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INDEX

TOMUS XIX

AHLBORN, T. J. and BHARGAVA, T. N., On topological spaces associated with digraphs 47
BAMBAH, R. P. and WOODS, A. C., On the minimal density of maximal packing of the plane by convex bodies 103
BAMBAH, R. P. and WOODS, A. C., On minimal density of plane coverings by circles 337
BHARGAVA, T. N. and AHLBORN, T. J., On topological spaces associated with digraphs 47
BOLLOBÁS, B., A problem of the theory of communication networks 75
CHATTERJEA, S. K., Operational results connected with some classical polynomials 53
CLAY, J. R., The group of left distributive multiplications on an Abelian group 221
EILHAUER, R., Zur Theorie der Halbkörper. I 23
ENNOLA, V., On the greatest common divisor of two values of a polynomial 329
ERDŐS, P., HAJNAL, A. and MILNER, E. C., On sets of almost disjoint subsets of a set 223
ERDŐS, P. and TURÁN, P., On some problems of a statistical group-theory. IV 413
FIALA, F., Standard-Ultraantifilter im Verband aller Kompenenten einer *l*-Gruppe 405
FREUD, G., Über starke Approximation mit Hilfe einer Klasse von Interpolationspolynomen .. 215
GHOSAL, A., Some remarks about the duality relation in queues 177
Гольдберг, А. А., О распределении значений целой функции по аргументам 205
GRILLET, P. A., Translations and congruences in lattices 161
GUY, R. K. and MILNER, E. C., Graphs defined by coverings of a set 7
HAJNAL, A., ERDŐS, P. and MILNER, E. C., On sets of almost disjoint subsets of a set 223
HALÁSZ, G., Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen 365
HARARY, F. and PALMER, E. M., On the point-group and line-group of a graph 263
KÁTAI, I., Some problems on the iteration of multiplicative number-theoretical functions 441
KEMPERMAN, J. H. B., Idempotent signed measures on a discrete group 135
KRAUSS, P. H., Representation of conditional probability measures on Boolean algebras 229
LEINDLER, L., On a problem of strong summability of Fourier series 87
LOVÁSZ, L., On chromatic number of finite set-systems 59
MILNER, E. C. and GUY, R. K., Graphs defined by coverings of a set 7
MILNER, E. C., ERDŐS, P. and HAJNAL,A ., On sets of almost disjoint subsets of a set 223
MONK, G. S., One-sided ideals in the endomorphism ring of an Abelian *p*-group 185
MURTY, U. S. R., On some extremal graphs 69
NEUMAN, F., Relation between the distribution of the zeros of the solutions of a 2nd order linear differential equation and the boundedness of these solutions 1
PALMER, E. M. and HARARY, F., On the point-group and line-group of a graph 263
PATTERSON, E. M., Modules for pseudo-rings 451
QUERRÉ, J., Plus grand groupe image homomorphe et isotone d'un monoïde ordonné: caractérisations des anneaux «clos» 143

ROTMAN, B., On the comparison of order types 311
RUZSA, I., Ein neues formales System der deontischen Logik 287
SREEHARI, M. S., A limit theorem for the maximum of cumulative sums..... 117
STEINFELD, O., Eine Charakterisierung der primitiven Ideale eines Ringes 233
STEINFELD, O., Primelemente und Primradikale in gewissen verbandsgeordneten algebraischen Strukturen 243
SZABADOS, J., Rational approximation in certain classes of functions 81
SZABADOS, J., Structural properties of continuous functions connected with the order of rational approximation. II 95
SZÁSZ, G., Halbgruppen, deren Elemente durch Primideale trennbar sind 201
TANDORI, K., Bemerkung zur starken Summation der Fourierreihen 271
TURÁN, P. and ERDŐS, P., On some problems of a statistical group-theory. IV 413
TURNER, R. E. L., An extremal problem 437
WIEGANDT, R., Radical and semi-simplicity in catagories 345
WOODS, A. C. and BAMBAH, R. P., On the minimal density of maximal packing of the plane by convex bodies 103
WOODS, A. C. and BAMBAH, R. P., On minimal density of plane coverings by circles..... 337

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RELATION BETWEEN THE DISTRIBUTION OF THE ZEROS OF THE SOLUTIONS OF A 2nd ORDER LINEAR DIFFERENTIAL EQUATION AND THE BOUNDEDNESS OF THESE SOLUTIONS

By
F. NEUMAN (Brno, Czechoslovakia)
(*Presented by* P. TURÁN)

1. Let us consider the differential equation

$$(q) \qquad y'' = q(t)y,$$

where $q(t)$ is a continuous function on $(-\infty, \infty)$. Denote by $C^n(i)$ the set of continuous functions defined on the interval i and having, on the latter, continuous derivatives up to and including the order n; $C^n = C^n((-\infty, \infty))$. By a solution of the differential equation (q) on the interval i we understand every function $y(t) \in C^2(i)$ satisfying, for every $t \in i$, the equation (q). The trivial solution of the equation (q) is generally excluded from our considerations. The function $f(t)$ complying, for every $t \in i$, with the relation $f(t+d) = -f(t)$ is called a half-periodic function with period d. If we speak about a solution or a half-periodic function without mentioning their domain of definition, then the latter is understood to be the entire interval $(-\infty, \infty)$. Likewise: when refering to the boundedness of a function without indicating the interval, we mean that the function is bounded on its entire domain of definition.

Let y_1 and y_2 be solutions of a periodic equation (q), $q(t+\pi) = q(t)$, such that $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 0$. Obviously, the Wronskian of the pair y_1, y_2 equals $W = y_1 y_2' - y_1' y_2 = -1$. Denote $A = y_1'(\pi) + y_2(\pi)$. Then, by Floquet's theory (see, e.g. [6] p. 95),

$$\lambda^2 - A\lambda + 1 = 0$$

is the characteristic (fundamental) equation of the differential equation (q). For $|A| > 2$, every non-trivial solution of the equation (q) is unbounded (see, e.g. [5]); for $|A| < 2$, every solution of the differential equation (q) is bounded, (see [2]), the general solution being, in this case, given in [7]. If $|A| = 2$, then the solutions of the differential equation (q) may be either bounded or not. To determine which of these cases will occur, we require a further condition, indicated in [8]. In the present paper we shall show, for the case $|A| = 2$, that, under an additional condition on a solution y of the differential equation (q) and the solution $\bar{y}$ of the differential equation $(\bar{q})$ with the same zeros, the solutions of the differential equations (q) and $(\bar{q})$ are already of the same character, i.e., all they are bounded either for both (q) and $(\bar{q})$ or for neither of them.

At the end of this paper, under the assumption of a certain distribution of zeros of solutions, the analogous statement even for oscillatory (not only periodic) differential equations (q) and $(\bar{q})$ will be proved.

2. If a differential equation (q) has a periodic (half-periodic) non-trivial solution $y(t)$ with period π then $q(t)$ is periodic with period π and, according to Floquet's theory, the corresponding $A = 2$ $(A = -2)$.

Since we shall, in the following, often refer to the results of paper [8], let some of them be introduced:

For $A=2$ $(A=-2)$, a periodic differential equation (q), $q(t+\pi)=q(t)$, has at least one periodic (half-periodic) non-trivial solution $y(t)$ with period π. Let $a_1<a_2<\ldots<a_n$ be all the zeros of the solution $y(t)$ on the interval $[0, \pi)$. Denote

$$r(t)=\begin{cases}\sum_{i=1}^{n}\dfrac{1}{y'^2(a_i)\sin^2(t-a_i)}, & \text{for } n\geqq 1\\ 0, & \text{for } n=0.\end{cases} \tag{1}$$

Then, if $y^*(t)$ is an arbitrary solution of the differential equation (q), independent of $y(t)$ and such that the Wronskian y, y^* is equal to 1, y^* is of the form

$$y^*(t)=p(t)+\frac{1}{\pi}\int_0^{\pi}[y^{-2}(\sigma)-r(\sigma)]\,d\sigma\cdot t\cdot y_1(t),$$

where $p(t)\in C^2$ is a periodic (half-periodic) function with period π. Thus, in particular, every solution of the differential equation (q) is bounded (on $(-\infty, \infty)$) just when

$$\int_0^{\pi}[y^{-2}(t)-r(t)]\,dt=0. \tag{2}$$

In the simplest case:

Let the differential equation (q) have one solution $y(t)$ which is half-periodic with period π, and has exactly one zero a in the interval $[0, \pi)$. Then

$$y=y'(a)e^{f(t)}\sin(t-a),$$

where $f(t)\in C^2, f(t+\pi)=f(t), f(a)=f'(a)=0$. Every solution y^* of the differential equation (q), independent of y, is bounded (and hence half-periodic with period π) exactly when

$$\left(\int_0^{\pi}[y^{-2}(t)-r(t)]\,dt=\right)\int_0^{\pi}\frac{e^{-2f(t)}-1}{\sin^2(t-a)}\,dt=0. \tag{3}$$

If (3) applies, then every non-trivial solution of the equation (q) has, in every interval $[b, b+\pi)$, exactly one zero.

Moreover, all the differential equations having their solutions half-periodic with period π and such that every solution has, on any interval $[b, b+\pi)$, just one zero, are given by the relation

$$y''=[f''(t)+f'^2(t)+2f'(t)\operatorname{cotg}(t-a)-1]y, \tag{4}$$

where $f(t)\in C^2, f(t+\pi)=f(t), f(a)=f'(a)=0$, ($a$ being an arbitrary or fixed number).

3. Let us now consider the differential equation $y''=q(t)y$, which will again be denoted by (q), but where $q(t)$ is a continuous function defined on $(-\infty, \infty)$ (not necessarily periodic) and such that the differential equation (q) is oscillatory, i.e., every solution of the equation (q) has an infinite number of zeros both to the left and to the right of an arbitrary number.

Let $\varphi(t)$ stand for the basic central dispersion of the 1st kind (further, briefly: dispersion) of the differential equation (q), i.e. the function defined as follows (see [3]): If t_0 is a zero of a non-trivial solution of (q) then $\varphi(t_0)$ is the first zero of the solution lying on the right of t_0. Let y_1 and y_2 stand for two independent solutions of the equations (q). A continuous function $\alpha(t)$ satisfying the relation

$$\operatorname{tg} \alpha(t) = \frac{y_1(t)}{y_2(t)}$$

wherever the latter has a meaning, is called the 1st phase of the equation (q). The general solution of the equation (q) then is

$$y = k_1 \frac{\sin [\alpha(t) + k_2]}{\sqrt{|\alpha'(t)|}}.$$

These and further relations may be found in [4]. As E. BARVÍNEK has shown, in [1], for $\varphi(t) = t + \pi$ (i.e., for every non-trivial solution to have the subsequent zeros at the same distances equal to π) it is necessary and sufficient that some (then, consequently, every) 1st phase of this equation be in the form $\alpha(t) = p(t) + \varepsilon \cdot t$, where $p(t) \in C^3$, $p(t+\pi) = p(t)$, $p'(t) \neq -\varepsilon$, $\varepsilon = 1$ or -1. With regard to the form of the general solution, every solution of the differential equation (q) with the dispersion $\varphi(t) = t + \pi$ must be a half-periodic function with period π. Every solution of this kind has just one zero in every interval $[b, b+\pi)$. Therefore, on the basis of the previous section, we state:

THEOREM 1. *The differential equation* (q) *has every solution with subsequent zeros at the same distances equal to* π *exactly when* (q) *has a half-periodic solution* $y(t)$, *with period* π *and with just one zero* $a \in [0, \pi)$, *for which* (3) *applies. All the differential equations of this kind are given in* (4), *where* $f(t)$ *complies with the given conditions.*

Thus we have again arrived at the result derived before in [9]. We may even state:

THEOREM 2. *The differential equation* (q) *has the dispersion* $\varphi(t) = t + \pi$ *exactly when there exists one half-periodic solution with period* π *and neighbouring zeros at the distances* π *and when all the solutions of this differential equation are bounded on* $(-\infty, 0)$ (*or on* $(0, \infty)$ *or* $(-\infty, \infty)$).

The differential equations (q) with the dispersion $\varphi(t) = t + \pi$ are therefore, among all the differential equations with a half-periodic solution with period π and zeros at equal distances π, characterized by that each of their solutions is bounded on $(-\infty, 0)$ (then, of course, even on $(-\infty, \infty)$).

This characteristic may be useful when studying the boundedness of the solutions of the equation (q) in case of $A = -2$ from the distribution of the zeros of these solutions.

But the boundedness of all the solutions may also be inferred from the distribution of the zeros of one solution and an additional condition on the latter:

THEOREM 3. *Let the differential equations* (q) *and* ($\bar{q}$) *have half-periodic solutions* y *and* $\bar{y}$ *whose zeros* $a + k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$) *are the same. Let, moreover, every*

solution of the equation (q) *be bounded. Then every solution of the differential equation* ($\bar{q}$) *is bounded exactly when*

$$\int_0^\pi \left[\frac{y'^2(a)}{y^2(t)} - \frac{\bar{y}'^2(a)}{\bar{y}^2(t)}\right] dt = 0. \tag{5}$$

PROOF. Every solution of the differential equation ($\bar{q}$) is bounded exactly when

$$\int_0^\pi \left[\frac{\bar{y}'^2(a)}{\bar{y}^2(t)} - \frac{1}{\sin^2(t-a)}\right] dt = 0, \tag{6}$$

see Sec. 2. Since, by the same section, one has:

$$\int_0^\pi \left[\frac{y'^2(a)}{y^2(t)} - \frac{1}{\sin^2(t-a)}\right] dt = 0,$$

(6) holds exactly when (5) applies.

Similar statements may be made even for a non-equidistant distribution of the zeros. The dispersions of such two differential equations need by far not coincide as it did in the preceding case, since there holds:

THEOREM 4. *Let the differential equations* (q) *and* ($\bar{q}$) *have periodic (half-periodic) solutions* y *and* $\bar{y}$ *with period* π *and the same zeros equal to* $a_i + k\pi$ ($a_i \in [0, \pi)$; $i = 1, 2, \ldots, n$; $n \geqq 0$; $k = 0, \pm 1, \pm 2, \ldots$). *Let, at all the zeros, the quotient* $y'(a_i)/\bar{y}'(a_i)$ $(= K)$ *be constant. Let every solution of the equation* (q) *be bounded. Then every solution of the differential equation* ($\bar{q}$) *is bounded exactly when*

$$\int_0^\pi \left[\frac{K^2}{y^2(t)} - \frac{1}{\bar{y}^2(t)}\right] dt = 0. \tag{7}$$

PROOF. Every solution of the differential equation (q) is, according to Sec. 2, bounded exactly when

$$\int_0^\pi \left[\frac{1}{\bar{y}^2(t)} - \bar{r}(t)\right] dt = 0 \tag{8}$$

where $\bar{r}(t)$ is defined by (1). Since, furthermore, $r(t) = \frac{1}{K^2}\,\bar{r}(t)$, (7) applies exactly when (8) holds.

REMARK. When the dispersions $\varphi(t)$ and $\bar{\varphi}(t)$ of the differential equations (q) and ($\bar{q}$) coincide, then, of course, the postulate of the constant quotient $y'(a_i)/\bar{y}'(a_i)$, indicated in the previous theorem, is satisfied for all the solutions y and $\bar{y}$ where the zeros coincide, since

$$\frac{y'^2(a_i)}{y'^2(a_{i+1})} = \varphi'(a_i) = \bar{\varphi}'(a_i) = \frac{\bar{y}'^2(a_i)}{\bar{y}'^2(a_{i+1})};$$

see [3], p. 210. And since $y'(a_i)\cdot y'(a_{i+1})<0$ and $\bar{y}'(a_i)\cdot\bar{y}'(a_{i+1})<0$, one always has

$$\frac{y'(a_i)}{\bar{y}'(a_i)}=\frac{y'(a_{i+1})}{\bar{y}'(a_{i+1})}\ (=K).$$

4. If $\alpha(t)$ stands for some 1st phase of the differential equation (q), then the general solution of this equation is of the form

$$y=k_1\sin[\alpha(t)+k_2]/\sqrt{|\alpha'(t)|}.$$

We state

LEMMA 1. *Every solution of the differential equation* (q) *is bounded* (*bounded on* $(0,\infty)$ *or* $(-\infty,0)$) *just when the function* $1/\sqrt{|\alpha'(t)|}$ *is bounded on the same interval.*

PROOF. a) If $1/\sqrt{|\alpha'(t)|}$ is bounded (always on the corresponding interval), then for fixed k_1 and k_2 even y is bounded here. b) If $1/\sqrt{|\alpha'(t)|}$ is unbounded on the interval $(0,\infty)$ $((-\infty,0))$, then there exists a simple sequence $\{t_j\}_{j=(-)1}^{(-)\infty}$ such that

$$\lim_{j\to(-)\infty}t_j=(-)\infty,\quad i\overset{<}{(>)}j\Rightarrow 1/\sqrt{|\alpha'(t_i)|}<1/\sqrt{|\alpha'(t_j)|}$$

and $\lim_{j\to(-)\infty}1/\sqrt{|\alpha'(t_j)|}=\infty$. Let the sequence $\{k_2^{(j)}\}_{j=(-)1}^{(-)\infty}$ be chosen so that $\alpha(t_j)+k_2^{(j)}=2m\pi$, m integer, $k_2^{(j)}\in[0,2\pi)$. Let $k_2^{(-)}$ be an accumulation point of this sequence and $\{k_2^{(j_i)}\}_{i=(-)1}^{(-)\infty}$ a subsequence of $\{k_2^{(j)}\}_{j=(-)1}^{(-)\infty}$ converging to $k_2^{(-)}$. Set

$$y^{(-)}=\sin[\alpha(t)+\pi/2+k_2^{(-)}]/\sqrt{|\alpha'(t)|}.$$

Choose an arbitrary $M>0$. Then there exists an i_1 such that for $i\overset{>}{(<)}i_1$ one has $1/\sqrt{|\alpha'(t_{j_i})|}>2M$. Let us, moreover, choose an i_2 such that for $i\overset{>}{(<)}i_2$

$$\sin[\alpha(t_{j_i})+\pi/2+k_2^{(-)}]=\cos[\alpha(t_{j_i})+k_2^{(-)}]>1/2$$

holds. Then for $i>i_0=\max\{i_1,i_2\}$ $(i<i_0=\min\{i_1,i_2\})$ one has

$$y(t_{j_i})=\sin[\alpha(t_{j_i})+\pi/2+k_2^{(-)}]/\sqrt{|\alpha'(t_{j_i})|}>M.$$

Consequently, there exists a solution $y^{(-)}$ of the differential equation (q) which is not bounded on the interval $(0,\infty)$ $((-\infty,0))$, qu. e. d.

If $\varphi(t)$ stands for the dispersion of the same differential equation, then

$$\alpha(\varphi(t))=\alpha(t)+\varepsilon\pi,\ \varepsilon=\operatorname{sign}\alpha',\quad \text{for all}\quad t.$$

Let us, further, denote $\varphi^{(1)}(t)=\varphi(t)$, $\varphi^{(n+1)}(t)=\varphi(\varphi^{(n)}(t))$ $(n=\pm1,\pm2,\ldots)$. Then $\alpha(\varphi^{(n)}(t))=\alpha(t)+\varepsilon n\pi$, so that

$$\sqrt{[\varphi^{(n)}(t)]'}/\sqrt{|\alpha'(t)|}=1/\sqrt{|\alpha'(\varphi^{(n)}(t))|}.$$

If $t \in [t_0, \varphi(t_0)]$ and $n = 0, \pm 1, \pm 2, \ldots$, then $\varphi^{(n)}(t)$ acquires all the values of the interval $(-\infty, \infty)$ in the manner that $\lim_{n \to \pm\infty} \varphi^{(n)}(t) = \pm\infty$. Hence:

LEMMA 2. *Every solution of an oscillatory differential equation* (q) *is bounded* (*on* $(0, \infty)$ *or* $(-\infty, 0)$) *exactly when the function* $[\varphi^{(n)}(t)]'$ *is bounded for* $t \in [t_0, \varphi(t_0)]$ *and all the integers* n ($n = 1, 2, \ldots$ *or* $n = -1, -2, \ldots$).

The direct consequence of the above result is the following

THEOREM 5. *Let the oscillatory differential equations* (q) *and* $(\bar{q})$ *have the same dispersion (i.e. the same distribution of the zeros of every solution). Then the statement: „Every solution of the differential equation is bounded (bounded on* $(0, \infty)$ *or bounded on* $(-\infty, 0)$*)" holds either for both the equations* (q) *and* $(\bar{q})$ *or for neither of them.*

The above considerations may be generalized, since, for the last statement there is by far no need for the dispersions of the differential equations (q) and $(\bar{q})$ to coincide. Let $\varphi(t)$ be the dispersion of the oscillatory differential equation (q). If $[\varphi^{(nk)}(t)]'$ is bounded for a fixed, non-zero integer k and for $t \in [t_0, \varphi^{(k)}(t_0)]$ $(k > 0)$ or for $t \in [\varphi^{(k)}(t_0), t_0]$ $(k < 0)$, and all the $n = 0, \pm 1, \pm 2, \ldots$ then $[\varphi^{(n)}(t)]'$ is bounded for $t \in [t_0, \varphi(t_0)]$ and all the $n = 0, \pm 1, \pm 2, \ldots$ and vice versa. Therefore:

THEOREM 6. *Let* $\varphi(t)$ *and* $\bar{\varphi}(t)$ *be dispersions of the oscillatory differential equations* (q) *and* $(\bar{q})$, *respectively. Let, for non-zero integers* k *and* $\bar{k}$ *as well as for all the* t *hold* $\varphi^{(k)}(t) = \bar{\varphi}^{(\bar{k})}(t)$. *Then the statement: "Every solution of the difierential equation is bounded" holds either for both* (q) *and* $(\bar{q})$ *or for neither of them.*

It is easily understood what an analogous statement will be like in case of restricting our consideration to boundedness on the interval $(0, \infty)$ or $(-\infty, 0)$ only. Let us also note that the numbers k and $\bar{k}$ need not be of the same sign.

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References

[1] E. BARVÍNEK, O rozložení nulových bodů řešení lineární diferenciální rovnice $y'' = Q(t)y$ a jejich derivací, *Acta Fac. Nat. Univ. Comenian V*, **(8—10)** Mat. (1961), pp. 465—474.

[2] R. BELLMAN, *Stability theory of differential equations* (New York—Toronto—London, 1953).

[3] O. BORŮVKA, On oscillatory integrals of linear differential equations of the second order, *Czechoslovak Math. J.*, **3 (78)** (1953), pp. 199—251. (Russian).

[4] O. BORŮVKA, Sur les transformations différentielles linéaires complètes du second ordre, *Ann. Mat. Pura ed Appl.*, **49** (1960), pp. 229—251.

[5] L. CESARI, *Asymptotic behavior and stability problems in ordinary differential equations* (Springer—Verlag, 1959).

[6] J. HORN, *Gewöhnliche Differentialgleichungen* (Berlin, 1937).

[7] F. NEUMAN, Bounded non-periodic solutions of second-order linear differential equations with periodic coeffcients, to appear in *Math. Nachrichten.*

[8] F. NEUMAN, Criterion of the periodicity of the solutions of a certain differential equation with a periodic coefficient, *Ann. Mat. Pura ed Appl.*, pp. 385—396.

[9] F. NEUMAN, Sur les équations différentielles linéaires oscillatoires du deuxième ordre avec la dispersion fondamentale $\varphi(t) = t + \pi$, *Buletinul Inst. Polit. Iaşi (Roumania)*, **X(XIV)** (1964), pp. 37—42.

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GRAPHS DEFINED BY COVERINGS OF A SET

By

R. K. GUY (Calgary, Canada) and E. C. MILNER (Reading, England)

(*Presented by* P. ERDŐS)

1. Introduction. Capital letters denote sets and small letters denote positive integers unless we state otherwise. The set theoretic difference of A and B is $A \setminus B$. The cardinal of A is $|A|$ and $[A]^l$ denotes the set of all subsets of A with l elements, i.e.

$$[A]^l = \{X: X \subset A; |X| = l\}.$$

Note that $[A]^l = \emptyset$ if $|A| < l$. We write $[p, q]$ to denote the set $\{i: p \leqq i \leqq q\}$. Also, instead of $[[p, q]]^l$ we omit one pair of brackets and simply write $[p, q]^l$.

We use the symbol

$$(m, n)^k \to p \tag{1}$$

to indicate that the truth of the following statement*: *Let* $A_1, \ldots, A_m$ *be any* m *sets with the property that the union of any* n *of them covers the union of them all*, i.e.

$$S = \bigcup_{i=1}^{m} A_i = \bigcup_{i \in N} A_i \quad \textit{whenever} \quad N \in [1, m]^n. \tag{2}$$

Then

$$S = \bigcup_{j=1}^{p} C_j, \quad \textit{where} \quad [C_j]^k \subset \bigcup_{i=1}^{m} [A_i]^k \qquad (1 \leqq j \leqq p). \tag{3}$$

If (1) is false we write $(m, n)^k \not\to p$. To avoid uninteresting pathological cases we only define the symbol (1) and its negation if $m \geqq n \geqq p$ and $k \geqq 2$ and we tacitly assume that the variables satisfy these conditions whenever we write these symbols. It is an immediate consequence of the definition of (1) that if $m_1 \geqq m$, $n_1 \leqq n$ and $p_1 \geqq p$, then (1) implies that $(m_1, n_1)^k \to p_1$.

The statement defining (1) can be interpreted in a graph-theoretic sense. A *k-graph* is an ordered pair $G_k = (S, E)$, where $E \subset [S]^k$. The elements of S and E are the *vertices* and *edges (k-edges)* of G_k, respectively. An ordinary graph is a 2-graph but we shall use the term *graph* when the context implies that we mean a k-graph. A *complete subgraph* of the k-graph $G_k = (S, E)$ is a set $T \subset S$ such that $[T]^k \subset E$. Thus (1) means that if S is the union of m sets $A_1, \ldots, A_m$ which satisfy (2), then S is the union of p complete subgraphs of the k-graph $\left(S, \bigcup_{i=1}^{m} [A_i]^k\right)$.

ERDŐS, HAJNAL and MILNER [2] proved that

$$(m, n)^k \to p \quad \textit{if} \quad m \geqq k(n-p)+p \tag{4}$$

* We have adopted here a slightly different notation from that used in [2] where this kind of relation was first studied. Our symbol (1) is written in [2] as $(m, n)^k \to p+1$.

and conjectured that this is best possible, i.e. that

(5) $$(?) \quad (kn-kp+p-1, n)^k \nrightarrow p.$$

They proved the following special cases of (5)

(6) $$(kn-k, n)^k \nrightarrow 1,$$

(7) $$(2n-3, n)^2 \nrightarrow 2,$$

(8) $$(p+3, p+2)^2 \nrightarrow p$$

and also that

(9) $$(3n-5, n)^3 \nrightarrow 2$$

holds for $n=3$, 4 and 5.

In this paper we show that the conjecture (5) is true in a number of other cases. We establish that (9) holds for all $n \geqq 3$ and also that

(10) $$(p+5, p+3)^2 \nrightarrow p,$$

(11) $$(p+5, p+2)^3 \nrightarrow p,$$

(12) $$(k+p-1, p+1)^k \nrightarrow p.$$

We also prove the indicated special cases of the following relations which are not included in any of the earlier results

(13) $$(p+7, p+4)^2 \nrightarrow p \qquad (p=3),$$

(14) $$(2n-4, n)^2 \nrightarrow 3 \qquad (n=8),$$

(15) $$(4n-7, n)^4 \nrightarrow 2 \qquad (n=4).$$

If we could prove (13) for $p \leqq 29$ then we could establish the relation for all p by an induction argument similar to our proof of (10). We have in fact proved (13) for $p=4$ but the details are too laborious to include here.

In establishing the above relations we prove a slightly stronger proposition since we always consider graphs of a particular kind. We write $S(m, n)$ for the set $[1, m]^{m-n+1}$ and $G_k(m, n)$ for the k-graph $(S(m, n), E_k(m, n))$, where

$$E_k(m, n) = \bigcup_{i=1}^{m} [A_i]^k$$

and

(16) $$A_i = \{X: i \in X \in S(m, n)\} \qquad (1 \leqq i \leqq m).$$

Since each element of $S(m, n)$ is a member of exactly $m-n+1$ of the sets A_i it follows that (2) holds. Therefore, in order to prove that $(m, n)^k \nrightarrow p$ for particular values of m, n, k and p it is enough to show that $S(m, n)$ is not the union of p complete subgraphs of $G_k(m, n)$. The complementary graph of $G_k(m, n)$ is $G_k^*(m, n) = (S(m, n), E_k^*(m, n))$ with k-edges

$$E_k^*(m, n) = [S(m, n)]^k \setminus E_k(m, n).$$

Thus, if $X_1, \ldots, X_k$ are distinct elements of $S(m, n)$, then $\{X_1, \ldots, X_k\}$ is an edge of $G_k^*(m, n)$ iff $X_1 \cap \ldots \cap X_k = \emptyset$. A k-graph is *r-colourable* if the vertices can be partitioned into r classes so that no class contains an edge of the graph; a graph is *r-chromatic* if it is r-colourable but not $(r-1)$-colourable. The assertion that $S(m, n)$ is or is not the union of p complete subgraphs of $G_k(m, n)$ is equivalent to saying that $G_k^*(m, n)$ is or is not p-colourable. Since $(kn-kp+p-1, n)^k \to p+1$ by (4), it follows that $G_k^*(kn-kp+p-1, n)$ is $(p+1)$-colourable. The conjecture that

(17) (?) $G_k^*(kn-kp+p-1, n)$ is $(p+1)$-*chromatic*

implies (5). We can show that (5) and (17) are in fact equivalent when $k=2$ but we do not know if this is the case for $k>2$. In the special case $k=2$ (17) is a conjecture of KNESER [4]. Our proofs of (9)—(16) confirm the corresponding cases of the conjecture (17).

2. Combinatorial theorems on systems of intersecting sets. Theorem 1 is a special case of a result proved by HILTON and MILNER [3].

THEOREM 1. *Let* $1 \leqq l \leqq \frac{1}{2}m$; $A_1, \ldots, A_t \subset [1, m]$; $|A_i| \geqq l$ $(1 \leqq i \leqq t)$; $A_i \cap A_j$ $(i \neq j)$; $A_i \cap A_j \neq \emptyset$ $(1 \leqq i < j \leqq t)$; $A_1 \cap \ldots \cap A_t = \emptyset$. *Then*

$$t \leqq 1 + \binom{m-1}{l-1} - \binom{m-l-1}{l-1}. \tag{18}$$

It is easy to see that (18) is best possible since there is equality if $A_1 = [1, l]$ and $A_2, \ldots, A_t$ are the distinct sets $A \in [1, m]^l$ such that $m \in A$ and $A \cap A_1 \neq \emptyset$.*

We will prove a similar type of result about systems of sets having at most four elements such that every triple have a non-empty intersection. We prove rather more than we need for our application of Theorem 2.

THEOREM 2. *Let* $m \geqq 5$; $A_i \subset [1, m]$, $|A_i| \leqq 4$ $(1 \leqq i \leqq t)$; $A_i \not\subset A_j$ $(i \neq j)$; $A_i \cap A_j \cap A_k \neq \emptyset$ $(1 \leqq i, j, k \leqq t)$; $A_1 \cap \ldots \cap A_t = \emptyset$. *Then*

$$t \leqq 4m - 15. \tag{19}$$

If $m > 5$ *and* $|A_i| < 4$ *for some* $i \in [1, t]$, *then there is strict inequality in* (19).

REMARKS. 1. We cannot prove the corresponding result if the condition $|A_i| \leqq 4$ is replaced by a more general condition $|A_i| \geqq l$. For a general conjecture in this direction see [3].

2. The inequality (19) is best possible since equality obtains if $A_1, \ldots, A_t$ are all the sets $A \in [1, m]^4$ such that $|A \cap \{1, 2, 3, 4\}| \geqq 3$.

3. The condition $m > 5$ is essential for the last part of the theorem since the five sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4, 5\}$ and $\{2, 3, 4, 5\}$ satisfy the conditions of the theorem when $m = 5$.

We first prove a simple lemma.

LEMMA. *If* $A_1, \ldots, A_t$ *are distinct sets such that* $|A_i| = 3$, $A_i \cap A_j \cap A_k \neq \emptyset$ $(1 \leqq i, j, k \leqq t)$ *and* $A_1 \cap \ldots \cap A_t = \emptyset$, *then* $t \leqq 4$.

* This example disproves a conjecture in [1].

PROOF. If $A_i \cap A_j$ is a singleton $\{x\}$ for some $i, j \in [1, t]$, then there is a $k \in [1, t]$ such that $x \notin A_k$ and we have the contradiction $A_i \cap A_j \cap A_k = \emptyset$. Therefore,

$$|A_i \cap A_j| \geqq 2 \qquad (1 \leqq i, j \leqq t). \tag{20}$$

We can assume that $A_1 = \{1, 2, 3\}$, $A_2 = \{1, 2, 4\}$. Since the intersection of all the sets is empty we can assume that $1 \notin A_3$ and $2 \notin A_4$ (note that $2 \in A_3$ by (20)). Thus $A_3 = \{2, 3, 4\}$ and $A_4 = \{1, 3, 4\}$. If $t > 4$, then $A_5 \subset [1, 4]$ and hence $|A_5 \cap A_i| < 2$ for some $i \in [1, 4]$. This contradicts (20) and proves the lemma.

The way we prove Theorem 2 closely resembles a method used by ERDŐS, CHAO KO and RADO in [1].

PROOF OF THEOREM 2. *Case* 1. $|A_i| = 4$ $(1 \leqq i \leqq t)$. If $m = 5$, then $t \leqq \binom{5}{4} = 4m - 15$. We now assume that $m > 5$ and use induction on m.

For fixed values of m and t we can assume that the set of sets $F = \{A_1, \ldots, A_t\}$ satisfies the additional condition that

$$\varphi(F) = \{i: 1 \leqq i \leqq t;\ m \in A_i\}$$

is minimal.

Suppose there are A and λ such that $m \in A \in F$, $\lambda \in [1, m] \setminus A$ and $(A \cup \{\lambda\}) \setminus \{m\} \notin F$. By relabelling the suffixes of the A_i, we can assume that

$$m \in A_i, \quad \lambda \notin A_i, \quad A_i' = (A_i \cup \{\lambda\}) \setminus \{m\} \notin F \qquad (1 \leqq i \leqq t_1),$$

$$m \in A_i, \quad \lambda \notin A_i, \quad (A_i \cup \{\lambda\}) \setminus \{m\} \in F \qquad (t_1 < i \leqq t_2),$$

$$m \in A_i, \ \lambda \in A_i \qquad (t_2 < i \leqq t_3),$$

$$m \notin A_i \qquad (t_3 < i \leqq t),$$

where $1 \leqq t_1 \leqq t_2 \leqq t_3 \leqq t$.

Put $A_i' = A_i$ for $i \in [t_1 + 1, t]$. Then

$$A_i' \neq A_j' \qquad (1 \leqq i < j \leqq t). \tag{21}$$

This is obvious if $i > t_1$ or if $j \leqq t_1$; if $i \leqq t_1 < j$, then (21) holds since $A_i' \notin F$ and $A_j' \in F$. We now show that

$$A_i' \cap A_j' \cap A_k' \neq \emptyset \qquad (1 \leqq i < j < k \leqq t). \tag{22}$$

If $i > t_1$, then $A_i' \cap A_j' \cap A_k' = A_i \cap A_j \cap A_k \neq \emptyset$. Therefore, we may assume that $1 \leqq i \leqq t_1$. If $t_1 < j \leqq t_2$, then there is a $\mu \in A_i \cap \big((A_j \cup \{\lambda\}) \setminus \{m\}\big) \cap A_k$ and $\mu \neq m$ or λ. In this case $\mu \in A_i' \cap A_j' \cap A_k' \neq \emptyset$. Therefore, we may assume that $j \notin [t_1 + 1, t_2]$ and similarly that $k \notin [t_1 + 1, t_2]$. Suppose $k > t_3$. Then there is a $\nu \in A_i \cap A_j \cap A_k$; $\nu \neq m$ or λ and hence $\nu \in A_i' \cap A_j' \cap A_k' \neq \emptyset$. Therefore, we may assume that $\{i, j, k\} \subset [1, t_1] \cup [t_2 + 1, t_3]$ and in this case $\lambda \in A_i' \cap A_j' \cap A_k' \neq \emptyset$. This proves (22).

It follows from (21) and (22) that the system of sets $F' = \{A_1', \ldots, A_t'\}$ also satisfies the hypothesis of the theorem. Since $\varphi(F') = \varphi(F) - t_1 < \varphi(F)$, this contradicts the minimal property of $\varphi(F)$. This proves

$$\textit{if} \quad m \in A \in F \quad \textit{and} \quad \lambda \in [1, m] \setminus A, \quad \textit{then} \quad (A \cup \{\lambda\}) \setminus \{m\} \in F. \tag{23}$$

We now assume that

$$m \in A_i \qquad (1 \leqq i \leqq t_0),$$
$$m \notin A_i \qquad (t_0 < i \leqq t),$$

where $0 \leqq t_0 \leqq t$. If $t_0 = 0$, then $A_i \subset [1, m-1]$ $(1 \leqq i \leqq t)$ and the induction hypothesis implies that $t \leqq 4(m-1) - 15$. Therefore, we can assume $t_0 \neq 0$. Also, $t_0 \neq t$ since $A_1 \cap \ldots \cap A_t = \emptyset$.

Suppose there are $i, j \in [1, t]$ such that $A_i \cap A_j$ is a singleton $\{\mu\}$. Then we obtain the contradiction that $\mu \in A_i \cap A_j \cap A_k \subset A_k$ $(1 \leqq k \leqq t)$. Therefore,

$$|A_i \cap A_j| \geqq 2 \qquad (1 \leqq i < j \leqq t). \tag{24}$$

Suppose that $|A_i \cap A_j| = 2$ for some pair $i, j \in [1, t_0]$. Then $A_i = \{x, y, z, m\}$ and $A_j = \{x, u, v, m\}$, where x, y, z, u, v, m are all distinct. It now follows from (23) that the sets

$$B_1 = \{x, y, z, u\}, \quad B_2 = \{x, y, z, v\}, \quad B_3 = \{x, u, v, y\} \quad \text{and} \quad B_4 = \{x, u, v, z\}$$

are all members of F. There is some $B \in F$ such that $x \notin B$ and since $A_i \cap A_j \cap B$, $A_i \cap B_\nu \cap B$ $(\nu = 3, 4)$ and $A_j \cap B_\nu \cap B$ $(\nu = 1, 2)$ are all non-empty, it follows that $\{m, y, z, u, v\} \subset B$. This is a contradiction since $|B| = 4$. Hence, we have that

$$|A_i \cap A_j| = 3 \qquad (1 \leqq i < j \leqq t_0). \tag{25}$$

We now show that

$$t_0 \leqq 4. \tag{26}$$

Put $C_i = A_i \setminus \{m\}$ $(1 \leqq i \leqq t_0)$. Then $|C_i| = 3$ and $|C_i \cap C_j| = 2$ $(1 \leqq i < j \leqq t_0)$ by (25). Therefore,

$$C_i \cap C_j \cap C_k \neq \emptyset \qquad (1 \leqq i, j, k \leqq t_0).$$

Put $C = C_1 \cap \ldots \cap C_{t_0}$. We consider four possible cases. (i) If $C = \emptyset$, then the lemma implies (26). (ii) If $C = \{x\}$, put $D_i = C_i \setminus \{x\}$ $(1 \leqq i \leqq t_0)$. Then $|D_i| = 2$, $D_i \cap D_j \neq \emptyset$ $(1 \leqq i, j \leqq t_0)$ and $D_1 \cap \ldots \cap D_{t_0} = \emptyset$. Therefore, by Theorem 1 (or it can be seen directly), $t_0 \leqq 3$. (iii) If $C = \{x, y\}_{\neq}$, then there are $p, q \in [t_0 + 1, t]$ such that $x \notin A_p$ and $y \notin A_q$. If $i \in [1, t_0]$, then $A_i = \{x, y, m, u\}$ and $u \in A_i \cap A_p \cap A_q \subset A_p \cap A_q$. Therefore, $t_0 \leqq |A_p \cap A_q| \leqq 4$. (iv) If $C = \{x, y, z\}_{\neq}$, then trivially $t_0 = 1$. This proves (26).

If $A_{t_0+1} \cap \ldots \cap A_t = \emptyset$, then the induction hypothesis implies that $t - t_0 \leqq$ $\leqq 4(m-1) - 15$ and (19) follows from (26). Therefore, we can assume that there is a μ such that $\mu \in A_i$ $(t_0 < i \leqq t)$. There is some $A \in \{A_1, \ldots, A_{t_0}\}$ such that $\mu \notin A$. Since $|A| = 4$, $m \in A$, $m \notin A_i$, $\mu \in A_i$, $|A \cap A_i| \geqq 2$ (by (24)) for $i \in [t_0 + 1, t]$, it follows that $t - t_0 \leqq 3(m-5) + 1$. Since $t_0 \leqq 4 \leqq m - 1$, it follows that $t \leqq 4m - 15$. This completes the proof of (19) in Case 1.

Case 2. $|A| < 4$ for some $A \in F$. If $A = \{x, y\}$ then there are $i, j \in [1, t]$ such that $x \notin A_i$, $y \notin A_j$ and we have the contradiction $A \cap A_i \cap A_j = \emptyset$. Therefore, we can assume that

$$|A_1| = |A_2| = \ldots = |A_u| = 3,$$
$$|A_{u+1}| = \ldots = |A_t| = 4,$$

where $1 \leqq u \leqq t$.

Let $B_1, \ldots, B_v$ be all the different sets $B \subset [1, m]$ such that $|B| = 4$ and $B \supset A_i$ for some $i \in [1, u]$. The inequality (24) holds in this case also. Therefore, if $1 \leqq i < j \leqq u$, then $|A_i \cup A_j| = 4$ and hence there is just one of the sets B which contains any given pair of the sets $A_1, \ldots, A_u$. Since every such A_i is contained in exactly $m-3$ B's, we have that

$$v = (m-3)u - \binom{u}{2}. \tag{27}$$

Since $B_1 \cap \ldots \cap B_v = A_1 \cap \ldots \cap A_u$, it follows that the system of sets $\{B_1, \ldots, B_v, A_{u+1}, \ldots, A_t\}$ also satisfies the hypothesis of the theorem. Therefore, by Case 1

$$v + (t-u) \leqq 4m - 15. \tag{28}$$

If $u \leqq 3$, then (27) implies that

$$v \geqq u \quad \text{if} \quad m = 5,$$

$$v > u \quad \text{if} \quad m > 5.$$

It now follows from (28) that $t \leqq 4m - 15$ and there is strict inequality if $m > 5$. Therefore, we can assume that $u > 3$.

By symmetry and (24) we can suppose that $A_1 = \{1, 2, 3\}$, $A_2 = \{1, 2, 4\}$ and $A_3 = \{1, 2, 5\}$ or $\{1, 3, 4\}$. Suppose $A_3 = \{1, 2, 5\}$. Then $A_4 = \{1, 2, 6\}$ by (24). There are i and j such that $1 \notin A_i$, $2 \notin A_j$. Since $A_i \cap A_j \cap A_k \neq \emptyset$ for $1 \leqq k \leqq 4$, it follows that $A_i = \{3, 4, 5, 6\}$ and we have $|A_i \cap A_1| < 2$ a contradiction against (24). Therefore, $A_3 = \{1, 3, 4\}$ and $A_4 = \{2, 3, 4\}$. Since $|A_i \cap A_j| \geqq 2$ $(1 \leqq i \leqq 4 < j \leqq t)$, it follows that $A_j = \{1, 2, 3, 4\}$ $(4 < j \leqq t)$. Therefore, $t \leqq 5 \leqq 4m - 15$ and there is strict inequality if $m > 5$. This completes the proof of Theorem 2.

The following simple result may be known.

THEOREM 3. *If $1 \leqq l \leqq t$ and $X_1, \ldots, X_t$ are sets such that $|X_i| = l - 1$ $(1 \leqq i \leqq t)$ and*

$$\bigcap_{i \in L} X_i \neq \emptyset \quad (L \in [1, t]^l),$$

then $\bigcap_{i=1}^{t} X_i \neq \emptyset$.

PROOF. We can assume that $X_1 = [2, l]$. If $\bigcap_{i=1}^{t} X_i = \emptyset$, then there are indices (not necessarily distinct) $i_j \in [1, t]$ such that $j \notin X_{i_j}$ $(1 < j \leqq l)$. Since $t \geqq l$, there is an $L \in [1, t]^l$ such that $\{1, i_2, \ldots, i_l\} \subset L$ and we have the contradiction $\bigcap_{i \in L} X_i = \emptyset$.

3. Proofs of (9)—(15). We prove (10), (11) and (12) by essentially the same method using respectively Theorems 1, 2 and 3 of the last section.

THEOREM 4. $(p+5, p+3)^2 \to p$.

PROOF. Put $k = 2$, $m = p + 5$ and $n = p + 3$. We want to show that

$$S(m, n) \text{ is not the union of } p \text{ complete subgraphs of } G_k(m, n). \tag{29}$$

In the case $p = 1$, (29) is obvious since $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are elements of $S(6, 4)$ not joined by an edge of $G_2(6, 4)$. Assume $p > 1$ and use induction on p.

Suppose $S(m, n) = C_1 \cup \ldots \cup C_p$, where each C_i $(1 \leqq i \leqq p)$ is a complete subgraph of $G_2(m, n)$. The elements of $S(m, n)$ are all triples from $[1, m]$ and we can assume that

$$|C_p| \geqq \frac{\binom{m}{3}}{p} > 3m - 8$$

since

$$\frac{1}{p}\binom{m}{3} - (3m-8) = \frac{1}{6p}(p^3 - 6p^2 + 5p + 60) > 0.$$

If $X, Y \in C_p \subset [1, m]^3$, then $X \cap Y \neq \emptyset$ since C_p is a complete subgraph of $G_2(m, n)$ Therefore, by Theorem 1 (with $l = 3$), we have that

$$\bigcap_{X \in C_p} X \neq \emptyset$$

and, by symmetry, we can assume that m belongs to each element of C_p. Then $S(m-1, n-1) = [1, m-1]^3 \subset C_1 \cup \ldots \cup C_{p-1}$. Since the restriction of the graph $G_2(m, n)$ to $S(m-1, n-1)$ is $G_2(m-1, n-1)$, it follows that $S(m-1, n-1)$ is the union of the $p-1$ complete subgraphs $C_j' = C_j \cap S(m-1, n-1)$ $(1 \leqq j < p)$ of $G_2(m-1, n-1)$. This contradicts the induction hypothesis and completes the proof of (29).

THEOREM 5. $(p+5, p+2)^3 \nrightarrow p$.

PROOF. We want to prove (29) in this case when $k = 3$, $m = p+5$ and $n = p+2$. In the case $p = 1$, $m = 6$, $n = 3$, (29) holds since $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$ and $\{3, 4, 5, 6\}$ are elements of $S(6, 3)$ which do not form a 3-edge of $G_3(6, 3)$. Assume $p > 1$ and use induction.

Suppose $S(m, n) = [1, m]^4 = C_1 \cup \ldots \cup C_p$ is the union of p complete subgraphs of $G_3(m, n)$. Every three quartets in C_p have a non-empty intersection and we can assume that

$$|C_p| \geqq \frac{\binom{m}{4}}{p} > 4m - 15.$$

It now follows from Theorem 2 that all the elements of C_p have a non-empty intersection. By symmetry we can assume that m belongs to each member of C_p. As in the proof of the last theorem this implies that $S(m-1, n-1)$ is the union of $p-1$ complete subgraphs of $G_3(m-1, n-1)$ contradicting the induction hypothesis.

Theorem 6 is the only instance of conjecture (5) that we have proved which involves two parameters.

THEOREM 6. $(k+p-1, p+1)^k \nrightarrow p$.

PROOF. In this case we have to prove (29) with $m = k+p-1$, $n = p+1$ and $k \geqq 2$. If $p = 1$, $S(m, n)$ is the set of k elements $[1, k]^{k-1}$ and these do not form a k-edge of $G_k(m, n)$ since they have an empty intersection. The remainder of the

proof follows in the same way as the proofs of the previous two theorems except that we use Theorem 3 (with $l=k$) and the observation that*

$$|C_p| \geqq \frac{\binom{k+p-1}{k-1}}{p} > k-1.$$

THEOREM 7. $(3n-5, n)^3 \nrightarrow 2$.

PROOF. The result has already been established in [2] in the case $n \leqq 5$. Suppose $n \geqq 5$.

We write $m=3n-5$. Suppose that $S(m, n)=[1, m]^{2n-4}=C_1 \cup C_2$, where C_i $(i=1, 2)$ is a complete subgraph of $G_3(m, n)$. We can assume that $C_1 \cap C_2=\emptyset$. Since $X=[1, 2n-4]$, $Y=[n, m]$ and $Z=[1, n-1] \cup [2n-3, 3n-7]$ are elements of $S(m, n)$ which do not form a 3-edge of $G_3(m, n)$, it follows that $C_i \neq \emptyset$ $(i=1, 2)$. We shall consider all the n elements of $S(m, n)$ which contain a particular set $P=\{\alpha_1, \ldots, \alpha_{n-4}, \beta_1, \ldots, \beta_{n-1}\} \in [1, m]^{2n-5}$ and distinguish two cases.

Case 1. At least two of the elements of $S(m, n)$ which contain P are in each of C_1, C_2. We write $[1, m] \setminus P=\{\gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \ldots, \varepsilon_{n-4}\}$. From now on, for brevity, we omit the braces and commas normally used to denote sets. Suppose that

$$\alpha_1 \ldots \alpha_{n-4} \beta_1 \ldots \beta_{n-1} \gamma_i \in C_1, \tag{30}$$

$$\alpha_1 \ldots \alpha_{n-4} \beta_1 \ldots \beta_{n-1} \delta_i \in C_2. \tag{31}$$

Here, and throughout the proof, *i, j etc.* are suffixes chosen arbitrarily from the relevant domains, $\alpha(1, 2, \ldots, n-4)$, $\beta(1, 2, \ldots, n-1)$, $\gamma(1, 2)$ and $\delta(1, 2)$. By symmetry we may suppose

$$\alpha_1 \ldots \alpha_{n-4} \gamma_1 \gamma_2 \delta_1 \delta_2 \varepsilon_1 \ldots \varepsilon_{n-4} \in C_1. \tag{32}$$

Since any three members of the same complete subgraph must have a common element, (30) and (32) imply

$$[\beta_1 \ldots \beta_{n-1} \gamma_j \delta_1 \delta_2 \varepsilon_1 \ldots \varepsilon_{n-4}]^{2n-4} \subset C_2. \tag{33}$$

(31) and (33) imply

$$[\alpha_1 \ldots \alpha_{n-4} \beta_1 \beta_2 \gamma_1 \gamma_2 \delta_j \varepsilon_1 \ldots \varepsilon_{n-4}]^{2n-4} \subset C_1. \tag{34}$$

Suppose that $1 \leqq q < n-3$ and that

$$[\alpha_q \ldots \alpha_{n-4} \beta_1 \ldots \beta_{q+1} \gamma_1 \gamma_2 \delta_j \varepsilon_1 \ldots \varepsilon_{n-4}]^{2n-4} \subset C_1. \tag{35}$$

(30) and (35) imply

$$[\alpha_1 \ldots \alpha_q \beta_{q+2} \ldots \beta_{n-1} \gamma_j \delta_1 \delta_2 \varepsilon_1 \ldots \varepsilon_{n-4}]^{2n-4} \subset C_2. \tag{36}$$

(31) and (36) imply

$$[\alpha_{q+1} \ldots \alpha_{n-4} \beta_1 \ldots \beta_{q+2} \gamma_1 \gamma_2 \delta_j \varepsilon_1 \ldots \varepsilon_{n-4}]^{2n-4} \subset C_1 \tag{37}$$

* If $r, s \geqq 1$, it follows by induction on $r+s$ that $\binom{r+s}{r} > rs$.

which is the same as (35) with q replaced by $q+1$. It follows by induction that (35) holds with $q=n-3$, *i.e.*

$$[\beta_1 \dots \beta_{n-2} \gamma_1 \gamma_2 \delta_j \varepsilon_1 \dots \varepsilon_{n-4}]^{2n-5} \subset C_1. \tag{38}$$

By (30), (32) and (38) we have that

$$\alpha_1 \dots \alpha_{n-4} \beta_1 \dots \beta_{n-1} \gamma_1,$$

$$\alpha_1 \dots \alpha_{n-4} \gamma_1 \gamma_2 \delta_1 \delta_2 \varepsilon_1 \dots \varepsilon_{n-4},$$

$$\beta_1 \dots \beta_{n-2} \gamma_2 \delta_2 \varepsilon_1 \dots \varepsilon_{n-4}$$

are elements of C_1 and this is a contradiction since these three sets have empty intersection.

Case 2. Only one of the elements of $S(m, n)$ which contain P belongs to C_2. We now write $[1, m] \setminus P = \{\gamma_1, \dots, \gamma_{n-1}, \delta\}$ and suppose

$$\alpha_1 \dots \alpha_{n-4} \beta_1 \dots \beta_{n-1} \gamma_i \in C_1 \qquad (1 \leqq i < n), \tag{39}$$

$$\alpha_1 \dots \alpha_{n-4} \beta_1 \dots \beta_{n-1} \delta \in C_2. \tag{40}$$

We first dispose of the hypothesis

$$\alpha_1 \dots \alpha_{n-4} \gamma_1 \dots \gamma_{n-1} \delta \in C_1. \tag{41}$$

(39) and (41) imply

$$[\beta_1 \dots \beta_{n-1} \gamma_2 \dots \gamma_{n-1} \delta]^{2n-4} \subset C_2. \tag{42}$$

(40) and (42) imply

$$[\alpha_1 \dots \alpha_{n-4} \beta_1 \beta_2 \gamma_1 \dots \gamma_{n-1}]^{2n-4} \subset C_1. \tag{43}$$

Let $1 \leqq q < n-3$ and suppose that

$$[\alpha_q \dots \alpha_{n-4} \beta_1 \dots \beta_{q+1} \gamma_1 \dots \gamma_{n-1}]^{2n-4} \subset C_1. \tag{44}$$

Using (39) and (40) successively, we deduce from (44) that

$$[\alpha_1 \dots \alpha_q \beta_{q+2} \dots \beta_{n-1} \gamma_2 \dots \gamma_{n-1} \delta]^{2n-4} \subset C_2,$$

and

$$[\alpha_{q+1} \dots \alpha_{n-4} \beta_1 \dots \beta_{q+2} \gamma_1 \dots \gamma_{n-1}]^{2n-4} \subset C_1.$$

It follows by induction that (44) holds with $q=n-3$, and hence

$$\beta_1 \dots \beta_{n-2} \gamma_1 \dots \gamma_{n-2} \in C_1.$$

This, together with (41) and (39) with $i=n-1$, contradicts the fact that C_1 is a complete subgraph of $G_3(m, n)$. So (41) is false and

$$\alpha_1 \dots \alpha_{n-4} \gamma_1 \dots \gamma_{n-1} \delta \in C_2. \tag{45}$$

(40) and (45) give

$$[\beta_1 \dots \beta_{n-1} \gamma_1 \dots \gamma_{n-1}]^{2n-4} \subset C_1. \tag{46}$$

(39) and (46) imply

$$[\alpha_1 \ldots \alpha_{n-4} \beta_1 \beta_2 \gamma_2 \ldots \gamma_{n-1} \delta]^{2n-4} \subset C_2. \tag{47}$$

Suppose that $1 \leqq q < n-3$ and

$$[\alpha_q \ldots \alpha_{n-4} \beta_1 \ldots \beta_{q+1} \gamma_2 \ldots \gamma_{n-1} \delta]^{2n-4} \subset C_2. \tag{48}$$

(40) and (48) imply

$$[\alpha_1 \ldots \alpha_q \beta_{q+2} \ldots \beta_{n-1} \gamma_1 \ldots \gamma_{n-1}]^{2n-4} \subset C_1. \tag{49}$$

(39) and (49) imply

$$[\alpha_{q+1} \ldots \alpha_{n-4} \beta_1 \ldots \beta_{q+2} \gamma_2 \ldots \gamma_{n-1} \delta]^{2n-4} \subset C_2.$$

It follows that (48) holds with $q = n-3$ and therefore

$$\beta_1 \ldots \beta_{n-2} \gamma_2 \ldots \gamma_{n-1} \in C_2.$$

which contradicts (40) and (45).

Since $C_i \neq \emptyset$ $(i = 1, 2)$, there is some set $P \in [1, m]^{2n-5}$ such that either Case 1 or Case 2 holds, but, as we have seen, these both lead to contradiction. This completes the proof.

Our proofs of (13), (14) and (15) use special arguments and for convenience in writing we use X, E, T for 10, 11 and 12, respectively.

THEOREM 8. $(10,7)^2 \nrightarrow 3$.

PROOF. Suppose that $S(10,7) = [1, 10]^4$ is covered by three complete subgraphs C_1, C_2, C_3 of $G_2(10,7)$. There are seven elements of $S(10,7)$ which contain $\{1, 2, 3\}$ and we may assume without loss that

$$1234,\ 1235,\ 1236 \in C_1. \tag{50}$$

We will first demonstrate the falsity of

$$149X \in C_1. \tag{51}$$

Since each pair of quartets in C_1 have a non-empty intersection, it follows from (50) and (51) that 2378, $459X$, $469X$ do not belong to C_1. Moreover, 2378 does not belong to the same complete subgraph as either $459X$ or $469X$. Without loss we may assume that $2378 \in C_2$ and that $459X$, $469X \in C_3$. The argument now follows Table 1,

Table 1:

1236	2378	(459X)
1235	2378	(469X)
149X	(2368)	459X
149X	(2357)	469X
149X	(2578)	469X
149X	(3678)	459X
1236	2368	(4579)
1235	2357	(468X)
(1259)	3678	468X
(136X)	2578	4579
1259	(368X)	4579
136X	368X	468X

where members of the three columns are in C_1, C_2, C_3 respectively. Members in parentheses belong to the subgraphs indicated by virtue of the already known membership of the other two entries in the same row. Finally, the entries in the last row show that 2579 is not in any C_i and therefore (51) is false.
Since $149X$ and 5678 belong to C_2, C_3 in some order we will assume $5678 \in C_2$ and $149X \in C_3$. In the argument so far 1 and 2 (or 1 and 3) may be interchanged. So $249X$ and $349X$ also belong to C_3. We next dispose of the hypothesis

$$4578 \in C_3 \tag{52}$$

by means of the tabulated argument in Table 2. The last row of this table gives the contradiction that 1258 is not in any C_i and disproves (52).

Table 2:

$(129X)$	5678	4578
$(139X)$	5678	4578
$(239X)$	5678	4578
$129X$	(3578)	$149X$
$139X$	(2678)	$149X$
$(269X)$	3578	4578
$269X$	2678	(1345)
1234	$(689X)$	1345
$269X$	$689X$	(1457)
$239X$	(1578)	$349X$
1234	$(679X)$	1345
$(369X)$	1578	1457
$369X$	$679X$	$349X$

Since (52) is false, we must have $4578 \in C_2$. Interchanging 5 and 6, we see that 4678 also belongs to C_2. The symmetrical role of 4, 5, 6 in (50) and

$$4578,\ 4678,\ 5678 \in C_2 \tag{53}$$

shows that $159X$, $169X$, $259X$, $269X$, $359X$, $369X$ all belong to C_3 in addition to $149X$, $249X$ and $349X$.

Finally, the hypotheses

$$1479 \in C_1 \quad \text{and} \quad 1479 \in C_2$$

are disposed in turn by the short arguments in tables 3 and 4 respectively. The last rows in these leave no place for 4579 and 1249 respectively. Therefore, $1479 \in C_3$.

Table 3:

1479	4578	$(236X)$
1479	(2368)	$149X$
1236	2368	$236X$

Table 4:

1234	1479	$(568X)$
(3568)	1479	$149X$
3568	5678	$568X$

However, the symmetry of the pairs (1,2), (4,5), (7,8), (9, X) then implies that $258X$ is also in C_3 and this is impossible.

THEOREM 9. $(12,8)^2 \nrightarrow 3$.

PROOF. Suppose that $S(12,8) = [1, 12]^5$ is the union of three complete subgraphs C_1, C_2, C_3 of $G_2(12,8)$.

Case 1. Suppose the elements of $S(12,8)$ which contain some fixed quartet are distributed over all three C_i, say $\{i, 4, 5, 6, 7\} \in C_i$ $(i = 1, 2, 3)$. Then $89XET$ cannot be in any C_i.

Case 2. Suppose the eight elements of $S(12,8)$ which contain some fixed quartet belong to at most two of the C_i and that there are at least two in each of them. One of the C_i contains at least four of these elements. *E.g.* suppose 12345, 12346, 12347, $12348 \in C_1$ and 12349, $1234X \in C_2$. Then all the members of the set*

$$[[5678]^3 \times [9X]^1 \times [ET]^2]^5$$

are in C_3 and members of $[[1234]^2 \times [5678]^2 \times [9X]^1]^5$ are not in C_3. Since 12569, $3478X$ are not in C_3 and not in the same C_i, we can assume $12569 \in C_1$, $3478X \in C_2$. If $135XE \in C_1$, then the last row of the argument displayed in table 5 excludes $137XE$ from any C_i.

Table 5:

$135XE$	$(2469T)$	$578XE$
12569	$2469T$	$(378XE)$
$(2569T)$	$3478X$	$378XE$
$2569T$	$2469T$	$5689T$

Therefore $135XE \in C_2$ (since $6789T \in C_3$). The situation is symmetrical in respect of the pairs (1, 2), (3, 4), (5, 6) and (E, T), so similar arguments show that

$$[[12]^1 \times [34]^1 \times [56]^1 \times [X]^1 \times [ET]^1]^5 \subset C_2.$$

Therefore,

$$[[12]^1 \times [34]^1 \times [56]^1 \times [9]^1 \times [ET]^1]^5 \subset C_1.$$

Thus $1359E \in C_1$, $135XE \in C_2$, and so that $2468T \in C_3$. But this is impossible by Case 1, as we have $2469T$, $246XT$, $2468T$ containing the same quartet $246T$ and in different C_i.

Case 3. Suppose the elements of $S(12,8)$ which contain a fixed quartet belong to two of the C_i and that one of the C_i contains just one of these elements. *E.g.* assume $1234j \in C_1$ $(5 \leqq j \leqq E)$ and $1234T \in C_2$. Then $[56789XE]^5 \subset C_3$. Consider all those elements of $S(12,8)$ which contain the quartet 5678. Three of these are in C_3 and therefore at least seven of them must be in C_3 by Case 2. Therefore at least three of $i5678$ $(i = 1, 2, 3, 4)$ are in C_3. Similarly, at least three of $i5679$ and three of $i5689$ $(i = 1, 2, 3, 4)$ are in C_3. By the symmetry of 1, 2, 3, 4 we can assume that 15678, 15679, $15689 \in C_3$. Since there are two elements of C_3 which contain 1567 and two which contain 1568, it follows, again by Case 2, that C_3

* We write $[[A]^a \times [B]^b \times [C]^c]^n$ to denote the set of all sets $X \in [A \cup B \cup C]^n$ such that $|X \cap A| \leqq a$, $|X \cap B| \leqq b$, $|X \cap C| \leqq c$.

contains at least seven elements of each type. Hence at least two of $1i567$ and two of $1i568$ ($i=2, 3, 4$) are in C_3. Therefore, we can assume that $12567, 12568 \in C_3$. Therefore C_3 contains at least seven members of $S(12,8)$ which contain 1256. Thus either $12356 \in C_3$ or $12456 \in C_3$ and this is incompatible with $789XE \in C_3$.

Case 4. All quintets which contain a particular quartet are in the same C_i. This implies that $S(12,8)$ is contained in the same C_i which is impossible since 12345, $6789X$ are not joined by an edge of $G_2(12,8)$.

THEOREM 10. $(9,4)^4 \nrightarrow 2$.

PROOF. Suppose $S(9,4)=[1,9]^6$ is the union of two complete subgraphs C_1, C_2 of $G_4(9,4)$. We consider the possible ways that the four elements of $S(9,4)$ which contain a particular quintet are distributed over C_1 and C_2.

Case 1. Each C_i contains two of the four elements, say $123456, 123457 \in C_1$ and $123458, 123459 \in C_2$. By symmetry, we can assume $126789 \in C_1$. Then $[3,9]^6 \in C_2$ and hence $136789, 246789 \in C_1$. This contradicts the fact that C_1 is a complete subgraph.

Case 2. One of the C_i contains only one of the four elements, say $12345i \in C_1$ ($i=6, 7, 8$) and $123459 \in C_2$. Consider the elements $1j6789$ ($j=2, 3, 4, 5$). By Case 1 these cannot be distributed two in each C_i. If C_1 contains three of these, then a simple argument shows that all the elements of C_1 must contain 1. Then $[2, 9]^6 \subset C_2$ and this is impossible since $A=[2,7]$, $B=[2,5] \cup [8,9]$, $C=[2,3] \cup [6,9]$ and $D=[4,9]$ do not form a 4-edge of the graph. Therefore C_2 contains at least three of the $1j6789$. Similarly, C_2 contains at least three of $2j6789$ ($j=1, 3, 4, 5$). It is now easy to see that every element of C_2 contains 9. This implies that $[1, 8]^6 \subset C_1$ and C_1 is not a complete subgraph.

Case 3. The elements of $S(9,4)$ which contain a particular quintet are all in the same C_i. Then $[1,9]^6$ is contained in this particular C_i which is clearly impossible.

4. Concluding remarks. In our proofs of (9)—(15) we showed that, for the relevant values of m, n, k and p,

(54) $\qquad S(m, n)$ *is not the union of* p *complete subgraphs of* $G_k(m, n)$.

As we have already remarked, (54) implies

(55) $$(m, n)^k \nrightarrow p.$$

To see that (55) does not imply (54) consider the following example. By Theorem 6 we have that $(6, 3)^5 \nrightarrow 2$ and this implies that $(5,3)^5 \nrightarrow 2$. On the other hand,

$$S(5, 3) = A \cup B,$$

where $A=\{X: 1 \in X \in [1, 5]^3\}$ and $B=\{X: 1 \notin X \in [1, 5]^3\}$; A is a complete subgraph of $G_5(5, 3)$ and so is B by virtue of the fact that $[B]^5=\emptyset$. It would be interesting to know if (54) and (55) are equivalent in the critical case when $m=k(n-p)+p-1$.

We say that the set $C \subset S(m, n)$ is a *trivial complete subgraph* of $G_k(m, n)$ if $0<|C|<k$ and $\bigcap_{X \in C} X=\emptyset$. For example, the set B described above is a trivial

complete subgraph of $G_5(5, 3)$. The next theorem shows that (54) and (55) are equivalent if we restrict ourselves only to *non-trivial* complete subgraphs.

THEOREM 11. *If $m \geqq n$ and $S(m, n)$ is the union of p non-trivial complete subgraphs of $G_k(m, n)$, then $(m, n)^k \to p$.*

REMARK. The example above shows that we cannot omit the qualification 'non-trivial' in the statement of the theorem. In the special case $k = 2$ every complete subgraph of $G_k(m, n)$ is non-trivial. Therefore (54) and (55) are equivalent in this case and the conjecture (5) is equivalent to (17) (i.e. KNESER's conjecture) when $k = 2$.

PROOF OF THEOREM 11. Suppose $(m, n)^k \not\to p$. Then there are sets $X_1, \ldots, X_m$ such that $X = X_1 \cup \ldots \cup X_m$ is covered by the union of any n of them and such that X is not the union of p complete subgraphs of the k-graph $G = \left(X, \bigcup_{i=1}^{m} [X_i]^k\right)$.

If $x \in X$, let $P(x) = \{i: 1 \leqq i \leqq m, x \in X_i\}$ and let $Q(x) = [1, m] \setminus P(x)$. Since $x \notin \bigcup_{i \in Q(x)} X_i$, it follows that $|Q(x)| < n$ and $|P(x)| \geqq m - n + 1$. The hypothesis of the theorem implies that

$$S(m, n) = C_1 \cup \ldots \cup C_p,$$

where each C_j is a non-trivial complete subgraph of $G_k(m, n)$. Put

$$Y_j = \{x \in X: P(x) \supset A \quad \textit{for some} \quad A \in C_j\} \qquad (1 \leqq j \leqq p).$$

Then

$$X = Y_1 \cup \ldots \cup Y_p$$

since $|P(x)| \geqq m - n + 1$ for all $x \in X$. Let $1 \leqq j \leqq p$ and suppose that $\{x_1, \ldots, x_k\} \subset Y_j$. There are sets $A_1, \ldots, A_k$ (not necessarily different) such that $P(x_i) \supset A_i \in C_j$ $(1 \leqq i \leqq k)$. Since C_j is a non-trivial complete subgraph of $G_k(m, n)$, it follows that there is some $e \in [1, m]$ such that $e \in A_i$ $(1 \leqq i \leqq k)$. Therefore, $x_i \in X_e$ $(1 \leqq i \leqq k)$ and so Y_j is a complete subgraph of G. This contradiction proves the theorem.

As a corollary of Theorem 11 we give another proof of the positive result (4) of ERDŐS, HAJNAL and MILNER.

COROLLARY. *If $m \geqq k(n - p) + p$, then $(m, n)^k \to p$.*

PROOF. By the theorem it is enough to show that $S(m, n)$ is the union of p non-trivial complete subgraphs of $G_k(m, n)$. Let the sets A_i be defined by (16) $(1 \leqq i \leqq m)$ and put $C = S(m, n) \setminus (A_1 \cup \ldots \cup A_{p-1})$. If $X \in C$, then $|[p, m] \setminus X| = n - p$. Therefore, if $1 \leqq r \leqq k$ and $X_1, \ldots, X_r \in C$, then

$$\left|[p, m] \setminus \bigcup_{j=1}^{r} ([p, m] \setminus X_j)\right| \geqq m - p + 1 - r(n - p) \geqq 1.$$

Therefore there is $e \in [p, m]$ such that $e \in X_j$ $(1 \leqq j \leqq r)$. This shows that C is a non-trivial complete subgraph of $G_k(m, n)$ and the result follows since each A_i is a non-trivial complete subgraph and $S(m, n) = A_1 \cup \ldots \cup A_{p-1} \cup C$.

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References

[1] P. ERDŐS, CHAO KO and R. RADO, Intersection theorems for systems of finite sets, *Oxford Q. J.*, (2), **12** (1961), pp. 313—360.
[2] P. ERDŐS, A. HAJNAL and E. C. MILNER, On the complete subgraphs of graphs defined by systems of sets, *Acta Math. Acad. Sci. Hung.*, **17** (1966), pp. 159—229.
[3] A. J. W. HILTON and E. C. MILNER, Some intersection theorems for systems of finite sets, to appear in *Oxford Q. J.* (1968).
[4] M. KNESER, Aufgabe 360, *Jahresberichte d. Deutschen Math. Vereinigung*, **58** (2) (1955).

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (1—2), (1968), pp. 23—45.

ZUR THEORIE DER HALBKÖRPER. I

Von
R. EILHAUER (Potsdam, DDR)
(*Vorgelegt von* L. RÉDEI)

Einleitung

Halbringe und insbesondere Halbkörper sind in den letzten Jahren mehrfach untersucht worden (vgl. etwa WEINERT [7]—[11], KOCH [2], STEINFELD [4], WIEGANDT [12] sowie die dort genannte Literatur). Entsprechende Begriffsbildungen und Ergebnisse haben auch schon Eingang in die Lehrbuchliteratur gefunden (vgl. z. B. RÉDEI [3]). Die vorliegende Arbeit ist weiteren Untersuchungen über Halbkörper gewidmet. Im Vordergrund stehen dabei Fragen der isomorphen Einbettung von Halbkörpern in der Körper K der reellen Zahlen bzw. in den Halbkörper H der positiven reellen Zahlen sowie des Zusammenhanges zwischen Halbkörpern und Ordnungen von Körpern. In den Begriffsbildungen und Bezeichnungen richten wir uns weitgehend nach WEINERT [7]—[10], wobei wir jedoch weniger Geläufiges (z. T. schon in der Einleitung) nochmals erläutern.

Im Sinne der Problemstellung betrachten wir nur Halbringe bzw. Halbkörper, bei denen sowohl die Addition als auch die Multiplikation kommutativ und regulär sind. Allerdings reichen diese Einschränkungen für die Einbettbarkeit eines Halbkörpers $\mathfrak{H}$ in einen Körper noch nicht aus. Der (stets existierende) Differenzenring $\mathfrak{R}=D(\mathfrak{H})$ von $\mathfrak{H}$ kann nämlich Nullteiler enthalten (für ein Beispiel vgl. WEINERT [9], Seite 187). Falls sich jedoch die multiplikative Regularität auf $D(\mathfrak{H})$ überträgt (ein notwendiges und hinreichendes Kriterium findet man bei WEINERT [8], § 6, Zusatz 3), existiert auch der Quotientenkörper $\mathfrak{K}=Q(\mathfrak{R})$ von $\mathfrak{R}=D(\mathfrak{H})$. Er ist dann der bis auf Isomorphie eindeutig bestimmte kleinste Körper, der $\mathfrak{H}$ enthält. Der Kürze halber werden wir $\mathfrak{K}$ *den von* $\mathfrak{H}$ *erzeugten Körper* bzw. $\mathfrak{H}$ *einen zu* $\mathfrak{K}$ *gehörigen Halbkörper* nennen. Natürlich kann dabei schon $\mathfrak{K}=D(\mathfrak{H})$ gelten, so daß wir dann vom Differenzenkörper von $\mathfrak{H}$ sprechen können.

Weiterhin hat ein echter Halbkörper $\mathfrak{H}$, der in einen Körper einbettbar ist, stets die Charakteristik 0.[1] Wir können also $\mathfrak{H}$ bzw. $\mathfrak{K}$ immer als Oberstrukturen des Halbkörpers H_0 der positiven rationalen Zahlen bzw. des Körpers K_0 der rationalen Zahlen annehmen.

Jeder in einem Körper $\mathfrak{K}$ gelegene echte Halbkörper tritt sozusagen in zwei Exemplaren auf, nämlich einmal mit dem Nullelement 0 von $\mathfrak{K}$ und einmal ohne dieses Element. Wir werden daher (insbesondere bei Aufzählungen) jeweils nur die echten Halbkörper ohne 0 angeben.

Bei der Untersuchung von Halbringen spielt die folgende Eigenschaft gewisser Halbringe $\mathfrak{R}$ oft eine wichtige Rolle, die wir wie in WEINERT [7], § 1, mit $(*)$ bezeichnen wollen:

$(*)$ Zu je zwei Elementen $a\neq b$ aus $\mathfrak{R}$ ist wenigstens eine der Gleichungen $a+x_1=b$, $b+x_2=a$ in $\mathfrak{R}$ lösbar.

[1] Vgl. WEINERT [7], § 2.

Für echte Halbkörper mit der Eigenschaft (✳) gilt dann sogar, daß jeweils genau eine der angegebenen Gleichungen lösbar ist. Bekanntlich sind gerade diese Halbkörper $\mathfrak{H}$ die Positivbereiche der Ordnungen ihrer Differenzenkörper $D(\mathfrak{H})=\mathfrak{K}$. Die entsprechende Ordnung des Körpers $\mathfrak{K}$ wollen wir mit $\mathfrak{O}_{\mathfrak{H}}$ bezeichnen und $\mathfrak{K}$ dann *mit* $\mathfrak{O}_{\mathfrak{H}}$ (oder *durch* $\mathfrak{H}$) *geordnet* nennen. Im übrigen verwenden wir im Hinblick auf Ordnungen und Halbordnungen Begriffsbildungen und Ergebnisse aus FUCHS [1], wobei wir uns aber der irreflexiven Schreibweise bedienen.

An dieser Stelle wollen wir noch den in der Arbeit verwendeten Begriff eines *archimedischen Positivbereiches* definieren. Wir verstehen darunter einen echten Halbkörper $\mathfrak{h}$ mit der Eigenschaft (✳), der für den Differenzenkörper $D(\mathfrak{h})$ eine archimedische Ordnung definiert.

Schließlich wird der Begriff der *Halbkörpererweiterung* mehrfach gebraucht. Dabei verstehen wir unter einer *Erweiterung* $\mathfrak{h}(\alpha)$ bzw. $\mathfrak{h}(\mathfrak{A})$ *eines Halbkörpers* $\mathfrak{h}$ *mit einem Element* α bzw. *mit einer Menge* $\mathfrak{A}$, wobei α bzw. $\mathfrak{A}$ zusammen mit $\mathfrak{h}$ in einem gemeinsamen Oberkörper[2] liegen sollen, den Durchschnitt aller Unterhalbkörper des betrachten Oberkörpers, die $\mathfrak{h}$ und α bzw. $\mathfrak{A}$ enthalten. Man überlegt sich leicht, wie sich die Elemente einer solchen Halbkörpererweiterung darstellen lassen; z. B. haben die Elemente von $\mathfrak{h}(\alpha)$ folgende Gestalt:

$$\frac{\sum_{i=0}^{n} a_i \alpha^i}{\sum_{j=0}^{m} b_j \alpha^j} \quad (a_i, b_j \in \mathfrak{h} \cup \{0\}),$$

wobei nicht alle a_i und nicht alle b_j gleich 0 sind. Ein Element α nennen wir *algebraisch über* $\mathfrak{h}$, wenn α Nullstelle eines Polynoms $f(x) \neq 0$ von $D(\mathfrak{h})[x]$ ist. Entsprechend heißt ein Halbkörper $\mathfrak{H}$ mit $\mathfrak{h} \subseteq \mathfrak{H}$ eine *algebraische Erweiterung* von $\mathfrak{h}$, wenn alle Elemente von $\mathfrak{H}$ algebraisch über $\mathfrak{h}$ sind.

In der vorliegenden Arbeit werden wir im wesentlichen die folgenden Ergebnisse erhalten. Zunächst geben wir in § 1 einige allgemeine, vorbereitende Sätze, wie z. B.: Wenn der Differenzenring eines Halbkörpers $\mathfrak{h}$ ein Körper ist, so gilt dies auch für jeden algebraischen Erweiterungshalbkörper von $\mathfrak{h}$ (Satz 1). Jeder zu einer einfachen algebraischen Körpererweiterung $\mathfrak{K}=\mathfrak{k}(\alpha)$ gehörige Halbkörper enthält ein primitives Element von $\mathfrak{K}/\mathfrak{k}$ (Satz 2).

Sodann zeigen wir (vgl. Satz 4), daß für einen algebraischen Erweiterungskörper $\mathfrak{K}$ eines reellen Körpers $\mathfrak{k}$ ein eineindeutiger Zusammenhang zwischen den bzgl. $\mathfrak{k}$ relativen Isomorphismen von $\mathfrak{K}$ in K und den Ordnungen von $\mathfrak{K}$ besteht, welche die übliche Ordnung von $\mathfrak{k}$ fortsetzen. Man kann daher gewisse Aussagen über Halbkörper wahlweise mit solchen Isomorphismen oder mit Ordnungen formulieren, wovon wir je nach Bedarf Gebrauch machen werden.

Zusammenhänge zwischen Halbkörpern und Ordnungen von Körpern werden auch in den weiteren Sätzen von § 1 deutlich. Es zeigt sich nämlich u. a.: Ob eine einfache algebraische Halbkörpererweiterung $\mathfrak{h}(\alpha)$ eines archimedischen Positivbereiches $\mathfrak{h}$ ein Körper ist oder nicht, hängt im wesentlichen von dem Verhalten des Elementes α bei gewissen Ordnungen des Körpers $D(\mathfrak{h}(\alpha))$ ab (vgl. Satz 6). In analoger Weise entscheidet die Existenz gewisser Ordnungen über das Vorhanden-

[2] Für einen allgemeineren Ansatzpunkt der betrachteten Begriffsbildung vgl. WEINERT [9], § 9.

sein bestimmter Halbkörper in einer einfachen algebraischen Erweiterung $\mathfrak{K}=\mathfrak{k}(\alpha)$ eines Körpers $\mathfrak{k}$ (vgl. Satz 7).

In § 2 beschäftigen wir uns mit der isomorphen Einbettbarkeit von Halbkörpern in den Halbkörper H der positiven reellen Zahlen. Für echte Halbkörper $\mathfrak{H}$, deren Elemente über einem reellen Körper $\mathfrak{k}$ algebraisch sind, gibt WEINERT [9] (§ 9) ein notwendiges und hinreichendes Kriterium für $\mathfrak{H} \rightarrowtail \overline{\mathfrak{H}} \subseteq H$ an, nämlich die Existenz eines echten Oberhalbkörpers $\mathfrak{H}^*$ von $\mathfrak{H}$ mit $(*)$ und $\mathfrak{H}^* \cap \mathfrak{k} \subseteq H$. Wir werden u. a. zeigen, daß für den absolut algebraischen Fall ein solcher Halbkörper $\mathfrak{H}^*$ stets existiert. Es gilt nämlich sogar (Satz 10): Jeder algebraische Erweiterungshalbkörper eines archimedischen Positivbereiches ist isomorph in H einbettbar.

Dem Problem der isomorphen Einbettbarkeit von Halbkörpern in den Körper K der reellen Zahlen wenden wir uns in § 3 zu. Zunächst definieren wir analog zum Körperfall den *Transzendenzgrad* $t(\mathfrak{H})$ eines Halbkörpers $\mathfrak{H}$ über dem Halbkörper H_0 der positiven rationalen Zahlen. Die Bedingung, daß $t(\mathfrak{H})$ die Mächtigkeit $\mathfrak{c}$ des Kontinuums nicht übersteigt, ist notwendig und hinreichend für die isomorphe Einbettbarkeit in den Körper $K(i)$ der komplexen Zahlen (Satz 11), reicht aber für die Einbettbarkeit in K, selbst im Falle eines echten Halbkörpers, nicht aus (vgl. Satz 12). Mit den Sätzen 13 und 14 geben wir sodann zwei hinreichende Kriterien für die isomorphe Einbettbarkeit in K an.

Das Ziel von § 4 ist die Gewinnung einer Übersicht über alle algebraischen Halbkörpererweiterungen eines archimedischen Positivbereiches. Hinsichtlich dieser Fragestellung findet man Teilergebnisse bereits in WEINERT [11] und in KOCH [2]. Es wird sich herausstellen (vgl. Satz 18), daß die echten Halbkörper $\mathfrak{H}$, die algebraische Erweiterungen eines archimedischen Positivbereiches $\mathfrak{h}$ sind, mit den Durchschnitten von Positivbereichen übereinstimmen, die zu Ordnungen algebraischer Erweiterungskörper $\mathfrak{K}$ von $\mathfrak{k}=D(\mathfrak{h})$ gehören, welche die Ordnung $\mathfrak{O}_\mathfrak{h}$ von $\mathfrak{k}$ fortsetzen. Darüber hinaus ist ein solcher Halbkörper $\mathfrak{H}$ genau dann eine einfache Erweiterung von $\mathfrak{h}$, wenn der Differenzenkörper $D(\mathfrak{H})$ von $\mathfrak{H}$ eine einfache Erweiterung von $\mathfrak{k}=D(\mathfrak{h})$ ist (vgl. Satz 16 bzw. die anschließende Folgerung).

In § 5 stellen wir abschließend die Frage nach dem Zusammenhang der formal-reellen Körper mit den durch echte Halbkörper erzeugbaren Körpern. Sie wird nahegelegt durch die Ergebnisse von § 2, nach denen im absolut algebraischen Fall die genannten Charakterisierungen von Körpern gleichwertig sind. Im allgemeinen Fall ist jedoch zunächst nur bekannt, daß jeder formalreelle Körper von einem echten Halbkörper erzeugt werden kann. Das Problem der Umkehrbarkeit dieser Aussage führen wir auf eine einfachere Fragestellung zurück, nämlich auf die Untersuchung einfacher algebraischer Erweiterungen von Körpern, die über K_0 rein transzendent sind und einen endlichen Transzendenzgrad besitzen. Hinsichtlich solcher Körper geben wir auch ein notwendiges und hinreichendes Kriterium für die Erzeugbarkeit durch echte Halbkörper an.

In einer Fortsetzung dieser Arbeit (Zur Theorie der Halbkörper. II) werden wir Halbkörper in einfachen transzendenten Körpererweiterungen untersuchen.

An dieser Stelle möchte ich Gelegenheit nehmen, Herrn Prof. Dr. WEINERT für die vielfältigen Anregungen und die mir bei der Anfertigung dieser Arbeit gewährte Unterstützung recht herzlich zu danken.

§ 1 Einige vorbereitende Sätze

Wie in der Einleitung erwähnt, betrachten wir in dieser Arbeit nur Halbkörper, die sich in einen Körper einbetten lassen. Der kleinste Oberkörper $\mathfrak{K}$ eines echten Halbkörpers $\mathfrak{H}$ ist dann bis auf Isomorphie eindeutig bestimmt, nämlich als Quotientenkörper $Q(\mathfrak{R})$ des Differenzenringes $\mathfrak{R}=D(\mathfrak{H})$. Dabei ist der zweite Schritt natürlich überflüssig, falls der Differenzenring $\mathfrak{R}=D(\mathfrak{H})$ des Halbkörpers $\mathfrak{H}$ selbst schon ein Körper ist. Diese Eigenschaft eines Halbkörpers, die z. B. trivialerweise aus $(*)$ folgt,[3] überträgt sich nun auf jeden algebraischen Erweiterungshalbkörper:

SATZ 1. *Es sei* $\mathfrak{h}$ *ein Halbkörper, für den der Differenzenring* $D(\mathfrak{h})$ *ein Körper ist, und* $\mathfrak{H}$ *ein algebraischer Erweiterungshalbkörper von* $\mathfrak{h}$. *Dann ist* $D(\mathfrak{H})$ *ebenfalls ein Körper.*

BEWEIS. Es sei $\alpha\neq 0$ ein Element des Ringes $\mathfrak{R}=D(\mathfrak{H})$. Wir zeigen $\alpha^{-1}\in\mathfrak{R}$. Wegen $\mathfrak{R}=D(\mathfrak{H})$ gilt $\alpha=h_1-h_2$ mit Elementen $h_1, h_2\in\mathfrak{H}$. Der Differenzenring $\mathfrak{R}'=D(\mathfrak{H}')$ des Halbkörpers $\mathfrak{H}'=\mathfrak{h}(h_1, h_2)\subseteq\mathfrak{H}$ enthält dann alle Elemente des Körpers $\mathfrak{k}=D(\mathfrak{h})$ und h_1, h_2. Da die Elemente des Körpers $\mathfrak{k}(h_1, h_2)$ durch ganzrationale Operationen mit h_1, h_2 und Elementen aus $\mathfrak{k}$ entstehen, gilt $\mathfrak{k}(h_1, h_2)\subseteq\subseteq\mathfrak{R}'\subseteq\mathfrak{R}$. Wegen $\alpha^{-1}\in\mathfrak{k}(h_1, h_2)$ ist somit $\alpha^{-1}\in\mathfrak{R}$.

Wir werden später mehrfach einen Satz über die primitiven Elemente einer einfachen algebraischen Körpererweiterung verwenden, den wir an dieser Stelle beweisen wollen:

SATZ 2. *Jeder zu einem einfachen algebraischen Erweiterungskörper* $\mathfrak{K}=\mathfrak{k}(\alpha)$ *eines Körpers* $\mathfrak{k}$ *gehörige Halbkörper* $\mathfrak{H}$ *enthält ein primitives Element von* $\mathfrak{K}/\mathfrak{k}$.

BEWEIS. Wegen $\mathfrak{K}=Q(D(\mathfrak{H}))$ gilt $\alpha=(h_1-h_2)(h_3-h_4)^{-1}$ mit Elementen $h_i\in\mathfrak{H}$ und somit $\mathfrak{K}=\mathfrak{k}(h_1, h_2, h_3, h_4)$. Die Elemente $\beta=c_1h_1+\ldots+c_4h_4$ mit $c_i\in\mathfrak{k}$ sind dann, mit Ausnahme von endlich vielen, primitive Elemente von $\mathfrak{K}/\mathfrak{k}$. Wenn $\mathfrak{H}=\mathfrak{K}$ gilt, ist $\beta\in\mathfrak{H}$ trivialerweise erfüllt. Für einen echten Halbkörper $\mathfrak{H}$ enthält der Halbkörper $\mathfrak{h}=\mathfrak{H}\cap\mathfrak{k}$ wenigstens H_0,[4] also jedenfalls unendlich viele Elemente. Damit sind die Elemente c_i aus $\mathfrak{h}$ wählbar, so daß β in $\mathfrak{H}$ liegt.

BEMERKUNG. Wenn $\mathfrak{K}=\mathfrak{k}(\alpha)$ eine einfache transzendente Erweiterung von $\mathfrak{k}$ ist, so gilt dieser Satz nicht mehr, worauf wir in der erwähnten Fortsetzung dieser Arbeit zurückkommen werden.

Als nächstes stellen wir fest, daß jeder echte Halbkörper mit der Eigenschaft $(*)$ strukturell in H höchstens einmal auftritt:

SATZ 3. *Für zwei in* H *gelegene Halbkörper* $\mathfrak{h}_1, \mathfrak{h}_2$ *mit der Eigenschaft* $(*)$ *ist jeder Isomorphismus von* $\mathfrak{h}_1$ *auf* $\mathfrak{h}_2$ *notwendig die Identität.*

BEWEIS. Es sei φ ein Isomorphismus von $\mathfrak{h}_1$ auf $\mathfrak{h}_2$. Das Bild $\varphi(\alpha)=\bar{\alpha}$ von α ist dann Element von $\mathfrak{h}_2$ für jedes $\alpha\in\mathfrak{h}_1$. Wir nehmen nun an, daß ein $\alpha\in\mathfrak{h}_1$ existiert mit $\alpha\neq\bar{\alpha}$. O. B. d. A. sei $\alpha>\bar{\alpha}$. Es existiert dann ein Element $a\in H_0$ mit $\alpha>a>\bar{\alpha}$. Wegen $a, \alpha\in\mathfrak{h}_1$; $a, \bar{\alpha}\in\mathfrak{h}_2$ und der vorausgesetzten Eigenschaft $(*)$ gilt $\alpha-a\in\mathfrak{h}_1$,

[3] Für ein hinreichendes und notwendiges Kriterium vgl. WEINERT [8]; § 6, Zusatz 4.

[4] Wir bezeichnen stets mit H_0 bzw. H den Halbkörper der positiven rationalen bzw. positiven reellen Zahlen und mit K_0 bzw. K den Körper der rationalen bzw. reellen Zahlen sowie mit $K(i)$ den Körper der komplexen Zahlen.

$a-\bar{\alpha}\in\mathfrak{h}_2$. Nach der anfangs gemachten Feststellung gilt auch $\varphi(\alpha-a)=$ $=\varphi(\alpha)-a\in\mathfrak{h}_2$, also $\bar{\alpha}-a\in\mathfrak{h}_2$. Die Elemente $a-\bar{\alpha}$ und $-(a-\bar{\alpha})$ können aber nicht beide in H liegen. Somit ist die Abbildung φ die Identität.

FOLGERUNG. *Ein echter Halbkörper* $\mathfrak{h}$ *mit der Eigenschaft* $(*)$, *der einen nichtidentischen Automorphismus besitzt, ist nicht in* H *isomorph einbettbar, definiert also als Positivbereich des Körpers* $\mathfrak{k}=D(\mathfrak{h})$ *eine nichtarchimedische Ordnung.*

BEWEIS. Nach Satz 3 hat jeder Unterhalbkörper von H mit der Eigenschaft $(*)$ nur den identischen Automorphismus, woraus schon die erste Behauptung folgt. Die zweite ergibt sich daraus, daß zu jedem archimedisch geordneten Körper $\mathfrak{k}$ ein ähnlicher Isomorphismus von $\mathfrak{k}$ in K existiert,[5] der $\mathfrak{h}$ als archimedischen Positivbereich von $\mathfrak{k}$ isomorph in H einbetten würde.

Weiter folgt aus Satz 3 unmittelbar, daß zwei in K enthaltene verschiedene Körper $\mathfrak{k}_1$ und $\mathfrak{k}_2$ niemals bzgl. der üblichen Ordnung in K ähnlich isomorph sind, da ein solcher Isomorphismus einen Isomorphismus der dazugehörigen Positivbereiche $\mathfrak{h}_i=\mathfrak{k}_i\cap H$ induzieren würde. Umgekehrt läßt sich auf diese Körperaussage natürlich auch die Aussage von Satz 3 zurückführen.

Wir wenden uns nun dem Zusammenhang zwischen Halbkörpern und Ordnungsfragen zu und beweisen den folgenden Satz, wovon Teilaussagen schon in der genannten Literatur zu finden sind:

SATZ 4. *Es sei* $\mathfrak{K}$ *ein algebraischer Erweiterungskörper eines Körpers* $\mathfrak{k}\subseteq K$ *und* $\mathfrak{h}=\mathfrak{k}\cap H$ *der übliche Positivbereich von* $\mathfrak{k}$. *Dann ergibt die Zuordnung*

$$\varphi\to\mathfrak{H}^*=\{\varkappa\,|\,\varkappa\in\mathfrak{K}\quad\text{und}\quad\varphi(\varkappa)>0\} \tag{1}$$

eine eineindeutige Abbildung der Menge aller reellen Isomorphismen φ *von* $\mathfrak{K}/\mathfrak{k}$ *auf die Menge der Positivbereiche sämtlicher Ordnungen von* $\mathfrak{K}$, *welche die übliche Ordnung von* $\mathfrak{k}$ *fortsetzen, wobei diese Menge wiederum gerade aus allen zu* $\mathfrak{K}$ *gehörigen echten Halbkörpern* $\mathfrak{H}$ *besteht, die* $(*)$ *und* $\mathfrak{h}\subseteq\mathfrak{H}$ *erfüllen. Im Falle einer einfachen algebraischen Erweiterung* $\mathfrak{K}=\mathfrak{k}(\alpha)$ *ist also insbesondere die Anzahl dieser Halbkörper* $\mathfrak{H}$ *gleich der Anzahl* r *der reellen Nullstellen des Minimalpolynoms von* α *über* $\mathfrak{k}$.

BEWEIS. Jede der Mengen $\mathfrak{H}^*$ von (1) ist der Positivbereich derjenigen Ordnung von $\mathfrak{K}$, die folgendermaßen definiert werden kann:

$$\alpha<\beta\quad\text{genau dann, wenn}\quad\varphi(\alpha)<\varphi(\beta)\qquad(\alpha,\beta\in\mathfrak{K}).$$

Da die Elemente von $\mathfrak{h}$ bei dem Isomorphismus φ fest bleiben, ist die genannte Ordnung eine Fortsetzung der üblichen Ordnung von $\mathfrak{k}$ und $\mathfrak{H}^*$ ein zu $\mathfrak{K}$ gehöriger echter Halbkörper, der $(*)$ und $\mathfrak{h}\subseteq\mathfrak{H}^*$ erfüllt. Zum Nachweis der Voreindeutigkeit der in (1) gegebenen Zuordnung betrachten wir zwei reelle Isomorphismen φ_1, φ_2 von $\mathfrak{K}/\mathfrak{k}$, für die $\mathfrak{H}_1^*=\mathfrak{H}_2^*$ gilt. Der Isomorphismus $\psi=\varphi_2\varphi_1^{-1}$ [6] von $\varphi_1(\mathfrak{H}_1^*)=$ $=\overline{\mathfrak{H}}_1^*\subseteq H$ auf $\varphi_2(\mathfrak{H}_1^*)=\overline{\mathfrak{H}}_2^*\subseteq H$ ist dann nach Satz 3 die Identität, so daß $\varphi_1(\varkappa)=\varphi_2(\varkappa)$ für alle $\varkappa\in\mathfrak{H}_1^*$, also auch für alle $\varkappa\in\mathfrak{K}$ gilt. Daraus folgt $\varphi_1=\varphi_2$. Es bleibt nun noch zu zeigen, daß jeder Halbkörper $\mathfrak{H}'$ mit den im Satz genannten Eigenschaften auf der rechten Seite von (1) auftritt. Jedenfalls ist $\mathfrak{H}'$ dann Positivbereich einer

[5] Vgl. etwa RÉDEI [3], Satz 387.

[6] Wir legen $\varphi\psi(a)=\varphi(\psi(a))$ für das Produkt von Isomorphismen φ, ψ zugrunde.

Ordnung $\mathfrak{O}_{\mathfrak{H}}$ von $\mathfrak{K}$, die Fortsetzung der üblichen Ordnung von $\mathfrak{k}$ ist. Mit der letzteren ist dann auch $\mathfrak{O}_{\mathfrak{H}'}$ archimedisch, da $\mathfrak{K}$ algebraisch über $\mathfrak{k}$ ist.[7] Es gibt also einen ähnlichen Isomorphismus φ des mit $\mathfrak{O}_{\mathfrak{H}'}$ geordneten Körpers $\mathfrak{K}$ in den Körper K, wobei u. a. $\varphi(\mathfrak{h})=\overline{\mathfrak{h}}\subseteq H$ gilt. Aus Satz 3 folgt nun, daß die Elemente von $\mathfrak{h}$ und damit auch von $\mathfrak{k}$ bei φ fest bleiben, so daß φ einen bzgl. $\mathfrak{k}$ relativen Isomorphismus von $\mathfrak{K}$ in K darstellt. Ersichtlich gilt auch $\mathfrak{H}'=\{\varkappa|\varkappa\in\mathfrak{K}\text{ und }\varphi(\varkappa)>0\}$.

Für den Fall einer einfachen algebraischen Erweiterung $\mathfrak{K}=\mathfrak{K}(\alpha)$ eines reellen Körpers $\mathfrak{k}$ besagt Satz 4, daß es genau dann keine zu $\mathfrak{K}$ gehörigen echten Halbkörper $\mathfrak{H}^*$ mit $(*)$ und $\mathfrak{h}=\mathfrak{k}\cap H\subseteq\mathfrak{H}^*$ gibt, wenn $r=0$ gilt. Darüber hinaus ergibt sich sogar:

SATZ 5. *Es sei $\mathfrak{K}=\mathfrak{k}(\alpha)$ eine einfache algebraische Erweiterung eines Körpers $\mathfrak{k}\subseteq K$, $\mathfrak{h}=\mathfrak{k}\cap H$ und r die Anzahl der reellen Nullstellen des Minimalpolynoms von α über $\mathfrak{k}$. Dann gilt $r=0$ genau dann, wenn es überhaupt keinen zu $\mathfrak{K}$ gehörigen echten Halbkörper $\mathfrak{H}$ mit $\mathfrak{h}\subseteq\mathfrak{H}$ gibt.*

BEWEIS. Es ist noch zu zeigen, daß für $r=0$ kein zu $\mathfrak{K}$ gehöriger echter Halbkörper $\mathfrak{H}$ mit $\mathfrak{h}\subseteq\mathfrak{H}$ existiert. Wir schließen indirekt und nehmen die Existenz eines solchen Halbkörpers $\mathfrak{H}$ an. Nach Satz 2 enthält dann $\mathfrak{H}$ ein primitives Element β von $\mathfrak{K}/\mathfrak{k}$, und $\mathfrak{h}(\beta)$ ist wegen $\mathfrak{h}(\beta)\subseteq\mathfrak{H}$ ein echter Halbkörper. Wir ziehen nun einen Teil von Satz 8 aus WEINERT [9], § 9, heran, der besagt, daß eine einfache Halbkörpererweiterung $\mathfrak{h}(\gamma)$ (mit den im Satz über $\mathfrak{h}$ und $\mathfrak{k}$ gemachten Voraussetzungen) genau dann ein Körper ist, wenn γ algebraisch über $\mathfrak{h}$ ist und das Minimalpolynom $f(x)\in\mathfrak{k}[x]$ von γ keine positive reelle Nullstelle besitzt. Das Minimalpolynom von β über $\mathfrak{k}$ hat also mindestens eine (positive) reelle Nullstelle. Wegen $\mathfrak{k}(\alpha)=\mathfrak{k}(\beta)$ gilt somit $r\geqq 1$, im Widerspruch zur Voraussetzung.

BEMERKUNG. Satz 5 ist für einen beliebigen echten Halbkörper $\mathfrak{h}$ mit $Q(D(\mathfrak{h}))=\mathfrak{k}\subseteq K$ nicht mehr richtig. Wir betrachten dazu folgendes Beispiel:

Es sei $\mathfrak{k}=K_0(\alpha)$ eine einfache Erweiterung des Körpers K_0 der rationalen Zahlen mit einer über K_0 transzendenten positiven reellen Zahl α. Nach Satz 4 aus WEINERT [9], § 9, ist jede einfache transzendente Halbkörpererweiterung eines echten Halbkörpers wieder ein echter Halbkörper, der außerdem die Eigenschaft $(*)$ nicht erfüllt. Diese Eigenschaften hat also auch der Halbkörper $\mathfrak{h}=H_0(-\alpha)$, für den ersichtlich $Q(D(\mathfrak{h}))=\mathfrak{k}$ gilt. Der einfache algebraische Erweiterungskörper $\mathfrak{K}=\mathfrak{k}(i\sqrt{\alpha})$ $(\sqrt{\alpha}\in K$, i eine Nullstelle von $x^2+1\in K_0[x])$ hat als definierendes Polynom $x^2+\alpha\in\mathfrak{k}[x]$, das keine reelle Nullstelle besitzt. Für $\mathfrak{K}/\mathfrak{k}$ gilt also $r=0$. Es existiert aber ein zu $\mathfrak{K}$ gehöriger echter Halbkörper $\mathfrak{H}$ mit $\mathfrak{h}\subseteq\mathfrak{H}$. Man nehme etwa den Halbkörper $\mathfrak{H}=\mathfrak{h}(i\sqrt{\alpha})=H_0(i\sqrt{\alpha})$, der wegen der Transzendenz von $i\sqrt{\alpha}$ über K_0 ein echter Halbkörper ist und für den sich ersichtlich $Q(D(\mathfrak{H}))=\mathfrak{K}$ ergibt.

Wir wollen nun einen später wieder benötigten Satz beweisen, der eine Verallgemeinerung des im Beweis von Satz 5 erwähnten Satzes 8 aus WEINERT [9], § 9, darstellt.

SATZ 6. *Für einen archimedischen Positivbereich*[8] *$\mathfrak{h}$ ist ein einfacher Erweiterungshalbkörper $\mathfrak{h}(\alpha)$ genau dann ein Körper, wenn α algebraisch über $\mathfrak{h}$ ist und keine Ordnung*

[7] Vgl. etwa RÉDEI [3], Satz 376.

[8] Die Definition wurde bereits in der Einleitung gegeben.

des Körpers $D(\mathfrak{h}(\alpha))$ existiert, die Fortsetzung der Ordnung $\mathfrak{O}_\mathfrak{h}$ von $D(\mathfrak{h})$ ist und bei der α positiv wird.

BEWEIS. Wir setzen zunächst voraus, daß $\mathfrak{h}(\alpha)$ ein Körper ist. Wegen des in der Bemerkung zu Satz 5 erwähnten Satzes 4 aus WEINERT [9], § 9, ist dann α algebraisch über $\mathfrak{h}$. Darüber hinaus kann α natürlich bei keiner der betrachteten Ordnungen des Körpers $D(\mathfrak{h}(\alpha))=\mathfrak{h}(\alpha)$ positiv werden, da sonst alle Elemente dieses Körpers positiv wären.

Wir zeigen nun, daß aus den im Satz an das Element α gestellten Bedingungen die Körpereigenschaft von $\mathfrak{h}(\alpha)$ folgt. Dazu schließen wir indirekt und nehmen an, daß $\mathfrak{h}(\alpha)$ ein echter Halbkörper ist. Da $\mathfrak{h}$ für den Körper $\mathfrak{k}=D(\mathfrak{h})$ eine archimedische Ordnung $\mathfrak{O}_\mathfrak{h}$ definiert, existiert ein Isomorphismus φ von $\mathfrak{k}$ auf einen Körper $\bar{\mathfrak{k}}\subseteq K$, der $\mathfrak{h}$ auf den Halbkörper $\bar{\mathfrak{h}}=\bar{\mathfrak{k}}\cap H$ abbildet. Für den Körper $\mathfrak{K}=D(\mathfrak{h}(\alpha))=\mathfrak{k}(\alpha)$ gibt es dann einen Isomorphismus Φ, der Fortsetzung von φ ist und $\mathfrak{K}=\mathfrak{k}(\alpha)$ auf einen einfachen algebraischen Erweiterungskörper $\bar{\mathfrak{K}}=\bar{\mathfrak{k}}(\bar{\alpha})$ von $\bar{\mathfrak{k}}$ abbildet, wobei $\Phi(\alpha)=\bar{\alpha}$ gilt. Der nach Annahme echte Halbkörper $\mathfrak{h}(\alpha)$ wird dabei auf den ebenfalls echten Halbkörper $\bar{\mathfrak{h}}(\bar{\alpha})$ abgebildet, so daß das Minimalpolynom von $\bar{\alpha}$ über $\bar{\mathfrak{k}}$ nach Satz 8 aus WEINERT [9], § 9, eine positive reelle Nullstelle besitzt. Wegen Satz 4 existiert dann eine Ordnung von $\bar{\mathfrak{K}}$, die Fortsetzung der Ordnung $\mathfrak{O}_{\bar{\mathfrak{h}}}$ von $\bar{\mathfrak{k}}$ ist und bei der $\bar{\alpha}$ positiv wird. Unter Verwendung von Φ^{-1} läßt sich dann für $\mathfrak{K}$ eine Ordnung definieren, die Fortsetzung der Ordnung $\mathfrak{O}_\mathfrak{h}$ von $\mathfrak{k}$ ist und bei der α positiv wird. Das ist ein Widerspruch zu der in diesem Beweisabschnitt über α gemachten Voraussetzung. Also ist $\mathfrak{h}(\alpha)$ ein Körper.

Mit Hilfe von Satz 6 können wir nun den Satz 5 folgendermaßen verallgemeinern:

SATZ 7. *Es sei $\mathfrak{K}=\mathfrak{k}(\alpha)$ eine einfache algebraische Erweiterung eines beliebigen Körpers $\mathfrak{k}$ und $\mathfrak{h}$ ein zu $\mathfrak{k}$ gehöriger archimedischer Positivbereich. Es gibt genau dann keine zu $\mathfrak{K}$ gehörigen echten Halbkörper $\mathfrak{H}$ mit $\mathfrak{h}\subseteq\mathfrak{H}$, wenn $\mathfrak{K}$ keine Ordnung besitzt, die Fortsetzung der Ordnung $\mathfrak{O}_\mathfrak{h}$ von $\mathfrak{k}$ ist.*

BEWEIS. Aus der Existenz einer Ordnung von $\mathfrak{K}$, die Fortsetzung von $\mathfrak{O}_\mathfrak{h}$ ist, folgt natürlich die Existenz eines zu $\mathfrak{K}$ gehörigen echten Halbkörpers $\mathfrak{H}$ mit $\mathfrak{h}\subseteq\mathfrak{H}$. Der Positivbereich der Ordnung ist ein solcher Halbkörper. Es bleibt nun noch zu zeigen, daß für einen Körper $\mathfrak{K}$, der keine Ordnung besitzt, welche Fortsetzung von $\mathfrak{O}_\mathfrak{h}$ ist, auch keine zu $\mathfrak{K}$ gehörigen echten Halbkörper $\mathfrak{H}$ mit $\mathfrak{h}\subseteq\mathfrak{H}$ existieren. Wir schließen indirekt und nehmen an, daß ein solcher Halbkörper $\mathfrak{H}$ existiert. Dieser enthält nach Satz 2 ein primitives Element β von $\mathfrak{K}/\mathfrak{k}$ und $\mathfrak{h}(\beta)$ ist ein echter Halbkörper. Nach Satz 6 gibt es dann eine Ordnung von $\mathfrak{K}$, die Fortsetzung von $\mathfrak{O}_\mathfrak{h}$ ist, im Widerspruch zur Voraussetzung.

§ 2 Isomorphe Einbettbarkeit von Halbkörpern in den Halbkörper H der positiven reellen Zahlen

In WEINERT [9] (§ 9, Satz 2 bzw. in der darauf folgenden Bemerkung) wird für den Fall eines echten Halbkörpers $\mathfrak{H}$, dessen Elemente algebraisch über einem reellen Körper $\mathfrak{k}$ sind, folgendes Kriterium angegeben:

$\mathfrak{H}$ ist genau dann isomorph in H einbettbar, wenn ein echter Halbkörper $\mathfrak{H}^*\supseteq\mathfrak{H}$ mit der Eigenschaft $(*)$ existiert, der außerdem $\mathfrak{H}^*\cap\mathfrak{k}\subseteq H$ erfüllt.

Wir werden in diesem Paragraphen u.a. zeigen, daß die Existenz eines solchen Halbkörpers $\mathfrak{H}^*$ stets dann gesichert ist, wenn $\mathfrak{H}$ den Halbkörper $\mathfrak{h}=\mathfrak{k}\cap H$ enthält, daß also etwa ein echter, absolut algebraischer Halbkörper stets in H isomorph eingebettet werden kann. Allgemeiner wird sich sogar ergeben (Satz 10), daß für die Einbettbarkeit eines beliebigen echten Halbkörpers $\mathfrak{H}$ in H folgende Bedingung hinreichend ist: $\mathfrak{H}$ ist eine algebraische Erweiterung eines archimedischen Positivbereiches.

Vorbereitend dazu betrachten wir Halbkörper, die in einer einfachen algebraischen Körperweiterung $\mathfrak{K}=\mathfrak{k}(\alpha)$ eines reellen Körpers $\mathfrak{k}$ enthalten sind. Es sei r die Anzahl der reellen Nullstellen des Minimalpolynoms von α über $\mathfrak{k}$. Für $r\geqq 1$ bezeichnen wir mit $\varphi_1, \ldots, \varphi_r$ die Isomorphismen von $\mathfrak{K}/\mathfrak{k}$ in K. Gemäß Satz 4 entspricht jedem φ_i gerade ein zu $\mathfrak{K}$ gehörender echter Halbkörper $\mathfrak{H}_i^*$, der $(*)$ und $\mathfrak{k}\cap H=\mathfrak{h}\subseteqq\mathfrak{H}_i^*$ erfüllt, nämlich der Positivbereich der durch φ_i definierten Ordnung von $\mathfrak{K}$:

$$\mathfrak{H}_i^*=\{\varkappa\,|\,\varkappa\in\mathfrak{K} \quad \text{und} \quad \varphi_i(\varkappa)>0\} \qquad (i=1,\ldots,r).$$

Zunächst beweisen wir einen Hilfssatz, der auch in § 4 Verwendung finden wird:

HILFSSATZ. *Es sei $\mathfrak{H}$ ein Unterhalbkörper von $\mathfrak{K}=\mathfrak{k}(\alpha)$ und $\beta\in\mathfrak{H}$ ein primitives Element von $\mathfrak{K}/\mathfrak{k}$, so daß $\varphi_j(\beta)<0$ für ein $j\in R=\{1,\ldots,r\}$ gilt. Dann existiert auch ein (primitives) Element $\gamma\in\mathfrak{H}$ mit $\varphi_j(\gamma)<0$ und $\varphi_i(\gamma)>0$ für $i\in R\setminus\{j\}$.*

BEWEIS. Wegen der Primitivität [9] von β sind alle $\varphi_i(\beta)$, $i\in R$, paarweise verschieden. Es sei μ das Minimum ihrer Abstände. Wir wählen nun ein Element $a\in H_0$ mit

$$\varphi_j(\beta) < -a < \varphi_j(\beta)+\mu.$$

Dann ist $\beta_1=\beta+a\in\mathfrak{H}$ ein primitives Element mit $\varphi_j(\beta_1)<0$, während für alle $i\neq j$ $\varphi_i(\beta_1)>0$ oder $\varphi_i(\beta_1)<\varphi_j(\beta_1)$ gilt. Daraus folgt $\varphi_j(\beta_1^{-1})<0$ und $\varphi_j(\beta_1^{-1})<\varphi_i(\beta_1^{-1})$ für alle $i\neq j$. Wir verfahren nun mit β_1^{-1} wie oben mit β. Dabei ergibt sich die Existenz eines Elementes $b\in H_0$, so daß für das primitive Element $\gamma=\beta_1^{-1}+b\in\mathfrak{H}$ gilt: $\varphi_j(\gamma)<0$, $\varphi_i(\gamma)>0$ für alle $i\neq j$.

Zur Klärung der Einbettungsfrage benötigen wir nun für jeden zu $\mathfrak{K}=\mathfrak{k}(\alpha)$ gehörigen echten Halbkörper $\mathfrak{H}$ mit $\mathfrak{k}\cap H=\mathfrak{h}\subseteqq\mathfrak{H}$ die Existenz eines Oberhalbkörpers $\mathfrak{H}^*$ mit der Eigenschaft $(*)$. Wir beweisen aber sogleich wesentlich mehr, was sich später als nützlich erweisen wird:

SATZ 8. *Es sei $\mathfrak{k}$ ein reeller Körper, $\mathfrak{h}=\mathfrak{k}\cap H$ und $\mathfrak{K}=\mathfrak{k}(\alpha)$ ein einfacher algebraischer Erweiterungskörper von $\mathfrak{k}$. Dann existieren für jeden zu $\mathfrak{K}$ gehörenden echten Halbkörper $\mathfrak{H}$ mit $\mathfrak{h}\subseteqq\mathfrak{H}$ genau eine Menge M mit $\emptyset\neq M\subseteqq R=\{1,\ldots,r\}$ und ein primitives Element β von $\mathfrak{K}/\mathfrak{k}$, so daß gilt:*

$$\mathfrak{h}(\beta)\subseteqq\mathfrak{H}\subseteqq\bigcap_{i\in M}\mathfrak{H}_i^*, \qquad \beta\notin\mathfrak{H}_j^* \quad \textit{für alle} \quad j\in R\setminus M.$$

BEWEIS. Es sei $\mathfrak{H}$ ein zu $\mathfrak{K}$ gehörender echter Halbkörper mit $\mathfrak{h}\subseteqq\mathfrak{H}$. Aus der Existenz eines solchen Halbkörpers $\mathfrak{H}$ folgt nach Satz 5, daß die Menge der reellen Isomorphismen von $\mathfrak{K}/\mathfrak{k}$ nicht leer ist, also $r\geqq 1$ gilt. Nach Satz 2 enthält $\mathfrak{H}$ mindestens

[9] „Primitiv" werden wir in vorliegender Arbeit stets im Sinne von „primitiv in Bezug auf $\mathfrak{K}/\mathfrak{k}$" gebrauchen.

ein primitives Element von $\mathfrak{K}/\mathfrak{k}$. Jedes solche primitive Element $\varkappa$ wird bei wenigstens einem reellen Isomorphismus von $\mathfrak{K}/\mathfrak{k}$ in H abgebildet (sonst wäre $\mathfrak{h}(\varkappa)$ ein Körper,[10] im Widerspruch zu $\mathfrak{h}(\varkappa)\subseteq\mathfrak{H}$). Sei nun $\beta\in\mathfrak{H}$ ein primitives Element von $\mathfrak{K}/\mathfrak{k}$, für das die Anzahl der Isomorphismen φ_i $(i\in R)$ mit $\varphi_i(\beta)>0$ ein Minimum ist. Diese eindeutig bestimmte Anzahl bezeichnen wir mit m. Nach den obigen Überlegungen ist $1\leqq m\leqq r$. Der Einfachheit halber seien die reellen Isomorphismen von $\mathfrak{K}/\mathfrak{k}$ so durchnumeriert, daß gilt:

$$\varphi_i(\beta)>0 \quad (i\in M), \qquad \varphi_j(\beta)<0 \quad (j\in R\setminus M), \tag{1}$$

wobei wir $M=\{1, \ldots, m\}$ gesetzt haben. Wir weisen nun nach, daß β und M die Behauptung von Satz 8 erfüllen. Aus Satz 4 folgt zunächst für β:

$$\beta\in\bigcap_{i\in M}\mathfrak{H}_i^*, \qquad \beta\notin\mathfrak{H}_j^* \quad (j\in R\setminus M).$$

Es bleibt nun noch zu zeigen, daß $\mathfrak{H}\subseteq\bigcap_{i\in M}\mathfrak{H}_i^*$ erfüllt ist, also für ein beliebiges $\gamma\in\mathfrak{H}$ gilt:

$$\varphi_i(\gamma)>0 \quad (i\in M).$$

Zu diesem Zweck nehmen wir für ein $\gamma\in\mathfrak{H}$ an, daß etwa $\varphi_m(\gamma)<0$ gilt und leiten daraus einen Widerspruch ab. Es existiert zunächst ein Element $a\in H_0$ mit $a>-\varphi_m(\beta\gamma^{-1})$, so daß $\gamma'=\beta+a\gamma\in\mathfrak{H}$ ein primitives Element von $\mathfrak{K}/\mathfrak{k}$ ist, für das $\varphi_m(\gamma')<0$ gilt. Nach dem Hilfssatz gibt es nun ein Element $\bar{\gamma}\in\mathfrak{H}$ mit

$$\varphi_m(\bar{\gamma})<0, \qquad \varphi_i(\bar{\gamma})>0 \quad \text{für alle} \quad i\in R\setminus\{m\}. \tag{2}$$

Wir bilden das Element $\delta=\beta\bar{\gamma}\in\mathfrak{H}$, für das wegen (1) und (2) gilt:

$$\varphi_1(\delta)>0, \ldots, \varphi_{m-1}(\delta)>0; \qquad \varphi_m(\delta)<0, \ldots, \varphi_r(\delta)<0.$$

Falls δ nicht selbst primitiv ist, gibt es jedenfalls ein Element $b\in H_0$ mit $b>-\varphi_m(\beta\delta^{-1})$, so daß $\bar{\delta}=\beta+b\delta\in\mathfrak{H}$ ein primitives Element von $\mathfrak{K}/\mathfrak{k}$ ist. Für $\bar{\delta}$ gilt ebenfalls

$$\varphi_1(\bar{\delta})>0, \ldots, \varphi_{m-1}(\bar{\delta})>0; \qquad \varphi_m(\bar{\delta})<0, \ldots, \varphi_r(\bar{\delta})<0.$$

Das ist aber ein Widerspruch zur Minimalität von m. Es gilt also für jedes $\gamma\in\mathfrak{H}$: $\varphi_i(\gamma)>0$ für alle $i\in M$.

Um nun den am Anfang von § 2 angekündigten Satz beweisen zu können, verallgemeinern wir noch einen Teil von Satz 8 und zeigen:

SATZ 9. *Es sei $\mathfrak{K}=\mathfrak{k}(\alpha)$ eine einfache algebraische Erweiterung eines Körpers $\mathfrak{k}$, $\mathfrak{h}$ ein zu $\mathfrak{k}$ gehörender archimedischer Positivbereich und $\mathfrak{H}$ ein zu $\mathfrak{K}$ gehörender echter Halbkörper mit $\mathfrak{h}\subseteq\mathfrak{H}$. Dann existiert eine Ordnung von $\mathfrak{K}$, bei der alle Elemente von $\mathfrak{H}$ positiv sind.*

BEWEIS. Da die Ordnung $\mathfrak{O}_\mathfrak{h}$ von $\mathfrak{k}$ archimedisch ist, existiert ein Isomorphismus φ von $\mathfrak{k}$ auf einen Körper $\bar{\mathfrak{k}}\subseteq K$ mit $\varphi(\mathfrak{h})=\bar{\mathfrak{h}}=\bar{\mathfrak{k}}\cap H$, der sich zu einem Isomorphismus Φ von $\mathfrak{K}=\mathfrak{k}(\alpha)$ auf einen einfachen algebraischen Erweiterungskörper $\bar{\mathfrak{k}}(\bar{\alpha})$ von $\bar{\mathfrak{k}}$ fortsetzen läßt. Für den echten Halbkörper $\bar{\mathfrak{H}}=\Phi(\mathfrak{H})$ von $\bar{\mathfrak{k}}(\bar{\alpha})$ existiert dann

[10] Vgl. WEINERT [9], § 9, Satz 8.

nach Satz 8 (unter Verwendung von Satz 4) mindestens eine Ordnung $\overline{\mathfrak{O}}$ von $\bar{\mathfrak{k}}(\bar{\alpha})$, die Fortsetzung der Ordnung $\mathfrak{O}_{\bar{\mathfrak{h}}}$ von $\bar{\mathfrak{k}}$ ist und bei der alle Elemente von $\overline{\mathfrak{H}}$ positiv sind. Mittels Φ^{-1} und $\overline{\mathfrak{O}}$ läßt sich dann eine den Behauptungen von Satz 9 genügende Ordnung von $\mathfrak{K}$ definieren.

Wir sind nun in der Lage, den angekündigten Satz über die isomorphe Einbettbarkeit von Halbkörpern in den Halbkörper H der positiven reellen Zahlen zu beweisen:

SATZ 10. *Jeder echte Halbkörper $\mathfrak{H}$, der eine algebraische Erweiterung eines archimedischen Positivbereiches $\mathfrak{h}$ ist, läßt sich isomorph in H einbetten.*

BEWEIS. Es sei $\mathfrak{K}$ der von $\mathfrak{H}$ erzeugte Körper, für den also gemäß Satz 1 $\mathfrak{K}=D(\mathfrak{H})$ gilt. Wir betrachten die Menge $\mathcal{M}$ aller echten Halbkörper von $\mathfrak{K}$, die $\mathfrak{H}$ enthalten. In $\mathcal{M}$ hat jede Kette $\mathfrak{H}_1\subset\mathfrak{H}_2\subset\ldots$ eine obere Schranke, nämlich die Vereinigung $\tilde{\mathfrak{H}}$ der Halbkörper $\mathfrak{H}_1, \mathfrak{H}_2, \ldots$. Nach dem speziellen Lemma von KURATOWSKI—ZORN enthält daher $\mathcal{M}$ einen maximalen Halbkörper $\mathfrak{H}^*$. Wir wollen nun zeigen, daß $\mathfrak{H}^*$ die Eigenschaft $(*)$ erfüllt. Zu diesem Zweck nehmen wir das Gegenteil an, also die Existenz eines Elementes $\alpha\in\mathfrak{K}$ $(\alpha\neq 0)$ mit

$$\alpha\notin\mathfrak{H}^* \quad \text{und} \quad -\alpha\notin\mathfrak{H}^*.$$

Die Halbkörper $\mathfrak{H}_1=\mathfrak{H}^*(\alpha)$ und $\mathfrak{H}_2=\mathfrak{H}^*(-\alpha)$ sind dann echte Erweiterungen von $\mathfrak{H}^*$, also Körper. Demzufolge gilt

$$0=\frac{\sum_{i=0}^{n} a_i\alpha^i}{\sum_{i=0}^{n'} b_i\alpha^i} \quad \text{und} \quad 0=\frac{\sum_{i=0}^{m} c_i(-\alpha)^i}{\sum_{i=0}^{m'} d_i(-\alpha)^i}$$

mit $a_i, b_i, c_i, d_i\in\mathfrak{H}^*\cup\{0\}$, wobei jeweils nicht alle a_i, b_i, c_i bzw. d_i verschwinden. Daraus ergeben sich zwei Darstellungen des Nullelementes:

$$p_1\alpha^{s_1}+\ldots+p_k\alpha^{s_k}=0, \tag{3}$$

$$q_1(-\alpha)^{t_1}+\ldots+q_l(-\alpha)^{t_l}=0 \tag{4}$$

mit Koeffizienten p_i bzw. q_j aus $\mathfrak{H}^*$. Wegen $\mathfrak{K}=D(\mathfrak{H})$ existieren Elemente $h_1, h_2\in\mathfrak{H}$ mit $\alpha=h_1-h_2$. Wir betrachten nun den Erweiterungshalbkörper $\overline{\mathfrak{H}}=\mathfrak{h}(p_1,\ldots,p_k, q_1,\ldots,q_l, h_1, h_2)$, der ein echter Halbkörper ist, denn es gilt $\overline{\mathfrak{H}}\subseteq\mathfrak{H}^*$. Da $\mathfrak{K}$ eine algebraische Erweiterung des Körpers $\mathfrak{k}=D(\mathfrak{h})$ darstellt, ist der kleinste Oberkörper $\overline{\mathfrak{K}}=D(\overline{\mathfrak{H}})$ von $\overline{\mathfrak{H}}$, also $\overline{\mathfrak{K}}=\mathfrak{k}(p_1,\ldots,p_k, q_1,\ldots,q_l, h_1, h_2)$, algebraisch und damit einfach über $\mathfrak{k}$: $\overline{\mathfrak{K}}=\mathfrak{k}(\beta)$. Nach Satz 9 besitzt $\overline{\mathfrak{K}}$ eine Ordnung, bei der alle Elemente von $\overline{\mathfrak{H}}$ positiv sind. Wegen $h_1, h_2\in\overline{\mathfrak{H}}$ gilt $\alpha\in\overline{\mathfrak{K}}$ und $-\alpha\in\overline{\mathfrak{K}}$. Da die Elemente $p_i\in\overline{\mathfrak{H}}$ und $q_j\in\overline{\mathfrak{H}}$ bei der genannten Ordnung von $\overline{\mathfrak{K}}$ positiv sind und von α und $-\alpha$ genau eines positiv ist, ergibt sich entweder aus (3) oder aus (4) ein Widerspruch. Damit gilt für jedes $\alpha\neq 0$ aus $\mathfrak{K}$ entweder $\alpha\in\mathfrak{H}^*$ oder $-\alpha\in\mathfrak{H}^*$, so daß also $\mathfrak{H}^*$ die Eigenschaft $(*)$ erfüllt und in $\mathfrak{K}=D(\mathfrak{H}^*)$ eine Ordnung $\mathfrak{O}_{\mathfrak{H}^*}$ definiert. Wegen $\mathfrak{h}\subseteq\mathfrak{H}^*$ ist $\mathfrak{O}_{\mathfrak{H}^*}$ eine Fortsetzung der archimedischen Ordnung $\mathfrak{O}_{\mathfrak{h}}$ von $\mathfrak{k}=D(\mathfrak{h})$, also ebenfalls eine archimedische Ordnung, da $\mathfrak{K}$ algebraisch über $\mathfrak{k}$ ist. Damit gibt es einen ähnlichen Isomorphismus des mit $\mathfrak{O}_{\mathfrak{H}^*}$ geordneten

Körpers $\mathfrak{K}$ in den Körper K der reellen Zahlen. Dabei wird $\mathfrak{H}^*$ in H abgebildet, so daß also ein Isomorphismus von $\mathfrak{H}$ in H induziert wird.

Aus Satz 10 ergibt sich unmittelbar:

FOLGERUNG. *Ein echter, absolut algebraischer Halbkörper ist stets in H isomorph einbettbar.*

§ 3 Isomorphe Einbettbarkeit von Halbkörpern in den Körper K der reellen Zahlen

Wir werden uns zunächst überlegen, daß für Halbkörper, die überhaupt in einen Körper einbettbar sind, der „Transzendenzgrad" über die Einbettbarkeit in den Körper $K(i)$ der komplexen Zahlen entscheidet. Dazu definieren wir in Anlehnung an den Begriff des Transzendenzgrades für Körper einen analogen Begriff für Halbkörper:

DEFINITION. Unter dem *Transzendenzgrad* $t(\mathfrak{H})$ eines Halbkörpers $\mathfrak{H}$ über H_0 verstehen wir die Mächtigkeit einer maximalen, über $K_0 = D(H_0)$ algebraisch unabhängigen Teilmenge [11] $\mathfrak{S} \subseteq \mathfrak{H}$.

Der so definierte Transzendenzgrad $t(\mathfrak{H})$ ist eine Invariante von $\mathfrak{H}$. Er ist nämlich gleich dem Transzendenzgrad des von $\mathfrak{H}$ erzeugten Körpers $\mathfrak{K}$ über K_0, der nach dem zweiten Hauptsatz von STEINITZ eine Invariante von $\mathfrak{K}$ ist. Die Gleichheit von $t(\mathfrak{H})$ mit dem Transzendenzgrad von $\mathfrak{K}$ ergibt sich daraus, daß für eine maximale, über K_0 algebraisch unabhängige Teilmenge $\mathfrak{S} \subseteq \mathfrak{H}$ der Körper $\mathfrak{K}$ wegen $\mathfrak{K} = Q(D(\mathfrak{H}))$ über $K_0(\mathfrak{S})$ algebraisch ist. $\mathfrak{S}$ ist also eine Transzendenzbasis von $\mathfrak{K}$ über K_0.

SATZ 11. *Jeder Halbkörper, dessen Transzendenzgrad* $t(\mathfrak{H}) = \mathfrak{t}$ *die Mächtigkeit* $\mathfrak{c}$ *des Kontinuums nicht übersteigt, ist isomorph in* $K(i)$ *einbettbar und umgekehrt.*

BEWEIS. Der von $\mathfrak{H}$ erzeugte Körper $\mathfrak{K}$ hat ebenfalls den Transzendenzgrad $\mathfrak{t}$. Da der Transzendenzgrad des Körpers K der reellen Zahlen gleich $\mathfrak{c}$ ist,[12] existiert eine über K_0 algebraisch unabhängige Teilmenge $\mathfrak{S}' \subseteq K$ von der Mächtigkeit $\mathfrak{t}$. Sei nun $\mathfrak{S} \subseteq \mathfrak{H}$ eine maximale, über K_0 algebraisch unabhängige Teilmenge von $\mathfrak{H}$. Dann existiert ein Isomorphismus von $K_0(\mathfrak{S})$ auf $K_0(\mathfrak{S}') \subseteq K$. Dieser Isomorphismus ist fortsetzbar zu einem Isomorphismus des algebraischen Abschlusses $\overline{K_0(\mathfrak{S})}$ von $K_0(\mathfrak{S})$ auf $K(i)$. Wegen $\mathfrak{H} \subseteq \overline{K_0(\mathfrak{S})}$ wird $\mathfrak{H}$ dabei isomorph in $K(i)$ abgebildet. Die Umkehrung ist trivial.

Hinsichtlich der isomorphen Einbettbarkeit eines Halbkörpers $\mathfrak{H}$ in K ist nun trivialerweise klar, daß dafür $t(\mathfrak{H}) \leqq \mathfrak{c}$ notwendig ist. Diese Bedingung ist aber selbst für echte Halbkörper keineswegs hinreichend:

SATZ 12. *Es gibt n* $K(i)$ *echte Halbkörper, die nicht in K isomorph einbettbar sind.*

BEWEIS. Es sei α eine über K_0 transzendente reelle Zahl. Die Teilmenge $\mathfrak{h}^*$ des Körpers $\mathfrak{k} = K_0(\alpha)$, die aus Elementen der Form

$$\frac{a_0 + a_1\alpha + \ldots + a_n\alpha^n}{b_0 + b_1\alpha + \ldots + b_m\alpha^m}, \quad a_i, b_j \in K_0, \quad a_n \cdot b_m > 0,$$

[11] Für die verwendeten Begriffsbildungen vgl. etwa RÉDEI [3], Seite 510 ff.
[12] Vgl. etwa RÉDEI [3], Seite 604.

besteht, ist ein echter Halbkörper mit der Eigenschaft $(*)$,[13] der $\mathfrak{k}$ erzeugt. Die durch $\mathfrak{h}^*$ definierte Ordnung von $\mathfrak{k}$ ist nichtarchimedisch, denn es gilt z.b. $\alpha - n > 0$ für alle natürlichen Zahlen n. Zu dem so geordneten Körper $\mathfrak{k}$ gibt es nun einen reell abgeschlossenen algebraischen Erweiterungskörper $\mathfrak{K}$ (o. B. d. A. $\mathfrak{K} \subseteq K(i)$), dessen (einzige) Ordnung Fortsetzung der Ordnung von $\mathfrak{k}$ ist.[14] Der Positivbereich $\mathfrak{H}^*$ von $\mathfrak{K}$ ist ein echter Halbkörper in $K(i)$, der nicht in K isomorph eingebettet werden kann, da $\mathfrak{K}$ nur eine nichtarchimedische Ordnung besitzt.

Wir werden nun zwei hinreichende Bedingungen dafür angeben, daß ein Halbkörper isomorph in K eingebettet werden kann. Dazu definieren wir zunächst den Begriff des „algebraischen Grades" für solche Halbkörper $\mathfrak{H}$, in denen eine maximale, über K_0 algebraisch unabhängige Teilmenge $\mathfrak{S} \subseteq \mathfrak{H}$ existiert, so daß der von $\mathfrak{H}$ erzeugte Körper $\mathfrak{K}$ eine einfache algebraische Erweiterung des Körpers $\mathfrak{k} = K_0(\mathfrak{S})$ ist. Den Grad von $\mathfrak{K}/\mathfrak{k}$ wollen wir mit $g_{\mathfrak{S}}$ bezeichnen. Wir betrachten nun die Menge $\{g_{\mathfrak{S}}\}$ aller auf diese Weise entstehenden Gradzahlen und definieren:

DEFINITION. Unter dem *algebraischen Grad* $g(\mathfrak{H})$ eines Halbkörpers $\mathfrak{H}$ verstehen wir für den Fall, daß $\{g_{\mathfrak{S}}\}$ nicht leer ist, das Minimum von $\{g_{\mathfrak{S}}\}$.

Der so definierte algebraische Grad $g(\mathfrak{H})$ ist eine Invariante von $\mathfrak{H}$ und es gilt:

SATZ 13. *Jeder Halbkörper $\mathfrak{H}$ mit $t(\mathfrak{H}) \leqq \mathfrak{c}$, für den $g(\mathfrak{H})$ existiert und ungerade ist, läßt sich isomorph in K einbetten.*

BEWEIS. Aus den Voraussetzungen des Satzes folgt, daß eine maximale, algebraisch unabhängige Teilmenge $\mathfrak{S} \subseteq \mathfrak{H}$ existiert, deren Mächtigkeit die Kontinuumsmächtigkeit nicht übersteigt und für die außerdem gilt, daß der von $\mathfrak{H}$ erzeugte Körper $\mathfrak{K}$ eine einfache algebraische Erweiterung $\mathfrak{K} = K_0(\mathfrak{S})(\alpha)$ von $K_0(\mathfrak{S})$ vom Grade $g = g(\mathfrak{H})$ ist. Es gibt also eine algebraisch unabhängige Teilmenge $\overline{\mathfrak{S}}$ von K, die zu $\mathfrak{S}$ gleichmächtig ist. Eine dazugehörige Abbildung induziert einen Isomorphismus φ von $K_0(\mathfrak{S})$ auf $K_0(\overline{\mathfrak{S}})$. Das Minimalpolynom $f(x)$ von α über $K_0(\mathfrak{S})$ geht bei Anwendung von φ über in ein irreduzibles Polynom $\bar{f}(x) \in K_0(\overline{\mathfrak{S}})[x]$ vom Grade g. Da g ungerade ist, existiert ein Element $\bar{\alpha} \in K$ mit $\bar{f}(\bar{\alpha}) = 0$, so daß sich φ vermöge der Zuordnung $\alpha \to \bar{\alpha}$ zu einem Isomorphismus von $K_0(\mathfrak{S})(\alpha)$ auf $K_0(\overline{\mathfrak{S}})(\bar{\alpha}) \subseteq K$ fortsetzen läßt. Dabei wird $\mathfrak{H}$ isomorph in K abgebildet.

Ein weiteres Kriterium für die Einbettbarkeit in K liefert der folgende

SATZ 14. *Es sei $\mathfrak{H}$ ein Halbkörper mit $t(\mathfrak{H}) \leqq \mathfrak{c}$. Wenn zu jedem echten Halbkörper $\mathfrak{H}^*$ mit $(*)$ und $\mathfrak{H}^* \subseteq \mathfrak{K}' \subset \mathfrak{K} = Q(D(\mathfrak{H}))$ ($\mathfrak{K}'$ Unterkörper von $\mathfrak{K}$) ein echter Halbkörper $\overline{\mathfrak{H}}$ mit $\mathfrak{H}^* \subset \overline{\mathfrak{H}} \subseteq \mathfrak{K}$ existiert, so ist $\mathfrak{H}$ isomorph in K einbettbar.*

BEWEIS. In dem Halbkörper $\mathfrak{H}$ existiert eine maximale, über K_0 algebraisch unabhängige Teilmenge $\mathfrak{S}$ (die auch leer sein kann), so daß $\mathfrak{H} = H_0(\mathfrak{S})(\mathfrak{A})$ eine algebraische Erweiterung des Halbkörpers $H_0(\mathfrak{S})$ ist. Wegen $t(\mathfrak{H}) \leqq \mathfrak{c}$ existiert eine über K_0 algebraisch unabhängige Teilmenge $\overline{\mathfrak{S}} \subseteq K$ und eine eineindeutige Abbildung von $\mathfrak{S}$ auf $\overline{\mathfrak{S}}$, die einen Isomorphismus φ von $K_0(\mathfrak{S})$ auf $K_0(\overline{\mathfrak{S}})$ induziert. Wir betrachten nun die Menge M aller Isomorphismen von Unterkörpern von $\mathfrak{K}$ in K, die Fortsetzungen von φ sind, und führen in M eine irreflexive Halbordnung ein:

Es sei $\xi \prec \eta$ ($\xi, \eta \in M$) genau dann, wenn η Fortsetzung von ξ ist und $\eta \neq \xi$ gilt.

[13] Vgl. WEINERT [9], Seite 186.

[14] Vgl. etwa V. D. WAERDEN [5], § 72, Satz 8.

Wir zeigen, daß M ein maximales Element besitzt. Es sei $\xi_1 < \xi_2 < \ldots < \xi_i < \ldots$ $(i \in I, I$ passende Indexmenge) eine Kette in M. Dann korrespondieren dazu zwei Ketten von Körpern

$$\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \ldots \subset \mathfrak{A}_i \subset \ldots \subseteqq \mathfrak{K},$$

$$\overline{\mathfrak{A}}_1 \subset \overline{\mathfrak{A}}_2 \subset \ldots \subset \overline{\mathfrak{A}}_i \subset \ldots \subseteqq K,$$

wobei $\xi_i(\mathfrak{A}_i) = \overline{\mathfrak{A}}_i$ $(i \in I)$ gilt. Für die Körper

$$\mathfrak{A} = \bigcup_{i \in I} \mathfrak{A}_i \subseteqq \mathfrak{K} \quad \text{und} \quad \overline{\mathfrak{A}} = \bigcup_{i \in I} \overline{\mathfrak{A}}_i \subseteqq K$$

läßt sich ein Isomorphismus ξ definieren: Für ein beliebiges Element $\varkappa \in \mathfrak{A}$ gelte

$$\xi(\varkappa) = \xi_j(\varkappa) \in \overline{\mathfrak{A}}, \quad \text{wenn} \quad \varkappa \in \mathfrak{A}_j.$$

Dann gilt $\xi \in M$ und $\xi_i \leqq \xi$ für alle $i \in I$, d. h., ξ ist eine obere Schranke der betrachteten Kette. Nach dem Lemma von KURATOWSKI—ZORN hat dann M ein maximales Element μ: $\mathfrak{B} \to \overline{\mathfrak{B}} \subseteqq K$, wobei $\mathfrak{B}$ ein Unterkörper von $\mathfrak{K}$ ist. Wir zeigen nun, daß $\mathfrak{B} = \mathfrak{K}$ gilt. Dazu schließen wir indirekt und nehmen $\mathfrak{B} \subset \mathfrak{K}$ an. Der Halbkörper $\mathfrak{b} = \mu^{-1}(\overline{\mathfrak{B}} \cap H)$ ist ein echter Halbkörper mit der Eigenschaft $(*)$. Es gibt also nach der Voraussetzung des Satzes ein Element $\alpha \in \mathfrak{K}$ $(\alpha \notin \mathfrak{B})$, so daß $\mathfrak{b}(\alpha)$ ein echter Halbkörper ist. Das Minimalpolynom $f(x)$ von α über $\mathfrak{B}$ [15] geht bei Anwendung von μ über in ein Minimalpolynom $\bar{f}(x) \in \overline{\mathfrak{B}}[x]$. Der Isomorphismus μ läßt sich dann vermöge $\alpha \to \bar{\alpha}$ $(\bar{f}(\bar{\alpha}) = 0)$ fortsetzen zu einem Isomorphismus μ': $\mathfrak{B}(\alpha) \to \overline{\mathfrak{B}}(\bar{\alpha})$, der $\mathfrak{b}(\alpha)$ auf den ebenfalls echten Halbkörper $\overline{\mathfrak{b}}(\bar{\alpha})$ $(\overline{\mathfrak{b}} = \overline{\mathfrak{B}} \cap H)$ abbildet. Nach WEINERT [9], § 9, Satz 8 hat somit $\bar{f}(x)$ mindestens eine reelle Nullstelle; $\bar{\alpha}$ sei bereits eine solche. Dann ist μ' ein Isomorphismus von $\mathfrak{B}(\alpha) \subseteqq \mathfrak{K}$ in K, also ein Element von M mit $\mu < \mu'$. Das ist ein Widerspruch zur Maximalität von μ in M. Es gilt damit $\mathfrak{B} = \mathfrak{K}$, so daß μ einen Isomorphismus von $\mathfrak{H} \subseteqq \mathfrak{K}$ in K induziert.

§ 4 Algebraische Halbkörpererweiterungen von archimedischen Positivbereichen

In diesem Paragraphen werden wir einen Überblick über diejenigen Halbkörper $\mathfrak{H}$ gewinnen, die algebraisch sind über einem archimedischen Positivbereich $\mathfrak{h}$, also zu einem über $\mathfrak{k} = D(\mathfrak{h})$ algebraischen Körper $\mathfrak{K}$ gehören. Für den Fall, daß $\mathfrak{h} \subseteqq H$ gilt und $\mathfrak{K}/\mathfrak{k}$ quadratisch ist, findet man bereits in WEINERT [11] eine Lösung dieses Problems, während der Fall $\mathfrak{h} = H_0$ und $\mathfrak{K} = K_0(\alpha)$ in KOCH [2] behandelt wird.

Zunächst wenden wir uns einigen vorbereitenden Aussagen zu. Es sei $\mathfrak{K} = \mathfrak{k}(\alpha)$ eine einfache algebraische Erweiterung eines Körpers $\mathfrak{k} \subseteqq K$ und $\mathfrak{h} = \mathfrak{k} \cap H$. Wie in § 2 bezeichnen wir mit $\varphi_1, \ldots, \varphi_r$ die Isomorphismen von $\mathfrak{K}/\mathfrak{k}$ in K und setzen $R = \{1, \ldots, r\}$. Die Halbkörper $\mathfrak{H}_i^* = \{\varkappa | \varkappa \in \mathfrak{K}$ und $\varphi_i(\varkappa) > 0\}$ sind gemäß Satz 4 paarweise verschieden. Für jede Teilmenge $X \subseteqq R$ sei

$$\mathfrak{P}(X) = \left\{ \varkappa \,|\, \varkappa \in \mathfrak{K} \quad \text{und} \quad \begin{matrix} \varphi_i(\varkappa) > 0 & \text{für} & i \in X \\ \varphi_j(\varkappa) < 0 & \text{für} & j \in R \setminus X \end{matrix} \right\},$$

[15] $\mathfrak{K}$ ist über $K_0(\mathfrak{S})$, also erst recht über $\mathfrak{B}$ algebraisch.

so daß dann für jede Menge M mit $\emptyset \neq M \subseteq R$ gilt:

$$\bigcap_{i \in M} \mathfrak{H}_i^* = \bigcup_{M \subseteq X \subseteq R} \mathfrak{P}(X).$$

Für $M = R$ entsteht dabei insbesondere der Halbkörper

$$\bigcap_{i \in R} \mathfrak{H}_i^* = \mathfrak{P}(R),$$

den wir mit $\overline{\mathfrak{H}}$ bezeichnen.

SATZ 15. *Das Mengensystem* $\{\mathfrak{P}(X)\}$, *wobei* X *alle Teilmengen von* R *durchläuft, stellt eine Klasseneinteilung von* $\mathfrak{K} \setminus \{0\}$ *dar. Die Mengen* $\mathfrak{P}(X)$ *sind gerade die Nebenklassen der multiplikativen Körpergruppe von* $\mathfrak{K}$ *nach der multiplikativen Gruppe des Halbkörpers* $\overline{\mathfrak{H}} = \mathfrak{P}(R)$.

BEWEIS. Für den ersten Teil des Satzes genügt es zu zeigen, daß $\mathfrak{P}(X) \neq \emptyset$ für alle $X \subseteq R$ gilt. Nun existiert für jedes $j \in R$ nach Satz 2 ein primitives Element $\bar{\beta}_j \in \mathfrak{H}_j^*$ von $\mathfrak{K}/\mathfrak{k}$, für das also $\varphi_j(\bar{\beta}_j) > 0$ gilt. Das Element $\beta_j = -\bar{\beta}_j$ ist dann ein primitives Element von $\mathfrak{K}/\mathfrak{k}$ mit $\varphi_j(\beta_j) < 0$. Nach dem Hilfssatz von § 2 existiert also ein Element $\gamma_j \in \mathfrak{K}$ mit $\varphi_j(\gamma_j) < 0$, $\varphi_i(\gamma_j) > 0$ für alle $i \in R \setminus \{j\}$. Für jedes $X \subseteq R$ gilt dann

$$\delta = 1 \cdot \prod_{j \in R \setminus X} \gamma_j \in \mathfrak{P}(X)$$

wegen $\varphi_k(\delta) > 0$ für alle $k \in X$ und $\varphi_l(\delta) < 0$ für alle $l \in R \setminus X$. In bezug auf den zweiten Teil des Satzes ist zu zeigen: Wenn $\varkappa \in \mathfrak{P}(X)$ gilt, so ist $\varkappa\overline{\mathfrak{H}} = \mathfrak{P}(X)$ erfüllt. Dabei ist $\varkappa\overline{\mathfrak{H}} \subseteq \mathfrak{P}(X)$ trivial. Für die Umkehrung sei γ ein beliebiges Element von $\mathfrak{P}(X)$. Dann existiert ein $\xi \in \mathfrak{K} \setminus \{0\}$ mit $\gamma = \varkappa\xi$. Wegen $\gamma, \varkappa \in \mathfrak{P}(X)$ gilt $\varphi_i(\xi) > 0$ für alle $i \in R$. Also ist $\xi \in \overline{\mathfrak{H}}$ und $\gamma \in \varkappa\overline{\mathfrak{H}}$.

FOLGERUNG. *Für jeden (echten) Halbkörper* $\mathfrak{H}$ *mit* $\overline{\mathfrak{H}} \subseteq \mathfrak{H} \subset \mathfrak{K}$ *gilt* $D(\mathfrak{H}) = \mathfrak{K}$, *und es gibt genau eine Menge* M *mit* $\emptyset \neq M \subseteq R$ *und* $\mathfrak{H} = \bigcap_{i \in M} \mathfrak{H}_i^*$.

BEWEIS. Zunächst folgt $D(\mathfrak{H}) = \mathfrak{K}$ sofort aus $D(\overline{\mathfrak{H}}) = \mathfrak{K}$ und letzteres daraus, daß $\overline{\mathfrak{H}}$ alle Quadrate von Elementen $\varkappa \neq 0$ aus $\mathfrak{K}$ enthält und für $\xi \in \mathfrak{K}$ gilt: $\xi = \left(\frac{\xi+1}{2}\right)^2 - \left(\frac{\xi-1}{2}\right)^2$. Für das Weitere können wir $\overline{\mathfrak{H}} \subset \mathfrak{H}$ annehmen. Nach Satz 8 gibt es dann genau eine Menge M mit $\emptyset \neq M \subset R$ und ein primitives Element β von $\mathfrak{K}/\mathfrak{k}$, so daß gilt:

$$\beta \in \mathfrak{H} \subseteq \bigcap_{i \in M} \mathfrak{H}_i^* = \bigcup_{M \subseteq X \subseteq R} \mathfrak{P}(X), \qquad \beta \notin \mathfrak{H}_j^* \text{ für } j \in R \setminus M.$$

Es ist nun noch zu zeigen, daß für jedes X mit $M \subseteq X \subseteq R$ gilt: $\mathfrak{P}(X) \subseteq \mathfrak{H}$. Nach dem Hilfssatz aus § 2 gibt es zu jedem $j \in R \setminus X$ ein $\gamma_j \in \mathfrak{H}$ mit $\varphi_j(\gamma_j) < 0$, $\varphi_i(\gamma_j) > 0$ für alle $i \in R \setminus \{j\}$, denn $\beta \in \mathfrak{H}$ ist ein primitives Element von $\mathfrak{K}/\mathfrak{k}$ mit $\varphi_j(\beta) < 0$. Es gilt nun

$$\gamma = \prod_{j \in R \setminus X} \gamma_j \in \mathfrak{H} \quad \text{und} \quad \gamma \in \mathfrak{P}(X).$$

Wegen $\gamma\overline{\mathfrak{H}} = \mathfrak{P}(X)$ gilt daher $\mathfrak{P}(X) \subseteq \mathfrak{H}$.

Wir wenden uns nun dem eigentlichen Problem dieses Paragraphen zu, betrachten aber zunächst nur Halbkörper, für die der kleinste Oberkörper $\mathfrak{K}$ eine einfache algebraische Erweiterung eines Körpers $\mathfrak{k}$ ist.

SATZ 16. *Es sei* $\mathfrak{h}$ *ein archimedischer Positivbereich und* $\mathfrak{K}=\mathfrak{k}(\alpha)$ *eine einfache algebraische Erweiterung des Körpers* $\mathfrak{k}=D(\mathfrak{h})$. *Der Körper* $\mathfrak{K}$ *besitze eine Ordnung, welche Fortsetzung der Ordnung* $\mathfrak{O}_{\mathfrak{h}}$ *von* $\mathfrak{k}$ *ist.*[16] *Dann gibt es endlich viele Ordnungen dieser Art. Die Durchschnitte von Positivbereichen* $\mathfrak{H}_i^*$ $(i\in R=\{1, 2, \ldots, r\})$ *dieser Ordnungen sind paarweise verschieden und stellen gerade die zu* $\mathfrak{K}$ *gehörigen echten Halbkörper* $\mathfrak{H}$ *mit* $\mathfrak{h}\subseteqq\mathfrak{H}$ *dar. Zu jedem solchen Halbkörper* $\mathfrak{H}$ *existiert also genau eine Menge* M *mit* $\emptyset\neq M\subseteqq R$ *und*

$$\mathfrak{H}=\bigcap_{i\in M}\mathfrak{H}_i^*.$$

Darüber hinaus ist die Menge der primitiven Elemente β *von* $\mathfrak{K}/\mathfrak{k}$ *mit*

$$\beta\in\mathfrak{H}_i^* \quad (i\in M), \qquad \beta\notin\mathfrak{H}_j^* \quad (j\in R\setminus M)$$

nicht leer und es gilt für jedes Element β *dieser Art*

$$\mathfrak{H}=\mathfrak{h}(\beta).$$

BEWEIS. Da $\mathfrak{h}$ für den Körper $\mathfrak{k}=D(\mathfrak{h})$ eine archimedische Ordnung definiert, gibt es einen Isomorphismus φ von $\mathfrak{k}$ auf einen Körper $\mathfrak{k}'\subseteqq K$ mit $\varphi(\mathfrak{h})=\mathfrak{h}'=\mathfrak{k}'\cap H$. Dieser Isomorphismus läßt sich fortsetzen zu einem Isomorphismus Φ von $\mathfrak{K}$ auf einen einfachen algebraischen Erweiterungskörper $\mathfrak{k}'(\alpha')$ von $\mathfrak{k}'$, wobei $\Phi(\alpha)=\alpha'$ gilt. Jeder zu $\mathfrak{K}$ gehörige echte Halbkörper $\mathfrak{H}$ mit $\mathfrak{h}\subseteqq\mathfrak{H}$ entspricht dann eineindeutig einem zu $\mathfrak{K}'=\mathfrak{k}'(\alpha')$ gehörigen echten Halbkörper $\mathfrak{H}'$ mit $\mathfrak{h}'\subseteqq\mathfrak{H}'$. Analog gilt dies für die Ordnungen von $\mathfrak{K}$ und $\mathfrak{K}'$, welche Fortsetzungen der Ordnung $\mathfrak{O}_{\mathfrak{h}}$ von $\mathfrak{k}$ bzw. $\mathfrak{O}_{\mathfrak{h}'}$ von $\mathfrak{k}'$ sind.

Damit genügt es also, die Behauptungen des Satzes für den Fall $\mathfrak{h}\subseteqq H$ nachzuweisen. Unter Verwendung von Satz 4 können wir den noch nachzuweisenden Sachverhalt wie folgt formulieren:

Es sei $\mathfrak{k}$ ein reeller Körper und $\mathfrak{K}=\mathfrak{k}(\alpha)$ eine einfache algebraische Erweiterung von $\mathfrak{k}$ mit $r\geqq 1$ (r Anzahl der reellen Nullstellen des Minimalpolynoms von α über $\mathfrak{k}$). Dann sind die Halbkörper

$$\mathfrak{H}=\bigcap_{i\in M}\mathfrak{H}_i^* \quad \text{mit} \quad \emptyset\neq M\subseteqq R=\{1, \ldots, r\}$$

paarweise verschieden und gerade die zu $\mathfrak{K}$ gehörenden echten Halbkörper mit $\mathfrak{h}=\mathfrak{k}\cap H\subseteqq\mathfrak{H}$. Die Menge der primitiven Elemente β von $\mathfrak{K}/\mathfrak{k}$ mit $\beta\in\mathfrak{P}(M)$ ist dabei stets nicht leer und für jedes dieser Elemente β gilt

$$\mathfrak{H}=\mathfrak{h}(\beta).$$

Wegen der Folgerung nach Satz 15 ist für den ersten Teil der Behauptung zu zeigen, daß für jeden zu $\mathfrak{K}$ gehörenden echten Halbkörper $\mathfrak{H}$ mit $\mathfrak{h}\subseteqq\mathfrak{H}$ gilt:

$$\bigcap_{i\in R}\mathfrak{H}_i^*=\overline{\mathfrak{H}}\subseteqq\mathfrak{H}. \tag{1}$$

[16] Für den Fall, daß keine solche Ordnung von $\mathfrak{K}$ existiert, vgl. Satz 7.

Für die einfache Erzeugbarkeit braucht dann nur noch Satz 8 herangezogen zu werden, der angewendet auf den Halbkörper $\bigcap_{i\in M}\mathfrak{H}_i^*$ für eine beliebige Menge M mit $\emptyset\neq M\subseteq R$ die Existenz eines primitiven Elementes β von $\mathfrak{K}/\mathfrak{k}$ mit $\varphi_i(\beta)>0$ für $i\in M$ und $\varphi_j(\beta)<0$ für $j\in R\setminus M$, also mit $\beta\in\mathfrak{P}(M)$ lehrt. Außerdem gilt nun für jedes Element β dieser Art

$$\mathfrak{h}(\beta)\subseteq\bigcap_{i\in M}\mathfrak{H}_i^* \quad\text{und}\quad \mathfrak{h}(\beta)=\bigcap_{i\in N}\mathfrak{H}_i^* \quad\text{mit}\quad \emptyset\neq N\subseteq R,$$

letzteres unter Verwendung von (1) auf Grund der Folgerung nach Satz 15. Wegen $\bigcap_{i\in N}\mathfrak{H}_i^*\subseteq\bigcap_{i\in M}\mathfrak{H}_i^*$ gilt $N\supseteq M$ was sich durch indirekten Schluß unter Verwendung von $\mathfrak{P}(N)\neq\emptyset$ sofort zeigen läßt. Aus $\beta\notin\mathfrak{H}_j^*$ für $j\in R\setminus M$ folgt daher $N=M$ und somit auch $\bigcap_{i\in M}\mathfrak{H}_i^*=\mathfrak{h}(\beta)$.

Wir bezeichnen nun die Menge der echten Halbkörper $\mathfrak{H}$, die zu $\mathfrak{K}$ gehören und $\mathfrak{h}=\mathfrak{k}\cap H$ enthalten, mit $\mathscr{H}$ und zeigen:

Wenn kein $\mathfrak{H}'\in\mathscr{H}$ existiert mit $\mathfrak{H}'\subset\overline{\mathfrak{H}}$, so gilt $\overline{\mathfrak{H}}\subseteq\mathfrak{H}$ für alle $\mathfrak{H}\in\mathscr{H}$.

Es sei $\mathfrak{H}$ ein beliebiger Halbkörper aus $\mathscr{H}$. Wir betrachten den echten Halbkörper $\overline{\mathfrak{h}}=\mathfrak{H}\cap\overline{\mathfrak{H}}$, der natürlich $\mathfrak{h}$ umfaßt, und zeigen, daß $\overline{\mathfrak{h}}$ ein primitives Element γ von $\mathfrak{K}/\mathfrak{k}$ enthält, daß er also zu $\mathfrak{K}$ gehört. Wegen $D(\mathfrak{H})=\mathfrak{K}$ existiert nach Satz 2 ein primitives Element $\delta\in\mathfrak{H}$. Wählt man nun ein $a\in H_0$ mit $a>-d$, $d=\min_{i\in R}\varphi_i(\delta)$, dann ist $\gamma=\delta+a$ ein primitives Element mit $\gamma\in\overline{\mathfrak{H}}$. Wegen $\gamma\in\mathfrak{H}$ ist auch $\gamma\in\overline{\mathfrak{h}}$. Damit gilt $\overline{\mathfrak{h}}\in\mathscr{H}$ und $\overline{\mathfrak{h}}\subseteq\overline{\mathfrak{H}}$. Aus der Voraussetzung folgt dann $\overline{\mathfrak{h}}=\overline{\mathfrak{H}}$, also $\overline{\mathfrak{H}}\subseteq\mathfrak{H}$.

Es ist also insgesamt noch folgende Aussage zu beweisen: *Für jedes primitive Element $\beta\in\overline{\mathfrak{H}}$ von $\mathfrak{K}/\mathfrak{k}$ gilt*

$$\mathfrak{h}(\beta)=\overline{\mathfrak{H}}.$$

Beim Nachweis dieser Behauptung werden wir den Approximationssatz [17] verwenden, den wir für unsere Zwecke durch den folgenden Zusatz verschärfen:

APPROXIMATIONSSATZ. *Es seien $\psi_1,\ldots,\psi_m$ inäquivalente reelle Bewertungen eines Körpers $\mathfrak{K}$. Dann gibt es zu jedem System $\gamma_1,\ldots,\gamma_m$ von Elementen aus $\mathfrak{K}$ und jedem $\varepsilon\in H$ ein $\varrho\in\mathfrak{K}$, so daß*

$$\psi_i(\gamma_i-\varrho)<\varepsilon\quad(i=1,\ldots,m)$$

gilt.

ZUSATZ. *Wenn die γ_i Elemente eines Halbkörpers $\mathfrak{H}\subseteq\mathfrak{K}$ sind, der zu jedem geordneten Paar $[k,l]$, $1\leqq k,\ l\leqq m$, $k\neq l$, ein Element $\varkappa$ mit*

$$\psi_k(\varkappa)>1>\psi_l(\varkappa) \tag{2}$$

enthält, so kann $\varrho\in\mathfrak{H}$ gewählt werden.

Die Richtigkeit des Zusatzes ergibt sich unmittelbar aus dem Beweis des Approximationssatzes. Das Element ϱ entsteht nämlich als Linearkombination $\varrho=\delta_1\gamma_1+\ldots+\delta_m\gamma_m$ mit Elementen $\delta_i\in\mathfrak{K}$. Diese Elemente δ_i werden sukzessiv durch Addition, Multiplikation und Division aus Elementen $\varkappa$ mit (2) aufgebaut, so daß mit $\gamma_1,\ldots,\gamma_m,\varkappa\in\mathfrak{H}$ auch $\delta_i\in\mathfrak{H}$ und $\varrho\in\mathfrak{H}$ gilt.

[17] Vgl. etwa V. D. WAERDEN [6], Seite 3.

Wir zeigen nun als erstes:

I. *Zu jedem* $\omega \in \overline{\mathfrak{H}}$ *und jedem* $\delta \in H$ *existiert ein* $\varrho \in \mathfrak{h}(\beta)$, *so daß gilt:*

$$|\varphi_l(\omega) - \varphi_l(\varrho)| < \delta \qquad (l = 1, \ldots, n).$$

Dabei bedeuten $\varphi_1, \ldots, \varphi_n$ *die Isomorphismen von* $\mathfrak{K}/\mathfrak{k}$ *in* $K(i)$.

BEWEIS. Es genügt der Nachweis für $l = 1, \ldots, m$, wobei wir annehmen, daß $\{\varphi_1, \ldots, \varphi_r\}$ das System der reellen Isomorphismen und $\{\varphi_{r+1}, \ldots, \varphi_m\}$ $(n = r + 2k,$ $m = r + k)$ ein Halbsystem (nichtreeller) komplexer Isomorphismen von $\mathfrak{K}/\mathfrak{k}$ darstellen. Da $\omega, \beta \in \overline{\mathfrak{H}}$ gilt und β primitiv ist, erhält man

$$\varphi_i(\omega), \varphi_i(\beta) \in H \quad \text{für} \quad i = 1, \ldots, r,$$

$$\varphi_j(\omega) \in K(i), \quad \varphi_j(\beta) \in K(i) \setminus K \quad \text{für} \quad j = r+1, \ldots, m.$$

Die Halbkörper $\mathfrak{h}(\varphi_i(\beta))$ bzw. $\mathfrak{h}(\varphi_j(\beta))$ liegen dicht in H bzw. $K(i)$. Der erste Fall ist trivial, da H_0 dicht in H liegt. Der zweite Fall ergibt sich daraus, daß der Halbkörper $H(z)$ mit einem $z \in K(i) \setminus K$ nach WEINERT [9] (§ 9, Satz 8) ein Körper ist (das Minimalpolynom von z über K besitzt ja keine reelle Nullstelle). Es ist also $H(z) = K(i)$, so daß $H_0(z)$ dicht in $K(i)$ liegt. Wegen $\varphi_l(h(\beta)) = h(\varphi_l(\beta))$ $(l = 1, \ldots, m)$ für jedes Element $h(\beta) \in \mathfrak{h}(\beta)$, existiert damit zu dem System $\varphi_1(\omega), \ldots, \varphi_m(\omega)$ für jedes $\delta' \in H$ ein System $\gamma_1, \ldots, \gamma_m \in \mathfrak{h}(\beta)$ mit

$$|\varphi_l(\omega) - \varphi_l(\gamma_l)| < \delta'.$$

Auf $\gamma_1, \ldots, \gamma_m$ soll nun der Approximationssatz zusammen mit dem Zusatz angewandt werden. Dies ist aus folgenden Gründen möglich:

a) Durch die Festsetzung

$$\psi_l(\varkappa) = |\varphi_l(\varkappa)|, \quad \varkappa \in \mathfrak{K}; \quad l = 1, \ldots, m,$$

wird ein System von inäquivalenten reellen Bewertungen $\psi_1, \ldots, \psi_m$ von $\mathfrak{K}$ definiert. Die Inäquivalenz ergibt sich folgendermaßen: Wären ψ_1, ψ_2 äquivalent, so müßte $|\varphi_1(\varkappa)| = |\varphi_2(\varkappa)|^p$ mit einem festen $p \in H$ für alle $\varkappa \in \mathfrak{K}$ gelten.[18] Da alle Elemente von $\mathfrak{k}$ fest bleiben, kommt nur $p = 1$ in Frage. Es gibt aber Elemente $\eta \in \mathfrak{K}$ mit $|\varphi_1(\eta)| \neq |\varphi_2(\eta)|$. Also sind die ψ_i paarweise inäquivalent.

b) Der Halbkörper $\mathfrak{h}(\beta)$ erfüllt die im Zusatz über $\mathfrak{H}$ gemachten Voraussetzungen: Man überlegt sich leicht, daß es zu jedem geordneten Paar $[k, l]$, $1 \leqq k, l \leqq m$, $k \neq l$ ein Element $\varkappa' \in \mathfrak{h}(\beta)$ gibt mit $|\varphi_k(\varkappa')| \neq |\varphi_l(\varkappa')|$. Wir nehmen nun $|\varphi_k(\varkappa')| > |\varphi_l(\varkappa')|$ an (sonst Verwendung von $\varkappa'^{-1}$), so daß ein $a \in H_0$ existiert mit $|\varphi_k(\varkappa')| > a^{-1} > |\varphi_l(\varkappa')|$. Dann gilt für $\varkappa = a\varkappa'$ $|\varphi_k(\varkappa)| > 1 > |\varphi_l(\varkappa)|$ und (2) ist erfüllt.

Die Anwendung des Approximationssatzes ergibt, daß zu $\gamma_1, \ldots, \gamma_m \in \mathfrak{h}(\beta)$ für jedes $\delta'' \in H$ ein $\varrho \in \mathfrak{h}(\beta)$ existiert mit

$$\psi_l(\gamma_l - \varrho) < \delta'' \qquad (l = 1, \ldots, m).$$

[18] Vgl. etwa RÉDEI [3], Satz 398.

Wählt man nun $\delta' = \delta'' = \frac{\delta}{2}$ für vorgegebenes $\delta \in H$, so gilt für $l = 1, 2, \ldots, m$ wegen $\psi_l(\gamma_l - \varrho) = |\varphi_l(\gamma_l) - \varphi_l(\varrho)|$

$$|\varphi_l(\omega) - \varphi_l(\varrho)| = |\varphi_l(\omega) - \varphi_l(\gamma_l) + \varphi_l(\gamma_l) - \varphi_l(\varrho)| \leqq$$
$$\leqq |\varphi_l(\omega) - \varphi_l(\gamma_l)| + |\varphi_l(\gamma_l) - \varphi_l(\varrho)| < \delta' + \delta'',$$

also
$$|\varphi_l(\omega) - \varphi_l(\varrho)| < \delta.$$

Aus I folgt mit üblichen Schlüssen:

II. *Zu jeder Körperbasis* $\{\omega_1, \ldots, \omega_n\} \subseteqq \overline{\mathfrak{H}}$ *und jedem* $\varepsilon \in H$ *existiert eine Körperbasis* $\{\varrho_1, \ldots, \varrho_n\} \subseteqq \mathfrak{h}(\beta)$, *so daß für jedes Element* $\varkappa = \sum_{i=1}^{n} k_i \omega_i = \sum_{i=1}^{n} \bar{k}_i \varrho_i$ $(k_i \in \mathfrak{h}, \bar{k}_i \in \mathfrak{k})$ *stets* $|k_i - \bar{k}_i| < \varepsilon$ *erfüllt ist, für hinreichend kleines* ε *also* $\bar{k}_i \in \mathfrak{h}$ *gilt.*

Wir zeigen nun:

III. *Jedes Element* $\varkappa \in \overline{\mathfrak{H}}$, *für das es eine Körperbasis* $\{\omega_1, \ldots, \omega_n\} \subseteqq \overline{\mathfrak{H}}$ *gibt, so daß* $\varkappa = \sum_{k=1}^{n} q_k \omega_k$ *mit* $q_k \neq 0$ $(k = 1, \ldots, n;\ q_k \in \mathfrak{k})$ *gilt, ist in* $\mathfrak{h}(\beta)$ *enthalten.*

BEWEIS. Zunächst zeigen wir, daß unter der gemachten Voraussetzung auch eine Körperbasis $\{\sigma_1, \ldots, \sigma_n\} \subseteqq \overline{\mathfrak{H}}$ mit $\varkappa = \sum_{k=1}^{n} s_k \sigma_k$, $s_k \in \mathfrak{h}$, existiert. Das Element $\varkappa$ können wir in folgender Form annehmen:

$$\varkappa = c_1 \omega_1 + \ldots + c_{t-1} \omega_{t-1} - c_t \omega_t - \ldots - c_n \omega_n,$$

wobei $c_k \in \mathfrak{h}$ $(k = 1, \ldots, n)$ und $2 \leqq t \leqq n$ gilt ($t \neq 1$, da $\varkappa \in \overline{\mathfrak{H}}$; $t - 1 \neq n$, da sonst nichts mehr zu beweisen). Wegen $\varkappa \in \overline{\mathfrak{H}}$ ist für alle $i = 1, 2, \ldots, r$

$$\varphi_i(\varkappa) = \sum_{k=1}^{t-1} c_k \varphi_i(\omega_k) - \sum_{k=t}^{n} c_k \varphi_i(\omega_k) > 0.$$

Somit gilt

$$\sum_{k=1}^{t-1} c_k \varphi_i(\omega_k) > c_t \varphi_i(\omega_t) \quad (i = 1, \ldots, r). \tag{3}$$

Für jedes i zerlegen wir nun c_k $(k = 1, \ldots, t-1)$ folgendermaßen:

$$c_k = \varepsilon_{ki} + \lambda_{ki}, \quad \varepsilon_{ki} \in H_0, \quad \lambda_{ki} \geqq 0. \tag{4}$$

Es ist also

$$\sum_{k=1}^{t-1} c_k \varphi_i(\omega_k) = \sum_{k=1}^{t-1} \varepsilon_{ki} \varphi_i(\omega_k) + \sum_{k=1}^{t-1} \lambda_{ki} \varphi_i(\omega_k).$$

Für hinreichend kleine ε_{ki} gilt dann wegen (3) auch

$$\sum_{k=1}^{t-1} \lambda_{ki} \varphi_i(\omega_k) > c_t \varphi_i(\omega_t) \quad (i = 1, \ldots, r). \tag{5}$$

Es sei nun $L_k = \max_{i=1,\ldots,r} \lambda_{ki}$. Dann folgt aus (4) und (5) $c_k > L_k$ $(k=1, \ldots, t-1)$ bzw.

$$\sum_{k=1}^{t-1} L_k \varphi_i(\omega_k) > c_t \varphi_i(\omega_t) \quad (i = 1, \ldots, r). \tag{6}$$

Wegen (4) ist jedes λ_{ki} und damit jedes L_k Element von $\mathfrak{k}$. Also folgt aus $c_k > L_k$, daß $c_k^* = c_k - L_k \in \mathfrak{h}$ gilt. Das Bild $\varphi_i(\varkappa)$ von $\varkappa$ hat also die Darstellung

$$\varphi_i(\varkappa) = \sum_{k=1}^{t-1} c_k^* \varphi_i(\omega_k) + \left[\sum_{k=1}^{t-1} L_k \varphi_i(\omega_k) - c_t \varphi_i(\omega_t)\right] - \sum_{k=t+1}^{n} c_k \varphi_i(\omega_k) =$$

$$= \varphi_i\left(\sum_{k=1}^{t-1} c_k^* \omega_k + \left[\sum_{k=1}^{t-1} L_k \omega_k - c_t \omega_t\right] - \sum_{k=t+1}^{n} c_k \omega_k\right).$$

Wegen $c_t \neq 0$ ist das System $\{\overline{\omega}_1, \ldots, \overline{\omega}_n\} \subseteq \mathfrak{R}$ mit $\overline{\omega}_k = \omega_k$ $(k \neq t)$, $\overline{\omega}_t = \sum_{k=1}^{t-1} L_k \omega_k - c_t \omega_t$ wieder eine Basis von $\mathfrak{R}/\mathfrak{k}$. Außerdem gilt $\{\overline{\omega}_1, \ldots, \overline{\omega}_n\} \subseteq \overline{\mathfrak{H}}$, da $\overline{\omega}_k = \omega_k \in \overline{\mathfrak{H}}$ $(k \neq t)$ und $\varphi_i(\overline{\omega}_t) > 0$ $(i=1, \ldots, r)$ wegen (6) erfüllt ist. In der sich daraus ergebenden Darstellung $\varkappa = \sum_{k=1}^{n} \bar{q}_k \overline{\omega}_k$ von $\varkappa$ sind dann t Koeffizienten positiv.

Bei Wiederholung dieses Verfahrens erhält man nach endlich vielen Schritten eine Darstellung von $\varkappa$ mit nur positiven Koeffizienten.

Aus II folgt dann sofort, daß $\varkappa \in \mathfrak{h}(\beta)$ gilt.

Abschließend zeigen wir nun:

IV. *Für ein beliebiges Element* $\varkappa \in \overline{\mathfrak{H}}$ *gilt* $\varkappa \in \mathfrak{h}(\beta)$.

BEWEIS. Für eine beliebige Körperbasis $\{\omega_1, \ldots, \omega_n\} \subseteq \overline{\mathfrak{H}}$ gelte

$$\varkappa = \sum_{k=1}^{n} c_k \omega_k, \quad c_k \in \mathfrak{k}.$$

Wir betrachten das Element $\gamma = \omega_1 + \ldots + \omega_n \in \overline{\mathfrak{H}}$. Wegen III gilt $\gamma \in \mathfrak{h}(\beta)$. Für ein Element $p \in H_0$ mit $p \neq c_k$ $(k=1, \ldots, n)$ und $p < m = \min_{i=1,\ldots,r} \varphi_i(\varkappa\gamma^{-1})$ $(m>0$ wegen $\varkappa\gamma^{-1} \in \overline{\mathfrak{H}})$ hat das Element

$$\delta = \varkappa - p\gamma = (c_1 - p)\omega_1 + \ldots + (c_n - p)\omega_n$$

folgende Eigenschaften:

$$\delta = \sum_{k=1}^{n} d_k \omega_k, \qquad 0 \neq d_k \in \mathfrak{k} \quad (k = 1, \ldots, n),$$

$$\varphi_i(\delta) > 0 \quad \text{für} \quad i = 1, \ldots, r.$$

Wegen III gilt also $\delta \in \mathfrak{h}(\beta)$ und somit auch

$$\delta + p\gamma = \varkappa \in \mathfrak{h}(\beta).$$

Damit ist Satz 16 bewiesen.

FOLGERUNG. *Ein algebraischer Erweiterungshalbkörper* $\mathfrak{H}$ *eines archimedischen Positivbereiches* $\mathfrak{h}$ *ist genau dann eine einfache Halbkörpererweiterung* $\mathfrak{h}(\alpha)$ *von* $\mathfrak{h}$,

wenn der Differenzenkörper $\mathfrak{K} = D(\mathfrak{H})$ *von* $\mathfrak{H}$ *eine einfache Erweiterung des Körpers* $\mathfrak{k} = D(\mathfrak{h})$ *ist.*

BEWEIS. Wenn $\mathfrak{K}$ keine einfache Erweiterung von $\mathfrak{k}$ ist, kann natürlich auch $\mathfrak{H} = \mathfrak{h}(\alpha)$ nicht eintreten, da sonst $\mathfrak{K}$ der kleinste Körper ist, der $\mathfrak{k}$ und α enthält, also $\mathfrak{K} = \mathfrak{k}(\alpha)$ erfüllt. Für den Fall, daß $\mathfrak{K}$ eine einfache Erweiterung von $\mathfrak{k}$ ist, können wir Satz 16 heranziehen, wobei wir uns nur noch zusätzlich überlegen müssen, daß sich auch $\mathfrak{K}$ selbst als einfache Halbkörpererweiterung von $\mathfrak{h}$ gewinnen läßt. Dazu genügt es nach Satz 6 zu zeigen, daß es ein primitives Element β von $\mathfrak{K}/\mathfrak{k}$ gibt, das bei keiner Ordnung von $\mathfrak{K}$, welche Fortsetzung der Ordnung $\mathfrak{O}_{\mathfrak{h}}$ von $\mathfrak{k}$ ist, positiv wird. Dies ist trivial, wenn keine solche Ordnung von $\mathfrak{K}$ existiert. Im anderen Falle gibt es nach Satz 16 ein primitives Element γ von $\mathfrak{K}/\mathfrak{k}$ mit $\gamma \in \mathfrak{H}_i^*$ für alle $i \in R$, so daß also $\beta = -\gamma$ ein primitives Element von $\mathfrak{K}/\mathfrak{k}$ mit der gewünschten Eigenschaft ist.

