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ON THE SOLUTIONS OF THE DIFFERENTIAL EQUATION $y''+q(x)y=0$, WHERE $[q(x)]^{\nu}$ IS CONCAVE. I

By

Á. ELBERT (Budapest)

It is well-known that if the function $q(x)$ is positive (except at $x=0$ where it may vanish) and non-decreasing on the interval $[0, \infty)$, then each solution of the differential equation

$$(1) \qquad y''+q(x)y=0 \qquad (x \geqq 0)$$

has an infinity of zeros on $[0, \infty)$ and a finite number of them on each interval $[0, x_0]$; moreover, $y^2(x)$ is bounded and the values at successive maximum of $y^2(x)$ form a decreasing sequence. Our purpose is to examine how rapidly this sequence can decrease if for $q(x)$ only some class of functions is taken into account.

DEFINITION. The function $q(x)$ belongs to $C_\nu[a, b]$ $(0<\nu<\infty)$ if $q(x)$ is a non-negative continuous function on $[a, b]$ and $[q(x)]^\nu$ is concave (no point of an arc lies below the corresponding chord). For the sake of simplicity we shall write $C_\nu = C_\nu[0, \infty)$.

Let $y_c = y_c(x)$ be the solution of (1) with the initial conditions $y_c(c)=1$, $y_c'(c)=0$ $(c \geqq 0)$ and denote the roots of $y_c'(x)=0$ in the order of succession by $x_0(c)= =c, x_1(c), x_2(c), \dots, x_n(c), \dots$, and let us write

$$(2) \qquad r_n(q, c) = y_c^2(x_n(c)).$$

With the form $r_n(q, c)$ we wish to express the fact that the maximum of the solution y_c^2 depends on the function $q(x)$.

We shall prove the following

THEOREM. *If $q(x)$ belongs to C_ν then there exist the values*

$$(3) \qquad \varrho_n^{(\nu)} = \min_{q \in C_\nu} r_n(q, 0) \qquad (n=1, 2, \dots),$$

these minima are attained for $q = Cx^{\frac{1}{\nu}}$ (C is any positive constant) and the relations

$$(4) \qquad \varrho_n^{(\nu)} > \frac{\alpha_\nu}{n^{\frac{1}{1+2\nu}}} \qquad (0<\alpha_\nu<1)$$

hold.

The proof of this theorem is based upon the idea that if $q(x)$ is not a function of the form $C(x+\gamma)^{\frac{1}{\nu}}$ (which can be easily handled), a family of functions of C_ν

can be found leading to a function with the form $C(x+\gamma)^{\frac{1}{\nu}}$ with some C and γ giving a smaller value to r_n.

PROOF. The theorem will be proved for a fixed n ($n=1, 2, \ldots$). Let $q(x)$ be an element of C_ν. At first assume that $q(x)$ has continuous first and second derivatives. Let

(5) $$r(x) = y_0^2(x) + \frac{y_0'^2(x)}{q(x)} \qquad (x \geqq 0),$$

then from (2) and (5) we have

(2′) $$r_n(q, 0) = r(x_n(0)).$$

Differentiation of (5) and (1) give

(6) $$r' = -\frac{q'}{q^2} y_0'^2,$$

hence $r(x)$ increases if $q(x)$ decreases, thus for our purpose it is sufficient to take into account only non-decreasing $q(x)$, i.e. $q'(x) \geqq 0$. Further we may assume $q'(x) > 0$ for $0 < x < x^{**}$ and $q'(x) = 0$ for $x \geqq x^{**}$, hence $q'(x) > 0$ on $(0, x^*)$ where $x^* = \min\{x^{**}, x_n(0)\}$.

Let $\varphi(x)$ be a continuous function defined by

(7) $$\cos\varphi = \frac{y'}{\sqrt{qr}}, \qquad \sin\varphi = \frac{y}{\sqrt{r}}$$

and

(8) $$\varphi(0) = \frac{\pi}{2}.$$

These definitions imply the relations $\varphi(x_n(0)) = \frac{\pi}{2} + n\pi$. By (6) and (7) we have

(9) $$\varphi' = \sqrt{q} + \frac{1}{4} \cdot \frac{q'}{q} \sin 2\varphi$$

and

(10) $$\frac{r'}{r} = -\frac{q'}{q} \cos^2 \varphi.$$

The function $q(x)$ satisfies the inequality

(11) $$(1-\nu)q'^2 - qq'' \geqq 0$$

because $q \in C_\nu$ gives $[q^\nu]'' = \nu q^{\nu-2}[(\nu-1)q'^2 + qq''] \leqq 0$. From a result of [2] we know that $\varphi' > 0$, hence φ is an increasing function of x.

By (10)

(12) $$\log r_n(q, 0) = \log r(x_n(0)) = -\int_0^{x_n(0)} \frac{q'}{q} \cos^2 \varphi \, dx.$$

Instead of $\sqrt{q(x)}$ introduce the functions $s_\xi(x)$ $(0 < \xi < x^*)$:

(13) $$s_\xi(x) = \begin{cases} \sqrt{q(x)} & 0 \leqq x \leqq \xi \\ C(x+\gamma)^{\frac{1}{2\nu}} & x \geqq \xi \end{cases}$$

where $C=C(\xi)$ and $\gamma=\gamma(\xi)$ are determined by $s_\xi(\xi)=\sqrt{q(\xi)}$ and $s'_\xi(\xi)=[\sqrt{q(x)}]'_{x=\xi}$. One can easily see the validity of the inequalities $C>0$ and $\gamma\geqq 0$. Now it can be stated: $s_\xi^2(x)\in C_\nu$, $s_\xi(x)\geqq\sqrt{q(x)}$ and $s_\xi(x)$ has continuous first and second derivatives except at $x=\xi$ where the second derivative may be discontinuous.

Let $\psi_\xi(x)$ be defined by the differential equation

$$\psi'_\xi = s_\xi+\frac{1}{2}\frac{s'_\xi}{s_\xi}\sin 2\psi_\xi \qquad (x\geqq 0) \tag{14}$$

with the initial condition $\psi_\xi(0)=\varphi(0)=\frac{\pi}{2}$. From (13) it is clear, that $\varphi(x)=\psi_\xi(x)$ on $[0,\xi]$ and $\psi_\xi(x)\geqq\varphi(x)$ on $[\xi, x_n(0)]$.

Let $x_{i,\xi}$ be defined by

$$\psi_\xi(x_{i,\xi}) = \frac{\pi}{2}+i\pi, \tag{15}$$

then we shall prove that the integral

$$I_n(\xi) = -\int_0^{x_{n,\xi}} 2\frac{s'_\xi(x)}{s_\xi(x)}\cos^2\psi_\xi(x)\,dx \tag{16}$$

is a non-decreasing function of ξ.

The function $\tau(x)$ is defined by

$$\tau(x) = 2\left(\frac{q}{q'}-\nu x\right) = \frac{\sqrt{q}}{(\sqrt{q})'}-2\nu x \qquad (0<x<x^*). \tag{17}$$

From (11) we have $\tau'(x)\geqq 0$, and if $\tau'(x)\equiv 0$, then $q = C(x+\gamma)^{\frac{1}{\nu}}$ with some C and γ. The function $\tau(x)$ is non-negative: $\tau(x)\geqq\lim_{x\to+0}\tau(x) = 2\lim_{x\to+0}\frac{q(x)}{q'(x)}\geqq 0$. Assume that $\tau'(\xi)>0$. Let $0<\eta<\xi<x^*$ then $\psi_\xi(x)\leqq\psi_\eta(x)$ because $s_\xi(x)\leqq s_\eta(x)$, hence $x_{n,\eta}\leqq x_{n,\xi}$. The value $\Delta=\tau(\xi)-\tau(\eta)$ tends to 0 if η tends to ξ.

We will prove the inequality

$$\lim_{\eta\to\xi}\frac{1}{\Delta}[I_n(\xi)-I_n(\eta)] = \frac{dI_n(\xi)}{d\tau(\xi)}>0. \tag{18}$$

For this the following limits are needed:

$$\lim_{\eta\to\xi}\frac{1}{\Delta}\left[\frac{s'_\eta(x)}{s_\eta(x)}-\frac{s'_\xi(x)}{s_\xi(x)}\right] = \left[\frac{s'_\xi(x)}{s_\xi(x)}\right]^2 \qquad (x\geqq\xi) \tag{19}$$

$$\lim_{\eta\to\xi}\frac{1}{\Delta}[s_\eta(x)-s_\xi(x)] = s_\xi(x)\int_\xi^x\left[\frac{s'_\xi}{s_\xi}\right]^2 dx \qquad (x\geqq\xi). \tag{20}$$

From (13) and (18) we get

$$\frac{s_\xi(\xi)}{s'_\xi(\xi)} = 2\nu(\xi+\gamma) = \left.\frac{\sqrt{q}}{(\sqrt{q})'}\right|_{x=\xi}$$

and

$$\frac{s_\xi(x)}{s'_\xi(x)} = 2\nu(x+\gamma) = 2\nu x - 2\nu\xi + 2\nu(\xi+\gamma) =$$

$$= 2\nu x - 2\nu\xi + \left.\frac{\sqrt{q}}{(\sqrt{q})'}\right|_{x=\xi} = 2\nu x + \tau(\xi) \qquad (x \geqq \xi),$$

hence

$$\frac{1}{\Delta}\left[\frac{s'_\eta(x)}{s_\eta(x)} - \frac{s'_\xi(x)}{s_\xi(x)}\right] = \frac{1}{\Delta}\left[\frac{1}{\tau(\eta)+2\nu x} - \frac{1}{\tau(\xi)+2\nu x}\right] = \frac{1}{[\tau(\xi)+2\nu x][\tau(\eta)+2\nu x]}.$$

On the one hand, this implies the relation (19), on the other hand, by integrating this over $[\xi, x]$ we get

$$(21) \qquad \frac{1}{\Delta}\left[\log\frac{s_\eta(x)}{s_\eta(\xi)} - \log\frac{s_\xi(x)}{s_\xi(\xi)}\right] = \int_\xi^x \frac{dx}{[\tau(\xi)+2\nu x][\tau(\eta)+2\nu x]} \qquad (x \geqq \xi).$$

From (17) we obtain

$$\frac{(\sqrt{q})'}{\sqrt{q}} = \frac{1}{\tau(x)+2\nu x} \qquad (x \geqq 0)$$

and

$$\frac{1}{\Delta}\log\frac{s_\eta(\xi)}{s_\xi(\xi)} = \frac{1}{\Delta}\left[\log\frac{s_\eta(\xi)}{s_\eta(\eta)} - \log\frac{\sqrt{q(\xi)}}{\sqrt{q(\eta)}}\right] =$$

$$= \frac{1}{\Delta}\int_\eta^\xi \left[\frac{1}{\tau(\eta)+2\nu x} - \frac{1}{\tau(x)+2\nu x}\right] dx = \frac{1}{\Delta}\int_\eta^\xi \frac{\tau(x)-\tau(\eta)}{[\tau(\eta)+2\nu x][\tau(x)+2\nu x]}\, dx,$$

but $0 \leqq \dfrac{\tau(x)-\tau(\eta)}{\Delta} \leqq 1$ in $[\eta, \xi]$, therefore $\lim\limits_{\eta\to\xi} \dfrac{1}{\Delta}\log\dfrac{s_\eta(\xi)}{s_\xi(\xi)} = 0$, and (21) involves (20).

It is obvious that $\lim\limits_{\eta\to\xi} \psi_\eta(x) = \psi_\xi(x)$ and by (14) the function

$$\frac{1}{\Delta}[\psi_\eta(x) - \psi_\xi(x)]$$

is the solution of the linear differential equation

$$(22) \qquad y' = \frac{s_\eta - s_\xi}{\Delta} + \frac{1}{2}\frac{1}{\Delta}\left[\frac{s'_\eta}{s_\eta} - \frac{s'_\xi}{s_\xi}\right]\sin 2\psi_\eta + \frac{s'_\xi}{s_\xi}\cos(\psi_\xi+\psi_\eta)\frac{\sin(\psi_\eta - \psi_\xi)}{\psi_\eta - \psi_\xi}\, y$$

with the initial condition $y(\eta) = 0$. By (19) and (20) there exists the limit function

$$(23) \qquad \Phi_\xi = \lim_{\eta\to\xi}\frac{1}{\Delta}[\psi_\eta(x) - \psi_\xi(x)],$$

which is the solution of the linear differential equation

$$\Phi'_\xi = s_\xi \int_\xi^x \left[\frac{s'_\xi}{s_\xi}\right]^2 dx + \frac{s'_\xi}{s_\xi}\cos 2\psi_\xi \cdot \Phi_\xi + \frac{1}{2}\left[\frac{s'_\xi}{s_\xi}\right]^2 \sin 2\psi_\xi \qquad (x \geqq \xi), \tag{24}$$

with the initial condition $\Phi_\xi(\xi)=0$.

By (16) we obtain

$$\frac{1}{\Delta}[I_n(\xi)-I_n(\eta)] = \frac{1}{\Delta}\int_\eta^{x_{n,\eta}} 2\left[\frac{s'_\eta}{s_\eta}-\frac{s'_\xi}{s_\xi}\right]\cos^2\psi_\eta\, dx + \tag{25}$$

$$+\frac{1}{\Delta}\int_\eta^{x_{n,\xi}} 2\frac{s'_\xi}{s_\xi}(\cos^2\psi_\eta - \cos^2\psi_\xi)\,dx - \frac{1}{\Delta}\int_{x_{n,\eta}}^{x_{n,\xi}} 2\frac{s'_\xi}{s_\xi}\cos^2\psi_\eta\, dx.$$

The third term on the right hand side tends to zero if $\eta\to\xi$ because (15) and (23) give

$$\psi_\eta(x_{n,\eta})-\psi_\xi(x_{n,\eta}) = \psi_\xi(x_{n,\xi})-\psi_\xi(x_{n,\eta}) = \psi'_\xi(x')[x_{n,\xi}-x_{n,\eta}] \qquad (x_{n,\xi}<x'<x_{n,\eta})$$

hence

$$\lim_{\eta\to\xi}\frac{x_{n,\xi}-x_{n,\eta}}{\Delta} = \frac{\Phi_\xi(x_{n,\xi})}{\psi'_\xi(x_{n,\xi})},$$

and $\cos\psi_\eta(x_{n,\eta})=0$ and $|\psi'_\eta(x)|$ is bounded on $[x_{n,\eta}, x_{n,\xi}]$. Taking into consideration the limits (19) and (20) we get from (25)

$$\frac{dI_n(\xi)}{d\tau(\xi)} = 2\int_\xi^{x_{n,\xi}}\left[\frac{s'_\xi}{s_\xi}\right]^2\cos^2\psi_\xi\, dx - \int_\xi^{x_{n,\xi}} 2\frac{s'_\xi}{s_\xi}\sin 2\psi_\xi\cdot\Phi_\xi\, dx. \tag{26}$$

We will express the functions $\Phi_\xi(x)$ and $\dfrac{dI_n(\xi)}{d\tau(\xi)}$ in a simpler form by the functions

$$\varrho(x) = Y'^2+s^2Y^2 \qquad (x\geqq 0) \tag{27}$$

and

$$t(x) = \varrho\frac{s}{s'}+YY' \qquad (x\geqq 0), \tag{28}$$

where $s=s_\xi(x)$ and $Y=Y_\xi(x)$ is the solution of the differential equation

$$Y''+s_\xi^2 Y = 0 \qquad (x\geqq 0) \tag{29}$$

with the initial conditions $Y_\xi(0)=1$, $Y'_\xi(0)=0$.

This definition and (13)—(14) make it clear that the relations

$$\varrho' = 2ss'Y^2 \qquad (x\geqq 0) \tag{30}$$

$$\sin\psi_\xi = \frac{sY}{\sqrt{\varrho}}, \qquad \cos\psi_\xi = \frac{Y'}{\sqrt{\varrho}} \qquad (x\geqq 0) \tag{31}$$

$$t' = \frac{1+\mu}{\mu}\varrho \qquad (x\geqq\xi) \quad \left(\mu=\frac{1}{2\nu}\right) \tag{32}$$

$$\psi' = \frac{s'}{\varrho}t \qquad (x\geqq 0) \tag{33}$$

hold.

First we solve the differential equation (24). By (27), (30), (31) we have

$$\frac{s'}{s}\cos 2\psi = \frac{\varrho}{s}\left(\frac{s}{\varrho}\right)',$$

hence equation (24) may be written as

$$\left(\frac{\Phi}{\frac{s}{\varrho}}\right)' = \varrho\int_{\xi}^{x}\left(\frac{s'}{s}\right)^2 du + \left(\frac{s'}{s}\right)^2 YY' \qquad (x \geqq \xi),$$

therefore

$$\Phi_\xi(x) = \frac{s}{\varrho}\int_{\xi}^{x}\left[\varrho\int_{\xi}^{u}\left(\frac{s'}{s}\right)^2 dv + \left(\frac{s'}{s}\right)^2 YY'\right] du \qquad (x \geqq \xi) \tag{34}$$

Now this integral can be calculated. By (13) and (29) we have for $u \geqq \xi$

$$\int_{\xi}^{u}\left[\frac{s'}{s}\right]^2 dv = \int_{\xi}^{u}\frac{\mu^2}{(v+\gamma)^2}\,dv = -\mu\left[\frac{s'}{s}\right]_{\xi}^{u}$$

and

$$\int_{\xi}^{x}\left(\frac{s'}{s}\right)^2 YY'du = \left[-\mu\frac{s'}{s}YY'\right]_{\xi}^{x} + \mu\int_{\xi}^{x}\frac{s'}{s}(Y'^2 - s^2Y^2)\,du.$$

Substituting these into (34) and making use of the relations (27)—(32) we obtain

$$\Phi_\xi(x) = \frac{s}{\varrho}\left\{-\mu\int_{\xi}^{x}\varrho\frac{s'}{s}\,du + \mu\left(\frac{s'}{s}\right)_{u=\xi}\int_{\xi}^{x}\varrho\,du - \mu\left[\frac{s'}{s}YY'\right]_{\xi}^{x} + \mu\int_{\xi}^{x}\frac{s'}{s}(Y'^2 - s^2Y^2)\,du\right\} =$$

$$= \frac{s}{\varrho}\left\{-\mu\int_{\xi}^{x}2ss'Y^2du + \frac{\mu^2}{1+\mu}\left(\frac{s'}{s}\right)_{u=\xi}\Big[t\Big]_{\xi}^{x}\mu - \left[\frac{s'}{s}YY'\right]_{\xi}^{x}\right\} =$$

$$= \frac{\varrho}{s}\left\{\frac{\mu^2}{1+\mu}\left(\frac{s'}{s}\right)_{u=\xi}\Big[t\Big]_{\xi}^{x} - \mu\left[\varrho + \frac{s'}{s}YY'\right]_{\xi}^{x}\right\} =$$

$$= \frac{s}{\varrho}\left[\frac{\mu^2}{1+\mu}\left(\frac{s'}{s}\right)_{u=\xi}t - \mu\frac{s'}{s}t + \frac{\mu}{1+\mu}\left(\frac{s'}{s}t\right)_{u=\xi}\right\}.$$

By a similar way we have for the first term on the right side of (26) by (33)

$$\int_{\xi}^{x_{n,\xi}} 2\left(\frac{s'}{s}\right)^2\cos^2\psi\,dx = -2\mu\left[\frac{s'}{s}\cos^2\psi\right]_{\xi}^{x_{n,\xi}} - 2\int_{\xi}^{x_{n,\xi}}\mu\frac{s'}{s}\sin 2\psi\,\frac{s'}{\varrho}\,t\,dx,$$

hence

$$\frac{dI_n(\xi)}{d\tau(\xi)} = -2\mu\left[\frac{s'}{s}\cos\psi\right]_\xi^{x_{n,\xi}} -$$

$$-2\mu\int_\xi^{x_{n,\xi}} \frac{s'}{\varrho s}\sin 2\psi\left[s't+\frac{\mu}{1+\mu}\left(\frac{s'}{s}\right)_{u=\xi} ts - s't + \frac{1}{1+\mu}\left(\frac{s'}{s}t\right)_{u=\xi} s\right]dx =$$

$$= -2\mu\left[\frac{s'}{s}\cos^2\psi\right]_\xi^{x_{n,\xi}} - \frac{2\mu}{1+\mu}\int_\xi^{x_{n,\xi}} \sin 2\psi\left[\mu\left(\frac{s'}{s}\right)_\xi \frac{s't}{\varrho} + \left(\frac{s'}{s}t\right)_\xi \frac{s'}{\varrho}\right]dx,$$

where by (33)

$$\int_\xi^{x_{n,\xi}} \sin 2\psi\,\frac{s't}{\varrho}\,dx = \int_\xi^{x_{n,\xi}} \sin 2\psi\cdot\psi'\,dx = [-\cos^2\psi]_\xi^{x_{n,\xi}}$$

and

$$\int_\xi^{x_{n,\xi}} \sin 2\psi\,\frac{s'}{\varrho}\,dx = \int_\xi^{x_{n,\xi}} \frac{\sin 2\psi\cdot\psi'}{t}\,dx = \left[\frac{-\cos^2\psi}{t}\right]_\xi^{x_{n,\xi}} - \int_\xi^{x_{n,\xi}} \frac{\cos^2\psi}{t^2}\,t'\,dx.$$

Finally by (15)

$$\frac{dI_n(\xi)}{d\tau(\xi)} = -2\mu\left[\frac{s'}{s}\cos^2\psi\right]_\xi^{x_{n,\xi}} - \frac{2\mu^2}{1+\mu}\left(\frac{s'}{s}\right)_\xi[-\cos^2\psi]_\xi^{x_{n,\xi}} -$$

$$-\frac{2\mu}{1+\mu}\left(\frac{s't}{s}\right)_\xi\left[-\frac{\cos^2\psi}{t}\right]_\xi^{x_{n,\xi}} + \frac{2\mu}{1+\mu}\left(\frac{s't}{s}\right)_\xi\int_\xi^{x_{n,\xi}} \frac{\cos^2\psi}{t^2}\,t'\,dx =$$

$$= \frac{2\mu}{1+\mu}\left(\frac{s't}{s}\right)_\xi\int_\xi^{x_{n,\xi}} \frac{\cos^2\psi}{t^2}\,t'\,dx,$$

consequently by (32)

$$\frac{dI_n(\xi)}{d\tau(\xi)} = 2\,\frac{s'_\xi(\xi)}{s_\xi(\xi)}\,t_\xi(\xi)\int_\xi^{x_{n,\xi}} \frac{\cos^2\psi_\xi(x)}{t_\xi^2(x)}\,\varrho_\xi(x)\,dx > 0, \tag{35}$$

because $t_\xi(x)$ is increasing in view of (32) and $t_\xi(0)\geqq 0$ involves $t_\xi(\xi)>0$.

The derivative $\dfrac{dI_n}{d\tau}$ can be defined by (35) not only for ξ with $\tau'(\xi)>0$ but for all $\xi\in(0, x^*)$. Taking into account the fact that the functions $s_\xi(x)$ tend uniformly to $\sqrt{q(x)}$ on $[0, x_n(0)]$ if $\xi\to x^*$ and utilising (12), (16) and (35), we obtain for any $0<\xi_1<x^*$

$$\log r(x_n(0)) = I_n(\xi_1) + \int_{\xi_1}^{x^*} \frac{dI_n(\xi)}{d\tau(\xi)}\,\tau'(\xi)\,d\xi. \tag{36}$$

Let $\sigma_\xi(x)$ be defined for $0<\xi<x^*$ by

$$\sigma_\xi(x) = C(x+\gamma)^{\frac{1}{2\nu}} \qquad (x \geqq 0) \tag{37}$$

where the values C and γ are the same as in (13). It is clear that $\sigma_\xi = s_\xi$ for $x \geqq \xi$ and $\sigma_\xi \geqq s_\xi$ for $0 \leqq x \leqq \xi$. Let $\chi_\xi(x)$ be the solution of the differential equation

$$\chi_\xi' = \sigma_\xi + \frac{1}{2}\frac{\sigma_\xi'}{\sigma_\xi}\sin 2\chi_\xi \qquad (x \geqq 0) \tag{38}$$

with the initial condition $\chi_\xi(\xi) = \psi_\xi(\xi)$ and let $X(\xi)$ be defined by $\chi_\xi(X(\xi)) = \frac{\pi}{2}$. The inequality $\sigma_\xi \geqq s_\xi$ on $[0, \xi]$ gives $\chi_\xi \leqq \psi_\xi$, therefore $X(\xi) \geqq 0$. But

$$\lim_{\xi \to 0}\left[I_n(\xi) + \int_{X(\xi)}^{x_{n,\xi}} 2\frac{\sigma_\xi'}{\sigma_\xi}\cos^2\chi_\xi\, dx\right] = 0,$$

hence by (36) a small value $\xi = \xi_0$ can be found such that the inequality

$$\log r(x_n(0)) > -\int_{X(\xi_0)}^{x_{n,\xi_0}} 2\frac{\sigma_{\xi_0}'}{\sigma_{\xi_0}}\cos^2\chi_{\xi_0}\, dx = \log r_n(\sigma_{\xi_0}^2, X(\xi_0)) \tag{39}$$

holds, provided that $\tau'(x)$ is not identically 0.

If $\gamma_0 = \gamma(\xi_0) + X(\xi_0) > 0$ then the domain of the definition of the differential equation (38) can be extended from $[0, \infty)$ to $[-\gamma_0, \infty)$. Let $\chi^{(0)}(x)$ be the solution of this extended equation with the initial condition $\chi^{(0)}(-\gamma_0) = \frac{\pi}{2}$. The uniqueness of the solutions involves

$$\chi^0(x) > \chi_{\xi_0}(x) \quad \text{for} \quad x \geqq X(\xi),$$

thus for the inverse functions $x^{(0)}(\chi)$ and $x_{\xi_0}(\chi)$ of the functions $\chi = \chi^{(0)}(x)$ and $\chi = \chi_{\xi_0}(x)$ the relation

$$x^{(0)}(\chi) < x_{\xi_0}(\chi) \qquad \left(\chi \geqq \frac{\pi}{2}\right) \tag{40}$$

holds. By (37), (38) we get for $\xi = \xi_0$

$$\log r_n(\sigma_{\xi_0}^2, X(\xi_0)) = -\int_{X(\xi_0)}^{x_{n,\xi_0}} 2\frac{\sigma_{\xi_0}'}{\sigma_{\xi_0}}\cos^2\chi_{\xi_0}\, dx =$$

$$= -\int_{X(\xi_0)}^{x_{n,\xi_0}} 2\frac{\cos\chi_{\xi_0}}{2\nu C_0 + (x\gamma_0)^{\frac{1}{2\nu}+1} + \frac{1}{2}\sin 2\chi_{\xi_0}}\chi_{\xi_0}'\, dx =$$

$$= -2\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+n\pi} \frac{\cos^2\chi}{2\nu C_0[x_{\xi_0}(\chi)+\gamma_0]^{\frac{1}{2\nu}+1} + \frac{1}{2}\sin 2\chi}\, d\chi,$$

where $C_0=C(\xi_0)$. Similarly

$$\log r_n(\sigma^2_{\xi_0}, -\gamma_0) = -2\int\limits_{\frac{\pi}{2}}^{\frac{\pi}{2}+n\pi} \frac{\cos^2\chi}{2\nu C_0[x^{(0)}(\chi)+\gamma_0]^{\frac{1}{2\nu}+1}+\frac{1}{2}\sin 2\chi}\,d\chi,$$

hence by (40)

$$r_n\big(\sigma^2_{\xi_0}, X(\xi_0)\big) > r_n\big(C_0^2[x+\gamma_0]^{\frac{1}{\nu}}, -\gamma_0\big) = r_n(C_0^2 x^{\frac{1}{\nu}}, 0). \tag{41}$$

By the substitution $x=\lambda t$, $\lambda=C_0^{-\frac{2\nu}{1+2\nu}}$ the differential equation

$$y''+C_0^2 x^{\frac{1}{\nu}} y = 0$$

turns into an equation of the form

$$z''+t^{\frac{1}{\nu}} z = 0,$$

therefore

$$r_n(C_0^2 x^{\frac{1}{\nu}}, 0) = r_n(x^{\frac{1}{\nu}}, 0).$$

Hence from (39) and (41) we obtain

$$r_n(q, 0) > r_n(x^{\frac{1}{\nu}}, 0), \tag{42}$$

provided q is not a function of the form $cx^{\frac{1}{\nu}}$.

If the function $q(x)\in C_\nu$ does not have continuous first or second derivative then we consider the functions

$$q_\varepsilon(x) = \left[\frac{1}{\varepsilon^2}\int\limits_x^{x+\varepsilon}\left(\int\limits_u^{u+\varepsilon} q^\nu\, dv\right) du\right]^{\frac{1}{\nu}} \qquad (x\geqq 0, \quad \varepsilon>0).$$

It is obvious that $q_\varepsilon\in C_\nu$; q_ε has continuous first and second derivatives and

$$\lim_{\varepsilon\to 0} q_\varepsilon(x) = q(x).$$

By (42) we get

$$r_n(q_\varepsilon, 0) > r_n\left(x^{\frac{1}{\nu}}, 0\right),$$

hence

$$r_n(q, 0) \geqq r_n\left(x^{\frac{1}{\nu}}, 0\right) \quad \text{for} \quad q\in C_\nu, \tag{43}$$

therefore

$$\varrho_n^{(\nu)} = r_n\left(x^{\frac{1}{\nu}}, 0\right),$$

and the assertions of our theorem except (4) have been proved.

To prove (4) we will deal with the case $q=x^{\frac{1}{\nu}}$. As we have seen above, the value $\varrho_n^{(\nu)}=r_n\left(x^{\frac{1}{\nu}},0\right)$ can be expressed by the function $\chi=\chi(x)$ or its inverse $x=x(\chi)$:

$$\log \varrho_n^{(\nu)} = -2\int\limits_{\frac{\pi}{2}}^{\frac{\pi}{2}+n\pi} \frac{\cos^2\chi}{2\nu x(\chi)^{1+\frac{1}{2\nu}}+\frac{1}{2}\sin 2\chi}\,d\chi, \tag{44}$$

where χ is the solution having the initial condition $\chi(0)=\frac{\pi}{2}$ of the differential equation

$$\chi' = x^{\frac{1}{2\nu}}+\frac{1}{4\nu x}\sin 2\chi. \tag{45}$$

Integration of (45) gives

$$\chi(x) = \frac{\pi}{2}+\frac{2\nu}{1+2\nu}x^{1+\frac{1}{2\nu}}+\int\limits_0^x \frac{1}{4\nu x}\sin 2\chi\,dx. \tag{46}$$

Denote the third term on the right side by $F(x)$. Corresponding to (45) the differential equation

$$y''+x^{\frac{1}{\nu}}y = 0$$

has a solution $y(x)$ with the initial conditions $y(0)=1$, $y'(0)=0$, $y(x)$ vanishes at $x=x(k\pi)$ $(k=1,2,\ldots)$ and has absolute maximum at $x=x\left(k\pi+\frac{\pi}{2}\right)$. Integrating (45) over $[x(k\pi),x((k+1)\pi)]$ we get

$$\pi = \int\limits_{x(k\pi)}^{x((k+1)\pi)} x^{\frac{1}{2\nu}}\,dx+F\big(x((k+1)\pi)\big)-F\big(x(k\pi)\big).$$

By a result of E. Makai [1] it is known that

$$\int\limits_{x(k\pi)}^{x((k+1)\pi)} x^{\frac{1}{2\nu}}\,dx<\pi,$$

hence

$$F\big(x((k+1)\pi)\big)>F\big(x(k\pi)\big) \qquad (k=1,2,\ldots). \tag{47}$$

On the other hand by making use of (45)

$$F\left(x\left(k\pi+\frac{\pi}{2}\right)\right)-F\left(x\left(k\pi-\frac{\pi}{2}\right)\right) = \int\limits_{\left(k-\frac{1}{2}\right)\pi}^{\left(k+\frac{1}{2}\right)\pi} \frac{\sin 2\chi}{2\nu x(\chi)^{1+\frac{1}{2\nu}}+\frac{1}{2}\sin 2\chi}\,d\chi =$$

$$= \int\limits_0^{\frac{\pi}{2}} \sin 2\chi\left[\frac{1}{2\nu x(k\pi+\chi)^{1+\frac{1}{2\nu}}+\frac{1}{2}\sin 2\chi}-\frac{1}{2\nu x(k\pi-\chi)^{1+\frac{1}{2\nu}}-\frac{1}{2}\sin 2\chi}\right]d\chi<0,$$

hence

$$F\left(x\left(k\pi-\frac{\pi}{2}\right)\right) > F\left(x\left(k\pi+\frac{\pi}{2}\right)\right) \qquad (k=1,2,\dots). \tag{48}$$

The function $F(x)$ has a local maximum at $x\left(k\pi-\frac{\pi}{2}\right)$ $(k=1,2,\dots)$ and local minimum at $x(k\pi)$, therefore by (47) and (48) we have

$$F(x(\pi)) \leqq F(x) \leqq F(0) = 0 \qquad (x \geqq 0).$$

Now it can be written

$$x(\chi)^{1+\frac{1}{2\nu}} = \frac{1+2\nu}{2\nu}\chi + O(1)$$

where $O(1)$ denotes a bounded function. Therefore by (44)

$$\log \varrho_n^{(\nu)} = -\frac{2}{1+2\nu}\int_{\frac{\pi}{2}}^{\left(n+\frac{1}{2}\right)\pi} \frac{\cos^2\chi}{\chi+O(1)}\,d\chi = -\frac{2}{2\nu+1}\int_{\frac{\pi}{2}}^{\left(n+\frac{1}{2}\right)\pi}\frac{\cos^2\chi}{\chi}\,d\chi + O(1) =$$

$$= -\frac{\log n}{1+2\nu} + O(1),$$

hence

$$\varrho_n^{(\nu)} = e^{-\frac{\log n}{1+2\nu}+O(1)} = \frac{e^{O(1)}}{n^{\frac{1}{1+2\nu}}},$$

which agrees with (4) if we take into account the inequalities $e^{O(1)} > \alpha_\nu > 0$ and $\varrho_1^{(\nu)} < \varrho_0^{(\nu)} = 1$.

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REMARKS ON DARBOUX FUNCTIONS

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§1. Notations. Introduction. In this paper we are going to consider real valued functions defined on the real line. A function f will be briefly said to be a Darboux function if $f(I)$ is an interval for every interval I. Our main aim is to give a necessary and sufficient condition for f to be Darboux, and a necessary and sufficient condition for a pair $g \leqq h$ of functions to be the lower and upper boundary functions of a Darboux function f. (The lower and upper boundary functions of f are defined as usual $\min\{f(x), \varliminf_{t\to x} f(t)\}$, $\max\{f(x), \varlimsup_{t\to x} f(t)\}$ respectively.) See Theorems 1 and 2. The proofs of both theorems are based on our main Lemma 3.

In § 3, Theorem 3 we give a probably well known necessary and sufficient condition for a Darboux function being open and in Theorem 4 we characterize the pairs of functions $g \leqq h$ which are the lower and upper boundary functions of an open Darboux function.

Let U be the class of functions which are uniform limits of Darboux functions. This class has been characterized in [1]. In § 4, we consider the wider class U_0 defined in [1] and prove certain inclusions for some subclasses of U and U_0 respectively.

It will be clear from the context that all our theorems are valid for Darboux functions defined on intervals too.

The set of finite real and rational numbers will be denoted by V and R respectively.

$|X|$ denotes the cardinality of the set X, $|V| = 2^{\aleph_0}$, $|R| = \aleph_0$.

$\overline{X}$ is the closure of the set X.

If f is a function, let $f_{\leqq c} = \{x \in V: f(x) \leqq c\}$ and similarly for $<$, $\geqq$ and $>$.

§2. Darboux functions. DEFINITION 1. The function f is said to be *strongly lower (upper) semi-continuous* at the point x_0, if

$$\varliminf_{x\to x_0} f(x) = f(x_0) \qquad \left[\varlimsup_{x\to x_0} f(x) = f(x_0)\right]$$

respectively.

To have a brief notation, we put

$$f_+(x) = \varliminf_{\substack{t>x\\ t\to x}} f(t), \qquad f^+(x) = \varlimsup_{\substack{t>x\\ t\to x}} f(t), \qquad f_-(x) = \varliminf_{\substack{t<x\\ t\to x}} f(t), \qquad f^-(x) = \varlimsup_{\substack{t<x\\ t\to x}} f(t),$$

$$f_*(x) = \max\{f_+(x), f_-(x)\}, \qquad f^*(x) = \min\{f^+(x), f^-(x)\}.$$

c will always denote a real number. If $g(x) \leqq h(x)$ are two arbitrary functions, we put

$$A_c = \{x : g(x) < c < h(x)\},$$

$$B_c = \{x : g_*(x) \leqq c \leqq h^*(x)\},$$

$$C_c = \{x : g_*(x) = c = h^*(x)\}.$$

DEFINITION 2. Let $g(x) \leqq h(x)$ be two arbitrary functions. The function f is said to be *spread between g and h* if $A_c \subseteqq \overline{f^{-1}(c)}$ for every c, and $g(x) \leqq f(x) \leqq h(x)$ holds for every x.

We will briefly say that f is a *spreading function* if it is spread between its boundary functions.

We have:

(1) Assume that $g(x) \leqq h(x)$ for every x, g and h are lower (upper) semi-continuous respectively and let f be spread between g and h. Then g and h are the boundary functions of f, thus f is a spreading function and g and h are strongly semi-continuous.

PROOF.

$$g(x) \leqq \varliminf_{t \to x} g(t) \leqq \varliminf_{t \to x} f(t) \leqq \varlimsup_{t \to x} f(t) \leqq \varlimsup_{t \to x} h(t) \leqq h(x).$$

Assume $g(x) < \varliminf_{t \to x} f(t)$ and let $g(x) < c < \varliminf_{t \to x} f(t)$, $c \neq f(x)$. Then $x \in A_c \subseteqq \overline{f^{-1}(c)}$, hence $\varliminf_{t \to x} f(t) \leqq c$ a contradiction.

Hence $g(x) = \varliminf_{t \to x} g(t) = \varliminf_{t \to x} f(t)$ and similarly we obtain

$$h(x) = \varlimsup_{t \to x} h(t) = \varlimsup_{t \to x} f(t).$$

We need three lemmas.

LEMMA 1. *Let f be an arbitrary function. Put $H = \{x : f^+(x) \neq f^-(x)$ or $f_+(x) \neq f_-(x)\}$. Then H is countable.*

Lemma 1 is well known, see e.g. [2], p. 261.

COROLLARY 1. *Let g be strongly lower (upper) semi-continuous. Then*

$$|\{x : g(x) \neq g_*(x)\}| \leqq \aleph_0 \qquad (|\{x : g(x) \neq g^*(x)\}| \leqq \aleph_0)$$

respectively.

PROOF. (E. g. for the first case.) By the assumption we have $g(x) = \min \{g_+(x), g_-(x)\}$. Hence $\{x : g(x) \neq g_*(x)\} = \{x : g_+(x) \neq g_-(x)\}$.

LEMMA 2. *Let g, h be functions such that $g(x) \leqq g_*(x) \leqq h^*(x) \leqq h(x)$ for every x. A necessary and sufficient condition for the existence of a function $f(x)$ satisfying $g_*(x) \leqq f(x) \leqq h^*(x)$ for every x and spread between g and h is that there exists a system $\{E_c\}_{c \in V}$ of mutually disjoint subsets of V such that*

$$E_c \subseteqq B_c, \; A_c \subseteqq \overline{E}_c \quad \textit{for every} \quad c.$$

PROOF. Assume that f satisfies the requirements of Lemma 2. Put $E_c=f^{-1}(c)$ for every c.

Then $E_c \subseteqq B_c$ since $g_*(x) \leqq f(x) \leqq h^*(x)$ and $A_c \subseteqq \bar{E}_c$ since f is spread between g and h. Hence the condition is necessary. Assume on the other hand that the condition is satisfied. Let f be equal to c for $x \in E_c$ and let $f(x)=d$ for an arbitrary $g_*(x) \leqq d \leqq h^*(x)$ if $x \notin \bigcup_c E_c$.

By the assumption $E_c \subseteqq B_c$ we have $g_*(x) \leqq f(x) \leqq h^*(x)$ for every x. Considering that $E_c \subseteqq f^{-1}(c)$ the assumption $A_c \subseteqq \bar{E}_c$ implies that f is spread between g and h.

Hence the condition is sufficient as well.

REMARK. In the second part of the proof f can be chosen so that $g_*(x) < f(x) < h^*(x)$ holds for every $x \notin \bigcup_c E_c$ provided $g_*(x) \neq h^*(x)$.

The next lemma is the main tool for proving our theorems.

LEMMA 3. *Let g and h be strongly lower and upper semicontinuous functions respectively. Assume that $g_*(x) \leqq h^*(x)$ for every x. Let $\mathcal{I}$ be an arbitrary open interval, c a real number and put*

$$U_1 = \mathcal{I} \cap \operatorname{Int}(g_{\geqq c}), \qquad U_2 = \mathcal{I} \cap \operatorname{Int}(h_{\leqq c}), \qquad K = \mathcal{I} - (U_1 \cup U_2).$$

Then one of the following conditions (i)—(iv) *holds:*

(i) $\mathcal{I} = U_1$,

(ii) $\mathcal{I} = U_2$,

(iii) $C_c \cap \mathcal{I} \neq 0$,

(iv) $|B_c \cap \mathcal{I}| = 2^{\aleph_0}$ *and* $A_c \cap \mathcal{I} \neq 0$.

PROOF. Assume that (i), (ii) and (iii) are false. Then c is finite. If $x \in U_1 \cap U_2$, then $g(x)=c=h(x)$, hence $C_c \cap \mathcal{I}$ being empty, $U_1 \cap U_2 = 0$. $\mathcal{I}$ being connected, $K \neq 0$. We prove that K is dense in itself. Assume that x_0 is an isolated point of K. Then there are open intervals I_1, I_2 such that $I_1 \cup \{x\} \cup I_2$ is an open interval again, $x_1 < x < x_2$ for every $x_i \in I_i$ $(i=1,2)$ and $I_1 \cup I_2 \subseteqq U_1 \cup U_2$. Using again that both I_1 and I_2 are connected we obtain that there are $1 \leqq \varepsilon(1), \varepsilon(2) \leqq 2$ such that

$$I_i \subseteqq U_{\varepsilon(i)} \quad \text{for} \quad i=1,2.$$

If $\varepsilon(1)=\varepsilon(2)=1$ [or 2], then

$$\varliminf_{t\to x_0} g(t) = g(x_0) \geqq c \quad \Big(\text{or } \varlimsup_{t\to x_0} h(t) = h(x_0) \leqq c\Big)$$

and $x_0 \in \operatorname{Int}(g_{\geqq c})$ or $x_0 \in \operatorname{Int}(h_{\leqq c})$ respectively: a contradiction.

If $\varepsilon(1) \neq \varepsilon(2)$, say $\varepsilon(1)=1, \varepsilon(2)=2$, then $g_-(x_0) \geqq c$ and $h^+(x_0) \leqq c$, hence $g_*(x_0)=c=h^*(x_0)$ and thus $x_0 \in C_c$ in contradiction with the assumption that (iii) is false. Hence K has no isolated points, and being non-empty and relatively closed in $\mathcal{I}$, it is relatively perfect as well and has power $2^{\aleph_0}$.

Considering that g and h being upper and lower semicontinuous respectively, both $g_{>c}$, $h_{<c}$ are open sets, we have

$$g_{>c} \subseteqq \operatorname{Int}(g_{\geqq c}), \quad h_{<c} \subseteqq \operatorname{Int}(h_{\leqq c}).$$

Hence $g(x) \leqq c \leqq h(x)$ holds for every $x \in K$.

Thus the set $K-\{x: g_*(x)\neq g(x)$ or $h^*(x)\neq h(x)\}$ is a subset of $B_c\cap\mathscr{I}$ and by Corollary 1 it has power $2^{\aleph_0}$. Hence the first part of (iv) is proved.

Assume that the second part of (iv) is false, i.e. $A_c\cap\mathscr{I}=0$. Put $K_1=K\cap g_{=c}$, $K_2=K\cap h_{=c}$. Then by the remark made above $K=K_1\cup K_2$.

First we prove $K_1\subseteqq\overline{K}_2$. Let $x_0\in K_1$ and let $I\subseteqq\mathscr{I}$ be an open interval containing x_0. $I\nsubseteq g_{\geqq c}$ for if not the $x_0\in U_1$. Hence there is $y\in I$ such that $g(y)<c$. We may assume $y\notin K_2$. Since $y\notin K_1$, $y\in U_1\cup U_2$, hence $y\in U_2$. Let I' be the component of U_2 containing y. Then $x_0\notin I'$ since $x_0\in K_1$. It follows that exactly one endpoint of I', let us say y' belongs to the interval $[x_0, y)$ or $(y, x_0]$, hence $y'\in I$. By the definition of I', $y'\notin U_2$, and U_1 and U_2 being disjoint, $y'\notin U_1$, hence $y'\in K$. Then $h^+(y')\leqq c$ or $h^-(y')\leqq c$ respectively, i.e. $h^*(y')\leqq c$. If $g(y')=c$ then $y'\in C_c$, hence $g(y')<c$ and $h(y')=c$. Thus $y'\in K_2$, $y'\in I$. This proves that $K_1\subseteqq\overline{K}_2$. Similarly we obtain $K_2\subseteqq\overline{K}_1$. It follows $\overline{K}_1\cap\mathscr{I}=\overline{K}_2\cap\mathscr{I}=K$.

K being of the second category in itself the same holds for one of the K_i's, say for K_1. $K_1=K\cap h_{>c}$, hence $K_1=\bigcup_{n=1}^{\infty}K\cap h_{\geqq c+\frac{1}{n}}$.

Then for at least one n there is an interval $\mathscr{I}'\subseteqq\mathscr{I}$ such that $K\cap h_{\geqq c+\frac{1}{n}}$ is dense in $K\cap\mathscr{I}'\neq 0$. But h being upper semi-continuous $h_{\geqq c+\frac{1}{n}}$ is closed, hence $K\cap\mathscr{I}'\subseteqq\subseteqq K\cap h_{\geqq c+\frac{1}{n}}$ in contradiction with the fact that K_2 is dense in K too. Hence the second part of (iv) is proved as well.

Remark. The proof of the first part of Lemma 3 gives immediately that in (iv) $|B_c\cap\mathscr{I}|=2^{\aleph_0}$ can be replaced by the stronger statement

$$\left|\left(B_c-\operatorname{Int}(g_{\geqq c})-\operatorname{Int}(h_{\leqq c})\right)\cup\mathscr{I}\right| = 2^{\aleph_0}.$$

Now we prove

Theorem 1. *The function $f(x)$ is a Darboux function if and only if the following two conditions hold:*

(i) *f is a spreading function,*

(ii) *$g_*(x)\leqq f(x)\leqq h^*(x)$ for every x, where g and h are the lower and upper boundary functions of f respectively.*

Proof. First we prove that the conditions are necessary. Assume $h^*(x_0)<f(x_0)$. We may assume that e.g. $h^*(x_0)=h^+(x_0)$. Let $\varepsilon>0$ such that $h^+(x_0)+\varepsilon<f(x_0)$. We choose a $\delta>0$ such that $h(x)<h^+(x_0)+\varepsilon$ for $x_0<x<x_0+\delta$. Then $f(x)<<h^+(x_0)+\varepsilon$ for $x_0<x<x_0+\delta$, hence f omits the value $h^+(x_0)+\varepsilon$ in each interval (x_0, y), $y\in(x_0, x_0+\delta)$ and thus f is not Darboux. By symmetry $f(x_0)<g_*(x_0)$ yields the same. Hence if f is Darboux, (ii) holds.

Let $x_0\in A_c$ and let I be an arbitrary open interval containing x_0. By the definition of g and h there are $x_1, x_2\in I$ such that $f(x_1)<c<f(x_2)$. If f is Darboux, there is an $x_3\in I$ for which $f(x_3)=c$. I being arbitrary we get that $x_0\in\overline{f^{-1}(c)}$, hence (i) holds too.

Now we prove that the conditions are sufficient. By the statement (1) g and h are strongly semi-continuous. Assume $f(a)<c<f(b)$.

We may assume $a<b$. Put $\mathscr{I}=(a, b)$,

$$U_1 = \mathscr{I}\cup\operatorname{Int}(g_{\geqq c}), \qquad U_2 = \mathscr{I}\cup\operatorname{Int}(h_{\leqq c}).$$

The conditions of Lemma 3 hold, hence one of the conditions (i)—(iv) stated in Lemma 3 is fulfilled. $U_1 \neq \mathcal{I}$ and $U_2 \neq \mathcal{I}$ for if not then by (ii) $c \leqq g_+(a) \leqq g_*(a) \leqq f(a)$ or $f(b) \leqq h^*(b) \leqq h^-(b) \leqq c$ in contradiction with the assumption $f(a) < c < f(b)$. Hence either $C_c \cap \mathcal{I} \neq 0$ or $A_c \cap \mathcal{I} \neq 0$. If $x \in C_c \cap \mathcal{I}$ then $f(x) = c$ by (ii). If $x \in A_c \cap \mathcal{I}$ then by (i) $x \in \overline{f^{-1}(c)}$, hence $f^{-1}(c) \cap \mathcal{I} \neq 0$ in both cases. It results that f is Darboux.

REMARK. Both (i) and (ii) can be formulated as local conditions fulfilled by a Darboux function at every point x. (It is easy to see that they are independent.) A characterization of Darboux functions with local conditions was first given in a paper of Á. CSÁSZÁR [3]. Though the conditions given in [3] are similar to ours we were unable to prove directly that they are implied by ours.

THEOREM 2. *The functions g and h are the upper and lower boundary functions of a Darboux function f if and only if the following two conditions hold:*

(i) *g and h are strongly lower and upper semi-continuous respectively,*

(ii) *$g_*(x) \leqq h^*(x)$ for every x.*

PROOF. The necessity of the conditions follows from (1) and Theorem 1.

We prove that they are sufficient as well.

Let $V = \{c_\alpha\}_{\alpha < \varphi}$ be a well-ordering of type φ of the set of real numbers where the ordinal number φ is the initial number of $2^{\aleph_0}$.

We are going to define a sequence $\{D_\alpha\}_{\alpha < \varphi}$ of type φ of subsets of V by transfinite induction on α as follows. Let $\alpha < \varphi$, and assume that D_β is defined for every $\beta < \alpha$.

Let D_α be a denumerable subset of $B_{c_\alpha} - \bigcup_{\beta < \alpha} D_\beta$ dense in this set. Thus the sequence is defined for every $\alpha < \varphi$.

Put briefly $D_c = D_\alpha$ if $c = c_\alpha$, $E_c = C_c \cup D_c$. We have $E_c \subseteqq B_c$, and thus the sets D_c being mutually disjoint the same holds for the sets E_c too, since if $c_1 \neq c_2$ then $\mathbf{C}_{c_1} \cap B_{c_2} = 0$.

We prove that $A_c \subseteqq \bar{E}_c$. Let $x_0 \in A_c$ and let I be an arbitrary open interval containing it. We may assume $I \cap \mathbf{C}_c = 0$. Put $U_1 = I \cap \operatorname{Int}(g_{\geqq c})$, $U_2 = I \cap \operatorname{Int}(h_{\leqq c})$. Then $U_1 \neq I$, $U_2 \neq I$. Applying Lemma 3 we obtain that $|B_c \cap I| = 2^{\aleph_0}$. Put $c = c_\alpha$. Considering that $|D_\beta| \leqq \aleph_0$ for $\beta < \alpha$, $|\alpha| < 2^{\aleph_0}$, we have $|I \cap (B_{c_\alpha} - \bigcup_{\beta < \alpha} D_\beta)| = 2^{\aleph_0}$. Thus by the definition of $D_\alpha = D_c$, $D_\alpha \cap I \cap (B_{c_\alpha} - \bigcup_{\beta < \alpha} D_\beta) \neq 0$, hence $D_c \cap I \neq 0$. I being arbitrary it results that $A_c \subseteqq \bar{E}_c$ for every c. By Lemma 2 there exists a function f spread between g and h and satisfying $g_*(x) \leqq f(x) \leqq h^*(x)$ for every x. Then by (1) f is a spreading function and g and h are its boundary functions. Hence by Theorem 1 f is Darboux and satisfies the requirement of our theorem.

REMARKS. 1. It would be quite easy to prove without referring to Theorem 1 (and thus to the involved argument of the proof of the second part of Lemma 3) that the function f constructed in the second part of the proof is a Darboux function. However, if one has already proved Theorem 1, the argument is shorter this way.

2. Applying the stronger form of Lemma 3 mentioned in the remark made after its proof, and the remark made after the proof of Lemma 2, one obtains the following slightly stronger statement. If g and h satisfy the conditions (i) and (ii)

of Theorem 2, there exists a function f satisfying the requirements of the theorem and the additional condition: $g_*(x) < f(x) < h^*(x)$ if $x \notin \bigcup_c E_c$ where $E_c = C_c \cup D_c$ and $D_c \cap \text{Int}(g_{\geqq c}) = 0$, $D_c \cap \text{Int}(h_{\leqq c}) = 0$.

§ 3. Open Darboux-functions. THEOREM 3. *A Darboux function is open if and only if it has no local extrema (in the wide sense).*

PROOF. $f(x_0)$ is a local maximum (minimum) of f if and only if $f(x_0) = c$ and $x_0 \in \text{Int}(f_{\leqq c})$ $(x_0 \in \text{Int}(f_{\geqq c}))$ respectively. Hence if f has a local extremum, f is not open. On the other hand, if f is a Darboux function then $f(I)$ is connected for every open interval I, and if f has no local extremum $f(I)$ must be an open interval since $f(x) \in \text{Int} f(I)$ for every $x \in I$.

THEOREM 4. *The functions g and h are the lower and upper boundary functions of an open Darboux function f if and only if the conditions* (i), (ii) *of Theorem* 2 *hold and*

(iii) $g(x)$ *has no local minimum and* $h(x)$ *has no local maximum at the points* x *of* C_c *where* $g(x) = g_*(x) = c$ *or* $h(x) = h^*(x) = c$ *holds respectively.*

PROOF. The necessity of the conditions follows immediately from Theorems 1 and 3 considering that, if $x \in C_c$, $g(x) = g_*(x) = c$, $x \in \text{Int}(g_{\geqq c})$ (or $h(x) = h^*(x) = c$, $x \in \text{Int}(h_{\leqq c})$, imply that $f(x) = c$, $x \in \text{Int}(f_{\geqq c})$ (or $f(x) = c$, $x \in \text{Int}(f_{\leqq c})$) respectively. To prove that the conditions are sufficient, observe that by the second remark, made after the proof of Theorem 2 the conditions imply the existence of a spreading function f spread between g and h such that $g_*(x) \leqq f(x) \leqq h^*(x)$ and $g_*(x) < f(x) < h^*(x)$ if $x \notin \bigcup_c E_c$ where $E_c = C_c \cup D_c$ and $D_c \cap \text{Int}(g_{\geqq c}) = 0$, $D_c \cap \text{Int}(h_{\leqq c}) = 0$.

We prove that f has no local extremum. By definition of the lower boundary function, it results that $\text{Int}(f_{\geqq c}) = \text{Int}(g_{\geqq c})$.

If $f(x) = c$ were a local minimum of f, then it would be $x \in \text{Int}(g_{\geqq c})$, $g(x) = c$. This is impossible if $x \in D_c$ because of $D_c \cap \text{Int}(g_{\geqq c}) = 0$. If $x \in C_c$, then this implies $g(x) = g_*(x) = c$, hence the indirect assumption contradicts (iii). Finally, if $x \notin \bigcup_c E_c$ then $g(x) \leqq g^*(x) < f(x) = c$ and $g(x) \neq c$. Hence f has no local minimum.

We obtain similarly that f has no local maximum either. Theorem 3 implies that f is open.

§ 4. Classes of Darboux functions. DEFINITION 3. Let U be the class of functions f satisfying the *condition*

(2) $\overline{f((a,b) - H)} \supseteqq ((f(a), f(b)))$ for every open interval (a, b) and for every $H \subseteqq (a, b)$, $|H| < 2^{\aleph_0}$. (Here $((f(a), f(b)))$ denotes the interval $(f(a), f(b))$ if $f(a) \leqq f(b)$ and the interval $(f(b), f(a))$ if $f(b) < f(a)$.)

The class U was characterized in [1] as follows: $f \in U$ if and only if f is the uniform limit of a sequence of Darboux functions.

The wider class U_0 is defined as follows.

DEFINITION 4. Let U_0 be the class of functions satisfying the condition

(3) $\overline{f((a, b))} \supseteqq ((f(a), f(b)))$ for every open interval (a, b).

THEOREM 5. *f is Darboux if and only if $f \in U_0$ and f is spreading.*

PROOF. The necessity of the conditions is obvious by Theorem 1. To prove that the conditions are sufficient, by Theorem 1 it is enough to see that $g_*(x) \leqq f(x) \leqq h^*(x)$ holds for every x where g and h are the lower and upper boundary functions of f. Assume $f(x_0) < g_*(x_0)$ for some x_0. We may assume e.g. $f(x_0) < g_+(x_0)$. But then, for suitable finite c and $\delta > 0$, $f(x) \geqq g(x) > c > f(x_0)$ for $x_0 < x < x_0 + \delta$. Hence no interval (x_0, x), $x_0 < x < x_0 + \delta$ satisfies (3).

COROLLARY 2. *The function f is Darboux if and only if it is spreading and is the uniform limit of a sequence of Darboux functions.*

Corollary 2 expresses the fact that in the class of spreading functions the properties (2) and (3) characterizing U and U_0 are equivalent. We are going to give an independent proof of the fact that this is true in a wider class of functions.

DEFINITION 5. Let f be a function, g, h its lower and upper boundary functions respectively. f is said to be weakly spreading if $A_c \subseteqq \overline{f^{-1}(c)}$ holds for a $2^{\aleph_0}$-dense set of c's, i.e. if $\Lambda = \{c: A_c \subseteqq \overline{f^{-1}(c)}\}$ then $|\Lambda \cap I| = 2^{\aleph_0}$ for every open interval I. Let $S_{2^{\aleph_0}}$ denote the class of weakly spreading functions.

THEOREM 6.

$$S_{2^{\aleph_0}} \cap U_0 \subseteqq U.$$

PROOF. Let $H \subseteqq (a, b)$, $|H| < 2^{\aleph_0}$ and assume $f \in S_{2^{\aleph_0}} \cap U_0$ and that there is an $\mathscr{I} \subseteqq (f(a), f(b))$ such that $\mathscr{I} \cap f((a, b) - H) = 0$. Put $K = f^{-1}(\mathscr{I}) \cap (a, b)$. Then $K \subseteqq H$, hence $|K| < 2^{\aleph_0}$, $K \neq 0$, f being an element of U_0. Assume $x_0 \in K$ and let g, h be the boundaries of f. If $g(x_0) < h(x_0)$ then $(g(x_0), h(x_0)) \cap \mathscr{I} \neq 0$. Considering $|f(K)| < 2^{\aleph_0}$, $f \in S_{2^{\aleph_0}}$ there is a $c \in (g(x_0), h(x_0)) \cap \mathscr{I} - f(K)$ such that $A_c \subseteqq \overline{f^{-1}(c)}$. This is a contradiction since $x_0 \in A_c$, $f^{-1}(c) \cap (a,b) = 0$. Hence $g(x_0) = h(x_0)$. Then $f(x)$ is continuous in x_0. But then $f(x_0) \in \mathscr{I}$ implies that there is a $I \subseteqq (a,b)$, $x_0 \in I$ such that $f(I) \subseteqq \mathscr{I}$ i.e. $I \subseteqq K$. This contradicts $|K| < 2^{\aleph_0}$. Hence $f \in U$.

As a corollary of Theorem 6 we have $S_{2^{\aleph_0}} \cap U_0 = S_{2^{\aleph_0}} \cap U$.

Finally we mention that the stronger statement $S_{2^{\aleph_0}} \cap U_0 = U$ is not valid, as is shown by the following

EXAMPLE. As it is well-known, there exists a sequence H_n, $n = 1, \ldots$ of mutually disjoint subsets of $(0, 1)$ such that $\bigcup_{n=1}^{\infty} H_n = (0, 1)$ and H_n is $2^{\aleph_0}$-dense in $(0, 1)$. Let r_n be the sequence of the rational numbers of $(0, 1)$. Let further $x_0 \in (0, 1)$ be arbitrary.

Assume $0 < x < 1$ and put

$$f(x) = \begin{cases} r_n & \text{if } x \in H_n \text{ and } |x - x_0| \geqq \dfrac{1}{n}, \\ 0 & \text{if } x \in H_n \text{ and } |x - x_0| < \dfrac{1}{n}. \end{cases}$$

We prove $f \in U$. Assume $(a, b) \subseteqq (0, 1)$, $H \subseteqq (a, b)$, $|H| < 2^{\aleph_0}$. Then $((a,b) - H) \cap H_n \neq 0$ for $n = 1, \ldots$. If $n > \dfrac{2}{b-a}$ then $(a, b) - H$ contains an element x such that

$f(x)=r_n$. It follows that $f[(a, b)-H]$ is dense in (0, 1). This proves $f\in U$ and implies that $g(x)\equiv 0$, $h(x)\equiv 1$ for the boundaries of f. Thus $A_c=(0, 1)$ for every $0<c<1$ and $A_c \not\subseteq \overline{f^{-1}(c)}$ for any c in (0, 1), since $f^{-1}(c)=0$ if c is irrational and $f^{-1}(r_n)\cap \cap \left(x_0-\frac{1}{n}, x_0+\frac{1}{n}\right)=0$ if r_n is rational. This shows that $f\in U\subseteqq U_0$ is not only not weakly spreading but we even have

$$(0, 1)\cap\{c: A_c \subseteqq \overline{f^{-1}(c)}\} = 0.$$

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ON THE ALMOST SUMMABILITY OF A TRIGONOMETRIC SEQUENCE

By

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1. Let $A=(a_{n,k})$ be an infinite matrix of real or complex numbers and $\{s_k\}$ be any sequence of real numbers. With every sequence $\{s_k\}$ we associate the sequence $\{\sigma_n\}$ given by

(1.1) $$\sigma_n = \sum_{k=0}^{\infty} a_{n,k} s_k,$$

provided the series on the right converges for all n. The sequence $\{\sigma_n\}$ is called an A-transform of $\{s_k\}$.

If $\sigma_n \to s$ as $n\to\infty$, we say that the sequence $\{s_k\}$ is summable A to s.

The matrix A is called regular if

$$s_k \to s \Rightarrow \sigma_n \to s.$$

A bounded sequence $\{s_k\}$ is said to be almost convergent to l if

$$\lim_{p\to\infty} \frac{s_n + s_{n+1} + \dots s_{n+p-1}}{p} = l,$$

uniformly in n (see [2]).

It is easy to see that a convergent sequence is almost convergent and the limits are the same.

A bounded sequence $\{s_k\}$ is said to be almost A-summable to s if the A-transform of $\{s_k\}$ is almost convergent to s (see [1]) and the matrix A is said to be almost regular if $s_k \to s$ implies that $\{\sigma_n\}$ is almost convergent to s. The necessary and sufficient conditions for the matrix A to be almost regular are (see [1]):

(1.2) $$\operatorname{Sup}\left\{\sum_{k=0}^{\infty} \frac{1}{p}\left|\sum_{j=n}^{n+p-1} a_{j,k}\right|\right\} < M \qquad (n=1,2,\dots),$$

where M is a positive constant and p is a positive integer,

(1.3) $$\lim_{p\to\infty} \frac{1}{p} \sum_{j=n}^{n+p-1} a_{j,k} = 0 \text{ uniformly in } n, \qquad k=0,1,2,\dots,$$

(1.4) $$\lim_{p\to\infty} \frac{1}{p} \sum_{j=n}^{n+p-1} \sum_{k=0}^{\infty} a_{j,k} = 1, \text{ uniformly in } n.$$

Let $f(x)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(0, 2\pi)$. Let

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

be the Fourier series of $f(x)$ and let

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)=\sum_{n=1}^{\infty}B_n(x)$$

be its conjugate series.

We write $\sigma_n^*(x)=\sum_{k=0}^{\infty}a_{n,k}kB_k(x)$ and

$$\psi(t)=\psi(f,t)=f(x+t)-f(x-t),\quad 0<t<\pi,\qquad \psi(0)=f(x+0)-f(x-0).$$

2. Generalizing a theorem of FEJÉR, J. A. SIDDIQI [3] proved the following theorem for summability (Λ), where $\Lambda=(\lambda_{n,k})$ is a triangular matrix of real or complex numbers and the Λ-transform of $\{s_k\}$ is given by $\sum_{k=0}^{n}\Delta\lambda_{n,k}s_k$, $\Delta\lambda_{n,k}=\lambda_{n,k}-\lambda_{n,k+1}$.

THEOREM A. *Let $f(x)\in \mathrm{BV}\,[0,2\pi]$ and periodic with period 2π. If (Λ) is regular and if*

$$\lim_{n\to\infty}\sum_{k=0}^{n}|\Delta^2\lambda_{n,k}|=0,\quad \Delta^2\lambda_{n,k}=\Delta\lambda_{n,k}-\Delta\lambda_{n,k+1} \tag{2.1}$$

then the sequence $\{nB_n(x)\}$ is summable (Λ) to

$$\frac{D(x)}{\pi}=\pi^{-1}\{(f(x+0)-f(x-0))\}.$$

Later on he [4] obtained necessary and sufficient condition on Λ for the validity of Theorem A and derived certain consequences for the Fourier coefficients of continuous functions of bounded variation. His main theorem is as follows:

THEOREM B. *If (Λ) is regular, then for every $f(x)\in \mathrm{BV}\,[0,2\pi]$ and for every $x\in[0,2\pi]$,*

$$\lim_{n\to\infty}\sum_{k=0}^{n}\Delta\lambda_{n,k}kB_k(x)=\frac{D(x)}{\pi}$$

iff

$$\lim_{n\to\infty}\sum_{k=0}^{n}\Delta\lambda_{n,k}\cos kt=0, \tag{2.2}$$

in every $0<\delta\leqq t\leqq\pi$.

The object of this note is to obtain necessary and sufficient condition in order that the sequence $\{kB_k(x)\}$ be almost A-summable to $\dfrac{D(x)}{\pi}$. We shall prove the following theorem.

THEOREM. *If A is almost regular, then for every $f(x)\in \mathrm{BV}\,[0, 2\pi]$ and for every $x\in[0, 2\pi]$,*

$$\lim_{p\to\infty}\frac{1}{p}\sum_{r=0}^{p-1}\sigma_{n+r}^{*}=\frac{D(x)}{\pi}\quad \textit{uniformly in } n,$$

iff

$$\lim_{p\to\infty}\frac{1}{p}\sum_{r=0}^{p-1}\sum_{k=0}^{\infty}a_{n+r,k}\cos kt=0, \tag{2.3}$$

uniformly in n for every $0<\delta\leqq t\leqq\pi$.

3. PROOF OF THE THEOREM. We have

$$\frac{1}{p}\sum_{r=0}^{p-1}\sigma_{n+r}^{*}=\frac{1}{p}\sum_{r=0}^{p-1}\sum_{k=0}^{\infty}a_{n+r,k}kB_k(x)=\frac{1}{p}\sum_{r=0}^{p-1}\sum_{k=0}^{\infty}a_{n+r,k}\frac{1}{\pi}\int_0^{\pi}k\sin kt\cdot\psi(t)\,dt=$$

$$=\frac{D(x)}{\pi}\cdot\frac{1}{p}\sum_{r=0}^{p-1}\sum_{k=0}^{\infty}a_{n+r,k}+\frac{1}{\pi}\int_0^{\pi}d\psi(t)\,\frac{1}{p}\sum_{r=0}^{p-1}\sum_{k=0}^{\infty}a_{n+r,k}\cos kt=I_1+I_2,$$

say. By virtue of the condition (1.4) $I_1\to\dfrac{D(x)}{\pi}$, as $p\to\infty$ uniformly in n. It, therefore, suffices to show that

$$I_2=\frac{1}{\pi}\int_0^{\pi}d\psi(t)\cdot K_{n,p}(t)\to 0,\quad p\to\infty, \tag{3.1}$$

uniformly in n, where

$$K_{n,p}(t)=\frac{1}{p}\sum_{r=0}^{p-1}\sum_{k=0}^{\infty}a_{n+r,k}\cos kt. \tag{3.2}$$

We shall first show that condition (3.1) is equivalent to the following condition:

$$\frac{1}{\pi}\int_{\delta}^{\pi}K_{n,p}(t)\,d\psi(t)\to 0,\qquad p\to\infty, \tag{3.3}$$

uniformly in n, for every $0<\delta\leqq\pi$, for every $f\in \mathrm{BV}\,[0, 2\pi]$ and for every x in $[0, 2\pi]$.

Suppose that (3. 1) holds. If $f\in \mathrm{BV}\,[0, 2\pi]$ and $x\in[0, 2\pi]$, then for every $0<\delta\leqq\pi$ we can construct a function $g(x)\in \mathrm{BV}\,[0, 2\pi]$ such that $g(x)$ is constant in $[x-\delta, x+\delta]$ and g coincides with f elsewhere (see [4]). Since

$$\int_{\delta}^{\pi}K_{n,p}(t)\,d\psi(f,t)=\int_0^{\pi}K_{n,p}(t)\,d\psi(g,t)$$

and hence by virtue of (3. 1) the left hand integral tends to zero uniformly in n as

$p \to \infty$. Thus (3. 1) implies (3. 3). Suppose now that (3. 3) holds. If $f \in \mathrm{BV}\ [0,2\pi]$ and $x \in [0, 2\pi]$, given an y$\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_0^\delta |d\psi(t)| < \frac{\varepsilon}{2M}. \tag{3. 4}$$

By virtue of condition (1. 2) we have

$$|K_{n,p}(t)| = \left| \frac{1}{p} \sum_{r=0}^{p-1} \sum_{k=0}^{\infty} a_{n+r,k} \cos kt \right| \leqq \tag{3. 5}$$

$$\leqq \frac{1}{p} \sum_{k=0}^{\infty} |\cos kt| \left| \sum_{j=n}^{n+p-1} a_{j,k} \right| \leqq \sum_{k=0}^{\infty} \frac{1}{p} \left| \sum_{j=n}^{n+p-1} a_{j,k} \right| < M$$

uniformly in n and so that

$$\left| \int_0^\delta K_{n,p}(t)\, d\psi(t) \right| < \frac{\varepsilon}{2}$$

uniformly in n. Since (3. 3) holds, there exists a p_0 such that for $p \geqq p_0$

$$\left| \int_\delta^\pi K_{n,p}(t)\, d\psi(t) \right| < \frac{\varepsilon}{2}$$

and hence we have

$$\int_\delta^\pi K_{n,p}(t)\, d\psi(t) \to 0 \quad \text{as} \quad p \to \infty,$$

uniformly in n. Thus (3. 3) implies (3. 1).

By a simple modification of a familiar theorem on weak convergence of sequences in the Banach space of all continuous functions defined in a finite closed interval, it follows that (3. 3) holds iff

(i) $|K_{n,p}(t)| \leqq K$ for all n, p and $t \in [\delta, \pi]$ for every $\delta > 0$,
(ii) (2. 3) holds.

Since by virtue of (3. 5) (i) holds it follows that (3. 3) holds iff (2. 3) holds.

This completes the proof of the theorem.

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SOME REMARKS ON A PROPERTY OF TOPOLOGICAL CARDINAL FUNCTIONS

By

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Introduction

In the paper [1] one of the authors has introduced the concept of the Darboux property of topological cardinal functions. In [1] several results and problems were stated. The main aim of this paper is to give some further results and simpler proofs for the results of [1].

In § 2 Theorems 1 and 2 give some information on the Darboux property of the weight function on the classes of T_1- and T_5-spaces respectively. However the results are still incomplete.

The rest of the theorems in this § deal with the density function and give an almost complete discussion of its behaviour on the different classes of spaces. We point out Problem 2 which remains unsolved.

In § 3 we prove Theorem 5 concerning linearly ordered spaces which settles the Darboux property of the weight (and density) function on these spaces. Without giving exact references we mention that at least in special cases the result must be contained in some theorems of W. Sierpiński and D. Kurepa concerning the Suslin problem.

In § 4 we introduce a new class of spaces lying between T_2- and T_3-spaces, called strongly Hausdorff spaces, and we prove a special result relevant to a problem stated by J. DE GROOT [2].

§ 1. Notations. Definitions

$|H|$ denotes the cardinality of the set H. We assume that each ordinal is the set of all smaller ordinals.

$\xi, \eta, \zeta, \ldots$ denote ordinals;

$\alpha, \beta, \varphi, \psi, \ldots$ denote cardinals (i.e. initial ordinals);

λ will always denote a limit cardinal.

α^+ denotes the immediate successor of the cardinal α.

If η is a limit ordinal, $\operatorname{cf}(\eta)$ is the least cardinal, which is cofinal with η.

The cardinal α is said to be *regular* if $\operatorname{cf}(\alpha)=\alpha$ and *singular* otherwise.

A regular limit cardinal is said to be *inaccessible.*

A limit cardinal λ is said to be a *strong limit cardinal* if $\alpha<\lambda$ implies $2^\alpha<\lambda$. (We sometimes write $\exp \alpha$ for 2^α.)

A strong limit inaccessible cardinal is called *strongly inaccessible.*

We will often make use of the generalized continuum hypothesis which will be briefly referred to as G.C.H. ω_ξ denotes the increasing sequence of infinite cardinals, $\omega_0=\omega$.

Capital letters $K, H, \ldots, X, Y, \ldots$ denote sets, $R, S, D, \ldots$ denote topological spaces.

The class of all topological spaces will be denoted by $\mathcal{T}$, while the class of T_i-spaces will be denoted by $\mathcal{T}_i$, $i=0, \ldots, 5$, respectively. $\mathcal{L}$ denotes the class of linearly ordered spaces provided with the usual interval topology.

A *topological cardinal function* is a function defined on a certain class of topological spaces with cardinal values.

In this paper we will consider the following cardinal functions.

The *weight function* w, defined as usual by

$$w(R) = \max \{\omega, \min \{|\mathfrak{B}|: \text{ for the open bases } \mathfrak{B} \text{ of } R\}\}.$$

The *density function* d:

$$d(R) = \max \{\omega, \min \{|S|: \text{ for } S \subset R, \bar{S} = R\}\}.$$

The *spread function* s, where

$$s(R) = \max \{\omega, \sup \{|D|: D \subset R \text{ where } D \text{ is a discrete subspace of } R\}\}.$$

The space R will be said *left separated (right separated)* if there exist a well-ordering $\{x_\xi\}_{\xi<\varphi} = R$ of the points of R and a sequence $\{U_\xi\}_{\xi<\varphi}$ of type φ of open subsets of R such that $x_\xi \in U_\xi$ and $x_\eta \notin U_\xi$ for $\eta < \xi$ [$x_\eta \notin U_\xi$ for $\eta > \xi$] for every $\xi < \varphi$ respectively.

To have a brief notation we introduce the following symbols.

Let Φ be a cardinal function defined on the class $\mathcal{C}$ of topological spaces;

$$(*) \qquad (\Phi, \mathcal{C}) \to \alpha$$

denotes that the following statement is true.

For each $R \in \mathcal{C}$, $\Phi(R) > \alpha$ implies that there exists a subspace $S \subset R$ such that $\Phi(S) = \alpha$.

$(\Phi, \mathcal{C}) \nrightarrow \alpha$ denotes the negation of the above statement.

If $(\Phi, \mathcal{C}) \to \alpha$ holds for every [regular] α then Φ is said to have the [regular] Darboux property on $\mathcal{C}$.

$$(**) \qquad [\Phi, \mathcal{C}] \to \lambda$$

denotes that the following statement is true:

If for each $\alpha < \lambda$ there exists a subspace $S \subset R$ with $\alpha \leq \Phi(S) < \lambda$ then there exists a subspace $S_0 \subset R$ with $\Phi(S_0) = \lambda$.

$[\Phi, \mathcal{C}] \nrightarrow \lambda$ denotes the negation of this statement.

If Φ has the Darboux property on $\mathcal{C}$ and $[\Phi, \mathcal{C}] \to \lambda$ holds for every λ then Φ is said to possess the *closed Darboux property* on $\mathcal{C}$.

The concepts of (regular, closed) Darboux properties were formulated in [1]. The introduction of the symbols $(*)$ and $(**)$ depending on the parameters α, λ enables us to give a more detailed analysis of these properties.

§ 2. The Darboux properties of the weight and density functions

First we are going to deal with the weight function w. In this case we know negative results only, except some trivial positive facts.

THEOREM 1. *If λ is a singular cardinal, then*

$$(w, \mathcal{T}_1) \not\twoheadrightarrow \lambda.$$

PROOF. Let H be a set of potency λ, provided with the topology whose non-trivial (i.e. different from H) closed sets are exactly those of cardinality not greater than $\mathrm{cf}(\lambda)$. As every one-point set is closed in H, it is a T_1-space, indeed.

For each $K \subset H$, let $w^*(K)$ be the smallest cardinal β such that there exists a system $\mathfrak{L}$, $|\mathfrak{L}| = \beta$ of non-trivial closed subsets of K, with the property that every non-trivial closed subset of K is contained in one of the elements of $\mathfrak{L}$. Since each base for the closed sets in K has this property we get immediately

$$w^*(K) \leqq w(K).$$

On the other hand, if $\mathfrak{L}$ is the above mentioned system of power $w^*(K)$, let $\mathfrak{B}$ be the system of all sets of the form $Z \setminus \{x\}$, where $Z \in \mathfrak{L}$ and $x \in K$. Then $|\mathfrak{B}| \leqq |\mathfrak{L}| \cdot \mathrm{cf}(\lambda) = w^*(K)\, \mathrm{cf}(\lambda)$, because $Z \in \mathfrak{L}$ implies $|Z| \leqq \mathrm{cf}(\lambda)$. At the same time $\mathfrak{B}$ is a base for the closed sets in K for if S is an arbitrary non-trivial closed set in K, then there is a set $Z \in \mathfrak{L}$ with $S \subset Z$, and so we get

$$S = \bigcap_{x \in Z \setminus S} (Z \setminus \{x\}).$$

These considerations show immediately

$$w(K) \leqq \mathrm{cf}(\lambda) \cdot w^*(K)$$

and so $w^*(K) \geqq \mathrm{cf}(\lambda)$ implies

$$w(K) = w^*(K)$$

and $w^*(K) < \mathrm{cf}(\lambda)$ implies $w(K) \leqq \mathrm{cf}(\lambda) < \lambda$. Now assume $w^*(K) \geqq \mathrm{cf}(\lambda)$. We will prove that $w^*(K) = w(K) \neq \lambda$. Assume on the contrary, that $w^*(K) = \lambda$ and let $\mathfrak{L}$ be the required set-system of power λ. Let the cardinals α_ξ be chosen for each $\xi < \mathrm{cf}(\lambda)$ such that $\alpha_\xi < \alpha_\eta$ if $\xi < \eta$ and

$$\sum_{\xi < cf(\lambda)} \alpha_\xi = \lambda.$$

The system $\mathfrak{L}$ can be represented in the form

$$\mathfrak{L} = \bigcup_{\xi < cf(\lambda)} \mathfrak{L}_\xi,$$

where $\xi < \eta$ implies $\mathfrak{L}_\xi \subset \mathfrak{L}_\eta$ and $|\mathfrak{L}_\xi| = \alpha_\xi$. Then $|\mathfrak{L}_\xi| < w^*(K)$ for each $\xi < \mathrm{cf}(\lambda)$, and so one can find a non-trivial closed subset of K, say S_ξ, with $S_\xi \not\subset Z$ for each $Z \in \mathfrak{L}_\xi$. Let

$$S = \bigcup_{\xi < cf(\lambda)} S_\xi.$$

Then, by definition, $|S_\xi| \leqq \mathrm{cf}(\lambda)$ and so $\left|\bigcup_{\xi < \mathrm{cf}(\lambda)} S_\xi\right| = |S| \leqq \mathrm{cf}(\lambda)$, i.e. S is a closed subset of K. Now if Z is an arbitrary element of $\mathfrak{L}$, then $Z \in \mathfrak{L}_\xi$ for some $\xi < \mathrm{cf}(\lambda)$, and so

$$S_\xi \subset S \not\subset Z,$$

which contradicts the definition of $\mathfrak{L}$, consequently

$$w^*(K) = w(K) \neq \lambda.$$

Finally, we have to prove $w(H) > \lambda$. Let, indeed, $\mathfrak{B}$ be an arbitrary family of non-trivial subsets of H with $|\mathfrak{B}| < \lambda$. Then $|\bigcup \mathfrak{B}| \leqq |\mathfrak{B}| \, \mathrm{cf}(\lambda) < \lambda$, hence $w(H) \geqq \lambda$. This obviously implies $w(H) > \lambda$.

THEOREM 2. $2^\alpha > \alpha^+$ *implies*

$$(w, \mathscr{T}_5) \not\rightarrow \alpha^+.$$

PROOF. Let D_α be the discrete topological space of power α and βD_α its Stone—Čech compactification. It has been proved by B. POSPÍŠIL (see [3]) that there exists a point $p \in \beta D_\alpha \setminus D_\alpha$ whose every base of neighbourhoods has the cardinality 2^α, in other words, the character $\chi(p, \beta D_\alpha)$ of p in βD_α equals to 2^α. It follows from this that the character $\chi(p, R)$ of p in $D_\alpha \cup \{p\} = R$ is also 2^α because βD_α is regular and R is dense in it. Trivially R belongs to $\mathscr{T}_5$. Now let $A \subset D_\alpha$, $\bar{A}$ its closure in R and $\bar{A}^\beta$ its closure in βD_α. It is well-known that $\bar{A}^\beta$ is open-and-closed in βD_α and so $\bar{A} = \bar{A}^\beta \cap R$ is also open in R. But then $p \in \bar{A}$ implies

$$\chi(p, \bar{A}) = \chi(p, R) = 2^\alpha.$$

Let $S \subset R$ be an arbitrary subspace of R. Then there are three possibilities: (i), (ii) and (iii).

(i) $p \notin S$, then S is obviously discrete, and so $w(S) = |S| \leqq \alpha$.

(ii) $p \in S$ but $p \notin \overline{S \setminus \{p\}}$; then S is discrete, too, thus $w(S) = |S| \leqq \alpha$.

(iii) $p \in S$ and $p \in \overline{S \setminus \{p\}}$; it means that $S \setminus \{p\}$ is dense in S, so (as we have seen above)

$$\chi(p, S) = \chi(p, \overline{S \setminus \{p\}}) = 2^\alpha$$

which immediately gives us $w(S) = 2^\alpha$. Hence every subspace of R has a weight either at most α or 2^α. This proves Theorem 2.

COROLLARY. *If G. C. H. fails then w does not possess the regular Darboux property on $\mathscr{T}_5$.*

After this manuscript had been completed we obtained a result saying $(w, \mathscr{T}_2) \not\rightarrow \alpha^+$, if $\alpha^+ = 2^\alpha$. This result is going to be published in our joint paper "On hereditarily α-separable and α-Lindelöf spaces" in the *Annales Univ. Sci. Budapest*, **11** (1968).

From this result, together with the above Theorem 2 we can get easily that for any α, which is not strong limit or inaccessible, $(w, \mathscr{T}_2) \not\rightarrow \alpha$ holds. However, we still do not know the answer to the following problem.

PROBLEM 1. Is $(w, \mathscr{T}_i) \rightarrow 2^\alpha$ true for $i \geqq 3$, $\alpha \geqq \omega$?

The following cardinal function we shall consider is the density. In this case at least assuming G.C.H. we can give a rather complete discussion of the symbols $(*)$ and $(**)$. The only problem left open is the one stated on p. 34.

The following Lemma 1 was first published in [6] (Theorem II). We give here a new proof of it, which does not make use of transfinite induction. The same idea will be used in the proof of Theorem 5.

LEMMA 1. *Each $R\in\mathcal{T}$ contains a left separated subspace $S\subset R$ with $|S|\geqq d(R)$.*

PROOF. Let

$$R=\{q_\xi\colon \xi<\mu\} \tag{1}$$

be an arbitrary well-ordering of R. A point $q\in R$ will be called minimal if it has a neighbourhood U_q, whose minimal element in the above well-ordering is q. Let

$$S=\{p_\xi\colon \xi<\varrho\} \tag{2}$$

be the well-ordering of the set of all minimal points of R induced by the well-ordering (1). Then S is dense in R and so $d(R)\leqq|S|$. Indeed, if G is an arbitrary non-void open set in R then there exists a point $q\in G$ with a minimal suffix in the well-ordering (1). Hence, by definition, $q\in S$ and so $G\cap S\neq\emptyset$.

On the other hand it is trivial, that if $p_\xi\in S$ and U_ξ is the neighbourhood of p_ξ whose first element is p_ξ then U_ξ does not contain any predecessors of p_ξ.

COROLLARY. *If α is regular then*

$$(d,\mathcal{T})\to\alpha.$$

PROOF. Let, indeed, $R\in\mathcal{T}$ and $d(R)>\alpha$. According to Lemma 1 there is a sequence $S=\{p_\xi\colon \xi<\varrho\}$ of points of R such that $\varrho\geqq\alpha$ and every p_ξ has a neighbourhood U_ξ not containing any points p_η, $\eta<\xi$. Let

$$T=\{p_\xi\in S\colon \xi<\alpha\}.$$

We state that $d(T)=\alpha$. $d(T)\leqq\alpha$ is trivial since $|T|=\alpha$. On the other hand, if $K\subset T$ and $|K|<\alpha$, then there exists an ordinal $\eta<\alpha$ such that $\xi<\eta$ for each $p_\xi\in K$, because of the regularity of α. But then $p_\xi\notin U_\eta$ for each $p_\xi\in K$, which shows that K is not dense in T and so $d(T)\geqq\alpha$ i.e. $d(T)=\alpha$.

LEMMA 2. *If λ is a strong limit cardinal, $|R|=\lambda$ and $R\in\mathcal{T}_2$ then $d(R)=\lambda$.*

PROOF. It is well-known (see e.g. [4]) that $R\in\mathcal{T}_2$ implies $|R|\leqq\exp\exp d(R)$. Since λ is a strong limit cardinal $d(R)<\lambda$ would imply $|R|\leqq\exp\exp d(R)<\lambda$, which is impossible. So $d(R)=\lambda$.

COROLLARY 1. *For each strong limit cardinal λ*

$$(d,\mathcal{T}_2)\to\lambda$$

holds.

COROLLARY 2. *If λ is a strong limit cardinal then*

$$[d,\mathcal{T}_2]\to\lambda.$$

PROOF. If for every $\alpha<\lambda$ there exists a subspace S of the space R for which $d(R)\geqq\alpha$, then obviously $|R|\geqq\lambda$.

THEOREM 3. *If* $\mathrm{cf}(\lambda)=\omega$ *then*

$$(d, \mathcal{T}_2)\to\lambda.$$

PROOF. Let, indeed, $R\in\mathcal{T}_2$, $d(R)>\lambda$, then according to Lemma 1 there exists a left separated subset $R'\subset R$ of the power λ^+. In what follows we are going to consider only this subspace R'. Let $\mathfrak{G}$ be the system of all sets $G\subset R'$ being open in R' and having a cardinality not greater than λ. We will distinguish two cases (i) and (ii):

(i) $|\bigcup\mathfrak{G}|=\lambda^+$. Then we define a sequence $\{q_\xi: \xi<\lambda^+\}$ of points of R' by transfinite induction on ξ as follows. Let $R'=\{p_\nu: \nu<\lambda^+\}$ be a well-ordering of R' and let U_ν be a neighbourhood of p_ν not containing any predecessors of p_ν.

Now let $p_{\nu_0}=q_0$ be the first element of $\bigcup\mathfrak{G}$ and let G_0 be an arbitrary element of $\mathfrak{G}$ with $q_0\in G_0$. Assume that the points q_η and their neighbourhoods G_η are defined already for all η less than some $\xi<\lambda^+$. Then

$$\left|\bigcup_{\eta<\xi} G_\eta\right| \leqq |\xi|\cdot\lambda = \lambda$$

and so $\bigcup\mathfrak{G}\setminus\bigcup\{G_\eta: \eta<\xi\}\neq\emptyset$; we choose the first element p_{ν_ξ} of the above non-void set as q_ξ. G_ξ will be an arbitrary element of $\mathfrak{G}$ containing q_ξ. Put $D=\{q_\xi: \xi<\lambda^+\}$. We prove that D is discrete. Let us consider the neighbourhood $V_\xi=U_{\nu_\xi}\cap G_\xi$ of $q_\xi(=p_{\nu_\xi})$ for $\xi<\lambda^+$. Since by definition $q_\eta\notin U_{\nu_\xi}$ if $\eta<\xi$ and $q_\eta\notin G_\xi$ if $\eta>\xi$, $V_\xi\cap D=\{q_\xi\}$ for $\xi<\lambda^+$. Hence D is discrete. Thus R' and so R also contain a discrete subspace of potency λ, which is of density λ, too.

(ii) $|\bigcup\mathfrak{G}|<\lambda^+$. Then let $R''=R'\setminus\bigcup\mathfrak{G}$. Obviously each non-void open subset of R'' has the cardinality λ^+.

Now because of $\mathrm{cf}(\lambda)=\omega$ there are regular cardinals α_k $(k<\omega)$ such that

$$\lambda=\sum_{k<\omega}\alpha_k.$$

Since every infinite T_2-space contains infinitely many pairwise disjoint, non-void, open sets we can choose non-void subsets G_k $(k<\omega)$ open in R'' such that $G_k\cap G_l=\emptyset$ if $k\neq l$. As we have seen above

$$|G_k|=\lambda^+$$

for each $k<\omega$. By Lemma 1 and the proof of its Corollary for every $k<\omega$ there exists an $S_k\subset G_k$ with $d(S_k)=|S_k|=\alpha_k$ (because α_k is regular).

Now let $S=\bigcup_{k<\omega} S_k$. Since $|S|=\lambda$ it is sufficient to prove $d(S)\geqq\lambda$. Let $M\subset S$ be an arbitrary dense subset of S. Then $M\cap S_k$ is dense in S_k, too, because $M\setminus(M\cap S_k)\subset\bigcup_{l\neq k} G_l\subset R''\setminus G_k$, and so none of the points of $S_k\subset G_k$ is a cluster point of $M\setminus(M\cap S_k)$. But then $d(S_k)=\alpha_k$ implies $|M\cap S_k|=\alpha_k$ and so

$$|M|=\sum_{k<\omega}|M\cap S_k|=\sum_{k<\omega}\alpha_k=\lambda$$

which proves our statement.

Theorem 3 is one of the new results of this paper. The problem stated originally in [1] still remains open for singular cardinals λ with $\mathrm{cf}(\lambda)>\omega$. The simplest unsolved problem is

PROBLEM 2. Is $(d, \mathscr{T}_2)\to\omega_{\omega_1}$ true?

(Note that assuming G.C.H. the answer is yes by Corollary 1 of Lemma 2.) Corollary 2 of Lemma 2 implies assuming G.C.H. that d has the closed Darboux property on $\mathscr{T}_2$. We will point out that without assuming G.C.H. we cannot solve the following

PROBLEM 3. Is $[d, \mathscr{T}_2]\to\omega_\omega$ true?

This should be compared with the remark made after the proof of Theorem 7.

The following theorem shows that for T_1-spaces the above result does not remain true.

THEOREM 4. *For every singular* λ

$$(d, \mathscr{T}_1)\not\to\lambda \quad \text{and} \quad [d, \mathscr{T}_1]\not\to\lambda.$$

PROOF. Let us consider the topology on the set λ^+ whose non-void open sets are exactly those of the form $[\varrho, \lambda^+)\setminus\{\varrho_1, \dots, \varrho_k\}$ where

$$[\varrho, \lambda^+)=\{\sigma\colon \varrho\leqq\sigma<\lambda^+\} \quad \text{and} \quad \varrho, \varrho_1, \dots, \varrho_k<\lambda^+, k<\omega.$$

Let R denote this space which is obviously a T_1-space.

Let now S be an arbitrary infinite subspace of R and let $\tau(S)$ be its order-type as a subset of λ^+. It is well-known that $\tau(S)$ has a unique decomposition

$$\tau(S)=\zeta(S)+k(S),$$

where $\zeta(S)$ is a limit ordinal and $k(S)<\omega$.

Now let H be the set of the last $k(S)$ elements of S and let C be an arbitrary cofinal subset of $S\setminus H$. Then it is obvious that for every $\xi\in S\setminus H$ and $\xi_1, \dots, \xi_l<\lambda^+$

$$S\cap([\xi, \lambda^+)\setminus\{\xi_1, \dots, \xi_l\})\cap(C\cup H)\neq\emptyset,$$

i.e. $C\cup H$ is dense in S.

On the other hand, if T is dense in S, then $T\setminus H$ must be cofinal with $S\setminus H$, because $\varrho\in S\setminus H$ and $\varrho>\sigma$ for every $\sigma\in T\setminus H$ would imply

$$T\cap([\varrho, \lambda^+)\setminus H)=\emptyset.$$

Hence we have got the result $d(S)=\mathrm{cf}(\zeta(S))$. So e.g. $d(R)=\mathrm{cf}(\zeta(R))=\mathrm{cf}(\lambda^+)=\lambda^+$, and since $\mathrm{cf}(\zeta)$ is always a regular cardinal, none of the subspaces of R have the density λ.

On the other hand, since for every regular cardinal $\alpha<\lambda$ there is a subspace of the density α, the same example shows that the second statement of our Theorem holds, as well.

§ 3. A theorem on ordered spaces

The main aim of this section is to prove Theorem 5.

THEOREM 5. *If $R \in \mathscr{L}$, then for each $\alpha < d(R)$ there exists a discrete subspace of R, which is of power α. Hence $d(R) \leqq s(R)^+$.*

PROOF. Let $\alpha < s(R)$ be arbitrary, and assume that R does not contain a discrete subspace of power α. The original order relation of R will be denoted by $<$ while $\prec$ is chosen to denote an arbitrary well-ordering of R.

As usual a set

$$(x, y) = \{z \in R \colon x < z < y\}$$

is called an open interval of R.

An element $p \in R$ is called normal if there exists an open interval (x, y) containing p, such that

$$p \prec z \quad \text{for every} \quad z \in (x, y) \setminus \{p\}.$$

It is easy to see that the set N of all normal elements is dense in R. Let, indeed, (x, y) be an arbitrary non-void interval of R, and p be the first element of (x, y) with respect to the well-ordering $\prec$. Then p is a normal element by definition. Thus $|N| \geqq d(R) > \alpha$.

For every $p \in N$ let I_p denote the maximal convex set containing p as the first element with respect to $\prec$. Of course, I_p contains p in its interior.

Now let N^* be the collection of all such sets I_p for $p \in N$. It is trivial that $p \neq q$, $p, q \in N$ implies $I_p \neq I_q$ since the least elements of I_p resp. I_q are different. We define a partial ordering $<^*$ of N^* as follows,

$$I_q <^* I_p \quad \text{iff} \quad p \in I_q.$$

As $p \in I_q$ obviously implies $q \prec p$ and so $q \prec z$ for every $z \in I_p$, $I_q <^* I_p$ implies $I_p \subset I_q$, since $I_p \cap I_q \neq 0$ and so $I_p \cup I_q$ is a convex set containing q as its first element, and therefore $I_p \cup I_q \subset I_q$. From this remark we get immediately that the relation $<^*$ is transitive.

Let now $q_1, q_2, p \in N$, $q_1 \prec q_2$ and $I_{q_1} <^* I_p$ and $I_{q_2} <^* I_p$. Then $p \in I_{q_1} \cap I_{q_2}$ and so $I_{q_1} \cup I_{q_2}$ is a convex set containing both q_1 and q_2. But then clearly $I_{q_1} \cup I_{q_2} \subset I_{q_1}$ which implies

$$I_{q_1} <^* I_{q_2}.$$

From these considerations it follows immediately that for every $I_p \in N^*$ the segment

$$S_p = \{I_q \in N^* \colon I_q <^* I_p\}$$

is well-ordered by the relation $<^*$, because for $I_{q_1}, I_{q_2} \in S_p$

$$q_1 \prec q_2 \Leftrightarrow I_{q_1} <^* I_{q_2}.$$

So the partially ordered set $(N^*, <^*)$ is a ramification system (or tree) in the sense of [7]. Let now $A \subset N$ be a set for which any two elements of the set-system

$$A^* = \{I_p \colon p \in A\}$$

are not comparable with respect to $<^*$. Then A is a discrete subspace of R since

I_p is a neighbourhood of p which — by definition — does not contain any other points of A. Consequently we obtain $|A|=|A^*|<\alpha$.

Now it is very easy to see that every ramification system of power greater than or equal to α^+, and not containing α pairwise incomparable elements contains a chain of length α, i.e. a set of power α, every two elements of which are comparable (see e.g. [7]). Let $C^* \subset N^*$ be a chain of length α and

$$C = \{p \in N: I_p \in C^*\}.$$

We can assume that the order-type of C (by $\prec$) is α. For every $p \in C$ let p^+ be the successor of p in C with respect to $\prec$. For every $p \in C$ let us choose an element

$$x_p \in I_p \setminus I_{p^+} \neq \emptyset.$$

Since I_{p^+} is convex, either $x_p > z$ or $z > x_p$ for each $z \in I_{p^+}$; in the first case we call x_p a right point and in the second case a left one. The set of all right points is denoted by H^r and that of the left points by H^l. Of course $|H^l \cup H^r| = |C| = \alpha$ and thus either $|H^r| = \alpha$ or $|H^l| = \alpha$.

Assume, for instance, $|H^l| = \alpha$. Then for $x_{p_1}, x_{p_2} \in H^l$ we get

$$x_{p_1} < x_{p_2} \Leftrightarrow p_1 \prec p_2.$$

Let indeed $p_1 \prec p_2$, then $p_1^+ \leqq p_2$ so $I_{p_2} \subset I_{p_1^+}$ i.e. $x_{p_2} \in I_{p_1^+}$ which implies

$$x_{p_1} < x_{p_2}$$

by the definition of H^l.

Thus we have got a subset of R of potency α, whose original ordering $<$ coincides with the well-ordering $\prec$. According to our assumption R can not contain α isolated points and so we have a subset $H \subset H^l$ of power α not containing any isolated points and whose *induced* ordering $<$ is a well-ordering. We shall denote by x^+ the successor of $x \in H$ in H with respect to $<$.

Since H does not contain any isolated points, one of the intervals (x, x^+) and (x^+, x^{++}) is not void. Consequently there are α distinct non-void open intervals of the form (x, x^+), $x \in H$ which is in contradiction to our assumption, since these intervals are pairwise disjoint as well. Hence we have proved the existence of a discrete subspace of power α in R.

An analogous consideration leads to the same result, if $|H^r| = \alpha$, however then $<$ coincides with the converse of $\prec$.

As an immediate consequence of Theorem 5 we get

COROLLARY 1. *The cardinal function d has the Darboux property on $\mathscr{L}$.*

We also prove

COROLLARY 2. *For every singular cardinal λ*

$$[d, \mathscr{L}] \to \lambda.$$

PROOF. If for cofinally many $\alpha < \lambda$ there exist subspaces of $R \in \mathscr{L}$ of the corresponding density, then by Theorem 5 one can find discrete subspaces, whose cardinalities are cofinal with λ, too. But it is easy to see that if a linearly ordered space contains an infinite discrete subspace, then it contains as many pairwise disjoint open intervals as the cardinality of this discrete subspace.

On the other hand, ERDŐS and TARSKI [5] proved that *in every topological space* the least cardinal for which the space does not contain as many pairwise disjoint open subsets, is always regular. So if R contains α disjoint open intervals for each $\alpha<\lambda$, then it contains λ disjoint open intervals, too. Hence R contains a discrete subspace of cardinality (or density) λ, too.

We do not know whether d has the closed Darboux property on $\mathscr{L}$ since we cannot solve the foll owing.

PROBLEM 4. Does Corollary 2 of Theorem 5 hold for inaccessible λ's as well?

Note that in [5] an example is given showing that the theorem we used for the proof of Corollary does not remain true for inaccessible cardinals greater than ω.

In order to get similar results about the Darboux property of the weight function on $\mathscr{L}$, we have to make some preliminary remarks about the relation between the weight and density of the ordered spaces.

Let $R\in\mathscr{L}$. A pair $\langle x, y\rangle$ of two distinct points $x, y\in R$ is called a gap in R, if the open interval (x, y) is empty (i.e. y is the successor of x), but neither x nor y is isolated. Let $U(R)$ be the set of all gaps in R, and $g(R)=|U(R)|$.

LEMMA 3. *If* $R\in\mathscr{L}$, *then* $w(R) = d(R)+g(R)$.

PROOF. We can assume $|R|\geqq\omega$. Now let $S\subset R$ be a dense subset of cardinality $d(R)$ and H be the set of the endpoints of all the gaps in R. First we will show that the open intervals (a, b), where a, b belong to $S\cup H$, plus the isolated points, which of course all belong to S, constitute a base for R, which evidently implies

$$w(R)\leqq|S\cup H|\leqq d(R)+g(R).$$

Let, indeed, $x\in R$ be not isolated, and (p, q) be any open interval containing x. Now, if x does not have either a predecessor or a successor, then we can find $a\in S$ with $a\in(p, x)$, and $b\in S$ with $b\in(x, q)$, hence $x\in(a, b)\subset(p, q)$. If x has a predecessor, say a, then a certainly belongs to $S\cup H$, because either it is isolated, or it constitutes a gap with x, since the latter is not isolated. Since in this case x has no successor, we can find $b\in S$ with $b\in(x, q)$, and then $x\in(a, b)\subset(p, q)$ holds again. The case, when x has a successor can be settled quite analogously.

In order to get the converse inequality

$$w(R)\geqq d(R)+g(R),$$

it is obviously enough to prove $w(R)\geqq g(R)$. This follows, however, immediately from the observation that, if $\langle x, y\rangle\in U(R)$ is an arbitrary gap, then any base of R has to contain a set with x as last element and a set with y as first element. Thus our Lemma is proved.

COROLLARY 3. *w has the Darboux property on $\mathscr{L}$.*

PROOF. Let $R\in\mathscr{L}$ and $w(R)>\alpha$. Then we have the following two possibilities a) and b), respectively.

a) $w(R)=d(R)$. In this case it follows immediately from Theorem 5 that R contains a discrete subspace of cardinality — hence of weight — α.

b) $w(R)=g(R)$. In this case let $U_1\subset U(R)$ be a set of gaps with $|U_1|=\alpha$, and H_1 be the set of all endpoints of the gaps belonging to U_1. Obviously, $|H_1|=\alpha$

as well. One can see easily that the subspace $H_1 \subset R$ is of the weight α, and this completes the proof of the corollary.

Since the weight function is monotone, and for monotone cardinal functions the Darboux property and the closed Darboux property are equivalent (see e.g. [1]), we also have that w has the closed Darboux property on $\mathcal{L}$.

§ 4. Strongly Hausdorff spaces

We will say that a Hausdorff space R is strongly Hausdorff if for each infinite subset $S \subset R$ one can select a sequence $\{x_i\}_{i<\omega}$ of points of S and a sequence $\{U_i\}_{i<\omega}$, $x_i \in U_i$ of neighbourhoods such that $i \neq j < \omega$ implies $U_i \cap U_j = \emptyset$. The following theorem shows that this class of spaces is wide enough.

THEOREM 6. *Every Uryson space, hence every regular Hausdorff space, is strongly Hausdorff.*

PROOF. Let R be an Uryson space and let $S \subset R$ be an arbitrary infinite subspace of it. Let x_0 and y_0 be two arbitrary points of S and U_0 and V_0 a closed neighbourhood of x_0 and y_0 respectively that are disjoint. We can assume $S \setminus U_0$ is infinite. Assume that the points $x_i \in S$ and their neighbourhoods U_i have been already defined for each $i < k$ $(k > 0)$ in such a way that $S \setminus \left(\bigcup_{i<k} \overline{U} \right)$ is infinite. Then we can choose two points $x_k, y_k \in S \setminus \left(\bigcup_{i<k} \overline{U}_i \right)$, and two disjoint neighbourhoods U_k and V_k of x_k and y_k, respectively which are contained in the open set $R \setminus \bigcup_{i<k} \overline{U}_i$, and which have disjoint closures in R. We can also assume that

$$\left(S \setminus \bigcup_{i<k} \overline{U}_i \right) \setminus \overline{U}_k = S \setminus \bigcup_{i<k+1} \overline{U}_i$$

is infinite since

$$S \setminus \bigcup_{i<k} \overline{U}_i = \left[\left(S \setminus \bigcup_{i<k} \overline{U}_i \right) \setminus \overline{U}_k \right] \cup \left[\left(S \setminus \bigcup_{i<k} \overline{U}_i \right) \setminus \overline{V}_k \right],$$

and the roles of x_k and y_k are perfectly symmetric. The sequence $\{x_i\}_{i<\omega}$ defined by induction on k obviously satisfies the requirements having the pairwise disjoint neighbourhoods U_i.

On the other hand, the following example shows that there are Hausdorff spaces which are not strongly Hausdorff.

EXAMPLE. Let the set R consists of two kinds of elements: $R = P \cup H$, where $P \cap H = \emptyset$. Both P and H are countable, the elements of P are denoted by $x_0, \ldots, x_k, \ldots$ $(k < \omega)$, while H is regarded as the set of all quadruples (j, l, m, n) where $j, l, m, n < \omega$. For the topology in R, the points of H are assumed to be isolated and a neighbourhood base $\mathfrak{B}_k = \{V_{r,s}^{(k)} : r, s < \omega\}$ for x_k is defined as follows:

$$V_{r,s}^{(k)} = \{x_k\} \cup \{(k, l, m, n) : l > r\} \cup \{(j, l, m, k) : j < k, \quad l \leqq k, \quad m > s\}.$$

It is easy to see that

$$V_{r_1, s_1}^{(k)} \cap V_{r_2, s_2}^{(k)} = V_{r,s}^{(k)},$$

where $r = \max\{r_1, r_2\}$ and $s = \max\{s_1, s_2\}$.

Furthermore

$$\bigcap_{r,s<\omega} V_{r,s}^{(k)} = \{x_k\},$$

since for every $(j, l, m, n) \in H$ either $j=k$ and then $(j, l, m, n) = (k, l, m, n) \notin V_{l}^{(k)}$ or $k \neq j$ and then $(j, l, m, n) \notin V_{r,m}^{(k)}$.

Finally if $k_1 < k_2$ then x_{k_1} and x_{k_2} have disjoint neighbourhoods since for example

$$V_{k_2,s}^{(k_1)} \cap V_{r,s}^{(k_2)} = \emptyset$$

for every $r, s < \omega$.

This altogether shows that R is a Hausdorff space. But R is not strongly Hausdorff, indeed, since if $\{x_{k_t}: t < \omega\}$ is an arbitrary sequence of points from P, $k_{t_1} < k_{t_2}$ if $t_1 < t_2$, and $V_{r,s}^{(k_0)}$ is an arbitrary neighbourhood of x_{k_0}, then

$$V_{r,s}^{(k_0)} \cap V_{p,q}^{(k_t)} \neq \emptyset$$

for each $p, q < \omega$ whenever $t > 0$ and $k_t > r$, because then for example

$$(k_0, k_t, q+1, k_t) \in V_{r,s}^{(k_0)} \cap V_{p,q}^{(k_t)}.$$

Finally we are going to show an application of the notion introduced above. First we need a lemma, which is, however, interesting in itself, too.

LEMMA 4. *Let R be an arbitrary topological space with $|R| = \alpha > \omega$, and $\beta < \alpha$. Then either R contains a discrete subspace of power α, or the set S_β of all points $x \in R$ having a neighbourhood U_x with $|U_x| < \beta$ is of cardinality less than α.*

PROOF. Assume $|S_\beta| = \alpha$. Then we can define a set mapping F on S_β as follows:

$$F(x) = U_x \setminus \{x\}.$$

Thus $|F(x)| < \beta < \alpha$ holds for all $x \in S_\beta$, hence a theorem proved by A. Hajnal (which is also known as Ruziewicz' conjecture, see e.g. [8]) can be applied, and we can get a free subset $S \subset S_\beta$ with $|S| = \alpha$. This means, however, that $x \notin U_y$ holds for each pair of distinct points $x, y \in S$, i.e. S is a discrete subspace of power α.

THEOREM 7. *Let* $\mathrm{cf}(\lambda) = \omega$, $\lambda > \omega$, *and assume that for each $\alpha < \lambda$ the strongly Hausdorff space R contains a discrete (or right separated, or left separated, respectively) subspace, of cardinality α. Then there exists a discrete (or right separated; or left separated, resp.) subspace of power λ in R as well.*

PROOF. Let $\{\alpha_k: k < \omega\}$ be such a strictly increasing sequence of regular cardinal numbers, for which

$$\sum_{k<\omega} \alpha_k = \lambda, \quad \text{and} \quad \alpha_0 > \omega.$$

Let R_k be a discrete (or right separated, or left separated) subspace of R with $|R_k| = \alpha_k$ $(k < \omega)$, and let

$$R' = \bigcup_{k<\omega} R_k.$$

Let us apply now Lemma 4 to R' with $\beta = \alpha_k$. We get then that we can assume, for each $k < \omega$, less than λ points of R' have neighbourhoods of cardinality $< \alpha_k$. Indeed, otherwise we should know the existence of a discrete subspace of power λ, and our theorem would be proved.

We shall define a sequence of pairwise distinct elements of R' by induction as follows: Let x_0 be any point in R', every neighbourhood of which is of power $\geqq \alpha_0$. (The existence of such a point is assured by the foregoing remark.) Assume, x_l has already been defined for each $l<k<\omega$. Then we can choose such a point $x_k \in R' \setminus \{x_0, \dots, x_{k-1}\}$, every neighbourhood of which has a cardinality $\geqq \alpha_k$, analogously as x_0 was chosen.

Since R is strongly Hausdorff, we can select such an infinite subsequence $\{x_{k_l}\}_{l<\omega} \subset \{x_k\}_{k<\omega}$, whose elements have pairvise disjoint open neighbourhoods (in R, hence in R' as well).

Let U_l be the neighbourhood of x_{k_l} in R', mentioned above. Hence $|U_l| \geqq \alpha_{k_l}$, according to the construction of the x_k's. Now

$$U_l = U_l \cap R' = U_l \cap \Big(\bigcup_{k<\omega} R_k \Big) = \bigcup_{k<\omega} (U_l \cap R_k),$$

hence there exists a $k_0 < \omega$ with

$$|U_l \cap R_{k_0}| \geqq \alpha_{k_l}.$$

In other words: U_l contains a discrete (or right separated, or left separated, resp.) subspace S_l of cardinality $\geqq \alpha_{k_l}$. But then $S = \bigcup_{l<\omega} S_l$ is a discrete (or right separated, or left separated, resp.) subspace of cardinality λ, which completes the proof.

Let us denote the class of strongly Hausdorff spaces by $\mathscr{T}_2^*$. Then a similar reasoning as in the proofs of the above theorem and lemma would yield us the following relation:

$$[d, \mathscr{T}_2^*] \to \lambda \ \big(\mathrm{cf}(\lambda) = \omega\big).$$

Note that J. DE GROOT [2] stated the problem whether each T_2-space R contains a right separated or discrete subspace of maximal cardinality. Thus Theorem 7 is a partial answer to his question.

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SYMMETRISCHE n-ORBIFORMEN KLEINSTEN INHALTS

Von

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Einleitung

Unter einer n-Orbiform wollen wir einen ebenen konvexen Bereich verstehen, der sich tangierend in einem regulären n-Eck drehen läßt, so daß also die Seiten des n-Ecks stets Stützgeraden des konvexen Bereichs sind. Diese konvexen Bereiche stellen eine direkte Verallgemeinerung der schon von EULER [1] betrachteten Orbiformen dar, welche auch als Gleichdicke oder Bereiche konstanter Breite bezeichnet werden und sich hier als 4-Orbiformen einordnen (vergl. etwa [2], [3], [4]). Mit den allgemeinen n-Orbiformen befaßten sich zuerst MEISSNER [5], der die Fourierdarstellung der n-Orbiformen angab, und später FUJIWARA [6], [7] und HAYASHI [8]. FUJIWARA und KAKEYA [9] stellten verschiedene Extremalprobleme für n-Orbiformen, so die interessante Frage nach der n-Orbiform kleinsten Flächeninhalts. Für $n=3$ erhielten sie das Kreisbogenzweieck als Lösung, für $n=4$ war schon vorher von BLASCHKE [10] und LEBESGUE [11], [12] das Reuleaux-Dreieck als das Gleichdick mit dem kleinsten Flächeninhalt nachgewiesen worden. Für $n>4$ blieb das Problem aber ungelöst, da sich die für $n\leqq 4$ verwendeten Beweismethoden nicht übertragen ließen. Weitere Untersuchungen über n-Orbiformen findet man erst in neuerer Zeit, so die Arbeiten von GOLDBERG [13], [14], der auch noch weitergehende Verallgemeinerungen der Orbiformen betrachtet [15], und die Arbeit von FOCKE [16], die sich mit einer technischen Anwendung der n-Orbiformen befaßt. Unabhängig von obigen Arbeiten wäre noch KAMENEZKI [17] zu nennen.

In der vorliegenden Arbeit wollen wir zu dem oben genannten Extremalproblem einen Beitrag leisten. Wir betrachten für jedes beliebige $n\geqq 3$ speziell die n-Orbiformen mit $(ln\pm 1)$-zähliger Symmetrie, die also bei Drehung um den Winkel $2\pi/(ln\pm 1)$ in sich übergehen, und stellen dann die Aufgabe, die $(ln\pm 1)$-zählig symmetrische n-Orbiform mit dem kleinsten Flächeninhalt zu bestimmen. Zur Behandlung dieser Aufgabe wird zunächst eine neue Darstellung für n-Orbiformen entwickelt, welche dann das gestellte Extremalproblem in eine quadratische Optimierungsaufgabe überführt. Diese kann durch geignete Abschätzungen gelöst werden.

1. Darstellung der n-Orbiformen

Gegeben sei ein konvexer (abgeschlossener, beschränkter) Bereich $\mathfrak{K}$ durch seine Stützfunktion $h(\varphi)$ mit dem Polarwinkel φ bezogen auf den inneren Punkt O und eine feste Richtung $\vec{e}$. $h(\varphi)$ ist als Stützfunktion stets stetig und periodisch mit 2π. An $\mathfrak{K}$ legen wir die Stützgeraden S_0, S_1 und S_2 mit den Außennormalenrichtungen φ, $\varphi+\delta$, $\varphi+2\delta$, wobei $\delta=2\pi/n$. Von O fällen wir auf die Stützgeraden die Lote und erhalten die Fußpunkte F_0, F_1 und F_2 (Abb. 1). Die Länge der Lote wird durch $h_0=h(\varphi)$, $h_1=h(\varphi+\delta)$ und $h_2=h(\varphi+2\delta)$ bestimmt. Wir wollen nun die auf S_1 ausgeschnittene Seitenlänge $s=A_0A_1=A_0F_1+F_1A_1$ berechnen. Dazu benutzen wir folgenden elementargeometrischen

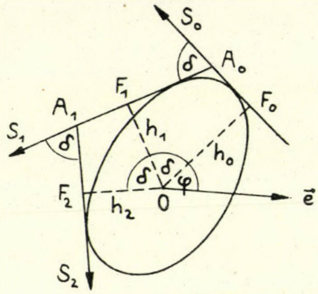


Abb. 1

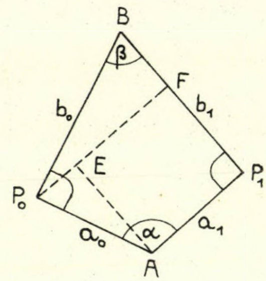


Abb. 2

HILFSSATZ. *In einem Viereck mit zwei gegenüberliegenden rechten Winkeln* (*Abb. 2*) *gelten die Formeln*

$$(1.1)\qquad \begin{aligned} a_0 \sin\beta &= b_1-b_0\cos\beta, & b_0\sin\alpha &= a_1-a_0\cos\alpha,\\ a_1\sin\beta &= b_0-b_1\cos\beta, & b_1\sin\alpha &= a_0-a_1\cos\alpha. \end{aligned}$$

BEWEIS. Nehmen wir zunächst etwa $0<\beta\leqq\frac{\pi}{2}$ an und fällen die Lote P_0F auf b_1 und AE auf P_0F (Abb. 2). Dann gilt in den rechtwinkligen Dreiecken AEP_0 und P_0FB

$$a_0\sin\beta=EA,\qquad a_0\cos\beta=EP_0,\qquad b_0\sin\beta=P_0F,\qquad b_0\cos\beta=BF.$$

Hieraus folgt unter Beachtung von $\alpha+\beta=\pi$

$$\begin{aligned} a_1 &= P_0F-P_0E = b_0\sin\beta-a_0\cos\beta=b_0\sin\alpha+a_0\cos\alpha,\\ b_1 &= AE+FB = a_0\sin\beta+b_0\cos\beta, \end{aligned}$$

und damit die erste Zeile von (1. 1). Durch Vertauschung von $_0$ und $_1$ ergibt sich die zweite Zeile. Da der ganze Formelsatz bei Vertauschung von a_i und b_i und entsprechend α und β in sich übergeht, gilt er unabhängig von der oben gemachten Annahme $\beta\leqq\frac{\pi}{2}$.

Wir wenden nun diesen Hilfssatz auf die Vierecke $OF_0A_0F_1$ und $OF_1A_1F_2$ in Abb. 1 an und erhalten

$$A_0F_1\cdot\sin\delta = h_0-h_1\cos\delta,\qquad F_1A_1\cdot\sin\delta = h_2-h_1\cos\delta,$$

und damit

$$(1.2)\qquad s\cdot\sin\delta = h(\varphi)-2h(\varphi+\delta)\cos\delta+h(\varphi+2\delta).$$

Bleibt in (1. 2) die rechte Seite und damit auch s für alle φ konstant, so kann man das starre »Geradenmaul« S_0, S_1, S_2 mit der Basis s um $\mathfrak{K}$ drehen. Nimmt man n solche kongruente »Geradenmäuler«, jeweils um den Winkel δ versetzt, so entsteht ein reguläres n-Eck, welches sich dann ebenfalls um $\mathfrak{K}$ drehen läßt, bzw. in welchem sich $\mathfrak{K}$ tangierend drehen läßt. Wir erhalten also den

SATZ 1. *Der konvexe Bereich $\mathfrak{K}$ läßt sich genau dann tangierend in einem regulären n-Eck der Seitenlänge s drehen, d.h. ist eine n-Orbiform, falls für seine Stützfunktion $h(\varphi)$ gilt*

(1. 3) $$h(\varphi-\delta)+h(\varphi+\delta)-2h(\varphi)\cos\delta = c$$

für alle φ mit $c=s\cdot\sin\delta$, $\delta=2\pi/n$.

Wir bilden nun die Fourierentwicklung von $h(\varphi)$,

(1. 4) $$h(\varphi)\sim\sum_{\nu=-\infty}^{\infty}\alpha_\nu e^{i\nu\varphi} \quad \text{mit} \quad \alpha_\nu=\frac{1}{2\pi}\int_0^{2\pi}h(\varphi)e^{-i\nu\varphi}d\varphi.$$

Dann ist

$$h(\varphi\pm\delta)\sim\sum_{\nu=-\infty}^{\infty}\alpha_\nu e^{\pm i\nu\delta}e^{i\nu\varphi},$$

und (1. 3) ist genau dann erfüllt, wenn folgender Koeffizientenvergleich gilt:

(1. 5) $$\begin{aligned} 2\alpha_0-2\alpha_0\cos\delta &= 2\alpha_0(1-\cos\delta) = c = s\cdot\sin\delta, \\ \alpha_\nu e^{-i\nu\delta}+\alpha_\nu e^{i\nu\delta}-2\alpha_\nu\cos\delta &= 2\alpha_\nu(\cos\nu\delta-\cos\delta) = 0, \qquad \nu\neq 0. \end{aligned}$$

Die erste Zeile liefert

(1. 6) $$\alpha_0 = \frac{s}{2}\cdot\frac{\sin\delta}{1-\cos\delta} = r,$$

wobei r der Inkreisradius des regulären n-Ecks ist. Die zweite Zeile von (1. 5) ist dann und nur dann erfüllt, falls für jedes ν gilt $\alpha_\nu=0$ oder $\cos\nu\delta-\cos\delta=0$. Die letzte Bedingung ist mit $\nu\delta=2k\pi\pm\delta$, also $\nu=kn\pm1$ äquivalent. Damit erhalten wir den Satz von MEISSNER [5].

SATZ 2. *Der konvexe Bereich $\mathfrak{K}$ läßt sich genau dann tangierend in einem regulären n-Eck mit Inkreisradius r drehen, ist also eine n-Orbiform, wenn seine Stützfunktion $h(\varphi)$ nur nichtverschwindende Fourierkoeffizienten α_ν für $\nu=0$ und $\nu=kn\pm1$ mit k ganz und $\alpha_0=r$ besitzt.*

Orbiformen haben nun nur reguläre Stützgeraden, wie man aus kinematischen Gründen sofort einsieht, ihre Stützfunktion $h(\varphi)$ ist deshalb stetig differenzierbar (vergl. [2], S. 26); wegen der Periodizität ist also insbesondere

(1. 7) $$h(0)=h(2\pi), \quad h'(0)=h'(2\pi).$$

Wir betrachten überdies nunmehr nur konvexe Bereiche $\mathfrak{K}$ mit zweimal differenzierbarer Stützfunktion $h(\varphi)$ mit quadratisch integrabler zweiten Ableitung $h''(\varphi)$.

Dann besitzt die Randkurve von $\mathfrak{K}$ einen Krümmungsradius $\varrho(\varphi)$, und es besteht der Zusammenhang (vergl. [2], S. 65)

$$\varrho(\varphi) = h(\varphi) + h''(\varphi) \geqq 0. \tag{1.8}$$

Dabei ist ϱ wegen der Konvexität nichtnegativ und auch quadratisch integrabel und periodisch. Umgekehrt ist bekanntlich jede periodische, zweimal differenzierbare Funktion $h(\varphi)$ mit $h+h'' \geqq 0$ die Stützfunktion eines konvexen Bereiches. Diese können wir dann durch Vorgabe von $\varrho(\varphi) \geqq 0$, quadratisch integrabel, und Integration von (1. 8) konstruieren,

$$h(\varphi) = a \cos \varphi + b \sin \varphi + \int_0^{\varphi} \varrho(\alpha) \sin(\varphi - \alpha) d\alpha, \tag{1.9}$$

falls noch (1. 7) erfüllt wird durch die »Schließbedingung«

$$\int_0^{2\pi} \varrho(\varphi) \sin \varphi \, d\varphi = 0, \qquad \int_0^{2\pi} \varrho(\varphi) \cos \varphi \, d\varphi = 0. \tag{1.10}$$

$h(\varphi)$ wird dabei nur bis auf $a \cos \varphi + b \sin \varphi$ eindeutig bestimmt, was einer Translation des zugehörigen konvexen Bereiches entspricht. Wir stellen nun auch zu $\varrho(\varphi)$ die Fourierentwicklung auf,

$$\varrho(\varphi) \sim \sum_{\nu=-\infty}^{\infty} \gamma_\nu e^{i\nu\varphi} = \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\varphi + b_\nu \sin \nu\varphi) \tag{1.11}$$

mit

$$\gamma_\nu = \frac{1}{2\pi} \int_0^{2\pi} \varrho(\varphi) e^{-i\nu\varphi} d\varphi, \quad \gamma_1 = \gamma_{-1} = 0,$$

$$a_\nu - ib_\nu = 2\gamma_\nu, \quad a_0 = 2\gamma_0.$$

Durch Vergleich mit (1. 4) ergibt sich gemäß (1. 8) der Zusammenhang

$$\alpha_0 = \gamma_0, \quad \alpha_\nu = -\frac{\gamma_\nu}{\nu^2 - 1}, \quad \nu = \pm 2, \pm 3, \dots . \tag{1.12}$$

Für diese ν verschwindet also α_ν mit γ_ν und umgekehrt. Damit erhalten wir folgende Darstellung der n-Orbiformen:

SATZ 3. *Zu jeder quadratisch integrablen, 2π-periodischen, nichtnegativen Funktion $\varrho(\varphi)$, welche nur nichtverschwindende Fourierkoeffizienten γ_ν für $\nu = 0$ und $\nu = kn \pm 1$ ($k \neq 0$, ganz) besitzt, gibt es eine eindeutig (bis auf Translation) bestimmte n-Orbiform mit $\varrho(\varphi)$ als Krümmungsradius, welche sich im regulären n-Eck mit Inkreisradius $r = \gamma_0$ tangierend drehen läßt. Umgekehrt kann man jede n-Orbiform mit quadratisch integrablem Krümmungsradius auf diese Weise konstruieren.*

Trotz ihrer Evidenz liefert diese Darstellung keinen befriedigenden Überblick über sämtliche n-Orbiformen, da sich die Bedingung $\varrho(\varphi) \geqq 0$ nicht in brauchbarer Weise in den Fourierkoeffizienten γ_ν ausdrücken läßt (vergl. [18], S. 244). Wir wollen deshalb eine für unsere Zwecke geeignetere Darstellungsmethode für n-Orbiformen angeben.

Wegen (1. 8) genügt auch $\varrho(\varphi)$ der Differenzengleichung (1. 3),

$$\varrho(\varphi-\delta)+\varrho(\varphi+\delta)-2\varrho(\varphi)\cos\delta = c. \tag{1. 13}$$

Nach den obigen Ausführungen ist dann (1. 13) und (1. 10) zusammen mit $\varrho(\varphi)\geqq 0$ für $\varrho(\varphi)$ charakteristisch. Nun läßt sich der Winkel φ entsprechend

$$\varphi = j\delta+\psi \quad \text{mit} \quad 0\leqq\psi<\delta, \quad j \text{ ganz}, \quad \delta=2\pi/n \tag{1. 14}$$

eindeutig darstellen, wir können also setzen

$$\varrho(\varphi) = \varrho(j\delta+\psi) = \varrho_j(\psi). \tag{1. 15}$$

Wegen der Periodizität von $\varrho(\varphi)$ gilt

$$\varrho_{j+n}(\psi)=\varrho_j(\psi). \tag{1. 16}$$

Umgekehrt ist durch die $\varrho_j(\psi)$ mit (1. 16) $\varrho(\varphi)$ bestimmt. Somit geht (1. 13) über in

$$\varrho_{j-1}(\psi)+\varrho_{j+1}(\psi)-2\varrho_j(\psi)\cos\delta = c. \tag{1. 17}$$

Wie man leicht verifiziert, lautet die allgemeine Lösung dieser Differenzengleichung mit unbestimmten Funktionen $u(\psi)$ und $v(\psi)$

$$\varrho_j(\psi) = u(\psi)\cos j\delta+v(\psi)\sin j\delta+r, \tag{1. 18}$$

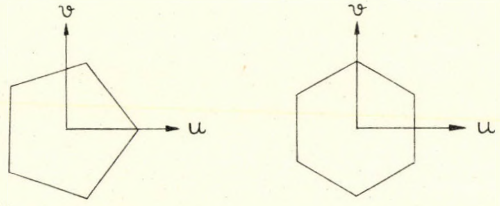
wobei r entsprechend (1. 6) wieder der Inkreisradius ist. (1. 18) genügt von selbst (1. 16). Die Bedingung $\varrho(\varphi)\geqq 0$ ist dann gleichbedeutend mit

$$u(\psi)\cos j\delta+v(\psi)\sin j\delta+r\geqq 0, \quad j = 0,1 \ldots, n-1. \tag{1. 19}$$

Diese Forderung gestattet eine interessante geometrische Deutung. Die n Ungleichungen

$$u\cos j\delta+v\sin j\delta+r \geqq 0, \quad j=0, 1, \ldots, n-1 \tag{1. 20}$$

definieren nämlich in der uv-Ebene als Durchschnitt der durch sie bestimmten n abgeschlossenen Halbebenen gerade den (abgeschlossenen) regulären n-Eckbereich $\mathfrak{P}_n$ mit Inkreisradius r, welcher so zentrisch zum Ursprung gelegen ist, daß die Innernormale einer Seite ($j=0$) in Richtung der positiven u-Achse zeigt (Abb. 3).



$\mathfrak{P}_5$ $\mathfrak{P}_6$

Abb. 3

Dann besagt die Bedingung (1. 19), daß der Punkt $(u(\psi), v(\psi))$ für $0 \leqq \psi < \delta$ in $\mathfrak{P}_n$ verbleibt. Wir wollen nun die uv-Ebene als komplexe $w = u + iv$-Ebene auffassen und setzen $u(\psi) + iv(\psi) = w(\psi)$. Damit folgt aus (1. 18)

$$(1.21)\qquad \varrho_j(\psi) = \tfrac{1}{2}\left(w(\psi)e^{-ij\delta} + \overline{w}(\psi)e^{ij\delta}\right) + r = \operatorname{Re}\left(w(\psi)e^{-ij\delta}\right) + r,$$

und wir können umrechnen

$$(1.22)\qquad \int_0^{2\pi} \varrho(\varphi)e^{i\varphi}\,d\varphi = \sum_{j=0}^{n-1} \int_{j\delta}^{(j+1)\delta} \varrho(\varphi)e^{i\varphi}\,d\varphi = \sum_{j=0}^{n-1} \int_0^{\delta} \varrho_j(\psi)e^{i(j\delta+\psi)}\,d\psi =$$

$$= \frac{1}{2}\sum_{j=0}^{n-1} \int_0^{\delta} \left(w(\psi) + \overline{w}(\psi)e^{2ij\delta}\right)e^{i\psi}\,d\psi = \frac{n}{2}\int_0^{\delta} w(\psi)e^{i\psi}\,d\psi,$$

so daß die Schließbedingung (1. 10) übergeht in

$$(1.23)\qquad \int_0^{\delta} w(\psi)e^{i\psi}\,d\psi = 0.$$

Damit erhalten wir eine neue Darstellung für n-Orbiformen.

SATZ 4. *Zu jeder quadratisch integrablen komplexwertigen Funktion* $w(\psi)$, $0 \leqq \psi \leqq \delta$, *deren Werte in dem durch* (1. 20) *definierten regulären n-Eckbereich* $\mathfrak{P}_n$ *verbleiben, und welche die Schließbedingung* (1. 23) *erfüllt, gibt es eine eindeutig bis auf Translation bestimmte n-Orbiform mit durch* (1. 21) *und* (1. 15) *gegebenem Krümmungsradius* $\varrho(\varphi)$. *Umgekehrt kann jede n-Orbiform mit quadratisch integrablem Krümmungsradius auf diese Weise konstruiert werden.*

Unsere Darstellung gestattet sofort eine Abschätzung für ϱ herzuleiten. Zunächst ist für alle $w \in \mathfrak{P}_n$ (vergl. Abb. 3)

$$(1.24)\qquad \begin{aligned} -r \leqq \operatorname{Re} w \leqq r \quad &\text{für } n \text{ gerade},\\ -r \leqq \operatorname{Re} w \leqq r^* \quad &\text{für } n \text{ ungerade}, \end{aligned}$$

wobei $r^* = r/\cos\frac{1}{2}\delta$ der Umkreisradius von $\mathfrak{P}_n$ ist. Mit $w \in \mathfrak{P}_n$ ist aber auch $we^{-ij\delta} \in \mathfrak{P}_n$ für alle j, und deshalb folgt wegen (1. 21) aus (1. 24)

$$(1.25)\qquad \begin{aligned} 0 \leqq \varrho(\varphi) \leqq 2r \quad &\text{für } n \text{ gerade},\\ 0 \leqq \varrho(\varphi) \leqq r + r^* \quad &\text{für } n \text{ ungerade}. \end{aligned}$$

Zum Abschluß wollen wir noch den Zusammenhang mit der Darstellung von $\varrho(\varphi)$ durch Fourierentwicklung herstellen und die Fourierkoeffizienten γ_ν nach (1. 15) und (1. 21) berechnen. Für $\nu \neq 0$ ist zunächst

$$\gamma_\nu = \frac{1}{2\pi}\int_0^{2\pi} \varrho(\varphi)e^{-i\nu\varphi}\,d\varphi = \frac{1}{2\pi}\sum_{j=0}^{n-1}\int_{j\delta}^{(j+1)\delta} \varrho(\varphi)e^{-i\nu\varphi}\,d\varphi = \frac{1}{2\pi}\sum_{j=0}^{n-1}\int_0^{\delta} \varrho_j(\psi)e^{-i\nu(j\delta+\psi)}\,d\psi =$$

$$(1.26)\qquad = \frac{1}{4\pi}\sum_{j=0}^{n-1}\left(e^{-i(\nu+1)j\delta}\int_0^{\delta} w(\psi)e^{-i\nu\psi}\,d\psi + e^{-i(\nu-1)j\delta}\int_0^{\delta} \overline{w}(\psi)e^{-i\nu\psi}\,d\psi\right),$$

und mit

$$\text{(1.27)}\qquad \sum_{j=0}^{n-1} e^{-i(\nu+1)j\delta} = \begin{cases} n & \text{für } \nu = -1+kn \\ 0 & \text{für } \nu \neq -1+kn \end{cases}$$

$$\sum_{j=0}^{n-1} e^{-i(\nu-1)j\delta} = \begin{cases} n & \text{für } \nu = 1+kn \\ 0 & \text{für } \nu \neq 1+kn \end{cases}$$

erhalten wir schließlich

$$\gamma_{kn-1} = \frac{n}{4\pi}\int_0^\delta w(\psi)e^{-i(kn-1)\psi}d\psi,$$

$$\text{(1.28)}\qquad \gamma_{kn+1} = \frac{n}{4\pi}\int_0^\delta \bar{w}(\psi)e^{-i(kn+1)\psi}d\psi = \bar{\gamma}_{-kn-1},$$

$$\gamma_\nu = 0 \quad \text{für} \quad \nu \neq kn \pm 1,$$

$$\gamma_0 = r.$$

Es ergeben sich zusammen mit der Schließbedingung $\gamma_{-1}=0$ genau die in Satz 3 an die Fourierkoeffizienten γ_ν gestellten Forderungen. Die γ_{kn-1} lassen sich auch als Fourierkoeffizienten von $w(\psi)$ über das Intervall $(0, \delta)$ auffassen. Wenn wir ansetzen

$$\text{(1.29)}\qquad w(\psi)e^{i\psi} \sim \sum_{k=-\infty}^{\infty} \beta_k e^{ikn\psi}, \quad \beta_k = \frac{1}{\delta}\int_0^\delta w(\psi)e^{-i(kn-1)\psi}d\psi,$$

so zeigt der Vergleich mit (1. 28)

$$\text{(1.30)}\qquad \beta_k = 2\gamma_{kn-1}, \quad \beta_0 = 2\gamma_{-1} = 0.$$

Nach der Vollständigkeitsrelation ist dann

$$\text{(1.31)}\qquad \sum_{k=-\infty}^{\infty} |\beta_k|^2 = \frac{1}{\delta}\int_0^\delta |w(\psi)|^2 d\psi.$$

Wenn wir die linke Seite nach (1. 30) und (1. 11) auf a_ν und b_ν umrechnen und wegen $w \in \mathfrak{P}_n$ rechts mit $|w| \leqq r^*$ abschätzen, so erhalten wir für die Fourierkoeffizienten einer Orbiform die Bedingung

$$\text{(1.32)}\qquad \sum_{\nu=1}^{\infty}(a_\nu^2 + b_\nu^2) \leqq r^{*2}.$$

2. Symmetrische n-Orbiformen

Wir wollen nun unsere Darstellung auf $(ln\pm 1)$-zählig symmetrische n-Orbiformen für $l=1, 2, \ldots$ spezialisieren. Für diese muß der Krümmungsradius die Periode ω besitzen,

$$\varrho(\varphi+\omega) = \varrho(\varphi) \quad \text{mit} \quad \omega = \frac{2\pi}{ln\pm 1}. \tag{2.1}$$

Mit dem Winkel σ,

$$\sigma = \frac{2\pi}{n(ln\pm 1)} = \frac{\delta}{ln\pm 1} = \frac{\omega}{n}, \tag{2.2}$$

also auch

$$(ln\pm 1)\sigma = l\omega\pm\sigma = \delta, \qquad \mp l\omega = \sigma\mp\delta, \tag{2.3}$$

muß dann für den Krümmungsradius gelten

$$\varrho(\varphi\mp l\omega) = \varrho(\varphi+\sigma\mp\delta) = \varrho(\varphi), \tag{2.4}$$

oder nach (1. 14) und (1. 15) umgeformt

$$\varrho_{j\mp 1}(\psi+\sigma) = \varrho_j(\psi) \quad \text{für} \quad 0\leqq\psi<\psi+\sigma<\delta, \qquad j=0, 1, \ldots. \tag{2.5}$$

Hier und im weiteren sind jeweils zugleich die oberen bzw. unteren Vorzeichen zu nehmen. Setzen wir in (2. 5) die Darstellung (1. 21) ein, so erhalten wir

$$\operatorname{Re}\big(w(\psi+\sigma)e^{-i(j\mp 1)\delta}\big)+r = \operatorname{Re}\big(w(\psi)e^{-ij\delta}\big)+r. \tag{2.6}$$

Da (2. 6) für alle j gilt, folgt die Bedingung

$$w(\psi+\sigma)e^{\pm i\delta} = w(\psi) \quad \text{für} \quad 0\leqq\psi<\psi+\sigma<\delta \tag{2.7}$$

für $(ln\pm 1)$-zählig symmetrische n-Orbiformen. Wir stellen nun den Winkel ψ entsprechend

$$\psi=\nu\sigma+\alpha, \quad \nu=0, 1, \ldots, (ln\pm 1)-1, \quad 0\leqq\alpha<\sigma \tag{2.8}$$

dar. Dann geht (2. 7) über in

$$w\big((\nu+1)\sigma+\alpha\big) = w(\nu\sigma+\alpha)e^{\mp i\delta} \quad \text{für} \quad \nu = 0, 1, \ldots, (ln\pm 1)-2. \tag{2.9}$$

Wenn wir $w(\alpha)=z(\alpha)$ setzen, so ergibt sich hieraus für $w(\psi)$ die Darstellung

$$w(\psi) = w(\nu\sigma+\alpha) = z(\alpha)e^{\mp i\nu\delta}, \quad \nu=0, 1, \ldots, (ln\pm 1)-1, \tag{2.10}$$

und damit für den Krümmungsradius

$$\begin{gathered}\varrho(\varphi) = \varrho(j\delta+\nu\sigma+\alpha) = \operatorname{Re}\big(z(\alpha)e^{-i(j\pm\nu)\delta}\big)+r \\ \text{mit } j=0, 1, \ldots, n-1, \ \nu=0,1, \ldots, (ln\pm 1)-1,\end{gathered} \tag{2.11}$$

wobei auch für die Funktionswerte $z(\alpha)$ gilt $z(\alpha)\in\mathfrak{P}_n$. Um die $(ln\pm 1)$-zählige Symmetrie, bzw. die Periode ω explizit zum Ausdruck zu bringen, stellen wir den Winkel φ andererseits in der Form

$$\begin{gathered}\varphi = \mu\omega+m\sigma+\alpha \quad \text{mit} \quad \mu = 0, 1, \ldots, (ln\pm 1)-1, \\ m = 0, 1, \ldots, n-1, \ 0\leqq\alpha<\sigma\end{gathered} \tag{2.12}$$

dar. Der Vergleich mit (1. 14) und (2. 8) liefert dann den Zusammenhang

$$\pm(\mu n+m-jln) = j\pm v, \tag{2. 13}$$

und die Darstellung (2. 11) geht unter Beachtung von $n\delta=2\pi$ über in

$$\varrho(\varphi) = \varrho(\mu\omega+m\sigma+\alpha) = \operatorname{Re}\left(z(\alpha)e^{\mp im\delta}\right)+r. \tag{2. 14}$$

Da die rechte Seite von μ unabhängig ist, muß $\varrho(\varphi)$ die Periode ω besitzen. Daraus folgt, daß umgekehrt jede quadratisch integrable Funktion $z(\alpha)$ mit $z(\alpha)\in\mathfrak{P}_n$ gemäß (2. 10) und Satz 4 eine $(ln\pm1)$-zählig symmetrische n-Orbiform erzeugt, denn mit $z(\alpha)\in\mathfrak{P}_n$ ist nach (2. 10) auch $w(\psi)\in\mathfrak{P}_n$, und die Schließbedingung (1. 23) ist von selbst erfüllt, wie wir noch zeigen wollen. Dazu berechnen wir gleich sämtliche Fourierkoeffizienten β_k aus (1. 29),

$$\begin{aligned}\beta_k &= \frac{1}{\delta}\int_0^\delta w(\psi)e^{-i(kn-1)\psi}d\psi = \frac{1}{\delta}\sum_{v=0}^{ln\pm1-1}\int_0^\sigma z(\alpha)e^{\mp iv\delta}e^{-i(kn-1)(v\sigma+\alpha)}d\alpha = \\ &= \frac{1}{\delta}\sum_{v=0}^{ln\pm1-1} e^{-i[(kn-1)\sigma\pm\delta]v}\int_0^\sigma z(\alpha)e^{-i(kn-1)\alpha}d\alpha.\end{aligned} \tag{2. 15}$$

Hierin ergibt sich nach (2. 2) und (2. 3)

$$[(kn-1)\sigma\pm\delta] = (kn-1)\sigma\pm(ln\pm1)\sigma = (k\pm l)n\sigma = (k\pm l)\frac{2\pi}{ln\pm1},$$

und damit

$$\sum_{v=0}^{ln\pm1-1} e^{-i[(kn-1)\sigma\pm\delta]v} = \begin{cases} ln\pm1 & \text{für } \dfrac{k\pm l}{ln\pm1} = p, \text{ ganz} \\ 0 & \text{ansonsten.}\end{cases} \tag{2.16}$$

Es existieren also nur nichtverschwindende Fourierkoeffizienten für

$$k=(ln\pm1)p\mp l. \tag{2. 17}$$

Für diese Werte folgt dann

$$\begin{aligned} kn-1 &= (ln\pm1)pn\mp(ln\pm1) = \\ &= (ln\pm1)(pn\mp1) = (ln\pm1)q, \quad q \text{ ganz.}\end{aligned} \tag{2. 18}$$

Es treten also in der Fourierentwicklung von $\varrho(\varphi)$ nur Vielfache der Zähligkeit $(ln\pm1)$ als Frequenzen auf, wie es wegen der vorausgesetzten $(ln\pm1)$-zähligen Symmetrie auch sein muß. Diese Frequenzen können aber auch alle auftreten, denn ist umgekehrt

$$kn-1 = (ln\pm1)q, \quad q \text{ ganz,} \tag{2. 19}$$

so ergibt sich $kn = lnq\pm(q\pm1)$, also ist $q\pm1$ durch n teilbar. Wir können dann mit ganzzahligem p ansetzen $q\pm1 = pn$, und es folgt

$$k = (ln\pm1)p\mp l. \tag{2. 20}$$

Damit erhalten wir nach (2. 14) und (2. 16) die Fourierkoeffizienten

$$(2.21)\qquad \beta_k = \begin{cases} \dfrac{1}{\sigma}\displaystyle\int_0^{\sigma} z(\alpha)\,e^{-i(kn-1)\alpha}d\alpha & \text{für } \dfrac{kn-1}{ln\pm 1} = q, \text{ ganz} \\ 0 & \text{ansonsten.} \end{cases}$$

Insbesondere ist also $\beta_0 = 0$, d.h. die Schließbedingung (1. 23) ist erfüllt. Wir können dann unsere Resultate zusammenfassen zu

SATZ 5. *Jede quadratisch integrable komplexwertige Funktion $z(\alpha)$, $0 \leqq \alpha \leqq \sigma$, mit $z(\alpha) \in \mathfrak{P}_n$ stellt eine $(ln\pm 1)$-zählig symmetrische n-Orbiform dar mit durch* (2. 14) *gegebenem Krümmungsradius $\varrho(\varphi)$, und jede solche n-Orbiform kann auf diese Weise konstruiert werden.*

In Anwendung dieses Satzes wollen wir für jedes $n \geqq 3$ eine Folge von speziellen $(ln\pm 1)$-zählig symmetrischen n-Orbiformen $O_n^{n-1}, O_n^{n+1}, O_n^{2n-1}, O_n^{2n+1}, O_n^{3n-1}, O_n^{3n+1}, \ldots$ konstruieren; und zwar machen wir für jedes $O_n^{ln\pm 1}$ den Ansatz

$$(2.22)\qquad z(\alpha) = r^* e^{\mp i(\pi + \frac{1}{2}\delta)} = \text{const.},$$

welcher $z(\alpha) \in \mathfrak{P}_n$ erfüllt, da $z(\alpha)$ konstant einen Eckpunktwert von $\mathfrak{P}_n$ annimmt. Nach (2. 14) ergibt sich dann der Krümmungsradius

$$(2.23)\qquad \varrho(\varphi) = \varrho(\mu\omega + m\sigma + \alpha) = r - r^* \cos(m + \tfrac{1}{2})\,\delta = r\left(1 - \frac{\cos(m+\frac{1}{2})\,\delta}{\cos\frac{1}{2}\delta}\right).$$

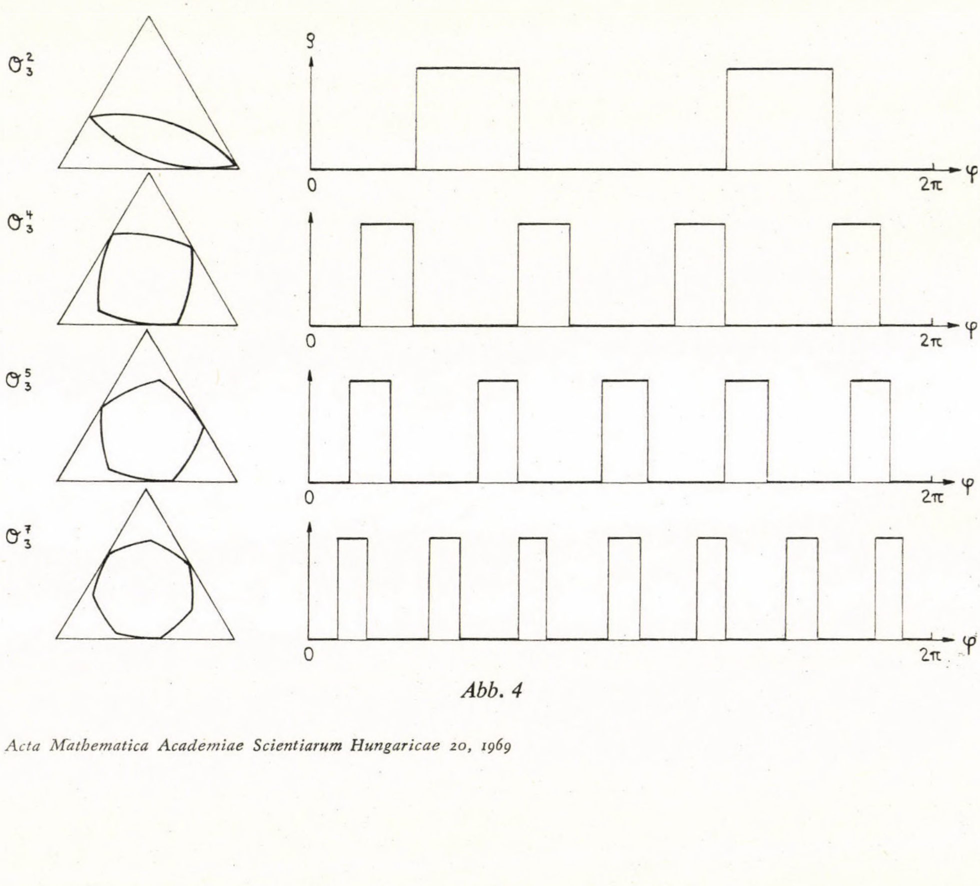


Abb. 4

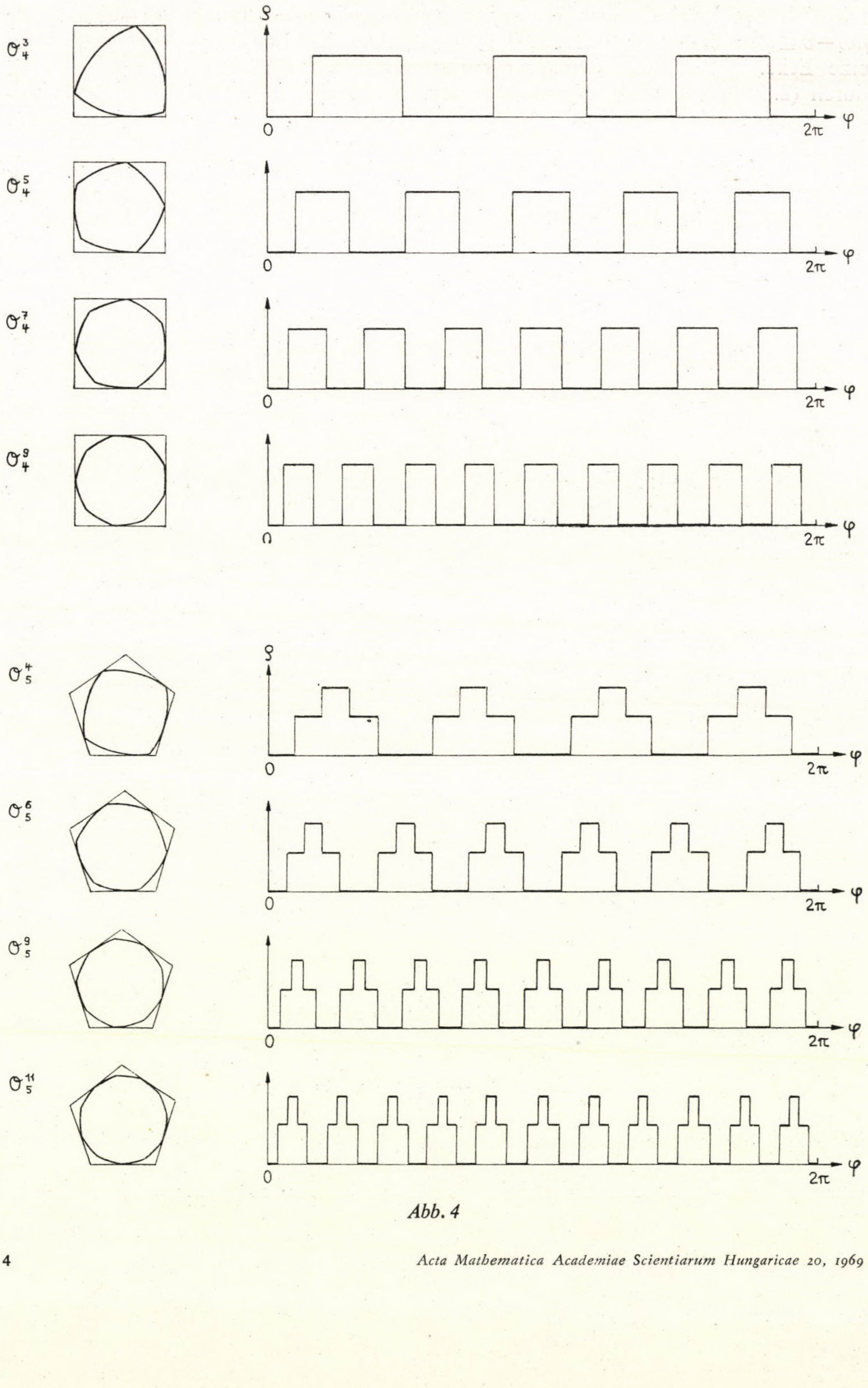


Abb. 4

Dieser ist stückweise konstant und verschwindet für $m=0$ und $m=n-1$, also für $\mu\omega-\sigma\leqq\varphi<\mu\omega+\sigma$, $\mu=0, 1, \dots, (ln\pm 1)-1$. Die Orbiform besitzt dann dort eine Ecke. $O_n^{ln\pm 1}$ stellt demnach ein regelmäßiges Kreisbogen-$(ln\pm 1)$-eck mit durch (2. 23) bestimmten Korbbögen dar. Abbildung 4 zeigt einige dieser Orbiformen bzw. deren Krümmungsverlauf. O_4^3 ist speziell das Reulaux-Dreieck (vergl. etwa [3]), O_5^4 wurde zuerst von FUJIWARA [7] angegeben und die O_n^{n-1} und O_n^{n+1} stimmen mit den von GOLDBERG [14] auf kinematische Weise erzeugten »Trammelrotors« überein. Wir wollen noch die Fourierkoeffizienten der $O_n^{ln\pm 1}$ berechnen.

Nach (2. 21) ist zunächst

$$(2.24)\quad \beta_k=\frac{1}{\sigma}r^*e^{\mp i(\pi+\frac{1}{2}\delta)}\int_0^\sigma e^{-i(kn-1)\alpha}d\alpha=\frac{1}{\sigma}r^*e^{\mp i\frac{\delta}{2}}\frac{1}{i(kn-1)}(e^{-i(kn-1)\sigma}-1)$$

für $kn-1=q(ln\pm 1)=(pn\mp 1)(ln\pm 1)$. Damit wird

$$(2.25)\quad \begin{aligned}\beta_k&=\frac{1}{\sigma}r^*e^{\pm i\frac{\delta}{2}}\frac{1}{i(kn-1)}(e^{\pm i\delta}-1)=\pm\frac{1}{\sigma}r^*\frac{2}{kn-1}\sin\frac{\delta}{2}=\\&=\pm r\frac{2}{\delta}\operatorname{tg}\frac{\delta}{2}\cdot\frac{ln\pm 1}{kn-1},\quad \text{für}\quad \frac{kn-1}{ln\pm 1}=q,\ \text{ganz.}\end{aligned}$$

Die übrigen β_k verschwinden. Die β_k sind hier reell, also nach (1. 30) auch die γ_{kn-1}. Wir können dann auf (1. 11) entsprechend

$$a_{kn-1}=2\gamma_{kn-1}=\beta_k$$

$$a_{kn+1}=2\gamma_{kn+1}=2\bar{\gamma}_{-kn-1}=2\gamma_{-kn-1}=\beta_{-k},$$

umrechnen und erhalten die Fourierdarstellung von $O_n^{ln\pm 1}$

$$(2.26)\quad \varrho(\varphi)=r+\sum_{\nu=1}^{\infty}a_\nu\cos\nu\varphi,$$

mit

$$a_{kn-1}=\pm r\frac{n}{\pi}\operatorname{tg}\frac{\pi}{n}\cdot\frac{ln\pm 1}{kn-1},\quad\text{für}\quad\frac{kn-1}{ln\pm 1}\ \text{ganz,}$$

$$a_{kn+1}=\mp r\frac{n}{\pi}\operatorname{tg}\frac{\pi}{n}\cdot\frac{ln\pm 1}{kn+1},\quad\text{für}\quad\frac{kn+1}{ln\pm 1}\ \text{ganz,}$$

$$a_\nu=0,\ \text{ansonsten.}$$

Zum Schluß wollen wir noch die Bedingung (1. 32) für die Fourierkoeffizienten überprüfen, in welcher hier das Gleichheitszeichen gelten muß. Nach (2. 26) und (2. 17) bis (2. 20) ist

$$a_{kn-1}=\pm r\frac{n}{\pi}\operatorname{tg}\frac{\pi}{n}\cdot\frac{1}{pn\mp 1}\quad\text{mit}\quad k=(ln\pm 1)p\mp l,$$

mit $p=1, 2, 3, \dots$ für die oberen und $p=0, 1, 2, \dots$ für die unteren Vorzeichen und

$$(2.27)\quad a_{kn+1}=\mp r\frac{n}{\pi}\operatorname{tg}\frac{\pi}{n}\cdot\frac{1}{pn\pm 1}\quad\text{mit}\quad k=(ln\pm 1)p\pm l,$$

mit $p=0, 1, 2, \dots$ für die oberen und $p=1, 2, 3, \dots$ für die unteren Vorzeichen. Damit berechnet sich

$$\sum_{\nu=1}^{\infty} a_\nu^2 = \left(r \frac{n}{\pi} \operatorname{tg} \frac{\pi}{n}\right)^2 \sum_{p=-\infty}^{\infty} \left(\frac{1}{pn+1}\right)^2. \tag{2.28}$$

Diese Summe läßt sich mittels der Partialbruchentwicklung

$$\left(\frac{x\pi}{\sin x\pi}\right)^2 = \sum_{\nu=-\infty}^{\infty} \left(\frac{x}{x+\nu}\right)^2 \tag{2.29}$$

für $x=1/n$ auswerten, und wir erhalten

$$\sum_{\nu=1}^{\infty} a_\nu^2 = \left(r \frac{n}{\pi} \operatorname{tg} \frac{\pi}{n}\right)^2 \left(\frac{n}{\pi} \sin \frac{\pi}{n}\right)^{-2} = \left(\frac{r}{\cos \frac{1}{2}\delta}\right)^2 = r^{*2} \tag{2.30}$$

also genau die Bedingung (1. 32) mit dem Gleichheitszeichen.

3. Der Flächeninhalt

Der Umfang L und der Flächeninhalt F eines konvexen Bereiches mit der Stützfunktion $h(\varphi)$ und dem Krümmungsradius $\varrho(\varphi) = h(\varphi)+h''(\varphi)$ berechnet sich nach [2], S. 65 zu

$$L = \int_0^{2\pi} h(\varphi)d\varphi = \int_0^{2\pi} \varrho(\varphi)d\varphi, \tag{3.1}$$

$$F = \tfrac{1}{2} \int_0^{2\pi} h(\varphi)\varrho(\varphi)\, d\varphi. \tag{3.2}$$

Wenn wir $h(\varphi)$ und $\varrho(\varphi)$ entsprechend (1. 4) und (1. 11) in Fourierreihen entwickeln, so lassen sich L und F unter Beachtung der Vollständigkeitsrelation durch die Fourierkoeffizienten ausdrücken,

$$L = 2\pi\alpha_0 = 2\pi\gamma_0, \tag{3.3}$$

$$F = \pi \sum_{\nu=-\infty}^{\infty} \alpha_\nu \bar{\gamma}_\nu = \pi |\gamma_0|^2 - 2\pi \sum_{\nu=2}^{\infty} \frac{1}{\nu^2-1} |\gamma_\nu|^2. \tag{3.4}$$

Für unsere n-Orbiformen folgt dann nach Satz 3 und nach (1. 30)

$$L = 2\pi r, \tag{3.5}$$

$$F = \pi r^2 - \Delta, \tag{3.6}$$

mit

$$\Delta = 2\pi \sum_{\substack{\nu=kn\pm 1\\ k=1,2,\dots}} \frac{1}{\nu^2-1} |\gamma_\nu|^2 = \frac{\pi}{2} \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{1}{kn(kn-2)} |\beta_k|^2.$$

Aus (3. 5) ergibt sich der bekannte Sachverhalt: Alle n-Orbiformen, die zu demselben regulären n-Eck gehören, haben den gleichen Umfang. Die Darstellung (3. 6) zeigt wie beim Hurwitzschen Beweis der isoperimetrischen Ungleichung, daß der Kreis unter allen n-Orbiformen den größten Flächeninhalt hat. Die Frage nach den symmetrischen n-Orbiformen mit dem kleinsten Flächeninhalt dürfte von der Darstellung (3. 6) aus dagegen kaum angreifbar sein, da der gemäß $\varrho \geqq 0$ zulässige Bereich der Fourierkoeffizienten, wie schon erwähnt, sich kaum überblicken läßt. Wir wollen deshalb F im Anschluß an Satz 4 in geschlossener Form angeben.

Auf Grund der Differenzengleichung (1. 13) bzw. (1. 17) konnten wir $\varrho(\varphi)$ nach (1. 15) und (1. 18) in der Gestalt

$$\varrho(\varphi) = \varrho(j\delta+\psi) = \varrho_j(\psi) = u(\psi)\cos j\delta + v(\psi)\sin j\delta + r \quad \text{mit} \quad j=0,1,\ldots,n-1 \tag{3. 7}$$

schreiben, wobei nach Satz 4 $w(\psi) = u(\psi)+iv(\psi) \in \mathfrak{P}_n$ ist. Da nun $h(\varphi)$ derselben Differenzengleichung genügt, können wir analog darstellen

$$h(\varphi) = h(j\delta+\psi) = h_j(\psi) = U(\psi)\cos j\delta + V(\psi)\sin j\delta + r. \tag{3. 8}$$

$\varrho(\varphi)$ und $h(\varphi)$ hängen gemäß (1. 9) eindeutig bis auf die zwei Integrationskonstanten a und b zusammen. Nun ist nach (3. 7) und (3. 8)

$$\varrho_0(\psi) = u(\psi)+r, \quad h_0(\psi) = U(\psi)+r; \tag{3. 9}$$

andererseits gilt wegen $\varrho = h+h''$ auch

$$\varrho_0(\psi) = h_0(\psi)+h_0''(\psi), \tag{3. 10}$$

also folgt zusammen

$$u(\psi) = U(\psi)+U''(\psi). \tag{3. 11}$$

Setzen wir dies in

$$\varrho-(h+h'') = [u-(U+U'')]\cos j\delta + [v-(V-V'')]\sin j\delta = 0 \tag{3. 12}$$

ein, so ergibt sich auch für $V(\psi)$

$$v(\psi) = V(\psi)+V''(\psi). \tag{3. 13}$$

Durch Integration von (3. 11) und (3. 13) erhalten wir mit den Integrationskonstanten C_1, C_2, C_3, C_4

$$U(\psi) = C_1\cos\psi + C_2\sin\psi + \int_0^{\psi} u(\psi')\sin(\psi-\psi')\,d\psi', \tag{3. 14}$$

$$V(\psi) = C_3\cos\psi + C_4\sin\psi + \int_0^{\psi} v(\psi')\sin(\psi-\psi')\,d\psi', \tag{3. 15}$$

was sich auch mit zwei komplexen Konstanten A und B zusammenfassen läßt zu

$$W(\psi) = U(\psi)+iV(\psi) = Ae^{i\psi}+Be^{-i\psi} + \int_0^{\psi} w(\psi')\sin(\psi-\psi')\,d\psi'. \tag{3. 16}$$

Nun ist aber bei gegebenem $\varrho(\varphi)$ die Stützfunktion $h(\varphi)$ bis auf zwei (reelle) Integrationskonstanten bestimmt, also muß noch eine (komplexe) Abhängkeitsrelation bestehen. Diese ergibt sich aus der für $h(\varphi)$ und $h'(\varphi)$ zu fordernden Stetigkeit. Nach (3. 8) muß dann gelten

$$h_j(\delta-0) = h_{j+1}(0), \quad h'_j(\delta-0) = h'_{j+1}(0). \tag{3.17}$$

Wenn wir (3. 8) in komplexer Form

$$h_j(\psi) = \operatorname{Re}\left(W(\psi)e^{-ij\delta}\right)+r \tag{3.18}$$

schreiben, so lauten die Forderungen (3. 17)

$$\begin{aligned} \operatorname{Re}\left(W(\delta-0)e^{-ij\delta}\right) &= \operatorname{Re}\left(W(0)e^{-i(j+1)\delta}\right),\\ \operatorname{Re}\left(W'(\delta-0)e^{-ij\delta}\right) &= \operatorname{Re}\left(W'(0)e^{-i(j+1)\delta}\right). \end{aligned} \tag{3.19}$$

Dabei berechnet sich nach (3. 16) unter Beachtung der Schließbedingung (1. 23)

$$\begin{aligned} W(\delta-0) &= Ae^{i\delta}+Be^{-i\delta}+\frac{1}{2i}e^{i\delta}\int_0^\delta w(\psi)e^{-i\psi}d\psi,\\ W'(\delta-0) &= iAe^{i\delta}-iBe^{-i\delta}+\frac{1}{2}e^{i\delta}\int_0^\delta w(\psi)e^{-i\psi}d\psi,\\ W(0) &= A+B, \quad W'(0) = iA-iB. \end{aligned} \tag{3.20}$$

Hiermit lauten die Bedingungen (3. 19) für $j=0$

$$\begin{aligned} \operatorname{Re}\left(Ae^{i\delta}-Ae^{-i\delta}+\frac{1}{2i}e^{i\delta}\int_0^\delta w(\psi)e^{-i\psi}d\psi\right) &= 0,\\ \operatorname{Re}\left(iAe^{i\delta}-iAe^{-i\delta}+\frac{1}{2}e^{i\delta}\int_0^\delta w(\psi)e^{-i\psi}d\psi\right) &= 0. \end{aligned} \tag{3.21}$$

Dies besagt aber gerade

$$Ae^{i\delta}-Ae^{-i\delta}+\frac{1}{2i}e^{i\delta}\int_0^\delta w(\psi)e^{-i\psi}d\psi = 0, \tag{3.22}$$

also gilt für die Konstante A,

$$A = \frac{1}{4\sin\delta}e^{i\delta}\int_0^\delta w(\psi)e^{-i\psi}d\psi, \tag{3.23}$$

während B frei wählbar bleibt. Zur Berechnung des Flächeninhaltes setzen wir nun (3. 7) und (3. 8) in (3. 2) ein,

$$F = \tfrac{1}{2}\int_0^{2\pi} h(\varphi)\varrho(\varphi)d\varphi = \tfrac{1}{2}\sum_{j=0}^{n-1}\int_0^\delta h_j(\psi)\varrho_j(\psi)d\psi. \tag{3.24}$$

Hierin berechnet sich nach (3. 18) und (1. 21) in komplexer Form

$$\begin{aligned} 4h_j\varrho_j &= (We^{-ij\delta}+\overline{W}e^{ij\delta}+2r)(we^{-ij\delta}+\overline{w}e^{ij\delta}+2r) =\\ &= Wwe^{-2ij\delta}+\overline{W}\overline{w}e^{2ij\delta}+W\overline{w}+\overline{W}w+2r(We^{-ij\delta}+\overline{W}e^{ij\delta}+we^{-ij\delta}+\overline{w}e^{ij\delta})+4r^2. \end{aligned} \tag{3.25}$$

Nun gilt für $\delta=2\pi/n$

$$\sum_{j=0}^{n-1} e^{ij\delta}=0, \qquad \sum_{j=0}^{n-1} e^{2ij\delta}=0, \tag{3.26}$$

so daß wir beim Einsetzen von (3. 25) in (3. 24) erhalten

$$F=\frac{1}{8}n\int_0^\delta \big(W(\psi)\overline{w}(\psi)+\overline{W}(\psi)w(\psi)\big)\,d\psi+\pi r^2. \tag{3.27}$$

Mit $W(\psi)$ entsprechend (3. 16) ergibt sich dann

$$\begin{aligned} F=\frac{1}{8}n\int_0^\delta \Big[&A\overline{w}(\psi)e^{i\psi}+B\overline{w}(\psi)e^{-i\psi}+\overline{A}w(\psi)e^{-i\psi}+\overline{B}w(\psi)e^{i\psi}+\\ &+\int_0^\psi \big(\overline{w}(\psi)w(\psi')+w(\psi)\overline{w}(\psi')\big)\sin(\psi-\psi')\,d\psi'\Big]\,d\psi+\pi r^2. \end{aligned} \tag{3.28}$$

Da sich nun F in eindeutiger Weise allein durch $\varrho(\varphi)$ bestimmt, müssen sich die Integrationskonstanten A und B aus (3. 28) eliminieren lassen. Tatsächlich fällt infolge der Schließbedingung (1. 23) B und $\overline{B}$ sofort heraus, und A bzw. $\overline{A}$ ist nach (3. 23) einzusetzen. Wenn wir noch die iterierten Integrale in Gebietsintegrale über das Rechteck $\mathfrak{R}=\{0\leqq\psi\leqq\delta,\ 0\leqq\psi'\leqq\delta\}$ umformen, so ergibt sich

$$\begin{aligned} F=\pi r^2+\frac{n}{16}\operatorname{ctg}\delta\iint_{\mathfrak{R}} e^{-i(\psi-\psi')}w(\psi)\overline{w}(\psi')\,d\psi\,d\psi'+\\ +\frac{n}{8}\iint_{\mathfrak{R}} \sin|\psi-\psi'|\,w(\psi)\overline{w}(\psi')\,d\psi\,d\psi'. \end{aligned} \tag{3.29}$$

Schreiben wir auch die Schließbedingung (1. 23) durch Bildung des Betragsquadrates in Form eines Gebietsintegrals

$$\iint_{\mathfrak{R}} e^{i(\psi-\psi')}w(\psi)\overline{w}(\psi')\,d\psi\,d\psi'=0, \tag{3.30}$$

so können wir damit (3. 29) noch etwas umformen und erhalten schließlich für den Flächeninhalt,

$$F=\pi r^2-\Delta(w,\overline{w}), \tag{3.31}$$

mit

$$\Delta(w,\overline{w})=-\frac{n}{8\sin\delta}\iint_{\mathfrak{R}}\cos\big(|\psi-\psi'|-\delta\big)\,w(\psi)\overline{w}(\psi')\,d\psi\,d\psi'.$$

Der Flächeninhalt F wird somit durch die hermitesche Integralform $\Delta(w,\overline{w})$ mit stetiger Kernfunktion im Hilbertraum der quadratisch integrablen komplexwertigen

Funktionen $w(\psi)$, $0 \leqq \psi \leqq \delta$ geliefert, und diese ist auf der durch (1. 23) bestimmten Hyperebene positiv definit, wie der Vergleich mit (3. 6) zeigt.

