrho$. We want β_ϱ, D_ϱ and φ_ϱ to satisfy the following conditions (4) and (5):

(4) For every $\sigma < \varrho$ and for every $\nu < \varrho$, there is a $\zeta \in R_\nu$ such that $\beta_\varrho(\zeta) = \sigma$.

(5) If $\zeta \in D_\sigma \cap D_\varrho$, $\beta_\varrho(\zeta) = \sigma$ for $\sigma < \varrho$ then $k(\varphi_\sigma(\zeta)) > k(\varphi_\varrho(\zeta)$, and $k(\varphi_\varrho(\zeta)) < k(\zeta)$ for $\zeta < \alpha$.

We define $\varphi_\varrho^{-1}(\mu)$ by transfinite induction on μ. Let $\mathscr{H} = \{R_\nu : \nu < \varrho\}$. If $\mathscr{H} = 0$ or $\varrho = 0$ put $D_\varrho = \beta_\varrho = \varphi_\varrho = 0$. If $\mathscr{H} \neq 0$, $\varrho > 0$, $\varphi_\varrho^{-1}(\mu)$ will be defined for every $\mu < \alpha$.

Let $\{\langle P_\mu, \sigma_\mu\rangle\}_{\mu<\alpha}$ be a sequence containing all the elements of $\mathscr{H}\times\varrho$ (may be with repetitions) and suppose that $\varphi_\varrho^{-1}(\tau)$ is defined for every $\tau<\mu$ for some $\mu<\alpha$.

(6) If $|P_\mu\cap D_{\sigma_\mu}|<\alpha$, then $|P_\mu-D_{\sigma_\mu}-\{\varphi_\varrho^{-1}(\tau)\}_{\tau<\mu}|=\alpha$. It follows from (3) that there exists a $\zeta\in P_\mu-D_{\sigma_\mu}-\{\varphi_\varrho^{-1}(\tau)\}_{\tau<\mu}$ such that $k(\zeta)>k(\mu)$. Put $\zeta=\varphi_\varrho^{-1}(\mu)$ for this ζ.

(7) If $|P_\mu\cap D_{\sigma_\mu}|=\alpha$, then φ_{σ_μ} being one-to-one by (3) $P_\mu\cap D_{\sigma_\mu}-\{\varphi_\varrho^{-1}(\tau)\}_{\tau<\mu}$ has an element ζ such that $k(\varphi_{\sigma_\mu}(\zeta))>k(\mu)$. Put $\varphi_\varrho^{-1}(\mu)=\zeta$ for this ζ. Thus $\varphi_\varrho^{-1}(\mu)$ is defined for every $\mu<\alpha$. Put

(8) $\mathscr{R}(\varphi_\varrho^{-1})=D_\varrho$. Then it $\mathscr{H}\neq 0$, $\varrho>0$ φ_ϱ is a one-to-one mapping of D_ϱ onto α. Put

(9) $\beta_\varrho(\zeta)=\sigma$ if $\zeta\in D_\varrho$, $\zeta=\varphi_\varrho^{-1}(\mu)$, $\sigma_\mu=\sigma$. This defines $\beta_\varrho(\zeta)$ for $\zeta\in D_\varrho$. Assume $\sigma<\varrho$, $\nu<\varrho$. Then $R_\nu\in\mathscr{H}$ hence there is a $\mu<\alpha$ such that $P_\mu=R_\nu$, $\sigma_\mu=\sigma$. Put $\zeta=\varphi_\varrho^{-1}(\mu)$. Then by (8) and (9) $\zeta\in D_\varrho$, $\beta_\varrho(\zeta)=\sigma$. By (6) and (7) $\zeta\in R_\nu=P_\mu$.

Thus (4) is satisfied.

Assume $\zeta\in D_\sigma\cap D_\varrho$ for some $\sigma<\varrho$, and $\beta_\varrho(\zeta)=\sigma$. Then by (8) and (9) $\zeta=\varphi_\varrho(\mu)$ for a $\mu<\alpha$, $\sigma=\sigma_\mu$ for this μ and by (6) $|P_\mu\cap D_\sigma|=\alpha$, hence by (7) $k(\varphi_\sigma(\zeta))>$ $>k(\varphi_\varrho(\zeta)=k(\mu)$. This proves the first statement of (5). To prove the second statement we use transfinite induction on ϱ. Assume that $k(\varphi_\sigma(\zeta))<k(\zeta)$ for every $\zeta\in D_\sigma$ for every $\sigma<\varrho$. Let $\zeta\in D_\varrho$. Then by (8) and (9) there is a $\mu<\alpha$ such that $\zeta=\varphi_\varrho^{-1}(\mu)$ and $\beta_\varrho(\xi)=\sigma_\mu$. If $|P_\mu\cap D_{\sigma_\mu}|<\alpha$ then by (6) $k(\zeta)>k(\mu)=k(\varphi_\varrho(\zeta))$. If $|P_\mu\cap D_{\sigma_\mu}|=\alpha$ then by (7) $\zeta\in D_{\sigma_\mu}$ and by the first statement of (5) already proved we have

$$k(\varphi_{\sigma_\mu}(\zeta))>k(\varphi_\varrho(\zeta));$$

hence the statement follows from the induction hypothesis.

It remains to show that the I_ζ defined by (2) and the I_0^ζ, I_1^ζ, defined by (1) satisfy the requirements of our theorem.

By (2) for every pair $\zeta<\alpha$, $\varrho<\alpha^+$ there is at most one $\sigma<\varrho$ for which $\{\sigma,\varrho\}\in I_\zeta$; this means that the I_ζ are forests, i.e. $I_0^\zeta\notin B_{\alpha^+}^3$ for $\zeta<\alpha$.

Using the above property of the I_ζ if it contains a path of length $2k-1$ then it contains an increasing path of length k. We will show that if $k(\zeta)=k$, then I_ζ does not contain an increasing path of length k. Assume $\varrho_0<\dots<\varrho_k<\alpha^+$ and $\{\varrho_i,\varrho_{i+1}\}\in I_\zeta$ for $i<k$. Then by (8) and (9) we have $\varrho_i=\beta_{\varrho_{i+1}}(\zeta)$ for $i<k$ and $\zeta\in D_{\varrho_i}\cap D_{\varrho_{i+1}}$ for $0<i<k$. Then by (5) $k>k(\varphi_{\varrho_0}(\zeta))>\dots>k(\varphi_{\varrho_k}(\zeta))\geqq 0$ a contradiction. Hence $I_0^\zeta\notin\mathscr{B}_{\alpha^+,k}^9$ and then by the above remark $I_0^\zeta\in\mathscr{B}_{\alpha^+,2k-1}^8$ for $k=k(\zeta)$, hence $I_0^\zeta\notin\mathscr{B}_{\alpha^+}^{10}$ for every $\zeta<\alpha$.

Let now $X\in\mathscr{B}_\alpha^1$, $C\subset\alpha$, $|C|=\alpha$ the statement that $X\not\subset\bigcup_{\xi\in C} I'_\xi$ follows from (4) literally the same way as in the proof of Theorem 4. 1.

Now we turn to the investigation of the case of edge disjoint forests

THEOREM 4. 7. *Assume G.C.H. Let $\alpha\geqq\omega$. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix}\nrightarrow\begin{pmatrix}1 & & 1 & & 2 & & \alpha\\ \mathscr{B}_{\alpha^+}^3 & \vee & \mathscr{B}_{\alpha^+}^7 & \vee & \mathscr{B}_{\alpha^+}^2, & & \mathscr{B}_{\alpha^+,\alpha,\alpha^+}\end{pmatrix}^{1,2}.$$

THEOREM 4. 8. *Assume G.C.H. and* $\alpha>\omega$. *Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & & 1 & & 2 & & \alpha^+\\ \mathscr{B}^3_{\alpha^+} & \vee & \mathscr{B}^8_{\alpha^+} & \vee & \mathscr{B}^2_{\alpha^+} & , & \mathscr{B}_{\alpha^+,\alpha,\alpha^+}\end{pmatrix}^{1,2}.$$

Both theorems are generalizations of Theorem 4. 3. Theorem 4. 2 shows that on the right hand side in $B_{\alpha^+,\alpha,\alpha^+}$ α can not be replaced by anything smaller. Theorem 4. 8 is sharper than Theorem 4. 7 but for $\alpha=\omega$ it is false by Theorem 4. 5.

We describe here the proof of Theorem 4. 3. The proofs of Theorems 4. 7 and 4. 8 can be obtained from this proof with a sligth modification using the tricks of Theorem 4. 1 and 4. 4 respectively. We will omit this.

PROOF OF THEOREM 4. 3. As in the preceeding proofs we will define a sequence I_ξ, $\xi<\alpha^+$; $I_\xi\subset[\alpha^+]^2$ with the intention that the partitions

(1) $I_0^\xi\cup I_1^\xi=[\alpha^+]^2$, $I_0^\xi=I_\xi$, $I_1^\xi=[\alpha^+]^2-I_0^\xi$ $\xi<\alpha^+$

should establish the required counterexample.

For each $\alpha\leqq\varrho<\alpha^+$ we will define a function $\beta_\varrho\in{}^\varrho\varrho$ and we put

(2) $I_\xi=\{\{\beta_\varrho(\xi),\varrho\}: \xi<\varrho, \alpha\leqq\varrho<\alpha^+\}$ for every $\xi<\alpha^+$.

Let $\{R_\nu\}_{\nu<\alpha^+}=[\alpha^+]^\alpha$ be a well-ordering of type α^+ of $[\alpha^+]^\alpha$. For every fixed ϱ, $\alpha\leqq\varrho<\alpha^+$ we will define β_ϱ so that it should satisfy the following requirements (3) and (4):

(3) Assume $\nu,\mu<\varrho$, $R_\nu\subset\varrho$, $R_\mu\subset\varrho$. Then there is a $\xi<\varrho$ such that $\beta_\varrho(\xi)\in R_\nu$, $\xi\in R_\mu$

(4) β_ϱ is one-to-one.

To do this we need the following lemma. If H is a set of power $\alpha\geqq\omega$, $\mathscr{H}_0,\mathscr{H}_1\subset[H]^\alpha$; $|\mathscr{H}_0|,|\mathscr{H}_1|\leqq\alpha$ then there is an $f\in{}^HH$ such that f is one-to-one and for every $H_0\in\mathscr{H}_0$, $H_1\in\mathscr{H}_1$ there are $h_0\in H_0$, $h_1\in H_1$ such that $f(h_0)=h_1$. The proof can be carried out by an easy transfinite induction; we omit it.

Applying this for $H=\varrho$, $\mathscr{H}_0-\mathscr{H}_1=\{R_\nu: R_\nu\subset\varrho\wedge\nu<\varrho\}$ we obtain an f and put $f=\beta_\varrho$. We prove that I_0^ξ, I_1^ξ satisfy the requirements of our theorem. If $\sigma<\varrho$, by (4), there is at most one $\xi<\alpha^+$ for which $\sigma=\beta(\xi)$, hence by (2) the I_ξ are disjoint, i.e. there is no $X\in\mathscr{B}^2_{\alpha^+}$, $X\subset I_{\xi_0}\cap I_{\xi_1}$ for $\xi_0\neq\xi_1<\alpha^+$. By (2) for each $\xi,\varrho<\alpha^+$. There is at most one $\sigma<\varrho$ for which $\beta_\varrho(\xi)=\sigma$. It follows that the I_ξ are forests, i.e. $I_\xi\notin\mathscr{B}^3_{\alpha^+}$.

Let now $X=[B,D]\in\mathscr{B}_{\alpha^+,\alpha,\alpha^+}$, $C\subset\alpha^+$, $|C|=\alpha$. Then there are $\nu,\mu<\alpha^+$ such that $B=R_\nu$, $C=R_\mu$. There is a $\varrho>\max[\nu,\mu,\alpha]$ such that $\varrho\in D$, R_μ, $R_\nu\subset\varrho$. Then by (3) there is a $\xi\in R_\mu$ such that $\beta_\varrho(\xi)\in R_\nu$. Hence there is a $\xi\in C$ such that $\{\beta_\varrho(\xi),\varrho\}\subset[B,D)$ i.e. by (2) $X\cap I_\xi\neq 0$.

By (1) this means

$$X\subset\bigcap_{\xi\in C} I_1^\xi.$$

§ 5. A remark on matrices of sets of Ulam type

Whenever we have a negative relation $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \nrightarrow \begin{pmatrix}\beta_{0,0} & \beta_{01}\\ \beta_{1,0}, & \beta_{11}\end{pmatrix}^{1,2}$ there are at least three natural ways to obtain matrices of sets having certain properties.

We give one example. As a corollary of Theorem 4. 1 we have

(1) Assume G.C.H. $\alpha \geqq \omega$. There exists a sequence I_ξ, $\xi < \alpha^+$; $I_\xi \subset [\alpha^+]^2$ satisfying the following conditions.

(2) For each $\varrho, \xi < \alpha^+$ there is at most one $\sigma < \varrho$ for which $\{\sigma, \varrho\} \in I_\xi$.

(3) Whenever $X \subset [\alpha^+]^2$, $C \subset \alpha^+$, $|X| = \alpha^+$, $|C| = \alpha$ then there is a $\xi \in C$ such that

$$X \cap I_\xi \neq 0.$$

We define a matrix $\{A_{\xi,\eta}\}_{\xi<\alpha^+, \eta<\alpha^+}$ of subsets of α^+ by the following stipulation:

(4) $A_{\xi,\eta} = \{\zeta : \eta < \zeta < \alpha^+ \wedge \{\eta, \zeta\} \in I_\xi\}$.

We obtain

COROLLARY 5. 1. *Assume G.C.H., $\alpha \geqq \omega$. There exists a matrix*

$$\{A_{\xi,\eta}\}_{\xi<\alpha^+, \eta<\alpha^+} \quad \textit{of subsets of} \quad \alpha^+$$

satisfying the following conditions.

(5) *For every $\xi < \alpha^+$ the sets $A_{\xi,\eta}$, $\eta < \alpha^+$ are disjoint.*

(6) *For every $C \subset \alpha^+$, $|C| = \alpha$, $f \in {}^C\alpha^+$*

$$|\alpha^+ - \bigcup_{\xi \in C} A_{\xi, f(\xi)}| < \alpha^+.$$

In fact if I_ξ, $\xi < \alpha^+$ satisfies (2) *and* (3) *then $\{A_{\xi,\eta}\}_{\xi<\alpha^+, \eta<\alpha^+}$ defined by* (4) *satisfies* (5) *and* (6), *respectively.*

Corollary 5. 1 has been stated and proved in a paper of P. ERDŐS and S. ULAM [4] independently.

§ 7. Positive results and some further counterexamples

In [3], Problem 59, we stated in an other notation the following problems:

(1)
$$\begin{pmatrix}\omega_1\\ \omega\end{pmatrix} \to \begin{pmatrix}1 & \omega\\ 4, & \omega\end{pmatrix}^{1,2} ?$$

(2)
$$\begin{pmatrix}\omega\\ \omega_1\end{pmatrix} \to \begin{pmatrix}1 & \omega\\ 4, & \omega\end{pmatrix}^{1,2} ?$$

We mentioned that if 4 is replaced by 3 we can prove a positive result.

Both problems are solved now.

As to (2) one can prove the following:

THEOREM 6. 1. *Assume* $\alpha \geqq \omega$ *and* α *is* 0, 1-*measurable*, $\beta < \alpha$. *Then*

$$\begin{pmatrix} \alpha \\ \alpha^+ \end{pmatrix} \to \begin{pmatrix} \alpha & 1 & \alpha \\ \beta & \vee & \alpha, \alpha \end{pmatrix}^{1,2}.$$

In fact one can prove the following slightly stronger result:

THEOREM 6. 2. *Assume* $\alpha \geqq \omega$, α *is* 0, 1-*measurable. Let* I_ζ, $\zeta < \alpha$ *be an arbitrary sequence* $I_\zeta \subset [\alpha^+]^2$, $\xi < \alpha$. *Then one of the following conditions* (3), (4), (5) *holds.*

(3) *There are* $X \subset \alpha^+$, $C \subset \alpha$, $|X| = \beta$, $|C| = \alpha$ *such that*

$$[X]^2 \subset \bigcap_{\xi \in C} I_\xi.$$

i.e. there are α I_ξ *whose intersection contains a complete* β.

(4) *There are* $X \subset \alpha^+$, $C \subset \alpha$, $|X| = \alpha^+$, $|C| < \alpha$ *such that*

$$[X]^2 \subset \bigcup_{\xi \in C} I_\xi,$$

i.e. there are fewer than α I_ξ *whose union contains a complete* α^+ *(and as a corollary one of them contains a complete* α*) (note that* $\alpha \to (\alpha)^2_\gamma$ *for* $\gamma < \alpha$ *if* α *is measurable).*

(5) *There are* $X \subset \alpha^+$, $C \subset \alpha$, $|X| = |C| = \alpha$ *such that*

$$[X]^2 \cap \bigcup_{\xi \in C} I_\xi = 0,$$

i.e. there are α I_ξ *such that the intersection of the complement of them contains a complete* α.

As to the problem (1) we proved

(6) Assume $\alpha \geqq \omega$, α is 0, 1-measurable, $\beta < \alpha$, then

$$\begin{pmatrix} \alpha^+ \\ \alpha \end{pmatrix} \to \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}^{1,2}_{\beta}.$$

F. Galvin proved the following generalization of (6) for $\alpha = \omega$:

THEOREM OF GALVIN.

$$\begin{pmatrix} \omega_1 \\ \omega \end{pmatrix} \to \begin{pmatrix} \omega \\ \omega \end{pmatrix}^{1,r}_{k} \quad \textit{for every} \quad k, r < \omega;$$

but this method of proof breaks down for $\alpha > \omega$ α 0, 1-measurable.

One can prove the following generalization of Galvin's theorem.

THEOREM 6. 3. *Assume* $\alpha \geqq \omega$, α *is* 0, 1-*measurable. Then*

$$\begin{pmatrix} \alpha^+ \\ \alpha \end{pmatrix} \to \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}^{1,r}_{\beta} \quad \textit{for} \quad r < \omega, \ \beta < \alpha.$$

DEFINITION 6. 1. Let $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma$ cardinals. $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}_{\gamma}^{1,<\omega}$ denotes that the following statement is true. Assume that for every $r<\omega$ $\alpha_0 X[\alpha_1]^r = \bigcup_{\nu<\gamma} I_\nu^r$. Then there are $A_0 \subset \alpha$, $A_1 \subset \alpha$, $f \in {}^\omega\gamma$ such that $|A_0| = \beta_0$, $|A_1| = \beta_1$ and

$$A_0 \times [A_1]^r \subset I_{f(r)}^r, \quad \text{for} \quad r<\omega.$$

The symbol defined above is a common generalization of the symbols defined in 1. 1. 2 and 1. 1. 3.

Galvin conjectured that as a generalization of the author's results [5] that for $\alpha > \omega$, α 0, 1-measurable $\alpha \to (\alpha)_\beta^{<\omega}$ $(\beta<\alpha)$ the following result will hold.

THEOREM 6. 4. *Assume $\alpha > \omega$, α is* 0, 1*-measurable. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha\end{pmatrix} \to \begin{pmatrix}\alpha\\ \alpha\end{pmatrix}^{1,<\omega}.$$

This was proved by the second author. The proofs of the theorems 6. 1, 6. 2, 6. 3, 6. 4 will appear in a forthcoming paper [6] in the *Fundamenta Mathematicae* containing the results of the lecture given by the second author on the symposium held in Warsaw August 27—September 2, 1968.

We now give some counterexamples to show that Theorems 6. 1, 6. 3 cannot be improved in certain directions.

THEOREM 6. 5. *Assume $\alpha \geqq \omega$, α is a strong limit cardinal. Then*

$$\begin{pmatrix}\alpha\\ 2^\alpha\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \alpha\\ \alpha & 2\end{pmatrix}^{1,2}.$$

PROOF. We define a sequence I_ζ, $\zeta<\alpha$; $I_\zeta \subset [{}^\alpha 2]^2$ as follows.

(7) Assume $f \neq g \in {}^\alpha 2$. Put

$$\{f, g\} \in I_\zeta \quad \text{iff} \quad \min\{\xi : f(\xi) \neq g(\xi)\} < \zeta.$$

Assume $[X]^2 \subset I_\zeta$ for some $X \subset {}^\alpha 2$, $\zeta<\alpha$. Then, by (7), for $f \in X$ $f: \to f \restriction \zeta$ is a one-to-one mapping of X into ${}^\zeta 2$. Hence, α being strong limit, $|X| \leqq 2^{|\zeta|} < \alpha$. That means none of the I_ζ contains a complete α.

Assume now that $\{f, g\} \in [{}^\alpha 2]^2$. Then $f(\xi) \neq g(\xi)$ for some $\xi<\alpha$ and by (7) $\{f, g\} \in I_\zeta$ for every $\xi<\zeta<\alpha$. This proves the theorem.

Assuming G.C.H., Theorem 6. 5 says that $\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \alpha\\ \alpha & 2\end{pmatrix}^{1,2}$ if α is a limit cardinal.

Strangely enough this very weak counter-example cannot be proved if α is a successor cardinal.

THEOREM 6. 7. *Assume G.C.H., $\alpha \geqq \omega$. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^{++}\end{pmatrix} \to \begin{pmatrix}1 & \alpha^+\\ \alpha^+ & 2\end{pmatrix}^{1,2}$$

and $\alpha^{++} \to [\alpha^+]^2_{\alpha^+,\alpha}$ are equivalent.

Note that $\alpha^{++}\to[\alpha^+]^2_{\alpha^+,\alpha}$ is known to be independent of the axioms of set theory and the G.C.H. (see e.g.[7]).

PROOF OF THEOREM 6. 7. Assume $\alpha^{++}\nrightarrow[\alpha^+]^2_{\alpha^+,\alpha}$ and let $I_\xi<[\alpha^{++}]^2$, $\xi<\alpha^+$ be a sequence establishing this negative relation. Put $I_0^\xi=\bigcup_{\zeta<\xi} I_\zeta$, $I_1^\xi=[\alpha^{++}]^2-I_0^\xi$ for $\xi<\alpha^+$. Then I_0^ξ obviously does not contain a complete α^+ and $\bigcap_{\xi\in C} I_1^\xi=0$ for $C\subset\alpha^+$, $|C|=\alpha^+$ since $\bigcup_{\xi<\alpha^+} I_\xi=[\alpha^{++}]^2$. Hence the partitions $[\alpha^{++}]^2=I_0^\xi\cup I_1^\xi$ $\xi<\alpha^+$ establish

$$\begin{pmatrix}\alpha^+\\ \alpha^{++}\end{pmatrix}\nrightarrow\begin{pmatrix}1 & \alpha^+\\ \alpha^+, & 2\end{pmatrix}^{1,2}.$$

On the other hand let $[\alpha^{++}]^2=I_0^\xi\cup I_1^\xi$, $\xi<\alpha^+$ be a sequence of partitions establishing the negative polarized partition relation.

Then for each $X\in[\alpha^{++}]^2$ there is a $\xi(X)<\alpha^+$ such that $X\in I_0^\xi$ for $\xi(X)\leqq\xi<\alpha^+$. Put $I_\xi=\{X\in[\alpha^{++}]^2,\ \xi(X)\leqq\xi\}$ for $\xi<\alpha^+$. Then $I_\xi\subset I_0^\xi$ and the I_ξ obviously establish $\alpha^{++}\nrightarrow[\alpha^+]^2_{\alpha^+,\alpha}$.

We think it is relevant to mention here the following negative result.

THEOREM 6. 8. *Assume G.C.H.,* $\alpha>\omega$. *Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix}\nrightarrow\begin{pmatrix}1 & 2\\ \mathscr{B}_{\alpha^+,\alpha,\alpha^+}, & \mathscr{B}^2_{\alpha^+}\end{pmatrix}^{1,2}.$$

This is a trivial reformation of Theorem 17/A of [2] saying $\alpha^+\to[\mathscr{B}_{\alpha^+,\alpha,\alpha^+}]^2_{\alpha^+}$ where this is a self explanatory modification of the symbol defined in 1. 1. 4.

The following theorem shows that assuming G.C.H. in Theorem 6. 1 the $\frac{1}{\alpha}$ cannot be replaced by $\mathscr{B}^1_{\alpha^+,\alpha,\alpha^+}$ even if $\beta=2$.

THEOREM 6. 9. *Assume G.C.H.,* $\alpha\geqq\omega$, $\alpha\to(\alpha)^2_2$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix}\nrightarrow\begin{pmatrix}\alpha & 1 & 2\\ 2 & \vee\ \mathscr{B}_{\alpha^+,\alpha,\alpha^+}, & \alpha\end{pmatrix}^{1,2}.$$

PROOF. By the assumption α is a (strong) limit cardinal. Hence by Theorem 6. 5 there exists a sequence $I_0^{0,\xi}$, $I_1^{0,\xi}$ $\xi<\alpha$ establishing $\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix}\nrightarrow\begin{pmatrix}\alpha & 1\\ 2, & \alpha\end{pmatrix}^{1,2}$. By Theorem 6. 8 there exists a sequence $I_0^{1,\xi}$, $I_1^{1,\xi}$, $\xi<\alpha$ establishing $\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix}\nrightarrow\begin{pmatrix}1 & 2\\ \mathscr{B}_{\alpha^+,\alpha,\alpha^+}, & \mathscr{B}^2_{\alpha^+}\end{pmatrix}^{1,2}$. Put $I_0^\xi=I_0^{0,\xi}\cap I_0^{1,\xi}$, $I_1^\xi=[\alpha^+]^2-I_0^\xi$. It is obvious from the construction that each $\{\zeta,\eta\}\in[\alpha^+]^2$ is contained only in less than α I_0^ξ, and that none of the I_0^ξ contain an $X\in\mathscr{B}_{\alpha^+,\alpha,\alpha^+}$.

Assume $X\subset\alpha_1^+$ $|X|=\alpha$ and let $\xi\neq\zeta<\alpha$. Considering that $I_1^{1,\xi}\cap I_1^{1,\zeta}=0$ it follows from the assumption $\alpha\to(\alpha)^2_2$, that there exists a $Y\subset X$, $|Y|=\alpha$ such that either $[Y]^2\subset I_0^{1,\xi}$ or $[Y]^2\subset I_0^{1,\zeta}$. Hence $[X]^2\subset I_1^\xi\cap I_1^\zeta$ would imply either $[Y]^2\subset I_1^{0,\xi}$ or $[Y]^2\subset I_1^{0,\zeta}$, a contradiction. Thus $X\subset\alpha$, $|X|=\alpha$ implies $[X]^2\not\subset I_1^\xi\cap I_1^\zeta$ for every pair $\xi\neq\zeta<\alpha$ and the theorem is proved.