Bevor wir nun zu dem Fall übergehen, daß $\mathfrak{K}/\mathfrak{k}$ eine beliebige algebraische Erweiterung ist, wollen wir erst noch die Frage nach der Isomorphie der in Satz 16 betrachteten echten Halbkörper $\mathfrak{H}$ klären.

SATZ 17. *Der Körper* $\mathfrak{K} = \mathfrak{k}(\alpha)$ *sei eine einfache algebraische Erweiterung eines Körpers* $\mathfrak{k}$. *Zwei zu* $\mathfrak{K}$ *gehörende echte Halbkörper* $\mathfrak{H}_1, \mathfrak{H}_2$, *die den gleichen archimedischen Positivbereich* $\mathfrak{h}$ *mit* $D(\mathfrak{h}) = \mathfrak{k}$ *enthalten, sind genau dann bzgl.* $\mathfrak{h}$ *relativ isomorph, wenn zwei Elemente* α_1, α_2 *mit gleichem Minimalpolynom über* $\mathfrak{k}$ *existieren, so daß* $\mathfrak{H}_1 = \mathfrak{h}(\alpha_1)$, $\mathfrak{H}_2 = \mathfrak{h}(\alpha_2)$ *gilt.*

BEWEIS. Wir nehmen zunächst für $\mathfrak{H}_i = \mathfrak{h}(\alpha_i)$ [19] an, daß α_1, α_2 das gleiche Minimalpolynom über $\mathfrak{k}$ besitzen. Da α_1, α_2 primitive Elemente von $\mathfrak{K}/\mathfrak{k}$ sind, existiert ein bzgl. $\mathfrak{k}$ relativer Automorphismus φ von $\mathfrak{K}$ mit $\varphi(\alpha_1) = \alpha_2$. Dabei wird $\mathfrak{H}_1$ isomorph auf $\mathfrak{H}_2$ abgebildet, so daß $\mathfrak{h}$ fest bleibt. Sei nun umgekehrt ψ ein Isomorphismus von $\mathfrak{H}_1 = \mathfrak{h}(\alpha_1)$ auf $\mathfrak{H}_2$, bei dem $\mathfrak{h}$ fest bleibt. Wir setzen $\psi(\alpha_1) = \alpha_2 \in \mathfrak{H}_2$. Es ist dann $\mathfrak{H}_2 = \mathfrak{h}(\alpha_2)$ und ψ ein Isomorphismus von $\mathfrak{h}(\alpha_1)$ auf $\mathfrak{h}(\alpha_2)$. Da $\mathfrak{K}$ der Differenzenkörper dieser beiden Halbkörper ist, läßt sich ψ zu einem bzgl. $\mathfrak{k}$ relativen Automorphismus Ψ des Körpers $\mathfrak{K}$ fortsetzen. Wegen $\Psi(\alpha_1) = \alpha_2$ besitzen α_1, α_2 dasselbe Minimalpolynom über $\mathfrak{k}$.

FOLGERUNG. *Zwei zu* $\mathfrak{K} = \mathfrak{k}(\alpha)$ *gehörende echte Halbkörper* $\mathfrak{H}_1, \mathfrak{H}_2$, *die den gleichen archimedischen Positivbereich* $\mathfrak{h}$ *von* $\mathfrak{k}$ *enthalten, für die also nach Satz* 16 $\mathfrak{H}_1 = \bigcap_{i \in M_1} \mathfrak{H}_i^*$, $\mathfrak{H}_2 = \bigcap_{i \in M_2} \mathfrak{H}_i^*$ $(\mathfrak{h} \subseteq \mathfrak{H}_i^*)$ *gilt, sind sicher dann nicht relativ isomorph bzgl.* $\mathfrak{h}$, *wenn* M_1 *und* M_2 *nicht die gleiche Anzahl von Elementen besitzen.*

BEWEIS. Wir schließen indirekt und nehmen an, daß M_1 und M_2 nicht die gleiche Anzahl von Elementen enthalten, aber $\mathfrak{H}_1$ und $\mathfrak{H}_2$ bzgl. $\mathfrak{h}$ relativ isomorph sind. Nach Satz 16 bzw. Satz 17 existiert dann eine Darstellung $\mathfrak{H}_1 = \mathfrak{h}(\beta_1)$, $\mathfrak{H}_2 = \mathfrak{h}(\beta_2)$, wobei β_1 und β_2 das gleiche Minimalpolynom über $\mathfrak{k}$ besitzen. Wir betrachten nun den zu Beginn des Beweises von Satz 16 angegebenen Isomorphismus Φ von $\mathfrak{K} = \mathfrak{k}(\alpha)$ auf $\mathfrak{K}' = \mathfrak{k}'(\alpha')$ mit $\Phi(\mathfrak{k}) = \mathfrak{k}' \subseteq K$ und $\Phi(\mathfrak{h}) = \mathfrak{h}' = \mathfrak{k}' \cap H$. Die Elemente $\beta_1' = \Phi(\beta_1)$ und $\beta_2' = \Phi(\beta_2)$ haben über $\mathfrak{k}'$ ebenfalls das gleiche Minimalpolynom. Gemäß Satz 4 ist die Anzahl der positiven reellen Konjugierten von β_1' bzw. β_2' bzgl. $\mathfrak{k}'$ gleich

[19] Nach Satz 16 sind die betrachteten Halbkörper $\mathfrak{H}_i$ stets einfache Erweiterungen von $\mathfrak{h}$.

der Anzahl der Elemente von M_1 bzw. M_2. Die Gleichheit der Minimalpolynome von β_1' und β_2' über $\mathfrak{k}'$ steht dann im Widerspruch zur Verschiedenheit der Elementanzahlen von M_1 und M_2.

Wir wenden uns nun dem angekündigten allgemeinen Fall zu. Dabei wird sich u.a. zeigen, daß sämtliche absolut algebraischen Halbkörper völlig bestimmt sind durch die Ordnungen der absolut algebraischen Körper. Sie sind nämlich gerade die Positivreiche dieser Ordnungen und deren Durchschnitte.

Wir werden aber sogleich folgenden, wesentlich allgemeineren Satz beweisen:

SATZ 18. *Jeder echte Halbkörper* $\mathfrak{H}$, *der eine algebraische Erweiterung eines archimedischen Positivbereiches* $\mathfrak{h}$ *ist, erfüllt entweder selbst die Eigenschaft* $(*)$ *oder ist der Durchschnitt von echten Halbkörpern mit* $(*)$.

BEWEIS. Bekanntlich definiert jeder echte Halbkörper $\mathfrak{H}$ in dem Körper $\mathfrak{K}=Q(D(\mathfrak{H}))$ eine Halbordnung.[20] Nun ist der Positivbereich $\mathfrak{H}$ dieser Halbordnung genau dann Durchschnitt von Positivbereichen gewisser Ordnungen in $\mathfrak{K}$, wenn $\mathfrak{H}$ alle Quadrate von Elementen $\varkappa\neq 0$ aus $\mathfrak{K}$ enthält.[21] Wir zeigen nun, daß für jedes $\varkappa\neq 0$ aus $\mathfrak{K}$ gilt: $\varkappa^2\in\mathfrak{H}$. Nach Voraussetzung ist $\mathfrak{K}$ algebraisch über $\mathfrak{k}=D(\mathfrak{h})$. Daher gilt $\mathfrak{K}=D(\mathfrak{H})$, und es gibt zu einem beliebigen $\varkappa\neq 0$ aus $\mathfrak{K}$ Elemente $h_1, h_2\in\mathfrak{H}$ mit $\varkappa=h_1-h_2$. Der Körper $\bar{\mathfrak{k}}=\mathfrak{k}(h_1, h_2)$ ist dann eine einfache Erweiterung $\bar{\mathfrak{k}}=\mathfrak{k}(\alpha)$ von $\mathfrak{k}$, und es gilt $\varkappa\in\bar{\mathfrak{k}}$. Der Halbkörper $\bar{\mathfrak{h}}=\bar{\mathfrak{k}}\cap\mathfrak{H}$ enthält h_1, h_2, so daß außer $\mathfrak{h}\subseteqq\bar{\mathfrak{h}}$ auch $D(\bar{\mathfrak{h}})=\bar{\mathfrak{k}}$ gilt. Nach Satz 16 ist $\bar{\mathfrak{h}}$ der Durchschnitt der Positivbereiche gewisser Ordnungen von $\bar{\mathfrak{k}}$. Damit enthält $\bar{\mathfrak{h}}$ die Quadrate aller Elemente $\bar{\varkappa}\neq 0$ von $\bar{\mathfrak{k}}$. Wegen $\varkappa\in\bar{\mathfrak{k}}$ gilt also $\varkappa^2\in\bar{\mathfrak{h}}\subseteqq\mathfrak{H}$. Damit ist Satz 18 bewiesen.

FOLGERUNG. *Es sei* $\mathfrak{H}$ *ein echter Halbkörper, der eine algebraische Erweiterung eines archimedischen Positivbereiches darstellt. Dann ist die durch* $\mathfrak{H}$ *definierte Halbordnungsrelation* $R\subseteqq\mathfrak{K}\times\mathfrak{K}$ *in* $\mathfrak{K}=D(\mathfrak{H})$ *Durchschnitt von Ordnungsrelationen* $R_i\subseteqq\mathfrak{K}\times\mathfrak{K}$.

§ 5. Formal-reelle Körper und Körper, die von einem echten Halbkörper erzeugt werden können

Da jeder formal-reelle Körper von einem echten Halbkörper erzeugt werden kann, erhebt sich die Frage nach der Umkehrbarkeit dieser Aussage. Für den absolut algebraischen Fall ist sie in der Tat gewährleistet, wie sich aus der Folgerung nach Satz 10 ergibt. In voller Allgemeinheit konnte jedoch das Problem nicht geklärt werden. Wir werden es lediglich auf eine einfachere Fragestellung reduzieren.

SATZ 19. *Wenn jeder einfache algebraische Erweiterungskörper* $\mathfrak{k}(\alpha)$ *eines über* K_0 *rein transzendenten Körpers* $\mathfrak{k}=K_0(\mathfrak{S})$ [22] *mit einer endlichen Menge* $\mathfrak{S}$, *welcher von einem echten Halbkörper* $\mathfrak{H}\supseteqq H_0(\mathfrak{S})$ *erzeugt werden kann, formal-reell ist, so ist jeder von einem echten Halbkörper erzeugbare Körper formal-reell.*

[20] Allerdings läßt sich nicht jede Halbordnung auf diese Weise definieren, wie man durch Beispiel zeigen kann.

[21] Vgl. etwa FUCHS [1], Corollary 21, Seite 124.

[22] D. h., $\mathfrak{S}$ ist eine über K_0 algebraisch unabhängige Menge.

BEWEIS. Wir schließen indirekt und nehmen (unter den über die Körper $\mathfrak{k}(\alpha)$ mit $\mathfrak{k}=K_0(\mathfrak{S})$ gemachten Voraussetzungen) an, daß es einen echten Halbkörper $\mathfrak{H}$ gibt, für den der Körper $Q(D(\mathfrak{H}))=\mathfrak{K}$ nicht formal-reell ist. Dann gibt es Elemente $a_i\in\mathfrak{K}$, $a_i\neq 0$, mit $a_1^2+a_2^2+\ldots+a_n^2=0$. Außerdem existiert stets eine über K_0 algebraisch unabhängige Teilmenge $\overline{\mathfrak{S}}\subseteq\mathfrak{H}$, so daß $\mathfrak{K}$ algebraisch über $\overline{\mathfrak{k}}=K_0(\overline{\mathfrak{S}})$ ist. Jedes der Elemente $a_i\in\mathfrak{K}$ ist dabei schon von einer endlichen Teilmenge $\mathfrak{S}_i$ von $\overline{\mathfrak{S}}$ algebraisch abhängig (über K_0), so daß das System $\{a_1, \ldots, a_n\}$ algebraisch ist über dem Körper $\mathfrak{k}=K_0(\mathfrak{S})$ mit der endlichen Menge $\mathfrak{S}=\bigcup_{i=1}^{n}\mathfrak{S}_i$. Wir betrachten nun den algebraischen Erweiterungskörper $\overline{\mathfrak{K}}=\mathfrak{k}(a_1, a_2, \ldots, a_n)\subseteq\mathfrak{K}$, der sich auch als einfache Erweiterung $\overline{\mathfrak{K}}=\mathfrak{k}(\beta)$ gewinnen läßt. Wegen $\mathfrak{K}=Q(D(\mathfrak{H}))$ gilt dabei $\beta=(h_1-h_2)(h_3-h_4)^{-1}$ mit Elementen $h_i\in\mathfrak{H}$, also $\overline{\mathfrak{K}}\subseteq\mathfrak{k}(h_1, \ldots, h_4)=\mathfrak{k}(\alpha)$ mit einem Element $\alpha\in\mathfrak{K}$. Der Halbkörper $\mathfrak{H}'=H_0(\mathfrak{S})(h_1, \ldots, h_4)\subseteq\mathfrak{k}(\alpha)$ ist wegen $\mathfrak{H}'\subseteq\mathfrak{H}$ ein echter Halbkörper, der außerdem $Q(D(\mathfrak{H}'))=\mathfrak{k}(\alpha)$ erfüllt. Somit ist nach Voraussetzung $\mathfrak{k}(\alpha)$ formal-reell. Wegen $\overline{\mathfrak{K}}\subseteq\mathfrak{k}(\alpha)$ gilt aber $a_1, a_2, \ldots, a_n\in\mathfrak{k}(\alpha)$ mit $a_1^2+a_2^2+\ldots+a_n^2=0$ und $a_i\neq 0$. Damit sind wir zu einem Widerspruch gelangt.

Da jeder in Satz 19 genannte Körper $\mathfrak{k}(\alpha)$ durch ein Minimalpolynom über $\mathfrak{k}$ definiert werden kann, erhebt sich natürlich die Frage, in welcher Weise sich die Tatsache, daß $\mathfrak{k}(\alpha)$ von einem echten Halbkörper erzeugbar ist, in den Eigenschaften eines definierenden Polynoms widerspiegelt. Dazu beweisen wir folgenden Satz:

SATZ 20. *Ein einfacher algebraischer Erweiterungskörper $\mathfrak{K}=\mathfrak{k}(\alpha)$ eines über K_0 rein transzendenten Körpers $\mathfrak{k}=K_0(\mathfrak{S})$ ist genau dann von einem echten Halbkörper $\mathfrak{H}\supseteq H_0(\mathfrak{S})$ erzeugbar, wenn es ein definierendes Polynom $f(x)\in\mathfrak{k}[x]$ von $\mathfrak{K}/\mathfrak{k}$ gibt, so daß für jedes von Null verschiedene Polynom $a_0+a_1x+\ldots+a_nx^n\in(f(x))$ mindestens ein a_i nicht in $H_0(\mathfrak{S})\cup\{0\}$ enthalten ist.*

BEWEIS. Wir verwenden den Satz, daß ein einfacher Erweiterungshalbkörper $\mathfrak{H}=\mathfrak{h}(\alpha)$ eines echten Halbkörpers $\mathfrak{h}$ genau dann ein Körper ist, wenn α Nullstelle eines Polynoms mit Koeffizienten aus $\mathfrak{h}\cup\{0\}$ ist (vgl. WEINERT [9], § 9, Satz 3). Als erstes nehmen wir an, daß $\mathfrak{K}=\mathfrak{k}(\alpha)$ von einem echten Halbkörper $\mathfrak{H}\supseteq H_0(\mathfrak{S})$ erzeugt werden kann: $\mathfrak{k}(\alpha)=Q(D(\mathfrak{H}))$. Nach Satz 2 enthält dann $\mathfrak{H}$ ein primitives Element β von $\mathfrak{K}/\mathfrak{k}$. Das Minimalpolynom von β sei $f(x)\in\mathfrak{k}[x]$. Da $H_0(\mathfrak{S})(\beta)$ ein echter Halbkörper ist, gibt es nach dem oben genannten Satz kein Polynom $g(x)$ mit $0\neq g(x)\in\mathfrak{k}[x]$ und $g(\beta)=0$, das nur Koeffizienten aus $H_0(\mathfrak{S})\cup\{0\}$ besitzt. Die Gesamtheit aller Polynome $g(x)$ mit $g(\beta)=0$ ist aber gerade das Hauptideal $(f(x))$. Es sei nun $f(x)\in\mathfrak{k}[x]$ ein definierendes Polynom von $\mathfrak{K}/\mathfrak{k}$, wofür das Ideal $(f(x))$ die angegebenen Eigenschaften hat. Dann gibt es ein Element γ mit $\mathfrak{K}=\mathfrak{k}(\gamma)$ und $f(\gamma)=0$. Da nach Voraussetzung für kein Polynom $g(x)\in\mathfrak{k}[x]$ mit $g(\gamma)=0$ sämtliche Koeffizienten in $H_0(\mathfrak{S})\cup\{0\}$ liegen, ist $\mathfrak{H}=H_0(\mathfrak{S})(\gamma)$ ein echter Halbkörper, für den außerdem $Q(D(\mathfrak{H}))=\mathfrak{K}$ gilt.

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Literaturverzeichnis

[1] L. FUCHS, *Partially ordered algebraic systems* (Oxford—London—New York—Paris, 1963).
[2] H. KOCH, Über Halbkörper, die in algebraischen Zahlkörpern enthalten sind, *Acta Math. Acad. Sci. Hung.*, **15** (1964), S. 439—444.
[3] L. RÉDEI, *Algebra*, 1. Teil, deutsche Ausgabe (Leipzig, 1959).
[4] O. STEINFELD, Über Semiringe mit multiplikativer Kürzungsregel, *Acta Sci. Math. Szeged*, **24** (1963), S. 190—195.
[5] B. L. V. D. WAERDEN, *Moderne Algebra I* (3. Auflage) (Berlin—Göttingen—Heidelberg, 1950).
[6] B. L. V. D. WAERDEN, *Algebra II* (4. Auflage) (Berlin—Göttingen—Heidelberg, 1959).
[7] H. J. WEINERT, Über Halbringe und Halbkörper. I, *Acta Math. Acad. Sci. Hung.*, **13** (1962), S. 365—378.
[8] H. J. WEINERT, Über Halbringe und Halbkörper. II, *Acta Math. Acad. Sci. Hung.*, **14** (1963), S. 209—227.
[9] H. J. WEINERT, Über Halbringe und Halbkörper. III, *Acta Math. Acad. Sci. Hung.*, **15** (1964), S. 177—194.
[10] H. J. WEINERT, Ein Struktursatz für idempotente Halbkörper, *Acta Math. Acad. Sci. Hung.*, **15** (1964), S. 289—295.
[11] H. J. WEINERT, Unterhalbkörper quadratischer Zahlkörper, *Wiss. Zeitschr. d. Päd. Hochsch. Potsdam*, **8** Heft 1/2 (1964).
[12] R. WIEGANDT, Über die Struktursätze der Halbringe, *Ann. Univ. Sci. Budapest*, **5** (1962), S. 51—68.

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Tomus 19 (1—2), (1968), pp. 47—52.

ON TOPOLOGICAL SPACES ASSOCIATED WITH DIGRAPHS*

By
T. N. BHARGAVA and T. J. AHLBORN (Kent, Ohio, U.S.A.)
(*Presented by* A. RÉNYI)

Let A be a set of points, $A=\{i: i\in I\}$, where I is an index set. A *directed graph* (or simply digraph) consists of a set A, and a subset E of the cartesian product $A\times A$, $\Phi\subseteqq E\subseteqq A\times A$, and is denoted by $\Gamma(A, E)$ (or simply $\Gamma(A)$, if the subset E is fixed). $\Gamma(\mathscr{A}, E\cap\mathscr{A}\times\mathscr{A})$ is said to be a *subdigraph* of digraph $\Gamma(A, E)$ if $\mathscr{A}\subseteqq A$. An ordered pair (i, j) is said to be a *directed edge* from i to j, and an L-tuple $((i, i_1), (i_1, i_2), \ldots, (i_{L-1}, j))$ is said to be a *directed path* of length L from i to j; points in A are generally called *vertices* of the digraph $\Gamma(A, E)$. Point j is said to be *accessible* from point i, if there exists a directed path from i to j, and is denoted by $\mathbf{A}(i, j)$. A digraph $\Gamma(A, E)$ is said to be (i) strongly connected if $\mathbf{A}(i, j)$ holds for every $i, j\in A$, (ii) unilaterally connected if for every $i, j\in A$ either $\mathbf{A}(i, j)$, or $\mathbf{A}(j, i,)$ or both, (iii) weakly connected if $\Gamma(A, E\cup E')$, where $E'=\{(j, i): (i, j)\in E\}$, is strongly connected, and (iv) disconnected if it is not even weakly connected. For details on graph theory, we refer to [3], [4] and [5]. Definitions and results from topology are taken from [6].

1. Topological space $(A, \mathbf{T}_E)$

DEFINITION 1. 1. Let $\Gamma(\mathscr{A}, E\cap\mathscr{A}\times\mathscr{A})$ be a subdigraph of the digraph $\Gamma(A, E)$, $\mathscr{A}\subseteqq A$. The subset $\mathscr{A}$ is said to define an open set if for every pair of points (i, j) with j in $\mathscr{A}$ and i not in $\mathscr{A}$, $(i, j)\notin E$.

With the above definition of an open set, the following theorem is obvious:

THEOREM 1. 1. *Each digraph $\Gamma(A, E)$ determines a unique topological space $(A, \mathbf{T}_E)$ where $\mathbf{T}_E=\{\mathscr{A}: \mathscr{A}\in 2^A, \mathscr{A}$ open$\}$; and the topology $\mathbf{T}_E$ has the property of completely additive closure (that is, the intersection of any number of open sets is open).*

LEMMA 1. 2. *Let $(i, j)\in A\times A$. A necessary and sufficient condition that the point j be accessible from point i, is that for all subsets $\mathscr{A}\in 2^A$ which contain i but not j there exists $k, l\in A$ such that $(k, l)\in\{\mathscr{A}\times(A\sim\mathscr{A})\}\cap E$.*

PROOF. For $i=j$, the lemma is vacuously true. For $i\neq j$, first we assume $\mathbf{A}(i, j)$, and let $\mathscr{A}$ be such that $i\in\mathscr{A}, j\notin\mathscr{A}$. Then there exists an ordered L-tuple $((i=i_0, i_1), (i_1, i_2), \ldots, (i_{L-1}, i_L=j))$, $1\leqq L<\infty$, such that each of its members belongs to E, and there also exists a K, $1\leqq K\leqq L$, such that $i_1, i_2, \ldots, i_{K-1}\in\mathscr{A}$, but $i_K\notin\mathscr{A}$. Thus $(i_{K-1}, i_K)\in\mathscr{A}\times(A\sim\mathscr{A})$.

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Next we assume that there exists $(k, l) \in \{\mathscr{A} \times (A \sim \mathscr{A})\} \cap E$. Let $\mathscr{A}_i = \{h: \mathbf{A}(i, h), h \in A\}$, and $j \notin \mathscr{A}_i$. Then there exists $(k, l) \in \{\mathscr{A}_i \times (A \sim \mathscr{A}_i)\} \cap E$. But $k \in \mathscr{A}_i$ implies $\mathbf{A}(i, k)$, which in turn implies $\mathbf{A}(i, l)$. Thus $l \in \mathscr{A}_i$ which is a contradiction, and hence $j \in \mathscr{A}_i$.

THEOREM 1. 3. *Let $(i, j) \in A \times A$. A necessary and sufficient condition for point j to be accessible from point i is that each closed set conntaining i also contains j (equivalently, each open set containing j also contains i).*

PROOF. Follows directly from Lemma 1. 2.

THEOREM 1. 4. *The topological space $(A, \mathbf{T}_E)$ satisfies* (a) *the separation axiom T_0 iff no subdigraph of $\Gamma(A, E)$ is strongly connected, and* (b) *the seppration axiom T_1 iff the set $\{(i, j): (i, j) \in E, i \neq j\}$ is empty.*

PROOF. (a) Let (i, j) be an arbitrary element in $A \times A$, and $i \neq j$. Assume $\mathbf{A}(i, j)$. Then by Theorem 1. 3 for every open set $\mathscr{A}$ such that $j \in \mathscr{A}$, i also $\in \mathscr{A}$; and therefore $(A, \mathbf{T}_E)$ cannot be a T_0-space. The other side is an immediate consequence of the definition of a T_0-space.

(b) Assume that the set $\{(i, j); (i, j) \in E, i \neq j\}$ is an empty set, that is for $i \neq j$, $(i, j) \notin E$. Each single-element set (consisting of a single element only) in 2^A is closed, and $(A, \mathbf{T}_E)$ is a T_1-space. The other side follows directly from the definition of a T_1-space.

COROLLARY 1. 5. *If $(A, \mathbf{T}_E)$ satisfies the axiom T_1, then it is a discrete topological space.*

PROOF. Theorem 1. 4 (b), and the property of completely additive closure (Theorem 1. 1).

2. Compactness

Let $O(i) = \bigcap_{\alpha} \{O_\alpha: i \in O_\alpha, O_\alpha \text{ open}\}$, be the minimal open set containing i.

DEFINITION 2. 1. Given $i, j \in A$, we say that $i \geqq j$ if $O(i) \supseteq O(j)$.

THEOREM 2. 1. *For the topological space $(A, \mathbf{T}_E)$, the set A is compact iff there exist a finite number of vertices $i_1, i_2, \ldots, i_k$ in A such that $A = \bigcup_{j=1}^{k} O(i_j)$.*

PROOF. Suppose A is compact, and let $\{O(i): i \in A\}$ be an open cover of A. By definition of compactness there must exist a finite open cover of A.

Next, we note that if A is covered by a finite number of open sets $O(i)$, then each open covering of A must have open sets in it containing each of $O(i)$; and hence a finite subcovering.

LEMMA 2. 2. *Suppose $i_1 \geqq i_2 \geqq \ldots$ is an infinite sequence of distinct vertices with $O(i_1) \subseteq O(i_2) \subseteq \ldots$. If there exist i and j, $i \neq j$ such that $\bigcup_k O(i_k) \subseteq O(i) \cup O(j)$, then $\bigcup_k O(i_k) \subseteq O(i)$ or $\bigcup_k O(i_k) \subseteq O(j)$ (or both).*

PROOF. If $O(i_k) \subseteqq O(i)$, then we have $\mathbf{A}(i_k, i)$ i.e. there is a path from i_k to i (Theorem 1. 3). Now consider

$$U_i = \{O(i_k) : O(i_k) \subseteqq O(i)\},$$

and

$$U_j = \{O(i_k) : O(i_k) \subseteqq O(j)\}.$$

At least one of U_i and U_j has an infinite number of sets, and the conclusion follows.

THEOREM 2. 3. *Let $(A, \mathbf{T}_E)$ be compact. For any sequence $\{i_k\} \subseteqq A$ (the number of distinct points infinite or finite) there exists $i \in A$ such that $i \geqq i_k$ for all k.*

PROOF. There exist $i_{\alpha_1}, i_{\alpha_2}, \ldots, i_{\alpha_n}$ by Theorem 2. 1 such that $A = \bigcup_{j=1}^{n} O(i_{\alpha_j})$. Since $\bigcup_k O(i_k) \subseteqq A$, there exists an index α_j, $j = 1, 2, \ldots, n$, such that $\bigcup_k O(i_k) \subseteqq O(i_{\alpha_j})$, by Lemma 2. 2.

Finally, since $\geqq$ is a partial ordering on A, there exist maximal elements (Zorn's lemma) in A. Using the results in this section we immediately obtain the following theorem, which gives the uniqueness property for the upper bound in the case when the digraph topology $(A, \mathbf{T}_E)$ is compact.

THEOREM 2. 4. *Let $(A, \mathbf{T}_E)$ be compact. If there exist points $j \neq k$ in A such that $j \geqq i$, $k \geqq i$ for every i in A, then $O(k) = O(j)$. Furthermore the union of the minimal open sets of the maximal elements is A.*

3. Connectedness

We present now a result showing that the graph-theoretic property of connectedness (of a digraph) is, in a certain sense, consistent with the notion of topological connectedness.

THEOREM 3. 1. *The digraph $\Gamma(A, E)$ is*

(i) *strongly connected iff $(A, \mathbf{T}_E)$ is an indiscrete topological space,*
(ii) *unilaterally connected iff $\mathbf{T}_E$ is linearly ordered by the relation "inclusion of sets",*
(iii) *weakly connected iff $(A, \mathbf{T}_E)$ is topologically connected, and*
(iv) *disconnected iff $(A, \mathbf{T}_E)$ is not topologically connected.*

PROOF. (i) By definition, if $\Gamma(A, E)$ is strongly connected, then we have $\mathbf{A}(i, j)$ for every $i, j \in A$. Thus, by Theorem 1. 3, each open set in $\mathbf{T}_E$ containing j also contains i. Hence, the only non-void open set in $\mathbf{T}_E$ is the whole set A, i. e., $\mathbf{T}_E = \{\Phi, A\}$. On the other hand, if $\mathbf{T}_E = \{\Phi, A\}$, and i, j two arbitrary points in A, then obviously each open set in $\mathbf{T}_E$ containing j contains i, so that we have $\mathbf{A}(i, j)$ for every $i, j \in A$.

(ii) $\mathbf{T}_E$ is linearly ordered by inclusion iff for $O_1, O_2 \in \mathbf{T}_E$, either $O_1 \subseteqq O_2$ or $O_2 \subseteqq O_1$. Thus, for arbitrary $i, j \in A$, there does not exist $O \in \mathbf{T}_E$ such that O contains only one of these points; this will be true iff there exists an open set $O \in \mathbf{T}_E$ such that both $i, j \in O$. Thus, either $\mathbf{A}(i, j)$ or $\mathbf{A}(j, i)$ and hence $\Gamma(A, E)$ is unilaterally connected.

(iii), (iv) $(A, \mathbf{T}_E)$ is topologically connected iff $A \neq O_1 \cup O_2$, where $O_1 \cap O_2 = \emptyset$ and $O_1, O_2 \in \mathbf{T}_E$. Thus, every non-void $B \subseteqq A$ is neither open nor closed, and the only open sets in $(A, \mathbf{T}_{E \cup E'})$ are the empty set and the whole set, which is true iff $\Gamma(A, E \cup E')$ is strongly connected. Hence, $\Gamma(A, E)$ is weakly connected.

Let us denote by $\mathscr{P}$ the property "digraph-connectedness implies topological connectedness," and by $\tilde{\mathscr{P}}$ the property "topological connectedness implies digraph-connectedness." Given topologies $\mathbf{T}_1, \mathbf{T}_2, \ldots$, let a maximal topology of the topologies $\{\mathbf{T}_i\}$ be denoted by $\mathbf{T}_{\max}$, and given by a least upper bound of the topologies $\{\mathbf{T}_i\}$ in the lattice of topologies $\mathbf{T}_i$, $i = 1, 2, \ldots$. Similarly, let $\mathbf{T}_{\min}$ be the greatest lower bound of the topologies $\{\mathbf{T}_i\}$ in the lattice of topologies $\mathbf{T}_i$, $i = 1, 2, \ldots$.

PROPOSITION 3. 1* (a) *If* $\mathbf{T}_1, \mathbf{T}_2, \ldots$ *be topologies satisfying* $\mathscr{P}$, *then the maximal topology,* $\mathbf{T}_{\max}$, *of the topologies* $\{\mathbf{T}_i, i = 1, 2, \ldots\}$ *is the topology* $\bigcup_i \mathbf{T}_i$, *and*

(b) *if* $\mathbf{T}_1, \mathbf{T}_2, \ldots$ *be topologies satisfying* $\tilde{\mathscr{P}}$, *then minimal topology,* $\mathbf{T}_{\min}$, *of the topologies is the topology* $\bigcap_i \mathbf{T}_i$.

4. Topological operators

In this section we investigate the digraph connectedness in terms of some topological operators.

Let $\mathscr{A} \subseteqq A$. We denote by $\mathrm{C1}(\mathscr{A})$, the closure of set $\mathscr{A}$; by $\mathrm{K1}(\mathscr{A})$, the kernel of set $\mathscr{A}$; by $\mathbf{K}(\mathscr{A})$, the core of set $\mathscr{A}$, [2].

DEFINITION 4. 1. (i) $\mathrm{C1}(\mathscr{A}) = \bigcap_j \{\mathscr{A}_j : \mathscr{A}_j \text{ closed}, \ \mathscr{A} \subseteqq \mathscr{A}_j \subseteqq A\}$

(ii) $\mathrm{K1}(\mathscr{A}) = \bigcap_j \{\mathscr{A}_j : \mathscr{A}_j \text{ open}, \ \mathscr{A} \subseteqq \mathscr{A}_j \subseteqq A\}$

(iii) $\mathbf{K}(\mathscr{A}) = \bigcap_j \{\mathscr{A}_j : \mathscr{A}_j \text{ open or closed}, \ \mathscr{A} \subseteqq \mathscr{A}_j \subseteqq A\}$.

We note that while $\mathrm{C1}(\mathscr{A})$ is always closed, $\mathrm{K1}(\mathscr{A})$ is open in our particular topology due to the property of completely additive closure. We also note that $\mathbf{K}(\mathscr{A}) = \mathrm{C1}(\mathscr{A}) \cap \mathrm{K1}(\mathscr{A})$.

It may also be remarked here that for a given digraph $\Gamma(A, E)$, and its corresponding topology $(A, \mathbf{T}_E)$, $\mathrm{C1}(i)$, $i \in A$, is equivalent to Berge's "transitive closure" of the point i, in terms of digraphs, [3].

The following theorem is obvious:

THEOREM 4. 1. *Let* i *be an arbitrary point in* A, *and let* $\mathscr{A}$ *be an arbitrary subset of* A.

(i) $\mathrm{C1}(\{i\}) = \{j : \mathbf{A}(i, j)\}$,

$\mathrm{K1}(\{i\}) = \{j : \mathbf{A}(j, i)\}$,

$\mathbf{K}(\{i\}) = \{j : \mathbf{A}(i, j) \text{ and } \mathbf{A}(j, i)\}$.

* Thanks are due to Professor K. KRICKEBERG for a helpful discussion on this proposition.

(ii) $\mathrm{C1}(\mathscr{A}) = \{j\colon \mathbf{A}(i, j)$ *for some* $i \in \mathscr{A}\}$,

$\mathrm{K1}(\mathscr{A}) = \{j\colon \mathbf{A}(j, i)$ *for some* $i \in \mathscr{A}\}$,

$\mathbf{K}(\mathscr{A}) = \{j\colon \mathbf{A}(i, j)$ *for some* $i \in \mathscr{A}$, *and* $\mathbf{A}(j, k)$ *for some* $k \in \mathscr{A}\}$.

The following corollaries are an immediate consequence of Theorem 4. 1.

COROLLARY 4. 2. *The subset* $\mathscr{A}$ *is dense in* $(A, \mathbf{T}_E)$ *iff for each point* $j \in A$, *there exists a point* $i \in A$ *such that in the digraph* $\Gamma(A, E)$, *we have* $\mathbf{A}(i, j)$.

COROLLARY 4. 3. *The subsets* $\mathscr{A}_1, \mathscr{A}_2 \subseteq A$ *are separated in* $(A, \mathbf{T}_E)$ *iff there do not exist* $i \in \mathscr{A}_1$, *and* $j \in \mathscr{A}_2$ *for which either* $\mathbf{A}(i, j)$ *or* $\mathbf{A}(j, i)$.

COROLLARY 4. 4. *For any* $i \in A$, $\mathbf{K}(\{i\})$ *is the maximum subset of* A *containing* i, *such that in the subdigraph* $\Gamma(\mathscr{A}, E \cap \mathscr{A} \times \mathscr{A})$, *we have* $\mathbf{A}(i, j)$ *for every* $(i, j) \in E \cap \cap \mathscr{A} \times \mathscr{A}$.

THEOREM 4. 5. *The digraph* $\Gamma(A,E)$ *is* (i) *strongly connected iff each point of* A *is dense in the topological space* $(A, \mathbf{T}_E)$, (ii) *weakly connected, or disconnected, iff the topological space* $(A, \mathbf{T}_E)$ *contains two non-void separated subsets of* A. *Moreover,* $\Gamma(A, E)$ *is disconnected iff the set* A *can be expressed as the union of two non-void separated subset of* A.

PROOF. (i) $\Gamma(A, E)$ strongly connected iff $\mathbf{T}_E = \{\Phi, A\}$, i.e., iff $\mathrm{C1}(\{i\}) = A$ for each $i \in A$.

(ii) $\Gamma(A, E)$ weakly connected, or disconnected iff there exist i, j in A, $i \neq j$, such that $\mathbf{A}(i, j)$ as well as $\mathbf{A}(j, i)$. Thus using Theorem 4. 1, $i \notin \mathrm{C1}(\{i\})$, and $j \notin \mathrm{C1}(\{j\})$ and $\{i\}$ and $\{j\}$ are two non-void separated subsets of A. Next, let $\mathscr{A}_1, \mathscr{A}_2$ be two separated subsets of A, $\mathscr{A}_1 \neq \emptyset$, $\mathscr{A}_2 \neq \emptyset$. Then by Corollary 4. 3, there do not exist $i \in \mathscr{A}_1, j \in \mathscr{A}_2$ such that either $\mathbf{A}(i, j)$ or $\mathbf{A}(j, i)$. However, since both $\mathscr{A}_1, \mathscr{A}_2$ are non-empty, there exist points $k \in \mathscr{A}_1$, $l \in \mathscr{A}_2$ such that neither $\mathbf{A}(k, l)$ nor $\mathbf{A}(l, k)$. Hence, $\Gamma(A, E)$ is either weakly connected, or disconnected.

Lastly $\Gamma(A, E)$ is disconnected iff there exists $\mathscr{A} \subseteq A$ such that $A = \mathscr{A} \cup (A \sim \mathscr{A})$, that is there does not exist $(i, j) \in \{\mathscr{A} \times (A \sim \mathscr{A})\} \cup \{(A \sim \mathscr{A}) \times \mathscr{A}\}$. Hence, $\mathrm{C1}(\mathscr{A}) = \mathscr{A}$, and $\mathrm{C1}((A \sim \mathscr{A})) = (A \sim \mathscr{A})$.

5. Certain class of topologies

Let $\mathbf{E} = \{E\colon \Phi \subseteq E \subseteq A \times A\}$, $\mathbf{D} = \{\Gamma(A, E)\colon E \in \mathbf{E}\}$, and $\mathbf{T} = \{(A, \mathbf{T}_E)\colon E \in \mathbf{E}$, and $(A, \mathbf{T}_E)$ has completely additive closure$\}$.

Let Φ be a mapping of $\mathbf{D}$ into $\mathbf{T}$, i.e., $\Phi\colon \mathbf{D} \to \mathbf{T}$.

THEOREM 5. 1. *For any* $D \in \mathbf{D}$, *the topology* $\Phi(D)$ *has family* $\{\{i\colon \mathbf{A}(i, j), i \in A\}\colon j \in A\}$ *as a base, and* Φ *is many-to-one mapping of* $\mathbf{D}$ *onto* $\mathbf{T}$.

PROOF. Theorems 4. 1 and 1. 1.

Let $\mathbf{D} \supseteq \mathbf{D}_T = \{\Gamma(A, E)\colon (i, j) \in E$ iff $\mathbf{A}(i, j);\ E \in \mathbf{E}\}$.

DEFINITION 5. 1. A digraph $\Gamma(A, E) \in \mathbf{D}_T$ is called a transitive digraph, and is such that (i) $\{(i, i)\colon i \in A\} \subseteq E$, and (ii) for every $i, j, k \in A$, $(i, j) \in E$, $(j, k) \in E$ implies $(i, k) \in E$.

THEOREM 5. 2. *Let* $(\Phi|\mathbf{D}_T)$ *denote the mapping* Φ *restricted to the family* $\mathbf{D}_T \subseteq \mathbf{D}$. (i) *The topology* $(\Phi|\mathbf{D}_T)(D_T)$, $D_T \in \mathbf{D}_T$ *has family* $\{\{i: (i, j) \in E, i \in A\}; j \in A\}$ *as a base, and* $(\Phi|\mathbf{D}_T)$ *is a one-to-one mapping of* $\mathbf{D}_T$ *onto* $\mathbf{T}$.

PROOF. Theorem 5. 1 with Φ replaced by $(\Phi|\mathbf{D}_T)$.

We find, thus, that $(\Phi|\mathbf{D}_T)$ is a one-to-one mapping of the class $\mathbf{D}_T$ of all transitive digraphs, with loops, onto the family, $\mathbf{T}$, of all topologies with completely additive closure.

In general, however, Φ is many-to-one and as such many distinct digraphs $\Gamma(A, E_i)$, $E_i \in \mathbf{E}$, $i = 1, 2, \ldots$ may determine identical topologies $(A, \mathbf{T}_E)$. We have the following results regarding such digraphs and arbitrary topologies with completely additive closure.

DEFINITION 5. 2. $\Gamma(A, E)$, $E \in \mathbf{E}$ is said to be the maximum (minimum) digraph determining the topology $(A, \mathbf{T}_E)$, if (i) $\Phi(\Gamma(A, E)) = (A, \mathbf{T}_E)$, and (ii) $E'(E) \subseteq E(E')$, for all E' such that $\Phi(\Gamma(A, E')) = (A, \mathbf{T}_E)$.

THEOREM 5. 3. (i) *The digraph* $\Gamma(A, E)$, *with* $E = \{(i, j): j \in \mathrm{Cl}(\{i\})$ *in* $(A, \mathbf{T}_E)\}$ *is the maximum digraph determining an arbitrary topology with completely additive closure;* (ii) *there does not necessarily exist a minimum digraph determining an arbitrary topology with completely additive closure.*

PROOF. (i) Consider the digraph $\Gamma(A, E)$ and let $i \in A$. Then $\mathrm{Cl}(\{i\}) = \{j: (i, j) \in E\}$. Let $\Gamma(A, E')$ be another digraph, and $k \in A$. Then $\mathrm{Cl}(\{k\}) \supseteq \{j: (k, j) \in E'\}$. Thus, if $E' = E \cup F$, $F \neq \emptyset$, $F \neq E$, and if $(r, s) \in F$, then $\mathrm{Cl}(\{r\})$ in $\mathbf{T}_{E'}$ is not the same as in $\mathbf{T}_E$. Hence, $(A, \mathbf{T}_{E'})$ cannot be the same topology as $(A, \mathbf{T}_E)$.

(ii) Let $A = \{i: i \leqq \omega$, the first non-finite ordinal$\}$. Let $\mathbf{T} = \{A\} \cup \{\{j: j < k\}; k \in A\}$, and let $\Gamma(A, E)$ be an arbitrary digraph determining $(A, \mathbf{T})$. Then the only open set containing ω is A. Thus, $\mathrm{K1}(\{\omega\}) = A$ and by Theorem 4. 1, $\mathbf{A}(k, \omega)$ for every $k \in A$; that is, the point ω is accessible from every $k \in (A \sim \{\omega\})$ in only finitely many steps, and there exist countably many edges from the set $(A \sim \{\omega\})$ to $\{\omega\}$. Let i denote the first point in A such that $(i, \omega) \in E$. Define $E^* = (E \sim \{(i, \omega)\})$. Then $\Phi(\Gamma(A, E^*)) = (A, \mathbf{T})$ where $E^* \subseteq E$, and $E^* \neq E$. Hence the theorem.

Further discussion on these lines may be found in [1].

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References

[1] T. J. AHLBORN, *On Directed Graphs and Related Topological Spaces*. M. A. Thesis, Kent State University (August 1964).

[2] C. E. AULL and W. J. THRON, Separation Axioms Between T_0 and T_1. *Indagationes Math.*, **24** (1962), pp. 26—37.

[3] C. BERGE (translated by A. DOIG), *The Theory of Graphs and Its Applications*. Wiley and Sons, New York (1962).

[4] T. N. BHARGAVA, Time Changes in a Digraph, *Journal of Applied Probability*, Vol. 2, No. 1 (1965).

[5] F. HARARY, Unsolved Problems in the Enumeration of Graphs. *Publications of the Hungarian Academy of Sciences*, Vol. V, Series A (1960).

[6] J. L. KELLEY, *General Topology*. Van Nostrand, New Jersey (1955).

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Tomus 19 (1—2), (1968), pp. 53—58.

OPERATIONAL RESULTS CONNECTED WITH SOME CLASSICAL POLYNOMIALS

By
S. K. CHATTERJEA (Calcutta, India)
(*Presented by* P. TURÁN)

1. In general there are three kinds of classical orthogonal polynomials, viz., Laguerre, Hermite and Jacobi polynomials. One may note that the Jacobi polynomials include Legendre and ultraspherical (also Tchebichef) polynomials as special cases. Now the Rodrigues' formulas of these three kinds of polynomials are the following:

$$L_n^{(\alpha)}(x) = \frac{1}{n!} e^x x^{-\alpha} D^n(e^{-x} x^{n+\alpha}) \tag{1.1}$$

$$H_n(x) = (-1)^n e^{x^2} D^n(e^{-x^2}) \tag{1.2}$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(-\frac{1}{2})^n}{n!} (1-x)^{-\alpha}(1+x)^{-\beta} D^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}]. \tag{1.3}$$

KRALL and FRINK [1] studied another class of orthogonal polynomials, known as generalized Bessel polynomials, and they gave the following analogue of Rodrigues' formula for their polynomials:

$$y_n(x, a, b) = b^{-n} x^{2-a} e^{b/x} D^n(x^{2n+a-2} e^{-b/x}). \tag{1.4}$$

Recently the study of these polynomials by operational methods has received much attention.

L. CARLITZ [2] has found the operational formula for the Laguerre polynomials. His formulas are

$$\prod_{j=1}^{n} (xD - x + \alpha + j) Y = x^{-\alpha} e^x D^n(x^{\alpha+n} e^{-x} Y), \tag{1.5}$$

$$\prod_{j=1}^{n} (xD - x + \alpha + j) = n! \sum_{r=0}^{n} \frac{x^r}{r!} L_{n-r}^{(\alpha+r)}(x) D^r, \tag{1.6}$$

where Y is an arbitrary (differentiable) function of x. BURCHNALL [3] has derived the operational formula for the Hermite polynomials:

$$(D - 2x)^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} H_{n-r}(x) D^r. \tag{1.7}$$

R. P. SINGH [4] has shown that

$$D^n[(1-x^2)^n(1-x)^\alpha(1+x)^\beta Y] = \tag{1.8}$$

$$= (1-x)^\alpha(1+x)^\beta \prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+\beta-\alpha\}Y,$$

$$\prod_{j=1}^{n}\{(1-x^2)D-(\alpha+\beta+2j)x+(\beta-\alpha)\} = \tag{1.9}$$

$$= \sum_{k=0}^{n}\frac{(-2)^{n-k}n!}{k!}(1-x^2)^k P_{n-k}^{(\alpha+k,\,\beta+k)}(x)D^k.$$

The author [5] of the present paper has proved that

$$x^{-a}D^n(x^{kn+a}Y) = \prod_{j=1}^{n}\{x^{k-1}(\delta+a+kj)\}Y, \tag{1.10}$$

$$\sum_{j=1}^{n}\{x^2D+(a+2j)x+b\} = \sum_{r=0}^{n}\binom{n}{r}b^{n-r}x^{2r}y_{n-r}(x,a+2r+2,b)D^r, \tag{1.11}$$

where $\delta \equiv xD$.

The main object of the present paper is to find out a way to combine all the operational formulas involving Laguerre, Hermite, Jacobi and Bessel polynomials into a single result. Actually we have found that these operational results fall out as special cases of a single operational statement. Our operational result is of a rather general nature and is very useful in proving the corresponding operational representations involving the generalizations of Hermite (by GOULD and HOPPER [6]) and Laguerre (by the present author [7]) polynomials.

2. For our purpose we define

$$p_n(x) = \frac{1}{K_n\omega(x)}D^n[\omega(x)X^n] \tag{2.1}$$

and

$$q_n(x,a) = \frac{1}{K_n\omega(x)}D^n[\omega(x)X^{n+a}]. \tag{2.2}$$

We remark that the definition like (2. 1) is well known in the theory of classical orthogonal polynomials and such formula is known as Rodrigues' formula. It may be noted that for the classical orthogonal polynomials and Bessel polynomials, K_n is a constant, X is a polynomial in x (of degree at most two) whose coefficients are independent of n and $\omega(x)$ is a non-negative weight function in a suitably chosen region.

Also we note from (2. 1) and (2. 2) that

$$q_n(x,0) = p_n(x).$$

Next we notice that [8]:

$$\Phi(D)\,\xi\eta = \xi\Phi\left(D+\frac{\xi'}{\xi}\right)\eta, \tag{2.3}$$

$$D^n\,\xi^n = (\xi D+\xi')\ldots(\xi D+(n-1)\,\xi')(\xi D+n\xi'); \tag{2.4}$$

where $\Phi(D)$ is a polynomial operator with variable coefficients and ξ, η are functions of x ($\xi'=d\xi/dx$). Now we have

$$D^n[\omega(x)\,X^nY] = \prod_{j=1}^{n}(XD+jX')\,\omega(x)Y = \omega(x)\prod_{j=1}^{n}\left\{X\left(D+\frac{\omega'(x)}{\omega(x)}\right)+jX'\right\}Y. \tag{2.5}$$

Again we get

$$D^n[\omega(x)\,X^nY] = \sum_{s=0}^{n}\binom{n}{s}D^{n-s}(\omega(x)\,X^n)\,D^sY = \tag{2.6}$$

$$= \omega(x)\sum_{s=0}^{n}\binom{n}{s}K_{n-s}q_{n-s}(x,s)\,D^sY.$$

Thus it follows from (2. 5) and (2. 6) our desired result:

$$\prod_{j=1}^{n}\left\{X\left(D+\frac{\omega'(x)}{\omega(x)}\right)+jX'\right\}Y = \sum_{s=0}^{n}\binom{n}{s}K_{n-s}q_{n-s}(x,s)\,D^sY. \tag{2.7}$$

It is interesting to note that in particular, when $Y=1$, we have

$$\prod_{j=1}^{n}\left\{X\left(D+\frac{\omega'(x)}{\omega(x)}\right)+jX'\right\}\cdot 1 = K_np_n(x). \tag{2.8}$$

From (2. 1) we obtain

$$K_1p_1(x)-X' = X\cdot\frac{\omega'(x)}{\omega(x)}. \tag{2.9}$$

Thus it follows from (2. 8)

$$K_np_n(x) = \prod_{j=1}^{n}\{XD+K_1p_1(x)+(j-1)\,X'\}\cdot 1. \tag{2.10}$$

Also from (2. 7) we note the following generalization of (2. 10):

$$\prod_{j=1}^{n}\{XD+K_1p_1(x)+(j-1)\,X'\}\,Y = \sum_{s=0}^{n}\binom{n}{s}K_{n-s}q_{n-s}(x,s)\,D^sY, \tag{2.11}$$

where

$$q_n(x,a) = \frac{1}{K_n\omega(x)}D^n[\omega(x)\,X^{n+a}]$$

and $q_1(x,0)=p_1(x)$.

It may be noted that in this section it is not necessary to assume that $p_n(x)$ are orthogonal polynomials or that $\omega(x)$ is a weight function.

Also we note that the operators used in (2. 10) and (2. 11) do not commute except when X is a constant or a monomial.

3. Special cases:

For the Laguerre polynomials $L_n^{(\alpha)}(x)$, we have

$$K_n = n!, \quad \omega(x) = x^\alpha e^{-x}, \quad X = x \qquad (\alpha > -1).$$

Thus from (2. 7) we derive

$$\prod_{j=1}^{n} \left\{ x\left(D + \frac{\alpha}{x} - 1\right) + j \right\} Y = \sum_{s=0}^{n} \binom{n}{s} x^{-\alpha} e^x D^{n-s}[x^\alpha e^{-x} x^n].$$

Therefore by (1. 1) we have

$$\prod_{j=1}^{n} (xD + \alpha - x + j)\, Y = n! \prod_{s=0}^{n} \frac{x^s}{s!} L_{n-s}^{(\alpha+s)}(x) D^s Y,$$

which is (1. 6).

For the Hermite polynomials $H_n(x)$, we have

$$K_n = (-1)^n, \quad \omega(x) = e^{-x^2}, \quad X = 1.$$

Thus from (2. 7) we derive

$$(D - 2x)^n Y = \sum_{s=0}^{n} \binom{n}{s} e^{x^2} D^{n-s}(e^{-x^2}) D^s Y.$$

Therefore by (1. 2)

$$(D - 2x)^n Y = \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} H_{n-s}(x) D^s Y,$$

which is (1. 7).

For the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$, we have

$$K_n = (-2)^n n!, \quad \omega(x) = (1-x)^\alpha (1+x)^\beta, \quad X = 1 - x^2 \qquad (\alpha > -1, \beta > -1).$$

Thus from (2. 7) we derive

$$\prod_{j=1}^{n} \{(1-x^2) D + (\beta - \alpha) - (\alpha + \beta + 2j) x\}\, Y =$$

$$= \sum_{s=0}^{n} \binom{n}{s} (1-x)^{-\alpha} (1+x)^{-\beta} D^{n-s}[(1-x)^\alpha (1+x)^\beta (1-x^2)^n] D^s Y.$$

Therefore by (1. 3)

$$\prod_{j=1}^{n} \{(1-x^2) D + (\beta - \alpha) - (\alpha + \beta + 2j) x\}\, Y =$$

$$= \sum_{s=0}^{n} \frac{(-2)^{n-s} n!}{s!} (1-x^2)^s P_{n-s}^{(\alpha+s, \beta+s)}(x) D^s Y,$$

which is (1. 9).

Lastly for the Bessel polynomials $y_n(x, a, b)$, we notice from (1. 4) and (2. 1) that

$$K_n = b^n, \quad \omega(x) = x^{\alpha-2} e^{-b/x}, \quad X = x^2.$$

Thus from (2. 7) we derive

$$\prod_{j=1}^{n} \{x^2 D + b + (a+2j-2)x\} Y = \sum_{s=0}^{n} \binom{n}{s} x^{2-a} e^{b/x} D^{n-s} [x^{a-2} e^{-b/x} x^{2n}] D^s Y.$$

Therefore by (1. 4)

$$\prod_{j=1}^{n} \{x^2 D + b + (a+2j-2)x\} Y = \sum_{s=0}^{n} \binom{n}{s} b^{n-s} x^{2s} y_{n-s}(x, a+2s, b) D^s Y,$$

which is actually the same as (1. 11).

4. Two other interesting special cases:

GOULD and HOPPER [6, p. 52] generalized the Hermite polynomials by the relation

$$H_n^r(x, a, p) = (-1)^n x^{-a} e^{px^r} D^n (x^a e^{-px^r}). \tag{4. 1}$$

For this function $H_n^r(x, a, p)$, we notice

$$K_n = (-1)^n, \quad \omega(x) = x^a e^{-px^r}, \quad X = 1.$$

Thus from our formula (2. 7) we derive

$$\left(D + \frac{a}{x} - prx^{r-1}\right)^n Y = \sum_{s=0}^{n} \binom{n}{s} x^{-a} e^{px^r} D^{n-s} [x^a e^{-px^r}] D^s Y.$$

Now by (4. 1) we have the GOULD and HOPPER's formula [6, p. 53]

$$\left(D + \frac{a}{x} - prx^{r-1}\right)^n = \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} H_{n-s}^r(x, a, p) D^s, \tag{4. 2}$$

which is a special case of our formula (2. 7).

Next the author [7, p. 181] of the present paper generalized the Laguerre polynomials $L_n^{(\alpha)}(x)$ by the relation

$$T_{kn}^{(\alpha)}(x, p) = \frac{1}{n!} x^{-\alpha} e^{px^k} D^n (x^{\alpha+n} e^{-px^k}), \tag{4. 3}$$

where k is a natural number. For this polynomial $T_{kn}^{(\alpha)}(x, p)$, of degree kn, we notice

$$K_n = n!, \quad \omega(x) = x^\alpha e^{-px^k}, \quad X = x.$$

Thus from (2. 7) we have

$$\prod_{j=1}^{n} \left\{x\left(D + \frac{\alpha}{x} - pkx^{k-1}\right) + j\right\} Y = \sum_{s=0}^{n} \binom{n}{s} x^{-\alpha} e^{px^k} D^{n-s} [x^{\alpha+n} e^{-px^k}] D^s Y.$$

Now by (4. 3)

$$\prod_{j=1}^{n} (xD + \alpha - pkx^k + j) = n! \sum_{s=0}^{n} \frac{x^s}{s!} T_{k(n-s)}^{(\alpha+s)}(x, p) D^s, \tag{4. 4}$$

which is author's formula in [7, p. 183].

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References

[1] H. KRALL and O. FRINK, A new class of orthogonal polynomials: The Bessel polynomials' *Trans. Amer. Math. Soc.*, **65** (1949), pp. 100—115.
[2] L. CARLITZ, A note on the Laguerre polynomials, *Michigan Math. J.*, **7** (1960), pp. 219—223.
[3] J. L. BURCHNALL, The Bessel polynomials, *Canad. J. Math.*, **3** (1951), pp. 62—68.
[4] R. P. SINGH, Operational formulae for Jacobi and other polynomials, *Rend. Sem. Mat. Univ. Padova*, **35** (1965), pp. 237—244.
[5] S. K. CHATTERJEA, Operational formulae for certain classical polynomials. I, *Quart. J. Math. (Oxford)*, **14** (1963), pp. 241—246.
[6] H. W. GOULD and A. T. HOPPER, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.*, **29** (1962), pp. 51—64.
[7] S. K. CHATTERJEA, On a generalization of Laguerre polynomials, *Rend. Sem. Math. Univ. Padova*, **34** (1964), pp. 180—190.
[8] T. CHAUNDY, *Differential Calculus* (Oxford), p. 388.

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Tomus 19 (1—2), (1968), pp. 59—67.

ON CHROMATIC NUMBER OF FINITE SET-SYSTEMS

By
L. LOVÁSZ (Budapest)
(Presented by G. HAJÓS)

Introduction

A finite set-system is an ordered pair $\langle h, H\rangle$ where H is a system of subsets of the finite set h. This notion is a natural generalization of the notion of non-directed graphs containing no loops and multiple edges, since such a graph may be considered to be an ordered pair $\langle g, G\rangle$ where g and G are the sets of vertices and edges of the graph, respectively.

The notion of chromatic number of a graph can be generalized for finite set-systems too ([4]). The aim of this paper is to generalize some theorems concerning about chromatic number of graphs, further to show an application of finite set systems in constructing a graph the existence of which has been proved by ERDŐS by probabilistic methods.

§ 1 contains some simple observations, which will be used later.

In § 2 we prove a generalization of the theorem of BROOKS. This theorem has a consequence in the theory of graphs too (see 2. 2). The method of the proof is similar to the method of BROOKS's original proof (see [1]).

TUTTE and ZYKOV proved, that for every natural number n there exists a graph of chromatic number n containing no triangles ([2], [6]). This theorem was generalized by ERDŐS ([3]). He showed, that for any pair of positive integers n, s there exists a graph of chromatic number n the circuits of which are longer than s. His proof uses the so-called probabilistic method, it does not give any concrete example. Such an example was given by ZYKOV for $s=5$ and by NEŠETŘIL for $s=7$ ([5], [6]).

ERDŐS and HAJNAL generalized this theorem of ERDŐS for finite set-systems, using also probabilistic methods ([3]). In § 4 we prove the theorem of ERDŐS and HAJNAL by a direct construction. This gives a construction for n chromatic graph the circuits of which are longer than s. It is to be mentioned, that I can not describe this construction for graphs using only graphs and no set-systems.

§ 1

We repeat the definition of finite set-systems: a *finite set-system* is an ordered pair $\langle h, H\rangle$ where h is a finite set and H is a set of subsets of h. The elements of h and H are the *vertices* and *edges* of $\langle h, H\rangle$ respectively.

The case when $\langle h, H\rangle$ contains an „isolated vertex" is uninteresting from the point of view of the chromatic number. We shall suppose, that for any $x\in h$ there exists an $E\in H$ such that $x\in E$. Thus, it does not cause any misunderstanding if we speak about the set-system H instead of the set-system $\langle h, H\rangle$. Put $P(H)=\bigcup_{E\in H} E$,

then the set-system H is actually the set-system $\langle P(H), H\rangle$. Further we shall suppose, that the edges of a set-system have at least two elements.

A finite set-system is said to be a *graph* if its edges are sets of two elements. A set-system is a *uniform k-system* if its edges are k-tuples. Thus a graph is a uniform 2-system. A *complete uniform k-system* of r vertices is a set-system consisting of all k-tuples of r given elements ($2 \leqq k \leqq r$). A subset H' of H is said to be a *sub-system* of H.

Let $x \neq y$ be two elements of $P(H)$. We say, that x is in *relation* with y (in notation $x \sim y$) if there exists an edge E of H such that $x \in E$, $y \in E$.

A sequence of vertices of H is said to be a *chain*, if any two neighbouring elements are in relation (the elements of this sequence are not required to be different). If $(a_1, \ldots, a_r) = V$ is a chain and $a_1 = a_r$, then V is said to *belong to* a_1. If the chains $(a_1, \ldots, a_r) = V$ and $(b_1, \ldots, b_s) = W$ have $a_r = b_1$, then we define $V \cdot W$ as the chain $(a_1, \ldots, a_{r-1}, b_1, \ldots, b_s)$. This multiplication is associative, and thus we may define $V^k = \underset{1}{V} \cdot \underset{2}{V} \ldots \underset{k}{V}$. Let $(a_1, \ldots, a_r) = V$ be an arbitrary chain, then we put $\tilde{V} = (a_r, \ldots, a_1)$.

If the elements of a sequence forming a chain are different, then this chain will be called a *path*. The *length* of a path $(a_1, \ldots, a_r)$ is $r-1$. Note, that by this definition a path is directed, the path $(a_1, \ldots, a_r)$ differs from the path $(a_r, \ldots, a_1)$.

The *different* vertices $a_1, \ldots, a_r$ and the (not necessarily different) edges $E_1, \ldots, E_r$ are said to form a *circuit*, if $a_i \in E_i$ for $i = 1, \ldots, r$, $a_{i+1} \in E_i$ for $i = 1, \ldots, r-1$ and $a_1 \in E_r$. This circuit will be denoted by $\langle (a_1, \ldots, a_r), (E_1, \ldots, E_r)\rangle$. The *length* of this circuit is r.

A circuit is *non-trivial*, if it has at least two different edges, and it is *simple*, if all of its edges are different.

(1. 1) *A shortest non-trivial circuit is simple.*

PROOF. Let $C = \langle (a_1, \ldots, a_r), (E_1, \ldots, E_r)\rangle$ be the shortest non-trivial circuit. Let us suppose, that C is not simple. Then let for instance $E_1 = E_i$ ($1 < i \leqq r$). There exists a j such that $E_1 \neq E_j$. We may suppose, that $1 < j < i$. Then the non-trivial circuit $\langle (a_2, \ldots, a_i), (E_2, \ldots, E_i)\rangle$ is shorter than r, which is a contradiction.

A finite set-system is connected, if the set of its edges can not be split into the union of two (non empty) sets H_1, H_2 such that $P(H_1) \cap P(H_2) = \emptyset$.

Let H be a finite set-system. We construct a bipartite graph G_H. Let $P(H) \cup H$ be the set of vertices of G_H. Two vertices of G_H are joined by an edge of G_H if (a) one of them is a vertex of H, the other an edge of H, (b) if x, y are these two vertices of G_H and $x \in H$, $y \in P(H)$, then $y \in x$.

(1. 2) *G_H is connected if and only if H is connected.*

PROOF. a) Let us suppose, that H is connected, and let us split $P(G_H)$ into the union of the classes g_1, g_2 ($g_1 \neq \emptyset$, $g_2 \neq \emptyset$). We prove, that there is an edge (x, y) of G_H such that $x \in g_1$ and $y \in g_2$.

A) Let us suppose, that one of the classes g_1, g_2, say g_1, does not contain any edge of H. Let x be an element of g_1, then $x \in P(H)$. If F is an edge of H containing x, then $F \in g_2$ and (x, F) is an edge of H.

B) Thus we may suppose, that $H_1 = H \cap g_1$ and $H_2 = H \cap g_2$ are not empty. Since H is connected there exist two edges E, F of H such that $E \in H_1$, $F \in H_2$ and

$E \cap F$ is not empty. Let x be an element of $E \cap F$. We may suppose, that $x \in g_1$. Then (x, F) is an edge of G_H for which $x \in g_1$ and $F \in g_2$.

b) Let us suppose, that H is not connected, i. e. there exist H_1, H_2 such that $H_1 \cup H_2 = H$ and $P(H_1) \cap P(H_2) = \emptyset$. Then there is no edge in G_H going between the classes $g_1 = H_1 \cup P(H_1)$, $g_2 = H_2 \cup P(H_2)$, consequently G_H is not connected.

(1. 3) *H contains a non trivial circuit if and only if G_H contains a (non-trivial) circuit.*

PROOF. a) Let $\langle (a_1, \ldots, a_r), (E_1, \ldots, E_r) \rangle$ be a shortest non-trivial circuit in H. By (1. 1) this circuit is simple. Then $(a_1, E_1, a_2, E_2, \ldots, a_r, E_r)$ is a circuit in G_H.

b) Let $(b_1, \ldots, b_{2r})$ be a circuit in G_H (by (a) G_H is bipartite). We may suppose, that $b_{2i} \in H$, $b_{2i+1} \in P(H)$ for $i = 1, 2, \ldots, s$. Then $\langle (b_1, b_3, \ldots, b_{2s-1}), (b_2, \ldots, b_{2s}) \rangle$ is a non-trivial circuit in H.

Further it is obvious, that

(1. 4) *The number of vertices of G_H is $|H| + |P(H)|$, the number of edges of G_H is $\sum_{E \in H} |E|$.*

The following theorem of the theory of graphs characterizes the notion of the tree:

(1. 5) *Let G be a graph. Any two of the following conditions imply the third one:*
G is connected,
G contains no circuit,
the number of edges of G is by 1 less, than the number of vertices of G.

Applying (1. 5) on the graph G_H we obtain by (1. 2), (1. 3), (1. 4)

(1. 6) *Let H be a finite set-system. Then any two of the following conditions imply the third one:*
H is connected,
H contains no non-trivial circuit,

$$|P(H)| + |H| = 1 + \sum_{E \in H} |E|.$$

Similarly it follows from these observations that

(1. 7) *If H contains no non-trivial circuit, then* $|P(H)| + |H| \geqq \sum_{E \in H} |E| + 1$.

Further it is obvious, that

(1. 8) *For a set-system H consisting of the edges of a non-trivial circuit* $|P(H)| + |H| \leqq \sum_{E \in H} |E|$.

Let x be a vertex of a set-system H. We denote the set of edges of H containing x by H_x. Let $E, F \in H_x$. E and F are said to be *quasi-disjoint* if $E \cap F = \{x\}$. We denote the maximal number of quasi-disjoint edges of H_x by $\varphi(x)$. A set of quasi-disjoint edges of H_x is *maximal* if it has $\varphi(x)$ elements. The vertex x is *normal* if for any maximal set H' of quasi-disjoint edges of H_x $P(H') = P(H_x)$. This $\varphi(x)$ may be considered as a generalization of the notion of valency. Actually it has been intro-

duced implicitely by A. HAJNAL and P. ERDŐS in [1] (Def. 13. 7). Theorems (1. 11) and (2. 1) will show, that this notion is a useful generalization of the notion of valency.

A vertex of a graph is clearly always normal.

Let $(A_1, ..., A_n)$ be a decomposition of $P(H)$ into the union of the disjoint classes A_j. This decomposition is said to be a *colouring* of H if for any $F \in H$ and any A_i $F \not\subset A_i$. $A_1, ..., A_n$ are the *colour classes*. We define the *chromatic number* of H as the smallest possible value of n.

Obviously, if H has chromatic number n and $(A_1, ..., A_n)$ is a colouring of H then $A_i \cup A_j$ contains an edge of H for any i, j $(i \neq j)$.

Let x be a vertex of H. A colouring of H is *x-critical*, if the set $\{x\}$ is one of the colour classes. An x-critical colouring will be denoted by $(A_x, A_1, ..., A_m)$, where $A_x = \{x\}$.

A finite set-system H is *n-critical*, if

(i) it has chromatic number n,

(ii) if $H' \subset H$ and $H' \neq H$ then the chromatic number of H' is $< n$.

Obviously any set-system of chromatic number n contains an n-critical subsystem.

We enumerate some simple properties of n-critical finite set-systems. We omit the simple proofs.

(1. 9) *An n-critical set-system is connected.*

(1. 10) *If H is n-critical and $x \in P(H)$, then there exists an x-critical colouring of H with n classes.*

(1. 11) *If H is n-critical and $x \in P(H)$ then $\varphi(x) \geqq n-1$.*

§ 2

We are going to prove the following theorem:

(2. 1) *Let H be a finite set-system of the following properties:*
for any $x \in P(H)$ $\varphi(x) \leqq n$,
if $\varphi(x) = n$ then x is normal,
H does not contain a complete uniform k-system of $n(k-1)+1$ vertices for any $k \geqq 2$,
if $n = 2$ then H does not contain a graph which is an odd circuit;
then H has chromatic number $\leqq n$.

A trivial consequence of (2. 1) is

(2. 2) *Let $k \geqq 2$, $n \geqq 2$. Let H be a uniform k-system containing no complete uniform k-system of $n(k-1)+1$ vertices and in the case $k = n = 2$ not even an odd circuit; further let $|P(H_x)| \leqq n(k-1)+1$ for every $x \in P(H)$. Then the chromatic number of H is $\leqq n$.*

For graphs we obtain the following theorem already mentioned in the introduction:

(2. 3) *Let the valencies of vertices of a graph G be* $\leqq n(k-1)$ $(k \geqq 2, n \geqq 2)$. *Let us suppose, that G does not contain a complete* $[n(k-1)+1]$*-gon and in the case* $n=k=2$ *not even an odd circuit. Then there exists a decomposition of $P(G)$ into n classes such that no one of them contains a complete k-gon.*

Theorem (2. 3) can be proved applying (2. 2) on the set-system consisting of those k-tuples of $P(G)$ which span a complete graph. This theorem includes the theorem of Brooks for $k=2$.

We are going to prove (2. 1) by proving that if $n \geqq 2$ and

(i) H is $(n+1)$-critical,

(ii) for any $x \in P(H)$ $\varphi(x)=n$,

(iii) the vertices of H are normal,

then

a) there exists a $k \geqq 2$ for which H is a complete uniform k-system of $n(k-1)+1$ vertices, or

b) $n=2$ and H is a graph, and furthermore an odd circuit.

Let $(A_x, A_1, \ldots, A_n)$ be an x-critical colouring of H. Put $E_i=(A_i \cap P(H_x)) \cup \{x\}$, for $i=1, \ldots, n$. $A_i \cup A_x$ contains an edge E_i'. Obviously $E_i' \subset E_i$. But because of (iii) $\bigcup_i E_i' = \bigcup_i E_i$, hence $E_i'=E_i$. Thus we have proved, that

(2. 4) *If $(A_x, A_1, \ldots, A_n)$ is an x-critical colouring of H then* $(A_i \cap P(H_x)) \cup \{x\} = E_i \in H$ *and if* $F \in H$, $F \subset E_i$ *then* $F=E_i$.

Let $\sigma=(h_1, \ldots, h_l)$ be an arbitrary decomposition of $P(H)$ into l classes, and let π be a permutation of $P(H)$. The decomposition $(h_1\pi, \ldots, h_l\pi)$ will be denoted by $\sigma\pi$.

Let $V=(a_1, a_2, \ldots, a_r)$ be a chain in H. We denote the permutation $(a_1a_2)(a_2a_3)\ldots(a_{r-1}a_r)$ of $P(H)$ by π_V. It is obvious, that $\pi_{VW}=\pi_V\pi_W$, $\pi_{\tilde{V}}=\pi_V^{-1}$.

(2. 5) *Let $V=(a_1, \ldots, a_r)$ be an arbitrary chain in H and let $\sigma=(A_{a_1}, A_1, \ldots, A_n)$ be an a_1-critical colouring of H. Then $\sigma\pi_V$ is an a_r-critical colouring of H.*

PROOF. We use induction on r. For $r=1$ π_V may be considered to be the identity, consequently the conclusion is true.

Suppose that putting $W=(a_1, \ldots, a_{r-1})$ $\sigma\pi_W=(A_{a_{r-1}}, A_1', \ldots, A_n')$ is an a_{r-1}-critical colouring. It is enough to prove that $\sigma\pi_V$ is a colouring, since $A_{a_1}\pi_V = \{a_r\}$.

Let i be the index for which $a_r \in A_i'$. Put $A_{a_r}=\{a_r\}$, $A_i^*=(A_i' \cup \{a_{r-1}\}) \setminus \{a_r\}$. Then $\sigma\pi_V=(A_{a_r}, A_1', \ldots, A_i^*, \ldots, A_n')$ is a colouring. Really, if one of these classes contains an edge, this may be only A_i^*. But $F \subset A_i^*$ implies $a_{r-1} \in F$ (since $F \not\subset A_i'$), i.e. $F \in H_{a_{r-1}}$, $F \subset (P(H_{a_{r-1}}) \cap A_i') \cup \{a_{r-1}\} = E$. By (2. 4) we obtain $F=E$, which is a contradiction since $a_r \in E$ but $a_r \notin F$. Thus (2. 5) is proved.

Let $W=(b_1, b_2, \ldots, b_s)$ be a path of H of maximal length. Then $P(H_{b_1}) \subset W$[1] because of the maximality of W. Put $|P(H_{b_1})|=\mu+1$, $P(H_{b_1})=\{b_1, b_{i_1}, \ldots, b_{i_\mu}\}$ $(1<i_1<\ldots<i_\mu \leqq s)$, and $V=(b_2, \ldots, b_{i_\mu})$.

(2. 6) *Let N, M be two subsets of V such that* $|N|=|M|=\nu \leqq \mu-1$. *Then there exists a chain U belonging to b_1 such that $N\pi_U=M$.*

[1] A subset of a sequence is a subset of the set of its elements.

PROOF. Let L denote the set $\{b_2, \ldots, b_{\nu+1}\}$. It is enough to prove that there exists a chain U_1 belonging to b_2 such that $M\pi_{U_1} = L$. Since similarly there exists also a chain U_2 belonging to b_1 such that $N\pi_{U_2} = L$ and then putting $U = U_2 \cdot \tilde{U}_1$ $N\pi_U = M$.

Let us consider the sets $\{b_{j_1}, \ldots, b_{j_\nu}\}$ $(j_1 < \ldots < j_\nu)$ which can be written in the form $M\pi_U$. The chain U_1 belonging to b_1 will be said to be better than the chain U_2 if putting $M\pi_{U_1} = \{b_{j_1}, \ldots, b_{j_\nu}\}$, $M\pi_{U_2} = \{b_{j'_1}, \ldots, b_{j'_\nu}\}$ we have $j_\xi < j'_\xi$ for the greatest ξ for which j_ξ and j'_ξ are different. Let U_0 be one of the best chains, $K = M\pi_{U_0} = \{b_{j_1}, \ldots, b_{j_\nu}\}$. We are going to show, that $K = L$.

Let $i_\psi < j_\nu \leqq i_{\psi+1}$. We are going to show that if $\eta \leqq i_\psi$ then $b_\eta \in K$. Let us suppose that $i_\xi < \eta \leqq i_{\xi+1}$, $\xi < \psi$. Put $U_1 = (b_1, b_{i_\xi}, \ldots, b_{i_{\xi+1}}, b_1)$, $U_2 = (b_1, b_{i_\xi}, \ldots, b_{i_\mu}, b_1)$. If $b_\eta \notin K$ then the "last" element of $M\pi_U$, where $U = U_0 U_1^{\eta - i_\xi} U_2$, is $b_{j_\nu - 1}$. Really, the "last" element of $M\pi_{U_0 U^{\eta - i_\xi}} = M'$ is b_{j_ν}, since π_{U_1} has not changed this vertex. Further b_{i_ψ} is the only vertex, for which $b_{i_\psi} \notin \{b_{j_\nu}, \ldots, b_s\}$, but $b_{i_\psi}\pi_{U_2} \in \in \{b_{j_\nu}, \ldots, b_s\}$, and $b_{i_\psi} \notin M'$.

Thus we have proved, that $b_\eta \in K$ for $2 \leqq \eta \leqq i_\psi$. This implies, that $\psi \leqq i_\psi - 1 \leqq \leqq \nu - 1 \leqq \mu - 2$.

Now we prove, that $b_\eta \in K$ holds also for $i_\psi < \eta < j_\nu$. Let us suppose indirectly, that there are indices for which this is not true and η is the greatest index of this type. Putting $U_1 = (b_1, b_{i_\mu}, \ldots, b_{i_\psi}, b_1)$ and $U_2 = (b_1, b_{i_{\psi+1}}, \ldots, b_{i_\mu}, b_1)$ it can be seen very easy, that $U_0 \cdot U_1^{i_{\psi+1} - \eta} U_2 \tilde{U}_1^{(i_{\psi+1} - \eta)}$ is better than U_0.

The proof of (2. 6) is now complete.

Let us consider a b_1-critical colouring $\sigma = (A_{b_1}, A_1, \ldots, A_n)$ of H and the quasi-disjoint set $\{E_1, E_2, \ldots, E_n\}$ defined in (2. 4). Let $|E_1| = k_{W,\sigma}$ be the smallest among the numbers $|E_1|, \ldots, |E_n|$, then

$$\mu \geqq \sum_{i=1}^{n} (|E_i| - 1) \geqq n(|E_1| - 1) = n(k_{W,\sigma} - 1). \tag{1}$$

(2. 7) *For* $F \in H_{b_1}$ $|F| \geqq k_{W,\sigma}$.

PROOF. Let us suppose indirectly, that $|F| < k_{W,\sigma}$. Let G be a subset of $P(H_{b_1})$ of $k_{W,\sigma}$ elements containing F. By (2. 6) there exists a chain U belonging to b_1 such that $E_1\pi_U = G$. By (2. 5) $\sigma\pi_U$ is a b_1-critical colouring, and this is a contradiction because of (2. 4).

(2. 7) implies, that $k_{W,\sigma}$ is independent on σ, thus we may write simply k_W instead of $k_{W,\sigma}$. Thus the fact, that in (1) the equality holds, i.e. $\mu = n(k_W - 1)$, does not depend on σ.

We shall say, that W is of type A, if $\mu = n(k_W - 1)$, and of type B otherwise. It is an obvious consequence of (1) that

(2. 8) *If* W *is of type* A *then* $P(H_{b_1}) = V \cup \{b_1\}$ *and* $|E_i| = k_W$ $(1 \leqq i \leqq n)$.

In what follows we distinguish two cases:

a) Let us suppose that every maximal path of H is of type A. Let $(b_1, \ldots, b_s) = W$ be a maximal path. We show, that every k_W-tuple of $P(H_{b_1})$ is an edge of H. Then H does not contain any more edge by (i), and thus we have got the case a) of the theorem.

Let $\{c_1, \ldots, c_{k_W}\}$ be an arbitrary k_W-tuple of $P(H_{b_1})$, $c_1 = b_\xi$. We may suppose, that $\xi \neq \mu + 1$. By (2. 8) $b_{\xi+1} \sim b_1$ and hence $W' = (b_\xi, \ldots, b_1, b_{\xi+1}, \ldots, b_s)$ is a maximal path of H. Let $\sigma' = (A'_{b_\xi}, A'_1, \ldots, A'_n)$ be a b_ξ-critical colouring and $b_\xi \in A_i$, $b_1 \in A'_j$. By (2. 4) $A_i \cap P(H_{b_1}) \cup \{b_1\} = E_i \in H_{b_1}$ and $A'_j \cap P(H_{b_\xi}) \cup \{b_\xi\} = E'_j \in H_{b_\xi}$. (2. 7) implies that $|E_i| \leqq |E'_j|$ and $|E_i| \geqq |E'_j|$, hence $|E_i| = |E'_j| = k_W$ and $P(H_{b_1}) = P(H_{b_\xi})$.

Put $C = E'_j \setminus \{b_\xi\}$. By (2. 6) there exists a chain U belonging to b_1 such that $C\pi_U = \{C_2, \ldots, C_{k_W}\}$. Then $\sigma'\pi_U$ is a b_1-critical colouring by (2. 5) and thus (2. 4) implies $\{C_1, C_2, \ldots, C_{k_W}\} \in H$.

b) Let us suppose, that there exists a maximal path $W = (b_1, \ldots, b_s)$ of type B. Let $(A_{b_1}, A_1, \ldots, A_n) = \sigma$ be a b_1-critical colouring. (1) implies, that there is an i $(1 \leqq i \leqq n)$ such that $|A_i \cap V| \geqq k_W$. Put $E'_1 = A_1 \cap P(H_{b_1})$.

We show, that $k_W \geqq |P(H_{b_1})| - 1$.

Let us suppose indirectly, that $|P(H_{b_1})| > k_W + 1$. Let $z \in P(H_{b_1})$, $z \notin E_1$. By (2. 6) there exists a chain U belonging to b_1 such that $(A_i \cap V)\pi_U \supset E'_1 \cup \{z\}$. Since by (2. 5) $\sigma\pi_U$ is a b_1-critical colouring, this contradicts (2. 4). $k_W \geqq |P(H_{b_1})| - 1$ implies $k_W \geqq k_W n - n$, i.e. $k_W = n = 2$. But $2 = k_W \geqq |P(H_{b_1})| - 1 \geqq 2$, hence

(2. 9) *If* $W = (b_1, \ldots, b_s)$ *is of type* B, *then* $n = 2$, $|P(H_{b_1})| = 3$. *Let* b_ξ *be the third element of* $P(H_{b_1})$, *then* $(b_1 ; b_\xi) \in H$ *and* $(b_1 ; b_2) \in H$.

Let $W = (b_1, \ldots, b_s)$ be a maximal path of type B and $\sigma = (A_{b_1}, A_1, \ldots, A_n)$ a b_1-critical colouring. Since W is of type B, putting $P(H_{b_1}) = \{b_1, b_2, b_\xi\}$, $\xi \geqq 4$ and therefore $b_{\xi-1} \nsim b_1$. The maximal path $(b_{\xi-1}, \ldots, b_1, b_\xi, \ldots, b_s)$ is of type B by (2. 8) and thus by (2. 9) $(b_\xi ; b_{\xi-1}) \in H$.

$\{(b_1 ; b_\xi), (b_{\xi-1} ; b_\xi)\}$ is a maximal quasi-disjoint set in H_{b_ξ} by (ii) and thus by (iii) $b_{\xi+1}$ "does not exist", i.e. $\xi = s$.

To have a shorter terminology we define b_j for every integer j so that $b_i = b_j$ if and only if $i \equiv j \pmod{s}$.

For $2 \leqq \tau \leqq s-1$ $W_\tau = (b_\tau, \ldots, b_1, b_s, \ldots, b_{\tau+1})$ is a maximal path. It can be seen very easy by induction on τ that W_τ is of type B and using (2. 8) we obtain

(2. 10) $P(H_{b_\tau}) = \{b_{\tau-1}, b_\tau, b_{\tau+1}\}$, $(b_{\tau-1} ; b_\tau) \in H$.

Put $(b_{\tau-1}, b_\tau) = F_\tau$ for $\tau = 1, \ldots, s$. We show, that $H = \{F_1, \ldots, F_s\}$. Let us suppose indirectly, that there are some other edges of H. By (i) and (1. 9) there is an F among them such that $b_\tau \in F$ for some τ. By (iii) and (2. 10) we get $F \subset \{b_{\tau-1}, b_\tau, b_{\tau+1}\}$. Let say $b_{\tau-1} \in F$, then $F \subset \{b_{\tau-2}, b_{\tau-1}, b_\tau\}$ consequently $F = (b_{\tau-1}, b_\tau) = F_\tau$. This is a contradiction.

Thus H equals to the graph which consists of the vertices $b_1, \ldots, b_s$ and edges $F_1, \ldots, F_s$. By (i) this circuit is of odd length, q. e. d.

§ 3

(3. 1) *Let* $k \geqq 2$, s, n *be natural numbers. We can construct a finite set-system* $H(k, s, n) = H$ *with the following properties:*

(i) H *is a uniform* k*-system,*

(ii) *the non-trivial circuits of* H *are longer than* s,

(iii) H *has chromatic number* $\geqq n + 1$.

In [1] ERDŐS and HAJNAL defined the property that the circuits of a uniform k-system are longer than s in an other way as we did in § 1. They said, that a uniform k-system is s-circuitless if for every $1 \leqq t \leqq s$ and for every $H' \subset H$, $|H'| = t$ $|P(H')| \geqq (k-1)t+1$. (1. 7) and (1. 8) show the equivalence of the two definitions.

PROOF. We construct $H(k, s, n)$ by induction on n. For $n=1$ a set-system consisting of a single k-tuple satisfies the requirements (i), (ii), (iii).

Let us suppose, that $H(k, s, n-1)$ is already defined. We take some disjoint copies of $H(k, s, n-1)$. Let $C_1, \ldots, C_\eta$ denote these copies and put $C = \bigcup_i C_i$, $E_i = P(C_i)$ $(1 \leqq i \leqq \eta)$, $H_2 = \{E_1, \ldots, E_\eta\}$ and $|E_i| = l$. We are going to carry out the construction by adding some further k-tuples of $P(C)$. Let H_1 denote the set of these "new" k-tuples.

Let us investigate what kind of properties the set-system $H_1 \cup H_2$ must have to guarantee that $C \cup H_1$ satisfies the requirements (i), (ii) and (iii).

(i) is clearly satisfied.

Put $\bar{F} = F$ if $F \in H_1$ and $\bar{F} = E_i$ if $F \in C_i$. Thus if $F \in C \cup H_1$ then $\bar{F} \in H_1 \cup H_2$ and $\bar{F} \supset F$.

Let $D = \langle (a_1, \ldots, a_r), (F_1, \ldots, F_r) \rangle$ be a non-trivial circuit in $C \cup H_1$. Then $\bar{D} = \langle (a_1, \ldots, a_r), (\bar{F}_1, \ldots, \bar{F}_r) \rangle$ is a circuit in $H_1 \cup H_2$.

If $\bar{D}$ is trivial, then $\bar{F}_1 = \ldots = \bar{F}_r = E_i$ and thus D has been a non-trivial circuit in C_i, and therefore $r > s$.

Thus, if we require the non-trivial circuits of $H_1 \cup H_2$ to be longer than s, then (ii) is satisfied.

We want $C \cup H_1$ to be at least $(n+1)$-chromatic. Let us split the set $P(C \cup H_1)$ into the union of n disjoint classes $A_1, \ldots, A_n$. If there is an $E_i \in H_2$ such that $E_i \cap A_1 = \emptyset$, then by the induction hypothesis one of the classes $A_2, \ldots, A_n$ contains an edge of C_i. (iii) is satisfied if otherwise A_1 contains an edge of H_1, i.e. if we require that if $M \subset P(C)$ and $M \cap E_i \neq \emptyset$ for any $E_i \in H_2$ then there exists an $F \in H_1$ such that $F \subset M$.

Collecting the required properties of $H_1 \cup H_2$ it is enough to show that

(3. 2) *We can construct a finite set-system $K(k, l, s)$ for any $k, l, s \geqq 1$ such that*

a) *$K(k, l, s) = H_1 \cup H_2$ where the edges of H_1 and H_2 are k-tuples and l-tuples respectively, and $H_1 \cap H_2 = \emptyset$,*

b) *if $E_1 \in H_2$ and $E_2 \in H_2$ then $E_1 \cap E_2 = \emptyset$,*

c) *$K(k, l, s)$ contains non-trivial circuits only longer than s,*

d) *if M is a subset of $P(K(k, l, s))$ such that for any $E \in H_2$ $M \cap E \neq \emptyset$, then there exists an $F \in H_1$ such that $F \subset M$.*

We prove this lemma by induction on s and simultaneously on k.

It will be more comfortable to allow the case $k=1$ too. In this case $E = \{1, \ldots, l\}$, $H_1 = \{\{1\}, \ldots, \{l\}\}$, $H_2 = \{E\}$, $K(1, l, s) = H_1 \cup H_2$ satisfies the requirements of the lemma for any l and s.

For $s=1$ c) is insignificant. Let $E_1, \ldots, E_k$ be disjoint l-tuples and let H_1 consist of all k-tuples of $E_1 \cup \ldots \cup E_k$. Putting $H_2 = \{E_1, \ldots, E_k\}$ $H_1 \cup H_2$ satisfies the other requirements.

We may suppose that we have constructed $K(k-1, l, s)$ and $K(k_0, l, s-1)$ for any k_0, and we are going to construct $K(k, l, s)$. We know, that $K(k-1, l, s) = K' = H_1' \cup H_2'$ where H_1' consists of $(k-1)$-tuples and H_2' consists of disjoint

l-tuples. Put $k_0=|H_1'|$. Then $K(k_0, l, s-1)=K''$ exists and $K''=H_1''\cup H_2''$ where the elements of H_1'' and H_2'' are k_0-tuples and l-tuples, respectively. For every $F\in H_1''$ we take a copy K_F' of K' such that the copies ordered to different elements of H_1'' are disjoint. Put $K_F'=H_{F1}'\cup H_{F2}'$ where H_{F1}' and H_{F2}' are the sets of the $(k-1)$-tuples and l-tuples of K_F' respectively. To every $F'\in H_{F1}'$ we order an element $x_{F'}$ of F, to different F'-s we order different elements $x_{F'}$. Put $F'^*=F'\cup\{x_{F'}\}$. We form the set-system H_{F1} of the F'^*-s for every $F'\in H_{F1}'$ and we shall prove, that putting $H_1=\bigcup_{F\in H_1''} H_{F1}$ and $H_2=\bigcup_{F\in H_1''} H_{F2}'\cup H_2''$, $K=H_1\cup H_2$ satisfies the requirements a)—d).

a) H_1 consists of k-tuples and H_2 consists of l-tuples.

b) It is also obvious that the elements of H_2 are disjoint.

c) Put $\bar{E}=E$ if $E\in H_2''$ and $\bar{E}=F$ if $E\in H_{F1}\cup H_{F2}'$.

Let $C=\langle(a_1, \ldots, a_r), (E_1, \ldots, E_r)\rangle$ be a shortest non-trivial circuit in K. By (1. 1) C is simple.

α) Let us suppose, that $\bar{E}_1=\bar{E}_2=\ldots=\bar{E}_r=F$. Obviously $F\in H_1''$, since in case $F\in H_2''$ $E_i=F$ $(1\leqq i\leqq r)$ and C is trivial. There is no vertex of K'' among the a_i-s. Really, $a_i\in P(K'')$ implies $E_{i-1}=E_i$, which is a contradiction since C is simple.

Further $E_i'=E_i\setminus P(K'')\in K_F'$ and thus the circuit $\langle(a_1, \ldots, a_r), (E_1', \ldots, E_r')\rangle$ is a non-trivial circuit in K_F', hence $r\geqq s+1$.

β) Let us suppose now, that there are two different edges among the $\bar{E}_i$-s, say $\bar{E}_1\neq\bar{E}_r$. Putting $i_1=1$ we have

$$\bar{E}_{i_1}=\ldots=\bar{E}_{i_2-1}\neq\bar{E}_{i_2}=\ldots=\bar{E}_{i_3-1}\neq\ldots\neq\bar{E}_{i_t}=\ldots=\bar{E}_r$$

for some $i_1<i_2<\ldots<r$. This implies, that $a_{i_j}\in P(K'')$ $(1\leqq j\leqq t)$.

Further $\bar{E}_1\neq\bar{E}_r$ implies, that one of these edges, say $E_1\notin H_2''$. Then $a_2\notin P(K'')$ and thus $t\leqq r-1$. $\langle(a_{i_1}, \ldots, a_{i_t}), (\bar{E}_{i_1}, \ldots, \bar{E}_{i_t})\rangle$ is a non-trivial circuit in K'', hence $t\geqq s$, and thus $r>t\geqq s$.

d) Let M be a subset of $P(K)$ such that $M\cap E\neq\emptyset$ for any $E\in H_2$. By the induction hypothesis then there exists an $F\in H_1''$ such that $F\subset M$, further there exists an $F'\in H_{F1}'$ such that $F'\subset M$. For $F_0=F'\cup\{x_{F'}\}$ we have $F_0\in H_1$ and $F_0\subset M$.

Thus we have constructed $K(k, l, s)$. As we have seen, if we have $H(k, s, n-1)$ and putting $l=|H(k, s, n-1)|$ we take a copy H_F' of $H(k, s, n-1)$ for every $F\in H_2$ such that the vertices of H_F' are the vertices of F, then $\bigcup_{F\in H_2} H_F'\cup H_1$ satisfies the requirements (i), (ii), (iii).

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References

[1] R. L. Brooks, On colouring the nodes of a network, *Proc. Cambridge Phil. Soc.*, **37** (1941), pp. 194—197.

[2] Blanche Descartes, A three colour problem, *Eureka* (April 1947; solution March 1948) and Solution to Advanced Problem No. 4526, *Amer. Math. Monthy*, **61** (1954), p. 352.

[3] P. Erdős, Graph theory and probability, *Canad. J. Math.*, **11** (1959), pp. 34—38.

[4] P. Erdős and A. Hajnal, On chromatic number of graphs and set-systems, *Acta Math. Acad. Sci. Hung.*, **17** (1966), pp. 61—99.

[5] J. Nešetřil, k-хроматические графы без циклов длинны $\leqq 7$, *Comment. Math. Univ. Carolina*, **7** (1966), pp. 373—376.

[6] A. A. Zykov, On some properties of linear complexes, *Russian Math. Sbornik* ,N. S. **24 (66)** (1949), pp. 163—188.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (1—2), (1968), pp. 69—74.

ON SOME EXTREMAL GRAPHS

By
U. S. R. MURTY (Calcutta. India)
(*Presented by* P. ERDŐS)

§ 1. Introduction. A graph is called k-accessible if the diameter of the graph is $\leqq k$. Let n, k, l and s be positive integers such that $n > l \geqq k$ and $n > s$. Then $G_V(n, k, l, s)$ denotes a k-accessible graph on n-vertices such that all subgraphs obtained by suppressing any s or less number of vertices are l-accessible. A graph with minimum possible number of edges within the class $G_V(n, k, l, s)$ is denoted by Min $G_V(n, k, l, s)$ and the minimum possible number of edges is denoted by $M_V(n, k, l, s)$. $G_E(n, k, l, s)$ denotes a k-accessible graph with the property that all its partial graphs obtained by suppressing any s or less number of edges are l-accessible. A graph with minimum possible number of edges within the class $G_E(n, k, l, s)$ is denoted by Min $G_E(n, k, l, s)$ and the minimum possible number of edges is denoted by $M_E(n, k, l, s)$. It was proved in [2] that there exists a function $f_1(s)$ of s such that whenever $n > f_1(s)$ we have $M_V(n, 2, 2, s) = (s+1)(n-s-1)$ and the structures of Min $G_V(n, 2, 2, s)$ are complete bipartite graphs with $s+1$ vertices in one set and $n-s-1$ in the other set. It was proved in [3] that there exists a function $f_2(s)$ of s such that whenever $n > f_2(s)$ we have $M_E(n, 2, 2, s) = \frac{s(s+1)}{2} + (s+1)(n-s-1)$. The structures of the extremal graphs in this case are obtained by completing the set of $s+1$ vertices in the Min $G_V(n, 2, 2, s)$. This paper contains a discussion about the structures of Min $G_V(n, 2, l, 1)$ and Min $G_E(n, 2, l, 1)$ for $l \geqq 3$.

The study of such extremal graphs is of very great interest in the theory of communication networks. The applications were discussed in detail by the author in [3].

§ 2. The structures of Min $G_V(n, 2, l, 1)$. We shall first observe by actual construction that

$$M_V(n, 2, l, 1) \leqq 2n-5 \quad \text{for} \quad n \geqq 5 \quad \text{and} \quad l \geqq 3.$$

$B_n^{(1)}$ denotes the class of graphs on n vertices ($n \geqq 5$) which are obtained in the following way. We start with a cycle $(x_1, x_2, x_3, x_4, x_5, x_1)$ of length five. Some of the other (possibly none) $n-5$ vertices are simultaneously joined to both x_2 and x_5 and all the remaining vertices (possibly none) together are joined to both x_1 and x_4 or both x_3 and x_5. Figures 1 and 2 represent two typical members of $B_7^{(1)}$.

It is easy to observe that a member of $B_n^{(1)}$ contains $2n-5$ edges and the class $B_n^{(1)}$ is contained within the class $G_V(n, 2, l, 1)$ for $l \geqq 3$. Hence

$$M_V(n, 2, l, 1) \leqq 2n-5 \quad \text{for all} \quad l \geqq 3 \quad \text{and} \quad n \geqq 5.$$

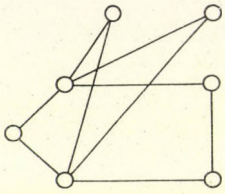

Fig. 1

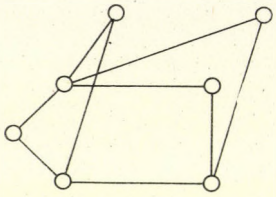

Fig. 2

We will in fact prove under theorem 1 that $M_V(n, 2, l, 1) = 2n-5$ for all $n \geqq 5$ and $l \geqq 3$, but $B_n^{(1)}$ will not* exhaust all the extremal graphs in the class Min $G_V(n, 2, l, 1)$.

The following observations are crucial for the proof of theorem 1.

(1) If u and v are any two vertices of degree two in a $G_V(n, 2, l, 1)$, both connected to p and q, then the subgraph obtained by suppressing v is a $G_V(n-1, 2, l, 1)$; and conversely

(2) if u is of degree two with $\Gamma_u = \{p, q\}$ in a $G_V(n, 2, l, 1)$, then by adding a new vertex z with $\Gamma_z = \{p, q\}$ we obtain a $G_V(n+1, 2, l, 1)$.

Now we shall prove that

(3) if $n \geqq 5$ then any Min $G_V(n, 2, l, 1)$ will contain a vertex of degree 2, for $l \geqq 3$. From the observations made earlier it is clear that

$$M_V(n, 2, l, 1) \leqq 2n-5.$$

Now if there is a vertex of degree 2 in a Min $G_V(n, 2, l, 1)$ then let

$$\min_{x \in X} |\Gamma_x| = m > 2$$

where X stands for the set of vertices of the Min $G_V(n, 2, l, 1)$.

Since the degree of each vertex is at least m, we have

$$\text{(A)} \qquad \frac{nm}{2} \leqq 2n-5 \quad \text{or} \quad m \leqq \frac{4n-10}{n}.$$

As for $n \geqq 5$ (A) implies that $m \leqq 3$, so if $m \geqq 2$ then $m = 3$. If $n < 10$ then

$$\frac{3n}{2} > 2n-5$$

and hence Min $G_V(n, 2, l, 1)$ cannot but contain a vertex of degree two for $n < 10$. If $n > 10$, then let x be any vertex of degree 3. (X, Γ_x) accounts for 3 edges. Each vertex belonging to $X - x - \Gamma_x$ should be joined to some vertex in Γ_x. These account for $n-4$ edges. Further, the degree of each vertex in $X - x - \Gamma_x$ being at least 3, the graph would contain at least $n-4$ more edges. But with this bare minimum we have a cycle in $X - x - \Gamma_x$ and this forces the diameter of the graph to be $\geqq 3$ although we have already been forced to use $2n-5$ edges. And m should therefore be equal to 2 in the extremal case.

* This was pointed out to me by Mr. BOLLOBÁS.

From observations (1), (2) and (3) it follows that we can restrict our structural study of extremal graphs to graphs which do not have configuration like $\Gamma_u = \Gamma_v = \{p, q\}$, for, in that case we can omit a vertex like v and if we have a basic solution, we can add new vertices like z. During the proof of the theorem 1 concerning the minimal structures in $G_V(n, 2, l, 1)$ we shall assume that the minimal graphs are free from configuration of the above type. We shall call this the assumption regarding the absence of rectangles.

The cycle of length five is the first basic graph that will prove useful in the discussion of extremal structures Min $G_V(n, 2, l, 1)$. The other basic graph is the one described below.

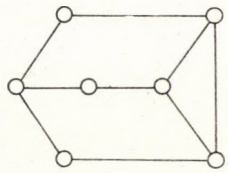

Fig. 3

This graph on 7 vertices has 9 edges and the graphs on n vertices ($n \geqq 7$) obtained by sequentially adding $n-7$ new vertices and joining the new vertex at every stage to two of the old vertices that happen to be the vertices adjacent to a vertex of degree two, we obtain the class $B_n^{(2)}$. Clearly $B_n^{(2)}$ are also $G_V(n, 2, l, 1)$. We shall in fact prove under theorem 1 that $B_n^{(1)}$ and $B_n^{(2)}$ provide the extremal structures Min $G_V(n, 2, l, 1)$.

THEOREM 1. *$M_V(n, 2, l, 1) = 2n-5$ and $B_n^{(1)}$ and $B_n^{(2)}$ provide the extremal structures* Min $G_V(n, 2, l, 1)$ *when* $n \geqq 5$ *and* $l \geqq 3$.

PROOF. By observation (3) there is a vertex of degree 2 in a Min $G_V(n, 2, l, 1)$, whenever $n \geqq 5$. Let x be any vertex of degree 2. Let a and b be the vertices in Γ_x. Let Y denote the set $X-x-a-b$. Let Y_a, Y_b and Y_{ab} be the disjoint subsets of Y that are joined to a, to b, to both a and b, respectively. We shall write

$$|Y_{ab}| = r, \quad |Y_a| = k, \quad |Y_b| = n-r-3-k.$$

First let us examine the possibility $|Y_a| = 0$. In this case each vertex $y_b \in Y_b$ would have to be joined to some vertex belonging to Y_{ab} by means of a chain not passing through b and in this case we have to have at least $2n-4$ edges and hence would not lead us to the best. Similarly we cannot have $|Y_b| = 0$.

Hence we may suppose that $|Y_a| \neq 0$ and $|Y_b| \neq 0$. There should be a chain of length two between any $y_a \in Y_a$ and any $y_b \in Y_b$ and these edges should be in Y. This means that there must be a connected component of Y, containing $Y_a \cup Y_b$. But if $Y_a \cup Y_b$ is a connected component then we have already accounted for at least $2n-5$ edges and any further edge will take us away from the best. Hence a and b cannot be joined and $Y_a \cup Y_b$ must be by itself a connected component and in fact a tree in $Y_a \cup Y_b$. Now because of the assumption regarding the absence of rectangles in the minimal structure we will have $|Y_{ab}| = r = 0$. The tree in $Y_a \cup Y_b$ cannot have a vertex d connected to two pendant vertices e and f of the tree, for (1) if e and f belong to the same set, namely Y_a or Y_b, then by our assumption

e could be omitted (2) if $d, e \in Y_a$ and $f \in Y_b$, then the distance between e and b is 3, and the diameter of the graph would not any more be 2. Consequently the tree $Y_a \cup Y_b$ is a simple chain. This chain cannot be of length two, for if it is of length two then we have a diameter 3 for the graph. The length of the chain cannot be more than three either, for if it is so, we again have a diameter strictly greater than 2. Therefore there are two possibilities: either the chain is of length 1, or length 3, giving the following two basic graphs.

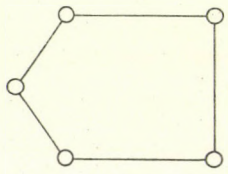

Fig. 4

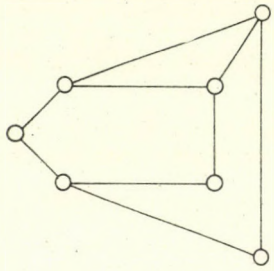

Fig. 5

The graph in figure 5 is the same as the graph in figure 3. Now from these two basic graphs we develop the minimal structures. This completes the proof that $B_n^{(1)}$ and $B_n^{(2)}$ provide the extremal structures Min $G_V(n, 2, l, 1)$ for $n \geqq 5$.

REMARK. Since $M_V(n, 2, l, 1) = 2n - 5$ for all $l \geqq 3$, it follows that a biconnected graph of diameter two would have to have at least $2n - 5$ edges when $n \geqq 5$.

§ 3. The structures of Min $G_E(n, 2, l, 1)$. We shall first observe by actual construction that

$$M_E(n, 2, \lambda, 1) \leqq n - 1 + \left[\frac{n}{2}\right] \quad \text{for} \quad n > 8 \quad \text{and} \quad \lambda \geqq 3.$$

H_n $(n > 6)$ is the class of graphs on n vertices defined in the following manner. We arbitrarily divide the set of vertices into two sets, one set containing a single vertex and the other set containing the remaining $n - 1$ vertices. The single vertex is joined to each vertex of the other set. Further the vertices of the larger set are joined within themselves by a set of $[n/2]$ edges.

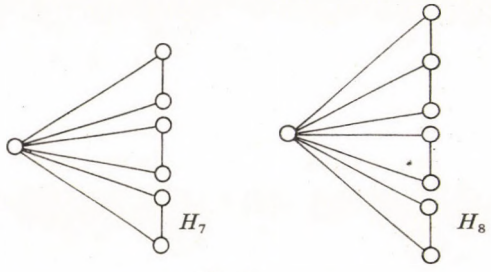

Fig. 6

Figure 6 represents a member of H_7 and H_8.

A member of H_n contains $n-1+[n/2]$ edges and the class H_n is contained within the class $G_E(n, 2, l, 1)$, for all $l \geqq 3$. Under theorem 2 we shall prove that in fact H_n coincides with Min $G_E(n, 2, l, 1)$ for $l \geqq 3$ and n sufficiently large.

THEOREM 2. *$M_E(n, 2, l, 1) = n-1+[n/2]$ and* Min $G_E(n, 2, l, 1)$, *coincide with H_n, for all $l \geqq 3$ and $n > 8$.*

PROOF. First we observe that there is a vertex of degree two in a Min $G_E(n, 2, l, 1)$, for all $l \geqq 3$, for, if this is not the case, there would have to be at least $3n/2$ edges contradicting the observetion that $M_E(n, 2, l, 1) \leqq n-1+[n/2]$. Consider a Min $G_E(n, 2, l, 1)$ for some $l \geqq 3$ and $n > 8$. Let V denote the set of its vertices. Let x be a vertex with degree 2 and let a and b be the two vertices in Γ_x. Let Y denote the set $V-x-a-b$. Let Y_a, Y_b and Y_{ab} be the subsets of Y that are joined to a, to b, to both a and b, respectively. We shall write

$$|Y_{ab}| = r, \quad |Y_a| = k, \quad |Y_b| = n-r-3-k.$$

Without loss of generality we can assume that $k \leqq \dfrac{n-r-3}{2}$.

The proof of theorem 2 will be divided into two parts. In the first part we shall suppose that a and b are not joined and in the second part the assumption will be that a and b are joined.

Part I. Here a and b are not joined. In this case it is not possible to have $|Y_{ab}| = 0$. For if $|Y_{ab}| = 0$, Y would have to be connected in the subgraph spanned by itself and this requires $|Y|-1 = n-4$ edges, and in all we require at least $2n-5$ edges. But $2n-5 > n-1+[n/2]$ when $n > 8$. Again, $|Y_a|$ and $|Y_b|$ cannot be simultaneously zero, for, when $|Y_a|$ and $|Y_b| = 0$, the number of edges is at least $2n-4$ and $2n-4 > n-1+[n/2]$ when $n > 8$. Now let $|Y_a| = 0$, $|Y_b| \neq 0$, $|Y_{ab}| \neq 0$. Each Y_b should be joined to some Y_{ab} and this would mean that the graph would require at least $(n+r-1)+(n-r-3) = 2n-4$ edges, which does not lead us to the best when $n > 8$. By symmetry we can rule out the possibility $|Y_a| \neq 0$, $|Y_b| = 0$ and $|Y_{ab}| \neq 0$. Finally suppose that none of $|Y_a|$, $|Y_b|$ and $|Y_{ab}|$ is zero. Here, again each Y_a should be joined to some vertex in $Y - Y_a$ and each Y_b should be joined to some vertex in $Y - Y_b$. Using the edges within Y we should be able to trace chains of length two between any Y_a and any Y_b. This would mean that the edges within Y would at least be $n-r-4$ in number and therefore the graph would have to have $(n+r-1)+$ $+(n-r-4) = 2n-5$ edges at least, and it cannot be the best for $n > 8$. Hence we can conclude that in a Min $G_E(n, 2, l, 1)$ the descendents of a vertex of degree two are joined to each other.

Part II. Here we examine the situation when a and b are joined. From the conclusion of the previous part of the proof we know that in a Min $G_E(n, 2, l, 1)$ a and b are joined. If $M_E(n, 2, l, 1) \leqq n-1+[n/2]$, then the edges in the subgraph spanned by Y together with the edges in $(Y, \{a, b\})$ should not be more than $n-3+\left[\dfrac{n-2}{2}\right]$ in number. And it can be seen that at least these many edges are always necessary and exactly these many lead to a $G_V(n, 2, l, 1)$ when $|Y_a| = 0$, $|Y_{ab}| = 0$ and $|Y_b| = n-3$, or $|Y_b| = 0$, $|Y_{ab}| = 0$ and $|Y_a| = n-3$, or when n is

even, $|Y_a|=n-4$, $|Y_{ab}|=1$ and $|Y_b|=0$, or $|Y_b|=n-4$, $|Y_{ab}|=1$ and $|Y_a|=0$. These are no other graphs than the members of H_n. This completes the proof that Min $G_E(n, 2, l, 1)=H_n$ for all $l\geqq 3$ when $n>8$.

REMARK. It follows from theorem 2 that a bicoherent graph on n vertices of diameter 2 will have to have at least $n-1+[n/2]$ edges.

Several other extremal structures in this direction have been recently studied by MR. BOLLOBÁS. The author is thankful to MR. BOLLOBÁS for pointing out several mistakes and inadequacies in the proofs in an earlier version of the paper. He expresses his deep gratitude to Professor P. ERDŐS with whom he had useful discussions about the problem.

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Added in proof (1 April 1968). When $n=10$, $M_V(n, 2, l, 1)$ can have a vertex of degree 3. PETERSEN's graph is the exceptional one in this category.

References

[1] C. BERGE, *Theory of graphs and its applications* (Mathnen, London).
[2] U. S. R. MURTY and K. VIJAYAN, On accessibility in graphs, *Sankhyā*, Series A, Vol. **26** (1964), pp. 279—302.
[3] U. S. R. MURTY, A structural study of graphs with respect to accessibility (unpublished).

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Tomus 19 (1—2), (1968), pp. 75—80.

A PROBLEM OF THE THEORY OF COMMUNICATION NETWORKS

By
B. BOLLOBÁS (Budapest)
(*Presented by* P. ERDŐS)

The present paper is strongly related to the previous paper of MURTY [3]. For further references see [1] and [2]. Using the notations of [3], the aim of this paper is to determine $M_V(n, 3, \lambda, 1)$ for $\lambda \geqq 4$.

For the sake of completeness we repeat the problem and the notations. The theory of communication networks motivated the following graph theoretical problem. Consider the graphs of n vertices and diameter at most k, having the property that by suppressing any v of the vertices (e edges respectively), the subgraphs (partial graphs respectively) obtained have at most diameter λ. The class of these graphs is denoted by $G_V(n, k, \lambda, v)$ ($G_E(n, k, \lambda, e)$ respectively).[1] The problem is to determine the minimum possible number of edges within the class $G(n, k, \lambda, v)$, which will be denoted by $M(n, k, \lambda, v)$, and find at least some — but possibly all the — extremal graphs. This obviously corresponds to the problem of constructing "economical" networks, which are "reliable" in a sense.

In the following the cases $k=3$, $\lambda=4$, $v=1$ and $k=3$, $\lambda>4$, $v=1$ will be investigated.

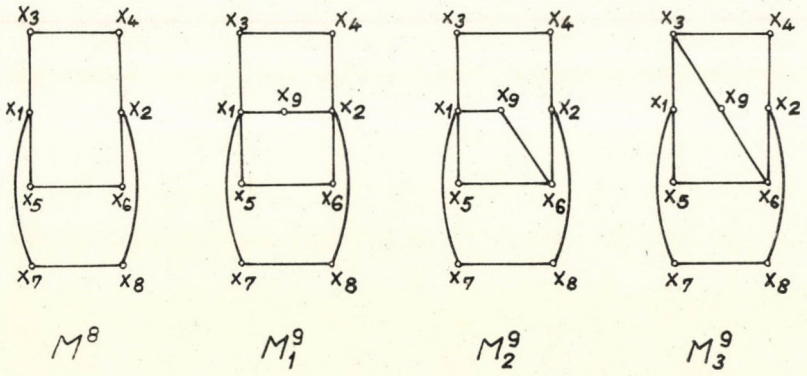

Fig. 1

Denote by M^{2m} the following graph ($m \geqq 3$): the vertices are $x_1, x_2, \ldots, x_{2m}$ and the graph contains the edges x_1x_{2i+1}, $x_{2i+1}x_{2i+2}$, $x_{2i+2}x_2$ ($i=1, 2, \ldots, m-1$). The graphs M_1^{2m+1}, M_2^{2m+1} and M_3^{2m+1} are obtained from M^{2m} by adding the vertex x_{2m+1} together with the edges x_1x_{2m+1}, $x_{2m+1}x_2$, the edges x_1x_{2m+1}, $x_{2m+1}x_4$

[1] As only the class G_V will be studied here, the subscript V will be omitted in the following.

and the edges x_3x_{2m+1}, $x_{2m+1}x_6$, respectively. It is easily seen that M^{2m} and M_i^{2m+1} $(i=1, 2, 3)$ belong to the class $G(n, 3, 4, 1)$ for $n=2m$ and $n=2m+1$, respectively, and they have $\left[\frac{3n-5}{2}\right]$ edges[2] (see Fig. 1). It will be proved that these are the best possible constructions.

THEOREM 1. *If* $n \geqq 5$, *then* $M(n, 3, 4, 1) = \left[\frac{3n-5}{2}\right]$.

The above constructed graphs show that the right hand side is greater than or equal to the left hand side, so it is sufficient to prove the opposite inequality.

We shall use induction on n, though it is not essential, only it helps to avoid a few simple cases.

The theorem is true for $n=5$. Suppose it is true for graphs of at least 5 and at most $n-1$ vertices.

Let G be a graph of $G(n, 3, 4, 1)$ with $M(n, 3, 4, 1)=M$ edges.

G is connected and it has no vertex of articulation, since if it had, by omitting it the graph obtained would be disconnected. So every vertex of G has degree at least 2. On the other hand $M \leqq \left[\frac{3n-5}{2}\right] < \frac{3}{2}n$, consequently G contains at least one vertex of degree 2. Suppose α is one of these and α is connected to x and y.

If G contains the edge xy, then the subgraph obtained by suppressing the vertex α belongs to the class $G(n-1, 3, 4, 1)$, and has $M-2$ edges. According to the inductional hypothesis $M-2 \geqq \left[\frac{3(n-1)-5}{2}\right]$, so $M \geqq \left[\frac{3n-5}{2}\right]$, which is the required inequality.

If the vertices x and y are not connected, then by omitting the vertex α and adding the edge xy, the graph obtained belongs to $G(n-1, 3, 4, 1)$, and hence $M-1 \geqq \left[\frac{3(n-1)-5}{2}\right]$. If n is even, this is exactly the required inequality, so in the following n can be supposed to be odd.

If there are at least two more vertices, different from α, connected to x and y then, by a simple argument it can be shown that, the graph obtained by omitting the vertex α belongs to $G(n-1, 3, 4, 1)$. Consequently $M \geqq \left[\frac{3(n-1)-5}{2}\right]+2 = \left[\frac{3n-5}{2}\right]$ (n is odd), so it can be supposed that G does not contain two such vertices.

In the following two cases will be studied separately, though because of the similarity only the first case will be examined in details.

Case a) $n=2m+1 \geqq 7$, *and* G *has two vertices,* x *and* y, *for which there is exactly one vertex, say vertex* α, *connected to* x *and* y, *and* α *is of degree* 2.

Denote by X the set of vertices of G different from α and connected to x, Y is similarly defined for y. According to the conditions $X \cap Y = \emptyset$. Put $R=G-\alpha-x-y-X-Y$ (Fig. 2). If R is empty, $|X|+|Y|=2m-2$,[3] and as the vertices

[2] $[x]$ denotes the integer part of x.

[3] $|S|$ denotes the number of elements (vertices) in S.

of G have degree at least 2, the subgraph spanned by $X \cup Y$ has at least $\frac{2m-2}{2} = m-1$ edges, hence G contains at least $2+(2m-2)+(m-1)=3m-1=\left[\frac{3(2m+1)-5}{2}\right]$ edges. From this follows that R can be supposed to be not empty. Denote the components of R by $R_1, R_2, \ldots, R_j$ and let U be the set of vertices of $X \cup Y$, not connected with any vertex of $X \cup Y$. (I. e. $X \cup Y - U = V$ is a maximal subset of $X \cup Y$, spanning a subgraph without an isolated vertex.)

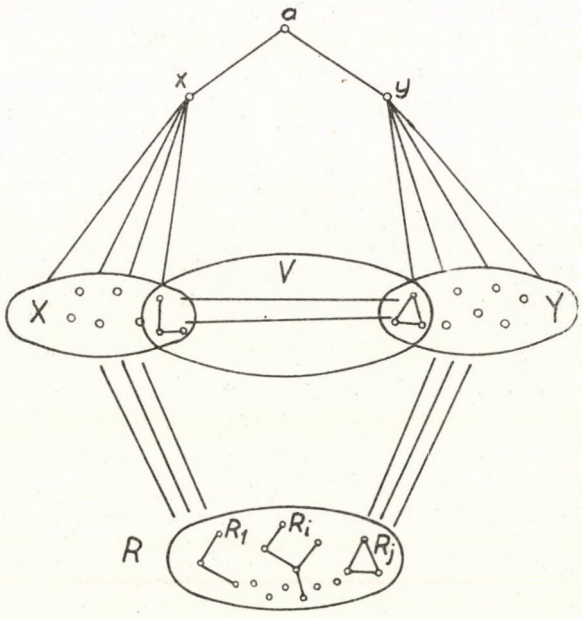

Fig. 2

To give reason for the argument below, we remark the following. In G there are approximately three edges for two vertices. Even if only the edges enumerated above are joining to the vertices of V, by omitting the vertices of V, at least $\frac{3}{2}|V|$ edges will be omitted, so V can not be expected to have many more edges than the enumerated ones. This is why the edges going to the vertices of the set U will be investigated more closely.

Two inequalities will be proved below for the components R_i, corresponding to different conditions, one of which holds for each R_i.

Let r_i denote the number of vertices of R_i, e_i the number of edges. The number of edges connecting a vertex of R_i with a vertex of $X \cup Y$ will be denoted by f_i. Let U_i be the set of those vertices of U which are connected to R_i, and put $|U_i| = u_i$.

(1) *If no edge connects a vertex of* R_i *to a vertex of* V, *then*

$$\frac{u_i - 1}{2} + \frac{3}{2} r_i \leqq e_i + f_i. \tag{1'}$$

The diameter of G is at most three, and so each vertex of R_i is connected to $X \cup Y$, since there is a path of length at most three between α and the vertices of R_i. Consequently $r_i \leqq f_i$ and obviously $u_i \leqq f_i$. R_i is a component of R, so it is connected and hence $e_i \geqq r_i - 1$. Adding the first two and the double of the third inequalities, the inequality

$$r_i + u_i + 2r_i - 2 \leqq 2e_i + 2f_i$$

is obtained.

Hence

$$\frac{u_i-1}{2}+\frac{3}{2}r_i-\frac{1}{2}\leqq e_i+f_i.$$

Saving the case $r_i=f_i$, $u_i=f_i$ and $r_i-1=e_i$, this implies $\frac{u_i-1}{2}+\frac{3}{2}r_i-\frac{1}{2}<$ $<e_i+f_i$, which is equivalent to (1′).

So to prove inequality (1′) it is sufficient to show that the following situation can not arise:

(∗) *R_i is a tree, each vertex of which is connected to exactly one vertex of $X\cup Y$, and this vertex belongs to U, furthermore there is no vertex of $X\cup Y$ which is connected to two vertices of R_i.*

Suppose (∗) is satisfied for R_i.

$r_i=1$ is trivially impossible.

In the case $r_i=2$ R_i has two vertices, say b_1 and b_2, and the edges b_1c, b_1b_2 and b_2d are adjacent to b_1, b_2 (where $c, d\in U$, $c\neq d$). It can be supposed that $c\in X$. A path of length at most three connects b_1 to y, hence $d\in Y$. But then by suppressing c the distance of b_1 and x is greater than four.

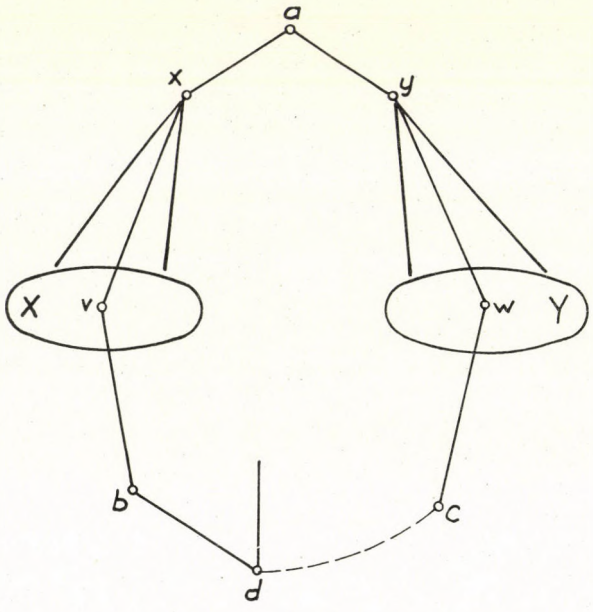

Fig. 3

If $r_i\geqq 3$, first it will be shown that if b and c are two non-adjacent vertices of R_i, and G contains the edges bv and cw, then either $v\in X$ and $w\in X$ or $v\in Y$ and $w\in Y$. R_i contains a unique bc path (since R_i is a tree), let bd be the first edge of it. Suppress the vertex d. It is easily seen that because of (∗) every path connecting b and c has at least five edges (Fig. 3).

From this statement above it follows that R_i is a star. For if b is an endvertex of R_i, joined to X, and if it is connected to the vertex c of R_i, then b is not connected to the vertices of R_i different from b and c. Consequently all the vertices of R_i except c are joined to the set X, and so R_i must be a star with centre c. As there exists a path of at most three between b and y, c is connected to a vertex of Y. But then if this vertex is suppressed, c and y can not be connected by a path of length at most four.

This proves that there is no graph with property (∗), and so (1) holds.

(2) *If R_i contains a vertex connected with a vertex of V, then*

$$\frac{u_i}{2}+\frac{3}{2}r_i \leqq e_i+f_i. \tag{2'}$$

To prove this, the proof of inequality (1′) can be repeated almost without change.

Having obtained the inequalities (1′) and (2′), we shall prove the required inequality, i. e. that $M \geqq \left[\frac{3n-5}{2}\right] = 3m-1$.

Put $|U|=u$, $|V|=v$, $|R|=r$. Let the components of R be $R_1, R_2, \ldots, R_j$. It can be supposed that the notation is such that condition (1) holds for the components $R_1, R_2, \ldots, R_k$, and condition (2) holds for $R_{k+1}, R_{k+2}, \ldots, R_j$ $(0 \leqq k \leqq j)$. As any vertex of R_i is connected by a path of at most three to any vertex of R_l, $U_i \cap U_l$ can not be empty if $1 \leqq i \leqq k$. Consequently

$$1+\sum_1^j (u_i-1) \geqq u, \tag{3}$$

if $k=j$, and

$$\sum_1^k (u_i-1)+\sum_{k+1}^j u_i \geqq u \tag{4}$$

if $k<j$.

The subgraph spanned by V contains at least $v/2$ edges, hence simply by counting the number of edges:

$$M \geqq 2+u+v+\frac{v}{2}+\sum_1^j (e_i+f_i). \tag{5}$$

If $j=k$, the inequalities (5), (1′) and (3) give

$$M \geqq 2+u+v+\frac{v}{2}+\sum_1^j \frac{u_i-1+3r_i}{2} \geqq 2+u+\frac{3}{2}v+\frac{u-1+3r}{2} = 3m-\frac{3}{2},$$

for $u+v+r=2m-2$. As M is an integer, $M \geqq 3m-1$ also holds.

In the case $k<j$ by the application of the inequalities (5), (1′), (2′), (3) and (4) the following inequality is obtained:

$$M \geqq 2+u+v+\frac{v}{2}+\sum_1^k \frac{u_i-1+3r_i}{2}+\sum_{k+1}^j \frac{u_i+3r_i}{2} \geqq 2+u+\frac{3}{2}v+\frac{u}{2}+\frac{3}{2}r = 3m-1.$$

This completes the proof of theorem 1 in Case a).

Case b) $n=2m+1$, *and if α is any vertex of degree* 2 *(say connected with x and y), then G has another vertex, say b, which is also connected with x and y (i. e. every vertex of degree* 2 *is contained in a quadrangle).*

Let U be the set of vertices of $X \cup Y$, which are different from b and are not connected to vertices of $X \cup Y$, otherwise keep the notations of the previous case. According to the hypothesis $X \cap Y = \{b\}$.

The proof of inequality (1) can be carried out without any change. However, by using the condition b), it is easy to show that equality can not hold in inequality

(2'). From this $M \geqq 3m-1$ follows obviously, moreover one can prove that equality can not hold in this case.

The proof of theorem 1 is thus complete.

It seems to be very likely that the graphs M^{2m}, M_i^{2m+1} $(i=1, 2, 3)$ are the extremal graphs, but to prove this by our method one ought to accomplish a rather tiresome discussion.

THEOREM 2. *If* $\lambda \geqq 5$ *and* $n \geqq 6$, *then* $M(n, 3, \lambda, 1) = \left[\frac{3n}{2}\right] - 3$.

In other words the theorem states that if the diameter of a graph is at most three and it has no vertex of articulation, then the graph has at least $\left[\frac{3n}{2}\right] - 3$ edges.

Let $m \geqq 3$ be an arbitrary number and put $N^{2m} = M^{2m}$, furthermore denote by N^{2m+1} the following graph: $x_0, x_1, x_2, \ldots, x_{2m}$ are the vertices, and the graph

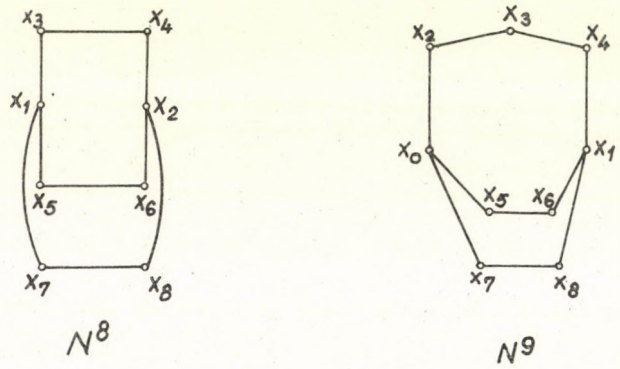

Fig. 4

contains the edges x_0x_2, x_2x_3, x_3x_4, x_4x_1; x_0x_5, x_5x_6, x_6x_1; ...; x_0x_{2m-1}, $x_{2m-1}x_{2m}$, $x_{2m}x_1$ (see Fig. 4). It is immediate that for $n=2m$ and $n=2m+1$, N^{2m} and N^{2m+1} respectively belong to the class $G(n, 3, \lambda, 1)$, and they have $\left[\frac{3n}{2}\right] - 3$ edges. This shows that $M(n, 3, \lambda, 1)$ is at most $\left[\frac{3n}{2}\right] - 3$.

The proof of the opposite inequality, which uses similar arguments to that of the proof of theorem 1, is omitted here.

It is easily seen that N^{2m} is *not* the unique extremal graph, but probably N^{2m+1} is the only extremal graph.

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References

[1] U. S. R. MURTY and K. VIJAYAN, On accessibility in graphs, *Sankhya*, series A, Vol. **26**, pp. 279—302.
[2] U. S. R. MURTY, A structural study of graphs with respect to accessibility (unpublished).
[3] U. S. R. MURTY, On some extremal graphs, *Acta. Math. Acad. Sci. Hung*, **19** (1968), pp. 69—74.

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RATIONAL APPROXIMATION IN CERTAIN CLASSES OF FUNCTIONS

By
J. SZABADOS (Budapest)
(*Presented by* P. TURÁN)

1.

Let

$$-1 = \xi_0 < \xi_1 < \ldots < \xi_s = +1 \tag{1.1}$$

be a fixed subdivision of the interval $[-1, +1]$. In this paper we consider the rational approximation of such continuous functions which have better properties in the intervals

(1.2)

$$I_k(\varepsilon) = [\xi_k + \varepsilon, \xi_{k+1} - \varepsilon] \quad (k = 0, 1, \ldots, s-1) \quad \text{for all} \quad 0 < \varepsilon < \min_{0 \leqq k \leqq s-1} \frac{\xi_{k+1} - \xi_k}{2}$$

than in the whole $[-1, +1]$. It will be proved that the rational approximation is generally better for these functions than the polynomial one.

First of all we quote a lemma which will be used later.

LEMMA. *Let $f(x)$ be a continuous function in $[-1, +1]$ and* $\max_{-1 \leqq x \leqq +1} |f(x)| = M$. *Further let* $-1 = \xi_n^{\{0\}} < \xi_n^{\{1\}} < \ldots < \xi_n^{\{s(n)\}} = +1$,

$$\delta_n = \min_{1 \leqq k \leqq s(n)} (\xi_n^{\{k\}} - \xi_n^{\{k-1\}}) \quad (n = 2, 4, 6, \ldots) \tag{1.3}$$

where $s(n)$ $(n = 2, 4, 6, \ldots)$ are some kind of integers. If we have such polynomials $p_n^{\{k\}}(x)$ of degree n at most that

$$\max_{1 \leqq k \leqq s(n)} \max_{\xi_n^{\{k-1\}} \leqq x \leqq \xi_n^{\{k\}}} |f(x) - p_n^{\{k\}}(x)| = \varepsilon_n \quad (n = 2, 4, 6, \ldots); \qquad \lim_{n \to \infty} \varepsilon_n = 0$$

then there exist rational functions $r_N(x)$ of degree N at most for which

$$|f(x) - r_N(x)| \leqq 4\varepsilon_n \quad (-1 \leqq x \leqq +1, N \geqq N_0) \tag{1.4}$$

where n is the greatest even number satisfying the inequality

$$s(n)\left(3n + \log^2 \frac{57 M^2 n^2 s(n)}{\delta_n \varepsilon_n^3}\right) \leqq N. \tag{1.5}$$

For the proof see [1].

2.

The first class of functions of above mentioned type was introduced by P. SZÜSZ and P. TURÁN [2]. Using the notations (1. 1), (1. 2), they considered that functions $f(x)$ for which

$$(2.1)\quad |f(x_1)-f(x_2)| \leqq \begin{cases} C(\varepsilon)|x_1-x_2|^{\beta} & \text{if } x_1, x_2 \in I_k(\varepsilon) \quad (k=0,1,\ldots,s-1), \\ K|x_1-x_2|^{\alpha} & \text{if } x_1, x_2 \in [-1,+1] \end{cases}$$

holds, where $0<\alpha<\beta\leqq 1$, $K>0$ is a constant depending only on $f(x)$, $c(\varepsilon)>0$ a monoton decreasing continuous function for which

$$(2.2)\quad \lim_{\varepsilon=+0} c(\varepsilon) = +\infty$$

can hold. I. e., $f(x)\in \mathrm{Lip}\,\alpha$ in the interval $[-1, +1]$ but $f(x)\in \mathrm{Lip}\,\beta$ in the intervals $I_k(\varepsilon)$. Let us denote this class of functions by $C(c(\varepsilon), \alpha, \beta, s)$.

The order of polynomial approximation in this class is generally not better than $O(N^{-\alpha})$. However, P. SZÜSZ and P. TURÁN stated in [2] that the rational approximation gives the order $O(N^{-\beta})$. They did not prove this result. Later G. FREUD [3] proved that for $C\left(\frac{1}{\varepsilon^{\alpha}}, \alpha, \beta, s\right)(\beta\leqq 2\alpha)$ the order of rational approximation is $O(N^{-\beta})$. (In fact, he proved this result under a little stronger condition than (2. 1).)

Now we prove the following

THEOREM 1. *Let $f(x)\in C(c(\varepsilon), \alpha, \beta, s)$ and denote by a_n the solution of the equation*

$$(2.3)\quad c(a_n) = Ka_n^{\alpha} n^{\beta-\alpha}$$

for sufficiently large n's. Then there exist rational functions $r_N(x)$ of degree N at most for which

$$(2.4)\quad |f(x)-r_N(x)| \leqq 96K(2s+1)^{\alpha}\left(\frac{a_n}{N}\right)^{\alpha} \quad \left(n=2\left[\frac{N}{8(2s+1)}\right], -1\leqq x\leqq +1, N\geqq N_1\right).$$

PROOF. The equation (2. 3) evidently has a unique solution for sufficiently large n's (n, as an index depending on N will be defined later). Because of (2. 2) obviously

$$(2.5)\quad \lim_{n\to\infty} a_n = 0$$

holds. Thus we may apply our Lemma so that the $\xi_n^{(k)}$'s are $-1=\xi_0, \xi_0+a_n, \xi_1-a_n, \xi_1+a_n, \ldots, \xi_s-a_n, \xi_s=+1$. Therefore (see (1. 3))

$$s(n) = 2s+1 \text{ (fixed)}, \quad \delta_n = a_n.$$

By Jackson's theorem and (2. 1), we can approximate $f(x)$ by polynomials of degree n at most in the intervals $[\xi_0, \xi_0+a_n]$, $[\xi_k-a_n, \xi_k+a_n]$ $(k=1, \ldots, s-1)$ and $[\xi_s-a_n, \xi_s]$ with an error $12Ka_n^{\alpha}n^{-\alpha}$ at most, and in the intervals $[\xi_k+a_n, \xi_{k+1}-a_n]$ $(k=0, 1, \ldots, s-1)$ with an error $12c(a_n)n^{-\beta}$ at most. Hence by (2. 3), we may put

$$(2.6)\quad \varepsilon_n = 12Ka_n^{\alpha}n^{-\alpha}.$$

But because of (2. 2) and (2. 3) $a_n^{\alpha} \geqq n^{\alpha-\beta}$ and so for sufficiently large n's the condition (1. 5) gives

$$(2s+1)\left(3n+\log^2 \frac{57M^2 n^2(2s+1)}{a_n a_n^{3\alpha} n^{-3\alpha}}\right) \leqq 4(2s+1)n \leqq N.$$

Correspondingly, we may choose

$$n = 2\left[\frac{N}{8(2s+1)}\right].$$

Hence (1. 4) and (2. 6) gives (2. 4).

REMARKS. Because of (2. 5), *the order of rational approximation is* $o(N^{-\alpha})$ *at least, in all* $C(c(\varepsilon), \alpha, \beta, s)$. It can be seen also that the SZÜSZ—TURÁN's result cannot be reached in this manner. I do not know what can be the order of divergence in (2. 2) but I mention three special cases which can be of interest:

$$\max_{-1 \leqq x \leqq +1} |f(x)-r_N(x)| = \begin{cases} O\left(s^{\beta}\dfrac{\log^{\gamma} N}{N^{\beta}}\right) & \text{if} \quad f(x) \in C(|\log \varepsilon|^{\gamma}, \alpha, \beta, s) \\ O\left(\left(\dfrac{s}{N}\right)^{\alpha \frac{\beta+\gamma}{\alpha+\gamma}}\right) & \text{if} \quad f(x) \in C(\varepsilon^{-\gamma}, \alpha, \beta, s) \\ O\left(\left(\dfrac{s\log s}{N\log N}\right)^{\alpha}\right) & \text{if} \quad f(x) \in C(e^{\frac{\gamma}{\varepsilon}}, \alpha, \beta, s) \quad (\gamma < \beta-\alpha). \end{cases}$$

3.

The second type of functions mentioned in the introduction is the following. Let $f(x)$ be analytic in every $I_k(\varepsilon)$'s (see (1. 1) and (1. 2)) but not necessarily analytic in the $I_k(0)$'s. This means that

$$\max_{0 \leqq k \leqq s-1} \limsup_{n \to \infty} \frac{\sqrt[n]{\max_{x \in I_k(\varepsilon)} |f^{(n)}(x)|}}{n} \leqq c(\varepsilon) \tag{3. 1}$$

where $c(\varepsilon) > 0$ is a monoton decreasing continuous function for which (2. 2) holds. Denote this class of functions by $D(c(\varepsilon), s)$. The order of polynomial approximation for this class is generally not better than $O\left(\omega\left(f, \dfrac{1}{N}\right)\right)$ where $\omega(f, \delta)$ is the module of continuity of $f(x)$. In contrary, we prove for $D\left(\dfrac{c}{\varepsilon}, s\right)$ ($c > 0$ constant) the following

THEOREM 2. *Let* $f(x) \in D\left(\dfrac{c}{\varepsilon}, s\right)$. *Then there exist rational functions* $r_N(x)$ *of degree* N *at most for which*

$$|f(x)-r_N(x)| = O\left(\omega\left(f, e^{-\sqrt[3]{N\frac{\log 1/q}{5s}}}\right)\right) \quad (-1 \leqq x \leqq +1) \tag{3. 2}$$

where

$$q = \frac{\sqrt{c^2e^2+4ec}-ce}{2} \quad (0<q<1). \tag{3.3}$$

PROOF. Again we want to apply our Lemma. Let the $\xi_n^{\{k\}}$'s be

$$\frac{\xi_k+\xi_{k+1}}{2} \pm \frac{\xi_k-\xi_{k+1}}{2}(1-q^i) \quad (i=0,1,\ldots,n;\ k=0,1,\ldots,s-1)$$

(n will be defined later). Then

$$s(n) = (n+1)s, \quad \delta_n = q^n \min_{0\leqq k\leqq s-1} \frac{\xi_{k+1}-\xi_k}{2}. \tag{3.4}$$

We approximate $f(x)$ by its Taylor-series of degree $n-1$ in that intervals

(3.5)

$$\left[\frac{\xi_k+\xi_{k+1}}{2}+\frac{\xi_k-\xi_{k+1}}{2}(1-q^{i+1}), \frac{\xi_k+\xi_{k+1}}{2}+\frac{\xi_k-\xi_{k+1}}{2}(1-q^i)\right],$$
$$\left[\frac{\xi_k+\xi_{k+1}}{2}-\frac{\xi_k-\xi_{k+1}}{2}(1-q^i), \frac{\xi_k+\xi_{k+1}}{2}-\frac{\xi_k-\xi_{k+1}}{2}(1-q^{i+1})\right] \quad \begin{pmatrix} i=0,1,\ldots,n-1; \\ k=0,1,\ldots,s-1 \end{pmatrix}$$

which belong to $[-1, +1]$. Here the error-bounds are (denoting the intervals (3.5) by J_{ik})

$$\frac{\max\limits_{x\in J_{ik}}|f^{(n)}(x)|}{n!}\left[\frac{\xi_{k+1}-\xi_k}{2}q^i(1-q)\right]^n \leqq \frac{\left(\dfrac{cn}{\dfrac{\xi_{k+1}-\xi_k}{2}q^{i+1}}\right)^n}{\left(\dfrac{n}{e}\right)^n}\left[\frac{\xi_{k+1}-\xi_k}{2}q^i(1-q)\right]^n =$$

$$= \left[\frac{ce(1-q)}{q}\right]^n = q^n$$

(see (3.1) and (3.3)). On the other hand, we approximate $f(x)$ by linear functions in the intervals

$$\left[\xi_0, \xi_0+\frac{\xi_1-\xi_0}{2}q^n\right], \left[\xi_k-\frac{\xi_k-\xi_{k-1}}{2}q^n, \xi_k+\frac{\xi_{k+1}-\xi_k}{2}q^n\right] \quad (k=1,\ldots,s-1),$$

$$\left[\xi_s-\frac{\xi_s-\xi_{s-1}}{2}q^n, \xi_s\right].$$

Here the error-bounds are (by Jackson's theorem)

$$12\omega\left(f, \frac{\dfrac{\xi_{k+1}-\xi_k}{2}q^n}{2}\right) \leqq 12\omega(f, q^n).$$

Thus we may apply our Lemma with (3. 4) and

$$\varepsilon_n = O\left(\omega\left(f, q^n\right)\right) \tag{3. 6}$$

being $q^n = O(\omega(q^n))$. Condition (1. 5) gives for sufficiently large n's

$$(n+1)s\left[3n+\log^2 \frac{57M^2n^2(n+1)s}{q^n \min\limits_{0\leqq k\leqq s-1} \frac{\xi_{k+1}-\xi_k}{2}\cdot c_1\omega(f,q^n)}\right] \leqq 5sn^3\log^2 q \leqq N$$

i. e. we may choose

$$n = 2\left[\frac{1}{2}\sqrt[3]{\frac{N}{5s\log^2 q}}\right].$$

Hence and from (1. 4) and (3. 6) we obtain

$$|f(x)-r_N(x)| = O\left(\omega\left(f, q^{\sqrt[3]{\frac{N}{5s\log^2 q}}}\right)\right) = O\left(\omega\left(f, e^{-\sqrt[3]{N\frac{\log 1/q}{5s}}}\right)\right)$$

qu. e. d.

REMARKS. It is easy to see that $|x|^\alpha \in D\left(\frac{1}{e\varepsilon}, 2\right)$ with $\omega(f, \delta)=\delta^\alpha$ and as a special case Theorem 2 gives essentially Theorem 2 from [1]. Similarly, $x\log|x| \in D\left(\frac{1}{e\varepsilon}, 2\right)$ with $\omega(f, \delta)=\delta|\log\delta|$. For this function the order of polynomial and rational approximation are $O\left(\frac{\log N}{N}\right)$ and $O\left(e^{-d\sqrt[3]{N}}\right)$ $(d>0)$, respectively. Further examples can be found in [4].

There remains an open question: whether $c(\varepsilon)=\frac{c}{\varepsilon}$ holds for all $f(x)\in D(c(\varepsilon), s)$?

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References

[1] G. FREUD—J. SZABADOS, Rational approximation to x^α, *Acta Math. Acad. Sci. Hung.*, **18** (1967), pp. 393—399.

[2] P. SZÜSZ—P. TURÁN, A konstruktív függvénytan egy újabb irányáról, *MTA III. Osztály Közleményei*, **16** (1966), pp. 33—45.

[3] G. FREUD, A contribution to the rational approximation of functions, *Studia Math. Hung.*, **2** (1967).

[4] J. SZABADOS, Remarks on a paper of A. A. Gončar and some connected problems in the theory of rational approximation, *Annales Univ. Sci. Budapest, Sectio Math.* (to appear).

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (1—2), (1968), pp. 87—94.

ON A PROBLEM OF STRONG SUMMABILITY OF FOURIER SERIES

By
L. LEINDLER (Szeged)
(*Presented by* G. ALEXITS)

Introduction

Let $f(x)$ be a Lebesgue-integrable and periodic function with period 2π and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

It is well known (cf. ZYGMUND [4] pp. 180—188) that

$$\lim_{n\to\infty} \left\{ \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu(x,f) - f(x)|^k \right\}^{1/k} = 0$$

for arbitrary $k>0$, almost everywhere, where $s_k(x,f)$ denotes the k-th partial sum of the Fourier series of $f(x)$; it actually holds at $x=x_0$ if $f(x)$ is continuous at the point x_0.

P. TURÁN ([3]) raised the following problem: Let $\{\lambda_n\}$ be a sequence of numbers increasing monotonely to infinity. Is then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu(x,f) - f(x)|^{\lambda_n} = 0$$

true? He gave a negative answer for this conjecture, and at the same time proved that *for any sequence* $\lambda_n \nearrow +\infty$ *a continuous function can be given having the property*

$$\overline{\lim_{n\to\infty}} \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu(x_0,f) - f(x_0)|^{\lambda_n} = +\infty$$

at a suitable point $x_0 \in [0, 2\pi]$.

In connection with this negative result G. ALEXITS (cf. *On approximation theory*, Birkhauser Verlag, Basel 1964, New and unsolved problems, Problem 3, pp. 179—180) raised the problem: What kind of functions have the property

$$\lim_{n\to\infty} h_n(x,f;\lambda) \equiv \lim_{n\to\infty} \left\{ \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu(x,f) - f(x)|^{\lambda_n} \right\}^{1/\lambda_n} = 0 \tag{1}$$

either at a fixed point $x=x_0$ or in an interval.

Recently, D. KRÁLIK [1] has given a sufficient condition for a function having the property (1). To explain his theorem, we require the following definition: A positive, increasing to infinity, differentiable $F(x)$ $(x>0)$ is said to belong to the class Λ if the function $x^{1/2\lambda_n} F^{-1}(x)$ is monotone increasing for $x > n^{1/2\lambda_n}$ in case of sufficiently large n. D. KRÁLIK [1] proved:

If $f(x)\in L^{\lambda_n}[-\pi,\pi]$ *in case of sufficiently large* n *and if at* $x_0\in[-\pi,\pi]$

$$f(x_0+t)+f(x_0-t)-2f(x_0)=O\left(F^{-1}\left(\frac{1}{t}\right)\right)\quad(t\to 0) \tag{2}$$

is fulfilled with any $F(x)\in\Lambda$, *furthermore* $\lambda_n=o(\min(\log n, F(n)))$, *then* (1) *holds. If* (2) *holds uniformly in* $[c,d]\subset[-\pi,\pi]$ *then* (1) *is fulfilled also uniformly in an arbitrary internal subinterval of* (c,d).

In the present paper we are going to prove that the statement (1), even the more general statement

$$\lim_{n\to\infty}\left\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_\nu(x,f)-f(x)|^{\lambda_n}\right\}^{1/\lambda_n}=0\quad\left(A_n=\sum_{\nu=0}^{n}\alpha_{n\nu}\right)$$

is true for a larger class Λ of functions $F(x)$, where $\left\|\frac{\alpha_{n\nu}}{A_n}\right\|$ is a suitable matrix defined later, and furthermore, we prove our statements for integrable functions $f(x)$.

At the same time, D. KRÁLIK proved a similar theorem for a uniformly bounded orthonormal system of polynomials. We generalize this theorem, too.

It is easy to see that it is sufficient to examine the sequence $\{\lambda_n\}$ having the property $\lambda_n=o(\log n)$. Namely, if $\lambda_n\neq o(\log n)$ then there exists a positive number c and a sequence of indices $\{\nu_n\}$ such that $\lambda_{\nu_n}\geqq c\log\nu_n$ and then even if only one index μ exists for which $|s_\mu(x,f)-f(x)|\neq 0$, we can see that (1) is not fulfilled (see [1]). We can also confine us to the case $\lambda_n=o(\log n)$ indeed.

Let us define a class $\Lambda(\varrho)$ of functions in the following way: A positive, increasing to infinity $F(x)$ $(x>0)$ is said to belong to the class $\Lambda(\varrho)$ if $\lambda_n=o(F(n))$ and there exists a positive number $\varrho<1$ such that the function $x^{\varrho/\lambda_n}F^{-1}(x)$ is monotone non-decreasing for sufficiently large n.

We consider a regular summation method T_n determined by a triangular matrix $\left\|\frac{\alpha_{nk}}{A_n}\right\|$ $\left(\alpha_{nk}\geqq 0 \text{ and } A_n=\sum_{k=0}^{n}\alpha_{nk}\right)$, i. e. if s_k tends s, then

$$T_n=\frac{1}{A_n}\sum_{k=0}^{n}\alpha_{nk}s_k\to s.$$

THEOREM 1. *If there exists a* $p>1$ *such that*

$$\left\{\sum_{k=1}^{n}\alpha_{nk}^{p}\right\}^{1/p}\leqq Kn^{\frac{1-p}{p}}\sum_{k=0}^{n}\alpha_{nk} \tag{3}$$

and if the function $f(x)$ *is integrable, moreover at a fixed point* $x_0\in[-\pi,\pi]$ *the relation*

$$\varphi_{x_0}(t)\equiv f(x_0+t)+f(x_0-t)-2f(x_0)=O\left(F^{-1}\left(\frac{1}{t}\right)\right)\quad(t\to 0) \tag{4}$$

is satisfied with any $F(x)\in\Lambda(\varrho)$, *where* $\varrho<\dfrac{p-1}{p}$, *then*

$$\lim_{n\to\infty}h_n(\alpha,\lambda,f;x_0)\equiv\lim_{n\to\infty}\left\{\frac{1}{A_n}\sum_{k=0}^{n}\alpha_{nk}|s_k(x_0,f)-f(x_0)|^{\lambda_n}\right\}^{1/\lambda_n}=0 \tag{5}$$

holds.

If (4) *is fulfilled uniformly in any interval* $[c, d] \subset [-\pi, \pi]$, *then* (5) *is valid uniformly in any interval internal to* (c, d).

It is easy to verify that in the cases:

$$\alpha_{nk} = A_{n-k}^{(\alpha-1)} \quad (0 < \alpha \leqq 1) \quad \text{and} \quad 1 < p < \frac{1}{1-\alpha} \quad \left(A_n^{(\alpha)} = \binom{n+\alpha}{n}\right),$$

$$\alpha_{nk} = A_{n-k}^{(\alpha-1)} \quad (\alpha > 1) \quad \text{and} \quad 1 < p < \infty,$$

$$\alpha_{nk} = k^{\beta} \quad \text{and} \quad p\beta > -1$$

then condition (3) is satisfied. From this we have e. g.

COROLLARY 1. *If the function* $f(x)$ *is integrable, moreover at a fixed point* $x_0 \in [-\pi, \pi]$ *the relation* (4) *is satisfied with any* $F(x) \in \Lambda(\varrho)$ *for some* $\varrho < \alpha$, *then*

$$\lim_{n\to\infty} \left\{ \frac{1}{A_n^{(\alpha)}} \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)} |s_k(x_0, f) - f(x_0)|^{\lambda_n} \right\}^{1/\lambda_n} = 0 \tag{6}$$

holds.

If (4) *is fulfilled uniformly in any interval* $[c, d] \subset [-\pi, \pi]$, *then* (6) *is valid uniformly in any interval internal to* (c, d).

It is clear that this Corollary in the special case $\alpha = 1$ includes the theorem of KRÁLIK.

For a uniformly bounded orthonormal system $\{p_n(x)\}$ of polynomials we have the following theorem. For the sake of simplicity, let us denote also now the k-th partial sum of the Fourier-series

$$f(x) \sim \sum_{n=0}^{\infty} c_n p_n(x)$$

and its k-th strong mean defined as in (5), by $s_k(x, f)$ and $h_k(\alpha, f, \lambda; x)$, respectively.

THEOREM 2. *Denote* $\{p_n(x)\}$ *the orthonormal system of polynomials generated by a non-negative weight function* $\mu(x)$ *on* $[a, b]$. *If* $|p_n(x)| \leqq M$ $(a \leqq x \leqq b;\ n = 0, 1, \ldots)$, $\mu(x) \leqq M_1$ *on an interval* $[c, d]$ *internal to* (a, b) *and* $f(x) \in L_{\mu}(a, b)$, *moreover if at a point* x_0 *internal to* (c, d)

$$|f(t) - f(x_0)| = O(F^{-1}(|t - x_0|^{-1})) \qquad (t \to x_0) \tag{7}$$

is satisfied with any $F(x) \in \Lambda(\varrho)$ *for some* $\varrho < \dfrac{p-1}{p}$, *then* (5) *holds for any matrix having the property* (3).

If (7) *is fulfilled uniformly in* $[c, d]$ *then* (5) *holds uniformly in any interval* $[c + \delta, d - \delta]$.

COROLLARY 2. *Under the conditions of Theorem* 2 *in the case* $\varrho < \alpha$, *the statement* (6) *holds either in the point* $x = x_0$ *or uniformly in some interval* $[c + \delta, d - \delta]$.

Corollary 2 generalizes the theorem of KRÁLIK concerning the system of polynomials.

1. Proof of Theorem 1

Applying Hölder's inequality, we obtain by (3)

$$h_n(\alpha,\lambda,f;x_0)=\left\{\frac{1}{A_n}\sum_{k=0}^{n}\alpha_{nk}|s_k(x_0,f)-f(x_0)|^{\lambda_n}\right\}^{1/\lambda_n}\leqq$$

$$\leqq\left\{\frac{1}{A_n}\left(\sum_{k=0}^{n}(\alpha_{nk})^p\right)^{1/p}\left(\sum_{k=0}^{n}|s_k(x_0,f)-f(x_0)|^{\lambda_n q}\right)^{1/q}\right\}^{1/\lambda_n}\leqq$$

$$\leqq 2K\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x_0,f)-f(x_0)|^{\lambda_n q}\right\}^{1/q\lambda_n}$$

where $q=\dfrac{p}{p-1}$.

By (4), we can find $\delta>0$ such that the relation (4) is fulfilled in the interval $(0,\delta]$ everywhere. Let $f(x)=f'(x)+f''(x)$, where $f'(x)=f(x)$ in $[x_0-\delta, x_0+\delta]$ and $f'(x)=0$ elsewhere. For the suitable partial sums we have $s_k(x_0,f)=s_k(x_0,f')+s_k(x_0,f'')$ and so by (1. 1) we obtain

$$\frac{1}{2K}h_n(\alpha,\lambda,f;x_0)\leqq\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x_0,f')-f'(x_0)|^{\lambda_n q}\right\}^{1/\lambda_n q}+$$

$$+\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x_0,f'')|^{\lambda_n q}\right\}^{1/\lambda_n q}.$$

Since $f''(x)=0$ in $[x_0-\delta, x_0+\delta]$, $|s_k(x_0,f'')|$ tends to zero. From this and from $\lambda_n=o(\log n)$ it follows that the second sum tends to zero.

Let $\lambda_n q=q_n$. Applying Minkowski's inequality, we get

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x_0,f')-f'(x_0)|^{q_n}\right\}^{1/q_n}=$$

$$=\left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{1}{\pi}\int_0^{\pi}\varphi'_{x_0}(t)\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}dt\right|^{q_n}\right\}^{1/q_n}\leqq$$

$$\leqq\left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{1}{\pi}\int_0^{n^{-1}}\varphi'_{x_0}(t)\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}dt\right|^{q_n}\right\}^{1/q_n}+$$

$$+\left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{1}{\pi}\int_{n^{-1}}^{\pi}\frac{\varphi'_{x_0}(t)}{2\operatorname{tg}\frac{t}{2}}\sin kt\,dt\right|^{q_n}\right\}^{1/q_n}+$$

$$+\left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{1}{2\pi}\int_{n^{-1}}^{\pi}\varphi'_{x_0}(t)\cos kt\,dt\right|^{q_n}\right\}^{1/q_n}\equiv\Sigma_1+\Sigma_2+\Sigma_3,$$

where $\varphi'_{x_0}(t)=f'(x_0+t)+f'(x_0-t)-2f'(x_0)$.

If $n>\delta^{-1}$, then by (4) we have

$$\Sigma_1 = O(1)\left\{\frac{1}{n+1}\sum_{k=0}^{n}\left[\int_0^{n^{-1}}\frac{1}{F(n)}(2k+1)\,dt\right]^{q_n}\right\}^{1/q_n} = O\left(\frac{1}{F(n)}\right) = o(1).$$

The integral in Σ_2 is the k-th sine coefficient of the following function

$$\Psi_{x_0,n}(t) = \begin{cases} \dfrac{\varphi'_{x_0}(t)}{2\operatorname{tg}\dfrac{t}{2}} & \text{for} \quad n^{-1} \leqq t \leqq \pi, \\ 0 & \text{for} \quad -\pi < t < n^{-1}. \end{cases}$$

By Hausdorff—Young's inequality we obtain

$$\left\{\sum_{k=0}^{n}\left|\frac{1}{\pi}\int_{n^{-1}}^{\pi}\Psi_{x_0,n}(t)\sin kt\,dt\right|^{q_n}\right\}^{\frac{1}{q_n}} \leqq \left\{\int_{-\pi}^{\pi}|\Psi_{x_0,n}(t)|^{\frac{q_n}{q_n-1}}\,dt\right\}^{\frac{q_n-1}{q_n}} \leqq$$

$$\leqq \left\{\int_{n^{-1}}^{\pi}\left|\frac{\varphi'_{x_0}(t)}{t}\right|^{\frac{q_n}{q_n-1}}dt\right\}^{\frac{q_n-1}{q_n}} \leqq \left\{\int_{n^{-1}}^{n^{-\varrho/\lambda_n}}\left|\frac{\varphi'_{x_0}(t)}{t}\right|^{\frac{q_n}{q_n-1}}dt\right\}^{\frac{q_n-1}{q_n}} +$$

$$+\left\{\int_{n^{-\varrho/\lambda_n}}^{\pi}\left|\frac{\varphi'_{x_0}(t)}{t}\right|^{\frac{q_n}{q_n-1}}dt\right\}^{\frac{q_n-1}{q_n}} \equiv I_1+I_2.$$

Let us suppose that $n^{\varrho/\lambda_n}>\delta^{-1}$, then by (4), considering that $F(x)\in\Lambda(\varrho)$, we can estimate I_1 in the following way:

$$I_1^{\frac{q_n}{q_n-1}} = O(1)\int_{n^{-1}}^{n^{-\varrho/\lambda_n}}\left[F^{-1}\left(\frac{1}{t}\right)t^{-1}\right]^{\frac{q_n}{q_n-1}}dt =$$

$$= O(1)\int_{n^{\varrho/\lambda_n}}^{n}[F^{-1}(x)\,x]^{\frac{q_n}{q_n-1}}\frac{dx}{x^2} = O(1)\int_{n^{\varrho/\lambda_n}}^{n}[F^{-1}(x)\,x^{\frac{2}{q_n}-1}]^{\frac{q_n}{q_n-1}}dx =$$

$$= O(1)\int_{n^{\varrho/\lambda_n}}^{n}[F^{-1}(x)\,x^{\varrho/\lambda_n}x^{\frac{2}{q_n}-1-\frac{q}{\lambda_n}}]^{\frac{q_n}{q_n-1}}dx =$$

$$= O(1)[F^{-1}(n)\,n^{\varrho/\lambda_n}]^{\frac{q_n}{q_n-1}}\int_{n^{\varrho/\lambda_n}}^{n}x^{\left(\frac{2}{q_n}-1-\frac{q\varrho}{q_n}\right)\frac{q_n}{q_n-1}}dx =$$

$$= O(1)[F^{-1}(n)n^{\varrho/\lambda_n}]^{\frac{q_n}{q_n-1}}\frac{q_n-1}{1-q\varrho}n^{\frac{1-q\varrho}{q_n-1}}.$$

Hence

$$I_1 = O(1)\frac{q_n}{1-q\varrho}n^{1/q_n}F^{-1}(n). \tag{1.2}$$

For I_2 we obtain

$$I_2 = O(1)\left\{\int_{n-\varrho/\lambda_n}^{\pi} n^{\frac{\varrho q}{q_n-1}}\,dt\right\}^{\frac{q_n-1}{q_n}} = O(n^{\varrho/\lambda_n}).$$

From this and (1. 2), by $\lambda_n = o(F(n))$ and $\varrho q < 1$, we have

$$\Sigma_2 = O\left(\frac{\lambda_n}{F(n)}\right) + O\left(n^{\frac{\varrho}{\lambda_n}-\frac{1}{q_n}}\right) = o(1).$$

As in the estimation of Σ_2, we have

$$\Sigma_3 = O(n^{-\frac{1}{q_n}})\left\{\int_{n^{-1}}^{\pi} |\varphi'_{x_0}(t)|^{\frac{q_n}{q_n-1}}\,dt\right\}^{\frac{q_n-1}{q_n}} = O\left(e^{-\frac{\log n}{\lambda_n}}\right) = o(1).$$

From the above analysis we obtain the first statement of Theorem 1.

It is easy to see from the proof, that if (4) is satisfied uniformly in any interval $[c, d]$, then (5) is fulfilled also uniformly in any interval internal to (c, d).

2. Proof of Theorem 2

The proof is similar to that of Theorem 1. As in the proof of Theorem 1, we have

$$h_n(\alpha, \lambda, f; x_0) \leqq 2K\left\{\frac{1}{n+1}\sum_{k=0}^{n} |s_k(x_0, f) - f(x_0)|^{q_n}\right\}^{1/q_n}$$

with $q_n = \lambda_n q \quad \left(q = \dfrac{p}{p-1}\right)$.

By (7), we can find a $\delta > 0$ such that (7) is satisfied in $(0, \delta]$ everywhere. Let $f(x) = f'(x) + f''(x)$, where $f'(x) = f(x)$ in $[x_0 - \delta, x_0 + \delta]$ and $f'(x) = 0$ elsewhere. Applying Minkowski's inequality, we obtain

$$\frac{1}{2K} h_n(\alpha, \lambda, f; x_0) \leqq \left\{\frac{1}{n}\sum_{k=0}^{n} |s_k(x_0, f') - f'(x_0)|^{q_n}\right\}^{1/q_n} + \tag{2. 1}$$

$$+ \left\{\frac{1}{n}\sum_{k=0}^{n} |s_k(x_0, f'') - f''(x_0)|^{q_n}\right\}^{1/q_n}.$$

Since $f''(x) = 0$ in $[x_0 - \delta, x_0 + \delta]$ and the polynomials $p_n(x)$ are uniformly bounded on $[a, b]$, $|s_k(x_0, f'') - f''(x_0)|$ tends to zero (see e.g. [2], p. 286); from this and $\lambda_n = o(\log n)$ it follows that the second sum tends to zero.

Since

$$s_k(x_0, f') - f'(x_0) = \int_a^b \{f'(t) - f'(x_0)\}\,\mu(t) \sum_{\nu=0}^{k} p_\nu(t)\, p_\nu(x_0)\,dt \equiv$$

$$\equiv \int_{x_0-n^{-1}}^{x_0+n^{-1}} + \left(\int_c^{x_0-n^{-1}} + \int_{x_0+n^{-1}}^{d}\right) + \left(\int_a^c + \int_d^b\right),$$

we get, using Minkowski's inequality and the Christoffel—Darboux formula

$$\left\{\frac{1}{n}\sum_{k=0}^{n}|s_k(x_0,f')-f'(x_0)|^{q_n}\right\}^{1/q_n} \leqq$$

$$\leqq \left\{\frac{1}{n}\sum_{k=0}^{n}\left|\int_{x_0-n^{-1}}^{x_0+n^{-1}}\{f'(t)-f'(x_0)\}\mu(t)\sum_{\nu=0}^{k}p_\nu(t)p_\nu(x_0)\,dt\right|^{q_n}\right\}^{1/q_n} +$$

$$+\left\{\frac{1}{n}\sum_{k=0}^{n}\left|\left(\int_{c}^{x_0-n^{-1}}+\int_{x_0+n^{-1}}^{d}\right)\right.\right. \tag{2.2}$$

$$\left.\left.\{f'(t)-f'(x_0)\}\mu(t)\alpha_k\frac{p_{k+1}(t)p_k(x_0)-p_{k+1}(x_0)p_k(t)}{t-x_0}\,dt\right|^{q_n}\right\}^{1/q_n} +$$

$$+\left\{\frac{1}{n}\sum_{k=0}^{n}\left|\left(\int_{a}^{c}+\int_{d}^{b}\right)\{f'(t)-f'(x_0)\}\mu(t)\alpha_k\frac{p_{k+1}(t)p_k(x_0)-p_{k+1}(x_0)p_k(t)}{t-x_0}\,dt\right|^{q_n}\right\}^{1/q_n} \equiv$$

$$\equiv S_1+S_2+S_3.$$

Let us suppose that $n^{\varrho/\lambda_n}>\delta^{-1}$ and $c<x_0-n^{-\frac{\varrho}{\lambda_n}}<x_0<x_0+n^{-\frac{\varrho}{\lambda_n}}<d$ are satisfied, then we get, by (7),

$$S_1 \leqq \left\{\frac{1}{n}\sum_{k=0}^{n}\left(\frac{2}{n}F^{-1}(n^{-1})M_1(k+1)M^2\right)^{q_n}\right\}^{1/q_n} = O\left(F^{-1}(n^{-1})\right)=o(1). \tag{2.3}$$

Let

$$\Psi_{x_0,n}(t)=\begin{cases}\dfrac{f'(t)-f'(x_0)}{t-x_0} & \text{for} \quad t\in[c,x_0-n^{-1}],\\ 0 & \text{elsewhere.}\end{cases}$$

It is easy to see that in order to prove $S_2=o(1)$ it suffices to verify that for instance

$$\Sigma_1 \equiv \left\{\frac{1}{n}\sum_{k=0}^{n+1}(|\alpha_k||p_k(x_0)||\beta_k|)^{q_n}\right\}^{1/q_n}=o(1),$$

where β_k denotes the k-th Fourier-coefficient of $\Psi_{x_0,n}(t)$ with respect to $\{p_k(t)\}$ belonging to the weight function $\mu(t)$. Since $f'(x)$ is bounded, $\Psi_{x_0,n}(t)$ is also bounded and so it is in L^p for all p. Applying the theorem of RIESZ ([4], p. 102), we obtain

$$\Sigma_1 \leqq O(n^{-\frac{1}{q_n}})\left\{\int_{c}^{x_0-n^{-1}}\left|\frac{f'(t)-f'(x_0)}{t-x_0}\right|^{\frac{q_n}{q_n-1}}\mu(t)\,dt\right\}^{\frac{q_n-1}{q_n}} =$$

$$= O(n^{-\frac{1}{q_n}})\left\{\int_{c}^{x_0-n^{-\varrho/\lambda_n}}+\int_{x_0-n^{-\varrho/\lambda_n}}^{x_0-n^{-1}}\right\}^{\frac{q_n-1}{q_n}} = O(n^{-\frac{1}{q_n}})(I_1+I_2).$$

For I_1, we get

$$I_1 = O(1)\left\{\int_c^{x_0-n^{-\varrho/\lambda_n}} (n^{\varrho/\lambda_n})^{\frac{q_n}{q_n-1}}\,dt\right\}^{\frac{q_n-1}{q_n}} = O(n^{\varrho/\lambda_n}).$$

Let $x_0-t=v$. From $F(x)\in\Lambda(\varrho)$ and (7) as in the proof of Theorem 1, it follows

$$I_2 = O(1)\left\{\int_{n-1}^{n-\varrho/\lambda_n} [F^{-1}(v^{-1})\,v^{-1}]^{\frac{q_n}{q_n-1}}\,dv\right\}^{\frac{q_n-1}{q_n}} = O(1)\,\frac{q_n}{1-\varrho q}\,n^{1/q_n}F^{-1}(n).$$

From these estimations in view of $\lambda_n=o(F(n))$ and $\varrho q<1$, we obtain

$$\Sigma_1 = O\left(\frac{\lambda_n}{F(n)}+n^{\frac{\varrho}{\lambda_n}-\frac{1}{q_n}}\right) = o(1).$$

Since the other sums appearing in S_2 are similar to Σ_1, we have

$$S_2 = o(1). \tag{2.4}$$

Let

$$\overline{\chi(t)} = \begin{cases} \dfrac{f'(t)-f'(x_0)}{t-x_0} & \text{for } t\in[a,c]\cup[d,b], \\ 0 & \text{elsewhere.}\end{cases}$$

Since x_0 is a point internal to (c, d), the function $\chi(t)$ is bounded independently of n, thus it belongs to L_μ^p for any $p\geqq 1$. By the cited theorem of RIESZ, we have

$$S_3 = O(n^{-\frac{1}{q_n}})\left\{\int_a^b |\chi(t)|^{\frac{q_n}{q_n-1}}\,\mu(t)\,dt\right\}^{\frac{q_n-1}{q_n}} =$$

$$= O(n^{-\frac{1}{q_n}})\left\{\int_a^b |\chi(t)|^2\,\mu(t)\,dt\right\}^{1/2} = o(1),$$

and this proves Theorem 2 (by (2. 2), (2. 3) and (2. 4)).

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References

[1] D. KRÁLIK, Über ein Problem der starken Summierbarkeit von Fourierreihen, *Acta Math. Acad. Sci. Hung.*, **17** (1966), pp. 303—312.
[2] I. P. NATANSZON, *Constructive theory of function* (Washington, 1961; translation from Russian).
[3] P. TURÁN, On the strong summability of Fourier series, *Journal of the Indian Math. Soc. (N. S.)* **12** (1948), pp. 8—12.
[4] A. ZYGMUND, *Trigonometric series II.* (Cambridge, 1959).

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (1—2), (1968), pp. 95—102.

STRUCTURAL PROPERTIES OF CONTINUOUS FUNCTIONS CONNECTED WITH THE ORDER OF RATIONAL APPROXIMATION. II

By
J. SZABADOS (Budapest)
(*Presented by* P. TURÁN)

§ 1

We shall use the same notations as in Part I (see [1]), i.e. let

$$E_1 = [-1, +1], \quad E_2 = \operatorname{int} E_1 = (-1, +1),$$

furthermore, if z_{kn} $(k=1, 2, \ldots, m \leqq n;\ n=0, 1, \ldots)$ are the different poles of the rational function $r_n(x)$ (the suffix n always denotes the degree of $r_n(x)$) then

$$\delta(r_n) = \min\left(1;\ \min_{1 \leqq k \leqq m} \min_{x \in E_1} |x - z_{kn}|\right), \tag{1.1}$$

$$\bar{\delta}(r_n, p) = \min\left(\delta(r_n),\ \min_{0 \leqq i \leqq q} \delta(r_{[np]+i})\right) \quad \left(p \geqq 1, \quad q = \begin{cases} 0 \text{ if } p \text{ integer} \\ p \text{ otherwise} \end{cases}\right), \tag{1.2}$$

$$\log_1 n = \log n, \quad \log_\nu n = \log(\log_{\nu-1} n) \quad (\nu = 2, 3, \ldots).$$

Finally, let $r_0(x), r_1(x), \ldots, r_n(x), \ldots$ be a sequence of approximating rational functions to a continuous function $f(x)$ in E_1 (not necessarily the best approximating one), i. e.

$$R_n(f) \overset{\text{def}}{=} \max_{x \in E_1} |f(x) - r_n(x)| \to 0 \quad (n \to \infty). \tag{1.3}$$

Notice that for an approximating rational function $r_n(x)$ we have $\delta(r_n) > 0$, and if $r_n(x)$ is especially a polynomial then $\delta(r_n) = 1$.

In what follows the O-sign always regards to $n \to \infty$.

§ 2

THEOREM 1. *If there exist $k \geqq 1$, $\nu \geqq 2$ integers, $\varepsilon > 0$ and $p > 1$ real numbers such that*

$$R_n(f) = O\left(\frac{\bar{\delta}(r_n, p)^k}{n^k \log_1 n \ldots \log_{\nu-1} n \log_\nu^{1+\varepsilon} n}\right) \tag{2.1}$$

then $f^{(k)}(x)$ exists and is continuous in E_2.

THEOREM 2. *If*

$$\delta(r_n) \geqq \delta(r_{n+1}) \quad (n = 0, 1, \ldots) \tag{2.2}$$

and there exist $k \geqq 1$ *integer,* $0 < \alpha < 1$, $\varepsilon > 0$ *and* $p > 1$ *real numbers such that*

$$R_n(f) = O\left(\frac{\bar{\delta}(r_n, p)^{k+\alpha}}{n^{k+\alpha+\varepsilon}}\right) \tag{2.3}$$

then $f^{(k)}(x)$ *exists in* E_2 *and belongs to the class* $\operatorname{Lip}\alpha$ *in every closed subinterval of* E_2.

THEOREM 3. *If there exists a* $p > 1$ *real number such that*

$$\lim_{n\to\infty} \frac{n^k R_n(f)}{\bar{\delta}(r_n, p)^k} = 0 \quad (k = 1, 2, \ldots)$$

then $f(x)$ *is infinitely many times differentiable in* E_2.

REMARKS. Analogous theorems hold for the interval $(-\infty, +\infty)$ but we omit the details.

As regards to the necessity of the above conditions, we only point it out that (2. 1) with $k = p = 1$, $\varepsilon = 0$, i. e. (see (1. 2))

$$R_n(f) = O\left(\frac{\delta(r_n)}{n \log_1 n \ldots \log_\nu n}\right)$$

does not ensure the differentiability of $f(x)$ in E_2. This can be easily seen by the example

$$f(x) = \sum_{m=m(\nu)}^{\infty}{}' \frac{xm \log_1 m \ldots \log_\nu m}{4^m x^2 + (m \log_1 m \ldots \log_\nu m)^2}$$

(compare [1], Theorem 3).

§ 3

In order to prove our theorems we state

LEMMA 1. *If* $\max_{x\in E_1} |r_n(x)| \leqq M$ *then*

$$\max_{|x|\leqq 1-t} |r_n^{(k)}(x)| \leqq \left[\frac{12\sqrt{2k}\,n}{\sqrt{t}\,\delta(r_n)}\right]^k k!\, M \quad \left(0 < t \leqq \frac{1}{2};\ k = 1, 2, \ldots\right). \tag{3.1}$$

PROOF. We use Lemma 2 from [1] in a little modified form, which regards to the interval $[-a, +a] \subseteq E_1$:

$$\max_{|x|\leqq b} |r_n'(x)| \leqq \frac{12\sqrt{2}\,n}{\sqrt{a^2 - b^2}\,\delta(r_n)} \max_{|x|\leqq a} |r_n(x)| \quad (0 < b < a). \tag{3.2}$$

Let

$$a_i = 1 - \frac{i}{k} t \geqq \frac{1}{2} \quad (i = 0, 1, \ldots, k) \tag{3.3}$$

and apply (3. 2) in the intervals $[-a_i, +a_i]$ for $r_n^{(i+1)}(x)$ $(i=0, 1, \ldots, k-1)$. Using the obvious relations

$$\operatorname{gr} r_n^{(i)}(x)=(i+1)n, \quad \delta(r_n^{(i)})=\delta(r_n) \qquad (i=1, \ldots, k)$$

we obtain

$$\max_{x\leqq 1-t} |r_n^{(k)}(x)| \leqq \left[\frac{12\sqrt{2}\,n}{\delta(r_n)}\right]^k \cdot \frac{k!\,M}{\sqrt{\prod_{i=0}^{k-1}(a_i^2-a_{i+1}^2)}}. \tag{3.4}$$

But from (3. 3)

$$\prod_{i=0}^{k-1}(a_i^2-a_{i+1}^2) \geqq \prod_{i=0}^{k-1}(a_i-a_{i+1}) = \left(\frac{t}{k}\right)^k.$$

Hence and by (3. 4) we get (3. 1).