Wir wollen nun die Formel für den Flächeninhalt (3. 31) auf $(ln\pm 1)$-zählig symmetrische n-Orbiformen spezialisieren. Wir teilen das Gebietsintegral gemäß (2. 8) auf und setzen (2. 10) ein,

$$(3.32)\quad \Delta = -\frac{n}{8\sin\delta}\sum_{\nu,\nu'=0}^{ln\pm 1-1}\int_0^\sigma\int_0^\sigma \cos\left(|(\nu-\nu')\sigma+\alpha-\alpha'|-\delta\right)z(\alpha)\bar{z}(\alpha')e^{\mp i(\nu-\nu')\delta}d\alpha\, d\alpha'.$$

Da die Summanden nur von $\nu-\nu'$ abhängen, können wir über $\nu-\nu'=0$, $\nu-\nu'=\tau$ und $\nu-\nu' = -\tau$ mit $\tau=1,2,\ldots, ln\pm 1-1$ summieren und erhalten

$$(3.33)\qquad \Delta = -\frac{n}{8\sin\delta}\left\{(ln\pm 1)\int_0^\sigma\int_0^\sigma \cos\left(|\alpha-\alpha'|-\delta\right)z(\alpha)\bar{z}(\alpha')d\alpha\, d\alpha' + S\right\},$$

mit

$$S = \sum_{\tau=1}^{ln\pm 1-1}(ln\pm 1-\tau)\int_0^\sigma\int_0^\sigma [\cos(\tau\sigma+\alpha-\alpha'-\delta)e^{\mp i\tau\delta}+\cos(-\tau\sigma+\alpha-\alpha'+\delta)e^{\pm i\tau\delta}]\times$$
$$\times z(\alpha)\bar{z}(\alpha')d\alpha\, d\alpha' =$$
$$= \sum_{\tau=1}^{ln\pm 1-1}(ln\pm 1-\tau)\tfrac{1}{2}[e^{i\tau(\sigma\mp\delta)}e^{-i\delta}+e^{-i\tau(\sigma\mp\delta)}e^{i\delta}]\int_0^\sigma\int_0^\sigma e^{i(\alpha-\alpha')}z(\alpha)\bar{z}(\alpha')d\alpha\, d\alpha' +$$
$$+\sum_{\tau=1}^{ln\pm 1-1}(ln\pm 1-\tau)\tfrac{1}{2}[e^{-i\tau(\sigma\pm\delta)}e^{i\delta}+e^{i\tau(\sigma\pm\delta)}e^{-i\delta}]\int_0^\sigma\int_0^\sigma e^{-i(\alpha-\alpha')}z(\alpha)\bar{z}(\alpha')d\alpha\, d\alpha'.$$

Unter Benutzung der Summationsformel

$$(3.34)\qquad \sum_{\nu=1}^{N-1}(N-\nu)q^\nu = \frac{Nq}{1-q}-\frac{(1-q^N)q}{(1-q)^2} \quad \text{für} \quad q\neq 1$$

können wir zunächst die erste Summe berechnen, wobei nach (2. 3) noch $\sigma\mp\delta = \mp l\omega$ und $\sigma\pm\delta = \mp(l\omega-2\delta)$ gilt,

$$\sum_{\tau=1}^{ln\pm 1-1}(ln\pm 1-\tau)[e^{i\tau(\sigma\mp\delta)}e^{-i\delta}+e^{-i\tau(\sigma\mp\delta)}e^{i\delta}] = (ln\pm 1)\left[\frac{e^{\mp il\omega}e^{-i\delta}}{1-e^{\mp il\omega}}+\frac{e^{\pm il\omega}e^{i\delta}}{1-e^{\pm il\omega}}\right] =$$
$$= (ln\pm 1)\frac{e^{-i\delta}-e^{\pm il\omega}e^{i\delta}}{e^{\pm il\omega}-1} = (ln\pm 1)\frac{e^{\mp i(l\omega-\delta\pm\delta)}+e^{\pm i(l\omega-\delta\pm\delta)}-e^{-i(\delta\pm\delta)}-e^{i(\delta\pm\delta)}}{(e^{\pm il\omega}-1)(e^{\mp i(l\omega-\delta)}-e^{\mp i\delta})} =$$

$$(3.35)\qquad = (ln\pm 1)\frac{\cos(\delta-\sigma)-\cos(\delta\pm\delta)}{\cos\delta-\cos\sigma}.$$

Zur Berechnung der zweiten Summe kehren wir im hinteren Term den Indexdurchlauf um und können dann beide Terme zusammenfassen und summieren,

$$(3.36)\qquad \sum_{\tau=1}^{ln\pm 1-1} (ln\pm 1-\tau)[e^{-i\tau(\sigma\pm\delta)}e^{i\delta}+e^{i\tau(\sigma\pm\delta)}e^{-i\delta}] =$$

$$= \sum_{\tau=1}^{ln\pm 1-1} [(ln\pm 1-\tau)e^{\pm i\tau(l\omega-2\delta)}e^{i\delta}+\tau e^{\mp i(ln\pm 1-\tau)(l\omega-2\delta)}e^{-i\delta}] =$$

$$= \sum_{\tau=1}^{ln\pm 1-1} [(ln\pm 1-\tau)e^{\pm i\tau(l\omega-2\delta)}e^{i\delta}+\tau e^{\pm i\tau(l\omega-2\delta)}e^{i\delta}] =$$

$$= (ln\pm 1)e^{i\delta}\sum_{\tau=1}^{ln\pm 1-1} e^{\pm i\tau(l\omega-2\delta)} = (ln\pm 1)\frac{e^{-i\delta}-e^{\pm i(l\omega-2\delta)}e^{i\delta}}{e^{\pm i(l\omega-2\delta)}-1} =$$

$$= (ln\pm 1)\frac{e^{\mp i(l\omega-\delta\pm\delta)}+e^{\pm i(l\omega-\delta\pm\delta)}-e^{-i(\delta\mp\delta)}-e^{i(\delta\mp\delta)}}{(e^{\pm i(l\omega-2\delta)}-1)(e^{\mp i(l\omega-\delta)}-e^{\pm i\delta})} =$$

$$= (ln\pm 1)\frac{\cos(\delta-\sigma)-\cos(\delta\mp\delta)}{\cos\delta-\cos\sigma}.$$

Setzen wir (3. 35) und (3. 36) in (3. 33) ein, so ergibt sich schließlich nach einigen Umformungen

$$(3.37)\quad \Delta = \Delta(z,\bar z) = \frac{1}{8}\frac{n(ln\pm 1)}{\cos\sigma-\cos\delta}\int_0^\sigma\int_0^\sigma [\cos\delta\sin|\alpha-\alpha'|\pm i\sin\delta\sin(\alpha-\alpha')+$$
$$+\sin(\sigma-|\alpha-\alpha'|)]\,z(\alpha)\bar z(\alpha')\,d\alpha\,d\alpha'.$$

Damit ist jetzt der Flächeninhalt F der $(ln\pm 1)$-zählig symmetrischen n-Orbiformen durch die positiv definite hermitesche Integralform $\Delta(z,\bar z)$ mit stetiger Kernfunktion im Hilbertraum der quadratisch integrablen komplexwertigen Funktionen $z(\alpha)$, $0\leqq\alpha\leqq\sigma$, dargestellt.

4. Approximation durch Kreisbogenorbiformen

Wir betrachten im Hilbertraum der Funktionen $z(\alpha)$, $0\leqq\alpha\leqq\sigma$, mit $\|z\|^2 = \int_0^\sigma |z(\alpha)|^2 d\alpha$ die konvexe Menge

$$(4.1)\qquad \mathfrak{M} = \{z \mid z(\alpha)\in\mathfrak{P}_n\}$$

und wählen hieraus die Teilmenge $\mathfrak{T}\subset\mathfrak{M}$ aller stückweise konstanten Funktionen (Treppenfunktionen) $t(\alpha)$ mit den Sprungstellen

$$(4.2)\qquad \alpha_0,\alpha_1,\dots,\alpha_K \quad\text{mit}\quad \alpha_\varkappa=\varkappa\varepsilon,\quad \varepsilon=\frac{\sigma}{K},\qquad K=1,2,4,8,\dots .$$

Die $t(\alpha)\in\mathfrak{T}$ beschreiben dann Kreisbogenorbiformen, deren Randkurven sich stückweise aus Kreisbögen mit Radien gemäß (2. 14) zusammensetzen. Es gilt dann der

APPROXIMATIONSSATZ. *Die Menge $\mathfrak{T}$ der Treppenfunktionen liegt in $\mathfrak{M}$ dicht, d.h. jede $(ln\pm 1)$-zählig symmetrische n-Orbiform kann durch ebensolche Kreisbogenorbiformen beliebig approximiert werden und zwar bezüglich der Krümmung im Mittel und bezüglich der Stützfunktion gleichmäßig.*

Obwohl dieser Satz vom funktionalanalytischen Standpunkt aus ziemlich naheliegend erscheint, so ist er doch nicht völlig evident, da die Menge $\mathfrak{M}$ keine inneren Punkte besitzt. Es ist zunächst bekannt, daß die Treppenfunktionen (auch nur diejenigen mit den Sprungstellen (4. 2)) im Hilbertraum dicht liegen. Wir können also zu $z\in\mathfrak{M}$ und zu beliebig kleinem $\eta>0$ eine Treppenfunktion $t_1(\alpha)$ mit Sprungstellen nach (4. 2) für ein von η abhängiges K angeben, so daß

$$\|z-t_1\|<\eta, \tag{4. 3}$$

t_1 braucht aber nicht aus $\mathfrak{M}$ zu sein. Wir bilden nun die Treppenfunktion aus $\mathfrak{M}$

$$t(\alpha)=\begin{cases} t_1(\alpha) & \text{für } \alpha \text{ mit } t_1(\alpha)\in\mathfrak{P}_n, \\ S(t_1(\alpha)) & \text{für } \alpha \text{ mit } t_1(\alpha)\notin\mathfrak{P}_n, \end{cases} \tag{4. 4}$$

wobei $S(t_1(\alpha))$ der Schnittpunkt des Halbstrahles von O nach t_1 mit dem Rand von $\mathfrak{P}_n$ in der komplexen z-Ebene ist. Dann gilt für alle α

$$|z(\alpha)-t(\alpha)|\leqq 2\,|z(\alpha)-t_1(\alpha)|,$$

und die gegebene Funktion $z(\alpha)$ wird durch $t(\alpha)\in\mathfrak{M}$ beliebig approximiert,

$$\|z-t\|\leqq 2\|z-t_1\|<2\eta. \tag{4. 5}$$

Damit ist nach (2. 14) der Approximationssatz bezüglich der mittleren Approximation der Krümmung bewiesen. Hieraus folgt aber wegen der Darstellung (1. 9) sofort die gleichmäßige Approximation der Stützfunktion.

Wir wollen nun den Flächeninhalt der symmetrischen Kreisbogenorbiformen entsprechend (3. 37) angeben. Für jedes feste K lassen sich die Treppenfunktionen $z(\alpha)\in\mathfrak{T}$ in der Form

$$z(\alpha)=z_\varkappa \quad \text{für} \quad \alpha_{\varkappa-1}\leqq\alpha<\alpha_\varkappa, \qquad \varkappa=1,2,\dots,K \tag{4. 6}$$

schreiben und als Punkte eines K-dimensionalen komplexen Euklidschen Raumes R^K auffassen. Setzen wir (4. 6) in (3. 37) ein, so erhalten wir für den Flächendefekt Δ eine positiv definite hermitesche Form

$$\Delta=\Delta(z_\varkappa,\bar z_\varkappa)=\frac{1}{8}\,\frac{n(ln\pm1)}{\cos\sigma-\cos\delta}\sum_{\varkappa,\varkappa'=1}^{K}c_{\varkappa\varkappa'}z_\varkappa\bar z_{\varkappa'} \tag{4. 7}$$

mit

$$c_{\varkappa\varkappa'}=\int_0^\varepsilon\!\!\int_0^\varepsilon[\cos\delta\sin|(\varkappa-\varkappa')\varepsilon+\alpha-\alpha'|\pm i\sin\delta\sin((\varkappa-\varkappa')\varepsilon+\alpha-\alpha')+$$
$$+\sin(\sigma-|(\varkappa-\varkappa')\varepsilon+\alpha-\alpha'|)]\,d\alpha\,d\alpha'.$$

Für $\varkappa > \varkappa'$ berechnet sich

$$(4.8)\quad c_{\varkappa\varkappa'} = e^{\pm i\delta}\,\mathrm{Im}\, e^{i(\varkappa-\varkappa')\varepsilon}\int_0^\varepsilon\int_0^\varepsilon e^{i(\alpha-\alpha')}\,d\alpha\,d\alpha' + \mathrm{Im}\, e^{i(\sigma-(\varkappa-\varkappa')\varepsilon)}\int_0^\varepsilon\int_0^\varepsilon e^{i(\alpha-\alpha')}\,d\alpha\,d\alpha' =$$

$$= 2(1-\cos\varepsilon)\left[e^{\pm i\delta}\sin(\varkappa-\varkappa')\varepsilon + \sin\left(\sigma-(\varkappa-\varkappa')\varepsilon\right)\right],$$

und für $\varkappa = \varkappa'$

$$(4.9)\quad c_{\varkappa\varkappa} = \cos\delta\int_0^\varepsilon\int_0^\varepsilon \sin|\alpha-\alpha'|\,d\alpha\,d\alpha' + \int_0^\varepsilon\int_0^\varepsilon \sin(\sigma-|\alpha-\alpha'|)\,d\alpha\,d\alpha' =$$

$$= 2\cos\delta\int_0^\varepsilon d\alpha\int_0^\alpha \sin(\alpha-\alpha')\,d\alpha' + 2\int_0^\varepsilon d\alpha\int_0^\alpha \sin(\sigma-\alpha+\alpha')\,d\alpha' =$$

$$= 2\cos\delta\int_0^\varepsilon (1-\cos\alpha)\,d\alpha + 2\int_0^\varepsilon \left(-\cos\sigma+\cos(\sigma-\alpha)\right)d\alpha =$$

$$= 2(1-\cos\varepsilon)\sin\sigma + 2(\sin\varepsilon-\varepsilon)(\cos\sigma-\cos\delta).$$

In Anwendung dieser Formel können wir jetzt auch leicht den Flächeninhalt der in (2. 22) erklärten speziellen Orbiformen $O_n^{ln\pm 1}$ angeben. Wir wählen $K=1$ also $\varepsilon=\sigma$ und haben dann in (4. 6) nur eine Komponente z_1, für welche nach (2. 22) $|z_1| = r^*$ gilt. Nach (3. 31), (4. 7) und (4. 9) folgt dann

$$(4.10)\quad F = \pi r^2 - \frac{1}{4}\,\frac{n(ln\pm 1)r^{*2}}{\cos\sigma-\cos\delta}\left[(\sigma-\sin\sigma)\cos\delta + (\sin\sigma-\sigma\cos\sigma)\right].$$

Die hermitesche Form (4. 7) gestattet noch eine für die weiteren Untersuchungen wichtige Umformung. Da die Koeffizienten $c_{\varkappa\varkappa'}$ nur von $\varkappa-\varkappa'$ abhängen, dürfen wir

$$(4.11)\quad c_{\varkappa\varkappa'} = c_{\varkappa-\varkappa'} = c_\lambda, \quad -(K-1)\leqq\lambda\leqq K-1$$

setzen, und da die $c_{\varkappa\varkappa'}$ eine hermitesche Matrix bilden, folgt

$$(4.12)\quad c_{-\lambda} = \bar{c}_\lambda.$$

Wir können uns also auf die Werte $\lambda = 0, 1, \ldots, K-1$ beschränken, und für diese gilt nach (4. 9) und (4. 8)

$$(4.13)\quad \begin{aligned} c_0 &= 2(1-\cos\varepsilon)\sin\sigma + 2(\sin\varepsilon-\varepsilon)(\cos\sigma-\cos\delta), \\ c_\lambda &= 2(1-\cos\varepsilon)[e^{\pm i\delta}\sin\lambda\varepsilon + \sin(K-\lambda)\varepsilon], \qquad \lambda = 1, \ldots, K-1. \end{aligned}$$

Hieraus ergibt sich für $\lambda = 1, 2, \ldots, K-1$ noch der Zusammenhang

$$(4.14)\quad \bar{c}_{K-\lambda} = e^{\mp i\delta}c_\lambda.$$

Wir ordnen nunmehr die Doppelsumme in (4. 7) nach den Diagonalen um,

$$(4.15)\quad \sum_{\varkappa,\varkappa'=1}^{K} c_{\varkappa\varkappa'} z_\varkappa \bar{z}_{\varkappa'} = c_0\sum_{\lambda=1}^{K}|z_\lambda|^2 + \sum_{\lambda=1}^{K-1} c_\lambda \sum_{\varkappa=1}^{K-\lambda} z_{\varkappa+\lambda}\bar{z}_\varkappa + \sum_{\lambda=1}^{K-1} c_{-\lambda}\sum_{\varkappa=1}^{K-\lambda} z_\varkappa \bar{z}_{\varkappa+\lambda}.$$

Unter Beachtung von (4. 12) und (4. 14) und Umkehrung des Indexdurchlaufes folgt für die letzte Summe

$$\sum_{\lambda=1}^{K-1} c_{-\lambda} \sum_{\varkappa=1}^{K-\lambda} z_\varkappa \bar{z}_{\varkappa+\lambda} = \sum_{\lambda=1}^{K-1} e^{\mp i\delta} c_{K-\lambda} \sum_{\varkappa=1}^{K-\lambda} z_\varkappa \bar{z}_{\varkappa+\lambda} = \sum_{\lambda=1}^{K-1} e^{\mp i\delta} c_\lambda \sum_{\varkappa=1}^{\lambda} z_\varkappa \bar{z}_{\varkappa+K-\lambda} =$$

(4. 16) $$= \sum_{\lambda=1}^{K-1} c_\lambda \sum_{\varkappa=K-\lambda+1}^{K} z_{\varkappa-K+\lambda} \bar{z}_\varkappa e^{\mp i\delta} = \sum_{\lambda=1}^{K-1} c_\lambda \sum_{\varkappa=K-\lambda+1}^{K} z_{\varkappa+\lambda} \bar{z}_\varkappa .$$

Hierbei haben wir noch die Komponenten $z_1, \ldots, z_K$ gemäß

(4. 17) $$z_{\varkappa+K} = z_\varkappa e^{\mp i\delta}$$

über den Index K hinaus foıtgesetzt. Man darf dann auch zyklisch vertauschen

(4. 18) $$\sum_{\varkappa=1}^{K} z_{\varkappa+\lambda} \bar{z}_\varkappa = \sum_{\varkappa=1}^{K} z_{\varkappa+\lambda+1} \bar{z}_{\varkappa+1} .$$

Berücksichtigen wir (4. 16) in (4. 15), so ergibt sich

(4. 19) $$\sum_{\varkappa,\varkappa'=1}^{K} c_{\varkappa\varkappa'} z_\varkappa \bar{z}_{\varkappa'} = c_0 \sum_{\lambda=1}^{K} |z_\lambda|^2 + \sum_{\lambda=1}^{K-1} c_\lambda \sum_{\varkappa=1}^{K} z_{\varkappa+\lambda} \bar{z}_\varkappa .$$

Wir führen nun die reellen Koeffizienten B_λ, $\lambda = 1, \ldots, K-1$ ein,

(4. 20) $$B_\lambda = \frac{e^{\pm i\delta} \bar{c}_\lambda - e^{\mp i\delta} c_\lambda}{e^{\pm i\delta} - e^{\mp i\delta}} = \frac{c_{K-\lambda} - \bar{c}_{K-\lambda}}{e^{\pm i\delta} - e^{\mp i\delta}} = 2(1-\cos\varepsilon) \sin(K-\lambda)\varepsilon .$$

Es gilt dann umgekehrt

(4. 21) $$B_\lambda + e^{\pm i\delta} B_{K-\lambda} = c_\lambda .$$

Setzen wir (4. 21) in (4. 19) ein, so folgt unter Beachtung von (4. 17) und (4 .18)

(4. 22) $$\sum_{\lambda=1}^{K-1} c_\lambda \sum_{\varkappa=1}^{K} z_{\varkappa+\lambda} \bar{z}_\varkappa = \sum_{\lambda=1}^{K-1} B_\lambda \sum_{\varkappa=1}^{K} z_{\varkappa+\lambda} \bar{z}_\varkappa + \sum_{\lambda=1}^{K-1} e^{\pm i\delta} B_{K-\lambda} \sum_{\varkappa=1}^{K} z_{\varkappa+\lambda} \bar{z}_\varkappa =$$

$$= \sum_{\lambda=1}^{K-1} B_\lambda \sum_{\varkappa=1}^{K} z_{\varkappa+\lambda} \bar{z}_\varkappa + \sum_{\lambda=1}^{K-1} B_\lambda \sum_{\varkappa=1}^{K} z_{\varkappa+K-\lambda} \bar{z}_\varkappa e^{\pm i\delta} = \sum_{\lambda=1}^{K-1} B_\lambda \sum_{\varkappa=1}^{K} z_{\varkappa+\lambda} \bar{z}_\varkappa + \sum_{\lambda=1}^{K-1} B_\lambda \sum_{\varkappa=1}^{K} z_\varkappa \bar{z}_{\varkappa+\lambda} ,$$

und wir erhalten zusammen mit (4.19) und (4. 20) für (4. 7) die gewünschte Darstellung

(4. 23) $$\Delta = \frac{1}{8} \frac{n(ln \pm 1)}{\cos\sigma - \cos\delta} \left[c_0 \sum_{\lambda=1}^{K} |z_\lambda|^2 + 4(1-\cos\varepsilon) \sum_{\lambda=1}^{K-1} \sin(K-\lambda)\varepsilon \sum_{\varkappa=1}^{K} \operatorname{Re} z_\varkappa \bar{z}_{\varkappa+\lambda} \right] .$$

Für $K=1$ entfällt der dann sinnlose zweite Summenterm.

5. Das Optimierungsproblem

Wir stellen nun die Aufgabe, die $(ln\pm 1)$-zählig symmetrischen n-Orbiformen kleinsten Flächeninhalts F bzw. größten Flächendefekts Δ zu bestimmen. Nach (3. 6), (1. 11) und (2. 21) würde diese Aufgabe lauten, bestimme

$$(5.1)\quad \operatorname{Max}\Delta = \operatorname{Max}\left\{\frac{\pi}{2}\sum\frac{1}{\nu^2-1}(a_\nu^2+b_\nu^2)\,\middle|\, r+\sum(a_\nu\cos\nu\varphi+b_\nu\sin\nu\varphi)\sim\varrho(\varphi)\geqq 0\right\}.$$

Dabei sind die Summen über alle $\nu = kn-1$ und $\nu = kn+1$, $k=1,2,3,\ldots$ die Vielfache von $ln\pm 1$ sind, zu erstrecken. In dieser Form dürfte die Aufgabe aber kaum lösbar sein, da sich der gemäß $\varrho\geqq 0$ zulässige Bereich der Fourierkoeffizienten nur schwer überblicken läßt. Nach Satz 5 und Darstellung (3. 37) können wir jedoch unter Benutzung von (4. 1) diese Aufgabe als Optimierungsproblem im Hilbertraum der Funktionen $z(\alpha)$ formulieren,

$$(5.2)\qquad \operatorname*{Max}_{z\in\mathfrak{M}}\Delta(z,\bar z).$$

Wegen der Stetigkeit von $\Delta(z,\bar z)$ und der Dichtheit der Menge $\mathfrak{T}$ der Treppenfunktionen in $\mathfrak{M}$ reduziert sich dieses Problem sofort auf

$$(5.3)\qquad \operatorname*{Max}_{z\in\mathfrak{T}}\Delta(z,\bar z).$$

Betrachten wir zunächst die Teilmenge $\mathfrak{T}^K\subset\mathfrak{T}$ der Treppenfunktionen, welche nach (4. 2) bzw. (4. 6) zu einem festen $K=2^m$, $m\geqq 1$, gehören. Diese lassen sich, wie schon bemerkt, als Punkte eines komplexen Euklidschen Raumes R^K auffassen, und

$$(5.4)\qquad \mathfrak{T}^K=\{(z_1,z_2,\ldots,z_K)\,|\,z_\varkappa\in\mathfrak{P}_n\}$$

ist dort ein konvexes Polyeder. Dieses hat entsprechend dem durch (1. 20) definierten regulären n-Eckbereich $\mathfrak{P}_n$ die Eckpunkte

$$(5.5)\qquad (z_1,\ldots,z_K) = (r^*e^{i(\pi+\frac{1}{2}\delta+\nu_1\delta)},\ldots,r^*e^{i(\pi+\frac{1}{2}\delta+\nu_K\delta)}),\quad \nu_\varkappa \text{ ganz.}$$

Als Teilproblem von (5. 3) ergibt sich dann das quadratische Optimierungsproblem

$$(5.6)\qquad \operatorname*{Max}_{z\in\mathfrak{T}^K}\Delta(z_\varkappa,\bar z_\varkappa)$$

mit der positiv definiten hermiteschen Form (4. 7). Damit liegt hier gerade ein Gegenstück zu der bei der quadratischen Programmierung behandelten Problemstellung vor, bei welcher bekanntlich das Minimum einer positiv definiten quadratischen Form in einem konvexen Bereich gesucht wird. Dabei ist jedes lokale Minimum zugleich globales Minimum. Zu seiner Bestimmung können also lokale Methoden herangezogen werden. Hier bei unserem Maximumproblem sind die Verhältnisse in gewissem Sinne gerade umgekehrt; ein lokales Maximum, also auch das globale Maximum, kann nur in einem Eckpunkt des konvexen Polyeders $\mathfrak{T}^K$ angenommen werden, was unmittelbar aus der Konvexität der positiv definiten hermiteschen Form folgt, aber nicht jedes lokale Maximum braucht das globale Maximum zu sein. Damit »vereinfacht« sich einmal die Suche nach dem Maximum auf die

Inspektion der endlich vielen Eckpunkte, zum anderen sind aber keine lokalen Methoden mehr anwendbar. Überdies ist die genannte Reduktion auf ein endliches Problem in unserem Fall nur scheinbar, da wir (5. 6) für allgemeines K behandeln müssen. Hierfür dürfte keine allgemeine Methode zur Verfügung stehen. Wir werden deshalb zur Lösung von (5. 6) in spezieller Weise vorgehen, und zwar werden wir versuchen, $\Delta(z_\varkappa, \bar{z}_\varkappa)$ auf den Eckpunkten von $\mathfrak{T}^K$ scharf nach oben abzuschätzen, so daß die erhaltene obere Schranke das gesuchte Maximum wird.

Zu diesem Zweck bilden wir Δ nach (4. 23) an den Eckpunkten (5. 5),

$$(5.7)\qquad \Delta = \frac{1}{8}\frac{n(ln\pm 1)r^{*2}}{\cos\sigma-\cos\delta}\left[c_0K+4(1-\cos\varepsilon)\sum_{\lambda=1}^{K-1}\sin(K-\lambda)\varepsilon\sum_{\varkappa=1}^{K}\cos(v_{\varkappa+\delta}-v_\varkappa)\delta\right].$$

Hierbei sind die ganzzahligen $v_1, \ldots, v_K$ gemäß (4. 17) fortzusetzen,

$$(5.8)\qquad v_{\varkappa+K} = v_\varkappa \mp 1.$$

Wir transformieren auf die ganzzahligen Variablen

$$(5.9)\qquad p_\varkappa = v_{\varkappa+1} - v_\varkappa,$$

welche sich nach (5. 8) gemäß

$$(5.10)\qquad p_{\varkappa+K} = p_\varkappa$$

fortsetzen und der Nebenbedingung

$$(5.11)\qquad p_1+p_2+\ldots+p_K = \mp 1$$

genügen. Damit erhalten wir für die inneren Summen in (5. 7)

$$(5.12)\qquad \begin{aligned}
\Phi_1 &= \sum_{\varkappa=1}^{K}\cos(v_{\varkappa+1}-v_\varkappa)\delta = \sum_{\varkappa=1}^{K}\cos p_\varkappa\delta\\
\Phi_2 &= \sum_{\varkappa=1}^{K}\cos(v_{\varkappa+2}-v_\varkappa)\delta = \sum_{\varkappa=1}^{K}\cos(p_\varkappa+p_{\varkappa+1})\delta\\
\Phi_3 &= \sum_{\varkappa=1}^{K}\cos(v_{\varkappa+3}-v_\varkappa)\delta = \sum_{\varkappa=1}^{K}\cos(p_\varkappa+p_{\varkappa+1}+p_{\varkappa+2})\delta\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\Phi_{K-1} &= \sum_{\varkappa=1}^{K}\cos(v_{\varkappa+K-1}-v_\varkappa)\delta = \sum_{\varkappa=1}^{K}\cos(p_\varkappa+p_{\varkappa+1}+p_{\varkappa+2}+\ldots+p_{\varkappa+K-2})\delta.
\end{aligned}$$

Der Flächendefekt Δ ist dann über alle ganzzahligen $p_\varkappa$ mit den Bedingungen (5. 10) und (5. 11) abzuschätzen. Wir führen noch die Bezeichnung

$$(5.13)\qquad \Psi_\lambda = \begin{cases}\Phi_1+\Phi_3+\Phi_5+\ldots+\Phi_\lambda & \text{für } \lambda \text{ ungerade}\\ \Phi_2+\Phi_4+\Phi_6+\ldots+\Phi_\lambda & \text{für } \lambda \text{ gerade}\end{cases}$$

ein und formen mittels der Summenformeln

$$(5.14)\qquad \begin{aligned}
\sin(2m+1)x &= 2\sin x\left[\tfrac{1}{2}+\cos 2x+\cos 4x+\ldots+\cos 2mx\right],\\
\sin 2mx &= 2\sin x\left[\cos x+\cos 3x+\ldots+\cos(2m-1)x\right]
\end{aligned}$$

die Darstellung (5. 7) noch etwas um,

$$\sum_{\lambda=1}^{K-1} \sin (K-\lambda)\varepsilon \sum_{\varkappa=1}^{K} \cos (\nu_{\varkappa+\lambda}-\nu_\varkappa)\delta =$$

$$= \sum_{\tau=1}^{\frac{1}{2}K-1} \sin (K-2\tau)\varepsilon\cdot \Phi_{2\tau} + \sum_{\tau=1}^{\frac{1}{2}K} \sin (K-2\tau+1)\varepsilon\cdot \Phi_{2\tau-1} =$$

$$(5.15)\qquad = 2\sin\varepsilon \sum_{\tau=1}^{\frac{1}{2}K-1} [\cos\varepsilon+\cos 3\varepsilon+\ldots+\cos (K-2\tau-1)\varepsilon]\, \Phi_{2\tau} +$$

$$+2\sin\varepsilon \sum_{\tau=1}^{\frac{1}{2}K} [\tfrac{1}{2}+\cos 2\varepsilon+\cos 4\varepsilon+\ldots+\cos (K-2\tau)\varepsilon]\, \Phi_{2\tau-1} =$$

$$= 2\sin\varepsilon\, [\Psi_1 \cos (K-2)\varepsilon+\Psi_2\cos (K-3)\varepsilon+\ldots+\Psi_{K-2}\cos\varepsilon+\tfrac{1}{2}\Psi_{K-1}].$$

Damit erhalten wir für den Flächendefekt

$$(5.16)\quad \Delta = \frac{n(ln\pm 1)r^{*2}}{\cos\sigma-\cos\delta}\left[\frac{1}{8}c_0K+(1-\cos\varepsilon)\sin\varepsilon\left(\sum_{\lambda=1}^{K-2}\Psi_\lambda\cos (K-\lambda-1)\varepsilon+\frac{1}{2}\Psi_{K-1}\right)\right].$$

Wir werden nun die Ψ_λ einzeln nach oben abschätzen. Dazu benötigen wir folgendes

Lemma. *Für ganzzahlige* $x_1, x_2, \ldots, x_N$, $N\geqq 2$ *und* $\delta=\dfrac{2\pi}{n}$ *gilt*

$$(5.17)\qquad \operatorname*{Max}_{x_1+x_2+\ldots+x_N=\mp 1} \sum_{i=1}^{N} \cos x_i\delta = N-1+\cos\delta.$$

Der Beweis ist fast trivial. Die Summanden können nur die Werte 1, $\cos\delta$, $\cos 2\delta, \ldots, \cos\frac{N}{2}\delta$ bzw. $\cos\frac{N-1}{2}\delta$ annehmen. Ohne Berücksichtigung der Nebenbedingung ist der größte Wert der Summe dann N, der zweitgrößte $N-1+\cos\delta$. Dieser Wert wird aber für $x_1=\ldots=x_{N-1}=0$, $x_N=\mp 1$ angenommen, stellt also das Maximum unter der Nebenbedingung dar. Wir zeigen zunächst für die ungeraden λ folgende

Abschätzung 1. *Für* $\lambda=1, 3, 5, \ldots, K-1$; $K=2^m$, *gilt unter den Bedingungen* (5. 10) *und* (5. 11) *stets*

$$(5.18)\qquad \Psi_\lambda(p_1,\ldots,p_K) \leqq \frac{1}{2}(\lambda+1)K-\frac{1}{4}(\lambda+1)^2(1-\cos\delta).$$

Beweis. Für $\lambda=1$ ist nach unserem Lemma

$$(5.19)\quad \Psi_1=\Phi_1=\sum_{\varkappa=1}^{K}\cos p_\varkappa\delta \leqq \operatorname*{Max}_{p_1+\ldots+p_K=\mp 1}\sum_{\varkappa=1}^{K}\cos p_\varkappa\delta = K-(1-\cos\delta),$$

also ist die Abschätzung (5. 18) für $\lambda=1$, bzw. für »alle« ungeraden λ mit $\lambda<2^1$ richtig. Als Induktionsvoraussetzung sei nun die Abschätzung (5. 18) für alle ungeraden λ mit $\lambda < 2^\mu$, $1 \leqq \mu < m$ richtig. Dann folgt für die ungeraden λ mit $2^\mu < \lambda < 2^{\mu+1} \leqq 2^m = K$ zunächst die Umformung

$$(5.20)\qquad \Psi_\lambda = \Phi_1 + \Phi_3 + \Phi_5 + \ldots + \Phi_\lambda =$$

$$= (\Phi_\lambda + \Phi_{2^{\mu+1}-\lambda}) + (\Phi_{\lambda-2} + \Phi_{2^{\mu+1}-\lambda+2}) + \ldots + (\Phi_{2^\mu+1} + \Phi_{2^\mu-1}) + \Psi_{2^{\mu+1}-\lambda-2},$$

wobei das letzte Glied für $\lambda = 2^{\mu+1}-1$ nicht existiert und durch Null zu ersetzen ist. Nach Induktionsvoraussetzung erhalten wir dann für dieses in jedem Fall die Abschätzung

$$(5.21)\qquad \Psi_{2^{\mu+1}-\lambda-2} \leqq \frac{1}{2}(2^{\mu+1}-\lambda-1)K - \frac{1}{4}(2^{\mu+1}-\lambda-1)^2(1-\cos\delta).$$

Die $\frac{1}{2}(\lambda - 2^\mu + 1)$ Klammersummanden in (5. 20) sind alle vom gleichen Typ

$$(5.22)\qquad \begin{aligned}(\Phi_\tau + \Phi_{2^{\mu+1}-\tau}) &= \sum_{\varkappa=1}^{K} \cos(p_\varkappa + p_{\varkappa+1} + \ldots + p_{\varkappa+\tau-1})\delta + \\ &+ \sum_{\varkappa=1}^{K} \cos(p_\varkappa + p_{\varkappa+1} + \ldots + p_{\varkappa+2^{\mu+1}-\tau-1})\delta\end{aligned}$$

für $\tau = \lambda, \lambda-2, \ldots, 2^\mu+1$ und lassen sich einzeln abschätzen. Nach (5. 10) kann man Summen dieser Art mit beliebigem ganzem s zyklisch umrechnen,

$$(5.23)\qquad \sum_{\varkappa=1}^{K} \cos(p_\varkappa + p_{\varkappa+1} + \ldots + p_{\varkappa+T})\delta = \sum_{\varkappa=1}^{K} \cos(p_{\varkappa+s} + p_{\varkappa+s+1} + \ldots + p_{\varkappa+s+T})\delta.$$

Verschieben wir auf diese Weise die zweite Summe in (5. 22) um $s = \tau$ so folgt

$$(5.24)\qquad \begin{aligned}\Phi_\tau + \Phi_{2^{\mu+1}-\tau} &= \sum_{\varkappa=1}^{K} [\cos(p_\varkappa + p_{\varkappa+1} + \ldots + p_{\varkappa+\tau-1})\delta + \\ &+ \cos(p_{\varkappa+\tau} + p_{\varkappa+\tau+1} + \ldots + p_{\varkappa+2^{\mu+1}-1})\delta].\end{aligned}$$

Im Falle $\mu+1 < m$ rechnen wir weiterhin (5.24) mit $s = 2^{\mu+1}, 2\cdot 2^{\mu+1}, 3\cdot 2^{\mu+1}, \ldots,$ $\ldots, (2^{m-\mu-1}-1)\cdot 2^{\mu+1} = K - 2^{\mu+1}$ um,

$$(5.25)\qquad \Phi_\tau + \Phi_{2^{\mu+1}-\tau} =$$

$$= \sum_{\varkappa=1}^{K} [\cos(p_{\varkappa+2^{\mu+1}} + \ldots + p_{\varkappa+2^{\mu+1}+\tau-1})\delta + \cos(p_{\varkappa+2^{\mu+1}+\tau} + \ldots + p_{\varkappa+2\cdot 2^{\mu+1}-1})\delta] =$$

$$= \sum_{\varkappa=1}^{K} [\cos(p_{\varkappa+2\cdot 2^{\mu+1}} + \ldots + p_{\varkappa+2\cdot 2^{\mu+1}+\tau-1})\delta + \cos(p_{\varkappa+2\cdot 2^{\mu+1}+\tau} + \ldots + p_{\varkappa+3\cdot 2^{\mu+1}-1})\delta]$$

. . . .

$$= \sum_{\varkappa=1}^{K} [\cos(p_{\varkappa+K-2^{\mu+1}} + \ldots + p_{\varkappa+K-2^{\mu+1}+\tau-1})\delta + \cos(p_{\varkappa+K-2^{\mu+1}+\tau} + \ldots + p_{\varkappa+K-1})\delta],$$

und erhalten durch Addition von (5. 24) und (5. 25) in jedem Falle

(5. 26)

$$2^{m-\mu-1}(\Phi_\tau+\Phi_{2^{\mu+1}-\tau})=\sum_{\varkappa=1}^{K}\{\cos(p_\varkappa+\dots+p_{\varkappa+\tau-1})\delta+\cos(p_{\varkappa+\tau}+\dots+p_{\varkappa+2^{\mu+1}-1})\delta+$$

$$+\cos(p_{\varkappa+2^{\mu+1}}+\dots+p_{\varkappa+2^{\mu+1}+\tau-1})\delta+\cos(p_{\varkappa+2^{\mu+1}+\tau}+\dots+p_{\varkappa+2\cdot 2^{\mu+1}-1})\delta+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\cos(p_{\varkappa+K-2^{\mu+1}}+\dots+p_{\varkappa+K-2^{\mu+1}+\tau-1})\delta+\cos(p_{\varkappa+K-2^{\mu+1}+\tau}+\dots+p_{\varkappa+K-1})\delta\}.$$

Setzen wir für festes $\varkappa$

(5. 27)
$$\begin{aligned} p_\varkappa+\dots+p_{\varkappa+\tau-1} &= q_1,\\ p_{\varkappa+\tau}+\dots+p_{\varkappa+2^{\mu+1}-1} &= q_1',\\ p_{\varkappa+2^{\mu+1}}+\dots+p_{\varkappa+2^{\mu+1}+\tau-1} &= q_2,\\ p_{\varkappa+2^{\mu+1}+\tau}+\dots+p_{\varkappa+2\cdot 2^{\mu+1}-1} &= q_2',\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ p_{\varkappa+K-2^{\mu+1}}+\dots+p_{\varkappa+K-2^{\mu+1}+\tau-1} &= q_{2^{m-\mu-1}},\\ p_{\varkappa+K-2^{\mu+1}+\tau}+\dots+p_{\varkappa+K-1} &= q'_{2^{m-\mu-1}}, \end{aligned}$$

wobei dann wegen (5. 10) und (5. 11) für die q gilt

(5. 28)
$$q_1+q_1'+q_2+q_2'+\dots+q_{2^{m-\mu-1}}+q'_{2^{m-\mu-1}}=$$
$$=p_\varkappa+p_{\varkappa+1}+\dots+p_{\varkappa+K-1}=p_1+p_2+\dots+p_K=\mp 1,$$

so ergibt sich für jede geschweifte Klammer von (5.26) nach Lemma (5.17) die Abschätzung

(5. 29)
$$\{\dots\}=\cos q_1\delta+\cos q_1'\delta+\cos q_2\delta+\cos q_2'\delta+\dots$$
$$+\cos q_{2^{m-\mu-1}}\delta+\cos q'_{2^{m-\mu-1}}\delta\leqq 2^{m-\mu}-1+\cos\delta.$$

Damit folgt für (5.26) selbst

(5. 30)
$$2^{m-\mu-1}(\Phi_\tau+\Phi_{2^{\mu+1}-\tau})\leqq K(2^{m-\mu}-1+\cos\delta).$$

Setzen wir nun (5. 30) und (5. 21) in (5. 20) ein, so erhalten wir die Abschätzung

(5. 31)
$$\Psi_\lambda\leqq\frac{1}{2}(\lambda-2^\mu+1)2^{\mu+1}(2^{m-\mu}-1+\cos\delta)+\frac{1}{2}(2^{\mu+1}-\lambda-1)K-$$
$$-\frac{1}{4}(2^{\mu+1}-\lambda-1)^2(1-\cos\delta)=\frac{1}{2}(\lambda+1)K-\frac{1}{4}(\lambda+1)^2(1-\cos\delta),$$

also genau (5. 18) für $2^\mu<\lambda<2^{\mu+1}$. Damit ist nach vollständiger Induktion die Abschätzung (5. 18) für alle ungeraden $\lambda<K$ und beliebiges $K=2^m$ richtig.

Wir müssen nun noch die Ψ_λ mit geradem λ abschätzen. Wir führen dazu die ganzzahligen Variablen

(5.32) $$p_i' = p_{2i-1}+p_{2i}, \qquad p_i'' = p_{2i}+p_{2i+1}$$

ein, für welche nach (5. 10) und (5. 11) die Bedingungen

(5.33) $$p_{i+\frac{1}{2}K}' = p_i', \qquad p_{i+\frac{1}{2}K}'' = p_i''$$

und

(5.34) $$p_1'+p_2'+\ldots+p_{\frac{1}{2}K}' = \mp 1, \qquad p_1''+p_2''+\ldots+p_{\frac{1}{2}K}'' = \mp 1$$

gelten, also die gleichen Bedingungen wie für p_λ nur statt für K für $\frac{1}{2}K$ genommen. Damit können wir umrechnen

(5.35) $$\Phi_{2j} = \sum_{\varkappa=1}^{K} \cos(p_\varkappa+p_{\varkappa+1}+\ldots+p_{\varkappa+2j-1})\delta =$$

$$= \sum_{i=1}^{\frac{1}{2}K} \cos(p_{2i-1}+p_{2i}+\ldots+p_{2(i+j-1)})\delta + \sum_{i=1}^{\frac{1}{2}K} \cos(p_{2i}+p_{2i+1}+\ldots+p_{2(i+j-1)+1})\delta =$$

$$= \sum_{i=1}^{\frac{1}{2}K} \cos(p_i'+p_{i+1}'+\ldots+p_{i+j-1}')\delta + \sum_{i=1}^{\frac{1}{2}K} \cos(p_i''+p_{i+1}''+\ldots+p_{i+j-1}'')\delta = \Phi_j'+\Phi_j'',$$

wobei die Φ_j' und Φ_j'' die gleiche Bedeutung bezüglich $\frac{1}{2}K$ wie Φ_λ bezüglich K haben. Wenn man diese Zerlegung in (5. 13) einsetzt, so folgt

(5.36) $$\Psi_{2j} = \Psi_j'+\Psi_j''+\Psi_{j-1}'+\Psi_{j-1}'',$$

wobei die Ψ_j' und Ψ_j'' analog zu (5.13) in den Φ_j' und Φ_j'' erklärt sind und die gleiche Bedeutung bezüglich $\frac{1}{2}K$ haben wie die Ψ_λ bezüglich K. Für $j=1$ fallen Ψ_{j-1}' und Ψ_{j-1}'' weg. Wir zeigen nun die

ABSCHÄTZUNG 2. *Für* $2j=2, 4, \ldots K-2$ *gilt unter den Bedingungen* (5. 10) *und* (5. 11) *stets*

(5. 37) $$\Psi_{2j}(p_1, \ldots, p_K) \leqq jK-(1+j)j(1-\cos\delta).$$

BEWEIS. Wir führen jetzt einen Induktionsbeweis über $K=2^2, 2^3, 2^4, \ldots$. Für $K=2^2$ tritt nur Ψ_2 auf. Für dieses gilt nach (5. 36) $\Psi_2=\Psi_1'+\Psi_1''$, was man nach (5. 18), genommen für $\frac{1}{2}K$ abschätzen kann durch

(5. 38) $$\Psi_2 = \Psi_1'+\Psi_1'' \leqq 2[2-(1-\cos\delta)].$$

Also ist für $K=2^2$ und »alle« j (5. 37) richtig. Sei nun als Induktionsvoraussetzung die Abschätzung (5. 37) für $K=2^m$ richtig, dann folgt für $K=2^{m+1}$ nach (5. 36)

(5.39) $$\Psi_2 = \Psi_1'+\Psi_1'',$$

(5.39′) $$\Psi_{2j} = \Psi_j'+\Psi_j''+\Psi_{j-1}'+\Psi_{j-1}'' \quad \text{für} \quad j = 2, 3, \ldots, \tfrac{1}{2}K-1.$$

Schätzen wir (5. 39) wieder nach (5. 18), genommen für $\frac{1}{2}K$ ab,

(5.40) $$\Psi_2 = \Psi_1'+\Psi_1'' \leqq 2[\tfrac{1}{2}K-(1-\cos\delta)].$$

Zur Abschätzung von (5. 39′) wenden wir im Falle j ungerade (5. 18) auf Ψ_j' und

Ψ_j'' an und nach Induktionsvoraussetzung (5. 37) auf Ψ_{j-1}' und Ψ_{j-1}'' jeweils genommen für $\frac{1}{2}K$,

$$\Psi_{2j} \leqq 2\left[\frac{1}{2}(j+1)\frac{1}{2}K-\frac{1}{4}(j+1)^2(1-\cos\delta)\right]+ \tag{5.41}$$
$$+2\left[\frac{j-1}{2}\cdot\frac{K}{2}-\left(1+\frac{j-1}{2}\right)\frac{j-1}{2}(1-\cos\delta)\right]=jK-(j+1)j(1-\cos\delta).$$

Im Falle j gerade verfahren wir umgekehrt,

$$\Psi_{2j} \leqq 2\left[\frac{j}{2}\cdot\frac{K}{2}-\left(1+\frac{j}{2}\right)\frac{j}{2}(1-\cos\delta)\right]+2\left[\frac{j}{2}\cdot\frac{K}{2}-\frac{1}{4}j^2(1-\cos\delta)\right]= \tag{5.42}$$
$$=jK-(j+1)j(1-\cos\delta).$$

Damit ist für $K=2^{m+1}$ und alle j die Abschätzung 2 richtig, also nach vollständiger Induktion für jedes $K=2^2, 2^3, \dots$.

Um die gesuchte Abschätzung für Δ zu erhalten, müssen wir nun (5. 18) und (5. 37) in (5. 16) einsetzen. Berechnen wir zunächst die dabei auftretenden Summen,

$$(1-\cos\varepsilon)\sin\varepsilon\left\{\sum_{\lambda=1}^{K-2}\Psi_\lambda\cos(K-\lambda-1)\varepsilon+\frac{1}{2}\Psi_{K-1}\right\}\leqq \tag{5.43}$$
$$\leqq\frac{1}{2}(1-\cos\varepsilon)\,2\sin\varepsilon\left\{\sum_{j=1}^{\frac{1}{2}K-1}[jK-j^2(1-\cos\delta)]\cos(K-2j)\varepsilon+\right.$$
$$\left.+\sum_{j=1}^{\frac{1}{2}K-1}[jK-(1+j)j(1-\cos\delta)]\cos(K-2j-1)\varepsilon+\frac{1}{4}K^2-\frac{1}{8}K^2(1-\cos\delta)\right\}.$$

Wenden wir hierauf die Zerlegung $2\sin\alpha\cos\beta=\sin(\beta+\alpha)-\sin(\beta-\alpha)$ an und verschieben die Summationsindizes geeignet, so geht die rechte Seite von (5. 43) über in

$$\frac{1}{2}(1-\cos\varepsilon)\left\{\sum_{j=1}^{\frac{1}{2}K-1}[jK-j^2(1-\cos\delta)]\sin(K-2j+1)\varepsilon-\right. \tag{5.44}$$
$$-\sum_{j=1}^{\frac{1}{2}K}[(j-1)K-(j-1)^2(1-\cos\delta)]\sin(K-2j+1)\varepsilon+$$
$$+\sum_{j=1}^{\frac{1}{2}K}[jK-(1+j)j(1-\cos\delta)]\sin(K-2j)\varepsilon-\sum_{j=1}^{\frac{1}{2}K}[(j-1)K-j(j-1)\times$$
$$\left.\times(1-\cos\delta)]\sin(K-2j)\varepsilon+2\sin\varepsilon\left[\frac{1}{4}K^2-\frac{1}{8}K^2(1-\cos\delta)\right]\right\}=$$
$$=\frac{1}{2}(1-\cos\varepsilon)\left\{\sum_{j=1}^{\frac{1}{2}K}[K-(2j-1)(1-\cos\delta)]\sin(K-2j+1)\varepsilon+\right.$$
$$\left.+\sum_{j=1}^{\frac{1}{2}K}[K-2j(1-\cos\delta)]\sin(K-2j)\varepsilon\right\}=$$
$$=\frac{1}{4}(2-2\cos\varepsilon)\left\{\sum_{\lambda=1}^{K-1}[K-\lambda(1-\cos\delta)]\sin(K-\lambda)\varepsilon\right\}.$$

Wir zerlegen nochmals entsprechend $2\cos\alpha\sin\beta=\sin(\beta+\alpha)+\sin(\beta-\alpha)$ und erhalten bei geeigneter Verschiebung der Summationsindizes aus (5. 44)

$$\frac{1}{4}\left\{\sum_{\lambda=1}^{K-1}[K-\lambda(1-\cos\delta)]\,2\sin(K-\lambda)\varepsilon-\sum_{\lambda=0}^{K-2}[K-(\lambda+1)(1-\cos\delta)]\sin(K-\lambda)\varepsilon-\right.$$

$$\left.-\sum_{\lambda=2}^{K}[K-(\lambda-1)(1-\cos\delta)]\sin(K-\lambda)\varepsilon\right\}= \tag{5.45}$$

$$=\frac{1}{4}\{K\sin(K-1)\varepsilon+[K-K(1-\cos\delta)]\sin\varepsilon-[K-(1-\cos\delta)]\sin K\varepsilon\}.$$

Gehen wir hiermit in (5. 16) ein und berücksichtigen noch $c_0=c_{\varkappa\varkappa}$ entsprechend (4. 9), so ergibt sich schließlich für $\Delta(z_\varkappa, \bar{z}_\varkappa)$ auf den Eckpunkten von $\mathfrak{T}^K$ und damit auf ganz $\mathfrak{T}^K$ die Abschätzung

$$\Delta \leqq \frac{1}{4}\frac{n(ln\pm1)r^{*2}}{\cos\sigma-\cos\delta}[(\sigma-\sin\sigma)\cos\delta+\sin\sigma-\sigma\cos\sigma]. \tag{5.46}$$

Diese obere Schranke ist jetzt völlig unabhängig von K, sie gilt also für alle $\mathfrak{T}^K$, $K=2, 4, 8, 16, \ldots$, also auf ganz $\mathfrak{T}$. Damit ist aber wegen unseres Approximationssatzes die Abschätzung (5. 46) für alle $z\in\mathfrak{M}$ gültig. Nun nimmt Δ nach (4. 10) für die speziellen $(ln\pm1)$-zählig symmetrischen n-Orbiformen $O_n^{ln\pm1}$ von Kap. 2 aber gerade diesen Schrankenwert an, also lösen die $O_n^{ln\pm1}$ unser Maximumproblem (5. 2), sie sind die $(ln\pm1)$-zählig symmetrischen n-Orbiformen mit dem kleinsten Flächeninhalt.

Betrachten wir nach (4. 10) den Flächeninhalt der $O_n^{ln\pm1}$ für festes n in Abhängigkeit von l bzw. von $\sigma=\dfrac{\delta}{ln\pm1}$ für $0<\sigma\leqq\dfrac{\delta}{n-1}=\dfrac{2\pi}{n(n-1)}\leqq\dfrac{\pi}{3}$,

$$F=\pi r^2-\frac{1}{2}\frac{\pi r^{*2}}{\cos\sigma-\cos\delta}\left[\left(1-\frac{\sin\sigma}{\sigma}\right)\cos\delta+\left(\frac{\sin\sigma}{\sigma}-\cos\sigma\right)\right]. \tag{5.47}$$

Da hier $\dfrac{1}{\cos\sigma-\cos\delta}$, $\left(1-\dfrac{\sin\sigma}{\sigma}\right)$ und $\left(\dfrac{\sin\sigma}{\sigma}-\cos\sigma\right)$ mit σ monoton wachsen, nimmt F für $\sigma=\dfrac{\delta}{n-1}$ den kleinsten Wert an. Unter den $O_n^{ln\pm1}$ hat also für festes n die Orbiform O_n^{n-1} den kleinsten Flächeninhalt. Da nun bei einer konvexen Linearkombination von zwei n-Orbiformen der Wert von Δ wegen der Konvexität nicht größer, der Flächeninhalt also nicht kleiner werden kann, erhalten wir folgenden Sachverhalt:

Für jedes feste $n\geqq3$ besitzt O_n^{n-1} in der abgeschlossenen konvexen Hülle aller $(ln\pm1)$-zählig symmetrischen n-Orbiformen, $l=1, 2, \ldots,$ den kleinsten Flächeninhalt.

In diesem Sinne sind wir damit dem Problem der n-Orbiform mit dem kleinsten Flächeninhalt nähergekommen, wobei aber die Frage noch offen bleibt, inwieweit diese abgeschlossene Hülle die Menge aller n-Orbiformen ausschöpft.

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ON THE DISTRIBUTION OF ARITHMETICAL FUNCTIONS

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§ 1.

1. 1. In this paper we give some generalization of a theorem of ERDŐS—WINTNER concerning the distribution of additive number-theoretical functions [1].

1. 2. Let $f(n); f_1(n), \dots, f_s(n)$ denote real-valued additive number-theoretical functions. Let $F(n)$; $F_1(n), \dots, F_s(n)$ be positive integer-valued polynomials with integer coefficients. Suppose that they are not divisible by the square of any irreducible polynomial. Suppose that $F_i(n)$, $F_j(n)$ are relatively primes for $i \neq j$. Let v_j denote the degree of $F_j(n)$. Suppose, that $F(x) \neq x$, $F_i(x) \neq x$.

1. 3. Let $\varrho_i(d)$ denote the number of incongruent solutions of the congruence $F_i(n) \equiv 0 \pmod d$; further let $\lambda_i(d)$ denote the number of those solutions for which $(n, d) = 1$.

Let $\varrho(d_1, \dots, d_s)$ be the number of solutions of the congruence system

$$F_i(n) \equiv 0 \pmod{d_i}, \qquad i = 1, \dots, s$$

and $\lambda(d_1, \dots, d_s)$ the number of those solutions, for which $(n, d) = 1$.

1. 4. In what follows the letters $p, q, p_1, p_2, \dots; q_1, q_2, \dots$ stand for primes.

1. 5. Let $g(n)$, $g_j(n)$, $g(n; K)$ denote complex-valued multiplicative functions' and let $h(n)$, $h_j(n)$; $h(n; K)$ denote their Moebius-transform, i.e. for example

$$h(n) = \sum_{d \mid n} \mu(d) g\left(\frac{n}{d}\right).$$

1. 6. For a $K > 0$ let

$$f(n; K) = \sum_{\substack{p^\alpha \| n \\ p^\alpha \leqq K}} f(p^\alpha); \qquad g(n; K) = \prod_{\substack{p^\alpha \| n \\ p^\alpha \leqq K}} g(p^\alpha).$$

1. 7. Let $N_1(F; x, y_1, y_2, \alpha)$ denote the number of solutions of the congruences

$$F(n) \equiv 0 \pmod{p^\alpha}$$

where n runs over the natural numbers not exceeding x, and p over the primes in the interval $y_1 \leqq p \leqq y_2$.

Similarly, let $N_2(F; x, y_1, y_2, \alpha)$ denote the number of solutions of the congruences

$$F(q) \equiv 0 \pmod{p^\alpha}$$

where p, q run over the primes in the intervals $1 < q \leqq x$, $y_1 \leqq p \leqq y_2$.

1. 8. Let

$$f^+(p^\alpha) = \begin{cases} f(p^\alpha), & \text{when } |f(p^\alpha)| \leqq 1, \\ 1, & \text{when } |f(p^\alpha)| > 1. \end{cases}$$

1. 9. We introduce the following abbreviations:

$A(f)$: $\sum_p \frac{f^+(p)}{p}$ converges,

$A^+(f_i(F_i))$: $\sum_p \frac{f^+(p)\varrho_i(p)}{p}$ converges,

$a(g)$: $\sum_p \frac{(g(p)-1)}{p}$ converges,

$a(g_i(F_i))$: $\sum_p \frac{(g_i(p)-1)\varrho_i(p)}{p}$ converges,

$B(f)$: $\sum_p \frac{f^{+2}(p)}{p} < \infty$

$B(f_i(F_i))$: $\sum_p \frac{f_i^{+2}(p)\varrho_i(p)}{p} < \infty$

$\sum A(f_i(F_i))$: $\sum_p \sum_{i=1}^{s} \frac{f_i^+(p)\varrho_i(p)}{p}$ converges,

$\sum a(g_i(F_i))$: $\sum_p \sum_{i=1}^{s} \frac{(g_i(p)-1)\varrho_i(p)}{p}$ converges,

$b(g)$: $\sum_p \frac{|g(p)-1|^2}{p} < \infty$,

$b(g_i(F_i))$: $\sum_p \frac{|g_i(p)-1|^2\varrho_i(p)}{p} < \infty$,

$C(f)$: $\sum_{f(p)\neq 0} \frac{1}{p} = \infty$,

$C(f_i(F_i))$: $\sum_{f_i(p)\varrho_i(p)\neq 0} \frac{1}{p} = \infty$.

1. 10. Let

$$a_j(n) = \sum_{p<n} \frac{f_j^+(p)}{p}; \qquad a_j'(n) = \sum_{p<n} \frac{f_j^+(p)}{p-1};$$

$$\tilde{a}_j(n) = \sum_{p<n} \frac{f_j^+(p)\varrho_j(p)}{p}; \qquad \tilde{a}_j'(n) = \sum_{p<n} \frac{f_j^+(p)\varrho_j(p)}{p-1}.$$

1.11. Let

$$H(n) = \prod_{i=1}^{s} g_i(F_i(n)); \qquad H(n;K) = \prod_{i=1}^{s} g_i(F_i(n);K);$$

$$U(x) = \sum_{n\leqq x} H(n); \qquad V(x) = \sum_{p\leqq x} H(p);$$

$$U(x;K) = \sum_{n\leqq x} H(n;K); \qquad V(x;K) = \sum_{p\leqq x} H(p;K).$$

1.12. Let D_γ denote the set of those s-tuples $\{d_1, \ldots, d_s\}$ of natural numbers, for which all the primfactors of d_i do not exceed γ. γ will be so chosen that $\varrho(d_1, \ldots, d_s)=0$, if $\{d_1, \ldots, d_s\} \notin D_\gamma$ and $(\prod_{p>\gamma} p, \prod_{i\neq j}(d_i, d_j))>1$. The existence of such γ follows from Lemma 1.

1.13. Let $N(\ \ldots\ )$ denote the number of those numbers, which satisfy the conditions stated in the brackets.

1.14. We shall say, that the s-tuples $\{t_1(n), \ldots, t_s(n)\}$ of arithmetical functions have a distribution, if

$$\frac{N(n \leqq x; t_1(n) < c_1, \ldots, t_s(n) < c_s)}{x}$$

tends to an s-dimensional distribution function $G(c_1, \ldots, c_s)$ as $x\to\infty$; for all continuity points of it.

Similarly, we say, that $\{t_1(n), \ldots, t_s(n)\}$ have a distribution for prime-values, if

$$\frac{N(p \leqq x; t_1(p) < c_1, \ldots, t_s(p) < c_s)}{\operatorname{li} x}$$

tends to a distribution function $G'(c_1, \ldots, c_s)$ in similar meaning.

Let $\varphi_G = \varphi_G(u_1, \ldots, u_s)$ denote the characteristic function of the distribution function G.

§ 2. Results

THEOREM 1. *Let $F_j(n)$ $(j=1, \ldots, s)$ be linear polynomials, satisfying the conditions stated in* 1. 2.

Let $g_j(n)$ $(j=1, \ldots, s)$ be such multiplicative functions for which $|g_j(n)|\leqq 1$ $(n=1, 2, \ldots, j=1, \ldots, s)$ hold, and the series

(1. 1) $$\sum_p \frac{1}{p} \sum_{j=1}^{s} (g_j(p)-1)$$

converges. Then

(1. 2) $$\frac{1}{x}\sum_{n\leqq x} H(n) \to M(g),$$

(1. 3) $$\frac{1}{\operatorname{li} x}\sum_{p\leqq x} H(p) \to N(g),$$

as $x\to\infty$. $M(g)$ *and* $N(g)$ *are defined as follows:*

$$M(g) = M_1(g)N_1(g); \qquad N(g) = N_1(g)N_2(g), \tag{1.4)—(1.5}$$

$$M_1(g) = \sum_{\{d_1,\dots,d_s\}\in D_\gamma} \frac{h_1(d_1)\dots h_s(d_s)}{[d_1,\dots,d_s]}\varrho(d_1,\dots,d_s) \tag{1.6}$$

$$N_1(g) = \sum_{\{d_1,\dots,d_s\}\in D_\gamma} \frac{h_1(d_1)\dots h_s(d_s)}{\varphi([d_1,\dots,d_s])}\lambda(d_1,\dots,d_s); \tag{1.7}$$

$$M_2(g) = \prod_{p>\gamma} m(p); \qquad N_2(g) = \prod_{p>\gamma} n(p), \tag{1.8)—(1.9}$$

$$m(p) = 1+\sum_{\alpha=1}^{\infty}\frac{1}{p^\alpha}\sum_{i=1}^{s}(g_i(p^\alpha)-1) \tag{1.10}$$

$$n(p) = 1+\sum_{\alpha=1}^{\infty}\frac{1}{p^{\alpha-1}(p-1)}\sum_{i=1}^{s}(g_i(p^\alpha)-1). \tag{1.11}$$

The sums (1. 6)—(1. 7) *and the products* (1. 8)—(1. 9) *are convergent. The convergence is uniform in the variables* $g_i(n)$ *if the convergence of* (1. 1) *is uniform.*

THEOREM 2. *Let* $F_j(n)$ $(j=1,\dots,s)$ *be linear polynomials, satisfying the conditions stated in* 1. 2. *Let* $f(n)$ *be additive arithmetical function for which the condition* $B(f)$ *holds* (*see* 1. 9).

Then the distributions of the $(s-1)$*-tuples*

$$\{f(F_1(n))-f(F_2(n)),\dots,f(F_{s-1}(n))-f(F_s(n))\}, \tag{2.1}$$

$$\{f(F_1(p))-f(F_2(p)),\dots,f(F_{s-1}(p))-f(F_s(p))\} \tag{2.2}$$

exist.

If $C(f)$ *holds, then the distribution functions are continuous. The characteristic functions are* $M(g)$, $N(g)$, *respectively, putting into* (1. 6)—(1. 11) *the functions*

$$g_1(n) = e^{iu_1f(n)}, \quad g_l(n) = e^{i(u_l-u_{l-1})f(n)} \quad (l=2,\dots,s-1), \quad g_s(n) = e^{-iu_{s-1}f(n)}.$$

THEOREM 3. *Let* $F_j(n)$ $(j=1,\dots,s)$ *be linear polynomials satisfying the conditions stated in* 1. 2. *Let* $f_j(n)$ $(j=1,\dots,s)$ *be additive functions for which the conditions* $A(f_j)$, $B(f_j)$ $(j=1,\dots,s)$ *are satisfied* (1. 9). *Then the distribution functions of the s-tuples*

$$\{f_1(F_1(n)),\dots,f_s(F_s(n))\}; \quad \{f_1(F_1(p)),\dots,f_s(F_s(p))\} \tag{3.1)—(3.2}$$

exist.

If $C(f_j)$ *hold for every* $j=1,\dots,s$, *then the distribution functions are continuous.*

The characteristic functions are $M(g)$ *and* $N(g)$, *resp., putting into* (1. 6)—(1. 11) *the functions*

$$g_j(n) = e^{iu_jf_j(n)} \qquad (j=1,\dots,s).$$

THEOREM 4. *Let* $F_j(n)$ $(j=1,\dots,s)$ *be linear polynomials, satisfying the conditions stated in* 1. 2. *Let* $f_j(n)$ $(j=1,\dots,s)$ *be additive functions for which* $B(f_j)$ $(j=1,\dots,s)$ *hold* (1. 9).

Then the distribution functions of s-tuples

$$\{f_1(F_1(n))-a_1(n), \dots, f_s(F_s(n))-a_s(n)\},$$
$$\{f_1(F_1(p))-a_1'(p), \dots, f_s(F_s(p))-a_s'(p)\}$$

exist, (see 1. 10). *If $C(f_j)$ hold for $j=1, \dots, s$, then the distribution functions are continuous.*

Our results for polynomials with degree greater than one are less complete.

THEOREM 5. *Let $F_j(n)$ $(j=1, \dots, s)$ be such polynomials as in* 1. 2. *Let $g_j(n)$ be multiplicative functions for which $|g_j(n)|\leqq 1$ and $a(g_j(F_j))$ $(j=1, \dots, s)$ hold.*
Supposing that

$$(g_j(p^\alpha)-1)\varrho_j(p^\alpha)\to 0 \quad as \quad p\to\infty, \tag{5. 1}$$

for $\alpha=1$, when $v_j\geqq 2$ and for $\alpha=1, \dots, v_j-2$, when $v_j\geqq 3$, we have

$$\frac{1}{x}\sum_{n\leqq x} H(n)\to M(g), \quad as \quad x\to\infty. \tag{5. 2}$$

Supposing, additionally, that

$$(g_j(p^\alpha)-1)\varrho_j(p^\alpha)\to 0 \tag{5. 3}$$

for $\alpha=v_j-1$, when $v_j\geqq 2$, we have

$$\frac{1}{\operatorname{li} x}\sum_{p\leqq x} H(p)\to N(g). \tag{5. 4}$$

$M(g)$ and $N(g)$ are defined as follows.
The definition of $M_1(g)$, $N_1(g)$ are formally the same as under (1. 6)—(1. 7).

(5. 5)—(5. 6) $$M(g)=M_1(g)M_2(g); \qquad N(g)=N_1(g)N_2(g).$$

(5. 7)—(5. 8) $$M_2(g)=\prod_{p>\gamma}\tilde{m}(p), \qquad N_2(g)=\prod_{p>\gamma}\tilde{n}(p),$$

$$\tilde{m}(p)=1+\sum_{\alpha=1}^{\infty}\frac{1}{p^\alpha}\sum_{i=1}^{s}(g_i(p^\alpha)-1)\varrho_i(p^\alpha), \tag{5. 9}$$

$$\tilde{n}(p)=1+\sum_{\alpha=1}^{\infty}\frac{1}{p^{\alpha-1}(p-1)}\sum_{i=1}^{s}(g_i(p^\alpha)-1)\varrho_i(p^\alpha). \tag{5. 10}$$

The sums (1. 6)—(1. 7) *and the products* (5. 7)—(5. 8) *are convergent. The convergence is uniform in the variables g_j when the convergence of* (5. 1) *and the convergence under $a(g_j(F_j))$ is uniform.*

THEOREM 6. *Let $F_j(n)$ be such polynomials as in* 1. 2. *Let $f_j(n)$ be additive arithmetical functions for which the conditions $B(f_j(F_j))$ hold. We apply the notions defined in* 1. 10. *Supposing, that*

$$f_j(p^\alpha)\varrho_j(p^\alpha)\to 0, \tag{6. 1}$$

for $\alpha=1$, *when* $\nu_j \geqq 2$, *and for* $\alpha=1, \ldots, \nu_j-2$, *when* $\nu_j \geqq 3$ *the distribution of the* $(s-1)$*-tuples*

$$\{f_1(F_1(n))-f_2(F_2(n))-\tilde{a}_1(n)+\tilde{a}_2(n), \ldots, f_{s-1}(F_{s-1}(n))-f_s(F_s(n))-\tilde{a}_{s-1}(n)+\tilde{a}_s(n)\}$$

exists.

Supposing, additionally, that (6. 1) *holds for* $\alpha=\nu_j-1$, *when* $\nu_j \geqq 2$, *the distribution of the* $(s-1)$*-tuples*

$$\{f_1(F_1(p))-f_2(F_2(p))-\tilde{a}_1'(p)+\tilde{a}_2'(p), \ldots, f_{s-1}(F_{s-1}(p))-f_s(F_s(p))-\tilde{a}_{s-1}'(p)+\tilde{a}_s'(p)\}$$

exists.

If $C(f_i(F_i))$ *hold for* $i=1, \ldots, s$, *then the distribution functions are continuous.*

THEOREM 7. *Suppose, that the conditions stated in Theorem* 6 *are satisfied. Then the distribution functions of the s-tuples*

$$\{f_1(F_1(n))-\tilde{a}_1(n), \ldots, f_s(F_s(n))-\tilde{a}_s(n)\}, \quad \{f_1(F_1(p))-\tilde{a}_1'(p), \ldots, f_s(F_s(p))-\tilde{a}_s'(p)\}$$

exist. If $C(f_j(F_j))$ *hold, then the distribution functions are continuous.*

THEOREM 8. *Suppose, that the conditions stated in Theorem* 6 *and in addition the conditions* $A(f_i(F_i))$ *are satisfied. Then the distribution function of the s-tuples*

$$\{f_1(F_1(n)), \ldots f_s(F_s(n))\}, \{f_1(F_1(p)), \ldots, f_s(F_s(p))\}$$

exist. If $C(f_j(F_j))$ *hold for* $j=1, \ldots, s$, *then the distribution functions are continuous.*

The characteristic functions are $M(g)$, $N(g)$ *(see Theorem* 5) *putting into* (5. 5)—(5. 10) *the functions*

$$g_j(n) = e^{iu_j f_j(n)} \qquad (j=1, \ldots, s).$$

THEOREM 9. *Let* $f_1(n), \ldots, f_s(n)$ *be integer valued additive functions,* $F_1(n), \ldots, F_s(n)$ *be polynomials satisfying the conditions stated in* 1. 2.

Let us suppose, that

$$f_j(p^\alpha)\,\varrho_j(p^\alpha)=0$$

for $\alpha=1$, *and for* $\alpha=1, \ldots, \nu_j-2$, *when* $\nu_j \geqq 3$.

Then for integer values $k_1, \ldots, k_s$

$$\lim_{x\to\infty} \frac{1}{x} N(n \leqq x; f_1(F_1(n))=k_1, \ldots, f_s(F_s(n))=k_s) = \tau(k_1, \ldots, k_s),$$

where $\tau(k_1, \ldots, k_s)$ *can be determined by the relation*

$$\sum_{k_1, \ldots, k_s=-\infty}^{\infty} \tau(k_1, \ldots, k_s) e^{i(u_1k_1+\ldots+u_sk_s)} = M(g).$$

Supposing additionally, that

$$\varrho_j(p^\alpha) f_j(p^\alpha)=0$$

for $\alpha = \nu_j - 1$, *we have*

$$\lim_{x\to\infty} \frac{1}{\operatorname{li} x} N(p \leqq x, f_1(F_1(p)) = k_1, \ldots, f_s(F_s(p)) = k_s) = \varkappa(k_1, \ldots, k_s),$$

and

$$\sum_{k_1,\ldots,k_s=-\infty}^{\infty} \varkappa(k_1, \ldots, k_s) e^{i(u_1k_1+\ldots+u_sk_s)} = N(g).$$

The definition of $M(g)$ *and* $N(g)$ *see in Theorem* 5, *by* $g_j(n) = e^{iu_j f_j(n)}$.

THEOREM 10. *Supposing, that the conditions stated for* $f_j(n)$; $F_j(n)$ *in Theorem* 9 *hold, we have for integer* $k_1, \ldots, k_s$ *the existence of the limits*

$$\lim_{x\to\infty} \frac{1}{x} N(n \leqq x; f_1(F_1(n)) - f_2(F_2(n)) = k_1, \ldots, f_{s-1}(F_{s-1}(n)) - f_s(F_s(n)) = k_{s-1}) = \\ = \eta(k_1, \ldots, k_{s-1}),$$

$$\lim_{x\to\infty} \frac{1}{\operatorname{li} x} N(p \leqq x; f_1(F_1(p)) - f_2(F_2(p)) = k_1, \ldots, f_{s-1}(F_{s-1}(p) - f_s(F_s(p)) = k_{s-1}) = \\ = \eta'(k_1, \ldots, k_{s-1}),$$

where η, η' *can be determined by the relations*

$$\sum_{k_1,\ldots,k_{s-1}=-\infty}^{\infty} \eta(k_1, \ldots, k_{s-1}) e^{i(u_1k_1+\ldots+u_{s-1}k_{s-1})} = M(g),$$

$$\sum_{k_1,\ldots,k_{s-1}=-\infty}^{\infty} \eta'(k_1, \ldots, k_{s-1}) e^{i(u_1k_1+\ldots+u_{s-1}k_{s-1})} = N(g),$$

where $M(g)$, $N(g)$ *are defined as in Theorem* 5 *choosing*

$$g_1(n) = e^{iu_1 f_1(n)}, \qquad g_l(n) = e^{i(u_l - u_{l-1}) f_l(n)} \quad (l = 2, \ldots, s-1), \qquad g_s(n) = e^{-iu_{s-1} f_s(n)}.$$

§ 3. Remarks

1. The relation (1. 2) in the one-dimensional case was proved by H. DELANGE [2]. I proved the relation (1. 3) for this case in [3] applying the large sieve (see Lemma 4), and deduced hence the Theorem 3.

2. The existence of the distribution of (3. 1) in the one-dimensional case was proved by P. ERDŐS and A. WINTNER in [1].

3. The existence of the distribution of (2. 1) and (3. 1) for $F_j(n) = n + j$ was proved by A. SCHINZEL and P. ERDŐS in [4]. Here it was stated without proof, that there exists a distribution for

$$\left\{ \frac{\varphi(p+1)}{p+1}, \ldots, \frac{\varphi(p+s)}{p+s} \right\}.$$

4. Theorem 9 for $s = 1$, $F(n) = n$, $f(p^\alpha) = \alpha - 1$ was investigated at first by A. RÉNYI [5]. Later J. KUBILIUS [6], LEVIN and FAJNLEIB [11] gave some generalizations and improvements of it.

5. The basic lemmas in our investigations are Lemma 4 and Lemma 6. Probably the relations

$$N_1(F; x, y_1, \infty, 2) = o(x), \qquad N_2(F; x, y_1, \infty, 2) = o\left(\frac{x}{\log x}\right)$$

for $y_1(x) \to \infty$ hold, for every polynomial which is a product of mutually different irreducible polynomials. If these relations were true, then the conditions (5. 3), (6.3) would be superfluous in the Theorems 5—10 for $\alpha \geqq 2$.

6. The method used in this paper can be applied to prove assertions for the accuracy of convergence to the limit distributions.

§ 4. Lemmas

LEMMA 1. *If $F_1(m)$ and $F_2(m)$ are relative prime polynomials with integer coefficients, then the congruences*

$$F_1(m) \equiv 0 \pmod a, \quad F_2(m) \equiv 0 \pmod a$$

have common roots at most for finitely many a-s. (See [7].*)*

LEMMA 2. *Let $F(m)$ be arbitrary primitive polynomial of degree v with integer coefficients, and with discriminant D. Let $D \neq 0$. Then the number of solutions of the congruence $F(m) \equiv 0 \pmod{p^\alpha}$ is $\varrho(p)$, when $p \nmid D$, and smaller than vD^2 when $p|D$. Further*

$$\varrho(ab) = \varrho(a)\varrho(b), \quad \text{when} \quad (a, b) = 1$$

and $\varrho(p^\alpha) \leqq c$, where c depends only on F.

Let $\pi(x, k, l)$ denote the number of primes not exceeding x in the arithmetical progression $\equiv l \pmod k$. Let further

$$\pi(x, k) = \max_{(l,k)=1} \pi(x, k, l).$$

LEMMA 3 (BRUN—TITCHMARSH [8]).

$$\pi(x, k) < C_\delta \frac{x}{\log x}$$

uniformly for every $k \leqq x^{1-\delta}$. $\delta > 0$ constant.

LEMMA 4 (BOMBIERI [12]).

$$\sum_{D \leqq Y} \max_{(l, D)=1} \left| \pi(x, D, l) - \frac{\operatorname{li} x}{\varphi(D)} \right| \ll \frac{x}{(\log x)^A},$$

where $Y = x^{\frac{1}{2}}(\log x)^{-B}$, $B \geqq 2A + 25$, A is a sufficiently large constant.

Let $N_k(x, a, b)$ denote the number of solutions of $ap + b = kq$, in primes $p, q \colon p \leqq x$.

LEMMA 5.

$$N_k(x, a, b) < c\frac{x}{\varphi(k)\log^2\frac{x}{k}},$$

where c is a constant depending only on a and b. (*See* [8]).

LEMMA 6. *For a polynomial $F(n)$ of degree $\nu \geqq 2$, satisfying the conditions stated in* 1. 2, *we have the relations:*

(1) $$N_1(F; x, y_1, \infty, \nu-1) = o(x),$$

(2) $$N_2(F; x, y_1, \infty, \nu) = o\left(\frac{x}{\log x}\right),$$

when y_1 tends to infinity as $x \to \infty$.

The proof of (4. 2) goes with a simple argument of the eratosthenian sieve as follows. It is enough to prove the relation for irreducible polynomial only because of Lemma 1. The number of solutions of $F(q) \equiv 0 \pmod{p^\nu}$, $q \leqq x$; $y_1 < p$ is smaller than

$$\sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3$$

where

$$\sum\nolimits_1 = \sum_{y_1 < p < x^{1/2}} \sum_{\substack{F(q)\equiv 0(p^2)\\ q \leqq x}} 1 \ll \frac{x}{\log x} \sum_{y_1 < p < x^{1/2-\varepsilon}} \frac{\varrho(p^2)}{p^2} + \sum_{x^{1/2-\varepsilon} < p < x^{1/2}} \sum_{\substack{F(q)\equiv 0(p)\\ q \leqq x}} 1 \ll$$

$$\ll o(\operatorname{li} x) + \varepsilon \operatorname{li} x = o(\operatorname{li} x),$$

$$\sum\nolimits_2 = \sum_{x^{1/2} < p < \frac{x}{\log\log x}} \sum_{F(q)\equiv 0(p^\nu)} 1 \ll \pi\left(\frac{x}{\log\log x}\right) = o\left(\frac{x}{\log x}\right).$$

$$\sum\nolimits_3 \ll \sum_{u < c(\log\log x)^\nu} \sum_{\substack{F(q) = up^k\\ q \leqq x}} 1.$$

Since the $F(q)$'s in the sum have no primdivisors in the interval $[(\log\log x)^\nu, \log x]$, so by eratosthenian sieve we obtain that $\sum_3 = o\left(\frac{x}{\log x}\right)$, and hence (2) follows.

The relation (4. 1) was recently proved by C. HOOLEY [9].

LEMMA 7. *Let $\tilde{f}(n)$ be an additive number-theoretical function defined for prime powers as follows:*

$$\tilde{f}(p^\alpha) = \begin{cases} f^+(p), & \textit{when} \quad u < p < v, \\ 0 & \textit{otherwise.} \end{cases}$$

Let $F(n)$ be an integer valued polynomial having no square-divisor. Then

$$\sum_{n \leqq x}\left\{\tilde{f}(F(n)) - \sum_q \frac{\tilde{f}(q)\varrho(q)}{q}\right\}^2 \ll x\sum_q \frac{\tilde{f}^2(q)\varrho(q)}{q} + o(x),$$

when $v < cx$, *and*

$$\sum_{p\leq x}\left\{\tilde{f}(F(q))-\sum_{q}\frac{\tilde{f}(q)\varrho(q)}{q-1}\right\}^2 \ll \operatorname{li} x\sum_{q}\frac{\tilde{f}^2(q)\varrho(q)}{q}+o(\operatorname{li} x),$$

when $v<x^{1/5}$, *say.*

The proof of these inequalities goes by the method of P. TURÁN, using additionally the *Bombieri*'s theorem in the proof of the later.

LEMMA 8. *Let* $g_j(n)$ *be multiplicative functions for which* $|g_j(n)|\leq 1$, *and* $F_j(n)$ *be polynomials satisfying the conditions stated in* 1. 2. *Let* $K_1=\frac{1}{5s}\log x$. *Then the following relations hold:*

$$\frac{U(x, K_1)}{x}=\sum_{\{d_1,\dots,d_s\}\in D_\gamma}\frac{h_1(d_1;K_1)\dots h_s(d_s;K_1)\varrho(d_1,\dots,d_s)}{[d_1,\dots,d_s]}\cdot \tag{3}$$

$$\cdot\prod_{p>\gamma}\left\{1+\sum_{\alpha=1}^{\infty}\frac{1}{p^\alpha}\sum_{i=1}^{s}h_i(p^\alpha;K_1)\varrho_i(p^\alpha)\right\}+o(1),$$

$$\frac{V(x, K_1)}{\operatorname{li} x}=\sum_{\{d_1,\dots,d_s\}\in D_\gamma}\frac{h_1(d_1;K_1)\dots h_s(d_s;K_1)\lambda(d_1,\dots,d_s)}{\varphi([d_1,\dots,d_s])}\cdot \tag{4}$$

$$\cdot\prod_{p>\gamma}\left\{1+\sum_{\alpha=1}^{\infty}\frac{1}{p^{\alpha-1}(p-1)}\sum_{i=1}^{s}h_i(p^\alpha;K_1)\lambda_i(p^\alpha)\right\}+o(1).$$

PROOF.

$$U(x,K_1)=\sum_{n\leq x}H(n;K_1)=\sum_{n\leq x}\prod_{j=1}^{s}\left\{\sum_{d_j|F_j(n)}h_j(d_j;K_1)\right\}=$$

$$=\sum_{d_1,\dots,d_s}h_1(d_1;K_1)\dots h_s(d_s,K_1)\sum_{\substack{F_i(n)\equiv 0(d_i)\\ n\leq x\\ i=1,\dots,s}}1=$$

$$=x\sum_{\{d_1,\dots,d_s\}}h_1(d_1;K_1)\dots h_s(d_s;K_1)\frac{\varrho(d_1,\dots,d_s)}{[d_1,\dots,d_s]}+R=x\sum\nolimits_1+R$$

and

$$|R|\ll\sum\tau(d_1)\dots\tau(d_s)\,\varrho_1(d_1)\dots\varrho_s(d_s)\ll\{\sum\tau^c(d)\}^s,$$

where the sums are extended over those d, d_i all primdivisor of which are smaller than K_1. Since $d\leq\prod_{p^\alpha\leq K_1}p^\alpha\ll x^{\frac{1}{5s}}$, so $|R|\ll\{x^{\frac{1}{5s}}\log^{c_1}x\}^s\ll x^{1/4}$.

Further using Lemma 1 we have that $\sum_1$ equals to the main term on the right hand side of (3), and so the relation (3) holds.

The proof of (4) is similar, using Lemma 4.

LEMMA 9. *Let* $F_j(n)$ *be such polynomials as in* 1. 2. *Suppose, that*

$$|g_j(p)|\leq 1,\quad \textit{if}\quad \varrho_j(p)\neq 0,\quad \textit{for}\quad j=1,\dots,s\quad \textit{and for every } p, \tag{5}$$

and that the series

$$\sum_p \frac{1}{p} \sum_{j=1}^{s} (g_j(p)-1)\varrho_j(p) \tag{6}$$

converges.

Then for every positive constant c

$$\sum_{|\arg g_j(p)|>c} \frac{\varrho_j(p)}{p} < \infty \qquad (j=1, \ldots, s), \tag{7}$$

further

$$\sum_p \frac{|g_j(p)-1|^2 \varrho_j(p)}{p} < \infty, \qquad (j=1, \ldots, s). \tag{8}$$

From (8) *it follows*

$$\sum_{x^{1/2}<p<x} \frac{|g_j(p)-1|\varrho_j(p)}{p} = o(1) \quad as \quad x \to \infty. \tag{9}$$

PROOF. Using the Cauchy-inequality and the boundedness of $\varrho_j(p)$ we have that (9) follows from (8).

Let $|g_j(p)| = r_j(p)$, $\arg g_j(p) = \vartheta_j(p)$, where $|\vartheta_j(p)| \leqq \pi$.

From (6) it follows, that the series

$$\sum_p \frac{(1-\operatorname{Re} g_j(p))\varrho_j(p)}{p}$$

are absolutely convergent, whence (7) immediately follows. Further

$$\sum_p \frac{|g_j(p)-1|^2 \varrho_j^2(p)}{p} \leqq 2 \sum_p \frac{|\operatorname{Re}(1-g_j(p))|^2 \varrho_j^2(p)}{p} +$$

$$+2 \sum_p \frac{|\operatorname{Im} g_j(p)|^2 \varrho_j^2(p)}{p} = 2\sum\nolimits_1 + 2\sum\nolimits_2 .$$

$$\sum\nolimits_1 \ll \sum_{\substack{p \\ |\vartheta_j(p)| \geqq \frac{1}{2}}} \frac{\varrho_j(p)}{p} + \sum_{|\vartheta_j(p)|<\frac{1}{2}} \frac{\operatorname{Re}(1-g_j(p))\varrho_j(p)}{p} < \infty .$$

Finally we prove, that $\sum_2$ converges, and hence (8) follows.

$$\sum\nolimits_2 = \sum_p \frac{r_j^2(p)\sin^2 \vartheta_j(p) \varrho_j^2(p)}{p} \leqq \sum_{|\vartheta_j(p)| \leqq \frac{1}{8}} + \sum_{|\vartheta_j(p)| > \frac{1}{8}} = \sum\nolimits_3 + o(1).$$

Using that for $|\vartheta_j(p)| \leqq \frac{1}{8}$

$$r_j^2(p)\sin^2 \vartheta_j(p) \leqq r_j^2(p) \sin^2 \frac{1}{2} \vartheta_j(p) \leqq 1 - r_j(p) \cos \vartheta_j(p)$$

holds we have

$$\sum_3 \ll \sum_p \frac{\operatorname{Re}(1-g_j(p))\varrho_j(p)}{p} \qquad (<\infty).$$

LEMMA 10. *Let $F_j(n)$ be polynomials as in* 1.2. *Let $f_j(n)$ be additive functions for which $B(f_i(F_i))$ hold.* (*See* 1.9.) *Let*

$$g_j(n) = e^{iu_j f_j(n)} \qquad (j = 1, \dots, s)$$

where $u_1, \dots, u_s$ are arbitrary real numbers.

We use the notations of 1.10. *Let for the sake of brevity*

$$\psi_j = \tilde{a}_j(K_2) - \tilde{a}_j(K_1); \qquad \psi'_j = \tilde{a}'_j(K_2) - \tilde{a}'_j(K_1);$$

$$\tau(y) = \tau(y, u_1, \dots, u_s) = \sum_{j=1}^{s} u_j \tilde{a}_j(y);$$

$$\tau'(y) = \tau'(y, u_1, \dots, u_s) = \sum_{j=1}^{s} u_j \tilde{a}'_j(y).$$

The following relations hold:

$$|U(x, K_2)e^{-i\tau(K_2)} - U(x, K_1)e^{-i\tau(K_1)}| = o(x), \tag{10}$$

when $K_1 < K_2 < cx$, further

$$|V(x, K_3)e^{-i\tau'(K_3)} - V(x, K_1)e^{-i\tau'(K_1)}| = o(\operatorname{li} x), \tag{11}$$

when $K_3 = x^{1/5}$.

The inequalities (4. 10), (4. 11) *are satisfied uniformly in* $\max |u_j| < c$.

PROOF. We prove the relation (11), since the proof of (10) is very similar. It is evident, that

$$|V(x, K_3)e^{-i\tau'(K_3)} - V(x, K_1)e^{-i\tau'(K_1)}| \leqq \sum_{j=1}^{s} S_j,$$

where

$$S_j = \sum_{p \leqq x} |g_j(F_j(p); K_3) - e^{i\psi_j u_j} g_j(F_j(p); K_1)|.$$

We shall prove, that $S_j = o(\operatorname{li} x)$. For the sake of simplicity we omit the indices.

Using the inequality $|1 - e^{iu}| \leqq \min(2, |u|)$ we have

$$S \leqq R = \sum_{p \leqq x} \min\left(2, u|f(F(p); K_3) - f(F(p); K_1) - \psi'|\right).$$

Now we subdivide the p's into three classes as follows:

$p \in \mathscr{A}$, if there exist q, $q | F(p)$, such that $K_1 < q < K_3$, $|f(q)| \geqq 1$.
$p \in \mathscr{B}$, if there exist q, $q^2 | F(p)$, such that $K_1 < q \leqq K_3$.
$p \in \mathscr{C}$, if $p \notin \mathscr{A} \cup \mathscr{B}$.

Then

$$R = \sum\nolimits_{\mathscr{A}} + \sum\nolimits_{\mathscr{B}} + \sum\nolimits_{\mathscr{C}}$$

where $\sum_{\mathscr{A}}, \ldots$ are defined by this partition. From Lemma 3 we have

$$\sum_{\mathscr{A}} < \sum_{\substack{K_1<q<K_3 \\ |f(q)|\geqq 1}} \pi(x, q)\varrho(q) \ll \operatorname{li} x \sum_{\substack{K_1<q \\ |f(q)|\geqq 1}} \frac{\varrho(q)}{q} = o(\operatorname{li} x) \quad \text{for} \quad K_1 \to \infty.$$

Similarly, $\sum_{\mathscr{B}} = o(\operatorname{li} x)$.

If $p \in \mathscr{C}$, then

$$f(F(p); K_3) - f(F(p); K_1) = \sum_{\substack{q|F(p) \\ K_1<q<K_3}} f^+(q)$$

using Lemma 7 and choosing $u=K_1$, $v=K_3$. Hence by the cited Lemma we have

$$\sum_{\mathscr{C}} \leqq \sum_{p\leqq x} |\tilde{f}(F(p)) - \psi'| \leqq (\operatorname{li} x)^{1/2} \Big\{\sum_{p\leqq x} |\tilde{f}(F(p)) - \psi'|^2\Big\}^{1/2} \ll$$

$$\ll \operatorname{li} x \sum_{K_1<q<K_3} \frac{f^{+2}(q)\varrho(q)}{q} = o(\operatorname{li} x)$$

and so the inequality (11) holds. The uniformity of (11) is evident.

LEMMA 11. *Let now $F_1(n), \ldots, F_s(n)$ be linear polynomials. Then by the conditions and notations of Lemma* 10 *we have*

$$(12 \qquad |V(x)e^{-i\tau'(x)} - V(x, K_3)e^{-i\tau'(K_3)}| = o(\operatorname{li} x).$$

PROOF. Let $K_4 = x^{1-\delta}$, $\delta > 0$. At first we prove, that

$$L \overset{\text{def}}{=} |V(x, K_4)e^{-i\tau'(K_4)} - V(x, K_3)e^{-i\tau'(K_3)}| < C_\delta \cdot o(\operatorname{li} x),$$

where the constant in the o term does not depend on δ. As in Lemma 10 we have $L \leqq \sum_{j=1}^{s} L_j$, where

$$L_j = \sum_{p\leqq x} |\exp\{iu_j\{f_j(F_j(p); K_4) - f_j(F_j(p); K_3) - a_j'(K_4) + a_j'(K_3)\} - 1| \ll$$

$$\ll \sum_{\substack{K_3<q<K_4 \\ |f_j(q)|>1}} \varrho_j(q)\pi(x, q) + \sum_{K_3<q<K_4} \sum_{\substack{F_j(p)\equiv 0(q^2) \\ p\leqq x}} 1 +$$

$$+ \sum_{p\leqq x} |f_j^+(F_j(p); K_4) - f_j^+(F_j(p); K_3)| + |a_j'(K_4) - a_j'(K_3)| \operatorname{li} x =$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + o(\operatorname{li} x).$$

Using Lemma 3 we have

$$\sum\nolimits_1 \ll C_\delta \operatorname{li} x \sum_{|f(q)|>1} \frac{1}{q} < C_\delta o(\operatorname{li} x).$$

From Lemma 6 $\sum_2 = o(\mathrm{li}\, x)$ follows. Further

$$\sum_3 \ll \sum_{K_3<q<K_4} \frac{|f_j^+(q)|\varrho_j(q)}{q}\pi(x,q) \ll C_\delta \,\mathrm{li}\, x \sum_{K_3<q<K_4} \frac{f_j^+(q)\varrho_j(q)}{q} \ll$$

$$\ll C_\delta \,\mathrm{li}\, x \left(\sum_{K_3<q<K_4} \frac{1}{q}\right)^{1/2} \left(\sum_{K_3<q<K_4} \frac{f_j^+(q)\varrho_j(q)}{q}\right)^{1/2} \ll C_\delta \,\mathrm{li}\, x \cdot o(1),$$

by the conditions $B(f_j(F_j))$. Hence $L = C_\delta o(\mathrm{li}\, x)$. Now we prove, that

$$|V(x)e^{-i\tau'(x)} - V(x, K_4)e^{-i\tau(K_4)}| = O(\delta \,\mathrm{li}\, x) + o(\mathrm{li}\, x).$$

Since $|V(x)| \ll \mathrm{li}\, x$ and

$$|\tau(x) - \tau(K_4)| \leqq \sum_{j=1}^{s} |u_j| \left| \sum_{K_4<q<x} \frac{f_j^+(q)\varrho_j(q)}{q} \right| \ll$$

$$\ll \max|u_j| \sum_{j=1}^{s} \left\{ \sum_{K_4<q<x} \frac{f_j^{+2}(q)\varrho_j(q)}{q} \right\}^{1/2} = o(1),$$

so it is enough to prove, that

$$T = |V(x) - V(x, K_4)| = O(\delta \,\mathrm{li}\, x).$$

We have $T \leqq \sum_{j=1}^{s} T_j$, where T_j denotes the number of those p, $p \leqq x$, for which there exists a primedivisor q of $F_j(p)$ in the interval $x^{1-\delta} \leqq q < cx$.

Let $F_j(p) = a_j p + b_j$. So T_j is smaller than the total number of the equalities

$$a_j p + b_j = kq, \qquad 1 \leqq p \leqq x$$

in prime numbers p, q, and in $1 \leqq k \leqq cx^\delta$. Using Lemma 5 we have

$$T_j \leqq \sum_{k<cx^\delta} N_k(x, a_j, b_j) < c\frac{x}{\log^2 x} \sum_{k<x^\delta} \frac{1}{\varphi(k)} \ll \delta \frac{x}{\log x}.$$

Hence $T = O(\delta \log x)$ follows. If δ tends now to zero sufficiently slowly, we obtain (12).

LEMMA 12. *Let $g_j(n)$ be multiplicative functions for which $|g_j(n)| \leqq 1$ and $\sum a(g_i(F_i))$ holds. Let $F_j(n)$ be polynomials as in* 1. 2. *Then*

(13) $$|U(x, x) - U(x, K_1)| = o(x),$$

(14) $$|V(x, x^{1-\delta}) - V(x, K_1)| = o(C_\delta \,\mathrm{li}\, x).$$

If $F_j(n)$ are linear polynomials, then

(15) $$|V(x) - V(x, K_1)| = o(\mathrm{li}\, x).$$

PROOF. The proof of (13)—(15) is very similar to the proofs of the assertions stated in Lemmas 10—11. We shall prove (14) only.

Using Lemma 6 the number of those p's for which there exist q, $q^2|F_j(q)$, $K_1 < q < x$ is at most $o(\mathrm{li}\, x)$.

It is enough to prove, that

$$E_1 \stackrel{\text{def}}{=} \sum_{p\le x} |H_1(p)-1| = o(\operatorname{li} x), \qquad \sum\nolimits_2 = \sum_{p\le x} |H_2(p)-1| = o(c_\delta \operatorname{li} x),$$

where

$$H_1(p) = \prod_{j=1}^{s} \prod_{\substack{q|F_j(p)\\ K_1<q<x^{1/5}}} g_j(q); \qquad H_2(p) = \prod_{j=1}^{s} \prod_{\substack{q|F_j(p)\\ x^{1/5}<q<x^{1-\delta}}} g_j(q).$$

Let us now subdivide the p's. p belongs to $\mathfrak{A}_1$, if there exists q, in the interval $K_1<q<x^{1-\delta}$, satisfying $q|F_j(p)$, and $|\arg g_j(q)|\ge\frac{\pi}{2}$ for at least one j. Let $\mathfrak{A}_2$ be the set of the remaining primes.

We have

$$E_{11} \stackrel{\text{def}}{=} \sum_{\substack{p\le x\\ p\in\mathfrak{A}_1}} |H_1(p)-1| \ll \sum_{j=1}^{s} \sum_{|\arg g_j(q)|\ge\frac{\pi}{2}} \varrho_j(q)\pi(x,q) \ll C_\delta \operatorname{li} x \sum_j \sum_{\substack{|\arg g_j(q)|>\frac{\pi}{2}\\ K_1<q}} \frac{1}{q} =$$

$$= o(C_\delta \operatorname{li} x),$$

by the inequality (7) in Lemma 9.

If $|z|\le 1$, $|\arg z|\le\frac{\pi}{2}$, then $\log(1+z)=z+O(|z|^2)$, and $\exp(z+O(|z|^2))=$ $=1+z+O(|z|^2)$. Using these relations we have for $p\in\mathfrak{A}_2$:

$$H_2(p)-1 = \sum_{j=1}^{s} \sum_{\substack{q|F_j(p)\\ K_1<q<x^{1/5}}} \{h_j(q)+o(|h_j(q)|^2)\}.$$

Hence

$$E_{12} \stackrel{\text{def}}{=} \sum_{p\in\mathfrak{A}_2} |H_1(p)-1| \le E_{1,2,1}+O(E_{1,2,2}),$$

where

$$E_{1,2,1} = \sum_{p\le x} \left| \sum_{j=1}^{s} \sum_{q|F_j(p)} h_j(q) \right| \le (\operatorname{li} x)^{1/2} \left\{ \sum_{p\le x} \left| \sum_{j=1}^{s} \sum_{q|F_j(p)} h_j(q) \right|^2 \right\}^{1/2} =$$

$$= (\operatorname{li} x)^{1/2} \sum\nolimits_1^{1/2},$$

further

$$\sum\nolimits_1 = \sum_{j_1=1}^{s} \sum_{j_2=1}^{s} \sum_{q_1,q_2} h_{j_1}(q_1)\bar{h}_{j_2}(q_2) \sum_{\substack{p\le x\\ q_i|F_{j_i}(p)\\ i=1,2}} 1 =$$

$$= \operatorname{li} x \sum_{j_1,j_2} \sum_{q_1\ne q_2} \frac{h_{j_1}(q_1)\bar{h}_{j_2}(q_2)\varrho_{j_1}(q_1)\varrho_{j_2}(q_2)}{(q_1-1)(q_2-1)} + O\left(\frac{x}{\log^2 x}\right) +$$

$$+ O\left(\operatorname{li} x \sum_j \sum_{K_1<q} |h_j(q)|^2 \frac{\varrho_j(q)}{q}\right) = \operatorname{li} x \sum\nolimits_2 + o(\operatorname{li} x).$$

Here we used the Lemma 4, further that $F_{j_1}(p)\equiv 0 \pmod q$, $F_{j_2}(p)\equiv 0 \pmod q$, is impossible, when $q>\gamma$, and (9) in Lemma 9.

Further we have

$$\sum_2 = \left| \sum_{K_1 < q < x^{1/5}} \sum_j \frac{h_j(q)\varrho_j(q)}{q-1} \right|^2 + O\left(\sum_{K_1 < q < x^{1/5}} \sum_{j_1, j_2} \frac{|h_{j_1}(q) h_{j_2}(q)| \varrho_{j_1}(q) \varrho_{j_2}(q)}{(q-1)^2} \right) =$$

$$= |\sum_3|^2 + o(1).$$

Using the convergence of (4. 6) hence $\sum_3 = o(1)$, and so $E_{1,2,1} = o(\mathrm{li}\, x)$ follows. Further

$$E_{1,2,2} = \sum_j \sum_{K_1 < q < x^{1/5}} |h_j(q)|^2 \sum_{\substack{F_j(p) \equiv 0(p) \\ p \leqq x}} 1 \ll \mathrm{li}\, x \sum_j \sum_{K_1 < q < x^{1/5}} \frac{|h_j(q)|^2 \varrho_j(q)}{q} = o(\mathrm{li}\, x),$$

and so

$$E_1 = o(\mathrm{li}\ x)$$

holds.

By Lemma 9 we have

$$E_{2,1} \stackrel{\text{def}}{=} \sum_{\substack{p \leqq x \\ p \in \mathfrak{A}_1}} |H_2(p) - 1| \ll C_\delta o(\mathrm{li}\, x).$$

For $p \in \mathfrak{A}_2$

$$|H_2(p) - 1| \leqq \sum_{j=1}^{s} \sum_{\substack{q | F_j(p) \\ x^{1/5} < q < x^{1-\delta}}} \{|h_j(q)| + O(|h_j(q)|^2)\}$$

holds. Hence by Lemma

$$E_{2,2} \stackrel{\text{def}}{=} \sum_{\substack{p \leqq x \\ p \in \mathfrak{A}_2}} |H_2(p) - 1| \ll C_\delta \,\mathrm{li}\, x \sum_j \left\{ \sum_{x^{1/5} < q < x^{1-\delta}} \frac{|h_j(q)| \varrho_j(q)}{q} + \sum_q \frac{|h_j(q)|^2 \varrho_j(q)}{q} \right\}$$

follows. Using (8), (9) in Lemma 9 we have the inequality

$$E_{22} = C_\delta \, o(\mathrm{li}\ x).$$

Hence (14) follows.

For linear polynomials the relation (15) follows in the same way, as in Lemma 11. (See the estimation of T_j's.)

LEMMA 12 ([13]). *Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent random variables, with discrete distribution function. Let us suppose, that the sum*

$$\sum_1^\infty X_k$$

converges almost everywhere to a random variable X. Let

$$d_k = \operatorname{supr}_x P(X_k = x).$$

Then the distribution function of X is continuous if and only if $\prod_{k=1}^{\infty} d_k = 0$.

LEMMA 13 ([14]). *Let $G_N(x_1, \dots, x_s)$ be a sequence of s-dimensional distribution functions, with characteristic functions $\varphi_N(u_1, \dots, u_s)$. If for all real s-tuples $\{u_1, \dots, u_s\}$ the relation*

$$\lim_{N\to\infty} \varphi_N(u_1, \dots, u_s) = \varphi(u_1, \dots u_s)$$

holds, and $\varphi(u_1, \dots, u_s)$ is continuous in the point $\{u_1, \dots, u_s\}=\{0, \dots, 0\}$, then φ is the characteristic function of a distribution function $G(x_1, \dots, x_s)$, and

$$\lim_{N\to\infty} G_N(x_1, \dots, x_s) = G(x_1, \dots, x_s)$$

in all continuity points of G.

§ 5. The proof of theorems 1—10

5. 1. PROOF OF THEOREM 1. From Lemma 12 it follows, that

$$U(x) = U(x, K_1)+o(x), \quad V(x) = V(x, K_1)+o(\mathrm{li}\, x).$$

Using Lemma 8 we have

$$U(x, K_1) = x\bigl(1+o(1)\bigr) \sum_{\{d_1, \dots, d_s\}\in D_\gamma} \frac{h_1(d_1; K_1) \dots h_s(d_s; K_1)}{[d_1, \dots, d_s]} \times$$

$$\times \varrho(d_1, \dots, d_s) \prod_{p>\gamma} \left\{1+\sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \sum_{i=1}^{s} h_i(p^\alpha; K_1)\right\},$$

$$V(x, K_1) = \mathrm{li}\, x\bigl(1+o(1)\bigr) \sum_{\{d_1, \dots, d_s\}\in D_\gamma} \frac{h_1(d_1; K_1) \dots h_s(d_s; K_1)\lambda(d_1, \dots, d_s)}{\varphi([d_1, \dots, d_s])} \times$$

$$\times \prod_{p>\gamma} \left\{1+\sum_{\alpha=1}^{\infty} \frac{1}{\varphi(p^\alpha)} \sum_{i=1}^{s} h_i(p^\alpha; K_1)\right\}.$$

From (1. 1) it follows, that the products tend to $M_2(g)$, $N_2(g)$, respectively. Similarly, the sums tend to $M_1(g)$, $N_1(g)$. Hence (1. 2)—(1. 3) follow. The uniformity of the convergence in (1. 2)—(1. 3) follows from the uniformity of the convergence in the cited lemmas.

5. 2. PROOF OF THEOREM 5. From the conditions (5. 1)—(5. 2) and from Lemma 6 it follows, that

$$U(x) = U(x, x^\tau)+o(x), \qquad V(x) = V(x, x^\tau)+o(\mathrm{li}\, x)$$

where $\tau > 0$ is an arbitrary constant. From the conditions $a\bigl(g_j(F_j)\bigr)$ it follows the convergence of the products $M_2(g)$, $N_2(g)$ (see (5. 7)—(5. 8)). Using now Lemma 12 and Lemma 8, as before, we obtain the Theorem.

5. 3. PROOF OF THEOREMS 3 AND 8. Here we use Lemma 13, Theorem 1 and 5, choosing

$$g_j(n) = e^{iu_j f_j(n)} \qquad (j=1, \dots, s).$$

The validity of the conditions (1. 1) in Theorem 1 and $a(g_j(F_j))$ in Theorem 5 follows from the conditions $A(f_j(F_j))$, $B(f_j(F_j))$.

5. 4. PROOF OF THEOREMS 4 AND 7. We investigate only the case

$$\{f_1(F_1(n))-a_1(n), \ldots, f_s(F_s(n))-a_s(n)\}.$$

Since $a_j(n)-a_j(x)\to 0$ for almost every $n\leqq x$, so

$$L_x \overset{\text{def}}{=} \frac{1}{x}\sum_{n\leqq x}\prod_{j=1}^{s} e^{iu_j(f_j(F_j(n))-a_j(n))} = \frac{1}{x}\left(\sum_{n\leqq x}\prod_{j=1}^{s} e^{iu_jf_j(F_j(n))}\right) e^{-i\sum_{j=1}^{s}u_ja_j(x)} + o(1)$$

holds.

Using the Lemmas 8, 10, 11, we have

$$L_x = e^{-i\sum_{j=1}^{s}u_ja_j(\gamma)} \sum_{\{d_1,\ldots,d_s\}\in D_\gamma} \frac{h_1(d_1)\ldots h_s(d_s)\varrho(d_1,\ldots,d_s)}{[d_1,\ldots,d_s]}\times$$

$$\times \prod_{\gamma<p<K_1}\left[\left\{1+\sum_{\alpha=1}^{\infty}\frac{1}{p^\alpha}\sum_{j=1}^{s}(e^{iu_jf_j(p^\alpha)}-1)\right\} e^{-\sum_{j=1}^{s}u_j\frac{f_j(p^\alpha)}{p}}\right]+o(1).$$

From the conditions $B(f_j(F_j))$ it follows, that the product converges for $K_1\to\infty$. In the remaining part the same reasoning leads to Theorem 4 and 7.

5. 5. PROOF OF THEOREMS 2 AND 6. We choose

$$g_1(n)=e^{iu_1f(n)},\quad g_l(n)=e^{i(u_l-u_{l-1})f(n)}\quad (l=2,\ldots,s-1),\quad g_s(n)=e^{-iu_{s-1}f(n)}$$

in Theorem 2, and

$$g_1(n)=e^{iu_1f_1(n)},\quad g_l(n)=e^{i(u_l-u_{l-1})f_l(n)}\quad (l=2,\ldots,s-1),\quad g_s(n)=e^{-iu_{s-1}f_s(n)}$$

in Theorem 6 and argue further as above.

5. 6. Theorems 9 and 10 are straightforward consequences of Theorem 5.

5. 7. The assertions concerning the continuity of the distribution functions in Theorems 2—4, 6—8 follow from the multiplicativity of $M_2(g)$, $N_2(g)$ using Lemma 12, as in [3].

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ON SETS OF INTEGERS CONTAINING NO FOUR ELEMENTS IN ARITHMETIC PROGRESSION

By

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In what follows we use capital letters to denote sequences of integers, $A+B$ to denote the sum of two sets of integers formed elementwise, and $A \urcorner B$ to denote the complement of the set B with respect to the set A.

Let us for convenience call an arithmetic progression of k (distinct) terms a *k-progression.*

If a set A contains no k-progression we say that A is k-free.

The maximal number of elements a k-free set $A \subseteqq [0, n)$ can have is denoted by $\tau_k(n)$. Furthermore we set

$$\gamma_k = \overline{\lim_{n\to\infty}} \frac{\tau_k(n)}{n}.$$

Actually we can replace $\overline{\lim}$ on the right hand side by lim. For, given $\varepsilon > 0$ and n, we can find arbitrarily large m so that $\tau_k(m) \geqq (\gamma_k - \varepsilon)m$; in particular we may assume that $qn < m \leqq (q+1)n$ holds for a positive integer q. In other words there is a k-free set $A \subseteqq [0, m)$ with cardinality $|A| \geqq (\gamma_k - \varepsilon)m$. Now $[0, m)$ can be split into $(q+1)$ subintervals of length at most n. One of these must contain at least $\left(\frac{1}{q+1}\right)|A|$ elements of A which clearly form a k-free set.

Hence

$$\tau_k(n) \geqq \left(\frac{1}{q+1}\right)|A| \geqq (\gamma_k - \varepsilon)\frac{m}{q+1} \geqq (\gamma_k - \varepsilon)\frac{q}{q+1}\, n.$$

Since ε can be taken arbitrarily small and q arbitrarily large, we have

$$\tau_k(n) \geqq \gamma_k n,$$

whence

$$\gamma_k = \lim \frac{\tau_k(n)}{n}.$$

Clearly $\gamma_k \leqq 1 - \frac{1}{k}$, and $\gamma_3 \leqq \gamma_4 \leqq \ldots$. It has been proved by F. BEHREND* that either all γ_k are zero, or $\gamma_k \to 1$ as $k \to \infty$.

* On sequences of integers containing no arithmetic progression, *Časopis Mat. Fis. Praha*, **67** (1938), pp. 235—239.

In 1953 ROTH* proved that $\gamma_3=0$. In fact he proved more than that, namely

$$\tau_3(n) \ll \frac{n}{\log\log n}.$$

Roth's proof uses estimates of exponential sums.

In this paper we shall prove the following

THEOREM.

$$\gamma_4=0, \quad i.e. \quad r_4(n)=o(n).$$

The proof is elementary. The problem of $\gamma_5, \gamma_6, \ldots$ is left open.

The proof is indirect, so from now on we assume that

$$\gamma_4>0.$$

For convenience we write

$$\gamma=\gamma_4.$$

We shall formulate in this section the two main lemmas and deduce the theorem from them.

We write $Q(b, c, d, e)$ for the system

$$b-2c+d = c-2d+e = 0,$$

which means that either b, c, d, e form an arithmetic progression, or they are identical.

Throughout the paper $n_4(\varepsilon)$ shall mean a number (for example the smallest one) with the property that for $n \geqq n_4(\varepsilon)$ a 4-free set $A \subseteqq [0, n)$ cannot contain more than $(\gamma+\varepsilon)n$ elements. Occasionally we use the analogue meaning for $n_3(\varepsilon)$ as well.

Let $B, C, D \subseteqq [0, q)$. We regard B and C as fixed while D varies. We then define

$$D^* = \{e;\ e\in[0, q) \text{ and there are } b\in B,\ c\in C,\ d\in D \text{ such that } Q(b, c, d, e)\}.$$

With this notation we shall prove

LEMMA $(H_0, \ldots, H_k)$.** *There are absolute constants* $\varepsilon_0>0$, $\gamma'>0$, k_0 *and* q_0 *with the following property: If*

$$q \geqq q_0, \quad 3|q,$$

and if B, C *are* 4*-free sets contained in* $[0, q)$, $|B| \geqq (\gamma-\varepsilon_0)q$, $|C| \geqq (\gamma-\varepsilon_0)q$, *then there are disjoint sets*

$$H_0, \ldots, H_k, \qquad k \leqq k_0,$$

such that

$$\bigcup_{K=0}^{k} H_K = \left[\frac{1}{3}q, \frac{2}{3}q\right),$$

$$|H_0| \leqq \frac{1}{12}\gamma q; \qquad |H_K^*| \geqq \gamma' q \quad for \quad K = 1, 2, \ldots, k,$$

* On certain sets of integers. I; II, *J. Lond. Math. Soc.*, **28** (1953), pp. 104—109; **29** (1953), pp. 20—26.

** The full force of the hypothesis that (say) C is 4-free is not needed for the proof of this lemma: see the footnote on page 95.

and such that if for some $K \neq 0$

$$G \subseteqq H_K, \qquad |G| \geqq \frac{1}{2}\gamma |H_K|,$$

then

$$|G^*| \geqq \left(1 - \frac{1}{2}\gamma\right) |H_K^*|.$$

The other main lemma is

LEMMA *BCDE. Let* $\varepsilon_1 \in (0, \gamma)$ u *and* q_0 *be given. Then there is a* $q \geqq q_0$ *and there are sets*

$$B_0, C_0, D_1, \ldots, D_u, E_1, \ldots, E_u \subseteqq [0, q),$$

all 4*-free, all with at least* $(\gamma - \varepsilon_1)q$ *elements, such that* $Q(b, c, d, e)$ *with* $b \in B_0$, $c \in C_0$, $d \in D_i$, $e \in E_i$ *is insolvable for all* $i = 1, \ldots, u$, *and such that for each* $x \in [0, q)$ *the set of all i's for which* $x \in E_i$ *holds is* 4*-free.*

We now prove the theorem using these two lemmas.

Let ε_0, γ and k_0 have the meaning of lemma $(H_0, \ldots, H_k)$. Put

$$\varepsilon_1 = \min\left(\varepsilon_0, \frac{\gamma}{20}, \frac{\gamma\gamma'}{6}\right)$$

and

$$t = n_4(\varepsilon_1).$$

Van der Waerden's Theorem* gives a number

$$u = N(k_0, t)$$

such that in any partition of $[0, u)$ into at most k_0 classes there is at least one class which contains a t-progresssion.

We apply lemma *BCDE* with this ε_1, and u, and with

$$q_0 = 3n_4(\varepsilon_1).$$

From $|D_i| \geqq (\gamma - \varepsilon_1)q$, $\frac{1}{3} q \geqq n_4(\varepsilon_1)$ we see that

$$\left|D_i \cap \left[\frac{1}{3}q, \frac{2}{3}q\right)\right| = |D_i| - \left|D_i \cap \left[0, \frac{1}{3}\right)\right| - \left|D_i \cap \left[\frac{2}{3}q, q\right)\right| \geqq$$

$$\geqq (\gamma - \varepsilon_1)q - 2(\gamma + \varepsilon_1)\frac{1}{3}q \geqq (\gamma - 5\varepsilon_1)\frac{1}{3}q.$$

We now define the sets H_K by lemma $(H_0, \ldots, H_k)$, using B_0, C_0 for B, C respectively.

For each $i \in (0, u]$ there is a $j = j(i) \in (0, k]$ such that

$$|D_i \cap H_j| \geqq \frac{1}{2}\gamma |H_j|.$$

* Beweis einer Baudetschen Vermutung, *Nienn. Arch. Wiskunde*, **15** (1927), pp. 212—216.

For otherwise we should get the contradiction

$$(\gamma-5\varepsilon_1)\frac{1}{3}q \leqq \left|D_i \cap \left[\frac{1}{3}q,\ \frac{2}{3}q\right)\right| = \sum_{j=0}^{k} |D_i \cap H_j| <$$

$$< |H_0| + \frac{1}{2}\gamma \sum_{j=1}^{k} |H_j| \leqq \left(\frac{1}{4}\gamma + \frac{1}{2}\gamma\right)\frac{1}{3}q \leqq (\gamma-5\varepsilon_1)\frac{1}{3}q$$

since $\varepsilon_1 \leqq \frac{1}{20}\gamma$.

Attaching such a $j(i)$ to each i, it gives a partition of the i's into k classes. Since $u=N(k_0, t)$ and $k \leqq k_0$ one of these classes contains a t-progression. In other words, there is a j_0 and an arithmetic progression $i_1, \dots, i_t$ such that

$$|D_i \cap H_{j_0}| \geqq \frac{1}{2} g |H_{j_0}| \quad \text{for} \quad i = i_1, \dots, i_t.$$

From lemma $(H_0, \dots, H_k)$ we then have that

$$|(D_i \cap H_{j_0})^*| \geqq \left(1 - \frac{1}{2}\gamma\right) |H_{j_0}^*|$$

where the * is taken with respect to B_0 and C_0. With the trivial relation $(U \cap V)^* \subseteqq U^* \cap V^*$ this implies that

$$|D_i^* \cap H_{j_0}^*| \geqq \left(1 - \frac{1}{2}\gamma\right) |H_{j_0}^*|.$$

Now $D_i^* \cap E_i = \emptyset$, for this is merely a restatement of the fact that the relations $Q(b, c, d, e)$ with $b \in B_0$, $c \in C_0$, $d \in D_i$, $e \in E_i$ are impossible.

Hence

$$|E_i \cap H_{j_0}^*| + |D_i^* \cap H_{j_0}^*| \leqq |H_{j_0}^*|,$$

so that

$$|E_i \cap H_{j_0}^*| \leqq \frac{1}{2}\gamma |H_{j_0}^*|$$

for $i=i_1, \dots, i_t$.

Put

$$|H_{j_0}^*| = \alpha \cdot q, \quad [0, q) - H_{j_0}^* = M.$$

We notice that M is not empty, since otherwise the last inequality would imply that $|E_i| \leqq \frac{1}{2}\gamma q$, in contradiction with the fact that

$$|E_i| \geqq (\gamma - \varepsilon_1)q \geqq \left(\gamma - \frac{1}{20}\gamma\right)q.$$

Furthermore, lemma $(H_0, \dots, H_k)$ shows that $\alpha \geqq \gamma'$. Therefore

$$\frac{|E_i \cap M|}{|M|} = \frac{|E_i| - |E_i \cap H_{j_0}^*|}{q - |H_{j_0}^*|} \geqq \frac{\gamma - \varepsilon_1 - \frac{1}{2}\gamma\alpha}{1-\alpha} = \gamma + \frac{\frac{1}{2}\gamma\alpha - \varepsilon_1}{1-\alpha} \geqq$$

$$\geqq \gamma + \frac{1}{2}\gamma\alpha - \varepsilon_1 \geqq \gamma + \frac{1}{2}\gamma\gamma' - \varepsilon_1 \geqq \gamma + 2\varepsilon_1$$

for $i=i_1, \ldots, i_t$. Summing over these i's we see that

$$\sum_{\tau=1}^{t} |E_{i_\tau} \cap M| \geqq (\gamma+2\varepsilon_1)t|M|.$$

We conclude that there is at least one $x \in M$ which occurs in not less than $(\gamma+2\varepsilon_1)t$ of the sets E_{i_τ}. By lemma *BCDE* those i_τ's for which $x \in E_{i_\tau}$ form a 4-free set. They are contained in an arithmetic progression of t terms and by the choice of $t=n_4(\varepsilon_1)$, there cannot be more than $(\gamma+\varepsilon_1)t$ numbers i_τ for which $x \in E_{i_\tau}$. Thus we have reached a contradiction and the theorem is proved.

In this section we shall prove lemma $(H_0, \ldots, H_k)$. For this we need three other lemmas. The first is almost obvious. We call it therefore

THE SIMPLE LEMMA. *Let $A \subseteqq [0, n)$ be 4-free and $|A| \geqq (\gamma-\varepsilon)n$. Let $M \subseteqq [0, n)$ have a complement that is the union of disjoint arithmetic progressions P_ϱ, $\varrho=1, \ldots, r$ each of length $|P_\varrho| \geqq n_4(\varepsilon')$. Then we have*

$$|A \cap M| \geqq \gamma|M| - (\varepsilon+\varepsilon')n.$$

PROOF. Each $A \cap P_\varrho$ as a 4-free subset of a progression fulfils

$$|A \cap P_\varrho| \leqq (\gamma+\varepsilon')|P_\varrho|.$$

Hence we have the following inequalities:

$$|A \cap M| = |A| - \sum_\varrho |A \cap P_\varrho| \geqq (\gamma-\varepsilon)n - (\gamma+\varepsilon')\sum_\varrho |P_\varrho| =$$

$$= (\gamma-\varepsilon)n - (\gamma+\varepsilon')(n-|M|) = (\gamma+\varepsilon')|M| - (\varepsilon+\varepsilon')n \geqq \gamma|M| - (\varepsilon+\varepsilon')n.$$

LEMMA $p(\delta, l)$. *For any real $\delta \in (0, 1)$ and any natural number l there exists a number $p(\delta, l)$ with the following property: If*

$$u \geqq p(\delta, l), \quad G \subseteqq [0, u), \quad |G| \geqq \delta u,$$

then G contains a set S_l of the form

$$S_l = \{y\} + \{0, x_1\} + \ldots + \{0, x_l\}$$

with natural numbers $x_1, \ldots, x_l$.

PROOF. The proof goes by complete induction and uses the box principle. The case $l=1$ is trivial, since it states only that there is a pair of elements of G. A suitable choice of $p(\delta, 1)$ is $\left[1+\frac{1}{\delta}\right]$ since this exceeds $\frac{1}{\delta}$ so that the hypothesis concerning G shows that

$$|G| \geqq \delta u > 1.$$

Now take $l \geqq 2$ and assume the case $l-1$ has been already proved. We set

$$q = p\left(\frac{\delta}{2}, l-1\right).$$

Any number u can be represented as

$$u = kq + r, \quad 0 \leqq r < q.$$

We choose $p(\delta, l)$ so that $u \geqq p(\delta, l)$ implies that

$$k > \frac{4}{\delta^2}, \quad \frac{\delta}{2} k > (q-1)^{l-1}.$$

A possible choice is, for example

$$p(\delta, l) = \max\left(\left[1+\frac{4}{\delta^2}\right] q, \ \left[1+\frac{2}{\delta}\right] q^l\right).$$

Let R be the number of those sets

$$G_K = G \cap [(K-1)q, Kq], \qquad K = 1, \ldots, k$$

for which $|G_K| \geqq \frac{\delta}{2} q$. Then $R \geqq \frac{\delta}{2} k$, otherwise

$$\delta kq \leqq \delta u \leqq |G| \leqq q + \sum_{K=1}^{k} |G_K| \leqq (1+R)q + (k-R)\frac{\delta}{2} q =$$

$$= \left(1-\frac{\delta}{2}\right) Rq + \left(1+\frac{k\delta}{2}\right) q < \left(1-\frac{\delta}{2}\right)\frac{\delta}{2} kq + \left(1+\frac{k\delta}{2}\right) q =$$

$$= \delta kq - \left(\frac{\delta^2 k}{4} - 1\right) q < \delta kq.$$

By the introduction hypothesis, in each of the sets G_K a set of the type S_{l-1} can be found. In each S_{l-1} we have $1 \leqq x_1, \ldots, x_{l-1} \leqq q-1$. Thus there are not more than $(q-1)^{l-1}$ different choices of $x_1, \ldots, x_l$. Since $R \geqq \frac{\delta}{2} k > (q-1)^{l-1}$ there are two sets G_K containing S_{l-1} and S'_{l-1} formed with the same numbers $x_1, \ldots, x_l$ but different y, y', say with $y' > y$. Then with $x_l = y' - y$ we have

$$G \supseteqq S_{l-1} \cup S'_{l-1} = S_{l-1} \cup (S_{l-1} + x_l) = S_l.$$

LEMMA $|G^*|$. *There are absolute constants* $\varepsilon_0 > 0$ *and* $\gamma' > 0$ *and a function* $g_0(\delta)$ *for* $0 < \delta < 1$ *with the following property:*

If $q \geqq q_0(\delta)$, $8 | q$, $B, C \subseteqq [0, q)$ *are both* 4-*free*,

$$|B| \geqq (\gamma - \varepsilon_0) q, \quad |C| \geqq (\gamma - \varepsilon_0) q, \quad G \subseteqq \left[\frac{1}{3} q, \frac{2}{3} q\right) \quad |G| \geqq \frac{\delta q}{3},$$

then

$$|G^*| \geqq \gamma' q.$$

REMARK. An analogous lemma can be similarly proved with $\gamma = \gamma_3$ (instead of $\gamma = \gamma_4$) on the assumption that $\gamma_3 > 0$. We then easily arrive at a contradiction, which proves Roth's theorem $\gamma_3 = 0$. For this purpose choose a $q \geqq 3n_3(\varepsilon)$. Next choose a 3-free set $A \subseteqq [0, 3q)$ with $|A| \geqq 3\gamma q$ and represent it as

$$A = B \cup (C+q) \cup (D+2q)$$

with $B, C, D \subseteqq [0, q)$; and finally set

$$G = D \cap \left[\frac{1}{3} q, \frac{2}{3} q\right).$$

One easily obtains the inequalities $|B| \geqq (\gamma - 2\varepsilon)g$, $|C| \geqq (\gamma - 2\varepsilon)q$, $|G| \geqq (\gamma - 8\varepsilon)\frac{q}{3}$. If we take $\varepsilon \leqq \frac{1}{2}\varepsilon_0$, $\varepsilon \leqq \frac{1}{16}\gamma$ and q large enough, we can apply the lemma with $\delta = \frac{1}{2}\gamma$ and get

$$|G^*| \geqq \gamma' q > 0$$

which means that there is a triplet (b, c, d) with

$$b - 2c + d = 0.$$

But $(b, c+q, d+2q)$ is then a 3-progression in A, a set that was supposed to be 3-free.

Proof of lemma $|G^*|$. Set

$$\varepsilon_0 = \frac{1}{100}\gamma^2, \quad m = n_4(\varepsilon_0),$$

and fix an l such that $l \geqq 24\frac{m}{\gamma}$, say

$$l = \left[\frac{25m}{\gamma}\right].$$

We shall prove the lemma with

$$q_0(\delta) = 3p(\delta, l) + 3m, \quad \gamma = \frac{\gamma^2}{50 \cdot 2^l}.$$

With these choices we have $\frac{q}{3} \geqq p(\delta, l)$ and can therefore find a set of type S_l in G. We consider

$$S_i = \{y\} + \{0, x_1\} + \ldots + \{0, x_i\}$$

for all $i = 0, 1, \ldots, l$; where we take $S_0 = \{y\}$. For each i we define

$$L_i = \left\{2c - s;\quad c \in C \cap \left[\frac{1}{3}q, \frac{2}{3}q\right),\ s \in S_i\right\}.$$

Since $S_i \subseteqq \left[\frac{1}{3}q, \frac{2}{3}q\right)$ one has $L_i \subseteqq [0, q)$.

With $|C| \geqq (\gamma - \varepsilon_0)q$ and $\frac{1}{3}q > m = n_4(\varepsilon_0)$ we obtain

$$|L_0| = \left|C \cap \left[\frac{1}{3}q, \frac{2}{3}q\right)\right| \geqq (\gamma - 5\varepsilon_0)\frac{q}{3} \geqq \frac{1}{4}\gamma q,^*$$

since $5\varepsilon_0 < \frac{1}{4}\gamma$.

* The derivation of this inequality is the only extent to which we use the hypothesis that C is 4-free.

From the fact that $|L_l| \leqq q$ and $L_0 \subseteqq L_1 \subseteqq \dots$ we infer that there is some $i \leqq l$ such that

$$|L_i| - |L_{i-1}| \leqq \frac{q}{l}.$$

We decompose this L_{i-1} into maximal progression (mod x_i). We shall denote by $\bar{L}$ the union of those of these progressions which have $3m$ or more elements, and by $\bar{\bar{L}}$ the union of the remaining ones. From

$$S_i = S_{i-1} \cup (S_{i-1} + x_i)$$

one sees that

$$L_i = L_{i-1} \cup (L_{i-1} - x_i).$$

Each maximal progression (mod x_i) in L_{i-1} produces therefore one and only one new element in L_i. Hence

$$|\bar{\bar{L}}| \leqq 3m(|L_i| - |L_{i-1}|) \leqq 3m \frac{q}{l},$$

and

$$|\bar{L}| = |L_{i-1}| - |\bar{\bar{L}}| \geqq |L_0| - |\bar{\bar{L}}| \geqq \left(\frac{\gamma}{4} - \frac{3m}{l}\right) q \geqq \frac{1}{8} \gamma q$$

since by our choice of l we have $l \geqq \frac{24m}{\gamma}$.

Now let us drop m elements from each end of each of the progressions (mod x_i) composing $\bar{L}$, and denote the remaining set by M. Since every progression in $\bar{L}$ has a length of at least $3m$ we have

$$|M| \geqq \frac{1}{3} |\bar{L}| \geqq \frac{\gamma}{24} q.$$

By construction $[0, q) - M$ can be represented as the union of disjoint progressions (mod x_i) each of length at least m. Thus we can apply the Simple Lemma with $\varepsilon = \varepsilon' = \varepsilon_0$ and obtain

$$|L_l \cap B| \geqq |\bar{L} \cap B| \geqq |M \cap B| \geqq \gamma |M| - 2\varepsilon_0 q \geqq \frac{\gamma^2}{24} q - 2\varepsilon_0 q \geqq \frac{\gamma^2}{50} q,$$

since ε_0 has been chosen suitably.

By definition, $L_l \cap B$ is the set of those b in B which have a representation

$$b = 2c - s, \quad s \in S_l \quad c \in C \cap \left[\frac{1}{3} q, \frac{2}{3} q\right).$$

In S_l there are at most 2^l elements. Therefore at least one y contained in S_l has the property that the equation

$$b - 2c + y = 0$$

has at least $\frac{\gamma^2 q}{50 \cdot 2^l}$ solutions (b, c). In another notation this means that

$$|\{y\}^*| \geqq \gamma' q,$$

where we have put

$$\gamma' = \frac{\gamma^2}{50\cdot 2^l}.$$

The statement of lemma $|G^*|$ is now immediate. From $y\in S_l\subseteqq G$ we see that

$$|G^*|\geqq|\{y\}^*|\geqq\gamma' q.$$

PROOF OF LEMMA ($H_0, \ldots, H_k$). We first fix some number h such that $\left(1-\frac{\gamma}{2}\right)^h<\gamma'$, for example

$$h=\left[1+\frac{\log\gamma'}{\log\left(1-\frac{\gamma}{2}\right)}\right].$$

We now start from some $G_0\subseteqq\left[\frac{1}{3}q, \frac{2}{3}q\right)$ with $|G_0|\geqq\frac{1}{12}\gamma q$ and put $g_0=|G^*|$. Next we define by recursion for $i=1, \ldots, h$

$$\Gamma_i=\left\{G, G\subseteqq G_{i-1}, |G|\geqq\frac{\gamma}{2}|G_{i-1}|\right\}, \qquad g_i=\min_{G\in\Gamma_i}|G^*|$$

and fixe one G_i in Γ_i for which $|G_i^*|=g_i$.

From $G_i\in\Gamma_i$ we see that

$$|G_i|\geqq\frac{\gamma}{2}|G_{i-1}|\geqq\ldots\geqq\left(\frac{\gamma}{2}\right)^i|G_0|\geqq\left(\frac{\gamma}{2}\right)^i\frac{\gamma}{12}q,$$

$$|G_i|\geqq\frac{1}{6}\left(\frac{\gamma}{2}\right)^{h+1}q.$$

Thus, if we take $\delta=\frac{1}{2}\left(\frac{\gamma}{2}\right)^{h+1}$ and $q_0=q_0(\delta)$ we can apply lemma $|G^*|$ for all $q\geqq q_0$ and obtain

$$g_i=|G_i^*|\geqq\gamma' q, \quad \text{for} \quad i=1, 2, \ldots, h.$$

Since clearly $g_0\leqq q$ there is a $j\leqq h$ such that

$$g_j\geqq\left(1-\frac{\gamma}{2}\right)g_{j-1},$$

otherwise we should have the contradiction

$$\gamma' q\leqq g_h<\left(1-\frac{\gamma}{2}\right)^h g_0\leqq\left(1-\frac{\gamma}{2}\right)^h q<\gamma' q.$$

Set with this j $H = G_{j-1}$. From the meaning of g_j and g_{j-1} it follows that if $G \subseteq H$, and $|G| \geqq \frac{\gamma}{2} |H|$, then $G \in \Gamma_j$ and therefore

$$|G^*| \geqq g_j \geqq \left(1-\frac{\gamma}{2}\right) g_{j-1} = \left(1-\frac{\gamma}{2}\right) |H^*|.$$

Moreover we have

$$|H| = |G_{j-1}| \geqq \frac{1}{6}\left(\frac{\gamma}{2}\right)^{h+1} q.$$

At first we apply this process to $G_0 = \left[\frac{1}{3}q, \frac{2}{3}q\right)$ and call the set H obtained H_1. Then we take $G_0 = \left[\frac{1}{3}q, \frac{2}{3}q\right) \urcorner H$, and if this set contains at least $\frac{1}{12}\gamma q$ elements we obtain a set H_2 from it. Next we take $G_0 = \left[\frac{1}{3}q, \frac{2}{3}q\right) \urcorner (H_1 \cup H_2)$ to get a set H_3, and so on. As soon as we are left with

$$\left|\left[\frac{1}{3}q, \frac{2}{3}q\right) \urcorner (H_1 \cup H_2 \cup \ldots \cup H_k)\right| < \frac{\gamma}{12} q$$

we stop the procedure and call this remaining set H_0.

Since the sets H_K are obviously disjoint and

$$|H_K| \geqq \frac{1}{6}\left(\frac{\gamma}{2}\right)^{h+1} q \quad \text{for} \quad K = 1, 2, \ldots, k$$

this occurs certainly after a finite number of steps. To be precise, we see that

$$k \leqq \frac{1}{3} q \left(\frac{1}{6}\left(\frac{\gamma}{2}\right)^{h+1} q\right)^{-1} = 2\left(\frac{2}{\gamma}\right)^{h+1}.$$

By construction $H_0 \cup H_1 \cup \ldots \cup H_k = \left[\frac{1}{3}q, \frac{2}{3}q\right)$ and if $G \subseteq H_K$, $|G| \geqq \frac{\gamma}{2} |H_K|$ then $|G^*| \geqq \left(1-\frac{\gamma}{2}\right) |H_K^*|$ for all $K = 1, 2, \ldots, k$. This is precisely the statement of lemma $(H_0, \ldots, H_k)$.

Proof of Lemma *BCDE*. Let us take n and q to be integers so that $nq \geqq 6n_4\left(\frac{\varepsilon}{3}\right)$ and let A be a 4-free set contained in $[0, 4nq)$ which satisfies

$$|A| \geqq \gamma 4nq.$$

Then we can decompose A into

$$A = B \cup (C+nq) \cup (D+2nq) \cup (E+3nq)$$

with $B, C, D, E \subseteq [0, nq)$ and (in an obvious notation)

$$B = \bigcup_{x<n} (B+xq) \quad \text{with} \quad B_x \subseteq [0, q),$$

similarly for C, D, E. For their respective cardinalities we get easily the estimates

$$|B|,\ |C|,\ |D|,\ |E| \geqq (\gamma-\varepsilon)nq.$$

That A is 4-free is reflected in the fact that $Q(b, c, d, e)$ has no solutions with $b\in B$, $c\in C$, $d\in D$, $e\in E$. More precisely: If $Q(x, y, z, w)$ holds, then $Q(b, c, d, e)$ is insolvable with $b\in B_x$, $c\in C_y$, $d\in D_z$, $e\in E_w$. Moreover all of the sets B_x, C_y, D_z, E_w are 4-free.

Let us call a set B etc. $\subseteqq [0, q)$ *full* if $|B| \geqq (\gamma-\varepsilon_1)q$, and *poor* otherwise.

Clearly lemma $BCDE$ will be proved if we can show that there are u quadruples (b, c, d, e) such that all B_b's are equal, all C_c's are equal, all B_b's, C_c's, D_d's, E_e's are full, and the e's form an arithmetic progression.

We shall use all the ideas from the proof of lemma $|G^*|$ but not only these, moreover the technique will be more involved.

We can easily provide a set $\mathfrak{B}$ with positive density (about 2^{-q}) such that all B_b for $b\in\mathfrak{B}$ are equal and full. Similarly we find a dense set $\mathfrak{C}$ with all C_c for $c\in\mathfrak{C}$ equal and full. We have then a set of type S_e in $\mathfrak{C}$ through which we 'project' $\mathfrak{B}$ onto the levels of D and E. The points e defined by $Q(b, s, *, e)$ are plentiful and are arranged into long progressions. Hence it can be shown that almost all E_e with these e's are full. The same could be done for the sets D_d with d from $Q(b, s, d, *)$ but unfortunately not in the necessary simultaneous way, since the relation between the e's and the d's is not unique and this relationship weakens the larger l is taken.

The idea which overcomes this difficulty is to use not only one set $\mathfrak{C}$, but a large number of them, $\mathfrak{C}_0, \mathfrak{C}_1, \ldots, \mathfrak{C}_{r-1}$ generated from one of them by shifting $\mathfrak{C}_\varrho = \mathfrak{C}_0 + \varrho$, such that $C_c = C_{c'}$, if c and c' belong to the same set $\mathfrak{C}_\varrho$. This again introduces long progressions on the levels of D and E, which can be exploited independently of the former ones. As a result we get u quadruples of the required type for at least one ϱ with $b\in\mathfrak{B}$ and all $C\in\mathfrak{C}_\varrho$, and so all B_b as well as C_c coincide.

We shall use the following simple counting argument a couple of times: If $\sum_{x=1}^{n} a_x \geqq (\gamma-\varepsilon_3)n$ and $a_x \leqq (\gamma+\varepsilon_2)$ for all x, then the number R of terms a_x which satisfy $a_x \leqq (\gamma-\varepsilon_1)$ is

$$R \leqq \frac{\varepsilon_2+\varepsilon_3}{\varepsilon_1}\, n.$$

PROOF.

$$(\gamma-\varepsilon_3)n \leqq (\gamma-\varepsilon_1)R + (\gamma+\varepsilon_2)(n-R), \qquad (\varepsilon_1+\varepsilon_2)R \leqq (\varepsilon_2+\varepsilon_3)n.$$

We list now the parameters used in the proof, in the order of their dependence. The reader may check them as they occur.

ε, u and q_0 are supposed to be given,

$$\varepsilon_2 = \frac{\varepsilon_1}{16u}, \qquad l = 75m\cdot 2^q,$$

$$q = \max(q_0, n_4(\varepsilon_2)), \qquad \varepsilon_4 = \frac{\varepsilon_2^2}{600\cdot 2^{q+2l}},$$

$$\varepsilon_3 = \frac{\varepsilon_2}{150\cdot 2^q}, \qquad r = n_4(\varepsilon_4),$$

$$\varepsilon = \text{sufficiently small}$$

$$m = \max(2u, n_4(\varepsilon_3)), \qquad n = \text{sufficiently large},\ 6r|n.$$

We can safely dispense with specifying ε and n since there is no feedback to the other parameters. A small ε only demands a large n.

By an already repeatedly used argument we get

$$\sum_{x<\frac{n}{6}} |B_x| = \left| B \cap \left[0, \frac{1}{6} nq\right) \right| \geqq (\gamma - \varepsilon) \frac{nq}{6}$$

$$\sum_{\frac{n}{6} \leqq y < \frac{n}{3}} |C_y| = \left| C \cap \left[\frac{nq}{6}, \frac{nq}{3}\right) \right| \geqq (\gamma - \varepsilon) \frac{nq}{6}$$

provided only that n is large enough. We set $\varepsilon_2 = \frac{\varepsilon_1}{16u}$, and take $q \geqq n_4(\varepsilon_2)$, so that we then have for all x, y, z, w,

$$|B_x|, |C_y|, |D_z|, |E_w| \leqq (\gamma + \varepsilon_2) q.$$

By the above counting argument the number of poor B_x, $0 \leqq x < \frac{n}{6}$ and the number of poor C_y, $\frac{n}{6} \leqq y < \frac{n}{3}$ is each at most $\left(\frac{1}{16u} + \frac{\varepsilon}{\varepsilon_1}\right) \frac{n}{6} \leqq \frac{1}{8} \cdot \frac{n}{6}$ if ε is small enough. Consequently more than half of the B_x are full.

There are only 2^q subsets of $(0, q)$, so there is a full $B_{(0)} \subseteqq (0, q)$ such that

$$B_b = B_{(0)} \quad \text{for} \quad b \in \mathfrak{B} \subseteqq \left[0, \frac{n}{6}\right), \quad \text{with} \quad |\mathfrak{B}| \geqq \frac{n}{12 \cdot 2^q}.$$

We next look at C, and assuming that $r|n$ we consider the r-tuples

$$(C_{mr}, C_{mr+1}, \ldots, C_{mr+r-1}), \quad \frac{n}{6r} \leqq m < \frac{n}{3r}.$$

Since not more than $\frac{1}{8}$ of the C_j are poor, not more than $\frac{1}{2}$ of the r-tuples contain more than $\frac{1}{4}$ poor sets. There are only 2^{qr} different r-tuples, so we find

$$C_{(0)}, \ldots, C_{(r-1)},$$

not more than $\frac{1}{4}$ of them being poor, and $\mathfrak{B} \subseteqq \left[\frac{n}{6}, \frac{n}{3}\right)$ so that

$$C_{c+\varrho} = C_{(\varrho)} \quad \text{for} \quad c \in \mathfrak{C} \quad \text{and} \quad \varrho \in [0, r), \qquad |\mathfrak{C}| \geqq \frac{n}{12r \cdot 2^{qr}}.$$

By lemma $p(\delta, l)$ we see that $\mathfrak{C}$ contains a subset of type

$$S_l = \{y\} + \{0, x_1\} + \ldots + \{0, x_l\}.$$

With the sets

$$S_i = \{y\} + \{0, x_1\} + \ldots + \{0, x_i\}$$

we form

$$L_i = \{35 - 2b; s \in S_i, b \in \mathfrak{B}\}.$$

Then we have

$$L_i \subseteq \left[\frac{n}{6}, n\right), \quad |L_0| = |\mathfrak{B}| \geqq \frac{n}{12\cdot 2^q},$$

$$L_i = L_{i-1} \cup (L_{i-1} + 3x_i).$$

For a suitable $i \leqq l$ we have

$$|L_i| - |L_{i-1}| \leqq \frac{n}{l}.$$

We decompose L_{i-1} into maximal progression (mod $3x_i$), collect those progressions which are longer than $3m$ into $\bar{L}$, and the remaining ones into $\bar{\bar{L}}$; as in the proof of lemma $|G^*|$ we get

$$|\bar{\bar{L}}| \leqq 3m(|L_i| - |L_{i-1}|) \leqq \frac{3mn}{l},$$

$$|\bar{L}| \geqq |L_0| - |\bar{\bar{L}}| \geqq \left(\frac{1}{12\cdot 2^q} - \frac{3m}{l}\right) n \geqq \frac{n}{25\cdot 2^q}.$$

(Here we have taken $l \geqq 72m\cdot 2^q$). Dropping the first m and the last m elements of each of the progressions collected into $\bar{L}$, we obtain a set we shall call $\mathcal{E}$. Then

$$|\mathcal{E}| \geqq \frac{1}{3}|\bar{L}| \geqq \frac{n}{75\cdot 2^q}$$

and $[0, n) \urcorner \mathcal{E}$ is the union of disjoint progressions (mod $3x_i$), none of which contains fewer than m elements.

If we start from $S_l + \varrho \subseteq \mathfrak{C} + \varrho$ instead of S_l, $0 \leqq \varrho < r$ we get $\mathcal{E} + 3\varrho$ instead of $\mathcal{E}$. Thus the complement of $\mathcal{E} + 3\varrho$ too is composed of disjoint progressions, each of length not less than m.

We now show that if m is large enough then almost all E_e with $e \in \mathcal{E}$ (or $\mathcal{E} + 3\varrho$) are full. In particular we show that the following conditions are sufficient:

$$m \geqq n_4(\varepsilon_3), \quad \text{where} \quad \varepsilon_3 = \frac{\varepsilon_2}{150\cdot 2^q}.$$

The set

$$M = \bigcup_{e\in\mathcal{E}} [eq, (e+1)q)$$

has the property of the set M in the Simple Lemma. (The progressions have the modulus $3qx_i$ and are each of length at least m; $\varepsilon' = \varepsilon_3$). Therefore

$$\sum_{e\in\mathcal{E}} |E_e| = |E \cap M| \geqq \gamma|M| - (\varepsilon + \varepsilon_3)qn = \gamma q|\mathcal{E}| - (\varepsilon + \varepsilon_3)qn \geqq$$
$$\geqq \gamma q|\mathcal{E}| - 2\varepsilon_3 qn \geqq (\gamma - 150\cdot 2^q \varepsilon_3)q|\mathcal{E}| = (\gamma - \varepsilon_2)q|\mathcal{E}|.$$

Since $|E_e| \leqq (\gamma + \varepsilon_2)q$ for all e, the 'counting argument' applies, showing that the number of poor E_e, $e \in \mathcal{E}$ is at most

$$\frac{2\varepsilon_2}{\varepsilon_1}|\mathcal{E}| = \frac{1}{8u}|\mathcal{E}|.$$

More generally, for each $\varrho=0, \dots, r-1$ there are at most $\frac{1}{8u}|\mathcal{E}|$ poor sets $E_{e+3\varrho}$, $e\in\mathcal{E}$.

Each $e\in\mathcal{E}$ by construction occurs in at least one quadruple (b, s, d, e) with $b\in\mathfrak{B}$ and $s\in S_l$. To each $e\in\mathcal{E}$ we attach one such quadruple making the d, as well as the b and the s, a function of e, $d=\varphi(e)$. Let

$$\mathcal{D}=\{\varphi(e);\ e\in\mathcal{E}\}.$$

Since S_l has at most 2^l elements any particular d in D can arise as a value $\varphi(e)$ at most 2^l times.

We consider the quadruples

$$(b, s+\varrho, \varphi(e)+2\varrho, e+3\varrho),\ e\in\mathcal{E},\ \varrho\in[0, r).$$

We want now to show that for at least one ϱ

$C_{s+\varrho}$ is full (independent of e since $C_{s+\varrho}=C_{(\varrho)}$),

and

almost all $D_\varphi(e)+2\varrho$ are full (counted with multiplicity).

We do this by considering all the ϱ together. The basic tool is again the Simple Lemma. Before applying it, however, we have to remove the multiplicities with which the $C\varphi(e)+\varrho$ occur. There are two sources of multiplicity: the mapping $\varphi(e)=d$, and the forming of the sum $d+\varrho$. We deal first with the case when φ is one to one, where only one of these sources is present.

Set

$$\mathcal{D}'=\left\{d;\ d\in\mathcal{D},\ \sum_{\varrho=0}^{r-1}|D_{d+2\varrho}|\leqq(\gamma-\varepsilon_2)qr\right\}.$$

We construct a subset $\mathcal{D}''\subseteqq\mathcal{D}'$ with the property that consecutive elements have a difference of at least $4r$, but

$$|\mathcal{D}''|\geqq\frac{1}{4r}|\mathcal{D}'|.$$

For this purpose we may go from left to right retaining for our set $\mathcal{D}''$ the first element not ruled out by the restriction upon the differences. Since we exclude at most $4r-1$ elements for each one which we keep we obtain the stated inequality.

Now, each element in

$$\mathcal{D}'''=\mathcal{D}''+\{0, 2, 4, \dots, 2(r-1)\}$$

is uniquely represented. Therefore we have

$$\Big|\bigcup_{x\in\mathcal{D}'''}D_x\Big|=\sum_{d\in\mathcal{D}''}\sum_{\varrho=0}^{r-1}|D_{d+2\varrho}|\leqq(\gamma-\varepsilon_2)rq\,|\mathcal{D}''|.$$

By construction the complement of $\mathcal{D}'''$ consists of progressions (mod 2), each of length at least r. (No difficulty arises when considering elements to the left of the first and to the right of the last elements in $\mathcal{D}'''$, respectively, since $\mathcal{D}+2\varrho\subseteqq\subseteqq\left[\frac{1}{6}n, \frac{2}{3}n\right)\Big)$. Therefore the left hand side can be estimated by the Simple Lemma.

We take

$$M = \bigcup_{x \in \mathcal{D}'''} [xq, (x+1)q), \quad r = n_4(\varepsilon_4), \quad \varepsilon_4 \leqq \frac{\varepsilon_2^2}{600 \cdot 2^q}$$

and obtain

$$|\bigcup_{x \in \mathcal{D}'''} D_x| = |D \cap M| \geqq \gamma q |\mathcal{D}'''| - (\varepsilon + \varepsilon_4) qn =$$

$$= \gamma qr |\mathcal{D}''| - (\varepsilon + \varepsilon_4) qn \geqq \gamma qr |\mathcal{D}''| - z\varepsilon_4 qn.$$

Putting these estimates together gives

$$\varepsilon_2 r |\mathcal{D}''| \leqq 2\varepsilon_4 n, \quad |\mathcal{D}'| \leqq 4r |\mathcal{D}''| \leqq 8 \frac{\varepsilon_4}{\varepsilon_2} n.$$

Next we have the estimate

$$(*) \qquad \sum_{d \in \mathcal{D}} \sum_{\varrho=0}^{r-1} |D_{d+2\varrho}| \geqq \sum_{d \in \mathcal{D} \neg \mathcal{D}'} \sum_{\varrho=0}^{r-1} |D_{d+2\varrho}| \geqq$$

$$\geqq (|\mathcal{D}| - |\mathcal{D}'|)(\gamma - \varepsilon_2) rq \geqq (\gamma - \varepsilon_2) \left(|\mathcal{D}| - 8 \frac{\varepsilon_4}{\varepsilon_2} n \right) rq.$$

In the present special case we have $|\mathcal{D}| = |\mathcal{E}| \geqq \frac{n}{75 \cdot 2^q}$. We therefore get the further inequality

$$\sum_{d \in \mathcal{D}} \sum_{\varrho=0}^{r-1} |D_{d+2\varrho}| \geqq (\gamma - \varepsilon_2) \left(1 - 8 \cdot 75 \cdot 2^q \frac{\varepsilon_4}{\varepsilon_2} \right) rq |\mathcal{D}| \geqq$$

$$\geqq (\gamma - \varepsilon_2)(1 - \varepsilon_2) rq |\mathcal{D}| \geqq (\gamma - 2\varepsilon_2) rq |\mathcal{D}|.$$

By the 'counting argument' we infer that not more than $3 \frac{\varepsilon_2}{\varepsilon_1} r |D| = \frac{3}{16u} r |\mathcal{E}|$ sets $D_{d+2\varrho}$, taken with their multiplicity, are poor. For at most one half of the ϱ's can we have more than $\frac{3}{8u} |\mathcal{E}|$ poor sets $D_{d+2\varrho}$.

If we drop these numbers ϱ, of which there at most $\frac{1}{2} r$, and also those ϱ for which $C_{(\varrho)}$ is poor, there being no more than $\frac{1}{4} r$ of them, some of the numbers ϱ remain. So far we have proved:

There is a number $o \in [0, r)$ such that $C_{s+o} = C_{(o)}$ is full, at most $\frac{3}{8u} |\mathcal{E}|$ of the sets $D_{\varphi(e)+2o}$, $e \in \mathcal{E}$ are poor, and at most $\frac{1}{8u} |\mathcal{E}|$ of the sets E_{e+3o} are poor.

Hence for at most $\frac{1}{2u} |\mathcal{E}|$ elements $e \in \mathcal{E}$ we have either E_{e+3o} or $D_{\varphi(e)+2o}$ poor. We call these $e \in \mathcal{E}$ 'bad'. The density of the bad elements in $\mathcal{E}$ is at most $\frac{1}{2u}$. Now recall that $\mathcal{E}$ is composed of disjoint arithmetic progressions of length at least m.

We can take $m \geqq 2u$. If one of every u consecutive elements of such a progression were a bad one, the density of bad elements in any particular progression in $\mathscr{E}$ would be at least

$$\frac{2}{3u-1} > \frac{2}{3u}$$

and so therefore would be the density of bad elements in the whole of $\mathscr{E}$. Since we have disproved this there exists an arithmetic progression of at least u good elements in $\mathscr{E}$, q.e.d.

Rather little has to be changed in the general case when the elements $d \in \mathscr{D}$ are taken with the multiplicities of $d=\varphi(e)$ not necessarily all equal to one.

Set

$$\mathscr{D}^i = \{d;\ d=\varphi(e) \quad \text{for exactly } i \text{ elements} \quad e \in \mathscr{E}\}.$$

Each $\mathscr{D}^i$ can be treated in exactly the same way that $\mathscr{D}$ was until we reach the formula $(*)$. However, in order to make the formula useful this time we must take a smaller ε_4 (and therefore a larger r):

$$\varepsilon_4 = \frac{\varepsilon^2}{600 \cdot 2^{q+2l}}, \qquad r = n_4(\varepsilon_4).$$

We have then

$$\sum_{d \in \mathscr{D}^i} \sum_{\varrho=0}^{r-1} |D_{d+2\varrho}| \geqq (\gamma - \varepsilon_2)\left(|\mathscr{D}^i| - 8\frac{\varepsilon_4}{\varepsilon_2} n\right) rq.$$

Multiplying by i and summing gives

$$\sum_{e \in \mathscr{E}} \sum_{\varrho=0}^{r-1} |D_{\varphi(e)+2\varrho}| \geqq (\gamma - \varepsilon_2)\left(|\mathscr{E}| - 8\frac{\varepsilon_4}{\varepsilon_2} n \sum_{i=1}^{2^l} i\right) rq \geqq$$

$$\geqq (\gamma - \varepsilon_2)\left(1 - 8\frac{\varepsilon_4}{\varepsilon_2} \cdot 2^{2l} \cdot 75 \cdot 2^q\right) rq\,|\mathscr{E}| \geqq (\gamma - 2\varepsilon_2) rq\,|\mathscr{E}|.$$

The counting argument again shows that there is an $o \in [0, r)$ such that for at most $\frac{3}{8u}|\mathscr{E}|$ elements $e \in \mathscr{E}$ the sets $D_{\varphi(e)+2o}$ are full, and the proof is finished as above.

We have now completed the proof of lemma *BCDE* and with it the proof of the theorem.

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NEIGHBOURINGLY NORMAL ARCHIMEDEAN ORDERED SEMIGROUPS

By
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KOWALSKI [5] proved the following

THEOREM. *Let S be a nonperiodic archimedean ordered semigroup. Then there exists an o-homomorphism of S into the additive semigroup of positive numbers such that two distinct elements of S have the same image if and only if they form an anomalous pair.*

The purpose of this note is to give a theorem (Theorem 1) which generalizes the Kowalski's Theorem such as includes a certain kind of periodic archimedean ordered semigroups.

By an *ordered semigroup*, we mean a semigroup S with a simple order which is compatible with the semigroup operation:

$$a, b, c \in S \text{ and } a \leqq b \text{ imply } ac \leqq bc \text{ and } ca \leqq cb.$$

An element a of an ordered semigroup S is called *positive* if $a^2 > a$ and is called *nonnegative* if $a^2 \geqq a$. An element a of S is called *positive (nonnegative) in the strict sense* if $ax > a$ and $xa > a$ ($ax \geqq a$ and $xa \geqq a$) for every $x \in S$. An ordered semigroup S is called *positively (nonnegatively) ordered (in the strict sense)*, if every element of S is positive (nonnegative) (in the strict sense). The number of distinct powers of an element a of S is called the order of a. A nonnegatively ordered semigroup S is called *archimedean* if, for each pair of elements a, b of S, there exists a natural number n such that $a \leqq b^n$.

REMARK. There are some slight differences between the above terminologies and those in FUCHS [2]. An ordered semigroup is a fully ordered semigroup in Fuchs' sense. An element is nonnegative in the strict sense when it is positive in Fuchs' sense. A nonnegatively ordered semigroup is archimedean when it is archimedean without identity in Fuchs' sense. We treat exclusively nonnegatively ordered semigroups, and so we say simply an archimedean ordered semigroup in the place of an archimedean nonnegatively ordered semigroup.

The importance of archimedean ordered semigroups was pointed out in our previous paper [6], from which now we give some lemmas.

LEMMA 1 ([6] Lemma 2. 2). *An archimedean ordered semigroup is nonnegatively ordered in the strict sense.*

LEMMA 2 ([6] Lemma 2. 3). *For an archimedean ordered semigroup S, the following conditions are equivalent:*

(1) *S contains an idempotent;*
(2) *S has the greatest element;*
(3) *S has the zero element;*
(4) *Every element of S is an element of finite order;*
(5) *S contains an element of finite order.*
Moreover, under these conditions, an idempotent of S is the greatest element and also the zero element of S.

LEMMA 3 ([6] Corollary 2. 4). *An archimedean ordered semigroup contains at most one idempotent.*

If an archimedean ordered semigroup S satisfies any one of the conditions in Lemma 2, then S is called a *periodic* archimedean ordered semigroup. Otherwise S is called a *nonperiodic* archimedean ordered semigroup.

Following ALIMOV [1], two distinct elements a, b of a nonnegatively ordered semigroup S are said to form an *anomalous pair* if $a^n < b^{n+1}$ and $b^n < a^{n+1}$ for every natural number n. Now we say that two distinct elements a, b of S form a *neighbouring pair* if they satisfy the following two conditions:

(1) If $x^k \leqq a^l \neq a^{l+1}$ and $l/k < n/m$ for some $x \in S$ and some natural numbers k, l, m, n, then $x^m \leqq b^n$;

(2) If $x^k \leqq b^l \neq b^{l+1}$ and $l/k < n/m$ for some $x \in S$ and some natural numbers k, l, m, n, then $x^m \leqq a^n$.

LEMMA 4. *Let a and b be two distinct elements of an archimedean ordered semigroup S. If they form an anomalous pair, then they form a neighbouring pair. If S is nonperiodic, then the converse holds also.*

PROOF. First we suppose that a and b form an anomalous pair and that $x^k \leqq a^l \neq a^{l+1}$ and $l/k < n/m$. Then

$$x^{km} \leqq a^{lm} < b^{lm+1} \leqq b^{kn} \quad \text{and so} \quad x^m < b^n.$$

We can similarly prove that, if $x^k \leqq b^l \neq b^{l+1}$ and $l/k < n/m$, then $x^m < a^n$. Next we suppose that S is nonperiodic and that a and b form a neighbouring pair. Then, putting x, k, l, m, n by $a, 2n, 2n, 2n, 2n+1$, respectively, in (1) of the definition of a neighbouring pair, we have $a^{2n} \leqq b^{2n+1}$, and also $b^{2n+1} < b^{2n+2}$, since S is nonperiodic. Therefore $a^n < b^{n+1}$. We can prove $b^n < a^{n+1}$ in a similar way.

An archimedean ordered semigroup S is called *neighbouringly normal* or *n-normal*, if, for the nonidempotent element a of S, the set of natural numbers k for which there exist $x \in S$ and a natural number l such that $x^k \leqq a^l \neq a^{l+1}$ is unbounded above.

LEMMA 5. *The above definition of neighbouring normality is determined irrespective of the choice of an element a in an archimedean ordered semigroup S. S is n-normal if and only if either*

(1) *S is nonperiodic, or*

(2) *S is periodic and, for every $a \in S$ and every natural number k, there exists an element $x \in S$ such that $x^k \leqq a$.*

The condition (2) *can be replaced by*

(2′) *S is periodic and, for every $a \in S$, there exists an element $x \in S$ such that $x^2 \leqq a$.*

PROOF. First we suppose that S is an n-normal periodic archimedean ordered semigroup and that a is the element satisfying the condition in the definition of the neighbouring normality. We denote by e the greatest element of S. We take $b \in S$ with $b < e$ and a natural number k arbitrarily. Since S is archimedean, there exists a natural number m such that $b^m = e$. Also by the definition of n-normality, there exist $x \in S$ and natural numbers l and n such that $x^n \leqq a^l < e$ and $mk \leqq n$. Then we have

$$x^{mk} \leqq x^n \leqq a^l < e = b^m \quad \text{and so} \quad x^k < b.$$

Thus we have proved that, if S is n-normal, then S satisfies (1) or (2). It is easily verified that, if S satisfies (1) or (2), then S is n-normal. This proves the second assertion, from which the first assertion is trivial. (2) implies (2′) trivially. Finally we assume the condition (2′). We take $a \in S$ and a natural number k arbitrarily. Choosing a natural number n such that $k \leqq 2^n$, we can find, by (2′), a sequence of elements $x_1, \dots, x_n$ such that

$$x_1^2 \leqq a, \; x_2^2 \leqq x_1, \; \dots, \; x_n^2 \leqq x_{n-1}.$$

Then we have $x_n^k \leqq x_n^{2^n} \leqq a$. Thus (2) holds.

A nonnegatively ordered semigroup S is called *naturally ordered*, if $a, b \in S$ and $a < b$ imply the existence of $c, d \in S$ such that $ac = b$ and $da = b$.

LEMMA 6. *An archimedean naturally ordered semigroup S is either cyclic (that is, S is generated by a single element) or n-normal.*

PROOF. By Lemma 5, we suppose that S is periodic. First we suppose that S has the least element a. For every $x \in S$, there exists a natural number n such that $a^n \leqq x \leqq a^{n+1}$. If S were not cyclic, then, for some $x \in S$, we have $a^n < x < a^{n+1}$ and so there exists $c \in S$ such that $a^n c = x$. Hence we have $a^{n+1} \leqq a^n c = x < a^{n+1}$, which is a contradiction. Next we suppose that S has not the least element. We take an arbitrary element $a \in S$. Then we can find an element $x \in S$ such that $x < a$ and so there exists $y \in S$ such that $xy = a$. Putting $z = \min\{x, y\}$, we have $z^2 \leqq a$ and so, by Lemma 5, S is n-normal.

LEMMA 7. *Let S be an n-normal periodic archimedean ordered semigroup with the greatest element e, and let $x^p \leqq y^q$, $y^r \leqq x^s$, $x^p < e$, $y^r < e$ for some $x, y \in S$ and some natural numbers p, q, r, s. Then $r/q \leqq s/p$.*

PROOF. We choose the natural numbers h and k arbitrarily. Since S is n-normal, we can find $u, v \in S$ such that $u^h \leqq x$ and $v^k \leqq y$. We put $z = \min\{u, v\}$. Since S is archimedean and $x < e$, $y < e$, we have $z^l \leqq x < z^{l+1}$, $z^m \leqq y < z^{m+1}$ for some $l \geqq h$ and $m \geqq k$. Hence

$$z^{pl} \leqq x^p \leqq y^q \leqq z^{q(m+1)}, \quad z^{rm} \leqq y^r \leqq x^s \leqq z^{s(l+1)}.$$

Since $x^p < e$, we have $z^{pl} < e$ and so, since $z^{pl} \leqq z^{q(m+1)}$, we have $pl \leqq q(m+1)$. Similarly $rm \leqq s(l+1)$. Therefore $pr/qs \leqq ((m+1)/m)((l+1)/l)$. The numbers l and m become sufficiently large with h and k, and so, from the above inequality, we obtain $pr/qs \leqq 1$ and $r/q \leqq s/p$.

Let S and T be two ordered semigroups. A mapping ω of a subset S^* of S into T is called a *partial o-homomorphism*, if it satisfies the following two conditions:

(1) for every $s_1, s_2 \in S^*$ such that $s_1 s_2 \in S^*$ in S, we have $\omega(s_1 s_2) = \omega(s_1)\omega(s_2)$;

(2) for every $s_1, s_2 \in S^*$ such that $s_1 \leqq s_2$, we have $\omega(s_1) \leqq \omega(s_2)$.

THEOREM 1. *Let S be an n-normal archimedean ordered semigroup and let S^* be the set of non-idempotents of S. (Thus, if S is nonperiodic, then $S^*=S$, and, if S is periodic with the greatest element e, then $S^*=S\backslash e$.) Then there exists a partial o-homomorphism of S^* into the additive semigroup P of positive real numbers such that two distinct elements of S^* have the same image if and only if they form a neighbouring pair.*

PROOF. In the case when S is nonperiodic, the notion of a neighbouring pair is equivalent to that of an anomalous pair. Thus the assertion is nothing but the Kowalski's Theorem. Hence, in the rest of the proof, we assume that S is periodic with the greatest element e. We fix an element a of S^*. For $b\in S^*$, we define $\omega(b)$ as the infimum of quotients l/k of two natural numbers l and k such that $x^k\leqq a$ and $b\leqq x^l$ for some $x\in S^*$.

1° *For every $b\in S^*$, we have $\omega(b)>0$.*

In fact, for an arbitrarily chosen real number $\varepsilon>0$, there exist $x\in S^*$ and natural numbers k and l such that

$$x^k\leqq a,\quad b\leqq x^l,\quad \omega(b)\leqq l/k<\omega(b)+\varepsilon.$$

We put $c=\min\{a, b\}$. Since S is archimedean, there exists a natural number n such that $a\leqq c^n$. Hence

$$x^k\leqq a\leqq c^n,\quad c\leqq b\leqq x^l,\quad c<e,\quad x^k<e.$$

Therefore, by Lemma 7, we have $1/n\leqq l/k<\omega(b)+\varepsilon$. Hence we have $\omega(b)\geqq 1/n>0$.

2° *If $b, c, bc\in S^*$, then $\omega(bc)\leqq\omega(b)+\omega(c)$.*

In fact, for an arbitrarily chosen real number $\varepsilon>0$, there exist $x, y\in S^*$ and natural numbers k, l, m, n such that

$$x^k\leqq a,\quad b\leqq x^l,\quad y^m\leqq a,\quad c\leqq y^n,\quad \omega(b)\leqq l/k<\omega(b)+\varepsilon,\quad \omega(c)\leqq n/m<\omega(c)+\varepsilon.$$

We take a natural number p such that $1/p<\varepsilon$ and then, since S is n-normal, take z_1, z_2 of S such that $z_1^p\leqq x$ and $z_2^p\leqq y$. We put $z=\min\{z_1, z_2\}$. Since S is archimedean and $x<e$, there exists a natural number q such that $z^q\leqq x<z^{q+1}$. Then, since $z^p\leqq z_1^p\leqq x$, we have $p\leqq q$ and also

$$z^{qk}\leqq x^k\leqq a,\quad b\leqq x^l\leqq z^{l(q+1)}.$$

Similarly, for some $p\leqq r$, we have

$$z^{mr}\leqq a,\quad c\leqq z^{n(r+1)}.$$

Therefore

$$z^{\max\{qk,\, mr\}}\leqq a,\quad bc\leqq z^{l(q+1)+n(r+1)}.$$

Hence, by definition, we have

$$\omega(bc)\leqq\big(l(q+1)+n(r+1)\big)/\max\{qk, mr\}\leqq\big(l(q+1)/qk\big)+\big(n(r+1)/mr\big)=$$

$$=\big((l/k)(1+(1/q))\big)+\big((n/m)(1+(1/r))\big)<\big(\omega(b)+\varepsilon\big)(1+\varepsilon)+\big(\omega(c)+\varepsilon\big)(1+\varepsilon).$$

Hence we have $\omega(bc)\leqq\omega(b)+\omega(c)$.