§ 7. A result for singular strong limit cardinals

DEFINITION 7. 1. Let $\mathscr{A}=(A_\xi)_{\xi<\beta}$ be a sequence of disjoint sets. Put $A=\bigcup_{\xi<\beta} A_\xi$. Let $X, Y\subset A$. Put $X|\!-\!|_{\mathscr{A}}Y$ if $|X\cap A_\xi|=|Y\cap A_\xi|$ for each $\xi<\beta$. Let $H\subset\mathscr{P}(A)$. H is said to be *canonical with respect to* $\mathscr{A}$ if for every $X, Y\subset A$ we have

$$X\in H \quad \text{iff} \quad Y\in H.$$

Let $I=(I_\nu)_{\nu<\gamma}$ be an r-partition of A. I is said to be *canonical with respect to* $\mathscr{A}$ if I_ν is canonical with respect to $\mathscr{A}$ for every $\nu<\gamma$.

CANONIZATION LEMMA. *Let α be a singular strong limit cardinal. Put $\beta=\mathrm{cf}(\alpha)$. Let $r<\omega$, $\gamma<\alpha$ and let $[\alpha]^r=\bigcup_{\nu<\gamma} I_\nu$ be an r-partition of type γ of α. Let further $(B_\xi)_{\xi<\beta}$ be a sequence of disjoint subsets of α such that the cardinality of B_ξ increase rapidly enough e.g. satisfies the following conditions.*

$$(1) \qquad \sum_{\xi<\beta}|B_\xi|=\alpha \quad \textit{and} \quad |B_\xi|\geqq \exp_{r-1}\ (|\bigcup_{\zeta<\xi}B_\zeta|)^+$$

where $\exp_0(\alpha)=\alpha$, $\exp_{s+1}(\alpha)=2^{\exp_s(\alpha)}$ $s<\omega$. Then there exists a sequence $\mathscr{A}=(A_\xi)_{\xi<\beta}$ of subsets of α such that

$$A_\xi\subset B_\xi,\quad |A_\xi|\leqq|A_\zeta| \quad \textit{for} \quad \xi<\zeta<\beta,\quad |A|=\alpha \quad \textit{for} \quad A=\bigcup_{\xi<\beta}A_\xi$$

and the r-partition is canonical with respect to $\mathscr{A}$.

The canonization lemma is proved in [2] assuming G.C.H. for every singular α, but the proof yields the result as stated above. A detailed proof will appear in a forthcoming book of the three of us. As a corollary we will prove the following

LEMMA. *Let α be a singular strong limit cardinal. Put $\beta=\mathrm{cf}(\alpha)$. Let $r<\omega$ and I_ν, $\nu<\alpha$ be a sequence such that $I_\nu\subset[\alpha]^r$ for $\nu<\alpha$. Then there are sequences $\mathscr{A}=(A_\xi)_{\xi<\beta}$, $\mathscr{C}=(C_\xi)_{\xi<\beta}$ of disjoint subsets of α satisfying the following conditions: For $A=\bigcup_{\xi<\beta}A_\xi$, $C=\bigcup_{\xi<\beta}C_\xi$, $|A_\xi|\leqq|A_\zeta|$, $|C_\xi|\leqq|C_\zeta|$ for $\xi<\zeta<\alpha$, $|A|=|C|=\alpha$. I_ν is canonical with respect to $\mathscr{A}$ for every $\nu\in C$ and $I_\nu\cap[A]^r=I_\mu\cap[A]^r$ for $\mu,\nu\in C_\xi$ for every $\xi<\beta$.*

PROOF. Considering that α is strong limit there is a sequence $(B_\xi)_{\xi<\beta}$ of type β of disjoint subsets of α satisfying the cardinality condition (1) of the canonization lemma. Put $B=\bigcup_{\xi<\beta}B_\xi$. Then $|B|=\alpha$. We define an $r+1$ partition J of type 2 of B as follows.

Let $X\in[B]^{r+1}$. Assume first that the following condition (2) holds:

$$(2) \qquad X=Y\cup\{\nu\},\quad Y\in[\bigcup_{\xi<\beta,\,\xi\,\text{even}}B_\xi]^r,\quad \nu\in B_{\xi_0} \text{ and } \xi_0 \text{ is odd.}$$

Put $X\in J_0$ iff $Y\in I_\nu$. If (2) is false we put $X\in J_0$. $J_1=[B]^2-J_0$. By the canonization lemma there is a sequence $\mathscr{A}'=(A'_\xi)_{\xi<\beta}$ of disjoint subsets of α satisfying the following conditions: $|\bigcup_{\xi<\beta}A'_\xi|=\alpha$, $|A'_\xi|\leqq|A'_\zeta|$ and $A'_\xi\subset B_\xi$ for $\xi<\zeta<\alpha$, and the partition J is canonical with respect to $\mathscr{A}'$. Put $A_\xi=A'_{\xi\cdot 2}$, $C_\xi=A'_{\xi 2+1}$, $A=\bigcup_{\xi<\beta}A_\xi$, $C=\bigcup_{\xi<\beta}C_\xi$.

Considering that the cardinality of the A'_ξ increase we have $|A|=|C|=\alpha$. Assume $X, Y\in[A]^r$, $X|-|_{\mathscr{A}}Y$ and let $\nu, \mu\in C_\xi$ for a $\xi<\beta$. Then $X\cup\{\nu\}$, $Y\cup\{\mu\}\in\in[A']^{r+1}$, $X\cup\{\nu\}|-|_{\mathscr{A}}$, $Y\cup\{\mu\}$ and $X\cup\{\nu\}$, $Y\cup\{\mu\}$ both satisfy (2). By the canonicity of J we have $X\cup\{\nu\}\in J_0$ iff $Y\cup\{\mu\}\in J_0$. Hence by the definition of J $X\in I_\nu$ iff $Y\in I_\mu$. Applying this for $\nu=\mu$ we get that I_ν is canonical with respect to $\mathscr{A}$ for $\nu\in C$. Applying the above result for $X=Y$ we get that $I_\nu\cap[A]^r=I_\mu\cap[A]^r$ for every $\nu, \mu\in C_\xi$ for every $\xi<\beta$.

Using the Lemma we can reduce a number of problems concerning singular strong limit cardinals to problems for regular cardinals already discussed. Before doing this as a converse of the Lemma we describe a construction for definings partition canonically.

DEFINITION 8. 2. Let α be a singular limit cardinal $\beta=\mathrm{cf}(\alpha)<\alpha$. Let $(\alpha_\xi)_{\xi<\beta}$ be a sequence of cardinals less than α tending to α, $\alpha=\bigcup_{\xi<\beta}A_\xi$ is a disjoint partition of α where $|A_\xi|=\alpha_\xi$ for $\xi<\beta$.

Let further $[\beta]^2=I_0^\xi\cup I_1^\xi$ be a sequence of type β of 2-partitions of β. We define two sequences of canonical partitions of α as follows:

(1) The sequence $[\alpha]^2=I_0^{*,\xi}\cup I_1^{*,\xi}$, $\xi<\beta$ of type β is defined by the following stipulation. For every $\sigma, \varrho<\alpha$, $\xi<\beta$ $\{\sigma,\varrho\}\in I_0^{*,\xi}$ iff $\sigma\in A_\zeta$, $\varrho\in A_\eta$ and $\{\eta,\zeta\}\in I_0^\xi$ $\{\sigma,\varrho\}\in I_1^{*,\xi}$ iff $\sigma\in A_\zeta$ $\varrho\in A_\eta$ and $\{\eta,\zeta\}\in I_1^\xi$ or $\zeta=\eta$.

(2) The sequence $[\alpha]^2=I_0^{**,\nu}\cup I_1^{**,\nu}$ $\nu<\alpha$ of type α is defined by the stipulation

$$I_i^{**,\nu}=I_i^{*,\xi}\quad\text{for}\quad i<2,\quad \nu\in A_\xi.$$

THEOREM 7. 1. *Let α be a singular limit cardinal,* $\mathrm{cf}(\alpha)=\beta$. *Assume that* $\begin{pmatrix}\beta\\ \beta\end{pmatrix}\nrightarrow\begin{pmatrix}1 & 1 & \beta\\ \gamma & \vee\, \mathscr{B}^1_\alpha, & \beta\end{pmatrix}^{1,2}$ *holds for some γ. Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\nrightarrow\begin{pmatrix}1 & 1 & \alpha\\ \gamma & \vee\, \mathscr{B}^1_\alpha, & \alpha\end{pmatrix}^{1,2}$$

holds as well.

PROOF. Let $[\beta]^2=I_0^\xi\cup I_1^\xi$ $\xi<\beta$ be a sequence of 2-partitions of β establishing the assumed negative relation. Then the second canonical sequence $I_0^{**,\nu}, I_1^{**,\nu}$ defined in 8. 2 satisfies the requirement of the theorem.

Considering that $\begin{pmatrix}\beta\\ \beta\end{pmatrix}\nrightarrow\begin{pmatrix}1 & 1 & \beta\\ \beta & \vee\, \mathscr{B}^1_\beta, & \beta\end{pmatrix}$ holds trivially e.g. by 2. 4 for every $\beta\geqq\omega$. Theorem 7. 1 yields a proof of 2. 6.

The following theorem is the main result of this §.

THEOREM 7. 2. *Let α be a singular strong limit cardinal. Put* $\mathrm{cf}(\alpha)=\beta$. *Let* $\gamma<\mathrm{cf}(\alpha)$, $\delta<\alpha$. *Assume*

$$\begin{pmatrix}\beta\\ \beta\end{pmatrix}\rightarrow\begin{pmatrix}1 & 1 & \beta\\ \gamma & \vee\, \mathscr{B}_{\beta,1,\beta}, & \beta\end{pmatrix}^{1,2}.$$

Then

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\rightarrow\begin{pmatrix}1 & 1 & \alpha\\ \gamma & \vee\, \mathscr{B}_{\alpha,\delta,\alpha}, & \alpha\end{pmatrix}^{1\;2}$$

holds as well.

PROOF. Let $[\alpha]^2 = I_0^\nu \cup I_1^\nu$, $\nu < \alpha$ be a sequence of type α of 2-partitions of α. We may assume $I_0^\nu \cap I_1^\nu = 0$ for $\nu < \alpha$. By the Lemma there exist sequences $\mathscr{A} = (A_\xi)_{\xi < \beta}$, $C = (C_\xi)_{\xi < \beta}$ satisfying the requirements of the Lemma. Put $A = \bigcup_{\xi < \beta} A_\xi$, $C = \bigcup_{\xi < \beta} C_\xi$. Considering that the cardinality of A_ξ is increasing and tends to α we may assume that $|A_\xi| > \max[\gamma, \delta]$ for every $\xi < \beta$. Assume that

(1) $X \subseteq \alpha$, $|X| = \gamma$ implies $[X]^2 \subset I_0^\nu$ and $Y \subset I_0^\nu$ implies that $Y \notin B_{\alpha,\delta,\alpha}$ for $\nu < \alpha$.

By the canonicity for every $\nu \in C$, $\xi < \beta$ we have either $[A_\xi]^2 \subset I_0^\nu$ or $[A_\xi]^2 \subset I_1^\nu$, hence by (1) we have

(2) $[A_\xi]^2 \subset I_1^\nu$ for every $\nu \in C$.

Using again the canonicity we define a sequence $[\beta]^2 = \tilde{I}_0^\xi \cup \tilde{I}_1^\xi$, $\xi < \beta$ of type β of disjoint 2-partitions of β by the following stipulation. For every $\zeta, \eta, \xi < \beta$ $i < 2$

(3) $\{\zeta, \eta\} \in \tilde{I}_i^\xi$ iff $[A_\zeta, A_\eta] \subset I_i^\mu$ for every $\mu \in C_\xi$.

Considering that $|A_\xi| > \delta$ it follows from (1) that $X \in B_{\beta,1,\beta}$ implies $X \not\subset \tilde{I}_0^\xi$ for $\xi < \beta$. On the other hand (1) and (2) obviously imply that $\tilde{I}_0^\xi$ does not contain a complete γ for $\xi < \beta$.

Thus it follows from the assumption $\begin{pmatrix}\beta\\ \beta\end{pmatrix} \to \begin{pmatrix}1 & 1 & \beta\\ \beta & \vee\, B_{\beta,1,\beta}, & \beta\end{pmatrix}^{1,2}$ that there are $U, V \subset \beta$, $|U| = |V| = \beta$ such that

(4) $[U]^2 \subset \tilde{I}_1^\xi$ for every $\xi \in V$.
Put

$$X = \bigcup_{\xi \in U} A_\xi, \quad Y = \bigcup_{\xi \in V} C_\xi.$$

Then $|X| = |Y| = \alpha$, (2) and (4) imply that $[X]^2 \subset I_1^\nu$ for every $\nu \in Y$. This proves the theorem.

We obtain from Theorem 1. 1 and 7. 2 the following

COROLLARY 7. 3. *Assume* cf(α) *is* 0, 1-*measurable and* α *is a singular strong limit cardinal,* $\gamma < \text{cf}(\alpha)$, $\delta < \alpha$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix} \to \begin{pmatrix}1 & 1 & \alpha\\ \gamma & \vee\, \mathscr{B}_{\alpha,\delta,\alpha}, & \alpha\end{pmatrix}^{1,2}.$$

We mention one more (very easy) positive result

THEOREM 7. 4. *Let* α *be a singular strong limit cardinal,* $\gamma < \alpha$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix} \to \begin{pmatrix}1 & \alpha\\ \mathscr{B}_{\alpha,\gamma,\gamma}, & \alpha\end{pmatrix}^{1,2}.$$

Using the same as in the proof of Theorem 7. 2, Theorem 7. 4 follows trivially from the lemma. We omit the details.

A short list of special notations

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ON CONTINUOUS ENDOMORPHISMS OF MIKUSINSKI'S OPERATOR FIELD

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Introduction

In this paper we shall deal with a generalization of the Efros-transformation. The definition of the Efros-transformation is based on the following theorem of E.M. Efros:

THEOREM E. *Let $f(x)$ be a Laplace transformable function with* $\operatorname{Re} p > \sigma_c$ *convergence-half-plane and let $k(x, t)$ be continuous in both variables and*

$$\int_0^\infty k(x, t) \exp(-pt)\, dt = \bar{a}(p) \exp(-x\bar{q}(p)) \tag{1}$$

if $\operatorname{Re} p > \sigma_c$, *and* $\operatorname{Re} \bar{q}(p) > \sigma_c$. *Then*

$$\mathcal{L}\left\{\int_0^\infty k(x, t) f(x)\, dx\right\} = \bar{a}(p) \bar{f}(\bar{q}(p)) \tag{2}$$

for $\operatorname{Re} p > \sigma_c$ *where* $\bar{f}(p) = \mathcal{L}\{f(x)\}$ *for* $\operatorname{Re} p > \sigma_c$.

A linear transformation defined for Laplace-transformable functions is called *Efros-transformation* if it is of the form

$$\int_0^\infty k(x, t) f(x)\, dx = Ef \tag{3}$$

such that (1) holds. Writing the kern function $k(x, t)$ of (3) in the operator terms, it will be

$$\{k(x, t)\} = \bar{a}(s) \exp(-x\bar{q}(s))$$

where s is the operator of differentiation. If $\bar{a}(p) \equiv 1$ then E is *multiplicative,* i.e.

$$E(f_1 \cdot f_2) = Ef_1 \cdot Ef_2 \tag{4}$$

where the product indicates the convolution. In this way E defines a *homomorphism* of the Laplace-transformable functions into M.

In fact, E is an isomorphism, too, whenever $\bar{q}(p)$ is not a constant as it can be shown.

On the basis of the precedings if $\mathcal{F}$ is an isomorphism of a subring M_0 of M into M then the linear transformation $a\mathcal{F}$ will be called a *the generalized Efros-transformation*, where a is an arbitrary non-zero operator.

Let us consider the algebra of piecewise polynomial functions and denote it by $\tilde{H}$. *The purpose of this paper is to show that every continuous endomorphism of Mikusinski's field is determined by defining them on* $\tilde{H}$, *that is, every continuous isomorphism of* $\tilde{H}$ *into* M *has a unique continuous extension being a continuous endomorphism of* M.

For the generalized Efros-transformation, there is also a unique extension from $\tilde{H}$ onto M.

The Efros-transformation has applications in the investigation of boundary value problems of partial differential equations (see in [4]). Similar applications of the generalized Efros-transform in Mikusinski's field will be dealt with in a subsequent paper.

1. §

The operator field will be denoted by M. In general we follow the terminology of Mikusinski's book [6]. Definitions which are not included or are different to that of [6] are as follows:

1.1. Let $H=\{H_\alpha(t)\colon -\infty<\alpha<+\infty\}$ where

$$H_\alpha(t)=\begin{Bmatrix} 0 & t<\alpha \\ 1 & \alpha\leqq t<\infty \end{Bmatrix}.$$

Moreover let $\tilde{H}$ be *the convolution module*, over the complex numbers, generated by H. It is easy to realize that if $f\in\tilde{H}$ then there exists a finite cover of I, say $\bigcup_{j=1}^{n} J_j$ for every finite or infinite interval I such that every J_j is an interval or a half-line and f is a polynomial over J_i for all i; obviously $\tilde{H}\subset M$.

1.2. Let $\mathscr{C}_1$ be an arbitrary subring of M and denote by $[\mathscr{C}_1]$ its *convolution quotient field*. We define the $\mathscr{C}$-convergence in $\mathscr{C}_1$ as the usual *almost uniform convergence*. (See [6].) If $\mathscr{F}$ is a (linear) mapping of $\mathscr{C}_1$ into M then it is called a $[\mathscr{C}_1]$-*continuous mapping in the point* x of $\mathscr{C}_1$ if every sequence $\{x_n\}$ $\mathscr{C}_1$-converging to $x\in\mathscr{C}_1$ then the sequence $\{\mathscr{F}(x_n)\}$ M-converges to $\mathscr{F}(x)$. A (linear) mapping $\mathscr{F}$ will be called $[\mathscr{C}_1]$-*strong continuous* if $\mathscr{F}\big(g(\lambda)\big)=f(\lambda)$ is a continuous operator function whenever $g(\lambda)$ is a continuous operator function having values in $\mathscr{C}_1$. It has been proved by E. GESZTELYI (in [5]) that strong continuity implies continuity. This is also valid for $\mathscr{C}_1$-strong continuity with the same proof. If $\mathscr{F}$ is a $[\mathscr{C}_1]$-(strong) continuous mapping of $\mathscr{C}_1$ into M and an isomorphism of the algebra $\mathscr{C}_1$ then it is called a $[\mathscr{C}_1]$-*(strong) continuous isomorphism*. $\mathscr{F}$ is a (strong) continuous endomorphism of M if $\mathscr{F}$ is a (strong) continuous isomorphism of M into itself.

2. § The extension theorem

Let $\mathscr{F}$ be a continuous isomorphism from $\tilde{H}$ into M. We shall extend it step by step to a endomorphism of M. The proof of the extension theorem is based on the following two theorems:

THEOREM (M). *Let $g(t)$ be a given function which vanishes identically for $t<0$, is continuous for $t\geqq 0$ and does not vanish identically in any right neighbourhood $0<t<\delta$ of the origin. Then every function $f\in\mathscr{C}$ (not necessarily vanishing at 0) can be approximated almost uniformly in $[0, \infty)$ by sums*

$$\sigma(t) = \sum_{i=1}^{n} \lambda_i g(t-\tau_i). \tag{M}$$

For the proof see J. MIKUSINSKI [7].

THEOREM (B). *Let g_n be a sequence in $\mathscr{C}$. A necessary and sufficient condition for the existence of a non-zero g in $\mathscr{C}$ such that for each factor g_n of $g=g_n f_n$, $f_n\in\mathscr{C}$ holds, is that there exists an initial interval $[0, T]$ such that g_n does not vanish on $[0, T]$ for any n.*

For the proof see T. K. BOEHME [1].

The main difficulty in the extension theorem is that we can not use the "closure of the closure" argument which is the standard method to prove a topological extension theorem. It is not valid in the operator field because the convergence in the operator field cannot be generated by any topology. The task of theorem (M) is to extend the isomorphism $\mathscr{F}$ from $\tilde{H}$ to M and theorem (B) is the key of the proof of the continuity of the extended $\mathscr{F}$.

The following simple propositions will be used:

B. 1. If $\{x_n\}$ is an M-converging (to x) operator-sequence then it can be written in the form:

$$\frac{p_n}{q} \ldots M \to \frac{p}{q} \quad p, p_n, q\in\mathscr{C}$$

with common denominator.

Indeed, the definition of the convergence, will immediately result B1.

B. 2. Let $\{a_n\}$ be a sequence in $\mathscr{C}$ M-converging to $a\in\mathscr{C}$. Then there exists an element $g\in\mathscr{C}$ such that g does not vanish identically in any of the neighbourhoods of the origin and $ga_n\to ga$ in $\mathscr{C}$.

Indeed, by the definition of M-convergence there is a g_1 in $\mathscr{C}$ such that $g_1 a_n \overset{\mathscr{C}}{\to} g_1 a$. If T_0 is the largest number to allowing g_1 to vanish identically in $[0, T_0]$ then $g=h^{-T_0} g_1\in\mathscr{C}$ is the desired function since in that case $g_1 a_n$, $n=1, 2, \ldots$ and $g_1 a$ are also vanishing in $[0, T_0]$.

B. 3. Let x be an arbitrary operator. Then x can be written in the *normal form*

$$x = \frac{p}{q} h^{\lambda}$$

where the continuous functions p and q do not vanish identically in any of the neighbourhoods of the origin. Indeed, if $x=\frac{a}{b}$ and T_a and T_b are the largest numbers for which a vanishes in $[0, T_a]$, b vanishes in $[0, T_b]$ then the desired normal form of x is given by choosing $p=h^{-T_a}a$, $q=h^{-T_b}b$, $\lambda=T_b-T_a$.

By the Titchmarsh's theorem *the number λ is unique in the normal form.*

LEMMA. *Let $\{x_n\}$ be an operator sequence converging to x. If the sequence is written in a normal form $\left\{\frac{p_n}{q_n} h^{\lambda_n}\right\}$ and $p/q \cdot h^{\lambda}$ then the relation $0 > \liminf \lambda_n > \lambda$ does not hold.*

PROOF. By hypothesis the sequence q_n, $n=1, 2, \ldots$ satisfies the conditions of Boehme's theorem, hence, there is a non-zero element $\tilde{q}$ of $\mathscr{C}$ being an equimultiple of q_n, $n=1, 2, \ldots$. It can be ensured that $\tilde{q}$ should not vanish identically in any of the neighbourhoods of the zero. Indeed, if $\tilde{q}$ vanishes in $[0, T]$ then the factor-pairs f_n of each q_n will do the same. Shifting $\tilde{q}$ and the f_n's to the origin we obtain a required function of $\mathscr{C}$ being a common multiple. Multiplying the sequence $\{x_n\}$ and x by $\tilde{q}h^{-\lambda}$ we obtain a sequence in $\mathscr{C}$ M-converging to an element of $\mathscr{C}$. According to B2, we may choose a convergence factor not vanishing identically in any of the neighbourhoods of the origin, say q^*, hence we have the functions $p_n/q_n \cdot aq^* h^{\lambda_n - \lambda}$ vanishing in $[0, \lambda - \lambda_n]$ and $\mathscr{C}$-converging to $\frac{p}{q} \tilde{q} q^*$ which does not vanish on any of the neighbourhoods of zero. Hence $\liminf \lambda_n - \lambda \geqq 0$, q.e.d.

THEOREM 1. *(Extension.) If $\mathscr{F}$ is a $[\tilde{H}]$-continuous isomorphism of $\tilde{H}$ into M then there exists a unique M-continuous extension $\tilde{\mathscr{F}}$ of $\mathscr{F}$ such that $\tilde{\mathscr{F}}$ is an M-continuous endomorphism of M.*

PROOF. The proof of the theorem consists of several steps.

a) *There exists a unique $[\tilde{H}]$-continuous extension $\tilde{\mathscr{F}}_1$ of $\mathscr{F}$ such that $\tilde{\mathscr{F}}_1$ is a continuous isomorphism of $[\tilde{H}]$ into M.*

Indeed, if

$$\tilde{\mathscr{F}}_1(x) \overset{\text{def}}{=} \frac{\mathscr{F}(a)}{\mathscr{F}(b)}, \tag{2. 1}$$

$x \in [H]$, $x = a/b$ and $a, b \in [\tilde{H}]$, then $\tilde{\mathscr{F}}_1$ is an extension of $\mathscr{F}$ as an isomorphism from $[\tilde{H}]$ into M. $\tilde{\mathscr{F}}_1$ is continuous since if $x_n \in [\tilde{H}]$ and the sequence $\{x_n\}$ is $[\tilde{H}]$-converging to $x \in [\tilde{H}]$ then there exists $q \in \tilde{H}$ such that qx_n $\mathscr{C}$-converges to qx and hence

$$\mathscr{F}(qx_n) \; M \to \mathscr{F}(qx),$$

i.e. there exists $p \in M$ such that

$$p\mathscr{F}(qx_n) \quad \mathscr{C} \to p\mathscr{F}(qx).$$

Moreover

$$p\mathscr{F}(qx) = p\mathscr{F}(q)\,\mathscr{F}(x) \quad \text{and} \quad p\mathscr{F}(qx_n) = p\mathscr{F}(q)\,\mathscr{F}(x_n)$$

since $\mathscr{F}$ is an isomorphism and hence $\tilde{\mathscr{F}}_1(x_n)$ M-converges to $\tilde{\mathscr{F}}_1(x)$. The uniqueness of the extension $\mathscr{F}_1$ is obvious. (See in [4].) q.e.d.

b) *There exists a unique $\mathscr{C}$-continuous extension $\tilde{\mathscr{F}}_2$ of $\mathscr{F}$ such that $\mathscr{F}_2$ is a $\mathscr{C}$-continuous isomorphism of $\mathscr{C}$ into M.*

In the operator term the approximation (M) of $f \in \mathscr{C}$ will be

$$\sigma_m = \{\sigma_m(t)\} = \sum_{i=1}^{m} \lambda_i h^{\tau_i} g = k_m g \tag{2. 2}$$

where h is the shift operator (see [6]). Obviously $k_m g \in \tilde{H}$ if $g \in \tilde{H}$ $k_m \in [\tilde{H}]$. If $f \in \mathscr{C}$ and $\{k_m g\}$ is a sequence $\mathscr{C}$-converging to f by theorem (M) then it will be shown that $\{\mathscr{F}(k_m g)\}$ is M-convergent and hence

$$\tilde{\mathscr{F}}_2(f) \stackrel{\text{def}}{=} M - \lim_{m\to\infty} \mathscr{F}(k_m g) \tag{2. 3}$$

for every $f \in \mathscr{C}$.

b) 1. Indeed, if $k_m g \in \tilde{H}$ and $k_m g$ $\mathscr{C}$-converges to f then $k_m g - k_{m+r} g$ $\tilde{H}$-converges to zero $(m \to \infty)$ will follow for every integer $r > 0$. Thus from the continuity of $\mathscr{F}$ we obtain the following:

$\mathscr{F}(k_m g) - \mathscr{F}(k_{m+r} g)$ M-converges to zero, i.e. there is a $q \in \mathscr{C}$ such that

(2. 4) $q\mathscr{F}(k_m g - k_{m+r} g)$ $\mathscr{C}$-converges to zero.