PROOF OF THEOREM 1. Let $0<t\leqq\frac{1}{2}$ be arbitrary and

$$\bar{r}_0(x)=r_{[p]}(x), \quad \bar{r}_m(x)=r_{[p^{m+1}]}(x)-r_{[p^m]}(x) \quad (m=1, 2, \ldots) \tag{3.5}$$

then by (1. 3)

$$f(x)=\sum_{m=0}^{\infty}\bar{r}_m(x) \tag{3.6}$$

where the right hand series converges uniformly in E_1. Clearly

$$|\bar{r}_m(x)| \leqq |r_{[p^{m+1}]}(x)-f(x)|+|f(x)-r_{[p^m]}(x)| \leqq 2R_{[p^m]}(f) \quad (x\in E_1; m=1, 2, \ldots).$$

Thus by Lemma 1, (2. 1) and (1. 2)

(3.7)

$$|\bar{r}_m^{(k)}(x)| \leqq \left[\frac{12\sqrt{2}\,2p^{m+1}}{\sqrt{t}\,\min\left(\delta(r_{[p^{m+1}]}), \delta(r_{[p^m]})\right)}\right]^k k!\,2O\left[\frac{\bar{\delta}(r_{[p^m]}, p)^k}{p^{mk}\,m\log_1 m \ldots \log_{\nu-2} m\cdot\log_{\nu-1}^{1+\varepsilon} m}\right]=$$

$$= c_1(k, p, t)\cdot\frac{1}{m\log_1 m \ldots \log_{\nu-2} m\cdot\log_{\nu-1}^{1+\varepsilon} m} \quad \left(|x| \leqq 1-t, 0<t\leqq\frac{1}{2}, m\geqq m(\nu)\right).$$

But

$$\sum_{m=m(\nu)}^{\infty}\frac{1}{m\log_1 m \ldots \log_{\nu-2} m\cdot\log_{\nu-1}^{1+\varepsilon} m}$$

converges, i.e. $\sum_{m=0}^{\infty}\bar{r}_m^{(k)}(x)$ uniformly converges for $|x|\leqq 1-t$ and by (3. 6)

$$f^{(k)}(x)=\sum_{m=0}^{\infty}\bar{r}_m^{(k)}(x) \tag{3.8}$$

is continuous in $[-1+t, 1-t]$.

PROOF OF THEOREM 2. We have (3. 5) and (3. 6). Similarly as in (3. 7) we obtain from (2. 3)

$$|\bar{r}_m^{(k)}(x)| \leqq \left[\frac{12\sqrt{2k}\,2p^{m+1}}{\sqrt{t}\cdot\delta(r_{[p^{m+1}]})}\right]^k k!\,2O\left[\frac{\delta(r_{[p^m]p]+p})^{k+\alpha}}{p^{m(k+\alpha+\varepsilon)}}\right] \leqq \tag{3.9}$$

$$\leqq c_2(k, p, t)\frac{\delta(r_{[p^{m+1}]})^\alpha}{p^{m(\alpha+\varepsilon)}} \leqq \frac{c_2(k, p, t)}{(p^\alpha)^m} \quad (|x|\leqq 1-t).$$

Being $\sum_m \frac{1}{(p^\alpha)^m}$ convergent, (3. 8) holds in $[-1+t, 1-t]$.

$$R_{(k+1)[p^m]}(x) = r^{(k)}_{[p^m]}(x) = \sum_{i=0}^{m-1} \bar{r}_i^{(k)}(x)$$

evidently is a rational function of degree $(k+1)[p^m]$ at most, and

$$\delta^*(R_{(k+1)[p^m]}) \leqq \delta(r^{(k)}_{[p^m]}) = \delta(r_{[p^m]}). \tag{3.10}$$

Here the star indicates that defining δ^* we put $[-1+t, 1-t]$ instead of E_1 in (1. 1). We have by (3. 9) and (2. 2)

$$R^*_{(k+1)[p^m]}(f^{(k)}) \overset{\text{def}}{=} \max_{|x|\leqq 1-t} |f^{(k)}(x) - R_{(k+1)[p^m]}(x)| \leqq \sum_{i=m}^{\infty} \max_{|x|\leqq 1-t} |\bar{r}_i^{(k)}(x)| \leqq$$

$$\leqq c_3(k,p,t)\frac{\delta(r_{[p^{m+1}]})^\alpha}{p^{m(\alpha+\varepsilon)}}. \tag{3.11}$$

Now let $n \geqq (k+1)p$ be an arbitrary integer,

$$(k+1)[p^m] \leqq n < (k+1)[p^{m+1}],$$

say. For these n's put

$$R_n(x) = R_{(k+1)[p^m]}(x), \quad \bar{\delta}^*(R_n, \sqrt{p}) \leqq \delta^*(R_{(k+1)[p^m]}). \tag{3.12}$$

Thus by (3. 12), (3. 11) and (3. 10)

$$R^*_n(f^{(k)}) = \max_{|x|\leqq 1-t} |f^{(k)}(x) - R_n(x)| = R^*_{(k+1)[p^m]}(f^{(k)}) \leqq c_3(k,p,t)\frac{\delta(r_{[p^{m+1}]})^\alpha}{p^{m(\alpha+\varepsilon)}} \leqq$$

$$\leqq c_4(k,p,t)\frac{\bar{\delta}^*(R_n, \sqrt{p})^\alpha}{n^{\alpha+\varepsilon}}.$$

Therefore we may apply Theorem 5 from [1] (in the interval $[-1+t, 1-t]$ instead of E_1) according to which $f^{(k)}(x) \in \mathrm{Lip}\,\alpha$ in every closed subinterval of $[-1+t, 1-t]$. But $0 < t \leqq \frac{1}{2}$ is arbitrary, qu. e. d.

Theorem is evidently a consequence of Theorem 1.

§ 4

Now we turn to the analytic and entire functions. For this purpose we introduce the following notations. Let $z = x+iy$ be a complex number,

$$E(\varrho) = \left\{ z : \frac{x^2}{\frac{1}{4}\left(\varrho + \frac{1}{\varrho}\right)^2} + \frac{y^2}{\frac{1}{4}\left(\varrho - \frac{1}{\varrho}\right)^2} \leqq 1 \right\} \quad (\varrho > 1), \quad E(1) = E_1$$

an ellipse-domain with focuses -1 and $+1$. The sum of half-axes of this ellipse is ϱ. Instead of (1. 1) we put

$$d(r_n) = \sup_{\varrho > 1} \{\varrho : z_{kn} \notin E(\varrho)\ (k = 1, \ldots, m)\}.$$

This is also a measure of the distance between E_1 and the poles of $r_n(x)$. It seems to be more suitable than (1. 1) in our investigations.

Notice that for approximating rational functions $d(r_n)>1$, and if $r_n(x)$ is especially a polynomial then $d(r_n)=\infty$.

THEOREM 4. *If*

$$\limsup_{n\to\infty} \sqrt[n]{R_n(f)} \leqq q < 1 \tag{4.1}$$

and there exists a sequence of indices $n_1<n_2<\ldots$ *such that* $n_{k+1}\leqq \lambda n_k$ $(k=1, 2, \ldots)$ *and*

$$\liminf_{k\to\infty} d(r_{n_k}) \geqq d > 1 \tag{4.2}$$

then there exists a function $F(z)$ *analytic in the interior of the ellipse* $E(\varrho)$ *where*

$$\varrho = \frac{d+q^{\frac{1}{1+\lambda}}}{dq^{\frac{1}{1+\lambda}}+1} \tag{4.3}$$

and $F(z)$ *coincides with* $f(x)$ *in* E_1.

THEOREM 5. *If*

$$\lim_{n\to\infty} \sqrt[n]{R_n(f)} = 0 \tag{4.4}$$

and there exists a sequence of indices $n_1<n_2<\ldots$ *such that* $n_{k+1}\leqq \lambda n_k$ $(k=1, 2, \ldots)$ *and*

$$\lim_{k\to\infty} d(r_{n_k}) = \infty \tag{4.5}$$

then $f(x)$ *is an entire function.*

REMARKS. Of course, conditions (4. 1) and (4. 4) are necessary, correspondingly to BERNSTEIN's theory of polynomial approximation. Condition (4. 2) means that the poles of certain approximating rational functions may not come too close to E_1. Easy to see that this is not necessary for the analyticity of $f(x)$. I belive that if we take the *best* rational approximation of an analytic function then (4. 2) holds.

Prof. P. TURÁN asked (oral communication): what can be the connection between the greatest ellipse of analyticity of $f(x)$ and (4. 1), if we take the best approximating rational functions. Theorem 4 gives partly an answer to this question, but I do not state that (4. 3) is the greatest ellipse in which $f(x)$ is analytic.

Similarly, (4. 5) is generally not necessary for $f(x)$ being an entire function. If the $r_n(x)$'s are especially the best approximating polynomials then (4. 5) automatically holds and we obtain a well-known BERNSTEIN's theorem.

In the proofs of our Theorems we shall follow a similar way to that of the corresponding BERNSTEIN's theorems.

§ 5

For the proofs we need the following

LEMMA 2. *If* $\max_{x\in E_1}|r_n(x)|\leqq M$ *then*

$$\max_{z\in E(\delta)}|r_n(z)|\leqq M\left(\frac{d(r_n)\,\delta-1}{d(r_n)-\delta}\right)^n \quad (1\leqq\delta<d(r_n)).$$

This is an easy consequence of Lemma 1 in [2] (p. 250), because the function $z=\frac{1}{2}\left(w+\frac{1}{w}\right)$ maps the domain $\{w:|w|>1\}$ onto the domain $\{z:z\notin E_1\}$ such that $z(\infty)=\infty$ and the circle $|w|=\varrho$ corresponds to the ellipse

$$\frac{x^2}{\frac{1}{4}\left(\varrho+\frac{1}{\varrho}\right)^2}+\frac{y^2}{\frac{1}{4}\left(\varrho-\frac{1}{\varrho}\right)^2}=1 \quad (\varrho>1).$$

PROOF OF THEOREM 4. Let $q<q_1<1$ and $1<d_1<d$ be arbitrary. Then for a sufficiently large k_1 we have by (4. 1) and (4. 2)

$$R_n(f)\leqq q_1^n \quad (n\geqq n_{k_1})$$

and

$$d(r_{n_k})>d_1 \quad (k\geqq k_1). \tag{5. 1}$$

Further let

$$\bar r_{k_1}(x)=r_{n_{k_1}}(x),\quad \bar r_k(x)=r_{n_k}(x)-r_{n_{k-1}}(x) \quad (k\geqq k_1+1) \tag{5. 2}$$

then

$$\max_{z\in E(d)}|\bar r_{k_1}(z)|<\infty \tag{5. 3}$$

and

$$|\bar r_k(x)|\leqq 2R_{n_{k-1}}(f)\leqq 2q_1^{n_{k-1}} \quad (x\in E_1,\ k\geqq k_1+1). \tag{5. 4}$$

Obviously, by (5. 2) and (5. 1)

$$d(\bar r_k)\geqq\min\left(d(r_{n_k}),d(r_{n_{k-1}})\right)>d_1 \quad (k\geqq k_1+1).$$

Thus, applying Lemma 2 we obtain by (5. 4)

$$\max_{z\in E(\delta)}|\bar r_k(z)|\leqq 2q_1^{n_{k-1}}\left(\frac{d(\bar r_k)\,\delta-1}{d(\bar r_k)-\delta}\right)^{n_k+n_{k-1}}\leqq 2\left[q_1\left(\frac{d_1\,\delta-1}{d_1-\delta}\right)^{\lambda+1}\right]^{n_{k-1}} \tag{5. 5}$$

$$(1<\delta<d_1,\ k\geqq k_1+1).$$

Now let $\delta>1$ be so near to 1 that

$$q_1\left(\frac{d_1\,\delta-1}{d_1-\delta}\right)^{\lambda+1}<1, \quad \text{i. e.} \quad \delta<\frac{d_1+q_1^{\frac{1}{1+\lambda}}}{d_1\,q_1^{\frac{1}{1+\lambda}}+1}$$

holds. Then $\sum_k \left[q_1\left(\frac{d_1\,\delta-1}{d_1-\delta}\right)^{\lambda+1}\right]^{n_k-1}$ converges, and by (5. 5) and (5. 3) $\sum_{k=k_1}^{\infty} \bar{r}_k(z)$ converges uniformly in $E(\delta)$,

$$F_{d_1,\,q_1}(z)=\sum_{k=k_1}^{\infty}\bar{r}_k(z) \quad (z\in E(\delta)), \tag{5.6}$$

say. But $1<\delta<\dfrac{d_1+q_1^{\frac{1}{1+\lambda}}}{d_1q_1^{\frac{1}{1+\lambda}}+1}$ is arbitrary and so the uniform convergence of (5. 6) holds in every closed subdomain of $E\left(\dfrac{d_1+q_1^{\frac{1}{1+\lambda}}}{d_1q_1^{\frac{1}{1+\lambda}}+1}\right)$. By Weierstrass' theorem, $F_{d_1,q_1}(z)$ is analytic in these subdomains (being every $\bar{r}_k(z)$ $(k\geqq k_1)$ analytic in $E(d_1)\supset E\left(\dfrac{d_1+q_1^{\frac{1}{1+\lambda}}}{d_1q_1^{\frac{1}{1+\lambda}}+1}\right)$). But by (5. 2) and (1. 3)

$$f(x)=\sum_{k=k_1}^{\infty}\bar{r}_k(x) \quad (x\in E_1) \tag{5.7}$$

i. e. $f(x)$ is analytic and $f(x)=F_{d_1,q_1}(x)$ $(x\in E_1)$.

Finally, if $q_1\to q$ and $d_1\to d$, we get a function $F(z)$ being defined and analytic in the interior of $E\left(\dfrac{d+q^{\frac{1}{1+\lambda}}}{dq^{\frac{1}{1+\lambda}}+1}\right)$ and $f(x)=F(x)$ if $x\in E_1$.

REMARK. We can see from the proof that if $r_n(x)=p_n(x)$ are especially polynomials then we may take $1/\lambda$ instead of $\dfrac{1}{1+\lambda}$ in (4. 3). Putting $d\to\infty$, $\lambda\to 1$, we obtain the ellipse of analyticity $E(1/q)$ where

$$q=\limsup_{n\to\infty}\sqrt[n]{\max_{x\in E_1}|f(x)-p_n(x)|}\,.$$

This is the well-known BERNSTEIN's theorem.

PROOF OF THEOREM 5. It is sufficient to show that if $\varrho>1$ arbitrary then there exists a function $F_\varrho(z)$ being analytic in the interior of $E(\varrho)$, and

$$F_\varrho(x)=f(x) \quad \text{for} \quad x\in E_1\,. \tag{5.8}$$

Define $d>1$ by the equation

$$\frac{1}{2}\left(d+\frac{1}{d}\right)=\varrho$$

and let $k(\varrho)$ be the index for which

$$\sqrt[n]{R_n(f)} \leqq \frac{1}{d^{1+\lambda}} \quad (n \geqq n_{k(\varrho)})$$

and

$$d(r_{n_k}) \geqq d \quad (k \geqq k(\varrho))$$

hold. This is possible because of (4. 4) and (4. 5). Thus we may apply Theorem 4 with our d, λ and $q = \frac{1}{d^{1+\lambda}}$. We obtain a function $F_\varrho(z)$ being analytic in the interior of the ellipse

$$E\left(\frac{d+\frac{1}{d}}{d\frac{1}{d}+1}\right) = E\left(\frac{1}{2}\left(d+\frac{1}{d}\right)\right) = E(\varrho)$$

and (5. 8) holds, qu. e. d.

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References

[1] J. SZABADOS, Structural properties of continuous functions connected with the order of rational approximation. I, *Annales Univ. Sci. Budapest, Sectio Math.*, **10** (1967), pp. 95—102.

[2] J. L. WALSH, *Interpolation and approximation by rational functions in the complex domain* (American Math. Soc., Colloquium Publ., 1960).

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Tomus 19 (1—2), (1968), pp. 103—116.

ON THE MINIMAL DENSITY OF MAXIMAL PACKINGS OF THE PLANE BY CONVEX BODIES

By

R. P. BAMBAH and A. C. WOODS (Columbus, Ohio, U. S. A.)

(*Presented by* L. FEJES TÓTH)

1. Introduction. Let R_2 denote the euclidean plane with a prescribed cartesian coordinate system, O being the origin. The usual point-vector notation is used. A convex body in R_2 is a closed bounded strictly convex set of points of R_2 that is symmetric in an interior point of the set. Let K be a convex body that is symmetric in the origin O. With K there is associated a unique distance function $F(X)$ defined and continuous for all X in R_2 such that

(i) $F(tX)=|t|F(X)$ for all real t,

(ii) $F(X)>0$ for all X other than O,

(iii) $F(X+Y)\leqq F(X)+F(Y)$ for all X, Y and with equality only if X, Y are collinear with O,

(iv) K is the set of points X for which $F(X)\leqq 1$.

Let Z be in R_2 and let R be a positive real number. Denote by $K(Z, R)$ the set of points X for which $F(X-Z)\leqq R$. The set $K(Z, R)$ is called the homothetic translate of K with centre Z and radius R. By a maximal K-packing of minimal radius R is meant a collection M of homothetic translates of K of possibly different radii but all at least R such that

(a) no two distinct members of M have an interior point in common, and

(b) any homothetic translate of K of radius R that is not in M meets at least one member of the collection M.

Let M be a maximal K-packing and let $V(M, k)$ be the area of the point set union of the members of M intersected with the circular disc whose centre is O and whose radius is $k>0$. The number $\liminf_{k\to\infty} V(M, k)/\pi k^2$ is called the lower density of M. Let $k^*=k^*(K)$ denote the greatest lower bound of the lower densities of all maximal K-packings of minimal radius $r>0$. L. FEJES TÓTH [see, e.g. 4] had conjectured that if K is a circle, then there is a maximal K-packing of lower density k^*, in which every member is a circle of radius R. This was recently proved by EGGLESTON [1]. Our object in this paper is to obtain the analogous result when K is a strictly convex domain with a centre. It may, however, be mentioned that FEJES TÓTH made this conjecture also for saturated systems of circles, which may not form a packing. EGGLESTON's theorem and our analog here are proved only for the situations where the sets form a packing. In another paper we have proved the FEJES TÓTH conjecture for circles which may overlap, provided, however, that their centres form a discrete set and their radii are bounded above by some finite number. We have not been able to extend this result to all centro-symmetric convex domains. Our main theorem is

THEOREM A. *There exists a maximal K-packing of minimal radius r whose lower density is* $k^*=k^*(K)$ *and every member of which has radius r. If* $d(K)$ *denotes*

the density of the thinnest covering of the plane by the translates of K and $t(K)$ denotes the area of the largest triangle contained in K then

$$k^*(K) = \frac{1}{4} d(K) = V(K)/8t(K).$$

As with EGGLESTON theorem A is a consequence of two theorems namely

THEOREM 1. *Let M be a maximal K-packing of minimal radius r. There exists a triangulation of the plane such that*

(i) *each vertex of each triangle is the centre of a member of M,*

(ii) *if abc is any one of the triangles and $K(a)$, $K(b)$, $K(c)$ are the members of M with centres a, b, c respectively then there exists a point p such that the homothetic translate of K centred at p of radius r meets each of $K(a)$, $K(b)$, $K(c)$.*

THEOREM 2. *Let $K(a)$, $K(b)$, $K(c)$ be nonoverlapping homothetic translates of K centred at the vertices abc of a nondegenerate triangle each of radius at least r and each meeting a homothetic translate of K of radius r. Then*

$$\frac{V(abc \cap (K(a) \cup K(b) \cup K(c)))}{V(abc)} \geqq \frac{V(K)}{2t(K)}.$$

It is known that

$$1 < d(K) = V(K)/2t(K) \leqq 2\pi/3\sqrt{3}.$$

The left hand inequality is obvious while that on the right follows from a theorem of SAS [3] (see e.g., BAMBAH and ROGERS [2]). It is also known that the equality on the left is needed only for ellipses. We therefore have also

THEOREM 3. $\frac{1}{4} < k^*(K) \leqq \pi/6\sqrt{3}$ *with equality on the right for ellipses only.*

2. LEMMA 1. *Let M be a maximal K-packing with minimal radius r. Let C be any fixed circle in R_2. There exist at most finitely many members of M that intersect C.*

PROOF. Assume by way of contradiction that C is such that infinitely many members of M meet C. Let $K(a, R(a)) \in M$ intersect C. If it meets C only in inner points of $K(a, R(a))$ then C is completely contained in the interior of $K(a, R(a))$ and then no other member of M can meet C without destroying the packing property. Thus $K(a, R(a))$ meets C in a point x lying on the boundary of $K(a, R(a))$. Therefore $F(x-a) = R(a)$. So in particular

$$F((1 - rR^{-1}(a))(x-a)) = R(a) - r.$$

Put $P = (1 - rR^{-1}(a))(x-a) + a$. Now

$$K(P, r) \subset K(a, R(a))$$

since the inequality $F(x-P) \leqq r$ implies that

$$F(x-a) - F((1 - rR^{-1}(a))(x-a)) \leqq r$$

or

$$F(x-a) \leqq R(a).$$

Furthermore, $K(P, r)$ meets C in the point x. If we repeat this construction for each $K(a, R(a))$ that intersects C we obtain infinitely many bodies $K(P, r)$ all of radius r and intersecting C such that no two overlap. Let D be the diameter of $K(P, r)$. We have therefore obtained infinitely many $K(P, r)$ packed in the disc centre of C and radius that of C plus D, an obvious absurdity which proves the lemma.

LEMMA 2. *There exists a positive constant N depending only on r such that if M is a maximal K-packing of minimal radius r then there is another maximal K-packing M^* of minimal radius r such that*

(i) *the lower density of M^* does not exceed the lower density of M,*

(ii) *if $K(a, R(a)) \in M^*$ then $R(a) < N$.*

PROOF. Denote by L a densest lattice packing of the plane by bodies $K(x, r)$. For $k > 0$ let $L(k)$ denote the point set union of the elements of L that are completely contained in $K(0, k)$. Further let $V(L(k))$ denote the area of $L(k)$. As is well-known, the strict convexity of K implies that

$$\limsup_{k\to\infty} V(L(k))/V(K(0, k)) = t < 1.$$

Select t' so that $t < t' < 1$ and take N_1 so large that

$$V(L(k))/V(K(0, k)) < t' \quad \text{if} \quad k \geqq N_1.$$

Since

$$\lim_{R\to\infty} \frac{(R+2r)^2 - (R-4r)^2}{R^2} = 0,$$

we may take N_2 so large that this expression does not exceed $1-t'$ provided only that $R \geqq N_2$. Put $N = \max(N_1, N_2)$. We show that with this value of N the lemma holds.

If M contains no body of radius at least N then there is nothing to prove so assume that there exists a body $K(a, R) \in M$ with $R \geqq N$. Removing $K(a, R)$ from M we will not disturb the packing property if we put in all the bodies $K(x, r)$ that make up $L(R)$ each translated by a. Thus we obtain the K-packing

$$P = (M - \{K(a, R)\}) \cup (L(R) + a)$$

where in this context $L(R)$ is understood to be the collection of bodies $K(x, r)$ within $K(0, R)$ rather than their point set union. By construction, P has minimal radius r but may not be maximal. We complete it to a maximal packing by adding as many sets $K(x, r)$ as possible without destroying the packing property. Call the packing thus obtained M_1. Suppose that $K(x, r) \in M_1 - P$. Then in particular $K(x, r)$ cannot overlap any member of P. Therefore x is confined to the band $K(a, R+r) - K(a, R-3r)$ for if x lies outside $K(a, R+r)$ then $K(x, r)$ lies completely outside $K(a, R)$ which contradicts the maximality of M. On the other hand if x lies inside $K(a, R-3r)$ then $K(x, r)$ is contained in $K(a, R-2r)$ which implies that L is not maximal. But it is well-known that L is a maximal packing of minimal radius r so this is also not possible. It follows that all the sets of $M_1 - P$ can cover at most the band $K(a, R+2r) - K(a, R-4r)$. Outside $K(a, R+2r)$ the packings M, M_1 agree. Inside $K(a, R+2r)$ the packing M covers

at least a set of area $V(K(a, R)) = R^2 V(K)$; whereas M_1 covers a set of area at most A where

$$A = V(K(a, R+2r) - V(a, R-4r)) + t' V(K(0, R))$$

since $R \geqq N_1$. But

$$A = ((R+2r)^2 - (R-4r)^2 + t' R^2) V(K) \leqq R^2 V(K)$$

since $R \geqq N_2$. Therefore the lower density of M_1 does not exceed that of M. If M contains at most finitely many bodies $K(a, R(a))$ for which $R(a) \geqq N$ then clearly by repeating the above construction a finite number of times we arrive at a maximal packing M^* which satisfies the lemma. However, if M contains infinitely many such bodies, M_1 is constructed as above but first $K(a, R)$ is chosen in M to be a body with $R \geqq N$ for which a is closest to 0. M_2 is constructed from M_1 in a like manner. Repeating this process we obtain a sequence of packings $M_1, M_2, \ldots$ all maximal with minimal radius R with the property that given any disc D centred at the origin there corresponds a suffix n_0 such that for $m, n > n_0$ the members of M_m that intersect D are precisely the members of M_n that intersect D as follows from lemma 1. Therefore there is a limiting collection which satisfies the requirements.

3. PROOF OF THEOREM 1. The proof is given here as a series of lemmas. The idea is a standard one of constructing Dirichlet regions and using the vertices of these regions to form the triangles of centres belonging to a vertex.

Let a, b, c be three noncollinear points of R_2. We denote by $H(a, b; c)$ the closed half-plane containing the point c whose boundary is the straight line though the points a band b.

LEMMA 3. *If* $x \in H(a, b; c) \cap H(a, c; 2a-b) \cap H(b, c; a)$ *then*

$$F(x-a) - F(x-b) \leqq F(c-a) - F(c-b)$$

with strict inequality if $x \neq c$.

PROOF. The line segment xb meets the line segment ac in a point y say. Then by convexity

$$F(c-y) + F(y-b) \geqq F(c-b)$$

and

$$F(y-a) + F(x-y) \geqq F(x-a).$$

Adding these two inequalities we obtain the result. If $x \neq c$ then y is an inner point of the line segment ac so that y, b, c are noncollinear. It follows from the strict convexity of K that the first of the above pair of inequalities is a strict inequality. This completes the proof of the lemma.

Now let M be a maximal packing of minimal radius r. Let $K(a, R(a)) \in M$ and for an arbitrary point x of the plane define

$$d_a(x) = F(x-a) - R(a).$$

Further denote by $S(a)$ the set of points x of the plane with the property that $d_a(x) \leqq \leqq d_b(x)$ for all members $K(b, R(b))$ of M.

LEMMA 4. *If $x \in S(a)$ then all points except possibly x of the line segment ax are interior points of $S(a)$.*

PROOF. Let $z \neq x$ be a point of ax. Further, let $K(b, R(b)) \in M$ with $a \neq b$. We show first that $d_a(z) < d_b(z)$. If a, b, z are not collinear then $z \in H(a, b; x) \cap \cap H(a, x; 2a-b) \cap H(b, x; a)$ and $z \neq x$. By lemma 3 it follows that

$$F(z-a) - F(z-b) < F(x-a) - F(x-b).$$

Hence $d_a(z) - d_b(z) < d_a(x) - d_b(x) \leqq 0$ by hypothesis. Thus $d_a(z) < d_b(z)$.

If on the other hand a, b, z are collinear then

$$d_a(z) = d_a(x) - F(x-z) \leqq d_b(x) - F(x-z) \leqq d_b(z)$$

and for the equality sign to hold we must have $F(z-b) = F(x-b) - F(x-z)$. Therefore z is between x and b, so that z is not on the line segment ab. If z is on the opposite side of a to b then

$$F(z-b) = F(z-a) + F(a-b) \geqq F(z-a) + r + R(b)$$

so

$$d_b(z) = d_a(z) + r + R(a),$$

thus the equality sign cannot hold. By a symmetrical argument if z is on the opposite side of b to a the equality sign again cannot hold. Therefore in all cases $d_a(z) < d_b(z)$. Assume by way of contradiction that z is not an interior point of $S(a)$, then there exists a sequence of points $\{z_n\}$ with the limit z and a sequence of bodies $\{K(b_n, R(b_n))\}$ in M and all different from $K(a, R(a))$ such that $z_n \in S(b_n)$ for every n. Since M is maximal so $S(b_n) \subset K(b_n, R(b_n) + r)$ so that all the bodies $K(b_n, R(b_n) + r)$ intersect some fixed disc centred at O. Hence all the bodies $K(b_n, R(b_n))$ intersect some fixed disc centred at O. By lemma 1 there are at most finitely many distinct b_n. Thus there is a subsequence $\{b_{n_m}\}$ of centres all equal to c say. We then have by the continuity of F, $d_c(z) \leqq d_a(z)$ a contradiction which proves the lemma.

LEMMA 5. *If a, b are distinct centres of members of M then*

$$S(a) \subset K(a, F(b-a)).$$

PROOF. Since $K(a, R(a))$, $K(b, R(b))$ do not overlap it follows that

$$F(b-a) \geqq r + R(a).$$

If $x \in S(a)$ then $r \geqq d_a(x) = F(x-a) - R(a)$ and so $F(x-a) \leqq r + R(a) \leqq F(b-a)$, and the lemma follows.

LEMMA 6. *If a, b, c are centres of members of M such that they are not collinear then $S(a)$, $S(b)$ cannot have any point in common in the closed set $H(a, c; 2a-b) \cap \cap H(b, c; 2c-a)$.*

PROOF. Assume by way of contradiction that there is a point x of $S(a) \cap S(b)$ in the sector $H(a, c; 2a-b) \cap H(b, c; 2c-a)$. Since $d_c(c) = -R(c) < 0$ and no two bodies in M overlap so c is an interior point of $S(c)$. Therefore $x \neq c$, so that c is a point of the triangle abx other than a vertex. By lemma 5 and the strict con-

vexity of K it follows that every point except possibly x is in the interior either of $K(a, F(x-a))$ or of $K(b, F(x-b))$. Hence either

$$F(c-a) < F(x-a) \quad \text{or} \quad F(c-b) < F(x-b).$$

In the first case $F(c-a) < d_a(x) + R(a) \leqq r + R(a)$ since $x \in S(a)$. This implies that the bodies $K(a, R(a))$ and $K(c, R(c))$ overlap which is impossible. Hence $F(c-b) < F(x-b)$, but a similar argument shows that the bodies $K(b, R(b))$ and $K(c, R(c))$ overlap, this contradiction proves the lemma.

We now define a vertex of M to be a point v of the plane such that v lies in at least three distinct sets $S(a)$, $S(b)$, $S(c)$ where a, b, c are centres of distinct members of M. We further say that a, b, c belong to v.

LEMMA 7. *Any three centres belong to at most one vertex.*

PROOF. Let v, w be vertices common to $S(a)$, $S(b)$, $S(c)$ with $v \neq w$. By lemma 5 a, b, c are not collinear. By lemma 6, neither v nor w can lie in the closed regions $H(a, c; 2a-b) \cap H(b, c; 2c-a)$, $H(c, a; 2c-b) \cap H(b, a; 2a-c)$ and $H(b, c; 2b-a) \cap H(a, b; 2b-c)$. The line segments va, vb, vc lie apart from v in the interiors of $S(a)$, $S(b)$, $S(c)$ respectively so that in particular w cannot lie on any of these line segments. The line segment wc cannot cross either line segment va or vb since all points of wc apart from w are in the interior of $S(c)$. By similar arguments, wb cannot cross either va or vc and wa cannot cross either vb or vc. But these conditions exhaust all possibilities for w producing a contradiction that proves the lemma.

COROLLARY. *The vertices of M form a discrete set.*

PROOF. Assume by way of contradiction that there is a limit point x say of vertices. By lemma 7 there are an infinite number of the sets $S(a)$ that intersect the disc centre x of radius 1. It follows that an infinite number of members of M intersect the disc centre x of radius $1+r$ which contradicts lemma 1.

LEMMA 8. *At most a finite number of centres of M belong to one vertex.*

PROOF. Assume by way of contradiction that there is a vertex v for which infinitely many centres of members of M belong to v. Then there are infinitely many members of M intersecting the disc centred at v of radius r, again a contradiction to lemma 1.

LEMMA 9. *If x is any point of the plane then there exists a vertex v such that x belongs to the convex hull $P(v)$ of all the centres of members of M that belong to v.*

PROOF. By definition the sets $S(a)$ cover the plane without overlapping. By lemma 4 $S(a)$ is a bounded star body and so its boundary is a simple closed curve any point of which belongs to at least one other such set $S(b)$. If the complete boundary of $S(a)$ belonged to $S(b)$ then $S(b)$ would not be a star body. Hence $S(a)$ meets at least two such sets. By lemma 1 $S(a)$ meets at most finitely many such sets. Hence the boundary of $S(a)$ can be represented as the union of finitely many closed sets each belonging to a set $S(b)$ different from $S(a)$, there being at least two such sets. Hence there exists a point v of $S(a)$ that is contained in at least two such sets $S(b)$. The point v is then a vertex of M. Thus there exists a vertex. Now let v be any vertex and $P(v)$ the convex hull of all the centres belonging to v. By lemma 8,

$P(v)$ is a polygon. Let a, b be centres belonging to v such that ab is a proper edge of $P(v)$. $P(v)$ has an interior since any three centres belonging to v cannot be collinear. No centre can lie in the triangle abv since the sets $K(a, R(a)+r)$, $K(b, R(b)+r)$ together cover abv completely apart from a, b and v. Denote by $B(a)$ that part of the boundary of $S(a)$ with beginning point v and endpoint v' where v' is the point of intersection of the boundary of $S(a)$ and the ray ab produced past a, and which passes through, at least in part, the side of ab opposite to that containing $P(v)$. Each point of $B(a)$ lies in at least one set of the form $S(c)$ other than $S(a)$. Since any three centres define at most one vertex by lemma 7, and since $S(a)$ meets at most finitely many $S(c)$, so there exists a neighbourhood of v such that apart from v the portion of $B(a)$ lying in the neighbourhood belongs to both $S(a)$ and $S(b)$ and to no other such set, for, assume this assertion is false, then there exists a set $S(c) \neq S(b)$ or $S(a)$ such that $S(c)$ has points other than v in common with $B(a)$ arbitrarily close to v. Therefore in particular v belongs to $S(c)$ so that c is a centre belonging to v. Thus a, b, c cannot be collinear and c lies on the same side of ab as $P(v)$. Let $d \neq v$ belong to $B(a) \cap S(c)$. The line segment cd consists apart from d of inner points of $S(c)$ hence cd cannot cross either av or bv. As d may be chosen arbitrarily close to v this implies that v must lie on the opposite side of ab to $P(v)$ and further that c is confined to one of the two regions $H(a, b; 2b-v) \cap$ $\cap H(b, v; 2b-a)$ or $H(a, b; 2a-v) \cap H(a, v; 2a-b)$. But then v lies in one of the two regions $H(a, b; 2b-c) \cap H(b, c; 2b-a)$ or $H(a, b; 2a-c) \cap H(a, c; 2a-b)$. Either of these is a contradiction to lemma 6. Thus there is some neighbourhood of v such that apart from v the portion of $B(a)$ in the neighbourhood belongs to both $S(a)$ and $S(b)$ and to no other such set. Hence there is a first point of $B(a)$ other than v that lies in some $S(c)$ other than $S(a)$ and $S(b)$ since the endpoint of $B(a)$ does not lie in $S(b)$. This point w say is then a vertex of M different from v, and a, b and c are centres belonging to w. Again a, b, c cannot be collinear and exactly as above lemma 6 implies that c must be on the opposite side of ab to $P(v)$. It follows that $P(v) \cup P(w)$ contains the segment ab as an interior diagonal. The union of all the polygons $P(v)$ is a closed set so that its complement is open. If not empty this complement must have a boundary composed of line segments each segment being on the boundary of a $P(v)$. This is in contradiction to has just been proved. It follows that the $P(v)$ cover the plane. This concludes the proof of lemma 9.

LEMMA 10. *If v and w are distinct vertices then $P(v)$, $P(w)$ have no interior points in common.*

PROOF. Let v, w be distinct vertices such that the centres a, b, c belong to v and centres d, e, f belong to w. By lemma 7 at least one of the centres d, e, f is distinct from all of a, b, c. We show first that the triangles abc and def cannot have any interior points in common. Assume, by way of contradiction, that they overlap. The union of the sets $K(a, R(a)+r)$, $K(b, R(b)+r)$ and $K(c, R(c)+r)$ contain the triangle abc, apart from its vertices and v, in its interior. Thus abc and similarly def can contain no centre apart from their vertices. It follows that at least one edge of abc intersects an edge of def in a point interior to both edges. Denote these edges by xy and zt respectively. Further, denote by $C(x, y)$ the path consisting of the two line segments xv and vy. Similarly denote by $C(z, t)$ the path consisting of the two line segments zw and wt. Apart from the point v the path $C(x, y)$ lies in the union of the interiors of $S(x)$ and $S(y)$ whereas, apart from the point w, the path

$C(z, t)$ lies in the union of the interiors of $S(z)$ and $S(t)$. Since $v \neq w$ so $C(x, y)$ and $C(z, t)$ have no point in common. On the other hand, by lemma 6, w cannot lie in the region $H(x, z; 2x-t) \cap H(x, t; 2x-z)$ or in the region $H(y, z; 2y-t) \cap \cap H(y, t; 2y-z)$. Hence $C(z, t)$ cannot intersect either of these regions. It follows that $C(z, t)$ intersects the line segment xy in an interior point. By a similar argument $C(x, y)$ must intersect zt in an interior point. Therefore $C(x, y)$ and $C(z, t)$ must have a point in common which is impossible. This contradiction proves that the triangles *abc* and *def* cannot overlap. It follows at once that $P(v)$ and $P(w)$ cannot overlap and the lemma is proved.

Theorem 1 follows from lemmas 9 and 10 since each polygon $P(v)$ can be triangulated as required.

4. PROOF OF THEOREM 2. Let a, b, c be the vertices of a proper triangle. Further let $K(a, R(a))$, $K(b, R(b))$, $K(c, R(c))$ be such that no two overlap and in addition such that $K(a, R(a)+r)$, $K(b, R(b)+r)$, $K(c, R(c)+r)$ all have a point in common. Put

$$A = \inf \frac{V(K(a, R(a)) \cup K(b, R(b)) \cup K(c, R(c)) \cap T(a, b, c))}{V(T(a, b, c))}.$$

Our object is to show that $\inf A$ is achieved by a configuration for which $R(a) = = R(b) = R(c) = r$. The proof parallels that of EGGLESTON [1] for the case when K is a circle though we felt that amplification of some of the details was desirable.

We begin with a reduction. Thus, let N be the set of all non-negative numbers n for which the set

$$K(a, R(a)+r-n) \cap K(b, R(b)+r-n) \cap K(c, R(c)+r-n)$$

is not empty. N is not empty since by hypothesis 0 is in N. Further, N is bounded above by r. Hence N has a least upper bound say m which lies in N since the bodies are closed. By the strict convexity of K there is a unique point t in the set

$$K(a, R(a)+r-m) \cap K(b, R(b)+r-m) \cap K(c, R(c)+r-m).$$

We show first that t is a point of the triangle $T(a, b, c)$. Write K_a, K_b, K_c for $K(a, R(a)+r-m)$, $K(b, R(b)+r-m)$, $K(c, R(c)+r-m)$ respectively. By strict convexity at least one pair of K_a, K_b, K_c overlap. Without loss of generality we may assume that K_a, K_b overlap. The convex set $K_a \cap K_b$ is a lenslike body and K_c is tangent to it at the point t. Either t is one of the two 'vertices' of the lens or it is not. If it is not then K_c is tangent to either K_a or K_b and we may suppose in this case that it is tangent to K_a from which it follows that t is an inner point of the line segment ac and therefore in the triangle $T(a, b, c)$. If on the other hand t is a vertex of the lens $K_a K_b$ denote by S the region $H(a, t; 2t-b) \cap H(b, t; 2t-a)$. If c lies in S then t lies in $T(a, b, c)$. Now if c does not lie in $H(a, t; 2t-b)$ denote by u the point of the boundary of K_a on the line segment ac. Then u belongs to K_c so that either the major arc or minor arc ut of the boundary of K_a lies in K_c. The major arc ut cannot lie in K_c for then the centre a lies in K_c which is impossible. Hence the minor arc ut is in K_c from which it follows that K_c overlaps $K_a \cap K_b$ which is again impossible by the construction of the number m. Therefore c lies in $H(a, t; 2t-b)$. Similarly c lies in $H(b, t; 2t-a)$. Thus c is in S so that t is in the triangle $T(a, b, c)$. This completes the proof that t is in $T(a, b, c)$.

Take coordinates such that t is the origin O. With $s=r-m$ it follows that $s>0$ since at least two of K_a, K_b, K_c must overlap. Further the bodies $K(a, R(a))$, $K(b, R(b))$, $K(c, R(c))$ all touch $K(0, s)$ externally unless O lies on one of the sides, say ab, so that $K(a, R(a))$, $K(b, R(b))$ touch $K(0, s)$ externally while $K(c, R(c))$ and $K(0, s)$ have common interior points. In this case move c along Oc produced to the point c' such that $K(c', R(c))$ touches $K(0, s)$ externally. Since $F(a)=R(a)+$ $+s\leqq R(a)+r\leqq R(a)+R(c)\leqq F(c-a)$ it follows that $F(c-a)\leqq F(c'-a)$ so that $K(a, R(a))$ and $K(c', R(c))$ do not overlap. Similarly $K(b, R(b))$ and $K(c', R(c))$ do not overlap. Let $K(a, R(a))$ meet ac at the point A. Then AB, the segment parallel to Oc intercepted by the triangle $T(a, b, c)$ lies in $K(a, R(a))$ and comparing the lengths of segments parallel to Oc intercepted by $K(a, R(a))\cap T(a, b, c)$ and by $K(a, R(a))\cap T(a, b, c')$ we find that as in EGGLESTON [1],

$$\frac{V(K(a, R(a))\cap T(a, b, c'))}{V(K(a, R(a))\cap T(a, b, c))}\leqq\frac{V(T(a, b, c'))}{V(T(a, b, c))}.$$

Similarly

$$\frac{V(K(b, R(b))\cap T(a, b, c'))}{V(K(b, R(b))\cap T(a, b, c))}\leqq\frac{V(T(a, b, c'))}{V(T(a, b, c))}.$$

Now angle $ac'O<$angle acO and angle $bc'O<$angle bcO hence

$$V(K(c', R(c))\cap T(a, b, c'))\leqq V(K(c, R(c))\cap(T(a, b, c)).$$

It therefore follows that

$$\frac{V((K(a, R(a))\cup K(b, R(b))\cup K(c, R(c)))\cap T(a, b, c))}{V(T(a, b, c))}$$

does not increase in the movement of c to c'. We can therefore suppose that $K(a, R(a))$, $K(b, R(b))$, $K(c, R(c))$ all touch $K(0, s)$ externally and our reduction is complete.

At this point we change the class of configurations under discussion. We denote by I that class of triangles every member of which has a vertex on each of the line segments Oa, Ob, Oc excluding the origin O. Thus with x on Oa, y on Ob and z on Oz and none the origin O the triangle $T(x, y, z)$ is a member of I. We define K_x, K_y, K_z to be the bodies $K(x, R(x))$, $K(y, R(y))$, $K(z, R(z))$ respectively, where the numbers $R(x)$, $R(y)$, $R(z)$ are such that K_x, K_y, K_z touch $K(0, s)$ externally. Theorem 2 will follow if we can prove

THEOREM 4. $\operatorname{Inf}\dfrac{V((K_x\cup K_y\cup K_z)\cap T(x, y, z))}{V(T(x, y, z))}$ *extended over all members of the subclass* I_0 *of* I *for which* $R(x)\geqq s$, $R(y)\geqq s$, $R(z)\geqq s$ *(and* K_x, K_y, K_z *do not overlap) is attained by a member of* I_0 *for which* $R(x)=R(y)=R(z)=s$.

Write $W(T)$ for $V((K_x\cup K_y\cup K_z)\cap T(x, y, z))$. Amongst the members of I_0 there is a sequence of triangles $T(x_n, y_n, z_n)=T_n$, $n=1, 2, \ldots$, such that $W(T_n)/V(T_n)\to\inf_{T\in I_0} W(T)/V(T)$. By hypothesis $0<2s\leqq F(x_n)\leqq F(a)$, $0<2s\leqq$ $\leqq F(y_n)\leqq F(b)$, $0<2s\leqq F(z_n)\leqq F(c)$. Therefore by compactness the sequence of triangles T_n contains a convergent subsequence. Thus there exists a triangle

T_0 of I_0 such that $W(T_0)/V(T_0) = \inf_{T \in I_0} W(T)/V(T)$. Let the vertices of T_0 be u, v, w respectively. Without loss of generality we may assume that $F(u) \geqq F(v) \geqq F(w)$. Theorem 3 will follow if we can prove $F(u) = 2s$. We assume that $F(u) > 2s$ and prove that this to a contradiction.

Write $X(uvw)$ for $(K_u \cup K_v \cup K_w) \cap T(u, v, w)$ and $Y(uvw)$ for that part of $T(u, v, w)$ not in $X(uvw)$. Let the intersection of $Y(uvw)$ with the line segments uv and uw be $Z(uvw)$ and let $W(uvw)$ be the part of $T(uvw)$ that lies on the lines parallel to Ou through points of $Z(uvw)$. Let the parts of $Y(uvw)$ and $W(uvw)$ on the same side of Ou as v be written $Y_v(uvw)$ and $W_v(uvw)$ while those parts on the same side of Ou as w are written $Y_w(uvw)$ and $W_w(uvw)$. The main part of the proof lies in showing that $V(W(uvw)) < V(Y(uvw))$. It suffices to show that $V(W_v(uvw)) \leqq \leqq V(Y_v(uvw))$ and $V(W_w(uvw)) \leqq V(Y_w(uvw))$ with strict inequality in at least one case. In fact we will prove that $V(W_w(uvw)) \leqq V(Y_w(uvw))$ with strict inequality provided that O is not on the line segment uw. If $V(W_w(uvw)) = 0$ this is clear so from now on we assume that this is not the case. We require the following two lemmas:

LEMMA 11. *Let A, B and X be distinct points of the plane such that $F(X-A) = F(X-B)$. If $C \in H(A, X; 2X-B) \cap H(B, X; A)$ then $F(C-A) \leqq F(C-B)$. The result holds with strict inequality provided $C \neq A$.*

PROOF. There are two cases:

(i) $C \in H(A, B; X)$.

The line segment BC meets AX in a point D say, so that

$$F(B-C) = F(B-D) + F(D-C).$$

Now

$$F(C-A) \leqq F(D-C) + F(A-D).$$

Further

$$F(X-B) \leqq F(D-X) + F(D-B)$$

and one of these inequalities is strict provided that C is different from X. Therefore adding these two inequalities we obtain

$$F(C-A) \leqq F(C-B)$$

since by hypothesis $F(X-B) = F(X-A)$.

(ii) $C \in H(A, B; 2A-X)$.

The segment BC meets the line XA produced in the point D say, so that

$$F(C-D) + F(D-B) = F(C-B).$$

Now

$$F(C-A) \leqq F(C-D) + F(D-A)$$

and

$$F(D-X) < F(X-B) + F(D-B).$$

Adding these two inequalities we again obtain the desired result. This completes the proof of the lemma.

Let K_u meet the segments uw and Ou in the points e, f respectively. Further let K_w meet the segment uw in the point e'. Let f' be the point such that ff' is equal

and parallel to ee' and drawn from f towards the segment vw. Denote by P the curvilinear parallelogram bounded by the line segments ee', ff' and the arc A bounding K_u with endpoints e, f that lies in T, together with the translation A' of this arc so that the endpoints of the translation are e', f'.

LEMMA 12. *P does not overlap any one of the bodies K_u, K_v, K_w.*

PROOF. By the construction it cannot overlap K_u. Further since K is symmetric and strictly convex the arc A' meets K_w in the single point e'. Therefore P cannot overlap K_w. It remains to show that P does not overlap K_v.

Let g be the point in which the boundary of K_w meets the line segment Ow. The boundary of $K(0, s)$ meets the line Ou in the two points f and $-f$ and meets the line segment Ow in g. Denote by B that arc of the boundary of $K(0, s)$ with endpoints g and $-f$ that lies in the sector $H(0, u; g) \cap H(0, w; -u)$. Now denote by N the region bounded by the line segments $-fu$, uw, wg together with the arc B. We claim first that N does not overlap K_v. Evidently N is the union of the triangle Ouw together with the sector bounded by the segments $O(-f)$, Og and the arc B. The latter sector lies in $K(0, s)$ and so does not overlap K_v. On the other hand the segments Ou and Ow apart from the points f and g lie in the interior of $K_u \cup K_w \cup K(0, s)$. Therefore K_w cannot cross these segments and so cannot overlap the triangle Ouw. It follows that K_v does not overlap N. We now claim that P is contained in N. Assume by way of contradiction that this is false. The segment ee' and the arc A lie in N since K_u cannot cross Ow. If f' lies in N then either it lies in the triangle Ouw in which case ff' is in N or it lies in $K(0, s)$ and again it is in N by the convexity of $K(0, s)$. If A' also lies in N then P is contained in N which is contrary to assumption so that either $f' \notin N$ or $A' \not\subset N$. Thus in either case $A' \not\subset N$. The arc A' must therefore cross the line segment Ow at least once. Let h be the point on Ow such that the part of A' with endpoints e' and h lies in the triangle Ouw. By the strict convexity of K, h must lie in the interior of $K(0, s)$. Hence A' crosses the boundary of $K(0, s)$ in a point x say that lies in the triangle Ouw. Since A' does not lie completely in N so A' crosses the boundary of $K(0, s)$ in a point $y \neq x$ that lies outside the triangle Ouw. The points x, y satisfy the equality $F(x) = F(y)$ since they are on the boundary of $K(0, s)$. On the other hand the translation of e to e' maps u into a point u' on the segment uw and then A' is part of the boundary of $K(u', R(u))$ and so $F(x-u') = F(y-u')$. But $u' \in H(0, x; 2x-y) \cap H(0, y; x)$ and $u' \neq 0$. Hence by lemma 11, $F(x-u') < F(y-u')$ the contradiction that proves the lemma.

We are now in a position to complete the proof that $V(W_w(uvw)) \leqq V(Y_w(uvw))$. Produce the line segment Ou to a point v_1 such that O is the midpoint of uv_1. Let the line through uv_1 meet vw in p. Since $F(u) = F(v_1)$ and $F(w) \leqq F(u)$ so w is contained in the body $K(0, F(u))$. Further $F(v) \leqq F(u)$ so that since p is on the line segment vw so $F(p) \leqq F(u)$. Hence v_1 is either outside $T(uvw)$ or on its boundary. We may assume that O is not on the segment uw for otherwise $W_w(uvw)$ is the empty set contrary to hypothesis. $W_w(uvw)$ is bounded by two lines parallel to Ou. Of these let that nearest to Ou meet wv_1 in j and uw in e. Let the other meet uw in e'. Note that e and e' are the same points as defined previously. Draw jq parallel to uw to meet uv_1 in q. Then the triangle $T(uwv_1)$ is a dilation of the triangle $T(v_1qj)$ about v_1. Hence

$$F(q-u)\, F(w-v_1) = F(j-w)\, F(v_1-u).$$

Further $T(uwv_1)$ is a dilation of $T(ewj)$ about w. Therefore also

$$F(j-w)F(u-w) = F(e-w)F(v_1-w).$$

It follows that

$$F(q-u) = F(e-w)F(u-v_1)/F(u-w).$$

Since $F(w) \leqq F(u)$ so $F(u-w) < 2F(u) = F(u-v_1)$. Therefore

$$\big(F(u-w)-F(u-e)\big)F(u-e) < \big(F(u-v_1)-F(u-e)\big)F(u-e).$$

As $\frac{1}{2}F(u-v_1) = F(u) = s + F(u-e)$, it follows that

$$F(w-e)F(u-e) < \big(2s+F(u-e)\big)F(u-e).$$

Hence $F(w-e)\big(2s+2F(u-e)\big) < \big(2s+F(u-e)\big)\big(F(u-e)+F(w-e)\big)$, and

$$F(w-e)F(u-v_1)/F(u-w) < 2s+F(u-e).$$

Therefore $F(q-u) < 2s + F(u-e)$ so that if q is on the line segment Ov_1 then $F(q) < s$ and $q \in K(0, s)$. We distinguish two cases:

(i) q lies on the line segment uf.

We map the curvilinear parallelogram P into a set containing $W_w(uvw)$ by an area preserving transformation as follows. Divide P into a collection of line segments parallel to ee'. Take such a line segment and map it onto the line segment that is collinear with it and has endpoints on the parallel lines through the segments ej and $e'j'$ where j' is the point on wv_1 such that $e'j'$ is parallel to ej. If for an individual segment the direction of the translation is from u towards w then any point of the image that lies in $T(uvw)$ has a preimage that also lies in $T(uvw)$. If however the direction of the translation is from w towards u then the arc A' must cross the ray from e' to j' in a point k different from e'. The point k is unique since A' is a convex curve. This however is impossible since that part of $e'j'$ that lies in the triangle Ouw lies also in K_w. To see this let m be the point on Ow where $e'j'$ meets Ow. Then since $F(e-u) \geqq F(e'-w)$ so $F(m) \geqq F(m-w)$. If m lies in K_w there is nothing to prove whereas if m does not lie in K_w then it lies in $K(0, s)$ so that $F(m) \leqq s$. Hence $F(m-w) \leqq s \leqq R(w)$ and m does lie in K_w, the contradiction that proves the assertion. We have produced an area preserving map whose image contains $W_w(uvw)$ and such that any point of $W_w(uvw)$ has a preimage lying in $P \cap T(uvw)$ which in turn is contained in $Y_w(uvw)$ by lemma 12. Hence $V\big(W_w(uvw)\big) \leqq V\big(Y_w(uvw)\big)$. Let ej meet pw in k and let $e'j'$ meet pw in k'. The trapezium $ee'j'j$ is in the image of the above map and part of this trapezium of positive area must have a preimage lying in $P \cap T(uvw)$. It follows that $V\big(W_w(uvw)\big) < V\big(Y_w(uvw)\big)$.

(ii) q does not lie on the segment uf.

It follows from what we have proved that $q \in K(0, s)$. Let the line through ff' meet v_1w in the point m. Let n be that point of the segment fm so that qn is parallel to v_1w. The triangle fqn is similar to the triangle uwv_1 and fq is along uv_1 in such a way that $F(f) \geqq F(q)$. Hence $F(n) \leqq F(f)$ since $F(w) \leqq F(u)$. That is to say f, q, n are all in $K(0, s)$ so that by convexity the triangle fqn is contained in $K(0, s)$.

Let the line segment ej meet vw in the point r and let fm meet vw in the point t. If ft lies completely between the segments up and er then the proof may be completed exactly as in case (i). So we may assume that ft intersects er in a point z within $T(uvw)$. The triangle fqn is congruent to the triangle zjm. Hence fqn can be translated

so as to cover zjm. Since fqn is contained in $T(uvw)$ that part of $Y_w(uvw)$ that is not between fm and uw has area greater than that part of $W_w(uvw)$ that is not between fm and uw. On the other hand arguing exactly as in case (i) the area of $P \cap T(uvw)$ is at least as great as the part of $W_w(uvw)$ between fm and uw. It follows that $V(W_w(uvw)) < V(Y_w(uvw))$.

This completes the proof that in all cases $V(W_w(uvw)) < V(Y_w(uvw))$. Theorem 4 follows quite rapidly. We vary $T(uvw)$ by replacing u by u_1 lying on the segment uO and near to u so that $uO > u_1O > 2s$. Then $T(u_1vw) \in I_0$. Write $T(uvw) = T_0$, $T(u_1vw) = T_1$, $W(uvw) = W_0$, $W(u_1vw) = W_1$, $Y(uvw) = Y_0$ and $Y(u_1vw) = Y_1$. Let the ratio of the distance up to the distance u_1p be d. Then

$$V(T_0) = dV(T_1) \quad \text{and} \quad V(W_0) = dV(W_1).$$

Since $K_{u_1} \subset K_u$ so

$$V(W_0) - V(W_1) > V(Y_0) - V(Y_1) + O(d-1)^2$$

Also we have proved $V(W_0) < V(Y_0)$. Thus

$$\frac{V(Y_0)}{V(T_0)} - \frac{V(Y_1)}{V(T_1)} = \frac{V(Y_0) - dV(Y_1)}{V(T_0)} <$$

$$< \frac{V(W_0) - V(W_1) + V(Y_1) - dV(Y_1) + O(d-1)^2}{V(T_0)} =$$

$$= \frac{(d-1)\big(V(W_1) - V(Y_1)\big) + O(d-1)^2}{V(T_0)}.$$

Now as $d \to 1+$, $V(W_1) \to V(W_0)$, $V(Y_1) \to V(Y_0)$ and thus for d sufficiently near 1 we have

$$\frac{V(Y_0)}{V(T_0)} - \frac{V(Y_1)}{V(T_1)} < 0,$$

and this is a contradiction to T_0 being a minimal configuration. This completes the proof of theorem 4. Theorem 2 quickly follows for since the vertices of T_0 lie on the boundary of $K(0, 2s)$ and $W(T_0) = \frac{1}{2} V(K(0, s))$ it follows that

$$\frac{W(T_0)}{V(T_0)} \geqq \frac{1}{2} \frac{V(K(0,s))}{t(K(0,2s))} = \frac{1}{8} \frac{V(K)}{t(K)}.$$

It remains to prove theorem A. Let M be a maximal K-packing of minimal radius r. By lemma 2, we can replace M by a maximal packing M^* in which the radii of all bodies $K(a, R(a))$ are bounded by a finite constant N say. Then there exists a constant N' depending only on N, K and r with the following property. For any circle C with centre O and radius k, the union of the triangles of the configuration given by theorem 1 that lie completely in C contains a concentric circle of radius $k - N'$. Then

$$\frac{V(M^*, k)}{\pi k^2} \geqq \frac{1}{8} \frac{V(K)}{t(K)} \frac{\pi(k-N')^2}{\pi k^2}$$

and taking lower limits as $k \to \infty$, we find the lower density of M^* and hence of M exceeds $\frac{1}{8} V(K)/t(K)$. This implies that

$$k^*(K) \geqq \frac{1}{8} V(K)/t(K).$$

To complete the proof of the main theorem it is enough to produce a maximal K-packing with radii of all its members equal to r which has density $\frac{1}{8} V(K)/t(K)$. To do this let abc be the largest triangle in area contained in $K(0, 2r)$. It is clear that O lies in abc for otherwise by replacing an edge by its image in O one can obtain a larger triangle in K. We next assert that $F(a-b) \geqq 2r$, $F(b-c) \geqq 2r$, $F(c-a) \geqq 2r$. Suppose for example that $F(a-b) < 2r$. Then the points $a-b$, $b-a$ lie in the interior of $K(0, 2r)$. In the triangle with vertices c, $a-b$, $b-a$ the side opposite to c has length $2ab$ and its height from c to the opposite side is at least $\frac{1}{2}$ the height of abc. Therefore this triangle has area not less than the area of abc. Since $a-b$ and $b-a$ lie in the interior of $K(0, 2r)$ so $K(0, 2r)$ contains larger triangles which is impossible.

Thus $K(a, r)$, $K(b, r)$, $K(c, r)$ do not overlap and touch $K(0, r)$ externally. Since $K(a, r)$, for example, cannot cross the segment bOc, none of the three bodies about a, b, c can intersect the side of abc opposite to its centre. The sides of abc therefore generate a lattice such that the bodies K of radius r about the points of this lattice give a maximal K-packing with all bodies of radius r and lower density $V(K)/8t(K)$. This completes the proof.

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References

[1] H. G. EGGLESTON, A minimal density plane covering problem, *Mathematika*, **12** (1965), pp. 226—234.

[2] R. P. BAMBAH and C. A. ROGERS, Covering the plane with convex sets, *Jour. Lond. Math. Soc.*, **27** (1952), pp. 304—314.

[3] E. SAS, Über eine Extremumeigenschaft der Ellipsen, *Compositio Math.*, **6** (1939), pp. 468—470.

[4] L. FEJES TÓTH, Packings and coverings in the plane, *Proc. Coll. Convexity*, Copenhagen, 1965 (1967), pp. 78—87.

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A LIMIT THEOREM FOR THE MAXIMUM OF CUMULATIVE SUMS

By
M. SREEHARI (Mysore, India)
(*Presented by* A. RÉNYI)

§ 1. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables (r. v's) defined on a probability space $(\Omega, \mathscr{B}, \mathbf{P})$. Let $N_1, N_2, \ldots$ be a sequence of positive integer valued r. v's defined on $(\Omega, \mathscr{B}, \mathbf{P})$ such that

$$\frac{N_n}{\psi(n)} \to N \quad \text{in probability,}$$

where N is a r. v. satisfying $\mathbf{P}(N=0)=0$ and $\psi(n)$ increases monotonically to ∞. Let $\mathbf{E}(X_1)=0$ and $\mathbf{V}(X_1)=1$.

Define $S_k = \sum_{j=1}^{k} X_j$ and $Z_n = \max_{1 \leqq k \leqq n} S_k$. It is known [1] that

$$\mathbf{P}\left(Z_n \leqq x\sqrt{n}\right) \to \Lambda(x) = \begin{cases} 0 & \text{if} \quad x \leqq 0 \\ \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{t^2}{2}}\, dt & \text{if} \quad x > 0. \end{cases}$$

The object of this paper is to prove the following two theorems.

THEOREM 1. $\mathbf{P}\left(Z_{N_n} \leqq x\sqrt{N_n}\right) \to \Lambda(x) \quad (-\infty < x < \infty).$

THEOREM 2. $\mathbf{P}\left(Z_{N_n} \leqq x\sqrt{\psi(n)}\right) \to \int_0^\infty \Lambda\left(\frac{x}{\sqrt{t}}\right) d\mathbf{P}(N \leqq t) \quad (-\infty < x < \infty)$

as $n \to \infty$.

§ 2. For $A, B \in \mathscr{B}$ denote the conditional probability of A given B by $\mathbf{P}(A|B)$. If $\mathbf{P}(B)=0$ then $\mathbf{P}(A|B)=\mathbf{P}(A)$. We will need the following two lemmas, the first of which is lemma (1) in [2].

LEMMA 1. *Let* $\{k_n\}$ *and* $\{m_n\}$ *with* $k_n \leqq m_n$ *be two increasing sequences of positive integers and let* $\{A_n\}$ *be a sequence of events such that* A_n *depends only on* $X_{k_n}, \ldots, X_{m_n}$. *If* A *is any event, then*

$$\mathbf{P}(A_n|A) - \mathbf{P}(A_n) \to 0$$

as $n \to \infty$.

LEMMA 2. *For any event* A

$$\mathbf{P}\left(Z_n \leqq x\sqrt{n}\,|\,A\right) - \mathbf{P}\left(Z_n \leqq x\sqrt{n}\right) \to 0 \tag{2.1}$$

as $n \to \infty$.

PROOF. Let $Z_n^* = \max_{k_n \leqq k \leqq n} \{S_k - S_{k_n}\}$. Observe that, by lemma 1

$$\mathbf{P}(Z_n^* \leqq x\sqrt{n} \mid A) - \mathbf{P}(Z_n^* \leqq x\sqrt{n}) \to 0$$

as $n \to \infty$, for any inreasing sequence of numbers $\{k_n\}$ such that $k_n \leqq n$. If k_m is sufficiently slowly increasing, say as $[\log n]$, then

$$\mathbf{P}(Z_n^* \leqq x\sqrt{n}) \to \Lambda(x)$$

as $n \to \infty$. It is therefore sufficient to prove that

$$\mathbf{P}(Z_n \leqq x\sqrt{n} \mid A) - \mathbf{P}(Z_n^* \leqq x\sqrt{n} \mid A) \to 0$$

which follows by observing that

$$\mathbf{P}\left(\left\{\max_{k_n \leqq k \leqq n} S_k \leqq x\sqrt{n}\right\} \cap \{Z_n^{*\prime} > x\sqrt{n}\}\right) \to 0$$

as $n \to \infty$.

§ 3. PROOF OF THEOREM 1. Without loss of generality we assume that $x > 0$ for, otherwise

$$\mathbf{P}(Z_{N_n} \leqq x\sqrt{N_n}) \leqq \mathbf{P}(Z_{N_n} \leqq 0) \leqq$$

$$\leqq \mathbf{P}\left(\left|\frac{N_n}{\psi(n)} - N\right| > \varepsilon\right) + \mathbf{P}(N \leqq \delta) + \mathbf{P}(Z_{[\psi(n)(\delta-\varepsilon)]} \leqq 0) \quad (\delta > \varepsilon).$$

Let $n \to \infty$ and choose $\delta > 0$ small. Then $\lim_{n\to\infty} \mathbf{P}(Z_{N_n} \leqq x\sqrt{N_n}) = 0$.

Let i_0 and k be two positive integers.

$$\mathbf{P}(Z_{N_n} \leqq x\sqrt{N_n}) = A_n + B_n + \sum_{i=i_0}^{\infty} C_{ni}, \tag{3.1}$$

where

$$A_n = \mathbf{P}\left[Z_{N_n} \leqq x\sqrt{N_n}, \left|\frac{N_n}{\psi(n)} - N\right| > \frac{1}{k}\right],$$

$$B_n = \mathbf{P}\left[Z_{N_n} \leqq x\sqrt{N_n}, \left|\frac{N_n}{\psi(n)} - N\right| \leqq \frac{1}{k}, N \leqq \frac{i_0}{k}\right]$$

and

$$C_{ni} = \mathbf{P}\left[Z_{N_n} \leqq x\sqrt{N_n}, \left|\frac{N_n}{\psi(n)} - N\right| \leqq \frac{1}{k}, \frac{i}{k} < N \leqq \frac{i+1}{k}\right].$$

Writing $n_{1i} = \left[\frac{\psi(n)(i-1)}{k}\right]$ and $n_{2i} = \left[\frac{\psi(n)(i+2)}{k}\right]$ we observe that

$$C_{ni} \leqq \mathbf{P}[Z_{n_{1i}} \leqq x\sqrt{n_{2i}} \mid M_i]\,\pi_i \tag{3.2}$$

where $M_i = \left\{\frac{i}{k} < N \leqq \frac{i+1}{k}\right\}$ and $\pi_i = \mathbf{P}(M_i)$.

$$C_{ni} \geqq \mathbf{P}[Z_{n_{2i}} \leqq x\sqrt{n_{1i}} \mid M_i]\,\pi_i - \mathbf{P}\left[\left|\frac{N_n}{\psi(n)} - N\right| > \frac{1}{k}, M_i\right]. \tag{3.3}$$

Choose i_0 so large that for every $\varepsilon>0$,

$$(1+\varepsilon)\Lambda(x) \geqq \Lambda\left(x\sqrt{\frac{i+2}{i-1}}\right), \quad \text{for every} \quad i \geqq i_0$$

and

$$(1-\varepsilon)\Lambda(x) \leqq \Lambda\left(x\sqrt{\frac{i-1}{i+2}}\right), \quad \text{for every} \quad i \geqq i_0.$$

Then choose k large enough such that $\mathbf{P}\left(N \leqq \frac{i_0}{k}\right) < \varepsilon$. Finally choose n_0 so large that whenever $n \geqq n_0$

$$\mathbf{P}\left[\left|\frac{N_n}{\psi(n)} - N\right| > \frac{1}{k}\right] < \varepsilon.$$

We have from (3. 1), (3. 2) and (3. 3), whenever $n \geqq n_0$

$$\sum_{i=i_0}^{\infty} \mathbf{P}\left[Z_{n_{2i}} \leqq x\sqrt{n_{1i}} \middle| M_i\right] \pi_i - \varepsilon \leqq \mathbf{P}\left[Z_{N_n} \leqq x\sqrt{N_n}\right] \leqq$$

$$\leqq \sum_{i=i_0}^{\infty} \mathbf{P}\left[Z_{n_{1i}} \leqq x\sqrt{n_{2i}} \middle| M_i\right] \pi_i + 2\varepsilon.$$

By lemma 2 and the result [1], using SLUTSKY's theorem (section 20. 6 in [3]) we get

$$\sum_{i=i_0}^{\infty} \Lambda\left(x\sqrt{\frac{i-1}{i+2}}\right) \pi_i - \varepsilon \leqq \overline{\lim}\, \mathbf{P}\left[Z_{N_n} \leqq x\sqrt{N_n}\right] \leqq \sum_{i=i_0}^{\infty} \Lambda\left(x\sqrt{\frac{i+2}{i-1}}\right) \pi_i + 2\varepsilon$$

in case the summations converge, i.e.

$$(1-\varepsilon)^2 \Lambda(x) - \varepsilon \leqq \overline{\lim}\, \mathbf{P}\left[Z_{N_n} \leqq x\sqrt{N_n}\right] \leqq (1+\varepsilon)\Lambda(x) + 2\varepsilon.$$

Since ε is arbitrary, theorem 1 follows.

PROOF OF THEOREM 2. As in theorem 1 we shall consider $x>0$. Taking $i_0=2$ and proceeding as before we arrive at

$$\sum_{i=2}^{\infty} \Lambda\left(x\sqrt{\frac{k}{i+2}}\right) \pi_i - \varepsilon \leqq \overline{\lim}\, \mathbf{P}\left[Z_{N_n} \leqq x\sqrt{\psi(n)}\right] \leqq \sum_{i=2}^{\infty} \Lambda\left(x\sqrt{\frac{k}{i-1}}\right) \pi_i + 2\varepsilon.$$

Since π_0 and π_1 are each less than ε we have the result

$$\lim_{n\to\infty} \mathbf{P}\left[Z_{N_n} \leqq x\sqrt{\psi(n)}\right] = \int_0^{\infty} \Lambda\left(\frac{x}{\sqrt{t}}\right) d\mathbf{P}(N \leqq t).$$

§ 4. It is known [1] that

$$\lim_{n\to\infty} \mathbf{P}\left[\max_{1\leqq k\leqq n} |S_k| \leqq x\sqrt{n}\right] = \Lambda^*(x) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \exp\left\{-\frac{(2m+1)^2\pi^2}{8x^2}\right\} \quad (x \geqq 0).$$

Proceeding on similar lines one proves the theorem

$$\mathbf{P}\left[\max_{1\leqq k\leqq N_n} |S_k| \leqq x\sqrt{N_n}\right] \to \Lambda^*(x) \quad (x \geqq 0)$$

and

$$\mathbf{P}\left[\max_{1\leqq k\leqq N_n} |S_k| \leqq x\sqrt{\psi(n)}\right] \to \int_0^\infty \Lambda^*\left(\frac{x}{\sqrt{t}}\right) d\mathbf{P}(N \leqq t) \quad (x \geqq 0).$$

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References

[1] P. ERDŐS and KAC, On certain limit theorems of the theory of probability, *Bull. of Amer. Math. Soc.*, **62** (1946).

[2] O. BARNDORFF-NIELSEN, On the limit distribution of the maximum of a random number of independent random variables, *Acta Math. Acad. Sci. Hung.*, **15** (1964).

[3] H. CRAMER, *Mathematical methods in Statistics.*

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IDEMPOTENT SIGNED MEASURES ON A DISCRETE GROUP

By
J. H. B. KEMPERMAN (Rochester, New York, U.S.A.)
(*Presented by* L. RÉDEI)

1. Introduction. Let G be a discrete group and let $L_1(G)$ denote the group algebra of G. Thus, $x \in L_1(G)$ if and only if x is a complex-valued function on G having a finite norm

$$\|x\| = \sum_{g \in G} |x(g)|.$$

The multiplication in $L_1(G)$ is the usual convolution multiplication

$$(xy)(g) = \sum_{g_1 \in G} x(g_1) y(g_1^{-1} g).$$

Given $x \in L_1(G)$, the set

$$S(x) = \{g \in G : x(g) \neq 0\} \quad (S(x) \subset G),$$

is called the support of x. The subgroup of G generated by $S(x)$ is called the support group of x.

We shall be interested in the idempotents on G, that is, the elements $x \in L_1(G)$ satisfying $x^2 = x$. Observe that $\|x\| \geqq 1$ for each non-zero idempotent, while $\|x\| = 1$ for each non-zero nonnegative idempotent. For the case that G is abelian, such idempotents were completely characterized by RUDIN [3] p. 199; the analogous problem for an arbitrary locally compact abelian group was solved by COHEN [1], compare [4].

It turns out that in the (discrete) abelian case every idempotent has a finite support and even a finite support group. Whether or not G is abelian, an idempotent with $\|x\| = 1$ always has a finite support group, see [5] p. 227. The same is true [2] for an idempotent x satisfying $x(gh) = x(hg)$, for all $g, h \in G$.

On the other hand, RUDIN and SCHNEIDER [5], [6] gave many examples of groups G admitting an idempotent x of infinite support. By the above remarks, G must be an infinite non-abelian group while x as a measure on G must be a signed measure of total variation $\|x\| > 1$. The problem was raised ([5], p. 226) to find an example of a self-adjoint idempotent of infinite support. Here, an element $x \in G$ is said to be self-adjoint if $x^* = x$. The adjoint x^* of x is defined by

$$x^*(g) = \overline{x(g^{-1})},$$

a bar denoting complex conjugation.

In the present paper we largely restrict ourselves to the case of a non-abelian group G which is the *splitting extension* of an abelian group H by an arbitrary finite group K. A simple device (Theorem 3. 1) is given for constructing a self-

adjoint idempotent in $L_1(G)$. In many cases this idempotent turns out to be of infinite support. In fact, many groups seem to possess a great abundance of self-adjoint idempotents of infinite support.

2. Notations. It will be convenient to regard each element x in the group algebra $L_1(G)$ as a linear combination

$$x = \sum_{g \in G} x(g)g$$

of the elements of g with complex coefficients $x(g)$ such that $\sum |x(g)| < \infty$. In particular,

$$xy = \Big[\sum_{g \in G} x(g)g\Big]\Big[\sum_{h \in G} y(h)h\Big] = \sum_{k \in G} \Big\{\sum_{gh=k} x(g)y(h)\Big\} k.$$

From now on, let us *suppose* that G contains a pair of subgroups H, K such that H is a normal subgroup of G, $HK = G$, $H \cap K = \{1\}$. The pair H, K will be kept fixed from now on. Each element $g \in G$ has a *unique* representation as $g = hk$ with $h \in H$ and $k \in K$. If $h_1, h_2 \in H$ and $k_1, k_2 \in K$ then

$$h_1 k_1 h_2 k_2 = h_1 h_2^{k_1} k_1 k_2, \tag{1}$$

where

$$h^k = khk^{-1}. \tag{2}$$

Here, $h^k \in H$ for each choice of $h \in H$ and $k \in K$. In this way we have an automorphism $h \to h^k$ of H, one for each $k \in K$, such that

$$(h^k)^{k'} = h^{k'k} \tag{3}$$

whenever $h \in H$, $k, k' \in K$ (note the order reversal).

Conversely, given a pair of abstract groups H and K and a mapping of K into the automorphism group $A(H)$ of H satisfying (3), the pairs $\langle h, k\rangle$ with $h \in H$, $k \in K$ form a group G of the above type under the multiplication

$$\langle h_1, k_1\rangle \langle h_2, k_2\rangle = \langle h_1 h_2^{k_1}, k_1 k_2\rangle.$$

One calls G the *splitting extension* of H by K corresponding to the given "homomorphism" of K into $A(H)$.

In a natural way, the automorphism (2) of the group H is extended to its group algebra $L_1(H)$; namely, if

$$\xi = \sum_{h \in H} \xi(h)h \quad \text{then} \quad \xi^k = \sum_{h \in H} \xi(h)h^k.$$

With this definition of ξ^k one has $\xi^1 = \xi$, $(\xi^k)^{k'} = \xi^{k'k}$,

$$(\xi\eta)^k = \xi^k \eta^k, \quad (a\xi + b\eta)^k = a\xi^k + b\eta^k,$$

for any choice of $\xi, \eta \in L_1(H)$, $k, k' \in K$ and complex numbers a and b.

Each $x \in L_1(G)$ can be written as

$$x = \sum_{k \in K} \sum_{h \in H} x(hk)hk = \sum_{k \in K} \xi_k k \quad \Big(\sum_{k \in K} \|\xi_k\| = \|x\|\Big),$$

where

$$\xi_k = \sum_{h \in H} x(hk)h \in L_1(H)$$

describes the behavior of x on the coset $Hk=kH$ of H. If $y=\sum\eta_k k$ $(\eta_k\in L_1(H), k\in K)$ is a further element of $L_1(G)$ then the product xy is given by

$$xy=\sum_{k_1,k_2}\xi_{k_1}(\eta_{k_2})^{k_1}k_1k_2.$$

In particular, x is an *idempotent* in $L_1(G)$ if and only if

$$\sum_{j\in K}\xi_j(\xi_{j^{-1}k})^j=\xi_k \quad \text{for each} \quad k\in K. \tag{4}$$

In other words, if and only if the matrix $A=(A_{jk}=\xi^j_{j^{-1}k};\ j,\ k\in K)$ with elements $A_{jk}\in L_1(H)$ is an idempotent matrix, $A^2=A$. In the special case that G is equal to the direct product of H and K (that is, $h^k=h$ for all $h\in H$, $k\in K$) then (4) says that $\{\xi_k,\ k\in K\}$ defines an idempotent measure on K which has its values in $L_1(H)$.

The adjoint x^* of the element $x=\sum\xi_k k$ in $L_1(G)$ is given by

$$x^*=\sum(\xi_k k)^*=\sum k^{-1}\xi_k^*=\sum(k\xi^*_{k^{-1}}k^{-1})k=\sum(\xi^*_{k^{-1}})^k k.$$

In particular, x is self-adjoint if and only if

$$(\xi_{k^{-1}})^k=\xi_k^* \quad \textit{for each} \quad k\in K. \tag{5}$$

This is equivalent to the requirement that the above matrix A be hermitean, $A^*_{kj}=A_{jk}$.

3. Idempotents in a splitting extension. Let G be a splitting extension as in Section 2 of a group H by a group K.

THEOREM 3. 1. *Suppose that the group H is abelian. Let $u=\sum u(k)k$ be an arbitrary idempotent in $L_1(K)$ and let η be an arbitrary element in $L_1(H)$. Then choosing*

$$\xi_k=u(k)e^{i(\eta^k-\eta)} \quad \textit{for each} \quad k\in K, \tag{6}$$

$(\xi_1=u(1))$, *one obtains an idempotent $x=\sum\xi_k k$ in $L_1(G)$. Moreover, the idempotent x is self-adjoint as soon as both $u\in L_1(K)$ and $\eta\in L_1(H)$ are self-adjoint.*

PROOF. Let ξ_k be as in (6). In view of

$$(e^\zeta)^j=j\left[\sum_{n=0}^{\infty}\zeta^n/n!\right]j^{-1}=e^{\zeta^j},$$

we have

$$(\xi_{j^{-1}k})^j=u(j^{-1}k)e^{i(\eta^k-\eta^j)},$$

(compare (3)). By the criterion (4), $x=\sum\xi_k k$ is an idempotent in $L_1(G)$ if and only if

$$\sum_{j\in K}u(j)u(j^{-1}k)e^{i(\eta^j-\eta)}e^{i(\eta^k-\eta^j)}=u(k)e^{i(\eta^k-\eta)},$$

for each $k\in K$. This is indeed the case since H was assumed to be abelian, thus, $e^{\alpha+\beta}=e^\alpha e^\beta$, for any $\alpha,\ \beta\in L_1(H)$. Finally, in view of

$$(\xi_{k^{-1}})^k=u(k^{-1})e^{-i(\eta^k-\eta)},$$

if both u and η are self-adjoint then (5) holds, that is, $x=\sum\xi_k k$ is self-adjoint.

From now on, we will take G as a splitting extension of an infinite *abelian* group H by a group K of *finite* order m. In this case,

$$u(k) = \frac{1}{m} \quad \text{for each} \quad k \in K$$

yields a self-adjoint idempotent in $L_1(K)$; (there may be many other self-adjoint idempotents in $L_1(K)$, see [5], p. 226). By Theorem 3. 1, for each $\eta \in L_1(H)$ the formulae

$$\xi_k = \frac{1}{m} e^{i(\eta^k - \eta)} \quad \text{for each} \quad k \in K; \quad x = \sum \xi_k k$$

yield an idempotent $x \in L_1(G)$. If η is self-adjoint then so is x. If η is purely imaginary then x is real. Thus, replacing $i\eta$ by η,

$$\xi_k = \frac{1}{m} e^{\eta^k - \eta} \tag{7}$$

yields a *real* and self-adjoint idempotent $x = \sum \xi_k k$ in $L_1(G)$ as soon as

$$\eta = \sum_{h \in H} \eta(h) h \in L_1(H)$$

satisfies

$$\eta(h) \quad \text{real}, \quad \eta(h^{-1}) = -\eta(h) \quad \text{for each} \quad h \in H. \tag{8}$$

There remains the problem of choosing η in such a way that this idempotent x has an infinite support, equivalently, such that for at least one $k \in K$ the element ξ_k in $L_1(H)$ has an infinite support $S(\xi_k)$.

LEMMA 3. 2. *Let H be an abelian group and let $\zeta \in L_1(H)$ be of finite support. Then the element e^{ζ} in $L_1(H)$ has an infinite support precisely when the support group $H(\zeta)$ of ζ is infinite.*

PROOF. The necessity is obvious, thus, suppose that $H(\zeta)$ is infinite. Since the group $H(\zeta)$ is generated by the finite support $S(\zeta)$ of ζ, we have that $H(\zeta)$ is equal to the direct product of finitely many cyclic subgroups, at least one isomorphic to the additive group N of integers. Thus, $H(\zeta) = F \otimes N$, where F may be finite or infinite. We can write ζ as a finite sum $\zeta = \sum_{n \in N} \zeta_n \cdot n$, where ζ_n denotes an element of $L_1(F)$ of finite support. Since $S(\zeta)$ generates $H(\zeta)$, we have $\zeta_n \neq 0$ for at least one $n \neq 0$.

Now suppose that e^{ζ} were of finite support, $e^{\zeta} = \sum \varrho_n \cdot n$, a *finite* sum with $\varrho_n \in L_1(F)$ of finite support. Taking Fourier transforms, we have

$$\exp\left[\sum_n \hat{\zeta}_n e^{itn}\right] = \sum_n \hat{\varrho}_n e^{itn} \quad (n \in N),$$

for *each* real number t. Here, $\hat{\zeta}_n$ and $\hat{\varrho}_n$ are bounded functions on the dual $\hat{F}$ of F. Introducing the complex variable $e^{it} = z$, and using analytic continuation, it would follow that for each $n \neq 0$ we have $\hat{\zeta}_n \equiv 0$, hence, $\zeta_n = 0$.

REMARK. It is essential that ζ be of finite support. For, let $\xi \in L_1(H)$ be of finite support, $\|\xi\| < 1$, and consider the element $\zeta = \log(1 + \xi)$.

Combining Theorem 3. 1 and Lemma 3. 2, we have that (7) *defines a real self-adjoint idempotent* $x=\sum\xi_k k$ *of infinite support as soon as:*

(i) The element $\eta\in L_1(H)$ satisfies (8) and is of finite support.

(ii) We have for at least one $k\in K$ that the support $S(\zeta_k)$ of $\zeta_k=\eta^k-\eta$ contains at least one element $h\in H$ of infinite order.

As an illustration, let us restrict ourselves to the case that each element $h\neq 1$ in H is of infinite order. For the moment, let us also assume that G is not equal to the direct product of H and K. Then there exists an element $h\in H$ such that $h^k\neq h$ for at least one $k\in K$; let h be fixed. Clearly, $h\neq 1$ hence h is of infinite order. Take c as a non-zero real number, $\eta=-c(h-h^{-1})$. Then, for each $k\in K$,

$$\zeta_k=\eta^k-\eta=c(h-h^{-1})-c\big(h^k-(h^k)^{-1}\big), \tag{9}$$

(possibly $h^k=h^{-1}$). The above conditions (i) and (ii) are satisfied, consequently,

$$x=\frac{1}{m}\sum_{k\in K}(e^{\zeta_k})k \tag{10}$$

represents a real self-adjoint idempotent in $L_1(G)$ having an infinite support. Since $\zeta_1=0$ this idempotent is such that $x(h)=0$ for each $h\in H$, $h\neq 1$, while $x(1)=\frac{1}{m}$.

In this connection, it may be interesting to observe that

$$e^{c(h-h^{-1})}=\sum_{r=-\infty}^{+\infty}J_r(2c)h^r. \tag{11}$$

Here,

$$J_r(2c)=\sum_{n=0}^{\infty}(-1)^n\frac{c^{2n+r}}{n!(n+r)!}$$

is the well-known Bessel function, $\big(J_{-r}=(-1)^rJ_r$ while $J_r(2c)\sim c^r/r!$ when $r\to+\infty.\big)$

4. Specific examples. (I). Put $\omega=e^{2\pi i/3}=-\frac{1}{2}+\frac{1}{2}i\sqrt{3}$, $(\omega^3=1,\ 1+\omega+\omega^2=0)$. Let L denote the lattice in the complex plane consisting of all points $z=m\omega+n\omega^2$, where m and n run through the integers. Finally, let G be the group of all the rigid motions of L onto itself which preserve orientation . Each member of G is of the form

$$g_{p,q}^{(\lambda)}:\ z\to\omega^\lambda z+p\omega+q\omega^2,$$

where p, q and λ are integers, $\lambda\in\{0,\ 1,\ 2\}$. The translations

$$h_{p,q}:\quad z\to z+p\omega+q\omega^2$$

form a commutative subgroup H of G of index 3. Let further $K=\{1,\ k,\ k^2\}$ denote the subgroup of G of order 3 generated by the rotation k: $z\to\omega z$. The elements $g_{p,q}^{(\lambda)}$ of G have a unique representation as $g=hk$ with $h\in H$ and $k\in K$, namely, $g_{p,q}^{(\lambda)}=h_{p,q}k^\lambda$. Thus, G is equal to a splitting extension of H by K; in fact, (1) holds with

$$(h_{p,q})^{k^\lambda}:\quad z\to z+p\omega^{\lambda+1}+q\omega^{\lambda+2},$$

$(\omega^3=-\omega-\omega^2)$. If we identify the translation $h_{p,q}$ with the point $z=p\omega+q\omega^2\in L$ then the automorphism $h\to h^k$ of H may be interpreted as a rotation of L about an angle $2\pi/3$.

Let us apply (9) and (10) with $h=h_{1,0}$. Then $h^k=h_{0,1}$ while $h^{k^2}=h_{-1,-1}$. In this way, one obtains

$$x = \frac{1}{3}+\frac{1}{3}[\exp\{(h_{1,0}-h_{-1,0})-(h_{0,1}-h_{0,-1})\}c]k+$$

$$+\frac{1}{3}[\exp\{(h_{1,0}-h_{-1,0})+(h_{1,1}-h_{-1,-1})\}c]k^2$$

as an example of a real self-adjoint idempotent in $L_1(G)$ having infinite support; here, c is any non-zero real constant. Using (11), one easily obtains the more explicit formula

$$x(g_{p,q}^{(1)}) = \frac{1}{3}J_p(2c)J_q(-2c), \quad x(g_{p,q}^{(2)}) = \frac{1}{3}J_{p-q}(2c)J_q(2c),$$

while $x(g_{p,q}^{(0)})=0$ except that $x(1)=\frac{1}{3}$.

(II). Let N denote the lattice of integers and let G be the group of rigid motions (not necessarily preserving orientation) of N onto itself. Each member of G is of the form

$$g_p^{(\lambda)}: \quad z\to(-1)^\lambda z+p,$$

where $\lambda\in\{0, 1\}$ and p is any integer. The translations

$$h_p: \quad z\to z+p \quad (p=0, \pm1, \pm2, \ldots),$$

form a subgroup H of G of index 2. Let further $K=\{1, k\}$ denote the subgroup of G of order 2 generated by the reflection k which sends z into $-z$. Clearly, G is a splitting extension of H by K. In fact, $g_p^{(\lambda)}=h_pk^\lambda$, while (1) holds with

$$(h_p)^k = h_p^{-1}: \quad z\to z-p.$$

Using (9) and (10), one finds

$$x = \frac{1}{2}+\frac{1}{2}[\exp\{2c(h_p-h_p^{-1})\}]k$$

as an example of a real self-adjoint idempotent in $L_1(G)$ having infinite support. Here, $c\neq0$ is a fixed real constant, p a fixed positive integer.

The collection of *all* self-adjoint idempotents can in principle be obtained from (4) and (5). In the present special case, introducing the Fourier transforms

$$\Phi(t) = \sum_{p=-\infty}^{+\infty} x(h_p)e^{ipt}, \quad \Psi(t) = \sum_{p=-\infty}^{+\infty} x(h_pk)e^{ipt},$$

these conditions become

$$\Phi(t) = \Phi(t)^2+\Psi(t)\Psi(-t), \quad \Psi(t) = \Phi(t)\Psi(t)+\Phi(-t)\Psi(t), \tag{12}$$

and $\overline{\Phi(t)}=\Phi(t)$, $\overline{\Psi(t)}=\Psi(-t)$, respectively. Excluding the trivial case of an idem potent with support $\{1\}$, we have that $\Psi(t)\not\equiv0$. If $\Psi(t)=0$ then $\Phi(\pm t)=0$ or 1.

If $\Psi(t) \neq 0$ then $\Phi(t)+\Phi(-t)=1$. Since the function Φ is continuous, we conclude that $\Phi(t)+\Phi(-t)=1$ for all t. Equivalently,

$$\Phi(t) = \frac{1}{2}+\frac{1}{2}\varphi(t), \tag{13}$$

where $\varphi(t)$ is a function of the form

$$\varphi(t) = \sum_{r=1}^{\infty} c_r \sin rt, \quad \text{where} \quad \sum|c_r| < \infty, \quad c_r \quad \text{real.} \tag{14}$$

Condition (12) now reduces to

$$4|\Psi(t)|^2 = 4\Psi(t)\Psi(-t) = 1-\varphi(t)^2, \quad \text{thus} \quad |\varphi(t)| \leqq 1.$$

Clearly, there is a multitude of solutions. The general solution may be obtained as follows.

(i) Choose $\varphi(t)$ as a real-valued odd and continuous function, $\varphi(\pm\pi)=0$, such that $|\varphi(t)| \leqq 1$ while the Fourier series (14) of $\varphi(t)$ is absolutely convergent. Let $A = \{t: |\varphi(t)|=1\}$.

(ii) Choose the periodic function $g(t)$, $g(t+2\pi)=g(t)$, such that, for $t \notin A$, $|g(t)|=1$ and $g(-t)=\overline{g(t)}$ and that, moreover, the function defined by

$$\Psi(t) = \frac{1}{2} g(t)\,[1-\varphi(t)^2]^{1/2} \tag{15}$$

is a continuous function with an absolutely convergent Fourier series. Finally, defining Φ as in (13), one obtains a solution $\{\Phi, \Psi\}$ and every solution can be obtained in this way.

As to condition (ii), it *suffices* that $g(t)$ be of the form

$$g(t) = \sum_{r=-\infty}^{+\infty} d_r e^{irt} \quad \text{with} \quad \sum|d_r| < \infty, \quad d_r \quad \text{real,} \tag{16}$$

and that further $|g(t)|=1$ whenever $t \notin A$. This follows from the fact that $[1-z]^{\frac{1}{2}} = \sum a_j z^j$ for $|z| \leqq 1$ with $\sum|a_j| < \infty$. By the Wiener—Lévy theorem, a representation of the form (16) is also necessary in the special case that A is empty (where $1-\varphi(t)^2 \neq 0$ for all t).

As a very special illustration, choose $\varphi(t) \equiv 0$ and

$$\Psi(t) = \frac{1}{2}(e^{it}-c)(ce^{it}-1)^{-1}.$$

Here, c denotes a real number with $|c|<1$. In this way one obtains the self-adjoint idempotent x defined by $x(1)=\frac{1}{2}$, $x(1\cdot k)=\frac{1}{2}c$, $x(h_p k) = -\frac{1}{2}(1-c^2)c^{p-1}$ for $p \geqq 1$, while $x(g)=0$ otherwise.

(III). From now on, let us assume that G is the *direct product* of an infinite (not necessarily abelian) group H and a finite non-abelian group K. It follows from a result of RUDIN and SCHNEIDER (Theorem 4. 3 in [6]) that $L_1(G)$ does contain an idempotent of infinite support. We shall now give an example where there even exists a self-adjoint idempotent of infinite support. Observe that the construction

given in Section 3 does not apply here since $h^k = h$ for all $h \in H$, $k \in K$ to the effect that the element ξ_k defined by (6) is supported by a single point.

Observe that $x = \sum \xi_k k$ (where $\xi_k \in L_1(H)$) is a self-adjoint idempotent precisely when ξ_k as a function of k is a self-adjoint idempotent on K (taking values in $L_1(H)$). If H is abelian this is equivalent to the property that, for each element v in the dual $\hat{H}$ of H, the function $\hat{\xi}_k(v)$ is an (ordinary) self-adjoint idempotent on K. Here, $\hat{\xi}_k(v)$ denotes the Fourier transform of ξ_k.

We will now take K as the group S_3 of all the six permutations (123), (12), etc. of $\{1, 2, 3\}$. Choose $\gamma \in L_1(H)$ such that γ is real and $\|\gamma\| < \frac{1}{4}$. Choose further

$$\xi_1 = \frac{1}{2}, \quad \xi_{(13)} = 0, \quad \xi_{(123)} = i\gamma, \quad \xi_{(132)} = -i\gamma,$$

and

$$\xi_{(12)} = \frac{1}{4}[1 + \sqrt{1 - 16\gamma^2}], \quad \xi_{(23)} = \frac{1}{4}[1 - \sqrt{1 - 16\gamma^2}].$$

One easily verifies that $x = \sum \xi_k k$ is a (non-real) self-adjoint idempotent in $L_1(G)$. It has infinite support as soon as γ has an infinite support.

One may also notice that S_3 is a splitting extension of $H' = \{1, (123), (132)\}$ by $K' = \{1, (13)\}$. Thus Theorem 3. 1 would suggest that we try to obtain a self-adjoint idempotent with $\xi_1 = \frac{1}{2}$, $\xi_{(123)} = \xi_{(132)} = 0$. This idea works indeed, namely, with

$$\xi_{(13)} = \frac{1}{2} - 2\gamma^2, \quad \xi_{(12)} = \gamma^2 + \gamma\sqrt{1 - 3\gamma^2}, \quad \xi_{(23)} = \gamma^2 - \gamma\sqrt{1 - 3\gamma^2}.$$

Here, γ denotes any real element in $L_1(H)$ with $\|\gamma\| < 3^{-\frac{1}{2}}$. Choosing γ such that γ^2 has an infinite support, one obtains in this way a real and self-adjoint idempotent on $G = H \otimes S_3$ having an infinite support.

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References

[1] P. J. COHEN, On a conjecture of Littlewood and idempotent measures, *Amer. J. Math.*, **82** (1960), pp. 191—212.
[2] D. RIDER, Central idempotents in group algebras, *Bull. Amer. Mat. Soc.*, **72** (1966), pp. 100—102.
[3] W. RUDIN, Idempotent measures on abelian groups, *Pacific J. Math.*, **9** (1959), pp. 195—209.
[4] W. RUDIN, *Fourier analysis on groups* (John Wiley and Sons, 1962).
[5] W. RUDIN, Idempotents in group algebras, *Bull. Amer. Math. Soc.*, **69** (1963), pp. 224—227.
[6] W. RUDIN and H. SCHNEIDER, Idempotents in group rings, *Duke J. Math.*, **31** (1964), pp. 585—602.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (1—2), (1968), pp. 129—146.

PLUS GRAND GROUPE IMAGE HOMOMORPHE ET ISOTONE D'UN MONOÏDE ORDONNÉ: CARACTÉRISATIONS DES ANNEAUX «CLOS»

Par
J. QUERRÉ (Brest, France)
(*Présenté par* L. RÉDEI)

Je rappelle ici quelques définitions. Un *monoïde* est un ensemble muni d'une opération interne associative. On dit aussi demi-groupe. Un monoïde est ordonné s'il est muni d'une relation d'ordre compatible avec l'opération c'est-à-dire si

$$a \leqq b \quad \Rightarrow \quad ax \leqq bx \quad \text{et} \quad xa \leqq xb$$

quel que soit a, b, x éléments du monoïde.

Soient D_1 et D_2 deux monoïdes ordonnés. Une surjection $\varphi: D_1 \to D_2$ sera dite un *épimorphisme croissant* si

$$\varphi(ab) = \varphi(a)\varphi(b)$$

$$a \leqq_1 b \quad \Rightarrow \quad \varphi(a) \leqq_2 \varphi(b).$$

On dit aussi 0-épimorphisme.

H étant une partie non vide c'est-à-dire un complexe d'un monoïde D on note $H.^{\cdot}a = \{x \in D \text{ tel que } ay \in H\}$.

Un complexe H est net si quel que soit $x \in D$ $H.^{\cdot}x$ et $H^{\cdot}.x$ ne sont pas vides.

Le complexe H sera dit *fort* si quel que soit a et b

$$(H.^{\cdot}a) \cap (H.^{\cdot}b) \neq \emptyset \quad \Rightarrow \quad H.^{\cdot}a = H.^{\cdot}b$$

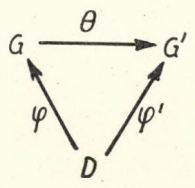
$$(H^{\cdot}.a) \cap (H^{\cdot}.b) \neq \emptyset \quad \Rightarrow \quad H^{\cdot}.a = H^{\cdot}.b$$

Idéaux fractionnaires
Anneau intégralement clos
Anneau complètement clos
Anneau de Dedekind
Anneau de Krull

} Pour ces définitions voir (BOURBAKI ou SAMUEL ZARISKI *Commutative Algebra.*)

Soit D un monoïde ordonné. Convenons d'associer à D catégorie $\Gamma(D)$ dont les objets sont les épimorphismes croissants de D sur les groupes ordonnés.

Si $\varphi: D \to G$ et $\varphi': D \to G'$ sont deux objets, une flèche $\theta \in \mathrm{Mor}(\varphi, \varphi')$ est un épimorphisme croissant $\theta: G \to G'$, tel que l'on ait le diagramme commutatif



On se propose de déterminer les conditions d'existence d'un objet initial de cette catégorie, c'est-à-dire les conditions d'existence d'un groupe ordonné Γ et d'un épimorphisme croissant $\varphi\colon D\to\Gamma$ tel que, pour tout épimorphisme croissant ψ de D sur un groupe G, on ait un, et un seul, épimorphisme croissant $\theta\colon \Gamma\to G$ avec $\theta\circ\varphi=\psi$. On dira que Γ est le *plus grand groupe image homomorphe et isotone de D*.

On sait que le monoïde de leurs idéaux fractionnaires (fini ou non) permet de caractériser les anneaux «clos», c'est-à-dire les anneaux intégralement clos, complètement intégralement clos, KRULL et DEDEKIND. Il est remarquable que ces caractérisations sont liées à l'existence d'un plus grand groupe image homomorphe et isotone du monoïde des idéaux fractionnaires. Aussi conviendrons-nous d'appeler *monoïde ordonné clos* tout monoïde ordonné D dont la catégorie associée $\Gamma(D)$ admet un objet initial.

1. Théorème fondamental

Soit D un monoïde ordonné. Une partie H de D sera dite *anticône* de D si elle vérifie les propriétés suivantes:

1° H est un sous-monoïde de D,

2° H est héréditaire ($x\in H$ et $y<x$ entraîne $y\in H$),

3° H est réflectif ($xy\in H$ entraîne $yx\in H$),

4° H est fortement net (pour tout a, il existe un a' tel que $a'a\in H$ et tel que $xa'a\in H$ entraîne $x\in H$).

On peut énoncer le résultat suivant [1]:

Il y a bijection entre l'ensemble des groupes images homomorphes et isotones du monoïde ordonné D et l'ensemble $\mathscr{C}$ des anticônes de D.

Notons $\mathscr{R}$ l'ensemble des équivalences principales [5] associées aux anticônes

$$H\in\mathscr{C},\quad a\equiv b(R_H)\quad\Leftrightarrow\quad H:a=H:b.$$

Chaque groupe image homomorphe et isotone de D est donné par D/R_H.

LEMME. *Le monoïde ordonné D est clos si, et seulement si, $\mathscr{R}$ admet un plus petit élément.*

Soit R_H le plus petit élément de $\mathscr{R}$; $\varphi\colon D\to D/R_H$ l'épimorphisme canonique. Tout groupe homomorphe et isotone à D est de la forme D/R_K avec $R_K\in\mathscr{R}$ et $R_H\subseteqq R_K$. Notons $\psi\colon D\to D/R_K$ l'épimorphisme canonique. Considérons la correspondance

$$\theta: D/R_H\to D/R_K\quad\text{tel que}\quad\varphi(a)\to\psi(a).$$

C'est une application car

$$\varphi(a)=\varphi(b)\quad\Leftrightarrow\quad a\equiv b(R_H)\quad\Leftrightarrow\quad a\equiv b(R_K),$$

d'où

$$\psi(a)=\psi(b).$$

Il est clair que c'est un épimorphisme croissant tel que $\psi=\theta\circ\varphi$. La réciproque est évidente.

Dans un monoïde ordonné D, on appelle *quasi-résiduels* de a par b les ensembles

$$\langle a\cdot b\rangle=\{x;\ bx\leqq a\};\quad\langle a\cdot b\rangle=\{a;\ xb\leqq a\}.$$

Un monoïde ordonné, tel que, pour tout couple (a, b) d'éléments, les quasi-résiduels $\langle a^{\cdot}.b\rangle$ et $\langle a.^{\cdot}b\rangle$ ne sont pas vides, est dit *quasi-résidué*. Si les quasi-résiduels de a par b possèdent chacun un plus grand élément, ces plus grands éléments seront respectivement notés $a^{\cdot}.b$ et $a.^{\cdot}b$ et appelés résiduels de a par b. Si pour tout couple (a, b) d'éléments, les résiduels $a^{\cdot}.b$ et $a.^{\cdot}b$ existent, le monoïde sera dit *résidué*.

Les quasi-résiduels d'un élément a par lui-même sont, s'ils existent, des monoïdes héréditaires. En effet:

$$x, y \in \langle a.^{\cdot}a\rangle = \begin{cases} ax \leqq a \\ ay \leqq a \end{cases}$$

d'où $axy \leqq ay < a$ et $xy \in \langle a.^{\cdot}a\rangle$, $x \in \langle a.^{\cdot}a\rangle$ et $y < x \Rightarrow ax \leqq a$, puis $ay < ax \leqq a$, d'où $y \in \langle a.^{\cdot}a\rangle$. Ces sous-monoïdes héréditaires seront dits *sous-monoïdes principaux* de D, et on notera $\mathscr{C}$ l'ensemble des sous-monoïdes principaux non vides de D, et P l'union de ces sous-monoïdes principaux.

Soient K un anticône de D et $\langle a.^{\cdot}a\rangle$ un sous-monoïde principal non vide, $x \in \langle a.^{\cdot}a\rangle$ entraîne $ax < a$. Mais, il existe a' tel que $a'a \in K$, d'où $a'ax < a'a$; K est héréditaire, donc $a'ax \in K$, d'où $x \in K$ et $\langle a.^{\cdot}a\rangle \subset K$, donc K contient tous les sous-monoïdes principaux, c'est-à-dire:

$$P \subset K \quad \text{quel que soit} \quad K \in \mathscr{C}.$$

THÉORÈME 1. *Un monoïde ordonné, dont l'union P des sous-monoïdes principaux est un complexe net, est clos si, et seulement si, $\mathscr{C}$ admet un plus petit élément.*

Supposons que $\mathscr{C}$ admette un plus petit élément H. Considérons l'équivalence principale R_H sur D, et soit K un anticône quelconque de D, distinct de H. On a $H \subset K$. Les classes unités des groupes D/R_H et D/R_K sont respectivement

$$E = \{u; H : u = H\} \quad \text{et} \quad F = \{v, K : v = K\}.$$

Soient u un élément de E, et $x \in D$ tel que $x \in K : u$. On a

$$P \subset H = H : u \subset K : u.$$

Il existe donc u' tel que $uu' \in K$, $u' \in P$, et w tel que $u'w < w$ ce qui entraîne

$$xuu'w < xuw.$$

Soit w' tel que $ww' \in K$, ce qui permet d'écrire $xuu'ww' < xuww'$. Or $xu \in K$, donc $xuww' \in K$ et, puisque K est héréditaire, $xuu'ww' \in K$; enfin, K étant fortement net, $x \in K$. Ainsi $K : u \subset K$. Inversement, soit $y \in K$, P est réflectif, car, si $xy \in P$, il existe w tel que $xyw < w$, ce qui entraîne

$$yxyw < yw,$$

d'où

$$yx \in \langle yw^{\cdot}.yw\rangle \subset P,$$

donc $yx \in P$. Si P est net, et $u \in E$, $P.^{\cdot}u \neq \emptyset$, et on a

$$P : u \subset H = H : u \subset K.$$

On peut donc trouver u_1 tel que $uu_1 \in P$, $u_1 \in K$, et w tel que $uu_1 w < w$, puis $yuu_1 w < yw$.

Soit w' tel que $ww' \in K$, on a $yuu_1ww' < yww'$. Puisque K est anticône, $yu \in K$ et $K \subset K : u$. Finalement $K : u = K$, c'est-à-dire $E \subset F$. Donc

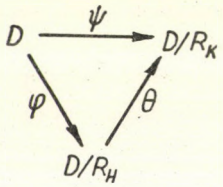
$$E : a \subset F : a \quad \text{et} \quad E : b \subset F : b.$$

Si $E : a = E : b$, $(F : a) \cap (F : b) \neq \emptyset$, F étant un complexe fort,

$$F : a = F : b.$$

Ainsi $R_E \subset R_F$, c'est-à-dire $R_H \subset R_K$, R_H est donc la plus fine congruence de D fournissant une image qui est un groupe homomorphe et isotone de D; D est clos, $D/R_H \simeq D/R_E$ est donc le plus grand groupe homomorphe et isotone, image de D.

Réciproquement, supposons que D admette un plus grand groupe image homomorphe et isotone. Notons H l'anticône correspondant, E la classe unité du groupe D/R_H. Soit K un anticône quelconque, F la classe unité du groupe D/R_K. Il est clair que $E \subset F$. Considérons le diagramme commutatif



φ et ψ désignant les épimorphismes canoniques.

Si $a \in H \Rightarrow \varphi(a) \leqq E$, d'où $\theta \circ \varphi(a) \leqq F$, soit $\psi(a) \leqq F$, c'est-à-dire $a \in K$, donc $H \subseteqq K$, et H est le plus petit élément de $\mathscr{C}$.

COROLLAIRE. *D est clos si P est un sous-monoïde fortement net.*

On a vu que P est réflectif. Il est héréditaire comme union des sous-monoïdes héréditaires; P est donc un anticône en vertu des hypothèses; c'est nécessairement le plus petit anticône de D.

2. Immersion d'un monoïde ordonné dans un monoïde clos

Soient D un monoïde ordonné unitaire et N l'ensemble des entiers naturels. Munissons $T = N \times D \times N$ de la loi

$$(m, d, n)(m', d', n') = (m + [m' - n], f(n - m'; d, d'), n' + [n - m'])$$

$$[x] = \begin{cases} x \geqq 0 & x \\ x < 0 & 0 \end{cases} \qquad f(x, d, d') = \begin{cases} x > 0 & d \\ x = 0 & dd' \\ x < 0 & d'. \end{cases}$$

R. H. BRUCK [4] a montré que, muni de cette loi, T est un monoïde simple ayant (0, 1, 0) pour élément unité et dans lequel D peut être immergé.

Considérons dans T la relation

$$(m, d, n) \leqq (m', d', n') \Leftrightarrow \begin{matrix} m = m' \\ d \leqq d' \\ n = n' \end{matrix} \quad \text{(dans } D\text{)}.$$

Il est clair que c'est une relation d'ordre dans T. Vérifions que cette relation est compatible avec la loi de T

$$(x, y, z)\,(m, d, n) = (x+[m-z], f(z-m; y, d), n+[z-m])$$

$$(x, y, z)\,(m', d', n') = (x+[m'-z], f(z-m'; y, d'), n'+[z-m']).$$

Si $(m, d, n) \leqq (m', d', n')$, on a $m = m'$; $n = n'$ et $d \leqq d'$. Dans ces conditions

$$x+[m-z] = x+[m'-z]$$

$$n+[z-m] = n'+[z-m']$$

d'autre part

$$\begin{matrix} z > m \\ z = m \\ z < m \end{matrix} \qquad f(z-m; yd) = \begin{cases} f(z-m'; y, d') = y \\ yd \leqq yd' = f(z-m'; y, d') \\ d \leqq d' = f(z-m'; y, d') \end{cases}$$

donc

$$(x, y, z)\,(m, d, n) \leqq (x, y, z)\,(m', d', n').$$

On montrerait de même:

$$(m, d, n)\,(x, y, z) \leqq (m', d', n')\,(x, y, z).$$

L'application $D \to T$ telle que $d \to (0, d, 0)$ est un monomorphisme croissant.

Soit H un anticône de D. Considérons l'ensemble

$$H^* = \{(x, h, x);\ x \in N,\ h \in H\}.$$

H^* est un sous-monoïde de T:

$$(x, h, x)\,(y, h', y) = (x+[y-x], f(x-y; h, h'), y+[x-y])$$

où $x+[y-x] = y+[x-y]$ et $f(x-y; h, h') = h$ ou hh' ou h'.

H^* est héréditaire

$$(y, u, z) \leqq (x, h, x) \quad \Rightarrow \quad x = y \quad \text{et} \quad z = x,$$

d'où $y = z$ et $u \leqq h$, donc puisque H est héréditaire, $u \in H$ et

$$(y, u, z) \in H^*.$$

H^* est net. Soit $a = (x, d, y)$ un élément quelconque de T. Il existe $d' \in D$ tel que $d'd \in H$. Posons $a' = (y, d', x)$, d'où

$$a'a = (y+[x-x], f(x-x; d', d), y+[x-x]) = (y, d'd, y) \in H^*.$$

H^* n'est pas en général fortement net. Supposons que $ba'a \in H^*$ avec $b=(u, v, w)$,

$$(u, v, w)(y, d'd, y) = (u+[y-w], f(w-y, v, d'd), y+[w-y])$$

$$ba'a \in H^* \quad \Rightarrow \quad u+[y-w] = y+[w-y]$$

$$\left.\begin{array}{ll} y > w, & u+y-w = y \\ y = w, & u = y \\ y < w, & u = y+w-y \end{array}\right\} \Rightarrow u = w \quad f(w-y; v, d'd) = \begin{cases} v \\ vd'd \\ d'd. \end{cases}$$

Mais v reste un élément quelconque de D, donc $b \notin H^*$ sauf si $H=D$, de plus

$$D^* = \{(x, d, x);\ x \in N;\ d \in D\}$$

est réflectif car, si $(m, d, n)(m', d', n') \in D^*$, on a

$$m+[m'-n] = n'+[n-m'] \quad \Rightarrow \quad m+m' = n'+n$$

condition symétrique qui entraîne $(m', d', n')(m, d, n) \in D^*$. D^* est donc un anticône de T; c'est même un sous-monoïde normal de T car il est unitaire.

Soit K^* un anticône de T contenu dans D^*, donc les éléments de K^* sont de la forme (x, k, x). Notons

$$K = \{k \in D;\ \text{il existe}\ \ x \in N\ \ \text{tel que}\ \ (x, k, x) \in K^*\}.$$

K est un sous-monoïde de D. Si $k, k' \in K$,

$$(x, k, x)(x, k', x) = (x, kk', x) \in K^*,$$

donc $kk' \in K$.

K est héréditaire. Soit $d \leqq k$, $k \in K$. On a $(x, d, x) \leqq (x, k, x)$, mais $(x, k, x) \in K^*$, donc K^* étant héréditaire $(x, d, x) \in K^*$ et $d \in K$.

K est réflectif. Supposons $dd' \in K$, donc $(m, d, n)(n, d', m) \in K^*$, mais K^* étant réflectif

$$(n, d', m)(m, d, n) = (n, d'd, n) \in K^*,$$

d'où $d'd \in K$.

K est net. Soit d un élément quelconque de D, $a=(m, d, n)$ est un élément quelconque de T; puisque K^* est net, il existe $a'=(m', d', n')$ tel que $aa' \in K^*$, donc

$$m+[m'-n] = n'+[n-m'] \quad \Rightarrow \quad m+m' = n'+n;$$

de plus, $f(n-m'; d, d') \in K$ si $n \neq m'$, ceci imposerait d ou $d' \in K$, et $K^*=D^*$, ce qui est contraire à l'hypothèse donc $n=m'$, d'où $m=n'$ et $f(n-m'; d, d') \in K$, c'est-à-dire $dd' \in K$.

K est fortement net. Soit $y \in D$ tel que $ydd' \in K$,

$$(m, y, m)(m, dd', m) = (m, ydd', m) \in K^*,$$

d'où $(m, y, m) \in K^*$ et $y \in K$.

Ainsi donc K est un anticône de D. Il y a donc contradiction avec les résultats obtenus plus haut c'est-à-dire que D^* est le plus petit anticône de T. On peut donc énoncer le théorème suivant.

THÉORÈME 2. *Tout monoïde ordonné unitaire peut être plongé dans un monoïde ordonné unitaire simple et clos.*

3. Monoïdes réguliers clos

Rappelons qu'un monoïde D est dit *régulier* si, pour tout $a \in D$, il existe $x \in D$ tel que $axa = a$. En posant $a' = xax$, on obtient

$$a = aa'a \quad \text{et} \quad a' = a'aa'.$$

Un tel élément a' est appelé un inverse généralisé ou, plus brièvement, un inverse de a. Deux types de monoïdes réguliers retiendront ici notre attention:

(a) les monoïdes réguliers, tels que chaque élément a admet un inverse unique noté a^{-1}, appelés *monoïdes inverses de Preston.*

(b) les monoïdes, tels que, pour tout a, il existe x vérifiant

$$axa = a \quad \text{et} \quad ax = xa,$$

appelés *monoïdes complètement réguliers.*

Soit D un monoïde inverse de Preston. Il est immédiat que la relation

$$a < b \quad \Leftrightarrow \quad aa^{-1} = ab^{-1} \quad \Leftrightarrow \quad aa^{-1} = ba^{-1}$$

est une relation d'ordre compatible avec l'opération du monoïde, relation d'ordre qui sera dite naturelle. Dans un tel monoïde, les quasi-résiduels existent car

$$ab^{-1} \in \langle a \cdot b \rangle \quad \text{et} \quad b^{-1}a \in \langle a \cdot b \rangle.$$

Soit E l'ensemble des idempotents. Considérons l'ensemble

$$\bar{E} = \{x \in D; \quad \text{il existe} \quad \alpha \in E \quad \text{tel que} \quad \alpha \leqq x\}.$$

Il est évident que $\bar{E}$ est un sous-monoïde de D et que $E \subset \bar{E}$. On peut aussi remarquer que si $u \in E$, $ua \leqq a$ et $au \leqq a$ quel que soit $a \in D$. On a en effet,

$$(ua)(ua)^{-1} = uaa^{-1}u = uaa^{-1}$$

donc E est un minorant de $\mathscr{P}$, ensemble des sous-monoïdes principaux.

THÉORÈME 3. *Tout monoïde inverse de Preston est clos pour la relation d'ordre naturelle.*

D'une part, $\bar{E}$ est fortement net car, quel que soit $x \in D$, il existe x^{-1} tel que $xx^{-1} \in E \subset \bar{E}$. De plus, si $yxx^{-1} \in \bar{E}$, il existe $v \in E$ tel que $v < yxx^{-1}$. Mais $yxx^{-1} < y$, car $xx^{-1} \in E$, donc $v < y$ et $y \in E$.

$\bar{E}$ est réflectif; supposons $u \in E$ tel que $u < xy$, donc $yu < yxy$, puis

$$yuy^{-1} < yxyy^{-1} < yx,$$

car $yy^{-1} \in E$; or $yuy^{-1} \in E$, donc $yx \in \bar{E}$. Enfin $\bar{E}$ est héréditaire. Soit $x \in \bar{E}$, donc il existe $u \in E$ tel que $u < x$ et soit $y < x$,

$$u < x \quad \Leftrightarrow \quad uu^{-1} = ux^{-1} = ux;$$

$$y < x \quad \Leftrightarrow \quad yy^{-1} = yx^{-1} \quad \Leftrightarrow \quad y^{-1}y = y^{-1}x,$$

d'où $y^{-1}yu = y^{-1}xu$, mais $xu \in E$, donc $y^{-1}yu < y^{-1}$ et $y^{-1} \in \bar{E}$. Si l'on remarque que $a < b \Leftrightarrow a^{-1} < b^{-1}$, on aura aussi $y^{-1}yu < y$, et donc $y \in \bar{E}$. $\bar{E}$ est donc un anticône

et contient P, union des sous-monoïdes principaux. Soit $x \in \bar{E}$, il existe donc $\alpha \in E$ tel que $\alpha < x$, c'est-à-dire $\alpha = x\alpha$, donc $x \in \langle \alpha \cdot . \alpha \rangle \subseteq P$, c'est-à-dire $\bar{E} \subseteq P$. Il en résulte $\bar{E} = P$.

HOWIE [8] a montré que la congruence $x \equiv y\,(R) \Leftrightarrow xy^{-1} \in \bar{E}$ fournit le plus grand groupe homomorphe à D. Mais $xy^{-1} \in \bar{E}$ entraîne $x^{-1}y \in \bar{E}$. Si $ax \in \bar{E}$, on a $ax^{-1}y \in \bar{E}$, puis $xx^{-1}ya \in \bar{E}$ et, puisque E est fortement net, $ya \in \bar{E}$, donc $\bar{E}:x \subseteq \bar{E}:y$, d'où l'égalité, par symétrie, soit $x \equiv y\,(R_E)$. Réciproque immédiate. D/R_E est donc le plus grand groupe homomorphe à D; compte tenu de la relation d'ordre et du fait que $\bar{E}$ est un anticône, c'est aussi le plus grand groupe image homomorphe et isotone de D.

Soit D un monoïde ordonné; $a \nparallel b$ signifiera «a comparable à b». On appelle équivalence «zig-zag» sur D l'équivalence

$$a \equiv a'\ (\varrho) \Leftrightarrow \text{il existe } a_1, \ldots, a_n \text{ en nombre fini}$$

tels que $a \nparallel a_1, \ldots, a_i \nparallel a_{i+1}, \ldots, a_n \nparallel a'$.

Il est clair que (ϱ) est compatible avec l'opération du monoïde, donc D/ϱ muni de la loi-quotient est aussi un monoïde image homomorphe de D. Définissons dans D/ϱ la relation d'ordre par l'égalité. Soit $f : D \to D/\varrho$ l'épimorphisme canonique; $a < b \Rightarrow \dot{a} = \dot{b}$, donc cet épimorphisme est croissant. D/ϱ peut-il être un groupe-image homomorphe et isotone de D? L'existence des quotients est assurée si D est quasi-résidué car, pour tout couple $(\dot{a}, \dot{b})$, on peut trouver $\dot{x}$ tel que $\dot{b}\dot{x} = \dot{a}$, et $\dot{y}$ tel que $\dot{y}\dot{b} = \dot{a}$. Il suffit de prendre $x \in \langle a \cdot . b \rangle$ et $y \in \langle a' \cdot . b \rangle$. M. L. DUBREIL-JACOTIN [6] a montré que la règle de simplification pour ϱ est également vraie si D est «fortement quasi-résidué». Ce résultat tient aussi dans un monoïde inverse de Preston ordonné par la relation d'ordre naturelle. Soit, en effet,

$$\begin{cases} x \\ y \end{cases} \in \langle a \cdot . b \rangle, \quad \text{d'où} \quad \begin{matrix} bx < a \\ by < a \end{matrix}$$

d'où $a^{-1}bx < a^{-1}a$ et $a^{-1}by < a^{-1}a$. On peut donc supposer $a \in E$,

$$\begin{aligned} bx < a \quad &\Rightarrow \quad e = (bx)^{-1}(bx) = abx < a \quad e \in E \\ by < a \quad &\Rightarrow \quad f = (by)^{-1}(by) = aby < a \quad f \in E \end{aligned} \qquad \Rightarrow \quad e \equiv f(\varrho)$$

$$b^{-1}ae = (b^{-1}a)(ab)x < x \quad \Rightarrow \quad x \nparallel b^{-1}ae$$

$$b^{-1}af = (b^{-1}a)(ab)y < y \quad \Rightarrow \quad y \nparallel b^{-1}af$$

mais $b^{-1}ae \equiv b^{-1}af\ (\varrho)$, d'où $x \equiv y\ (\varrho)$.

Soit donc $zx \equiv zy\ (\varrho)$; on a $\widehat{zx} = \widehat{zy} = t$ avec $t = zx$. Posons $\dot{u}$ la classe contenant $\langle t \cdot . z \rangle$, donc $x \in \dot{u}$ et $\dot{x} = \dot{u}$. De même, $\dot{y} = \dot{u}$, d'où $x \equiv y\ (\varrho)$. Etude analogue à droite. D/ϱ est donc un groupe homomorphe et isotone de D et on peut énoncer le théorème ci-après.

THÉORÈME 4. *Si D est un monoïde inverse de Preston, ordonné par la relation d'ordre naturelle, et ϱ l'équivalence «zig-zag», il existe un épimorphisme croissant*

$$D/R_E \to D/\varrho.$$

SAITÔ [12] a caractérisé les monoïdes totalement ordonnés complètement réguliers. On peut donner à cette caractérisation la forme suivante:

Soit Λ un monoïde ordonné dont tous les éléments sont idempotents et l'ordre total. On note T_Λ le demi-treillis qui lui est associé, et $\Gamma(\Lambda)$ la catégorie tel que $\mathrm{ob}\,\Gamma(\Lambda) = T_\Lambda$; $\mathscr{F}_l(\Gamma(\Lambda))$ est le graphe de la relation d'ordre de T_Λ; donc une flèche de $\Gamma(\Lambda)$ est un couple d'éléments de T_Λ tel que $\alpha \leqq \beta$. Enfin, $\Gamma(G)$ est la catégorie des groupes totalement ordonnés.

On considère un foncteur contravariant $F\colon \Gamma(\Lambda) \to (\Gamma(G))^*$.

$$\begin{aligned} F_{\mathrm{ob}}\colon \quad & T_\Lambda \to \mathrm{ob}\,\Gamma(G) \\ & \alpha \to G(\alpha) \\ F_{\mathscr{F}_l}\colon \quad & \mathscr{F}_l(\Gamma(\Lambda)) \to \mathscr{F}_l(\Gamma(G))^* \\ & (\alpha, \beta) \to \varphi_\alpha^\beta\colon G(\beta) \to G(\alpha) \\ & \alpha \leqq \beta. \end{aligned}$$

Notons $\Lambda(\alpha)$ la $\mathscr{D}$-classe d'équivalence qui correspond à $\alpha \in T_\Lambda$, et $D(\alpha)$ l'ensemble des couples (g, f) avec $g \in G(\alpha)$, $f \in \Lambda(\alpha)$. Enfin,

$$D = \bigcup_{\alpha \in T_\Lambda} D(\alpha).$$

Cet ensemble munit du produit

$$\begin{matrix} (g,f) \in D(\alpha), \\ (g',f') \in D(\beta), \end{matrix} \quad (g,f)(g',f') = (\varphi^\alpha_{\alpha\circ\beta}(g)\varphi^\beta_{\alpha\circ\beta}(g'), ff')$$

et de la relation d'ordre

$$(y,f) \leqq (g',f') \quad \Leftrightarrow \quad \begin{cases} \varphi^\alpha_{\alpha\circ\beta}(g) < \varphi^\beta_{\alpha\circ\beta}(g') \\ \text{ou} \\ \varphi^\alpha_{\alpha\circ\beta}(g) = \varphi^\alpha_{\alpha\circ\beta}(g'), \quad f \leqq f' \end{cases}$$

est un monoïde totalement ordonné complètement régulier et tout monoïde de ce type peut s'obtenir de cette manière. Λ et F sont donc des invariants caractéristiques.

Les idempotents de D sont de la forme (e_α, f), où e_α est l'élément neutre de $G(\alpha)$. L'ensemble E des idempotents est un sous-monoïde de D. On notera que (g^{-1}, f) est un inverse de (g, f).

LEMMES.

(a) $$(e_\alpha, f)(k, l) \leqq (k, l);\ (k, l) \in D(\beta).$$

(b) $$(g, f)(e_\beta, f')(g^{-1}, f) \in E;\ (g, f) \in D(\alpha).$$

(c) $$(e_\alpha, f)(g, h)(e_\alpha, f) = (e_\alpha, f)(g, h);\ (g, h) \in D(\beta).$$

La démonstration de ces lemmes n'offre aucune difficulté.

THÉORÈME 5. *Tout monoïde totalement ordonné complètement régulier est clos. L'union des sous-monoïdes principaux est le plus petit anticône.*

Soient D un monoïde totalement ordonné complètement régulier, E le monoïde des idempotents, et P l'union des sous-monoïdes principaux.

D'une part, P est un sous-monoïde de D. En effet, si $x \in P$, il existe u tel que $ux < u$, et on peut toujours choisir $u \in E$ (en multipliant par un inverse u': $u'u \in E$); soit aussi $y \in P$, tel que $vy < v$, $v \in E$. On a $uxy < uy < y$, puis $vuxy < vy < v$, ce qui entraîne $vuxyu < vu$, d'où $xy \in P$. Même résultat si $y \in P$ tel que $yv < v$.

D'autre part, P est fortement net: P est net puisqu'il contient E, de plus, si x' est un inverse de x et $yxx' \in P$, il existe $u \in E$ tel que $yxx'u < u$, ce qui entraîne

$$xx'yxx'u < xx'u \quad \text{or} \quad xx'yxx' = yxx',$$

car $xx' \in E$, donc $yxx'u < xx'u$ et $y \in P$.

P est donc un anticône, c'est évidemment le plus petit anticône, et l'équivalence principale R_P fournit le plus grand groupe image homomorphe et isotone de D.

4. Monoïde totalement clos

Soit D un monoïde ordonné tel que $\mathscr{P}$ possède un plus grand élément $H = \langle w^{\cdot}.w \rangle$ fortement net

$$a \equiv b(R_H) \quad \Leftrightarrow \quad H:a = H:b$$

$$\langle w^{\cdot}.w \rangle : a = \{x;\ ax \in \langle w^{\cdot}.w \rangle\} = \{x;\ axw < w\}.$$

Mais H est réflectif, donc

$$\langle w^{\cdot}.w \rangle : a = \{x;\ xaw < w\} = \langle w^{\cdot}.aw \rangle.$$

Ainsi

$$a \equiv b\ (R_H) \quad \Leftrightarrow \quad \langle w^{\cdot}.aw \rangle = \langle w^{\cdot}.bw \rangle.$$

L'application f de D, dans l'ensemble des parties de D,

$$a \to \langle w^{\cdot}.aw \rangle,$$

est donc un homomorphisme croissant dont l'image $f(D)$, munie de la loi

$$\langle w^{\cdot}.aw \rangle * \langle w^{\cdot}.bw \rangle = \langle w^{\cdot}.abw \rangle$$

et de la relation d'ordre inverse de l'inclusion, et le plus grand groupe homomorphe et isotone à D (corollaire du théorème 1).

Toujours avec les mêmes hypothèses, supposons que de plus H admette un plus grand élément ξ. Cette propriété supplémentaire se traduit par:

$$\begin{cases} xw \leqq w & \Rightarrow \quad x < \xi \\ \xi w \leqq w. \end{cases}$$

Donc $\xi^2 w \leqq \xi w \leqq w$ et $\xi^2 \leqq \xi$. D'autre part,

$$xw \leqq w \quad \Rightarrow \quad x \leqq \xi \quad \Rightarrow \quad x\xi \leqq \xi^2 \leqq \xi,$$

donc $\langle w^{\cdot}.w \rangle = H \leqq \langle \xi^{\cdot}.\xi \rangle$. Mais H est le plus grand des sous-monoïdes principaux, c'est-à-dire que $H = \langle \xi^{\cdot}.\xi \rangle$.

On établira sans peine:

$$H = \langle \xi \cdot . \xi \rangle = \langle \xi . \cdot \xi \rangle = \xi^{\leqq}$$

$\xi^{\leqq}$ désignant la section commençante de ξ, ensemble des éléments de D qui sont $\leqq \xi$. De plus,

$$\langle w \cdot . aw \rangle = \langle \xi \cdot . \xi \rangle = \langle \xi \cdot . a \rangle.$$

L'équivalence d'homomorphie s'écrit donc

$$a \equiv b\,(R_H) \quad \Leftrightarrow \quad \langle \xi \cdot . a \rangle = \langle \xi \cdot . b \rangle.$$

Il est clair que $\langle \xi \cdot . a \rangle = \langle \xi . \cdot a \rangle$.

On convient de poser $R_H = \mathscr{A}_\xi$. L'équivalence $\mathscr{A}_\xi$ sera dite *équivalence d'Artin généralisée relative à* ξ.

Nous dirons qu'un monoïde, tel que l'ensemble des monoïdes principaux possède un plus grand élément H fortement net, H ayant lui-même un plus grand élément ξ, est *totalement clos*; ξ est appelé *élément bimaximum* de D.

Nous complétons ainsi un résultat mis en évidence par M.-L. DUBREIL-JACOTIN [6] et L. FUCHS [7].

THÉORÈME 6. *Si D est un monoïde totalement clos, ξ l'élément bimaximum, et $\mathscr{A}_\xi$, l'équivalence généralisée d'Artin, $D/\mathscr{A}_\xi$ est le plus grand groupe image homomorphe et isotone de D.*

EXEMPLES. 1° Soient A un anneau intègre unitaire et $I_f(A)$ le monoïde unitaire des idéaux fractionnaires de type fini. On sait que A est intégralement clos si, et seulement si, $\mathfrak{m} : \mathfrak{m} = A$ pour tout $\mathfrak{m} \in I_f(A)$. Ainsi dans $I_f(A)$ les quasi-résiduels $\langle \mathfrak{m} : \mathfrak{m} \rangle$ existent et admettent un plus grand élément $\langle A : A \rangle$, ensemble des idéaux entiers de type fini. Ce plus grand sous-monoïde principal de $I_f(A)$ admet un plus grand élément A. On peut donc énoncer la caractérisation suivante:

A est un anneau intégralement clos si, et seulement si, le monoïde $I_f(A)$ des idéaux fractionnaires de type fini est totalement clos, d'élément bimaximum A.

2° Soient G un groupe ordonné, et E un monoïde dont les éléments sont des idempotents et qui sera ordonné par $h \prec f \Leftrightarrow hf = fh = h$.

$D = G \times E$ est un monoïde pour la loi

$$(g, \alpha)\,(g', \alpha') = (gg', \alpha).$$

Il est ordonné par

$$(g, \alpha) \leqq (g', \alpha') \quad \Leftrightarrow \quad \begin{cases} g < g' \\ \text{ou} \\ g = g', \quad \alpha < \alpha'. \end{cases}$$

L'application $\varphi : D \to G$ $((g, \alpha) \to g)$ est un épimorphisme croissant, dont l'anticône associé est

$$H = \{(g, \alpha); g < e, \alpha \in E\},$$

e élément neutre de G. Il est clair que si $(g, \alpha) \in H$:

$$(g, \alpha)\,(e, \alpha) < (e, \alpha)$$

d'où

$$(g, \alpha) \in \langle (e, \alpha) \cdot . (e, \alpha) \rangle.$$

Mais $H \supseteq P$ (union des monoïdes principaux), donc

$$H = \langle (e, \alpha)^{\cdot}.(e, \alpha) \rangle \quad \text{quel que soit} \quad \alpha \in E.$$

H est donc le plus grand élément de $\mathscr{P}$, ensemble des monoïdes principaux.

Si E contient un élément unité $\alpha < 1$ pour tout $\alpha \in E$, donc $(e, 1)$ est le plus grand élément de H. D est donc totalement clos, d'élément bimaximum $(e, 1)$ et

$$D/\mathscr{A}_{(e,1)} \cong G.$$

Les monoïdes résidués totalement clos (on dit aussi $\mathscr{A}$-nomal) ont été étudiés dans [9] et [11]. Soit D un monoïde résidué. On appellera ordre dans D, tout $m \in D$ tel que $m^2 < m$. Quel que soit $x \in D$, $x.^{\cdot}x$ et $x^{\cdot}.x$ sont des ordres, appelés respectivement ordres à droite et à gauche associés à x. Si $x \in D$, on posera

$$x^{-1} = (x^{\cdot}.x).^{\cdot}x.$$

Le monoïde D est totalement clos si l'ensemble des ordres associés aux éléments de D admet un plus grand élément ε qui est l'élément bimaximum; ε est équirésiduel. L'équivalence

$$x \equiv y \ (\mathscr{A}_\varepsilon) \quad \Leftrightarrow \quad \varepsilon : x = \varepsilon : y$$

fournit pour la loi induite le groupe quotient $D/\mathscr{A}_\varepsilon$ plus grand groupe image homomorphe et isotone de D. Ce groupe est isomorphe a

$$D_\varepsilon = \{\varepsilon : x, x \in D\}$$

muni de la loi

$$(\varepsilon : x) \circ (\varepsilon : y) = \varepsilon : yx.$$

L'épimorphisme croissant $D \to D_\varepsilon$ $(x \to \varepsilon : (\varepsilon : x))$ est ici une fermeture α_ε, donc $D_\varepsilon \subset D$; en particulier D_ε est ordonné par l'ordre de D.

THÉORÈME 7. *Soient D un monoïde résidué, R une équivalence sur D tel que D/R soit un groupe homomorphe et isotone à D; R est l'équivalence d'Artin et D est totalement clos si, et seulement si, l'une des classes* mod R *possède un plus grand élément r.*

Soit ε_1 un élément de la classe unité mod R; $\varepsilon_1 r \equiv r \ (R)$, d'où

$$\varepsilon_1 r < r \quad \text{et} \quad \varepsilon_1 < r^{\cdot}.r.$$

On en déduit $\varepsilon_1 r < (r^{\cdot}.r)r < r$, donc par convexité $\varepsilon_1 r \equiv (r^{\cdot}.r)r \ (R)$ et, puisque D/R est un groupe, $\varepsilon_1 \equiv (r^{\cdot}.r) \ (R)$. On établirait de même que

$$\varepsilon_1 \equiv (r.^{\cdot}r) \ (R).$$

Ainsi $\varepsilon = r.^{\cdot}r = r^{\cdot}.r$ est le plus grand élément de la classe unité mod R.

Soit a un élément d'une classe A mod R et $\bar{a}$ un élément de la classe inverse A^{-1}, $a\bar{a} \equiv \varepsilon \ (R)$ entraîne $a\bar{a} \leqq \varepsilon$, puis $\bar{a} < \varepsilon.^{\cdot}a$, c'est-à-dire

$$a\bar{a} < a[\varepsilon.^{\cdot}a] < \varepsilon \ (R),$$

donc par convexité

$$a[\varepsilon.^{\cdot}a] \equiv \varepsilon \ (R).$$

Cette relation montre que si $a \in A$, $\varepsilon .\cdot a \in A^{-1}$. On montrerait de même que $\varepsilon \cdot . a \in A^{-1}$, et finalement

$$\varepsilon \cdot .(\varepsilon .\cdot a) \in (A^{-1})^{-1} = A; \quad a < \varepsilon \cdot .(\varepsilon .\cdot a)$$

entraîne

$$a \cdot . a < \varepsilon \cdot .(\varepsilon .\cdot a) a.$$

Or $\varepsilon .\cdot a \in A^{-1}$, donc $(\varepsilon .\cdot a) a \in E$, classe unité, et $\varepsilon \cdot .(\varepsilon .\cdot a) a \in E^{-1} = E$, donc $\varepsilon \cdot .(\varepsilon .\cdot a) a < \varepsilon$, puisque ε est le plus grand élément de la classe unité. Donc $a \cdot . a < \varepsilon$ quel que soit $a \in D$, c'est-à-dire que ε est élément bimaximum et D totalement clos. $a \equiv b$ (R) entraîne

$$a(\varepsilon : a) \equiv b(\varepsilon : a) \equiv \varepsilon \, (R),$$

d'où $b(\varepsilon : a) \leqq \varepsilon$, donc $\varepsilon : a < \varepsilon : b$. De même $\varepsilon : a > \varepsilon : b$, d'où

$$a \equiv b \; (\mathscr{A}_\varepsilon).$$

Ainsi $R \subset \mathscr{A}_\varepsilon$; mais on sait que l'équivalence d'Artin est plus fine puisque fournissant le plus grand groupe image homomorphe et isotone, donc $R = \mathscr{A}_\varepsilon$.

On retiendra le résultat suivant:

Un monoïde résidué unitaire est totalement clos si et seulement s'il contient un plus grand idempotent.

Un monoïde D résidué totalement clos, d'élément bimaximum ε, sera dit *nomalement clos* ($\mathscr{A}$-nomalement fermé) si, quel que soit $x \in D$, on a

$$\varepsilon : (x \cdot . \varepsilon) = \varepsilon : (x .\cdot \varepsilon) = \varepsilon : x.$$

Si D est réticulé, $\mathscr{A}_\varepsilon$ est alors régulier pour l'intersection. Si le monoïde admet un élément neutre e, il est nomalement clos si, et seulement si,

$$\varepsilon : (e .\cdot \varepsilon) = \varepsilon : (e \cdot . \varepsilon) = \varepsilon.$$

Un monoïde D résidué totalement clos, d'élément bimaximum ε, sera dit *intégralement clos* ($\mathscr{A}$-totalement fermé) si tous les ordres associés sont égaux à ε. Si l'élément neutre existe, il suffira que $x \cdot . x = e$ ou $x .\cdot x = e$.

Un monoïde résidué D sera dit *B-totalement clos* ($\mathscr{B}$-nomal) s'il existe un élément β vérifiant la condition $\beta^{-1} = \beta \cdot . \beta = \beta .\cdot \beta$ et tel que, pour tout $x \in D$,

$$\beta = x(\beta .\cdot x) = (\beta \cdot . x) x,$$

β est dit *B-élément*. Un B-élément, s'il existe, est unique, c'est un idempotent appartenant au centre de D.

Un monoïde D, B-totalement clos, dont β est le B-élément, sera dit *B-nomalement clos* si, quel que soit $x \in D$, on a

$$(x \cdot . x) \cdot . \beta = (x .\cdot x) .\cdot \beta = \beta^{-1}.$$

Si l'élément neutre existe, il suffira que

$$e \cdot . \beta = e .\cdot \beta = \beta^{-1}.$$

Notons qu'un monoïde nomalement clos est B-nomalement clos si l'une des classes modulo $\mathscr{A}_\varepsilon$ possède un plus petit élément. Or dans un monoïde B-nomalement clos, le B-élément est le plus petit élément de la classe de l'élément bimaximum.

Comme tous les idempotents appartiennent à cette classe, le B-élément est le plus petit des idempotents.

Un monoïde intégralement clos, B-totalement clos, sera dit *B-intégralement clos*.

Un monoïde B-totalement clos, avec élément unité, est B-intégralement clos si, et seulement si, l'élément bimaximum coïncide avec l'élément unité.

Enfin, un monoïde résidué unitaire est un groupe ordonné si, et seulement si, l'élément neutre est B-élément.

EXEMPLES. 1° Soit A un anneau commutatif intègre et unitaire, $I(A)$ le monoïde de ses idéaux fractionnaires. A est complètement clos si, et seulement si, $\mathfrak{m}: \mathfrak{m} = A$ quel que soit $\mathfrak{m} \in I(A)$, c'est-à-dire si, et seulement si, le monoïde unitaire $I(A)$ est intégralement clos ([3], ex. 6, p. 82).

2° A est un anneau de Dedekind si, et seulement si, A est B-élément du monoïde $I(A)$.

3° BLYTH a montré [2] que, si D est un monoïde résidué et ϱ l'équivalence «zig-zag», D/ϱ est un groupe image homomorphe et isotone de D. Si D est totalement clos, d'élément bimaximum ε, il existe un épimorphisme croissant

$$\theta : D/\mathscr{A}_\varepsilon \to D/\varrho \quad \text{et} \quad D/\mathscr{A}_\varepsilon/\operatorname{Ker}\theta \cong D/\varrho.$$

Il est clair que θ est un isomorphisme si, et seulement si, D contient un élément maximal qui sera le plus grand élément dans sa classe mod ϱ.

4° Soit N^* l'ensemble des entiers naturels strictement positifs; $N^* \times N^*$, muni de la loi

$$(i, j)\,(k, s) = \big(i + k - \inf(j, k),\, j + s - \inf(j, k)\big),$$

est appelé monoïde *bicyclique*. On vérifiera que c'est un monoïde inverse de Preston. Il est clair que $(i, j)^{-1} = (j, i)$ et que les idempotents sont de la forme (i, i).

De plus la relation d'ordre naturelle est:

$$(i, j) \leqq (k, s) \quad \Leftrightarrow \quad \begin{cases} j \geqq s \\ i - j = k - s. \end{cases}$$

Ce monoïde bicyclique est résidué:

$$[(c, d)\cdot(a, b) = \begin{cases} d > b & (c, d - b + a) \\ b \geqq d \begin{cases} c - d + b - a \geqq 0 & (c - d + b - a + 1, 1) \\ c - d + b - a < 0 & (1, d - c + a - b + 1) \end{cases} \end{cases}$$

$$(c, d)\cdot(a, b)] = \begin{cases} c > a & (c - a + b, d) \\ a \leqq c \begin{cases} d - c + a - b \geqq 0 & (1, d - c + a - b + 1) \\ d - c + a - b < 0 & (c - d + b - a + 1, 1). \end{cases} \end{cases}$$

(1, 1) est élément neutre et plus grand idempotent, ce monoïde est donc intégralement clos. Il en est toujours ainsi pour un monoïde inverse de Preston résidué et unitaire car l'élément unité est le plus grand idempotent pour la relation d'ordre naturelle.

5. Anneau nomal

Un anneau d'intégrité A commutatif unitaire sera dit *nomal* si le monoïde résidué unitaire $I(A)$ de ses idéaux fractionnaires est totalement clos. Un tel anneau est donc caractérisé par l'existence d'un plus grand idéal fractionnaire idempotent $\mathfrak{e}$, élément bimaximum de $I(A)$. Il est clair que si K est le corps des fractions de A, $\mathfrak{e}$ est un ordre de K tel que $\mathfrak{e} \in I(A)$.

THÉORÈME 8. *Un ordre A d'un corps K est nomal si et seulement s'il existe un ordre $\mathfrak{e}$ de K complètement intégralement clos, appartenant à $I(A)$.*

Soit $\mathfrak{e} \in I(A)$, ordre de K complètement intégralement clos; $\mathfrak{e}$ est un idempotent fractionnaire de $I(A)$. Soit $\mathfrak{m}$ un autre idempotent de $I(A)$ et $m \in \mathfrak{m}$, donc $m^\lambda \in \mathfrak{m}$ quel que soit l'entier naturel $\lambda \geqq 0$ et il existe $d \in A$ tel que $dm^\lambda \in A$. Mais $A \subset \mathfrak{e}$, donc $dm^\lambda \in \mathfrak{e}$, et comme $\mathfrak{e}$ est complètement intégralement clos, $m \in \mathfrak{e}$ et $\mathfrak{m} \subseteqq \mathfrak{e}$; $\mathfrak{e}$ est donc le plus grand idempotent, $I(A)$ est totalement clos, et donc A est nomal.

Réciproquement, si A est nomal d'élément bimaximum $\mathfrak{e}$, $\mathfrak{e}$ est un ordre de K tel que $\mathfrak{e} \in I(A)$. Il est clair que $I(\mathfrak{e}) \subset I(A)$; $\mathfrak{e}$ plus grand idempotent de $I(A)$ est aussi plus grand idempotent de $I(\mathfrak{e})$, c'est aussi l'élément unité de $I(\mathfrak{e})$. Ce monoïde est donc intégralement clos, c'est-à-dire $\mathfrak{m} : \mathfrak{m} = \mathfrak{e}$ quel que soit $\mathfrak{m} \in I(\mathfrak{e})$; $\mathfrak{e}$ est complètement intégralement clos.

COROLLAIRE 1. *Pour que la clôture intégrale $\bar{A}$ d'un anneau noethérien intègre A soit un A-module de type fini, il faut et il suffit que A soit nomal.*

Soit A noethérien. Si $\bar{A}$ est un sous-A-module de type fini, c'est un idempotent de $I(A)$. On sait que $\bar{A}$ est un anneau de KRULL ([10], p. 118 en particulier) complètement intégralement clos. Donc selon le théorème ci-dessus, A est nomal.

Réciproquement, supposons A noethérien nomal d'idéal bimaximum $\mathfrak{e}$, $A \subset \mathfrak{e}$ entraîne $\bar{A} \subset \mathfrak{e} = \bar{\mathfrak{e}}$. Si $\mathfrak{m}$ est un idempotent quelconque de $I(A)$ et $m \in \mathfrak{m}$, alors $m^\alpha \in \mathfrak{m}$ pour tout entier $\alpha > 0$, et il existe $d \neq 0$ de A tel que $dm^\alpha \in A \subset \bar{A}$, d'où $\bar{A}$ complètement intégralement clos $m \in \bar{A}$ et $\mathfrak{m} \subset \bar{A}$; en particulier, $\mathfrak{e} \subset \bar{A}$, d'où $\mathfrak{e} = \bar{A}$ et $\bar{A} \in I(A)$. Ainsi A, idéal fractionnaire d'un anneau intègre noethérien, est un A-module de type fini.

COROLLAIRE 2. *Soient A un anneau nomal, A' la fermeture intégrale de A dans une extension de K, les anneaux A', $A_{[x]}$ et $A_{[[x]]}$ sont nomaux.*

C'est clair en utilisant le théorème.

COROLLAIRE 3. *Si A est un anneau noethérien nomal, et S une partie multiplicative telle que $0 \notin S$, alors $S^{-1}A$ est un anneau noethérien nomal.*

Si $\bar{A}$ est la clôture intégrale de A, $S^{-1}\bar{A}$ est la clôture intégrale de $S^{-1}A$. $S^{-1}A$ est noethérien, et $S^{-1}\bar{A}$ est un idéal fractionnaire de $S^{-1}A$, donc de type fini, il en résulte que $S^{-1}A$ est nomal.

EXEMPLES. 1° Soit d un entier rationnel non divisible par un carré dans Z. Si $d \equiv 1 \pmod 4$ les éléments du corps $\sum = Q(\sqrt{d})$, qui sont entiers sur Z, sont

de la forme

$$\frac{a+b\sqrt{d}}{2}; \quad [(a, b)\in Z\times Z; a\equiv b\ (2)].$$

Notons E l'anneau des entiers, et soit

$$Z_{[\sqrt{d}]} = \{u+v\sqrt{d}\,; u, v\in Z\},$$

$Z_{[\sqrt{d}]}$ est un anneau intègre dont $\sum$ est le corps des quotients; de plus,

$$2E \subseteqq Z_{[\sqrt{d}]} \subseteqq E,$$

donc E est un idéal fractionnaire de l'anneau $Z_{[\sqrt{d}]}$; E est un anneau de Dedekind a fortiori complètement intégralement clos, donc $Z_{[\sqrt{d}]}$ est nomal.

2° Soit m l'idéal de $C_{[X,Y]}$ engendré par le polynôme Y^2-X^3. On pose $A=C_{[X,Y]/m}$. Soit σ la surjection canonique $C_{[X,T]}\to A$; si $\sigma(X)=\xi$ et $\sigma(Y)=\eta$, en posant $t=\eta/\xi$, il est clair que

$$A = C_{[t^2,t^3]} \subseteqq C_{[t]},$$

donc A est intègre. Soit L le corps des fractions de A, $t\in L$, de plus $t^2-\xi=0$, donc t appartient à $\bar{A}$, clôture intégrale de A. Mais $t\notin A$, ce qui veut dire que A n'est pas intégralement clos. $A\subset C_{[t]}$, où $C_{[t]}$ est principal, donc intégralement clos, t étant entier sur A, cela implique

$$C_{[t]} = \bar{A}.$$

Selon le corollaire 1, A est donc nomal.

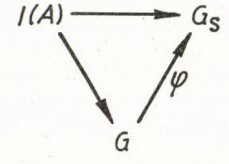
3° A est un anneau noethérien nomal, $\mathfrak{p}$ un idéal premier de hauteur 1 dans A. Alors $A_{\mathfrak{p}}$ est intégralement clos, sauf pour un nombre fini d'idéaux $\mathfrak{p}$ ([10], p. 212).

4° On trouvera dans [10], p. 211, un exemple d'anneau noethérien non nomal A, mais intègre, tel que si $\mathfrak{p}$ est un idéal premier, $A_{\mathfrak{p}}$ soit nomal.

REMARQUE. Si A est un anneau noethérien et S une partie multiplicative telle que $0\notin S$ l'application $\mathfrak{m}\in I(A)\to S^{-1}\mathfrak{m}\in I(S^{-1}A)$ est une surjection; c'est même un épimorphisme croissant. Si $S^{-1}A$ est un anneau nomal et G_S le plus grand groupe image homomorphe et isotone de $I(S^{-1}A)$ on a

$$I(A)\to I(S^{-1}A)\to G_S.$$

Donc G_S est aussi un groupe image homomorphe et isotone de $I(A)$. Tout particulièrement si A est nomal et si G est le plus grand groupe image homomorphe et isotone de $I(A)$ on a le diagramme commutatif:



φ étant un épimorphisme de groupe: $G_S\cong G/\mathrm{Ker}\,\varphi$.

Un élément $x \in K$ est dit *quasi-entier* sur A s'il existe $d \in A$, $d \neq 0$, tel que $dx^n \in A$ pour tout entier $n \geqq 0$. Il est clair que l'ensemble des éléments de K qui sont quasi-entiers sur A est un anneau A^* ordre de K qui sera nommé *quasi-clôture intégrale* de A.

THÉORÈME 9. *Un ordre A d'un corps K est nomal si et seulement si sa quasi-clôture intégrale appartient à $I(A)$.*

Soit $\mathfrak{m} \in I(A)$ et $x \in \mathfrak{m} : \mathfrak{m}$ donc $x\mathfrak{m} \subseteqq \mathfrak{m}$ puis $x^n\mathfrak{m} \subseteqq \mathfrak{m}$. Or $\mathfrak{m} : \mathfrak{m}$ est un élément de $I(A)$ et il existe d tel que $d(\mathfrak{m} : \mathfrak{m}) \subseteqq A$ $d \in A$, $d \neq 0$ c'est-à-dire que $dx^n \in A$ quelque soit $n > 0$. Ainsi $x \in A^*$ quasi-clôture intégrale de A, et $\mathfrak{m} : \mathfrak{m} \subseteqq A^*$. Si $A^* \in I(A)$, $I(A)$ est totalement clos d'élément bimaximum A^* donc A est nomal et A^* complètement intégralement clos. Si A est nomal d'élément bimaximum $\mathfrak{e} : \mathfrak{e} = \mathfrak{e} : \mathfrak{e} \subseteqq A^*$. Or $x \in A^*$ implique qu'il existe $d \in A$, $d \neq 0$ tel que $dx^n \in A \subset \mathfrak{e}$; $\mathfrak{e}$ étant complètement intégralement clos $x \in \mathfrak{e}$ et $A^* \subseteqq \mathfrak{e}$ d'où $A^* = \mathfrak{e}$ et $A^* \in I(A)$.

COROLLAIRE 4. *Un ordre A d'un corps K est nomal si et seulement s'il existe un ordre B de K nomal dont le conducteur dans A n'est pas vide.*

Si A est nomal on peut adopter pour B la quasi-clôture intégrale. Réciproquement si B est nomal tel que $A : B \neq \emptyset$ il existe $B^* \in I(B)$ complètement intégralement clos donc il existe $\mu \in B$ avec $\mu B^* \subset B^*$. Mais $A : B \neq \emptyset$ implique qu'il existe $\lambda \in A$ tel que $\lambda B \subset A$ d'où $\lambda\mu B^* \subset \lambda B \subset A \subset B^*$ donc $B^* \in I(A)$ et A est nomal.

COROLLAIRE 5. *Si A est un ordre nomal d'un corps K, tout ordre B de K entier sur A est nomal.*

Soit A nomal et A^* sa quasi-clôture intégrale; $A^* \in I(A)$ donc il existe $\mu \in A$ tel que $\mu A^* \subset A \subset A^*$. Notons $\bar{A}$ la clôture intégrale de A. On a $\bar{A} \subset A^*$ et puisque B est entier sur A $\mu A^* \subset A \subset B \subset \bar{A} \subset A^*$. Ainsi donc $A^* \in I(B)$ A^* est complètement intégralement clos donc B est nomal.

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Bibliographie

[1] A. BIGARD, Sur les images homomorphes d'un demi-groupe ordonné, *C. R. Acad. Sc. Paris*, **260** (1965), p. 5987—5988.

[2] T. C. BLYTH, *Contribution à la théorie de la résiduation dans les structures algébriques ordonnées* (Thèse Sc. math. Paris, 1963).

[3] N. BOURBAKI, *Algèbre commutative*, Chap. 7: Diviseurs. (Paris, Hermann, 1965.) (Act. Scient. et ind. 1314; Bourbaki 31).

[4] R. H. BRUCK, *A survey of binary systems*, 2nd edition. (Berlin, Springer-Verlag, 1966.) (Ergebnisse der Mathematik, 20).

[5] P. DUBREIL, *Algèbre*, t. 1, 2e édition. (Paris, Gauthier-Villars, 1954.) (Cahiers scientifiques, 20).

[6] MARIE-LUISE DUBREIL-JACOTIN, Sur les images homomorphes d'un demi-groupe ordonné, *Bull. Soc. Math. France*, **92** (1964), p. 101—115.

[7] L. FUCHS, On group homomorphic images of partially ordered semigroups, *Acta Scient. Math. Szeged*, **25** (1964), p. 139—142.

[8] J. M. HOWIE, The maximum idempotent-separating congruence on an inverse semi-group, *Proc. Edinburgh math. Soc.*, Series 2, t. **14** (1964), p. 71—79.

[9] I. MOLINARO, Demi-groupes résidutifs, *J. Math. pures et appl.*, **39** (1960), p. 319—356 et **40** (1961), p. 43—110 (Thèse Sc. math. Paris, 1956).
[10] M. NAGATA, *Local rings* (New York, Interscience-Publishers, 1962) (Interscience Tracts in pure and applied Mathematics, 13).
[11] J. QUERRE, Contribution à la théorie des structures ordonnées et des systèmes d'idéaux, *Annali di Mat. pura ed appl.*, Serie **4**, t. **66** (1964), p. 265—389 (Thèse Sc. math. Paris, 1963).
[12] T. SAITO, Ordered completely regular semigroups, *Pacific J. of Math.*, **14** (1964), p. 295—308.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (1—2), (1968), pp. 147—162.

TRANSLATIONS AND CONGRUENCES IN LATTICES

By
P. A. GRILLET (Manhattan, Kansas, U.S.A.)
(*Presented by* L. RÉDEI)

As far as recent litterature is concerned, the study of congruences on a non-distributive lattice L seems to require either distributivity conditions (as in [4]) or the consideration of translations, thus of conditions or concepts in which are involved finitely many (but possibly very many) elements of L (for instance, weak modularity [3]). Even though these methods have proved very valuable, we have felt that an individual investigation of the translations could bring interesting results. We have concentrated on the characterization of simple lattices. We do find such characterization under both chain conditions and the assumption that every translation is product of at most n elementary translations, where n is a small, fixed integer ($n=3, 4$ in our results). This latter condition can be considered as an extremely loose form of distributivity, although it is not very weak for these small values of n.

In the first part, we classify the translations of a lattice L, then study the smallest n (if it exists) such that any translation of L is product of at most n elementary translations. We call it the translational dimension (tr. dim) of L. Then we study the behavior of tr. dim L under homomorphic images and direct products. We also have the following results: (i) if tr. dim $L \leqq 2$, then L is distributive (the converse holds almost trivially); (ii) if tr. dim $L \leqq 3$ and if L is modular, then L is distributive; (iii) if L is modular and generated by at most three elements, then tr. dim $L \leqq 4$.

In the second part we turn to the characterization of the kernels of congruences and the construction of the largest congruence with a given kernel. In spite of some simplifications, both are essentially infinite, but become of course finite if tr. dim L is. We shortly investigate a sufficient distributivity condition for an ideal to be a kernel; it was suggested to us by the examination of the general conditions, and does not seem to have been considered before. Then we come back to our problem with another characterization of kernels (under both chain conditions); this leads to the aforementioned characterizations of simple lattices. Some unsolved questions conclude the paper (including the behavior of tr. dim L in the sublattices of L).

We are glad to thank here Miss POINSIGNON and Dr. VARLET for some helpful remarks, and especially our referee, Prof. G. SZÁSZ for his very valuable suggestions.

I. Translations and translational dimension

1. Description of the translations. In any lattice L we shall denote the elementary translations defined by $a \in L$ by $\varrho_a: x \to x \wedge a$ and $\sigma_a: x \to x \vee a$. A translation is then the composition of finitely such mappings. We shall write all translations

as operators on the right. (The word translation is used by Szász with a different meaning (cf. [7]); except for the identity, we follow the terminology used e.g. in [2]).

Since $\varrho_a\varrho_b=\varrho_{a\wedge b}$, $\sigma_a\sigma_b=\sigma_{a\vee b}$ identically, any finite product of elementary translations can be reduced to one of the following forms:

(A) $\varrho_{c_1}\sigma_{c_2}\varrho_{c_3}\sigma_{c_4}\cdots\varrho_{c_{n-1}}\sigma_{c_n}$, $n\geqq 1$, n even;

(B) $\varrho_{c_1}\sigma_{c_2}\varrho_{c_3}\sigma_{c_4}\cdots\varrho_{c_n}$, $n\geqq 1$, n odd;

(C) $\sigma_{c_1}\varrho_{c_2}\sigma_{c_3}\varrho_{c_4}\cdots\sigma_{c_{n-1}}\varrho_{c_n}$, $n\geqq 1$, n even;

(D) $\sigma_{c_1}\varrho_{c_2}\sigma_{c_3}\varrho_{c_4}\cdots\sigma_{c_n}$, $n\geqq 1$, n odd.

Such reduction is unique, although the translation itself can be written in possibly more than one of the forms above.

For any positive integer n, we denote by $R_n(L)$ the set of all translations which can be expressed under the form (A) or (B) (with the same n); $S_n(L)$ is defined similarly with the forms (C), (D). We set $T_n(L)=R_n(L)\cup S_n(L)$, so that the set $T(L)$ of all translations of L is the union of all $T_n(L)$. We shall write these sets R_n, S_n, T_n, T if there is only one lattice under consideration.

It is convenient to call *order* of a translation τ the smallest integer n such that $\tau\in T_n$. It is also the smallest integer p such that τ can be written as a product of p elementary translations. For example, for any $a\in L$, the constant mapping $\varkappa_a: x\to a$ can be written as $\varkappa_a=\varrho_a\sigma_a=\sigma_a\varrho_a$ (absorption law) and is a translation of order 2 unless a is a greatest, or least, element of L.

The translations of L are order-preserving. They are also bounded, except possibly for the elementary translations: if for instance $\tau=\tau'\varrho_c\sigma_d$, where $\tau'\in T$ or is the identity, then $x\tau=(x\tau'\wedge c)\vee d$ and $d\leqq x\tau\leqq c\vee d$ for all $x\in L$. On the other hand, the translations of R_1 (for example) are bounded above, but are bounded below if and only if L has a least element.

PROPOSITION 1. 1. *For any $n\geqq 2$, $R_n\subseteq R_{n+1}$ and $S_n\subseteq S_{n+1}$, so that $T_n\subseteq T_{n+1}$. The corresponding inclusions for $n=1$ hold if and only if L is bounded. In this case $T_n\subseteq R_{n+1}\cap S_{n+1}$ for all n.*

PROOF. Take $\tau\in R_n$, and assume that $n\geqq 2$ or that L is bounded. Then we can find $a, b\in L$ such that $a\leqq x\tau\leqq b$ for all $x\in L$. Now $\tau=\tau\varrho_b=\tau\sigma_a$, and one of $\tau\varrho_b$, $\tau\sigma_a$ is in R_{n+1}. In the case when L is bounded, with least element 0, we also have $\tau=\sigma_0\tau\in S_{n+1}$. Dually, $S_n\subseteq S_{n+1}$ and, if L is bounded, $S_n\subseteq R_{n+1}$. This proves all the inclusions. If conversely $T_1\subseteq T_2$, then every elementary translation of L is bounded, hence L is bounded.

The next results concern distributive lattices.

PROPOSITION 1. 2. *If L is distributive, then $R_2=S_2$, so that $T=T_1\cup T_2$.*

PROOF. The identity $(x\wedge a)\vee b=(x\vee b)\wedge(a\vee b)$ shows that $R_2\subseteq S_2$; dually $S_2\subseteq R_2$. From this follows inductively that $T_n\subseteq T_2$ for all $n\geqq 2$; indeed it holds for $n=2$ and, if $T_p\subseteq T_2$, where $p\geqq 2$, then $R_{p+1}=R_1S_p\subseteq R_1R_2\subseteq R_2$ and dually $S_{p+1}\subseteq T_2$. Therefore $T=T_1\cup T_2$.

PROPOSITION 1. 3. *If L is distributive, then* Card $T_2=$ Card $\mathscr{R}$, *where* $\mathscr{R}=\{(a, b);\ a, b\in L,\ a\leqq b\}$.

PROOF. (Observe that $\mathscr{R}$ is another notation for $\leqq$.) Consider the mapping $\sigma: \mathscr{R} \to T_2$ defined by $x\sigma_{a,b} = (x \vee a) \wedge b$. If $(x \vee a) \wedge b = (x \vee c) \wedge d$ for all $x \in L$, where $a \leqq b$, $c \leqq d$, then $a = c$ (let $x = a \wedge c$) and $b = d$ (let $x = b \vee d$); therefore σ is one-to-one. If conversely $\tau \in T_2$, then $\tau = \varrho_c \sigma_d$ for some $c, d \in L$ by 1. 2 and, for all $x \in L$:

$$x\tau = (x \wedge c) \vee d = (x \vee d) \wedge (c \vee d) = x\sigma_{d, c \vee d};$$

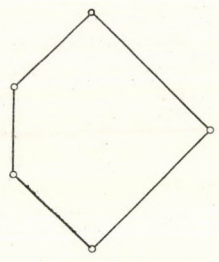
therefore σ is onto. This completes the proof.

This allows to calculate Card T; if L is bounded, then Card $T =$ Card T_2 by 1. 1 and 1. 2; if L is not bounded, then the number of non-bounded elementary translations is readily found and added to Card T_2 to give Card T. For example, let L be a finite Boolean lattice with n atoms (and 2^n elements); an easy combinatorial reasoning shows that $T(L)$ has 3^n elements.

2. Translational dimension. If a lattice L is such that $T(L) = T_1(L) \cup T_n(L)$ for some n, then the smallest such integer is, by definition, the *translational dimension* (tr. dim) of L. If no such n exists, we define tr. dim $L = \infty$.

Equivalently, tr. dim L is the largest of, or l.u.b. of, all the orders of the translations of L. If it is finite it also the smallest integer n such that any translation of L can be written as a product of at most n elementary translations.

For example a distributive lattice has tr. dim 1 or 2, by 1. 2. Any finite lattice has finite tr. dim. We present below some other examples, for which we have



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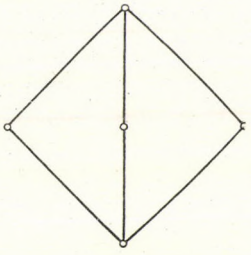

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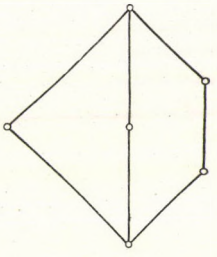

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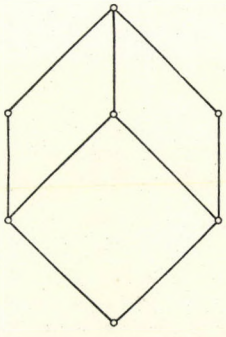

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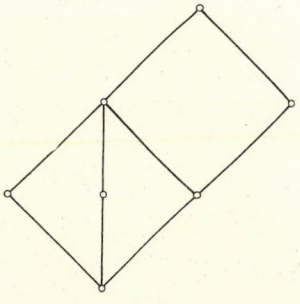

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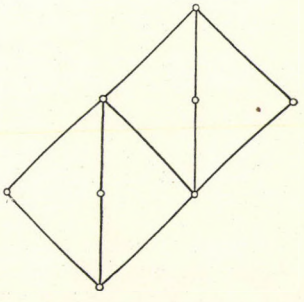

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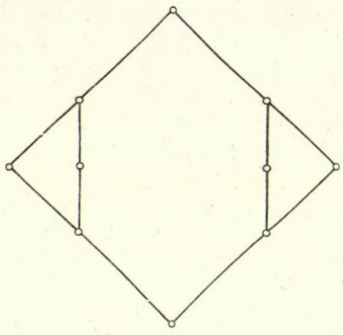

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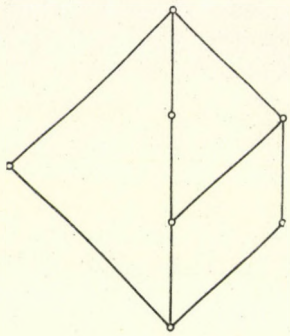


8

been able to calculate all the translations as well as the tr. dim. These tedious, sometimes long calculations cannot be reproduced here.

EXAMPLE 2. 1 (the non-modular 5 element lattice). Card $T=18$, tr. dim $=3$.

EXAMPLE 2. 2 (the modular, non-distributive 5 element lattice). Simple, 3 generators. Card $T=42$, tr. dim $=4$.

EXAMPLE 2. 3. Non-modular, contains 2. 2 as a sublattice. Card $T=64$, tr. dim $=4$.

EXAMPLE 2. 4. Semimodular but not modular. Card $T=42$, tr. dim $=4$. Also $R_4 \subset S_4$.

EXAMPLE 2. 5. Modular, non-distributive, 3 generators. Card $T=62$, tr. dim $=4$.

EXAMPLE 2. 6. Modular, non-distributive, 4 generators, simple. Card $T=136$, tr. dim $=6$. This example was suggested by Dr. VARLET.

EXAMPLE 2.7. Non-modular, does not contain 2. 2 as a sublattice. Card $T=79$, tr. dim $=3$.

EXAMPLE 2. 8. Non-modular, simple. Card $T=83$, tr. dim $=7$.

EXAMPLE 2. 9. Same as 2. 2 but for more than 3 elements of height 1. An explicit description of the translations can be given even if the lattice is infinite. Yet tr. dim $=4$.

EXAMPLE 2. 10. The free modular lattice with three generators, which is described in [1]. Card $T=403$, tr. dim $=4$.

3. Some properties of the translational dimension. PROPOSITION 3. 1. (a) tr. dim $L \leqq n$ *if and only if* $T_{n+1}(L) \subseteqq T_n(L)$; (b) *if* $R_n(L)=S_n(L)$, *then* tr. dim $L \leqq n$; (c) *if* tr. dim $L \leqq n$ *and if* L *is bounded, then* $R_{n+1}(L)=S_{n+1}(L)$.

PROOF. Take a lattice L and suppose first that tr. dim $L=m \leqq n$. Then $T_{n+1} \subseteqq \subseteqq T=T_1 \cup T_m \subseteqq T_1 \cup T_n$ by 1. 1. If L is bounded or if $n=1$, then $T_{n+1} \subseteqq T_n$. Otherwise, the translations of T_1 which are bounded are in T_2 hence in T_n, the others are not in T_{n+1}, so that again $T_{n+1} \subseteqq T_n$. If conversely $T_{n+1} \subseteqq T_n$, then $T_{n+p} \subseteqq T_n$ for all p, by induction; indeed it holds for $p=1$ and, if $T_{n+p} \subseteqq T_n$, then $T_{n+p+1} \subseteqq \subseteqq T_{n+p}T_1 \subseteqq T_nT_1 \subseteqq T_{n+1} \subseteqq T_n$. Therefore $T=T_1 \cup T_n$ and tr. dim $L \leqq n$, which completes the proof of (a).

To prove (b), assume that $R_n=S_n$. Then $R_{n+1}=R_1S_n=R_1R_n \subseteqq R_n$ and dually $S_{n+1} \subseteqq S_n$. Therefore $T_{n+1} \subseteqq T_n$ and tr. dim $L \leqq n$ by (a).

Finally, if L is bounded and if tr. dim $L \leqq n$, then by (a) $T_{n+1} \subseteqq T_n$ and by 1. 1:

$$R_{n+1} \subseteqq T_n \subseteqq S_{n+1} \subseteqq T_n \subseteqq R_{n+1}$$

which establishes (c) and completes the proof.

Let L be a bounded lattice such that tr. dim $L = m \neq \infty$. By 3. 1 we see that $R_n = S_n$ does not hold for $n < m$, but does for $n > m$; it may or may not hold for $n = m$. Generally we shall say that a lattice L is *transitionally regular* if tr. dim $L = n \neq \infty$ with $R_n = S_n$. Distributive lattices (with more than 2 elements) are transitionally regular by 1. 2, but most lattices are not. For instance $R_n \not\subseteqq S_n$ and $S_n \not\subseteqq R_n$ for each example of section 2 except 2. 4 (with $n =$ tr. dim).

PROPOSITION 3. 2. *Let f be a lattice homomorphism of L onto L'. Then* tr. dim $L' \leqq$ tr. dim L.

PROOF. Let tr. dim $L = n \neq \infty$ (there is nothing to prove if tr. dim $L = \infty$). Any $\tau' \in T_{n+1}(L')$ is product of $n+1$ elementary translations defined, say, by $a'_1, \ldots, a'_{n+1} \in L'$. For each i we can find $a_i \in L$ such that $f(a_i) = a'_i$; then the translation $\tau \in T_{n+1}(L)$ similarly defined by $a_1, \ldots, a_{n+1}$ is such that $f(x\tau) = (f(x))\tau'$ for all $x \in L$. But $T_{n+1}(L) \subseteqq T_n(L)$ by 3. 1, so that τ is product of n elementary translations defined, say, by $b_1, \ldots, b_n \in L$. Since f is onto, τ' may now be written as product of n elementary translations defined by $f(b_1), \ldots, f(b_n)$. Therefore $\tau' \in T_n(L')$. This shows that $T_{n+1}(L') \subseteqq T_n(L')$ and tr. dim $L' \leqq n =$ tr. dim L by 3. 1.

PROPOSITION 3. 3. *Let $(L_\iota)_{\iota \in I}$ be a family of lattices, L the direct product of the L_ι and $n = \sup_{\iota \in I}$ tr. dim L_ι. Then* tr. dim $L \geqq n$.

PROOF. This follows from 3. 2.

It may happen that tr. dim $L > n$; take for instance L_1, L_2 with two elements; then tr. dim $L_1 =$ tr. dim $L_2 = 1$; yet tr. dim $L_1 \times L_2 = 2$. We can describe this phenomenon with more precision if all the L_ι are bounded.

PROPOSITION 3. 4. *Let $(L_\iota)_{\iota \in I}$ be a family of bounded lattices, L the direct product of the L_ι and $n = \sup_{\iota \in I}$ tr. dim L_ι. Assume that $n \neq \infty$. Then:*

(a) $n \leqq$ tr. dim $L \leqq n+1$;

(b) *if* tr. dim $L = n+1$, *then L is transitionally regular*;

(c) tr. dim $L = n$ *if and only if*

(i) $R_n(L_\iota) \not\subseteqq S_n(L_\iota) \Rightarrow S_n(L_{\iota'}) \subseteqq R_n(L_{\iota'})$ *for all* $\iota' \neq \iota$;

(ii) $S_n(L_\iota) \not\subseteqq R_n(L_\iota) \Rightarrow R_n(L_{\iota'}) \subseteqq S_n(L_{\iota'})$ *for all* $\iota' \neq \iota$.

PROOF. It rests on the description of $T(L)$. If we choose $\tau_\iota \in R_p(L_\iota)$ for every ι, then the direct product $\tau: L \to L$ of the τ_ι is in $R_p(L)$; conversely every translation in $R_p(L)$ can be obtained in this fashion. Dually $S_p(L)$ has a similar description.

Now $R_{n+1}(L_\iota) = S_{n+1}(L_\iota)$ for all ι by 3. 1, since every L_ι is bounded. From this follows that $R_{n+1}(L) = S_{n+1}(L)$. By 3. 1 again, tr. dim $L \leqq n+1$ and (a) follows by 3. 3; and if tr. dim $L = n+1$, then L is transitionally regular, which is (b).

Assume now that tr. dim $L = n$ and take $\tau_\iota \in R_n(L_\iota) - S_n(L_\iota)$. For every $\iota' \neq \iota$, choose any $\tau_{\iota'} \in S_n(L_{\iota'})$. As each $\tau_{\iota'}$ is in $R_{n+1}(L_{\iota'})$ their direct product τ is in $R_{n+1}(L)$, hence either in $R_n(L)$ or in $S_n(L)$. But $\tau \in S_n(L)$ contradicts the choice

of τ_ι, so that $\tau \in R_n(L)$ and $\tau_{\iota'} \in R_n(L_{\iota'})$ for all $\iota' \neq \iota$. Therefore (i) holds. Dually (ii) holds.

Assume conversely that (i) holds, and take $\tau \in T_{n+1}(L)$ direct product of the $\tau_\iota \in T_{n+1}(L_\iota)$. By 3. 1, $\tau_\iota \in T_n(L_\iota)$ for all ι. If $\tau_\iota \in S_n(L_\iota)$ for all ι, then $\tau \in S_n(L)$. If $\tau_\iota \notin S_n(L_\iota)$ for some ι, then by (i) $\tau_{\iota'} \in R_n(L_{\iota'})$ for all $\iota' \neq \iota$ and since $\tau_\iota \in R_n(L_\iota)$ we conclude that $\tau \in R_n(L)$. Therefore $T_{n+1}(L) \subseteq T_n(L)$ and tr. dim $L \leqq n$ by 3. 1. This completes the proof.

Incidentally we have shown that (i) implies (ii); since dually (ii) implies (i), these two conditions are equivalent. As a further improvement, notice that $R_n(L_\iota) = S_n(L_\iota)$ holds for all ι such that tr. dim $L_\iota < n$; in conditions (i), (ii) we may thus consider only the values of ι for which tr. dim $L_\iota = n$.