3° *If $b, c, bc\in S^*$, then $\omega(b)+\omega(c)\leqq\omega(bc)$.*

In fact, for an arbitrarily chosen real number $\varepsilon > 0$, there exist $x \in S^*$ and natural numbers k and l such that

$$x^k \leqq a, \quad bc \leqq x^l, \quad \omega(bc) \leqq l/k < \omega(bc) + \varepsilon.$$

We take a natural number p such that $1/p < \varepsilon$. Then, since S is n-normal, there exists $y \in S^*$ such that $y^p \leqq x$. Since S is archimedean, there exist natural numbers p', q, r, s such that

$$y^{p'} \leqq x < y^{p'+1}, \quad y^q \leqq b < y^{q+1}, \quad y^r \leqq c < y^{r+1}, \quad y^s \leqq a < y^{s+1}.$$

Then we have $p \leqq p'$ and so, since $y^p \leqq y^{p'} \leqq x \leqq x^k \leqq a$, we obtain $p \leqq s$. Now we get

$$x^k \leqq a < y^{s+1}, \quad y^{q+r} \leqq bc \leqq x^l,$$

and so, by Lemma 7,

$$(q+r)/(s+1) \leqq l/k < \omega(bc) + \varepsilon.$$

By definition, we have

$$\omega(b) \leqq (q+1)/s, \quad \omega(c) \leqq (r+1)/s$$

and so

$$\omega(b) + \omega(c) \leqq (q+r+2)/s \leqq \big((q+r)/(s+1)\big)\big(1 + (1/s)\big) + 2/s <$$
$$< \big(\omega(bc) + \varepsilon\big)(1+\varepsilon) + 2\varepsilon.$$

Hence $\omega(b) + \omega(c) \leqq \omega(bc)$.

4° *If* $b, c \in S^*$ *and* $b \leqq c$, *then* $\omega(b) \leqq \omega(c)$.

In fact, for an arbitrarily chosen real number $\varepsilon > 0$, there exist $x \in S^*$ and natural numbers k and l such that

$$x^k \leqq a, \quad c \leqq x^l, \quad \omega(c) \leqq l/k < \omega(c) + \varepsilon.$$

Then, since $b \leqq c \leqq x^l$, we have $\omega(b) \leqq l/k < \omega(c) + \varepsilon$, and so $\omega(b) \leqq (c)$.

5° *If* $b, c \in S^*$ *and* b *and* c *form a neighbouring pair, then* $\omega(b) = \omega(c)$.

In fact, for a sufficiently large arbitrary natural number k, we take $x \in S^*$ such that $x^k \leqq b$. Also we take a natural number k' such that $x^{k'} \leqq b < x^{k'+1}$. Then $k \leqq k'$. Also we have

$$x^{k'} \leqq b < e, \quad 1/k' < 1/(k'-1),$$

and so, since b and c form a neighbouring pair, we have $x^{k'-1} \leqq c$. By 1°—4°,

$$\omega(x), \omega(b), \omega(c) > 0, \quad \omega(b) \leqq (k'+1)\omega(x),$$

$$(k'-1)\omega(x) \leqq \omega(c),$$

and so

$$\omega(c)/\omega(b) \geqq (k'-1)/(k'+1).$$

Hence $\omega(c) \geqq \omega(b)$. We can similarly prove that $\omega(b) \geqq \omega(c)$.

6° *If* $b, c \in S^*$ *and* $\omega(b) = \omega(c)$, *then* b *and* c *form a neighbouring pair.*

In fact, we suppose that $x^k \leqq b^l < e$ and $l/k < n/m$. Then we have $k\omega(x) \leqq$ $\leqq l\omega(b)$ and so

$$\omega(x)/\omega(c) = \omega(x)/\omega(b) \leqq l/k < n/m.$$

If $c^n = e$, then $x^m \leqq c^n$ trivially. Next we suppose that $c^n < e$. Then $\omega(x) < (n/m)\omega(c) =$

$=(1/m)\omega(c^n)$, and so, by the definition of $\omega(x)$, there exist $y \in S^*$ and natural numbers p and q such that

$$y^p \leqq a, \quad x \leqq y^q, \quad \omega(x) \leqq q/p < (1/m)\omega\,(c^n).$$

By the way of contradiction, we suppose $c^n < x^m$. Then we have $y^p \leqq a$ and $c^n < x^m \leqq \leqq y^{qm}$, and so, by definition, $\omega(c^n) \leqq mq/p$. Hence $(1/m)\omega(c^n) \leqq q/p$, which is a contradiction. Thus $x^m \leqq c^n$. Similarly, if $x^k \leqq c^l < e$ and $l/k < n/m$, then $x^m \leqq b^n$.

This completes the proof of the theorem.

Finally we give an example which shows that Theorem 1 does not hold in general without the assumption of n-normality.

EXAMPLE. S is a system with the multiplication

	a	b	c	d	f	g	h	e
a	c	d	f	g	h	h	e	e
b	d	f	g	h	h	e	e	e
c	f	g	h	h	e	e	e	e
d	g	h	h	e	e	e	e	e
f	h	h	e	e	e	e	e	e
g	h	e	e	e	e	e	e	e
h	e	e	e	e	e	e	e	e
e	e	e	e	e	e	e	e	e

and with the order relation $a<b<c<d<f<g<h<e$. Then it is verified that S is a periodic archimedean ordered semigroup with the maximal element e. But there is no partial o-homomorphism of $S^* = S \backslash e$ into the additive semigroup of positive real numbers. In fact, by the way of contradiction, if ω is a partial o-homomorphism, then, since $a^3 = b^2 = f < e$ and $a^4 = b^3 = h < e$, we have $3\omega(a) = 2\omega(b)$ and $4\omega(a) = 3\omega(b)$ at the same time, which is absurd.

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ON THE RING OF BITRANSLATIONS OF CERTAIN RINGS

By

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1. Introduction. In the theory of ring extensions, the notion of a bitranslation (our terminology, for the definition see below) plays the role of automorphisms in group extensions. Bitranslations were studied by HOCHSCHILD [5] for algebras („multiplications", they are also linear transformations); by CLIFFORD [2] for semigroups („pairs of linked left and right translations", where only the multiplicative part of the definition is retained); by RÉDEI [9] („Doppelhomothetismen") and MAC LANE [7] for rings („bimultiplications"), among others.

Under a natural addition and multiplication, the set of bitranslations of a ring itself forms a ring. RÉDEI [9] defines a holomorph of a ring R as a (natural) split extension of R by a maximal ring of permutable bitranslations. It turns out that R may have more than one holomorph but that otherwise these have several properties analogous to the properties of the holomorph of a group. A number of properties of a ring holomorph have been established by RÉDEI [8], [9], SZENDREI [10], VAN LEEUWEN [6], and WEINERT and EILHAUER [11].

The purpose of this paper is to study the ring $\Omega(R)$ of bitranslations of a ring R for which $\alpha\beta=\beta\alpha=0$ for all $\beta\in R$ implies $\alpha=0$ (see [5], [7], [11]). In this case $\Omega(R)$ is a ring of permutable bitranslations and the split extension of R by $\Omega(R)$ is the unique holomorph of R. The analogy with the group holomorph is in this case much stronger than in the case of an arbitrary ring. Section 2 contains most of the necessary definitions and notation. We are mainly concerned with rings R satisfying the above condition; for such R, in Section 3, we establish several properties of $\Omega(R)$, and in Section 4, characterize $\Omega(R)$ in several ways. We conclude in Section 5 by constructing $\Omega(R)$ for a ring R which is of a special kind but need not satisfy the above condition, and derive several consequences of this result.

It is of interest that the theory of ideal extensions of semigroups is quite similar to the theory of ring extensions, at least as far as multiplication is concerned (see [2] and 4. 4, [3]). Even though the multiplicative part of a ring is a semigroup, this still seems surprising in view of very different definitions of extensions in semigroups and rings.

From the results of Sections 3 and 4, we see that for a ring R satisfying the above condition, $\Omega(R)$ plays the role of a „holomorph" of R (cf. § 54, [8]).

2. Terminology and notation. We use the results and follow the terminology and notation of RÉDEI [8] (§§ 52—54) with some exceptions. Throughout, R denotes an arbitrary ring unless specified otherwise.

A *bitranslation* a of R is a double operator on R (i.e., a pair of functions mapping R into R, one of which is written on the left and the other one on the right, both denoted by a) with the properties

(1) $$a(\alpha+\beta) = a\alpha+a\beta, \quad (\alpha+\beta)a = \alpha a+\beta a$$

(2) $$a(\alpha\beta)=(a\alpha)\beta, \quad (\alpha\beta)a=\alpha(\beta a)$$

(3) $$(\alpha a)\beta=\alpha(a\beta)$$

for all $\alpha, \beta\in R$ (Rédei includes the condition $a(\alpha a)=(a\alpha)a$; for the purposes of this paper, it is more convenient to adopt Mac Lane's definition). The pair of identical transformations is always a bitranslation, as well as the *inner bitranslation* induced by $\alpha\in R$, denoted by $[\alpha]$, and defined by

(4) $$[\alpha]\beta=\alpha\beta, \quad \beta[\alpha]=\beta\alpha$$

for all $\beta\in R$. We denote by $\Omega(R)$ *the set of all bitranslations of R and by* $\Pi(R)$ *the set of all inner bitranslations of R.*

A *biendomorphism* of R^+ is a double operator on R consisting of two endomorphisms of the additive group R^+ of R. On the set $\mathscr{E}_2(R^+)$ of biendomorphisms of R^+, define addition and multiplication by

$$(a+b)\alpha = a\alpha+b\alpha, \qquad \alpha(a+b) = \alpha a+\alpha b$$

$$(ab)\alpha=a(b\alpha), \qquad \alpha(ab)=a(\alpha b)$$

for all $\alpha\in R$. Then $\mathscr{E}_2(R^+)$ is a ring isomorphic to the direct sum of the endomorphism ring of R^+ and its opposite ring. In light of (1), $\Omega(R)$ *is a subring of* $\mathscr{E}_2(R^+)$ *with identity* consisting of the pair of identical transformations.

We say that $a, b\in\Omega(R)$ are *permutable* (MAC LANE [7]) if

(5) $$(a\alpha)b=a(\alpha b)$$

for all $\alpha\in R$. Any nonempty set of permutable bitranslations generates a subring of $\mathscr{E}_2(R^+)$ which is itself a ring of permutable bitranslations, and every such ring is contained in a maximal one. $\Pi(R)$ is an ideal of every maximal ring of permutable bitranslations and of $\Omega(R)$.

A *holomorph* of R is the split extension of R by a maximal ring B of permutable bitranslations of R, with operations defined by

(6) $$\begin{cases}(a, \alpha)+(b, \beta) = (a+b, \alpha+\beta), \\ (a, \alpha)(b, \beta) = (ab, \alpha b+a\beta+\alpha\beta), \quad (a, b\in B; \quad \alpha, \beta\in R).\end{cases}$$

If D is any subset of $\Omega(R)$, let

$$D\cdot R=\{(a, \alpha)\,|\,a\in D, \alpha\in R\}$$

under (6) if $D\cdot R$ is closed under these operations. We will write $\Omega\cdot R$ instead of $\Omega(R)\cdot R$.

Let S be an extension of R by Q, i.e., R is an ideal of S and $S/R\cong Q$. We write the elements of S in the form (a, α) where $a\in Q$, $\alpha\in R$, and as usual identify $(0, \alpha)$ with α, $(a, 0)+R$ with a (for multiplication, addition, and conditions to be satisfied see Theorem 112, [8]); in the case of a split extension, the operations in S are given by (6), where $\alpha\to a\alpha$, $\alpha\to\alpha a$ is the action of the bitranslation of R induced by $a\in Q$.

The function $\pi\colon R \to \Pi(R)$ defined by $\alpha\pi = [\alpha]$, is an onto homomorphism with the kernel

$$\mathfrak{A}(R) = \{\alpha \in R \mid \alpha\beta = \beta\alpha = 0 \quad \text{for all} \quad \beta \in R\}.$$

We will be mainly interested in the rings for which $\mathfrak{A}(R) = 0$. The *center* of R is denoted by $\mathfrak{Z}(R)$. Further let

$$\Gamma(R) = \{a \in \Omega(R) \mid a\alpha = \alpha a \quad \text{for all} \quad \alpha \in R\}.$$

3. Properties of $\Omega(R)$ when $\mathfrak{A}(R) = 0$. We first prove a sequence of lemmas for an arbitrary ring R, and then use them to establish a number of properties of $\Omega(R)$ for the case $\mathfrak{A}(R) = 0$. Note that some of these lemmas have independent interest and that their proofs depend only on multiplicative parts (2) and (3) of the definition of a bitranslation.

LEMMA 1. *If $a, b \in \Omega(R)$ and $\alpha \in R$, then $(a\alpha)b - a(\alpha b) \in \mathfrak{A}(R)$.*

PROOF. For any $\beta \in R$, we have

$$\{(a\alpha)b\}\beta = (a\alpha)(b\beta) = a\{\alpha(b\beta)\} = a\{(\alpha b)\beta\} = \{a(\alpha b)\}\beta$$

and dually $\beta\{(a\alpha)b\} = \beta\{a(\alpha b)\}$ so that $(a\alpha)b - a(\alpha b) \in \mathfrak{A}(R)$.

LEMMA 2. *If S is an extension of R, $\alpha \in \mathfrak{Z}(R)$, and $\beta \in S$, then $\alpha\beta - \beta\alpha \in \mathfrak{A}(R)$.*

PROOF. *If $\gamma \in R$, then*

$$(\alpha\beta)\gamma = \alpha(\beta\gamma) = (\beta\gamma)\alpha = \beta(\gamma\alpha) = \beta(\alpha\gamma) = (\beta\alpha)\gamma$$

and dually $\gamma(\alpha\beta) = \gamma(\beta\alpha)$ so that $\alpha\beta - \beta\alpha \in \mathfrak{A}(R)$.

LEMMA 3. *For $a \in \Omega(R)$ and $\alpha \in R$, we have $a[\alpha] = [\alpha]a$ if and only if $a\alpha - \alpha a \in \mathfrak{A}(R)$.*

PROOF. If $a[\alpha] = [\alpha]a$, then for any $\beta \in R$,

$$(a\alpha)\beta = a(\alpha\beta) = a([\alpha]\beta) = (a[\alpha])\beta = ([\alpha]a)\beta = [\alpha](a\beta) = \alpha(a\beta) = (\alpha a)\beta$$

and dually $\beta(a\alpha) = \beta(\alpha a)$, which implies $a\alpha - \alpha a \in \mathfrak{A}(R)$. Conversely, if $a\alpha - \alpha a \in \mathfrak{A}(R)$, then for any $\beta \in R$,

$$(a[\alpha])\beta = a([\alpha]\beta) = a(\alpha\beta) = (a\alpha)\beta = (\alpha a)\beta = \alpha(a\beta) = [\alpha](a\beta) = ([\alpha]a)\beta$$

and dually $\beta(a[\alpha]) = \beta([\alpha]a)$ so that $a[\alpha] = [\alpha]a$.

LEMMA 4. *If $a \in \Gamma(R)$ and $b \in \Omega(R)$ are permutable, then $ab = ba$.*

PROOF. For any $\alpha \in R$, we obtain

$$(ab)\alpha = a(b\alpha) = (b\alpha)a = b(\alpha a) = b(a\alpha) = (ba)\alpha,$$

and dually $\alpha(ab) = \alpha(ba)$ so that $ab = ba$.

LEMMA 5. *For $(a, \alpha) \in \Omega \cdot R$, we have*

$$(a, \alpha)([\beta], \gamma) = ([\beta], \gamma)(a, \alpha) \quad \textit{for all} \quad \beta, \gamma \in R \tag{7}$$

if and only if $a \in \Gamma(R)$ and $\alpha \in \mathfrak{Z}(R)$.

PROOF. Suppose that (7) holds. Then

$$\alpha\beta + a\gamma + \alpha\gamma = \beta\alpha + \gamma a + \gamma\alpha \quad \text{for all} \quad \beta, \gamma \in R. \tag{8}$$

For $\gamma = 0$, (8) yields $\alpha\beta = \beta\alpha$ for all $\beta \in R$ so that $\alpha \in \mathfrak{Z}(R)$. But then (8) becomes $a\gamma = \gamma a$ for all $\gamma \in R$ and thus $a \in \Gamma(R)$. Conversely, suppose that $a \in \Gamma(R)$ and $\alpha \in \mathfrak{Z}(R)$. For $\beta, \gamma \in R$, we obtain

$$(a[\beta])\gamma = a(\beta\gamma) = (a\beta)\gamma = (\beta a)\gamma = \beta(a\gamma) = ([\beta]a)\gamma$$

and dually $\gamma(a[\beta]) = \gamma([\beta]a)$. Thus $a[\beta] = [\beta]a$; the hypothesis implies (8) directly so that (7) also holds.

LEMMA 6. *If $a \in \Omega(R)$ and $\alpha \in \mathfrak{Z}(R)$, then $a\alpha - \alpha a \in \mathfrak{A}(R)$.*

PROOF. If $\beta \in R$, then

$$(a\alpha)\beta = a(\alpha\beta) = a(\beta\alpha) = (a\beta)\alpha = \alpha(a\beta) = (\alpha a)\beta$$

and dually $\beta(a\alpha) = \beta(\alpha a)$ so that $a\alpha - \alpha a \in \mathfrak{A}(R)$.

We are now ready to consider the case $\mathfrak{A}(R) = 0$. Note that items a) and e) below were established in [11], item b) generalizes Corollary to Theorem 2, [10], and item c) Theorem 5, [9].

THEOREM 1. *In a ring R for which $\mathfrak{A}(R) = 0$, the following statements hold.*
a) *Any two elements of $\Omega(R)$ are permutable and thus $\Omega \cdot R$ is the unique holomorph of R.*
b) *If S is an extension of R, then $\mathfrak{Z}(R) = \mathfrak{Z}(S) \cap R$.*
c) *If D is any subring of $\Omega(R)$ containing $\Pi(R)$, then $\mathfrak{Z}(D) = \Gamma(R) \cap D$ and $\mathfrak{Z}(D \cdot R) = \mathfrak{Z}(D) \cdot \mathfrak{Z}(R)$.*
d) $\mathfrak{Z}(\Omega(R)) = \Gamma(R)$, $\mathfrak{Z}(\Omega \cdot R) = \Gamma(R) \cdot \mathfrak{Z}(R)$.
e) *If R is commutative, $\Omega \cdot R$ is also.*

PROOF. Item a) follows from Lemma 1 and b) from Lemma 2.
c) If $a \in \mathfrak{Z}(D)$, then $a[\alpha] = [\alpha]a$ for all $\alpha \in R$ since $\Pi(R) \subseteqq D$ which by Lemma 3 implies $a\alpha = \alpha a$ for all $\alpha \in R$. Thus $\mathfrak{Z}(D) \subseteqq \Gamma(R) \cap D$; the opposite inclusion follows from Lemma 4. If $(a, \alpha) \in \mathfrak{Z}(D \cdot R)$, then $(a, \alpha)([\beta], \gamma) = ([\beta], \gamma)(a, \alpha)$ for all $\beta, \gamma \in R$ since $\Pi(R) \subseteqq D$ which by Lemma 5 implies $a \in \Gamma(R)$ and $\alpha \in \mathfrak{Z}(R)$. Conversely, let $a \in \Gamma(R) \cap D$, $\alpha \in \mathfrak{Z}(R)$, and $(b, \beta) \in D \cdot R$. Then for any $\gamma \in R$,

$$(ab)\gamma = a(b\gamma) = (b\gamma)a = b(\gamma a) = b(a\gamma) = (ba)\gamma$$

and dually $\gamma(ab) = \gamma(ba)$. Hence $ab - ba \in \mathfrak{A}(R) = 0$. Also by Lemma 6, $b\alpha - \alpha b \in \mathfrak{A}(R) = 0$ since $\alpha \in \mathfrak{Z}(R)$. Thus

$$(a, \alpha)(b, \beta) = (ab, \alpha b + a\beta + \alpha\beta) = (ba, b\alpha + \beta a + \beta\alpha) = (b, \beta)(a, \alpha)$$

and hence $(a, \alpha) \in \mathfrak{Z}(D \cdot R)$. Therefore

$$\mathfrak{Z}(D \cdot R) = \{(a, \alpha) \mid a \in \Gamma(R) \cap D,\ \alpha \in \mathfrak{Z}(R)\} = \mathfrak{Z}(D) \cdot \mathfrak{Z}(R)$$

since $\mathfrak{Z}(D) = \Gamma(R) \cap D$.

Item d) follows from c) by letting $D=\Omega(R)$.

e) Let $(a, \alpha)\in\Omega\cdot R$. Since $\mathfrak{Z}(R)=R$ and $\mathfrak{A}(R)=0$, Lemma 6 implies $a\in\Gamma(R)$, which by d) implies $(a, \alpha)\in\mathfrak{Z}(\Omega(R))$. Thus $\Omega\cdot R$ is commutative.

We will repeatedly make use of the following lemma without express mention.

LEMMA 7. *For $a\in\Omega(R)$ and $\alpha\in R$, we have $a[\alpha]=[a\alpha]$ and $[\alpha]a=[\alpha a]$.*

PROOF. For any $\beta\in R$, we have

$$(a[\alpha])\beta=a([\alpha]\beta)=a(\alpha\beta)=(a\alpha)\beta=[a\alpha]\beta,$$

$$\beta(a[\alpha])=(\beta a)[\alpha]=(\beta a)\alpha=\beta(a\alpha)=\beta[a\alpha]$$

so that $a[\alpha]=[a\alpha]$; dually $[\alpha]a=[\alpha a]$.

Since $\Pi(R)$ is a ring, we obtain the following

COROLLARY. *If D is any ring of bitranslations of R, then $D\cap\Pi(R)$ is an ideal of D.*

An analogue of Theorem 121, [8] is provided by

THEOREM 2. *An ideal I of a ring R with $\mathfrak{A}(R)=0$ is characteristic if and only if the image $\pi(I)$ of I in $\Pi(R)$ is an ideal of $\Omega(R)$.*

PROOF. Let I be a characteristic ideal of R. Then $\pi(I)$ is a characteristic ideal of $\Pi(R)$ since π is an isomorphism of R onto $\Pi(R)$. But then $\pi(I)$ is an ideal of $\Omega(R)$. Conversely, let $\pi(I)$ be an ideal of $\Omega(R)$ and S be an extension of R. If $\alpha\in I$, $\beta\in S$, then $\alpha\beta=\alpha[\beta]=\alpha a$, where a is the restriction of $[\beta]$ to R, and $a\in\Omega(R)$. Since $\pi\colon \alpha a\to[\alpha a]=[\alpha]a\in\pi(I)$, it follows that $\alpha a\in I$. Hence I is a right ideal; dually I is also a left ideal.

Examples of characteristic ideals in any ring R:

a) Semiprime ideals of R; for if I is a semiprime ideal of R, S an extension of R, and $\alpha\in I$, $\beta\in S$, $\gamma\in R$, then $(\alpha\beta)\gamma(\alpha\beta)=\alpha(\beta\gamma\alpha\beta)\in I$ so that $(\alpha\beta)R(\alpha\beta)\subseteqq I$ and thus $\alpha\beta\in I$. In particular, prime ideals (Theorem 4, [9]) and the prime radical of R are characteristic.

b) Any ideal with the property $I^2=I$ (I^2 is the set of all finite sums of elements of the form $\alpha\beta$ with $\alpha, \beta\in I$).

c) $\mathfrak{A}(R)$ (Theorem 2, [11]) and R^2.

d) The Amitsur—Kuroš radical of R.

4. Characterizations of the ring of bitranslations of R for $\mathfrak{A}(R)=0$. First let R be an arbitrary ring and S an extension of R. The function $\tau\colon S\to\Omega(R)$ defined by $\tau\colon a\to a^\tau$ where

$$a^\tau\alpha=a\alpha,\quad \alpha a^\tau=\alpha a$$

for all $\alpha\in R$, (note that heretofore we have tacitly used the notation $a^\tau=a$) has the following properties:

α) τ is a homomorphism extending $\pi\colon R\to\Omega(R)$ (recall that $\pi\colon\alpha\to[\alpha]$ is defined as a function from R onto $\Pi(R)$, it is considered here as a mapping of R into $\Omega(R)$). The kernel of τ is the set

$$\mathfrak{A}_S(R)=\{a\in S\mid a\alpha=\alpha a=0\quad\text{for all}\quad \alpha\in R\}.$$

β) An element $a \in S$ induces a bitranslation $b \in \Omega(R)$ in R if and only if $a^\tau = b$.

γ) If $\mathfrak{A}(R) = 0$, then τ is the unique extension of π to S. For suppose that σ is another extension of π. Then for any $a \in S$, $\alpha \in R$, we have

$$[a^\sigma \alpha] = a^\sigma[\alpha] = a^\sigma \alpha^\sigma = (a\alpha)^\sigma = [a\alpha] = [a^\tau \alpha]$$

so that $a^\sigma \alpha = a^\tau \alpha$; dually $\alpha a^\sigma = \alpha a^\tau$ which implies $a^\sigma = a^\tau$.

The next two lemmas will be used for establishing several characterizations of $\Omega(R)$ when $\mathfrak{A}(R) = 0$; they seem also to be of independent interest. We will use the notation introduced above.

LEMMA 8. *When* $\mathfrak{A}(R) = 0$, *the following statements are equivalent:*

a) τ *is a monomorphism;*

b) *every bitranslation on* R *is induced by at most one element of* S;

c) *if* A *is an ideal of* S *such that* $A \cap R = 0$, *then* $A = 0$;

d) R *is not a direct summand in any subring of* S *(different from* R*).*

PROOF. a)$\Rightarrow$b) by β) above.

b)$\Rightarrow$c). If $a \in A$, then $A \cap R = 0$ implies that a and 0 induce the same bitranslation on S which then implies that $a = 0$.

c)$\Rightarrow$d). If $R \oplus T$ is a subring of S, where $T \subseteqq S$, then $T \subseteqq \mathfrak{A}_S(R)$. Since $\mathfrak{A}_S(R) \cap R = \mathfrak{A}(R) = 0$, and $\mathfrak{A}_S(R)$ is an ideal, c) implies that $\mathfrak{A}_S(R) = 0$ and thus also $T = 0$. Hence $R \oplus T = R$.

d)$\Rightarrow$a). Since $\mathfrak{A}_S(R) \cap R = \mathfrak{A}(R) = 0$, the sum $\mathfrak{A}_S(R) + R$ is direct, whence $\mathfrak{A}_S(R) = 0$, that is, τ is one-to-one.

LEMMA 9. *In any ring* R, a′) *and* b′) *are equivalent,* c′) *and* d′) *are equivalent. If* $\mathfrak{A}(R) = 0$, *then also* b′) *implies* c′).

a′) τ *maps* S *onto* $\Omega(R)$;

b′) *every bitranslation of* R *is induced by at least one element of* S;

c′) *for every extension* D *of* R *properly containing* S, *there is an ideal* A *of* D *such that* $A \cap R = 0$ *and* $A \neq 0$;

d′) *for every extension* D *of* R *properly containing* S, R *is a direct summand of a subring of* D *different from* R.

PROOF. a′)$\Leftrightarrow$b′) follows from β) above, while c′)$\Leftrightarrow$d′) is clear. Suppose that $\mathfrak{A}(R) = 0$ and that b′) holds. Let $a \in D$, $a \notin S$; a induces some bitranslation b of R. By b′), there is $c \in S$ which also induces b. Thus for every $\alpha \in R$, $(a-c)\alpha = \alpha(a-c) = 0$ and hence $a - c \in \mathfrak{A}_D(R)$ where $a - c \neq 0$. Since $\mathfrak{A}_D(R) \cap R = \mathfrak{A}(R) = 0$, $\mathfrak{A}_D(R)$ is the desired ideal of D, and c′) holds.

THEOREM 3. *Let* $\mathfrak{A}(R) = 0$ *and let* S *be an extension of* R. *Then* $\pi\colon R \to \Omega(R)$ *can be extended (uniquely) to an isomorphism of* S *onto* $\Omega(R)$ *if and only if any one of the conditions in Lemma* 8 *and any one of the conditions in Lemma* 9 *are satisfied.*

PROOF. By Lemmas 8 and 9, it remains only to show that a) together with d′) implies a′). Suppose that a) holds and a′) does not. Since τ is a monomorphism, we may identify S with its image in $\Omega(S)$ under τ. Hence $\Omega(R)$ is an extension of R (here identified with $\Pi(R)$) properly containing S in which R is not a direct summand of any subring different from R (by d)). Consequently d′) does not hold.

For a double operator a on a ring R define the *conjugate* a^* of a as the double operator defined by: $a^*\alpha=\alpha a$, $\alpha a^*=a\alpha$ for all $\alpha\in R$. If A is any set of double operators on R, we let $\mathscr{K}(A)=\{a^*|a\in A\}$. For double operators a, b on R and any $\alpha\in R$,

$$(ab)^*\alpha=\alpha(ab)=(\alpha a)b=(a^*\alpha)b=b^*(a^*\alpha)=(b^*a^*)\alpha$$

and dually $\alpha(ab)^*=\alpha(b^*a^*)$ so that $(ab)^*=b^*a^*$. Similarly $(a^*)^*=a$ and $(a+b)^*=$ $=a^*+b^*$. Clearly $\Gamma(R)=\{a\in\Omega(R)|a=a^*\}$. By $C(\Pi(R))$ denote the *centralizer* of $\Pi(R)$ in the ring $\mathscr{E}_2(R^+)$ of biendomorphisms of R (see Section 2).

A theorem of Gluskin for semigroups (1. 3, [4]; for the proof see the correction) and the usual characterization of the holomorph of a group as the normalizer of its right regular representation have the following analogue for rings R with $\mathfrak{A}(R)=0$.

THEOREM 4. *Let R be a ring with $\mathfrak{A}(R)=0$. Then $\Omega(R)$ is the unique maximal subring of $\mathscr{K}(C(\Pi(R)))$ containing $\Pi(R)$ as an ideal.*

PROOF. By definition

$$C=C(\Pi(R))=\{a\in\mathscr{E}_2(R^+)|a[\alpha]=[\alpha]a \quad \text{for all } \alpha\in R\}.$$

Clearly C is a subring of $\mathscr{E}_2(R^+)$, and by the remarks preceding the theorem, the function $a\to a^*$ is an antiisomorphism of C onto $\mathscr{K}(C)$. It follows immediately that $\mathscr{K}(C)$ is a subring of $\mathscr{E}_2(R^+)$. Further,

$$\mathscr{K}(C)=\{a^*|a[\alpha]=[\alpha]a \text{ for all } \alpha\in R\}=\{a|a^*[\alpha]=[\alpha]a^* \text{ for all } \alpha\in R\}.$$

If $a\in\Omega(R)$ and $\alpha, \beta\in R$, then

$$(a^*[\alpha])\beta=a^*(\alpha\beta)=(\alpha\beta)a=\alpha(\beta a)=[\alpha](a^*\beta)=([\alpha]a^*)\beta$$

and dually $\beta(a^*[\alpha])=\beta([\alpha]a^*)$. Thus $a^*[\alpha]=[\alpha]a^*$ and hence $\Omega(R)\subseteqq\mathscr{K}(C)$. Let

$$M=\{a\in\mathscr{K}(C)|a[\alpha], [\alpha]a\in\Pi(R) \quad \text{for all } \alpha\in R\}.$$

Since $a[\alpha]=[a\alpha]$ and $[\alpha]a=[\alpha a]$ for all $a\in\Omega(R)$ and $\alpha\in R$, and $\Omega(R)\subseteqq\mathscr{K}(C)$, it follows that $\Omega(R)\subseteqq M$. Conversely, let $a\in M$ and $\alpha, \beta, \gamma\in R$. Then $a[\beta]=[\delta]$ for some $\delta\in R$. Hence

$$\{(\alpha a)\beta\}\gamma=\{\alpha(a[\beta])\}\gamma=(\alpha[\delta])\gamma=\alpha\delta\gamma=\alpha([\delta]\gamma)=\alpha\{(a[\beta])\gamma\}=$$
$$=\alpha\{a(\beta\gamma)\}=\{\alpha(a\beta)\}\gamma$$

and dually $\gamma\{(\alpha a)\beta\}=\gamma\{\alpha(a\beta)\}$. Consequently $(\alpha a)\beta=\alpha(a\beta)$ since $\mathfrak{A}(R)=0$. Furthermore, if $a\in\mathscr{K}(C)$ and $\alpha, \beta\in R$, then

$$a(\alpha\beta)=(\alpha[\beta])a^*=\alpha([\beta]a^*)=\alpha(a^*[\beta])=(\alpha a^*)[\beta]=(a\alpha)\beta$$

and dually $(\alpha\beta)a=\alpha(\beta a)$. It follows that $M\subseteqq\Omega(R)$ and thus $M=\Omega(R)$. Maximality of $\Omega(R)$ follows obviously from the definition of M.

5. Bitranslations of a special kind of ring. Let A be a ring satisfying the following conditions:

(i) if for all $\beta\in A$, $\alpha\beta=0$ or for all $\beta\in A$, $\beta\alpha=0$, then $\alpha=0$;

(ii) $A^2=A$ (i.e., every element of A can be written as a finite sum of elements of the form $\gamma\delta$).

Let B be a zero ring (i.e., the product of any two elements of B is equal to zero), and let $R=A\oplus B$. We show in this section that bitranslations of R can be conveniently expressed in terms of bitranslations of A and B, respectively, and then derive several consequences of this result. First note that $\Omega(B)=\mathscr{E}_2(B^+)$ (see Section 2). In the next theorem and the first two corollaries, R has the meaning just laid down.

THEOREM 5 (*cf. Theorems* 3 and 4, [11]). $\Omega(R)=\{(a, b)|a\in\Omega(A), b\in\Omega(B)\}$ *where*

$$(a, b)(\alpha, \beta)=(a\alpha, b\beta),\ (\alpha, \beta)(a, b)=(\alpha a, \beta b)$$

for every $(\alpha, \beta)\in R$. *Thus*

$$\Omega(R)\cong\Omega(A)\oplus\Omega(B).$$

PROOF. Let $c\in\Omega(R)$ and let $\alpha, \alpha'\in A$, $\beta, \beta'\in B$. Then

$$c(\alpha, \beta)=\big(d(\alpha, \beta), e(\alpha, \beta)\big)$$

for some functions $d\colon R\to A$, $e\colon R\to B$. Hence

$$\big(d(\alpha\alpha', 0), e(\alpha\alpha', 0)\big)=c(\alpha\alpha', 0)=c\{(\alpha, \beta)(\alpha', \beta')\}=$$
$$=\{c(\alpha, \beta)\}(\alpha', \beta')=\big(d(\alpha, \beta), e(\alpha, \beta)\big)(\alpha', \beta')=\big(d(\alpha, \beta)\alpha', 0\big)$$

which implies

(9) $$d(\alpha\alpha', 0)=d(\alpha, \beta)\alpha'$$

(10) $$e(\alpha\alpha', 0)=0.$$

Further,

$$\big(d(\alpha+\alpha', \beta+\beta'), e(\alpha+\alpha', \beta+\beta')\big) = c(\alpha+\alpha', \beta+\beta') = c\{(\alpha, \beta)+(\alpha', \beta')\} =$$
$$= c(\alpha, \beta)+c(\alpha', \beta') = \big(d(\alpha, \beta), e(\alpha, \beta)\big)+\big(d(\alpha', \beta'), e(\alpha', \beta')\big) =$$
$$=\big(d(\alpha, \beta)+d(\alpha', \beta'), e(\alpha, \beta)+e(\alpha', \beta')\big)$$

which implies

(11) $$d(\alpha+\alpha', \beta+\beta')=d(\alpha, \beta)+d(\alpha', \beta'),$$

(12) $$e(\alpha+\alpha', \beta+\beta') = e(\alpha, \beta)+e(\alpha', \beta').$$

For $\gamma\in A$, by (ii) above, there exist $\alpha_i, \alpha_i', \in A$ such that $\gamma=\alpha_1\alpha_1'+\ldots+\alpha_n\alpha_n'$. Using (12) and (10), we obtain

$$e(\gamma, 0)=e(\alpha_1\alpha_1'+\ldots+\alpha_n\alpha_n', 0)=e(\alpha_1\alpha_1', 0)+\ldots+e(\alpha_n\alpha_n', 0)=0$$

so that by (12)

$$e(\alpha, \beta)=e(\alpha, 0)+e(0, \beta)=e(0, \beta),$$

and we may write $e\beta$ instead of $e(0, \beta)$ where now $e\colon B\to B$. It follows from (12) that $e\in\mathscr{E}(B^+)$ (endomorphisms of the additive group of B).

In (9), the left hand side is independent of β, so for any $\beta'\in B$

$$d(\alpha, \beta)\alpha'=d(\alpha, \beta')\alpha'$$

which by (i) above yields $d(\alpha, \beta)=d(\alpha, \beta')$ and we may write $d\alpha$ instead of $d(\alpha, \beta)$, where now $d\colon A\to A$. By (9) and (11), we have $d(\alpha\alpha')=(d\alpha)\alpha'$ and $d\in\mathscr{E}(A^+)$.

Dually, we define f and g by

$$(\alpha, \beta)c = ((\alpha, \beta)f, (\alpha, \beta)g)$$

and similarly as above deduce that g and f can be considered as operators on the right on A and B, respectively, and that $g \in \mathscr{E}^0(B^+)$ (the opposite ring of $\mathscr{E}(B^+)$), $(\alpha\alpha')f = \alpha(\alpha' f)$, $f \in \mathscr{E}^0(A^+)$.

Moreover,

$$((\alpha f)\alpha', (\beta g)\beta') = (\alpha f, \beta g)(\alpha'\beta') = [(\alpha, \beta)c](\alpha', \beta') =$$
$$= (\alpha, \beta)[c(\alpha', \beta')] = (\alpha, \beta)(d\alpha', e\beta') = (\alpha(d\alpha'), \beta(e\beta'))$$

so that

$$(\alpha f)\alpha' = \alpha(df'), \quad (\beta g)\beta' = \beta(e\beta').$$

By a denote the pair (d, f) and by b the pair (e, g). Then a satisfies (1), (2), (3) and hence $\alpha \in \Omega(A)$; b satisfies (1), (2), while (3) is trivially satisfied so that $b \in \Omega(B)$.

The proof that for $a \in \Omega(A)$, $b \in \Omega(B)$ we have $(a, b) \in \Omega(R)$ is a simple calculation and is omitted (it needs no restrictions on A or B).

Recall that by Theorem 1, (i) implies that any two bitranslations of A are permutable.

COROLLARY 1. *Maximal rings of permutable bitranslations of R are precisely the rings*

(13) $$\hat{D} = \{(a, b) \mid a \in \Omega(A),\ b \in D\}$$

where D is a maximal ring of commuting biendomorphisms of B^+, and $\hat{D} \cong \Omega(A) \oplus D$.

COROLLARY 2 (*cf. Theorem* 6, [9]). *The following conditions on R are equivalent:*

a) *$\Omega(R)$ is a ring of permutable bitranslations;*

b) *$\Omega(B)$ is a ring of permutable bitranslations;*

c) *any two endomorphisms of B^+ commute.*

For example, if B is the zero ring with B^+ the additive group of integers mod 3, then B^+ satisfies c), and thus for any A satisfying (i) and (ii) above, $R = A \oplus B$ has a unique holomorph. We also have $\mathfrak{A}(R) \neq 0$, $R^2 \neq R$, and if A is not a zero ring, R is not a zero ring. Thus neither of the conditions $\mathfrak{A}(R) = 0$, $R^2 = R$ is necessary for the uniqueness of the holomorph even for a non zero ring, while each of them is sufficient (see [11]; for the case $R^2 = R$, see [6]).

COROLLARY 3. *Let A be a ring with identity, B be a zero ring, and $R = A \oplus B$. Then*

$$\Omega(R) = \{([\alpha], b) \mid \alpha \in A,\ b \in \Omega(B)\} \cong A \oplus \Omega(B),$$

and $\hat{D} \cong A \oplus D$ (see Corollary 1).

COROLLARY 4. *Let R be a ring satisfying the descending chain condition for right ideals and suppose that R^2 is a regular ring. Then*

$$\Omega(R) \cong R^2 \oplus \Omega(B)$$

where

$$B = \{\gamma \in R \mid \gamma\alpha\beta = \alpha\beta\gamma = 0 \quad \text{for all} \quad \alpha, \beta \in R\}$$

is a zero ring, and $\hat{D} \cong R^2 \oplus D$ (see Corollary 1).

PROOF. Let R satisfy the above conditions, and let M be the maximal regular ideal of R. Since $M \subseteqq R^2$ in any ring, and R^2 is itself regular, it follows that $M = R^2$. By Theorem 7, [1], $R \cong R^2 \oplus B$ where $B = \{\gamma \in R | \gamma R^2 = R^2 \gamma = 0\}$, and R^2 is semi-simple. But then R^2 has an identity and since B is clearly a zero ring, we may apply Corollary 3.

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EMBEDDING OF A SEMIRING INTO A SEMIRING WITH IDENTITY

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A semiring R is a set together with two associative operations called multiplication and addition and denoted by $(\cdot)$ and $(+)$ respectively such that the multiplication is distributive with respect to the addition. We call left (right) translation of R a finite pointwise sum of mappings of the form: $x \to ax$ $(x \to xa)$ or $x \to x$ for $a \in R$. Examples show that a semiring is not always embeddable into a semiring with identity. Our main result is that a semiring is embeddable into a semiring with identity if and only if every translation is a homomorphism. The proof makes use of the construction of the universal semiring with identity of an arbitrary semiring.

1. Construction of the universal semiring with identity

Given any semiring R, we want to construct a semiring with identity R^1 together with a homomorphism φ of R into R^1 such that, for any homomorphism ψ of R into a semiring with identity R', there exists a unique homomorphism ψ' of R^1 into R' such that $\psi'_0\varphi = \psi$ and that $\psi'(1) = 1$.

We consider the set Ω of all words of the form: $w = (w_1, w_2, \ldots, w_n)$, where $n \in N$ (set of positive integers) and $w_i \in R \cup N$ for all i, such that two consecutive letters w_i, w_{i+1} are never of the same kind: one is in R, the other in N. Such words can be written under one of the following forms:

(1) $(n_1, a_2, n_3, \ldots, n_{2p+1})$

(2) $(n_1, a_2, n_3, \ldots, n_{2p-1}, a_{2p})$

(3) $(a_1, n_2, a_3, \ldots, a_{2p+1})$

(4) $(a_1, n_2, a_3, \ldots, a_{2p-1}, n_{2p})$ $\quad (a_i \in R,\ n_i \in N)$.

We define an addition in Ω by setting for any two elements $w = (w_1, w_2, \ldots, w_n)$ and $w' = (w'_1, w'_2, \ldots, w'_p)$ of Ω:

$$w + w' = (w_1, w_2, \ldots, w_n, w'_1, w'_2, \ldots, w'_p)$$

if w_n and w'_1 are of different kinds;

$$w + w' = (w_1, w_2, \ldots, w_{n-1}, w_n + w'_1, w'_2, \ldots, w'_p)$$

if w_n and w'_1 are of the same kind.

This addition is clearly associative. Furthermore any word of Ω can be written as sum of its letters (considered as words of one letter). Conversely if the sum $\sum_{i=1}^{i=n} w_i$ of one letter words represents the word $(w_1, w_2, \ldots, w_n)$, we shall say that the sum is *reduced*.

We naturally identify $R(N)$ with the additive subsemigroup of Ω consisting of all the words of one letter $a \in R\,(n \in N)$.

LEMMA 1. *R is a consistent additive subsemigroup of Ω.*

PROOF. It is clear that R is an additive subsemigroup of Ω and all we have to show is that $w+w' \in R$ implies $w \in R$ and $w' \in R$, which follows immediately from the definition of the addition in Ω.

We define a multiplication in Ω by:

$$(w_1, w_2, \ldots, w_n)(w'_1, w'_2, \ldots, w'_n) = \sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p} w_i w'_j\right)$$

for any $w = (w_1, w_2, \ldots, w_n)$, $w' = (w'_1, w'_2, \ldots, w'_p) \in \Omega$.

Clearly, when restricted to R or N, this multiplication coincides with the multiplication in R or N. Moreover the multiplication of $a \in R$ and $n \in N$ gives $na \in R$. Also 1 acts as an identity and, when restricted to products of one letter words, the multiplication is associative.

LEMMA 2. *R is a prime ideal of $(\Omega, \cdot)$.*

PROOF. For any $w=(w_1, w_2, \ldots, w_n) \in \Omega$ and $a \in R$, $wa = \sum_{i=1}^{i=n} w_i a \in R$ and dually $aw \in R$. Let now $w=(w_1, w_2, \ldots, w_n)$, $w' = (w'_1, w'_2, \ldots, w'_p) \in \Omega$ be such that $ww' = \sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=n} w_i w'_j\right) \in R$. By lemma 1, $w_i w'_j \in R$ for all i, j. Suppose that w is not in R; then $w_{i_0} \notin R$ for some i_0. Since for all j $w_{i_0} w'_j \in R$, we must have $w'_j \in R$ for all j, so that $w' \in R$.

We shall now construct on Ω a suitable congruence $\mathscr{C}$ such that the induced operations on $\Omega/\mathscr{C}$ determine on $\Omega/\mathscr{C}$ a structure of semiring.

We define a binary relation $\mathscr{R}$ on Ω as the set of all pairs having one of the following forms:

(A) $\big(w(w'+w''),\ ww'+ww''\big)$

(B) $\big((w'+w'')w,\ w'w+w''w\big)$

for all $w, w', w'' \in \Omega$.

We denote by $\mathscr{C}$ the smallest congruence on Ω containing $\mathscr{R}$.

LEMMA 3. *Let $\mathscr{C}'$ be any transitive relation on Ω containing $\mathscr{R}$. Then for any $w, w', w'' \in \Omega$, we have:*

$$\big(w(w'w''),\ (ww')w''\big) \in \mathscr{C}'.$$

PROOF. Let $w=(w_1, w_2, \ldots, w_n)$, $w' = (w'_1, w'_2, \ldots, w'_p)$, $w'' = (w''_1, w''_2, \ldots, w''_p) \in \Omega$. Then

$$w(w'w'') = \left(\sum_{i=1}^{i=n} w_i\right)\left(\sum_{j=1}^{j=p}\left(\sum_{k=1}^{k=q} w'_j w''_k\right)\right)$$

and $\left(w(w'w''),\ \sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p}\left(\sum_{k=1}^{k=q} w_i(w'_j w''_k)\right)\right)\right)\in\mathscr{C}'$ since the multiplication is distributive on the right modulo $\mathscr{C}'$ in view of form (B) of $\mathscr{R}$, by transitivity of $\mathscr{C}'$. Similarly using distributivity on the left of the multiplication modulo $\mathscr{C}'$,

$$(ww')w'' = \left(\sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=n} w_i w'_j\right)\right)\left(\sum_{k=1}^{k=q} w''_k\right)$$

and $\left((ww')w'',\ \sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p}\left(\sum_{k=1}^{k=q}(w_i w'_j)w''_k\right)\right)\right)\in\mathscr{C}'$. Since the product of one letter words is associative, the result follows.

LEMMA 4. *Let $\mathscr{B}$ be the set of all pairs $\big(u(wv), u(w'v)\big)$ where $(w, w')\in\mathscr{R}$ and $u, v\in\Omega$. Let Ω^0 be the additive semigroup resulting from the adjunction of a formal zero to $(\Omega, +)$. Define successively the binary relations $\mathscr{B}^*$, $\mathscr{T}$ by:*

$$(w, w')\in\mathscr{B}^*\Leftrightarrow(w, w')\in\mathscr{B} \quad or \quad (w', w)\in\mathscr{B} \quad or \quad w=w';$$

$$\mathscr{T}=\{(x+w+y,\ x+w'+y);\ (w, w')\in\mathscr{B}^*,\ x, y\in\Omega^0\}.$$

Then $\mathscr{C}$ is the transitive closure of $\mathscr{T}$.

PROOF. Any congruence containing $\mathscr{R}$ contains also $\mathscr{B}$, $\mathscr{B}^*$, $\mathscr{T}$ and therefore the transitive closure $\mathscr{C}'$ of $\mathscr{T}$. Thus $\mathscr{C}'\subseteq\mathscr{C}$. Conversely $\mathscr{T}$ is reflexive and symmetric, so that $\mathscr{C}'$ is an equivalence relation. Also $(w, w')\in\mathscr{T}$ implies $(x+w, x+w')\in\mathscr{T}$ and $(w+x, w'+x)\in\mathscr{T}$ for all $x\in\Omega$, whence $\mathscr{C}'$ is an additive congruence.

By the construction of $\mathscr{B}$, we have: $(w, w')\in\mathscr{B}$ implies $(wx, w'x)\in\mathscr{B}$ and $(xw, xw')\in\mathscr{B}$ for all $x\in\Omega$. Clearly $\mathscr{B}^*$ has the same property. Let now $(x+w+y, x+w'+y)\in\mathscr{T}$, where $(w, w')\in\mathscr{B}^*$ and $x, y\in\Omega^0$. We treat only the case when $x, y\in\Omega$, the cases when $x=0$ or $y=0$ being simpler. We have, for any $u\in\Omega$:

$$\big(u(x+w+y),\ u(x+w)+uy\big)\in\mathscr{C}'$$

$$\big(u(x+w),\ ux+uw\big)\in\mathscr{C}'$$

since $\mathscr{R}\subseteq\mathscr{C}'$; since $\mathscr{C}'$ is an additive congruence, $\big(u(x+w+y),\ ux+uw+uy\big)\in\mathscr{C}'$. Similarly $\big(u(x+w'+y),\ ux+uw'+uy\big)\in\mathscr{C}'$. Now $(uw, uw')\in\mathscr{C}'$ so that $(ux+uw+uy,\ ux+uw'+uy)\in\mathscr{C}'$. Therefore we have $\big(u(x+w+y),\ u(x+w'+y)\big)\in\mathscr{C}'$. Similarly $\big((x+w+y)u,\ (x+w'+y)u\big)\in\mathscr{C}'$.

From this property of $\mathscr{T}$ follows that $\mathscr{C}'$ is a multiplicative congruence, which completes the proof.

LEMMA 5. *$(w, w')\in\mathscr{C}$ and $w\in R$ implies $w'\in R$.*

PROOF. We shall prove this property successively for $\mathscr{R}$, $\mathscr{B}$, $\mathscr{B}^*$, $\mathscr{T}$. If $(x, y)\in\mathscr{R}$, for instance has the form

(A): $x=w(w'+w'')$, $y=ww'+ww''$,

and if $x\in R$, then, by lemma 1 and 2, either $w\in R$ or $w', w''\in R$; if $y\in R$, then ww' and ww'' are in R and either $w\in R$ or $w', w''\in R$ so that $x\in R$. The case (B) is dual.

If $(x, y) \in \mathscr{B}$, then $x = u(wv)$, $y = u(w'v)$, where $u, v \in \Omega$ and $(w, w') \in \mathscr{R}$; if $x \in R$, then u or v or w is in R and by the first part of the proof $y \in R$; if $y \in R$, then $x \in R$ similarly; therefore $(w, w') \in \mathscr{B}^*$ and $w \in R$ implies $w' \in R$.

If $(u, v) \in \mathscr{T}$, then $u = x + w + y$, $v = x + w' + y$ where $x, y \in \Omega^0$ and $(w, w') \in \mathscr{B}^*$. If $u \in R$, then $x, y, w \in R$ unless $x = 0$ or $y = 0$; thus $y = x + w' + y \in R$ by the previous part.

If finally $(w, w') \in \mathscr{C}$, then by lemma 4 there exist $x_1, x_2, \dots, x_n \in \Omega$ such that $(x_{i-1}, x_i) \in \mathscr{T}$ for all $i = 2, \dots, n$ and $x_1 = w$, $x_n = w'$. If $w \in R$, then $x_2 \in R$ by the above and by induction on i, $w' = x_n \in R$.

THEOREM 6. *$R^1 = \Omega/\mathscr{C}$ is a semiring with identity. The canonical mapping φ of R into R^1 is a homomorphism. Furthermore $\varphi(R)$ is a prime ideal of R^1 and a consistent subset of $(R^1, +)$. Finally for any homomorphism ψ of R into a semiring with identity R', there exists a unique homomorphism ψ' of R^1 into R' such that $\psi' \circ \varphi = \psi$ and that $\psi'(1) = 1$.*

DEFINITION 7. We call R^1 the *universal semiring with identity* of R.

REMARK. The universal property characterizes R^1 up to isomorphism.

PROOF. We shall denote by $\bar{w}$ the class of $w \in \Omega$ modulo $\mathscr{C}$. The operations of R^1 are well defined by: $\bar{w} + \bar{w}' = \overline{w + w'}$, $\bar{w}\bar{w}' = \overline{ww'}$. Looking at the forms (A) and (B) of the pairs in $\mathscr{R}$, we conclude that the multiplication is distributive with respect to the addition. By lemma 3, the multiplication is associative, so that R^1 is a semiring.

It is clear that the canonical mapping $\varphi: a \to \bar{a}$ from R to R^1 is a homomorphism.

By lemma 2, $\varphi(R)$ is an ideal of R^1. Also $\bar{w}\bar{w}' = \varphi(a) = \bar{a}$, where $a \in R$, implies $(ww', a) \in \mathscr{C}$, $ww' \in R$ by lemma 5, $w \in R$ or $w' \in R$ by lemma 2 and $\bar{w} \in \varphi(R)$ or $\bar{w}' \in \varphi(R)$. Therefore $\varphi(R)$ is a prime ideal of R^1. Using lemmas 1 and 5, one proves similarly that $\varphi(R)$ is a consistent subset of $(R^1, +)$.

Let now ψ be a homomorphism of R into a semiring with identity R'. First we extend ψ to Ω in the following way. Let ξ be the mapping of Ω into R' defined by: $\xi(w_1, w_2, \dots, w_n) = \sum_{i=1}^{i=n} \tilde{w}_i$, where $\tilde{w}_i = \psi(w_i)$ if $w_i \in R$, and $\tilde{w}_i = w_i 1$ if $w_i \in N$. This mapping is obviously an additive homomorphism; observe also that $\xi(w_i) = \tilde{w}_i$ and therefore ξ is a multiplicative homomorphism when restricted to words of one letter.

From this we deduce that ξ is a multiplicative homomorphism. Indeed for any $w = (w_1, w_2, \dots, w_n)$, $w' = (w_1', w_2', \dots, w_p') \in \Omega$:

$$\xi(ww') = \xi\left(\sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p} w_i w_j'\right)\right) = \sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p} \xi(w_i w_j')\right) = \sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p} \xi(w_i)\xi(w_j')\right) =$$

$$= \left(\sum_{i=1}^{i=n} \xi(w_i)\right)\left(\sum_{j=1}^{j=p} \xi(w_j')\right) = \xi(w)\xi(w').$$

Consequently the equivalence relation induced by ξ on Ω is a congruence. It contains all the pairs of the forms (A) and (B) since R' is a semiring. Therefore it contains $\mathscr{C}$ and there exists a homomorphism ψ' of R^1 into R' such that: $\psi'(\bar{w})=\xi(w)$ for all $w\in\Omega$. Now ψ' is such that $\psi'(\varphi(a))=\psi'(\bar{a})=\xi(a)=\psi(a)$ for all $a\in R$ and $\psi'(1)=\psi(\bar{1})=\xi(1)=1$.

The uniqueness of ψ' with these properties results from the fact that $\varphi(R)\cup\{1\}$ generates R^1. This completes the proof.

2. Embedding of a semiring into a semiring with identity

By the previous theorem we see that R is embeddable into a semiring with identity R' if and only if it is embeddable into its universal semiring with identity. In this part we shall give a necessary and sufficient condition.

Let us consider the set Λ (P) of all inner left (right) translations of the multiplicative semigroup $(R, \cdot)$.

DEFINITION 8. We call *left (right) translations* of a semiring R the elements of the additive subsemigroup $\bar{\Lambda}(\bar{\mathrm{P}})$ generated in the additive semigroup of all mappings of R into R by $\Lambda^1=\Lambda\cup\{\varepsilon\}$ $(\mathrm{P}^1=\mathrm{P}\cup\{\varepsilon\})$, where ε is the identity mapping of R.

A trivial example where $\bar{\Lambda}$ and $\bar{\mathrm{P}}$ are semirings is when R has an identity, for in this case $\bar{\mathrm{P}}=\mathrm{P}^1=\mathrm{P}$, $\bar{\Lambda}=\Lambda^1=\Lambda$. Then we have also $\bar{\Lambda}\bar{\mathrm{P}}=\bar{\mathrm{P}}\bar{\Lambda}$. But the following example shows that in general $\bar{\Lambda}$ and $\bar{\mathrm{P}}$ are not semirings and do not commute element by element.

EXAMPLE 9. Consider the semiring given by the Cayley tables

+	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	c	c	c	c
d	c	d	a	b

·	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	a	a
d	a	a	a	a

We see that

$$(\varepsilon+\varepsilon)(\varepsilon+\varrho_c)(d) = 2(d+dc) = c$$

$$(\varepsilon+\varrho_c)(\varepsilon+\varepsilon)(d) = 2d+2dc = a.$$

Therefore $\varepsilon+\varepsilon\in\bar{\Lambda}$ and $\varepsilon+\varrho_c\in\bar{\mathrm{P}}$ do not commute. Also

$$(\varepsilon+\varepsilon)(\varepsilon+\varrho_c)(d) = c\neq a = ((\varepsilon+\varepsilon)+(\varepsilon+\varepsilon)\varrho_c)(d)$$

whence $\bar{\mathrm{P}}$ is not a semiring, since ε, $\varepsilon+\varepsilon$, $\varrho_c\in\bar{\mathrm{P}}$.

It is easy to verify that the following holds:

LEMMA 10. *If every element of $\bar{\Lambda}$, $\bar{\mathrm{P}}$ is an additive homomorphism, $\bar{\Lambda}$ and $\bar{\mathrm{P}}$ are semirings and commute element by element.*

The following result gives a necessary condition for a semiring to be embeddable into a semiring with identity. It shows in particular that a semiring is not always embeddable into a semiring with identity: example 9 fails to have this property.

LEMMA 11. *If R is embeddable into a semiring with identity R', then the elements of $\bar{\Lambda}$ and $\bar{\mathrm{P}}$ are additive homomorphisms of R.*

PROOF. We may as well assume that R is a subsemiring of R'. Let $\lambda=\sum_{i=1}^{i=n}\lambda_i$ be a left translation of R, with $\lambda_i\in\Lambda^1$ for all i. We extend λ to a left translation of R' as follows: let $\lambda'=\sum_{i=1}^{i=n}\lambda'_i$ where $\lambda'_i=\varepsilon'$ if $\lambda_i=\varepsilon$, $\lambda'_i=\lambda'_{a_i}$ if $\lambda_i=\lambda_{a_i}$. Clearly the restriction of λ' to R is λ and $\lambda'\in\Lambda'$ since R' has an identity. Thus $\lambda(x+y)=$ $=\lambda'(x+y)=\lambda'(x)+\lambda'(y)=\lambda(x)+\lambda(y)$ for all $x,y\in R$. Dually any $\varrho\in\bar{\mathrm{P}}$ is an additive homomorphism.

THEOREM 12. *The following conditions are equivalent:*
(i) *R is embeddable into a semiring with identity.*
(ii) *R is embeddable as a prime ideal and additively consistent subset into its universal semiring with identity.*
(iii) *Every translation of R is an additive homomorphism.*

PROOF. Trivially (ii)⇒(i); (i)⇒(iii) is lemma 11. With the notations of the preceding section, it is enough to show that, if (iii) holds, then the restriction of $\mathscr{C}$ to R is the equality; for then φ will be one-to-one and (iii)⇒(ii) in view of theorem 6.

LEMMA 13. *For any $w\in\Omega$, there exist $\lambda_w\in\bar{\Lambda}$ and $\varrho_w\in\bar{\mathrm{P}}$ such that: $wa=\lambda_w(a)$, $aw=\varrho_w(a)$, for all $a\in R$. Furthermore, we have for all $w,w'\in\Omega$:*

(5) $\lambda_w+\lambda_{w'}=\lambda_{w+w'},\quad \varrho_w+\varrho_{w'}=\varrho_{w+w'}$;

(6) $\lambda_w\circ\lambda_{w'}=\lambda_{ww'}$

(7) *if $\bar{\mathrm{P}}$ is a semiring, $\varrho_w\circ\varrho_{w'}=\varrho_{w'w}$.*

PROOF. Define first ε_n by $\varepsilon_n(a)=na$ for all $a\in R$, $n\in N$ so that we have $\varepsilon_n=$ $=n\varepsilon\in\bar{\Lambda}\cap\bar{\mathrm{P}}$.

If $w\in\Omega$ has form (1), $w=(n_1,a_2,\dots,n_{2p+1})$, $wa=n_1a+a_2a+\dots+n_{2p+1}=$ $=(\varepsilon_{n_1}+\lambda_{a_2}+\dots+\varepsilon_{n_{2p+1}})(a)=\lambda_w(a)$, if $\lambda_w=\varepsilon_{n_1}+\lambda_{a_2}+\dots+\varepsilon_{n_{2p+1}}\in\bar{\Lambda}$. Similarly $aw=\varrho_w(a)$ with $\varrho_w=\varepsilon_{n_1}+\varrho_{a_2}+\dots+\varepsilon_{n_{2p+1}}\in\bar{\mathrm{P}}$. We have a similar proof if w has form (2), (3) or (4). It is possible to resume as follows: for any reduced sum

$$w=\sum_{i=1}^{i=n}w_i,\qquad \lambda_w=\sum_{i=1}^{i=n}\lambda_{w_i},\qquad \varrho_w=\sum_{i=1}^{i=n}\varrho_{w_i},$$

where λ_{w_i}, ϱ_{w_i} have the usual meaning if $w_i\in R$, and $\lambda_{w_i}=\varrho_{w_i}=\varepsilon_{n_i}$ if $w_i=n_i\in N$.

To prove (5), let $w=(w_1,w_2,\dots,w_n)$, $w'=(w'_1,w'_2,\dots,w'_p)$ be in Ω. Then $\lambda_w=\sum_{i=1}^{i=n}\lambda_{w_i}$, $\lambda_{w'}=\sum_{j=1}^{j=p}\lambda_{w'_j}$. If w_n, w'_1 are of different kinds: $w+w'=(w_1,w_2,\dots,w_n,$ $w'_1,w'_2,\dots,w'_p)$ so that $\lambda_{w+w'}=\lambda_w+\lambda_{w'}$. If w_n and w'_1 are of the same kind: $w+w'=$ $=(w_1,w_2,\dots,w_{n-1},w_n+w'_1,w'_2,\dots,w'_p)$ so that

$$\lambda_{w+w'}=\sum_{i=1}^{i=n-1}\lambda_{w_i}+(\lambda_{w_n}+\lambda_{w'_1})+\sum_{j=2}^{j=p}\lambda_{w'_j}.$$

Since w_n and w_1' are either both in R or both in N, we have $\lambda_{w_n}+\lambda_{w_1'}=\lambda_{w_n+w_1'}$, whence in this case also $\lambda_{w+w'}=\lambda_w+\lambda_{w'}$. Similarly $\varrho_{w+w'}=\varrho_w+\varrho_{w'}$.

Formula (6) holds trivially when w, w' are words of one letter. Suppose now that $w=(w_1, w_2, \ldots, w_n)$, $w'=(w_1', w_2', \ldots, w_p')\in\Omega$. Then $\lambda_w=\sum_{i=1}^{i=n}\lambda_{w_i}$, $\lambda_{w'}=\sum_{j=1}^{j=p}\lambda_{w_j'}$ so that

$$\lambda_w\circ\lambda_{w'}=\sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p}\lambda_{w_i}\circ\lambda_{w_j'}\right)=\sum_{j=1}^{i=n}\left(\sum_{j=1}^{j=p}\lambda_{w_i w_j'}\right)=\lambda_{ww'}$$

by the definition of the addition in $\bar{\Lambda}$ and (5).

Formula (7) holds trivially when w, w' are words of one letter. Suppose now that $\bar{\mathrm{P}}$ is a semiring and let $w=(w_1, w_2, \ldots, w_n)$, $w'=(w_1', w_2', \ldots, w_p')\in\Omega$. Then

$$\varrho_w=\sum_{i=1}^{i=n}\varrho_{w_i},\qquad \varrho_{w'}=\sum_{j=1}^{j=p}\varrho_{w_j'}$$

so that

$$\varrho_w\circ\varrho_{w'}=\sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p}\varrho_{w_i}\circ\varrho_{w_j'}\right)$$

by definition of the addition in P. Also $w'w=\sum_{j=1}^{j=p}\left(\sum_{i=1}^{i=n}w_j'w_i\right)$, whence

$$\varrho_{w'w}=\sum_{j=1}^{j=p}\left(\sum_{i=1}^{i=n}\varrho_{w_j'w_i}\right)=\sum_{j=1}^{j=p}\left(\sum_{i=1}^{i=n}\varrho_{w_i}\circ\varrho_{w_j'}\right).$$

Then by distributivity in P,

$$\sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p}\varrho_{w_i}\circ\varrho_{w_j'}\right)=\sum_{j=1}^{j=p}\left(\sum_{i=1}^{i=n}\varrho_{w_i}\circ\varrho_{w_j'}\right),$$

and formula (7) follows.

LEMMA 14. *Suppose that any translation of R is an additive homomorphism. Then*

(i) $(u, v)\in\mathscr{R}$ *and u or v in R implies* $u=v$.

(ii) $(u, v)\in\mathscr{R}$ *implies* $\lambda_u=\lambda_v$, $\varrho_u=\varrho_v$.

PROOF. We shall consider the pairs of form (A). For the pairs of form (B) the proof is dual.

Let $u=w(w'+w'')$, $v=ww'+ww''\in\Omega$. If $u, v\in R$, then either $w\in R$ or $w', w''\in R$ by lemmas 1 and 2. If $w=a\in R$, we have: $u=a(w'+w'')=\varrho_{w'+w''}(a)=\varrho_{w'}(a)+\varrho_{w''}(a)=aw'+aw''=v$ by formula (5). If $w'=b\in R$, $w''=c\in R$, then $u=w(b+c)=\lambda_w(b+c)=\lambda_w(b)+\lambda_w(c)=wb+wc=v$ since λ_w is an additive homomorphism.

Let us now prove that $\lambda_u=\lambda_v$. By lemma 13:

$$\lambda_u=\lambda_{w(w'+w'')}=\lambda_w(\lambda_{w'}+\lambda_{w''})=\lambda_{ww'}+\lambda_{ww''}=\lambda_v$$

since λ_w is an additive homomorphism. Furthermore:

$$\varrho_u = \varrho_{w(w'+w'')} = (\varrho_{w'} + \varrho_{w''})\varrho_w = \varrho_{w'}\varrho_w + \varrho_{w''}\varrho_w = \varrho_v$$

by lemma 13. This proves (ii).

LEMMA 15. *Suppose that any translation of R is an additive homomorphism. Then the restriction of $\mathscr{C}$ to R is the equality.*

PROOF. We shall prove successively for $\mathscr{B}$, $\mathscr{B}^*$ and $\mathscr{T}$ the property (i) of lemma 14. By lemma 5, $s \in R$ if and only if $t \in R$ when $(s, t) \in \mathscr{C}$, for any $s, t \in \Omega$.

a) Assume $(s, t) \in \mathscr{B}$ and $s, t \in R$. Then $s = u(wv)$ and $t = u(w'v)$ where $u, v \in \Omega$, $(w, w') \in \mathscr{R}$. If $u = c \in R$, then:

$$s = \varrho_{wv}(c) = \varrho_v(\varrho_w(c)) = \varrho_v(\varrho_{w'}(c)) = t$$

by lemma 14 applied to w and w', and formula (6). If $v \in R$, then $s = t$ dually. If $w, w' \in R$, then by lemma 14 $w = w'$ and $s = t$.

b) Suppose $(s, t) \in \mathscr{B}^*$ and $s, t \in R$. Then $s = t$ by a).

c) Suppose $(s, t) \in \mathscr{T}$ and $s, t \in R$. Then $s = x + w + y$, $t = x + w' + y$, where $x, y \in \Omega^0$ and $(w, w') \in \mathscr{B}^*$. By lemma 1, $w, w' \in R$ which by b)implie s $w = w'$ and $s = t$.

If finally $(s, t) \in \mathscr{C}$ and $s, t \in R$, then, by lemma 4, there exist $w_1, w_2, \ldots, w_n \in \Omega$ such that $w_1 = s$, $w_n = t$ and $(w_{i-1}, w_i) \in \mathscr{T}$ for $i = 2, \ldots, n$. By induction on i, $w_i \in R$ by lemma 5 and $w_{i-1} = w_i$ by c), for all i. Therefore $s = t$.

This completes the proof of theorem 12.

REMARK. It is possible, in the case of a commutative semiring, to generalize the method used for embedding of a ring into a ring with identity. This gives a much simpler construction, however the universal properties will not be respected.

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ON ODD CIRCUITS IN CHROMATIC GRAPHS

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Abstract

It has been conjectured by P. Erdős that any graph G with chromatic number $\chi(G)=5$ contains as subgraphs either two odd independent (i.e. vertex-disjoint) circuits or a $\langle 5\rangle$. (For any integer r, $\langle r\rangle$ will denote any complete graph with r vertices.) In this paper we shall prove this conjecture; indeed we shall prove some stronger results, in part also conjectured by P. Erdős.

1. Introduction

For any graph G, let $t(G)$ denote the maximal integer m such that G contains an $\langle m\rangle$; let $c(G)$ denote the maximal number of independent odd circuits in G. We shall prove the following theorems:

THEOREM 1. *Let* $\chi(G)>t(G)$ *and* $\chi(G)>4$. *Then*

$$c(G)\geqq\left[\frac{\chi(G)+2}{3}\right].$$

THEOREM 2. *Let* $t(G)<4$. *Then* $c(G)\geqq\left[\frac{2\chi(G)-3}{3}\right]$.

THEOREM 3. *Let* $t(G)>2$. *Then* $c(G)\geqq\left[\frac{2\chi(G)-t(G)}{3}\right]$.

2. Preliminary results

Let G be a graph with vertex set V. For $V'\subseteqq V$ $\mathfrak{G}(V')$ will denote the restriction graph defined by V' (i.e. the maximal subgraph of G with vertex set V'). If G', G'' are subgraphs of G with vertex sets V', V'' we may write $G-V'$ or $G-G'$ instead of $\mathfrak{G}(V-V')$, and $\mathfrak{G}(G'\cup G'')$ instead of $\mathfrak{G}(V'\cup V'')$.

(2. 1) LEMMA. *Let H be a non-empty proper restriction graph of G. Suppose that at most $\chi(G-H)-1$ vertices of $G-H$ are connected to more than $\chi(H)-1$ vertices of H. Then $\chi(G)\leqq\chi(H)+\chi(G-H)-1$.*

PROOF. Let colourings of H and $G-H$ be given in colours c_i, d_j $(1\leqq i\leqq\chi(H);$ $1\leqq j\leqq\chi(G-H))$ respectively. Suppose that each vertex coloured d_1 is connected to at most $\chi(H)-1$ vertices of H. Then each of these vertices coloured d_1 can be recoloured with one of the $\chi(H)$ colours $c_1, \ldots, c_{\chi(H)}$ which does not appear among its neighbours in H.

(2. 2) COROLLARY. *Let H be a complete subgraph of G. Then either* $\chi(G) \leqq \chi(G-H)+|H|-1$ *or there exist at least* $\chi(G)-|H|$ *vertices in* $G-H$ *each connected to all vertices of* H. (For a graph G with vertex set V, $|G|$ and $|V|$ both will denote the number of vertices in V.)

(2. 3) COROLLARY. *Let H be an odd circuit of minimal length in a graph G containing no* $\langle 4 \rangle$. *Then* $\chi(G-H) \geqq \chi(G)-2$ *or* $G=H$.

PROOF. Since $\chi(H)=3$, it suffices to prove that no vertex x of $G-H$ can be connected to three vertices of H. H is not a triangle, and therefore G does not contain any triangle. Hence, if x is connected to three vertices of H, these separate H into three arcs of length greater than one; at least one of these arcs has odd length, which cannot exceed $|H|-4$. This arc, together with the edges joining its ends to x, forms a shorter odd circuit.

(2. 4) LEMMA. *Let* $\chi(G)=n \geqq 4$, $t(G)=n-1$ *or* $n-2$. *Then G contains an* $\langle n-2 \rangle$ H, *such that* $\chi(G-H) \geqq 3$.

PROOF. Suppose that for any $\langle n-2 \rangle$ H, $\chi(G-H)=2$. Choose some $\langle n-2 \rangle$ H. By (2. 2) there exist in $G-H$ vertices x, y $(x \neq y)$ each connected to all vertices of H; since $t(G)<n$, x and y are not adjacent. Let w, z be distinct vertices in H. Agani by (2. 2) there must exist a vertex u distinct from w connected to all vertices of the $\langle n-2 \rangle$ defined by $H-\{w\}$ and y. Then the triangle $\mathfrak{G}(\{u, y, z\})$ is disjoint from the $\langle n-2 \rangle$ defined by $H-\{z\}$ and x, a contradiction.

(2. 5) LEMMA. *If A and B are subsets of the vertex set of a graph G, such that* $\mathfrak{G}(A)$ *and* $\mathfrak{G}(B)$ *are complete graphs, then* $\chi(\mathfrak{G}(A \cup B)) = t(\mathfrak{G}(A \cup B))$.[1]

PROOF. We may assume $A \cap B = \emptyset$ and $\mathfrak{G}(A \cup B)=G$. Furthermore we may assume $t(G)=|A|$; otherwise we can construct a graph G' from G by adjoining $t(G)-|A|$ new vertices $c_1, \ldots$ each connected to the others and to every $a \in A$, ensuring $t(G)=t(G')=|A \cup \{c_1, \ldots\}|$. For $b \in B$ we define $T(b)$ to be the set of all $a \in A$ which are not connected to b. Then for different vertices $b_1, \ldots, b_k$ (k being any integer) the set $T(b_1) \ldots T(b_k)$ cannot contain less than k elements, as that would imply that $b_1, \ldots, b_k$ together with those vertices in A which are not in the union form a $\langle t(G)+1 \rangle$. Hence, by a well known theorem of P. HALL [1], there exists a system of distinct representatives for these sets $T(b)$ $(b \in B)$ yielding a colouring of G in $t(G)$ colours.

We conclude this section with a proof of the original conjecture of Erdős.

(2. 6) LEMMA. *Let G be a graph such that* $\chi(G)=5$ *and* $t(G)<5$. *Then* $c(G) \geqq 2$.

PROOF. The assertion follows from (2. 3) when $t(G) \leqq 3$; and from (2. 4) when $t(G) \geqq 3$.

[1] Lemma (2. 5) was expressed in terms of the complement graph at first by D. KÖNIG (see e. g. [3], [4]).

3. Proofs of the theorems

Our proof of Theorem 1 will require the following

(3. 1) LEMMA. *Let* $\chi(G) = n \geqq 7$, *and* $t(G) = n-1$ *or* $n-2$. *Then there exist, disjoint in* G, *an* $\langle n-4\rangle$ *and two odd circuits.*

PROOF. Suppose that the conclusion is false for some graph G. Let $V(G)$ denote the set of vertices of G. Among all pairs $\{X, C\}$ of disjoint subsets $X, C \subseteqq V(G)$ such that $\mathfrak{G}(X)$ is an $\langle n-2\rangle$ and $\mathfrak{G}(C)$ is an odd circuit, select one for which $|C|$ is minimal; its existence is ensured by (2. 4). For any two vertices $y, z \in X$ we have, by hypothesis, $\chi(G-(X-\{y, z\})-C) \leqq 2$ hence

$$(*) \qquad \chi(G(X \cup C-\{y, z\})) \geqq n-2.$$

Case 1. $C=\{a, b, c\}$. Since G contains no $\langle n\rangle$ we may suppose that there exist in X two distinct vertices v, w such that a is not adjacent to v, and b is not adjacent to w. Choosing distinct vertices y and z in $X-\{v, w\}$ we arrive at a contradiction by $(*)$ and (2. 5).

Case 2. $|C| \geqq 5$. For $y, z \in X$ $(y \neq z)$ let $U(y, z)$ denote the set of all vertices in $V(G)-X$ connected to all vertices in $X-\{y, z\}$. By hypothesis $U(y, z) \subseteqq C$; and, since $|C|>3$, $G-(X-\{y, z\})$ contains no $\langle 5\rangle$. Hence, by (2. 6), $\chi(G-(X-\{y, z\})) \leqq 4$. By (2. 2), $\chi(G(X-\{y, z\})) = n-4$ implies $|U(y, z)| \geqq 2$.

Case 2a. Suppose at most one vertex $a \in V(G)-X$ is connected to all vertices in X. We choose distinct vertices y, z in X, and b (distinct from a if any) in $U(y, z)$. Since $\chi(G(X \cup \{b\})) = n-2$, and, by assumption $\chi(G-X-\{b\}) \leqq 2$, by (2. 1) we can find two different vertices c and d in $V(G)-X-\{b\}$ each of which is connected to at least $n-2$ vertices of $X \cup \{b\}$. One of these, say c, is distinct from a. Hence c is joined to b and to $n-3$ vertices of X. Since $c \in U(v, w)$ for some $v, w \in X$, we find by a similar argument a vertex e not contained in $X \cup \{a\}$ which is joined to c and to $n-3$ vertices of X. A $\langle n-2\rangle$, Y is formed by c, e and some set $X-\{r, s\}$. Choose distinct vertices p, q in $X-\{r, s\}$ and f in $U(p, q)$. Then $c \neq f$ and $e \neq f$. Hence Y and the $\langle 3\rangle$ formed by r, s, f are disjoint contradicting $|C| \geqq 5$.

Case 2b. Suppose a and b are distinct vertices connected to all vertices in X. Since $t(G)<n$, a and b are not adjacent. Let y and z be distinct vertices in X, and define D to be the set of all vertices in C which are connected to at least $n-5$ elements of $X-\{y, z\}$. Since $n \geqq 7$, $2(n-5)+2>n-2$. Hence, if two vertices c and d of D were adjacent we could find a triangle spanned by them and some vertex x of X; this triangle would be disjoint from $X \cup \{a\}-\{x\}$ or from $X \cup \{b\}-\{x\}$. Thus it is possible to define an $(n-3)$-colouring of $\mathfrak{G}(X \cup D-\{y, z\})$ such that all vertices in D are coloured the same. Every vertex in $C-D$ is not adjacent to at least two elements of $X-\{y, z\}$. Thus this $(n-3)$-colouring can be extended by proceeding successively along the arcs of $\mathfrak{G}(C-D)$, a contradiction to $(*)$.

(3. 2) PROOF OF THEOREM 1. Define $\chi(G)=n$. For all $n \equiv 0 \pmod 3$ the theorem is trivial. For $n=5$ it follows from (2. 6). If $t(G) \geqq n-2 \geqq 5$ it follows from (3. 1) that $c(G) \geqq 2+[(n-4)/3]$; the case $t(G)<4$ follows easily from (2. 3). Hence we may suppose that $n \geqq t(G)+3 \geqq 7$ and that the theorem has been proved for all G' such that $|G'|<|G|$. If for some triangle H in G, $\chi(G-H) \geqq n-2$, our induction hypothesis applied to $G-H$ yields $c(G) \geqq 1+[n/3]$; we may thus assume that

$\chi(G-H) = n-3$ for all triangles H in G. Let a, b, c be vertices of a triangle. By (2. 2) there exist $n-3$ vertices each connected to each of these vertices. Since $t(G)<n$, at least two of them, say d and e are not adjacent. Similarly we can find some vertex f distinct from a which is adjacent to b, c, and d; obviously $f \neq e$. The triangles formed by $U=\{a, b, e\}$ and $V=\{c, d, f\}$ are disjoint and $\chi(G(U \cup V)) \leqq 5$. If $\chi(G-U-V) \geqq n-4$, $c(G) \geqq 2+[(n-4)/3] = [(n+2)/3]$; we may thus assume $\chi(G-U-V) = n-5$ and $\chi(G(U \cup V)) = 5$. It follows by (2. 5) that $n \geqq t(G)+ +3 \geqq 8$. If $t(G-U)<n-3$ the induction hypothesis applied to $G-U$ yields $c(G) \geqq 1+[(n-1)/3]$. In the remaining case let W be the vertex set of an $\langle n-3 \rangle$ in $G-U$. By (2. 5) $\chi(G(U \cup W)) = t(G)$, hence $\chi(G-U-W) \geqq 3$. But now $c(G) \geqq \geqq 2+[(n-3)/3]$.

(3. 3) PROOF OF THEOREM 2. Define $\chi(G)=n$.

1. The theorem is evidently true for $n<5$. Thus we can prove it by induction by showing that for $n \geqq 5$ there exist two independent odd circuits C, C' such that $\chi(G-C-C') \geqq n-3$. As this is obvious (by (2. 5)) if G contains two independent triangles, we assume that situation does not occur.

2. Let C be a minimal odd circuit of G with cyclically ordered vertices $v_1, v_2, \dots$ (indices modulo $|C|$). Let $\mathfrak{F}$ be the 3-colouring which assigns to v_1 the colour 3 and to the remaining vertices the colours 1, 2 so that v_2 is coloured 2. Let A denote the set of vertices of $G-C$ connected to two vertices of C; (no vertex can be connected to more than two such vertices, cf. proof of (2. 3)). The two vertices of C connected to a vertex x in A are ends of a path of length 2 in C; denote the intermediate vertex of this path by $\varphi(x)$. We observe that if x and y are adjacent vertices in A, x (respectively y) is connected to $\varphi(y)$ (respectively $\varphi(x)$); for otherwise there would exist in G either a circuit shorter than C or a $\langle 4 \rangle$, according as $|C|$ exceeds or is equal to 3.

3. Let C' be a circuit without diagonals in $G-C$. Fix some orientation in C'. We shall prove that there exists a 4-colouring $\mathfrak{F}'$ (in colours 1, 2, 3, and 4) which extends $\mathfrak{F}$ to the graph spanned by C and C', and has the following properties:

(i) If $A \cap C' = \emptyset$, exactly one vertex is coloured 4.

(ii) If $A \cap C' \neq \emptyset$, $\mathfrak{F}'(x)=4$ if and only if $x \in A$ and the successor of x in C' is not contained in A.

(iii) If for $x \in C' \cap A$, $\mathfrak{F}'(x) \neq 4$, then $\mathfrak{F}'(x)=\mathfrak{F}(\varphi(x))$.

(iv) Let y be the successor of x in C' and $y \in A$, $x \notin A$.

If $\mathfrak{F}'(y) \neq 3$ and if x is not adjacent to v_1, then $\mathfrak{F}'(x)=3$.

$\mathfrak{F}'$ is defined as follows. We first colour the vertices of $A \cap C'$ by the rule $\mathfrak{F}'(x) = =\mathfrak{F}(\varphi(x))$. We then recolour with the colour 4 any vertex of C' already coloured whose successor is not contained in A. If $A \cap C' = \emptyset$ an arbitrary vertex of C' is coloured 4. Thus far conditions (i), (ii), (iii) are satisfied. The vertices 4 coloured separate C' into oriented arcs the initial vertex of each of which is adjacent to a vertex coloured 4. Working backwards from the last vertex of the arc which has not been coloured we can extend the colouring using only the colours 1, 2, and 3; at the first step we can obviously satisfy (iv).

4. Let A' denote the set of vertices in $G-C$ which are connected to two vertices of C bearing different colours. Among all odd circuits in $G-C$ of minimal length select one, C', for which $|C' \cap A'|$ is maximal. We shall deduce a contradiction from $\chi(G-C-C') \leqq n-4$. Let $\mathfrak{F}'$ be defined as above. There exists, by (2. 1)

a vertex x in $G-C-C'$ joined to vertices of each colour 1, 2, 3, 4 in $G(C\cup C')$. Exactly two vertices of C', say a, b and exactly two vertices of C are connected to x. Let c be the vertex of C' adjacent to a and b. The circuit C'' constructed from C' by deleting c and adding x has minimal length; hence c, like x, belongs to A'. Let $\mathfrak{F}'(a)=4$ and $\mathfrak{F}'(b)\neq 4$. Then $b\notin A$ for otherwise x is connected to $\varphi(b)$ and b which are coloured alike. From $\mathfrak{F}'(c)\neq\mathfrak{F}'(b)$ we conclude that x is connected to some vertex d on C with $\mathfrak{F}(d)=\mathfrak{F}'(c)$, hence c is not connected to d which gives $\varphi(c)\neq\varphi(x)$. Now a is connected to $\varphi(c)$ and $\varphi(x)$. Therefore $\varphi(a)$ is connected to $\varphi(c)$ and $\varphi(x)$. From $x, c\in A'$ follows that $\varphi(x)$ and $\varphi(c)$ must occur among the vertices v_0, v_1 and v_2.

α) In the case $|C|>3$, we deduce $\{\varphi(c), \varphi(x)\}=\{v_0, v_2\}$ and $\varphi(a)=v_1$. From $\varphi(c)\neq\varphi(a)=v_1$ we get $\mathfrak{F}'(c)=\mathfrak{F}'(\varphi(c))\neq 3$. b is not connected to v_1 for otherwise x, v_1 and b would define a triangle. Hence (iv) in 3 gives $\mathfrak{F}'(b)=3$. But then the neighbours of x on C are coloured 1 and 2 which implies $\varphi(x)=v_1$, contradicting $\varphi(x)\neq\varphi(a)$.

β) In the case $|C|=3$ the vertices $x, a, \varphi(c)$ and $c, \varphi(a), \varphi(x)$ would define disjoint triangles in G. Our proof of Theorem 3 will require the following two lemmas:

(3. 4) LEMMA. *Let H and K be disjoint complete subgraphs of G such that $|K|=t(G)$. Then there exists a subgraph K' of K such that $|K'|=|H|$ and $\chi(\mathfrak{G}(H\cup K'))=|H|$.*

PROOF. By (2. 5) $\mathfrak{G}(H\cup K)$ is $t(G)$-colourable. For each vertex x of H there is exactly one vertex y of K having the same colour as x. Define K' to be the subgraph of K spanned by the vertices bearing the colours which appear in H.

(3. 5) LEMMA. *Let G be a graph such that $t(G)<\chi(G)$ and let m be an integer such that $1\leqq m<t(G)$. Suppose that for every $\langle m\rangle$, H, in G, $\chi(G-H)=\chi(G)-m$. Then for every $\langle t(G)\rangle$ in G there exists a disjoint $\langle m+1\rangle$.*

PROOF. Let H be a $\langle t(G)\rangle$ in a graph G satisfying the hypotheses above. We shall prove by induction on k the stronger assertion that for every integer k such that $0\leqq k\leqq m+1$ there exists an $\langle m+1\rangle$, K_k, such that $|H\cap K_k|=m+1-k$. The case $k=0$ is trivial. Let K_k be an $\langle m+1\rangle$ in G such that $|H\cap K_k|=m+1-k>0$. Let a be a vertex of $H\cap K_k$. By hypothesis, $\chi(G-(K_k-\{a\}))=\chi(G)-m$; hence by (2. 2) there exist $\chi(G)-m$ vertices each adjacent to every vertex of $K_k-\{a\}$. As $t(G)<\chi(G)$ these vertices do not form a complete graph. Thus some vertex b of $G-H$ is connected to all vertices of $K_k-\{a\}$. $K_{k+1}=K_k\cup\{b\}-\{a\}$ is the desired set for $k+1$.

(3. 6) PROOF OF THEOREM 3. Let $\chi(G)=n$, $t(G)=m$. Our proof will be by induction on n and m; the case $m=3$ is covered by Theorem 2. As the theorem is trivial for $n=m$ we shall assume that $4\leqq m<n$ and that the theorem is true for all graphs G' for which $|G'|<|G|$.

Case 1. Suppose there are disjoint triangles H, K in G for which $\chi(\mathfrak{G}(H\cup K))=3$. Let $G'=G-H-K$. Then $\chi(G')\geqq n-3$. By the induction hypothesis, or by Theorem 2, $c(G)\geqq 2+\left[\frac{2(n-3)-m}{3}\right]$.

Case 2. Suppose that for some triangle H in G, $\chi(G-H) \geqq \chi(G)-2$. If $t(G-H)=m$ we are led by (3. 4) to Case 1. If $3 \leqq t(G-H)<m$, $c(G) \geqq 1+\left[\frac{2(n-2)-(m-1)}{3}\right]$ by the induction hypothesis. If $t(G-H) \leqq 2$, $c(G) \geqq 1+\left[\frac{2(n-2)-3}{3}\right]$ by Theorem 2.

Case 3. Suppose that for all triangles H in G, $\chi(G-H)=n-3$. By (3. 5) there exist disjoint in G a $\langle 3 \rangle$ and an $\langle m \rangle$. By (3. 4) we are led to Case 1.

4. Two examples

That Theorem 3 is not true for $t(G)=2$ is shown by the following

(4. 1) EXAMPLE. Let the graph G have vertices x_i, y_i, z, and edges $x_i x_{i+1}$, $y_i x_{i-1}$, $y_i x_{i+1}$, $y_i z$ ($i=1, \ldots, 5$; subscripts modulo 5). Here $\chi(G)=4$, $t(G)=2$, but $c(G)=1$.

It is *not* claimed that Theorem 3 is best possible for all values of $t(G)$. While this is obviously true for $t(G)=\chi(G)$ it is in fact false for $t(G)=\chi(G)-1>5$ since, by (3. 1) $c(G) \geqq \left[\frac{\chi(G)+2}{3}\right]$ in that case. This last result *is* best possible, as seen from the following

(4. 2) EXAMPLE. Let G be the graph formed by connecting all vertices in a 5-gon with all vertices in an $\langle n-3 \rangle$. Then $\chi(G)=n$, $t(G)=n-1$, but $c(G) \leqq [|G|/3]=[(n+2)/3]$.

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ASYMMETRISCHE REGULÄRE GRAPHEN

Von

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1. Es wird untersucht für welche Paare (n, k) von natürlichen Zahlen zusammenhängende einfache ungerichtete asymmetrische reguläre Graphen vom Grad k mit n Punkten existieren. Dabei verstehen wir unter einem einfachen Graphen einen Graphen ohne Mehrfachkanten und Schlingen. Besteht die Automorphismengruppe eines Graphen nur aus der Identität, so spricht man von einem asymmetrischen Graphen. Inzidiert jeder Knotenpunkt mit k Kanten, so handelt es sich um einen regulären Graphen vom Grad k.

Die Anregung zur Untersuchung asymmetrischer regulärer Graphen verdanken wir den Arbeiten von ERDŐS und RÉNYI [1], sowie QUINTAS [5]. In der zweiten Arbeit werden für jede Punktanzahl die minimale und maximale Kantenanzahl asymmetrischer und zusammenhängender asymmetrischer Graphen bestimmt. Da bei regulären Graphen mit vorgegebener Punktezahl die Anzahl der Kanten dem Grad proportional ist, lautet das entsprechende Problem hier bei vorgegebener Punktezahl den minimalen und maximalen Grad asymmetrischer regulärer Graphen zu bestimmen. Es erweist sich, daß die Beschränkung auf zusammenhängende Graphen die Extremwerte nicht ändert.

Es wird auch das Problem behandelt bei vorgegebenei Gradzahl die minimale und maximale Punktzahl asymmetrischer regulärer Graphen zu bestimmen; dabei wird gezeigt, daß es für jedes $k \geqq 3$ ein $N(k)$ mit folgender Eigenschaft gibt: Existieren für $n \geqq N(k)$ reguläre Graphen vom Grad k mit n Punkten, so gibt es darunter einen asymmetrischen. Dies ist eine Verschärfung eines Resultates von IZBICKI [4], aus welchem folgt, daß für jedes $k \geqq 3$ unendlich viele asymmetrische reguläre Graphen vom Grad k existieren.

2. Die Menge der Knotenpunkte eines Graphen X bezeichnen wir mit $V(X)$ und die Menge der Kanten mit $E(X)$. Die Kanten können als ungeordnete Paare von Punkten aufgefaßt werden. Ist $(P, Q) \in E(X)$, so sagen wir die Kante (P, Q) verbindet die gegenseitigen Nachbarn P und Q. Unter einem die Punkte P_0 und P_n verbindenden Weg verstehen wir eine Folge von Kanten $\{(P_0, P_1), (P_1, P_2), \ldots, \ldots, (P_{n-2}, P_{n-1}), (P_{n-1}, P_n)\}$. Sind je zwei Punkte eines Graphen durch einen Weg verbunden, so nennt man den Graphen zusammenhängend. Im andern Fall spricht man von einem unzusammenhängenden oder nichtzusammenhängenden Graphen, und seine zusammenhängenden Bestandteile heißen Komponenten. Eine Brücke ist eine Kante, nach deren Entfernung ein zusammenhängender Graph in zwei Komponenten zerfällt. Entsteht durch Herausnahme von zwei Kanten, die keine Endpunkte gemeinsam haben, ein nichtzusammenhängender Graph, so bezeichnet man diese zwei Kanten als Doppelbrücke; war der ursprüngliche

Graph brückenlos, so entsteht dabei offensichtlich ein unzusammenhängender Graph mit zwei Komponenten.

Das Komplement $\bar{X}$ eines Graphen X wird folgendermaßen definiert: $V(\bar{X})=V(X)$ und $(A, B)\in E(\bar{X})$ genau dann, wenn $A\neq B$ und $(A, B)\notin E(X)$ ist.

Ein Automorphismus von X ist eine Permutation π auf $V(X)$, bei der $(\pi P, \pi Q)$ dann und nur dann in $E(X)$ liegt, falls $(P, Q)\in E(X)$ ist. Die Automorphismen von X bilden die Automorphismengruppe $G(X)$. Wir bezeichnen eine Teilmenge $A\subset V(X)$ als Block in bezug auf eine Permutationsgruppe H auf $V(X)$, falls $\pi A=A$ ist für alle $\pi\in H$. Besteht A nur aus einem Punkt P, so nennt man P einen Fixpunkt. Wir werden später folgende Regeln für Blöcke verwenden:

a) Mit A und B ist auch $A\cap B$ ein Block.

b) Mit A und B ist auch $A-B$ ein Block.

c) Mit A ist auch die Nachbarschaft von A (das ist die Menge aller Punkte $P\notin A$, die Nachbarn von mindestens einem Punkt aus A sind) ein Block.

Mit den obigen Begriffen können wir folgendes Lemma formulieren:

LEMMA 1. *Bilden die Kanten (P, Q) und (U, V) die einzige Doppelbrücke des zusammenhängenden Graphen X, so ist $\{P, Q, U, V\}$ ein Block.*

BEWEIS. Da durch jeden Automorphismus Doppelbrücken in Doppelbrücken übergeführt werden, bilden ihre Endpunkte einen Block.

LEMMA 2. *Zerfällt ein zusammenhängender Graph X durch Wegnahme seiner einzigen Doppelbrücke in die nichtisomorphen Komponenten X_1 und X_2, so bilden $V(X_1)$ und $V(X_2)$ Blöcke von $G(X)$.*

BEWEIS. Es sei (A_1, A_2) und (B_1, B_2) die einzige Doppelbrücke von X, wobei $A_i, B_i\in V(X_i)$. $V(X_1)$ ist die Menge aller Punkte, die von A_1 auf Wegen erreichbar sind, die keine der Doppelbrückenkanten enthalten. Wird A_1 durch einen Automorphismus auf einen Punkt in $V(X_2)$ abgebildet, so ist $V(X_2)$ die Bildmenge von $V(X_1)$, nämlich die Menge aller Punkte, die vom Bild von A_1 auf Wegen erreichbar sind, welche keine Doppelbrückenkanten enthalten. Das widerspricht aber der vorausgesetzten Nichtisomorphie von X_1 und X_2. Also bilden $V(X_1)$ und $V(X_2)$ je einen Block.

LEMMA 3. *Ist X ein zusammenhängender regulärer Graph vom Grad $k\geqq 3$, und zerfällt X durch Wegnahme von zwei Kanten, so enthält X mindestens $2k+2$ Punkte.*

BEWEIS. Es sei Y der Graph, der aus X durch Entfernung dieser beiden Kanten entsteht. Y besteht aus mindestens zwei Komponenten. Ist Y_1 eine dieser Komponenten, so gibt es darin mindestens einen Endpunkt der weggenommenen Kanten, der einen von diesen Endpunkten verschiedenen Nachbarn T in Y_1 hat. T hat in Y_1 k Nachbarn, also enthält Y_1 mindestens $k+1$ Punkte. Dasselbe gilt für jede andere Komponente, woraus die Behauptung folgt.

LEMMA 4. *Ist X ein zusammenhängender brückenloser regulärer Graph vom Grad $k\geqq 3$ und zerfällt X nach Entfernung der Doppelbrücke (A_i, B_i), $i=1, 2$ so in zwei Komponenten X_1 und X_2 mit A_i, $B_i\in V(X_i)$, daß X_1 höchstens $2k$ Punkte enthält und A_1 nicht mit B_1 verbunden ist, so ist keine Kante aus X_1 in einer Doppelbrücke von X enthalten.*

BEWEIS. Sei Y der Graph, der aus X_1 durch Hinzunahme der Kante (A_1, B_1) entsteht. Ist (P, Q), (R, S) eine Doppelbrücke von X mit $(P, Q) \in E(X_1)$, so gilt nach Lemma 3 $(R, S) \notin E(X_1)$. Dann bilden aber auch (P, Q), (A_1, B_1) eine Doppelbrücke von Y, im Widerspruch zu Lemma 3.

Wir definieren nun Graphen, die wir später zur Konstruktion von asymmetrischen regulären Graphen verwenden. Die bei den Knotenpunkten auftretenden Indizes sind jeweils modulo r zu nehmen.

$G_{r,k}$ für $k \neq 5$ und $r \geqq k \geqq 3$:

$$V(G_{r,k}) = \{P_0, \ldots, P_{r-1}, Q_0, \ldots, Q_{r-1}\}$$

$$E(G_{r,k}) = \{(P_i, Q_{i+j}) | i=0, \ldots, r-1;\ j=0, \ldots, k-3\} \cup$$
$$\cup \{(P_i, P_{i+1}), (Q_i, Q_{i+1}) |\ i=0, \ldots, r-1\},$$

$G_{r,5}$: $$V(G_{r,5}) = \{P_0, \ldots, P_{r-1}, Q_0, \ldots, Q_{r-1}\}$$

$$E(G_{r,5}) = \{(P_i, Q_{i+j}) |\ i=0, \ldots, r-1;\ j=0, 1, 3\} \cup$$
$$\cup \{(P_i, P_{i+1}), (Q_i, Q_{i+1}) |\ i=0, \ldots, r-1\}.$$

LEMMA 5. *Jede Kante von $G_{r,k}$, $k \geqq 5$, liegt in mindestens zwei sonst elementfremden Kreisen.*

BEWEIS. Die folgende Tabelle behandelt alle möglichen Fälle.

	Kante	Kreis 1	Kreis 2
	(P_i, P_{i+1})	$(P_0, \ldots, P_{r-1})$	(P_i, P_{i+1}, Q_{i+1})
	(Q_i, Q_{i+1})	$(Q_0, \ldots, Q_{r-1})$	(Q_i, Q_{i+1}, P_i)
	(P_i, Q_i)	(P_{i-1}, P_i, Q_i)	(P_i, Q_i, Q_{i+1})
$j = 1, \ldots, k-4$	(P_i, Q_{i+j})	(P_i, P_{i+1}, Q_{i+j})	$(P_i, Q_{i+j-1}, Q_{i+j})$
$k = 5$	(P_i, Q_{i+3})	$(P_i, Q_{i+1}, Q_{i+2}, Q_{i+3})$	$(P_i, Q_{i+3}, P_{i+2}, P_{i+1})$
$k \geqq 6$	(P_i, Q_{i+k-3})	$(P_i, P_{i+1}, Q_{i+k-3})$	$(P_i, Q_{i+k-4}, Q_{i+k-3})$

LEMMA 6. *Ist τ ein Automorphismus von $G_{r,k}$, $k \geqq 5$, mit $\tau P_i = P_i$ für $i=0, 1$, so ist τ die Identität.*

BEWEIS. Wir wenden die Regeln a)—c) für Blöcke an, wobei die betrachtete Gruppe H aus allen Automorphismen von $G_{r,k}$ besteht, die P_0 und P_1 fix lassen.

Es sei zunächst $k=5$. Wir erhalten folgende Blöcke: $\alpha = \{P_{r-1}, P_1, Q_0, Q_1, Q_3\}$, $\beta = \{P_0, P_2, Q_1, Q_2, Q_4\}$, $\alpha \cap \beta = Q_1$, $\gamma = \{P_{r-2}, P_0, P_1, Q_0, Q_2\}$, $\beta \cap \gamma = \{P_0, Q_2\}$, also Q_2, $\delta = \{P_{r-1}, P_1, P_2, Q_1, Q_3\}$, $\beta \cap \delta = \{P_2, Q_1\}$, also P_2. Es sind daher mit P_0 und P_1 auch Q_0 und P_2 fix. Durch wiederholte Anwendung dieses Arguments ergibt sich die Richtigkeit der Behauptung für $k=5$.

Im Fall $k \geqq 6$ erhalten wir folgende Blöcke: $\alpha = \{P_{r-1}, P_1, Q_0, \ldots, Q_{k-3}\}$, $\beta = \{P_0, P_2, Q_1, \ldots, Q_{k-2}\}$, $\gamma = \alpha \cap \beta = \{Q_1, \ldots, Q_{k-3}\}$, $\beta - \gamma = \{P_0, P_2, Q_{k-2}\}$, also $\{P_2, Q_{k-2}\}$. P_2 hat in γ die Nachbarn $Q_2, \ldots, Q_{k-3}$, also mindestens zwei; hingegegen hat Q_{k-2} in γ nur den Nachbarn Q_{k-3}. Daher sind P_2 und Q_{k-2} Fixpunkte. Durch wiederholte Anwendung des Arguments wird der Beweis des Lemmas abgeschlossen.

LEMMA 7. *Ist τ ein Automorphismus von $G_{r,k}$, $r \geqq k+2$, $k \geqq 3$, der Q_0 und P_1 fix läßt, so ist τ die Identität.*

BEWEIS. Wir wenden wie vorhin die Regeln für Blöcke an, wobei die zugelassene Permutationsgruppe H aus jenen Automorphismen von $G_{r,k}$ besteht, die Q_0 und P_1 fix lassen. Die Fälle $k \geqq 5$ führen wir auf Lemma 6 zurück.

Es sei zunächst $k \neq 5$. Wir erhalten die Blöcke $\alpha = \{P_{r-k+3}, \ldots, P_0, Q_{r-1}, Q_1\}$, $\beta = \{P_0, P_2, Q_1, \ldots, Q_{k-2}\}$ und $\alpha \cap \beta = \{P_0, Q_1\}$.

Für $k=3$ ist $\beta = \{P_0, Q_1, P_2\}$ und daher P_2 ein Fixpunkt. Dies gibt den Block $\{P_1, Q_2, P_3\}$, also auch $\{Q_2, P_3\}$ und dessen Nachbarschaft $\gamma = \{Q_1, P_2, Q_3, P_4\}$, sowie $\alpha \cap \beta \cap \gamma = Q_1$. Damit sind Q_1 und P_2 fix, und man kann das Argument wiederholen.

Für die übrigen Werte von k, $k \neq 5$, liegen $k-3 \neq 2$ Nachbarn von P_0 in β, während Q_1 mit genau zwei Punkten aus β verbunden ist. P_0 und Q_1 sind daher Fixpunkte und wir können für $k \geqq 6$ Lemma 6 auf P_0 und P_1 anwenden.

Für $k=4$ haben wir bereits die vier Fixpunkte P_0, P_1, Q_0 und Q_1. Die Nachbarschaft von Q_1 gibt den Block $\delta = \{P_0, P_1, Q_0, Q_2\}$. Damit ist auch $\beta - \delta = \{Q_1, P_2\}$ ein Block, also P_2 fix. Nun wird das Argument mit Q_1 und P_2 wiederholt.

Ist $k=5$, so ist $\alpha = \{P_{r-3}, P_{r-1}, P_0, Q_{r-1}, Q_1\}$, $\beta = \{P_0, P_2, Q_1, Q_2, Q_4\}$ und $\alpha \cap \beta = \{P_0, Q_1\}$. P_0 hat in β nur einen Nachbarn, nämlich Q_1, während Q_1 die zwei Nachbarn P_0 und Q_2 in β hat. P_0 ist somit Fixpunkt und wir können Lemma 6 auf P_0 und P_1 anwenden.

3. Es gibt offensichtlich keine asymmetrischen regulären Graphen mit den Graden 1 oder 2. In der Dissertation von K. FESSL [2] wurden alle regulären Graphen mit höchstens neun Punkten aufgestellt und untersucht. Es gibt darunter keine asymmetrischen, außer dem einpunktigen Graphen. Die Untersuchung wurde von G. Baron auf die zehnpunktigen regulären Graphen erweitert, wobei es sich herausstellte, daß nur asymmetrische reguläre Graphen vierten Grades auftreten. Die Veröffentlichung dieser Untersuchung ist an anderer Stelle vorgesehen.

4. Nach dem obigen ist die kleinste mögliche Punktzahl für asymmetrische kubische Graphen (das sind reguläre Graphen vom Grad drei) gleich zwölf. Ein solcher Graph ist in der Arbeit [3] von FRUCHT angegeben. Wir zeigen nun, daß es für jedes gerade $m \geqq 12$ mindestens einen asymmetrischen kubischen Graphen mit m Punkten gibt, und zwar den in Figur 1 dargestellten Graphen X_{2n}. X_{12} ist der oben erwähnte Graph aus der Arbeit von Frucht.

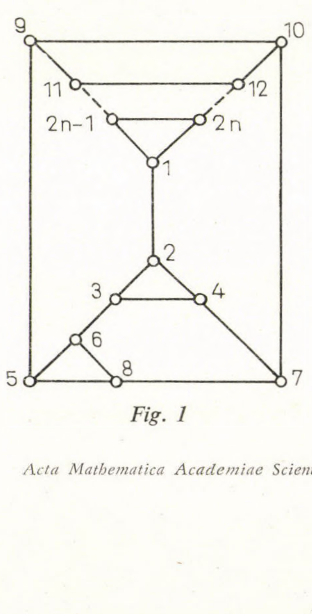

Fig. 1

LEMMA 8. *Der Graph X_{2n} ist asymmetrisch für $n \geqq 6$.*

BEWEIS. Das Dreieck $(1, 2n-1, 2n)$ grenzt mit der Seite $(2n-1, 2n)$ an ein Viereck und ist das einzige Dreieck mit dieser Eigenschaft. Daher ist 1 ein Fixpunkt und $\{2n-1, 2n\}$ ein Block. Gibt es einen Automorphismus τ, der Punkt $2n-1$ in den Punkt $2n$ überführt, so muß τ auch die Nachbarn $2n-3$ und $2n-2$ dieser beiden Punkte ineinander überführen. Durch wiederholte Anwendung dieses Arguments ergibt sich, daß auch Punkt 9 mit Punkt 10, bzw. Punkt 5 mit Punkt 7 durch

τ vertauscht werden muß. Da Punkt 5 in einem Dreieck liegt, Punkt 7 aber nicht, sind 5 und 7 fix, und damit auch alle oben genannten Punkte. Man sieht nun leicht, daß auch alle anderen Punkte Fixpunkte sind.

5. Wie schon erwähnt, gibt es keinen asymmetrischen regulären Graphen vom Grad 4 mit weniger als 10 Punkten. Wir zeigen, daß es für jedes $m \geqq 10$ mindestens einen asymmetrischen regulären Graphen vom Grad 4 mit m Punkten gibt. Es sei G der in Figur 2, und H_n der n-punktige in Figur 3 dargestellte Graph. Durch Identifizierung der Punkte $1, 2, n-1$ und n von H_n mit den Punkten $1', 2', 3'$ und $4'$ von G entsteht ein regulärer Graph X_s vom Grad vier mit $s = n+5$ Punkten.

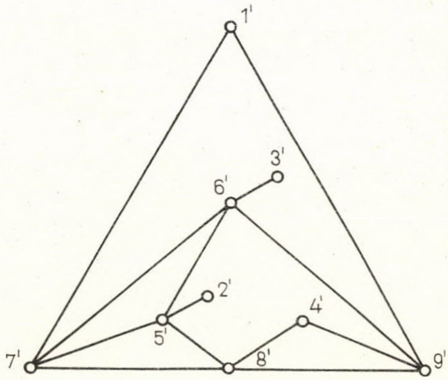

Fig. 2

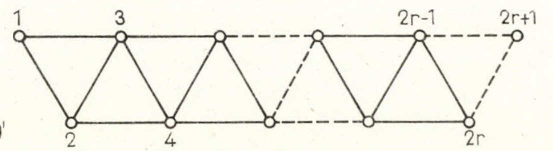

Fig. 3

LEMMA 9. *X_s ist asymmetrisch für $s \geqq 10$.*

BEWEIS. Die Punkte $3, 4, \ldots, n-2$ sind die einzigen Punkte von X_s, die in drei Dreiecken liegen. Sie bilden daher einen Block α. Die Nachbarschaft von α bildet den Block $\beta = \{1', 2', 3', 4'\}$; damit ist auch $\alpha \cup \beta = V(H_n)$ ein Block. Da $4'$ der einzige Punkt aus β ist, der in G in einem Dreieck liegt, ist $4'$ fix, und somit auch $3'$, der einzige Nachbar von $4'$ in β. $1'$ hat in G zwei Nachbarn, $2'$ nur einen, also zerfällt β in vier Fixpunkte. Daher sind auch die einzigen Nachbarn von $2'$ und $3'$ in G, nämlich $5'$ und $6'$ Fixpunkte. Als einziger gemeinsamer Nachbar von $5'$ und $6'$ ist $7'$ fix und damit auch $8'$ und $9'$.

Es bleibt zu zeigen, daß alle Punkte in α fix sind. 1 und 2 sind Fixpunkte und auch ihre in H_n liegenden Nachbarn 3 und 4, da 3 mit 1 und 2, 4 aber nur mit 2 verbunden ist. Auf diese Art kann man zeigen, daß alle Punkte in α Fixpunkte sind.

6. Die Lemmata 8 und 9 ermöglichen, zusammen mit der Erhaltung der Regularität und der Asymmetrie beim Übergang zum komplementären Graphen, die folgende Aussage:

SATZ 1. *Sei $m(n)$, bzw. $M(n)$ der minimale, bzw. der maximale Grad von regulären asymmetrischen Graphen mit $n > 0$ Punkten, so gilt:*

(i) $m(1) = M(1) = 0$,
(ii) *$m(n)$ und $M(n)$ sind für $2 \leqq n \leqq 9$ nicht definiert,*
(iii) $m(10) = 4$, $M(10) = 5$,
(iv) *$m(n) = 4$, $M(n) = n-5$ für ungerade $n > 10$,*
(v) *$m(n) = 3$, $M(n) = n-4$ für gerade $n > 10$.*

7. Aus Satz 1 folgt, daß es für jedes $k>2$ einen asymmetrischen regulären Graphen vom Grad k gibt, also auch einen mit minimaler Punktanzahl, welche in folgendem Satz angegeben ist:

SATZ 2. *Für $k \geqq 0$ sei $p(k)$ die minimale Punktanzahl der asymmetrischen regulären Graphen vom Grad k. Es gilt:*

(i) $p(0)=1$,
(ii) $p(k)$ *ist nicht definiert für* $1 \leqq k \leqq 2$,
(iii) $p(3)=12$,
(iv) $p(4)=p(5)=10$,
(v) $p(6)=11$,
(vi) $p(k) = k+4$ *für gerade* $k>6$,
(vii) $p(k) = k+5$ *für ungerade* $k>6$.

BEWEIS. Die Fälle (i)—(v) sind klar. Aus Satz 1 und $k>6$ folgt $p(k)>10$. Da das Komplement eines regulären Graphen vom Grad k mit $p(k)$ Punkten den Grad $p(k)-k-1$ hat und dieser Graph asymmetrisch sein soll, gilt $p(k)-k-1 \geqq 3$, also $p(k) \geqq k+4$. Ist k gerade, so auch $k+4$, und nach Satz 1, (v) gibt es einen asymmetrischen regulären Graphen vom Grad k mit $k+4$ Punkten. Ist k ungerade, so muß $p(k)$ gerade sein, also gilt $p(k) \geqq k+5$ und nach Satz 1, (iv) gibt es einen asymmetrischen regulären Graphen vom Grad k mit $k+5$ Punkten.

BEMERKUNG. Aus Lemma 9 ergibt sich durch Komplementbildung, daß für gerade $k>6$ nicht nur ein asymmetrischer regulärer Graph vom Grad k mit $k+4$ Punkten existiert, sondern auch einer mit der nächsthöheren Punktzahl $k+5$. Um auch für den Grad 6 zwei asymmetrische reguläre Graphen mit aufeinanderfolgenden Punktanzahlen zur Verfügung zu haben, geben wir einen solchen mit zwölf Punkten an. Nämlich den Graphen K mit $V(K)=\{1, 2, \ldots, 12\}$ und $E(K)= =\{(1,6), (1,7), (1,8), (1,9), (1,10), (1,11), (2,3), (2,4), (2,5), (2,8),(2,11), (2,12), (3,4), (3,5), (3,8), (3,9), (3,12), (4,5), (4,10), (4,11), (4,12), (5,6), (5,8), (5,10), (6,7), (6,8), (6,9), (6,10), (7,9), (7,10), (7,11), (7,12), (8,11), (9,11), (9,12), (10,12)\}$. Die Asymmetrie dieses Graphen läßt sich leicht beweisen.

8. Ist X ein regulärer Graph vom Grad k, der eine Kante (A, B) enthält, so bilden wir aus X und $G_{r,k}$ einen Graphen $D(X, r)$, indem wir aus X die Kante (A, B), aus $G_{r,k}$ die Kante (P_0, P_1) entfernen und die beiden Graphen mit Hilfe der Kanten (A, P_0) und (B, P_1) verbinden. Ebenso bilden wir aus X und $\bar{G}_{r,2r-k-1}$ einen Graphen $E(X, r)$ durch Entfernung von (A, B) und (Q_0, P_1), sowie durch Hinzufügung von (A, Q_0) und (B, P_1).

LEMMA 10. *Ist X ein asymmetrischer regulärer Graph vom Grad k mit $m \leqq 2k$ Punkten, so sind für $r \geqq k$ die Graphen $D(X, r)$ asymmetrische reguläre Graphen vom Grad k mit $m+2r$ Punkten.*

BEWEIS. Klarerweise ist $D(X, r)$ regulär vom Grad k mit $m+2r$ Punkten. Es bleibt zu zeigen, daß $D(X, r)$ asymmetrisch ist. Wegen Lemma 4 und 5 ist (A, P_0), (B, P_1) die einzige Doppelbrücke von $D(X, r)$, und wegen Lemma 2 bilden $V(X)$ und $V(G_{r,k})$ Blöcke von $D(X, r)$. Da X asymmetrisch ist, sind A, B, und somit auch P_0 und P_1 Fixpunkte. Aus Lemma 6 folgt nun die Asymmetrie von $D(X, r)$.

LEMMA 11. *Ist X ein asymmetrischer regulärer Graph vom Grad k mit $m \leqq 2k$ Punkten, so sind für $2(k-1) \geqq 2r \geqq k+4$ die Graphen $E(X, r)$ asymmetrische reguläre Graphen vom Grad k mit $m+2r$ Punkten.*

BEWEIS. Da die Regularität leicht zu zeigen ist, bleibt wieder diebehauptete Asymmetrie zu beweisen. Durch Anwendung von Lemma 4 auf X und $\bar{G}_{r,2r-k-1}$ ergibt sich, daß (A, Q_0), (B, P_1) die einzige Doppelbrücke von $E(X, r)$ bilden. Wie vorhin folgt nun die Fixpunkteigenschaft von Q_0 und P_1. Eine Anwendung von Lemma 7 ergibt sodann die Asymmetrie von $E(X, r)$.

SATZ 3. *Für die in der Spalte* NE *der folgenden Tabelle angegebenen Werte n gibt es keinen asymmetrischen regulären Graphen mit n Punkten und dem in der entsprechenden Zeile angegebenen Grad k. Hingegen gibt es für die in der Spalte* E *angegebenen Werte n solche Graphen. Für ungerade Werte k sind allerdings nur gerade Werte n in der Spalte* E *zulässig, da es in diesen Fällen überhaupt keinen regulären Graphen mit ungerader Punktanzahl gibt.*

Grad k	NE	E
3	$n \leqq 11$	$12 \leqq n$
4	$n \leqq 9$	$10 \leqq n$
5	$n \leqq 9$	$10 \leqq n \leqq 12$, $20 \leqq n$
6	$n \leqq 10$	$11 \leqq n \leqq 12$, $21 \leqq n$
$7 \leqq k \leqq 13$, ungerade	$n \leqq k+4$	$n = k+5$, $2k+10 \leqq n$
$k \geqq 15$, ungerade	$n \leqq k+4$	$k+5 \leqq n \leqq 2k-6$, $2k+10 \leqq n$
$k \geqq 8$, gerade	$n \leqq k+3$	$k+4 \leqq n \leqq \max(k+5, 2k-7)$, $2k+8 \leqq n$

BEWEIS. Die in der Spalte NE enthaltenen Aussagen folgen aus Satz 2. Um die Aussagen der Spalte E zu beweisen, beschränken wir uns zunächst auf die Fälle $n \geqq 2k+1$ und zeigen die Existenz von asymmetrischen regulären Graphen für die anderen Werte von n durch Komplementbildung. Für $k=3$, bzw. 4 folgt das Resultat aus Lemma 8, bzw. 9. Für $k=5$, $n=12$ ist das Komplement des nach Satz 2 angegebenen Graphen vom Grad 6 mit 12 Punkten ein Beispiel. Um die Ungleichungen der Form $n_0 \leqq n$ für $k \geqq 5$ zu beweisen, benützen wir Lemma 10 und 11, wobei wir für X folgende Graphen wählen, deren Existenz durch Satz 2 und die darauffolgende Bemerkung gesichert ist:

a) Für ungerade $k \geqq 5$ asymmetrische reguläre Graphen vom Grad k mit $k+5$ Punkten.

b) Für $k=6$ asymmetrische reguläre Graphen vom Grad 6 mit 11, bzw. 12 Punkten.

c) Für gerade $k \geqq 8$ asymmetrische reguläre Graphen vom Grad k mit $k+4$, bzw. $k+5$ Punkten.

Die angegebenen Werte für n_0 ergeben sich nun direkt aus Lemma 11 und den Punktanzahlen der Graphen X.

Die Fälle $n \leqq 2k$ sind für $k \leqq 13$ durch Satz 2 und Bemerkung erledigt. Es bleiben also für $k \geqq 14$ die Ungleichungen der Form $n_1 \leqq n \leqq n_2$ zu beweisen.

Ist $k \geqq 15$ ungerade, so gibt es für gerade $n \geqq 2(n-k-1)+8$ einen asymmetrischen regulären Graphen vom Grad $n-k-1$ mit n Punkten, falls $n-k-1 \geqq 4$ ist. Sein Komplement ist ein asymmetrischer regulärer Graph vom Grad k und n Punkten. Damit ist die Existenz für $k+5 \leqq n \leqq 2k-6$ gesichert. Für gerade $k \geqq 14$ gibt es asymmetrische reguläre Graphen vom Grad $n-k-1 \geqq 3$ wenn $n \geqq 2(n-k-1)+8$ und gerade ist, da dann $n-k-1$ ungerade ist. Daraus folgt die Existenz für $k+4 \leqq n \leqq 2k-7$.

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ON A PROPERTY OF INFINITELY DIVISIBLE DISTRIBUTIONS IN A HILBERT SPACE

By

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In the present note we study some properties of probability distributions in a Hilbert space. Let $\mathscr{H}$ be a real separable Hilbert space. Then the characteristic functional of a probability distribution μ in $\mathscr{H}$ is given by the formula (cf. [2], [3])

$$\chi(f) = \int_{\mathscr{H}} e^{i(x,f)}\, d\mu. \tag{1}$$

Here (x, f) is the inner product of the elements $x \in \mathscr{H}$ and the linear functional $f \in \mathscr{H}$. The relation (1) holds for every $f \in \mathscr{H}$.

We restrict our to study the class of probability distributions μ in $\mathscr{H}$ for which

$$\int_{\mathscr{H}} \|x\|^2 d\mu < \infty. \tag{2}$$

Following PROHOROV [3], we introduce the operator $S\mu$ by the formula

$$(S\mu f, f) = \int_{\mathscr{H}} (x, f)^2 d\mu$$

for every $f \in \mathscr{H}$ and call such an operator an S-operator. An S-operator is linear, self-adjoint, non-negative and completely continuous with a finite trace.

Let $\{\xi_n\}$ be a sequence of independent random variables with values in the Hilbert space $\mathscr{H}$. Let $\eta_N = \sum_{n=1}^{N} \xi_n$ and let μ_N denote the probability distribution of η_N $(N=1, 2, \dots)$. If the sequence $\{\mu_N\}$ converges weakly to a probability distribution μ in $\mathscr{H}$, then we say that the series $\sum_{n=1}^{\infty} \xi_n$ converges.

The mathematical expectation of a random variable ξ in $\mathscr{H}$ is an element $M\xi \in \mathscr{H}$ such that for every linear functional $f \in \mathscr{H}$, we have

$$M(\xi, f) = (M\xi, f). \tag{3}$$

THEOREM 1. *Let $\{\xi_n\}$ be a sequence of independently and identically distributed random variables with values in the Hilbert space $\mathscr{H}$ each having a mathematical expectation $M\xi_n = \theta$, where θ is the null element in $\mathscr{H}$. Let $\{A_n\}$ be a sequence of bounded, linear, self-adjoint operators such that $\sum_{n=1}^{\infty} \|A_n\|^2 < \infty$.*

Then the series $\sum_{n=1}^{\infty} A_n \xi_n$ converges.

PROOF. We construct the sequence of random variables $\eta_N=\sum_{n=1}^{N} A_n\xi_n$ and we denote the corresponding probability distribution by μ_N $(N=1, 2, \ldots)$. Let S_{μ_N} be the S-operator corresponding to μ_N. Then it is easy to verify that for every $f\in\mathscr{H}$, we have

$$(S\mu_N f, f)\leqq\alpha\|f\|^2\sum_{n=1}^{N}\|A_n\|^2$$

where $\alpha=\int_{\mathscr{H}}\|x\|^2d\mu$ and μ is the common probability distribution of ξ_n. Let $\{e_j\}$ be a complete orthonormal sequence of elements in $\mathscr{H}$. Then we have for all N,

$$(S\mu_N e_j,\ e_j)\leqq\alpha\lambda$$

where $\lambda=\sum_{n=1}^{\infty}\|A_n\|^2<\infty$. Using the compactness criteria of PROHOROV (cf. [3], Theorem 1. 13), we conclude that the sequence $\{\mu_N\}$ is weakly compact.

Now we take an arbitrary functional f in $\mathscr{H}$ and construct the sequence of real random variables $\{\xi_n^*\}$ as

$$\xi_n^*=\xi_n^*(f)=(A_n\xi_n, f)=(\xi_n, A_n f).$$

It is then easy to verify that $\{\xi_n^*\}$ is a sequence of independently distributed random variables with mean $M\xi_n^*=0$ and variance $M\xi_n^{*2}=M(\xi_n, A_n f)^2=\sigma_n^2(f)$ where $\sum_{n=1}^{\infty}\sigma_n^2(f)<\infty$ for any fixed f. Hence the series $\sum_{n=1}^{\infty}\xi_n^*$ converges a.e. (cf. [1], p. 197).

We next construct the sequence of real random variables $\eta_N^*=(\eta_N, f)$ where $\eta_N=\sum_{n=1}^{N}A_n\xi_n$ so that $\eta_N^*=\sum_{n=1}^{N}\xi_n^*$. It follows from above that the sequence $\{\eta_N^*\}$ converges a.e. to a real random variable $\eta^*=\eta^*(f)$ for any fixed $f\in\mathscr{H}$.

Let $\varphi_N(u; f)$ and $\varphi(u; f)$ denote the characteristic functions of η_N^* and η^* respectively. In view of the continuity theorem, the sequence $\{\varphi_N(u; f)\}$ converges to $\varphi(u; f)$ for all real u. But we have

$$\varphi_N(u; f)=\chi_N(f)$$

where $\chi_N(f)$ denotes the characteristic functional of η_N. We set $u=1$ and thus conclude that the sequence of characteristic functionals $\chi_N(f)$ converges to some function $\chi(f)$ for every $f\in\mathscr{H}$. Then it follows at once from Lemma 1. 6 [3] that the sequence of probability distributions $\{\mu_N\}$ converges weakly, as $N\to\infty$, to a probability distribution. This completes the proof of Theorem 1.

THEOREM 2. *Let $\{\xi_n\}$ be a sequence of independently and identically distributed random variables with values in the Hilbert space $\mathscr{H}$ each having mathematical expectation $M\xi_n=\theta$. Let $\{A_n\}$ be a sequence of bounded, linear, self-adjoint operators such that*

(i) $$\sum_{n=1}^{\infty}\|A_n\|^2<\infty,$$

(ii) $$\sup_n\|A_n\|<1.$$

Suppose that the sum $\sum_{n=1}^{\infty} A_n\xi_n$ *has the same distribution as* ξ_1. *Then the common probability distribution of random variable* ξ_n *is infinitely divisible.*

PROOF. Let $\chi(f)=\int_{\mathscr{H}} e^{i(x,f)}d\mu$ denote the characteristic functional of the common probability distribution μ. Then the characteristic functional of the random variable $A_n\xi_n$ is given by

$$\int_{\mathscr{H}} e^{i(A_n x,f)}\,d\mu = \int_{\mathscr{H}} e^{i(x,A_n f)}\,d\mu = \chi(A_n f). \tag{4}$$

Hence we have the relation

$$\chi(f) = \prod_{n=1}^{\infty} \chi(A_n f) \tag{5}$$

where the product on the right hand side converges uniformly in every bounded sphere $\|f\|\leqq\varrho$ ($\varrho>0$).

Let $\varepsilon>0$ be an arbitrary positive number. Since $\sum_{n=1}^{\infty}\|A_n\|^2<\infty$, we can choose an $N=N(\varepsilon)$ such that $\sum_{n=N+1}^{\infty}\|A_n\|^2<\varepsilon$. With the value of N thus determined, we introduce function $\varphi_N(f)$ as

$$\varphi_N(f) = \prod_{n=N+1}^{\infty} \chi(A_n f). \tag{6}$$

Then clearly $\varphi_N(f)$ is the characteristic functional of the sum $\sum_{n=N+1}^{\infty} A_n\xi_n$. Let μ_N be the corresponding probability distribution. Then we can verify easily that

$$(S\mu_N f,f) \leqq \alpha\|f\|^2 \prod_{n=N+1}^{\infty} \|A_n\|^2 < \alpha\|f\|^2\varepsilon \tag{7}$$

for every $f\in\mathscr{H}$ where $\alpha=\int_{\mathscr{H}}\|x\|^2d\mu<\infty$.

We can then rewrite (4) in the form

$$\chi(f)=\chi(A_1 f)\chi(A_2 f)\dots\chi(A_N f)\varphi_N(f). \tag{8}$$

We replace f by $A_j f$ in (8) and thus obtain

$$\chi(A_j f)=\chi(A_1 A_j f)\chi(A_2 A_j f)\dots\chi(A_N A_j f)\varphi_N(A_j f) \tag{9}$$

for $j=1,2,\dots,N$. Then combining (8) and (9), we get

$$\chi(f) = \prod_{j=1}^{N}\chi(A_j^2 f)\cdot\prod_{j\neq k}\chi(A_j A_k f)\prod_{j=1}^{N}\varphi_N(A_j f)\cdot\varphi_N(f).$$

We repeat this process n times and thus obtain

$$\chi(f) = \prod\chi(A_{j_1}A_{j_2}\dots A_{j_n}f)\times\prod_{k=1}^{n-1}\prod\varphi_N(A_{j_1}A_{j_2}\dots A_{j_{n-k}}f)\times\varphi_N(f). \tag{10}$$

Here the product on the right hand side consists of $N^n+N^{n-1}+\dots+N+1$ factors, each of the subscripts $j_1, j_2, \dots, j_n$ taking any one of the values $1, 2, \dots, N$ and the repetition being allowed. The formula (10) indicates that the random variable ξ with a characteristic functional $\chi(f)$ can be expressed as the sum of $k_n=\sum_{k=0}^{n} N^{n-k}$ independent random variables $\xi_{n,k}$ $(k=1, 2, \dots, k_n)$ each having the zero mathematical expectation, that is

$$\xi = \sum_{k=1}^{k_n} \xi_{n,k}$$

for every n where $k_n=N^n+N^{n-1}+\dots+N^2+1$.

We denote the characteristic functional of the random variable $\xi_{n,k}$ by $\chi_{n,k}(f)$ $(k=1, 2, \dots, k_n)$. Then we can rewrite (10) in the form

$$\chi(f) = \prod_{k=1}^{k_n} \chi_{n,k}(f). \tag{11}$$

We shall now prove that each of the factors on the right hand side of (11) satisfy the condition

$$\lim_{n\to\infty} \sup_{1\leqq k\leqq k_n} \sup_{\|f\|\leqq\varrho} |\chi_{n,f}-1| = 0 \tag{12}$$

for every $\varrho>0$.

First we note that for any real β, we have the inequality

$$|e^{i\beta}-1-i\beta| \leqq \frac{1}{2}\beta^2. \tag{13}$$

We now proceed to estimate the factors on the right hand side of (10). Since the random variable $\sum_{n=N+1}^{\infty} A_n\xi_n$ has zero mathematical expectation, we can easily verify using (7) and (13) that

$$\sup_{\|f\|\leqq\varrho} |\varphi_N(f)-1| \leqq \frac{1}{2}\alpha\varrho^2\varepsilon \tag{14}$$

for any $\varrho>0$. Since $\sup_n \|A_n\|<1$, it is also easy to see that any factor of the form $\varphi_N(A_{j_1} \dots A_{j_{n-k}}f)$ has the same estimate

$$\sup_{\|f\|\leqq\varrho} |\varphi_N(A_{j_1}, \dots, A_{j_{n-k}}f)-1| \leqq \frac{1}{2}\alpha\varrho^2\varepsilon. \tag{15}$$

Finally we consider the factors of the form $\chi(A_{j_1}A_{j_2} \dots A_{j_n}f)$ where the subscripts $j_1, j_2, \dots, j_n$ can take any one of the values $1, 2, \dots, N$. We denote $\sup_n \|A_n\|=\gamma$ and verify easily that any such factor has the estimate

$$\sup_{\|f\|\leqq\varrho} |\chi(A_{j_1}A_{j_2}\dots A_{j_n}f)-1| \leqq \frac{1}{2}\alpha\varrho^2\gamma^{2n}.$$

Since $\gamma < 1$, for a given ε, we can always select an $n_0 = n_0(\varepsilon)$ sufficiently large such that for $n > n_0$, we have

$$(16) \qquad \sup_{\|f\| \leqq \varrho} |\chi(A_{j_1} \dots A_{j_n} f) - 1| \leqq \frac{1}{2} \alpha \varrho^2 \varepsilon .$$

Thus from the estimates obtained in (14), (15) and (16), we conclude that all the factors on the right hand side of (11) satisfy the condition (12). Therefore the sequences of independent random variables $\xi_{n,k}$ $(k = 1, 2, \dots, k_n;\ n = 1, 2, \dots)$ are uniformly infinitesimal and consequently the probability distribution of the sum $\xi = \sum_{k=1}^{k_n} \xi_{n,k}$ is infinitely divisible in view of Theorem 7. 7 [4].

This completes the proof of Theorem 2.

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ÜBER DIE WIRKFELDER VERALLGEMEINERTER ABELSCHER LIMITIERUNGSVERFAHREN

Von

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1. Verfahren, die das Abelsche Verfahren einschließen

Wir definieren in folgender Weise eine Klasse von Limitierungsverfahren, die mit dem Abelschen Verfahren verwandt sind:

Ist H eine für $s>0$ erklärte Funktion, und ist $\{\lambda_n\}$ eine Folge reeller Zahlen mit $0<\lambda_1<\lambda_2<\lambda_3<\ldots$, $\lambda_n\to\infty$, dann soll eine Reihe Σa_n durch das Verfahren $A\{H, \lambda_n\}$ zum Werte a limitierbar heißen, wenn

$$A_H(s) = \sum_{n=1}^{\infty} a_n H(s\lambda_n)$$

für $s>0$ konvergiert mit $\lim\limits_{s\to+0} A_H(s)=a$.

Im Falle $H(s)=e^{-s}$ erhalten wir Abelsche Verfahren $A\{e^{-s}, \lambda_n\}$ mit zugehörigen Dirichletschen Reihen

$$A(s) = \sum_{n=1}^{\infty} a_n e^{-s\lambda_n}.$$

Vergleiche HARDY [1], S. 71. Dabei ergibt $\lambda_n=n$ das einfachste und bekannteste Abelsche Verfahren A.

Für $H(s) = \dfrac{se^{-s}}{1-e^{-s}}$, $\lambda_n=n$ erhalten wir zum Beispiel das Lambeitsche Verfahren L.

Im ersten Teil dieser Arbeit wollen wir hinreichende und notwendige Eigenschaften der Funktion H dafür angeben, daß derartige Limitierungsverfahren $A\{H, \lambda_n\}$ stärker als die entsprechenden Abelschen Verfahren $A\{e^{-s}, \lambda_n\}$ sind. Dabei schreiben wir bezüglich der Wirkfelder dieser Verfahren

$$A\{e^{-s}, \lambda_n\}\subseteqq A\{H, \lambda_n\}, \tag{1}$$

falls jede durch $A\{e^{-s}, \lambda_n\}$ limitierbare Reihe auch durch das Verfahren $A\{H, \lambda_n\}$ limitiert wird. (Wir wollen hierbei jedoch nicht verlangen, daß die durch diese Verfahren erhaltenen Grenzwerte gleich sind.)

Ferner soll auch die Frage untersucht werden, inwieweit sich bei der Einschließung (1) die Konvergenzgeschwindigkeit überträgt. Wir schreiben diesbezüglich

$$A^{(k)}\{e^{-s}, \lambda_n\}\subseteqq A^{(k)}\{H, \lambda_n\} \tag{2}$$

für ein reellwertiges k, falls $A(s)=o(s^k)$ $(s\to+0)$ stets $A_H(s)=o(s^k)$ $(s\to+0)$ impliziert.

Es zeigt sich, daß die Herleitung äquivalenter Bedingungen für die Gültigkeit der Einschließung (1) oder (2) zu der tieferliegenden Frage führt, wie „dicht" die als Dirichletsche Reihen $\Sigma a_n e^{-s\lambda_n}$ darstellbaren Funktionen im Raum der stetigen Funktionen liegen.

Als Hauptergebnis erhalten wir die folgenden zwei Sätze, die in einer früheren Untersuchung des Verfassers nur für den Spezialfall $\lambda_n = n$ unter stärkeren, zusätzlichen Voraussetzungen bewiesen wurden. Siehe HOISCHEN [2], S. 53—55.

SATZ 1. *Die Folge* $\{\lambda_n\}$, $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$, $\lambda_n \to \infty$ *besitze eine Teilfolge* $\{\lambda_{n_l}\}$ $(l = 1, 2, 3, \dots, n_1 = 1)$ *mit* $\sum_{l=1}^{\infty} \frac{1}{\lambda_{n_l}} = \infty$ *und* $\lambda_{n_{l+1}} - \lambda_{n_l} \geqq q > 0$. *Es sei außerdem* $H(s) = \int_1^{\infty} e^{-st} dK(t)$ *für* $s > 0$ *absolut konvergent. Dann ist die Bedingung*

$$\int_1^{\infty} |dK(t)| < \infty$$

hinreichend und notwendig für die Einschließung

$$A\{e^{-s}, \lambda_n\} \subseteqq A\{H, \lambda_n\}.$$

Entsprechend gilt für die Konvergenzgeschwindigkeit

SATZ 2. *Unter den Voraussetzungen von Satz* 1 *ist*

$$A^{(k)}\{e^{-s}, \lambda_n\} \subseteqq A^{(k)}\{H, \lambda_n\}, \ k \text{ reellwertig}$$

äquivalent mit $\int_1^{\infty} t^k |dK(t)| < \infty$.

Insbesondere lassen sich mit Satz 1 und 2 die Verfahren $A\{H, \lambda_n\} \supseteqq A\{e^{-s}, \lambda_n\}$ sehr einfach charakterisieren, wenn H als Funktion von $z = e^{-s}$ im Inneren des Einheitskreises analytisch ist. Wir erhalten speziell im Falle $\lambda_n = n$ für das einfachste Abelsche Verfahren A

KOROLLAR. *Es sei* $H(s) = \sum_{n=1}^{\infty} b_n e^{-sn}$ *für* $s > 0$ *konvergent. Dann ist*

$$A \subseteqq A\{H, n\}$$

äquivalent mit $\sum_{n=1}^{\infty} |b_n| < \infty$. *Für reellwertiges k ist ferner*

$$A^{(k)} \leqq A^{(k)}\{H, n\}$$

mit $\sum_{n=1}^{\infty} n^k |b_n| < \infty$ *äquivalent.*

Dieses Korollar konnte in [2], S. 54—55 nur unter der zusätzlichen Voraussetzung $b_n = O\left(\frac{1}{n}\right)$ beziehungsweise $b_n = O\left(\frac{1}{n^{1+k}}\right)$ bewiesen werden.

Satz 1 und 2 lassen sich leicht aus dem folgenden, allgemeineren Ergebnis herleiten.

2. Ein Satz über Stieltjes-Integraltransformationen

SATZ 3. *Gegeben sei die Stieltjes-Integraltransformation*

$$h(s) = \int_1^\infty g(e^{-st})\,dK(t) \tag{3}$$

mit $\int_1^\infty e^{-st}dK(t)$ *absolut konvergent für* $s>0$. *Ferner besitze die Folge* $\{\lambda_n\}$, $0<\lambda_1<<\lambda_2<\lambda_3<\ldots$, $\lambda_n\to\infty$ *eine Teilfolge* $\{\lambda_{n_l}\}$ $(l=1,2,3,\ldots,n_1=1)$ *mit* $\sum_{l=1}^\infty \frac{1}{\lambda_{n_l}}=\infty$ *und* $\lambda_{n_{l+1}}-\lambda_{n_l}\geqq q>0$. *Dann gilt:*

(a) *Die Bedingung*

$$\int_1^\infty |dK(t)|<\infty \tag{4}$$

ist hinreichend und notwendig dafür, daß durch (3) *alle für* $t>0$ *konvergenten Dirichletschen Reihen* $g(e^{-t}) = \sum_{n=1}^\infty a_n e^{-t\lambda_n}$ *mit* $g(e^{-t}) = c+o(1)$ $(t\to+0)$ *wieder in Funktionen* h *mit* $h(s) = d+o(1)$ $(s\to+0)$ *transformiert werden, wobei* c *und* d *Konstanten sind.*

(b) *Die Bedingung*

$$\int_1^\infty t^k|dK(t)|<\infty,\ k \text{ reelwertig} \tag{5}$$

ist hinreichend und notwendig dafür, daß durch (3) *alle für* $t>0$ *konvergenten Dirichletschen Reihen* $g(e^{-t}) = \sum_{n=1}^\infty a_n e^{-t\lambda_n}$ *mit* $g(e^{-t}) = o(t^k)$ $(t\to+0)$ *wieder in Funktionen* h *mit* $h(s)=o(s^k)$ $(s\to+0)$ *transformiert werden.*

ANMERKUNG. In Satz 3, (a) kann die Konvergenz von $g(e^{-t})$ und $h(s)$ für $t\to+0$ beziehungsweise $s\to+0$ auch durch Beschränktheit ersetzt werden.

Ferner bleibt Satz 3, (b) gültig, falls $g(e^{-t})=o(t^k)$ und $h(s)=o(s^k)$ durch $g(e^{-t})=O(t^k)$ und $h(s)=O(s^k)$ ersetzt werden.

Satz 3 stellt eine Erweiterung des Satzes von Toeplitz und Schur (siehe HARDY [1], S. 43) auf Integraltransformationen dar. Ein Beweis dieses Satzes wäre mit allgemeineren, funktionalanalytischen Hilfsmitteln über lineare Operatoren (Satz von Banach und Steinhaus) sehr einfach, wenn durch (3) ein vollständiger Funktionenraum, zum Beispiel der Raum aller im Intervall [0, 1] stetigen Funktionen, in einen Raum von konvergenten Funktionen h abgebildet würde. Die wesentliche Schwierigkeit dieses Satzes liegt jedoch gerade darin, daß (3) lediglich eine Transformation des nicht vollständigen Unterraumes aller konvergenten beziehungsweise beschränkten Dirichletschen Reihen darstellt. Daher sind zum Beweise speziellere Methoden und geeignete Approximationsuntersuchungen für Dirichletsche Reihen erforderlich.

Dem Beweise seien zunächst drei Hilfssätze vorangestellt.

LEMMA a. *Ist* $\int_1^\infty e^{-st}|dK(t)|<\infty$ *für* $s>0$, *dann gilt für jedes feste* $s>0$

$$\sup_f \int_1^\infty f(e^{-st})e^{-st}\,dK(t)=\int_1^\infty e^{-st}|dK(t)|.$$

Dabei wird das Supremum über alle im Intervall $[0, 1]$ *stetigen Funktionen* f *mit* $|f(x)|\leqq 1$ $(x\in[0, 1])$ *gebildet.*

Siehe zum Beispiel RIESZ und NAGY [5], S. 102.

LEMMA b. *Es sei* f *in* $[0, 1]$ *stetig. Dann streben die verallgemeinerten Bernsteinschen Polynome*

$$P_n^f(x)=\sum_{l=0}^{n}x^{w_l}\sum_{v=0}^{l}\frac{f(b_{n,v})(-1)^{n-v}\prod_{k=v+1}^{n}w_k}{(w_l-w_v)(w_l-w_{v+1})\dots(w_l-w_{l-1})(w_l-w_{l+1})\dots(w_l-w_n)}$$

mit $n\to\infty$ *gleichmäßig im Intervall* $[0, 1]$ *gegen* $f(x)$, *wobei*

$$b_{n,v}=\left[\left(1-\frac{w_1}{w_{v+1}}\right)\left(1-\frac{w_1}{w_{v+2}}\right)\dots\left(1-\frac{w_1}{w_n}\right)\right]^{1/w_1},\qquad 0\leqq v<n,\quad b_{n,n}=1$$

ist und außerdem

$$w_0=0<w_1<w_2<w_3<\dots,\qquad w_n\to\infty,\qquad \sum_{n=1}^{\infty}\frac{1}{w_n}=\infty$$

vorausgesetzt wird.

Siehe LORENTZ [4], S. 47.

LEMMA c. *Es sei* $w_0=0<w_1<w_2<w_3<\dots$, $w_n\to\infty$, $\sum_{n=1}^{\infty}\frac{1}{w_n}=\infty$ *und* $w_{n+1}-w_n\geqq$ $\geqq q>0$,

$$b_{n,v}=\left[\left(1-\frac{w_1}{w_{v+1}}\right)\left(1-\frac{w_1}{w_{v+2}}\right)\dots\left(1-\frac{w_1}{w_n}\right)\right]^{1/w_1},\qquad 0\leqq v<n,\quad b_{n,n}=1.$$

Ferner werde $0<a_i<a_{i+1}<1$ $(i=1, 2, 3, \dots)$ *mit* $\lim_{i\to\infty}a_i=1$ *vorausgesetzt, und es sei* $\{n_i\}$ *eine Folge natürlicher Zahlen, die so schnell wachsen, daß*

$$b_{n_i,n_{i-1}}<a_i\qquad (i=1, 2, 3, \dots)$$

gilt. (Nach der Definition der $b_{n,v}$ *lassen sich die* n_i *mit dieser Eigenschaft wählen, da* $\Sigma\frac{1}{w_n}$ *divergiert.) Dann gilt für die Koeffizienten* A_m, *die für* $n_{i-1}<m\leqq n_i$ *durch*

$$A_m=\sum_{\substack{v\leqq m\\ b_{n_i,v}\geqq a_i}}\left[\left(\frac{w_m}{w_v}-1\right)\left(\frac{w_m}{w_{v+1}}-1\right)\dots\left(\frac{w_m}{w_{m-1}}-1\right)\left(1-\frac{w_m}{w_{m+1}}\right)\dots\left(1-\frac{w_m}{w_{n_i}}\right)\right]^{-1}$$

definiert sind und sonst gleich 0 *gesetzt werden:*

$$\overline{\lim_{m\to\infty}} \frac{\log |A_m|}{w_m} \leqq 0.$$

Ein Beweis dieses Hilfssatzes ist in den Abschätzungen von [2], S. 22—24 enthalten.

Beweis zu Satz 3. Wir werden nur Satz 3, (a) in allen Einzelheiten beweisen. Der Beweis zu Satz 3, (b) verläuft im wesentlichen analog und soll nur kurz angedeutet werden.

Der hinreichende Teil ergibt sich sehr einfach, wenn wir $g(e^{-t}) = c+\varepsilon(t)$ mit $\varepsilon(t)=o(1)$ $(t\to+0)$ in (3) einsetzen. Denn wir erhalten dann nach (4)

$$h(s) = c\int_1^\infty dK(t) + \int_1^\infty \varepsilon(st)\, dK(t) =$$

$$= d+o(1)\int_1^{[\sqrt{s}]^{-1}} |dK(t)| + O(1) \int_{[\sqrt{s}]^{-1}}^\infty |dK(t)| = d+o(1).$$

Das eigentliche Kernstück von Satz 3, (a) ist die Notwendigkeit der Bedingung (4).

Wir setzen $w_n = \lambda_{n+1}-\lambda_1$ $(n=0, 1, 2, 3, \ldots)$. Dann genügt es zu zeigen, daß sich unter den Voraussetzungen

$$w_0=0<w_1<w_2<w_3<\ldots,\quad w_n\to\infty,\quad \sum_{n=1}^\infty \frac{1}{w_n} = \infty,\quad w_{n+1}-w_n \geqq q > 0$$

und

$$\int_1^\infty |dK(t)| = \infty \tag{6}$$

eine für $t>0$ konvergente Dirichletsche Reihe

$$g(e^{-t}) = \sum_{n=0}^\infty a_n e^{-tw_n} = c+o(1) \qquad (t\to+0)$$

mit

$$\overline{\lim_{s\to+0}} \int_1^\infty g(e^{-st}) e^{-st\lambda_1}\, dK(t) = \infty \tag{7}$$

konstruieren läßt.

Wir können ferner annehmen, daß

$$\int_1^\infty P(e^{-st}) e^{-st\lambda_1}\, dK(t) = O(1) \qquad (s\to+0) \tag{8}$$

für alle Polynome der Gestalt $P(x) = \sum_{l=0}^n ax^{w_l}$ ist, da wir andernfalls bereits eine Funktion g mit den gewünschten Eigenschaften gefunden hätten.

Wir substituieren $x=e^{-s}$ und bestimmen in einer noch genauer festzulegenden Weise eine Folge von paarweise punktfremden Intervallen $I_i=[a_i, b_i]$, $0<a_i<b_i<<a_{i+1}<1$ $(i=1, 2, 3, \dots)$ mit $\lim_{i\to\infty} a_i=1$. Zu diesen Intervallen I_i werden wir auf $[0, 1]$ stetige Funktionen f_i wählen, die außerhalb von I_i verschwinden und auf I_i dem Betrage nach kleiner als 2^{-i} sind. Diese Funktionen f_i werden wir dann nach Lemma b durch Bernsteinsche Polynome $P_{n_i}^{f_i}(x)$ approximieren, um durch Summation dieser Polynome schließlich eine Funktion g mit der Eigenschaft (7) zu erhalten.

Nach (6) strebt $\int\limits_1^\infty x^{t\lambda_1}|dK(t)|$ mit $x\to 1-0$ gegen ∞. Daher folgt aus Lemma a nach einfachen Überlegungen, daß sich ein Intervall $I_1=[a_1, b_1]$ sowie eine auf $[0, 1]$ stetige Funktion f_1, die außerhalb von I_1 verschwindet, mit den Eigenschaften angeben lassen:

$$|f_1(x)| < 2^{-1} \quad \text{für} \quad x\in[0, 1] \quad \text{und} \quad \int\limits_1^\infty f_1(b_1^t)b_1^{t\lambda_1}\, dK(t) > 1.$$

Zu f_1 wählen wir nach Lemma b ein Bernsteinsches Polynom $P_{n_1}^{f_1}(x)$ so, daß

$$|P_{n_1}^{f_1}(x)| < 2^{-1} \quad \text{für} \quad x\in[0, 1], \quad |P_{n_1}^{f_1}(x)| < \varepsilon_1 \quad \text{für} \quad x\in[0, 1]-I_1$$

und

$$\int\limits_1^\infty P_{n_1}^{f_1}(b_1^t)b_1^{t\lambda_1}\, dK(t) > 1 \text{ ist mit } 0<\varepsilon_1<1.$$

Nach (8) gilt

$$\left|\int\limits_1^\infty P_{n_1}^{f_1}(x^t)x^{t\lambda_1}\, dK(t)\right| \leqq k_1 \quad \text{für} \quad x\in[0, 1)$$

mit einer geeigneten Konstanten k_1. Sind nun die Intervalle $I_1, \dots, I_{i-1}$, die stetigen, außerhalb dieser Intervalle verschwindenden Funktionen $f_1, \dots, f_{i-1}$ sowie die zugehörigen Bernsteinschen Polynome $P_{n_1}^{f_1}(x), \dots, P_{n_{i-1}}^{f_{i-1}}(x)$ bereits festgelegt mit

$$\left|\int\limits_1^\infty P_{n_j}^{f_j}(x^t)x^{t\lambda_1}\, dK(t)\right| \leqq k_j \quad \text{für} \quad x\in[0, 1), \quad j=1, 2, \dots, i-1,$$

dann wählen wir durch ganz entsprechende Überlegungen nach Lemma a ein Intervall $I_i=[a_i, b_i]$, $b_{i-1}<a_i<b_i<1$ und eine auf $[0, 1]$ stetige, außerhalb von I_i verschwindende Funktion f_i mit

$$|f_i(x)| < 2^{-i} \quad \text{für} \quad x\in[0, 1] \quad \text{und} \quad \int\limits_1^\infty f_i(b_i^t)b_i^{t\lambda_1}\, dK(t) > i+\sum_{j=1}^{i-1} k_j.$$

Zu f_i können wir nach Lemma b ein Bernsteinsches Polynom $P_{n_i}^{f_i}(x)$ so bestimmen, daß

$$(9) \qquad |P_{n_i}^{f_i}(x)| < 2^{-i} \quad \text{für} \quad x\in[0, 1], \quad |P_{n_i}^{f_i}(x)| < \varepsilon_i \quad \text{für} \quad x\in[0, 1]-I_i$$

und

$$\int\limits_1^\infty P_{n_i}^{f_i}(b_i^t)b_i^{t\lambda_1}\, dK(t) > i+\sum_{j=1}^{i-1} k_j$$

ist. Dabei wählen wir die $\varepsilon_i > 0$ so klein, daß

(10) $$\varepsilon_i < \frac{\varepsilon_{i-1}}{2} \quad \text{und} \quad \varepsilon_i \int_1^\infty b_{i-1}^{t\lambda_1} |dK(t)| < 1$$

gilt. Wir setzen nun

(11) $$g(x) = \sum_{i=1}^\infty P_{n_i}^{f_i}(x).$$

Aus (9) folgt leicht, daß $\lim_{x\to 1-0} g(x)$ existiert. Außerdem besitzt g die Eigenschaft (7). Denn es ist für $x = b_l$ nach (9), (10)

$$\int_1^\infty g(b_l^t) b_l^{t\lambda_1}\, dK(t) = \int_1^\infty P_{n_l}^{f_l}(b_l^t) b_l^{t\lambda_1}\, dK(t) + \sum_{i\neq l} \int_1^\infty P_{n_i}^{f_i}(b_l^t) b_l^{t\lambda_1}\, dK(t) >$$

$$> l + \sum_{j=1}^{l-1} k_j - \sum_{i\neq l} \left| \int_1^\infty P_{n_i}^{f_i}(b_l^t) b_l^{t\lambda_1}\, dK(t) \right| \geqq l + \sum_{j=1}^{l-1} k_j - \sum_{i=1}^{l-1} k_i - \sum_{i=l+1}^\infty \varepsilon_i \int_1^\infty b_l^{t\lambda_1} |dK(t)| >$$

$$> l - \sum_{i=0}^\infty 2^{-i},$$

und der letzte Term wächst für $l \to \infty$, das heißt für $b_l \to 1-0$ über alle Grenzen.

Als wesentlicher Teil des Beweises muß noch gezeigt werden, daß $g(e^{-s})$ in (11) eine Dirichletsche Reihe mit einer Konvergenzabszisse $\sigma \leqq 0$ darstellt.

Nach Lemma b ist

(12) $$P_{n_i}^{f_i}(x) = \sum_{l=0}^{n_i} x^{w_l} \sum_{v=0}^{l} \frac{f_i(b_{n_i,v})(-1)^{n_i-v} \prod_{k=v+1}^{n_i} w_k}{(w_l - w_v)(w_l - w_{v+1}) \dots (w_l - w_{l-1})(w_l - w_{l+1}) \dots (w_l - w_{n_i})}.$$

Wir wählen die Grade $n_i > n_{i-1}$ dieser Polynome so groß, daß

(13) $$b_{n_i, n_{i-1}} < a_i \qquad (i = 1, 2, 3, \dots)$$

ist. Somit folgt aus (12) und (13)

$$P_{n_i}^{f_i}(x) = \sum_{l=n_{i-1}+1}^{n_i} x^{w_l} \sum_{\substack{v \leqq l \\ b_{n_i,v} \geqq a_i}} \frac{f_i(b_{n_i,v})(-1)^{n_i-v} \prod_{k=v+1}^{n_i} w_k}{(w_l - w_v) \dots (w_l - w_{l-1})(w_l - w_{l+1}) \dots (w_l - w_{n_i})},$$

da $f_i(x) = 0$ für $x < a_i$ ist, sodaß sich die Polynome $P_{n_i}^{f_i}(x)$ nicht »überlappen«. Wir erhalten daher als Koeffizienten A_m in $g(x) = \sum_{l=0}^\infty A_l x^{w_l}$

$$A_m = \sum_{\substack{v \leqq m \\ b_{n_i,v} \geqq a_i}} \frac{f_i(b_{n_i,v})(-1)^{n_i-v} \prod_{k=v+1}^{n_i} w_k}{(w_m - w_v) \dots (w_m - w_{m-1})(w_m - w_{m+1}) \dots (w_m - w_{n_i})}$$

für $n_{i-1} < m \leqq n_i$. Somit ist

$$|A_m| \leqq \frac{w_m}{w_1} \sum_{\substack{v \leqq m \\ b_{n_i,v} \geqq a_i}} \left[\left(\frac{w_m}{w_v} - 1\right)\left(\frac{w_m}{w_{v+1}} - 1\right) \cdots \left(\frac{w_m}{w_{m-1}} - 1\right)\left(1 - \frac{w_m}{w_{m+1}}\right) \cdots \left(1 - \frac{w_m}{w_{n_i}}\right)\right]^{-1}$$

(14)

für $n_{i-1} < m \leqq n_i$.

Nach VALIRON [7] fallen die gewöhnliche und die absolute Konvergenzabszisse zusammen und besitzen den Wert

$$\sigma = \overline{\lim_{m \to \infty}} \frac{\log |A_m|}{w_m},$$

falls $\frac{\log n}{w_n} = o(1)$ ist. Die letzte Bedingung ist hier wegen $w_{n+1} - w_n \geqq q > 0$ erfüllt. Daher folgt $\sigma \leqq 0$ aus (13), (14) und Lemma c, womit Satz 3, (a) in allen Teilen bewiesen ist.

Der Beweis zu Satz 3, (b) verläuft entsprechend. Die Polynome $P_{n_i}^{f_i}$ aus (9) müssen dann jedoch für $k \geqq 0$ so gewählt werden, daß $P_{n_i}^{f_i}(e^{-s}) = o(s^k)$ und $|P_{n_i}^{f_i}(e^{-s})| < s^k 2^{-i}$ ist. Dieses läßt sich dadurch erreichen, daß man die stetigen Funktionen f_i zunächst durch p-mal stetig-differenzierbare Funktionen $(p > k)$ hinreichend gut approximiert und zu den p-ten Ableitungen dieser Funktionen nach Lemma b geeignete Bernsteinsche Polynome wählt. p-malige Integration ergibt dann Polynome mit den gewünschten Eigenschaften.

BEWEIS ZU SATZ 1 UND 2. Satz 1 und Satz 2 folgen nun unmittelbar aus Satz 3, (3)—(5), da zwischen $A_H(s)$ und $A(s)$ der Zusammenhang besteht

$$A_H(s) = \sum_{n=1}^{\infty} a_n H(s\lambda_n) = \int_1^{\infty} \sum_{n=1}^{\infty} a_n e^{-st\lambda_n} \, dK(t) = \int_1^{\infty} A(st) \, dK(t).$$

Dabei ist die Vertauschung von Summation und Integration wegen der absoluten Konvergenz erlaubt.

3. Eine Anwendung auf die Lambert-Summierbarkeit

Das eingangs erwähnte Lambertsche Limitierungsverfahren L mit $H(s) = \frac{se^{-s}}{1 - e^{-s}}$, $\lambda_n = n$ ist bekanntlich schwächer als das Abelsche Verfahren A. Siehe hierzu HARDY [1], S. 373. In [3] wurde gezeigt, daß die Einschließung $L \subseteqq A$ äquivalent ist mit $\sum_{n=1}^{\infty} \frac{|N(n)|}{n} < \infty$, wobei $N(t) = \sum_{n \leqq t} \frac{\mu(n)}{n}$ und μ die Möbiussche Funktion ist. Dieses Ergebnis ist ebenfalls in Satz 3, (a) enthalten, wie eine einfache Rechnung zeigt. Die aus der Zahlentheorie bekannte Eigenschaft $\sum_{n=1}^{\infty} \frac{|N(n)|}{n} < \infty$ liegt etwas tiefer als der Primzahlsatz.

Mit Hilfe von Satz 3, (b) läßt sich nun ein Ergebnis über die Konvergenzgeschwindigkeit bei diesen Verfahren L und A herleiten, das in [2], S. 58 nur bezüglich einer gewissen »gleichmäßigen« Limitierbarkeit bewiesen werden konnte. Wir erhalten

SATZ 4. *Die Riemannsche Vermutung ist äquivalent mit der Gültigkeit von* $L^{(k)} \subseteqq A^{(k)}$ *für* $0 \leqq k < \frac{1}{2}$.

BEWEIS. Die bisher unbewiesene Riemannsche Vermutung, das heißt die Annahme, daß die analytische Fortsetzung der Funktion $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ im Streifen $\frac{1}{2} < \operatorname{Re}(z) < 1$ keine Nullstelle besitzt, ist äquivalent mit $N(t) = O\left(t^{-\frac{1}{2}+\varepsilon}\right)$ für jedes $\varepsilon > 0$ und ebenfalls äquivalent mit $\sum_{n=1}^{\infty} \frac{|N(n)|}{n^{1-k}} < \infty$ für $0 \leqq k < \frac{1}{2}$. Siehe hierzu TITCHMARSH [6], S. 315 sowie HOISCHEN [2], S. 58. Mit

$$L(s) = \sum_{n=1}^{\infty} a_n \frac{sne^{-sn}}{1-e^{-sn}}, \quad A(s) = \sum_{n=1}^{\infty} a_n e^{-sn}$$

erhalten wir nach kurzer Rechnung (siehe HARDY [1], S. 374)

$$A(s) = \sum_{n=1}^{\infty} N(n) \int_{sn}^{s(n+1)} \frac{L(t)}{t} dt = \int_{0}^{\infty} \frac{L(t)}{t} N\left(\frac{t}{s}\right) dt. \tag{15}$$

Beachtet man, daß sich jede im Inneren des Einheitskreises analytische Funktion in eine Lambertsche Reihe der Gestalt $\sum_{n=1}^{\infty} a_n \frac{e^{-sn}}{1-e^{-sn}}$ entwickeln läßt, dann folgt aus (15) und Satz 3, (b), (5) die Äquivalenz von $L^{(k)} \subseteqq A^{(k)}$ mit $\int_0^{\infty} \frac{|N(t)|}{t^{1-k}} dt < \infty$, das heißt mit $\sum_{n=1}^{\infty} \frac{|N(n)|}{n^{1-k}} < \infty$. Damit ist Satz 4 bewiesen.

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RATIONAL APPROXIMATION OF ANALYTIC FUNCTIONS WITH FINITE NUMBER OF SINGULARITIES ON THE REAL AXIS

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Denote by $R_n(f; a, b)$ the best rational approximation of a continuous function $f(x)$ on the finite interval $[a, b]$, and let $\omega_f(a, b; h)$ be the module of continuity of $f(x)$ on $[a, b]$. In [1], A. A. GONČAR proved the following theorem:

Let $f(z)$ be analytic and bounded on the circle $C_A = \left\{z: \left|z-\frac{A}{2}\right| < \frac{A}{2}\right\}$ $(A > 1)$ of the complex plane, and continuous on $[0, 1]$. Then*

$$R_n(f; 0, 1) = O\left[\inf_{1<t<\infty}\left(t \cdot e^{-\frac{c_1 n}{t}} + \omega_f(0, 1; e^{-t})\right)\right].$$

This estimate gives $R_n(f; 0, 1) = O(\omega_f(0, 1; e^{-c_2\sqrt{n}}))$ at least in each cases, while the order of polynomial approximation is generally not better than $O\left(\omega_f\left(0, 1; \frac{1}{n}\right)\right)$.

In this paper, using the GONČAR's method, we intend to generalize the above mentioned theorem. The first step in this direction will be the following

THEOREM 1. *Let $f(x)$ be analytic and bounded in the circle $C_1 = \left\{z: \left|z-\frac{1}{2}\right| < \frac{1}{2}\right\}$, and continuous on* $[0, 1]$. *Then*

$$R_n(f; 0, 1) = O\left[\inf_{\sqrt[3]{n}<t<\infty}\left(t \cdot e^{-\frac{c_3 n}{t}} + \omega_f(0, 1; e^{-t})\right)\right].$$

REMARKS. Theorem 1 states that the GONČAR's result remains true if we allow 1 also to be a singularity point of $f(z)$. The restriction $t > \sqrt[3]{n}$ is not significant because at every reasonable choice of t we have $\frac{\sqrt{n}}{t} = O(1)$. Of course, our theorem holds in arbitrary finite interval $[a, b]$ instead of $[0, 1]$, too.

PROOF. The proof is a slight modification of the GONČAR's proof. He shows that for the rational function

$$s_n(z) = \prod_{k=1}^{n} \frac{z - \varepsilon^{k/n}}{z + \varepsilon^{k/n}} \qquad \left(0 < \varepsilon < \frac{1}{2},\ n = 1, 2, \ldots\right)$$

* $c_1, c_2, \ldots$ denote explicite positive numerical constants.

we have

(1) $$|s_n(x)| = O\left(e^{-\frac{c_4 n}{\log 1/\varepsilon}}\right) \qquad (\varepsilon \leqq x \leqq 1)$$

and

(2) $$|s_n(z)| = 1 \qquad (\mathrm{Re}\, z = 0).$$

Now let

(3) $$0 < h < e^{-\sqrt[3]{n}}$$

and

$$w(z) = -\frac{h(1-h)(2z-h)}{(2-3h)(2z-2+h)}$$

then

$$w\left(\frac{h}{2}\right) = 0, \quad w(1-h) = 1-h, \quad w\left(1-\frac{h}{2}\right) = \infty.$$

Thus $w(z)$ maps the circle $C_2 = \left\{z: \left|z-\frac{1}{2}\right| = \frac{1-h}{2}\right\}$ into the line $\mathrm{Re}\, z = 0$, i.e. by (2)

(4) $$|s_n(w(z))| = 1 \qquad (z \in C_2).$$

If we put

(5) $$\varepsilon = w(h) = \frac{1-h}{(2-3h)^2} h^2$$

in (1), then by $\varepsilon > \frac{1}{8} h^2$ and $w([h, 1-h]) = [\varepsilon, 1-h]$ we have

(6) $$|s_n(w(x))| = O\left(e^{-\frac{c_3 n}{\log 1/h}}\right) \qquad (h \leqq x \leqq 1-h).$$

The roots of $s_n(z)$, $\varepsilon^{k/n}$ $(k = 1, 2, \ldots, n)$ are in $[\varepsilon, 1)$, and so, by

$$\varepsilon^{1/n} \leqq h^{2/n} \leqq e^{-\frac{2}{\sqrt[3]{n^2}}} \leqq 1 - \frac{1}{\sqrt[3]{n^2}} \leqq 1 - e^{-\sqrt[3]{n}} \leqq 1-h$$

(see (5) and (3)), the roots of $s_n(w(z))$ are in $[h, 1-h]$. Let $R_n(z)$ be the rational function of degree at most n which interpolates $f(z)$ in the roots of $s_n(w(z))$ and in $\frac{1}{2}$, and which has the same poles as $s_n(w(z))$. Then

$$f(z) - R_n(z) = \frac{s_n(w(z))(2z-1)}{2\pi i} \int_{C_2} \frac{f(\xi)\, d\xi}{s_n(w(\xi))(2\xi-1)(\xi-z)}.$$

Using (6) and (4), we get by simple estimates

$$|f(x) - R_n(x)| = O\left(e^{-\frac{c_3 n}{\log 1/h}}\right) \cdot \int_{C_2} \frac{|d\xi|}{|\xi - x|} = O\left(\log \frac{1}{h} \cdot e^{-\frac{c_3 n}{\log 1/h}}\right)$$

$$(h \leqq x \leqq 1-h).$$

Substituting $t=\log 1/h>\sqrt[3]{n}$ (by (3)) and $x=(1-2h)u+h$, we obtain

$$(7)\qquad |f((1-2h)u+h)-R_n((1-2h)u+h)| = O\left(t\cdot e^{-\frac{c_3 n}{t}}\right) \qquad (0\leqq u\leqq 1).$$

But

$$(8)\qquad |f(u)-f((1-2h)u+h)| \leqq \omega_f(0,1;h|2u-1|) \leqq \omega_f(0,1;h) = $$
$$= \omega_f(0,1;e^{-t}) \qquad (0\leqq u\leqq 1).$$

(7) and (8) gives the theorem.

THEOREM 2. *Let* $-1=\xi_0<\xi_1<\ldots<\xi_s=+1$,

$$(9)\qquad \delta = \min_{1\leqq i\leqq s}(\xi_i-\xi_{i-1}),$$

and assume that $f(z)$ *is analytic and bounded on the circles*

$$\left\{z: \left|z-\frac{\xi_{i-1}+\xi_i}{2}\right| < \frac{\xi_i-\xi_{i-1}}{2}\right\} \qquad (i=1,2,\ldots,s)$$

and continuous on $[-1,+1]$. *Then*

$$R_n(f;-1,+1) = s\cdot O\left[\inf_{\sqrt[3]{n}<t<\infty}\left(t\cdot e^{-\frac{c_5\delta\cdot n}{s\cdot t}}+\omega_f(-1,+1;e^{-t})\right)\right].$$

REMARKS. When $f(x)$ is analytic on the *closed* intervals $[\xi_{i-1},\xi_i]$ $(i=1,2,\ldots,s)$ then

$$R_n(f;-1,+1) = O\left(e^{-d\sqrt{n}}\right)$$

where $d>0$ depends on δ and on the analyticity-domains of $f(z)$ (see e.g. P. TURÁN [2]). If the analyticity-condition takes the form

$$(10)\qquad \sup_{0<\varepsilon<\delta/2}\left[\varepsilon\cdot\limsup_{n\to\infty}\frac{\sqrt[n]{\max\limits_{1\leqq i\leqq s}\max\limits_{\xi_{i-1}+\varepsilon\leqq x\leqq\xi_i-\varepsilon}|f^{(n)}(x)|}}{n}\right] \leqq \frac{c}{e} \qquad (c>0 \text{ fixed})$$

then (see [3])

$$R_n(f;-1,+1) = O\left(\omega_f\left(-1,+1;e^{-\sqrt[3]{\frac{n}{s}\cdot\log\frac{1}{q}}}\right)\right) \qquad \left(q=\frac{\sqrt{c^2+4c}-c}{2}\right)$$

holds which is weaker than Theorem 2. But, on the other hand, the condition (10) means (as easy to see) that $f(z)$ is analytic only in certain rhombuses of the complex plane whose two opposite vertices are ξ_{i-1} and ξ_i $(i=1,2,\ldots,s)$.