Since $\mathscr{C}$ is a complete space for the $\mathscr{C}$-convergence, there exists $p^* \in \mathscr{C}$ such that $\mathscr{C} - \lim q\mathscr{F}(k_m g) = p^*$ which means that

$$M - \lim_{m\to\infty} \mathscr{F}(k_m g) = \frac{p^*}{q}. \tag{2. 5}$$

Obviously $\tilde{\mathscr{F}}_2$ is unique and the value $\tilde{\mathscr{F}}_2(f)$ is independent of the choice of the sequence $\{k_m g\}$. Indeed, if $\{\tilde{k}_m g\}$ and $\{k_m g\}$ are two sequences $\mathscr{C}$-converging to f then the unified sequence $\{\tilde{k}_m g, k_m g\}$ also $\mathscr{C}$-converges to f, hence the $\mathscr{F}$ range of the joined sequence is M-convergent. So (2. 3) is well defined.*

b) 2. For every $a, b \in \mathscr{C}$ and complex number λ

$$\tilde{\mathscr{F}}_2(a+b) = \tilde{\mathscr{F}}_2(a) + \tilde{\mathscr{F}}_2(b),$$
$$\tilde{\mathscr{F}}_2(ab) = \tilde{\mathscr{F}}_2(a)\tilde{\mathscr{F}}_2(b),$$
$$\tilde{\mathscr{F}}_2(\lambda a) = \lambda\tilde{\mathscr{F}}_2(a)$$

which can be easily proved by 2. 3.

We now have only to show that

b) 3. $\tilde{\mathscr{F}}_2$ *is continuous*. Let $p_n \in \mathscr{C}$ and the sequence $\{p_n\}$ $\mathscr{C}$-converges to $p \in \mathscr{C}$. If $\tilde{\mathscr{F}}_2$ is continuous then $\tilde{\mathscr{F}}_2(p_n)$ M-converges to $\tilde{\mathscr{F}}_2(p)$. That is, there exists a $q \in \mathscr{C}$ such that $q \neq 0$ and $q\tilde{\mathscr{F}}_2(p_n)$ $\mathscr{C}$-converges to $q\tilde{\mathscr{F}}_2(p)$. By the definition of $\mathscr{C}$-convergence we have to show for every fixed $[0, T]$ the existence of $n(\varepsilon)$ for every $\varepsilon > 0$, such that

$$|q\tilde{\mathscr{F}}_2(p_n)(t) - q\tilde{\mathscr{F}}_2(p)(t)| < \varepsilon \tag{2. 6}$$

for $n > n(\varepsilon)$, $t \in [0, T]$.

* If $u_n \in \tilde{H}$ and $u_n\, [\tilde{H}] \to u \in \mathscr{C}$ then $\mathscr{F}_1(u_n) M \to \tilde{\mathscr{F}}_2(u)$. Indeed, there is a non-zero element $r \in \tilde{H}$ such that $ru_n \in H$ and $ru_n\ \mathscr{C} \to ru$, thus by (2. 3)

$$\mathscr{F}_1(ru_n) = \mathscr{F}(ru_n) M \to \mathscr{F}_2(ru) = \mathscr{F}(r)\tilde{\mathscr{F}}_2(u)$$

since $\mathscr{F}(r) \neq 0$ it follows $\mathscr{F}_1(u_n) M \to \mathscr{F}_2(u)$.

By the definition 2. 3 of $\tilde{\mathscr{F}}_2(p)$ there exists a sequence $\{\sigma_n\}$ in $\tilde{H}$ $\mathscr{C}$-converging to p.

Now we can make the following table of functions:

$$\begin{array}{cccccc}
\sigma_1 & k_2^1\dots & & & & \mathscr{C}\to p_1 \\
\sigma_2 & \sigma_2 & k_3^2\dots & & & \mathscr{C}\to p_2 \\
\vdots & & \vdots & & & \\
\sigma_n & & \sigma_n\dots & k_{n+1}^n\dots & & \mathscr{C}\to p_n \\
\mathscr{C} & & \mathscr{C} & \mathscr{C} & & \mathscr{C} \\
\downarrow & & \downarrow & & \searrow & \downarrow \\
p\dots & & p & \mathscr{C}\to & & p
\end{array}$$

Table 1

where $k_m^n = \begin{Bmatrix} \sigma_m & t\in[m,\infty) \\ \hat{\sigma}_m & t\in[0,m) \end{Bmatrix}$ such that

$$\|\hat{\sigma}_m - p_n\|_m \leqq \min\left\{\|\sigma_n - p_n\|_m, \frac{1}{m}\right\} \quad \text{and} \quad \hat{\sigma}_m \in \tilde{H}.$$

Evidently $k_m^n \in \tilde{H}$.* Moreover $\hat{\sigma}_m$ required above exists always by theorem (M).

B. 4. The following statements are true

I. $$k_m^n \mathscr{C}\to p_n \quad \text{as} \quad m\to\infty.$$

Furthermore

II. $$k_{m_j}^{n_j} \mathscr{C}\to p \quad \text{as} \quad n_j\to\infty.$$

Indeed, if $n > n_0(\varepsilon, T)$ then $\| p_n - p\|_T < \varepsilon$. Now let ε and $[0, T]$ be fixed so we can get by $n_j \geqq m_j$ (it can be assumed because of Table 1) with $m_{j_0} \geqq \max\{T, 2/\varepsilon\}$

$$\|k_{m_j}^{n_j} - p\|_T \leqq \|k_{m_j}^{n_j} - p_{n_j}\|_T + \|p_{n_j} - p\|_T \leqq \|k_{m_j}^{n_j} - p_{n_j}\|_{m_{j_0}} + \|p_{n_j} - p\|_T < \frac{1}{m_{j_0}} + \frac{\varepsilon}{2} \leqq \varepsilon$$

whenever $m_j \geqq m_{j_0}$ and $n_j > n_0$ $(\varepsilon/2, T)$.

III. Now let us consider the different sequences of Table 1 converging to p with

$$n_{i+1} > n_i \quad \text{and} \quad m_{i+1} > m_i.$$

Evidently there is at least one sequence having the above properties (e.g.: $\{\sigma_n\}$) and no more than countable.

* $\|f\|_m = \sup\limits_{t\in[0,m]} |f(t)|$ and σ_n can be produced as a linear combination of the $H_\alpha(t)'$ s, $\hat{\sigma}_m$ is also, thus the functions

$$\sigma_m^* = \begin{Bmatrix} 0 & t\in[0,\ m] \\ \sigma_n & t\in(m,\ \infty) \end{Bmatrix} \quad \text{and} \quad \hat{\sigma}_m^* = \begin{Bmatrix} \hat{\sigma}_m & t\in[0,\ m] \\ 0 & t\in(m,\ \infty) \end{Bmatrix} \quad \text{are in } \tilde{H}.$$

Now, we apply the mapping $\tilde{\mathscr{F}}_2$ to the elements of Table 1 and write the members of the range $\tilde{\mathscr{F}}_2$ in a normal form, i.e.

$$\tilde{\mathscr{F}}_2(k_m^n) = \frac{\tilde{u}_{nm}}{\tilde{q}_{nm}} h^{\lambda_{nm}} \qquad (n, m = 1, 2, \ldots)$$

$$\tilde{\mathscr{F}}_2(p_n) = \frac{\tilde{u}_n}{\tilde{q}_n} h^{\lambda_n} \qquad (n = 1, 2, \ldots)$$

and

$$\tilde{\mathscr{F}}_2(p) = \frac{\tilde{u}_0}{\tilde{q}_0} h^{\lambda}.$$

If every λ_{nm} and every λ_n are positive numbers then $\tilde{u}_{nm}h^{\lambda_{nm}}, \tilde{u}_n h^{\lambda_n} \in \mathscr{C}$ and by theorem (B) there is a common-denominator for range of the table. Moreover, when there are only finite operators having negative λ then we can left them without loss of generality. Finally, if every $\lambda_{n,m}$, λ_n are negative numbers then we can suppose $\liminf_m \lambda_{n,m} \leqq \lambda_n$ because of the Lemma.

Since $h^\lambda = s\{H_\lambda(t)\}$ $(\lambda > 0)$ we put

$$\frac{\tilde{u}_{nm}}{\tilde{q}_{n,m}} h^{\lambda_{nm}} = \frac{\tilde{u}_{nm}\, l}{\tilde{q}_{nm}\{\mathscr{H}_{\lambda_{nm}}(t)\}}$$

where $s = \dfrac{1}{l}$. Since every element is in a normal form so Theorem (B) and the lemma imply that there is an equimultiple Q of $\{\tilde{q}_n, \tilde{q}_{nm}\}_{n,m=1}^{\infty}$ not vanishing in any $[0, \delta]$. Multiplying by Q the elements of the $\tilde{\mathscr{F}}_2$-range of Table 1 we get an infinite matrix of elements

$$\frac{a_{nm}}{\{H_{-\lambda_{nm}}(t)\}} \qquad n, m = 1, 2, \ldots$$

$$\frac{a_n}{\{H_{-\lambda_n}(t)\}} \qquad n = 1, 2, \ldots$$

and $\dfrac{a}{\{H_{-\lambda}(t)\}}$ where $a_{n,m}$, a_n and a do not vanish for all n, m in any $[0, \delta]$. Moreovert here are functions $\tilde{q}(n)$ in $\mathscr{C}$ such that

$$\tilde{q}(n) \frac{a_{nm}}{\{H_{-\lambda_{nm}}(t)\}} \mathscr{C} \xrightarrow[m\to\infty]{} \tilde{q}(n) \frac{a_n}{\{H_{-\lambda_n}(t)\}}.$$

Now we have to show that there exists a non-zero common multiple q for every $\tilde{q}(n)$. So the common denominator of the range of Table 1 will be Q.

Since $\tilde{q}(n) \dfrac{a_{nm}}{\{H_{-\lambda_{nm}}(t)\}}$ and $\tilde{q}(n) \dfrac{a_n}{\{H_{-\lambda_n}(t)\}}$ are in $\mathscr{C}$, by the Titchmarsh's theorem it will be deduced, that every $\tilde{q}(n)$ have to vanish identically on $[0, -\lambda_{nm}]$ $m = 1, 2, \ldots$ and on $[0, -\lambda_n]$. Because of the convergence of these sequences and the lemma it results that $\liminf_m \lambda_{nm} \leqq \lambda_n$, $n = 1, 2, \ldots$. This means that it may be assumed $\bigcup_m [0, -\lambda_{nm}] \supset [0, -\lambda_n]$. Since $\tilde{q}(n) \neq 0$ there exists $[0, T_n]$ such that $[0, T_n] \supset \bigcup_m [0, -\lambda_{nm}]$.

If there exists an interval $[0, T]$ where $\tilde{q}(n)$ does not vanish for all n identically then by Boehme's theorem there is a common multiple q of $\{\tilde{q}(n)\}_{n=1}^{\infty}$. Let us note that, the existence of the above interval $[0, T]$ is $\bigcup_{n,m} [0, -\lambda_{nm}] \subset [0, T]$. If this relation does not hold so we may assume that $\tilde{q}(n)$ has to be chosen vanishing identically in $[0, \eta_n]$ for every integer n, where $\eta_n \to \infty$. Then there is a $\{H_{-\lambda_{nm_i}}(t)\}$ with $-\lambda_{nm_i} \geqq T$ for every n. As a consequence of

$$\left\{\frac{a_{nm_i}}{\{H_{-\lambda_{n,m_i}}(t)\}}\right\} M \to \frac{a}{\{H_{-\lambda}(t)\}}$$

there exists a non-zero element $\tilde{q}$ in $\mathscr{C}$ such that

$$\tilde{q}\,\frac{a_{nm_i}}{\{H_{-\lambda_{n,m_i}}(t)\}}\,\mathscr{C} \to \tilde{q}\,\frac{a}{\{H_{-\lambda}(t)\}}.$$

(The convergence of this sequence follows from Property II of Table 1.)

But, as we have seen above, $\tilde{q}$ has to vanish identically in $[0, -\lambda_{n,m_i}]$ for every (n, m_i), that is, $\tilde{q} \equiv 0$, a contradiction.

Hence there is a equimultiple for all $\tilde{q}(n)$.

On the basis of the precedings the common denominator of $\tilde{\mathscr{F}}_2$ range of Table 1 is denoted by q. Multiplying the range by q we have an infinite matrix of elements being in $\mathscr{C}$ which has the same convergence properties as the $\tilde{\mathscr{F}}_2$ range of Table 1. By B2, there are convergence multipliers not vanishing identically in any of the neighbourhoods of the zero; thus by Theorem (B) there is an equiconvergence multiplier. It can also be supposed that this common multiplier is good for all sequences

$$\{q\tilde{\mathscr{F}}_2(k_{m_l}^{n_l})\}_{l<1}^{\infty} \quad \text{with} \quad n_{l+1} > n_l; \quad m_{l+1} > m_l$$

which are M-converning to $q\tilde{\mathscr{F}}_2(p)$. This is a consequence of B1, B2, B4 and Theorem (B).

Let $[0, T]$ be an arbitrary but fixed interval and $\varepsilon > 0$. Then we can get the numbers $m(n)$ such that

$$|q\mathscr{F}(k_m^n)(t) - q\tilde{\mathscr{F}}_2(p_n)(t)| < \frac{\varepsilon}{2} \tag{2.7}$$

for every $m > m(n)$ and $t \in [0, T]$. We may suppose $m(1) \leqq m(2) \leqq \ldots \leqq m(n) \leqq \ldots$. Then let us choose the numbers $\overline{m}(n)$ with

$$\overline{m}(n) > m(n)$$

$$\overline{m}(1) < \overline{m}(2) < \overline{m}(3) < \ldots < \overline{m}(n) \ldots$$

$$\overline{m}(n) \to \infty$$

and the sequence $\{k^n_{\overline{m}(n)}\}$ which is evidently $\mathscr{C}$-converging to p according to Property II of Table 1. Let n_0 be a positiv integer with

$$(2.8) \qquad |q\tilde{\mathscr{F}}_2(k^n_{\overline{m}(n)})(t) - q\tilde{\mathscr{F}}_2(p)(t)| < \frac{\varepsilon}{2}$$

for every $n > n_0$ and $t \in [0, T]$. Thus we have by 2. 7 and 2. 8 that

$$|q\tilde{\mathscr{F}}_2(p_n)(t) - q\tilde{\mathscr{F}}_2(p)(t)| \leqq |q\tilde{\mathscr{F}}_2(p_n)(t) - q\mathscr{F}(k^n_{\overline{m}(n)})(t)| +$$
$$+ |q\mathscr{F}(k^n_{\overline{m}(n)})(t) - q\tilde{\mathscr{F}}_2(p)(t)| < \varepsilon$$

for every $n > n_0$, (because $\overline{m}(n) > \overline{m}(n_0) > m(n_0)$) $t \in [0, T]$, that is, (2. 6) is satisfied.)

b) 4. *There is no other continuous extension of $\mathscr{F}$ as $\tilde{\mathscr{F}}_2$.* If $\mathscr{F}^*(f) = \tilde{\mathscr{F}}_2(f) = \mathscr{F}(f)$ for all $f \in \tilde{H}$ then as a consequence of the continuity of $\mathscr{F}^*$ and $\tilde{\mathscr{F}}_2$ for every sequence $\{g_n\} \in \tilde{H}$ $\mathscr{C}$-converging to $p \in \mathscr{C}$ $\lim \mathscr{F}^*(g_n) = \lim \tilde{\mathscr{F}}_2(g_n) = \mathscr{F}^*(p) = \tilde{\mathscr{F}}_2(p)$ holds. But by Theorem (M) there exists a sequence $\{g_n\}$ in $\tilde{H}$ $\mathscr{C}$-converging to p for all p in $\mathscr{C}$.

b) 5. $\tilde{\mathscr{F}}_2$ *is an isomorphism.* That means $\operatorname{Ker} \tilde{\mathscr{F}}_2 = \{0\}$. Let us assume the existence of a function $u \in \mathscr{C}$ with $\mathscr{F}_2(u) = 0$ and $u \neq 0$. We can suppose u does not vanish in any of the neighbourhoods of the origin since for $u \neq 0$ there exists a $T \geqq 0$ such that $u(t) = 0$ $t < T$ and $u(t) \neq 0$ $T < t < T + \delta$ for some $\delta > 0$, then $h^{-T} \in \tilde{H}$ is an operator and $h^{-T}u \in \mathscr{C}$ is the desired function, and $\tilde{\mathscr{F}}_2(h^{-T}u) = \tilde{\mathscr{F}}_1(h^{-T})\tilde{\mathscr{F}}_2(u) = 0$. Applying Theorem (M) with the function u to a non-zero function g of $\tilde{H}$, we have from the continuity of $\tilde{\mathscr{F}}_2$ that $M - \lim\limits_{n \to \infty} \tilde{\mathscr{F}}_2(k_n u) = \tilde{\mathscr{F}}_2(g) = \mathscr{F}(g)$ whenever $k_n u$ $\mathscr{C}$-converges to g. But if $g \neq 0$ and $g \in \tilde{H}$ then $\tilde{\mathscr{F}}_2(g) = \mathscr{F}(g) \neq 0$ by the conditions on $\mathscr{F}$, thus $\tilde{\mathscr{F}}_2(k_n u) = \tilde{\mathscr{F}}_1(k_n)\tilde{\mathscr{F}}_2(u) = 0$, a contradiction.

c) $\tilde{\mathscr{F}}(x) \stackrel{\text{def}}{=} \dfrac{\tilde{\mathscr{F}}_2(a)}{\tilde{\mathscr{F}}_2(b)}$ for every $x = \dfrac{a}{b} \in M$; $a, b \in \mathscr{C}$. $\tilde{\mathscr{F}}$ is a *unique extension* of $\mathscr{F}$.

This obviously follows from the uniqueness of the extension $\tilde{\mathscr{F}}_2$. $\tilde{\mathscr{F}}$ is an *M-continuous endomorphism of M.* It can be proved by same way as is the part a) of the proof. The proof of the theorem is complete.

(2. 9) REMARK. It is evident that the M-continuity of $\mathscr{F}$ implies its $[\tilde{H}]$-continuity. The other way round it is easy to get from the part b) 2 of the proof of extension theorem that if $\mathscr{F}$ is $[\tilde{H}]$-continuous the $\mathscr{F}$ is M-continuous, too.

Thus for every subalgebra $\mathscr{C}_1$ of M containing $\tilde{H}$ can be proved the following

THEOREM 1. a. *If $\mathscr{F}$ is a $[\mathscr{C}_1]$-continuous isomorphism of $\mathscr{C}_1$ into M then there exists a unique extension $\tilde{\mathscr{F}}$ of $\mathscr{F}$ such that $\tilde{\mathscr{F}}$ is an M-continuous endomorphism of M.*

As we have mentioned in the introduction, Gesztelyi has proved that every $[\tilde{H}]$-strong continuous isomorphism is $[\tilde{H}]$ continuous isomorphism. We can get a unique extension for every $[\tilde{H}]$-strong continuous isomorphism:

THEOREM 2. *If $\mathscr{F}$ is a $[\mathscr{C}_1]$-strong continuous isomorphism of $\mathscr{C}_1$ into M then there exists a unique extension $\tilde{\mathscr{F}}$ of $\mathscr{F}$ such that $\tilde{\mathscr{F}}$ is an M-continuous endomorphism* *of M. Moreover if there exists an M-strong continuous extension* $\mathcal{F}^*$ *of* $\mathcal{F}$ *then* $\mathcal{F}^* = \tilde{\mathcal{F}}$ *on M.*

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ON THE DISTRIBUTION OF ROOTS OF RIEMANN ZETA AND ALLIED FUNCTIONS. II*

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1. Let $L(s, k, \chi)$ denote Dirichlet L-functions corresponding to a character χ belonging to the modulus k, $s=\sigma+it$ and $N(\alpha, T, k, \chi)$ stand for the number of its zeros in the parallelogram (with multiplicity)

$$\text{(1. 1)} \qquad \begin{array}{ll} \sigma \geqq \alpha, & 0<t \leqq T \\ \frac{1}{2} \leqq \alpha \leqq 1, & T \geqq 2. \end{array}$$

As RÉNYI [2] and BOMBIERI [3] discovered, in several problems of analytic number theory essential role is played by the quantity

$$\text{(1. 2)} \qquad S(\alpha, T, X) \overset{\text{def}}{=} \sum_{k \leqq X} \sum_{\chi \bmod k}^{*} N(\alpha, T, k, \chi)$$

where Σ^* stands for summation with respect to primitive characters mod k only; this means of course the total number of zeros of all L-functions belonging to a modulus $\leqq X$ in the domain (1. 1). The best known theorem valid for $\frac{1}{2} \leqq \alpha \leqq 1$ is due to BOMBIERI [3], which in a form perfected by DAVENPORT—HALBERSTAM [4] asserts

$$\text{(1. 3)} \qquad S(\alpha, T, X) \ll T(X^2+XT)^{\frac{4(1-\alpha)}{3-2\alpha}} \log^{10}(X+T).$$

BOMBIERI stated in [3] a further conjecture, according to which for $\varepsilon>0$, $T \geqq 2$, $\frac{1}{2} \leqq \alpha \leqq 1$ the inequality

$$\text{(1. 4)} \qquad S(\alpha, T, X) < c(\varepsilon) X^{4(1-\alpha)+\varepsilon} T^{1+\varepsilon}$$

holds; c stands generally throughout this paper for unspecified positive constant; dependence on parameter will always be explicitly stated. Since (1. 4) is, as he stated justly l.c., a new type of density theorems, we set the task to explore as a matter of first orientation, whether or not it is deducible from a natural analogon of Lindelöf's conjecture. Such an analogon is the inequality

$$\text{(1. 5)} \qquad |L(s, k, \chi)| < c_1(T, \lambda) k^{\lambda^2}$$

for arbitrarily small positive λ, uniformly for

$$\text{(1. 6)} \qquad \sigma \geqq \tfrac{1}{2}, \quad |t| \leqq T;$$

* The first paper of this series is [1].

$c(T, \lambda)$ being a positive function of its arguments. We have found that this is indeed the case. More exactly we assert the

THEOREM I. *For the total number* $S(\alpha, T, X)$ *the inequality*

$$S(\alpha, T, X) < c_2(\varepsilon, T)X^{4(1-\alpha)+\varepsilon} \tag{1. 7}$$

holds for $\frac{1}{2} \leqq \alpha \leqq 1$, $T \geqq 2$ *if only* (1. 5)—(1. 6) *is true.*

Supposing (1. 5) only for $\sigma \geqq \vartheta$, $|t| \leqq T$ with an $\frac{1}{2} < \vartheta < 1$ (1. 7) would hold for $\vartheta \leqq \alpha \leqq 1$. Though this is not at all an insignificant generalization, we shall not go into details.

We have found further that for the half-plane $\sigma > \frac{3}{4}$ much stronger conclusion can be drawn from (1. 5)—(1. 6). We assert namely

THEOREM II. *For* $T \geqq 2$, *arbitrarily small positive* ε, δ *and* $X > c(\delta, \varepsilon, T)$ *the inequality*

$$S\left(\frac{3}{4} + \delta, T, X\right) < X^{\varepsilon}$$

holds if only (1. 5)—(1. 6) *holds.*

The point of this theorem is of course the fact that the supposition (1. 5)—(1. 6) is so strong that throwing away at most X^{ε} "bad" k-moduli not exceeding X, for *all* L-functions belonging to the remaining "good" k-moduli the Riemann—Piltz conjecture holds at least for

$$\sigma \geqq \frac{3}{4} + \delta, \quad |t| \leqq T$$

if only X is sufficiently large. In this range our Theorem II gives also an orientation as to the new type conjecture of BARBAN (see [5]) according to which

$$\sum_{k \leqq X} \max_{\chi^*} N(\alpha, T, k, \chi^*) \ll X^{2(1-\alpha)+\varepsilon}$$

if only $X > c_2(\varepsilon, T)$, $\frac{1}{2} \leqq \alpha \leqq 1$.

2. The proofs of the above two theorems will be based on the same two basic ideas used in [1] with appropriate modifications however. (See [6] resp. [7].) Combining these ideas with the method used in [8] we can get an orientation on the strength of an assumption much weaker than that in (1. 5)—(1. 6). Restricting ourselves to the case of Riemann zeta-function this assumption says that having in the parallelogram

$$\sigma \geqq \frac{1}{2}, \qquad 0 \leqq t \leqq T$$

a zero-free parallelogram

$$\sigma \geqq \alpha, \qquad |t - \tau| \leqq \log T \tag{2. 1}$$

(so that with a fixed $\eta > 0$, $\frac{1}{2}+\eta \leqq \alpha \leqq 1$ and $\log T \leqq \tau \leqq T - \log T$) then the number U of zeros of $\zeta(s)$ in the half circle

$$(2.\,2) \qquad |s-(\alpha+i\tau)| \leqq \delta \qquad 0<\delta<\frac{\eta}{2}$$
$$\sigma \leqq \alpha$$

cannot exceed the quantity

$$(2.\,3) \qquad o(\delta) \log T.$$

for $\delta \to 0$ and $T > c(\delta, \eta)$. This is obviously weaker than Lindelöf's conjecture and it is easy to prove actually e.g. the inequality

$$U \leqq \delta \log T.$$

As was shown in [8] this assumption implies the density hypothesis in the form

$$N(\alpha, T) < c(\varepsilon) T^{(2+\varepsilon)(1-\alpha)},$$
$$\tfrac{1}{2} \leqq \alpha \leqq 1$$

where $N(\alpha, T)$ denotes the number of zeros of $\zeta(s)$ in the parallelogram $\sigma \geqq \alpha$, $0 < t \leqq T$. Now the above mentioned combination of ideas leads — under the weak supposition (2. 1)—(2. 2)—(2. 3) — to the inequality

$$(2.\,4) \qquad N(\alpha, T) < c(\varepsilon) T^{1-\alpha+\varepsilon}$$

for $\frac{11}{12} \leqq \alpha \leqq 1$. We shall not enter into details.

For Theorem I we give two proofs, the first one operating with the large sieve. Giving both we offer a possibility to compare it with our method; this is one of the reasons for the arrangement of our proofs. Nevertheless there is much to do to get a complete comparison of the two methods.