If every L_ι is transitionally regular, then (i) holds and by 3. 4 tr. dim $L = n$; furthermore the description of $R_n(L)$ and $S_n(L)$ shows that L is also transitionally regular.

4. Small values of the translational dimension. PROPOSITION 4. 1. *For any lattice* L, tr. dim $L = 1$ *if and only if* L *has one or two elements.*

PROOF. If L has more than two elements, then there is a constant translation which is not elementary; hence tr. dim $L \geqq 2$. Conversely, it is readily verified that, if L has one or two elements, every translation of L is elementary.

THEOREM 4. 2. *A lattice* L *is distributive if and only if* tr. dim $L \leqq 2$.

PROOF. If L is distributive, then tr. dim $L \leqq 2$ by 1. 2. Conversely, assume that L is not distributive. Then L contains a five element non-distributive sublattice.

If L contains a sublattice $\{u, a, b, c, v\}$, where $a \wedge b = a \wedge c = u$, $a \vee b = a \vee c = v$, $b < c$, then the translation $x \to ((x \wedge a) \vee b) \wedge c$ has order 3. Indeed, assume that

$$((x \wedge a) \vee b) \wedge c = (x \wedge d) \vee e$$

for all $x \in L$. For $x = b \wedge d \wedge e$ we obtain $b = b \wedge c = e$. For $x = a$ we obtain

$$c = (a \vee b) \wedge c = (a \wedge d) \vee e = (a \wedge d) \vee b.$$

Therefore $a \wedge d \leqq c$. Since $a \wedge d \leqq a$, also $a \wedge d \leqq u < b$ and $c = b$, a contradiction.

Assume that

$$((x \wedge a) \vee b) \wedge c = (x \vee d) \wedge e$$

for all $x \in L$. For $x = a \vee e$ we obtain $c = (a \vee b) \wedge c = e$. For $x = c$ we thus obtain

$$b = ((c \wedge a) \vee b) \wedge c = (c \vee d) \wedge e = (c \vee d) \wedge c = c,$$

another contradiction. Therefore our translation has order 3. (Incidentally we have proved that example 2. 1 has tr. dim at least 3.)

Similarly, if L contains a sublattice $\{u, a, b, c, v\}$, where $a \wedge b = b \wedge c = c \wedge a = u$, $a \vee b = b \vee c = c \vee a = v$, then the translation $x \to ((x \wedge a) \vee b) \wedge c$ has order 3. Indeed, assume that

$$((x \wedge a) \vee b) \wedge c = (x \wedge d) \vee e$$

for all $x \in L$. For $x = b \wedge e$ we obtain $u = b \wedge c = e$. Now $x = a$ yields:

$$c = (a \vee b) \wedge c = (a \wedge d) \vee e = (a \wedge d) \vee u \leqq a,$$

which is impossible.

Similarly, assume that

$$((x \wedge a) \vee b) \wedge c = (x \vee d) \wedge e$$

for all $x \in L$. For $x = a \vee e$ we obtain $c = (a \vee b) \wedge c = e$. For $x = c$ we obtain

$$u = ((c \wedge a) \vee b) \wedge c = (c \vee d) \wedge e = (c \vee d) \wedge c = c$$

which is again impossible. (Incidentally we have proved that example 2. 2 has tr. dim at least 3.)

In either case L has a translation of order 3, which completes the proof.

THEOREM 4. 3. *If* tr. dim $L \leqq 3$ *and if* L *is modular, then* L *is distributive.*

PROOF. If L is modular but not distributive, then L contains a sublattice $\{u, a, b, c, v\}$, where $a \wedge b = b \wedge c = c \wedge a = u$, $a \vee b = b \vee c = c \vee a = v$. We shall prove that the translation $\tau: x \to (((x \wedge a) \vee b) \wedge c) \vee a$ has order 4 (incidentally showing that example 2. 2 has tr. dim at least 4). This will follow from six lemmas in which we keep the previous assumptions except possibly that L is modular.

LEMMA 4. 4. *It is not possible that, for all* $x \in L$:

$$(((x \wedge a) \vee b) \wedge c) \vee a = ((x \wedge d) \vee e) \wedge f.$$

PROOF. If such identity holds for all $x \in L$, then for $x = b \wedge e$ we obtain $a = (b \wedge c) \vee a = e \wedge f$ and for $x = a$ we obtain

$$v = ((a \vee b) \wedge c) \vee a = ((a \wedge d) \vee e) \wedge f.$$

Therefore $v \leqq f$ and $v \leqq (a \wedge d) \vee e \leqq a \vee e = e$. Hence $v \leqq e \wedge f = a$, which is not possible.

In the following five lemmas, we assume that

$$x\tau = (((x \wedge a) \vee b) \wedge c) \vee a = ((x \vee d) \wedge e) \vee f \tag{1}$$

holds for all $x \in L$. We want an eventual contradiction.

LEMMA 4. 5. *One may assume that* $a \leqq f$.

PROOF. Since $x\tau \geqq a$ for all $x \in L$, (1) remains true if σ_a is applied to both sides. Hence we may replace f by $f \vee a \geqq a$.

In the next four lemmas we shall assume that $a \leqq f$. Then:

LEMMA 4. 6. *The following holds:* $a = f$, $d \wedge e \leqq a$, $a \vee e = v \leqq a \vee d$.

PROOF. If $x = b \wedge d$, (1) reads: $a = (b \wedge c) \vee a = (d \wedge e) \vee f$. Then $f \leqq a$ so that $f = a$; also $d \wedge e \leqq a$. For $x = a \vee e$, (1) gives $v = ((a \vee b) \wedge c) \vee a = e \vee f = a \vee e$. Finally (1) also gives, if $x = a$:

$$v = ((a \vee b) \wedge c) \vee a = ((a \vee d) \wedge e) \vee f \leqq a \vee d \vee f = a \vee d.$$

LEMMA 4. 7. $b \wedge e \leqq u$.

PROOF. For $x = b$, (1) yields

$$a = (((b \wedge a) \vee b) \wedge c) \vee a = ((b \vee d) \wedge e) \vee f.$$

Therefore $b \wedge e \leqq (b \vee d) \wedge e \leqq a$. Since also $b \wedge e \leqq b$, we have $b \wedge e \leqq u$.

The two last lemmas use modularity of L; for, if L is modular then $x \leqq y$, $x \vee z = y \vee z$, $x \wedge z = y \wedge z$ implies $x = y$ (and conversely; see [1]).

LEMMA 4. 8. *If L is modular:* $(e \wedge a) \vee b = v$.

PROOF. For $x = e$, (1) gives

$$(((e \wedge a) \vee b) \wedge c) \vee a = ((e \vee d) \wedge e) \vee f = e \vee a = v = c \vee a.$$

But also

$$u = (b \wedge c) \wedge a \leqq (((e \wedge a) \vee b) \wedge c) \wedge a \leqq c \wedge a = u$$

so that $(((e \wedge a) \vee b) \wedge c) \wedge a = c \wedge a$. By modularity, $((e \wedge a) \vee b) \wedge c = c$ so that $c \leqq (e \wedge a) \vee b$. Finally

$$v = b \vee c \leqq (e \wedge a) \vee b \leqq a \vee b = v.$$

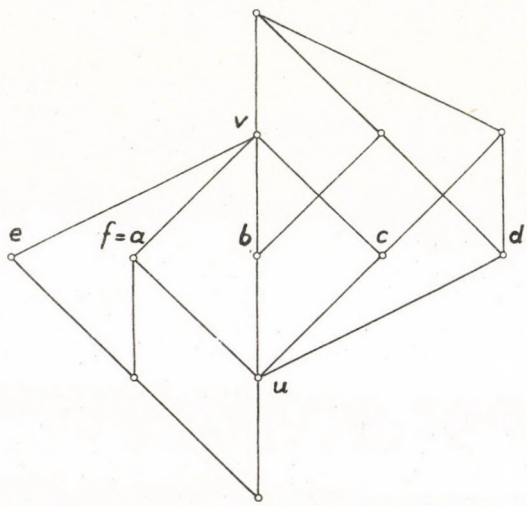
LEMMA 4. 9. *If L is modular, $e \leqq a$.*

PROOF. By 4. 8, $(e \wedge a) \vee b = v$. Since also $e \leqq v$ by 4. 6, we have $v = (e \wedge a) \vee b \leqq$ $\leqq e \vee b \leqq v$ and $(e \wedge a) \vee b = e \vee b$. Also $(e \wedge a) \wedge b = e \wedge b$ since $e \wedge b \leqq a$ by 4. 7. By modularity $e \wedge a = e$, and $e \leqq a$.

We are now in position to conclude the proof of the theorem: by 4. 6 and 4. 9, $v = a \vee e = a$, which is impossible.

Observe that, if L is not modular, the equality (1) may hold for all $x \in L$ with a suitable choice of d, e, f. This is the case in the following lattice:

EXAMPLE 4.10:



COROLLARY 4. 11. *If L is modular, then* tr. dim $L \neq 3$.

PROOF. If L is modular with tr. dim $L=3$, then L is distributive, whence tr. dim $L\leqq 2$, a contradiction.

On the other hand, examples 2. 1 and 2. 7 have tr. dim 3.

THEOREM 4. 12. *A modular lattice L generated by* 3 *elements has translational dimension* 2 *or* 4.

PROOF. Since L has at least 3 elements, tr. dim $L\geqq 2$ by 4. 1. Also L is homomorphic image of the free modular lattice on three generators, given as example 2. 10, whose tr. dim is 4; therefore tr. dim $L\leqq 4$ by 3. 2. Finally tr. dim $L\neq 3$ by 4. 11.

II. Congruences

The congruences on an arbitrary lattice L are the equivalence relations on L which admit all translations. It is necessary that we recall at this point some classical results concerning equivalence relations which admit a given semigroup of mappings T; we may call them T-congruences. First the largest T-congruence $\mathscr{F}_S$ for which a given subset S is a union of classes is given by: $x\mathscr{F}_S y$ if and only if x, y either are both in S or both not in S, and, for any $\tau\in T$, $x\tau\in S\Leftrightarrow y\tau\in S$. Consequently S is a class of some T-congruence if and only if, for any $\tau\in T$, $S\tau\cap S\neq\emptyset$ implies $S\tau\subseteq S$; then $\mathscr{F}_S$ is the largest T-congruence with S as a class. (It is known also the description of the smallest such congruence.)

These results were first established for semigroups [8] and can be used for the congruences of a tas [5] or of any universal algebra — in particular of any lattice L. In the latter case we take of course $T=T(L)$; since T contains all the elementary translations it is readily seen that the description of $\mathscr{F}_S$ reduces to: $x\mathscr{F}_S y$ if and only if, for any $\tau\in T$, $x\tau\in S\Leftrightarrow y\tau\in S$.

In the following sections we investigate these general results a little more closely. We also assume that L is bounded, though some of the results would hold without the existence of a greatest element.

5. Characterization of the kernels. If I is an ideal of the bounded lattice L, then $I\tau\cap I\neq\emptyset$ is equivalent to $0\tau\in I$, for any $\tau\in T$. If indeed $0\tau\in I$, then $0\tau\in I\tau\cap I$; if conversely $x\tau\in I$ for some $x\in I$, then $0\tau\leqq x\tau$ is also in I. Therefore

PROPOSITION 5. 1. *An ideal I of L is a kernel if and only if it satisfies*

(K) *for any* $\tau\in T$, $0\tau\in I$ *implies* $I\tau\subseteq I$.

This condition (K) decomposes into the infinitely many following conditions:

(K_n) for any $\tau\in R_n$, $0\tau\in I$ implies $I\tau\subseteq I$;

(K'_n) for any $\tau\in S_n$, $0\tau\in I$ implies $I\tau\subseteq I$.

We shall study these conditions.

LEMMA 5. 2. *For any $n\geqq 2$, (K_n) implies (K_{n-1}) and (K'_{n-1}); similarly (K'_n) implies (K_{n-1}) and (K'_{n-1}).*

PROOF. This follows from $T_{n-1}\subseteq R_n\cap S_n$ (cf. 1. 1).

LEMMA 5. 3. *For any* n, (K'_n) *implies* (K_{n+1}).

PROOF. Let $\tau \in R_{n+1}$ be such that $0\tau \in I$. Then $\tau = \varrho_c \tau'$ for some $\tau' \in S_n$ and $0\tau' = 0\varrho_c\tau' = 0\tau \in I$. If I satisfies (K'_n), then $x\tau' \in I$ for all $x \in I$, and also $x\tau = x\varrho_c\tau' \in I$ since $x\varrho_c \leqq x$ is also in I. Therefore I satisfies (K_{n+1}).

LEMMA 5. 4. *For any* p, (K'_{2p}) *implies* (K'_{2p+1}).

PROOF. Let $\tau \in S_{2p+1}$ be such that $0\tau \in I$. Then $\tau = \tau'\sigma_c$ for some $\tau' \in S_{2p}$ and $0\tau' \in I$, $c \in I$ since $0\tau' \vee c = 0\tau \in I$. If I satisfies (K'_{2p}), then, for any $x \in I$, $x\tau' \in I$ and $x\tau = x\tau' \vee c \in I$. Therefore I satisfies (K'_{2p+1}).

LEMMA 5. 5. *The condition* (K'_1) *is trivial.*

PROOF. For any ideal I, $c = 0\sigma_c \in I$ implies $x\sigma_c = x \vee c \in I$ for any $x \in I$. Hence I satisfies (K'_1).

Incidentally this lemma shows that any ideal is a kernel when L is distributive; for then (K) is equivalent to (K_2) by 1. 2 and also to (K'_1) by 5. 2, 5. 3, so that any ideal satisfies (K). More generally the kernels can be characterized by a single condition when tr. dim L is finite. For example:

COROLLARY 5. 6. *If* tr. dim $L \leqq 5$, *an ideal* I *of* L *is a kernel if and only if:*

$$x \in I \quad \text{and} \quad ((a \wedge b) \vee c) \wedge d \in I \quad \text{implies} \quad (((x \vee a) \wedge b) \vee c) \wedge d \in I.$$

PROOF. In this case we have only to consider (K_n) and (K'_n) for $n \leqq 5$. By 5. 2, (K'_4) implies these conditions when $n \leqq 3$; it also implies (K_5) by 5. 3, (K'_5) by 5. 4 and now (K_4) by 5. 2. Thus (K) is equivalent to (K'_4), which is precisely the condition of 5. 6.

In the general case, the kernels are characterized in a similar fashion by all the (K'_{2p}); eventually any infinite set of (K_n), (K'_n) characterizes them as well.

6. Largest congruence with a given kernel. We go into the same details for the slightly more general task of describing $\mathscr{F}_I$ for any ideal I of the bounded lattice L. As we know, $x \mathscr{F}_I y$ if and only if $x\tau \in I \Leftrightarrow y\tau \in I$ for any $\tau \in T$. Hence $\mathscr{F}_I$ is the intersection of infinitely many equivalence relations, namely:

$$\mathscr{F}_I^n: \quad \text{for any} \quad \tau \in R_n, \quad x\tau \in I \Leftrightarrow y\tau \in I;$$

$$\mathscr{F}_I'^n: \quad \text{for any} \quad \tau \in S_n, \quad x\tau \in I \Leftrightarrow y\tau \in I.$$

LEMMA 6. 1. *For any* $n \geqq 2$, $\mathscr{F}_I^n \subseteqq \mathscr{F}_I^{n-1} \cap \mathscr{F}_I'^{n-1}$ *and* $\mathscr{F}_I'^n \subseteqq \mathscr{F}_I^{n-1} \cap \mathscr{F}_I'^{n-1}$.

PROOF. This follows from 1. 1.

LEMMA 6. 2. *For any* p, $\mathscr{F}_I^{2p+1} \subseteqq \mathscr{F}_I^{2p+2}$, $\mathscr{F}_I'^{2p} \subseteqq \mathscr{F}_I'^{2p+1}$.

PROOF. Assume that $x \mathscr{F}_I^{2p+1} y$ $(x \mathscr{F}_I'^{2p} y)$ and that $x\tau \in I$ for some $\tau \in R_{2p+2}$ (S_{2p+1}). Then $\tau = \tau'\sigma_c$ for some $\tau' \in R_{2p+1}$ (S_{2p}) and $x\tau' \vee c = x\tau \in I$. Hence $x\tau' \in I$, $c \in I$, $y\tau' \in I$ by the assumption on x, y and $y\tau = y\tau' \vee c \in I$. Similarly $y\tau \in I$ implies $x\tau \in I$.

COROLLARY 6. 3. *For any ideal* I, $\mathscr{F}_I = \bigcap_{p \geqq 1} \mathscr{F}_I'^{2p}$.

PROOF. By 6. 1, 6. 2, $\mathscr{F}_I'^{2p+2} \subseteq \mathscr{F}_I^{2p+1} \subseteq \mathscr{F}_I'^{2p} \subseteq \mathscr{F}_I'^{2p+1}$, $\mathscr{F}_I'^{2p+2} \subseteq \mathscr{F}_I^{2p+1} \subseteq$ $\subseteq \mathscr{F}_I^{2p}$. (From 6. 3 follows that in fact any intersection of infinitely many $\mathscr{F}_I^n$, $\mathscr{F}_I'^n$ is also equal to $\mathscr{F}_I$.)

COROLLARY 6. 4. *If* tr. dim $L \leqq 3$, *then* $\mathscr{F}_I = \mathscr{F}_I^3$.

PROOF. From 3. 1 follows that $T = R_n = S_n$ for all $n > 3$. Hence $\mathscr{F}_I = \mathscr{F}_I^n = \mathscr{F}_I'^n$ for all $n > 3$. Then $\mathscr{F}_I = \mathscr{F}_I'^4$ by 6. 1, 6. 3. But $\mathscr{F}_I^3 \subseteq \mathscr{F}_I^4 = \mathscr{F}_I'^4 \subseteq \mathscr{F}_I^3$ by 6. 1, 6. 2. Therefore $\mathscr{F}_I = \mathscr{F}_I^3$.

COROLLARY 6. 5. *If* tr. dim $L \leqq 4$, *then* $\mathscr{F}_I = \mathscr{F}_I'^4$.

PROOF. As above, $\mathscr{F}_I'^4 \subseteq \mathscr{F}_I'^{2p}$ for all p, and the result follows from 6. 3.

Similarly it can be proved that, if tr. dim $L \leqq n$, then $\mathscr{F}_I = \mathscr{F}_I^n$ if n is odd and $\mathscr{F}_I = \mathscr{F}_I'^n$ if n is even.

In some cases the description of $\mathscr{F}_I$ can be simplified without such an assumption. For example $\mathscr{F}_{\{0\}} = \mathscr{F}_{\{0\}}^1$ if L is weakly complemented (then both are the equality) or 0-distributive [9]. Another condition (on I) is given in the next section.

7. The condition (K^*). It reads as follows:

$$(K^*) \qquad a \wedge c \in I,\ b \wedge c \in I \quad \text{implies} \quad (a \vee b) \wedge c \in I.$$

At first inspection, it seems to have no relation with neutrality or p-neutrality [6], nor with standardness [4]. That it implies (K'_{2p}) for all p (and thus (K)) can be easily proved by induction, but will follow from 7. 2.

To study the ideals satisfying (K^*), we use the residuals. For any I of L and any $c \in L$, define

$$I_c = \{x \in L;\ x \wedge c \in I\}.$$

It is clear that I satisfies (K^*) if and only if I_c is an ideal for all $c \in L$.

PROPOSITION 7. 1. *Every intersection of prime ideals satisfies* (K^*), *and the converse holds under the descending chain condition.*

PROOF. If $I = \bigcap_{\lambda \in \Lambda} P_\lambda$, where each P_λ is a prime ideal, and if $a \wedge c \in I$, $b \wedge c \in I$, then, for each $\lambda \in \Lambda$, either $c \in P_\lambda$ or $a, b \in P_\lambda$; in either case $(a \vee b) \wedge c \in P_\lambda$. Therefore I satisfies (K^*).

If conversely, I satisfies (K^*), all we have to show is that for any $a \in L - I$ there exists a prime ideal P such that $I \subseteq P$, $a \notin P$. For $a \in L - I$ the descending chain condition provides an element p of L, minimal with the properties $p \geqq a$, $p \notin I$. Take $P = I_p$. Then I_p is an ideal by (K^*), $I \subseteq I_p$; $a \notin I_p$ since $a \wedge p = p \notin I$. To show that I_p is prime, suppose that $x \wedge y \in I_p$, or $x \wedge y \wedge p \in I$. If $y \notin I_p$, then $y \wedge p \notin I$ and $y \wedge p = p$ by minimality of p; hence $x \wedge p = x \wedge y \wedge p \in I$ and $x \in I_p$. This completes the proof.

PROPOSITION 7. 2. *If* I *satisfies* (K^*), *then* $\mathscr{F}_I = \mathscr{F}_I^1$ *and* $a \mathscr{F}_I b$ *if and only if* $I_a = I_b$.

PROOF. By definition $a \mathscr{F}_I^1 b$ if and only if $I_a = I_b$; also $\mathscr{F}_I \subseteq \mathscr{F}_I^1$. Conversely, the binary relation $I_a = I_b$ is an equivalence relation. To show that it is a congruence, observe that $I_{a \wedge c} = (I_a)_c$ for all $a, c \in L$; also $I_{a \vee c} = I_a \cap I_c$ for all $a, c \in L$ by (K^*); for

$$(a \vee c) \wedge x \in I \Leftrightarrow a \wedge x \in I \quad \text{and} \quad c \wedge x \in I.$$

From these two formulae it follows that $I_a = I_b$ is a congruence, i.e. $\mathscr{F}_I^1$ is a congruence. But $I_a = I_b$ and $a \in I$ implies $I_a = L = I_b$ and $b \in I$, so that I is a union of classes of $\mathscr{F}_I^1$. Therefore $\mathscr{F}_I^1 \subseteq \mathscr{F}_I$, which completes the proof.

PROPOSITION 7. 3. *(K*) implies (K).*

PROOF. We have to show that an ideal I which satisfies (K*) is kernel of some congruence. It suffices to show that it is contained in a single class of $\mathscr{F}_I$. But, for any $a, b \in I$, $I_a = L = I_b$ and the result follows from 7. 2.

It is clear that any ideal of a distributive lattice satisfies (K*); therefore 7. 1, 7. 2, 7. 3 extend well-known properties of distributive lattices.

8. Second characterization of the kernels. THEOREM 8. 1. *Let L be a bounded lattice satisfying the descending chain condition. Then a principal ideal $(a]$ of L is not a kernel if and only if there exists a $b \in L$ and $\tau \in T$ such that $a \prec b$, $0\tau = a$, $a\tau = b$.*

REMARK. If, furthermore, L satisfies the ascending chain condition, then any ideal of L is principal and 8. 1 can be applied to any ideal.

PROOF. If an ideal $I = (a]$ satisfies this condition, and if $\mathscr{C}$ is a congruence on L such that $0 \mathscr{C} a$, then $a \mathscr{C} b$ and I is not the kernel of $\mathscr{C}$. Therefore I is not a kernel.

Assume, conversely, that I is not a kernel so that $0\tau \in I$, $x\tau \notin I$ for some $\tau \in T$, $x \in I$ (or else, (K) would hold). Since $I = (a]$, $0\tau \leqq a$, $x\tau \not\leqq a$ for some $x \leqq a$. Then $\tau' = \tau\sigma_a \in T$ is such that $0\tau' = 0\tau \vee a = a$, $a < x\tau \vee a = x\tau' \leqq a\tau'$. By the descending chain condition there exists $b \in L$ such that $a \prec b \leqq a\tau'$. Then $\tau'' = \tau' \varrho_b$ is such that $0\tau'' = 0\tau' \wedge b = a$, $a\tau'' = a\tau' \wedge b = b$. This completes the proof.

The negative condition in 8. 1 is satisfied if and only if one of the following conditions holds:

$$(N_n) \qquad (\exists b \in L)(\exists \tau \in R_n) \quad a \prec b, \quad 0\tau = a, \quad a\tau = b;$$

$$(N'_n) \qquad (\exists b \in L)(\exists \tau \in S_n) \quad a \prec b, \quad 0\tau = a, \quad a\tau = b.$$

LEMMA 8. 2. *For each n, (N_n) implies (N_{n+1}) and (N'_{n+1}), and (N'_n) implies (N_{n+1}) and (N'_{n+1}).*

PROOF. This follows from 1. 1.

LEMMA 8. 3. *For each p, (N_{2p+1}) implies (N'_{2p}).*

PROOF. Assume that $a \prec b$, $0\tau = a$, $a\tau = b$ for some $\tau \in R_{2p+1}$. Then $\tau = \varrho_c \tau'$ where $\tau' \in S_{2p}$, so that $0\tau' = 0\varrho_c\tau' = 0\tau = a$, $b = (a \wedge c)\tau' \leqq a\tau'$. Finally $\tau'' = \tau' \varrho_b \in S_{2p}$ is such that $0\tau'' = 0\tau' \wedge b = a$, $a\tau'' = a\tau' \wedge b = b$. Therefore (N'_{2p}) holds.

LEMMA 8. 4. *The conditions (N'_1), (N'_2), (N_1), (N_2), (N_3) are contradictory.*

PROOF. If (N'_2) holds, then $a \prec b$, $(0 \vee c) \wedge d = a$, $(a \vee c) \wedge d = b$ for some $b, c, d \in L$. Then $a \leqq c$ so that $b = (a \vee c) \wedge d = c \wedge d = a$, which is impossible. Therefore (N'_2) is contradictory. Also (N_3) is contradictory since it implies (N'_2) by 8. 3. Finally (N'_1), (N_1), (N_2) imply (N_3) by 8. 2 and are also contradictory.

LEMMA 8. 5. *The conditions (N'_3) and (N_4) are equivalent and equivalent to the following: there exists a $b \in L$ such that $a \prec b$ and there exists either $c \in L$ such that $a \wedge c = b \wedge c$, $a \vee c = b \vee c$, or $c', c'' \in L$ such that $a \vee c' = a \vee c'' = b$, $c' \wedge c'' \leqq a$.*

PROOF. First (N_3') implies (N_4) by 8. 2. Next we prove that (N_4) implies the above condition. If (N_4) holds then

$$a \prec b, \quad (((0 \wedge c) \vee d) \wedge e) \vee f = a, \quad (((a \wedge c) \vee d) \wedge e) \vee f = b$$

for some $b, c, d, e, f \in L$. Then $f \leqq a$; applying σ_a merely replaces f by a, so we may assume $f = a$. Similarly the equations do not change if we replace c by $c \wedge a$, so we may assume $c \leqq a$. The equations now read:

$$a \prec b, \quad c \leqq a, \quad (d \wedge e) \vee a = a, \quad ((c \vee d) \wedge e) \vee a = b.$$

Applying σ_d to the last equation, it gives

$$b \vee d = ((c \vee d) \wedge e) \vee (a \vee d) = a \vee d$$

since $c \leqq a$. If $b \wedge d = a \wedge d$, then the condition in the statement holds. If $b \wedge d \neq a \wedge d$, then $a \wedge d \prec b \wedge d$. In this case $c' = b \wedge d$ is such that $c' \wedge e \leqq d \wedge e \leqq a$, $c' \leqq b$ and $c' \not\equiv a$ (or else $b \wedge d = c' \leqq a \wedge d$ which is a contradiction). Therefore $a \prec a \vee c' \leqq b$ and $a \vee c' = b$. Furthermore $c'' = b \wedge e$ is such that $c' \wedge c'' \leqq c' \wedge e \leqq a$ and

$$b = ((c \vee d) \wedge e) \vee a = ((c \vee d) \wedge e \wedge b) \vee a \leqq c'' \vee a \leqq b$$

i.e. $a \vee c'' = b$, whence the condition in the statement holds again.

If finally that condition holds, then either $a \prec b$, $a \wedge c = b \wedge c$, $a \vee c = b \vee c$ for some $b, c \in L$ and then $((0 \vee c) \wedge b) \vee a = a$, $((a \vee c) \wedge b) \vee a = b$; or $a \prec b$, $a \vee c' = a \vee c'' = b$, $c' \wedge c'' \leqq a$ for some $b, c', c'' \in L$ and then $((0 \vee c') \wedge c'') \vee a = a$, $((a \vee c') \wedge c'') \vee a = b$. In either case (N_3') holds. This completes the proof.

LEMMA 8. 6. *The condition (N_4') is equivalent to the following: there exist $b, p, q, r \in L$ such that $a \prec b$, $a \leqq r$, $q \leqq a \vee p$, $b \leqq q \vee r$, $b \not\equiv (p \wedge q) \vee r$.*

PROOF. Assume that (N_4') holds, so that

$$a \prec b, \quad (((0 \vee c) \wedge d) \vee e) \wedge f = a, \quad (((a \vee c) \wedge d) \vee e) \wedge f = b$$

for some $b, c, d, e, f \in L$. Then $b \leqq f$; applying ϱ_b to these equations merely replaces f by b, so we may assume $f = b$. The equations become:

$$a \prec b, \quad ((c \wedge d) \vee e) \wedge b = a, \quad b \leqq ((a \vee c) \wedge d) \vee e.$$

By the second relation, $a \leqq (c \wedge d) \vee e$, $b \not\equiv (c \wedge d) \vee e$.

Put $r = e \vee a$; then $(c \wedge d) \vee r = (c \wedge d) \vee e$ so that $b \not\equiv (c \wedge d) \vee r$; also $b \leqq$ $\leqq ((a \vee c) \wedge d) \vee r$, $a \leqq r$.

Put $q = (a \vee c) \wedge d$. Then $q \leqq a \vee c$, $b \leqq q \vee r$, $b \not\equiv (c \wedge q) \vee r \leqq (c \wedge d) \vee r$. Hence the condition in 8. 6 holds (with $p = c$).

If, conversely, $a \prec b$, $a \leqq r$, $q \leqq a \vee p$, $b \leqq q \vee r$, $b \not\equiv (p \wedge q) \vee r$ for some $b, p, q, r \in L$, then $a \leqq ((p \wedge q) \vee r) \wedge b \prec b$ so that $(((0 \vee p) \wedge q) \vee r) \wedge b = a$; furthermore $(((a \vee p) \wedge$ $\wedge q) \vee r) \wedge b = (q \vee r) \wedge b = b$. Therefore (N_4') holds. This completes the proof.

REMARK. By the above conditions on b, p, q, r, $r \vee q = r \vee a \vee q \leqq r \vee a \vee p =$ $= r \vee p$, whence $b \leqq (p \vee r) \wedge (q \vee r)$, $a \leqq (p \wedge q) \vee r$, $b \not\equiv (p \wedge q) \vee r$. In particular $(p \wedge q) \vee r \prec (p \vee r) \wedge (q \vee r)$.

Gathering information from the previous lemmas, we obtain the following results.

THEOREM 8. 7. *Let L be a bounded lattice satisfying the descending chain condition and such that* tr. dim $L \leqq 3$. *Then a principal ideal $(a]$ is not a kernel if and only if there exists a $b \in L$ such that $a \prec b$ and there exists either a $c \in L$ such that $c \wedge a = c \wedge b$, $c \vee a = c \vee b$, or $c', c'' \in L$ such that $a \vee c' = a \vee c'' = b$, $c' \wedge c'' \leqq a$.*

PROOF. This condition is sufficient by 8. 5 and 8. 1. If conversely $(a]$ is not a kernel, then one of the conditions (N_n), (N'_n) holds, where we may take $n = 3$ since $T = T_3$. By 8. 4 the condition which holds is necessarily (N'_3) and necessity follows from 8. 5.

THEOREM 8. 8. *Let L be a bounded lattice satisfying the descending chain condition and such that* tr. dim $L \leqq 4$. *Then a principal ideal $(a]$ is not a kernel if and only if there exists $b, p, q, r \in L$ such that $a \prec b$, $a \leqq r$, $q \leqq a \vee p$, $b \leqq q \vee r$, $b \not\leqq (p \wedge q) \vee r$.*

PROOF. This condition is sufficient by 8. 6 and 8. 1. If, conversely, $(a]$ is not a kernel, then since $T = T_4$, one of the conditions (N_4), (N'_4) holds. But (N_4) implies (N'_4) by 8. 5, 8. 2 so that (N'_4) holds in either case. Necessity follows then from 8. 6.

REMARK. If L satisfies also the ascending chain condition, theorems 8. 7, 8. 8 give necessary and sufficient conditions for an arbitrary ideal not to be a kernel.

9. Simple lattices. A lattice L is simple when it has no proper congruence. It is clear that, if L has a 0, then L is simple if and only if L has no proper kernel and $\mathscr{F}_{\{0\}}$ is the equality.

This condition can be refined when L satisfies both chain conditions and tr. dim L is small, using the results of section 8. As for $\mathscr{F}_{\{0\}}$, it will be the equality if $\mathscr{F}^n_{\{0\}}$ or $\mathscr{F}'^n_{\{0\}}$ is for some n. This type of condition can be considered as a loose form of weak complementation; the two following lemmas explain some of them. We still assume both chain conditions.

LEMMA 9. 1. *$\mathscr{F}^3_{\{0\}}$ is the equality if and only if for any $a, b \in L$ such that $a \prec b$ there exist $p, q, r \in L$ such that $0 \prec p$, $q \leqq b$, $a \wedge q \leqq r$, $p \leqq q \vee r$, $p \not\leqq r$.*

PROOF. By definition, $a \mathscr{F}^3_{\{0\}} b$ if and only if

$$((a \wedge x) \vee y) \wedge z = 0 \Leftrightarrow ((b \wedge x) \vee y) \wedge z = 0$$

for any $x, y\, z \in L$. Since this relation is an $\wedge$-congruence, it is the equality if and only if no class contains two distinct comparable elements, if and only if no class contains two elements a, b such that $a \prec b$ (by the chain conditions).

If two elements $a < b$ of L are not equivalent modulo $\mathscr{F}^3_{\{0\}}$, there exist $x, y, z \in L$ such that

$$((a \wedge x) \vee y) \wedge z = 0, \quad ((b \wedge x) \vee y) \wedge z \neq 0.$$

By the descending chain condition there exist a $p \in L$ such that $0 \prec p \leqq ((b \wedge x) \vee y) \wedge z$. In particular $p \leqq z$, $p \leqq (b \wedge x) \vee y$; also $p \not\leqq (a \wedge x) \vee y$ in view of the first equation. Let $q = b \wedge x$; then $q \leqq b$, $p \leqq q \vee y$ and $p \not\leqq (a \wedge q) \vee y$ since $a \wedge q = a \wedge x$. Let $r = (a \wedge q) \vee y$; then $a \wedge q \leqq r$, $p \leqq q \vee y \leqq q \vee r$, $p \not\leqq r$.

If, conversely, there exist $p, q, r \in L$ such that $0 \prec p$, $q \leqq b$, $a \wedge q \leqq r$, $p \leqq q \vee r$, $p \not\leqq r$, then

$$((a \wedge q) \vee r) \wedge p = r \wedge p = 0 \quad \text{since} \quad 0 \leqq r \wedge p \prec p,$$
$$((b \wedge q) \vee r) \wedge p = (q \vee r) \wedge p = p \neq 0$$

so that a, b are not equivalent modulo $\mathscr{F}^3_{\{0\}}$.

Now the lemma follows if we take $a \prec b$.

LEMMA 9. 2. *$\mathscr{F}'^4_{\{0\}}$ is the equality if and only if for any $a, b \in L$ such that $a \prec b$, there exist $p, q, r, s \in L$ such that $0 \prec p$, $a \leqq s$, $q \leqq b \vee s$, $s \wedge q \leqq r$, $p \leqq q \vee r$, $p \not\leqq r$.*

PROOF. First $a \mathscr{F}'^4_{\{0\}} b$ if and only if $(a \vee c) \mathscr{F}^3_{\{0\}} (b \vee c)$ for all $c \in L$. Hence $\mathscr{F}'^4_{\{0\}}$ is a $\vee$-congruence and is the equality if and only if for any $a, b \in L$ such that $a \prec b$ there exists a $c \in L$ such that $a \vee c$ and $b \vee c$ are not equivalent modulo $\mathscr{F}^3_{\{0\}}$, if and only if there exists an $s \in L$ such that $a \leqq s$ and that s and $b \vee s$ are not equivalent modulo $\mathscr{F}^3_{\{0\}}$. Since $s \prec b \vee s$, the result follows by the calculations used to prove 9. 1.

THEOREM 9. 3. *Let L be a lattice satisfying the ascending chain condition and the descending chain condition, and such that* tr. dim $L \leqq 3$. *Then L is simple if and only if:*

(i) *for any $0 \prec a \prec 1$ there exists a $b \in L$ such that $a \prec b$ and either $c \in L$ such that $a \wedge c = b \wedge c$, $a \vee c = b \vee c$, or $c', c'' \in L$ such that $a \vee c' = a \vee c'' = b$, $c' \wedge c'' \leqq a$;*

(ii) *for any $a, b \in L$ such that $a \prec b$ there exist $p, q, r \in L$ such that $0 \prec p$, $q \leqq b$, $a \wedge q \leqq r$, $p \leqq q \vee r$, $p \not\leqq r$.*

PROOF. This follows from 6. 4, 9. 1 and 8. 7.

THEOREM 9. 4. *Let L be a lattice satisfying the ascending chain condition and the descending chain condition, and such that* tr. dim $L \leqq 4$. *Then L is simple if and only if:*

(i) *for any $0 \prec a \prec 1$ there exist $b, p, q, r \in L$ such that $a \prec b$, $a \leqq r$, $q \leqq a \vee p$, $b \leqq q \vee r$, $b \not\leqq (p \wedge q) \vee r$;*

(ii) *for any $a, b \in L$ such that $a \prec b$ there exist $p, q, r, s \in L$ such that $0 \prec p$, $a \leqq s$, $q \leqq b \vee s$, $s \wedge q \leqq r$, $p \leqq q \vee r$, $p \not\leqq r$.*

PROOF. This follows from 6. 5, 9. 2 and 8. 8.

REMARK. If furthermore L is weakly complemented then condition (i) becomes necessary and sufficient in 9. 3, and similarly for 9. 4; for then $\mathscr{F}_{\{0\}}$ is the equality and L is simple if and only if it has no proper kernel.

10. Some open questions. PROBLEM 1. In the proof of 1. 3 we found necessary and sufficient conditions for two translations of a distributive lattice to be equal. Is it possible to find similar conditions in any lattice? Is it possible to find necessary and sufficient conditions for a translation to have order at most n, n given? (For instance if, in a modular lattice, $\varrho_{c_1} \sigma_{c_2} \varrho_{c_3} \sigma_{c_4}$ has order 4, then any two consecutive c_i are uncomparable and $c_1 \not\leqq c_3$, $c_4 \not\leqq c_2$; but this is not sufficient.)

PROBLEM 2. What are the lattices of tr. dim 3?

PROBLEM 3. Is it true that tr. dim $L' \leqq$ tr. dim L whenever L' is a sublattice (convex sublattice) of L?

PROBLEM 4. If L contains a modular non-distributive sublattice of five elements, is then tr. dim $L \geqq 4$? (the lattice given after the proof of 4. 9 still has this property).

PROBLEM 5. Find a sufficient condition for a lattice L to have tr. dim $L \leqq 4$.

PROBLEM 6. Find a more sophisticated method to determine the tr. dim.

PROBLEM 7. Is any positive integer $n \neq 3$ the translational dimension of some modular lattice? (Examples 2. 2 and 2. 6 and a suitable use of 3. 4 shows that it is true when $n \leqq 7$.)

PROBLEM 8. Does (N_{n+1}) imply (N_n') for all n?

PROBLEM 9. Can 8. 6 be improved so as to facilitate verification on actual examples?

PROBLEM 10. In a simple lattice L with both chain conditions such that tr. dim $L \leqq 4$, a non-distributivity separates any element $0 < a < 1$ from some $b \succ a$ (see the remark following 8. 6). Can a similar condition be used to characterize simple lattices?

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References

[1] G. BIRKHOFF, *Lattice theory* (Amer. Math. Soc. Colloq., Publ. 25, New-York, 1948).
[2] P. M. COHN, *Universal algebra* (Harper and Row, New-York, 1965).
[3] G. GRÄTZER and E. T. SCHMIDT, Ideals and congruence relations in lattices, *Acta Math. Acad. Sci. Hung.*, **9** (1958), pp. 137—175.
[4] G. GRÄTZER and E. T. SCHMIDT, Standard ideals in lattices, *Acta Math. Acad. Sci. Hung.*, **12** (1961), pp. 17—86.
[5] P. A. GRILLET, Sur la notion de tas, *C. R. Acad. Sci. Paris*, **256** (1963), pp. 2739—2741.
[6] IQBALUNNISA, Neutral ideals and congruences, *Fund. Math.*, **48** (1960), pp. 343—353.
[7] G. SZÁSZ, Translationen der Verbände, *Acta Fac. Rerum Nat. Univ. Comenianae*, **5** (1961), pp. 53—57.
[8] M. TEISSIER, Sur les équivalences réqulières dans les demi-groupes, *C. R. Acad. Sci. Paris*, **232** (1951), pp. 1987—1989.
[9] J. VARLET, Congruences dans les treillis pseudo-complémentés, *Bull. Soc. Royale Sci. Liège*, **9** (1963), pp. 623—635.

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SOME REMARKS ABOUT THE DUALITY RELATION IN QUEUES

By
A. GHOSAL (New Delhi, India)*
(*Presented by* A. RÉNYI)

Abstract

The dual of a queue is derived by inter-changing the arrival and service processes. In this paper some general relations relating the dual queues to the parent queues have been brought out. The queue considered is of type GI/G/1.

1. Introduction

Suppose in a queue of type GI/G/1 the inter-arrival times of respective customers $\{t_n\}$ are independent and follow the identical d. f. (distribution function) $G(x)$ $(=\mathbf{P}(t_n \leqq x))$, their service times $\{s_n\}$ are also independent and follow the identical d. f. $H(x)$ $(=\mathbf{P}(s_n \leqq x))$, then in the dual queue the inter-arrival times $\{t_n^*\}$ follow the d. f. $H(x)$ and the service times $\{s_n^*\}$ follow the d. f. $G(x)$. The basic problem is to determine the waiting time d. f. and the d. f. of the queue size of the dual queue, from the knowledge of the corresponding d. f.'s of the parent queue. In the whole of the paper we shall be concerned only with the stationary conditions; in other words, we shall assume that for the parent queue the d. f. of the waiting time (W_n) is $F(y)$ $\left(=\lim_{n\to\infty} \mathbf{P}(W_n \leqq y)\right)$ and that the p. d. f. (probability distribution function) of the queue size Q_i at the time of departure of the n^{th} customer is $f_r = \lim_{i\to\infty} \mathbf{P}(Q_i = r)$.

The work on the duality relations in queues and dams has been done by various workers. A duality relation was derived by the present author (GHOSAL [4], [5]), and it will be shown subsequently that this led to the basic relationship of the dual principle. The dual principles relating to the waiting time d. f. were derived by FINCH [3] and PRABHU [11]. Later on LOYNES [10] made extensions to the study by considering more general random walks. Recently KEILSON [6] worked out a relationship between the queue size distribution of a transportation variant of M/G/1 system and that of a transportation variant of a GI/M/1 system when the waiting space is limited; it has been shown below that his relation holds true for more general systems and can be derived from the basic random walk relationship derived of the present author (GHOSAL [5]).

2. Conditions for existence of limiting d. f.'s

It may be worthwhile recapitulating the conditions on which the limiting distributions of the waiting time and the queue size exist. If the waiting time W_n is unbounded, i. e. it can range between 0 and ∞, then the limiting d. f. $F(y)$ of

* At present Monash University (Australia).

the waiting time exists only if $\mathbf{E}(u_n)<0$, where $u_n=s_n-t_n$; this means that the traffic intensity $\varrho=\mathbf{E}(s_n)/\mathbf{E}(t_n)$ will have to be less than unity (LINDLEY [9]). This condition is also necessary for the existence of a limiting d. f. of the queue size, if the waiting room has an infinite capacity (KENDALL [7]). If in the parent queue $\mathbf{E}(u_i)<0$ or $\varrho<1$, then in the dual queue $\mathbf{E}(u_i^*)=\mathbf{E}(-u_i)>0$ and $\varrho^*=1/\varrho>1$, so that a limiting d. f. of waiting time or queue size does not exist. If, however, the waiting time has an upper bound $(0\leqq W_n\leqq k)$, i. e. a customer does not wait beyond a time k $(<\infty)$ or if the size of a waiting room (N) is finite, then it is always possible to derive the d. f. of the waiting time or queue size for the dual queue so long $\mathbf{E}(u_i)<\infty$ or $\varrho<\infty$ (FINCH [1], [3]). We may make a remark here: while studying a queueing system we should consider the waiting time and the queue size together, and cannot afford to have conditions separately for the waiting time and queue size (SYSKI [11]). Thus we find that even if the waiting time is unbounded $(0\leqq W_n\leqq\infty)$, we can obtain a limiting d. f. for W_n, even for the case $\varrho\geqq 1$ or $\mathbf{E}(u_i)\geqq 0$, if the size of the waiting room (N) is finite $(N<\infty)$. Similarly if the waiting time is bounded $(0\leqq W_n\leqq k)$, we can obtain a limiting d. f. for the queue size, for the case $\infty>\varrho>1$, if the size of the waiting room is infinite. We therefore conclude that if either the waiting space (N) is finite or the waiting time is bounded, limiting d. f.'s both for the waiting time and the queue size exist for the condition $\varrho<\infty$ or $-\infty<\mathbf{E}(u_i)<\infty$; for these cases the limiting d. f.'s of the waiting time and the queue size for the dual queue exist.

3. Basic duality relations

The basic duality relations are derived from a random walk problem (GHOSAL [5]). A particle traverses a straight line path of finite length (say k); at the beginning of each time interval $(t, t+1)$, where $t=\ldots, -1, 2, \ldots$ the particle suffers a displacement u_t which is an r. v. (random variable) following the continuous d. f. $U(x)$, u_t's being mutually independent. Let Z_t be the position of the particle at time t $(0\leqq Z_t\leqq k)$; and let $F(y)$ be the steady state d. f. of Z_t. Let us define $V(y)$ as the probability that, given the initial position of the particle is y $(<k)$, the particle reaches the impenetrable barrier at 0 for the first time before reaching the barrier at k. If u_t be a discrete r. v. with p. d. f. g_j $(=\mathbf{P}(u_t=j))$, then the p. d. f. of Z_t is also discrete (say $F_i=\mathbf{P}(Z_t\leqq i)$ and $f_i^{(k)}=\mathbf{P}(Z_t=i)$); similarly let V_i be the discrete analogue of $V(y)$. Then we get the following theorem (for proof see GHOSAL, [5]).

THEOREM 1. *The probability function $V(y)$ can be expressed in terms of the d. f. of the position of the particle by the relation*

$$V(y) = F(k-y) \quad (0 \leqq y < k). \tag{1}$$

When Z_t follows a discrete d. f. we have

$$V_i^{(k+1)} = F_{k-i}^{(k)} \quad (0 < i < k), \tag{2}$$

k standing for the length of the path.

The equation (1) was derived by the present author (GHOSAL [4]) with respect to a finite dam for a particular class of input distributions, $V(y)$ representing the probability that, given the storage level is y, the dam becomes empty before filling

completely. In a queueing process Z_n may represent the waiting time of the n^{th} customer, so that $Z_{n+1}=\min[k, Z_n+u_n]^*$ where $u_n=s_n-t_n$. The existence of an upper bound (k) to Z_n suggests that a customer does not wait beyond time k. The function $V(y)$, for this process, implies the probability of termination of the busy period before ejecting any impatient customer (with $Z<k$), when the waiting time of the customer who has just now left has been y. The relation (2) can be applied in a queueing process with finite waiting space $(N=k)$; here we define F_i as the probability that the queue size is less than or equal to i.

We now consider a dual random walk $\{Z^*\}$ in which displacements are in the reverse direction, i. e. $-u_t$ $(=v_t$ say) corresponding to u_t at time interval $(t, t+1)$. Then we prove the following theorem.

THEOREM 2. *If $V^*(y)$ and $F^*(y)$ are the probability functions for the dual walk Z_t^* (corresponding to $V(y)$ and $F(y)$ respectively), then we get the following relations between the parent walk and its dual*

$$V(y) = 1-F^*(y), \tag{3}$$

$$F^*(y) = 1-F(k-y) \quad (0 \leqq y < k). \tag{4}$$

When Z_t and Z_t^* follow discrete d. f.'s, we get

$$V_i^{(k+1)} = 1-F_i^{*(k)}, \tag{5}$$

$$F_i^{*(k)} = 1-F_{k-i}^{(k)} \quad (0 < i < k). \tag{6}$$

PROOF. The proof of the relation (3) will be given by following a similar approach as adopted in my previous paper (GHOSAL [5]). Let A be the event that the walk Z_t starting from y reaches the impenetrable barrier at 0 before reaching k, and A_j the event that it does so in the j^{th} step. Then

$$A = \bigcup_{j=1}^{\infty} A_j.$$

We can express A_j as follows $(v_r = -u_r)$:

$$A_j = \bigcap_{s=1}^{j-1}\left(y+\sum_{r=1}^{s} u_r > 0\right)\bigcap_{s=1}^{j-1}\left(\sum_{r=1}^{s} u_r < k-y\right)\cap\left(y+\sum_{r=1}^{j} u_r \leqq 0\right) = \tag{7}$$

$$= \bigcap_{s=1}^{j-1}\left(y > \sum_{r=1}^{s} v_r\right)\bigcap_{s=1}^{j-1}\left(k+\sum_{r=1}^{s} v_r > y\right)\cap\left(y \leqq \sum_{r=1}^{j} v_r\right),$$

where v_r is the r^{th} step displacement in the dual walk Z^*. If in (7) we replace v_r everywhere by v_{-r} the probabilities will remain unchanged, and let us assume A_j to be equivalent to an event A_j^* for the walk Z^*. Then we have the event A_j^{**} through this replacement as

$$A_j^{**} \subset \bigcap_{s=1}^{j-1}\left(\sum_{r=1}^{s} v_{-r} \geqq y-k\right)\cap\left(\sum_{=1}^{j} v_{-r} \geqq y\right). \tag{8}$$

The set on the right-hand side of (8) is the event that the walk Y which has only one impenetrable barrier at k, starting from $Y_{-j}=0$ has a position Y_0 not less

than y. Since the walk Y is always to the left of the walk Z^*, it follows that $A_j^{**} \subset \subset \{Y_0 \geqq y\} \subset \{Z_0^* \geqq y\}$, and

$$A^{**} = \bigcup_{j=1}^{\infty} A_j^{**} \subset \{Z_0^* \geqq y\}. \tag{9}$$

Again let us write

$$\{Z_0^* \geqq y\} = \bigcup_{j=1}^{\infty} C_j$$

where C_j is the event that $Z_0^* \geqq y$ and Z_{-j}^* is the last Z_s before Z_0 to equal 0. If we consider the walk Y defined above and starting from 0 at step $-j$, it is obvious that the event C_j implies that either $Y_0 \geqq y$ or $Y_0 \neq Z_0^*$. But on C_j we must have $Y_0 = Z_0^*$ because the walk Z_n^* never reaches 0 for $-j < t \leqq 0$, so the barrier at has no effect. Thus

$$C_j \subset \{Y_0 \geqq y\} = \bigcap_{s=1}^{j-1} \left\{ \sum_{r=1}^{s} v_{-r} \geqq y-k \right\} \cap \left(\sum_{r=1}^{j} v_{-r} \geqq y \right). \tag{10}$$

If we replace v_{-r} uniformly by v_r, there will not be any change in probability; let C_j^1 be the event obtained through such replacement. Then we have

$$C_j^1 \subset \bigcap_{s=1}^{j-1} \left\{ \sum_{r=1}^{s} v_r \geqq y-k \right\} \cap \left(\sum_{r=1}^{j} v_r \geqq y \right).$$

Thus we prove $C_j^1 \subset A_j^*$, and since $A^* = \bigcup_{j=1}^{\infty} A_j^*$, we get

$$\mathbf{P}(A) = \mathbf{P}(A^*) = \mathbf{P}(Z_0^* \geqq y) \tag{11}$$

which proves (3) of the Theorem 1.

From (1) and (3) we prove the relation (4). It may be pointed out in this connection, that in the above proof we did not make a tacit assumption regarding mutual independence of u's and v's, so the relation holds true under weaker conditions.

For the discrete case, we write the basic equations of F_i and V_i (as in GHOSAL [5]):

$$F_i^{(k)} = \sum_{j=-\infty}^{i-k} g_j + \sum_{j=0}^{k-1} F_j^{(k)} g_{i-j} \quad (0 < i < k), \tag{12}$$

$$V_i^{(k)} = \sum_{j=-\infty}^{i} g_j + \sum_{j=1}^{k-1} V_j^{(k)} g_{j-i} \quad (0 < i < k). \tag{13}$$

Let for the dual walk $H_i^{(k)} = \mathbf{P}(Z^* \geqq i) = 1 - F_i^{*(k)}$, so that we have $(\mathbf{P}(u_n^* = i) = \mathbf{P}(u_n = -i) = g_{-i})$:

$$H_i^{(k)} = \sum_{j=-\infty}^{i} g_j + \sum_{j=1}^{k} H_j^{(k)} g_{j-i}. \tag{14}$$

By equating (12), (13) and (14), we get $H_i^{(k)} = V_i^{(k+1)} = F_{k-i}^{(k)}$. This proves the relations (5) and (6). From (6), we also get

$$f_i = f_{k-i}^* \quad (0 \leqq i \leqq k). \tag{15}$$

The relations (3) and (4) can be applied in relating the d. f. of the waiting time Z_n for the parent queue with that of the dual queue Z_n^*; while the relations (5) and (6) are applied in doing so for the queue size. In the waiting time model, it is assumed that a customer does not wait beyond time k; while in the queue size model we assume a finite waiting space ($N=k$). The relation (3) was obtained by PRABHU [11] only for $k=\infty$, while (4) was derived by FINCH [3].

So far we have considered only a random walk of finite length ($k<\infty$). In terms of queueing models we have considered waiting time d. f. in isolation with the queue size d. f. From Theorem 2, we can infer that the relations (3) and (5) hold true for $k=\infty$, provided stable d. f. for the dual ($F^*(y)$ or F_i^*) exists. We have also found that if the waiting space is limited ($N<\infty$) or the customer does not wait in the queue beyond a finite time ($k<\infty$), then both the waiting tinme d. f. and the queue size d. f. for the dual exist for the stationary condition. Hence the relations (3) and (5) will be applied for determining the d. f.'s $F^*(y)$ and F_i^* respectively. The random walks representing the waiting time and the queue size are different; one is of finite length and the other is of infinite length, but the displacement $u_n^{(1)}$ one is a function of the displacement $u_n^{(2)}$ of the other. Thus if $Z_n^{(1)}$ denotes the waiting time of the n^{th} customer, $u_n^{(1)}$ the difference between his service time and the inter-arrival time (between him and the next customer); $Z_n^{(2)}$ $(=Q_n)$ the queue size left by the n^{th} departing customer, ξ_n number of arrivals during the service time of the $(n+1)^{\text{st}}$ customer, $u_n^{(2)}=\xi_n-1$, then we have $u_n^{(2)}=\varphi(u_n^{(1)})$.

The above analogy leads us to conjecture the following property of conjugate random walks, which is stated here without a proof.

THEOREM 3. *If we have a series of random walks* $Z^{(1)}, Z^{(2)}, \ldots, Z^{(r)}$, *where* $Z_t^{(j)}$ *is the position of the particle on the* j^{th} *walk at time* t, *such that* $u_t^{(j)}$ $(j\neq 1)$ *is a function of* $u_t^{(1)}$ $(j=2, \ldots r)$, *then the stable d. f.'s for all the* r *walks exist if at least one walk is over a finite path, whether* $\mathbf{E}(u^{(j)})\gtrless 0$ $(j=1, \ldots, r)$.

4. Finite waiting space

We deal with the problem of the queueing system with finite waiting space in a little detail here. With the definition of Q_n, ξ_n the same as in Section 3, we have $Q_{n+1}=$ $=\min[N, Q_n+\xi_n-1]^+$. Let the waiting space be limited for N customers, and let $\mathbf{P}(Q_n=i)=f_i$ be the stable queue size d. f. Let Q_n^* be the dual walk corresponding to the walk Q_n; then we have $Q_{n+1}^*=\min[N, Q_n^*+1-\xi_n]^+$; in terms of the dual queue therefore Q_n^* represents asymptotically the queue size just before the n^{th} arrival epoch and ξ_n number of departures in the inter-arrival time just before the $(n+1)^{\text{th}}$ arrival. We can also express the recursive relations of the parent and the dual models in the following menner: we can write $Q_{n+1}=\min\{N, (Q_n-1)+\xi_n\}$, in which case for the dual model we write $Q_{n+1}^*=[\min\{N, Q_n^*+1\}-\xi_n^*]^+$. From Theorem 2 (relation (15)), then we get the following relation:

$$f_i=f_{N-i}^* \quad (0\leqq i\leqq N) \tag{16}$$

where $f_i^*=\mathbf{P}(Q_n^*=i)$. The relation (16) was obtained by KEILSON [6] only for the transportation variants of systems M/G/1 and GI/M/1. It is shown below that this relation is general and holds true asymptotically for any GI/G/1 and its dual.

In the above model for the parent queue, let s_n be the service time of the n^{th} customer; then within this time ξ_n customers have arrived, i.e. $\sum_{i=1}^{\xi_n} t_i \leqq s_n$ (if $\xi_n \geqq 1$), and $\mathbf{E}(\xi_n)$ = average number of arrivals per service time $= \mathbf{E}(s)/\mathbf{E}(t) = \varrho$. Similarly in the Q^* model on an average ϱ customers depart during an inter-arrival time, and $\sum_{i=1}^{n} s_i^* \leqq t_n^*$ (if $\xi_n > 1$). This is possible if $s_i^* = t_i$ and $t_i^* = s_i$.

Again, let in the queue model Q^* (above), let Q_n^{**} be the queue size left by the n^{th} departing customer, ξ_n the arrivals in the inter-departure period, then we have $Q_{n+1}^{**} = \min [N, Q_n^{**} + \eta_n - 1]$. Let $f_i^{**} = \mathbf{P}(Q^{**} = i)$ in the steady state; then it can be shown (FINCH [2] has proved the relation only for the case $N = \infty$) that

$$f_i^{**} = f_i^* \quad \text{(for every } i\text{)}. \tag{17}$$

It can be shown that $\mathbf{E}(\eta_n) = 1/\varrho$, where ϱ is the traffic intensity for the parent queue.

For the relation (16) to hold true, it is not necessary that the d. f. of $\{t_n\}$ should be identical with the d. f. of $\{s_n^*\}$, the service time of the dual queue, and the d. f. of $\{s_n\}$ of the parent queue to be identical with the d. f. of $\{t_n^*\}$, the inter-arrival time, of the dual queue. What is necessary is that the form of the d. f. of $\{t_n\}$ should be identical with that of the d. f. of $\{s_n^*\}$ and vice-versa, with the proviso that $\mathbf{E}(s_n^*)/\mathbf{E}(t_n) = c$ (a constant). (The special case $c = 1$ leads to a dual queue in the sense implied in Section 1 of this paper.) We enunciate a theorem which is relevant in this context and give an informal proof of the same.

THEOREM 4. *Suppose there are two queues for which $Q_n^{(i)}$ $(i = 1, 2)$ represents the queue size left by the n^{th} departing customer in the i^{th} queue, $\xi_n^{(i)}$ $(i = 1, 2)$ the number of arrivals during the interval between the departure times of the n^{th} and $(n+1)^{\text{th}}$ customers in the i^{th} queue, $t_n^{(i)}$ the inter-arrival times in the i^{th} queue $(i = 1,2)$, then the necessary condition for the d.f.'s of $Q_n^{(1)}$ and $Q_n^{(2)}$ to be the same is as follows: there exists $c > 0$ such that*

$$\mu_r(W_n^{(1)}) = c^r \mu_r(W_n^{(2)}), \tag{18}$$

$$\mu_r(s_n^{(1)}) = c^r \mu_r(s_n^{(2)}), \tag{19}$$

$$\mu_r(t_n^{(1)}) = c^r \mu_r(t_n^{(2)}), \tag{20}$$

where $\mu_r(.)$ is the r^{th} moment of $(.)$, and $W_n^{(i)}$ the waiting time of the n^{th} customer in the i^{th} queue.

PROOF. Let $R_k^{(i)}(n)$ represent the probability that the n^{th} customer in the i^{th} queue leaves k customers behind $(i = 1,2)$. Then we have, for a waiting room of size N (following the notation of FINCH [2]:

$$R_0^{(i)}(n) = \mathbf{P}(W_n^{(i)} + s_n^{(i)} \leqq t_n^{(i)}) \tag{21}$$

$$\sum_{k=j}^{N} R_k^{(i)}(n) = \mathbf{P}(W_n^{(i)} + s_n^{(i)} > t_n^{(i)} + t_{n+1}^{(i)} + \ldots + t_{n+j-1}^{(i)}). \tag{22}$$

For a waiting room of infinite size we put $N = \infty$.

From (21) and (22), we find that for $R_j^{(1)}(n) = R_j^{(2)}(n)$ for all j ($j \geqq 0$),

$$\text{Event } (W_n^{(1)} + s_n^{(1)} \leqq t_n^{(1)}) \subseteq \text{Event } (W_n^{(2)} + s_n^{(2)} \leqq t_n^{(2)}) \tag{23}$$

and

$$\text{Event } (W_n^{(1)} + s_n^{(1)} > \sum_{r=0}^{j-1} t_{n+r}^{(1)}) \supseteq \text{Event } (W_n^{(2)} + s_n^{(2)} > \sum_{r=0}^{j-1} t_{n+r}^{(2)}) \tag{24}$$

for all j's. We observe that (23) and (24) are possible only if we can transform $s_n^{(1)}$, $t_n^{(1)}$ into $s_n^{(2)}$, $t_n^{(2)}$ only by changing the scale; since the d. f. of $W_n^{(i)}$ depends upon the d. f.'s of $s_n^{(i)}$, $t_n^{(i)}$, $W_n^{(1)}$ and $W_n^{(2)}$ are also related in a similar fashion. Thus, for $c > 0$,

$$\mathbf{P}(W_n^{(1)} + s_n^{(1)} \leqq t_n^{(1)}) = \mathbf{P}(c\, W_n^{(1)} + c\, s_n^{(1)} \leqq c\, t_n^{(1)}),$$

$$\mathbf{P}(W_n^{(1)} + s_n^{(1)} > \sum_{-r=0}^{j+1} t_{n+r}^{(1)}) = \mathbf{P}(c W_n^{(1)} + c\, s_n^{(1)} > c \sum_{-r=0}^{j+1} t_{n+r}^{(1)}),$$

which are possible only if (19), (20) and hence (18) are true. It may however be noted that Theorem 4 is true if $0 < W_n^{(i)} < \infty$; what would be when $W_n^{(i)}$ is also bounded has not been investigated.

From what has been discussed above, it follows that as a special case, for a queueing system with finite waiting space (N) following M/G/1, its dual will be a corresponding GI/M/1 type, and the relation (16) holds good irrespective of whether the queueing systems are transportation variants in which there is no idle time of the server.

5. Conclusions

Some results on the duality properties of single-server queueing systems have been presented in this paper. Rigid analytical proofs have been avoided in this paper in many cases. There is further scope for detailed work on this aspect: further results of investigations which the present author has been doing will be presented in subsequent communications. An interesting result binding the waiting time of the parent queue with the idle time of a dual queue was derived by KINGMAN [8]. For finite systems (either waiting time with an upper limit or limited waiting space) it is always possible to derive the d. f. of the waiting time (or queue size) for the dual from that of the parent queue. For limited waiting space, the derivation of the waiting time d. f. for the dual from the knowledge of that of the parent queue involves complicated calculus in many cases. In simple M/M/1, however, the $F^*(x)$, the waiting time d. f. for the dual queue, can be obtained simply by substituting $\varrho^* = 1/\varrho$, $\lambda^* = \mu$ and $\mu^* = \lambda$ for the expression for $F(x)$ (see FINCH [1]).

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References

[1] P. D. FINCH, The effect of the size of the waiting room on a simple queue, *J. Roy. Statist. Soc.* **B 20** (1958), pp. 182—186.

[2] P. D. FINCH, A probability limit theorem with application to a generalization of a queueing theory, *Acta Math. Acad. Sci. Hung.*, **10** (1959), pp. 317—326 (see also *Acta Math. Acad. Sci. Hung.*, **10** (1959), pp. 327—336).

[3] P. D. FINCH, Deterministic customer impatience in the queueing system GI/M/1, *Biometrika*, **47**, (1960), pp. 45—62.

[4] A. GHOSAL, Emptiness in the finite dam, *Ann. Math. Statist.*, **31** (1960), pp. 803—808.

[5] A. GHOSAL, Some results in the theory of inventory, *Biometrika*, **51** (1964), pp. 487—490.

[6] J. KEILSON, The ergodic queue length distribution for queueing with finite capacity, *J. Roy. Statist. Soc.*, **B 28** (1966), pp. 190—201.

[7] D. G. KENDALL, Some problems in the theory of queues, *J. Roy. Statist. Soc.*, **B 13** (1951), pp. 151—185.

[8] J. F. C. KINGMAN, The heavy traffic approximation in the theory of queues, *Proceedings of the Symposium on Congestion Theory, Univ. of North Carolina*, 1966, pp. 137—169.

[9] D. V. LINDLEY, Theory of queues with a single server, *Proc. Camb. Phil. Soc.*, **48** (1952), pp. 277—289.

[10] R. M. LOYNES, On a property of the random walks describing simple queues and dams, *J. Roy. Statist. Soc.*, **B 27** (1965), pp. 125—129.

[11] N. U. PRABHU, Elementary methods for some waiting time problems, *Opns. Res.*, **10** (1962), pp. 599—606.

[12] R. SYSKI, Markovian queues, (Chap. 7), *Proceedings of the Symposium on Congestion Theory Univ. of North Carolina*, 1966, pp. 170—223.

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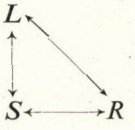
ONE-SIDED IDEALS IN THE ENDOMORPHISM RING OF AN ABELIAN p-GROUP

By
G. S. MONK (Seattle, Washington, U.S.A.)
(*Presented by* L. RÉDEI)

1. Introduction. In his book [2, Chap. V], R. BAER shows that, for a vector space V with endomorphism ring E, the lattice S of subspaces of V is isomorphic to the lattice R of finitely generated right ideals of E and dually isomorphic to the lattice L of finitely generated left ideals of E. (Note. BAER writes operators on the right. We will write them on the left.) Further, there is a dual isomorphism between R and L such that the following diagram commutes:



The object of this paper is to generalize those parts of BAER's theorem relating S and L and S and R to the case of abelian p-groups without elements of infinite height.

Unfortunately, BAER's proofs cannot be extended in a straightforward manner, since the correspondence between R and L plays an important role in them, and BAER himself showed in [1] that it cannot be generalized in the case of abelian groups to a form useful for our purposes. To circumvent this difficulty, we begin, in section 2, with the development of a new duality theorem for abelian groups in which all of the groups considered are p-groups without elements of infinite height. Since the PONTRYAGIN duality theorem does not have this property, which is crucial in later sections, a new duality theory is necessary. This is followed by a section, which may be of independent interest, relating the endomorphism rings of groups that are dual to one another. Section 4 is concerned with generalizing the correspondence between S and R to our class of groups, and in section 5 the results of the preceding sections are used to obtain a relation between the counterparts of S and L.

The notation and terminology for abelian groups will be those of FUCHS [3] and KAPLANSKY [6]. For topological abelian groups, the reader is referred to HEWITT and ROSS [4].

2. Character groups. The results of the next two sections hold in a larger context than p-groups without elements of infinite height taken with their p-adic topology. Thus we will consider topological abelian groups with open bases at 0 consisting of subgroups. Numerous authors have considered such objects, notably KAPLANSKY [5], LEPTIN [7, 8] and SCHÖNEBORN [10, 11].

DEFINITION 2. 1. The topological abelian group G with open basis $\mathscr{D}$ at 0 is an *S. group* if each element of $\mathscr{D}$ is a subgroup, and

$$\cap \mathscr{D} = \{0\}.$$

If the S-group G is also a p-group, then it will be called an *S. p-group.*

When there is no danger of confusion we will write G instead of the pair $(G, \mathscr{D})$.

We have the following examples of S. groups:

1. Given the abelian group G, let $\mathscr{D}$ be the set $\{0\}$. The resulting topology is, of course, the discrete topology.

2. Given the abelian p-group G without elements of infinite height, let $\mathscr{D}$ be $\{p^kG\}_{k=0}^{\infty}$. The group G is then said to have the *p-adic topology.*

3. If G is a group of homomorphisms of the group A into the abelian group B, for each subset X of A, define

$$N(G, X) = \{g \in G | gX = 0\}.$$

For the set $\mathscr{X}$ of subsets of A let

$$\mathscr{X}^0 = \{N(G, X) | X \in \mathscr{X}\}.$$

Then, letting $\mathscr{F}$ be the set of finite subsets of A, we have that $(G, \mathscr{F}^0)$ is an S. group. This topology on G is called the *finite topology.* The notations $N(G, X)$, $\mathscr{X}^0$, and $\mathscr{F}$ will be used throughout the paper.

If $(G, \mathscr{D})$ is an S. p-group, we can form $(G, \mathscr{D})^*$, the group of continuous homomorphisms of $(G, \mathscr{D})$ into $Z(p^\infty)$ (with the discrete topology). It is readily seen that $(G, \mathscr{D})^*$ consists of the homomorphisms of G into $Z(p^\infty)$ whose kernels contain elements of $\mathscr{D}$. The elements of $(G, \mathscr{D})^*$ will be called *characters* on $(G, \mathscr{D})$ and $(G, \mathscr{D})^*$ the *character group* of $(G, \mathscr{D})$.

When G has the discrete topology its character group will be denoted by G^d, and when G is a p-group without elements of infinite height, $(G, \{p^kG\}_{k=0}^{\infty})^*$ will be denoted by G^p. Since, for such a group G, and a homomorphism f of G into $Z(p^\infty)$, $f(p^kG) = 0$ if and only if $p^kf = 0$, we see that G^p is exactly the torsion subgroup of G^d.

NOTATION. Given an abelian group G, let $\mathscr{J}$ be the set of subgroups of finite index. If G is a p-group, let $\mathscr{K}$ be the set of subgroups H of G such that for each H there is an integer k such that H contains p^kG.

It is clear that, if G is a p-group without elements of infinite height, then $(G, \mathscr{K})$ is homeomorphic with $(G, \{p^kG\}_{k=0}^{\infty})$, so that $(G, \mathscr{K})^*$ is G^p. On the other hand, we have:

LEMMA 2. 2. *If G is a p-group without elements of infinite height, then $(G, \mathscr{J})$ is an S. p-group, and $(G, \mathscr{J})^*$ is G^p.*

PROOF. We infer from a theorem of ERDÉLYI [cf. 3, 24. 8] that, for any element $g \in G$, there is a subgroup D of G such that

$$G = [g] \oplus D.$$

Thus $(G, \mathscr{J})$ is an S. p-group. It is clear that $(G, \mathscr{J})^*$ is contained in G^p. If f is in G^p, then $p^kf = 0$, for some k, so that the image of f in $Z(p^\infty)$ is finite. Thus the kernel of f is of finite index and f is in $(G, \mathscr{J})^*$.

By "continuous character" on a topological group G, one usually means a continuous homomorphism of G into the circle group T. However, if G is a p-group, then the only values of characters that are possible are p^{th}-power roots of unity. It is well known that the group of p^{th}-power roots of unity is a realization of $Z(p^\infty)$. Further, if f is a continuous map of an $S.$ group G into T, then

$$E = \{g \in G \mid |f(g) - 1| < \sqrt{2}\} = \left\{g \in G \;\middle|\; -\frac{\pi}{2} < \arg(f(g)) < \frac{\pi}{2}\right\}$$

is an open neighbourhood of 0 and must, therefore, contain an open subgroup D. The image of D under f must be a subgroup of T. But clearly, there are no subgroups of T other than $\{0\}$ all of whose elements have argument in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus $f(D) = \{0\}$; f is continuous with respect to the discrete topology on T, and our notion of continuous character coincides with the usual one.

If $(G, \mathscr{D})$ is an $S.$ group, and K is a closed subgroup of $(G, \mathscr{D})$, then G/K is also an $S.$ group with the open basis at 0 taken to be $\{D + K | D \in \mathscr{D}\}$. It is well-known that the natural homomorphism of G onto G/K is continuous and open.

For an $S.$ p-group $(G, \mathscr{D})$ and a group H of continuous homomorphisms of $(G, \mathscr{D})$ into $Z(p^\infty)$, each element g of G can be viewed as a homomorphism of H into $Z(p^\infty)$ by defining

$$g(h) = h(g) \quad \text{for all} \quad h \in H.$$

Clearly, for any subgroup K of G,

$$N(G, N(H, K)) \supseteq K.$$

Further, $N(G, K)$ is closed in $(G, \mathscr{D})$ for any subset K of H, since it is the intersection of kernels of continuous maps. According to KAPLANSKY [5, Lemma 5], this characterizes the closed subgroups of G. That is:

LEMMA 2. 3. *If K is a subgroup of the $S.$ p-group G, then K is closed if and only if*

$$K = N(G, N((G, \mathscr{D})^*, K)).$$

It is well-known that if K, G and H are abelian groups, and the homomorphisms f_1 and f_2 are such that the sequence

$$0 \longrightarrow K \xrightarrow{f_1} G \xrightarrow{f_2} H \longrightarrow 0$$

is exact, then, for any abelian group D,

$$0 \longrightarrow \mathrm{Hom}(H, D) \xrightarrow{\bar{f}_2} \mathrm{Hom}(G, D) \xrightarrow{\bar{f}_1} \mathrm{Hom}(K, D) \tag{1}$$

is also exact, where

$$\bar{f}_i(r) = r f_i$$

for r in Hom (G, D) or Hom (H, D). It is readily seen that if the groups are topological, f_1 and f_2 are continuous, and f_2 is open, then (1) again holds, where by Hom (A, B) we mean the continuous homomorphisms of A into B. Thus

$$\mathrm{Hom}(H, D) \cong \{r \in \mathrm{Hom}(G, D) | r f_1 = 0\}.$$

If we take D to be $Z(p^\infty)$, G an $S.$ p-group, K a closed subgroup of G, f_1 the inclusion map, and f_2 the natural homomorphism of G onto G/K, we obtain the following:

THEOREM 2. 4. *Let $(G, \mathscr{D})$ be an $S.$ p-group and K a closed subgroup of $(G, \mathscr{D})$. Then*

$$(G/K, \{D+K \mid D \in \mathscr{D}\})^* \cong N((G, \mathscr{D})^*, K). \tag{2}$$

An important aspect of the PONTRYAGIN duality theorem is that every locally compact abelian group is the group of *all* continuous characters of another locally compact abelian group. It will be shown next that the counterpart of this holds for $S.$ p-groups. We first prove a preliminary result.

LEMMA 2.5. *If $(G, \mathscr{D})$ is an $S.p$-group, then $(G, \mathscr{D})^*$ is dense in G^d in the finite topology.*

PROOF. Given x in G^d and a finite subset F in G, we wish to find y in $(G, \mathscr{D})^*$ that agrees with x on F. Let D in $\mathscr{D}$ be such that $[F] \cap D = 0$, and let $H = [F] \oplus D$. Define the homomorphism y of H into $Z(p^\infty)$ by:

$$y(d) = 0 \quad \text{for every} \quad d \in D,$$

$$y(f) = x(f) \quad \text{for every} \quad f \in [F].$$

Since $Z(p^\infty)$ is divisible, there is an extension $\bar{y}$ of y to G. The map $\bar{y}$ is the desired element of $(G, \mathscr{D})^*$ since, by definition, it agrees with x on F and annihilates D.

In view of the third example of an $S.$ p-group, we have that, for an $S.p$-group $(G, \mathscr{D})$, $((G, \mathscr{D})^*, \mathscr{F}^0)$ is also an $S.$ p-group, so that its character group can be formed. We then have:

THEOREM 2. 6. *For an $S.$ p-group $(G, \mathscr{D})$,*

$$G \cong ((G, \mathscr{D})^*, \mathscr{F}^0)^*.$$

PROOF. First observe that, by the Pontryagin duality theorem,

$$G \cong (G^d, \mathscr{F}^0)^*.$$

But, by lemma 2. 5, $(G, \mathscr{D})^*$ is dense in G^d, and, by a remark in [4, p. 369], a continuous character on a dense subgroup of a group can be extended uniquely to a continuous character on the whole group. Thus, the group of characters on $((G, \mathscr{D})^*, \mathscr{F}^0)$ is identical with the group of characters on $(G^d, \mathscr{F}^0)$, and the theorem is established.

We have in the theorem, two $S.$ p-groups, each of which can be viewed as the character group of the other. This situation is given a name.

DEFINITION 2. 7. The $S.$ p-groups $(A, \mathscr{D})$ and $(B, \mathscr{E})$ are said to be *paired*, if each is the character group of the other in such a way that

$$a(b) = b(a) \quad \text{for all} \quad a \in A, \quad b \in B.$$

We will write, in this case, $[a, b]$ for $a(b)$ and $b(a)$.

We have then as a trivial consequence of 2. 3 the following:

THEOREM 2. 8. *Let* $(A_1, \mathscr{D}_1)$ *and* $(A_2, \mathscr{D}_2)$ *be paired S. p-groups. Then for* $i, j = 1, 2$, *distinct, and any subset* X *of* A_i, $N(A_j, N(A_i, X))$ *is the closed subgroup generated by* X, *and the correspondences* $N(A_1, \ )$ *and* $N(A_2, \ )$ *are reciprocal lattice dualities between the closed subgroups of* $(A_1, \mathscr{D}_1)$ *and* $(A_2, \mathscr{D}_2)$.

COROLLARY 2. 9. *If* $(A_1, \mathscr{D}_1)$ *and* $(A_2, \mathscr{D}_2)$ *are paired S. p-groups, then, for* $k = 0, 1, 2, \ldots$,

$$N(A_i, p^k A_j) = A_i[p^k] \quad i, j = 1, 2, \quad i \neq j,$$

$$N(A_i, A_j[p^k]) = (p^k A_i)^- \quad i, j = 1, 2, \quad i \neq j.$$

PROOF. Inasmuch as

$$[p^k a, b] = p^k [a, b] = [a, p^k b] \quad a \in A_i, \quad b \in A_j,$$

we have that

$$N(A_i, p^k A_j) = \{g \in A_i | [A_j, p^k g] = 0\} = \{g \in A_i | p^k g = 0\} = A_i[p^k].$$

On the other hand,

$$N(A_i, A_j[p^k]) = N(A_i, N(A_j, p^k A_i)) = (p^k A_i)^-,$$

as was to be shown.

Notice that an essential difference between our theory of paired groups and the PONTRYAGIN duality theorem is that within the class of locally compact abelian groups there is a canonical topology for all character groups such that every group is homeomorphic with the character group of its character group. We have no such canonical topology and can, therefore, make no such claim. However, we have the following theorem whose proof is obvious.

THEOREM 2. 10. *If* $(A_1, \mathscr{D}_1)$ *and* $(A_2, \mathscr{D}_2)$ *are paired S. p-groups, then, for each* $i = 1, 2$, $(A_i, \mathscr{D}_i)$ *is isomorphic and homeomorphic with*

$$((A_j, \mathscr{D}_j)^*, (\mathscr{D}_i^0)^0) \quad j = 1, 2, \quad j \neq i.$$

In general, the elements of $\mathscr{D}_i^0$ are not easily identified in $(A_j, \mathscr{D}_j)$ as the compact sets are in the PONTRYAGIN duality theorem. However, if G is a p-group without elements of infinite height, then $(\{p^k G\}_{k=0}^{\infty})^0$ is, by 2. 9 $\{G^p[p^k]\}_{k=0}^{\infty}$, while, for $\mathscr{J}$ in G, $\mathscr{J}^0$ in G^p is $\mathscr{F}$.

THEOREM 2. 11. *If* G *is a p-group without elements of infinite height, then, for* $\mathscr{K}$ *in* G, $\mathscr{K}^0$ *is the set of compact subgroups of* $(G^p, \mathscr{F}^0)$.

PROOF. Given a subgroup H of G containing $p^k G$,

$$N(G^d, H) \subseteq G^d[p^k] = G^p[p^k].$$

Inasmuch as $N(G^d, H)$ is compact in $(G^d, \mathscr{F}^0)$ and is contained in G^p, it is compact in $(G^p, \mathscr{F}^0)$. By a remark of KAPLANSKY [6, p. 55], G^d, and hence G^p, has no elements of infinite height. Thus, if H is a compact subgroup of $(G^p, \mathscr{F}^0)$, it is a compact torsion group with no elements of infinite height, whence, by problem 62 of KAPLANSKY [6], it is of bounded order. Hence H is contained in $G^p[p^k]$ for some k, and

$$H = N(G^p, N(G, H)),$$

where

$$N(G, H) \supseteq p^k G.$$

3. Endomorphism rings of paired groups. We turn next to the problem of relating the endomorphism rings of two groups that are paired.

DEFINITION 3. 1. Given S. p-groups $(A, \mathscr{D})$ and $(B, \mathscr{E})$ that are paired, they are said to be *strongly paired* if, given any continuous endomorphism f of $(A, \mathscr{D})$ $((B, \mathscr{E}))$, and any set X in $\mathscr{E}^0(\mathscr{D}^0)$, there is a set X' in $\mathscr{E}^0(\mathscr{D}^0)$ such that $f(X) \subseteq X'$.

THEOREM 3. 2. *Let $(A, \mathscr{D})$ and $(B, \mathscr{E})$ be strongly paired S. p-groups with rings of continuous endomorphisms R and S, respectively. For each element f of R define*

$$[a, f^s(b)] = [f(a), b] \quad \textit{for every} \quad a \in A, \quad b \in B.$$

Then the map s is an anti-isomorphism of R onto S.

PROOF. Clearly $f^s(b)$ is a homomorphism of A into $Z(p^\infty)$ which is continuous because it is the composite of the continuous maps f and b. Thus $f^s(b) \in B$. It is readily verified that f^s preserves addition. To show that f^s is in S, it remains to show that it is continuous. Choose E in $\mathscr{E}$ and let $X = N(A, E)$. Then there is an element X' in $\mathscr{E}^0$ such that $f(X) \subseteq X'$. Letting $E' = N(B, X')$, we have that

$$f(N(A, E)) \subseteq N(A, E'),$$

whence

$$f^s(E') \subseteq E.$$

Thus s is a map of R into S. Similarly, we have the map t of S into R given by

$$[f^t(a), b] = [a, f(b)] \quad \text{for every} \quad a \in A, \quad b \in B, \quad f \in S.$$

Given f in R,

$$[f^{st}(a), b] = [a, f^s(b)] = [f(a), b] \quad \text{for all} \quad a \in A, \quad b \in B,$$

so that $f^{st} = f$. Similarly $f^{ts} = f$ for all f in S, and we have that s is a one-to-one map of R onto S. Clearly s preserves addition. But

$$[a, (fg)^s(b)] = [fg(a), b] = [g(a), f^s(b)] = [a, g^s f^s(b)] \quad \text{for all} \quad a \in A, \quad b \in B,$$

whence $(fg)^s = g^s f^s$, and s is an anti-isomorphism.

It is obvious that, by the symmetry of the situation, there is no real difference between the maps s and t of the proof, and no confusion will arise by replacing them both by the notation $*$.

There are natural topologies on R and S which make this correspondence a homeomorphism.

DEFINITION 3. 3. If E is a ring of endomorphisms of a group G and $\mathscr{S}$ and $\mathscr{T}$ are families of subsets of G such that:

(a) Given S_1, S_2 in $\mathscr{S}$, there is an element S_3 in $\mathscr{S}$ such that $S_3 \supseteq S_1 \cup S_2$;

(b) Given $T_1, T_2 \in \mathscr{T}$, there is an element T_3 in $\mathscr{T}$ such that $T_3 \subseteq T_1 \cap T_2$;

(c) Every T in $\mathscr{T}$ is a subgroup of G, then the $\mathscr{S} - \mathscr{T}$ *topology* on E is the topology on E, as an additive group, given by a basis of open sets at 0 of the form

$$\{r \in E \mid r(S) \subseteq T\}$$

for S in $\mathscr{S}$, T in $\mathscr{T}$.

This does not, in general, make E a topological ring.

THEOREM 3. 4. *The map * is a homeomorphism of R with the $\mathscr{E}^0-\mathscr{D}$ topology onto S with the $\mathscr{D}^0-\mathscr{E}$ topology.*

PROOF. This is a direct consequence of the readily verified fact that, for closed subgroups M_1 and M_2 of A,

$$\{r\in R|rM_1 \subseteqq M_2\}^* = \{r\in S|rN(B, M_2) \subseteqq N(B, M_1)\}.$$

The following will be useful later.

COROLLARY 3. 5. *If H_1 and H_2 are subgroups of the S. p-group $(A, \mathscr{D})$ such that H_1 is open and $A=H_1\oplus H_2$, then*

$$B = N(B, H_1)\oplus N(B, H_2),$$

and $N(B, H_i)\cong H_j^*$ for $i, j=1, 2$ and $i\neq j$, where H_j^* is the character group with respect to the relative topology on H_j.

PROOF. Let π be the projection on H_1 with kernel H_2. For D in $\mathscr{D}$, there is an element D' in $\mathscr{D}$ such that $D'\subseteqq D\cap H_1$. Since

$$\pi(D') = D' \subseteqq D,$$

we infer that π is continuous, and hence is in R. Certainly $(\pi^*)^2=\pi^*$, so that π^* is a projection in S. We also have:

$$\ker\pi^* = \{b\in B|[A, \pi^*b] = 0\} = \{b\in B|[\pi A, b] = 0\} = N(B, H_1).$$

Similarly,

$$\ker(1-\pi^*) = N(B, H_2).$$

Therefore, π^* is a projection on $N(B, H_2)$ with kernel $N(B, H_1)$, and

$$B = N(B, H_1)\oplus N(B, H_2).$$

Since H_2 is the kernel of π, it is closed, and

$$H_1^* \cong (A/H_2)^* \cong N(B, H_2).$$

Similarly, $H_2^*\cong N(B, H_1)$, and we have the desired result.

4. The right ideals of an endomorphism ring. We shall study the relationship between the subgroups of a group and the right ideals of its endomorphism ring.

DEFINITION 4. 1. An *S.* group G is said to have the *property* (C) if:
Given a finite subset F of G, there is a an open subgroup D and a finite subgroup H of D containing F such that $G=H\oplus D$.

LEMMA 4. 2. *If G is a p-group without elements of infinite height, then* $(G, \{0\})$, $(G, \{p^kG\}_{k=0}^{\infty})$, *and* $(G^p, \mathscr{F}^0)$ *all have the property* (C).

PROOF. By the theorem of ERDÉLYI, in a p-group G without elements of infinite height, given the finite set F, there are subgroups H and D of G with H finite such that $G=H\oplus D$, and $F\subseteqq H$. Thus $(G, \{0\})$ has property (C). Since H is finite, p^kG is contained in D, for some k, whence D is open in $(G, \{p^kG\}_{k=0}^{\infty})$, and this group has property (C).

On the other hand, given F finite in G^p,

$$(G/N(G,[F]))^* \cong N(G^p, N(G,[F])) = [F],$$

so that $N(G,[F])$ is of finite index in G, whence, by a theorem of PIERCE [9, 16. 5], there are subgroups E and H of G such that E is finite, $G = E \oplus H$ and, H is contained in $N(G,[F])$. Inasmuch as E is finite, H is open in $(G, \{p^k G\}_{k=0}^{\infty})$, whence, by 3. 5,

$$G^p = N(G^p, E) \oplus N(G^p, H),$$

and $N(G^p, H) \cong E^*$. Since E is finite, and $H \subseteq N(G,[F])$, we have that $N(G^p, H)$ is finite and $N(G^p, H) \supseteq [F]$. Thus $N(G^p, H)$ and $N(G^p, E)$ are the desired subgroups.

Among the results proved by BAER in [2] are that the finitely generated right and left ideals in the endomorphism ring E of a vector space V are of the form

$$\{f \in E | f(M) = 0\}, \quad \{f \in E | f(V) \subseteq M\}$$

respectively, where M is a subspace of V. Our aim is to describe those one-sided ideals in the ring R of endomorphisms of a p-group G without elements of infinite height that are of the form

$$\{f \in R | f(A_i) \subseteq B_i \quad \text{for every} \quad i \in I\}, \tag{1}$$

where $\{(A_i, B_i)\}_{i \in I}$ is a set of pairs of subgroups of G. We will show that the set of such one-sided ideals is sufficiently large to include all one-sided ideals in case G is finite.

Defining the finite-open topology on the ring R of continuous endomorphisms of an S. p-group $(G, \mathscr{D})$ to be the $\mathscr{S}-\mathscr{T}$ topology of the previous section with $\mathscr{S}$ and $\mathscr{T}$ taken to be the families of finite subsets of G and $\mathscr{D}$ respectively, we first observe:

LEMMA 4. 3. *Given the subgroups $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ of the S. p-group $(G, \mathscr{D})$ such that B_i is closed for every i in I, the set*

$$\left| X = \{f \in R | f(A_i) \subseteq B_i \quad \text{for every} \quad i \in I\} \right|$$

is closed in the finite-open topology on R, where R is the ring of continuous endomorphisms of $(G, \mathscr{D})$.

PROOF. Given an endomorphism f in X^-, an element a in A_i, and a set D in $\mathscr{D}$, there is an element g in X such that

$$f(a) - g(a) = (f-g)(a) \in D.$$

Since g is in X, $g(a)$ is in B_i. Thus $f(a)$ is in B_i^-, whence, by the closedness of B_i, $f(a)$ is in B_i, and f is in X.

Our task, then, is to show that the closed one-sided ideals of R are of the form (1) for suitably chosen sets of pairs of subgroups $\{(A_i, B_i)\}_{i \in I}$.

DEFINITION 4. 4. The element c in the S. group $(G, \mathscr{D})$ is said to be *primitive* if there is an open subgroup H of G such that $G = [c] \oplus H$.

LEMMA 4. 5. *If G is an S. p-group, and c is a primitive element of G of exponent k, then $G[p^k] = Rc$, where R is the ring of continuous endomorphisms of G.*

PROOF. Clearly Rc is contained in $G[p^k]$. Given an element g of $G[p^k]$, if H is an open subgroup such that $G=[c]\oplus H$, there is a unique endomorphism f of G given by

$$f(c) = g, \quad f(h) = 0 \quad \text{for all} \quad h \in H.$$

The map f is continuous since H is open.

DEFINITION 4. 6. The *order type* of an S. p-group G is the ascending sequence of integers $\{k(i)\}_{i=0}^{\infty}$ occuring as exponents of primitive elements of G.

NOTE. Strictly speaking, we should denote the order type of a group by $\{k(i)\}_{i\in N}$, where N is some subset of the natural numbers, since we do not want to exclude groups of bounded order. However, since no confusion will arise by doing so, we will use the more convenient notation $\{k(i)\}_{i=0}^{\infty}$.

We observe, in passing, that it is not difficult to show that if the order type of $(G, \{p^kG\}_{k=0}^{\infty})$ is $\{k(i)\}_{i=0}^{\infty}$, the integers $\{k(i)-1\}_{i=0}^{\infty}$ are exactly those integers for which the Ulm invariants of G are not zero.

DEFINITION 4. 7. A *right ideal sequence* in an S. p-group G with order type $\{k(i)\}_{i=0}^{\infty}$ is a sequence of subgroups $\{H_i\}_{i=0}^{\infty}$ of G such that, for every i:

(a) $$H_i \subseteq G[p^{k(i)}];$$

(b) $$H_i \subseteq H_{i+1};$$

(c) $$p^{k(i+1)-k(i)}H_{i+1} \subseteq H_i.$$

If all of the subgroups in such a sequence are closed, it will be called a *closed right ideal sequence*.

It is readily verified that if $\{H_i\}_{i=0}^{\infty}$ and $\{K_i\}_{i=0}^{\infty}$ are closed right ideal sequences, then $\{(H_i+K_i)^-\}_{i=0}^{\infty}$ has the first two properties of a closed right ideal sequence, and that

$$p^{k(i+1)-k(i)}(H_{i+1}+K_{i+1}) \subseteq H_i+K_i \quad \text{for every} \quad i,$$

whence

$$\left(p^{k(i+1)-k(i)}(H_{i+1}+K_{i+1})\right)^- \subseteq (H_i+K_i)^-.$$

We then infer from the continuity of multiplication by p that

$$p^{k(i+1)-k(i)}(H_{i+1}+K_{i+1})^- \subseteq (H_i+K_i)^-,$$

so that $\{(H_i+K_i)^-\}_{i=0}^{\infty}$ is a closed right ideal sequence. It is evident that $\{H_i\cap K_i\}_{i=0}^{\infty}$ is also a closed right ideal sequence, whence the set of closed right ideal sequences forms a lattice under the operations:

$$\{H_i\}_{i=0}^{\infty}+\{K_i\}_{i=0}^{\infty} = \{(H_i+K_i)^-\}_{i=0}^{\infty},$$

$$\{H_i\}_{i=0}^{\infty}\cap\{K_i\}_{i=0}^{\infty} = \{H_i\cap K_i\}_{i=0}^{\infty}.$$

DEFINITION 4. 8. Let $(G, \mathscr{D})$ be an S. p-group of order type $\{k(i)\}_{i=0}^{\infty}$ with ring R of continuous endomorphisms. For a right ideal sequence $\{H_i\}_{i=0}^{\infty}$, define

$$\mathfrak{I}^r(R, \{H_i\}_{i=0}^{\infty}) = \{r\in R \mid rG[p^{k(i)}] \subseteq H_i \quad i = 0, 1, 2, \ldots\}.$$

For a right ideal I in R, define

$$\mathscr{H}^r(G, I) = \{IG[p^{k(i)}]\}_{i=0}^{\infty}.$$

When there is no danger of confusion, we will write $\mathscr{I}^r(\{H_i\}_{i=0}^{\infty})$ and $\mathscr{H}^r(I)$.

It is clear that $\mathscr{I}^r(\{H_i\}_{i=0}^{\infty})$ is a right ideal in R which, if H_i is closed for all i, is closed in the finite open topology on R. The set $IG[p^{k(i)}]$ is a subgroup of G since, for c a primitive element of order $p^{k(i)}$,

$$IG[p^{k(i)}] = IRc = Ic.$$

It is then easily checked that $\mathscr{H}^r(I)$ is a right ideal sequence of G.

THEOREM 4. 9. *Let $(G, \mathscr{D})$ be an S. p-group with property* (C) *and let R be its ring of continuous endomorphisms. Then $\mathscr{H}^r$ and $\mathscr{I}^r$ are reciprocal lattice isomorphisms between the lattice of closed right ideal sequences of $(G, \mathscr{D})$ and the lattice of right ideals of R closed in the finite-open topology.*

PROOF. It is clear that $\mathscr{H}^r$ and $\mathscr{I}^r$ preserve order. By 4. 3, we have that if $\{H_i\}_{i=0}^{\infty}$ is a closed right ideal sequence, then $\mathscr{I}^r(\{H_i\}_{i=0}^{\infty})$ is a closed right ideal. We will show that, for a closed right ideal I in R and a primitive element c, Ic is a closed subgroup of G, so that $\mathscr{H}^r(I)$ is a closed right ideal sequence of $(G, \mathscr{D})$.

Inasmuch as $G[p^n]$ is closed for every n, we infer that if x is in $(Ic)^-$, then x is of exponent no greater than that of c. Thus, there is an element r in R such that $x = rc$. If π is a projection in R on $[c]$, then $x = r\pi c$, so that, by the closedness of I, it suffices to prove that $r\pi$ is in I^-. Given D in $\mathscr{D}$, there is an element s of I such that $x - sc$ is in D, whence

$$(r\pi - s\pi)c \in D.$$

For a finite set F and g an element of F

$$g = \pi g + (1-\pi)g = mc + (1-\pi)g$$

for some integer m, so that

$$(r\pi - s\pi)g = (r\pi - s\pi)mc \in D,$$

and

$$(r\pi - s\pi)F \subseteqq D.$$

Since $s\pi$ is in I, we conclude that $r\pi$ is in I^-, as was to be shown.

We show next, that for a closed right ideal I,

$$\mathscr{I}^r\mathscr{H}^r(I) = I.$$

Certainly $\supseteqq$ holds. For the opposite inclusion it suffices to prove that if r in R is such that

$$rG[p^{k(i)}] \subseteqq IG[p^{k(i)}] \quad (i = 0, 1, 2, \ldots)$$

(where $\{k(i)\}_{i=0}^{\infty}$ is the order type of $(G, \mathscr{D})$), then, for any finite subset F and open subgroup D of G, there is an element s in I such that

$$(r-s)F \subseteqq D.$$

By property (C), there is a finite subgroup H containing F and an open subgroup D of G such that $G=H\oplus D$. Writing H as a direct sum of cyclic groups, we have

$$G = \oplus \sum_{i=1}^{n} [g_i] \oplus D,$$

and letting π_j be the projection on $[g_j]$ with kernel $\oplus \sum_{i\neq j} [g_i]\oplus D$ for each $j=1, 2, \ldots, n$, we have continuous projections $\{\pi_j\}_{j=1}^{n}$ such that

$$F \subseteqq \left(\sum_{j=1}^{n} \pi_j\right) G, \quad \pi_i\pi_j = 0 \quad (i \neq j).$$

Given i, g_i is primitive, and there is an integer j such that

$$rg_i \in rG[p^{k(j)}] \subseteqq IG[p^{k(j)}] = Ig_i.$$

We then obtain an element s_i of I such that $rg_i = s_i g_i$. Thus

$$rg_i = \left(\sum_{j=1}^{n} s_j\pi_j\right) g_i \quad (i = 1, 2, \ldots, n),$$

and

$$\left(r - \sum_{j=1}^{n} s_j\pi_j\right) F = \{0\} \subseteqq D.$$

Inasmuch as s_j is in I for each j, $\sum_{j=1}^{n} s_j\pi_j$ is in I and is the desired element.

Finally, we show that, for a closed right ideal sequence $\{H_i\}_{i=0}^{\infty}$,

$$\mathscr{H}^r\mathfrak{I}^r(\{H_i\}_{i=0}^{\infty}) = \{H_i\}_{i=0}^{\infty}.$$

Certainly $\subseteqq$ holds. Given h in H_i let c be a primitive element of exponent $k(i)$, and let D be an open subgroup of $(G, \mathscr{D})$ such that $G=[c]\oplus D$. Let r be an element of R such that $rc=h$, and let π be the continuous projection on $[c]$ with kernel D, so that $h=r\pi c$. We will show that h is in $\mathscr{H}^r\mathfrak{I}^r(\{H^i\}_{i=0}^{\infty})$, by showing that

$$r\pi G[p^{k(j)}] \subseteqq H_j \quad \text{for all} \quad j.$$

For each g in $G[p^{k(j)}]$ there is an element g' in D and an integer m such that

$$g = mc + g',$$

from which we conclude that $r\pi g = mh$. If $j \geqq i$, then

$$r\pi g = mh \in H_i \subseteqq H_j,$$

while if $j<i$, mc is in $G[p^{k(j)}]$, and $m=p^{k(i)-k(j)}m'$ for some integer m', whence

$$r\pi g = p^{k(i)-k(j)}m'h \in p^{k(i)-k(j)}H_i \subseteqq H_j,$$

as was to be shown.

The most direct application of this theorem is to $(G, \{0\})$ and $(G, \{p^kG\}_{k=0}^{\infty})$ where, in both cases, G is a p-group without elements of infinite height. The conclusion in the first case, then applies to the finite topology on the ring of all endomor-

phisms of G. The second case also applies to the ring of all endomorphisms of G, since, for any integer k and endomorphism f of G,

$$f(p^kG) \subseteq p^kG.$$

5. The left ideals of an endomorphism ring. To prove directly a theorem like 4. 9 for left ideals appears to be very difficult. However, by first proving a theorem relating the correspondence between subgroups of strongly paired groups given in section 2 and the correspondence between the one-sided ideals of their endomorphism rings given in section 3, we will show that such a theorem is a consequence of 4. 9.

LEMMA 5. 1. *The order types of strongly paired S. p-groups are equal.*

PROOF. Let $(A, \mathscr{D})$ and $(B, \mathscr{E})$ be strongly paired $S.$ p-groups. By symmetry, it suffices to show that for each primitive element a of A, there is a primitive element of B of the same order as a. Let D be an open subgroup of A such that

$$A = [a] \oplus D.$$

Then, by 3. 5,

$$B = N(B, [a]) \oplus N(B, D)$$

with $N(B, D) \cong [a]^*$. Inasmuch as $[a]$ is finite, $N(B, D)$ is isomorphic to $[a]$ and is therefore generated by an element of the same order as a. This element is primitive since $N(B, [a])$ is a closed subgroup of finite index and is consequently open.

DEFINITION 5. 2. In an $S.$ p-group $(G, \mathscr{D})$ of order type $\{k(i)\}_{i=0}^{\infty}$ a *left ideal sequence* is a sequence of subgroups $\{H_i\}_{i=0}^{\infty}$ such that:

(a) $$H_i \supseteq (p^{k(i)}G)^-;$$

(b) $$H_i \supseteq H_{i+1};$$

(c) $$p^{k(i+1)-k(i)}H_i \subseteq H_{i+1}.$$

If all of the subgroups in such a sequence are closed, it will be called a *closed left ideal sequence*. As with right ideal sequences, the closed left ideal sequences form a lattice under the operations:

$$\{H_i\}_{i=0}^{\infty} \cap \{K_i\}_{i=0}^{\infty} = \{H_i \cap K_i\}_{i=0}^{\infty},$$

$$\{H_i\}_{i=0}^{\infty} + \{K_i\}_{i=0}^{\infty} = \{(H_i + K_i)^-\}_{i=0}^{\infty}.$$

If E is the ring of continuous endomorphisms of $(G, \mathscr{D})$, we define, for a left ideal sequence,

$$\mathfrak{I}^l(E, \{H_i\}_{i=0}^{\infty}) = \{r \in R | r(H_i) \subseteq (p^{k(i)}G)^- \quad i = 0, 1, 2, \ldots\}.$$

For a left ideal I in E we define

$$\mathscr{H}^l(G, I) = \{\{g \in G | Ig \subseteq (p^{k(i)}G)^-\}_{i=0}^{\infty}\}.$$

It is clear that $\mathfrak{I}^l(E, \{H_i\}_{i=0}^{\infty})$ is a left ideal in E, while $\mathscr{H}^l(G, I)$ is a left ideal sequence in G.

The following can be routinely checked.

LEMMA 5. 3. *If $(A, \mathscr{D})$ and $(B, \mathscr{E})$ are paired S. p-groups, and $\{H_i\}_{i=0}^{\infty}$ is a sequence of subgroups of A, then $\{H_i\}_{i=0}^{\infty}$ is a closed left ideal sequence in $(A, \mathscr{D})$ if and only if $\{N(B, H_i)\}_{i=0}^{\infty}$ is a closed right ideal sequence in $(B, \mathscr{E})$.*

THEOREM 5. 4. *Let $(A, \mathscr{D})$ and $(B, \mathscr{E})$ be strongly paired S. p-groups with rings of continuous endomorphisms R and S respectively. If $\mathscr{H}^r(A, \ )$ and $\mathscr{I}^r(A, \ )$ are reciprocal lattice isomorphisms between the lattice $\mathscr{R}(R)$ of right ideals of R closed in the $\mathscr{E}^0 - \mathscr{D}$ topology and the lattice $\mathscr{R}(A)$ of closed right ideal sequences of $(A, \mathscr{D})$, then $\mathscr{H}^l(B, \ )$ and $\mathscr{I}^l(S, \ )$ are reciprocal lattice dualities between the lattice $\mathscr{L}(S)$ of left ideals of S closed in the $\mathscr{D}^0 - \mathscr{E}$ topology and the lattice $\mathscr{L}(B)$ of closed left ideal sequences of B. Furthermore, the following diagram commutes:*

$$\begin{array}{ccc} \mathscr{R}(R) & \overset{*}{\underset{*}{\rightleftarrows}} & \mathscr{L}(S) \\ \mathscr{H}^r(A, \ ) \downarrow \uparrow \mathscr{I}^r(R, \ ) & & \mathscr{H}^l(B, \ ) \downarrow \uparrow \mathscr{I}^l(S, \ ) \\ \mathscr{R}(A) & \underset{\{N(A, \ )\}}{\rightleftarrows} & \mathscr{L}(B) \end{array} \qquad \{N(B, \ )\}$$

PROOF. By assumption, the two maps on the left are reciprocal isomorphisms; by theorem 3. 3, those on the top are reciprocal isomorphisms and by 2. 8, those on the bottom are reciprocal dualities. Clearly, to prove the theorem, it suffices to show just two things: (1) For a left ideal I in S

$$\mathscr{H}_i^l(B, I) = N(B, \mathscr{H}_i^r(A, I^*)) \quad (i = 0, 1, 2, \ldots)$$

where, by $\mathscr{H}_i^l$, we mean the i^{th}-term of the sequence $\mathscr{H}^l$ and similarly with $\mathscr{H}_i^r$. (2) For a left ideal sequence $\{H_i\}_{i=0}^{\infty}$ in B,

$$\mathscr{I}^l(H_i) = (\mathscr{I}^r(\{N(A, H_i)\}_{i=0}^{\infty}))^*.$$

To prove (1), the right hand side is

$$\{b \in B | [\mathscr{H}_i^r(I^*), b] = 0\} = \{b \in B | [I^* A[p^{k(i)}], b] = 0\} =$$

$$= \{b \in B | [A[p^{k(i)}], Ib] = 0\} = \{b \in B | Ib \subseteq (p^{k(i)} B)^-\} = \mathscr{H}_i^l(B, I).$$

For (2), the right hand side is

$$\{r \in R | rA[p^{k(i)}] \subseteq N(A, H_i) \quad \text{for all} \quad i\}^* = \{r \in R | [rA[p^{k(i)}], H_i] = 0 \quad \text{for all} \quad i\} =$$

$$= \{r \in R | [A[p^{k(i)}], r^* H_i] = 0 \quad \text{for all} \quad i\}^* = \{r^* \in S | r^* H_i \subseteq (p^{k(i)} B)^- \quad \text{for all} \quad i\} =$$

$$= \mathscr{I}^l(R, \{H_i\}_{i=0}^{\infty}).$$

As a first consequence of this theorem, we can take $(A, \mathscr{D})$ to be any $S.$ p-group with property (C) of section 4 and $(B, \mathscr{E})$ to be $((A, \mathscr{D})^*, \mathscr{F}^0)$. However, as a more important result we will prove the counterpart of 4. 9 for left ideals.

THEOREM 5. 5. *Let G be a p-group without elements of infinite height and let R be its ring of endomorphisms. Then $\mathscr{I}^l(R, \ )$ and $\mathscr{H}^l(G, \ )$ are reciprocal lattice dualities between the lattice $\mathscr{L}(R)$ of left ideals of R closed in finite topology on R*

and the lattice $\mathscr{L}(G)$ of all left ideal sequences. Furthermore, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{R}(S) & \underset{*}{\overset{*}{\rightleftarrows}} & \mathscr{L}(R) \\ \mathscr{H}^r(G^p, \ ) \downarrow \uparrow \ \mathscr{I}^r(S, \ ) & & \mathscr{H}^l(G, \ ) \downarrow \uparrow \ \mathscr{I}^l(R, \ ) \\ & \{N(G, \ )\} & \\ \mathscr{R}(G^p) & \underset{\{N(G^p, \ )\}}{\rightleftarrows} & \mathscr{L}(G) \end{array}$$

where $\mathscr{R}(G^p)$ is the lattice of closed right ideal sequences of $(G^p, \mathscr{F}^0)$ and $\mathscr{R}(S)$ is the lattice of right ideals in the ring S of continuous endomorphisms of $(G^p, \mathscr{F}^0)$ closed in the finite-open topology.

PROOF. We have, from 2. 2 and 2. 6, that $(G, \mathscr{J})$ and $(G^p, \mathscr{F}^0)$ are paired S. p-groups. Inasmuch as $\mathscr{J}^0$ is the finite subset of G^p, we infer that $(G, \mathscr{J})$ and $(G^p, \mathscr{F}^0)$ are strongly paired. In view of 4. 2 and 4. 9, we can apply 5. 4 to obtain a diagram similar to the one we want except that it refers to the ring R' of continuous endomorphisms of $(G, \mathscr{J})$, the lattice of left ideals of R' closed in the $\mathscr{F}-\mathscr{J}$ topology, and the lattice of closed left ideal sequences of $(G, \mathscr{J})$.

It is evident that if f is an endomorphism of G, and H is a subgroup of finite index, then

$$\{g \in G | f(g) \in H\}$$

is also of finite index, so that R' is identical with R. Thus the remainder of the proof will consist of two parts: (1) To show that the left ideals of R closed in the finite topology are identical with the left ideals closed in $\mathscr{F}-\mathscr{J}$ topology. (2) To show that the subgroups of any left ideal sequence are closed in $(G, \mathscr{J})$.

For (1) it clearly suffices to prove that if r in R is in the $\mathscr{F}-\mathscr{J}$ closure of a left ideal I, then r is in the finite closure of I. Given a finite set F of G, there are, by property (C), subgroups K and D of G with K finite and D open such that

$$G = K \oplus D, \quad F \cup rF \subseteq K.$$

Given K, there is an element s of I such that $(r-s)K \subseteq D$. Let π be the projection on K with kernel D. For g in F, rg is in K, so that $\pi rg = rg$. Thus

$$(r - \pi s)F = \pi(r-s)F \subseteq \pi D = 0,$$

and r is in the finite closure of I. For (2), observe that if $p^k G \subseteq H$, then G/H is bounded. Thus G/H is the direct sum of cyclic groups $\{C_i\}_{i \in X}$. It is well-known that if

$$\overline{C}_j = \sum \{C_i | i \in X, i \neq j\} \quad \text{for every} \quad j \in X,$$

then

$$\cap \{\overline{C}_j | j \in X\} = 0.$$

Letting C_j' be the subgroup of G containing H such that $C_j'/H = \overline{C}_j$, we have that C_j' is of finite index in G for every j in X, and that

$$\cap \{C_j' | j \in X\} = H,$$

whence H is closed in $(G, \mathscr{J})$.

We observe, in conclusion, that for a finite abelian p-group, the finite topology on its endomorphism ring R coincides with the discrete topology, so that theorems 4. 9 and 5. 5, when applied to R, give all the left and right ideals in R.

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Bibliography

[1] R. BAER, Automorphism rings of primary abelian groups, *Ann. of Math.*, **44** (1943), pp. 192—227.
[2] R. BAER, *Linear algebra and projective geometry* (Academic Press, New York, 1952).
[3] L. FUCHS, *Abelian groups* (Publishing House of the Hungarian Academy of Sciences, Budapest, 1958).
[4] E. HEWITT and K. ROSS, *Abstract harmonic analysis* (Springer-Verlag, Berlin, 1963).
[5] I. KAPLANSKY, Dual modules over a valuation ring I, *Proc. Amer. Math. Soc.*, **4** (1953), pp. 213—219.
[6] I. KAPLANSKY, *Infinite abelian groups* (University of Michigan Press, Ann Arbor, 1954).
[7] H. LEPTIN, Über eine Klasse linear kompakter abelscher Gruppen I, II, *Abh. Math. Sem. Univ. Hamburg*, **19** (1954), pp. 23—40; (1955), pp. 221—243.
[8] H. LEPTIN, Abelsche Gruppen mit kompakten Charaktergruppen und Dualitätstheorie gewisser linear topologischer abelscher Gruppen, *Abh. Math. Sem. Univ. Hamburg*, **19** (1955), pp. 244—263.
[9] R. PIERCE, *Homomorphisms of primary abelian groups*, *Topics in abelian groups* (Scott, Foresman, and Co. Chicago, 1963).
[10] H. SCHÖNEBORN, Über gewisse Topologien in Abelschen Gruppen I, II, *Math. Z.*, **59** (1954), pp. 455—473; **60** (1954), pp. 17—30.
[11] H. SCHÖNEBORN, Über den Zusammenhang zwischen Dualitäts- und Vollständigkeitseigenschaften bei gewissen topologischen Abelschen Gruppen, *Math. Z.*, **65** (1956), pp. 429—441.

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HALBGRUPPEN, DEREN ELEMENTE DURCH PRIMIDEALE TRENNBAR SIND

Von
G. SZÁSZ (Nyíregyháza, Ungarn)
(*Vorgelegt von* L. RÉDEI)

1. Einleitung. Ein bekannter Satz von STONE [3] lautet folgendermaßen: Sind x und y Elemente eines distributiven Verbandes V und enthält das durch x in V erzeugte Hauptideal das Element y nicht, so gibt es ein Primideal P in V mit den Eigenschaften $P \supset x$, $P \not\supset y$. Da jedes Ideal (bzw. Hauptideal, Primideal) eines Verbandes noch mehr ein Ideal (bzw. Hauptideal, Primideal) seines Durchschnittshalbverbandes ist,[1] ergibt dieses Resultat als Korollar auch einen halbgruppentheoretischen Satz: man soll nichts anderes machen, als die Worte „eines distributiven Verbandes V" durch den Ausdruck „einer kommutativen idempotenten Halbgruppe V, die zum Durchschnittshalbverband eines distributiven Verbandes isomorph ist" zu ersetzen.

Ist a ein Element einer Halbgruppe H, so bezeichne (a) das durch a in H erzeugte Hauptideal. Wir sagen, daß die Elemente der Halbgruppe H *durch Primideale trennbar* sind, wenn man zu jedem Elementepaar a, b von H mit $(a) \not\ni b$ ein Primideal P in H findet, welches den Bedingungen $P \supset a$, $P \not\ni b$ genügt.

In dieser Arbeit zeigen wir, daß die einzigen Halbgruppen mit dieser Eigenschaft die intraregulären sind. Dabei nennt man eine Halbgruppe H *intraregulär*, wenn es zu jedem Element a von H gewisse Elemente x, y in H gibt, so daß die Gleichung $a = xa^2y$ besteht.

2. Vorbereitungen. Man nennt ein Ideal I der Halbgruppe H *halbprim*, wenn aus $a^2 \in I$ immer $a \in I$ folgt. Solche Ideale spielen bei der Charakterisierung der intraregulären Halbgruppen eine Hauptrolle. Es gilt nämlich das folgende

LEMMA 1. *Eine Halbgruppe ist dann und nur dann intraregulär, wenn jedes Ideal in ihr halbprim ist* ([1], Satz 4. 4).

Für halbprime Ideale gilt auch

LEMMA 2. *Ist I ein halbprimes Ideal der Halbgruppe H und $c^n \in I$ $(c \in H)$, so ist auch $c \in I$.*

BEWEIS. Man findet einen Beweis für diese Behauptung in [2], Seite 240; wir geben hier einen anderen Beweis. Man betrachte ein halbprimes Ideal I und wähle eine natürliche Zahl k mit $2^k \geqq n$. Aus $c^n \in I$ folgt dann $c^{2^k} \in I$. Daraus ergibt sich aber, wegen $c^{2^k} = (c^{2^{k-1}})^2$, daß auch $c^{2^{k-1}}$ in I enthalten ist. Durch k-malige Wiederholung dieser Überlegung erhält man $c \in I$.

Wir bezeichnen die durch c erzeugte zyklische Unterhalbgruppe mit $\langle c \rangle$. Als ein unmittelbares Korollar von Lemma 2 erhalten wir dann

[1] Für die hier nicht definierten Begriffe und Symbole der Halbgruppen- und Verbandstheorie siehe [1], bzw. [4].

LEMMA 3. *Enthält ein halbprimes Ideal I das Element c nicht, so ist der Durchschnitt $I \cap \langle c \rangle$ leer.*

Zum Beweis unseres obengenannten Satzes brauchen wir noch das folgende

LEMMA 4. *Für beliebige Elemente a, b einer intraregulären Halbgruppe gilt* $(a) \cap (b) = (ab)$ ([1], Satz 4. 4).

3. Satz und Beweis. Nunmehr beweisen wir unseren schon angekündigten

SATZ. *Die Elemente einer Halbgruppe H sind dann und nur dann durch Primideale trennbar, wenn H intraregulär ist.*

a) Als erstes zeigen wir, daß die Bedingung des Satzes notwendig ist. Zu diesem Zweck betrachten wir eine nicht intrareguläre Halbgruppe H. Nach Lemma 1 gibt es in H ein Ideal I und ein Element b, so daß $b \notin I$, aber $b^2 \in I$. Für das Element $a = b^2$ gilt also das Enthaltensein $(a) \subseteq I$, woraus sich offensichtlich $b \notin (a)$ ergibt. Andererseits ist $b^2 \in (a)$. Ist jetzt P ein Primideal von H, welches das Element a enthält, so folgt aus $b^2 \in (a) \subseteq P$ auch $b \in P$. So erhielten wir, daß jedes Primideal von H, dem das Element a gehört, notwendigerweise auch b enthält. Damit haben wir den Beweis für die Notwendigkeit der genannten Bedingung erbracht.

b) Wir zeigen, daß diese Bedingung auch hinreichend ist. Wir betrachten eine intrareguläre Halbgruppe H und die Elemente a, b von H; es sei $(a) \not\ni b$ vorausgesetzt. Nach Lemma 1 ist das Hauptideal (a) halbprim; deshalb ist, nach Lemma 3, der Durchschnitt $(a) \cap \langle b \rangle$ leer.

Man betrachte die Menge aller Ideale I von H, für die $I \supseteq (a)$ und $I \cap \langle b \rangle = \emptyset$ gelten. Für diese Menge sind die Voraussetzungen des Kuratowski—Zornschen Lemmas erfüllt. Deswegen gibt es ein Ideal P, so daß

$$P \supseteq (a), \quad P \cap \langle b \rangle = \emptyset \tag{1}$$

und für jedes Ideal P' von H

$$\text{aus} \quad P' \supset P \quad \text{folgt} \quad P' \cap \langle b \rangle \neq \emptyset. \tag{2}$$

Wegen (1) haben wir nur zu zeigen, daß dieses P prim ist.

Es seien $u, v \in H$, aber $u, v \notin P$. Dann ist $P \subset P \cup (u)$ und $P \subset P \cup (v)$, so daß nach (2) die Ideale $P' = P \cup (u)$ und $P'' = P \cup (v)$ einen nichtleeren Durchschnitt mit $\langle b \rangle$ haben. Es gilt also

$$b^m \in P' \quad \text{bzw.} \quad b^n \in P'' \tag{3}$$

für gewisse Potenzen von b. Da jedes Ideal von H halbprim ist, folgt $b \in P'$ und $b \in P''$ aus (3). Nach (1) ergibt sich aber daraus $b \in (u)$ und $b \in (v)$; folglich ist nach Lemma 4 das Element b auch im Hauptideal (uv) enthalten. Dementsprechend würde aus $P \ni uv$ auch $P \ni b$ folgen, was nach (1) unmöglich ist; es gilt also $P \not\ni uv$. Damit haben wir bewiesen, daß das Ideal P tatsächlich prim ist.

4. Bemerkungen. Ist jedes Element einer Halbgruppe H idempotent, so ist H offensichtlich intraregulär. Deshalb sind die Elemente jedes Halbverbandes durch Primideale trennbar.

Bekanntlich ist eine kommutative Halbgruppe dann und nur dann intraregulär, wenn sie (im Neumannschen Sinne) regulär ist. Folglich sind die Elemente einer kommutativen Halbgruppe genau dann durch Primideale trennbar, wenn die Halbgruppe regulär ist.

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Literatur

[1] A. H. CLIFFORD—G. B. PRESTON, *The algebraic theory of semigroups*, Bd. 1 (American Mathematical Society, Providence, 1961).
[2] E. S. LIAPIN (Е. С. Ляпин), *Полугруппы*, (Гос. Изд. Физико-Математической Литературы, Moskau, 1960).
[3] M. H. STONE, The theory of representations for Boolean algebras, *Trans. Amer. Math. Soc.*, **40** (1936), S. 37—111.
[4] G. SZÁSZ, *Einführung in die Verbandstheorie* (Akadémiai Kiadó, Budapest und Teubner Verlag, Leipzig, 1962).

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О РАСПРЕДЕЛЕНИИ ЗНАЧЕНИЙ ЦЕЛОЙ ФУНКЦИИ ПО АРГУМЕНТАМ

А. А. ГОЛЬДБЕРГ (Львов, СССР)

(*Представлено* П. Тураном)

Постановка задачи, рассматриваемой в этой статье, возникла из одного вопроса проф. Екатерины Реньи [1]. Хотя здесь будет дан ответ на первоначальный вопрос Е. Реньи, решение задачи в более общей постановке удалось нам только в довольно частном случае. Пусть $G(z)$ — целая функция, обозначим через $z_j(a)$ ее a-точки. Будем называть значение a вполне рассеянным, если множество $\{\arg z_j(a)\}$ всюду плотно на $[0, 2\pi]$. Под $\arg z$ понимаем ветвь аргумента z, определенную условием $0 \leqq \arg z < 2\pi$. Множество всех вполне рассеянных значений функции $G(z)$ обозначаем через $D(G)$. Задача заключается в следующем: установить, какие множества комплексных точек могут служить множеством $D(G)$ для некоторой целой функции $G(z)$. Приведем некоторые примеры.

Пример 1. $D(G)$ может быть всей конечной плоскостью. Это имеет место, например, для $G(z) = \sin(\sin z)$.

Пример 2. $D(G)$ может быть всей конечной плоскостью без одной точки a_0. За $G(z)$ можно взять $G(z) = a_0 + \exp(\sin z)$.

В настоящее время мы не можем указать целой функции $G(z)$, у которой множество $D(G)$ есть конечная плоскость без двух точек.

Пример 3. Для $G(z) = e^z$ множество $D(G)$ пусто.

Пример 4 (Е. Реньи [1]). Множество $D(G)$ может состоять из одной точки a_0. Действительно, легко построить целую функцию $f(z)$ порядка $\varrho < 1$ такую, что $0 \in D(f)$. Тогда множество $D(G)$ для функции $G(z) = a_0 + e^z f(z)$ состоит из одной точки a_0.

Е. Реньи спрашивала, может ли множество $D(G)$ состоять из двух точек. Положительный ответ на этот вопрос содержится в следующей теореме.

Теорема. *Пусть A — ограниченное не более чем счетное замкнутое множество. Тогда существует целая функция $G(z)$ такая, что $D(G) = A$.*

Наше доказательство опирается на следующую лемму.

Лемма. *Пусть T — произвольное не более чем счетное множество, лежащее в кольце $d \leqq |t| \leqq 1 - d$, где $0 < d < 1/2$. Существует функция $f(\zeta)$, аналитическая в полуплоскости $\{\operatorname{Im} \zeta > 0\}$, непрерывная в $\{\operatorname{Im} \zeta \geqq 0\}$ и обладающая следующими свойствами:*

1) $|f(\zeta)| < 1 - d/2$ *при* $\operatorname{Im} \zeta \geqq 0$;

2) *для всякого $\tau \in T$ существует такая последовательность τ-точек $\zeta_j(\tau)$ функции $f(\zeta)$, $\operatorname{Im} \zeta_j(\tau) \geqq 2$, $j = 1, 2, \ldots$, что множество $\{\arg \zeta_j(\tau)\}$ всюду плотно*

на $[0, \pi]$, *а на окружностях* $|\zeta - \zeta_j(\tau)| = 1$ *выполняется неравенство*

$$|f(\zeta) - \tau| \geqq M|\zeta_j(\tau)|^{-\frac{5}{4}}, \quad M > 0, \quad j \geqq j_0(M).$$

Доказательство леммы довольно громоздко, и мы приведем его в конце статьи.

Доказательство теоремы. Покажем, что существует последовательность односвязных ограниченных областей D_n, $n = 1, 2, \ldots$, такая, что $\bigcap_{n=1}^{\infty} D_n = A$. Пусть k — натуральное число. Известно, что множество A можно покрыть конечным числом замкнутых, попарно не пересекающихся многоугольников $\Pi_j^{(k)}$, $1 \leqq j \leqq n(k)$, с диаметрами, меньшими $1/k$, причем A пересекается с каждым $\Pi_j^{(k)}$ и не пересекается с границей $\Pi_j^{(k)}$ (см. [2], § 49; в этой книге речь идет о совершенных разрывных множествах, но каждое ограниченное не более чем счетное замкнутое множество легко дополнить до совершенного разрывного множества). Пусть $A \subset \{|w| < R\}$, $R > 1$. Соединим точки $w = \pm 2R$ жордановой дугой C, лежащей, кроме концов, в круге $|w| < 2R$, имеющей общие точки с внутренностью $\tilde{\Pi}_j^{(k)}$ каждого многоугольника $\Pi_j^{(k)}$ и ровно две общие точки с его границей. Дуга C разбивает круг $|w| < 2R$ на две области B_1 и B_2. Пусть

$$D_{2k-1} = B_1 \cup \left\{ \bigcup_{j=1}^{n(k)} \tilde{\Pi}_j^{(k)} \right\}, \quad D_{2k} = B_2 \cup \left\{ \bigcup_{j=1}^{n(k)} \tilde{\Pi}_j^{(k)} \right\}.$$

Так как области D_{2k-1} и D_{2k} односвязны,

$$D_{2k-1} \cap D_{2k} = \bigcup_{j=1}^{n(k)} \tilde{\Pi}_j^{(k)}, \quad \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{n(k)} \tilde{\Pi}_j^{(k)} = A,$$

то очевидно, что последовательность областей D_n является искомой.

Обозначим через $w = P_n(t)$ аналитическую функцию, конформно и однолистно отображающую круг $|t| < 1$ на D_n, $P_n(0) \in A$. Обозначим через T_n прообраз множества A при этом отображении, а через $\tau_n(a)$ — прообраз точки $a \in A$. Пусть

$$0 < d_n = \min \left\{ \min_{t \in T_n} |t|, 1 - \max_{t \in T_n} |t| \right\}.$$

Пусть функция $f_n(\zeta)$ удовлетворяет условиям леммы с $T = T_n$, $d = d_n$.

Обозначим через G_1 область $\{\operatorname{Im} z \leqq -1\}$, через G_n $(n \geqq 2)$ — образ $\{\operatorname{Im} z \geqq 0\}$ при отображении функцией

$$z = (-\zeta)^{2^{-n+1}} + n2^n \exp(i\pi 2^{-n+2}), \quad \pi \leqq \arg(-\zeta) \leqq 2\pi.$$

Пусть

$$\zeta = L_1(z) = -z - i, \quad z \in G_1,$$

$$\zeta = L_n(z) = -\{z - n2^n \exp(i\pi 2^{-n+2})\}^{2^{n-1}}, \quad z \in G_n, \quad n \geqq 2.$$

Будем обозначать через $\zeta_j^{(n)}(\tau)$ те τ-точки функции $f_n(\zeta)$, о которых шла речь в лемме, а через $C_j^{(n)}(\tau)$ — прообразы окружностей $|\zeta - \zeta_j^{(n)}(\tau)| = 1$ в G_n, $n \geqq 1$,

при отображении функцией $\zeta = L_n(z)$. Определим теперь в G_n функцию $w = F_n(z) = P_n(f_n(L_n(z)))$. Легко видеть, что при фиксированном $n \geqq 1$ и $j \to \infty$

$$K_j^{(n)}(\tau) = k_j^{(n)}(\tau) + o(1) = (\arg \zeta_j^{(n)}(\tau) + \pi) 2^{-n+1} + o(1),$$

где

$$K_j^{(n)}(\tau) = \max_{z \in C_j^{(n)}(\tau)} \arg z, \quad k_j^{(n)}(\tau) = \min_{z \in C_j^{(n)}(\tau)} \arg z.$$

Следовательно, при $\tau \in T_n$ последовательность $K_j^{(n)}(\tau)$; $j = 1, 2, \ldots$, всюду плотна на $[\pi 2^{-n+1}, \pi 2^{-n+2}]$, $n \geqq 1$. Очевидно, что для всякого $a \in A$ множества точек $K_j^{(n)}(\tau_n(a))$ и точек $k_j^{(n)}(\tau_n(a))$, $n = 1, 2, \ldots, j = 1, 2, \ldots$, всюду плотны на $[0, 2\pi]$ и при фиксированном n

$$\lim_{j \to \infty} \{K_j^{(n)}(\tau_n(a)) - k_j^{(n)}(\tau_n(a))\} = 0. \tag{1}$$

Обозначим через Λ ломаную, состоящую из отрезков, соединяющих последовательно точки $-i$, -8, $24i$, $\ldots$, $n\, 2^n \exp(i\pi 2^{-n+2})$, $\ldots$. Пусть $\varkappa$ — континуум, определяемый как $\varkappa = \Lambda \cup G_1 \cup G_2 \cup G_3 \cup \ldots$. Определим на $\varkappa$ функцию $w = F(z)$ следующим образом: $F(z) = F_n(z)$ при $z \in G_n$, а на Λ функцию $F(z)$ задаем произвольным образом так, чтобы $F(z)$ на $\varkappa$ была непрерывной.

Оценим теперь $|F(z) - a|$ при $z \in C_j^{(n)}(\tau_n(a))$, $a \in A$. Заметим, что в силу однолистности $P_n(t)$ при $|t| \leqq 1 - d_n/2$, $|t'| \leqq 1 - d_n/2$ выполняется $|(P_n(t) - P_n(t'))/(t - t')| \geqq \mu_n > 0$, где μ_n — постоянная, зависящая только от n. Поэтому при $a \in A$ и $j \geqq j_0(n)$ для $z \in C_j^{(n)}(\tau_n(a))$ выполняется

$$|F(z) - a| = |F_n(z) - a| \geqq \mu_n |f_n(\zeta) - \tau_n(a)| \geqq \tag{2}$$

$$\geqq \mu_n M_n |\zeta_j^{(n)}(\tau_n(a))|^{-\frac{5}{4}} \geqq M_n' |z|^{-\frac{5}{4} \cdot 2^{n-1}} \geqq M_n' |z|^{-2^n}, \quad M_n' > 0.$$

Очевидно, что любую точку дополнения к $\varkappa$ можно соединить с ∞ полупрямой. Поэтому к функции $F(z)$, непрерывной на $\varkappa$ и аналитической во внутренних точках $\varkappa$, можно применить теорему М. В. Келдыша [3], [4] и построить целую функцию $G(z)$ такую, что на континууме $\varkappa$ выполняется

$$|F(z) - G(z)| < \exp\left(-|z|^{\frac{1}{4}}\right), \quad z \in \varkappa. \tag{3}$$

Покажем, что для функции $G(z)$ справедливо $D(G) = A$. Запишем $G(z) - a = \{F(z) - a\} + \{G(z) - F(z)\}$, где $a \in A$, и заметим, что в силу (2) и (3) при фиксированном n и $j \geqq j_1(n) \geqq j_0(n)$ на $C_j^{(n)}(\tau_n(a))$ выполняется $|G(z) - F(z)| < |F(z) - a|$. Следовательно, по теореме Руше функция $G(z) - a$ имеет внутри $C_j^{(n)}(\tau_n(a))$ столько же нулей, сколько и функция $F(z) - a$, то есть внутри $C_j^{(n)}(\tau_n(a))$ имеется по крайней мере одна a-точка. Учитывая свойства последовательностей $K_j^{(n)}(\tau_n(a))$ и $k_j^{(n)}(\tau_n(a))$ и формулу (1), получаем, что $a \in D(G)$. Таким образом, доказано, что $A \subset D(G)$.

Покажем теперь, что если $a \bar{\in} A$, то $a \bar{\in} D(G)$. Тем самым будет показано, что $A = D(G)$. Так как $\bigcap_{n=1}^{\infty} D_n = A$, то найдется такое $n_0 = n_0(a)$, что $a \bar{\in} D_{n_0}$. Обозначим через D_{n_0}' образ замкнутого круга $|t| \leqq 1 - d_{n_0}/2$ при отображении $w_0 = P_n(t)$; расстояние между границами D_{n_0}' и D_{n_0} положительно. Образ G_{n_0} при отображении функцией $w = F_{n_0}(z)$ содержится в D_{n_0}'. Учитывая (3), полу-

чаем, что образ области $G_{n_0}(R_{n_0}) = G_{n_0} \cap \{|z| \geqq R_{n_0}\}$, где R_{n_0} достаточно велико, при отображении функцией $w = G(z)$ содержится в D_{n_0}. Следовательно, в области $G_{n_0}(R_{n_0})$ нет a-точек функции $G(z)$. Так как область $\{3 \cdot 2^{-n_0}\pi < \arg z < 2^{-n_0+2}\pi\} \setminus G_{n_0}(R_{n_0})$ ограничена, то в углу $\{3 \cdot 2^{-n_0}\pi < \arg z < 2^{-n_0+2}\pi\}$ функция $G(z)$ имеет конечное число a-точек. Следовательно, $a \bar{\in} D(G)$. Теорема доказана. Остается доказать лемму.

Доказательство леммы.[1] Пусть A — некоторое число такое, что $A > \exp\exp 4$; выбор A будет уточнен позднее. Пусть $\delta = (\ln\ln A)^{-1}$; ясно, что $\delta < 1/4$, $A^{\delta} > 4$ и $A^{\delta} \to \infty$ при $A \to \infty$. Возьмем последовательность $\zeta_n = r_n \exp(i\theta_n)$, $\operatorname{Im}\zeta_n \geqq 2$, $r_n^{-\frac{1}{4}} < \theta_n < \pi$. Будем требовать, чтобы последовательность θ_n была всюду плотной на $[0, \pi]$, а последовательность r_n выберем так, чтобы выполнялось: $r_1 = 1$, $r_2 = A$, $r_{n+1} > r_n^{1+2/\delta}$, $n = 1, 2, \ldots$. Очевидно, что $r_{n+1}/r_n \geqq A$, $r_n \geqq A^{n-1}$ при $n \geqq 1$. Выберем последовательность τ_n, $n = 1, 2, \ldots$, так, чтобы $\tau_n \in T$ и для каждого $\tau \in T$ существовала подпоследовательность τ_{n_j} такая, что $\tau_{n_j} = \tau$ и последовательность θ_{n_j} всюду плотна на $[0, \pi]$. Так как множество T не более чем счетно, то ясно, что всегда можно выбрать такую последовательность τ_n. Мы построим функцию $t = f(\zeta)$, аналитическую в $\{\operatorname{Im}\zeta > 0\}$, непрерывную в $\{\operatorname{Im}\zeta \geqq 0\}$ и удовлетворяющую условиям:

а) $|f(\zeta)| < 1 - d/2$ при $\operatorname{Im}\zeta \geqq 0$;

б) $f(\zeta_n) = \tau_n$, $n = 1, 2, \ldots$;

в) на окружностях $|\zeta - \zeta_n| = 1$ выполняется

$$|f(\zeta) - \tau_n| \geqq M r_n^{-\frac{5}{4}}, \quad M > 0, \quad n \geqq n_0.$$

Легко видеть, что такая функция $f(\zeta)$ обладает свойствами 1) и 2) леммы. Пусть

$$\varphi_n(\zeta) = \prod_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{\bar{\zeta}_j}{\zeta_j} \frac{\zeta - \zeta_j}{\zeta - \bar{\zeta}_j}.$$

При $\operatorname{Im}\zeta \geqq 0$ выполняется $|\varphi_n(\zeta)| \leqq 1$. С другой стороны,

$$|\varphi_n(\zeta_n)| = \prod_{\substack{j=1 \\ j \neq n}}^{\infty} \left|\frac{\zeta_n - \zeta_j}{\zeta_n - \bar{\zeta}_j}\right| \geqq \prod_{j=1}^{n-1} \frac{1 - r_j/r_n}{1 + r_j/r_n} \prod_{j=n+1}^{\infty} \frac{1 - r_n/r_j}{1 + r_n/r_j} \geqq \tag{4}$$

$$\geqq \left(\prod_{j=1}^{\infty} \frac{1 - A^{-j}}{1 + A^{-j}}\right)^2 = \Psi(A).$$

Ясно, что $\Psi(A) < 1$ и $\Psi(A) \to 1$ при $A \to \infty$.

Пусть

$$\Phi_n(\zeta) = \left(\frac{\zeta}{r_n}\right)^{\delta} \exp\left\{-\left(\frac{\zeta}{r_n}\right)^{\delta}\right\}, \quad 0 \leqq \arg\zeta \leqq \pi.$$

На дуге $\{|\zeta| = r, \operatorname{Im}\zeta \geqq 0\}$ выполняется

$$|\Phi_n(\zeta)| \leqq \left(\frac{r}{r_n}\right)^{\delta} \exp\left\{-\left(\frac{r}{r_n}\right)^{\delta} \cos\delta\pi\right\}, \tag{5}$$

[1] В этом доказательстве мы используем одну идею У. К. Хеймана [5]

а в полуплоскости $\{\operatorname{Im}\zeta \geqq 0\}$

$$|\Phi_n(\zeta)| \leqq \max_{0 \leqq r < \infty} r^\delta \exp(-r^\delta \cos \delta\pi) = (e \cos \delta\pi)^{-1}. \tag{6}$$

Кроме того,

$$\exp(-\cos \delta\pi) \geqq |\Phi_n(\zeta_n)| \geqq e^{-1}. \tag{7}$$

Отметим еще следующее элементарное неравенство: при $r>0$

$$r^\delta \exp(-r^\delta \cos \delta\pi) \leqq \frac{k!}{r^{\delta(k-1)} \cos^k \delta\pi}, \quad k = 0, 1, 2, \ldots. \tag{8}$$

Используя (5), (6) и (8) при $k=0, 2$, получаем при $\sqrt{r_{n-1} r_n} \leqq |\zeta| = r < \sqrt{r_n r_{n+1}}$, $\operatorname{Im}\zeta \geqq 0$, $r_0 = 0$, что

$$\sum_{j=1}^{\infty} |\Phi_j(\zeta)| \leqq \frac{1}{e \cos \delta\pi} + \sum_{j=1}^{n-1} |\Phi_j(\zeta)| + \sum_{j=n+1}^{\infty} |\Phi_j(\zeta)| \leqq$$

$$\leqq \frac{1}{e \cos \delta\pi} + \sum_{j=1}^{n-1} \frac{2}{\left(\frac{r}{r_j}\right)^\delta \cos^2 \delta\pi} + \sum_{j=n+1}^{\infty} \left(\frac{r}{r_j}\right)^\delta \leqq$$

$$\leqq \frac{1}{e \cos \delta\pi} + \left(\frac{r_{n-1}}{r}\right)^\delta \frac{2}{\cos^2 \delta\pi} \sum_{j=1}^{n-1} \left(\frac{r_j}{r_{n-1}}\right)^\delta + \left(\frac{r}{r_{n+1}}\right)^\delta \sum_{j=n+1}^{\infty} \left(\frac{r_{n+1}}{r_j}\right)^\delta \leqq$$

$$\leqq \frac{1}{e \cos \delta\pi} + \left(\frac{r_{n-1}}{r}\right)^\delta \frac{4}{\cos^2 \delta\pi} + 2\left(\frac{r}{r_{n+1}}\right)^\delta \leqq$$

$$\leqq \frac{1}{e \cos \delta\pi} + \left(\frac{r_{n-1}}{r_n}\right)^{\frac{\delta}{2}} \frac{4}{\cos^2 \delta\pi} + 2\left(\frac{r_n}{r_{n+1}}\right)^{\frac{\delta}{2}} \leqq$$

$$\leqq \frac{1}{e \cos \delta\pi} + \frac{4A^{-\frac{\delta}{2}}}{\cos^2 \delta\pi} + 2A^{-\frac{\delta}{2}} = \frac{\Psi_1(A)}{e}.$$

Отсюда следует, что при $\operatorname{Im}\zeta \geqq 0$ выполняется

$$\sum_{j=1}^{\infty} |\Phi_j(\zeta)| \leqq e^{-1} \Psi_1(A). \tag{9}$$

Легко видеть, что $\Psi_1(A) > 1$ и $\Psi_1(A) \to 1$ при $A \to \infty$.

Пусть $g_n(\zeta) = \Phi_n(\zeta)\varphi_n(\zeta)$. Очевидно, что $g_n(\zeta_j) = 0$ при $j \neq n$. Из (7) и (4) следует $|g_n(\zeta_n)| \geqq e^{-1}|\varphi_n(\zeta_n)| \geqq e^{-1}\Psi(A)$ и $|g_n(\zeta_n)| \leqq |\Phi_n(\zeta_n)| \leqq \exp(-\cos \delta\pi) < 1$. Пусть

$$h_n(\zeta) = g_n(\zeta)/g_n(\zeta_n).$$

Очевидно,

$$h_n(\zeta_j) = \begin{cases} 0, & j \neq n, \\ 1, & j = n. \end{cases}$$

Далее, учитывая (9), получаем при $\operatorname{Im}\zeta \geqq 0$

$$\sum_{j=1}^{\infty} |h_j(\zeta)| \leqq \frac{e}{\Psi(A)} \sum_{j=1}^{\infty} |g_j(\zeta)| \leqq \frac{e}{\Psi(A)} \sum_{j=1}^{\infty} |\Phi_j(\zeta)| \leqq \frac{\Psi_1(A)}{\Psi(A)}. \tag{10}$$

Так как $\Psi_1(A)/\Psi(A) \to 1$ при $A \to \infty$, то можно выбрать A столь большим, чтобы выполнялось

$$\frac{\Psi_1(A)}{\Psi(A)} < \frac{1-d/2}{1-d}. \tag{11}$$

Это значение A мы фиксируем.

Покажем, что функция

$$f(\zeta) = \sum_{j=1}^{\infty} \tau_j h_j(\zeta)$$

является искомой. Очевидно, что $f(\zeta)$ непрерывна в $\{\operatorname{Im}\zeta \geqq 0\}$ и аналитична в $\{\operatorname{Im}\zeta > 0\}$. При $\operatorname{Im}\zeta \geqq 0$

$$|f(\zeta)| \leqq \sum_{j=1}^{\infty} |\tau_j|\,|h_j(\zeta)| \leqq (1-d) \sum_{j=1}^{\infty} |h_j(\zeta)| < 1-d/2,$$

в силу (10) и (11). Таким образом, требование а) выполняется. Ясно также, что и требование б) выполнено. Чтобы показать выполнение требования в), нам потребуются дополнительные оценки.

Пусть $0 \leqq \omega \leqq \pi$. При $z \to 0$ равномерно относительно ω выполняется

$$\delta \ln(1+z) + e^{i\delta\omega}[1-(1+z)^{\delta}] = \delta(1-e^{i\delta\omega})z + O(z^2).$$

Предположим, что $\zeta = \zeta_n + e^{i\theta}$, $0 \leqq \theta < 2\pi$. Тогда

$$\ln h_n(\zeta) = \ln \frac{\Phi_n(\zeta)}{\Phi_n(\zeta_n)} + \ln \frac{\varphi_n(\zeta)}{\varphi_n(\zeta_n)} = \tag{12}$$

$$= \delta \ln\left(\frac{\zeta}{\zeta_n}\right) + \left(\frac{\zeta_n}{r_n}\right)^{\delta}\left[1-\left(\frac{\zeta}{\zeta_n}\right)^{\delta}\right] + \ln \frac{\varphi_n(\zeta)}{\varphi_n(\zeta_n)} =$$

$$= \delta \ln\left(1+\frac{e^{i\theta}}{\zeta_n}\right) + e^{i\delta\theta_n}\left[1-\left(1+\frac{e^{i\theta}}{\zeta_n}\right)^{\delta}\right] + \ln \frac{\varphi_n(\zeta)}{\varphi_n(\zeta_n)} =$$

$$= \delta(1-e^{i\delta\theta_n})\frac{e^{i\theta}}{\zeta_n} + O\left(\frac{1}{\zeta_n^2}\right) + \ln \frac{\varphi_n(\zeta)}{\varphi_n(\zeta_n)}.$$

Учитывая, что

$$2 \geqq |1-\exp(i\delta\theta_n)| \geqq \sin \delta\theta_n \geqq \frac{2\delta}{\pi} r_n^{-\frac{1}{4}},$$

имеем

$$\frac{2\delta}{r_n} \geqq \left|\delta(1-e^{i\delta\theta_n})\frac{e^{i\theta}}{\zeta_n}\right| \geqq \frac{2\delta^2}{\pi}\frac{1}{r_n^{5/4}}. \tag{13}$$

Далее

$$\ln \frac{\varphi_n(\zeta)}{\varphi_n(\zeta_n)} = \int_{\zeta_n}^{\zeta} d \ln \varphi_n(z) = \int_{\zeta_n}^{\zeta} \frac{\varphi_n'(z)}{\varphi_n(z)}\, dz, \tag{14}$$

где за путь интегрирования возьмем отрезок, соединяющий точки ζ_n и ζ. При $|z-\zeta_n| \leqq 1$ имеем

$$\left|\frac{\varphi_n'(z)}{\varphi_n(z)}\right| = \left|\sum_{\substack{j=1 \\ j\neq n}}^{\infty} \frac{\zeta_j - \bar{\zeta}_j}{(z-\zeta_j)(z-\bar{\zeta}_j)}\right| \leqq \sum_{j=1}^{n-1} \frac{|\zeta_j - \bar{\zeta}_j|}{|z-\zeta_j|\,|z-\bar{\zeta}_j|} +$$

$$+ \sum_{j=n+1}^{\infty} \frac{|\zeta_j - \bar{\zeta}_j|}{|z-\zeta_j|\,|z-\bar{\zeta}_j|} \leqq \frac{2}{|z|^2} \sum_{j=1}^{n-1} \frac{r_j}{\left(1-\frac{r_j}{|z|}\right)^2} + 2 \sum_{j=n+1}^{\infty} \frac{1}{r_j \left(1-\frac{|z|}{r_j}\right)^2} \leqq$$

$$\leqq \frac{8}{|z|^2} \sum_{j=1}^{n-1} r_j + 8 \sum_{j=n+1}^{\infty} r_j^{-1} \leqq 16 \left(\frac{r_{n-1}}{|z|^2} + \frac{1}{r_{n+1}}\right) =$$

$$= O\left(\frac{r_{n-1}}{r_n^2} + \frac{1}{r_{n+1}}\right) = O\left(\frac{r_n^{\frac{1}{3}}}{r_n^2} + \frac{1}{r_n^3}\right) = O\left(r_n^{-\frac{5}{3}}\right).$$

Из (14) получаем теперь, что

$$\left|\ln \frac{\varphi_n(\zeta)}{\varphi_n(\zeta_n)}\right| = O\left(r_n^{-\frac{5}{3}}\right). \tag{15}$$

Из (12), (13) и (15) следует, что

$$|\ln h_n(\zeta)| = K_n r_n^{-\frac{5}{4}} + O\left(r_n^{-\frac{5}{3}}\right),$$

где

$$2\delta r_n^{\frac{1}{4}} \geqq K_n \geqq 2\delta^2 \pi^{-1}.$$

Следовательно, при $n \to \infty$ на $|\zeta - \zeta_n| = 1$ выполняется

$$|h_n(\zeta) - 1| \sim |\ln h_n(\zeta)| \sim K_n r_n^{-\frac{5}{4}}. \tag{16}$$

Но

$$|f(\zeta) - \tau_n| \geqq |\tau_n|\,|h_n(\zeta) - 1| - \sum_{\substack{j=1 \\ j\neq n}}^{\infty} |\tau_j|\,|h_j(\zeta)| \geqq \tag{17}$$

$$\geqq d|h_n(\zeta) - 1| - \sum_{\substack{j=1 \\ j\neq n}}^{\infty} |h_j(\zeta)|.$$

При $|\zeta-\zeta_n|=1$, используя (5) и (8) с $k=0$ и с $k=p=[3/\delta]+2$ ($[x]$ — целая часть x), оценим

$$(18)\qquad \sum_{\substack{j=1\\ j\neq n}}^{\infty}|h_j(\zeta)| \leqq \frac{e}{\Psi(A)}\left\{\sum_{j=1}^{n-1}|\Phi_j(\zeta)|+\sum_{j=n+1}^{\infty}|\Phi_j(\zeta)|\right\}\leqq$$

$$\leqq \frac{e}{\Psi(A)}\left\{\sum_{j=1}^{n-1}\frac{p!}{\left(\frac{|\zeta|}{r_j}\right)^{\delta(p-1)}\cos^p\delta\pi}+\sum_{j=n+1}^{\infty}\left(\frac{|\zeta|}{r_j}\right)^{\delta}\right\}\leqq$$

$$\leqq \frac{e}{\Psi(A)}\left\{\frac{p!}{\cos^p\delta\pi}\frac{1}{|\zeta|^3}\sum_{j=1}^{n-1}r_j^3+|\zeta|^{\delta}\sum_{j=n+1}^{\infty}r_j^{-\delta}\right\}=$$

$$=O\left\{\left(\frac{r_{n-1}}{|\zeta|}\right)^3+\left(\frac{|\zeta|}{r_{n+1}}\right)^{\delta}\right\}=O\left\{\left(\frac{r_{n-1}}{r_n}\right)^3+\left(\frac{r_n}{r_{n+1}}\right)^{\delta}\right\}=O\left(\frac{1}{r_n^2}\right),$$

здесь мы воспользовались тем обстоятельством, что $r_{n+1}>r_n^{1+2/\delta}$ и $r_{n-1}<r_n^{1/3}$. Из (16), (17) и (18) следует, что при $|\zeta-\zeta_n|=1$ выполняется

$$|f(\zeta)-\tau_n|\geqq dK_n(1+o(1))r_n^{-5/4}+O(r_n^{-2})\geqq Mr_n^{-5/4},\quad n\geqq n_0,$$

где $M=\delta^2\pi^{-1}d>0$. Таким образом, лемма доказана.

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Примечание при корректуре (4. 12. 1967). Методом, использованным при доказательстве нашей теоремы, можно доказать следующее её обобщение. *Пусть A — произвольное ограниченное не более чем счетное множество. Тогда существует целая функция $G(z)$, такая, что $A\subset D(G)\subset\bar{A}$.* Действительно, считая A счетным множеством, можно показать, что существует последовательность односвязных конечнолистных римановых поверхностей D_n, накрывающих w-плоскость, без алгебраический точек ветвления, которые обладают следующими свойствами. Обозначим через π_n операцию проектирования D_n на w-плоскость. Тогда 1) $\bigcap_{n=1}^{\infty}\pi_n(D_n)=\bar{A}$; 2) на D_n существует такое счетное множество A_n, что расстояние от A_n до края D_n положительно и $\pi_n(A_n)=A$. Теперь с небольшими изменениями сохранится прежнее доказательство теоремы, лишь при определении функции $P_n(t)$ требуем, чтобы $\pi_n(P_n(0))\notin\bar{A}$, при определении множества T_n заменяем A на A_n, а функцию $F_n(z)$ определяем как $F_n(z)=\pi_n(P_n(f_n(L_n(z))))$. При определении величины μ_n требуем дополнительно, чтобы $|t-t'|\leqq\delta_n$, где величина $0<\delta_n<d_n/2$ выбрана так, что для любого t', $|t'|\leqq 1-d_n/2$, функция $P_n(t)$ однолистна в круге $|t-t'|\leqq\delta_n$.

Литература

[1] C. RÉNYI, Problem 3, Lecture notes prepared in connection with the Summer Institute on Entire Functions held at University of California, Amer. Math. Soc., 1966, P-1.
[2] Н. Н. Лузин, *Теория функций действительного переменного* (Учпедгиз, Москва, 1948).
[3] М. В. Келдыш, О приближении голоморфных функций целыми функциями, *ДАН СССР*, **47** (4), (1945), стр. 243—245.
[4] С. Н. Мергелян, Равномерные приближения функций комплексного переменного, *Успехи матем. наук*, **7** (2) (1952), стр. 31—122.
[5] W. K. HAYMAN, Interpolation by bounded functions, *Annales de l'Institut Fourier*, **8** (1) (1959), стр. 277—290.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (1—2), (1968), pp. 201—208.

ÜBER STARKE APPROXIMATION MIT HILFE EINER KLASSE VON INTERPOLATIONSPOLYNOMEN

Von
G. FREUD (Budapest)
(*Vorgelegt von* G. ALEXITS)

§ 1. Einleitung

Es sei $\alpha(x)$ $(-\infty < x < +\infty)$ eine nichtabnehmende Funktion mit unendlich vielen Wachstumspunkten, für welche die uneigentlichen Stieltjesschen Integrale

$$\int_{-\infty}^{+\infty} x^n \, d\alpha(x) = \lim_{\omega_1, \omega_2 \to \infty} \int_{-\omega_1}^{\omega_2} x^n \, d\alpha(x)$$

existieren und einen endlichen Wert haben. Es sei $p_n(d\alpha; x) = \gamma_n(d\alpha) x^n + \ldots$ $(n = 0, 1, \ldots)$ die Folge der normierten Orthogonalpolynome zu dieser Belegung. Es sei $\{A_n\}$ eine Folge reeller Zahlen und es seien $\xi_1^{(n)} > \xi_2^{(n)} > \ldots > \xi_n^{(n)}$ die Nullstellen von $\psi_n(x) = p_n(d\alpha; x) + A_n p_{n-1}(d\alpha; x)$; diese sind sämtlich reell und verschieden, und für ein beliebiges Polynom höchstens $2n-2$-ten Grades $\Pi_{2n-2}(x)$ gilt die Qadraturformel

$$\int_{-\infty}^{+\infty} \Pi_{2n-2}(x) \, d\alpha(x) = \sum_{\nu=1}^{n} \lambda_n(\xi_\nu^{(n)}) \Pi_{2n-2}(\xi_\nu^{(n)}) \tag{1}$$

mit

$$\lambda_n^{-1}(\xi) = \sum_{\nu=0}^{n-1} p_\nu^2(\xi) \quad \text{d. h.} \quad \lambda_n(\xi) > 0. \tag{2}$$

Es sei $\alpha_n(x)$ die Sprungfunktion mit den Sprungstellen $\xi_\nu^{(n)}$ $(\nu = 1, 2, \ldots, n)$ und den entsprechenden Sprüngen $\lambda_n(\xi_\nu^{(n)})$. Die Quadraturformel (1) lautet mit Hilfe dieser Bezeichnung

$$\int_{-\infty}^{+\infty} \Pi_{2n-2}(x) \, d\alpha(x) = \int_{-\infty}^{+\infty} \Pi_{2n-2}(x) \, d\alpha_n(x). \tag{3}$$

Es seien nun die natürlichen Zahlen μ und ν kleiner als n; dann folgt aus (3)

$$\int_{-\infty}^{+\infty} p_\mu(d\alpha; x) p_\nu(d\alpha; x) \, d\alpha_n(x) = \delta_{\mu\nu}$$

d. h. die ersten n Glieder der Folge

$$p_0(d\alpha; x), p_1(d\alpha; x), \ldots$$

sind nicht nur bezüglich der Belegung $d\alpha$, aber auch bezüglich der Belegung $d\alpha_n$ Orthogonal. (Vgl. J. SHOHAT [3].)

Ist $f(x)$ eine beliebige stetige reelle Funktion, dann hat es einen Sinn, von der (abbrechenden) Orthogonalentwicklung

$$f(x) \sim \sum_{\nu=0}^{n-1} a_\nu^{(n)}(f) p_\nu(d\alpha; x), \quad a_\nu^{(n)} = \int_{-\infty}^{+\infty} f(x) p_\nu(d\alpha; x) d\alpha_n(x) \tag{4}$$

zu sprechen.

Die Teilsumme vom Grade r ($r \leqq n-1$) dieser Entwicklung nennen wir $s_r^{(n)}(f; x)$. Gemäß der bekannten Eigenschaft der Teilsummen von Orthogonalentwicklungen gibt $\Pi_r(x) = s_r^{(n)}(f; x)$ das Minimum des Ausdruckes

$$\int_{-\infty}^{+\infty} [f(x) - \Pi_r(x)]^2 d\alpha_n(x) = \sum_{\nu=1}^{n} \lambda_n(\xi_\nu^{(n)}) [f(\xi_\nu^{(n)}) - \Pi_r(\xi_\nu^{(n)})]^2 \tag{5}$$

an, falls $\Pi_r(x)$ alle Polynome höchstens r-ten Grades durchläuft. Das ist eine interessante approximationstheoretische Deutung der Polynome $s_r^{(n)}(f; x)$. Wir möchten die Funktion $f(x)$ an den n Stellen $\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_n^{(n)}$ mit Hilfe eines Polynoms höchstens r-ten Grades interpolieren. Da wir — wenigstens für $r < n-1$ — zu viele Knotenpunkte haben, dürfen wir nicht fordern, daß unser Interpolationspolynom den Wert $f(\xi_\nu^{(n)})$ im Knotenpunkt $\xi_\nu^{(n)}$ genau annimmt. Es liegt nahe, daß man anstatt dessen verlangt, daß der Quadratische Mittel über die Knotenpunkte der Abweichung $f(x) - \Pi_r(x)$ klein wird. Der Fall $r = n-1$ führt zu dem Lagrangeschen Interpolationspolynom.

Man sieht aus Formel (5), daß dabei die Christoffelschen Zahlen $\lambda_n(\xi_\nu^{(n)})$ die Rolle natürlicher Gewichte spielen. Aufgabe der vorliegenden Arbeit ist die Untersuchung der starken Approximationseigenschaften dieser Polynome.

§ 2. Abschätzung der Lebesgue—Alexits-schen Funktionen

Zu der reellen Zahl x suchen wir die zwei benachbarten Knotenpunkte $\xi_\mu^{(n)}$ und $\xi_{\mu+1}^{(n)}$ auf, so daß $\xi_{\mu+1}^{(n)} \leqq x < \xi_\mu^{(n)}$ ist, und führen die Bezeichnungen

$$\lambda_n^-(x) = \lambda_n(\xi_{\mu+1}^{(n)}), \quad \lambda_n^+(x) = \lambda_n(\xi_\mu^{(n)})$$

ein. Für $x \geqq \xi_1^{(n)}$ sei $\lambda_n^+(x) = 0$ und $\lambda_n^{-1}(x) = \lambda_n(\xi_1^{(n)})$ bzw. für $x < \xi_n^{(n)}$ sei $\lambda_n^-(x) = 0$ und $\lambda_n^+(x) = \lambda_n(\xi_n^{(n)})$. Aus der Markoff—Stieltjes-schen Ungleichung folgt

$$\alpha_n(x) - \lambda_n^-(x) = \alpha(x) \leqq \alpha_n(x) + \lambda_n^+(x)$$

d. h.

$$\alpha(x) - \lambda_n^+(x) \leqq \alpha_n(x) \leqq \alpha(x) + \lambda_n^-(x). \tag{6}$$

Mit Hilfe der Christoffel—Darboux-schen Summenformel erhalten wir

$$s_r^{(n)}(f; x) = \int_{-\infty}^{+\infty} \sum_{\nu=0}^{r} p_\nu(d\alpha; x) p_\nu(d\alpha; t) f(t) d\alpha_n(t) = \tag{7}$$

$$= \frac{\gamma_r(d\alpha)}{\gamma_{r+1}(d\alpha)} \int_{-\infty}^{+\infty} [p_{r+1}(d\alpha; x) p_r(d\alpha; t) - p_r(d\alpha; x) p_{r+1}(d\alpha; t)] (x-t)^{-1} f(t) d\alpha_n(t).$$

Es sei $\|f\| = \sup_{-\infty<x<\infty} |f(t)|$.

Wir werden — zwecks späterer Anwendung — eine Abschätzung von

$$\text{(8)} \qquad \operatorname{Max}\operatorname{Sup}_{\|f\|\leqq 1} \frac{1}{R_2-R_1} \sum_{r=R_1}^{R_2-1} |s_r^{(n)}(f;x)| = M_n(R_1, R_2; x)$$

mit $1 \leqq R_1 < R_2 \leqq n-1$ ableiten.

In einer unlängst erschienenen Arbeit [1] zeigte G. ALEXITS, daß die Ausdrücke $M(R_1, R_2; x)$ in der Theorie der starken Approximation die gleiche Rolle spielen, wie die gewöhnlichen Lebesgueschen Funktionen in der Theorie der Konvergenz. Wir nennen sie deshalb die Lebesgue—Alexits-schen Funktionen der entsprechenden Approximationsfolge (in unserem Falle: der Folge $s_r^{(n)}(f)$; $r=0, 1, \ldots, n$). Es sei δ eine kleine positive Zahl, welche in einer später zu bestimmenden Weise von R_1 abhängt. Wir setzen

$$f_\delta(t) = \begin{cases} (x-t)^{-1} f(t) & \text{für } |x-t| > \delta \\ 0 & \text{für } |x-t| \leqq \delta . \end{cases}$$

Aus (7) folgt

$$\text{(9)} \qquad s_r^{(n)}(f;x) = \int_{x-\delta-0}^{x+\delta+0} \sum_{\nu=0}^{r} p_\nu(d\alpha; x) p_\nu(d\alpha; t) f(t)\, d\alpha_n(t) +$$

$$+ \frac{\gamma_r(d\alpha)}{\gamma_{r+1}(d\alpha)} [p_{r+1}(d\alpha; x) a_r^{(n)}(f_\delta) - p_r(d\alpha; x) a_{r+1}^{(n)}(f_\delta).$$

Unter der Voraussetzung $\|f(t)\| \leqq 1$ ist das erste Glied kleiner als

$$\int_{x-\delta-0}^{x+\delta+0} \left| \sum_{\nu=0}^{r} p_\nu(d\alpha; x) p_\nu(d\alpha; t) \right| d\alpha_n(t) \leqq$$

$$\leqq \sqrt{\int_{-\infty}^{+\infty} \left[\sum_{\nu=0}^{r} p_\nu(d\alpha; x) p_\nu(d\alpha; t) \right]^2 d\alpha_n(t)} \sqrt{\int_{x-\delta-0}^{x+\delta+0} d\alpha_n(t)} \leqq$$

$$\leqq \sqrt{\sum_{\nu=0}^{r} p_\nu^2(d\alpha; x) [\alpha_n(x+\delta+0) - \alpha_n(x-\delta-0)]} .$$

Infolge (6) ist also

$$\text{(10)} \qquad \int_{x-\delta-0}^{x+\delta+0} \left| \sum_{\nu=0}^{r} p_\nu(d\alpha; x) p_r(d\alpha; t) \right| d\alpha_n(t) \leqq$$

$$\leqq \sqrt{\lambda_r^{-1}(x) [\alpha(x+\delta+0) - \alpha(x-\delta-0) + \lambda_n^-(x+\delta) + \lambda_n^+(x-\delta)]} = \varepsilon_r^{(n)}(x, \delta).$$

Aus (9) folgt für $\|f\| \leqq 1$:

$$\text{(11)} \qquad |s_r^{(n)}(f;x)| \leqq \varepsilon_r^{(n)}(x, \delta) +$$

$$+ \frac{\gamma_r(d\alpha)}{\gamma_{r+1}(d\alpha)} [|p_{r+1}(d\alpha; x)|\, |a_r^{(n)}(f_\delta)| + |p_r(d\alpha; x)|\, |a_{r+1}^{(n)}(f_\delta)|].$$

Es sei

$$\Gamma(R_1, R_2) = \max_{R_1 \leqq r \leqq R_2} \frac{\gamma_r(d\alpha)}{\gamma_{r+1}(d\alpha)}; \tag{12}$$

dann folgt weiter (ebenfalls für $\|f\| \leqq 1$)

$$\sum_{r=R_1}^{R_2-1} |s_r^{(n)}(f; x)| \leqq \sum_{r=R_1}^{R_2-1} \varepsilon_r^{(n)}(x, \delta) + 2\Gamma(R_1, R_2) \sqrt{\lambda_{R_2}^{-1}(x) \sum_{r=R_1}^{R_2} [a_r^{(n)}(f_\delta)]^2}. \tag{13}$$

Aus der Besselschen Ungleichung ergibt sich

$$\sum_{r=R_1}^{R_2} [a_r^{(n)}(f_\delta)]^2 \leqq \sum_{r=0}^{n-1} [a_r^{(n)}(f_\delta)]^2 \leqq \int_{-\infty}^{x-\delta} \left[\frac{f(t)}{x-t}\right]^2 d\alpha_n(t) + \int_{x+\delta}^{\infty} \left[\frac{f(t)}{x-t}\right]^2 d\alpha_n(t),$$

also wegen $\|f\| \leqq 1$

$$\sum_{r=R_1}^{R_2} [a_r^{(n)}(f_\delta)]^2 \leqq \int_{-\infty}^{x-\delta} \frac{d\alpha_n(t)}{(x-t)^2} + \int_{x+\delta}^{\infty} \frac{d\alpha_n(t)}{(x-t)^2}. \tag{14}$$

Wir betrachten etwa den zweiten dieser Integrale. Mit Hilfe einer partiellen Integration ergibt sich

$$\int_{x+\delta}^{\infty} \frac{d\alpha_n(t)}{(x-t)^2} = \frac{\alpha_n(t)-\alpha_n(x)}{(t-x)^2}\Bigg|_{x+\delta}^{\infty} +$$

$$+2\int_{x+\delta}^{\infty} \frac{\alpha_n(t)-\alpha_n(x)}{(t-x)^3}\,dt \leqq 2\int_{x+\delta}^{\infty} \frac{\alpha_n(t)-\alpha_n(x)}{(t-x)^3}\,dt,$$

und unter Beachtung der Ungleichung (3)

$$\int_{x+\delta}^{\infty} \frac{d\alpha_n(t)}{(x-t)^2} \leqq 2\int_{x+\delta}^{\infty} \frac{\alpha(t)-\alpha(x)}{(t-x)^3}\,dt + 2\lambda_n^+(x)\int_{x+\delta}^{\infty} \frac{dt}{(t-x)^3} + 2\int_{x+\delta}^{\infty} \frac{\lambda_n^-(t)}{(t-x)^3}\,dt.$$

Genau so ergibt sich

$$\int_{-\infty}^{x-\delta} \frac{d\alpha_n(t)}{(x-t)^2} \leqq 2\int_{-\infty}^{x-\delta} \frac{\alpha(x)-\alpha(t)}{(x-t)^3}\,dt + 2\lambda_n^-(x)\int_{-\infty}^{x-\delta} \frac{dt}{(x-t)^3} + 2\int_{-\infty}^{x-\delta} \frac{\lambda_n^+(t)}{(t-x)^3}\,dt,$$

so daß aus (14)

$$\sum_{r=R_1}^{R_2} [a_r^{(n)}(f_\delta)]^2 \leqq \int_{-\infty}^{x-\delta} \frac{\alpha(x)-\alpha(t)}{(x-t)^3}\,dt + \tag{15}$$

$$+\int_{x+\delta}^{\infty} \frac{\alpha(t)-\alpha(x)}{(t-x)^3}\,dt + \delta^{-2}[\lambda_n^-(x)+\lambda_n^+(x)] + 2\int_{-\infty}^{x-\delta} \frac{\lambda_n^+(t)}{(x-t)^3}\,dt + 2\int_{x+\delta}^{\infty} \frac{\lambda_n^-(t)}{(t-x)^3}\,dt$$

folgt. Wir vereinigen jetzt die Abschätzungen (10), (13) und (15), und beachten, daß die rechte Seite von (11) infolge (2) eine wachsende Funktion von r ist:

$$(16)\qquad \sum_{r=R_1}^{R_2-1} |s_r^{(n)}(f;x)| \leqq (R_2-R_1)\lambda_{R_2}^{-1/2}(x)\cdot$$

$$\cdot[\alpha(x+\delta+0)-\alpha(x-\delta-0)+\lambda_n^-(x+\delta)+\lambda_n^+(x-\delta)]^{1/2}+$$

$$+2\Gamma(R_1,R_2)\lambda_{R_2}^{-1/2}(x)\left\{\int_{-\infty}^{x-\delta}\frac{\alpha(x)-\alpha(t)}{(x-t)^3}\,dt+\right.$$

$$\left.+\int_{x+\delta}^{\infty}\frac{\alpha(t)-\alpha(x)}{(t-x)^3}\,dt+\delta^{-2}[\lambda_n^-(x)+\lambda_n^+(x)]+2\int_{-\infty}^{x-\delta}\frac{\lambda_n^+(t)}{(x-t)^3}\,dt+2\int_{x+\delta}^{\infty}\frac{\lambda_n^-(t)}{(x-t)^3}\,dt\right\}^{1/2}.$$

Die Abschätzung (16) bildet die Grundlage unserer weiteren Untersuchung.

§ 3. Sätze über gleichmäßige starke Approximation durch $s_r^{(n)}(x)$

Es sei in diesem Paragraphen vorausgesetzt, daß die Belegung $d\alpha$ innerhalb eines endlichen Intervalls $[a, b]$ liegt, d. h. es ist

$$\alpha(x)=\alpha(a-0)\quad\text{für}\quad x<a\quad\text{und}\quad\alpha(x)=\alpha(b+0)\quad\text{für}\quad x>b.$$

Es sei weiter vorausgesetzt, daß $\alpha(x)$ in einem echten inneren Teilintervall $[c, d]\subset(a, b)$ absolut stetig und außerdem

$$(17)\qquad 0<m\leqq\alpha'(x)\leqq M\quad x\in[c,d]$$

gültig ist. Aus (17) folgt für jedes echte innere Teilintervall $[c_1, d_1]\subset(c, d)$

$$(18)\qquad \frac{K_1(c_1,d_1)}{\nu}\leqq\lambda_\nu(d\alpha;x)\leqq\frac{K_2(c_1,d_1)}{\nu}\quad\text{für}\quad x\in[c_1,d_1].$$

Wir berufen uns noch auf den bekannten Satz, daß — unter der Voraussetzung, daß die Belegung in einem endlichen Intervall liegt — die Knotenpunkte $\xi_\nu^{(n)}$ sich um jede Wachstumstelle von $\alpha(x)$ häufen. Aus (17) folgt dann weiter

$$(18a)\qquad \frac{K_3(c_1,d_1)}{n}\leqq\lambda_n^-(x)\leqq\frac{K_4(c_1,d_1)}{n}\quad\text{für}\quad x\in[c_1,d_1]$$

und

$$(18b)\qquad \frac{K_3(c_1,d_1)}{n}\leqq\lambda_n^+(x)\leqq\frac{K_4(c_1,d_1)}{n}\quad\text{für}\quad x\in[c_1,d_1].$$

Es gilt ferner

$$(19)\qquad \Gamma(R_1,R_2)\leqq\operatorname{Sup}\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\leqq\frac{b-a}{2}.$$

Es seien jetzt $\Theta_2>\Theta_1>1$ zwei feste, von n und ν unabhängige Zahlen.

Es sei ν eine positive ganze Zahl und ν^* eine von ν abhängende positive ganze Zahl, so daß die Ungleichungen

$$1 \leqq \nu < \Theta_1 \nu \leqq \nu^* \leqq \Theta_2 \nu \quad \text{und} \quad \nu^* < n \tag{20}$$

befriedigt sind.

Es sei jetzt $[c', d'] \subset (c, d)$ ein beliebiges echtes inneres Teilintervall von $[c, d]$ und wir wählen c_1, d_1 so, daß $c < c_1 < c'$ und $d' < d_1 < d$ erfüllt seien. Wir setzen in Formel (16) $\delta = \nu^{-1}$; es sei ferner $x \in [c', d']$. Dann gilt wegen (17) für $x + \nu^{-1} < d$

$$\int_{x+\nu^{-1}}^{\infty} \frac{\alpha(t)-\alpha(x)}{(t-x)^2}\,dt \leqq M \int_{x+\nu^{-1}}^{d} \frac{dt}{(t-x)^2} + [\alpha(b)-\alpha(a)] \int_{d}^{\infty} \frac{dt}{(t-x)^2} =$$

$$= M\nu + \frac{\alpha(b)-\alpha(a)}{d-x} \leqq M\nu + \frac{\alpha(b)-\alpha(a)}{d-d'}.$$

Für $x + \nu^{-1} \geqq d$ ergibt sich sogar, daß das Integral links kleiner als $\dfrac{\alpha(b)-\alpha(a)}{d-d'}$ ist, wir können also die Voraussetzung $x + \nu^{-1} < d$ streichen. Genau so ergibt sich

$$\int_{-\infty}^{x-\nu^{-1}} \frac{\alpha(x)-\alpha(t)}{(x-t)^3}\,dt \leqq M\nu + \frac{\alpha(b)-\alpha(a)}{c'-c}.$$

Wir kehren zur Abschätzung von

$$\int_{-\infty}^{x-\nu^{-1}} \frac{\lambda_n^+(t)}{(x-t)^3}\,dt.$$

Wir wenden neben (18b) die triviale Abschätzung

$$\lambda_n^+(t) \leqq \operatorname*{Max}_{\nu} \lambda_n(\xi_\nu^{(n)}) \leqq \sum_{\nu=1}^{n} \lambda_n(\xi_\nu^{(n)}) = \alpha(b)-\alpha(a)$$

an. Es ergibt sich für $x - \nu^{-1} > c_1$ unter Beachtung von $\nu < n$

$$\int_{-\infty}^{x-\nu^{-1}} \frac{\lambda_n^+(t)}{(x-t)^3}\,dt \leqq \frac{K_4(c_1, d_1)}{n} \int_{a_1}^{x-\nu^{-1}} \frac{dt}{(x-t)^3} + [\alpha(b)-\alpha(a)] \int_{-\infty}^{c_1} \frac{dt}{(x-t)^3} \leqq$$

$$\leqq 2K_4(c_1, d_1)\nu + [\alpha(b)-\alpha(a)](c'-c_1)^{-2};$$

für $x - \nu^{-1} \leqq c_1$ ist unser Integral links sogar kleiner als das zweite Glied rechts. Unsere letzte Formel gilt also für beliebige Werte von ν (nicht nur für $x + \nu^{-1} > c_1$). In der gleichen Weise erhält man

$$\int_{x+\nu^{-1}}^{\infty} \frac{\lambda_n^-(t)}{(t-x)^3}\,dt \leqq K_4(c_1, d_1)\nu + [\alpha(b)-\alpha(a)](d_1-d')^{-2}.$$

Indem wir $\delta = \nu^{-1}$, $R_1 = \nu$, $R_2 = \nu^*$ wählen und alle entsprechende Abschätzungen dieses Paragraphen in Formel (16) einsetzen, erhalten wir, daß für jede stetige

Funktion $f(x)$ mit $\|f(x)\| \leqq 1$, für jedes Paar ν, ν^* ganzer Zahlen, welche (20) befriedigen und an jeder Stelle $x \in [c', d']$

$$\frac{1}{\nu^* - \nu} \sum_{r=\nu}^{\nu^*-1} |s_r^{(n)}(f; x)| \leqq K_5(\Theta_1, \Theta_2, c', d') \tag{21}$$

gültig ist. Die Konstante an der rechten Seite ist von ν, ν^* und x unabhängig, vorausgesetzt, daß (20) und $x \in [c', d']$ befriedigt ist.