The proof of Theorem 2 is rather long, and it is based on some lemmas. In order to formulate the first of them, let

$$-1\leqq a<b\leqq+1, \quad m=2k+1 \qquad (k=1,2,\ldots),$$

$$C_{a,b} = \left\{z: \left|z-\frac{a+b}{2}\right| \leqq \frac{b-a}{2}\right\},$$

$T_m(z) = \cos(m \cdot \arc\cos z)$ the Chebyshev-polynomial of degree m and

(11) $$p_m(z) = (-1)^k \frac{b-a}{2} T_m \left(\frac{2\cos\frac{m-1}{m}\frac{\pi}{2}}{b-a} \left(z - \frac{a+b}{2} \right) \right) + \frac{a+b}{2}.$$

LEMMA 1. *The polynomial $p_m(z)$ maps the domain $C_{a,b}$ one-to-one into a domain $D_{a,b}$ which contains $C_{a,b}$.*

PROOF. Let $z \in C_{a,b}$ and

(12) $$u = \frac{2\cos\frac{m-1}{m}\frac{\pi}{2}}{b-a} \left(z - \frac{a+b}{2} \right)$$

then $|u| \leqq \cos\frac{m-1}{m}\frac{\pi}{2}$. Thus

$$\frac{m-1}{m}\frac{\pi}{2} \leqq \operatorname{Re}(\arc\cos u) \leqq \frac{m+1}{m}\frac{\pi}{2},$$

$$k\pi \leqq \operatorname{Re}(m \arc\cos u) \leqq (k+1)\pi.$$

Here one of the equality signs holds only for $u = \cos\frac{m-1}{m}\frac{\pi}{2}$ and $u = \cos\frac{m+1}{m}\frac{\pi}{2}$. Consequently, the mappings u, $\arc\cos u$, $m \cdot \arc\cos u$, $\cos(m \cdot \arc\cos u)$, $(-1)^k \frac{b-a}{2}\cos(m \cdot \arc\cos u) + \frac{a+b}{2}$ are all one-to-one in the corresponding domains.

Now we have to prove that $C_{a,b} \subseteqq D_{a,b}$. Let $\left| z - \frac{a+b}{2} \right| = \frac{b-a}{2}$ then by (12),

$$u = e^{i\varphi} \cdot \cos\frac{m-1}{m}\frac{\pi}{2} \qquad (0 \leqq \varphi < 2\pi).$$

Thus from (11)

$$\left| p_m(z) - \frac{a+b}{2} \right| = \frac{b-a}{2} \left| T_m \left(e^{i\varphi} \cdot \cos\frac{m-1}{m}\frac{\pi}{2} \right) \right| = \frac{b-a}{2} 2^{m-1} \cos\frac{m-1}{m}\frac{\pi}{2} \cdot$$

$$\cdot \prod_{j=1}^{k} \left| e^{2i\varphi} \cdot \cos^2\frac{m-1}{m}\frac{\pi}{2} - \cos^2\frac{2j-1}{m}\frac{\pi}{2} \right| = \frac{b-a}{2} 2^{m-1} \cos\frac{m-1}{m}\frac{\pi}{2} \cdot$$

$$\cdot \prod_{j=1}^{k} \left(\cos^4\frac{m-1}{m}\frac{\pi}{2} + \cos^4\frac{2j-1}{m}\frac{\pi}{2} - 2\cos 2\varphi \cos^2\frac{m-1}{m}\frac{\pi}{2}\cos^2\frac{2j-1}{m}\frac{\pi}{2} \right)^{1/2} \geqq$$

$$\geqq \frac{b-a}{2} 2^{m-1} \cos\frac{m-1}{m}\frac{\pi}{2} \prod_{j=1}^{k} \left(\cos^4\frac{m-1}{m}\frac{\pi}{2} + \cos^4\frac{2j-1}{m}\frac{\pi}{2} - \right.$$

$$\left. -2\cos\frac{m-1}{m}\frac{\pi}{2}\cos^2\frac{2j-1}{m}\frac{\pi}{2} \right)^{1/2} = \frac{b-a}{2} T_m \left(\cos\frac{m-1}{m}\frac{\pi}{2} \right) = \frac{b-a}{2}.$$

Consequently, by $p_m\left(\frac{a+b}{2}\right) = \frac{a+b}{2}$ we obtain $C_{a,b} \subseteqq D_{a,b}$, qu.e.d.

COROLLARY. *The unique inverse $q_m(z)$ of $p_m(z)$ exists for $z \in C_{a,b}$ and is analytic in the interior of $C_{a,b}$. Moreover, $q_m(z)$ maps (one-to-one) $C_{a,b}$ into a part of $C_{a,b}$;*

$$q_m(p_m(z)) = z \qquad (z \in C_{a,b}), \tag{13}$$

$$q_m(a) = a, \quad q_m(b) = b, \quad q_m([a, b]) = [a, b]. \tag{14}$$

LEMMA 2. *For the above defined function $q_m(x)$ we have*

$$\omega_{q_m}(a, b; h) \leqq \sqrt{8(b-a)h}.$$

PROOF. First of all we prove that

$$\frac{p_m(y) - p_m(a)}{(y-a)^2} \geqq \frac{1}{8(b-a)} \qquad (a < y \leqq b). \tag{15}$$

Let

$$d_m = \frac{\cos \frac{m + \frac{1}{2}}{m} \frac{\pi}{2}}{\sin \frac{\pi}{2m}}.$$

Being $T_m''(x) > 0$ monoton decreasing in $\left[-\sin \frac{\pi}{2m}, 0\right]$, we have for $a \leqq y \leqq d_m \frac{b-a}{2} + \frac{a+b}{2} \leqq \frac{a+b}{2}$ with a suitable $a < \eta < y$

$$\frac{p_m(y) - p_m(a)}{(y-a)^2} = \frac{1}{2} p_m''(\eta) = \frac{1}{2} (-1)^k \frac{b-a}{2} \frac{4 \sin^2 \frac{\pi}{2m}}{(b-a)^2} T_m'' \left(\frac{2 \sin \frac{\pi}{2m}}{b-a} \left(\eta - \frac{a+b}{2} \right) \right) \geqq$$

$$\geqq \frac{(-1)^k}{m^2(b-a)} T_m'' \left(d_m \sin \frac{\pi}{2m} \right) = \frac{(-1)^k}{m^2(b-a)} T_m'' \left(\cos \frac{m + \frac{1}{2}}{m} \frac{\pi}{2} \right) = \frac{(-1)^k}{m^2(b-a)} \cdot$$

$$\cdot\, m \frac{m(-1)^k \frac{\sqrt{2}}{2} - (-1)^k \frac{\sqrt{2}}{2} \operatorname{tg} \frac{\pi}{4m}}{\cos^2 \frac{\pi}{4m}} \geqq \frac{\sqrt{2}(m-1)}{2m(b-a)} > \frac{1}{6(b-a)}.$$

As regards for $d_m \frac{b-a}{2} + \frac{a+b}{2} \leqq y \leqq b$, we get

$$\frac{p_m(y) - p_m(a)}{(y-a)^2} \geqq \frac{\frac{b-a}{2} (-1)^k T_m \left(d_m \sin \frac{\pi}{2m} \right) + \frac{a+b}{2} - a}{(b-a)^2} =$$

$$= \frac{\frac{b-a}{2} (-1)^k (-1)^{k+1} \frac{\sqrt{2}}{2} + \frac{b-a}{2}}{(b-a)^2} = \frac{2 - \sqrt{2}}{4(b-a)} > \frac{1}{8(b-a)}$$

which proves (15). Hence by (15), (13) and (14) we have for $a \leqq x \leqq x+h \leqq b$ with the notation $y=q_m(a+h)$

$$|q_m(x+h)-q_m(x)| \leqq q_m(a+h)-q_m(a) = y-a \leqq \sqrt{8(b-a)[p_m(y)-p_m(a)]} = \\ = \sqrt{8(b-a)h},$$

qu.e.d.

LEMMA 3. *Let $f(z)$ be analytic and bounded in the interior of $C_{a,b}$ and continuous on $[a, b]$. Then there exists a continuous function $F(x)$ in $[-1, +1]$ for which*
$1°$ $F(x)=f(x)$ *for* $a \leqq x \leqq b$;
$2°$ $\omega_F(-1, +1; h) \leqq \omega_f(a, b; 11\sqrt{h})$;

$$3° \quad R_n(F; -1, +1) = O\left[\inf_{\sqrt[3]{n} \leqq t < \infty} \left(t \cdot e^{-\frac{c_6(b-a)n}{t}} + \omega_f(a, b; e^{-t})\right)\right].$$

PROOF. Let

$$m = 2\left[\frac{5}{b-a}\right]+1, \tag{16}$$

then easy to show that

$$\left|\frac{2\cos\frac{m-1}{m}\frac{\pi}{2}\left(x-\frac{a+b}{2}\right)}{b-a}\right| \leqq 1 \qquad (-1 \leqq x \leqq +1). \tag{17}$$

Therefore, by (11)

$$a \leqq p_m(x) \leqq b \qquad (-1 \leqq x \leqq +1). \tag{18}$$

Now let

$$F(x)=f\left(q_m(p_m(x))\right) \qquad (-1 \leqq x \leqq +1). \tag{19}$$

By (18) and the Corollary of Lemma 1, $F(x)$ is continuous on $[-1, +1]$, and using (13), we obtain $1°$.

Now let $-1 \leqq x \leqq x+h \leqq +1$ then by Lemma 2, (11), (16) and (18) we have

$$\left|f\left(q_m(p_m(x))\right)-f\left(q_m(p_m(x+h))\right)\right| \leqq \omega_f\left(a, b; \left|q_m(p_m(x))-q_m(p_m(x+h))\right|\right) \leqq \\ \leqq \omega_f\left(a, b; \sqrt{8(b-a)|p_m(x)-p_m(x+h)|}\right) \leqq \\ \leqq \omega_f\left(a, b; \sqrt{8(b-a)h\frac{b-a}{2}\frac{2\sin\frac{\pi}{2m}}{b-a}m^2}\right) \leqq \\ \leqq \omega_f\left(a, b; \sqrt{4\pi hm(b-a)}\right) \leqq \omega_f\left(a, b; 11\sqrt{h}\right).$$

By the Corollary of Lemma 1, the function $f\left(q_m(z)\right)$ is analytic in the interior of $C_{a,b}$. Moreover, for $a \leqq x \leqq x+h \leqq b$ we have by Lemma 2

$$\left|f(q_m(x))-f\left(q_m(x+h)\right)\right| \leqq \omega_f(a, b; |q_m(x)-q_m(x+h)|) \leqq \omega_f(a, b; 4\sqrt{h}).$$

Thus, applying Theorem 1 for $f(q_m(x))$ in $[a, b]$, we get a rational function $R_n(x)$ of degree at most n such that

$$|f(q_m(x))-R_n(x)| = O\left[\inf_{\sqrt[4]{n}<t<\infty}\left(t\cdot e^{-\frac{c_3 n}{t}}+\omega_f(a, b; e^{-\frac{t}{2}})\right)\right] \qquad (a\leqq x\leqq b).$$

Write this in the (form by (18))

$$|f(q_m(p_m(x)))-R_n(p_m(x))| = O\left[\inf_{\sqrt[3]{n}<t<\infty}\left(t\cdot e^{-\frac{c_7 n}{t}}+\omega_f(a, b; e^{-t})\right)\right]$$

$$(-1\leqq x\leqq +1).$$

Here $R_n(p_m(x))=r_{nm}(x)$ is a rational function of degree at most nm. Using (19), we have

$$|F(x)-r_{nm}(x)| = O\left[\inf_{\sqrt[3]{n}<t<\infty}\left(t\cdot e^{-\frac{c_7 n}{t}}+\omega_f(a, b; e^{-t})\right)\right] \qquad (-1\leqq x\leqq +1),$$

which, by (16), proves 3°, qu.e.d.

LEMMA 4. *Let* $-1\leqq\xi\leqq+1$ *and*

$$\Lambda_\xi(x) = \begin{cases} -\frac{1}{2} & \text{if } x\leqq\xi \\ +\frac{1}{2} & \text{if } x>\xi. \end{cases} \tag{20}$$

Then for all $t>1$ *there exist rational functions* $\lambda_n^{\{\xi\}}(x)$ *of degree n such that*

$$|\Lambda_\xi(x)-\lambda_n^{(\xi)}(x)| = O\left(e^{-\frac{c_8 n}{t\xi}}\right) \quad \text{if} \quad x\in([-1,\xi]\cup[\xi+6e^{-t}, 1])\cap[-1, +1] \tag{21}$$

and

$$|\lambda_n^{(\xi)}(x)| = O(1) \quad \text{if} \quad -1\leqq x\leqq +1. \tag{22}$$

PROOF. Let

$$\Lambda(x) = \begin{cases} 0 & \text{if } x\leqq 0 \\ 1 & \text{if } x>0. \end{cases}$$

The Lemma 3 in [1] can be formulated as follows: There exist rational functions $\lambda_n(x)$ such that

$$|\Lambda(x)-\lambda_n(x)| = O\left(e^{-\frac{c_8 n}{t}}\right) \qquad \text{for} \quad x\in[-1, -e^{-t}]\cup[e^{-t}, 1],$$

and

$$|\lambda_n(x)|=O(1) \quad \text{for} \quad -1\leqq x\leqq +1.$$

Putting

$$\Lambda_\xi(x) = \Lambda\left(\frac{x-\xi}{3}\right)-\frac{1}{2}, \quad \lambda_n^{\{\xi\}}(x) = \lambda_n\left(\frac{x-\xi}{3}-e^{-t}\right)-\frac{1}{2},$$

we obtain our Lemma 4.

PROOF OF THEOREM 2. By Lemma 3, to each $[\xi_{i-1}, \xi_i]$ $(i=1, 2, \dots, s)$ there exists a function $F_i(x)$ continuous in $[-1, +1]$ such that (see (9))

$$F_i(x) = f(x) \quad \text{for} \quad x \in [\xi_{i-1}, \xi_i] \qquad (i=1, 2, \dots, s); \tag{23}$$

$$\omega_{F_i}(-1, +1; h) \leqq \omega_f(\xi_{i-1}, \xi_i; 11\sqrt{h}) \leqq \omega_f(-1, +1; 11\sqrt{h}) \qquad (i=1, 2, \dots, s); \tag{24}$$

$$|F_i(x) - R_n^{\{i\}}(x)| = O\left[\inf_{\sqrt[3]{n}<t<\infty} \left(t \cdot e^{-\frac{c_6 \delta n}{t}} + \omega_f(-1, +1; e^{-t})\right)\right] \tag{25}$$

$$(-1 \leqq x \leqq +1;\ i=1, 2, \dots, s)$$

where $R_n^{\{i\}}(x)$ is a rational function of degree at most n. Evidently, by (20) and (23) we have

$$f(x) = \frac{F_1(x) + F_s(x)}{2} + \sum_{i=1}^{s-1} \Lambda_{\xi_i}(x)[F_{i+1}(x) - F_i(x)] \qquad (-1 \leqq x \leqq +1).$$

Let (see Lemma 4)

$$R_{n(3s-1)}(x) = \frac{R_n^{\{1\}}(x) + R_n^{\{s\}}(x)}{2} + \sum_{i=1}^{s-1} \lambda_n^{\{\xi_i\}}(x)[R_n^{\{i+1\}}(x) - R_n^{\{i\}}(x])$$

(this is a rational function of degree at most $n(3s-1)$). Then

$$f(x) - R_{n(3s-1)}(x) = \Delta_1 + \Delta_2 \tag{26}$$

where

$$\Delta_1 = \frac{F_1(x) - R_n^{\{1\}}(x) + F_s(x) - R_n^{\{s\}}(x)}{2} +$$

$$+ \sum_{i=1}^{s-1} \lambda_n^{\{\xi_i\}}(x)[F_{i+1}(x) - R_n^{\{i+1\}}(x) + R_n^{\{i\}}(x) - F_i(x)]$$

and

$$\Delta_2 = \sum_{i=1}^{s-1} [\Lambda_{\xi_i}(x) - \lambda_n^{\{\xi_i\}}(x)] \cdot [F_{i+1}(x) - F_i(x)].$$

From (25) and (22) we get

$$|\Delta_1| = s \cdot O\left[\inf_{\sqrt[3]{n}<t<\infty} \left(t \cdot e^{-\frac{c_6 \delta n}{t}} + \omega_f(-1, +1; e^{-t})\right)\right] \qquad (-1 \leqq x \leqq +1). \tag{27}$$

As regards to Δ_2, let $6e^{-t} \leqq \delta$, and first of all $\xi_{j-1} + 6e^{-t} \leqq x \leqq \xi_j$ $(1 \leqq j \leqq s)$. Then, using (21), we obtain

$$|\Delta_2| = s \cdot O\left(e^{-\frac{c_8 n}{t}}\right).$$

Finally, let $\xi_{j-1} \leqq x \leqq \xi_{j-1} + 6e^{-t}$ $(1 \leqq j \leqq s)$. Then, by (22), (24), (21) and $F_j(\xi_{j-1}) = F_{j-1}(\xi_{j-1})$ we have

$$\Delta_2 = |\Lambda_{\xi_{j-1}}(x) - \lambda_n^{\{\xi_{j-1}\}}(x)| \cdot [|F_j(x) - F_j(\xi_{j-1})| + |F_{j-1}(\xi_{j-1}) - F_{j-1}(x)|] +$$

$$\sum_{\substack{i=1 \\ i \neq j-1}}^{s-1} |\Lambda_{\xi_i}(x) - \lambda_n^{\{\xi_i\}}(x)| \cdot |F_{i+1}(x) - F_i(x)| = O\left(\omega_f(-1, +1; e^{-\frac{t}{2}})\right) + s \cdot O\left(e^{-\frac{c_8 n}{t}}\right).$$

Thus

$$\Delta_2 = O\left[\inf_{\log 6/\delta \leqq t < \infty} \left(s \cdot e^{-\frac{c_8 n}{t}} + \omega_f(-1, +; e^{-t})\right)\right] \qquad (-1 \leqq x \leqq +1).$$

From this, (27) and (26)

$$|f(x) - R_{n(3s-1)}(x)| = s \cdot O\left[\inf_{\sqrt[3]{n} < t < \infty} \left(t \cdot e^{-\frac{c_9 \delta n}{t}} + \omega_f(-1, +1; e^{-t})\right)\right]$$

$$(-1 \leqq x \leqq +1)$$

which proves Theorem 2.

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ON A PROBLEM OF KLEE

By

E. KNUTH (Budapest)

In June 1961 at the Seattle Symposium on Convexity, V. KLEE proposed the following problem: Given a planar domain D of constant width λ, is it always possible to inscribe in D a semicircle of diameter λ? BESICOWITH [1] gives an affirmative answer for a certain class of domains which he calls simple. In fact, he proves that for any simple domain of constant width it is always possible to inscribe three distinct semicircles of the required diameter. COOKE [2] solves the problem. He proves the same statement for the Reuleaux domains. With reference to the result of EGGLESTON [3] — that any domain of constant width can be approximated closely by an appropriately chosen Reuleaux polygon — COOKE proves at the same time a similar statement for the general case.

In this paper we shall give an immediate proof for the general case, and we produce all the required semicircles.

1. The existence of the required semicircles

First of all we introduce some notations. Let **D** be a (closed) domain of constant width, and D be its boundary. We can suppose, that the width of D is 1. Take a coordinate system in the plane and let t be an arbitrary real number. We introduce the following functions: the support strip (bordered by two parallel support lines) of D of direction-angle t is $S(t)$, the middle parallel of $S(t)$ is $k(t)$ ($k(t)$ is a directed line), $d(t)$ is the diameter of D perpendicular to $k(t)$, the midpoint of $d(t)$ is $L(t)$. The open half-lines of $k(t)$ of endpoint $L(t)$ are $k_1(t)$ and $k_2(t)$ (see Fig. 1). If t runs through an interval of length π, $L(t)$ describes a continuous closed curve. We denote it by L.

Let F be an arbitrary semicircle of diameter 1. Let be its diameter $d(F)$, $L(F)$ the midpoint of $d(F)$ and $k(F)$ the open half-line of the perpendicular bisector of $d(F)$ of endpoint $L(F)$, which does not intersect the semicircle F (see Fig. 2).

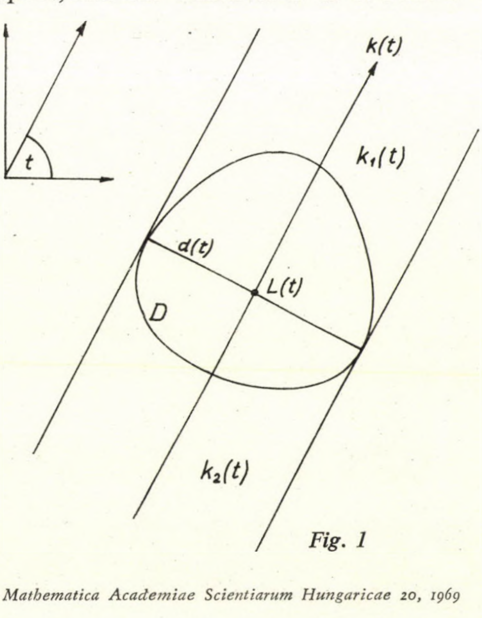

Fig. 1

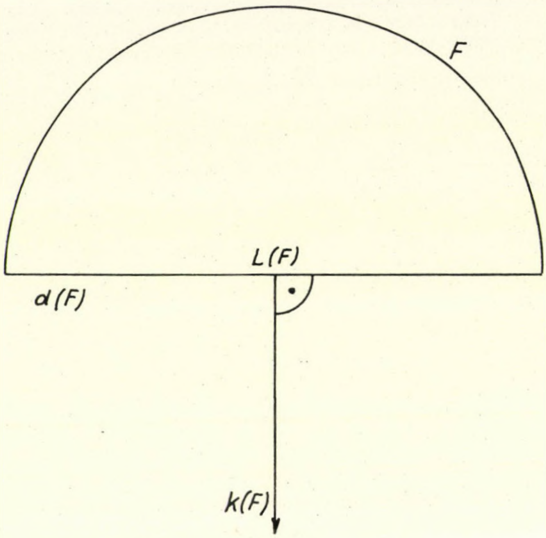

Fig. 2

If $F \subset \mathbf{D}$ then $d(F)$ is a diameter of D also — because each segment of length 1 included by D is a diameter of D — and consequently $L(F) \in L$. We have then $k(F) = k_1(t)$ or $K(F) = k_2(t)$ where t is the argument of the half-line $k(F)$. Hence to each semicircle $F \subset \mathbf{D}$ corresponds a parameter value t and an index i for which $L(F) = L(t)$ and $k(F) = k_i(t)$. Let $F(t, i)$ denote the semicircle F, for which $L(F) = L(t)$ and $k(F) = k_i(t)$.

We shall examine, which t and i values satisfy the relation $F(t, i) \subset \mathbf{D}$, and we shall prove the following

THEOREM 1. *$F(t, i) \subset \mathbf{D}$ if and only if $k_i(t) \cap L = \emptyset$.*

After this we shall prove, that at least three distinct (mod π) parameter values t with one or the other index i satisfy the condition of Theorem 1. More exactly we prove the following

THEOREM 2. *If the point $L(t)$ is on the convex hull of L, then $k_1(t) \cap L = \emptyset$ or $k_2(t) \cap L = \emptyset$.*

In fact it is evident that Theorem 2 furnishes at least three distinct semicircles $F \subset \mathbf{D}$, since if L is not a single point, then L has at least three distinct points on its convex hull, and if L is a single point, then all parameter values are suitable.

LEMMAS:

$$(1.1) \qquad L(t) = \lim_{t' < \to t} [k(t) \cap k(t')].$$

PROOF. Consider the parallelogramm determined by the support strips $S(t)$, $S(t')$ (see Fig. 3). The point $k(t) \cap k(t')$ is the midpoint

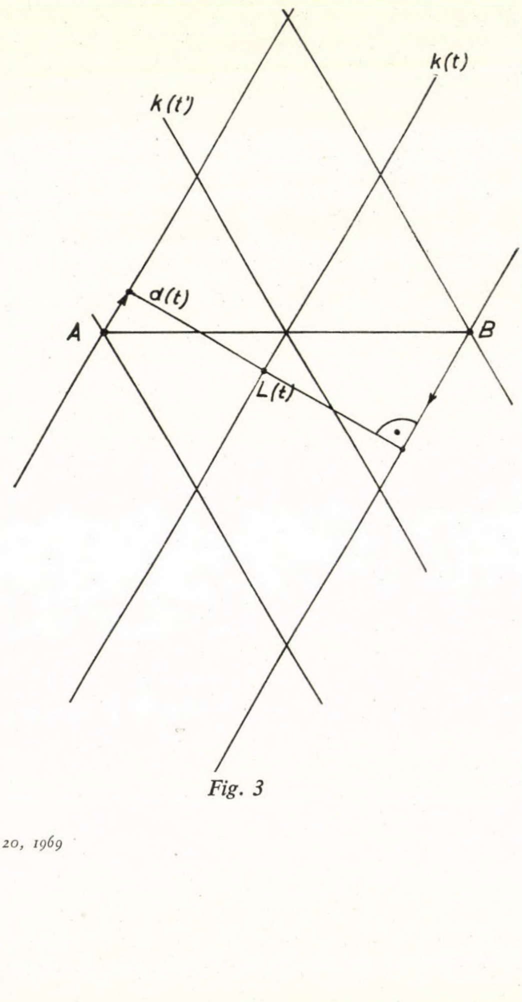

Fig. 3

of the parallelogramm. Since D is strictly convex, the diagonal AB converges to the diameter $d(t)$ if $t' \to t$. Hence the midpoint of the diagonal converges to the midpoint of $d(t)$.

Now we define a point set $H(t)$ on $k(t)$ (for any t) by

$$H(t) = L(t) \cup \bigcup_{t' \not\equiv t \, (\mathrm{mod}\ \pi)} [k(t) \cap k(t')].$$

(1. 2) *$H(t)$ is a closed segment.*

PROOF. Because of (1. 1) and the continuity of the function $k(t)$ the set $H(t)$ is closed, connected and bounded.

(1. 3) *The endpoints of $H(t)$ belong to L.*

PROOF. Let P be an endpoint of $H(t)$. If $P \neq L(t)$, then there exists a line $k(t')$ through P; $t' \not\equiv t$ (mod π). We prove that $P = L(t')$.

Suppose that $L(t')$ is outside of $k(t)$ (see Fig. 4). If $t'' < t' < t'''$ and the parameter values t'', t''' are near enough to t, then owing to (1. 1) the lines $k(t'')$, $k(t''')$ intersect $k(t')$ in the half-plane (determined by $k(t)$) which contains the point $L(t')$. Then the points $k(t) \cap k(t'')$ and $k(t) \cap k(t''')$ — which belong to $H(t)$ — intercept the point P. Hence P was not an endpoint of $H(t)$.

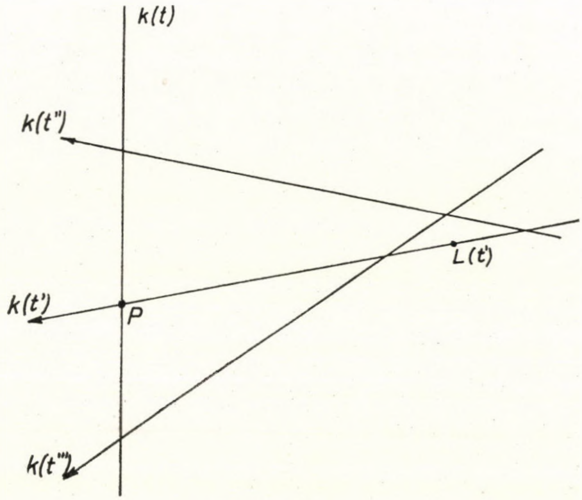

Fig. 4

(1. 4) *If $t_1 < t_2 < t_1 + \pi$ and $P = k(t_1) \cap k(t_2) \in k(t')$ for all parameter values $t' \in (t_1, t_2)$, then $P = L(t_1) = L(t_2)$.*

PROOF. Since $k(t_1)$ and $k(t_2)$ contain the point P, we have by (1. 1) $L(t_1) = \lim_{t' \to t_1 + 0} [k(t_1) \cap k(t')] = \lim_{t' \to t_1} P = P$. Similar argument holds for $L(t_2)$.

(1. 5) *If D is not a circle, then each line $k(t)$ cuts the curve L.*

PROOF. Take a line $k(t)$. We prove that both open half-planes determined by $k(t)$ contain a point of L.

We know that L does not consist of the single point $L(t)$, since if each line $k(t)$ passes through the same point, then the support stripes envelop a circle around this point and we excluded D to be a circle. Hence there exists a line which does not go through the point $L(t)$, correspondingly $H(t)$ cannot consist of a single

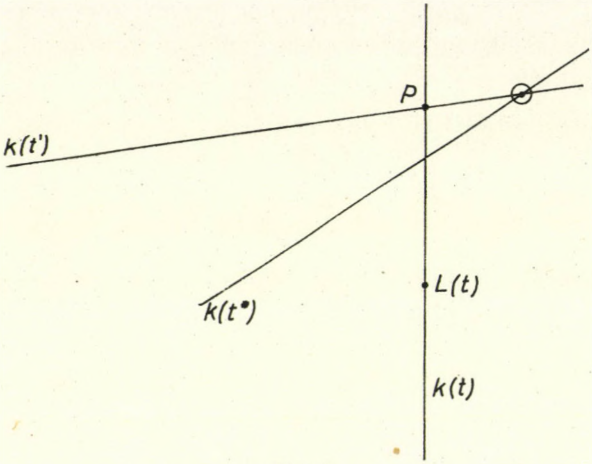

Fig. 5

point. One extremity P of $H(t)$ is certainly different from $L(t)$ (see Fig. 5). Then there exists a line $k(t')$ through the point P, $t' \not\equiv t \pmod{\pi}$ and we may assume $t' < t < t' + \pi$. Because of (1. 4) the interval (t', t) contains a parameter value t^* for which $P \notin k(t^*)$. The line $k(t^*)$ intersects one or the other open half-line of $k(t)$ with the endpoint P. Owing to the definition of the point P this is the half-line which contains the point $L(t)$. Hence the point of intersection $k(t^*) \cap k(t')$ is in the right open half-plane (see Fig. 5). Then $H(t'')$ has a point in the right open half-plane, and one of its endpoints is also in this half-plane. This point belongs by (1. 3) to L.

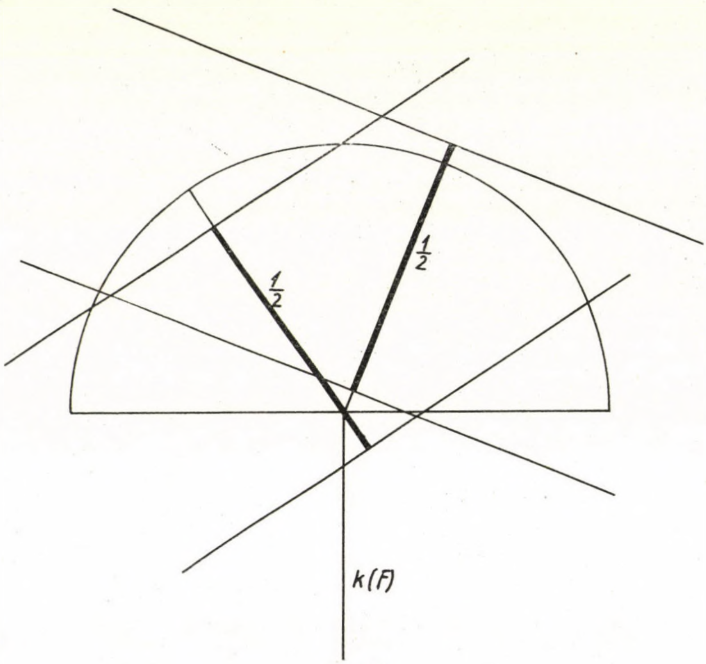

Fig. 6

Similarly if we start instead of t, t' from $t, t' + \pi$, respectively, we get a point of L in the left open half-plane of $k(t)$.

PROOF OF THEOREM 1 AND 2. It is evident that $F \subset \mathbf{D}$ holds if and only if there is no support line of D intersecting the semicircle F. Since the radius of the semicircle is $\frac{1}{2}$ and the distance of a support line from its parallel line $k(t)$ is also $\frac{1}{2}$, the above condition is satisfied if and only if there is no line $k(t)$ intersecting the open half-line $k(F)$ (see Fig. 6).

We have to investigate for which $F = F(t, i)$ this condition holds. Because of the lemmas (1. 2) and (1. 3) the half-line $k(F) = k_i(t)$ does not intersect the other $k(t')$ lines for each $t' \not\equiv t \pmod{\pi}$ if and only if $k_i(t)$ contains no point of L. This was the statement of Theorem 1.

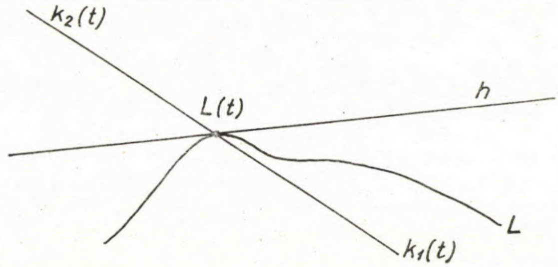

Fig. 7

Let the point $L(t)$ be now on the convex hull of L. Then there exists a line h through $L(t)$, such that the curve L is contained in one of the half-planes determined by h. Because of (1. 5) we have $h \neq k(t)$. Then $k_1(t)$ or $k_2(t)$ is contained in the open half-plane bounded by h which does not contain any point of the curve L. This proves the Theorem 2 (see Fig. 7).

2. A method of obtaining all the required semicircles

(K) Suppose for the time being that there exists no interval (t_1, t_2) $t_1 \not\equiv t_2 \pmod{\pi}$ for which all the lines $k(t)$ $t \in (t_1, t_2)$ go through the same point. This is not an essential restriction, but it simplifies much the drafting.

Now we shall classify the points of the curve L on the basis of their local properties. We say the point $L(t)$ to be of type (A) if there exists a number $\varepsilon > 0$ such that for any parameter value $t' \in (t-\varepsilon, t)$ and $t' \in (t, t+\varepsilon)$ we have $L(t) \notin k(t')$ and the point $L(t)$ is always on the same side of the directed line $k(t')$. We say the point $L(t)$ to be of type (B_1) if there exists a number $\varepsilon > 0$ such that for any parameter value $t' \in (t-\varepsilon, t)$ the point $L(t)$ is always on one side of the directed line $k(t')$ or $L(t) \in k(t')$, and for any parameter value $t'' \in (t, t+\varepsilon)$, the point $L(t)$ is always on the other

Fig. 8

side of the directed line $k(t'')$ or $L(t) \in k(t'')$. Finally, we say the point $L(t)$ to be of type (B_2) if $L(t)$ is neither of type (A) nor of type (B_1). Because of the condition (K) if the point $L(t)$ is of type (B_2) then for any number $\varepsilon > 0$ one of the intervals $(t-\varepsilon, t)$, $(t, t+\varepsilon)$ contains parameter values t', t'' for which the point $L(t)$ is on different sides of the directed lines $k(t')$, $k(t'')$ (see Fig. 8).

This classification is a generalization of the classification given by COOKE for the points of the curve L in case of the Reuleaux domains. COOKE calls the points of type (A) intermediate points, and those of type (B_1) cusp-points. In his case points of type (B_2) do not occur because of the finiteness of the number of arcs.

If we omit the condition (K) and if t is an inner-point of a contradicting interval, then we take the maximum of such intervals (if D is not a circle, this is always possible) and we have to add the above mentioned neighbourhoods to this interval. This does not cause any real difficulty, but it makes the drafting much more complicated.

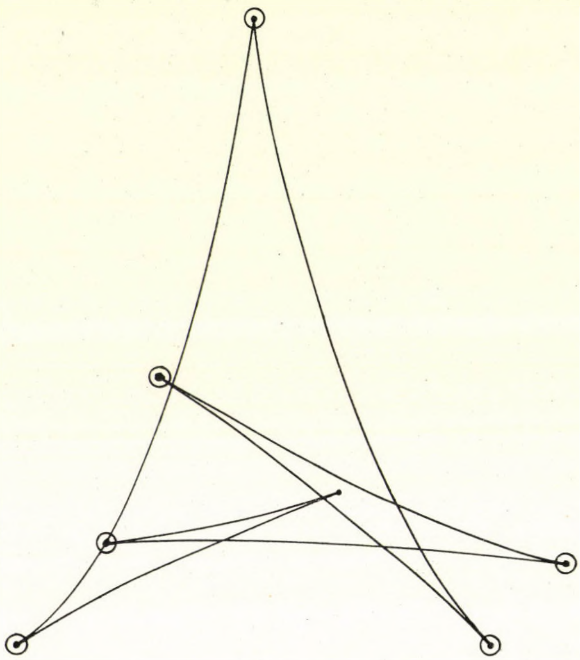

Fig. 9

COOKE proved that if D is a Reuleaux domain then the points of L which are on the boundary of his convex hull and are of type (B_1) are midpoints of required semicircles. In this paper we shall generalize this theorem. We prove for arbitrary domains of constant width the following

THEOREM 3. *$F(t, 1) \subset \mathbf{D}$ or $F(t, 2) \subset \mathbf{D}$ holds if and only if $L(t)$ is of type (B_1) and $L(t)$ is attainable from outside (i.e. $L(t)$ can be connected with a far away point by a polygon G for which $G \cap L = L(t)$; see Fig. 9).*

LEMMAS:

(2. 1) $L(t) = \lim_{t' \to t, t'' \to t} [k(t') \cap k(t'')]$.

This is a stronger form of (1. 1). The proof of (1. 1) proves also (2. 1).

Let us consider now the point set

$$M = \bigcup_{t', t''} [k(t') \cap k(t'')], \qquad t' \not\equiv t'' \pmod{\pi}$$

Let $[M]$ be the closure of M and $\langle M \rangle$ be the interior of M. M is a bounded set, because M is included by D.

(2. 2) $[M] \setminus M \subset L$.

PROOF. Let $P\in[M]\setminus M$. Because of $P\in[M]$ and of the continuity of $k(t)$ there exists a parameter value t for which $P\in k(t)$ and there exist points $P_1, P_2, P_3, \ldots$ $\ldots\in M$, $P_i\to P$, $P_i=k(t_i')\cap k(t_i'')$, $t_i'\not\equiv t_i''$ (mod π). Instead of taking subsequences we may suppose that the sequences $\{t_i'\}$, $\{t_i''\}$ are convergent. Let $\lim_{i\to\infty} t_i'=t'$,

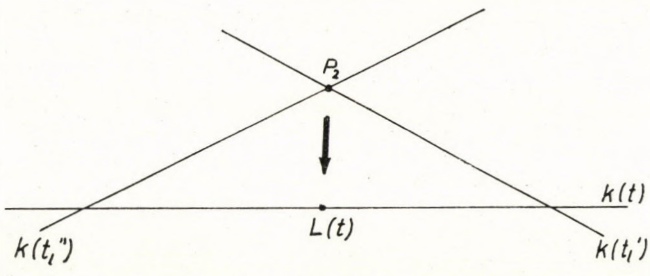

Fig. 10

$\lim_{i\to\infty} t_i''=t''$. Because of $P\notin M$ we have $t'\equiv t\equiv t''$. Hence by (2. 1) $P=\lim_{\substack{t'\to t\\ t''\to t}}[k(t')\cap$ $\cap k(t'')]=L(t)$ (see Fig. 10).

(2. 3) $M\setminus L\subset\langle M\rangle$.

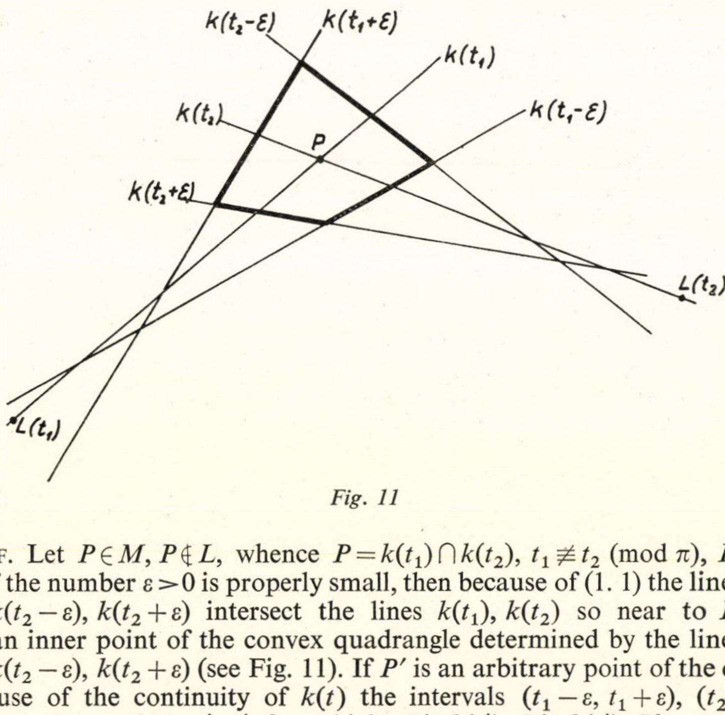

Fig. 11

PROOF. Let $P\in M$, $P\notin L$, whence $P=k(t_1)\cap k(t_2)$, $t_1\not\equiv t_2$ (mod π), $L(t_1)\neq P\neq$ $\neq L(t_2)$. If the number $\varepsilon>0$ is properly small, then because of (1. 1) the lines $k(t_1-\varepsilon)$, $k(t_1+\varepsilon)$, $k(t_2-\varepsilon)$, $k(t_2+\varepsilon)$ intersect the lines $k(t_1)$, $k(t_2)$ so near to $L(t_1)$, $L(t_2)$ that P is an inner point of the convex quadrangle determined by the lines $k(t_1-\varepsilon)$, $k(t_1+\varepsilon)$, $k(t_2-\varepsilon)$, $k(t_2+\varepsilon)$ (see Fig. 11). If P' is an arbitrary point of the quadrangle then because of the continuity of $k(t)$ the intervals $(t_1-\varepsilon, t_1+\varepsilon)$, $(t_2-\varepsilon, t_2+\varepsilon)$ contain parameter values t_1', t_2' for which $P'\in k(t_1')$, $P'\in k(t_2')$. If ε was properly

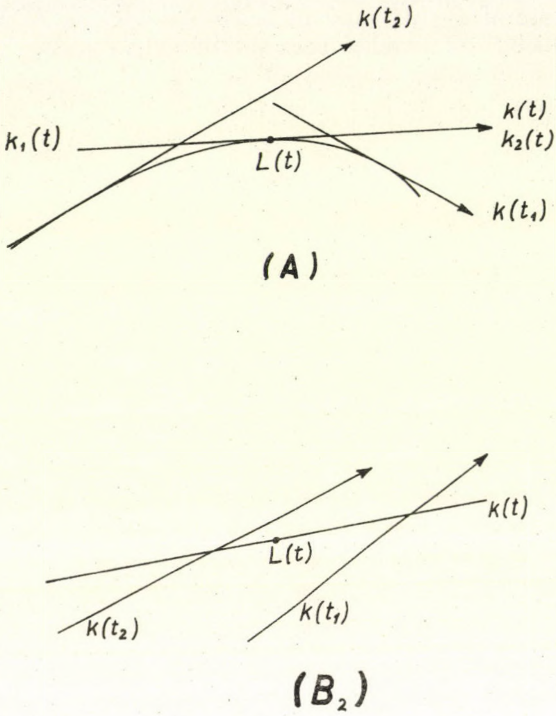

Fig. 12

small, then $t_1' \not\equiv t_2' \pmod{\pi}$ holds certainly. Hence the quadrangle is contained by M and the inner points of the quadrangle are inner points of M.

PROOF OF THEOREM 3. a) The necessity of the conditions: If $L(t)$ is of type (A) or (B_2) and $\varepsilon > 0$ is arbitrary, then there exist lines $k(t_1)$, $k(t_2)$, $|t_1 - t| < \varepsilon$, $|t_2 - t| < \varepsilon$ such that one of these intersects the half-line $k_1(t)$ and the other intersects the half-line $k_2(t)$. Hence the condition of Theorem 1 cannot be satisfied (see Fig. 12).

If the point $L(t)$ is of type (B_1) and not attainable from outside, then each polygon connecting $L(t)$ and a far away point, particularly the half-lines $k_1(t)$, $k_2(t)$ also contain a point of the curve L. Hence because of Theorem 1 $F(t, i) \subset \mathbf{D}$ cannot hold.

b) The sufficiency of the conditions: We must prove that if $L(t)$ is of type (B_1) and attainable from outside then one of the half-lines $k_1(t)$, $k_2(t)$ does not intersect the curve L.

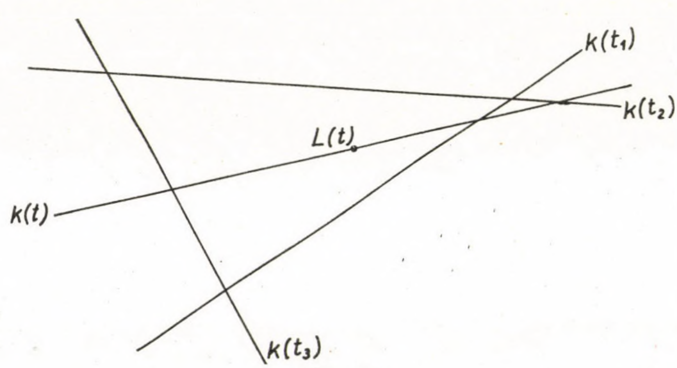

Fig. 13

Let $L(t)$ be a point of type (B_1). Then for any $\varepsilon > 0$ there exist lines $k(t_1)$, $k(t_2)$; $t_1 \in (t - \varepsilon, t)$, $t_2 \in (t, t + \varepsilon)$ which intersect the same half-line $k_{i_1}(t)$. Suppose that there exists a line $k(t_3)$ which intersects the other half-line $k_{i_2}(t)$ (i.e. $k_{i_2}(t)$ contains a point of the curve L). We shall prove that in this case any polygon connecting $L(t)$ and a far away point contains a point P of the curve L, $P \not\equiv L(t)$.

We consider such a polygon and let P be its last common point with $[M]$. $L(t)$ is an inner point of the triangle $k(t_1)$, $k(t_2)$, $k(t_3)$ (see Fig. 13). Hence we get (similarly as in (2. 3)) that $L(t)$ is an inner point of M. Therefore by (2. 3) $P \not\equiv L(t)$. If $P \in M$ then $P \in L$ since otherwise we should infer by (2. 3) $P \in \langle M \rangle$ and P could not be the last common point with $[M]$. If $P \notin M$ then by (2. 2) $P \in L$. This was to be proved.

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A CHARACTERIZATION OF THE CLOSED SUBGROUPS OF THE SCHUTZENBERGER GROUP

By

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0. Introduction. In Section 2 it is shown that if the initial semigroup is compact or discrete and card $H_a>1$, then to each closed subgroup of the Schutzenberger group, $a^{(-1)}H_a/\mathcal{S}(H_a, a)$, there corresponds a topologically equivalent subset A of H_a with the property that $A^{[-1]}A=A^{(-1)}A$; and, conversely, each closed subset A of H_a with that property generates a homeomorphic subgroup of the Schutzenberger group, namely, $\Phi(A^{[-1]}A)$ where Φ is the natural map for the Dubreil semigroups. In addition, it is shown that for a subset A of an $\mathcal{H}$-slice with cardinality >1 to be a component of a compact or discrete semigroup it is necessary that A be homeomorphic to the topological group, $\Phi(A^{[-1]}A)$. As a consequence, a homeomorphic copy of the unit interval cannot be a component of such a semigroup and a subset of an $\mathcal{H}$-slice simultaneously.

1. Preliminary material. Terminology and background results are presented in this section.

1. 1. Definition. A (topological) *semigroup* S is a nonnull Hausdorff space together with a continuous associative multiplication. Precisely, a semigroup is such a function $m\colon S\times S\to S$ that

(i) S is a nonnull Hausdorff space,
(ii) m is continuous, and
(iii) m is associative; i.e., for each x, y, z in S, $m(x, m(y, z))=m(m(x, y), z)$.

For brevity the multiplication is ordinarily denoted by juxtaposition, that is, $m(x, y)=xy$. Moreover, it is common usage to say that a semigroup S is compact if S is a compact space and to say that a subset of S is closed if it is closed in a topological sense.

1. 2. Definitions. The empty set will be designated by $\Box$. If X and Y are subsets of S, then $X^{(-1)}Y=\{w \text{ in } S;\ Xw\cap Y\neq\Box\}$, $X^{[-1]}Y=\{w \text{ in } S;\ Xw\subset Y\}$, $YX^{(-1)}=\{w \text{ in } S;\ wX\cap Y\neq\Box\}$, and $YX^{[-1]}=\{w \text{ in } S;\ wX\subset Y\}$.

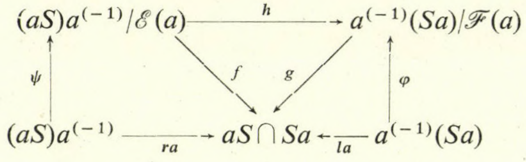
It is noted that if X is a singleton set, then $X^{(-1)}Y=X^{[-1]}Y$ and $YX^{(-1)}=$ $=YX^{[-1]}$. In addition, we will use the fact that if Y is closed, then $X^{[-1]}Y$ is closed.

1. 3. Definition. Letting Y be a subset of S and Δ be the diagonal of $Y\times Y$, then an equivalence relation $R\subset Y\times Y$ is a *closed congruence* on Y if and only if $\Delta R\cup R\Delta\subset R$ and R is closed in $Y\times Y$ with respect to the relative topology.

It may be shown that if S is compact or discrete and if R is a closed congruence on S, then S/R is a semigroup and the canonical map from S to S/R is continuous [6].

The algebraic form of the next result is of french origin, see [2] and [5], and topological generalizations appear in [1] and [4]:

1. 4. THEOREM (DUBREIL). *If S is compact or discrete, $a \in S$ and if we define $\mathscr{E}(a) = \{(x, y);\ x, y \in (aS)a^{(-1)}$ and $xa = ya\}$ and $\mathscr{F}(a) = \{(u, v);\ u, v \in a^{(-1)}(Sa)$ and $au = av\}$, then $\mathscr{E}(a)$ and $\mathscr{F}(a)$ are congruences on the semigroups $(aS)a^{(-1)}$ and $a^{(-1)}(Sa)$, respectively. If ψ and φ are the appropriate natural homomorphisms in the diagram and letting ra and la denote multiplication by a on the right and left, respectively, then f, g and h are such homeomorphisms that $f\psi = ra$, $g\Phi = la$ and $g^{-1}f = h$; moreover, h is an isomorphism.*



1. 5. DEFINITION. If S is a semigroup and A and T are subsets of S, then one defines $L(A, T) = A \cup TA$, $R(A, T) = A \cup AT$ and $H(A, T) = R(A, T) \cap L(A, T)$. When the context clearly indicates which subset T is under consideration, then reference to T is usually omitted, that is, we write $L(A, T) = L(A)$, etc. Moreover, for $T \subset S$, one defines the Relative Green (equivalence) Relations, $\mathscr{L} = \{(x, y);\ L(x) = L(y)\}$, $\mathscr{R} = \{(x, y);\ R(x) = R(y)\}$ and $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$. For $x \in S$, we will let $H_x(T)$ denote the $\mathscr{H}(T)$-class (or slice) containing x; here again reference to T is omitted if the context is clear.

It is true that $H_w^{(-1)} H_w = H_w^{[-1]} H_w = w^{(-1)} H_w$ for any $w \in S$ [6].

1. 6. DEFINITION. For any $A \subset S$ and $y \in S$, let us define $\mathscr{S}(A, y) = \{(u, v);\ u, v \in A^{[-1]}A$ and $yu = yv\}$ and $\mathscr{T}(A, y) = \{(u, v);\ u, v \in AA^{[-1]}$ and $uy = vy\}$.

It is well known that if a compact semigroup is algebraically a group, then it is a topological group. (It is noted that a generalization to the locally compact case appears in [3].) Using this fact one may show that the Schutzenberger—Wallace Theorem follows [6]:

1. 7. THEOREM (SCHUTZENBERGER—WALLACE). *If S is compact or discrete, if T is a closed subset of S and if y is an element of S such that* card $H_y > 1$, *then H_y is homeomorphic to the topological group, $y^{(-1)}H_y/\mathscr{S}(H_y, y)$, and the groups $y^{(-1)}H_y/\mathscr{S}(H_y, y)$ and $H_y y^{(-1)}/\mathscr{T}(H_y, y)$ are iseomorphic.*

The groups mentioned in (1. 7), namely $y^{(-1)}H_y/\mathscr{S}(H_y, y)$ and $H_y y^{(-1)}/\mathscr{T}(H_y, y)$, are commonly referred to as the Schutzenberger groups.

Lastly, we mention sufficient conditions in order for a semigroup in a compact (topological) group to be a closed subgroup [7]:

1. 8. THEOREM. *If S is such a semigroup in a compact group that S is either open or closed, then S is a closed subgroup.*

2. A characterization and a consequence. The principal result of this paper, namely, the characterization of the closed subgroups of the Schutzenberger groups, is embodied in (2. 3). The consequence is presented in (2. 5).

2. 1. PROPOSITION. *If b is an element of a semigroup S and A is a subset of S, then with regard to the statements*

(1) $b^{(-1)}A$ *is a semigroup*
(2) $b^{(-1)}A \subset A^{[-1]}A$
(3) $b^{(-1)}A \subset \{x \in S;\ Ax = A\}$
(4) $A = Cb$ *and* $C^2 b \subset Cb$ *where* $C \subset S$

the dependence is indicated by the diagram,

$$\begin{matrix} (3) & & \\ & \searrow & \\ & & (2) \rightarrow (1). \\ & \nearrow & \\ (4) & & \end{matrix}$$

Moreover, if $A \subset bS$, *then* (1) *implies* (2); *consequently, if* $b \in A \subset bS$ *and* $b^{(-1)}A$ *is a semigroup, then* $b^{(-1)}A = A^{[-1]}A$.

PROOF. If condition (2) holds and if $x, y \in b^{(-1)}A$, then $b(xy) = (bx)y \in Ay \subset A$ and so $b^{(-1)}A$ is a semigroup; moreover, since it is always the case that $\{x;\ Ax = A\} \subset \subset A^{[-1]}A$, it is clear that condition (3) implies condition (2). If condition (4) holds and $x \in b^{(-1)}A$, then $(Cb)x = C(bx) \subset CA = C(Cb) \subset Cb$ so that x is in $A^{[-1]}A$ and condition (2) is satisfied.

$b^{(-1)}A$ is a semigroup means that $(b^{(-1)}A)(b^{(-1)}A) \subset b^{(-1)}A$ so that, multiplying by b on the left and using the fact that $A \subset bS$, we obtain $A(b^{(-1)}A) \subset A$ and, consequently, $b^{(-1)}A \subset A^{[-1]}A$. If, in addition, b is in A, then it is clear that $b^{(-1)}A = = A^{[-1]}A$.

It is possible to indicate, as shown by the following examples, that the implications among conditions (1) through (4) of (2. 1) may not be reversed:

2. 2. EXAMPLES. (a) Consider the semigroup S defined by the multiplication table,

	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

and let $b = 1$ and $A = \{0, 2\}$. Then $b^{(-1)}A = \{0, 1\}$ which is clearly a subsemigroup so that condition (1) holds and yet $b^{(-1)}A$ is not a subset of $A^{[-1]}A$ which is $\{0, 2\}$.

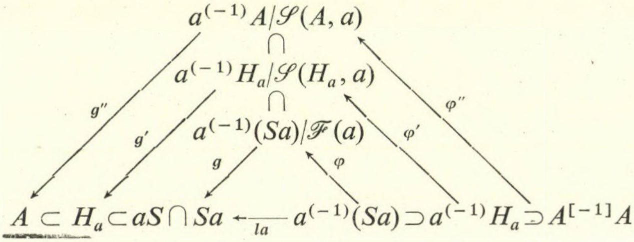
(b) Using the semigroup S defined in (a) and letting $b = 1$, $A = \{0, 1\}$ and $C = \{0, 2\}$ we have that $A = Cb$ and $C^2 b = Cb$ whereas $b^{(-1)}A = S$ and $\{x \in S;\ Ax = A\} = \{2\}$ so that neither condition (2) nor condition (4) implies condition (3).

(c) If we let a semigroup S be defined by the table

	0	1	2	3
0	0	0	2	2
1	1	1	3	3
2	2	2	0	0
3	3	3	1	1

and if $b = 1$ and $A = \{0, 1, 2\}$, then $\{x \in S;\ Ax = A\} = b^{(-1)}A = \{0, 1\}$; however, the only set C such that $A = Cb$ is A itself and, in this case, we find that $C^2 b = S$. Consequently, neither condition (2) nor condition (3) implies condition (4).

2. 3. THEOREM. *Let S be a compact or discrete semigroup, T be a closed subset of S and y be such an element of S that* card $H_y > 1$. *If G is a closed topological subgroup of the Schutzenberger group, $y^{(-1)}H_y/\mathscr{S}(H_y, y)$, and if w is an element of H_y, then, letting $A_w = w(\varphi^{-1}G)$ where φ is the homomorphism of Dubreil's diagram, it is true that $\varphi^{-1}G = A_w^{[-1]}A_w = A_w^{(-1)}A_w$ and A_w is topologically equivalent to G. Conversely, if A is a nonempty closed subset of H_y such that $A^{[-1]}A = A^{(-1)}A$, then for a in A the following diagram is commutative and, as a result, $A \stackrel{t}{=} \varphi(A^{[-1]}A)$ which is a subgroup of the Schutzenberger group:*



where the primes and double primes indicate that those functions are restrictions of φ and g.

PROOF. If G is contained in the Schutzenberger group, namely, $\Phi(w^{(-1)}H_y)$, for any $w \in H_y$, then $\Phi^{-1}G \subset w^{(-1)}H_y$, for if x is in $w^{(-1)}(Sw)$ and $\Phi(x) \in G$, then $\Phi(x) = \Phi(y)$ for some $y \in w^{(-1)}H_y$ so that $wx = wy \in H_y$ and $x \in w^{(-1)}H_y$; letting $A_w = w(\Phi^{-1}G)$ we have that $\Phi^{-1}G = w^{(-1)}A_w$. Since $w^{(-1)}w$ is nonempty and $H_y \subset L(w)$, it follows that $\Phi(w^{(-1)}w)$ is the identity of the Schutzenberger group and so $\Phi^{-1}G$ contains an element q such that $wq = w$. Consequently, $w \in A_w$ and then using the last part of (2. 1) we find that $\Phi^{-1}G = A_w{}^{[-1]}A_w$. Because $A_w \subset H_w \subset R(w)$, the restriction of the lw function of Dubreil's diagram to $A_w{}^{[-1]}A_w$ has as its image A_w and therefore, by Dubreil's result, $G = \Phi(A_w{}^{[-1]}A_w)$ is homeomorphic to A_w. If $x \in \Phi^{-1}G$ and $a \in A_w$, then card $H_y > 1$ implies that $wx = atx = wt'tx$ where $t \in T \cap w^{(-1)}H_y$ because $H_y{}^{(-1)}H_y = w^{(-1)}H_y$ and $t' \in T \cap \Phi^{-1}G$ because $\Phi^{-1}G = w^{(-1)}A_w$. Therefore, $\Phi(x) = \Phi(t'tx) = \Phi(t')\Phi(t)\Phi(x)$ and since $\Phi(t')$ and $\Phi(x)$ have group inverses in G, it follows, letting $\Phi(t')^{-1}$ be the inverse of $\Phi(t')$, that $\Phi(t')^{-1} = \Phi(t) \in G$ and so $t \in \Phi^{-1}G$ and, consequently, $w(\Phi^{-1}G) \subset a(\Phi^{-1}G)$; moreover, $ax = wt'x$ and so we have the reverse set inclusion, namely, $a(\varphi^{-1}G) \subset w(\varphi^{-1}G)$. As a result, for each $a \in A_w$, it follows that $\varphi^{-1}G = a^{(-1)}[a(\varphi^{-1}G)] = a^{(-1)}[w(\varphi^{-1}G)] = a^{(-1)}A_w$ and, consequently, $A_w^{[-1]}A_w = A_w^{(-1)}A_w$.

Conversely, if A is a nonempty closed subset of H_y such that $A^{[-1]}A = A^{(-1)}A$, then, for $a \in A$, $A^{[-1]}A = a^{(-1)}A \subset a^{(-1)}H_y \subset a^{(-1)}(Sa)$, the last inclusion being true because card $H_y > 1$, so that $\varphi(A^{[-1]}A) \subset \varphi(a^{(-1)}H_y)$, the Schutzenberger group, where φ is the appropriate canonical map in Dubreil's result.

We now consider two cases: (i) If S is compact, in view of (1. 8) it follows that $\Phi(A^{[-1]}A)$ is a group because A is closed. (ii) In the case that S is discrete we may select an element q of $a^{(-1)}a$ because $a^{(-1)}a$ is nonempty if and only if a is in aS. Then, since $A \subset L(a)$ we have $ax \in L(a)$ for each $x \in A^{[-1]}A$ and so it is easily verified that $ax = axq$. Consequently, $\Phi(q)$ is a right unit for $\Phi(x)$. If $a = ax$, we have $a = axx$ and if $a \neq ax$, then $a = axt$ for some $t \in T$ because $A \subset R(b)$ for all $b \in A$ so that in either circumstance there is an element x' such that $a = axx'$ and, consequently, $aq = axx'$. Clearly, x' is in $A^{[-1]}A$ and therefore $\Phi(x)$ has a right inverse. We have thus shown that $\Phi[A^{[-1]}A]$ is a group.

Lastly, since $A^{[-1]}A = a^{(-1)}A$ and $A \subset R(a)$, the restriction of the la function of Dubreil's diagram to $A^{[-1]}A$ has as its image A and, therefore, using Dubreil's result, $\Phi(A^{[-1]}A)$ is homeomorphic to A.

2. 4. Proposition. *If A and B are subsets of S such that A is nonvoid and connected and B is a component of S, then $A^{[-1]}B = A^{(-1)}B$.*

Proof. If y is an element of $A^{(-1)}B$, then Ay is connected and so it follows that $B \cup A(A^{(-1)}B)$ is connected. Then, since B is a component, $A(A^{(-1)}B) \subset B$ and we have that $A^{(-1)}B \subset A^{[-1]}B$. The fact that A is nonempty is used to ensure that $A^{[-1]}B \subset A^{(-1)}B$.

2. 5. Corollary. *If A is a component of S, then $A^{(-1)}A = A^{[-1]}A$. Consequently, if S is compact or discrete, then for a set A contained in an $\mathcal{H}$-slice having cardinality >1 to be a component of S it is necessary that A be homeomorphic to the topological group, $\Phi(A^{[-1]}A)$, where Φ is the canonical map of Dubreil's diagram.*

Proof. This result is immediate in view of the 2. 3 and 2. 4 Theorems.

In view of (2. 5) a homeomorphic copy of the unit interval cannot be a component of a compact, or discrete, semigroup and a subset of an $\mathcal{H}$-slice simultaneously.

If we consider a semigroup in which two distinct points a and b are contained in a component, then by letting $A = \{a\}$ it is easy to see that the converse of the first part of (2. 5) is not true. Moreover, if we look at a totally disconnected space with cardinality >1 which has the multiplication $xy = x$, it is easy to see that the weaker converse, namely, $A^{(-1)}A = A^{[-1]}A$ implies that A is connected, is also false because in such a semigroup equality holds for any nomempty subset.

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ON THE NUMBER OF SOLUTIONS OF A DIOPHANTINE SYSTEM

By

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1. Let $A_N(v_1, \ldots, v_l)$ denote the number of solutions of the diophantine equation system

$$\sum_{k=1}^{N} \varepsilon_k k^\nu - \sum_{k=1}^{N} \varepsilon'_k k^\nu = v_\nu \qquad (\nu = 1, 2, \ldots, l) \tag{1.1}$$

in unknowns $\varepsilon_1, \ldots, \varepsilon_N, \varepsilon'_1, \ldots, \varepsilon'_N$, where $\varepsilon_j, \varepsilon'_j$ assume the values 0 and 1.

Let

$$Q(\beta_1, \ldots, \beta_l) = \sum_{\mu=1}^{l} \sum_{\nu=1}^{l} \frac{\beta_\nu \beta_\mu}{\nu+\mu+1}, \tag{1.2}$$

$$\sigma_N(v_1, \ldots, v_l) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-2\pi i \left(\frac{\beta_1}{N^{\frac{3}{2}}} v_1 + \ldots + \frac{\beta_l}{N^{\frac{2l+1}{2}}} v_l\right)\right) \times \tag{1.3}$$

$$\times \exp\left(-\pi^2 Q(\beta_1, \ldots, \beta_l)\right) d\beta_1 \ldots d\beta_l.$$

Let

$$\tau_N(v_1, \ldots, v_l) = \sum \exp\left(-2\pi i \left(\sum_{j=1}^{l} \frac{a_j v_j}{q_j}\right)\right), \tag{1.4}$$

where in the right hand side we sum over those $a_1, q_1; a_2, q_2; \ldots; a_l, q_l$ for which the following two conditions hold:

1. $(a_j, q_j) = 1$ $(j = 1, \ldots, l)$,
2. *the polynomial* $\sum_{j=1}^{l} \frac{a_j}{q_j} x^j$ *assumes integer values for every integer* x.

The number of these polynomials is finite.

We shall prove the following

THEOREM 1.

$$A_N(v_1, \ldots, v_l) = 2^{2N} N^{-\frac{l(l+2)}{2}} \{\sigma_N(v_1, \ldots, v_l)\tau_N(v_1, \ldots, v_l) + O(N^{-c})\},$$

where c is a suitable positive constant.

REMARKS. (a) Let ν be a fixed positive integer and let $B_N(v)$ denote the number of solutions of the diophantine equation

$$\sum_{k=1}^{N} (\varepsilon_k - \varepsilon'_k) k^\nu = v$$

in unknowns $\varepsilon_1, \dots, \varepsilon_N, \varepsilon_1', \dots, \varepsilon_N'$, where $\varepsilon_k, \varepsilon_k'$ assume the values 0 and 1. Using the method of this paper we can prove the following

THEOREM 2.

$$B_N(v) = 2^{2N} N^{-\frac{2\nu+1}{2}} \{\varrho_N(v) + O(N^{-c})\} \quad (c>0, \textit{ constant}),$$

where

$$\varrho_N(v) = \int_{-\infty}^{\infty} \exp\left(-2\pi i N^{-\frac{2\nu+1}{2}} \beta v - \pi^2 \beta^2/(2\nu+1)\right) d\beta =$$

$$= \sqrt{\frac{2\nu+1}{\pi}} \exp(-(2\nu+1) v^2 N^{-(2\nu+1)}).$$

Hence we see, that $B_N(v)/2^{2N}$ is asymptotically equal to the density function of the Gauss-law with mean 0 and variance $\sigma^2 = \dfrac{N^{2\nu+1}}{2(2\nu+1)}$.

(b) ERDŐS noted in [1], that VAN LINT has found the asymptotic of $B_N(0)$ in the case $\nu=1$.

2. PROOF OF THEOREM 1. Set

$$F(x, \alpha_1, \dots, \alpha_l) = \sum_{\nu=1}^{l} \alpha_\nu x^\nu, \tag{2.1}$$

$$\psi_N(\alpha_1, \dots, \alpha_l) = \prod_{k=1}^{N} |1 + e^{2\pi i F(k, \alpha_1, \dots, \alpha_l)}|^2 = \tag{2.2}$$

$$= \prod_{k=1}^{N} (1 - \sin^2 \pi F(k, \alpha_1, \dots, \alpha_l)),$$

$$\mathscr{I}(\mathscr{A}) = \int \cdots \int_{\mathscr{A}} \exp(-2\pi i(\alpha_1 v_1 + \dots + \alpha_l v_l)) \psi_N(\alpha_1, \dots, \alpha_l)\, d\alpha_1 \dots d\alpha_l, \tag{2.3}$$

$$\mathscr{L}(\mathscr{A}) = \int \cdots \int_{\mathscr{A}} \psi_N(\alpha_1, \dots, \alpha_l)\, d\alpha_1 \dots d\alpha_l. \tag{2.4}$$

Let

$$I = \{(\alpha_1, \dots, \alpha_l);\ |\alpha_j| \leqq 1/2,\ j = 1, \dots, l\},$$

$$\tau_j = N^{j-\varkappa} \qquad (j = 1, \dots, l), \quad 0<\varkappa<1. \tag{2.5}$$

For coprime integers a_j, q_j let $\mathfrak{M} = \mathfrak{M}(a_1, \dots, q_l)$ be the set of those $(\alpha_1, \dots, \alpha_l)$, for which

$$\left|\alpha_j - \frac{a_j}{q_j}\right| \leqq \frac{1}{q_j \tau_j} \quad (j = 1, \dots, l). \tag{2.6}$$

Let

$$Q = [q_1, \dots, q_l];^* \quad Q_1 = [q_2, \dots, q_l], \tag{2.7)—(2.8}$$

$$\psi_j = \alpha_j - \frac{a_j}{q_j} \quad (j = 1, \dots, l). \tag{2.9}$$

* $[a, b, \dots]$ denotes the least common multiple of $a, b, \dots$.

Let $\mathcal{T}$ denote the set of all cubes $\mathfrak{M}(a_1, \dots, q_l)$ for which the polynomial

$$F\left(x, \frac{a_1}{q_1}, \dots, \frac{a_l}{q_l}\right) = \sum_{j=1}^{l} \frac{a_j}{q_j} x^j$$

assumes integer values for every integer x.

Since the number of these polynomials is finite ,so for $\mathfrak{M} \in \mathcal{T}$ we have $Q \leqq c_1$ (when c_1 is large enough).

Let $\mathfrak{N}$ denote the set of those points in I which do not belong to the set $\bigcup\limits_{\mathfrak{M} \in \mathcal{T}} \mathfrak{M}$.

By the Parseval-formula we have

$$A_N(v_1, \dots, v_l) = \mathcal{I}(I).$$

Since the sets $\mathfrak{M}$ in $\mathcal{T}$ are disjoint (if N is sufficiently large) we have

$$A_N(v_1, \dots, v_l) = \sum_{\mathfrak{M} \in \mathcal{T}} \mathcal{I}(\mathfrak{M}) + \mathcal{I}(\mathfrak{N}). \tag{2.10}$$

Let $\beta_j = N^{j+\frac{1}{2}} \alpha_j$ $(j = 1, \dots, l)$. We define the sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ as follows:

$$\mathcal{A}_1 = \{(\alpha_1, \dots, \alpha_l), \max_j |\beta_j| < N^{\frac{1}{2}-\Delta_1}\}, \tag{2.11}$$

$$\mathcal{A}_2 = \{(\alpha_1, \dots, \alpha_l), \max_j |\beta_j| \geqq N^{\frac{1}{2}-\Delta_1}, |\beta_j| < N^{j+\frac{1}{2}-\Delta_2}, j = 1, \dots, l\}, \tag{2.12}$$

$$\mathcal{A}_3 = \{(\alpha_1, \dots, \alpha_l), \tfrac{1}{2} \geqq \max |\alpha_j| \geqq N^{-\Delta_2}\}, \tag{2.13}$$

where $0 < \Delta_1 < \frac{1}{2}$, $0 < \Delta_2 < \frac{1}{2}$.

Estimation of $\mathcal{L}(\mathfrak{N})$. If $(\alpha_1, \dots, \alpha_l) \in \mathfrak{N}$, then $\max\limits_j |\beta_j| > N^{\frac{1}{2}}$, and so $\mathfrak{N} \subset \mathcal{A}_2 \cup \mathcal{A}_3$. We show that in the set $\mathfrak{N}$ the inequality

$$\psi_N(\alpha_1, \dots, \alpha_l) \ll 2^{2N} \exp(-N^c) \qquad (c > 0) \tag{2.14}$$

holds, whence

$$\mathcal{I}(\mathfrak{N}) \ll 2^{2N} \exp(-N^c) \tag{2.15}$$

follows immediately.

Case A: $(\alpha_1, \dots, \alpha_l) \in \mathcal{A}_2(\cap \mathfrak{N})$.

For any fixed $\alpha_1, \dots, \alpha_l$ let M denote the greatest integer, for which

$$\pi(|\alpha_1| M + \dots + |\alpha_l| M^l) \leqq \tfrac{1}{4},$$

when $M \leqq N$, and let $M = N$ otherwise.

Since $\sin \pi x \geqq x$ in $0 \leqq x \leqq \frac{1}{2}$ we have

$$L \stackrel{\text{def}}{=} \sum_{k=1}^{M} \sin^2 \pi F(k, \alpha_1, \dots, \alpha_l) \geqq \sum_{k=1}^{M} F^2(k, \alpha_1, \dots, \alpha_l) = \frac{1}{N} \sum_{\nu, \mu} \beta_\nu \beta_\mu \sum_{k=1}^{M} \frac{k^{\nu+\mu}}{N^{\nu+\mu}} \geqq$$

$$\geqq Q\left(\beta_1 \left(\frac{M}{N}\right)^{3/2}, \dots, \beta_l \left(\frac{M}{N}\right)^{\frac{2l+1}{2}}\right) - O\left(\frac{1}{N} Q\left(|\beta_1| \frac{M}{N}, \dots, |\beta_l| \left(\frac{M}{N}\right)^l\right)\right).$$

Since

$$Q(\beta_1, \dots, \beta_l) = \int_0^1 (\beta_1 x + \dots + \beta_l x^l)^2 \, dx,$$

so Q is a positive definite quadratic form, whence

$$c_2(\beta_1^2 + \dots + \beta_l^2) \leqq Q(\beta_1, \dots, \beta_l) \leqq c_3(\beta_1^2 + \dots + \beta_l^2)$$

follows with suitable positive constants c_2, c_3.

Hence we obtain, that

$$L \gg \sum_\nu \beta_\nu^2 \left(\frac{M}{N}\right)^{2\nu+1} \gg \max_\nu \beta_\nu^2 \left(\frac{M}{N}\right)^{2\nu+1}.$$

By the definition of M it follows, that $M \gg N^{\Delta_2/l}$ and so

$$L \gg N^{\Delta_3}, \quad \Delta_3 = \min(1 - 2\Delta_1, \Delta_2/l).$$

Using the well-known inequality $1 - x \leqq e^x$ for $0 \leqq x \leqq 1$ we have

$$\psi_N(\alpha_1, \dots, \alpha_l) \ll 2^{2N} \prod_{k=1}^{M} (1 - \sin^2 \pi F(k, \alpha_1, \dots, \alpha_l)) < 2^{2N} \exp(-L),$$

i.e. (2. 14) holds.

Case B: $(\alpha_1, \dots, \alpha_l) \in \mathscr{A}_3(\cap \mathfrak{N})$.

Let S denote a suitable subset of 1, 2, ..., N. Let $|S|$ denote the number of the elements of S.

We shall construct a set S with $|S| \gg N^c (c > 0)$ for which the sum

$$(T(S) =) \, T(S; \alpha_1, \dots, \alpha_l) \overset{\text{def}}{=} \sum_{k \in S} e^{2\pi i F(k, \alpha_1, \dots, \alpha_l)}$$

satisfies the inequality

$$|T(S)| < (1 - \delta)|S|, \qquad 0 < \delta < 1.$$

Hence the inequality

$$\psi_N(\alpha_1, \dots, \alpha_l) < 2^{2N} \prod_{k \in S} (1 - \sin^2 \pi F(k, \alpha_1, \dots, \alpha_l)) <$$

$$< 2^{2N} \exp\Big(-\sum_{k \in S} \sin^2 \pi F(k, \alpha_1, \dots, \alpha_l)\Big) =$$

$$= 2^{2N} \exp\left(-\frac{|S|}{2} - \frac{1}{2} \operatorname{Re} T(S)\right) < 2^{2N} \exp\left(-\frac{\delta}{2} N^c\right),$$

i.e. (2. 14) follows.

1. Suppose, that $q_1 \leqq N^{2l\Delta_4}$ and $q_j \leqq N^{\Delta_4}$ for $j \geqq 2$, $0 < \Delta_4 < 1$, Δ_4 is small enough. Then

$$\left| F(k, \alpha_1, \dots, \alpha_l) - F\left(k, \frac{a_1}{q_1}, \dots, \frac{a_l}{q_l}\right) \right| \leqq \varepsilon, \tag{2.16}$$

for $1 \leqq k \leqq M$, with $M = o(1) N^{1-\varkappa}$.

LEMMA.

$$\left|\sum_{k=1}^{Q} e^{2\pi i F\left(k, \frac{a_1}{q_1}, \ldots, \frac{a_l}{q_l}\right)}\right| < (1-\delta)|S|, \qquad 0<\delta<1,$$

whenever $F\left(k, \frac{a_1}{q_1}, \ldots, \frac{a_l}{q_l}\right)$ *assumes at least one non-integer value for some integer* k.

This lemma is a weak variant of a theorem of MORDELL (see [2], Ch. I). Hence by $S=[1, M]$ it follows that

$$|T(S, \alpha_1, \ldots, \alpha_l)| \leqq \varepsilon|S| + \left|T\left(S, \frac{a_1}{q_1}, \ldots, \frac{a_l}{q_l}\right)\right| \leqq$$

$$\leqq \varepsilon|S| + \left(\frac{|S|}{Q}+1\right) Q(1-\delta) < (1-\delta/2)|S|,$$

when $\varepsilon < \frac{\delta}{2}$, because in our case $\mathfrak{M}(a_1, \ldots, q_l) \notin \mathcal{T}$.

2. Suppose now that $q_j \geqq N^{\Delta_4}$ for at least one j in $2 \leqq j \leqq l$. Then using the well known result of I. M. VINOGRADOV concerning the Weyl-sums, it follows that

$$|T(S)| \ll N^{1-\alpha}, \quad 0<\alpha,$$

with $S=[1, N]$. (See [3], Ch. IV.)

3. Let finally $q_j \leqq N^{\Delta_4}$ for $j \geqq 2$ and $q_1 \geqq N^{2l\Delta_4}$. Then for $1 \leqq k \leqq M$ we have

$$\left|F(k, \alpha_1, \ldots, \alpha_l) - F\left(k, \frac{a_1}{q_1}, \ldots, \frac{a_l}{q_l}\right)\right| \leqq |\psi_1| M + \ldots + |\psi_l| M^l \leqq$$

$$\leqq \frac{1}{q_1}\frac{M}{\tau_1} + \frac{M^2}{\tau_2} + \ldots + \frac{M^l}{\tau_l} \leqq \varepsilon,$$

whenever

$$M = o(1)\min\left(N^{1-\varkappa/2}, N^{1-\varkappa+2l\Delta_4}\right).$$

Since in this case $Q_1 \leqq N^{l\Delta_4}$, $q_1 \leqq N^{1-\varkappa}$ so choosing

$$M_1 = \left[\frac{M}{Q_1}\right], \quad S = \{Q_1, 2Q_1, \ldots, M_1 Q_1\}$$

we have

$$\left|T\left(S, \frac{a_1}{q_1}, \ldots, \frac{a_l}{q_l}\right)\right| = \left|\sum_{k=1}^{M_1} e^{2\pi i F\left(Q_1 k, \frac{a_1}{q_1}, \ldots, \frac{a_l}{q_l}\right)}\right| =$$

$$= \left|\sum_{k=1}^{M_1} e^{2\pi i k \frac{a_1 Q_1}{q_1}}\right| \leqq q_1 \leqq N^{1-\varkappa} = o(|S|)$$

and $|S| \gg N^c$.

So the relation (2. 15) holds.

Estimation of $\psi_N(\alpha_1, \ldots, \alpha_l)$ *in the case* $(\alpha_1, \ldots, \alpha_l) \in \mathcal{A}_1$. Let

$$f(x) = \sum_{j=1}^{l} \beta_j x^j \quad \text{i. e.} \quad F(x, \alpha_1, \ldots, \alpha_l) = \frac{1}{\sqrt{N}} f\left(\frac{x}{N}\right).$$

From the definition of $\mathscr{A}_1$

$$\max_{k=1,\dots,N}\left|\frac{1}{\sqrt{N}} f\left(\frac{k}{N}\right)\right| = o(1) \qquad (N\to\infty)$$

follows. Hence

$$\psi_N(\alpha_1,\dots,\alpha_l) = 2^{2N}\exp(-V)$$

where

$$V = -\sum_{k=1}^{N}\sin^2\frac{\pi}{\sqrt{N}} f\left(\frac{k}{N}\right) + O\left(\sum_{k=1}^{N}\sin^4\frac{\pi}{\sqrt{N}} f\left(\frac{k}{N}\right)\right).$$

Furthermore

$$V = -\frac{\pi^2}{N}\sum_{k=1}^{N} f^2\left(\frac{k}{N}\right) + O(R) = -U + O(R),$$

$$R = \frac{1}{N^2}\sum_{k=1}^{N} f^4\left(\frac{k}{N}\right).$$

Since

$$\sum_{k=1}^{N} f^2\left(\frac{k}{N}\right) = \sum_{\nu,\mu=1}^{l}\frac{\beta_\nu\beta_\mu}{N^{\nu+\mu}}\sum_{k=1}^{N}k^{\nu+\mu} = NQ(\beta_1,\dots,\beta_l) + O\big(Q(|\beta_1|,\dots,|\beta_l|)\big)$$

we have

$$U = \pi^2 Q(\beta_1,\dots,\beta_l) + O\left(\frac{1}{N}Q(|\beta_1|,\dots,|\beta_l|)\right).$$

Further

$$|R| \leqq \frac{\max|\beta_j|^2}{N}\frac{1}{N}\sum_{k=1}^{N} f^2\left(\frac{k}{N}\right) \ll N^{-2\Delta_1}Q(\beta_1,\dots,\beta_l).$$

Hence for $\mathscr{B}\subset\mathscr{A}_1$

$$\mathscr{I}(\mathscr{B}) = 2^{2N}N^{-\frac{l(l+2)}{2}}\int\limits_{\mathscr{B}}\dots\int \exp\big(-2\pi i(\textstyle\sum\alpha_j v_j)\big)\exp(-H)\,d\beta_1\dots d\beta_l$$

follows, where

$$H = -\pi^2 Q(\beta_1,\dots,\beta_l)\big(1+Q(N^{-2\Delta_1})\big).$$

Let

$$\varrho(\mathscr{A}) = \int\limits_{\mathscr{A}}\dots\int \exp\big(-2\pi i(\textstyle\sum\alpha_j v_j) - \pi^2 Q(\beta_1,\dots,\beta_l)\big)\,d\beta_1\dots d\beta_l.$$

Let $\mathscr{A}(\Delta)$ denote the set of those $(\alpha_1,\dots,\alpha_l)$ for which $\max_{j=1,\dots,l}|\beta_j| < N^{\Delta}$. Suppose that Δ is a sufficiently small positive constant.

Since

$$\text{(2.17)}\qquad \int\limits_{\max|\beta_j|>N^{\Delta}}\dots\int \exp\left(-\frac{\pi^2}{2}Q(\beta_1,\dots,\beta_l)\right)d\beta_1\dots d\beta_l \ll \exp(-N^{\Delta/2})$$

and

$$\text{(2.18)}\qquad \int\limits_{\mathscr{A}(\Delta)}\dots\int \exp\big(-Q(\beta_1\dots\beta_l)\pi^2\big)O\big(N^{-c}Q(\beta_1,\dots,\beta_l)\big)\,d\beta_1\dots d\beta_l \ll$$

$$\ll N^{2(\Delta-\Delta_1)} \ll N^{-\Delta_1} \qquad (\Delta<\Delta_1/2)$$

we have

$$\mathcal{I}(\mathcal{B}) = 2^{2N} N^{-\frac{l(l+2)}{2}} \{\varrho(\mathcal{B} \cap \mathcal{A}(\Delta)) + O(N^{-\Delta_1})\}, \tag{2.19}$$

when $\mathcal{B} \subset \mathcal{A}_1$,

Further, from (2. 17)

$$\varrho(\mathcal{A}(\Delta)) = \sigma_N(v_1, \ldots, v_l) + O(\exp(-N^{\Delta/2})).$$

If $\mathfrak{M} \in \mathcal{T}$, then

$$\mathcal{I}(\mathfrak{M}) = \exp\left(-2\pi i\left(\sum \frac{a_j}{q_j} v_j\right)\right) \mathcal{I}(\mathfrak{M}^*), \tag{2.20}$$

where $\mathfrak{M}^*$ denotes the set of those $(\psi_1, \ldots, \psi_l)$ for which $|\psi_j| \leqq \frac{1}{q_j \tau_j}$ $(j=1, \ldots, l)$.

On the domain $\mathfrak{M}^*$ the same estimations hold as on $\mathcal{A}_1$. Further $\mathcal{A}(\Delta) \subset \subset \mathfrak{M}^* \subset \mathcal{A}_1$ and so

$$\mathcal{I}(\mathfrak{M}^*) = \mathcal{I}(\mathcal{A}(\Delta)) + O(2^{2N} \exp(-N^c)), \quad c > 0. \tag{2.21}$$

Hence

$$\sum_{\mathfrak{M} \in \mathcal{T}} \mathcal{I}(\mathfrak{M}) = 2^{2N} N^{-\frac{l(l+2)}{2}} \left\{\sum_{\mathfrak{M} \in \mathcal{T}} \exp\left(-\left(\sum \frac{a_j}{q_j} v_j\right) 2\pi i\right) \sigma_N(v_1, \ldots, v_l) + O(N^{-c})\right\}.$$

Taking into account the inequality (2. 15) our theorem follows.

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SOME FURTHER ESTIMATES CONCERNING SUMS OF POWERS OF COMPLEX NUMBERS

By

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1. Introduction

Let $z_1, \ldots, z_n$ be complex numbers, and wiite

$$s_k = \sum_{r=1}^{n} z_r^k \tag{1.1}$$

for their power sums. TURÁN [1, 2] has, along with many related problems, considered that of investigating

$$M_n = \min_{z_1, \ldots, z_n} \max (|s_1|, \ldots, |s_n|), \tag{1.2}$$

under the constraint that

$$\max (|z_1|, \ldots, |z_n|) = 1. \tag{1.3}$$

The problem is relevant to certain techniques for the numerical solution of algebraic equations [2, 4], and has recently been applied in connection with the maximum modulus theorem for a Banach space [6]. For further related work I cite [7].

It was shown in [8] that $M_n > 1/6$ for all n; earlier results [9] gave lower bounds for M_n which tended to zero as $n \to \infty$. In a technical report [10] it was shown that this lower bound could be raised to 1/3.

The purpose of the present paper is to give an improved version of this latter work. We show that the lower bound $\pi/8$ is valid for a large range of values of n; this bound is good at least for $n < 1{,}6 \cdot 10^3$. For large values of n, we obtain a lesser lower bound, the root of a certain transcendental equation.

2. Notation and conventions. We make use of the following functions:

$$g(\theta) = \sum_{1}^{n} m^{-1} s_m e^{mi\theta}, \tag{2.1}$$

$$h(\theta) = \sum_{n+1}^{\infty} m^{-1} e^{mi\theta}, \qquad -\pi \leqq \theta \leqq \pi, \quad \theta \neq 0, \tag{2.2}$$

$$\psi(\theta) = \sum_{1}^{n} s_m m^{-1}(1 - e^{mi\theta}) = g(\theta) - g(0), \tag{2.3}$$

$$\chi(\theta) = \sum_{1}^{n} 2m^{-1} |\sin \tfrac{1}{2} m\theta|, \tag{2.4}$$

$$\operatorname{Si} x = \int_0^x y^{-1} \sin y \, dy, \tag{2.5}$$

$$T(x) = \int_0^x y^{-1} |\sin y| \, dy. \tag{2.6}$$

The notation (2. 5) for the integral-sine is standard. We have in particular $\mathrm{Si}\,\pi = 1{,}8519\ldots$. Of course, $\mathrm{Si}\,x = T(x)$ for $0 \leqq x \leqq \pi$.

We write

$$s = \max\{|s_1|, \ldots, |s_n|\}, \tag{2. 7}$$

and will assume that $s < \frac{1}{2}$; our lower bounds for s, subject to (1. 3), will be less than $\frac{1}{2}$, such as $\pi/8$.

In place of (1. 3), it will be sufficient to consider the case that

$$z_1 = 1. \tag{2. 8}$$

We shall assume that $n \geqq 3$; the case $n = 2$ is explicitly soluble.

3. The basic inequality. Starting, as in many previous investigations, with the identity

$$\exp\left(-\sum_1^\infty m^{-1} s_m y^m\right) = \prod_1^n (1 - z_r y),$$

valid for small y, we proceed as in (8) to the result

$$e^{g(\theta)} = \prod_1^n (1 - z_r e^{i\theta}) + \sum_{n+1}^\infty c_m e^{mi\theta}. \tag{3. 1}$$

Here the Fourier coefficients c_m are given by

$$c_m = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-mi\theta + g(\theta)}\, d\theta, \qquad m > n. \tag{3. 2}$$

Putting $\theta = 0$ in (3. 1), we have by (2. 8) that

$$e^{g(0)} = \sum_{n+1}^\infty c_m. \tag{3. 3}$$

As in (8), the argument consists in showing that if s is too small, then the c_m are so small that (3. 3) cannot hold.

By a partial integration in (3. 2) we have that

$$c_m = (2\pi m i)^{-1} \int_{-\pi}^{\pi} e^{-mi\theta} g'(\theta) e^{g(\theta)}\, d\theta,$$

and so

$$c_m = (2\pi i)^{-1} \int_{-\pi}^{\pi} h(\theta) g'(\theta) e^{g(\theta)}.$$

On combining this with (3. 3) we have

$$1 = (2\pi i)^{-1} \int_{-\pi}^{\pi} h(\theta) g'(\theta) e^{\psi(\theta)}\, d\theta. \tag{3. 4}$$

Here we note that, by orthogonality,

$$\int_{-\pi}^{\pi} g'(\theta)h(\theta)\,d\theta = 0,$$

so that (3. 4) may be replaced by

$$1 = (2\pi i)^{-1} \int_{-\pi}^{\pi} g'(\theta)h(\theta)\,\{e^{\psi(\theta)}-1\}\,d\theta.$$

At this point we apply the Bunjakowski—Schwarz inequality, to give

$$(3.5)\qquad 1 \leqq (2\pi)^{-1} \int_{-\pi}^{\pi} |g'(\theta)|^2\,d\theta \cdot (2\pi)^{-1} \int_{-\pi}^{\pi} |h(\theta)\,\{e^{\psi(\theta)}-1\}|^2\,d\theta.$$

However,

$$\int_{-\pi}^{\pi} |g'(\theta)|^2\,d\theta = 2\pi \sum_{1}^{n} |s_m|^2 \leqq 2\pi n s^2.$$

Hence from (3. 5) we get

$$(3.6)\qquad 1 \leqq \frac{ns^2}{2\pi} \int_{-\pi}^{\pi} |e^{\psi(\theta)}-1|^2\,|h(\theta)|^2\,d\theta.$$

It is clear that this implies that s is bounded below from zero. We proceed to make this remark more precise by estimating the integrand in (3. 6).

4. Estimation of $h(\theta)$. We prove first

LEMMA 1. *For* $0<\theta\leqq\pi$,

$$(4.1)\qquad |h(\theta)| \leqq (2n+1)^{-1}\,(\sin\tfrac{1}{2}\theta)^{-1}.$$

We have in fact

$$h(\theta) = i\sum_{n+1}^{\infty} \int_{\theta}^{\theta-i\infty} e^{-mi\varphi}\,d\varphi = \int_{0}^{\infty} \frac{e^{-(n+1)i\theta-(n+1)t}}{1-e^{-i\theta-t}}\,dt$$

and so

$$(4.2)\qquad |h(\theta)| = \left|\int_{0}^{\infty} e^{-(n+\frac{1}{2})t}\{2\sin\tfrac{1}{2}(\theta-it)\}^{-1}\,dt\right|.$$

Here we note that

$$|\sin\tfrac{1}{2}(\theta-it)|^2 = \sin^2\tfrac{1}{2}\theta+\sinh^2\tfrac{1}{2}t \geqq \sin^2\tfrac{1}{2}\theta,$$

and so (4. 2) yields

$$|h(\theta)| \leqq (2\sin\tfrac{1}{2}\theta)^{-1}\int_{0}^{\infty} e^{-(n+\frac{1}{2})t}\,dt,$$

from which (4. 1) follows.

We need also

LEMMA 2. *For α, such that $0<\alpha\leqq\frac{1}{2}\pi$, we have*

$$(4.3)\qquad |h(\theta)|^2 \leqq (n+\tfrac{1}{2})^{-2}(\theta^{-2}+4\operatorname{cosec}^2\tfrac{1}{2}\alpha),\quad 0<\theta\leqq\alpha,$$

$$(4.4)\qquad |h(\theta)|^2 \leqq (2n+1)^{-2}\operatorname{cosec}^2\tfrac{1}{2}\alpha,\quad \alpha\leqq\theta\leqq\pi.$$

In the result

$$\operatorname{cosec}^2\tfrac{1}{2}\theta-4\theta^{-2} = 4\sum_{1}^{\infty}\{(2q\pi-\theta)^{-2}+(2q\pi+\theta)^{-2}\}$$

we note that the right-hand side increases as θ increases in $(0,\frac{1}{2}\pi)$. Thus, for $0<\theta\leqq\alpha$,

$$\operatorname{cosec}^2\tfrac{1}{2}\theta-4\theta^{-2}\leqq\operatorname{cosec}^2\tfrac{1}{2}\alpha-4\alpha^{-2}\leqq\operatorname{cosec}^2\tfrac{1}{2}\alpha,$$

and so (4. 3) follows from (4. 1).

We get (4. 4) from (4. 1) on noting that $\operatorname{cosec}\frac{1}{2}\theta$ decreases as θ increases in $\alpha\leqq\theta\leqq\pi$.

Since $h(\theta)$ is an even function of θ, we have similar results to (4. 1), (4. 3—4) for negative θ. In stating that $\alpha\leqq\frac{1}{2}\pi$, we have taken the greatest range of α-values of interest to us.

5. Second form of basic inequality. We apply these bounds for $h(\theta)$ in (3. 6), for some α, $0<\alpha\leqq\frac{1}{2}\pi$, to be chosen later, we apply (4. 3) over $(0,\alpha)$, and similarly over $(-\alpha,0)$. We apply also (4. 4) over (α,π), and likewise over $(-\pi,-\alpha)$. Combining the results we get

$$(5.1)\quad 1\leqq\frac{ns^2}{2\pi(n+\frac{1}{2})^2}\int_{-\alpha}^{\alpha}|e^{\psi(\theta)}-1|^2\theta^{-2}\,d\theta+\frac{ns^2}{2\pi(n+\frac{1}{2})^2\,4\sin^2\frac{1}{2}\alpha}\int_{-\pi}^{\pi}|e^{\psi(\theta)}-1|^2\,d\theta.$$

In the first term on the right, we use a pointwise bound for $\psi(\theta)$, valid for all $s_1,\dots,s_n$ satisfying $|s_r|\leqq s$, $r=1,\dots,n$. This is

$$|\psi(\theta)|\leqq s\chi(\theta),$$

from which it follows that

$$|e^{\psi(\theta)}-1|\leqq e^{s\chi(\theta)}-1.$$

Thus the first term on the right of (5. 1) does not exceed

$$(5.2)\qquad I_1=\frac{ns^2}{\pi(n+\frac{1}{2})^2}\int_0^{\alpha}|e^{s\chi(\theta)}-1|^2\theta^{-2}\,d\theta;$$

here we have used the fact that $\chi(\theta)$ is an even function of θ.

We denote the second term on the right of (5. 1) by I_2. As the second form of the basic inequality we have then

$$(5.3)\qquad 1\leqq I_1+I_2.$$

The next two sections are devoted to estimations in respect of I_1, by means of certain bounds for sums. We then use a different method to estimate I_2, and will then, in § 9, obtain a more explicit, though weaker, form of the basic inequality.

6. Some integral inequalities. In the next stage we find an upper bound for the sum (2. 4) in terms of the corresponding integral. We have to deal with a function, namely x^{-} $|\sin x|$, which is infinitely differentiable, except at certain points, namely multiples of π, and have to consider it over intervals which may contain one of these points, or may not. For the latter event we have

LEMMA 3. *In the real interval $a \leqq x \leqq b$ let the real-valued function $f(x)$ be continuously four times differentiable. Then*

$$\frac{1}{2}(b-a)\{f(a)+f(b)\} \leqq \int_a^b f(x)\,dx + 12^{-1}(b-a)^2\{f'(b)-f'(a)\} + \\ + 384^{-1}(b-a)^4 \int_a^b \max\{-f^{(iv)}(x), 0\}\,dx. \tag{6. 1}$$

The left of (6. 1) is the same as

$$\int_a^b f(x)\,dx + \int_a^b \{x-\tfrac{1}{2}(b+a)\} f'(x)\,dx. \tag{6. 2}$$

By integration by parts, the last integral is found to be

$$[\{\tfrac{1}{2}(x-\tfrac{1}{2}(a+b))^2 - 24^{-1}(b-a)^2\} f'(x)]_a^b - \\ - \int_a^b \{\tfrac{1}{2}(x-\tfrac{1}{2}(a+b))^2 - 24^{-1}(b-a)^2\} f''(x)\,dx. \tag{6. 3}$$

Here the first, integrated, term yields the second term on the right of (6. 1). In the second term in (6. 3) we integrate by parts again, getting

$$-[\{6^{-1}(x-\tfrac{1}{2}(a+b))^3 - 24^{-1}(x-\tfrac{1}{2}(a+b))(b-a)^2\} f''(x)]_a^b + \\ + \int_a^b \{6^{-1}(x-\tfrac{1}{2}(a+b))^3 - 24^{-1}(x-\tfrac{1}{2}(a+b))(b-a)^2\} f'''(x)\,dx. \tag{6. 4}$$

Here the first term vanishes. In the second, we integrate by parts again, getting

$$[24^{-1}(x-a)^2(x-b)^2 f'''(x)]_a^b - \int_a^b 24^{-1}(x-a)^2(x-b)^2 f^{(iv)}(x)\,dx. \tag{6. 5}$$

Here the integrated term vanishes. We get the result of the lemma on noting that, for $a \leqq x \leqq b$,

$$0 \leqq 24^{-1}(x-a)^2(x-b)^2 \leqq 384^{-1}(b-a)^4. \tag{6. 6}$$

We give next the modifications required to deal with the event that $x^{-1}|\sin x|$ has, in the interval concerned, a point at which its derivatives are discontinous; the function itself will still be continuous. We have

LEMMA 4. *For some c, $a<c<b$, let $f(x)$ be continuously four times differentiable in $a\leqq x\leqq c$, $c\leqq x\leqq b$, with one-sided derivatives at $x=c$. Let*

$$\begin{cases} f(c+0) = f(c-0),\ f'(c+0) = -f'(c-0)>0, \\ f''(c+0) = -f''(c-0)<0,\ f(c+0) = -f(c-0)<0. \end{cases} \tag{6. 7}$$

Then (6. 1) *remains in force, subject to the addition on right of the supplementary terms*

$$12^{-1}(b-a)^2 f'(c+0)+36^{-1}3^{-\frac{1}{2}}(b-a)^3\,|f''(c+0)|+192^{-1}(b-a)^4\,|f'''(c+0)|. \tag{6. 8}$$

For this case, (6. 2) is unaffected, since $f(x)$ is continuous at $x=c$. However the integrated terms in (6. 3—5) must now be supplemented by, altogether,

$$-\{\tfrac{1}{2}(c-\tfrac{1}{2}(a+b)^2-24^{-1}(b-a)^2\}\{f'(c+0)-f'(c-0)\}+$$
$$+\{6^{-1}(c-\tfrac{1}{2}(a+b))^3-24^{-1}(c-\tfrac{1}{2}(a+b)(b-a)^2\}\{f''(c+0)-f''(c-0)\}-$$
$$-24^{-1}(c-a)^2(c-b)^2\{f'''(c+0)-f'''(c-0)\}.$$

These are collectively bounded above by the terms in (6. 8), as we see on using the bound (6. 6), together with the bounds, for $a<c<b$,

$$(c-\tfrac{1}{2}(a+b))^2-24^{-1}(b-a)^2\geqq -24^{-1}(b-a)^2,$$
$$6^{-1}(c-\tfrac{1}{2}(a+b))^3-24^{-1}(c-\tfrac{1}{2}(a+b))(b-a)^2\geqq -72^{-1}3^{-\frac{1}{2}}(b-a)^3.$$

7. Estimation of $\chi(\theta)$. We now apply these last lemmas in the case $f(x)=$ $=x^{-1}|\sin x|$, $x>0$, $f(0)=1$, with, as the intervals (a, b), the sequence of intervals

$$(0, \tfrac{1}{2}\theta),\ \ldots,\ (\tfrac{1}{2}n\theta, \tfrac{1}{2}(n+1)\theta). \tag{7. 1}$$

We apply Lemma 3 if no multiple of π occurs in the interior of one of these intervals, and Lemma 4 otherwise. However, we may as well include the supplementary terms in Lemma 4 even when a multiple of π occurs at an end-point of one of these intervals, without essential loss of precision. It is easily verified that the conditions (6. 7) are satisfied; the derivatives will be calculated later.

Applying Lemma 3 or 4 as the case may be to the intervals (7. 1), and adding the results, we get

$$\tfrac{1}{4}\theta\left\{f(0)+2\sum_1^n f(\tfrac{1}{2}m\theta)+f(\tfrac{1}{2}(n+1)\theta)\right\}\leqq \tag{7. 2}$$
$$\leqq\int_0^{(n+2)\theta/2} f(x)\,dx+\frac{\theta^2}{48}\{f'(\tfrac{1}{2}(n+1)\theta-0)-f'(0)\}+$$
$$+\sum_{0<q\pi<(n+1)\theta/2}\left\{\frac{\theta^2}{48}f'(q\pi+0)+\frac{\theta^3}{288\sqrt{3}}\,|f''(q\pi+0)|+\frac{\theta^4}{3072}\,|f'''(q\pi+0)|\right\}+$$
$$+\frac{\theta^4}{6144}\int_0^{(n+1)\theta/2}\max\{-f^{(\mathrm{iv})}(x), 0\}\,dx.$$

In clarifying this we use the facts that

$$f(0)=1,\quad f\left(\tfrac{1}{2}(n+1)\theta\right)\geqq 0,\qquad f'(x)\leqq x^{-1},\quad (x>0),$$

$$f'(0)=0,\quad f'(q\pi+0)=(q\pi)^{-1},\qquad f''(q\pi+0)=-2(q\pi)^{-2},$$

and finally

$$f'''(q\pi+0)=-\{(q\pi)^{-1}-6\left((q\pi)^{-3}\right)\}>-(q\pi)^{-1}.$$

By means of these we deduce from (7. 2) that

$$\sum_{1}^{n} m^{-1}|\sin\tfrac{1}{2}m\theta|\leqq\int_{0}^{(n+1)\theta/2} f(x)\,dx-\tfrac{1}{4}\theta+\frac{\theta}{24(n+1)}+ \tag{7. 3}$$

$$+\frac{\theta^2}{48}\sum_{0<q\pi<(n+1)\theta/2}\left\{(q\pi)^{-1}+\frac{\theta}{3^{3/2}}(q\pi)^{-2}+\frac{\theta^2}{64}(q\pi)^{-1}\right\}+$$

$$+\frac{\theta^4}{6144}\int_{0}^{(n+1)\theta/2}\max\{-f^{(\mathrm{iv})}(x),0\}\,dx.$$

Let us now dispose of the last term. For this we note that

$$\int_{0}^{\infty}\max\{-f^{(\mathrm{iv})}(x),0\}\,dx<2. \tag{7. 4}$$

If $0\leqq x\leqq\pi$ we use the expression

$$f^{(\mathrm{iv})}(x)=1/5-x^2(2!\cdot 7)+x^4(4!\cdot 9)-\dots,$$

from which we may see that $f^{(\mathrm{iv})}(x)>0$ when $0\leqq x\leqq\sqrt{14/5}$, and so in particular for $0\leqq x\leqq\frac{1}{2}\pi$. For $\frac{1}{2}\pi\leqq x\leqq\pi$ we use the expression

$$f^{(\mathrm{iv})}(x)=(x^{-1}-12x^{-3}+24x^{-5})\sin x-(4x^{-2}-24x^{-4})\cos x. \tag{7. 5}$$

Here the coefficient of $\sin x$ is positive, except when $\sqrt{6-\sqrt{12}}<x<\sqrt{6+\sqrt{12}}$, when it satisfies

$$0\geqq x^{-1}-12x^{-3}+24x^{-5}\geqq-12x^{-5}.$$

Also, the coefficient $-(4x^{-2}-24x^{-4})$ of $\cos x$ is positive in $(\frac{1}{2}\pi,\pi)$, except when $x\geqq\sqrt{6}$. Hence we have

$$\int_{0}^{\pi}\max\{-f^{(\mathrm{iv})}(x),0\}\,dx\leqq\int_{\sqrt{6-\sqrt{12}}}^{\sqrt{6+\sqrt{12}}}12x^{-5}\,dx+\int_{\sqrt{6}}^{\pi}(4x^{-2}-24^{-4})\,dx.$$

In intervals of the form $2k\pi\leqq x\leqq(2k+1)\pi$, $k\geqq 1$, we may still use (7. 5), where now only the term in $\cos x$ can become negative. We thus get

$$\int_{2k\pi}^{(2k+1)\pi}\max\{-f^{(\mathrm{iv})}(x),0\}\,dx\leqq\int_{2k\pi}^{(2k+1)\pi}(4x^{-2}-24x^{-4})\,dx.$$

We get a similar bound for intervals of the form $(2k-1)\pi \leqq x \leqq 2k\pi$, noting that here the expression (7. 5) must be reserved in sign. Thus altogether the left of (7. 4) is bounded above by

$$\int_{\sqrt{6-\sqrt{12}}}^{\sqrt{6+\sqrt{12}}} 12x^{-5}\,dx + \int_{\sqrt{6}}^{\infty} (4x^{-2} - 24x^{-4})\,dx,$$

and this is easily seen to be less than 2.

We deal now with the case that the sum on the right of (7. 3) is empty. We have

LEMMA 5. *If*

$$0 \leqq \theta \leqq 2\pi/(n+1), \tag{7. 6}$$

then

$$\sum_1^n m^{-1} |\sin \tfrac{1}{2} m\theta| \leqq \operatorname{Si}\{\tfrac{1}{2}(n+1)\theta\}. \tag{7. 7}$$

In this case (7. 3) reduces to

$$\sum_1^n m^{-1} |\sin \tfrac{1}{2} m\theta| \leqq \int_0^{(n+1)\theta/2} f(x)\,dx - \tfrac{1}{4}\theta + \frac{\theta}{24(n+1)} + \frac{\theta^4}{3072}.$$

However (7. 6) implies that $\theta \leqq \frac{1}{2}\pi$, since we may assume that $n \geqq 3$, and so

$$\frac{\theta}{24(n+1)} + \frac{\theta^4}{3072} \leqq \tfrac{1}{4}\theta,$$

which proves (7. 7).

In the event that the sum in (7. 3) is not empty, we use the bounds

$$\sum_{1 \leqq q \leqq (n+1)\theta/(2\pi)} q^{-1} \leqq \log \frac{(n+1)\theta}{2\pi} + \frac{2\pi}{(n+1)\theta} + \eta,$$

and

$$\sum_{1 \leqq q \leqq (n+1)\theta/(2\pi)} q^{-2} < \pi^2/6,$$

where $\eta = 0{,}577215\ldots$ is Euler's constant. These give from (7. 3) that

$$\sum_1^n m^{-1} |\sin \tfrac{1}{2} m\theta| \leqq \int_0^{(n+1)\theta/2} f(x)\,dx - \tfrac{1}{4}\theta + \frac{\theta}{24(n+1)} + \tag{7.8}$$

$$+ \frac{\theta^2}{48\pi}\left(1 + \frac{\theta^2}{64}\right)\left\{\log \frac{(n+1)\theta}{2\pi} + \frac{2\pi}{(n+1)\theta} + \eta\right\} + \frac{\theta^3}{868\sqrt{3}} + \frac{\theta^4}{3072}.$$

In simplified form, we prove

LEMMA 6. *If*

$$2\pi/(n+1) \leqq \theta \leqq \tfrac{1}{2}\pi, \tag{7. 9}$$

then

(7. 10) $$\sum_{1}^{n} m^{-1}|\sin \tfrac{1}{2}m\theta| \leqq \int\limits_{0}^{(n+1)\theta/2} f(x)\,dx - \frac{\theta}{5} + \frac{\theta^2}{46\pi}\log\frac{(n+1)\theta}{2\pi}.$$

Since $\theta \leqq \frac{1}{2}\pi$, we have $(\theta^2/48)(1+\theta^2/64) \leqq \theta^2/46$, and so the right of (7. 8) is seen to be less than

$$\int\limits_{0}^{(n+1)\theta/2} f(x)\,dx - \tfrac{1}{4}\theta + \frac{\theta^2}{46\pi}\log\frac{(n+1)\theta}{2\pi} +$$
$$+\frac{\theta}{24(n+1)} + \frac{\theta}{23(n+1)} + \frac{\theta^2}{46\pi} + \frac{\theta^3}{864\sqrt{3}} + \frac{\theta^4}{3072}.$$

Again since $\theta \leqq \frac{1}{2}\pi$, the last five terms are less than

$$\theta\left\{\frac{2}{23(n+1)} + \frac{\eta}{92} + \frac{\pi^2}{3472\sqrt{3}} + \frac{\pi^3}{24576}\right\} < \frac{\theta}{20}.$$

This proves (7. 10).

We have the following specialisations.

LEMMA 7. *Let*

(7. 11) $$2\pi/(n+1) \leqq \theta \leqq \min\left\{\frac{46\pi}{5\log\{\frac{1}{4}(n+1)\}}, \tfrac{1}{2}\pi\right\}.$$

Then the last two terms in (7. 10) *can be omitted.*

It is a question of showing that the last two terms in (7. 10) are, in sum, negative or zero when θ satisfies (7. 11). We thus have to verify that

$$\frac{\theta}{46\pi}\log\frac{(n+1)\theta}{2\pi} < \frac{1}{5},$$

when θ satisfies (7. 11). Since $\theta \leqq \frac{1}{2}\pi$, the left does not exceed

$$\frac{\theta}{46\pi}\log\frac{n+1}{4},$$

and this is less than 1/5 by the other bound for θ in (7. 11).

In particular,

LEMMA 8. *If* $0 < \theta \leqq \frac{1}{2}\pi$, *and* $n < 3{,}9\cdot 10^8$, *the bound* (7. 10) *holds, with the last two terms omitted.*

We have (7. 7) for θ subject to (7. 6) for all $n \geqq 3$. It is a question of showing that the right of (7. 11) is $\frac{1}{2}\pi$ if $n < 3{,}9\cdot 10^8$, or that for such n we have

$$\frac{46\pi}{5\log\frac{1}{4}(n+1)} > \tfrac{1}{2}\pi.$$

This is easily checked.

8. Estimation of I_2. For this, the second term in (5. 1), we use a different method. We have

LEMMA 9. *For* $0<s<\frac{1}{2}$,

$$(8.1)\qquad (2\pi)^{-1}\int_{-\pi}^{\pi}|e^{\psi(\theta)}-1|^2\,d\theta \leqq (n+1)^{2s}e^{2\eta s}\frac{\Gamma(1-2s)}{\{\Gamma(1-s)\}^2}.$$

For the left-hand side does not exceed

$$\max\,(2\pi)^{-1}\int_{-\pi}^{\pi}\left|\exp\left\{\sum_1^n m^{-1}s_m(1-e^{mi\theta})\right\}-1\right|^2 d\theta,$$

where the s_m range over $|s_m|\leqq s$, $1\leqq m\leqq n$. This in turn does not exceed

$$(8.2)\qquad \max\,(2\pi)^{-1}\int_{-\pi}^{\pi}\left|\exp\left\{\sum_1^n m^{-1}(s_m+t_m e^{mi\theta})\right\}-1\right|^2 d\theta,$$

where the s_m, t_m range over $|s_m|\leqq s$, $|t_m|\leqq s$, $m=1,\ldots,n$. Let us write now

$$\exp\sum_1^n m^{-1}(s_m+t_m e^{mi\theta}) = \sum_0^\infty d_r e^{ri\theta},$$

where the Fourier coefficients d_r are independent of θ, and are expressible as power series in the s_m, t_m. Then (8.2) is equal to

$$(8.3)\qquad |d_0-1|^2+\sum_1^\infty |d_r|^2.$$

We now remark that in the expression of d_0-1 and $d_1, d_2, \ldots,$ as power series in the s_m, t_m, the coefficients are all real and non-negative. Thus (8. 3) will be maximised by putting $s_m=t_m=s$, $m=1,\ldots,n$.

For simplicity, we may replace d_0-1 in this maximum by d_0, at the cost of increasing the maximum. Thus we find that (8. 2) is less than

$$(8.4)\qquad (2\pi)^{-1}\int_{-\pi}^{\pi}\left|\exp\left\{s\sum_1^n m^{-1}(1+e^{mi\theta})\right\}\right|^2 d\theta =$$

$$=\exp\left\{2s\sum_1^n m^{-1}\right\}(2\pi)^{-1}\int_{-\pi}^{\pi}\left|\exp\left\{s\sum_1^n m^{-1}e^{mi\theta}\right\}\right|^2 d\theta.$$

Here we use the bound

$$\sum_1^n m^{-1} < \log(n+1)+\eta.$$

We also take the sum under the integral sign on the right from 1 to ∞; a Fourier argument, similar to that of (8. 2—3), shows that this increases the expression on the right of (8. 4). Thus (8. 2) is less than

$$(n+1)^{2s}e^{2\eta s}(2\pi)^{-1}\int_{-\pi}^{\pi}\left|\exp\left\{s\sum_{1}^{\infty}m^{-1}e^{mi\theta}\right\}\right|^{2}d\theta =$$

$$= (n+1)^{2s}e^{2\eta s}(2\pi)^{-1}\int_{-\pi}^{\pi}|1-e^{i\theta}|^{-2s}\,d\theta,$$

which gives the right of (8. 1). This completes the proof.

9. Third form of basic inequality. We now apply the bounds obtained in the last two sections to transform (5. 3). We start by estimating I_1 on the basis of the results of § 7. We take it that

$$2\pi/(n+1)\leqq\alpha\leqq\tfrac{1}{2}\pi. \tag{9. 1}$$

In I_1, as given in (5. 2), we write I_{11} for the contribution of the interval $(0, 2\pi/(n+1))$, and I_{12} for the contribution of $(2\pi/(n+1), \alpha)$.

From Lemma 5 we have

$$I_{11}\leqq\frac{ns^2}{\pi(n+\frac{1}{2})^2}\int_{0}^{2\pi/(n+1)}|\exp\{2s\,\mathrm{Si}\,(\tfrac{1}{2}(n+1)\theta)\}-1|^2\,\theta^{-2}\,d\theta\leqq \tag{9. 2}$$

$$\leqq\frac{ns^2(n+1)}{2\pi(n+\frac{1}{2})^2}\int_{0}^{\pi}|e^{2s\,\mathrm{Si}\,\varphi}-1|^2\,\varphi^{-2}\,d\varphi\leqq\frac{s^2}{2\pi}\int_{0}^{\pi}(e^{2s\,\mathrm{Si}\,\varphi}-1)^2\,\varphi^{-2}\,d\varphi.$$

In I_2 we use Lemma 7, or Lemma 8. We take

$$\alpha=\min\left\{\frac{46\pi}{5\log\frac{1}{4}(n+1)},\,\tfrac{1}{2}\pi\right\}, \tag{9. 3}$$

so that $\alpha=\frac{1}{2}\pi$ if $n<3{,}9\cdot10^8$. Then Lemma 7 gives, since here $T(x)=\int_{0}^{x}f(y)\,dy$,

$$I_{12}\leqq\frac{ns^2}{\pi(n+\frac{1}{2})^2}\int_{2\pi/(n+1)}^{\alpha}|\exp\{2sT(\tfrac{1}{2}(n+1)\theta)\}-1|^2\,\theta^{-2}\,d\theta\leqq \tag{9. 4}$$

$$\leqq\frac{s^2}{2\pi}\int_{\pi}^{(n+1)\alpha/2}(e^{2sT(\varphi)}-1)^2\,\varphi^{-2}\,d\varphi.$$

In I_2, the last term in (5. 1), we use Lemma 9. This gives

$$I_2\leqq\frac{ns^2}{(n+\frac{1}{2})^2\,4\sin^2\frac{1}{2}\alpha}(n+1)^{2s}e^{2\eta s}\frac{\Gamma(1-2s)}{\{\Gamma(1-s)\}^2}\leqq\frac{s^2(n+1)^{2s-1}e^{2\eta s}\Gamma(1-2s)}{4\sin^2\frac{1}{2}\alpha\{\Gamma(1-s)\}^2}. \tag{9. 5}$$

We sum this up in the inequality

$$1 \leqq J_1 + J_2 + J_3, \tag{9. 6}$$

where J_1 is given in (9. 2), J_2 in (9. 4), and J_3 in (9. 5). This is our third version of the basic inequality.

At this point we can see, roughly, what can be achieved by the present method, in its present state. We have

THEOREM 1. *Let* s_0, $0 < s_0 < \frac{1}{2}$, *be such that*

$$\frac{s_0^2}{2\pi} \int_0^\infty \{\exp(2s_0 T(\varphi)) - 1\}^2 \varphi^{-2}\, d\varphi = 1. \tag{9. 7}$$

Then

$$\liminf_{n\to\infty} M_n \geqq s_0. \tag{9. 8}$$

It is here being assumed that the integral

$$\int_0^\infty \{\exp(2sT(\varphi)) - 1\}^2 \varphi^{-2}\, d\varphi \tag{9. 9}$$

converges at $\varphi = \infty$ for $s = s_0$. As will be verified in the next section, (9. 9) converges when $s < \pi/8$. Thus the root s_0 of (9. 7) will be less than $\pi/8$, and the ultimate lower bound for M_n, yielded by the present method, will be less than $\pi/8$. However, as we show later, the lower bound $\pi/8$ holds for a substantial range of n.

For the proof of Theorem 1, we note that

$$J_1 + J_2 = s^2 (2\pi)^{-1} \int_0^{(n+1)\alpha/2} \{\exp(2sT(\varphi)) - 1\}^2 \varphi^{-2}\, d\varphi,$$

and also that, as $n \to \infty$, with α given by (9. 3), $(n+1)\alpha/2 \to \infty$, $(n+1)^{2s-1} \operatorname{cosec}^2 \frac{1}{2}\alpha \to 0$, where s is fixed in $0 < s < \frac{1}{2}$. Thus, for such s, if (9. 9) converges, we have, as $n \to \infty$,

$$J_1 + J_2 + J_3 \to s^2 (2\pi)^{-1} \int_0^\infty \{\exp(2sT(\varphi)) - 1\}^2 \varphi^{-2}\, d\varphi,$$

and so the right-hand side must be at least as great as 1. This proves the result.

10. Estimation of $T(\varphi)$. In order to investigate the convergence of (9. 9), and to obtain numerical information, we need to compare the integrals

$$\int x^{-1} |\sin x|\, dx, \quad \int x^{-1} (2/\pi)\, dx; \tag{10. 1}$$

here we are motivated by the fact that $2/\pi$ is the average value of $\sin x$. We have, with the notation (2. 6),

LEMMA 10. *For* $\varphi \geqq \pi$,

$$T(\varphi) \leqq \operatorname{Si} \pi + (2/\pi) \log(\varphi/\pi) + 1/(7\pi). \tag{10. 2}$$

It is clearly sufficient to prove that

$$\int_{\pi}^{\varphi} x^{-1} |\sin x| \, dx \leqq (2/\pi) \log (\varphi/\pi) + 1/(7\pi). \tag{10.3}$$

We consider first integrals between multiples of π, and prove that

$$\int_{k\pi}^{k\pi+\pi} x^{-1} |\sin x| \, dx \leqq (2/\pi) \int_{k\pi}^{k\pi+\pi} x^{-1} \, dx, \quad k \geqq 1. \tag{10.4}$$

We suppose first that k is even, say $k=2p$, $p \geqq 1$, so that $|\sin x| = \sin x$ in $(k\pi, k\pi+\pi)$. We introduce the functions

$$\beta_1(x) = 4p + 1 - 2x/\pi - \cos x, \tag{10.5}$$

$$\beta_2(x) = \tfrac{1}{2}\pi - \pi^{-1}(\tfrac{1}{2}\pi + 2p\pi - x)^2 - \sin x, \tag{10.6}$$

which have the properties

$$\beta_1'(x) = \sin x - 2/\pi, \quad \beta_2'(x) = \beta_1(x), \tag{10.7}$$

and also

$$\beta_1(2p\pi) = \beta_1(2p\pi+\pi) = \beta_2(2p\pi) = \beta_2(2p\pi+\pi) = 0. \tag{10.8}$$

Thus repeated integration by parts gives

$$\int_{2p\pi}^{2p\pi+\pi} x^{-1}(\sin x - 2/\pi) \, dx = 2 \int_{2p\pi}^{2p\pi+\pi} \beta_2(x) x^{-3} \, dx, \tag{10.9}$$

and (10. 4) now follows from the fact that

$$\beta_2(x) \leqq 0, \quad 2p\pi \leqq x \leqq 2p\pi + \pi; \tag{10.10}$$

the latter is easily verified by means of (10. 7—8). The proof of (10. 5) for odd k is similar.

It remains to consider the case of a partial interval $(k\pi, \varphi)$, where $k\pi < \varphi < k\pi + \pi$, and to show that in this case

$$\int_{k\pi}^{\varphi} x^{-1}(|\sin x| - 2/\pi) \, dx \leqq 1/(7\pi). \tag{10.11}$$

Again, the case that k is odd is similar to the case of k even, and we consider the latter only, putting $k=2p$. The above argument gives

$$\int_{2p\pi}^{\varphi} x^{-1}(\sin x - 2/\pi) \, dx = \varphi^{-1} \beta_1(\varphi) + \int_{2p\pi}^{\varphi} x^{-2} \beta_1(x) \, dx =$$

$$= \varphi^{-1} \beta_1(\varphi) + \varphi^{-2} \beta_2(\varphi) + 2 \int_{2p\pi}^{\varphi} x^{-3} \beta_2(x) \, dx.$$

Hence, by (10. 10), we have

$$\int_{2p\pi}^{\varphi} x^{-1}(\sin x - 2/\pi)dx \leqq \varphi^{-1}\beta_1(\varphi). \tag{10. 12}$$

We now use the facts that

$$\beta_1(x) < \tfrac{1}{4}, \quad 2p\pi \leqq x \leqq 2p\pi + \pi,$$

and that the left of (10. 12) attains a maximum when $\sin x = 2/\pi$ in the interval $(2p\pi + \frac{1}{2}\pi, 2p\pi + \pi)$. Thus, when k is even,

$$\int_{k\pi}^{\varphi} x^{-1}(|\sin x| - 2/\pi)\, dx \leqq \tfrac{1}{4}\{k\pi + \pi - \sin^{-1}(2/\pi)\}^{-1}, \tag{10. 13}$$

where $\sin^{-1}(2/\pi)$ has its principal value; the result is proved similarly when k is odd. Actually, the right-hand side is least, for varying positive integral k, when $k=1$, and is then less than $1/(7\pi)$, since $\sin^{-1}(2/\pi) < \pi/4$. This completes the proof of the Lemma.

11. Convergence of an integral. For the purposes of Theorem 1, it was assumed that the integral (9. 9) converged at least for some s, $0 < s < \frac{1}{2}$. We now fill in this gap with

LEMMA 11. *The integral* (9. 9) *converges for* $0 < s < \pi/8$.

We have, for $\varphi \geqq \pi$,

$$1 \leqq \exp(2sT(\varphi)) \leqq \exp\left\{2s\left(\frac{2}{\pi}\log\frac{\varphi}{\pi} + \mathrm{Si}\,\pi + \frac{1}{7\pi}\right)\right\} = \zeta^s(\varphi/\pi)^{4s/\pi}, \tag{11. 1}$$

say, where

$$\zeta = \exp(2\,\mathrm{Si}\,\pi + 2/(7\pi)). \tag{11. 2}$$

Thus

$$\{e^{2sT(\varphi)} - 1\}^2\varphi^{-2} \leqq \{\zeta^s(\varphi/\pi)^{4s/\pi} - 1\}^2\varphi^{-2} \leqq \tag{11.3}$$

$$\leqq \zeta^{2s}\pi^{-8s/\pi}\varphi^{(8s/\pi)-2}. \tag{11.4}$$

If the latter be integrated over (π, ∞), the resulting integral converges if $(8s/\pi) - 2 < -1$, or if $s < \pi/8$, as was asserted.

12. Fourth form of basic inequality. We now replace (9. 6) by an inequality

$$1 \leqq J_1 + J_4 + J_3, \tag{12. 1}$$

where J_1, J_3 are given as before by (9. 2) and (9. 5), and J_4 is derived from (9. 4) with the aid of the bound (11. 3). This gives

$$J_4 \leqq \frac{s^2}{2\pi} \int_{\pi}^{(n+1)\alpha/2} \{\zeta^s(\varphi/\pi)^{4s/\pi} - 1\}^2 \varphi^{-2}\, d\varphi =$$

$$(12.2) \qquad = \frac{s^2}{2\pi^2} \int_{1}^{(n+1)\alpha/(2\pi)} \{\zeta^s t^{4s/\pi} - 1\}^2 t^{-2}\, dt <$$

$$(12.3) \qquad < \frac{s^2}{2\pi^2} \zeta^{2s} \int_{1}^{(n+1)\alpha/(2\pi)} t^{(8s/\pi)-2}\, dt.$$

The term J_1 can, it seems, only be usefully investigated numerically. Since however Si $\varphi < \varphi$, $\varphi > 0$, we have the crude bound, used in (10), that

$$(12.4) \qquad J_1 < \tfrac{1}{2} s^2 \left\{\frac{e^{2s \operatorname{Si} \pi} - 1}{\operatorname{Si} \pi}\right\}^2.$$

Concerning J_3 we remark only that if $\alpha = \frac{1}{2}\pi$, then

$$(12.5) \qquad J_3 = \frac{s^2(n+1)^{2s-1} e^{2\eta s} \Gamma(1-2s)}{2\{\Gamma(1-s)\}^2}.$$

This choice of α is permissible for $n < 3{,}9 \cdot 10^8$, though it need not be the most advantageous choice.

13. The lower bound $\pi/8$. As follows from Lemma 10, the argument based on the inequality (12. 1) must fail for large n with $s = \pi/8$, since the term J_4, or at least its bounds (12. 2—3), become infinite as $n \to \infty$. Nevertheless, this integral diverges relatively slowly, and it turns out that the lower bound $\pi/8$ for M_n can be established for a certain range of values of n.

We start by setting $s = \pi/8$ in (12. 1), and will also take $\alpha = \frac{1}{2}\pi$. This latter is permissible if $n < 3{,}9 \cdot 10^8$, and we are concerned only with a much smaller range.

Putting $s = \pi/8$ in (9. 2) we find that

$$(13.1) \qquad J_1 = \frac{\pi^2}{128} \int_0^{\pi} \{\exp(\tfrac{1}{4}\pi \operatorname{Si} \varphi) - 1\}^2 \varphi^{-2}\, d\varphi.$$

By numerical means, sharper than (12. 4), we find that

$$(13.2) \qquad J_1 < 0{,}13.$$

Turning to J_4, we have

$$\zeta^{2s} = \exp\{\tfrac{1}{2}\pi \operatorname{Si} \pi + 14^{-1}\} < 20.$$

Hence, with the bound (12. 2)

$$(13.3)\qquad J_4 \leqq \frac{1}{128} \int_1^{(n+1)/4} \{(20t)^{\frac{1}{2}} - 1\}^2 t^{-2}\, dt \leqq$$

$$(13.4)\qquad \leqq \frac{5}{32} \log \frac{n+1}{4}.$$

For J_3 we have in this case

$$J_3 = \frac{\pi^2}{128} e^{\frac{1}{4}\pi n} \frac{\Gamma(1-\pi/4)}{\{\Gamma(1-\pi/8)\}^2} (n+1)^{\frac{1}{4}\pi-1}$$

and so

$$(13.5)\qquad J_3 \leqq 0{,}24(n+1)^{\frac{1}{4}\pi-1}.$$

On the basis of these results we prove

THEOREM 2. *For* $3 \leqq n < 1{,}6 \cdot 10^3$, *we have* $M_n > \pi/8$.

We prove this first for $3 \leqq n \leqq 500$. Using the bound (13. 4), we can replace (12. 1) by

$$(13.6)\qquad 1 \leqq 0{,}13 + \frac{5}{32} \log \frac{n+1}{4} + 0{,}24(n+1)^{\frac{1}{4}\pi-1}.$$

Here the right-hand side is an increasing function of n, so that if (13. 6) is false for some n, it will be false for lower n-values. We then remark that (13. 6) is false when $n = 500$, so that it is false also for $3 \leqq n \leqq 500$. Thus, the assumption that $s = \pi/8$ leads to a contradiction, if $3 \leqq n \leqq 500$; the same is true if $s < \pi/8$, since the functions J_1, J_2 and J_3 are increasing functions of s.

When dealing with the case $500 < n \leqq 1600$, we use the sharper form (13. 3) in preference to (13. 4), and have in place of (13. 6) the inequality

$$(13.7)\qquad 1 \leqq 0{,}13 + \frac{1}{128}\left\{20 \log \frac{n+1}{4} - 4\sqrt{20}\left(1 - \sqrt{\frac{4}{n+1}}\right) + \left(1 - \frac{4}{n+1}\right)\right\} + \\ + 0{,}24(n+1)^{\frac{1}{4}\pi-1}.$$

Here the right-hand side is an increasing function of n in the range considered. We then verify that (13. 7) is false when $n = 1600$, and so also when $500 < n \leqq 1600$. The conclusion that $M_n > \pi/8$ for such n then follows.

14 An improvement of Theorem 1. We return to the topic of lower bounds for M_n, valid for all large n. We have

THEOREM 3. *Let* s_0, $0 < s_0 < \pi/8$, *be the root of* (9. 7). *Then* $M_n > s_0$, *for sufficiently large* n.

We write (9. 6) in the form

$$(14.1)\qquad 1 \leqq \frac{s^2}{2\pi} \int_0^{\infty} (e^{2sT(\varphi)} - 1)^2 \varphi^{-2}\, d\varphi - \frac{s^2}{2\pi} \int_{(n+1)\alpha/2}^{\infty} (e^{2sT(\varphi)} - 1)^2 \varphi^{-2}\, d\varphi + J_3,$$

where J_3 is given in (9. 5). Since n is to be indefinitely large, we have from (9. 3) that

$$\alpha = \frac{46\pi}{5\log\frac{1}{4}(n+1)}. \tag{14. 2}$$

The proof consists in the observation that the last two terms in (14. 1) are, taken together, negative, for large n and $s<\pi/8$, so that (14. 1) becomes impossible when $s=s_0$ and n is sufficiently large.

In fact, it follows from (9. 5) that as $n\to\infty$, with s fixed,

$$J_3 = O(n^{2s-1}\alpha^{-2}) = O\big(n^{2s-1}(\log n)^2\big). \tag{14. 3}$$

On the other hand, it is easily shown that the last integral in (14. 1) is of a higher order of magnitude. By the argument of § 10, in particular from (10. 9), one may show that

$$T(\varphi) \geqq (2/\pi)\log\varphi - A_1, \quad \varphi \geqq \pi,$$

for some absolute constant A_1. Denoting other such constants by $A_2, \ldots$, we may then deduce that for $n>A_2$, we have that

$$\int_{(n+1)\alpha/2}^{\infty} (e^{2sT(\varphi)}-1)^2\varphi^{-2}\,d\varphi$$

is greater than

$$A_3 \int_{(n+1)\alpha/2}^{\infty} \varphi^{(8s/\pi)-2}\,d\varphi, \quad A_3>0,$$

and so, by (14. 2), is greater than

$$A_4(n/\log n)^{8s/\pi-1},$$

for some $A_4>0$. Since $8s/\pi>2s$, we see that this term will predominate over the term J_3, and so we have

$$1 < \frac{s^2}{2\pi}\int_0^{\infty} (e^{2sT(\varphi)}-1)^2\varphi^{-2}\,d\varphi,$$

for sufficiently large n. This implies that $s>s_0$, for such n, as was to be proved.

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DIE LÖSUNG EINES PROBLEMS BEZÜGLICH DES DURCHSCHNITTES ZWEIER MODULARER RECHTSIDEALE IN EINEM RING

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In der Theorie des Jacobsonschen Radikales eines (assoziativen) Ringes spielen die modularen maximalen Rechtsideale und die quasimodularen maximalen Rechtsideale eine wichtige Rolle. Das Jacobsonsche Radikal eines Ringes stimmt nämlich sowohl mit dem Durchschnitt aller modularen maximalen Rechtsideale als auch mit dem Durchschnitt aller quasimodularen maximalen Rechtsideale des Ringes überein (vgl. JACOBSON [1, Theorem 1. 6. 1 (1)], KERTÉSZ [2, Satz 5. 24 (g)]).

Bekanntlich wird ein Rechtsideal R eines Ringes A modular (bzw. quasimodular) in A genannt, wenn es ein Element $a \in A$ mit $x - ax \in R$ für jedes $x \in A$ gibt (bzw. $R: A \subseteq R$ gilt, wobei $R: A = [y; y \in A, Ay \subseteq R]$ ist). Offenbar ist jedes modulare Rechtsideal auch quasimodular im Ring. Das Problem 3 des Buches [2] von KERTÉSZ lösend hat Verfasser [6] die Existenz eines Ringes mit einem quasimodularen maximalen, aber nicht modularen Rechtsideal gezeigt.[1] Es soll bemerkt werden, daß nach JACOBSON [1, Proposition 3. 6. 1 (2)] der Durchschnitt von endlich vielen modularen *maximalen* Rechtsidealen in einem Ring stets modular ist. Mit ähnlichen Methoden, wie JACOBSON [1] gezeigt hat, hat KERTÉSZ [2, Satz 5. 2] bewiesen, daß der Durchschnitt $R_1 \cap R_2$ zweier modularer Rechtsideale R_1 und R_2 des Ringes A ebenfalls modular ist, wenn die Bedingung $R_1 + R_2 = A$ gilt. Diese Tatsachen sind bekanntlich für einen eleganten Beweis des Wedderburn—Artinschen Satzes über die Struktur der Ringe ohne von Null verschiedenes (Jacobsonsches) Radikal und mit Minimalbedingung für Rechtsideale wichtig.

Bezüglich des Durchschnittes der modularen maximalen Rechtsideale lautet das Problem 2 des Buches [2] von A. KERTÉSZ folgendermaßen:

Ist der Durchschnitt zweier modularer Rechtsideale eines Ringes stets modular?

Dieses Problem war für die Algebraiker auch früher bekannt, aber es wurde im Druck erst von KERTÉSZ [2] aufgeworfen.

Das Ziel dieser Arbeit ist nun zweifach: Einerseits geben wir einige Beispiele der Ringe an, in denen der Durchschnitt zweier modularer Rechtsideale nicht modular ist (Satz 1), und somit zeigt diese Lösung, daß die Antwort für das Problem 2 des Buches [2] von KERTÉSZ im allgemeinen »nein« ist. Andererseits betrachten

[1] Nennt man ein Ideal P eines Ringes primitiv (bzw. quasiprimitiv) im Ring A, wenn es ein modulares (quasimodulares) maximales Rechtsideal R von A mit $P = R:A$ gibt, so stimmt das Jacobsonsche Radikal nach Jacobson [1] (bzw. Verfasser [7]) mit dem Durchschnitt aller primitiven (bzw. quasiprimitiven) Ideale überein. Offenbar ist jedes primitive Ideal quasiprimitiv, und umgekehrt ist auch jedes quasiprimitive Ideal nach STEINFELD [5] und Verfasser [8] primitiv. In [8] gibt es eine Verschärfung des Resultates von [5]. Weiterhin stimmt das Jacobsonsche Radikal nach KERTÉSZ [3] mit $\Phi_r(A):A$ überein, wobei $\Phi_r(A)$ den Durchschnitt aller maximalen Rechtsideale des Ringes A bezeichnet.

wir einige, stärkere, hinreichende Bedingungen für einen Ring, die eine positive Antwort des Problems 2 von [2] in speziellen Ringen garantieren (Satz 3).

Die Lösung des Problems 2 des Buches [2] liefert[2] der folgende

SATZ 1. *Es gibt für jede endliche Mächtigkeit* $2^{m\cdot n}$ $(m \geqq 2, n \geqq 2)$ *einen Ring* A *mit* $2A=0$ *und mit* $2^{m\cdot n}$ *Elementen derart, daß der Ring* m *solche modulare Rechtsideale hat, aus denen der Durchschnitt beliebiger zweier modularer Rechtsideale nicht modular in* A *ist. Weiterhin gibt es für jede unendliche Mächtigkeit* $\aleph_\alpha$ *einen Ring* A *mit* $2A=0$ *und mit* $\aleph_\alpha$ *Elementen derart, daß der Ring* $\aleph_\alpha$ *solche modulare Rechtsideale hat, aus denen der Durchschnitt beliebiger zweier modularer Rechtsideale nicht modular in* A *ist.*

BEWEIS. Es seien K_2 der Primkörper mit zwei Elementen und A die über K_2 durch die Elemente t_α erzeugte Algebra mit der Multiplikation

$$t_\alpha^{k_\alpha} t_\beta^{k_\beta} = t_\alpha^{k_\beta} + t_\beta^{k_\beta} + t_\alpha^{k_\alpha + k_\beta}.$$

Ist die Mächtigkeit der Menge der verschiedenen Symbole t_α eine endliche Zahl $m \geqq 2$, so gelte $t_\alpha^{n+1} = t_\alpha^n$ für jedes t_α mit einer festgewählten Zahl $n \geqq 2$. Ist aber die Mächtigkeit der Menge der verschiedenen Symbole t_α eine unendliche Mächtigkeit $\aleph_\alpha$, so dürfen entweder $t_\alpha^{n+1} = t_\alpha^n$ für jedes t_α mit einer festgewählten Zahl $n \geqq 2$ gelten, oder alle Potenzen von t_α für jedes t_α voneinander verschieden sein. Dann ist die Multiplikation in A wegen $x+x=0$ für jedes $x \in A$ und wegen

$$t_\alpha^{k_\alpha}(t_\beta^{k_\beta} t_\gamma^{k_\gamma}) = t_\alpha^{k_\alpha}(t_\beta^{k_\gamma} + t_\gamma^{k_\gamma} + t_\beta^{k_\beta + k_\gamma}) = t_\alpha^{k_\gamma} + t_\beta^{k_\gamma} + t_\alpha^{k_\alpha + k_\gamma} + t_\alpha^{k_\gamma} + t_\gamma^{k_\gamma} + t_\alpha^{k_\alpha + k_\gamma} +$$

$$+ t_\alpha^{k_\beta + k_\gamma} + t_\beta^{k_\beta + k_\gamma} + t_\alpha^{k_\alpha + k_\beta + k_\gamma} = t_\alpha^{k_\gamma} + t_\gamma^{k_\gamma} + t_\alpha^{k_\beta + k_\gamma} + t_\beta^{k_\gamma} + t_\gamma^{k_\gamma} + t_\beta^{k_\beta + k_\gamma} + t_\alpha^{k_\gamma} + t_\gamma^{k_\gamma} +$$

$$+ t_\alpha^{k_\alpha + k_\beta + k_\gamma} = (t_\alpha^{k_\beta} + t_\beta^{k_\beta} + t_\alpha^{k_\alpha + k_\beta}) t_\gamma^{k_\gamma} = (t_\alpha^{k_\alpha} t_\beta^{k_\beta}) t_\gamma^{k_\gamma}$$

offenbar assoziativ, und es gilt $(t_\alpha + t_\beta)^2 = 0$ für jedes t_α, t_β. Weiterhin hat jedes Element des Ringes die Gestalt

$$\sum_{i=1}^{k} f_i(t_{\alpha_i}) = \sum_{i=1}^{k} \sum_{j=1}^{k_i} a_{ij} t_{\alpha_i}^j$$

wobei $f_i(t_{\alpha_i})$ Polynome in t_{α_i}, mit konstantem Glied 0 sind. Es gilt dabei dann und nur dann $f_i(t_{\alpha_i}) = f_j(t_{\alpha_j})$ für $t_{\alpha_i} \neq t_{\alpha_j}$, wenn $f_i(t_{\alpha_i}) = f_j(t_{\alpha_j}) = 0$ ist. Wegen $x+x=0$ für jedes $x \in A$ erhält man $(1-x)A = (1+x)A$ für das modulare Rechtsideal $(1-x)A = [y - xy;\ y \in A]$. Wegen $(1+t_\alpha)t_\beta = (1+t_\alpha)t_\alpha$ für jedes t_β ergibt sich, daß das modulare Rechtsideal $(1-t_\alpha)A$ genau aus den Polynomen der Gestalt $(1+t_\alpha)f(t_\alpha)$ besteht, wobei das konstante Glied von $f(t_\alpha)$ verschwindet. Hiernach erhält man nach der vorigen Bemerkung gewiß

$$(1-t_{\alpha_i})A \cap (1-t_{\alpha_j})A = 0$$

für $t_{\alpha_i} \neq t_{\alpha_j}$.

[2] Bezüglich der Resultate in Zusammenhang mit den Problemen 1 und 3 des Buches [2] von KERTÉSZ siehe die Arbeiten [6] und [9] des Verfassers.

Dieser Durchschnitt, da er gleich Null ist, ist aber dann und nur dann modular im Ring A, wenn A ein Linkseinselement hat. Wir werden aus der Voraussetzung, daß der Ring A ein Linkseinselement hat, einen Widerspruch ableiten.

Hat nämlich der Ring A ein Linkseinselement e, so besitzt e eine Gestalt

$$e = \sum_{i=1}^{k} \sum_{j=1}^{s} a_{ij} t_{\alpha_i}^j.$$

Wegen $et_{\alpha_m} = t_{\alpha_m}$ und $t_\beta^l t_\alpha = t_\alpha + t_\beta + t_\beta^{l+1}$ ergibt sich aus der Darstellung von e nach einem Vergleich der Koeffizienten von t_{α_m} und t_{α_i} der folgende Widerspruch:

$$1 = \sum_{\substack{i=1\\ i\neq m}}^{k} \sum_{j=1}^{n} a_{ij}; \qquad \sum_{\substack{j=1\\ i\neq m}}^{n} a_{ij} = 0$$

woraus $1=0$ folgt, und somit folgt, daß A kein Linkseinselement hat. Daher ist der Durchschnitt

$$(1-t_{\alpha_i})A \cap (1-t_{\alpha_j})A = 0$$

für $t_{\alpha_i} \neq t_{\alpha_j}$ gewiß nicht modular in A.

Die Mächtigkeit der modularen Rechtsideale der Gestalt $(1-t_\alpha)A$ ist entweder $m(\geqq 2, <\aleph_0)$ oder $\aleph_\alpha$. Weiterhin ist die Mächtigkeit der Elemente des Ringes A entweder die endliche Zahl $2^{m \cdot n}$ $(m \geqq 2,\ n \geqq 2)$, oder $\aleph_\alpha$.

Damit ist der Satz 1 bewiesen.

Bemerkung 2. Es kann erwähnt werden, daß der im Beweis des Satzes 1 betrachtete Ring A für $m=2$ und $n=2$ dem durch die Matrizen

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}; \qquad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

über dem Primkörper K_2 erzeugten Ring isomorph ist. In diesem Matrixring des Typs 5×5 gelten dann:

$$M_1^2 = M_1^3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}; \qquad M_2^2 = M_2^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_1 M_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}; \qquad M_2 M_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$M_1+M_2=\begin{pmatrix}0&1&0&1&0\\0&1&0&1&0\\0&1&0&1&0\\0&1&0&1&0\\0&1&0&1&0\end{pmatrix}$$

$$M_1+M_1^2=\begin{pmatrix}0&1&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&1&1&1\\0&1&1&1&1\end{pmatrix};\quad M_2+M_2^2=\begin{pmatrix}0&0&0&1&1\\0&1&1&1&1\\0&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

Dieser Ring ist keine monomiale Algebra im Sinne von Rédei [4], und er hat folgende Multiplikationstabelle:

	M_1	M_2	M_1^2	M_2^2
M_1	M_1^2	$M_1+M_2+M_1^2$	M_1^2	M_2^2
M_2	$M_1+M_2+M_2^2$	M_2^2	M_1^2	M_2^2
M_1^2	M_1^2	$M_1+M_2+M_1^2$	M_1^2	M_2^2
M_2^2	$M_1+M_2+M_2^2$	M_2^2	M_1^2	M_2^2

Jetzt betrachten wir einige hinreichende Bedingungen, aus denen folgt, daß der Durchschnitt zweier modularer Rechtsideale des Ringes A ebenfalls modular in A ist.

Es gilt der

Satz 3. *Gilt eine der folgenden Bedingungen in einem Ring A, so ist der Durchschnitt von beliebigen zwei modularen Rechtsidealen von A ebenfalls modular in A:*

a) *A hat ein Linkseinselement*

b) *Die Mengen $Q_a=[a+x-ax;\, x\in A]$ und $Q_b=[b+y-by;\, y\in A]$ haben für jedes feste Paar der Elemente a, b von A einen nichtleeren Durchschnitt, wobei x, y alle Elemente von A überlaufen.*

Ergänzung 4. Wichtige Unterfälle sind, in welchen anstatt b) eine der folgenden Bedingungen erfüllt ist:

b_1) Es gilt $a-b\in(1-a)A+(1-b)A$ für jedes $a, b\in A$;

b_2) A ist ein Radikalring im Sinne von Jacobson [1];

b_3) Für jedes $a, b\in A$ gibt es Elemente $q_a\in Q_a$, $q_b\in Q_b$ mit $q_aq_b=q_bq_a$;

b_4) A ist kommutativ;

b_5) Gilt $q_a t_a = q_a$ für ein Element $q_a \in Q_a$ und für ein Element $t_a \in A$, so gilt auch $q_b t_a = q_b$ für ein Element $q_b \in Q_b$ für jedes $b \in A$;

b_6) A hat ein Rechtseinselement;

b_7) Es gilt $a - ab \in (1-ab)A$ für jedes Paar der Elemente $a, b \in A$.

BEWEIS DES SATZES 3. Hat A ein Linkseinselement e, so ist wegen $(1-e)A = 0 \subseteqq R$ jedes Rechtsideal, und somit jeder Durchschnitt von Rechtsidealen modular in A.

Haben nun die Mengen Q_a und Q_b für jedes Element $a \in A$, $b \in A$ einen nichtleeren Durchschnitt, und bestehen $(1-a)A \subseteqq R_a$ und $(1-b)A \subseteqq R_b$ für die modularen Rechtsideale R_a und R_b von A, so erhält man $q_a = q_b \in Q_a \cap Q_b$. Daher gibt es Elemente $x \in A$, $y \in A$ mit

$$q = a + x - ax = b + y - by \in Q_a \cap Q_b,$$

woraus man

$$(1-q)A = (1-a)(1-x)A = (1-b)(1-y)A \subseteqq (1-a)A \cap (1-b)A \subseteqq R_a \cap R_b,$$

also die Modularität von $R_a \cap R_b$ in A erhält.

BEWEIS DER ERGÄNZUNG 4.

b_1) Gilt $a - b \in (1-a)A + (1-b)A$ für jedes $a \in A$, $b \in A$, so gibt es Elemente $x, y \in A$ mit

$$a - b = (1-a)(-x) + (1-b)y,$$

woraus

$$q_a = a + x - ax = b + y - by = q_b \in Q_a \cap Q_b$$

folgt.

b_2) Ist A ein Radikalring im Sinne von Jacobson, so gilt $(1-a)A = A$ für jedes $a \in A$, und somit gilt auch die Bedingung b_1).

b_3) Gibt es Elemente $q_a \in Q_a$, $q_b \in Q_b$ mit $q_a q_b = q_b \cdot q_a$ für jedes $a \in A$, $b \in A$, so gilt auch

$$q_a + q_b - q_a q_b = q_b + q_a - q_b q_a,$$

woraus wegen der Assoziativität der Verknüpfung $x \circ y = x + y - xy$ offenbar folgt, daß $Q_a \cap Q_b$ nicht leer ist.

b_4) Ist A kommutativ, so gilt die Bedingung b_3) und somit auch b).

b_5) Bestehen gleichzeitig $q_a = q_a t_a$ und $q_b = q_b t_a$ $(t_a \in A)$, so ergibt sich $t_a = q_a + t_a - q_a t_a = q_b + t_a - q_b t_a = a + (x + t_a - xt_a) - a(x + t_a - xt_a) = b + (y + t_a - yt_a) - b(y + t_a - yt_a)$ und somit gilt die Bedingung b).

b_6) Enthält A ein Rechtseinselement e, so gilt wegen $q_a e = q_a$ und $q_b e = q_b$ die Bedingung b_5) und somit auch b).

b_7) Gilt $a - ab \in (1-ab)A$ für jedes Paar der Elemente $a \in A$, $b \in A$, so gibt es ein Element $c = c_{a,b} \in A$ mit $a - ab = c - abc$. Daraus folgt $c = a - ab + abc$. Mit der Bezeichnung $d = b - bc$ ergibt sich daher

$$c = a - ad.$$

Nun erhält man einerseits $a + d - ad \in Q_a$, andererseits $b + c - bc \in Q_b$, und wegen der Identität

$$a + d - ad = a + (b - bc) - ad = b - bc + c$$

auch $a + b - bc - ad \in Q_a \cap Q_b$, folglich die Gültigkeit der Bedingung b).

Zum Schluß erwähnen wir einige offene Fragen:

PROBLEM 1. Gibt es einen Ring A ohne die Bedingungen a) und b), in dem der Durchschnitt zweier modularer Rechtsideale stets modular ist?

PROBLEM 2. Ist der Durchschnitt zweier quasimodularer Rechtsideale in einem Ring stets quasimodular?

Da der in der Bemerkung 2 betrachtete Ring A keinen von Null verschiedenen Rechtsannihilator enthält, ist der Durchschnitt $(1+M_1)A \cap (1+M_2)A$ ein quasimodulares (aber weder modulares noch maximales) Rechtsideal von A.

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RANDOM CENTRAL LIMIT THEOREMS FOR MARTINGALES

By

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1. Introduction. Suppose $\{X_n\}$ is a strictly stationary ergodic process such that

(1) $$\mathbf{E}(X_1) = 0, \ \mathbf{E}(X_n|X_1, \dots, X_{n-1}) = 0, \ n \geqq 2$$

and

(2) $$\mathbf{E}(X_1^2) = 1.$$

Let

(3) $$\zeta_n = X_1 + X_2 + \dots + X_n, \quad \zeta_0 = 0;$$

and

(4) $$\eta_n = n^{-\frac{1}{2}} \sum_{i=1}^{n} X_i.$$

It is known that η_n is asymptotically normal with mean 0 and variance 1 from a theorem of BILLINGSLEY [1]. Under a strong mixing condition on $\{X_n\}$, we shall now obtain a random version of this theorem similar to the theorems obtained by RÉNYI [5] and BLUM, HANSON, ROSENBLATT [2] for sums of independent random variables.

Suppose that $\{\nu_n\}$ is a sequence of positive integer valued random variables such that $\{\nu_n\}$ is independent of $\{X_n\}$ and $\nu_n \to +\infty$ in probability. Then, it follows, from the arguments in RÉNYI [5], that

(5) $$\lim_{n \to \infty} \mathbf{P}(\eta_{\nu_n} \leqq x) = \Phi(x),$$

where

(6) $$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

We shall now assume that the stationary process $\{X_n\}$ satisfies the following modified version of the strong mixing condition of ROSENBLATT [7]. Let m_a^b denote the σ-field generated by the random variables X_n, $a \leqq n \leqq b$. Then

(7) $$\sup_{A \in m_1^k, \, B \in m_{k+n}^{\infty}} |\mathbf{P}(B|A) - \mathbf{P}(B)| \leqq \alpha(n) \quad \text{where} \quad \alpha(n) \to 0 \quad \text{as} \quad n \to \infty.$$

THEOREM 1. *Under condition* (7), *if* $\{\nu_n\}$ *is a sequence of positive integer valued random variables such that* $n^{-1}\nu_n$ *converges in probability to a positive random variable* μ *having a discrete distribution, then* (5) *holds.*

THEOREM 2. *Under condition* (7), *if* $\{v_n\}$ *is a sequence of positive integer valued random variables such that* $n^{-1}v_n$ *converges in probability to a positive random variable* μ, *then* (5) *holds.*

Note that theorem 1 is a special case of theorem 2. We shall prove theorem 1 and make use of it in the proof of theorem 2. Proofs of the above theorems follow the corresponding proofs of RÉNYI [5], BLUM, HANSON and ROSENBLATT [2] for sums of independent random variables.

2. Some lemmas. In this section, we shall state and prove some lemmas which will be used later.

LEMMA 1. *If* $\{\tau_n\}$ *is a sequence of random variables with* $\mathbf{E}(\tau_k|\tau_1, \ldots, \tau_{k-1})=0$ *and* $\sigma_k^2 = \operatorname{var}[\tau_1+\tau_2+\ldots+\tau_k]^2<\infty$ *for* $1\leqq k\leqq n$, *then for any* $\varepsilon>0$

$$\mathbf{P}\big(\max_{1\leqq k\leqq n}|\tau_1+\tau_2+\ldots+\tau_k|>\varepsilon\big) \leqq \frac{\sigma_n^2}{\varepsilon^2}. \tag{8}$$

Proof of this lemma can be found in DOOB [4].

LEMMA 2. *Let* W_n, X_{mn}, $Y_{m,n}^{(j)}$ *and* $Z_{m,n}^{(j)}$ *be random variables for* $m, n=1, 2, \ldots$ *and* $j=1, \ldots, k$. *Suppose*

$$W_n = X_{m,n}+\sum_{j=1}^{k} Y_{m,n}^{(j)} Z_{m,n}^{(j)}$$

and

(a) $\lim\limits_{m\to\infty} \limsup\limits_{n} \mathbf{P}(|Y_{m,n}^{(j)}|>\varepsilon) = 0$ *for every* $\varepsilon>0$ *and* $1\leqq j\leqq k$;

(b) $\lim\limits_{M\to\infty} \limsup\limits_{m} \limsup\limits_{n} \mathbf{P}(|Z_{m,n}^{(j)}|>M)=0$ *for* $j=1, \ldots, k$;

(c) *the distributions of* $\{X_{m,n}\}$ *converge to the distribution function* F *for each fixed* m.

Then the distribution function of $\{W_n\}$ *converges to* F.

Proof of this lemma can be found in BLUM, HANSON and ROSENBLATT [2].

DEFINITION. A sequence of random variables $\{\eta_n\}$ is said to be mixing in the sense of Rényi if for any event A with $\mathbf{P}(A)>0$, and for any real number x,

$$\lim_{n} \mathbf{P}(A_n|A) = \lim_{n} \mathbf{P}(A_n)$$

where $A_n = [\eta_n\leqq x]$.

LEMMA 3. *Under the strong mixing condition on the process* $\{X_n\}$, *the sequence* $\{\eta_n\}$ *is mixing.*

PROOF. Let $A_k=(\eta_k\leqq x)$. By Theorem 2 of RÉNYI [6] it is enough to prove that for any A_k,

$$\lim_{n} \mathbf{P}(A_n|A_k) = \lim_{n} \mathbf{P}(A_n).$$

Let $j_n=[n^{1/4}]$ and consider the sequence of random variables $n^{-1/2}\zeta_{k+j_n}$. Sincy $\mathbf{E}(\zeta_{k+j_n})=0$ and $\operatorname{Var}(\zeta_{k+j_n})=(k+j_n)$, $n^{-1/2}\zeta_{k+j_n}$ converges to zero in probabilite as n tends to ∞. Therefore

$$\lim_{n} \mathbf{P}(A_n|A_k) = \lim_{n} \mathbf{P}(n^{-1/2}\{\zeta_n-\zeta_{k+j_n}\}\leqq x|A_k). \tag{9}$$

Now note that the event $(n^{-1/2}\{\zeta_n-\zeta_{k_n}\}_{+j}\leqq x)\in m^n_{k+j_n+1}$ and $A_k\in m^k_1$. Hence by the strong mixing condition (7),

$$\lim_n \mathbf{P}([n^{-1/2}(\zeta_n-\zeta_{k+j_n})\leqq x]|A_k) = \lim_n \mathbf{P}(n^{-1/2}(\zeta_n-\zeta_{k+j_n})\leqq x) \tag{10}$$

since $j_n\to\infty$ as $n\to\infty$. Again using the fact that $n^{-1/2}\zeta_{k+j_n}\to 0$ in probability as $n\to\infty$, we get that

$$\lim_n \mathbf{P}(n^{-1/2}(\zeta_n-\zeta_{k+j_n})\leqq x) = \lim_n \mathbf{P}(A_n). \tag{11}$$

Combining (9), (10) and (11), we obtain that the sequence $\{\eta_n\}$ is mixing in the sense of Rényi.

LEMMA 4. *Any sequence of events $A_n\in m^{b_n}_{a_n}$ where $a_n\to\infty$ and $b_n\to\infty$ is a mixing sequence in the sense that for any event A with* $\mathbf{P}(A)>0$, $\lim_n \mathbf{P}(A_n|A)=\lim_n \mathbf{P}(A_n)$.

PROOF. It follows again by Theorem 2 of RÉNYI [6] that it is sufficient to show that for any fixed k,

$$\lim_n \mathbf{P}(A_n|A_k) = \lim_n \mathbf{P}(A_n). \tag{12}$$

Since $A_k\in m^{b_k}_{a_k}$, $A_k\in m^{b_k}_1$ and similarly $A_n\in m^{\infty}_{a_n}$. Furthermore $a_n\to\infty$ as $n\to\infty$. Hence (12) follows by the strong mixing condition (7).

LEMMA 5. *Suppose that $\nu_n=[n\mu]$ where μ is a positive random variable having a discrete distribution. Then* (5) *holds under* (7).

PROOF. Suppose that $p_k=\mathbf{P}(\mu=l_k)$ and $\sum_1^\infty p_k=1$. Then

$$\mathbf{P}(\eta_{\nu_n}\leqq x) = \sum_1^\infty \mathbf{P}(\eta_{[nl_k]}\leqq x|\mu = l_k)p_k. \tag{13}$$

But $\mathbf{P}(\eta_{[nl_k]}\leqq x|\mu=l_k)\to\Phi(x)$ as $n\to\infty$ for every k, since the sequence $\{\eta_n\}$ is mixing by Lemma 3 which proves this lemma by (13).

3. PROOF OF THEOREM 1. Now

$$\eta_{\nu_n} = \eta_{[\mu n]}+\left(\frac{\zeta_{\nu_n}-\zeta_{[\mu n]}}{\sqrt{\mu n}}\right)\sqrt{\frac{\mu n}{\nu_n}}+\frac{\zeta_{[\mu n]}}{\sqrt{[\mu n]}}\left(\sqrt{\frac{[\mu n]}{\nu_n}}-1\right). \tag{14}$$

By lemma 5, $\eta_{[\mu n]}=\dfrac{\zeta_{[\mu n]}}{\sqrt{[\mu n]}}$ is asymptotically normal with mean 0 and variance 1. Furthermore

$$\sqrt{\frac{[\mu n]}{\nu_n}}-1\to 0 \text{ in probability.}$$

Hence by Slutsky's theorem (see CRAMER [3]), it is enough to prove that

$$\frac{\zeta_{\nu_n}-\zeta_{[\mu n]}}{\sqrt{\mu n}}\to 0 \text{ in probability.} \tag{15}$$

Suppose that $p_k = \mathbf{P}(\mu = l_k)$, $k \geqq 1$ and $\sum_1^\infty p_k = 1$. Define for any $\varepsilon > 0$, $\varrho > 0$,

$$B_n(\varrho) = (|\nu_n - [\mu n]| < n\varrho), \tag{16}$$

$$C_{nk} = \left(|\zeta_{\nu_n} - \zeta_{n_k}| > \varepsilon\sqrt{n_k}\right) \quad \text{where} \quad n_k = [nl_k], \tag{17}$$

and

$$A_k = [\mu = l_k], \; D_k = [\mu \geqq l_k], \tag{18}$$

as was done in RÉNYI [5]. It is easy to see that

$$\mathbf{P}\left(\left|\frac{\zeta_{\nu_n} - \zeta_{[\mu n]}}{\sqrt{[\mu n]}}\right| > \varepsilon\right) \leqq \sum_{k=1}^{\infty} \mathbf{P}(A_k B_n(\varrho) C_{nk}) + \mathbf{P}(\overline{B_n(\varrho)}). \tag{19}$$

Now

$$\mathbf{P}(A_k B_n(\varrho) C_{nk}) \leqq \mathbf{P}\left(\operatorname*{Max}_{|l - n_k| \leqq n\varrho} \frac{|\zeta_l - \zeta_{n_k}|}{\sqrt{n_k}} > \varepsilon\right) \leqq \mathbf{P}\left(\operatorname*{Max}_{n_k \leqq l \leqq n_k + n\varrho} \frac{|\zeta_l - \zeta_{n_k}|}{\sqrt{n_k}} > \varepsilon\right) + \tag{20}$$

$$+ \mathbf{P}\left(\operatorname*{Max}_{n_k - n\varrho \leqq l \leqq n_k} \frac{|\zeta_l - \zeta_{n_k}|}{\sqrt{n_k}} > \varepsilon\right).$$

By stationarity of the process $\{X_n\}$, and by lemma 1

$$\mathbf{P}\left(\operatorname*{Max}_{n_k \leqq l \leqq n_k + n\varrho} \frac{|X_{n_k+1} + \ldots + X_l|}{\sqrt{n_k}} > \varepsilon\right) = \tag{21}$$

$$= \mathbf{P}\left(\operatorname*{Max}_{1 \leqq l \leqq n\varrho} \frac{|X_1 + \ldots + X_l|}{\sqrt{n_k}} > \varepsilon\right) \leqq \frac{\operatorname{Var}(X_1 + \ldots + X_{n\varrho})}{\varepsilon^2 n_k} \leqq \frac{\varrho}{\varepsilon^2 l_k}.$$

Similarly we have

$$\mathbf{P}\left(\operatorname*{Max}_{n_k - n\varrho \leqq l \leqq n_k} \frac{|\zeta_l - \zeta_{n_k}|}{\sqrt{n_k}} > \varepsilon\right) \leqq \frac{\varrho}{\varepsilon^2 l_k}. \tag{22}$$

Combining (19)—(22), we get that

$$\mathbf{P}(|\zeta_{\nu_n} - \zeta_{[\mu n]}| > \varepsilon\sqrt{[\mu n]}) \leqq \mathbf{P}(D_M) + \frac{2\varrho}{\varepsilon^2} \sum_{k=1}^{M-1} \frac{1}{l_k} + \mathbf{P}(\overline{B_n(\varrho)}). \tag{23}$$

Let $\delta > 0$. Choose M such that $\mathbf{P}(D_M) < \frac{\delta}{3}$ and $\varrho > 0$ such that

$$\frac{2\varrho}{\varepsilon^2} \sum_{k=1}^{M-1} \frac{1}{l_k} < \frac{\delta}{3}.$$

Now choose n_0 such that for every $n \geqq n_0$,

$$\mathbf{P}(\overline{B_n(\varrho)}) < \frac{\delta}{3}.$$

It now follows from (23) that for every $n \geqq n_0$, $\mathbf{P}(|\zeta_{\nu_n} - \zeta_{[\mu n]}| > \varepsilon\sqrt{[\mu n]}) \leqq \delta$ which proves theorem 1.

4. PROOF OF THEOREM 2. Define $\mu_m=\frac{k}{2^m}$ when $\frac{k-1}{2^m}\leqq\mu<\frac{k}{2^m}$ and $\mu_{mn}=$ $=\nu_n+[n(\mu_m-\mu)]$ where $[x]$ stands for the greater integer less than or equal to x. It is sufficient to show that the random variables $\left\{\frac{\zeta_{\nu_n}-\zeta_{\mu_{mn}}}{\sqrt{n\mu_m}}\right\}$ satisfy condition (a) of lemma 2. Then an application of Theorem 1 and Lemma 2 will complete the proof of the theorem by arguments similar to those of BLUM, HANSON and ROSENBLATT [2]. It is easily seen again from the same proofs that

$$\limsup_m \limsup_n \mathbf{P}\left(\left|\frac{\zeta_{\nu_n}-\zeta_{\mu_{mn}}}{\sqrt{n\mu_m}}\right|>\varepsilon\right)\leqq \tag{24}$$

$$\leqq \limsup_m \sum_{k=m}^{m2^m} \limsup_n 2\mathbf{P}\left(\max_{n(k-3)2^{-m}<r<n(k+3)2^{-m}}|\zeta_r-\zeta_t|>\frac{\varepsilon}{2}\sqrt{\frac{nk}{2^m}}\,\middle|\,\frac{k-1}{2^m}\leqq\mu<\frac{k}{2^m}\right)\cdot$$

$$\cdot\mathbf{P}\left(\frac{k-1}{2^m}\leqq\mu<\frac{k}{2^m}\right)$$

where $t=[n(k-3)2^{-m}]$. Now consider

$$\mathbf{P}\left(\max_{n(k-3)2^{-m}<r<n(k+3)2^{-m}}|\zeta_r-\zeta_t|>\frac{\varepsilon}{2}\sqrt{\frac{nk}{2^m}}\right)=\mathbf{P}\left(\max_{0\leqq r\leqq 6n2^{-m}}|\zeta_r-\zeta_0|>\frac{\varepsilon}{2}\sqrt{\frac{nk}{2^m}}\right) \tag{25}$$

by stationarity

$$\leqq\frac{24n}{\varepsilon^2 nk}\qquad\text{by Lemma 1.}$$

Therefore

$$\limsup_n \mathbf{P}\left(\max_{n(k-3)2^{-m}<r<n(k+3)2^{-m}}|\zeta_r-\zeta_t|\geqq\frac{\varepsilon}{2}\sqrt{\frac{nk}{2^m}}\right)\leqq\frac{24}{\varepsilon^2 k}\leqq\frac{24}{\varepsilon^2 m} \tag{26}$$

for $k\geqq m$. By Lemma 4, (24) and (26) imply that

$$\limsup_m \limsup_n \mathbf{P}\left(|\zeta_{\nu_n}-\zeta_{\mu_{mn}}|>\varepsilon\sqrt{n\mu_m}\right)\leqq\limsup_m \sum_{k=m}^{m2^m}\frac{48}{\varepsilon^2 m}\mathbf{P}\left(\frac{k-1}{2^m}\leqq\mu<\frac{k}{2^m}\right)=0.$$

This completes the proof of Theorem 2.

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ON WEIGHTED POLYNOMIAL APPROXIMATION ON THE WHOLE REAL AXIS

By

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A. Introduction. Let $Q(x)>0$ be an even continuously differentiable function on $(-\infty, +\infty)$ and let $xQ'(x)$ be increasing for $x>0$ and $Q'(x)\to\infty$ for $x\to\infty$. Let us further denote by q_n the unique positive solution of the equation $xQ'(x)=n$. It follows $q_n=o(n)$.