In [1] we had the (as far as we know the first) unconditional proof that the function

$$(2.\,5) \qquad \mu_1(\alpha) \overset{\text{def}}{=} \varlimsup_{T \to +\infty} \frac{\log\left(1+N(\alpha, T)\right)}{\log T}$$

touches the α-axis at $\alpha = 1$ (and even a much stronger theorem). Here we could utilize the deep theorem of I. M. Vinogradov—Korobov—Richert (see [9])

$$(2.\,6) \qquad \zeta(\sigma+it) \ll |t|^{100(1-\sigma)^{3/2}} \log^{2/3} |t|.$$

As to $L(s, k, \chi)$ the best known result is for $\sigma \geqq \frac{1}{2}$, $|t| \leqq 10$, say

$$(2.\,7) \qquad L(s, k, \chi) \ll k^{\frac{3}{8}(1-\sigma)+\varepsilon}$$

due to Burgess (see [10]). We could deduce from an assumption

$$(2.\,8) \qquad |L(s, k, \chi)| < c(T) k^{100 \frac{1-\sigma}{\log \frac{1}{1-\sigma}}}$$
$$\sigma \geqq \frac{1}{2}, \quad |t| \leqq T$$

that

$$(2.9)\qquad \sum_{k \leqq X} \sum_{\chi \bmod k}^{*} N(\alpha, T, k, \chi) < c(T) X^{c \frac{1-\alpha}{\log \frac{1}{1-\alpha}}},$$

say. Again we will not go into details.

3. We mention an easy corollary of Theorem II in the prime number theory. Let as usual $P(k, l)$ stand for the smallest prime $\equiv l \bmod k$. Then it is known (see [11]) that from the extended Riemann hypothesis the inequality

$$(3.1)\qquad P(k, l) < c(\varepsilon)\varphi(k) \log^{2+\varepsilon} k$$

follows for *all* k moduli and *almost all* $(l, k)=1$ i.e. with exception of at most $o(\varphi(k))$ l-values. As we learned from an oral communication of Prof. H. Halberstam, P. D. F. A. Elliott proved without any conjectures that the inequality

$$(3.2)\qquad P(k, l) < c(\varepsilon)\varphi(k) \log^{1+\varepsilon} k$$

holds for *almost all* l-residue classes belonging to *almost all* k-moduli.* Now Theorem II admits an easy proof of the following

COROLLARY. *The conjecture* (1. 5) *implies that apart from* $O(X^{\eta})$ *"bad" moduli* $\leqq X$ *the remaining "good" k-moduli have the property that for all l residue classes with* $(l, k)=1$ *the inequality*

$$(3.3)\qquad P(k, l) < c(\eta)\varphi(k)^{4+\eta}$$

holds for an arbitrarily small $\eta > 0$.

Note the different character of (3. 1), (3. 2) and (3. 3); they hold for *all* k's and *almost all* l-values, for *almost all* k's and *almost all* l-values, for *almost all* k's (in a very strong sense) and *all* l-values, respectively. For the sake of orientation we remark that the inequality

$$P(k, l) < k^{c}$$

with a positive (large) constant c was first proved without any conjectures by JU. V. LINNIK [13]; the best known unconditional inequality

$$P(k, l) \ll k^{630}$$

for all k and l will be contained in the Turku-thesis of M. Jutila. Finally ERDŐS (see [14]) proved using Brun's method that the inequality

$$P(k, l) < A\varphi(k) \log k$$

(A is a sufficiently large positive constant) holds for all k's and a positive $c=c(A)$ percentage of residue classes mod k.

* Somewhat weaker first results of the same character are contained in the paper of A. RÉNYI [12].

4. First we deduce shortly the Corollary from Theorem II. For this sake let k be a "good" modulus, $0<\eta<\frac{1}{100}$, $T=T(\eta)$ to be determined later

(4. 1) $$\omega=[A\log\varphi(k)],\quad Y=\varphi(k)^B,\qquad B>2,$$

A, B constants to be determined later, further for $(l,k)=1$

(4. 2) $$f(s)=-\frac{1}{\varphi(k)}\sum_{\chi \bmod k}\bar\chi(l)\frac{L'}{L}(s,k,\chi)$$

and we start from the integral

(4. 3) $$J=\frac{1}{2\pi i}\int\limits_{\left(1+\frac{1}{\log Y}\right)}\frac{Y^s}{s^{\omega+1}}f(s)\,ds.$$

On one hand this is

(4. 4) $$=\frac{1}{\omega!}\sum_{\substack{n\leqq Y\\ n\equiv l(k)}}\Lambda(n)\log^{\omega}\frac{Y}{n}.$$

On the other hand we deform the line of integration to the broken line V defined by

a) $\sigma=1+\frac{1}{\log Y}$ for $t\geqq T$

b) $t=T$ for $\frac{3}{4}+\frac{\eta}{40}\leqq\sigma\leqq 1+\frac{1}{\log Y}$

c) $\sigma=\frac{3}{4}+\frac{\eta}{40}$ for $|t|\leqq T$

d) $t=-T$ for $\frac{3}{4}+\frac{\eta}{40}\leqq\sigma\leqq 1+\frac{1}{\log Y}$

e) $\sigma=1+\frac{1}{\log Y}$ for $t\leqq -T$.

This gives

$$I=\varphi(k)^{B-1}+\frac{1}{2\pi i}\int\limits_{(V)}\frac{Y^s}{s^{\omega+1}}f(s)\,ds.$$

The contribution of a), b), d) and e) is together absolutely

(4. 5) $$<c\,\frac{Y\log k}{T^{A\log\varphi(k)}}=c\varphi(k)^{B-A\log T}\log k<c\varphi(k)^{B-\frac{A}{2}\log T}$$

that of c) absolutely

(4. 6) $$<c(\eta)\,Y^{\frac{3}{4}+\frac{\eta}{40}}\log k\left(\frac{4}{3}\right)^{A\log\varphi(k)}=c(\eta)\,\varphi(k)^{\left(\frac{3}{4}+\frac{\eta}{30}\right)B+A\log\frac{4}{3}}.$$

In order to have $B-1>\frac{3}{4}B$ we choose

$$B=4+\eta \tag{4. 7}$$

then A so small that

$$A\log\frac{4}{3}=\frac{\eta}{30}$$

and finally T so that

$$A\log T=4.$$

All these give with (4. 5), (4. 6) and (4. 4)

$$\frac{1}{\omega!}\sum_{\substack{n\leqq\varphi(k)^{4+\eta}\\ n\equiv l(k)}}\Lambda(n)\log^{\omega}\frac{\varphi(k)^{4+\eta}}{n}>c(\eta)\varphi(k)^{3+\eta}.$$

As one can see at once the contribution of all prime powers p^{α} with $\alpha\geqq2$ is at most $c(\eta)\varphi(k)^{2+\eta}$ which completes the proof.

5. Next we turn to the proof of Theorem II. Let d be an arbitrary positive number and we shall investigate the total number of zeros of L-functions with moduli $\leqq X$ in

$$\sigma\geqq\frac{3}{4}+\delta,\quad |t-d|\leqq1. \tag{5. 1}$$

We shall use (1. 5)—(1. 6) in the form that for an arbitrarily small $\lambda>0$ and all Dirichlet L-functions the inequality

$$|L(s,k,\chi)|\leqq c(d,\lambda)k^{\lambda^2} \tag{5. 2}$$

holds for

$$\sigma\geqq\tfrac{1}{2},\qquad |t-d|\leqq3. \tag{5. 3}$$

Let

$$s_j=2+i\left(d+\frac{j}{\log^3 X}\right)\overset{\text{def}}{=}2+it_j\qquad j=0,\pm1,\dots,\pm[\log^3 X]. \tag{5. 4}$$

The integer ν should be restricted at present only by the inequality

$$\frac{\lambda+6\lambda^2}{\log\frac{5}{4}}\log X\leqq\nu\leqq\left(\frac{\lambda}{\log\frac{5}{4}}+40\lambda^2\right)\log X. \tag{5. 5}$$

Let $L(s,k,\chi)$ denote an arbitrary Dirichlet L-function where χ is a *primitive* character, belonging to a modulus $k\leqq X$. Then we define for each fixed ν the functions

$$f_\nu(s,k,\chi)=(-1)^{\nu+1}\frac{L'}{L}(s,k,\chi)^{(\nu)}. \tag{5. 6}$$

Fixing also j we consider those among them which are for $s=s_j$ absolutely $\geqq\nu!X^{-\lambda}$; if they — in some order — belong to the primitive character $\chi_h \bmod k_h$ then $M=M(\nu,j,X,\lambda)$ is defined by

$$|f_\nu(s_j,k_h,\chi_h)|\geqq\nu!X^{-\lambda}\qquad h=1,2,\dots,M. \tag{5. 7}$$

LEMMA. *Under the restriction* (5. 5) *the inequality*

$$M < X^{2\lambda+10\lambda^2}$$

holds for $X > c(d, \lambda)$ *and* λ *sufficiently small.*

6. For the proof of this lemma we get first after summation with respect to h the inequality

$$(6.\ 1)\qquad M \cdot \nu!\, X^{-\lambda} \leqq \sum_{h=1}^{M} |f_\nu(s_j, k_h, \chi_h)| = \left|\sum_{h=1}^{M} \eta_h f_\nu(s_j, k_h, \chi_h)\right|$$

with suitable η_h's, $|\eta_h| = 1$. Inserting the series*

$$(-1)^{\nu+1} \frac{L'}{L}(s, k, \chi)^{(\nu)} = \sum_n \frac{\Lambda(n)\chi(n, k)\log^\nu n}{n^s}$$

(6. 1) gives (see (5. 4))

$$(6.\ 2)\qquad M \cdot \nu!\, X^{-\lambda} \leqq \left|\sum_n \frac{\Lambda(n)\log^\nu n}{n^{s_j}} \sum_{h=1}^{M} \chi_h(n, k_h)\eta_h\right| \leqq$$

$$\leqq \sum_n \frac{\log^{\nu+1} n}{n^2}\left|\sum_{h=1}^{M} \chi_h(n, k_h)\eta_h\right| \stackrel{\text{def}}{=} \sum_n \frac{\log^{\nu+1} n}{n^2} s_n.$$

Writing

$$\frac{\log^{\nu+1} n}{n^2} s_n = \left(\frac{\log n}{n^{\frac{1}{2}+\lambda}}\right)\left(\frac{\log^\nu n}{n^{\frac{3}{2}-\lambda}} s_n\right)$$

(6. 2) gives

$$(6.\ 3)\qquad M^2 \cdot \nu!^2 X^{-2\lambda} \leqq \left(\sum_n \frac{\log^2 n}{n^{1+2\lambda}}\right)\left(\sum_n \frac{\log^{2\nu} n}{n^{3-2\lambda}} |s_n|^2\right) \leqq$$

$$\leqq c(\lambda)\left|\sum_n \frac{\log^{2\nu} n}{n^{3-2\lambda}} \sum_{h_1=1}^{M}\sum_{h_2=1}^{M} \eta_{h_1}\bar\eta_{h_2}\chi_{h_1}(n, k_{h_1})\bar\chi_{h_2}(n, k_{h_2})\right| =$$

$$= c(\lambda)\left|\sum_{h_1=1}^{M}\sum_{h_2=1}^{M} \eta_{h_1}\bar\eta_{h_2} \sum_n \frac{\chi_{h_1}(n, k_{h_1})\bar\psi_{h_2}(n, k_{h_2})\log^{2\nu} n}{n^{3-2\lambda}}\right| \leqq$$

$$\leqq c(\lambda)\sum_{h_1=1}^{M}\sum_{h_2=1}^{M}\left|\sum_n \chi_{h_1}(n, k_{h_1})\bar\chi_{h_2}(n, k_{h_2})\frac{\log^{2\nu} n}{n^{3-2\lambda}}\right| \leqq c(\lambda)\{M|\zeta^{(2\nu)}(3-2\lambda)|\} +$$

$$+ \sum_{1\leqq h_1 \neq h_2 \leqq M}\left|\sum_n \chi_{h_1}(n, k_{h_1})\bar\chi_{h_2}(n, k_{h_2})\frac{\log^{2\nu} n}{n^{3-2\lambda}}\right|.$$

Obviously $\chi_{h_1}(n, k_{h_1})\bar\chi_{h_2}(n, k_{h_2})$ is a nonprincipal character $\chi_{h_1,h_2}(n, k')$ belonging to a modulus $k' \leqq X^2$; hence the last sum is

$$|L^{(2\nu)}(3-2\lambda, k', \chi_{h_1,h_2})|.$$

* Generally, the dependence of the character on the modulus is not indicated; we shall use instead of $\chi(n)$ the notation $\chi(n, k)$ wherever misunderstanding can arise.

Applying Cauchy's coefficient estimation the last quantity cannot exceed (using (5. 2) with $d=0$)

$$(2\nu)!\,\frac{c(\lambda)X^{2\lambda^2}}{\left(\frac{5}{2}-2\lambda\right)^{2\nu}}, \tag{6. 4}$$

whereas

$$|\zeta^{(2\nu)}(3-2\lambda)| < \frac{c(2\nu)!}{(2-2\lambda)^{2\nu}}. \tag{6. 5}$$

Putting (6. 4) and (6. 5) into (6. 3) we get

$$M^2\cdot\nu!^2X^{-2\lambda} \leqq c(\lambda)(2\nu)!\left\{\frac{M}{(2-2\lambda)^{2\nu}}+M^2\frac{X^{2\lambda^2}}{\left(\frac{5}{2}-2\lambda\right)^{2\nu}}\right\}$$

and a fortiori

$$M^2\cdot X^{-2\lambda} \leqq c(\lambda)\left\{\frac{M}{(1-\lambda)^{2\nu}}+\frac{M^2X^{2\lambda^2}}{\left(\frac{5}{4}-\lambda\right)^{2\nu}}\right\}. \tag{6. 6}$$

Now (5. 5) gives for $X > c(d,\lambda)$

$$\frac{c(\lambda)X^{2\lambda^2}}{\left(\frac{5}{4}-\lambda\right)^{2\nu}} < \frac{1}{2}X^{-2\lambda}$$

and hence from (6. 6) and (5. 5)

$$M \leqq c(\lambda)\frac{X^{2\lambda}}{(1-\lambda)^{2\nu}} < X^{2\lambda+10\lambda^2}$$

indeed.

7. Hence if ν and j runs over their permitted values in (5. 5) resp. (5. 4) the number of these "exceptional" L-functions cannot exceed

$$2X^{2\lambda+10\lambda^2}\log^4 X < X^{2\lambda+11\lambda^2}$$

for $X > c(\lambda)$. By other words throwing away at most $X^{2\lambda+11\lambda^2}$ of our L-functions for the remaining ones the inequality

$$|f_\nu(s_j,k,\chi)| \leqq \nu!\,X^{-\lambda} \tag{7. 1}$$

holds for all permitted values of ν and j. The L-functions thrown away will be the "bad" ones; the total number of their zeros in the parallelogram (5. 1) cannot exceed

$$X^{2\lambda+11\lambda^2}c\log(kd) < X^{2\lambda+12\lambda^2} \tag{7. 2}$$

for $X > c(\lambda, d)$. Hence if we succeed to shown that remaining "good" L-functions do not vanish in (5. 1) the proof of Theorem II will be finished.

8. We shall use again the following functiontheoretical lemma due essentially to Landau (see the paper [1], formula (8. 5)).

If $G(z)$ is regular for $|z|\leqq R$, $G(0)\neq 0$ and here the inequality

$$\left|\frac{G(z)}{G(0)}\right| \leqq U \tag{8. 1}$$

holds and all zeros of $G(z)$ on the disk $|z|\leqq \vartheta R$ with $0<\vartheta<1$ are $z_1, z_2, \dots$ then for all positive integer ν-values the inequality

$$\left|\frac{1}{\nu!}\frac{G'}{G}(z)^{(\nu)}_{z=0}+(-1)^\nu \sum_{|z_j|\leqq \vartheta R}\frac{1}{z_j^{\nu+1}}\right| \leqq \frac{2(\nu+1)\log U}{(\vartheta R)^{\nu+1}}\left(1+\frac{1}{\log\frac{1}{\vartheta}}\right) \tag{8. 2}$$

holds.

We shall apply (8. 2) with $G(z)=L(s_j+z, k, \chi)$, L "good" and fixed, $R=\frac{3}{2}$, $\vartheta=\frac{14}{15}$ and taking for ν any integer satisfying (5. 5). Using for U the trivial

$$U=ckd$$

this gives for our L-function

$$\left|\frac{1}{\nu!}f_\nu(s_j, k, \chi)-\sum_{|\varrho-s_j|\leqq\frac{7}{5}}\frac{1}{(s_j-\varrho)^{\nu+1}}\right| < c\left(\frac{5}{7}\right)^\nu \log^2 k < X^{-1,02\lambda}\log^2 X < X^{-1,01\lambda}$$

for $X>c(\lambda)$; here $\varrho=\sigma_\varrho+it_\varrho$ stand for the nontrivial zeros of $L(s, k, \chi)$. This and (7. 1) give for all of our j's and ν's

$$\left|\sum_{|s_j-\varrho|\leqq\frac{7}{5}}\frac{1}{(s_j-\varrho)^{\nu+1}}\right| \leqq 2X^{-\lambda}\leqq X^{-\lambda+\lambda^2}. \tag{8. 3}$$

9. Now we shall estimate the sum in (8. 3) from below by a proper choice of ν satisfying (5. 5). This is done by the following theorem (see [7], Satz X).

If m is an arbitrary positive number, $w_1, w_2, \dots, w_n$ arbitrary complex numbers where $n\leqq N^*$ then there exists an integer ν_0 satisfying

$$m\leqq \nu_0\leqq m+N^* \tag{9. 1}$$

so that

$$\left|\sum_{b=1}^n w_b^{\nu_0}\right| \geqq \left(\frac{N^*}{8e(m+N^*)}\right)^{N^*}\max_b |w_b|^{\nu_0}$$

and thus a fortiori

$$\left|\sum_{b=1}^n w_b^{\nu_0}\right| \geqq \left(\frac{N^*}{8e(m+N^*)}\right)^{N^*}|w_a|^{\nu_0} \tag{9. 2}$$

for an *arbitrary* a-index.

In applying this theorem let j be fixed and

(9. 3) $$m = \frac{\lambda + 7\lambda^2}{\log \frac{5}{4}} \log X.$$

In order to obtain N^* we apply Jensen's formula to $L(s, k, \chi)$ and the circle $|s - s_j| \leqq \frac{3}{2}$; this gives

$$\sum_{|s_j - \varrho| \leqq \frac{7}{5}} \log \frac{1,5}{|s_j - \varrho|} \leqq \sum_{|s_j - \varrho| \leqq \frac{3}{2}} \log \frac{1,5}{|s_j - \varrho|} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \frac{L\left(s_j + \frac{3}{2} e^{i\vartheta}, k, \chi\right)}{L(s_j, k, \chi)} \right| d\vartheta$$

and hence on using (5. 2)

(9. 4) $$N^* = 16\lambda^2 \log X$$

if only $X > c(d, \lambda)$. Thus choosing our $\nu + 1$ in (8. 3) as ν_0, in (9. 1)—(9. 2) the restriction (5. 5) is fulfilled. The role of the numbers w_b will be played of course by the numbers $\frac{1}{s_j - \varrho}$ (j fixed); as index a in (9. 2) we choose the one which corresponds to the zero $\varrho^* = \sigma^* + it^*$ with the maximal real part in the strip

(9. 5) $$|t - t_j| \leqq \frac{1}{2 \log^3 X}$$

(if there is a zero in this strip at all). Then we have

(9. 6) $$\left(\frac{N^*}{8e(m + N^*)}\right)^{N^*} > \left(\frac{\log \frac{5}{4}}{e} \lambda\right)^{16\lambda^2 \log X} > X^{-32\lambda^2 \log \frac{1}{\lambda}} > X^{-32\lambda^{3/2}}.$$

Further

$$\frac{1}{|s_j - \varrho^*|^{\nu_0}} \geqq \left(\frac{1}{(2 - \sigma^*)^2 + \frac{1}{4 \log^6 X}}\right)^{\frac{1}{2}\left(\frac{\lambda}{\log \frac{5}{4}} + 40\lambda^2\right) \log X} >$$

$$> \frac{1}{2} \left(\frac{1}{2 - \sigma^*}\right)^{\left(\frac{\lambda}{\log \frac{5}{4}} + 40\lambda^2\right) \log X} > \left(\frac{1}{2 - \sigma^*}\right)^{\left(\frac{\lambda}{\log \frac{5}{4}} + 41\lambda^2\right) \log X}.$$

Putting this and (9. 6) into (8. 3) we get

$$\left(\frac{1}{\log \frac{5}{4}} + 41\lambda\right) \log \frac{1}{2 - \sigma^*} < -1 + \lambda + 32\sqrt{\lambda} < -1 + 33\sqrt{\lambda}$$

or

$$(1+41\lambda)\log(2-\sigma^*) > \log\frac{5}{4}(1-33\sqrt{\lambda}),$$

$$(2-\sigma^*) > \left(\frac{5}{4}\right)^{1-34\sqrt{\lambda}} > \frac{5}{4}(1-9\sqrt{\lambda})$$

which proves Theorem II choosing λ so small that

$$12\sqrt{\lambda} < \delta, \quad 2\lambda + 12\lambda^2 < \frac{\varepsilon}{2}$$

and X so large that beside the previous requirements of type $X > c(d, \lambda) = c(T, \lambda)$ also

$$T < X^{\frac{\varepsilon}{2}}$$

should hold.

10. Next we turn to the first proof of Theorem I. We shall use the large sieve method in Bombieri's form

$$\sum_{k \leqq X} \sum_{\chi \bmod k}^{*} \left| \sum_{n=M+1}^{M+\omega} a_n \chi(n, k) \right|^2 \ll (X^2+\omega) \sum_{n=M+1}^{M+\omega} |a_n|^2 \tag{10. 1}$$

for arbitrary complex a_n coefficients.

Let first $\sigma > 4$, X, ν arbitrary positive integers and we consider the sum

$$Z \overset{\text{def}}{=} \sum_{k \leqq X} \sum_{\chi \bmod k}^{*} \left| \frac{L'}{L}(s, k, \chi)^{(\nu)} \right|^2. \tag{10. 2}$$

This can be written as

$$Z = \sum_{k \leqq X} \sum_{\chi \bmod k}^{*} \left| \sum_{d=0}^{\infty} \sum_{n=dX^2+1}^{(d+1)X^2} \frac{\Lambda(n)\log^{\nu} n}{n^s} \chi(n, k) \right|^2. \tag{10. 3}$$

Writing the inner double sum as

$$\sum_{d=0}^{\infty} \left\{ \frac{1}{(d+1)^{\frac{1}{2}+\varepsilon}} \right\} \left\{ (d+1)^{\frac{1}{2}+\varepsilon} \sum_{n=dX^2+1}^{(d+1)X^2} \frac{\Lambda(n)\log^{\nu} n}{n^s} \chi(n, k) \right\}$$

we get from (10. 3)

$$Z \leqq c(\varepsilon) \sum_{k \leqq X} \sum_{\chi \bmod k}^{*} \sum_{d=0}^{\infty} (d+1)^{1+2\varepsilon} \left| \sum_{n=dX^2+1}^{(d+1)X^2} \frac{\Lambda(n)\log^{\nu} n}{n^s} \chi(n, k) \right|^2 =$$

$$= c(\varepsilon) \sum_{d=0}^{\infty} (d+1)^{1+2\varepsilon} \left\{ \sum_{k \leqq X} \sum_{\chi \bmod k}^{*} \left| \sum_{n=dX^2+1}^{(d+1)X^2} \frac{\Lambda(n)\log^{\nu} n}{n^s} \chi(n, k) \right|^2 \right\}.$$

Now applying (10. 1) to the expression in the curly bracket we get

$$Z \leqq c(\varepsilon)X^2 \sum_{d=0}^{\infty}(d+1)^{1+2\varepsilon} \sum_{n=dX^2+1}^{(d+1)X^2} \frac{\log^{2\nu+2} n}{n^{2\sigma}} = \tag{10. 4}$$

$$= c(\varepsilon)X^2 \left\{ \sum_{n \leqq X^2} \frac{\log^{2\nu+2} n}{n^{2\sigma}} + \sum_{d=1}^{\infty}(d+1)^{1+2\varepsilon} \sum_{n=dX^2+1}^{(d+1)X^2} \frac{\log^{2\nu+2} n}{n^{2\sigma}} \right\}.$$

The first sum is easily

$$< c \int_1^{\infty} x^{-2\sigma} \log^{2\nu+2} x \, dx = c \frac{(2\nu+2)!}{(2\sigma-1)^{2\nu+3}}. \tag{10. 5}$$

The last double-sum in (10. 4) is evidently

$$< X^2 \sum_{d=1}^{\infty}(d+1)^{1+2\varepsilon} \frac{\log^{2\nu+2}(d+1)X^2}{d^{2\sigma}X^{4\sigma}} = X^{2-4\sigma} \sum_{d=1}^{\infty} \frac{\log^{2\nu+2}(d+1)X^2}{(d+1)^{2\sigma-1-2\varepsilon}} \left(\frac{d+1}{d}\right)^{2\sigma} <$$

$$< 4^{\sigma} X^{2-4\sigma} \sum_{d=2}^{\infty} \frac{\log^{2\nu+2}(dX^2)}{d^{2\sigma-1-2\varepsilon}} < 4^{\sigma} \sum_{d=2}^{\infty} \frac{\log^{2\nu+2}(dX^2)}{(dX^2)^{2\sigma-1-2\varepsilon}}.$$

If

$$2\sigma \log X < \nu < 40\sigma \log X \tag{10. 6}$$

then — since

$$\max_{I \geqq 1} y^{1+2\varepsilon-2\sigma} \log^{2\nu+2} y = \left(\frac{2\nu+2}{e(2\sigma-1-2\varepsilon)}\right)^{2\nu+2}$$

the last sum cannot exceed

$$\frac{1}{X^2}\left\{\int_1^{\infty} y^{1+2\varepsilon-2\sigma} \log^{2\nu+2} y \, dy + 2X^2 \left(\frac{2\nu+2}{e(2\sigma-1-2\varepsilon)}\right)^{2\nu+2}\right\} < \tag{10. 7}$$

$$< \frac{(2\nu+2)!}{X^2(2\sigma-2-2\varepsilon)^{2\nu+3}} \left\{1 + c\sigma X^2 \left(\frac{2\sigma-2-2\varepsilon}{2\sigma-1-2\varepsilon}\right)^{2\nu+2}\right\}.$$

Using (10. 6)

$$X^2 \left(\frac{2\sigma-2-2\varepsilon}{2\sigma-1-2\varepsilon}\right)^{2\nu+2} < e^{\frac{\nu}{\sigma}} \cdot e^{-\frac{2\nu+2}{2\sigma-1-2\varepsilon}} < e^{-\frac{\nu}{2\sigma^2}} < \frac{1}{\sigma} \tag{10. 8}$$

if only $\nu > 2\sigma^3$, say, which is certainly fulfilled owing to (10. 6) if

$$X > e^{\sigma^2}. \tag{10. 9}$$

Putting (10. 5), (10. 7) and (10. 8) into (10. 4) we get

$$Z \leqq c(\varepsilon)X^2(2\nu+2)! \left\{\frac{1}{(2\sigma-1)^{2\nu+3}} + \frac{4^{\sigma}}{X^2} \frac{1}{(2\sigma-2-2\varepsilon)^{2\nu+3}}\right\} < \tag{10. 10}$$

$$< c(\varepsilon)4^{\sigma} \cdot \nu!^2 \log^2 X \left\{\frac{X^2}{\left(\sigma-\frac{1}{2}\right)^{2\nu}} + \frac{1}{(\sigma-1-\varepsilon)^{2\nu}}\right\} < \frac{c(\varepsilon)4^{\sigma} \cdot \nu!^2 \log^2 X}{(\sigma-1-\varepsilon)^{2\nu}}$$

owing to (10. 6).