Wir bezeichnen — wie üblich — mit $E_n(f) = E_n(f, a, b)$ die beste Approximation von $f(x)$ durch Polynome höchstens n-ten Grades, d. h.

$$E_n(f) = \operatorname{Min} \|f(x) - \Pi_n(x)\|$$

falls $\Pi_n(x)$ alle Polynome höchstens n-ten Grades durchläuft. Aus dem Satze von G. ALEXITS [1] folgt dann

SATZ I. *Der Träger der Belegung $d\alpha$ liege im endlichen Intervall $[a, b]$, es sei $\alpha(x)$ in $[c, d] \subset (a, b)$ absolut stetig und es sei* (17) *erfüllt. Dann gilt für jedes Paar ganzer Zahlen ν, ν^*, welche* (20) *befriedigen und jede in $[a, b]$ stetige Funktion $f(x)$*

$$\frac{1}{\nu^* - \nu} \sum_{r=\nu}^{\nu^*-1} |f(x) - s_r^{(n)}(f; x)| \leqq K_6(\Theta_1, \Theta_2; c', d') E_\nu(f) \tag{22}$$

für $x \in [c', d']$.

Aus dem Satze I folgt[1]

SATZ II. *Unter den gleichen Voraussetzungen bezüglich $\alpha(x)$, wie in Satz* 1 *gilt für jedes $s > 0$ und $k < n$*

$$\sum_{\nu=0}^{k} (\nu+1)^{s-1} |f(x) - s_\nu^{(n)}(f; x)| \leqq \tag{23}$$

$$\leqq K_7(s; c'\, d') \sum_{\nu=0}^{k} (\nu+1)^{s-1} E_\nu(f), \quad \textit{für} \quad x \in [c', d'].$$

Aus dem Satz 1 folgt mit Hilfe eines Resultates des Verfassers (siehe [2], Satz 2):

SATZ III. *Es seien die Voraussetzungen des Satzes* I *erfüllt, ferner sei $f(x)$ auch m-mal stetig differenzierbar in $[a, b]$, dann gilt*

$$\frac{1}{\nu^* - \nu} \sum_{r=\nu}^{\nu^*-1} |f^{(m)}(x) - [s_r^{(n)}(f; x)]^{(m)}| \leqq \tag{24}$$

$$\leqq K_8(m; \Theta_1, \Theta_2; c', d') E_\nu(f^{(m)}) \quad \textit{für} \quad x \in [c', d'].$$

Hier bedeutet $[s_r^{(n)}(f; x)]^{(m)}$ die m-te Ableitung von $s_r^{(n)}(f; x)$.

Aus dem Satze 3 folgt[1]

[1] Vgl. etwa den Hilfssatz in der Arbeit [2] des Verfassers.

SATZ IV. *Bezüglich $\alpha(x)$ seien die Voraussetzungen des Satzes* I *befriedigt, und es sei $f(x)$ in $[a, b]$ m-mal stetig differenzierbar; dann gilt für jedes $s>0$ und $k<n$*

$$\sum_{\nu=0}^{k}(\nu+1)^{s-1}\,|f^{(m)}(x)-[s_\nu^{(n)}(f;x)]^{(m)}|\leqq \tag{25}$$

$$\leqq K_9(s;m;c',d')\sum_{\nu=0}^{k}(\nu+1)^{s-1}E_\nu(f) \quad \textit{für} \quad x\in[c',d'].$$

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Literaturverzeichnis

[1] G. ALEXITS, Einige Beiträge zur Approximationstheorie, *Acta Sci. Math. (Szeged)*, **26** (1965), S. 211—224.

[2] G. FREUD, Über starke Approximation durch differenzierte Folgen von approximierenden Polynomen, *Studia Sci. Math. Hung.*, **2** (1967), S. 221—226.

[3] J. SHOHAT, Application of orthogonal Tchebycheff polynomials to Lagrangean interpolation and to the general theory of polynomials, *Annali di Mat.*, **18** (1939), S. 201—238.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (1—2), (1968), pp. 209—218.

ON SETS OF ALMOST DISJOINT SUBSETS OF A SET

By
P. ERDŐS (Budapest), member of the Academy, A. HAJNAL (Budapest)
and E. C. MILNER (Reading, England)

1. Introduction

The cardinal power of a set A is denoted by $|A|$. Two sets A_1, A_2 are said to be *almost disjoint* if

$$|A_1 \cap A_2| < |A_i| \quad (i = 1, 2).$$

We call B a *transversal* of the disjoint non-empty sets A_v $(v \in M)$ if $B \subset \bigcup_{v \in M} A_v$ and B intersects each A_v $(v \in M)$ in a singleton.

An old and well known theorem of W. SIERPINSKI is that an infinite set of power **m** contains more than **m** subsets of power **m** which are pairwise almost disjoint and A. TARSKI obtained various generalizations and extensions of this in [1] and [2]. It is easy to see that Sierpinski's result is equivalent to the following statement: *If* A_v $(v \in M)$ *are* **m** *disjoint sets of power* **m**, *then there are more than* **m** *almost disjoint transversals of the* A_v. In § 3 we prove some new results which are analogous to this formulation of Sierpinski's theorem.

We will denote the following statement by $\mathscr{H}$: There are $\aleph_2$ almost disjoint transversals of $\aleph_1$ disjoint denumerable sets. In view of recent axiomatic results $\mathscr{H}$ is independent of the usual axioms of set theory and the generalized continuum hypothesis. In § 4 we show that $\mathscr{H}$ implies a certain unsolved problem of [3].

In § 5 we consider another question about sets of almost disjoint subsets of a set which was raised by F. S. CATER [4].

2. Notation

Capital letters always denote sets and $\mathscr{F}$ denotes a set whose members are sets. We write $\bigcup \mathscr{F}$ to denote the union of all the members of $\mathscr{F}$. The set-theoretic difference of A and B is $A - B$. Bold lower case latin letters denote cardinals and greek letters denote ordinal numbers. If S is a well-ordered set of type α, then the cardinal of α is the same as the cardinal of S and is denoted by $|\alpha|$. The smallest ordinal number with cardinal **m** is denoted by $\omega(\mathbf{m})$. As is customary we write ω_α instead of $\omega(\aleph_\alpha)$ and ω instead of ω_0. The set of ordinal numbers $\{v: \alpha \leqq v < \beta\}$ is denoted by $[\alpha, \beta)$. The obliterator sign $\hat{}$ written above any symbol indicates that that symbol is to be disregarded. For example, we sometimes write $A_0 \cup \ldots \cup \hat{A}_\lambda$ instead of $\bigcup_{v<\lambda} A_v$.

The smallest cardinal greater than **m** is called the *successor* of **m** and is denoted by $\mathbf{m}^+$. If **a** is not a successor cardinal (i. e. $\mathbf{a} \neq \mathbf{b}^+$ for any **b**), then **a** is called a *limit* cardinal. The *cofinality cardinal* of **a**, denoted by **a′**, is the smallest cardinal **m** which is such that **a** can be expressed as the sum of **m** cardinals each less than **a**. In the

notation of Tarski, $\aleph'_\alpha = \aleph_{cf(\alpha)}$. $\mathbf{a}$ is *regular* if $\mathbf{a}' = \mathbf{a}$ and *singular* if $\mathbf{a}' < \mathbf{a}$. A cardinal is *inaccessible* if it is a regular limit number. It is not known if there are inaccessible cardinals greater than $\aleph_0$ but the assumption that there are not is known to be consistent with the axioms of set theory.

If B is a set of ordinal numbers we call β a limit point of B if β is the limit of an increasing sequence of members of B. B is *closed in* A if all the limit points of B which are in A are also in B. B is a *cofinal* subset of $[0, \lambda)$ if for any $\nu < \lambda$ there is $\beta \in B$ such that $\nu \leqq \beta < \lambda$. B is a *band* in $[0, \lambda)$ if it is a closed cofinal subset. If S is a set of ordinal numbers and f is an ordinal-valued function on S such that $f(\nu) < \nu$ for all arguments $\nu (\neq 0)$ in S, then f is called a *regressive* function on S. A *stationary value* of such a function is an ordinal number Θ such that $|\{\nu: \nu \in S, f(\nu) = \Theta\}| = |S|$. A well known result of ALEXANDROFF and URYSOHN is that, if $\mathbf{m}$ is a regular cardinal greater than $\aleph_0$, then any regressive function on $[0, \omega(\mathbf{m}))$ has a stationary value. A more general theorem of W. NEUMER [5] is the following: *Let* $\mathbf{m} = \mathbf{m}' > \aleph_0$ *and let* S *be a subset of* $[0, \omega(\mathbf{m}))$ *of power* $\mathbf{m}$. *Then every regressive function on* S *has a stationary value if and only if the complement* $[0, \omega(\mathbf{m})) - S$ *contains no band of* $[0, \omega(\mathbf{m}))$. A set satisfying this condition is said to be *stationary*.

The theorem of Sierpinski stated in § 1 does not depend for its proof on the generalized continuum hypothesis (g. c. h.) that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ — in fact, not even the axiom of choice is required in the case $\mathbf{m} = \aleph_0$. In this paper we always assume the axiom of choice and sometimes we use the g. c. h. or some weaker hypothesis, but we always indicate when this hypothesis is employed.

3. Transversals of disjoint sets

THEOREM 1. *Let* A_ν $(\nu < \omega_{\alpha+1})$ *be* $\aleph_{\alpha+1}$ *disjoint sets each of power* $\aleph_\alpha$. *Then there is a set,* $\mathscr{F}$, *of transversals of the* A_ν *such that* $|\mathscr{F}| = \aleph_{\alpha+1}$ *and*

$$|F \cap F'| < \aleph_\alpha \quad (F \neq F'; \quad F, F' \in \mathscr{F}). \tag{1}$$

PROOF. We can assume that $A_\nu = \{\xi_{\nu\mu}: \mu \leqq \max\{\nu, \omega_\alpha\}\}$ $(\nu < \omega_{\alpha+1})$. Let $\lambda < \omega_{\alpha+1}$ and suppose the transversals F_ϱ have already been defined for $\varrho < \lambda$. Put $\pi = \min\{\lambda, \omega_\alpha\}$ and let f be a 1—1 map of $[0, \pi)$ onto $[0, \lambda)$. If $\varrho < \omega_\alpha$, then $T_\varrho = A_{f(\varrho)} - \bigcup_{\sigma<\varrho} F_{f(\sigma)} \neq \emptyset$ and we can choose $x_{f(\varrho)} \in T_\varrho$. Now put

$$F_\lambda = \{x_{f(\varrho)}: \varrho < \pi\} \cup \{\xi_{\nu\lambda}: \nu \in [\lambda, \omega_{\alpha+1})\}.$$

This defines F_λ for $\lambda < \omega_{\alpha+1}$ by induction. It is clear from the construction that F_λ is a transversal of the A_ν. Also, if $\mu < \lambda < \omega_{\alpha+1}$ then $\mu = f(\varrho)$ for some $\varrho < \pi$ and

$$F_\lambda \cap F_\mu \subset \{x_{f(0)}, \ldots, x_{f(\varrho)}\},$$

i. e. $|F_\lambda \cap F_\mu| < \aleph_\alpha$. Thus the set $\mathscr{F} = \{F_\lambda: \lambda < \omega_{\alpha+1}\}$ has the properties described.

A. TARSKI [1] proved: *If* $\mathscr{F}$ *is a set of subsets of a set of power* $\mathbf{m}$ *and if* $|F \cap F'| < p$ *for distinct members* $F, F' \in \mathscr{F}$, *then* $|\mathscr{F}| \leqq \mathbf{m}^{\mathbf{p}}$. It follows from this and the g. c. h. that if $\mathscr{F}$ is any set of transversals of $\aleph_{\alpha+1}$ sets of power $\aleph_\alpha$ such that (1) holds, then $|\mathscr{F}| \leqq \aleph_{\alpha+1}^{\aleph_\alpha} = \aleph_{\alpha+1}$. In this sense Theorem 1 is the best possible result. The total number of different transversals of the A_ν is $\aleph_\alpha^{\aleph_{\alpha+1}} \geqq \aleph_{\alpha+2}$.

Our next theorem has some relevance to problem $\mathscr{H}$.

THEOREM 2. *Let A_ν $(\nu<\omega_{\alpha+1})$ be $\aleph_{\alpha+1}$ disjoint sets of power $\aleph_\alpha$. Then there is a maximal set, $\mathscr{F}$, of $\aleph_{\alpha+1}$ almost disjoint transversals of the A_ν, i.e. if B is any transversal of the A_ν, then there is some $F\in\mathscr{F}$ such that $|F\cap B|=\aleph_{\alpha+1}$.*

PROOF. Since $|A_\nu|=\aleph_\alpha$ we can assume that

$$A_\nu=\{\xi_{\nu\mu}:\mu<\max\{\nu,\omega_\alpha\}\}\quad(\nu<\omega_{\alpha+1}).$$

For $\lambda<\omega_{\alpha+1}$ put

$$F_\lambda=\{\xi_{\nu 0}:\nu\leqq\lambda\}\cup\{\xi_{\nu\lambda}:\lambda<\nu<\omega_{\alpha+1}\}.$$

Then $\mathscr{F}=\{F_\lambda:\lambda<\omega_{\alpha+1}\}$ is a maximal set of almost disjoint transversals of the A_ν. To see this consider any transversal B. By the definition of the F_λ we have that

$$A_\nu\subset\bigcup_{\lambda<\nu}F_\lambda\quad\textit{if}\quad\nu\in S=[\omega_\alpha,\omega_{\alpha+1}).$$

Therefore, for each $\nu\in S$, there is $f(\nu)<\nu$ such that

$$B\cap F_{f(\nu)}\cap A_\nu\neq\emptyset.$$

Since f is regressive on S, there is $\gamma<\omega_{\alpha+1}$ such that $N_\gamma=\{\nu:\nu\in S,f(\nu)=\gamma\}$ has power $\aleph_{\alpha+1}$. Since the A_ν are disjoint it follows that $|B\cap F_\gamma|\geqq|N_\gamma|=\aleph_{\alpha+1}$.

The remaining theorems in this section are concerned with almost disjoint transversals of sets A_ν which do not necessarily have the same power. Let $\lambda=\omega(\mathbf{m}')$ and let $A_0,\ldots,\hat{A}_\lambda$ be $\mathbf{m}'$ disjoint sets which satisfy

$$0<|A_0|\leqq|A_1|\leqq\ldots\leqq|\hat{A}_\lambda|\leqq\mathbf{m}=\lim_{\nu<\lambda}|A_\nu|. \tag{2}$$

By Kőnig's theorem, the total number of transversals is

$$|A_0|\cdot|A_1|\ldots|\hat{A}_\lambda|>|A_0\cup A_1\cup\ldots\cup\hat{A}_\lambda|=\mathbf{m}.$$

In Theorem 3 we show that there are $\mathbf{m}^+$ almost disjoint transversals *if* $\mathbf{m}'=\aleph_0$. The corresponding statement in the case $\mathbf{m}'>\aleph_0$ is not true. In this case the existence or non-existence of $\mathbf{m}^+$ almost disjoint transversals depends upon whether or not some extra condition on the cardinals $|A_\nu|$ $(\nu<\lambda)$ is satisfied (Theorems 5, 6).

THEOREM 3. *Let $\mathbf{m}'=\aleph_0$ and let A_ν $(\nu<\omega)$ be $\aleph_0$ disjoint sets which satisfy the condition* (2). *Then there are $\mathbf{m}^+$ almost disjoint transversals of the A_ν.*

PROOF. There are cardinals $\mathbf{m}_\nu<\mathbf{m}$ $(\nu<\omega)$ such that

$$1\leqq\mathbf{m}_0\leqq\mathbf{m}_1\leqq\ldots<\mathbf{m}=\mathbf{m}_0+\mathbf{m}_1+\ldots.$$

If $\theta<\omega$, then by (2) there is $\nu_\theta<\omega$ such that

$$|A_\nu|>\mathbf{m}_0+\ldots+\hat{\mathbf{m}}_\theta=\mathbf{n}_\theta\quad(\nu_\theta\leqq\nu<\omega), \tag{3}$$

and we can assume that $0=\nu_0<\nu_1<\ldots$.

We shall define $\mathbf{m}^+$ almost disjoint transversals F_μ $(\mu<\omega(\mathbf{m}^+))$ by induction. Let $\mu<\omega(\mathbf{m}^+)$ and suppose that we have already defined the transversals F_ϱ $(\varrho<\mu)$.

Since $|\mu| \leqq \mathbf{m}$, we may write

$$\{F_0, \ldots, \hat{F}_\mu\} = \mathscr{F}_0 \cup \mathscr{F}_1 \cup \ldots \cup \hat{\mathscr{F}}_\omega,$$

where $|\mathscr{F}_\nu| \leqq \mathbf{m}_\nu$ $(\nu < \omega)$. By (3) there is

$$x_\nu \in A_\nu - \bigcup_{\varphi<\theta} \cup \mathscr{F}_\varphi \quad (\nu_\theta \leqq \nu < \nu_{\theta+1}; \quad \theta < \omega).$$

Then $F_\mu = \{x_\nu : \nu < \omega\}$ is a transversal which intersects each F_ϱ $(\varrho < \mu)$ in a finite set. This completes the proof.

Assuming the hypothesis (4) (which is true if $\mathbf{m} = \aleph_0$ and is implied by the g. c. h. if $\mathbf{m} > \aleph_0$), the next theorem shows that, if $\mathbf{m}' = \aleph_0$ then no set of $\mathbf{m}$ almost disjoint transversals of denumerably many disjoint sets is maximal. Theorem 3 can easily be deduced from this. We cannot prove Theorem 4 without the hypothesis (4).

THEOREM 4. *Suppose that* $\mathbf{m}' = \aleph_0$ *and*

$$2^{\mathbf{n}} < \mathbf{m} \quad \textit{if} \quad \mathbf{n} < \mathbf{m}. \tag{4}$$

Let $\mathscr{F}$ *be any set of power* $\mathbf{m}$ *of almost disjoint transversals of* $\aleph_0$ *disjoint sets* A_ν $(\nu < \omega)$. *Then* $\mathscr{F}$ *is not maximal, i. e. there is a transversal which is almost disjoint from each member of* $\mathscr{F}$.

PROOF. If $\theta < \omega$ we will show that there is $\nu_\theta < \omega$ such that (3) holds. Suppose this is false. Then there is an infinite set $I \subset [0, \omega)$ such that

$$|A_\nu| \leqq \mathbf{n}_\theta \quad (\nu \in I). \tag{5}$$

Case 1. $\mathbf{m} = \aleph_0$. Then n_θ is finite. Let $\mathscr{F}^*$ be a subset of $\mathbf{n}_{\theta+1}$ members of $\mathscr{F}$. Since $\mathscr{F}^*$ is a finite set of almost disjoint transversals, there is $\pi < \omega$ such that

$$F \cap A_\nu \neq F' \cap A_\nu$$

if $\pi \leqq \nu < \omega$ and F, F' are different members of $\mathscr{F}^*$. There is $\gamma \in I$ such that $\gamma > \pi$ and we have the contradiction that

$$|A_\gamma| \geqq |A_\gamma \cap \cup \mathscr{F}^*| > \mathbf{n}_\theta.$$

Case 2. $\mathbf{m} > \aleph_0$. Since $\mathbf{n}_\theta < \mathbf{m}$, it follows from (4) and (5) that the total number of distinct transversals of the sets A_ν $(\nu \in I)$ is at most $\mathbf{n}_\theta^{\aleph_0} \leqq 2^{\mathbf{n}_\theta \aleph_0} < \mathbf{m}$. Therefore, there are distinct members F, $F' \in \mathscr{F}$ such that $F \cap A_\nu = F' \cap A_\nu$ $(\nu \in I)$ and this contradicts the fact that the members of $\mathscr{F}$ are almost disjoint. The theorem now follows by precisely the same argument used in the last paragraph of the proof of Theorem 3.

There is no analogue of Theorem 4 for cardinals not cofinal with $\aleph_0$. For example, if $A_\nu = \{\xi_{\nu\mu} : \mu < \omega_\nu\}$ $(\nu < \omega_1)$ and

$$F_\sigma = \{\xi_{\nu 0} : \nu \leqq \varrho\} \cup \{\xi_{\nu\sigma} : \varrho < \nu < \omega_1\} \quad (\omega_\varrho \leqq \sigma < \omega_{\varrho+1}; \quad \varrho < \omega_1),$$

then there is a maximal set of almost disjoint transversals, $\mathscr{F}$, which contains the set $\{F_\sigma : \omega \leqq \sigma < \omega_{\omega_1}\}$, i. e. $|\mathscr{F}| \geqq \aleph_{\omega_1}$. The sets A_ν $(\nu < \omega_1)$ do not have the pro-

perty $\mathscr{P}(\aleph_1)$ defined below and so, by Theorem 6 and the g. c. h., $|\mathscr{F}| \leqq \aleph_{\omega_1}$. On the other hand, if $|B_\nu| = \aleph_{\nu+1}$ $(\nu < \omega_1)$, then the sets B_ν do have the property $\mathscr{P}(\aleph_1)$ and therefore, by Theorem 5, there is a set of $\aleph_{\omega_1+1}$ almost disjoint transversals of the B_ν.

A set of cardinal numbers $M = \{\mathbf{m}_0, \ldots, \mathbf{m}_\lambda\}_<$* is *closed* if it contains the limits of all increasing sequences in M, i. e. if ϱ is a limit number and $\nu_\sigma < \lambda$ $(\sigma < \varrho)$, then

$$\lim_{\sigma<\varrho} \mathbf{m}_{\nu_\sigma} = \mathbf{m}_\nu,$$

where $\nu = \lim_{\sigma<\varrho} \nu_\sigma$. We prove the following simple lemma.

LEMMA. *Let* $\mathbf{m}$ *be a limit cardinal not cofinal with* $\aleph_0$, *let* $\lambda = \omega(\mathbf{m}')$ *and let* $\{\mathbf{m}_0, \ldots, \hat{\mathbf{m}}_\lambda, \mathbf{m}\}_<$ *and* $\{\mathbf{n}_0, \ldots, \hat{\mathbf{n}}_\lambda, \mathbf{m}\}_<$ *be closed sets of cardinals. Then*

$$B = \{\nu : \nu < \lambda, \quad \mathbf{m}_\nu = \mathbf{n}_\nu\}$$

is a band in $[0, \lambda)$.

PROOF. It is clear that B is closed in $[0, \lambda)$. Suppose there is $\nu_0 < \lambda$ such that $\mathbf{m}_\nu \neq \mathbf{n}_\nu$ for $\nu > \nu_0$. There are ordinals $\nu_\varrho < \lambda$ $(\varrho < \omega)$ such that $\nu_0 < \nu_1 < \nu_2 < \ldots$ and

$$\mathbf{m}_{\nu_0} < \mathbf{n}_{\nu_1} < \mathbf{m}_{\nu_2} < \mathbf{n}_{\nu_3} < \ldots .$$

If $\nu = \lim_{\varrho<\omega} \nu_\varrho$, then $\nu < \lambda$ since $\mathbf{m}' > \aleph_0$ and

$$\mathbf{m}_\nu = \lim_{\varrho<\omega} \mathbf{m}_{\nu_\varrho} = \lim_{\varrho<\omega} \mathbf{n}_{\nu_\varrho} = \mathbf{n}_\nu .$$

This contradiction proves that B is cofinal, and hence a band, in $[0, \lambda)$.

Let $\mathbf{m}$ be any limit cardinal and let $\lambda = \omega(\mathbf{m}')$. Then there are cardinals $\mathbf{m}_0, \ldots, \hat{\mathbf{m}}_\lambda$ such that

$$\mathbf{m}_0 < \mathbf{m}_1 < \ldots < \hat{\mathbf{m}}_\lambda < \mathbf{m} = \lim_{\nu<\lambda} \mathbf{m}_\nu$$

and we can assume that the set $\{\mathbf{m}_0, \ldots, \hat{\mathbf{m}}_\lambda, \mathbf{m}\}_<$ is closed (if not, the closure of the set also contains $\mathbf{m}'$ cardinals). Let A_ν $(\nu < \lambda)$ be $\mathbf{m}'$ disjoint sets. We say that these sets have the property $\mathscr{P}(\mathbf{m})$ if (2) holds and the set

$$C = \{\nu : 1 \leqq \nu < \lambda, \quad |A_\nu| \leqq \mathbf{m}_\nu\}$$

is non-stationary in $[0, \lambda)$. This definition of $\mathscr{P}(\mathbf{m})$ does not depend upon the particular choice of the $\mathbf{m}_\nu$ $(\nu < \lambda)$. For suppose that $\{\mathbf{n}_0, \ldots, \hat{\mathbf{n}}_\lambda, \mathbf{m}\}_<$ is another closed set of cardinals and

$$C' = \{\nu : 1 \leqq \nu < \lambda, \quad |A_\nu| \leqq \mathbf{n}_\nu\}.$$

If C is stationary then so is C' since $C' \supset C \cap B$, where B is the set defined in the lemma, and the intersection of a stationary set and a band is also stationary.** Conversely, C is stationary if C' is.

* The symbol $\{\mathbf{m}_0, \ldots, \mathbf{m}_\lambda\}_<$ indicates that $\mathbf{m}_0 < \mathbf{m}_1 < \ldots < \mathbf{m}_\lambda$.

** See H. BACHMANN [6] page 41.

THEOREM 5. *Let* $\mathbf{m}$ *be any limit cardinal not cofinal with* $\aleph_0$ *and let* $\lambda=\omega(\mathbf{m}')$. *If the disjoint sets* A_ν $(\nu<\lambda)$ *have the property* $\mathscr{P}(\mathbf{m})$, *then there are* $\mathbf{m}^+$ *almost disjoint transversals of the* A_ν.

PROOF. Since $C=\{\nu\colon \nu<\lambda, |A_\nu|\leqq\mathbf{m}_\nu\}$ is non-stationary, there is a band $\{\nu_0, \nu_1, \ldots, \hat{\nu}_\lambda\}_<$ in the complement $[0,\lambda)-C$. We can assume that $\nu_0=0$. Then, if $\mu<\lambda$, there is $\varrho(\mu)<\lambda$ such that

$$\nu_{\varrho(\mu)} \leqq \mu < \nu_{\varrho(\mu)+1}.$$

Let $\theta<\omega(\mathbf{m}^+)$ and suppose the transversals F_φ $(\varphi<\theta)$ of the A_ν have already been defined. Since $|\theta|\leqq\mathbf{m}$, we may write

$$\{F_0, \ldots, \hat{F}_\theta\} = \mathscr{F}_0 \cup \ldots \cup \hat{\mathscr{F}}_\lambda,$$

where $|\mathscr{F}_\nu|\leqq\mathbf{m}_\nu$ $(\nu<\lambda)$. If $\mu<\lambda$, then

$$|A_\mu| \geqq |A_{\nu_{\varrho(\mu)}}| > \mathbf{m}_{\nu_{\varrho(\mu)}}$$

since $\nu_{\varrho(\mu)}\notin C$. Also,

$$\Big|A_\mu \cap \bigcup_{\nu<\nu_{\varrho(\mu)}} \cup\mathscr{F}_\nu\Big| \leqq |\nu_{\varrho(\mu)}|\cdot\mathbf{m}_{\nu_{\varrho(\mu)}} = \mathbf{m}_{\nu_{\varrho(\mu)}}.$$

Therefore, we can choose

$$x_\mu\in A_\mu - \bigcup_{\nu<\nu_{\varrho(\mu)}} \cup\mathscr{F}_\nu \quad (\mu<\lambda).$$

Then $F_\theta=\{x_0, \ldots, \hat{x}_\lambda\}$ is a transversal of the A_ν. If $\varphi<\theta$, then there is $\sigma<\lambda$ such that $F_\varphi\in\mathscr{F}_\sigma$ and, by the definition of F_θ,

$$F_\theta\cap F_\varphi\subset\{x_\mu : \mu < \nu_{\varrho(\sigma)+1}\}.$$

Therefore, F_θ is almost disjoint from all the sets F_φ $(\varphi<\theta)$. Theorem 5 now follows by induction.

The next theorem shows that (if we assume (6) which is weaker than the g. c. h.) the condition $\mathscr{P}(\mathbf{m})$ in Theorem 5 is a necessary one for the existence of $\mathbf{m}^+$ almost disjoint transversals in the case when $\mathbf{m}$ is a singular limit number. We do not know if a similar result holds for inaccessible cardinals.

THEOREM 6. *Let* $\mathbf{m}>\mathbf{m}'>\aleph_0$ *and suppose that*

$$\mathbf{n}^{\mathbf{m}'} < \mathbf{m}^+ \quad (\mathbf{n}<\mathbf{m}). \tag{6}$$

Let $\lambda=\omega(\mathbf{m}')$ *and suppose that the disjoint sets* A_ν $(\nu<\lambda)$ *satisfy* (2) *but do not have the property* $\mathscr{P}(\mathbf{m})$. *Then any set of almost disjoint transversals of the* A_ν *has power less than or equal to* $\mathbf{m}$.

PROOF. We will assume that $\mathscr{F}$ is a set of $\mathbf{m}^+$ almost disjoint transversals of the A_ν and deduce a contradiction.

By hypothesis, the set $C=\{\nu\colon 1\leqq\nu<\lambda, |A_\nu|\leqq\mathbf{m}_\nu\}$ is stationary in $[0,\lambda)$. Therefore L, the set of limit ordinals in C, is also stationary. For each $\nu\in L$ we can assume

$$A_\nu\subset\{(\nu,\varrho)\colon \varrho<\omega(\mathbf{m}_\nu)\}.$$

If $F \in \mathscr{F}$ and $\nu \in L$, then there is $\varrho(F, \nu) < \omega(\mathbf{m}_\nu)$ such that

$$(\nu, \varrho(F, \nu)) \in F.$$

Since ν is a limit ordinal less than λ and $\{\mathbf{m}_0, \ldots, \hat{\mathbf{m}}_\lambda, \mathbf{m}\}_<$ is closed, there is $f_F(\nu) < \nu$ such that

$$|\varrho(F, \nu)| < \mathbf{m}_{f_F(\nu)}.$$

Since f_F is regressive on the stationary set L, there are $\theta_F < \lambda$ and $N_F \subset L$ such that $|N_F| = \mathbf{m}'$ and

$$f_F(\nu) = \theta_F \quad (\nu \in N_F).$$

It follows from (6) that there are $\theta < \lambda$, $N \subset L$ and $\mathscr{F}^* \subset \mathscr{F}$ such that $|\mathscr{F}^*| = \mathbf{m}^+$ and

$$\theta_F = \theta, \quad N_F = N \quad (F \in \mathscr{F}^*).$$

Put

$$A_\nu^* = A_\nu \cap \{(\nu, \sigma) : \sigma < \omega(\mathbf{m}_0)\} \quad (\nu \in N).$$

Then each $F \in \mathscr{F}^*$ meets each A_ν^* $(\nu \in N)$ in a singleton. By (6) the number of distinct transversals of the A_ν^* $(\nu \in N)$ is at most $\mathbf{m}_\theta^{\mathbf{m}'} < \mathbf{m}^+$. Therefore, there are distinct members $F, F' \in \mathscr{F}^*$ such that

$$F \cap A_\nu^* = F' \cap A_\nu^* \quad (\nu \in N).$$

This is a contradiction since $|N| = \mathbf{m}'$ and the members of $\mathscr{F}$ are pairwise almost disjoint.

4. A deduction from $\mathscr{H}$

One of the unsolved problems mentioned in [3] is to to prove or disprove the following statement. $\mathscr{S}$: *Let S be a set of power $\aleph_2$ and let E be the set of all unordered distinct pairs in S.* Then there is a partition of E*

$$E = E_0 \cup \ldots \cup \hat{E}_{\omega_1} \tag{7}$$

into $\aleph_1$ disjoint sets E_ν $(\nu < \omega_1)$ such that for every subset $S' \subset S$ of power $\aleph_1$

$$|\{\nu : \nu < \omega_1, \quad E' \cap E_\nu \neq \emptyset\}| = \aleph_1, \tag{8}$$

where E' is the set of all pairs in S'.

Let A_ν $(\nu < \omega_1)$ be $\aleph_1$ disjoint denumerable sets. Assuming that $\mathscr{H}$ is true, we take S to be a set of $\aleph_2$ almost disjoint transversals of the A_ν. If F, F' are distinct elements of S, then there is $\varrho_{FF'} < \omega_1$ such that

$$F \cap A_\nu \neq F' \cap A_\nu \quad (\varrho_{FF'} \leqq \nu < \omega_1).$$

If E is the set of all unordered pairs of S, put

$$E_\nu = E \cap \{\{F, F'\} : \varrho_{FF'} = \nu\} \quad (\nu < \omega_1).$$

* i.e. (S, E) is a complete graph.

Then (7) holds and $E_\mu \cap E_\nu = \emptyset$ $(\mu < \nu < \omega_1)$. Let S' be any subset of S of power $\aleph_1$ and let E' be the set of pairs in S'. If (8) is false, then there is $\varrho < \omega_1$ such that

$$E' \subset E_0 \cup \ldots \cup \hat{E}_\varrho,$$

i. e. $\varrho_{FF'} < \varrho$ for all distinct pairs $\{F, F'\} \subset S'$. This implies that different members of S' meet A_ϱ in different points, and therefore $|A_\varrho| \geqq |S'| = \aleph_1$. This contradiction proves that $\mathscr{H}$ implies $\mathscr{S}$.

We do not know if $\mathscr{S}$ also implies $\mathscr{H}$.

5. A problem of F. S. Cater

F. S. CATER [4] noted the following extension of Sierpinski's theorem. *If* $\mathbf{m}$ *is a regular cardinal and* $|S| = \mathbf{m}$, *then there is a set,* $\mathscr{F}$, *of almost disjoint subsets of power* $\mathbf{m}$ *of* S *such that* $|\mathscr{F}| > \mathbf{m}$ *and, in addition,*

(9) *if* $\mathscr{F}' \subset \mathscr{F}$ *and* $|\mathscr{F}'| = \mathbf{m}$, *then there is* $F \in \mathscr{F} - \mathscr{F}'$ *such that* $|F \cap \cup \mathscr{F}'| = \mathbf{m}$.

If $\mathscr{F}$ is any *maximal* set of almost disjoint subsets of power $\mathbf{m}$ of S and $|\mathscr{F}| > \mathbf{m}$, then it can easily be seen that (9) follows if $\mathbf{m}$ is regular. The g. c. h. is not used in the proof just outlined but it only works for regular $\mathbf{m}$. Cater asked if the result is true for singular numbers. We will show (Theorem 8) with the aid of the g. c. h. that Cater's result holds for arbitrary $\mathbf{m} \geqq \aleph_0$ and that (9) can be replaced by the stronger condition

(10) *if* $\mathscr{F}' \subset \mathscr{F}$ *and* $|\mathscr{F}'| = \mathbf{m}$, *then there is* $F \in \mathscr{F} - \mathscr{F}'$ *such that* $F \subset \cup \mathscr{F}'$.

We cannot prove this result without the g. c. h. even in the case of regular $\mathbf{m}$.

We call B a *weak transversal* of the $\mathbf{m}$ disjoint sets A_ν $(\nu \in M)$ if

$$B \subset \bigcup_{\nu \in M} A_\nu, \quad |B| = \mathbf{m} \quad \text{and} \quad |B \cap A_\nu| \leqq 1 \quad (\nu \in M).$$

The next theorem shows that Theorem 4 can be extended to arbitrary cardinals if we consider weak transversals instead of transversals.

THEOREM 7. *Suppose that* $\mathbf{m}$ *is an infinite cardinal and that**

(11) $$\mathbf{n}^{\mathbf{m}'} < \mathbf{m} \quad \text{if} \quad \mathbf{m}' < \mathbf{n} < \mathbf{m}.$$

If $\mathscr{F}$ *is any set of almost disjoint weak transversals of* $\mathbf{m}'$ *disjoint sets* A_ν $(\nu < \omega(\mathbf{m}'))$ *such that* $|\mathscr{F}| = \mathbf{m}$, *then* $\mathscr{F}$ *is not maximal.*

PROOF. Put $\lambda = \omega(\mathbf{m}')$.

Case 1. $\mathbf{m} = \mathbf{m}'$. Let $\mathscr{F} = \{F_0, \ldots, \hat{F}_\lambda\}$. Let $\varrho < \lambda$ and suppose that $\nu_\sigma < \lambda$ and $x_\sigma \in A_{\nu_\sigma}$ have been defined for $\sigma < \varrho$. Let $N = [0, \lambda) - \{\nu_0, \ldots, \hat{\nu}_\varrho\}$. If

$$A_\nu \subset F_0 \cup \ldots \cup \hat{F}_\varrho \quad \text{for all} \quad \nu \in N,$$

* (11) is satisfied vacuously if $\mathbf{m}$ is regular.

then $|F_\varrho \cap F_\sigma| = \mathbf{m}$ for some $\sigma < \varrho$ since $\mathbf{m} = \mathbf{m}'$ and $|F_\varrho \cap \bigcup_{\nu \in N} A_\nu| = \mathbf{m}$. This contradiction shows that there is $\nu_\varrho \in N$ such that $A_{\nu_\varrho} \not\subset F_0 \cup \ldots \cup \hat{F}_\varrho$ and we can choose $x_\varrho \in A_{\nu_\varrho} - F_0 \cup \ldots \cup \hat{F}_\varrho$. The set $\{x_\varrho : \varrho < \lambda\}$ defined by induction is a weak transversal of the A_ν which is almost disjoint from all the members of $\mathscr{F}$.

Case 2. $\mathbf{m} > \mathbf{m}'$. Let $\mathscr{F} = \mathscr{F}_0 \cup \ldots \cup \hat{\mathscr{F}}_\lambda$, where $|\mathscr{F}_\nu| = \mathbf{m}_\nu < \mathbf{m}$ $(\nu < \lambda)$. Put $\mathscr{F}_\nu^* = \mathscr{F}_0 \cup \ldots \cup \hat{\mathscr{F}}_\nu$ $(\nu < \lambda)$ and let

$$N_\nu = \{\mu : \mu < \lambda, \quad A_\mu \not\subset \bigcup \mathscr{F}_\nu^*\} \quad (\nu < \lambda).$$

If $|N_\nu| < \mathbf{m}'$ for some $\nu < \lambda$, then there is $\theta < \lambda$ such that $A_\mu \subset \bigcup \mathscr{F}_\nu^*$ $(\mu \in [\theta, \lambda))$. Therefore, $|A_\mu| \leqq \mathbf{m}_0 + \ldots + \hat{\mathbf{m}}_\nu = \mathbf{n}_\nu < \mathbf{m}$ and the number of distinct weak transversals of the A_μ $(\mu \in [\theta, \lambda))$ is at most $2^{\mathbf{m}'} \cdot \mathbf{n}_\nu^{\mathbf{m}'} < \mathbf{m}$. Therefore, there are distinct elements $F, F' \in \mathscr{F}$ which meet in a common weak transversal of the A_μ $(\mu \in [\theta, \lambda))$. This contradiction proves that

$$|N_\nu| = \mathbf{m}' \quad (\nu < \lambda).$$

Let $\varrho < \lambda$ and suppose $\nu_\sigma < \lambda$ and $x_\sigma \in A_{\nu_\sigma}$ have been defined for $\sigma < \varrho$. Then we can choose $\nu_\varrho \in N_\varrho - \{\nu_0, \ldots, \hat{\nu}_\varrho\}$ and $x_\varrho \in A_{\nu_\varrho} - \bigcup \mathscr{F}_\varrho^*$. The set $B = \{x_\varrho : \varrho < \lambda\}$ is a weak transversal of the A_ν $(\nu < \lambda)$. Also, if $F \in \mathscr{F}_\gamma$, then $B \cap F \subset \{x_0, \ldots, x_\gamma\}$, i. e. B is almost disjoint from all members of $\mathscr{F}$. This completes the proof of Theorem 7.

THEOREM 8. *Let* $\mathbf{m} \geqq \aleph_0$ *and assume that*

$$2^{\mathbf{m}} = \mathbf{m}^+ \quad \textit{and} \quad \mathbf{n}^{\mathbf{m}'} < \mathbf{m}^+ \quad \textit{if} \quad \mathbf{m}' < \mathbf{n} < \mathbf{m}.$$

Let $|S| = \mathbf{m} \geqq \mathbf{p} \geqq \mathbf{p}' = \mathbf{m}'$. *Then there is a set,* $\mathscr{F}$, *of almost disjoint subsets of* S *each of power* $\mathbf{p}$ *such that* $|\mathscr{F}| = \mathbf{m}^+$ *and* (10) *holds.*

PROOF. Throughout the proof we write $\lambda = \omega(\mathbf{m}')$, $\alpha = \omega(\mathbf{m})$ and $\beta = \omega(\mathbf{m}^+)$.

Case 1. $\mathbf{p} = \mathbf{m}'$. The number of distinct subsets of $[0, \beta)$ of power $\mathbf{m}$ is $(\mathbf{m}^+)^{\mathbf{m}} = 2^{\mathbf{m} \cdot \mathbf{m}} = \mathbf{m}^+$ and we assume that these are the sets $N_0, \ldots, \hat{N}_\beta$. Let A_ν $(\nu < \lambda)$ be $\mathbf{m}'$ disjoint subsets of S of power $\mathbf{m}$. Also, let $F_0, \ldots, \hat{F}_\alpha$ be any $\mathbf{m}$ mutually disjoint transversals of the A_ν.

Let $\theta \in [\alpha, \beta)$. Suppose that we have already defined ordinals $\tau_\varphi < \beta$ for $\alpha \leqq \varphi < \beta$ and also the weak transversals of the A_ν $(\nu < \lambda)$ $F_0, \ldots, \hat{F}_\theta$ which are pairwise almost disjoint. Since $[0, \theta)$ contains $\mathbf{m}^{\mathbf{m}} = \mathbf{m}^+$ subsets of power $\mathbf{m}$, we can choose τ_θ to be the least ordinal $\tau < \beta$ such that $\tau \neq \tau_\varphi$ $(\varphi < \theta)$ and such that $N_\tau \subset [0, \theta)$. Put $\mathscr{F}_\theta = \{F_\varrho : \varrho \in N_{\tau_\theta}\}$, $A_\nu^* = A_\nu \cap \bigcup \mathscr{F}_\theta$ $(\nu < \lambda)$ and let $F_\varrho^* = F_\varrho \cap \bigcup_{\nu < \lambda} A_\nu^*$ $(\varrho < \theta)$. Now put $\mathscr{F}_\theta^* = \{F_\varrho^* : \varrho < \theta, |F_\varrho^*| = \mathbf{m}'\}$. Then $\mathscr{F}_\theta^*$ is a set of almost disjoint weak transversals of the A_ν^* $(\nu < \lambda)$ and $|\mathscr{F}_\theta^*| = \mathbf{m}$ since $\mathscr{F}_\theta \subset \mathscr{F}_\theta^*$. By Theorem 7 there is a weak transversal F_θ of the A_ν^* which is almost disjoint from each member of $\mathscr{F}_\theta^*$. Since

$$F_\theta \subset \bigcup \mathscr{F}_\theta = \bigcup_{\varrho \in N_{\tau_\theta}} F_\varrho \tag{12}$$

it follows that F_θ is also almost disjoint from each F_φ $(\varphi < \theta)$. This defines F_θ and τ_θ for $\alpha \leqq \theta < \beta$. By the construction, it is clear that $\mathscr{F} = \{F_\theta : \theta < \beta\}$ is a set of $\mathbf{m}^+$ almost disjoint weak transversals of the A_ν $(\nu < \lambda)$.

In order to complete the proof in this case we only need to observe that as θ ranges from α to β so τ_θ assumes all ordinal values less than β. If this were not the case there is a least $\varrho<\beta$ such that $\tau_\theta\neq\varrho$ $(\alpha\leqq\theta<\beta)$. There is $\pi<\beta$ such that $N_\varrho\subset[0,\pi)$ and the definition of τ_θ implies that $\tau_\theta<\varrho$ for $\pi\leqq\theta<\beta$. But this is impossible since $\tau_\theta\neq\tau_\varphi$ if $\theta\neq\varphi$. It follows that if $\mathscr{F}'$ is any subset of $\mathscr{F}$ of power $\mathbf{m}$, then there is some $\theta\in[\alpha,\beta)$ such that $\mathscr{F}'=\{F_\varrho\colon \varrho\in N_{\tau_\theta}\}$, i. e. $\mathscr{F}'=\mathscr{F}_\theta$ and (10) follows from (12) and the fact that $\theta\notin N_{\tau_\theta}$.

Case 2. $\mathbf{p}>\mathbf{m}'$. There are cardinals $\mathbf{p}_\nu$ $(\nu<\lambda)$ such that $\mathbf{p}_0<\mathbf{p}_1<\ldots<\hat{\mathbf{p}}_\lambda<\mathbf{p}=\lim_{\nu<\lambda}\mathbf{p}_\nu$. The result in this case easily follows from the last case if we replace each element of A_ν $(\nu<\lambda)$ by a subset of S of cardinal $\mathbf{p}_\nu$. Since the weak transversals meet $\mathbf{m}'$ different A_ν they will, in this case, be subsets of S of power $\mathbf{p}$.

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References

[1] A. TARSKI, Sur la decomposition des ensembles en sous-ensembles presque disjoints, *Fund. Math.*, **12** (1928), pp. 188—205.
[2] A. TARSKI, Sur la decomposition des ensembles en sous-ensembles presque disjoints, *Fund. Math.*, **14** (1929), pp. 205—215.
[3] P. ERDŐS, A. HAJNAL and R. RADO, Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hung.*, **16** (1965), pp. 93—196.
[4] F. S. CATER, Advanced problem 5370, *American M. M.*, **73** (1966), p. 311.
[5] W. NEUMER, Verallgemeinerung eines Satzes von Alexandroff und Urysohn, *Math. Z.*, **54** (1951), pp. 254—261.
[6] H. BACHMANN, *Transfinite Zahlen* (Springer-Verlag, Berlin, 1955).

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (1—2), (1968), pp. 219—220.

EINE CHARAKTERISIERUNG DER PRIMITIVEN IDEALE EINES RINGES

Von
O. STEINFELD (Budapest)
(*Vorgelegt von* L. RÉDEI)

Es ist bekannt, daß die primitiven Ideale eines assoziativen Ringes R eine wichtige Rolle in der Ringtheorie spielen. Z. B. ist das Jacobsonsche Radikal des Ringes R der Durchschnitt aller primitiven Ideale von R. Lösend ein Problem von F. SZÁSZ [5] geben wir hier eine neue Charakterisierung der primitiven Ideale. Ein Rechtsideal $\mathfrak{r}$ eines assoziativen Ringes R wird nach [4], [5] *quasimodular* genannt, wenn $\mathfrak{r}::R \subseteq \mathfrak{r}$ gilt, wo $\mathfrak{r}::R$ die Menge aller Elemente $x \in R$ mit $Rx \subseteq \mathfrak{r}$ bezeichnet.[1] Es ist leicht einzusehen, daß $\mathfrak{r}::R$ ein Ideal von R ist. Ein Ideal $\mathfrak{p}$ des Ringes R heißt in der Arbeit [5] *quasiprimitiv*, wenn es ein quasimodulares maximales Rechtsideal $\mathfrak{r}$ von R mit $\mathfrak{p} = \mathfrak{r}::R$ existiert. Da die modularen Rechtsideale von R quasimodular sind ([1] Corollary I. 3), ist jedes primitive Ideal von R nach [1] Corollary I. 4 quasiprimitiv. In der Arbeit [5] ist das Problem aufgeworfen, ob die Begriffe der primitiven und quasiprimitiven Ideale von R übereinstimmen oder nicht. Die Antwort ist darauf unerwartet[2] bejahend. Es gilt nämlich

SATZ. *Jedes quasiprimitive Ideal eines assoziativen Ringes R ist primitiv.*[3]

Zum Beweis wenden wir das folgende Resultat von F. SZÁSZ [5] an.

BEHAUPTUNG 1. Ist $\mathfrak{r}$ ein quasimodulares maximales Rechtsideal von R, so ist der Quotient $\mathfrak{r}::b$ für jedes Element $b \in R$, $b \notin \mathfrak{r}$ ein modulares maximales Rechtsideal von R.

BEWEIS DES SATZES. Es sei $\mathfrak{r}$ ein quasimodulares maximales Rechtsideal von R und sei b ein Element von R mit $b \notin \mathfrak{r}$. Nach der Behauptung 1 ist $\mathfrak{r}_b = \mathfrak{r}::b$ ein modulares maximales Rechtsideal von R. (Zum Beweis ist schon die Tat genügend, daß mindestens ein solches Element $b \in R$ existiert, wofür $\mathfrak{r}::b$ ein modulares maximales Rechtsideal von R ist.) Wir werden zeigen, daß $\mathfrak{p} = \mathfrak{r}::R = \mathfrak{r}_b::R$ gilt; d. h. das quasiprimitive Ideal $\mathfrak{p}$ ist primitiv.

Zuerst zeigen wir die Gültigkeit[4] von

$$\mathfrak{r}_b::R = (\mathfrak{r}::b)::R = \mathfrak{r}::bR. \tag{1}$$

[1] Für ein Rechtsideal $\mathfrak{r}$ und eine Teilmenge T des Ringes R bezeichnet der *Quotient* $\mathfrak{r}::T$ die Menge aller Elemente x von R mit $Tx \subseteq \mathfrak{r}$. Offenbar ist $\mathfrak{r}::T$ ein Rechtsideal von R.

[2] Lösend ein Problem von A. KERTÉSZ [2] hat F. SZÁSZ [4] die Existenz eines solchen Ringes gezeigt, der ein quasimodulares, nicht aber modulares maximales Rechtsideal enthält.

[3] In der Arbeit [5] von F. SZÁSZ ist schon auf meinem Resultat hingewiesen. Inzwischen hat er in seiner Arbeit [6] unabhängig von mir eine schärfere Behauptung bewiesen.

[4] Man kann im allgemeinen $(\mathfrak{r}::T_1)::T_2 = \mathfrak{r}::T_1T_2$ für ein beliebiges Rechtsideal $\mathfrak{r}$ und beliebige Teilmengen T_1, T_2 von R beweisen.

Aus $b[R((\mathfrak{r}::b)::R)] \subseteq b[\mathfrak{r}::b] \subseteq \mathfrak{r}$ folgt $(\mathfrak{r}::b)::R \subseteq \mathfrak{r}::bR$. Anderseits $bR(\mathfrak{r}::bR) \subseteq \mathfrak{r}$ impliziert $R(\mathfrak{r}::bR) \subseteq \mathfrak{r}::b$, woraus $\mathfrak{r}::bR \subseteq (\mathfrak{r}::b)::R$ folgt, womit (1) bewiesen ist.

Nach Behauptung 1 ist $\mathfrak{r}_b = \mathfrak{r}::b$ ein von R verschiedenes Rechtsideal von R, folglich muß $bR \nsubseteq \mathfrak{r}$ bestehen. Da $\mathfrak{r}$ ein maximales Rechtsideal von R ist, gilt $\mathfrak{r} + bR = R$. Daraus folgt

$$R(\mathfrak{r}::bR) = (\mathfrak{r}+bR)(\mathfrak{r}::bR) = \mathfrak{r}(\mathfrak{r}::bR)+bR(\mathfrak{r}::bR) \subseteq \mathfrak{r}.$$

Es gilt also

$$\mathfrak{r}::bR \subseteq \mathfrak{r}::R = \mathfrak{p}. \tag{2}$$

Anderseits gilt wegen $bR\mathfrak{p} = bR(\mathfrak{r}::R) \subseteq R(\mathfrak{r}::R) \subseteq \mathfrak{r}$ die Bedingung

$$\mathfrak{p} \subseteq \mathfrak{r}::bR. \tag{3}$$

Aus (1), (2) und (3) folgt unsere Behauptung $\mathfrak{p} = \mathfrak{r}_b::R$, womit der Satz bewiesen ist.

Mit Hilfe dieses Satzes können wir einen einfachen Beweis der folgenden bekannten Resultate[5] geben.

BEHAUPTUNG 2. ([1], Proposition VIII. 2. 1.) *Ist $\mathfrak{p}$ ein primitives Ideal des Ringes R, so ist $\mathfrak{p}$ ein Primideal von R.*

BEHAUPTUNG 3. (Vgl. [1] Theorem 1. 4. 1.) *Es sei $\mathfrak{r}$ ein modulares maximales Rechtsideal des Ringes R und sei $\mathfrak{p} = \mathfrak{r}::R$. Der Faktorring $R/\mathfrak{p}$ ist dann und nur dann ein Schiefkörper, wenn $\mathfrak{p} = \mathfrak{r}$ gilt.*

BEHAUPTUNG 4. (Vgl. [1] Proposition IX. 2. 2 und IX. 2. 3.) *Es sei $\mathfrak{a}$ ein Ideal von R. Ist $\mathfrak{p}$ ein primitives Ideal von R mit $\mathfrak{p} \not\supseteq \mathfrak{a}$, so ist $\mathfrak{a} \cap \mathfrak{p}$ ein primitives Ideal von $\mathfrak{a}$. Umgekehrt ist $\mathfrak{p}_1$ ein primitives Ideal von $\mathfrak{a}$, so ist $\mathfrak{p}_1::\mathfrak{a}$ ein primitives Ideal von R mit $(\mathfrak{p}_1::\mathfrak{a}) \cap \mathfrak{a} = \mathfrak{p}_1$.*

Ich möchte meinem Kollegen, Herrn dr. F. Szász danken, daß er das ursprüngliche Manuskript seiner Arbeit [5] noch vor dem Erscheinen mir zu Verfügung gestellt hat.

(Eingegangen am 16. Mai 1967.)

Literaturverzeichnis

[1] N. JACOBSON, *Structure of Rings* (Amer. Math. Soc. Colloqu. Publ., Providence, 1956).
[2] A. KERTÉSZ, *Vorlesungen über Artinsche Ringe* (Akadémiai Kiadó, Budapest, 1968).
[3] O. STEINFELD, Über Zerlegungssätze für teilweise geordnete Halbgruppen mit bedingten Distributivitätsregeln, *MTA Matem. Kut. Int. Közl.*, **9A** (1964), S. 313—330.
[4] F. SZÁSZ, Lösung eines Problems bezüglich einer Charakterisierung des Jacobsonschen Radikals, *Acta Math. Acad. Sci. Hung.*, **18** (1967), S. 261—272.
[5] F. SZÁSZ, Eine Charakterisierung des Jacobsonschen Radikales eines Ringes, *Bull. Acad. Polon. Sciences, Sér. Sci. math. astronom et phys.*, **14** (1966), S. 53—56.
[6] F. SZÁSZ, The sharpening of a result concerning primitive ideals of an associative ring, *Proc. Amer. Math. Soc.*, **18** (1967), S. 910—912.

[5] Wir können einen Teil dieser Resultate für die — in einer Verbandshalbgruppe [3] definierten — quasiprimitiven Absorbenten verallgemeinern.

INDEX

Neuman, F., Relation between the distribution of the zeros of the solutions of a 2nd order linear differential equation and the boundedness of these solutions 1
Guy, R. K. and *Milner, E. C.*, Graphs defined by coverings of a set 7
Eilhauer, R., Zur Theorie der Halbkörper. I 23
Bhargava, T. N. and *Ahlborn, T. J.*, On topological spaces associated with digraphs 47
Chatterjea, S. K., Operational results connected with some classical polynomials 53
Lovász, L., On chromatic number of finite set-systems 59
Murty, U. S. R., On some extremal graphs 69
Bollobás, B., A problem of the theory of communication networks 75
Szabados, J., Rational approximation in certain classes of functions 81
Leindler, L., On a problem of strong summability of Fourier series 87
Szabados, J., Structural properties of continuous functions connected with the order of rational approximation. II 95
Bambah, R. P. and *Woods, A. C.*, On the minimal density of maximal packing of the plane by convex bodies 103
Sreehari, M., A limit theorem for the maximum of cumulative sums 117
Kemperman, J. H. B., Idempotent signed measures on a discrete group 121
Querré, J., Plus grand groupe image homomorphe et isotone d'un monoïde ordonné: caractérisations des anneaux «clos» 129
Grillet, P. A., Translations and congruences in lattices 147
Ghosal, A., Some remarks about the duality relation in queues 163
Monk, G. S., One-sided ideals in the endomorphism ring of an Abelian p-group 171
Szász, G., Halbgruppen, deren Elemente durch Primideale trennbar sind 187
Гольдберг, А. А., О распределении значений целой функции по аргументам 191
Freud, G., Über starke Approximation mit Hilfe einer Klasse von Interpolationspolynomen 201
Erdős, P., *Hajnal, A.* and *Milner, E. C.*, On sets of almost disjoint subsets of a set 209
Steinfeld, O., Eine Charakterisierung der primitiven Ideale eines Ringes 219

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THE GROUP OF LEFT DISTRIBUTIVE MULTIPLICATIONS ON AN ABELIAN GROUP[1]

By

J. R. CLAY (Tucson, Arizona, U.S.A.)

(*Presented by* L. RÉDEI)

The problem of characterizing all rings R, whose additive group R^+ is isomorphic to a given Abelian group A, gives motivation to the study of Mult A, the set of binary operations on A satisfying both distributive laws. (See Chapter XII of [2].) Similarly, the problem of constructing all the near-rings on an arbitrary group $(G, +)$, not necessarily Abelian, gives motivation to the study of $\text{Mult}_L G$, the set of binary operations on G left distributive with respect to $+$.

If $(A, +)$ is an Abelian group, then we think of $\text{Mult}_L A$ as the set of all mappings $\mu: A \times A \to A$ such that $\mu(x, y+z) = \mu(x, y) + \mu(x, z)$ for arbitrary $x, y, z \in A$. Such a mapping will be called a *multiplication*. Two multiplications, μ and π, may be added pointwise; i.e. $(\mu+\pi)(x, y) = \mu(x, y) + \pi(x, y)$. It is easy to see that if $(A, +)$ is an Abelian group, then $(\text{Mult}_L A, +)$ is also Abelian group.

This paper is concerned with the structure of $\text{Mult}_L A$, some of the properties $(\text{Mult}_L A, +)$ inherits from $(A, +)$, and some of the properties $(A, +)$ inherits from $(\text{Mult}_L A, +)$.

Throughout the sequel, group means Abelian group. The terminology and notation is that of [2] unless otherwise defined herein. If A and B are groups, then Map (A, B) denotes all functions with domain A and range contained in B.

It is well known that Map (A, B) is an Abelian group under pointwise addition. Hence Map $(A,$ End $A)$ is an Abelian group.

1. Basic results.

THEOREM 1. 1. $\text{Mult}_L A \cong \text{Map}\,(A,\ \text{End}\ A)$.

PROOF. Theorem 1. 1 of [1] shows that there is a one-to-one correspondence between $\text{Mult}_L A$ and Map $(A,$ End $A)$. This correspondence can be defined by the mapping $F: \text{Mult}_L A \to \text{Map}\,(A,\ \text{End}\ A)$ where $F(\Phi) = f^{\Phi}, f^{\Phi}(x) = f_x^{\Phi}$, and $f_x^{\Phi}(y) = \Phi(x, y) (x, y \in A,\ \Phi \in \text{Mult}_L A)$. The proof that F is an isomorphism is direct.

COROLLARY 1. 2. $\text{Mult}_L A \cong \sum_{|A|}^{*} \text{Hom}\,(A, A)$.

PROOF. It is easy to show that $\sum_{|A|}^{*} \text{Hom}\,(A, A) = \sum_{|A|}^{*} \text{End}\ A \cong \text{Map}\,(A,\ \text{End}\ A)$.

COROLLARY 1. 3. *If* $\text{Hom}\,(A, A) \cong A$, *then* $\text{Mult}_L A \cong \sum_{|A|}^{*} A$.

Note that Theorem 1. 1 generalizes Theorem 67. 1. (3) of [2] which states that $\text{Mult}\ A \cong \text{Hom}\,(A,\ \text{End}\ A)$.

[1] Some of the results of this paper are included in the author's doctoral dissertation at the University of Washington.

THEOREM 1. 4. *Let A be a group. If $f_k: A \to A$, defined by $f_k(a) = ka$ where k is an integer, is an automorphism, then $F_k: \mathrm{Mult}_L A \to \mathrm{Mult}_L A$, defined by $F_k(\Phi) = k\Phi$, is an automorphism.*

PROOF. For $\Phi, \pi \in \mathrm{Mult}_m A$, $F_k(\Phi+\pi) = k(\Phi+\pi) = k\Phi + k\pi = F_k(\Phi) + F_k(\pi)$. So F_k is an endomorphism. Suppose $F_k(\Phi) = F_k(\pi)$; i.e. $k\Phi = k\pi$. Then for $a, b \in A$, $0 = (k\Phi)(a, b) - (k\pi)(a, b) = k[\Phi - \pi](a, b) = f_k([\Phi - \pi](a, b))$. But f_k is an automorphism, so $[\Phi - \pi](a, b) = 0$, and therefore $\Phi = \pi$. Hence F_k is a monomorphism. For $\Phi \in \mathrm{Mult}_L A$, define π by $\pi(a, b) = f_k^{-1}(\Phi(a, b))(a, b \in A)$, an element of $\mathrm{Mult}_L A$. Then F_k is an epimorphism.

THEOREM 1. 5. *For $f, g \in \mathrm{Hom}(A, A)$, define $F_{f,g}: \mathrm{Mult}_L A \to \mathrm{Mult}_L A$ by $F_{f,g}(\Phi) = \pi$ where $\pi(x, y) = g[\Phi(x, f(y))]$ and $x, y \in A$. Then $F_{f,g}$ is a homomorphism. If $f = g = I$ is the identity map on A, then $F_{I,I}$ is the identity map on $\mathrm{Mult}_L A$. If either f or g is $0 \in \mathrm{Hom}(A, A)$, then $F_{f,g} = 0 \in \mathrm{Hom}(\mathrm{Mult}_L A, \mathrm{Mult}_L A)$.*

PROOF. It follows easily that $\pi \in \mathrm{Mult}_L A$, and that $F_{f,g}$ is an endomorphism. Since $F_{I,I}(\Phi) = \pi$ and $\pi(x, y) = I[\Phi(x, I(y))] = \Phi(x, y)$, it must be that $F_{I,I}$ is the identity map. Similarly, if f or g is $0 \in \mathrm{Hom}(A, A)$, then $F_{f,g} = 0 \in \mathrm{Hom}(\mathrm{Mult}_L A, \mathrm{Mult}_L A)$.

COROLLARY 1. 6. *If $f \in \mathrm{Hom}(A, A)$ is an epimorphism and I is the identity map on A, then $F_{f,I}$ is a monomorphism.*

PROOF. Suppose $F_{f,I}(\mu) = \zeta$, the zero multiplication. We shall see that $\mu = 0$. Let $x, y \in A$, and let $t \in A$ be such that $f(t) = y$. Then $0 = \zeta(x, t) = I[\mu(x, f(t))] = \mu(x, y)$.

COROLLARY 1. 7. *If $f \in \mathrm{Hom}(A, A)$ is an automorphism, then $F_{f,I}$ is an automorphism.*

PROOF. From Corollary 1. 6, we have that $F_{f,I}$ is a monomorphism. Let $\pi \in \mathrm{Mult}_L A$ and define $\Phi: A \times A \to A$ by $\Phi(x, t) = \pi(x, f^{-1}(t))$. Hence $\Phi \in \mathrm{Mult}_L A$ and $F_{I,f}(\Phi) = \pi$. So $F_{f,I}$ is an epimorphism.

COROLLARY 1. 8. *If $f \in \mathrm{Hom}(A, A)$ is defined by $f(x) = nx$ for some positive integer n, then $F_{f,I}(\Phi) = n\Phi$.*

PROOF. We have $F_{f,I}(\Phi) = \pi$ and $\pi(x, y) = I[\Phi(x, f(y))] = \Phi(x, ny) = n\Phi(x, y)$, so $\pi = n\Phi$.

2. **Hereditary properties.** We have already noted that $\mathrm{Mult}_L A$ is an Abelian group if A is.

THEOREM 2.1. *If A is bounded by the positive integer n, then so is $\mathrm{Mult}_L A$, and conversely.*

PROOF. The endomorphism $\theta(x) = nx$ is the zero endomorphism since $nA = 0$. Hence, by Theorem 1. 5, $F_{\theta,I}$ is the zero endomorphism of $\mathrm{Mult}_L A$. But $F_{\theta,I}$ takes $\Phi \to n\Phi$ by Corollary 1.8. So $n\Phi = 0$ for all $\Phi \in \mathrm{Mult}_L A$, and therefore, $\mathrm{Mult}_L A$ is bounded by n.

The mapping $\Phi(x, y) = y$ defines $\Phi \in \mathrm{Mult}_L A$. So $ny = n\Phi(x, y) = (n\Phi)(x, y) = 0$. Hence A is bounded by n.

THEOREM 2. 2. *If* $A[n]=\{0\}$, *then* $(\mathrm{Mult}_L A)[n]=\{0\}$.

PROOF. If $\Phi\in\mathrm{Mult}_L A$ and $n\Phi=0$, then $0=(n\Phi)(a,b)=n[\Phi(a,b)]$. But $A[n]=\{0\}$ implies that $\Phi(a,b)=0$. So $\Phi=0$.

THEOREM 2. 3. *If A is torsion free, then* $\mathrm{Mult}_L A$ *is torsion free.*

PROOF. We have $A[n]=\{0\}$ for each positive integer n. So by Theorem 2. 2, $(\mathrm{Mult}_L A)[n]=\{0\}$ for each positive integer n. Hence $\mathrm{Mult}_L A$ is torsion free.

THEOREM 2. 4. *If n is a positive integer, and* $nA=A$, *then* $(\mathrm{Mult}_L A)[n]=\{0\}$.

PROOF. The mapping $f(a)=na$ is an epimorphism since $nA=A$. Consequently, by Corollary 1. 6, $F_{f,I}$ is a monomorphism. But $F_{f,I}(\Phi)=n\Phi$ by Corollary 1. 8, so $n\Phi=0$ if and only if $\Phi=0$. So $(\mathrm{Mult}_L A)[n]=\{0\}$.

THEOREM 2. 5. *If n is a positive integer such that* $nA=A$ *and* $A[n]=\{0\}$, *then* $n\,\mathrm{Mult}_L A=\mathrm{Mult}_L A$ *and* $(\mathrm{Mult}_L A)[n]=\{0\}$.

PROOF. The mapping $f(a)=na$ is an automorphism of A since $nA=A$ and $A[n]=\{0\}$. So $F_{f,I}$ is an automorphism. Also $F_{f,I}(\Phi)=n\Phi$, so $n\,\mathrm{Mult}_L A=\mathrm{Mult}_L A$ and $(\mathrm{Mult}_L A)[n]=\{0\}$.

COROLLARY 2. 6. *If A is torsion free and divisible, then* $\mathrm{Mult}_L A$ *is torsion free and divisible.*

THEOREM 2. 7. *If A is divisible, then* $\mathrm{Mult}_L A$ *is torsion free.*

PROOF. We have $nA=A$ for each positive integer n, so by Theorem 2. 4, $(\mathrm{Mult}_L A)[n]=\{0\}$ for each positive integer n. Hence, $\mathrm{Mult}_L A$ is torsion free.

THEOREM 2. 8. *If* $n\,\mathrm{Mult}_L A=\mathrm{Mult}_L A$, *then* $nA=A$.

PROOF. For $a\in A$, we must show the existence of an $x\in A$ such that $nx=a$. Now $\Phi(x,y)=y$ defines $\Phi\in\mathrm{Mult}_L A$, and there is a $\pi\in\mathrm{Mult}_L A$ such that $n\pi=\Phi$. So $(n\pi)(y,a)=\Phi(y,a)=a$ and $(n\pi)=n[\pi(y,a)]$. Let $x=\pi(y,a)$.

THEOREM 2. 9. *Let* $A=\sum_m C(n)$ *where m and n are positive integers. Then* $\mathrm{Mult}_L A=\sum_{m^2 n^m} C(n)\cong\sum_{mn^m} A$.

PROOF. We know from Corollary 1. 2 that $\mathrm{Mult}_L A\cong\sum_{n^m}\mathrm{Hom}(A,A)$. Since $\mathrm{Hom}(A,A)\cong\sum_{m^2} C(n)$ we have the desired results.

If $f: A\to B$ is an isomorphism between Abelian groups A and B, and if $\Phi\in\mathrm{Mult}_L A$, then $\Phi^*\in\mathrm{Mult}_L B$ where $\Phi^*(x,y)=f[\Phi(f^{-1}(x),f^{-1}(y))]$. Similarly, if $\Phi^*\in\mathrm{Mult}_L B$, then $\Phi\in\mathrm{Mult}_L A$ where $\Phi(a,b)=f^{-1}[\Phi^*(f(a),f(b))]$. It is easy to see that $\Phi\leftrightarrow\Phi^*$ is an isomorphism between $\mathrm{Mult}_L A$ and $\mathrm{Mult}_L B$. The next three results show that if $\mathrm{Mult}_L A\cong\mathrm{Mult}_L B$, then we cannot conclude that $A\cong B$.

THEOREM 2. 10. *If m and n are positive integers, and if* $A=\sum_m C(\infty)$, $B=\sum_n C(\infty)$, *then* $\mathrm{Mult}_L A\cong\mathrm{Mult}_L B\cong\sum_{\aleph_0}^{*} C(\infty)$.

PROOF. Since

$$\mathrm{Mult}_L A \cong \sum_{|A|}^{*} \mathrm{Hom}(A, A) \cong \sum_{\aleph_0}^{*} \mathrm{Hom}(A, A),$$

and

$$\mathrm{Hom}(A, A) \cong \sum_{m} \mathrm{Hom}(C(\infty), A),$$

and

$$\mathrm{Hom}(C(\infty), A) \cong A,$$

we have

$$\mathrm{Hom}(A, A) \cong \sum_{m} A = \sum_{m^2} C(\infty),$$

so

$$\mathrm{Mult}_L A \cong \sum_{\aleph_0}^{*} \mathrm{Hom}(A, A) = \sum_{\aleph_0}^{*} \sum_{m^2} C(\infty) = \sum_{\aleph_0}^{*} C(\infty).$$

Similarly, we get $\mathrm{Mult}_L B = \sum_{\aleph_0}^{*} C(\infty)$.

THEOREM 2. 11. *If* $A \cong \sum_{\aleph_0}^{*} C(\infty)$, *then* $\mathrm{Mult}_L A \cong \sum_{\aleph_0}^{*} \sum_{\aleph_0} C(\infty)$.

PROOF. Since $|A| = \aleph_0$, then $\mathrm{Mult}_L A \cong \sum_{\aleph_0} \mathrm{Hom}(A, A)$. But

$$\mathrm{Hom}(A, A) \cong \sum_{\aleph_0}^{*} \mathrm{Hom}(A, C(\infty)),$$

and by Exercise 11, page 228 of [2],

$$\mathrm{Hom}(A, C(\infty)) \cong \sum_{\aleph_0} C(\infty).$$

Consequently

$$\mathrm{Hom}(A, A) \cong \sum_{\aleph_0}^{*} \sum_{\aleph_0} C(\infty).$$

Thus

$$\mathrm{Mult}_L A \cong \sum_{\aleph_0}^{*} \sum_{\aleph_0}^{*} \sum_{\aleph_0} C(\infty) \cong \sum_{\aleph_0}^{*} \sum_{\aleph_0} C(\infty).$$

THEOREM 2.12. *If* $A \cong \sum_{\aleph_0} C(\infty)$, *then* $\mathrm{Mult}_L A \cong \sum_{\aleph_0}^{*} \sum_{\aleph_0} C(\infty)$.

PROOF. We have $\mathrm{Hom}(C(\infty), A) \cong A$, so $\mathrm{Hom}(A, A) \cong \sum_{\aleph_0}^{*} \mathrm{Hom}(C(\infty), A) \cong$ $\cong \sum_{\aleph_0}^{*} \sum_{\aleph_0} C(\infty)$. Consequently,

$$\mathrm{Mult}_L A \cong \sum_{\aleph_0}^{*} \sum_{\aleph_0}^{*} \sum_{\aleph_0} C(\infty) \cong \sum_{\aleph_0}^{*} \sum_{\aleph_0} C(\infty).$$

From Corollary 1. 3, if $\mathrm{Hom}(A, A) \cong A$, then $\mathrm{Mult}_L A \cong \sum_{|A|}^{*} A$. The following two theorems show that the converse of this is not true.

THEOREM 2. 13. *If* $A = \sum_{\aleph_0} C(n)$, *then* $\mathrm{Mult}_L A \cong \sum_{\aleph_0}^{*} A \cong \sum_{|A|}^{*} A$.

PROOF. Since $\mathrm{Hom}(A, A) = \sum_{\aleph_0}^{*} \mathrm{Hom}(C(n), A)$ and $\mathrm{Hom}(C(n), A) \cong A[n] = A$, we have $\mathrm{Hom}(A, A) \cong \sum_{\aleph_0}^{*} A$. Consequently,

$$\mathrm{Mult}_L A \cong \sum_{|A|}^{*} \mathrm{Hom}(A, A) = \sum_{\aleph_0}^{*} \sum_{\aleph_0}^{*} A = \sum_{\aleph_0}^{*} A = \sum_{|A|}^{*} A.$$

THEOREM 2. 14. *Let* $A \cong \sum_{\Lambda} C(\infty)$ *be a free group. Then* $\mathrm{Mult}_L A \cong \sum^*_{|A|} \sum^*_{\Lambda} A \cong$ $\cong \sum^*_{|A|} A$. *Consequently, if* A *is free, then* $\mathrm{Mult}_L A$ *is not free.*

PROOF. We have $\mathrm{Hom}\,(C(\infty), A) \cong A$, so $\mathrm{Hom}\,(A, A) \cong \sum^*_{\Lambda} \mathrm{Hom}\,(C(\infty), A) \cong$ $\cong \sum^*_{\Lambda} A$, so $\mathrm{Mult}_L A \cong \sum^*_{|A|} \mathrm{Hom}\,(A, A) \cong \sum^*_{|A|} \sum^*_{\Lambda} A$.

If $\Lambda \geqq \aleph_0$, then $|A| = \Lambda \cdot \aleph_0 = \Lambda$, so $\sum^*_{|A|} \sum^*_{\Lambda} A = \sum^*_{|A|} A$. If $\Lambda < \aleph_0$, then $|A| = \aleph_0$, so $\sum^*_{|A|} \sum^*_{\Lambda} A = \sum^*_{|A|} A$.

We shall now give a partial answer to the question: when can $\mathrm{Mult}_L (A \oplus B)$ be described entirely in terms of $\mathrm{Mult}_L A$ and $\mathrm{Mult}_L B$?

THEOREM 2.15. $\mathrm{Mult}_L (A \oplus B) \cong \sum^*_{|B|} \mathrm{Mult}_L A \oplus \sum^*_{|A|} \mathrm{Mult}_L B \oplus \sum_{|A|\cdot|B|} [\mathrm{Hom}\,(A, B)$ $\oplus \mathrm{Hom}\,(B, A)]$.

PROOF. By Corollary 1. 3, $\mathrm{Mult}_L (A \oplus B) \cong \sum^*_{|A|\cdot|B|} \mathrm{Hom}\,(A \oplus B, A \oplus B)$. But $\mathrm{Hom}\,(A \oplus B, A \oplus B) \cong \mathrm{Hom}\,(A, A) \oplus \mathrm{Hom}\,(B, B) \oplus \mathrm{Hom}\,(A, B) \oplus \mathrm{Hom}\,(B, A)$.

Hence,

$$\mathrm{Mult}_L (A \oplus B) \cong \sum^*_{|A|\cdot|B|} \mathrm{Hom}\,(A, A) \oplus \sum^*_{|A|\cdot|B|} \mathrm{Hom}\,(B, B) \oplus$$

$$\oplus \sum^*_{|A|\cdot|B|} [\mathrm{Hom}\,(A, B) \oplus \mathrm{Hom}\,(B, A)] = \sum^*_{|B|} \sum^*_{|A|} \mathrm{Hom}\,(A, A) \oplus \sum^*_{|A|} \sum^*_{|B|} \mathrm{Hom}\,(B, B) \oplus$$

$$\oplus \sum^*_{|A|\cdot|B|} [\mathrm{Hom}\,(A, B) \oplus \mathrm{Hom}\,(B, A)] = \sum^*_{|B|} \mathrm{Mult}_L A \oplus \sum^*_{|A|} \mathrm{Mult}_L B \oplus$$

$$\oplus \sum^*_{|A|\cdot|B|} [\mathrm{Hom}\,(A, B) \oplus \mathrm{Hom}\,(B, A)].$$

COROLLARY 2. 16. *If* $\mathrm{Hom}\,(A, B) \cong \mathrm{Hom}\,(B, A) \cong \{0\}$, *then* $\mathrm{Mult}_L (A \oplus B) \cong$ $\cong \sum^*_{|B|} \mathrm{Mult}_L A \oplus \sum^*_{|A|} \mathrm{Mult}_L B$.

COROLLARY 2. 17. *For* $\mathrm{Mult}_L (A \oplus B)$ *to be expressed entirely in terms of* $\mathrm{Mult}_L A$ *and* $\mathrm{Mult}_L B$, *it is sufficient that* $\mathrm{Hom}\,(A, B)$ *and* $\mathrm{Hom}\,(B, A)$ *each be isomorphic to one of the groups* $\{0\}$, $\mathrm{Hom}\,(A, A)$ *or* $\mathrm{Hom}\,(B, B)$.

COROLLARY 2. 18. *If* p *is a prime and* m, n *are positive integers such that* $n < m$, *then*

$$\mathrm{Mult}_L \left(C(p^n) \oplus C(p^m)\right) \cong \sum_{3p^m} \mathrm{Mult}_L C(p^n) \oplus \sum_{p^n} \mathrm{Mult}_L C(p^m).$$

PROOF. This follows from Theorem 2. 15 since

$$\mathrm{Hom}\left(C(p^n), C(p^m)\right) \cong \mathrm{Hom}\left(C(p^m), C(p^n)\right) \cong C(p^n).$$

The condition in Corollary 2.16 is sufficient for the result, but it is not necessary. For if $A = B = C(\infty)$, then $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(B, A) \cong C(\infty) \cong \mathrm{Hom}(A, A) \cong \cong \mathrm{Hom}(B, B) \neq \{0\}$. Using Theorem 2.15 we get

$$\mathrm{Mult}_L(A \oplus B) \cong \sum_{|B|}^{*} \mathrm{Mult}_L A \oplus \sum_{|A|}^{*} \mathrm{Mult}_L B \oplus \sum_{\aleph_0}^{*} [\mathrm{Hom}(A, A) \oplus \mathrm{Hom}(B, B)] =$$

$$= \sum_{\aleph_0}^{*} \mathrm{Mult}_L A \oplus \sum_{\aleph_0}^{*} \mathrm{Mult}_L B \oplus \sum_{|A|}^{*} \mathrm{Hom}(A, A) \oplus \sum_{|B|}^{*} \mathrm{Hom}(B, B) =$$

$$= \sum_{\aleph_0}^{*} \mathrm{Mult}_L A \oplus \sum_{\aleph_0}^{*} \mathrm{Mult}_L B \oplus \mathrm{Mult}_L A \oplus \mathrm{Mult}_L B = \sum_{|B|}^{*} \mathrm{Mult}_L A \oplus \sum_{|A|}^{*} \mathrm{Mult}_L B.$$

THEOREM 2.19. *If* $A \neq \{0\}$, *then* $\mathrm{Hom}(\mathrm{Mult}_L A, A) \neq \{0\}$.

PROOF. For $a \in A$, define a homomorphism $F_a: \mathrm{Mult}_L A \to A$ by $F_a(\Phi) = \Phi(a, a)$ where $\Phi \in \mathrm{Mult}_L A$. The multiplication $\pi \in \mathrm{Mult}_L A$ defined by $\pi(x, y) = y$ is non-trivial. So if $a \in A$ and $a \neq 0$, then $F_a(\pi) = \pi(a, a) = a \neq 0$. So $F_a \neq 0$.

THEOREM 2.20. *If* $|A| < \aleph_0$ *and* $\mathrm{Hom}(A, A) \cong A$, *then* $\mathrm{Hom}(\mathrm{Mult}_L A, A) \cong \cong \mathrm{Mult}_L A$.

PROOF. $\mathrm{Mult}_L A \cong \sum_{|A|}^{*} \mathrm{Hom}(A, A) \cong \sum_{|A|} A$. Consequently,

$$\mathrm{Hom}(\mathrm{Mult}_L A, A) \cong \sum_{|A|} \mathrm{Hom}(A, A) \cong \sum_{|A|} A = \sum_{|A|}^{*} \mathrm{Hom}(A, A) \cong \mathrm{Mult}_L A.$$

THEOREM 2.21. *If* $A \cong \sum_{i=1}^{n} G_{p_i}$ *where each* G_{p_i} *is a* p_i*-group for the distinct primes* p_i, *then* $\mathrm{Mult}_L A \cong \sum_{|A|}^{*} \sum_{i=1}^{n} \mathrm{Hom}(G_{p_i}, G_{p_i})$.

PROOF. We have $\mathrm{Hom}(A, A) \cong \sum_{i=1}^{n} \mathrm{Hom}(G_{p_i}, A) \cong \sum_{i=1}^{n} \mathrm{Hom}(G_{p_i}, G_{p_i})$, since $\mathrm{Hom}(G_{p_i}, G_{p_j}) \cong \{0\}$ if $i \neq j$. Consequently, $\mathrm{Mult}_L A \cong \sum_{|A|}^{*} \mathrm{Hom}(A, A) \cong$

$$\cong \sum_{|A|}^{*} \sum_{i=1}^{n} \mathrm{Hom}(G_{p_i}, G_{p_i}).$$

THEOREM 2.22. *Suppose* $A \cong C(\infty) \oplus G$ *where* G *is a torsion group and* $|G| = |A|$. *Then* $\mathrm{Mult}_L A \cong \mathrm{Mult}_L G \oplus \sum_{|A|}^{*} A$.

PROOF. We have $\mathrm{Hom}(A, A) \cong \mathrm{Hom}(C(\infty), C(\infty)) \oplus \mathrm{Hom}(C(\infty), G) \oplus \oplus \mathrm{Hom}(G, C(\infty)) \oplus \mathrm{Hom}(G, G)$. But $\mathrm{Hom}(C(\infty), C(\infty)) \cong C(\infty)$, $\mathrm{Hom}(C(\infty), G) \cong \cong G$, and $\mathrm{Hom}(G, C(\infty)) \cong \{0\}$ since G is torsion and $C(\infty)$ is torsion free. Consequently, $\mathrm{Hom}(A, A) = A \oplus \mathrm{Hom}(G, G)$. This together with $\mathrm{Mult}_L A \cong \cong \sum_{|A|}^{*} \mathrm{Hom}(A, A)$ yields the desired results.

As a result of this study, there are three questions the author would like to have answered. These are:

1. Which groups have the property that $\mathrm{Mult}_L A \cong \sum_{|A|}^{*} A$?

2. What are necessary and sufficient conditions for $\mathrm{Mult}_L (A \oplus B)$ to be xepressed entirely in terms of $\mathrm{Mult}_L A$ and $\mathrm{Mult}_L B$?

3. What are necessary and sufficient conditions on A and B such that $\mathrm{Mult}_L (A \oplus B) \cong \sum_{|B|}^{*} \mathrm{Mult}_L A \oplus \sum_{|A|}^{*} \mathrm{Mult}_L B$?

The author wishes to thank Prof. Ross A. Beaumont who pointed out many shorter and more elegant proofs to many of the theorems herein.

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References

[1] J. R. Clay, The near-rings on groups of low order, *Math. Zeitschr.*, **104** (1968), pp. 364—371.
[2] L. Fuchs, *Abelian Groups* (Publishing House of the Hungarian Academy, Budapest, 1958)

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (3—4), (1968), pp. 229—241.

REPRESENTATION OF CONDITIONAL PROBABILITY MEASURES ON BOOLEAN ALGEBRAS

By

P. H. KRAUSS[1] (Atlanta, Georgia, U.S.A.)

(*Presented by* A. RÉNYI)

1. Introduction

Throughout this paper $\mathscr{B}$ will be a Boolean algebra, and we use the symbols $0, 1, \vee, \wedge, -, \leqq$ with their ordinary Boolean meaning in $\mathscr{B}$. Simultaneously and clearly separated by context we use the standard set theoretic symbolism $\emptyset, \cup, \cap, \sim, \subseteq, \bigcup, \bigcap$. In particular we use the symbol $\subset$ for proper set inclusion. In all matters of Boolean algebras we refer the reader to HALMOS [3] and SIKORSKI [9]. A σ-additive (finitely additive) probability on $\mathscr{B}$ is a σ-additive (finitely additive) normed measure on $\mathscr{B}$. By a probability on $\mathscr{B}$ we mean either a σ-additive or a finitely additive probability on $\mathscr{B}$. By a measure on a Boolean ring we mean a not necessarily bounded, real valued measure which is either finitely additive or σ-additive. We recall that an ideal in a Boolean algebra is a Boolean ring.

A real valued function $p(\cdot|\cdot)$ on $\mathscr{B}\times(\mathscr{B}\sim\{0\})$ is called a *σ-additive (finitely additive) conditional probability* on $\mathscr{B}$ if (I) for every $y\in\mathscr{B}\sim\{0\}$, $p(\cdot|y)$ is a σ-additive (finitely additive) probability on $\mathscr{B}$ such that $p(y|y)=1$; (II) for every $x, y, z\in\mathscr{B}$, if $x\wedge z\neq 0$ then $p(x\wedge y|z)=p(x|z)\cdot p(y|x\wedge z)$.

By a conditional probability on $\mathscr{B}$ we mean either a σ-additive or a finitely additive conditional probability on $\mathscr{B}$. Throughout this paper p will be a conditional probability on $\mathscr{B}$ unless we give more explicit specifications.

Axiom systems for conditional probability measures have been introduced and discussed by several authors, and elementary properties of conditional probabilities are well-known. As far as we could ascertain the only systematic study of conditional probabilities has been undertaken by RÉNYI [7], [8] and CSÁSZÁR [1], and for a detailed exposition of elementary results we refer the reader to their work. However, these authors investigate a still more general concept of conditional probability, where p is defined on cartesian products $\mathscr{B}\times\mathscr{A}$, and $\mathscr{A}$ is an *arbitrary* non-empty subset of $\mathscr{B}\sim\{0\}$. Many of their rather subtle results are concerned with difficulties arising from this more general definition. In particular they develop a theory of representation of conditional probabilities in terms of "dimensionally ordered" sets of measures on $\mathscr{B}$. We use Rényi's basic idea of the natural ordering of $\mathscr{B}$ induced by a conditional probability p, and on the basis of these definitions we then can considerably simplify and generalize Rényi's analysis of the structure of conditional probabilities. In Section 2 we thus arrive at a *representation theorem* for conditional probabilities on Boolean algebras, and we study conditional probabilities in terms of this representation. In Section 3 we show that *finitely additive*

[1] This paper is essentially a part of the authors doctoral dissertation submitted in September 1966 to the University of California at Berkeley. It was prepared under the guidance of Professor DANA SCOTT of Stanford University, to whom the author is indebted for many ideas and suggestions.

conditional probabilities can be represented by strictly positive, finitely additive probabilities with values in (non-archemedian) extension fields of the reals. We use this representation to prove that every finitely additive conditional probability on a subalgebra of a Boolean algebra can be extended to the whole algebra.

2. Representation of conditional probabilities

We begin with a definition which is due to RÉNYI [8]. For every $x, y \in \mathscr{B} \sim \{0\}$ define

$$x \preceq y \quad \text{iff} \quad p(y|x \vee y) > 0;$$

$$x \prec y \quad \text{iff} \quad p(x|x \vee y) = 0.$$

Then we have the following basic properties of $\preceq$ and $\prec$:

LEMMA 2.1. *Let* $x, y, z \in \mathscr{B} \sim \{0\}$. *Then*

(i) *if* $x \leqq y$ *then* $x \preceq y$;

(ii) *if* $x \preceq y$ *and* $y \preceq z$ *then* $x \preceq z$;

(iii) $x \preceq y$ *or* $y \preceq x$;

(iv) *if* $x \preceq y$ *then* $x \vee y \preceq y$;

(v) $x \prec y$ *iff* $x \preceq y$ *and* $y \not\preceq x$.

PROOF. (i), (iii), (iv) and (v) are obvious. To prove (ii), suppose $p(y|x \vee y)$, $p(z|y \vee z) > 0$. We consider two cases. If $p(y|x \vee y \vee z) = 0$, then by Axiom (II), $p(y|x \vee y) \cdot p(x \vee y|x \vee y \vee z) = 0$. By hypothesis $p(x \vee y|x \vee y \vee z) = 0$. Thus $p(z|x \vee y \vee z) = 1$, and therefore $p(z|x \vee z) = 1$. If $p(y|x \vee y \vee z) > 0$, then $p(y \vee z|x \vee y \vee z) > 0$. By Axiom (II), $p(z|x \vee y \vee z) = p(z|y \vee z) \cdot p(y \vee z|x \vee y \vee z)$. By hypothesis $p(z|x \vee y \vee z) > 0$, and therefore $p(z|x \vee z) > 0$. Thus in either case $x \preceq z$.

LEMMA 2.2. *Let* $x_i, y_i \in \mathscr{B}$, $y_i \neq 0$ *for* $i = 1, 2$. *If* $x_1 \vee x_2 \leqq y_1 \wedge y_2$ *and if* $p(x_2|y_1), p(x_2|y_2) > 0$ *then*

$$\frac{p(x_1|y_1)}{p(x_2|y_1)} = \frac{p(x_1|y_2)}{p(x_2|y_2)}.$$

PROOF. By hypothesis $y_1 \preceq x_2$ and $y_2 \preceq x_2$. Also $x_2 \preceq y_1$ and $x_2 \preceq y_2$, using Lemma 2.1 (i). By Lemma 2.1 (ii), $y_1 \preceq y_2$ and $y_2 \preceq y_1$. Thus Lemma 2.2 follows from Axiom (II):

$$\frac{p(x_1|y_1 \vee y_2)}{p(x_2|y_1 \vee y_2)} = \frac{p(x_1|y_1) \cdot p(y_1|y_1 \vee y_2)}{p(x_2|y_1) \cdot p(y_1|y_1 \vee y_2)} = \frac{p(x_1|y_2) \cdot p(y_2|y_1 \vee y_2)}{p(x_2|y_2) \cdot p(y_2|y_1 \vee y_2)}.$$

From Lemma 2.1 we can derive purely algebraically:

COROLLARY 2.3. *Let* $x, y, z \in \mathscr{B} \sim \{0\}$. *Then*

(i) *if* $x \preceq z$ *and* $y \preceq z$ *then* $x \vee y \preceq z$;

(ii) *if* $x \prec z$ *and* $y \prec z$ *then* $x \vee y \prec z$.

PROOF. (i) Suppose $x \preceq z$ and $y \preceq z$. By 2. 1, (iii) and (iv), either $x \vee y \preceq x$ or $x \vee y \preceq y$. By 2. 1 (ii) in both cases $x \vee y \preceq z$.

(ii) Suppose $x \prec z$ and $y \prec z$. By (i), $x \vee y \preceq z$. Suppose $z \preceq x \vee y$. As before $x \vee y \preceq x$ or $x \vee y \preceq y$. By 2. 1 (ii), $z \preceq x$ or $z \preceq y$, contradicting 2. 1 (v) and the hypothesis. Thus $z \not\preceq x \vee y$, and therefore $x \vee y \prec z$ by 2. 1 (v).

For every $x \in \mathscr{B} \sim \{0\}$ define

$$\mathscr{I}(x) = \{y \in \mathscr{B} \sim \{0\} \mid y \preceq x\} \cup \{0\}.$$

Then we have as an elementary consequence of Lemma 2. 1 and Corollary 2. 3

LEMMA 2. 4. (i) *For every* $x \in \mathscr{B} \sim \{0\}$, $\mathscr{I}(x)$ *is an ideal in* $\mathscr{B}$.

(ii) *The family* $\{\mathscr{I}(x) \mid x \in \mathscr{B} \sim \{0\}\}$ *is linearly ordered by set inclusion, and for every* $x, y \in \mathscr{B} \sim \{0\}$

$$\mathscr{I}(x) \subset \mathscr{I}(y) \quad \textit{iff} \quad x \prec y,$$

where we use $\subset$ *for proper set inclusion.*

(iii) *For every* $x \in \mathscr{B} \sim \{0\}$,

$$x \in \mathscr{I}(x) \sim \cup \{\mathscr{I}(y) \mid \mathscr{I}(y) \subset \mathscr{I}(x)\}.$$

For every $x \in \mathscr{B} \sim \{0\}$ *and* $y \in \mathscr{I}(x)$ *define*

$$m_x(y) = \frac{p(y|x \vee y)}{p(x|x \vee y)}.$$

LEMMA 2. 5. (i) *For every* $x \in \mathscr{B} \sim \{0\}$, m_x *is a (non-trivial) finitely additive measure on* $\mathscr{I}(x)$. *If* p *is* σ*-additive, then* m_x *is* σ*-additive.*

(ii) *For every* $y \in \mathscr{I}(x) \sim \{0\}$, $m_x(y) = 0$ *iff* $\mathscr{I}(y) \subset \mathscr{I}(x)$.

(iii) *For every* $y, z \in \mathscr{I}(x)$,

$$m_x(y \wedge z) = p(y|z) \cdot m_x(z).$$

(iv) *For every* $x, y \in \mathscr{B} \sim \{0\}$, *if* $\mathscr{I}(x) = \mathscr{I}(y)$ *then*

$$m_x = \frac{p(y|x \vee y)}{p(x|x \vee y)} \cdot m_y.$$

PROOF. (i) Let $x \in \mathscr{B} \sim \{0\}$, let $y, z \in \mathscr{I}(x)$ and assume $y \wedge z = 0$. Then by Lemma 2. 2

$$m_x(y \vee z) = \frac{p(y \vee z|x \vee y \vee z)}{p(x|x \vee y \vee z)} = \frac{p(y|x \vee y \vee z)}{p(x|x \vee y \vee z)} + \frac{p(z|x \vee y \vee z)}{p(x|x \vee y \vee z)} =$$

$$= \frac{p(y|x \vee y)}{p(x|x \vee y)} + \frac{p(z|x \vee z)}{p(x|x \vee z)} = m_x(y) + m_x(z).$$

A similar argument establishes the second part of (i), and (ii) is obvious.

(iii) Let $y, z \in \mathscr{I}(x)$. If $m_x(z) = 0$ then the equation holds trivially. If $m_x(z) > 0$ then $p(z|x \vee z) > 0$, and therefore by Lemma 2. 2

$$m_x(y \wedge z) = \frac{p(y \wedge z|x \vee y \vee z)}{p(x|x \vee y \vee z)} = \frac{p(y \wedge z|x \vee z)}{p(x|x \vee z)} =$$

$$= \frac{p(y \wedge z|x \vee z)}{p(z|x \vee z)} \cdot \frac{p(z|x \vee z)}{p(x|x \vee z)} = p(y|z) \cdot m_x(z).$$

(iv) Let $x, y \in \mathscr{B} \sim \{0\}$ and suppose that $\mathscr{I}(x) = \mathscr{I}(y)$. Then, by Lemma 2. 2, for every $z \in \mathscr{I}(x)$

$$m_x(z) = \frac{p(z|x \vee z)}{p(x|x \vee z)} = \frac{p(z|x \vee y \vee z)}{p(x|x \vee y \vee z)}$$

$$m_y(z) = \frac{p(z|y \vee z)}{p(y|y \vee z)} = \frac{p(z|x \vee y \vee z)}{p(y|x \vee y \vee z)}.$$

Thus, by another application of Lemma 2. 2,

$$m_x(z) = \frac{p(y|x \vee y \vee z)}{p(x|x \vee y \vee z)} m_y(z) = \frac{p(y|x \vee y)}{p(x|x \vee y)} \cdot m_y(z).$$

Collecting our results, we have the basic *Representation Theorem:*

THEOREM 2. 6. *Let $\mathscr{B}$ be a Boolean algebra, and let p be a conditional probability on $\mathscr{B}$. Then there exists a family $\{\langle \mathscr{I}_i, m_i \rangle | i \in I\}$ such that*

(i) *for every $i \in I$, $\mathscr{I}_i$ is an ideal in $\mathscr{B}$, and the family $\{\mathscr{I}_i | i \in I\}$ is linearly ordered by set inclusion;*

(ii) *for every $x \in \mathscr{B} \sim \{0\}$ there exists $i \in I$ such that $x \in \mathscr{I}_i \sim \bigcup \{\mathscr{I}_j | \mathscr{I}_j \subset \mathscr{I}_i\}$;*

(iii) *for every $i \in I$, m_i is a (non-trivial) finitely additive measure on $\mathscr{I}_i$ such that for every $x \in \mathscr{I}_i$, $m_i(x) = 0$ iff $x \in \bigcup \{\mathscr{I}_j | \mathscr{I}_j \subset \mathscr{I}_i\} \cup \{0\}$. Moreover, if p is σ-additive then m_i is σ-additive;*

(iv) *for every $i \in I$ and every $x, y \in \mathscr{I}_i$*

$$m_i(x \wedge y) = p(x|y) \cdot m_i(y).$$

Conversely, every family $\{\langle \mathscr{I}_i, m_i \rangle | i \in I\}$ which satisfies conditions (i)—(iii) *and the compatibility condition:*

(v) *for every $i, j \in I$, if $\mathscr{I}_i = \mathscr{I}_j$ then there exists a real number $r > 0$ such that $m_i = r \cdot m_j$;*

uniquely determines a conditional probability p on $\mathscr{B}$ such that (iv) *holds. Moreover, if for every $i \in I$, m_i is σ-additive then p is σ-additive.*

PROOF. We only have to comment on the converse. If the family $\{\langle \mathscr{I}_i, m_i \rangle | i \in I\}$ satisfies conditions (i)—(iii) and (v), consider $x, y \in \mathscr{B}$, where $y \neq 0$. By (ii) there exists $i \in I$ such that $y \in \mathscr{I}_i \sim \bigcup \{\mathscr{I}_j | \mathscr{I}_j \subset \mathscr{I}_i\}$. By (iii), $m_i(y) > 0$. Thus define

$$p(x|y) = \frac{m_i(x \wedge y)}{m_i(y)}.$$

By (v) p is well-defined, and it is easy to check that p is the desired conditional probability on $\mathscr{B}$. The uniqueness of p is obvious.

A family $\{\langle \mathscr{I}_i, m_i \rangle | i \in I\}$ satisfying conditions (i)—(iv) of Theorem 2. 6 is called a *chain representation* of p. Chain representations are essentially unique, as the next theorem shows.

THEOREM 2. 7. *If $\{\langle \mathscr{I}_i, m_i \rangle | i \in I\}$ is a chain representation of p then $\{\mathscr{I}_i | i \in I\} = \{\mathscr{I}(x) | x \in \mathscr{B} \sim \{0\}\}$, and for every $i \in I$ and $x \in \mathscr{B} \sim \{0\}$, if $\mathscr{I}_i = \mathscr{I}(x)$ then there exists a real number $r > 0$ such that $m_i = r \cdot m_x$.*

PROOF. Let $i \in I$. By (iii) there exists $x \in \mathscr{I}_i \sim \bigcup \{\mathscr{I}_j | \mathscr{I}_j \subset \mathscr{I}_i\}$ such that $m_i(x) > 0$. We will show that $\mathscr{I}_i = \mathscr{I}(x)$. Indeed, suppose $y \in \mathscr{I}_i$. By (i), $x \vee y \in \mathscr{I}_i$ and by (iv), $m_i(x) = p(x|x \vee y) \cdot m_i(x \vee y)$. Thus $p(x|x \vee y) > 0$, and therefore $y \in \mathscr{I}(x)$. Now suppose $y \notin \mathscr{I}_i$. By (ii) there exists $k \in I$ such that $y \in \mathscr{I}_k \sim \bigcup \{\mathscr{I}_j | \mathscr{I}_j \subset \mathscr{I}_k\}$. Thus, by (i), $\mathscr{I}_i \subset \mathscr{I}_k$ and therefore, by (iii), $m_k(y) > 0$ and $m_k(x) = 0$. By (i), $x \vee y \in \mathscr{I}_k$ and by (iv), $m_k(x) = p(x|x \vee y) \cdot m_k(x \vee y)$. Thus $p(x|x \vee y) = 0$, and therefore $y \notin \mathscr{I}(x)$. This proves that $\mathscr{I}_i = \mathscr{I}(x)$.

Now let $x \in \mathscr{B} \sim \{0\}$. By (ii) there exists $i \in I$ such that $x \in \mathscr{I}_i \sim \bigcup \{\mathscr{I}_j | \mathscr{I}_j \subset \mathscr{I}_i\}$, and the same argument establishes that $\mathscr{I}(x) = \mathscr{I}_i$. This proves the first part of the assertion.

Finally, let $i \in I$ and $x \in \mathscr{B} \sim \{0\}$ and suppose $\mathscr{I}_i = \mathscr{I}(x)$. Then, by (iv), for every $y \in \mathscr{I}_i$

$$m_i(y) = p(y|x \vee y) \cdot m_i(x \vee y).$$

Thus

$$m_i(y) = m_i(x) \cdot m_x(y)$$

where $m_i(x) > 0$, and the proof is complete.

Theorem 2. 7 shows that different chain representations of a conditional probability p all have the same chain of ideals. However, the measures on these ideals are only determined up to a multiplicative constant. Note also that in a given chain representation an ideal may occur with many different measures. Of course we can choose representatives, however the arbitrariness of such a choice may as well be reflected in the definition of the representation.

We wish to utilize chain representations for the construction of conditional probabilities on Boolean algebras. For this purpose we will isolate what might be called the "constructive part" of the chain representation. Let $x \in \mathscr{B}$. Then x is called a *p-annihilator* if either $x = 0$ or there exist $y_n \in \mathscr{B}$, for each $n < \omega$, such that $x \leq y_0$ and for every $n < \omega$, $y_n \prec y_{n+1}$. Let $\mathscr{N}$ be the set of p-annihilators. $\mathscr{N}$ determines a "cut" in the chain of ideals:

LEMMA 2. 8. (i) *$\mathscr{N}$ is an ideal in $\mathscr{B}$ which either equals $\{0\}$ or is infinite.*

(ii) *For every $x \in \mathscr{B} \sim \{0\}$, either $\mathscr{I}(x) \subset \mathscr{N}$ or $\mathscr{N} \subset \mathscr{I}(x)$.*

(iii) *The family $\{\mathscr{I}(x) | x \in \mathscr{B} \sim \{0\}, \mathscr{N} \subset \mathscr{I}(x)\}$ is inversely well-ordered by set inclusion such that*

(a) *if $\mathscr{I}(y)$ is the successor-ideal of $\mathscr{I}(x)$ then*

$$\mathscr{I}(y) = \{z \in \mathscr{I}(x) | m_x(z) = 0\};$$

(b) *if $\mathscr{I}(y)$ is a limit-ideal then*

$$\mathscr{I}(y) = \bigcap \{\mathscr{I}(x) | \mathscr{I}(y) \subset \mathscr{I}(x)\};$$

(c) *if $\mathscr{I}(x)$ is the last ideal then*

$$\mathscr{N} = \{y \in \mathscr{I}(x) | m_x(y) = 0\};$$

(d) *if there exists no last ideal then*

$$\mathscr{N} = \bigcap \{\mathscr{I}(x) | \mathscr{N} \subset \mathscr{I}(x)\}.$$

(iv) $\mathscr{N} = \bigcup \{\mathscr{I}(x) | \mathscr{I}(x) \subset \mathscr{N}\}$.

PROOF. (i) By definition, if $x \leqq y \in \mathcal{N}$ then $x \in \mathcal{N}$. Thus by Lemma 2. 1, if $x \leqq y \in \mathcal{N}$ then $x \in \mathcal{N}$. Suppose $0 \neq x, x' \in \mathcal{N}$. Then there exist $y_n, y'_n \in \mathcal{B}$, for each $n < \omega$, such that $x \leqq y_0$, $x' \leqq y'_0$ and for every $n < \omega$, $y_n \prec y_{n+1}$ and $y'_n \prec y'_{n+1}$. For each $n < \omega$, let $z_n = y_n \vee y'_n$. Then $x \vee x' \leqq z_0$ and by Corollary 2. 3, for each $n < \omega$, $z_n \prec z_{n+1}$. Thus $x \vee x' \in \mathcal{N}$, and $\mathcal{N}$ is an ideal in $\mathcal{B}$. The last part of (i) is obvious.

(ii) Let $x \in \mathcal{B} \sim \{0\}$. Clearly $\mathcal{I}(x) \neq \mathcal{N}$. If $x \in \mathcal{N}$ then $\mathcal{I}(x) \subset \mathcal{N}$. If $x \notin \mathcal{N}$ then for every $0 \neq z \in \mathcal{N}$, $x \not\leqq z$. Thus $z \leqq x$, and therefore $\mathcal{N} \subset \mathcal{I}(x)$.

(iii) Since for every $x, y \in \mathcal{B} \sim \{0\}$, $x \prec y$ iff $\mathcal{I}(x) \subset \mathcal{I}(y)$ (Lemma 2. 4), by the definition of $\mathcal{N}$ there exists no infinite strictly increasing chain in the family $\{\mathcal{I}(x) | x \in \mathcal{B} \sim \{0\}, \mathcal{N} \subset \mathcal{I}(x)\}$. Therefore this family is inversely well-ordered by set inclusion. If $\mathcal{I}(y)$ is the successor-ideal of $\mathcal{I}(x)$ then $\mathcal{I}(y) = \bigcup \{\mathcal{I}(z) | \mathcal{I}(z) \subset \mathcal{I}(x)\}$ and (a) follows from Lemma 2. 5. If $\mathcal{I}(y)$ is a limit-ideal then $\mathcal{I}(y) \subset \bigcap \{\mathcal{I}(x) | \mathcal{I}(y) \subset \subset \mathcal{I}(x)\}$ contradicts Lemma 2. 4. This establishes (b). If $\mathcal{I}(x)$ is the last ideal, then by (ii) and (iv), $\mathcal{N} = \bigcup \{\mathcal{I}(y) | \mathcal{I}(y) \subset \mathcal{I}(x)\}$, and (c) follows from Lemma 2. 5. If there exists no last ideal, then $\mathcal{N} \subset \bigcap \{\mathcal{I}(x) | \mathcal{N} \subset \mathcal{I}(x)\}$ contradicts Lemma 2. 4. This establishes (d).

(iv) is obvious and the proof is complete.

We call $\mathcal{N}$ the *ideal* of *p-annihilators*, and if $\mathcal{N} = \{0\}$, p is called *well-founded*.

THEOREM 2. 9. (i) *Every Boolean algebra $\mathcal{B}$ carries a well-founded finitely additive conditional probability p.*

(ii) *A Boolean algebra $\mathcal{B}$ is finite iff every conditional probability p on B is well-founded.*

PROOF. (i) Let $\mathcal{B}$ be a Boolean algebra. We construct an ordinal $\alpha > 0$ and a sequence $\{\langle \mathcal{I}_\xi, m_\xi \rangle | \xi < \alpha\}$ such that for each $\xi < \alpha$, $\mathcal{I}_\xi$ is an ideal of $\mathcal{B}$ and m_ξ is a (bounded) non-trivial finitely additive measure on $\mathcal{I}_\xi$, as follows: Let $\mathcal{I}_0 = \mathcal{B}$, and let m_0 be a finitely additive probability on $\mathcal{B}$. Suppose $\eta > 0$ and $\{\langle \mathcal{I}_\xi, m_\xi \rangle | \xi < \eta\}$ has been constructed. If $\eta = \xi + 1$ and $\{x \in \mathcal{I}_\xi | m_\xi(x) = 0\} = \{0\}$, let $\alpha = \eta$. If $\{x \in \mathcal{I}_\xi | m_\xi(x) = 0\} \neq \{0\}$, let $\mathcal{I}_\eta = \{x \in \mathcal{I}_\xi | m_\xi(x) = 0\}$ and let m_η be a (bounded) non-trivial finitely additive measure on $\mathcal{I}_\eta$. If $\eta = \bigcup_{\xi < \eta} \xi$ and $\bigcap_{\xi < \eta} \mathcal{I}_\xi = \{0\}$, let $\alpha = \eta$. If $\bigcap_{\xi < \eta} \mathcal{I}_\xi \neq \{0\}$, let $\mathcal{I}_\eta = \bigcap_{\xi < \eta} \mathcal{I}_\xi$ and let m_η be a (bounded) non-trivial finitely additive measure on $\mathcal{I}_\eta$. By a cardinality argument this construction has to terminate at some ordinal α. Clearly the sequence $\{\langle \mathcal{I}_\xi, m_\xi \rangle | \xi < \alpha\}$ satisfies conditions (i) and (iii) of Theorem 2. 6. Let $x \in \mathcal{B} \sim \{0\}$. If $x \in \bigcap_{\xi < \alpha} \mathcal{I}_\xi$, then for some $\xi < \alpha$, $\alpha = \xi + 1$ and condition (ii) of Theorem 2. 6 is satisfied. If $x \notin \bigcap_{\xi < \alpha} \mathcal{I}_\xi$, let $\xi < \alpha$ be the first ordinal such that $x \notin \mathcal{I}_\xi$. Then for some $\eta < \xi$, $\xi = \eta + 1$ and condition (ii) of Theorem 2. 6 is again satisfied. Thus by Theorem 2. 6, there exists a unique conditional probability p on $\mathcal{B}$ such that condition (iv) of Theorem 2. 6 holds. It is also clear from the construction that p is well-founded.

(ii) Let $\mathcal{B}$ be a Boolean algebra. If $\mathcal{B}$ is finite then, by Lemma 2. 8 (i), every conditional probability p on $\mathcal{B}$ is well-founded. Conversely, assume $\mathcal{B}$ is infinite. Then there exist $b_n \in \mathcal{B}$, for each $n < \omega$, such that $b_0 = 0$ and for every $0 < n < \omega$, $0 < b_n < b_{n+1} < 1$. For every $n < \omega$, let $\mathcal{K}_n = \{x \in \mathcal{B} | x \leqq b_n\}$. Then $\mathcal{K}_0 = \{0\}$, and for every $0 < n < \omega$, $\mathcal{K}_0 \subset \mathcal{K}_n \subset \mathcal{K}_{n+1} \subset \mathcal{B}$. Let $\mathcal{K}_\omega = \bigcup_{n < \omega} \mathcal{K}_n$, and let $\mathcal{K}_{\omega+1} = \mathcal{B}$.

Then $\mathscr{K}_\omega \subset \mathscr{K}_{\omega+1}$, since $1 \notin \mathscr{K}_n$ for $n<\omega$. As in the proof of part (i) we now fill in the gaps by the "top-down construction". For every $\xi+1 \leqq \omega+1$ we define an ordinal $\alpha_{\xi+1}>0$ and a sequence $\{\langle \mathscr{I}_{\xi+1,\eta}, m_{\xi+1,\eta}\rangle | \eta<\alpha_{\xi+1}\}$ such that $\mathscr{I}_{\xi+1,0}= \mathscr{K}_{\xi+1}$, and for each $\eta<\alpha_{\xi+1}$, $\mathscr{I}_{\xi+1,\eta}$ is an ideal of $\mathscr{B}$ such that $\mathscr{K}_\xi \subset \mathscr{I}_{\xi+1,\eta}$, and $m_{\xi+1,\eta}$ is a (bounded) non-trivial finitely additive measure on $\mathscr{I}_{\xi+1,\eta}$ such that $m_{\xi+1,\eta}(x)=0$ for all $x \in \mathscr{K}_\xi$. Moreover, if $\eta+1<\alpha_{\xi+1}$ then $\mathscr{I}_{\xi+1,\eta+1}= \{x \in \mathscr{I}_{\xi+1,\eta} | m_{\eta+1,\eta}(x)=0\}$; if $\bigcup_{\zeta<\eta} \zeta = \eta < \alpha_{\xi+1}$ then $\mathscr{I}_{\xi+1,\eta} = \bigcap_{\zeta<\eta} \mathscr{I}_\zeta$; if $\alpha_{\xi+1} = \beta+1$ then $\mathscr{K}_\xi = \{x \in \mathscr{I}_{\xi+1,\beta} | m_{\xi+1,\beta}(x) = 0\}$; and if $\alpha_{\xi+1} = \bigcup_{\beta<\alpha_{\xi+1}} \beta$ then $\mathscr{K}_\xi = \bigcup_{\beta<\alpha_{\xi+1}} \mathscr{I}_\beta$. It is now easy to check that the family $\{\langle \mathscr{I}_{\xi+1,\eta}, m_{\xi+1,\eta}\rangle | \xi+1 \leqq \leqq \omega+1, \eta<\alpha_{\xi+1}\}$ satisfies conditions (i)—(iii) of Theorem 2. 6. Thus there exists a unique conditional probability p on $\mathscr{B}$ such that (iv) of Theorem 2. 6 holds. Finally it is obvious that $\mathscr{N}=\mathscr{K}_\omega$, that is p is not well-founded and the proof is complete.

We can also show that our analysis of the order type of chain representations of conditional probabilities is as fine as possible. Indeed, the family $\{\mathscr{I}(x) | x \in \mathscr{B} \sim \{0\}\}$ can be constructed together with a suitable Boolean algebra $\mathscr{B}$ to be linearly ordered in *any* type $\tau+1$. To see this, let I be an index set linearly ordered in type τ, and let $\mathscr{B}$ be the Boolean algebra with set of free generators $\{b_i | i \in I\}$. For every $i \in I$, let $\mathscr{I}_i$ be the ideal generated by $\{b_j | j \leqq i\}$. Then clearly for every $i, j \in I$, $\mathscr{I}_i \subset \mathscr{I}_j$ iff $i<j$. Moreover, $\bigcup\{\mathscr{I}_i | i \in I\}$ is a prime ideal in $\mathscr{B}$. Indeed, if $x \in \mathscr{B} \sim \bigcup\{\mathscr{I}_i | i \in I\}$, then x is of the form $[(1-b_{i_0}) \wedge \ldots \wedge (1-b_{i_{n-1}})] \vee a$. Thus $1-x$ is of the form $[b_{i_0} \vee \ldots \vee b_{i_{n-1}}] \wedge (1-a)$, and therefore $1-x \in \bigcup\{\mathscr{I}_i | i \in I\}$. Assume $0 \notin I$, and define $\mathscr{I}_0 = \mathscr{B}$ and

$$m_0(x) = \begin{cases} 1 & \text{if} \quad x \notin \bigcup\{\mathscr{I}_i | i \in I\} \\ 0 & \text{if} \quad x \in \bigcup\{\mathscr{I}_i | i \in I\} \end{cases}$$

for every $x \in \mathscr{B}$. Similarly we see that for every $i \in I$, $\bigcup\{\mathscr{I}_j | \mathscr{I}_j \subset \mathscr{I}_i\}$ is maximal in $\mathscr{I}_i$. Thus define for every $x \in \mathscr{I}_i$

$$m_i(x) = \begin{cases} 1 & \text{if} \quad x \notin \bigcup\{\mathscr{I}_j | \mathscr{I}_j \subset \mathscr{I}_i\} \\ 0 & \text{if} \quad x \in \bigcup\{\mathscr{I}_j | \mathscr{I}_j \subset \mathscr{I}_i\} \end{cases}.$$

Finally we define $0>i$ for every $i \in I$ to obtain a family $\{\langle \mathscr{I}_k, m_k\rangle | k \in I \cup \{0\}\}$ which satisfies conditions (i)—(iii) of Theorem 2. 6.

In our constructions thus far we have only produced *finitely additive* conditional probabilities. This is not accidental. Often not even the "top-down construction", which yields well-founded probabilities, is a useful method to construct σ-additive conditional probabilities. The question of which Boolean rings carry a non-trivial, real valued σ-additive measure has not yet been settled in full generality and appears to be very difficult. Of course, there exist Boolean σ-algebras which cannot carry a σ-additive conditional probability, and the Borel algebra of [0, 1] modulo sets of first category is a well-known example.

The standard result of measure theory, that a σ-additive probability on a Boolean σ-algebra $\mathscr{B}$ is already determined by its values on a subalgebra $\mathscr{A}$ of $\mathscr{B}$ which σ-generates $\mathscr{B}$, also fails for σ-additive conditional probabilities. To see this, let $\mathscr{A}$ be the field of subsets of [0, 1] generated by the left closed, right open intervals,

and let λ be Lebesgue measure. Define a σ-additive conditional probability p on $\mathscr{A}$ by

$$p(A|B) = \frac{\lambda(A \cap B)}{\lambda(B)}$$

for all $A, B \in \mathscr{A}$ where $B \neq \emptyset$. Let $\mathscr{B}$ be the Borel field of [0, 1] and define:

$$\mathscr{I}_{00} = \mathscr{I}_{10} = \mathscr{B}, \quad m_{00} = m_{10} = \lambda, \quad \mathscr{I}_{01} = \mathscr{I}_{11} = \{A \in \mathscr{B} | \lambda(A) = 0\}.$$

Now choose $\emptyset \neq A_{01}, A_{11} \in \mathscr{I}_{01}$ such that $A_{01} \cap A_{11} = \emptyset$, and pick $a_{i1} \in A_{i1}$, where $i = 1, 2$. Define for every $B \in \mathscr{I}_{01}$

$$m_{i1}(B) = \begin{cases} 1 & \text{if} \quad a_{i1} \in B \\ 0 & \text{if} \quad a_{i1} \notin B \end{cases}$$

where $i = 1, 2$. If we complete the "top-down construction" by defining two chains of σ-ideals $\{\mathscr{I}_{i\xi} | \xi < \alpha_i\}$, where $i = 0, 1$, each of which is determined by a point, we obtain two σ-additive, well-founded extensions p_1 and p_2 of p defined on $\mathscr{B}$ such that $p_1 \neq p_2$.

Interesting σ-additive conditional probabilities on Boolean σ-algebras therefore seem to be rather difficult to define. On the other hand, for finitely additive conditional probabilities we can further improve our results. This will be done in Section 3.

3. Representation of finitely additive conditional probabilities

We first state some well-known facts about ordered extension fields of the reals. Let R be the set of real numbers, and let F be an ordered extension field of R. Let I be the set of infinitesimals of F, that is the set of $\xi \in F$ such that for every $n > 0, |\xi| < 1/n$. Let D be the set of $\xi \in F$ such that for some $n > 0$, $|\xi| < n$. Note that $R \subseteq D \subseteq F$, and D is an integral domain. Every $\xi \in D$ determines a cut in R. For $\xi \in D$, let $\mathrm{Re}\,(\xi)$ be the real cut-number determined by ξ.

LEMMA 3. 1. (i) *I is a maximal ideal in D.*
(ii) *$D/I \cong R$, and for every $\xi, \eta \in D$*

$$\mathrm{Re}\,(\xi + \eta) = \mathrm{Re}\,(\xi) + \mathrm{Re}\,(\eta), \qquad \mathrm{Re}\,(\xi \cdot \eta) = \mathrm{Re}\,(\xi) \cdot \mathrm{Re}\,(\eta).$$

PROOF. (i) If $\xi, \eta \in I$ then for every $n > 0$

$$|\xi + \eta| \leqq |\xi| + |\eta| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

If $\xi \in I, \eta \in D$ then for some $N > 0$, $|\eta| < N$. Thus for every $n > 0$

$$|\xi \cdot \eta| = |\xi| \cdot |\eta| < N \frac{1}{N \cdot n} = \frac{1}{n}.$$

Therefore I is an ideal in D. If for some ideal J in D, $I \subset J \subseteq D$, choose $\xi \in J \sim I$. We may assume that for some $n > 0$, $1/n < \xi$. Thus $\mathrm{Re}\,(\xi) > 0$. Since $\xi - \mathrm{Re}\,(\xi) \in I$ and $\xi \in J$, $\mathrm{Re}\,(\xi) \in J$. Thus $1 \in J$, and therefore $J = D$.

(ii) is an obvious consequence of (i).

As is well-known, not every Boolean algebra carries a strictly positive, finitely additive probability. On the other hand every Boolean algebra carries a strictly positive, finitely additive probability measure with values in an ordered extension field of the reals (see NIKODYM [6]), and TARSKI observed that this fact has an easy proof by means of the Compactness Theorem of finitary first order logic. LUXEMBURG [5] subsequently gave a proof based on an ultra power construction, and we will apply this method to prove a new representation theorem for finitely additive conditional probabilities in terms of strictly positive, finitely additive probability measures with values in an ordered extension field of the reals. For a detailed discussion of ultra power constructions we refer the reader to FRAYNE, MOREL and SCOTT [2].

Let $m: \mathscr{B} \to F$ be a strictly positive, finitely additive probability measure on $\mathscr{B}$ with values in F. For $x, y \in \mathscr{B}$, $y \neq 0$ define

$$p(x|y) = \operatorname{Re} \frac{m(x \wedge y)}{m(y)}.$$

THEOREM 3. 2. *p is a finitely additive conditional probability on $\mathscr{B}$.*

PROOF. Let $x, y, z \in \mathscr{B}$, where $z \neq 0$ and $x \wedge y = 0$. By Lemma 3. 1

$$p(x \vee y|z) = \operatorname{Re} \frac{m((x \wedge z) \vee (y \wedge z))}{m(z)} =$$

$$= \operatorname{Re} \frac{m(x \wedge z) + m(y \wedge z)}{m(z)} = \operatorname{Re} \frac{m(x \wedge z)}{m(z)} + \operatorname{Re} \frac{m(y \wedge z)}{m(z)} = p(x|z) + p(y|z).$$

To check the multiplication axiom, let $x, y, z \in \mathscr{B}$, where $x \wedge z \neq 0$. By Lemma 3.1

$$p(x \wedge y|z) = \operatorname{Re} \frac{m(x \wedge y \wedge z)}{m(z)} = \operatorname{Re} \frac{m(x \wedge y \wedge z)}{m(x \wedge z)} \cdot \frac{m(x \wedge z)}{m(z)} =$$

$$= \operatorname{Re} \frac{m(x \wedge y \wedge z)}{m(x \wedge z)} \cdot \operatorname{Re} \frac{m(x \wedge z)}{m(z)} = p(x|z) \cdot p(y|x \wedge z).$$

This completes the proof.

We also can characterize the ordering $\preceq$ in terms of the F-valued measure m:

LEMMA 3. 3. (i) *For every $x, y \in \mathscr{B} \sim \{0\}$, $x \preceq y$ iff for some $n > 0$, $m(x) \leqq n \cdot m(y)$.*
(ii) *For every $x, y \in \mathscr{B} \sim \{0\}$, $x \prec y$ iff for some $\xi \in I$, $m(x) = \xi \cdot m(y)$.*

PROOF. (i) Let $x, y \in \mathscr{B} \sim \{0\}$. Then $x \preceq y$ iff $p(y|x \vee y) > 0$, iff $\operatorname{Re} \frac{m(y)}{m(x \vee y)} > 0$, iff for some $n > 0$, $\frac{m(y)}{m(x \vee y)} \geqq \frac{1}{n}$. If $n \cdot m(y) \geqq m(x \vee y)$, then $n \cdot m(y) \geqq m(x)$. If $n \cdot m(y) \geqq m(x)$, then $(n+1) m(y) \geqq m(x) + m(y) \geqq m(x \vee y)$.

(ii) Let $x, y \in \mathscr{B} \sim \{0\}$. Then, by part (i), $x \prec y$ iff for all $n > 0$, $n \cdot m(x) < m(y)$, iff $\frac{m(x)}{m(y)} \in I$.

We now prove the converse of Theorem 3. 2 to obtain the desired *Representation Theorem* for finitely additive conditional probabilities.

THEOREM 3. 4. Let $\mathscr{B}$ be a Boolean algebra, and let p be a finitely additive conditional probability on $\mathscr{B}$. Then there exists an ordered extension field F of R and a strictly positive, finitely additive probability measure m on $\mathscr{B}$ with values in F such that for all $x, y \in \mathscr{B}$, $y \neq 0$

$$p(x|y) = \mathrm{Re}\frac{m(x \wedge y)}{m(y)}.$$

PROOF. In a first step we construct an ultra power H of the reals which contains elements ξ_i, for each $i < \omega$, such that

(1) $$1 > n \cdot \xi_0 \quad \text{for all} \quad 0 < n < \omega,$$

(2) $$\xi_i > n \cdot \xi_{i+1} \quad \text{for all} \quad i < \omega \quad \text{and} \quad 0 < n < \omega.$$

Such constructions are well-known by now and we only sketch the basic ideas. Let $R^{(\omega)}$ be the set of finite sequences of reals, and for $s \in R^{(\omega)}$ let $|s|$ be the length of s. For each $0 < r < \omega$ and $0 < n < \omega$ define $J_{rn} = \{s \in R^{(\omega)} | r \leqq |s|$ and $1 > n \cdot s_0$ and $s_i > n \cdot s_{i+1}$ for all $i < |s| - 1\}$. Then it is easy to check that the family $\{J_{rn} | 0 < r < \omega, 0 < n < \omega\}$ has the finite intersection property, and therefore is contained in an ultrafilter $\mathfrak{U}$ on $R^{(\omega)}$. Let H be the ultrapower $R^{(\omega)}/\mathfrak{U}$. Then H is an ordered extension field of R. For each $i < \omega$ and $s \in R^{(\omega)}$ we define

$$\xi_i(s) = \begin{cases} s(i) & \text{if} \quad i < |s| \\ 0 & \text{otherwise}. \end{cases}$$

Then $\xi_i \in R^{R^{(\omega)}}$, and we set $\xi_i^* = \xi_i/\mathfrak{U}$. Now it is an immediate consequence of our definitions that the sequence $\langle \xi_i^* | i < \omega \rangle$ satisfies (1) and (2).

Next we construct the desired extension field F of R as an ultrapower of H. For each finite subalgebra $\mathscr{A} \subseteqq \mathscr{B}$, let $J_{\mathscr{A}}$ be the set of all strictly positive, finitely additive probability measures m with values in H whose domain includes $\mathscr{A}$, and for all $x, y \in \mathscr{A}$, $y \neq 0$

(3) $$p(x|y) = \mathrm{Re}\frac{m(x \wedge y)}{m(y)}.$$

Let $M = \cup \{J_{\mathscr{A}} | \mathscr{A} \subseteqq \mathscr{B}$ a finite subalgebra$\}$. To show that the family $\{J_{\mathscr{A}} | \mathscr{A} \subseteqq \mathscr{B}$ a finite subalgebra$\}$ has the finite intersection property, it suffices to show that for every finite subalgebra $\mathscr{A} \subseteqq \mathscr{B}$, $J_{\mathscr{A}} \neq \emptyset$.

So let $\mathscr{A} \subseteqq \mathscr{B}$ be a finite subalgebra. Then for every $x \in \mathscr{A}$, the ideal $\mathscr{I}(x)$ is principal. From each ideal $\mathscr{I}(x)$ we choose its largest element h. By Theorem 2. 6 we obtain a chain representation of p on $\mathscr{A}$:

$$\emptyset \subset \mathscr{I}(h_{r-1}) \subset \ldots \subset \mathscr{I}(h_1) \subset \mathscr{I}(h_0) = \mathscr{A}$$

where

$$0 < h_{r-1} < \ldots < h_1 < h_0 = 1,$$

and for every $i < r$, h_i is the largest element of $\mathscr{I}(h_i)$. Let

$$\begin{aligned} k_0 &= h_0 - h_1, \\ &\vdots \\ k_{r-2} &= h_{r-2} - h_{r-1}, \\ k_{r-1} &= h_{r-1}. \end{aligned}$$

Then by Theorem 2. 6, for every $i<r$, $p(k_i|h_i)=1$ and for every $x\in\mathscr{A}$, if $x\wedge k_i\neq 0$ then $p(x|h_i)>0$. There exist elements $\xi_0, \ldots, \xi_{r-1}\in H$ such that $\xi_i>n\cdot\xi_{i+1}>0$ for all $0<n<\omega$ and $i<r-1$, and $\xi_0+\ldots+\xi_{r-1}=1$. Indeed such elements can be easily constructed from the sequence satisfying (1) and (2). Define an H-valued finitely additive measure m on $\mathscr{A}$ by the equation

$$m(x)=\sum_{i<r}\xi_i\cdot p(x|h_i),$$

for all $x\in\mathscr{A}$. If $x\in\mathscr{A}\sim\{0\}$ then for some $i<r$, $x\wedge k_i\neq 0$. Thus $m(x)\geqq\xi_i p(x|h_i)>0$, and m is strictly positive. Finally $m(1)=\sum_{i<n}\xi_i=1$, so that m is a probability. It is left to show that (3) is satisfied. Let $x, y\in\mathscr{A}$, $y\neq 0$. Then for some $j<r$, $y\leqq h_j$ and $y\not\leqq h_i$ for all $i>j$. Thus for every $0\leqq i<j$, $p(y|h_i)=0$ and $p(y|h_j)>0$. Thus

$$\frac{m(x\wedge y)}{m(y)}=\frac{\xi_j p(x\wedge y|h_j)+\sum_{j<i<r}\xi_i p(x\wedge y|h_i)}{\xi_j p(y|h_j)+\sum_{j<i<r}\xi_i p(y|h_i)}.$$

Since for all $j<i<r$ and all $n>0$, $\xi_j>n\xi_i$ we have $\frac{\xi_i}{\xi_j}<\frac{1}{n}$ for all $n>0$. Thus for some $\eta_1, \eta_2\in I$

$$\frac{m(x\wedge y)}{m(y)}=\frac{p(x\wedge y|h_j)+\eta_1}{p(y|h_j)+\eta_2}=\frac{p(x\wedge y|h_j)}{p(y|h_j)}+\frac{\eta_1-\eta_2\dfrac{p(x\wedge y|h_j)}{p(y|h_j)}}{p(y|h_j)+\eta_2}.$$

Since $0<p(y|h_j)\in R$, $p(y|h_j)+\eta_2\notin I$. Thus

$$\frac{\eta_1-\eta_2\dfrac{p(x\wedge y|h_j)}{p(y|h_j)}}{p(y|h_j)+\eta_2}\in I$$

and therefore

$$\mathrm{Re}\,\frac{m(x\wedge y)}{m(y)}=\frac{p(x\wedge y|h_j)}{p(y|h_j)}=p(x|y).$$

Thus the family $\{J_{\mathscr{A}}|\mathscr{A}\subseteqq\mathscr{B}$ a finite subalgebra$\}$ has the finite intersection property, and therefore is contained in an ultrafilter $\mathfrak{U}$ on M. Let F be the ultrapower $H^M/\mathfrak{U}$. Then F is an ordered extension field of H, and thus of R. For each $x\in\mathscr{B}$ and $m\in M$ define

$$\mu(x)(m)=\begin{cases}m(x) & \text{if } x\in\mathrm{dom}\,m\\ 0 & \text{otherwise.}\end{cases}$$

Then $\mu(x)\in H^M$, and we set $\mu^*(x)=\mu(x)/\mathfrak{U}$. Now it is again an immediate consequence of our definitions that μ^* is a strictly positive, finitely additive probability measure on $\mathscr{B}$ with values in F. Finally let $x, y\in\mathscr{B}$, $y\neq 0$. Then

$$p(x|y)=\mathrm{Re}\,\frac{\mu^*(x\wedge y)}{\mu^*(y)}.$$

Indeed, $p(x|y) = \operatorname{Re}\dfrac{\mu^*(x \wedge y)}{\mu^*(y)}$ iff for every $0 < n < \omega$,

$$\left|\frac{\mu^*(x \wedge y)}{\mu^*(y)} - p(x|y)\right| < \frac{1}{n}.$$

So let $0 < n < \omega$, and let $\mathscr{A} \subseteqq \mathscr{B}$ be a finite subalgebra such that $x, y \in \mathscr{A}$. Then

$$\left|\frac{\mu^*(x \wedge y)}{\mu^*(y)} - p(x|y)\right| < \frac{1}{n}$$

iff

$$\left\{m \in M \;\middle|\; \left|\frac{\mu(x \wedge y)(m)}{\mu(y)(m)} - p(x|y)\right| < \frac{1}{n}\right\} \in \mathfrak{U}.$$

But

$$J_{\mathscr{A}} \subseteqq \left\{m \in M \;\middle|\; \left|\frac{\mu(x \wedge y)(m)}{\mu(y)(m)} - p(x|y)\right| < \frac{1}{n}\right\}$$

and the proof is complete.

We now prove a slightly stronger version of Nikodym's original theorem which says that every Boolean algebra carries a strictly positive, finitely additive probability measure with values in an ordered extension field of the reals.

LEMMA 3. 5. *Let $\mathscr{B}$ be a Boolean algebra, and let $\mathscr{A} \subseteqq \mathscr{B}$ be a subalgebra. Let F be an ordered extension field of R, and let m be a strictly positive, finitely additive probability measure on $\mathscr{A}$ with values in F. Then there exists an ordered extension field G of F and a strictly positive, finitely additive probability measure n on $\mathscr{B}$ with values in G which extends m.*

PROOF. For each finite subalgebra $\mathscr{B}' \subseteqq \mathscr{B}$, let $J_{\mathscr{B}'}$ be the set of all strictly positive, finitely additive probability measures n with values in F whose domain includes $\mathscr{B}'$, and for all $x \in \mathscr{B}' \cap \mathscr{A}$, $n(x) = m(x)$. Let $M = \bigcup \{J_{\mathscr{B}'} | \mathscr{B}' \subseteqq \mathscr{B}$ a finite subalgebra$\}$. We follow the familiar pattern and show that for every finite subalgebra $\mathscr{B}' \subseteqq \mathscr{B}$, $J_{\mathscr{B}'} \neq \emptyset$. So let $\mathscr{B}' \subseteqq \mathscr{B}$ be a finite subalgebra. Let $\mathscr{A}' = \mathscr{A} \cap \mathscr{B}'$. Let $b_0, \ldots, b_{n-1} \in \mathscr{B}'$ be the atoms of $\mathscr{B}'$ and let

$$\begin{aligned} a_0 &= b_0 \vee \ldots \vee b_{i_0 - 1}, \\ a_1 &= b_{i_0} \vee \ldots \vee b_{i_1 - 1}, \\ &\vdots \\ a_{r-1} &= b_{i_{r-1}} \vee \ldots \vee b_{n-1} \end{aligned}$$

be the atoms of $\mathscr{A}'$. By refinement on the atoms the restriction of m to $\mathscr{A}'$ clearly can be extended to a strictly positive, finitely additive probability measure on $\mathscr{B}'$ with values in F.

This proves that the family $\{J_{\mathscr{B}'} | \mathscr{B}' \subseteqq \mathscr{B}$ a finite subalgebra$\}$ has the finite intersection property, and therefore is contained in an ultrafilter $\mathfrak{U}$ on M. Let G be the ultrapower $F^M/\mathfrak{U}$. Then G is an ordered extension field of F. For every $x \in \mathscr{B}$ and $n \in M$ define

$$\mu(x)(n) = \begin{cases} n(x) & \text{if } x \in \operatorname{dom} n \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu(x) \in F^M$, and we set $\mu^*(x) = \mu(x)/\mathfrak{U}$. Again it is clear that μ^* is a strictly positive, finitely additive probability measure on $\mathscr{B}$ with values in G, and it is easy to check that μ^* extends m. This completes the proof.

We now can prove the full analogue of the well-known theorem that every finitely additive probability on a subalgebra of a Boolean algebra can be extended to the whole algebra (See HORN—TARSKI [4]).

THEOREM 3. 6. *Let $\mathscr{B}$ a Boolean algebra, let $\mathscr{A} \subseteq \mathscr{B}$ be a subalgebra, and let p be a finitely additive conditional probability on $\mathscr{A}$. Then p can be extended to a finitely additive conditional probability q on $\mathscr{B}$.*

PROOF. By Theorem 3. 4, there exists an ordered extension field F of R and a strictly positive, finitely additive probability measure m on $\mathscr{A}$ with values in F such that for every $x, y \in \mathscr{A}$, $y \neq 0$

$$p(x|y) = \operatorname{Re} \frac{m(x \wedge y)}{m(y)}.$$

By Lemma 3. 5, there exists an ordered extension field G of F and a strictly positive, finitely additive probability measure n on $\mathscr{B}$ with values in G which extends m. Thus define for every $x, y \in \mathscr{B}$, $y \neq 0$

$$q(x|y) = \operatorname{Re} \frac{n(x \wedge y)}{n(y)}.$$

Theorem 3. 2 now proves the assertion.

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References

[1] Á. CSÁSZÁR, Sur la structure des espaces de probabilité conditionelle, *Acta Math. Acad. Sci. Hung.*, **6** (1955), pp. 337—361.
[2] A. FRAYNE, A. C. MOREL and D. S. SCOTT, Reduced direct products, *Fundamenta Mathematicae*, **51** (1962), pp. 195—228.
[3] P. R. HALMOS, *Lectures on Boolean algebras* (New York, 1963).
[4] A. HORN and A. TARSKI, Measures in Boolean algebras, *Transactions of the American Mathematical Society*, **64** (1948), pp. 467—497.
[5] W. A. J. LUXEMBURG, Two applications of the method of construction by ultrapowers to analysis, *Bulletin of the American Mathematical Society*, **68** (1962), pp. 416—419.
[6] O. NIKODÝM, Sur la mesure non Archimedienne effective sur une tribu de Boole arbitraire, *C. R. Acad. Sci. Paris*, **251** (1960), pp. 2113—2115.
[7] A. RÉNYI, On a new axiomatic theory of probability, *Acta Math. Acad. Sci. Hung.*, **6** (1955), pp. 285—335.
[8] A. RÉNYI, On conditional probability spaces generated by a dimensionally ordered set of measures, *Teoriia veraiatnostei i ee primeneniia*, **1** (1956), pp. 61—71.
[9] R. SIKORSKI, *Boolean Algebras* (Second Edition) (Berlin, 1964).

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (3—4), (1968), pp. 243—261.

PRIMELEMENTE UND PRIMRADIKALE IN GEWISSEN VERBANDSGEORDNETEN ALGEBRAISCHEN STRUKTUREN

Von

O. STEINFELD (Budapest)

(*Vorgelegt von* L. RÉDEI)

Einleitung

Es ist bekannt, daß die Menge $\mathscr{H}_1$ aller Ideale eines assoziativen Ringes $\mathscr{R}$ eine verbandsgeordnete Halbgruppe bildet. Man kann die bekannten Begriffe und Ergebnisse bezüglich der teilweise geordneten algebraischen Strukturen im Buch [5] von L. FUCHS finden. Da das Produkt $\mathfrak{ab}$ der Ideale $\mathfrak{a}$ und $\mathfrak{b}$ von $\mathscr{R}$ in dem Durchschnitt $\mathfrak{a} \cap \mathfrak{b}$ enthalten ist, ist $\mathscr{H}_1$ negativ geordnet. Man kann in $\mathscr{H}_1$ auch die rechts- und linksseitigen Idealquotienten definieren, wodurch $\mathscr{H}_1$ zu einer negativ und verbandsgeordneten Halbgruppe mit Quotienten wird.

Bezeichne $\mathscr{H}_2$ den vollständigen Verband aller Normalteiler einer beliebigen Gruppe $\mathscr{G}$. Wird das „Produkt" von $\mathscr{A}, \mathscr{B} \in \mathscr{H}_2$ als der Kommutator $[\mathscr{A}, \mathscr{B}]$ von $\mathscr{A}$ und $\mathscr{B}$ definiert, so ist $\mathscr{H}_2$ ein verbandsgeordnetes (kommutatives) Gruppoid. Da $[\mathscr{A}, \mathscr{B}] \subseteqq \mathscr{A} \cap \mathscr{B}$ gilt, ist $\mathscr{H}_2$ negativ geordnet. Auch in $\mathscr{H}_2$ kann man die Quotienten $\mathscr{A} : \mathscr{B}$ $(\mathscr{A}, \mathscr{B} \in \mathscr{H}_2)$ definieren.

Einige Eigenschaften von $\mathscr{H}_1$ und $\mathscr{H}_2$ verallgemeinernd definieren wir im § 1 eine teilweise geordnete algebraische Struktur $\mathscr{L}$, in der eine Operation f mit $n(\geqq 2)$ Variabeln definiert ist. $\mathscr{L}$ ist außerdem ein vollständiger Verband, deshalb nennen wir $\mathscr{L}$ einen vollständigen f-Verband oder kürzer vf-Verband. Man kann in einem vf-Verband $\mathscr{L}$ den Begriff des i-ten Quotienten $a :_i b$ $(a, b \in \mathscr{L}; i = 1, 2, \ldots, n)$ einführen und einige Eigenschaften desselben beweisen. (Siehe Behauptungen 1. 1—1. 4.)

Das Primideal eines Ringes und der durch E. SCHENKMAN [17] eingeführte Primnormalteiler einer Gruppe sind analoge Begriffe. Wir definieren in einem f-negativ geordneten vf-Verband die f-Primelemente, welche als Verallgemeinerungen der Primideale und Primnormalteiler zu betrachten sind. Ist jedes Element des f-negativ geordneten vf-Verbandes f-prim, so ist $\mathscr{L}$ eine Kette. (Siehe Behauptung 2. 1.)

Bezeichne 0 das kleinste Element des f-negativ geordneten vf-Verbandes $\mathscr{L}$, so ist das abgeschlossene Intervall $[0, a] = \mathscr{A}$ $(a \in \mathscr{L})$ ein f-negativ geordneter vf-Teilverband von $\mathscr{L}$. Ist $p(\neq a)$ ein f-Primelement von $[0, a] = \mathscr{A}$, so ist $p :_1 a = p :_2 a = \ldots = p :_n a$ das einzige f-Primelement von $\mathscr{L}$, dessen Durchschnitt mit a das Element p ist. Mit Hilfe dieses Ergebnisses können wir zwischen den f-Primelementen $p(\neq a)$ des Intervalls $[0, a]$ und den f-Primelementen $q(\neq a)$ von $\mathscr{L}$ eine eineindeutige Zuordnung angeben. (Siehe Satz 2. 2 und Korollar 2. 3.) Im § 2 sind einige Ergebnisse auch über die f-Semiprimelemente bewiesen. Wir erwähnen nur Satz 2. 5, welcher eine Charakterisierung der f-Semiprimelemente gibt.

Behauptung 2. 1 ist eine Verallgemeinerung eines halbgruppentheoretischen Ergebnisses von G. SZÁSZ [23]. Man kann diese Behauptung auch auf die Prim-

normalteiler einer Gruppe anwenden. (Siehe Behauptung 5. 2.) Satz 2. 2 und Korollar 2. 3 verallgemeineren einige frühere Resultate des Verfassers [18], [20]. Die Analogie zwischen den Primidealen eines assoziativen Ringes und den Primnormalteilern einer Gruppe ist nicht vollständig. Jedes Primideal eines Ideals des assoziativen Ringes $\mathscr{R}$, ist ein Ideal von $\mathscr{R}$, ein Primnormalteiler eines Normalteilers der Gruppe $\mathscr{G}$ ist aber im allgemeinen kein Normalteiler von $\mathscr{G}$. Trotzdem läßt sich Satz 2. 2 auf beliebige Gruppen übertragen. (S. Satz 5.5.) Weiterhin ist der Durchschnitt eines Primideales und eines Ideales $\mathfrak{a}$ von $\mathscr{R}$ ein Primideal von $\mathfrak{a}$, wir können jedoch das analoge Ergebnis für beliebige Gruppen nicht beweisen. In Behauptung 5. 4 zeigen wir, daß die erwähnten ringtheoretischen Ergebnisse in den metaabelschen Gruppen Analoga haben. Dem entsprechend können wir Korollar 2. 3 auf diese Gruppen anwenden.

Im § 3 ist $\mathscr{T}_f[a, b] =$ das $\mathscr{T}_f$-Radikal des Intervalls $[a, b]$ eines f-negativ geordneten vf-Verbandes $\mathscr{L}$ als Durchschnitt gewisser f-Primelemente des Intervalls $[a, b]$ definiert. Bezeichnet e das größte Element von $\mathscr{L}$ und hat das Element $a(\in \mathscr{L})$ die Eigenschaft

$$\mathscr{T}_f[0, a] = \mathscr{T}_f[0, e] \wedge a,$$

so kann man $\mathscr{T}_f[0, e]$ mit Hilfe von $\mathscr{T}_f[0, a]$ und $\mathscr{T}_f[a, e]$ bestimmen (siehe Satz 3. 1). Das Radikal $\mathscr{P}_f[a, b]$ ist ein wichtiger Spezialfall des $\mathscr{T}_f$-Radikals. Im Satz 3. 2 sind die wichtigsten Eigenschaften von $\mathscr{P}_f[a, b]$ gegeben. Hier sei nur das Folgende erwähnt: für jedes Element $a(\in \mathscr{L})$ gilt

$$\mathscr{P}_f[0, a] = \mathscr{P}_f[0, e] \wedge a.$$

Korollar 3. 3 ist ein Spezialfall des Satzes 3. 1. Man kann zeigen, daß das Radikal $\mathscr{P}_f[a, b]$ das kleinste f-Semiprimelement des Intervalls $[0, b]$ über dem Element a ist. (Siehe Satz 3. 4.)

Spezialisiert man das $\mathscr{T}_f$-Radikal für $\mathscr{H}_1$, so entsteht eine Verallgemeinerung z.B. des Baerschen, Jacobsonschen, Krull—McCoyschen Radikals eines assoziativen Ringes. Das $\mathscr{P}_f$-Radikal ist eine unmittelbare Verallgemeinerung des Krull—McCoyschen Radikals. Die Ergebnisse über die $\mathscr{T}_f$- und $\mathscr{P}_f$-Radikale lassen sich für die assoziativen Ringe spezialisieren. Satz 3. 4 ist z.B. eine Verallgemeinerung jenes berühmten Ergebnisses von LEVITZKI [11] und NAGATA [15], daß das Baersche untere Radikal und das Krull—McCoysche Radikal eines assoziativen Ringes übereinstimmen.

Man kann das $\mathscr{T}_f$-Radikal in $\mathscr{H}_2$ definieren und den Satz 3. 1 für eine beliebige Gruppe spezialisieren. Das $\mathscr{P}_f$-Radikal ist eine Verallgemeinerung des Kurataschen Radikals einer Gruppe und ein Teil des Satzes 3. 2 verallgemeinert einige Ergebnisse von KURATA [7]. Satz 5. 8 charakterisiert die Semiprimnormalteiler einer Gruppe mit Hilfe des $\mathscr{P}_f$-Radikals. Die Korollare 2. 3 und 3. 3 und Behauptung 5 des Satzes 3. 2 sind für die metaabelschen Gruppen spezialisiert. (Siehe Korollare 5. 9 und 5. 10.)

§ 1. Quotienten

Es sei $\mathscr{L}$ eine algebraische Struktur mit den folgenden Eigenschaften:

(α) in $\mathscr{L}$ ist eine Operation f mit $n\,(\geqq 2)$ Variabeln definiert;

(β) $\mathscr{L}$ ist ein vollständiger Verband bezüglich einer teilweisen Ordnung $\leqq$;

(γ) f ist isoton in jeder Variable und $\mathscr{L}$ ist sein Monotoniebereich, d.h. aus $a_i \leqq b_i$ $(1 \leqq i \leqq n)$ folgt

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \leqq f(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n) \quad (a_1, \ldots, a_n, b_i \in \mathscr{L}).$$

Eine algebraische Struktur $\mathscr{L}$ mit den Eigenschaften (α), (β), (γ) nennen wir einen *vollständigen f-Verband*, oder kürzer *vf-Verband*. In dieser Arbeit bezeichnen wir mit $\mathscr{L}$ immer einen *vf*-Verband.

Jetzt setzen wir voraus, daß $\mathscr{L}$ auch die folgende Eigenschaft besitzt: Für jedes Paar $a, b \in \mathscr{L}$ und für jedes $i\,(i = 1, 2, \ldots, n)$ gibt es ein Element $a \overset{i}{:} b$ von $\mathscr{L}$ derart, daß

$$x \leqq a \overset{i}{:} b \quad \text{zu} \quad f(b, b, \ldots, b, \overset{i}{\smile}{x}, b, \ldots, b) \leqq a \tag{1.1}$$

äquivalent ist. Wir sagen, daß das Element $a \overset{i}{:} b$ *i-ter f-Quotient, und $\mathscr{L}$ ein vf-Verband mit f-Quotienten ist.*

Ein ähnlicher Begriff der *f*-Quotienten wurde in FUCHS [4] eingeführt.

BEMERKUNGEN. 1. In der Definition der *i*-ten *f*-Quotienten spielt es keine Rolle, daß $\mathscr{L}$ ein Verband ist.

2. Der Begriff der *i*-ten *f*-Quotienten ist eine Verallgemeinerung der in einem teilweise geordneten Gruppoid definierten Rechts- und Linksquotienten.

Wegen der Isotonie der Operation f gelten:

$$\text{aus} \quad a_1 \leqq a_2 \quad \text{folgt} \quad a_1 \overset{i}{:} b \leqq a_2 \overset{i}{:} b \quad (a_1, a_2, b \in \mathscr{L}), \tag{1.2}$$

$$\text{aus} \quad b_1 \leqq b_2 \quad \text{folgt} \quad a \overset{i}{:} b_2 \leqq a \overset{i}{:} b_1 \quad (a, b_1, b_2 \in \mathscr{L}). \tag{1.3}$$

Die Verbandsoperationen seien in $\mathscr{L}$ durch $\wedge$ und $\vee$ bezeichnet, und es sei e bzw. 0 das größte bzw. das kleinste Element von $\mathscr{L}$.

BEHAUPTUNG 1.1. *Ist $\mathscr{L}$ ein vf-Verband mit f-Quotienten, so gilt die Distributivitätsregel*

$$f\left(x, \ldots, x, \overset{i}{\underset{\lambda \in \Lambda}{\vee}} y_\lambda, x, \ldots, x\right) = \bigvee_{\lambda \in \Lambda} f(x, \ldots, x, \overset{i}{\smile}{y_\lambda}, x, \ldots, x) \quad (i = 1, 2, \ldots, n). \tag{1.4}$$

BEWEIS. Wegen der Isotonie der Operation f besteht für jedes $\lambda\,(\in \Lambda)$ die Ungleichung $f(x, \ldots, x, \overset{i}{\smile}{y_\lambda}, x, \ldots, x) \leqq f(x, \ldots, x, \overset{i}{\underset{\lambda \in \Lambda}{\vee}} y_\lambda, x, \ldots, x)$. Nun nehmen wir an, daß $f(x, \ldots, x, \overset{i}{\smile}{y_\lambda}, x, \ldots, x) \leqq z\,(\in \mathscr{L})$ für jedes $\lambda\,(\in \Lambda)$ gilt. Nach (1.1) besteht $y_\lambda \overset{i}{\leqq} z : x\,(\lambda \in \Lambda)$, woraus $\bigwedge_{\lambda \in \Lambda} y_\lambda \overset{i}{\leqq} z : x$ und $f(x, \ldots, x, \overset{i}{\underset{\lambda \in \Lambda}{\vee}} y_\lambda, x, \ldots, x) \leqq z$ folgen.

Dies zeigt, daß das Element $\bigvee_{\lambda\in\Lambda} f(x, \ldots, x, \overset{i}{\smile}{y_\lambda}, x, \ldots, x)$ die kleinste obere Schranke der Elemente $f(x, \ldots, x, \overset{i}{\smile}{y_\lambda}, x, \ldots, x)$ ist; qu.e.d.

BEHAUPTUNG 1. 2. *Ist $\mathscr{L}$ ein vf-Verband mit f-Quotienten, so gilt für die Elemente $\bigwedge_{\gamma\in\Gamma} a_\gamma$ $(a_\gamma\in\mathscr{L})$ und b von $\mathscr{L}$*

$$\left(\bigwedge_{\gamma\in\Gamma} a_\gamma\right)\overset{i}{:} b = \bigwedge_{\gamma\in\Gamma}(a_\gamma \overset{i}{:} b) \quad (i = 1, 2, \ldots, n). \tag{1.5}$$

BEWEIS. Wegen $\bigwedge_{\gamma\in\Gamma} a_\gamma \leqq a_\gamma$ und (1. 2) besteht $\left(\bigwedge_{\gamma\in\Gamma} a_\gamma\right)\overset{i}{:} b \leqq a_\gamma \overset{i}{:} b$ für jedes $\gamma\in\Gamma$, woraus $\left(\bigwedge_{\gamma\in\Gamma} a_\gamma\right)\overset{i}{:} b \leqq \bigwedge_{\gamma\in\Gamma}(a_\gamma \overset{i}{:} b)$ folgt.

Wegen (1. 1) und der Isotonie der Operation f bekommt man die Ungleichung

$$f\left(b, \ldots, b, \bigwedge_{\gamma\in\Gamma}(a_\gamma \overset{i}{:} b), b, \ldots, b\right) \leqq \bigwedge_{\gamma\in\Gamma} f\left(b, \ldots, b, \overset{i}{\smile}{(a_\gamma \overset{i}{:} b)}, b, \ldots, b\right) \leqq \bigwedge_{\gamma\in\Gamma} a_\gamma,$$

was $\bigwedge_{\gamma\in\Gamma}(a_\gamma \overset{i}{:} b) \leqq \left(\bigwedge_{\gamma\in\Gamma} a_\gamma\right)\overset{i}{:} b$ impliziert. Damit ist (1. 5) bewiesen.

BEHAUPTUNG 1. 3. *Ist $\mathscr{L}$ ein vf-Verband mit Quotienten, so sind die folgenden zwei Voraussetzungen äquivalent:*

$$a \overset{i}{:} \left(\bigvee_{\delta\in\Delta} b_\delta\right) = \bigwedge_{\delta\in\Delta}(a \overset{i}{:} b_\delta) \qquad (a, b_\delta\in\mathscr{L};\quad i = 1, 2, \ldots, n); \tag{A}$$

$$f\left(\bigvee_{\omega\in\Omega} c_\omega, \ldots, \bigvee_{\omega\in\Omega} c_\omega, \overset{i}{\smile}{d}, \bigvee_{\omega\in\Omega} c_\omega, \ldots, \bigvee_{\omega\in\Omega} c_\omega\right) = \tag{B}$$

$$= \bigvee_{\omega\in\Omega} f(c_\omega, \ldots, c_\omega, \overset{i}{\smile}{d}, c_\omega, \ldots, c_\omega) \qquad (c_\omega, d\in\mathscr{L};\quad i = 1, 2, \ldots, n).$$

BEWEIS. $(A)\Rightarrow(B)$. Wegen der Isotonie der Operation f gilt

$$f(c_\omega, \ldots, c_\omega, \overset{i}{\smile}{d}, c_\omega, \ldots, c_\omega) \leqq f\left(\bigvee_\omega c_\omega, \ldots, \bigvee_\omega c_\omega, \overset{i}{\smile}{d}, \bigvee_\omega c_\omega, \ldots, \bigvee_\omega c_\omega\right).$$

Es sei

$$f(c_\omega, \ldots, c_\omega, \overset{i}{\smile}{d}, c_\omega, \ldots, c_\omega) \leqq m \quad (\in\mathscr{L}) \tag{1.6}$$

für jedes $\omega\in\Omega$. Infolge (1. 6) und (A) gelten $d \leqq m \overset{i}{:} c_\omega$ und $d \leqq \bigwedge_{\omega\in\Omega}(m \overset{i}{:} c_\omega) = m \overset{i}{:} \left(\bigwedge_{\omega\in\Omega} c_\omega\right)$, woraus

$$f\left(\bigvee_\omega c_\omega, \ldots, \bigvee_\omega c_\omega, \overset{i}{\smile}{d}, \bigvee_\omega c_\omega, \ldots, \bigvee_\omega c_\omega\right) \leqq m$$

folgt. Dies bedeutet, daß $f(\bigvee_{\omega} c_\omega, \ldots, \bigvee_{\omega} c_\omega, \overset{i}{\check{d}}, \bigvee_{\omega} c_\omega, \ldots, \bigvee_{\omega} c_\omega)$ die kleinste obere Schranke der Elemente $f(c_\omega, \ldots, c_\omega, \overset{i}{\check{d}}, c_\omega, \ldots, c_\omega)$ $(\omega \in \Omega)$ ist, d.h. (B) gilt.

$(B) \Rightarrow (A)$. Wegen (1. 3) gilt $a:(\bigvee_{\delta \in \Delta} b_\delta) \leqq \overset{i}{a}:b_\delta$ für jedes $\delta \in \Delta$. Es sei $k \leqq \overset{i}{a}:b_\delta$ für jedes $\delta \in \Delta$. Dann besteht

$$f(b_\delta, \ldots, b_\delta, \overset{i}{\check{k}}, b_\delta, \ldots, b_\delta) \leqq a \qquad (\delta \in \Delta),$$

woraus wegen (B)

$$f\left(\bigvee_{\delta} b_\delta, \ldots, \bigvee_{\delta} b_\delta, \overset{i}{\check{k}}, \bigvee_{\delta} b_\delta, \ldots, \bigvee_{\delta} b_\delta\right) = \bigvee_{\delta \in \Delta} f(b_\delta, \ldots, b_\delta, \overset{i}{\check{k}}, b_\delta, \ldots, b_\delta) \leqq a$$

folgt. Dies impliziert nach (1. 1)

$$k \leqq a:\overset{i}{\left(\bigvee_{\delta \in \Delta} b_\delta\right)}.$$

So ist $a:(\bigvee_{\delta \in \Delta} b_\delta)$ die größte untere Schranke der Elemente $\overset{i}{a}:b_\delta (\delta \in \Delta)$, womit (A) bewiesen ist.

Wir sagen, daß der *vf*-Verband $\mathfrak{L}$ *f-negativ geordnet* ist, wenn für jede $a_1, a_2, \ldots, a_n \in \mathfrak{L}$

$$f(a_1, a_2, \ldots, a_n) \leqq \bigwedge_{i=1}^{n} a_i \qquad (a_i \in \mathfrak{L};\quad i = 1, 2, \ldots, n) \tag{1.7}$$

gilt.

Aus den Definitionen folgt

BEHAUPTUNG 1. 4. *Ist der vf-Verband mit Quotienten $\mathfrak{L}$ f-negativ geordnet, so gilt*

$$a \leqq \overset{i}{a}:b \qquad (a, b \in \mathfrak{L};\quad i = 1, 2, \ldots, n). \tag{1.8}$$

§ 2. Primelemente

Setzen wir voraus, daß der *vf*-Verband $\mathfrak{L}$ *f*-negativ geordnet ist. Ein Element p von $\mathfrak{L}$ heißt *f-prim*, wenn aus

$$f'(a_1, \ldots, a_k) \leqq p \qquad (a_1, \ldots, a_k \in \mathfrak{L})$$

jeweils

$$a_i \leqq p \quad \text{für mindestens ein } i$$

folgt, wo f' eine aus f (durch Iteration) gebildete Operation[1] von $\mathfrak{L}$ ist. (Vgl. den Begriff der in FUCHS [4] definierten (f, Φ)-Primelemente.)

Natürlich ist e ein f-Primelement von $\mathfrak{L}$.

[1] Im Falle $n=3$ ist

$$f'(x_1, \ldots, x_{13}) = f(f(x_1, x_2, x_3), f(x_4, f(x_5, x_6, x_7), x_8), f(f(x_9, x_{10}, x_{11}), x_{12}, x_{13})).$$

ein Beispiel für die Operation f'.

BEHAUPTUNG 2. 1. *Ist jedes Element des f-negativ geordneten vf-Verbandes f-prim, so bilden die Elemente von $\mathscr{L}$ eine Kette.**

BEWEIS. Sind p, q zwei Elemente von $\mathscr{L}$, so gilt nach (1. 7)

$$f(\overset{1}{\smile}p, \ldots, p, q, \ldots, \overset{n}{\smile}q) \leqq p \wedge q.$$

Nach unserer Voraussetzung ist $p \wedge q$ ein f-Primelement, deshalb ist entweder $p \leqq p \wedge q \leqq q$ oder $q \leqq p \wedge q \leqq p$ gültig.

Diese Behauptung ist eine Verallgemeinerung eines Ergebnisses von G. SZÁSZ [23].

Es sei $\mathscr{L}$ ein f-negativ geordneter vf-Verband. Dann ist das abgeschlossene Intervall $[0, a] = \mathscr{A}$ für jedes Element a von $\mathscr{L}$ ein f-negativ geordneter vf-Teilverband von $\mathscr{L}$, also sind auch die f-Primelemente von $\mathscr{A}$ definiert. Jetzt beweisen wir einen Satz der eine eineindeutige Zuordnung zwischen den f-Primelementen $p(\neq a)$ des Intervalls $[0, a]$ $(a \in \mathscr{L})$ und den f-Primelementen $q(\not\geqq a)$ von $\mathscr{L}$ ermöglicht. Das folgende Ergebnis ist eine Verallgemeinerung der Sätze des Verfassers [18], [20].

SATZ 2. 2. *Es sei $\mathscr{L}$ ein f-negativ geordneter vf-Verband mit f-Quotienten und $[0, a] = \mathscr{A}$ $(a \in \mathscr{L})$ sei ein Intervall von $\mathscr{L}$. Ist p ein f-Primelement von $\mathscr{A}$, so ist $p \overset{1}{:} a$ ein f-Primelement von $\mathscr{L}$ und es gilt*

$$(p \overset{1}{:} a) \wedge a = p, \tag{2. 1}$$

$$p \overset{1}{:} a = p \overset{i}{:} a \qquad (i = 2, 3, \ldots, n). \tag{2. 2}$$

Im Falle $p \neq a$ ist $p \overset{1}{:} a$ das einzige f-Primelement von $\mathscr{L}$, dessen Durchschnitt mit a das Element p ist.

Besitzt das Element r von $\mathscr{L}$ die Eigenschaft $r \wedge a = p$, so gilt $r \leqq p \overset{1}{:} a$. Das f-Primelement p von $\mathscr{A}$ ist dann und nur dann ein f-Primelement von $\mathscr{L}$, wenn $p = p \overset{1}{:} a$ gilt.

KOROLLAR 2. 3. Es gibt eine eineindeutige Zuordnung zwischen den f-Primelementen $p(\neq a)$ des Intervalls $[0, a]$ von $\mathscr{L}$ und derjenigen q von $\mathscr{L}$ mit $q(\not\geqq a)$, und zwar

$$p \to p \overset{1}{:} a \quad \text{und} \quad q \to q \wedge a. \tag{2. 3}$$

BEWEIS. Im Falle $p = a$ ist $a \overset{1}{:} a = e$ und die Bedingungen (2. 1) und (2. 2) gelten trivialerweise. Es sei dann $p(\neq a)$ ein f-Primelement des Intervalls $[0, a] = \mathscr{A}$ $(a \in \mathscr{L})$. Gilt für die aus f gebildete Operation f' die Bedingung $f'(m_1, \ldots, m_k) \leqq p \overset{1}{:} a$ $(m_1, \ldots, m_k \in \mathscr{L})$, so besteht nach (1. 1)

$$f(f'(m_1, \ldots, m_k), a, \ldots, a) \leqq p.$$

* *Bemerkung bei der Korrektur (14. 8. 1968).* Inzwischen haben wir in der Arbeit [28] eine notwendige und hinreichende Bedingung dafür gegeben, daß alle Elemente von $\mathscr{L}$ f-prim seien.

Wegen $m_i \wedge a \leqq m_i$ und wegen der Isotonie der Operation f bekommt man

$$f(f'(m_1 \wedge a, \ldots, m_k \wedge a), a, \ldots, a) \leqq p. \tag{2.4}$$

Da $m_i \wedge a$ $(i = 1, 2, \ldots, n)$ Elemente von $\mathscr{A}$ sind, ist auch $f'(m_1 \wedge a, \ldots, m_k \wedge a)$ ein Element von $\mathscr{A}$. Nach unserer Voraussetzung ist p ein f-Primelement von $\mathscr{A}$, deshalb impliziert (2.4) wegen $a \not\leqq p$

$$f'(m_1 \wedge a, \ldots, m_k \wedge a) \leqq p,$$

woraus $m_i \wedge a \leqq p$ für mindestens ein i folgt. Da $\mathscr{A}$ f-negativ ist, gilt

$$f(m_i, a, \ldots, a) \leqq m_i \wedge a \leqq p,$$

also $m_i \leqq \overset{1}{p}{:}a$. Damit haben wir bewiesen, daß $\overset{1}{p}{:}a$ ein f-Primelement von $\mathscr{L}$ ist.

Nach Behauptung 1.4 besteht $p \leqq (\overset{1}{p}{:}a) \wedge a$, so haben wir statt (2.1) nur $(\overset{1}{p}{:}a) \wedge a \leqq p$ zu beweisen. Wegen $f((\overset{1}{p}{:}a) \wedge a, a, \ldots, a) \leqq f(\overset{1}{p}{:}a, a, \ldots, a) \leqq p$ und $a \not\leqq p$ gilt $(\overset{1}{p}{:}a) \wedge a \leqq p$.

Ist r ein Element von $\mathscr{L}$ mit der Eigenschaft $r \wedge a = p$, so schließen wir aus $f(r, a, \ldots, a) \leqq r \wedge a = p$ auf $r \leqq \overset{1}{p}{:}a$. Ist q ein *f-Primelement* von $\mathscr{L}$ mit $q \wedge a = p$, dann bekommen wir wegen $f(\overset{1}{p}{:}a, a, \ldots, a) \leqq p \leqq q$ und $a \not\leqq q$ die Beziehung $\overset{1}{p}{:}a \leqq q$, folglich besteht $q = \overset{1}{p}{:}a$.

Mit ähnlicher Methode kann man zeigen, daß $\overset{i}{p}{:}a$ für jedes i $(i = 2, 3, \ldots, n)$ ein f-Primelement von $\mathscr{L}$ mit der Eigenschaft $(\overset{i}{p}{:}a) \wedge a = p$ ist. So muß $\overset{i}{p}{:}a = \overset{1}{p}{:}a$ $(i = 2, \ldots, n)$ nach den vorigen bestehen.

Ist $p = \overset{1}{p}{:}a$, so ist p in der Tat ein f-Primelement von $\mathscr{L}$. Im Falle $p \neq \overset{1}{p}{:}a$ bestehen

$$f(\overset{1}{p}{:}a, a, \ldots, a) \leqq p, \quad \overset{1}{p}{:}a \not\leqq p \quad \text{und} \quad a \not\leqq p,$$

was zeigt, daß p kein f-Primelement von $\mathscr{L}$ ist.

Damit ist der Beweis vom Satz 2.2 beendet.

Korollar 2.3 folgt aus Satz 2.2 und aus der folgenden Behauptung, welche leicht einzusehen ist.

BEHAUPTUNG 2.4. Es sei $\mathscr{L}$ ein f-negativ geordneter *vf*-Verband. Ist q ein f-Primelement des Intervalls $[0, b] = \mathscr{B}$ $(b \in \mathscr{L})$, so ist $q \wedge a$ für jedes $a (\in \mathscr{B})$ ein f-Primelement des Intervalls $[0, a]$.

Das Element s des f-negativ geordneten *vf*-Verbandes heißt *f-semiprim*, wenn aus $f'(a, \ldots, a) \leqq s (a \in \mathscr{L})$ immer $a \leqq s$ folgt, wo f' eine aus f gebildete beliebige Operation von $\mathscr{L}$ bezeichnet.

Aus den Definitionen sieht man leicht, daß der Durchschnitt von f-Primelementen immer f-semiprim ist.

Wir sagen, daß der *vf*-Verband *f-verbandsgeordnet* ist, wenn in $\mathscr{L}$ die unendliche Distributivitätsregel

$$(2.5)\quad f\Big(x_1, x_2, \ldots, \bigvee_\lambda x_i^{(\lambda)}, \ldots, x_n\Big) = \bigvee_\lambda f(x_1, x_2, \ldots, x_i^{(\lambda)}, \ldots, x_n) \quad (i = 1, 2, \ldots, n)$$

gültig ist.

Man sieht, daß die unendliche Distributivitätsregel in einem f-verbandsgeordneten *vf*-Verband auch für jede aus f gebildete Operation f' richtig bleibt.

SATZ 2. 5. *Es sei $\mathscr{L}$ ein f-negativ und f-verbandsgeordneter vf-Verband. Das Element s von $\mathscr{L}$ ist dann und nur dann f-semiprim, wenn $s = \bigwedge_\alpha p_\alpha$ ist, wo die p_α alle f-Primelemente von $\mathscr{L}$ mit $s \leqq p_\alpha$ durchlaufen.*

BEWEIS. Gilt $s = \bigwedge p_\alpha$, so ist s offenbar f-semiprim.

Es sei nun s ein f-Semiprimelement. Für den Durchschnitt $u = \bigwedge_\alpha p_\alpha$ der f-Primelemente p_α mit $s \leqq p_\alpha$ besteht $s \leqq u$. Bezeichne $\mathscr{U}$ die Menge aller Elemente

$$u, f(u, \ldots, u), f\big(f(u, \ldots, u), u, \ldots, u\big), \ldots, f'(u, \ldots, u), \ldots$$

wo f' alle mögliche aus der Operation f gebildete Operationen von $\mathscr{L}$ durchläuft.

Jetzt setzen wir voraus, daß die Ungleichung $s < u$ im Widerspruch zu unserer Behauptung gültig ist. Dies impliziert wegen der Definition von s, daß die Menge $\mathscr{U}$ und das Intervall $[0, s]$ kein gemeinsames Element haben, das heißt

$$\mathscr{U} \cap [0, s] = \emptyset.$$

Nach dem Zornschen Lemma existiert ein Element m in $\mathscr{L}$, welches bezüglich der Eigenschaften $s \leqq m$ und $\mathscr{U} \cap [0, m] = \emptyset$ maximal ist. Wir zeigen, daß dieses Element m f-prim sein muß.

Wenn m kein f-Primelement wäre, so gäbe es eine aus f gebildete Operation f^* mit den Eigenschaften

$$(2.6)\qquad f^*(a_1, \ldots, a_k) \leqq m \quad \text{und} \quad a_i \not\leqq m \qquad (i = 1, 2, \ldots, k).$$

Dies hätte dann wegen $s \leqq m < m \vee a_i$ und der Definition von m

$$(2.7)\qquad \mathscr{U} \cap [0, m \vee a_i] = f_i'(u, \ldots, u)$$

für jedes $i = 1, 2, \ldots, k$ zur Folge. Aus (2. 6) und (2. 7) bekämen wir die folgende Ungleichungen

$$f^*\big(f_1'(u, \ldots, u), f_2'(u, \ldots, u), \ldots, f_k'(u, \ldots, u)\big) \leqq f^*(m \vee a_1, m \vee a_2, \ldots, m \vee a_k) =$$

$$= f^*(m, \ldots, m) \vee f^*(a_1, m, \ldots, m) \vee f^*(m, a_2, m, \ldots, m) \vee \ldots$$

$$\ldots \vee f^*(a_1, a_2, m, \ldots, m) \vee \ldots \vee f^*(a_1, a_2, \ldots, a_k) \leqq m.$$

Nach der Definition von $\mathscr{U}$ gilt $f^*\big(f_1'(u, \ldots, u), f_2'(u, \ldots, u), \ldots, f_k'(u, \ldots, u)\big) \in \mathscr{U}$, so widerspricht unsere letzte Ungleichung der Voraussetzung $\mathscr{U} \cap [0, m] = \emptyset$. Damit haben wir bekommen, daß m ein f-Primelement mit der Eigenschaft $s \leqq m$ und $u \not\leqq m$ ist.

Das ist aber wegen der Definition des Elements u unmöglich, folglich muß in der Tat $s=u$ gelten.

Damit ist Satz 2. 5 bewiesen.

Satz 2. 5 ist eine Verallgemeinerung des Korollars 4. 16 in McCoy [13].

BEHAUPTUNG 2. 6. *Ist ein vf-Verband $\mathscr{L}$ f-negativ und f-verbandsgeordnet, so ist $\mathscr{L}$ ein vf-Verband mit f-Quotienten.*

BEWEIS. Für die Elemente $a, b(\in\mathscr{L})$ bezeichne $a\overset{i}{|}b$ die Vereinigung $\bigvee\limits_{\delta\in\Delta} z_\delta$ aller Elemente z_δ von $\mathscr{L}$ $(\delta\in\Delta)$, die der Bedingung

$$f(b, \ldots, b, \overset{i}{\breve{z}}_\delta, b, \ldots, b) \leqq a$$

befriedigen. Da $\mathscr{L}$ f-negativ geordnet ist, gilt $f(b, \ldots, b, \overset{i}{\breve{a}}, b, \ldots, b)\leqq a$, folglich existiert das Element $a\overset{i}{|}b\in\mathscr{L}$. Wegen der Voraussetzung (2. 5) besteht

$$f\Big(b, \ldots, b, \bigvee_{\delta\in\Delta}\overset{i}{\breve{z}}_\delta, b, \ldots, b\Big) = \bigvee_{\delta\in\Delta} f(b, \ldots, b, \overset{i}{\breve{z}}_\delta, b, \ldots, b) \leqq a,$$

folglich stimmt das Element $a\overset{i}{|}b=\bigvee\limits_{\delta\in\Delta} z_\delta$ mit dem i-ten f-Quotienten $a\overset{i}{:}b$ überein.

Der folgende Satz ist mit Satz 2. 2 analog.

SATZ 2. 7. *Es sei $\mathscr{L}$ ein f-negativ und f-verbandsgeordneter vf-Verband und sei $[0, a]=\mathscr{A}$ $(a\in\mathscr{L})$ ein Intervall in $\mathscr{L}$. Ist s ein f-Semiprimelement von $\mathscr{L}$, so ist $s\overset{1}{:}a$ ein f-Semiprimelement von $\mathscr{L}$ mit den Eigenschaften*

$$(s\overset{1}{:}a)\wedge a = s, \tag{2. 8}$$

$$s\overset{1}{:}a = s\overset{i}{:}a \qquad (i=2, 3, \ldots, n). \tag{2. 9}$$

BEWEIS. Nach Satz 2. 5 besteht $s=\bigwedge\limits_\alpha p_\alpha$, wo die p_α alle f-Primelemente von $\mathscr{A}$ mit $s\leqq p_\alpha$ durchlaufen. Infolge Behauptung 2. 6 existiert der f-Quotient $s\overset{1}{:}a=$ $=\Big(\bigwedge\limits_\alpha p_\alpha\Big)\overset{1}{:}a$, und nach Behauptung 1. 2 gilt $\Big(\bigwedge\limits_\alpha p_\alpha\Big)\overset{1}{:}a=\bigwedge\limits_\alpha (p_\alpha\overset{1}{:}a)$. Da die Elemente $p_\alpha\overset{1}{:}a$ nach Satz 2. 2 f-Primelemente von $\mathscr{L}$ sind, ist $s\overset{1}{:}a=\bigwedge\limits_\alpha (p_\alpha\overset{1}{:}a)$ ein f-Semiprimelement von $\mathscr{L}$.

Die Eigenschaften (2. 8) und (2. 9) folgen aus $s\overset{1}{:}a=\bigwedge (p_\alpha\overset{1}{:}a)$ und aus den Bedingungen (2. 1) und (2. 2) von Satz 2. 2.