Finally, let $f(x)$ be a continuous function in $(-\infty, +\infty)$ so that for an integer r

$$\omega_r(f;\delta) = \sup_{\substack{|h|\leqq\delta \\ -\infty<x<+\infty}} \left| \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu f(x+\nu h) \right| \to 0 \tag{1}$$

for $\delta\to 0$.

Our aim is to prove the following

THEOREM. *There exists a sequence of polynomials $\{\Pi_n(x)\}$ so that the degree of $\Pi_n(x)$ is at most n and we have for $n>r$ and $q_n>1$*

$$|f(x)-\Pi_n(x)| \leqq \begin{cases} A_r\omega_r(f;n^{-1}q_n) & \text{for} \quad |x|\leqq 4q_n \\ e^{Q(x)}B_r(Q)e^{-K_r(Q)n}[\|f\|+\omega_r(f;r^{-1})] & \text{for} \quad |x|>4q_n \end{cases} \tag{2}$$

where A_r, $B_r(Q)$, $K_r(Q)$ are positive numbers depending on r, resp. on r and $Q(x)$ only, and

$$\|f\| = \max_{|x|\leqq 1} |f(x)|. \tag{3}$$

In what follows let $c_1(r), c_2(r), \ldots$ be positive constants, depending on r only. It is well-known, that $\omega_r(f;\delta)\geqq c_1(r)\omega_r(f;1)\delta^r$ so that the second term in (2) is asymptotically smaller than $e^{Q(x)}\omega_r(f;n^{-1}q_n)$.

Our theorem generalizes former results of M. M. DŽRBASIAN [2] in several aspects.

B. Lemmata. We are treating first some preliminary estimations.

LEMMA 1. *We have for every real x and natural number n*

$$|x|^n e^{-Q(x)} \leqq q_n^n \tag{4}$$

PROOF. A simple differentiation shows that the maximum of (4) is attained for $x=q_n$.

LEMMA 2. *We have for $N>r$ and $1\leqq|x|\leqq N/r$*

$$|f(x)|\leqq[c_2(r)\|f\|+c_3(r)]N^r\omega_r(f;r^{-1}). \tag{5}$$

PROOF. Let $\{a_k;\ k=0, 1, \ldots N\}$ be an arbitrary sequence of numbers, let us put $a_{-1}=a_{-2}=\ldots=a_{-r+1}=0$ and

$$\Delta^r a_k = \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu a_{k-\nu}.$$

We have then

$$|a_N| = \left| \sum_{k=0}^{N} \frac{(N-k+r-1)!}{(N-k)!\,(r-1)!} \Delta^r a_k \right| \leqq c_4(r) \max_{0 \leqq k \leqq r} |a_k| + \frac{(N+r)!}{N!\,r!} \max_{r \leqq k \leqq N} |\Delta^r a_k| \leqq$$

$$\leqq c_4(r) \max_{0 \leqq k \leqq r} |a_k| + c_5(r) N^r \max_{r \leqq k \leqq N} |\Delta^r a_k|.$$

We insert $a_k = f\left(k \dfrac{x}{N}\right)$ $(k=0, 1, \ldots, N)$ in this formula. In this way we obtain (5) by straightforward calculation.

LEMMA 3. *Let $\Pi_n(x)$ be an arbitrary polynomial of degree at most n, let $\varrho_n \geqq 1$ and*

(6) $$M_n = \max_{|x| \leqq \varrho_n} |\Pi_n(x)|$$

then we have for $|x| > \varrho_n$

(7) $$|\Pi_n(x)| \leqq \left(\frac{2|x|}{\varrho_n}\right)^n M_n$$

PROOF. From (6) it follows

$$|\Pi_n(\varrho_n x)| \leqq M_n \qquad (|x| \leqq 1).$$

If $T_n(x)$ is the n-th Chebycheff polynomial, then by a well-known theorem of P. L. CHEBYCHEFF it follows

$$|\Pi_n(\varrho_n x)| \leqq T_n(|x|) M_n \leqq (2|x|)^n M_n \qquad (|x| \geqq 1).$$

Replacing here $\varrho_n x$ by x, we obtain (7).

C. Proof of the theorem. Let $\varrho_n = 4q_n$ and $f_n(x) = f(\varrho_n x)$. By a generalization of Jackson's approximation theorem*

$$|f_n(x) - p_n(x)| \leqq c_6(r)\omega_r(f_n;\, n^{-1}) = c_6(r)\omega_r(f;\, \varrho_n n^{-1}) \leqq c_7(r)\omega_r(f;\, q_n n^{-1}) \quad (|x| \leqq 1).$$

Putting here $\Pi_n(x) = p_n\left(\dfrac{x}{\varrho_n}\right)$ and replacing x by x/ϱ_n we obtain the upper part of (2).

From Lemma 2 we obtain by taking $N = [rx]+1$

(8) $$|f(x)| \leqq c_8(r) \|f\| \cdot |x|^r \qquad (|x| > 1)$$

* The trigonometric case is treated by S. B. STECHKIN [4], see also A. F. TIMAN [3], § 5.1.32. The case just needed is included in the result of JU. A. BRUDNYĬ [1].

and we conclude that*

$$(9)\quad |\Pi_n(x)| \leqq |f(x)| + c_6(r)\omega_r(f_n; n^{-1}) \leqq c_9(r) \max_{|x| \leqq \varrho_n} |f(x)| \leqq c_{10}(r)[\|f\| + \omega_r(f; r^{-1})]\, q_n^r$$

$$(|x| \leqq 4q_n = \varrho_n).$$

We have then by Lemma 3 and Lemma 1

$$(10)\qquad e^{-Q(x)}|\Pi_n(x)| \leqq c_9(r)\|f\| q_n^r (2q_n)^{-n} |x|^n e^{-Q(x)} \leqq c_9(r)\|f\| q_n^r 2^{-n} \qquad (|x| \geqq 4q_n).$$

The function $x^r e^{-Q(x)}$ is decreasing for $x > q_r$ and a fortiori for $x > 4q_n$, so that we have for sufficiently large n

$$(11)\qquad x^r e^{-Q(x)} \leqq (4q_n)^r e^{-Q(4q_n)} \leqq 4^r q_r^n e^{-\int_{q_n}^{4q_n} \frac{xQ'(x)}{x}\,dx}\, dx \leqq 4^r n^r e^{-n}.$$

Using (10), resp. (8) and (11) we obtain the second part of (2), q.e.d.

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* We consider the fact that $\omega_r(f_n; n^{-1})$ is related to the interval $[-1, +1]$, so that

$$\omega_r(f_n; n^{-1}) \leqq 2^r \max_{|x| \leqq \varrho_n} |f(x)|.$$

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EINFACHER BEWEIS DES HAUPTSATZES VON HAJÓS–RÉDEI FÜR ELEMENTARE GRUPPEN VON PRIMZAHLQUADRATORDNUNG

Von

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Bei der Verallgemeinerung des Hauptsatzes von Hajós (s. RÉDEI [1]) auf normierte vollständige schlichte Komplexzerlegungen endlicher abelscher Gruppen wird überraschenderweise der Beweis für den Fall elementarer Gruppen von Primzahlquadratordnung sehr langwierig, weil zwei nicht leichte Sätze für Polynome über endlichen Primkörpern herangezogen werden. Eine erhebliche Vereinfachung ergibt sich allerdings dadurch, daß von den drei in RÉDEI [1], Satz 9, behandelten Fällen für die Anwendung nur der zweite und dritte Fall relevant sind.

In der vorliegenden Note geben wir einen weiter vereinfachten Beweis, dessen wesentlicher Schritt ein neuer Ansatz ist. Dadurch wird das Problem so reduziert, daß wir mit einem leicht beweisbaren Spezialfall von RÉDEI [1] Satz 8, auskommen können.

Mit Z_p bezeichnen wir im folgenden den Primkörper der Charakteristik p, mit Z_p^x die Menge der von Null verschiedenen Elemente von Z_p. Im übrigen folgen wir den Bezeichnungen der Arbeit RÉDEI [1].

Als Hilfsmittel benötigen wir folgendes Lemma, das einen Spezialfall von RÉDEI [1], Satz 9 darstellt.

LEMMA. *p sei eine Primzahl größer als 2. Wenn für $p-1$ Zahlen $g_1, \dots, g_{p-1}$ aus Z_p, von denen mindestens eine gleich Null ist, die Abbildungen $\varphi_y\colon Z_p^x \to Z_p$, $w \to g_w + yw$ für mindestens $\frac{p-1}{2}$ verschiedene $y \in Z_p^x$ Permutationen von Z_p^x sind, so gilt notwendig*

$$g_1 = g_2 = \dots = g_{p-1} = 0.$$

BEWEIS. Der Beweis stützt sich stark auf den Beweis von Satz 8 in RÉDEI [1]. Für Elemente $x_1, \dots, x_n$ aus einem Körper K bezeichnen wir mit

$$S_\nu(x_1, \dots, x_n) \qquad (\text{für } 1 \leqq \nu \leqq n)$$

die ν-te elementarsymmetrische Funktion. Nach Voraussetzung gilt dann für mindestens $\frac{p-1}{2}$ verschiedene $y \in Z_p^x$.

$$S_\nu(g_1 + y, g_2 + 2y, \dots, g_{p-1} + (p-1)y) = 0 \qquad (\nu = 1, \dots, p-2). \tag{1}$$

Die Polynome $P_\nu(x) = S_\nu(g_1 + x, g_2 + 2x, \dots, g_{p-1} + (p-1)x)$ haben für $\nu = 1, \dots, \dots, \frac{p-1}{2}$ offenbar einen Grad $< \frac{p-1}{2}$ und verschwinden somit wegen (1) identisch.

Insbesondere gilt

(2) $$P_\nu(0) = S_\nu(g_1, \ldots, g_{p-1}) = 0 \quad \text{für} \quad \nu = 1, \ldots, \frac{p-1}{2}.$$

Das Polynom

$$f(x) = \prod_{t \in Z_p^\times} (x - g_t)$$

welches nach Voraussetzung mindestens eine Nullstelle 0 besitzt, ist daher ein Lückenpolynom der Form

$$f(x) = x^{p-1} + jx^{\frac{p-3}{2}} + \ldots + vx.$$

Wir betrachten außerdem noch die Polynome

(3) $$g(x) = xf(x)$$

und

(4) $$h(x) = g(x) - (x^p - x).$$

$s_1, \ldots, s_t$ seien die verschiedenen Nullstellen von $g(x)$. Wir setzen

$$Q(x) = \prod_{\tau=1}^{t} (x - s_\tau).$$

Dann gelten (teils wegen $Q(r)|x^p - x$) die Teilbarkeiten

$$Q(x) \mid h(x) \quad \text{und} \quad \frac{g(x)}{Q(x)} \Big| g'(x).$$

Insgesamt ergibt sich hieraus, da wegen (4) $h'(x) = g'(x) + 1$ ist,

(5) $$g(x) | h(x)\big(h'(x) - 1\big).$$

Ein Gradvergleich zeigt, daß (5) nur haltbar ist, wenn die rechte Seite das Nullpolynom ist. Wegen (4) ist der Fall $h(x) \equiv 0$ ausgeschlossen, sodaß nur der Fall

(6) $$h'(x) = 1$$

übrig bleibt, woraus wieder wegen (4) $h(x) = x$ und $f(x) = x^{p-1}$ folgt. Damit ist die Behauptung bewiesen.

Nun ergibt sich leicht der

SATZ (RÉDEI [1]). *Wenn eine elementarabelsche Gruppe G von Primzahlquadratordnung p^2 $(p > 2)$ (nichttriviales) schlichtes Produkt $G = A \circ B$ zweier normierter Komplexe A, B $\big(O(A) = O(B) = p\big)$ ist, so ist notwendig A oder B eine Gruppe.*

BEWEIS. Es werde angenommen, daß weder A noch B eine Gruppe ist. Dann existiert zu jedem von ε verschiedenem $\gamma \in A$ eine natürliche Zahl $e(\gamma) < p - 1$ so daß

$$\gamma^n \in A \quad \text{für} \quad 0 \leqq n \leqq e(\gamma),$$

$$\gamma^n \notin A \quad \text{für} \quad n = e(\gamma) + 1.$$

Wir wählen ein $\alpha \in A$ $(\alpha \neq \varepsilon)$ mit maximalem $e(\alpha)=e$ aus. Die Menge

$$B_\alpha = \langle \delta \in B\colon \delta \notin \{\alpha\}\rangle \tag{7}$$

ist offenbar nichtleer. Zu jedem $\delta \in B_\alpha$ existiert analog eine natürliche Zahl $f(\delta) < p-1$ mit

$$\delta^m \in B \quad \text{für} \quad 0 \leqq m \leqq f(\delta)$$

$$\delta^m \notin B \quad \text{für} \quad m = f(\delta)+1.$$

Aus B_α werde ein β mit maximalem $f(\beta)=f$ ausgewählt. Wir entnehmen (7) $G = \{\alpha\} \times \{\beta\}$. Weiter ist

$$e < p-1, \quad f < p-1. \tag{8}$$

Für A und B haben wir mit den Abkürzungen $r = p-1-e(\neq 0)$ und $s = p-1-f(\neq 0)$ die Darstellungen

$$A = \langle \varepsilon, \alpha, \ldots, \alpha^e, \alpha^{a_1}\beta^{b_1}, \ldots, \alpha^{a_r}\beta^{b_r}\rangle \tag{9}$$

$$B = \langle \varepsilon, \beta, \ldots, \beta^f, \alpha^{c_1}\beta^{d_1}, \ldots, \alpha^{c_s}\beta^{d_s}\rangle \tag{10}$$

O.E.d.A. dürfen sämtliche Exponenten als Elemente aus Z_p aufgefaßt werden, was im folgenden stets geschehen soll.

Wir ziehen nun mehrfach das Hauptlemma von RÉDEI [1] heran, zunächst für einen nichtidentischen Charakter χ, der durch $\chi(\alpha)=\varrho$, $\chi(\beta)=1$ erklärt wird, wo ϱ eine beliebige primitive p-te Einheitswurzel ist. Wegen

$$\chi(B) = f+1+\varrho^{c_1}+\ldots+\varrho^{c_s} \neq 0$$

gehört χ dem Nullifikator $\mathfrak{N}(A)$ an, d.h.

$$\chi(A) = 1+\varrho+\ldots+\varrho^e+\varrho^{a_1}+\ldots+\varrho^{a_r} = 0.$$

Bei geeigneter Numerierung gilt somit

$$a_1 = e+1,\ a_2 = e+2, \ldots, a_r = e+r = p-1.$$

(9) reduziert sich nun auf

$$A = \langle \varepsilon, \alpha, \ldots, \alpha^e, \alpha^{e+1}\beta^{b_1}, \ldots, \alpha^{p-1}\beta^{b_r}\rangle. \tag{11}$$

Aus Symmetriegründen reduziert sich auch (10) bei geeigneter Numerierung auf

$$B = \langle \varepsilon, \beta, \ldots, \beta^f, \beta^{f+1}\alpha^{c_1}, \ldots, \beta^{p-1}\alpha^{c_s}\rangle. \tag{12}$$

Nun betrachten wir die paarweise verschiedenen $p-1$ Charaktere χ_n, welche durch

$$\chi_n(\alpha) = \varrho^n, \quad \chi_n(\beta) = \varrho \quad \text{für} \quad n \in Z_p^x$$

erklärt sind.

Wir setzen zur Abkürzung

$$M_A = \langle n\colon \ \chi_n \in \mathfrak{N}(A),\ n \in Z_p^x\rangle.$$

Analog werde M_B definiert.

Auf Grund des Hauptlemmas können wir zwei Fälle unterscheiden:

1. $O(M_A) \geqq \frac{p-1}{2}$.

2. $O(M_B) \geqq \frac{p-1}{2}$.

Wir betrachten zunächst Fall 1. Jedes $n \in M_A$ induziert wegen

$$1+\varrho^n+\ldots+\varrho^{ne}+\varrho^{n(e+1)+b_1}+\ldots+\varrho^{n(p-1)+b_r}=0$$

eine Permutation von Z_p^x, die sich mit

$$g_w = \begin{cases} 0 & w=1,\ldots,e \\ b_\varrho & w=e+\varrho, \quad \varrho=1,\ldots,r \end{cases}$$

in der Form $w \to g_w + nw$ schreiben läßt. Da $O(M_A) \geqq \frac{p-1}{2}$ und $g_1=0$, darf das Lemma angewendet werden. Alle g_w insbesondere $g_{e+1}=b_1$ verschwinden und wir haben einen Widerspruch zur Maximalität von e.

Im zweiten Fall kann man analog vorgehen und erhält einen Widerspruch zur Maximalität von f.

Unsere Annahme war daher falsch, d.h. von den Komplexen A, B ist mindestens einer eine Gruppe.

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INVERSE SEMIGROUP CONGRUENCEL ON REGULAR SEMIGROUPS

By

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Introduction. The object of this paper is to prove the existence of inverse semigroup congruences and Brandt congruences on certain regular semigroups. We first prove in Theorem 1 that if S is a $(1, n)$ regular Rees matrix semigroup with 0, then there exists a Brandt congruence ϱ such that S/ϱ is a Brandt semigroup. In Theorem 2 we show that there exists a minimum inverse semigroup congruence on a $(1, n)$ regular semigroup. If we combine a theorem of MUNN [6] with our Theorem 2 we shall readily arrive at a group (with 0) congruence on an inverse semigroup S/ϱ, where S is $(1, n)$ regular. As an application of Theorem 1 we establish a theorem (Theorem 3) concerning a Brandt congruence on a $(1, n)$ regular semigroup S with 0.

The significance of our Theorem 1 is that it enables us to prove the existence of a Brandt congruence on a certain *regular* semigroup S while MUNN proved in [7] that there exists a Brandt congruence on a certain *inverse* semigroup.

1. Definitions and notations. By a semigroup we mean a set which is closed under an associative binary operation. The basic definitions and results can be found in [1]. Throughout, this paper we shall use the following notation. S denotes a semigroup with 0. A congruence ϱ on a semigroup S is called a group congruence if S/ϱ is a group. It has been shown by MUNN [6] that there exists a minimum group congruence on every inverse semigroup.

A regular semigroup S with 0 is said to be (h, k) regular if every non-zero principal left (right) ideal of S contains exactly k (h) non-zero idempotents, where h and k are finite positive integers [2]. A congruence ϱ on a semigroup S is called an inverse semigroup (a Brandt) congruence if S/ϱ is an inverse (a Brandt) semigroup.

A semigroup S is called by W. D. MUNN [6] an *inverse semigroup*, if one of the following equivalent conditions is satisfied:

P_1. S is regular and its idempotents commute.

P_2. Every element of S has a unique inverse.

P_3. Each principal left ideal and each principal right ideal of S is generated by one and only one idempotent.

Such semigroups were first and independently studied by V. V. VAGNER (Generalized groups, *Doklady Akad. Nauk SSSR*, **84** (1952), pp. 1119—1122) and by G. B. PRESTON (Inverse semigroups, *Journal London Math. Soc.*, **29** (1954), pp. 396—403). It must be mentioned, that an *inverse* of an element a means here an element $a' \in S$ such that $a = aa'a$ and $a' = a'aa'$ are valid, and here S is called *regular*, if $a \in aSa$ holds for every $a \in S$. It is not difficult to see, that in a regular semigroup every element possesses an inverse in the above sense.

Moreover, a congruence ϱ of a semigroup S is called a *Brandt congruence*, if S/ϱ is a *Brandt semigroup*, which is frequently referred to as a semigroup, which means a structure consisting of a zero element adjoined to a Brandt groupoid (cf. CLIFFORD—PRESTON [1]). It can be shown, that a completely 0-simple inverse semigroup is a Brandt semigroup, and it is isomorphic with a regular Rees matrix semigroup of the form

$$M^0(G; I, I; \Delta),$$

where Δ is the identity $I \times I$ matrix over G^0 (see CLIFFORD—PRESTON [1], Theorem 3.9).

The cardinal number of a set T will be denoted by $|T|$. If A and B are subsets of S then $A \setminus B$ will denote the set of all elements of A which are not in B. Let $a \in S \setminus 0$. Define $V(a) = \{x \in S: axa = a \text{ and } xax = x\}$, $E_a = \{e \in S: e = e^2 \text{ and } ea = a\}$, $F_a = \{f \in S: f = f^2 \text{ and } af = a\}$. If $T \subseteqq S$, $E(T) = \{e \in T: e = e^2\}$. Let ϱ be a congruence on a semigroup S; the ϱ-class containing the element x of S will be denoted by x_ϱ and the natural homomorphism $x \to x_\varrho$ of S onto S/ϱ will be denoted by ϱ_*.

DEFINITION 1. A semigroup S with 0 is said to be homogeneous n regular if $|V(a)| = n$ for every $a \neq 0$ in S, where n is a fixed positive integer. An homogeneous n regular semigroup S is said to be (h, k) type homogeneous n regular if S is (h, k) regular.

DEFINITION 2. A congruence ϱ on a semigroup S with 0 is proper if $\{0\}$ is a ϱ-class of S.

2. We use the following known results.

(A) If R and L are 0-minimal right and left ideals respectively of a semigroup such that $LR \neq \{0\}$ and $RL \neq \{0\}$, then LR is a completely 0-simple semigroup and RL is a group with zero [8, Theorem B].

(B) If e is a non-zero primitive idempotent in a regular semigroup S then $eS\,(Se)$ is a 0-minimal right (left) ideal of S [9, Lemma 2].

(C) If R and L are respectively right and left ideals of a regular semigroup S such that $LR \neq \{0\}$ then $RL \neq \{0\}$ [9, Lemma 3].

(D) Let S be a completely 0-simple semigroup. Then the following conditions are equivalent.

(i) S is homogeneous n regular.

(ii) There exist positive integers h and k such that $hk = n$ and S is (h, k) regular [2, Theorem 1 (iv)].

(E) A regular semigroup S with 0 is (h, k) regular if and only if S is a union of its minimal ideals each of which is an (h, k) type homogeneous n regular and completely 0-simple semigroup, where $n = hk$ [2, Theorem 7].

(F) Let x, y and e be elements of a Brandt semigroup S such that $e = e^2$ and $ex = ey \neq 0$. Then $x = y$ [7, Lemma 2. 6].

3. Preliminary results. We need the following lemmas to prove our theorems.

LEMMA 1. *Every non-zero idempotent of an (h, k) regular semigroup S is primitive.*

PROOF. Let $e = e^2 \in S \setminus 0$ and suppose that e is not primitive; that is, assume that $f = f^2 \in S \setminus 0$ such that $f \leqq e$. Then $ef = fe = f$ and, consequently $Sf \subset Se$.

Moreover $e \in Se \setminus Sf$. Hence Sf can contain at most $k-1$ idempotents. This contradiction shows that e is primitive.

This lemma 1 was shown by N. R. Reilly.

LEMMA 2. *Let S be a $(1, n)$ regular semigroup with 0. If e and f are non-zero idempotents of S, and if $f \notin E(Se \setminus 0)$, then $fE(Se \setminus 0) = \{0\} = E(Se \setminus 0)f$.*

PROOF. If $f=0$, then Lemma 2 holds. Assume that $f \notin E(Se \setminus 0)$ and let $fe_i \neq 0$ for some $e_i \in E(Se \setminus 0)$. Setting $L = Sf$ and $R = e_i S$, by (B), L and R are 0-minimal left and right ideals of S, respectively; by (C), $LR \neq \{0\}$ implies that $RL \neq \{0\}$. Applying (A), RL is a group with 0. Let g be the identity of $RL = e_i Sf$ and let $g = e_i sf$ $(s \in S)$. By (B), $gS = e_i S$, whence $g = e_i$ by the $(1, n)$ regularity of S. It is easy to see that $Se_i = Sf = Se$, which implies that $f \in E(Se \setminus 0)$, contrary to the fact that $f \notin E(Se \setminus 0)$. We conclude that $fE(Se \setminus 0) = \{0\}$. A similar proof holds for $E(Se \setminus 0)f = \{0\}$. This completes the proof of Lemma 2.

MUNN [7] introduced the two following conditions C1 and C2 on a semigroup S with 0:

C1. If a, b, and c are elements of S such that $abc=0$, then either $ab=0$ or $bc=0$;

C2. If M and N are two non-zero ideals of S, then $M \cap N \neq \{0\}$.

LEMMA 3. *Let S be a $(1, n)$ regular Rees matrix semigroup with 0.*

(i) *If L is a 0-minimal left ideal of S, then $E(L \setminus 0)$ is a left zero semigroup. If L_i and L_j are arbitrary 0-minimal left ideals of S, then*

$$E(L_i)E(L_j) = E(L_j)E(L_i) = \begin{cases} \{0\} & \text{if } L_i \cap L_j = \{0\}, \\ E(L_i) & \text{otherwise.} \end{cases}$$

(ii) *S satisfies* C1.

(iii) *If $aeb=0$ then either $ab=0$ or $eb=0$, where e, a and b lie in S and $e=e^2 \neq 0$.*

PROOF. It is easy to check that (i) and (ii) hold.

(iii) Let a, b and e be elements in S with $e = ee \neq 0$ and $aeb=0$. By C1, either $ae=0$ or $eb=0$. Assume that $eb \neq 0$. We proceed to show that $ab=0$. If $a=0$ or $b=0$, then $ab=0$; hence assume that $a \neq 0 \neq b$. Then, by (D), there exist two sets $V(a) = \{x_i\colon ax_i a = a$ and $x_i a x_i = x_i;\ i=1, 2, \dots, n\}$ and $V(b) = \{y_j\colon by_j b = b$ and $y_j b y_j = y_j;\ j = 1, 2, \dots, n\}$. Putting $e_i = x_i a$ and $f_j = by_j$, e_i and f_j are non-zero idempotents. Since $ab = (ax_i a)(by_j b) = a(e_i f_j)b$, the proof will be concluded when we show that $e_i f_j = 0$ for some i and j. Suppose that $e_i f_j \neq 0$ for some i and j. By Corollary 2. 49 of [1], Lemma 2 and Lemma 3 (i), we have $0 \neq e_i f_j = e_i$. From $eb \neq 0$, it follows that $eb = e(by_j)b = ef_j b$ and $ef_j \neq 0$; again by (i), $ef_j = e$ and $f_j e = f_j$. Observe $e_i f_j = e_i(f_j e) = (e_i f_j)e = e_i e = (x_i a)e = x_i(ae) = x_i 0 = 0$, which contradicts the fact that $e_i f_j \neq 0$. Hence we conclude that $e_i f_j = 0$ for some i and j, and so that $ab=0$. This proves Lemma 3 (iii).

LEMMA 4. *Let S be a $(1, n)$ regular Rees matrix semigroup with 0. Let ϱ be a relation on S defined by $a \varrho b$ if and only if there exists a set $\{e_i\colon i = 1, 2, \dots, n\}$ of n non-zero idempotents such that $0 \neq e_i a = e_i b$ for all $i = 1, 2, \dots, n$.*

(i) *If ϱ is a proper congruence and f is a non-zero idempotent of a 0-minimal left ideal L of S, then $E(L \setminus 0)$ forms a ϱ-class containing f.*

(ii) *If ϱ is a proper congruence and if A is a non-zero idempotent of S/ϱ, then there exists a non-zero idempotent a in S such that $\varrho_*(a)=A$, where ϱ_* is the natural homomorphism of S onto S/ϱ.*

PROOF. (i) Suppose that ϱ is a proper congruence on S defined by the rule in the above. Let e be a non-zero idempotent in a 0-minimal left ideal L and let $E(L\setminus 0)=\{f_i: i=1,2,\dots,n\}$ by definition of $(1,n)$ regular. If f belongs to $E(L\setminus 0)$ then $f_if=f_i=f_ie$ $(i=1,2,\dots,n)$. Hence $E(L\setminus 0)$ is contained in e_ϱ, the ϱ-class containing e. On the other hand, if $g=g^2$ and $g\varrho e$, then there exists a set $\{g_i: i=1,2,\dots,n\}$ of n non-zero idempotents such that $0\neq g_ig=g_ie$ $(i=1,2,\dots,n)$. From this and Lemma 3 (i), we see that $g\in L$ as desired.

(ii) This follows from Lemma 2. 2 in [4].

REMARK. In the proof of Theorem 1, we shall see that the relation ϱ defined in Lemma 4 is a proper congruence on a $(1,n)$ regular Rees matrix semigroup S with 0.

4. Theorems. THEOREM 1. *Let S be a $(1,n)$ regular Rees matrix semigroup with 0. Define a relation ϱ on $S\setminus 0$ in such a way that $a\varrho b$ if and only if there exists a set $\{f_i: i=1,2,\dots n\}$ of n distinct non-zero idempotents such that $f_ia=f_ib\neq 0$, for every $i=1,2,\dots,n$. Then ϱ is an equivalence on $S\setminus 0$. If we extend ϱ to S by defining $\{0\}$ to be a ϱ-class, then ϱ is a proper Brandt congruence on S. Furthermore, if σ is any congruence on S with the property that S/σ is a Brandt semigroup, then $\sigma\supseteqq\varrho$ (in other words, ϱ is the minimum Brandt congruence on S).*

PROOF. We show first that ϱ is an equivalence on $S\setminus 0$. Let a be a fixed non-zero element of S and let b be any element in $S\setminus 0$ such that $ba\neq 0$. The existence of such b follows from $SaS=S$. By (D), there exists $F_b=\{f_i: i=1,2,\dots,n\}$ such that $bf_i=b$ $(i=1,2,\dots,n)$. The symmetric property is immediate. To show that ϱ is transitive, let $a\varrho b$ and $b\varrho c$, for a, b, c in $S\setminus 0$. Then there exist two sets $\{f_i: i=1,2,\dots,n\}$ and $\{e_i: i=1,2,\dots,n\}$ of non-zero idempotents of S such that $f_ia=f_ib\neq 0$ and $e_ib=e_ic\neq 0$ for every $i=1,2,\dots,n$. Now consider $e_if_ja=e_if_jb$ $(i=1,2,\dots,n)$. Suppose that $e_if_jb=0$, for some i and j. Then by Lemma 3 (iii), either $e_ib=0$ or $f_jb=0$, each of which provides a contradiction. Hence $e_if_jb\neq 0$ and $e_if_j\neq 0$ for every i and j. By Lemma 2 and Lemma 3 (i), $e_if_j=e_i$. By direct calculation $0\neq e_if_jb=(e_if_j)b=e_ib=e_ic$, $0\neq e_if_jb=e_i(f_jb)=e_i(f_ja)=(e_if_j)a=e_ia$, and thus we see that $e_ia=e_ic\neq 0$ $(i=1,2,\dots,n)$ and $a\varrho c$. Hence ϱ is an equivalence on $S\setminus 0$. Now let ϱ be extended to be an equivalence on S defining $\{0\}$ to be a ϱ-class. Let $a\varrho b$ (a, b in S) and let x be in S. To show that ϱ is a congruence on S we have to verify that $ax\varrho bx$ and that $xa\varrho xb$. If $a=b=0$ these results clearly hold. Hence we suppose that there is a set $\{e_i: i=1,2,\dots,n\}$ of non-zero idempotents such that $e_ia=e_ib\neq 0$ for every i, and so $xe_ia=xe_ib$. Consider first the case $xe_ia\neq 0$. Then $xe_i\neq 0$, which shows that e_i and x lie in the same 0-minimal left ideal of S. For if e_i and x are not contained in the same 0-minimal left ideal, assume that $e_i\in L_i$ and $x\in L_j$, where L_i and L_j are distinct 0-minimal left ideals. Let f be a non-zero idempotent of L_j, then $xf=x$. By Lemma 3 (i), $fe_i=0$, and so $xe_i=(xf)e_i=$ $=x(fe_i)=x0=0$, contrary to $xe_i\neq 0$. Since x and e_i are in the same 0-minimal left ideal, we have $xe_i=x$, which implies that $xa=(xe_i)a=x(e_ia)=x(e_ib)=(xe_i)b=$ $=xb\neq 0$. Choose a set $\{g_k: k=1,2,\dots,n\}$ of non-zero idempotents of S such that $g_k x \neq 0$ for every $k=1, 2, ..., n$. By C1 and Lemma 3 (ii), $0 \neq g_k xa = g_k xb$ for every $k=1, 2, ..., n$, and hence $xa\varrho xb$. Consider the other case $xe_i a = 0$. By Lemma 3 (iii), either $xa=0$ or $e_i a=0$. But the latter provides a contradiction, and so $xa=0$. Similarly, $xb=0$ and $xa\varrho xb$. We shall show that $ax\varrho bx$. Since $e_i a = e_i b$, $e_i ax = e_i bx$ $(i=1, 2, ..., n)$. If $e_i ax = 0$, then, by C1, either $ax=0$ or $e_i a=0$. But $e_i a \neq 0$, and so $ax=0$. Analogously, $bx=0$. If $e_i ax \neq 0$ then $e_i ax = e_i bx \neq 0$. Consequently, $ax\varrho bx$. Thus ϱ is a congruence on S, and it is proper, for if $0\varrho x$, then $x=0$. Clearly S/ϱ is regular. We claim that S/ϱ is an inverse semigroup. It suffices to show that idempotents of S/ϱ commute. Let A and B be two non-zero idempotents of S/ϱ. Then there exist a and b in $E(S\setminus 0)$ such that $\varrho_*(a)=A$ and $\varrho_*(b)=B$ by Lemma 4 (ii); by Lemma 4 (i), there exist 0-minimal left ideals L and L' of S containing a and b, respectively, such that $\varrho_*(a)=E(L\setminus 0)$ and $\varrho_*(b)=E(L'\setminus 0)$. By Lemma 3 (i), $AB=E(L\setminus 0)E(L'\setminus 0)=E(L'\setminus 0)E(L\setminus 0)=BA$; hence any two idempotents of S/ϱ commute with each other and S/ϱ is an inverse semigroup. By Lemma 3. 10 of [1], S/ϱ is completely 0-simple. Thus S/ϱ is a Brandt semigroup.

Finally, let σ be any proper Brandt congruence on S and let $a \neq 0$ and b be two elements in S with $a\varrho b$. Then, there exists a set $\{e_i : i=1, 2, ..., n\}$ of n distinct non-zero idempotents of S such that $e_i a = e_i b \neq 0$ $(i=1, 2, ..., n)$. Then $\sigma_*(e_i)\sigma_*(a) = \sigma_*(e_i)\sigma_*(b) \neq o$, which implies that $\sigma_*(a)=\sigma_*(b)$ by (F), where o is the zero of S/σ. Thus $a\sigma b$ and we have shown that $a\varrho b$ implies $a\sigma b$, or $\varrho \subseteqq \sigma$. This completes the proof of Theorem 1.

REMARK. In Theorem 1, we can replace $(1, n)$ and $e_i a = e_i b \neq 0$ with $(n, 1)$ and $ae_i = be_i \neq 0$, respectively.

We shall use Lemma 5 to prove Theorem 2.

LEMMA 5. *Let S be a $(1, n)$ regular semigroup with* 0.

(i) *For e in $E(S\setminus 0)$, SeS is a completely 0-simple and $(1, n)$ type homogeneous n regular semigroup.*

(ii) *S is a mutually annihilating sum of completely 0-simple semigroups* $\{SeS: e \in E(S\setminus 0)\}$ [3, Theorem 3].

PROOF. By (E) and Lemma 1, (i) follows. Hence (ii) is proved by (i) and Theorem 3 in [3].

THEOREM 2. *Let S be a $(1, n)$ regular semigroup with 0. If we define a relation ϱ on S by the rule that $a\varrho b$ if and only if there exists a set $\{e_i: i=1, 2, ..., n\}$ of n non-zero idempotents in S such that $e_i a = e_i b \neq 0$ for every $i=1, 2, ..., n$, then ϱ is an equivalence on $S\setminus 0$. If we extend ϱ to S by defining $\{0\}$ to be a ϱ-class, then ϱ is a proper inverse semigroup congruence on S. Furthermore, ϱ is the minimum such congruence.*

PROOF. By Lemma 1, every non-zero idempotent of S is primitive. By Lemma 5 (ii), $S = \bigcup \{SeS: e \in E(S\setminus 0)\}$ is the annihilating sum of $(1, n)$ regular and completely 0-simple semigroups. For each non-zero idempotent e, by Theorem 1, there exists a proper Brandt congruence ϱ_e on SeS defined by the rule that $a\varrho_e b$ (a, b in SeS) if and only if there exists a set $\{e_i: i=1, 2, ..., n\}$ with $e_i a = e_i b$ $(i=1, 2, ..., n)$. Hence if we define a relation ϱ on S by the rule as defined in the theorem, then ϱ is a proper congruence by Lemma 5(ii) and Theorem 1. By an application of Lemmas 5 (ii), 3 (i) and 4, we can show that idempotents of S/ϱ commute, and hence S/ϱ is an inverse semigroup by Theorem 1. 17 of [1].

Finally, we shall show that ϱ is the finest such congruence on S. To do this, let us assume that σ is any proper inverse semigroup congruence on S. Let $a\varrho b$ and $e_i a = e_i b \neq 0$ $(i = 1, 2, ..., n)$. Then $\sigma_*(e_i)\sigma_*(a) = \sigma_*(e_i)\sigma_*(b) \neq o$, where σ_* is the natural homomorphism of S onto S/σ and o is the zero of S/σ. By (E), let $V(a) = \{x_i\colon\ i = 1, 2, ..., n\}$ for $a \neq 0$. Since idempotents $\sigma_*(e_i)$ and $\sigma_*(ax_j)$ of S/σ commute, $\sigma_*(e_i)\sigma_*(a) = \sigma_*(e_i)\sigma_*(ax_j a) = \sigma_*(e_i)\sigma_*(ax_j)\sigma_*(a) = \sigma_*(ax_j)\sigma_*(e_i)\sigma_*(a) = = \sigma_*((ax_j)e_i)\sigma_*(a) = \sigma_*(ax_j)\sigma_*(a) = \sigma_*(a)$, because e_i and ax_j are contained in a left zero semigroup $E(L \setminus 0)$, for some 0-minimal left ideal L. Similarly, we can show that $\sigma_*(e_i)\sigma_*(b) = \sigma_*(b)$, and $\sigma_*(a) = \sigma_*(b)$. Thus $a\sigma b$. Hence $a\varrho b$ implies $a\sigma b$. This completes the proof of Theorem 2.

THEOREM 3. *Let S be a $(1, n)$ regular semigroup with* 0. *Then the condition* C2 *is necessary and suficient to ensure the existence of a proper Brandt congruence on S.*

PROOF. By Lemma 1, every non-zero idempotent of S is primitive. It is easy to see that S satisfies C1. If S satisfies C2, then $\bigcup\{SeS\colon\ e \in E(S \setminus 0)\} = SeS = S$ for $e \in E(S \setminus 0)$. The proof of the rest follows from Theorem 1 and [7, Theorem 1. 1].

Theorems 1 and 2 have appeared in [2, Theorems 4 and 5] without proofs.

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SOME REMARKS ON THE VECTOR SUBSPACES OF CYCLIC GALOIS EXTENSIONS

By

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Let F be a finite field and let E be a finite extension of F. Consider E as a vector space over F. In [1] the author proved two results on the vector subspaces of E. The object of this note is to show that these results hold in the more general setting of an arbitrary Galois extension with cyclic Galois group. The results, discussed in this paper, seem to be in connection with the "error correcting codes" from the information theory.

Let F be any field and let E be a cyclic extension of F of degree n. Suppose Φ is a generator of the Galois group of E/F. Consider E as a vector space over F and view Φ as a linear transformation of E into itself. Identify the element $\alpha \in E$ with the linear transformations of E into itself which sends β to $\alpha \cdot \beta$. We have the following.

THEOREM 1. *Every linear transformation of E into E can be written uniquely as $f(\Phi)$ where $f(x) \in E[x]$ is a polynomial of degree $\leqq n-1$.*

PROOF. Let $\xi_1, \xi_2, \dots, \xi_n$ be a field basis for E/F and consider the n^2 linear transformations $\xi_i \Phi^j$ $i=1, 2, \dots, n$; $j=0, 1, \dots, n-1$. Since the space of linear transformations of E into itself is of dimension n^2 over F, to prove the theorem it suffices to show that these n^2 linear transformations are independent over F. Thus suppose $a_{ij} \in F$ and $\sum_{i,j} a_{ij}\xi_i\Phi^j = 0$, i.e. for all $\alpha \in E$ $\sum_{i,j} a_{ij}\xi_i\Phi^j(\alpha) = 0$. Rewriting this as

$$\sum_{j=0}^{n-1} \left(\sum_{i=1}^{n} a_{ij}\xi_i \right) \Phi^i(\alpha) = 0,$$

we have by Dedekind's Theorem on the independence of isomorphisms that

$$\sum_{i=1}^{n} a_{ij}\xi_i = 0$$

for each $j=0, 1, \dots, n-1$. Since the ξ_i are a basis for E/F this gives $a_{ij}=0$ for all $i=1, 2, \dots, n$; $j=0, 1, \dots, n-1$.

We want now to prove some results on the vector subspaces of E/F. The following ideas which are introduced in [1] will be used. Denote by $E_\Phi[x]$ the set of polynomials $f(x) = a_0 + a_1 x + \dots + a_s x^s$ with coefficients in E with addition defined as usual but with multiplication defined by

$$x^i * x^j = x^{i+j} \qquad i, j = 0, 1, 2, \dots$$

$$x * a = \Phi(a)x \qquad a \in E.$$

Relative to these operations $E_\Phi[x]$ is a non-commutative (for $n>1$) integral domain. In addition it is easy to check that both a right and left division algorithm hold in $E_\Phi[x]$ and consequently right and left ideals in $E_\Phi[x]$ are principal. The center of $E_\Phi[x]$ consists of all polynomials of the form $a_0+a_1x^n+a_2x^{2n}+\ldots+a_sx^{sn}$ with $a_i\in F$. Note that the multiplication in $E_\Phi[x]$ is such that for $f(x)$,

$$g(x)\in E_\Phi[x], \quad \big(f(x)*g(x)\big)_{x=\Phi}=f(\Phi)\cdot g(\Phi).$$

We now have

THEOREM 2. *Let V be an s-dimensional subspace of E. Then there exists a polynomial $f(x)\in E_\Phi[x]$ of degree s such that* $\ker f(\Phi)\supset V$.

PROOF. Suppose first that V is a one dimensional subspace and choose a basis ξ of V. The polynomial $f(x)=x-\Phi(\xi)\cdot\xi^{-1}$ then has the required property. Now assume that the theorem is true for subspaces of dimension s and let V be an $(s+1)$-dimensional subspace. Write $V=U\oplus F\cdot\xi$ where U is of dimension s. By assumption there exists a polynomial $g(x)$ of degree s such that $\ker g(\Phi)\supset U$. If $g(\Phi)(\xi)=0$, then the polynomial $f(x)=x*g(x)$ is of degree $s+1$ and satisfies $\ker f(\Phi)\supset V$. If $g(\Phi)(\xi)\neq 0$, then the polynomial $f(x)=[x-(g(\Phi)(\xi))^{-1}\cdot\xi]*g(x)$ has the required property.

THEOREM 3. *Let V be an s-dimensional subspace of E. Then there exists a polynomial f of degree $\leqq s$ such that* $\ker f(\Phi)=V$.

PROOF. Consider the set I_V of all polynomials $h(x)$ which satisfy $h(\Phi)(\alpha)=0$ for all $\alpha\in V$. It is clear that I_V is a left ideal. Let $f(x)$ be a generator for I_V. By Theorem 2 $\deg f(x)\leqq s$. On the other hand since there always exists a linear transformation whose kernel is precisely V, Theorem 1 implies there exists $h(x)\in I_V$ such that $h(\Phi)$ has kernel V. Since $h(\Phi)=u(\Phi)\cdot f(\Phi)$ for some polynomial u, it follows that $\ker f(\Phi)=V$.

COROLLARY 4. *Let $f(x)\in E_\Phi[x]$ be a polynomial of degree s. Then the nullity of $f(\Phi)\leqq s$.*

PROOF. Let $V=\ker f(\Phi)$ and let $t=\dim V$. According to the proof of Theorem 3, I_V is generated by a polynomial $g(x)$ of degree $\leqq t$. Since $f(x)\in I_V$, $f(x)$ is a left multiple of $g(x)$ and therefore $t\leqq s$.

COROLLARY 5. *Let $\xi_1, \xi_2, \ldots, \xi_s$ be elements of E which are independent over F. Then the $s\times s$ matrix with entries $\Phi^i(\xi_j)$ $i=0, 1, \ldots, s-1$; $j=1, 2, \ldots, s$ is non-singular.*

PROOF. Suppose on the contrary that this matrix is singular. Then there exist $a_0, a_1, \ldots, a_{s-1}\in E$ not all zero such that

$$\sum_{i=0}^{s-1} a_i\Phi^i(\xi_j) = 0$$

for $j=1, 2, \ldots, s$. But then the non-zero polynomial $g(x)=a_0+a_1x+\ldots+a_{s-1}x^{s-1}$

which is of degree $\leqq s-1$ would have the property that $g(\Phi)$ annihilates the subspace of E spanned by $\xi_1, \xi_2, \dots, \xi_s$. This contradicts Corollary 4.

For $f, h \in E_\Phi[x]$ let us say that f is a *right divisor* of h if and only if there exists $g \in E_\Phi[x]$ such that $h = g * f$. We now state

THEOREM 6. *There is a 1—1 correspondence between the subspaces V of E of dimension s and the monic right divisors of x^n-1 of degree s. If f is such a monic right divisor, then this correspondence is described by assigning to f the subspace $V = \ker f(\Phi)$.*

PROOF. Let f be a monic right divisor of x^n-1 of degree s. Suppose $x^n-1 = g*f$. Thus $\deg g = r = n-s$. Set $V = \ker f(\Phi)$ and $U = \ker g(\Phi)$. By Corollary 5, $\dim V \leqq s$ and $\dim U \leqq r$.

Since $\dim V = \dim \ker f(\Phi) \leqq s$, we have $\dim \operatorname{im} f(\Phi) \geqq n-s = r$. On the other hand $\operatorname{im} f(\Phi) \subset U = \ker g(\Phi)$ which gives $\dim \operatorname{im} f(\Phi) \leqq r$. Therefore, we must have $\dim \operatorname{im} f(\Phi) = r$ and consequently $\dim V = \dim \ker f(\Phi) = s$.

Now let V be an s-dimensional subspace of E. We wish to show that there exists a monic polynomial $f(x)$ of degree s which is a right divisor of x^n-1 and which satisfies $V = \ker f(\Phi)$. Let I_V be the left ideal of polynomials which annihilate V and let $f(x)$ be a monic generator for I_V. From the proof of Theorem 3 we have that $\ker f(\Phi) = V$ and $\deg f(x) \leqq s$. On the other hand Corollary 4 implies $\deg f(x) \geqq s$. Since $x^n - 1 \in I_V$, $f(x)$ is a right divisor of x^n-1. Finally the description of f as the generator of I_V gives that f is the unique monic polynomial of degree s that annihilates V.

COROLLARY 7. *There exists a 1—1 correspondence between subspaces U of dimension s and monic left divisors g of x^n-1 of degree $r = n-s$. If g is such a monic left divisor, then this correspondence is described by assigning to g the subspace $U = \{g(\Phi)(\alpha) \colon \alpha \in E\}$.*

PROOF. Let g be a monic left divisor of x^n-1 of degree $r = n-s$ and let $U = g(\Phi)(E)$. Write $x^n - 1 = g*f$. Since $x^n-1 \in$ center of $E_\Phi[x]$, $g*f*g = (x^n-1)*g = g*(x^n-1)$ and therefore $f*g = x^n-1$. By Theorem 6 the kernel of $g(\Phi)$ is of dimension r and therefore $U = \operatorname{im} g(\Phi)$ is of dimension s.

Now let U be a subspace of dimension s. Choose f of degree s such that $U = \ker f(\Phi)$ and $x^n-1 = g*f$. Since $x^n-1 = f*g$, Theorem 6 gives that $\ker g(\Phi)$ is of dimension r and therefore $\operatorname{im} g(\Phi)$ is of dimension s. But $f(\Phi)\cdot g(\Phi) = 0$ implies $\operatorname{im} g(\Phi) \subset U$ and U is of dimension s. Therefore, $\operatorname{im} g(\Phi) = U$.

Finally suppose g and g' are both monic divisors of x^n-1 of degree r and that $g(\Phi)(E) = U = g'(\Phi)(E)$. Write $g*f = x^n-1 = g'*f'$. Then $f*g = x^n-1 = f'*g'$ and therefore $f(\Phi)g(\Phi)(E) = f(\Phi)(U) = 0 = f'(\Phi)(U) = f'(\Phi)g'(\Phi)(E)$ and by Theorem 6 $f = f'$ and therefore $g = g'$.

REMARKS. In conclusion we point out that the results of [1] are easily obtained from the above theorems and corollaries. Thus suppose that F is a finite field of q elements. It is only necessary to show that if $g*f = x^n-1$ where $f(x) = a_0 + a_1x + \dots + a_sx^s (a_s = 1)$, $g(x) = b_0 + b_1x + \dots + b_rx^r (b_r = 1)$ then the polynomials $\bar{f}(x) = a_0x + a_1x^q + \dots + a_sx^{q^s}$ and $\bar{g}(x) = b_0x + b_1x^q + \dots + b_rx^{q^r}$ split completely in E. Since for $\alpha \in E f(L)(\alpha) = \bar{f}(\alpha)$, $f(L)$ has a null space of dimension s, and $\bar{f}$

is of degree q^s, we have that $\bar{f}$ splits. On the other hand $g(L)\cdot f(L)=0$ implies $\bar{g}(\bar{f}(\alpha))=0$ for all $\alpha\in E$. Since the image of $f(L)$ has dimension r, $\bar{f}(\alpha)$ runs over a set of order q^r as α runs over E. Since $\bar{g}$ is of degree q^r and has the maximum possible number of roots, $\bar{g}$ also splits completely in E.

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NOTE ON A THEOREM OF ERDŐS AND RÉNYI

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Let $f(z)$ be an entire function. Let us denote by $N_k(f(z), 1)$ the number of zeros of $f^{(k)}(z)$, the k-th derivative of $f(z)$, in $|z| \leqq 1$. Let us denote by $\mu(r)$ the maximum term of the power series of $f(z)$ for $|z| = r$, and by $\nu(r)$ the rank of the maximum term.

The following Theorem has been proved by ERDŐS and RÉNYI ([2], Theorem A).

THEOREM A ([2], p. 223). *If $f(z)$ is an entire function of finite order $\varrho \geqq 1$, and $x = H(y)$ denotes the inverse function of $Y = \log M(x)$, where $M(r) = \max_{|z|=r} |f(z)|$, then $N_k(f(z), 1)$, denoting the number of zeros of $f^{(k)}(z)$ in $|z| \leqq 1$, satisfies the condition*

$$\liminf_{k \to \infty} \frac{N_k(f(z), 1) H(k)}{k} \leqq e^{2 - \frac{1}{\varrho}}. \tag{1}$$

As a consequence of the above theorem, ERDŐS *and* RÉNYI *have obtained, for entire functions $f(z)$ of exponential type τ,*

$$\liminf_{k \to \infty} N_k(f(z), 1) \leqq \tau e, \tag{2}$$

(see [2], p. 225, (15)). In this note, first we improve the inequality (1) of Theorem A by replacing order ϱ by lower order λ assumed to be such that $1 \leqq \lambda < \infty$ and allowing the case $\varrho = \infty$. Secondly we show that, for functions of irregular growth, we can have a better estimate than that of (2) as in (7) or (8). Lastly in Theorem III, we improve a theorem of PÓLYA in much the same way as we improve (1) to (6).

LEMMA 1 ([7], p. 220, (3)). *If $f(z)$ is an entire function of order $\varrho \geqq 0$, lower order $\lambda (0 \leqq \lambda \leqq \varrho \leqq \infty)$, then*

$$\liminf_{r \to \infty} \frac{\nu(r)}{\log \mu(r)} \leqq \lambda. \tag{3}$$

LEMMA 2. *If $f(z)$ is an entire function of order $\varrho\, (0 < \varrho < \infty)$, lower order $\lambda\, (0 \leqq \lambda \leqq \varrho)$, type τ $(0 \leqq \tau \leqq \infty)$, then*

$$\liminf_{r \to \infty} \frac{\nu(r)}{r^{\varrho}} \leqq \lambda \tau. \tag{4}$$

PROOF. Since

$$\frac{\nu(r)}{r^{\varrho}} = \frac{\nu(r)}{\log \mu(r)} \cdot \frac{\log \mu(r)}{r^{\varrho}},$$

we have at once

$$\liminf_{r\to\infty}\frac{\nu(r)}{r^\varrho} \leqq \liminf_{r\to\infty}\frac{\nu(r)}{\log\mu(r)}\cdot\limsup_{r\to\infty}\frac{\log\mu(r)}{r^\varrho}$$

which, along with lemma 1 and definition of type, gives us

$$\liminf_{r\to\infty}\frac{\nu(r)}{r^\varrho} \leqq \lambda\tau.$$

LEMMA 3 ([7], p. 220, (6)). *If $f(z)$ is an entire function of order ϱ $(0<\varrho<\infty)$, lower type ω, then*

(5) $$\liminf_{r\to\infty}\frac{\nu(r)}{r^\varrho} \leqq \varrho\omega.$$

THEOREM I. *In Theorem A, if the hypothesis $1\leqq$ order $\varrho<\infty$, is changed to $1\leqq$ lower order $\lambda<\infty$, the conclusion* (1) *will be true with λ instead of ϱ, i.e.*

(6) $$\liminf_{k\to\infty}\frac{N_k(f(z),1)H(k)}{k} \leqq e^{2-\frac{1}{\lambda}}.$$

Proof of Theorem I is exactly like that of Theorem A, with one difference. The former uses $\liminf_{r\to\infty}\frac{\nu(r)}{\log\mu(r)}\leqq\lambda$ of Lemma 1 where the latter uses $\liminf_{r\to\infty}\frac{\nu(r)}{\log\mu(r)}\leqq\varrho$ ([2], p. 224, (7)).

APPLICATIONS OF THEOREM I. Any result got by applying Theorem A can be improved by applying Theorem I instead. For instance, such an improvement, consisting in the replacement of finite $\varrho\geqq 1$ by finite $\lambda\geqq 1$, is possible in the case of the following known results of C. RÉNYI: [3], Theorem 1, (1. 1), Theorem 3, (2. 4); [4], Theorem II, (3), Theorem III, (6).

THEOREM II. *If $f(z)$ is an entire function of finite exponential type τ, lower type $\omega(0\leqq\omega\leqq\tau)$, lower order $\lambda(0\leqq\lambda\leqq\varrho=1)$, then*

(7) $$\liminf_{k\to\infty}N_k(f(z),1) \leqq \lambda\tau e.$$

(8) $$\liminf_{k\to\infty}N_k(f(z),1) \leqq \omega e.$$

PROOF. It has been shown ([1] p. 132 (30)) that, for any $\varepsilon>0$ and $r>r_0(\varepsilon)$

(9) $$N_{\nu(r)}(f(z),1)\leqq\frac{\nu(r)}{r}e(1+\varepsilon).$$

From (9) and Lemma 2 with $\varrho=1$, it follows that there is a sequence r_n $(n=1,2,3,\ldots)$ for which $r_n\Rightarrow\infty$ and

$$\liminf_{n\to\infty}N_{\nu(r_n)}(f(z),1)\leqq\lambda\tau e(1+\varepsilon),$$

$\varepsilon>0$ being arbitrary, this leads at once to (7). Similarly (9) and Lemma 3 together lead to (8).

REMARKS ON (7), (8). (7), (8) are improvements on the corresponding previously proved, inequality ([2], p. 225, (15)) with $\varrho=1$ instead of λ in (7) and τ instead of ω in (8). For instance it is known ([6], p. 1051) that there is an entire function of order $\varrho=1$, lower order $\lambda=\frac{1}{2}$. For this function (7) is an improvement on its previously proved version $\varrho=1$.

The next Theorem is given by PÓLYA ([5] p. 183, (4f)), with finite ϱ (=order of $f\geqq 1$) instead of λ(=lower order of $f\geqq 1$).

THEOREM III. *If $f(z)$ is an entire function of lower order $\lambda\geqq 1$, then, in the notation of Theorem* A,

$$\liminf_{k\to\infty}\frac{\log N_k(f(z),1)}{\log k}\leqq\frac{\lambda-1}{\lambda}. \tag{10}$$

PROOF. From (9), we get successively

$$\frac{\log N_{\nu(r)}(f(z),1)}{\log \nu(r)}\leqq\frac{\log \nu(r)-\log r+\log e(1+\varepsilon)}{\log \nu(r)}\qquad (r>r_0).$$

Now by taking lower limits on both sides we have,

$$\liminf_{r\to\infty}\frac{\log N_{\nu(r)}(f(z),1)}{\log \nu(r)}\leqq 1-\limsup_{r\to\infty}\frac{\log r}{\log \nu(r)}=1-\frac{1}{\lambda},$$

where we use the well known fact ([8], Theorem 1)

$$\limsup_{r\to\infty}\frac{\log r}{\log \nu(r)}=\frac{1}{\lambda}.$$

Hence, arguing as in the proof of (7) of Theorem II, we get the desired conclusion.

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ON THE CONVERGENCE OF STRONGLY MULTIPLICATIVE ORTHOGONAL SERIES

By

G. ALEXITS (Budapest), member of the Academy

1. The well known three series theorem in probability theory due to KOLMOGOROV can be expressed in the language of orthogonal series roughly as follows:

If $\{\varphi_n(x)\}$ *is a system of stochastically independent functions of mean value zero and* L^2*-norm* 1, *then the condition*

$$\sum_{n=1}^{\infty} c_n^2 < \infty \tag{1}$$

is sufficient for the convergence almost everywhere of the series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x). \tag{2}$$

If $\sum_{n=1}^{\infty} \operatorname{mes}(E_n) < \infty$, *where* $E_n = E(x: |c_n\varphi_n(x)| \geqq K)$ *for an arbitrary* $K>0$, *then* (1) *is also necessary for the convergence almost everywhere of* (2).

Some years ago we have introduced a notion (see [1], pp. 186—196), more general than the independence, calling $\{\varphi_n(x)\}$ a strongly multiplicatively orthogonal system (SMS), if

$$\int_a^b \varphi_{n_1}^{r_1}(x)\varphi_{n_2}^{r_2}(x)\ldots\varphi_{n_m}^{r_m}(x)\, d\mu(x) = 0$$

for every combination of $n_1 < n_2 < \cdots < n_m$, $r_k = 1$ or 2 and at least one $r_k = 1$. For a stochastically independent system of mean value zero, this condition is satisfied even for arbitrary integer r_k's, if at least one $r_k = 1$. An SMS is called equinormed, if also

$$\int_a^b \varphi_{n_1}^{2}(x)\varphi_{n_2}^{2}(x)\ldots\varphi_{n_m}^{2}(x)\, d\mu(x) = C$$

is satisfied with an appropriate constant C. Using an idea of TANDORI wehave proved the following theorem ([1], p. 189):

If $\{\varphi_n(x)\}$ *is a uniformly bounded and equinormed* SMS, *then* (1) *implies the convergence almost everywhere of* (2).

This theorem could be considered as a partial generalization of Kolmogorov's statement, if there did not occur the troublesome condition of equinorming. Indeed, on one hand, it is logically not clear why the convergence of (2) is influenced by the norms of the products of the functions $\varphi_n(x)$, on the other hand, it is not always possible to equinorm an SMS. Consider e.g. the following system defined in $[-1, 1]$:

$$\varphi_n(x) = \begin{cases} r_n(x) & \text{for} \quad 0 \leqq x \leqq 1, \\ 1 & \text{for} \quad -2^{-(n-1)} \leqq x \leqq -3\cdot 2^{-(n+1)}, \\ -1 & \text{for} \quad -3\cdot 2^{-(n+1)} < x < -2^{-n}, \\ 0 & \text{elsewhere,} \end{cases}$$

where $r_n(x)$ denotes the n-th Rademacher function. $\{\varphi_n(x)\}$ is obviously an SMS with $\mu(x)=x$. To equinorm it, we have to consider the function

$$\bar{\varphi}_n(x) = \sqrt{\frac{C}{1+2^{-n}}}\cdot\varphi_n(x)$$

instead of $\varphi_n(x)$, because only then is

$$\int_{-1}^{1} \bar{\varphi}_n^2(x)\,dx = C.$$

But taking, for instance, the product of two different $\bar{\varphi}_\nu^2(x)$, we obtain

$$\int_{-1}^{1} \bar{\varphi}_m^2(x)\bar{\varphi}_n^2(x)\,dx = \frac{C^2}{(1+2^{-m})(1+2^{-n})} \neq C.$$

This means just that the SMS $\{\varphi_n(x)\}$ can not be equinormed.

It seems to have some interest to get rid of the condition of equinorming without weakening the statement of our theorem, because then we would have actually a partial generalization of the three series theorem. Now we shall prove that the uniform boundedness of an SMS is enough for our statement without any condition about the norms. So we shall obtain the following

THEOREM 1. *If $\mu(x)$ is a positive, bounded, non-decreasing function and $\{\varphi_n(x)\}$ a uniformly bounded* SMS, *then* (1) *is a sufficient condition for the convergence almost everywhere of* (2).

If the $\varphi_n(x)$ are normed and we have also

$$\int_E \varphi_n^2(x)\,d\mu(x) \geqq K\cdot|E|_\mu \qquad (n=1, 2, \ldots) \tag{3}$$

with $K>0$ for every set E of μ-measure $|E|_\mu>0$, then (1) *is also a necessary condition for the convergence almost everywhere of* (2).

We have to remark that P. RÉVÉSZ ([2], pp. 92—96) has proved a convergence theorem for the series (2) under a weaker condition than the strong multiplicative orthogonality, but his postulate is stronger than (1).

After the proof of our Theorem 1, we shall consider applications to probability theory, especially, we shall obtain a strong law of large numbers (Corollary 2), which seems to be sharper than the classical one.

2. The proof of the announced theorem is based on following

LEMMA. *Let* $\{\varphi_n(x)\}$ *be an* SMS *having the properties*

$$|\varphi_n(x)| \leqq 1, \tag{4}$$

$$\liminf_{n\to\infty} \int_a^b \varphi_n^2(x)\, d\mu(x) > 0. \tag{5}$$

Then (1) *implies the convergence almost everywhere of* (2).

PROOF. Let $\{\psi_n(x)\}$ be the product system generated by $\{\varphi_n(x)\}$, i.e.

$$\psi_n(x) = \prod_{k=1}^{m} \varphi_{\nu_k+1}(x),$$

when the binary representation of n is $2^{\nu_1}+2^{\nu_2}+\dots 2^{\nu_m}$. For $n=0$ we put $\psi_0(x)\equiv 1$. Denoting by $s_n(x)$ the n-th partial sum of the series (2) and by $S_n(x)$ that of the series $\sum a_k\psi_k(x)$, where

$$a_k = \begin{cases} c_k & \text{for} \quad k = 2^\nu, \\ 0 & \text{for} \quad k \neq 2^\nu, \end{cases}$$

we have

$$S_{2^n-1}(x) = s_n(x).$$

Let $n(x)$ be the least index ν for which

$$s_{n(x)}(x) = \max_{1\leqq\nu\leqq n} s_\nu(x),$$

then $\{s_{n(x)}(x)\}$ is a non-decreasing sequence. Let

$$\|\psi_n\|^2 = \int_a^b \psi_n^2(x)\, d\mu(x)$$

and consider the sum

$$\sigma_{2^n-1}(x) = \sum_{k=1}^{2^n-1} \frac{a_k\psi_k(x)}{\|\psi_k\|^2}.$$

We get the following estimation:

$$\left|\int_a^b s_{n(x)}(x)\, d\mu(x)\right| = \left|\int_a^b S_{2^{n(x)}-1}(x)\, d\mu(x)\right| =$$

$$= \left|\int_a^b\int_a^b \sigma_{2^n-1}(t) \sum_{k=0}^{2^{n(x)}-1} \psi_k(t)\psi_k(x)\, d\mu(t)\, d\mu(x)\right| \leqq$$

$$\leqq \left\{\int_a^b \sigma_{2^n-1}^2(t)\, d\mu(t) \int_a^b \left(\int_a^b \sum_{k=0}^{2^n-1} \psi_k(t)\psi_k(x)\, d\mu(x)\right)^2 d\mu(t)\right\}^{\frac{1}{2}}.$$

By definition of a_k and $\psi_k(x)$ we have

$$\int_a^b \sigma_{2^n-1}^2(t)\,d\mu(t) = \sum_{k=0}^{2^n-1} \frac{a_k^2}{\|\psi_k\|^2} = \sum_{k=1}^{n} \frac{c_k^2}{\|\varphi_k\|^2}.$$

From (5) it follows the existence of a number $q>0$ such that $\|\varphi_k\|^2>q$, thus

$$\int_a^b \sigma_{2^n-1}^2(t)\,d\mu(t) \leqq \frac{1}{q}\sum_{k=1}^{\infty} c_k^2 \leqq K^2.$$

Therefore, introducing the symbol $n(x,y)=\min\big(n(x),n(y)\big)$ and applying the well known method of evaluation due to KOLMOGOROV—SELIVERSTOV and PLESSNER, we obtain

$$(6)\qquad \left|\int_a^b s_{n(x)}(x)\,d\mu(x)\right| \leqq K\left\{\int_a^b\left(\int_a^b \sum_{k=0}^{2^{n(x)}-1} \psi_k(t)\psi_k(x)\,d\mu(x)\right)^2 d\mu(t)\right\}^{\frac{1}{2}} =$$

$$= K\left\{\int_a^b\int_a^b\int_a^b \sum_{k=0}^{2^{n(x)}-1} \psi_k(t)\psi_k(x)\cdot \sum_{k=0}^{2^{n(y)}-1} \psi_k(t)\psi_k(y)\,d\mu(t)\,d\mu(x)\,d\mu(y)\right\}^{\frac{1}{2}} =$$

$$= K\left\{\int_a^b\int_a^b \sum_{k=0}^{2^{n(x,y)}-1} \|\psi_k\|^2\psi_k(x)\psi_k(y)\,d\mu(x)\,d\mu(y)\right\}^{\frac{1}{2}} \leqq$$

$$\leqq 2K\left\{\int_a^b\int_a^b \left|\sum_{k=0}^{2^{n(x)}-1} \|\psi_k\|^2\psi_k(x)\psi_k(y)\right| d\mu(x)\,d\mu(y)\right\}^{\frac{1}{2}}.$$

For the estimation of the last integral, remark first that

$$(7)\qquad \sum_{k=0}^{2^{n(x)}-1} \|\psi_k\|^2\psi_k(x)\psi_k(y) = \int_a^b \sum_{k=0}^{2^{n(x)}-1} \psi_k(t)\psi_k(x) \sum_{k=0}^{2^{n(x)}-1} \psi_k(t)\psi_k(y)\,d\mu(t).$$

On account of the connection between the system $\{\varphi_n(x)\}$ and its product system $\{\psi_n(x)\}$, one can see immediately that

$$\sum_{k=0}^{2^{n(x)}-1} \psi_k(t)\psi_k(x) = \prod_{k=1}^{n(x)} \big(1+\varphi_k(t)\varphi_k(x)\big).$$

Since, by (4), we have $1+\varphi_k(t)\varphi_k(x)\geqq 0$, the integrand in (7) in non-negative, hence

$$\sum_{k=0}^{2^{n(x)}-1} \|\psi_k\|^2\psi_k(x)\psi_k(y) \geqq 0.$$

Thus we can omit the sign of absolute value in the last integral of (6) and then we get

$$\left|\int_a^b s_{n(x)}(x)\,d\mu(x)\right| \leqq 2K\left\{\int_a^b\int_a^b \sum_{k=0}^{2^{n(x)}-1} \|\psi_k\|^2\psi_k(x)\psi_k(y)\,d\mu(x)\,d\mu(y)\right\}^{\frac{1}{2}}.$$

Integrating over y and considering that $\{\varphi_n(x)\}$ is an SMS, one has

$$\int_a^b \psi_k(y)\, d\mu(y) = 0 \qquad (k \geqq 1).$$

Thus we obtain finally

$$\left|\int_a^b s_{n(x)}(x)\, d\mu(x)\right| \leqq$$

$$\leqq 2K \left\{\int_a^b\int_a^b \|\psi_0\|^2 \psi_0(x)\psi_0(y)\, d\mu(x)\, d\mu(y)\right\}^{\frac{1}{2}} = 2K\{\mu(b)-\mu(a)\}^{\frac{1}{2}}.$$

The sequence $\{s_{n(x)}(x)\}$ being non-decreasing, it follows from B. Levy's theorem that the sequence $\{s_n(x)\}$ has a finite upper bound almost everywhere and the same is true for $\{-s_n(x)\}$. The convergence a.e. of the sequence $\{s_n(x)\}$ is a known consequence of the fact that the upper bound of $\{|s_n(x)|\}$ is finite a.e. Indeed, (1) implies $\sum c_n^2\mu_n^2 < \infty$ with $\mu_n^2 \nearrow \infty$. Hence, to show that $\sum c_n\varphi_n(x)$ converges a.e., we have to apply only an Abel transformation to the series $\sum c_n\mu_n\varphi_n(x)\cdot\mu_n^{-1}$ and our lemma is entirely proved.

3. Proof of Theorem 1. The necessity part is already proved, because we have shown ([1], p. 194) that, if $\sum c_n^2 = \infty$ and (3) is satisfied for a normed system $\{\varphi_n(x)\}$, then the series (2) is divergent a.e. and even not summable by any permanent method.

Concerning the sufficiency, extend first the function $\mu(x)$ to the interval $[b, b+1]$ by putting

$$\bar{\mu}(x) = \begin{cases} \mu(x) & \text{for} \quad a \leqq x \leqq b, \\ \mu(b)+x-b & \text{for} \quad b \leqq x \leqq b+1. \end{cases}$$

Then $\bar{\mu}(x)$ is positive, bounded, non-decreasing and, in $[b, b+1]$, equivalent to the Lebesgue measure. Let M be the upper bound of $|\varphi_n(x)|$ for $n=1, 2, \ldots$ and put

$$\Phi_n(x) = \begin{cases} \dfrac{\varphi_n(x)}{M} & \text{for} \quad a \leqq x \leqq b, \\ \bar{r}_n(x) & \text{for} \quad b < x \leqq b+1, \end{cases}$$

where $\bar{r}_n(x)$ denotes the n-th Rademacher function defined in $[b, b+1]$. Obviously $\{\Phi_n(x)\}$ is an SMS in $[a, b+1]$ according to the $\bar{\mu}$-measure. Moreover, we have

$$\Phi_n(x) \leqq 1 \qquad (a \leqq x \leqq b+1;\ n=1, 2, \ldots)$$

and

$$\int_a^{b+1} \Phi_n^2(x)\, d\bar{\mu}(x) \geqq \int_b^{b+1} \bar{r}_n^2(x)\, dx = 1.$$

Thus the conditions (4) and (5) of our lemma are satisfied, hence $\sum c_n\Phi_n(x)$ converges a. e. in the larger interval $[a, b+1]$. But $\Sigma c_n \bar{r}_n(x)$ being convergent a. e. in $[b, b+1]$, from

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) = \sum_{n=1}^{\infty} c_n \Phi_n(x) - \sum_{n=1}^{\infty} c_n \bar{r}_n(x)$$

it follows the convergence a. e. of $\sum c_n \varphi_n(x)$ in $[a, b]$, as we have stated.[1]

4. The foregoing result contains also the following generalization of a theorem proved by TANDORI and myself ([1], p. 189):

COROLLARY 1. *If* $\{\varphi_n(x)\}$ *is an* SMS *such that*

$$|\varphi_n(x)| \leqq M_n \qquad (n=1, 2, \ldots)$$

where $0<M_1 \leqq M_2 \leqq \ldots$, *then the convergence of* $\sum c_n^2 M_n^2$ *implies the convergence almost everywhere of the series* (2).

Indeed, we have to apply Theorem 1 to the series $\sum c_n M_n \bar{\varphi}_n(x)$ with $\bar{\varphi}_n(x) = \varphi_n(x)/M_n$ and our statement follows at once.

5. For the application of the foregoing results to probability theory, we have to introduce the notion of stochastic SMS. Denote by $\mathbf{E}(\xi)$ the expectation of the random variable ξ and by $\mathbf{P}(A)$ the probability of the event A. We call the sequence of random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ weakly quasi independent, if for every combination of indices $N=(n_1<n_2<\ldots n_m)$ there exists a number $K_N \neq 0$ depending on N such that

$$\mathbf{E}(\xi_{n_1}^{r_1} \xi_{n_2}^{r_2} \ldots \xi_{n_m}^{r_m}) = K_N \cdot \mathbf{E}(\xi_{n_1}^{r_1}) \mathbf{E}(\xi_{n_2}^{r_2}) \ldots \mathbf{E}(\xi_{n_m}^{r_m})$$

where $r_k = 1$ or 2. The sequence $\{\xi_n\}$ is a stochastic SMS, if it is weakly quasi independent and

$$\mathbf{E}(\xi_n) = 0 \qquad (n=1, 2, \ldots).$$

It is evident that Theorem 1 and Corollary 1 remain valid for a stochastic SMS, if we substitute $\mathbf{E}(\xi_n)$ for $\int \varphi_n(x) d\mu(x)$ and $\mathbf{P}\left(\sum_{k=1}^{n} c_k \xi_k \to \xi\right) = 1$ for the convergence a. e. of $\sum c_n \varphi_n(x)$. Thus we can pass over to the strong law of large numbers.

THEOREM 2. *Let* $\{\xi_n\}$ *be a stochastic* SMS *such that* $|\xi_n| \leqq M_n$ *where* $M_n \nearrow$. *Choosing two sequences of positive numbers* $\{p_n\}$ *and* $\{q_n\}$ *satisfying the conditions*

$$\sum_{n=1}^{\infty} \frac{p_n^2}{q_n^2} < \infty, \tag{8}$$

$$q_n < q_{n+1} \to \infty, \tag{9}$$

we have

$$\mathbf{P}\left(\frac{\sum_{k=1}^{n} p_k \xi_k}{q_n M_n} \to 0\right) = 1.$$

[1] The idea to ensure the validity of (5) by extension of $\varphi_n(x)$ to a larger interval arouse in course of a talk with P. Révész.

Indeed, we may apply Corollary 1 to the series $\sum c_n \xi_n$ with

$$c_n = \frac{p_n}{q_n M_n},$$

since $\sum c_n^2 M_n^2 < \infty$ by (8). Thus the series

$$\sum_{n=1}^{\infty} \frac{p_n \xi_n}{q_n M_n}$$

converges with probability 1. By (9) the sequence $\{q_n M_n\}$ tends to infinity monotonically, therefore it follows by a well known lemma of Kronecker that

$$\mathbf{P}\left(\sum_{k=1}^{n} p_k \xi_k = o(q_n M_n)\right) = 1,$$

as we asserted.

This theorem admits many different forms of the strong law of large numbers. We emphasize the following one, which seems to be a sharpening of the classical result.

COROLLARY 2. *Let $\{\xi_n\}$ be a stochastic* SMS *such that $|\xi_n| \leqq M_n$, where $M_n n^{-1} \searrow 0$ and $\sum M_n^2/n^2 < \infty$. Then*

$$\mathbf{P}\left(\frac{\sum_{k=1}^{n} \xi_k}{n} \to 0\right) = 1.$$

Put $p_n = 1$ and $q_n = n/M_n$. Then the hypothesis of Theorem 2 is satisfied and our statement follows.

6. For uniformly bounded systems $\{\xi_n\}$ Theorem 2 gets close to the theorem of iterated logarithm. We conjecture that this last one is valid actually in the following form: If $\{\xi_n\}$ is a uniformly bounded stochastic SMS, then there exists a constant $K > 0$ such that

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} \xi_k}{\sqrt{n \log\log n}} \leqq K\right) = 1.$$

Added in proof (30 October 1969). In the meantime we have proved the conjecture of § 6. The proof will be published in *Studia Sci Math. Hung.*

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GLEICHMÄSSIGE APPROXIMATION UND GLEICHMÄSSIGE STETIGKEIT

Von

Á. CSÁSZÁR (Budapest)

Herrn Prof. G. ALEXITS *zum 70. Geburtstag*

1. Es sei $\mathscr{U}$ eine uniforme Struktur auf einer Grundmenge E. Wir bezeichnen mit $C(\mathscr{U})$ bzw. $C^*(\mathscr{U})$ die Gesamtheit aller bezüglich $\mathscr{U}$ gleichmäßig stetigen bzw. beschränkten und gleichmäßig stetigen reellen Funktionen. Es ist leicht einzusehen, daß die Funktionenklasse $\Phi = C^*(\mathscr{U})$ einen, die konstanten Funktionen enthaltenden Ring, einen Vektorraum und einen Verband bildet und außerdem bezüglich der gleichmäßigen Konvergenz abgeschlossen ist. M. a. W. $\Phi = C^*(\mathscr{U})$ erfüllt die folgenden Bedingungen:

(1. 1) *Jede reelle Konstante gehört zu* Φ,

(1. 2) *Aus* $f, g \in \Phi$ *folgt* $f+g \in \Phi$,

(1. 3) *Aus* $f, g \in \Phi$ *folgt* $fg \in \Phi$,

(1. 4) *Aus* $f \in \Phi$ *folgt* $-f \in \Phi$,

(1. 5) *Aus* $f \in \Phi$ *und* $\alpha > 0$ *folgt* $\alpha f \in \Phi$,

(1. 6) *Aus* $f, g \in \Phi$ *folgen* $\max(f, g) \in \Phi$ *und* $\min(f, g) \in \Phi$,

(1. 7) *Ist* f *gleichmäßiger Limes einer Folge* $\{f_n\} \subset \Phi$, *so gehört* f *zu* Φ.

Selbstverständlich gilt für $\Phi = C^*(\mathscr{U})$ auch:

(1. 8) *Jede Funktion* $f \in \Phi$ *ist beschränkt.*

Es ist bekannt, daß einige dieser Eigenschaften genügen um die Funktionenklasse $C^*(\mathscr{U})$ zu charakterisieren. Den Ergebnissen der Note [5] entnimmt man nämlich den folgenden Satz:

(A) *Ist* Φ *eine, die Konstanten enthaltende und bezüglich der gleichmäßigen Konvergenz abgeschlossene Klasse beschränkter reeller Funktionen auf* E, *die noch einen Ring oder einen Vektorverband bildet* (d.h. sind die Bedingungen (1. 1), (1. 2), (1. 4), (1. 7), (1. 8) und außerdem entweder (1. 3) oder (1. 5) und (1. 6) erfüllt), *so gibt es eine uniforme Struktur* $\mathscr{U}$ *auf* E *mit der Eigenschaft* $\Phi = C^*(\mathscr{U})$.[1]

Der Beweis von (A) läßt sich in [5] auf einen Approximationssatz zurückführen.

[1] S. [5], Sätze 1 und 6.

Es sei nämlich zuerst Φ eine beliebige Klasse auf E definierter reeller Funktionen. Wir bilden für $f \in \Phi$ die Abstände (Pseudometriken)

$$\sigma_f(x, y) = |f(x)-f(y)|.$$

Diese Abstände induzieren eine uniforme Struktur $\mathcal{U}(\Phi)$ auf E, und zwar ist $\mathcal{U}(\Phi)$ offenbar die gröbste uniforme Struktur bezüglich welcher alle Funktionen $f \in \Phi$ gleichmäßig stetig sind. Setzt man voraus, daß die Funktionen $f \in \Phi$ beschränkt sind, so besteht also $\Phi \subset C^*(\mathcal{U}(\Phi))$. Erfüllt Φ noch (1. 7), so braucht man, um $\Phi = C^*(\mathcal{U}(\Phi))$ zu beweisen, nur zu zeigen, daß jede bezüglich $\mathcal{U}(\Phi)$ gleichmäßig stetige Funktion gleichmäßiger Limes einer Folge $\{f_n\} \subset \Phi$ ist. Genau diese Tatsache wird vom folgenden Satz behauptet:

(B) *Φ genüge den Bedingungen* (1. 1), (1. 2), (1. 4), (1. 8) *und außerdem entweder* (1. 3) *oder* (1. 5) *und* (1. 6), *und f sei bezüglich $\mathcal{U}(\Phi)$ gleichmäßig stetig, d. h. es sollen zu $\varepsilon > 0$ eine endliche Folge $\{g_1, \ldots, g_m\}$ und eine Zahl $\delta > 0$ existieren mit der Eigenschaft, daß $|f(x)-f(y)| < \varepsilon$ ist, sobald*

$$x, y \in E \quad \text{und} \quad |g_i(x)-g_i(y)| < \delta \qquad (i=1, \ldots, m).$$

Dann gibt es zu $\eta > 0$ eine Funktion $h \in \Phi$ mit

$$|f(x)-h(x)| \leqq \eta \qquad (x \in E).$$ [2]

J. CZIPSZER und der Verfasser dieser Note haben in [2] bewiesen, daß die Voraussetzung, nach der Φ ein Ring bzw. ein Vektorverband ist, in Satz (B) abgeschwächt werden kann, und zwar genügt es vorauszusetzen, daß Φ entweder ein *affiner Verband* ist, d. h. die Bedingungen (1. 4), (1. 5), (1. 6) und (1. 9) erfüllt mit

(1. 9) *Aus $f \in \Phi$ folgt $f + \alpha \in \Phi$ für jede reelle Konstante α,*

oder daß Φ ein *Gruppenverband* ist, d. h. (1. 2), (1. 4) und (1. 6) erfüllt. Genauer gesagt, der folgende Approximationssatz wurde bewiesen:

(C) *Φ genüge den Bedingungen* (1. 1), (1. 4), (1. 6), (1. 8) *und außerdem noch entweder* (1. 2) *oder* (1. 5) *und* (1. 9). *Ist f bezüglich $\mathcal{U}(\Phi)$ gleichmäßig stetig, so gibt es zu $\eta > 0$ eine Funktion $h \in \Phi$ mit*

$$|f(x)-h(x)| \leqq \eta \qquad (x \in E).$$ [3]

Nun führt derselbe Gedankengang, mit Hilfe dessen (A) aus (B) abgeleitet wurde, zum

SATZ 1. *Φ sei eine, die Konstanten enthaltende Klasse beschränkter reeller Funktionen auf E, die entweder ein affiner Verband oder ein Gruppenverband und bezüglich der gleichmäßigen Konvergenz abgeschlossen ist* (d. h. Φ genüge (1. 1), (1. 4), (1. 6), (1. 7), (1. 8) und entweder (1. 2) oder (1. 5) und (1. 9)). *Dann gilt $\Phi = C^*(\mathcal{U})$ mit einer passend gewählten uniformen Struktur $\mathcal{U}$.*

[2] S. [4], S. 190 und [5], S. 58, Korollar.
[3] S. [2], (12) und (19), und die Bemerkung unter [2].

Zusammenfassend kann man also sagen:

SATZ 2. *Die folgenden Gruppen von Eigenschaften bilden je eine notwendige und hinreichende Bedingung dafür, daß die Funktionenklasse Φ mit $C^*(\mathcal{U})$ für eine geeignete uniforme Struktur $\mathcal{U}$ zusammenfällt:*

(1. 1), (1. 2), (1. 3), (1. 4), (1. 7), (1. 8);

(1. 1), (1. 2), (1. 4), (1. 5), (1. 6), (1. 7), (1. 8);

(1. 1), (1. 4), (1. 5), (1. 6), (1. 7), (1. 8), (1. 9);

(1. 1), (1. 2), (1. 4), (1. 6), (1. 7), (1. 8).

2. Durch Satz 2 werden die Funktionenklassen der Gestalt $C^*(\mathcal{U})$ sogar auf mehrere Weisen charakterisiert. Um eine ähnliche Charakterisierung für die Klassen der Gestalt $C(\mathcal{U})$ zu erhalten, vermerken wir zuerst, daß jede Funktionenklasse $\Phi = C(\mathcal{U})$ die Eigenschaften (1. 1), (1. 2), (1. 4), (1. 5), (1. 6) und (1. 7) (natürlich auch (1. 9)) besitzt, (1. 3) aber im allgemeinen nicht gültig ist.

Um eine weitere notwendige Bedingung zu erhalten, führen wir folgende Definition ein:

DEFINITION. f_n $(n=1, 2, \ldots)$ und g_i $(i=1, \ldots, m)$ seien reelle Funktionen auf E. Die Folge $\{f_n\}$ heiße *kohärent* bezüglich der *Basis* $\{g_1, \ldots, g_m\}$, wenn es eine positive Zahl γ_0 gibt mit der Eigenschaft, daß aus $0 < \gamma \leqq \gamma_0$, $x, y \in E$ und

(2. 1) $$|g_i(x) - g_i(y)| \leqq \gamma \qquad (i=1, \ldots, m)$$

immer

(2. 2) $$|f_n(x) - f_n(y)| \leqq \gamma \qquad (n=1, 2, \ldots)$$

folgt.

Man sieht nun leicht:

(2. 3) *$\mathcal{U}$ sei eine uniforme Struktur auf E. Ist eine Folge $\{f_n\}$ bezüglich der Basis $\{g_1, \ldots, g_m\} \subset C(\mathcal{U})$ kohärent, so ist $\{f_n\} \subset C(\mathcal{U})$, und, wenn f_n sogar auf E punktweise gegen eine endliche Funktion f konvergiert, dann gilt auch $f \in C(\mathcal{U})$.*

In der Tat, zu $\varepsilon > 0$ gibt es eine Nachbarschaft $U \in \mathcal{U}$ mit der Eigenschaft, daß aus $(x, y) \in U$ die Ungleichungen (2. 1) mit $\gamma = \min(\gamma_0, \varepsilon)$ folgen. Aus (2. 2) folgt also

$$|f_n(x) - f_n(y)| \leqq \varepsilon \qquad ((x, y) \in U);$$

falls $f_n \to f$, so ergibt sich noch

$$|f(x) - f(y)| \leqq \varepsilon \qquad ((x, y) \in U).$$

Jede Klasse $\Phi = C(\mathcal{U})$ besitzt also die Eigenschaft:

(2. 4) *Konvergiert die bezüglich der Basis $\{g_1, \ldots, g_m\} \subset \Phi$ kohärente Folge $\{f_n\} \subset \Phi$ punktweise gegen eine endliche Funktion f, so gehört f zu Φ.*

Den Approximationssätzen (B) und (C) entspricht nun der folgende

SATZ 3. *Φ sei ein Gruppenverband, der die Konstanten enthält und* (2. 4) *erfüllt. Zu jede bezüglich $\mathcal{U}(\Phi)$ gleichmäßig stetige Funktion f und $\eta > 0$ gibt es eine Funktion $h \in \Phi$ mit*

$$|f(x) - h(x)| \leqq \eta \qquad (x \in E).$$[4]

Der Beweis von Satz 3 wird auf dem weiteren Satz 5 beruhen. Aus ihm ergibt sich mit Hilfe des üblichen Gedankenganges:

SATZ 4. *Eine Klasse Φ auf E definierter reeller Funktionen läßt sich genau dann in der Form $\Phi = C(\mathcal{U})$ mit einer geeigneten uniformen Struktur $\mathcal{U}$ darstellen, wenn Φ ein, die Konstanten enthaltender und die Bedingungen* (1. 7) *und* (2. 4) *erfüllender Gruppenverband ist* (d.h. wenn Φ (1. 1), (1. 2), (1. 4), (1. 6), (1. 7) und (2. 4) genügt).

Die Frage, ob in Satz 3 oder Satz 4 (1. 2) durch (1. 9) ersetzt werden kann, bleibt offen.

Satz 3 ergibt sich aus dem noch etwas allgemeineren

SATZ 5. *Φ erfülle die Bedingungen* (1. 1), (1. 2), (1. 6) *und* (2. 4). *Die auf E definierte reelle Funktion f sei so beschaffen, daß es zu $\varepsilon > 0$ eine endliche Folge $\{g_1, \dots, g_m\} \subset \Phi$ und ein $\delta > 0$ gibt mit der Eigenschaft, daß aus $x, y \in E$ und*

$$g_i(y) - g_i(x) < \delta \qquad (i = 1, \dots, m)$$

immer

$$f(y) - f(x) < \varepsilon$$

folgt. Dann gibt es zu $\eta > 0$ ein $h \in \Phi$ mit

$$|f(x) - h(x)| \leqq \eta \qquad (x \in E).$$[5]

Satz 3 folgt aus Satz 5. In der Tat, Φ erfülle (1. 1), (1. 2), (1. 4), (1. 6), (1. 7) und (2. 4). f sei bezüglich $\mathcal{U}(\Phi)$ gleichmäßig stetig. Dann gibt es zu $\varepsilon > 0$ eine endliche Folge $\{g_1, \dots, g_m\} \subset \Phi$ und ein $\delta > 0$ mit der Eigenschaft, daß aus $x, y \in E$ und

$$|g_i(x) - g_i(y)| < \delta \qquad (i = 1, \dots, m)$$

immer

$$|f(x) - f(y)| < \varepsilon$$

[4] Ein ähnlicher Approximationssatz wurde in [3], (4.1) bewiesen; dort lauten aber die unserer Voraussetzung (2.4) entsprechenden Bedingungen wesentlich komplizierter.

[5] Es ist leicht zu sehen, daß die hier für f aufgestellte Voraussetzung damit identisch ist, daß f gleichmäßig stetig sein soll in bezug auf die von Φ erzeugte, d.h. aus den Quasi-Abständen

$$\tau_f(x, y) = \max(f(y) - f(x), 0) \quad (f \in \Phi)$$

auf E abgeleitete quasi-uniforme Struktur und die aus dem Quasi-Abstand

$$\tau(u, v) = \max(v - u, 0)$$

auf der Zahlengeraden abgeleitete quasi-uniforme Struktur. Für die hier verwendete Terminologie s. [1], S. 78, 117, 175, 177.

folgt. Die Folge $\{g_1, \ldots, g_m, -g_1, \ldots, -g_m\}$ ist nach (1. 4) ebenfalls in Φ enthalten, und die Ungleichungen

$$g_i(y)-g_i(x) < \delta, \quad -g_i(y)-(-g_i(x)) < \delta \qquad (i=1, \ldots, m)$$

haben

$$f(y)-f(x) < \varepsilon$$

zur Folge, so das f die Voraussetzungen von Satz 5 erfüllt. Daher gibt es zu $\eta > 0$ ein $h \in \Phi$ mit

$$|f(x)-h(x)| \leqq \eta \qquad (x \in E).$$

Somit brauchen wir nur Satz 5 zu beweisen.

Satz 3 läßt aber eine Verallgemeinerung in eine andere Richtung ebenfalls zu. Im Aufsatz [5] wurde nämlich gezeigt, daß die Bedingung (1. 1) in Satz (B) durch (2. 5) ersetzbar ist:

(2. 5) *Aus* $f \in \Phi$, $\alpha > 0$ *folgt* $\min(f, \alpha) \in \Phi$,

wenn man zugleich von f außer der gleichmäßigen Stetigkeit bezüglich $\mathcal{U}(\Phi)$ noch folgendes voraussetzt:

(2. 6) *Zu* $\varepsilon > 0$ *gibt es ein* $\delta > 0$ *und Funktionen* $g_1, \ldots, g_m \in \Phi$ *mit der Eigenschaft, daß aus* $|g_i(x)| < \delta$ $(i=1, \ldots, m)$ *immer* $|f(x)| < \varepsilon$ *folgt.*

Eine ähnliche Verallgemeinerung von Satz 3 werden wir nun beweisen, nämlich die folgende:

Satz 6. *Φ erfülle die Bedingungen* (1. 2), (1. 4), (1. 6), (2. 4) *und* (2. 5). *Genügt die auf E definierte und bezüglich $\mathcal{U}(\Phi)$ gleichmäßig stetige Funktion f der Bedingung* (2. 6), *so gibt es zu $\eta > 0$ ein $h \in \Phi$ mit*

$$|f(x)-h(x)| \leqq \eta \qquad (x \in E).$$

Es handelt sich tatsächlich um eine Verallgemeinerung von Satz 3; ist nämlich (1. 1) erfüllt, so trifft offenbar dasselbe für (2. 5) zu, und jede Funktion f genügt der Bedingung (2. 6), was man sogleich einsieht, wenn man $m=1$, $g_1=1$, $0<\delta<1$ setzt.

3. Beweis von Sätzen 5 und 6. Wir führen die Beweise dieser Sätze parallel. Die Voraussetzungen von Satz 5 bzw. Satz 6 werden wir kurz als (5) bzw. (6) erwähnen.

f erfülle also entweder (5) oder (6). Im Falle (6) dürfen wir offenbar $f \geqq 0$ annehmen. Bei gegebenem $\eta > 0$ seien die Funktionen $g_1, \ldots, g_m \in \Phi$ und die Zahl $\delta = \dfrac{\eta}{r} > 0$ mit ganzem r so gewählt, daß

$$g_i(y)-g_i(x) \leqq \delta \qquad (i=1, \ldots, m)$$

immer

$$f(y)-f(x) < \eta$$

zur Folge hat, und im Falle (6) noch so, daß aus

$$|g_i(x)| \leqq \delta \qquad (i=1, \ldots, m)$$

immer

$$|f(x)| < \eta$$

folgt. Ersetzt man g_i durch $\frac{2\eta}{\delta} g_i = 2rg_i \in \Phi$, was nach (1. 2) gestattet ist, so kann man bereits $\delta = 2\eta$ wählen. Es werde

$$A_k = \{x \in E: f(x) < k\eta\}, \tag{3. 1}$$

$$B_{k_1,\dots,k_m} = \{x \in E: (k_i - 1)\eta \leqq g_i(x) < k_i\eta \quad (i = 1, \dots, m)\} \tag{3. 2}$$

für jedes System $k, k_1, \dots, k_m$ von ganzen Zahlen gesetzt. Nun definieren wir im Falle (5)

$$h_{k_1,\dots,k_m} = \min\left(\eta, \max\left(0, g_1 - k_1\eta, \dots, g_m - k_m\eta\right)\right), \tag{3. 3}$$

im Falle (6) aber

$$h_{k_1,\dots,k_m} = \min\left(\eta, \max\left(g_1 - g_1', \dots, g_m - g_m'\right)\right), \tag{3. 3'}$$

wobei

$$g_i' = \min(g_i, \alpha_i), \quad \alpha_i = \max(k_i\eta, 1).$$

Dann gilt $h_{k_1, \dots, k_m} \in \Phi$, und zwar im Falle (5) nach (1. 1), (1. 2) und (1. 6), im Falle (6) aber wegen (2. 5), (1. 4), (1. 2) und (1. 6), und man hat die Ungleichungen

$$0 \leqq h_{k_1,\dots,k_m}(x) \leqq \eta \qquad (x \in E), \tag{3. 4}$$

ferner

$$h_{k_1,\dots,k_m}(x) = 0 \qquad (x \in B_{k_1,\dots,k_m}). \tag{3. 5}$$

Es sei nun

$$p_k(x) = \inf\{h_{k_1,\dots,k_m}(x): A_k \cap B_{k_1,\dots,k_m} \neq \emptyset\}, \tag{3. 6}$$

falls die Beziehung $A_k \cap B_{k_1, \dots, k_m} \neq \emptyset$ für mindestens ein System $(k_1, \dots, k_m)$ zutrifft, d.h. wenn $A_k \neq \emptyset$ ist, denn

$$\bigcup B_{k_1,\dots,k_m} = E. \tag{3. 7}$$

Im Falle $A_k = \emptyset$ sei $p_k(x) = \eta$ für $x \in E$, falls (5) zutrifft; gilt aber (6), so sei $p_k(x) = \eta$ nur für $k < 0$, für $k \geqq 0$ sei dagegen

$$p_k = \min\left(\eta, t \max\left(g_1, \dots, g_m, -g_1, \dots, -g_m\right)\right),$$

wobei t eine ganze Zahl mit $t\delta > \eta$ bedeutet. Allerdings gilt also wegen (3.4)

$$0 \leqq p_k(x) \leqq \eta \qquad (x \in E), \tag{3. 8}$$

und

$$p_k(x) = 0 \qquad (x \in A_k), \tag{3. 9}$$

denn $x \in A_k$ hat nach (3. 7) $x \in A_k \cap B_{k_1, \dots, k_m}$ für mindestens ein System $(k_1, \dots, k_m)$ zur Folge, also ist nach (3. 5) $h_{k_1, \dots, k_m}(x) = 0$ und dann $p_k(x) = 0$. Ferner gilt

$$p_k(x) = \eta \qquad (x \in E - A_{k+1}). \tag{3. 10}$$

Diese Gleichung ist trivial, wenn $A_k = \emptyset$ und $k < 0$, und ebenfalls für $k \geqq 0$ im Falle (5). Im Falle (6) ist aber für $k \geqq 0$ und $x \in E - A_{k+1}$

$$f(x) \geqq (k+1)\eta \geqq \eta,$$

also nach (2. 6)

$$\max\left(g_1(x), \ldots, g_m(x), -g_1(x), \ldots, -g_m(x)\right) \geqq \delta$$

und wiederum $p_k(x) = \eta$. Ist nun $z \in A_k \cap B_{k_1, \ldots, k_m}$, so hat man nach (3. 1)

$$f(x) - f(z) > \eta,$$

also

$$g_i(x) - g_i(z) \geqq \delta = 2\eta$$

für mindestens ein i, somit

$$g_i(z) \geqq (k_i - 1)\eta,$$

also

$$g_i(x) \geqq (k_i + 1)\eta$$

und nach (3. 3)

(3. 11) $$h_{k_1, \ldots, k_m}(x) = \eta.$$

Daher gilt (3. 11) für jedes System $(k_1, \ldots, k_m)$ mit $A_k \cap B_{k_1, \ldots, k_m} \neq \emptyset$, und (3. 10) folgt daraus nach (3. 6).

Wir zeigen nun, daß $p_k \in \Phi$ im Falle (5) und auch im Falle (6) für $k \geqq 0$. Im letztgenannten Falle ergibt sich diese Behauptung aus (1. 4), (1. 6), (1. 2) und (2. 5), im Falle (5) mit $A_k = \emptyset$ aus (1. 1), so daß man den Fall (5) und $A_k \neq \emptyset$ voraussetzen darf. In der Tat, aus der Tatsache, daß aus

$$|u(x) - u(y)| \leqq \gamma, \quad |v(x) - v(y)| \leqq \gamma$$

immer

$$|\max\left(u(x), v(x)\right) - \max\left(u(y), v(y)\right)| \leqq \gamma,$$

$$|\min\left(u(x), v(x)\right) - \min\left(u(y), v(y)\right)| \leqq \gamma$$

folgen, ergibt sich leicht, daß

(3. 12) $$|g_i(x) - g_i(y)| \leqq \gamma \qquad (i = 1, \ldots, m)$$

immer

(3. 13) $$|h_{k_1, \ldots, k_m}(x) - h_{k_1, \ldots, k_m}(y)| \leqq \gamma$$

zur Folge hat. Ist nun f_n das Minimum der ersten n unter (3. 6) vorkommenden Funktionen $h_{k_1, \ldots, k_m}$, so folgt nach der obigen Bemerkung aus (3.12) noch

(3. 14) $$|f_n(x) - f_n(y)| \leqq \gamma \qquad (n = 1, 2, \ldots).$$

Daher ist die Folge $\{f_n\}$ bezüglich der Basis $\{g_1, \ldots, g_m\} \subset \Phi$ kohärent, und (2. 4) ergibt $p_k = \lim f_n \in \Phi$, denn $f_n \in \Phi$ nach (1. 6). Außerdem folgt aus (3. 12) noch

(3. 15) $$|p_k(x) - p_k(y)| \leqq \gamma$$

für jede ganze Zahl k.

Es sei nun

(3. 16) $$q_k(x) = \begin{cases} p_k(x) & \text{für} \quad k \geqq 0, \\ p_k(x) - \eta & \text{für} \quad k < 0. \end{cases}$$

Dann ist $q_k \in \Phi$ nach (1. 1) und (1. 2) im Falle (5), trivialerweise im Falle (6), $k \geqq 0$, und wegen $q_k = 0$ nach (1. 4) und (1. 2) im Falle (6), $k < 0$.

Wir betrachten ein $x \in A_{s+1} - A_s$. Es sei zuerst $s \geqq 0$. Dann hat man wegen

$$A_{k_1} \subset A_{k_2} \qquad (k_1 < k_2)$$

die folgenden Gleichungen bzw. Ungleichungen:

für $k < 0$: $A_{k+1} \subset A_s$, $x \in E - A_{k+1}$, $p_k(x) = \eta$ (vgl. (3. 10)), $q_k(x) = 0$;

für $0 \leqq k \leqq s-1$: $A_{k+1} \subset A_s$, $x \in E - A_{k+1}$, $p_k(x) = q_k(x) = \eta$;

für $k = s$: $0 \leqq q_k(x) \leqq \eta$ (vgl. (3. 8));

für $k \geqq s+1$: $x \in A_k$, $p_k(x) = q_k(x) = 0$ (vgl. (3. 9)).

Nun sei $s < 0$. Dann gelten:

für $k \leqq s-1$: $A_{k+1} \subset A_s$, $x \in E - A_{k+1}$, $p_k(x) = \eta$, $q_k(x) = 0$;

für $k = s$: $0 \leqq p_k(x) \leqq \eta$, $-\eta \leqq q_k(x) \leqq 0$;

für $s+1 \leqq k < 0$: $x \in A_k$, $p_k(x) = 0$, $q_k(x) = -\eta$;

für $k \geqq 0$: $x \in A_k$, $p_k(x) = q_k(x) = 0$.

Das zeigt, daß die Reihe

$$h(x) = \sum_{k=-\infty}^{\infty} q_k(x) \tag{3. 17}$$

für $x \in A_{s+1} - A_s$ konvergiert (tatsächlich verschwinden die Glieder bis auf eine endliche Anzahl von ihnen), und

$$s\eta \leqq h(x) \leqq (s+1)\eta \qquad (x \in A_{s+1} - A_s) \tag{3. 18}$$

sowohl für $s \geqq 0$ als auch für $s < 0$. Mit Rücksicht auf (3. 1) erhält man aus (3. 18)

$$|f(x) - h(x)| \leqq \eta \qquad (x \in E).$$

Es bleibt übrig zu zeigen, daß $h \in \Phi$. Für

$$s_n = \sum_{k=-n}^{n} q_k \tag{3. 19}$$

gilt allerdings $s_n \in \Phi$ nach (1. 2). Es genügt zu beweisen, daß die Folge $\{s_n\}$ bezüglich der Basis $\{2g_1, \ldots, 2g_m\}$ kohärent ist. Setzt man $\gamma_0 = 4\eta$, $0 < \gamma \leqq \gamma_0$ und

$$|2g_i(x) - 2g_i(y)| \leqq \gamma \qquad (i = 1, \ldots, m), \tag{3. 20}$$

so müssen die Ungleichungen

$$g_i(y) - g_i(x) \leqq 2\eta$$

und

$$g_i(x) - g_i(y) \leqq 2\eta$$

für $i=1, \ldots, m$ bestehen, so daß

$$f(y)-f(x) < \eta, \quad f(x)-f(y) < \eta$$

d. h.

$$|f(x)-f(y)| < \eta.$$

Ist die Bezeichnung so gewählt, daß $f(x) \leqq f(y)$, so gibt es eine ganze Zahl s mit

(3. 21) $$s\eta \leqq f(x) < (s+1)\eta, \quad \text{also} \quad x \in A_{s+1} - A_s,$$

und man hat

(3. 22) $$s\eta \leqq f(y) < (s+2)\eta, \quad \text{also} \quad y \in A_{s+2} - A_s.$$

Aus (3. 21) und (3. 22) sieht man nun, daß

$$q_k(x) = q_k(y) \quad \text{für} \quad k \leqq s-1,$$

$$q_k(x) = q_k(y) \quad \text{für} \quad k \geqq s+2,$$

mit Rücksicht auf die oben angegebenen Werte von q_k auf $A_{s+1}-A_s$ bzw. $A_{s+2}-A_{s+1}$; die Fälle $s \geqq 0$, $s < -1$ und $s = -1$ müssen getrennt betrachtet werden. Nach (3. 15) und (3. 20) erhält man dann

$$|q_k(x)-q_k(y)| \leqq \frac{\gamma}{2} \quad \text{für} \quad k = s, \; s+1.$$

Daraus ergibt sich also

$$|s_n(x)-s_n(y)| \leqq \gamma \qquad (n=1, 2, \ldots),$$

sobald (3. 20) mit $0 < \gamma \leqq \gamma_0$ besteht. Damit ist alles bewiesen.

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BEMERKUNG ZUR STARKEN SUMMATION DER WALSH—FOURIER-REIHE

Von

F. SCHIPP (Budapest)

Herrn Prof. G. ALEXITS *zum 70. Geburtstag*

Einleitung

Es sei $\{r_n(x)\}$ $(n=0, 1, 2, \ldots)$ das Rademachersche System, d.h.

(1) $$r_0(x) = \begin{cases} 1 & (0 \leqq x < \frac{1}{2}), \\ -1 & (\frac{1}{2} \leqq x < 1), \end{cases} \quad r_0(x+1) = r_0(x),$$

$$r_n(x) = r_0(2^n x) \qquad (n = 1, 2, \ldots).$$

Das Walshsche Orthogonalsystem $\{\psi_n(x)\}$ $(n=0, 1, 2, \ldots)$ ist folgenderweise definiert: $\psi_0(x) \equiv 1$ und für

$$n = 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_s} \quad (\nu_1 > \nu_2 > \cdots > \nu_s \geqq 0)$$

ist

(2) $$\psi_n(x) = r_{\nu_1}(x) r_{\nu_2}(x) \ldots r_{\nu_s}(x).$$

Wir bezeichnen mit

$$S_n(f; x) = \sum_{\nu=0}^{n-1} \psi_\nu(x) \left(\int_0^1 f(t) \psi_\nu(t)\, dt \right)$$

die n-te Partialsumme der Walsh—Fourier-Entwicklung von $f(x)$. In der Arbeit [1] haben wir die folgende Behauptung gezeigt:

Es sei $p > 0$. *Ist* $f(x) \in L[0, 1]$, *dann gilt*

$$\frac{1}{n} \sum_{\nu=1}^{n} |S_\nu(f; x) - f(x)|^p \to 0 \qquad (n \to \infty)$$

fast überall.

In dieser Arbeit beweisen wir, daß diese Behauptung im gewissen Sinne unverbesserbar ist. Es gilt nämlich der folgende Satz.

SATZ. *Es sei* $\{\lambda_n\}$ *eine Folge von positiven Zahlen mit* $\lambda_n \to \infty$ $(n \to \infty)$. *Dann gibt es eine Funktion* $f(x) \in L[0, 1]$ *derart, daß*

(3) $$\varlimsup_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} |S_\nu(f; x) - f(x)|^{\lambda_\nu} = \infty$$

fast überall gilt.

K. TANDORI [2] hat einen analogen Satz für Fourierreihen bewiesen.

Ich möchte Herrn Professor K. TANDORI für seine wertvollen Ratschlägebei Fertigstellung dieser Arbeit meinen aufrichtigen Dank aussprechen.

§ 1. Hilfssätze

Wir bezeichnen mit $x=0,x_0x_1\ldots x_n\ldots$ die dyadische Entwicklung der Zahl $x\in[0,1)$, wobei wir festsetzen, daß für $x=p\cdot 2^{-q}$ alle x_i $(i\geqq q)$ gleich 0 sind. Dann gilt

(1. 1) $$r_n(x)=(-1)^{x_n}\qquad(n=0,1,2,\ldots).$$

N. J. FINE [3] hatte die folgende Operation eingeführt: es sei

(1. 2) $$x\mathring{+}y=\sum_{n=0}^{\infty}\frac{|x_n-y_n|}{2^n}$$

für

$$x=0,x_0x_1\ldots x_n\ldots=\sum_{n=0}^{\infty}\frac{x_n}{2^{n+1}},\qquad y=0,y_0y_1\ldots y_n\ldots=\sum_{n=0}^{\infty}\frac{y_n}{2^{n+1}}.$$

Es ist leicht zu verifizieren, daß die Menge $R\subset[0,1)$ der dyadisch rationalen Zahlen mit dieser Operation eine Abelsche Gruppe ist, und daß die Relationen

(1. 3) $$x\mathring{+}x=0,\quad \psi_n(x\mathring{+}y)=\psi_n(x)\psi_n(y)\quad\big(x\in[0,1),\,y\in R,\ n=0,1,2\ldots\big)$$

gelten.

Wir bezeichnen mit G die Menge der nichtnegativen ganzen Zahlen. In der Menge G führen wir eine Operation $\oplus$ folgenderweise ein:

(1. 4) $$k\oplus l=\sum_{\nu=0}^{\infty}|k_\nu-l_\nu|2^\nu$$

für

$$k=\sum_{\nu=0}^{\infty}k_\nu 2^\nu,\quad l=\sum_{\nu=0}^{\infty}l_\nu 2^\nu\qquad(k_\nu,l_\nu=0,1).$$

Offensichtlich wird G mit der Operation $\oplus$ zu einer Abelschen Gruppe und die nichtnegativen ganzen Zahlen, die Kleiner als 2^n sind, bilden eine Untergruppe G_n von G von der Ordnung 2^n wobei noch

(1. 5) $$\psi_k(x)\psi_l(x)=\psi_{k\oplus l}(x),$$

$$\frac{k}{2^n}\mathring{+}\frac{l}{2^n}=\frac{k\oplus l}{2^n}\quad(k,l\in G_n)$$

besteht.

Für jede natürliche Zahl n bezeichnet $D_n(x)$ die n-te Walsch—Dirichletsche Kernfunktion:

(1. 6) $$D_n(x)=\sum_{\nu=0}^{n-1}\psi_\nu(x).$$

Es ist leicht zu zeigen, daß für $n=2^k+n'$ $(0<n'\leqq 2^k)$

(1. 7) $$D_n(x)=D_{2^k}(x)+r_k(x)D_{n'}(x)$$

gilt, woraus sich für $n'=2^k$ die Gleichung

$$(1.8)\qquad D_{2^{k+1}}(x)=\prod_{\nu=0}^{k}(1+r_\nu(x))=\begin{cases}2^{k+1} & \left(x\in\left[0,\frac{1}{2^{k+1}}\right)\right),\\ 0 & \left(x\in\left[\frac{1}{2^{k+1}},1\right)\right)\end{cases}$$

ergibt.

Zur Konstruktion der Funktion $f(x)$ werden wir einige Hilfssätze benutzen.

Hilfssatz I. *Es sei* $n=\sum_{i=0}^{s}n_i2^i$ $(n_i=0,1)$. *Dann besteht*

$$(1.9)\qquad D_n(x)=\psi_n(x)\sum_{i=0}^{s}n_ir_i(x)D_{2^i}(x).$$

(Siehe: [4], Hilfssatz I.)

Hilfssatz II. *Es seien*

$$(1.10)\qquad A_{-1}=\{0\},\quad A_\nu=\{l\colon 2^\nu\leqq l<2^{\nu+1},\ l\in G\}\qquad(\nu=0,1,2,\ldots),$$

$$P(s,n)=\{-1\}\cup\{\nu\colon 0\leqq\nu<s,\ n_\nu=0\},\quad P(n)=\{\nu\colon n_\nu=1\},$$

$$B(n,\nu)=\{k\colon k=n\oplus l,\ l\in A_\nu\}\qquad(\nu=-1,0,1,\ldots),$$

wobei

$$n=\sum_{\nu=0}^{s-1}n_\nu2^\nu\qquad(n_\nu=0,1),$$

und

$$C(n)=\{l\colon 0\leqq l<n,\ l\in G\}.$$

Dann bestehen

$$(1.11)\qquad \text{a)}\quad \bigcup_{\nu\in P(n)}B(n,\nu)=C(n),$$

$$\text{b)}\quad \bigcup_{\nu\in P(s,n)}B(n,\nu)=G_s-C(n).$$

Beweis von Hilfssatz II. Wegen

$$\bigcup_{\nu=-1}^{s-1}B(n,\nu)=\bigcup_{\nu=-1}^{s-1}A_\nu=G_s,\quad B(n,\nu)\cap B(n,\nu')=\varnothing\qquad(\nu\neq\nu')$$

folgt (1.11) b) aus (1.11) a). Da die Anzahl der Elemente von $\bigcup_{\nu\in P(n)}B(n,\nu)$ gleich $\sum_{n_\nu=1}2^\nu=\sum_{\nu=0}^{s-1}n_\nu2^\nu=n$ ist, deshalb folgt (1.11) a) aus Ungleichung $k<n$ $(k\in B(n,\nu),$ $\nu\in P(n))$. Für $\nu\in P(n)$ gilt $n_\nu=1$, und so ist $k=n\oplus l$ $(l\in A_\nu)$ und $l=\sum_{i=0}^{\nu-1}l_i2^i+2^\nu$ $(l_i=0,1)$. Daraus folgt:

$$k=n\oplus l=\sum_{i=0}^{\nu-1}|n_i-l_i|2^i+\sum_{i=\nu+1}^{s-1}n_i2^i\leqq 2^\nu-1+\sum_{i=\nu+1}^{s-1}n_i2^i\leqq\sum_{i=0}^{s-1}n_i2^i-1=n-1,$$

womit Hilfssatz II bewiesen ist.

HILFSSATZ III. *Für* $k \in G_{2m^2}$ *und* $k = \sum_{i=0}^{2m^2-1} k_i 2^{2m^2-i-1}$ $(k_i = 0,1)$ *sei*

$$(1.12) \quad a_k^{(m)} = \frac{k}{2^{2m^2}} + \frac{k_{2m^2}}{2^{2m^2+1}} = \sum_{i=0}^{2m^2} \frac{k_i}{2^{1+i}} \quad (k = 0, 1, \dots, 2^{2m^2}-1;\ m = 1, 2, \dots),$$

wobei $k_{2m^2} = \varphi_m(k)$ $(=0, 1)$ *folgenderweise definiert wird:*

$$(1.13) \quad \varphi_m(k) = k_0 \oplus k_2 \oplus \cdots \oplus k_{2m^2-2} \oplus j = j \oplus \sum_{i=0}^{m^2-1}{}^{\circ}\, k_{2i} \quad (k \in A_{2\nu+j},\ j = 0,1).$$

($\sum^{\circ}$ *bedeutet die modulo 2 genommene Summe.*) *Es seien weiterhin*

$$(1.14) \quad N_m = \sum_{i=0}^{m^2} 2^{2i}, \quad E_m = \{\nu\colon -1 \leqq \nu < 2m^2\} \quad (m = 1, 2, \dots),$$

und für $P \subset E_m$ *setzen wir*

$$(1.15) \quad U_m(P; x) = \sum_{\nu \in P} \sum_{k \in A_\nu} \psi_{N_m}(x \mathring{+} a_k^{(m)}) \sum_{i=0}^{m^2-1} r_{2i}(x \mathring{+} a_k^{(m)}) D_{2^{2i}}(x \mathring{+} a_k^{(m)}).$$

Dann gilt

$$(1.16) \quad |U_m(P; x)| \geqq \frac{2^{2m^2}}{6} \sigma(P) \quad \left(x \in \left[0, \frac{1}{2^{2m^2}}\right]\right),$$

wobei $\sigma(P)$ *die Anzahl der Elemente von* P *bedeutet.*

BEWEIS VON HILFSSATZ III. Auf Grund von (2), (1. 1), (1. 8), (1. 12), (1. 13) und (1. 14) für $x \in \left[0, \frac{1}{2^{2m^2}}\right)$, $k \in A_\nu$ $(\nu \in E_m)$ und $0 \leqq i \leqq 2m^2 - 2$ gelten die folgende Gleichungen:

$$\psi_{N_m}(a_k^{(m)}) = \prod_{s=0}^{m^2} (-1)^{k_{2s}} = (-1)^\nu, \quad r_i(a_k^{(m)}) = \begin{cases} 1 & (0 \leqq i \leqq 2m^2 - \nu - 2), \\ -1 & (i = 2m^2 - \nu - 1), \end{cases}$$

$$D_{2^i}(x \mathring{+} a_k^{(m)}) = D_{2^i}(a_k^{(m)}) = \begin{cases} 2^i & (0 \leqq i \leqq 2m^2 - \nu - 1), \\ 0 & (i \geqq 2m^2 - \nu), \end{cases}$$

$$r_i(x) = 1 \quad (0 \leqq i \leqq 2m^2 - 1).$$

Daraus nach (1. 3) und (1. 10) für $\nu = 2s + j$ $(j = 0, -1)$, $x \in \left[0, \frac{1}{2^{2m^2}}\right)$ und $k \in A_\nu$ ergibt sich

$$(1.17) \quad V_m(\nu; x) = \sum_{k \in A_\nu} \psi_{N_m}(x \mathring{+} a_k^{(m)}) \sum_{i=0}^{m^2-1} r_{2i}(x \mathring{+} a_k^{(m)}) D_{2^{2i}}(x \mathring{+} a_k^{(m)}) =$$

$$= (-1)^\nu \psi_{N_m}(x) \sum_{k \in A_\nu} \left(\sum_{i=0}^{m^2-s-1} 2^{2i} + j 2^{2m^2-2s} \right) =$$

$$= \psi_{N_m}(x) \sum_{k \in A_\nu} (-1)^j \left[\left(\frac{1}{3} + j \right) 2^{2m^2-2s} - \frac{1}{3} \right] =$$

$$= \psi_{N_m}(x) \frac{2^{2m^2-\nu} - (-1)^\nu}{3} \sum_{k \in A_\nu} 1 = \psi_{N_m}(x) \delta_m(\nu),$$

wobei

$$\delta_m(\nu) = \begin{cases} \dfrac{2^{2m^2} - (-2)^\nu}{3} & (\nu \geqq 0), \\ \dfrac{2^{2m^2+1} + 1}{3} & (\nu = -1). \end{cases}$$

Aus (1. 17) auf Grund von (1. 15) erhalten wir

$$U_m(P; x) = \sum_{\nu \in P} V_m(\nu; x) = \psi_{N_m}(x) \sum_{\nu \in P} \delta_m(\nu) \qquad \left(x \in \left[0, \frac{1}{2^{2m^2}}\right)\right),$$

woraus sich die zu bewiesende Ungleichung (1. 16) ergibt.

HILFSSATZ IV. *Es seien* $\mu \in G_{2m^2}$, $\varrho_1, \varrho_2 \in B(\mu, \nu)$ $(\nu \in P(\mu)$ *und* $k \in B(\mu, \nu')$ $(\nu' \in P(2m^2, \mu))$. *Dann gilt*

$$\varphi_m(\varrho_1 \oplus k) \oplus \varphi_m(\varrho_2 \oplus k) = \varphi_m(\varrho_1 \oplus \mu) \oplus \varphi_m(\varrho_2 \oplus \mu).$$

BEWEIS VON HILFSSATZ IV. Auf Grund von (1. 10) folgt $k \oplus \mu \in A_{\nu'}$, $\varrho_1 \oplus \mu$, $\varrho_2 \oplus \mu \in A_\nu$. Da $\nu \neq \nu'$ ist, deshalb gilt

$$(k \oplus \mu) \oplus (\varrho_1 \oplus \mu) = k \oplus \varrho_1 \in A_{\bar\nu}, \quad (k \oplus \mu) \oplus (\varrho_2 \oplus \mu) = k \oplus \varrho_2 \in A_{\bar\nu} \quad (\bar\nu = \max(\nu, \nu'))$$

Daraus und aus der Definition von φ_m mit $\nu = 2\nu^* + j^*$, $\bar\nu = 2\nu^{**} + j^{**}$ $(j^*, j^{**} = 0, 1)$ erhalten wir

$$\varphi_m(\varrho_1 \oplus k) \oplus \varphi_m(\varrho_2 \oplus k) = \sum_{i=0}^{m^2-1}{}^{\circ} (\varrho_{2i}^{(1)} \oplus k_{2i}) \oplus j^{**} \oplus \sum_{i=0}^{m^2-1}{}^{\circ} (\varrho_{2i}^{(2)} \oplus k_{2i}) \oplus j^{**} =$$

$$= \sum_{i=0}^{m^2-1}{}^{\circ} (\varrho_{2i}^{(1)} \oplus \mu_{2i}) \oplus j^* \oplus \sum_{i=0}^{m^2-1}{}^{\circ} (\varrho_{2i}^{(2)} \oplus \mu_{2i}) \oplus j^* = \varphi_m(\varrho_1 \oplus \mu) \oplus \varphi_m(\varrho_2 \oplus \mu),$$

womit Hilfssatz IV bewiesen ist.

Aus Hilfssatz IV folgt, daß für $\mu \in G_{2m^2}$, $\varrho_j \in B(\mu, \nu)$ $(\nu \in P(\mu), j = 1, 2, \ldots, 2s)$ und $k \in B(\mu, \nu')$ $(\nu' \in P(2m^2, \mu))$ die Summe

$$\sum_{j=1}^{2s}{}^{\circ} \varphi_m(\varrho_j \oplus k)$$

von k unabhängig ist.

Es sei

$$H_m = \left\{\mu : \mu = \sum_{i=0}^{2m^2-1} \mu_i 2^i \ (\mu_i = 0, 1), \ \sum_{i=0}^{2m^2-1} \mu_i = m^2\right\}. \tag{1.18}$$

Wir bezeichnen mit M_m die Anzahl der Elemente von H_m. Offensichtlich gilt $M_m = \binom{2m^2}{m^2}$, woraus sich auf Grund der Stirlingschen Formel die Ungleichung

$$M_m \geqq \frac{1}{4} \frac{2^{2m^2}}{m} \qquad (m > m_1) \tag{1.19}$$

ergibt.

Wir setzen für $\mu \in H_m$

$$(1.20)\quad I(\mu) = \left\{l\colon l = \sum_{i=0}^{\mu-1} l_{2m^2+2+i} 2^{2m^2+2+i} (l_s = 0, 1),\ \sum_{i\in B(\mu,\nu)}^{\circ} l_{2m^2+2+i} = 0 (\nu \in P(\mu)\right\}.$$

Auf Grund von (1. 11) a) und (1. 18) ist leicht zu zeigen, daß die Anzahl der Elemente von $I(\mu)$ gleich $2^{\mu-m^2}$ ist.

HILFSSATZ V. *Die Ziffern $\beta^{(i)}(k, m)$ der dyadisch rationalen Zahl*

$$(1.21)\qquad b_k^{(m)} = \sum_{i=0}^{2^{2m^2}-1} \frac{\beta^{(i)}(k, m)}{2^{2m^2+2+i+1}} \qquad (k \in G_{2m^2},\ m = 1, 2, \ldots)$$

seien folgendermaßen definiert:

$$(1.22)\qquad \beta^{(i)}(k, m) = \varphi_m(i) \oplus \varphi_m(k) \oplus \varphi_m(i \oplus k),$$

ferner sei

$$(1.23)\qquad c_k^{(m)} = a_k^{(m)} \mathring{+} b_k^{(m)},$$

$$Q_m(l, \mu; x) = r_{\mu+2m^2+2}(x) \psi_l(x) \psi_{N_m}(x) \sum_{s=0}^{m^2-1} r_{2s}(x) D_{2^{2s}}(x) \qquad (l \in I(\mu)).$$

Dann gilt

$$(1.24)\qquad \sum_{k=\mu}^{2^{2m^2}-1} Q_m(l, \mu\colon x \mathring{+} c_k^{(m)}) =$$

$$= \vartheta r_{\mu+2m^2+2}(x) \psi_l(x) U_m\left(P(2m^2,\mu);\ x \mathring{+} \frac{\mu}{2^{2m^2}}\right) \qquad (\vartheta = \pm 1).$$

BEWEIS VON HILFSSATZ V. Aus der Definition der $c_k^{(m)}$ und aus (1. 12) folgt

$$(1.25)\qquad r_{2m^2+2+\mu}(c_k^{(m)}) = (-1)^{\beta^{(\mu)}(k,m)} = (-1)^{\varphi_m(\mu)\oplus\varphi_m(k)\oplus\varphi_m(\mu\oplus k)} =$$
$$= r_{2m^2}(a_\mu^{(m)} \mathring{+} a_k^{(m)}) r_{2m^2}(a_{k\oplus\mu}^{(m)}).$$

Auf Grund von (2), (1. 11) a), (1. 20), (1. 21) und (1. 22) für $l \in I(\mu)$ und $k \geqq \mu$ ergibt sich

$$\psi_l(c_k^{(m)}) = \prod_{i=0}^{\mu-1} \left(r_{2m^2+2+i}(c_k^{(m)})\right)^{l_{2m^2+2+i}} = \prod_{\nu\in P(\mu)} \prod_{\varrho\in B(\mu,\nu)} \left(r_{2m^2+2+\varrho}(c_k^{(m)})\right)^{l_{2m^2+2+\varrho}} =$$

$$= \prod_{\nu\in P(\mu)} \prod_{\substack{\varrho\in B(\mu,\nu)\\ l_{2m^2+2+\varrho}=1}} (-1)^{\beta^{(\varrho)}(k,m)} = \prod_{\nu\in P(\mu)} \prod_{\substack{\varrho\in B(\mu,\nu)\\ l_{2m^2+2+\varrho}=1}} (-1)^{\varphi_m(\varrho)\oplus\varphi_m(k)\oplus\varphi_m(\varrho\oplus k)}.$$

Da nach (1. 20) der Anzahl der Faktoren von innere Produkt gerade ist, weiterhin wegen $k \geqq \mu$, nach (1. 11) b) $k \in B(\mu, \nu')$ $(\nu' \in P(2m^2,\mu))$ gilt, deshalb ist auf Grund von Hilfssatz IV $\psi_l(c_k^{(m)})$ von k unabhängig.

Aus (1. 3), (1. 5), (1. 23) und (1. 25) für $N_m = 2^{2m^2} + N'$ $(0 \leqq N' < 2^{2m^2})$ erhalten wir

$$Q_m(l, \mu: y \dotplus c_k^{(m)} \dotplus a_\mu^{(m)}) =$$

$$= r_{\mu+2m^2+2}(y)\psi_l(y)\psi_{N_m}(y) r_{\mu+2m^2+2}(c_k^{(m)}) r_{2m^2}(a_k^{(m)} \dotplus a_\mu^{(m)}) \cdot \psi_l(c_k^{(m)}) \cdot$$

$$\cdot \psi_{N_m}\left(\frac{k}{2^{2m^2}} \dotplus \frac{\mu}{2^{2m^2}}\right) \sum_{s=0}^{m^2-1} r_{2s}\left(y \dotplus \frac{\mu}{2^{2m^2}} \dotplus \frac{k}{2^{2m^2}}\right) D_{2^{2s}}\left(y \dotplus \frac{\mu}{2^{2m^2}} \dotplus \frac{k}{2^{2m^2}}\right) =$$

$$= r_{\mu+2m^2+2}(y)\psi_l(c_k^{(m)})\psi_l(y)\psi_{N_m}(y \dotplus a_{k\oplus\mu}^{(m)}) \sum_{s=0}^{m^2-1} r_{2s}(y \dotplus a_{k\oplus\mu}^{(m)}) D_{2^{2s}}(y \dotplus a_{k\oplus\mu}^{(m)}),$$

woraus sich auf Grund von Hilfssatz II und (1. 15)

$$\sum_{k=\mu}^{2^{2m^2}-1} Q_m(l, \mu: y \dotplus c_k^{(m)} \dotplus a_\mu^{(m)}) =$$

$$= r_{\mu+2m^2+2}(y)\psi_l(y \dotplus c_k^{(m)}) \sum_{\nu \in P(2m^2,\mu)} \sum_{k \in B(\mu,\nu)} \psi_{N_m}(y \dotplus a_{k\oplus\mu}^{(m)}) \cdot$$

$$\cdot \sum_{s=0}^{m^2-1} r_{2s}(y \dotplus a_{k\oplus\mu}^{(m)}) D_{2^{2s}}(y \dotplus a_{k\oplus\mu}^{(m)}) =$$

$$= \psi_l(y \dotplus c_k^{(m)}) r_{\mu+2m^2+2}(y) \sum_{\nu \in P(2m^2,\mu)} \sum_{l \in A_\nu} \psi_{N_m}(y \dotplus a_l^{(m)}) \cdot$$

$$\cdot \sum_{s=0}^{m^2-1} r_{2s}(y \dotplus a_l^{(m)}) D_{2^{2s}}(y \dotplus a_l^{(m)}) = \psi_l(y \dotplus c_k^{(m)}) r_{\mu+2m^2+2}(y) U_m(P(2m^2,\mu); y)$$

ergibt. Für $y = x \dotplus a_\mu^{(m)}$ erhalten wir so die zu bewiesende Gleichung (1. 24).

Aus (1. 24) folgt noch, indem man (1. 10), (1. 16) und (1. 18) berücksichtigt, daß für $\mu \in M_m$ die folgende Abschätzung gilt:

$$(1.26) \qquad \left|\sum_{k=\mu}^{2^{2m^2}-1} Q_m(l, \mu; x \dotplus c_k^{(m)})\right| = \left|U_m\left(P(2m^2,\mu); x \dotplus \frac{\mu}{2^{2m^2}}\right)\right| \geqq$$

$$\geqq \frac{2^{2m^2}\sigma(P(2m^2,\mu)}{6} \geqq \frac{m^2 2^{2m^2}}{6} \qquad \left(x \in \left[\frac{\mu}{2^{2m^2}}, \frac{\mu+1}{2^{2m^2}}\right)\right).$$

Die Untermenge von [0, 1) wird einfach genannt, wenn sie die Vereinigung von endlichvielen Intervallen ist, deren Endpunkt dyadische rationale Zahlen sind.

HILFSSATZ VI. *Für eine genügend große positive ganze Zahl $m(>m_0)$ gibt es ein Walsh-Polynom $T_m(x)$ und eine einfache Menge X_m mit folgenden Eigenschaften:*

Es gelten

$$\text{(1.27)}\qquad \text{a)}\quad \int_0^1 T_m(x)\,dx = 0,$$

$$\text{b)}\quad \int_0^1 |T_m(x)|\,dx \leqq 2,$$

$$\text{c)}\quad \operatorname{mes}(X_m) \geqq \frac{1}{16m}.$$

Weiterhin für jedes $x \in X_m$ *gibt es einen Index* $q = q(x)$ $(\geqq 1)$ *derart, daß*

$$\text{d)}\quad \frac{1}{q}\sum_{\varkappa=1}^{q} [S_\varkappa(T_m; x)]^{2m^2} \geqq (4m\log^2 m)^{2m^2}$$

besteht.

Beweis von Hilfssatz VI. Es sei

$$\text{(1.28)}\qquad T_m(x) = 2^{-2m^2} \sum_{k=0}^{2^{2m^2}-1} D_{2^{2m^2+k+3}}(x \overset{\circ}{+} c_k^{(m)}) - 1.$$

Dann besteht (1. 27) a) und b) auf Grund von (1. 8). Für $\mu \in H_m$ setzen wir

$$\text{(1.29)}\qquad K_m(\mu) = \{\varkappa \colon \varkappa = 2^{2m^2+2+\mu} + l + N_m,\ l \in I(\mu)\}.$$

Es sei nun $\varkappa = \sum_{i=0}^{2m^2+2+\mu} \varkappa_i 2^i \in K_m(\mu)$ $(\varkappa_i = 0,1)$. Aus (1. 28) erhalten wir

$$\text{(1.30)}\qquad S_\varkappa(T_m; x) =$$

$$= 2^{-2m^2} \sum_{k=0}^{\mu-1} D_{2^{2m^2+k+3}}(x \overset{\circ}{+} c_k^{(m)}) + 2^{-2m^2} \sum_{k=\mu}^{2^{2m^2}-1} S_\varkappa\big(D_{2^{2m^2+k+3}}(x \overset{\circ}{+} c_k^{(m)}); x\big) - 1 =$$

$$= 2^{-2m^2} \sum_{k=0}^{\mu-1} D_{2^{2m^2+k+3}}(x \overset{\circ}{+} c_k^{(m)}) + 2^{-2m^2} \sum_{k=\mu}^{2^{2m^2}-1} D_\varkappa(x \overset{\circ}{+} c_k^{(m)}) - 1 = s_1(x) + s_2(x) - 1,$$

wobei

$$\text{(1.31)}\qquad s_1(x) = \frac{1}{2^{2m^2}} \sum_{k=0}^{\mu-1} D_{2^{2m^2+k+3}}(x \overset{\circ}{+} c_k^{(m)}) = 0 \quad \left(x \in \left[\frac{\mu}{2^{2m^2}}, \frac{\mu+1}{2^{2m^2}}\right)\right)$$

gilt. Auf Grund von (1. 7), (1. 9) und (1. 23) ergibt sich

$$D_\varkappa(x) = \psi_\varkappa(x) \sum_{i=0}^{2m^2+2+\mu} \varkappa_i r_i(x) D_{2^i}(x) =$$

$$= r_{2m^2+2+\mu}(x)\psi_l(x)\psi_{N_m}(x) \sum_{i=0}^{m^2-1} r_{2i}(x) D_{2^{2i}}(x) + \psi_\varkappa(x) r_{2m^2}(x) D_{2^{2m^2}}(x) +$$

$$+ \psi_\varkappa(x) \sum_{i=2m^2+2}^{2m^2+2+\mu} \varkappa_i r_i(x) D_{2^i}(x) = Q_m(l, \mu; x) + s_3(x) + s_4(x),$$

woraus wir nach (1. 24) die Gleichung

$$(1.32)\qquad s_2(x) = \vartheta\cdot 2^{-2m^2} r_{2m^2+2+\mu}(x)\psi_l(x)\,U_m\left(P(2m^2,\mu);\, x \mathring{+} \frac{\mu}{2^{2m^2}}\right) +$$

$$+2^{-2m^2}\sum_{k=\mu}^{2^{2m^2}-1} s_3(x\mathring{+}c_k^{(m)}) + 2^{-2m^2}\sum_{k=\mu}^{2^{2m^2}-1} s_4(x\mathring{+}c_k^{(m)})$$

erhalten. Da

$$D_{2^{2m^2}}(x\mathring{+}c_k^{(m)}) = \begin{cases} 2^{2m^2} & \left(x\in\left[\frac{k}{2^{2m^2}},\frac{k+1}{2^{2m^2}}\right)\right),\\ 0 & \left(x\notin\left[\frac{k}{2^{2m^2}},\frac{k+1}{2^{2m^2}}\right)\right),\end{cases}$$

deshalb gilt

$$(1.33)\qquad 2^{-2m^2}\left|\sum_{k=\mu}^{2^{2m^2}-1} s_3(x\mathring{+}c_k^{(m)})\right| \leqq 1 \qquad (x\in[0,1)).$$

Für $x\in\left[\frac{\mu}{2^{2m^2}},\frac{\mu+1}{2^{2m^2}}\right)$ und $k>\mu$ besteht $s_4(x\mathring{+}c_k^{(m)})=0$. Da nach (1. 12), (1. 21) und (1. 23) $c_\mu^{(m)} < \frac{\mu+\frac{3}{4}}{2^{2m^2}}$ gilt, deshalb ist im Falle $k=\mu$

$$s_4(x\mathring{+}c_\mu^{(m)}) = 0 \quad (x\in Y_m(\mu)),$$

wobei $Y_m(\mu)$ das Intervall

$$(1.34)\qquad Y_m(\mu) = \left[\frac{\mu+\frac{3}{4}}{2^{2m^2}},\frac{\mu+1}{2^{2m^2}}\right)$$

bedeutet. So auf Grund von (1. 26), (1. 30), (1. 31), (1. 32) und (1. 33) ergibt sich

$$(1.35)\qquad |S_\varkappa(T_m;x)| \geqq 2^{-2m^2}\left|U_m\left(P(2m^2,\mu);\, x\mathring{+}\frac{\mu}{2^{2m^2}}\right)\right| - 2 \geqq \frac{m^2}{12}$$

$$(m\geqq m_2;\ \varkappa\in K_m(\mu),\ x\in Y_m(\mu)).$$

Wir setzen

$$(1.36)\qquad X_m = \bigcup_{\mu\in H_m} Y_m(\mu) \quad (m>m_3=\max(m_1,m_2)).$$

Offensichtlich können wir annehmen, daß die Menge X_m einfach ist, weiterhin auf Grund von (1. 19) und (1. 34) gilt noch

$$\operatorname{mes}(X_m) = \sum_{\mu\in H_m}\operatorname{mes}(Y_m(\mu)) \geqq \frac{M_m}{4}\cdot\frac{1}{2^{2m^2}} \geqq \frac{1}{16m} \qquad (m>m_3),$$

womit die Bedingung (1. 27) c) erfüllt ist.

Es sei nun $m > m_3$ und $x \in X_m$. Dann gibt es einen Index $\mu = \mu(x)$ $(\in H_m)$ derart, daß für jede $\varkappa \in K_m(\mu)$ die Ungleichung (1. 35) besteht. Wir setzen $q = 2^{2m^2+3+\mu(x)}$. Dann gilt nach (1. 20), (1. 29) und (1. 35)

$$(1.37)\quad \frac{1}{q}\sum_{\varkappa=1}^{q}(S_\varkappa(T_m;x))^{2m^2} \geqq \frac{1}{q}\sum_{\varkappa\in K_m(\mu)}(S_\varkappa(T_m;x))^{2m^2} \geqq \frac{1}{q}\left(\frac{m^2}{12}\right)^{2m^2}\sum_{\varkappa\in K_m(\mu)}1 =$$

$$= \frac{2^{\mu-m^2}}{q}\left(\frac{m^2}{12}\right)^{2m^2} = \frac{2^{\mu-m^2}}{2^{\mu+3+2m^2}}\left(\frac{m^2}{12}\right)^{2m^2} \geqq (4m\log^2 m)^{2m^2} \quad (m \geqq m_0 \geqq m_3),$$

womit auch (1. 27) d) bewiesen ist.

§ 2. Beweis des Satzes

Wir wenden den Hilfssatz VI für $m = m_0, m_0+1, \ldots$ an. Die entsprechenden Walsh-Polynome, bzw. die entsprechenden Mengen bezeichnen wir mit $T_m(x)$, bzw. mit X_m $(m \geqq m_0)$, es sei weiterhin $\varrho_s = 2s^2 + 2^{2s^2} + 2$ $(s \geqq m_0)$.

Wir betrachten eine Folge von ganzen Zahlen $(1 \leqq)\, \omega_{m_0} < \omega_{m_0+1} < \ldots$ für die

$$(2.1)\qquad \lambda_\nu \geqq 2m^2 \qquad (\nu \geqq \omega_m,\ m \geqq m_0)$$

besteht. (Wegen der Annahme $\lambda_\nu \to \infty$ $(\nu \to \infty)$ kann man eine solche Folge $\{\omega_m\}$ bestimmen.) Es sei weiterhin $n_m = \varrho_{s_m}$ $(m = m_0, m_0+1, \ldots)$ eine Teilfolgen von $\{\varrho_s\}$ mit $m_0 \leqq s_{m_0} < s_{m_0+1} < s_{m_0+2} < \ldots$ und

$$(2.2)\qquad 2^{n_m} > \omega_m \qquad (m \geqq m_0).$$

Wir setzen $F_m(x) = T_m(2^{n_m}x)$ $(m \geqq m_0)$. Aus (1. 27) folgt

$$\sum_{m=m_0}^{\infty}\int_0^1 |F_m(x)|\,dx/m\log^2 m < \infty.$$

Nach dem Satz von B. Levi ist also die Reihe

$$\sum_{m=m_0}^{\infty}\frac{F_m(x)}{m\log^2 m}$$

fast überall absolut konvergent, und ihre Summe $f(x)$ gehört zu $L[0, 1]$. Auf Grund der Ungleichung

$$(2.3)\qquad \varrho_m + n_m < n_{m+1} \qquad (m \geqq m_0)$$

und nach (1. 27) und (1. 28) enthalten die Walsh-Polynome $F_k(x)$ und $F_l(x)$ $(k \neq l)$ kein Glied mit derselben Walshen Funktion. So sind auf Grund des Lebesgueschen Satzes und der Definition von $T_m(x)$

$$(2.4)\qquad S_k(f;x) = \sum_{\nu=m_0}^{m-1}\frac{F_\nu(x)}{\nu\log^2\nu} + \frac{S_\varkappa(T_m;2^{n_m}x)}{m^2\log^2 m}$$

$$(k = \varkappa 2^{n_m} + s,\ \varkappa = 1, 2, \ldots, 2^{\varrho_m},\ s = 1, 2, \ldots, 2^{n_m};\ m \geqq m_0),$$

$$(2.5)\qquad S_{2^{n_m}}(f;x) = \sum_{\nu=m_0}^{m-1}\frac{F_\nu(x)}{\nu\log^2\nu} \qquad (m > m_0).$$

Wir bezeichnen mit Γ die Menge von $x(\in[0,1))$, für die $S_{2^{n_m}}(f;x)\to f(x)$ $(m\to\infty)$ gilt. Nach den obigen gilt

$$\text{(2. 6)}\qquad \operatorname{mes}(\Gamma)=1.$$

Es sei

$$\text{(2. 7)}\qquad \Gamma_m=\{x: 2^{n_m}x\in X_m,\ 0\leqq x<1\}\qquad (m\geqq m_0).$$

Aus (1. 27) folgt

$$\operatorname{mes}(\Gamma_m)>\frac{1}{16m}\qquad (m\geqq m_0),$$

und so ist

$$\text{(2. 8)}\qquad \sum_{m=m_0}^{\infty}\operatorname{mes}(\Gamma_m)=\infty.$$

Auf Grund der Definition von Γ_m und der Relationen (1. 34), (1. 36) und (2. 3) ist leich zu zeigen, daß die Mengen Γ_m $(m\geqq m_0)$ stochastisch unabhängig sind. Aus (2. 8), durch Anwendung des zweiten Borel—Cantellischen Lemmas ergibt sich

$$\text{(2. 9)}\qquad \operatorname{mes}(\overline{\lim_{m\to\infty}}\,\Gamma_m)=1.$$

Es sei $x\in\Gamma\cap\overline{\lim\limits_{m\to\infty}}\,\Gamma_m$. Dann gibt es ein Index $l=l(x)$ und unendlich viele $m(>l)$, derart, daß

$$\text{(2. 10)}\qquad |S_{2^{n_m}}(f;x)-f(x)|\leqq 1\quad (m>l),\qquad 2^{n_m}x\in Y_m(\mu(x))\quad (\mu(x)=\mu_m(x)\in H_m)$$

gilt. Aus (1. 35), (1. 36), (1. 37), (2. 4), (2. 5) und (2. 9) folgt

$$|S_k(f;x)-f(x)|\geqq |S_k(f;x)-S_{2^{n_m}}(f;x)|-|S_{2^{n_m}}(f;x)-f(x)|\geqq$$
$$\geqq\frac{|S_\varkappa(T_m;2^{n_m}x)|}{m\log^2 m}-1\geqq 1$$
$$(2^{n_m}x\in Y_m(\mu),\ k=\varkappa 2^{n_m}+s,\ \varkappa\in K_m(\mu),\ s=1,2,\dots,2^{n_m};\ m\geqq l).$$

So, auf Grund (2. 1) und (2. 2) ergibt sich

$$\text{(2. 11)}\qquad \sum_{\varkappa\in K_m(\mu)}\sum_{s=1}^{2^{n_m}}|S_{\varkappa\cdot 2^{n_m}+s}(f;x)-f(x)|^{\lambda_{\varkappa\cdot 2^{n_m}+s}}\geqq$$
$$\geqq\sum_{\varkappa\in K_m(\mu)}\sum_{s=1}^{2^{n_m}}(S_{\varkappa 2^{n_m}+s}(f;x)-f(x))^{2m^2}\qquad (2^{n_m}x\in Y_m(\mu);\ m\geqq l).$$

Aus der Minkowskischen Ungleichung erhalten wir

$$\left\{\sum_{\varkappa\in K_m(\mu)}\sum_{s=1}^{2^{n_m}}(S_{\varkappa 2^{n_m}+s}(f;x)-f(x))^{2m^2}\right\}^{\frac{1}{2m^2}}\geqq$$
$$\geqq\left\{\sum_{\varkappa\in K_m(\mu)}\sum_{s=1}^{2^{n_m}}(S_{\varkappa 2^{n_m}+s}(f;x)-S_{2^{n_m}}(f;x))^{2m^2}\right\}^{\frac{1}{2m^2}}-$$
$$-\left\{\sum_{\varkappa\in K_m(\mu)}\sum_{s=1}^{2^{n_m}}(S_{2^{n_m}}(f;x)-f(x))^{2m^2}\right\}^{\frac{1}{2m^2}}.$$

Daraus, auf Grund der Ungleichung $|\alpha+\beta|^\gamma \leqq 2^\gamma(|\alpha|^\gamma+|\beta|^\gamma)$ folgt

$$(2.12)\qquad \sum_{\varkappa\in K_m(\mu)} \sum_{s=1}^{2^{n_m}} (S_{\varkappa 2^{n_m}+s}(f;x)-f(x))^{2m^2} \geqq$$

$$= \sum_{\varkappa\in K_m(\mu)} \sum_{s=1}^{2^{n_m}} \left\{\frac{1}{2}(S_{\varkappa 2^{n_m}+s}(f;x)-S_{2^{n_m}}(f;x))\right\}^{2m^2} - \sum_{\varkappa\in K_m(\mu)} \sum_{s=1}^{2^{n_m}} (S_{2^{n_m}}(f;x)-f(x))^{2m^2}.$$

Weiterhin nach (2. 4) und (2. 5) ergibt sich

$$(2.13)\qquad \sum_{\varkappa\in K_m(\mu)} \sum_{s=1}^{2^{n_m}} (S_{\varkappa 2^{n_m}+s}(f;x)-S_{2^{n_m}}(f;x))^{2m^2} =$$

$$= 2^{n_m} \sum_{\varkappa\in K_m(\mu)} (S_\varkappa(T_m; 2^{n_m}x)/m\log^2 m)^{2m^2}.$$

Aus (1. 27), (1. 29), (1. 37), (2. 10), (2. 11), (2. 12) und (2. 13) erhalten wir

$$(2.14)\qquad \frac{1}{2^{n_m}q}\sum_{k=1}^{2^{n_m}q} |S_k(f;x)-f(x)|^{\lambda_k} \geqq$$

$$\geqq \frac{1}{2^{n_m}q} \sum_{\varkappa\in K_m(\mu)} \sum_{s=1}^{2^{n_m}} |S_{\varkappa.2^{n_m}+s}(f;x)-f(x)|^{\lambda_{\varkappa.2^{n_m}+s}} \geqq$$

$$\geqq \frac{1}{2^{n_m}q} \sum_{\varkappa\in K_m(\mu)} \sum_{s=1}^{2^{n_m}} (S_{\varkappa.2^{n_m}+s}(f;x)-f(x))^{2m^2} \geqq$$

$$\geqq \frac{1}{2^{n_m}q} \sum_{\varkappa\in K_m(\mu)} \sum_{s=1}^{2^{n_m}} \left\{\frac{1}{2}(S_{\varkappa.2^{n_m}+s}(f;x)-S_{2^{n_m}}(f;x))\right\}^{2m^2} - \frac{2^{n_m}\cdot 2^{\mu-m^2}}{2^{n_m}\cdot 2^{2m^2+\mu+3}} \geqq$$

$$\geqq \frac{1}{q} \sum_{\varkappa\in K_m(\mu)} (S_\varkappa(T_m; 2^{n_m}x)/2m\log^2 m)^{2m^2} - 1 > 2^{2m^2}-1 \quad (x\in Y_m(\mu),\ m\geqq l).$$

Da für $x\in\Gamma\cap\overline{\lim}_{m\to\infty}\Gamma_m$ die Ungleichung (2. 14) für unendlich viele m gilt, deshalb erfüllt die Funktion $f(x)$ in Punkt x die Relation (3). Aus (2. 6) und (2. 9) folgt endlich, daß (3) fast überall besteht.

Damit haben wir den Satz vollständig bewiesen.

(Eingegangen am 13. April 1968.)

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ÜBER DIE SÄTTIGUNGSKLASSE DER STARKEN APPROXIMATION DURCH TEILSUMMEN DER FOURIERSCHEN REIHE

Von
G. FREUD (Budapest)

Herrn Prof. G. ALEXITS *zum 70. Geburtstag*

Es sei $f(x)$ eine stetige 2π-periodische Funktion, die Fouriersche Reihe von $f(x)$ sei

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} [a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x] \tag{1}$$

und $s_k(f; x)$ $(k=0, 1, \ldots)$ sei die Teilsumme k-ter Ordnung dieser Reihe. Wir betrachten für ein $p>1$ die Ausdrücke

$$h_n(f; x; p) = \left\{\frac{1}{n+1} \sum_{k=0}^{n} |f(x) - s_k(f, x)|^p\right\}^{1/p}. \tag{2}$$

Das Problem der starken Approximation besteht in der Abschätzung der Ausdrücke (2); es wurde von G. ALEXITS [1] gestellt und in der wesentlichen Zügen gelöst. Es zeigte sich, daß aus $f \in \mathrm{Lip}\,\alpha$ im Falle $\alpha < \dfrac{1}{p}$ $h_n(f; x; p) = O(n^{-\alpha})$ folgt, aber für $\alpha = p^{-1}$ trifft das nicht mehr zu. In vorliegender Arbeit untersuchen wir die Sättigungsklasse (classe de saturation) der starken Approximation.[1]

Wir betrachten für ein festes $p>1$ die Folge (2); der Kürze halber bezeichnen wir es als „$h(p)$-Verfahren".

SATZ 1. *Die Sättigungsordnung des $h(p)$-Verfahrens ist gleich $n^{-1/p}$.*

BEWEIS. Es muß gezeigt werden, daß
a) aus $h_n(q; x; p) = o(n^{-1/p})$ folgt $q(x) \equiv$ Konstante;
b) es gibt nichtkonstante Funktionen $f(x)$ für welche

$$h_n(f; x; p) = O(n^{-1/p}) \tag{3}$$

gleichmäßig in x befriedigt ist.

Teil a): Aus $h_n(q; x; p) = o(n^{-1/p})$ ergibt sich

$$\lim_{n\to\infty} \sum_{k=0}^{n} |g(x) - s_k(g; x)|^p = 0,$$

woraus $g(x) \equiv s_0(x) \equiv \dfrac{a_0}{2}$ folgt.

[1] Begriff der Sättigungsklasse eines Approximationsverfahrens wurde von J. FAVARD [4] eingeführt; eine Darstellung verschiedener Ergebnisse findet man z. B. bei G. SUNOUCHI [5].

Teil b): Es sei $f(x)$ zweimal stetig differenzierbar. Es gilt dann $f(x)-s_k(f;x)=$ $=O\left(\frac{\log k}{k^2}\right)$, so daß

$$\sum_{k=0}^{\infty}|f(x)-s_k(f;x)|^p<\infty$$

gültig ist. Es folgt, daß (3) gleichmäßig in x erfüllt ist, w.z.b.w.[2]

Die Sättigungsklasse des $h(p)$-Verfahrens, d.h. die Menge der Funktionen $f(x)$, welche (3) gleichmäßig in x befriedigen, bezeichnen wir mit X_p. Wir schließen aus den Ungleichungen[3]

$$h_n(f_1+f_2;x;p)\leqq h_n(f_1;x;p)+h_n(f_2;x;p)$$

und

$$0\leqq h_n(f;x;p)\leqq K_p\max|f(x)|$$

daß X_p ein abgeschlossener linearer Teilraum von $C_{2\pi}$ (Banachscher Raum der 2π-periodischen stetigen Funktionen) ist.

SATZ 2. *Es ist $X_p\subset\operatorname{Lip}1/p$ und für ein $f\in X_p$ gilt für fast alle x*

(4) $$\lim_{h\to 0}|h|^{-1/p}[f(x+h)-f(x)]=0.$$

BEMERKUNG. Es gibt für jedes $p>1$ Funktionen $f\in\operatorname{Lip}1/p$, welche (4) für keine einzige Stelle x befriedigen (vgl. G. FREUD [3]). Die Frage ob für ein $f\in X_p$ sogar für jeden Punkt x (4) gültig ist, scheint ein interessantes ungelöstes Problem zu sein.

BEWEIS DES SATZES 2. Wegen $f\in X_p$ ist $h_n(f;x;p)\leqq K(n+1)^{-1/p}$, d.h.

(5) $$\sum_{k=0}^{\infty}|f(x)-s_k(f;x)|^p<K_0^p,$$

wo K_0 (und im weiteren $K_1, K_2, \ldots$) nicht von x abhängt (aber von der Wahl von f abhängen kann). Wir bezeichnen mit

(6) $$V_n(f;x)=\frac{1}{n}\sum_{k=n+1}^{2n}s_k(f;x)\qquad(n=1,2,\ldots)$$

die de la Vallée Poussinschen Mitteln von (1). Aus (5) folgt

(7) $$|f(x)-V_n(f;x)|\leqq\frac{1}{n}\sum_{k=n+1}^{2n}|f(x)-s_k(f;x)|\leqq$$

$$\leqq\frac{1}{n}n^{1-1/p}\left\{\sum_{k=n+1}^{2n}|f(x)-s_k(f;x)|^p\right\}^{1/p}\leqq K_0n^{-1/p}.$$

[2] Satz 1 ist für jedes $p>0$ gültig. Für uns ist aber nur der Fall $p>1$ von Interesse.

[3] Die erste dieser Ungleichungen folgt aus der Minkowskischen Ungleichung. Bezüglich der zweiten Ungleichung siehe G. ALEXITS [2].

Aus (7) folgt mit Hilfe des Bersteinschen Umkehrsatzes $f \in \mathrm{Lip}\, 1/p$; damit ist die erste Behauptung des Satzes bewiesen. Aus dem Beweisgang des Bernsteinschen Umkehrsatzes folgt

$$V_n'(f; x) \leqq K_1 n^{1-1/p}. \tag{8}$$

Es sei jetzt $\eta > 0$ beliebig. Mit Hilfe des Egoroffschen Satzes folgt aus (5), daß es eine perfekte Punktmenge $\mathfrak{M}_\eta \subset [0, 2\pi]$ gibt, deren Maß größer als $2\pi - \eta$ ist, so daß die Reihe (5) für $x \in \mathfrak{M}_\eta$ gleichmäßig konvergiert. Es sei $\mathfrak{M}_\eta^*$ die Teilmenge von $\mathfrak{M}_\eta$ in deren Punkten $\mathfrak{M}_\eta$ die asymptotische dichte 1 besitzt. Nach einem bekannten Satze von H. LEBESGUE ist $|\mathfrak{M}_\eta^*| = |\mathfrak{M}_\eta| > 2\pi - \eta$. Wir betrachten einen festen Punkt $x \in \mathfrak{M}_\eta^*$. Es sei ε eine feste, aber beliebig kleine positive Zahl. Wir wählen δ so klein, daß für jedes $0 < h \leqq \delta$ des Maß der Punktmengen $[x-h, x] \cap \mathfrak{M}_\eta$ und $[x, x+h] \cap \mathfrak{M}_\eta$ größer als $(1-\varepsilon)h$ ausfällt. Ferner wählen wir $N \geqq 2$ so groß, daß

$$\sum_{k=N+1}^{\infty} |f(x) - s_k(f; x)|^p < \varepsilon^p \qquad (x \in \mathfrak{M}_\eta) \tag{9}$$

gültig ist.

Indem man (7) bis zum vorletzten Teile anwendet, folgt aus (9)

$$|f(x) - V_n(f; x)| \leqq \varepsilon n^{-1/p} \qquad (x \in \mathfrak{M}_\eta,\ n > N). \tag{10}$$

Es sei $|x - \xi| \leqq \min\{\delta, N^{-1}\varepsilon\}$ und $n = n(\xi)$ sei die kleinste natürliche Zahl mit $n|\xi - x| \geqq \varepsilon$; dann ist $n > N$ und $n|\xi - x| < 2\varepsilon$. Wegen $|x - \xi| < \delta$ gibt es ein $\xi_1 \in \mathfrak{M}_\eta$ zwischen x und ξ, so daß $|\xi - \xi_1| \leqq \varepsilon |\xi - x|$ gültig ist. Unter Beachtung von (10) und (8) erhalten wir

$$|f(\xi) - f(x)| \leqq |f(\xi) - f(\xi_1)| + |f(\xi_1) - V_n(f, \xi_1)| + |f(x) - V_n(f, x)| +$$

$$+ |V_n(f, \xi_1) - V_n(f, x)| \leqq K_2 |\xi - \xi_1|^{1/p} + 2\varepsilon n^{-1/p} + K_1 n^{1-1/p} |\xi_1 - x| \leqq$$

$$\leqq K_2 \varepsilon^{1/p} |\xi - x|^{1/p} + 2\varepsilon \left(\frac{|\xi - x|}{\varepsilon}\right)^{1/p} + K_1 \left(\frac{2\varepsilon}{|\xi - x|}\right)^{1-1/p} |\xi - x| \leqq$$

$$\leqq [K_2 \varepsilon^{1/p} + 2(1 + K_1)\varepsilon^{1-1/p}] |\xi - x|^{1/p}.$$

Da ε beliebig war, schließen wir aus dieser Ungleichung, daß (4) für jeden Punkt $x \in \mathfrak{M}_\eta^*$ gültig ist. Die Menge $\mathfrak{N}(f)$ der Punkte $x \in [0, 2\pi]$ wo (4) nicht befriedigt ist, ist in dem Komplement von $\mathfrak{M}_\eta^*$ enthalten. Das äußere Maß von $\mathfrak{N}(f)$ ist also kleiner als η. Da $\eta > 0$ beliebig war, muß $|\mathfrak{N}(f)| = 0$ sein, w.z.b. w.

SATZ 3. *Es sei $p^{-1} + q^{-1} = 1$ und $f(x)$ eine 2π-periodische stetige Funktion. Aus $q \leqq 2$ und*

$$\int_{-\pi}^{\pi} \left| \frac{f(x+t) + f(x-t) - 2f(x)}{t} \right|^q dt \leqq K_3 \qquad (x \in [0, 2\pi]) \tag{11}$$

folgt

$$h_n(f; x; p) \leqq K_4 (n+1)^{-1/p}, \tag{12}$$

ist andererseits (12) *für ein $p \leqq 2$ befriedigt, dann ist* (11) *gültig.*

BEMERKUNG. *Für den Fall $p=2$ sind* (11) *und* (12) *gemäß Satz* 3 *äquivalent, d.h. X_2 besteht genau aus den Funktionen, für welche*

$$\int_{-\pi}^{+\pi} \left| \frac{f(x+t)+f(x-t)-2f(x)}{t} \right|^2 dt \leqq K_5$$

ist.

BEWEIS. Es sei

$$F_x(t) = \frac{1}{4} \operatorname{cotg} \frac{t}{2} [f(x+t)+f(x-t)-2f(x)]. \tag{13}$$

Aus der Dirichletschen Kernformel ergibt sich

$$s_n(f; x)-f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} F_x(t) \sin nt \, dt + \tag{14}$$

$$+\frac{1}{2\pi} \int_{-\pi}^{+\pi} \cos nt f(x+t) \, dt = b_n(F_x)+\frac{1}{2} [a_n(f) \cos nx + b_n(f) \sin nx],$$

wo $a_n(g)$, bzw. $b_n(g)$ den Koeffizienten von $\cos nt$ bzw. $\sin nt$ in der Fourierschen Reihe von $g(t)$ bedeutet. Wegen (13) ist (11) mit

$$\int_{-\pi}^{\pi} |F_x(t)|^q \, dt \leqq K_6 \tag{15}$$

gleichwertig. Aus (15) und der Stetigkeit von $f(t)$ folgt mit Hilfe der ersten Young—Hausdorffschen Ungleichung

$$\sum_{k=1}^{\infty} |b_k(F_x)|^p \leqq K_7 \tag{16}$$

und

$$\sum_{k=1}^{\infty} \left(|a_k|^p + |b_k|^p\right) \leqq K_8 . \tag{17}$$

Aus (14), (16) und (17) folgt

$$\sum_{k=0}^{\infty} |s_n(f; x)-f(x)|^p \leqq K_9 ,$$

und weiter

$$h_n(f; x; p) \leqq [K_9(n+1)]^{-1/p}.$$

Wir zeigten also die Gültigkeit der ersten Hälfte des Satzes 3. Der Beweis der zweiten Hälfte verläuft ähnlich. Aus (12) folgt

$$\sum_{k=0}^{\infty} |s_k(f; x)-f(x)|^p \leqq K_4^p$$

und wegen $a_k(f) \cos kx + b_k(f) \sin kx = [s_k(f; x)-f(x)]-[s_{k-1}(f; x)-f(x)]$

$$\sum_{k=1}^{\infty} |a_k(f) \cos kx + b_k(f) \sin kx|^p \leqq K_{10} .$$

Aus diesen beiden letzten Ungleichungen und (14) ergibt sich

$$\sum_{k=1}^{\infty} |b_k(F_x)|^p \leqq K_{11}. \tag{18}$$

Es ist $F_x(t)$ gemäß (13) eine ungerade Funktion von t, so daß

$$a_k(F_x) = 0 \qquad (k = 0, 1, \ldots)$$

ist. Aus (18) schließen wir mit Hilfe der zweiten Young—Hausdorffschen Ungleichung auf (15) und weiter auf (11). Wir haben auch die Gültigkeit der zweiten Hälfte von Satz 3 gezeigt, w.z.b.w.

SATZ 4. *Bezüglich $f(x)$, p und q seien die gleichen Foraussetzungen erfüllt, wie in Satz 3, und es sei $\tilde{f}(x)$ die zu $f(x)$ harmonisch konjugierte Funktion. Dann folgt aus $q \leqq 2$ und*

$$\int_{-\pi}^{\pi} \left| \frac{\tilde{f}(x+t) - \tilde{f}(x-t)}{t} \right|^q dt \leqq K_{12} \tag{19}$$

die Gültigkeit von (12). *Ist umgekehrt* (12) *für ein $p \leqq 2$ befriedigt, dann ist* (11) *erfüllt.*

BEMERKUNG. *Für $q = 2$ sind also* (11) *und* (19) *entweder beide gültig, oder sie sind beide falsch.*

BEWEIS. Man beweist Satz 4 genau so mit Hilfe der Formeln

$$s_n(x) - f(x) = a_n(F_x^*) + \frac{1}{2}\,[a_n(f)\sin nx - b_n(f)\cos nx] \tag{20}$$

mit

$$F_x^* = \frac{1}{4}\operatorname{cotg} t/2\,[\tilde{f}(x+t) - \tilde{f}(x-t)], \tag{21}$$

wie Satz 3 aus den Formeln (14) und (13) bewiesen wurde.

(Eingegangen am 7. Mai 1968.)

MTA MATEMATIKAI KUTATÓ INTÉZETE,
BUDAPEST, V., REÁLTANODA U. 13—15

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ÜBER DIE VERALLGEMEINERTE DE LA VALLÉE-POUSSINSCHE SUMMIERBARKEIT ALLGEMEINER ORTHOGONALREIHEN

Von
M. HEGEDŰS (Budapest)

Herrn Prof. G. ALEXITS *zum 70. Geburtstag*

Einleitung

Seien $\{\varphi_n(x)\}$ ein im Grundintervall (a, b) orthonormiertes Funktionensystem und $\lambda=\{\lambda_n\}$ eine monotone Zahlenfolge von natürlichen Zahlen mit $\lambda_{n+1}-\lambda_n\leqq 1$. Die n-te Partialsumme, bzw. das n-te verallgemeinerte de la Vallée-Poussinsche Mittel bezüglich λ der Orthogonalreihe

$$\sum_{n=0}^{\infty} c_n\varphi_n(x) \tag{1}$$

wird mit $S_n(x)$, bzw. $V_n(\lambda; x)$ bezeichnet, d.h. ist

$$S_n(x) = \sum_{\nu=1}^{n} c_\nu\varphi_\nu(x), \quad V_n(\lambda; x) = \frac{1}{\lambda_n}\sum_{\nu=n-\lambda_n+1}^{n} S_\nu(x).$$

Die Reihe (1) heißt an einer Stelle x (V, λ)-summierbar zur Funktion $f(x)$, die im Falle $\sum_{n=0}^{\infty} c_n^2<\infty$ durch den Riesz—Fischerschen Satz bis auf eine Nullmenge eindeutig bestimmt ist, wenn für $n\to\infty$

$$V_n(\lambda; x)\to f(x)$$

gilt. Sei $\mu_0=1$ und $\mu_n=\sum_{k=0}^{n-1}\lambda\mu_k$. L. LEINDLER [1] hat die folgenden Sätze bewiesen:

SATZ A. *Unter der Bedingung*

$$\sum_{n=1}^{\infty} C_n^2\log^2 n<\infty \quad \left(C_n^2=\sum_{\nu=\mu_n+1}^{\mu_{n+1}} c_\nu^2\right)$$

ist die Orthogonalreihe (1) *in* (a, b) *fast überall* (V, λ)*-summierbar.*

SATZ B. *Sichert die Bedingung*

$$\sum_{n=0}^{\infty} c_n^2 b_n^2<\infty \qquad (1\leqq b_n\leqq\log n) \tag{2}$$

die (V, λ)*-summierbarkeit der Reihe* (1) *für jede die Bedingung* (2) *erfüllenden Koeffizientenfolgen* $\{c_n\}$ *auf einer Menge* $E\subset(a, b)$, *so folgt aus den Bedingungen*

$$\sum_{n=0}^{\infty} c_n^2\delta_n^2<\infty, \quad \frac{b_n}{\delta_n}\leqq\frac{b_{n+1}}{\delta_{n+1}} \quad \text{und} \quad \frac{b_n}{\delta_n}\to\infty$$

für die Mittel $V_n(\lambda; x)$ der Reihe (1) *die Abschätzung*

$$V_n(\lambda; x) = o_x\left(\frac{b_n}{\delta_n}\right)$$

auf E fast überall.

Aus dem Satz A und aus dem Satz B ergibt sich unmittelbar:

FOLGERUNG. *Ist $\{\gamma_n\}$ eine positive, monoton zunehmende Zahlenfolge und genügen die nicht verschwindenen, sonst beliebigen Koeffizienten $c_0, c_1, \ldots$ der Bedingung*

$$\sum_{n=0}^{\infty} \frac{c_n^2}{\gamma_n^2 \sum_{\nu=0}^{n} c_\nu^2} < \infty,$$

so gilt fast überall

$$V_n(\lambda; x) = o_x\left(\left(\gamma_n \varrho_n \sum_{\nu=0}^{n} c_\nu^2\right)^{1/2}\right),$$

wobei $\varrho_k = \log(n+1)$ *für* $\mu_n < k \leqq \mu_{n+1}$ $(n=1, 2, \ldots)$ *ist.*

In dieser Note werden wir zuerst beweisen, daß die Folgerung nicht mehr verbessert werden kann. Hernach geben wir eine neue hinreichende Bedingung für die (V, λ)-Summierbarkeit von (1). Es gelten nämlich die folgenden Sätze:

SATZ I. *Es seien $\{a_n\}$ eine Koeffizientenfolge, $\{\gamma_n\}$ eine positive Zahlenfolge, für welche die Bedingungen*

$$\sum_{n=0}^{\infty} \frac{a_n^2}{\gamma_n^2 \sum_{\nu=0}^{n} a_\nu^2} = \infty, \tag{3}$$

$$\gamma_n \varrho_n \left(\sum_{\nu=0}^{n} a_\nu^2\right)^{1/2} \leqq \gamma_{n+1} \varrho_{n+1} \left(\sum_{\nu=0}^{n+1} a_\nu^2\right)^{1/2}, \quad \gamma_n \varrho_n \geqq 1 \quad (n \geqq n_0) \tag{4}$$

und

$$A_n^2 \geqq A_{n+1}^2 \quad \left(A_n^2 = \sum_{\nu=\mu_n+1}^{\mu_{n+1}} a_\nu^2\right) \tag{5}$$

erfüllt sind. Dann kann ein im Intervall (a, b) orthonormiertes, gleichmäßig beschränktes Funktionensystem $\{\Phi_n(x)\}$ derart angegeben werden, daß in (a, b) fast überall

$$\varlimsup_{n\to\infty} \frac{1}{\gamma_n \varrho_n \left(\sum_{\nu=0}^{n} a_\nu^2\right)^{1/2}} |V_n(\lambda; x)| = \infty \tag{6}$$

gilt.