11. Let $\omega=\omega(\varepsilon)$ be >4 and

(11. 1) $$s_j = \omega + \frac{j}{\log^3 X}\cdot i, \qquad j = 0, 1, \dots, [T\log^3 X].$$

Applying (10. 10) with a fixed j and ν we obtain that the number of L-functions belonging to primitive characters whose moduli $\leqq X$ and for which for an arbitrary fixed $0<\beta<1$ the inequality

(11. 2) $$\left|\frac{L'}{L}(s_j, k, \chi)^{(\nu)}\right| \geqq \frac{\nu!\, X^{-\beta}}{(\sigma-1-\varepsilon)^\nu}$$

holds, cannot exceed

(11. 3) $$c(\varepsilon)4^\omega X^{2\beta}\log^2 X.$$

Since the total number of j and ν-values cannot exceed

$$40\omega\, T\log^4 X,$$

throwing away at most

(11. 4) $$c(\varepsilon)8^\omega T\, X^{2\beta}\log^6 X$$

such L-functions, for the remaining "good" ones the inequality

(11. 5) $$\left|\frac{L'}{L}(s, k, \chi)^{(\nu)}\right|_{s=s_j} \leqq \frac{\nu!\, X^{-\beta}}{(\omega-1-\varepsilon)^\nu}$$

holds for *all* permitted ν and j values.

The contribution of "bad" L-functions to $S(\alpha, T, X)$ is owing to (11. 3) at most

(11. 6) $$c(\varepsilon)8^\omega T^2 X^{2\beta}\log^7 X.$$

Hence if we succeed in proving that the "good" L-functions do not vanish for

(11. 7) $$\sigma \geqq 1-\frac{\beta}{2}+\varepsilon, \quad |t| \leqq T,$$

then we will have for $0<\beta<1$

$$S\left(1-\frac{\beta}{2}+\varepsilon, T, X\right) < c(\varepsilon)\, T^2 X^{2\beta}\log^2 X$$

and replacing $1-\dfrac{2}{\beta}+\varepsilon$ by α the proof of Theorem I will be finished.

12. In order to complete the proof of Theorem I we apply (8. 2) to

$$G(z) = L(s_j+z, k, \chi), \ L \text{ "good" and fixed}$$

(12. 1) $$R = \omega-\frac{1}{2}, \quad \vartheta = \frac{\omega-1-\varepsilon}{\omega-\frac{1}{2}}\, e^{\frac{\beta}{2\omega}}.$$

It is easy to see that $\vartheta < 1$ if only $\omega > \omega_0(\varepsilon)$ and hence applying (8. 2), using only rough upper bound on the L-function

$$\left|\sum \frac{1}{(s_j-\varrho)^{\nu+1}}\right| \leqq \frac{c(\varepsilon)a^3\log^2 X}{(\omega-1-\varepsilon)^{\nu+1}}\left\{X^{-\beta}+X^{-\frac{\beta}{2\omega}\nu}\right\}, \tag{12. 2}$$

where the summation refers to the $\varrho = \sigma_\varrho + it_\varrho$ zeros of our L-function satisfying

$$|s_j-\varrho| \leqq (\omega-1-\varepsilon)e^{\frac{\beta}{2\omega}}. \tag{12. 3}$$

Using (10. 6) the inequality (12. 2) assumes for $X > c(\omega) = c(\varepsilon)$ the form

$$\left|\sum\left(\frac{\omega-1-\varepsilon}{s_j-\varrho}\right)^{\nu+1}\right| \leqq c(\varepsilon)X^{-\beta}\log^2 X < X^{-\beta+\varepsilon}. \tag{12. 4}$$

13. Now we shall apply theorem (9. 2) to determine ν restricted sofar only by (10. 6), i.e.

$$2\omega\log X \leqq \nu \leqq 40\omega\log X. \tag{13. 1}$$

We choose

$$m = 2\omega\log X. \tag{13. 2}$$

In order to determine N^* we shall use (1. 5) in the form

$$|L(s,k,\chi)| \leqq c(s,\omega)k^{\frac{1}{\omega}} \leqq c(s,\omega)X^{\frac{1}{\omega}} \tag{13. 3}$$

for $\sigma \geqq \frac{1}{2}$. As N^* we can choose any upper bound for the number of zeros of $L(s,k,\chi)$ in the parallelogram

$$\sigma \geqq \omega-(\omega-1-\varepsilon)e^{\frac{\beta}{2\omega}}, \quad |t-t_j| \leqq 2\sqrt{\omega}$$

if only $\omega > c(\varepsilon)$. Hence application of Jensen's inequality together with (13. 3) gives for $X > c(T,\omega) = c(T,\varepsilon)$

$$N^* = \frac{6}{\sqrt{\omega}}\log X \tag{13. 4}$$

and hence the theorem in (9. 2) can be applied indeed. We have

$$\left(\frac{N^*}{8e(m+N^*)}\right)^{N^*} = \left(\frac{\frac{6}{\sqrt{\omega}}}{8e\left(2\omega+\frac{6}{\sqrt{\omega}}\right)}\right)^{\frac{6}{\sqrt{\omega}}\log X} > X^{-\frac{40}{\sqrt{\omega}}}. \tag{13. 5}$$

As to the index a in (9. 2) we choose that one which corresponds to the zero $\varrho^* = \sigma^* + it^*$ with the greatest real part in the horizontal strip

$$|t-t_j| \leqq \frac{1}{2\log^3 X}.$$

Since very roughly*

(13. 6) $$2\omega \log X \leqq \nu_0 \leqq (2\omega + 20\sqrt{\omega}) \log X$$

we have

(13. 7) $$|w_a|^{\nu_0} \geqq \left(\frac{\omega - 1 - \varepsilon}{\sqrt{(\omega - \sigma^*)^2 + \frac{1}{4 \log^6 X}}} \right)^{(2\omega + 20\sqrt{\omega}) \log X} >$$

$$> \frac{1}{2} \left(\frac{\omega - 1 - \varepsilon}{\omega - \sigma^*} \right)^{(2\omega + 20\sqrt{\omega}) \log X}.$$

Hence

(13. 8) $$\left| \sum \left(\frac{\omega - 1 - \varepsilon}{s_j - \varrho} \right)^{\nu_0} \right| \geqq \frac{1}{2} X^{-\frac{40}{\sqrt{\omega}} - (2\omega + 20\sqrt{\omega}) \log \frac{\omega - \sigma^*}{\omega - 1 - \varepsilon}} >$$

$$> X^{-\frac{41}{\sqrt{\omega}} - (2\omega + 20\sqrt{\omega}) \log \frac{\omega - \sigma^*}{\omega - 1 - \varepsilon}}.$$

Putting it into (12. 4) we get

$$\beta - \varepsilon \leqq \frac{41}{\sqrt{\omega}} + (2\omega + 20\sqrt{\omega}) \log \left(1 + \frac{1 + \varepsilon - \sigma^*}{\omega - 1 - \varepsilon} \right) \leqq$$

$$\leqq \frac{70}{\sqrt{\omega}} + 2\omega \frac{1 + \varepsilon - \sigma^*}{\omega - 1 - \varepsilon} < \frac{71}{\sqrt{\omega}} + 2(1 + \varepsilon - \sigma^*).$$

Choosing ω so large that

$$\frac{71}{\sqrt{\omega}} < \varepsilon$$

we get

$$\frac{\beta}{2} \leqq (1 - \sigma^*) + 2\varepsilon.$$

Replacing ε by $\varepsilon/2$ (11. 7) is true indeed and hence the first proof of Theorem I is complete.

14. Now we turn to the second proof of Theorem I. Just as in the first proof let $\omega = \omega(\varepsilon)$ be a large positive number

$$s_j = \omega + \frac{j}{\log^3 X} i, \qquad j = 0, 1, \ldots, [T \log^3 X]$$

and M the number of L-functions with

$$\left| \frac{L'}{L}(s, k, \chi)^{(\nu)} \right|_{s = s_j} \geqq \nu! \frac{X^{-\beta}}{(\omega - 1 - \varepsilon)^{\nu}},$$

* Here we could require $\frac{7}{\sqrt{\omega}}$ instead of $20\sqrt{\omega}$; we write here $20\sqrt{\omega}$ in order to shorten the *second* proof.

j and $0 \leqq \beta \leqq \frac{3}{2}$ fixed, χ a primitive character belonging to the modulus $k \leqq X$ and ν restricted — a little differently as in (10. 6) — by

$$(2\omega + 10\sqrt{\omega}) \log X \leqq \nu \leqq 40\omega \log X. \tag{14. 1}$$

To estimate M we replace the large sieve method by the suitably changed argument of Section 6. This gives unchanged the inequality

$$M^2 \cdot \nu!^2 \frac{X^{-2\beta}}{(\omega - 1 - \varepsilon)^\nu} \leqq \tag{14. 2}$$

$$\leqq c(\varepsilon)\{M\zeta^{(2\nu)}(2\omega - 1 - 2\varepsilon) + M^2 \max |L^{(2\nu)}(2\omega - 1 - 2\varepsilon, k, \chi)|\}$$

the maximum taken over all nonprincipal characters belonging to moduli not exceeding X^2. Here we have

$$\zeta^{(2\nu)}(2\omega - 1 - 2\varepsilon) < c(\varepsilon) \frac{(2\nu)!}{(2\omega - 2 - 2\varepsilon)^{2\nu}}.$$

In order to estimate $|L^{(2\nu)}|$ we apply Cauchy's estimation to the disk

$$|s - (2\omega - 1 - 2\varepsilon)| \leqq 2\omega - 4 + \sqrt{\omega}.$$

In this disc we have

$$|L(s, k, \chi)| \leqq c(\omega) k^{\sqrt{\omega}} < c(\varepsilon) X^{2\sqrt{\omega}}$$

owing to the functional equation (for easy deduction see the book quoted in [13] VII, Satz 31. p. 219). Hence

$$|L^{(2\nu)}(2\omega - 1 - 2\varepsilon, k, \chi)| \leqq c(\varepsilon) \frac{(2\nu)!\, X^{2\sqrt{\omega}}}{(2\omega - 4 + 4\sqrt{\omega})^{2\nu}} \tag{14. 3}$$

and thus

$$M^2 \frac{X^{-2\beta}}{(\omega - 1 - \varepsilon)^{2\nu}} \leqq c(\varepsilon) \left\{ \frac{M}{(\omega - 1 - \varepsilon)^{2\nu}} + \frac{M^2 X^{2\sqrt{\omega}}}{\left(\omega - 2 + \frac{1}{2}\sqrt{\omega}\right)^{2\nu}} \right\}. \tag{14. 4}$$

Let us observe that — in contrary to the proof of Theorem II the "Lindelöf-conjecture" in (1. 5)—(1. 6) is *not* used in this step. The strong result of Theorem II is due to the *double* use of (1. 5)—(1. 6). Now the coefficient of M^2 on the right side of (14. 4) is less than the half of that on the left since

$$\left(\frac{\omega - 2 + \frac{1}{2}\sqrt{\omega}}{\omega - 1 - \varepsilon} \right)^{2\nu} > e^{\frac{1}{2} \cdot \frac{1}{\sqrt{\omega}} \left(1 - \frac{2}{\sqrt{\omega}}\right) 2\nu} > e^{\frac{1}{\sqrt{\omega}} \left(1 - \frac{2}{\sqrt{\omega}}\right)(2\omega + 10\sqrt{\omega}) \log X} >$$

$$> e^{(2\sqrt{\omega} + 4) \log X} > 2c(\varepsilon) X^{2\sqrt{\omega} + 2\beta}$$

using the first half of the restriction (1. 4. 1). Therefore

$$M \leqq c(\varepsilon) X^{2\beta}.$$

The bound is the same as in (11. 3) for the first proof so that from this point on the two proofs coincide (apart from the fact that as m in (13. 2)) we have to choose

$$m = \left(2\omega + 10\sqrt{\omega}\right) \log X.$$

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THE STRUCTURE OF LARGE SUBGROUPS OF PRIMARY ABELIAN GROUPS

By

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Section I
Introduction

The purpose of this paper is to investigate the relation between the structures of primary groups and their large subgroups. All groups are abelian. This research was motivated by a study of the monumental paper "Homomorphisms of Primary Abelian Groups" [8] by R. S. PIERCE in which the theory of large subgroups was introduced. We shall state some of the fundamental results of this theory in section I.

In section II we collect an assortment of facts, some of which appear in the literature, some of which must be classified as folklore and some of which are new. The principal result of this section is a new representation theorem for large subgroups of p-groups.

Section III is devoted to the problem: Give necessary and sufficient conditions for a p-group G to be quasi-isomorphic to every large subgroup of G. The results of this section show that unless the group G satisfies certain restrictive conditions on its Ulm invariants it contains a large subgroup L which is not quasi-isomorphic to G.

Although a large subgroup of a group G is not in general quasi-isomorphic to G the theorems of section IV show that the structure of a p-group G is in many cases preserved by its large subgroups. Some of our results can be stated as follows. If L is a large subgroup of a group G then: 1. L is a direct sum of cyclic groups if and only if G is a direct sum of cyclic groups; 2. L is totally projective if and only if G is totally projective; 3. L is torsion-complete if and only if G is torsion-complete; 4. L is quasi-closed if and only if G is quasi-closed.

Our notation is that of FUCHS [1] with the following exceptions. We use $A+B$ and $\Sigma_\lambda A_\lambda$ to denote group unions. Direct sums we denote by $A\oplus B$ and $\oplus_\lambda A_\lambda$. Direct products we denote by $\Pi_\lambda A_\lambda$. We use the symbol Zx to denote the cyclic group generated by the element x.

The following definitions and theorems were given by PIERCE in [8]. All groups are p-groups. A large subgroup L of a group G is a fully invariant subgroup of G such that $B+L=G$ for every basic subgroup B of G. It is easily shown that if L is a large subgroup of a group G then p^nL is a large subgroup of G for every non-negative integer n.

If x is an element of a group G define the U-sequence of x to be $U(x) = \langle h_G(x), h_G(px), h_G(p^2x), \ldots\rangle$. Our heights are either non-negative integers or the symbol ∞. Define a U-sequence for the group G to be any strictly increasing sequence of integers $\langle n_0, n_1, n_2, \ldots\rangle$ such that $n_i+1<n_{i+1}$ implies $f_G(n_i)\neq 0$. Following KAPLANSKY [5] we define the m-th Ulm invariant $f_G(m)$ by $f_G(m) = \dim (p^mG[p]/p^{m+1}G[p])$. The notion of U-sequences was introduced by KAPLANSKY

in [5] in developing his theory of characteristic submodules of modules over a complete discrete valuation ring.

The following result is fundamental. If G is any p-group and x, y are elements of G with $Zx \cap G^1 = 0 = Zy \cap G^1$ then there exists $f \in \text{End}\, G$ such that $fx = y$ if and only if $U(x) \leqq U(y)$. For a proof see either PIERCE [8] or KAPLANSKY [5]. This is used in [8] to obtain the following representation theorem for large subgroups. This representation theorem parallels that of KAPLANSKY's Theorem 25 [5]. If $\mathbf{n} = \langle n_0, n_1, n_2, \ldots \rangle$ and $\mathbf{m} = \langle m_0, m_1, m_2, \ldots \rangle$ are any two sequences define $\mathbf{n} \geqq \mathbf{m}$ if $n_i \geqq m_i$ for each $i \in Z^+$. For each U-sequence $\mathbf{n}$ for G define $G(\mathbf{n}) = \{x\colon U(x) \geqq \mathbf{n}\}$. Then $G(\mathbf{n})$ is a large subgroup of G. This provides a one-to-one order reversing correspondence between large subgroups of a group G and U-sequences for G.

The following results are also due to PIERCE. If H is a pure subgroup of a group G and L is a large subgroup of G then $L \cap H$ is a large subgroup of H. If B is a basic subgroup of G then $L \cap B$ is also a basic subgroup of L. If L is a large subgroup of a group G and L' is a large subgroup of L then L' is a large subgroup of G.

For some purposes the representation of large subgroups of G in the form $G(\mathbf{n})$ is somewhat difficult to handle. Thus RICHMAN [9] gives a representation for large subgroups of G in the form $L = \sum_{k=1}^{\infty} p^{n_k} G[p^k]$ where the sequence $\langle n_1, n_2, n_3, \ldots \rangle$ is non-decreasing. Our representation for large subgroups is similar to this one.

Section II
Fully invariant and large subgroups of p-groups

In this section we state further properties of fully invariant subgroups and large subgroups of p-groups. We provide proofs for those which we believe do not appear in the literature. Lemmas 2. 1, 2. 2 and 2. 3 hold for arbitrary abelian groups.

LEMMA 2. 1. *Let H be a subgroup of a group G and $f \in \text{End}\, G$ such that $f(H) \subseteq H$. Then f induces a homomorphism $f^* \in \text{End}\, G/H$ defined by $f^*(x+H) = f(x)+H$.*

LEMMA 2. 2. *Let F be a fully invariant subgroup of a group G and M/F be a fully invariant subgroup of G/F. Then M is a fully invariant subgroup of G.*

PROOF. Let $x \in M$, $f \in \text{End}\, G$ and f^* as in Lemma 2. 1 then $f^*(x+F) = f(x)+F \in M/F$. Therefore $f(x) \in M$ and M is fully invariant.

The converse of Lemma 2. 2 does not hold in general as the following example shows. Let $G = Zx \oplus Zy$ where $O(x) = p$ and $O(y) = p^2$, $F = pG$ and $M = G[p]$ then M/F is not a fully invariant subgroup of G/F.

LEMMA 2. 3. *Let F be a fully invariant subgroup of a group G and suppose $G = \oplus_\lambda G_\lambda$. Then $F = \oplus_\lambda (F \cap G_\lambda)$ where each $F \cap G_\lambda$ is a fully invariant subgroup of G_λ.*

Henceforth we restrict our attention to primary groups.

LEMMA 2. 4. *If L is a large subgroup of a group G then $L^1 = G^1$.*

PROOF. In [8] it was shown that every large subgroup of G contains G^1. Since p^nL is a large subgroup of G for every $n \in Z^+$ we have $L^1 \leqq G^1 \leqq \bigcap_{n=1}^{\infty} p^nL = L^1$, hence the desired equality.

LEMMA 2. 5. *Let L be a subgroup of a group G. Then L/G^1 is a large subgroup of G/G^1 if and only if L is a large subgroup of G. In fact $G(\mathbf{n})/G^1 = (G/G^1)(\mathbf{n})$.*

PROOF. From the characterization of large subgroups in [8] it suffices to prove the second statement. Let $f: G \to G/G^1$ be the canonical map. Then f is height preserving on G, hence $U(x) = U(fx)$ for all $x \in G$ and the result follows.

The following lemma yields an important special case of Corollary 2. 9 in [8].

LEMMA 2. 6. *Let G be a group without elements of infinite height and let $F \neq 0$ be a fully invariant subgroup of G. Then $F[p] = p^nG(p]$ for some $n \in Z^+$.*

PROOF. Let $n = \min(h(x): x \in F[p])$ and choose $z \in F[p]$ with $h(z) = n$. If $y \in G[p]$ with $h(y) \geqq n$ then there exists $f \in \operatorname{End} G$ with $fz = y$, hence $y \in F$ and we have $p^nG[p] = F[p]$.

COROLLARY. *Let L be a large subgroup of a group G. Then*

$$L[p] = p^nG[p] \quad \textit{for some} \quad n \in Z^+.$$

Our next lemma provides insight into one of the conditions for a subgroup of G to be large.

LEMMA 2. 7. *Let G be a group, H a subgroup of G such that $p^kH[p] \geqq p^{n_k}G[p]$ for $k = 0, 1, \ldots$ where the sequence of non-negative integers $n_0, n_1, \ldots$ is monotone increasing. Then $G = H + B$ for any basic subgroup B of G.*

PROOF. Let $B = \oplus B_m$ be a basic subgroup of G and S_n be the n-th partial sum in $\oplus B_m$. Then from Theorem 29. 3 in [1] we obtain $G[p] = S_n[p] \oplus p^nG[p]$ and therefore (1) $G[p] = S_{n_k}[p] + p^kH[p]$. By induction suppose $g \in G$ and $O(g) \leqq p^n$ implies $g \in B + H$ and let $g \in G$ $O(g) = p^{n+1}$ then $p^nG \in G[p]$ and letting $k = n$ in (1) we have

$$p^ng = b + h \quad \text{where} \quad b \in B \quad \text{and} \quad h \in p^nH[p]$$

therefore $h = p^nh'$ for some $h' \in H$ and from the purity of B we have $p^ng = p^nb' + p^nh'$ or $p^n(g - b' - h') = 0$ therefore $g - b' - h' \in B + H$ and $g \in B + H$. Thus we have $G = B + H$.

COROLLARY. *Let G be an unbounded group with no elements of infinite height. Then the large subgroups of G are precisely the unbounded fully invariant subgroups of G.*

PROOF. Let L be a large subgroup of G and let B be a proper basic subgroup of G. Then $G/B = (B + L)/B \cong L/(L \cap B)$ therefore L is unbounded. Conversely let L be an unbounded fully invariant subgroup of G. Then p^kL is fully invariant and non trivial for all $k \in Z^+$. The result then follows from an easy application of Lemmas 2. 6 and 2. 7.

We now collect some well known facts concerning fully invariant subgroups of direct sums of cyclic groups. Our objective is to give a representation for large subgroups L of a given group G which will display the Ulm invariants of L in terms of the Ulm invariants of G. These facts have straightforward proofs.

A) Let $B_k = \oplus C(p^k)$ and $x \in B_k$ such that $x \neq 0$. Then $U(x) = \langle r_0, r_1, \ldots, \ldots, r_{k-r_0-1}, \infty, \infty \rangle$ where $r_0 = h(x)$ and $r_i = r_0 + i$ for $0 \leqq i \leqq k - r_0 - 1$.

B) Let $B_k = \oplus C(p^k)$ with $x \in B_k$, $y \in B_k$. Then there exists $f \in \operatorname{End} B_k$ with $fx = y$ if and only if $h(x) \leqq h(y)$.

C) Let F be a fully invariant subgroup of $B_k = \oplus C(p^k)$. Then $F = p^{n_k} B_k$ where $n_k \leqq k$. If $F = 0$, $n_k = k$ and if $F \neq 0$ then $n_k = \min(h(x) : x \in F)$.

D) Let $B_k = \oplus C(p^k)$, $B_{k+r} = \oplus C(p^{k+r})$, $x \in B_k$ and $y \in B_{k+r}$. Then

(i) there exists $f \in \operatorname{Hom}(B_k, B_{k+r})$ such that $fx = y$ if and only if $O(x) \geqq O(y)$.

(ii) there exists $g \in \operatorname{Hom}(B_{k+r}, B_k)$ such that $gy = x$ if and only if $h(x) \geqq h(y)$.

Before we prove the next theorem we note that it was observed by MEGIBBEN [6] that if $B = \oplus B_k$ and L is a large subgroup of B then $L = \oplus p^{n_k} B_k$. What is new here is the relation on the indices.

THEOREM 2. 8. *Let $B = \oplus B_k$ where each $B_k = \oplus C(p^k)$. Then L is a fully invariant subgroup of B if and only if $L = \oplus_k p^{n_k} B_k$ where*

1) $n_k \leqq k$ for all $k \in Z^+$,

2) $n_k \leqq n_{k+r} \leqq n_k + r$ for all $k \in Z^+$, $r \in Z^+$.

A fully invariant subgroup L of B is a large subgroup of B if and only if $L = \oplus_k p^{n_k} B_k$ where 1) *and* 2) *hold and the sequence $\langle 1 - n_1, 2 - n_2, 3 - n_3, \ldots \rangle$ is unbounded if B is unbounded.*

PROOF. Let L be a fully invariant subgroup of G. Then $L = \oplus (B_k \cap L) = \oplus p^{n_k} B_k$ by Lemma 2. 3 and the remark C. Here $n_k \leqq k$ for all $k \in Z^+$ so the first condition holds. If $L = 0$ then $p^{n_k} B_k = 0$ for every k so $n_k = k$ for every k and condition (2) holds. Now suppose $L \neq 0$ so there exists a least positive integer j with $p^{n_j} B_j \neq 0$. Then $p^{n_k} B_k \neq 0$ for all $k \geqq j$ such that $B_k \neq 0$. To see this observe that $B_j[p] = p^{j-1} B_j[p] \leqq L$ implies $p^{j-1} B[p] \leqq L[p]$ by Lemma 2. 6. Since $B_k[p] = p^{j-1} B_k[p]$ for $k \geqq j$ we have $B_k[p] \leqq L \cap B_k = p^{n_k} B_k$ which proves the assertion.

Now suppose $L \neq 0$ and $B_k \neq 0 \neq B_{k+r}$. As we have just seen, if $p^{n_{k+r}} B_{k+r} = 0$ then $p^{n_k} B_k = 0$ and $n_k = k$, $n_{k+r} = k + r = n_k + r$ so condition (2) holds. We therefore assume $p^{n_{k+r}} B_{k+r} \neq 0$. Let $x \in B_k$ such that $h(x) \geqq n_{k+r}$ and choose $y \in p^{n_{k+r}} B_{k+r}$ with $h(y) = n_{k+r}$. Then there exists $g \in \operatorname{End} B$ with $gy = x$ by an easy corollary to remark D (ii). Hence $x \in L$ and we have $p^{n_{k+r}} B_k \leqq L \cap B_k = p^{n_k} B_k$ so $n_k \leqq n_{k+r}$.