Dieses Ergebnis ist eine Verallgemeinerung des Satzes 2′ in [19].

§ 3. Primradikale

Der Begriff des f-Primelementes ermöglicht die Einführung einer Radikalklasse in f-negativ geordneten vf-Verbänden, welche als eine Verallgemeinerung gewißer wichtiger Radikale der assoziativen Ringe zu betrachten ist.

Es sei $\mathscr{L}$ ein f-negativ geordneter vf-Verband und seien a, b $(a<b)$ zwei Elemente von $\mathscr{L}$. Den (nichtleeren) Durchschnitt aller f-Primelemente des Intervalls $[0, b]$, die im Intervall $[a, b]$ liegen und eine gegebene Eigenschaft $(\mathscr{T})$ haben, nennen wir das *$\mathscr{T}_f$-Radikal des Intervalls* $[a, b]$ und bezeichnen es mit $\mathscr{T}_f[a, b]$. Hat aber das Intervall $[0, b]$ kein f-Primelement mit der Eigenschaft $(\mathscr{T})$ im Intervall $[a, b]$, so werde $\mathscr{T}_f[a, b]=b$ gesetzt.

Nach der Definition ist das Radikal $\mathscr{T}_f[a, b]$ ein f-Semiprimelement des Intervalls $[0, b]$ mit der Eigenschaft

$$a \leqq \mathscr{T}_f[a, b] \leqq b. \tag{3. 1}$$

SATZ 3. 1. *Es sei $\mathscr{L}$ ein f-negativ geordneter vf-Verband mit f-Quotienten. Ist für das $\mathscr{T}_f$-Radikal $\mathscr{T}_f[0, a]$ des Intervalls $[0, a]$ $(a \in \mathscr{L})$ die Bedingung*

$$\mathscr{T}_f[0, a] = \mathscr{T}_f[0, e] \wedge a, \tag{3. 2}$$

erfüllt, so gilt

$$\mathscr{T}_f[0, e] = (\mathscr{T}_f[0, a] \overset{1}{:} a) \wedge \mathscr{T}_f[a, e]. \tag{3. 3}$$

BEWEIS. Zuerst setzen wir voraus, daß $\mathscr{T}_f[0, e] \geqq a$ gilt. Dann ist $\mathscr{T}_f[0, e] = \mathscr{T}_f[a, e]$. Aus (3. 2) folgt $\mathscr{T}_f[0, a]=a$ in diesem Falle, und (3. 3) ist wegen $a \overset{1}{:} a = e$ in der Tat richtig.

Es sei nun $\mathscr{T}_f[0, e] \not\geqq a$. Ist $\mathscr{T}_f[0, e] = \bigwedge_{\lambda \in \Lambda} p_\lambda$, wo die p_λ alle f-Primelemente des Intervalls $[0, e]$ mit der Eigenschaft $(\mathscr{T})$ durchlaufen, so existiert mindestens ein f-Primelement $p_{\lambda'} (\lambda' \in \Lambda)$ mit $a \not\geqq p_{\lambda'}$. Bezeichne s den Durchschnitt $\bigwedge_{\lambda' \in \Lambda} p_{\lambda'}$ der f-Primelemente $p_{\lambda'}$ mit $a \not\geqq p_{\lambda'}$, das heißt $s = \bigwedge_{a \not\geqq p_{\lambda'}} p_{\lambda'} (\lambda' \in \Lambda)$. Da

$$\mathscr{T}_f[a, e] = \bigwedge_{a \geqq p_{\lambda''}} p_{\lambda''} \qquad (\lambda'' \in \Lambda)$$

gilt, können wir

$$\mathscr{T}_f[0, e] = \bigwedge_{\lambda \in \Lambda} p_\lambda = s \wedge \Big(\bigwedge_{a \geqq p_{\lambda''}} p_{\lambda''} \Big) = s \wedge \mathscr{T}_f[a, e] \tag{3. 3'}$$

schreiben. So haben wir statt (3. 3) die Beziehung

$$s = \mathscr{T}_f[0, a] \overset{1}{:} a$$

zu beweisen. Aus (3. 2) und (3. 3') schließen wir wegen $a \geqq \mathscr{T}_f[a, e]$ auf

$$\mathscr{T}_f[0, a] = \mathscr{T}_f[0, e] \wedge a = (s \wedge \mathscr{T}_f[a, e]) \wedge a = s \wedge a. \tag{3. 4}$$

Für ein beliebiges f-Primelement $p_{\lambda'} (\lambda' \in \Lambda;\ a \not\geqq p_{\lambda'})$ von $\mathscr{L}$ besteht $p_{\lambda'} = p_{\lambda'} \overset{1}{:} a$. Nach Behauptung 1. 4 gilt nämlich $p_{\lambda'} \leqq p_{\lambda'} \overset{1}{:} a$. Aus $f(p_{\lambda'} \overset{1}{:} a, a, \ldots, a) \leqq p_{\lambda'}$ und $a \not\geqq p_{\lambda'}$

folgt $p_{\lambda'}^{1}: a \leqq p_{\lambda'}$, womit $p_{\lambda'}^{1}: a = p_{\lambda'}$ bewiesen ist. Daraus und aus (3. 4), Behauptung 1. 2 und $s = \bigwedge_{a \not\leqq p_{\lambda'}} p_{\lambda'}$ bekommen wir

$$\mathscr{T}_f[0, a]^{1}: a = (s \wedge a)^{1}: a = (s^{1}: a) \wedge (a^{1}: a) = \left(\bigwedge_{a \not\leqq p_{\lambda'}} p_{\lambda'}\right)^{1}: a =$$

$$= \bigwedge_{a \not\leqq p_{\lambda'}} (p_{\lambda'}^{1}: a) = \bigwedge_{a \not\leqq p_{\lambda'}} p_{\lambda'} = s.$$

Damit haben wir unsere Behauptung $s = \mathscr{T}_f[0, a]^{1}: a$ und auch Satz 3. 1 bewiesen.

Satz 3. 1 ist eine Verallgemeinerung des Korollars 4 der Arbeit [19].

Jetzt beschäftigen wir uns mit einem wichtigen speziellen Fall des $\mathscr{T}_f$-Radikals. Bezeichne $\mathscr{P}_f[a, b]$ $(a, b \in \mathscr{L};\ a < b)$ den Durchschnitt aller f-Primelemente p_α des Intervalls $[0, b]$ mit $p_\alpha \in [a, b]$. (Man sieht, daß $\mathscr{P}_f[a, b]$ jenes spezielle $\mathscr{T}_f$-Radikal ist, wo die Eigenschaft $(\mathscr{T})$ die leere Menge ist.) Da das Element b ein f-Primelement von $[0, b]$ mit $b \in [a, b]$ ist, existiert $\mathscr{P}_f[a, b]$ immer. Der Begriff des $\mathscr{P}_f[a, b]$-Radikals ist eine Verallgemeinerung des durch MURATA [14] in gewissen teilweise geordneten Gruppoiden eingeführten Radikals.

SATZ 3. 2. *Es sei $\mathscr{L}$ ein f-negativ geordneter vf-Verband mit f-Quotienten. Das Radikal $\mathscr{P}_f[a, b]$ des Intervalls $[a, b]$ $(a, b \in \mathscr{L};\ a < b)$ hat die folgenden Eigenschaften:*

1. *Ist $a_1 \leqq a_2$ $(a_1, a_2 \in \mathscr{L})$, so ist $\mathscr{P}_f[a_1, b] \leqq \mathscr{P}_f[a_2, b]$ $(b \in \mathscr{L})$;*
2. *Ist $b_1 \leqq b_2$ $(b_1, b_2 \in \mathscr{L})$, so ist $\mathscr{P}_f[a, b_1] \leqq \mathscr{P}_f[a, b_2]$ $(a \in \mathscr{L})$;*
3. *$\mathscr{P}_f[\mathscr{P}_f[a, b], b] = \mathscr{P}_f[a, b]$ $(a, b \in \mathscr{L})$;*
4. *$\mathscr{P}_f[f(a_1, \ldots, a_n), b] = \mathscr{P}_f\left[\bigwedge_{i=1}^{n} a_i, b\right] = \bigwedge_{i=1}^{n} \mathscr{P}_f[a_i, b]$ $(a_1, \ldots, a_n \in \mathscr{L})$;*
5. *Ist $b_1 \leqq b_2$ $(b_1, b_2 \in \mathscr{L})$, so gilt $\mathscr{P}_f[a, b_1] = \mathscr{P}_f[a, b_2] \wedge b_1$ $(a \in \mathscr{L})$.*

BEWEIS. Behauptung 1 folgt unmittelbar aus der Definition von $\mathscr{P}_f[a, b]$. 2 ist eine Folgerung der Behauptung 5, deshalb zeigen wir zuerst 5. Ist p ein f-Primelement des Intervalls $[0, b_2]$ mit $p \in [a, b_2]$, so ist $p \wedge b_1 = p_1$ nach Behauptung 2. 4 ein f-Primelement des Intervalls $[0, b_1]$ mit $p_1 \in [a, b_1]$. Dies impliziert infolge (3. 1) die Ungleichung $\mathscr{P}_f[a, b_1] \leqq \mathscr{P}_f[a, b_2] \wedge b_1$.

Um die entgegengesetzte Ungleichung zu zeigen, betrachten wir das Intervall $[0, b_2] = \mathscr{B}_2$ in $\mathscr{L}$ und sei $\mathscr{P}_f[a, b_1] = \bigwedge_{\lambda \in \Lambda} p_\lambda$, wo die p_λ $(\lambda \in \Lambda)$ alle f-Primelemente des Intervalls $[0, b_1]$ $(b_1 \in \mathscr{L})$ mit $p_\lambda \in [a, b_1]$ durchlaufen. Man kann Satz 2. 2 auf den vf-Verband $\mathscr{B}_2 = [0, b_2]$ $(b_2 \in \mathscr{L})$ und auf sein Intervall $[0, b_1] = \mathscr{B}_1$ $(b_1 \in \mathscr{B}_2)$ anwenden. So bekommen wir, daß der f-Quotient $p_\lambda^{1}: b_1$ $(\lambda \in \Lambda)$ ein f-Primelement von $\mathscr{B}_2$ mit der Eigenschaft $(p_\lambda^{1}: b_1) \wedge b_1 = p_\lambda$ ist. Dem entsprechend gelten $\mathscr{P}_f[a, b_2] \leqq$ $\leqq \bigwedge_{\lambda \in \Lambda} (p_\lambda^{1}: b_1)$ und

$$\mathscr{P}_f[a, b_2] \wedge b_1 \leqq \left(\bigwedge_{\lambda \in \Lambda} (p_\lambda^{1}: b_1)\right) \wedge b_1 = \bigwedge_{\lambda \in \Lambda} (p_\lambda^{1}: b_1) \wedge b_1 = \bigwedge_{\lambda \in \Lambda} p_\lambda = \mathscr{P}_f[a, b_1],$$

womit 5 bewiesen ist.

Wegen 1 und $a \leqq \mathscr{P}_f[a, b]$ gilt $\mathscr{P}_f[a, b] \leqq \mathscr{P}_f[\mathscr{P}_f[a, b], b]$. Andererseits betrachten wir ein f-Primelement p des Intervalls $[0, b]$ mit $p \in [a, b]$. Wegen der Definition des Radikals $\mathscr{P}_f[a, b]$ gilt $\mathscr{P}_f[a, b] \leqq p$, deshalb muß $\mathscr{P}_f[\mathscr{P}_f[a, b], b] \leqq \mathscr{P}_f[a, b]$ bestehen. Damit ist auch Eigenschaft 3 bewiesen.

Um Eigenschaft 4 zu zeigen, seien $a_1, \ldots, a_n$ beliebige Elemente von $\mathscr{L}$. Da $\mathscr{L}$ f-negativ geordnet ist, gilt $f(a_1, \ldots, a_n) \leqq \bigwedge_{i=1}^{n} a_i$, was infolge von 1 für jedes Element b von $\mathscr{L}$ mit $a_i < b$ $(i = 1, 2, \ldots, n)$ die Ungleichung

$$\mathscr{P}_f[f(a_1, \ldots, a_n), b] \leqq \mathscr{P}_f\left[\bigwedge_{i=1}^{n} a_i, b\right] \leqq \bigwedge_{i=1}^{n} \mathscr{P}_f[a_i, b]$$

impliziert.

Wir werden jetzt die entgegengesetzte Ungleichung beweisen. Durchlaufe $p_\lambda (\lambda \in \Lambda)$ die f-Primelemente des Intervalls $[0, b]$ mit $f(a_1, \ldots, a_n) \leqq p_\lambda$, dann existiert für jedes $p_\lambda (\lambda \in \Lambda)$ mindestens ein $a_{j(\lambda)} (1 \leqq j(\lambda) \leqq n)$ mit der Eigenschaft $a_{j(\lambda)} \leqq p_\lambda$. Dies impliziert für jedes $\lambda \in \Lambda$

$$\bigwedge_{i=1}^{n} \mathscr{P}_f[a_i, b] \leqq \mathscr{P}_f[a_{j(\lambda)}, b] \leqq p_\lambda \qquad (1 \leqq j(\lambda) \leqq n; \quad \lambda \in \Lambda),$$

woraus

$$\bigwedge_{i=1}^{n} \mathscr{P}_f[a_i, b] \leqq \bigwedge_{\lambda \in \Lambda} p_\lambda = \mathscr{P}_f[f(a_1, \ldots, a_n), b]$$

folgt.

Damit ist der Beweis beendet.

Infolge Behauptung 5 des Satzes 3. 2 besteht

$$\mathscr{P}_f[0, a] = \mathscr{P}_f[0, e] \wedge a \tag{3. 2'}$$

für jedes Element a von $\mathscr{L}$ mit $a < e$, deshalb hat Satz 3. 1 die unmittelbare Folgerung

KOROLLAR 3. 3. Ist $\mathscr{L}$ ein f-negativ geordneter vf-Verband mit f-Quotienten, so besteht für das Radikal $\mathscr{P}_f[0, a]$ des Intervalls $[0, a]$ $(a \in \mathscr{L}; a < e)$ die Beziehung

$$\mathscr{P}_f[0, e] = (\mathscr{P}_f[0, a]\overset{1}{:} a) \wedge \mathscr{P}_f[a, e].$$

Wenn der vf-Verband $\mathscr{L}$ f-negativ und f-verbandsgeordnet ist, kann man das $\mathscr{P}_f$-Radikal folgenderweise charakterisieren.

SATZ 3. 4. *Ist $\mathscr{L}$ ein f-negativ und f-verbandsgeordneter vf-Verband, so stimmt das Radikal $\mathscr{P}_f[a, b]$ des Intervalls $[a, b]$ $(a, b \in \mathscr{L}; a < b)$ mit dem kleinsten f-Semiprimelement $s_0 (\in [a, b])$ des Intervalls $[0, b]$ überein.*

BEWEIS. Nach der Definition des $\mathscr{P}_f$-Radikals ist $\mathscr{P}_f[a, b]$ ein f-Semiprimelement des Intervalls $[0, b]$ mit der Eigenschaft $\mathscr{P}_f[a, b] \in [a, b]$, so muß infolge der Definition des Elements s_0

$$s_0 \leqq \mathscr{P}_f[a, b]$$

bestehen. Andererseits können wir nach Satz 2. 5 das Element s_0 in der Form $s_0 = \bigwedge_\alpha p_\alpha$

darstellen, wo die p_α alle f-Primelemente des Intervalls $[0, b]$ mit $s_0 \leqq p_\alpha \leqq b$ durchlaufen. Wegen $a \leqq s_0$ und der Definition von $\mathscr{P}_f[a, b]$ gilt

$$\mathscr{P}_f[a, b] \leqq \bigwedge_\alpha p_\alpha = s_0,$$

womit Satz 3. 4 bewiesen ist.

Aus Satz 3. 4 bekommt man das

KOROLLAR 3. 5. Es sei $\mathscr{L}$ ein f-negativ und f-verbandsgeordneter vf-Verband. Das Element s des Intervalls $[a, b]$ $(b \in \mathscr{L})$ ist dann und nur dann f-semiprim, wenn $s = \mathscr{P}_f[s, b]$ gilt.

§ 4. Über gewisse Radikale der Ringe

Jetzt wollen wir die Ergebnisse der vorigen Paragraphen auf die assoziativen Ringe anwenden. Es sei $\mathscr{L}_1$ die Menge aller Teilringe eines assoziativen Ringes $\mathscr{R}$. Definiert man in $\mathscr{L}_1$ die folgenden Operationen:

a) den mengentheoretischen Durchschnitt $\mathscr{A} \cap \mathscr{B}$ von $\mathscr{A}, \mathscr{B} (\in \mathscr{L}_1)$,

b) die Vereinigung $\mathscr{A} \cup \mathscr{B}$ von $\mathscr{A}, \mathscr{B} (\in \mathscr{L}_1)$, wo $\mathscr{A} \cup \mathscr{B}$ den durch $\mathscr{A}$ und $\mathscr{B}$ erzeugten Teilring von $\mathscr{R}$ bezeichnet;

c) das Produkt $\mathscr{A}\mathscr{B}$ von $\mathscr{A}, \mathscr{B} (\in \mathscr{L}_1)$, was den aus allen Elementen $ab (a \in \mathscr{A}; b \in \mathscr{B})$ erzeugten Teilring von $\mathscr{R}$ bezeichnet;

so bildet $\mathscr{L}_1$ ein teilweise geordnetes Gruppoid, welches ein vollständiger Verband ist. Es ist bekannt, daß die Menge $\overline{\mathscr{L}}_1$ aller Ideale eines assoziativen Ringes $\mathscr{R}$ bezüglich der in a), b), c) definierten Operationen eine negativ und verbandsgeordnete Halbgruppe mit Quotienten ist. In diesem Falle gehen die Begriffe des f-Primelementes, f-Semiprimelementes und des f-Quotienten in die Begriffe des Primideals, Semiprimideals und der links- und rechtsseitigen Idealquotienten über und die Ergebnisse der §§ 1, 2 sind Verallgemeinerungen der entsprechenden ringtheoretischen Sätze.

Man kann den entsprechenden Begriff des $\mathscr{T}_f$-Radikals in einem assoziativen Ring $\mathscr{R}$ definieren. Sind $\mathfrak{a}, \mathfrak{b}$ $(\mathfrak{a} \subset \mathfrak{b})$ zwei Ideale des Ringes $\mathscr{R}$, so nennen wir den nichtleeren Durchschnitt aller Primideale des Teilringes $\mathfrak{b}$, die das Ideal $\mathfrak{a}$ enthalten und eine gegebene Eigenschaft $(\mathscr{T})$ haben, das *$\mathscr{T}$-Radikal von $\mathfrak{b}$ über $\mathfrak{a}$*, und wir bezeichnen es mit $\mathscr{T}[\mathfrak{a}, \mathfrak{b}]$. Hat der Teilring $\mathfrak{b}$ kein das Ideal $\mathfrak{a}$ enthaltendes Primideal mit der Eigenschaft $(\mathscr{T})$, so sei $\mathscr{T}[\mathfrak{a}, \mathfrak{b}] = \mathfrak{b}$.

Dieser Begriff wurde in unserer Arbeit [19] eingeführt und dort sind auch einige bekannte $\mathscr{T}$-Radikale des Ringes $\mathscr{R}$ angegeben. Jetzt werden wir diese Beispiele ein wenig ergänzen.

Das *Krull—McCoysche Radikal* eines Ringes $\mathscr{R}$ ist der nichtleere Durchschnitt aller Primideale von $\mathscr{R}$. (Siehe MCCOY [12].) Das $\mathscr{P}_f$-Radikal ist die Verallgemeinerung dieses Begriffes und Satz 3. 2 verallgemeinert die wohlbekannten Eigenschaften dieses Radikals. (Siehe MCCOY [13].) Korollar 3. 3 läßt sich auf das Krull—McCoysche Radikal von $\mathscr{R}$ anwenden.

Das *Baersche untere Radikal* war als der nichtleere Durchschnitt aller Ideale $\mathfrak{a}$ des assoziativen Ringes $\mathscr{R}$ mit der Eigenschaft, daß $\mathscr{R}/\mathfrak{a}$ kein von Null verschiedenes nilpotentes Rechtsideal besitzt, definiert. (Siehe BAER [1].) Es ist leicht einzusehen, daß diese Ideale $\mathfrak{a}$ eben die Semiprimideale von $\mathscr{R}$ sind. LEVITZKI [11] und NAGATA [15]

haben bewiesen, daß das Baersche untere Radikal und das Krull—McCoysche Radikal eines assoziativen Ringes übereinstimmen. Dieses Ergebnis ist im Satz 3. 4 verallgemeinert.

Man kann einsehen, daß das *Brown—McCoysche Radikal* eines Ringes $\mathscr{R}$ ein spezielles $\mathscr{T}[0, \mathscr{R}]$-Radikal ist. (Siehe BROWN—MCCOY [2] und RÉDEI [16] Satz 186.)

F. SZÁSZ [22] definierte ein Radikal des Ringes $\mathscr{R}$ mit Hilfe beliebiger nichtnegativen ganzen Zahlen k, l, m, n. Dieses Radikal ist für gewisse k, l, m, n ein Durchschnitt von Primidealen und stimmt für einige k, l, m, n mit dem Brown—McCoyschen Radikal überein.

Das Fuchssche Zeroidradikal eines Ringes $\mathscr{R}$ ist auch ein Durchschnitt gewißer Primideale von $\mathscr{R}$ und die van Leeuwensche Verallgemeinerung dieses Radikals besitzt dieselbe Eigenschaft. (Siehe FUCHS [3] und VAN LEEUWEN [8].)

F. SZÁSZ [21] verallgemeinerte das Fuchssche Radikal für verbandsgeordnete Gruppoide und zeigte, daß jedes Semiprimideal eines Ringes als ein Szászsches Radikal zu betrachten ist.

Das *Jacobsonsche Radikal* des Ringes $\mathscr{R}$ ist der nichtleere Durchschnitt aller primitiven Ideale von $\mathscr{R}$. (Siehe JACOBSON [6].) Es ist bekannt, daß jedes primitive Ideal von $\mathscr{R}$ ein Primideal von $\mathscr{R}$ ist. (Siehe JACOBSON [6].)

Da ein vollständiges Primideal eines Ringes immer ein Primideal ist, ist der Durchschnitt Q aller vollständigen Primideale von $\mathscr{R}$ auch ein $\mathscr{T}[0, \mathscr{R}]$-Radikal. Der Begriff dieses Radikals wurde durch THIERRIN [26] eingeführt. THIERRIN [27] untersuchte jenen speziellen Fall dieses Radikals, wo Q der nichtleere Durchschnitt aller solchen Ideale $\mathfrak{q}_\alpha$ von $\mathscr{R}$ ist, für welche $\mathscr{R}/\mathfrak{q}_\alpha$ ein Schiefkörper ist.

Unter den erwähnten Radikalen haben das Baersche, das Krull—McCoysche, das Brown—McCoysche, das Jacobsonsche und das Thierrinsche Radikal die Eigenschaft:

$$\mathscr{T}[0, \mathfrak{a}] = \mathscr{T}[0, \mathscr{R}] \cap \mathfrak{a}, \tag{4.1}$$

wo $\mathfrak{a}$ ein beliebiges Ideal des Ringes $\mathscr{R}$ ist. Wir nennen ein $\mathscr{T}$-Radikal (in $\mathscr{R}$), welches auch die Bedingung (4. 1) befriedigt, $\mathscr{T}^*$-Radikal, und bezeichnen es mit $\mathscr{T}^*[\mathfrak{a}, \mathfrak{b}]$.

Satz 3. 1 läßt sich für das $\mathscr{T}$-Radikal von Ringen spezialisieren.

SATZ 4. 1. *Gilt die Bedingung* (4. 1) *für das Radikal* $\mathscr{T}[0, \mathfrak{a}]$ *des Ideals* $\mathfrak{a}$ *des assoziativen Ringes* $\mathscr{R}$, *so gilt*

$$\mathscr{T}[0, \mathscr{R}] = (\mathscr{T}[0, \mathfrak{a}] : \mathfrak{a}) \cap \mathscr{T}[\mathfrak{a}, \mathscr{R}]. \tag{4.2}$$

KOROLLAR 4. 2 (siehe Korollar 5 von [19]). Ist $\mathfrak{a}$ ein beliebiges Ideal des Ringes $\mathscr{R}$, so besteht die Formel

$$\mathscr{T}^*[0, \mathscr{R}] = (\mathscr{T}^*[0, \mathfrak{a}] : \mathfrak{a}) \cap \mathscr{T}^*[\mathfrak{a}, \mathscr{R}]. \tag{4.3}$$

BEMERKUNG. J. SZENDREI [24] hat ein ähnliches Ergebnis über das Jacobsonsche Radikal eines Ringes bewiesen.

§ 5. Über die Primnormalteiler und Primradikale von Gruppen

Es sei $\mathscr{L}_2$ die Menge aller Teilgruppen einer Gruppe $\mathscr{G}$. Man kann in $\mathscr{L}_2$ die folgenden Verknüpfungen definieren:

a′) den mengentheoretischen Durchschnitt $\mathscr{A} \cap \mathscr{B}$ von $\mathscr{A}, \mathscr{B} (\in \mathscr{L}_2)$;

b′) die Vereinigung $\mathscr{A} \cup \mathscr{B} = \mathscr{A}\mathscr{B}$ von $\mathscr{A}, \mathscr{B} (\in \mathscr{L}_2)$;

c′) den Kommutator $[\mathscr{A}, \mathscr{B}]$ als das „Produkt" von $\mathscr{A}, \mathscr{B} (\in \mathscr{L}_2)$.

Bezüglich dieser Operationen bildet $\mathscr{L}_2$ ein teilweise geordnetes kommutatives Gruppoid, welches ein vollständiger Verband ist. Die Menge $\overline{\mathscr{L}_2}$ aller Normalteiler von $\mathscr{G}$ bildet ein negativ und verbandsgeordnetes Gruppoid mit Quotienten bezüglich derselben Operationen. (Siehe FUCHS [5].) Dem entsprechend kann man in $\overline{\mathscr{L}_2}$ die in den vorigen Paragraphen eingeführten Begriffe definieren.

Es ist leicht das folgende Resultat zu beweisen.

BEHAUPTUNG 5. 1 (siehe Стеллецкий [25]). *Die Teilgruppe $\mathscr{A}$ der Gruppe $\mathscr{G}$ ist dann und nur dann ein Normalteiler von $\mathscr{G}$, wenn $[\mathscr{A}, \mathscr{G}] \subseteqq \mathscr{A}$ gilt.*

Aus der Behauptung 5.1 sieht man, daß das Ideal eines assoziativen Ringes und der Normalteiler einer Gruppe in $\mathscr{L}_1$ bzw. in $\mathscr{L}_2$ analoge Begriffe sind. Ist $\mathfrak{a}$ ($\mathscr{N}$) nämlich ein Ideal (ein Normalteiler) des Ringes $\mathscr{R}$ (der Gruppe $\mathscr{G}$), so gelten $\mathfrak{a}\mathscr{R} \subseteqq \mathfrak{a}$ und $\mathscr{R}\mathfrak{a} \subseteqq \mathfrak{a}$ ($[\mathscr{N}, \mathscr{G}] = [\mathscr{G}, \mathscr{N}] \subseteqq \mathscr{N}$).

Ein Normalteiler $\mathscr{P}$ einer Gruppe $\mathscr{G}$ heißt nach E. SCHENKMAN [17] *prim*, falls aus der Bedingung $[\mathscr{N}_1, \mathscr{N}_2] \subseteqq \mathscr{P}$ ($\mathscr{N}_1, \mathscr{N}_2 \in \overline{\mathscr{L}_2}$) entweder $\mathscr{N}_1 \subseteqq \mathscr{P}$ oder $\mathscr{N}_2 \subseteqq \mathscr{P}$ folgt.

Behauptung 2. 1 lautet in $\overline{\mathscr{L}_2}$ folgenderweise.

BEHAUPTUNG 5. 2. *Ist jeder Normalteiler der Gruppe $\mathscr{G}$ prim, so bilden die Normalteiler von $\mathscr{G}$ eine Kette.*

SCHENKMAN [17] zeigte einige Eigenschaften der Quotienten $\mathscr{N}_1 : \mathscr{N}_2$ der Normalteiler $\mathscr{N}_1$ und $\mathscr{N}_2$ der Gruppe $\mathscr{G}$. (Der Quotient $\mathscr{N}_1 : \mathscr{N}_2$ besteht aus allen Elementen x von $\mathscr{G}$, die der Bedingung $xn_2x^{-1}n_2^{-1} = [x, n_2] \in \mathscr{N}_1$ für jedes $n_2 \in \mathscr{N}_2$ befriedigen.) Behauptungen 1. 1—1. 4 verallgemeinern diese Ergebnisse von SCHENKMAN.

Vor der Anwendung der übrigen Begriffe und Ergebnisse auf $\overline{\mathscr{L}_2}$ müssen wir einige Bemerkungen bezüglich der Analogie zwischen den Primnormalteilern und Primidealen machen. Es ist bekannt, daß das Primideal $\mathfrak{p}$ des Ideals $\mathfrak{a}$ des assoziativen Ringes $\mathscr{R}$ ein Ideal von $\mathscr{R}$ ist. (NAGATA [15].) Weiterhin ist der Durchschnitt des Primideals $\mathfrak{q}$ und Ideals $\mathfrak{b}$ von $\mathscr{R}$ ein Primideal von $\mathfrak{b}$. (MCCOY [12].) Diese zwei Ergebnisse spielen eine wichtige Rolle in den Sätzen über die Primideale und Primradikale eines Ringes. Im Beweis dieser zwei Ergebnisse ist es sehr wichtig, daß $\mathscr{R}$ ein *assoziativer* Ring ist. Die analogen gruppentheoretischen Probleme lauten folgenderweise: 1. Ist der Primnormalteiler $\mathscr{P}$ eines Normalteilers $\mathscr{N}$ der Gruppe $\mathscr{G}$ ein Normalteiler von $\mathscr{G}$? 2. Ist der Durchschnitt eines Primnormalteilers Q und Normalteilers $\mathscr{M}$ der Gruppe $\mathscr{G}$ ein Primnormalteiler von $\mathscr{G}$? Wir werden zeigen, daß die Antwort auf diese zwei Fragen bejahend ist, falls $\mathscr{G}$ eine metaabelsche Gruppe ist, d.h. jeder Kommutator $[x, y]$ ($x, y \in \mathscr{G}$) liegt im Zentrum von $\mathscr{G}$.

Das folgende Resultat ist wohlbekannt. (Siehe LEVI [10] oder Курош [8].)

BEHAUPTUNG 5. 3. *Für eine Gruppe $\mathcal{G}$ sind die folgenden Bedingungen äquivalent:*
(i) *$\mathcal{G}$ ist metaabelisch;*
(ii) *es gilt* $[[x, y], z] = [x, [y, z]]$ *für jedes* $x, y, z \in \mathcal{G}$;
(iii) *es gilt* $[x, yz] = [x, y][x, z]$, $[xy, z] = [x, z][y, z]$ *für jedes* $x, y, z \in \mathcal{G}$;
(iv) *es gilt* $[[\mathcal{G}, \mathcal{G}], \mathcal{G}] = [\mathcal{G}, [\mathcal{G}, \mathcal{G}]] = 1$.

SATZ 5. 4. *Ist $\mathcal{G}$ eine metaabelsche Gruppe, so gelten die folgenden Ergebnisse.*
(A) *Für die Teilgruppen $\mathcal{A}, \mathcal{B}, \mathcal{C}$ von $\mathcal{G}$ bestehen*

$$[[\mathcal{A}, \mathcal{B}], \mathcal{C}] = [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \tag{5. 1}$$

und

$$[\mathcal{A} \cup [\mathcal{A}, \mathcal{G}], \mathcal{B}] = [\mathcal{A}, \mathcal{B}] \cup [[\mathcal{A}, \mathcal{G}], \mathcal{B}]; \tag{5. 2}$$

(B) *Der durch die Teilgruppe $\mathcal{A}$ erzeugte Normalteiler von $\mathcal{G}$ ist $\mathcal{A} \cup [\mathcal{A}, \mathcal{G}]$;*
(C) *Jeder Primnormalteiler $\mathcal{P}$ des Normalteilers $\mathcal{N}$ von $\mathcal{G}$ ist ein Normalteiler von $\mathcal{G}$;*
(D) *Der Durchschnitt des Primnormalteilers $\mathcal{Q}$ und Normalteilers $\mathcal{M}$ von $\mathcal{G}$ ist ein Primnormalteiler von $\mathcal{M}$.*

BEWEIS. Aus der Bedingung (iv) der Behauptung 5. 3 folgt (5. 1) unmittelbar.
Um (5. 2) zu zeigen, muß man nur einsehen, daß die linke Seite in der rechten Seite enthalten ist. Da jedes Element der Teilgruppe $\mathcal{A} \cup [\mathcal{A}, \mathcal{G}]$ ein endliches Produkt der Elemente von $\mathcal{A}$ und $[\mathcal{A}, \mathcal{G}]$ ist, sind die erzeugenden Elemente von $[\mathcal{A} \cup [\mathcal{A}, \mathcal{G}], \mathcal{B}]$ wegen der Bedingung (ii) der Behauptung 5. 3 in der Teilgruppe $[\mathcal{A}, \mathcal{B}] \cup [[\mathcal{A}, \mathcal{G}], \mathcal{B}]$ enthalten. Damit ist (A) bewiesen.
Wegen (5. 2) und (iv) gilt

$$[\mathcal{A} \cup [\mathcal{A}, \mathcal{G}], \mathcal{G}] = [\mathcal{A}, \mathcal{G}] \cup [[\mathcal{A}, \mathcal{G}], \mathcal{G}] = [\mathcal{A}, \mathcal{G}] \subseteq \mathcal{A} \cup [\mathcal{A}, \mathcal{G}],$$

was nach Behauptung 5. 1 bedeutet, daß $\mathcal{A} \cup [\mathcal{A}, \mathcal{G}]$ ein Normalteiler von $\mathcal{G}$ mit $\mathcal{A} \subseteq \mathcal{A} \cup [\mathcal{A}, \mathcal{G}]$ ist. Wenn $\mathcal{N}$ ein Normalteiler von $\mathcal{G}$ mit $\mathcal{A} \subseteq \mathcal{N}$ ist, so besteht infolge der Behauptung 5. 1 $\mathcal{A} \cup [\mathcal{A}, \mathcal{G}] \subseteq \mathcal{N} \cup [\mathcal{N}, \mathcal{G}] \subseteq \mathcal{N}$, womit (B) bewiesen ist.
Es sei $\mathcal{P}$ ein Primnormalteiler eines Normalteilers $\mathcal{N}$ von $\mathcal{G}$. Im Falle $\mathcal{N} = \mathcal{P}$ ist $\mathcal{P}$ ein Normalteiler von $\mathcal{G}$. Wenn $\mathcal{P} \subset \mathcal{N}$ ist, dann besteht nach (5. 1) und der Behauptung 5. 1

$$[\mathcal{N}, [\mathcal{G}, \mathcal{P}]] = [[\mathcal{N}, \mathcal{G}], \mathcal{P}] \subseteq [\mathcal{N}, \mathcal{P}] \subseteq \mathcal{P},$$

woraus wegen $\mathcal{N} \not\subseteq \mathcal{P}$ die Beziehung $[\mathcal{G}, \mathcal{P}] \subseteq \mathcal{P}$ folgt. Dies bedeutet nach Behauptung 5. 1, daß $\mathcal{P}$ in der Tat ein Normalteiler von $\mathcal{G}$ ist.[2]
Um Behauptung (D) zu zeigen, betrachten wir zwei Normalteiler $\mathcal{X}, \mathcal{Y}$ des Normalteilers $\mathcal{M}$ mit der Eigenschaft

$$[\mathcal{X}, \mathcal{Y}] \subseteq \mathcal{Q} \cap \mathcal{M}. \tag{5. 3}$$

Nach (B) ist $\mathcal{X} \cup [\mathcal{X}, \mathcal{G}]$ bzw. $\mathcal{Y} \cup [\mathcal{Y}, \mathcal{G}]$ der durch $\mathcal{X}$ bzw. $\mathcal{Y}$ erzeugte Normalteiler von $\mathcal{G}$. Infolge (iv), (5. 2), (5. 3) und Behauptung 5. 1 bekommt man

$$[\mathcal{X} \cup [\mathcal{X}, \mathcal{G}], \mathcal{Y} \cup [\mathcal{Y}, \mathcal{G}]] = [\mathcal{X}, \mathcal{Y} \cup [\mathcal{Y}, \mathcal{G}]] \cup [[\mathcal{X}, \mathcal{G}], \mathcal{Y} \cup [\mathcal{Y}, \mathcal{G}]] =$$
$$= [\mathcal{X}, \mathcal{Y}] \cup [\mathcal{X}, [\mathcal{Y}, \mathcal{G}]] \cup [[\mathcal{X}, \mathcal{G}], \mathcal{Y}] \cup [[\mathcal{X}, \mathcal{G}], [\mathcal{Y}, \mathcal{G}]] \subseteq \mathcal{Q} \cap \mathcal{M} \subseteq \mathcal{Q},$$

[2] Zum Beweis dieses Teiles benutzen wir nur die Egenschiaft (5.1) der Gruppe $\mathcal{G}$.

woraus nach der Definition von $\mathfrak{Q}$ entweder $\mathfrak{X} \subseteq \mathfrak{X} \cup [\mathfrak{X}, \mathfrak{G}] \subseteq \mathfrak{Q}$ oder $\mathfrak{Y} \subseteq \subseteq \mathfrak{Y} \cup [\mathfrak{Y}, \mathfrak{G}] \subseteq \mathfrak{Q}$ folgt. Dies impliziert $\mathfrak{X} \subseteq \mathfrak{Q} \cap \mathfrak{M}$ oder $\mathfrak{Y} \subseteq \mathfrak{Q} \cap \mathfrak{M}$.

Damit ist der Beweis beendet.

Wenn man die Begriffe und Ergebnisse der Paragraphen 2 und 3 auf eine Gruppe $\mathfrak{G}$ anwenden will, kann man unter zwei Möglichkeiten wählen. Entweder beschränkt man sich nur auf die metaabelschen Gruppen, oder muß man die Begriffe und Bedingungen geeigneterweise ändern.

Aus Satz 2. 2 ergibt sich

SATZ 5. 5. *Es sei $\mathfrak{N}$ ein Normalteiler der Gruppe $\mathfrak{G}$ und sei $\mathfrak{P}$ ein solcher Primnormalteiler von $\mathfrak{N}$, welcher Normalteiler von $\mathfrak{G}$ ist. Dann ist $\mathfrak{P}: \mathfrak{N}$ ein Primnormalteiler von $\mathfrak{G}$ mit der Eigenschaft*

$$(\mathfrak{P}: \mathfrak{N}) \cap \mathfrak{N} = \mathfrak{P}.$$

Ist $\mathfrak{P} \neq \mathfrak{N}$, dann ist $\mathfrak{P}: \mathfrak{N}$ der einzige Primnormalteiler von $\mathfrak{G}$, dessen Durchschnitt mit $\mathfrak{N}$ der Normalteiler $\mathfrak{P}$ ist.

$\mathfrak{P}$ ist dann und nur dann ein Primnormalteiler von $\mathfrak{G}$, wenn $\mathfrak{P} = \mathfrak{P}: \mathfrak{N}$ gilt.

Man kann das $\mathfrak{T}_f$-Radikal in $\overline{\mathfrak{L}_2}$ folgenderweise definieren. Es seien $\mathfrak{A}$ und $\mathfrak{B}$ zwei Normalteiler der Gruppe $\mathfrak{G}$. Der nichtleere Durchschnitt aller solcher Primnormalteiler $\mathfrak{P}_\lambda (\lambda \in \Lambda)$ von $\mathfrak{B}$,

(α) der ein Normalteiler von $\mathfrak{G}$ ist,

(β) in dem $\mathfrak{A}$ enthalten ist,

(γ) der eine gegebene Eigenschaft ($\mathfrak{T}$) besitzt, heißt das *$\mathfrak{T}$-Radikal von $\mathfrak{B}$ über $\mathfrak{A}$*, und es ist mit $\mathfrak{T}\overline{\mathfrak{A}, \mathfrak{B}}$ bezeichnet. Ist $\bigcap_{\lambda \in \Lambda} \mathfrak{P}_\lambda = \emptyset$, so sei $\mathfrak{T}\overline{\mathfrak{A}, \mathfrak{B}} = \mathfrak{B}$. (Wir müssen die ursprüngliche Bezeichnung des $\mathfrak{T}_f$-Radikals hier verändern.)

Wir können Satz 3. 1 für dieses $\mathfrak{T}$-Radikal spezialisieren.

SATZ 5. 6. *Bezeichne $\mathfrak{T}\overline{1, \mathfrak{N}}$ das $\mathfrak{T}$-Radikal eines Normalteilers $\mathfrak{N}$ der Gruppe $\mathfrak{G}$ über dem Einselement* 1. *Gilt*

$$\mathfrak{T}\overline{1, \mathfrak{N}} = \mathfrak{T}\overline{1, \mathfrak{G}} \cap \mathfrak{N},$$

so besteht

$$\mathfrak{T}\overline{1, \mathfrak{G}} = \left(\mathfrak{T}\overline{1, \mathfrak{N}} : \mathfrak{N}\right) \cap \mathfrak{T}\overline{\mathfrak{N}, \mathfrak{G}}.$$

Der folgende Begriff ist ein wichtiger Spezialfall des $\mathfrak{T}\overline{\mathfrak{A}, \mathfrak{B}}$-Radikals bzw. des $\mathfrak{P}_f[a, b]$-Radikals. Wir betrachten zwei Normalteiler $\mathfrak{A}, \mathfrak{B} (\mathfrak{A} \subset \mathfrak{B})$ einer Gruppe $\mathfrak{G}$. Das *Radikal* $\mathfrak{P}\overline{\mathfrak{A}, \mathfrak{B}}$ bezeichne den Durchschnitt aller solcher den $\mathfrak{A}$ enthaltenden Primnormalteiler von $\mathfrak{B}$, die Normalteiler von $\mathfrak{G}$ sind. Das durch Y. KURATA [7] eingeführte Radikal ist jener Spezialfall von $\mathfrak{P}\overline{\mathfrak{A}, \mathfrak{B}}$, wo $\mathfrak{B} = \mathfrak{G}$ ist.

Wir werden jetzt die Bedingung (3. 1) und die Behauptungen 1, 3 und 4 des Satzes 3. 2 für das Radikal $\mathfrak{P}\overline{\mathfrak{A}, \mathfrak{B}}$ spezialisieren, wodurch sich im wesentlichen die Behauptung 1. 14 von KURATA [7] ergibt.

BEHAUPTUNG 5. 7. *Für die Normalteiler* $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ $(\mathcal{A}_i \subset \mathcal{B};\ i=1,2)$ *einer Gruppe* $\mathcal{G}$ *gelten die folgenden Ergebnisse:*

1) $\mathcal{A}_1 \subseteqq \mathcal{P}\overline{\mathcal{A}_1, \mathcal{B}} \subseteqq \mathcal{B}$;

2) *ist* $\mathcal{A}_1 \subseteqq \mathcal{A}_2$, *so besteht* $\mathcal{P}\overline{\mathcal{A}_1, \mathcal{B}} \subseteqq \mathcal{P}\overline{\mathcal{A}_2, \mathcal{B}}$;

3) $\overline{\mathcal{P}\mathcal{P}\overline{\mathcal{A}_1, \mathcal{B}}, \mathcal{B}} = \mathcal{P}\overline{\mathcal{A}_1, \mathcal{B}}$;

4) $\mathcal{P}\overline{[\mathcal{A}_1, \mathcal{A}_2], \mathcal{B}} = \mathcal{P}\overline{\mathcal{A}_1 \cap \mathcal{A}_2, \mathcal{B}} = \mathcal{P}\overline{\mathcal{A}_1, \mathcal{B}} \cap \mathcal{P}\overline{\mathcal{A}_2, \mathcal{B}}$.

Ein Normalteiler $\mathcal{S}$ der Gruppe $\mathcal{G}$ heißt *semiprim*, wenn aus der Bedingung $[\mathcal{N}, \mathcal{N}] \subseteqq \mathcal{S}$ immer $\mathcal{N} \subseteqq \mathcal{S}$ folgt, wo $\mathcal{N}$ einen Normalteiler von $\mathcal{G}$ bezeichnet. Dieser Begriff ist ein Spezialfall des im § 2 definierten f-Semiprimelementes. Der Durchschnitt von Primnormalteilern der Gruppe $\mathcal{G}$ ist offenbar ein Semiprimnormalteiler von $\mathcal{G}$. Aus Satz 2. 5 und Korollar 3. 5 bekommt man den

SATZ 5. 8. *Für den Normalteiler* $\mathcal{S}$ *der Gruppe* $\mathcal{G}$ *sind die folgenden drei Bedingungen äquivalent:*

(i) $\mathcal{S}$ *ist semiprim*;

(ii) $\mathcal{S} = \bigcap_{\alpha} \mathcal{P}_\alpha$, *wo* $\mathcal{P}_\alpha$ *alle den* $\mathcal{S}$ *enthaltenden Primnormalteiler von* $\mathcal{G}$ *durchlaufen*;

(iii) $\mathcal{S} = \mathcal{P}\overline{\mathcal{S}, \mathcal{G}}$.

Will man die Korollare 2. 3 und 3. 3 und die Behauptung 5 des Satzes 3. 2 für $\overline{\mathcal{L}_2}$ spezialisieren, so benutzt man auch noch Satz 5. 4. Dem entsprechend bekommen wir

KOROLLAR 5. 9. Es sei $\mathcal{G}$ eine metaabelsche Gruppe.

Es gibt eine eineindeutige Zuordnung zwischen den Primnormalteilern $\mathcal{P}(\neq \mathcal{N})$ des Normalteilers $\mathcal{N}$ von $\mathcal{G}$ und den Primnormalteilern $\mathcal{Q}$ von $\mathcal{G}$ mit $\mathcal{Q}(\not\equiv \mathcal{N})$, u.zw.

$$\mathcal{P} \to \mathcal{P}: \mathcal{N} \quad \text{und} \quad \mathcal{Q} \to \mathcal{Q} \cap \mathcal{N}.$$

KOROLLAR 5. 10. Bezeichne $\mathcal{G}$ eine metaabelsche Gruppe. Für das Radikal $\mathcal{P}\overline{1, \mathcal{N}}$ des Normalteiler $\mathcal{N}$ von $\mathcal{G}$ über dem Einselement 1 bestehen die Eigenschaften

$$\mathcal{P}\overline{1, \mathcal{N}} = \mathcal{P}\overline{1, \mathcal{G}} \cap \mathcal{N}$$

und

$$\mathcal{P}\overline{1, \mathcal{G}} = \left(\mathcal{P}\overline{1, \mathcal{N}} : \mathcal{N}\right) \cap \mathcal{P}\overline{\mathcal{N}, \mathcal{G}}.$$

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Literaturverzeichnis

[1] R. Baer, Radical ideals, *Amer. J. Math.*, **65** (1943), S. 537—568.
[2] B. Brown—N. H. McCoy, Radicals and subdirect sums, *Amer. J. Math.*, **69** (1947), S. 46—58.
[3] L. Fuchs, On a new type of radical, *Acta Sci. Math.*, **16** (1955), S. 43—53.
[4] L. Fuchs, On partially ordered algebras. II, *Acta Sci. Math. Szeged*, **26** (1965), S. 35—41.
[5] L. Fuchs, *Teilweise geordnete algebraische Strukturen* (Budapest, 1966).
[6] N. Jacobson, *Structure of rings* (Providence, 1956).
[7] Y. Kurata, A decomposition of normal subgroups in a group, *Osaka Y. Math.*, **1** (1964), S. 201—229.
[8] А. Г. Курош, *Теория групп* (Москва, 1953).
[9] L. C. A. van Leeuwen, On the zeroid radical of a ring, *Indagationes Math.*, **21** (1959), S. 428—433.
[10] F. Levi, Groups in which the commutator operation satisfies certain algebraic conditions, *J. Indian Math. Soc.*, **6** (1942), S. 87—97.
[11] F. Levitzki, Prime ideals and the lower radical, *Amer. J. Math.*, **73** (1951), S. 25—29.
[12] N. H. McCoy, Prime ideals in general rings, *Amer. J. Math.*, **71** (1949), S. 823—833.
[13] N. H. McCoy, *The theory of rings* (New York—London, 1964).
[14] K. Murata, On nilpotent-free multiplicative systems, *Osaka Math. J.*, **14** (1962), S. 53—70.
[15] M. Nagata, On the theory of radicals in a ring, *J. Math. Soc. Japan*, **3** (1951), S. 330—344.
[16] L. Rédei, *Algebra I.* (Leipzig, 1959).
[17] E. Schenkman, The similarity between the properties of ideals in commutative rings and the properties of normal subgroups of groups, *Proc. Amer. Math. Soc.*, **9** (1958), S. 375—381.
[18] O. Steinfeld, On ideal-quotients and prime ideals, *Acta Math. Acad. Sci. Hung.*, **4** (1953), S. 289—298.
[19] O. Steinfeld, Verbandstheoretische Betrachtung gewisser idealtheoretischer Fragen, *Acta Sci. Math. Szeged*, **12** (1961), S. 136—149.
[20] O. Steinfeld, On residuals in partially ordered semigroups, *Publ. Math. Debrecen*, **12** (1965), S. 107—116.
[21] F. Szász, Verbandstheoretische Bemerkungen zum Fuchsschen Zeroidradikal der nichtassoziativen Ringe, *Archiv der Math.*, **12** (1961), S. 282—289.
[22] F. Szász, An observation on the Brown-McCoy radical, *Proc. of Japan Acad.*, **37** (1961), S. 413—416.
[23] G. Szász, Über Primideale von Halbgruppen. *Publ. Math. Debrecen*, **13** (1966), S. 39—41.
[24] J. Szendrei, On the Jacobson radical of a ring, *Publ. Math. Debrecen*, **4** (1955), S. 93—97.
[25] И. В. Стеллецкий, Нилпотентные структуры, *Труды Московского Мат. Общества*, **9** (1960), S. 211—235.
[26] G. Thierrin, Sur les idéaux complétement premiers d'un anneau quelconque, *Bull. Acad. Royal Belgique*, **43** (1957), S. 124—132.
[27] G. Thierrin, Sur le radical corpoidal d'un anneau, *Canadian J. Math.*, **12** (1960), S. 101—106.
[28] O. Steinfeld, Negatívan rendezett algebrai struktúrák prímelemeiről, *MTA. III. Oszt. Közl.*, **17** (1967), S. 467—472.

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ON THE POINT-GROUP AND LINE-GROUP OF A GRAPH[1]

By

F. HARARY (Ann Arbor, Michigan, U.S.A.) and E. M. PALMER
(East Lansing, Michigan, U. S. A.)

(Presented by G. HAJÓS)

One of the most important concepts associated with a graph G is its group of automorphisms $\Gamma(G)$, a permutation group which acts on the points of G. This group induces another permutation group $\Gamma_1(G)$ which acts on the lines of G. Now any permutation group can, of course, be considered simply as an abstract group. Thus it is natural to ask the questions in which the terminology follows CARMICHAEL [1]: when are $\Gamma(G)$ and $\Gamma_1(G)$ *isomorphic* as abstract groups and when are they *identical* as permutation groups? These questions are answered in Theorems 1 and 2.

When A and B are permutation groups acting on the sets X and Y respectively, we will write $A \cong B$ to mean that A and B are isomorphic groups. However $A \equiv B$ indicates not only isomorphism but that A and B are identical permutation groups. More specifically $A \cong B$ if there is a $1-1$ map $h: A \leftrightarrow B$ such that for all α_1, α_2 in A, $h(\alpha_1\alpha_2) = h(\alpha_1)h(\alpha_2)$. To define $A \equiv B$ precisely, we also require another $1-1$ map $f: X \leftrightarrow Y$ such that for all x in X and α in A

$$f(\alpha x) = h(\alpha) f(x).$$

A *graph* G consists of a finite set $V(G)$ of *points* $v_1, \ldots, v_p$, together with a prescribed set $X(G)$ of unordered pairs of distinct points of $V(G)$. Each such pair of points u and v is a *line* $x = uv$ of the graph G. In this case we say that the points u and v are *adjacent* and that the line x is *incident* with each of the points u and v. The *complete graph* K_p of p points has every two distinct points adjacent. The graph $K_p - x$ is obtained from K_p by removing any line x. The graph $K_m \cdot K_n$ is obtained by identifying any point of K_m with any point of K_n.

An *automorphism* of G is a permutation of the set $V(G)$ of points of G which preserves adjacency. Let $\Gamma(G)$ be the group of all automorphisms of G. Usually $\Gamma(G)$ is called the group of G, but here it will be called the *point-group* of G. Of course $\Gamma(G)$ induces a group $\Gamma_1(G)$ of permutations acting on the set $X(G)$ of lines of G. We will call $\Gamma_1(G)$ the *line-group* of G.

We now use $K_4 - x$ to illustrate the difference between the point-group and the line-group. In Figure 1 we have a copy of $K_4 - x$ with points labelled v_1, v_2, v_3, v_4 and lines x_1, x_2, x_3, x_4, x_5.

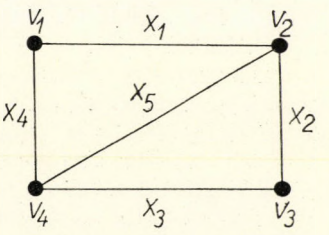

Fig. 1

[1] Invited address at the Bolyai Mathematical Society in Budapest on May 19th, 1967.

The point-group $\Gamma(K_4-x)$ consists of the four permutations

$$(v_1)(v_2)(v_3)(v_4)$$
$$(v_1)(v_3)(v_2v_4)$$
$$(v_2)(v_4)(v_1v_3)$$
$$(v_1v_3)(v_2v_4).$$

The identity permutation of the point-group induces the identity permutation on the lines, while $(v_1)(v_3)(v_2v_4)$ induces a permutation on the lines which fixes x_5, interchanges x_1 with x_4 and x_2 with x_3. Thus the line-group $\Gamma_1(K_4-x)$ consists of the following permutations, induced respectively by the above members of the point-group:

$$(x_1)(x_2)(x_3)(x_4)(x_5)$$
$$(x_1x_4)(x_2x_3)(x_5)$$
$$(x_3x_4)(x_1x_2)(x_5)$$
$$(x_1x_3)(x_2x_4)(x_5).$$

Of course the line-group and the point-group of K_4-x are isomorphic. But they are certainly not identical permutation groups since $\Gamma_1(K_4-x)$ has degree 5 and $\Gamma(K_4-x)$ has degree 4. Note that the line x_5 is fixed by every member of the line-group. Even the permutation group obtained from $\Gamma_1(K_4-x)$ by restricting its object set to x_1, x_2, x_3, x_4 only is not identical with $\Gamma(K_4-x)$, since these two isomorphic permutation groups of the same degree have different cycle structure. Furthermore, even when two permutation groups have the same degree and the same cycle structure, they still need not be identical.

SABIDUSSI [4] has demonstrated the sufficiency of the following conditions using group theoretic methods. This proof uses very elementary graphical concepts.

THEOREM 1. *The line-group and the point-group of a graph G are isomorphic if and only if G has at most one isolated point and K_2 is not a component of G.*

PROOF. Let α' be the permutation in $\Gamma_1(G)$ which is induced by the permutation α in $\Gamma(G)$. By the definition of multiplication in $\Gamma_1(G)$, we have $\alpha'\beta'=(\alpha\beta)'$ for all α, β in $\Gamma(G)$. Thus the mapping $\alpha\to\alpha'$ is a group homomorphism from $\Gamma(G)$ onto $\Gamma_1(G)$. Hence $\Gamma(G)\cong\Gamma_1(G)$ if and only if the kernel of this mapping is trivial.

To prove the necessity, assume $\Gamma(G)\cong\Gamma_1(G)$. Then $\alpha\neq 1$ (the identity permutation) implies $\alpha'\neq 1$. If G has distinct isolated points v_1 and v_2, we can define $\alpha\in\Gamma(G)$ by $\alpha(v_1)=v_2$, $\alpha(v_2)=v_1$ and $\alpha(v)=v$ for all $v\neq v_1, v_2$. Then $\alpha\neq 1$ but $\alpha'=1$. If K_2 is a component of G, take the line of K_2 to be $x=v_1v_2$ and define $\alpha\in\Gamma(G)$ exactly as above to obtain $\alpha\neq 1$ but $\alpha'=1$.

To prove the sufficiency, assume that G has at most one isolated point and and that K_2 is not a component of G. If $\Gamma(G)$ is trivial, then obviously $\Gamma_1(G)$ fixes every line and hence $\Gamma_1(G)$ is trivial. Therefore suppose there exists $\alpha\in\Gamma(G)$ with $\alpha(u)=v\neq u$. Then the degree of u is equal to the degree of v. Since u and v are not isolated, this degree is not zero.

Case 1. u is adjacent to v. Let $x=uv$. Since K_2 is not a component, the degreet of both u and v are greater than one. Hence there is a line $y\neq x$ which is incidens with u and $\alpha'(y)$ is incident with v. Therefore $\alpha'(y)\neq y$ and so $\alpha'\neq 1$.

Case 2. u is not adjacent to v. Let x be any line incident with u. Then $\alpha'(x)\neq x$ and so $\alpha'\neq 1$, completing the proof.

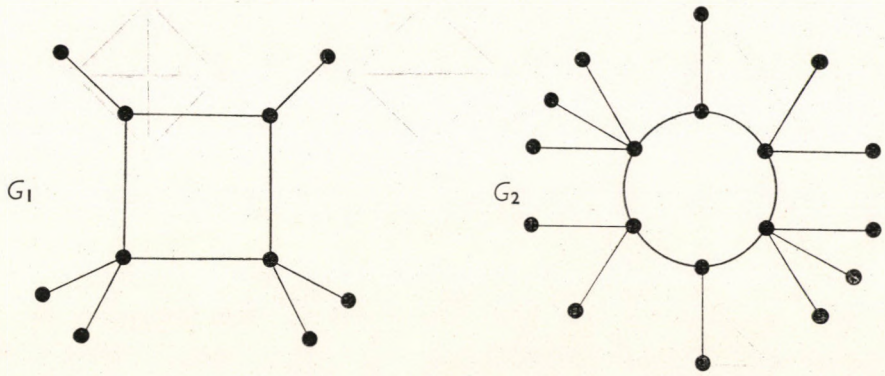

Fig. 2

Let G be a connected graph with exactly one cycle $z=v_1v_2...v_nv_1$; we call G a *unicyclic graph*. A *reflection* of G is an automorphism of G which does not fix all of the points of the cycle Z but which does fix at least one point or line of Z. The automorphisms of G which do not fix any points or lines of Z are called *rotations*. In Figure 2, it is clear that $\Gamma(G_1)$ contains reflections but no rotations and $\Gamma(G_2)$ contains rotations but no reflections.

Let T be a rooted tree with root v. Then by definition, every permutation in $\Gamma(T)$ leaves v fixed. Called $\Gamma(T, v)$ the permutation group which acts on the set $V(T)-\{v\}$ and which is induced by $\Gamma(T)$. The line-group of T, $\Gamma_1(T)$, is the permutation group which acts on the set of lines $X(T)$ and is induced by $\Gamma(T)$. In the proof of Theorem 2, we require the following observation.

LEMMA. *If T is a rooted tree with root v, then $\Gamma(T, v)\equiv\Gamma_1(T)$.*

PROOF. Define Φ: $\big(V(T)-\{v\}\big)\to X(T)$ by $\Phi(u)=x$, where x is the line which is both incident with u and on the $u-v$ path of T. Clearly Φ provides the necessary map which shows that the two groups are identical.

From now on for convenience we will denote $\Gamma_1(T)$ by $\Gamma_1(T, v)$ to emphasise the fact that T is rooted at the point v. Hence the lemma may be written $\Gamma(T, v)\equiv$ $\equiv\Gamma_1(T, v)$.

The *line-graph* of G is that graph $L(G)$ whose points correspond to the lines of G and in which two points are adjacent if and only if the corresponding lines of G are adjacent, i.e. have exactly one common point. We now note that the line-group $\Gamma_1(G)$ and the group of the line-graph $\Gamma\big(L(G)\big)$ are not always isomorphic. This is a consequence of WHITNEY's results [5] on the line-graph, which show that there are exactly three connected graphs G with $p\geqq 3$ points such that $\Gamma\big(L(G)\big)$

and $\Gamma(G)$ are not isomorphic. These three graphs, shown along with their line-graphs in Figure 3, are $K_3 \cdot K_2$, $K_4 - x$ and K_4.

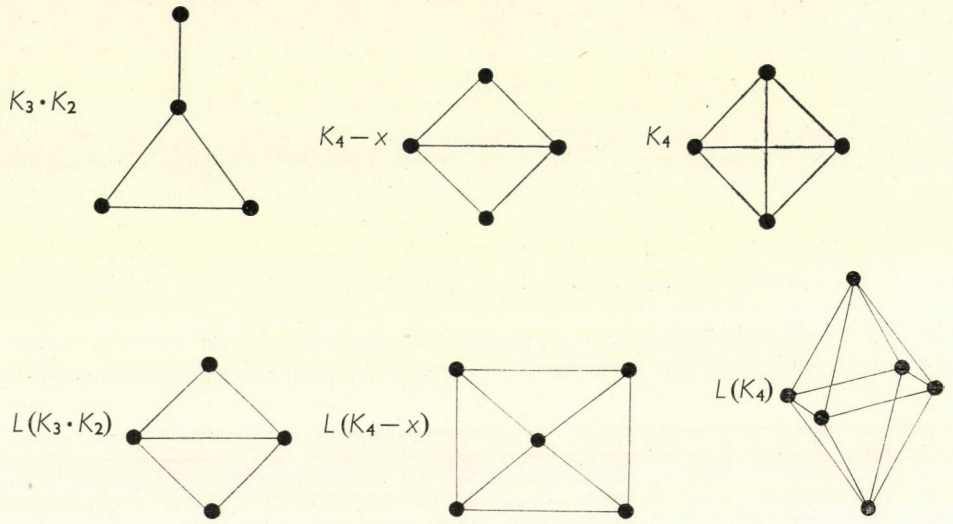

Fig. 3

When G is connected with $p \geqq 3$ points, $\Gamma(G)$ and $\Gamma_1(G)$ are always isomorphic by Theorem 1. Thus $K_3 \cdot K_2$, $K_4 - x$ and K_4 are the only connected graphs G with $p \geqq 3$ points such that $\Gamma(L(G))$ and $\Gamma_1(G)$ are not isomorphic.

In particular $K_3 \cdot K_2$ is the only connected unicyclic graph G such that $\Gamma(L(G))$ and $\Gamma_1(G)$ are not isomorphic. Because of this exception, it is more convenient to use the line-group of G instead of the point-group of the line-graph in the statement of the theorem which follows.

THEOREM 2. *The point-group and line-group of a connected graph G are identical fi and only if G is unicyclic and either its cycle is odd, or its cycle is even and*

(i) *there are no reflections of G which fix more than two points,*

(ii) *the number of reflections which fix exactly two points is the same as the number of reflections which fix exactly two lines.*

We first prove the necessity. By a classical theorem of Kirchhoff (see KÖNIG [3]), the number m of independent cycles of a connected graph with p points and q lines is given by $m = q - p + 1$. Since the degree of $\Gamma(G)$ is equal to the degree of $\Gamma_1(G)$, $q = p$ and hence G has exactly one cycle, say $Z = v_1 v_2 \ldots v_n v_1$.

Now assume that n is even but $\Gamma(G)$ contains a reflection which fixes more than two points. Let N be the maximum number of points fixed by a reflection of G. Since $N > 2$, no reflection can fix N lines.

Let r_k be the number of elements of $\Gamma(G)$ which fix exactly k points and let m_k be the number of elements of $\Gamma_1(G)$ which fix exactly k lines. Since $\Gamma(G) \equiv \Gamma_1(G)$, we have $r_k = m_k$ for all k. Let r'_k be the number of elements of $\Gamma(G)$ which fix exactly

k points including the n points of Z and let m'_k be the number in $\Gamma_1(G)$ which fix exactly k lines including the n lines of Z. For any unicyclic graph we have $r'_k = m'_k$. Hence $r_k - r'_k = m_k - m'_k$. But $r_k - r'_k$ is the number of reflections which fix k points and $m_k - m'_k$ is the number of reflections which fix k lines. By definition of N, we have $r_N - r'_N \geqq 1$ and $m_N - m'_N = 0$.

Therefore G satisfies (i). Hence if a reflection fixes exactly two lines, the two lines must be in Z. Also the only elements of $\Gamma(G)$ which fix exactly two points or exactly two lines must be reflections. Therefore G satisfies (ii).

To prove the sufficiency, assume that G is unicyclic. Then G can be constructed by identifying the distinguished points v_i of a collection of rooted trees $\{(T_i, v_i)\}$ with the points v_i of the cycle Z for $i = 1, \ldots, n$. By Theorem 1, the mapping from $\Gamma(G)$ onto $\Gamma_1(G)$ given by $\alpha \to \alpha'$ is an isomorphism.

If n is odd, we define a $1-1$ map $\Phi: V(G) \to X(G)$ as follows:

(a) If v is a point of (T_i, v_i) and $v \neq v_i$, let $\Phi(v) = x$ as in the lemma.

(b) If $v = v_i$, let $\Phi(v_i) = x_i$ where x_i is the line of Z which is diametrically opposite v_i. Then $\Phi\alpha(v) = \alpha'\Phi(v)$ for all $\alpha \in \Gamma(G)$ and all $v \in V(G)$. Hence $\Gamma(G) \equiv \Gamma_1(G)$.

If n is even and G satisfies (i) and (ii), then only the four following mutually exclusive cases can occur.

(1) $G = Z$.

(2) Every element of $\Gamma(G)$ fixes all of the points of Z.

(3) $\Gamma(G)$ contains rotations but no reflections.

(4) $G \neq Z$ but the number of reflections which fix exactly two points is the same as the number of reflections which fix exactly two lines.

Note that only in Cases (2) and (3), $\Gamma(G)$ contains no reflections. Define a $1-1$ map $\Phi: V(G) \to X(G)$ as follows:

(a) If v is a point of (T_i, v_i) and $v \neq v_i$, let $\Phi(v) = x$ as in the lemma.

(b) If $v = v_i$, let $\Phi(v_i) = x_i$ where $x_i = v_i v_{i+1}$, and the subscripts are taken modulo n.

In Cases (1), (2) or (3), we have $\Phi\alpha(v) = \alpha'\Phi(v)$ for all $\alpha \in \Gamma(G)$ and all $v \in V(G)$. Hence $\Gamma(G) \equiv \Gamma_1(G)$.

We will only sketch the details of Case (4). Let G_1 and G_2 be two disjoint copies of G. In G_1 we can assume that the point v_1 of the cycle Z is fixed by a reflection. And starting with v_2 and proceeding in cyclic order to v_n we can label the trees of G_1 by $T_1, T_2, \ldots, T_k$, where such a tree is attached only to those points of Z not fixed by any reflection of G. In G_2 the lines of Z can be labelled $x_1 x_2 \ldots x_n x_1$ with x_i adjacent to x_{i+1} and such that x_1 is fixed by a reflection. Let r be the number of points of Z which are fixed by some reflection and set $m = \dfrac{n-r}{2r}$. Now starting with x_1 and proceeding in cyclic order to x_n, we can label the trees of G_2 with the same labels used in G_1 but the order is changed to $T_m, \ldots, T_1, T_{2m}, \ldots, T_{m+1}; \ldots; T_k, \ldots, T_{k-m+1}$.

Under this correspondence of trees in G_1 with trees in G_2, any labelling of the non-cyclic points of the trees in G_1, say $v_{n+1}, \ldots, v_p$, induces a labelling of the lines of the trees in G_2, say $x_{n+1}, \ldots, x_p$ with v_j corresponding to x_j. Then for every $\alpha \in \Gamma(G_1)$ we can define $\alpha^* \in \Gamma_1(G_2)$ by $\alpha^*(x_i) = x_j$ if and only if $\alpha(v_i) = v_j$.

This homomorphism of $\Gamma(G_1)$ onto $\Gamma_1(G_2)$ together with the correspondence $v_i \leftrightarrow x_i$ between the points of G_1 and the lines of G_2 show that $\Gamma(G_1) \equiv \Gamma_1(G_2)$ and so $\Gamma(G) \equiv \Gamma_1(G)$.

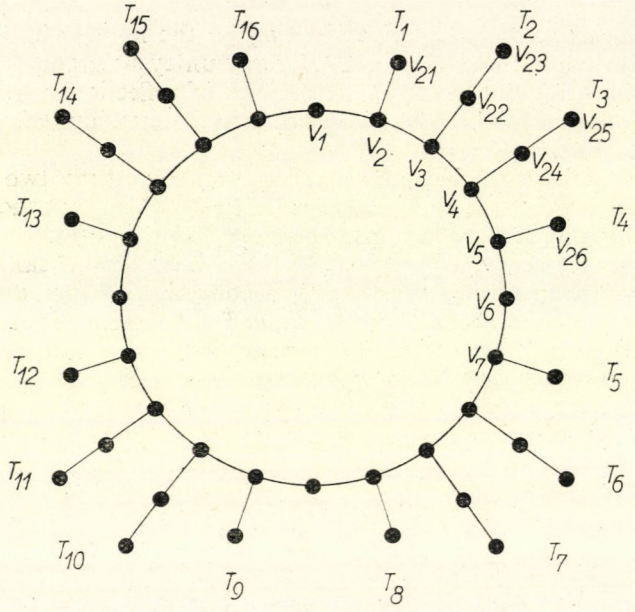
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Fig. 4

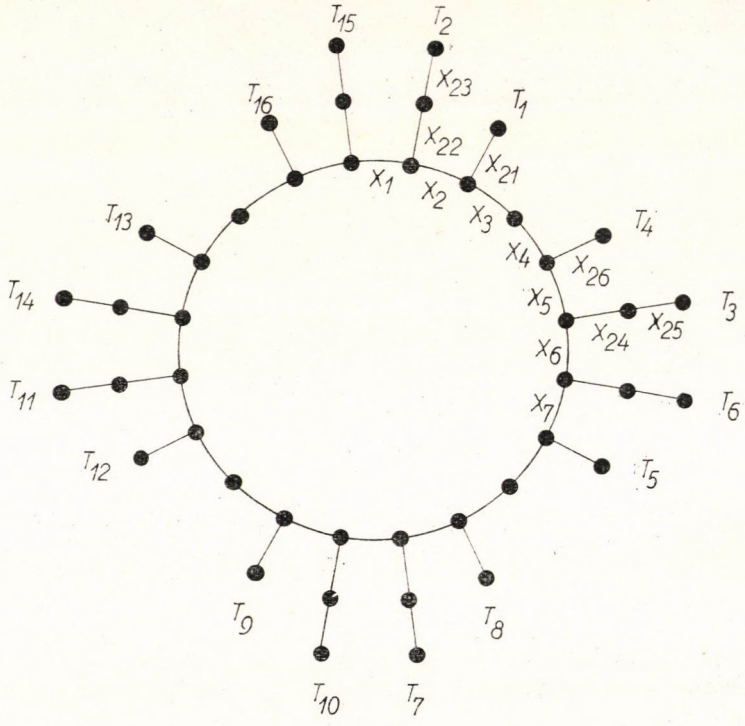
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Fig. 5

We now present a detailed example for Case (4) in order to illustrate the constructions for performing the $1-1$ correspondence between the points and lines of G. Figures 4 and 5 show two isomorphic copies G_1 and G_2 of G. Clearly this graph G satisfies Case (4) since there are just two reflections which fix exactly two points and the same number fixing exactly two lines. In this example $r=4$ and $n=20$, so $m=2$. The labels for all the trees $T_1, T_2, \ldots, T_k$ (where $k=n-r=16$) are indicated in the figure. We observe that if an automorphism α of G_1 sends tree T_i to T_j, then the corresponding permutation α^* acting on the lines of G_2 also sends T_i to T_j.

We conclude by applying Theorem 2 to the case of oriented graphs. By a semicycle of an oriented graph we mean a subgraph which is a cycle in the underlying undirected graph. The definitions used in the following corollaries may be found in [2].

COROLLARY 2. 1. *If G is a connected oriented graph, then $\Gamma(G)\equiv\Gamma_1(G)$ if and only if G is unicyclic and $\Gamma(G)$ contains no reflctions.*

PROOF. Any reflection in $\Gamma(G)$ always fixes exactly two points on the cycle.

COROLLARY 2. 2. *If D is a weakly connected digraph which is not an oriented graph then $\Gamma(D)\equiv\Gamma_1(D)$ if and only if D contains exactly one semicycle and it is a symmetric pair of lines.*

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References

[1] R. CARMICHAEL, *Introduction to the theory of groups of finite order* (New York, 1937), p. 22.
[2] F. HARARY, R. NORMAN and D. CARTWRIGHT, *Structural Models: an introduction to the theory of directed graphs* (New York, 1965).
[3] D. KÖNIG, *Theorie der endlichen und unendlichen Graphen* (Leipzig, 1936); reprinted N. Y., 1950, p. 53.
[4] G. SABIDUSSI, Loewy-groupoids related to linear graphs, *Amer. J. Math.*, **76** (1954), pp. 477—487.
[5] H. WHITNEY, Congruent graphs and the connectivity of graphs, *Amer. J. Math.*, **54** (1932), pp. 150—168.

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BEMERKUNG ZUR STARKEN SUMMATION DER FOURIERREIHEN

Von

K. TANDORI (Szeged), korrespondierendes Mitglied der Akademie

Einleitung

A. ZYGMUND [2] hat die folgende Behauptung bewiesen:

Es sei $r>0$. *Ist* $f(x)\in L(0, 2\pi)$, *dann gilt*

$$\frac{1}{n+1}\sum_{k=0}^{n}|s_k(f;x)-f(x)|^r\to 0 \qquad (n\to\infty)$$

fast überall, wobei $s_k(f;x)$ *die k-te Partialsumme der Fourierreihe der Funktion* $f(x)$ *bezeichnet.*

In dieser Note beweisen wir, daß diese Behauptung in gewissem Sinne unverbesserbar ist. Es gilt nämlich der folgende Satz.

SATZ. *Es sei* $\{\lambda_n\}$ *eine Folge von positiven Zahlen mit* $\lambda_n\to\infty$ $(n\to\infty)$. *Dann gibt es eine Funktion* $f(x)\in L(0, 2\pi)$ *derart, daß*

$$\overline{\lim_{n\to\infty}}\frac{1}{n+1}\sum_{k=0}^{n}|s_k(f;x)-f(x)|^{\lambda_k}=\infty \tag{1}$$

fast überall gilt.

Mit diesem Satz geben wir Antwort auf ein Problem von P. TURÁN. Der Beweis dieses Satzes gründet sich auf das bekannten Beispiel von A. N. KOLMOGOROFF [1].

§ 1. Hilfssätze

Im Folgenden bezeichnen $c_1, c_2, \ldots$ positive absolute Konstanten.

HILFSSATZ I. *Es sei* $\{b_n\}_1^N$ *eine abnehmende Folge von positiven Zahlen mit*

$$nb_n\leqq A \qquad (n=1, \ldots, N), \tag{2}$$

dann gilt

$$\left|\sum_{n=1}^{N}b_n\sin nx\right|\leqq c_1 A \qquad (-\infty<x<\infty).$$

BEWEIS DES HILFSSATZES I. Der Beweis dieses Hilfssatzes geht mit bekannter Methode. (Siehe z.B. A. ZYGMUND [3], Vol. I., S. 182—183.) Es sei $0<x\leqq\pi/4$,

und $v=[1/x]+1$. ($[1/x]$ bezeichnet den ganzen Teil von $1/x$.) Dann ist $v>1$ und $1/v<x\leqq 1/(v-1)$. Wir setzen

$$\sum_{n=1}^{N} b_n \sin nx = \sum_{n=1}^{v-1} b_n \sin nx + \sum_{n=v}^{N} b_n \sin nx = A_1(x)+A_2(x). \tag{3}$$

Auf Grund der Ungleichung $|\sin nx|\leqq n|x|$ $(-\infty<x<\infty;\ n=1, 2, \ldots)$ und von (2) folgt

$$|A_1(x)| \leqq x \sum_{n=1}^{v-1} nb_n \leqq A. \tag{4}$$

Durch Abelsche Umformung erhalten wir

$$A_2(x) = \sum_{n=v}^{N-1} (b_n - b_{n-1})\left(\tilde{D}_n(x) - \tilde{D}_{v-1}(x)\right) + b_N\left(\tilde{D}_N(x) - \tilde{D}_{v-1}(x)\right), \tag{5}$$

wobei

$$\tilde{D}_n(x) = \sin x + \ldots + \sin nx = \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

die n-te konjugierte Dirichletsche Kernfunktion bezeichnet. Auf Grund der bekannten Ungleichung

$$|\tilde{D}_n(x)| \leqq \frac{c_2}{|x|} \qquad (|x|\leqq\pi;\quad n=1, 2, \ldots)$$

und der Monotonität der Folge $\{b_n\}$ aus (2) und (5) ergibt sich

$$|A_2(x)| \leqq \frac{c_3}{x}\left(\sum_{n=v}^{N-1}(b_n - b_{n+1}) + b_N\right) \leqq c_3 v b_v \leqq c_3 A. \tag{6}$$

Für $\pi/4<x\leqq\pi$ gilt $|\tilde{D}_n(x)|\leqq c_4$ $(|x|\leqq\pi; n=1, 2, \ldots)$, und durch Abelsche Umformung erhalten wir

$$\left|\sum_{n=1}^{N} b_n \sin nx\right| \leqq \sum_{n=1}^{N-1}(b_n - b_{n+1})\,|\tilde{D}_n(x)| + b_N|\tilde{D}_N(x)| \leqq c_4 A. \tag{7}$$

Aus (3), (4), (6) und (7) ergibt sich die Behauptung für $0<x\leqq\pi$ mit $c_1=\max(1+c_3, c_4)$. Da die Behauptung für $x=0$ mit jeder positiven Zahl c_1 gilt und die abgeschätzte Summe eine ungerade Funktion ist, besteht die Behauptung überall.

Damit ist der Hilfssatz I bewiesen.

Es sei λ eine positive ganze Zahl. Durch partielle Integration erhalten wir

$$I_\lambda(m) = \int_0^{2\pi} (\sin mx)^{2\lambda}\,dx = \int_0^{2\pi} (\cos mx)^{2\lambda}\,dx = \frac{1.3\ldots(2\lambda-1)}{2.4\ldots(2\lambda)}\,2\pi.$$

Durch Anwendung der Stirlingschen Formel

$$\lim_{\lambda\to\infty}\frac{\lambda!}{\sqrt{2\pi\lambda}\left(\frac{\lambda}{e}\right)^{\lambda}}=1$$

ergibt sich

$$\frac{c_5}{\sqrt{\lambda}}\leqq I_\lambda(m)\leqq\frac{c_6}{\sqrt{\lambda}}\qquad(m=1,2,\ldots;\quad\lambda\geqq\lambda_0). \tag{8}$$

Es seien $M_1, M_2 (M_1<M_2)$ positive ganze Zahlen und

$$E_1(\lambda, M_1, M_2)=\left\{x:\sum_{m=M_1+1}^{M_2}(\sin mx)^{2\lambda}\geqq\frac{c_5(M_2-M_1)}{4\pi\sqrt{\lambda}};\quad 0\leqq x<2\pi\right\},$$

$$E_2(\lambda, M_1, M_2)=\left\{x:\sum_{m=M_1+1}^{M_2}(\cos mx)^{2\lambda}\geqq\frac{4c_6(M_2-M_1)}{c_5};\quad 0\leqq x<2\pi\right\}.$$

HILFSSATZ II. *Für alle positiven ganzen Zahlen* λ, M_1, M_2 $(\lambda\geqq\lambda_0, M_1<M_2)$ *gelten*

$$\operatorname{mes}\left(E_1(\lambda, M_1, M_2)\right)\geqq\frac{c_5}{2\sqrt{\lambda}},\qquad \operatorname{mes}\left(E_2(\lambda, M_1, M_2)\right)\leqq\frac{c_5}{4\sqrt{\lambda}}.$$

BEWEIS DES HILFSSATZES II. Aus (8) folgt

$$\frac{c_5(M_2-M_1)}{\sqrt{\lambda}}\leqq\sum_{m=M_1+1}^{M_2}I_\lambda(m)\leqq\frac{c_6(M_2-M_1)}{\sqrt{\lambda}}. \tag{9}$$

Daraus ergibt sich $\operatorname{mes}\left(E_2(\lambda, M_1, M_2)\right)4c_6(M_2-M_1)/c_5\leqq c_6(M_2-M_1)/\sqrt{\lambda}$, woraus die zweite Ungleichung des Hilfssatzes II folgt. Weiterhin aus (9) und aus der Abschätzung

$$\sum_{n=M_1+1}^{M_2}(\sin mx)^{2\lambda}\leqq M_2-M_1\qquad(0\leqq x\leqq 2\pi)$$

erhalten wir

$$\frac{c_5(M_2-M_1)}{\sqrt{\lambda}}\leqq\operatorname{mes}\left(E_1(\lambda, M_1, M_2)\right)(M_2-M_1)+\frac{2\pi c_5(M_2-M_1)}{4\pi\sqrt{\lambda}},$$

woraus die erste Ungleichung des Hilfssatzes II sich ergibt.

Damit haben wir den Hilfssatz II bewiesen.

Für eine positive ganze Zahl m bezeichne $n=n(m)$ die größte positive ganze Zahl mit

$$m^2=\left[\frac{\log n}{16\log\log n}\right]. \tag{10}$$

(Im Folgenden bezeichnet $\log x$ den Logarithmus mit der Basis e.) Offensichtlich gilt $n\to\infty$ für $m\to\infty$.

HILFSSATZ III. *Es sei c eine positive Zahl. Ist m genügend groß, dann gilt die Abschätzung*

$$\int_c^n \frac{(\log x)^{2m^2}}{x^2}\,dx \geqq \frac{(2m^2)!}{2c} \qquad (m \geqq \lambda_0).$$

BEWEIS DES HILFSSATZES III. Durch partielle Integration bekommen wir

$$J_{2\lambda} = \int_c^n \frac{(\log x)^{2\lambda}}{x^2}\,dx = 2\lambda J_{2\lambda-1} - \frac{(\log x)^{2\lambda}}{x}\Bigg|_c^n \qquad (\lambda = 1, 2, \ldots).$$

Durch vollständige Induktion ergibt sich daraus

$$J_{2\lambda} = -(2\lambda)!\left(\frac{1}{x} + \frac{1}{1!}\frac{\log x}{x} + \ldots + \frac{1}{(2\lambda)!}\frac{(\log x)^{2\lambda}}{x}\right)\Bigg|_c^n \qquad (\lambda = 1, 2, \ldots),$$

und so gilt

$$I_{2m^2} \geqq (2m^2)!\left(\frac{1}{c} - \frac{1}{n}\left(1 + \frac{\log n}{1!} + \ldots + \frac{(\log n)^{2m^2}}{(2m^2)!}\right)\right). \tag{11}$$

Auf Grund von (10) gilt

$$\log \frac{4m^2(\log n)^{2m^2}}{n} = \log 4 + \log m^2 + 2m^2 \log\log n - \log n \leqq$$

$$\leqq \log 4 + \log\log n + \frac{\log n}{4} - \log n \leqq \log \frac{1}{2c},$$

und so ist

$$\frac{1}{n}\left(1 + \frac{\log n}{1!} + \ldots + \frac{(\log n)^{2m^2}}{(2m^2)!}\right) \leqq \frac{4m^2(\log n)^{2m^2}}{n} \leqq \frac{1}{2c} \tag{12}$$

für genügend großes m.

Die Behauptung des Hilfssatzes III folgt aus (11) und (12).

Eine Untermenge von $[0, 2\pi)$ wird einfach genannt, wenn sie die Vereinigung endlichvieler Intervalle ist, derer Endpunkte rationale Vielfachen von π sind.

HILFSSATZ IV. *Für eine genügend große positive ganze Zahl $m (m \geqq m_0)$ gibt es ein trigonometrisches Polynom $T_m(x)$ und eine einfache Menge G_m mit folgenden Eigenschaften:*

Es gelten

$$\int_0^{2\pi} T_m(x)\,dx = 0, \tag{13}$$

$$\int_0^{2\pi} |T_m(x)|\,dx \leqq c_7, \tag{14}$$

$$\operatorname{mes}(G_m) > \frac{c_8}{m}. \tag{15}$$

Weiterhin für jedes $x \in G_m$ gibt es einen Index $\mu = \mu(x) (\geqq 1)$ derart, daß

$$\frac{1}{\mu+1} \sum_{l=0}^{\mu} (s_l(T_m; x))^{2m^2} \geqq c_9 (4m \log^2 m)^{2m^2} \tag{16}$$

besteht.

BEWEIS DES HILFSSATZES IV. Für jede natürliche Zahl i bezeichnet $D_i^*(x)$ bzw. $K_i(x)$ die i-te modifizierte Dirichletsche Kernfunktion bzw. die i-te Fejérsche Kernfunktion:

$$D_i^*(x) = \frac{1}{2} + \cos x + \ldots + \cos (i-1)x + \frac{\cos ix}{2} = \frac{\sin ix}{2 \operatorname{tg} \frac{x}{2}},$$

$$K_i(x) = \frac{1}{2} + \sum_{l=1}^{i} \left(1 - \frac{l}{i+1}\right) \cos lx = \frac{1}{2(i+1)} \left(\frac{\sin (n+1) \frac{x}{2}}{\sin \frac{x}{2}} \right)^2 .$$

Für $D_i^*(x)$ bzw. für $K_i(x)$ gelten die folgenden Abschätzungen

$$|D_i^*(x)| \leqq c_{10} i \qquad (-\infty < x < \infty), \tag{17}$$

$$|D_i^*(x)| \leqq \frac{c_{11}}{|x|} \quad (|x| \leqq \pi), \qquad |D_i^*(x)| \leqq c_{12} i \quad \left(|x| \leqq \frac{\pi}{2i}\right), \tag{18}$$

$$0 \leqq K_i(x) \leqq \frac{c_{13}}{ix^2} \qquad (|x| \leqq \pi), \tag{19}$$

$$\int_0^{2\pi} K_i(x)\, dx = \pi \tag{20}$$

für jedes $i = 1, 2, \ldots$.

Es sei

$$P_{2i}(x) = \frac{1}{2} + \sum_{l=1}^{i} \cos lx + \sum_{l=i+1}^{2i} \left(1 - \frac{l-i}{i+1}\right) \cos lx = \frac{2i+1}{i+1} K_{2i}(x) - \frac{i}{i+1} K_i(x)$$

die $2i$-te de la Vallée Poussinsche Kernfunktion ($i = 1, 2, \ldots$). Aus (19) und (20) folgt

$$\int_0^{2\pi} |P_{2i}(x)|\, dx \leqq c_{14} \qquad (i = 1, 2, \ldots). \tag{21}$$

Es sei

$$a = c_5/16mn, \tag{22}$$

und $\{\nu(i)\}_1^n$ eine Folge von positiven ganzen Zahlen mit

$$\nu(i)+2 \leqq \nu(i+1) \qquad (i=1, \ldots, n-1), \tag{23}$$

$$2^{\nu(1)}a^2 \geqq 1. \tag{24}$$

Wir setzen

$$T_m(x) = \frac{1}{n}\sum_{i=1}^{n} P_{2\cdot 2^{\nu(i)}n}\left(x-\frac{\pi}{2n}i\right)-\frac{1}{2}.$$

Nach der Definition von $P_{2i}(x)$ gilt (13) offensichtlich. Weiterhin folgt (14) aus (21).

Durch einfache Rechnung erhalten wir

$$s_l(T_m; x) = \frac{1}{n}\sum_{i=1}^{i_0} P_{2\cdot 2^{\nu(i)}n}\left(x-\frac{\pi}{2n}i\right)+\frac{1}{n}\sum_{i=i_0+1}^{n} D_l^*\left(x-\frac{\pi}{2n}i\right)+ \tag{25}$$

$$+\frac{1}{2}\left(\sum_{i=i_0+1}^{n}\cos l\left(x-\frac{\pi}{2n}i\right)-\frac{1}{2}\right) = B_1(x)+B_2(x)+B_3(x)$$

$$(2.2^{\nu(i_0)}n < l \leqq 2^{\nu(i_0+1)}n; \quad 1 \leqq i_0 \leqq n).$$

Offensichtlich gilt

$$|B_3(x)| \leqq 1 \qquad (-\infty < x < \infty). \tag{26}$$

Ist $\pi(i_0-1)/2n+a \leqq x \leqq \pi i_0/2n-a$, dann folgt

$$|B_1(x)| \leqq \frac{c_{13}}{n}\sum_{i=1}^{i_0}\frac{1}{2.2^{\nu(i)}n\left(x-\frac{\pi}{2n}i\right)^2} \leqq$$

$$\leqq c_{15}\left(\sum_{i=1}^{i_0-2}\frac{1}{i^2}+\frac{1}{2^{\nu(i_0-1)}a^2}+\frac{1}{2^{\nu(i_0)}a^2}\right) \qquad (i_0>1)$$

$\left(\text{im Falle } i_0=2 \text{ soll man } \sum_{i=1}^{i_0-2}\frac{1}{i^2}=0 \text{ setzen}\right)$ bzw.

$$|B_1(x)| \leqq \frac{c_{13}}{2^{\nu(1)}a^2} \qquad (i_0=1)$$

aus (19), und so gilt

$$|B_1(x)| \leqq c_{16} \qquad \left(\frac{\pi(i_0-1)}{2n}+a \leqq x \leqq \frac{\pi i_0}{2n}-a\right) \tag{27}$$

wegen (23) und (24).

Durch einfache Rechnung bekommen wir

$$B_2(x) = \frac{1}{2n} \sum_{i=i_0+1}^{n} \frac{\sin l\left(x-\frac{\pi}{2n}i\right)}{\operatorname{tg}\frac{1}{2}\left(x-\frac{\pi}{2n}i\right)} = \tag{28}$$

$$= \frac{1}{2n} \sin\left(x-\frac{\pi}{2n}i_0\right) \sum_{i=i_0+1}^{n} \frac{\cos(i-i_0)l\frac{\pi}{2n}}{\operatorname{tg}\frac{1}{2}\left(x-\frac{\pi}{2n}i\right)} -$$

$$-\frac{1}{2n}\cos\left(x-\frac{\pi}{2n}i_0\right) \sum_{i=i_0+1}^{n} \frac{\sin(i-i_0)l\frac{\pi}{2n}}{\operatorname{tg}\frac{1}{2}\left(x-\frac{\pi}{2n}i\right)} = C_1(x)+C_2(x).$$

Im Falle $\pi(i_0-1)/2n \leqq x \leqq \pi i_0/2n$ sind die Bedingungen des Hilfssatzes I für die Folge

$$b_j = -\frac{1}{\operatorname{tg}\frac{1}{2}\left(x-\frac{\pi}{2n}i_0-\frac{\pi}{2n}j\right)} \qquad (j=1, \ldots, n-i_0)$$

mit $A=c_{17}n$ offensichtlich erfüllt, und so erhalten wir

$$|C_2(x)| \leqq c_{18} \qquad \left(\frac{\pi(i_0-1)}{2n} \leqq x \leqq \frac{\pi i_0}{2n}\right). \tag{29}$$

Für $l=4np+q$ $(p \geqq 1, 0 \leqq q < 4n)$ ergibt sich durch Abelsche Umformung

$$\sum_{i=i_0+1}^{n} \frac{\cos(i-i_0)l\frac{\pi}{2n}}{\operatorname{tg}\frac{1}{2}\left(x-\frac{\pi}{2n}i\right)} = \sum_{i=i_0+1}^{n} \frac{\cos(i-i_0)q\frac{\pi}{2n}}{\operatorname{tg}\frac{1}{2}\left(x-\frac{\pi}{2n}i\right)} = \tag{30}$$

$$= \sum_{i=i_0+1}^{n} \left(\frac{1}{\operatorname{tg}\frac{1}{2}\left(x-\frac{\pi}{2n}i\right)} - \frac{1}{\operatorname{tg}\frac{1}{2}\left(x-\frac{\pi}{2n}(i+1)\right)}\right)\left(D^*_{i-i_0}\left(\frac{q\pi}{2n}\right)-\frac{1}{2}+\frac{1}{2}\cos(i-i_0)\frac{q\pi}{2n}\right)+$$

$$+\frac{1}{\operatorname{tg}\frac{1}{2}\left(x-\frac{\pi}{2}\right)}\left(D^*_{n-i_0}\left(\frac{q\pi}{2n}\right)-\frac{1}{2}+\frac{1}{2}\cos(n-i_0)\frac{q\pi}{2n}\right) = S_1(x)+S_2(x).$$

Auf Grund von (17) folgt leicht

$$|S_2(x)| \leqq c_{19}n \qquad \left(\frac{\pi(i_0-1)}{2n} \leqq x \leqq \frac{\pi i_0}{2n};\quad 1 \leqq i_0 \leqq \frac{n}{2}\right). \tag{31}$$

Weiterhin erhalten wir durch einfache Rechnung

$$(32)\qquad c_{20}\frac{n}{(i-i_0)^2} \leqq \left| \frac{1}{\operatorname{tg}\frac{1}{2}\left(x-\frac{\pi}{2n}i\right)} - \frac{1}{\operatorname{tg}\frac{1}{2}\left(x-\frac{\pi}{2n}(i+1)\right)} \right| \leqq c_{21}\frac{n}{(i-i_0)^2}$$

$$\left(\frac{\pi(i_0-1)}{2n} \leqq x \leqq \frac{\pi i_0}{2n};\quad i_0 < i \leqq n\right).$$

Für genügend großes n/q aus (18) und (32) ergibt sich

$$(33)\qquad |S_1(x)| \geqq c_{22}n \sum_{i=i_0+1}^{i_0+[n/q]} \frac{1}{i-i_0} - c_{23}\frac{n^2}{q} \sum_{i=i_0+[n/q]+1}^{n-1} \frac{1}{(i-i_0)^2} -$$

$$-c_{24}n \sum_{i=i_0+1}^{n} \frac{1}{(i-i_0)^2} \geqq c_{25}n\log\frac{n}{q}$$

$$\left(l = 4np+q;\quad p \geqq 1;\quad 1 \leqq q < 4n;\quad \frac{n}{q} \geqq \bar{c};\quad \frac{\pi(i_0-1)}{2n} \leqq x \leqq \frac{\pi i_0}{2n}\right).$$

Endlich folgt aus (25), (26), (27), (28), (29), (30), (31) und (33)

$$(34)\qquad |s_l(T_m; x)| \geqq c_{26}\left|\sin l\left(x-\frac{\pi}{2n}i_0\right)\right|\log\frac{n}{q} - c_{27}$$

$$\left(2.2^{\nu(i_0)}n < l \leqq 2^{\nu(i_0+1)}n;\quad l = 4np+q;\quad 1 \leqq q < 4n;\quad \frac{n}{q} \geqq \bar{c};\right.$$

$$\left.\frac{\pi(i_0-1)}{2n}+a \leqq x \leqq \frac{\pi i_0}{2n}-a;\quad 1 \leqq i_0 \leqq \frac{n}{2}\right).$$

Ist die positive ganze Zahl c genügend groß, dann gilt

$$(35)\qquad \frac{c_5}{4\pi}\left(\frac{\cos\frac{\pi}{2c}}{2}\right)^{2m^2}\frac{1}{m} - \frac{4c_6}{c_5}\left(\sin\frac{\pi}{2c}\right)^{2m^2} \geqq r^{2m^2} \qquad (m = 1, 2, \ldots)$$

mit einer positiven Zahl r, wobei c_5 und c_6 die Konstanten im Hilfssatz II bezeichnen. Wir können $c \geqq \max(\bar{c}, 3)$ annehmen. Offensichtlich gilt

$$(36)\qquad \left|\sin(4np+q)\left(x-\frac{\pi}{2n}i_0\right)\right| \geqq \cos\frac{\pi}{2c}\,|\sin 4npx| - \sin\frac{\pi}{2c}\,|\cos 4npx|$$

$$\left(\frac{\pi(i_0-1)}{2n} \leqq x \leqq \frac{\pi i_0}{2n};\quad p \geqq 1;\quad 1 \leqq q \leqq \frac{n}{c}\right).$$

Mit den Bezeichnungen des Hilfssatzes II setzen wir

$$E = E_1(m^2, 2^{\nu(i_0)-1}, 2^{\nu(i_0+1)-2}) - E_2(m^2, 2^{\nu(i_0)-1}, 2^{\nu(i_0+1)-2}).$$

(Wegen (23) ist diese Definition richtig.) Nach dem Hilfssatz II gelten

$$\sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}} (\sin 4npx)^{2m^2} \leqq \frac{c_5}{2\pi m}(2^{\nu(i_0+1)-1} - 2^{\nu(i_0)}),$$

$$\sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}} (\cos 4npx)^{2m^2} \leqq \frac{4c_6}{2c_5}(2^{\nu(i_0+1)-1} - 2^{\nu(i_0)})$$

für $4nx \in E$ und für genügend großes $m (m \geqq \lambda_0)$. Aus diesen Ungleichungen erhalten wir auf Grund von (35)

$$\sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}} (\sin 4npx)^{2m^2}\left(\frac{\cos\frac{\pi}{2c}}{2}\right)^{2m^2} - \sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}} (\cos 4npx)^{2m^2}\left(\sin\frac{\pi}{2c}\right)^{2m^2} \geqq$$

$$\geqq (2^{\nu(i_0+1)-1} - 2^{\nu(i_0)})\left(\frac{c_5}{2\pi m}\left(\frac{\cos\frac{\pi}{2c}}{2}\right)^{2m^2} - \frac{4c_6}{c_5}\left(\sin\frac{\pi}{2c}\right)^{2m^2}\right) \geqq$$

$$\geqq c_{28} 2^{\nu(i_0+1)} r^{2m^2}$$

für $4nx \in E$ und für $m \geqq \lambda_0$. Daraus folgt

$$\sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}}{}' \left(\left(\frac{\cos\frac{\pi}{2c}}{2}\sin 4npx\right)^{2m^2} - \left(\sin\frac{\pi}{2c}\cos 4npx\right)^{2m^2}\right) \geqq \tag{37}$$

$$\geqq c_{28} 2^{\nu(i_0+1)} r^{2m^2} \qquad (4nx \in E; \quad m \geqq \lambda_0),$$

wobei $\sum'$ bedeutet, daß es nur für solche p zu summieren ist, für die $2^{\nu(i_0)-1} < p \leqq 2^{\nu(i_0+1)-2}$ und $\frac{1}{2}\cos\frac{1}{2c}|\sin 4npx| \geqq \sin\frac{\pi}{2c}|\cos 4npx|$ bestehen. Auf Grund von (36) und der Ungleichung

$$|\alpha+\beta|^s \leqq 2^s(|\alpha|^s + |\beta|^s) \qquad (s > 0) \tag{38}$$

erhalten wir

$$\left(\sin(4np+q)\left(x - \frac{\pi}{2n}i_0\right)\right)^{2m^2} \geqq \left(\frac{\cos\frac{\pi}{2c}}{2}\sin 4npx\right)^{2m^2} - \left(\sin\frac{\pi}{2c}\cos 4npx\right)^{2m^2}$$

$$\left(\frac{\pi(i_0-1)}{2n} \leqq x \leqq \frac{\pi i_0}{2n}; \quad p \geqq 1; \quad 1 \leqq q \leqq \frac{n}{c}\right).$$

Daraus und aus (37) folgt

$$(39)\qquad \sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}} \left(c_{26}\sin(4np+q)\left(x-\frac{\pi}{2n}i_0\right)\log\frac{n}{q}\right)^{2m^2} \geqq$$

$$\geqq \sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}}{}' \left(c_{26}\sin(4np+q)\left(x-\frac{\pi}{2n}i_0\right)\log\frac{n}{q}\right)^{2m^2} \geqq$$

$$\geqq c_{28}2^{\nu(i_0+1)}\varrho^{2m^2}\left(\log\frac{n}{q}\right)^{2m^2}$$

$$\left(\frac{\pi(i_0-1)}{2n}\leqq x\leqq\frac{\pi i_0}{2n};\quad 4nx\in E;\quad m\geqq\lambda_0;\quad 1\leqq q\leqq\frac{n}{c}\right),$$

wobei $\varrho=r/c_{26}$.

Wir setzen

$$E_i=\left\{x: 4nx\in E;\quad \frac{\pi(i-1)}{2n}\leqq x\leqq\frac{\pi i}{2n}\right\}\qquad (i=1,\ldots,n).$$

Auf Grund des Hilfssatzes II ist

$$(40)\qquad \operatorname{mes}(E_i)=\frac{\operatorname{mes}(E)}{n}\geqq\frac{c_5}{4mn}\qquad (i=1,\ldots,n).$$

Aus (39) folgt

$$(41)\qquad \sum_{q=1}^{[n/c]}\sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}}\left(\frac{1}{2}c_{26}\sin(4np+q)\left(x-\frac{\pi}{2n}i_0\right)\log\frac{n}{q}\right)^{2m^2}\geqq$$

$$\geqq c_{28}2^{\nu(i_0-1)}\left(\frac{\varrho}{2}\right)^{2m^2}\sum_{q=1}^{[n/c]}\left(\log\frac{n}{q}\right)^{2m^2}$$

$$(x\in E_{i_0};\quad m\geqq\lambda_0).$$

Ist $m\geqq m_1=\dfrac{\log 2c}{16\log\log 2c}$, dann gilt

$$(42)\qquad \sum_{q=1}^{[n/c]}\left(\log\frac{n}{q}\right)^{2m^2}\geqq\int_1^{[n/c]}\left(\log\frac{n}{t}\right)^{2m^2}dt\geqq\frac{n}{2}\int_c^n\frac{(\log x)^{2m^2}}{x^2}dx.$$

Aus (41) und (42) ergibt sich durch Anwendung des Hilfssatzes II

$$(43)\qquad \sum_{q=1}^{[n/c]}\sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}}\left(\frac{1}{2}c_{26}\sin(4np+q)\left(x-\frac{\pi}{2n}i_0\right)\log\frac{n}{q}\right)^{2m^2}\geqq$$

$$\geqq c_{29}2^{\nu(i_0+1)}n\left(\frac{\varrho}{2}\right)^{2m^2}(2m^2)!$$

$$(x\in E_{i_0};\quad m\geqq m_2=\max(\lambda_0,m_1)).$$

Für genügend großes m $(m \geqq m_3 \geqq m_2)$ gilt

$$c_{29}\left(\frac{\varrho}{2}\right)^{2m^2}(2m^2)! - c_{27}^{2m^2} \geqq \left((2m^2)!\right)^{3/4} \tag{44}$$

auf Grund der Stirlingschen Formel. Aus (43), (44) und aus der Ungleichung

$$\sum_{q=1}^{[n/c]} \sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}} c_{27}^{2m^2} \leqq c_{27}^{2m^2} 2^{\nu(i_0+1)} n$$

erhalten wir

$$\sum_{q=1}^{[n/c]} \sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}} \left(\frac{1}{2} c_{26} \sin(4np+q)\left(x-\frac{\pi}{2n} i_0\right)\log\frac{n}{q}\right)^{2m^2} -$$

$$- \sum_{q=1}^{[n/c]} \sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}} c_{27}^{2m^2} \geqq 2^{\nu(i_0+1)} n\left((2m^2)!\right)^{3/4}$$

und so ist

$$\sum_{q=1}^{[n/c]} \sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}}{}' \left(\left(\frac{1}{2} c_{26} \sin(4np+q)\left(x-\frac{\pi}{2n} i_0\right)\log\frac{n}{q}\right)^{2m^2} - c_{27}^{2m^2}\right) \geqq \tag{45}$$

$$\geqq 2^{\nu(i_0+1)} n\left((2m^2)!\right)^{3/4}$$

$$(x \in E_{i_0};\quad m \geqq m_3),$$

wobei es nur für solche p und q zu summieren ist, für die

$$\left(\frac{1}{2} c_{26} \sin(4np+q)\left(x-\frac{\pi}{2n} i_0\right)\log\frac{n}{q}\right)^{2m^2} \geqq c_{27}^{2m^2}$$

besteht. Aus (34), (38) und (39) erhalten wir

$$\sum_{q=1}^{[n/c]} \sum_{p=2^{\nu(i_0)-1}+1}^{2^{\nu(i_0+1)-2}} \left(s_{4np+q}(T_m; x)\right)^{2m^2} \geqq 2^{\nu(i_0+1)} n\left((2m^2)!\right)^{3/4} \tag{46}$$

$$\left(x \in E_{i_0};\quad \frac{\pi(i_0-1)}{2n} + a \leqq x \leqq \frac{\pi i_0}{2n} - a;\quad 1 \leqq i_0 \leqq \frac{n}{2};\quad m \geqq m_3\right).$$

Wir setzen

$$F_i = E_i \cap \left[\frac{\pi(i-1)}{2n} + a,\ \frac{\pi i}{2n} - a\right] \qquad (i = 1, \ldots, n)$$

und

$$G_m = \bigcup_{i=1}^{[n/2]} F_i.$$

Aus (22) und (40) folgt

$$\operatorname{mes}(G_m) \geqq \left(\frac{c_5}{4mn} - 2a\right)\left[\frac{n}{2}\right] > \frac{c_5}{32m}.$$

Also ist (15) mit $c_8 = c_5/32$ erfüllt.

Aus (46) ergibt sich

$$\frac{1}{2^{\nu(i_0+1)}n+1}\sum_{l=0}^{2^{\nu(i_0+1)}n}(s_l(T_m;x))^{2m^2}\geqq c_{30}((2m^2)!)^{3/4} \tag{47}$$

$$\left(x\in F_{i_0};\quad 1\leqq i_0\leqq\frac{n}{2};\quad m\geqq m_3\right).$$

Durch Anwendung der Stirlingschen Formel und auf Grund von (10) bekommen wir

$$\frac{3}{4}\log((2m^2)!)=\frac{3}{4}\log\sqrt{\pi}2m\left(\frac{2m^2}{e}\right)^{2m^2}+\frac{3}{4}\log\frac{(2m^2)!}{\sqrt{\pi}2m\left(\frac{2m^2}{e}\right)^{2m^2}}\geqq \tag{48}$$

$$\geqq\frac{9}{16}\log m\left(\frac{2m^2}{e}\right)^{2m^2}\geqq\frac{9}{4}m^2(\log m-2)\geqq$$

$$\geqq 2m^2(\log m+2\log\log m+4)=\log(4m(\log m)^2)^{2m^2}$$

für genügend großes m $(m\geqq m_0\geqq m_3)$. Aus (47) und (48) folgt

$$\frac{1}{2^{\nu(i_0+1)}n+1}\sum_{l=0}^{2^{\nu(i_0+1)}n}(s_l(T_m;x))^{2m^2}\geqq c_{30}(4m(\log m)^2)^{2m^2}$$

$$\left(x\in F_{i_0};\quad 1\leqq i_0\leqq\frac{n}{2};\quad m\geqq m_0\right).$$

Also ist (16) mit $c_9=c_{30}$ erfüllt.

Offensichtlich können wir annehmen, daß die Menge G_m einfach ist.

Damit haben wir Hilfssatz IV bewiesen.

§ 2. Beweis des Satzes

Wir wenden den Hilfssatz IV für $m=m_0, m_0+1, \ldots$ an. Die entsprechenden trigonometrischen Polynome bzw. die entsprechenden Mengen bezeichnen wir mit $T_m(x)$ bzw. mit G_m $(m\geqq m_0)$. Es sei $r(m)$ die Ordnung von $T_m(x)$ und

$$G_m=\bigcup_{j=1}^{R(m)}(a_j(m),b_j(m))\qquad(m\geqq m_0),$$

wobei $(a_j(m), b_j(m))$ $(1\leqq j\leqq R(m))$ für jedes $m(\geqq m_0)$ paarweise disjunkte Intervalle sind.

Es sei m^* so groß $(m^*\geqq m_0)$, daß $c_9 2^{2m^2}\geqq 4$ $(m\geqq m^*)$ gilt. Nach dem Hilfssatz IV gibt es für jedes $x\in G_m$ einen Index $\mu=\mu(x)$, für die (16) gilt. Es sei

$$S(m;x)=\left\{l: 1\leqq l\leqq\mu;\quad\frac{|s_l(T_m;x)|}{2m\log^2 m}\geqq 1\right\}.$$

Aus (16) folgt

$$\frac{1}{\mu+1}\sum_{l\in S(m;x)}\left(\frac{s_l(T_m;x)}{2m(\log m)^2}\right)^{2m^2}\leqq c_9\frac{2^{2m^2}}{2}. \tag{49}$$

Es sei $(1\leqq)u_{m^*}<u_{m^*+1}<\ldots$ eine Folge von ganzen Zahlen mit

$$\lambda_n\geqq 2m^2 \qquad (n\geqq u_m;\ m\geqq m^*). \tag{50}$$

(Wegen der Annahme $\lambda_n\to\infty$ $(n\to\infty)$ kann man eine solche Folge $\{u_m\}$ bestimmen.) Weiterhin definieren wir durch Rekursion eine Folge von ganzen Zahlen $(1\leqq)N_{m^*}<$ $<N_{m^*+1}<\ldots$, für die

$$r(m)N_m<N_{m+1} \qquad (m\geqq m^*), \tag{51}$$

$$N_m>u_m \qquad (m\geqq m^*) \tag{52}$$

bestehen, und für die $N_{m+1}a_j(m)$, $N_{m+1}b_j(m)$ $(1\leqq j\leqq R(m))$ ganzzahlige Vielfachen von 2π sind $(m\geqq m^*)$.

Wir setzen $Q_m(x)=T_m(N_mx)$ $(m\geqq m^*)$. Aus (14) folgt

$$\sum_{m=m^*}^{\infty}\int_0^{2\pi}\frac{|Q_m(x)|}{m(\log m)^2}<\infty.$$

Nach dem Satz von B. Levi ist also die Reihe

$$\sum_{m=m^*}^{\infty}\frac{Q_m(x)}{m(\log m)^2}$$

fast überall absolut konvergent, und ihre Summe $f(x)$ gehört zu $L(0,2\pi)$. Auf Grund von (13) und (51) haben die trigonometrischen Polynome $Q_k(x)$ und $Q_l(x)$ $(k\neq l)$ keine Glieder mit gemeinsamer Frequenz. So sind auf Grund des Lebesgueschen Satzes und der Definition von $Q_m(x)$

$$s_l(f;x)=\sum_{k=m^*}^{m-1}\frac{Q_k(x)}{k(\log k)^2}+\frac{s_s(T_m;N_mx)}{m(\log m)^2} \tag{53}$$

$$\left(l=sN_m+\sigma;\quad s=1,\ldots,r(m);\quad \sigma=0,\ldots,N_m-1;\quad m\geqq m^*\right),$$

$$s_{N_m-1}(f;x)=\sum_{k=m^*}^{m-1}\frac{Q_k(x)}{k(\log k)^2} \qquad (m\geqq m^*). \tag{54}$$

Wir bezeichnen mit H die Menge von jenen $x(\in[0,2\pi))$, für die $s_{N_m-1}(f;x)\to$ $\to f(x)$ $(m\to\infty)$ besteht. Nach den obigen gilt

$$\text{mes}\,(H)=2\pi. \tag{55}$$

Es sei

$$H_m=\{x:\ N_mx\in G_m;\ 0\leqq x<2\pi\} \qquad (m\geqq m^*).$$

Aus (15) folgt

$$\text{mes}\,(H_m)>c_8/m \qquad (m\geqq m^*),$$

und so ist

$$\sum_{m=m^*}^{\infty}\text{mes}\,(H_m)=\infty. \tag{56}$$

Auf Grund der Definitionen von N_m und H_m sind die Mengen $H_m (m \geqq m^*)$ stochastisch unabhängig. Aus (56), durch Anwendung des zweiten Borel—Cantellischen Lemmas, ergibt sich

$$\text{mes}\left(\overline{\lim_{m\to\infty}} H_m\right) = 2\pi. \tag{57}$$

Es sei $x \in H$. Dann gibt es einen Index $\nu = \nu(x)$ mit

$$|s_{N_m-1}(f; x) - f(x)| \leqq 1 \qquad (m \geqq \nu). \tag{58}$$

Aus der Definition von $S(m; x)$ und aus (53), (54) und (58) folgt

$$|s_{sN_m-1+\sigma}(f; x) - f(x)| \geqq |s_{sN_m+\sigma}(f; x) - s_{N_m-1}(f; x)| - |s_{N_m-1}(f; x) - f(x)| \geqq$$
$$\geqq |s_s(T_m; N_m x)| - 1 \geqq 1$$
$$(x \in H_m \cap H;\ s \in S(m; x);\ \sigma = 0, \ldots, N_m - 1;\ m \geqq \nu).$$

Daraus, auf Grund von (50) und (52), ergibt sich

$$\sum_{s\in S(m;\,x)} \sum_{\sigma=0}^{N_m-1} |s_{sN_m+\sigma}(f; x) - f(x)|^{\lambda_{sN_m+\sigma}} \geqq \tag{59}$$

$$\geqq \sum_{s\in S(m;\,x)} \sum_{\sigma=0}^{N_m-1} \left(s_{sN_m+\sigma}(f; x) - f(x)\right)^{2m^2} \qquad (x \in H_m \cap H;\quad m \geqq \nu).$$

Aus der Minkowskischen Ungleichung erhalten wir

$$\left\{\sum_{s\in S(m;\,x)} \sum_{\sigma=0}^{N_m-1} \left(s_{sN_m+\sigma}(f; x) - f(x)\right)^{2m^2}\right\}^{1/2m^2} \geqq$$

$$\geqq \left\{\sum_{s\in S(m;\,x)} \sum_{\sigma=0}^{N_m-1} \left(s_{sN_m+\sigma}(f; x) - s_{N_m-1}(f; x)\right)^{2m^2}\right\}^{1/2m^2} -$$

$$- \left\{\sum_{s\in S(m;\,x)} \sum_{\sigma=0}^{N_m-1} \left(s_{N_m-1}(f; x) - f(x)\right)^{2m^2}\right\}^{1/2m^2}.$$

Daraus, auf Grund der Ungleichung (38), folgt

$$\sum_{s\in S(m;\,x)} \sum_{\sigma=0}^{N_m-1} \left(s_{sN_m+\sigma}(f; x) - f(x)\right)^{2m^2} \geqq \tag{60}$$

$$\geqq \sum_{s\in S(m;\,x)} \sum_{\sigma=0}^{N_m-1} \frac{1}{2}\left(s_{sN_m+\sigma}(f; x) - s_{N_m-1}(f; x)\right)^{2m^2} -$$

$$- \sum_{s\in S(m;\,x)} \sum_{\sigma=0}^{N_m-1} \left(s_{N_m-1}(f; x) - f(x)\right)^{2m^2}.$$

Weiterhin ergibt sich nach (53) und (54)

$$\sum_{s\in S(m;\,x)} \sum_{\sigma=0}^{N_m-1} \left(s_{sN_m+\sigma}(f; x) - s_{N_m-1}(f; x)\right)^{2m^2} = N_m \sum_{s\in S(m;\,x)} \left(\frac{s_s(T_m; N_m x)}{m(\log m)^2}\right)^{2m^2}. \tag{61}$$

Aus (49), (58), (59), (60) und (61) erhalten wir

$$(62)\qquad \frac{1}{N_m\mu+1}\sum_{l=0}^{N_m\mu}{}'|s_l(f;x)-f(x)|^{\lambda_l}\geqq$$

$$\geqq \frac{1}{N_m\mu+1}\sum_{s\in S(m;x)}\sum_{\sigma=0}^{N_m-1}{}'\left(s_{sN_m+\sigma}(f;x)-f(x)\right)^{2m^2}\geqq$$

$$\geqq \frac{1}{N_m\mu+1}\sum_{s\in S(m;x)}\sum_{\sigma=0}^{N_m-1}{}'\frac{1}{2}\left(s_{sN_m+\sigma}(f;x)-s_{N_m-1}(f;x)\right)^{2m^2}-\frac{\mu N_m}{N_m\mu+1}\geqq$$

$$\geqq \frac{N_m}{N_m\mu+1}\sum_{s\in S(m;x)}{}'\left(\frac{s_s(T_m;N_mx)}{2m(\log m)^2}\right)^{2m^2}-1\geqq c_9\frac{2^{2m^2}}{4}.$$

Ist $x\in H\cap\overline{\lim_{m\to\infty}}H_m$, dann gilt (62) für unendlich viele $m(\geqq\nu)$, woraus folgt, daß für die Funktion $f(x)$ die Relation (1) in dem Punkt x erfüllt wird. Aus (55) und (56) folgt endlich, daß (1) fast überall besteht.

Damit haben wir den Satz vollständig bewiesen.

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Schriftenverzeichnis

[1] A. N. KOLMOGOROFF, Une série de Fourier-Lebesgue divergente partout, *Comptes Rendus de l'Acad. Sci. Paris*, **183** (1926), S. 1327—1328.

[2] A. ZYGMUND, On the convergence and summability of power series on the circle of convergence, *Proceedings of London Math. Soc.*, **47** (1941), S. 326—350.

[3] A. ZYGMUND, *Trigonometric series* (Cambridge, 1959).

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (3—4), (1968), pp. 287—309.

EIN NEUES FORMALES SYSTEM DER DEONTISCHEN LOGIK

Von

I. RUZSA (Budapest)
(*Vorgelegt von* L. KALMÁR)

In meinem Aufsatz [3] habe ich ein formales System der deontischen Logik dargestellt, welches ich dort System **D** genannt habe. Im vorliegenden Aufsatz werde ich eine Weiterentwickelung des Systems **D** vorlegen, die ich als System **D*** bezeichnen werde.

Die Hauptunterschiede des Systems **D*** im Vergleich mit **D** sind die folgenden:

(1) Die Form der Primformeln des Systems **D** ist $T\alpha$ bzw. $P\alpha$; sie formalisieren elementare deontische *Aussagen* der Form „die Handlung α ist ausgeführt" bzw. „zulässig". In **D*** haben die Primformeln die Form $T\alpha/t$ bzw. $P\alpha/t$; sie formalisieren *Prädikate* der Form „die Handlung α ist im Zeitpunkt t ausgeführt" bzw. „zulässig", mit veränderlichen t. *Auf die Zeitvariablen* t der Primformeln $T\alpha/t$ bzw. $P\alpha/t$ sind auch Quantoren anwendbar.

(2) In **D*** ist auch der Gebrauch von Primformeln der Form $t<u$ zulässig, falls t und u Zeitvariablen sind; sie formalisieren Aussagen der Form „der Zeitpunkt t geht dem Zeitpunkt u voraus". Auf Formeln der Form $t<u$ haben aber die Quantoren keine Wirkung. Das System **D*** erweist sich so als ein spezieller Teil des einstelligen Prädikatenkalküls.

(3) Die Aussagevariablen $p, q, r, \ldots$ sind in **D*** auch zulässige Primformeln. So ermöglicht **D*** eine Formalisierung von Aussagen, die deontische und nicht deontische Komponente enthalten. Kurz gesagt: der Aussagenkalkül ist ein echter Teil des Systems **D***.

Somit ermöglicht das System **D*** eine Formalisierung der feineren Struktur von Aussagen, die deontische Komponente enthalten. Ferner meine ich, daß einige Probleme, deren Lösung in **D** noch streitig ist, im System **D*** ihre befriedigende Lösung erhalten.

Überdies enthält das System **D*** einige syntaktische Vereinfachungen gegenüber **D**. Darauf und auf sonstige Unterschiede der beiden Systeme werde ich während der Beschreibung des Systems **D*** hinweisen.

Im wesentlichen Teil meines Aufsatzes berufe ich mich nicht auf meine Arbeit [3]. Die vergleichenden Auseinandersetzungen über die Systeme **D** und **D*** werden vom Hauptteil des Aufsatzes durch Kleinschrift unterschieden. Diese Teile sind für die Beschreibung des Systems **D*** entbehrlich.

Meines Wissens nach ist mein Aufsatz der erste Versuch einer Verwendung von Quantoren in der deontischen Logik. Auch dieser Aufsatz beschränkt sich nur auf eine limitierte Quantifikation der zeitlichen Veränderung. Auf die Probleme der weiteren Ausdehnung der Quantifikation in der deontischen Logik beabsichtige ich in einem weiteren Aufsatz zurückzukommen.

Teil I des Aufsatzes gibt die syntaktische Beschreibung des Systems **D***. Unter II wird das Entscheidungsverfahren von **D*** dargestellt. Endlich behandle ich unter III einige Probleme der Interpretation und der Anwendungen von **D***.

I. DIE SYNTAKTISCHE BESCHREIBUNG DES SYSTEMS **D***

Grundzeichen

Variablen:

(1) *Aussagevariablen*; ihre Mitteilungszeichen sind $p, q, r, \ldots$;
(2) *Handlungsvariablen* (Aktvariablen); ihre Mitteilungszeichen sind $a, b, c, \ldots$;
(3) *Zeitvariablen*; ihre Mitteilungszeichen sind $t, u, v, \ldots$.

Konstanten:

(1) Operatoren für Handlungen: $\llcorner$, $\cap$, $\cup$;
(2) Operatoren für Aussagen: T, P, $\prec$, $\neg$, $\wedge$, $\vee$, $\supset$, $\sim$, $\exists$;
(3) runde Klammern (ihr Gebrauch wird im allgemeinen vermieden, wo ihre Weglassung kein Mißverständnis verursacht);
(4) das Trennzeichen „/".

Regelmäßige Zeichenfolgen

Terme:

(1) Ist a eine Handlungsvariable, so ist a ein Term;
(2) ist α ein Term, so ist auch $(\llcorner\alpha)$ ein Term;
(3) sind α und β Terme, so sind auch $(\alpha \cap \beta)$ und $(\alpha \cup \beta)$ Terme. — Andere Terme gibt es nicht.

Primformeln:

(1) Ist p eine Aussagevariable, so ist p eine Primformel;
(2) sind t und u Zeitvariablen, so ist $t \prec u$ eine Primformel;
(3) ist α ein Term und t eine Zeitvariable, so ist $T\alpha/t$ bzw. $P\alpha/t$ eine Primformel. — Andere Primformeln gibt es nicht.

Formeln:

(1) Jede Primformel ist eine Formel;
(2) ist Γ eine Formel, so ist auch $(\neg\Gamma)$ eine Formel;
(3) sind Γ und Δ Formeln, so sind auch $(\Gamma \wedge \Delta)$, $(\Gamma \vee \Delta)$, $(\Gamma \supset \Delta)$ und $(\Gamma \sim \Delta)$ Formeln;
(4) ist Γ eine Formel und t eine Zeitvariable, so ist auch $(\exists t\Gamma)$ eine Formel. — Andere Formeln gibt es nicht.

Axiome

(A1) Ist Γ eine Tautologie des Aussagenkalküls und entsteht Γ^* aus Γ durch Einsetzen von Formeln des Systems **D*** in ihre Variablen, so ist Γ^* ein Axiom.
(A2) Ist $\Gamma[\alpha]$ eine Formel, in der der Term α vorkommt, entsteht $\Gamma[\beta]$ aus $\Gamma[\alpha]$

durch Ersetzung des Terms α in einem seiner Vorkommen durch β und ist $\alpha=\beta$ nach der Booleschen Algebra gültig, so ist $(\Gamma[\alpha]\sim\Gamma[\beta])$ ein Axiom.

Sind α und β beliebige Terme, und sind t, u, v Zeitvariablen, so sind die folgenden Formeln (A3)—(A10) Axiome:

(A3) $T\alpha\cap\beta/t\sim(T\alpha/t\wedge T\beta/t)$
(A4) $T\alpha/t\sim\neg T\llcorner\alpha/t$
(A5) $P\alpha\cup\llcorner\alpha/t$
(A6) $P\alpha\cup\beta/t\sim(P\alpha/t\vee P\beta/t)$
(A7) $\neg P\alpha\cap\llcorner\alpha/t$
(A8) $\neg t<t$
(A9) $t<u\supset\neg u<t$
(A10) $(t<u\wedge u<v)\supset t<v$.

Ist Γ eine beliebige Formel und t eine Zeitvariable, so ist
(A11) $\Gamma\supset\exists t\Gamma$
ein Axiom. — Andere Axiome gibt es nicht.

Herleitbare Formeln

Definition der aktiven bzw. passiven Zeitvariablen:

Eine Zeitvariable t wird als in der Formel Γ *aktiv* bezeichnet, wenn Γ eine Teilformel $T\alpha/t$ oder $P\alpha/t$ enthält (mit beliebigem Term α), die innerhalb Γ nicht im Wirkungsbereich eines Quantors $\exists t$ steht.

Ist die Zeitvariable t in der Formel Γ *nicht aktiv*, so wird sie als in Γ *passiv* bezeichnet.

BEMERKUNG. Die Begriffe „Teilformel einer Formel" und „Wirkungsbereich eines Quantors" sind hier in dem üblichen Sinn gebraucht. Die aktive bzw. passive Zeitvariable ist ein Analogon der freien bzw. gebundenen Individuenvariablen des Prädikatenkalküls.

Definition der unmittelbaren Folge:

(MP) Die Formel Δ ist eine unmittelbare Folge der Formeln Γ und $\Gamma\supset\Delta$. *(Modus ponens.)*

(QS) Sind Γ und Φ Formeln, und ist t eine Zeitvariable, die in Φ passiv ist, so ist $\exists t\Gamma\supset\Phi$ eine unmittelbare Folge von $\Gamma\supset\Phi$. *(Quantorschluß.)* — Andere Fälle als (MP) und (QS) der unmittelbaren Folge gibt es nicht.

Definition der Herleitung:

(1) Ist M eine (eventuell leere) endliche Menge von Formeln, und sind $\Gamma_1, \ldots, \Gamma_n$ Formeln ($n\geqq 1$), so wird die Folge $\Gamma_1, \ldots, \Gamma_n$ als *eine Herleitung aus der Prämissenmenge* M bezeichnet, falls für $i=1, \ldots, n$ Γ_i (a) ein Axiom oder (b) ein Element von M oder (c) eine unmittelbare Folge einer oder zweier der ihm in der Folge vorausgehenden Formeln ist. Γ_n wird als *die Endformel der Herleitung* bezeichnet.

(2) Eine Formel Γ wird *aus der Prämissenmenge* M *herleitbar* genannt, wenn es eine Herleitung aus M mit der Endformel Γ gibt; dies wird kurz $M\vdash\Gamma$,

oder — wenn $\Delta_1, \ldots, \Delta_m$ die Liste der Elemente von M ist — durch $\Delta_1, \ldots, \Delta_m \vdash \Gamma$ bezeichnet.

(3) Eine Formel Γ wird *herleitbar* genannt, wenn sie aus einer leeren Prämissenmenge herleitbar ist; dies wird kurz durch $\vdash \Gamma$ bezeichnet.

BEMERKUNG. Wie es aus der Definition des Terms und aus (A2) hervorgeht, die Terme mit den Operatoren $\llcorner$, $\cap$, $\cup$ bilden eine Boolesche Algebra.

Im System **D** ist auch die Zeichenfolge „$\alpha = \beta$" eine Primformel, falls α und β Terme sind. Dieser Fall der Primformel ist in **D*** als überflüssig eliminiert, durch die Modifikation des formalen Aufbaus.

Definierte Zeichen (Abkürzungen):

(D1) $O =_{\text{def}} \neg P \llcorner$. (Wenn $P\alpha/t$ bedeutet: α ist in der Zeit t zulässig, so bedeutet $O\alpha/t$: α ist in der Zeit t verpflichtend.)

(D2) $\forall t =_{\text{def}} \neg \exists t \neg$, falls t eine Zeitvariable ist.

Einige Herleitungsschemata

Die folgenden Schemata (A12)—(A18) sind im Aussagenkalkül wohlbekannt, sodaß ihre Herleitung hier nicht detailliert wird. In diesen Schemata sind $\Gamma, \Delta, \Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_n$ beliebige Formeln, und p ist eine Aussagevariable.

(A12) $\Gamma \supset \Delta, \Delta \supset \Gamma \vdash \Gamma \sim \Delta$.

(A13) $\vdash ((\Gamma \wedge \Delta) \supset \theta) \sim (\Delta \supset (\Gamma \supset \theta))$.

(A14) $\Gamma \vdash \Gamma \sim (p \vee \neg p)$.

(A15) $\Gamma \vdash \neg \Gamma \sim (p \wedge \neg p)$.

(A16) $(\Gamma_1 \vee \ldots \vee \Gamma_n) \supset \Delta \vdash \Gamma_i \supset \Delta; \quad i = 1, \ldots, n; \quad n \geqq 2$.

(A17) $\Gamma_1 \supset \Delta, \ldots, \Gamma_n \supset \Delta \vdash (\Gamma_1 \vee \ldots \vee \Gamma_n) \supset \Delta; \quad n \geqq 2$.

(A18) $\Gamma_1 \supset \Delta_1, \ldots, \Gamma_n \supset \Delta_n \vdash (\Gamma_1 \vee \ldots \vee \Gamma_n) \supset (\Delta_1 \vee \ldots \vee \Delta_n); \quad n \geqq 2$.

In den Schemata (A19)—(A28) sind Γ und Δ beliebige Formeln, t ist eine Zeitvariable und Φ eine Formel, in der t passiv ist.

(A19) $\Gamma \sim \Delta \vdash \exists t \Gamma \sim \exists t \Delta$. — Skizze der Herleitung:

(a) $\Gamma \sim \Delta$, Prämisse.
(b) $(\Gamma \sim \Delta) \supset (\Gamma \supset \Delta)$, nach (A1).
(c) $\Gamma \supset \Delta$, aus (a) und (b), durch (MP).
(d) $(\Gamma \supset \Delta) \supset ((\Delta \supset \exists t \Delta) \supset (\Gamma \supset \exists t \Delta))$, nach (A1).
(e) $\Gamma \supset \exists t \Delta$, aus (e), (A11) und (d), durch (2-malige) (MP).
(f) $\exists t \Gamma \supset \exists t \Delta$, aus (e), durch (QS).
(g) $\exists t \Delta \supset \exists t \Gamma$, nach (b)—(f), durch Vertausch von Γ und Δ.
$\exists t \Gamma \sim \exists t \Delta$, aus (f) und (g), durch (A12).

(A20) $\vdash \exists t \Gamma \supset \exists t (\Gamma \vee \Delta); \quad \vdash \exists t \Delta \supset \exists t (\Gamma \vee \Delta)$.

(a) $\Gamma \supset (\Gamma \vee \Delta)$, nach (A1).
(b) $(\Gamma \vee \Delta) \supset \exists t (\Gamma \vee \Delta)$, nach (A11).

(c) $(\Gamma \supset (\Gamma \vee \Delta)) \supset (((\Gamma \vee \Delta) \supset \exists t(\Gamma \vee \Delta)) \supset (\Gamma \supset \exists t(\Gamma \vee \Delta)))$, nach (A1).
(d) $\Gamma \supset \exists t(\Gamma \vee \Delta)$, aus (a), (b) und (c), durch (MP).
$\exists t\Gamma \supset \exists t(\Gamma \vee \Delta)$, aus (d), durch (QS).

Die Herleitung der anderen Scheme verläuft analog.

(A21) $\vdash \exists t(\Gamma \vee \Delta) \supset (\exists t\Gamma \vee \exists t\Delta)$.
(a) $(\Gamma \supset \exists t\Gamma) \supset ((\Delta \supset \exists t\Delta) \supset ((\Gamma \vee \Delta) \supset (\exists t\Gamma \vee \exists t\Delta)))$, nach (A1).
(b) $(\Gamma \vee \Delta) \supset (\exists t\Gamma \vee \exists t\Delta)$, aus (a) und (A11), durch (MP).
$\exists t(\Gamma \vee \Delta) \supset (\exists t\Gamma \vee \exists t\Delta)$, aus (b), durch (QS).

(A22) $\vdash (\exists t\Gamma \vee \exists t\Delta) \supset \exists t(\Gamma \vee \Delta)$.
(a) $\exists t\Gamma \supset \exists t(\Gamma \vee \Delta)$,
(b) $\exists t\Delta \supset \exists t(\Gamma \vee \Delta)$, } nach (A20).
(c) $(\exists t\Gamma \supset \exists t(\Gamma \vee \Delta)) \supset ((\exists t\Delta \supset \exists t(\Gamma \vee \Delta)) \supset ((\exists t\Gamma \vee \exists t\Delta) \supset \exists! t(\Gamma \vee \Delta)))$, nach (A1).
$(\exists t\Gamma \vee \exists t\Delta) \supset \exists t(\Gamma \vee \Delta)$, aus (a), (b) und (c), durch (MP).

(A23) $\vdash (\exists t\Gamma \vee \exists t\Delta) \sim \exists t(\Gamma \vee \Delta)$. — Aus (A21) und (A22), durch (A12).

(A24) $\vdash \exists t\Phi \sim \Phi$.
(a) $\Phi \supset \Phi$, nach (A1).
(b) $\exists t\Phi \supset \Phi$, aus (a), durch (QS).
(c) $\Phi \supset \exists t\Phi$, nach (A11).
$\exists t\Phi \sim \Phi$, aus (b) und (c), durch (A12).

(A25) $\vdash (\Phi \wedge \exists t\Delta) \supset \exists t(\Phi \wedge \Delta)$.
(a) $(\Phi \wedge \Delta) \supset \exists t(\Phi \wedge \Delta)$, nach (A11).
(b) $\Delta \supset (\Phi \supset \exists t(\Phi \wedge \Delta))$, aus (a), nach (A13).
(c) $\exists t\Delta \supset (\Phi \supset \exists t(\Phi \wedge \Delta))$, aus (b), durch (QS).
$(\Phi \wedge \exists t\Delta) \supset \exists t(\Phi \wedge \Delta)$, aus (c), nach (A13).

(A26) $\vdash \exists t(\Gamma \wedge \Delta) \supset (\exists t\Gamma \wedge \exists t\Delta)$.
(a) $(\Gamma \supset \exists t\Gamma) \supset ((\Delta \supset \exists t\Delta) \supset ((\Gamma \wedge \Delta) \supset (\exists t\Gamma \wedge \exists t\Delta)))$, nach (A1).
(b) $(\Gamma \wedge \Delta) \supset (\exists t\Gamma \wedge \exists t\Delta)$, aus (a) und (A11), durch (MP).
$\exists t(\Gamma \wedge \Delta) \supset (\exists t\Gamma \wedge \exists t\Delta)$, aus (b), durch (QS).

(A27) $\vdash \exists t(\Phi \wedge \Delta) \sim (\Phi \wedge \exists t\Delta)$.
(a) $(\exists t\Phi \sim \Phi) \supset ((\exists t(\Phi \wedge \Delta) \supset (\exists t\Phi \wedge \exists t\Delta)) \supset (\exists t(\Phi \wedge \Delta) \supset (\Phi \wedge \exists t\Delta)))$, nach (A1).
(b) $\exists t(\Phi \wedge \Delta) \supset (\Phi \wedge \exists t\Delta)$, aus (A24), (A26) und (a), durch (MP).
$\exists t(\Phi \wedge \Delta) \sim (\Phi \wedge \exists t\Delta)$, aus (b) und (A25), durch (A12).

(A28) $\Gamma \vdash \forall t\Gamma$.
(a) Γ, Prämisse.
(b) $\Gamma \supset (\neg\Gamma \supset (p \wedge \neg p))$, nach (A1).
(c) $\neg\Gamma \supset (p \wedge \neg p)$, aus (a) und (b), durch (MP).
(d) $\exists t\neg\Gamma \supset (p \wedge \neg p)$, aus (c), durch (QS).
(e) $(\exists t\neg\Gamma \supset (p \wedge \neg p)) \supset \neg\exists t\neg\Gamma$, nach (A1).
(f) $\neg\exists t\neg\Gamma$, aus (d) und (e), durch (MP).
$\forall t\Gamma$, aus (f), nach (D2).

SATZ 1. *(Satz der Ersetzung.) Ist $\Gamma[\Delta]$ eine Formel, in der die Formel Δ als ein Teil vorkommt, entsteht $\Gamma[\theta]$ aus $\Gamma[\Delta]$ durch Ersetzung von Δ in einem seiner Vorkommen durch θ und ist $\vdash \Delta \sim \theta$, so ist $\vdash \Gamma[\Delta] \sim \Gamma[\theta]$.*

Der Beweis des Satzes geht — wie gewöhnlich — durch Induktion bezüglich *der Länge der Formel* vor sich. *Die Länge der Formel* erklären wir — wie üblich — durch folgende induktive Definition: (1) Die Länge irgendeiner Primformel ist 0. (2) Ist n die Länge von Γ, so ist $n+1$ die Länge von $\neg\Gamma$, bzw. $\exists t\Gamma$. (3) Ist n die Länge von Γ, und m die von Δ, so ist $\max(n, m)+1$ die Länge von $\Gamma \wedge \Delta$, $\Gamma \vee \Delta$, $\Gamma \supset \Delta$ bzw. $\Gamma \sim \Delta$.

Zu dem erwähnten Induktionsbeweise brauchen wir die folgenden Hilfssätze:

(1) $$\vdash (\Gamma \sim \Delta) \supset (\neg\Gamma \sim \neg\Delta).$$

(2) $$\vdash (\Gamma \sim \Delta) \supset ((\Gamma \wedge \theta) \sim (\Delta \wedge \theta)).$$

(3) $$\vdash (\Gamma \sim \Delta) \supset ((\Gamma \vee \theta) \sim (\Delta \vee \theta)).$$

(4) $$\vdash (\Gamma \sim \Delta) \supset ((\Gamma \supset \theta) \sim (\Delta \supset \theta)).$$

(5) $$\vdash (\Gamma \sim \Delta) \supset ((\theta \supset \Gamma) \sim (\theta \supset \Delta)).$$

(6) $$\vdash (\Gamma \sim \Delta) \supset ((\Gamma \sim \theta) \sim (\Delta \sim \theta)).$$

(7) $$\Gamma \sim \Delta \vdash \exists t\Gamma \sim \exists t\Delta.$$

Die Hilfssätze (1)—(6) sind wahr schon nach (A1); (7) ist wahr nach (A19).

BEMERKUNG. Wir können den obigen Hilfssatz (1) selbstverständlich auch in der Form „$\Gamma \sim \Delta \vdash \neg\Gamma \sim \neg\Delta$" schreiben, aufgrund von (MP); eine ähnliche Feststellung ist auch für (2)—(6) gültig. — Die Ersetzung von Termen, die nach der Booleschen Algebra gleich sind, ist gültig schon nach (A2).

Die Ersetzung äquivalenter Formeln bzw. gleicher Terme ist in **D** *als Grundschlußregel* gültig.

Herleitungen unter Anwendung der Ersetzung

Sind $\Gamma_1, \ldots, \Gamma_n$ beliebige Formeln und ist t eine Zeitvariable, so gilt:

(A29) $\Gamma_1 \vee \ldots \vee \Gamma_n \vdash \exists t\Gamma_1 \vee \ldots \vee \exists t\Gamma_n$.
(a) $\Gamma_1 \vee \ldots \vee \Gamma_n$, Prämisse.
(b) $\exists t(\Gamma_1 \vee \ldots \vee \Gamma_n)$, aus (a), nach (A11), durch (MP).
$\exists t\Gamma_1 \vee \ldots \vee \exists t\Gamma_n$, aus (b), durch Ersetzungen nach (A23).

(A30) Sind $\alpha_1, \ldots, \alpha_n$ Terme, ist t eine Zeitvariable, a eine Handlungsvariable und ist $\alpha_1 \cup \ldots \cup \alpha_n = a \cup \llcorner a$ nach der Booleschen Algebra, so gilt: $\vdash P\alpha_1/t \vee \ldots \vee P\alpha_n/t$.
(a) $Pa \cup \llcorner a/t$, nach (A5).
(b) $P\alpha_1 \cup \ldots \cup \alpha_n/t$, aus (a) und der Voraussetzung, nach (A2).
$P\alpha_1/t \vee \ldots \vee P\alpha_n/t$, aus (b), durch Ersetzungen nach (A6).

In den Schemata (A31)—(A34) sind α und β beliebige Terme und t ist eine Zeitvariable.

(A31) $\vdash T\alpha \cup \beta/t \sim (T\alpha/t \vee T\beta/t)$.

(a) $T\alpha \cup \beta/t \sim \neg T\llcorner(\alpha \cup \beta)/t$, nach (A4).
(b) $T\alpha \cup \beta/t \sim \neg T\llcorner\alpha \cap \llcorner\beta/t$, aus (a), nach (A2).
(c) $T\llcorner\alpha \cap \llcorner\beta/t \sim (T\llcorner\alpha/t \wedge T\llcorner\beta/t)$, nach (A3).
(d) $T\alpha \cup \beta/t \sim \neg(T\llcorner\alpha/t \wedge T\llcorner\beta/t)$ aus (b), durch Ersetzung nach (c).
(e) $\neg(T\llcorner\alpha/t \wedge T\llcorner\beta/t) \sim (\neg T\llcorner\alpha/t \vee \neg T\llcorner\beta/t)$, nach (A1).
(f) $\neg(T\llcorner\alpha/t \wedge T\llcorner\beta/t) \sim (T\alpha/t \vee T\beta/t)$, aus (e), durch Ersetzungen nach (A4).
$T\alpha \cup \beta/t \sim (T\alpha/t \vee T\beta/t)$, aus (d), durch Ersetzung nach (f).

(A32) $\vdash \forall t(T\alpha \cup \beta/t \sim (T\alpha/t \vee T\beta/t))$. — Aus (A31), durch (A28).

Im System **D** ist $\vdash (T\alpha \vee T\beta) \supset T\alpha \cup \beta$, ihre Umkehrung jedoch nicht herleitbar. Entsprechend in $\mathbf{D}^*$ ist $\forall t T\alpha \cup \beta/t \supset (\forall t T\alpha/t \vee \forall t T\beta/t)$ nicht herleitbar (ihre Umkehrung aber ja).

(A33) $\vdash T\alpha \cup \llcorner\alpha/t$.

(a) $T\alpha/t \vee \neg T\alpha/t$, nach (A1).
(b) $\neg T\alpha/t \sim T\llcorner\alpha/t$, nach (A4) und (A2).
(c) $T\alpha/t \vee T\llcorner\alpha/t$, aus (a), durch Ersetzung nach (b).
$T\alpha \cup \llcorner\alpha/t$, aus (c), durch Ersetzung nach (A31).

(A34) $\vdash O\alpha \cup \llcorner\alpha/t$.

(a) $\neg P\alpha \cap \llcorner\alpha/t$, nach (A7).
(b) $\neg P\llcorner(\alpha \cup \llcorner\alpha)/t$, aus (a) und (A2).
$O\alpha \cup \llcorner\alpha/t$, aus (b), nach (D1).

In **D** ist $T\alpha \cup \llcorner\alpha$ ein Axiom, und $O\alpha$ eine unmittelbare Folge von $T\alpha$; somit ist $O\alpha \cup \llcorner\alpha$ herleitbar.

Das Deduktionstheorem

Ist $\Gamma_1, \ldots, \Gamma_n$ eine Herleitung aus der Prämissenmenge M und ist $i \leqq n$, so ist die Teilfolge $\Gamma_1, \ldots, \Gamma_i$ auch eine Herleitung aus M, nach der Definition der Herleitung.

Wir wollen sagen, daß in der Herleitung $\Gamma_1, \ldots, \Gamma_n$ die Formel Γ_j *von der Formel* Γ_i *abhängig* ist, wenn (a) Γ_i identisch mit Γ_j ist oder (b) Γ_j von einer Formel Γ_k der Herleitung abhängig ist und Γ_k eine unmittelbare Folge von Γ_i (und eventuell einer anderen Formel der Herleitung) ist.

Ist Γ_j von Γ_i in der Herleitung $\Gamma_1, \ldots, \Gamma_n$ nicht abhängig, so wird in der Herleitung Γ_j als *von* Γ_i *unabhängig* bezeichnet.

Die Zeitvariable t wird in einer Herleitung als Γ-*aktiv* bezeichnet, wenn (a) Γ eine Prämisse der Herleitung ist, (b) t in Γ aktiv ist und (c) in der Herleitung eine Formel $\Delta \supset \Phi$ und als ihre unmittelbare Folge nach (QS) die Formel $\exists t \Delta \supset \Phi$ vorkommt, wo $\Delta \supset \Phi$ von Γ abhängt.

SATZ 2. *(Deduktionstheorem.) Ist M eine Prämissenmenge, sind Γ und Δ Formeln und ist $M, \Gamma \vdash \Delta$ ohne Γ-aktive Zeitvariablen in der Herleitung, so ist $M \vdash \Gamma \supset \Delta$.*

Der Beweis des Satzes geht durch Induktion bezüglich der Länge der Herleitung vor sich. Unter *der Länge der Herleitung* verstehen wir — wie üblich — die Anzahl ihrer Gliedformeln. Der Verlauf des Induktionsbeweises ist folgender:

(1) Es ist zu beweisen, daß die Behauptung des Satzes wahr ist, wenn die Länge der Herleitung 1 ist.

(2) Es ist zu beweisen, daß wenn die Behauptung des Satzes für alle Herleitungen wahr ist, deren Länge $\leqq n$, so ist sie auch für solche Herleitungen wahr, deren Länge $=n+1$ ist.

(2a) Die Endformel der Herleitung ist eine der Prämissen oder ein Axiom oder eine unmittelbare Folge nach (MP) zweier Formeln, die in der Herleitung ihr vorausgehen.

(2b) Die Endformel der Herleitung ist eine unmittelbare Folge nach (QS) einer Formel, die in der Herleitung ihr vorausgeht.

Der Beweis in (1) und im Falle (2a) in (2) verläuft wie im Aussagenkalkül, im Falle (2b) aber wie folgt.

Fall von (2b). Die Form der Endformel ist $\exists t\theta \supset \Phi$, sie ist eine unmittelbare Folge von $\theta \supset \Phi$ nach (QS). (Selbstverständlich ist t passiv in Φ.)

Fall (2ba): $\theta \supset \Phi$ ist unabhängig von Γ. So:

$$M \vdash \theta \supset \Phi, \tag{3}$$

und weiter:

$$\theta \supset \Phi \vdash \exists t\theta \supset \Phi, \quad \text{nach (QS).} \tag{4}$$

$$\vdash (\exists t\theta \supset \Phi) \supset (\Gamma \supset (\exists t\theta \supset \Phi)), \quad \text{nach (A1).} \tag{5}$$

Aus (3), (4) und (5) folgt aber leicht:

$$M \vdash \Gamma \supset (\exists t\theta \supset \Phi),$$

was zu beweisen war.

Fall (2bb): $\theta \supset \Phi$ ist von Γ abhängig. Da $M, \Gamma \vdash \theta \supset \Phi$ ist, nach der Induktionsvoraussetzung

$$M \vdash \Gamma \supset (\theta \supset \Phi) \tag{6}$$

ist (es wurde angewendet, daß es in der Herleitung keine Γ-aktive Zeitvariable gibt). Und weiter:

$$\vdash (\Gamma \supset (\theta \supset \Phi)) \supset (\theta \supset (\Gamma \supset \Phi)), \quad \text{nach (A1).} \tag{7}$$

$$\theta \supset (\Gamma \supset \Phi) \vdash \exists t\theta \supset (\Gamma \supset \Phi), \quad \text{nach (QS),} \tag{8}$$

da t in $\Gamma \supset \Phi$ passiv ist.

$$\vdash (\exists t\theta \supset (\Gamma \supset \Phi)) \supset (\Gamma \supset (\exists t\theta \supset \Phi)), \quad \text{nach (A1).} \tag{9}$$

Aus (6)—(9) folgt wiederum:

$$M \vdash \Gamma \supset (\exists t\theta \supset \Phi),$$

womit der Beweis des Satzes 2 vollständig ist.

Das Deduktionstheorem ist im System **D** nur unter der Beschränkung gültig, daß die Schlußregel $T\alpha \vdash O\alpha$ in der Herleitung nicht angewendet wird (da $T\alpha \supset O\alpha$ in **D** nicht herleitbar ist). In **D*** ist $O\alpha/t$ aus $T\alpha/t$ nicht herleitbar.

SATZ 3. *Ist* $f(a_1, \ldots, a_n)$ *ein Term, sind* $a_1, \ldots, a_n$ *die in* $f(a_1, \ldots, a_n)$ *vorkommenden Handlungsvariablen, ist ferner* $f^*(Ta_1/t, \ldots, Ta_n/t)$ *eine Formel, die aus* $f(a_1, \ldots, a_n)$

durch Ersetzen der in ihm vorkommenden Operatoren $\llcorner$, $\cup$, $\cap$ *und Handlungsvariablen* $a_1, \ldots, a_n$ *der Reihe nach durch* $\neg, \vee, \wedge$; $Ta_1/t, \ldots, Ta_n/t$ *entsteht, so ist*

$$\vdash Tf(a_1, \ldots, a_n)/t \sim f^*(Ta_1/t, \ldots, Ta_n/t).$$

Den Nachweis des Satzes mögen wir durch Induktion bezüglich der Länge des Terms f führen. *Die Länge des Terms* erklären wir wie folgt: (a) Die Länge einer Handlungsvariablen ist 0. (b) Ist n die Länge eines Terms α, so ist $n+1$ die Länge von $\llcorner\alpha$. (c) Ist n bzw. m die Länge des Terms α bzw. β, so ist $\max(n, m)+1$ die Länge von $\alpha \cap \beta$ und $\alpha \cup \beta$.

Ist 0 die Länge von f, so ist die Behauptung des Satzes trivial. Setzen wir jetzt voraus, daß unsere Behauptung wahr ist für solche Fälle, wo die Länge von $f \leqq k$ $(k \geqq 0)$ ist. Wir haben zu beweisen, daß die Behauptung des Satzes 3 auch für ein beliebiges f von der Länge $k+1$ wahr ist.

Fall (1). Die Form von f ist $\llcorner f_1$. Die Länge von f_1 ist offenbar k, und f^* ist $\neg f_1^*$.

(a) $\vdash Tf_1/t \sim f_1^*$, nach der Induktionsvoraussetzung.

(b) $\vdash T\llcorner f_1/t \sim \neg Tf_1/t$, nach (A4) und (A2).
$\vdash T\llcorner f_1/t \sim \neg f_1^*$, aus (b), durch Ersetzung nach (a).

Fall (2). Die Form von f ist $g \cap h$. Die Länge von g und von h ist $\leqq k$, und f^* ist $g^* \wedge h^*$.

(a) $\left.\begin{matrix} Tg/t \sim g^* \\ Th/t \sim h^* \end{matrix}\right\}$ nach der Induktionsvoraussetzung.

(b) $Tg \cap h/t \sim (Tg/t \wedge Th/t)$, nach (A3).
$Tg \cap h/t \sim (g^* \wedge h^*)$, aus (b), durch Ersetzung nach (a).

Fall (3). Die Form von f ist $g \cup h$. Die Diskussion ist analog mit der des Falles (2). Wir haben aber (A31) statt (A3) anzuwenden.

BEMERKUNG. Infolge des Satzes 3 wäre der Operator T im System $\mathbf{D}^*$ vermeidbar unter der Vereinbarung, daß wir einfach $f^*(a_1/t, \ldots, a_n/t)$ statt $f^*(Ta_1/t, \ldots, Ta_n/t)$ setzen. So wurde man z. B. nach Satz 3 $\neg(Ta/t \wedge Tb/t) \vee \neg Tc/t$ statt $T\llcorner(a \cap b) \cup \llcorner c/t$ setzen, dann alle T weglassen: $\neg(a/t \wedge b/t) \vee \neg c/t$.

II. DAS ENTSCHEIDUNGSVERFAHREN DES SYSTEMS $\mathbf{D}^*$

In diesem Teil des Aufsatzes werden die definierten Operatoren O und $\forall$ nicht benutzt.

Lineare Formeln

Ist Γ eine Formel und t eine Zeitvariable, so wird die Formel $\exists t\Gamma$ als eine $\exists$-*Formel* bezeichnet. — Eine $\exists$-Formel wird als in einer Formel Γ *frei* bezeichnet, wenn sie ein Teil von Γ ist, aber innerhalb Γ nicht im Wirkungsbereich eines $\exists$-Quantors steht.

Wir werden nun den $\exists$-*Grad einer Formel* erklären wie folgt:

(a) Gibt es in Γ keinen $\exists$-Quantor, so ist 0 der $\exists$-Grad von Γ. (b) Ist n der $\exists$-Grad von Γ, so ist n der $\exists$-Grad von $\neg\Gamma$. (c) Ist n bzw. m der $\exists$-Grad von Γ

bzw. von Δ, so ist $\max(n, m)$ der $\exists$-Grad der Formeln $\Gamma \wedge \Delta$, $\Gamma \vee \Delta$, $\Gamma \supset \Delta$ und $\Gamma \sim \Delta$. (d) Ist n der $\exists$-Grad von Γ, so ist $n+1$ der $\exists$-Grad von $\exists t\Gamma$.

Eine Formel Γ wird als *linear* bezeichnet, wenn (a) ihr $\exists$-Grad $\leqq 1$ ist und (b) ist $\exists t\Delta$ ein Teil von Γ, so kommen in Δ nur solche Primformeln vor, in welchen t aktiv ist.

Es ist nach den obigen Definitionen klar, daß eine Formel mit dem $\exists$-Grad 0 immer linear ist.

Ist Γ eine lineare Formel, so werden wir als *die Moleküle von* Γ die in ihr vorkommenden $\exists$-Formeln und jene Primformeln bezeichnen, die nicht im Wirkungsbereich eines $\exists$-Quantors stehen.

Wir werden weiter unten in (L1)—(L3) den Begriff „*der Linearform einer Formel*" induktiv definieren. Im Satz 4 wir werden beweisen, daß die Linearform irgendeiner Formel durch (L1)—(L3) bestimmt ist.

(L1) Ist Γ eine *lineare* Formel, so werden wir als *die Linearform von* $\exists t\Gamma$ bezeichnen:

(a) die Formel Γ, falls t in Γ passiv ist;

(b) die Formel $\exists t\Gamma$, falls t in allen Molekülen von Γ aktiv ist;

(c) die Formel $p \wedge \neg p$, falls weder (a) noch (b) erfüllt ist und *die perfekte disjunktive Normalform von* Γ *bezüglich ihrer Moleküle* (in dem üblichen Sinn) leer ist;

(d) die Formel

$$(\Phi_1 \wedge \exists t\Delta_1) \vee \ldots \vee (\Phi_n \wedge \exists t\Delta_n) \qquad (n \geqq 1),$$

falls weder (a) noch (b) noch (c) erfüllt ist, die perfekte disjunktive Normalform von Γ bezüglich ihrer Moleküle die Form $\theta_1 \vee \ldots \vee \theta_n$ hat, und Φ_i bzw. Δ_i die Konjunktion derer Konjunktionsglieder von θ_i ist, in denen t passiv bzw. aktiv ist.

Ist Γ_1 die Linearform von $\exists t\Gamma$ nach (L1), so ist

$$\vdash \Gamma_1 \sim \exists t\Gamma,$$

und zwar: im Falle (a) nach (A24), im Falle (b) nach (A1), im Falle (c) nach (A1) und (A24), im Falle (d) nach (A23) und (A27); ferner ist Γ_1 durch Γ und t „wesentlich eindeutig" bestimmt (abgesehen von der Reihenfolge der Disjunktions- und Konjunktionsglieder im Falle (d)), und Γ_1 ist eine *lineare* Formel.

(L2) Ist Γ eine Formel, die nicht linear ist, und ist Γ_1 die Linearform von Γ, so wird die Linearform von $\exists t\Gamma_1$ auch als *die Linearform von* $\exists t\Gamma$ bezeichnet.

(L3) Es sei Γ eine beliebige Formel. Ersetzen wir in Γ alle freie $\exists$-Teilformen durch ihre Linearformen. Die so entstandene Formel werden wir als *die Linearform von* Γ bezeichnen.

SATZ 4. *Ist* Γ *eine beliebige Formel, so ist durch* (L1)—(L3) *eine lineare Formel* Γ_1 *als die Linearform von* Γ „*wesentlich eindeutig*" *bestimmt, und es ist* $\vdash \Gamma \sim \Gamma_1$.

Die Wahrheit des Satzes können wir durch Induktion bezüglich des $\exists$-Grads der Formel Γ beweisen.

Ist 0 der $\exists$-Grad von Γ, so ist Γ eine lineare Formel und die Linearform ihrer selbst (nach (L3)), und es ist $\vdash \Gamma \sim \Gamma$.

Setzen wir nun voraus, daß der Satz wahr ist für Formeln, deren $\exists$-Grad $\leqq n$ ist ($n \geqq 0$), und daß $n+1$ der $\exists$-Grad von Γ ist. Infolge der Definition des $\exists$-Grads

ist der $\exists$-Grad aller freier $\exists$-Teilformel von $\Gamma \leqq n+1$, und Γ soll wenigstens einen Teil der Form von $\exists t\Delta$ (mit einer Zeitvariablen t) enthalten, dessen $\exists$-Grad $n+1$ ist. Der $\exists$-Grad von Δ ist nun n. Nach der Induktionsvoraussetzung gibt es eine lineare Formel Δ_1, die als die Linearform von Δ nach (L1)—(L3) bestimmt ist, und es ist $\vdash \Delta \sim \Delta_1$. Dann ist aber nach (L1) eine lineare Formel Δ_2 bestimmt, die die Linearform von $\exists t\Delta_1$ ist; Δ_2 ist — nach (L1), wenn Δ_1 identisch mit Δ ist (d.h. wenn Δ linear ist), und nach (L2), wenn Δ nicht linear ist — auch die Linearform von $\exists t\Delta$, und es ist $\vdash \Delta_2 \sim \exists t\Delta$. — Ist nun θ ein freier $\exists$-Teil von Γ, dessen $\exists$-Grad $\leqq n$ ist, so gibt es — nach der Induktionsvoraussetzung — eine lineare Formel θ_1, die als die Linearform von θ durch (L1)—(L3) bestimmt ist, und es ist $\vdash \theta \sim \theta_1$. So ist für eine jede freie $\exists$-Teilformel Ω von Γ eine lineare Formel Ω_1 durch (L1)—(L3) bestimmt, die die Linearform von Ω ist, und es ist $\vdash \Omega \sim \Omega_1$. Ersetzen wir in Γ alle freie $\exists$-Teile durch ihre Linearform, so bekommen wir eine lineare Formel Γ_1, die nach (L3) die Linearform von Γ ist, und es ist offenbar $\vdash \Gamma \sim \Gamma_1$.

Eine Herleitung wird als *linear* bezeichnet, wenn die in ihm vorkommenden Formeln alle linear sind.

SATZ 5. *Ist die Formelnfolge $\Gamma_1, \ldots, \Gamma_n, \Gamma$ eine Herleitung aus einer leeren Prämissenmenge, und ist Δ die Linearform von Γ, so gibt es eine lineare Herleitung $\Delta_1, \ldots, \Delta_m, \Delta$ aus der leeren Prämissenmenge.*

Den Beweis des Satzes werden wir durch Induktion bezüglich der Länge der Herleitung von Γ führen.

Ist 1 die Länge der Herleitung, d.h. ist Γ ein Axiom, so sind die folgenden Unterfälle möglich:

Fall (1). Γ ist ein Axiom nach (A1). Dann gibt es eine Formel Γ^*, in der als Primformel nur die Aussagevariablen $p_1, \ldots, p_k$ $(k \geqq 1)$ vorkommen, Γ^* ist ein Axiom nach (A1), und Γ entsteht aus Γ^* durch Einsetzen der Aussagevariablen $p_1, \ldots, p_k$ durch irgendwelche Formeln $\theta_1, \ldots, \theta_k$. Es sei nun Ω_1 bzw. ... bzw. Ω_k die Linearform von θ_1 bzw. ... bzw. θ_k. Ersetzen wir in $\Gamma^* p_i$ durch Ω_i $(i = 1, \ldots, k)$, so bekommen wir eine Formel Δ, die einerseits die Linearform von Γ, andererseits ein Axiom nach (A1) ist.

Fall (2). Γ ist ein Axiom nach (A2), d.h. seine Form ist $\theta[\alpha] \sim \theta[\beta]$. Es sei nun $\Omega[\alpha]$ die Linearform von $\theta[\alpha]$. Dann ist $\Omega[\alpha] \sim \Omega[\beta]$ die Linearform von Γ (da die Linearisierung ohne Veränderung der in Primformeln vorkommenden Terme vollzogen wird) und gleichzeitig ein Axiom nach (A2).

Fall (3). Γ ist ein Axiom nach (A3)—(A10). Diese Axiome sind alle lineare Formeln mit dem $\exists$-Grad 0, so ist ein jedes die Linearform seiner selbst.

Fall (4). Γ ist ein Axiom nach (A11), d.h. seine Form ist $\theta \supset \exists t\theta$. Es sei θ_1 die Linearform von θ und θ_2 die Linearform von $\exists t\theta_1$ (letztere ist nach (L2) auch die Linearform von $\exists t\theta$). Dann ist $\theta_1 \supset \theta_2$ die Linearform von $\theta \supset \exists t\theta$.

(a) Ist θ_2 identisch mit θ_1, oder ist ihre Form $p \wedge \neg p$ (laut (L1) (a) oder (c)), so ist $\theta_1 \supset \theta_2$ ein Axiom nach (A1).

(b) Ist θ_2 identisch mit $\exists t\theta_1$ (laut (L1) (b)), so ist $\theta_1 \supset \theta_2$ ein Axiom nach (A11).

(c) Ist

$$(\Phi_1 \wedge \exists t\Delta_1) \vee \ldots \vee (\Phi_k \wedge \exists t\Delta_k) \qquad (k \geqq 1)$$

die Form von θ_2 (laut (L1) (d)), dann ist $(\Phi_1 \wedge \Delta_1) \vee \ldots \vee (\Phi_k \wedge \Delta_k)$ die perfekte disjunktive Normalform von θ_1 bezüglich ihrer Moleküle. Die Form von $\theta_1 \supset \theta_2$ ist nun

$$\theta_1 \sim ((\Phi_1 \wedge \exists t \Delta_1) \vee \ldots \vee (\Phi_k \wedge \exists t \Delta_k)),$$

und eine lineare Herleitung dieser Formel ist die folgende:

(d) $\theta_1 \sim ((\Phi_1 \wedge \Delta_1) \vee \ldots \vee (\Phi_k \wedge \Delta_k))$, nach (A1).
(e) $(\Delta_i \supset \exists t \Delta_i)$, $i = 1, \ldots, k$; nach (A11).
(f) $(\Delta_i \supset \exists t \Delta_i) \supset ((\Phi_i \wedge \Delta_i) \supset (\Phi_i \wedge \exists t \Delta_i))$, $i = 1, \ldots, k$; nach (A1).
(g) $(\Phi_i \wedge \Delta_i) \supset (\Phi_i \wedge \exists t \Delta_i)$, $i = 1, \ldots, k$; aus (e) und (f), durch (MP).
(h) $((\Phi_1 \wedge \Delta_1) \vee \ldots \vee (\Phi_k \wedge \Delta_k)) \supset ((\Phi_1 \wedge \exists t \Delta_1) \vee \ldots \vee (\Phi_k \wedge \exists t \Delta_k))$, aus (g), durch (A18).
$\theta_1 \supset ((\Phi_1 \wedge \exists t \Delta_1) \vee \ldots \vee (\Phi_k \wedge \exists t \Delta_k))$, aus (h), durch Umsatz nach (d).

Setzen wir nun woraus, daß Satz 5 wahr ist, falls die Länge der Herleitung $\leqq n$ ist ($n \geqq 1$), und daß $n+1$ die Länge der Herleitung von Γ ist.

Fall (5). Γ ist ein Axiom. Dieser Fall ist schon ausführlich diskutiert.

Fall (6). Γ ist eine unmittelbare Folge nach (MP) der Formeln θ und $\theta \supset \Gamma$, die in der Herleitung ihm vorausgehen. So ist die Länge der Herleitung von θ und von $\theta \supset \Gamma$ $\leqq n$, und es haben nach der Induktionsvoraussetzung die Linearformen von θ und von $\theta \supset \Gamma$ eine lineare Herleitung. Bezeichnen wir die Linearform von θ bzw. Γ durch θ^* bzw. Γ^*, so ist $\theta^* \supset \Gamma^*$ die Linearform von $\theta \supset \Gamma$. Aber es ist nach (MP) $\theta^*, \theta^* \supset \Gamma^* \vdash \Gamma^*$; d.h. wenn die Folge $\Delta_1, \ldots, \Delta_m$ die lineare Herleitung von θ^* und von $\theta^* \supset \Gamma^*$ enthält, so ist die Folge $\Delta_1, \ldots, \Delta_m, \Gamma^*$ eine lineare Herleitung von Γ^*.

Fall (7). Γ ist eine unmittelbare Folge nach (QS) einer ihm vorausgehenden Formel. Die Form von Γ ist also $\exists t \theta \supset \Phi$, und die Form des erwähnten Vorgänger ist $\theta \supset \Phi$, wo t in Φ passiv ist. Da die Länge der Herleitung von $\theta \supset \Phi$ $\leqq n$ ist, gibt es nach der Induktionsvoraussetzung eine lineare Herleitung der Linearform von $\theta \supset \Phi$. Es sei nun θ^* bzw. Φ^* die Linearform von θ bzw. von Φ; dann ist $\theta^* \supset \Phi^*$ die Linearform von $\theta \supset \Phi$. Ist θ_1 die Linearform von $\exists t \theta^*$, dann ist $\theta_1 \supset \Phi^*$ die Linearform von $\exists t \theta \supset \Phi$. Man erhält nun die lineare Herleitung von $\theta_1 \supset \Phi^*$ aus der von $\theta^* \supset \Phi^*$ wie folgt.

Ist θ_1 identisch mit θ^*, dann ist die Herleitung mit der Formel $\theta^* \supset \Phi^*$ beendet.

Ist θ_1 identisch mit $\exists t \theta^*$, dann bekommt man $\theta_1 \supset \Phi^*$ aus $\theta^* \supset \Phi^*$ durch (QS).

Ist $p \wedge \neg p$ die Form von θ_1, so bekommt man $\theta_1 \supset \Phi^*$ aus $\theta^* \supset \Phi^*$ durch einen Umsatz (da in diesem Fall $\theta^* \sim (p \wedge \neg p)$ ein Axiom nach (A1) ist).

Ist die Form von θ_1

$$(\Phi_1 \wedge \exists t \Delta_1) \vee \ldots \vee (\Phi_k \wedge \exists t \Delta_k) \qquad (k \geqq 1),$$

dann ist (laut (L1) (d))

$$\vdash \theta^* \sim ((\Phi_1 \wedge \Delta_1) \vee \ldots \vee (\Phi_k \wedge \Delta_k)).$$

Nun bekommt man $\theta_1 \supset \Phi^*$ aus $\theta^* \supset \Phi^*$ folgendermaßen:

(a) $\theta^* \supset \Phi^*$, nach der Induktionsvoraussetzung.
(b) $\theta^* \sim ((\Phi_1 \wedge \Delta_1) \vee \ldots \vee (\Phi_k \wedge \Delta_k))$, nach (A1).
(c) $((\Phi_1 \wedge \Delta_1) \vee \ldots \vee (\Phi_k \wedge \Delta_k)) \supset \Phi^*$, aus (a), durch Umsatz nach (b).
(d) $(\Phi_i \wedge \Delta_i) \supset \Phi^*$, $i = 1, \ldots, k$; aus (c), durch (A16).

(e) $\Delta_i \supset (\Phi_i \supset \Phi^*)$, $i=1, \ldots, k$; aus (d), durch Umsatz nach (A13).
(f) $\exists t\Delta_i \supset (\Phi_i \supset \Phi^*)$, $i=1, \ldots, k$; aus (e), durch (QS).
(g) $(\Phi_i \wedge \exists t\Delta_i) \supset \Phi^*$, $i=1, \ldots, k$; aus (f), durch Umsatz nach (A13).
$((\Phi_1 \wedge \exists t\Delta_1) \vee \ldots \vee (\Phi_k \wedge \exists t\Delta_k)) \supset \Phi^*$, aus (g), durch (A17).

ZUSATZ ZUM SATZ 5. *Alle herleitbare lineare Formeln haben lineare Herleitungen.* (Da die Linearform einer linearen Formel sie selbst ist.)

Kanonische Formen

Wir wollen eine Zeitvariable t in einer Formel Γ als *P-aktiv* bzw. als *$\exists$-aktiv* bezeichnen, wenn sie in dem Wirkungsbereich eines P-Operators bzw. eines $\exists$-Quantors in Γ vorkommt. Ist t P-aktiv in Γ, so wollen wir alle Handlungsvariablen, die wenigstens in einer Primformel der Form $P\alpha/t$ von Γ vorkommen, als *die P_t-Variablen von Γ* bezeichnen. Ist t $\exists$-aktiv in Γ, so wollen wir alle Primformeln, die in dem Wirkungsbereich wenigstens eines $\exists t$-Quantors von Γ vorkommen, als *die $\exists_t$-Primformeln von Γ* bezeichnen.

(K1) Es sei die Bedeutung von $f, f^*, a_1, \ldots, a_n, t$ wie im Satz 3. Wir werden die Formel $f^*(Ta_1/t, \ldots, Ta_n/t)$ als *die kanonische Form der Primformel* $Tf(a_1, \ldots, a_n)/t$ bezeichnen. Nach Satz 3 ist $Tf(a_1, \ldots, a_n)/t \sim f^*(Ta_1/t, \ldots, Ta_n/t)$.

(K2) Es sei α ein Term, t eine Zeitvariable, θ eine lineare Formel, $P\alpha/t$ eine Teilformel von θ, und M die Menge der P_t-Variablen von θ. Wir werden als *die kanonische Form von $P\alpha/t$ bezüglich θ* bezeichnen:

(a) die Formel $p \wedge \neg p$ (mit beliebiger Aussagevariablen p), falls die perfekte disjunktive Normalform von α bezüglich M (in dem üblichen Sinn) leer ist,

(b) die Formel $P\alpha_1/t \vee \ldots \vee P\alpha_n/t$, falls $\alpha_1 \cup \ldots \cup \alpha_n$ $(n \geqq 1)$ die perfekte disjunktive Normalform von α bezüglich M ist.

Im Falle (a) ist $\alpha = \alpha \cap \llcorner \alpha$ nach der Booleschen Algebra, so ist $\vdash P\alpha/t \sim \sim P\alpha \cap \llcorner \alpha/t$ nach (A2); und daraus und aus (A7) ist $\vdash P\alpha/t \sim (p \wedge \neg p)$ durch (A15).

Im Falle (b) ist $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$ nach der Booleschen Algebra, und so ist $\vdash P\alpha/t \sim (P\alpha_1/t \vee \ldots \vee P\alpha_n/t)$ — nach (A2), (A6) und nach Satz 1.

(K3) Es sei θ eine lineare Formel, t eine Zeitvariable und $\exists t\Gamma$ eine Teilformel von θ. Bezeichnen wir durch Γ_1 bzw. durch θ_1 jene Formel, die aus Γ bzw. aus θ entsteht, wenn wir in θ alle Primformeln der Form $T\alpha/t$ durch ihre kanonische Form und alle Primformeln der Form $P\beta/t$ durch ihre kanonische Form bezüglich θ ersetzen. Offenbar ist (nach (K1) und (K2))

$$\vdash \Gamma \sim \Gamma_1 . \tag{1}$$

Es sei ferner M_1 die Menge der $\exists_t$-Primformeln von θ_1. Wir werden als *die kanonische Form von $\exists t\Gamma$ bezüglich θ* bezeichnen:

(a) die Formel $p \wedge \neg p$, falls die perfekte disjunktive Normalform von Γ_1 bezüglich M_1 leer ist.

(b) die Formel $\exists t\Delta_1 \vee \ldots \vee \exists t\Delta_k$, falls $\Delta_1 \vee \ldots \vee \Delta_k$ $(k \geqq 1)$ die perfekte disjunktive Normalform von Γ_1 bezüglich M_1 ist.

Im Falle (a) ist $\vdash \Gamma_1 \sim (p \wedge \neg p)$ nach (A1), und unter Berücksichtigung von (1) und (A24) ist $\vdash \exists t\Gamma \sim (p \wedge \neg p)$.

Im Falle (b) ist $\vdash \Gamma_1 \sim (\Delta_1 \vee \ldots \vee \Delta_k)$ nach (A1), und unter Berücksichtigung von (1), (A19) und (A23) ist $\vdash \exists t\Gamma \sim (\exists t\Delta_1 \vee \ldots \vee \exists t\Delta_k)$.

(K4) Es sei θ eine lineare Formel und Γ eine beliebige Teilformel von θ; es seien $t_1, \ldots, t_n$ bzw. $u_1, \ldots, u_k$ die in Γ vorkommenden P-aktiven bzw. $\exists$-aktiven Zeitvariablen ($n \geqq 0$, $k \geqq 0$). Bezeichnen wir mit Γ_1 bzw. mit θ_1 jene Formel, die aus Γ bzw. aus θ entsteht, wenn wir in θ alle Primformeln der Form $T\alpha/t$ (mit beliebiger Zeitvariablen t) durch ihre kanonische Form und alle Primformeln der Form $P\beta/t_i$ durch ihre kanonische Form bezüglich θ ersetzen (für $i=1, \ldots, n$). Offenbar ist (nach (K1) und (K2))

$$\vdash \Gamma \sim \Gamma_1. \tag{2}$$

Bezeichnen wir nun mit Γ_2 jene Formel, die aus Γ_1 entsteht, wenn wir in Γ_1 alle $\exists$-Moleküle der Form $\exists u_i \Delta$ durch ihre kanonische Form bezüglich θ_1 ersetzen (für $i=1, \ldots, k$). Dann ist (nach (K3))

$$\vdash \Gamma_1 \sim \Gamma_2. \tag{3}$$

Die Formel Γ_2 wollen wir als *die kanonische Form der Formel Γ bezüglich θ* bezeichnen. Nach (2) und (3) ist $\vdash \Gamma \sim \Gamma_2$.

Aus der Definitionen (K1)—(K4) folgt unmittelbar:

SATZ 6. *Ist Γ eine Teilformel einer linearen Formel θ, so ist eine Formel Γ_1 als die kanonische Form von Γ bezüglich θ durch* (K1)—(K4) *„wesentlich eindeutig" bestimmt, und ferner ist $\vdash \Gamma \sim \Gamma_1$.*

Die Wertung der kanonischen Formen

Es sei Γ eine kanonische Form einer linearen Formel. Ersetzen wir in Γ alle Moleküle der Form $\exists t \Delta$ (für beliebige Zeitvariable t):

(a) durch $p \wedge \neg p$, falls n die Anzahl der P_t-Variablen von Γ und 2^n die Anzahl der Konjunktionsglieder der Form $\neg P\alpha/t$ von Δ ist.

(b) durch $\Delta \vee \exists t \Delta$ sonst.