SATZ II. *Unter der Bedingung*

$$\sum_{n=1}^{\infty} c_n^2 (\log\log(1+\lambda_n)^{n-\lambda_n+1})^2 < \infty$$

ist die Orthogonalreihe (1) *in (a, b) fast überall (V, λ)-summierbar.*

§ 1. Beweis von Satz I

Zum Beweis benötigen wir die folgenden Hilfssätze:

HILFSSATZ I. (L. LEINDLER [1], Hilfssatz I.) *Sei* $\{l_n\}$ *eine positive, monotone Zahlenfolge mit* $l_{\mu_{n+1}} \leqq c l_{\mu_n}$, $1 \leqq c < \sqrt{2}$. *Dann folgt aus der Bedingung*

$$\sum_{n=0}^{\infty} c_n^2 l_n^2 < \infty$$

die Abschätzung

$$V_n(\lambda; x) - S_{\mu_n}(x) = o_x(l_{\mu_n}^{-1})$$

in (a, b) *fast überall.*

HILFSSATZ II. (L. LEINDLER—L. CSERNYÁK [2], Satz I.) *Es seien* $\{\nu_k\}$ $(0 = \nu_0 < \nu_1 < \ldots < \nu_k < \ldots)$ *eine Indexfolge und* $\{a_n\}$ *eine Koeffizientenfolge, für welche die Bedingungen*

$$\sum_{n=2}^{\infty} A_n^2 \log^2 n = \infty \quad \left(A_n^2 = \sum_{k=\nu_n+1}^{\nu_{n+1}} a_k^2\right), \quad A_n^2 \geqq A_{n+1}^2$$

erfüllt sind. Dann kann ein im Intervall (a, b) *orthonormiertes, gleichmäßig beschränktes Funktionensystem* $\{\Phi_n(x)\}$ *derart angegeben werden, daß die Folge der* ν_k*-ten Partialsummen der Orthogonalreihe* $\sum a_n \Phi_n(x)$ *im Intervall* (a, b) *fast überall divergiert.*

Seien $\{a_n\}$ eine Koeffizientenfolge und $\{\gamma_n\}$ eine positive Zahlenfolge, welche die Bedingungen (3), (4) und (5) erfüllen. Ist

$$\sum_{k=1}^{\infty} \frac{a_k^2}{\gamma_k \varrho_k \sum_{\nu=0}^{k} a_\nu^2} < \infty, \tag{1.1}$$

so sei $\tilde{\gamma}_k = \gamma_k$. Besteht (1.1) nicht, so sei $\gamma_k^* = \max\left(\gamma_k, \sqrt{\gamma_k \varrho_k}\right)$. Im Falle

$$\sum_{k=1}^{\infty} \frac{a_k^2}{\gamma_k^* \varrho_k \sum_{\nu=0}^{k} a_k^2} < \infty$$

setzen wir $\tilde{\gamma}_k = \gamma_k^*$, im entgegengesetzten Fall $\tilde{\gamma}_k = \max\left(\gamma_k, \sqrt{\gamma_k^* \varrho_k}\right)$.

Die so definierte Zahlenfolge $\{\tilde{\gamma}_k\}$ erfüllt offenbar die Bedingungen (4) und

$$\sum_{n=1}^{\infty} \frac{a_n^2}{\tilde{\gamma}_n^2 \sum_{\nu=0}^{n} a_\nu^2} = \infty. \tag{1.2}$$

Die Ungleichung

$$\sum_{n=1}^{\infty} \frac{a_n^2}{\tilde{\gamma}_n \varrho_n \sum_{\nu=0}^{n} a_\nu^2} < \infty \tag{1.3}$$

folgt in den ersten beiden Fällen unmittelbar aus der Definition von $\{\tilde{\gamma}_n\}$, im letzten Fall kann man sie mit der folgenden einfachen Rechnung beweisen. Nach (4) ist $\tilde{\gamma}_n \geqq 1$ für alle genügend große n, also gilt auf Grund von (5)

$$\sum_{k=\mu_1+1}^{\infty} \frac{a_k^2}{\tilde{\gamma}_k \varrho_k \sum_{\nu=0}^{k} a_\nu^2} = \sum_{n=1}^{\infty} \sum_{k=\mu_n+1}^{\mu_{n+1}} \frac{a_k^2}{\tilde{\gamma}_k \varrho_k \sum_{\nu=0}^{k} a_\nu^2} =$$

$$= O(1) \sum_{n=1}^{\infty} \sum_{k=\mu_n+1}^{\mu_{n+1}} \frac{a_k^2}{\varrho_k^{3/2} \sum_{\nu=0}^{k} a_\nu^2} = O(1) \sum_{n=1}^{\infty} \sum_{k=\mu_n+1}^{\mu_{n+1}} \frac{a_k^2}{\log_+^{3/2} n \cdot \sum_{\nu=0}^{k} a_\nu^2} =$$

$$= O(1) \sum_{n=1}^{\infty} \frac{1}{\log_+^{3/2} n \cdot \sum_{\nu=0}^{\mu_n} a_\nu^2} \sum_{k=\mu_n+1}^{\mu_{n+1}} a_k^2 = O(1) \sum_{n=1}^{\infty} \frac{1}{n \cdot \log_+^{3/2} n} < \infty,^*$$

womit (1. 3) in jedem Fall bewiesen ist. Wegen (1. 2) gibt es eine positive, stets wachsende Zahlenfolge $\{\eta_n\}$ mit $\eta_n \to \infty$ und

$$\sum_{u=1}^{\infty} \frac{a_n^2}{\tilde{\gamma}_n^2 \eta_n^2 \sum_{\nu=0}^{n} a_\nu^2} = \infty.$$

Wir setzen $\bar{\gamma}_n = \tilde{\gamma}_n \eta_n$ und

$$b_n^2 = \frac{a_n^2}{\bar{\gamma}_n^2 \varrho_n^2 \sum_{\nu=0}^{n} a_\nu^2}, \quad \beta_n^2 = \sum_{\nu=\mu_n+1}^{\mu_{n+1}} b_\nu^2.$$

Offenbar besteht die Ungleichung

$$\beta_n^2 \geqq \beta_{n+1}^2,$$

und mit einfacher Rechnung bekommen wir die Beziehung

$$\sum_{n=0}^{\infty} \beta_n^2 \log_+^2 n = \infty.$$

Nach dem Hilfssatz II ergibt sich ein orthonormiertes, gleichmäßig beschränktes Funktionensystem $\{\Phi_n(x)\}$, für welches die Folge der μ_n-ten Partialsummen der Reihe $\sum_{n=0}^{\infty} b_n \Phi_n(x)$ in (a, b) fast überall divergiert. Wir bekommen durch Abelsche Transformation mit der Abkürzung

$$t_n = \frac{1}{\bar{\gamma}_n \varrho_n \left(\sum_{\nu=0}^{n} a_\nu^2 \right)^{1/2}},$$

$$\sum_{n=2}^{\mu_N} b_n \Phi_n(x) = \sum_{n=2}^{\mu_N - 1} (t_n - t_{n+1}) \bar{S}_n(x) + t_{\mu_N} \bar{S}_{\mu_N}(x) - t_2 \bar{S}_1(x)$$

* Der Kürze halber schreiben wir immer $\log n \cdot U$ statt $(\log n) U$, weiterhin ist $\log_+ x = \max(1, \log x)$.

und so gilt

$$(1.4)\qquad t_{\mu_N}\bar{S}_{\mu_N}(x)=\sum_{n=2}^{\mu_N} b_n\Phi_n(x)-\sum_{n=2}^{\mu_N-1}(t_n-t_{n+1})\bar{S}_n(x)+t_2\bar{S}_1(x),$$

wobei $\bar{S}_n(x)$ die n-te Partialsumme der Reihe $\sum_{n=0}^{\infty} a_n\Phi_n(x)$ bezeichnet.

Mit einfacher Rechnung bekommen wir die Abschätzung:

$$(1.5)\qquad \sum_{n=2}^{\infty}(t_n-t_{n+1})\int_a^b|\bar{S}_n(x)|\,dx\leqq\sqrt{b-a}\sum_{n=2}^{\infty}(t_n-t_{n+1})\left(\sum_{\nu=0}^{n}a_\nu^2\right)^{1/2}.$$

Für jedes N ist mit $\alpha_n=\left(\sum_{\nu=0}^{n}a_\nu^2\right)^{1/2}$

$$\sum_{n=2}^{N}\alpha_n(t_n-t_{n+1})=\alpha_2t_2+\sum_{n=3}^{N}t_n(\alpha_n-\alpha_{n-1})+\alpha_Nt_{N+1},$$

woraus wegen $\alpha_Nt_{N+1}\to0$ $(N\to\infty)$ folgt:

$$(1.6)\qquad \sum_{n=2}^{\infty}\alpha_n(t_n-t_{n+1})=\alpha_2t_2+\sum_{n=3}^{\infty}t_n(\alpha_n-\alpha_{n-1}).$$

Wir erhalten auf Grund von (1. 3)

$$\sum_{n=3}^{\infty}t_n(\alpha_n-\alpha_{n-1})=\sum_{n=3}^{\infty}t_n\frac{\alpha_n^2-\alpha_{n-1}^2}{\alpha_n+\alpha_{n-1}}=O(1)\sum_{n=3}^{\infty}\frac{a_n^2}{\bar{\gamma}_n\varrho_n\sum_{\nu=0}^{n}a_\nu^2}<\infty,$$

woraus sich auf Grund von (1. 5), (1. 6) und dem B. Levischen Satz ergibt, daß die Reihe

$$\sum_{n=2}^{\infty}(t_n-t_{n+1})\bar{S}_n(x)$$

in (a, b) fast überall konvergiert. So divergiert die rechte Seite von (1. 4) nach obigen fast überall, und folglich ist fast überall

$$\frac{1}{\bar{\gamma}_{\mu_N}\varrho_{\mu_N}\left(\sum_{\nu=0}^{\mu_N}a_\nu^2\right)^{1/2}}\left|\sum_{\nu=0}^{\mu_N}a_\nu\Phi_\nu(x)\right|\neq o_x(1),$$

woraus nach der Definition von $\{\eta_n\}$ folgt:

$$(1.7)\qquad \overline{\lim_{N\to\infty}}\frac{1}{\tilde{\gamma}_{\mu_N}\varrho_{\mu_N}\left(\sum_{\nu=0}^{\mu_n}a_\nu^2\right)^{1/2}}\left|\sum_{\nu=0}^{\mu_N}a_\nu\Phi_\nu(x)\right|=\infty$$

fast überall im Intervall (a, b).

Wir setzen

$$l_n = \frac{1}{\tilde{\gamma}_n \varrho_n \left(\sum_{\nu=0}^{n} a_\nu^2\right)^{1/2}}.$$

Auf Grund von (1. 3) und von (4) erhalten wir

$$\sum_{n=1}^{\infty} a_n^2 l_n^2 = \sum_{n=1}^{\infty} \frac{a_n^2}{\tilde{\gamma}_n^2 \varrho_n^2 \sum_{\nu=0}^{n} a_\nu^2} < \infty,$$

woraus sich wegen des Hilfssatzes I ergibt, daß in (a, b) fast überall

(1. 8) $$\bar{S}_{\mu_n}(x) - V_{\mu_n}(\lambda; x) = o_x(l_{\mu_n}^{-1})$$

gilt.

Auf Grund von (1. 7) ergibt sich aus (1. 8), daß die Beziehung (6) für dieses Funktionensystem $\{\Phi_n(x)\}$ in (a, b) fast überall besteht, was zu beweisen war.

§ 2. Beweis von Satz II

Nach der Behauptung des Satzes A ist es hinreichend zu beweisen, daß für alle genügend große n und k mit $\mu_n < k \leqq \mu_{n+1}$

$$\log\log(1+\lambda_k)^{k-\lambda_k+1} \geqq \log n,$$

d.h.

$$(1+\lambda_k)^{k-\lambda_k+1} \geqq 2^n$$

gilt.

Auf Grund der Definition der Folge $\{\lambda_n\}$ ist es genügend zu beweisen, daß für alle genügend große n

$$(1+\lambda_{\mu_n})^{\mu_n-\lambda_{\mu_n}+1} \geqq 2^n$$

gilt.

Im Falle $\lambda_n = n$ $(n = 1, 2, \ldots)$ ist die Behauptung des Satzes II schon bekannt.

Im entgegengesetzten Fall sei n_0 die kleinste natürliche Zahl, für die die Bedingung $\lambda_{n_0} < n_0$ erfüllt ist.

Wir werden beweisen, daß für alle n mit $\mu_n \geqq n_0$

(2. 1) $$(1+\lambda_{\mu_{n+1}})^{\mu_{n+1}-\lambda_{\mu_{n+1}}+1} \geqq 2(1+\lambda_{\mu_n})^{\mu_n-\lambda_{\mu_n}+1}$$

gilt.

Sei n eine natürliche Zahl, mit $\mu_n \geqq n_0$. Nach der Definition von μ_n und der Bedingung über $\{\lambda_n\}$ gilt die Ungleichung $\lambda_{\mu_{m+1}} \leqq 2\lambda_{\mu_m}$ $(m = 0, 1, 2, \ldots)$.

Im Falle $\lambda_{\mu_{n+1}} < 2\lambda_{\mu_n}$ mit einfachener Rechnung bekommen wir, daß die Beziehung

$$(1+\lambda_{\mu_{n+1}})^{\mu_{n+1}-\lambda_{\mu_{n+1}}+1} = \{1+\lambda_{\mu_n}+(\lambda_{\mu_{n+1}}-\lambda_{\mu_n})\}^{\mu_n-\lambda_{\mu_n}+1} \cdot$$
$$\cdot \{1+\lambda_{\mu_n}+(\lambda_{\mu_{n+1}}-\lambda_{\mu_n})\}^{2\lambda_{\mu_n}-\lambda_{\mu_{n+1}}}$$

gilt, woraus (2. 1) unmittelbar folgt.

Im Falle $\lambda_{\mu_{n+1}}=2\lambda_{\mu_n}$ bekommen wir ähnlicherweise mit einfachener Rechnung die Abschätzung:

$$2<\left(\frac{3}{2}\right)^2\leqq\left(\frac{3}{2}\right)^{\mu_n-\lambda_{\mu_n}+1}\leqq\left(1+\frac{\lambda_{\mu_n}}{1+\lambda_{\mu_n}}\right)^{\mu_n-\lambda_{\mu_n}+1}\leqq\left(\frac{1+2\lambda_{\mu_n}}{1+\lambda_{\mu_n}}\right)^{\mu_n-\lambda_{\mu_n}+1},$$

womit der Satz II bewiesen ist.

(Eingegangen am 31. Mai 1968.)

EGYETEMI SZÁMÍTÓKÖZPONT,
BUDAPEST, IX., DIMITROV TÉR 8

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ON THE DUALIZATION OF SUBDIRECT EMBEDDINGS

By
F. SZÁSZ and R. WIEGANDT (Budapest)

To Professor G. ALEXITS *on his 70th birthday*

§ 1. Introduction

In the algebra there are several kinds of structure theorems which can be formulated without operations, using only homological tools. For instance, the well-known fact that any universal algebra can be subdirectly embedded in a direct product of subdirectly irreducible algebras, can be formulated in a pure category-theoretical manner. Now the question arises what its dual statement asserts. Our purpose is to give such a category which satisfies certain selfdual conditions, and making use of these, to prove structure theorems and their dual statements. The structure theorems themselves are, of course, well-known statements for algebraic stuctures. However, their duals yield some theorems of unusual type. About the possibility of the dualization there occurs some trouble. The most of the difficulties is at finding selfdual conditions being necessary to prove the theorems. So we must not make use of the condition 'every epimorphism is a normal one' which is fulfilled for groups, since its dual is false. Further the lattice of all congruence-relations of any universal algebra is a so-called compactly generated lattice. This fact plays a very important role in the proof of the theorem according to subdirect embeddings of universal algebras, nevertheless compactly generating is not a selfdual notion.

Applying the theorems proved for certain categories, we establish some particular theorems for rings, groups, modules, respectively.

In § 2 we give a detailed enumeration of the usual notions and assertions of the theory of categories with respect to the importance of the dual notions and assertions, moreover, we form a system of selfdual conditions which will be satisfied by the category we are dealing with. § 3 is devoted to the investigation of subdirect embeddings, subdirect irreducibility and to the dualization of those. In § 4 we applicate the results developed before for rings, groups, modules and abelian groups. Most of the applications are concerned with rings.

§ 2. Preliminaries

Let $\mathscr{C}$ be a category. The objects and maps of $\mathscr{C}$ will be denoted by small Latin and small Greek letters, respectively. By definition $\mathscr{C}$ satisfies the following conditions:

(C_1) *If* $\alpha:a\to b$ *and* $\beta:b\to c$ *are maps, then there is a uniquely defined map* $\alpha\beta:a\to c$, *which is called the product of the maps* α *and* β;

(C_2) *If* $\alpha:a\to b$, $\beta:b\to c$, $\gamma:c\to d$ *are maps, then* $(\alpha\beta)\gamma=\alpha(\beta\gamma)$ *holds.*

(C_3) *For each object* $a\in\mathscr{C}$ *there is a map* $\varepsilon_a:a\to a$, *called the identity map of a such that for each* $\alpha:b\to a$ *and* $\beta:a\to c$ *we have* $\alpha\varepsilon_a=\alpha$, $\varepsilon_a\beta=\beta$.

The *dual category* of the category $\mathscr{C}$, denoted by $\mathscr{C}^*$, consists of the same objects as $\mathscr{C}$, and $\alpha^*:b\to a$ is a map of $\mathscr{C}^*$ if and only if $\alpha:a\to b$ is a map of $\mathscr{C}$. Clearly $(\mathscr{C}^*)^*=\mathscr{C}$, and if a statement P is true for category $\mathscr{C}$, then there is a dual statement P^* which will be true for $\mathscr{C}^*$. In what follows we shall assume that the category $\mathscr{C}$ satisfies some additional assumptions. *These requirements will be selfdual which means that both of $\mathscr{C}$ and $\mathscr{C}^*$ satisfy them. So any statement P which can be proved for $\mathscr{C}$, will be true for $\mathscr{C}^*$ too. Hence statement P^* is true for $(\mathscr{C}^*)^*=\mathscr{C}$.*

Let $H(a, b)$ denote the class of all maps of $\mathscr{C}$ which map a into b. An object $o\in\mathscr{C}$ is said to be a *zero object* if for any object a of $\mathscr{C}$ both of the classes $H(a, o)$ and $H(o, a)$ contain only one map.

We assume that

(C_4) *$\mathscr{C}$ possesses zero objects.*

Obviously also $\mathscr{C}^*$ contains zero objects. We shall say that $\mathscr{C}$ is a *category with zero maps*, if for any ordered pair of objects a, b there is a map $\omega_{ab}:a\to b$ such that for any $\alpha:c\to a$, $\beta:b\to d$ we have $\alpha\omega_{ab}=\omega_{cb}$ and $\omega_{ab}\beta=\omega_{ad}$. If $\mathscr{C}$ possesses zero objects, then $\mathscr{C}$ is a category with zero maps (cf. KUROŠ—LIVŠITS—ŠULGEIFER—TSALENKO [8]). If there is no doubt between which objects the zero map operates, then that zero map will be shortly denoted by ω.

A map $\alpha:a\to c$ will be called a *monomorphism*, if for any maps $\varrho:b\to a$, $\sigma:b\to a$ from $\varrho\alpha=\sigma\alpha$ it follows $\varrho=\sigma$.

A map $\alpha:c\to a$ will be called an *epimorphism*, if for any maps $\varrho:a\to b$, $\sigma:a\to b$ from $\alpha\varrho=\alpha\sigma$ it follows $\varrho=\sigma$.

The notion of epimorphism is dual to that of monomorphism in the sense that α is a monomorphism of $\mathscr{C}$ if and only if α^* is an epimorphism of $\mathscr{C}^*$.

The product of two monomorphism (if it exists) is again a monomorphism. If $\alpha\beta$ is a monomorphism, then α is also a monomorphism.

The product of two epimorphisms (if it exists) is again an epimorphism. If $\beta\alpha$ is an epimorphism, then α is also an epimorphism.

The statements are well-known (cf. KUROŠ—LIVŠITS—ŠULGEIFER—TSALENKO [8], or MITCHELL [9]). Now we are going to give the definitions of some usual notions together with their duals.

Let $\beta_1:b_1\to a$ and $\beta_2:b_2\to a$ be monomorphisms. We shall say that $(b_2, \beta_2)\leqq(b_1, \beta_1)$, if there exists a map ϱ (which has to be a monomorphism) such that $\varrho\beta_1=\beta_2$. If both of $(b_2, \beta_2)\leqq(b_1, \beta_1)$ and $(b_1, \beta_1)\leqq(b_2, \beta_2)$ hold then the pairs (b_1, β_1) and (b_2, β_2) are said to be equivalent. If $(b_2, \beta_2)\leqq(b_1, \beta_1)$ but they are not equivalent, then we shall write $(b_2, \beta_2)<(b_1, \beta_1)$. The equivalence classes of the relation thus defined will be called the *subobjects* of a. For

Let $\beta_1:a\to b_1$ and $\beta_2:a\to b_2$ be epimorphisms. We shall say that $(\beta_2, b_2)\leqq(\beta_1, b_1)$ if there exists a map ϱ (which has to be an epimorphism) such that $\beta_1\varrho=\beta_2$. If both of $(\beta_2, b_2)\leqq(\beta_1, b_1)$ and $(\beta_1, b_1)\leqq(\beta_2, b_2)$ hold, then the pairs (β_1, b_1) and (β_2, b_2) are said to be equivalent. If $(\beta_2, b_2)\leqq(\beta_1, b_1)$ but they are not equivalent, then we shall write $(\beta_2, b_2)<(\beta_1, b_1)$. The equivalence classes of the relation thus defined will be called the *factor-objects* of a. For convenience the convenience the equivalence class represented by the pair (b, β) will also be denoted by (b, β).

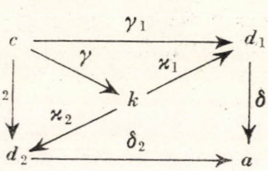
A commutative diagram

$$\begin{array}{ccc} k & \xrightarrow{\varkappa_1} & d_1 \\ {\scriptstyle\varkappa_2}\downarrow & & \downarrow{\scriptstyle\delta_1} \\ d_2 & \xrightarrow[\delta_2]{} & a \end{array}$$

is called a *pullback* for δ_1 and δ_2, if for any object $c \in \mathscr{C}$ and commutative diagram

$$\begin{array}{ccc} c & \xrightarrow{\gamma_1} & d_1 \\ {\scriptstyle\gamma_2}\downarrow & & \downarrow{\scriptstyle\delta_1} \\ d_2 & \xrightarrow{\delta_2} & a \end{array}$$

there exists a unique map $\gamma: c \to k$ such that the diagram



is again commutative.

A subobject $(k, \varkappa)$ of an object $a \in \mathscr{C}$ is said to be a *kernel* of the map $\alpha: a \to b$, if

$$\begin{array}{ccc} k & \longrightarrow & 0 \\ {\scriptstyle\varkappa}\downarrow & & \downarrow \\ a & \xrightarrow{\alpha} & b \end{array}$$

is a pullback diagram. Here the map $\varkappa$ has to be a monomorphism. Equivalently, the subobject $(k, \varkappa)$ is the kernel of α if (i) $\varkappa\alpha = \omega$; (ii) for each $\gamma: c \to a$ satisfying $\gamma\alpha = \omega$, there is a unique map $\gamma': c \to k$ such that $\gamma'\varkappa = \gamma$. If $(k, \varkappa)$ is a kernel of α, then we shall write $\operatorname{Ker} \alpha = (k, \varkappa)$, or only $\operatorname{Ker} \alpha = k$. The map $\varkappa$ is called a normal monomorphism and the subobject $(k, \varkappa)$ is a *normal subobject or* an *ideal* of a.

equivalence class represented by the pair (β, b) will also be denoted by (β, b).

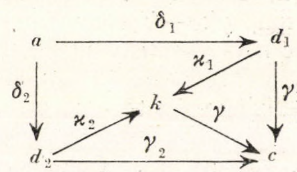
A commutative diagram

$$\begin{array}{ccc} a & \xrightarrow{\delta_1} & d_1 \\ {\scriptstyle\delta_2}\downarrow & & \downarrow{\scriptstyle\varkappa_1} \\ d_2 & \xrightarrow{\varkappa_2} & k \end{array}$$

is called a *pushout* for δ_1 and δ_2, if for any object $c \in \mathscr{C}$ and commutative diagram

$$\begin{array}{ccc} a & \xrightarrow{\delta_1} & d_1 \\ {\scriptstyle\delta_2}\downarrow & & \downarrow{\scriptstyle\gamma_1} \\ d_2 & \xrightarrow{\gamma_2} & c \end{array}$$

there exists a unique map $\gamma: k \to c$ such that



is again commutative.

A factorobject $(\varkappa, k)$ of an object $a \in \mathscr{C}$ is said to be a *cokernel* of the map $\alpha: b \to a$ if

$$\begin{array}{ccc} b & \xrightarrow{\alpha} & a \\ \downarrow & & \downarrow{\scriptstyle\varkappa} \\ 0 & \longrightarrow & k \end{array}$$

is a pusout diagram. Here the map $\varkappa$ has to be an epimorphism. Equivalently, the factorobject $(\varkappa, k)$ is the cokernel of α if (i) $\alpha\varkappa = \omega$; (ii) for each $\gamma: a \to c$ satisfying $\alpha\gamma = \omega$, there is a unique map $\gamma': k \to c$ such that $\varkappa\gamma' = \gamma$. If $(\varkappa, k)$ is a cokernel of α, then we shall write $\operatorname{Coker} \alpha = (\varkappa, k)$ or only $\operatorname{Coker} \alpha = k$. The map $\varkappa$ is called a *normal epimorphism* and the factorobject $(\varkappa, k)$ is a *normal factorobject* of a.

These definitions correspond to those of MITCHELL [9] and SULIŃSKI [13]. In KUROŠ—LIVŠITS—ŠULGEIFER—TSALENKO [8] ideals and normal subobjects (and so their duals) are not the same notions, but under conditions supposed below they coincide.

In the category of groups every epimorphism is a normal one, but not every monomorphism is a normal one (i.e. not every subgroup is a normal subgroup). The product of two normal monomorphisms need not be a normal one. Moreover, if α is a monomorphism, then Ker $\alpha=(o, \omega)$, but the converse statement does not hold.

If $\alpha\beta$ is a normal monomorphism and β is a monomorphism, then α is a normal monomorphism. (Cf. [8] § 8.3). The dual statement also holds for normal epimorphisms.

We assume that

(C_5) *Every map has a kernel and a cokernel.*

PROPOSITION 1. Ker Coker Ker $\alpha=$ Ker α.

PROOF. Let $\alpha:a\to b$ be a map, and put Ker $\alpha=(k, \varkappa)$ and Coker $\varkappa=(\lambda, l)$. We have to prove Ker $\lambda=(k, \varkappa)$. (i) Since $(\lambda, l)=$ Coker $\varkappa$, so by definition $\varkappa\lambda=\omega$ holds. (ii) Let $\gamma:c\to a$ be a map with $\gamma\lambda=\omega$. By definition of Coker $\varkappa$ there is a unique map $\gamma':c\to k$ such that $\gamma'\varkappa=\gamma$. Thus from $\gamma\lambda=\omega$ we get the existence of a unique map γ' satisfying $\gamma'\varkappa=\gamma$. Hence Ker $\lambda=(k, \varkappa)$ is valid.

Dualizing we get

PROPOSITION 1*. Coker Ker Coker $\alpha=$ Coker α.

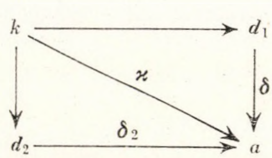
We suppose that

(C_6) *The class of all subobjects and factorobjects of any object a is a set, and it forms a complete lattice L_a and L_a^* with respect to the relation $\leqq$ defined for subobjects and factorobjects, respectively.*

(C_7) *For each object $a\in\mathscr{C}$ the set of all normal subobjects and normal factorobjects, forms a complete sublattice of L_a and L_a^*, respectively.*

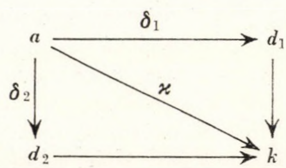
The intersection $\cap$ and union $\cup$ in the lattices L_a and L_a^* of the ideals and normal factorobjects of the objects a can be defined in the following way.

The intersection $(k, \varkappa)$ of two ideals (d_1, δ_1), (d_2, δ_2) is an ideal such that



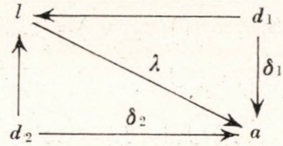
is a pullback diagram.

The intersection $(\varkappa, k)$ of two normal factorobjects (δ_1, d_1), (δ_2, d_2) is a normal factorobject such that

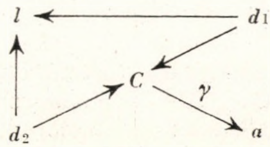


is a pushout diagram.

The union (l, λ) of two ideals (d_1, δ_1), (d_2, δ_2) means an ideal for which



is a commutative diagram, and for any monomorphism $\gamma: c \to a$ and diagram



there is a monomorphism $\lambda': l \to c$ such that $\lambda'\gamma = \lambda$, and the diagram becomes commutative.

The union of two normal factorobjects is defined in the dual way.

These definitions of the unions correspond to the relation $\leqq$ defined on L_a and L_a^*, respectively. However, in MITCHELL [9] the unions are defined in a somewhat different manner.

PROPOSITION 2. *The lattice L_a of the ideals of an object a is dually isomorphic to the lattice L_a^* of the normal factorobjects of a in the following sense. Any ideal $(k, \varkappa)$ of L_a is a kernel* Ker α *of a map α. The correspondence* Ker $\alpha \to$ Coker Ker $\alpha = (\lambda, l)$ *is one-to-one, further the relation $(k_1, \varkappa_1) \leqq (k_2, \varkappa_2)$ holds if and only if $(\lambda_1, l_1) \geqq (\lambda_2, l_2)$ is valid for their cokernels in L_a^*.*

PROOF. Proposition 1 implies that Ker $\alpha \to$ Coker Ker α is a one-to-one correspondence.

Assume $(k_1, \varkappa_1) \leqq (k_2, \varkappa_2) \in L_a$, and put Coker $\varkappa_i = (\lambda_i, l_i)$, $i = 1, 2$. By definition

$$\begin{array}{ccc} k_1 & \xrightarrow{\varkappa_1} & a \\ \downarrow & & \downarrow{\scriptstyle \lambda_1} \\ 0 & \longrightarrow & l_1 \end{array} \tag{1}$$

is a pushout diagram. Since $(k_1, \varkappa_1) \leqq (k_2, \varkappa_2)$, so there is a map $\varkappa': k_1 \to k_2$ such that $\varkappa'\varkappa_2 = \varkappa_1$. Thus

$$\begin{array}{ccc} k_1 & \xrightarrow{\varkappa'\varkappa_2} & a \\ \downarrow & & \downarrow{\scriptstyle \lambda_2} \\ 0 & \longrightarrow & l_2 \end{array}$$

is a commutative diagram, and since (1) is a pushout, therefore there is a map $\lambda': l_1 \to l_2$ such that $\lambda_1\lambda' = \lambda_2$. This means $(\lambda_1, l_1) \geqq (\lambda_2, l_2)$.

Dualizing, $(\lambda_1, l_1) \geqq (\lambda_2, l_2)$ implies $(k_1, \varkappa_1) \leqq (k_2, \varkappa_2)$. Thus proposition 2 is proved. One can formulate this statement as follows:

$$(k_1, \varkappa_1) \cap (k_2, \varkappa_2) = (c, \gamma),$$

$$(k_1, \varkappa_2) \cup (k_2, \varkappa_2) = (d, \delta)$$

are valid if and only if

$$(\lambda_1, l_1) \cup (\lambda_2, l_2) = \text{Coker } \gamma,$$

$$(\lambda_1, l_1) \cap (\lambda_2, l_2) = \text{Coker } \delta$$

are valid.

Let $\alpha: a \to b$ be a map. If $\mu: a \to m$ is an epimorphism and $v: m \to b$ a monomorphism with $\mu v = \alpha$, then the subobjects (m, v) of b will be called *an image of* α (with the epimorphism μ), (m, v) is said to be a *normal image*, if μ is a normal epimorphism.

Let $(k, \varkappa)$ be a subobject of the object a and let $\alpha: a \to b$ be an epimorphism. If (m, v) is an image of $\varkappa\alpha$, then (m, v) will be called an *image of* $(k, \varkappa)$ *by the epimorphism* α.

Let $\alpha: b \to a$ be a map. If $\mu: m \to a$ is a monomorphism and $v: b \to m$ is an epimorphism with $v\mu = \alpha$, then the factorobject (v, m) of b will be called a *coimage of* α (with the monomorphism μ), (v, m) is said to be a *normal coimage*, if μ is a normal monomorphism.

Let $(\varkappa, k)$ be a factorobject of the object a and let $\alpha: b \to a$ be a monomorphism. If (v, m) is a coimage of αk, then (v, m) *will be called a coimage of* $(\varkappa, k)$ *by the monomorphism* α.

A normal image (and normal coimage) is uniquely determined, but image (and coimage) is not (cf. KUROŠ—LIVŠITS—ŠULGEIFER—TSALENKO [8]). If (m, v) is an image of b such that for every image (m', v'), of b $(m, v) \leqq (m', v')$, then (m, v) will be denoted by $\text{Im}\,\alpha$. $\text{Coim}\,\alpha$ will denote the dual notion.

In the category of groups or rings, for any map α both of $\text{Im}\,\alpha$ and $\text{Coim}\,\alpha$ does exist, moreover, $\text{Im}\,\alpha$ is always a normal image, but $\text{Coim}\,\alpha$ need not be a normal coimage.

Let us assume that

(C_8) *For any map* α *there exist* $\text{Im}\,\alpha$ *and* $\text{Coim}\,\alpha$ *(they need not be normal).*

(C_9) *An im ageof an ideal by a normal epimorphism is always a normal ideal, and a coimage of a normal factorobject by a normal monomorphism is always a normal factorobject.*

Obviously all axioms (C_1)—(C_8) are satisfied in the category of groups or rings. This category satisfies clearly the first condition of axiom (C_9). Also the second condition is fulfilled. Consider the coimage (v, M) of a normal factorobject $(\varkappa, K)$ by a monomorphism $\alpha: B \to A$. Now the group (or ring) K is a factorgroup A/C and B is a subgroup of A. By the Second Isomorphism Theorem $B/B \cap C$ is isomorphic to a subgroup B'/C of A/C, and if B is a normal subgroup of A, then B'/C is also a normal one of A/C.

PROPOSITION 3. *If the map* α *has a normal image and* $\text{Ker}\,\alpha = (o, \omega)$, *then* α *is a monomorphism.*

This statement is proved in KUROŠ—LIVŠITS—ŠULGEIFER—TSALENKO [8] § 10,6.

Let a_i $(i \in I)$ be a family of objects of the category $\mathscr{C}$.

An object g is said to be a *direct product* of the objects a_i $(i \in I)$, if there are maps $\pi_i: g \to a_i$ $(i \in I)$ (called the *projections* of g onto a_i) such that for each object $h \in \mathscr{C}$ and for any system of maps $\alpha_i: h \to \alpha_i$ $(i \in I)$, there is a unique map (called the *canonical map*) $\gamma: h \to g$ such that $\gamma\pi_i = \alpha_i$ for all $i \in I$. g will be denoted by $g = \prod_{i \in I} a_i(\pi_i)$.

An object f is said to be a *free product* of the objects a_i $(i \in I)$ if there are maps $\varrho_i: a_i \to f$ $(i \in I)$ (called the *injections* of a_i into f) such that for each object $h \in \mathscr{C}$ and for any system of maps $\alpha_i: a_i \to h$ $(i \in I)$ there is a unique map (called the *canonical* map) $\gamma: f \to h$ such that $\varrho_i\gamma = \alpha_i$ for all $i \in I$. f will be denoted by $f = \sum_{i \in I} a_i(\varrho_i)$.

Assume that

(C_{10}) *Every family of objects has a direct product and a free product.*

Axiom (C_4) implies that all the projections π_i (injections ϱ_i) of a direct product $g = \prod_{i \in I} a_i(\pi_i)$ $\left(\text{free product } f = \sum_{i \in I} a_i(\varrho_i)\right)$ are epimorphism (monomorphisms). Moreover, to every projection π_i there is a normal monomorphism $\sigma_i: a_i \to g$ such that $\sigma_i\pi_i = \varepsilon_{a_i}$ and $\sigma_i\pi_j = \omega\,(i \neq j)$ hold, and so (a_i, σ_i) is an ideal of g (dually: to every injection ϱ_i there is a normal epimorphism $\tau_i: f \to a_i$ satisfying $\varrho_i\tau_i = \varepsilon_{a_i}$, $\varrho_j\tau_i = \omega\,(i \neq j)$). These facts are proved in [8].

PROPOSITION 4. *Let $(k_i, \varkappa_i)$ $(i \in I)$ be a family of ideals of an object $a \in \mathscr{C}$, and let $\alpha_i: a \to a_i$ be epimorphisms with* $\operatorname{Ker} \alpha_i = (k_i, \varkappa_i)$ $(i \in I)$. *Consider the direct product $g = \prod_{i \in I} a_i(\pi_i)$, and the canonical map $\gamma: a \to g\,(\gamma\pi_i = \alpha_i,\ i \in I)$. Then* $\operatorname{Ker} \gamma = \bigcap_{i \in I} (k_i, \varkappa_i)$ *is valid.*

For the proof we refer to SULIŃSKI [13], Proposition 2. 1. We omit to formulate the dual statement.

An object $a \in \mathscr{C}$ is said to be *subdirectly embedded* in the direct product $g = \prod_{i \in I} a_i(\pi_i)$ if there exists a monomorphism $\gamma: a \to g$ such that all maps $\alpha_i = \gamma\pi_i: a \to a_i$ $(i \in I)$ are normal epimorphisms (cf. [13]).

An object $a \in \mathscr{C}$ is said to be a *transfree image* of the free product $f = \sum_{i \in I} a_i(\varrho_i)$, if there exists an epimorphisms $\sigma: f \to a$ such that all maps $\beta_i = \varrho_i\sigma_i: a_i \to a$ $(i \in I)$ are normal monomorphisms.

Let us remark that according to this definition generally g can not be embedded subdirectly in itself, for the projections π_i need not be normal epimorphisms. The dual consideration holds for transfree images.

PROPOSITION 5. *An object $a\in\mathscr{C}$ can be subdirectly embedded in the direct product $g=\prod_{i\in I} a_i(\pi_i)$ if and only if there is a family of ideals $(k_i, \varkappa_i)$ $(i\in I)$ of a such that each of them is the kernel of the normal epimorphism $\alpha_i: a\to a_i$ $(i\in I)$ and $\bigcap_{i\in I}(k_i, \varkappa_i)=(o, \omega)$ holds.*

Dualizing we obtain

PROPOSITION 5*. *An object $a\in\mathscr{C}$ is a transfree image of the free product $f=\sum_{i\in I} a_i(\varrho_i)$ if and only if there is a family of normal factorobjects (λ_i, l_i) $(i\in I)$, of a such that each of them is the cokernel of the normal monomorphism $\beta_i: a_i\to a$ $(i\in I)$ and $\bigcap_{i\in I}(\lambda_i, l_i)=(\omega, o)$ holds.*

The statement of Proposition 5 is proved in SULIŃSKI [13] (Theorem 2, 3), assumed that every epimorphism is a normal one. Thus we give a modificated proof of this assertion.

Let a be subdirectly embedded in g by a monomorphism $\gamma: a\to g$. Now every $\alpha_i=\gamma\pi_i$ $(i\in I)$ is a normal epimorphism. If $(k_i, \varkappa_i)=\operatorname{Ker}\alpha_i$, then by Proposition 4 we get $\operatorname{Ker}\gamma=\bigcap_{i\in I}(k_i, \varkappa_i)$. Since γ is a monomorphism, therefore $\bigcap_{i\in I}(k_i; \varkappa_i)=(o, \omega)$ is valid.

Conversely, let $(k_i, \varkappa_i)$ be a family of ideals of a such that $(k_i, \varkappa_i)=\operatorname{Ker}\alpha_i$ where $\alpha_i: a\to a_i$ are normal epimorphisms and $\bigcap_{i\in I}(k_i, \varkappa_i)=(o, \omega)$ holds. Then there is a map $\gamma: a\to g$ such that $\gamma\pi_i=\alpha_i$ for $i\in I$. Applying Proposition 4, we get $\operatorname{Ker}\gamma=\bigcap_{i\in I}(k_i, \varkappa_i)=(o, \omega)$. By Proposition 2 we obtain $\bigcup_{i\in I}(\alpha_i, a_i)=\operatorname{Coker}\omega=(\varepsilon_a, a)$. Consider $\operatorname{Im}\gamma=(m, \nu)$ with the epimorphism μ (i.e. ν is a monomorphism and $\gamma=\mu\nu$). Since $\alpha_i=\mu\nu\pi_i$ and α_i $(i\in I)$ is an epimorphism, so $\nu\pi_i$ is also an epimorphism. Thus $(\mu, m)\geqq(\alpha_i, a_i)$ holds for every $i\in I$. Therefore we have $(\mu, m)\geqq\bigcup_{i\in I}(\alpha_i, a_i)=(\varepsilon_a, a)$. So (μ, m) is equivalent to (ε_a, a), and μ is a normal epimorphism. Therefore Proposition 3 implies that γ is a monomorphism, and Proposition 5 is proved.

An object $a\in\mathscr{C}$ is said to be *subdirectly irreducible*, if the intersection all of its non-zero ideals is a non-zero ideal.

An object $a\in\mathscr{C}$ is said to be *transfreely irreducible*, if the intersection all of its non-zero normal factorobjects is a non-zero normal factorobject.

According to Proposition 2, an object $a\in\mathscr{C}$ is transfreely irreducible if and only if the join of all its ideals $\neq(a, \varepsilon_a)$ differs from (a, ε_a).

Finally, let us mention that the categories of all rings and groups, respectively, and their dual categories fulfill axioms (C_1)—(C_{10}).

§ 3. Subdirect embeddings and transfree images

It is well-known that any universal algebra can be subdirectly embedded in a direct product of subdirectly irreducible universal algebras (G. BIRKHOFF [4]). In the proof there is making use of the fact that the lattice of congruence-relations of any universal algebra is compactly generated.

Let L be a complete lattice. An element $k \in L$ is said to be a *compact element*, if $k \leqq \bigcup_{i \in I} l_i$ implies $k \leqq \bigcup_{j \in J} l_j$ for some finite $J \subseteqq I$. The lattice is called *compactly generated*, if L is complete and every element of L is a union of (an infinite number of) compact elements.

In his paper [13] SULIŃSKI asked whether every object of a category satisfying somewhat stronger conditions than (C_1)—(C_{10}), can be subdirectly embedded in a direct product of subdirectly irreducible objects. Concerning this problem for a category $\mathscr{C}$ satisfying axioms (C_1)—(C_{10}) we present

THEOREM 1. *If the lattice L_a of all ideals of an object $a \in \mathscr{C}$ is compactly generated, then a can be subdirectly embedded in a direct product $g = \prod_{i \in I} a_i(\pi_i)$ by a monomorphism γ such a way that every $\gamma\pi_i = \alpha_i$ $(i \in I)$ is a normal epimorphism. A normal factorobject a_i of this decomposition is subdirectly irreducible if and only if the following condition holds:*

(I) *For any normal factorobject $(\chi, m) \neq (\varepsilon_{a_i}, a_i)$ of a_i (which is clearly a factorobject (χ_1, m) of a) there exists a normal factorobject (δ, d) of a such that $(\alpha_i, a_i) > > (\delta, d) \geqq (\chi_1, m)$.*

REMARK. Condition (I) seems to be complicated, but in the category of groups and rings, respectively, (I) is trivially fulfilled, for $\operatorname{Im} \alpha$ is always a normal image. However, its dual will be a rather natural condition in Theorem 1*. By Proposition 2 condition (I) means that for any ideal $(m', \chi') \neq (o, \omega)$ of a_i, there exists an ideal $(d', \delta') \geqq \operatorname{Ker} \alpha_i$ of a such that for its image (n', ν') by α_i we have $(o, \omega) \neq \neq (n', \nu') \leqq (n', \chi')$.

PROOF. Let $(k, \varkappa) \neq (o, \omega)$ be a compact element of the lattice L_a of all ideals of an object $a \in \mathscr{C}$. Consider the set $S_k = \{(l_j, \lambda_j)\}_{j \in J}$ of all ideals of a for which $(k, \varkappa) \cap (l_j, \lambda_j) < (k, \varkappa)$. Let $(l_1, \lambda_1) < (l_2, \lambda_2) < \cdots < (l_n, \lambda_n) < \cdots$ an ascending chain of ideals from S_k, and denote $\bigcup_n (l_n, \lambda_n)$ by (l_0, λ_0). We will show that $(k, \varkappa) \cap \cap (l_0, \lambda_0) < (k, \varkappa)$. Otherwise it would be $(k, \varkappa) = (l_0, \lambda_0)$ and since $(k, \varkappa)$ is a compact element of L_a, so for an index n_0 a relation $(k, \varkappa) \leqq (l_{n_0}, \lambda_{n_0})$ would hold in contradiction to the assumption. Making use of Zorn's lemma we obtain the existence of a maximal element $(\bar{l}, \bar{\lambda})$ of S_k.

To any compact element $(k_i, \varkappa_i)$ $(i \in I)$ of L_a, consider a maximal element $(\bar{l}_i, \bar{\lambda}_i)$ of S_{k_i}. Now we shall show $\bigcap_{i \in I} (\bar{l}_i, \bar{\lambda}_i) = (o, \omega)$. On the contrary, suppose $(l', \lambda') = = \bigcap_{i \in I} (\bar{l}_i, \bar{\lambda}_i) \neq (o, \omega)$. Since L_a is compactly generated, so (l', λ') is a union $\bigcup_{t \in T} (k_t, \varkappa_t)$ of compact elements $(k_t, \varkappa_t) \neq (o, \omega)$. The maximal elements $(\bar{l}_t, \bar{\lambda}_t)$ of S_{k_t} belonging to $(k_t, \varkappa_t)$ occur in the intersection representation of (l', λ'). Thus we get $(k_t, \varkappa_t) \leqq$

$\leqq(l', \lambda')\leqq(\bar{l}_t, \bar{\lambda}_t)$ which implies $(k_t, \varkappa_t)\cap(\bar{l}_t, \bar{\lambda}_t)=(k_t, \varkappa_t)$ contradicting the choice of $(\bar{l}_t, \bar{\lambda}_t)$.

Now, consider $(\alpha_i, a_i)=\text{Coker}\,\bar{\lambda}_i$. Since $(\bar{l}_i, \bar{\lambda}_i)$ is an ideal, therefore by Proposition 1 we have $\text{Ker}\,\alpha_i=(\bar{l}_i, \bar{\lambda}_i)=\text{Ker Coker}\,\bar{\lambda}_i$, further α_i is a normal epimorphism for all $i\in I$. Hence by virtue of Proposition 5 a can be subdirectly embedded in a direct product $g=\prod_{i\in I} a_i(\pi_i)$ by a monomorphism $\gamma: a\to g$ such that every map $\gamma\pi_i=\alpha_i$ is a normal epimorphism.

Finally, assume (I) for an object a_i. Since $(\alpha_i, a_i)\succ(\delta, d)$ so by Proposition 2 for their kernels we obtain $(\bar{l}_i, \bar{\lambda}_i)=\text{Ker}\,\alpha_i<\text{Ker}\,\delta=(d', \delta')$. By the choice of $(\bar{l}_i, \bar{\lambda}_i)$ it follows $(k_i, \varkappa_i)\leqq(d', \delta')$ where $(k_i, \varkappa_i)$ denotes the compact element of L_a belonging to $(\bar{l}_i, \bar{\lambda}_i)$. Thus for the intersection (d_0', δ_0') of all ideals $(d', \delta')\succ(\bar{l}_i, \bar{\lambda}_i)$ we have $(k_i, \varkappa_i)\leqq(d_0', \delta_0')$. Again, by Proposition 2 for $\text{Coker}\,\varkappa_i=(\varkappa_0, k_0)$ and $\text{Coker}\,\delta_0'=(\delta_0, d_0)$ we get $(\varkappa_0, k_0)\geqq(\delta_0, d_0)$ and $(\alpha_i, a_i)\geqq(\delta_0, d_0)$. Hereby

$$(\varkappa_0, k_0)\cup(\alpha_i, a_i)\succ(\varkappa_0, k_0)\geqq(\delta_0, d_0)$$

and so $(\alpha_i, a_i)\succ(\delta_0, d_0)$ follows. On the other hand for any normal factorobject (χ, m) of a_i being a factorobject (χ_1, m) of a the relation

$$(\chi_1, m)\leqq(\delta, d)\leqq(\delta_0, d_0)<(\alpha_i, a_i)$$

is valid. Therefore the union of all normal factorobjects differs from (ε_{a_i}, a_i) and so a_i is indeed subdirectly irreducible. If a_i is subdirectly irreducible, then (I) is trivially fulfilled.

REMARK. From Theorem 1 one can easily obtain BIRKHOFF's well-known theorem mentioned at the beginning of this chapter.

Dualizing Theorem 1 we obtain

THEOREM 1*. *If the lattice L_a^* of all normal factorobjects of an object $a\in\mathscr{C}$ is compactly generated, then a is a transfree image of a free product $f=\sum_{i\in I} a_i(\varrho_i)$ by an epimorphism γ such a way that every map $\varrho_i\gamma=\alpha_i: a_i\to a$ $(i\in I)$ is a normal monomorphism. A factor a_i of this decomposition is transfreely irreducible if and only if the following condition holds:*

(I*) *For any ideal $(m, \chi)\neq(a_i, \varepsilon_{a_i})$ of a_i (which is clearly a subobject (m, χ_1) of a) there exists an ideal (d, δ) of a such that $(m, \chi_1)\leqq(d, \delta)<(a_i, \alpha_i)$.*

Condition (I*) means that *for the ideal (a_i, α_i) the object a_i has exactly one maximal ideal (d, δ') (and (d, δ) is an ideal of a $(\delta=\delta'\alpha_i)$).*

To give an interpretation of Theorem 1* we introduce the following concept. An element k of a complete lattice L is called a *co-compact element*, if $k\geqq\bigcap_{i\in I} l_i$ implies $k\geqq\bigcap_{j\in J} l_j$ for some finite $J\subseteq I$. The lattice L is said to be co-compactly generated, if L is complete and every element of L is an intersection of co-compact elements. Hence by Proposition 2 the condition 'the lattice L_a^* of all normal factorobjects of a is compactly generated' should read '*the lattice L_a of all ideals of a is co-compactly generated*'. For comparison we mention that the lattice of all ideals of a ring need not be co-compactly generated and the same holds for groups too.

§ 4. Some applications

In what follows $\mathscr{C}_R$ will denote the category of rings. As it was mentioned before, $\mathscr{C}_R$ satisfies axioms (C_1)—(C_{10}) and condition (I), furter the lattice L_A of all ideals of a ring $A \in \mathscr{C}_R$ is compactly generated. Moreover, in $\mathscr{C}_R$ every map has a normal image. This means that Theorems 1 and 1* hold for $\mathscr{C}_R$. On the other hand, L_A has not to be co-compactly generated and condition (I*) is generally not fulfilled. However, there are some special but usual conditions which involve the validity of (I*) or that L_A is co-compactly generated. Thus Theorem 1* yields some theorems of unusual type for rings.

First of all we remark that instead of a free product of rings we speak about a *free sum* of rings. Further, if A_i $(i \in I)$ is a family of rings, then their free sum F is defined as the ring F consisting of all formal finite sums $\sum n_r \varphi_r$ where n_r is an integer and φ_r is a product of a finite number of elements from some A_i. For commutative rings, as it is well-known, free sum means the tensor product.

First, we show the existence of a ring the ideals of which do not form a co-compactly generated lattice.

EXAMPLE. Let A be a commutative principal ideal-ring with unity and without divisors of zero. (Such a ring is e.g. the ring of rational integers.) For any ideal $J \neq 0$ of A there exists an element $a \in A$ with $(a) = I$. According to RÉDEI [10], Satz 188 and 189 in A there exist g.c.d. and irreducible elements. Let $p \neq 1$ be an irreducible element with $(a, p) = 1$. Now $(a) \supset 0 = \bigcap_{k=1}^{\infty} (p^k)$ is valid, but obviously $(a) \not\supseteq (p^k)$ for any finite k. Thus $(a) = I$ is not a co-compact element of the lattice L_A of all ideals of A. Since I was chosen arbitrarily, so L_A is not co-compactly generated.

Now we give some sufficient conditions which guarantee that a lattice L should be co-compactly generated.

PROPOSITION 6. *Every element l of a lattice L is co-compact if and only if L satisfies the descending chain condition. In particular, the lattice L_A of a ring, abelian group and R-module A satisfying the minimum condition for ideals, subgroups and R-modules, respectively, is co-compactly generated.*

PROOF. Assume that each element of L is co-compact, and consider a descending chain $l_1 \supseteq l_2 \supseteq l_3 \supseteq \dots$ in L. Since also $l_0 = \bigcap l_n$ is co-compact, so there exists an index n_0 with $l_0 = l_{n_0}$ and the chain is finite. The inverse statement is trivial.

PROPOSITION 7. *If the ring A is a discrete direct sum of rings A_i $(i \in I)$ with minimum condition for ideals and each A_i has either a left or a right unity, then the ideal lattice L_A of A is co-compactly generated.*

PROOF. At first we prove that any ideal B of A is a discrete direct sum $B = \sum_{i \in I} \oplus B_i$ where B_i is an ideal of A_i for all $i \in I$. Let b be an arbitrary element of B, then b is a finite sum $b = \sum_{i \in J} a_i$ of elements $a_i \in A_i$. Let e_i be , for instance, a left unity of A_i. Then we obtain $e_i b = a_i \in B \cap A_i$ and obviously $B_i = B \cap A_i$ is an ideal of A. Thus we have $B = \sum \oplus B_i$. Since A_i fulfils the minimum condition for ideals, so there is a finite number of ideals of A containing $K_i = B_i + \sum_{i \neq j \in I} \oplus A_j$. Therefore K_i is a co-compact element of the lattice L_A. Further we have $B = \sum_{i \in I} \oplus B_i = \bigcap_{i \in I} K_i$ which means that L_A is co-compactly generated.

We remark that a ring satisfying the condition of this proposition need not fulfil the minimum condition for ideals.

Let us list some types of rings which fulfil condition (I*).

1) *Every accessible subring of the ring is an ideal.* A subring S is called accessible in the ring A, if there exists a finite ascending chain of subrings $S = S_1 \subseteq \subseteq S_2 \subseteq \ldots \subseteq S_n = A$ where each S_i is an ideal of S_{i+1} $(i = 1, 2, \ldots, n-1)$. Since any ideal of an ideal is any accessible subring, so this condition involves condition (I*) trivially. (Cf. ANDERSON—DIVINSKY—SULIŃSKI [1]).

2) *Every subring of the ring is an ideal* (Cf. RÉDEI [11]).

3) *The ring is completely reducible*, i.e. it is a discrete direct sum of simple rings. In such a ring every ideal is a direct summand. Since any ideal of a direct summand is an ideal also in the ring, so it follows condition (I*) (Cf. JACOBSON [7] Chapter IV. 1).

4) *Every subring of the ring is a direct summand* (cf. F. SZÁSZ [14]).

5) *Every ideal of the ring is idempotent.* Let A be, namely, such a ring and K an ideal of the ideal I of A. By a varied form of a lemma of ANDRUNAKIEVIČ [2] (see also DIVINSKY [6], Lemma 61), we obtain

$$\bar{K} = \bar{K}^3 \subseteq I \cdot \bar{K} \cdot I = I(K + KA + AK + AKA)I \subseteq K \subseteq \bar{K},$$

where $\bar{K}$ denotes the ideal of A generated by the subring K. Thus K is an ideal in A too.

Important subcases of 5) are the following:

6) *The ring A is regular in the sense of* VON NEUMANN, i.e. for any $a \in A$ there exists an element $x \in A$ with $a = axa$. By definition, it is clear that the ideals of such a ring are idempotent.

7) *The ring A is weakly regular*, i.e. every right ideal of A is idempotent (Cf. BROWN—MCCOY [5]).

8) *The ring A is biregular*, i.e. every principal two-sided ideal of A can be generated by a central idempotent element (Cf. ARENS—KAPLANSKY [3], BROWN—MCCOY [5] and ANDRUNAKIEVIČ [2]). If I is an arbitrary ideal of the ring A and $a \in I$, then there is a central idempotent element $c \in A$ such that $a \in (a) = (c)$. Hence from $c \in (c)^2 = (a)^2 \subseteq I^2$ we obtain $a \in I^2$ for every $a \in I$. Thus the ideals of A are idempotent.

Theorem 1* yields immediately

THEOREM 2. *Let A be a ring of one of the types* 1)—8). *If the ring A is either a ring with minimum condition for ideals or a discrete direct sum of rings with left or right unity elements and the direct components satisfy the minimum condition for ideals, then there exist ideals A_i $(i \in I)$ of A such that*

(i) *A_i has exactly one maximal ideal which is an ideal also of A for each $i \in I$.*

(ii) *every A_i is of the same type as A,*

(iii) *A is a homomorphic image of a free sum $\sum_{i \in I} B_i$, where $B_i \cong A_i$ holds for all $i \in I$.*

The statement that rings having one of the properties 1)—8) satisfy condition (ii), is almost trivial.

As another application, consider the category $\mathscr{C}_G$ of all groups. Now conditions (C_1)—(C_{10}) are satisfied. Condition (I*) is fulfilled, for instance, if any normal subgroup of a normal subgroup is a normal subgroup of the group, or briefly: normality is a transitive relation among the subgroups of a group. (Cf. D. S. ROBINSON [12]). From Theorem 1* it follows immediately

THEOREM 3. *Let L_G denote the lattice of all normal subgroups of a group G. If L_G is co-compactly generated, and normality of subgroups of G is a transitive relation, then there exist normal subgroups G_i $(i\in I)$ of G such that*

(i) *each G_i has exactly one maximal normal subgroup,*

(ii) *G is a homomorphic image of a free product $\prod_{i\in I}^* F_i$ where $F_i\cong G_i$ holds for every $i\in I$.*

Let R be a ring, and consider the category $\mathscr{C}_R$ of all R-modules. $\mathscr{C}_R$ fulfils conditions (C_1)—(C_{10}) as well as (I) and (I*). In $\mathscr{C}_R$ free sum means discrete direct sum. Hence from Theorem 1* we obtain

THEOREM 4. *If the lattice L_M of submodules of an R-modul M is co-compactly generated, then there exist submodules M_i $(i\in I)$ of M such that M is a homomorphic image of a discrete direct sum $\sum_{i\in I}\oplus N_i$ where N_i is isomorphic to M_i and N_i has exactly one maximal submodule for each $i\in I$.*

Since any abelian group can be regarded as a *module over the integers, so the analogous statement to that of Theorem 4 is valid for abelian groups too.*

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A METRIC CHARACTERIZATION OF SYMMETRIC SPACES

By
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To Professor G. ALEXITS *on his 70th birthday*

Let $\mathscr{C}$ be a class of metric spaces and $\mathscr{S}$ a system of axioms referring to a metric space (R, ϱ) by postulating properties that can be defined solely in terms of its distance function ϱ. If $\mathscr{S}$ is necessary and sufficient in order that (R, ϱ) be an element of $\mathscr{C}$, then $\mathscr{S}$ is called a *metric characterization* of $\mathscr{C}$. Congruent metric spaces are to be considered here as identical. The first systematic study of metric characterization problems was carried out by K. MENGER.[1] Since then results in this topic have formed a theory which embodies metric characterizations for every standard space of geometry.[2] MENGER and others however have applied metric methods to differential geometric problems as well, and in this respect a wider concept of metric characterization seems to be useful. Let V be a Riemannian manifold, $x, y \in V$, and denote by $\varrho(x, y)$ the infimum of length of piecewise differentiable curves of class C^1 joining x and y, then ϱ is a distance function on V, and (V, ϱ) is called the *induced metric space* of V.[3] Riemannian manifolds which have congruent induced metric spaces are known to be isometric.[4] This justifies the following definition: Let $\mathscr{C}$ be a class of Riemannian manifolds, $\overline{\mathscr{C}}$ the class of their induced metric spaces; if $\mathscr{S}$ is a metric characterization of $\overline{\mathscr{C}}$, then it is called a metric characterization of $\mathscr{C}$ too. The problem to give a metric characterization of the class of all 2-dimensional Riemannian manifolds has been completely solved.[5] In this paper a metric characterization of the class of all symmetric Riemannian manifolds is given.

1. The induced metric space of a Riemannian manifold

Some definitions and well-known facts are listed below concerning the induced metric space of a Riemannian manifold, which is assumed to be of class C^∞ for convenience.

If any bounded infinite subset of a metric space has at least one point of accumulation, then the space is called *finitely compact*.[6] The induced space of a complete Riemannian manifold is finitely compact.[7] Let x, y, z be different points of a metric space (R, ϱ) with $\varrho(x, z) + \varrho(z, y) = \varrho(x, y)$, then z is said *to be between* x and y,

[1] MENGER [7].
[2] BLUMENTHAL [1].
[3] KOBAYASHI—NOMIZU [6], 157—158.
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[5] WALD [13], RESETNIAK [17].
[6] BUSEMANN [3], 6.
[7] KOBAYASHI—NOMIZU [6], 172—176.

in notation: xzy. A subset of a metric space is called *convex* if it contains with any two different points a further point being between them.[8] The induced space of a complete Riemannian manifold is convex.[9] A distance preserving map of a compact interval of the real numbers into a metric space is called a *metric segment*.[10] If (R, ϱ) is a finitely compact convex metric space, and $x, y \in R$ are different points, then there is a metric segment $\varphi:[\alpha, \beta] \to R$ with $\varphi(\alpha)=x$, $\varphi(\beta)=y$.[11] A metric segment of the induced metric space of a complete Riemannian manifold is a minimizing geodesic of the latter.[12]

Let x be a point of a finitely compact convex metric space (R, ϱ). If there is an $\varepsilon > 0$ such that to any two different points a, b of its open ε-neighbourhood $U_\varepsilon(x)=\{p:\varrho(p, x)<\varepsilon\}$ a point c with abc exists, the the metric segments are said *to be prolongable* at x. If the metric segments are prolongable at all points of the space, then the *condition of local prolongability* is said to hold for (R, ϱ).[13] If x, y, z_1, z_2 are points of (R, ϱ) such that xyz_1, xyz_2, $\varrho(x, z_1)=\varrho(x, z_2)$, $z_1 \neq z_2$, then y is called a *ramification point*. The axiom of local prolongability holds for the induced space of a Riemannian manifold, and it has no ramification points.[14] A map of the real numbers into a metric space is called a *geodesic* if it is distance preserving in some neighbourhood of every number.[15] The metric segments of a finitely compact and convex metric space can be uniquely extended to geodesics, if the axiom of local prolongability is valid and the space has no ramification points. The geodesics of a complete Riemannian manifold are identical with those of its induced metric space.

A closed convex subset of a metric space is called *strictly convex* if any point being between two points of the set is its interior point.[16] A metric space (R, ϱ) is said to be *regular* at the point $x \in R$, if there exist real numbers $\delta > 0$, $\varkappa > 0$ such that the closed ξ-neighbourhood $\overline{U_\xi(z)}=\{p: \varrho(p, z) \leqq \xi\}$ is strictly convex for $z \in U_\delta(x)$, $0 < \xi \leqq \varkappa$. A metric space is called regular if it is regular at all of its points. A Riemannian manifold induces regular metric space.[17]

If $\{a, b, c\}$ is a point-triple of a metric space, then there is a triple of points $\{A, B, C\}$ of the euclidean plane, which is congruent with the former. If $a \neq b, c$, the measure of $\sphericalangle BAC$ is called the *metric angle* of $\{a, b, c\}$ at a and it is denoted by $\gamma(a; b, c)$. Let $\varphi, \psi:[0, \eta] \to R$ be continuous curves of (R, ϱ) with $\varphi(0)=\psi(0)=x \in R$ and $\varphi(t), \psi(t) \neq x$, for $0 < t \leqq \eta$. If $\gamma = \lim_{t, t' \to 0} \gamma(x; \varphi(t), \psi(t'))$ exists, the curves φ, ψ are said *to have an angle* γ at x.[18] In a Riemannian manifold the angle of two curves exists if and only if it exists in the induced metric space of the manifold, and then the two angles are equal.[19] A metric space is said to be

[8] MENGER [7].
[9] KOBAYASHI—NOMIZU [6], 172—174.
[10] BUSEMANN [3], 27—28.
[11] BUSEMANN [3], 29—30.
[12] KOBAYASHI—NOMIZU [6], 168.
[13] BUSEMANN [3], 33—34.
[14] KOBAYASHI—NOMIZU [6], 168.
[15] BUSEMANN [3], 32.
[16] BUSEMANN [3], 117.
[17] KOBAYASHI—NOMIZU [6], 162—167, WHITEHEAD [14].
[18] RINOW [11], 296—297, WILSON [15].
[19] RINOW [11], 302—308.

euclidean at its point x, if any two metric segments issuing from x have an angle there.

The isometric transformations of a Riemannian manifold are identical with the distance preserving transformations of its induced metric space.[20]

Frequent use will be made of the following axioms:

A1 (R, ϱ) *is finitely compact.*

A2 (R, ϱ) *is convex.*

A3 (R, ϱ) *satisfies the local prolongability condition.*

A4 (R, ϱ) *has no ramification points.*

A5 (R, ϱ) *is regular.*

A6 (R, ϱ) *is euclidean at all of its points.*

The first four of these axioms have been introduced by H. BUSEMANN as a starting point for his theory of G-spaces.[21]

2. The group of distance preserving transformations

The group of isometric transformations of a Riemannian manifold is a Lie group and it is identical with the group of distance preserving transformations of the induced space of the Riemannian manifold.[22] To generalize this fact it will be shown below that the group of distance preserving transformations of a metric space satisfying **A1—5** has no small subgroups.

Obvious applications of standard methods yield the following:

2. 1. LEMMA. *The distance preserving transformations of a finitely compact metric space* (R, ϱ) *form with the compact-open topology a locally compact topological transformation group* Γ *of the space.*[23]

The following theorem will prove useful at subsequent developments:

2. 2. THEOREM. *The group* Γ *of distance preserving transformations of a metric space* (R, ϱ) *satisfying* **A1—5** *has no small subgroups.*[24]

Assume that the assertion is not valid. If $A \subset R$ is a compact subset and $\alpha > 0$, then there is a subgroup Ξ of Γ with $\varrho(u, \varphi(u)) < \alpha$ for $u \in A$, $\varphi \in \Xi$. Since A is compact there are points v_i, $i=1, \dots, k$ with $A \subset \bigcup_{i=1}^{k} U_\beta(v_i)$, $\beta = \frac{1}{3}\alpha$. Since $[\{v_i\}, U_\beta(v_i)] = \{\varphi : \varphi \in \Gamma, \varphi(v_i) \in U_\beta(v_i)\}$ is a neighbourhood of the identity in Γ,

[20] KOBAYASHI—NOMIZU [6], 169—172.
[21] BUSEMANN [3], 37.
[22] MYERS—STEENROD [8].
[23] BUSEMANN [3], 16—18.
[24] SZENTHE [12].

there exists a non-trivial subgroup Ξ with $\Xi \subset \bigcap_{i=1}^{k} [\{v_i\}, U_\beta(v_i)]$ by the indirect assumption. Hence $\varrho(u, \varphi(u)) \leqq \varrho(u, v_i) + \varrho(v_i, \varphi(v_i)) + \varrho(\varphi(v_i), \varphi(u)) < \alpha$ for any $u \in A$, $\varphi \in \Xi$ with some $1 \leqq i \leqq k$.

According to **A3** and **A5** to any $x \in R$ there are numbers $\varepsilon > 0$ and $\delta, \varkappa > 0$ satisfying the requirements of these axioms. Let $x \in R$ be fixed and put $\mu = \min(\varepsilon, \delta)$, $\nu = \min(\varkappa, \frac{1}{3}\mu)$, $\sigma = \frac{1}{4}\nu$ for the $\varepsilon, \delta, \varkappa$ corresponding to this x. There exists a non-trivial closed subgroup Φ of Γ with $\varrho(u, \varphi(u)) < \sigma$ for $u \in \overline{U_\mu(x)}$, $\varphi \in \Phi$. If the set of fixed points of a distance preserving transformation of (R, ϱ) has a non-empty interior then this transformation is the identity.[25] Therefore there is a $z \in U_\nu(x)$ and a $\psi \in \Phi$ with $z \neq \psi(z)$. By **A1—4** there is a point w with $wz\psi(z)$ and $\varrho(w, z) = \sigma$. Let $S(\alpha)$ for $\alpha > 0$ be defined by $S(\alpha) = \bigcap_{\varphi \in \Phi} \overline{U_\alpha(\varphi(w))}$.

$S(\alpha)$ is strictly convex for $0 < \alpha \leqq \nu$. It suffices to show that c is an interior point of $S(\alpha)$ if $a, b \in S(\alpha)$ and acb. Since $\overline{U_\alpha(\varphi(w))}$ is strictly convex and $a, b \in \overline{U_\alpha(\varphi(w))}$ for $\varphi \in \Phi$, $0 < \alpha \leqq \nu$, there is a maximal $\delta_\varphi > 0$ with $U_{\delta_\varphi}(c) \subset \overline{U_\alpha(\varphi(w))}$ for any $\varphi \in \Phi$. Put $\lambda = \inf\{\delta_\varphi : \varphi \in \Phi\}$, then $\lambda > 0$. In fact $\lambda = 0$ would imply the existence of a sequence $\{\varphi_n\}_{n=1,2\dots}$ with $\lim_{n\to\infty} \varphi_n = \bar\varphi \in \Phi$ and $\lim_{n\to\infty} \delta_{\varphi_n} = 0$. Therefore a positive integer k would exist with $\varrho(\varphi_k(w), \varphi(w)) < \frac{1}{2}\delta_{\bar\varphi}$, $\delta_{\varphi_k} < \frac{1}{2}\delta_{\bar\varphi}$. But $\varrho(\bar\varphi(w), c) = \alpha - \delta_{\bar\varphi}$ and $\varrho(\varphi_k(w), c) = \alpha - \delta_{\varphi_k}$ by **A1—4**. Hence $\varrho(\varphi_k(w), c) > \alpha - \frac{1}{2}\delta_{\bar\varphi}$ and $\varrho(\bar\varphi(w), \varphi_k(w)) \geqq |\varrho(\bar\varphi(w), c) - \varrho(\varphi_k(w), c)| > \alpha - \frac{1}{2}\delta_{\bar\varphi} - (\alpha - \delta_{\bar\varphi}) = \frac{1}{2}\delta_{\bar\varphi}$ would be valid, the latter being in contradiction with a former inequality. Therefore $\lambda > 0$ and consequently $S(\alpha)$, $0 < \alpha \leqq \nu$ is strictly convex.

If $\chi \in \Phi$, then $\chi(S(\alpha)) = S(\alpha)$. This is shown by the following identities:

$$\chi(S(\alpha)) = \chi\Big(\bigcap_{\varphi \in \Phi} \overline{U_\alpha(\varphi(w))}\Big) = \bigcap_{\varphi \in \Phi} \overline{U_\alpha((\chi \circ \varphi)(w))} = \bigcap_{\varphi \in \Phi} \overline{U_\alpha(\varphi(w))} = S(\alpha).$$

Put $\xi = \inf\{\alpha : z \in S(\alpha'), \alpha' \geqq \alpha\}$. Then $z \in S(\xi)$, for in case of $z \notin S(\xi)$ there would be a $\xi' > \xi$ with $z \notin S(\xi')$, because $S(\xi) = \bigcap_{\varphi \in \Phi} \overline{U_\xi(U\varphi(w))}$ and $\overline{U_\xi(\varphi(w))}$ is closed. It follows from $z \in S(\xi)$, $\varphi \in \Phi$ and $\chi(S(\xi)) = S(\xi)$ for $\chi \in \Phi$ that $\psi(z) \in S(\xi)$. Further $w \in S(\xi)$, because in case of $w \notin S(\xi)$, a $\varphi' \in \Phi$ would exist with $\xi < \varrho(w, \varphi'(w)) < \sigma$ in contradiction with $\varrho(z, w) = \sigma \leqq \xi$. But z is not an interior point of $S(\xi)$, since $U_\tau(z) \subset S(\xi)$ for a $\tau > 0$ and **A1—4** would imply that $z \in S(\alpha)$ for $\alpha \geqq \xi - \tau$ contradicting the definition of ξ. Thus a strictly convex set $S(\xi)$ is given and points $w, z, \psi(z) \in S(\xi)$ with $wz\psi(z)$, where z is not an interior point of $S(\xi)$. This is a contradiction.

3. A metric characterization of the symmetric Riemannian manifolds

If p is isolated fixed point of an involutoric distance preserving transformation σ of a metric space, then σ is called a *symmetry* in p. Let (R, ϱ) be a metric space satisfying **A1—4**, σ a symmetry in $p \in R$ and $\varphi: R^1 \to R$ a geodesic with $\varphi(0) = p$, then $\sigma(\varphi(t)) = \varphi(-t)$ for $t \in R^1$. Thus (R, ϱ) cannot have more than one symmetry

[25] BUSEMANN [3].

in a point.[26] Let S be the set of points of (R, ϱ) which admit a symmetry, and σ_p the symmetry in p for $p \in S$.

3. 1. Lemma. *If (R, ϱ) satisfies* **A1—4** *and Γ is the group of its distance preserving transformations, then $\Sigma: S \to \Gamma$, defined by $\Sigma(p) = \sigma_p$ for $p \in S$, is continuous.*

Let $p \in S$ and let $[C, O]$ be any subbasic neighbourhood of σ_p in the compact-open topology of Γ. Assume that there is no neighbourhood U of p in S with $\sigma_q(C) \subset O$ for $q \in U$. Then there exist sequences $\{q_n\}_{n=1,2,\dots}$, $\{x_n\}_{n=1,2,\dots}$ with $\lim_{n\to\infty} q_n = p$, $\lim_{n\to\infty} x_n = x$, $x_n \in C$, $\sigma_{q_n}(x_n) \notin O$ for $n = 1, 2, \dots$. Hence by **A1—4** $\sigma_p(x) = \lim_{n\to\infty} \sigma_{q_n}(x_n) \notin O$ in contradiction with $\sigma_p(C) \in O$.

If σ_p, σ_q are symmetries of a metric space, then the distance preserving transformation $\tau_{pq} = \sigma_q \circ \sigma_p$ is called a *transvection.*

3. 2. Lemma. *Let (R, ϱ) satisfy* **A1—4** *and $\varphi: R^1 \to R$ be a geodesic with $\varphi(R^1) \subset S$. If $\alpha < \beta < \gamma < \delta$, $\beta - \alpha = \delta - \gamma$ and $\varphi(\alpha) = a$, $\varphi(\beta) = b$, $\varphi(\gamma) = c$, $\varphi(\delta) = d$,* then $\tau_{ab} = \tau_{cd}$.[27]

If $\beta = \gamma$, then $\tau_{ab}(x) = \tau_{cd}(x)$ for any $x \in R$. For if $\tau_{ab}(x') \neq \tau_{cd}(x')$ for some $x' \in R$, then $\sigma_b(\tau_{ab}(x')) = \sigma_a(x') \neq \sigma_b(\tau_{cd}(x'))$. But $\sigma_b(d) = a$, $\sigma_b(\sigma_c(x') = x'$, therefore $\sigma_b(\tau_{cd}(x')) = \sigma_b(\sigma_d(\sigma_c(x'))) = \sigma_a(x')$, which is a contradiction. Hence $\tau_{ad} = \sigma_d \circ \sigma_c \circ \circ \sigma_b \circ \sigma_a = \tau_{ab}^2$, when $\beta = \gamma$.

If $\beta \neq \gamma$, but $(\beta - \alpha)/(\gamma - \beta)$ is rational let $\alpha = \alpha_0 < \alpha_1 < \dots < \alpha_r = \delta$ be a subdivision of $[\alpha, \delta]$ with $\alpha_i - \alpha_{i-1} = (\delta - \alpha)/r$, such that $\beta = \alpha_k$, $\gamma = \alpha_1$ for some $1 < k < l < r$. By the preceeding assertions $\tau_{ab} = \tau^k_{\varphi(\alpha_0)\varphi(\alpha_1)} = \tau^k_{\varphi(\alpha_l)\varphi(\alpha_{l+1})} = \tau_{cd}$.

In the general case there are sequences $\{\alpha_n\}_{n=1,2,\dots}$, $\{\beta_n\}_{n=1,2,\dots}$, $\{\gamma_n\}_{n=1,2,\dots}$, $\{\delta_n\}_{n=1,2,\dots}$, with $\alpha_n < \beta_n < \gamma_n < \delta_n$, $\beta_n - \alpha_n = \delta_n - \gamma_n$ and $(\beta_n - \alpha_n)/(\gamma_n - \beta_n)$ rational for $n = 1, 2, \dots$, such that $\lim_{n\to\infty} \alpha_n = \alpha$, $\lim_{n\to\infty} \beta_n = \beta$, $\lim_{n\to\infty} \gamma_n = \gamma$. In consequence of 3.1. lemma

$$\tau_{ab} = \lim_{n\to\infty} \tau_{\varphi(\alpha_n)\varphi(\beta_n)} = \lim_{n\to\infty} \tau_{\varphi(\gamma_n)\varphi(\delta_n)} = \tau_{cd}.$$

3. 3. Lemma. *Let (R, ϱ) satisfy* **A1—4** *and $\varphi: R^1 \to R$ be a geodesic with $\varphi(R^1) \subset S$, then $\tau_\varphi: R^1 \to \Gamma$ defined by $\tau_\varphi(t) = \tau_{\varphi(0)\varphi(t/2)}$, $t \in R^1$ is a 1-parameter subgroup of Γ.*

If $u, v \in R^1$, then by the preceeding lemma $\tau_\varphi(u+v) = \tau_{\varphi(u/2)\varphi(u+v/2)} \circ \circ \tau_{\varphi(0)\varphi(u/2)} = \tau_\varphi(v) \circ \tau_\varphi(u)$. Thus $\tau_\varphi: R^1 \to \Gamma$ is a homomorphism; its continuity being implied by 3. 1. Lemma.

In such case τ_φ is called the *1-parameter subgroup of transvections corresponding to the geodesic φ.*

If $S = R$, then (R, ϱ) is called *symmetric.* The identity component of the group of distance preserving transformations of a symmetric metric space satisfying **A1—4** is transitive on the space by 3. 3. Lemma.

3. 4. Theorem. *Let (R, ϱ) be a symmetric metric space satisfying* **A1—5**, *then Γ_0 the identity component of its distance preserving transformations is a Lie group and R can be endowed with an analytic structure such that the map $\pi: \Gamma_0 \times R \to R$ defined by $\pi(\alpha, x) = \alpha(x)$, $\alpha \in \Gamma_0$, $x \in R$ is analytic.*

[26] Busemann [3], 345.
[27] Cartan [4], Part I, Volume 1, 93—94.

In consequence to 2. 2. Theorem the group of distance preserving transformations of (R, ϱ) has no small subgroups, therefore by the theorem of GLEASON and YAMABE[28] Γ_0 is a Lie group. Since Γ_0 is transitive on R the theory of transformation groups yields that R is a manifold which has a unique analytic structure for which $\pi:\Gamma_0 \times R \to R$ is analytic.[29]

In what follows the symmetric metric space satisfying **A1—5** will be considered with the above analytic structure as an analytic manifold too.

COROLLARY. *If $\varphi:R^1 \to R$ is a geodesic of the symmetric metric space (R, ϱ) satisfying* **A1—5**, *then φ is analytic curve of the analytic manifold R, the tangent vectors $\varphi_*(t)$, $t \in R^1$ are not zero-vectors, and if $\tau_\varphi:R^1 \to \Gamma_0$ is the 1-parameter subgroup of transvections corresponding to φ, then $\varphi_*(t)=\pi_*(\tau'_{\varphi_*}(t))$, $t \in R^1$, where $\tau'_{\varphi_*}(t)$ is the tangent vector of the curve $(\tau_\varphi(t), \varphi(0))$, $t \in R^1$ in $\Gamma_0 \times R$.*

If (R, ϱ) satisfies **A1—4**, then $\mathcal{G}$ the set of its geodesics, with the compact-open topology is called the space of its geodesics. Let $\{\varphi_n\}_{n=0,1,\dots}$ be a sequence of geodesics of (R, ϱ), then $\lim_{n\to\infty} \varphi_n = \varphi_0$ in $\mathcal{G}$ if and only if $\varphi_n(t) \to \varphi_0(t)$ for values of t taken from a compact interval of R^1. If $\mathcal{A} \in \mathcal{G}$ and $A=\{\varphi(0):\varphi \in \mathcal{A}\}$ is bounded, then $\mathcal{A}$ has a compact closure.[30]

3. 5. LEMMA. *Let $\{\varphi_n\}_{n=0,1,\dots}$ be a sequence of geodesics of the symmetric metric space satisfying* **A1—5** *with $\lim_{n\to\infty} \varphi_n = \varphi_0$, then $\lim_{n\to\infty} \varphi_{n_*}(0) = \varphi_{0_*}(0)$ holds in the topology of the tangent bundle TR of the analytic manifold R.*

Let $\{\tau_{\varphi_n}\}_{n=0,1,\dots}$ be the sequence of 1-parameter subgroups of transvections corresponding to the sequence of geodesics. By 3. 1. Lemma $\lim_{n\to\infty} \tau_{\varphi_n}(t) = \tau_{\varphi_0}(t)$ for $t \in R^1$. The existence of a canonical coordinate system of the first kind[31] in Γ_0 implies that $\lim_{n\to\infty} \tau_{\varphi_n*}(0) = \tau_{\varphi_0*}(0)$ holds in the tangent space of Γ_0 at the identity. Since $\lim_{n\to\infty} \tau'_{\varphi_n*}(0) = \tau'_{\varphi_0*}(0)$ in $T(\Gamma_0 \times R)$, $\varphi_{n*}(0) = \pi_*(\tau'_{\varphi*})$ and π_* is continuous, the assertion follows.

3. 6. LEMMA. *Let (R, ϱ) be a symmetric metric space satisfying* **A1—5**, *if X is a non-zero vector in the tangent bundle TR of the analytic manifold R, then there is a geodesic $\varphi:R^1 \to R$ of (R, ϱ) such that $X = \lambda\varphi_*(0)$ with $\lambda > 0$.*

Let X be a non-zero tangent vector of R with $X \in R_x$, $x \in R$, and $\varkappa:U_\gamma \to R^m$ a coordinate system of the analytic manifold R, defined on the spherical neighbourhood U_γ, $\gamma > 0$ of x, such that $\varkappa(x) = (\underset{1}{0}, \underset{2}{0}, \dots, \underset{m}{0})$. If $\mathcal{G}_x = \{\varphi:\varphi \in \mathcal{G},\ \varphi(0) = x\}$, then the curves, $\varkappa \circ \varphi$, $\varphi \in \mathcal{G}_x$ are analytic and cover the neighbourhood $\varkappa(U_\gamma)$ of $\varkappa(x)$ in R^m. If $(\varkappa \circ \varphi)(t) = (x^1_\varphi(t), \dots, x^m_\varphi(t))$, $0 \leq |t| < \gamma$, $\varphi \in \mathcal{G}_x$, then there is a $\gamma' > 0$ such that each analytic function x^i_φ, $i = 1, \dots, m$, $\varphi \in \mathcal{G}_x$ is represented by power series for $0 \leq |t| \leq \gamma'$. To verify the last assertion observe that since $\pi:\Gamma_0 \times R \to R$

[28] GLEASON [5], YAMABE [16].
[29] PONTRJAGIN [10], II. 95—97.
[30] BUSEMANN [3], 42.
[31] PONTRJAGIN [10], II. 65.

is analytic, if $(\alpha^1, \ldots, \alpha^r)$ is a canonical coordinate system of the first kind of Γ_0, then there is a neighbourhood W of the identity element of Γ and a neighbourhood $U \subset U_\gamma$ of x such that $\pi(\alpha, y) = \alpha(y) = z \in U_\gamma$ for $\alpha \in W$, $y \in U$ and if $z^i = \pi^i(\alpha^1, \ldots, \alpha^r; y^1, \ldots, y^m)$, $i = 1, \ldots, m$, z^i being the coordinates of z with respect to $\varkappa$, then each function π^i $(i = 1, \ldots, m)$ is represented by a power series for values of the variables which correspond to elements $\alpha \in W$, $y \in U$. By 3. 1. Lemma there is a $\gamma' > 0$ such that $\varphi(t) \in U$ and $\tau_\varphi(t) \in W$ for $0 \leq |t| \leq \gamma'$, $\varphi \in \mathcal{G}_x$. If $\big(\tau_\varphi^1(t), \ldots, \tau_\varphi^r(t)\big)$ are coordinates of $\tau_\varphi(t)$ with respect to the canonical coordinate system, then $\tau_\varphi^i(t) = \lambda_\varphi^i t$ for $0 \leq |t| \leq \gamma'$, $\varphi \in \mathcal{G}_x$, $i = 1, \ldots, m$. Therefore $x_\varphi^i(t) = \pi^i(\lambda_\varphi^1 t, \ldots, \lambda_\varphi^r t; \underbrace{0}_{1}, \underbrace{0}_{2}, \ldots, \underbrace{0}_{m})$ can be rearranged to give a power series of t for $0 \leq |t| \leq \gamma'$. Assume now that the assertion of the lemma is false; then there is a $(\xi^1, \ldots, \xi^m) \neq (\underbrace{0}_{1}, \underbrace{0}_{2}, \ldots, \underbrace{0}_{m})$ such that $(\lambda\xi^1, \ldots, \lambda\xi^m)$ for $\lambda > 0$ are not coordinates with respect to $\varkappa$ of a tangent vector $\varphi_*(0)$, $\varphi \in \mathcal{G}_x$. Let $\{p_n\}_{n=1,2,\ldots}$ be a sequence of points with $\lim_{n\to\infty} p_n = x$, $\varkappa(p_n) = (p_n^1, \ldots, p_n^m)$ and $\lim_{n\to\infty} \lambda_n p_n^i = \xi^i$, $i = 1, \ldots, m$ for a sequence $\{\lambda_n\}_{n=1,2,\ldots}$ of positive numbers. Let $\varphi_n \in \mathcal{G}_x$ be a geodesic with $\varphi_n(t_n) = p_n$, $\varrho(p_n, x) = t_n$ for $n = 1, 2, \ldots$. Since $\mathcal{G}_x$ is compact there is no loss of generality by assuming that $\lim_{n\to\infty} \varphi_n = \varphi_0 \in \mathcal{G}_x$. Hence $\lim_{n\to\infty} \varphi_n(t) = \varphi_0(t)$ and consequently $x_{\varphi_0}^i(t) = \lim_{n\to\infty} x_{\varphi_n}^i(t)$, $i = 1, \ldots, m$ for $0 \leq |t| \leq \gamma'$, and this convergence can be assumed to be uniform without loss of generality. Hence

$$\lim_{n\to\infty} \big(x_{\varphi_n}^i(t_n) - x_{\varphi_n}^i(0)\big)/t_n = \lim_{n\to\infty} p_n^i/t_n = \dot{x}_0^i(0).$$

But $\xi^i = \lim_{n\to\infty} \lambda_n p_n^i = \big(\lim_{n\to\infty} \lambda_n t_n\big)\dot{x}_0^i(0)$ gives a contradiction.

3. 7. LEMMA. *Let (R, ϱ) be a symmetric metric space satisfying* **A1—5**, *$\varkappa: U \to R^m$ a coordinate system of the analytic manifold R defined on a neighbourhood U of $x \in R$ and d the distance function of R^m. Then there is a neighbourhood $V \subset U$ of x such that $d\big(\varkappa(a), \varkappa(b)\big)/\varrho(a, b) \geq \mu$ for $a, b \in V$, $a \neq b$, with some $\mu > 0$.*

Assume that the above assertion is not valid, then there are sequences $\{a_n\}_{n=1,2,\ldots}$, $\{b_n\}_{n=1,2,\ldots}$ with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$, $a_n \neq b_n$ and

$$\lim_{n\to\infty} d(\varkappa(a_n), \varkappa(b_n))/\varrho(a_n, b_n) = 0.$$

A sequence of geodesics $\{\varphi_n\}_{n=1,2,\ldots}$ exists with $\varphi_n(0) = a_n$, $\varphi_n(t_n) = b_n$, $t_n = \varrho(a_n, b_n)$ and there is no loss of generality by assuming that $\lim_{n\to\infty} \varphi_n = \varphi_0$. Since $\pi: \Gamma_0 \times R \to R$ is analytic, if $(\alpha^1, \ldots, \alpha^r)$ is a canonical coordinate system of the first kind of Γ_0, then there is a neighbourhood W of the identity of Γ and a neighbourhood $U' \subset U$ of x such that $\pi(\alpha, y) = \alpha(y) = z \in U$ for $\alpha \in W$, $y \in U'$, and if $z^i = \pi^i(\alpha^1, \ldots, \alpha^r; y^1, \ldots, y^m)$, $i = 1, \ldots, m$, z^i, y^j being coordinates of z, y with respect to $\varkappa$, then each function π^i is represented by power series for values of the variables which correspond to $\alpha \in W$, $y \in U'$. By 3. 1. Lemma there exists a neighbourhood $U'' \subset U'$ of x with $\tau_{pq} \in W$ for $p, q \in U''$, consequently there is a $\gamma > 0$ and a N such that $\tau_{\varphi_n}(t) \in W$

for $0 \leqq |t| < \gamma$ if $n=0$, or $n \geqq N$. Thus $\tau_{\varphi_n}(t)$ is represented by $(\lambda_n^1 t, \dots, \lambda_n^r t)$ for $0 \leqq |t| < \gamma$ in the canonical coordinate system if $n=0$, or $n \geqq N$, and if $(\varkappa \circ \varphi_n)(t) = (x_n^1(t), \dots, x_n^m(t))$, then $x_n^i(t) = \pi^i(\lambda_n^1 t, \dots, \lambda_n^r t; x_n^1(0), \dots, x_n^m(0))$ for $0 \leqq |t| < \gamma$, $i=1, \dots, m$; $n=0$, or $n \geqq N$. Therefore $x_n^i(t)$, $i=1, \dots, m$, $n=0$, or $n \geqq N$ is represented by a power series of t for $0 \leqq |t| < \gamma$. By the Corollary to 3. 4. Theorem

$$\lim_{t \to 0} d(\varkappa \circ \varphi_0(t), \varkappa \circ \varphi_0(0))/|t| = \lim_{t \to 0} \sqrt{\sum_{i=1}^m (x_0^i(t) - x_0^i(0))^2} \Big/ |t| = \sqrt{\sum_{i=0}^m (\dot{x}_0^i(0))^2} \neq 0.$$

The same argument as applied at the preceeding lemma gives $\lim_{n \to \infty} (x_n^i(t_n) - x_0^i(0))/t_n = \dot{x}_0^i(0)$. Hence

$$\sqrt{\sum_{i=1}^m \dot{x}_0^i(0)^2} = \lim_{n \to \infty} \sqrt{\sum_{i=1}^m (x_n^i(t_n) - x_n^i(0))^2} \Big/ t_n = \lim_{n \to \infty} d(\varkappa(a_n), \varkappa(b_n))/\varrho(a_n, b_n) = 0,$$

which contradicts the former assertion.

3. 8. *Let (R, ϱ) be a symmetric metric space satisfying* **A1—5**, $x \in R$ *and* $\varphi_n \in \mathscr{G}_x$, $n=1, \dots, m$ *geodesics such that* $\{\varphi_{n*}(0)\}_{n=1,2,\dots,m}$ *is a base of the tangent space* R_x. *Then there is a coordinate system* $\varkappa: U_\gamma \to R^m$ *of the analytic manifold R, defined on a spherical neighbourhood* U_γ, $\gamma > 0$ *of x such that* $(\varkappa \circ \varphi_n)(t) = (\underbrace{0}_{1}, \dots, \underbrace{0}_{n-1}, \underbrace{t}_{n}, \underbrace{0}_{n+1}, \dots, \underbrace{0}_{m})$ *if* $0 \leqq |t| \leqq \gamma$ *for* $n=1, \dots, m$.

Let $\tau_{\varphi_n}: R^1 \to \Gamma_0$ be the 1-parameter subgroup of transvections corresponding to φ_n for $n=1, \dots, m$ and $\gamma_1, \dots, \gamma_k: R^1 \to H_x$ a maximal system of 1-parameter subgroups of the isotropic subgroup H_x such that $\gamma_{1*}(0), \dots, \gamma_{k*}(0)$ are linearly independent. The vectors $\tau_{\varphi_{1*}}(0), \dots, \tau_{\varphi_{m*}}(0), \gamma_{1*}(0), \dots, \gamma_{k*}(0)$ are linearly independent too. Let $\tau_{\varphi_{1*}}(0), \dots, \tau_{\varphi_{m*}}(0), X_1, \dots, X_j, \gamma_{1*}(0), \dots, \gamma_{k*}(0)$ be a base of the tangent space of Γ_0 at the identity, and $(t^1, \dots, t^m, v^1, \dots, v^j, s^1, \dots, s^k)$ the corresponding coordinate system of the second kind[32] of Γ_0 defined on a neighbourhood V of the identity by $\alpha = \tau_{\varphi_1}(t^1) \circ \dots \circ \tau_{\varphi_m}(t^m) \circ \xi_1(v^1) \circ \dots \circ \xi_j(v^j) \circ \gamma_1(s^1)^0 \dots \circ \gamma_k(s^k)$ for $\alpha \in V$. This canonical coordinate system yields a coordinate system $\tilde{\varkappa}: \tilde{U} \to R^{m+j}$ of Γ_0/H_x, which is defined on a neighbourhood $\tilde{U}$ of Γ_0/H_x by $\tilde{\varkappa}(\alpha H_x) = (t^1, \dots, t^m, v^1, \dots, v^j)$ for $\alpha \in V' \subset V$, if $\alpha = \tau_{\varphi_1}(t^1) \circ \dots \circ \tau_{\varphi_m}(t^m) \circ \xi_1(v^1) \circ \dots \circ \xi_j(v^j)$.[33] But for any $y \in R$ the set $C_y = \{\alpha: \alpha \in \Gamma_0, \alpha(x) = y\}$ is a left coset of H_x, and $\iota_x(y) = C_y$ is a homeomorphism $\iota_x: R \to \Gamma_0/H_x$, which inducing the analytic structure of R is analytic.[34] Therefore a coordinate system $\varkappa' = \tilde{\varkappa} \circ \iota_x: U \to R^{m+j}$ is defined on $U = \iota_x^{-1}(\tilde{U})$. Hence $j=0$ and $(\tilde{\varkappa} \circ \varphi_n)(t) = (\tilde{\varkappa} \circ \iota_x)(\varphi_n(t)) = (\tilde{\varkappa} \circ \iota_x)(\tau_{\varphi_n}(t)(x)) = \tilde{\varkappa}(\tau_{\varphi_n}(t) H_x) = (\underbrace{0}_{1}, \dots, \underbrace{0}_{n-1}, \underbrace{t}_{n}, \underbrace{0}_{n+1}, \dots, \underbrace{0}_{m})$. Let $U_\gamma \subset U$ be with $\gamma > 0$, and $\varkappa$ the restriction of $\varkappa'$ to U_γ.

3. 9. Lemma. *Let (R, ϱ) be a symmetric metric space satisfying* **A1—5**, $x \in R$, $\varphi \in \mathscr{G}_x$ *and* $\psi: [0, \alpha] \to R$, $\psi(0) = x$ *a continuous curve of the analytic manifold R differentiable at* $t=0$. *If* $\psi_*(0) = \lambda \varphi_*(0)$, $\lambda > 0$, *then* $\lambda = \lim_{t \to 0} \varrho(\psi(t), \psi(0))/|t|$.

[32] Pontrjagin [10], II. 71.
[33] Pontrjagin [10], II. 81—82.
[34] Pontrjagin [10], II. 91—97.

By 3. 8. Lemma there is a coordinate system $\varkappa: U_\gamma \to R^m$ of the analytic manifold R defined on the spherical neighbourhood $U_{\gamma'}$, $\gamma' > 0$ of x such that $(\varkappa \circ \varphi)(t) = (\underbrace{t}_{1}, \underbrace{0}_{2}, \ldots, \underbrace{0}_{m})$ for $0 \leqq |t| < \gamma'$. If d is the distance function of R^m, then

$$\lim_{t \to +0} \frac{d((\varkappa \circ \varphi)(t), (\varkappa \circ \varphi)(0))}{\varrho(\varphi(t), \varphi(0))} = \lim_{t \to 0} \frac{d((\varkappa \circ \psi)(t), (\varkappa \circ \psi)(0))}{\varrho(\psi(t), \psi(0))} = 1.$$

To verify the last assertion observe at first that

$$\frac{d((\varkappa \circ \varphi)(t), (\varkappa \circ \varphi)(0))}{\varrho(\varphi(t), \varphi(0))} = 1$$

if $|t|$ is small enough. Further there is a $0 < \beta \leqq \alpha$ such that to any $0 < t \leqq \beta$ there exists a $0 < \bar{t} \leqq \beta$ with $d(\varkappa(x), (\varkappa \circ \psi)(t)) = d(\varkappa(x), (\varkappa \circ \varphi)(\bar{t})) = \varrho(x, \varphi(\bar{t})) = \bar{t}$. Hence

$$\left| \frac{\varrho(x, \varphi(t))}{d(\varkappa(x), (\varkappa \circ \varphi)(\bar{t}))} - \frac{\varrho(\psi(t), \varphi(\bar{t}))}{\varrho(x, \varphi(\bar{t}))} \right| \leqq \frac{\varrho(x, \psi(t))}{d(\varkappa(x), (\varkappa \circ \psi)(t))} \leqq$$

$$\leqq \frac{\varrho(x, \varphi(\bar{t}))}{d(\varkappa(x), (\varkappa \circ \varphi)(\bar{t}))} + \frac{\varrho(\psi(t), \varphi(\bar{t}))}{\varrho(x, \varphi(\bar{t}))}$$

for $0 < t < \beta$. Since $\psi_*(0) = \lambda \varphi_*(0)$, if $(\varkappa \circ \psi)(t) = (\psi^1(t), \ldots, \psi^m(t))$ for $\psi(t) \in U_{\gamma'}$, then $\dot{\psi}^1(0) = \lambda$, $\dot{\psi}^2(0) = \cdots = \dot{\psi}^m(0) = 0$. Thus

$$\lim_{t \to 0} \frac{d((\varkappa \circ \psi)(\bar{t}), (\varkappa \circ \varphi)(\bar{t}))}{d(\varkappa(x), (\varkappa \circ \varphi)(\bar{t}))} = \lim_{t \to 0} \frac{\sqrt{(\psi^1(t) - \bar{t})^2 + (\psi^2(t))^2 + \ldots + (\psi^m(t))^2}}{\bar{t}} =$$

$$= \lim_{t \to 0} \sqrt{2\left(1 - \frac{\psi^1(t)}{\bar{t}}\right)} = 0.$$

By 3. 7. Lemma there are $\gamma, \mu > 0$ with $\dfrac{d((\varkappa \circ \psi)(t), (\varkappa \circ \varphi)(\bar{t}))}{\varrho(\psi(t), \varphi(\bar{t}))} > \mu$ for $0 < t < \gamma$. Therefore

$$\limsup_{t \to 0} \frac{\varrho(\psi(t), \varphi(\bar{t}))}{\varrho(x, \varphi(\bar{t}))} =$$

$$= \limsup_{t \to 0} \frac{\varrho(\psi(t), \varphi(\bar{t}))}{d((\varkappa \circ \psi)(t), (\varkappa \circ \varphi)(\bar{t})} \cdot \limsup_{t \to 0} \frac{d((\varkappa \circ \psi)(t), (\varkappa \circ \varphi)(\bar{t}))}{d(\varkappa(x), (\varkappa \circ \varphi)(\bar{t}))} = 0$$

and consequently

$$\lim_{t \to 0} \frac{\varrho(x, \psi(t))}{d(\varkappa(x), (\varkappa \circ \psi)(t))} = 1.$$

These imply the validity of the equality in question, from which the assertion of the lemma follows.

COROLLARY. *Let (R, ϱ) be a symmetric metric space satisfying* **A1—5**, *$x \in R$ and $\varphi_1, \varphi_2 \in \mathcal{G}_x$ with $\varphi_{1*}(0) = \lambda \varphi_{2*}(0)$, $\lambda > 0$, then $\lambda = 1$.*

If (R, ϱ) is a symmetric metric space satisfying **A1—5**, then a real valued function $F:TR\to R^1$ can be defined on the tangent bundle TR of the analytic manifold R. If $x\in R$ and $O\in R_x$, let $F(O)=0$; in case of $X\in R_x$, $X\neq O$ there is a $\varphi\in\mathcal{G}_x$ with $X=\lambda\varphi_*(0)$, $\lambda>0$ by 3. 6. Lemma, put $F(X)=\lambda$ then. Obviously $F(-X)=F(X)$ for any $X\in TR$.

3. 10. LEMMA. *Let (R, ϱ) be a symmetric metric space satisfying* **A1—5**, *then $F:TR\to R^1$ is continuous.*

Let $\{x_n\}_{n=0,1,\dots}\subset R$ and $X_n\in R_{x_n}$, $n=0, 1, \dots$ with $\lim\limits_{n\to\infty} X_n=X_0$. Assume that a subsequence $\{X_{n_i}\}_{i=1,2,\dots}$ exists with $F(X_0)\neq\lim\limits_{i\to\infty} F(X_{n_i})\leqq\infty$. There is no loss of generality by assuming that there exists a sequence of geodesics $\{\varphi_{n_i}\}_{i=1,2,\dots}$ with $X_{n_i}=\lambda_{n_i}\varphi_{n_i*}(0)$, $\lambda_{n_i}\geqq 0$, $i=1, 2, \dots$, and $\lim\limits_{n\to\infty}\varphi_{n_i}=\varphi\in\mathcal{G}_{x_0}$. By 3. 5 Lemma $\lim\limits_{i\to\infty}\varphi_{n_i*}(0)=\varphi_*(0)$, therefore $X_0=\lim\limits_{i\to\infty}\lambda_{n_i}\varphi_{n_i*}(0)=\lim\limits_{i\to\infty}\lambda_{n_i}\varphi_*(0)=\lambda\varphi_*(0)$ and $F(X_0)=\lambda=\lim\limits_{i\to\infty}\lambda_{n_i}=\lim\limits_{i\to\infty}F(X_{n_i})$ contradicting the above assumption. Hence $F(X_0)=\lim\limits_{n\to\infty}F(X_n)$.

The function F is called the *arc-element* of the distance function ϱ. This terminology is justified by the following

3. 11. LEMMA. *Let (R, ϱ) be a symmetric metric space satisfying* **A1—5** *and $F:TR\to R^1$ its arc-element. Then any curve $\varphi:[0, \alpha]\to R$ of class C^1 of the analytic manifold R is a rectifiable curve of the metric space (R, ϱ) and its arc length is given by $\int\limits_0^\alpha F(\varphi_*(t))dt$.*

If $0<\alpha'<\alpha$, then for a sufficiently large integer n_0 all the functions $f_n(t)=n\varrho(\varphi(t+1/n), \varphi(t))$, $n\geqq n_0$ are defined in $[0, \alpha']$. They are continuous and $F(\varphi_*(t))$, $t\in[0, \alpha]$ is continuous by 3. 10. Lemma. In consequence of 3. 9. Lemma $\lim\limits_{n\to\infty} f_n(t)=F(\varphi_*(t))$, $t\in[0, \alpha']$; therefore by the theorem of ARZELA this convergence is quasi-uniform. This means, that to any $\varepsilon>0$ there are integers $n_1, \dots, n_k\geqq n_0$ with $|F(\varphi_*(t))-f_{n_i}(t)|<\varepsilon/2\alpha$ for some $1\leqq i\leqq k$ for any $t\in[0, \alpha']$. For $t_0=0$ let n_{i_0} be an integer corresponding to t_0, and for $t_1=t_0+1/n_{i_0}$ let n_{i_1} be an integer corresponding to t_1; by continuing this process there will be a first step with $t_m\geqq\alpha'$. Put $\Delta t_j=t_j-t_{j-1}$, $j=1, \dots, m$, then

$$\left|\sum_{j=1}^{m-1}\varrho(\varphi(t_{j-1}), \varphi(t_j))+\varrho(\varphi(t_{m-1}), \varphi(\alpha'))-\right.$$
$$\left.-\left(\sum_{j=1}^{m-1}F(\varphi_*(t_{j-1}))\Delta t_j+F(\varphi_*(t_{m-1}))(\alpha'-t_{m-1})\right)\right|\leqq$$
$$\leqq\sum_{j=1}^{m-1}\left|\frac{\varrho(\varphi(t_{j-1}), \varphi(t_j))}{\Delta t_j}-F(\varphi_*(t_{j-1}))\right|\Delta t_j+$$
$$+|\varrho(\varphi(t_{m-1}), \varphi(\alpha'))-F(\varphi_*(t_{m-1}))(\alpha'-t_{m-1})|<\varepsilon/2+\varrho(\varphi(t_{m-1}), \varphi(\alpha'))+$$
$$+F(\varphi_*(t_{m-1}))(\alpha'-t_{m-1})<\varepsilon$$

if n_0 is large enough. This implies that $\varphi:[0, \alpha'] \to R$ is a rectifiable curve of (R, ϱ), and its length is given by $\int_0^{\alpha'} F(\varphi_*(t))dt$. The assertion follows now obviously for the whole interval as well.

If M is a differentiable manifold of class C^1 and $F:TM \to R^1$ is a non-negative continuous real-valued function defined on the tangent bundle TM of M, such that $F(X)=0$ if and only if $X=O\in TM$ and $F(\lambda X)=\lambda F(X)$ for $X\in TM$, $\lambda>0$, then (M, F) is called a *Finsler-manifold*.[35] The induced metric space of a Finsler-manifold can be defined analogously to that of a Riemannian manifold.

3. 12. LEMMA. *If (R, ϱ) is a symmetric metric space satisfying* **A1—5**, *then its arc-element $F:TR \to R^1$ defines a Finsler-manifold (R, F) which induces the metric space (R, ϱ) and has convex indicatrix in every tangent space of R.*

That (R, ϱ) has an arc-element F which defines a Finsler-manifold (R, F) follows by 3. 9, and 3. 10. Lemmas. The induced space of (R, F) is (R, ϱ) by 3. 11. Lemma. The same lemma and a result of BUSEMANN and MAYER[36] imply that F has convex indicatrix in every tangent space of the analytic manifold R.

3. 13. LEMMA. *Let (R, ϱ) be a symmetric metric space satisfying* **A1—6**, *then (R, F) is a Riemannian manifold.*

It suffices to show that F as norm defines a euclidean vectorspace on each tangent space of the analytic manifold R. To this end let $X, Y\in R_x$, $x\in R$, $F(X)=F(Y)=1$ be linearly independent vectors and $\bar\varphi, \bar\psi:R^1 \to R_x$ defined by $\bar\varphi(t)=tX$, $\bar\psi(t)=tY$, $t\in R^1$. By 3. 6. and 3. 9. Lemma there are geodesics $\varphi, \psi\in\mathcal{G}_x$ with $\varphi_*(0)=X$, $\psi_*(0)=Y$, and in consequence of 3. 8. Lemma there exists a coordinate system $\varkappa:U_\gamma \to R^m$ which is defined on a spherical neighbourhood U_γ, $\gamma>0$ of x and satisfies $(\varkappa\circ\varphi)(t)=(\underset{1}{t}, \underset{2}{0}, \dots, \underset{m}{0})$, $(\varkappa\circ\psi)(t)=(\underset{1}{0}, \underset{2}{t}, \underset{3}{0}, \dots, \underset{m}{0})$ for $0\leq|t|<\gamma$. If $0<t, t'<\gamma$, then

$$\varepsilon(t, t') = \left|\cos\gamma\big(x; \varphi(t), \psi(t')\big)-\cos\gamma\,(0; tX, t'Y)\right| =$$

$$=\left|\frac{t^2+t'^2-\big(\varrho(\varphi(t), \psi(t'))\big)^2}{2t\cdot t'}-\frac{t^2+t'^2-\big(F(tX-t'Y)\big)^2}{2t\cdot t'}\right|\leq$$

$$\leq\frac{1}{2}\left|1-\frac{\varrho\big(\varphi(t), \psi(t')\big)}{F(tX-t'Y)}\right|\cdot\left(1+\frac{\varrho\big(\varphi(t), \psi(t')\big)}{F(tX-t'Y)}\right)(1+t'/t)(1+t/t').$$

But
$$\frac{\varrho\big(\varphi(t), \psi(t')\big)}{F(tX-t'Y)} \to 1$$

as $t, t'\to 0$.[37] Hence $\varepsilon(t, t')\to 0$, if t/t' is fixed and $t, t'\to 0$. As (R, ϱ) satisfies **A6**, the angle $\gamma(\varphi, \psi) = \lim_{t, t'\to 0}\gamma\big(x; \varphi(t), \psi(t')\big)$ exists. Since the norm F defines a Min-

[35] BUSEMANN—MAYER [2].
[36] BUSEMANN—MAYER [2], BUSEMANN [3], 83.
[37] BUSEMANN—MAYER [2].

kowskian geometry in the tangent space R_x these assertions imply that $\gamma(\bar{\varphi}, \bar{\psi}) = \lim_{t, t' \to 0} \gamma(0; tX, t' Y)$ exists as well.[38] Hence the norm F defines a euclidean vector space on R_x.

3. 14. LEMMA. *If (R, ϱ) is a symmetryc metric space satisfying* **A1 – 6**, *then the Riemannian manifold (R, F) is symmetric.*

Since the induced space of (R, F) is (R, ϱ) and every distance preserving transformation of (R, ϱ) is an isometric transformation of (R, F),[39] the latter is a symmetric Riemannian manifold.

The above assertions imply the following metric characterization of symmetric Riemannian manifolds:

3. 15. THEOREM. *Let (R, ϱ) be a symmetric metric space satisfying* **A1—6**, *then there is a unique symmetric Riemannian manifold which induces (R, ϱ).*

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EINE BEMERKUNG ZUM KONVERGENZPROBLEM DER ORTHOGONALREIHEN

Von

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Herrn Professor G. ALEXITS *zum 70. Geburtstag gewidmet*

Einleitung

Das Konvergenzproblem der Orthogonalreihen wurde von mehreren Autoren diskutiert. D. MENCHOFF [3] und H. RADEMACHER [6] haben gezeigt, daß im Falle

$$(1)\qquad \sum_{k=1}^{\infty} a_k^2 \log^2 k < \infty$$

die Orthogonalreihe

$$(2)\qquad \sum_{k=1}^{\infty} a_k \varphi_k(x)$$

für jedes im Grundintervall (0, 1) orthonormierte System $\{\varphi_n(x)\}$ fast überall konvergiert. D. MENCHOFF [3] hat auch bewiesen, daß die Bedingung (1) im allgemeinen unverbesserbar ist. Ist nämlich $\{w(k)\}$ eine positive Folge mit $w(k)=o(\log k)$, dann gibt es eine Koeffizientenfolge $\{a_k\}$ und ein orthonormiertes System $\{\varphi_k(x)\}$ derart, daß

$$(3)\qquad \sum_{k=1}^{\infty} a_k^2 w^2(k) < \infty$$

gilt und die Reihe (1) fast überall divergiert. D. MENCHOFF [4] hat auch gezeigt, daß in dieser Behauptung das System $\{\varphi_k(x)\}$ mit einer Konstante $K>1$ gleichmäßig beschränkt gewählt werden kann. Aus der Konstruktion von D. MENCHOFF kann man leicht einsehen, daß diese Behauptung mit einer monoton abnehmenden Koeffizientenfolge $\{a_k\}$ besteht. Also ist (1) für gleichmäßig beschränkte orthonormierte Systeme und für monoton abnehmende Koeffizientenfolgen auch unverbesserbar (siehe auch K. TANDORI [7], [8]).

Das im Grundintervall (0, 1) orthonormierte System $\{\varphi_k(x)\}$ nennen wir „vorzeichensartig", wenn $|\varphi_k(x)|=1$ ($k=1, 2, \dots;\ 0\leqq x\leqq 1$) fast überall besteht. Ob (1) für vorzeichensartige orthonormierte Systeme $\{\varphi_k(x)\}$ unverbesserbar ist, ist noch eine offene Frage. Es ist nur das folgende von A. KOLMOGOROFF und D. MENCHOFF [2] stammende Resultat bekannt: ist $\{w(k)\}$ eine positive Folge mit $w(k)=o(\sqrt{\log k})$, dann gibt es eine Koeffizientenfolge $\{a_k\}$ mit (3) und ein vorzeichensartiges orthonormiertes System $\{\varphi_k(x)\}$ derart, daß die Reihe (2) fast überall divergiert. Also im allgemeinen kann die Bedingung

$$(4)\qquad \sum_{k=1}^{\infty} a_k^2 \log k < \infty$$

nicht abgeschwächt werden, damit die Reihe (2) für jedes vorzeichensartige orthonormierte System $\{\varphi_k(x)\}$ fast überall konvergiert. Es soll noch bemerkt werden, daß es auch problematisch ist, ob in dieser Behauptung die Koeffizientenfolge $\{a_k\}$ monoton abnehmend gewählt werden kann.

Man sagt, daß die Reihe (2) fast überall unbedingt konvergiert, wenn sie in jeder Anordnung ihrer Glieder fast überall konvergiert. (Die Menge von Divergenzpunkten kann von der Anordnung abhängen.) Die unbedingte Konvergenz der Orthogonalreihen ist im allgemeinen Fall vollständig untersucht. K. TANDORI [9] hat nämlich gezeigt, daß

$$\sum_{\nu=0}^{\infty} \sqrt{\sum_{k=2^{2^\nu}+1}^{2^{2^{\nu+1}}} a_k^{*2} \log^2 k} < \infty \tag{5}$$

notwendig und hinreichend ist, damit die Reihe (2) für jedes orthonormierte System $\{\varphi_k(x)\}$ fast überall unbedingt konvergiert, wobei $\{a_k^*\}$ eine dem absoluten Wert nach monoton abnehmende Anordnung der Folge $\{a_k\}$ bezeichnet. (Im Falle $a_k \not\to 0$ soll man unter der linkseitigen Summe $+\infty$ verstehen.) Ob die Bedingung (5) für die unbedingte Konvergenz fast überall der Reihe (2) für gleichmäßig beschränkte oder für vorzeichensartige orthonormierte Systeme $\{\varphi_k(x)\}$ im allgemeinen unverbesserbar ist, ist noch eine offene Frage. Aus (5) ergibt sich durch Anwendung der Tschebyscheffschen Ungleichung die folgende hinreichende Bedingung: gilt

$$\sum_{k=2}^{\infty} a_k^2 \log^2 k\,(\log\log k)^{1+\varepsilon} < \infty \qquad (\varepsilon > 0), \tag{6}$$

dann ist die Reihe (2) für jedes orthonormierte System $\{\varphi_k(x)\}$ fast überall unbedingt konvergent. (Das ist ein spezieller Fall eines Satzes von W. ORLICZ [5].)

In dieser Note werden wir die folgende Behauptung beweisen:

SATZ. *Es sei $\varepsilon > 0$ beliebig. Dann gibt es eine Koeffizientenfolge $\{a_k\}$ mit*

$$\sum_{k=2}^{\infty} a_k^2 \log k\,(\log\log k)^{1-\varepsilon} < \infty \tag{7}$$

und ein vorzeichensartiges orthonormiertes System $\{\varphi_k(x)\}$ derart, daß die Reihe (2) *in einer gewissen Anordnung ihrer Glieder fast überall divergiert.*

Offensichtlich gibt es eine Analogie zwischen (1), (4) und (6), (7).

§ 1. Hilfssatz

Gewisse Ideen des Verf. werden benützt (K. TANDORI [10]). Es sei $r_n(x) = \operatorname{sign} \sin 2^n \pi x$ die n-te Rademachersche Funktion. Das Walshsche System $\{w_n(x)\}$ ist folgenderweise definiert: $w_0(x) \equiv 1$ in (0, 1); ist $n = 2^{\nu_1} + \dots + 2^{\nu_p}$ $(\nu_1 < \dots < \nu_p)$ die diadysche Darstellung von $n \geqq 1$, so sei $w_n(x) = r_{\nu_1+1}(x) \dots r_{\nu_p+1}(x)$. Bekanntlich ist das Walshsche System in (0, 1) orthonormiert. (Siehe z. B. G. ALEXITS [1], S. 59—62.)

Es sei a eine positive ganze Zahl. Wir setzen

$$\varphi_a\left(\frac{l}{2^{a+1}};x\right)=\frac{1}{2^a}\prod_{k=1}^{a}\left(1+r_k\left(\frac{l}{2^a}+\frac{1}{2^{a+1}}\right)r_k(2x)\right)\qquad (l=0,\dots,2^a-1).$$

$\varphi_a\left(\frac{l}{2^{a+1}};x\right)$ ist die Linearkombination von Funktionen $w_0(2x), w_1(2x), \dots, w_{2^a-1}(2x)$; es gilt

$$\varphi_a\left(\frac{l}{2^{a+1}};x\right)=\begin{cases}1 & \left(\frac{l}{2^{a+1}}<x<\frac{l+1}{2^{a+1}},\ \frac{1}{2}+\frac{l}{2^{a+1}}<x<\frac{1}{2}+\frac{l+1}{2^{a+1}}\right),\\ 0 & \text{sonst}\end{cases}$$

und

$$\int_0^1 \varphi_a^2\left(\frac{l}{2^{a+1}};x\right)dx=\frac{1}{2^a}. \tag{8}$$

Wir setzen

$$\Phi_1(0;x)=\varphi_1(0;x),$$

$$\Phi_1(1;x)=r_3(x)\varphi_1(0;x),\quad \Phi_2(1;x)=-r_3(x)r_1(x)\varphi_1(0;x),$$

$$\Phi_1(2;x)=r_4(x)\varphi_2(0;x),\quad \Phi_2(2;x)=-r_4(x)r_1(x)\varphi_2(0;x),$$

$$\Phi_3(2;x)=r_4(x)\varphi_2\left(\frac{1}{2^3};x\right),\quad \Phi_4(2;x)=-r_4(x)r_1(x)\varphi_2\left(\frac{1}{2^3};x\right),$$

und im allgemeinen

$$\Phi_{2j+1}(k;x)=r_{k+2}(x)\varphi_k\left(\frac{j}{2^{k+1}};x\right)\qquad (j=0,\dots,[(2^k-1)/2]),\,[1]$$

$$\Phi_{2j}(k;x)=-r_1(x)\Phi_{2j-1}(k;x)\qquad (j=1,\dots,2^{k-1}).$$

Offensichtlich ist $\Phi_i(k;x)$ $(k=0,1,\dots;\ i=1,\dots,2^k)$ eine Linearkombination der 2^k Funktionen von der Form $w_p(x)w_q(2x)$:

$$\Phi_i(k;x)=\sum_{u=0}^{2^k-1}c(u,i,k)\,w_{p(u,i,k)}(x)\,w_u(2x). \tag{9}$$

Gehört $w_{p_s}(x)w_s(2x)$, bzw. $w_{p_\sigma}(x)w_\sigma(2x)$ zu der Zerlegung in (9) angegebenen von $\Phi_{2j-1}(k;x)$, bzw. zu der Zerlegung von $\Phi_{2j}(k;x)$, dann gilt weiterhin

$$\int_0^1 w_{p_s}(x)w_s(2x)w_{p_\sigma}(x)w_\sigma(2x)\,dx=0.$$

Aus (8) folgt weiterhin

$$\int_0^1 \Phi_i^2(k;x)\,dx=\frac{1}{2^k}\qquad (k=0,1,\dots;\ i=1,\dots,2^k). \tag{10}$$

[1] $[\alpha]$ bezeichnet den ganzen Teil von α.

Es sei N eine natürliche Zahl. Wir betrachten ein Intervall $I_s = \left(\frac{s}{2^{N+1}}, \frac{s+1}{2^{N+1}}\right)$ $(s = 0, \dots, 2^{N+1}-1)$. Es sei Z_s die Menge der Indexpaare (i, k) $(k=0, \dots, N;$ $i=1, \dots, 2^k)$ mit $\Phi_i(k; x) \equiv 0$ $(x \in I_s)$; die Elemente von Z_s in einer gewissen Anordnung bezeichnen wir mit $(i_1, k_1), (i_2, k_2), \dots$. Wir setzen

$$\chi(s, i, k; x) = \begin{cases} r_{N+1+t}(x) & (x \in I_s; (i, k) = (i_t, k_t),\ t = 1, 2, \dots), \\ 1 & (x \in I_s, (i, k) \notin Z_s), \\ 0 & \text{sonst.} \end{cases}$$

Mit den Funktionen $w_{p(u,i,k)}(x) w_u(2x)$ $(u=0, \dots, 2^k-1)$, welche zu der Zerlegung (9) von $\Phi_i(k; x)$ gehören, bilden wir die Funktionen

$$\psi_1(x) = \sum_{s=0}^{2^{N+1}-1} \chi(s, 1, 0; x), \quad \psi_2(x) = \sum_{s=0}^{2^{N+1}-1} \chi(s, 1, 1; x) r_1(2x),$$

$$\psi_{2+4+\dots+4^{k-1}+(i-1)2^k+u+1}(x) = \sum_{s=0}^{2^{N+1}-1} \chi(s, i, k; x) w_{p(u,i,k)}(x) w_u(2x)$$

$$(u = 0, 1, \dots, 2^k-1;\ k = 1, 2, \dots).$$

Nach obigem sind die Funktionen $\psi_l(x)$ $(l = 1, \dots, 2+4+\dots+4^N)$ Treppenfunktionen (d.h. man kann das Intervall $(0, 1)$ in endlichviele Teilintervalle zerlegen derart, daß jede Funktion $\psi_l(x)$ in jedem Teilintervall konstant ist) und bilden ein vorzeichensartiges orthonormiertes System. Es gilt

$$\int_0^1 \psi_l(x)\, dx = 0 \qquad (l = 1, \dots, 2+4+\dots+4^N).$$

Weiterhin erhalten wir aus (9) die Zerlegung

$$\Phi_i(k; x) = \sum_{u=0}^{2^k-1} c(u, i, k) \psi_{2+4+\dots+4^{k-1}+(i-1)2^k+u+1}(x).$$

Für die Funktionen $\Phi_i(k; x)$ gelten offensichtlich die folgenden Behauptungen: Jede Funktion $\Phi_i(k; x)$ ist eine Linearkombination von Funktionen $\psi_l(x)$; verschiedene $\Phi_i(k; x)$ haben in ihren Darstellungen keine gemeinsame $\psi_l(x)$; für die Darstellungen von $\Phi_i(k; x)$ $(k=0, \dots, N;\ i=1, \dots, 2^k)$ benötigen wir $2+4+\dots+4^N$ Funktionen $\psi_l(x)$.

Es sei nun $N=2^n$ und a eine positive ganze Zahl $(2 \leqq a < n)$. Wir betrachten die Summe

$$(11) \qquad S_n(x) = \frac{1}{a2^a}\left(\Phi_1(0; x) + \sum_{k=1}^{2^a} \sum_{j=0}^{2^{k-1}-1} (\Phi_{2j+1}(k; x) + 2\Phi_{2j+2}(k; x)) + \right.$$

$$\left. + \sum_{\sigma=a+1}^{n} \frac{1}{\sigma 2^{\sigma-1}} \sum_{k=2^{\sigma-1}+1}^{2^\sigma} \sum_{j=0}^{2^{k-1}} (\Phi_{2j+1}(k; x) + 2\Phi_{2j+2}(k; x)) \right) =$$

$$= A_0 \Phi_1(0; x) + \sum_{k=1}^{2^n} A_k \sum_{j=0}^{2^{k-1}-1} (\Phi_{2j+1}(k; x) + 2\Phi_{2j+2}(k; x)),$$

wobei

$$A_k = \frac{1}{a2^a} \qquad (k = 0, \dots, 2^a),$$

$$A_k = \frac{1}{\sigma 2^{\sigma-1}} \qquad (2^{\sigma-1} < k \leqq 2^\sigma;\ \sigma = a+1, \dots, n)$$

ist. Es gelten die folgenden Vorstellungen:

$$A_0 \Phi_1(0; x) = \sum_{l=1}^{2} d_l(n)\psi_l(x)$$

$$A_k\big(\Phi_{2j+1}(k; x) + 2\Phi_{2j+2}(k; x)\big) = \sum_{l=2+4+\dots+4^{k-1}+2j2^k+1}^{2+4+\dots+4^{k-1}+2(j+1)2^k} d_l(n)\psi_l(x)$$

$$(j = 0, \dots, 2^{k-1}-1;\ k = 0, 1, \dots, 2^n).$$

Nach obigem folgt aus (10)

$$\sum_{i=1}^{2^k} \int_0^1 \Phi_i^2(k; x)\,dx = 1 \qquad (k = 0, 1, \dots),$$

und so erhalten wir für ein $\varepsilon\,(>0)$

$$\sum_{l=1}^{2+4+\dots+4^N} d_l^2(n) \log(l+a)\big(\log\log(l+a)\big)^{1-\varepsilon} \leqq C_1 \sum_{v=a}^{n} \frac{1}{v^{1+\varepsilon}} \leqq C_2 \frac{1}{a^\varepsilon}.$$

($C_1, C_2, \dots$ bezeichnen positive absolute Konstanten.)

Wir definieren eine Anordnung der Glieder der Summe $S_n(x)$. Es sei

$$\sigma_1(x) = A_0\Phi_1(0; x) + A_1\Phi_1(1; x) + 2A_1\Phi_2(1; x),$$

$$\sigma_2(x) = A_0\Phi_1(0; x) + A_1\Phi_1(1; x) + A_2\Phi_1(2; x) + 2A_2\Phi_2(2; x) +$$

$$+ 2A_1\Phi_2(1; x) + A_2\Phi_3(2; x) + 2A_2\Phi_4(2; x),$$

u.s.w. Die Summe $\sigma_{\mu+1}(x)$ erhalten wir derart, daß wir in $\sigma_\mu(x)$ nach dem Glied $A_\mu\Phi_{2j+1}(\mu; x)$, bzw. nach dem Glied $2A_\mu\Phi_{2j+2}(\mu; x)$ die Summe

$$A_{\mu+1}\Phi_{2^2 j+1}(\mu+1; x) + 2A_{\mu+1}\Phi_{2^2 j+4}(\mu+1; x),$$

bzw. die Summe $A_{\mu+1}\Phi_{2^2 j+3}(\mu+1; x) + 2A_{\mu+1}\Phi_{2^2 j+4}(\mu+1; x)$ einschreiben. Durch vollständige Induktion erhalten wir die Summe $\sigma_{2^n}(x)$, die eine Anordnung der Glieder von $S_n(x)$ definiert. Nach der Definition von $\Phi_i(k; x)$ ist es klar, daß das Maximum der Partialsummen dieser Anordnung von $S_n(x)$ in $(0, 1/4)$ — abgesehen von endlichvielen Punkten — nicht kleiner als

$$\sum_{v=a}^{n} \frac{1}{v} \geqq C_3 \log \frac{n}{a}$$

ist. Damit haben wir den folgenden Hilfssatz bewiesen:

HILFSSATZ. *Es seien n und a positive ganze Zahlen* ($2 \leqq a < n$) *und* $\varepsilon > 0$. *Dann gibt es eine Koeffizientenfolge* $\{d_l(n)\}$ ($l = 1, \dots, 2+4+\dots+4^{2^n} = M_n$) *und ein vorzeichensartiges System von Treppenfunktionen* $\psi_l(n; x)$ ($l = 1, \cdots, M_n$) *mit folgenden Eigenschaften.*

Es gilt

$$\sum_{l=1}^{M_n} d_l^2(n) \log(l+a)(\log\log(l+a))^{1-\varepsilon} \leqq C_2 \frac{1}{a^\varepsilon}.$$

Weiterhin gibt es eine einfache Menge E_n *(d.h.* E_n *ist die Vereinigung endlichvieler Intervalle) mit dem Mass* $|E_n| = 1/4$ *und eine Anordnung*

$$\sum_{i=1}^{M_n} d_{l_i}(n) \psi_{l_i}(n; x)$$

der Summe

$$\sum_{l=1}^{M_n} d_l(n) \psi_l(n; x)$$

derart, daß

$$\max_{1 \leqq s \leqq M_n} \sum_{i=1}^{s} d_{l_i}(n) \psi_{l_i}(n; x) \geqq C_3 \log \frac{n}{a} \qquad (x \in E_n)$$

besteht. Weiterhin gilt

$$\int_0^1 \psi_l(n; x)\, dx = 0 \qquad (l = 1, \dots, M_n).$$

§ 2. Beweis des Satzes

Durch Anwendung dieses Hilfssatzes können wir den Satz leicht beweisen. Es sei $n(1) = 4$, $a(1) = 2$. Durch Rekursion definieren wir die ganzen Zahlen $n(r)$ ($r = 2, 3, \dots$) mit folgenden Eigenschaften

$$(12) \qquad 2[\log\log(M_{n(1)} + \dots + M_{n(r)})] = a(r) < n(r+1) \qquad (r = 1, 2, \dots),$$

$$(13) \qquad \sum_{r=2}^{\infty} \frac{1}{a(r)^\varepsilon} < \infty,$$

$$(14) \qquad \log \frac{n(r+1)}{a(r)} \geqq 1 \qquad (r = 1, 2, \dots).$$

Wir werden eine Koeffizientenfolge $\{a_k\}$, eine Folge von einfachen Mengen $G_r (r = 1, 2, \dots)$ und ein vorzeichensartiges orthonormiertes System von Treppenfunktionen $\varphi_k(x)$ ($k = 1, 2, \dots$) mit folgenden Eigenschaften konstruieren:

Die Mengen G_r sind stochastisch unabhängig und für jedes r besteht

$$(15) \qquad |G_r| = 1/4.$$

Es gilt für jedes r

$$(16)\qquad \sum_{k=B_r+1}^{B_{r+1}} a_k^2 \log k\,(\log\log k)^{1-\varepsilon} \leqq C_2 \frac{1}{a(r)^\varepsilon} \quad (B_r = M_{n(1)} + \dots + M_{n(r)}).$$

Weiterhin gibt es für jedes r eine Anordnung

$$\sum_{i=B_r+1}^{B_{r+1}} a_{k_i} \varphi_{k_i}(x)$$

der Summe

$$\sum_{k=B_r+1}^{B_{r+1}} a_k \varphi_k(x)$$

derart, daß

$$(17)\qquad \max_{B_r < j \leqq B_{r+1}} \left| \sum_{i=1}^{j} a_{k_i} \varphi_{k_i}(x) \right| \geqq C_3 \log \frac{n(r+1)}{a(r)} \quad (x \in G_r)$$

besteht.

Mit $a=a(1)$ und $n=n(1)$ wenden wir den Hilfssatz an. Wir setzen

$$a_k = d_k(n(1)), \quad \varphi_k(x) = \psi_k(n(1); x) \quad (k=1, \dots, a(1)), \quad G_1 = E_{n(1)}.$$

Es sei $r_0(\geqq 1)$ eine ganze Zahl. Wir nehmen an, daß die Koeffizienten a_k $(1 \leqq k \leqq B_{r_0})$, die einfachen Mengen G_r $(r=1, \dots, r_0)$ und die Treppenfunktionen $\varphi_k(x)$ $(k=1, \dots, B_{r_0})$ schon definiert sind derart, daß diese Funktionen ein vorzeichensartiges orthonormiertes System bilden, diese Mengen stochastisch unabhängig sind, weiterhin (15), (16) und (17) für $r=1, \dots, r_0$ bestehen.

Dann gibt es eine Zerlegung des Intervalls (0, 1) in endlichviele Intervalle $J_1, \dots, J_\varrho$ derart, daß jede Funktion $\varphi_k(x)$ $(1 \leqq k \leqq B_{r_0})$ in jedem J_k konstant ist, weiterhin jede Menge G_r $(r=1, \dots, r_0)$ die Vereinigung gewisser J_k ist. Die Endpunkte des Intervalls J_k bezeichnen wir mit α_k, β_k. Wir wenden den Hilfssatz mit $a=a(r_0)$ und $n=n(r_0+1)$ an. (Wegen (12) ist das möglich.) Wir setzen

$$\varphi_{k+B_{r_0}}(x) = \sum_{k=1}^{\varrho} \Psi_k(n(r_0+1); J_k; x) \qquad (k = 1, \dots, B_{r_0+1}),$$

$$G_{r_0+1} = \bigcup_{\varkappa=1}^{\varrho} E_{n(r_0+1)}(J_\varkappa),$$

wobei

$$\psi_k(n(r_0+1); J_\varkappa; x) = \begin{cases} \psi_k\left(n(r_0+1); \dfrac{x-\alpha_\varkappa}{\beta_\varkappa-\alpha_\varkappa}\right) & (\alpha_\varkappa < x < \beta_\varkappa), \\ 0 & \text{sonst} \end{cases}$$

ist und $E_{n(r_0+1)}(J_\varkappa)$ die Menge bezeichnet, welche aus der Menge $E_{n(r_0+1)}$ durch der linearen Transformation $y = (\beta_\varkappa - \alpha_\varkappa)x + \alpha_\varkappa$ entsteht.

Die Funktionen $\psi_k(x)$ $(B_{r_0} < k \leqq B_{r_0+1})$ sind Treppenfunktionen, die Menge G_{r_0+1} ist einfach. Ferner folgt auf Grund des Hilfssatzes, daß die Funktionen $\varphi_k(x)$ $(k=1, \dots, B_{r_0+1})$ ein vorzeichensartiges, orthonormiertes System bilden, die Mengen G_r $(r=1, \dots, r_0+1)$ stochastisch unabhängig sind, und (15), (16) und (17) für $r = r_0+1$ bestehen. Das Funktionensystem $\{\varphi_k(x)\}$, die Koeffizientenfolge $\{a_k\}$ und die Mengenfolge $\{G_r\}$ mit den erwähnten Eigenschaften erhalten wir somit durch Induktion.

Aus (13) und (16) ergibt sich (7). Es sei $G = \overline{\lim_{r\to\infty}} G_r$. Auf Grund der stochastischen Unabhängigkeit der Mengen G_r erhalten wir aus (15)

$$(18) \qquad |G| = 1$$

durch Anwendung des zweiten Borel—Cantellischen Lemmas. Aus (14) und (17) ersieht man, daß die mit diesen a_k und $\varphi_k(x)$ gebildete Reihe (2) eine Anordnung ihrer Glieder besitzt, welche in jedem Punkt von G divergiert. Wegen (18) divergiert sie also in dieser Anordnung fast überall.

Damit haben wir den Satz bewiesen.

(Eingegangen am 2. Juli 1968.)

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SZEGED, ARADI VÉRTANÚK TERE 1

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A PROBLEM ON WELL ORDERED SETS

By

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To Professor G. ALEXITS *on his 70th birthday*

1. Introduction. In this paper we settle one of the questions left open in [1] concerning the symbol

(1) $$\alpha \Rightarrow [\beta, \gamma]_m .$$

By definition, (1) means that the following statement is true: *If S is well ordered set of order type α and if $\mathscr{F} = (F_\mu : \mu \in M)$ is any family of $m = |M|$ subsets of S such that each F_μ $(\mu \in M)$ has order type less than β, then S contains a subset C of type γ which is disjoint from m sets F_μ of the family $\mathscr{F}$*, i.e.

$$|\{\mu : \mu \in M;\ F_\mu \cap C = \emptyset\}| = m.$$

The set C is said to be $(\mathscr{F}, m)$-free. The negation of (1) is written as

$$\alpha \not\Rightarrow [\beta, \gamma]_m .$$

We proved ([1], Theorem 10.6) that

(2) $$\omega_{\nu+2}\alpha \Rightarrow [\omega_{\nu+1}^{\omega}, \omega_{\nu+2}\alpha]_{\aleph_{\nu+2}} \qquad (\alpha < \omega_{\nu+1}).$$

So that, in particular,

(3) $$\omega_2 \alpha \Rightarrow [\omega_1^{\omega}, \omega_2 \alpha]_{\aleph_2}$$

holds for all $\alpha < \omega_1$. The condition $\alpha < \omega_{\nu+1}$ in (2) is necessary since, for example ([1], Theorem 10. 1) assuming $2^{\aleph_1} = \aleph_2$,

$$\omega_2\omega_1 \not\Rightarrow [\omega_1 + 1, \omega_2\omega_1]_{\aleph_2}.$$

By using a result of [2] on set mappings (see [1], Theorem 6. 2) it is very easily seen that

$$\omega_2 n \Rightarrow [\beta, \omega_2 n]_{\aleph_2} \qquad (n < \omega;\ \beta < \omega_2),$$

and this is stronger than (3) when $\alpha < \omega$. We asked in [1] (Problem 5) whether (3) is best possible when $\alpha = \omega$, i.e. does

(4) $$\omega_2 \omega \not\Rightarrow [\omega_1^{\omega} + 1, \omega_2 \omega]_{\aleph_2}$$

hold?

Using the generalized continuum hypothesis (more precisely, using $2^{\aleph_1} = \aleph_2$) we can now show that (4) holds. In fact, the following theorem shows that (3) is best possible in the sense that ω_1^{ω} cannot be replaced by any larger ordinal.

THEOREM. *If* $2^{\aleph_1}=\aleph_2$ *and* $\omega\leqq\alpha<\omega_1$, *then*

(5) $$\omega_2\alpha \mathrel{\not\to} [\omega_1^\omega+1, \omega_2\alpha]_{\aleph_2}.$$

2. Notation and preliminary results. Capital letters denote sets and small letters denote ordinal numbers unless stated otherwise. The cardinal of X is $|X|$. The obliterator sign ^ written above a symbol means that that symbol should be disregarded. For example,

$$\{x_0, \ldots, \hat{x}_\alpha\} = \{x_\nu : \nu<\alpha\}.$$

We write $S=\{x_0, \ldots, \hat{x}_\alpha\}_<$ if the set $S=\{x_0, \ldots, \hat{x}_\alpha\}$ is simply ordered by $<$ so that $x_\mu<x_\nu$ for $\mu<\nu<\alpha$. For any α, β we write $[\alpha, \beta)=\{\nu: \alpha\leqq\nu<\beta\}$.

The order type of the well ordered set A is denoted by tp A. If the sets A_ν $(\nu<\alpha)$ are disjoint and ordered, we write

$$S = A_0\cup\ldots\cup\hat{A}_\alpha\,(\mathrm{tp})$$

to indicate that S is the union of the A_ν and also that S is ordered in such a way that the order relations in each A_ν are preserved and $x<y$ if $x\in A_\mu$, $y\in A_\nu$ and $\mu<\nu<\alpha$. T is a *cofinal* subset of the ordered set S if for each $x\in S$ there is some $y\in T$ so that $x\leqq y$. For $\alpha>0$, co (α) denotes the smallest ordinal β such that $[0, \alpha)$ contains a cofinal subset of type β. Thus co (α) is either 1 or an initial ordinal. If α is such that $\beta+\gamma<\alpha$ whenever $\beta<\alpha$ and $\gamma<\alpha$, then α is said to be *indecomposable*. The indecomposable ordinals are 0, 1 and powers of ω.

An ordinal valued function f defined on the set of ordinal numbers A is *regressive* if $f(\alpha)<\alpha$ $(\alpha\in A; \alpha\neq 0)$. $B\subset A$ is *closed* (w.r.t. A) if B contains the limit of any increasing sequence of elements of B which is also in A. $S\subset[0, \omega_\alpha)$ is *stationary* if $[0, \omega_\alpha)-S$ does not contain a closed subset cofinal with $[0, \omega_\alpha)$. It is easily seen (see [3]) that the set

$$\{\alpha: \alpha<\omega_2;\ \mathrm{co}(\alpha)=\omega_1\}$$

is stationary. It is well known that if $\aleph_\alpha(>\aleph_0)$ is regular and f is a regressive function defined on the stationary set $S\subset[0, \omega_\alpha)$, then f has a *stationary value*, i.e. there is some θ such that

$$|\{\alpha: \alpha\in S; f(\alpha)=\theta\}|=\aleph_\alpha.$$

It has been proved in [4] that if S is a well ordered set and tp $S<\omega_{\alpha+1}$, then there is a partition of S into countably many (small) sets,

(6) $$S = S_0\cup S_1\cup\ldots\cup\hat{S}_\omega$$

with tp $S_n\leqq\omega_\alpha^n$ $(n<\omega)$. We shall use this in the special case $\alpha=1$ and refer to (6) as a *paradoxical decomposition* of S.

3. Lemmas. To prove our theorem we need the following two lemmas.

LEMMA 1. *Let* $A=[0, \alpha_0)$, *where* $\omega\leqq\alpha_0<\omega_1$ *and* α_0 *is indecomposable. Let* $S_\nu^\gamma=\{(\nu, \delta):\delta<\gamma\}$ $(\nu\in A;\ \gamma<\omega_2)$ *and let*

$$S = \bigcup_{\nu\in A}\ \bigcup_{\gamma<\omega_2} S_\nu^\gamma$$

be ordered lexicographically. If $S' \subset S$ and $\operatorname{tp} S' = \omega_2\alpha_0$, *then there are* $\gamma < \omega_2$ *and* $N \subset A$ *such that* $\operatorname{co}(\gamma) = \omega_1$, *$N$ is cofinal with A and $S' \cap S_\nu^\gamma$ is cofinal with S_ν^γ for all* $\nu \in N$.

PROOF. Suppose the lemma is false. Then for each

$$\gamma \in M = \{\varrho : \varrho < \omega_2 : \operatorname{co}(\varrho) = \omega_1\}$$

the set

$$N_\gamma = \{\nu : \nu \in A;\ S' \cap S_\nu^\gamma \text{ is cofinal with } S_\nu^\gamma\}$$

is not cofinal with A. Therefore, for $\gamma \in M$, there is $\nu_\gamma \in A$ so that

$$S' \cap S_\nu^\gamma \text{ is not cofinal with } S_\nu^\gamma \qquad (\nu_\gamma \leqq \nu < \alpha_0).$$

Thus for $\gamma \in M$ and $\nu_\gamma \leqq \nu < \alpha_0$, there is $\theta_\nu < \gamma$ such that

$$S' \cap \{(\nu, \delta) : \theta_\nu < \delta < \gamma\} = \emptyset.$$

Also, since $|A| = \aleph_0$ and $\operatorname{co}(\gamma) = \omega_1$ for $\gamma \in M$, it follows that there is $f(\gamma) < \gamma$ such that

$$\theta_\nu < f(\gamma) \qquad (\gamma \in M;\ \nu_\gamma \leqq \nu < \alpha_0).$$

Since by NEUMER's Theorem M is stationary, the regressive function f has a stationary value $\theta < \omega_2$, i.e. there is $M_1 \subset M$ such that $|M_1| = \aleph_2$ and

$$f(\gamma) = \theta \qquad (\gamma \in M_1).$$

Since $\nu_\gamma < \alpha_0$ $(\gamma \in M)$, there is $M_2 \subset M_1$ such that $|M_2| = \aleph_2$ and

$$\nu_\gamma = \xi \qquad (\gamma \in M_2).$$

If $\gamma \in M_2$ and $\xi \leqq \nu < \alpha_0$, then

$$S' \cap \{(\nu, \delta) : \theta \leqq \delta < \gamma\} = \emptyset.$$

This holds for each $\gamma \in M_2$ and as $|M_2| = \aleph_2$, it follows that

$$S' \cap \{(\nu, \delta) : \theta \leqq \delta < \omega_2\} = \emptyset \qquad (\xi \leqq \nu < \alpha_0).$$

We now have the contradiction

$$\operatorname{tp} S' \leqq \omega_2\xi + \theta\alpha_0 < \omega_2\alpha_0.$$

This proves Lemma 1.

LEMMA 2. *Let* $1 \leqq n < \omega$ *and let* $P = \{\alpha_\varrho : \varrho < \omega_1^n\}_<$ *be a set of ordinal numbers with*

$$\alpha_\varrho < \omega_2, \operatorname{co}(\alpha_\varrho) = \omega_1 \qquad (\varrho < \omega_1^n).$$

For $\varrho < \omega_1^n$, *let* $C_{\varrho 0}, C_{\varrho 1}, \ldots, \hat{C}_{\varrho\omega_1}$ *be* $\aleph_1$ *sets which are all cofinal subsets of* $[0, \alpha_\varrho)$. *Then there is a set* C^* *such that* $\operatorname{tp} C^* \leqq \omega_1^{n+1}$ *and*

$$C^* \cap C_{\varrho\nu} \neq \emptyset \quad (\varrho < \omega_1^n;\ \nu < \omega_1).$$

PROOF. For $\varrho < \omega_1^n$, we define β_ϱ in the following way. $\beta_0 = 0$. If $\varrho = \sigma + 1$, put $\beta_\varrho = \alpha_\sigma$; if ϱ is a limit number put

$$\beta_\varrho = \lim_{\sigma < \varrho} \alpha_\sigma.$$

Note that $\beta_\varrho < \alpha_\varrho$ if $\mathrm{co}(\varrho) = 1$ or ω, since $\mathrm{co}(\alpha_\varrho) = \omega_1$.

We will first prove, by induction on n, that there is a regressive function f defined on P so that

$$|\{\varrho : \varrho_0 < \varrho < \omega_1^n;\ f(\alpha_\varrho) < \alpha_{\varrho_0}\}| \leqq \aleph_0 \qquad (\varrho_0 < \omega_1^n). \tag{7}$$

If $n = 1$, the function $f(\alpha_\varrho) = \beta_\varrho$ $(\varrho < \omega_1)$ obviously satisfies (7). Now suppose $n > 1$. Let $Q = \{\alpha_\sigma : \sigma < \omega_1^n;\ \mathrm{co}(\sigma) = \omega_1\}$. Then

$$\{\alpha_{\omega_1(\sigma+1)} : \sigma < \omega_1^{n-1}\} \subset Q \subset \{\alpha_{\omega_1\sigma} : \sigma < \omega_1^{n-1}\}$$

and so Q has order type ω_1^{n-1}. By the induction hypothesis, there is a regressive function g defined on Q so that

$$|\{\sigma : \sigma_0 < \sigma;\ \alpha_\sigma \in Q;\ g(\alpha_\sigma) < \alpha_{\sigma_0}\}| \leqq \aleph_0 \qquad (\alpha_{\sigma_0} \in Q).$$

Now define f in the following way:

$$f(\alpha_\varrho) = g(\alpha_\varrho) \qquad (\alpha_\varrho \in Q),$$

$$f(\alpha_\varrho) = \beta_\varrho \qquad (\alpha_\varrho \in P - Q).$$

Clearly f is regressive. We have to verify that (7) holds. Let $\varrho_0 < \omega_1^n$. It follows from the definition of the β_ϱ that, if $\varrho_0 < \varrho < \omega_1^n$ and $\alpha_\varrho \in P - Q$, then $\alpha_{\varrho_0} \leqq f(\alpha_\varrho)$. Therefore,

$$R = \{\varrho : \varrho_0 < \varrho < \omega_1^n;\ f(\alpha_\varrho) < \alpha_{\varrho_0}\} = \{\varrho : \varrho_0 < \varrho, \alpha_\varrho \in Q, f(\alpha_\sigma) < \alpha_{\varrho_0}\}.$$

Let σ_0 be the least ordinal such that $\varrho_0 \leqq \omega_1 \sigma_0$. Then

$$R \subset \{\sigma : \sigma_0 < \sigma;\ \alpha_\sigma \in Q, g(\alpha_\sigma) < \alpha_{\sigma_0}\}$$

which is countable. Therefore (7) holds.

We now prove the substantive part of the lemma.

Let $\varrho < \omega_1^n$ and suppose we have already defined $x_{\sigma\nu}$ for $\sigma < \varrho$ and $\nu < \omega_1$. Since $C_{\varrho 0}$ is cofinal with $[0, \alpha_\varrho)$, we can choose $x_{\varrho 0} \in C_{\varrho 0}$ so that

$$x_{\varrho 0} > f(\alpha_\varrho).$$

More generally, by induction on ν, since $C_{\varrho\nu}$ is cofinal with $[0, \alpha_\varrho)$ we can define elements $x_{\varrho\nu} \in C_{\varrho\nu}$ $(\nu < \omega_1)$ so that

$$f(\alpha_\varrho) < x_{\varrho\nu} < x_{\varrho\mu} \qquad (\nu < \mu < \omega_1)$$

and $C_\varrho^* = \{x_{\varrho\nu} : \nu < \omega_1\}$ is a cofinal subset of $[f(\alpha_\varrho), \alpha_\varrho)$. Now put

$$C^* = \bigcup_{\varrho < \omega_1^n} C_\varrho^*.$$

Then $C^* \cap C_{\varrho\nu} \neq \emptyset$ $(\varrho < \omega_1^n;\ \nu < \omega_1)$. To prove the lemma we must show that $\mathrm{tp}\, C^* \leqq \omega_1^{n+1}$.

For $\sigma < \omega_1^n$, put $B_\sigma = [\beta_\sigma, \alpha_\sigma)$. Then

$$\bigcup_{\sigma<\omega_1^n} [0, \alpha_\sigma) = \bigcup_{\sigma<\omega_1^n} B_\sigma \,(\text{tp}).$$

If $\varrho < \sigma$, then $C_\varrho^* \cap B_\sigma = \emptyset$. If $\varrho = \sigma$, then $C_\varrho^* \cap B_\sigma$ is either empty (if $\beta_\sigma = \alpha_\sigma$) or it is a cofinal subset of B_σ of order type ω_1. By (7) there are only countably many values of $\varrho > \sigma$ such that $C_\varrho^* \cap B_\sigma \neq \emptyset$ and for every such ϱ, $C_\varrho^* \cap B_\sigma$ is countable since C_ϱ^* is cofinal with α_ϱ $(>\alpha_\sigma)$ and has order type ω_1. Thus we see that, if $D_\sigma = C^* \cap B_\sigma$, then

$$\text{tp}\, D_\sigma \leqq \omega_1 \qquad (\sigma < \omega_1^n).$$

Since $C^* = \bigcup_{\sigma<\omega_1^n} D_\sigma(\text{tp})$, we have the desired conclusion that $\text{tp}\, C^* \leqq \omega_1^{n+1}$.

4. Proof of Theorem. First we observe that it is enough to prove (5) in the case of indecomposable ordinals, i.e. that

$$(8) \qquad \omega_2\alpha_0 \not\to [\omega_1^\omega + 1, \omega_2\alpha_0]_{\aleph_2}$$

holds if α_0 is indecomposable and $\omega \leqq \alpha_0 < \omega_1$. Let $\omega \leqq \alpha < \omega_1$. Then $\alpha = \alpha_0 + \alpha_1$ where α_0 is indecomposable and $\alpha_1 < \alpha$. Let $S = S_0 \cup S_1$ (tp), tp $S_i = \omega_2\alpha_i$ $(i<2)$. If (8) holds, then there is a family $\mathcal{F} = (F_\mu : \mu < \omega_2)$ of subsets of S_0 such that tp $\mathcal{F}_\mu \leqq \omega_1^\omega$ $(\mu < \omega_2)$ and such that S_0 does not contain any $(\mathcal{F}, \aleph_2)$-free subset of type $\omega_2\alpha_0$. Therefore, if S' is any $(\mathcal{F}, \aleph_2)$-free subset of S, we have that

$$\text{tp}\, S' = \text{tp}\,(S' \cap S_0) + \text{tp}\,(S' \cap S_1) \leqq \gamma + \omega_2\alpha_1,$$

where $\gamma < \omega_2\alpha_0$. Therefore, tp $S' < \omega_2\alpha$. Thus (5) follows from (8).

We now assume that α_0 is indecomposable and that $\omega \leqq \alpha_0 < \omega_1$. Let $A = [0, \alpha_0)$,

$$S_\nu^\gamma = \{(\nu, \delta) : \delta < \gamma\} \qquad (\nu \in A;\ \gamma < \omega_2),$$

and let $S_\nu = \bigcup_{\gamma<\omega_2} S_\nu^\gamma$. Then the set

$$S = \bigcup_{\nu \in A} S_\nu$$

ordered lexicographically has order type $\omega_2\alpha_0$. Since α_0 is indecomposable and $\omega \leqq \alpha_0 < \omega_1$, there are sets $A_n \neq \emptyset$ $(n < \omega)$ such that

$$A = A_0 \cup A_1 \cup \ldots \cup \hat{A}_\omega \,(\text{tp}).$$

If $\gamma < \omega_2$ and N is cofinal with A, the set $\bigcup_{\nu \in N} S_\nu^\gamma$ has power $\aleph_1$. Therefore, by the hypothesis $2^{\aleph_1} = \aleph_2$, it follows that there are only $\aleph_2$ sets $B \subset S$ which are such that

$$B \subset \bigcup_{\nu \in N} S_\nu^\gamma$$

for some $\gamma = \gamma(B) < \omega_2$ and $N = N(B) \subset A$ with co $(\gamma) = \omega_1$ and N cofinal with A, and which have the further property that

$$B \cap S_\nu^\gamma \text{ is cofinal with } S_\nu^\gamma \qquad (\nu \in N(B)).$$

Let $B_0, B_1, \ldots, \hat{B}_{\omega_2}$ be a well ordering of all such sets B.

We are going to define a family $\mathcal{F}=(F_\mu\colon \mu<\omega_2)$ of subsets of S such that

(9) $$\operatorname{tp} F_\mu \leqq \omega_1^\omega \qquad (\mu<\omega_2),$$

(10) $$F_\mu \cap B_\nu \neq \emptyset \qquad (\nu<\mu<\omega_2).$$

This will prove (8). For suppose the F_μ $(\mu<\omega_2)$ satisfy (9) and (10). If $S'\subset S$ and $\operatorname{tp} S'=\omega_2\alpha_0$, then by Lemma 1, $S'\supset B_\nu$ for some $\nu<\omega_2$. Therefore, by (10),

$$\{\mu\colon F_\mu\cap S'=\emptyset\}\subset[0,\nu)$$

and so S' is not $(\mathcal{F}, \aleph_2)$-free.

Let $\mu<\omega_2$.

Put $C_\mu=\{\gamma(B_\nu)\colon \nu<\mu\}$. Since $\operatorname{tp} C_\mu<\omega_2$, there is a paradoxical decomposition of C_μ,

$$C_\mu = C_{\mu 0}\cup\ldots\cup \hat{C}_{\mu\omega},$$

so that $\operatorname{tp} C_{\mu n}\leqq\omega_1^n$ $(n<\omega)$. Thus we may write

$$C_{\mu n}=\{\gamma_{\mu n\delta}\colon \delta<\delta_{\mu n}\}_<,$$

where

$$\delta_{\mu n}\leqq\omega_1^n \qquad (n<\omega).$$

For $\delta<\delta_{\mu n}$, the set $M_{\mu n\delta}=\{\nu\colon\nu<\mu;\ \gamma(B_\nu)=\gamma_{\mu n\delta}\}$ is nonempty and has cardinal power less than or equal to $\aleph_1$. Therefore, there is a sequence $(\nu_{\mu n\delta\sigma})_{\sigma<\omega_1}$ (whose terms are not necessarily distinct) such that

$$M_{\mu n\delta}=\{\nu_{\mu n\delta\sigma}\colon \sigma<\omega_1\}.$$

Let $C_{\mu n\delta\sigma}=\{\gamma\colon(\varrho,\gamma)\in B_{\nu_{\mu n\delta\sigma}}$ *for some* $\varrho\in A-(A_0\cup\ldots\cup A_n)\}$. Then the sets $C_{\mu n\delta\sigma}$ are cofinal with $[0,\gamma_{\mu n\delta})$ for $\sigma<\omega_1$ and $\delta<\delta_{\mu n}\leqq\omega_1^n$. By Lemma 2, there is a set $C^*_{\mu n}$ such that

(11) $$C^*_{\mu n}\cap C_{\mu n\delta\sigma}\neq\emptyset \qquad (\sigma<\omega_1;\ \delta<\delta_{\mu n})$$

and

(12) $$\operatorname{tp} C^*_{\mu n}\leqq\omega_1^{n+1}$$

Put $G_{\mu n}=\{(\varrho,\gamma)\colon\ \gamma\in C^*_{\mu n},\ \varrho\in A-(A_0\cup\ldots\cup A_n)\}$. Then

(13) $$\operatorname{tp}(G_{\mu n}\cap S_\varrho)\leqq\omega_1^{n+1} \qquad (\varrho\in A_m,\ n<m<\omega),$$

(14) $$G_{\mu n}\cap S_\varrho=\emptyset \qquad (\varrho\in A_m,\ m\leqq n<\omega).$$

Also, by (11),

(15) $$G_{\mu n}\cap B_\nu\neq\emptyset \qquad (n<\omega;\ \nu\in M_{\mu n\delta};\ \delta<\delta_{\mu n}).$$

Now put $F_\mu=\bigcup_{n<\omega} G_{\mu n}$. Then, by (15) and the definition of the sets $M_{\mu n\delta}$, we have that

$$F_\mu\cap B_\nu\neq\emptyset \qquad (\nu<\mu),$$

i.e. (10) holds. If $m<\omega$ and $\varrho\in A_m$, then by (13) and (14)

$$\operatorname{tp}(F_\mu\cap S_\varrho)=\operatorname{tp}\Big(\bigcup_{n<m} G_{\mu n}\cap S_\varrho\Big)\leqq\omega_1^{m+1}.$$

Therefore

$$\operatorname{tp}\Big(F_\mu\cap\bigcup_{\varrho\in A_m} S_\varrho\Big)<\omega_1^{m+2}\qquad(m<\omega).$$

Since $A=A_0\cup A_1\cup\ldots\cup\hat{A}_\omega$ (tp), it follows that

$$\operatorname{tp}F_\mu\leqq\sum_{m<\omega}\omega_1^{m+2}=\omega_1^\omega.$$

This proves (9) and completes the proof of the theorem.

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SUR LA NORME DES FONCTIONS DE CERTAINS OPÉRATEURS

Par

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Hommage à M. G. Alexits à son 70^e anniversaire

1. En connexion avec ses recherches sur l'exposant critique des opérateurs, V. Pták vient d'obtenir un résultat sur les normes des itérés des opérateurs linéaires dans un espace euclidien complexe de dimension finie, opérateurs vérifiant une «équation de Cayley—Hamilton» prescrite [1], [2]. Afin de l'énoncer, on a besoin des définitions suivantes:

Soit $p(z)$ un polynôme de degré n, à coefficients complexes et ayant tous ses zéros <1 en module. Désignons par $\mathscr{A}_p$ la classe des contractions T (c'est-à-dire des opérateurs linéaires de norme $\leqq 1$) dans l'espace euclidien complexe de dimension n, telles que $p(T)=O$.[1]

Soit $K=l^2$ l'espace des suites infinies $x=(x_0, x_1, \ldots)$ de nombres complexes avec $\|x\|=\left[\sum_0^\infty |x_i|^2\right]^{1/2}<\infty$, et désignons par S la «translation en arrière» dans K, définie par

$$S(x_0, x_1, \ldots)=(x_1, x_2, \ldots); \tag{1}$$

S est une contraction dans K. Posons

$$K_p=\{x\colon x\in K, p(S)x=0\}: \tag{2}$$

c'est évidemment un sous-espace de K, de dimension n, invariant pour S. Soit S_p l'opérateur induit dans K_p par S, donc soit

$$S_p = S|K_p. \tag{3}$$

On a alors $q(S_p)=q(S)|K_p$ pour tout polynôme q, en particulier $p(S_p)=p(S)|K_p=O$, donc $S_p\in\mathscr{A}_p$.

Cela étant, le théorème de Pták peut être énoncé de la manière suivante:

Soit m fixé, $m\geqq n$. Le maximum de la norme de T^m pour $T\in\mathscr{A}_p$ est atteint lorsque $T=S_p$.

2. La restriction aux exposants $m\geqq n$ fait l'impression que la validité de ce théorème est essentiellement liée des espaces de dimension finie. Or, cela n'en est pas ainsi. En effet, nous allons étendre le théorème aux opérateurs des espaces de Hilbert

[1] $p(T)=O$ entraîne que les valeurs propres de T sont des zéros de $p(z)$, d'où il dérive que les valeurs propres de T sont inférieures à 1 en module. Cela entraîne, à son tour, que $T^m\to O$ lorsque $m\to\infty$, par conséquent T est complètement non-unitaire. — Un opérateur linéaire d'un espace de Hilbert (de dimension quelconque) s'appelle *complètement non-unitaire* s'il n'est réduit par aucun sous-espace non nul à un opérateur unitaire.

de dimension quelconque, finie ou infinie. On obtiendra, en particulier, que la restriction aux exposants m dans le théorème ci-dessus est superflue.

Dans cette généralisation on considérera notamment des contractions T complètement non-unitaires des espaces de Hilbert. Pour telles T, il existe un calcul fonctionnel (cf. [3], Chap. III) portant sur toutes les fonctions $\psi(z)$ holomorphes et bornées dans le disque unité ouvert du plan des nombres complexes, c'est-à-dire pour toute fonction de classe H^∞. Notamment, si $\psi(z) = \sum_0^\infty a_n z^n$, on définit:

$$\psi(T) = \lim_{r \to 1-0} \sum_0^\infty a_n r^n T^n.$$

Fixons une fonction $\varphi \in H^\infty$, $\varphi \not\equiv 0$, et désignons par $\mathscr{A}_\varphi$ la classe des contractions complètement non-unitaires T des espaces de Hilbert (de dimension quelconque), telles que

$$\varphi(T) = O. \tag{4}$$ [2]

Soit S l'opérateur de l'espace $K = l^2$, défini plus haut. Puisque $S^n \to O$ lorsque $n \to \infty$, S est complètement non-unitaire, donc $\psi(S)$ a un sens pour $\psi \in H^\infty$. On peut donc définir, à l'analogie de (2) et (3)

$$K_\varphi = \{x\colon x \in K,\ \varphi(S)x = 0\} \quad \text{et} \quad S_\varphi = S|K_\varphi, \tag{5}$$

notons que K_φ est un sous-espace invariant pour S. Il dérive de (5) que $\varphi(S_\varphi) = \varphi(S)|K_\varphi = O$. Donc $S_\varphi \in \mathscr{A}_\varphi$.

THÉORÈME. *Soit $\psi \in H^\infty$. Le maximum de la norme de $\psi(T)$ pour $T \in \mathscr{A}_\varphi$ est atteint lorsque $T = S_\varphi$.*

DÉMONSTRATION. Soit T un opérateur de classe $\mathscr{A}_\varphi$, dans l'espace de Hilbert $\mathfrak{H}$. On a alors $T^n \to O$ $(n \to \infty)$, [3] d'où

$$\|h\|^2 = \sum_{n=0}^\infty [\|T^n h\|^2 - \|T^{n+1} h\|^2] \quad \text{pour} \quad h \in \mathfrak{H}. \tag{6}$$

Comme $\|T\| \leqq 1$, on peut définir $D = (I - T^*T)^{1/2}$ et on observe alors que

$$\|g\|^2 - \|Tg\|^2 = \|Dg\|^2 \quad \text{pour tout} \quad g \in \mathfrak{H}.$$

Ainsi, (6) entraîne

$$\|h\|^2 = \sum_{n=0}^\infty \|DT^n h\|^2 \quad \text{pour tout} \quad h \in \mathfrak{H}. \tag{7}$$

[2] Sauf pour les fonctions „extérieures", c'est-à-dire pour lesquelles

$$\varphi(z) = k \cdot \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log|\varphi(e^{it})|\, dt\right], \quad |k| = 1,$$

la classe $\mathscr{A}_\varphi$ n'est pas banale, c'est-à-dire contient des opérateurs dans des espaces $\neq \{0\}$, cf. [3], Proposition III. 3.2.

[3] Cf. [3], Proposition III. 4.2.

Envisageons le sous-espace $\mathfrak{D}=\overline{D\mathfrak{H}}$ de $\mathfrak{H}$, et construisons l'espace de Hilbert $\mathbf{K}=l^2(\mathfrak{D})$ des suites $\mathbf{x}=(\mathbf{x}_0, \mathbf{x}_1, \dots)$ d'éléments $\mathbf{x}_n \in \mathfrak{D}$ $(n=0, 1, \dots)$ avec

$$\|\mathbf{x}\| = \left(\sum_0^\infty \|\mathbf{x}_n\|^2\right)^{1/2} < \infty.$$

Dans $\mathbf{K}$ on définit la translation en arrière $\mathbf{S}$ à l'analogie de (1), c'est-à-dire par

$$\mathbf{S}(\mathbf{x}_0, \mathbf{x}_1, \dots) = (\mathbf{x}_1, \mathbf{x}_2, \dots);$$

Comme $\mathbf{S}^n \to \mathbf{0}$ $(n\to\infty)$, $\mathbf{S}$ est une contraction complètement non-unitaire de $\mathbf{K}$. Posons, à l'analogie de (5),

(8) $$\mathbf{K}_\varphi = \{\mathbf{x} \colon \mathbf{x}\in\mathbf{K},\ \varphi(\mathbf{S})\mathbf{x}=0\} \quad \text{et} \quad \mathbf{S}_\varphi = \mathbf{S}|\mathbf{K}_\varphi;$$

notons que $\mathbf{K}_\varphi$ est un sous-espace invariant pour $\mathbf{S}$ et qu'on a $\psi(\mathbf{S}_\varphi) = \psi(\mathbf{S})|\mathbf{K}_\varphi$ pour tout $\psi\in H^\infty$, d'où il s'ensuit en particulier que $\mathbf{S}_\varphi \in \mathscr{A}_\varphi$.

On déduit de (7) qu'on peut plonger l'espace $\mathfrak{H}$ dans l'espace $\mathbf{K}$ d'une manière linéaire et isométrique, en identifiant les éléments

$$h\in\mathfrak{H} \quad \text{et} \quad (Dh, DTh, DT^2h, \dots)\in\mathbf{K};$$

$\mathfrak{H}$ devient ainsi un sous-espace de $\mathbf{K}$. L'élément Th s'identifiera à l'élément $(DTh, DT^2h, DT^3h, \dots)$, qui est égal à $\mathbf{S}(Dh, DTh, DT^2h, \dots)$, donc on aura

(9) $$T = \mathbf{S}|\mathfrak{H}.$$

Cette relation entraîne $\psi(T) = \psi(\mathbf{S})|\mathfrak{H}$ pour toute function $\psi\in H^\infty$, on aura donc en particulier

$$O = \varphi(T) = \varphi(\mathbf{S})|\mathfrak{H},$$

d'où il dérive que $\mathfrak{H}\subset\mathbf{K}_\varphi$. Ainsi, on a pour toute function $\psi\in H^\infty$:

(10) $$\psi(T) = \psi(\mathbf{S})|\mathfrak{H} \subset \psi(\mathbf{S})|\mathbf{K}_\varphi = \psi(\mathbf{S}_\varphi),$$

d'où

(11) $$\|\psi(T)\| \leqq \|\psi(\mathbf{S}_\varphi)\|.$$

Choisissons dans $\mathfrak{D}$ une base orthonormale $\{e_\alpha\}_{\alpha\in A}$ (où A est un ensemble d'indices, de cardinalité égale à $d=\dim\mathfrak{D}$). Elle engendre une décomposition de l'espace $\mathbf{K}=l^2(\mathfrak{D})$ en somme orthogonale de d répliques de l'espace $K=l^2$:

(12) $$\mathbf{K} = \bigoplus_{\alpha\in A} K^{(\alpha)},$$

et une décomposition correspondante de $\mathbf{S}$ en somme orthogonale de α répliques de S:

(13) $$\mathbf{S} = \bigoplus_{\alpha\in A} S^{(\alpha)}.$$

(13) entraîne

$$\psi(\mathbf{S}) = \bigoplus_{\alpha\in A} \psi(S^{(\alpha)}) \quad \text{pour tout} \quad \psi\in H^\infty.$$

Il en dérive en particulier que si $\mathbf{x} = \bigoplus_{\alpha\in A} x^{(\alpha)}$, on a

$$\varphi(\mathbf{S})\mathbf{x} = 0 \Leftrightarrow \psi(S^{(\alpha)})x^{(\alpha)} = 0 \qquad (\alpha\in A),$$

d'où on conclut:

$$\mathbf{K}_\varphi = \bigoplus_{\alpha \in A} K_\varphi^{(\alpha)} \tag{14}$$

et par conséquent

$$\mathbf{S}_\varphi = \bigoplus_{\alpha \in A} S_\varphi^{(\alpha)},$$

ou, d'une manière générale,

$$\psi(\mathbf{S}_\varphi) = \bigoplus_{\alpha \in A} \psi(S^{(\alpha)}) \qquad (\psi \in H^\infty), \tag{15}$$

les $K_\varphi^{(\alpha)}$ et les $S_\varphi^{(\alpha)}$ étant des répliques de K_φ et de S_φ, selon les cas. De (15) on obtient

$$\|\psi(\mathbf{S}_\varphi)\| = \|\psi(S_\varphi)\|;$$

ensemble avec (11) cela fournit:

$$\|\psi(T)\| \leqq \|\psi(S_\varphi)\|.$$

Puisque S_φ appartient aussi à $\mathscr{A}_\varphi$, cela achève la démonstration.