To obtain the second inequality of (2) first suppose $p^{n_k} B_k = 0$. Then $n_k = k$ so $n_{k+r} \leqq k + r = n_k + r$ immediately. We therefore assume $p^{n_k} B_k \neq 0$ and let $y \in B_{k+r}$ such that $h(y) \geqq n_k + r$. Choose $x \in B_k$ so that $h(x) = n_k$. Then $O(x) = k - n_k$ and $O(y) \leqq k + r - (n_k + r) = k - n_k$ so by an easy corollary to remark D (i) there exists $f \in \operatorname{End} B$ with $fx = y$. Thus $y \in L$ and we have $p^{n_k + r} B_{k+r} \leqq L \cap B_{k+r} = p^{n_{k+r}} B_{k+r}$. It follows that $n_{k+r} \leqq n_k + r$.

We now have the inequality $n_k \leqq n_{k+r} \leqq n_k + r$ whenever $B_k \neq 0 \neq B_{k+r}$. But for those $B_k = 0$ we are free to define n_k so that this inequality holds for all k. Thus all fully invariant subgroups of B can be given the above representation. If L is a large subgroup of B and B is unbounded then so is L by the corollary to Lemma 2. 7. Hence the sequence $\langle 1 - n_1, 2 - n_2, 3 - n_3, \ldots \rangle$ must also be unbounded.

We now prove the sufficiency of these conditions. Let $L=\oplus p^{n_k}B_k$ where $n_k\leqq k$ for all $k\in Z^+$ and $n_k\leqq n_{k+r}\leqq n_k+r$ for all $k\in Z^+, r\in Z^+$. To show that L is a fully invariant subgroup of B it suffices to show for arbitrary $k\in Z^+$ that if $x\in p^{n_k}B_k$ and $f\in \operatorname{End} B$ then $fx\in L$. So let $x\neq 0$, $fx=\sum_{i=1}^{m} x_i$ where $x_i\in B_i$ and observe that $h(x)\leqq h(fx)=\min(h(x_i): 1\leqq i\leqq m)$ and $O(x)\geqq O(fx)=\max(O(x_i): :1\leqq i\leqq m)$. If $i\leqq k$ then $h(x_i)\geqq h(x)\geqq n_k$ so $x_i\in p^{n_k}B_i\leqq p^{n_i}B_i$ since $n_i\leqq n_k$, hence $x_i\in L$. If $i=k+r$ where $r\in Z^+$ then $O(x_i)\leqq O(x)\leqq k-n_k=(k+r)-(n_k+r)\leqq \leqq k+r-n_{k+r}$ since $n_{k+r}\leqq n_k+r$. Thus $x_i\in B_{k+r}[p^{k+r-n_{k+r}}]=p^{n_{k+r}}B_{k+r}\subseteq L$. Thus L is a fully invariant subgroup of B. In case B is unbounded and the sequence $\langle 1-n_1, 2-n_2, 3-n_3, \dots\rangle$ is also unbounded then L is unbounded and is therefore a large subgroup of B by the corollary to Lemma 2. 7. This completes the proof.

COROLLARY 1. *If L is a large subgroup of a group G then G/L is a direct sum of cyclic groups.*

PROOF. Let B be any basic subgroup of G and observe that $G/L=(B+L)/L\cong \cong B/(B\cap L)=\oplus_k B_k/\oplus_k p^{n_k}B_k\cong \oplus_k(B_k/p^{n_k}B_k)$.

Recall that a subgrouph H of a group G is relatively closed in the p-adic topology on G if and only if $(G/H)^1=0$. We thus have another corollary to Theorem 2. 8.

COROLLARY 2. *Let L be a large subgroup of a group G. Then L is a relatively closed subgroup of G.*

If H is any subgroup of a group G we now define the relative closure K of H in G to be that subgroup K of G such that $K/H=(G/H)^1$. This is an agreement with the topological definition.

THEOREM 2. 9. *Let H be a pure subgroup of a group G and L be a large subgroup of H. Then there exists a large subgroup L^* of G such that $L^*\cap H=L$, in fact, if $L=H(\mathbf{n})$ then it suffices to choose $L^*=G(\mathbf{n})$. If G/H is divisible then L^* is the relative closure of L in G and is therefore uniquely determined by L and $G/L^*\cong H/L$.*

PROOF. Let $L=H(\mathbf{n})$ and $L^*=G(\mathbf{n})$. Since H is a pure subgroup of G and $\mathbf{n}$ is a U-sequence for H we have that $\mathbf{n}$ is a U-sequence for G. Thus $L^*=G(\mathbf{n})$ is a large subgroup of G. To see that $L^*\cap H=L$ we observe that if $x\in L$ then $U_G(x)= =U_H(x)\geqq\mathbf{n}$ so $x\in L^*\cap H$ and $L\leqq L^*\cap H$. But if $y\in L^*\cap H$ then $U_H(y)= =U_G(y)\geqq \mathbf{n}$ implies $y\in L$, hence $L^*\cap H\leqq L$. Thus the first assertion is proved.

Now let G/H be divisible and suppose L^* is a large subgroup of G with $L^*\cap H=L$. Then $L^*/L=L^*/(L^*\cap H)\cong(H+L^*)/H=G/H$ and L^*/L is divisible. Since $G/L^*\cong(G/L)/(L^*/L)$ where L^*/L is a direct summand of G/L we have $G/L\cong \cong(L^*/L)\oplus(G/L^*)$ where G/L^* is a direct sum of cyclic groups by corollary 1 to Theorem 2. 8. Hence $(G/L)^1=L^*/L$ and L^* is the relative closure in G of L. To see that $G/L^*\cong H/L$ note that $G/L^*=(H+L^*)/L^*\cong H/(H\cap L^*)=H/L$.

The following lemma is well known.

LEMMA 2. 10. *Let $\{F_\lambda\subseteq G: \lambda\in\lambda\}$ be any collection of fully invariant subgroups of a group G. Then $\Sigma_\lambda F_\lambda$ is a fully invariant subgroup of G.*

We are now prepared to prove our representation theorem for large subgroups of p-groups.

THEOREM 2. 11. *Let L be a large subgroup of a group G. Then L is a large subgroup of G if and only if* $L=\sum_{k-1}^{\infty} p^{n_k}G[p^{k-n_k}]$ *where*

1) $n_k \leqq k$ *for all* $k \in Z^+$,

2) $n_k \leqq n_{k+1} \leqq n_k+1$ *for all* $k \in Z^+$,

3) *the sequence* $\langle 1-n_1, 2-n_2, 3-n_3, \ldots \rangle$ *is unbounded if G is unbounded. In this representation the Ulm invariants of L are given by* $f_L(n) = \Sigma_k\bigl(f_G(k-1): :k-n_k-1 = n\bigr)$ *for all non-negative integers n.*

PROOF. Let $L=\sum_{k=1}^{\infty} p^{n_k}G[p^{k-n_k}]$ as above and observe that L is a fully invariant subgroup of G by Lemma 2. 10 and the fact that each $p^{n_k}G[p^{k-n_k}]$ is a fully invariant subgroup of G. If G is bounded then L is already large, so assume G is unbounded. Then for each $j \in Z^+$ there exists $i=i_j \in Z^+$ such that $i-n_i>j$ by condition 3. Thus $i>j+n_i$ so $p^iG[p] \leqq p^{j+n_i}G[p] \leqq p^jL[p]$. To see this last inequality observe that if $x \in p^{j+n_i}G[p]$ then there exists $y \in G$ such that $x=p^{j+n_i}y$ where $px=0$. Hence $y \in G[p^{j+n_i+1}]$ where $i-n_i>j$ implies $i-n_i \geqq j+1$, hence $i \geqq j+n_i+1$ and $y \in G[p^i]$. So $p^{n_i}y \in p^{n_i}G[p^{i-n_i}] \leqq L$ and $x \in p^jL$ since $x=p^{j+n_i}y=p^j(p^{n_i}y)$.

Thus by Lemma 2. 7, $L+B = G$ for every basic subgroup B of G and L is a large subgroup of G.

Now suppose L is a large subgroup of G. Then if B is any basic subgroup of G we have $L \cap B$ is a large subgroup of B so by Theorem 2. 8 $L \cap B = \oplus_k p^{n_k}B_k$ where conditions 1), 2) and 3) hold for the indices $n_k, k \in Z^+$. But under these conditions $\oplus_k p^{n_k}B_k = \Sigma_k p^{n_k}B[p^{k-n_k}]$. To see this observe that $p^{n_j}B_j = p^{n_j}B_j[p^{j-n_j}] \leqq \leqq p^{n_j}B[p^{j-n_j}]$ for each $j \in Z^+$. And for each $j \in Z^+$ $p^{n_j}B[p^{j-n_j}] \leqq \Sigma_k p^{n_k}B[p^{k-n_k}]$ so $\oplus_k p^{n_k}B_k \leqq \Sigma_k p^{n_k}B[p^{k-n_k}]$.

To obtain the reverse inequality we observe that for each $j \in Z^+$, $p^{n_j}B[p^{j-n_j}] \leqq \leqq \oplus_k p^{n_k}B_k$. To see this let $x \in p^{n_j}B[p^{j-n_j}]$ and write $x=\sum_{i=1}^{m} x_i$ where $x_i \in B_i$. Then $h(x_i) \geqq h(x) \geqq n_j$ for all x_i, $1 \leqq i \leqq m$ and $O(x_i) \leqq O(x) \leqq j-n_j$ for all x_i, $1 \leqq \leqq i \leqq m$. Hence for $i \leqq j$ we have $h(x_i) \geqq n_j \geqq n_i$ and $x_i \in p^{n_i}B_i$. If $i=j+r$ for $r \in Z^+$ then $O(x_i) \leqq j-n_j = j+r-(n_j+r) \leqq j+r-n_{j+r}$ since $n_{j+r} \leqq n_j+r$. Thus $x_i \in B_{j+r}[p^{j+r-n_{j+r}}] = p^{n_{j+r}}B_{j+r} = p^{n_i}B_i$. Thus $p^{n_j}B[p^{j-n_j}] \leqq \oplus_k p^{n_k}B_k$.

Now let $L' = \Sigma_k p^{n_k}G[p^{k-n_k}]$ and observe that $L' \cap B = \Sigma_k p^{n_k}B[p^{k-n_k}] = L \cap B$. Since B is a pure subgroup of G with G/B divisible we have $L' = L$ by Theorem 2. 9.

To find the Ulm invariants of L we observe that $L \cap B$ is a basic subgroup of L, hence $f_L(n) = f_{L\cap B}(n)$ for all non-negative integers n. If we write $L \cap B = \oplus_n (L \cap B)_n$ in the usual manner, that is, where each $(L \cap B)_n = \oplus C(p^n)$, then

$$f_{L\cap B}(n) = \operatorname{rank}(L\cap B)_{n+1} = \operatorname{rank}\bigl(\oplus_k(p^{n_k}B_k\colon p^{n_k}B_k = \oplus C(p^{n+1}))\bigr) =$$

$$= \operatorname{rank}\bigl(\oplus_k(p^{n_k}B_k\colon k-n_k = n+1)\bigr) = \Sigma_k\bigl(\operatorname{rank}(p^{n_k}B_k)\colon k-n_k = n+1\bigr) =$$

$$= \Sigma_k\bigl(\operatorname{rank} B_k\colon k-n_k-1 = n\bigr) = \Sigma_k\bigl(f_B(k-1)\colon k-n_k-1 = n\bigr) =$$

$$= \Sigma_k\bigl(f_G(k-1)\colon k-n_k-1 = n\bigr).$$

Section III
A quasi-isomorphism theorem

Primary groups G and H are said to be quasi-isomorphic if there exist subgroups A of G and B of H such that G/A and H/B are bounded and $A \cong B$.

We now consider the following problem: Give necessary and sufficient conditions for a group G to be quasi-isomorphic to each of its large subgroups. In studying this problem the following theorem due to HILL [2] is important.

THEOREM. *If G and H are p-groups such that G/G^1 and H/H^1 are direct sums of cyclic groups then G is quasi-isomorphic to H if and only if*

1) *there exists $k \geqq 0$ such that for all $r \geqq 0$, $n \geqq 0$*

$$\sum_{j=n+h}^{n+h+r} f_G(j) \leqq \sum_{j=n}^{n+2h+r} f_H(j) \quad \text{and} \quad \sum_{j=n+h}^{n+h+r} f_H(j) \leqq \sum_{j=n}^{n+2h+r} f_G(j),$$

and

2) *G^1 is isomorphic to H^1.*

In order to obtain necessary conditions for a group to be quasi-isomorphic to every large subgroup we use the following

LEMMA 3. 1. *Let G be a group of infinite length and let $m_0, m_1, \dots$ be a strictly increasing sequence of integers with $m_0 = -1$. Then there exists a large subgroup L of G such that*

$$f_L(n) = \sum_{k=m_n+1}^{m_{n+1}} f_G(k).$$

PROOF. For each k there exists a unique i_k such that

$$m_{i_k}+2 \leqq k \leqq m_{i_k+1}+1,$$

so we define $n_k = k-i_k-1$. Since $i_k<k$, $0 \leqq i_{k+1}-i_k \leqq 1$ and i_k is unbounded, it follows that n_k satisfies the three conditions of Theorem 2. 11. Therefore

$$L = \sum_{k=1}^{\infty} p^{n_k} G[p^{k-n_k}]$$

is a large subgroup of G. Since for a fixed n, $k-n_k-1 = n$ if and only if $n = i_k$ if and only if $m_n+2 \leqq k \leqq m_{n+1}+1$ we have by Theorem 2. 11 again

$$f_L(n) = \sum_{k=m_n+1}^{m_{n+1}} f_G(k).$$

COROLLARY 1. *Let G be a group of infinite length. Then for any sequence of integers $k_1, k_2, k_3, \dots$ there exists a large subgroup L of G with $f_L(n) > k_n$ for all $n \in Z^+$.*

PROOF. Let $m_0 = -1$ and when m_n has been defined choose m_{n+1} so that $\sum_{i=m_n+1}^{m_{n+1}} f_G(i) > k_n$. Since G is of infinite length this can always be done. Now apply Lemma 3. 1.

COROLLARY 2. *Let G be any group. Then there exists a non-negative integer N such that for every strictly increasing sequence of integers* $r_1=N, r_2, r_3, \ldots$ *there exists a large subgroup L of G with* $f_L(n) \geqq \sum_{j=N}^{r_n} f_G(j)$ *for all* $n \in Z^+$.

PROOF. If G has finite length choose N so that $p^N G = G^1$ and let $L=G^1$. If G has infinite length and there exists $r \in Z^+$ such that $f_G(n)$ is finite for $n \geqq r$ then choose $N=r$. Define $k_n = \sum_{j=n}^{r_n} f_G(j)$ for each $n \in Z^+$ and use the previous Corollary. If $f_G(n)$ is infinite for infinitely many integers n choose N so that rank $p^N G$ = final rank G. Then define integers $m_0, m_1, m_2, \ldots$ by $m_0 = -1$, $m_1 = N-1$ and when m_n has been defined choose m_{n+1} so that $m_{n+1} > m_n$ and $f_G(m_{n+1})$ is an infinite cardinal number such that $f_G(m_{n+1}) \geqq \max\,(f_G(j) \colon N \leqq j \leqq r_n)$. Then use Lemma 3. 1 and the fact that $f_G(m_{n+1}) \geqq \sum_{j=N}^{r_n} f_G(j)$.

THEOREM 3. 2. *If G is a p-group such that* G/G^1 *is a direct sum of cyclic groups then all of the large subgroups of G are quasi-isomorphic if and only if*

a) $G = B \oplus D$ *where B is bounded and D is divisible, or*

b) *there exists a sequence of positive integers* $m_1, m_2, m_3, \ldots$ *and an infinite cardinal number* m^* *such that* $f_G(m_i) = m^*$ *for all i,* $f_G(n) \leqq m^*$ *if* $n \geqq m_1$, *and there exists an integer* $d > 0$ *such that* $m_{i+1} - m_i \leqq d$ *for all i.*

PROOF. The sufficiency of condition (a) is clear. We now assume that condition (b) holds for the group G. Let L be a large subgroup of G so that we may represent $L = \sum_{k=1}^{\infty} p^{n_k} G[p^{k-n_k}]$ according to Theorem 2. 11. Choose $h = \max\,(n_1, d)$ so we have

1) $f_G(j) \leqq m^*$ if $j \geqq h$,

2) for any n there exists m_i with $n \leqq m_i \leqq n+h$.

The first statement is clear and the second is shown as follows. If $n \leqq m_1$ then $h \geqq m_1$ implies $n \leqq m_1 \leqq h \leqq n+h$. For $n > m_1$ let i be the least positive integer such that $n \leqq m_i$. Then $m_{i-1} < n \leqq m_i \leqq m_{i-1} + d < n+d \leqq n+h$.

From statement (1) it follows immediately that for $n \geqq 0$, $r \geqq 0$,

$$\text{a)} \qquad \sum_{j=n+k}^{n+h+r} f_G(j) \leqq m^*,$$

and from statement (2) we have for any $n \geqq 0$ there exists j with $n \leqq j \leqq n+h$ such that $f_G(j) = m^*$, hence

$$\text{b)} \qquad \sum_{j=n}^{n+2h+r} f_G(j) \geqq m^* \quad \text{for all} \quad n \geqq 0,\ r \geqq 0.$$

We also obtain readily the inequality

$$\text{c)} \qquad \sum_{j=n+h}^{n+h+r} f_L(j) \leqq m^* \quad \text{for all} \quad n \geqq 0,\ r \geqq 0.$$

For we have $f_L(j) = \Sigma_k\big(f_G(k)\colon k-n_{k+1} = j\big)$ so if $j\geqq h$ and $k-n_{k+1} = j$ then $k\geqq h$ and $f_G(k)\leqq m^*$. Hence $f_L(j)\leqq m^*$ whenever $n+h\leqq j\leqq n+h+r, n = 0, 1, 2, \dots$ and the desired inequality follows.

To complete the proof it suffices to show

$$\text{d)} \qquad \left[\sum_{j=n}^{n+2h+r} f_L(j)\right] \geqq m^* \quad \text{for all} \quad n\geqq 0,\ r\geqq 0.$$

This is accomplished by showing that for any $n\geqq 0$ there exists m_i with $n\leqq m_i - n_{m_i+1}\leqq n+2h$. For then we have $f_L(m_i-n_{m_i+1}) = \Sigma_k\big(f_G(k)\colon k-n_{k+1} = m_i-n_{m_i+1}\big)\geqq f_G(m_i) = m^*$. So for each $n\geqq 0$ let $i=i_n$ be the least positive integer such that $m_i-n_{m_i+1}\geqq n$. Since the sequence $1-m_1, 2-m_2, 3-m_3, \dots$ is unbounded such an integer i exists. If $i=1$ then we have $0\leqq n\leqq n_1-m_{n_1+1}\leqq n_1\leqq h<n+2h$. If $i>1$ then we have $n_{i-1}-m_{n_{i-1}+1}\leqq n\leqq n_i-m_{n_i+1}\leqq n_{i-1}-m_{n_{i-1}+1}+(n_i-n_{i-1})\leqq n+d\leqq n+2h$. Thus we have proved (d) which concludes the proof of sufficiency.

To prove the necessity of these conditions we will show that if G is a group of infinite length which does not satisfy condition (b) then there exists a large subgroup L of G such that for any $h\geqq 0$ there exists $n>0$ with $f_L(n+h)>\sum_{j=n}^{n+2h} f_G(j)$. We therefore consider these cases.

Case 1. The Ulm invariants $f_G(n)$ are finite for all but finitely many $n\in Z^+$.

Case 2. For all but finitely many $n\in Z^+$ we have $f_G(n)<$final rank G, and $f_G(n)$ is infinite for infinitely many $n\in Z^+$.

Case 3. For infinitely many $n\in Z^+$ we have $f_G(n) =$ final rank G but if $n_k, n_2, n_3, \dots$ is the sequence of integers for which $f_G(n_i)=$ final rank G then the sequence $n_2-n_1,\ n_3-n_2,\ n_4-n_3, \dots$ of first differences is unbounded.

In Case 1 we use Corollary 1 to Lemma 3. 1 and choose $N\in Z^+$ according to that Corollary. Now define integers $k_1, k_2, k_3, \dots$ by $k_n = \sum_{j=N}^{2n+N} f_G(j)$ where we assume $f_G(j)$ is finite for $j\geqq N$. Then there exists a large subgroup L of G such that for all $n\in Z^+$ we have $f_L(n)>k_n$. But then for all $h\geqq 0$ we have $f_L(n+h)>k_{n+h} = \sum_{j=N}^{2(n+h)+N} f_G(j)\geqq \sum_{j=n}^{n+2h} f_G(j)$ for $n\geqq N$.

In Case 2 we use Corollary 2 to Lemma 3. 1 and choose $N\in Z^+$ accordingly. We may assume $f_G(n)<$final rank G for $n\geqq N$. We now define the sequence $r_1, r_2, r_3, \dots$ by $r_1=N$ and when r_n has been defined we define r_{n+1} to be the least positive integer r such that $r>\max(2n+2, r_n)$ and $f_G(r)>\max\big(f_G(j)\colon N\leqq j\leqq r-1\big)$. The conditions of Case 2 guarantee that such an r exists. Now define a large subgroup L so that for all $n\in Z^+$ we have $f_L(n)=\sum_{j=N}^{r_n} f_G(j)$. Then for every $h\geqq 0$ and every $n\geqq N$ we have $f_L(n+h) = \sum_{j=N}^{r_{n+h}} f_G(j)\geqq f_G(r_{n+h})>\sum_{j=n}^{n=2h} f_G(j)$.

In Case 3 we observe that Corollary 2 to Lemma 3. 1 guarantees the existence of a large subgroup L of G with $f_L(n)=$final rank G for all $n\in Z^+$. But for every $h \geqq 0$ there exists an $n > 0$ such that $f_G(j) <$ final rank G for $n \leqq j \leqq n+2h$. Thus $\sum_{j=h}^{n+2h} f_G(j) <$ final rank $G = f_L(n+h)$ as claimed. This concludes the proof.

COROLLARY. *If G is any p-group of infinite length which does not satisfy condition* (b) *then there exists a large subgroup of G which is not quasi-isomorphic to G.*

Section IV
The structure of large subgroups

This section is devoted to producing a number of examples where the structure of a group G is preserved by its large subgroups. We conjecture that there are many other such examples. The theorems here take the form "If some large subgroup of a group G has property P then every large subgroup of G has property P". Our proofs, however, take the equivalent form "Let L be a large subgroup of a group G. Then G has property P if and only if L has property P". This is reminiscent of a number of well known theorems of the form "G has property P if and only if p^nG has property P". The theorems of this section are evidently generalizations of such theorems for p^nG.

We begin with a theorem which is essentially proved in [8]. Recall that the length of a group G is the first ordinal α such that $p^\alpha G = p^{\alpha+1}G$ where the subgroups $p^\beta G$ are defined inductively by $p^\beta G = p(p^{\beta-1}G)$ if β has a predecessor and $p^\beta G = \bigcap_{\alpha<\beta} p^\alpha G$ if β is a limit ordinal.

THEOREM 4. 1. *If some large subgroup L of a group G has finite length then every large subgroup of G has finite length.*

PROOF. If G has finite length, then all of its subgroups have finite length. If some large subgroup L of G has finite length then $L = B \oplus D$ where B is bounded and D is divisible and there exists $n \in Z^+$ such that $p^nL = D$. Since p^nL is a large subgroup of G we have for any basic subgroup B' of G that $B' + p^nL = G$. But $B' + p^nL = B' \oplus D$ and if B' had a proper basic B'' then $G \neq B'' \oplus D = B'' + p^nL$. Hence B' has no proper basic subgroups and must be bounded. That is, $G = B' \oplus D$ where there exists $m \in Z^+$ with $p^mB' = 0$. Hence $p^mG = p^{m+1}G$ and G has finite length.

COROLLARY. *If some large subgroup L of a group G is bounded then G is bounded.*

We now consider another sort of boundedness condition.

LEMMA 4. 2. *Let L be a large subgroup of a group G and H be any subgroup of G. Then there exists $n \in Z^+$ such that $H \cap p^nL = 0$ if and only if there exists $m \in Z^+$ such that $H \cap p^mG = 0$.*

PROOF. Observe that $p^mL \leqq p^mG$ so if $H \cap p^mG = 0$, then $H \cap p^mL = 0$. Now suppose $H \cap p^nL = 0$. Since p^nL is a large subgroup of G there exists $m \in Z^+$ such that $p^nL[p] = p^mG[p]$ by lemma 2. 6. Hence $(H \cap p^mG)[p] = H[p] \cap p^mG[p] = H[p] \cap p^nL[p] = 0$ and $H \cap p^mG = 0$.

We turn now to our first major result in this section.

THEOREM 4. 3. *If some large subgroup of a group G is a direct sum of cyclic groups then every large subgroup of G is a direct sum of cyclic groups.*

PROOF. If G is a direct sum of cyclic groups then every subgroup of G is a direct sum of cyclic groups. Now suppose some large subgroup L of G is a direct sum of cyclic groups. Let $L[p]=p^nG[p]$ and write $G=S_n\oplus G_n$ according to the Baer decomposition Theorem 29. 3 [1]. It suffices to show that G_n is a direct sum of cyclic groups. But $G_n[p]=p^nG[p]=L[p]$ where $L[p]=\bigcup_{k=1}^{\infty}H_k$ such that there exists $n_k\in Z^+$ with $H_k\cap p^{n_k}L=0$ by the Kulikov criterion. Then there exist $m_k\in Z^+$ such that $H_k\cap p^{m_k}G=0$ for $k=1, 2, 3, \dots$ hence $H_k\cap p^{m_k}G_n=0$ for $k=1, 2, 3, \dots$ and $G_n[p]=L[p]=\bigcup_{k=1}^{\infty}H_k$. Hence by the Kulikov criterion and Lemma 4. 2 again we have that G_n is a direct sum of cyclic groups.

In the paper *Direct Sums of Countable Groups and Related Concepts* by IRWIN and RICHMAN [4] it was shown that a group G is a direct sum of countable groups if and only if G^1 is a direct sum of countable groups and G/G^1 is a direct sum of cyclic groups. We use this result in the proof of the next theorem.

THEOREM 4. 4. *If some large subgroup L of G is a direct sum of countable groups then all large subgroups of G are direct sums of countable groups.*

PROOF. If $G=\Sigma\oplus G_\lambda$ where each G_λ is countable then $L=\Sigma\oplus(G_\lambda\cap L)$ must also be a direct sum of countable groups. Now suppose some large subgroup L of G is a direct sum of countable groups, $L=\Sigma\oplus L_\lambda$. Then $G^1=L^1=\Sigma\oplus(L_\lambda\cap L^1)$ is a direct sum of countable groups and L/L^1 must be a direct sum of cyclic groups. But L/L^1 is a large subgroup of G/G^1 so G/G^1 is a direct sum of cyclic groups. Hence G must be a direct sum of countable groups.