Die so entstandene Formel wird *die $\exists$-Reduzierte von Γ* genannt und durch Γ^0 bezeichnet.

Im Falle (a) ist $\neg P\alpha_1/t \wedge \ldots \wedge \neg P\alpha_k/t \wedge \theta$ die Form von Δ, wo $k=2^n$, und θ eine (eventuell leere) Konjunktion von Gliedern der Form Ta/t bzw. $\neg Ta/t$ ist; ferner ist

$$\alpha_1 \cup \ldots \cup \alpha_k = a \cup \llcorner a$$

nach der Booleschen Algebra. Dann ist aber:

(1) $\vdash P\alpha_1/t \vee \ldots \vee P\alpha_k/t$, nach (A30);
(2) $\vdash \neg(\neg P\alpha_1/t \wedge \ldots \wedge \neg P\alpha_k/t) \sim (P\alpha_1/t \vee \ldots \vee P\alpha_k/t)$, nach (A1);
(3) $\vdash \neg(\neg P\alpha_1/t \wedge \ldots \wedge \neg P\alpha_k/t)$ aus (1) und (2);
(4) $\vdash (\neg P\alpha_1/t \wedge \ldots \wedge \neg P\alpha_k/t) \sim (p \wedge \neg p)$, aus (3), nach (A15);
(5) $\vdash (p \wedge \neg p \wedge \theta) \sim (p \wedge \neg p)$, nach (A1);
(6) $\vdash \Delta \sim (p \wedge \neg p)$, aus (4) und (5);
(7) $\vdash \exists t \Delta \sim (p \wedge \neg p)$, aus (6), nach (A24).

Im Falle (b) betrachten wir nun das folgende:

(8) $\vdash \exists t \Delta \supset (\Delta \vee \exists t \Delta)$, nach (A1);
(9) $\vdash (\Delta \supset \exists t \Delta) \supset ((\Delta \vee \exists t \Delta) \supset \exists t \Delta)$, nach (A1);
(10) $\vdash (\Delta \vee \exists t \Delta) \supset \exists t \Delta$, aus (9) und (A11), durch (MP);
(11) $\vdash \exists t \Delta \sim (\Delta \vee \exists t \Delta)$, aus (8) und (10), nach (A12).

Aus (7) und (11) folgt, daß $\vdash \Gamma \sim \Gamma^0$ ist.

Ersetzen wir in Γ^0 alle ihre Moleküle durch eine der Ziffern 0, 1 beliebig, aber unter Berücksichtigung der folgenden Einschränkungen (W1)—(W4):

(W1) Ist t eine Zeitvariable, so darf $t<t$ nicht durch 1 ersetzt werden.

(W2) Sind t und u Zeitvariablen, und ist $t<u$ durch 1 ersetzt, so darf $u<t$ nicht durch 1 ersetzt werden.

(W3) Sind t, u und v Zeitvariablen, und ist $t<u$ und $u<v$ durch 1 ersetzt, so darf $t<v$ nicht durch 0 ersetzt werden.

(W4) Ist n für eine Zeitvariable t die Anzahl der P_t-Variablen von Γ^0 und 2^n die Anzahl der in Γ^0 vorkommenden verschiedenen Moleküle der Form $P\alpha/t$, so dürfen diese Moleküle nicht alle durch 0 ersetzt werden.

Eine so entstandene Zeichenfolge wollen wir als *„eine Wertung von Γ^0"* bezeichnen. Jeder Wertung von Γ^0 werden wir einen „Wert" nach den „Wertungsregeln" von (W5) zuordnen. (Diese Wertungsregeln decken sich mit den Wertungsregeln des Aussagenkalküls: dem logischen Wert „wahr" entspricht 1, dem Wert „falsch" entsprich 0.)

(W5) Der Wert einer Wertung von Γ^0 wird nach den folgenden Regeln sukzessiv berechnet:

$$\neg x = 1-x,$$

$$x \wedge y = x \cdot y,$$

$$x \vee y = x+y-x \cdot y,$$

$$x \supset y = 1-x+x \cdot y,$$

$$x \sim y = 1-|x-y|.$$

In meinem Aufsatz [3] habe ich eine ähnliche Wertung „W" für das System **D** gegeben. Jene Wertung ist aber nicht adäquat für **D**, trotz der Behauptung des letzten Absatzes von [3], denn es fehlt in ihr ein Analogon der hiesigen Regel (W4). Dieser Fehler ist aber natürlich leicht korrigierbar.

Ist 1 der Wert aller möglichen Wertungen von Γ^0, so werden wir sagen, daß *„der Wert von Γ^0 identisch* 1 ist"; abgekürzt: „$w(\Gamma^0) \equiv 1$".

SATZ 7. *Ist Γ eine kanonische Form einer linearen Formel und ist $w(\Gamma^0) \equiv 1$, so ist Γ^0 und damit auch Γ herleitbar.*

BEWEIS. Hat man bei der Berechnung keine der Einschränkungen (W1)—(W4) anzuwenden, so ist Γ^0 ein Axiom nach (A1) (da die Regeln von (W5) identisch mit den Wertungsregeln des Aussagenkalküls sind). In anderen Fällen verläuft der Beweis wie folgt.

Wir werden jeder Anwendung der Regeln (W1)—(W4) eine sog. *„kritische Formel"* zuordnen, und zwar folgenderweise:

Zu (W1): $\neg t<t$; dies ist herleitbar nach (A8).

Zu (W2): $t<u \supset \neg u<t$; dies ist herleitbar nach (A9).

Zu (W3): $(t<u \wedge u<v) \supset t<v$; dies ist herleitbar nach (A10).

Zu (W4): $P\alpha_1/t \vee \ldots \vee P\alpha_k/t$, wo $P\alpha_1/t, \ldots, P\alpha_k/t$ die Liste aller Moleküle der Form $P\alpha/t$ ist; diese Formel ist herleitbar nach der Voraussetzung von (W4) und nach (A30).

Es seien nun $\Gamma_1, \ldots, \Gamma_n$ alle diese kritischen Formeln, die bei der Berechnung der Werte von Γ^0 vorkommen. Dann ist

$$\vdash \Gamma_1 \wedge \ldots \wedge \Gamma_n. \tag{1}$$

Betrachten wir die Formel $(\Gamma_1 \wedge \ldots \wedge \Gamma_n) \supset \Gamma^0$. Sie ist eine lineare Formel und eine kanonische Form ihrer selbst, ihre $\exists$-Reduzierte ist sie selbst, und ihre Moleküle sind dieselbe als die von Γ^0. Lassen wir nun die Beschränkungen (W1)—(W4) weg und berechnen wir die Werte von $(\Gamma_1 \wedge \ldots \wedge \Gamma_n) \supset \Gamma^0$ bloß nach (W5), d.h. wie im Aussagenkalkül. Verletzt man bei einer Wertung eine von (W1)—(W4), so ist (wie leicht einzusehen ist) 0 der Wert der zugeordneten kritischen Formel — d.h. einer von $\Gamma_1, \ldots \Gamma_n$ — nach (W5), und so ist 1 der Wert von $(\Gamma_1 \wedge \ldots \wedge \Gamma_n) \supset \Gamma^0$ nach (W5). Verletzt man aber bei einer Wertung keine der Beschränkungen (W1)—(W4), so ist 1 der Wert von Γ^0 schon nach (W5) (unserer Voraussetzung nach) und so ist 1 der Wert von $(\Gamma_1 \wedge \ldots \wedge \Gamma_n) \supset \Gamma^0$ widerum nach (W5). So ist $w((\Gamma_1 \wedge \ldots \wedge \Gamma_n) \supset \Gamma^0) \equiv 1$ *bloß nach* (W5), d.h. nach den Wertungsregeln des Aussagenkalküls. Dann ist aber diese Formel ein Axiom nach (A1), d.h. es ist

$$\vdash (\Gamma_1 \wedge \ldots \wedge \Gamma_n) \supset \Gamma_*^0. \tag{2}$$

Aus (1) und (2) folgt — durch (MP) — daß $\vdash \Gamma^0$ ist. Damit ist Satz 7 bewiesen.

SATZ 8. *Ist θ eine lineare Formel, Γ eine Teilformel von θ, ist Γ_1 bzw. Γ_2 die kanonische Form von Γ bezüglich Γ bzw. bezüglich θ, so ist $w(\Gamma_1^0) \equiv 1$ dann und nur dann, wenn $w(\Gamma_2^0) \equiv 1$ ist.*

Der Beweis dieses Satzes kann auf die folgenden Hilfssätze gegründet werden:

HILFSSATZ 1. *Ist t eine P-aktive Zeitvariable und $P\alpha/t$ eine Primformel von Γ, ist $\{a_1, \ldots, a_n\}$ die Menge der P_t-Variablen von Γ, ist $P\alpha_1/t \vee \ldots \vee P\alpha_k/t$ die Normalform von $P\alpha/t$ bezüglich $\{a_1, \ldots, a_n\}$ (d.h. bezüglich Γ), und ist b eine von einem jeden der $a_1, \ldots, a_n$ verschiedene Handlungsvariable, so entsteht die kanonische Form von $P\alpha/t$ bezüglich $\{a_1, \ldots, a_n, b\}$ aus der kanonischen Form bezüglich $\{a_1, \ldots, a_n\}$ durch Ersetzen der Moleküle $P\alpha_i/t$ für $i = 1, \ldots, k$ durch $P\alpha_i \cap b/t \vee P\alpha_i \cap \llcorner b/t$.*

HILFSSATZ 2. *Ist t eine $\exists$-aktive Zeitvariable und $\exists t\Delta$ eine Teilformel von Γ, ist $\exists t\Delta_1 \wedge \ldots \wedge \exists t\Delta_k$ die kanonische Form von $\exists t\Delta$ bezüglich Γ, ist $\{A_1, \ldots, A_n\}$ die Menge der in $\Delta_1, \ldots, \Delta_k$ vorkommenden Primformeln und ist B eine von einem jeden der $A_1, \ldots, A_n$ verschiedene Primformel der Form Ta/t oder $P\alpha/t$, so entsteht die kanonische Form von $\exists t\Delta$ bezüglich $\{A_1, \ldots, A_n, B\}$ aus der kanonischen Form bezüglich $\{A_1, \ldots, A_n\}$ durch Ersetzen der Moleküle $\exists t\Delta_i$ für $i = 1, \ldots, k$ durch $\exists t(\Delta_i \wedge B) \vee \exists t(\Delta_i \wedge \neg B)$.*

Der Beweis der Hilfssätze verläuft wie der des entsprechenden Satzes im Aussagenkalkül (aber unter Benutzung von (A6) und (A22)). Eine ähnliche Feststellung ist auch für den Beweis des Satzes 8 gültig; so wird dieser Beweis hier nicht detailliert.

In meinem Aufsatz [3] ist Satz (8) der entsprechende Satz für das System **D**. In [3] findet sich ein ausführlicherer Beweis dieses Satzes.

SATZ 9. *Ist Γ eine herleitbare lineare Formel und ist Γ_k die kanonische Form von Γ bezüglich Γ, so ist $w(\Gamma_k^0) \equiv 1$.*

BEWEIS. Aufgrund Satzes 5 und seines Zusatzes ist es hinreichend, die folgenden Hilfssätze nachzuweisen:

(Der Index k weist in diesen Hilfssätzen immer auf die kanonischen Form der Formel *bezüglich der Formel selbst* hin.)

HILFSSATZ 1. *Ist* Γ *ein lineares Axiom, so ist* $w(\Gamma_k^0) \equiv 1$.

HILFSSATZ 2. *Sind* Γ *und* Δ *lineare Formeln, ist ferner* $w(\Gamma_k^0) \equiv 1$ *und* $w((\Gamma \supset \Delta)_k^0) \equiv 1$, *so ist* $w(\Delta_k^0) \equiv 1$.

HILFSSATZ 3. *Sind* Γ, $\exists t\Gamma$ *und* Φ *lineare Formeln, ist* t *eine Zeitvariable, die in* Φ *passiv ist, und ist* $w((\Gamma \supset \Phi)_k^0) \equiv 1$, *so ist* $w((\exists t\Gamma \supset \Phi)_k^0) \equiv 1$.

BEWEIS DES HILFSSATZES 1. *Fall* (1): Γ ist ein lineares Axiom nach (A1). Es sei Γ^* wie im Falle (1) des Beweises des Satzes 5. Nun ist die kanonische Form von Γ^* sie selbst, und Γ_k entsteht aus Γ^* durch Einsetzen ihrer Aussagevariablen durch irgendeine Formel. Da Γ^* eine Tautologie des Aussagenkalküls ist und da (W5) identisch mit den Wertungsregeln des Aussagenkalküls ist, ist $w(\Gamma^*) \equiv 1$. Folglich ist $w(\Gamma_k^0) \equiv 1$, da ja $\Gamma_{k]}^0$ aus Γ^* durch Einsetzen ensteht.

Fall (2): Γ ist ein lineares Axiom nach (A2), d.h. sie hat die Form $\theta[\alpha] \sim \theta[\beta]$. Ist nun θ_1 die kanonische Form von $\theta[\alpha]$ bezüglich $\theta[\alpha] \sim \theta[\beta]$, so ist θ_1 auch die kanonische Form von $\theta[\beta]$ bezüglich $\theta[\alpha] \sim \theta[\beta]$, da nach der Booleschen Algebra $\alpha = \beta$ ist. Dann ist aber $\theta_1 \sim \theta_1$ die Form von $(\theta[\alpha] \sim \theta[\beta])_k$; und offensichtlich ist $w((\theta_1 \sim \theta_1)^0) \equiv 1$.

Fall (3): Γ ist ein Axiom nach (A3), (A4) oder (A6). Alle erwähnten Axiome haben die kanonische Form der Form $\theta \sim \theta$, und offensichtlich ist $w((\theta \sim \theta)^0) \equiv 1$.

Fall (4): Γ ist ein Axiom nach (A5). Nach (W4) und (W5) ist $w(\Gamma_k^0) \equiv 1$.

Fall (5): Γ ist ein Axiom nach (A7). Dann hat Γ_k die Form $\neg(p \wedge \neg p)$ (nach (K2)) und es ist $w(\neg(p \wedge \neg p)) \equiv 1$.

Fall (6): Γ ist ein Axiom nach (A8), (A9) oder (A10). Dann ist $w(\Gamma_k^0) \equiv 1$ nach (W1), (W2), (W3) und (W5).

Fall (7): Γ ist ein Axiom nach (A11). Die Form von Γ ist nun $\theta \supset \exists t\theta$, wo θ und $\exists t\theta$ lineare Formeln sind und in θ nur Primformeln der Form $T\alpha/t$ und $P\beta/t$ vorkommen. Somit ist θ_k bzw. $(\exists t\theta)_k$ gleichzeitig die kanonische Form von θ bzw. $\exists t\theta$ bezüglich $\theta \supset \exists t\theta$, und so ist Γ_k die Formel $\theta_k \supset (\exists t\theta)_k$. Ist $\Delta_1 \vee \ldots \vee \Delta_n$ $(n \geqq 1)$ die perfekte disjunktive Normalform von θ_k, so ist

$$\exists t\Delta_1 \vee \ldots \vee \exists t\Delta_n$$

die Form von $(\exists t\theta)_k$. Da $\theta_k \sim (\Delta_1 \vee \ldots \vee \Delta_n)$ ein Axiom nach (A1) ist, so ist der Wert von θ_k für alle Wertungen identisch mit dem Wert von $(\Delta_1 \vee \ldots \vee \Delta_n)$. Ist 1 der Wert von θ_k für eine Wertung, so ist 1 der Wert von Δ_i für irgendein i, und deshalb ist 1 der Wert von $\Delta_i \vee \exists t\Delta_i$, woraus der Wert von $(\exists t\theta)_k^0 = 1$ folgt. Folglich kann der Fall, daß „der Wert von θ_k" $= 1$ und „der Wert von $(\exists t\theta)_k^0$" $= 0$ ist (für eine und dieselbe Wertung), nicht auftreten, somit ist $w(\Gamma_k^0) \equiv 1$. Ist aber die perfekte disjunktive Normalform von θ_k leer, so ist (nach (K3)) $p \wedge \neg p$ die Form von $(\exists t\theta)_k$ und ist $\theta_k \sim (p \wedge \neg p)$ ein Axiom nach (A1). Dann ist aber $\theta_k \supset (p \wedge \neg p)$ (d.h. Γ_k) auch ein Axiom nach (A1).

NACHWEIS DES HILFSSATZES 2. Es seien Γ_1 bzw. Δ_1 die kanonische Formen von Γ bzw. Δ bezüglich $\Gamma \supset \Delta$; dann ist $\Gamma_1 \supset \Delta_1$ die Form von $(\Gamma \supset \Delta)_k$. Da nach der Voraussetzung $w(\Gamma_{k1}^0) \equiv 1$ ist, ist nach Satz 8 auch $w(\Gamma_1^0) \equiv 1$. Daraus und aus der Voraussetzung „$w((\Gamma_1 \supset \Delta_1)^0) \equiv 1$" folgt $w(\Delta_1^0) \equiv 1$, und widerum nach Satz 8 ist $w(\Delta_k^0) \equiv 1$.

NACHWEIS DES HILFSSATZES 3. Es sei Γ_1 bzw. Φ_1 die kanonische Form von Γ bzw. Φ bezüglich der Formel $\exists t\Gamma \supset \Phi$. Dann ist $\Gamma_1 \supset \Phi_1$ eine kanonische Form von $\Gamma \supset \Phi$ (und zwar bezüglich $\exists t\Gamma \wedge (\Gamma \supset \Phi)$) und $(\Gamma_1 \supset \Phi_1)^0$ ist $\Gamma_1 \supset \Phi_1^0$ (da in Γ und in Γ_1 kein $\exists$-Quantor vorkommt). Nach der Voraussetzung und nach Satz 8 ist $w(\Gamma_1 \supset \Phi_1^0) \equiv 1$. Ist die perfekte disjunktive Normalform von Γ_1 bezüglich ihrer Moleküle leer, so ist die Behauptung des Hilfssatzes trivial. Setzen wir also voraus, daß diese Normalform

$$\Delta_1 \vee \ldots \vee \Delta_n \qquad (n \geqq 1)$$

ist. Dann ist

$$\exists t\Delta_1 \vee \ldots \vee \exists t\Delta_n$$

die kanonische Form von $\exists t\Gamma$ bezüglich $\exists t\Gamma \supset \Phi$, und

$$(\exists t\Delta_1 \vee \ldots \vee \exists t\Delta_n) \supset \Phi_1$$

die Form von $(\exists t\Gamma \supset \Phi)_k$. Sind nun $\exists t\Omega_1$ und $\exists t\Omega_2$ Teile (d.h. Moleküle) dieser Formel, so müssen (nach (K3) und (K4)) in Ω_1 und Ω_2 dieselben Primformeln vorkommen.

Ist der Wert von Δ_i (für ein beliebiges, aber fixiertes i) nicht identisch 0, so gibt es eine einzige Wertung „W_i", für welche der Wert von $\Delta_i = 1$, aber im Falle $j \neq i$ der Wert von $\Delta_j = 0$ ist. Da offensichtlich für alle Wertungen der Wert von Γ_1 identisch mit dem Wert von $\Delta_1 \vee \ldots \vee \Delta_n$ ist, folgt es aus unserer Voraussetzung, daß der Wert von Φ_1^0 für „W_i" 1 ist, wie immer man die Moleküle von Φ_1^0 bewertet, deren Werte durch „W_i" noch nicht bestimmt sind. Daraus folgt, daß (abgesehen vom Trivialfall $w(\Phi_1^0) \equiv 1$) die Moleküle von Δ_i in Φ_1^0 vorkommen müssen. Da t in Φ_1 passiv, in allen Molekülen von Δ_i aber aktiv ist, kann keine Molekül von Δ_i in Φ_1 *als Molekül* vorkommen. So können die Moleküle von Δ_i nur in Φ_1^0 vorkommen, woraus folgt, daß Φ_1^0 einen Teil der Form $\Omega \vee \exists t\Omega$ — d.h. Φ_1 einen Teil der Form $\exists t\Omega$ — hat, und daß die Wertung „W_i" den Wert von Φ_1^0 durch den Wert einer oder mehrerer solchen Teilformeln eindeutig bestimmt.

Ist 0 der Wert von Ω, so kann der Wert von $\Omega \vee \exists t\Omega$ beliebig 0 oder 1 sein, d.h. der Wert von $\Omega \vee \exists t\Omega$ durch „W_i" ist nur in dem Falle eindeutig bestimmt, wenn 1 der Wert von Ω ist (dann ist 1 auch der Wert von $\Omega \vee \exists t\Omega$). Das ist aber für die Wertung „W_i" nur in dem Falle möglich, wenn Δ_i und Ω identisch sind; da ja $\exists t\Delta_i$ und $\exists t\Omega$ in $(\exists t\Gamma \supset \Phi)_k$ beide vorkommen, und so — nach einer früheren Bemerkung — die Moleküle von Δ_i und Ω dieselben sind.

Wir gewannen somit, daß $\exists t\Delta_i$ ein Teil von Φ_1 ist, falls der Wert von Δ_i nicht identisch 0 ist. Letzteres kann aber nur vorkommen, wenn der Fall (a) der Definition der $\exists$-Reduzierten für Δ_i erfüllt ist; dann wird aber $\exists t\Delta_i$ kein Teil von $(\exists t\Delta_1 \vee \ldots \vee \exists t\Delta_n)^0$ sein. So wird jeder Teil der Form $\Omega \vee \exists t\Omega$ von $(\exists t\Delta_1 \vee \ldots \vee \exists t\Delta_n)^0$ auch ein Teil von Φ_1^0 sein. Ferner gewannen wir, daß, wenn 1 der Wert von $\Omega \vee \exists t\Omega$ ist — wo $\Omega \vee \exists t\Omega$ ein Teil von Φ_1^0 ist —, so 1 auch der Wert von Φ_1^0 ist.

Ist nun 1 der Wert von $(\exists t\Delta_1 \vee \ldots \vee \exists t\Delta_n)^0$, so muß einer ihrer Teile $\Omega \vee \exists t\Omega$

den Wert 1 haben, dann ist 1 aber auch der Wert von Φ_1^0. So kann der Fall „der Wert von $(\exists t\Gamma \supset \Phi)_k^0 = 0$" nicht auftreten; d.h. es ist $w((\exists t\Gamma \supset \Phi)_k^0) \equiv 1$.

SATZ 10. *Ist Γ eine Formel, Γ_1 die Linearform von Γ und Γ_2 die kanonische Form von Γ_1 bezüglich Γ_1, so ist $\vdash\Gamma$ dann und nur dann, wenn $w(\Gamma_2^0) \equiv 1$ ist.*

NACHWEIS. Nach Satz 4 ist

(a) $$\vdash \Gamma \sim \Gamma_1$$

und nach Satz 6 ist $\vdash \Gamma_1 \sim \Gamma_2$, woraus

(b) $$\vdash \Gamma \sim \Gamma_2$$

folgt. Ist nun Γ herleitbar, so ist nach (a) auch Γ_1 herleitbar, und nach Satz 5 hat Γ_1 eine lineare Herleitung. Dann ist aber nach Satz 9 $w(\Gamma_2^0) \equiv 1$. Ist nun umgekehrt $w(\Gamma_2^0) \equiv 1$, so ist nach Satz 7 $\vdash \Gamma_2$ und somit ist nach (b) $\vdash \Gamma$.

Satz 10 zeigt, daß im System **D*** die Bildung der Linearform und der kanonischen Form nebst dem Wertungsverfahren ein *Entscheidungsverfahren* ergibt.

SATZ 11. *Das System* **D*** *ist widerspruchsfrei in dem folgenden Sinne:*

(1) *ist* $\vdash\Gamma$, *so ist* $\neg\Gamma$ *nicht herleitbar,*
(2) *ist* $\vdash T\alpha/t$, *so ist* $T\llcorner\alpha/t$ *nicht herleitbar,*
(3) *ist* $\vdash \neg P\llcorner\alpha/t$, *so ist* $\neg P\alpha/t$ *nicht herleitbar.*

Dieser dreifache Sinn der Widerspruchsfreiheit des deontischen Systems ist zuerst im System **D** eingeführt.

NACHWEIS. (1) ist eine unmittelbare Folge des Satzes 10. (2) ist eine unmittelbare Folge von (1), da $\vdash T\llcorner\alpha/t \sim \neg T\alpha/t$ nach Satz 3. — Wäre nun $\vdash \neg P\llcorner\alpha/t$ und $\vdash \neg P\alpha/t$, so wäre (nach (A1)) auch $\vdash \neg P\llcorner\alpha/t \wedge \neg P\alpha/t$. Aber:

(a) $\vdash (\neg P\llcorner\alpha/t \wedge \neg P\alpha/t) \sim \neg(P\llcorner\alpha/t \vee P\alpha/t)$, nach (A1).
(b) $\vdash (P\llcorner\alpha/t \vee P\alpha/t) \sim P\alpha \cup \llcorner\alpha/t$, nach (A6).
$\vdash (\neg P\llcorner\alpha/t \wedge \neg P\alpha/t) \sim \neg P\alpha \cup \llcorner\alpha/t$, aus (a) und (b).

Also aus $\vdash \neg P\llcorner\alpha/t$ und $\vdash \neg P\alpha/t$ folgt $\vdash \neg P\alpha \cup \llcorner\alpha/t$. Da aber $\vdash P\alpha \cup \llcorner\alpha/t$ (nach (A5)) ist, kann nach (1) $\neg P\alpha \cup \llcorner\alpha/t$ nicht herleitbar sein.

III. ÜBER DIE INTERPRETATION UND DIE ANWENDUNGEN DES SYSTEMS **D***

Es scheint zweckgemäß zu sein, die folgenden definierten Zeichen einzuführen:

(D3) $F\alpha/t =_{\text{def}} \neg P\alpha/t$. („$\alpha$ ist *verboten*.")
(D4) $I\alpha/t =_{\text{def}} P\alpha/t \wedge P\llcorner\alpha/t$. („$\alpha$ ist *indifferent*.")
(D5) $\alpha/t \Rightarrow \beta/u =_{\text{def}} T\alpha/t \supset \forall u(t < u \supset O\beta/u)$. („Durch α ist β *deontisch impliziert*.")
(D6) α/t Cond $\beta/u =_{\text{def}} P\beta/u \sim \exists t(t < u \wedge T\alpha/t)$. („$\alpha$ ist *die Bedingung* von β.")
(D7) α/t Incq $\beta/u =_{\text{def}} P\alpha/t \wedge (\alpha/t \Rightarrow \beta/u) \wedge \forall u(t < u \supset F\beta/u)$. („$\alpha$ ist *inkonsequent* bezüglich β.")
(D8) α/t Confl $\beta/u =_{\text{def}} O\alpha/t \wedge (\alpha/t \Rightarrow \beta/u) \wedge \forall u(t < u \supset F\beta/u)$. ($\alpha$ *konfligiert* gegen β.")

Die folgende Liste von Herleitungsschemata (nebst den bisherigen Schemata (A12)—(A34)) und die nachstehende Liste von nicht herleitbaren Formeln veranschaulicht die Anwendbarkeit des Systems **D***.

Herleitungsschemata

(In den Schemata (A35)—(A57) sind α, β und γ beliebige Terme, t, u und v beliebige Zeitvariablen.)

(A35) $\vdash P\alpha/t \vee P\llcorner\alpha/t$
(A36) $\vdash T\alpha/t \vee T\llcorner\alpha/t$
(A37) $\vdash O\alpha/t \supset P\alpha/t$
(A38) $\vdash F\alpha \cap \llcorner\alpha/t$
(A39) $\vdash \neg O\alpha/t \sim P\llcorner\alpha/t$
(A40) $\vdash O\alpha \cap \beta/t \sim (O\alpha/t \wedge O\beta/t)$
(A41) $\vdash (O\alpha/t \vee O\beta/t) \supset O\alpha \cup \beta/t$
(A42) $\vdash P\alpha \cap \beta/t \supset (P\alpha/t \wedge P\beta/t)$
(A43) $\vdash (O\alpha/t \wedge O\llcorner\alpha \cup \beta/t) \supset O\beta/t$
(A44) $\vdash (P\alpha/t \wedge O\llcorner\alpha \cup \beta/t) \supset P\beta/t$
(A45) $\vdash F\beta/t \wedge O\llcorner\alpha \cup \beta/t) \supset F\alpha/t$
(A46) $\vdash \alpha/t \Rightarrow \alpha/t$
(A47) $\vdash (T\alpha/t \wedge (\alpha/t \Rightarrow \beta/u)) \supset \forall u(t < u \supset O\beta/u)$
(A48) $\vdash (T\alpha/t \wedge \exists u(t < u \wedge F\beta/u)) \supset \neg(\alpha/t \Rightarrow \beta/u)$
(A49) $\vdash \forall t O\alpha/t \supset (\beta/u \Rightarrow \alpha/t)$
(A50) $\vdash (\alpha/t \Rightarrow \beta/u) \supset (T\alpha/t \supset \forall u(t < u \supset P\beta/u))$
(A51) $\vdash (T\alpha/t \wedge (\alpha \cap \beta/t \Rightarrow \gamma/u)) \supset (\beta/t \Rightarrow \gamma/u)$
(A52) $(O\beta/u \wedge (\beta/u \Rightarrow \gamma/v)) \supset \forall v(u < v \supset O\gamma/v) \vdash$
$((\alpha/t \Rightarrow \beta/u) \wedge (\beta/u \Rightarrow \gamma/v)) \supset ((t < u \wedge u < v) \supset (\alpha/t \Rightarrow \gamma/v))$
(A53) $(O\alpha/t \wedge (\alpha/t \Rightarrow \beta/u)) \supset \forall u(t < u \supset O\beta/u) \vdash \neg(\alpha/t \text{ Confl } \beta/u)$
(A54) $\alpha/t \text{ Cond } \beta/u \wedge P\beta/u \vdash \neg(\alpha/t \text{ Confl } \beta/u)$
(A55) $\alpha/t \text{ Confl } \beta/u \vdash \alpha/t \text{ Incq } \beta/u$
(A56) $(P\alpha/t \wedge (\alpha/t \Rightarrow \beta/u)) \supset \forall u(t < u \supset P\beta/u) \vdash \neg(\alpha/t \text{ Incq } \beta/u)$
(A57) $(P\alpha/t \wedge (\alpha/t \Rightarrow \beta/u)) \supset \forall u(t < u \supset P\beta/u) \vdash \neg(\alpha/t \text{ Confl } \beta/u)$

BEMERKUNGEN. Die sinngemäßen Aequivalenten der Formeln (A35), (A37)—(A42) sind in den meisten Systemen der deontischen Logik auch herleitbar. (A43) ist die sinngemäße Aequivalente der sog. „derivierten Verpflichtung" (englisch: „derived obligation"; gewöhnlich in der Form $(O\alpha \wedge O(\alpha \supset \beta)) \supset O\beta$ formalisiert); ähnlich ist die Bedeutung von (A44) und von (A45). In **D*** dient aber eine „strengere" Formel für den Ausdruck der derivierten Verpflichtung; diese ist als „deontische Implikation" bezeichnet (s. (D5)); ihre wichtigeren Eigenschaften sind durch (A46)—(A51) zusammengefaßt. (A46) hat keine inhaltliche Bedeutung, da in ihrem Hinterglied als Vorderglied $t < t$ vorkommt (vgl. (D5)), die nach (A8) immer „falsch" ist. (Siehe auch (N14), S. 307.) (A48) besagt, daß eine *ausgeführte* Handlung eine verbotene Handlung nie implizieren kann. — Der Abschluß der Formeln (A35)—(A51) und (A54) durch geeigneten Allquantor ist auch herleitbar (nach (A28)). Nach Satz 2 kann man in den Schemata (A52), (A53), (A55)—(A57) auch „$\supset$" statt „$\vdash$" setzen. Diese Schemata bleiben auch gültig, wenn entweder die Prämissen oder die Konklusionen oder beide mit geeigneten Allquantoren abgeschlossen werden (dann darf man aber nicht in jedem Falle „$\supset$" statt „$\vdash$" setzen!).

Im System **D** ist die der Formel (A36) entsprechende Formel $T\alpha \vee T\llcorner\alpha$ nicht herleitbar. — Die „strikte" deontische Implikation ist auch zuerst in **D** eingeführt.

Nicht herleitbare Formeln

(In den Formeln (N1)—(N22) sind a, b und c *verschiedene* Handlungsvariablen, t, u und v *verschiedene* Zeitvariablen.)

(N1) $Ta/t \supset Oa/t$
(N2) $Oa/t \supset Ta/t$
(N3) $Ta/t \supset Pa/t$
(N4) $Fa/t \supset \neg Ta/t$
(N5) $Oa/t \vee O\llcorner a/t$ (Vgl. (A35)—(A36).)
(N6) $Oa \cup b/t \supset (Oa/t \vee Ob/t)$ (Vgl. (A41).)
(N7) $(Pa/t \wedge Pb/t) \supset Pa \cap b/t$ (Vgl. (A42).)
(N8) $\forall t Fa/t \supset (a/t \Rightarrow b/u)$ (Vgl. (A49).)
(N9) $\forall u Fb/u \wedge (a/t \Rightarrow b/u)$
(N10) $\neg(\forall u Fb/u \wedge (a/t \Rightarrow b/u))$ } (Vgl. (A48).)
(N11) $(Oa/t \wedge (a/t \Rightarrow b/u)) \supset \forall u(t < u \supset Ob/u)$ (Vgl. (A43).)
(N12) $(Pa/t \wedge (a/t \Rightarrow b/u)) \supset \forall u(t < u \supset Pb/u)$ (Vgl. (A44).)
(N13) $((a/t \Rightarrow b/u) \wedge \exists u(t < u \wedge Fb/u)) \supset Fa/t$ (Vgl. (A45).)
(N14) $a/t \Rightarrow a/u$ (Vgl. (A46).)
(N15) $(a/t \Rightarrow b/u) \supset ((Oa/t \wedge t < u) \supset \forall u Pb/u)$ (Vgl. (A50).)
(N16) $(\llcorner a/t \Rightarrow a/u) \supset \forall u(t < u \supset Oa/u)$
(N17) a/t Cond a/u
(N18) a/t Confl a/u
(N19) $\neg(a/t$ Confl $a/u)$
(N20) $((a/t \Rightarrow b/u) \wedge (b/u \Rightarrow c/v)) \supset ((t < u \wedge u < v) \supset (a/t \Rightarrow c/v))$
(Deontischer Kettenschluß.)
(N21) $\neg(a/t$ Incq $b/u)$
(N22) $\neg(a/t$ Confl $b/u)$

BEMERKUNGEN. (N1)—(N7) zeigen, daß **D*** die Minimalforderungen der deontischen Logik erfüllt. (N8) zeigt, daß das PRIORsche Paradox der derivierten Verpflichtung (siehe [2] und [5]) in der deontischen Implikation nicht auftritt. (Das andere PRIORsche Paradox, dessen Aequivalente in **D*** (A49) ist, ist — meiner Ansicht nach — kein Paradox: was verpflichtend ist, ist und bleibt verpflichtend, was immer du tust.) Die Nichtherleitbarkeit von (N11)—(N13), (N15), (N20) mag bedenklich erscheinen. Meines Erachtens, ermöglicht die Nichtherleitbarkeit dieser Formeln eine ausgedehnte Anwendbarkeit des Systems; das möchte ich teils durch folgende Beispiele, teils durch einige spätere Auseinandersetzungen beleuchten.

(N11) drückt aus, daß die Konsequenz einer verpflichtenden Handlung auch „in sich" verpflichtend ist. Erklärt man aber die Pflicht als solche, deren Nichterfüllung eine Sanktion nach sich zieht, so sieht man sofort, daß (N11) nicht eine allgemein anerkannte Rechtsnorm ausdrückt. Beispiel: Die Pflicht des Schulkindes ist, sich um 8 Uhr in die Schule einzustellen, und wenn es um 8 in der Schule angekommen ist, so ist seine weitere Pflicht, seine Hausaufgabe dem Lehrer vorzuzeigen. Ist es um 8 in die Schule gegangen, doch zeigt es seine Hausaufgabe dem Lehrer nicht, so bekommt es gerecht eine Strafe. Ist es um 8 nicht in der Schule angekommen — und zeigt es deshalb seine Hausaufgabe nicht vor —, so bekommt es gerecht eine Strafe wegen seinem Wegbleiben, nicht aber wegen Nichtvorzeigen der Hausaufgabe. — Diese Auseinandersetzungen zeigen auch, daß die Bedenken von

MCLAUGHLIN (siehe [5]) nicht völlig unbegründet sind, und daß $\mathbf{D}^*$ eine befriedigende Lösung dieses Problem gibt.

(N12) drückt aus, daß die Konsequenz einer zulässigen Handlung auch zulässig ist. Dies ist auch nicht eine allgemein anerkannte Rechtsnorm. Es ist bekannt z.B., daß Sekten gibt, die ihren Mitgliedern das Waffenergreifen verboten, und es ist in einigen Staaten die Mitgliedschaft solcher Sekte zulässig, jedoch die Verweigerung des Waffenergreifens verboten. — (N13) ist mit (N12) äquivalent und (N15) ist eine Folgerung aus (N12). (N20) ist eine Folgerung aus (N11), wie es aus (A52) zu sehen ist. (N20) zeigt, daß in $\mathbf{D}^*$ „das Paradox des barmherzigen Samaritaners" (wie PRIOR eine Folgerung einer sinngemäßen Aequivalenten des deontischen Kettenschlußes bezeichnet hat, siehe in [1] die Formeln 53 und 54) nicht auftreten kann.

Deontische Kodexe

Für die Anwendungen ist es zweckmäßig, den Begriff des deontisches Kodexes einzuführen.

Eine Menge $\mathbf{K}$ von Formeln des Systems $\mathbf{D}^*$ wird als ein *deontischer Kodex* bezeichnet, falls (a) eine unmittelbare Folge irgendeines Elements bzw. zweier Elemente von $\mathbf{K}$ auch ein Element von $\mathbf{K}$ ist, und (b) $\mathbf{K}$ kein Widerspruch enthält (im Sinne des Satzes 11).

Man kann einen deontischen Kodex z.B. durch eine Prämissenmenge (die als „die Generatormenge des Kodexes" bezeichnet werden kann) angeben. Eine Aufnahme von „konstanten Handlungen" und von sich auf sie beziehenden „Axiomen" kann auch zweckmäßig sein. (Über einige konsistenten Erweiterungen der Booleschen Algebra durch Aufnahme konstanter Elemente siehe z.B. meinen Aufsatz [4].) Ist etwa A eine konstante Handlung, so kann z.B. die folgende Formel ein sich auf sie beziehendes Axiom sein:

$$(A/t \Rightarrow \alpha/u) \supset \big(PA/t \wedge \forall u(t < u \supset P\alpha/u)\big),$$

wo α ein beliebiger Term ist, t und u Zeitvariablen sind. Wir können diese Formel als „A ist absolut zulässig" interpretieren, und mit „Abs A" abkürzen. Ersetzt man α durch $a \cup \llcorner a$, so kann man leicht einsehen, daß

$$\text{Abs } A \vdash \forall t PA/t$$

ist.

Ein Kodex $\mathbf{K}$ wird als *strikt* bzw. als *homogen* bezeichnet, wenn alle Formeln der Form

$$\big(O\alpha/t \wedge (\alpha/t \Rightarrow \beta/u)\big) \supset \forall u(t < u \supset O\beta/u)$$

bzw.

$$\big(P\alpha/t \wedge (\alpha/t \Rightarrow \beta/u)\big) \supset \forall u(t < u \supset P\beta/u)$$

Elemente von $\mathbf{K}$ sind.

Ein Kodex $\mathbf{K}$ wird als *inkonsequent* bzw. als *konfligierend* bezeichnet, falls für irgendwelche Terme α, β die Formel

$$\alpha/t \text{ Incq } \beta/u \quad \text{bzw.} \quad \alpha/t \text{ Confl } \beta/u$$

ein Element von $\mathbf{K}$ ist. Ein inkonsequenter bzw. konfligierender Kodex muß nicht unbedingt einen Widerspruch enthalten, wie es (N18) bzw. (N19) zeigt.

In einem strikten Kodex gilt der „deontische Kettenschluß" (vgl. (N17)), wie es aus (A52) hervorgeht. Ein strikter Kodex kann nach (A53) nicht konfligierend sein.

Ein homogener Kodex kann nach (A56) und (A57) weder inkonsequent noch konfligierend sein. Ein konfligierender Kodex ist zugleich auch inkonsequent, wie es (A55) zeigt.

Eine ausführliche Untersuchung der deontischen Kodexe ist aber eine Aufgabe, die schon über den Rahmen dieses Aufsatzes hinausgeht.

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Literaturverzeichnis

[1] A. N. PRIOR, Escapism: the logical basis of ethics, *Essays in Moral Philosophy* (ed. A. I. Melden), University of Washington Press, Seattle 1958, S. 135—146.
[2] A. N. PRIOR, The paradoxes of derived obligation, *Mind*, **63** (1954), S. 64—65.
[3] I. RUZSA, Axiomatischer Aufbau eines Systems der deontischen Logik, *Acta Sci. Math. Szeged*, **26** (1965), S. 253—267.
[4] I. RUZSA, Über einige Erweiterungen des formalen Systems der elementaren Booleschen Algebra, *Annales Univ. Sci. Budapest, Sectio Mathematica*, **8** (1965), S. 163—180.
[5] G. H. VON WRIGHT, A note on Deontic logic and derived obligation, *Mind*, **65** (1956), S. 507—509.

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ON THE COMPARISON OF ORDER TYPES

By

B. ROTMAN (Bristol)

(*Presented by* P. ERDŐS)

Introduction

Our purpose here is to study the comparison of order types (isomorphism types of ordered sets) with respect to the following relation of embeddability: if A, B are ordered sets with types α, β respectively then $\alpha \leqq \beta$ means that A can be embedded in B by an order preserving isomorphism. If we identify α, β when $\alpha \leqq \beta \leqq \alpha$ then any set of order types is partially-ordered by embeddability. In the special case when the order types are those of well-ordered sets, i.e. ordinals, the embeddability relation is the usual ordering relation between ordinals and any set of ordinals is well-ordered by it. In the general case of arbitrary order types, however, there exist incomparable types, i.e. types α, β such that $\alpha \nleqq \beta \nleqq \alpha$ and it is the existence of such types which complicates the investigation of the embeddability relation.

The above definition of embeddability, which is a natural one, seems to have been first suggested for order types by FRAÏSSÉ in [4]. In [4], FRAÏSSÉ announces certain results and poses several questions concerning the embeddability relation for denumerable order types. In view of the fact that the methods used to investigate the embeddability relation in the case of denumerable types differ markedly from those used in the case of non-denumerable types we have restricted ourselves here to one case only, namely the case of non-denumerable types; in a later publication (see footnote 13 below) we shall present some results on denumerable order types.

The embeddability relation for non-denumerable types has so far been investigated only in the case of types of the power of the continuum; first by SIERPINSKI in [14] and then by GINSBURG in [7], [8], [9]. In fact both authors confine themselves to types of the power of the continuum which can be embedded in the continuum. Our main concern here is to show that if m is an arbitrary infinite cardinal and if the Generalised Continuum Hypothesis is assumed whenever necessary then almost all the results of Sierpinski and Ginsburg hold for certain types of power m^+ which can be described roughly as those which have subtypes dense in them of smaller power.

Preliminaries

The cardinal number of a set X will be denoted by $|X|$ and its order-type if it is ordered by $\mathrm{tp}(X)$. Inclusion between sets will be denoted by $\subset$ in the strict sense and by $\subseteq$ in the wider sense. The letters i, j, k will denote ordinals and m, n, r cardinals; no distinction will be made between finite ordinals and cardinals. Lower case Greek letters will denote arbitrary order-types where we reserve λ for the type of the continuum and η for the type of the rationals. A *similarity mapping* is an order preserving (1, 1) function. If $(A, <)$ is an ordered set of type α then the reserve type α^* denotes the type of the ordered set $(A, >)$. If X is an ordered

set then, as usual, $\sum_{x\in X}\alpha_x$ denotes the type of the ordered union $\bigcup_{x\in X}A_x$ where $\mathrm{tp}\,(A_x)=\alpha_x$. By a *section* X of an ordered set A we mean a subset X of A such that if $p, q\in X$ then $x\in X$ for all $x\in A$ such that $p<x<q$. By an *interval* of an ordered set A we mean any one of the sets $\{x\in A|x<a\}$, $\{x\in A|x\leqq a\}$ or their complements or (a, b), $(a, b]$, $[a, b)$, $[a, b]$ for $a, b\in A$ where the last four have their normal meanings, e.g. $(a, b]=\{x\in A|a<x\leqq b\}$. If I is the set of ordinals less than the ordinal k then the set $\{x_i|i\in I\}$ enumerated in a well-ordered sequence $x_0, x_1, \ldots, x_j, \ldots$ $(j<k)$ will be denoted by $(x_i|i<k)$. If X, Y are subsets of the ordered set A and $x<y$ for all $x\in X$, $y\in Y$ then we write $X<Y$. If A, B are ordered sets such that $\mathrm{tp}\,(A)=\alpha$, $\mathrm{tp}\,(B)=\beta$ and there exists a similarity mapping of A into B then we write $\alpha\leqq\beta$ and say that α is a *subtype* of β. If $\alpha\leqq\beta$ and $\beta\not\leqq\alpha$ then we write $\alpha<\beta$ and say that α is a *proper subtype* of β. If $\alpha\leqq\beta$ and $\beta\leqq\alpha$ then we say that α and β are *equivalent* and write $\alpha\equiv\beta$. If $\alpha\not\leqq\beta$ and $\beta\not\leqq\alpha$ then we say that α and β are *incomparable* and write $\alpha\|\beta$. The Generalised Continuum Hypothesis $\aleph_{k+1}=2^{\aleph_k}$ will be denoted by GCH, and its use in the enunciation and proof of a theorem will be indicated at the beginning of the theorem.

§ 1. The types λ_k

We shall introduce in this section a class of continuous types λ_k for ordinals $k\geqq 1$ which are to serve as generalisations of the type λ of the continuum. Before doing so, however, we consider some of the basic notions and assertions which will be needed both in this section and in the sequel.

An ordered set A is *dense* if (a, b) is non-empty for all $a, b\in A$. An element x of an ordered set A is an *isolated point of* A if $\{x\}=(a, b)$ for some $a, b\in A$; $x\in A$ is a *semi-isolated point* if $\{x\}=[a, b)$ or $\{x\}=(a, b]$ for some $a, b\in A$. A subset B of an ordered set A is *weakly dense in* A if every non-empty open interval of A contains a point of B. A subset B of an ordered set A is *dense in* A if B is weakly dense in A and B contains all the semi-isolated points of A. Of course, A need not be dense in order to have subsets which are either dense in A or weakly dense in A and if A is dense then the two notions are the same.

If A is an ordered set then a *Dedekind cut* in A is a partition[1] of A into two non-empty sections X, Y written $[X, Y]$ such that $X<Y$. The cut $[X, Y]$ is a *gap* if X has no last and Y no first member. For every gap $G=[X, Y]$ we introduce a gap filling element Z_G such that $X<Z_G<Y$. The *Dedekind completion* A^+ of A is the set A together with $\{Z_G|G$ is a gap in $A\}$ ordered in the obvious way. We assume that a suitable convention is adopted so that Z_G is uniquely determined by G, and A^+ is uniquely determined by A.

The two theorems which follow summarise the facts that we shall need (mostly in § 2) about the notions just introduced.

THEOREM 1.1. (a) *If* $A\subseteqq B$ *then* $A^+\subseteqq B^+$ *for any* A, B.

(b) $A^{++}=A^+$ *for any* A.

(c) *If* A, B *are similar then so are* A^+, B^+.

[1] We shall always use the term *partition* to mean disjoint partition.

(d) *If B is weakly dense in A then B is weakly dense in A^+ so that A is weakly dense in A^+ for any A.*

(e)[2] *If B is dense in A then $B^+ = A^+$.*

(f) *Any similarity map of A onto B can be extended to a similarity map of A^+ onto B^+.*

PROOF. (a), (b) are immediate from the definition of completion, and (c) is a special case of (f) proved below.

For (d): Let (p, q) be a non-empty interval of A^+ where we can assume that $p \in A^+ - A$. Now p can not have an immediate successor in A since P is a gap filling element, so that there are infinitely many members of A between p and q. Therefore (p, q) contains a non-empty interval of A and hence a point of B.

For (e): Let $[X, Y]$ be a gap in A and let $p \in A^+$ fill this gap. Put

$$X_B = \{y \in B | y \leqq \text{ some } x \in X\} \quad \text{and} \quad Y_B = \{y \in B | y \geqq \text{ some } x \in Y\}.$$

Clearly, X_B has no last element and Y_B no first since B is dense in A, so that $[X_B, Y_B]$ is a gap in B. Let $\bar{p} \in B^+$ fill this gap. We show that $\bar{p} = p$. If $\bar{p} < p$ then the interval $(\bar{p}, p)$ of A^+ must be empty for otherwise, by (d), it would contain a point of B contradicting the definition of $\bar{p}$. Thus $(\bar{p}, p)$ is an empty interval of A^+ and so $\{\bar{p}\} = [\bar{p}, p)$, i.e. $\bar{p}$ is a semi-isolated point of A^+ and hence of A. Thus $\bar{p} \in B$ by hypothesis. But this contradicts the definition of $\bar{p}$ and so $\bar{p} \not< p$. Similarly $p \not< \bar{p}$, and so $\bar{p} = p$. We have therefore shown that every gap filling element of A, i.e. every member of $A^+ - A$, belongs to B^+.

For (f): Suppose $[X_p, Y_p]$ is a gap in A filled by $p \in A^+ - A$. Let f be a similarity map of A onto B. It is clear that $f(X_p) < f(Y_p)$ in B and that $f(X_p)$ has no last element and $f(Y_p)$ no first, so that $[f(X_p), f(Y_p)]$ is a gap in B. Let $\bar{p} \in B^+ - B$ fill this gap and define a similarity map g of A^+ onto B^+ by $g(p) = \bar{p}$ for each such $p \in A^+ - A$ and $g(x) = f(x)$ for $x \in A$. It is clear that g extends f.

THEOREM 1. 2. (a) *If B is weakly dense in A then every subset of A has a set of power $\leqq |B|$ weakly dense in it.*

(b) *If B is dense in A then every subset of A has a set of power $\leqq |B|$ dense in it.*

(c) *If B is dense in A then any subset of A has at most $2^{|B|}$ similar copies of itself in A.*

PROOF. (a) is a proposition of GILLMAN[3] [5; 4. 2], we give a proof here for convenience.

Let E be any subset of A. There are $\leqq |B|^2 = |B|$ open intervals in A with end points in B. Let F be the set of all such intervals which meet E. For $X \in F$ put $f(X) \in X \cap E$ and $D_1 = \{f(X) | X \in F\}$, then $|D_1| \leqq |B|$. Let $D_2 = \{x_i | i \in I\}$ be the set of isolated poits of E. We show that $|D_2| \leqq |B|$.

By definition $\{x_i\} = (x_i^0, y_i^0)$ in E for some $x_i^0, x_i^1 \in E$, and so the open interval (x_i, x_i) in A is non-empty; and so $y_i \in (x_i^0, x_i^1)$ in A for some $y_i \in B$, which shows that $|D_2| \leqq |B|$ since $y_i \neq y_j$ for $i \neq j$.

[2] Simple examples show that one cannot replace 'dense in A' by 'weakly dense in A' in this proposition.

[3] See footnote 5.

We now put $D=D_1 \cup D_2$, and show that D is weakly dense in E. Let (a, b) be any non-empty interval in E. There exists $c \in E$ such that $a<c<b$ and we may suppose that $c \notin D_1$. Thus either (a, c) in E or (c, b) in E is infinite. Since the argument is symmetrical we may suppose the latter. The fact that B is weakly dense in A implies that (c, b) in A contains a point $y \in B$. If z is any member of B such that $z>b$ then (y, z) in A has end points in B and meets the interval (c, b) of E, and so there exists $t \in D_1$ such that $c<t<b$. Thus the arbitrary non-empty open interval (a, b) of E contains a point of D.

For (b): Let E be any subset of A and E_1 the set of semi-isolated points of E. Suppose $x \in E_1$ and that $\{x\}=[x, y)$ in E. If $[x, y)$ as an interval of A is still $\{x\}$ then x is semi-isolated in A and so $x \in B$ by assumption on B. If $x \in E_1$ is not a semi-isolated point of A then there exists $m_x \in B$ such that $x<m_x<y$ and moreover $m_x \neq m_z$ for any $z \in E_1$ such that $z=[z, t)$. Thus there are at most $|B|$ members of E_1 of the form $[x, y)=\{x\}$. Similarly there are at most $|B|$ members of E of the form $(y, x]=\{x\}$ and so $|E_1| \leqq |B|$. If now E_2 is any set of power $\leqq |B|$ weakly dense in E as given by (a) then $E_1 \cup E_2$ is dense in E and is of power $\leqq |B|$.

For (c): Let a, b be fixed members of $A-B$ and f a fixed similarity mapping of B into A. Assume $a<b$ and let F be the set of all similarity mappings of A into itself which extend f. Put

$$F(a)=\{x \in A \mid x=g(a) \quad \text{for some} \quad g \in F\}$$

and let $H(a)$ be the section of A determined by $F(a)$, i.e.

$$H(a)=\{x \in A \mid p<x<q \quad \text{for} \quad p, q \in F(a)\}.$$

We show first that $H(a)<H(b)$ in A. Since $a \in A-B$ we know that a is not semi-isolated in A and therefore that there exists $x \in B$ such that $a<x<b$. If $h, g \in F$ then $h(a)<h(x)$, $g(x)<g(b)$. But $h(x)=g(x)=f(x)$ since $x \in B$, so that

$$h(a)<h(x)=g(x)<g(b),$$

that is, $F(a)<F(b)$ and so $H(a)<H(b)$. Thus $H=\{H(x) \mid x \in A-B\}$ is a set of mutually disjoint sections of A. Now if $|H(x)|=2$ then both members of $H(x)$ are semi-isolated in A and so belong to B; if $|H(x)|>2$ then $H(x)$ contains a member of B. It follows therefore that H can have at most $|B|$ members $H(x)$ satisfying $|H(x)|>1$ and so $|F| \leqq \prod_{x \in X} |H(x)|$ where $X=\{x \in A \mid |H(x)|>1\}$. Thus $|F| \leqq |A|^{|B|}$ since $|H(x)| \leqq |A|$ for all $x \in A$. But B is dense in A implies that $|A| \leqq 2^{|B|}$ by virtue of 1. 1 (e) and the definition of Dedekind completion, and so $|F| \leqq (2^{|B|})^{|B|} \leqq 2^{|B|}$.

We have thus shown that the number of extensions to A of a given similarity mapping f of B into A is at most $2^{|B|}$. But the number of distinct such f's is at most the number of subsets of A of power $|B|$, i.e. at most $|A|^{|B|} \leqq 2^{|B|}$. Thus the number of distinct similarity mappings of A into itself is at most $(2^{|B|})^2=2^{|B|}$. The conclusion now follows from the fact that the above reasoning can be applied to an arbitrary subset of A by virtue of 1. 2 (b).

We observe that if the hypothesis of 1. 2 (c) is weakened by requiring only that B be weakly dense in A then the conclusion is false as the following example

shows.[4] Let A be the ordered set obtained from the real numbers by replacing each irrational x by $\{x^0, x^1\}$ with the ordering $x^0 < x^1 < y^0 < x^1$ if $x < y$. Now the rationals are weakly dense in A and yet A has 2^c ($c = 2^{\aleph_0}$) distinct copies of the irrationals, obtained by selecting one and only one member from each set $E_x = \{x^0, x^1\}$.[5]

We need now some more definitions. Let k be an ordinal $\geqq 0$. After Hausdorff we define an ordered set A to be an *η_k-set* if whenever X, $Y \subseteq A$ are such that $|X| < \aleph_k$, $|Y| < \aleph_k$ and $X < Y$ then there exists $a \in A$ such that $X < a < Y$.

If m is an infinite cardinal number then an ordered set A is *m-universal* if every ordered set of power m can be embedded in A.

The following properties of η_k-sets are all due to HAUSDORFF [12; p. 180ff.].

THEOREM 1. 3. (a) *Any two η_k-sets of power $\aleph_k$ are similar.*

(b) *Any η_k-set is $\aleph_k$-universal.*
(c) *Any open interval of an η_k-set is an η_k-set.*
(d) *If $\aleph_k$ is singular there exists no η_k-set of power $\aleph_k^1$.*
(e) *Any η_{k+1}-set has power $\geqq 2^{\aleph_k}$ for all $k \geqq 0$.*

Let k be an arbitrary ordinal. We make the following definitions.

U_{k+1} = set of all sequences $x = (x_i | i < \omega_k)$ where $x_i = 0$ or 1 ordered lexicographically.
$U^1_{k+1} = \{x \in U_{k+1} | x_i = 1$ for all $i > i_0$ for some $i_0 < \omega_k\}$
$U^0_{k+1} = \{x \in U_{k+1} | x_i = 0$ for all $i > i_0$ for some $i_0 < \omega_k\}$
$R_{k+1} = U_{k+1} - U^1_{k+1} - 0$ where $0 =$ the ω_k sequence of zeros.
$\lambda_{k+1} = \mathrm{tp}\,(R_{k+1})$.

We observe that $\lambda_1 \equiv \lambda$ and that for any $k \geqq 0$ $\lambda_{k+1} \equiv \mathrm{tp}\,(U_{k+1})$.

The following properties of U^0_{k+1} and R_{k+1} are due to SIERPINSKI [15; pp. 460—6] with the exception of (b) for singular $\aleph_k$ which is due to GILLMAN [6].

THEOREM 1. 4. (a) *U^0_{k+1} is an η_k-set of power $\sum_{j<k} 2^{\aleph_j}$ if $\aleph_k$ is regular.*

(b) *U^0_{k+1} is $\aleph_k$-universal for all $k \geqq 0$.*
(c) *U^0_{k+1} is dense in $R_{k+1} = (U^0_{k+1})^+$.*
(d) *R_{k+1} contains no well-ordered or reverse well-ordered subset of power $\aleph_{k+1}$.*
(e) *$|U^0_{k+1}| = \aleph_k$ for all k if GCH is assumed.*

We now prove several propositions about the types λ_k all of which are well known for λ. The first proposition is a corollary of the following theorem.

THEOREM 1. 5. *If for a given ordinal k, A is dense in an η_k-set H then A contains an η_k-set. Consequently, if for a given k the ordered set A contains no η_k-set then the Dedekind completion A^+ of A contains no η_k-set.*

[4] I am grateful to F. ROWBOTTOM for correcting an obscurity in an earlier draft of this section and for providing the counter-example given here.

[5] It should be mentioned here that theorem 4.3 of [5] states 'If A is dense in B then B has at most $2^{|B|}$ subsets of any given order type.' The definition of 'A dense in B' adopted in [5; p. 442] is the same as 'A weakly dense in B' adopted here so that the example given above is in fact a counterexample to 4.3 of [5].

PROOF. For $x, y \in A$ put $x \wedge y$ if and only if there exists no $h \in H$ such that $x < h < y$. It is clear that $\wedge$ is an equivalence relation and that the equivalence classes $[a]$ of $\wedge$ are sections of A; so that if $D = \{x | x \in [a]\}$ is a set of distinct representatives then D is totally ordered as a subset of A. We show that D is an η_k-set. Observe first that D and H are dense in each other. For if $h_1, h_2 \in H$, $h_1 < h_2$ then there exists $a \in A$ such that $h_1 < a < h_2$ since A is dense in H. Plainly $h_1 < y < h_2$ for all $y \in [a]$ and so $h_1 < d < h_2$ where $d \in D$ is the representative of $[a]$; thus D is dense in H. Now let x, y be distinct members of D with $x < y$. If we consider x, y as members of A then their distinctness as members of D implies not $x \wedge y$, i.e. there exists $h \in H$ such that $x < h < y$ and so H is dense in D.

Suppose now that D_1, D_2 are subsets of D such that $D_1 < D_2$ and $|D_1| < \chi_k$, $|D_2| < \chi_k$. For each pair of members x, y of D_1 and of D_2, let $h(x, y) \in H$ be such that $x < h(x, y) < y$; this is possible as H is dense in D. Put

$$H_1 = \{h(x, y) | x, y \in D_1\}, \quad H_2 = \{h(x, y) | x, y) | x, y \in D_2\}.$$

Clearly $H_1 < H_2$ and $|H_i| \leqq |D_i|^2 < \chi_k$ for $i = 1, 2$, so that since H is an η_k-set there exist $z_1, z_2 \in H$ such that $H_1 < z_1 < z_2 < H_2$. Now $D_1 \leqq z_1$ for otherwise there would exist $y \in D_1$ such that $z_1 < y$ and so $z_1 < h(z_1, y) \in H$ contrary to $H_1 < z_1$. Similarly $z_2 \leqq D_2$. Thus $D_1 \leqq z_1 < z_2 \leqq D_2$. But D is dense in H so that there exists $z \in D$ such that $z_1 < z < z_2$. We have therefore shown that if D_1, D_2 are neighbouring subsets of D each of power $< \chi_k$, there exists z satisfying $D_1 < z < D_2$. Thus D is an η_k-set. To see the last part of the theorem we observe that if A^+ contains an η_k-set H then A would be dense in H, since A is dense in A^+, and so A would, by the first part, contain an η_k-set contrary to hypothesis.

COROLLARY. *If A is dense in R_{k+1} then A is $\aleph_k$-universal.*

PROOF. R_{k+1} is dense and so therefore is A. Hence $A^+ = R_{k+1}$ by 1. 1 (e). But R_{k+1} contains the η_k-set U^0_{k+1} and so by the theorem A contains an η_k-set which makes A $\aleph_k$-universal by 1. 3 (b).

THEOREM 1. 6. *If $a, b \in R_{k+1}$ then* $\mathrm{tp}\,(a, b) \equiv \lambda_{k+1}$ *for all* $k \geqq 0$.

PROOF. Put $a = A0x$, $b = A1y$ where A is a sequence (of 0's, 1's) of length $< \omega_k$ and $x, y \in U_{k+1}$. Now $x_i = 0$ for some $i < \omega_k$ for otherwise $a \in U^1_{k+1}$. Thus we can put $x = C0z$ where C is a sequence, possibly empty, of length $< \omega_k$ of 1's and $z \in R_{k+1}$. Now let p be an arbitrary member of R_{k+1}, and define $c_p = A0C1p$. Then

$$a = A0x = A0C0z < A0C1p < A1y = b,$$

i.e. $a < c_p < b$. Moreover, if $p < q$ then $c_p < c_q$. Thus (a, b) contains a copy, namely $\{c_p | p \in R_{k+1}\}$, of R_{k+1}.

THEOREM 1. 7. *If* $\mathrm{tp}\,(A) = \lambda_{k+1}$ *then A is the union of $2^{\aleph_k}$ disjoint sets each of which has order type* $\equiv \lambda_{k+1}$.

PROOF. We can without loss of generality take $A = R_{k+1}$. For $a \in R_{k+1}$ let $S(a) = \{x \in R_{k+1} | x_{j2} = a_{j2} \text{ for all } j < \omega_k\}$. It is immediate that $S(a) \cap S(b) \neq \emptyset$ if and only if $a \in S(b)$ so that there are $2^{\aleph_k}$ disjoint subsets of R_{k+1} of the form $S(a)$ for varying a. For $x \in S(a)$ put $\bar{x} = (x_1, x_3, \ldots, x_{j2+1}, \ldots)$ where $j < \omega_k$, and $\overline{S(a)} = \{\bar{x} | x \in S(a)\}$. If $\overline{S(a)}$ is ordered lexicographically then it is similar to the set

U_{k+1}. So that $S(a)$ is similar to U_{k+1} since for $x, y \in S(a)$ we have $x < y$ if and only if $\bar{x} < \bar{y}$. Thus there are $2^{\aleph_k}$ disjoint subsets of R_{k+1} each of type $\equiv \lambda_{k+1}$, and the result follows immediately.

COROLLARY. *If* $\mathrm{tp}(A) = \lambda_{k+1}$ *and* $|B| < |A|$ *then* $\mathrm{tp}(A - B) \equiv \lambda_{k+1}$.

If we put $k = 0$ in the corollary then we obtain a proposition of SIERPINSKI [14; p. 261]. Our proof differs from that of Sierpinski.

THEOREM 1. 8. *The subtypes of* λ_{k+1} *are precisely the order-types of chains of subsets of a set of power* $\aleph_k$ *whose subsets are partially ordered by inclusion.*

PROOF. Denote by $[A]$ the set of all subsets of A and by $\mathrm{Ch}([A], \subseteq)$ the set of all order-types of chains which occur in $[A]$ when it is partially ordered by inclusion. Put $A = (a_i | < \omega_k)$. For $B \in [A]$ denote by $x(B)$ the characteristic function of the set B; so that $x(B) = (x(B)_i | i < \omega_k)$ where $x(B)_i = 1$ if $a_i \in B$ and $x(B)_i = 0$ if $a_i \notin B$. Suppose $B, C \in [A]$ with $C \subseteq B$. Then for some $i < \omega_k$ we must have $x(B)_i = 1$, $x(C)_i = 0$; let i_0 be the least such i. Then $x(B)_i = x(C)_i$ for all $i < i_0$ and so $x(C) < x(B)$ in the lexicographic ordering of U_{k+1}. Thus any member of $\mathrm{Ch}([A), \subseteq)$ is order isomorphic to a subset of U_{k+1}; and since $\mathrm{tp}(U_{k+1}) \equiv \lambda_{k+1}$, this proves that $\mathrm{Ch}([A], \subseteq)$ is contained in the set of subtypes of λ_{k+1}. Conversely, since $R_{k+1} = (U^0_{k+1})^+$, every $x \in R_{k+1}$ is uniquely determined by its predecessors in U^0_{k+1}. Thus if $P(a) = \{x \in U^0_{k+1} | x \leqq a\}$ for each $a \in R_{k+1}$ then R_{k+1} is order isomorphic to the family of sets $\{P(a) | a \in R_{k+1}\}$ partially ordered by inclusion.

We shall want to refer to the following well-known result of Bernstein—Sierpinski. A proof can be found for example in SIERPINSKI [15; p. 455].

THEOREM 1. 9. *If* F *is a family of* m *sets each of power* m *then there exists a set* G *which meets every member of* F *in* m *points but does not contain any member of* F.

We now establish the main theorem of this section.

THEOREM 1. 10 (GCH). *If* $\{\alpha_x | x \in I\}$ *is a set of types such that* (i) $\alpha_x < \lambda_{k+1}$, $|\alpha_x| = \aleph_{k+1}$ *and* (ii) $|I| \leqq \aleph_{k+1}$ *then there exists a type* α *such that* (a) $\alpha < \lambda_{k+1}$, $|\alpha| = \aleph_{k+1}$, (b) α *is incomparable with* α_x *for all* $x \in I$, (c) α *is dense and* $\aleph_k$*-universal.*

PROOF. For each α_x, let $T(\alpha_x) = \{X \subseteq R_{k+1} | \mathrm{tp}(X) = \alpha_x\}$. By 1. 2 (c), GCH and 1. 4 (e) it follows that $|T(\alpha_x)| \leqq \chi_{k+1}$. So that by (ii), $T = \bigcup_{x \in I} T(\alpha_x)$ has power $\leqq \chi_{k+1}$, and by (i) each member of T has power χ_{k+1}. If L is the set of all open intervals of R_{k+1} then $|L| = \chi_{k+1}$ and each member of L has power χ_{k+1} by 1. 6. Let $(T_i | i < \omega_{k+1})$ be an enumeration of T, $(L_i | i < \omega_{k+1})$ an enumeration of L and $(f_i | i < \omega_{k+1})$ an enumeration of all the similarity mappings of R_{k+1} into itself where each such mapping occurs χ_{k+1} times. We are going to define a set A by transfinite induction such that $\mathrm{tp}(A) = \alpha$ has the properties (a)—(c). Observe first that

$$(*) \qquad |R_{k+1} - f_i^{-1}(T_j)| = \chi_{k+1} \quad \text{for all} \quad i, j < \omega_{k+1}.$$

To see this note that if $|R_{k+1} - f_i^{-1}(T_j)| < \chi_{k+1}$ then by the corollary to 1. 7 it follows that $\mathrm{tp}(f_i^{-1}(T_j)) \equiv \lambda_{k+1}$, i.e. $\mathrm{tp}(T_j) \equiv \lambda_{k+1}$ contrary to (i). Let $T_0, L_0, \ldots, T_i, L_i, \ldots$ $(i < \omega_{k+1})$ be an enumeration of $T \cup L$ and let $e < \omega_{k+1}$. Suppose we have defined for all $i < e$ elements x_{ij}, a_i, b_i, c_i subject to the conditions

(1) $x_{ij}\in R_{k+1}-f_i^{-1}(T_j)$ for $j\leqq i$, (2) $a_i, b_i\in T_i$, (3) $c_i\in L_i$, (4) $a_i\neq b_j$, $b_i\neq a_j$ for $j\leqq i$, (5) $x_{ij}\neq b_n$ for $n<i$, (6) $c_i\neq b_n$ for $n<i$. Put $A_e=\{a_i\,|\,i<e\}$, $B_e=\{b_i\,|\,i<e\}$, $C_e=\{c_i\,|\,i<e\}$, $X_e=\bigcup_{i<e}\{x_{ij}\,|\,j<i\}$. Clearly $|A_e|<\chi_{k+1}$, $|B_e|<\chi_{k+1}$, $|C_e|<\chi_{k+1}$, $|X_e|<\chi_{k+1}$. The sets $D_{ej}=R_{k+1}-f_e^{-1}(T_j)-B_e$ each have power χ_{k+1} for $j\leqq e$ by $(*)$, so there exists $x_{ej}\in D_{ej}$ for $j\leqq e$ such that x_{ej} satisfies (1) and (5). Further the set L_e-B_e has power χ_{k+1} so there exists $c_e\in L_e-B_e$ satisfying (3) and (6). Finally $T_e-B_e-A_e$ has power χ_{k+1} so that there exist $a_e, b_e\in T_e-B_e-A_e$ which satisfy (2) and (4). Thus we can assume that (1)—(6) hold for all $i<\omega_{k+1}$ and put $A=\bigcup_{j\leqq i<\omega_{k+1}}\{x_{ij}\}\cup\bigcup_{i<\omega_{k+1}}\{a_i\}\cup\bigcup_{i<\omega_{k+1}}\{c_i\}$.

Let $\mathrm{tp}\,(A)=\alpha$. Suppose $\alpha\cong\alpha_x$ for some $x\in I$, then A contains a copy of α_x, i.e. $A\supseteq T_j$ for some $T_j\in T(\alpha_x)$. But $b_j\in T_j$ by (2) and $b_j\notin A$ by (4)—(6) and so $\alpha\ncong\alpha_x$ for any $x\in I$. To show that $\alpha\not<\alpha_x$ for any $x\in I$ we observe first that A is dense in R_{k+1} since the condition $c_i\in A$ for all $i<\omega_{k+1}$ together with (3) means that A meets every open interval of R_{k+1}. Now suppose $\alpha<\alpha_x$ for some $x\in I$. Then there exists a similarity mapping g from A into T_j for some $T_j\in T(\alpha_x)$. By 1.1 (f) and the fact that A is dense in R_{k+1} we know that g is the restriction of f_i to A for some $i<\omega_{k+1}$, i.e. $f_i(A)\subseteqq T_j$, and since each mapping f_i occurs χ_{k+1} times in the given enumeration it follows that we may assume that $j\leqq i$. But this means that $f_i^{-1}(T_j)$ contains x_{ij} by definition of A, which contradicts (1). Thus $\alpha\not<\alpha_x$ for any $x\in I$. We have therefore shown that α is incomparable with all α_x for $x\in I$. It is obvious that $|\alpha|=\chi_{k+1}$, and $\alpha<\lambda_{k+1}$ follows at once from the fact that if $\alpha\ncong\alpha_x$ for some $\alpha_x<\lambda_{k+1}$ then α cannot be equivalent to λ_{k+1}. Thus (a), (b) hold. Finally (c) holds by virtue of the corollary to 1. 5.

REMARK. If $k=0$ then the Continuum Hypothesis is not used in the above proof and the resulting theorem is essentially a proposition of GINSBURG [8; theorem 4. 2].

ERDŐS and RADO [3; theorem 29] establish the following[6] result:

$$\lambda\nrightarrow(\alpha_0, \alpha_1, \ldots)^1_m \text{ where } |\alpha_i|=|\lambda| \text{ for } i<m \text{ and } 2\leqq m\leqq|\lambda|.$$

The following corollary of theorem 1. 10 is a straightforward generalization of this result.

COROLLARY (GCH). *$\lambda_{k+1}\nrightarrow(\alpha_0, \alpha_1, \ldots)^1_m$ where $|\alpha_i|=\aleph_{k+1}$ for $i<m$ and $2\leqq m\leqq\aleph_{k+1}$.*

PROOF. It is obvious from the definition of the partition symbol that we need only consider the case when $\alpha_i\leqq\lambda_{k+1}$ for all $i<m$. Further, that we can assume that $\alpha_i<\lambda_{k+1}$ since if $\lambda_{k+1}\nrightarrow(\beta_0, \beta_1, \ldots)^1_m$ and $\beta_i\leqq\alpha_i$ for all $i<m$ then $\lambda_{k+1}\nrightarrow(\alpha_0, \alpha_1, \ldots)^1_m$. Thus α_i for $i<m$ satisfy the conditions of theorem 1. 10. Let A be the set constructed in the proof of 1. 10 and let $\{P_i|i<m\}$ be a partition of R_{k+1} which is a refinement of the partition $\{A, R_{k+1}-A\}$. If for any $i<m$, P_i contains

[6] If θ, α_i are order types for $i<m$ then the symbol $\theta\rightarrow(\alpha_0, \alpha_1, \ldots)^i_m$ means that if H is a set of type θ which is partitioned into m sets then at least one of these sets will contain a set of type α_i for some $i<m$. $\theta\nrightarrow(\alpha_0, \alpha_1, \ldots)^i_m$ denotes the negation of $\theta\rightarrow(\alpha_0, \alpha_1, \ldots)^1_m$. See [3] for further details.

a subset of type α_j then P_i contains a member of $T(\alpha_j)$ and therefore so does A or $R_{k+1}-A$ which contradicts the definition of A.

It is possible to establish the following refinement of theorem 1. 9: *If F is a family of m sets each of power m then every set which meets every member of F in m points but contains no member of F is the union of 2^m almost disjoint sets*[7] *with this property.* If we apply this result to the set A constructed in the proof of 1. 10 we obtain

THEOREM 1. 11 (GCH). *If $\{\alpha_x|x\in I\}$ is a set of types such that* (i) $\alpha_x<\lambda_{k+1}$, $|\alpha_x|=\aleph_{k+1}$ *and* (ii) $|I|\leqq\aleph_{k+1}$ *and if E is an ordered set of type λ_{k+1} then E is the union of $\aleph_{k+2}$ almost disjoint sets each of which has order-type incomparable with all the α_x.*

REMARK. By using the methods of this section it is easy to produce types satisfying the hypothesis of theorems 1. 10, 1. 11, i.e. types which are $<\lambda_{k+1}$; and to produce, for any given $\alpha<\lambda_{k+1}$ such that $|\alpha|=\aleph_{k+1}$, types β, γ such that $\beta<\alpha<\gamma<\lambda_{k+1}$ with $|\beta|=\aleph_{k+1}$. Much stronger results than these, however, will be established in the next section.

§ 2. Heterogeneous sets

An ordered set H is *heterogeneous* if no two disjoint subjects of H of power $|H|$ are similar.

The notion of a heterogeneous set was first introduced by DUSHNIK & MILLER [2] who established the existence of a heterogeneous subset of the continuum of power $2^{\aleph_0}$. Their result was extended by GINSBURG[8] [9; theorem 5] who showed that any subset of the continuum of power $2^{\aleph_0}$ has a heterogeneous subset of power $2^{\aleph_0}$. Theorem 2. 1 below, which is the main result of this section is a further extension of these results.[9]

THEOREM 2. 1 (GCH). *If M is an infinite ordered set with a subset dense in it of power $<|M|$ then M (and consequently any subset of M of power $|M|$) has a heterogeneous subset of power $|M|$.*

PROOF. If M has a set of power $n<m=|M|$ dense in it then $m\leqq 2^n$ and so by GCH,[10] $m=n^+=2^n$. Let $[M]^m$ denote the set of all subsets of M of power m. Call a member A of $[M]^m$ *M-bad* if A is not similar to any $B\in[M]^m$, with $B\cap A=\emptyset$, and *M-good* otherwise. If the number of M-good subsets of M is less than m then since $[M]^m$ has m disjoint members it follows that one of these, B say, will contain no M-good set; i.e. every member of $[B]^m$ will be M-bad and hence B-bad and so B would be a heterogeneous subset of M of power m. Thus we can suppose that $[M]^m$ has at least m M-good members. Let $K=\{A^+|A\in[M]^m\}$. By 1. 2 (b) each $A\in[M]^m$ has a set A^* of power at most n dense in it, and $(A^*)^+=A^+$ by 1. 1 (e). Thus each member of K is the Dedekind completion of a set of power n and so

[7] Two sets A, B are *almost disjoint* if $|A\cap B|<|A|, |B|$.

[8] GINSBURG [9; p. 344] uses the term *property (A)* for heterogeneous.

[9] BUCHI [1] has also generalised the original theorem of Dushnik and Miller, but in a direction different from that considered here.

[10] m^+ denotes the next cardinal after m.

$|K| \leqq |[M]^n| = 2^n = m$. Let T be the set of all similarity maps of members of K into M^+. Then by theorem 1. 2 (c) and GCH, $|T| \leqq m$. For each $A^+ \in K$, let $L(A^+)$ be the subset of T consisting of those maps f: $A^+ \to M^+$ such that $\{x \in A^+ \cap M | f(x) \neq x\}$ has power m. Put

$$S = \{(A^+, f) | A^+ \in K, f \in L(A^+)\}.$$

We show that $|S| = m$. Since $|T| \leqq m$ it follows that $|S| \leqq m^2 = m$. On the other hand $|S| \geqq m$ since if A is an M-good subset of M then $(A^+, h) \in S$ for some h. To see this, let $A \in [M]^m$ be M-good. Then $g(A) \cap A$ is empty for some similarity map g of A into M; and so the set $\{x \in A^+ \cap M | h(x) \neq x\}$ has power m, where h is the similarity map from A^+ into M^+ extending g, the existence of which is given by theorem 1. 1 (f).

Let Ω be the least ordinal of power m. We enumerate M as $(x_i | i < \Omega)$ and $S = ((A_i^+, f_i) | i < \Omega)$. Let $J_i = \{x \in M \cap A_i^+ | f_i(x) \neq x\}$ for $i < \Omega$. Observe that $|J_i| = m$ by definition of f_i. We are going to define a set $P = (p_i | i < \Omega)$ by induction. Let $p_0 \in J_0$ and suppose the members p_i of $P_{j_0} = (p_i | i < j_0)$ have been defined subject to the condition that $p_j \in J_j - D_j$ for $j < j_0$ where

$$D_j = P_j \cup \bigcup_{i<j} f_i(P_j) \cup \bigcup_{i<j} f_i^{-1}(P_j).$$

Let $p_{j_0} \in J_{j_0} - D_{j_0}$. It is clear that $|D_{j_0}| < m$, so that p_{j_0} exists. We show that $P = (p_i | i < \Omega)$ is heterogeneous. Suppose F, G are disjoint subsets of P of power m, and f is a similarity map of F onto G. By theorem 1. 1 (f) f can be extended to a similarity map g of F^+ onto G^+. Now $F^+ \in K$ and $g \in L(F^+)$ since

$$\{x \in F^+ \cap M | g(x) \neq x\} \supseteq \{x \in F \cap M | g(x) \neq x\} \supseteq g(F) = G,$$

so that $\{x \in F^+ \cap M | g(x) \neq x\}$ is of power m. Thus $F^+ = A_{k_0}^+$ and $g = f_{k_0}$ for some $k_0 < \Omega$. Put $f_{k_0}(p_i) = q_i$ and consider q_i when $p_i \in F$ and $i > k_0$.

(i) $q_i \neq p_i$ since $F \cap G$ is empty.

(ii) $q_i \neq p_{i+k}$ for any $k \geqq 0$ since $p_i \in P_{i+k}$ and $p_{i+k} \notin f_{k_0}(P_{i+k})$ by construction.

(iii) $q_i \neq p_j$ for $j < i$ for otherwise $p_i = f_{k_0}^{-1}(p_j) \in f_{k_0}^{-1}(P_i)$ which contradicts the definition of p_i.

Thus if $i > k_0$ and $p_i \in F$ then $f_{k_0}(p_i) \notin P$. So that $|f_{k_0}(F) \cap P| < m$ which contradicts $f_{k_0}(F) = G$, $G \subseteq P$, $|G| = m$. Therefore our assumption of the existence of the map f of F onto G has led to a contradiction, which proves that P is heterogeneous.

COROLLARY (GCH). *If α is a type such that $\alpha < \lambda_{k+1}$, $|\alpha| = \aleph_{k+1}$ then there exists a type $\beta < \alpha$ such that $|\beta| = \aleph_{k+1}$ and β is heterogeneous.*

We observe that if $k = 0$ and we replace $\aleph_1$ by $2^{\aleph_0}$ then the above corollary remains valid and refers to subsets of the continuum and moreover the GCH is not used in its proof.

We need now some more definitions. An ordered set A is *m-dense* if every interval of A has power m ($\aleph_0$-dense is the same as dense). A point p of an ordered set A is a *fixed point* of A if every similarity map f of A into itself is such that $f(p) = p$. If every point of A is fixed then we say that A is *rigid.*[11]

[11] GINSBURG [9; p. 341] uses the term *exact* for rigid.

THEOREM 2. 2. *If A is heterogeneous and $|A|$-dense then A is rigid.*

PROOF. Let $p \in A$ and f a similarity map of A into itself. Suppose $f(p) > p$. Then for $x \in A$ such that $p < x < f(p)$ we have $f(x) > f(p)$ so that the interval $B = (p, f(p))$ of A has power $|A|$ since A is $|A|$-dense, and B is disjoint from $f(B)$ contrary to the heterogeneity of A. Thus $f(p) \not> p$ and similarly $f(p) \not< p$, so that every point of A is fixed.

The next result that we need is due to GLEYZAL [10; theorem 22] and independently to ERDŐS and RADO [3 lemma, p. 446].

LEMMA 2. 3. *If M is an ordered set which is such that $|M| = \aleph_k$ is regular and $\omega_k, \omega_k^* \not\equiv \mathrm{tp}\,(M)$ then M contains an $\aleph_k$-dense subset.*

THEOREM 2. 4 (GCH). *If M is an ordered set with a subset dense in it of power $< |M|$ then M (and consequently any subset of M of power $|M|$) has a rigid $|M|$-dense subset.*

PROOF. The hypothesis with the GCH implies that $|M| = \aleph_{k+1}$ for some k, and hence $|M|$ is regular. Now M has a set dense in it of power $\aleph_k$ and so any set of disjoint intervals of M is of power $\aleph_k$ which implies that $\omega_k, \omega_k^* \not\equiv \mathrm{tp}\,(M)$. Hence by 2. 3 any subset A of M of power $\aleph_{k+1}$ has an $\aleph_{k+1}$-dense subset $B \subseteq A$. If now A is a heterogeneous subset of M given by theorem 2. 1 then it follows from theorem 2. 2 that B is rigid.

From theorem 2. 4 we immediately obtain

THEOREM 2. 5 (GCH). *If α is a type such that $\alpha < \lambda_{k+1}$, $|\alpha| = \aleph_{k+1}$ then there exists a rigid $\aleph_{k+1}$-dense type $\beta < \alpha$.*

If $k = 0$ and $\aleph_1$ is replaced by $2^{\aleph_0}$ then theorem 2. 5 refers to subsets of the continuum. It does not then depend on the continuum hypothesis and is a proposition of GINSBURG [9; theorem 4]. The same remarks apply to the next theorem (see GINSBURG [7; theorem 2. 4]).

THEOREM 2. 6 (GCH). *If $\alpha < \lambda_{k+1}$ then there exists a rigid $\aleph_{k+1}$-dense type β such that $\alpha \leqq \beta < \lambda_{k+1}$.*

PROOF. Let $(f_i | i < \omega_{k+1})$ be an enumeration of the similarity maps other than the identity of R_{k+1} into itself where each map occurs $\aleph_{k+1}$ times. If $E_i = \{x \in R_{k+1} | f_i(x) \neq x\}$ then $R_{k+1} - E_i$ can not be dense in R_{k+1} for otherwise f_i would be the identity map by theorems 1. 1 (f), 1. 1 (e); so that E_i contains an interval I_i of R_{k+1}. Let E be a subset of R_{k+1} such that $\mathrm{tp}\,(E) = \alpha$. It follows from theorem 1. 6, the corollary to 1. 7 and the fact that $\mathrm{tp}\,(E) < \lambda_{k+1}$ that $I_0 - f_1^{-1}(E)$ has power $\aleph_{k+1}$. We are going to define a set $X = (x_i | i < \omega_{k+1})$ by induction. Put $x_0 \in I_0 - f_0^{-1}(E)$, $y_0 = f_0(x_0)$. Suppose that x_i, y_i have been defined for all $i < j$. Put $P_j = \{x_i, y_i | i < j\}$. By the corollary to 1. 7, $(\mathrm{tp}\,(E \cup P_j) < \lambda_{k+1}$ so that $I_j - f_j^{-1}(E \cup P_j) - P_j$ has power $\aleph_{k+1}$. Let $x_j \in I_j - f_j^{-1}(E \cup P_j) - P_j$, $y_j = f_j(x_j)$. We show first that $X = (x_i | i < \omega_{k+1})$ is dense in R_{k+1}. If I is an interval of R_{k+1} then I is not rigid so that there exists a similarity map h of I onto a proper subset of itself. If a map f is defined as the identity on $R_{k+1} - I$ and as h on I then f is a similarity map of R_{k+1} into itself, and so $f = f_i$ for some $i < \omega_{k+1}$. But this means, by the definition of f_i that $x_i \in I$. Thus every interval of I contains points of X, i.e. X is dense in R_{k+1}.

We note further that since each map f_i occurs $\aleph_{k+1}$ times in the original enumeration the interval I contains $\aleph_{k+1}$ members of X; this together with the fact that X is dense in R_{k+1} shows that X is $\aleph_{k+1}$-dense. We show now that X is rigid. If g is a similarity map of X onto a proper subset of itself then by theorems 1. 1 (f), 1. 1 (e) and the above we can extend g to a similarity map f_j of R_{k+1} into itself. But $f_j(x_j) = g(x_j)$ and $f_j(x_j) \notin X$ by construction, so that the assumption of the existence of g leads to a contradiction and therefore X is rigid. If we put $\mathrm{tp}(X) = \beta$ then β is as required.

THEOREM 2. 7 (GCH). *If α is a type such that $\alpha < \lambda_{k+1}$, $|\alpha| = \aleph_{k+1}$ then there exist* (a) *$\aleph_{k+2}$ mutually incomparable types ϱ_i such that $\alpha < \varrho_i < \lambda_{k+1}$ and* (b) *$\aleph_{k+2}$ mutually incomparable types μ_i such that $\mu_i < \alpha$ and $|\mu_i| = \aleph_{k+1}$.*

PROOF. *For* (a): Let $(\alpha_i | i < \omega_{k+2})$ be any set of mutually incomparable subtypes of λ_{k+1} guaranteed by theorem 1. 10, and let ϱ be a rigid type such that $\alpha \leqq \varrho < \lambda_{k+1}$ given by theorem 2. 6. Put $\varrho_i = \alpha_i + \varrho + \alpha_i$ for all $i < \omega_{k+2}$. We show that the types ϱ_i are as required. Suppose that $(A_i | i < \omega_{k+2})$, $(B_i | i < \omega_{k+2})$ are two families of sets such that $A_i \cap A_j = \emptyset = B_i \cap B_j$ if $i \neq j$, $A_i \cap B_j = \emptyset$ for all $i, j < \omega_{k+2}$ and $\mathrm{tp}(A_i) = \alpha_i = \mathrm{tp}(B_i)$ and that P_1, P_2 are disjoint sets which are disjoint from the A_i, B_i such that $\mathrm{tp}(P_1) = \mathrm{tp}(P_2) = \varrho$. Let f be a similarity map of the ordered union $A_i \cup P_1 \cup B_i$ into $A_j \cup P_2 \cup B_j$ for $i \neq j$. Now $f(A_i) \subsetneqq A_j$, $f(B_i) \subsetneqq B_j$ since $\alpha_i \not\leqq \alpha_j$. Hence $f(P_1) \subset P_2$ which is impossible since ϱ is rigid. Thus $\varrho_i \not\leqq \varrho_j$. Similarly $\varrho_j \not\leqq \varrho_i$ and so ϱ_i is incomparable with ϱ_j for $i \neq j$. Finally $\varrho_i < \lambda_{k+1}$ since the sum of three proper subtypes of λ_{k+1} is a proper subtype of λ_{k+1} (Cf. theorem 3. 1 below).

For (b): By theorem 2. 1 there exists a heterogeneous subset A of R_{k+1} of power $\aleph_{k+1}$ such that $\mathrm{tp}(A) \leqq \alpha$. Now A has $\aleph_{k+2}$ almost disjoint subsets of power $\aleph_{k+1}$ (see [16]) whose types μ_i for $i < \omega_{k+2}$, as is easily shown from the definition of heterogeneity, are mutually incomparable.

The next two theorems both generalise propositions of GINSBURG concerning subtypes of the continuum ([9; p. 345]). The proofs we give are also based on Ginsburg's ideas.

THEOREM 2. 8 (GCH). *If θ is an arbitrary subtype of λ_{k+2} and β is such that $\beta < \lambda_{k+1}$, $|\beta| = \aleph_{k+1}$ then there exists a set of subtypes of β each of power $\aleph_{k+1}$ which form a totally ordered set with respect to embeddability whose order type is θ.*

PROOF. By theorems 2. 1, 2. 3 there exists an $\aleph_{k+1}$-dense heterogeneous subtype γ of β. Let G be a subset of R_{k+1} of type γ. By applying theorem 1. 9 to the intervals of G we see that G is the union of two disjoint $\aleph_{k+1}$-dense sets G_1, G_2 each dense in G. If $A, B \subseteqq G_2$ then $G_1 \cup A$, $G_1 \cup B$ are heterogeneous and $\aleph_{k+1}$-dense and therefore, by theorem 2. 2, rigid. Moreover, if $A \subset B$ then $G_1 \cup A \subset G_1 \cup B$ and this implies, since both sets are rigid, that $\mathrm{tp}(G_1 \cup A) < \mathrm{tp}(G_1 \cup B)$. Thus any chain of subsets of G_2 gives rise in this way to a totally-ordered set of types order-isomorphic to it. The result now follows from theorem 1. 8.

A minor modification of the argument of 2. 8 yields

THEOREM 2. 9 (GCH). *If θ is an arbitrary subtype of λ_{k+2} and $\beta < \lambda_{k+1}$ then there exists a set of subtypes of λ_{k+1} each of power $\aleph_{k+1}$ and each $> \beta$ which form a totally-ordered set with respect to embeddability whose order-type is θ.*

§ 3. Sums of types

We turn now to considerations of a slightly different nature and consider sums of types over an index set whose type is $<\lambda_{k+1}$. We show first that if the summands are all $<\lambda_{k+1}$ then so will the sum be, if there are not too many types.

THEOREM 3. 1 (GCH). *Let* $\{\alpha_x | x \in X\}$ *be a set of types such that* $\alpha_x < \lambda_{k+1}$, $|\alpha_x| = \aleph_{k+1}$. *If* X *is ordered of type* $\beta < \lambda_{k+1}$ *and* $\sigma = \sum_{x \in X} \alpha_x$ *then* $\sigma < \lambda_{k+1}$ *if* $|X| \leqq \aleph_k$, *and* $\sigma \| \lambda_{k+1}$ *if* $|X| > \aleph_k$.

PROOF. If $|X| \leqq \aleph_k$ then $\mathrm{tp}(X) = \beta \leqq \mathrm{tp}(U^0_{k+1})$ since U^0_{k+1} is $\aleph_k$-universal. In fact $2\beta \leqq \mathrm{tp}(U^0_{k+1})$ so that U^0_{k+1} and therefore R_{k+1} has a set of disjoint intervals $\{I_x | x \in E\}$ of type β, i.e. $\mathrm{tp}(E) = \beta$. Now by theorem 1. 6, $\mathrm{tp}(I_x) \equiv \lambda_{k+1}$ so that I_x contains a set A_x of type α_x for $x \in X$. Thus the ordered union $\bigcup_{x \in E} A_x$ is a subset of R_{k+1} and so $\sigma = \mathrm{tp}\left(\bigcup_{x \in E} A_x\right) \leqq \lambda_{k+1}$. But $\sigma \equiv \lambda_{k+1}$ is ruled out since this would either imply that $\beta \equiv \lambda_{k+1}$ which is not so by hypothesis, or it would imply, by virtue of 1. 6, that $\alpha_x \equiv \lambda_{k+1}$ for at least one $x \in X$, which is again not so by hypothesis. If $|X| > \aleph_k$ then $\sigma \not\equiv \lambda_{k+1}$ since λ_{k+1} has at most $\aleph_k$ mutually disjoint intervals. If $\sigma > \lambda_{k+1}$ then since $\beta \not\equiv \lambda_{k+1}$ we would have to have, by 1. 6, $\lambda_{k+1} \leqq \alpha_x$ for some $x \in X$ which is not so by hypothesis. Thus $\sigma \| \lambda_{k+1}$.

COROLLARY. *Any set of* $\aleph_k$ *proper subtypes of* λ_{k+1} *is bounded above by a proper subtype of* λ_{k+1}.

For the case $k = 0$ the above corollary is weaker than the following result of GINSBURG [9; p. 350]. Any set of $2^{\aleph_0}$ proper subtypes of λ is bounded above by a proper subtype of λ.

We were unable to obtain this more general result for subtypes of λ_{k+1} when $k > 0$. We therefore leave the question open as to whether the corollary to theorem 3. 1 is valid if $\aleph_k$ is replaced by $\aleph_{k+1}$, where $k > 0$.

We mention that the problem of finding *lower* bounds for sets of subtypes of λ_{k+1} is of a different nature from that of finding upper bounds. In fact, there exist types $\alpha, \beta < \lambda_{k+1}$ with $|\alpha| = |\beta| = \aleph_{k+1}$ which are such that if $\gamma < \alpha, \beta$ then $|\gamma| \leqq \aleph_k$. To see this (Cf. SIERPINSKI [14; théorème 10]) let $\alpha = \mathrm{tp}(A)$, $\beta = \mathrm{tp}(B)$ where A, B are any two disjoint subsets of power $\aleph_{k+1}$ of a heterogeneous subset H of R_{k+1} where $|H| = \aleph_{k+1}$.

The next lemma enables us to produce new mutually incomparable types by summation.

LEMMA 3. 2. *If* $\alpha = \sum_{x \in X} \alpha_x$, $\beta = \sum_{y \in Y} \beta_y$, $\mathrm{tp}(X) \| \mathrm{tp}(Y)$ *and* $\alpha_x \| \beta_y$ *for all pairs* x, y *then* $\alpha \| \beta$.

PROOF. Let $\{A_x | x \in X\}$, $\{B_y | y \in Y\}$ be families of disjoint sets such that $\mathrm{tp}(A_x) = \alpha_x$, $\mathrm{tp}(B_y) = \beta_y$. Suppose that f is a similarity map from the ordered union $\bigcup_{x \in X} A_x$ into the ordered union $\bigcup_{y \in Y} B_y$. For a given $x \in X$, we know that $f(A_x) \not\subseteq B_y$ for any $y \in Y$ since $\alpha_x \not\equiv \beta_y$ for any $y \in Y$. Therefore the set $E_x = \{y \in Y | f(A_x) \cap B_y \neq \emptyset\}$ has more than one member. Moreover, $|E_{x_1} \cap E_{x_2}| \leqq 1$ for $x_1 \neq x_2$ so that we can

pick a member $g(x)$ of E_x for each $x \in X$ as follows. If $|E_x| \geqq 3$ then let $g(x)$ be any member of E_x satisfying $a < g(x) < b$ for some $a, b \in E_x$. If $|E_x| = 2$ so that $E_x = \{a, b\}$ with $a < b$ then put $g(x) = b$. Clearly g is a similarity map of X onto a subset of Y which contradicts $\mathrm{tp}(X) \not\leqq \mathrm{tp}(Y)$. Thus there is no similarity map of $\bigcup_{x \in X} A_x$ into $\bigcup_{y \in Y} B_y$, i.e. $\alpha \not\leqq \beta$. In a similar way, $\beta \not\leqq \alpha$ and so $\alpha \| \beta$.

It is a consequence of theorem 3. 1 that no set E of proper subtypes of λ_{k+1} which contains types of power $\aleph_{k+1}$ is closed with respect to β-sums if β is allowed to vary over E. We now consider the 'closure' of E with respect to such sums. Following HAUSDORFF [11; p. 454] we define the *type ring* $\langle T \rangle$ generated by a set T of types as follows. Put $T_0 = T$ and

$$T_i = \left\{ \alpha = \sum_{x \in X} \alpha_x | \alpha_x \in \bigcup_{j<i} T_j, \mathrm{tp}(X) \in T \right\}$$

for an ordinal i. If $\Omega(T)$ is the smallest ordinal i such that $T_i = T_{i+1}$, then $T_{\Omega(T)}$, which we denote by $\langle T \rangle$, is defined to be the *type ring generated* by T. We note that the restriction on $\mathrm{tp}(X)$ in the definition of T_i for $i > 0$, namely that $\mathrm{tp}(X) \in T$ rather than $\mathrm{tp}(X) \in \bigcup_{j<i} T_j$, is an unnecessary one. More precisely if we put $T_0 = T^0$ and define

$$T^i = \left\{ \alpha = \sum_{x \in X} \alpha_x | \alpha_x, \mathrm{tp}(X) \in \bigcup_{j<i} T^j \right\}$$

then it is easily shown by induction on i that $T_{i+2j-1} = T^{i+j}$ and $T^i = T_i$ if i is a limit ordinal and $1 \leqq j < \omega$. So that $T_{\Omega(T)} = T^{\Omega(T)} = \langle T \rangle$, and thus $\langle T \rangle$ is the smallest set of types which contains T and which contains $\sum_{x \in X} \alpha_x$ whenever it contains α_x and $\mathrm{tp}(X)$.

Certain properties of sets of types are preserved in the passage from T to $\langle T \rangle$ and we now discuss one such property concerned with incomparable types. A set T of types has *property* $\Phi(m)$, where m is a cardinal, if every $\alpha \in T$ has m mutually incomparable supertypes in T, m mutually incomparable subtypes in T and is incomparable with m mutually incomparable members of T.

THEOREM 3. 3. *If T has $\Phi(m)$ for some m then $\langle T \rangle$ has $\Phi(m)$.*

PROOF. Write $T = T_0$, $\langle T \rangle = \bigcup_{k<\Omega(T)} T_k$ where T_k is as above. We shall show that each T_k has $\Phi(m)$ from which the theorem follows immediately. T_0 has $\Phi(m)$ by hypothesis, so that suppose that T_j has $\Phi(m)$ for all $j < k$. Let $\alpha \in T_k$, then $\alpha = \sum_{x \in X} \alpha_x$ where $\alpha_x \in \bigcup_{j<k} T_j$ and $\mathrm{tp}(X) \in T_0$. By the induction hypothesis each α_x has a set $S(\alpha_x)$ of m incomparable supertypes where $S(\alpha_x) \subseteqq \bigcup_{j<k} T_j$. Let β_x^1, β_x^2 be two distinct members of $S(\alpha_x)$ for each $x \in X$ and let X_1, X_2 be two sets whose types belong to $S(\mathrm{tp}(X))$ where for $\alpha \in T_0$ we have denoted by $S(\alpha)$ a set of m incomparable supertypes of α in T. The types

$$\beta^1 = \sum_{x \in X_1} \beta_x^1 = \beta_x^1 \, \mathrm{tp}(X_1)$$

and

$$\beta^2 = \sum_{x \in X_2} \beta_x^2 = \beta_x^2 \, \mathrm{tp}(X_2)$$

are incomparable by lemma 3. 2. Moreover, they are supertypes of α and belong to T_k. By varying the choice of β_x^1, β_x^2 from $S(\alpha_x)$ and noting that $|S(\alpha_x)|=$ $=|S(\mathrm{tp}\,(X))|=m$ we see that α has m mutually incomparable supertypes in T_k. The argument that α is incomparable with m incomparable members of T_k follows from an appropriate induction hypothesis and lemma 3. 2 in just the same way. The existence of m incomparable subtypes of α is immediate from the definition of T_k and the fact that T_0 has $\Phi(m)$.

THEOREM 3. 4 (GCH). *The type ring generated by the proper subtypes of* λ_{k+1} *of power* $\aleph_{k+1}$ *has* $\Phi(\aleph_{k+2})$ *for* $k\geqq 0$.

PROOF. Immediate from 3. 3, 2. 7 and 1. 10.

§ 4. Singular cardinals

The results of the previous sections suffer from two important shortcomings. The first is that we have only considered order-types of sets with successor powers, i.e. types α such that $|\alpha|=\aleph_{k+1}$ for some ordinal k. The second is that even with this restriction on the cardinality of the types we have considered only a small subset of the set of all types of power $\aleph_{k+1}$ namely those $<\lambda_{k+1}$, or those obtainable from such types by summation (Theorem 3.4). These restrictions are natural to the methods used here and it would seem that new methods are needed both to give an adequate description of the set of *all* types of power $\aleph_{k+1}$ and to investigate types with limit cardinals. For certain limit cardinals we are however able to obtain

THEOREM 4. 1 (GCH). *If* m *is a singular cardinal which is less than the first weakly inaccessible cardinal* (=*regular limit cardinal*) *and which is not cofinal with* $\aleph_0$ *then there exists a rigid set of power* m.

PROOF. The conditions on m together with the GCH mean that $m=\aleph_{\omega_{k+1}}$ for some $k\geqq 0$. Let X be an $\aleph_{k+1}$-dense rigid set guaranteed by theorem 2. 4. Let $(x_i|i<\omega_{k+1})$ be a fixed enumeration of X and, for each $i<\omega_{k+1}$, let A_{x_i} be a rigid set of power $\aleph_{i+1}$ guaranteed by 2. 4. We shall show that the ordered union $A=\bigcup_{x\in X} A_x$ is rigid. Let f be a supposed similarity map of A into itself. If I is an interval of X then $|I|=\aleph_{k+1}$ and so $|A|=m$ since $\{i|x_i\in I\}$ in the above enumeration of X is cofinal with ω_{k+1}. It therefore follows that

$$(*)\qquad f\Big(\bigcup_{x\in I} A_x\Big) \not\subseteq A_y \quad \text{for any} \quad y\in X$$

since $|A_y|<m$ for all $y\in X$. Let $E_x=\{y\in X|f(A_x)\cap A_y\neq\emptyset\}$ for each $x\in X$. We show that $E_{x_1}\cap E_{x_2}=\emptyset$ for $x_1\neq x_2$. Suppose first that $x_1<x_2$ then, since f is a similarity map, $f(x)<f(y)$ for $x\in A_{x_1}$, $y\in A_{x_2}$, i.e. $f(A_{x_1})<f(A_{x_2})$. Now if $E_{x_1}\cap E_{x_2}$ is not empty then $y\in E_{x_1}\cap E_{x_2}$ and $f(A_1)\cap A_y, f(A_{x_2})\cap A_y$ are both non-empty. Since X is dense there exists x_3 such that $x_1<x_3<x_2$ and since $f(A_{x_1})<f(A_{x_3})<$ $<f(A_{x_2})$ it follows that $f(A_{x_3})\subseteq A_y$. This means that $f\big(\bigcup_{x\in I} A_x\big)\subseteq A_y$ where I is the interval (x_1, x_2) in X, which contradicts $(*)$ above. Thus $E_{x_1}\cap E_{x_2}=\emptyset$ for $x_1\neq x_2$. In fact, $E_{x_1}<E_{x_2}$ in X if $x_1<x_2$. We can now show that A is rigid. Let

$g(x)$ be an arbitrary member of E_x for each $x \in X$. Then g is a similarity map of X into itself which, since X is rigid, must be the identity. From the definition of $g(x)$ as an arbitrary member of E_x it follows that $E_x = \{x\}$. Thus $f(A_x) \subseteq A_x$ for all $x \in X$. But A_x is rigid so that f restricted to A_x is the identity, and so f is the identity on the whole of A.

COROLLARY (GCH). *If m is as in* 4. 1 *then there exist m mutually incomparable types of power m.*

PROOF. Let M be a rigid set of power m and put $M_x = M - \{x\}$. We show that $\mathrm{tp}\,(M_x) \| \mathrm{tp}\,(M_y)$ for $x \neq y$. Suppose not, i.e. let f be a similarity map of M_x into M_y and take $x < y$. Let $A = \{a \in M_x | a < x\}$ and $B = \{b \in M_y | b \geqq x\}$. We distinguish two cases.

Case 1. $x \in f(A)$. Then $x < f(M_x - A)$ in M_y, i.e. $f(M_x - A) \subset \{a \in M_y | a > x\}$ and so the rigid set $M_x - A$ is mapped into a proper subset of itself, which is a contradiction.

Case 2. $x \notin f(A)$. Then $f(A) \subset M_y - B = A$ so the rigid set A is mapped into a proper subset of itself, which is a contradiction.

Thus neither case is possible and therefore $\mathrm{tp}\,(M_x) \not\leqq \mathrm{tp}\,(M_y)$. Similarly, $\mathrm{tp}\,(M_y) \not\leqq \mathrm{tp}\,(M_x)$ which gives the result.

We shall now summarise the facts established so far on the existence of sets of mutually incomparable types of a given cardinality.

For cardinal numbers m, n let $P(m, n)$ denote the statement that there exist n mutually incomparable order types of power m.

The theorems of § 2 establish[12] $P(m^+, m^{++})$, assuming GCH, for any infinite m. The above corollary establishes $P(m, m)$ for singular m not cofinal with $\aleph_0$. Thus the simplest open problems (asuming GCH) are

PROBLEM 1. $P(\aleph_\omega, \aleph_\omega)$?

PROBLEM 2. $P(\aleph_{\omega_1}, \aleph_{\omega_2})$?

Finally our results tell us nothing about $P(m, m)$ in the case that m is a regular limit cardinal or $m = \aleph_0$.[13]

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[12] This result and some of the other results of §2 were announced in [13].

[13] It can be shown that $P(\aleph_0, \aleph_0)$ implies $P(\aleph_0, 2^{\aleph_0})$. This, together with other results on the comparison of denumerable order types, will appear in a forthcoming paper by the author, F. ROWBOTTOM and R. PARIKH.

References

[1] J. R. BÜCHI, On the existence of totally heterogeneous spaces, *Fund. Math.*, **41** (1955), pp. 97—102.
[2] B. DUSHNIK & E. W. MILLER, Partially ordered sets, *Amer. J. of Math.*, **63** (1941), pp. 601—610.
[3] P. ERDŐS & R. RADO, A partition calculus in set theory, *Bull. Amer. Math. Soc.*, **62** (1957), pp. 427—489.
[4] R. FRAÏSSE, Sur la comparaison des types d'ordres, *C. R. Acad. Sci. Paris*, **226** (1948), p. 1330.
[5] L. GILLMAN, On intervals of ordered sets, *Annals of Math.*, **56** (1952), pp. 440—459.
[6] L. GILLMAN, Some remarks on η_α-sets, *Fund. Math.*, **43** (1956), pp. 77—82.
[7] S. GINSBURG, Some remarks on order types and decompositions of sets, *Trans. Amer. Math. Soc.*, **74** (1953), pp. 514—535.
[8] S. GINSBURG, Further results on order types and decompositons of sets, *Trans. Amer. Math. Soc.*, **77** (1954), pp. 122—150.
[9] S. GINSBURG, Order types and similarity transformations, *Trans. Amer. Math. Soc.*, **79** (1955), pp. 341—361.
[10] A. GLEYZAL, Order types and structures of orders, *Trans. Amer. Math. Soc.*, **48** (1940), pp. 451—466.
[11] F. HAUSDORFF, Grundzüge einer Theorie der geordneten Mengen, *Math. Ann.*, **65** (1908), pp. 435—505.
[12] F. HAUSDORFF, *Grundzüge der Mengenlehre*, (Leipzig, 1914).
[13] B. ROTMAN, On incomparable order types, *Amer. Math. Soc. Notices*, **13** (1966), pp. 633—634.
[14] W. SIERPINSKI, Sur les types d'ordre des ensembles lineaires, *Fund. Math.*, **37** (1950), pp. 253—264.
[15] W. SIERPINSKI, *Cardinal and ordinal numbers* (Warsaw, 1958).
[16] A. TARSKI, Sur la decomposition des ensembles en sous-ensembles presque disjoint, *Fund. Math.*, **12** (1928), pp. 186—205.

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Tomus 19 (3—4), (1968), pp. 329—335.

ON THE GREATEST COMMON DIVISOR OF TWO VALUES OF A POLYNOMIAL

By

V. ENNOLA (Turku, Finland)

(*Presented* by P. TURÁN)

Let $P(x)=c_0x^k+c_1x^{k-1}+\ldots+c_k$ be an arbitrary irreducible polynomial of degree $k\geqq 2$ with integral coefficients such that $P(x)>0$ for $x>0$. Put

$$g(n)=\sum_{m=1}^{n-1}(P(m), P(n)),$$

where (a, b) denotes the greatest common divisor of a and b, and

$$G(N)=\sum_{n=1}^{N}\frac{g(n)}{P(n)}.$$

Using a modification of a method of ERDŐS [3] we shall prove that

$$G(N)\ll N, \tag{1}$$

where, as always in this paper, the notation $\ll$ indicates an inequality with an unspecified constant factor depending only on the polynomial $P(x)$. The question of finding an estimate for $G(N)$ arises in connection with some metrical theorems on diophantine approximation (see [4] and [5]) and our result enables us to improve one of these theorems. We shall return to this topic later.

At first we state some preliminary lemmas. Let $d(a)$ denote the number of positive divisors of a. Let $N\geqq 2$.

LEMMA 1.
$$\sum_{x=1}^{N}d(P(x))\ll N\log N.$$

This has been proved by ERDŐS [3].

LEMMA 2. *There exists a constant A such that*

$$\sum_{x=1}^{N}(d(P(x)))^2<N\log^A N.$$

This is a result of VAN DER CORPUT [2].

LEMMA 3. *Let δ be any positive real number. Then*

$$\sum_{x=1}^{N}\sum_{a\mid P(x)}a^{-\delta}<C(\delta)N,$$

where $C(\delta)$ is a constant depending on δ and the polynomial $P(x)$.

This result is due to BELLMAN [1]. For a short proof, see Lemma 7 of [5].

For any natural number a, let $\varrho(a)$ denote the number of integers x such that, $1 \leqq x \leqq a$ and $P(x) \equiv 0 \pmod a$. For any positive real number z, let $\varrho_z(a)$ denote the number of integers x such that $1 \leqq x \leqq z$ and $P(x) \equiv 0 \pmod a$. Then, clearly

$$\varrho_z(a) \leqq \left(\frac{z}{a}+1\right)\varrho(a). \tag{2}$$

LEMMA 4. *We have*

(i) $\varrho(ab)=\varrho(a)\varrho(b)$ *if* $(a, b)=1$,

(ii) $\varrho(p^\alpha) \leqq B$ *for every prime power* p^α,

(iii) $\sum_{z<p\leqq z^2} \frac{\varrho(p)}{p} < 1$ *if* z *is sufficiently large.*

Here (i) and (ii) are well known. A proof of (iii) can be found in [3].

LEMMA 5. *The number of positive integers* $x \leqq N$ *such that* $P(x)$ *is divisible by a prime power* p^α, *with* $\alpha > 1$ *and*

$$\log^{2A} N < p^\alpha \leqq N,$$

is $\ll N \log^{-A} N$.

This is essentially Lemma 5 in [3].

By Schwarz's inequality, Lemma 2 easily implies

LEMMA 6. *Let* $x_1, x_2, \ldots, x_v$ *be distinct positive integers, all* $\leqq N$, *and let* $v \ll N \log^{-A} N$. *Then*

$$\sum_{i=1}^{v} d(P(x_i)) \ll N.$$

We now begin the estimation of $G(N)$. Put

$$s_{mn}=(P(m), P(n))$$

for any natural numbers m, n $(m<n)$. Clearly

$$\frac{1}{2}x^k < P(x) < x^{k+1} \tag{3}$$

if x is sufficiently large. $P(x)$ is monotonically increasing for $x \geqq x_0$, say. Since the values of $m < x_0$ contribute only $O(1)$ to each $g(n)$, we may assume that $P(x)$ is monotonically increasing for all the values of m and n in question. We also assume that N is sufficiently large, i.e. greater than a suitable constant depending only on $P(x)$.

Suppose now that, for fixed n, $s_{mn}=s$. Then $P(m)=st$, where t can take any of the values $1, 2, \ldots, \frac{P(n)}{s}$ at most once, since $P(x)$ is monotonically increasing. Hence the number of positive integers $m<n$ such that $s_{mn}=s$ is at most $\frac{P(n)}{s}$.

We thus have

$$g(n) \leqq \sum_{s|P(n)} s\frac{P(n)}{s} = P(n)\sum_{s|P(n)} 1 = P(n)\,d(P(n)). \tag{4}$$

Lemma 1 then implies

$$G\left(\frac{N}{\log N}\right) \ll N,$$

so that we may restrict ourselves to values of n satisfying

$$\frac{N}{\log N} < n \leqq N. \tag{5}$$

Let $\mathscr{A}$ be the set of integers n such that $n \leqq N$ and $d(P(n)) > \log^A N$. Lemma 2 implies that the number of elements in $\mathscr{A}$ is less than $N \log^{-A} N$, so that (4) and Lemma 6 imply

$$\sum_{n\in\mathscr{A}} \frac{g(n)}{P(n)} \ll N.$$

Hence we may assume that $n \notin \mathscr{A}$, i.e.

$$d(P(n)) \leqq \log^A N. \tag{6}$$

Similarly, Lemma 5 shows that we may further restrict ourselves to values of n such that

$$P(n) \not\equiv 0 \pmod{p^\alpha} \quad \text{if} \quad \alpha > 1 \quad \text{and} \quad \log^{2A} N < p^\alpha \leqq N. \tag{7}$$

Let $\omega(a)$ denote the number of different prime factors of a. Then (6) implies

$$2^{\omega(P(n))} \leqq d(P(n)) \leqq \log^A N,$$

so that

$$\omega(P(n)) \leqq \frac{\log(\log^A N)}{\log 2} < 2A \log\log N. \tag{8}$$

If s is an arbitrary positive divisor of $P(n)$, we thus have, by (8) and Lemma 4, (i), (ii),

$$\varrho(s) \leqq B^{\omega(s)} \leqq B^{\omega(P(n))} < \log^E N, \tag{9}$$

where $E = 2A \log B$.

Put

$$X = N^{\frac{1}{8A \log\log N}}.$$

We claim that

$$\prod_{\substack{p^\alpha \| P(n) \\ p < X}} p^\alpha < N^{\frac{1}{4}}, \tag{10}$$

where $p^\alpha \| P(n)$ indicates that $p^\alpha | P(n)$, $p^{\alpha+1} \nmid P(n)$. Suppose, namely, that the product in (10) is $\geqq N^{\frac{1}{4}}$. If at least one of the prime powers p^α occuring in the product (10) is $\geqq X$, then we must have $\alpha > 1$. Determine an integer β such that $2 \leqq \beta \leqq \alpha$ and $\log^{2A} N < p^\beta \leqq N$. Then, however, $P(n) \equiv 0 \pmod{p^\beta}$, which contra-

dicts (7). If, on the other hand, every prime power in (10) is $<X$, then the number of p's must be $>2A\log\log N$, which contradicts (8).

For fixed n, the values of $m<n$ such that $s_{mn}\leqq n$ contribute to $g(n)$ less than $n^2\ll P(n)$. We may thus assume that

$$s_{mn}>n. \tag{11}$$

Let s be an arbitrary positive divisor of $P(n)$ greater than n. Then, by (2) and (9), the number of positive integers $m<n$ such that $s_{mn}=s$ is less than

$$\varrho_n(s)\leqq\left(\frac{n}{s}+1\right)\varrho(s)\leqq 2\varrho(s)<2\log^E N. \tag{12}$$

Consider now the values of s such that $\dfrac{P(n)}{s}\geqq\log^{2E}N$. Then, by (12),

$$\varrho_n(s)<2\left(\frac{P(n)}{s}\right)^{\frac{1}{2}},$$

so that the contribution to $g(n)$ is less than

$$2\sum_{s\mid P(n)}s\left(\frac{P(n)}{s}\right)^{\frac{1}{2}}=2P(n)\sum_{t\mid P(n)}t^{-\frac{1}{2}}.$$

By Lemma 3, the total contribution to $G(N)$ is $\ll N$. For each n we may thus restrict ourselves to values of $m<n$ such that

$$\frac{P(n)}{s_{mn}}<\log^{2E}N. \tag{13}$$

Let $P(n)=p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_v^{\alpha_v}$, where $p_1<p_2<\ldots<p_v$, and define j by

$$p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_j^{\alpha_j}\leqq N\log^{-k}N<p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_j^{\alpha_j}p_{j+1}^{\alpha_{j+1}}. \tag{14}$$

Put

$$a_n=p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_j^{\alpha_j},\qquad b_n=\frac{P(n)}{a_n}.$$

Then, by (3), (5), (14),

$$b_n>\frac{1}{2}\frac{n^k}{a_n}>\frac{1}{2}N^{k-1}\geqq\frac{1}{2}N\geqq\frac{1}{2}n. \tag{15}$$

By (10), we have

$$p_i\geqq X\qquad(j+1\leqq i\leqq v),$$

so that (13) implies

$$p_i\nmid\frac{P(n)}{s_{mn}}\qquad(j+1\leqq i\leqq v).$$

Hence $b_n\mid s_{mn}$, so that, by (2), (15), and Lemma 4, the number of $m<n$ with $s_{mn}=s$ is at most

$$\varrho_n(s)\leqq\varrho_n(b_n)\leqq 3\varrho(b_n)\ll B^{v-j}. \tag{16}$$

Consider at first the values of n such that $p_{j+1} > N^{\frac{1}{16}}$. By (3), we must then have $v-j<16(k+1)$, so that, by (16), $\varrho_n(s)=O(1)$ for every $s=s_{mn}$. Hence the contribution to $G(N)$ of the values of n of this kind is, by Lemma 3,

$$\ll \sum_{n=1}^{N} \frac{1}{P(n)} \sum_{s|P(n)} s = \sum_{n=1}^{N} \sum_{t|P(n)} t^{-1} \ll N.$$

We may thus assume that $p_{j+1} \leqq N^{\frac{1}{16}}$. By (7), $P(n)$ cannot be divisible by p_{j+1}^2, so that $\alpha_{j+1}=1$, and we have, by (14),

$$a_n > N \log^{-k} N \, p_{j+1}^{-1} > N^{\frac{1}{2}}. \tag{17}$$

For every integer r such that $16 \leqq r \leqq 8A \log\log N$, let $\mathscr{B}_r$ be the set of positive integers $n \leqq N$ satisfying

$$N^{\frac{1}{r+1}} < p_{j+1} \leqq N^{\frac{1}{r}}. \tag{18}$$

Then every integer n under consideration belongs to some set $\mathscr{B}_r$. Let r be fixed and let $\mathscr{I}_t$ denote the half-open interval

$$\left(N^{\frac{1}{r2^{t+1}}}, N^{\frac{1}{r2^t}}\right],$$

where $t=0, 1, \ldots, \tau$ and τ is the largest integer for which $r2^\tau \leqq 8A \log\log N$. Put

$$K_t = \left[\frac{r(t+1)}{16}\right] + 1.$$

As in the paper of ERDŐS ([3], p. 12), we can conclude that, if $n \in \mathscr{B}_r$, then a_n must have at least K_t prime factors in at least one of these intervals. For each h $(0 \leqq h \leqq \tau)$, let $\mathscr{B}_r^{(h)}$ be the set of $n \in \mathscr{B}_r$ such that h is the least integer for which a_n has at least K_h prime factors in $\mathscr{I}_h$. Suppose that $n \in \mathscr{B}_r^{(h)}$ and write $K=K_h$. By (7), a_n cannot be divisible by the square of any prime in $\mathscr{I}_h$. Let $\mathscr{U}$ be the set of all integers of the form $u=q_1 q_2 \ldots q_K$, where the q_i's are different primes belonging to $\mathscr{I}_h$. Since $n \in \mathscr{B}_r^{(h)}$, there exists an $u \in \mathscr{U}$ such that $u|a_n$ and hence $P(n) \equiv 0 \pmod{u}$. For fixed $u<N$, the number of integers $n \leqq N$ satisfying this condition, is less than

$$\left(\frac{N}{u}+1\right)\varrho(u) < 2N\frac{\varrho(u)}{u}.$$

Hence the number of elements in $\mathscr{B}_r^{(h)}$ is less than

$$2N \sum_{u \in \mathscr{U}} \frac{\varrho(u)}{u}.$$

On the other hand, we have, by Lemma 4,

$$\sum_{u \in \mathscr{U}} \frac{\varrho(u)}{u} \leqq \frac{1}{K!}\left(\sum_{p \in \mathscr{I}_h} \frac{\varrho(p)}{p}\right)^K < \frac{1}{K!}.$$

Thus the number of elements in $\mathscr{B}_r^{(h)}$ is less than $2N/K!$ and the number of elements in $\mathscr{B}_r$ is less than

$$2N\sum_{h=0}^{\infty}(K_h!)^{-1} \ll N\left(\left[\frac{r}{16}\right]!\right)^{-1}. \tag{19}$$

Suppose now that $n\in\mathscr{B}_r$. It then follows from (18) and (3) that $v-j<(k+1)\cdot\cdot(r+1)$. Therefore (16) implies that, for any $s=s_{mn}$,

$$\varrho_n(s) \ll B^{(k+1)(r+1)}. \tag{20}$$

Consider at first the values of s such that

$$\frac{P(n)}{s} \geqq B^{2(k+1)(r+1)}.$$

Then the number of positive integers $m<n$ such that $s_{mn}=s$ is, by (20), at most

$$\varrho_n(s) \ll \left(\frac{P(n)}{s}\right)^{\frac{1}{2}}.$$

Therefore the values of s of this kind contribute to $g(n)$

$$\ll \sum_{s|P(n)} s\left(\frac{P(n)}{s}\right)^{\frac{1}{2}} = P(n)\sum_{t|P(n)} t^{-\frac{1}{2}}. \tag{21}$$

Consider next the values of s such that

$$\frac{P(n)}{s} < B^{2(k+1)(r+1)}.$$

These values contribute to $g(n)$, by (20), at most

$$\sum_{\substack{s|P(n)\\ \frac{P(n)}{s}<B^{2(k+1)(r+1)}}} s\varrho_n(s) \ll P(n)B^{(k+1)(r+1)} \sum_{\substack{t|P(n)\\ t<B^{2(k+1)(r+1)}}} t^{-1} \ll \tag{22}$$

$$\ll P(n)B^{(k+1)(r+1)}(r+1) < P(n)B^{2(k+1)(r+1)}.$$

By (19), (21), (22), and Lemma 3, the total contribution to $G(N)$ is

$$\ll \sum_r \sum_{n\in\mathscr{B}_r} \sum_{t|P(n)} t^{-\frac{1}{2}} + N\sum_r B^{2(k+1)(r+1)}\left(\left[\frac{r}{16}\right]!\right)^{-1} <$$

$$< \sum_{n=1}^{N} \sum_{t|P(n)} t^{-\frac{1}{2}} + N\sum_{r=16}^{\infty} B^{2(k+1)(r+1)}\left(\left[\frac{r}{16}\right]!\right)^{-1} \ll N.$$

This concludes the proof.

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References

[1] R. BELLMAN, Ramanujan sums and the average value of arithmetic functions, *Duke Math. J.*, **17** (1950), pp. 159—168.

[2] J. G. VAN DER CORPUT, Une inégalité relative au nombre des diviseurs, *Proc. K. Neder. Akad. van Wet.*, **42** (1939), pp. 547—553.

[3] P. ERDŐS, On the sum $\sum_{k=1}^{x} d(f(k))$, *J. London Math. Soc.*, **27** (1952), pp. 7—15.

[4] W. J. LE VEQUE, On the frequency of small fractional parts in certain real sequences III, *J. Reine Angew. Math.*, **202** (1959), pp. 215—220.

[5] W. M. SCHMIDT, Metrical theorems on fractional parts of sequences, *Trans. Amer. Math. Soc.*, **110** (1964), pp. 493—518.

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ON MINIMAL DENSITY OF PLANE COVERINGS BY CIRCLES

By

R. P. BAMBAH and A. C. WOODS (Columbus, Ohio, U.S.A.)

(*Presented by* L. FEJES TÓTH)

1. We say that a set of closed circular discs in the euclidean plane is saturated if and only if $r = \inf s$ extended over all the radii s of the circles is positive and any circle of radius r has at least one point in common with a circle of the set. For any measurable set S we let $V(S)$ denote the area of S. If S denotes the point set union of the discs and $S(k)$ the part of S inside the disc whose centre is the origin and radius k then by the lower density of the covering we mean $\liminf_{k\to\infty} V(S(k))/\pi k^2$. L. FEJES TÓTH [see, e.g., (3)] conjectured that the lower bound of the lower densities for saturated systems of circles is $\pi/6\sqrt{3}$, which is attained by a saturated set of equal circles, all of radius r. EGGLESTON [2] has recently proved this conjecture under the restriction that the circles of the saturated set do not overlap. Our object in this paper is to prove that FEJES TÓTH's conjecture is valid also for saturated sets with overlapping circles provided, however, that they satisfy a uniformity condition, namely that

(a) any bounded region of the plane contains at most a finite number of centres of the circles of the set, and

(b) there exists a constant N such that no circle of the set has radius exceeding N.

Accordingly we say that a set of circles is uniform if it satisfies (a) and (b). The authors [1] have shown that if circle is replaced by strictly convex body then an analogous result to that of Eggleston is true and a number of results used in that paper are applicable here. Thus where the proofs require no modification we shall merely refer to [1].

As with Eggleston our theorem is a consequence of two results namely

THEOREM 1. *Let M be a uniform saturated set of circles. Then there exists a triangulation of the plane such that*

(i) *each vertex of each triangle is the centre of a member of M,*

(ii) *if abc is any one of the triangles and $K(a)$, $K(b)$, $K(c)$ are the members of M with centres a, b, c respectively then there exists a point p such that the circle centred at p of radius r meets each of $K(a)$, $K(b)$, $K(c)$.*

THEOREM 2. *Let $K(a)$, $K(b)$, $K(c)$ be circles with centres a, b, c respectively each of radius at least r and all meeting a fixed circle of radius r. Then*

$$\frac{V(abc \cap (K(a) \cup K(b) \cup K(c)))}{V(abc)} \geqq \pi/6\sqrt{3}$$

with equality when the circles are of equal radius placed at the vertices of an equilateral triangle of side length $2\sqrt{3}r$ and p is the centroid of the triangle.

Theorem 2 is stated as an auxiliary problem by EGGLESTON [2] but the authors feel that his proof requires considerable amplification. The case when the circles do not overlap has been dealt with in [1]. We therefore give a proof for the case when some of the circles overlap. The approach used is that of Eggleston.

2. PROOF OF THEOREM 1. Let a, b, c be three noncollinear points of R_2. We denote by $H(a, b; c)$ the closed half plane containing the point c whose boundary is the straight line through the points a and b.

LEMMA 1. *If* $x \in H(a, b; c) \cap H(a, c; 2a-b) \cap H(b, c; a)$ *then*

$$|x-a| - |x-b| \leqq |c-a| - |c-b|$$

with strict inequality if $x \neq c$.

PROOF. See lemma 3 of [1].

Now let M be a uniform saturated set of circles of minimal radius r. Denote by $K(a, R(a))$ the circle with centre a and radius $R(a)$. Let $K(a, R(a)) \in M$ and for an arbitrary point x of the plane define

$$d_a(x) = |x-a| - R(a).$$

Denote by $S(a)$ the set of all points x of the plane with the property that $d_a(x) \leqq d_b(x)$ for all $K(b, R(b)) \in M$.

LEMMA 2. *If* $x \in S(a)$ *then all points except possibly* x *of the line segment* ax *are interior points of* $S(a)$.

PROOF. See lemma 4 of [1]. Together with lemma 1 the proof uses the property that only finitely many members of M meet any given circle. This is obviously the case for uniform sets.

We now reduce M as follows. If any circle $K(a, R(a))$ of M is such that $K(a, R(a)+r)$ is contained within a union of circles $K(b, R(b)+r)$ for which $K(b, R(b))$ is in M then it is clear that omitting the circle $K(a, R(a))$ from M does not destroy the saturation property and cannot increase the minimal density of M. Since M is uniform such circles may be omitted in a systematic way to produce a uniform saturated set of circles with minimal radius r with minimal density no greater than that of M. Furthermore proving theorem 1 for the new set clearly implies theorem 1 for M. Hence there is no loss of generality in assuming from now on that M is such that for any $K(a, R(a))$ in M the circle $K(a, R(a)+r)$ is not contained in any union of circles $K(b, R(b)+r)$ for which $K(b, R(b))$ is in M and $a \neq b$.

LEMMA 3. *If* a, b, c *are centres of members of* M *such that they are not collinear then* $S(a)$, $S(b)$ *cannot have a common point in the closed set*

$$H(a, c; 2a-b) \cap H(b, c; 2c-a).$$

PROOF. Assume by way of contradiction that there exists a point d in $S(a) \cap S(b)$ such that $d \in H(a, c; 2a-b) \cap H(b, c; 2c-a)$. Then

$$d_a(d) = d_b(d) \leqq d_c(d).$$

Then

$$|a-d| = R(a)+t, \quad |b-d| = R(b)+t, \quad |c-d| \geqq R(c)+t$$

where t is a real number such that $t \leqq r$. We will show that the circle $K(c, R(c)+t)$ is contained in $K(a, R(a)+t) \cup K(b, R(b)+t)$. With this aim in view let $x \in K(c, R(c)+t)$. We distinguish two cases.

Case (i) $x \in H(a, b; c)$. Here x lies in one of three regions namely

(a) $H(b, c; d) \cap H(b, d; 2b-c) \cap H(c, d; b)$
(b) $H(a, c; d) \cap H(a, d; 2a-c) \cap H(c, d; a)$
(c) the triangle *abd*.

If x lies in the set (a) then by lemma 1,

$$|x-b| - |x-c| \leqq |d-b| - |d-c|.$$

Hence

$$|x-b| \leqq |x-c| + |d-b| - |d-c| \leqq (R(c)+t) + (R(b)+t) - (R(c)+t) = R(b)+t,$$

and x lies in the circle $K(b, R(b)+t)$.

If x lies in the set (b) then by a similar argument it follows that x lies in the circle $K(a, R(a)+t)$. If finally x lies in the triangle *abd* then it must belong to $K(a, R(a)+t) \cup K(b, R(b)+t)$ since these two circles intersect at d and therefore contain the whole triangle *abd* in their union. This verifies that in case (i) x lies in $K(a, R(a)+t) \cup K(b, R(b)+t)$.

Case (ii) $x \in H(a, b; 2a-c)$. Reflection in the line through a and b leaves the circles $K(a, R(a)+t)$, $K(b, R(b)+t)$ invariant and the image of that part of $K(c, R(c)+t)$ in $H(a, b; c)$ covers the part of $K(c, R(c)+t)$ in $H(a, b, 2a-c)$ since the centre c is in $H(a, b; c)$. Therefore by case (i) the result follows.

Thus in all cases $x \in K(a, R(a)+t) \cup K(b, R(b)+t)$ from which it follows that

$$K(c, R(c)+t) \subset K(a, R(a)+t) \cup K(b, R(b)+t). \tag{1}$$

We now claim that $K(c, R(c)+r) \subset K(a, R(a)+r) \cup K(b, R(b)+r)$. For let x be a point on the boundary of $K(c, R(c)+r)$ so that

$$|x-c| = R(c)+r.$$

If $R(c)+t \geqq 0$ let y be that point of the line segment xc for which $|y-c| = R(c)+t$. If $R(c)+t < 0$ let $y = c$. Then since c lies in the triangle *abd* so y has been chosen so that

$$y \in K(a, R(a)+t) \cup K(b, R(b)+t)$$

by (1).

Also $|c-y| \geqq R(c)+t$. Now

$$|y-x| = |x-c| - |c-y| \leqq R(c)+r-(R(c)+t) = r-t.$$

If we assume that $y \in K(a, R(a)+t)$ then

$$|x-a| \leqq |y-a| + |y-x| \leqq R(a)+t+(r-t) = R(a)+r$$

and x lies in $K(a, R(a)+r)$. If on the other hand $y \in K(b, R(b)+t)$ then in a similar fashion $x \in K(b, R(b)+r)$. Therefore

$$K(c, R(c)+r) \subset K(a, R(a)+r) \cup K(b, R(b)+r)$$

which is a contradiction that proves the lemma.

COROLLARY. *Let a and b be centres of members of M and let $x \in S(a) \cap S(b)$. Then no point of the closed triangle abx other than a, b can be the centre of a member of M.*

PROOF. Suppose that c is the centre of a member of M different from a or b such that c lies in the closed triangle abx. Assume first that x does not lie on the line through a and b. It follows at once from lemma 3 that c must lie on the line segment ab. But then exactly as in the proof of lemma 3 one can show that the circle $K(c, R(c)+r)$ lies in the union $K(a, R(a)+r) \cup K(b, R(b)+r)$ which is a contradiction. Hence x lies on the line through a and b and therefore also c lies on this line. If x does not lie on the line segment ab then either $K(a, R(a)+r) \subset$ $\subset K(b, R(b)+r)$ or $K(b, R(b)+r) \subset K(a, R(a)+r)$ which is a contradiction. Thus x lies on the line segment ab and so c lies either on the line segment ax or the segment xb. Without loss of generality c lies on the segment ax. But then $K(a, R(a)+r) \supset$ $\supset K(c, R(c)+r)$ a contradiction that proves the corollary.

We now define a vertex of M to be a point v of the plane such that v lies in at least three distinct sets $S(a)$, $S(b)$, $S(c)$ where a, b, c are centres of distinct members of M. We further say that a, b, c belong to v.

LEMMA 4. *Any three centres belong to at most one vertex.*

PROOF. See lemma 7 of [1]. The fact that no three centres belonging to a vertex can be collinear follows from the corollary to lemma 3.

COROLLARY. *The vertices of M form a discrete set.*

LEMMA 5. *At most a finite number of centres of M belong to one vertex.*

PROOF. See lemma 8 of [1].

LEMMA 6. *If x is any point of the plane then there exists a vertex v such that x belongs to the convex hull $P(v)$ of all the centres of members of M that belong to v.*

PROOF. See lemma 9 of [1].

LEMMA 7. *If v and w are distinct vertices then $P(v)$, $P(w)$ have no interior points in common.*

PROOF. See lemma 10 of [1]. We have only to verify that if a, b, c are centres of members of M then no centre of a member of M can lie in the triangle abc apart from its vertices. Assume by way of contradiction that d is a centre different from a, b, c in the triangle abc. By the corollary to lemma 3, d must lie in the interior of the triangle. Let v be the vertex defined by a, b, and c. Then by lemma 3, v cannot lie in any one of the closed regions

$$H(a, x; 2a-b) \cap H(b, x, 2x-a),$$

$$H(c, x; 2c-b) \cap H(b, x; 2x-c),$$

$$H(c, x; 2c-a) \cap H(a, x; 2x-c),$$

which together comprise the whole plane. This contradiction proves the assertion.

Theorem 1 follows from lemmas 6 and 7 since each polygon $P(v)$ can be triangulated as required.

PROOF OF THEOREM 2. Let abc be a triangle of the theorem. If $K(a) \cap K(b) \cap K(c)$ is not empty then abc is contained in $K(a) \cup K(b) \cup K(c)$ and

$$\varrho(abc) = \frac{V((K(a) \cup K(b) \cup K(c)) \cap abc)}{V(abc)} = 1 > \pi/6\sqrt{3}.$$

Therefore we may suppose that $K(a) \cap K(b) \cap K(c)$ is empty.

If no two of $K(a)$, $K(b)$, $K(c)$ overlap then theorem 2 is a special case of theorem 2 of [1]. We therefore need consider only triangles abc for which

(1) $K(a) \cap K(b) \cap K(c)$ is empty,

(2) Some of $K(a)$, $K(b)$, $K(c)$ overlap.

As in [1] we begin with a reduction. We let N be the set of all nonnegative numbers n for which

$$K(a, R(a)+r-n) \cap K(b, R(b)+r-n) \cap K(c, R(c)+r-n)$$

is not empty. Hence r is not in N. Let m be the supremum of all the numbers in N. Then since the circles are closed it follows that $m \in N$, $0 \leqq m < r$ and therefore also $0 < s = r - m \leqq r$.

Since the circles are strictly convex there exists one point t in $K(a, R(a)+s) \cap \cap K(b, R(b)+s) \cap K(c, R(c)+s)$. As in [1] t lies in the triangle abc. If t lies within abc then $K(a)$, $K(b)$, $K(c)$ all touch $K(t, s)$ externally. If however t lies on a side of abc say bc then $K(t, s)$ touches $K(b)$, $K(c)$ externally and meets $K(a)$. If $K(t, s)$ does not touch $K(a)$ externally, then as in EGGLESTON [2] (p. 230) we reduce the radius of $K(a)$ until either $K(a)$ touches $K(t, s)$ with $R(a) \geqq r$ or $K(a)$ has radius r and meets $K(t, s)$. In the latter case $at < r+s$, $bt \geqq r+s$ and $ct \geqq r+s$. Therefore $at < bt$ and $at < ct$ and so the perpendicular from a to bc meets the segment bc in an interior point h say. As in EGGLESTON ([1], p. 230) we can move a along ha to a point a_1 such that $K(a_1, r)$ touches $K(t, s)$ and $\varrho(a_1bc) \leqq \varrho(abc)$. Changing the origin to t we can suppose

(i) $K(a)$, $K(b)$, $K(c)$ touch $K(0, s)$ externally,

(ii) O lies in the triangle abc,

(iii) some of $K(a)$, $K(b)$, $K(c)$ overlap.

As in (1) we now consider the class I of triangles xyz where x, y, z lie on the segments Oa, Ob, Oc and Ox, Oy, Oz are each of length at least $2s$. For each xyz in I define K_x, K_y, K_z to be the circles with centres x, y, z and radii $Ox - s$, $Oy - s$, $Oz - s$ respectively. Theorem 2 will follow from

THEOREM 3. $$\inf_{T \in I} \varrho(xyz) = \inf \frac{V((K_x \cup K_y \cup K_z) \cap xyz)}{V(xyz)} \geqq \pi/6\sqrt{3}.$$

As in [1] $\inf \varrho(xyz)$ is attained at a triangle $T_0 = uvw \in I$. There is no loss of generality in assuming that $Ou \geqq Ov \geqq Ow$. We assert that

(A) $Ou = Ov = Ow = 2s$.

By way of contradiction we assume that (A) is false so that in particular $Ou > 2s$. We first prove

LEMMA 8. *Let K_u and K_w be nonoverlapping. Further let K_u meet uw at e and Ou at f. Similarly let K_w meet uw at e' and Ow at f'. Then $K(0, s)$ meets Ou, Ow*

at f, f' respectively and K_v does not meet the region $\tilde{P}$ bounded by the segment ee', the arc $e'f'$ of K_w, the arc $f'f$ of $K(0, s)$ and the arc fe of K_u, where these arcs all lie in the triangle Ouw.

PROOF. Since K_v does not overlap $K(0, s)$ and O lies in the triangle uvw so K_v has no point in common with the arc ff' of $K(0, s)$ lying in the triangle Ouw. If K_v contains a point of $\tilde{P}$ then K_v contains a point of either (i) the line segment ee', (ii) the arc ef or (iii) the arc $e'f'$. In case (i) by convexity K_v must meet either the arc ef or the arc $e'f'$ since it cannot meet the arc ff' of $K(0, s)$. Thus either case (ii) or case (iii) must hold. In case (ii) the point f is the closest point of the arc ef to the point v hence f lies in K_v which is impossible since K_v touches $K(0, s)$ in a point distinct from f. Therefore case (iii) must hold. But by a similar argument the point f' lies in K_v again a contradiction which proves the lemma.

As with EGGLESTON [2] we now write $X(uvw)$ for $(K_u \cup K_v \cup K_w) \cap uvw$ and $Y(uvw)$ for the part of uvw not in $X(uvw)$. Let the intersection of $Y(uvw)$ with segments uv and uw be $Z(uvw)$ and let $W(uvw)$ be the part of uvw which lies on lines parallel to Ou through points of $Z(uvw)$. Let the parts of $Y(uvw)$ and of $W(uvw)$ on the same side of uO as v be written $Y_v(uvw)$ and $W_v(uvw)$ whilst those parts on the same side of Ou as w are written $Y_w(uvw)$ and $W_w(uvw)$. We assert that if O is on neither of the segments uv or uw then $V(W_v(uvw))V(Y_v(uvw))$ and $V(W_w(uvw)) < V(Y_w(uvw))$ whereas if O is on one of the segments uv, uw then one of these inequalities becomes an equality while the other remains a strict inequality so that always $V(W(uvw)) < V(Y(uvw))$. The proof of this assertion follows that of the corresponding result in [1] where we use lemma 8 in the proof of lemma 12 of [1] if K_u, K_w or K_u, K_w do not overlap. In case they overlap the assertion is obvious since then one of the regions is empty while the other must have positive area. Finally to prove assertion (A) we proceed exactly as in EGGLESTON [2] pp. 233—34 using the inequality $V(W(uvw)) < V(Y(uvw))$ to obtain the desired contradiction.

To complete the proof of theorem 3 and hence the main theorem it is enough to prove

THEOREM 4. *Let abc be a triangle, and let O lie in abc. Further let $Oa = Ob = Oc = 2s$ and let K_a, K_b, K_c be circles centred at a, b, c respectively each of radius s. Then*

$$\varrho(abc) = \frac{V(abc \cap (K_a \cup K_b \cup K_c))}{V(abc)} \geqq \pi/6\sqrt{3}\,.$$

PROOF. If no one of the lengths ab, bc, ca is less than $2s$ then the result is well known. So we may assume that $bc < 2s$. Thus the angle bOc is not more than $\pi/3$ so that in particular both the angles aOb and aOc are obtuse from which it follows that ab and ac are both greater than $2s$. Thus $K_a \cap K_c$ and $K_a \cap K_b$ are empty. Put the angle bOc equal to 2θ so that $0 < \theta \leqq \pi/6$. Then

$$V((K_a \cup K_b \cup K_c) \cap abc) = s^2(\psi + \sin\psi \cos\psi)$$

where $\sin\psi = 2 \sin\theta$. For fixed θ the triangle abc achieves its maximum area when the angles Oac and Oab are both $\frac{1}{2}\theta$. Hence

$$V(abc) \leqq 4s^2 \sin\theta\,(1 + \cos\theta)$$

from which it follows that

$$\varrho(abc) \geqq \frac{\psi + \sin\psi\cos\psi}{2\sin\psi(1+\cos\theta)} = f(\psi),$$

say, where $2\sin\theta = \sin\psi$ and since $0 < \theta \leqq \pi/6$ so $0 < \psi \leqq \pi/2$. Now we wish to show that $f(\psi)$ is nonincreasing in this range. Thus

$$f'(\psi)\,(2\sin\psi(1+\cos\theta))^2 =$$

$$= 2\cos^2\psi\, 2\sin\psi(1+\cos\theta) - 2(\psi+\sin\psi\cos\psi)\left(\cos\psi(1+\cos\theta) - \frac{\sin^2\psi\cos\psi}{4\cos\theta}\right) =$$

$$= 2\cos\psi\left\{\left(\sin\psi\cos\psi(1+\cos\theta) - (\psi+\sin\psi\cos\psi)(1+\cos\theta - \frac{\sin^2\psi}{4\cos\theta}\right)\right\}.$$

Thus $f'(\psi) \leqq 0$ will be true if

$$(\psi+\sin\psi\cos\psi)\left((1+\cos\theta - \frac{\sin^2\psi}{4\cos\theta}\right) \geqq 2\sin\psi\cos\psi(1+\cos\theta)$$

hence also if

$$\sin\psi(1+\cos\psi)\left(1+\cos\theta - \frac{\sin^2\psi}{4\cos\theta}\right) \geqq 2\sin\psi\cos\psi(1+\cos\theta),$$

which is true if

$$(1+\cos\theta)(1-\cos\psi) \geqq \frac{(1+\cos\psi)(1-\cos^2\psi)}{4\cos\theta}$$

or

$$4\cos\theta(1+\cos\theta) \geqq (1+\cos\psi)^2 = 2+2\cos\psi-\sin^2\psi = 2+2\cos\psi-4\sin^2\theta,$$

or

$$2+4\cos\theta \geqq 2\cos\psi.$$

Therefore $f(\psi)$ is nonincreasing and $f(\psi) \geqq f(\pi/2) = \pi/(4+2\sqrt{3}) \geqq \pi/6\sqrt{3}$.

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References

[1] R. P. BAMBAH and A. C. WOODS, On the minimal density of maximal packings of the plane by convex bodies, *Acta Math. Acad. Sci. Hung.*, **19** (1968), pp. 103—116.

[2] H. G. EGGLESTON, A minimal density plane covering problem, *Mathematika*, **12** (1965), pp. 226—234.

[3] L. FEJES TÓTH, Packings and coverings in the plane, *Proc. Coll. Convexity, Copenhagen*, 1965 (1967), pp. 78—87.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (3—4), (1968), pp. 345—364.

RADICAL AND SEMI-SIMPLICITY IN CATEGORIES

By

R. WIEGANDT (Budapest)

(*Presented by* L. RÉDEI)

Introduction

In the recent papers [2] and [12] general theories of radicals were established. HOEHNKE has treated the radical theory of universal algebras, the radicals occured in [2] are certain congruence relations. SULIŃSKI developed a general theory of the *Brown—McCoy radical* in categories, in [12] the radical of an object is an ideal. In both of the papers the point of view is to establish such types of radicals that the semi-simple objects should be subdirectly embedded into the direct products of simple objects.

In this paper, following SULIŃSKI, it will be assumed that the category occured here satisfies some additional requirements.

The purpose of this paper is to discuss the possibility of the representation of semi-simple objects as a direct product of simple objects. For this reason we define a type of radical which will be called $\mathscr{M}$-radical, and this turns out to be an ideal of an object.

After two preliminary sections in § 3 we present an exposition of $\mathscr{M}$-radical which will be defined as an intersection of certain ideals of an object. Particularly, it yields a new approach of the Brown—McCoy radical introduced by SULIŃSKI [12]. It will be proved that the $\mathscr{M}$-radical has the radical properties in the sense of NAGATA and MARANDA ([7], [6]), but it fails to be a radical in the sense of AMITSUR and KUROŠ ([1], [3]). However, the $\mathscr{M}$-radical is a rather general type of radical, it is suitable to obtain a satisfactory description of the $\mathscr{M}$-semi-simple objects as a direct product of $\mathscr{M}$-objects.

For this reason we use the concept of inverse limit in categories; § 4 is devoted to develop theorems concerning the inverse limit. The results of this section serve as tools for obtaining theorems in the subsequent section.

In § 5 we establish necessary and sufficient conditions that a) an $\mathscr{M}$-semi-simple object should be a direct product of $\mathscr{M}$-objects, b) each $\mathscr{M}$-semi-simple object should be a direct product of $\mathscr{M}$-objects. We shall prove namely that

a) an $\mathscr{M}$-semi-simple object is a direct product of $\mathscr{M}$-objects if and only if it is $\mathscr{M}$-compact, and

b) every $\mathscr{M}$-semi-simple object is a direct product of $\mathscr{M}$-objects if and only if every object of the category is $\mathscr{M}$-compact, further, every $\mathscr{M}$-semi-simple object is a direct product of simple $\mathscr{M}$-objects if and only if the discrete direct product coincides with the direct product.

Particularly, this later statement means that in the category of rings there does not exist such a radical that all the semi-simple rings should be complete direct

sums of simple rings (except the case when every ring is considered a radical ring).* But there are categories in which the discrete direct product and the direct product are the same concepts.

§ 1. Fundamental concepts

In this § we shortly recall the fundamental concepts which we shall need later. For details we refer to [4] and [12].

Let $\mathscr{C}$ be a category. The objects and maps of $\mathscr{C}$ will be denoted by small Latin and small Greek letters, respectively. By definition $\mathscr{C}$ satisfies the following conditions:

(c_1) *If* $\alpha: a \to b$ *and* $\beta: b \to c$ *are maps, then there is a uniquely defined map* $\alpha\beta: a \to c$ *which is called the product of the maps* α *and* β;

(c_2) *If* $\alpha: a \to b$, $\beta: b \to c$, $\gamma: c \to d$, *then* $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

(c_3) *For each object* $a \in \mathscr{C}$ *there is a map* $\varepsilon_a: a \to a$, *called the identity map of* a, *such that for any* $\alpha: b \to a$ *and* $\beta: a \to b$ *we have* $\alpha\varepsilon_a = \alpha$, $\varepsilon_a\beta = \beta$.

In this paper we suppose that the category satisfies some additional requirements, too. They were adopted from SULIŃSKI [12] and will be introduced successively after the notions necessary to formulate them have been explained.

The objects a and b are said to be *equivalent* if there are maps $\xi: a \to b$ and $\xi^{-1}: b \to a$ such that $\xi\xi^{-1} = \varepsilon_a$ and $\xi^{-1}\xi = \varepsilon_b$. This fact will be denoted by $a \sim b$ (ξ, ξ^{-1}).

A map $\mu: a \to b$ is said to be a *monomorphism* if for any maps $\alpha: c \to a$, $\beta: c \to a$ from $\alpha\mu = \beta\mu$ it follows $\alpha = \beta$. The product of two monomorphisms (if it exists) is also a monomorphism, and if μ is a monomorphism and $\mu = \varrho\sigma$, then ϱ is a monomorphism.

Consider an object $a \in \mathscr{C}$ and all pairs of the form (b, μ) where $\mu: b \to a$ is a monomorphism. We say that $(b; \mu) \leqq (c, \nu)$ if there is a $\varrho: b \to c$ such that $\varrho\nu = \mu$ holds. If $(b, \mu) \leqq (c, \nu)$ and $(c, \nu) \leqq (b, \mu)$ are valid, then the pairs (b, μ) and (c, ν) will be said to be equivalent. The classes of equivalent pairs will be called the *subobjects* of the object a and for convenience the equivalence class determined by the pair (b, μ) will also be denoted by (b, μ). If (b, μ) is a subobject of an object $a \in \mathscr{C}$, and the object $c \in \mathscr{C}$ is equivalent to b, then we shall say that the object c *can be embedded* in the object a.

An object $0 \in \mathscr{C}$ is said to be a *zero object* if for any object $a \in \mathscr{C}$ there is only one map which maps a into 0, and only one map which maps 0 into a. The zero objects are always equivalent.

We assume that

(c_4) $\mathscr{C}$ *possesses zero objects.*

A map $\omega_{ab}: a \to b$ is called a *zero map* if for any $\alpha: c \to a$ and $\beta: b \to d$ we have $\alpha\omega_{ab} = \omega_{cd}$, $\omega_{ab}\beta = \omega_{ad}$. If the category $\mathscr{C}$ possesses zero objects, then for every objects a, b there exists a zero map ω_{ab}. If there is no doubt between which objects the zero map operates, then the zero map will be briefly denoted by ω.

* This solves a problem raised by F. SZÁSZ during a personal discussion.

A subobject $(k, \varkappa)$ of an object $a \in \mathscr{C}$ is said to be a *kernel* of the map $\alpha: a \to b$, if (1) $\varkappa\alpha = \omega$; (2) for each $\gamma: c \to a$ satisfying the condition $\gamma\alpha = \omega$ there is a $\delta: c \to k$ such that $\delta\varkappa = \gamma$. If $(k, \varkappa)$ is a kernel of $\alpha: a \to b$, then the subobject $(k, \varkappa)$ of a will be called an *ideal* (or normal subobject), of a, and we shall write $\operatorname{Ker}\alpha = (k, \varkappa)$.

We suppose that

(c_5) *Every map has a kernel.*

A map μ is a monomorphism if and only if $\operatorname{Ker}\mu = (0, \omega)$. A map α is a zero map if and only if $\operatorname{Ker}\alpha = (a, \varepsilon_a)$. A non-zero object a will be called *simple* if its only ideals are $(0, \omega)$ and (a, ε_a).

A map $\tau: a \to b$ with the kernel $(k, \varkappa)$ is said to be a *normal epimorphism* if for each $\gamma: a \to c$ satisfying the condition $\varkappa\gamma = \omega$ there is a unique map $\delta: b \to c$ such that $\tau\delta = \gamma$. In this paper for convenience we shall shortly say *epimorphism* instead of normal epimorphism. If $\tau: a \to b$ is an epimorphism, then from $\tau\alpha = \tau\beta$ it follows $\alpha = \beta$. The product of two epimorphisms is again an epimorphism. If τ and ϱ are epimorphisms and $\tau = \varrho\sigma$, then σ is also an epimorphism.

A map $\xi: a \to b$ is an equivalence (i.e. such a map that $a \sim b(\xi, \xi^{-1})$) if and only if ξ is simultaneously a monomorphism and an epimorphism.

If a map $\alpha: a \to b$ can be represented in the form $\alpha = \mu\nu$, where $\mu: a \to m$ is an epimorphism and $\nu: m \to b$ is a monomorphism, then the triplet (μ, m, ν) will be called the *image* of the map α, and we shall write $\operatorname{Im}\alpha = (\mu, m, \nu)$.

We assume that

(c_6) *Every map has an image.*

If (μ, m, ν) is the image of a map $\alpha: a \to b$, then α is an epimorphism if and only if $(m, \nu) = (b, \varepsilon_b)$. Let $(k, \varkappa)$ be a subobject of an object a and let $\alpha: a \to b$ be an epimorphism. If (μ, m, ν) is the image of the map $\varkappa\alpha: k \to b$, then the subobject (m, ν) of b will be called the *image of* $(k, \varkappa)$ *by the epimorphism* α.

We suppose that

(c_7) *An image of an ideal by an epimorphism is always an ideal.*

Let I denote a set of indices, and let a_i $(i \in I)$ be a family of objects of $\mathscr{C}$. An object $g \in \mathscr{C}$ is said to be a *direct product* of the objects a_i $(i \in I)$ if there are maps $\pi_i: g \to a_i$ $(i \in I)$ such that for each object $h \in \mathscr{C}$ and for any system of maps $\alpha_i: h \to a_i$ $(i \in I)$, there is a unique map $\gamma: h \to g$ such that $\gamma\pi_i = \alpha_i$ holds for all $i \in I$. This product g will be denoted by $g = \prod_{i \in I} a_i(\pi_i)$.

The maps $\pi_i (i \in I)$ are called *projections* of g onto a_i, and they are epimorphisms. Moreover, for each $i \in I$ there is a uniquely defined monomorphism $\varrho_i: a_i \to g$ such that $\varrho_i\pi_i = \varepsilon_{a_i}$ and $\varrho_i\pi_j = \omega$ for $i \neq j$. Therefore the direct product g will sometimes be denoted by $g = \prod_{i \in I} a_i(\pi_i, \varrho_i)$. In the case of two factors the direct product will be denoted by $g = a_1 \times a_2(\pi_1, \pi_2; \varrho_1, \varrho_2)$, and then $\operatorname{Ker}\pi_1 = (a_2, \varrho_2)$, $\operatorname{Ker}\pi_2 = (a_1, \varrho_1)$ hold.

The dual definition of the direct product leads to the concept of free product.

We assume further:

(c_8) *Every family of objects has a direct product and a free product.*

(c_9) *The class of all subobjects of any object is a set.*

(c_{10}) *For each object $a \in \mathscr{C}$ the set of all ideals of a is a complete lattice.*

The axioms (c_1)—(c_{10}), however, seem to be not independent. In view of a theorem of TSALENKO [4] from (c_4), (c_6), (c_8) and (c_9) it follows (c_5). But in this paper it is not the aim to give an independent system of axioms.*

Let $\alpha: a \to b$ be an epimorphism and let (m, χ) be an ideal of b. If (d, δ) is an ideal of a such that $(d, \delta) \cong (k, \varkappa) = \operatorname{Ker} \alpha$, and (m, χ) is the image of (d, δ) by α, then the ideal (d, δ) will be called the *complete counterimage* of (m, χ) by α. If $\alpha: a \to b$ is an epimorphism and (m, χ) is an ideal of b, then there exists a unique (up to equivalence) complete counterimage of (m, χ) by α.

We recall yet some theorems proved in [10] and [12] respectively.

THEOREM 1. 1. *Let $(k_1, \varkappa_1)$, $(k_2, \varkappa_2)$ be two ideals of an object a, let $\alpha_1: a \to b$ be an epimorphism such that $(k_1, \varkappa_1) = \operatorname{Ker} \alpha_1$, and let (m, χ) be the image of $(k_2, \varkappa_2)$ by α_1. Then the union $(k_1, \varkappa_1) \cup (k_2, \varkappa_2)$ is the complete counterimage of $(m, \chi)$by α_1.*

THEOREM 1. 2. *Let $\alpha: a \to b$ be an epimorphism and let (m_i, χ_i) $(i = 1, 2)$ be two ideals of b. If $(m_1, \chi_1) \leqq (m_2, \chi_2)$, then $(d_1, \delta_1) \leqq (d_2, \delta_2)$ holds where (d_i, δ_i) $(i = 1, 2)$ is the complete counterimage of (m_i, χ_i) by α.*

PROPOSITION 1. 3. *Let $(k, \varkappa) \leqq (d_1, \delta_1) \leqq (d_2, \delta_2)$ be ideals of an object a and let $\alpha: a \to b$ be an epimorphism with $\operatorname{Ker} \alpha = (k, \varkappa)$. If (μ_i, m_i, χ_i) denotes the image of $\delta_i \alpha$, then $(m_1, \chi_1) \leqq (m_2, \chi_2)$ holds.*

PROOF. Let $\gamma: d_1 \to d_2$ be a monomorphism satisfying $\gamma \delta_2 = \delta_1$, and denote the image of $\gamma \mu_2$ by (ν, n, φ). Now (n, φ) is a subobject of m_2 and $\operatorname{Im} \delta_1 \alpha = (\nu, n, \varphi\chi_2)$ holds. Hence $(n, \varphi\chi_2) = (m_1, \chi_1)$ and so $(m_1, \chi_1) \leqq (m_2, \chi_2)$.

THEOREM 1. 4. *Let $(k_1, \varkappa_1)$, $(k_2, \varkappa_2)$ be ideals of a such that $(k_1, \varkappa_1) \cap (k_2, \varkappa_2) = (0, \omega)$ and $(k_1, \varkappa_1) \cup (k_2, \varkappa_2) = (a, \varepsilon_a)$. If $\alpha_i: a \to a_i$ $(i = 1, 2)$ is an epimorphism such that $(k_i, \varkappa_i) = \operatorname{Ker} \alpha_i$, then $a_1 \sim k_2(\xi, \xi^{-1})$, $a_2 \sim k_1(\eta, \eta^{-1})$ and the object a is a direct product of a_1 and a_2.*

The proof can be found in SULIŃSKI [12].

§ 2. Preliminary theorems

In our investigations we need the following theorem which corresponds to the so called First Isomorphism Theorem.

THEOREM 2. 1. *Let $(k, \varkappa) \leqq (d, \delta)$ be two ideals of an object a and let $\alpha: a \to b$ be an epimorphism with $\operatorname{Ker} \alpha = (k, \varkappa)$, further consider the image $\operatorname{Im} \delta\alpha = (\mu, m, \chi)$ and an epimorphism $\beta: b \to c$ with $\operatorname{Ker} \beta = (m, \chi)$. In other words, consider the commutative diagram*

$$\begin{array}{ccccc} k & \xrightarrow{\varkappa_1} & d & \xrightarrow{\mu} & m \\ & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \chi} \\ k & \xrightarrow{\varkappa} & a & \xrightarrow{\alpha} & b \\ & & & & \downarrow{\scriptstyle \beta} \\ & & & & c \end{array}$$

* Axiomatic investigations are considered e. g. in the paper of D. PUPPE: Über die Axiome für abelsche Kategorien, *Archiv der Math.*, **18** (1967), pp. 217—222.

where $\varkappa_1$, $\varkappa$ *and* χ *are monomorphisms and* μ, α *and* β *are epimorphisms with* $\operatorname{Ker}\alpha=(k,\varkappa)$, $\operatorname{Ker}\beta=(m,\chi)$. *Then* $\operatorname{Ker}\mu=(k,\varkappa_1)$, *and there exists an epimorphism* $\gamma: a\to c$ *such that* $\operatorname{Ker}\gamma=(d,\delta)$, *moreover* $\gamma=\alpha\beta$.

Theorem 2. 1 is essentially Corollary 16.2 of B. MITCHELL's book: *Theory of Categories* (New York, 1965). This theorem was proved there for exact categories (i.e. for categories in which every monomorphism and epimorphism are normal). Since in this paper the categories have not to be exact, therefore we reprove this theorem.

PROOF. We show that $\operatorname{Ker}\mu=(k,\varkappa_1)$. On the one hand we have $\varkappa_1(\mu\chi)=\varkappa_1(\delta\alpha)=(\varkappa_1\delta)\alpha=\varkappa\alpha=\omega$. Since χ is a monomorphism, so also $\varkappa_1\mu=\omega$ holds. On the other hand let $\nu: n\to d$ be an arbitrary map with $\nu\mu=\omega$. Then from $\omega=(\nu\mu)\chi=\nu(\delta\alpha)=(\nu\delta)\alpha$ and from $\operatorname{Ker}\alpha=(k,\varkappa)$ it follows that there exists a map $\tau: n\to k$ with $\tau\varkappa=\nu\delta$. Thus we have $\tau\varkappa_1\delta=\nu\delta$ and so also $\tau\varkappa_1=\nu$. Hence $\operatorname{Ker}\mu=(k,\varkappa_1)$.

Now we turn to prove that $\operatorname{Ker}\gamma=(d,\delta)$ is valid for $\gamma=\alpha\beta$. Obviously $\delta(\alpha\beta)=(\mu\chi)\beta=\omega$ holds. Moreover, let $\sigma: s\to a$ be an arbitrary map with $\sigma(\alpha\beta)=\omega$. We have to prove that there is a map $\sigma': s\to d$ with $\sigma'\delta=\sigma$. Since $(\sigma\alpha)\beta=\omega$, and β is an epimorphism with $\operatorname{Ker}\beta=(m,\chi)$, so by the definition of the kernel there is a map $\varrho: s\to m$ such that $\varrho\chi=\sigma\alpha$. Let us denote the image of ϱ, σ and $\sigma_1\alpha$ by $(\varrho_0, m', \varrho_1)$, (σ_0, a', σ_1) and (τ_0, n, τ_1), respectively. Now we have

$$(\varrho_0, m', \varrho_1\chi)=\operatorname{Im}\varrho\chi=\operatorname{Im}\sigma_0\sigma_1\alpha=(\sigma_0\tau_0, n, \tau_1)$$

and so $(m', \varrho_1\chi)$ is the image of both of (s,σ) and (a', σ_1); further $(m', \varrho_1\chi)\leqq(m,\chi)$ holds. Applying Theorems 1. 1 and 1. 2 for the complete counterimages by α we get

$$(a', \sigma_1)\leqq(a', \sigma_1)\cup(k,\varkappa)\leqq(d,\delta).$$

Hence there is a map $\sigma'': a'\to d$ such that $\sigma''\delta=\sigma_1$. Then the map $\sigma'=\sigma_0\sigma''$ maps s into d and $\sigma'\delta=\sigma$ holds. Thus the theorem is proved.

COROLLARY 2. 2. *Consider the following commutative diagram*

$$\begin{array}{ccccc} k & \xrightarrow{\varkappa_1} & d & \xrightarrow{\mu} & m \\ & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\chi} \\ k & \xrightarrow{\varkappa} & a & \xrightarrow{\alpha} & b \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ & & c_1 & & c_2 \end{array}$$

where $\varkappa_1$, $\varkappa$, δ *and* χ *are monomorphisms and* μ, α, γ *and* β *are epimorphisms with* $\operatorname{Ker}\alpha=(k,\varkappa)$, $\operatorname{Ker}\gamma=(d,\delta)$ *and* $\operatorname{Ker}\beta=(m,\chi)$. *Then there exists an equivalence map* ξ, *such that* $c_1\sim c_2(\xi,\xi^{-1})$.

By Theorem 2. 1 we have $\operatorname{Ker}\gamma=(d,\delta)=\operatorname{Ker}\alpha\beta$. Hence by the definition of the epimorphism there exists a unique $\xi: c_1\to c_2$ such that $\gamma\xi=\alpha\beta$, and a unique $\xi^{-1}: c_2\to c_1$ such that $\alpha\beta\xi^{-1}=\gamma$. Thus we have $\gamma\xi\xi^{-1}=\gamma$ and $\alpha\beta\xi^{-1}\xi=\alpha\beta$. But γ and $\alpha\beta$ are epimorphisms, so the map ξ is indeed an equivalence map.

Now we prove the following generalization of Theorem 1. 4.

THEOREM 2. 3. *Let* (d_i, δ_i) $(i=1, 2)$ and $(k, \varkappa)$ *be ideals of an object* a *such that* $(d_1, \delta_1) \cap (d_2, \delta_2) = (k, \varkappa)$ *and* $(d_1, \delta_1) \cup (d_2, \delta_2) = (a, \varepsilon_a)$. *Consider the epimorphisms* $\alpha: a \to b$ *and* $\gamma_i: a \to b_i$ *with* $\operatorname{Ker} \alpha = (k, \varkappa)$ *and* $\operatorname{Ker} \gamma_i = (d_i, \delta_i)$. *Then the object* b *is a direct sum of the objects* b_1 *and* b_2.

PROOF. Denote the image of $\delta_i \alpha$ by $\operatorname{Im} \delta_i \alpha = (\mu_i, m_i, \chi_i)$, and consider the commutative diagram

$$\begin{array}{ccccc} k & \longrightarrow & d_i & \longrightarrow & m_i \\ & & \downarrow{\scriptstyle \delta_i} & & \downarrow{\scriptstyle \chi_i} \\ k & \xrightarrow{\varkappa} & a & \xrightarrow{\alpha} & b \\ & & & & \downarrow{\scriptstyle \beta_i} \\ & & & & b_i' \end{array}$$

where β_i $(i=1, 2)$ is an epimorphism with $\operatorname{Ker} \beta_i = (m_i, \chi_i)$. From Theorem 2. 1 it follows that the epimorphism $\alpha_i' = \alpha\beta_i$ has the kernel $\operatorname{Ker} \alpha_i' = (d_i, \delta_i)$, therefore $b_i \sim b_i'(\xi, \xi^{-1})$ holds. On the other hand, according to Proposition 1. 3, we have

$$(m_1, \chi_1) \cap (m_2, \chi_2) = (0, \omega),$$

and

$$(m_1, \chi_1) \cup (m_2, \chi_2) = (b, \varepsilon_b).$$

So applying Theorem 1. 4, we get that b is a direct product of b_1 and b_2.

§ 3. $\mathcal{M}$-radical

Let $\mathcal{M}$ be a class of objects of $\mathcal{C}$. We assume that

(i) *if* $a \in \mathcal{M}$ *and* $a \sim b(\xi, \xi^{-1})$ *then* $b \in \mathcal{M}$;

(ii) *if* $a, b \in \mathcal{M}$ *and* $\alpha: a \to b$ *is an epimorphism, then either* $\alpha = \omega$, *or* $a \sim b(\alpha, \alpha^{-1})$.

The objects of $\mathcal{M}$ will be called $\mathcal{M}$-*objects*.

To illustrate condition (ii) we mention that any class of simple objects satisfying (i) satisfies also condition (ii). Thus we may regard the objects of $\mathcal{M}$ as „$\mathcal{M}$-simple" objects.

An ideal $(k, \varkappa)$ will be called an $\mathcal{M}$-*ideal* if $(k, \varkappa)$ is the kernel of an epimorphism $\alpha: a \to b$ with $b \in \mathcal{M}$.

PROPOSITION 3. 1. *Every* $\mathcal{M}$-*ideal* $(k, \varkappa) \neq (a, \varepsilon_a)$ *is an* $\mathcal{M}$-*maximal ideal, i.e. for any epimorphism* $\gamma: a \to c$ *with* $\operatorname{Ker} \gamma = (d, \delta) \cong (k, \varkappa)$ *and* $c \in \mathcal{M}$ *it follows either* $(d, \delta) = (k, \varkappa)$ *or* $(d, \delta) = (a, \varepsilon_a)$.

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccc} k & \xrightarrow{\varkappa_1} & d & \xrightarrow{\mu} & m \\ & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \chi} \\ k & \xrightarrow{\varkappa} & a & \xrightarrow{\alpha} & b \\ & & & \searrow^{\gamma} & \downarrow{\scriptstyle \beta} \\ & & & & c \end{array}$$

where $\operatorname{Im}\delta\alpha=(\mu, m, \chi)$ and β is an epimorphism with $\operatorname{Ker}\beta=(m, \chi)$. Since $b, c\in\mathscr{M}$, so according to (ii) we have either $\beta=\omega$ or $b\sim c(\beta, \beta^{-1})$. If $\beta=\omega$, then we have $\gamma=\alpha\beta=\omega$, and so $(d, \delta)=\operatorname{Ker}\omega=(a, \varepsilon_a)$. In the case $b\sim c(\beta, \beta^{-1})$ we get $(m, \chi)=\operatorname{Ker}\beta=(0, \omega)$, and so $\omega=\mu\chi=\delta_\alpha$. Hence from $\operatorname{Ker}\alpha=(k, \varkappa)$ it follows $(d, \delta)\leqq(k, \varkappa)$. Thus $(k, \varkappa)$ is an $\mathscr{M}$-maximal ideal.

Consider the class of all $\mathscr{M}$-ideals $(d, \delta)\neq(a, \varepsilon_a)$ of an object $a\in\mathscr{C}$. By (c_9) this class is a set. This set will be denoted by M_a and will be called the *structure $\mathscr{M}$-space* of a.

A subset Δ of M_a is called an *independent $\mathscr{M}$ system of a*, if for every epimorphism $\alpha: a\to b$ with $\operatorname{Ker}\alpha=(k, \varkappa)=\bigcap_{i=1}^{n}(d_i, \delta_i)$ $((d_i, \delta_i)\in\Delta;\ i=1, \ldots, n)$, there are epimorphisms $\alpha_i: a\to b_i\in\mathscr{C}$ such that $\operatorname{Ker}\alpha_i=(d_i, \delta_i)$ $(i=1, \ldots, n)$, and b is a direct product of the $\mathscr{M}$-objects $[b_1, \ldots, b_n$: $b=\prod_{i=1}^{n} b_i(\pi_i)$ with $\alpha\pi_i=\alpha_i$.

PROPOSITION 3. 2. *Every object $a\in\mathscr{C}$ has a maximal independent $\mathscr{M}$-system.*

PROOF. If M_a is the void set, then this is a maximal independent $\mathscr{M}$-system. Let

$$\Delta_1\subset\Delta_2\subset\ldots\subset\Delta_i\subset\ldots \qquad (i\in I)$$

be independent $\mathscr{M}$-systems. If $(d_1, \delta_1), \ldots, (d_n, \delta_n)\in\bigcup_{i\in I}\Delta_i$, then there exists an index $i\in I$ such that $(d_1, \delta_1), \ldots, (d_n, \delta_n)\in\Delta_i$. Thus $\Delta_0=\bigcup_{i\in I}\Delta_i$ is also an independent $\mathscr{M}$-system. So applying Zorn's lemma we get that there is a maximal independent $\mathscr{M}$-system Δ of a.

THEOREM 3. 3. *Let $\Delta_1=\{(d_{1i}, \delta_{1i})\,|\,i\in I\}$ and $\Delta_2=\{(d_{2j}, \delta_{2j})\,|\,j\in J\}$ be two maximal independent $\mathscr{M}$-systems of the object $a\in\mathscr{C}$. Then*

$$(c_1, \gamma_1)=\bigcap_{i\in I}(d_{1i}, \delta_{1i})=\bigcap_{j\in J}(d_{2j}, \delta_{2j})=(c_2, \gamma_2)$$

holds.

PROOF. Suppose that the statement is not true. Without loss of generality we may assume $(c_1, \gamma_1)\not\leqq(c_2, \gamma_2)$. Now there exists an ideal $(d_2, \delta_2)\in\Delta_2$ with $(c_1, \gamma_1)\not\leqq(d_2, \delta_2)$. For any finite subset $(d_{11}, \delta_{11}), \ldots, (d_{1n}, \delta_{1n})$ of Δ_1 obviously there holds also

$$(d_1, \delta_1)=\bigcap_{i=1}^{n}(d_{1i}, \delta_{1i})\not\leqq(d_2, \delta_2).$$

Consider the ideal $(k, \varkappa)=(d_1, \delta_1)\cap(d_2, \delta_2)$, and the commutative diagram

$$\begin{array}{ccccc} k & \longrightarrow & d_i & \xrightarrow{\mu_i} & m_i \\ & & \downarrow{\scriptstyle\delta_i} & & \downarrow{\scriptstyle\chi_i} \\ k & \xrightarrow{\varkappa} & a & \xrightarrow{\alpha} & b \\ & & & \searrow^{\gamma_i} & \downarrow{\scriptstyle\beta_i} \\ & & & & b_i \end{array}$$

where χ_i $(i=1, 2)$ is a monomorphism and μ_i, α, β_i and γ_i $(i=1, 2)$ are epimorphisms

such that $\operatorname{Ker}\alpha=(k,\varkappa)$, $\operatorname{Ker}\beta_i=(m_i,\chi_i)$ and $\operatorname{Im}\delta_i\alpha=(\mu_i,m_i,\chi_i)$. By virtue of Theorem 2. 1 we have $\operatorname{Ker}\gamma_i=(d_i,\delta_i)$, and from Theorem 2. 3 it follows that b is a direct product of b_1 and b_2. But because of $\operatorname{Ker}\gamma_1=(d_1,\delta_1)$ the object b_1 is a direct product of finite many $\mathcal{M}$-objects, further b_2 itself is also an $\mathcal{M}$-object. Thus $\Delta_0=\{\Delta_1,(d_2,\delta_2)\}$ is an independent $\mathcal{M}$-system which contradicts the fact that Δ_1 was a maximal independent $\mathcal{M}$-system.