(Reçu le 10 Septembre 1968)

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NOTE ON THE DIVERGENCE OF TRIGONOMETRIC INTERPOLATION

By

J. SZABADOS (Budapest)

To Professor G. ALEXITS *on his 70th birthday*

1. Notations. Let

$$X = \begin{Bmatrix} x_{0,0} & & & \\ x_{0,1}, & x_{1,1}, & x_{2,1} & \\ \cdots & \cdots & \cdots & \\ x_{0,n}, & x_{1,n}, & \ldots, & x_{2n,n} \\ \cdots & \cdots & \cdots & \cdots \end{Bmatrix}$$

be an infinite triangular matrix where

$$0 \leqq x_{0,n} < x_{1,n} < \cdots < x_{2n,n} < 2\pi \qquad (n=0, 1, 2, \ldots),$$

and

$$\delta_n = \min_{0 \leqq k \leqq 2n} (x_{k+1,n} - x_{k,n}) \qquad (n = 0, 1, 2, \ldots)$$

(here $x_{2n+1,n} = x_{0,n} + 2\pi$). The interpolating polynomial of degree n of a 2π-periodical continuous function $f(x)$ based on the nodes of X will be the following:

$$L_n(f, x, X) = \sum_{k=0}^{2n} f(x_{k,n}) l_{k,n}(x, X) \qquad (n = 0, 1, 2, \ldots)$$

where $l_{k,n}(x, X)$ are the fundamental polynomials of the trigonometric interpolation. The quantity

$$\lambda_n(X) = \max_x \sum_{k=0}^{2n} |l_{k,n}(x, X)| \qquad (n = 0, 1, 2, \ldots)$$

is said to be the Lebesgue-costant of the interpolation. The continuity modulus $\omega(f, h)$ of a continuous, 2π-periodical function is defined by

$$\omega(f, h) = \max_{\substack{0 \leqq t \leqq h \\ -\infty < x < +\infty}} |f(x+t) - f(x)| \qquad (0 \leqq h \leqq 2\pi).$$

If $\omega(h)$ is such a modulus of continuity then denote by $C(\omega)$ the class of continuous, 2π-periodical functions $f(x)$ for which

$$\sup_{0 < h \leqq 2\pi} \frac{\omega(f, h)}{\omega(h)} < \infty$$

holds.

2. A divergence condition. The following problem arises: whether the relation

(1) $$\lim_{n\to\infty} \max_{n} |f(x)-L_n(f, x, X)| = 0$$

or

(2) $$\limsup_{n\to\infty} \max_{x} |f(x)-L_n(f, x, X)| > 0$$

holds? There is a lot of results in which conditions are stated in connection with (1)—(2). These conditions are expressed by ω and $\lambda_n(X)$. In his paper [1] O. KIS has given such conditions in which the above defined δ_n figures as well. His theorems deal with divergence problems. The first of them asserts that if $C(\omega)\neq \text{Lip } 1$ and $\limsup_{n\to\infty} \omega(\delta_n)\lambda_n(X)>0$ then there exists a function $f(x)\in C(\omega)$ such that (2) holds.

3. The Zygmund-class. The above mentioned theorem excludes the class Lip 1. In connection with this class the problem seems to be more difficult. In this paper we prove a theorem for a little wider class, namely, for the so-called Zygmund-class (Z-class). We shall say that $f(x)\in Z$ if $f(x)$ is continuous, 2π-periodical, and

(3) $$\sup_{\substack{0<h\leqq 2\pi \\ -\infty<x<+\infty}} \frac{|f(x+h)-2f(x)+f(x-h)|}{h} < \infty.$$

Evidently

$$\text{Lip } 1 \subset Z \subset C(\omega)$$

for all $\omega(h)\neq O(h)$.

THEOREM. *If*

(4) $$\limsup_{n\to\infty} \delta_n \lambda_n(X) > 0$$

holds for a matrix X then there exists an $f(x)\in Z$ *such that* (2) *holds.*

PROOF. Let z_n be a point such that

$$\lambda_n(X) = \sum_{k=0}^{2n} |l_{k.n}(z_n, X)| \qquad (n = 0, 1, 2, \ldots).$$

Define the function $g_n(x)$ $(n=0, 1,2, \ldots)$ for all x as follows. Let

(5) $$g_n(x_{k,n})=\operatorname{sign} l_{k,n}(z_n, X) \qquad (k=0, 1, \ldots, 2n);$$

furthermore, if $g_n(x_{k,n})=g_n(x_{k+1,n})$ then

$$g_n(x)=g_n(x_{k,n}) \quad \text{for} \quad x\in[x_{k,n}, x_{k+1,n}],$$

and if $g_n(x_{k,n}) = -g_n(x_{k+1,n})$ then

$$g_n(x) = \frac{\operatorname{sign} g_n(x_{k,n})}{x_{k+1,n}-x_{k,n}}\left[\frac{4\left(x-\dfrac{x_{k,n}+x_{k+1,n}}{2}\right)^3}{(x_{k+1,n}-x_{k,n})^2}-3\left(x-\frac{x_{k,n}+x_{k+1,n}}{2}\right)\right]$$

for $x \in [x_{k,n}, x_{k+1,n}]$. Evidently, $g_n(x)$ is a 2π-periodical continuous function for which

(6) $$\max_x |g_n(x)| = 1 \qquad (n = 0, 1, 2, \ldots)$$

and, by (5) and the choosing of z_n

(7) $$L_n(g_n, x, X) \leqq \lambda_n(X) = L_n(g_n, z_n, X) \qquad (n = 0, 1, 2, \ldots).$$

Moreover, the first derivative of $g_n(x)$ exists ewerywhere, and so by the Lagrangean mean-value theorem

$$|g_n(x+h) - 2g_n(x) + g_n(x-h)| = h|g_n'(\xi) - g_n'(\eta)| \qquad (h > 0,\ |\xi - \eta| < 2h).$$

Now, if $h \leqq \delta_n$ then

$$|g_n'(\xi) - g_n'(\eta)| \leqq |\xi - \eta| \cdot \max_{\substack{x_{k,n} < x < x_{k+1,n} \\ 0 \leqq k \leqq 2n}} |g_n''(x)| \leqq 2h \cdot \frac{12}{\delta_n^2} = \frac{24h}{\delta_n^2}$$

and if $h > \delta_n$ then

$$|g_n'(\xi) - g_n'(\eta)| \leqq 2 \max_x |g_n'(x)| \leqq 2 \cdot \frac{3}{\delta_n} < \frac{6h}{\delta_n^2}.$$

Thus in both cases

(8) $$|g_n(x+h) - 2g_n(x) + g_n(x-h)| \leqq \frac{24}{\delta_n^2} h^2 \qquad (h > 0,\ n = 0, 1, 2, \ldots).$$

We may assume that

(9) $$\lim_{n \to \infty} \max_x |g_m(x) - L_n(g_m, x, X)| = 0 \qquad (m = 0, 1, 2, \ldots)$$

because in the contrary case a $g_m(x) \in Z$ (by (8)) would satisfy (2). Define the sequence of indices $n_1 < n_2 < \ldots$ as follows:

$$\left.\begin{array}{lc} (10) & \lambda_{n_1}(X) \geqq \frac{2}{9}, \quad \lambda_{n_{k+1}}(X) \geqq 3\lambda_{n_k}(X) \\ (11) & \delta_{n_k} \geqq 2\delta_{n_{k+1}} \\ (12) & \delta_{n_k}\lambda_{n_k}(X) \geqq c > 0 \end{array}\right\} \qquad (k = 1, 2, \ldots),$$

(13) $$\max_x |g_{n_k}(x) - L_{n_j}(g_{n_k}, x, X)| \leqq 1 \qquad (j = k+1, k+2, \ldots).$$

These are possible by the relations $\lim_{n \to \infty} \lambda_n(x) = \infty$, $\lim_{n \to \infty} \delta_n = 0$, by (4) and (9). Let

$$f(x) = \sum_{k=1}^{\infty} \frac{g_{n_k}(x)}{\lambda_{n_k}(X)}.$$

Here the right hand side series converges for all x, by (6) and (10). Clearly, $f(x)$ is a continuous 2π-periodical function. We show that $f(x) \in Z$. Let $0 < h < \delta_{n_1}$ be arbitrary,

$$\delta_{n_{j+1}} < h \leqq \delta_{n_j},$$

say. Then by (8), (6), (12) and (11) we have

$$|f(x+h)-2f(x)+f(x-h)| \leqq \sum_{k=1}^{\infty} \frac{|g_{n_k}(x+h)-2g_{n_k}(x)+g_{n_k}(x-h)|}{\lambda_{n_k}(X)} =$$

$$= \sum_{k=1}^{j} + \sum_{k=j+1}^{\infty} \leqq \sum_{k=1}^{j} \frac{24h^2}{\delta_{n_k}^2 \lambda_{n_k}(X)} + \sum_{k=j+1}^{\infty} \frac{4}{\lambda_{n_k}(X)} \leqq \frac{24}{c} h^2 \sum_{k=1}^{j} \frac{1}{\delta_{n_k}} + \frac{4}{c} \sum_{k=j+1}^{\infty} \delta_{n_k} \leqq$$

$$\leqq \frac{24h^2}{c\delta_{n_j}} \sum_{k=1}^{j} 2^{k-j} + \frac{4\delta_{n_{j+1}}}{c} \sum_{k=j+1}^{\infty} 2^{j+1-k} \leqq \frac{24h}{c} \cdot 2 + \frac{4h}{c} \cdot 2 = \frac{56}{c} \cdot h,$$

i.e. (3) holds. Now, by (7), (6) and (10) we get

$$L_{n_j}(f, z_{n_j}, X) - f(z_{n_j}) = \sum_{k=1}^{\infty} \frac{L_{n_j}(g_{n_k}, z_{n_j}, X) - g_{n_k}(z_{n_j})}{\lambda_{n_k}(X)} \geqq 1 - \frac{1}{\lambda_{n_j}(X)} -$$

$$- \sum_{k=1}^{j-1} \frac{1}{\lambda_{n_k}(X)} - \sum_{k=j+1}^{\infty} \frac{\lambda_{n_j}(X)}{\lambda_{n_k}(X)} \geqq 1 - \frac{1}{\lambda_{n_1}(X)} \sum_{k=1}^{\infty} \frac{1}{3^{k-1}} - \sum_{k=j+1}^{\infty} \frac{1}{3^{k-j}} \geqq$$

$$\geqq 1 - \frac{2}{9} \cdot \frac{3}{2} - \frac{1}{2} = \frac{1}{6},$$

i. e.

$$\limsup_{j \to \infty} \max_{x} |f(x) - L_{n_j}(f, x, X)| \geqq \frac{1}{6},$$

q.e.d.

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ЗАМЕЧАНИЯ О ПОРЯДКЕ ПОГРЕШНОСТИ ИНТЕРПОЛЯЦИИ

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Посвящается академику Д. АЛЕКСИЧУ *к семидесятилетию со дня рождения*

§ 1. Введение

Обозначим через $\omega(t)$ модуль непрерывности некоторой фиксированной вещественной функции, непрерывной на отрезке $[-1, +1]$ или непрерывной и 2π-периодической на всей вещественной оси, а через $C(\omega)$ множество таких же функций, модуль непрерывности которых равен $O(\omega(t))$. В статьях [1]—[7] изучался следующий вопрос: каков должен быть порядок роста чисел Лебега Лагранжева или тригонометрического интерполирования, чтобы интерполяционный процесс равномерно сходился для всех функций класса $C(\omega)$ или не сходился равномерно для некоторой функции этого множества? В работе [8] результаты статьи [2] были обобщены на тот случай, когда вместо модуля непрерывности рассматривается модуль гладкости любого порядка. В заметке [9] было получено условие расходимости тригонометрического интерполирования, в котором кроме модуля непрерывности $\omega(t)$ и чисел Лебега интерполяции фигурирует еще наименьшее расстояние между соседними узлами интерполяции. В работе [12] этот результат был перенесен на тот случай, когда множество $C(\omega)$ заменяется на класс тех 2π-периодических непрерывных функций, модуль гладкости второго порядка которых равен $O(t)$. Наконец, в статьях [10] и [11] изучался порядок погрешности тригонометрического интерполирования функций из множества $C(\omega)$.

В настоящей заметке результаты статьи [10] обобщаются на тот случай, когда вместо модуля непрерывности рассматривается модуль гладкости любого порядка. При этом изучается как случай Лагранжева, так и случай тригонометрического интерполирования. В следующем параграфе дается оценка сверху погрешности интерполяции. В § 3 формулируется основной результат работы, оценивающий снизу погрешность интерполяции некоторых функций, и приводятся некоторые следствия этой теоремы, доказательству которой посвящен последний параграф.

§ 2. Верхняя оценка погрешности интерполирования

Пусть $\omega(t)$ есть определенная при $t \geqq 0$ неубывающая непрерывная функция, причем $\omega(0)=0$, $\omega(t)>0$, если $t>0$, $t^m/\omega(t)$ не убывает и

$$(1) \qquad \lim_{t=+0} \frac{t^m}{\omega(t)} = 0,$$

где m любое, но фиксированное натуральное число. Обозначим через C пространство непрерывных и 2π-периодических на всей вещественной оси функций. Пусть $C_m(\omega)$ есть подпространство тех элементов из C, для которых

$$\omega_m(f, t) \leqq a_m(f)\omega(t), \tag{2}$$

где $\omega_m(f, t)$ модуль гладкости m-ого порядка функции $f(x)$, а $a_m(f)$ не зависящее от t число. Пусть $L_n(f, x)$ есть тригонометрический многочлен n-ого порядка, интерполирующий функцию $f(x)$ в $2n+1$ различных точках отрезка $[0,2\pi)$, а λ_n соответствующее число Лебега интерполяции.

Теорема 1. *В случае тригонометрического интерполирования для всех функций из $C_m(\omega)$ выполняется неравенство*

$$\|L_n(f, x)-f(x)\| \leqq b_m(f)(1+\lambda_n)\omega\left(\frac{1}{n}\right) \qquad (n = 1, 2, 3, \ldots), \tag{3}$$

где число $b_m(f)$ не зависит от n.

Доказательство. Известно, что

$$\|L_n(f, x)-f(x)\| \leqq (1+\lambda_n)E_n(f) \qquad (n = 1, 2, 3, \ldots), \tag{4}$$

где $E_n(f)$ наилучшее равномерное приближение функции $f(x)$ тригонометрическими многочленами n-ого порядка. Здесь (см., например, [13], стр. 274)

$$E_n(f) \leqq c_m\omega_m\left(f, \frac{1}{n}\right) \qquad (n = 1, 2, 3, \ldots), \tag{5}$$

где число c_m не зависит от n. Полагая

$$b_m(f) = c_m a_m(f),$$

получаем (3) из (2), (4) и (5).

Пусть теперь C обозначает множество непрерывных на отрезке $[-1, +1]$ функций, алгебраический многочлен n-той степени $L_n(f, x)$ совпадает с функцией $f(x)$ в $n+1$ различных точках отрезка $[-1, +1]$, а λ_n есть число Лебега интерполяции Лагранжа по этим узлам. Так как неравенство (4) остается в силе, если $E_n(f)$ есть наилучшее равномерное приближение функции $f(x)$ алгебраическими многочленами n-той степени, а оценка (5) выполняется для $m=1$ и $m=2$ (см. [13], стр. 269 и 281), то *при $m=1$ и $m=2$ теорема* 1 *справедлива и в случае Лагранжева интерполирования.*

§ 3. Нижняя оценка погрешности интерполирования некоторых функций

Обозначим через $x_k = x_{k,n}$ узлы Лагранжева интерполирования и пусть

$$-1 \leqq x_0 < x_1 < \cdots < x_n \leqq +1 \qquad (n = 1, 2, 3, \ldots). \tag{6}$$

Обозначим через d_n наименьшее расстояние между соседними узлами интерполирования, то есть пусть

$$d_n = \min_{1 \leqq k \leqq n} (x_k - x_{k-1}) \qquad (n = 1, 2, 3, \ldots). \tag{7}$$

Теорема 2. *В случае Лагранжевой интерполяции для некоторой функции* $f(x)$ *из* $C_m(\omega)$

$$\|L_n(f, x)-f(x)\| \geqq \lambda_n \omega(d_n) \qquad (n = n_1, n_2, n_3, \ldots), \tag{8}$$

где n_k *строго возрастающая последовательность натуральных чисел.*

Если $x_k = x_{k,n}$ суть узлы тригонометрического интерполирования, причем

$$0 \leqq x_0 < x_1 < \cdots < x_{2n} < 2\pi \qquad (n = 1, 2, 3, \ldots) \tag{9}$$

и

$$x_{2n+1} = x_0 + 2\pi, \quad d_n = \min_{0 \leqq k \leqq 2n} (x_{k+1} - x_k) \qquad (n = 1, 2, 3, \ldots), \tag{10}$$

то теорема 2 *справедлива и в случае тригонометрического интерполирования.*

Мы предположили, что функция $\omega(t)$ не убывает, поэтому имеет место

Следствие 1. *Если*

$$d_n \geqq \frac{1}{n} \qquad (n = 1, 2, 3, \ldots), \tag{11}$$

то как в случае алгебраического, так и в случае тригонометрического интерполирования для некоторой функции $f(x)$ *из* $C_m(\omega)$

$$\|L_n(f, x)-f(x)\| \geqq \lambda_n \omega\left(\frac{1}{n}\right) \qquad (n = n_1, n_2, n_3, \ldots). \tag{12}$$

Таким образом, порядок оценки (3) не может быть улучшен для всех функций из $C_m(\omega)$ ни в случае параболического, ни в случае тригонометрического интерполирования.

В [2] было доказано, что для любых узлов алгебраического интерполирования

$$d_n > \frac{1}{n^2 \lambda_n} \qquad (n = 1, 2, 3, \ldots). \tag{13}$$

Поэтому из теоремы 2 можно получить

Следствие 2. *В случае Лагранжева интерполирования для некоторой функции* $f(x)$ *из* $C_m(\omega)$

$$\|L_n(f, x)-f(x)\| \geqq \lambda_n \omega\left(\frac{1}{n^2 \lambda_n}\right) \qquad (n = n_1, n_2, n_3, \ldots). \tag{14}$$

В [3] было доказано, что для любых узлов тригонометрического интерполирования

$$d_n > \frac{1}{n \lambda_n} \qquad (n = 1, 2, 3, \ldots). \tag{15}$$

Поэтому *в случае тригонометрического интерполирования для некоторой функции* $f(x)$ *из* $C_m(\omega)$

(16) $$\|L_n(f, x)-f(x)\| \geqq \lambda_n \omega\left(\frac{1}{n\lambda_n}\right) \qquad (n = n_1, n_2, n_3, \ldots).$$

Порядок этой оценки, вообще говоря, нельзя улучшить, так как в [11] было доказано, что при $m=1$ для некоторых узлов интерполирования и всех функций из $C_1(\omega)$

$$\|L_n(f, x)-f(x)\| = O\left[\lambda_n \omega\left(\frac{1}{n\lambda_n}\right)\right] \qquad (n = 1, 2, 3, \ldots).$$

В качестве примера рассмотрим случай

(17) $$\omega(t) = t^\alpha \qquad (0 < \alpha < m).$$

Если предположить, что

(18) $$\lambda_n = O(n^\beta) \qquad (\beta > 0),$$

то в силу теоремы 1

(19) $$\|L_n(f, x)-f(x)\| = O(n^{\beta-\alpha})$$

для всех $f(x)$ из $C_m(\omega)$. Интерполяционный процесс равномерно сходится, если $\alpha > \beta$. Если выполняется условие (11) и

(20) $$n^\gamma = O(\lambda_n) \qquad (\gamma > 0),$$

то ввиду следствия 1 для некоторой $f(x)$ из $C_m(\omega)$

(21) $$n^{\gamma-\alpha} = O(\|L_n(f, x)-f(x)\|) \qquad (n = n_1, n_2, n_3, \ldots).$$

Интерполяционный процесс не сходится равномерно, если $\alpha \leqq \gamma$. Если $\alpha < 1$, то в силу следствия 2 в случае параболического интерполирования для некоторой $f(x)$ из $C_m(\omega)$

(22) $$n^{\gamma(1-\alpha)-2\alpha} = O(\|L_n(f, x)-f(x)\|) \qquad (n = n_1, n_2, n_3, \ldots).$$

Интерполяционный процесс не сходится равномерно, если $\gamma > 2\alpha/(1-\alpha)$. Если $\alpha \geqq 1$, то

(23) $$n^{\beta(1-\alpha)-2\alpha} = O(\|L_n(f, x)-f(x)\|) \qquad (n = n_1, n_2, n_3, \ldots).$$

Заменяя 2α на α, получаем соответствующие неравенствам (22) и (23) результаты для случая тригонометрического интерполирования.

§ 4. Доказательство теоремы 2

Рассмотрим фундаментальные многочлены Лагранжева интерполирования

(24) $$l_k(x) = l_{k,n}(x) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x-x_i}{x_k-x_i} \qquad (k = 0, 1, \ldots, n)$$

и точку z_n, в которой функция Лебега

$$\sum_{k=0}^{n} |l_k(x)| \tag{25}$$

принимает свое наибольшее значение

$$\lambda_n = \sum_{k=0}^{n} |l_k(z_n)| \qquad (n = 1, 2, 3, \ldots). \tag{26}$$

Определим для каждого натурального числа n некоторую функцию $g_n(x)$. Сначала определим ее значения в узлах x_k:

$$g_n(x_k) = \operatorname{sign} l_k(z_n) \qquad (k = 0, 1, \ldots, n). \tag{27}$$

Затем определим ее вне отрезка $[x_0, x_n]$:

$$g_n(x) = \begin{cases} g_n(x_0), & \text{если} \quad x < x_0, \\ g_n(x_n), & \text{если} \quad x > x_n. \end{cases} \tag{28}$$

Наконец, определим ее на отрезках (x_k, x_{k+1}): там она совпадает с тем интерполяционным многочленом Эрмита $2m-1$-ой степени, который принимает в паре точек x_k и x_{k+1} значения $g_n(x_k)$ и $g_n(x_{k+1})$ и первые $m-1$ производные которого исчезают в этих точках. Если $g_n(x_{k+1}) = g_n(x_k)$, то этот многочлен, очевидно, совпадает с постоянной $g_n(x_k)$. Если же $g_n(x_{k+1}) = -g_n(x_k)$, то, воспользовавшись равенством

$$\int_{-1}^{+1} (1-t^2)^{m-1}\, dt = 2\,\frac{(2m-2)!!}{(2m-1)!!} \qquad (m = 1, 2, 3, \ldots), \tag{29}$$

можно представить этот многочлен так:

$$g_n(x) = g_n(x_k)\left[1 - \frac{(2m-1)!!}{(2m-2)!!} \int_{-1}^{2\frac{x-x_k}{x_{k+1}-x_k}-1} (1-t^2)^{m-1}\, dt\right] \quad (x_k \leqq x \leqq x_{k+1}). \tag{30}$$

Отметим некоторые свойства функций $g_n(x)$.

Очевидно, каждая функция $g_n(x)$ имеет $m-1$-ую непрерывную произодную, модуль непрерывности которой не превосходит $B_m d_n^{-m} t$, где B_m некоторое положительное число, зависящее от m, но не зависящее от n. Отсюда (см. [13], стр. 116) следует неравенство

$$\omega_m(g_n, t) \leqq B_m\, d_n^{-m}\, t^m \qquad \left(0 < t \leqq \frac{2}{m}\right). \tag{31}$$

Поэтому и ввиду (1) имеет место (2), то есть $g_n(x) \in C_m(\omega)$.

Очевидно,

$$|g_n(x)| \leqq 1 \qquad (n = 1, 2, 3, \ldots), \tag{32}$$

поэтому при любых n и N

(33) $$|L_n(g_N, x)| = \left|\sum_{k=0}^{n} g_N(x_k) l_k(x)\right| \leqq \sum_{k=0}^{n} |l_k(x)| \leqq \lambda_n.$$

Заметим, что ввиду (27) и (26)

(34) $$L_n(g_n, z_n) = \sum_{k=0}^{n} g_n(x_k) l_k(z_n) = \sum_{k=0}^{n} |l_k(z_n)| = \lambda_n \qquad (n = 1, 2, 3, \ldots).$$

Если для некоторого фиксированного натурального числа N и строго возрастающей последовательности натуральных чисел n_k

(35) $$\|L_n(g_N, x) - g_N(x)\| \geqq \omega(d_n)\lambda_n \qquad (n = n_1, n_2, n_3, \ldots),$$

то утверждение теоремы выполняется для функции $g_N(x)$. Поэтому можно считать, что

(36) $$\|L_n(g_N, x) - g_N(x)\| < \omega(d_n)\lambda_n, \quad \text{если} \quad n > M(N),$$

где N любое натуральное число и $M(N)$ зависящее лишь от N натуральное число.

Выберем строго возрастающую последовательность натуральных чисел n_i так, чтобы выполнялись условия

(37) $$\omega(d_{n_1}) \leqq \frac{1}{6}, \quad \omega(d_{n_{i+1}}) \leqq \frac{1}{6}\omega(d_{n_i}) \qquad (i = 1, 2\ 3, \ldots),$$

(38) $$\sum_{i=1}^{j-1} \omega(d_{n_i}) d_{n_i}^{-m} \leqq \omega(d_{n_j}) d_{n_j}^{-m} \qquad (j = 2, 3, 4, \ldots),$$

(39) $$n_{i+1} > M(n_i) \qquad (i = 1, 2, 3, \ldots), \qquad \lambda_{n_i} > 5 \qquad (i = 1, 2, 3, \ldots).$$

Это возможно, так как $d_n \leqq 2/n$ и поэтому $d_n \to 0$, если $n \to \infty$; мы предположили, что $\omega(t) \to 0$, если $t \to +0$; выполняется (1) и $\lim\limits_{n\to\infty} \lambda_n = \infty$.

Рассмотрим функцию

(40) $$f(x) = 3\sum_{i=1}^{\infty} \omega(d_{n_i}) g_{n_i}(x).$$

Ряд справа равномерно сходится ввиду (37) и (32).

Докажем, что $f(x) \in C_m(\omega)$. Пусть $0 < t \leqq \frac{2}{m}$. Очевидно,

(41) $$\omega_m(f, t) \leqq 3\sum_{i=1}^{\infty} \omega(d_{n_i}) \omega_m(g_{n_i}, t).$$

Обозначим через j тот индекс, для которого выполняется условие

(42) $$d_{n_{j+1}} < t \leqq d_{n_j}$$

(ввиду (37) и монотонности функции $\omega(t)$ последовательность d_{n_i} строго убывает; если $t>d_{n_1}$, то полагаем $j=0$). В силу (31) при $j>0$

(43) $$\sum_{i=1}^{j} \omega(d_{n_i})\omega_m(g_{n_i}, t) \leqq B_m t^m \sum_{i=1}^{j} \omega(d_{n_i}) d_{n_i}^{-m}.$$

Принимая во внимание (38), получаем отсюда:

(44) $$\sum_{i=1}^{j} \omega(d_{n_i})\omega_m(g_{n_i}, t) \leqq 2B_m t^m \omega(d_{n_j})\, d_{n_j}^{-m}.$$

Так как функция $t^m/\omega(t)$ не убывает, то из (42) и (44) следует:

(46) $$\sum_{i=1}^{j} \omega(d_{n_i})\omega_m(g_{n_i}, t) \leqq 2B_m \omega(t).$$

Кроме того, ввиду (32), (37), (42) и монотонности функции $\omega(t)$

(47) $$\sum_{i=j+1}^{\infty} \omega(d_{n_i})\omega_m(g_{n_i}, t) \leqq 2^m \sum_{i=j+1}^{\infty} \omega(d_{n_i}) \leqq 1{,}2\cdot 2^m \omega(d_{n_{j+1}}) \leqq 1{,}2\cdot 2^m \omega(t).$$

Ввиду (41), (46) и (47)

(48) $$\omega_m(f, t) \leqq 3(2B_m + 1{,}2\cdot 2^m)\omega(t)$$

и поэтому $f(x)\in C_m(\omega)$, что и требовалось доказать.

Нам остается доказать неравенство (8). Пусть $n=n_k$. Очевидно,

(49) $$L_n(f, z_n) - f(z_n) = 3\sum_{i=1}^{\infty} \omega(d_{n_i})[L_n(g_{n_i}, z_n) - g_{n_i}(z_n)].$$

Ввиду (39), (36) и (37) при $k>1$

(50) $$\sum_{i=1}^{k-1} \omega(d_{n_i})|L_n(g_{n_i}, z_n) - g_{n_i}(z_n)| \leqq \omega(d_n)\lambda_n \sum_{i=1}^{k-1} \omega(d_{n_i}) \leqq 0{,}2\omega(d_n)\lambda_n.$$

В силу (34)

(51) $$\omega(d_n) L_n(g_n, z_n) = \omega(d_n)\lambda_n.$$

Исходя из (33) и (37), получаем:

(52) $$\sum_{i=k+1}^{\infty} \omega(d_{n_i})|L_n(g_{n_i}, z_n)| \leqq \lambda_n \sum_{i=k+1}^{\infty} \omega(d_{n_i}) \leqq 0{,}2\lambda_n \omega(d_n).$$

Наконец, ввиду (32) и (37)

(53) $$\sum_{i=k}^{\infty} \omega(d_{n_i})|g_{n_i}(z_n)| \leqq 1{,}2\omega(d_n).$$

Резюмируя (49)—(53), получаем:

$$L_n(f, z_n) - f(z_n) \geqq \omega(d_n)(1{,}8\lambda_n - 3{,}6), \tag{54}$$

отсюда следует доказываемое неравенство (8), так как $\lambda_n > 5$.

Закончив доказательство теоремы 2, заметим, что ее тригонометрический аналог доказывается совершенно аналогичным образом.

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ON STRONG SUMMABILITY OF FOURIER SERIES. II

By
L. LEINDLER (Szeged)

Dedicated to G. ALEXITS *on his 70th birthday*

1. Let $f(x)$ be a continuous and periodic function with period 2π and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. Let $s_n(x) = s_n(f; x)$ denote the n-th partial sum of (1).

Concerning the strong summability, FREUD [1] recently proved that *if a function $f(x)$ has the property*

$$\left\{\frac{1}{n} \sum_{k=0}^{n} |s_k(x) - f(x)|^p\right\}^{1/p} \leqq \frac{K}{n^{1/p}} \tag{2}$$

for all x, where $p > 1$ and K is an absolute constant, then*

$$\lim_{h \to 0} h^{-1/p}(f(x+h) - f(x)) = 0 \tag{3}$$

holds for almost every x.

FREUD, in his paper, raised also the following problem: Does (3) hold for all x, if (2) is satisfied?

We ([2]) answered this problem negatively; that is, we can give a function such that the estimation (3) is not fulfilled in $x = 0$. Our counterexample is

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{1+1/p}}.$$

The purpose of this paper is to generalize these results.

We consider a regular summation method T_n determined by a triangular matrix $\|\alpha_{nk}\| \left(\alpha_{nk} = \frac{\lambda_k}{\Lambda_n},\ \lambda_k > 0;\ n = 0, 1, \ldots;\ k = 0, \ldots, n \text{ and } \Lambda_n = \sum_{k=0}^{n} \lambda_k\right)$, i.e. if s_k tends to s, then

$$T_n = \frac{1}{\Lambda_n} \sum_{k=0}^{n} \lambda_k s_k \to s.$$

Let $\Lambda(x)$ be an increasing function, between n and $n+1$ linear, such that $\Lambda(n) = \Lambda_n$.

* $K, K_1, K_2, \ldots$ will always denote positive constants not necessarily the same at each occurence.

THEOREM 1. *Let $p>1$. Suppose that the triangular matrix $\|\alpha_{nk}\|$ has the following properties: $n\Lambda_n^{-1/p}$ is increasing,*

$$\sum_{k=0}^{n} \Lambda_k^{-1/p} \leqq Kn\Lambda_n^{-1/p}, \tag{4}$$

$$\sum_{k=n}^{\infty} k^{-1}\Lambda_k^{-1/p} \leqq K\Lambda_n^{-1/p} \tag{5}$$

and

$$\left\{\sum_{k=n}^{2n} \lambda_k^{1/(1-p)}\right\}^{p-1} \leqq Kn^p\Lambda_n^{-1}. \tag{6}$$

Then, if

$$\left\{\frac{1}{\Lambda_n}\sum_{k=0}^{n} \lambda_k |s_k(x)-f(x)|^p\right\}^{1/p} \leqq K\Lambda_n^{-1/p} \tag{7}$$

for all x, then

$$f(x+h)-f(x) \leqq K_1\Lambda^{-1/p}\left(\frac{1}{h}\right) \tag{8}$$

for all x, furthermore for almost every x

$$\lim_{h\to 0} \Lambda^{1/p}\left(\frac{1}{h}\right)(f(x+h)-f(x)) = 0 \tag{9}$$

holds.

We can also prove that the estimates (8) and (9) can not be strengthened. This is stated by the following theorem.

THEOREM 2. *Suppose that the matrix $\|\alpha_{nk}\|$ has the same properties as in Theorem 1. Then there exists a function $\bar{f}(x)$ such that*

$$\left\{\frac{1}{\Lambda_n}\sum_{k=0}^{n} \lambda_k |s(\bar{f}; x)-\bar{f}(x)|^p\right\}^{1/p} \leqq K_2\Lambda_n^{-1/p} \tag{10}$$

for all x, but

$$\bar{f}\left(\frac{\pi}{2^n}\right)-\bar{f}(0) > \frac{1}{4}\Lambda^{-1/p}\left(\frac{2^n}{\pi}\right) \tag{11}$$

for all $n \geqq 6$.

It is easy to verify that, for example, in the cases

$$\lambda_k = k^{\beta-1}, \qquad 0<\beta<p,$$

$$\lambda_k = \frac{\log k}{k^\beta}, \qquad 0<\beta<1$$

the conditions (4), (5) and (6) are satisfied and the matrix $\|\alpha_{nk}\|$ is regular.

It is of some interest to remark that the strong (C, γ) means

$$\sigma_n(f, \gamma, p; x) = \left\{\frac{1}{A_n^{(\gamma)}}\sum_{\nu=0}^{n} A_{n-\nu}^{(\gamma-1)} |s_k(x)-f(x)|^p\right\}^{1/p} \quad \left(A_n^{(\gamma)} = \binom{n+\gamma}{n}\right)$$

do not belong to the means satisfying the conditions of Theorem 1, however *if* $1 \leqq \gamma < p$ *and*

$$\sigma_n(f, \gamma, p; x) \leqq Kn^{-\gamma/p} \tag{12}$$

for all x, then we have the statements

$$f(x+h)-f(x) \leqq K_1 h^{\gamma/p} \tag{13}$$

for all x, and

$$\lim_{h\to 0} h^{-\gamma/p}\big(f(x+h)-f(x)\big) = 0 \tag{14}$$

for almost every x.

This follows from (12) which implies

$$\sum_{k=0}^{\infty} k^{\gamma-1}|s_k(x)-f(x)|^p \leqq K_2$$

and if we set $\lambda_k = k^{\gamma-1}$ $(1 \leqq \gamma < p)$, then the conditions of Theorem 1 are satisfied, so by using this theorem we obtain the statements (13) and (14).

If $p=1$, then we can prove a similar theorem.

THEOREM 3. *Suppose that the triangular matrix* $\|\alpha_{nk}\|$ *has the following properties: the sequence* $\{\lambda_k\}$ *is positive and non-increasing,*

$$\sum_{k=0}^{n} \Lambda_k^{-1} \leqq Kn\Lambda_n^{-1}, \tag{15}$$

$$\sum_{k=n}^{\infty} k^{-1}\Lambda_k^{-1} \leqq K\Lambda_n^{-1}$$

and

$$\Lambda_n \leqq Kn\lambda_n.$$

Then, if

$$\frac{1}{\Lambda_n}\sum_{k=0}^{n} \lambda_k |s_k(x)-f(x)| \leqq \frac{K}{\Lambda_n}$$

for all x, thus

$$f(x+h)-f(x) \leqq K_1 \Lambda^{-1}\left(\frac{1}{h}\right)$$

for all x, and

$$\lim_{h\to 0} \Lambda\left(\frac{1}{h}\right)\big(f(x+h)-f(x)\big) = 0$$

for almost every x holds.

Furthermore there exists a function $f^*(x)$ having the properties:

$$\sum_{k=0}^{\infty} \lambda_k |s_k(f^*; x)-f^*(x)| \leqq K_2$$

for all x, but

$$f^*\left(\frac{\pi}{2^n}\right)-f^*(0) > \frac{1}{4}\Lambda^{-1}\left(\frac{2^n}{\pi}\right)$$

for all $n \geqq 6$.

The proof of Theorem 3 follows exactly the same lines as in Theorem 1 and 2. The counterexample is

$$f^*(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n\Lambda_n}.$$

It is easy to see that if $\lambda_k = 1$ $(k = 0, 1, 2, \ldots)$, then the condition (15) is not fulfilled. Thus, from Theorem 3, it does not follow that $f(x) \in \mathrm{Lip}\,1$, if

$$h_n(f; x) = \frac{1}{n+1} \sum_{k=0}^{n} |s_k(x) - f(x)| \leqq \frac{K}{n}.$$

The following example shows that the condition $h_n(f; x) = O\left(\frac{1}{n}\right)$ does not ensure that $f(x) \in \mathrm{Lip}\,1$, indeed. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^2}.$$

It is well-known that $f(x) \notin \mathrm{Lip}\,1$, but an easy computation gives that

$$\sum_{k=1}^{\infty} |s_k(x) - f(x)| \leqq K.$$

Namely, if $\frac{1}{N+1} \leqq |x| < \frac{1}{N}$, then

$$\left(\sum_{k=1}^{N} + \sum_{k=N+1}^{\infty}\right) \left| \sum_{n=k+1}^{\infty} \frac{\sin nx}{n^2} \right| \leqq$$

$$\leqq \sum_{k=1}^{N} \left| \sum_{n=k+1}^{N+1} \frac{\sin nx}{n^2} \right| + \sum_{k=1}^{N} \left| \sum_{n=N+2}^{\infty} \frac{\sin nx}{n^2} \right| + \sum_{k=N+1}^{\infty} \left| \sum_{n=k+1}^{\infty} \frac{\sin nx}{n^2} \right|$$

and these sums are less than an absolute constant.

2. Proof of Theorem 1. We set

$$V_n(x) = \frac{1}{n} \sum_{k=n+1}^{2n} s_k(x)$$

and

$$U_n(x) = V_{2^n}(x) - V_{2^{n-1}}(x) \qquad (n = 0, 1, 2, \ldots),$$

where $V_{2^{-1}}(x) \equiv 0$. Then we have

$$f(x) = \sum_{n=0}^{\infty} U_n(x). \tag{2.1}$$

Since

$$|U_n(x)| \leqq |V_{2^n}(x) - f(x)| + |f(x) - V_{2^{n-1}}(x)|$$

and, by (6) and (7),

$$\sum_{k=2^{n-1}+1}^{2^n} |f(x) - s_k(x)| \leqq$$

$$\leqq \left\{ \sum_{k=2^{n-1}+1}^{2^n} \lambda_k |f(x) - s_k(x)|^p \right\}^{1/p} \left\{ \sum_{k=2^{n-1}+1}^{2^n} \lambda_k^{1/(1-p)} \right\}^{\frac{p-1}{p}} \leqq K^2 2^n \Lambda_{2^n}^{-1/p},$$

so we get

$$|U_n(x)| \leqq K_2 \Lambda_{2^n}^{-1/p}. \tag{2.2}$$

Hence, by (2. 1) and (5), if $2^{-m} < h \leqq 2^{-m+1}$, we obtain

$$|f(x+h)-f(x)| \leqq \sum_{n=0}^{m-1} |U_n(x+h)-U_n(x)| + K_3 \sum_{n=m}^{\infty} \Lambda_{2^n}^{-1/p} \leqq \tag{2.3}$$

$$\leqq \sum_{n=0}^{m-1} h \max_x |U_n'(x)| + K_4 \Lambda_{2^m}^{-1/p}.$$

Using the well known Bernstein's inequality, we get by (4), (2. 2) and (2. 3)

$$|f(x+h)-f(x)| \leqq K_5 h \sum_{n=0}^{m-1} 2^n \Lambda_{2^n}^{-1/p} + K_4 \Lambda_{2^m}^{-1/p} \leqq K_6 \Lambda_{2^m}^{-1/p} \leqq K_6 \Lambda^{-1/p}\left(\frac{1}{h}\right).$$

This proves the inequality (8).

Now we prove the inequality (9). The proof is similar to that of FREUD. Let η be an arbitrary positive number. By Jegorov's theorem and (7) there exists a perfect set $\mathfrak{M}_\eta \subset [0, 2\pi]$ such that $\mu(\mathfrak{M}_\eta) > 2\pi - \eta$* and on the set $\mathfrak{M}_\eta$ the series

$$\sum_{k=0}^{\infty} \lambda_k |s_k(x) - f(x)|^p \tag{2.4}$$

converges uniformly. By a known theorem of LEBESGUE there is a subset $\mathfrak{M}_\eta^*$ of $\mathfrak{M}_\eta$ such that $\mu(\mathfrak{M}_\eta^*) = \mu(\mathfrak{M}_\eta)$ and the points of $\mathfrak{M}_\eta^*$ are of density 1. Let us consider an arbitrary fixed point x of $\mathfrak{M}_\eta^*$. Let $\varepsilon_1 > 0$ be an arbitrary fixed number. Let us choose a positive integer μ such that $\mu^{-1} < \varepsilon_1$. Let $\varepsilon_2 = 2^{-\mu}$. Since $x \in \mathfrak{M}_\eta^*$, there exists a positive δ such that if $0 < h \leqq \delta$, then

$$\mu([x-h, x] \cap \mathfrak{M}_\eta) > (1-\varepsilon_2)h$$

and

$$\mu([x, x+h] \cap \mathfrak{M}_\eta) > (1-\varepsilon_2)h.$$

Since the series (2. 4) converges uniformly on $\mathfrak{M}_\eta$, there exists an integer $N(\geqq 4)$ such that, for all $t \in \mathfrak{M}_\eta$,

$$\sum_{k=N+1}^{\infty} \lambda_k |s_k(t) - f(t)|^p \leqq \frac{\varepsilon_2^p}{K}.$$

Hence, by (6) we obtain that if $t \in \mathfrak{M}_\eta$, and $n > N$, then

$$|f(t) - V_n(t)| \leqq \frac{1}{n} \left\{ \sum_{k=n+1}^{2n} \lambda_k |s_k(t) - f(t)|^p \right\}^{1/p} \left\{ \sum_{k=n+1}^{2n} \lambda_k^{1/(1-p)} \right\}^{\frac{p-1}{p}} \leqq \varepsilon_2 \Lambda_n^{-1/p}. \tag{2.5}$$

Let us choose ξ such that $|x-\xi| < \min\left(\delta, \frac{\varepsilon_2}{N}\right)$. Let $\nu = \nu(\xi)$ be the smallest natural number with $\varepsilon_2 \leqq \nu |x-\xi| < 2\varepsilon_2$. It is clear that $\nu > N$. Since $|x-\xi| < \delta$, there is a point $\xi_1 \in \mathfrak{M}_\eta$ lying between x and ξ such that $|\xi - \xi_1| \leqq \varepsilon_2 |x-\xi|$.

* $\mu(H)$ denotes the Lebesgue measure of H.

From (8) and (2. 5) it follows that

$$|f(\xi)-f(x)| \leqq |f(\xi)-f(\xi_1)|+|f(\xi_1)-V_\nu(\xi_1)|+|V_\nu(\xi_1)-V_\nu(x)|+ \tag{2. 6}$$

$$+|V_\nu(x)-f(x)| \leqq K_1 \Lambda^{-1/p}\left(\frac{1}{|\xi-\xi_1|}\right)+2\varepsilon_2 \Lambda_\nu^{-1/p}+|\xi_1-x| \max_t |V_\nu'(t)|.$$

By (5), the first term in (2. 6) can be estimated as follows.

$$\Lambda^{-1/p}\left(\frac{1}{|\xi-\xi_1|}\right) \leqq \Lambda^{-1/p}\left(\frac{1}{\varepsilon_2|x-\xi|}\right) = \Lambda^{-1/p}\left(\frac{2^\mu}{|x-\xi|}\right) \leqq \frac{1}{\mu}\sum_{k=i_0+1}^{i_0+\mu} \Lambda_{2^k}^{-1/p} \leqq$$

$$\leqq \frac{1}{\mu}\Lambda^{-1/p}\left(\frac{1}{|x-\xi|}\right) \leqq \varepsilon_1 \Lambda^{-1/p}\left(\frac{1}{|x-\xi|}\right),$$

where $i_0 = \left[\overset{2}{\log}\frac{1}{|x-\xi|}\right]$ and $[\alpha]$ denotes the integer part of α.

In the estimate of the second term in (2. 6) we use the condition (4) and the monotonicity of the sequence $\{n\Lambda_n^{-1/p}\}$. If $2^m < \nu \leqq 2^{m+1}$, then

$$\varepsilon_2 \Lambda_\nu^{-1/p} \leqq \nu|x-\xi|\Lambda_\nu^{-1/p} \leqq |x-\xi|2^{m+1}\Lambda_{2^{m+1}}^{-1/p} \leqq |x-\xi|\frac{1}{\mu}\sum_{k=m+1}^{m+\mu} 2^k \Lambda_{2^k}^{-1/p} \leqq$$

$$\leqq \frac{K}{\mu}|x-\xi|2^{m+\mu}\Lambda_{2^{m+\mu}}^{-1/p} \leqq \frac{K}{\mu}|x-\xi|\nu 2^\mu \Lambda_{\nu 2^\mu}^{-1/p} \leqq$$

$$\leqq \frac{K}{\mu}|x-\xi|\frac{2}{|x-\xi|}\Lambda^{-1/p}\left(\frac{2}{|x-\xi|}\right) \leqq \frac{2K}{\mu}\Lambda^{-1/p}\left(\frac{1}{|x-\xi|}\right) \leqq 2K\varepsilon_1 \Lambda^{-1/p}\left(\frac{1}{|x-\xi|}\right).$$

In order to estimate the third term in (2. 6) we set

$$V_\nu'(t) = V_\nu'(t) - V_{2^m}'(t) + \sum_{k=0}^{m}\left(V_{2^k}'(t) - V_{2^{k-1}}'(t)\right).$$

Hence, using Bernstein's inequality, the condition (4) and the estimate (2. 2), we obtain

$$|V_\nu'(t)| \leqq K_7\left(2^m \Lambda_{2^m}^{-1/p} + \sum_{k=0}^{m} 2^k \Lambda_{2^k}^{-1/p}\right) \leqq K_8 \nu \Lambda_\nu^{-1/p}.$$

From this it follows, as before, that

$$|\xi_1 - x|\max_t |V_\nu'(t)| \leqq K_9|\xi_1-x|\nu\Lambda_\nu^{-1/p} \leqq K_9|x-\xi|\nu\Lambda_\nu^{-1/p} \leqq K_{10}\varepsilon_1\Lambda^{-1/p}\left(\frac{1}{|x-\xi|}\right).$$

Summing up, we obtain

$$|f(x)-f(\xi)| \leqq K_{12}\varepsilon_1 \Lambda^{-1/p}\left(\frac{1}{|\xi-x|}\right). \tag{2. 7}$$

Since ε_1 was arbitrary, so (2. 7) implies (9) for all $x \in \mathfrak{M}_\eta^*$. Let $\mathfrak{N}(f)$ denote the subset of $[0, 2\pi]$, where (9) is not fulfilled. It is obvious that $\mathfrak{N}(f) \subset [0, 2\pi] - \mathfrak{M}_\eta^*$, so the exterior measure of $\mathfrak{N}(f)$ is less than η. Since η was arbitrary, we have that the measure of $\mathfrak{N}(f)$ is zero, that is, the statement (9) is also proved.

3. PROOF OF THEOREM 2. Let

$$\bar{f}(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k\Lambda_k^{1/p}}.$$

First we show that this function has the property (10), that is, for all x

$$\sum_{k=0}^{\infty} \lambda_k |s_k(\bar{f}; x) - \bar{f}(x)|^p \leqq K. \tag{3.1}$$

If $x=0$, then (3. 1) is obvious. Since $\bar{f}(x)$ is an odd function it is enough to verify (3. 1) for positive x. Let x (>0) be an arbitrary fixed point. Let us choose N such that

$$\frac{\pi}{N+2} < x \leqq \frac{\pi}{N+1}$$

should be satisfied. Then we split the sum in (3. 1) into two parts:

$$\sum_{k=0}^{\infty} = \sum_{k=0}^{N} + \sum_{k=N+1}^{\infty}. \tag{3.2}$$

The first sum can easily be estimated:

$$\sum_{k=0}^{N} \lambda_k |s_k(\bar{f}; x) - \bar{f}(x)|^p = \sum_{k=0}^{N} \lambda_k \left| \sum_{n=k+1}^{\infty} \frac{\sin nx}{n\Lambda_n^{1/p}} \right|^p \leqq$$

$$\leqq 2^p \left(\sum_{k=0}^{N} \lambda_k \left| \sum_{n=k+1}^{N+1} \frac{\sin nx}{n\Lambda_n^{1/p}} \right|^p + \sum_{k=0}^{N} \lambda_k \left| \sum_{n=N+2}^{\infty} \frac{\sin nx}{n\Lambda_n^{1/p}} \right|^p \right) \equiv 2^p(\Sigma_1 + \Sigma_2).$$

Since, by (4),

$$\Sigma_1 \leqq \sum_{k=0}^{N} \lambda_k \left| \sum_{n=k+1}^{N+1} \frac{nx}{n\Lambda_n^{1/p}} \right|^p \leqq x^p \sum_{k=0}^{N} \lambda_k \left(\sum_{n=1}^{N+1} \Lambda_n^{-1/p} \right)^p \leqq$$

$$\leqq K_2 x^p \sum_{k=0}^{N} \lambda_k (N\Lambda_N^{-1/p})^p \leqq K_2 x^p N^p \leqq K_3$$

and

$$\Sigma_2 \leqq \sum_{k=0}^{N} \lambda_k \left| \sum_{n=N+2}^{\infty} \frac{1}{n\Lambda_n^{1/p}} \right|^p \leqq \sum_{k=0}^{N} \lambda_k K^p \Lambda_N^{-1} = K^p,$$

so the first sum in (3. 2) is uniformly bounded.

The estimation of the second sum in (3. 3) runs as follows:

$$\sum_{k=N+1}^{\infty} \lambda_k |s_k(\bar{f}; x) - \bar{f}(x)|^p = \sum_{k=N+1}^{\infty} \lambda_k \left| \sum_{n=k+1}^{\infty} \frac{\sin nx}{n\Lambda_n^{1/p}} \right|^p \leqq$$

$$\leqq K_4 \sum_{k=N+1}^{\infty} \lambda_k \left| \frac{1}{xk\Lambda_k^{1/p}} \right|^p = K_4 \sum_{k=N+1}^{\infty} \lambda_k \frac{1}{x^p k^p \Lambda_k} \leqq$$

$$\leqq \frac{K_4}{x^p} \sum_{i=0}^{\infty} \sum_{k=2^i N}^{2^{i+1}N} \lambda_k \frac{1}{(2^i N)^p \Lambda_{2^i N}} \leqq \frac{K_5}{x^p} \frac{1}{N^p} \sum_{i=0}^{\infty} \frac{1}{2^{ip}} \leqq K_6.$$

Collecting our results, we obtain that the function $\bar{f}(x)$ satisfies (10).

In order to prove (11) we set $h_m = \frac{\pi}{2^{m+4}}$ and $N_m = 2^{m+4}$. If $m \geqq 2$, then we have

$$(3.3)\qquad \bar{f}(h_m) = \sum_{n=1}^{\infty} \frac{\sin nh_m}{n\Lambda_n^{1/p}} = \left(\sum_{n=1}^{\frac{N_m}{4}} + \sum_{n=\frac{N_m}{4}+1}^{N_m} + \sum_{n=N_m+1}^{2N_m} + \sum_{k=2}^{\infty} \sum_{n=kN_m+1}^{(k+1)N_m} \right) \frac{\sin nh_m}{n\Lambda_n^{1/p}}.$$

It is clear that for $l \geqq 1$

$$\sum_{n=2lN_m+1}^{(2l+1)N_m} \frac{\sin nh_m}{n\Lambda_n^{1/p}} > \left| \sum_{n=(2l+1)N_m+1}^{(2l+2)N_m} \frac{\sin nh_m}{n\Lambda_n^{1/p}} \right|,$$

therefore the sum

$$\sum_{k=2}^{\infty} \sum_{n=kN_m+1}^{(k+1)N_m} \frac{\sin nh_m}{n\Lambda_n^{1/p}}$$

is positive. Furthermore an easy computation gives that

$$\sum_{n=\frac{N_m}{4}+1}^{N_m} \frac{\sin nh_m}{n\Lambda_n^{1/p}} > \left| \sum_{n=N_m+1}^{2N_m} \frac{\sin nh_m}{n\Lambda_n^{1/p}} \right|,$$

namely

$$\sum_{n=\frac{N_m}{2}+1}^{N_m} \frac{\sin nh_m}{n\Lambda_n^{1/p}} > \left| \sum_{n=N_m+1}^{\frac{3}{2}N_m-1} \frac{\sin nh_m}{n\Lambda_n^{1/p}} \right|$$

and

$$\sum_{n=\frac{N_m}{4}+1}^{\frac{N_m}{2}} \frac{\sin nh_m}{n\Lambda_n^{1/p}} \geqq \frac{\sqrt{2}}{2} \sum_{n=2^{m+2}+1}^{2^{m+3}} \frac{1}{n\Lambda_n^{1/p}} \geqq \frac{\sqrt{2}}{2} 2^{m+2} \frac{1}{2^{m+3}\Lambda_{2^{m+3}}^{1/p}} \geqq$$

$$\geqq \frac{1}{3}(2^{m+3}+1) \frac{1}{2^{m+3}\Lambda_{2^{m+3}}^{1/p}} \geqq \sum_{n=3\cdot 2^{m+3}}^{2^{m+5}} \frac{1}{3\cdot 2^{m+3}\Lambda_{2^{m+3}}^{1/p}} \geqq \sum_{n=3\cdot 2^{m+3}}^{2^{m+5}} \frac{1}{n\Lambda_n^{1/p}} \geqq$$

$$\geqq \left| \sum_{n=3\cdot 2^{m+3}}^{2^{m+5}} \frac{\sin nh_m}{n\Lambda_n^{1/p}} \right| \equiv \left| \sum_{n=\frac{3}{2}N_m}^{2N_m} \frac{\sin nh_m}{n\Lambda_n^{1/p}} \right|.$$

Combining these results and (3.3), we obtain that

$$\bar{f}(h_m) \geqq \sum_{n=1}^{\frac{N_m}{4}} \frac{\sin nh_m}{n\Lambda_n^{1/p}} \geqq \frac{2}{\pi} h_m \sum_{n=1}^{2^{m+2}} \frac{1}{\Lambda_n^{1/p}} > \frac{1}{2^{m+3}} \cdot \frac{2^{m+1}}{\Lambda_{2^{m+2}}^{1/p}} > \frac{1}{4} \Lambda^{-1/p}\left(\frac{1}{h_m}\right),$$

that is,

$$\bar{f}(h_m)-\bar{f}(0) > \frac{1}{4}\Lambda^{-1/p}\left(\frac{1}{h_m}\right),$$

which proves (11).

The proof is thus completed.

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A REMARK ON LINEAR DIFFERENTIAL EQUATIONS

By

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To G. Alexits *on his 70th birthday*

1. *General* inequalities in the theory of linear differential equations can very often be reduced to the case when the coefficient functions are *constants*. Actually the theory of the later ones contains — at least in the case of smooth coefficient functions — "in germ" the general theory. However only such inequalities can be useful which depend only loosely on the coefficients. Let

$$y^{(n)}+a_1y^{(n-1)}+\dots+a_ny=0,\quad y=y(z),\quad a_\nu \quad \text{constants} \tag{1. 1}$$

be our equation, further

$$\varphi(\omega)=\omega^n+a_1\omega^{n-1}+\dots+a_n=0 \tag{1. 2}$$

the corresponding characteristic equation; suppose that all the zeros $\omega_1, \omega_2, \dots, \omega_n$ of the equation (1. 2) are in the half-plane

$$\operatorname{Re} z \geqq A. \tag{1. 3}$$

Then we are going to prove that for fixed

$$\alpha>\beta,\quad \delta>0 \tag{1. 4}$$

the quotient

$$\frac{\int\limits_{\alpha}^{\alpha+\delta}|y(t)|^2\,dt}{\int\limits_{\beta}^{\beta+\frac{\delta}{2}}|y(t)|^2\,dt}$$

can be limited from below by A alone. More exactly we assert the following

Theorem. *With the conventions* (1. 3)—(1. 4) *the inequality*

$$\int\limits_{\alpha}^{\alpha+\delta}|y(t)|^2\,dt \geqq \left(\frac{\delta}{2e(2\alpha-2\beta+\delta)}\right)^{n^2} e^{-2(\alpha-\beta+\delta)|A|}\int\limits_{\beta}^{\beta+\frac{\delta}{2}}|y(t)|^2\,dt \tag{1. 5}$$

holds for all solutions of (1. 1).

Probably the exponent n^2 on the right side can be replaced by cn with a numerical positive constant c.

In the case when all zeros of (1. 2) are distinct and lie on the imaginary axis the theorem gives the inequality*

$$(1.6)\qquad \int_{\alpha}^{\alpha+\delta}\left|\sum_{\nu=1}^{n}c_\nu e^{i\lambda_\nu t}\right|^2 dt \geqq \left(\frac{\delta}{2e(2|\alpha-\beta|+\delta)}\right)^{n^2}\int_{\beta}^{\beta+\frac{\delta}{2}}\left|\sum_{\nu=1}^{n}c_\nu e^{i\lambda_\nu t}\right|^2 dt$$

for $0=\lambda_1<\lambda_2<\cdots<\lambda_n$, $\delta>0$. This is a trigonometrical inequality, totally independent of the coefficients as well as of the exponents and has interesting consequences in the theory of trigonometrical series, a subject enriched by many important contributions of George Alexits. The inequality (1. 6) can be compared with the well-known inequality of N. Wiener,** according which if $\lambda_{\nu+1}-\lambda_\nu\geqq\gamma>0$ $(\nu=1,2,\dots)$ then the inequality

$$\sum_{\nu=1}^{n}|c_\nu|^2 \leqq \frac{3(\pi+\varepsilon)}{2\varepsilon}\int_{-\frac{\pi+\varepsilon}{\gamma}}^{\frac{\pi+\varepsilon}{\gamma}}\left|\sum_{\nu=1}^{n}c_\nu e^{i\lambda_\nu t}\right|^2 dt$$

holds. To all consequences, partly of quasianalytic nature, I shall return at another occasion.

As Mr. G. Halász remarked a more elegant form of the inequality (1. 6) can be given, replacing β by $\beta+\delta/2$; then addition leads to the inequality

$$(1.7)\qquad \int_{\beta}^{\beta+\delta}\left|\sum_{\nu=1}^{n}c_\nu e^{i\lambda_\nu t}\right|^2 dt \leqq 2\left\{\left(2\frac{|\alpha-\beta|}{\delta}+\frac{3}{2}\right)2e\right\}^{n^2}\int_{\alpha}^{\alpha+\delta}\left|\sum_{\nu=1}^{n}c_\nu e^{i\lambda_\nu t}\right|^2 dt.$$

2. The proof of our theorem will easily follow from the theory developed in my book "Eine neue Methode in der Analysis und deren Anwendungen."*** The relevant theorem reads as follows.

If $z_1, z_2, \dots, z_N$ are complex numbers so that

$$(2.1)\qquad \min_{j=1,\dots,N}|z_j| \geqq 1$$

and m is an arbitrary given positive integer, then for arbitrarily given complex numbers b_j the inequality

$$(2.2)\qquad \max_{\nu=m+1,\dots,m+N}\left|\sum_{j=1}^{N}b_j z_j^\nu\right| \geqq \left(\frac{N}{2e(m+N)}\right)^N\left|\sum_{j=1}^{N}b_j\right|$$

holds.

* Here the restriction $\alpha>\beta$ is obviously unnecessary.

** A class of gap-theorems, *Annali di Pisa* (2) (1934), pp. 367—372. This sharper form is due to A. E. Ingham.

*** Akadémiai Kiadó, Budapest 1953. A completely rewritten English edition will follow in the Interscience Tracts.

If m_1 is an arbitrary positive number then applying (2. 1)—(2. 2) with $m=[m_1]$ we get at once

$$\max_{\substack{m_1 \leqq \nu \leqq m_1+N \\ \nu \text{ integer}}} \left| \sum_{j=1}^{N} b_j z_j^{\nu} \right| \geqq \left(\frac{N}{2e(m_1+N)} \right)^N \left| \sum_{j=1}^{N} b_j \right|. \tag{2. 3}$$

Next let $A>0$, $D>0$, $\alpha_1, \alpha_2, \ldots, \alpha_n$ arbitrary complex numbers with

$$\min_{\nu=1,2,\ldots,N} \operatorname{Re} \alpha_\nu = 0 \tag{2. 4}$$

and apply (2. 3) with

$$m_1 = \frac{AN}{D}, \quad z_j = e^{\alpha_j \frac{D}{N}}, \qquad j = 1, 2, \ldots, N.$$

This gives

$$\max_{\substack{A \leqq \frac{\nu D}{N} \leqq A+D \\ \nu \text{ integer}}} \left| \sum_{j=1}^{N} b_j e^{\alpha_j \frac{\nu D}{N}} \right| \geqq \left(\frac{D}{2e(A+D)} \right)^N \left| \sum_{j=1}^{N} b_j \right|,$$

i.e. a fortiori

$$\max_{A \leqq x \leqq A+D} \left| \sum_{j=1}^{N} b_j e^{\alpha_j x} \right| \geqq \left(\frac{D}{2e(A+D)} \right)^N \left| \sum_{j=1}^{N} b_j \right|, \tag{2.5}$$

if only (2. 4) holds.

3. Next let $a>0$, $d>0$

$$f(t) = \sum_{j=1}^{n} c_j e^{\beta_j t} \tag{3. 1}$$

with

$$\min_{j=1,\ldots,n} \operatorname{Re} \beta_j = 0. \tag{3. 2}$$

Let further

$$g(x) = \int_{x}^{x+\frac{d}{2}} |f(t)|^2 \, dt \tag{3. 3}$$

and suppose first

$$\min_{j=1,\ldots,n} \operatorname{Re} \beta_j > 0. \tag{3. 4}$$

Then (3. 3) has the form

$$g(x) = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} c_\mu \bar{c}_\nu \frac{e^{(\beta_\mu + \bar{\beta}_\nu)\frac{d}{2}} - 1}{\beta_\mu + \bar{\beta}_\nu} e^{(\beta_\mu + \bar{\beta}_\nu) x}$$

and we can apply (2. 5) with

$$A = a, \quad N = n^2, \quad D = \frac{d}{2}$$

as α_j's with the numbers $(\beta_n+\bar\beta_\nu)$, as coefficients b_j the numbers

$$c_\mu \bar c_\nu \frac{e^{(\beta_\mu+\bar\beta_\nu)\frac{d}{2}}-1}{\beta_\mu+\bar\beta_\nu}.$$

This gives that

$$\max_{a\leqq x\leqq a+\frac{d}{2}} |g(x)| \geqq \left(\frac{d}{2e(2a+d)}\right)^{n^2} |g(0)|.$$

The left side is evidently

$$\leqq \int_a^{a+d} |f(t)|^2\, dt.$$

Hence we get from (3. 1)—(3. 2)

$$\text{(3. 5)} \qquad \int_a^{a+d} |f(t)|^2\, dt \geqq \left(\frac{d}{2e(2a+d)}\right)^{n^2} \int_0^{\frac{d}{2}} |f(t)|^2\, dt.$$

If we have instead of (3. 2) only (1. 3), more exactly

$$\text{(3. 6)} \qquad \min_{j=1,\dots,n} \operatorname{Re}\beta_j = \Lambda$$

then applying (3. 5) to

$$f(t)e^{-\Lambda t}$$

the inequality (3. 5) takes the form

$$\text{(3. 7)} \qquad \int_a^{a+d} |f(t)|^2\, dt \geqq \left(\frac{d}{2e(2a+d)}\right)^{n^2} e^{-2(a+d)|\Lambda|} \int_0^{\frac{d}{2}} |f(t)|^2\, dt.$$

Replacing a by $\alpha-\beta$, d by δ and $f(t)$ by $f(t-\beta)$ this gives

$$\text{(3. 8)} \qquad \int_\alpha^{\alpha+\delta} |f(t)|^2\, dt \geqq \left(\frac{\delta}{2e(2\alpha-2\beta+\delta)}\right)^{n^2} e^{-2(\alpha-\beta+\delta)|\Lambda|} \int_\beta^{\beta+\frac{\delta}{2}} |f(t)|^2\, dt$$

if only $\alpha>\beta$, $\delta>0$.

Then it is easy to complete the proof of the theorem. Let first be the equation (1. 1) such that the equation (1. 2) has only simple zeros. Then each solution has the form $\sum_{\nu=1}^{n} c_\nu e^{\omega_\nu t}$ and thus the inequality (3. 8) is applicable. In the general case a trivial passage to limit completes the proof.

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ÜBER DIE APPROXIMATIONSTHEORETISCHE CHARAKTERISIERUNG GEWISSER FUNKTIONENKLASSEN MIT HILFE DER RIESZSCHEN MITTEL VON FOURIERREIHEN

Von

D. KRÁLIK (Budapest)

Professor G. ALEXITS *anläßlich seines 70. Geburtstages mit aufrichtiger Verehrung und Dankbarkeit gewidmet*

1. Einleitung

Ein Weg zur Untersuchung verschiedener Fragen der trigonometrischen Approximation beruht auf folgendem sehr einfachem Gedanken: Kennt man die Beschränktheitsverhältnisse gewisser Mittel der beliebigen Reihe Σa_n, so trachtet man auf die Annäherungsgeschwindigkeit der betreffenden Mittel der Reihe $\Sigma \lambda_n a_n$ zu schließen, wenn $\{\lambda_n\}$ eine entsprechend gewählte Zahlenfolge ist. Dieser Gedanke wurde von G. ALEXITS schon vor langem eingeführt und auf verschiedene Fragen der trigonometrischen Approximation angewendet (vgl. [1], [2], [3], [4]). Ebenfalls diese einfache reihentheoretische Methode führte uns zu einem sehr einfachen Beweis der sonst tiefliegenden Sätze von HARDY und LITTLEWOOD über die Integrale gebrochener Ordnung ([7]); später konnten wir mit derselben Methode den Begriff des Integrals gebrochener Ordnung erweiternd die erwähnten Hardy—Littlewoodschen Sätze verallgemeinern, ferner diese reihentheoretische Methode auch auf Approximationsfragen allgemeiner Orthogonalentwicklungen anwenden ([5]).

Trotz der Vorteile, welche die Anwendung dieser sehr einfachen reihentheoretischen Methode anbietet, konnte sie bisher nur auf die approximationstheoretische Charakterisierung von Stetigkeitsklassen angewandt werden, nicht aber auf Differenzierbarkeitsklassen, obwohl u.a. die Charakterisierung eben dieser Klassen immer ein Hauptziel der klassischen Approximationstheorie war. Die Schwierigkeit bestand darin, daß die nötigen allgemeinen reihentheoretischen Hilfsmittel nur für die Cesàroschen Mittel ausgearbeitet wurden, diese aber zu genügend rascher Annäherung nicht brauchbar sind.

Wir werden im folgenden die reihentheoretischen Hilfsmittel für die Rieszschen typischen Mittel ausarbeiten und erreichen dadurch die Charakterisierung der Klassen von beliebig oft differenzierbaren Funktionen, u. zw. in lokalisierter Form.

Wir gewinnen u. a. als einfache Folgerung unserer reihentheoretischen Rechnungen die approximationstheoretische Charakterisierung derjenigen Funktionenklasse, deren Funktionen in einem Teilintervall $[a, b]\subset[-\pi, \pi]$ eine zur

Klasse Lip $(1, p)$[1] mit $1 \leqq p \leqq +\infty$ gehörige $(r-1)$-te Derivierte besitzen ($r \geqq 1$, ganz).

Ebenso einfach erhalten wir die Charakterisierung zweier weiterer Funktionenklassen: die eine Klasse besteht aus allen denjenigen 2π-periodischen Funktionen, deren $(r-1)$-te Derivierte im ganzen Intervall $[-\pi, \pi]$ absolut stetig ist, die andere Klasse bilden die Funktionen, deren r-te Derivierte überall stetig ist.

2. Allgemeine Sätze über die Transformierten der Rieszschen Mittel unendlicher Reihen

Sei $\sum_{k=0}^{\infty} a_k$ eine beliebige Reihe, deren Glieder Elemente irgendeines Banachschen Raumes B mit der Norm $\| \circ \|$ sind. Betrachten wir die Rieszschen Mittel r-ter Ordnung

$$R_n^{(r)} = \sum_{k=0}^{n} \left(1 - \frac{k^r}{(n+1)^r}\right) a_k \qquad (n = 0, 1, 2, \dots;\ r \geqq 1) \tag{1}$$

der Reihe Σa_k, sowie die entsprechenden Mittel

$$R_n^{(r)}(\lambda) = \sum_{k=0}^{n} \left(1 - \frac{k^r}{(n+1)^r}\right) \lambda_k a_k \tag{2}$$

der transformierten Reihe $\Sigma \lambda_k a_k$. Wählen wir für $\{\lambda_k\}$ die Zahlenfolge $\{1/k^r\}$ und bezeichnen wir der Kürze halber die Mittel (2) für diesen Fall mit $\bar{R}_n^{(r)}$, so gilt der

SATZ 1. *Die Rieszschen Mittel* (1) *der Reihe* Σa_k *sind in der Norm von* B *genau dann beschränkt:*

$$\|R_n^{(r)}\| = O(1),$$

wenn die Mittel

$$\bar{R}_n^{(r)} = \sum_{k=1}^{n} \left(1 - \frac{k^r}{(n+1)^r}\right) \frac{a_k}{k^r}$$

in der Metrik von B *gegen ein Element* $s \in B$ *mit der Approximationsgeschwindigkeit*

$$\|\bar{R}_n^{(r)} - s\| = O\left(\frac{1}{n^r}\right)$$

konvergieren.

[1] Die 2π-periodische Funktion $f(x) \in L[-\pi, \pi]$ gehört zur Funktionenklasse Lip $(1, p)$ mit $1 \leqq p \leqq +\infty$, wenn für sie die Beziehung

$$\left\{ \int_{-\pi}^{\pi} |f(x+h) - f(x)|^p \, dx \right\}^{1/p} = \|f(x+h) - f(x)\|_{L^p[-\pi, \pi]} = O(h)$$

gültig ist, wo wir unter $L^\infty[-\pi, \pi]$ den Raum $C[-\pi, \pi]$ der stetigen Funktionen verstehen. Wir sagen, die Funktion $f(x) \in L[-\pi, \pi]$ gehöre im Teilintervall $[a, b] \subset [-\pi, \pi]$ zur Funktionenklasse Lip$(1, p)$, wenn für jedes, ganz im Inneren von (a, b) liegendes Subintervall $[a', b']$ die Beziehung

$$\|f(x+h) - f(x)\|_{L^p[a', b']} = O(h)$$

gilt, wo die Konstante im O im allgemeinen von a' und b' abhängt.

BEWEIS. Für $m > n$ haben wir:

$$\bar{R}_m^{(r)} - \bar{R}_n^{(r)} = \sum_{k=n+1}^{m} (\bar{R}_k^{(r)} - \bar{R}_{k-1}^{(r)}) =$$

$$= \sum_{k=n+1}^{m} \left\{ \sum_{\nu=1}^{k} \left(1 - \frac{\nu^r}{(k+1)^r}\right) \frac{a_\nu}{\nu^r} - \sum_{\nu=1}^{k-1} \left(1 - \frac{\nu^r}{k^r}\right) \frac{a_\nu}{\nu^r} \right\} = \sum_{k=n+1}^{m} \sum_{\nu=1}^{k} \left(\frac{1}{k^r} - \frac{1}{(k+1)^r}\right) a_\nu =$$

$$= \sum_{k=n+1}^{m} \frac{(k+1)^r - k^r}{k^r(k+1)^r} \sum_{\nu=1}^{k} a_\nu = \sum_{k=n+1}^{m} \frac{(k+1)^r - k^r}{k^r(k+1)^r} (s_k - a_0) =$$

$$= -a_0 \sum_{k=n+1}^{m} \frac{(k+1)^r - k^r}{k^r(k+1)^r} + \sum_{k=n+1}^{m} \frac{(k+1)^r - k^r}{k^r(k+1)^r} s_k,$$

wo

$$s_k = \sum_{\nu=0}^{k} a_\nu \quad \text{und} \quad \left\| -a_0 \sum_{k=n+1}^{m} \frac{(k+1)^r - k^r}{k^r(k+1)^r} \right\| = O\left(\frac{1}{n^r}\right)$$

ist. Durch Abel-Transformation erhalten wir für die letzte Summe

$$\sum_{k=n+1}^{m-1} \left[\frac{1}{k^r(k+1)^r} - \frac{1}{(k+1)^r(k+2)^r}\right] (k+1)^r \sum_{\nu=0}^{k} \frac{(\nu+1)^r - \nu^r}{(k+1)^r} s_\nu +$$

$$+ \frac{1}{m^r} \sum_{\nu=0}^{m} \frac{(\nu+1)^r - \nu^r}{(m+1)^r} s_\nu - \frac{1}{(n+2)^r} \sum_{\nu=0}^{n} \frac{(\nu+1)^r - \nu^r}{(n+1)^r} s_\nu.$$

Da

$$\sum_{\nu=0}^{k} \frac{(\nu+1)^r - \nu^r}{(k+1)^r} s_\nu = R_k^{(r)}$$

ist, haben wir endlich

$$\bar{R}_m^{(r)} - \bar{R}_n^{(r)} = \sum_{k=n+1}^{m-1} \frac{(k+2)^r - k^r}{k^r(k+2)^r} R_k^{(r)} + \frac{R_m^{(r)}}{m^r} - \frac{R_n^{(r)}}{(n+2)^r} - a_0 \sum_{k=n+1}^{m} \frac{(k+1)^r - k^r}{k^r(k+1)^r}.$$

Daraus erhalten wir auf Grund unserer Annahme $\|R_n^{(r)}\| = O(1)$ die Beziehung

$$\|\bar{R}_m^{(r)} - \bar{R}_n^{(r)}\| = O(1) \sum_{k=n+1}^{m-1} k^{-r-1} + O(m^{-r}) + O(n^{-r}) = O(n^{-r}),$$

womit wir die erste Hälfte des Satzes 1 bewiesen haben.

Um auch die andere Hälfte unseres Satzes zu beweisen, nehmen wir die Mittel

$$\bar{R}_{n,2}^{(r)} = \sum_{\nu=1}^{n} \left(1 - \frac{\nu^r}{(n+1)^r}\right)\left(1 - \frac{\nu^r}{(n+2)^r}\right) \frac{a_\nu}{\nu^r} \tag{3}$$

zu Hilfe. Nach leichter Rechnung ergibt sich

$$\bar{R}_n^{(r)} - \bar{R}_{n,2}^{(r)} = \frac{1}{(n+2)^r} \sum_{\nu=0}^{n} \left(1 - \frac{\nu^r}{(n+1)^r}\right) a_\nu - \frac{a_0}{(n+2)^r} = \frac{R_n^{(r)}}{(n+2)^r} - \frac{a_0}{(n+2)^r}. \tag{4}$$

Für die Mittel (3) erhalten wir durch Abel-Transformation, indem wir

$$\bar{s}_k = \sum_{\nu=1}^{k} \frac{a_\nu}{\nu^r} \quad \text{mit} \quad \bar{s}_0 = 0$$

schreiben:

$$\bar{R}_{n,2}^{(r)} = \sum_{k=0}^{n} \left[\frac{(k+1)^r - k^r}{(n+1)^r} + \frac{(k+1)^r - k^r}{(n+2)^r} - \frac{(k+1)^{2r} - k^{2r}}{(n+1)^r (n+2)^r} \right] \bar{s}_k =$$

$$= \sum_{k=0}^{n} \left[\frac{1}{(n+1)^r} + \frac{1}{(n+2)^r} - \frac{(k+1)^r + k^r}{(n+1)^r (n+2)^r} \right] [(k+1)^r - k^r] \bar{s}_k =$$

$$= \sum_{k=0}^{n-1} (b_k - b_{k+1})(k+1)^r \sum_{\nu=0}^{k} \frac{(\nu+1)^r - \nu^r}{(k+1)^r} \bar{s}_\nu + (n+1)^r b_n \sum_{\nu=0}^{n} \frac{(\nu+1)^r - \nu^r}{(n+1)^r} \bar{s}_\nu,$$

wo wir zur Abkürzung

$$b_k = \frac{1}{(n+1)^r} + \frac{1}{(n+2)^r} - \frac{(k+1)^r + k^r}{(n+1)^r (n+2)^r}$$

geschrieben haben. Die Summen

$$\sum_{\nu=0}^{k} \frac{(\nu+1)^r - \nu^r}{(k+1)^r} \bar{s}_\nu$$

sind die Rieszschen Mittel $\bar{R}_k^{(r)}$ der Reihe $\Sigma a_\nu / \nu^r$, so daß wir endlich für $\bar{R}_{n,2}^{(r)}$ den Ausdruck erhalten:

$$\bar{R}_{n,2}^{(r)} = \frac{1}{(n+1)^r (n+2)^r} \sum_{k=0}^{n} [(k+2)^r - k^r](k+1)^r \bar{R}_k^{(r)}.$$

Es ist nun

$$\frac{1}{(n+1)^r (n+2)^r} \sum_{k=0}^{n} [(k+2)^r - k^r](k+1)^r = 1,$$

so daß wir für ein beliebiges Element $s \in B$ die Beziehung gewinnen:

$$\bar{R}_{n,2}^{(r)} - s = \frac{1}{(n+1)^r (n+2)^r} \sum_{k=0}^{n} [(k+2)^r - k^r](k+1)^r [\bar{R}_k^{(r)} - s]. \tag{5}$$

Gilt nun die Relation

$$\|\bar{R}_k^{(r)} - s\| = O\left(\frac{1}{k^r}\right),$$

so ergibt sich nach (5) für die $\bar{R}_{n,2}^{(r)}$-Mittel die Approximationsgrößenordnung

$$\|\bar{R}_{n,2}^{(r)} - s\| = O(n^{-2r}) \sum_{k=1}^{n} k^{r-1} = O\left(\frac{1}{n^r}\right).$$

Der weitere Verlauf unseres Beweises ist nun sehr einfach. Nach unseren Erörterungen haben wir nämlich

$$\|\bar{R}_n^{(r)}-\bar{R}_{n,2}^{(r)}\| \leqq \|\bar{R}_n^{(r)}-s\|+\|s-\bar{R}_{n,2}^{(r)}\| = O\left(\frac{1}{n^r}\right),$$

so daß wir nach der Relation (4) die erwünschte Beziehung

$$\|R_n^{(r)}\| = O(1)$$

erhalten. Damit haben wir unseren Satz 1 vollständig bewiesen.

Der Spezialfall $r=1$ ist der schon unter [1] und [3] zitierte, auf die $(C, 1)$-Mittel bezogene reihentheoretische Satz von ALEXITS.

Wir wollen nun von der Formel (2) ausgehend einen Ausdruck für die transformierten $R_n^{(r)}(\lambda)$-Mittel ableiten. Sind $\{\lambda_n\}$ und $\{\mu_n\}$ beliebige Zahlenfolgen, so haben wir für die Differenzen der Produktfolge $\{\mu_n\lambda_n\}$:

$$\Delta\{\mu_n\lambda_n\} = \Delta\mu_n\lambda_n+\mu_{n+1}\Delta\lambda_n.$$

Wählen wir in der Formel (2) $\mu_k = 1-\frac{k^r}{(n+1)^r}$, so erhalten wir aus (2) nach einer Abel-Transformation:

$$R_n^{(r)}(\lambda) = \sum_{k=0}^{n}\left(1-\frac{k^r}{(n+1)^r}\right)\lambda_k a_k = \sum_{k=0}^{n}\frac{(k+1)^r-k^r}{(n+1)^r}\lambda_k s_k+$$

$$+\sum_{k=0}^{n-1}\left(1-\frac{(k+1)^r}{(n+1)^r}\right)\Delta\lambda_k s_k = \mathrm{I}+\mathrm{II}$$

mit

$$s_k = \sum_{\nu=0}^{k} a_\nu.$$

Durch Abel-Transformation formen wir I bzw. II, indem wir

$$\sum_{\nu=0}^{k}\frac{(\nu+1)^r-\nu^r}{(k+1)^r}s_\nu = R_k^{(r)}$$

schreiben, folgenderweise um:

$$\mathrm{I} = \frac{1}{(n+1)^r}\sum_{k=0}^{n-1}\Delta\lambda_k(k+1)^r R_k^{(r)}+\lambda_n R_n^{(r)},$$

bzw.

$$\mathrm{II} = \sum_{k=0}^{n}\left(1-\frac{(k+1)^r}{(n+1)^r}\right)\Delta\lambda_k\frac{(k+1)^r-k^r}{(k+1)^r-k^r}s_k = \sum_{k=0}^{n-1}\Delta\left\{\frac{1-\frac{(k+1)^r}{(n+1)^r}}{(k+1)^r-k^r}\Delta\lambda_k\right\}(k+1)^r R_k^{(r)}.$$

Nach dem Mittelwertsatz der Differentialrechnung ist

$$(k+1)^r-k^r = r(k+\vartheta_k)^{r-1} \qquad (0<\vartheta_k<1)$$

so daß sich endlich für $R_n^{(r)}(\lambda)$ durch wiederholte Abel-Transformation der folgende Ausdruck ergibt:

$$R_n^{(r)}(\lambda) = \frac{1}{r}\sum_{k=0}^{n-1}\left(1-\frac{(k+1)^r}{(n+1)^r}\right)\frac{(k+1)^r}{(k+\vartheta_k)^{r-1}}\Delta^2\lambda_k R_k^{(r)} + \tag{6}$$

$$+\frac{1}{(n+1)^r}\sum_{k=0}^{n-1}(k+1)^r\Delta\lambda_k R_k^{(r)}+$$

$$+\frac{1}{r}\sum_{k=0}^{n-1}(k+1)^r\left[\frac{1-\dfrac{(k+1)^r}{(n+1)^r}}{(k+\vartheta_k)^{r-1}}-\frac{1-\dfrac{(k+2)^r}{(n+1)^r}}{(k+1+\vartheta_{k+1})^{r-1}}\right]\Delta\lambda_{k+1}R_k^{(r)}+\lambda_n R_n^{(r)}.$$

Die Formel (6) wollen wir nun zum Beweis eines weiteren reihentheoretischen Satzes verwenden. Zu diesem Behufe betrachten wir eine positive Zahlenfolge $\{\eta_n\}$ mit den Eigenschaften:

$$\eta_n \geqq \eta_{n+1} \quad \text{für} \quad n=0,1,2,\ldots \quad \text{und} \quad n^r\eta_n \uparrow +\infty. \tag{7}$$

Es sei weiterhin $\{\lambda_n\}$ eine beliebige *konvexe* Nullfolge mit positiven Gliedern. Es gilt nun der

SATZ 2. *Ist* $\|R_n^{(r)}\| = O(\eta_n)$ *für die Mittel* (1), *so gilt für die transformierten Mittel* (2) *die Relation*

$$\|R_m^{(r)}(\lambda)-R_n^{(r)}(\lambda)\| = o(\eta_n).$$

BEWEIS. Aus der Formel (6) erhalten wir für $m>n$:

$$R_m^{(r)}(\lambda)-R_n^{(r)}(\lambda) = \underbrace{\frac{1}{r}\sum_{k=0}^{n-1}\left\{\frac{1}{(n+1)^r}-\frac{1}{(m+1)^r}\right\}\frac{(k+1)^{2r}}{(k+\vartheta_k)^{r-1}}\Delta^2\lambda_k R_k^{(r)}+}_{=S_1}$$

$$\underbrace{+\frac{1}{r}\sum_{k=n}^{m-1}\left(1-\frac{(k+1)^r}{(m+1)^r}\right)\frac{(k+1)^r}{(k+\vartheta_k)^{r-1}}\Delta^2\lambda_k R_k^{(r)}+}_{=S_2}$$

$$\underbrace{+\sum_{k=0}^{n-1}\left\{\frac{1}{(m+1)^r}-\frac{1}{(n+1)^r}\right\}(k+1)^r\Delta\lambda_k R_k^{(r)}}_{=S_3}+\underbrace{\frac{1}{(m+1)^r}\sum_{k=n}^{m-1}(k+1)^r\Delta\lambda_k R_k^{(r)}+}_{=S_4}$$

$$\underbrace{+\frac{1}{r}\sum_{k=0}^{n-1}(k+1)^r\Delta\lambda_{k+1}R_k^{(r)}\left[\frac{1}{(n+1)^r}-\frac{1}{(m+1)^r}\right]\left[\frac{(k+1)^r}{(k+\vartheta_k)^{r-1}}-\frac{(k+2)^r}{(k+1+\vartheta_{k+1})^{r-1}}\right]+}_{=S_5}$$

$$\left.\begin{aligned}&+\frac{1}{r}\sum_{k=n}^{m-1}(k+1)^r\Delta\lambda_{k+1}R_k^{(r)}\left[\frac{1}{(k+\vartheta_k)^{r-1}}-\frac{1}{(k+1+\vartheta_{k+1})^{r-1}}+\right.\\&\left.+\frac{1}{(m+1)^r}\left(\frac{(k+2)^r}{(k+1+\vartheta_{k+1})^{r-1}}-\frac{(k+1)^r}{(k+\vartheta_k)^{r-1}}\right)\right]+\end{aligned}\right\}=S_6$$

$$+\underbrace{\lambda_m R_m^{(r)}-\lambda_n R_n^{(r)}}_{=S_7} = S_1+S_2+S_3+S_4+S_5+S_6+S_7.$$

Um die Größenordnung von $\|R_m^{(r)}(\lambda)-R_n^{(r)}(\lambda)\|$ zu berechnen, werden wir die Normen $\|S_i\|$ $(i=1, 2, \ldots, 7)$ einzeln abschätzen.

Wir bemerken zuerst, daß für unsere Folge $\{\lambda_n\}$, da sie eine positive konvexe Nullfolge ist, alle Differenzen $\Delta\lambda_n$ und $\Delta^2\lambda_n$ nicht-negativ sind, weiterhin die Konvergenzbeziehungen

$$(8) \qquad \sum_{k=0}^{\infty} \Delta\lambda_k < +\infty \quad \text{und} \quad \sum_{k=0}^{\infty} (k+1)\Delta^2\lambda_k < +\infty$$

bestehen. Für die Normen $\|S_i\|$ erhalten wir nun der Reihe nach:

$$\|S_1\| = O(n^{-r}) \sum_{k=0}^{n-1} k^{r+1}\Delta^2\lambda_k \|R_k^{(r)}\| = O(n^{-r}) \sum_{k=0}^{n-1} k^r \eta_k (k+1)\Delta^2\lambda_k.$$

Unsere letzte Summe ist nach einem bekannten Satz von KRONECKER wegen (7) und (8) von der Größenordnung $o(n^r\eta_n)$, so daß für $\|S_1\|$ gilt:

$$\|S_1\| = o(\eta_n).$$

Aus demselben Grund haben wir weiter:

$$\|S_3\| = o(\eta_n), \quad \text{sowie} \quad \|S_5\| = o(\eta_n).$$

Zur Abschätzung der Summen mit geraden Indizes ergibt sich zunächst

$$\|S_2\| = O(1) \sum_{k=n}^{m-1} (k+1)\Delta^2\lambda_k \eta_k = O(\eta_n) \sum_{k=n}^{m-1} (k+1)\Delta^2\lambda_k = o(\eta_n)$$

wegen (7) und (8). Aus demselben Grund erhalten wir

$$\|S_4\| = O(1) \sum_{k=n}^{m-1} \Delta\lambda_k \eta_k = O(\eta_n) \sum_{k=n}^{m-1} \Delta\lambda_k = o(\eta_n).$$

Bei S_6 haben wir zuerst den Klammerausdruck

$$\left\{\frac{1}{(k+\vartheta_k)^{r-1}} - \frac{1}{(k+1+\vartheta_{k+1})^{r-1}}\right\} + \frac{1}{(m+1)^r}\left\{\frac{(k+2)^r}{(k+1+\vartheta_{k+1})^{r-1}} - \frac{(k+1)^r}{(k+\vartheta_k)^{r-1}}\right\} = \mathrm{I} + \mathrm{II}$$

abzuschätzen. Nach dem Mittelwertsatz der Differentialrechnung erhalten wir für I:

$$|\mathrm{I}| \leqq 2(r-1)(k+\tau_k)^{-r} \quad \text{mit} \quad 0 < \tau_k < 2,$$

also es ist $|\mathrm{I}| = O(k^{-r})$. Für II erhalten wir auf Grund unserer vorigen Abschätzung die Größenordnung

$$\mathrm{II} = O(m^{-r}) = O(k^{-r}) \quad \text{für} \quad k \leqq m.$$

Insgesamt ergibt sich also wieder nach (7) und (8) für S_6 die Größenordnung:

$$\|S_6\| = O(1) \sum_{k=n}^{m-1} \Delta\lambda_{k+1}\eta_k = O(\eta_n) \sum_{k=n}^{m-1} \Delta\lambda_{k+1} = o(\eta_n).$$

Endlich gilt für S_7:

$$\|S_7\| = o(\eta_n),$$

da $\{\lambda_n\}$ eine Nullfolge ist. Damit haben wir unseren Satz 2 in allen Einzelheiten bewiesen.

Für den Spezialfall $r=1$ und $\eta_n=1$ $(n=0, 1, 2, \ldots)$ ergibt sich aus Satz 2 der schon zitierte Satz von ALEXITS ([2], 2. 2).

SATZ 3. *Ist für die Mittel* (1) $\|R_n^{(r)}\|=o(1)$, *so existiert eine geeignete, von unten konkave, positive Zahlenfolge* $\lambda_n \uparrow +\infty$, *für welche die Beziehung*

$$\|R_m^{(r)}(\lambda)-R_n^{(r)}(\lambda)\| = o(1)$$

besteht.

BEWEIS. Es sei nun $\mu_n \uparrow +\infty$ eine positive, von unten konkave Zahlenfolge mit den folgenden Eigenschaften:

$$1^\circ \quad \|R_n^{(r)}\| \leqq \frac{1}{\mu_n^2}, \qquad 2^\circ \quad \Delta\mu_n = O\left(\frac{1}{n}\right) \quad \text{und} \quad 3^\circ \quad \Delta^2\mu_n = O\left(\frac{\Delta\mu_n}{n}\right).$$

Es ist leicht ersichtlich, daß diese Bedingungen erfüllbar sind (vgl. z. B. [2], 2. 3). Betrachten wir zuerst die mit der Folge $\{\mu_n\}$ transformierten Mittel $R_n^{(r)}(\mu)$. Aus (6) folgt

$$\|R_n^{(r)}(\mu)\| \leqq \frac{1}{r}\sum_{k=0}^{n-1}\left(1-\frac{(k+1)^r}{(n+1)^r}\right)\frac{(k+1)^r}{(k+\vartheta_k)^{r-1}}|\Delta^2\mu_k|\,\|R_k^{(r)}\| +$$

$$+\frac{1}{(n+1)^r}\sum_{k=0}^{n-1}(k+1)^r|\Delta\mu_k|\,\|R_k^{(r)}\|+$$

$$+\frac{1}{r}\sum_{k=0}^{n-1}(k+1)^r\left[\frac{1-\dfrac{(k+1)^r}{(n+1)^r}}{(k+\vartheta_k)^{r-1}}-\frac{1-\dfrac{(k+2)^r}{(n+1)^r}}{(k+1+\vartheta_{k+1})^{r-1}}\right]|\Delta\mu_{k+1}|\,\|R_k^{(r)}\|+\mu_n\|R_n^{(r)}\| =$$

$$= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.$$

Auf Grund von 1° und 3° ergibt sich für I die Abschätzung:

$$\mathrm{I} = O(1)\sum_{k=1}^{n-1} k\frac{|\Delta\mu_k|}{k}\frac{1}{\mu_k^2} = O(1)\sum_{k=1}^{n}\frac{|\Delta\mu_k|}{\mu_{k-1}\mu_k} =$$

$$= O(1)\sum_{k=1}^{\infty}\left(\frac{1}{\mu_{k-1}}-\frac{1}{\mu_k}\right) = O\left(\frac{1}{\mu_0}\right) = O(1).$$

Für II erhalten wir nach 2°, da $\{1/\mu_k^2\}$ eine Nullfolge ist:

$$\mathrm{II} = O\left(\frac{1}{n}\right)\sum_{k=1}^{n}\frac{(k+1)^{r-1}}{(n+1)^{r-1}}\frac{1}{\mu_k^2} = O(1)\sum_{k=0}^{n}\frac{1/\mu_k^2}{n+1} = o(1).$$

Bei III ergibt sich für den Klammerausdruck nach leichter Rechnung die Größenordnung $O(k^{-r})$, da $k\leqq n$ ist, so daß für III gilt:

$$\mathrm{III} = O(1)\sum_{k=1}^{n}\frac{|\Delta\mu_k|}{\mu_k^2} = O(1)\sum_{k=1}^{\infty}\left(\frac{1}{\mu_{k-1}}-\frac{1}{\mu_k}\right) = O(1).$$

Da für IV die Größenordnung $o(1)$ unmittelbar ersichtlich ist, gilt für $R_n^{(r)}(\mu)$ die Relation

$$\|R_n^{(r)}(\mu)\| = O(1).$$

Bezeichne nun $\{\bar{\lambda}_n\}$ eine solche konvexe positive Nullfolge, für die $\{\bar{\lambda}_n \mu_n\}$ konkav und $\bar{\lambda}_n \mu_n \uparrow +\infty$ ist. Es sei endlich

$$\lambda_n = \bar{\lambda}_n \mu_n.$$

Wählen wir im Satz 2 statt der Reihe Σa_n die Reihe $\Sigma \mu_n a_n$ und die Größen η_n alle gleich 1, so erhalten wir nach Satz 2 für die Mittel $R_n^{(r)}(\lambda)$ der transformierten Reihe $\Sigma \bar{\lambda}_n \mu_n a_n = \Sigma \lambda_n a_n$ die Beziehung

$$\|R_m^{(r)}(\lambda) - R_n^{(r)}(\lambda)\| = o(1),$$

womit wir auch unseren Satz 3 bewiesen haben.

3. Anwendungen der Sätze 1—3 auf die Charakterisierung von Funktionenklassen

Sei $f(x) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) = \sum_{\nu=0}^{\infty} A_\nu(x)$ die Fourierreihe der 2π-periodischen Funktion $f(x)$, $\sum_{\nu=1}^{\infty} (a_\nu \sin \nu x - b_\nu \cos \nu x) = \sum_{\nu=1}^{\infty} B_\nu(x)$ die konjugierte Reihe. Aus dem Satz 1 ergibt sich unmittelbar als einfaches Korollar eine approximationstheoretische Charakterisierung der Klasse aller derjenigen 2π-periodischen Funktionen $f(x)$, deren $(r-1)$-te Derivierte ($r \geqq 1$, ganz) überall existiert und zur Funktionenklasse Lip $(1, p)$ mit $1 \leqq p \leqq +\infty$ gehört. Hierbei bemerken wir, daß unter Lip $(1, \infty)$ die gewöhnliche Lip 1 Klasse, während unter Lip $(1, 1)$ die Klasse der Funktionen endlicher Variation zu verstehen ist.

KOROLLAR. *Es ist* $f^{(r-1)}(x) \in \text{Lip}\,(1, p)$ $(1 \leqq p \leqq +\infty)$ *genau dann, wenn für gerade* r *die Beziehung*

$$\|R_n^{(r)}(f; x) - f(x)\|_{L^p[-\pi,\pi]} = O\left(\frac{1}{n^r}\right), \tag{9}$$

für ungerade r *die Beziehung*

$$\|\tilde{R}_n^{(r)}(f; x) - \tilde{f}(x)\|_{L^p[-\pi,\pi]} = O\left(\frac{1}{n^r}\right) \tag{10}$$

besteht, wo $R_n^{(r)}(f; x)$ *bzw.* $\tilde{R}_n^{(r)}(f; x)$ *die Rieszschen Mittel bzw. die konjugierten Rieszschen Mittel der Fourierreihe von* $f(x)$ *bedeuten.*

Ist nämlich r eine gerade Zahl und $f^{(r-1)}(x) \in \text{Lip}\,(1, p)$, so sind die Rieszschen Mittel der r-mal differenzierten Fourierreihe von $f(x)$, d.h. der Reihe $\pm \sum_{\nu=1}^{\infty} \nu^r A_\nu(x)$ in der Norm des Raumes $L^p[-\pi, \pi]$ beschränkt.[2] Nach Satz 1 konvergieren daher

[2] Vgl. hierzu [9], Volume I, 136—138, sowie 144—145. Die zitierten Beziehungen zwischen den $(C, 1)$-Mitteln der Fourierreihe einer Funktion $f(x)$ sowie der Zugehörigkeit von $f(x)$ zu irgendeiner Funktionenklasse $L^p[-\pi, \pi]$ bleiben bestehen, wenn wir statt der $(C, 1)$-Mittel die entsprechenden $R_n^{(r)}(f; x)$-Mittel der Fourierreihe von $f(x)$ betrachten.

die n-ten Rieszschen Mittel der Reihe $\Sigma A_\nu(x)$ in der L^p-Metrik mit der Approximationsgeschwindigkeit $O(n^{-r})$ zu $f(x)$.

Gilt umgekehrt (9), so folgt wieder aus dem Satz 1, daß die Rieszschen Mittel der Reihe $\Sigma \nu^r A_\nu(x)$ in der L^p-Metrik beschränkt sind, woraus sich bekanntlich $f^{(r-1)}(x) \in \mathrm{Lip}\,(1, p)$ folgt. Im Fall, wenn r ungerade ist, verfahren wir wörtlich, wie vorher, nur haben wir überall die entsprechenden konjugierten Größen zu betrachten.

Wir bemerken, daß dieses Resultat schon früher von M. ZAMANSKY ([8]) gefunden worden ist, jedoch auf eine ganz andere Weise. (Vgl. diesbezüglich auch [6].)

Ist $r=1$, so ist unser Korollar identisch mit dem unter [3] zitierten Ergebnis von ALEXITS über die Charakterisierung der Funktionenklasse Lip $(1, p)$.

Wir wollen nun unser im Korollar enthaltenes Ergebnis auch in lokalisierter Form aussprechen.

SATZ 4. *Die 2π-periodische Funktion $f(x) \in L[-\pi, \pi]$ habe in $[-\pi, \pi]$ eine absolut stetige $(r-1)$-te Derivierte $f^{(r-1)}(x)$. Diese Derivierte $f^{(r-1)}(x)$ gehört im Teilintervall $[a, b] \subset [-\pi, \pi]$ zur Funktionenklasse* Lip $(1, p)$ $(1 \leqq p \leqq +\infty)$ *genau dann, wenn für jedes, ganz im Inneren von (a, b) liegendes Subintervall $[a', b']$ bzw. die folgenden Relationen gelten:*

$$\|R_n^{(r)}(f; x) - f(x)\|_{L^p[a', b']} = O\left(\frac{1}{n^r}\right), \tag{11}$$

wenn r eine gerade Zahl ist, bzw.

$$\|\tilde{R}_n^{(r)}(f; x) - \tilde{f}(x)\|_{L^p[a', b']} = O\left(\frac{1}{n^r}\right) \tag{12}$$

wenn r eine ungerade Zahl ist. Dabei hängt die Konstante im O im allgemeinen von a' und b' ab.

BEWEIS. Betrachten wir zuerst die Hilfsfunktion $g(x)$, die in dem Teilintervall $[a'', b'']$ mit $a < a'' < a'$ und $b' < b'' < b$ mit der Funktion $f(x)$ identisch ist und in den Intervallen $[-\pi, a'']$ und $[b'', \pi]$ so glatt geformt ist, daß $g^{(r-1)}(x)$ im ganzen Intervall $[-\pi, \pi]$ zur Funktionenklasse Lip $(1, p)$ gehört. Infolgedessen gilt die Beziehung

$$\|R_n^{(r)}(g^{(r)}; x)\|_{L^p[-\pi, \pi]} = O(1),$$

umsomehr gilt also

$$\|R_n^{(r)}(g^{(r)}; x)\|_{L^p[a', b']} = O(1).$$

Da im Intervall $[a'', b'']$ $f^{(r)}(x) - g^{(r)}(x) = 0$ ist, konvergiert die Differenz $R_n^{(r)}(f^{(r)}; x) - R_n^{(r)}(g^{(r)}; x)$ im Subintervall $[a', b']$ gleichmäßig gegen Null, so daß auch

$$\|R_n^{(r)}(f^{(r)}; x) - R_n^{(r)}(g^{(r)}; x)\|_{L^p[a', b']} = o(1)$$

gilt. Wir haben also die Beziehung

$$\|R_n^{(r)}(f^{(r)}; x)\|_{L^p[a', b']} = O(1),$$

woraus sich auf Grund des Satzes 1 die Approximationsbeziehung (11), bzw. (12) ergibt.

Nehmen wir umgekehrt an, daß für gerade r (11) gilt. Nach Satz 1 gilt dann auch die Beziehung

$$\|R_n^{(r)}(f^{(r)}; x)\|_{L^p[a',b']} = O(1).$$

Wir haben also auf Grund der Minkowskischen Ungleichung:

$$\|R_n^{(r)}(f^{(r-1)}; x+h) - R_n^{(r)}(f^{(r-1)}; x)\|_{L^p[a',b']} \leqq \left\{\int_{a'}^{b'} \left|\int_0^h R_n^{(r)}(f^{(r)}; x+t)\,dt\right|^p dx\right\}^{1/p} \leqq$$

$$\leqq \int_0^h \left\{\int_{a'}^{b'} |R_n^{(r)}(f^{(r)}; x+t)|^p\,dx\right\}^{1/p} dt = \int_0^h \|R_n^{(r)}(f^{(r)}; x+t)\|_{L^p[a',b']}\,dt = O(h),$$

wenn nur h genügend klein ist. Die Rieszschen Mittel $R_n^{(r)}(f^{(r-1)}; x)$ konvergieren aber in $[a', b']$ gleichmäßig zu $f^{(r-1)}(x)$, daher erhalten wir endlich die Beziehung

$$\|f^{(r-1)}(x+h) - f^{(r-1)}(x)\|_{L^p[a',b']} = O(h).$$

Für ungerade r verfahren wir ähnlich, überall die entsprechenden konjugierten Größen betrachtend. Damit haben wir den Satz 4 vollständig bewiesen.

SATZ 5. *Die $(r-1)$-te Derivierte $f^{(r-1)}(x)$ der 2π-periodischen Funktion $f(x)$ ($r \geqq 1$ ganz) ist genau dann absolut stetig, wenn eine geeignete von unten konkave positive Zahlenfolge $\lambda_n \uparrow +\infty$ existiert, so daß bzw. die Beziehungen bestehen:*

$$\|R_n^{(r)}(\lambda; f; x) - f_\lambda(x)\|_{L[-\pi,\pi]} = O\left(\frac{1}{n^r}\right) \quad (r \text{ gerade}),$$

$$\|\tilde{R}_n^{(r)}(\lambda; f; x) - \tilde{f}_\lambda(x)\|_{L[-\pi,\pi]} = O\left(\frac{1}{n^r}\right) \quad (r \text{ ungerade}),$$

mit

$$R_n^{(r)}(\lambda; f; x) = \sum_{\nu=0}^{n}\left(1 - \frac{\nu^r}{(n+1)^r}\right)\lambda_\nu A_\nu(x),$$

$$\tilde{R}_n^{(r)}(\lambda; f; x) = \sum_{\nu=1}^{n}\left(1 - \frac{\nu^r}{(n+1)^r}\right)\lambda_\nu B_\nu(x),$$

und $f_\lambda(x)$, bzw. $\tilde{f}_\lambda(x)$ den Limes im Raum $L[-\pi, \pi]$ von $\{R_n^{(r)}(\lambda; f; x)\}$, bzw. von $\{\tilde{R}_n^{(r)}(\lambda; f; x)\}$ bedeuten.

Es sei nämlich r eine gerade Zahl und $f^{(r-1)}(x)$ absolut stetig. Dann ist (abgesehen vom Vorzeichen) die Reihe $\Sigma\, \nu^r A_\nu(x)$ die Fourierreihe der integrierbaren Funktion $f^{(r)}(x)$. Es sind also die Rieszschen Mittel der Reihe $-f^{(r)}(x) + \sum_{\nu=1}^{\infty} \nu^r A_\nu(x)$ in der Metrik des Raumes $L[-\pi, \pi]$ von der Größenordnung $o(1)$. Nach Satz 3 existiert eine positive, von unten konkave Zahlenfolge $\lambda_n \uparrow +\infty$, für welche die Rieszschen Mittel $R_n^{(r)}(\lambda; f^{(r)}; x)$ der λ-transformierten Reihe $\Sigma \lambda_\nu \nu^r A_\nu(x)$ die Relation

$$\|R_m^{(r)}(\lambda; f^{(r)}; x) - R_n^{(r)}(\lambda; f^{(r)}; x)\|_{L[-\pi,\pi]} = o(1)$$

erfüllen. Es ist also

$$\|R_n^{(r)}(\lambda; f^{(r)}; x)\|_{L[-\pi,\pi]} = O(1),$$

woraus nach unserem Satz 1 die Approximationsbeziehung

$$\|R_n^{(r)}(\lambda; f; x) - f_\lambda(x)\|_{L[-\pi,\pi]} = O\left(\frac{1}{n^r}\right)$$

folgt.

Wenn umgekehrt die letzte Abschätzung gilt, so haben wir nach Satz 1 die Beziehung

$$\|R_n^{(r)}(\lambda; f^{(r)}; x)\|_{L[-\pi,\pi]} = O(1),$$

wo $R_n^{(r)}(\lambda; f^{(r)}; x)$ das n-te Rieszsche Mittel der trigonometrischen Reihe $\Sigma\lambda_\nu \nu^r A_\nu(x)$ bedeutet. Da jetzt $\{1/\lambda_n\}$ eine konvexe Nullfolge ist, folgt aus Satz 2 mit $\eta_n = 1$ die Relation

$$\|R_m^{(r)}(f^{(r)}; x) - R_n^{(r)}(f^{(r)}; x)\|_{L[-\pi,\pi]} = o(1).$$

Die Reihe $\Sigma\nu^r A_\nu(x)$ ist also eine Fourierreihe, und zwar die Fourierreihe der Derivierten $f^{(r)}(x)$ (abgesehen vom Vorzeichen), so daß $f^{(r-1)}(x)$ absolut stetig ist.

Für den Fall, wenn r eine ungerade Zahl ist, verfahren wir wörtlich wie oben, wir haben nur überall die entsprechenden konjugierten Größen zu betrachten.

Wenn wir in unserem Beweis statt der Normen $\|\circ\|_L$ die Normen $\|\circ\|_C$ im Raum $C[-\pi, \pi]$ der stetigen Funktionen betrachten, erhalten wir den folgenden

SATZ 6. *Die r-te Derivierte $f^{(r)}(x)$ der 2π-periodischen Funktion $f(x)$ ist genau dann überall stetig, wenn eine geeignete, von unten konkave Zahlenfolge $\lambda_n \uparrow +\infty$ existiert, so daß bzw. die Relationen bestehen:*

$$\|R_n^{(r)}(\lambda; f; x) - f_\lambda(x)\|_C = O\left(\frac{1}{n^r}\right),$$

wenn r gerade, bzw.

$$\|\tilde{R}_n^{(r)}(\lambda; f; x) - \tilde{f}_\lambda(x)\|_C = O\left(\frac{1}{n^r}\right),$$

wenn r ungerade ist.

(Eingegangen am 18. Dezember 1968.)

BUDAPESTI MŰSZAKI EGYETEM,
VEGYÉSZMÉRNÖKI KAR,
MATEMATIKAI TANSZÉK
BUDAPEST, XI., STOCZEK U. 2

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SCHEIBENPACKUNGEN KONSTANTER NACHBARNZAHL

Von

L. FEJES TÓTH (Budapest), korrespondierendes Mitglied der Akademie

Herrn Professor G. ALEXITS *zu seinem 70. Geburtstag gewidmet*

§ 1. Einleitung. Wir legen unseren Betrachtungen die Euklidische Ebene zu Grunde. Unter einer *Scheibe* verstehen wir eine beschränkte offene konvexe Punktmenge. Eine Menge disjunkter Scheiben heißt eine *Packung*. Zwei Scheiben, die einen gemeinsamen Randpunkt haben, nennen wir *Nachbarn*. Hat in einer Packung jede Scheibe eine konstante Anzahl n von Nachbarn, so sprechen wir von einer *Packung konstanter Nachbarnzahl* oder von einer *n-Nachbarnpackung*.

Zerlegen wir einen Kreis in $n+1$ kongruente Sektoren, so haben wir ein Beispiel einer n-Nachbarnpackung mit einem beliebigen nicht negativen ganzzahligen Wert von n vor uns. Als zweites Beispiel betrachten wir eine n-Nachbarnpackung aus endlich vielen glatten Scheiben. Auf Grund des Eulerschen Polyedersatzes kommen hier nur die Werte $n=0, 1, 2, 3, 4$ und 5 vor. Es scheint nun ein interessantes Forschungsgebiet zu sein, unter verschiedenen Bedingungen die zulässigen Werte von n zu bestimmen. In diesem Aufsatz möchten wir auf einige derartige Probleme hinweisen.

Wir werden uns auf einige Konstruktionen beschränken, die unter den betreffenden Bedingungen mögliche Werte von n liefern. Die Frage, ob die Aufzählung dieser Werte vollständig ist, wird nicht behandelt, so daß die meisten hier aufgeworfenen Probleme noch ungelöst sind. Als Hauptergebnis kann die im § 4 angegebene Konstruktion angesehen werden, die alle Möglichkeiten erschöpft und damit das betreffende Problem erledigt.

§ 2. Parkettierungen. Wir nennen eine Packung, bei der jeder Punkt der Ebene entweder zu einer Scheibe oder zum Rand einer Scheibe gehört, eine *Parkettierung* oder *Pflasterung*. Es erhebt sich folgendes Problem: Gesucht werden diejenigen Werte von n, die eine n-Nachbarnpflasterung aus kongruenten konvexen Polygonen gestatten.

Die Figuren 1 bis 9 stellen Beispiele für $n=6, 7, 8, 9, 10, 12, 14, 16$ und 21 dar. Gibt es Beispiele auch mit anderen Werten von n? Diese Frage ist noch nich beantwortet. Es steht nicht einmal fest, ob es für n eine universale obere Schranke gibt oder nicht.

§ 3. Bewegungsstabile Packungen. Eine Packung wird relativ stabil, schlechthin *stabil* oder auch *bewegungsstabil* genannt, wenn jede Scheibe durch ihre Nachbarn gegen Bewegungen fixiert wird. Runden wir die Ecken der in Fig. 6 dargestellten Dreiecke durch kleine kongruente Kreisbogen ab, so erhalten wir eine stabile Packung kongruenter Scheiben mit der konstanten Nachbarnzahl $n=3$. Auf ähnliche Weise erhalten wir durch Abrundung der Ecken vom Winkel 90° in den Figuren 3, 2, 7 und 8 Beispiele für $n=4, 5, 13$ bzw. 15. Ein weiteres einfaches Beispiel hat

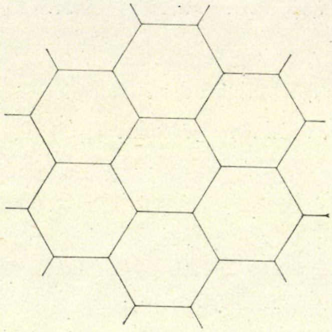

Fig. 1

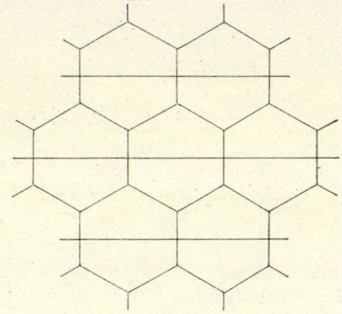

Fig. 2

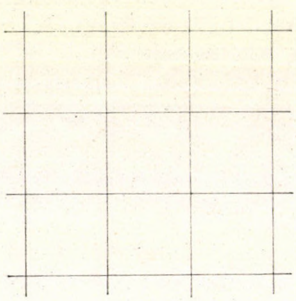

Fig. 3

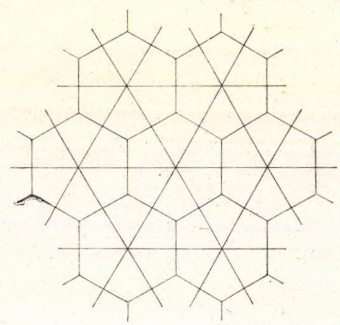

Fig. 4

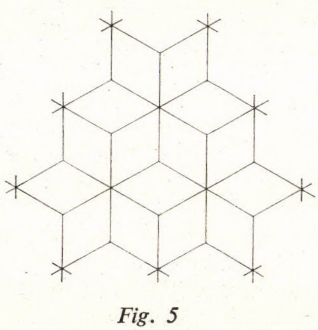

Fig. 5

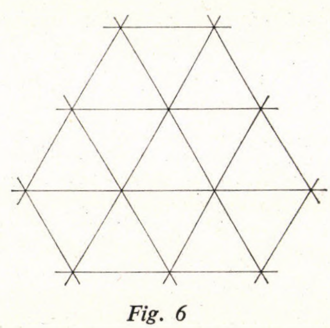

Fig. 6

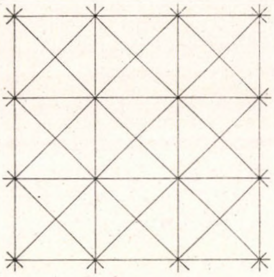

Fig. 7

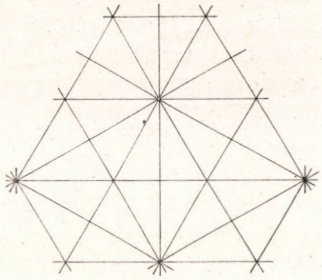

Fig. 8

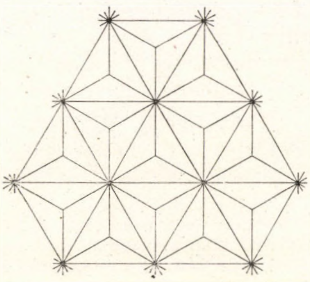

Fig. 9

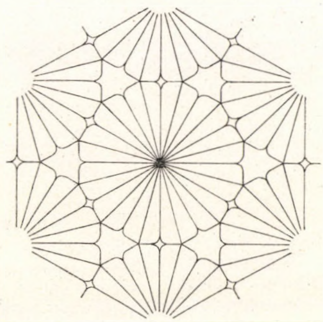

Fig. 10

G. Fejes Tóth gefunden: man zerlege im Archimedischen Mosaik (12, 3, 3) die 12-Ecke in je 24 kongruente Dreiecke und runde die rechten Winkel ab (Fig. 10). Hier ist die Nachbarnzahl $n=25$. Zusammen mit den oben betrachteten Parkettierungen haben wir also Beispiele für stabile Packungen kongruenter Scheiben mit den konstanten Nachbarnzahlen $n=3$ bis 10, 12 bis 16, 21 und 25.

Wir können hier ganz analoge Fragen stellen wie im § 2. Doch lassen die in den §§ 4 und 7 enthaltenen Ergebnisse vermuten, daß hier schon die Entscheidung der Frage, ob es für n eine endliche obere Schranke gibt oder nicht, äußerst schwierig sein muß.

§ 4. Verschiebungsstabile Packungen. Wird in einer Packung jede Scheibe durch ihre Nachbarn gegen Translationen fixiert, so wird die Packung *verschiebungsstabil* genannt. Es sei hier gezeigt:

Zu jeder ganzen Zahl $n>2$ gibt es eine verschiebungsstabile Packung kongruenter Scheiben mit einer konstanten Nachbarnzahl n.

Da bei den Nachbarnzahlen 3, 4 und 5 sogar bewegungsstabile Packungen existieren, können wir voraussetzen, daß $n>5$ ist.

Es seien A und B zwei Punkte mit einem Abstand größer als 1 und kleiner als $\sqrt{2}$. Ferner sei a_0 ein gleichschenkliges Dreieck mit der Spitze A in einer solchen Lage, daß es die Halbgerade AB in zwei kongruente Teile schneidet. Die Schenkellänge von a_0 sei 1 und die Basislänge p. Jetzt konstruieren wir sukzessiv weitere zu

a_0 kongruente, nicht übereinandergreifende Dreiecke $b_1, a_1, ..., b_k, a_k$, die abwechselnd B und A als Spitzen haben, so daß b_1 mit a_0, a_1 mit $b_1, ..., a_k$ mit b_k benachbart sei. Ist p genügend klein, so läßt sich diese Konstruktion für jede natürliche Zahl k so durchführen, daß die Symmetrieachse von a_k einen beliebig kleinen Winkel mit AB einschließt. Nähern wir die Punkte A und B, so nehmen die Winkel, die die Symmetrieachsen der Dreiecke $b_1, ..., a_k$ mit AB einschließen, stetig zu, so daß bei einem wohlbestimmten Abstand $\overline{AB}$ die Basis von b_k auf einem Schenkel von a_k liegen wird (Fig. 11). Wie man leicht nachrechnet, gilt in dieser Lage von a_k und b_k die Ungleichung $\overline{AB} < \sqrt{2-p}$, die wegen der Voraussetzung $\overline{AB} < \sqrt{2}$ für genügend kleine Werte von p erfüllt ist.

Wir betrachten jetzt die Bewegungsgruppe, die durch die Spiegelung bezüglich AB, die Translation $\overline{AB}$ und die Spiegelung bezüglich des Basismittelpunktes von a_k erzeugt wird. Ergänzen wir die betrachteten Dreiecke mit ihren Bildern unter dieser Gruppe, so entsteht eine Packung mit der konstanten Nachbarnzahl $4k+2$.

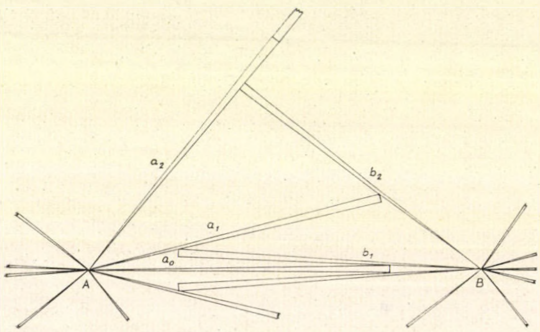

Fig. 11

Anstatt von den obigen Dreiecken gehen wir jetzt von den ähnlich konstruierten Dreiecken $a_0, b_1, a_1, ..., b_k, a_k, b_{k+1}$ aus, wobei die Basis von a_k auf einem Schenkel von b_{k+1} liegt. Gleichzeitig ersetzen wir bei der Bildung der Bewegungsgruppe die Spiegelung bezüglich des Basismittelpunktes von a_k durch die Spiegelung bezüglich des Basismittelpunktes von b_{k+1}. Die erhaltene Packung hat die konstante Nachbarnzahl $4k+4$.

Man sieht leicht ein, daß a_0, b_k und a_k, bzw. a_0, a_k und b_{k+1}, sowie ihre Transformierten sogar gegen Bewegungen, die übrigen Dreiecke aber nur gegen Translationen fixiert sind. Damit sind die Fälle mit einem geraden Wert von n erledigt.

Bei einem ungeraden Wert von n legen wir a_0 so, daß ein Schenkel von a_0 auf der Strecke AB liegt. Wir konstruieren in derselben Weise wie oben die Dreiecke $b_1, ..., a_k$, bzw. $b_1, ..., a_k, b_{k+1}$, wobei b_1 auf derselben Seite von AB liegt wie a_0. Die Ergänzung dieser Dreiecke zu einer Packung konstanter Nachbarnzahl folgt ebenfalls wie oben. Die Nachbarnzahl ist $4k+3$ bzw. $4k+5$. Es sei noch bemerkt, daß hier statt der Spiegelung bezüglich der Geraden AB auch die Spiegelung bezüglich des Mittelpunktes der Strecke AB entsprechen würde.

§ 5. Maximalpackungen. Wir definieren die *Newtonsche Zahl* einer Scheibe s als die Maximalzahl der kongruenten Exemplare von s, die ohne übereinander zu greifen mit s in Berührung gebracht werden können. Hat in einer Packung kongruenter Scheiben jede Scheibe genau soviele Nachbarn wie ihre Newtonsche Zahl, so sprechen wir von einer *Maximalpackung* [1].

Das einfachste Beispiel für eine Maximalpackung ist eine 6-Nachbarnpackung kongruenter Kreise. Im allgemeinen ist aber die Bestimmung der Newtonschen

Zahl einer Scheibe, und folglich die Entscheidung, ob eine Packung maximal ist oder nicht, ein schwieriges Problem [vgl. 2]. Es ist bekannt [3], daß die Newtonsche Zahl eines Quadrats 8 und die Newtonsche Zahl eines gleichseitigen Dreiecks 12 beträgt. Deshalb bilden die Flächen eines regulären Vier- und Dreiecksmosaiks (Fign. 3 und 6) je eine Maximalpackung. Es ist nicht ausgeschlossen, daß die in Fig. 9 dargestellte Dreieckpackung mit der konstanten Nachbarnzahl 21 ebenfalls maximal ist. Obwohl dies nicht feststeht, läßt diese Packung vermuten, daß in einer Maximalpackung die Nachbarnzahl erheblich größer sein kann als 12. Es ist andererseits kaum zu bezweifeln, daß bei Maximalpackungen die Nachbarnzahl unter einer universalen Schranke bleibt. Wir lassen für diese naheliegende Vermutung einen Beweisansatz folgen.

Es sei s eine Scheibe mit dem Durchmesser 1, der Dicke d und der Newtonschen Zahl N. Ferner seien A und B zwei Randpunkte von s vom Abstand $\overline{AB}=1$, p der Parallelbereich von s vom Abstand 1/2 und P der Parallelbereich von s vom Abstand 1. Wir stellen uns N sehr groß vor. Dann ist d offensichtlich sehr klein, so daß s praktisch mit der Strecke AB zusammenfällt. Deshalb wollen wir den Mittelpunkt M der Strecke AB den Mittelpunkt von s nennen.

Wir setzen jetzt voraus, daß sich s durch Hinzufügung weiterer Scheiben zu einer Maximalpackung ergänzen läßt. Es leuchtet intuitiv ein, daß dann die Mittelpunkte der mit s benachbarten Scheiben in der Nähe des Randes von p, und die Mittelpunkte der übrigen Scheiben außerhalb p liegen müssen. Hieraus würde folgen, daß der Mindestabstand zwischen M und den übrigen Scheibenmittelpunkten angenähert 1/2 ist. Wir setzen aber nur voraus, daß für genügend kleine Werte von d dieser Mindestabstand, also zugleich der Abstand zwischen je zwei Scheibenmittelpunkten größer ist als etwa 1/4. Da aber einerseits die N Mittelpunkte der mit s benachbarten Scheiben sicher in P liegen, andererseits in P nur eine geringe Anzahl von Punkten mit einem Mindestabstand $>1/4$ Platz hat, kann N im Gegensatz zu unserer Voraussetzung nicht groß sein.

§ 6. Endliche Packungen. Unter dem *Winkel* einer Scheibe s in einem Randpunkt A verstehen wir den Winkel des kleinsten Winkelbereiches mit der Ecke A, der s enthält. Man sieht leicht ein, daß die zu verschiedenen Randpunkten gehörigen Winkel ein Minimum haben, das wir *Minimalwinkel* nennen. Ist der Minimalwinkel gleich π, so heißt die Scheibe *glatt*.

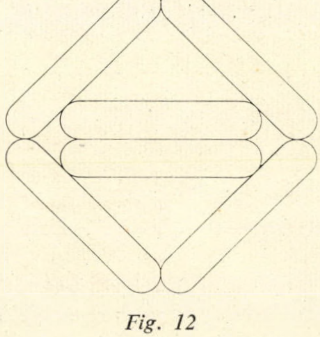

Fig. 12

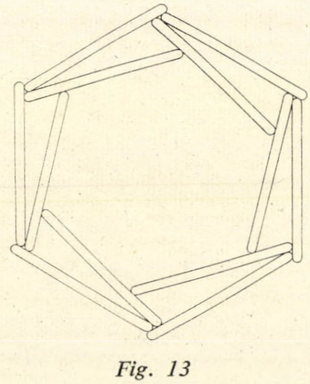

Fig. 13

Wir bertachten eine n-Nachbarnpackung von endlich vielen kongruenten Scheiben vom Minimalwinkel $2\pi/h$ mit einem reellen Wert von $h \geqq 2$. Für drei einander gegenseitig berührende glatte Scheiben gilt $n = 2$. Ferner gilt für $[h]$ Scheiben, die in einem Punkt zusammentreffen, $n = [h-1]$. Dies sind aber im allgemeinen nicht die größtmöglichen Werte, die n für $h < 3$ bzw. $h \geqq 3$ erreichen kann. Es gibt nämlich Packungen von endlich vielen kongruenten glatten Scheiben mit der konstanten Nachbarnzahl 3 und 4 (Fign. 12 und 13). Es läßt sich aber vermuten, daß die gesuchte Maximalzahl Max $(4, [h-1])$ ist. Hieraus würde natürlich folgen, daß endlich viele kongruente glatte Scheiben keine 5-Nachbarnpackung bilden können.

Für unendliche Packungen läßt sich die analoge obere Schranke in der Form Max $(6, f(h))$ schreiben. Die in den Figuren 3, 6, 7, 9 und 10 dargestellten Beispiele zeigen, daß $f(4) \geqq 8$, $f(6) \geqq 12$, $f(8) \geqq 14$, $f(12) \geqq 21$ und $f(24) \geqq 25$ ausfällt.

§ 7. Normalpackungen. Verzichten wir auf die Kongruenz der Scheiben, so werden mehrere der obigen Probleme uninteressant. Man kann nämlich zu einem beliebigen konvexen n-Eck mit $n > 6$ weitere konvexe n-Ecke so sukzessiv hinzufügen, daß eine n-Nachbarnpflasterung entsteht. Runden wir die Ecken der n-Ecke in geeigneter Weise ab, so erhalten wir eine bewegungsstabile Packung glatter Scheiben mit der konstaten Nachbarnzahl n. In diesen Beispielen müssen aber entweder immer längere oder immer dünnere Scheiben vorkommen.

Wir richten hier unsere Aufmerksamkeit auf *Normalpackungen*, die wir dadurch definieren, daß die Radien der Scheibeninkreise und Scheibenumkreise eine positive untere und eine endliche obere Grenze haben.

Es scheint nicht einfach zu sein, außer für die Zahlen $n = 6, 7, 8, 9, 10, 12, 14, 16$ und 21, die bereits bei Parkettierungen mit kongruenten Vielecken erwähnt wurden, für andere Werte von n normale n-Nachbarnpflasterungen zu konstruieren. Das einzige mir bekannte derartige Beispiel ist eine von G. Fejes Tóth herrührende normale 11-Nachbarnpflasterung (Fig. 14). Trotzdem läßt die folgende Konstruktion vermuten, daß für jede ganze Zahl $n > 5$ normale n-Nachbarnpflasterungen existieren.

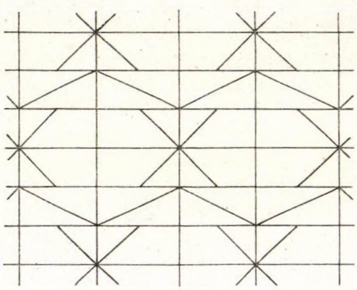
Fig. 14

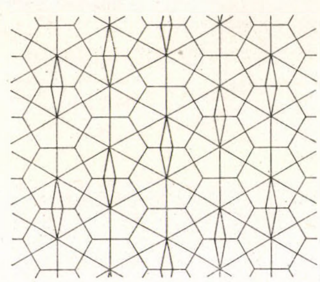
Fig. 15

Wir zerlegen die Seiten eines regulären Sechsecks in je $k \geqq 2$ kongruente Strecken und verbinden die $6(k-1)$ Teilungspunkte mit dem Mittelpunkt des Sechsecks. Dadurch zerfällt das Sechseck in 6 Deltoide und $6(k-2)$ Dreiecke. Zerlegen wir in dieser Weise alle Flächen eines regulären Sechseckmosaiks, so entsteht eine Normalpflasterung, wo zwar die Nachbarnzahl nicht konstant ist, aber nur die nacheinander folgenden Werte $6k-4$ und $6k-3$ annimmt. Für $k = 2$ ist die Anzahl der Dreiecke Null, so daß jetzt unsere Konstruktion die in Fig. 4 dargestellte 9-Nachbarnpflasterung ergibt.

Man kann auch zu einer beliebig vorgegebenen Zahl $n \geqq 9$ solche Normalpflasterungen konstruieren, wo die Scheiben nur n oder $n+1$ Nachbarn haben. Fig. 15 zeigt den Fall $n=9$.

Wird nur Stabilität verlangt, so läßt sich das entsprechende Problem leicht erledigen: *Zu jeder ganzen Zahl $n>2$ gibt es bewegungsstabile Normalpackungen mit der konstanten Nachbarnzahl n.* Zerlegen wir nämlich ein gleichseitiges Dreieck D in n Teildreiecke, die im Mittelpunkt von D zusammentreffen, runden die an den Seiten von D liegenden Ecken der Teildreiecke ab und nehmen zu den entstehenden Scheiben ihre Transformierten unter derjenigen Bewegungsgruppe hinzu, die durch die Spiegelungen an den Seiten von D erzeugt wird, so haben wir die gewünschte Packung vor uns.

In einer endlichen Packung mit der konstanten Nachbarnzahl n sei der Minimalwinkel sämtlicher Scheiben $\geqq 2\pi/h$. Dann gilt vermutlich $n \leqq \mathrm{Max}\,(5, [h-1])$. Es sei bemerkt, daß durch eine geeignete stereographische Projektion der Flächeninkreise des regulären dodekaedrischen Mosaiks eine 5-Nachbarnpackung von 12 Kreisen entsteht.

Zum Schluß betrachten wir eine Normalpackung mit der konstanten Nachbarnzahl n. Ist der Minimalwinkel sämtlicher Scheiben $\geqq 2\pi/h$, so gilt vermutlich $n \leqq \mathrm{Max}\,(6, f(h))$, wo $f(h)$ mit der am Ende von §6 definierten Funktion identisch ist.

(Eingegangen am 21. Dezember 1968.)

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BUDAPEST, V., REÁLTANODA U. 13—15

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ON THE SUMMABILITY OF THE FOURIER SERIES OF L^2 INTEGRABLE FUNCTIONS. IV

By

E. MAKAI (Budapest)

To G. ALEXITS *on his seventieth birthday*

1. Let π_n denote the space of the complex trigonometric polynomials

$$f(x) = \sum_{-n}^{n} c_\nu e^{i\nu x} = \frac{a_0}{2} + \sum_{1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

of order n and let C_n be defined by

$$C_n = \max_{f\in\pi_n} \left| \int_0^{2\pi} \max_{0\le k\le n} s_k(x;f)\, dx \right| \Big/ \left\{ \int_0^{2\pi} |f(x)|^2\, dx \right\}^{1/2} \tag{1. 1}$$

where $s_k(x;f)$ is the partial sum of $f(x)$ of order k.

The theorem of Carleson, namely that every $L^2(0, 2\pi)$ integrable function can be expanded into an almost everywhere convergent Fourier series is equivalent to the statement that the sequence $\{C_n\}_{n=0}^{\infty}$ is bounded [2].

The more difficult question of the behaviour of the quantities

$$A_n = \max_{f\in\pi_n} \int_0^{2\pi} \max_{0\le k\le n} |s_k(x;f)|^2\, dx \Big/ \int_0^{2\pi} |f(x)|^2\, dx \qquad (n=1, 2, \dots) \tag{1. 2}$$

was settled recently by R. A. HUNT [1]: he proved that the infinite sequence $A_1, A_2, \dots$ is bounded.

Another question is the following. Let the complex function $\varphi(x) \in L^2(-\pi, \pi)$ be of unit norm: $\|\varphi\| = \int_{-\pi}^{\pi} |\varphi(x)|^2 dx = 1$ and let $\{\varkappa_r\}_{r=-\infty}^{\infty}$ be a sequence of non-negative numbers not exceeding π. Let us consider the quantity

$$E(\varphi; \{\varkappa_r\}) = \sum_{r=-\infty}^{\infty} \left| \int_{-\varkappa_r}^{\varkappa_r} \varphi(x) e^{-irx}\, dx \right|^2. \tag{1. 3}$$

Do the $E(\varphi; \{\varkappa_r\})$'s have a finite bound if φ ranges over all $L^2(-\pi, \pi)$ integrable functions of unit norm and $\{\varkappa_r\}$ is a fixed sequence? (Of course, if $\varkappa_r = \pi$ for every r, then by Parseval's formula $E(\varphi; \{\varkappa_r\}) = 2\pi$). This would certainly be true if the quantity

$$E(\varphi) = \sum_{r=-\infty}^{\infty} \max_{0\le\varkappa_r\le\pi} \left| \int_{-\varkappa_r}^{\varkappa_r} \varphi(x) e^{-irx}\, dx \right|^2 \tag{1. 4}$$

had a finite bound the $\varphi(x)$'s ranging over all $L^2(-\pi, \pi)$ integrable functions of unit norm.

The purpose of this paper is to show the equivalence of Hunt's result and of the problem mentioned in the last paragraph: *from $A_n < A$ it follows that*

$$\sup_{\|\varphi\|=1} E(\varphi) < \infty; \tag{1. 5}$$

conversely (1. 5) *implies Hunt's result.*

2. Let $x_1, x_2, \ldots, x_m$ be a strictly increasing sequence of real numbers and $k_1, k_2, \ldots, k_m$ non-negative integers not exceeding n. Let further $\mathbf{x}$ and $\mathbf{k}$ denote the m-vectors $\{x_1, \ldots, x_m\}$ and $\{k_1, \ldots, k_m\}$, respectively. Finally we shall use the notation

$$|||f||| = \left|\left|\left|\frac{a_0}{2} + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)\right|\right|\right| = \left\{\frac{|a_0|^2}{2} + \sum_{\nu=1}^{n} \left(|a_\nu|^2 + |b_\nu|^2\right)\right\}^{1/2}.$$

In part I of this paper [3] I have introduced the quantities

$$\Lambda(\mathbf{x}, \mathbf{k}) = \max_{f \in \pi_n} \sum_{r=1}^{m} |s_{k_r}(x_r; f)|^2 \Big/ |||f|||^2 \tag{2. 1}$$

and have shown that they are equal to the greatest eigenvalue of the matrix

$$[D_{\min(k_p, k_q)}(x_p - x_q)]_{p,q=1}^{m} \tag{2. 2}$$

where $D_l(x) = 1/2 + \cos x + \ldots + \cos lx$ is Dirichlet's kernel.

We shall call a function $f^* = a_0^*/2 + \Sigma(a_\nu^* \cos \nu x + b_\nu^* \sin \nu x)$ an extremal function of the maximum problem (2. 1) if one has

$$\Lambda(\mathbf{x}, \mathbf{k}) = \sum_{r=1}^{m} |s_{k_r}(x_r; f^*)|^2 \Big/ |||f^*|||^2. \tag{2. 3}$$

The existence of these extremal functions is obvious and we state

LEMMA 1. *There exists an extremal function of* (2. 1) *with real Fourier coefficients* a_ν^*, b_ν^*.

Indeed by introducing the notations $a_0/\sqrt{2} = \zeta_0$, $a_\nu = \zeta_\nu$, $b_\nu = \zeta_{-\nu}$ $(\nu = 1, 2, \ldots, n)$ we see that the numerator of the right-hand side of (2. 1) is an Hermitian form of the quantities $\zeta_{-n}, \zeta_{-n+1}, \ldots, \zeta_n$ with real coefficients. $\Lambda(\mathbf{x}, \mathbf{k})$ is the greatest eigenvalue of the corresponding symmetric matrix, and the quantities

$$\zeta_\nu = \zeta_\nu^* = \begin{cases} a_\nu^* & (\nu = 1, 2, \ldots, n), \\ a_0^*/\sqrt{2}, & \\ b_{-\nu}^* & (\nu = -1, -2, \ldots, -n) \end{cases}$$

are the components of the real eigenvector corresponding to the eigenvalue $\Lambda(\mathbf{x}, \mathbf{k})$

We quote another result of part I:

$$(2.4)\qquad A_n(\mathbf{x}) = \max_{f\in\pi_n} \frac{1}{m} \sum_{r=1}^{m} \max_{l=0,1,\dots,n} |s_l(x_r;f)|^2 \Big/ |||f|||^2 = \frac{1}{m} \max_{\substack{k_r=0,1,\dots,n\\ r=1,2,\dots,m}} \Lambda(\mathbf{x};\mathbf{k}).$$

In the special case when $x_r = x_r^* = 2\pi r/m$ $(r=1,2,\dots,m)$ the quantity $A_n(\mathbf{x})$ was denoted by $A_n^{(m)}$ and it was shown in part I that

$$(2.5)\qquad 2A_n^{(m)} \geqq A_n.$$

A counterpart of this inequality is

LEMMA 2.

$$(2.6)\qquad 2A_n^{(m)} \leqq \left(1+4\pi\frac{n}{m}\right) A_n.$$

Indeed let $k_1^*, k_2^*, \dots, k_m^*$ be an extremal sequence corresponding to the maximum problem (2. 3), i.e.

$$A_n^{(m)} = \frac{1}{m} \Lambda_n(k_1^*, k_2^*, \dots, k_m^*)$$

and a_ν^*, b_ν^* should denote, as before, the Fourier coefficients of the real extremal function f^* in (2. 4), only that now $k_1 = k_1^*, \dots, k_m = k_m^*$. So, if

$$(2.7)\qquad |||f^*||| = \frac{1}{\pi}, \quad \text{i.e.} \quad \int_{-\pi}^{\pi} \{f^*(x)\}^2\,dx = 1,$$

we have

$$(2.8)\qquad A_n^{(m)} = \frac{\pi}{m} \sum_{r=1}^{m} s_{k_r^*}^2\left(\frac{2\pi}{m} r, f^*\right).$$

On the other hand, by (1. 2), if $x_r^* = 2\pi r/m$,

$$A_n \geqq \int_{-\pi}^{\pi} \max_{0\leqq k\leqq n} |s_k(x;f^*)|^2\,dx \geqq \sum_{r=1}^{m} \int_{x_{r-1}^*}^{x_r^*} s_{k_r^*}^2(x;f^*)\,dx$$

and if

$$\min_{x_{r-1}^*\leqq x\leqq x_r^*} s_{k_r^*}^2(x;f^*) = s_{k_r}^2(x_r^* - \eta_r; f^*)$$

then *a fortiori*

$$(2.9)\qquad A_n \geqq \sum_{r=1}^{m} \int_{x_{r-1}^*}^{x_r^*} s_{k_r^*}^2(x_r^*-\eta_r;f^*)\,dx = \frac{2\pi}{m} \sum_r s_{k_r^*}^2(x_r^*-\eta_r;f^*).$$

Let us now introduce the continuous function

$$g(\xi) = \frac{2\pi}{m} \sum_r s_{k_r^*}^2(x_r^* - \xi\eta_r, f^*).$$

By (2. 5) and (2. 7) $g(0)\geqq A_n$ and by (2. 9) $g(1)\leqq A_n$. Hence, there exists a ϑ, $0\leqq\vartheta\leqq 1$, such that $g(\vartheta)=A_n$ and by (2. 7)

$$2A_n^{(m)}-A_n=\frac{2\pi}{m}\sum_r\{s_{k_r^*}^2(x_r^*;f^*)-s_{k_r^*}^2(x_r^*-\vartheta\eta_r;f^*)\}=$$

$$=\frac{2\pi}{m}\sum_r\int_{x_r^*-\vartheta\eta_r}^{x_r^*}\frac{d}{dx}s_{k_r^*}^2(x;f^*)\,dx=\frac{4\pi}{m}\sum_r\int_{x_r^*-\vartheta\eta_r}^{x_r^*}s_{k_r^*}(x;f^*)\frac{d}{dx}s_{k_r^*}(x;f^*)\,dx.$$

Taking into regard that $(d/dx)s_{k_r^*}(x_r;f^*)=s_{k_r^*}(x_r,f^{*\prime})$ where

$$f^{*\prime}=\sum\nu(b_\nu^*\cos\nu x-a_\nu^*\sin\nu x) \tag{2. 10}$$

and using in turn Schwarz's inequality and $x_{r-1}^*\leqq x_r^*-\vartheta\eta_r$ we have

$$2A_n^{(m)}-A_n<\frac{4\pi}{m}\sum_r\left\{\int_{x_{r-1}^*}^{x_r^*}s_{k_r^*}^2(x;f^*)\,dx\right\}^{1/2}\left\{\int_{x_{r-1}^*}^{x_r^*}s_{k_r^*}^2(x;f^{*\prime})\,dx\right\}^{1/2}.$$

Again, by Cauchy's inequality

$$2A_n^{(m)}-A_n\leqq\frac{4\pi}{m}\left\{\sum_r\int_{x_{r-1}^*}^{x_r^*}s_{k_r^*}^2(x;f^*)\,dx\sum_r\int_{x_{r-1}^*}^{x_r^*}s_{k_r^*}^2(x;f^{*\prime})\,dx\right\}^{1/2}\leqq$$

$$\leqq\frac{4\pi}{m}\left\{\int_0^{2\pi}\max_{0\leqq k\leqq n}s_k^2(x;f^*)\,dx\cdot\int_0^{2\pi}\max_{0\leqq k\leqq n}s_k^2(x;f^{*\prime})\,dx\right\}^{1/2}\leqq$$

$$\leqq\frac{4\pi}{m}\left\{A_n\int_0^{2\pi}[f^*(x)]^2\,dx\cdot A_n\int_0^{2\pi}[f^{*\prime}(x)]^2\,dx\right\}^{1/2}$$

by the definition (1. 2) of A_n. Finally by (2. 8) and (2. 10)

$$2A_n^{(m)}-A_n\leqq\frac{4\pi}{m}A_n\cdot n \tag{2. 11}$$

since by (2. 7)

$$\int_0^{2\pi}[f^{*\prime}(x)]^2\,dx=\frac{1}{\pi}\sum_1^n\nu^2(a_\nu^{*2}+b_\nu^{*2})\leqq n^2.$$

We remark that from (2. 5) and (2. 6) we have

$$1\leqq\frac{2A_n^{(2n)}}{A_n}\leqq 1+2\pi \tag{2. 12}$$

i.e. A_n and $A_n^{(2n)}$ have the same order of magnitude.

Let now in (2. 4) x_r be equal to $\pi r/m$. We shall denote the corresponding quantity $A_n(\mathbf{x})$ by $a_n^{(m)}$ and prove

LEMMA 3. $a_n^{(m)}\leqq 2A_n^{(2m)}\leqq 2a_n^{(m)}$.

It is trivial that $a_n^{(m)}$ may be defined in the more general way (cf. § 2 of Part I)

$$(2.13)\qquad a_n^{(m)} = \max_{|||f|||=1} \frac{1}{m} \sum_{r=1}^{m} \max_{k=0,1,\dots,n} \left| s_k\left(\frac{\pi r}{m}+C;f\right)\right|^2$$

where C is any real constant.

So we have, if $|||f|||=1$

$$\max_f \sum_{r=1}^{m} \max_k \left| s_k\left(\frac{\pi r}{m};f\right)\right|^2 \leqq \max_f \sum_{r=1}^{2m} \max_k \left| s_k\left(\frac{\pi r}{m};f\right)\right|^2 \leqq$$

$$\leqq \max_f \sum_{r=1}^{m} \max_k \left| s_k\left(\frac{\pi r}{m};f\right)\right|^2 + \max_f \sum_{r=m+1}^{2m} \max_k \left| s_k\left(\frac{\pi r}{m};f\right)\right|^2$$

and by (2. 13) the two terms on the right-hand side are equal. Dividing by m yields Lemma 3.

3. We now deal with the following maximum problem. Let $k_1, \dots, k_m$ be non-negative numbers, not necessarily integers, $\mathbf{z}$ an m-vector with complex components $z_1, \dots, z_m$ and let the vectors $\mathbf{x}$ and $\mathbf{k}$ have the same meaning as before. Our problem is to find the maximum $\mu(\mathbf{z}; \mathbf{x}, \mathbf{k})$ of

$$\left| \sum_{r=1}^{m} z_r \int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt \right|^2$$

if $\varphi(t)$ ranges over all $L^2(-\infty, \infty)$ functions of unit norm. We shall evaluate the quantity $\mu(\mathbf{z}, \mathbf{x}, \mathbf{k})$ in a way analogous to the solving of Problem 2a in Part I.

Using the notation

$$\varepsilon_k(t) = \begin{cases} 1 & \text{if} \quad |t| \leqq k, \\ 0 & \text{if} \quad t > k \end{cases}$$

we have

$$\left| \sum_r z_r \int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt \right|^2 = \left| \sum_r z_r \int_{-\infty}^{\infty} \varepsilon_{k_r}(t) \varphi(t) e^{itx_r}\, dt \right|^2 =$$

$$= \left| \int_{-\infty}^{\infty} \varphi(t) \sum_r z_r \varepsilon_{k_r}(t) e^{itx_r}\, dt \right|^2 \leqq \int_{-\infty}^{\infty} |\varphi(t)|^2\, dt \int_{-\infty}^{\infty} \left|\sum_r z_r \varepsilon_{k_r}(t) e^{itx_r}\right|^2 dt =$$

$$= \int_{-\infty}^{\infty} \sum_{p,q=1}^{m} z_p \bar{z}_q \varepsilon_{k_p}(t) \varepsilon_{k_q}(t) e^{it(x_p - x_q)}\, dt =$$

$$= \sum_{p,q=1}^{m} z_p \bar{z}_q \frac{2 \sin \min(k_p, k_q)(x_p - x_q)}{x_p - x_q} = \mu(\mathbf{z}; \mathbf{x}, \mathbf{k})$$

and the sign of equality stands if $\varphi(t) = \varphi(t, \mathbf{z}) = c \sum z_r \varepsilon_{k_r}(t) e^{itx_r}$ where c is the norming constant

$$\left\{ \int_{-\infty}^{\infty} \left|\sum z_r \varepsilon_{k_r}(t) e^{itx_r}\right|^2 dt \right\}^{-1/2} = \{\mu(\mathbf{z}, \mathbf{x}, \mathbf{k})\}^{-1/2}.$$

Summing up we have

$$(3.1)\qquad \left|\sum_r z_r \int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt\right|^2 \leqq \left|\sum_r z_r \int_{-k_r}^{k_r} \varphi(t, \mathbf{z}) e^{itx_r}\, dt\right|^2 =$$

$$= \mu(\mathbf{z}, \mathbf{x}, \mathbf{k}) \leqq M(\mathbf{x}, \mathbf{k}) \sum |z_r|^2, \quad \text{if} \quad \int_{-\infty}^{\infty} |\varphi(t)|^2\, dt = 1$$

where $M(\mathbf{x}, \mathbf{k})$ is the greatest eigenvalue of the matrix of the positive semidefinite form $\mu(\mathbf{z}; \mathbf{x}, \mathbf{k})$. We now state

LEMMA 4.

$$\max_{\|\varphi(t)\|=1} \sum_r \left| \int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt\right|^2 = M(\mathbf{x}, \mathbf{k}).$$

Indeed substituting

$$z_r = \overline{\int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt}$$

into (3. 1) we get by division with $\sum |z_r|^2$

$$(3.2)\qquad \sum \left| \int_{-k_r}^{k_r} \varphi(t) e^{itx_r}\, dt\right|^2 \leqq M(\mathbf{x}, \mathbf{k}) \quad \text{if} \quad \|\varphi(t)\| = 1.$$

To show that the sign of equality is valid here for some φ let $z_1^*, \ldots, z_m^*$ be the components of a vector $\mathbf{z}^*$ for which

$$(3.3)\qquad \mu(\mathbf{z}^*; \mathbf{x}, \mathbf{k}) = \left|\sum z_r^* \int_{-k_r}^{k_r} \varphi(t, \mathbf{z}^*) e^{ik_r t}\, dt\right|^2 = M(\mathbf{x}, \mathbf{k}) \sum |z_r^*|^2.$$

Such a vector exists and is an eigenvector of the matrix of the Hermitian form $\mu(\mathbf{z}; \mathbf{x}, \mathbf{k})$ belonging to its greatest eigenvalue $M(\mathbf{x}, \mathbf{k})$. We state that

$$(3.4)\qquad z_r^* = \gamma\, \overline{\int_{-k_r}^{k_r} \varphi(t, \mathbf{z}^*) e^{itx_r}\, dt}$$

where γ is independent of r. Indeed if it were not so we should have by Cauchy's inequality and by (3. 3)

$$\sum_r \left| \int_{-k_r}^{k_r} \varphi(t, \mathbf{z}^*) e^{itx_r}\, dt\right|^2 > \left|\sum_r z_r^* \int_{-k_r}^{k_r} \varphi(t, \mathbf{z}^*) e^{itx_r}\, dt\right|^2 \Big/ \sum_r |z_r^*|^2 = M(\mathbf{x}, \mathbf{k})$$

in contradiction to (3. 2). Substituting now (3. 4) into (3. 3) and dividing by $\sum |z_r^*|^2$

we get

$$\sum_r \left| \int_{-k_r}^{k_r} \varphi(t, \mathbf{z}^*) e^{itx_r}\, dt \right|^2 = M(\mathbf{x}, \mathbf{k})$$

thus completing the proof of Lemma 4.

4. We now introduce the m-vectors $\boldsymbol{\xi}$ and $\boldsymbol{\varkappa}$ with components $\xi_r = \sigma^{-1} x_r$ and $\varkappa_r = \sigma k_r$ and state

$$M(\boldsymbol{\xi}, \boldsymbol{\varkappa}) = \sigma M(\mathbf{x}, \mathbf{k}). \tag{4. 1}$$

This follows from the fact that the matrix of the form $\mu(\mathbf{z}, \xi, \boldsymbol{\varkappa})$ is σ times the matrix of $\mu(\mathbf{z}; \mathbf{x}, \mathbf{k})$.

The quantity

$$\alpha_n(\mathbf{x}) = \sup_{\substack{0 \leqq k_r \leqq m \\ r=1,2,\dots,n}} \frac{1}{m} M(\mathbf{x}, \mathbf{k}) \tag{4. 2}$$

is analogous to the $A_n(\mathbf{x})$ defined by (2. 4). (Cf. Lemma 4.) Hence it follows from (4. 1)

LEMMA 5. $\alpha_{\sigma n}(\boldsymbol{\xi}) = \sigma \alpha_n(\mathbf{x})$.

5. In this section let x_r be $\pi(r - [m/2])/m$, k_r any non-negative number and $[\mathbf{k}]$ the m-vector of components $[k_1], \dots, [k_m]$, where $[\beta]$ means the greatest integer contained in β. By comparing $M(\mathbf{x}, \mathbf{k})$ and $\Lambda(\mathbf{x}, [\mathbf{k}])$ defined in Section 2 we shall prove

LEMMA 6. $|M(\mathbf{x}, \mathbf{k}) - 2\Lambda(\mathbf{x}, [\mathbf{k}])| < 2m$.

It follows from (2. 2) that if

$$d_{pp} = k_p + \frac{1}{2}, \quad d_{pq} = \frac{\sin\{\min([k_p], [k_q]) + \frac{1}{2}\}(x_p - x_q)}{2 \sin \frac{1}{2}(x_p - x_q)}$$

then

$$\Lambda(\mathbf{x}, [\mathbf{k}]) = \max_{\Sigma |z_r|^2 = 1} \sum_{p,q=1}^{m} d_{pq} z_p \bar{z}_q \tag{5. 1}$$

in perfect analogy with

$$\frac{1}{2} M(\mathbf{x}, \mathbf{k}) = \max_{\Sigma |z_r|^2 = 1} \sum_{p,q=1}^{m} \delta_{pq} z_p \bar{z}_q \tag{5. 2}$$

where $\delta_{pp} = k_p$ and if $p \neq q$, $\delta_{pq} = \{\sin \min(k_p, k_q)(x_p - x_q)\}/(x_p - x_q)$. Let now $\Delta_{pq} = d_{pq} - \delta_{pq}$. We state that $|\Delta_{pq}| < 1$. This is obvious for the diagonal elements Δ_{pp}. For the non-diagonal elements the inequalities

$$0 < |x_p - x_q| < \pi \quad \text{and} \quad \left| \min([k_p], [k_q]) + \frac{1}{2} - \min(k_p, k_q) \right| \leqq \frac{1}{2}$$

hold. Now under the assumptions $0<|x|<\pi$ and $|k-k_1|<1/2$ one has

$$\left|\frac{\sin k_1 x}{2\sin\frac{x}{2}}-\frac{\sin kx}{x}\right| \leqq \left|\frac{\sin k_1 x}{2\sin\frac{x}{2}}-\frac{\sin kx}{2\sin\frac{x}{2}}\right|+\left|\frac{\sin kx}{2\sin\frac{x}{2}}-\frac{\sin kx}{x}\right| \leqq$$

$$\leqq \left|\cos\frac{k_1+k}{2}x\right|\left|\frac{\sin\frac{k_1-k}{2}x}{\sin\frac{x}{2}}\right|+|\sin kx|\left|\frac{1}{2\sin\frac{x}{2}}-\frac{1}{x}\right| \leqq$$

$$\leqq \left|\frac{\sin\frac{x}{4}}{\sin\frac{x}{2}}\right|+\left|\frac{1}{2\sin\frac{x}{2}}-\frac{1}{x}\right|<1.*$$

Hence $|\Delta_{pq}|<1$ and

$$\left|\sum_{p,q=1}^{m}\Delta_{pq}z_p\bar{z}_q\right| \leqq \sum_{p,q=1}^{m}|\Delta_{pq}|\frac{|z_p|^2+|z_q|^2}{2}<m\sum_{1}^{m}|z_p|^2. \tag{5. 3}$$

A well known reasoning yields now Lemma 6: by (5. 1), (5. 2) and (5. 3)

$$\Lambda(\mathbf{x},[\mathbf{k}]) = \max_{\Sigma|z_r|^2=1}\left(\sum\Delta_{pq}z_p\bar{z}_q+\sum\delta_{pq}z_p\bar{z}_q\right) \leqq$$

$$\leqq \max\sum\Delta_{pq}z_p\bar{z}_q+\max\sum\delta_{pq}z_p\bar{z}_q = m+\frac{1}{2}M(\mathbf{x},\mathbf{k})$$

and similarly $\frac{1}{2}M(\mathbf{x},\mathbf{k}) \leqq m+\Lambda(\mathbf{x},\mathbf{k})$.

By taking into account the definitions of $a_n^{(m)}$ and $\alpha_n(\mathbf{x})$ at the end of Sections 2 and 4, respectively, it follows

LEMMA 7. *If* $x_r=\pi(r-c)/m$ $(r=1,2,\ldots,m)$ *then*

$$|\alpha_n(\mathbf{x})-2\alpha_n^{(m)}|<2.$$

We now choose $m=n$, $c=[n/2]$ and $\sigma=\pi/n$ in Lemma 5. Combining Lemmas 5 and 7 we get

$$\left|\frac{n}{\pi}\alpha_\pi(\xi)-2\alpha_n^{(n)}\right|<2 \tag{5. 4}$$

* Indeed for $|x|<\pi$ one has $\{\sin(x/4)\}/\{\sin(x/2)\}=\{2\cos(x/4)\}^{-1}\leqq\{2\cos(\pi/4)\}^{-1}<0.71$ and if $t^2<6$, the Maclaurin series of $\sin t$ is of Leibniz' type, hence for $0\leqq t\leqq\pi/2$ $(<6^{1/2})$ one has $t>\sin t>t-t^3/6$, or if $0<x<2\pi$

$$\frac{x}{2}>\sin\frac{x}{2}>\frac{x}{2}\left(1-\frac{x^2}{24}\right)$$

from which

$$0<\frac{1}{2\sin(x/2)}-\frac{1}{x}<\frac{1}{x(1-x^2/24)}-\frac{1}{x}=\frac{x}{24-x^2}<\frac{\pi}{24-\pi^2}<\frac{1}{4}.$$

if $\xi_r = r-[n/2]$ $(r=1, 2, \ldots, n)$. Now from (4. 2)

$$(5.\,5)\qquad \frac{n}{\pi}\alpha_\pi(\xi) = \frac{1}{\pi}\sup_\varphi \sum_{r=1}^{n} \max_{0\leqq\varkappa_r\leqq\pi} \left| \int_{-\varkappa_r}^{\varkappa_r} \varphi(t)e^{i(r-[m/2])t}\,dt \right|^2 \Big/ \int_{-\infty}^{\infty} |\varphi(t)|^2\,dt =$$

$$= \frac{1}{\pi}\sup_\varphi \sum_{s=1-[n/2]}^{n-[n/2]} \max_{0\leqq\varkappa_s\leqq\pi} \left| \int_{-\varkappa_s}^{\varkappa_s} \varphi(t)e^{ist}\,dt \right|^2 \Big/ \int_{-\infty}^{\infty} |\varphi(t)|^2\,dt$$

and the last supremum evidently remains unaltered if we restrict ourselves to functions $\varphi(t)$ with $\varphi(t)\equiv 0$, if $|t|>\pi$.

6. Now we are ready to prove the statement at the end of Section 1. First we suppose that $A_n < A$ for each integer n.

Then using in turn (5. 4), Lemmas 3 and 2 we have

$$\frac{n}{\pi}\alpha_\pi(\xi) < 2a_n^{(n)}+2 < 4A_n^{(2n)}+2 < (2+4\pi)A_n+2 < (2+4\pi)A+2$$

for every n, so by (5. 5)

$$\sup_\varphi \sum_{s=-\infty}^{\infty} \max_{0\leqq\varkappa_s\leqq\pi} \left| \int_{-\varkappa_s}^{\varkappa_s} \varphi(t)e^{ist}\,dt \right|^2 \Big/ \int_{-\pi}^{\pi} |\varphi(t)|^2\,dt < 2\pi\{(1+2\pi)A+1\},$$

or by (1. 4) $E(\varphi)$ is bounded if $\|\varphi\|=1$.

If, however $n\pi^{-1}\alpha_\pi(\xi)<\alpha$ for each integer n, then by (5. 4), Lemma 3 and (2. 5)

$$\alpha \geqq E(\varphi) \geqq n\pi^{-1}\alpha_\pi(\xi) > 2a_n^{(n)}-2 > 2A_n^{(2n)}-2 > A_n-2,$$

hence A_n is bounded.

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ON THE LAGRANGE INTERPOLATION BASED ON THE ZEROS OF THE ORTHONORMAL LEGENDRE POLYNOMIALS

By

G. RÓNA (Budapest)

To Professor G. ALEXITS *on his 70th anniversary*

Let $f(x)$ be a continuous function in the interval $[-1, 1]$ and $L_n(f, x, x_{kn})$ its Lagrange interpolating polynomial of degree at most $n-1$, which coincides $f(x)$ in the points x_{kn} where $1 > x_{1n} > \ldots > x_{kn} > \ldots > x_{nn} > -1$ are the zeros of the orthonormal Legendre polynomials of order n. The explicit expression for the Lagrange interpolating polynomials is very simple. If we denote by

$$l_{kn}(x, x_{kn}) = \prod_{\substack{i=1\\ k\neq i}}^{n} \frac{x - x_{in}}{x_{kn} - x_{in}}$$

the fundamental polynomials of Lagrange interpolation based on the nodes x_{kn}, the desired polynomials are

$$L_n(f, x, x_{kn}) = \sum_{i=1}^{n} f(x_{kn}) l_{kn}(x, x_{kn}) \tag{1}$$

and plainly

$$L_n(f, x_{in}, x_{kn}) = f(x_{in}).$$

For sake of brevity let us denote x_{kn} by x_k, $l_{kn}(x, x_{kn})$ by $l_k(x)$ and $L_n(f, x, x_{kn})$ by $L_n(f, x)$. It is well known that, if $f(x) \in \mathrm{Lip}\, \varrho\ (\frac{1}{2} < \varrho < 1)$ then

$$|L_n(f, x) - f(x)| \leqq c_0 n^{\frac{1}{2}-\varrho} \tag{2}$$

where $x \in [-1, 1]$ and henceforth all $c_i > 0$ are constants independent of n. In their paper [2] G. GRÜNWALD and P. TURÁN give estimation for the Lebesgue-function of Lagrange interpolation. From this result we have

$$\lambda_n(x, x_{kn}) \leqq c_1 n^{\frac{1}{2}} \quad \text{if} \quad x \in (-1, 1) \tag{3}$$

where

$$\lambda_n(x, x_{kn}) = \sum_{i=1}^{n} |l_i(x)|$$

and

$$\sum_{k=1}^{n} |l_k(x)| = \sum_{k=1}^{n} \left| \frac{P_n^{(\alpha)}(x)}{P_n^{(\alpha)\prime}(x_k)(x - x_k)} \right|$$

where $P_n^{(\alpha)}(x)$ are the orthonormal ultrasferical polynomials of order n. From (8. 9. 2) in [1] it follows that

$$\sum_{k=1}^{n} |l_k(1)| \sim n^{\alpha} \sum_{k=1}^{n} (1 - x_k)^{-1} |P_n^{(\alpha)}(x_k)|^{-1}.$$

The contribution of the positive zeros x_k in this sum is

$$\sim n^{\alpha} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{-2} k^{\alpha+\frac{3}{2}} n^{-\alpha-2} = \sum_{k=1}^{n} k^{\alpha-\frac{1}{2}}$$

which is $\sim n^{\alpha+\frac{1}{2}}$. The contribution of the negativ zeros is

$$\sim n^{\alpha} \sum_{k=1}^{n} |P_n^{(\alpha)\prime}(x_k)|^{-1} \sim n^{\alpha} \sum k^{\alpha+\frac{3}{2}} n^{-\alpha-2} \sim n^{\alpha+\frac{1}{2}}$$

but in our case $\alpha = 0$, thus

$$\sum_{k=1}^{n} |l_k(1)| \sim n^{\frac{1}{2}}. \tag{4}$$

The purpose of this paper is to give a function $f(x) \in \operatorname{Lip} \varrho$ for which the estimation (2) is exact, i.e. for which the next theorem holds.

THEOREM. *There exist* $f(x) \in \operatorname{Lip} \varrho$ *and* $c_2 > 0$ *for which*

$$|L_n(f, 1) - f(1)| > c_2 n^{\frac{1}{2}-\varrho} \tag{5}$$

holds.

PROOF. Let us denote by $g_n(x)$ the next function:

$$g_n(x) = 0, \quad \text{if} \quad |x| > x_m \quad \text{where} \quad m = \left[\frac{n}{d}\right], \; d > 1 \tag{6}$$

$$g_n(x_i) = -1, \quad \text{if} \quad i \text{ is even and} \quad m < i < n-m \tag{7}$$

$$g_n(x_i) = 1, \quad \text{if} \quad i \text{ is odd and} \quad m < i < n-m \tag{8}$$

$$g_n(x) \quad \text{is linear if} \quad x_i \leqq x \leqq x_{i-1}. \tag{9}$$

The functions $g_n(x)$ have the undermentioned properties:

$$|g_n(x)| \leqq 1, \tag{10}$$

$$g_n(1) = 0. \tag{11}$$

By Theorem 6. 21. 3 in [1] we get that (using the notation $x_k = \cos \Theta_k$)

$$\left(k - \frac{1}{2}\right) \frac{\pi}{n} \leqq \Theta_k \leqq k \frac{\pi}{n+1}$$

holds. From this it follows, that

$$\Theta_{k+1} - \Theta_k \geqq \frac{\pi}{2n}.$$

Easy to see, that

$$\arccos x_{k+1} - \arccos x_k = u < \frac{2}{\pi} \sin u \leqq c \frac{x_k - x_{k+1}}{\sqrt{1 - x_k^2}}$$

holds, where c is an absolute constant. Thus

$$x_k - x_{k+1} \geqq c \frac{\sqrt{1-x_k^2}}{n}.$$

But, if we choose the constant $d>1$ so, that

$$\sqrt{1-x_k^2} = \sin \Theta_k \geqq \sin \left(k - \frac{1}{2}\right) \frac{\pi}{n} \geqq \sin \left(m - \frac{1}{2}\right) \frac{\pi}{n} \geqq \sin \frac{\pi}{2d} > 0$$

holds, then

$$x_k - x_{k+1} \geqq \frac{c}{n}$$

if $m<k<n-m$. Thus

$$|g_n(x+h) - g_n(x)| \leqq c^* hn. \tag{12}$$

Furthermore, by definition

$$L_n(g_n, x) = \sum_{i=1}^{n} g_n(x_i) l_i(x) = \sum_{i=m}^{n-m} (-1)^{i+1} l_i(x) \tag{13}$$

thus

$$L_n(g_n, 1) = \sum_{i=m}^{n-m} (-1)^{i+1} l_i(1) = \sum_{i=m}^{n-m} |l_i(1)|$$

because easy to see, that

$$\operatorname{sign} l_i(1) = (-1)^{i+1}.$$

But

$$\sum_{i=m}^{n-m} |l_i(1)| \sim c^{**} n^{\frac{1}{2}}$$

because

$$\sum_{k=m}^{n-m} k^{-\frac{1}{2}} \sim c' n^{\frac{1}{2}} \quad \text{and} \quad \sum_{k=m}^{n-m} k^{\frac{3}{2}} n^{-2} \sim c'' n^{\frac{1}{2}}.$$

From this it follows, that there exists $c_3>0$, so that

$$|L_n(g_n, 1) > c_3 n^{\frac{1}{2}}. \tag{14}$$

Being

$$L_N(g_n, x) = \sum_{i=1}^{N} g_n(x_i) l_i(x) \tag{15}$$

thus, from (2), (11) and (12)

$$|L_n(g_n, 1)| \leqq c_4 n N^{-\frac{1}{2}}. \tag{16}$$

Furthermore, from (3), (10) and (15) we have

$$|L_N(g_n, 1)| \leqq c_5 N^{\frac{1}{2}}. \tag{17}$$

By $\lim_{n\to\infty} n^{-\varrho} = 0$ and $\frac{1}{2} < \varrho < 1$, we can choose n_k such that

$$n_{k+1}^{-\varrho} \leqq c_6 n_k^{-\varrho} \quad \text{where} \quad c_6 < 1 \tag{18}$$

and

$$n_k^{1-\varrho} \geqq c_7 \sum_{i=1}^{k-1} n_i^{1-\varrho}. \tag{19}$$

Then from (18)

$$\sum_{i=k}^{\infty} n_i^{-\varrho} \leqq \frac{1}{1-c_6} n_k^{-\varrho}.$$

Let

$$f(x) = \sum_{i=1}^{\infty} \omega\left(\frac{1}{n_i}\right) g_{n_i}(x) = \sum_{i=1}^{\infty} n_i^{-\varrho} g_{n_i}(x)$$

where $\omega(h) = c|h|^{\varrho}$. This function fulfils the requirement (5). The definition is correct, because from $\frac{1}{2} < \varrho < 1$ and (10) it follows, that the right hand side series has a convergent numerical majorant series. From (11)

$$f(1) = 0. \tag{20}$$

Evidently

$$c' n_i^{-\varrho} = \omega\left(\frac{1}{n_i}\right) = \omega\left(\frac{h}{hn_i}\right) \leqq \frac{1}{hn_i}\omega(h) \leqq c'' \cdot \frac{1}{hn_i}|h|^{\varrho},$$

thus, from (12)

$$|f(x+h)-f(x)| \leqq \sum_{i=1}^{\infty} n_i^{-\varrho} |g_{n_i}(x+h) - g_{n_i}(x)| \leqq c_8 |h|^{\varrho} \tag{21}$$

i.e. $f(x) \in \text{Lip}\, \varrho$. For the proof of our theorem it is enough to show that

$$L_{n_k}(f, 1) \geqq c_9 n_k^{\frac{1}{2}-\varrho} \tag{22}$$

holds, because $f(1) = 0$ and from (22) it follows, that (5) is also true. But

$$L_{n_k}(f, 1) = \sum_{i=1}^{\infty} n_i^{-\varrho} L_{n_k}(g_{n_i}, 1) \tag{23}$$

and from (14)

$$L_{n_k}(g_{n_k}, 1) \geqq c_3 n^{\frac{1}{2}}, \tag{24}$$

$$n_k^{-\varrho} L_{n_k}(g_{n_k}, 1) \geqq c_{10} n_k^{-\varrho} n_k^{\frac{1}{2}} = c_{10} n_k^{\frac{1}{2}-\varrho}. \tag{25}$$

On the other hand, we can choose the constants c_i to satisfy the next inequalities:

$$c_7 = \frac{3c_0}{2c_3}; \quad c_9 = \frac{c_3}{3}; \quad c_{12} \leqq \frac{c_3}{3}; \quad c_{14} \leqq \frac{c_3}{3}.$$

Thus using (16) and (19), we get

$$\sum_{i=1}^{k-1} n_i^{-\varrho} |L_{n_k}(g_{n_i}, 1)| \leqq c_{11} n_k^{-\frac{1}{2}} \sum_{i=1}^{k-1} n_i^{1-\varrho} \leqq c_{12} n_k^{\frac{1}{2}-\varrho} \tag{26}$$

and from (17), (18), (20)

$$\sum_{i=k}^{\infty} n_i^{-\varrho} |L_{n_k}(g_{n_i}, 1)| \leqq c_{13} n_k^{-\frac{1}{2}} \sum_{i=k}^{\infty} n_i^{-\varrho} \leqq c_{14} n_k^{\frac{1}{2}-\varrho}. \tag{27}$$

From (24), (26), (27), (23) and (24) we get (22). Q.e.d.