In *Homology and Direct Sums of Countable Groups* [7], NUNKE has given the generalization for direct sums of cyclic groups and direct sums of countable groups. He defines a group G to be totally projective if $p^\alpha \operatorname{Ext}(G/p^\alpha G, K)=0$ for every ordinal α and every group K. Evidently a group G is a direct sum of cyclic groups if and only if length $G\leqq\omega$ and G is totally projective. It is also shown in [7] that G is a direct sum of countable groups if and only if length $G\leqq\Omega$ and G is totally projective. Another result to be found in [7] is that for every ordinal β, G is totally projective if and only if $p^\beta G$ and $G/p^\beta G$ are both totally projective. We use this result in the proof of our next theorem.

THEOREM 4. 5. *If some large subgroup of G is totally projective then every large subgroup of G is totally projective.*

PROOF. If G is totally projective then G/G^1 is a direct sum of cyclic groups and G^1 is totally projective. If L is any large subgroup of G then $L^1=G^1$ is totally projective and $L/L^1\leqq G/G^1$ where G/G^1 is a direct sum of cyclic groups. Hence L/L^1 is also totally projective and L is totally projective. If L is totally projective then $L^1=G^1$ is totally projective and L/L^1 is a direct sum of cyclic groups which is a large subgroup of G/G^1 by Lemma 2. 5. Thus by Theorem 4. 3 we have that G/G^1 is a direct sum of cyclic groups, hence G is totally projective.

We use Theorem 4. 3 again in the proof of the next theorems. We also use the following useful lemma.

LEMMA 4. 6. *Let G be a group, H a subgroup of G and B a pure subgroup of H such that $G/B=(H/B)\oplus(K/B)$. Then K is a pure subgroup of G.*

PROOF. Suppose $ng\in K$ where $g\in G$ and $n\in Z^+$. We have $g+B = h+B+k+ +B$ for some $h\in H$ and $k\in K$ so, for some $b\in B$, $g-h-k = b$ and $nh = ng-nk- -nb\in H\cap K = B$. Since B is a pure subgroup of H there exists $b'\in B$ such that $nh=nb'$. Therefore $ng = n(b'+b+k)$ where $b'+b+k\in K$, hence K is a pure subgroup of G.

THEOREM 4. 7. *Let L be a large subgroup of a group G and C be a pure dense subgroup of L. Then there exists a pure dense subgroup B of G such that $B\cap L=C$. If C is a basic subgroup of L then B is a basic subgroup of G.*

PROOF. Since L/C is divisible we write $G/C=(L/C)\oplus(B/C)$. Since C is pure in L we have B pure in G by the previous lemma. Clearly $B\cap L=C$ and $G/B\cong L/C$ is divisible. If C is a basic subgroup of L then C is a direct sum of cyclic groups which is large in B. Hence by Theorem 4. 3 we have that B is a direct sum of cyclic groups whence B is a basic subgroup of G.

At this point we review some facts. Let H be a pure subgroup of a group G. If L is a large subgroup of G then $L\cap H$ is a large subgroup of H. If B is a basic subgroup of G then $L\cap B$ is a basic subgroup of L. These results were proved by PIERCE in [8]. We proved in section II that if L is a large subgroup of H then there exists a large sugroup L^* of G such that $L^*\cap H=L$, and if G/H is divisible then L^* is precisely the relative closure of L in G. We shall have more to say on this below. Now we have Theorem 4. 7 which completes this set of results. Unfortunately, it is not true in general that if C is a pure subgroup of L then there exists a pure subgroup B of G such that $B\cap L=C$. Let $G=Zx\oplus Zy$ where $O(x)=p$, $O(y)=p^3$ and let $L=G[p^2]$. If $C= Z(x+py)$ then C is a pure subgroup of L but there exists no pure subgroup B of G such that $B\cap L=C$.

LEMMA 4. 8. *Let G be a group with $G^1=0$ and denote the torsion completion of G by $\bar{B}$. If L is a large subgroup of G then the natural image in $\bar{B}$ of the torsion-completion of L is the same as the closure of L in $\bar{B}$.*

PROOF. Let B denote the appropriate basic subgroup of G so that G is embedded in $(\pi_k B_k)_t$. Since $L\cap B= \oplus_k p^{n_k}B_k$ is basic in L we have that the torsion completion of L is $\bar{L}=(\Pi_k p^{n_k}B_k)_t$. But then $f\colon \bar{L}\to\bar{B}\colon \langle p^{n_1}b_1, p^{n_2}b_2, p^{n_3}b_3, \ldots\rangle\to\langle p^{n_1}b_1, p^{n_2}b_2, p^{n_3}b_3, \ldots\rangle$ is the desired isomorphism.

THEOREM 4. 9. *A large subgroup of G is torsion complete if and only if all large subgroups of G are torsion-complete.*

PROOF. If G is torsion complete and L is a large subgroup of G then L is a relatively closed subgroup of G. But by the previous lemma we then have that L is torsion complete. Now suppose that L is torsion complete. Then $L^1=0$, hence $G^1=0$ and $L[p]=p^nG[p]=G_n[p]$ for some integer n. If $\bar{B}$ is the torsion completion of G_n then $\bar{B}[p]=\overline{(G_n[p])}=G_n[p]$ where G_n is a pure subgroup of $\bar{B}$. Hence by Lemma 12, Kaplansky, we have $G_n=\bar{B}$, hence $G=S_n\oplus G_n$ is also torsion complete.

COROLLARY. *If G is a direct sum of torsion-complete groups then every large subgroup of G is a direct sum of torsion-complete groups.*

PROOF. If L is a large subgroup of G where $G=\oplus_\lambda \bar{B}_\lambda$ then $L=\oplus_\lambda L\cap \bar{B}_\lambda$ by Lemma 2. 3 where each $L\cap \bar{B}_\lambda$ is large in $\bar{B}_\lambda$ by Pierce. Hence by Theorem 4. 7 we have each $L\cap \bar{B}_\lambda$ is torsion-complete.

The following lemma is of general usefulness.

LEMMA 4. 10. *Let H be a pure, dense subgroup of G where $G^1=0$. Then for each $x\in G$ there exists a Cauchy sequence $x_1, x_2, x_3, \ldots$ in H such that $U(x_n)=U(x)$ for every n.*

PROOF. Choose any basic subgroup B of H and embed G in $\bar{B}$. We may then write $x=\langle b_1, b_2, b_3 \ldots\rangle$ where $b_n\in B_n$. Now observe there exists m so that $j>m$ implies $U(b_j)>U(x)$. That is, $h(p^r b_j)>h(p^r x)$ for all r for which $p^r x\neq 0$ and $p^r b_j=0$, if $p^r x=0$. Let $y_j=\langle 0, \ldots, 0, b_j, b_{j+1}, \ldots\rangle$ so that, again, $U(y_j)>U(x)$ for $j>m$ and define $x_n=x-y_{m+n}$ for $n>0$. Then $x_n\to x$ and $U(x_n)=U(x)$ for all $n>0$.

If H is a subgroup of G, then H^- denotes the closure of H with respect to the p-adic topology on G. Note by Corollary 2 to Theorem 2. 8, we have that if H is a subgroup of L, the L-closure and the G-closure of H are the same.

THEOREM 4. 11. *If A is a pure subgroup of G and L is a large subgroup of G then $(A\cap L)^-$ is a pure subgroup of L if and only if A^- is a pure subgroup of G.*

PROOF. If A^- is pure in G then $A^-\cap L$ is pure in L and we need to show that $A^-\cap L=(A\cap L)^-$. So let $\bar{a}\in A^-\cap L$ and observe that since A is a pure dense subgroup of A^- there exists a sequence $a_1, a_2, a_3, \ldots$ in A such that $U(a_n)=U(\bar{a})$ and $\lim_n a_n=\bar{a}$. But $\bar{a}\in L$ so every $a_n\in L$ and $\bar{a}\in(A\cap L)^-$. Thus $A^-\cap L\leqq(A\cap L)^-\leqq A^-\cap L$, the second inequality being trivial, and $A^-\cap L=(A\cap L)^-$ is a pure subgroup of L.

Now suppose $(A\cap L)^-$ is a pure subgroup of L. Since $A\cap L$ is a large subgroup of A we may write $A=S_n\oplus A_n$ where $A_n[p]=(A\cap L)[p]$ and since $A^-=S_n\oplus A_n^-$, we may as well assume that $A[p]=(A\cap L)[p]$. Then $(A[p])^-=(A\cap L)[p]^-=$ $=(A\cap L)^-[p]\leqq(A^-\cap L)[p]\leqq A^-[p]=(A[p])^-$ where the last equality holds by the purity of A in G. Hence $(A\cap L)^-[p]=(A^-\cap L)[p]$ where $(A\cap L)^-$ is a pure subgroup of $A^-\cap L$ so $A^-\cap L=(A\cap L)^-$. In order to show that A^- is a pure subgroup of G it suffices to show that A^-/A is divisible. But $(A\cap L)^-/A\cap L$ is divisible since $(A\cap L)^-$ is pure in L. Now let $x\in A^-[p]=(A\cap L)^-[p]$. Then there exists $x_n\in(A\cap L)^-$ with $p^n x_n+A\cap L=x+A\cap L$ hence $p^n x_n-x=a_n$ where $a_n\in A\cap L$. But then $p^n x_n+A=x+A$ is also satisfied and $h(x+A)=\infty$. By Lemma 8 Kaplansky, we have that A^-/A is divisible, hence A^- is a pure subgroup of G.

In their paper on quasi-closed groups [3] HILL and MEGIBBEN showed that a group G is quasi-closed if and only if $G[p]+\bar{S}=\bar{B}[p]$ for every non-discrete subgroup S of $G[p]$. We use this to prove our concluding theorem.

THEOREM 4. 12. *If some large subgroup of G is quasi-closed then so is every large subgroup of G.*

PROOF. We show that a large subgroup L of G is quasi-closed if and only if G is. If A is any pure subgroup of G and L is quasi-closed then $A \cap L$ is a pure subgroup of L and $(A \cap L)^-$ is a pure subgroup of L, so A^- must be a pure subgroup of G. Therefore G is quasi-closed.

Now suppose that G is quasi-closed and let L be a large subgroup of G. Then there exists an integer n so that $G_n[p] = p^n G[p] = L[p]$. Let S be any non-discrete subsocle of $L[p]$. Then S is non-discrete in G_n and since G_n is quasi-closed we have $G_n[p] + \bar{S} = \bar{B}[p]$ where $\bar{B}$ is the torsion completion of G_n. If $\bar{L}$ is the torsion completion of L we have by the results of part 1 that $\bar{L}[p] = \bar{B}[p]$ so that $L[p] + \bar{S} = \bar{L}[p]$. Thus L is quasi-closed.

Section V
Unanswered questions

There are a number of apparently difficult questions which we have left unanswered. We list those which we believe are interesting.

PROBLEM 1. Give necessary and sufficient conditions for all of the large subgroups of a group G to be quasi-isomorphic.

Observe that we only solved this problem in case G/G^1 is a direct sum of cyclic groups.

PROBLEM 2. If L is a large subgroup of a group G and H is a pure subgroup of L does there exist a large subgroup L' of L and a pure subgroup K of G such that $K \cap L' = H \cap L'$?

PROBLEM 3. If L is a large subgroup of a group G and H is a direct summand of L, under what conditions does there exist a direct summand K of G such that $K[p] = H[p]$?

PROBLEM 4. Let L be a large subgroup of G. Is it true that L is a direct sum of torsion complete groups if and only if G is a direct sum of torsion-complete groups?

We have shown that if G is a direct sum of torsion-complete groups then L is.

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SUR LES FONCTIONS RECURSIVES PRIMITIVES DE RAMIFICATIONS

Par

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1. Introduction

Dans [1] Madame R. PÉTER fait l'étude des fonctions récursives primitives pour un binoïde libre $\hat{V}$ (pour la définition d'un binoïde libre voir [1, II] ou [2]) et montre que cette étude se rattache à la théorie générale des ensembles holomorphes libres [3]. Les fonctions de base choisies dans [1] sont:

1 — La fonction constante $\wedge$ (où $\wedge$ désigne la ramification vide)

2 — Pour tout $a \in V$ la fonction f_a de deux variables définie par:

$$(\forall x_1, x_2 \in \hat{V}) \qquad f_a(x_1, x_2) = x_1 \to a \uparrow x_2$$

3 — Le prédicat d'égalité.

Les fonctions récursives primitives sont les fonctions obtenues à partir des fonctions de base par un nombre fini de substitutions (Substitution) et de récurrences (primitive Rekursion) du type:

$$\text{(A)} \quad \begin{cases} f(\wedge, u_1, \dots, u_r) = g(u_1, \dots, u_r) \\ \text{et pour } a \in V: \\ f(f_a(x_1, x_2), u_1, \dots, u_r) = \\ \qquad = g_a(x_1, x_2, u_1, \dots, u_r, f(x_1, u_1, \dots, u_r), f(x_2, u_1, \dots, u_r)). \end{cases}$$

Dans ce qui suit on démontre qu'il n'est pas nécessaire de prendre le prédicat d'égalité parmi les fonctions de base. On peut faire le méme geme de démonstration pour le cas du monoïde libre.

2. Notations

Désormais, nous appelerons fonctions récursives primitives les fonctions déduites des seules fonctions 1 et 2 par substitutions et récurrences.

Nous aurons besoin de deux ramifications particulières de $\hat{V}$; l'une, notée 0, sera utilisée pour représenter la valeur «faux» des prédicats (la valeur «vrai» étant représentée par la ramification vide $\wedge$); l'autre, notée σ, servira à introduire une «représentation parenthésée» d'une ramification (paragraphe 3) et, au lieu de comparer directement deux ramifications, nous comparerons leurs représentations (paragraphe 4); a_0 étant un élément choisi dans V, nous poserons:

$$\sigma = a_0 \uparrow a_0, \quad 0 = a_0 \uparrow (a_0 \to a_0).$$

$\hat{V}$, muni de la loi $\to$, est un monoïde; nous notons V^* (resp. $(V \cup \{\sigma\})^*$) le sous-monoïde de $\hat{V}$ engendré par V (resp. $V \cup \{\sigma\}$); les éléments de ce sous-monoïde sont appelés *mots* sur V (resp. $V \cup \{\sigma\}$). Un mot sur V est donc une suite finie d'éléments de V et la *longueur* de ce mot est le nombre d'éléments de la suite. Le signe $\rightarrow$ sera parfois omis lorsqu'il portera sur un mot (par exemple, on écrira $a_1 a_0 a_1$ au lieu de $a_1 \rightarrow a_0 \rightarrow a_1$, pour $a_1, a_0 \in V$).

On dit qu'un mot β est *facteur gauche* (resp. *facteur gauche strict*) d'un mot α s'il existe γ (resp. s'il existe γ, $\gamma \neq \wedge$) tel que:

$$\alpha = \beta\gamma,$$

γ est alors le *quotient à gauche* de α par β.

3. Mot attaché à une ramification

Une ramification peut être écrite de manière unique, sous forme d'une combinaison d'élément de V par $\uparrow$ et $\rightarrow$; par exemple la ramification x de la

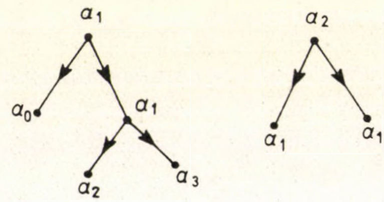

Fig. 1

figure 1 s'écrit:

$$x = a_1 \uparrow (a_0 \rightarrow a_1 \uparrow (a_2 \rightarrow a_3) \rightarrow a_2 \uparrow (a_1 \rightarrow a_1),$$

en convenant que le signe opératoire $\uparrow$ a priorité sur le signe $\rightarrow$.

On peut, sans créer d'ambiguité, remplacer $a_i \uparrow ($ par la seule lettre a_i à condition de conserver la parenthèse fermante correspondante que nous pouvons coder par σ, et remplacer de même les a_i qui ne précèdent pas $\uparrow$ par $a_i \sigma$; de plus les $\rightarrow$ peuvent être supprimés; ainsi à partir de x on obtient le mot:

$$\mu(x) = a_1 a_0 \sigma a_1 a_2 \sigma a_3 \sigma \sigma \sigma a_2 a_1 \sigma a_1 \sigma \sigma.$$

DÉFINITION. La fonction μ de $\hat{V}$ dans $(V \cup \{\sigma\})^* \subset \hat{V}$ définie par

$$\mu(\wedge) = \wedge \quad \mu(f_a(x_1, x_2)) = \mu(x_1) a \mu(x_2) \sigma$$

est récursive primitive; $\mu(x)$ est appelé *mot attaché* à la ramification x.

Nous allons montrer que deux ramifications qui ont le même mot attaché sont égales. Pour cela étudions l'ensemble P des mots «bien parenthésés» dans $(V \cup \{\sigma\})^*$. Cet ensemble P sera défini en introduisant une transformation ι qui associe à tout mot α de $(V \cup \{\sigma\})^*$, le mot «réduit» obtenu en supprimant toutes les «parenthèses»

$a \in V$ et σ qui se correspondent dans α: ainsi P sera l'ensemble des mots qui sont réductibles au mot vide:

DÉFINITION. Soit l'application ι de $(V \cup \{\sigma\})^*$ dans $V^* \cup \{0\}$ définie par récurrence sur la longueur d'un mot $\alpha \in (V \cup \{\sigma\})^*$:

(i) $\iota(\Lambda) = \Lambda$.

(ii) $\forall a \in V$ $\iota(\alpha a) =$ SI $\iota(\alpha) = 0$ ALORS 0 SINON $\iota(\alpha)\, a$.

(iii) SI $\iota(\alpha) = 0$ ou Λ ALORS $\iota(\alpha\sigma) = 0$;

SI $\iota(\alpha) = \beta b$ ($\beta \in V^*$ et $b \in V$) ALORS $\iota(\alpha\sigma) = \beta$.

P est l'ensemble $\iota^{-1}(\Lambda)$.

LEMME 1. *Quels que soient* $\alpha, \beta \in (V \cup \{\sigma\})^*$

$$\iota(\alpha) \neq 0 \quad et \quad \iota(\beta) \neq 0 \Rightarrow \iota(\alpha\beta) = \iota(\alpha)\, \iota(\beta)$$

$$\iota(\alpha) = 0 \Rightarrow \iota(\alpha\beta) = 0.$$

Ce résultat se démontre par récurrence sur la longueur de β.

LEMME 2. *Le mot attaché à toute ramification sur V appartient à P.*

Se démontre par récurrence sur la ramification donnée, à l'aide du lemme 1. Réciproquement, tout mot appartenant à P est attaché à une ramification sur V. Nous n'utiliserons pas ici ce résultat.

PROPOSITION. *Deux ramifications sont égales si, et seulement si, elles ont même mot attaché.*

Il suffit de prouver que:

$$(\forall x, y \in \hat{V}) \quad \mu(x) = \mu(y) \Rightarrow x = y;$$

procédons par récurrence sur la longueur de $\mu(x)$:

1) Si $\mu(x) = \mu(y) = \Lambda$, alors $x = y = \Lambda$.

2) Supposons $\mu(x)$ et $\mu(y)$ différents de Λ, alors x et y ne sont pas vides et s'écrivent:

$$x = x_1 \rightarrow a \uparrow x_2 \quad y = y_1 \rightarrow b \uparrow y_2.$$

Le plus grand facteur gauche strict de $\mu(x) = \mu(x_1) a\, \mu(x_2) \sigma$ appartenant à P est $\mu(x_1)$; en effet un facteur gauche strict plus grand que $\mu(x_1)$ s'écrit $\mu(x_1) a\, \gamma$ où γ est facteur gauche de $\mu(x_2)$; d'après le lemme 1:

$$\iota(\mu(x_2)) \neq 0 \Rightarrow \iota(\gamma) \neq 0$$

et

$$\iota(\mu(x_1) a \gamma) = \iota(\mu(x_1)) a \iota(\gamma) = a \iota(\gamma) \neq \Lambda;$$

ainsi $\mu(x_1)\ a\ \gamma$ n'est pas dans P. De même, le plus grand facteur gauche strict de

$\mu(y)$ appartenant à P est $\mu(y_1)$ et donc:

$$\mu(x)=\mu(y) \text{ entraîne } \mu(x_1)=\mu(y_1)$$
$$\text{d'où } a=b, \ \mu(x_2)=\mu(y_2)$$

et, par hypothèse de récurrence, $x_1=y_1, x_2=y_2$, soit finalement $x=y$.

4. Predicat d'égalité

THÉORÈME. *Le prédicat d'égalité défini par:*

$$(\forall x, y \in \hat{V}) \ eq(x, y) = SI \ x=y \ ALORS \wedge \ SINON \ 0$$

est récursif primitif.

Remarquons que toute ramification x différente de $\wedge$ s'écrit de façon unique:

$$x = a \uparrow x_1 \to x_2 \quad (a \in V).$$

Soit f'_a la fonction de deux variables définie par:

$$(\forall x_1, x_2 \in \hat{V}) \quad f'_a(x_1, x_2) = a \uparrow x_1 \to x_2;$$

on peut introduire un nouvel opérateur de récurrence:

$$\text{(B)} \quad \begin{cases} f(\wedge, u_1, \dots, u_r) = g(u_1, \dots, u_r) \\ \text{et pour } a \in V: \\ f(f'_a(x_1, x_2), u_1, \dots, u_r) = \\ \qquad = g_a(x_1, x_2, u_1, \dots, u_r, f(x_1, u_1, \dots, u_r), f(x_2, u_1, \dots, u_r)). \end{cases}$$

LEMME 3. *Soit une fonction f obtenue par une récurrence de type* (B) *à partir des fonctions g et g_a (pour tout $a \in V$); si g et les g_a sont récursives primitives f l'est aussi.*

En effet à une fonction h on peut associer $\bar{h}$ définie par:

$$\bar{h}(x, u_1, \dots, u_r) = h(\text{rev}\,(x), u_1, \dots, u_r);^*$$

si h est récursive primitive $\bar{h}$ l'est aussi; or, si f est obtenue par une récurrence de type (B), $\bar{f}$ est obtenue par une récurrence de type (A), donc la fonction $f=\bar{\bar{f}}$ est récursive primitive.

LEMME 4. *Pour tout élément a de $V \cup \{\sigma\}$ il existe une fonction récursive primitive d_a telle que*

$$(\forall y \in \hat{V}) \ (\forall b \in V \cup \{\sigma\}) \ d_a(b \to y) = SI \ b=a \ ALORS \ y \ SINON \ 0.$$

* rev désigne la fonction définie par: rev $(\wedge)=\wedge$, rev $(f_a(x_1, x_2)) = a \uparrow \text{rev}\,(x_2) \to \text{rev}\,(x_1)$ (*reversierte Allee* dans [1], *ramification réfléchie* dans [2]).

Remarquons que la fonction de trois variables cond définie par

$$\text{cond}(x_1, x_2, x_3) = \text{SI } x_1 = \wedge \text{ ALORS } x_2 \text{ SINON } x_3$$

est récursive primitive. Les fonctions d_a répondant à la question sont alors définies par des récurrences de type (B):

1) pour $a \in V$ $\quad d_a(\wedge) = 0$

$$d_a(f_b'(x_1, x_2)) = 0 \text{ pour tout } b \in V, \text{ différent de } a$$

$$d_a(f_a'(x_1, x_2)) = \text{SI } x_1 = \wedge \text{ ALORS } x_2 \text{ SINON } 0.$$

2) $\quad d_\sigma(\wedge) = 0$

$$d_\sigma(f_a'(x_1, x_2)) = 0 \qquad \text{pour tout } a \in V, \text{ différent de } a_0$$

$$d_\sigma(f_{a_0}'(x_1, x_2)) = \text{SI eqa}_0(x_1) = \wedge \text{ ALORS } x_2 \text{ SINON } 0,$$

où $\text{eqa}_0(x)$ est la fonction récursive primitive définie par:

$$\text{eqa}_0(\wedge) = 0$$

$$\text{eqa}_0(f_a(x_1, x_2)) = 0 \quad \text{pour tout } a \in V, \text{ différent de } a_0$$

$$\text{eqa}_0(f_{a_0}(x_1, x_2)) = x_1 \rightarrow x_2;$$

c'est-à-dire que $\quad \text{eqa}_0(x) = \wedge \Leftrightarrow x = a_0.$

Lemme 5. *Il existe une fonction récursive primitive δ de deux variables telle que, lorsque x et y sont dans $(V \cup \{\sigma\})^*$, $\delta(x, y)$ soit le quotient à gauche de y par x si x est facteur gauche de y et 0 sinon.*

Il suffit de définir δ par récurrence de la façon suivante:

$$\delta(\wedge, y) = y$$

$$\delta(f_a(x_1, x_2), y) = d_a(\delta(x_1, y)) \quad \text{pour } a \in V, \ a \neq a_0,$$

$$\delta(fa_0(x_1, x_2), y) = \text{SI eqa}_0(x_2) = \wedge \text{ ALORS } d_\sigma(\delta(x_1, y)) \text{ SINON } d_{a_0}(\delta(x_1, y)).$$

Par récurrence sur la longueur de $x \in (V \cup \{\sigma\})^*$, on montre facilement que δ satisfait à la propriété annoncée.

Finalement la fonction $eq(x, y)$ est définie par:

$$eq(x, y) = \text{SI } \delta(\mu(x), \mu(y)) = \wedge \text{ ALORS } \wedge \text{ SINON } 0$$

c'est-à-dire que $eq(x, y)$ est récursive primitive.

5. Généralisation

Toute relation binaire sur V peut-être prolongée en une relation sur $\hat{V}$: il suffit de convenir que deux ramifications r et s sont en relation si les graphes (forêts) sous-jacents à r et s sont isomorphes et si les points «correspondants» dans ces graphes ont des «étiquettes» (ou «couleurs») en relation dans V; plus précisément:

DÉFINITION ET PROPOSITION. *Soit Γ un prédicat à deux variables sur V. Prolongeons Γ en un prédicat $\hat{\Gamma}$ sur $\hat{V}$ en posant*

$$\hat{\Gamma}(\wedge,\wedge)=\wedge$$

$$\hat{\Gamma}(f_a(x_1, x_2), f_{a'}(x_1', x_2'))=\wedge \Leftrightarrow \Gamma(a, a')=\wedge \text{ et } \hat{\Gamma}(x_1, x_1')=\wedge \text{ et } \hat{\Gamma}(x_2, x_2')=\wedge$$

Le prédicat $\hat{\Gamma}$ est récursif primitif.

Il suffit de reprendre la démonstration précédente en remplaçant d_a, pour $a \in V$, par $d_{a,\Gamma}$ définie par:

$$d_{a,\Gamma}(\wedge)=0$$

$$d_{a,\Gamma}(f_b'(x_1, x_2))=0 \text{ pour tout } a \text{ tel que } \Gamma(b, a)=0$$

$$d_{a,\Gamma}(f_b'(x_1, x_2))=\text{SI } x_1=\wedge \text{ ALORS } x_2 \text{ SINON } 0$$

pour tout a tel que $\Gamma(b, a)=\wedge$.