REMARK. In aforesaid way easy to prove, that using the zeros of the orthonormal ultrasferical polynomials $P_n^{(\alpha)}(x)$, $\alpha > -\frac{1}{2}$ we can choose a function $h(x)$ in Lip ϱ, $\frac{1}{2} < \varrho < 1$ for which

$$|L_n(h, 1, x_{kn}^{(\alpha)}) - h(1)| > cn^{\alpha+\frac{1}{2}-\varrho}$$

holds. The case $\alpha = -\frac{1}{2}$ was treated by O. KIS in [3].

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ON THE DIVERGENCE OF THE SEQUENCE OF LINEAR OPERATORS

By

P. O. H. VÉRTESI (Budapest)*

To Professor G. ALEXITS *on his 70th anniversary*

1. Introduction

Let

$$0 \leqq x_{0n} < x_{1n} < \ldots < x_{2nn} < 2\pi \qquad (n=0, 1, 2, \ldots) \tag{1}$$

be an infinite system of points. Let us denote by $C(\omega)$ the class of continuous, 2π-periodical functions $f(x)$ for which

$$\omega(f;\, t) \leqq a(f)\omega(t) \tag{2}$$

holds. Here $\omega(f;\, t)$ is the modulus of continuity of $f(x)$, $\omega(t)$ is an arbitrary modulus of continuity, $a(f)$ depends only on $f(x)$.

Let

$$L_n(f;\, x) = \sum_{k=0}^{2n} f(x_{kn}) l_{kn}(x) \qquad (n = 0, 1, 2, \ldots) \tag{3}$$

be the trigonometric interpolating polynomial of degree n of the function $f(x) \in C(\omega)$ based on the nodes of system (1); $l_{kn}(x)$ are the fundamental polynomials of the trigonometrical interpolation. At last let

$$\lambda_n(x) = \sum_{k=0}^{2n} |l_{kn}(x)|, \quad \lambda_n = \max_{0 \leqq x < 2\pi} \lambda_n(x) \qquad (n = 0, 1, 2, \ldots) \tag{4}$$

be the Lebesgue function and the Lebesgue constant of the interpolation, respectively.

We want to investigate, how the difference

$$L_n(f;\, x) - f(x)$$

is expressible by the quantities $\omega(t)$ and λ_n, and if we know these, we are able in many cases to decide whether the interpolation process is convergent or not, that is the relation

$$\|L_n(f;\, x) - f(x)\| \to 0 \qquad (n \to \infty)$$

is true or not. (Here $\|g\| = \max_{0 \leqq x < 2\pi} |g(x)|$, $g(x)$ is a 2π-periodical continuous function.)

We could search the similar question in the theory of the Lagrange interpolation, it is easy to define the corresponding expression.

There is a lot of results about the above mentioned problem (see e.g. the references in [1]).

Our aim is to generalize the problem and some results proved by O. Kis.

* Some of my papers is written under the name P. Vértesi.

2. Generalizations

2. 1. Let us denote now by $\omega(t)$ the function having the following properties:

(i) $\omega(t)>0$ if $t>0$, $\omega(0)=0$, $\omega(T)\geqq\omega(t)$ if $T\geqq t\geqq 0$; $\omega(t)$ is a continuous function.

(ii) $\dfrac{t^m}{\omega(t)}$ is a monotone increasing function.

(iii) $\lim\limits_{t=+0}\dfrac{t^m}{\omega(t)}=0$ (m is a fixed integer, $m\geqq 1$).

Let us denote by $\tilde{C}_m(\omega)$ the class of continuous, 2π-periodical functions $f(x)$ for which

(5) $$\omega_m(f;\,t)\leqq a_m(f)\omega(t)$$

holds. Here $\omega_m(f;\,t)$ is the modulus of continuity of order m of $f(x)$, $a_m(f)$ depends only on $f(x)$, $\omega(t)$ is defined by (i), (ii) and (iii).

Now we define a very important quantity. Let

(6) $$d_n=\min_{0\leqq k\leqq 2n}(x_{k+1,n}-x_{k,n})\qquad (n=0,1,2,\dots).$$

(Here $x_{2n+1,n}=x_{0,n}+2\pi$; x_{in} is defined by (1).)

O. Kis proved in his paper [1] the following

THEOREM (O. KIS). *There exists an $f(x)\in\tilde{C}_m(\omega)$ for which the relation*

$$\|L_{n_k}(f;\,x)-f(x)\|\geqq\lambda_{n_k}\omega(d_{n_k})\qquad (k=1,2,3,\dots)$$

is valid, where $0<n_1<n_2<n_3<\dots$ are integers. An analogous theorem is true in the case of the Lagrange-interpolation.

2. 2. To generalize this theorem we define the expressions $\bar{l}_{kn}(x)$, $\bar{L}_n(f;\,x)$, $\bar{\lambda}_n(x)$ and $\bar{\lambda}_n$.

Let $[a,b]$ be an arbitrary finite interval,

(7) $$\begin{cases} a\leqq\bar{x}_{1n}<\bar{x}_{2n}<\dots<\bar{x}_{pn}\leqq b & (n=1,2,3,\dots),\ \text{where}\\ p=p(n)\ \text{and}\ \varlimsup\limits_{n=\infty}p(n)=\infty & (\text{e.g. } p=n\ \text{or}\ p=2n+1),\\ \qquad p\ \text{is integer}; & \end{cases}$$

(8) $$\bar{l}_{kn}(x)\qquad (n=1,2,3,\dots;\ k=1,2,\dots,p(n))$$

are continuous functions on the interval $[a,b]$.

If $f(x)$ is a continuous function on $[a,b]$, then

(9) $$\bar{L}_n(f;\,x)=\sum_{k=1}^{p}f(\bar{x}_{kn})\bar{l}_{kn}(x),$$

(10) $$\bar{\lambda}_n(x)=\sum_{k=1}^{p}|\bar{l}_{kn}(x)|,\quad \bar{\lambda}_n=\max_{a\leqq x\leqq b}\bar{\lambda}_n(x).$$

Denoting by $C_m^{[a,b]}(\omega)$ the class of continuous on $[a, b]$ functions $f(x)$ for which (5) is valid, we can prove the following

THEOREM I. *If* $\lim_{n=\infty} \bar{\lambda}_n \neq 1$,* *then there exists an* $f(x) \in C_m^{[a,b]}(\omega)$ *for which*

(11) $$\|\bar{L}_{n_k}(f; x) - f(x)\| > \bar{\lambda}_{n_k} \omega(\bar{d}_{n_k}) \qquad (k = 1, 2, 3, \dots),$$

where $0 < n_1 < n_2 < n_3 < \dots$ *are integers and*

(12) $$\bar{d}_n = \min_{1 \leqq k \leqq p-1} (\bar{x}_{k+1,n} - \bar{x}_{k,n}) \qquad (n = 1, 2, 3, \dots).$$

We can see that *we suppose only that $l_{kn}(x)$ are continuous functions, and the theorem is valid for „almost every" sequence of the numbers $\bar{\lambda}_n$.*

PROOF. Our proof is similar to the proof in [1]. For the sake of brevity let us denote $\bar{x}_{kn}$ by x_k, $\bar{l}_{kn}(x)$ by $l_k(x)$ etc. (Convention (M)).

First of all let us consider the very simple case when

(α) $$\underline{\lim}_{n=\infty} \lambda_n = 1 - 2\varepsilon \qquad (\varepsilon > 0).$$

Now we can choose the sequence $n_1 < n_2 < \dots$ such that

$$\lambda_{n_k} \leqq 1 - \varepsilon \qquad (k = 1, 2, 3, \dots).$$

If $f(x) \equiv 1$, then evidently $f(x) \in C_m^{[a,b]}(\omega)$ and

$$f(x) - L_{n_k}(f; x) = 1 - \sum_{k=1}^{p(n_k)} l_k(x) \geqq 1 - (1 - \varepsilon) = \varepsilon.$$

If $\omega(t) < \varepsilon$ (this is true when $t < \delta(\varepsilon)$ — see (i)), then let n_1 so large that $d_{n_1} \leqq t$, and let $d_{n_1} > d_{n_2} > \dots$. These are possible by the relations $\overline{\lim}_{n=\infty} p(n) = \infty$ and $d_n \leqq \dfrac{b-a}{p(n)}$. But then

$$f(x) - L_{n_k}(f; x) \geqq \varepsilon > \omega(t) \geqq \omega(d_{n_k}) > \omega(d_{n_k})(1 - \varepsilon) \geqq \omega(d_{n_k})\lambda_{n_k} \qquad (k = 1, 2, 3, \dots),$$

as we stated.

We shall consider the more complicated case

(β) $$\overline{\lim}_{n=\infty} \lambda_n = \infty \quad \text{or} \quad \overline{\lim}_{n=\infty} \lambda_n = 1 + 2\varepsilon \qquad (\varepsilon > 0).$$

Let z_n be a point such that

(13) $$\lambda_n = \lambda_n(z_n) = \sum_{k=1}^{p} |l_k(z_n)| \qquad (n = 1, 2, 3, \dots).$$

Let us define the function $g_n(x)$ $(n = 1, 2, 3, \dots)$ for all $x \in [a, b]$ as follows. Let

(14) $$\begin{cases} g_n(x_k) = \operatorname{sign} l_k(z_n) & (k = 1, 2, \dots, p), \\ g_n(x_k) = 1 \quad \text{if} \quad l_k(z_n) = 0 & (k = 1, 2, \dots, p), \end{cases}$$

* The existence of $\lim_{n=\infty} \bar{\lambda}_n$ is not assumed.

furthermore

$$g_n(x) = \begin{cases} g_n(x_1) & \text{if} \quad a \leqq x < x_1, \\ g_n(x_p) & \text{if} \quad b \geqq x > x_p. \end{cases} \tag{15}$$

We now define $g_n(x)$ for $x \in (x_k, x_{k+1})$ $(k=1, 2, \ldots, p-1)$. Let $g_n(x)$ be in this interval the Hermite-interpolating polynomial of degree $2m-1$ which is equal to $g_n(x_k)$ or $g_n(x_{k+1})$ at the end-points, respectively, and in these end-points $g_n'(x_k) = g_n''(x_k) = \ldots = g_n^{(m-1)}(x_k) = g_n'(x_{k+1}) = g_n''(x_{k+1}) = \ldots = g_n^{(m-1)}(x_{k+1}) = 0$. Evidently, if $g_n(x_k) = g_n(x_{k+1})$ then this polynomial is equal to the constant $g_n(x_k)$.

Let us suppose that $g_n(x_k) = -g_n(x_{k+1})$. Then we can easily prove that

$$g_n(x) = g_n(x_k)\left[1 - \frac{(2m-1)!!}{(2m-2)!!} \int\limits_{-1}^{2\cdot\frac{x-x_k}{x_{k+1}-x_k}-1} (1-t^2)^{m-1}\, dt\right] \quad (x_k \leqq x \leqq x_{k+1}).^* \tag{16}$$

Let us consider some properties of $g_n(x)$. Evidently it has the continuous derivative $g_n^{(m-1)}(x)$ and

$$\omega(g_n^{(m-1)}; t) \leqq B_m d_n^{-m} t. \tag{17}$$

Here the constant $B_m > 0$ depends only on the fixed number m. But then (see e.g. [2], p. 116)

$$\omega_m(g_n; t) \leqq B_m d_n^{-m} t^m \qquad \left(0 < t \leqq \frac{b-a}{n}\right) \tag{18}$$

and from this by the formula (iii), we can say that $g_n(x) \in C_m^{[a,b]}(\omega)$ (see (5)).

The following properties are evident:

$$|g_n(x)| \leqq 1 \qquad (n = 1, 2, 3, \ldots), \tag{19}$$

$$L_n(g_N; x) = \left|\sum_{k=1}^{p} g_N(x_k) l_k(x)\right| \leqq \sum_{k=1}^{p} |l_k(x)| \leqq \lambda_n \qquad (n, N = 1, 2, 3, \ldots), \tag{20}$$

$$L_n(g_n; z_n) = \sum_{k=1}^{p} g_n(x_k) l_k(z_n) = \sum_{k=1}^{p} |l_k(z_n)| = \lambda_n, \tag{21}$$

(see (14), (13)).

If there exist the fixed number N and the sequence $0 < n_1 < n_2 < n_3 < \ldots$ such that

$$\|L_n(g_N; x) - g_N(x)\| > \omega(d_n)\lambda_n \qquad (n = n_1, n_2, n_3, \ldots), \tag{22}$$

* We used the equality

$$\int\limits_{-1}^{+1} (1-t^2)^{m-1}\, dt = 2\,\frac{(2m-2)!!}{(2m-1)!!} \qquad (m = 1, 2, 3, \ldots).$$

then the theorem is true; so we can suppose that

$$\|L_n(g_N; x) - g_N(x)\| \leqq \omega(d_n)\lambda_n, \quad \text{if} \quad n > M(N) \tag{23}$$

(N is an arbitrary integer, $N > 0$; $M(N)$ is an integer also depending only on n).

We now define the sequence of indices $0 < n_1 < n_2 < n_3 < \dots$ as follows:

$$\lambda_{n_i} \geqq 1 + \varepsilon \qquad (i = 1, 2, 3, \dots), \tag{24}$$

$$\begin{cases} \omega(d_{n_1}) \leqq q, \quad \omega(d_{n_{i+1}}) \leqq q\omega(d_{n_i}) \qquad (i = 1, 2, 3, \dots), \\ \text{where} \quad 0 < q < 1, \end{cases} \tag{25}$$

$$\sum_{i=1}^{j-1} \omega(d_{n_i})\, d_{n_i}^{-m} \leqq \omega(d_{n_j})\, d_{n_j}^{-m} \qquad (j = 2, 3, 4, \dots), \tag{26}$$

$$n_{i+1} > M(n_i) \qquad (i = 1, 2, 3, \dots). \tag{27}$$

These are possible by the relations (β), $\overline{\lim\limits_{n=\infty}}\, p(n) = \infty$, $d_n \leqq \dfrac{b-a}{p(n)}$, (i) and (iii).

Let

$$f(x) = Q \sum_{i=1}^{\infty} \omega(d_{n_i}) g_{n_i}(x) \qquad (Q > 1). \tag{28}$$

Here the right hand side series converges for all x, by (19) and (25). Clearly, $f(x)$ is a continuous function on $[a, b]$. We show that $f(x) \in C_m^{[a,b]}$. Let $0 < h \leqq t \leqq \dfrac{b-a}{m}$ be arbitrary. Evidently

$$\omega_m(f; t) \leqq Q \sum_{i=1}^{\infty} \omega(d_{n_i}) \omega_m(g_{n_i}; t). \tag{29}$$

Now, if

$$d_{n_{j+1}} < t \leqq d_{n_j} \tag{30}$$

($d_{n_1} > d_{n_2} > d_{n_3} > \dots$ by (25) and (i); if $d_{n_1} < t, j = 0$), then by (18); (26); (ii) and (30) we get

$$\sum_{i=1}^{j} \omega(d_{n_i}) \omega_m(g_{n_i}; t) \leqq B_m t^m \sum_{i=1}^{j} \omega(d_{n_i})\, d_{n_i}^{-m} \leqq 2B_m t^m \omega(d_{n_j})\, d_{n_j}^{-m} \leqq 2B_m \omega(t). \tag{31}$$

(If $j = 0$, then this part is omitted.)

Furthermore by (19), (24) and (30) we have

$$\sum_{i=j+1}^{\infty} \omega(d_{n_i}) \omega_m(g_{n_i}; t) \leqq 2^m \sum_{i=j+1}^{\infty} \omega(d_{n_i}) \leqq 2^m \frac{1}{1-q} \omega(d_{n_{j+1}}) \leqq \frac{2^m}{1-q} \omega(t). \tag{32}$$

That is, by (31) and (32) we can say, that $f(x) \in C_m^{[a,b]}(\omega)$. Now we shall prove our theorem, the inequality (11). It is clear that

$$(33)\quad L_{n_k}(f; z_{n_k}) - f(z_{n_k}) = \sum_{j=1}^{p(n_k)} \left[Q \sum_{i=1}^{\infty} \omega(d_{n_i}) g_{n_i}(x_j) \right] l_j(z_{n_k}) - Q \sum_{i=1}^{\infty} \omega(d_{n_i}) g_{n_i}(z_{n_k}) =$$

$$= Q \sum_{i=1}^{\infty} \omega(d_{n_i}) \left[\sum_{j=1}^{p(n_k)} g_{n_i}(x_j) l_j(z_{n_k}) - g_{n_i}(z_{n_k}) \right] = Q \sum_{j=1}^{\infty} \omega(d_{n_i}) [L_{n_k}(g_{n_i}; z_{n_k}) - g_{n_i}(z_{n_k})] =$$

$$= Q \left\{ \sum_{i=1}^{k-1} \omega(d_{n_i}) [L_{n_k}(g_{n_i}; z_{n_k}) - g_{n_i}(z_{n_k})] + \omega(d_{n_k}) L_{n_k}(g_{n_k}; z_{n_k}) + \right.$$

$$\left. + \sum_{i=k+1}^{\infty} \omega(d_{n_i}) L_{n_k}(g_{n_i}; z_{n_k}) - \sum_{i=k}^{\infty} \omega(d_{n_i}) g_{n_i}(z_{n_k}) \right\} \geqq$$

$$\geqq Q \left[\omega(d_{n_k}) L_{n_k}(g_{n_k}; z_{n_k}) - \sum_{i=1}^{k-1} \omega(d_{n_i}) |L_{n_k}(g_{n_i}; z_{n_k}) - g_{n_i}(z_{n_k})| - \right.$$

$$\left. - \sum_{i=k+1}^{\infty} \omega(d_{n_i}) |L_{n_k}(g_{n_i}; z_{n_k})| - \sum_{i=k}^{\infty} \omega(d_{n_i}) |g_{n_i}(z_{n_k})| \right].$$

Let us estimate the parts figuring here.

$$(34)\qquad \omega(d_{n_k}) L_{n_k}(g_{n_k}; z_{n_k}) = \omega(d_{n_k}) \lambda_{n_k},$$

$$(35)\qquad \sum_{i=1}^{k-1} \omega(d_{n_i}) |L_{n_k}(g_{n_i}; z_{n_k}) - g_{n_i}(z_{n_k})| \leqq \omega(d_{n_k}) \lambda_{n_k} \sum_{i=1}^{k-1} \omega(d_{n_i}) \leqq \frac{q}{1-q} \omega(d_{n_k}) \lambda_{n_k},$$

$$(36)\qquad \sum_{i=k+1}^{\infty} \omega(d_{n_i}) |L_{n_k}(g_{n_i}; z_{n_k})| \leqq \lambda_{n_k} \sum_{i=k+1}^{\infty} \omega(d_{n_i}) \leqq \lambda_{n_k} \omega(d_{n_k}) \frac{q}{1-q},$$

$$(37)\qquad \sum_{i=k}^{\infty} \omega(d_{n_i}) |g_{n_i}(z_{n_k})| \leqq \omega(d_{n_k}) \frac{1}{1-q}.$$

At these estimations we applied the formulae (31); (23), (25); (20), (25); (19) and (25).

That is, by (33), (34), (35), (36) and (37) we have

$$(38)\qquad L_{n_k}(f; z_{n_k}) - f(z_{n_k}) \geqq \omega(d_{n_k}) \left(Q\lambda_{n_k} - 2\lambda_{n_k} \frac{Qq}{1-q} - \frac{Q}{1-q} \right).$$

To finish our considerations, we have to prove that there exist Q and q for which

$$(39)\qquad Q\lambda_{n_k} - 2\lambda_{n_k} \frac{Qq}{1-q} - \frac{Q}{1-q} > \lambda_{n_k}.$$

Evidently, if

$$\lambda_{n_k} \left(Q - 1 - \frac{2Qq}{1-q} \right) > \frac{Q}{1-q},$$

then (39) is valid.

Using $\lambda_{n_k} \geqq 1+\varepsilon$, we can restrict ourselves to verify

$$(1+\varepsilon)\left(Q-1-\frac{2Qq}{1-q}\right) > \frac{Q}{1-q},^{*}$$

i.e.

$$Q\left[1+\varepsilon-\frac{(1+\varepsilon)2q}{1-q}-\frac{1}{1-q}\right] > 1+\varepsilon. \tag{40}$$

But evidently the part in brackets >0 if q is small enough, so (40) is valid for a large enough Q. That is, (39) is true, also.

So we perfectly proved our statement. Q.e.d.

2. 3. Evidently our theorem is valid in the periodic case, too (see notations (7)—(10)):

THEOREM II. *If $\bar{l}_{kn}(x)$ are 2π-periodic continuous functions, $\lim_{n=\infty} \bar{\lambda}_n \neq 1$, then there exists a 2π-periodic continuous function $f(x) \in \tilde{C}_m(\omega)$ for which*

$$\|\bar{L}_{n_k}(f; x)-f(x)\| > \bar{\lambda}_{n_k}\omega(\bar{d}_{n_k}) \qquad (k=1, 2, 3, \ldots), \tag{41}$$

where $0<n_1<n_2<n_3<\ldots$ are integers and

$$\bar{d}_n = \min_{1\leqq k\leqq p}(x_{k+1,n}-x_{k,n}), \quad x_{p+1,n} = x_{1,n}+2\pi.$$

The proof is analogous to that of Theorem I.

2. 4. *We can easily see that Theorem I and Theorem II will be true if in the estimations* (11) *and* (41) *on the right hand side we write $c\cdot\omega(\bar{d}_{n_k})\bar{\lambda}_{n_k}$, where $c>0$ is an arbitrary fixed number.*

Indeed, we have to consider $cf(x)$ instead of $f(x)$:

$$\|\bar{L}_{n_k}(cf; x)-cf(x)\| = c\,\|\bar{L}_{n_k}(f; x)-f(x)\| > c\bar{\lambda}_{n_k}\omega(\bar{d}_{n_k}).$$

2. 5. Now the problem arises whether in the estimations (11) and (41) the order of estimation cannot be improved.

We can see that — not considering the case $\lim_{n=\infty} \bar{\lambda}_n = 1$ — the estimations (11) and (41) are the best, apart from some kind of constant number, depending maybe on m.

Indeed, let us consider a special case, e.g. the trigonometric interpolation. O. Kis [1] has proved that for *every* $f(x) \in \tilde{C}_m(\omega)$

$$\|L_n(f; x)-f(x)\| \leqq b_m(f)(1+\lambda_n)\omega\left(\frac{1}{n}\right) \qquad (n=1, 2, 3, \ldots) \tag{42}$$

is valid (see notations (3)—(6)).

* Here $Q-1-\dfrac{2Qq}{1-q}>0$, if e.g. $Q \geqq 2$, $q \leqq 0.1$.

Let $d_n \geqq \frac{c}{n}$ then $\left(1+\frac{1}{c}\right)\omega(d_n) > \omega\left(\frac{1}{n}\right)$, so the order of our lower estimation cannot be improved. (Naturally, we can consider some other special cases, too.)

2. 6. *What is the situation when* $\lim\limits_{n=\infty} \lambda_n = 1$? In this case I was unable to prove the relations like (11) or (41), so it is maybe true or not (I think it is true), but *in a special case a formula like* (42) *is valid.*

For this aim let us consider the class $C_1^{[a,b]}(\omega)$, that is, the continuous functions for which

$$\omega(f; t) \leqq a_1(f)\omega(t).$$

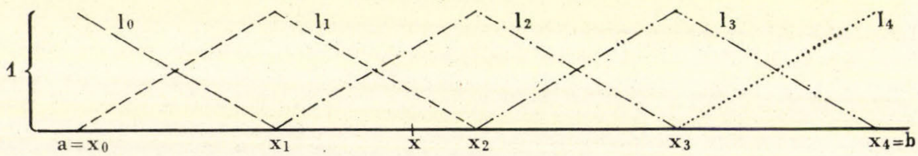
Let

$$p = n+1, \quad x_{k_n} = k\frac{b-a}{n} \qquad (k = 0, 1, 2, \dots, n).$$

Then

$$d_n = \frac{b-a}{n} \qquad (n = 1, 2, 3, \dots).$$

We define the functions $l_k(x)$ as follows ($n=4$)



That is

$$l_k(x) = \begin{cases} 1 & \text{if} \quad x = x_k \\ 0 & \text{if} \quad x = x_{k+1} \quad \text{or} \quad x = x_{k-1} \\ & \text{linear function on } [x_{k-1}, x_k] \quad \text{or} \quad [x_k, x_{k+1}] \\ 0 & \text{elsewhere} \end{cases} \qquad (k = 1, 2, \dots, n-1),$$

$$l_0(x) = \begin{cases} 1 & \text{if} \quad x = x_0 \\ 0 & \text{if} \quad x = x_1 \\ & \text{linear function on } [x_0, x_1] \\ 0 & \text{elsewhere} \end{cases}$$

$$l_n(x) = \begin{cases} 1 & \text{if} \quad x = x_n \\ 0 & \text{if} \quad x = x_{n-1} \\ & \text{linear function on } [x_{n-1}, x_n] \\ 0 & \text{elsewhere.} \end{cases}$$

By these definitions we can easily prove the relation

(43) $$\|L_n(f; x) - f(x)\| \leqq a_1(f)\lambda_n\omega(d_n) \qquad (n=1, 2, 3, \dots)$$

for every $f(x) \in C_1^{[a,b]}(\omega)$.

Indeed, evidently for every point $x \in [a, b]$

$$\lambda_n(x) = \sum_{k=0}^{n} |l_k(x)| = \sum_{k=0}^{n} l_k(x) = \lambda_n = 1 \qquad (n = 1, 2, 3, \dots),$$

so clearly $\lim\limits_{n=\infty} \lambda_n = 1$.

If $x \in [x_i, x_{i+1}]$ $(i = 0, 1, \dots, n-1)$ and $h = \dfrac{x_{i+1}-x}{d_n}$, then we have

$$|L_n(f;x)-f(x)| = \left|\sum_{k=0}^{n} f(x_k) l_k(x) - f(x)\right| = |hf(x_i)+(1-h)f(x_{i+1})-f(x)| =$$

$$= |h[f(x_i)-f(x)]+(1-h)[f(x_{i+1})-f(x)]| \leqq (h+1-h)\omega(f;d_n) \leqq a_1(f)\lambda_n\omega(d_n),$$

as it was stated. (By these considerations the restrictions $a=x_0$, $b=x_n$ can be omitted in a very simple way and for d_n the condition $d_n \geqq \frac{c}{n}$ (c fixed) is enough.)

The relation (43) shows that if (11) or (41) is true with $\lim_{n=\infty} \lambda_n = 1$, then the order of the estimation cannot be improved neither in this case.

3. An application

3. 1. Now we give an application of our Theorem I. In his paper [3] O. KIS has found a theorem about the trigonometric (0, 2) interpolation. His results were generalized by A. SHARMA and A. K. VARMA in [4].

The problem was to obtain the explicit form of the trigonometric polynomial $R_n(x)$ of order n and to establish their uniqueness in the $(0, M)$ case, that is, when

$$(44)\qquad R_n(x_{kn}) = \alpha_{kn},\quad R_n^{(M)}(x_{kn}) = \beta_{kn},\quad x_{kn} = \frac{2k\pi}{n} \qquad (k = 0, 1, \dots, n-1)$$

are prescribed, M being a fixed positive integer $\geqq 1$. (For the sake of simplicity we shall throughout write x_k, α_k, β_k for $x_{k_n}, \alpha_{k_n}, \beta_{k_n}$ respectively.) Here we consider only the case when M is even.

If the trigonometric polynomial $R_n(x)$ has the form

$$(45)\qquad R_n(x) = \sum_{k=0}^{n-1} \alpha_k F(x-x_k) + \sum_{k=0}^{n-1} \beta_k G(x-x_k),$$

$F(x)$ and $G(x)$ are defined in [4] by (4), (5) and (6), then $R_n(x)$ satisfies the condition (44) (see [4]).

Let

$$(46)\qquad R_n(x) = \sum_{k=0}^{n-1} f(x_k) F(x-x_k) + \sum_{k=0}^{n-1} \beta_k G(x-x_k).$$

THEOREM (SHARMA and VARMA). (M even, $\geqq 2$). *If $f(x)$ is a continuous 2π-periodic function and satisfies the Zygmund condition*

$$(\lambda)\qquad f(x+h)-2f(x)+f(x-h) = o(h)$$

and if $\beta_k = o(n^{M-1})$ $(k=0, 1, \dots, n-1)$, n odd, where $R_n(x)$ is given by (46) *and M is even, then $R_n(x)$ converges uniformly to $f(x)$ on every finite interval on the x-axis.*

Even if all β_k are zero, the condition (λ) cannot be replaced by a Lipschitz condition of order $\alpha < 1$.

This theorem is a generalization of the O. Kis' theorem in [3]; he proved the case $M=2$.

Now we prove the following

THEOREM III. *In the above mentioned theorem of Sharma and Varma the condition* (λ) *cannot be replaced by the condition*

$$f(x+h)-2f(x)+f(x-h) = O(h), \tag{47}$$

even if all β_k *are zero, that is, in this case there exist a continuous* 2π*-periodic function* $f(x)$ *and a sequence* $0<n_1<n_2<n_3<\dots$ *such that*

$$\|R_{n_k}(x)-f(x)\|_{[0,2\pi)} \geqq c \qquad (k=1, 2, 3, \dots; c>0). \tag{48}$$

PROOF. *If* $\beta_k=0$ $(k=0, 1, \dots, n-1)$, then by (46)

$$R_n(x) = \sum_{k=0}^{n-1} f(x_k)F(x-x_k).$$

Let $\omega(t)=t$, then for every $f(x)\in \breve{C}_2(\omega)$

$$\omega_2(f; t)\leqq a_1(f)t.$$

Let us consider the Theorem II, taking $l_k(x) = F(x-x_k)$,

$$L_n(f; x) = R_n(x) \quad \text{and} \quad \lambda_n = \max_{0\leqq x<2\pi} \sum_{k=0}^{n-1} |F(x-x_k)|.$$

Using $\lambda_n\geqq c_1 n$ (see (31) in [4]) and considering that in the case $f(x)\in \breve{C}_2(\omega)$ $f(x)$ satisfies (47), we have an $f(x)$ for which (47) is valid and

$$\|L_{n_k}(f; x)-f(x)\| > \omega(d_{n_k})\lambda_{n_k} = \omega\left(\frac{2\pi}{n_k}\right)\lambda_{n_k} > \frac{2\pi}{n_k}c_1 n_k = 2c_1\pi = c,$$

as we stated. Q.e.d.

3. 2. We shall investigate some further applications in another paper.

Finally, I wish to give many thanks to O. Kis for his help rendered at these problems.

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DIRECT THEOREMS IN THE THEORY OF APPROXIMATION

By

G. SUNOUCHI (Sendai)

Dedicated to Professor G. ALEXITS *on his 70th birthday*

1. Let $f(x)$ be a continuous and periodic function with period 2π and let its Fourier series be

$$f(x) \sim a_0/2 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(x). \tag{1. 1}$$

Let $\gamma_k(n)$ $(k=0, 1, 2, \ldots)$ $(\gamma_0(n)=1)$ be summating functions and consider a family of approximation of $f(x)$ by the summation of (1. 1) by the matrix $\gamma_k(n)$, that is to say,

$$P_n(f; x) \sim \sum_{k=0}^{\infty} \gamma_k(n) A_k(x), \tag{1. 2}$$

where the parameter n need not be discrete. Of course we mean that the series in (1. 2) is the Fourier series of $P_n(f; x)$ which is also supposed to be continuous for each n. By a theorem of (C, C)-multiplier, the last fact is equivalent to the following. There is a family of even functions $\Phi_n(t)$ of bounded total variation such that

$$P_n(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t)\, d\Phi_n(t), \tag{1. 3}$$

where

$$\gamma_k(n) = \frac{2}{\pi} \int_0^{\pi} \cos kt\, d\Phi_n(t).$$

In these circumstances, a general theorem of saturation is as follows (G. SUNOUCHI [8], [9]).

THEOREM 1. *Suppose that*

$$\lim_{n\to\infty} \frac{1-\gamma_k(n)}{\varphi(n)\psi(k)} = c \neq 0 \qquad (k = 1, 2, \ldots) \tag{1. 4}$$

where $\varphi(n)$ is positive non-increasing and $\psi(k)$ is non-vanishing. Then

$$\|f-P_n\| = o\{\varphi(n)\} \text{ [1]} \tag{1°}$$

if and only if $f(x)$ is a constant.

[1] Norm is the uniform norm.

If

(2°) $$\|f-P_n\| = O\{\varphi(n)\}$$

then the formal trigonometric series

(1. 5) $$\sum_{k=1}^{\infty} \psi(k) A_k(x)$$

represents the Fourier series of a function of L^∞.
(3°) *When* (1. 5) *is the Fourier series of a function of* L^∞, *and*

(1. 6) $$\lambda_k(n) = \frac{1-\gamma_k(n)}{\varphi(n)\psi(k)}$$

is an (L^∞, L^∞)*-multiplier uniform in n, then* $\|f-P_n\| = O\{\varphi(n)\}$.

The proposition (2°) is the so-called converse theorem of saturation and (3°) is the direct theorem. The above converse theorem is satisfactory and it was given originally by G. SUNOUCHI—C. WATARI [7] and F. HARSILADZE [3]. At first the direct theorem was considered less important, because many direct problems were solved by the estimation of kernels of transformation (1. 3) directly or by another devices for special approximation processes without appealing to the proposition (3°). After Professor G. ALEXITS' work [1] as a pioneer, there are many contributions for these directions.

The object of this paper is to study the condition of (3°) for a comparatively broad class of approximation processes which contains traditional linear approximation processes. Firstly we shall prove that the condition of (3°) is not always satisfied even under (1. 4) and

$$\int_{-\pi}^{\pi} |d\Phi_n(t)| \leqq M.$$

This fact shows that Theorem 1 of A. H. TURECKIĬ [11] is incomplete. Theorem 2 of I. P. NATANSON [6] is also incomplete, because he uses the theorem of Tureckiĭ. In the subsequent sections, we give a simple sufficient criterion for validity of the condition of (3°) and give several applications of that criterion.

2. If $\Phi_n(t)$ are absolutely continuous and $\Phi_n'(t) = \varphi_n(t)$ a.e., then (1. 3) becomes

(2. 1) $$P_n(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t)\varphi_n(t)\, dt.$$

In particular we suppose here that $\varphi_n(t)$ have the simple form such as

(2. 2) $$\gamma_k(n) = \gamma\left(\frac{k}{n}\right) = \frac{2}{\pi}\int_0^{\pi} \cos kt\, \varphi_n(t)\, dt.$$

THEOREM 2. *There exists an even function* $\varphi_n(t)$ *such that*

(2. 3) $$\lim_{n\to\infty} \frac{1-\gamma(k/n)}{k/n} = c \neq 0 \qquad (k = 1, 2, \ldots)$$

and

(2. 4)
$$\int_{-\pi}^{\pi} |\varphi_n(t)|\, dt \leqq M$$

where $\varphi_n(t)$ *and* $\gamma(k/n)$ *are connected by* (2. 2), *nevertheless*

(2. 5)
$$\| f - P_n \| \neq O(1/n),$$

for a function $f(x)$ *such as* $\tilde{f}' \in L^\infty$.

For the proof of Theorem 2, we need two lemmas. For simplicity, we write

$$\|f\|^* = \int_{-\infty}^{\infty} |d\mu|, \quad \text{when} \quad f(t) = \int_{-\infty}^{\infty} e^{-iut}\, d\mu(u).$$

LEMMA 1 (P. MALLIAVIN). *There exists a function* $h(t)$, *which is the Fourier transform of a function of* L, *such that* $h(t)/t$ *is continuous and* $\|h(t)/t\|^* = \infty$.

Professor Malliavin showed this lemma several years ago. For the sake of completeness, we reproduce his proof.

PROOF. Let $u_n(t)$ be a sequence of functions which have their supports in $[-1, 1]$ and satisfy

(2. 6)
$$\max |u_n(t)| \to 0 \quad \text{as} \quad n \to \infty$$

(2. 7)
$$\|u_n\|^* \to \infty \quad \text{as} \quad n \to \infty \quad \text{and} \quad \|u_n\|^* < n^2.$$

The existence of such $u_n(t)$ is observed by the following consideration.
Let

$$\Delta_k(t) = \begin{cases} 1 - \dfrac{|t|}{k} & (|t| \leqq k) \\ 0 & (|t| > k), \end{cases}$$

then

$$\hat{\Delta}_k(x) = \frac{1}{\pi} \frac{1 - \cos kx}{kx^2}, \quad \text{and} \quad \|\Delta_k\|^* = 1.$$

Let us set for $0 < k < 1$

$$w_k(x) = \frac{1}{2k} \{\hat{\Delta}_1(x) - (1-2k)\hat{\Delta}_{1-2k}(x)\} = \frac{1}{\pi k x^2} \sin(1-k)x \sin kx.$$

Since $\lim\limits_{k \to 0} \dfrac{1}{\pi k x^2} \sin(1-k)x \sin kx = \dfrac{1}{\pi} \dfrac{\sin x}{x}$,

$$\lim_{k \to 0} \frac{1}{\pi k} \int_{-\infty}^{\infty} \frac{|\sin(1-k)x \sin kx|}{x^2}\, dx = \infty$$

by Fatou's lemma. $\hat{w}_k(t) = \dfrac{1}{2k} [\Delta_1(t) - (1-2k)\Delta_{1-2k}(t)]$ is positive and continuous with their supports in $[-1, 1]$ and $\max |\hat{w}_k(t)| = 1$. It is ready to get $u_n(t)$ by modifying $\hat{w}_k(t)$.

Let

$$h(t) = \sum_{n=1}^{\infty} e^{-n} u_n(te^{2n} - e^n) \tag{2. 8}$$

then $\|h\|^* \leqq \Sigma n^2 e^{-n} < \infty$ and $g(t) = h(t)/t \to 0$ as $t \to 0$, because the support of every $u_n(te^{2n} - e^n)$ is contained in interval $[e^{-n} - e^{-2n}, e^{-n} + e^{-2n}]$ which is mutually disjoint and the maximum of u_n tends to zero as $n \to \infty$, by (2. 6).

Let $V(x) = \hat{w}_{1/4}(x)$, $V_n(x) = V\left(\frac{xe^{2n} - e^n}{2}\right)$ and $K(x) = \frac{x}{1+x}$, then $V_n g = \{u_n(te^{2n} - e^n)\}\{1 - K(te^n - 1)\}$. Since we can find a function K_1 and a number r such that

$$K_1(x) = K(x), \quad \text{if} \quad |x| < r \quad \text{and} \quad \|K_1\|^* < 1/2,$$

we have

$$\|V_n g\|^* > \|u_n\|^* - \|K_1 u_n\|^*$$

if $e^{-n} < r$. However

$$\|V_n g\|^* < \|V_n\|^* \|g\|^* < 4\|g\|^*, \quad \|K_1 u_n\|^* < \|K_1\|^* \|u_n\|^* < \tfrac{1}{2} \|u_n\|^*,$$

and we get by (2. 7)

$$4\|g\|^* > \tfrac{1}{2} \|u_n\|^* \to \infty,$$

which contradicts $\|g\|^* < \infty$.

The following lemma is given in Katznelson's book [4, p. 135]. Here we give a simple proof. For the sake of simplicity, we write $f(t) \in S(\mathbf{R})$ (or $f(k) \in S(\mathbf{Z})$) if $f(t)$ (or $f(k)$) is a Fourier—Stieltjes transform (or Fourier—Stieltjes coefficients) of a function of bounded total variation.

LEMMA 2. *If $g(t)$ is continuous and $g(k/n) \in S(\mathbf{Z})$ with uniformly bounded total variation as $n \to \infty$, then $g(t) \in S(\mathbf{R})$.*

PROOF. By the hypothesis

$$\int_{-\pi}^{\pi} \left| \sum_{|k| \leqq nR} g\left(\frac{k}{n}\right)\left(1 - \frac{|k|}{nR}\right) e^{ikx} \right| dx \leqq M$$

uniformly in n and R. Changing the variable

$$\int_{-\pi n}^{\pi n} \left| \frac{1}{n} \sum_{|k| \leqq nR} g\left(\frac{k}{n}\right)\left(1 - \frac{|k|}{nR}\right) e^{ixk/n} \right| dx \leqq M.$$

Since the inner sum is an approximate Riemann sum of an integral, letting $n \to \infty$, we have

$$\int_{-\infty}^{\infty} \left| \int_{-R}^{R} g(y)\left(1 - \frac{|y|}{R}\right) e^{iyx}\, dy \right| dx \leqq M$$

which is the required relation.

PROOF OF THEOREM 2. If we take any well behaved even function $h_1(t)$ such that

$$\frac{1 - h_1(t)}{t} \to c \neq 0 \quad \text{and} \quad \left\| \frac{1 - h_1(t)}{t} \right\|^* < \infty$$

and adding to even extension of $h(t)$ of Lemma 1 set

$$\gamma(t) = h(t) + h_1(t),$$

then $\gamma(t)$ is even and the Fourier transform of a function of L, and moreover $\lim_{t\to\infty} \{1-\gamma(t)\}/|t| = c \neq 0$,

$$\| \{1-\gamma(t)\}/|t| \|^* = \infty. \tag{2. 9}$$

Set $\gamma(k/n) = \gamma_k(n)$, then $\gamma(k/n)$ are Fourier coefficients of integrable functions $\varphi_n(t)$ such that $\int_{-\pi}^{\pi} |\varphi_n(t)| dt \leqq M$ by Poisson's summation formula. Hence

$$\|P_n(f)\| = \max \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t)\varphi_n(t)\, dt \right| \leqq M \|f\|.$$

Of course

$$\lim_{n\to\infty} \frac{1-\gamma(k/n)}{k/n} = c \neq 0.$$

If $\{1-\gamma(k/n)\}/(k/n)$ were Fourier—Stieltjes coefficients of functions of bounded total variations which are uniformly bounded in n, then by Lemma 2, $\{1-\gamma(x)\}/|x| \in \in S(\mathbf{R})$, which contradicts (2. 9).

From (1. 1) and (1. 2), we have

$$n\{f(x) - P_n(f; x)\} \sim \sum_{k=1}^{\infty} \frac{1-\gamma(k/n)}{k/n} kA_k(x).$$

By a theorem of (L^∞, L^∞)-multiplier, there is a function with mean value zero such as $\sum_{k=1}^{\infty} kA_k(x) \in L^\infty$, nevertheless $\|f - P_n\| \neq O(1/n)$. Thus Theorem 2 is proved completely.

3. In spite of Theorem 2, if $P_n(f; x)$ is a positive operator, that is,

$$d\Phi_n(x) \sim \frac{1}{2} + \sum_{k=1}^{\infty} \gamma_k(n) \cos kx \geqq 0 \tag{3. 1}$$

and

$$\lim_{n\to\infty} \frac{1-\gamma_k(n)}{\varphi(n)k^2} = c \neq 0 \qquad (k = 1, 2, \ldots) \tag{3. 2}$$

then, the condition of (3°) in Theorem 1 is automatically satisfied as many persons have already mentioned. For example, A. H. TURECKIĬ [10] argues as follows. The fact that

$$\sum_{k=1}^{\infty} k^2 (a_k \cos kx + b_k \sin kx)$$

is a Fourier series of a function of L^∞, is equivalent to

(3. 3) $$|f(x+t)+f(x-t)-2f(x)|\leqq ct^2.$$

Hence, by (3. 1),

$$|P_n(f;x)-f(x)| \leqq \frac{1}{\pi}\int_0^\pi |f(x+t)+f(x-t)-2f(x)|\,d\Phi_n(t) \leqq \frac{c}{\pi}\int_0^\pi t^2\,d\Phi_n(t) \leqq$$

$$\leqq c\pi\int_0^\pi \sin^2\frac{t}{2}\,d\Phi_n(t) = c\pi\int_0^\pi\left(\frac{1-\cos t}{2}\right)d\Phi_n(t) = \frac{c\pi^2}{4}\{1-\gamma_1(n)\} = O(\varphi(n)).$$

Thus, under (3. 1) and (3. 2), the saturation problem is completely solved. This class of approximation processes contains methods of Gauss—Weierstrass, de la Vallée Poussin, Riemann and Jackson.

4. For another class of approximation processes, the direct theorems are somewhat delicate. There are several criteria for a function to be the Fourier transform of an integrable function such as the criteria of Beurling, Young—Kolmogoroff and others. Here we shall give another simple criterion analogous to that given in K. Kojima—G. Sunouchi [5] and solve saturation problems for several approximation processes unifiedly and simply. Of course many of these problems have already been solved by diverse methods. But our method is more advantageous, because it is extensible to the several-dimensional approximation.

Theorem 3. *If*

(4. 1) $$1-\gamma(t) = r\int_0^t (t^r-\tau^r)^m s(\tau)\tau^{r-1}\,d\tau$$

for some $m\geqq 0$, $r>0$ *where* $s(t)\in S(\mathbf{R})$, *then for* $a\neq 0$,

(4. 2) $$\left\{1-\gamma\left(\frac{k}{an+b}\right)\right\}\Big/\left(\frac{k}{an+b}\right)^{r(m+1)}\in S(\mathbf{Z})$$

uniformly in n.

Proof. Changing the variable $\tau\to tu$ in the right hand side of (4. 1),

$$r\int_0^t (t^r-\tau^r)^m s(\tau)\tau^{r-1}\,d\tau = rt^{r(m+1)}\int_0^1 (1-u^r)^m s(tu)u^{r-1}\,du =$$

$$= t^{r(m+1)}\lim_{n\to\infty}\frac{1}{n^r}\sum_{\nu=1}^n\left\{1-\left(\frac{\nu}{n}\right)^r\right\}^m s\left(\frac{t\nu}{n}\right)\nu^{r-1}$$

by the approximate sum of Riemann integral. The total variation of the function which has the last formula as its Fourier—Stieltjes transform is less than

$$\frac{r}{n^r}\sum_{\nu=1}^n\left\{1-\left(\frac{\nu}{n}\right)^r\right\}^m \|s\|^*\nu^{r-1} \leqq \|s\|^*\frac{r}{n^r}\sum_{\nu=1}^n \nu^{r-1}\leqq \|s\|^*.$$

Hence, by Lévy's continuity theorem

$$\frac{1-\gamma(t)}{t^{r(m+1)}} = \lambda(t) \in S(\mathbf{R}),$$

and

$$\left\{1-\gamma\left(\frac{k}{an+b}\right)\right\}\Big/\left(\frac{k}{an+b}\right)^{r(m+1)} = \lambda\left(\frac{k}{an+b}\right) \qquad (a \neq 0) \in S(\mathbf{Z})$$

uniformly in n, by Poisson's summation formula.

5. We use Theorem 3 to verify the condition of direct part of Theorem 1.

(1°) The Abel—Poisson method. Since

$$\gamma_k(n) = \exp(-k/n), \quad \lim_{n\to\infty} \frac{1-\exp(-k/n)}{k/n} = 1$$

and for $x > 0$

$$1-e^{-x} = \int_0^x e^{-u}\,du, \quad \text{and} \quad e^{-|u|} \in S(\mathbf{R}),$$

the condition of Theorem 3 is satisfied by $m=0$ and $r=1$.

(2°) The Bernstein—Rogosinski method. Since

$$\gamma_k(n) = \cos\frac{k\pi}{2n+1} \quad \text{for} \quad k \leqq n \quad \text{and,} \quad = 0 \quad \text{for} \quad k > n,$$

$$\lim_{n\to\infty} \frac{1-\cos k\pi/(2n+1)}{(k/n)^2} = \pi/2 \neq 0, \quad 1-\cos x = \int_0^x (x-u)\cos u\,du,$$

where $x \leqq \pi/2$, and $s(u) = \cos u(|u| \leqq \pi/2) = 0$ $(|u| \geqq \pi/2)$ belongs to the class $S(\mathbf{R})$, the hypothesis of Theorem 3 is satisfied by $m=1$, and $r=1$.

(3°) The Riesz method. Since

$$\gamma_k(n) = R_\lambda^\varrho\left(\frac{k}{n}\right) = \left\{1-\left(\frac{k}{n}\right)^\lambda\right\}^\varrho \quad \text{if} \quad k \leqq n \quad \text{and} \quad = 0 \quad \text{if} \quad k > n$$

where λ, $\varrho > 0$, we have

$$\lim_{n\to\infty} \frac{1-\gamma_k(n)}{(k/n)^\lambda} = \varrho \neq 0,$$

$$1-(1-x^\lambda)^\varrho = \lambda\varrho \int_0^x (1-u^\lambda)^{\varrho-1} u^{\lambda-1}\,du, \quad |x| \leqq 1,$$

that is to say, $1-R_\lambda^\varrho(u) = \lambda\varrho \int_0^1 R_\lambda^{\varrho-1}(u) u^{\lambda-1}\,du$, where the integrand $R_\lambda^{\varrho-1}(u)$ belongs to $S(\mathbf{R})$, provided that $\varrho > 1$.

However since[2]

$$(1-|x|^\lambda)^\varrho = (1-|x|^\lambda)^{\varrho+1} + |x|^\lambda(1-|x|^\lambda)^\varrho \quad |x| \leqq 1,$$

and

$$\frac{1-(1-|x|^\lambda)^\varrho}{|x|^\lambda} = \frac{1-(1-|x|^\lambda)^{\varrho+1}}{|x|^\lambda} - (1-|x|^\lambda)^\varrho,$$

if $\varrho > 0$, the right hand side belongs to $S(\mathbf{R})$. Thus the direct theorem is proved for all $\lambda, \varrho > 0$.

6. The above method of proof may be generalized to more general operators such as a Taylor expansion of $\gamma(k/n)$ or a linear combination of kernels. Since the generalization is straightforward, we mention only several examples.

The first is the saturation of higher order. We take the Abel—Poisson method for an example. Let us set in the complex form

$$V_t(f; x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{-|k|t} e^{ikx}$$

where

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx,$$

and write $f^{\{j\}}(x) = f^{(j)}(x)$ or $\tilde{f}^{(j)}(x)$ as j is even or odd.

PROPOSITION 1. *If $f, f^{\{1\}}, \ldots, f^{\{r-1\}} \in C$, then*

$$\left\| V_t(f; x) - \sum_{j=0}^{r-1} (-1)^{[(j+1)/2]} \frac{t^j}{j!} f^{\{j\}}(x) \right\| = O(t^r)$$

as $t \to 0$, if and only if $f^{\{r\}} \in L^\infty$.

PROOF. Set

$$V_t(f; x) - \sum_{j=0}^{r-1} (-1)^{[(j+1)/2]} \frac{t^j}{j!} f^{\{j\}}(x) = (U_t * f)(x),$$

then the Fourier coefficients of $U_t(x)$ are

$$\hat{U}_t(k) = e^{-|k|t} - \sum_{j=0}^{r-1} (-1)^j \frac{(|k|\,t)^j}{j!}.$$

Since

$$\lim_{t\to 0} \frac{\hat{U}_t(k)}{(|k|\,t)^r} = \frac{(-1)^r}{r!}$$

and by Taylor's theorem

$$e^{-t} - \sum_{j=0}^{r-1} (-1)^j \frac{t^j}{j!} = \frac{(-1)^r}{r!} \int_0^t (t-\tau)^r e^{-\tau}\, d\tau$$

[2] This device is suggested by H. Berens.

where $e^{-t} \in S(\mathbf{R})$, the analogue of the hypothesis of Theorem 3 is satisfied. Hence the proposition is proved.

Concerning the Gauss—Weierstrass integral and another approximation processes, we can prove analogous theorems by the Taylor expansion of their summating functions.

Another example is approximation by a linear combination of kernels. We consider the derivatives of Riemann type as an example. Let r is an integer and

$$\Delta^r f(x;t) = \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} f(x+\nu t)$$

then $\lim_{t\to 0} \Delta^r f(x;t)/t^r$ is the unsymmetric derivative of Riemann type.

PROPOSITION 2. *$\|\Delta^r f(x;t)\| = O(t^r)$ if and only if $f^{(r)}(x) \in L^\infty$.*

PROOF. Since $f(x+t)$ is representable by a convolution of f and δ-function, the k-th Fourier coefficient of $\Delta^r f(x;t)$ is

$$\sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} e^{ik\nu t} \hat{f}(k) = (e^{ikt}-1)^r \hat{f}(k).$$

$$\lim_{t\to 0} \frac{e^{ikt}-1}{kt} = i, \quad e^{ix}-1 = i\int_0^x e^{iu}\,du \quad \text{and} \quad e^{iu} \in S(\mathbf{R}).$$

Hence $(e^{it}-1)/t \in S(\mathbf{R})$ and its products also $\{(e^{it}-1)/t\}^r \in S(\mathbf{R})$ by the convolution.

A correct form of Natanson's theorem [6] is also treated by the same idea.

PROPOSITION 3. *Let*

$$Q_n(f;x) = \sum_{\nu=1}^{p+1} (-1)^{\nu-1} \binom{p+1}{\nu} P_n^{[\nu]}(f;x)$$

where $P_n^{[1]} = P_n$, $P_n^{[k+1]} = P_n[P_n^{[k]}]$. If

$$\lim_{n\to\infty} \frac{1-\gamma_k(n)}{k^r \varphi(n)} = c \neq 0 \quad \text{and} \quad \frac{1-\gamma_k(n)}{k^r \varphi(n)} \in S(\mathbf{Z})$$

uniformly in n, then the process $Q_n(f;x)$ is saturated with the order $[\varphi(n)]^{p+1}$ and the class $\{f \mid \Sigma k^{r(p+1)} A_k(x)\} \in L^\infty$.

PROOF. Let us develop $P_n(f;x)$ and $Q_n(f;x)$ in Fourier series,

$$P_n(f;x) \sim \sum_{k=0}^{\infty} \gamma_k(n) A_k(x), \quad Q_n(f;x) \sim \sum_{k=0}^{\infty} \delta_k(n) A_k(x),$$

then $\delta_k(n) = 1-(1-\gamma_k(n))^{p+1}$. Hence

$$\frac{1-\delta_k(n)}{[k^r \varphi(n)]^{p+1}} = \left\{\frac{1-\gamma_k(n)}{k^r \varphi(n)}\right\}^{p+1} \to c^{p+1} \neq 0.$$

Since

$$\frac{1-\gamma_k(n)}{k^r\varphi(n)} \in S(\mathbf{Z}),$$

we have by the convolution rule,

$$\left\{\frac{1-\gamma_k(n)}{k^r\varphi(n)}\right\}^{p+1} \in S(\mathbf{Z}).$$

7. In the last section, we shall give a characterization of the saturation classes. The special case is treated in [8].

Let us set

$$\Delta^{2s}f(x;y) = \sum_{j=0}^{2s}(-1)^j\binom{2s}{j}f(x+(s-j)y).$$

THEOREM 4. *The class*

$$\sum_{k=1}^{\infty}|k|^r A(x) \in L^\infty$$

for any positive r, is equivalent to

$$\left\| \int_{|y|\geqq\varepsilon} \frac{\Delta^{2s}f(x;y)}{|y|^{1+r}}\,dy \right\| = O(1) \quad 0<r<2s, \tag{7.1}$$

uniformly in $\varepsilon(\varepsilon\to 0)$.

PROOF. We can write

$$\int_{|y|\geqq\varrho^{-1}} \frac{\Delta^{2s}f(x;y)}{|y|^{1+r}}\,dy = 2c\varrho^r\left\{\frac{\varrho}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-y)k(\varrho y)\,dy - f(x)\right\} \tag{7.2}$$

where

$$c = (-1)^{s+1}\binom{2s}{s}\frac{1}{r},$$

$$k(x) = \begin{cases} \dfrac{1}{c}\sqrt{2\pi}\,c_j\dfrac{1}{|x|^{1+r}}, & j\leqq|x|\leqq j+1, \quad j=0,1,\dots,s-1,\\[2ex] \dfrac{1}{c}\sqrt{2\pi}\,c_s\dfrac{1}{|x|^{1+r}} & |x|\geqq s,\end{cases}$$

$$c_j = \sum_{k=0}^{j}(-1)^{s-k}\binom{2s}{s-k}k^r \qquad (0\leqq j\leqq s).$$

Hence (7. 1) reduces to determine the class of saturation and so we can apply Theorem 1 and Theorem 3. Since

$$k(x)\in L(-\infty,\infty), \quad \int_{-\infty}^{\infty}k(x)\,dx = \sqrt{2\pi},$$

the integral transform (7. 2) is developed in the form of Fourier series by Poisson's summation formula. By Theorem 1 and Theorem 3, it is sufficient to prove

$$\lim_{t\to 0}\frac{\hat{k}(t)-1}{t^r}=c\neq 0,\quad\text{and}\quad \frac{\hat{k}(t)-1}{t^r}\in S(\mathbf{R}).$$

The former is shown by

$$\frac{\hat{k}(t)-1}{t^r}=a\int_t^\infty \frac{\sin^{2s}(x/2)}{x^{1+r}}\,dx\to a'\int_0^\infty \frac{\sin^{2s}x}{x^{1+r}}\,dx\neq 0,$$

where a and a' are constants. In the following, $a, b, \ldots$ denote constants, also.

Since

$$\hat{k}(t)-1=b\int_1^\infty \sum_{j=0}^{2s}(-1)^j\binom{2s}{j}\frac{e^{i(s-j)yt}-e^{-i(s-j)yt}}{y^{1+r}}\,dy,$$

if we could write

$$\hat{k}(t)-1=r\int_0^t h(\tau)\tau^{r-1}\,d\tau,$$

where $h(\tau)\in S(\mathbf{R})$, the proof would be complete. We suppose momently that this was true, then since $\hat{k}(t)$ is differentiable, we should have

$$h(t)=c\int_t^\infty \sum_{j=0}^{2s}(-1)^j\binom{2s}{j}(s-j)\frac{e^{i(s-j)y}-e^{-i(s-j)y}}{y^r}\,dy.$$

In the sense of distribution,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(|t|)e^{-itx}\,dt=$$

$$=d\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}(s-j)\frac{1}{x}\int_0^\infty \frac{1}{|y|^r}[\cos\{x+(s-j)y\}-\cos\{x-(s-j)y\}]\,dy.$$

By a formula in a book of GELFAND and SILOV [2, p. 359, p. 361], the last

$$=\frac{d}{x}\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}(s-j)\{|x+(s-j)|^{r-1}-|x-(s-j)|^{r-1}\}\qquad (r\neq 1,3,5,\ldots),$$

$$=\frac{1}{x}\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}(s-j)\{e(|x+(s-j)|^{2m}-|x-(s-j)|^{2m})+$$

$$+e'(|x+(s-j)|^{2m}\log|x+(s-j)|-|x-(s-j)|^{2m}\log|x-(s-j)|)\}$$

$$(r=2m+1,\ m=0,1,2,\ldots).$$

Since the last term is absolutely integrable if $0<r<2s$, $h(t)\in S(\mathbf{R})$ and the proof is finished.

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ON THE ORDER OF MAGNITUDE OF THE LEBESGUE FUNCTIONS FOR STRONGLY MULTIPLICATIVE SYSTEMS

By

F. MÓRICZ (Szeged)

To G. ALEXITS *on his 70th birthday*

Introduction

G. ALEXITS [2] introduced the following concept. The sequence of real measurable functions $\varphi_1(x), \varphi_2(x), \dots$ defined in the interval $(0, 1)$ is called an equinormed strongly multiplicative system (in abbreviation: ESMS) if the following conditions are satisfied:

$$(1) \qquad \int_0^1 \varphi_n(x)\,dx = 0, \quad \int_0^1 \varphi_n^2(x)\,dx = 1 \qquad (n = 1, 2, \dots);$$

$$\int_0^1 \varphi_{n_1}^{\alpha_1}(x)\varphi_{n_2}^{\alpha_2}(x)\dots\varphi_{n_k}^{\alpha_k}(x)\,dx = \int_0^1 \varphi_{n_1}^{\alpha_1}(x)\,dx \int_0^1 \varphi_{n_2}^{\alpha_2}(x)\,dx \dots \int_0^1 \varphi_{n_k}^{\alpha_k}(x)\,dx$$
$$(1 \leqq n_1 < n_2 < \dots < n_k;\ k = 2, 3, \dots),$$

where the exponents $\alpha_1, \alpha_2, \dots, \alpha_k$ can be equal to 1 or 2. In other words, the sequence $\{\varphi_n(x)\}$ is called an ESMS if the system $\{\varphi_{n_1}(x)\varphi_{n_2}(x)\dots\varphi_{n_k}(x)\}$ $(1 \leqq n_1 < n_2 < \dots < n_k;$ $k = 1, 2, \dots)$ is orthonormal.

Evidently a sequence of independent functions with mean value 0 and dispersion 1 is an ESMS. In this special case the relations in (1) hold for arbitrary positive integer values of the exponents $\alpha_1, \alpha_2, \dots, \alpha_k$. Another example is a strongly lacunary sequence of trigonometric functions, i.e. $\{\sqrt{2}\sin 2\pi m_k x\}$ if $m_{k+1}/m_k \geqq 3$ $(k = 1, 2, \dots)$.[1]

The behaviour of the series arising from the functions of an ESMS resembles, in many respects, that of series of independent functions. As to convergence properties, see ALEXITS [2], ALEXITS and TANDORI [3]. To investigate the central limit theorem and a law of iterated logarithm for ESMS, see RÉVÉSZ [9] (Chapter 3), [10]. The present author [6] also studied what properties of the independent functions remain valid for ESMS. In our cited paper we proved, among others, the following form of the central limit theorem. Before stating it in an explicit form, let us introduce the notations

$$S_N(x) = \sum_{n=1}^{N} c_n\varphi_n(x), \quad C_N^2 = \sum_{n=1}^{N} c_n^2 \qquad (N = 1, 2, \dots),$$

where $\{c_n\}$ is an arbitrary sequence of real numbers.

[1] Our results remain valid if the interval $(0, 1)$ with the Lebesgue measure is replaced by an arbitrary probability space $\{\Omega, \mathscr{S}, \mathbf{P}\}$ and the sequence $\{\varphi_n(x)\}$ is replaced by a sequence $\{\xi_n\}$ of random variables defined on Ω. Then in the assertions, of course, the term "almost everywhere" must be understood in the sense of the probability measure $\mathbf{P}$ in question. (In detail, see RÉVÉSZ [10].)

THEOREM A. *Let* $\{\varphi_n(x)\}$ *be a uniformly bounded* ESMS. *If*

(2) $$C_N \to \infty \quad \text{and} \quad c_N = o(C_N) \qquad (N \to \infty),$$

then the distribution functions

$$F_N(y) = \operatorname{mes}\left(\left\{x\colon \frac{S_N(x)}{C_N} < y\right\}\right)^2 \qquad (-\infty < y < \infty)$$

tend pointwise to the Gaussian distribution function

(3) $$G(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}}\, dx.$$

If the series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x)$$

is the Fourier series of a square integrable function $f(x)$, i.e.

(4) $$\sum_{n=1}^{\infty} c_n^2 < \infty,$$

then it converges almost everywhere. (See ALEXITS [2].) Setting

$$R_N(x) = f(x) - S_{N-1}(x), \quad D_N^2 = \sum_{n=N}^{\infty} c_n^2 \qquad (N = 1, 2, \dots;\ S_0(x) \equiv 0),$$

we can obtain the following result — an obvious supplement to Theorem A.

THEOREM B. *Let* $\{\varphi_n(x)\}$ *be a uniformly bounded* ESMS, *and let* $\{c_n\}$ *be a sequence of coefficients satisfying* (4). *If*

(5) $$c_N = o(D_N),$$

then the distribution functions

$$G_N(y) = \operatorname{mes}\left(\left\{x\colon \frac{R_N(x)}{D_N} < y\right\}\right) \qquad (-\infty < y < \infty)$$

tend pointwise to the Gaussian distribution function (3).

The proof of Theorem B does not require new ideas and follows the same pattern as that of Theorem A in our cited paper [6]. For the analogue in the special case of lacunary trigonometric series,[3] see SALEM and ZYGMUND [11].

[2] mes (E) denotes the Lebesgue measure of the set E.

[3] $\sum_{k=1}^{\infty} (a_k \cos m_k x + b_k \sin m_k x)$ is called lacunary if $m_{k+1}/m_k \geqq q > 1$ $(k = 1, 2, \dots)$.

Using these theorems we are able to deduce simple but exact estimates for the mean of $S_N(x)$ and $R_N(x)$ in absolute value.

THEOREM 1. *Let* $\{\varphi_n(x)\}$ *be a uniformly bounded* ESMS. *If the sequence* $\{c_n\}$ *of coefficients satisfies the conditions in* (2), *then*

$$\int_0^1 |S_N(x)|\,dx = \left[\sqrt{\frac{2}{\pi}}+o(1)\right] C_N.$$

In particular

$$\int_0^1 \left|\sum_{n=1}^N \varphi_n(x)\right| dx = \left[\sqrt{\frac{2}{\pi}}+o(1)\right] \sqrt{N}.$$

THEOREM 2. *Let* $\{\varphi_n(x)\}$ *be a uniformly bounded* ESMS. *If the sequence* $\{c_n\}$ *of coefficients satisfies conditions* (4) *and* (5), *then*

$$\int_0^1 |R_N(x)|\,dx = \left[\sqrt{\frac{2}{\pi}}+o(1)\right] D_N.$$

We note that a less complete form of these estimates was previously proved in our paper [6]; more exactly, for arbitrary coefficients c_n we have

$$K_1 C_N \leqq \int_0^1 |S_N(x)|\,dx \leqq C_N,$$

where K_1 denotes a positive constant. Here the upper estimate is an easy consequence of Schwarz's inequality.

It is less obvious that the central limit theorem, stated as Theorem A, can be used in the investigation of the order of magnitude of the Lebesgue functions. It is well known that the Nth Lebesgue function of a system $\{\varphi_n(x)\}$ is defined by

$$L_N(x) = \int_0^1 \left|\sum_{n=1}^N \varphi_n(x)\varphi_n(t)\right| dt \qquad (0 \leqq x \leqq 1;\ N = 1, 2, \ldots).$$

The order of magnitude of the Lebesgue functions may, in many cases, be decisive for the convergence problems. (See ALEXITS [1], p. 175.) It turns out, however, that the method of the Lebesgue functions is not adequate to treat the convergence properties of ESMS.

More specifically, our main result can be stated as follows:

THEOREM 3. *Let* $\{\varphi_n(x)\}$ *be a uniformly bounded* ESMS. *Then the quotient* $L_N(x)/\sqrt{N}$ *converges in measure to* $\sqrt{2/\pi}$. [4]

[4] A sequence $\{f_n(x)\}$ of measurable functions defined in (0, 1) is said to converge in measure to the measurable function $f(x)$ if, for every $\varepsilon > 0$, $\lim_{n\to\infty} \text{mes}\ (\{x \in (0,1): |f_n(x)-f(x)| \geqq \varepsilon\}) = 0$; in symbols $\lim_{n\to\infty} \text{i. m.}\, f_n(x) = f(x)$.

We observe that Theorem 3 immediately implies the following two consequences.

COROLLARY 1. *If* $\{\varphi_n(x)\}$ *is a uniformly bounded* ESMS *then the estimate*

$$\overline{\lim_{N\to\infty}} \frac{L_N(x)}{\sqrt{N}} \geqq \sqrt{\frac{2}{\pi}} \tag{6}$$

holds almost everywhere.

COROLLARY 2. *Let* $\{\varphi_n(x)\}$ *be a uniformly bounded* ESMS. *Then for every set* $E\subset(0, 1)$ *of positive measure we have*

$$\frac{1}{\operatorname{mes}(E)} \int_E \frac{L_N(x)}{\sqrt{N}}\, dx \to \sqrt{\frac{2}{\pi}} \qquad (N\to\infty).$$

These corollaries come in a natural way from the definition of convergence in measure and (in the case of Corollary 2) from the fact that the quotients $L_N(x)/\sqrt{N}$ are uniformly bounded. In fact, from Schwarz's inequality we can infer the estimate

$$L_N(x) \leqq \left\{\int_0^1 dt \cdot \int_0^1 \left[\sum_{n=1}^N \varphi_n(x)\varphi_n(t)\right]^2 dt\right\}^{1/2} = \left\{\sum_{n=1}^N \varphi_n^2(x)\right\}^{1/2} \leqq K\sqrt{N},$$

where K denotes a common bound of $\{\varphi_n(x)\}$.

Corollary 1 shows that the order of magnitude of the Lebesgue functions in the case of an arbitrary uniformly bounded ESMS is as bad as it could be for uniformly bounded orthonormal systems. To our knowledge, the estimate (6) (in a sharper form) has so far been shown only for the Rademacher system [7]. Corollary 1 generalizes in a certain sense this classical estimate not only for systems consisting of independent functions but also for uniformly bounded ESMS.

We also remark that Corollary 1 trivially implies that for an arbitrary uniformly bounded ESMS there is no arrangement of its terms into a new system that improves the order of magnitude of the Lebesgue functions. The reason it is interesting is that the convergence properties of orthogonal series in L^2 can get essentially better by (almost every) rearrangement of the terms of the series in question. (See GARSIA [5].)

The proof of Theorem 3 makes essential use of the above form of the central limit theorem stated in Theorem A, and of an important result which is similar to the weak laws of large numbers in probability theory. We state this auxiliary assertion in the form of a theorem.

THEOREM 4. *Let* $\{\varphi_n(x)\}$ *be a uniformly bounded* ESMS, *with bound K, and let* $\{c_n\}$ *be a sequence of coefficients satisfying* (2). *Then the relation*

$$\lim_{N\to\infty} \text{ i. m. } \frac{1}{C_N^2} \sum_{n=1}^N c_n^2 \varphi_n^2(x) = 1 \tag{7}$$

holds; in particular, we have

$$\lim_{N\to\infty} \text{ i. m. } \frac{1}{N} \sum_{n=1}^N \varphi_n^2(x) = 1.$$

The idea of investigating the order of magnitude of the Lebesgue functions by means of the central limit theorem is originated, as far as we know, from ERDŐS [4] (concerning the trigonometric system).

§ 1. Lemmas

We require two lemmas to prove our theorems.

LEMMA 1. *Let $\{b_n\}$ be a sequence of non-negative real numbers. If*

$$s_N = \sum_{n=1}^{N} b_n \to \infty \quad \textit{and} \quad b_N = o(s_N) \qquad (N \to \infty),$$

then for an arbitrary real number $\alpha\,(>1)$, we have

$$\sum_{n=1}^{N} b_n^\alpha = o(s_N^\alpha).$$

This lemma was proved in our earlier paper [6]. (See there Lemma 6.)

LEMMA 2. *Let $F_1(x)$, $F_2(x)$, ... be the distribution functions of the square integrable functions $f_1(x), f_2(x)$, ... with*

$$\int_0^1 f_n^2(x)\,dx = 1 \qquad (n = 1, 2, \ldots).$$

If $F_n(x)$ converges to the distribution function $F(x)$ of a square integrable function $f(x)$ (at the points of continuity of $F(x)$), then

$$\int_0^1 |f_n(x)|\,dx \to \int_0^1 |f(x)|\,dx \qquad (n \to \infty).$$

PROOF. Given an arbitrary $\varepsilon>0$, select an $\omega>0$ such that $F(x)$ is continuous at $\pm\omega$ and

$$F(-\omega)<\varepsilon^2, \quad 1-F(\omega)<\varepsilon^2.$$

Then $F_n(x)$ satisfies the same inequalities for n large enough. Now

$$\int_0^1 |f_n(x)|\,dx - \int_0^1 |f(x)|\,dx = \int_{-\infty}^{\infty} |x|\,dF_n(x) - \int_{-\infty}^{\infty} |x|\,dF(x) = \int_{-\infty}^{\infty} |x|\,d\{F_n(x)-F(x)\}$$

can be written

$$\int_{|x|\leqq\omega} |x|\,d\{F_n(x)-F(x)\} + \int_{|x|\geqq\omega} |x|\,dF_n(x) - \int_{|x|\geqq\omega} |x|\,dF(x) = P+Q+R,$$

say. By Schwarz's inequality we find that

$$|R| \leqq \Big\{ \int_{|x|\geqq\omega} x^2\,dF(x) \cdot \int_{|x|\geqq\omega} dF(x)\Big\}^{1/2} \leqq \{[1-F(\omega)]+F(-\omega)\}^{1/2} \leqq \sqrt{2}\,\varepsilon,$$

where we took into consideration that

$$\int_{|x| \geqq \omega} x^2\, dF(x) \leqq \int_{-\infty}^{\infty} x^2\, dF(x) = \int_0^1 f^2(x)\, dx = 1.$$

Similarly we obtain that $|Q| \leqq \sqrt{2\varepsilon}$ for n large enough.

Integrating by parts we can see that

$$P = \{[F_n(x) - F(x)] \cdot |x|\}_{-\omega}^{\omega} - \int_{|x| \leqq \omega} [F_n(x) - F(x)]\, d|x|.$$

Since $F_n(\pm\omega) \to F(\pm\omega)$ $(n \to \infty)$, the first term on the right-hand side tends to 0. The second term does not exceed

$$\int_{|x| \leqq \omega} |F_n(x) - F(x)|\, dx$$

in absolute value, and so tends to 0.

Collecting results, we finally conclude that

$$\left| \int_0^1 |f_n(x)|\, dx - \int_0^1 |f(x)|\, dx \right| \leqq 4\varepsilon,$$

if n is large enough. This proves our assertion.

§ 2. Proofs of the theorems

If we have a sequence of functions $f_n(x)$ defined in $(0, 1)$, and if their distribution functions, $F_n(y)$ say, converge to the Gaussian distribution function (3), we say that the $f_n(x)$ are asymptotically Gauss distributed.

PROOFS OF THEOREM 1 AND THEOREM 2. It is sufficient to deal with the proof of Theorem 1, as that of Theorem 2 essentially resembles it.

According to Theorem A the $S_N(x)/C_N$ are asymptotically Gauss distributed. Since the square integral of these functions is equal to 1, on account of Lemma 2, we have

$$\int_0^1 \left| \frac{S_N(x)}{C_N} \right| dx \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{x^2}{2}}\, dx = \sqrt{\frac{2}{\pi}} \quad (N \to \infty),$$

in accordance with our statement.

PROOF OF THEOREM 3. Let us temporarily take for granted that Theorem 4 is true and consider the series

$$\sum_{n=1}^{\infty} \varphi_n(x) \varphi_n(t) \tag{8}$$

as a function of t, where the "coefficients" $\varphi_n(x)$ depend on x. According to Theorem 4 the squares of the coefficients $\varphi_n(x)$ of (8) form a divergent series up to a set of measure zero of the points x. Hence, owing to the uniform boundedness of the system $\{\varphi_n(x)\}$, the conditions of Theorem A are satisfied for almost every x. Thus the functions

$$\frac{1}{\left\{\sum_{n=1}^{N}\varphi_n^2(x)\right\}^{1/2}}\sum_{n=1}^{N}\varphi_n(x)\varphi_n(t) \qquad (0\leqq t\leqq 1;\ N=1,2,\ldots)$$

are asymptotically Gauss distributed for almost every x in (0, 1). Since the square integral of these functions with respect to t is equal to 1, by virtue of Lemma 2, we have

$$\frac{1}{\left\{\sum_{n=1}^{N}\varphi_n^2(x)\right\}^{1/2}}\int_0^1\left|\sum_{n=1}^{N}\varphi_n(x)\varphi_n(t)\right|dt\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|e^{-\frac{x^2}{2}}\,dx=\sqrt{\frac{2}{\pi}}$$

for almost every x.

Finally let us consider the representation

$$\frac{L_N(x)}{\sqrt{N}}=\frac{L_N(x)}{\left\{\sum_{n=1}^{N}\varphi_n^2(x)\right\}^{1/2}}\cdot\left\{\frac{\sum_{n=1}^{N}\varphi_n^2(x)}{N}\right\}^{1/2}.$$

The first factor on the right converges (in the ordinary sense) to $\sqrt{2/\pi}$ almost everywhere and the second factor, by virtue of Theorem 4, converges in measure to 1; consequently their product converges in measure to $\sqrt{2/\pi}$. This completes the proof of Theorem 3.

PROOF OF THEOREM 4. In the proof we shall use the following elementary inequality: for every real number and every positive integer l, we have (see [8], p. 365)

$$\left|e^{iu}-\sum_{k=0}^{l-1}\frac{(iu)^k}{k!}\right|\leqq\frac{|u|^l}{l!}. \tag{9}$$

We make use of the classical method of characteristic functions. Denote by $\psi_N(\lambda)$ the characteristic function of the distribution function

$$G_N(y)=\text{mes}\left(\left\{x:\frac{1}{C_N^2}\sum_{n=1}^{N}c_n^2\varphi_n^2(x)<y\right\}\right)\quad(-\infty<y<\infty),$$

i.e. $\psi_N(\lambda)$ is defined by

$$\psi_N(\lambda)=\int_{-\infty}^{\infty}e^{i\lambda y}\,dG_N(y)\qquad(-\infty<\lambda<\infty).$$

To prove (7), on the basis of a classical result of probability theory, it suffices to prove that for any fixed λ $\psi_N(\lambda)$ converges to the characteristic function of the distribution function belonging to the constant function $f(x)\equiv 1$, that is

$$\psi_N(\lambda)\to e^{i\lambda}\qquad(N\to\infty). \tag{10}$$

It is obvious that

$$\psi_N(\lambda) = \int_0^1 \exp\left\{\frac{i\lambda}{C_N^2} \sum_{n=1}^N c_n^2 \varphi_n^2(x)\right\} dx.$$

Applying (9) with $l=2$, we obtain that

$$\psi_N(\lambda) = \int_0^1 \prod_{n=1}^N \left\{\left(1+\frac{i\lambda c_n^2 \varphi_n^2(x)}{C_N^2}\right) - \Theta_n \frac{\lambda^2 c_n^4 \varphi_n^4(x)}{2C_N^4}\right\} dx, \tag{11}$$

where Θ_n also depends on N, and $|\Theta_n| \leqq 1$ $(n=1, 2, \ldots, N)$.

We show that the integral on the right-hand side of (11) can be replaced by the simpler integral

$$\int_0^1 \prod_{n=1}^N \left(1+\frac{i\lambda c_n^2 \varphi_n^2(x)}{C_N^2}\right) dx, \tag{12}$$

in the sense that for every fixed λ the difference of (11) and (12) tends to 0 as $N \to \infty$. For the sake of brevity we shall use the notations:

$$P_n(x) = 1+\frac{i\lambda c_n^2 \varphi_n^2(x)}{C_N^2} \quad \text{and} \quad R_n(x) = -\Theta_n \frac{\lambda^2 c_n^4 \varphi_n^4(x)}{2C_N^4},$$

where we do not indicate the dependence on N. Using the identity

$$\prod_{n=1}^N (p_n+r_n) - \prod_{n=1}^N p_n = \sum_{n=1}^N r_n \left(\prod_{k=1}^{n-1} p_k\right)\left(\prod_{k=n+1}^N (p_k+r_k)\right),$$

where the empty product equals 1 (see [8], p. 367), we find

$$\left|\exp\left\{\frac{i\lambda}{C_N^2} \sum_{n=1}^N c_n^2 \varphi_n^2(x)\right\} - \prod_{n=1}^N \left(1+\frac{i\lambda c_n^2 \varphi_n^2(x)}{C_N^2}\right)\right| = \tag{13}$$

$$= \left|\prod_{n=1}^N (P_n(x)+R_n(x)) - \prod_{n=1}^N P_n(x)\right| \leqq$$

$$\leqq \sum_{n=1}^N |R_n(x)| \left(\prod_{k=1}^{n-1} |P_k(x)|\right)\left(\sum_{k=n+1}^N (|P_k(x)|+|R_k(x)|)\right).$$

A simple calculation shows that

$$|P_n(x)| = \left\{1+\frac{\lambda^2 c_n^4 \varphi_n^4(x)}{C_N^4}\right\}^{1/2} \leqq 1+\frac{|\lambda| c_n^2 K^2}{C_N^2},$$

and

$$|R_n(x)| \leqq \frac{\lambda^2 c_n^4 K^4}{2C_N^4} \qquad (n = 1, 2, \ldots, N),$$

where K denotes a common bound of the system $\{\varphi_n(x)\}$.

Hence we obtain that the right-hand side of (13) does not exceed

$$\sum_{n=1}^N \frac{\lambda^2 c_n^4 K^4}{2C_N^4} \left\{\prod_{k=1}^{n-1} \left(1+\frac{|\lambda| c_k^2 K^2}{C_N^2}\right) \prod_{k=n+1}^N \left(1+\frac{|\lambda| c_k^2 K^2}{C_N^2}+\frac{\lambda^2 c_k^4 K^4}{2C_N^4}\right)\right\}.$$

Using the inequality $1+u\leqq e^u$ $(u\geqq 0)$, the last sum is not greater than

$$\sum_{n=1}^{N}\frac{\lambda^2 c_n^4 K^4}{2C_N^4}\exp\left\{\sum_{k=1}^{N}\frac{|\lambda|\, c_k^2 K^2}{C_N^2}+\sum_{k=n+1}^{N}\frac{\lambda^2 c_k^4 K^4}{2C_N^4}\right\}\leqq$$

$$\leqq\frac{\lambda^2 K^4}{2}\exp\left\{|\lambda|\, K^2+\frac{\lambda^2 K^4}{2}\right\}\frac{1}{C_N^4}\sum_{n=1}^{N}c_n^4.$$

By (2) the conditions of Lemma 1 are satisfied by the sequence $\{c_n^2\}$. Thus, by virtue of Lemma 1, this last sum, and consequently the difference of the integrands (11) and (12), tends to 0 $(N\to\infty)$ uniformly in x $(0\leqq x\leqq 1)$ if λ is fixed.

Now carry out the multiplication in the integrand (12) and integrate term by term. Then

$$\int_0^1\prod_{n=1}^{N}\left(1+\frac{i\lambda c_n^2\varphi_n^2(x)}{C_N^2}\right)dx=$$

$$=1+\sum_{(1\leqq)n_1<n_2<\ldots<n_k(\leqq N)}\frac{(i\lambda)^k}{C_N^{2k}}c_{n_1}^2 c_{n_2}^2\ldots c_{n_k}^2\int_0^1\varphi_{n_1}^2(x)\,\varphi_{n_2}^2(x)\ldots\varphi_{n_k}^2(x)\,dx,$$

where the sum $\sum$ is extended over all systems of integer values $(1\leqq)n_1<n_2<\ldots<n_k(\leqq N)$ $(1\leqq k\leqq N)$. By (1) it follows that the integral (12) equals

$$1+\sum_{(1\leqq)n_1<n_2<\ldots<n_k(\leqq N)}\frac{(i\lambda)^k}{C_N^{2k}}c_{n_1}^2 c_{n_2}^2\ldots c_{n_k}^2=\prod_{n=1}^{N}\left(1+\frac{i\lambda c_n^2}{C_N^2}\right).$$

Finally, we make use of the following inequality:

$$1+z=e^{z+o(|z|)},[5]$$

if the complex number z tends to 0. By this we obtain that

$$\prod_{n=1}^{N}\left(1+\frac{i\lambda c_n^2}{C_N^2}\right)=\exp\left\{\sum_{n=1}^{N}\frac{i\lambda c_n^2}{C_N^2}+o\left(\sum_{n=1}^{N}\frac{|\lambda|\, c_n^2}{C_N^2}\right)\right\}=e^{i\lambda+o(1)},$$

whence we deduce that

$$\prod_{n=1}^{N}\left(1+\frac{i\lambda c_n^2}{C_N^2}\right)\to e^{i\lambda}\qquad(N\to\infty)$$

holds for every fixed λ. Combining this result with the above, we can see that (10) is true. This completes the proof of Theorem 4.

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[5] This equality follows from $\log(1+z)=z+o(|z|)$ $(z\to 0)$ which is clear from the Taylor series of $\log(1+z)$ in the neighbourhood of the point $z=0$.

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M-MIXING SYSTEMS. I

By
P. RÉVÉSZ (Budapest)

To Professor G. ALEXITS *on his 70th birthday*

Introduction

The properties of mixing sequences of random variables were investigated by a number of authors. Their aim was to generalize theorems (first of all limit theorems) valid for independent random variables to a class of weakly dependent random variables. In order to obtain such theorems slightly different concepts of mixing were introduced. However the essential idea of these definitions is a condition saying that "the future is independent from the long past". More precisely let $\xi_1, \xi_2, \dots$ be a sequence of random variables and let $\mathscr{B}_m^n$ $(m \leqq n)$ denote the smallest σ-algebra with respect to which the random variables $\xi_m, \xi_{m+1}, \dots, \xi_n$ are measurable. Then a mixing condition says that the elements of $\mathscr{B}_1^k$ are nearly independent from the elements of $\mathscr{B}_{k+l}^\infty$ if l is large enough, i.e. we assume

$$(1) \qquad |\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)| \leqq f(l)$$

where $A \in \mathscr{B}_1^k$, $B \in \mathscr{B}_{k+l}^\infty$ and $f(l)$ is a function converging to 0 with a certain rate.

A quite different way to define a concept of weak dependence is due to G. ALEXITS (see [1] and [2]).

He introduced the following:

DEFINITION. A sequence $\xi_1, \xi_2, \dots$ of random variables is called an equinormed strongly *multiplicative* system (ESMS) if

$$(2) \qquad \begin{aligned} &\mathbf{E}(\xi_i) = 0, \quad \mathbf{E}(\xi_i^2) = 1 \qquad (i = 1, 2, \dots) \\ &\mathbf{E}(\xi_{i_1}^{r_1}\xi_{i_2}^{r_2}\dots\xi_{i_n}^{r_n}) = \mathbf{E}(\xi_{i_1}^{r_1})\mathbf{E}(\xi_{i_2}^{r_2})\dots\mathbf{E}(\xi_{i_n}^{r_n}) \qquad (i_1 < i_2 < \dots < i_n;\ n = 1, 2, \dots) \end{aligned}$$

where r_j $(j = 1, 2, \dots, n)$ can be equal to 1 or 2. (The existence of the mentioned expectations is assumed.)

Alexits himself and others obtained results showing that in some sense this condition is able to substitute the condition of independence.

In the present paper we try to give a common genaralization of the ESMS and the systems with mixing property.

Namely we introduce the following

DEFINITION. A sequence $\xi_1, \xi_2, \dots$ of random variables is called *M-mixing* if there is a function $f(l)$ $(l = 1, 2, \dots)$ convering to 0 such that

$$|\mathbf{E}(\xi_{i_1}^{r_1}\xi_{i_2}^{r_2}\dots\xi_{i_n}^{r_n}\xi_{j_1}^{s_1}\xi_{j_2}^{s_2}\dots\xi_{j_m}^{s_m}) - \mathbf{E}(\xi_{i_1}^{r_1}\xi_{i_2}^{r_2}\dots\xi_{i_n}^{r_n})\mathbf{E}(\xi_{j_1}^{s_1}\xi_{j_2}^{s_2}\dots\xi_{j_m}^{s_m})| \leqq f(j_1 - i_n)$$

where $i_1 < i_2 < \dots < i_n < j_1 < j_2 < \dots < j_m$; $n = 1, 2, \dots$; $m = 1, 2, \dots$ and r_l and s_k $(l = 1, 2, \dots, n;\ k = 1, 2, \dots, m)$ can be equal to 1 or 2.

The fact that an ESMS is an M-mixing sequence is obvious.

The following theorem of IBRAGIMOV ([3], Theorem 17. 2. 2) shows the connection between the mixing sequences and the M-mixing systems.

THEOREM OF IBRAGIMOV. *Let $\xi_1, \xi_2, \dots$ be a sequence of random variables obeying condition* (1). *Further let ξ and η be random variables measurable with respect to $\mathcal{B}_1^k$ and $\mathcal{B}_{k+l}^\infty$ respectively, for which there exist positive numbers δ, c_1, c_2 such that*

$$\mathbf{E}(|\xi|^{2+\delta}) < c_1 \quad \text{and} \quad \mathbf{E}(|\eta|^{2+\delta}) < c_2.$$

Then we have

$$|\mathbf{E}(\xi\eta) - \mathbf{E}(\xi)\mathbf{E}(\eta)| \leqq \left(4 + 3\left(c_1^{\frac{1}{1+\delta}} c_2^{\frac{1+\delta}{2+\delta}} + c_1^{\frac{1+\delta}{2+\delta}} c_2^{\frac{1}{1+\delta}}\right)\right)(f(l))^{\frac{\delta}{2+\delta}}.$$

In § 1 we give some known theorems for mixing sequences, in § 2 the known results of ESMS are repeated. The aim of these two paragraphs is just to give a comparison to the new results.

In § 3 a convergence theorem (and a strong law of large numbers) are formulated for M-mixing sequences. § 4 contains the proofs.

The paper "M-mixing systems. II" will contain a central limit theorem and a law of iterated logarithm for M-mixing sequences.

§ 1. Mixing systems

The investigation of mixing systems started by the paper of ROSENBLATT ([4]). He proved a central limit theorem for such sequences. A detailed treatment can be found in [3]. (See also [5].) In [3] it is assumed that the sequence of random variables is not only mixing but it is a stationary sequence too. This second restriction generally can be dropped (or replaced by a weaker one) without any difficulty.

A typical result of this type is the following

THEOREM MI—1 ([3], Theorem 18. 5. 3). *Let $\xi_1, \xi_2, \dots$ be a stationary sequence obeying condition* (1). *Assume that there exists a $\delta > 0$ such that*

$$\mathbf{E}(\xi_j^{2+\delta}) < \infty$$

and

$$\sum_{n=1}^{\infty} (f(n))^{\frac{\delta}{2+\delta}} < \infty.$$

Then

$$\sigma^2 = \mathbf{E}(\xi_1^2) + 2\sum_{j=2}^{\infty} \mathbf{E}(\xi_i \xi_j) < \infty$$

and

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\sum_{j=1}^{n} \xi_j}{\sigma\sqrt{n}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$$

provided that $\sigma > 0$.

A strong law of large numbers for mixing systems is given in [6]. (See also [7] Theorem 8. 2. 1.) In this paper the concept of mixing is defined in a different way, namely the condition says that (only) the present is independent from the long past. More precisely it is assumed that

$$|\mathbf{P}(AB)-\mathbf{P}(A)\mathbf{P}(B)| \leqq f(l)$$

where $A \in B_1^k$, $B \in \mathscr{B}_{k+l}^{k+l}$ and $f(l)$ is a function converging to 0 ($\mathscr{B}_a^b$ is defined in the Introduction). A sequence of this type is called $*$-mixing.

For this type of mixing systems the following theorem is proved.

THEOREM MI—2 ([6]). *Let $\xi_1, \xi_2, \ldots$ be a $*$-mixing sequence such that* $\mathbf{E}(\xi_n)=0$, $\mathbf{E}(\xi_n^2)<\infty$ $(n=1,2,\ldots)$. *Suppose that* $\mathbf{E}(|\xi_n|)\leqq K$ $(n=1,2,\ldots;$ $K>0$ *is constant) and*

$$\sum_{n=1}^{\infty}\frac{\mathbf{E}(\xi_n^2)}{n^2}<\infty. \tag{3}$$

Then

$$\mathbf{P}\left(\frac{\xi_1+\xi_2+\ldots+\xi_n}{n}\to 0\right)=1.$$

§ 2. Multiplicative systems

The fundamental theorem of ESMS was obtained by ALEXITS and TANDORI.

THEOREM MU—1 ([1], [2]). *Let $\xi_1, \xi_2, \ldots$ be a uniformly bounded* ESMS, *further let $c_1, c_2, \ldots$ be a sequence of real numbers for which*

$$\sum_{k=1}^{\infty} c_k^2<\infty.$$

Then

$$\sum_{k=1}^{\infty} c_k\xi_k$$

is convergent almost everywhere.

A central limit theorem and a law of iterated logarithm for ESMS were obtained by the author [8] (see also [9] and [10]).

THEOREM MU—2 ([8]). *If $\xi_1, \xi_2, \ldots$ is a uniformly bounded* ESMS, *then*

$$\lim_{n\to\infty}\mathbf{P}\left(\frac{\xi_1+\xi_2+\ldots+\xi_n}{\sqrt{n}}<x\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}\,dt.$$

THEOREM MU—3 ([8] and [7] Theorem 3. 3. 3). *If $\xi_1, \xi_2, \ldots$ is a uniformly bounded* ESMS, *then*

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\xi_1+\xi_2+\ldots+\xi_n}{\sqrt{n\log\log n}}\leqq 7\right)=1.$$

A generalization of the concept of ESMS was studied by the author, namely the systems were investigated in which condition (2) holds if $n \leqq 4$. For this type of systems we obtained

THEOREM MU—4 ([7], Theorem 3. 3. 4). *Let $\xi_1, \xi_2, \ldots$ be a sequence of random variables for which*

$$\mathbf{E}(\xi_i^6) \leqq K \qquad (i = 1, 2, \ldots)$$

$$\mathbf{E}(\xi_i^2 \xi_j \xi_k) = \mathbf{E}(\xi_i^2 \xi_j) = \mathbf{E}(\xi_i \xi_j \xi_k \xi_l) = \mathbf{E}(\xi_i \xi_j \xi_k) = \mathbf{E}(\xi_i \xi_j) = \mathbf{E}(\xi_i) = 0$$

where the indices i, j, k, l are different and K is a positive constant. Further let $c_1, c_2, \ldots$ be a sequence of real numbers and suppose that there exists an integer r (depending on $\{c_k\}$) such that

$$\sum_{k=1}^{\infty} c_k^2 l_r^2(k) < \infty$$

where[1]

$$l(x) = l_1(x) = \begin{cases} \log x & \text{if} \quad x \geqq 2 \\ 1 & \text{if} \quad 0 < x < 2 \end{cases}$$

and $l_r(x)$ is the r-th iterated of $l(x)$ i.e. $l_r(x) = l(l_{r-1}(x))$. Then the series

$$\sum_{k=1}^{\infty} c_k \xi_k$$

is convergent almost everywhere.

§ 3. A convergence theorem

Now we formulate our main

THEOREM MM—1.[2] *Let $\xi_1, \xi_2, \ldots$ be a sequence of random variables obeying the following conditions*

(i) $$\mathbf{E}(\xi_i) = 0, \quad \mathbf{E}(\xi_i^4) \leqq K \qquad (i = 1, 2, \ldots)$$

(ii) $$|\mathbf{E}(\xi_i \xi_j)| \leqq f(j-i) \qquad (i < j)$$

(iii) $$|\mathbf{E}(\xi_i \xi_j \xi_k \xi_l)| \leqq \min\big(f(l-k), f(j-i)\big) \qquad (i < j < k < l)$$

(iv) $$|\mathbf{E}(\xi_i^2 \xi_j \xi_k)| \leqq f(k-j) \qquad (i < j < k)$$

(v) $$|\mathbf{E}(\xi_i \xi_j^2 \xi_k)| \leqq \min\big(f(k-j), f(j-i)\big) \qquad (i < j < k)$$

(vi) $$|\mathbf{E}(\xi_i \xi_j \xi_k^2)| \leqq f(j-i) \qquad (i < j < k)$$

(vii) $$\mathbf{E}(\xi_i^2 \xi_j^2) \leqq 1 + f(j-i) \qquad (i < j)$$

where K is a positive constant greater than 1, and $f(k)$ is a decreasing function defined on the integers for which there exists a positive constant d such that

(4) $$f(k) \leqq e^{-dk}.$$

[1] Here and in what follows $\log x$ means the logarithm with base 2.
[2] This Theorem clearly contains Theorem MU—4.

Further let $c_1, c_2, \ldots$ *be a sequence of real numbers and suppose that there exists an integer r (depending on* $\{c_k\}$*) such that*

$$\sum_{k=1}^{\infty} c_k^2 l_r^2(k) < \infty \tag{5}$$

where

$$l(x) = l_1(x) = \begin{cases} \log x & \text{if } x \geqq 2 \\ 1 & \text{if } 0 < x < 2 \end{cases}$$

and $l_r(x)$ *is the r-th iterated of* $l(x)$ *i.e.* $l_r(x) = l(l_{r-1}(x))$. *Then the series*

$$\sum_{k=1}^{\infty} c_k \xi_k$$

is convergent almost everywhere.

Making use of the Kronecker lemma (see e.g. [7] Theorem 1. 2. 2) Theorem MM—1 implies the following strong law of large numbers.

THEOREM MM—2. *Let* $\xi_1, \xi_2, \ldots$ *be a sequence of random variables obeying the conditions of Theorem* MM—1, *further let* $c_1, c_2, \ldots$ *be a sequence of real numbers and suppose that there exists an integer r such that*

$$\sum_{k=1}^{\infty} \frac{c_k^2}{k^2} l_r^2(k) < \infty \tag{6}$$

(where $l_r(k)$ *is defined in Theorem* MM—1*).*
Then

$$\mathbf{P}\left(\frac{c_1\xi_1 + c_2\xi_2 + \ldots + c_n\xi_n}{n} \to 0\right) = 1.$$

It can be seen that in some sense this theorem is stronger than Theorem MI—2 but in some sense this is the weaker one. Namely in this theorem there is a more strict restriction about the meaning of the „long past" (condition (4)) but there is no restriction about the whole long past only about two or three members of it and now we do not take into consideration all events of the long past, just the moments (see Theorem of Ibragimov). Since in condition (6) the integer r can be as large as we wish, this condition is not much stronger than condition (3).

The proof of Theorem MM—1 is based on an inequality analogous to the Rademacher—Mensov inequality (see e.g. [7] Theorem 3. 1. 1).

THEOREM MM—3. *Let* $\xi_1, \xi_2, \ldots, \xi_n$ *be a sequence of random variables, obeying the conditions* (i)—(vii) *of Theorem* MM—1, *where (now)* $f(k)$ *is a decreasing function for which*

$$\sum_{k=1}^{n} k f(k) \leqq \frac{1}{8}.$$

Then

$$\mathbf{E}\left(\max_{1 \leqq k \leqq n}\left(\sum_{j=1}^{k} c_j \xi_j\right)^4\right) \leqq 24K(\log 4n)^4 \left(\sum_{j=1}^{n} c_j^2\right)^2$$

where $\{c_j\}_{j=1}^{n}$ *is an arbitrary sequence of real numbers.*

§ 4. Proofs

First of all we give four lemas.

LEMMA 1. *Under the conditions of Theorem* MM—3 *we have*

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \mathbf{E}^2(\xi_i^3\xi_j) \leqq \frac{8}{3}K \quad (i=1,2,\ldots,n).$$

PROOF. Let $\mathbf{E}(\xi_i^3\xi_j)=d_{ij}$ and consider the inequality

$$0\leqq \mathbf{E}\left[\left(\xi_i^2-\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}\xi_i\xi_j\right)^2\right] = \mathbf{E}(\xi_i^4)+\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}^2\mathbf{E}\,[(\xi_i^2\xi_j^2)]-2\sum_{j=1}^{n} d_{ij}^2+$$

$$+2\sum_{\substack{1\leqq k<l\leqq n\\ k\neq i;\ l\neq i}} d_{il}d_{ik}\mathbf{E}(\xi_i^2\xi_k\xi_l) \leqq K+\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}^2\big(1+f(j-i)\big)-2\sum_{j=1}^{n} d_{ij}^2+$$

$$+\sum_{\substack{1\leqq k<l\leqq n\\ k\neq i;\ l\neq i}} (d_{il}^2+d_{ik}^2)\mathbf{E}(\xi_i^2\xi_k\xi_l) \leqq K+\frac{9}{8}\sum_{j=1}^{n} d_{ij}^2-2\sum_{j=1}^{n} d_{ij}^2+$$

$$+\sum_{\substack{1\leqq k<l\leqq n\\ k\neq i;\ l\neq i}} d_{il}^2\max\big(f(k-l),f(k-i)\big)+\sum_{\substack{1\leqq k<l\leqq n\\ k\neq i;\ l\neq i}} d_{ik}^2\max\big(f(k-l),f(k-i)\big)\leqq$$

$$\leqq K+\frac{9}{8}\sum_{j=1}^{n} d_{ij}^2-2\sum_{j=1}^{n} d_{ij}^2+\frac{4}{8}\sum_{j=1}^{n} d_{ij}^2 = K-\frac{3}{8}\sum_{j=1}^{n} d_{ij}^2$$

which implies our lemma.

LEMMA 2. *Under the conditions of Theorem* MM—3 *we have*

$$\mathbf{E}\left[\left(\sum_{j=1}^{n} c_j\xi_j\right)^4\right]\leqq 24K\left(\sum_{j=1}^{n} c_j^2\right)^2.$$

PROOF. We have

$$\mathbf{E}\left[\left(\sum_{j=1}^{n} c_j\xi_j\right)^4\right] = \sum_{j=1}^{n} c_j^4\mathbf{E}(\xi_j^4)+6\sum_{1\leqq i<j\leqq n} c_i^2c_j^2\mathbf{E}(\xi_i^2\xi_j^2)+4\sum_{1\leqq i<j\leqq n} c_i^3c_j\mathbf{E}(\xi_i^3\xi_j)+$$

$$+4\sum_{1\leqq i<j\leqq n} c_ic_j^3\mathbf{E}(\xi_i\xi_j^3)+12\sum_{1\leqq i<j<k\leqq n} c_i^2c_jc_k\mathbf{E}(\xi_i^2\xi_j\xi_k)+$$

$$+12\sum_{1\leqq i<j<k\leqq n} c_ic_j^2c_k\mathbf{E}(\xi_i\xi_j^2\xi_k)+12\sum_{1\leqq i<j<k\leqq n} c_ic_jc_k^2\mathbf{E}(\xi_i\xi_j\xi_k^2)+$$

$$+24\sum_{1\leqq i<j<k<l\leqq n} c_ic_jc_kc_l\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)\leqq$$

$$\leqq K\sum_{j=1}^{n} c_j^4+6K\sum_{1\leqq i<j\leqq n} c_i^2c_j^2+8\sum_{i=1}^{n} c_i^3\sqrt{\sum_{j=1}^{n} c_j^2\sum_{\substack{j=1\\ j\neq i}}^{n}\mathbf{E}^2(\xi_i^3\xi_j)}+$$

$$+6\sum_{1\le i<j<k\le n} c_i^2(c_j^2+c_k^2)f(k-j)+6\sum_{1\le i<j<k\le n} c_j^2(c_i^2+c_k^2)|\mathbf{E}(\xi_i\xi_j^2\xi_k)|+$$

$$+6\sum_{1\le i<j<k\le n}(c_i^2+c_j^2)c_k^2 f(j-i)+6\sum_{1\le i<j<k<l\le n}(c_i^2+c_j^2)(c_k^2+c_l^2)|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|\le$$

$$\le 3K\left(\sum_{j=1}^n c_j^2\right)^2+14K\left(\sum_{i=1}^n c_i^2\right)^2+\sum_{1\le i<j\le n} c_i^2c_j^2+\sum_{1\le i<k\le n} c_i^2c_k^2+$$

$$+6\sum_{1\le i<j<k\le n} c_i^2c_j^2 f(k-j)+6\sum_{1\le i<j<k\le n} c_j^2c_k^2 f(j-i)+\sum_{1\le i<k\le n} c_i^2c_k^2+$$

$$+\sum_{1\le j<k\le n} c_j^2c_k^2+6\sum_{1\le i<j<k<l\le n}(c_i^2+c_j^2)(c_k^2+c_l^2)|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|\le$$

$$\le 20K\left(\sum_{j=1}^n c_j^2\right)^2+6\sum_{1\le i<j<k<l\le n} c_i^2c_k^2|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|+6\sum_{1\le i<j<k<l\le n} c_i^2c_l^2|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|+$$

$$+6\sum_{1\le i<j<k<l\le n} c_j^2c_k^2|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|+6\sum_{1\le i<j<k<l\le n} c_j^2c_l^2|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|.$$

The last four members of the last sum can be estimated in the same way. As an example the estimation of

$$I_1=\sum_{1\le i<j<k<l\le n} c_i^2c_k^2|\mathbf{E}(\xi_i\xi_j\xi_k\xi_l)|$$

will be given. We take into consideration two possible cases.

Case 1. $j-i\le l-k$,

Case 2. $j-i>l-k$.

The members of I_1 for which the restriction of Case 1 holds can be estimated by

$$\sum_{1\le i<k\le n} c_i^2c_k^2\sum_{l=k+1}^{\infty}\sum_{j=i+1}^{i+l-k} f(l-k)=\sum_{1\le i<k\le n} c_i^2c_k^2\sum_{l=k+1}^{\infty}(l-k)f(l-k)\le$$

$$\le\frac{1}{8}\sum_{1\le i<k\le n} c_i^2c_k^2.$$

The sum of the members of I_1 obeying the restriction of Case 2 can be estimated in the same way, so we have

$$I_1\le\frac{1}{4}\sum_{1\le i<k\le n} c_i^2c_k^2.$$

Hence we obtained

$$\mathbf{E}\left[\left(\sum_{j=1}^n c_j\xi_j\right)^4\right]\le 24K\left(\sum_{j=1}^n c_j^2\right)^2$$

which is the statement of our Lemma.

LEMMA 3. *If $\xi_1, \xi_2, \dots$ is a sequence of random variables obeying the conditions* (i)—(vii) *of Theorem* MM—1 *with a function $f_1(k)$ and if $m_1 < m_2 < \dots$ is a sequence of integers then the sequence*

$$\psi_k = \begin{cases} \frac{1}{\alpha_k} \sum_{j=m_k+1}^{m_{k+1}} c_j \xi_j & \text{if} \quad \alpha_k > 0 \\ 0 & \text{if} \quad \alpha_k = 0 \end{cases}$$

where $\alpha_k = \left[\sum_{j=m_k+1}^{m_{k+1}} c_j^2 \right]^{1/2}$ *is obeying the condition* (i)—(vii) *with a function $f_2(k)$ for which*

$$f_2(k) \leqq 8 \sum_{l=k}^{\infty} l f_1(l).$$

PROOF. Condition (i) is proved by Lemma 1. The others can be proved in the same way. So as an example we prove that

$$|\mathbf{E}(\psi_i \psi_j \psi_k \psi_l)| \leqq 8 \sum_{j=l-k}^{\infty} j f_1(j)$$

provided that $i < j < k < l$.

We have

$$|\mathbf{E}(\psi_i \psi_j \psi_k \psi_l)| = \left| \mathbf{E} \left[\sum_{\varkappa=m_i+1}^{m_{i+1}} \frac{c_\varkappa}{\alpha_i} \xi_\varkappa \sum_{\lambda=m_j+1}^{m_{j+1}} \frac{c_\lambda}{\alpha_j} \xi_\lambda \sum_{\mu=m_k+1}^{m_{k+1}} \frac{c_\mu}{\alpha_k} \xi_\mu \sum_{\nu=m_l+1}^{m_{l+1}} \frac{c_\nu}{\alpha_l} \xi_\nu \right] \right| \leqq$$

$$\leqq \sum_\varkappa \sum_\lambda \sum_\mu \sum_\nu \left(\frac{c_\varkappa^2}{\alpha_i^2} + \frac{c_\lambda^2}{\alpha_j^2} \right) \left(\frac{c_\mu^2}{\alpha_k^2} + \frac{c_\nu^2}{\alpha_l^2} \right) |\mathbf{E}(\xi_\varkappa \xi_\lambda \xi_\mu \xi_\nu)| \leqq$$

$$\leqq \sum_\varkappa \sum_\mu \frac{c_\varkappa^2}{\alpha_i^2} \frac{c_\mu^2}{\alpha_k^2} \Big[\sum_\nu f_1(\nu-\mu)(\nu-\mu) + \sum_{\{\lambda:\, \lambda-\varkappa > \nu-\mu\}} f_1(\lambda-\varkappa)(\lambda-\varkappa) \Big] +$$

$$+ \sum_\varkappa \sum_\nu \frac{c_\varkappa^2}{\alpha_i^2} \frac{c_\nu^2}{\alpha_l^2} \Big[\sum_\mu f_1(\nu-\mu)(\nu-\mu) + \sum_{\{\varkappa:\, \lambda-\varkappa > \nu-\mu\}} f_1(\lambda-\varkappa)(\lambda-\varkappa) \Big] +$$

$$+ \sum_\lambda \sum_\mu \frac{c_\lambda^2}{\alpha_j^2} \frac{c_\mu^2}{\alpha_k^2} \Big[\sum_\nu f_1(\nu-\mu)(\nu-\mu) + \sum_{\{\lambda:\, \lambda-\varkappa > \nu-\mu\}} f_1(\lambda-\varkappa)(\lambda-\varkappa) \Big] +$$

$$+ \sum_\lambda \sum_\nu \frac{c_\lambda^2}{\alpha_j^2} \frac{c_\nu^2}{\alpha_l^2} \Big[\sum_\mu f_1(\nu-\mu)(\nu-\mu) + \sum_{\{\varkappa:\, \lambda-\varkappa > \nu-\mu\}} f_1(\lambda-\varkappa)(\lambda-\varkappa) \Big] \leqq$$

$$\leqq 8 \sum_{j=m_l+1-m_{k+1}}^{\infty} j f_1(j) \leqq 8 \sum_{j=l-k}^{\infty} j f_1(j).$$

LEMMA 4. *If $c_1, c_2, \dots$ is a sequence of real numbers for which*

$$\sum_{k=1}^{\infty} c_k^2 l_r^2(k) < \infty$$

then there exists a sequence $n_1 < n_2 < \dots$ *of integers for which*

$$\sum_{k=1}^{\infty}\left(\sum_{j=n_k+1}^{n_{k+1}} c_j^2\right) l_{r-1}^2(k) < \infty$$

and

$$\sum_{k=1}^{\infty}\left(\sum_{j=n_k+1}^{n_{k+1}} c_j^2\right)^2 l^4(n_{k+1}-n_k) < \infty.$$

See the proof in [7] Lemma 3. 3. 3.

PROOF OF THEOREM MM—3. First of all we assume that $n=2^\nu$ $(\nu=1, 2, \dots)$ and introduce the following notations

$$\sigma_j = c_1\xi_1+c_2\xi_2+\dots+c_j\xi_j,$$
$$\psi_{\alpha\beta} = c_{\alpha+1}\xi_{\alpha+1}+c_{\alpha+2}\xi_{\alpha+2}+\dots+c_\beta\xi_\beta$$

where

$$\alpha = \mu 2^k;\quad \beta=\beta(\alpha)=(\mu+1)2^k;\qquad \mu=0, 1, 2, \dots, 2^{\nu-k}-1;\ k=0, 1, 2, \dots, \nu.$$

Consider the random variable σ_j as the sum of some $\psi_{\alpha\beta}$. Let

$$\sigma_j = \sum_i \psi_{\alpha_i\beta_i}$$

where $\beta_1-\alpha_1 > \beta_2-\alpha_2 > \dots$. Clearly the number of the members of the sum $\sum_i \psi_{\alpha_i\beta_i}$ is less than ν. Therefore by the Cauchy inequality we have

$$\sigma_j^4 = \left(\sum_i \psi_{\alpha_i\beta_i}\right)^4 \leqq \nu^2\left(\sum_i \psi_{\alpha_i\beta_i}^2\right)^2 \leqq \nu^3 \sum_i \psi_{\alpha_i\beta_i}^4$$

which implies

$$\int_\Omega \max_{1\leqq j\leqq 2^\nu} \sigma_j^4\, d\mathbf{P} \leqq \nu^3 \sum_{\alpha,\beta}\int_\Omega \psi_{\alpha\beta}^4\, d\mathbf{P} \leqq \nu^3 \sum_{k=0}^{\nu}\sum_{\mu=0}^{2^{\nu-k}-1}\int_\Omega\left(\sum_{i=\mu 2^k+1}^{(\mu+1)2^k} c_i\xi_i\right)^4 d\mathbf{P}$$

where α and $\beta=\beta(\alpha)$ run through all their possible values.

Making use of Lemma 1 we have

$$\mathbf{E}\left(\max_{1\leqq j\leqq 2^\nu}\sigma_j^4\right) \leqq (\nu+1)^4 24K\left(\sum_{j=1}^{2^\nu} c_j^2\right)^2.$$

This inequality proves our statement in the case $n=2^\nu$. If $2^\nu \leqq n < 2^{\nu+1}$ then

$$\mathbf{E}\left(\max_{1\leqq j\leqq n}\sigma_j^4\right) \leqq 24K(\nu+2)^4\left(\sum_{j=1}^{n} c_j^2\right)^2 \leqq 24K(\log 4n)^4\left(\sum_{j=1}^{n} c_j^2\right)^2$$

which completes our proof.

PROOF OF THEOREM MM—1. First of all choose an integer $S\geqq 2$ such that

$$\sum_{k=s}^{\infty} ke^{-dk} \leqq \frac{1}{2}.$$

Now we prove that

$$\sum_{k=1}^{\infty} c_{ks+t}\xi_{ks+t} \quad (t = 0, 1, 2, \dots, s-1)$$

is convergent almost everywhere, which implies our Theorem.

Put

$$c_{ks+t} = \gamma_k, \quad \xi_{ks+t} = \eta_k.$$

As a first step we prove the almost everywhere convergence of the series $\sum_{k=1}^{\infty} \gamma_k \eta_k$ under the condition

$$\sum_{k=1}^{\infty} c_k^2 l^2(k) < \infty.$$

Set

$$\vartheta_n = \sum_{k=n}^{\infty} \gamma_k \eta_k.$$

Then

$$\mathbf{E}(\vartheta_n^2) \leqq \frac{3}{2} \sum_{k=n}^{\infty} \gamma_k^2 \leqq \frac{3}{2} \frac{A}{l^2(n)} \quad \left(\text{where } A = \sum_{k=1}^{\infty} \gamma_k^2 l^2(k)\right)$$

and

$$\sum_{n=1}^{\infty} \mathbf{E}(\vartheta_{2^n}^2) \leqq \frac{3A}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

By Beppo Levi theorem this fact implies

$$\vartheta_{2^n} \to 0.$$

By Theorem MM—3 we have

$$\mathbf{E}\left(\max_{2^n \leqq k < 2^{n+1}} \left(\sum_{j=2^n}^{k} \gamma_j \eta_j\right)^4\right) \leqq 24Kl^4(2^{n+2}) \left(\sum_{j=2^n+1}^{2^{n+1}} \gamma_j^2\right)^2 = 24K(n+2)^4 \left(\sum_{j=2^n+1}^{2^{n+1}} \gamma_j^2\right)^2$$

which is less than

$$24K\left(\sum_{j=2^n+1}^{2^{n+1}} \gamma_j^2 \log^2 j\right)^2$$

if n is large enough. Hence

$$\sum_{n=1}^{\infty} \mathbf{E}\left(\max_{2^n \leqq k < 2^{n+1}} \left|\sum_{j=2^n}^{k} \gamma_j \eta_j\right|^4\right) < \infty$$

and

$$\max_{2^n \leqq k < 2^{n+1}} \left|\sum_{j=2^n}^{k} \gamma_j \eta_j\right|^4 \to 0$$

which proves the convergence of the series

$$\sum_{k=1}^{\infty} \gamma_k \eta_k$$

in the case when

$$\sum_{k=1}^{\infty} \gamma_k^2 \log^2 k < \infty.$$

Now we prove our Theorem by induction. Suppose (as the condition of our induction) that if $\{a_k\}$ is a sequence of real numbers for which

$$\sum_{k=1}^{\infty} a_k^2 l_{r-1}^2(k) < \infty$$

and ψ_k is a sequence of random variables obeying the conditions of Theorem MM—1. Then

$$\sum_{k=1}^{\infty} a_k \psi_k$$

is convergent.

Now let $\{\overset{*}{b}_k\}$ be a sequence of real numbers for which

$$\sum_{k=1}^{\infty} b_k^2 l_r^2(k) < \infty$$

and denote by $\{n_k\}$ a sequence of integers for which

$$\sum_{k=1}^{\infty} \left(\sum_{j=n_k+1}^{n_{k+1}} b_j^2 \right) l_{r-1}^2(k) < \infty$$

$$\sum_{k=1}^{\infty} \left(\sum_{j=n_k+1}^{n_{k+1}} b_j^2 \right)^2 l^4(n_{k+1} - n_k) < \infty.$$

By Lemma 2 the sequence

$$\psi_k = \begin{cases} \dfrac{1}{\alpha_k} \sum\limits_{j=n_k+1}^{n_{k+1}} b_j \eta_j & \text{if} \quad \alpha_k > 0 \\ 0 & \text{if} \quad \alpha_k = 0 \end{cases}$$

$\left(\text{where } \alpha_k = \left[\sum\limits_{j=n_k+1}^{n_{k+1}} b_j^2 \right]^{1/2} \right)$ is obeying the conditions of Theorem MM—1. This fact implies — by the condition of our induction — that

$$\sum_{k=1}^{\infty} \alpha_k \psi_k$$

is convergent almost everywhere.

In order to prove our theorem it is enough to show that

$$\sum_{k=1}^{\infty} \mathbf{E} \left[\max_{n_k+1 \leq t < n_{k+1}} \left(\sum_{j=n_k+1}^{t} b_j \eta_j \right)^4 \right] < \infty.$$

However this fact follows immediately from Theorem MM—3.

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SUR LE RESTE DANS LA FORMULE DE QUADRATURE D'EVERETT

Par

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Dédié à M. G. ALEXITS à l'occasion de son 70e anniversaire

1. Dans son mémoire sur les formules de quadrature, R. v. MISES [5] appelle formule de quadrature d'Everett, la formule sommatoire

$$\int_0^n f(x)\,dx = \sum_{\alpha=0}^{n} f(\alpha) + \sum_{\alpha=0}^{m-1} d_\alpha[f(\alpha)+f(n-\alpha)] + R_n[f] \tag{1}$$

où m est un nombre naturel, n un nombre entier nonnégatif, les coefficients d_α, indépendants de la fonction $f(x)$, étant déterminés de manière que le reste $R_n[f]$ soit de degré d'exactitude $\geqq m$, donc qu'il s'annule pour tout polynome de degré m.

Nous nous proposons de trouver une expression du reste $R_n[f]$ en faisant les hypothèses suivantes:

H. 1. *m est impair.*

H. 2. *La fonction $f(x)$ est continue sur un intervalle I de l'axe réel contenant les points* $0, 1, \ldots, n$ *et* $\alpha, n-\alpha, \alpha=1, 2, \ldots, m-1$.

Le reste $R_n[f]$ est une fonctionnelle linéaire (additive et homogène) et pour trouver son expression désirée nous allons rappeler la définition de la simplicité d'une telle fonctionnelle.

2. Considérons une fonctionnelle linéaire (donc additive et homogène) $R[f]$, définie sur un ensemble linéaire S de fonctions (réelles et) *continues* $f(x)$ définies sur un intervalle donné I (de longueur non nulle) de l'axe réel. Nous supposons toujours que S contient tous les polynomes. Lorsqu'il est nécessaire on peut encore préciser la structure de l'ensemble S.

Le *degré d'exactitude* de $R[f]$ (s'il existe) est l'entier $m \geqq -1$ qui jouit de la propriété

$$\begin{cases} R[1] \neq 0 \quad \text{si} \quad m=-1, \\ R[1]=R[x]=\ldots=R[x^m]=0, \quad R[x^{m+1}]\neq 0 \quad \text{si} \quad m \geqq 0. \end{cases} \tag{2}$$

Le degré d'exactitude, s'il existe, est bien déterminé. Lorsque seules les égalités (2) sont vérifiées ($m \geqq 0$) nous dirons que $R[f]$ est de degré d'exactitude au moins m (ou que son degré d'exactitude est $\geqq m$). Ceci est équivalent au fait que la fonctionnelle linéaire $R[f]$ s'annule sur tout polynome de degré m. Pour que le degré d'exactitude soit égal à m, il faut et il suffit que $R[f]$ soit de plus différent de zéro sur un polynome de degré $m+1$ au moins.

Rappelons la définition suivante de la simplicité de la fonctionnelle linéaire $R[f]$:

La fonctionnelle linéaire $R[f]$ *est dite de la forme simple s'il existe un entier* $m \geqq -1$, *indépendant de la fonction* $f(x)$, *tel que l'on ait, pour* $f(x) \in S$,

$$R[f] = K \cdot [\xi_1, \xi_2, \ldots, \xi_{m+2}; f] \tag{3}$$

où ξ_α, $\alpha = 1, 2, \ldots, m+2$ *sont* $m+2$ *points distincts de l'intervalle* I, *dépendant en général de la fonction* $f(x)$ *et* K *est une constante différente de zéro, indépendante de la fonction* $f(x)$.

Le nombre m est complètement déterminé et il est précisément le degré d'exactitude de $R[f]$.

On a aussi $K = R[x^{m+1}] = R[x^{m+1} + P]$ $(\neq 0)$ où $P(x)$ est un polynome de degré m qui, dans certains cas concrets, peut être choisi convenablement.

Dans la formule (3) nous désignons par $[y_1, y_2, \ldots, y_r; f]$ la différence divisée, d'ordre $r-1$, de la fonction $f(x)$ sur les noeuds (distincts) $y_1, y_2, \ldots, y_r$.

3. Nous avons alors le

THÉORÈME 1. *Pour que la fonctionnelle linéaire* $R[f]$, *de degré d'exactitude* m, *soit de la forme simple, il faut et il suffit que l'on ait* $R[f] \neq 0$, *pour toute fonction* $f(x) \in S$ *convexe d'ordre* m.

Pour la notion et les propriétés des fonctions convexes (non-concaves, non-convexes, concaves) d'ordre m et pour la démonstration du théorème 1, le lecteur peut consulter mes travaux antérieurs. La fonction $f(x)$ est dite *convexe d'ordre* m sur I si toutes ses différences divisées d'ordre $m+1$, sur des noeuds distincts, sont positives. En particulier dans mes mémoires de „Mathematica" [7, 8] on peut trouver diverses applications et diverses généralisations de la notion de simplicité d'une fonctionnelle linéaire.

Si $m \geqq 0$ on peut même affirmer que les points ξ_α, $\alpha = 1, 2, \ldots, m+2$ de la formule (3) sont *à l'intérieur* de l'intervalle I.

Si $m \geqq 0$ et si $f(x)$ a une dérivée d'ordre $m+1$ à l'intérieur de I, nous avons, grâce à une formule de la moyenne importante de A. CAUCHY [1],

$$R[f] = K \frac{f^{(m+1)}(\xi)}{(m+1)!} \qquad (K = R[x^{m+1}]) \tag{4}$$

en supposant que $R[f]$ est de degré d'exactitude m et qu'il est de la forme simple, ξ étant un point à l'intérieur de l'intervalle I.

La formule (4) peut, en particulier, servir à donner une borne supérieure de $R[f]$ lorsqu'on connaît $f^{(m+1)}(x)$, la dérivée $(m+1)^{\text{ième}}$ de la fonction $f(x)$.

4. Revenons à la formule de quadrature (1). Nous allons démontrer d'abord que, sous l'hypothèse H. 1 et en supposant que $R_n[f]$ soit de degré d'exactitude m, *les coefficients* d_α, $\alpha = 0, 1, \ldots, m-1$ *sont déterminés indépendamment de* n.

Calculons $R_n[x^k]$. En utilisant la théorie bien connue des nombres B_α et des polynomes $B_\alpha(x)$ de Bernoulli, telle qu'elle est exposée dans le traité classique de N. E. NÖRLUND [6], nous avons, pour k entier $\geqq 0$,

$$\int_0^n x^k\,dx - \sum_{\alpha=0}^{n} \alpha^k = \frac{1}{k+1}\left[\sum_{\alpha=1}^{k}(-1)^{k-\alpha}\binom{k+1}{\alpha}B_{k+1-\alpha}n^\alpha + \left(1+(-1)^k\right)B_{k+1}\right]. \tag{5}$$

Si nous posons

$$s_k = \sum_{\alpha=0}^{m-1} \alpha^k d_\alpha, \qquad k = 0, 1, \ldots \tag{6}$$

$(s_0 = d_0 + d_1 + \ldots + d_{m-1})$, nous avons

$$\sum_{\alpha=0}^{m-1} d_\alpha[\alpha^k + (n-\alpha)^k] = \sum_{\alpha=1}^{k}(-1)^{k-\alpha}\binom{k}{\alpha}s_{k-\alpha}n^\alpha + [1+(-1)^k]s_k. \tag{7}$$

En comparant les formules (5), (7) il découle que, si nous posons

$$s_k = \sum_{\alpha=0}^{m-1} \alpha^k d_\alpha = \frac{B_{k+1}}{k+1} \qquad (k = 0, 1, \ldots, m-1), \tag{8}$$

nous avons $R_n[x^k] = 0$, $k = 0, 1, \ldots, m$. L'exactitude de la dernière égalité ($R_n[x^m] = 0$) est assurée par l'hypothèse H. 1 (l'imparité de m).

Le système (8) détermine complètement et indépendamment de n les coefficients d_α, $\alpha = 0, 1, \ldots, m-1$. Le fait que le reste $R_n[f]$ est effectivement de degré d'exactitude m résultera de ce qui suit.

5. Nous allons maintenant démontrer le

THÉORÈME 2. *Si $m+n>1$ et si les coefficients d_α, $\alpha = 0, 1, \ldots, m-1$ sont déterminés par les équations* (8), *sous les hypothèses* H. 1, H. 2, *le reste $R_n[f]$ est de degré d'exactitude m et il est de la forme simple, c'est-à-dire que*

$$R_n[f] = R_n[x^{m+1}][\xi_1, \xi_2, \ldots, \xi_{m+2}; f] \tag{9}$$

où ξ_α, $\alpha = 1, 2, \ldots, m+2$ sont $m+2$ points distincts à l'intérieur de l'intervalle I (et dépendent en général de la fonction $f(x)$).

La condition $m+n>1$ est essentielle. En effet si $m+n=1$ on a nécessairement $m=1$, $n=0$ et alors $R_0[f]=0$, quelle que soit la fonction $f(x)$.

La démonstration se fait maintenant par étapes en démontrant successivement les lemmes suivants.

LEMME 1. *Si $f(x)$ est une fonction convexe d'ordre m, on a $R_n[f] - R_{n-1}[f] < 0$, $n = 1, 2, \ldots$.*

C'est une conséquence du *critère de simplicité de Steffensen* [8]. On peut l'obtenir d'ailleurs facilement de la manière suivante. La différence $R[f] = R_n[f] - R_{n-1}[f]$ est le reste de la formule de quadrature $(n>0)$

$$\int_{n-1}^{n} f(x)\,dx = f(n) + \sum_{\alpha=0}^{m-1} d_\alpha[f(n-\alpha) - f(n-\alpha-1)] + R[f]$$

qui est de degré d'exactitude $\geqq m$. C'est alors nécessairement la formule de Cotes dans l'intervalle $[n-1, n]$ relativement aux noeuds $n-\alpha$, $\alpha=0, 1, \dots, m$. Nous avons donc

$$R[f] = \int_{n-1}^{n} [f(x) - L(n, n-1, \dots, n-m; f|x)]\, dx$$

où nous désignons par $L(y_1, y_2, \dots, y_r; f|x)$ le polynome d'interpolation de Lagrange de la fonction $f(x)$ sur les noeuds $y_1, y_2, \dots, y_r$. On sait que (pour x différent d'un noeud),

$$f(x) - L(n, n-1, \dots, n-m; f|x) = \prod_{\alpha=0}^{m} (x-n+\alpha)\cdot[n, n-1, \dots, n-m, x; f]$$

et le lemme résulte du fait que le polynome $\prod_{\alpha=0}^{m} (x-n+\alpha)$ est négatif sur l'intervalle ouvert $]n-1, n[$ et que le second facteur du second membre, la différence divisée d'ordre $m+1$ est, par hypothèse, positive.

En particulier si $m=1$ et si $f(x)$ est une fonction convexe d'ordre 1, on a $R_1[f]<0$.

LEMME 2. *Si $m>1$ et si $f(x)$ est une fonction convexe d'ordre m, on a*

$$R_{m-1}[f]<0.$$

Dans ce cas la formule (1) est la formule de Cotes relative aux m noeuds $0, 1, \dots, m-1$. La propriété résulte alors de la simplicité du reste de cette formule [8].

LEMME 3. *Si $m>1$ et si $f(x)$ est une fonction convexe d'ordre m, on a*

$$R_{m-2}[f]>0.$$

La démonstration est encore basée sur le critère de Steffensen qui découle d'ailleurs des importants résultats de J. F. STEFFENSEN [9] sur le reste des formules du type Cotes. En suivant l'exposé de J. F. STEFFENSEN nous pouvons démontrer le lemme 3 en remarquant d'abord qu'on peut écrire

$$R_{m-2}[f] = A[f]+B[f] \tag{10}$$

où $A[f]$ est le reste dans la formule de Cotes dans l'intervalle $[m-3, m-2]$ et $B[f]$ le reste dans la formule de Cotes dans l'intervalle $[0, m-3]$, tous les deux sur les noeuds $-1, 0, 1, \dots, m-1$.

Alors $A[f]$ est l'intégrale de $m-3$ à $m-2$ de la différence (pour x différent d'un noeud)

$$f(x) - L(-1, 0, 1, \dots, m-1; f|x) = \prod_{\alpha=0}^{m} (x+1-\alpha)\cdot[-1, 0, 1, \dots, m-1, x; f] \tag{11}$$

et le polynome $\prod_{\alpha=0}^{m} (x+1-\alpha)$ est positif sur l'intervalle $]m-3, m-2[$. Il en résulte que si $f(x)$ est convexe d'ordre m, on a

$$A[f]>0. \tag{12}$$

Lorsque $m=3$ on a $B[f]=0$, quel que soit $f(x)$.

Si $m>3$, $B[f]$ est l'intégrale de 0 à $m-3$ de la même différence (11). En suivant toujours un raisonnement de J. F. STEFFENSEN [9] remarquons maintenant que la différence (11) peut s'écrire (pour x différent d'un noeud)

$$\prod_{\alpha=0}^{m-1}(x+1-\alpha)\{[-1,0,1,\dots,m-2,x;f]-[-1,0,1,\dots,m-1;f]\}$$

et il s'ensuit que $B[f]$ est le reste de la formule de Cotes dans l'intervalle $[0, m-3]$ sur les noeuds $-1, 0, 1, \dots, m-2$. On déduit des considérations faites par J. F. STEFFENSEN [9] sur le polynome $P(x)=\prod_{\alpha=0}^{m-1}(x+1-\alpha)$ que le polynome $\int_0^x P(t)dt$ est négatif sur l'intervalle ouvert $]0, m-3[$ et est nul pour $x=m-3$. On en déduit que si $f(x)$ est une fonction convexe d'ordre m, on a

$$B[f]>0. \tag{13}$$

Les formules (10), (12) et (13) démontrent le lemme 3.

On obtient maintenant facilement le théorème 2. On peut conclure de la formule

$$R_n[f]=R_{m-1}[f]+\sum_{\alpha=0}^{n-m}\{R_{m+\alpha}[f]-R_{m+\alpha-1}[f]\}$$

où $n\geqq m$ et des lemmes 1, 2 que

$$R_n[f]<0,\qquad n\geqq m-1 \tag{14}$$

pour toute fonction $f(x)$ convexe d'ordre m.

Si $m>1$, il vient de la formule

$$R_n[f]=R_{m-2}[f]-\sum_{\alpha=0}^{m-n-3}\{R_{m-2-\alpha}[f]-R_{m-3-\alpha}[f]\}$$

où $n\leqq m-3$ et des lemmes 1, 3 que

$$R_n[f]>0,\qquad n\leqq m-2 \tag{15}$$

pour toute fonction $f(x)$ convexe d'ordre m.

La fonction x^{m+1} est convexe d'ordre m et alors les formules (14), (15) montrent que $R_n[f]$ est effectivement de degré d'exactitude m. Le théorème 2 est donc une conséquence du théorème 1.

6. Les considérations précédentes permettent aussi de calculer, sous diverses formes, le facteur $R_n[x^{m+1}]$ qui figure dans la formule (9). Compte tenu de l a formule (5), de la notation (6) et de l'hypothèse H. 1, nous avons $(m>1)$

$$R_n[x^{m+1}]=\frac{1}{m+2}\left[\sum_{\alpha=1}^{m+1}(-1)^{m+1-\alpha}\binom{m+2}{\alpha}B_{m+2-\alpha}n^\alpha\right]-$$
$$-\sum_{\alpha=1}^{m+1}(-1)^{m+1-\alpha}\binom{m+1}{\alpha}s_{m+1-\alpha}n^\alpha-2s_{m+1}.$$

Nous obtenons ainsi, en vertu de (8),

$$R_n[x^{m+1}] = [(m+1)s_m - B_{m+1}]n - 2s_{m+1} = \lambda n + \mu,$$

expression linéaire par rapport à n, λ, μ étant des coefficients numériques indépendants de n.

Il en résulte qu'on a aussi

$$R_n[x^{m+1}] = (n-m+2)R_{m-1}[x^{m+1}] - (n-m+1)R_{m-2}[x^{m+1}]$$

et dans cette formule $R_{m-1}[x^{m+1}]$, $R_{m-2}[x^{m+1}]$ peuvent être obtenus en suivant la démonstration des lemmes 2, 3.

L'interprétation de $R_{m-1}[f]$ donne

$$R_{m-1}[x^{m+1}] = \int_0^{m-1} \left[x + \frac{m(m-1)}{2}\right] \prod_{\alpha=0}^{m-1} (x-\alpha)dx$$

et celle de $R_{m-2}[f]$ que

$$R_{m-2}[x^{m+1}] = \int_0^{m-3} \left[x + \frac{m(m-3)}{2}\right] \prod_{\alpha=0}^{m-1} (x+1-\alpha)dx + \int_{m-3}^{m-2} \prod_{\alpha=0}^{m} (x+1-\alpha)dx.$$

Dans le cas $m=1$ nous avons $\lambda = -\frac{1}{6}$, $\mu=0$ et

$$R_n[f] = -\frac{n}{6}[\xi_1, \xi_2, \xi_3; f]$$

est le reste de la formule du trapèze

$$\int_0^n f(x)\,dx = \frac{1}{2}f(0) + f(1) + \ldots + f(n-1) + \frac{1}{2}f(n) + R_n[f].$$

Si $m>1$, l'analyse précédente nous montre que $\lambda<0$ et $\mu>0$, $m-2 < \frac{\mu}{-\lambda} < m-1$.

7. Lorsque la fonction $f(x)$ a une dérivée d'ordre $m+1$ à l'intérieur de l'intervalle I, on a

$$R_n[f] = R_n[x^{m+1}]\frac{f^{(m+1)}(\xi)}{(m+1)!},$$

ξ étant un point à l'intérieur de I.

Ce résultat, pour $m=3, 5, 7$ a été obtenu, d'une autre manière, par D. V. Ionescu [2] et D. V. Ionescu et A. Cotiu [3, 4].

Lorsque $|f^{(m+1)}(x)| \leqq M(m+1)!$ pour $x \in I$, on obtient la délimitation

$$|R_n[f]| \leqq |R_n[x^{m+1}]|M,$$

M étant un nombre réel non-négatif. Une telle borne supérieure du reste existe encore si la fonction $f(x)$ est à $(m+1)^{\text{ième}}$ différence divisée en valeur absolue par M. Un exemple d'une telle fonction est fourni par tout $f(x)$ qui à une $m^{\text{ième}}$ dérivée $f^{(m)}(x)$ vérifiant une condition de Lipschitz ordinaire.

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ON A TRIGONOMETRIC CONVOLUTION OPERATOR WITH KERNEL HAVING TWO ZEROS OF SIMPLE MULTIPLICITY

By

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Dedicated to Professor G. ALEXITS *on the occasion of his 70th birthday, January 5, 1969*

1. Introduction

If $T_n(f; x)$ is a positive (more exactly: non-negative) bounded linear trigonometric polynomial operator of degree n, then, as P. P. KOROVKIN [8] has shown, the optimal order of approximation of functions $f \in C_{2\pi}$ by such operators is $O(n^{-2})$, $n \to \infty$; this order of approximation cannot be improved by supposing f to be arbitrarily smooth. In particular, let the operator T_n be a singular integral of convolution type defined by $T_n(f; x) = (f * p_n)(x) = (1/\pi) \int_{-\pi}^{\pi} f(x-u)p_n(u)du$ having an even kernel $p_n(x) \in C_{2\pi}$. The question then arises whether the order of approximation by $f * p_n$ can be improved if the kernel is allowed to be negative for some x.

In this respect, P. P. KOROVKIN [10], cf. [9], (investigating the analogous algebraic case, only remarking that all results are valid in the periodic case as well) and independently P. L. BUTZER—R. J. NESSEL—K. SCHERER [5] have shown that if the (even) kernel $p_n(x)$ has $2m$ changes of sign in the interval $(-\pi, \pi]$ — the number of changes being independent of n —, then the optimal order of approximation cannot exceed $O(n^{-2m-2})$ provided such a kernel exists. A. I. KOVALENKO [11] has actually stated a scheme in how to construct such operators $f * p_n$ which approximate with order $O(n^{-2m-2})$ if $f \in C_{2\pi}^{(2m+2)}$; (for an algebraic analogon, see G. N. VINOGRADOVA [15]).

The purpose of this note is to give an explicit example of a convolution operator approximating with order $O(n^{-4})$ provided e.g. $f \in C_{2\pi}^{(4)}$. Thus, we shall not be content with a scheme but will in fact construct the associated kernel $N_{n-1,4}(x)$, cf. Theorem 1. As the general case in [11] is very complicated, we prefer a direct approach in which the choice of the function $\varphi^*(t) = \sin^3 \pi t$, $t \in [0, 1]$, is essential; this function as a special example satisfies the conditions given in [11].

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2. Basic results

Let $C_{2\pi}$, $L_{2\pi}^p$ $(1 \leq p < \infty)$ be the space of all 2π-periodic functions which are continuous on the whole real line or Lebesgue integrable to the pth power over $(-\pi, \pi)$, respectively. $X_{2\pi}$ denotes one of these spaces endowed with the usual norm. $C_{2\pi}^{(s)}$ denotes the space of functions $f \in C_{2\pi}$ whose derivatives of order s are continuous. $C(a, b)$ means those functions continuous on (a, b). n, m are natural numbers.

To be more precise let us define the classes $\mathfrak{S}_{2m}$ and S_{2m}, cf. [9], [11].

A sequence of trigonometric polynomials $\{t_n(x)\}$ *of degree n is said to belong to the class* $\mathfrak{S}_{2m}$ *if* $t_n(x)$ *has for* $x\in(-\pi, \pi]$ *and all n exactly 2m changes of sign, i.e. real zeros of odd multiplicity (and all further zeros are of even multiplicity or complex). In this case the singular integral* $I_n(p; f; x)\equiv(f * p_n)(x)$ *having kernel* $p_n\in\mathfrak{S}_{2m}$ *belongs to the* KOROVKIN-*class* S_{2m}.

If such an even kernel $p_n(x)\in\mathfrak{S}_{2m}$ exists a result in [5; p. 70] on the optimal order of approximation of the corresponding singular integral reads:

If $p_n(0)>0$ *for all n sufficiently large, then at least one of the* $(m+2)$ *sequences*

$$\{n^{2m+2}\|I_n(p; \cos ku; x)-\cos kx\|_{X_{2\pi}}\} \qquad (0\leqq k\leqq m+1)$$

does not tend to zero as $n\to\infty$.

To build up our kernel as announced, we begin with the well-known fact that it is possible to construct a non-negative polynomial by means of a generating function, cf. e.g. [7], [11].

LEMMA 1. *Let* $\varphi(t)$ *be defined for* $t\in[0, 1]$ *such that for all n*

$$\Phi_n=\sum_{s=0}^{n}\varphi^2\left(\frac{s}{n}\right)>0. \tag{1}$$

Then φ *generates via*

$$P_n(\varphi; t)=\frac{1}{2\Phi_n}\left|\sum_{s=0}^{n}\varphi\left(\frac{s}{n}\right)e^{ist}\right|^2 \tag{2}$$

and even and non-negative trigonometric polynomial in its closed form of degree at most n; the representation as a normalized polynomial is given by

$$P_n(\varphi; t)=\frac{1}{2}+\sum_{k=1}^{n}\lambda_{k,n}(\varphi)\cos kt\geqq 0 \tag{3}$$

with convergence factors

$$\lambda_{k,n}(\varphi)=\frac{1}{\Phi_n}\sum_{s=0}^{n-k}\varphi\left(\frac{s}{n}\right)\varphi\left(\frac{s+k}{n}\right). \tag{4}$$

LEMMA 2. *A normalized and even trigonometric polynomial of degree n of class* $\mathfrak{S}_2$ *with (nonconstant) symmetrical zeros at* $\pm\alpha$, $\alpha\in(0,\pi)$, *has the representation*

$$N_n(\varphi; t)=\nu(n)\cdot(\cos t-\cos\alpha)\cdot P_{n-1}(\varphi; t) \tag{5}$$

where $P_{n-1}(\varphi; t)$ *is a non-negative polynomial of degree* $(n-1)$ *generated by a suitable* φ *and* $\nu(n)$ *is such that* $\int_{-\pi}^{\pi}N_n(\varphi; t)dt=\pi$ *for all n.*

By the representation (5) it is obvious that $N_n(\varphi; t)\in\mathfrak{S}_2$, cf. [11]; by the decomposition lemma for general polynomials of class $\mathfrak{S}_{2m}$ in [5, p. 89] (where the positive factor kernel need not be given by a φ-function) it follows that this representation is also unique.

Later on, we shall see that in our example α is completely determined by φ and depends only upon n.

LEMMA 3. *Let $P_n(t)$ be a polynomial of degree n given by* (2) *or* (3), *respectively, with convergence factors $\lambda_{k,n}$. Then $P^*_{n+1}(t)\in\mathfrak{S}_2$ with zeros at $\pm\alpha$ defined by*

$$(6)\qquad P^*_{n+1}(t) = \mu(n)\cdot(\cos t-\cos\alpha)\cdot P_n(t) =$$

$$(7)\qquad = \frac{1}{2}+\sum_{k=1}^{n+1}\varrho_{k,n+1}\cos kt$$

has normalization constant

$$(8)\qquad \mu(n) = \frac{1}{\lambda_{1,n}-\cos\alpha}$$

and convergence factors

$$(9)\qquad \varrho_{k,n+1} = \frac{\lambda_{k-1,n}-2\lambda_{k,n}\cos\alpha+\lambda_{k+1,n}}{2(\lambda_{1,n}-\cos\alpha)}.$$

The verification of (8) and (9) follows by straightforward computation; for $k=0$ we have $\varrho_{0,n+1}=1$ by formally setting $\lambda_{-1,n}=\lambda_{1,n}$.

3. The positive factor kernel

In order to apply Lemma 2 in constructing an operator which approximates with order $O(n^{-4})$ we use the function

$$(10)\qquad \varphi^*(t) = \begin{cases}\sin^3\pi t, & t\in[0,1]\\ 0, & \text{otherwise.}\end{cases}$$

This is suggested by results of A. I. KOVALENKO on constructing such operators which give conditions upon a generating function φ of the positive factor kernel in (5). For an arbitrary m these conditions are given by, see [11, p. 600],

(11) (i) $\varphi(t)\equiv 0,\ t\notin[0,1]$,

(ii) $\varphi^{(2m)}(t)\in C(-\infty,\infty)$, (iii) $\varphi^{(2m+2)}(t)\in C(0,1)$,

(iv) $|\varphi^{(s)}(t)|\leqq M,\ t\in(0,1),\ s=0,1,\dots,2m+2.$

In case $m=1$ the particular function (10), cf. [11, p. 616], satisfies (11). As a matter of fact, the powers of $\sin\pi t$ play an important rôle in the construction and investigation of linear polynomial operators, see furthermore [12], [14].

LEMMA 4. *If $\varphi^*(t)=\sin^3\pi t$, then for $n\geqq 4$*

$$(12)\qquad P_{n-2}(\varphi^*;t) = \frac{8}{5n}\sin^6\frac{\pi}{n}\,\frac{\left(\cos t+2\cos\frac{\pi}{n}\right)^2\cos^2 n\frac{t}{2}}{\left(\cos t-\cos\frac{\pi}{n}\right)^2\left(\cos t-\cos\frac{3\pi}{n}\right)^2}$$

is the polynomial of degree $(n-2)$ generated by φ^* with convergence factors

$$\lambda_{k,n-2}(\varphi^*) = \frac{1}{10n}\left\{(n-k)\left(9\cos\frac{k\pi}{n}+\cos\frac{3k\pi}{n}\right)+\right. \tag{13}$$

$$\left.+3\cot\frac{\pi}{n}\sin\frac{k\pi}{n}\left(3+2\cos\frac{2k\pi}{n}\right)-6\cot\frac{2\pi}{n}\sin\frac{2k\pi}{n}\cos\frac{k\pi}{n}+\cot\frac{3\pi}{n}\sin\frac{3k\pi}{n}\right\}.$$

PROOF. We first note that in our case the polynomial given by (2) is actually of degree $(n-2)$; this is easily seen by taking $k=n-2,\ n-1,\ n$ in (4) and making use of $\varphi^*(t) = \varphi^*(1-t)$.

In order to prove (13), we set $\lambda_{k,n-2}(\varphi^*)=\tau_{k,n}/\tau_{0,n}$; then in view of (4) we have for $0\leqq k\leqq n-2$

$$\tau_{k,n} = \sum_{s=0}^{n-k}\left(\sin\frac{s\pi}{n}\sin\frac{(s+k)\pi}{n}\right)^3.$$

Making use of the identities

$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha),\quad \cos^3\alpha = \frac{1}{4}(\cos 3\alpha+3\cos\alpha) \tag{14}$$

it follows that

$$\tau_{k,n} = \frac{1}{8}\sum_{s=0}^{n-k}\left(\cos\frac{k\pi}{n}-\cos\frac{(k+2s)\pi}{n}\right)^3 =$$

$$= \frac{1}{8}\left\{(n-k+1)\left(\cos^3\frac{k\pi}{n}+\frac{3}{2}\cos\frac{k\pi}{n}\right)-3\left(\cos^2\frac{k\pi}{n}+\frac{1}{4}\right)\sum_{s=0}^{n-k}\cos\frac{(k+2s)\pi}{n}+\right.$$

$$\left.+\frac{3}{2}\cos\frac{k\pi}{n}\sum_{s=0}^{n-k}\cos\frac{2(k+2s)\pi}{n}-\frac{1}{4}\sum_{s=0}^{n-k}\cos\frac{3(k+2s)\pi}{n}\right\}.$$

For $j=1, 2, 3$ we need the further identity (cf. [13; p. 31])

$$\sum_{s=0}^{n-k}\cos\frac{j(k+2s)\pi}{n} = -\frac{\sin\frac{j(k-1)\pi}{n}}{\sin\frac{j\pi}{n}} = \cot\frac{j\pi}{n}\sin\frac{jk\pi}{n}-\cos\frac{jk\pi}{n} \tag{15}$$

which is valid for $n>j$. Applying (15) and again (14) it turns out that for $n>3$

$$\tau_{k,n} = \frac{1}{32}\left\{(n-k)\left(9\cos\frac{k\pi}{n}+\cos\frac{3k\pi}{n}\right)+3\cot\frac{\pi}{n}\sin\frac{k\pi}{n}\left(3+2\cos\frac{2k\pi}{n}\right)-\right.$$

$$\left.-6\cot\frac{2\pi}{n}\sin\frac{2k\pi}{n}\cos\frac{k\pi}{n}+\cot\frac{3\pi}{n}\sin\frac{3k\pi}{n}\right\}$$

and together with (1)

$$\tau_{0,n} \equiv \Phi_n = \frac{5}{16}n. \tag{16}$$

The proof of (12) may be indicated as follows. From (2) it follows that

$$\left|\sum_{k=0}^{n-1} \sin^3 \frac{k\pi}{n} e^{ikt}\right|^2 = \frac{1}{64}\left|\sum_{k=0}^{n-1} \left(e^{i\frac{k\pi}{n}} - e^{-i\frac{k\pi}{n}}\right)^3 e^{ikt}\right|^2 \equiv \frac{1}{64}|S|^2,$$

say. Using the formula for geometric progression and various trigonometric identities we have

$$S = -4i(e^{int}-1)\frac{\sin^3 \frac{\pi}{n}\left(\cos t + 2\cos\frac{\pi}{n}\right)}{\left(\cos t - \cos\frac{\pi}{n}\right)\left(\cos t - \cos\frac{3\pi}{n}\right)}.$$

From this and (16) we obtain (12).

REMARK. Here we wish to indicate why the kernel of FEJÉR—KOROVKIN cannot take the place of $P_{n-2}(\varphi^*; t)$. That kernel is generated by the particular related function $\bar{\varphi}(t) = \sin \pi t$ and given by, cf. [8],

(17) $$K_{n-2}(\bar{\varphi}; t) = \frac{\sin^2 \frac{\pi}{n}}{n} \frac{\cos^2 n \frac{t}{2}}{\left(\cos t - \cos \frac{\pi}{n}\right)^2}$$

with

$$\varrho_{1,n-2}(\bar{\varphi}) = \cos\frac{\pi}{n}.$$

In view of the following extremal properties, (17) is a suitable comparison kernel for other positive factor kernels.

Thus, $\varrho_{1,n}(\bar{\varphi}) = \cos\frac{\pi}{n+2}$ attains the *maximum* value of the first convergence factor for any even positive kernels of degree n (which are not necessarily generated by a φ-function). Furthermore,

$$\lim_{n\to\infty} n^2(1-\varrho_{k,n-2}(\bar{\varphi})) = \frac{\pi^2}{2}k^2$$

where the constant $c(\bar{\varphi}) = \pi^2/2$ is a *minimum* value for optimal positive kernels; cf. [6].

For the kernel (12) we have

$$\lambda_{1,n-2}(\varphi^*) = \frac{1}{10}\left(9\cos\frac{\pi}{n} + \cos\frac{3\pi}{n}\right)$$

and

(18) $$\lim_{n\to\infty} n^2(1-\lambda_{k,n-2}(\varphi^*)) = \frac{9\pi^2}{10}k^2$$

this being easily checked by a theorem in [7] only using $\varphi^*(t)$.

Thus, $\lambda_{1,n-2}(\varphi^*) < \varrho_{1,n-2}(\bar{\varphi})$ and $c(\varphi^*) = 9\pi^2/10 > c(\bar{\varphi})$ revealing that (12) — as an optimal positive kernel — is in some sense not as good as (17).

Note that in view of (17) we may represent (12) as the product

$$P_{n-2}(\varphi^*; t) = \frac{8}{5}\sin^4\frac{\pi}{n}\frac{\left(\cos t + 2\cos\frac{\pi}{n}\right)^2}{\left(\cos t - \cos\frac{3\pi}{n}\right)^2}\cdot K_{n-2}(\bar{\varphi}; t).$$

Concerning the distribution of zeros in $(0, \pi)$ of both kernels under consideration, it is of interesto note that the following situation might be responsible for the fact that, in spite of the above behaviour, (12) is a suitable approximation improving factor kernel in Lemma 2 whereas (17) is not so.

For, the denominator of (17) has a zero at $\bar{t}_0=\pi/n$ thus producing an indeterminate form such that the actual first zero of (17) is given by $\bar{t}_1=3\pi/n$. The zeros of the denominator of (12) at $t_0^*=\pi/n$, $t_1^*=3\pi/n$ effect that the actual first zero of (12) is shifted even to $t_2^*=5\pi/n$.

As we shall see later, the zero in the final kernel causing the change of sign is in fact situated in the initial interval $(0, t_2^*)$ which is exempt from zeros of the factor kernel.

4. The trigonometric convolution operator of class S_2

Combining Lemma 3 and 4, we have

LEMMA 5. *The polynomial of degree $(n-1)$ of class $\mathfrak{S}_2$ which is generated by φ^* with exactly two changes of sign at $\pm\alpha$ is given for $n \geqq 4$ by*

$$N_{n-1}(\varphi^*; t) = \frac{16 \sin^6 \frac{\pi}{n}}{n\left(9 \cos \frac{\pi}{n} + \cos \frac{3\pi}{n} - 10 \cos \alpha\right)} \cdot \tag{19}$$

$$\cdot \frac{(\cos t - \cos \alpha)\left(\cos t + 2 \cos \frac{\pi}{n}\right)^2 \cos^2 n \frac{t}{2}}{\left(\cos t - \cos \frac{\pi}{n}\right)^2 \left(\cos t - \cos \frac{3\pi}{n}\right)^2} = \frac{1}{2} + \sum_{k=1}^{n-1} \varrho_{k,n-1}(\varphi^*) \cos kt$$

and convergence factors

$$\varrho_{k,n-1}(\varphi^*) = \frac{Z(k, n)}{N(n)} \tag{20}$$

with

$$N(n) = n\left(9 \cos \frac{\pi}{n} + \cos \frac{3\pi}{n} - 10 \cos \alpha\right) \tag{21}$$

$$\begin{aligned} Z(k, n) = (n-k)&\left\{9 \cos \frac{k\pi}{n}\left(\cos \frac{\pi}{n} - \cos \alpha\right) + \cos \frac{3k\pi}{n}\left(\cos \frac{3\pi}{n} - \cos \alpha\right)\right\} + \\ &+ 9 \sin \frac{k\pi}{n} \sin \frac{\pi}{n} + \sin \frac{3k\pi}{n} \sin \frac{3\pi}{n} + \\ &+ 3 \cot \frac{\pi}{n}\left\{\sin \frac{3k\pi}{n} \cos \frac{3\pi}{n} + \sin \frac{k\pi}{n}\left[\cos \frac{\pi}{n} - \left(3 + 2 \cos \frac{2k\pi}{n}\right) \cos \alpha\right]\right\} - \\ &- 3 \cot \frac{2\pi}{n}\left\{\sin \frac{3k\pi}{n} \cos \frac{3\pi}{n} + \sin \frac{k\pi}{n}\left[\cos \frac{\pi}{n} - 2\left(1 + \cos \frac{2k\pi}{n}\right) \cos \alpha\right]\right\} + \\ &+ \cot \frac{3\pi}{n} \sin \frac{3k\pi}{n}\left[\cos \frac{3\pi}{n} - \cos \alpha\right]. \end{aligned} \tag{22}$$

By $N_{n-1}(\varphi^*; t)$ we have thus given an explicit example of a kernel of $\mathfrak{S}_2$ generated by φ^*. This kernel will now be used to define a trigonometric convolution operator which approximates with order $O(n^{-4})$, the best possible order of approximation for operators of S_2. To this end we must determine the constant α in (19) which is so far a free parameter such that this order is actually attained. A successful method is to demand that $1-\varrho_{1,n-1}(\varphi^*)=O(n^{-4})$ which is suggested by the fact that $\|I_{n-1}(\varphi^*; \cos u; x)-\cos x\|_{X_{2\pi}} = |1-\varrho_{1n-1}(\varphi^*)|$, cf. [5; p. 95]. For non-negative kernels this postulate may also be compared with the expression $1-\varrho_{1,n}=O(n^{-2})$ which, in the optimal case, determines the approximation behaviour of the corresponding singular integral (cf. [4], [6]).

LEMMA 6. *Under the condition*

$$\varrho_{1,n-1}(\varphi^*) = 1+O(n^{-4}), \quad n\to\infty, \tag{23}$$

the changes of sign in (19) *are uniquely determined by* φ^* *and given by*

$$\alpha_n(\varphi^*) = \pm\sqrt{5}\,\frac{\pi}{n}. \tag{24}$$

PROOF. From (9) and (13) it follows that

$$1-\varrho_{1,n-1}(\varphi^*) = \frac{2\lambda_{1,n-2}(\varphi^*)-1-\lambda_{2,n-2}(\varphi^*)-2(1-\lambda_{1,n-2}(\varphi^*))\cos\alpha}{2(\lambda_{1,n-2}(\varphi^*)-\cos\alpha)}.$$

Using the Taylor series expansions of $\sin t$, $\cos t$, and

$$\cot t = \frac{1}{t}\left(1-\sum_{j=1}^{\infty}\frac{2^{2j}}{(2j)!}|B_{2j}|\,t^{2j}\right), \quad |t|<\pi, \tag{25}$$

with $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, cf. [13; p. 35, 413], we have

$$\lambda_{1,n-2}(\varphi^*) = 1-\frac{9\pi^2}{10}\frac{1}{n^2}+\frac{3\pi^4}{8}\frac{1}{n^4}+O(n^{-6}), \quad n\to\infty,$$

$$\lambda_{2,n-2}(\varphi^*) = 1-4\frac{9\pi^2}{10}\frac{1}{n^2}+6\pi^4\frac{1}{n^4}+O(n^{-6}), \quad n\to\infty,$$

$$\cos\alpha = 1-\frac{1}{2}\alpha^2+\frac{1}{24}\alpha^4+O(\alpha^6), \quad \alpha\to 0.$$

Using these expansions, we have

$$1-\varrho_{1,n-1}(\varphi^*) = \frac{\frac{9}{20}\pi^2\frac{1}{n^2}\alpha^2-\frac{9}{4}\pi^4\frac{1}{n^4}+O(n^{-6}+\alpha^2n^{-4}+\alpha^4n^{-2})}{\frac{1}{2}\alpha^2-\frac{9\pi^2}{10}\frac{1}{n^2}+O(n^{-4}+\alpha^4)}, \quad n\to\infty, \quad \alpha\to 0.$$

In order that the leading difference of the numerator vanishes whereas the corresponding expression of the denominator must not be zero, we set $\alpha=\gamma\pi/n$. This leads to

$$\frac{9\pi^4}{4n^4}\left(\frac{\gamma^2}{5}-1\right)=0;$$

since this quadratic equation has the solutions $\gamma=\pm\sqrt{5}$ we obtain (24), and thus (23) is satisfied.

REMARK. According to [5, p. 97] a necessary condition upon the zeros of a kernel of $\mathfrak{S}_2$ such that the corresponding operator not only belongs to S_2 but also approximates better than an optimal positive operator is that the zeros tend to 0 as $n\to\infty$. In our case, the condition $\alpha_n(\varphi^*)=O(n^{-1})$, $n\to\infty$, ensures this fact.

Consequently, we may formulate

THEOREM 1. *The normalized even trigonometric polynomial* $N_{n-1,4}(\varphi^*;t)\in\mathfrak{S}_2$ *of degree* $(n-1)$ *with convergence factors* $\varrho_{k,n-1}^{(4)}(\varphi^*)$ *which is generated by* φ^* *of* (10) *such that* $\varrho_{1,n-1}^{(4)}(\varphi^*)=1+O(n^{-4})$, $n\to\infty$, *is given for* $n\geqq 4$ *by* $N_{n-1}(\varphi^*;t)$ *of* (19) *or* $\varrho_{k,n-1}(\varphi^*)$ *of* (20), *respectively, with changes of sign at* $\alpha=\alpha_n(\varphi^*)=$ $=\pm\sqrt{5}\,\pi n^{-1}$.

LEMMA 7. *The Lebesgue constants of the polynomial* $N_{n-1,4}(\varphi^*;t)$ *are uniformly bounded in n.*

PROOF. On account of the normalization of $N_{n-1,4}(\varphi^*;t)$ the Lebesgue constants are determined by

$$L_n(\varphi^*)\equiv\int_0^{\pi}|N_{n-1,4}(\varphi^*;t)|\,dt=\int_0^{\alpha_n(\varphi^*)}N_{n-1,4}(\varphi^*;t)\,dt-\int_{\alpha_n(\varphi^*)}^{\pi}N_{n-1,4}(\varphi^*;t)\,dt=$$
$$=2\int_0^{\alpha_n(\varphi^*)}N_{n-1,4}(\varphi^*;t)\,dt-\frac{\pi}{2}.$$

Since $N_{n-1,4}(\varphi^*;t)$ has positive absolute maximum in $t=0$ and

$$N_{n-1,4}(\varphi^*;0)=\frac{16\sin^6\frac{\pi}{n}\left(1-\cos\frac{\sqrt{5}\pi}{n}\right)\left(1+2\cos\frac{\pi}{n}\right)^2}{n\left(9\cos\frac{\pi}{n}+\cos\frac{3\pi}{n}-10\cos\frac{\sqrt{5}\pi}{n}\right)\left(1-\cos\frac{\pi}{n}\right)^2\left(1-\cos\frac{3\pi}{n}\right)^2}=$$
$$=\frac{40}{9\pi^2}n+o(n),\quad n\to\infty,$$

it follows that

$$\int_0^{\alpha_n(\varphi^*)}N_{n-1,4}(\varphi^*;t)\,dt<F\equiv N_{n-1,4}(\varphi^*;0)\cdot\alpha_n(\varphi^*)=O(1)$$

for all n sufficiently large.

LEMMA 8. *For* $\varrho_{k,n-1}^{(4)}(\varphi^*)$ *we have*

$$\lim_{n\to\infty} n^4\big(1-\varrho_{k,n-1}^{(4)}(\varphi^*)\big) = \frac{3}{8}\pi^4(k^4+k^2). \tag{26}$$

The proof follows again by means of a Taylor series expansion using (25).

LEMMA 9. *Let* $f\in X_{2\pi}$; *the singular integral of convolution type*

$$I_{n-1,4}(\varphi^*;f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-u)N_{n-1,4}(\varphi^*;u)\,du \tag{27}$$

belongs to the class S_2 *and*

$$\lim_{n\to\infty} \|I_{n-1,4}(\varphi^*;f;x)-f(x)\|_{X_{2\pi}} = 0. \tag{28}$$

PROOF. By construction of the respective kernel it is obvious that $I_{n-1,4}(\varphi^*;f;x)\in S_2$. From Lemma 7 it follows that the norm of the operator (27) is uniformly bounded; furthermore, $\lim_{n\to\infty}\varrho_{k,n-1}^{(4)}(\varphi^*)=1$ for all fixed $k=1,2,3,\dots$ by (26). Hence (28) is a consequence of the theorem of BANACH—STEINHAUS; cf. [4].

Note that the proof of (28) also follows immediately by the fact that the kernel $N_{n-1,4}(\varphi^*;t)$ is even an *approximate identity*, cf. [4], i.e.

$$\text{(i)}\quad \|N_{n-1,4}(\varphi^*;t)\|_{L^1_{2\pi}} \leqq M,$$

$$\text{(ii)}\quad \lim_{n\to\infty}\Big[\sup_{0<\delta\leqq t\leqq\pi} |N_{n-1,4}(\varphi^*;t)|\Big] = 0. \tag{29}$$

Condition (29) is easily verified since by (6), (8), (13) and (24)

$$\frac{\cos t-\cos\dfrac{\sqrt{5}\pi}{n}}{\lambda_{1,n-2}(\varphi^*)-\cos\dfrac{\sqrt{5}\pi}{n}} = O(1),\quad n\to\infty,$$

and since the positive factor kernel $P_{n-2}(\varphi^*;t)$ itself satisfies the respective conditions (i), (ii).

Finally, as a further consequence of Lemma 8 we may consider for our singular integral a general problem which was investigated for the first time by G. ALEXITS [1], cf. [2], and which was later on called the *saturation problem* of an approximation process. For the exact definition etc. see e.g. [3], [4, Ch. 12].

For this purpose we need the following notation

$$\mathsf{V}[\mathsf{X}_{2\pi};\psi(k)] = \begin{cases}\{f\in \mathsf{C}_{2\pi};\quad \psi(k)f^\wedge(k)=g^\wedge(k),\ g\in \mathsf{L}^\infty_{2\pi}\}\\ \{f\in \mathsf{L}^1_{2\pi};\quad \psi(k)f^\wedge(k)=\mu^\vee(k),\ \mu\in \mathsf{BV}_{2\pi}\}\\ \{f\in \mathsf{L}^p_{2\pi};\quad \psi(k)f^\wedge(k)=g^\wedge(k),\ g\in \mathsf{L}^p_{2\pi},\ 1<p<\infty\}\end{cases}$$

where $\mathsf{L}^\infty_{2\pi}$, $\mathsf{BV}_{2\pi}$ are the spaces of all 2π-periodic functions which are essentially bounded or of bounded variation, respectively; $f^\wedge(k)=(1/\pi)\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$ are the (complex) Fourier coefficients of f, whilst $\mu^\vee(k)$ are the respective Fourier—Stieltjes coefficients of μ; $\psi(k)$ is a function defined for $k=0, 1, 2, \ldots$ with $\psi(0)=0$, $\psi(k)\neq 0$ for $k\neq 0$.

THEOREM 2. *Let $f\in X_{2\pi}$; the singular integral (27) is saturated in $X_{2\pi}$ with order $O(n^{-4})$, $n\to\infty$, and f belongs to the Favard (saturation) class $F[X_{2\pi}; I_{n-1,4}]$ if and only if $f\in V[X_{2\pi}; \psi(k)]$ with $\psi(k) = \frac{3}{8}\pi^4(k^4+k^2)$.*

PROOF. Condition (26) gives the order of saturation as $O(n^{-4})$, $n\to\infty$, and furthermore it follows that (cf. [4]) $\|I_{n-1,4}(\varphi^*; f; x)-f(x)\|_{X_{2\pi}} = O(n^{-4})$ implies $f\in V[X_{2\pi}; \psi(k)]$.

To prove the direct part of the saturation theorem let us set

$$\Psi_n(k) = \begin{cases} 1, & k=0 \\ \dfrac{\varrho_{k,n-1}^{(4)}(\varphi^*)-1}{n^{-4}\psi(k)}, & k=1, 2, 3, \ldots \end{cases} \tag{30}$$

and $\Delta^2\Psi_n(k) = \Psi_n(k-1)-2\Psi_n(k)+\Psi_n(k+1)$; then a lengthy calculation shows that

$$\sum_{k=1}^{\infty} k\Delta^2\Psi_n(k) = O(1), \quad n\to\infty \tag{31}$$

giving the quasi-convexity of (30) uniformly in n which is sufficient for (30) to be a multiplier, cf. [3, p. 349]. But (26) and the additional condition (31) are then sufficient for $f\in V[X_{2\pi}; \psi(k)]$ to belong to $F[X_{2\pi}; I_{n-1,4}]$.

By the way, we may remark that as a consequence of (18) for the (positive) singular integral

$$I_{n-2}(\varphi^*; f; x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-u)P_{n-2}(\varphi^*; u)\,du, \quad f\in X_{2\pi},$$

all results in [6] concerning the saturation problem, the theorem of VORONOVSKAJA-type, and the measure of approximation are valid.

Though in our example for $m=1$ the generating function φ^* of the kernel of $\mathfrak{S}_2$ is very simple, the representation of the convergence factors becomes unwieldy, whereas the kernel itself has a relatively concrete closed form. In a further paper we shall investigate other conditions upon positive factor kernels (which are not necessarily generated by a suitable φ-function) such that they produce kernels for operators of class S_{2m}, $m\geqq 1$, which improve the approximation order.

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