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A GENERALIZATION OF KŐNIG'S THEOREM

By
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Introduction. A well known theorem of D. Kőnig states that the maximum number of (pairwise) independent edges of a bipartite graph G equals to the minimum number of vertices covering all edges of G. This theorem has a lot of generalizations, equivalents and applications in different branches of mathematics. We are going to give a generalization related to the original problem concerning bipartite graphs. As an application of this we prove a conjecture of P. ERDŐS concerning chromatic number of finite set-systems.

§ 1. Let A, B be two disjoint finite sets. We consider bipartite graphs the edges of which join vertices of A to vertices of B. If we say bipartite graph we always think of a graph of this type. If G is a bipartite graph and $X \subseteq A$ then let X^G denote the set of those vertices which are joined to a vertex of X by an edge of G.

ORE gave the following equivalent of Kőnig's theorem: *The maximum number of independent edges of a bipartite graph G is*

$$|A| - \max_{X \subseteq A} \{|X| - |X^G|\}.$$

In other words, a bipartite graph critical concerning the property that for any $X \subseteq A$

$$|X^G| \geqq |X| - \delta,$$

consists of independent edges.

In connexion with this theorem the following question arises. We order an integer $f(X)$ to every $X \subseteq A$. Let K denote the class of those bipartite graphs G for which every $X \subseteq A$ satisfies

$$|X^G| \geqq f(X). \tag{1}$$

An element G of K is said to be *critical* if no proper subgraph of it belongs to K. Which graphs are critical?

I do not know the answer in the general case, only supposing

$$f(X \cup Y) + f(X \cap Y) \geqq f(X) + f(Y) \quad \text{if} \quad X \cap Y \neq \varnothing. \tag{2}$$

$$f(X \cup Y) \leqq f(X) + f(Y) \quad \text{if} \quad X \cap Y = \varnothing. \tag{3}$$

An example for a function satisfying (2) and (3) is

$$f(X) = \begin{cases} a|X| + b, & \text{if} \quad X \neq \varnothing \quad (a, b \geqq 0), \\ 0, & \text{if} \quad X = \varnothing. \end{cases}$$

THEOREM. *A $G \in K$ is critical if and only if every $x \in A$ has in G valency $f(\{x\})$.*

Let G be an arbitrary graph of K and let α_G denote the system of those subsets of A for which

$$|X^G| = f(X).$$

We may suppose, that $\emptyset \in \alpha_G$, i.e. $f(\emptyset) = 0$, since otherwise $K = \emptyset$. Our theorem can be formulated in the form that G is critical if and only if $\{x\} \in \alpha_G$ for every $x \in A$.

We need the following

LEMMA. *If* $X, Y \in \alpha_G$ *then* $X \cap Y \in \alpha_G$.

PROOF OF THE LEMMA. If $X \cap Y = \emptyset$, then it is trivial. Otherwise we may use (2):

$$|(X \cup Y)^G| + |(X \cap Y)^G| \leqq |X^G \cup Y^G| + |X^G \cap Y^G| =$$

$$= |X^G| + |Y^G| = f(X) + f(Y) \leqq f(X \cup Y) + f(X \cap Y).$$

Since $G \in K$, here the equality must hold. This proves $X \cap Y \in \alpha_G$.

PROOF OF THE THEOREM. The "if" part is trivial. Suppose that G is a critical element of K. Let $x \in A$ and let $y_1, \ldots, y_\varphi$ be the vertices of B joined to x by the edges $E_1, \ldots, E_\varphi$, respectively. If we omit E_i $(1 \leqq i \leqq \varphi)$ then the remaining graph G_i does not belong to K and hence there is an $X_i \subseteqq A$ such that

$$|X_i^{G_i}| < f(X_i),$$

i.e.

$$X_i \in \alpha_G, \quad x \in X_i, \quad y_i \notin X_i^{G_i}. \tag{4}$$

By our lemma $Y = X_0 \cap \ldots \cap X_\varphi \in \alpha_G$. Put $Y_0 = Y - \{x\}$, then by (3) and (4)

$$|Y^G| = \varphi + |Y_0^G| \geqq f(\{x\}) + f(Y_0) \geqq f(Y).$$

Since here the equality must hold because of $Y \in \alpha_G$, we obtain $\varphi = f(\{x\})$, Q.E.D.

§ 2. Let $\mathscr{H}$ be a finite set-system. We put $P(\mathscr{H}) = \bigcup_{E \in \mathscr{H}} E$. The elements of $P(\mathscr{H})$ and $\mathscr{H}$ are called the *vertices* and *simplices* of $\mathscr{H}$, respectively. If the simplices of $\mathscr{H}$ have the same cardinality β then we say that $\mathscr{H}$ is a *uniform β-system.*

The notion of a finite set-system is a generalization of the notion of finite undirected graphs without isolated vertices; since we can identify a graph with the system of its edges. The concepts and theorems of graph theory have more possible generalizations for set-systems. In [3] and [4] some questions of this type are detailed.

An *α-colouring* of $\mathscr{H}$ means a function defined on $P(\mathscr{H})$ the range of which is a set of natural numbers $\leqq \alpha$. Such a colouring is *correct* if every simplex of $\mathscr{H}$ contains vertices to which different natural numbers are ordered. It is *strictly correct* if the integers ordered to the elements of any simplex of $\mathscr{H}$ cover the interval $[1, \alpha]$.

The *chromatic number* of a set-system $\mathscr{H}$ is the least natural number α for which a correct α-colouring of $\mathscr{H}$ exists. The *strict chromatic number* of $\mathscr{H}$ is the greatest integer β for which a strictly correct β-colouring of $\mathscr{H}$ exists. It can be easily seen, that a set-system has strict chromatic number $\geqq 2$ if and only if it has chromatic number $\leqq 2$.

Let $\mathscr{H}$ be a graph (i.e. a uniform 2-system). It is well known that the following properties are equivalent (they characterize the *forests*):

(i) $\mathscr{H}$ contains no circiuts;
(ii) for every $\mathscr{H}' \subseteq \mathscr{H}$, $\mathscr{H}' \neq \emptyset$ there exists an $E \in \mathscr{H}'$ such that $|E \cap P(\mathscr{H}' - \{E\})| \leqq 1$;
(iii) for any $\mathscr{H}' \subseteq \mathscr{H}$, $\mathscr{H}' \neq \emptyset$ $|\mathscr{H}'| + |P(\mathscr{H}')| \geqq \sum_{E \in \mathscr{H}'} |E| + 1$;
(iv) for any $\mathscr{H}' \subseteq \mathscr{H}$, $\mathscr{H}' \neq \emptyset$ $|P(\mathscr{H}')| \geqq |\mathscr{H}'| + 1$.

It can be easily seen, that (i), (ii) and (iii) are equivalent for an arbitrary set-system (see [3]; we do not define here the notion of a circuit, since we shall not need it). Furthermore, if the simplices of $\mathscr{H}$ have at least two elements then (iii) implies (iv).

It is a trivial fact that a forest has chromatic number $\leqq 2$. ERDŐS conjectured that

if a finite set-system has property (iv) *then it has chromatic number* $\leqq 2$. [1]

Requiring property (iii) instead of (iv) the statement is obvious. We are going to prove the following generalization of Erdős' conjecture:

THEOREM. *Suppose that a finite set-system* $\mathscr{H}$ *has the property*

$$|P(\mathscr{H}')| \geqq (\beta - 1)|\mathscr{H}'| + 1$$

holds for every $\mathscr{H}' \subseteq \mathscr{H}$, $\mathscr{H}' \neq \emptyset$. *Then* $\mathscr{H}$ *has strict chromatic number* $\geqq \beta$.

PROOF. [2] I. First we prove the theorem in the case $\mathscr{H}$ is a uniform β-system In this case $\mathscr{H}$ has properties (i)—(iii). Using (ii) a strictly correct β-colouring of $\mathscr{H}$ can be constructed by induction on $|\mathscr{H}|$.

II. Let now $\mathscr{H}$ be an arbitrary set-system satisfying the supposition of the theorem. Put $\mathscr{H} = A$, $P(\mathscr{H}) = B$,

$$f(X) = \begin{cases} (\beta - 1)|X| + 1, & \text{if } X \neq \emptyset, \\ 0, & \text{if } X = \emptyset, \end{cases}$$

and let us construct a bipartite graph G by joining $x \in A$ to $y \in B$ if and only if $y \in x$. By our supposition

$$|X^G| \geqq f(X)$$

holds for any $X \subseteq A$. The theorem of §1. gives that there exists a subgraph G' of G such that for $X \subseteq A$

$$|X^{G'}| \geqq f(X)$$

and every $x \in A$ has in G' valency β. We form the uniform β-system $\mathscr{H}'$ the simplices of which are the β-tuples of B being joined to the same $x \in A$ in G'. The first part of our proof gives that there exists a strictly correct β-colouring of $\mathscr{H}'$. Extending that β-colouring arbitrarily over $P(\mathscr{H})$ we obtain a strictly correct β-colouring of $\mathscr{H}$.

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[1] Erdős formulated his conjecture only for uniform 3-systems, because in this case it is sharp, while for uniform 4-systems it could be probably sharpened.
[2] A different proof of Erdős' conjecture is given in [4].

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A NOTE ON RADEMACHER'S INEQUALITY

By
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Introduction

In the present note we improve slightly an inequality for uncorrelated random variables attributed to RADEMACHER [3] and independently proved by MENSOV [2]. The method of proof is the same with the one given in the books of DOOB [1], RÉVÉSZ [4], etc., but the bound is improved due to the representation of integers in the triadic system instead of the dyadic.

Inequalities

The main inequality is given in the following

THEOREM. *Let the sequence of random variables $X_1, \dots, X_n$ such that*

$$\mathbf{E}(X_i) = 0, \quad \mathbf{E}(X_iX_j) = 0 \quad (i \neq j), \mathbf{E}(X_i^2) = \sigma_i^2 < \infty \quad \textit{for} \quad i, j = 1, 2, \dots, n.$$

Then

$$\mathbf{E}\big(\max_{1 \leqq k \leqq n} (X_1 + \dots + X_k)^2\big) \leqq \left(\frac{\log n}{\log 3} + 2\right)^2 (\sigma_1^2 + \dots + \sigma_n^2).$$

PROOF. Every positive integer $k \leqq n$ can be written uniquely in the triadic system as

$$k = \sum_{i=0}^{n} I_i 3^i$$

where $I_i = \pm 1$ or 0 and $3^{r-1} < n \leqq 3^r$ i.e.,

$$\frac{\log n}{\log 3} \leqq r < \frac{\log n}{\log 3} + 1. \tag{1}$$

Now set $X_{n+1} = \dots = X_{3^r} = 0$ and define the random variables,

$$Y_{h,i} = \sum_{j=1}^{3^h} X_{i3^h+j} \qquad (h = 0, 1, \dots, r, \ i = 0, 1, \dots, 3^{r-h} - 1) \tag{2}$$

i.e., we define $r+1$ classes A_h, $h = 0, 1, \dots, r$ and each class has 3^{r-h} groups $Y_{h,i}$ $(i = 0, 1, \dots, 3^{r-h} - 1)$. Hence we can write

$$X_1 + \dots + X_k = \sum_{h=0}^{r} Y_{h,i(h)} I_h \quad \text{for all} \quad 1 \leqq k \leqq n \tag{3}$$

where $Y_{h,i(h)}$ is a member of the class A_h and $I_h = \pm 1$ or 0, and

(4) $$(X_1 + \cdots + X_k)^2 \leqq (r+1) \sum_{h=0}^{r} Y_{h,i(h)}^2 I_h^2 \leqq (r+1) \sum_{h=0}^{r} \sum_{i=0}^{3^{r-h}-1} Y_{h,i}^2.$$

Therefore

(5) $$\max_{1 \leqq k \leqq n} (X_1 + \cdots + X_k)^2 \leqq (r+1) \sum_{h=0}^{r} \sum_{i=0}^{3^{r-h}-1} Y_{h,i}^2$$

and taking expectations we obtain

$$\mathbf{E}\big(\max_{1 \leqq k \leqq n} (X_1 + \cdots + X_k)^2\big) \leqq (r+1)^2 (\sigma_1^2 + \cdots + \sigma_n^2) \leqq \left(\frac{\log n}{\log 3} + 2\right)^2 (\sigma_1^2 + \cdots + \sigma_n^2),$$

q.e.d.

It should be noted that this is a slight improvement over the Rademacher's inequality which has log 2 instead of log 3, but the strong laws for orthogonal random variables based on this inequality are not affected by our improved bound.

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EXTENSION OF PARTIAL ORDERS ON A SEMIGROUP TO SEMIGROUPS OF QUOTIENTS

By

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Let G be a semigroup which contains at least one central and cancellable element. All these elements form a subsemigroup $C^*(G)$ of G. Let S be a subsemigroup of $C^*(G)$. Then there exists (cf. e.g. [3]), unique up to isomorphisms, a semigroup $G_S = Q(G, S)$ which contains G and consists of all quotients a/s, $a \in G$, $s \in S$, obeying the rules

$$a/s = b/t \Leftrightarrow at = bs \quad \text{and} \quad a/s \cdot b/t = ab/st.$$

B. M. PUTTASWAMAIAH has shown in [2], that a partial order on G with a supplementary condition (i.e. (1) in our Theorem 1a) can be extended to a partial order on G_S. In this paper we give a complete solution of the question of the connection of partial orders in G and G_S. We obtain that each partial order on G has an extension on G_S (Theorem 1c). Applying this to cancellative abelian semigroups we get as a corollary that every partial order of such a semigroup G can be extended to a full order on G if and only if G is torsion-free. Furthermore, our Theorems 1a—1c have generalizations to the essentially non-commutative case of semigroups $Q_r(G, S)$ of right quotients; we shall deal with this case in a continuing paper.

We recall that a partial order (briefly p.o.) O on a set M is defined to be a binary relation $\leqq$ which is reflexive, antisymmetric and transitive. Assume $M_1 \subseteqq M_2$ and let O_i $(i=1, 2)$ be a p.o. of M_i. Then O_2 is said to be an extension of O_1 if, for all $a, b \in M_1$, $a \leqq_1 b$ implies $a \leqq_2 b$. Especially, O_2 shall be called a *strict extension* of O_1 if O_1 is the restriction $O_2|M_1$ of O_2 on the subset M_1 of M_2, in other words: if $a \leqq_1 b$ is equivalent with $a \leqq_2 b$ for all $a, b \in M_1$. Of course, a p.o. O of a semigroup G is always assumed to obey the monotony law, i.e. $a \leqq b \Rightarrow ac \leqq bc$ and $ca \leqq cb$ for all $c \in G$. An element $c \in G$ is called positive (negative) in G for O, if $a \leqq ca$ and $a \leqq ac$ ($a \geqq ca$ and $a \geqq ac$) for any $a \in G$. If G has an identity 1, then $1 \leqq c$ ($1 \geqq c$) implies c positive (negative) and conversely.

THEOREM 1a. *Let G be a semigroup and let $F = G_S = Q(G, S)$ be a semigroup of quotients as described above with $S \subseteqq C^*(G)$. Then a p.o. O_G on G has a strict extension O_F on F if and only if*

(1) *$ax \leqq bx$ with $a, b \in G$, $x \in S$ implies $a \leqq b$. If this is the case, O_F is uniquely determined by O_G. Moreover, an element $c \in G$ is positive (negative) in G for O_G if and only if it is positive (negative) in F for O_F.*[1]

[1] The statement of Theorem 1 in [2], that each element of a generating set N of S and hence each element of S is positive in $F = G_S$ for O_F fails to be true, cf. [5].

PROOF. Let O_F (designed by $\leqq_F$) be any p.o. of the semigroup of quotients $F=Q(G, S)$ and let $O_G=O_F|G$ (designed by $\leqq$) be the restriction of O_F on G. Then we have (1), because $x\in S$ has its inverse in F and O_F obeys the monotony law in F. For the same reasons one has

$$(2) \qquad a/s \leqq_F b/t \leftrightarrow at \leqq_F bs \leftrightarrow at \leqq bs,$$

hence O_F is uniquely determined by its restriction $O_G=O_F|G$. Conversely, if a p.o. O_G on G obeys (1), we can define a relation $\leqq_F$ by (2) and prove that this gives a p.o. O_F on the semigroup F; this is just done in [2]. Clearly, the so defined O_F is a strict extension of the given O_G.

The remaining statements about positive elements follow by

$a\leqq ca$ for any $a\in G \leftrightarrow a/s \leqq_F c(a/s)$ for any $a/s\in F$,
$a\leqq ac$ for any $a\in G \leftrightarrow a/s \leqq_F (a/s)c$ for any $a/s\in F$,

and similarly for negative elements.

THEOREM 1b. *Let G be a semigroup, and let S be a subsemigroup of G such that each $x\in S$ is in the center of G and cancellable in G, i.e. $S\subseteq C^*(G)$. Then every p.o. O_G on G can be extended to a p.o. O'_G on G which obeys* (1). *Clearly, the smallest extension O'_G of O_G which obeys* (1) *is uniquely determined by O_G and S. Moreover, an element $c\in G$ which is positive (negative) for O_G remains so for O'_G, but the converse is not true.*

PROOF. Let us design the given p.o. O_G by $\leqq$. We define a relation $\leqq\cdot$ in G by

$$(3) \qquad a\leqq\cdot b \text{ if theree exists an element } x\in S \text{ with } ax\leqq bx.$$

This relation is well defined, because, if there exists an element $y\in S$ with $ay\geqq by$, by the properties of the elements of S we obtain $axy=bxy$ and hence $a=b$. It is an extension of O_G, because $a\leqq b$ implies $ax\leqq bx$ for each $x\in S$ and therefore $a\leqq\cdot b$. The reflexivity of $\leqq\cdot$ is included in this statement. To prove antisymmetry take $a\leqq\cdot b$ and $b\leqq\cdot a$, then by (3) $ax\leqq bx$ and $by\leqq ay$, hence $axy=bxy$ and $a=b$ as above. The transitivity and both rules for monotony are similarly verified, and the so defined p.o. O'_G on G obeys (1), as $ax\leqq\cdot bx$ implies $axy\leqq bxy$ for some $y\in S$ and hence $a\leqq\cdot b$.

For the statements about positive (negative) elements these properties are clearly preserved by each extension of a p.o. within the same semigroup. A counter-example for the other direction is obtained as follows: Take the semigroup G of natural numbers with the usual multiplication and define $a\leqq b$ if $a=b$ or if $3|a$ and $3|b$ and $a+x=b$ for some $x\in G$. Clearly this is a p.o. O_G without any positive element, but taking S generated by 3 we get O'_G as the usual order of G.

Combining both results we have (omitting the clear statements for positive and negative elements)

THEOREM 1c. *Let G be a semigroup and let $F=Q(G, S)$ be a semigroup of quotients with $S\subseteq C^*(G)$. Then each p.o. O_G on G has an extension O_F on F, and the smallest extension of this kind is uniquely determined by O_G as the strict extension of O'_G, defined by O_G according to* (3.)

In other words, there is a 1—1 correspondence between all p.o. O_F on $F=Q(G, S)$ and the classes of those p.o. O_G on G which have the same extension O'_G by (3). Clearly, O_F is a full order if and only if its restriction O'_G is a full order.

In this connection one remark may be made: As well in the condition (1) of Theorem la as in the definition (3) of O'_G occurs the denominator subsemigroup S of $F=Q(G, S)$. Now it is possible that there are different semigroups $S_i \subseteqq C^*(G)$ which give the same $F=Q(G, S_i)$, and (1) and (3) then have to work in the same way with each of these S_i. We see the reason for this using the following proposition, which is easily obtained from Theorem 2, § 4 of [4]:

PROPOSITION. *Among all semigroups $S_i \subseteqq C^*(G)$ with $F=Q(G, S_i)$ there is a maximal one, $\bar{S}$, containing all the others. $\bar{S}$ is the set of all elements of $C^*(G)$ which have an inverse in F and $\bar{S}$ is related to each S_i by*

$$x \in \bar{S} \Leftrightarrow x \in C^*(G) \quad \text{and} \quad xy \in S_i \quad \text{for some} \quad y \in C^*(G).$$

COROLLARY. *Let G be an abelian cancellative semigroup. Then $F=Q(G, G)$ is the smallest group containing G and each p.o. O_G on G has an extension O_F on F, the smallest one being uniquely determined by O_G. Moreover, every p.o. O_G can be extended to a full order on G if and only if G is torsion-free.*

Here the last statement comes from the same theorem for abelian groups, where it is well known (cf. [1] and the references there). Indeed, O_G can be extended to a full order on G if and only if the same is true for an extension O_F of O_G with respect to F, and with G clearly F is torsion-free and conversely.

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THE MATRIX EQUATION $A^k = E$ $(a_{ij} \geqq 0)$ OVER A STRICT PARTIALLY ORDERED INTEGRAL DOMAIN

By

L. RÉDEI, member of the Academy and ANNA LEE (Budapest)

1. Introduction

For any arbitrary positive integer k, all $n \times n$ matrix solutions of $A^k = E$ over the non-negative cone of a strict partially ordered integral domain $\mathscr{R}$ with unit will be determined. In the special case of $\mathscr{R}$ being the ring of the rational integers, the equation in question is connected with a sociological problem proposed by Professor Alexander Szalai.

2. Notations and preliminaries

Let $\mathscr{R}$ be a strict partially ordered integral domain (i.e. a commutative associative ring without zero-divisors) with unit 1, and let $\mathscr{S}$ be the semi-ring of the non-negative elements of $\mathscr{R}$. We suppose $1 \in \mathscr{S}$. $\mathscr{R}_n$ denotes the n-th full matrix ring over $\mathscr{R}$ and $\mathscr{S}_n$ is the semi-ring of all those matrices of $\mathscr{R}_n$ which have entries from $\mathscr{S}$ only.

Let π be an arbitrary permutation of the elements $1, 2, \ldots, n$ and ε the identity permutation. The image of i according to the permutation π is $\pi(i)$ $(i = 1, 2, \ldots, n)$. δ_{ij} is the Kronecker symbol. Then $E = [\delta_{ij}]$ will denote the unit matrix in $\mathscr{R}_n$ (thus $E \in \mathscr{S}_n$ by the above assumption) and $P = [\delta_{i\pi(j)}]$ will represent a permutation matrix, which rearranges the rows of an arbitrary matrix (by premultiplication) according to π.

3. The result

THEOREM. *For any positive integer k, all solutions of*

$$(1) \qquad A^k = E \qquad (A \in \mathscr{S}_n)$$

are given by

$$(2) \qquad A = [c_j \delta_{i\pi(j)}] \qquad (c_1, c_2, \ldots, c_n \in \mathscr{S} \setminus 0),$$

where the permutation π satisfies

$$(3) \qquad \pi^k = \varepsilon$$

and the elements $c_1, c_2, \ldots, c_n$ are restricted by the conditions

$$(4) \qquad c_i c_{\pi(i)} c_{\pi^2(i)} \cdots c_{\pi^{k-1}(i)} = 1 \qquad (i = 1, 2, \ldots, n).$$

REMARK. The conditions (4) can be transformed into several simple independent conditions. For this purpose we decompose π into the product of disjoint cycles. Let $\zeta = (i_1\, i_2 \ldots i_r)$ be one of these cycles, then by (3) necessarily $r | k$. Considering now an equation in (4) where i is equal to any of the indices $i_1, i_2, \dots, i_r$, then (4) becomes

$$(c_i c_{\zeta(i)} \cdots c_{\zeta^{r-1}(i)})^{\frac{k}{r}} = 1.$$

This can be written in the form

$$c_i c_{\zeta(i)} \cdots c_{\zeta^{r-1}(i)} = \sqrt[\frac{k}{r}]{1}, \tag{5}$$

where, of course, the radical on the right is meant in $\mathscr{S}$. Taking the corresponding condition (5) for each cycle ζ, we obtain a system of independent conditions equivalent to (4).

REMARK 2. In the special case (mentioned in the introduction) if $\mathscr{R}$ is the ring of the rational integers, the solutions of (1) can be permutation matrices only.

4. Proof of the theorem

Suppose that (1) is satisfied. Then $\det A \neq 0$ and thus it contains at least one non-zero term. Hence

$$A = PD + B \tag{6}$$

with suitable matrices $P, D, B \in \mathscr{S}_n$ where P denotes a permutation matrix and D a diagonal matrix with positive diagonals.

Let $\mathscr{N}(X)$ be the number of the non-zero elements of any matrix $X \in \mathscr{S}_n$, then by (6)

$$\mathscr{N}(AX) \geqq \mathscr{N}(PDX) = \mathscr{N}(DX) = \mathscr{N}(X).$$

Applying this inequality consecutively to the matrices $X = A, A^2, \dots$ we get

$$\mathscr{N}(A) \leqq \mathscr{N}(A^2) \leqq \dots$$

Hence

$$\mathscr{N}(A^k) \geqq \mathscr{N}(A).$$

By (1) the value on the left is equal to n and by (6) the value on the right is greater than or equal to n thus $\mathscr{N}(A) = n$. Therefore, again by (6) we must have $B = 0$, that is

$$A = PD. \tag{7}$$

We now introduce the notation

$$D_t = P^{-t} D P^t \qquad (t = 0, 1, 2, \dots; D_0 = D),$$

then by (1) and (7)

$$E = A^k = P^k D_{k-1} D_{k-2} \dots D_1 D_0.$$

Hence P^k must be a diagonal matrix, that is

$$P^k = E \tag{8}$$

and

$$D_{k-1} D_{k-2} \dots D_1 D_0 = E. \tag{9}$$

Conversely, it follows from (8) and (9) that the matrix (7) gives a solution of (1). Therefore we have only to show, that (7), (8), (9) can be transformed into (2), (3), (4). To do this we represent the matrices P and D as

$$P=[\delta_{i\pi(j)}], \qquad D=[c_i\delta_{ij}]$$

where, for the time being, π is an arbitrary permutation and $c_1, c_2, \ldots, c_n$ are arbitrary positive elements of $\mathscr{S}$. Then (7) and (8) transform to (2) and (3) resp. For the transformation of (9), too, we perform the calculation

$$D_1=P^{-1}DP=[\delta_{\pi(i)p}][c_p\delta_{pq}][\delta_{q\pi(j)}]=[\delta_{\pi(i)p}][c_p\delta_{p\pi(j)}]=[c_{\pi(i)}\delta_{ij}].$$

This gives in general

$$D_t=[c_{\pi^t(i)}\delta_{ij}] \qquad (t=0, 1, 2, \ldots)$$

and therefore (9) reproduces the conditions (4) as

$$[c_i c_{\pi(i)} c_{\pi^2(i)} \ldots c_{\pi^{k-1}(i)}\delta_{ij}]=E=[\delta_{ij}].$$

This completes the proof.

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