Denote the intersection of all $(d,\delta)\in M_a$ by $(k_M,\varkappa_M)$. From Theorem 3. 3 we obtain

COROLLARY 3. 4. *For any maximal independent $\mathcal{M}$-system $\Delta=\{(d_i,\delta_i)|i\in I\}$*

$$\bigcap_{i\in I}(d_i,\delta_i)=(k_M,\varkappa_M)$$

is valid.

Let $(k,\varkappa)=(a,\varepsilon_a)$ be an arbitrary $\mathcal{M}$-ideal of a. Obviously there is a maximal independent $\mathcal{M}$-system containing $(k,\varkappa)$. Using Theorem 3. 3 we have

$$(k_M,\varkappa_M)\leqq\bigcap_{i\in I}(d_i,\delta_i)\leqq(k_M,\varkappa_M).$$

DEFINITION 3. 5. The *$\mathcal{M}$-radical* of an object $a\in\mathcal{C}$ is the subobject

$$R(a)=\begin{cases}(k_M,\varkappa_M) & \text{if} \quad M_a\neq\varnothing\\ (a,\varepsilon_a) & \text{if} \quad M_a=\varnothing.\end{cases}$$

In the case $R(a)=(a,\varepsilon_a)$ the object a will be called an *$\mathcal{M}$-radical object*. If $R(a)=(0,\omega)$ then the object a is called *$\mathcal{M}$-semi simple*.

We remark that if $\mathcal{C}$ denotes the category of all rings and $\mathcal{M}$ consists of all simple rings with the unity element, then $R(a)$ becomes the *Brown—McCoy radical*. If $\mathcal{M}$ is the class of all fields, then we obtain the *Thierrin radical*.*

Next we show that the $\mathcal{M}$-radical possesses the radical properties in the sense of NAGATA [7] and MARANDA [6].

THEOREM 3. 6. *The $\mathcal{M}$-radical has the following properties:*

a) *For each object $a\in\mathcal{C}$ there is an $\mathcal{M}$-radical which is an ideal of a.*

b) *If $\alpha: a\to b$ is an epimorphism, and (m,χ) denotes the image of $R(a)$ by the map α, then $(m,\chi)\leqq R(b)$ is valid.*

c) *If $\alpha: a\to b$ is an epimorphism such that $R(a)=\operatorname{Ker}\alpha$, then the object b is $\mathcal{M}$-semi simple.*

PROOF. Property a) is clear by Definition 3. 5 and axiom (c_{10}).

b) First, suppose that $R(a)=(a,\varepsilon_a)$. Now M_a is the void set. Consider an $\mathcal{M}$-ideal (n,φ) of the object b and denote the complete counterimage of (n,φ) by (d,δ). By Theorem 2. 1 the diagram

$$\begin{array}{ccccc} k & \xrightarrow{\varkappa_1} & d & \xrightarrow{\nu} & n \\ & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varphi} \\ k & \xrightarrow{\varkappa} & a & \xrightarrow{\alpha} & b \\ & & & \searrow^{\gamma} & \downarrow{\scriptstyle\beta} \\ & & & & c \end{array}$$

* Cf.: N. J. DIVINSKY: *Rings and Radicals* (London, 1965).

(Ker $\nu=(k, \varkappa_1)$, Ker $\alpha=(k, \varkappa)$ and β is an epimorphism with Ker $\beta=(n, \varphi)$) is commutative, and Ker $\gamma=(d, \delta)$ holds. Since $(n, \varphi)\in M_b$, so $c\in\mathscr{M}$ and therefore $(d, \delta)\in M_a$ which is a contradiction.

If $M_a\neq\varnothing$, then by Corollary 2. 2 the complete counterimage of each $\mathscr{M}$-ideal (n, φ) of b is an $\mathscr{M}$-ideal (d', δ') of a which contains Ker α. Hence

$$R(a)\leqq\cap(d', \delta')$$

and so, in view of Proposition 1. 3

$$(m, \chi)\leqq R(b)$$

is valid where (m, χ) denotes the image of $R(a)$ by α.

c) If $R(a)=(a, \varepsilon_a)$ then the statement is clear. If $\bigcap\limits_{i\in I}(d_i, \delta_i)=(k_M, \varkappa_M)=$ $=R(a)\neq(a, \varepsilon_a)$, then for any $\mathscr{M}$-maximal ideal (d_i, δ_i) of a, by Corollary 2. 2, we have the commutative diagram

$$\begin{array}{ccccc} k_M & \longrightarrow & d_i & \xrightarrow{\mu_i} & m_i \\ & & \downarrow{\scriptstyle\delta_i} & & \downarrow{\scriptstyle\chi_i} \\ k_M & \xrightarrow{\varkappa_M} & a & \xrightarrow{\alpha} & b \\ & & & \searrow^{\gamma_i} & \downarrow{\scriptstyle\beta_i} \\ & & & & c_i \end{array}$$

where μ_i, β_i and γ_i are epimorphisms with Ker $\beta_i=(m_i, \chi_i)$ and Ker $\gamma=(d_i, \delta_i)$. Since $(d_i, \delta_i)\in M_a$, so c is an $\mathscr{M}$-object and (m_i, χ_i) is an $\mathscr{M}$-ideal of b. Therefore

$$R(b)\leqq\bigcap_{i\in I}(m_i, \varkappa_i)=(m_0, \varkappa_0)$$

holds. If (l, λ) and $(k_0, \varkappa_0)$ denotes the complete counterimage of $R(b)$ and (m_0, χ_0) by α respectively, then because of Theorem 1. 2 we get

$$(l, \lambda)\leqq(k_0, \varkappa_0)\leqq\bigcap_{i\in I}(d_i, \delta_i)=(k_M, \varkappa_M)=\text{Ker}\,\alpha,$$

and from Theorem 1. 1

$$(l, \lambda)=(k_M, \varkappa_M)=\text{Ker}\,\alpha$$

follows, so $R(b)=(0, \omega)$ holds.

In particular, if a is an $\mathscr{M}$-radical object, then property b) states the following:

b′) *If $\alpha: a\to b$ is an epimorphism, then b is also an $\mathscr{M}$-radical object.*

Generally, however, the $\mathscr{M}$-radical is not a radical in the sense of AMITSUR [1] and KUROŠ [3], since the $\mathscr{M}$-radical of an object $a\in\mathscr{C}$ need not be an $\mathscr{M}$-radical object.* But if $\mathscr{M}$ is a so-called modular class of objects, then the $\mathscr{M}$-radical possesses the radical-properties in the sense of AMITSUR and KUROŠ, moreover the $\mathscr{M}$-radical coincides with the *Brown—McCoy* radical introduced in categories by SULIŃSKI [12].

* F. SZÁSZ turned kindly my attention to the following example: in the category of abelian groups let the radical of an object be the Frattini subgroup. The cyclic group $C(p^2)$ of order p^2 has the radical $C(p)$, and the radical of $C(p)$ is 0.

A class $\mathscr{M}$ satisfying conditions (i) and (ii) is said to be a *modular class* if the following conditions hold:

(iii) *If* (p, π) *is an ideal of* a, *and* $p \in \mathscr{M}$, *then there is a unique ideal* $(m, \chi) \in M_a$ *such that* $(p, \pi) \cap (m, \chi) = (0, \omega)$.

(iv) *If* (l, λ) *is an ideal of* a *and* (q, ϑ) *is an* $\mathscr{M}$*-ideal of* l, *then* $(q, \vartheta\lambda)$ *is an ideal of* a.

We remark that condition (iii) seems to be somewhat weaker than the corresponding condition 4.5. I in SULIŃSKI [12], where it is supposed that the ideal (p, π) should be a retract of a. (An ideal (p, π) of a is said to be a retract of a if there is a map $\alpha: a \to p$ such that $\pi\alpha = \varepsilon_p$.)

THEOREM 3. 6. *If* $\mathscr{M}$ *is a modular class of objects, then the* $\mathscr{M}$*-radical of an object* $a \in \mathscr{C}$ *is an* $\mathscr{M}$*-radical object.*

PROOF. Let (q, ϑ) denote an $\mathscr{M}$-ideal of the radical $R(a) = (k, \varkappa)$ of an object $a \in \mathscr{C}$. By condition (iv) $(q, \vartheta\varkappa)$ is an ideal of a. Let $\alpha: a \to b$ be an epimorphism such that $\operatorname{Ker} \alpha = (q, \vartheta\varkappa)$, and let (p, π) denote the image of $(k, \varkappa)$ by α. Since $(q, \vartheta) \in M_k$, therefore $p \in \mathscr{M}$ holds. Hence by condition (iii) there exists a unique ideal $(m, \mu) \in \mathscr{M}_b$, such that $(m, \chi) \cap (p, \pi) = (0, \omega)$. If (d, δ) denotes the complete counterimage of (m, χ) by α, then in view of Corollary 2. 2 we have the following commutative diagram

$$\begin{array}{ccccc} q & \longrightarrow & d & \xrightarrow{\mu} & m \\ & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \chi} \\ q & \xrightarrow{\vartheta} & a & \xrightarrow{\alpha} & b \\ & & & \searrow^{\gamma} & \downarrow{\scriptstyle \beta} \\ & & & & c \end{array}$$

where β and γ are epimorphisms with $\operatorname{Ker} \beta = (m, \chi)$ and $\operatorname{Ker} \gamma = (d, \delta)$. Moreover, from $(m, \chi) \in M_b$ it follows $c \in \mathscr{M}$. Thus $(d, \delta) \in M_a$, and so we have $(k, \varkappa) \leqq (d, \delta)$. Using Proposition 1. 3 this implies $(p, \pi) \leqq (m, \chi)$, and so by condition (iii) $(p, \pi) = (0, \omega)$ is valid. But $(p, \pi) = (0, \omega)$ is the image of $(k, \varkappa)$ by α, therefore $(k, \varkappa) \leqq (q, \vartheta\varkappa)$. Hence k is an $\mathscr{M}$-radical object.

In view of Theorem 4. 8 of SULIŃSKI [12] we obtain

THEOREM 3. 7. *If* $\mathscr{M}$ *is a modular* class of simple objects, then the* $\mathscr{M}$*-radical is just the Brown—McCoy radical determined by* $\mathscr{M}$.

In the following we omit the assumptions (iii) and (iv). Thus our results are valid for rings, Lie rings and alternative rings, as well.

Added in proof (6 August 1968). ANDRUNAKIEVIČ and RIABUKHIN has shown that the class of simple Lie rings does not fulfill condition (iv) (*Dokl. Akad. Nauk. SSSR*, **179** (1968), 503—506).

* In the sense of SULIŃSKI [12].

§ 4. Inverse limit

To obtain a satisfactory description of $\mathscr{M}$-semi simple objects as a direct product of $\mathscr{M}$-objects we need the concept of invers limit* and we shall establish some theorems concerning this concept.

In what follows I will denote a partially ordered directed set of indices.

DEFINITION 4. 1. We shall call a system $\Omega=[a_i, \pi_j^i]$, of objects a_i and maps π_j^i $(i, j\in I;\ i\geqq j)$ an *inverse system*, if the following conditions hold:

1) $\pi_j^i : a_i \to a_j \qquad (i \geqq j)$,

2) $\pi_j^i \pi_k^j = \pi_k^i \qquad (i \geqq j \geqq k)$,

3) $\pi_i^i = \varepsilon_{a_i} \qquad (i\in I)$.

DEFINITION 4. 2. An object g is said to be an *inverse limit* of the inverse system Ω, if there are maps $\pi_i: g\to a_i$ (called the *projections* of g into a_i) such that $\pi_i\pi_j^i=\pi_j$ $(i, j\in I;\ i\geqq j)$ and for each object $h\in\mathscr{C}$ and for any system of maps $\alpha_i: h\to a_i$ satisfying $\alpha_i\pi_j^i=\alpha_j$, there is a unique map $\gamma: h\to g$ such that $\gamma\pi_i=\alpha_i$ for all $i\in I$. If g is an inverse limit of the inverse system Ω, then we write $g=\varprojlim \Omega$.

We assume, the category $\mathscr{C}$ satisfies axiom

(c_{11}) *Every inverse system has an inverse limit.***

PROPOSITION 4. 3. *If both of g_1 and g_2 are inverse limits of the inverse system Ω, then g_1 and g_2 are equivalent.*

PROOF. Let us denote the projections of g_1 and g_2 by π_i and ϱ_i $(i\in I)$ respectively. By Definition 4. 2 there exist uniquely defined maps $\gamma_1: g_2\to g_1$ and $\gamma_2: g_1\to g_2$ such that $\gamma_1\pi_i=\varrho_i$ and $\gamma_2\varrho_i=\pi_i$. Hence $\gamma_1\gamma_2\varrho_i=\varrho_i$ and $\gamma_2\gamma_1\pi_i=\pi_i$. Taking into account that γ_1 and γ_2 are uniquely determined we obtain $\gamma_1\gamma_2=\varepsilon_{g_2}$ and $\gamma_2\gamma_1=\varepsilon_{g_1}$. Hence $g_1\sim g_2(\gamma_2, \gamma_1)$.

Two inverse systems Ω_1 and Ω_2 are said to be *equivalent* if their inverse limits g_1 and g_2 are equivalent objects. An inverse system $\Omega'=[a_{i'}', \varrho_{j'}^{i'}]$ is called a *subsystem* of the inverse system $\Omega=[a_i, \pi_j^i]$ if $i'\in I'\subseteq I$ and $a_{i'}'=a_{i'}$, $\varrho_{j'}^{i'}=\pi_{j'}^{i'}$ $(i'\geqq j')$ are valid.

PROPOSITION 4. 4. *If Ω' is a cofinal subsystem of the inverse system Ω, then Ω' and Ω are equivalent.*

PROOF. Let us denote an inverse limit of Ω and Ω' by g and g' and the corresponding projections by π_i and ϱ_i, respectively. Obviously by Definition 4. 2 we have that there is a unique map $\gamma_1: g'\to g$ such that $\gamma_1\pi_i=\varrho_{i'}\pi_i^{i'}$ $(i\in I)$ holds. On the other hand, there is a unique map $\gamma_2: g\to g'$ such that $\gamma_2\varrho_{i'}=\pi_{i'}$ is valid for all $i'\in I'$. Hence we obtain $\gamma_2\gamma_1\pi_{i'}=\gamma_2\varrho_{i'}\pi_{i'}^{i'}=\pi_{i'}$ and $\gamma_1\gamma_2\varrho_{i'}=\gamma_1\pi_{i'}=\varrho_{i'}$. Taking into account the uniqueness of γ_1 and γ_2 we get $\gamma_2\gamma_1=\varepsilon_g$ and $\gamma_1\gamma_2=\varepsilon_{g'}$. Thus $g\sim g'(\gamma_2, \gamma_1)$.

* The concepts of direct and inverse limit were introduced in categories in the MITCHELL's book mentioned before.

** The category of universal algebras does satisfy this conditon (cf. P. M. COHN: *Universal Algebra* (Harper & Row, 1965)).

PROPOSITION 4. 5. *If* $\Omega=[a_i, \pi_j^i]$ *is an inverse system and* $g=\varprojlim \Omega$ *an inverse limit with projections* π_i $(i\in I)$, *then* $\bigcap_{i\in I} \operatorname{Ker} \pi_i=(0, \omega)$.

PROOF. Denote the kernel of π_i by $(k_i, \varkappa_i)$ and consider the ideal $(l, \lambda)=$ $=\bigcap_{i\in I}(k_i, \varkappa_i)$. For each $i\in I$ there is a map λ_i such that $\lambda_i\varkappa_i=\lambda$. Therefore we have $\lambda\pi_i=\lambda_i(\varkappa_i\pi_i)=\omega$. On the other hand $\omega\pi_i=\omega$ holds, so by Definition 4. 2 we get $\lambda=\omega$, and $(l, \lambda)=(0, \omega)$.

THEOREM 4. 6. *The inverse limit* $g=\varprojlim \Omega$ $(\Omega=[a_i, \pi_j^i])$ *is a subobject of the direct product* $\prod_{i\in I} a_i(\varrho_i)$.

PROOF. By the definition of the direct product there is a unique map γ such that $\gamma\varrho_i=\pi_i$ for all $i\in I$. If (l, λ) and $(k_i, \varkappa_i)$ denote the kernels of γ and α_i respectively, then $\lambda\alpha_i=(\lambda\gamma)\varrho_i=\omega$ holds. Therefore there is a map $\lambda_i: l\to k_i$ satisfying $\lambda_i\varkappa_i=\lambda$. Since λ is a monomorphism, so λ_i is also a monomorphism. Thus by Proposition 4. 5 we obtain

$$(l, \lambda) \leqq \bigcap_{i\in I}(k_i, \varkappa_i) = (0, \omega)$$

Hence γ is a monomorphism.

Consider an inverse system $\Omega=[a_i, \pi_j^i]$ and suppose that every map π_j^i is either an equivalence, or a zero map. The inverse map of π_j^i will be denoted by π_i^j. If for every π_j^i we have $\pi_j^i=\omega$, then obviously $\varprojlim \Omega=0$. If $\pi_j^i\neq\omega$, then $a_j\neq 0$ and so each epimorphism π_j^k $(k\geqq j)$ is an equivalence. By Proposition 4. 4 we can suppose without loss of generality that every π_j^i $(i, j\in I; i\geqq j)$ is an equivalence. Now any two objects a_j and a_k are equivalent, for there is an index $i\in I$ such that $i\geqq j, k$, and so $a_j\sim a_k(\pi_i^j\pi_k^i, \pi_i^k\pi_j^i)$ holds.

Let us denote an inverse limit of Ω by g, and choose an object a_i of Ω. Since $\alpha_j=\pi_k^i\pi_j^k: a_i\to a_j$ $(k\geqq i, j)$ holds for all $j\in I$, so by Definition 4. 2 there is a unique map $\gamma_i: a_i\to g$ such that $\gamma_i\pi_j=\alpha_j$ $(j\in I)$ is valid, where π_j denotes the projection $g\to a_j$ $(j\in I)$. In particular, for $j=i$ we get

$$\gamma_i\pi_i=\varepsilon_{a_i}. \tag{1}$$

Let us denote the kernel of π_i by (l, λ). We prove $\operatorname{Ker}\pi_j=(l, \lambda)$ for all $j\geqq i$. Obviously $\lambda\pi_j=(\lambda\pi_i)\pi_j^i=\omega$ holds. If $\delta: b\to g$ is a map satisfying the condition $\delta\pi_j=\omega$, then also $\delta\pi_i=(\delta\pi_j)\pi_i^j=\omega$ holds, therefore there exists a map $\delta': b\to g$ such that $\delta'\lambda=\delta$. Hence $\operatorname{Ker}\pi_j=(l, \lambda)$.

Using this fact for the subsystem $\Omega'=[a_j, \pi_j^k]$ $(k, j\geqq i)$ we obtain that Ω' is a cofinal subsystem of Ω such that $\bigcap_{j\geqq i} \operatorname{Ker}\pi_j=(l, \lambda)$. Hence from Proposition 4. 5 it follows $(l, \lambda)=(0, \omega)$, and so the epimorphism π_i is also a monomorphism. Thus, according to (1), π_i and γ_i are equivalences. Hence we obtain

THEOREM 4. 7. *If the maps* π_j^i $(i\geqq j; i, j\in I)$ *of the inverse system* $\Omega=[a_i, \pi_j^i]$. *are equivalences or zero maps, then* $\varprojlim \Omega\sim a_i(\pi_i, \gamma_i)$ *holds, where* $a_i\neq 0$ *(if the system* Ω *possesses such an object).*

DEFINITION 4. 8. The inverse system $\Omega=[a_i, \pi_j^i]$ will be called a *direct product* of the inverse systems $\Omega_k=[a_{ik}, \varrho_{jk}^i]$ $(i, j\in I, k\in K)$, if for each $i\in I$ a_i is a direct product $a_i=\prod_{k\in K} a_{ik}(\vartheta_{ik})$ and $\vartheta_{ik}\varrho_{jk}^i=\pi_j^i\vartheta_{jk}$ is valid. The direct product of the inverse systems Ω_k $(k\in K)$ will be denoted by $\prod_{k\in K}\Omega_k$.

THEOREM 4. 9. *If the inverse system $\Omega=[a_i, \pi_j^i]$ is a direct product of the inverse system $\Omega_k=[a_{ik}, \varrho_{jk}^i]$ $(k\in K)$, and their inverse limits are denoted by g and g_k respectively, then the equivalence*

$$g\sim\prod_{k\in K} g_k(\sigma_k)$$

holds.

PROOF. If ϱ_{ik} $(i\in I)$ denotes the projection $g_k\to a_{ik}$ for all $k\in K$, then $\sigma_k\varrho_{ik}$ maps $\prod_{k\in K} g_k(\sigma_k)$ into a_{ik} for all $k\in K$ and $i\in I$. Hence there is a unique map

$$\lambda_i: \prod_{k\in K} g_k(\sigma_k)\to\prod_{k\in K} a_{ik}(\vartheta_{ik}) = a_i$$

such that $\lambda_i\vartheta_{ik}=\sigma_k\varrho_{ik}$ holds for all $k\in K$ and fixed $i\in I$. Moreover, using Definition 4.8 we have

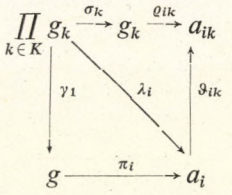
$$\lambda_i\pi_j^i\vartheta_{jk} = \lambda_i\vartheta_{ik}\varrho_{jk}^i = \sigma_k\varrho_{ik}\varrho_{jk}^i = \sigma_k\varrho_{jk} = \lambda_j\vartheta_{jk}$$

for all $k\in K$. But according to the definition of the direct product, λ_j is uniquely determined, so $\lambda_i\pi_j^i=\lambda_j$ is valid for all $i, j\in I$; $i\geqq j$. Hence by Definition 4. 2 there is a unique map $\gamma_1: \prod_{k\in K} g_k(\sigma_k)\to g$ such that $\gamma_1\pi_i=\lambda_i$ $(i\in I)$, where π_i $(i\in I)$ denotes the projection of g onto a_i.

This implies

$$\gamma_1\pi_i\vartheta_{ik}=\sigma_k\varrho_{ik}. \tag{2}$$

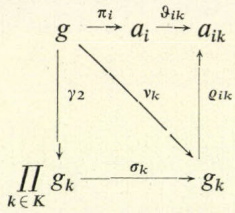
Thus we get the following commutative diagram



On the other hand, consider the epimorphisms $\pi_i\vartheta_{ik}: g\to a_{ik}$ $(i\in I, k\in K)$. Since by Definition 4.8 $\pi_i\vartheta_{ik}\varrho_{jk}^i=\pi_i\pi_j^i\vartheta_{jk}=\pi_j\vartheta_{jk}$ holds for all $i, j\in I$ $(i\geqq j)$, therefore because of Definition 4. 2 there exists a unique map $\nu_k: g\to g_k$ such that $\nu_k\varrho_{ik}=\pi_i\vartheta_{ik}$ holds for all $i\in I$. Hence there is also a unique map $\gamma_2: g\to\prod_{k\in K} g_k(\sigma_k)$ such that $\gamma_2\sigma_k=\nu_k$ holds for all $k\in K$. This implies

$$\gamma_2\sigma_k\varrho_{ik}=\pi_i\vartheta_{ik}. \tag{3}$$

Thus we have the following commutative diagram



By virtue of (2) and (3) we obtain

$$\gamma_2\gamma_1\pi_i\vartheta_{ik} = \pi_i\vartheta_{ik}$$

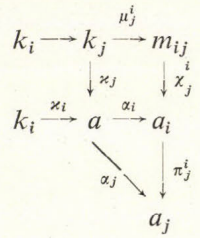
and

$$\gamma_1\gamma_2\sigma_k\varrho_{ik} = \sigma_k\varrho_{ik}.$$

Taking into account the uniqueness of γ_1 and γ_2 we get $\gamma_2\gamma_1 = \varepsilon_g$ and $\gamma_1\gamma_2 = \varepsilon_{g^*}$ where g^* denotes $\prod_{k\in K} g_k(\sigma_k)$. Thus the equivalence $g \sim g^*(\gamma_2, \gamma_1)$ is proved.

A system $\mathbf{F} = \{(k_i, \varkappa_i) \mid i \in I\}$ consisting of ideals of an object $a \in \mathscr{C}$ is called a *filter of a*, if for any $i, j \in I$ there is an index $l \in I$ such that $(k_l, \varkappa_l) \leqq (k_i, \varkappa_i) \cap (k_j, \varkappa_j)$. A filter $\mathbf{F}$ of an object $a \in \mathscr{C}$ induces an inverse system Ω as follows.

For each $(k_i, \varkappa_i) \in \mathbf{F}$ let us consider an epimorphism $\alpha_i: a \to a_i$ with $\operatorname{Ker} \alpha_i = (k_i, \varkappa_i)$. In view of Corollary 2. 2 for any ideals $(k_i, \varkappa_i) \leqq (k_j, \varkappa_j)$ of $\mathbf{F}$ there is an epimorphism π^i_j such that the diagram



is commutative, where $\operatorname{Im} \varkappa_j\alpha_i = (\mu^i_j, m_{ij}, \chi^i_j)$ and $\operatorname{Ker} \pi^i_j = (m_{ij}, \chi^i_j)$ are valid. Hence we have $\alpha_i\pi^i_j = \alpha_j$, moreover, for any $i \geqq j \geqq k$

$$\alpha_i\pi^i_j\pi^j_k = \alpha_j\pi^j_k = \alpha_k = \alpha_i\pi^i_k$$

holds. Since α_i is an epimorphism, so we get $\pi^i_j\pi^j_k = \pi^i_k$ $(\pi^i_i = \varepsilon_{a_i})$. Thus $\Omega = [a_i, \pi^i_j]$ is an inverse system.

By Definition 4. 2 there exists a uniquely determined map $\gamma: a \to \varprojlim \Omega$. This map will be called the *canonical map* of a belonging to the filter $\mathbf{F}$.

THEOREM 4. 10. *If* $\mathbf{F} = \{(k_i, \varkappa_i) \mid i \in I\}$ *is a filter of an object* $a \in \mathscr{C}$ *such that* $\bigcap_{i\in I} (k_i, \varkappa_i) = (0, \omega)$, *and* $\Omega = [a_i, \pi^i_j]$ *denotes the inverse system induced by* $\mathbf{F}$, *then the canonical map* γ *embeds* a *in* $g = \varprojlim \Omega$.

PROOF. For the canonical map $\gamma: a \to g$ we have $\gamma\pi_i = \alpha_i$ for all $i \in I$ where π_i denotes the projection $\pi_i: g \to a_i$ and $\operatorname{Ker} \alpha_i = (k_i, \varkappa_i) \in \mathbf{F}$. Let (l, λ) denote the

kernel of γ. Since $\lambda\alpha_i=(\lambda\gamma)\pi_i=\omega$, so there is a map λ' such that $\lambda'\varkappa_i=\lambda$. But λ is a monomorphism, so λ' is also a monomorphism. Hence $(l, \lambda)\leqq\bigcap_{i\in I}(k_i, \varkappa_i)=$ $=(0, \omega)$, therefore γ is a monomorphism.

To motivate the notion of filter, consider for instance, the category of all modules over a ring. Now we get the definition of filter given in LEPTIN [5]. If a filter $\mathbf{F}$ of a ring is chosen, then it is equipped with a linear topology. This topology is hausdorffian if and only if $\bigcap_{i\in I}(k_i, \varkappa_i)=(0, \omega)$ holds.

§ 5. $\mathcal{M}$-semi-simple objects

Let $\mathcal{M}$ be again a class of objects satisfying conditions (i) and (ii) of § 3. The structure $\mathcal{M}$-space M_a of an object $a\in\mathcal{C}$ induces a filter $\mathbf{F}_M=\{(k_i, \varkappa_i)\,|\,i\in I\}$ consisting of all finite intersections of $\mathcal{M}$-ideals. Let Ω_M denote the inverse system belonging to $\mathbf{F}_M$. As an immediate consequence of Theorem 4. 10 we get

THEOREM 5. 1. *If $a\in\mathcal{C}$ is an $\mathcal{M}$-semi-simple object, then the canonical map of a belonging to $\mathbf{F}_M$ embeds a in $\varprojlim\Omega_M$.*

If a maximal independent $\mathcal{M}$-system of an $\mathcal{M}$-semi-simple object possesses only finite many ideals, then by the definition of the independent $\mathcal{M}$-system we obtain:

Every $\mathcal{M}$-semi-simple object with a finite maximal independent $\mathcal{M}$-system is a direct product of $\mathcal{M}$-objects.

Let $\Delta=\{(d_r, \delta_r)\,|\,r\in P\}$ be a maximal independent $\mathcal{M}$-system of an object $a\in\mathcal{C}$, and let $\mathbf{F}_\Delta$ denote the filter consisting of all finite intersections of ideals from Δ. The inverse system induced by $\mathbf{F}_\Delta$ will be denoted by Ω_Δ.

PROPOSITION 5. 2. *If $\bigcap_{k=1}^{n}(d_{r_k}, \delta_{r_k})\leqq\bigcap_{l=1}(d_{p_l}, \delta_{p_l})$ is valid where (d_{r_1}, δ_{r_1}), ..., (d_{r_n}, δ_{r_n}) are different ideals of Δ and the same holds for (d_{p_1}, δ_{p_1}), ..., (d_{p_m}, δ_{p_m}), then for every p_l there is an r_k such that $(d_{p_l}, \delta_{p_l})=(d_{r_k}, \delta_{r_k})$ and $n\geqq m$ holds.*

PROOF. Since $\bigcap_{k=1}^{n}(d_{r_k}, \delta_{r_k})\leqq(d_{p_l}, \delta_{p_l})$ is valid for all p_l, so either $(d_{p_l}, \delta_{p_l})=$ $=(d_{r_k}, \delta_{r_k})$ holds for a suitable r_k or Δ is not independent. The statement $n\geqq m$ is trivial.

PROPOSITION 5. 3. *The inverse system Ω_Δ is a cofinal subsystem of Ω_M.*

PROOF. Any $(m, \mu)\in M_a$ belongs either to Δ or for some (d_1, δ_1), ..., $(d_n, \delta_n)\in\Delta$ the relation $\bigcap_{r=1}^{n}(d_r, \delta_r)\leqq(m, \mu)$ is valid. Hence for every finite intersection $\bigcap_{i=1}^{s}(m_i, \mu_i)$ with $(m_i, \mu_i)\in M_a$, there is a finite intersection $\bigcap_{r=1}^{t}(d_r, \delta_r)$ $((d_r, \delta_r)\in\Delta)$ such that $\bigcap_{r=1}^{t}(d_r, \delta_r)\leqq\bigcap_{i=1}^{s}(m_i, \mu_i)$ holds. Thus every ideal of $\mathbf{F}_\Delta$ is contained in $\mathbf{F}_M$, and so by the construction of Ω_Δ and Ω_M it follows that Ω_Δ is a cofinal subsystem of Ω_M.

Now we can prove

THEOREM 5. 4. *If Ω_M denotes the inverse system induced by the structure $\mathscr{M}$-space of an object $a \in \mathscr{C}$, then $\varprojlim \Omega_M$ is a direct product of $\mathscr{M}$-objects.*

PROOF. Let $\Delta = \{(d_k, \delta_k) \mid k \in K\}$ be a maximal independent $\mathscr{M}$-system of an object a. In view of Propositions 4. 4 and 5. 3 it is sufficient to prove that $\varprojlim \Omega_\Delta$ is a direct product of $\mathscr{M}$-objects. Since

$$\mathbf{F}_\Delta = \left\{ \bigcap_{r=1}^{n} (d_{k_r}, \delta_{k_r}) \mid (d_{k_r}, \delta_{k_r}) \in \Delta \right\},$$

therefore any object a_i of $\Omega_\Delta = [a_i, \pi_j^i]$ is the image of an epimorphism $\alpha_i : a \to a_i$ ($\operatorname{Ker} \alpha_i = \bigcap_{r=1}^{n} (d_{k_r}, \delta_{k_r})$, moreover $a_i \sim \prod_{r=1}^{n} b_{ik_r}(\vartheta_{ik_r})$ holds where $b_{ik_r} \in \mathscr{M}$. In the inverse system Ω_Δ, $i \geqq j$ means that

$$\bigcap_{r=1}^{n} (d_{ik_r}, \delta_{ik_r}) \leqq \bigcap_{r=1}^{m} (d_{jl_r}, \delta_{jl_r}),$$

therefore by Proposition 5. 2 we have $n \geqq m$ and $(d_{jl_r}, \delta_{jl_r}) = (d_{ik_r}, \delta_{ik_r})$ for a suitable k_r. Thus there is a map $\varrho_{jk}^i : b_{ik} \to b_{jk}$ which is either an equivalence or the zero map, such that $\vartheta_{ik} \varrho_{jk}^i = \pi_j^i \vartheta_{jk}$ holds for all $i, j \in I$, $k \in K$. Hence $\Omega_\Delta = \prod_{k \in K} \Omega_k (\Omega_k = [b_{ik}, \varrho_{jk}^i])$ is valid. Applying Theorems 4. 7 and 4. 9 we have

$$\varprojlim \Omega_\Delta = \varprojlim \prod_{k \in K} \Omega_k \sim \prod_{k \in K} \varprojlim \Omega_k \sim \prod_{k \in K} b_{ik}(\vartheta_k),$$

and every b_{ik} is an $\mathscr{M}$-object ($k \in K$).

An object $a \in \mathscr{C}$ is *subdirectly embedded* in the direct product $g = \prod_{k \in K} b_k(\pi_k)$, if there is a monomorphism $\gamma : a \to g$ such that all maps $\alpha_k = \gamma \pi_k$ ($k \in K$) are epimorphisms. According to SULIŃSKI [12], Theorem 2. 3, $a \in \mathscr{C}$ is subdirectly embedded in $g = \prod_{k \in K} b_k(\pi_k)$ if and only if there are epimorphisms $\alpha_k : a \to b_k$ ($k \in K$) such that $\bigcap_{k \in K} \operatorname{Ker} \alpha_k = (0, \omega)$ holds.

Taking into account Proposition 5. 1 and Theorem 5. 4 we obtain:

Every $\mathscr{M}$-semi-simple object can be subdirectly embedded in a direct product of $\mathscr{M}$-objects. This is just the statement of Theorem 4. 9 in SULIŃSKI [12].

DEFINITION 5. 5. Let Ω_M denote the inverse system induced by the structure $\mathscr{M}$-space of an object a. The object a is called to be *$\mathscr{M}$-compact* if the canonical map $\gamma : a \to \varprojlim \Omega_M$ is an epimorphism.

Let us remark that if $\mathscr{C}$ denotes the category of all modules over a ring R, in which the topology induced by a structure $\mathscr{M}$-space is hausdorffian, then by Definition 5. 5, we get 'in a narrow sense linearly compact' modules introduced by LEPTIN [5].

THEOREM 5. 6. *An object $a\in\mathscr{C}$ is an $\mathscr{M}$-compact $\mathscr{M}$-semi-simple object if and only if a is a direct product of $\mathscr{M}$-objects.*

PROOF. Suppose that $a\in\mathscr{C}$ is an $\mathscr{M}$-compact $\mathscr{M}$-semi-simple object. From Theorems 5. 1 and 5. 4 we obtain the statement.

Conversely, assume the equivalence $a\sim\prod_{k\in K}b_k(\vartheta_k)$ where b_k is an $\mathscr{M}$-object for every $k\in K$. Consider the direct product $c_k=\prod_{k\neq l\in K}b_l(\tau_l)$. Now there exists a unique map $\gamma_k\colon c_k\to a$ such that $\gamma_k\vartheta_l=\tau_l$ and $\gamma_k\vartheta_k=\omega$. If $(s,\sigma)=\operatorname{Ker}\gamma_k$ then $\sigma\tau_l=\sigma\gamma_k\vartheta_l=\omega$ holds. Thus there exists a map $\sigma'\colon s\to r_l$ such that $\sigma'\varrho_l=\sigma$, where $(r_l,\varrho_l)=\operatorname{Ker}\tau_l$. Since σ is a monomorphism, therefore σ' is also a monomorphism. This implies $(s,\sigma)\leqq(r_l,\varrho_l)$. But we have $\bigcap_{k\neq l\in K}(r_l,\varrho_l)=(0,\omega)$, and so γ_k is a monomorphism for all $k\in K$. Thus (c_k,γ_k) is a subobject of a. Moreover, we prove $(c_k,\gamma_k)\in M_a$. Obviously $\gamma_k\vartheta_k=\omega$ is valid. Let $\delta\colon h\to a$ be a map such that $\delta\vartheta_k=\omega$. Consider the maps $\delta\vartheta_l\colon h\to b_l$ $(l\neq k)$. By the definition of the direct product there is a unique map $\gamma'\colon h\to c_k$ such that $\gamma'\tau_l=\delta\vartheta_l$ holds. Hence we have $\gamma'\gamma_k\vartheta_l=\gamma'\tau_l=\delta\vartheta_l$, and so by the uniqueness of γ' and γ_k we get $\gamma'\gamma_k=\delta$. Thus we have proved $(c_k,\gamma_k)=\operatorname{Ker}\vartheta_k$, and therefore $(c_k,\gamma_k)\in M_a$ is valid.

If $(c,\gamma)=\bigcap_{k\in K}(c_k,\gamma_k)$, then there is a unique map φ such that the diagram

$$\begin{array}{ccc} c & \longrightarrow & c_k \\ {\scriptstyle\varphi}\downarrow & & \uparrow{\scriptstyle\omega} \\ a & \xrightarrow{\vartheta_k} & b_k \end{array}$$

is a commutative diagram for all $k\in K$. This implies $\varphi=\omega$, and so $\bigcap_{k\in K}(c_k,\gamma_k)=(0,\omega)$ is valid. This means that a is $\mathscr{M}$-semi-simple.

Since for any object $b\in\mathscr{M}$ we have $b\sim\varprojlim\Omega$ $(\Omega=[b,\varepsilon_b])$, therefore every $\mathscr{M}$-object is $\mathscr{M}$-compact. If $a\sim\prod_{k\in K}b_k(\vartheta_k)$ $(b_k\in\mathscr{M})$, then for $\Omega_k=[b_k,\varepsilon_{bk}]$ we have

$$a\sim\prod_{k\in K}\varprojlim\Omega_k\sim\varprojlim\prod_{k\in K}\Omega_k\sim\varprojlim\Omega.$$

Hence a is also $\mathscr{M}$-compact.

Theorem 5. 6 corresponds to LEPTIN's theorem [5] which is a generalization of the I. WEDDERBURN—ARTIN's *structure-theorem*. In this theorem LEPTIN has established: the linearly compact semi-simple rings are just the complete direct sums of (topologically) simple linearly compact rings.

Now the following question arises: *does there exist an $\mathscr{M}$-radical such that every $\mathscr{M}$-semi-simple object is a direct product of $\mathscr{M}$-objects*?

Concerning this question we yield some theorems.

THEOREM 5. 7. *Every $\mathscr{M}$-semi-simple object is a direct product of $\mathscr{M}$-objects if and only if every object of $\mathscr{C}$ is $\mathscr{M}$-compact.*

PROOF. If every object is $\mathscr{M}$-compact, then Theorem 5. 6 implies that every $\mathscr{M}$-semi-simple object is a direct product of $\mathscr{M}$-objects.

Suppose that the object $a \in \mathscr{C}$ is not $\mathscr{M}$-compact, and let $\alpha: a \to b$ be an epimorphism with $\operatorname{Ker} \alpha = R(a) = (k_M, \varkappa_M)$. By Theorem 3. 6 the object b is $\mathscr{M}$-semi-simple. Consider the commutative diagram

$$\begin{array}{ccccc} k_M & \longrightarrow & d_i & \xrightarrow{\mu_i} & m_i \\ & & \downarrow{\scriptstyle \delta_i} & & \downarrow{\scriptstyle \chi_i} \\ k_M & \xrightarrow{\varkappa_M} & a & \xrightarrow{\alpha} & b \\ & & & \searrow^{\gamma_i} & \downarrow{\scriptstyle \beta_i} \\ & & & & a_i \end{array}$$

where $(d_i, \delta_i) \in M_i$ and μ_i, γ_i, β_i are epimorphisms with $\operatorname{Ker} \gamma_i = (d_i, \delta_i)$ and $\operatorname{Ker} \beta_i = (m_i, \chi_i)$. Hence $(d_i, \delta_i) \leftrightarrow (m_i, \chi_i)$ induces a one-to-one correspondence between the ideals of $\mathbf{F}_{M_a}$ and $\mathbf{F}_{M_b}$. Thus the inverse systems induced by $\mathbf{F}_{M_a}$ and $\mathbf{F}_{M_b}$ are the same $\Omega = [a_i, \pi_j^i]$. Therefore by Definition 4. 2 for the canonical maps γ_1 and γ_2

$$\begin{array}{ccc} a & \xrightarrow{\alpha} & b \\ {\scriptstyle \gamma_1}\downarrow & {\scriptstyle \gamma_2}\swarrow & \downarrow{\scriptstyle \beta_i} \\ \varprojlim \Omega & \xrightarrow{\pi_i} & a_i \end{array}$$

is a commutative diagram. Since a is not $\mathscr{M}$-compact, so γ_1 is not an epimorphism. From $\gamma_1 = \alpha\gamma_2$ it follows that γ_2 is not an epimorphism, therefore b is not $\mathscr{M}$-compact, but $\mathscr{M}$-semi-simple.

In the following we need the notion of discrete direct product. Consider a direct product $a = \prod_{i \in I} a_i(\vartheta_i, \tau_i)$. Now (a_i, τ_i) is a subobject of a, moreover, we prove that (a_i, τ_i) is an ideal of a. Let us consider the direct product $b_i = \prod_{i \neq j \in I} a_j(\xi_j, \eta_j)$, and the projections $\vartheta_j: a \to a_j$ $(j \neq i)$. Hence there is a unique map $\varphi: a \to b_i$ such that $\varphi\xi_j = \vartheta_j$ holds for all $j \neq i$. We show that $\operatorname{Ker} \varphi = (a_i, \tau_i)$.

First of all we note

$$(\tau_i\varphi)\xi_j = \tau_i\vartheta_j = \omega. \tag{4}$$

Hence for the system $\omega_j: a_i \to a_j$ $(i \neq j)$, the map $\omega: a_i \to b_i$ is the unique map such that

$$\omega\xi_j = \omega. \tag{5}$$

(4) and (5) imply $\tau_i\varphi = \omega$.

If $\delta: c \to a$ is a map with $\delta\varphi = \omega$, then $\omega = \delta\varphi\xi_j = \delta\vartheta_j$ $(j \neq i)$. Thus δ is the unique map satisfying $\delta\vartheta_j = \omega$ and $\delta\vartheta_i = \delta'$. Now we have $(\delta'\tau_i)\vartheta_i = \delta\vartheta_i\tau_i\vartheta_i = \delta\vartheta_i$, and $\omega = (\delta'\tau_i)\vartheta_j = \delta\vartheta_j$, so there exists a unique map δ_0 such that $\delta_0\vartheta_i = \delta'\tau_i\vartheta_i$ and $\delta_0\vartheta_j = \delta'\tau_i\vartheta_j = \omega$. Hence $\delta = \delta_0 = \delta'\tau_i$, from which $\operatorname{Ker} \varphi = (a_i, \tau_i)$ follows.

DEFINITION 5. 8. The ideal $(c, \gamma) = \bigcup_{i \in I} (a_i, \tau_i)$ of $a = \prod_{i \in I} a_i(\vartheta_i, \tau_i)$ will be called a *discrete direct* product of the object a_i $(i \in I)$. (Cf. [4].)

Let $a \in \mathscr{C}$ be an object with a maximal independent $\mathscr{M}$-system $\{(d_i, \delta_i) \mid i \in I\}$, and consider the epimorphism $\alpha_i: a \to a_i$ with $\operatorname{Ker} \alpha_i = (d_i, \delta_i)$ for all $i \in I$.

PROPOSITION 5. 9. *If $\mathcal{M}$ consists of simple objects, then the discrete direct product* $(c, \gamma) = \bigcup_{i \in I} (a_i, \tau_i)$ *is an ideal of the $\mathcal{M}$-semi-simple object a.*

PROOF. Consider the ideal $(k_i, \varkappa_i) = \bigcap_{i \neq j \in I} (d_j, \delta_j)$. Since $\mathcal{M}$ consists of simple objects, so we have $(a, \varepsilon_a) = (k_i, \varkappa_i) \cup (d_i, \delta_i)$ and $(k_i, \varkappa_i) \cap (d_i, \delta_i) = R(a) = (0, \omega)$. Thus according to Theorem 1. 4 a is a direct product $b_1 \times b_2$ (β_1, β_2) of objects $b_1 \sim k_i(\xi, \xi^{-1})$ and $b_2 \sim d_i(\eta, \eta^{-1})$ such that $\beta_1 : a \to b_1$ and $\operatorname{Ker} \beta_1 = (d_i, \delta_i)$ hold. Hence we have $a_i \sim b_1(\zeta, \zeta^{-1})$, and so $(a_i, \zeta \xi \varkappa_i)$ is an ideal of a for all $i \in I$. Thus the discrete direct product $\bigcup_{i \in I} (a_i, \tau_i)$ can be embedded as an ideal in a.

As a consequence of Proposition 5. 9 we obtain

THEOREM 5. 10. *Let $\mathcal{M}$ be a class of simple objects. The $\mathcal{M}$-semi-simple objects are direct products of $\mathcal{M}$-objects if and only if every direct product* $\prod_{i \in I} a_i(\vartheta_i, \tau_i)$ *is equivalent to the discrete direct product* $\bigcup_{i \in I} (a_i, \tau_i)$ *for all sets of indices I and $a_i \in \mathcal{M}$ $(i \in I)$.*

PROOF. If the discrete direct products and the direct products of $\mathcal{M}$-objects are equivalent the because of Theorems 5. 1, 5. 4 and Proposition 5. 9, every $\mathcal{M}$-semi-simple object is a direct product of $\mathcal{M}$-objects.

If $a = \bigcup_{i \in I} (a_i, \tau_i) \nsim \prod_{i \in I} a_i(\vartheta_i, \tau_i)$, then a is obviously $\mathcal{M}$-semi-simple and it is easy to see that the system $\{ \bigcup_{i \neq j \in I} (a_j, \tau_j) \mid j \in I \}$ is an independent $\mathcal{M}$-system. Let Δ_1 denote a maximal independent $\mathcal{M}$-system containing this system. Assume that a is a direct product of $\mathcal{M}$-objects b_k $(k \in K)$: $a = \prod_{k \in K} b_k (\beta_k, \gamma_k)$. Since the inverse systems Ω_{Δ_1} and Ω_{Δ_2} induced by Δ_1 and $\Delta_2 = \{ \prod_{k \neq l \in K} b_l(\beta_l', \gamma_l') \mid l \in K \}$ respectively, are cofinal to Ω_{M_a}, therefore we get

$$a = \varprojlim \Omega_{\Delta_2} \sim \varprojlim \Omega_{M_a} \sim \varprojlim \Omega_{\Delta_1} = \prod_{i \in I} a_i(\vartheta_i, \tau_i)$$

which is a contradiction.

Theorem 5. 10 gives a negative answer to our question, for in general the discrete direct product differs from the direct one. For instance, if $\mathcal{C}$ is the category of all rings, then Theorem 5. 10 asserts that

there does not exist such a class $\mathcal{M}$ (not consisting only from the ring of one element) *of simple rings that every $\mathcal{M}$-semi-simple ring would be a (complete) direct sum of simple rings belonging to the class $\mathcal{M}$.*

We remark that if we restrict ourselves only to objects with finite maximal independent $\mathcal{M}$-systems and we suppose only the existence of direct product of finite myan factors instead of axiom (c_8), then, as it was mentioned before, every $\mathcal{M}$-semi-simple object is a direct product of $\mathcal{M}$-objects.

On the other hand, we mention that

there exist categories in which the concepts of discrete direct product and direct product coincide.

For instance, let us consider the category $\mathscr{C}_R$ in which the objects are the linearly compact unitary modules over a ring R and the maps are the continuous R-homomorphisms. In $\mathscr{C}_R$ the discrete direct sum coincides with the direct sum, for this latter is the closest linearly compact R-module containing its direct components. Thus, by Theorem 5. 10, every linearly compact unitary R-module, in which the intersection of all maximal closed submodules is 0, is a direct sum of irreducible R-modules. This fact is well-known.

It is remarkable that if in the category of all semigroups with zero element the direct product of objects $a_i\,(i \in I)$ is defined as the set theoretical union $\bigcup_{i \in I} a_i \left(a_i \cap \left(\bigcup_{i \neq j \in I} a_j\right) = \varnothing\right)$, then there is no difference between discrete direct and direct product. However, this definition of the direct product is not adaquate to our one, decomposition-theorems can be established according to this definition (cf. [8], [9]).

Applying the results of §§ 3 and 5, one can obtain structure theorems for several kinds of algebraic structures.

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Bibliography

[1] S. AMITSUR, A general theory of radicals I, II, III, *Amer. J. Math.*, **74** (1952), pp. 774—786; **76** (1954), pp. 100—125; **76** (1954), pp. 126—136.
[2] H. J. HOEHNKE, Radikale in allgemeinen Algebra, *Math. Nachr.*, **32** (1966), pp. 347—383.
[3] A. G. KUROŠ, Radicals of rings and algebras (Russian), *Mat. Sbornik*, **33** (1953), pp. 13—26.
[4] A. G. KUROSCH—A. CH. LIWSCHITZ—E. G. SCHULGEIFER—M. S. ZALENKO, *Zur Theorie der Kategorien* (Berlin, 1963).
[5] H. LEPTIN, Linear kompakte Moduln und Ringe, I, II, *Math. Zeitschr.*, **62** (1955), pp. 241—267; **66** (1957), pp. 289—327.
[6] J. M. MARANDA, Injective structures, *Trans. Amer. Math. Soc.*, (1964), pp. 98—135.
[7] M. NAGATA, On the theory of radicals in a ring, *J. Math. Soc. Japan*, **3** (1951), pp. 330—341.
[8] O. STEINFELD, On semigroups which are unions of completely 0-simple semigroups, *Czechosl. Math. J.*, **16** (1966), pp. 63—69.
[9] O. STEINFELD—R. WIEGANDT, Über die Verallgemeinerungen und Analoga der Wedderburn-Artinschen und Noetherschen Struktursätze, *Math. Nachr.*, **34** (1967), pp. 143—156.
[10] E. ŠULGEIFER, General theory of radicals in categories (Russian), *Math. Sbornik*, **51** (1960), pp. 93—100.
[11] E. ŠULGEIFER, On the lattice of ideals of an object a category (Russian), *Math. Sbornik*, **54** (1961), pp. 209—224.
[12] A. SULIŃSKI, The Brown—McCoy radical in categories, *Fund. Math.*, **59** (1966), pp. 23—41.

Acta Mathematica Academiae Scientiarum Hungaricae.
Tomus 19 (3—4), (1968), pp. 365—403.

ÜBER DIE MITTELWERTE MULTIPLIKATIVER ZAHLENTHEORETISCHER FUNKTIONEN

Von

G. HALÁSZ (Budapest)

(*Vorgelegt von* P. TURÁN)

Eine (komplexwertige) zahlentheoretische Funktion $f(n)$ heißt bekanntlich multiplikativ, wenn $f(nm)=f(n)f(m)$ wenigstens für teilerfremde n und m. Die Untersuchung der Mittelverte bedeutet hier, asymptotische Formel für die Summe

$$M(x) = \sum_{n \leqq x} f(n)$$

zu suchen.

In seinem Artikel [1] hat E. WIRSING viele interessante und tiefe Sätze in dieser Hinsicht bewiesen. Seine Beweise sind „elementar", jedoch sehr kompliziert. In der vorliegenden Arbeit entwickeln wir eine einfachere und wirksamere funktionentheoretische Methode, die mehrere gemeinsame Züge mit der Wirsingschen hat. Hier aber wollen wir sie zu neuen Beweisen seiner Sätze oder zu den sich im Zusammenhang mit ihnen erhebenden zahlreichen Problemen nicht verwenden, sondern neue Sätze beweisen, welche eben durch diese funktionentheoretische Methode ermöglicht werden.

Eine analytische Annäherung beruht natürlich auf der Dirichletschen Reihe

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \qquad (s = \sigma + it).$$

Man kann nämlich die Multiplikativität durch die Produktendarstellung

$$F(s) = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \ldots\right) \qquad (p \text{ prim})$$

ausnützen. Dieses Produkt wird im Folgenden stets in der Halbebene $\sigma = \mathrm{Re}\, s > 1$ absolut konvergieren. Die Reihenentwicklung von $F(s)$ ist also für $\sigma > 1$ ebenfalls absolut konvergent und stellt eine dort reguläre Funktion dar. Auch eine weitere Darstellung wird gebraucht:

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \int_1^{\infty} \frac{dM(x)}{x^s} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}}\, dx \qquad (\sigma > 1), \tag{1}$$

wo die partielle Integration auf dem unendlichen Intervall wegen $\frac{M(x)}{x^s} \to 0$ (eine Folge der Konvergenz) erlaubt ist. Nach diesen Vorbereitungen sind wir in der Lage, unsere Sätze zu formulieren und diskutieren.

SATZ 1. *$f(n)$ sei multiplikativ und dem absoluten Betrage nach nicht größer als* 1. *Die dadurch erzeugte analytische Funktion $F(s)$ erfülle die asymptotische Gleichung*

$$F(s) = \frac{C}{s-1} + o\left(\frac{1}{\sigma-1}\right) \qquad (s = \sigma + it, \quad \sigma \to 1+0)$$

mit einer komplexen Konstante C, gleichmäßig auf jeder endlichen Strecke $|t| \leqq K$. Dann gilt

$$M(x) = Cx + o(x) \qquad (x \to \infty).$$

Die Bedingung für $F(s)$ ist notwendig. Man braucht nur $M(x) = Cx + o(x)$ in seine Integraldarstellung (1) einzusetzen:

$$F(s) = s\int_1^\infty \frac{M(x)}{x^{s+1}}\,dx = s\int_1^\infty \frac{Cx}{x^{s+1}}\,dx + s\int_1^\infty \frac{o(x)}{x^{s+1}}\,dx = \frac{Cs}{s-1} + o\left(\frac{|s|}{\sigma-1}\right) = \tag{2}$$

$$= \frac{C}{s-1} + C + o\left(\frac{|s|}{\sigma-1}\right) = \frac{C}{s-1} + o\left(\frac{|s|}{\sigma-1}\right).$$

Interessant ist es also, daß diese trivial notwendige Bedingung nach Satz 1 schon hinreichend ist, wenn $f(n)$ multiplikativ ist und $|f(n)| \leqq 1$. Bemerken wir, daß die Multiplikativität nicht einfach weggelassen werden kann. Fordert man nur $|f(n)| \leqq 1$, so muß man von der Fehlerabschätzung von $F(s)$ viel mehr annehmen, als $o\left(\frac{1}{\sigma-1}\right)$, um auf eine Asymptotik für $M(x)$ schließen zu können.

Mit Rücksicht auf eine Anwendung beweisen wir zugleich den etwas allgemeineren

SATZ 1′. *Wie früher, sei $f(n)$ multiplikativ und $|f(n)| \leqq 1$, dagegen besitze $F(s)$ eine Asymptotik der Form*

$$F(s) = \frac{CL\left(\frac{1}{\sigma-1}\right)}{s-(1+ia)} + o\left(\frac{1}{\sigma-1}\right).$$

gleichmäßig auf jeder endlichen Strecke $|t| \leqq K$. Hier sind a reell, die Funktion $L(u)$ vom Betrage 1 *langsam oszillierend:*

$$|L(u)| = 1, \quad \frac{L(u_1)}{L(u)} \to 1 \quad \textit{gleichmäßig für} \quad u \to \infty, \quad u \leqq u_1 \leqq 2u, \tag{3}$$

während die anderen Größen dieselbe Bedeutung haben, wie im Satz 1. *In diesem Fall ist*

$$M(x) = \frac{CL(\log x)}{1+ia}\, x^{1+ia} + o(x).$$

Die erwähnte Anwendung ist folgende Vermutung von E. WIRSING [1]:

SATZ 2. *Ist $f(n)$ eine beliebige multiplikative Funktion mit $|f(n)| \leqq 1$, so existieren ein komplexes C_0, ein reelles a_0, und eine Funktion $L_0(u)$ mit den obigen Eigenschaften* (3), *so daß:*

$$M(x) = C_0 L_0(\log x) x^{1+ia_0} + o(x).$$

Diese Größen C_0, a_0 und $L_0(u)$ werden im Beweise durch eine einfache Formel gegeben werden. Aus dieser Formel ist es leicht zu ersehen, daß im Falle, wenn $f(n)$ auch reell ist, $a_0 = 0$, $L_0(u) \equiv 1$ sind. Man erhält insbesondere einen neuen Beweis zur alten Vermutung von P. ERDŐS (zuerst von E. WIRSING [1] bewiesen), wonach $\lim \frac{M(x)}{x}$ stets existiert, falls $f(n)$ nur die Werte ± 1 annimmt.

Die Sätze 1, 1′ und 2 sind O-Taubersätze. Die Abschätzung $M(x) = O(x)$ folgt aus $f(n) = O(1)$, wir müßen nur dies bis auf $o(x)$ bestimmen. Im folgenden Satz ist nicht einmal die O-Relation selbstverständlich.

SATZ 3. *Voraussetzungen:* a. *$f(n)$ ist multiplikativ,*

b. $f(p) = O(1)$ *oder nur* $\sum_p \frac{|f(p)|^{1+\delta}}{p^\sigma} \log p = O\left(\frac{1}{\sigma - 1}\right)$ *mit einem* $\delta > 0$,

c. $\sum_{\substack{p,k \\ k \geqq 2}} \frac{|f(p^k)|}{p^k} < +\infty$,

d. *keiner der Faktoren von $F(s)$*

$$1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots$$

verschwindet auf der Gerade $\operatorname{Re} s = 1$,

e. $$F(s) = \frac{C}{s-1} + o\left(\frac{|s|}{\sigma - 1}\right)$$

gleichmäßig für $\operatorname{Re} s = \sigma$ *(die notwendige Bedingung, s.* (2)*).*
Behauptung:

$$M(x) = Cx + o(x).$$

Die wesentliche Verallgemeinerung gegenüber Satz 1 liegt im Punkte b. Allein $f(p)$ muß (im Durchschnitt) beschränkt sein, und seine Schranke kann auch größer als 1 sein. Diese Tatsache, obwohl es sich nur um eine andere Konstante handelt, macht einen großen Unterschied, da für Zahlen n, welche aus vielen Primzahlen zusammengesetzt sind, $f(n)$ sehr groß sein kann. Einfach würde nur $M(x) = O(x \log^A x)$ folgen. Weiter kann man aber nicht gehen, in keiner Richtung der zwei Formen von b, wie es das folgende Beispiel zeigt.

SATZ 4. *Es sei eine wachsende, zum Unendlichen strebende positive Funktion* $\omega(x)$ *gegeben. Dann gibt es eine positive multiplikative Funktion* $f_0(n)$, *für welche statt* b:

$$0 \leqq f_0(p) \leqq \omega(p), \qquad \sum_p \frac{f_0(p)}{p^\sigma} \log p = O\left(\frac{1}{\sigma - 1}\right),$$

die aber c, d, *und* e *des vorigen Satzes genügt, dem letzten in der Form*

$$|F_0(s)| \leqq F_0(\sigma) = o\left(\frac{1}{\sigma - 1}\right),$$

doch:

$$M_0(x) = \sum_{n \leqq x} f_0(n) \neq O(x).$$

Die Konstruktion is eine Weiterentwicklung derjenigen von E. WIRSING [1].

In den Beweisen werden wir nach Vollständigkeit streben. Wenn wir trotzdem etwas (meistens Integralformeln und Eigenschaften der ζ-Funktion) als bekannt voraussetzen, so kann man dem in „Handbuch der Lehre von der Verteilung der Primzahlen" von Landau nachsehen.

$c_1, c_2, \ldots$ werden positive Konstanten bezeichnen, die von den Funktionen $f(n), F(s), \ldots$ abhängen dürfen, nicht aber von den Veränderlichen $n, x, s, \ldots$. Die Bezeichnung $O(\ )$ wird eine ähnliche Bedeutung haben.

Zur Vorbereitung wollen wir bemerken, daß Primzahlpotenzen von mindestens zweitem Grade hier keine große Rolle spielen und im Wesentlichen genügt es, uns mit streng multiplikativen Funktionen (d.h. wenn stets $f^*(nm) = f^*(n) f^*(m)$ oder was dasselbe besagt, $f^*(p^k) = [f^*(p)]^k$) zu beschäftigen, die bei den ersten wenigen Primzahlen verschwinden. Um dies einzusehen, verfertigen wir aus unserem $f(n)$ eine solche streng multiplikative Funktion $f^*(n)$ durch

$f^*(p^k) = 0$ für $p \leqq P$, $k = 1, 2, \ldots$

$f^*(p^k) = f^k(p)$ für $p > P$, $k = 1, 2, \ldots$ (P ist fest und später zu bestimmen).

Entsprechend finden wir

$$F^*(s) = \sum_{n=1}^{\infty} \frac{f^*(n)}{n^s} = \prod_p \left(1 + \frac{f^*(p)}{p^s} + \frac{f^*(p^2)}{p^{2s}} + \ldots\right) =$$

$$= \prod_{p > P} \left(1 + \frac{f(p)}{p^s} + \frac{f^2(p)}{p^{2s}} + \ldots\right) = \prod_{p > P} \frac{1}{1 - \dfrac{f(p)}{p^s}}.$$

Das Produkt, also auch die Reihe konvergiert für $\sigma > 1$ absolut, wenn z.B. $\dfrac{|f(p)|}{p} \leqq \dfrac{1}{2}$ $(p > P)$ und $\sum_p \dfrac{|f(p)|}{p^\sigma} < +\infty$ $(\sigma > 1)$. Im Falle $|f(n)| \leqq 1$ sind diese ohne weiteres klar, während sie im Satz 3 aus der Konvergenz von $\sum_p \dfrac{|f(p)|^{1+\delta}}{p^\sigma} \log p$

für $\sigma>1$ folgen (was in der Bedingung b stillschweigend angenommen ist); das erste schon mit $\sigma=1+\frac{\delta}{2}$ wegen

$$\frac{|f(p)|^{1+\delta}}{p^{1+\delta/2}} \leqq \sum_p \frac{|f(p)|^{1+\delta}}{p^{1+\delta/2}} \log p = c_1, \quad \left(\frac{|f(p)|}{p}\right)^{1+\delta} \leqq \frac{c_1}{p^{\delta/2}},$$

$$\frac{|f(p)|}{p} \leqq \frac{c_2}{p^{\delta/2(1+\delta)}} \overset{\text{def}}{=} \frac{c_2}{p^{\delta'}} \tag{4}$$

für genügend großes P, das zweite mittels der Hölderschen Ungleichung $\left(p=1+\delta, \frac{1}{p}+\frac{1}{q}=1, q=\frac{1+\delta}{\delta}\right)$

$$\sum_p \frac{|f(p)|}{p^{\sigma/p}} \cdot \frac{1}{p^{\sigma/q}} \leqq \left(\sum_p \frac{|f(p)|^{1+\delta}}{p^{\sigma}}\right)^{\frac{1}{1+\delta}} \left(\sum_p' \frac{1}{p^{\sigma}}\right)^{\frac{\delta}{1+\delta}} < +\infty, \tag{5}$$

wo p im Exponent mit der Summationsvariable p nicht zu verwechseln ist.

Vergleichen wir jetzt die ursprüngliche Funktion mit der neuen:

$$H(s) \overset{\text{def}}{=} \frac{F(s)}{F^*(s)} = \prod_p \left(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\dots\right) \prod_{p>P}\left(1-\frac{f(p)}{p^s}\right) =$$

$$= \prod_{p \leqq P}\left(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\dots\right) \prod_{p>P}\left(1+\frac{f(p^2)-f(p)f(p)}{p^{2s}}+\frac{f(p^3)-f(p)f(p^2)}{p^{3s}}+\dots\right). \tag{6}$$

Hier konvergieren das Produkt und deshalb die davon durch Ausmultiplizieren entstehende Reihe sogar für $\sigma=1$ absolut.* Wieder müßen wir dies nur für Satz 3 verifizieren. Jeder einzelne Faktor des ersten Teiles $p \leqq P$ tut es unter der Voraussetzung c über die höheren Primzahlpotenzen. Was das zweite Produkt $\prod_{p>P}$ betrifft, haben dieselbe Voraussetzung und die bereits bekannten Abschätzungen (4) und (5) zur Folge, daß die zwei Teile der abzuschätzenden Summe

$$\sum_{p>P}\left(\frac{|f(p^2)|}{p^2}+\frac{|f(p^3)|}{p^3}+\dots\right) \equiv \sum_{\substack{p>P \\ k \geqq 2}} \frac{|f(p^k)|}{p^k}$$

und

$$\sum_{p>P} \frac{|f(p)|}{p}\left(\frac{|f(p)|}{p}+\frac{|f(p^2)|}{p^2}+\dots\right) \leqq c_2 \sum_p \frac{1}{p^{\delta'}}\left(\frac{|f(p)|}{p}+\frac{|f(p^2)|}{p^2}+\dots\right) \leqq$$

$$\leqq c_2\left(\sum_p \frac{|f(p)|}{p^{1+\delta'}}+\sum_{\substack{p,k \\ k \geqq 2}} \frac{|f(p^k)|}{p^k}\right)$$

endlich sind.

* Hieraus wird insbesondere folgen, was wir bereits beim Satz 3 hätten erwähnen müssen, daß auch die Reihen- und Produktendarstellung von $F(s)=H(s)F^*(s)$ in der Halbebene $\sigma>1$ absolut konvergent sind.

Dann aber besteht folgendes elementares Lemma:
Ist für drei Dirichletsche Reihen $F(s)$, $F^*(s)$, $H(s)$

$$F(s)=H(s)F^*(s),$$

wobei die entsprechenden Koeffizienten

$$\sum_{n=1}^{\infty}\frac{|h(n)|}{n}<+\infty, \qquad M^*(x)=\sum_{n\leqq x}f^*(n)=Cx+o(x)$$

genügen, so ist

$$M(x)=\sum_{n\leqq x}f(n)=C\sum_{n=1}^{\infty}\frac{h(n)}{n}x+o(x)=CH(1)x+o(x).$$

Der Vollständigkeit halber geben wir den Beweis,
Der Zusammenhang zwischen den drei Funktionen führt zur Identität

$$M(x)=\sum_{n\leqq x}f(n)=\sum_{m\leqq x}h(m)\sum_{l\leqq\frac{x}{m}}f^*(l)=\sum_{m\leqq x}h(m)M^*\left(\frac{x}{m}\right).$$

Für eine vorläufig festes z zerlegen wir die Summe in zwei Teile je nachdem $m\leqq z$ oder $m>z$. Im ersten Teil benützen wir die Asymptotik von $M^*(x)$:

$$\sum_{m\leqq z}h(m)M^*\left(\frac{x}{m}\right)=Cx\sum_{m\leqq z}\frac{h(m)}{m}+o(x)\sum_{m\leqq z}\frac{|h(m)|}{m}=$$

$$=Cx\sum_{m=1}^{\infty}\frac{h(m)}{m}+O(x)\sum_{m>z}\frac{|h(m)|}{m}+o(x),$$

im zweiten Teil nur die entsprechende O-Relation:

$$\sum_{m>z}h(m)M^*\left(\frac{x}{m}\right)=O(x)\sum_{m>z}\frac{|h(m)|}{m},$$

so daß

$$|M(x)-CH(1)x|\leqq c_3x\sum_{m>z}\frac{|h(m)|}{m}+o(x),$$

$$\overline{\lim_{x\to\infty}}\frac{|M(x)-CH(1)x|}{x}\leqq c_3\sum_{m>z}\frac{|h(m)|}{m}.$$

Dies besteht für jedes feste z und $z\to\infty$ gibt das erwünschte Resultat.

Das reicht aus, sofern es sich um die Sätze 1 und 3 handelt. Im Falle der Sätze 1′ und 2 müßen wir von

(7) $$M^*(x)=CL(\log x)x^{1+ia}+o(x)$$

ausgehen. Die Abschätzung des zweiten Teiles $m>z$ verläuft ohne Änderung, nachdem auch hier $M^*(x)=O(x)$ ist. Im ersten Teil ergeben sich dieselben Restglieder und da für eine langsam oszillierende Funktion vom Betrage 1

$$L\left(\log\frac{x}{m}\right)=L(\log x)+o(1) \qquad (m=1,2,\ldots,[z])$$

ist, wird das Hauptglied

$$CL(\log x)x^{1+ia}\sum_{m=1}^{\infty}\frac{h(m)}{m^{1+ia}} = CH(1+ia)L(\log x)x^{1+ia}$$

und das Endresultat:

$$M(x)=C\,H(1+ia)\,L(\log x)x^{1+ia}+o(x). \tag{8}$$

Man kann also von $F^*(s)$, $M^*(x)$ des streng multiplikativen Falles zu den allgemeinen $F(s)$, $M(x)$ übergehen und diese Tatsache wird ermöglichen, daß wir uns eigentlich auf $F^*(s)$ beschränken können. Dieses letzte werden wir oft in der Form

$$F^*(s)=\prod_{p>P}\frac{1}{1-\frac{f(p)}{p^s}}=e^{\sum_{p>P}\log\frac{1}{1-\frac{f(p)}{p^s}}}=e^{\sum_{p>P}\left(\frac{f(p)}{p^s}+\frac{f^2(p)}{2p^{2s}}+\ldots\right)}=e^{\sum_{n=1}^{\infty}\frac{\lambda(n)f^*(n)}{n^s}} \tag{9}$$

schreiben, wo hier

$$\lambda(n)=\begin{cases}\dfrac{1}{k}, & \text{wenn} \quad n=p^k\cdot\\ 0 & \text{sonst.}\end{cases}$$

Wenn es kürzer sein wird, wird auch das Mangoldtsche Symbol

$$\Lambda(n)=\lambda(n)\log n=\begin{cases}\log p, & \text{wenn} \quad n=p^k\\ 0 & \text{sonst}\end{cases}$$

benützt.

Einiges Elementares folgt noch zur Vorbereitung des Satzes 3.

Die Höldersche Ungleichung $\left(\delta_1<\delta,\ p=\dfrac{1+\delta}{1+\delta_1},\ \dfrac{1}{p}+\dfrac{1}{q}=1,\ q=\dfrac{1+\delta}{\delta-\delta_1}\right)$

$$\sum_p\frac{|f(p)|^{1+\delta_1}\log p}{p^\sigma}=\sum_p\frac{|f(p)|^{1+\delta_1}\log^{1/p}p}{p^{\sigma/p}}\cdot\frac{\log^{1/q}p}{p^{\sigma/q}}\leqq$$

$$\leqq\left(\sum_p\frac{|f(p)|^{1+\delta}\log p}{p^\sigma}\right)^{1/p}\left(\sum_p\frac{\log p}{p^\sigma}\right)^{1/q}=O(1)\left(\frac{1}{\sigma-1}\right)^{1/p}\left(\sum_p\frac{\log p}{p^\sigma}\right)^{1/q}$$

zusammen mit der bekannten Abschätzung

$$\sum_p\frac{\log p}{p^\sigma}=O\left(\frac{1}{\sigma-1}\right)$$

zeigt uns, daß b auch für das kleinere δ_1 zutrifft. Wir werden b in der Form

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)|f^*(n)|^{1+\delta_1}}{n^\sigma}=O\left(\frac{1}{\sigma-1}\right)$$

anwenden. Dazu haben wir Recht, wenn wir (4) in Betracht nehmen.

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)|f^*(n)|^{1+\delta_1}}{n^\sigma}=\sum_{p>P}\log p\left(\frac{|f(p)|^{1+\delta_1}}{p^\sigma}+\frac{|f(p)|^{2(1+\delta_1)}}{p^{2\sigma}}+\ldots\right)$$

und wegen $|f(p)| \leqq c_2 p^{1-\delta'}$ hat die geometrische Reihe in der inneren Summe den Quotient (sogar für $\sigma = 1$)

$$\frac{|f(p)|^{1+\delta_1}}{p^\sigma} \leqq \frac{c_2^{1+\delta_1} p^{(1-\delta')(1+\delta_1)}}{p} \leqq \frac{1}{2} \qquad (p > P)$$

für genügend kleines δ_1 und genügend großes P. Die volle Summe kann daher

$$2 \sum_{p>P} \frac{|f(p)|^{1+\delta_1} \log p}{p^\sigma}$$

nicht übersteigen, worüber soeben gezeigt wurde, daß es $O\left(\frac{1}{\sigma-1}\right)$ ist. Statt δ_1 die Bezeichnung δ beibehaltend, ersetzen wir also die Bedingung b durch

$$\text{(10)} \qquad \text{b}'. \qquad \sum_{n=1}^{\infty} \frac{\Lambda(n)|f^*(n)|^{1+\delta}}{n^\sigma} = O\left(\frac{1}{\sigma-1}\right) \quad \text{mit einem} \quad 0 < \delta \leqq 1.$$

(Offenbar gilt die obige Überlegung auch mit $\delta_1 = 0$ und manchmal werden wir b′ mit $\delta = 0$ benützen.)

Die zweite Bemerkung geht der Funktion $H(s)$ an. Die Produktenentwicklung (6) von $H(s)$ ist, wie wir wissen, in der abgeschloßenen Halbebene $\sigma \geqq 1$ absolut konvergent. Wenn P genügend groß ist, so ist also das zweite Produkt über $p > P$ ebendort auch von 0 abgegrenzt. Auf der anderen Seite, nach Annahme d verschwinden keine der Faktoren des ersten Teiles $p \leqq P$ auf der Grenzlinie $\sigma = 1$. Da sie periodisch mit der Periode $\frac{2\pi i}{\log p}$ sind, können auch sie durch eine positive Zahl nach unten abgeschätzt werden, mindestens für zu 1 genügend nahes σ. Vergessen wir also nicht, daß im Satz 3

$$\text{(11)} \qquad \frac{1}{H(s)} = O(1) \quad \text{und stetig ist, gleichmäßig für} \quad 1 \leqq \sigma \leqq \sigma^*.$$

Beweis zum Satz 1′ für $F^*(s)$. Hier jedoch kann $H(1+it)$ verschwinden und es ist nicht ohne weiteres klar, daß auch $F^*(s)$ dieselbe asymptotische Eigenschaft besitzt, wie $F(s)$. Nehmen wir aber dessen ungeachtet an, daß dies jetzt der Fall ist.

Wir werden uns mit der Ableitung

$$F^{*\prime}(s) = -\sum_{n=1}^{\infty} \frac{f^*(n) \log n}{n^s}$$

und mit ihrer summatorischen Funktion

$$N^*(x) = \sum_{n \leqq x} f^*(n) \log n$$

beschäftigen. Wir müssen

$$N^*(x) = \frac{CL(\log x)}{1+ia} x^{1+ia} \log x + o(x \log x)$$

nachweisen, die Behauptung des Satzes wird sich dann aus der folgenden partiellen Integration ergeben:

$$M^*(x) = f^*(1) + \int_2^x \frac{dN^*(t)}{\log t} = 1 + \left[\frac{N^*(t)}{\log t}\right]_{2-0}^x + \int_2^x \frac{N^*(t)}{t \log^2 t}\, dt =$$

(12)

$$= \frac{CL(\log x)}{1+ia} x^{1+ia} + o(x) + O(1) \int_2^x \frac{dt}{\log t} = \frac{CL(\log x)}{1+ia} x^{1+ia} + o(x).$$

Von der durch die Dirichletsche Reihe hergestellten Funktion auf die Koeffizienten zurückzuschließen, ist durch die bekannte Formel ermöglicht, die für die Ableitung angewandt lautet:

$$\sum_{n \leqq x} f^*(n) \log n \log \frac{x}{n} = \frac{1}{2\pi i} \int_{(\sigma_0)} \frac{x^s}{s^2} [-F^{*\prime}(s)]\, ds,$$

wobei das Integral auf der Gerade $\operatorname{Re} s = \sigma_0 (> 1)$ zu nehmen ist. Eine leichte Rechnung zeigt, daß die linksseitige Summe, falls die erwünschte Asymptotik von $N^*(x)$ besteht,

$$\sum_{n \leqq x} f^*(n) \log n \log \frac{x}{n} = \frac{CL(\log x)}{(1+ia)^2} x^{1+ia} \log x + o(x \log x)$$

sein muß. Natürlich brauchen wir diese Rechnung nicht, werden aber diese asymptotische Relation herleiten und wollen jetzt zeigen, wie die erwünschte Asymptotik von $N^*(x)$ daraus folgt.

Schreiben wir diese letzte Gleichung außer x auch mit $x(1+\varepsilon)$ an und substrahieren wir die zwei Formeln voneinander. Da für jedes feste ε

$$L(\log x(1+\varepsilon)) = L(\log x) + o(1), \quad \log x(1+\varepsilon) = \log x + o(\log x)$$

sind, bekommen wir

$$\sum_{n \leqq x} f^*(n) \log n \log (1+\varepsilon) + \sum_{x < n \leqq x(1+\varepsilon)} f^*(n) \log n \log \frac{x(1+\varepsilon)}{n} =$$

$$= \frac{CL(\log x)}{(1+ia)^2} x^{1+ia} [(1+\varepsilon)^{1+ia} - 1] \log x + o(x \log x).$$

Die erste Summe ist nichts anderes als $N^*(x) \log (1+\varepsilon)$, für die zweite hat man wegen $|f^*(n)| \leqq 1$

$$\left| \sum_{x < n \leqq x(1+\varepsilon)} f^*(n) \log n \log \frac{x(1+\varepsilon)}{n} \right| \leqq \log x(1+\varepsilon) \log (1+\varepsilon) \sum_{x < n \leqq x(1+\varepsilon)} 1 =$$

$$= O(\varepsilon \log (1+\varepsilon) x \log x) = O(\varepsilon^2 x \log x)$$

und auf der rechten Seite durch Differenzieren

$$(1+\varepsilon)^{1+ia} - 1 = (1+ia)\varepsilon + O(\varepsilon^2).$$

Wenn wir diese in unsere Formel einsetzen, so nimmt sie die Gestalt

$$N^*(x)\log(1+\varepsilon)=\frac{CL(\log x)}{(1+ia)^2}x^{1+ia}(1+ia)\varepsilon\log x+O(\varepsilon^2 x\log x)+o(x\log x).$$

Beachtet man noch $\varepsilon=\log(1+\varepsilon)+O(\varepsilon^2)$, so gibt der Grenzübergang $x\to\infty$

$$\overline{\lim_{x\to\infty}}\frac{\left|N^*(x)-\frac{CL(\log x)}{1+ia}x^{1+ia}\log x\right|}{x\log x}\leqq c_4\varepsilon$$

und $\varepsilon\to 0$ die Behauptung.

Die Ergebnisse der bisherigen wohlbekannten elementaren Überlegungen zusammenfassend, bleibt also

$$\frac{1}{2\pi i}\int_{(\sigma_0)}\frac{x^{s-1}}{s^2}[-F^{*\prime}(s)]\,ds=\left(\frac{1}{x}\sum_{n\leqq x}f^*(n)\log n\log\frac{x}{n}\right)= \tag{13}$$

$$=\frac{CL(\log x)}{(1+ia)^2}x^{ia}\log x+o(\log x)$$

zu beweisen.

Die Abszisse des Integrationsweges sei in Abhängigkeit von x als $\sigma_0=1+\frac{1}{\log x}$ gewählt. In diesem Fall ist nämlich

$$|x^{s-1}|=x^{\sigma_0-1}=x^{\frac{1}{\log x}}=e$$

möglichst klein. Ohne es bei jedem Vorkommen zu erwähnen, wird x als genügend groß, also σ_0-1 genügend klein angenommen. Die Abschätzung des Integrals verläuft verschiedenartig auf verschiedenen Teilen der Gerade.

Zum ersten Schritt wird eine Strecke durch einen großen Winkelraum mit Spitzenpunkt in $1+ia$ ausgeschnitten:

$$I_0=\{\sigma_0+it\colon |t-a|\leqq K(\sigma_0-1)\},$$

wo K eine große, aber vorläufig feste Zahl ist. I_0 liefert das Hauptglied.

Auf dem Kreis

$$|z-s|\leqq\frac{1}{2}(\sigma_0-1)\qquad(s\in I_0\quad\text{fest})$$

haben wir die Abschätzung

$$F^*(z)-\frac{CL\left(\frac{1}{\sigma_0-1}\right)}{z-(1+ia)}=o\left(\frac{1}{\sigma_0-1}\right).$$

Hier durften wir $L\left(\frac{1}{\sigma_0-1}\right)$ statt $L\left(\frac{1}{\operatorname{Re}z-1}\right)$ schreiben, da für $1+\frac{1}{2}(\sigma_0-1)\leqq$ $\leqq\sigma\leqq 1+\frac{3}{2}(\sigma_0-1)$ gleichmäßig

$$L\left(\frac{1}{\sigma-1}\right)=L\left(\frac{1}{\sigma_0-1}\right)+o(1)$$

ist. Um eine entsprechende Beziehung für die Ableitung im Mittelpunkt $z=s$ zu gewinnen, zieht man die Cauchysche Abschätzung hervor:

$$|(\quad)'_{z=s}| = \left| F^{*\prime}(s) + \frac{CL\left(\frac{1}{\sigma_0-1}\right)}{[s-(1+ia)]^2} \right| \leqq \frac{1}{\frac{\sigma_0-1}{2}} \max_{|z-s|\leqq\frac{1}{2}(\sigma_0-1)} |(\quad)| = o\left(\frac{1}{(\sigma_0-1)^2}\right).$$

Es ist also

$$\frac{1}{2\pi i}\int_{I_0} \frac{x^{s-1}}{s^2}\,[-F^{*\prime}(s)]\,ds =$$

(14)

$$= \frac{CL\left(\frac{1}{\sigma_0-1}\right)}{2\pi i}\int_{I_0} \frac{x^{s-1}}{s^2}\,\frac{ds}{[s-(1+ia)]^2} + \int_{I_0} \frac{x^{\sigma_0-1}}{|s|^2}\,o\left(\frac{1}{(\sigma_0-1)^2}\right)|ds|.$$

Das zweite Integral kann sofort erledigt werden, wenn wir $x^{\sigma_0-1}=e$ und $|s|\geqq 1$ beachten. Integration über das Intervall der Länge $2K(\sigma_0-1)$ gibt daher

$$o\left(\frac{1}{(\sigma_0-1)^2}\,(\sigma_0-1)\right) = o\left(\frac{1}{\sigma_0-1}\right) = o(\log x).$$

Das erste Integral wird mittels des Residuumsatzes ausgerechnet. Fügen wir an I_0 einen Halbkreis nach links an:

$$\operatorname{Re} s \leqq \sigma_0, \quad |s-(\sigma_0+ia)| \leqq K(\sigma_0-1).$$

Für genügend kleines σ_0-1 ist die einzige Singularität ein Pol zweiter Ordnung bei $1+ia$ mit Residuum

$$\left(\frac{x^{s-1}}{s^2}\right)'_{s=1+ia} = \frac{x^{ia}\log x}{(1+ia)^2} - \frac{2x^{ia}}{(1+ia)^3} = \frac{x^{ia}\log x}{(1+ia)^2} + O(1).$$

Zur Abschätzung auf dem Halbkreis finden wir

$$|x^{s-1}| \leqq x^{\sigma_0-1} = e, \quad |s| \geqq \frac{1}{2},$$

$$|s-(1+ia)| \geqq (K-1)(\sigma_0-1) \geqq \frac{K}{2}\,(\sigma_0-1) \qquad (\text{für} \quad K \geqq 2)$$

und zusammen mit dem $O(1)$-Glied des Residuums ergibt sich das Restglied

$$\left|\frac{1}{2\pi i}\int_{I_0} \frac{x^{s-1}}{s^2}\,\frac{ds}{[s-(1+ia)]^2} - \frac{x^{ia}\log x}{(1+ia)^2}\right| \leqq O(1) + \frac{K(\sigma_0-1)\pi}{2\pi}\,\frac{e}{\left(\frac{1}{2}\right)^2\left[\frac{K}{2}\,(\sigma_0-1)\right]^2} \leqq$$

$$\leqq \frac{c_5}{K(\sigma_0-1)} = \frac{c_5}{K}\log x$$

und für das ursprüngliche Integral (14) von $F^{*\prime}(s)$

$$\left|\frac{1}{2\pi i}\int_{I_0}\frac{x^{s-1}}{s^2}[-F^{*\prime}(s)]\,ds-\frac{CL(\log x)}{(1+ia)^2}x^{ia}\log x\right|\leqq \tag{15}$$

$$\leqq\frac{c_6}{K}\log x+o(\log x)\leqq\frac{2c_6}{K}\log x.$$

Im nachfolgenden werden wir grob den absoluten Betrag des Integrandes nehmen und die Schwarzsche Ungleichung verwenden:

$$\left|\int_I\frac{x^{s-1}}{s^2}[-F^{*\prime}(s)]\,ds\right|\leqq e\int_I\frac{|F^{*\prime}(s)|}{|s|^2}|ds|=e\int_I\frac{\left|\frac{F^{*\prime}}{F^*}(s)\right|}{|s|^{3/4}}\frac{|F^*(s)|}{|s|^{5/4}}|ds|\leqq \tag{16}$$

$$\leqq e\sqrt{\int_I\frac{\left|\frac{F^{*\prime}}{F^*}(s)\right|^2}{|s|^{3/2}}|ds|\cdot\int_I\frac{|F^*(s)|^2}{|s|^{5/2}}|ds|}\,.$$

(I ist eine vorläufig beliebige Strecke auf (σ_0).) Das erste Integral wird immer auf die volle Gerade (σ_0) ausgedehnt werden und es sei auf einmal für die übrigbleibenden Fälle erledigt.

Wir haben in (1) im Falle von $F(s)$ gesehen, daß eine durch s dividierte Dirichletsche Reihe ein Laplacesches Integral ist. Aus der exponentiellen Form (9) von $F^*(s)$ ist zu ersehen, daß auch

$$\frac{F^{*\prime}}{F^*}(s)=\left(\sum_{n=1}^{\infty}\lambda(n)f^*(n)\right)'=-\sum_{n=1}^{\infty}\frac{\Lambda(n)f^*(n)}{n^s}$$

als Dirichletsche Reihe darstellbar ist, also wie in (1):

$$\frac{1}{s}\frac{F^{*\prime}}{F^*}(s)=-\int_1^{\infty}\frac{\sum_{n\leqq x}\Lambda(n)f^*(n)}{x^{s+1}}dx=-\int_0^{\infty}\frac{\sum_{n\leqq e^u}\Lambda(n)f^*(n)}{e^u}e^{-(s-1)u}\,du.$$

Dann aber können wir das Quadratintegral nach der Parsevalschen Formel mit der summatorischen Funktion ausdrücken:

$$\int_{(\sigma_0)}\left|\frac{F^{*\prime}}{F^*}(s)\right|^2\frac{|ds|}{|s|^2}=2\pi\int_0^{\infty}\frac{\left|\sum_{n\leqq e^u}\Lambda(n)f^*(n)\right|^2}{e^{2u}}e^{-2(\sigma_0-1)u}\,du.$$

Dies wird wegen $|f^*(n)|\leqq 1$ das größte sein, wenn alle $f^*(n)$ 1 sind, wenn also $F^*(s)$ die Riemannsche ζ-Funktion

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

ist. Auf Grund der bekannten elementaren Abschätzung

$$\sum_{n \leqq x} \Lambda(n) = O(x)$$

erhält man

$$\int_{(\sigma_0)} \left| \frac{F^{*\prime}}{F^*}(s) \right|^2 \frac{|ds|}{|s|^2} \leqq c_7 \int_0^\infty e^{-2(\sigma_0 - 1)u}\, du = \frac{c_8}{\sigma_0 - 1},$$

wo einstweilen statt des erwünschten $|s|^{3/2}$ noch $|s|^2$ steht. Vielmehr gilt aber die Ungleichung für das Teilintervall $|t| \leqq \frac{1}{2}$:

$$\int_{\substack{(\sigma_0) \\ |t| \leqq \frac{1}{2}}} \left| \frac{F^{*\prime}}{F^*}(s) \right|^2 |ds| < 4 \int_{\substack{(\sigma_0) \\ |t| \leqq \frac{1}{2}}} \left| \frac{F^{*\prime}}{F^*}(s) \right|^2 \frac{|ds|}{|s|^2} \leqq \frac{4c_8}{\sigma_0 - 1}.$$

Führt man diese Überlegung mit $F^*(s+ib)$ (mit Koeffizienten $f^*(n)n^{-ib}$) durch, so findet man

$$\int_{\substack{(\sigma_0) \\ b-\frac{1}{2} \leqq t \leqq b+\frac{1}{2}}} \left| \frac{F^{*\prime}}{F^*}(s) \right|^2 |ds| \leqq \frac{4c_8}{\sigma_0 - 1}$$

und die Gerade auf Einheitsintervalle zerlegend,

$$(17) \qquad \int_{(\sigma_0)} \left| \frac{F^{*\prime}}{F^*}(s) \right|^2 \frac{|ds|}{|s|^{3/2}} = \sum_{n=-\infty}^{\infty} \int_{\substack{(\sigma_0) \\ n-\frac{1}{2} \leqq t \leqq n+\frac{1}{2}}} \left| \frac{F^{*\prime}}{F^*}(s) \right|^2 \frac{|ds|}{|s|^{3/2}} <$$

$$< 4 \sum_{n=-\infty}^{\infty} \frac{1}{\left(|n| + \frac{1}{2}\right)^{3/2}} \int_{\substack{(\sigma_0) \\ n-\frac{1}{2} \leqq t \leqq n+\frac{1}{2}}} \left| \frac{F^{*\prime}}{F^*}(s) \right|^2 |ds| \leqq \frac{16c_8}{\sigma_0 - 1} \sum_{n=-\infty}^{\infty} \frac{1}{(|n|+1)^{3/2}} = \frac{c_9}{\sigma_0 - 1}.$$

Nachher kommen wir auf das Integral mit $|F^*(s)|^2$ in der Schwarzschen Ungleichung (16). Zuerst wird von den Endpunkten von I_0 aufwärts und abwärts bis zu einer großen festen Höhe integriert:

$$I = I_1 = \{\sigma_0 + it\colon a + K(\sigma_0 - 1) \leqq t \leqq K\}$$

oder

$$I = I_2 = \{\sigma_0 + it\colon -K \leqq t \leqq a - K(\sigma_0 - 1)\},$$

angenommen z.B. $K \geqq |a| + 1$. Da s jetzt „fern" von $1 + ia$ ist, wird die asymptotische Gleichung

$$|F^*(s)| = \left| \frac{CL\left(\frac{1}{\sigma_0 - 1}\right)}{s - (1+ia)} + o\left(\frac{1}{\sigma_0 - 1}\right) \right| \leqq \frac{|C|}{|s - (1+ia)|} + o\left(\frac{1}{\sigma_0 - 1}\right) \leqq$$

$$\leqq \frac{|C|}{K(\sigma_0 - 1)} + o\left(\frac{1}{\sigma_0 - 1}\right) \leqq \frac{c_{10}}{K(\sigma_0 - 1)}.$$

Nach Aushebung einer kleinen Potenz von unter dem Integralzeichen gibt dies

$$\int_{I_{1,2}} \frac{|F^*(s)|^2}{|s|^{5/2}}\,|ds| \leqq \max_{s\in I_{1,2}} |F^*(s)|^{1/2} \int_{I_{1,2}} \frac{|F^*(s)|^{3/2}}{|s|^{5/2}}\,|ds| \leqq$$

$$\leqq \left(\frac{c_{10}}{K(\sigma_0-1)}\right)^{1/2} \int_{I_{1,2}} \frac{|F^*(s)|^{3/2}}{|s|^{5/2}}\,|ds|. \tag{18}$$

In diesem Fall brauchen wir die große Potenz im Nenner noch nicht und wegen $|s|\geqq 1$ können wir $|s|^{1/2}$ weglassen. Das übrigbleibende Integral ist ein Quadratintegral $\left(\text{von } \frac{[F^*(s)]^{3/4}}{s}\right)$. Die exponentielle Darstellung (9) kann man zur 3/4-ten Potenz erheben:

$$[F^*(s)]^{3/4} = e^{\frac{3}{4}\sum_{n=1}^{\infty}\frac{\lambda(n)f^*(n)}{n^s}}$$

und e^z in seine Taylorsche Reihe entwickelnd, kann man sich davon überzeugen, daß auch $[F^*(s)]^{3/4}$ eine Dirichletsche Reihe ist. Die entstehenden Koeffizienten werden auch hier die größten sein, wenn $f^*(n)\equiv 1$ ist, d.h. wenn es sich wieder um die ζ-Funktion handelt. Wie früher, führt also die Parsevalsche Gleichung zu

$$\int_{I_{1,2}} \frac{|F^*(s)|^{3/2}}{|s|^{5/2}}\,|ds| \leqq \int_{(\sigma_0)} \frac{|[F^*(s)]^{3/4}|^2}{|s|^2}\,|ds| \leqq \int_{(\sigma_0)} \frac{|[\zeta(s)]^{3/4}|^2}{|s|^2}\,|ds|. \tag{19}$$

Dieses letzte Integral kann man mit Hilfe der Koeffizienten elementar behandeln, oder einfacher, wenn wir verwenden dürfen, daß die ζ-Funktion einen Pol erster Ordnung bei 1 hat und für großes t

$$\zeta(s) = O(\log|t|) \qquad (|t|\geqq 2, \text{ gleichmäßig für } \sigma>1)$$

ist:

$$\int_{(\sigma_0)} \frac{|\zeta(s)|^{3/2}}{|s|^2} \leqq c_{11}\left\{\int_{|t|\leqq 2} \frac{dt}{|\sigma_0+it-1|^{3/2}} + \int_{|t|\geqq 2} \frac{\log^{3/2}|t|}{|t|^2}\,dt\right\} \leqq \tag{20}$$

$$\leqq c_{11}\int_{-\infty}^{\infty} \frac{dt}{[(\sigma_0-1)^2+t^2]^{3/4}} + O(1) = \frac{c_{11}}{(\sigma_0-1)^{1/2}}\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{3/4}} + O(1) \leqq \frac{c_{12}}{(\sigma_0-1)^{1/2}}$$

(20) in (19), (19) in (18), (18) und (17) in die Schwarzsche Ungleichung (16) eingesetzt:

$$\left|\int_{I_{1,2}} \frac{x^{s-1}}{s^2}[-F^{*\prime}(s)]\,ds\right| \leqq e\sqrt{\frac{c_9}{\sigma_0-1}\left(\frac{c_{10}}{K(\sigma_0-1)}\right)^{1/2}\frac{c_{12}}{(\sigma_0-1)^{1/2}}} =$$

$$= \frac{c_{13}}{K^{1/4}(\sigma_0-1)} = \frac{c_{13}}{K^{1/4}}\log x. \tag{21}$$

Es bleiben noch die zwei unendlichen Strecken

$$I=I_3=\{\sigma_0+it\colon K\leqq t\},\quad I=I_4=\{\sigma_0+it\colon t\leqq -K\}$$

und wir müssen für

$$\int\limits_{I_{3,4}}\frac{|F^*(s)|^2}{|s|^{5/2}}\,|ds|$$

eine ähnliche Ungleichung ableiten, wie früher. Das damals weggelassene $|s|^{1/2}$ macht jetzt das Integral klein:

$$\int\limits_{I_{3,4}}\frac{|F^*(s)|^2}{|s|^{5/2}}\,|ds|\leqq\frac{1}{K^{1/2}}\int\limits_{(\sigma_0)}\frac{|F^*(s)|^2}{|s|^2}\,|ds|$$

und für das Quadratintegral können wir von neuem unsere Methode mit der Parsevalschen Gleichung benützen. Es wird auch hier durch die ζ-Funktion maximiert, und auf Grund ihrer erwähnten Eigenschaften erhält man

$$\int\limits_{(\sigma_0)}\frac{|F^*(s)|^2}{|s|^2}\,|ds|\leqq\int\limits_{(\sigma_0)}\frac{|\zeta(s)|^2}{|s|^2}\,|ds|\leqq c_{14}\left\{\int\limits_{|t|\geqq 2}\frac{dt}{|\sigma_0+it-1|^2}+\int\limits_{|t|\geqq 2}\frac{\log^2|t|}{|t|^2}\,dt\right\}\leqq$$

$$\leqq c_{14}\int\limits_{-\infty}^{\infty}\frac{dt}{(\sigma_0-1)^2+t^2}+O(1)=\frac{c_{14}}{\sigma_0-1}\int\limits_{-\infty}^{\infty}\frac{dt}{1+t^2}+O(1)\leqq\frac{c_{15}}{\sigma_0-1}.$$

Daher ist

$$\int\limits_{I_{3,4}}\frac{|F^*(s)|^2}{|s|^{5/2}}\,|ds|\leqq\frac{c_{15}}{K^{1/2}(\sigma_0-1)}$$

und endlich

$$(22)\qquad\left|\int\limits_{I_{3,4}}\frac{x^{s-1}}{s^2}\,[-F^{*\prime}(s)]\,ds\right|\leqq e\sqrt{\frac{c_9}{\sigma_0-1}\,\frac{c_{15}}{K^{1/2}(\sigma_0-1)}}=\frac{c_{16}}{K^{1/4}(\sigma_0-1)}=\frac{c_{16}}{K^{1/4}}\log x.$$

Die Restglieder können also sowohl in (15) für I_0, als auch in (21) für $I_{1,2}$ und in (22) für $I_{3,4}$ $\frac{c_{17}}{K^{1/4}}\log x$ nicht übersteigen:

$$\left|\frac{1}{2\pi i}\int\limits_{(\sigma_0)}\frac{x^{s-1}}{s^2}\,[-F^{*\prime}(s)]\,ds-\frac{CL(\log x)}{(1+ia)^2}\,x^{ia}\log x\right|\leqq\frac{5c_{17}}{K^{1/4}}\log x.$$

Wählt man K groß, so sieht man, daß es nichts anderes ist, als (13), was zu beweisen war.

Beweis zum Satz 2. Der Beweis wird daraus bestehen, daß wir eine komplexe Konstante C_0, eine reelle Konstante a_0 und eine langsam oszillierende Funktion $L_0(u)$ vom Betrage 1 bestimmen, so daß

$$F^*(s) = \frac{C_0 L_0\left(\frac{1}{\sigma-1}\right)}{s-(1+ia_0)} + o\left(\frac{1}{\sigma-1}\right)$$

gleichmäßig auf jedem $|t| \leqq M$ ist. Nach dem bereits bewiesenen Teil des Satzes 1′ wird daraus

$$M^*(x) = \frac{C_0 L_0(\log x)}{1+ia_0} x^{1+ia_0} + o(x)$$

folgen, woher man zu $M(x)$ übergehen kann, wie es in der Vorbereitung gezeigt wurde (s. (7)—(8)):

$$M(x) = \frac{C_0 H(1+ia_0) L_0(\log x)}{1+ia_0} x^{1+ia_0} + o(x).$$

Dies ist die Behauptung des Satzes, mit $\frac{C_0 H(1+ia_0)}{1+ia_0}$ als C_0.

Als erster Schritt müssen wir $|F^*(s)|$ mit $\frac{1}{\sigma-1}$ vergleichen. Statt $\frac{1}{\sigma-1}$ dürfen wir aber $\zeta(\sigma)$ nehmen, da bekanntlich

$$\frac{\zeta(\sigma)}{\frac{1}{\sigma-1}} \to 1 \qquad (\sigma \to 1+0)$$

ist. Sowohl $F^*(s)$ wie auch $\zeta(\sigma)$ werden hier in der exponentiellen Form (9) geschrieben:

$$\frac{|F^*(s)|}{\zeta(\sigma)} = \frac{\left| e^{\sum_{n=1}^{\infty} \frac{\lambda(n) f^*(n)}{n^{\sigma+it}}} \right|}{e^{\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^{\sigma}}}} = e^{-\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^{\sigma}} (1 - \mathrm{Re} f^*(n) n^{-it})}. \tag{23}$$

Wegen $|f^*(n)| \leqq 1$ hat die Reihe in dem Exponent nichtnegative Glieder und die Summe ist für jedes t eine monoton abnehmende nichtnegative Funktion von σ. Wir unterscheiden zwei Fälle.

1. Strebt die Summe für jedes t zu $+\infty$, wenn $\sigma \to 1+0$, so tut sie es nach einem bekannten reell-funktionentheoretischen Satz wegen der Monotonität und Stetigkeit gleichmäßig auf jedem endlichen Intervall. Wir haben also die gewünschte Asymptotik mit $C_0 = 0$, die Bestimmung der anderen Größen ist überflüßig.

2. Es gibt ein $t = a_0$ mit

$$\gamma = \lim_{\sigma \to 1+0} \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^{\sigma}} (1 - \mathrm{Re} f^*(n) n^{-ia_0}) < +\infty$$

oder was dasselbe ist

$$\gamma = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n} (1 - \mathrm{Re} f^*(n) n^{-ia_0}) < +\infty.$$

Ersetzt man $f^*(n)$ durch $f^*(n)n^{-ia_0}$ so kann man der Einfachheit halber $a_0=0$ annehmen. Später wird es sich herausstellen, daß nur ein einziges solches a_0 existiert.

Wie sie im vorigen Beweis verwendet wurde, beweisen wir die asymptotische Formel separat für $|t|\leqq K(\sigma-1)$ und separat für $K(\sigma-1)<|t|\leqq M$.

Untersuchen wir zuerst (23) mit $K(\sigma-1)<|t|\leqq M$. In der folgenden Kette von Ungleichungen werden

$$|1-z|^2\leqq 2(1-\operatorname{Re} z) \qquad (|z|\leqq 1),$$

$$|1-z|^2=2(1-\operatorname{Re} z) \qquad (|z|=1)$$

oft benützt.

Da die Summe in (23) für $t=0\,(=a_0)$ endlich bleibt,

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{\sigma}}|1-f^*(n)|^2\leqq 2\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{\sigma}}(1-\operatorname{Re} f^*(n))\leqq 2\gamma. \tag{24}$$

Dies bringt zum Ausdruck, daß $f^*(n)$ im Durchschnitt zu 1 nahe ist. Ebenso ist mit unserem t

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{\sigma}}|n^{it}-f^*(n)|^2=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{\sigma}}|1-f^*(n)n^{-it}|^2\leqq 2\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{\sigma}}(1-\operatorname{Re} f^*(n)n^{-it}). \tag{25}$$

Dies drückt aus, daß die abzuschätzende Summe auf der rechten Seite nur dann klein sein kann, wenn $f^*(n)$ im Durchschnitt auch zu n^{it} nahe ist. Wir zeigen aber, daß 1 und n^{it} voneinander entfernt sind:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{\sigma}}|1-n^{it}|^2&=2\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{\sigma}}(1-\operatorname{Re} n^{it})=2\log\frac{\zeta(\sigma)}{|\zeta(\sigma-it)|}\geqq\\ &\geqq 2\log\frac{\frac{c_{18}}{\sigma-1}}{\frac{c_{19}}{K(\sigma-1)}}=2\log K+2\log\frac{c_{18}}{c_{19}}\geqq\log K\end{aligned} \tag{26}$$

für genügend großes K und genügend kleines (auch von der Schranke $|t|\leqq M$ abhängendes) $\sigma-1$. Aus den drei vorangehenden Ungleichungen (24), (25), (26) folgt in der Tat wegen

$$|1-n^{it}|^2=|(1-f^*(n))+(f^*(n)-n^{it})|^2\leqq 2|1-f^*(n)|^2+2|f^*(n)-n^{it}|^2$$

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{\sigma}}(1-\operatorname{Re} f^*(n)n^{-it})\geqq\frac{1}{4}\log K-\gamma.$$

Diese Summe steht im Exponent unseres Quotientes (23):

$$\frac{|F^*(s)|}{\zeta(\sigma)}\leqq e^{-\frac{1}{4}\log K+\gamma},\qquad |F^*(s)|\leqq\frac{c_{20}}{K^{1/4}(\sigma-1)}\qquad (K(\sigma-1)<|t|\leqq M).$$

Auf der Strecke $|t| \leqq K(\sigma-1)$ müssen wir $F^*(s)$ selbst, und nicht nur den absoluten Betrag, mit $\frac{1}{s-1}$, also mit $\zeta(s)$ vergleichen:

$$\frac{F^*(s)}{\zeta(s)} = e^{\sum_{n=1}^{\infty} \frac{\lambda(n) f^*(n)}{n^s} - \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}} = e^{-\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}(1-f^*(n))}. \tag{27}$$

Der reelle Teil der Summe für $s=\sigma$ strebt zu γ und wir setzen

$$C_0 \overset{\text{def}}{=} e^{-\gamma} = e^{-\sum_{n=1}^{\infty} \frac{\lambda(n)}{n}(1-\operatorname{Re} f^*(n))},$$

während der imaginäre Teil die langsam oszillierende Funktion liefern wird:

$$L_0\left(\frac{1}{\sigma-1}\right) \overset{\text{def}}{=} e^{i \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^\sigma} \operatorname{Im} f^*(n)},$$

so daß zusammen

$$C_0 L_0\left(\frac{1}{\sigma-1}\right) \sim e^{-\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^\sigma}(1-f^*(n))} \tag{28}$$

ist. (Statt C_0 haben wir $e^{-\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^\sigma}(1-\operatorname{Re} f^*(n))}$ $(\to C_0)$ genommen, daher das Zeichen $\sim$). Wir müssen zeigen, daß erstens die Funktion tatsächlich langsam oszilliert, d.h.

$$\frac{L_0\left(\frac{1}{\sigma_1-1}\right)}{L_0\left(\frac{1}{\sigma-1}\right)} = e^{i \sum_{n=1}^{\infty} \lambda(n)\left(\frac{1}{n^{\sigma_1}} - \frac{1}{n^\sigma}\right) \operatorname{Im} f^*(n)} \to 1 \quad \text{für} \quad 1+\frac{\sigma-1}{2} \leqq \sigma_1 \leqq \sigma \to 1+0$$

und zweitens, daß $\frac{F^*(s)}{\zeta(s)}$ in (27) durch $C_0 L_0\left(\frac{1}{\sigma-1}\right)$ in (28) dividiert zu 1 strebt:

$$\frac{F^*(s)}{C_0 L_0\left(\frac{1}{\sigma-1}\right)\zeta(s)} \sim e^{-\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}(1-f^*(n)) + \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^\sigma}(1-f^*(n))} = e^{-\sum_{n=1}^{\infty} \lambda(n)\left(\frac{1}{n^s}-\frac{1}{n^\sigma}\right)(1-f^*(n))} \to 1.$$

Wenn wir gleichmäßig für $\sigma' \overset{\text{def}}{=} 1+\frac{\sigma-1}{2} \leqq \operatorname{Re} s \leqq \sigma$, $|\operatorname{Im} s| \leqq K(\sigma-1)$

$$\sum_{n=1}^{\infty} \lambda(n) \left| \frac{1}{n^s} - \frac{1}{n^\sigma} \right| |1-f^*(n)| \to 0$$

nachweisen, so werden wir beide bewiesen haben.

$$\left| \frac{1}{n^s} - \frac{1}{n^\sigma} \right| = \frac{1}{n^{\operatorname{Re} s}} |1-e^{-(\sigma-s)\log n}| \leqq \frac{1}{n^{\sigma'}} |s-\sigma| \log n \leqq \frac{1}{n^{\sigma'}} (K+1)(\sigma-1)\log n =$$

$$= 2(K+1)(\sigma'-1)\frac{\log n}{n^{\sigma'}}.$$

Was $|1-f^*(n)|$ anbelangt, wissen wir, daß die Reihe

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}|1-f^*(n)|^2 \leqq 2\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}\left(1-\mathrm{Re}\,f^*(n)\right) = 2\gamma$$

konvergiert. Folglich gibt es zu $\varepsilon>0$ ein N mit

$$\sum_{n=N+1}^{\infty}\frac{\lambda(n)}{n}|1-f^*(n)|^2 < \varepsilon. \tag{29}$$

Zerlegen wir also unsere abzuschätzende Summe:

$$\sum_{n=1}^{\infty}\lambda(n)\left|\frac{1}{n^s}-\frac{1}{n^\sigma}\right||1-f^*(n)| \leqq 2(K+1)(\sigma'-1)\sum_{n=1}^{\infty}\frac{\lambda(n)\log n}{n^{\sigma'}}|1-f^*(n)| \leqq$$

$$\leqq 2(K+1)(\sigma'-1)\sum_{n=1}^{N}\frac{\lambda(n)\log n}{n}|1-f^*(n)| +$$

$$+2(K+1)(\sigma'-1)\sum_{n=N+1}^{\infty}\frac{\lambda(n)\log n}{n^{\sigma'}}|1-f^*(n)|.$$

Das erste Glied strebt zu 0 wegen des Faktors $(\sigma'-1)$. Zur zweiten Summe wird die Schwarzsche Ungleichung angewandt:

$$\sum_{n=1}^{\infty}\frac{\lambda^{\frac{1}{2}}(n)\log n}{n^{\sigma'/2}}\frac{\lambda^{\frac{1}{2}}(n)|1-f^*(n)|}{n^{\sigma'/2}} \leqq \sqrt{\sum_{n=1}^{\infty}\frac{\lambda(n)\log^2 n}{n^{\sigma'}}\sum_{n=N+1}^{\infty}\frac{\lambda(n)|1-f^*(n)|^2}{n^{\sigma'}}}.$$

Nach der Wahl von N in (29) und wegen $\sigma'>1$ ist der zweite Faktor unter der Quadratwurzel kleiner als ε. Für den ersten ist bekanntlich

$$\sum_{n=1}^{\infty}\frac{\lambda(n)\log^2 n}{n^{\sigma'}} = \left(\frac{\zeta'}{\zeta}(s)\right)'_{s=\sigma'} \leqq \frac{c_{21}}{(\sigma'-1)^2}.$$

Wir haben also die Schranke

$$o(1)+2(K+1)(\sigma'-1)\sqrt{\frac{c_{21}}{(\sigma'-1)^2}\varepsilon} = o(1)+c_{22}(K+1)\sqrt{\varepsilon}$$

gefunden, die zusammen mit ε tatsächlich beliebig klein wird.

Zusammenfassend gilt für jedes feste K

$$\frac{F^*(s)}{C_0 L_0\left(\frac{1}{\sigma-1}\right)\zeta(s)} = 1+o(1) \qquad (|t| \leqq K(\sigma-1)),$$

d. h.

$$F^*(s) = \frac{C_0 L_0\left(\frac{1}{\sigma-1}\right)}{s-1}+o\left(\frac{1}{\sigma-1}\right) \qquad (|t| \leqq K(\sigma-1),\quad a_0=0),$$

während für $K(\sigma-1)<|t|\leqq M$

$$|F^*(s)| \leqq \frac{c_{20}}{K^{1/4}(\sigma-1)}.$$

Hier ist aber auch das Hauptglied

$$\left|\frac{C_0 L_0\left(\frac{1}{\sigma-1}\right)}{s-1}\right| \leqq \frac{|C_0|}{K(\sigma-1)}$$

verhältnismäßig klein, und wählt man K groß, so sieht man, daß die asymptotische Beziehung auf dem ganzen $|t|\leqq M$ besteht. Qu.e.d.

Vervollständigung des Beweises zum Satz 1'. Im Beweis von Satz 2 haben wir gezeigt, daß es stets ein C_0, ein a_0 und ein $L_0(u)$ mit

$$F^*(s) = \frac{C_0 L_0\left(\frac{1}{\sigma-1}\right)}{s-(1+ia_0)} + o\left(\frac{1}{\sigma-1}\right)$$

gibt, in welchem Fall:

$$M(x) = \frac{H(1+ia_0)C_0 L_0(\log x)}{1+ia_0} x^{1+ia_0} + o(x). \tag{30}$$

Für Satz 1' müssen wir aber dies mit den gegebenen C, a und $L(u)$ nachweisen. Wir haben daher

$$F(s) = H(s)F^*(s) = \frac{H(1+ia_0)C_0 L_0\left(\frac{1}{\sigma-1}\right)}{s-(1+ia_0)} + o\left(\frac{1}{\sigma-1}\right).$$

Nach der Voraussetzung genügt $F(s)$ einer ähnlichen Asymptotik, die entsprechenden Größen müssen also übereinstimmen:

$$a = a_0 \quad (\text{wenn } C \neq 0 \text{ ist}),$$

$$CL\left(\frac{1}{\sigma-1}\right) = H(1+ia_0)C_0 L_0\left(\frac{1}{\sigma-1}\right) \qquad (\text{abgesehen von einem } o(1)).$$

Die Asymptotik (30) von $M(x)$ kann also auch in der Form

$$M(x) = \frac{CL(\log x)}{1+ia} x^{1+ia} + o(x)$$

geschrieben werden, wie es behauptet wurde.

Beweis zum Satz 3. Wir haben früher gezeigt, daß $\frac{1}{H(s)}$ in der Nähe der Gerade $\mathrm{Re}\, s=1$ und auf ihr selbst beschränkt und stetig ist (s. (11)). Darum erfüllt auch

$$F^*(s) = \frac{F(s)}{H(s)}$$

selbstverständlich die Voraussetzung des Satzes mit $\frac{C}{H(1)}$ statt C. Im Hinblick auf das in der Vorbereitung bewiesene Lemma über den Übergang von $M^*(x)$ zu $M(x)$, können wir uns ohne weiteres auf $F^*(s)$ und $M^*(x)$ beschränken.

Der Übergang (12) von $N^*(x)$ zu $M^*(x)$ im Beweis des Satzes 1′ ist auch hier gültig, so daß es genügt

$$N^*(x) = \sum_{n \leqq x} f^*(n) \log n = Cx \log x + o(x \log x)$$

zu beweisen. Nicht aber die darauffolgende elementare Überlegung, wo $|f^*(n)| \leqq 1$ gebraucht wurde. Deshalb müssen wir von der etwas schwerer behandelbaren „genauen Koeffizientenformel"

$$N^*(x) = \sum_{n \leqq x} f^*(n) \log n = \frac{1}{2\pi i} \int_{(\sigma_0)} \frac{x^s}{s} [-F^{*\prime}(s)]\, ds$$

ausgehen, wobei das Integral als Cauchyscher Hauptwert existiert, falls x nicht ganz ist. In dem Folgenden sei x (und später y) stets eine um 1/2 erhöhte natürliche Zahl. Fixieren wir die Abszisse, wie früher, als $\sigma_0 = 1 + \frac{1}{\log x}$. Da das Integral nicht absolut konvergiert, wollen wir nur über $|t| \leqq T$ integrieren, wo T später geeignet gewählt wird:

$$\frac{1}{2\pi i} \int_{\substack{(\sigma_0) \\ |t| \leqq T}} \frac{x^s}{s} [-F^{*\prime}(s)]\, ds =$$

$$= \frac{1}{2\pi i} \int_{\substack{(\sigma_0) \\ |t| \leqq T}} \frac{x^s}{s} \sum_{n=1}^{\infty} \frac{f^*(n) \log n}{n^s}\, ds = \sum_{n=1}^{\infty} f^*(n) \log n \frac{1}{2\pi i} \int_{\substack{(\sigma_0) \\ |t| \leqq T}} \left(\frac{x}{n}\right)^s \frac{ds}{s}.$$

Ersetzen wir den Integrationsweg mit den anderen Seiten des Rechtecks $r \pm iT$, $\sigma_0 \pm iT$. Im Falle $n < x$ lassen wir r zum negativen Unendlichen streben. $s = 0$ ist die einzige Singularität mit Residuum 1, so daß der Residuumsatz als Hauptglied

$$\sum_{n<x} f^*(n) \log n = N^*(x)$$

gibt. Auf der senkrechten Strecke strebt das Integral mit $\left|\left(\frac{x}{n}\right)^s\right| = \left(\frac{x}{n}\right)^r$ zu 0 und die zwei Integrale auf den horizontalen Halbgeraden sind kleiner als

$$\int_{-\infty}^{\sigma_0} \left(\frac{x}{n}\right)^{\sigma} \frac{d\sigma}{|s|} \leqq \frac{1}{T} \int_{-\infty}^{\sigma_0} e^{\sigma \log \frac{x}{n}}\, d\sigma = \frac{1}{T \log \frac{x}{n}} \frac{x^{\sigma_0}}{n^{\sigma_0}} = \frac{ex}{T} \frac{1}{n^{\sigma_0} \log \frac{x}{n}}.$$

Für $n > x$ kann man aus demselben Grunde auf $r \to +\infty$ übergehen. Der Integrand

ist auf dem Rechteck ausnahmslos regulär, während die vorige Abschätzung durch

$$\int_{\sigma_0}^{\infty}\left(\frac{x}{n}\right)^{\sigma}\frac{d\sigma}{|s|}\leqq\frac{1}{T}\int_{\sigma_0}^{\infty}e^{-\sigma\log\frac{n}{x}}d\sigma=\frac{1}{T\log\frac{n}{x}}\frac{x^{\sigma_0}}{n^{\sigma_0}}=\frac{ex}{T}\frac{1}{n^{\sigma_0}\log\frac{n}{x}}$$

zu ersetzen ist. Summation über n gibt daher

$$\left|\frac{1}{2\pi i}\int_{\substack{(\sigma_0)\\|t|\leqq T}}\frac{x^s}{s}[-F^{*\prime}(s)]\,ds-N^*(x)\right|\leqq\frac{2ex}{2\pi T}\sum_{n=1}^{\infty}\frac{|f^*(n)|\log n}{n^{\sigma_0}\left|\log\frac{n}{x}\right|}. \tag{31}$$

Der Vollständigkeit halber haben wir diese bekannte Abschätzung hergeleitet, deren rechte Seite (auch allein das $[x]$-te Glied] jedoch manchmal zu groß wird. Man kann dem folgendermaßen helfen.

Wir werden sogleich zeigen, daß für ein genügend großes (festes) A

$$\sum_{x-\frac{x}{\log^A x}<n\leqq x}|f^*(n)|=o(x)$$

ist und daß es im Summationsintervall $\left(x-\frac{x}{\log^A x},\,x\right]$ ein y gibt, für welche die obige Abschätzung (31) statt x angewandt, die Summe auf der rechten Seite doch unter $\log^{A_1}x$ bleibt. Dann kann man z.B. $T=\log^{A_1}y\sim\log^{A_1}x$ wählen, in welchem Fall (31) für $x=y$ das Restglied

$$O\left(\frac{y}{T}\log^{A_1}x\right)=O\left(\frac{x}{\log^{A_1}x}\log^{A_1}x\right)=O(x)=o(x\log x)$$

gibt. Wenn es uns dann gelingt, das Integral auf der linken Seite mit $x=y$ bis auf $o(y\log y)=o(x\log x)$ zu bestimmen, so können wir das für $N^*(x)$ benützen, da

$$|N^*(x)-N^*(y)|=\left|\sum_{y<n\leqq x}f^*(n)\log n\right|\leqq\log x\sum_{x-\frac{x}{\log^A x}<n\leqq x}|f^*(n)|=o(x\log x)$$

ist.

Die Abschätzung von

$$\sum_{x-\frac{x}{\log^A x}<n\leqq x}|f^*(n)|$$

beruht auf der Hölderschen Ungleichung $\left(p=1+\delta,\,\frac{1}{p}+\frac{1}{q}=1,\,q=\frac{1+\delta}{\delta}\right)$

$$\sum_{x-\frac{x}{\log^A x}<n\leqq x}|f^*(n)|\leqq x^{\frac{\sigma_0}{1+\delta}}\sum_{x-\frac{x}{\log^A x}<n\leqq x}\frac{|f^*(n)|}{n^{\frac{\sigma_0}{1+\delta}}}\cdot 1\leqq$$

$$\leqq(ex)^{\frac{1}{1+\delta}}\left(\sum_{n=1}^{\infty}\frac{|f^*(n)|^{1+\delta}}{n^{\sigma_0}}\right)^{\frac{1}{1+\delta}}\left(\frac{x}{\log^A x}+1\right)^{\frac{\delta}{1+\delta}}\leqq\frac{c_{23}x}{\log^{A\frac{\delta}{1+\delta}}x}\left(\sum_{n=1}^{\infty}\frac{|f^*(n)|^{1+\delta}}{n^{\sigma_0}}\right)^{\frac{1}{1+\delta}}.$$

Wenn wir für die rechtsseitige Summe eine Abschätzung $O\left(\frac{1}{(\sigma_0-1)^{A_2}}\right)=O(\log^{A_2} x)$ finden, so kann man z.B. A so groß wählen, daß $A\frac{\delta}{1+\delta}=\frac{A_2}{1+\delta}+1$ ist, damit diese Schranke tatsächlich $O\left(\frac{x}{\log x}\right)=o(x)$ nicht übersteigt.

$|f^*(n)|^{1+\delta}$ ist ebenfalls eine streng multiplikative Funktion, so daß die exponentielle Darstellung (9)

$$\sum_{n=1}^{\infty}\frac{|f^*(n)|^{1+\delta}}{n^{\sigma_0}}=e^{\sum_{n=1}^{\infty}\frac{\lambda(n)|f^*(n)|^{1+\delta}}{n^{\sigma_0}}} \tag{32}$$

gültig ist. Nach dem die Voraussetzung b ersetzenden b′ (s. (10)):

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)|f^*(n)|^{1+\delta}}{n^{\sigma}}\equiv\sum_{n=1}^{\infty}\frac{\lambda(n)\log n\,|f^*(n)|^{1+\delta}}{n^{\sigma}}\leqq\frac{A_2}{\sigma-1}$$

oder von $\sigma=\sigma_0$ bis $\sigma=2$ integriert:

$$\sum_{n=1}^{\infty}\frac{\lambda(n)|f^*(n)|^{1+\delta}}{n^{\sigma_0}}\leqq A_2\log\frac{1}{\sigma_0-1}+O(1).$$

Eben dies brauchen wir im Exponent von (32):

$$\sum_{n=1}^{\infty}\frac{|f^*(n)|^{1+\delta}}{n^{\sigma_0}}\leqq e^{A_2\log\frac{1}{\sigma_0-1}+O(1)}\leqq c_{24}\left(\frac{1}{\sigma_0-1}\right)^{A_2}.$$

Was

$$\sum_{n=1}^{\infty}\frac{|f^*(n)|\log n}{n^{\sigma_0}\left|\log\frac{n}{y}\right|}$$

betrifft, summieren wir dies über die in Rede kommenden $y\in\left(x-\frac{x}{\log^A x},x\right]$. Wenn wir nachweisen können, daß diese Doppelsumme $O(x\log^{A_3}x)$ ist, so muß ein y mit

$$\sum_{n=1}^{\infty}\frac{|f^*(n)|\log n}{n^{\sigma_0}\left|\log\frac{n}{y}\right|}\leqq\log^{A_3+A+1}x \tag{33}$$

existieren, andernfalls wäre die Doppelsumme größer als

$$\sum_{x-\frac{x}{\log^A x}<y\leqq x}\log^{A_3+A+1}x\geqq\log^{A_3+A+1}x\left(\frac{x}{\log^A x}-1\right)\geqq\frac{1}{2}x\log^{A_3+1}x\gg x\log^{A_3}x.$$

(Es ist wahr, daß wir (33) nicht mit $\sigma_0=1+\frac{1}{\log x}$, sondern mit $1+\frac{1}{\log y}$ brauchen, aber wegen $y\leqq x$ ist $1+\frac{1}{\log y}\geqq 1+\frac{1}{\log x}$.)

Wechseln wir in der Doppelsumme Summation über y und n auf. Im Falle $\frac{1}{2}<\frac{n}{y}\leqq 2$ haben wir

$$\left|\log\frac{n}{y}\right| \geqq \log 2\left|\frac{n}{y}-1\right| = \log 2\,\frac{|n-y|}{y} \geqq \frac{\log 2}{x}|n-y|$$

und in den übrigen Fällen $\frac{n}{y}\leqq\frac{1}{2}$ oder $\frac{n}{y}>2$

$$\left|\log\frac{n}{y}\right| \geqq \log 2.$$

Summation über y dieses zweiten Typs gibt daher grob

$$\sum_y{}'' \frac{1}{\left|\log\frac{n}{y}\right|} \leqq \frac{1}{\log 2}\sum_y 1 < \frac{x}{\log 2}.$$

Für den Beitrag der y des ersten Typs findet man

$$\sum_y{}' \frac{1}{\left|\log\frac{n}{y}\right|} \leqq \frac{x}{\log 2}\sum_y \frac{1}{|n-y|} < \frac{x}{\log 2}\,2\sum_{m=0}^{[x]}\frac{1}{m+\frac{1}{2}} \leqq c_{25}\,x\log x$$

und zusammenfassend

$$\sum_y\sum_n \leqq c_{25}\,x\log x\sum_{n=1}^{\infty}\frac{|f^*(n)|\log n}{n^{\sigma_0}}.$$

Es bleibt noch also, die rechtsseitige Summe durch eine Potenz von $\log x$ abzuschätzen.

Wie wir bei (10) bemerkt haben, besteht die Bedingung b′ auch mit $\delta=0$. Eine frühere Überlegung Wort für Wort wiederholt:

$$-\frac{\varphi'(\sigma)}{\varphi(\sigma)} \overset{\text{def}}{=} \sum_{n=1}^{\infty}\frac{\lambda(n)\log n\,|f^*(n)|}{n^{\sigma}} < \frac{A_4}{\sigma-1} \qquad (\text{b}'\ \text{mit}\ \delta=0),$$

$$\log\varphi(\sigma) = \left(-\int\sum_{n=1}^{\infty}\frac{\lambda(n)\log n\,|f^*(n)|}{n^{\sigma}}\right) = \sum_{n=1}^{\infty}\frac{\lambda(n)\,|f^*(n)|}{n^{\sigma}} \leqq A_4\log\frac{1}{\sigma-1}+O(1),$$

$$\varphi(\sigma) = e^{\sum_{n=1}^{\infty}\frac{\lambda(n)|f^*(n)|}{n^{\sigma}}} = \sum_{n=1}^{\infty}\frac{|f^*(n)|}{n^{\sigma}} \leqq c_{26}\left(\frac{1}{\sigma-1}\right)^{A_4} \tag{34}$$

und das Produkt der ersten und dritten Linie für $\sigma=\sigma_0$ gibt eben

$$-\varphi'(\sigma_0) = \sum_{n=1}^{\infty}\frac{|f^*(n)|\log n}{n^{n_0}} \leqq A_4 c_{26}\left(\frac{1}{\sigma_0-1}\right)^{A_4+1} = c_{27}\log^{A_4+1}x,$$

was zu beweisen war.

Bisher haben wir erreicht, daß es genügt (die Bezeichnung x statt y beibehaltend, nach einer Division durch x),

$$\frac{1}{2\pi i}\int\limits_{\substack{(\sigma_0)\\ |t|\leqq T}} \frac{x^{s-1}}{s}[-F^{*\prime}(s)]\,ds = C\log x + o(\log x)$$

mit $T=\log^{A_1} x=\left(\frac{1}{\sigma_0-1}\right)^{A_1}$ zu beweisen. Will man die Methode von Satz 1′ anwenden, so ist es klar, daß man scharfe Abschätzungen von Integralen über die Potenzen von $\frac{F^{*\prime}}{F^*}(s)$ und $F^*(s)$ braucht, was jetzt die Hauptschwierigkeit verursacht. Dies wird in den nachfolgenden zwei Hilfssätzen enthalten sein.

Wie früher, bezeichnen A_i große feste Zahlen, dagegen B_i beliebig große Zahlen, die erst später zu einem A_i gewählt werden.

HILFSSATZ 1. *Es sei, wie immer,*

$$\text{(b′)} \qquad \sum_{n=1}^{\infty}\frac{\Lambda(n)|f^*(n)|^{1+\delta}}{n^{\sigma}} \leqq \frac{A_2}{\sigma-1} \qquad (0<\delta\leqq 1).$$

1. *Ist w eine beliebige positive Zahl, so ist das Maß der Menge Ω der t-Werte, für welche:*

$$\left|\frac{F^{*\prime}}{F^*}(\sigma+it)\right| \geqq \frac{w}{\sigma-1}, \qquad |t|\leqq\left(\frac{1}{\sigma-1}\right)^{B}$$

mit festem σ, kleiner als

$$c_{28}\cdot(\sigma-1) = c_{28}(B, w)(\sigma-1),$$

wobei c_{28} von σ nicht abhängt.

2. *Es gibt ein genügend großes $\alpha=\alpha(B_1)$ mit*

$$\int\limits_{\substack{(\sigma)\\ |t|\leqq\left(\frac{1}{\sigma-1}\right)^{B_1}}} \left|\frac{F^{*\prime}}{F^*}(s)\right|^{\alpha}|ds| \leqq \frac{c_{29}(B_1)}{(\sigma-1)^{\alpha-1}}.$$

BEWEIS. 1 folgt leicht aus 2, wir werden aber 2 durch eine strengere Form von 1 beweisen.

Nehmen wir an, wir haben N Punkte t_k unserer Menge Ω, die voneinander wenigstens um $|t_i-t_j|\geqq\eta(\sigma-1)$ entfernt sind; η wird später gegeben. Wir wollen eine obere Schranke für N ableiten.

Wenn t_k zu Ω gehört, so ist nach der Definition

$$\left|\sum_{n=1}^{\infty}\frac{\Lambda(n)f^*(n)}{n^{\sigma+it_k}}\right| = \left|\frac{F^{*\prime}}{F^*}(\sigma+it_k)\right| \geqq \frac{w}{\sigma-1}.$$

Man kann also N komplexe Zahlen ϱ_k mit $|\varrho_k|=1$ bestimmen, so daß

$$\sum_{k=1}^{N}\varrho_k\sum_{n=1}^{\infty}\frac{\Lambda(n)f^*(n)}{n^{\sigma+it_k}} \geqq \frac{Nw}{\sigma-1}$$

ist, folglich

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)|f^*(n)|}{n^{\sigma}} \left| \sum_{k=1}^{N} \varrho_k n^{-it_k} \right| \leqq \frac{Nw}{\sigma-1}. \tag{35}$$

Für die linke Seite wenden wir die Höldersche Ungleichung mit $p=1+\delta$, $q=\frac{1+\delta}{\delta}$ an:

$$\sum_{n=1}^{\infty} \frac{\Lambda^{1/p}(n)|f^*(n)|}{n^{\sigma/p}} \frac{\Lambda^{1/q}(n)\left|\sum_{k=1}^{N} \varrho_k n^{-it_k}\right|}{n^{\sigma/q}} \leqq$$

$$\leqq \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)|f^*(n)|^{1+\delta}}{n^{\sigma}} \right)^{1/p} \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \left| \sum_{k=1}^{N} \varrho_k n^{-it_k} \right|^q \right)^{1/q}.$$

Die erste Summe auf der rechten Seite ist nach der Voraussetzung kleiner als $\frac{A_2}{\sigma-1}$. In der zweiten ist $q \geqq 2$, weil $\delta \leqq 1$ angenommen wurde, und da jedenfalls

$$\left| \sum_{k=1}^{N} \varrho_k n^{-it_k} \right| \leqq N$$

ist, nur die zweite Potenz beibehaltend:

$$\left| \sum_{k=1}^{N} \varrho_k n^{-it_k} \right|^q \leqq N^{q-2} \left| \sum_{k=1}^{N} \varrho_k n^{-it_k} \right|^2.$$

Dieses letzte rechnen wir aus:

$$\left| \sum_{k=1}^{N} \varrho_k n^{-it_k} \right|^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \varrho_i \bar{\varrho}_j n^{-i(t_i-t_j)}.$$

Die volle Summe wird daher

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \left| \sum_{k=1}^{N} \varrho_k n^{-it_k} \right|^q \leqq N^{q-2} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \sum_{i=1}^{N} \sum_{j=1}^{N} \varrho_i \bar{\varrho}_j n^{-i(t_i-t_j)} =$$

$$= N^{q-2} \sum_{i=1}^{N} \sum_{j=1}^{N} \varrho_i \bar{\varrho}_j \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma+i(t_i-t_j)}} = N^{q-2} \sum_{i=1}^{N} \sum_{j=1}^{N} \varrho_i \bar{\varrho}_j \left[-\frac{\zeta'}{\zeta}(\sigma+i(t_i-t_j)) \right].$$

Die N Glieder mit $i=j$ geben

$$N^{q-2} N \left[-\frac{\zeta'}{\zeta}(\sigma) \right] \leqq N^{q-1} \frac{2}{\sigma-1}.$$

Im Falle der übrigbleibenden, wenigeren Glieder als N^2 ist aber $|t_i-t_j| \geqq \eta(\sigma-1)$ und der Beitrag kann

$$N^{q-2} N^2 \max_{\eta(\sigma-1) \leqq |t| \leqq 2\left(\frac{1}{\sigma-1}\right)^B} \left| \frac{\zeta'}{\zeta}(\sigma+it) \right|$$

nicht übersteigen. In diesem Bereich haben wir

$$\left|\frac{\zeta'}{\zeta}(\sigma+it)\right| \leqq \frac{c_{30}}{|t|} \leqq \frac{c_{30}}{\eta(\sigma-1)} \qquad (|t| \leqq 2)$$

wegen des Poles bei $s=1$ und

$$\left|\frac{\zeta'}{\zeta}(\sigma+it)\right| \leqq c_{31}\log|t| \leqq c_{31}\log 2\left(\frac{1}{\sigma-1}\right)^B \qquad (|t| \geqq 2)$$

nach der bekannten Abschätzung. Selbst wenn wir nur $|t|^\varepsilon$ statt $\log|t|$ wüßten, würde die für die kleinen t gültige Schranke dominieren, falls z.B.

$$\eta(\sigma-1) \leqq (\sigma-1)^{\frac{1}{2}}, \qquad \eta \leqq \frac{1}{(\sigma-1)^{\frac{1}{2}}}.$$

Für das obige Maximum hat man also

$$\max_{\eta(\sigma-1)\leqq|t|\leqq 2\left(\frac{1}{\sigma-1}\right)^B}\left|\frac{\zeta'}{\zeta}(\sigma+it)\right| \leqq \frac{c_{32}(B)}{\eta(\sigma-1)}$$

und damit wird unsere ursprüngliche Ungleichung (35)

$$\left(\frac{A_2}{\sigma-1}\right)^{1/p}\left(\frac{2N^{q-1}}{\sigma-1}+\frac{c_{32}(B)N^q}{\eta(\sigma-1)}\right)^{1/q} \geqq \frac{Nw}{\sigma-1},$$

$$c_{33}(B)\left(N^{q-1}+\frac{N^q}{\eta}\right)^{1/q} \geqq Nw, \qquad c_{34}(B)\left(N^{q-1}+\frac{N^q}{\eta}\right) \geqq N^q w^q.$$

Wählt man η in Abhängigkeit von w so, daß das Glied auf der linken Seite mit N^q die Hälfte der rechten Seite ausmacht:

$$\frac{c_{34}(B)}{\eta} = \frac{1}{2}w^q, \qquad \eta = \frac{2c_{34}(B)}{w^q}$$

so bekommt man schließlich

$$c_{34}(B)N^{q-1} \geqq \frac{1}{2}N^q w^q, \qquad N \leqq \frac{2c_{34}(B)}{w^q}.$$

Die Bedingung dafür war nur

$$\eta = \frac{2c_{34}(B)}{w^q} \leqq \frac{1}{(\sigma-1)^{\frac{1}{2}}}, \qquad w \geqq c_{35}(B)(\sigma-1)^{\frac{1}{2q}}.$$

Jetzt konstruieren wir eine Folge $\{t_k\}$. Als t_1 wählen wir die kleinste Zahl der abgeschlossenen Menge Ω, wenn sie nicht leer ist. t_2 sei die kleinste Zahl $t\in\Omega$ mit $t\geqq t_1+\eta(\sigma-1)$, t_3 die kleinste mit $t\geqq t_2+\eta(\sigma-1)$, u.s.w. Auf diese Art entstehen endlich viele, zu Ω gehörende Punkte der Strecke $|t|\leqq\left(\frac{1}{\sigma-1}\right)^B$, die nach der Konstruktion voneinander wenigstens um $\eta(\sigma-1)$ entfernt sind. Nach den soeben

Bewiesenen kann ihre Anzahl $N \frac{2c_{34}(B)}{w^q}$ nicht übersteigen. Die N Intervalle $[t_k, t_k+\eta(\sigma-1)]$ bedecken aber das ganze Ω, so daß für sein Maß:

$$|\Omega| \leqq N\eta(\sigma-1) \leqq \frac{2c_{24}(B)}{w^q}\frac{2c_{34}(B)}{w^q}(\sigma-1) = \frac{c_{36}(B)}{w^{2q}}(\sigma-1).$$

Dies für ein festes w angewandt, ist nichts anderes als die Behauptung 1 des Hilfssatzes. Die Ungleichung besteht aber gleichmäßig bis zu $w=c_{35}(B)(\sigma-1)^{\frac{1}{2q}}$.

Auf der anderen Seite haben wir

$$\left|\frac{F^{*\prime}}{F^*}(s)\right| \leqq \sum_{n=1}^{\infty}\frac{\Lambda(n)|f^*(n)|}{n^\sigma} < \frac{A_4}{\sigma-1}$$

(b′ mit $\delta=0$, s. die Bemerkung nach (10)) überall. Wählen wir $w=w_m=A_4\cdot 2^{-m}$, $m=0, 1, \ldots$, wo die Folge dann abbrechen muß, wenn

$$w_m = A_4\cdot 2^{-m} < c_{35}(B_1)(\sigma-1)^{\frac{1}{2q}} \tag{36}$$

wäre. Die entsprechenden monoton zunehmenden Mengen Ω bezeichnen wir mit Ω_m ($\Omega_0=\varnothing$) und diejenigen t des Integrationsintervalls $\left[-\left(\frac{1}{\sigma-1}\right)^{B_1}, \left(\frac{1}{\sigma-1}\right)^{B_1}\right]$, die sogar aus dem letzten Ω_m ausbleiben, mit Ω_∞. Das Integrationsintervall denken wir uns aus den Mengen $\Omega_m-\Omega_{m-1}$ ($m=1, 2, \ldots$) und Ω_∞ zusammengesetzt. Auf $\Omega_m-\Omega_{m-1}$ besteht die gegensätzliche Ungleichung, wie auf Ω_{m-1}:

$$\left|\frac{F^{*\prime}}{F^*}(\sigma+it)\right| < \frac{w_{m-1}}{\sigma-1} = \frac{A_4 2^{-m+1}}{\sigma-1},$$

während nach der obigen Abschätzung:

$$|\Omega_m-\Omega_{m-1}| \leqq |\Omega_m| \leqq \frac{c_{36}(B_1)}{w_m^{2q}}(\sigma-1) = c_{37}(B_1)2^{2qm}(\sigma-1).$$

Der Beitrag der Mengen $\Omega_m-\Omega_{m-1}$ ist also kleiner als

$$\sum_m \int_{\Omega_m-\Omega_{m-1}} \left|\frac{F^{*\prime}}{F^*}(s)\right|^\alpha |ds| \leqq \sum_{m=1}^{\infty}\left(\frac{A_4 2^{-m+1}}{\sigma-1}\right)^\alpha c_{37}(B_1)2^{2qm}(\sigma-1) =$$

$$= \frac{c_{38}(B_1,\alpha)}{(\sigma-1)^{\alpha-1}}\sum_{m=1}^{\infty}2^{-(\alpha-2q)m}.$$

Sorgt man nur für $\alpha>2q$, so ist die Reihe konvergent und dieser Ausdruck ist tatsächlich $\frac{c_{39}(B_1,\alpha)}{(\sigma-1)^{\alpha-1}}$. Es bleibt noch Ω_∞. Für sein Maß ist nichts besseres als

$$|\Omega_\infty| \leqq 2\left(\frac{1}{\sigma-1}\right)^{B_1}$$

bekannt. Die Funktion $\left|\frac{F^{*\prime}}{F^*}(s)\right|$ kann aber auf ihm durch $\frac{1}{\sigma-1}$, multipliziert mit dem letzten w_m abgeschätzt werden, das kleiner ist als die Doppelte der Grenze $c_{35}(B_1)(\sigma-1)^{\frac{1}{2q}}$ in (36):

$$\int_{\Omega_\infty}\left|\frac{F^{*\prime}}{F^*}(s)\right|^{\alpha}|ds| \leqq \left(\frac{2c_{35}(B_1)(\sigma-1)^{\frac{1}{2q}}}{\sigma-1}\right)^{\alpha} 2\left(\frac{1}{\sigma-1}\right)^{B_1} = \frac{c_{40}(B_1,\alpha)}{(\sigma-1)^{\left(1-\frac{1}{2q}\right)\alpha+B_1}}.$$

Da $1-\frac{1}{2q}<1$ fest ist, wird der Exponent im Nenner für genügend großes $\alpha=\alpha(B_1)$ kleiner als $\alpha-1$, und der Hilfssatz ist bewiesen.

HILFSSATZ 2. *Ist*

$$|F^*(s)| \leqq \frac{c_{41}|s|}{\sigma-1} \tag{37}$$

gleichmäßig für $\operatorname{Re} s=\sigma$, *so ist für jedes* $\beta>1$

$$\int\limits_{|t|\leqq\left(\frac{1}{\sigma_0-1}\right)^{B_2}}^{(\sigma_0)}\left|\frac{F^*(s)}{s}\right|^{\beta}|ds| \leqq \frac{c_{42}(B_2,\beta)}{(\sigma_0-1)^{\beta-1}}.$$

BEWEIS. Wir fixieren die Länge des Integrationsweges $2\left(\frac{1}{\sigma_0-1}\right)$, betrachten aber das Integral

$$U(\sigma) \overset{\text{def}}{=} \int\limits_{|t|\leqq\left(\frac{1}{\sigma_0-1}\right)^{B_2}}^{(\sigma)}\left|\frac{F^*(s)}{s}\right|^{\beta}|ds|$$

auch für $\sigma_0\leqq\sigma\leqq\sigma_1$ mit einem im Verhältniß zu σ_0 nicht zu großen σ_1. Der Einfachheit halber bezeichne I das genannte t-Intervall $|t|\leqq\left(\frac{1}{\sigma_0-1}\right)^{B_2}$.

Rechnen wir $U'(\sigma)$ aus. Dazu müssen wir den Integrand nach σ differenzieren. Eine einfache Tatsache ist

$$||g(x)|'| \leqq |g'(x)|,$$

also

$$\left|\frac{\partial}{\partial\sigma}\left|\frac{F^*(s)}{s}\right|^{\beta}\right| \leqq \left|\frac{\partial}{\partial\sigma}\left(\frac{F^*(s)}{s}\right)^{\beta}\right| = \left|\beta\left(\frac{F^*(s)}{s}\right)^{\beta-1}\left\{\frac{F^{*\prime}(s)}{s}-\frac{F^*(s)}{s^2}\right\}\right| \leqq$$

$$\leqq \beta\frac{|F^*(s)|^{\beta-1}|F^{*\prime}(s)|}{|s|^{\beta}}+\beta\frac{|F^*(s)|^{\beta}}{|s|^{\beta}},$$

wo ein $|s|$ auf Grund von $|s|\geqq 1$ im zweiten Glied weggelassen wurde. Integriert man dies über (σ), $t\in I$, so gewinnt man eine Schranke für $|U'(\sigma)|$. Das zweite Glied

auf der rechten Seite wird $\beta U(\sigma)$. Um das Integral des ersten Gliedes weiter abzuschätzen, ziehen wir 1, Hilfssatz 1 mit $w=\frac{\beta-1}{3\beta}$ hervor. Wenn $\frac{1}{2}|I|=\left(\frac{1}{\sigma_0-1}\right)^{B_2}$ eine feste Potenz $\left(\frac{1}{\sigma_1-1}\right)^{B_3}$ (und nicht nur eine von $\frac{1}{\sigma_0-1}$) nicht übersteigt, so können wir $B=B_3$ setzen, damit das dortige Intervall $|t|\leqq\left(\frac{1}{\sigma-1}\right)^{B}$ unser I für jedes $\sigma_0\leqq\sigma\leqq\sigma_1$ enthält. Die dortige Ungleichung $|\Omega|\leqq c_{28}(\sigma-1)$ besteht mit einem gemeinsamen $c_{28}=c_{28}(B_3,\beta)$ für $\sigma_0\leqq\sigma\leqq\sigma_1$, wenn auch Ω von σ abhängen kann.

Auf Ω benützen wir die vorausgesetzte Schranke (37) für $F^*(s)$. Wie wir schon einmal die Cauchysche Abschätzung für die Ableitung angewandt haben:

$$|F^*(z)|\leqq\frac{c_{41}(|s|+1)}{\frac{\sigma-1}{2}}\quad\text{für}\quad|z-s|\leqq\frac{\sigma-1}{2},$$

$$|F^{*\prime}(s)|\leqq\frac{1}{\frac{\sigma-1}{2}}\frac{c_{41}(|s|+1)}{\frac{\sigma-1}{2}}\leqq\frac{c_{43}|s|}{(\sigma-1)^2}.$$

Für den Beitrag von Ω ist also

$$\int\limits_{\substack{(\sigma)\\ t\in\Omega\cap I}}\frac{|F^*(s)|^{\beta-1}|F^{*\prime}(s)|}{|s|^\beta}|ds|\leqq\int\limits_{\substack{(\sigma)\\ t\in\Omega\cap I}}\frac{\left(\frac{c_{41}|s|}{\sigma-1}\right)^{\beta-1}\frac{c_{43}|s|}{(\sigma-1)^2}}{|s|^\beta}|ds|=$$

$$=\frac{c_{41}^{\beta-1}c_{43}}{(\sigma-1)^{\beta+1}}|\Omega\cap I|\leqq\frac{c_{44}(\beta)}{(\sigma-1)^{\beta+1}}c_{28}(B_3,\beta)(\sigma-1)=\frac{c_{45}(B_3,\beta)}{(\sigma-1)^\beta}.$$

Das Komplement von Ω in Bezug auf I ist aber durch

$$\left|\frac{F^{*\prime}}{F^*}(s)\right|<\frac{w}{\sigma-1}=\frac{\beta-1}{3\beta(\sigma-1)}$$

gekennzeichnet. Schreibt man also $F^{*\prime}(s)$ in der Form $\frac{F^{*\prime}}{F^*}(s)F^*(s)$, so sieht man

$$\int\limits_{\substack{(\sigma)\\ t\in I-\Omega}}\frac{|F^*(s)|^{\beta-1}|F^{*\prime}(s)|}{|s|^\beta}|ds|\leqq\frac{\beta-1}{3\beta(\sigma-1)}\int\limits_{\substack{(\sigma)\\ t\in I}}\left|\frac{F^*(s)}{s}\right|^\beta|ds|=\frac{\beta-1}{3\beta(\sigma-1)}U(\sigma)$$

und zusammenfassend

$$|U'(\sigma)|\leqq\beta\frac{\beta-1}{3\beta(\sigma-1)}U(\sigma)+\beta\frac{c_{45}(B_3,\beta)}{(\sigma-1)^\beta}+\beta U(\sigma).$$

Für genügend kleines $\sigma-1$ kann das letzte Glied in das erste einverleibt werden,

wenn wir den Koeffizient $\frac{\beta-1}{3}$ z.B. auf $\frac{\beta-1}{2} \overset{\text{def}}{=} \vartheta$ erhöhen. Benützt man nur $-U'(\sigma) \leqq |U'(\sigma)|$, so haben wir also die Differentialungleichung

$$-U'(\sigma) \leqq \frac{\vartheta}{\sigma-1} U(\sigma) + \frac{c_{46}(B_3, \beta)}{(\sigma-1)^{\beta}} \qquad \left(\vartheta = \frac{\beta-1}{2}\right)$$

zu lösen. Multipliziert man mit $(\sigma-1)^{\vartheta}$, so findet man, daß die Glieder mit $U'(\sigma)$ und $U(\sigma)$ zusammen eine Ableitung geben:

$$-[U(\sigma)(\sigma-1)^{\vartheta}]' \leqq \frac{c_{46}(B_3, \beta)}{(\sigma-1)^{\beta-\vartheta}}.$$

Integration über (σ_0, σ_1) gibt

$$-U(\sigma_1)(\sigma_1-1)^{\vartheta} + U(\sigma_0)(\sigma_0-1)^{\vartheta} \leqq c_{46}(B_3, \beta) \int_{\sigma_0}^{\infty} \frac{d\sigma}{(\sigma-1)^{\beta-\vartheta}} =$$

$$= \frac{c_{46}(B_3, \beta)}{\beta-\vartheta-1} \frac{1}{(\sigma_0-1)^{\beta-\vartheta-1}},$$

da $\beta-\vartheta = \beta - \frac{\beta-1}{2} = \frac{1+\beta}{2} > 1$ ist. Wir brauchen daraus $U(\sigma_0)$ auszudrücken:

$$U(\sigma_0) \leqq \frac{c_{47}(B_3, \beta)}{(\sigma_0-1)^{\beta-1}} + U(\sigma_1) \frac{(\sigma_1-1)^{\vartheta}}{(\sigma_0-1)^{\vartheta}}.$$

Das erste Glied ist die erwünschte Schranke. Wir müssen noch σ_1 geeignet bestimmen, damit auch das zweite unter einer solchen bleibt.

Es ist grob (wegen $\beta > 1$)

$$U(\sigma_1) = \int_{\substack{(\sigma_1) \\ t \in I}} \left| \frac{F^*(s)}{s} \right|^{\beta} |ds| \leqq \max_{s \in (\sigma_1)} |F^*(s)|^{\beta} \int_{(\sigma_1)} \frac{|ds|}{|s|^{\beta}} \leqq \left(\sum_{n=1}^{\infty} \frac{|f^*(n)|}{n^{\sigma_1}} \right)^{\beta} c_{48}(\beta).$$

Für die Summe haben wir bereits in (34) die Abschätzung $c_{26} \left(\frac{1}{\sigma_1-1}\right)^{A_4}$ abgeleitet. Wir finden also

$$U(\sigma_1) \frac{(\sigma_1-1)^{\vartheta}}{(\sigma_0-1)^{\vartheta}} \leqq c_{49}(\beta) \left(\frac{1}{\sigma_1-1}\right)^{A_4\beta} \frac{(\sigma_1-1)^{\vartheta}}{(\sigma_0-1)^{\vartheta}} \leqq c_{49}(\beta) \left(\frac{1}{\sigma_1-1}\right)^{A_4\beta} \frac{1}{(\sigma_0-1)^{\vartheta}}.$$

Wählt man σ_1 so, daß

$$\left(\frac{1}{\sigma_1-1}\right)^{A_4\beta} = \frac{1}{(\sigma_0-1)^{\vartheta/2}}$$

ist, so ist diese Schranke

$$c_{49}(\beta) \frac{1}{(\sigma_0-1)^{\vartheta/2}} \frac{1}{(\sigma_0-1)^{\vartheta}} = \frac{c_{49}(\beta)}{(\sigma_0-1)^{\frac{3}{2}\vartheta}} = \frac{c_{49}(\beta)}{(\sigma_0-1)^{\frac{3}{2}\frac{\beta-1}{2}}} \ll \frac{1}{(\sigma_0-1)^{\beta-1}}.$$

Wir müssen noch nachsehen, ob $\left(\frac{1}{\sigma_0-1}\right)^{B_2}$ tatsächlich eine feste Potenz von $\frac{1}{\sigma_1-1}$ nicht übersteigt. Das ist aber aus der Definition klar, und der Beweis ist beendet.

Im Besitz dieser zwei Hilfssätze können wir den Beweis nach dem Muster von Satz 1′ leicht zu Ende führen.

Das Hauptglied von

$$\frac{1}{2\pi i}\int\limits_{\substack{(\sigma_0)\\ |t|\leqq T}}\frac{x^{s-1}}{s}[-F^{*\prime}(s)]\,ds \qquad \left(T=\log^{A_1}x=\left(\frac{1}{\sigma_0-1}\right)^{A_1}\right)$$

wird, wie dort, von Integration über $I_0=\{t\colon |t|\leqq K(\sigma_0-1)\}$ geliefert. Die Überlegung Wort für Wort wiederholt,

$$\left|\frac{1}{2\pi i}\int\limits_{\substack{\sigma_0\\ t\in I_0}}\frac{x^{s-1}}{s}[-F^{*\prime}(s)]\,ds-C\log x\right|\leqq\frac{c_{50}}{K(\sigma_0-1)}=\frac{c_{50}}{K}\log x. \tag{38}$$

Auf $I_{1,2}=\{t\colon K(\sigma_0-1)\leqq|t|\leqq T\}$ schätzen wir den absoluten Betrag

$$\int\limits_{\substack{(\sigma_0)\\ t\in I_{1,2}}}\frac{x^{\sigma_0-1}}{|s|}|F^{*\prime}(s)||ds|=e\int\limits_{\substack{(\sigma_0)\\ t\in I_{1,2}}}\left|\frac{F^{*\prime}}{F^*}(s)\right|\left|\frac{F^*(s)}{s}\right||ds|$$

ab. α sei das zu unserem $T=\left(\frac{1}{\sigma_0-1}\right)^{A_1}$ gehörende $\alpha=\alpha(A_1)$ in 2, Hilfssatz 1, und schreiben wir die Höldersche Ungleichung mit α als p und $\frac{\alpha}{\alpha-1}$ als q an:

$$\int\limits_{\substack{(\sigma_0)\\ t\in I_{1,2}}}\left|\frac{F^{*\prime}}{F^*}(s)\right|\left|\frac{F^*(s)}{s}\right||ds|\leqq\left(\int\limits_{\substack{(\sigma_0)\\ |t|\leqq T}}\left|\frac{F^{*\prime}}{F^*}(s)\right|^{\alpha}|ds|\right)^{\frac{1}{\alpha}}\left(\int\limits_{\substack{(\sigma_0)\\ t\in I_{1,2}}}\left|\frac{F^*(s)}{s}\right|^{q}|ds|\right)^{\frac{1}{q}}. \tag{39}$$

Nach 2, Hilfssatz 1 ist das erste Integral $O\left(\frac{1}{(\sigma_0-1)^{\alpha-1}}\right)$. Nach Aushebung einer kleinen Potenz, z.B. der $\frac{q-1}{2}$-ten, im zweiten Integral, und dann die Integration auf das volle $|t|\leqq T$ ausgedehnt,

$$\int\limits_{\substack{(\sigma_0)\\ t\in I_{1,2}}}\left|\frac{F^*(s)}{s}\right|^{q}|ds|\leqq\max_{\substack{(\sigma_0)\\ t\in I_{1,2}}}\left|\frac{F^*(s)}{s}\right|^{\frac{q-1}{2}}\int\limits_{\substack{(\sigma_0)\\ |t|\leqq T}}\left|\frac{F^*(s)}{s}\right|^{\frac{q+1}{2}}|ds|.$$

Auf Grund der vorausgesetzten Asymptotik (Bedingung e)

$$F^*(s)=\frac{C}{s-1}+o\left(\frac{|s|}{\sigma_0-1}\right)$$

ist auf $I_{1,2}$

$$\left|\frac{F^*(s)}{s}\right| \leqq \frac{|C|}{K(\sigma_0-1)|s|}+o\left(\frac{1}{\sigma_0-1}\right) \leqq \frac{c_{51}}{K(\sigma_0-1)},$$

das vorherige Maximum ist also kleiner als

$$\frac{c_{52}}{K^{\frac{q-1}{2}}(\sigma_0-1)^{\frac{q-1}{2}}},$$

während auf der ganzen Gerade $\operatorname{Re} s=\sigma$:

$$|F^*(s)| \leqq \frac{|C|}{\sigma-1}+o\left(\frac{|s|}{\sigma-1}\right) \leqq \frac{c_{53}|s|}{\sigma-1}$$

so daß die Bedingung (37) für Hilfssatz 2 erfüllt wird, und wir können ihn mit $\beta=\frac{q+1}{2}>1$ anwenden:

$$\int\limits_{\substack{(\sigma_0)\\ |t|\leqq T}} \left|\frac{F^*(s)}{s}\right|^{\frac{q+1}{2}} |ds| \leqq \frac{c_{54}}{(\sigma_0-1)^{\frac{q-1}{2}}}.$$

Wenn wir unsere Abschätzungen zu (39) sammeln,

$$\int\limits_{\substack{(\sigma_0)\\ t\in I_{1,2}}} \frac{|F^{*\prime}(s)|}{|s|}|ds| \leqq \left(\frac{c_{55}}{(\sigma_0-1)^{\alpha-1}}\right)^{\frac{1}{\alpha}}\left(\frac{c_{52}}{K^{\frac{q-1}{2}}(\sigma_0-1)^{\frac{q-1}{2}}}\cdot\frac{c_{54}}{(\sigma_0-1)^{\frac{q-1}{2}}}\right)^{\frac{1}{q}}=$$

$$=\frac{c_{56}}{K^{\frac{q-1}{2q}}(\sigma_0-1)^{2-\frac{1}{\alpha}-\frac{1}{q}}}=\frac{c_{56}}{K^{\frac{q-1}{2q}}(\sigma_0-1)}=\frac{c_{55}}{K^{\frac{q-1}{2q}}}\log x.$$

Dies mit der Ungleichung (38) für I_0 vereinigt,

$$\left|\frac{1}{2\pi i}\int\limits_{\substack{(\sigma_0)\\ |t|\leqq T}} \frac{x^{s-1}}{s}[-F^{*\prime}(s)]\,ds-C\log x\right| \leqq \left(\frac{c_{50}}{K}+\frac{c_{56}}{K^{\frac{q-1}{2q}}}\right)\log x,$$

wo K beliebig groß sein kann und der Satz ist vollständig bewiesen.

Beweis zum Satz 4. Als erster Schritt wird eine nichtnegative Funktion $f_1(n)$ konstruiert, welche allen Forderungen mit Ausnahme der o-Beziehung entspricht, statt deren

$$\frac{c_{57}}{\sigma-1} \leqq F_1(\sigma) \leqq \frac{c_{58}}{\sigma-1}$$

sein wird. Aber auch die endgültige Funktion wird

$$f_1(p^k) = \frac{1}{k!} f_1^k(p)$$

genügen, was die die höheren Primzahlpotenzen betreffenden Bedingungen c und d sofort erledigt; die erste deshalb, weil für die Schranke von $f_0(p)$ z.B. $\omega(x) \leqq x^{1/3}$ angenommen werden kann und

$$\sum_{k=2}^{\infty} \frac{f_1(p^k)}{p^k} = \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{f_1(p)}{p}\right)^k < e\left(\frac{f_1(p)}{p}\right)^2 \qquad \left(\frac{f_1(p)}{p} \leqq \frac{p^{1/3}}{p} < 1\right),$$

$$\sum_p \left(\frac{f_1(p)}{p}\right)^2 \leqq \sum_p \left(\frac{p^{1/3}}{p}\right)^2 < +\infty,$$

die zweite wegen

$$F_1(s) = \prod_p \left(1 + \frac{f_1(p)}{p^s} + \frac{f_1(p^2)}{p^{2s}} + \ldots\right) = \tag{40}$$

$$= \prod_p \left(1 + \frac{1}{1!}\frac{f_1(p)}{p^s} + \frac{1}{2!}\left(\frac{f_1(p)}{p^s}\right)^2 + \ldots\right) = \prod_p e^{\frac{f_1(p)}{p^s}} = e^{\sum_p \frac{f_1(p)}{p^s}}$$

und wir können uns auf die ersten Primzahlpotenzen beschränken.

Zerlegen wir sie durch die Folge $x_m = e^{2^m}$. Mit Rücksicht darauf, daß $f_1(p)$ im Durchschnitt 1 sein muß, wollen wir erreichen, daß

$$\sum_{p \leqq x_m} \frac{f_1(p)}{p} = \log\log x_m = \log 2^m = m \log 2 \qquad (m \geqq m_0)$$

genau bestehen soll. An einer einzigen Stelle $f_1(p)$ geeignet gewählt, kann man dies für genügend großes m sichern, falls

$$\sum_{x_{m-1} < p \leqq x_m} \frac{f_1(p)}{p} = \log 2 \qquad (m \geqq m_0)$$

wäre. Später wird $f_1(p)$ mit dieser Eigenschaft explizit gegeben, zuerst wollen wir aber sehen, was bloß daraus folgt.

Es sei $\sigma_m = 1 + \dfrac{1}{\log x_m} = 1 + \dfrac{1}{2^m}$ und untersuchen wir den Exponent von $F_1(s)$ in (40) für $s = \sigma_m$. Man kann von $p > x_m$ absehen, da in einem Intervall $(x_{l-1}, x_l]$

$$\frac{1}{p^{\sigma_m}} = \frac{1}{p^{\sigma_m - 1}} \cdot \frac{1}{p} < \frac{1}{x_{l-1}^{\sigma_m - 1}} \cdot \frac{1}{p} = e^{-\frac{2^{l-1}}{2^m}} \cdot \frac{1}{p}$$

ist:

$$\sum_{l > m} \sum_{x_{l-1} < p \leqq x_l} \frac{f_1(p)}{p^{\sigma_m}} \leqq \sum_{l > m} e^{-\frac{2^{l-1}}{2^m}} \sum_{x_{l-1} < p \leqq x_l} \frac{f_1(p)}{p} = \tag{41}$$

$$= \sum_{l > m}^{\infty} e^{-\frac{2^{l-1}}{2^m}} \log 2 = \log 2 (e^{-1} + e^{-2} + e^{-4} + \ldots) = c_{60}.$$

Für die obere Abschätzung kann man für $p \leqq x_m$ $\sigma = 1$ nehmen:

$$\log F_1(\sigma_m) \leqq \sum_{p \leqq x_m} \frac{f_1(p)}{p} + c_{60} = \log\log x_m + c_{60} = \log \frac{1}{\sigma_m - 1} + c_{60},$$

$$F_1(\sigma_m) \leqq \frac{c_{61}}{\sigma_m - 1}.$$

Wenn wir zeigen, daß

$$\sum_{p \leqq x_m} \frac{f_1(p)}{p} \log p \leqq c_{62} \log x_m \tag{42}$$

ist, so kann man auch für die untere Abschätzung $\sigma = 1$ nehmen, da der Unterschied

$$\sum_{p \leqq x_m} f_1(p) \left(\frac{1}{p} - \frac{1}{p^{\sigma_m}} \right) = \sum_{p \leqq x_m} \frac{f_1(p)}{p} (1 - e^{-(\sigma_m - 1)\log p}) \leqq$$

$$\leqq (\sigma_m - 1) \sum_{p \leqq x_m} \frac{f_1(p)}{p} \log p = \frac{1}{\log x_m} \sum_{p \leqq x_m} \frac{f_1(p)}{p} \log p$$

beschränkt ist, folglich

$$\log F_1(\sigma_m) \geqq \sum_{p \leqq x_m} \frac{f_1(p)}{p^{\sigma_m}} \geqq \sum_{p \leqq x_m} \frac{f_1(p)}{p} - c_{62} = \log \frac{1}{\sigma_m - 1} - c_{62},$$

$$F_1(\sigma_m) \geqq \frac{c_{63}}{\sigma_m - 1}.$$

Wir haben diese und die gegensätzliche Ungleichung mit $\sigma = \sigma_m$ abgeleitet. Von hier kann man aber auf den allgemeinen Fall auf Grund der Monotonität von $F_1(\sigma)$ schließen, wenn man $\sigma > 1$ in ein Intervall $(\sigma_m, \sigma_{m-1}]$ einschließt und beachtet, daß $\sigma_m - 1$ nach 2-Potenzen abnimmt. Die Konstanten c_{63} und c_{61} müssen nur halbiert bzw. gedoppelt werden.

Was das oben benützte (42) betrifft,

$$\sum_{p \leqq x_m} \frac{f_1(p)}{p} \log p = \sum_{l \leqq m} \sum_{x_{l-1} < p \leqq x_l} \frac{f_1(p)}{p} \log p \leqq \sum_{l \leqq m} \log x_l \cdot \log 2 + O(1) = \tag{43}$$

$$= \sum_{l \leqq m} 2^l \log 2 + O(1) < 3 \log 2 \cdot 2^m = 3 \log 2 \cdot \log x_m.$$

Daraus folgt auch

$$\sum_{p} \frac{f_1(p)}{p^{\sigma}} \log p = O\left(\frac{1}{\sigma - 1} \right) \tag{44}$$

zuerst mit $\sigma = \sigma_m$, wenn man für $p \leqq x_m$ wieder durch $\sigma = 1$ abschätzt:

$$\sum_{p \leqq x_m} \frac{f_1(p)}{p^{\sigma_m}} \log p \leqq \sum_{p \leqq x_m} \frac{f_1(p)}{p} \log p \leqq 3 \log 2 \cdot \log x_m = \frac{3 \log 2}{\sigma_m - 1}$$

und für $p > x_m$ von neuem $\frac{1}{p^{\sigma_m}} < e^{-\frac{2^{l-1}}{2^m}} \cdot \frac{1}{p}$ $(x_{l-1} < p \leqq x_l)$ gebraucht:

$$\sum_{l>m} \sum_{x_{l-1}<p\leqq x_l} \frac{f_1(p)}{p^{\sigma_m}} \log p \leqq \sum_{l>m} \log x_l \cdot e^{-\frac{2^{l-1}}{2^m}} \sum_{x_{l-1}<p\leqq x_l} \frac{f_1(p)}{p} =$$

$$= \sum_{l>m} 2^l e^{-\frac{2^{l-1}}{2^m}} \log 2 = 2^m(2e^{-1}+4e^{-2}+8e^{-4}+\ldots)\log 2 = \frac{c_{64}}{\sigma_m - 1}.$$

Zusammenfassend gilt also

$$\sum_p \frac{f_1(p)}{p^\sigma} \log p \leqq \frac{3\log 2 + c_{64}}{\sigma - 1} = \frac{c_{65}}{\sigma - 1}$$

für $\sigma = \sigma_m$. Wegen der Monotonität besteht dies ebenfalls für alle $\sigma > 1$ mit der doppelten Konstante $2c_{65}$.

$f_1(n)$ hat also die gewünschten „guten" Eigenschaften. Das Wesentliche steht noch bevor, indem wir $f_1(p)$ so konstruieren müssen, daß es auch die „schlechte" Eigenschaft besitzt.

Wenn

$$\log x_{m-1} = 2^{m-1} < \log p \leqq (1-\varepsilon_m)\log x_m = (1-\varepsilon_m)2^m$$

ist, wo die abnehmende Nullfolge $\{\varepsilon_m\}$ bald gegeben wird, so sei $f_1(p) = 0$. Im anderen Teil

$$(1-\varepsilon_m)\log x_m = (1-\varepsilon_m)2^m < \log p \leqq \log x_m = 2^m$$

sei $f_1(p)$ eine positive (von m abhängende) Konstante. Damit die Summe $\log 2$ beträgt, muß diese Konstante

$$\frac{\log 2}{\sum_{(1-\varepsilon_m)2^m < \log p \leqq 2^m} \frac{1}{p}}$$

sein. Wir haben bekanntlich

$$\sum_{\log p \leqq Y} \frac{1}{p} = \log Y + C + O\left(\frac{1}{Y}\right),$$

woraus für den Nenner die untere Abschätzung

$$\log \frac{2^m}{(1-\varepsilon_m)2^m} + O\left(\frac{1}{2^m}\right) = \log \frac{1}{1-\varepsilon_m} + O\left(\frac{1}{2^m}\right) \geqq \frac{\varepsilon_m}{2} \qquad (m \geqq m_0)$$

folgt, wenn z.B. $\varepsilon_m \geqq \frac{1}{2^{m/2}}$. Wählt man also

$$\varepsilon_m = \max\left(\frac{2\log 2}{\omega(x_{m-1})}, \frac{1}{2^{\frac{m}{2}}}\right) \qquad (\to 0),$$

so wird nicht nur diese Bedingung erfüllt, sondern auch

$$f_1(p) = 0 \text{ oder } = \frac{\log 2}{\sum\limits_{(1-\varepsilon_m)2^m < \log p \leqq 2^m} \frac{1}{p}} \leqq \frac{\log 2}{\frac{\varepsilon_m}{2}} \leqq$$

$$\leqq \frac{2\log 2}{\frac{2\log 2}{\omega(x_{m-1})}} = \omega(x_{m-1}) \leqq \omega(p) \qquad (x_{m-1} < p \leqq x_m).$$

Untersuchen wir jetzt

$$\frac{F_1(\sigma)}{|F_1(\sigma+it)|} = e^{\sum\limits_p \frac{f_1(p)}{p^\sigma}(1-\cos t \log p)} \tag{45}$$

mit unserem $\sigma = \sigma_m = 1 + \frac{1}{\log x_m} = 1 + \frac{1}{2^m}$. Da jedenfalls $0 \leqq 1 - \cos t \log p \leqq 2$ ist, kann die Summation über $p > x_m$ für alle t weggelassen werden, wofür die Abschätzung $O(1)$ in (41) nachgewiesen wurde. Wenn wir jetzt $t = t_k = \frac{2\pi}{2^m} 2^k (0 \leqq k \leqq m)$ setzen, so kann man auch von $p \leqq x_{m-k}$ absehen, da nach (43)

$$\sum_{p \leqq x_{m-k}} \frac{f_1(p)}{p^\sigma}(1-\cos t_k \log p) \leqq \sum_{p \leqq x_{m-k}} \frac{f_1(p)}{p} t_k \log p \leqq t_k \, 3 \log 2 \log x_{m-k} =$$

$$= \frac{2\pi}{2^m} 2^k \cdot 3 \log 2 \cdot 2^{m-k} = c_{66}$$

ist. Betrachten wir einen übrigbleibenden Block $(x_{l-1}, x_l]$ $(m-k < l \leqq m)$. $f_1(p)$ verschwindet nur im Intervall

$$(1-\varepsilon_l)2^l < \log p \leqq 2^l$$

nicht, in welchem Fall für $t_k \log p = \frac{2\pi}{2^{m-k}} \log p$:

$$\frac{2\pi}{2^{m-k}}(1-\varepsilon_l)2^l < t_k \log p \leqq \frac{2\pi}{2^{m-k}} \cdot 2^l,$$

$$\frac{2^l}{2^{m-k}} 2\pi - \frac{2^l}{2^{m-k}} 2\pi\varepsilon_l < t_k \log p \leqq \frac{2^l}{2^{m-2}} 2\pi.$$

Der Multiplikator $\frac{2^l}{2^{m-k}}$ von 2π ist hier wegen $l > m-k$ eine ganze Zahl, so daß

$$|t_k \log p| \leqq \frac{2^l}{2^{m-k}} 2\pi\varepsilon_l \leqq \frac{2^m}{2^{m-k}} 2\pi\varepsilon_{m-k} = 2\pi \cdot 2^k \varepsilon_{m-k} \pmod{2\pi}$$

ist, d.h.

$$0 \leqq 1 - \cos t_k \log p \leqq 2\pi \, 2^k \varepsilon_{m-k}.$$

Für den fehlenden Teil $(x_{m-k}, x_m]$ in (45) findet man also

$$\sum_{x_{m-k}<p\leqq x_m} \frac{f_1(p)}{p^\sigma}(1-\cos t_k \log p) \leqq \sum_{m-k<l\leqq m} \sum_{x_{l-1}<p\leqq x_l} \frac{f_1(p)}{p}(1-\cos t_k \log p) \leqq$$

$$\leqq 2\pi 2^k \varepsilon_{m-k} \sum_{m-k<l\leqq m} \log 2 = 2\pi \log 2 \cdot 2^k k \varepsilon_{m-k} \leqq 2\pi \log 2 \cdot 4^k \varepsilon_{m-k}.$$

Wenn $4^k \varepsilon_{m-k} \leqq 1$, $k \leqq \dfrac{\log \dfrac{1}{\varepsilon_{m-k}}}{\log 4}$ wäre, so wäre auch dieser Teil beschränkt. Nehmen wir $k \leqq \dfrac{m}{2}$, d.h. $\varepsilon_{m-k} \leqq \varepsilon_{\left[\frac{m}{2}\right]}$ an. Die Ungleichung für k wird daher

$$0 \leqq k \leqq \min\left(\left[\frac{\log \dfrac{1}{\varepsilon_{\left[\frac{m}{2}\right]}}}{\log 4}\right], \left[\frac{m}{2}\right]\right) \stackrel{\text{def}}{=} N_m - 1.$$

Wir haben also N_m $(\to \infty)$ Punkte $t = t_k$

$$\frac{2\pi}{2^m}, \frac{2\pi}{2^m}\cdot 2, \frac{2\pi}{2^m}\cdot 4, \ldots, \frac{2\pi}{2^m} 2^{N_m - 1}$$

mit

$$\frac{F_1(\sigma_m)}{|F_1(\sigma_m + it_k)|} \leqq c_{67}, \qquad |F_1(\sigma_m + it_k)| \geqq \frac{F_1(\sigma_m)}{c_{67}} \geqq \frac{1}{c_{67}} \frac{c_{63}}{\sigma_m - 1} = \frac{c_{68}}{\sigma_m - 1}.$$

Da die logarithmische Ableitung auf der ganzen Gerade $\operatorname{Re} s = \sigma_m$ $O\left(\dfrac{1}{\sigma_m - 1}\right)$ ist (s. (44)), so besteht diese Ungleichung mit einem kleineren $c_{69} < c_{68}$ auch in einer Umgebung von Radius $\eta(\sigma_m - 1) = \dfrac{\eta}{2^m}$ dieser Punkte auf Grund von

$$\left|\log\left|\frac{F_1(\sigma + it_2)}{F_1(\sigma + it_1)}\right|\right| = \left|\operatorname{Re} \int_{\sigma + it_1}^{\sigma + it_2} \frac{F_1'}{F_1}(s)\, ds\right| \leqq |t_2 - t_1| \max_{s \in (\sigma)} \left|\frac{F_1'}{F_1}(s)\right|.$$

Wenn wir $\eta = \pi$ wählen, so werden diese Umgebungen disjunkt, und da auch das größte t_k zu 0 strebt, aber gewiß kleiner ist als 2π, finden wir auf Grund des Beitrages dieser Umgebungen

$$\int_{\substack{(\sigma_m) \\ |t| \leqq 2\pi}} |F_1(s)|^2 |ds| \geqq N_m \left(\frac{c_{69}}{\sigma_m - 1}\right)^2 2\eta(\sigma_m - 1) = N_m \frac{c_{70}}{\sigma_m - 1} \neq O\left(\frac{1}{\sigma_m - 1}\right).$$

Wenn trotzdem $M_1(x) = \sum_{n \leqq x} f_1(n) = O(x)$ wäre, so würde die Parsevalsche Formel

$$\int_{(\sigma_m)} \left|\frac{F_1(s)}{s}\right|^2 |ds| = O\left(\frac{1}{\sigma_m - 1}\right)$$

implizieren, was jedoch unserer gegensätzlichen Ungleichung widerspricht, (da der Faktor $|s|$ auf dem endlichen $|t| \leqq 2\pi$ nicht zählt).

Dieses Beispiel $F_1(s)$ entspricht dem Hilfssatz 2. Um daraus ein $F_0(s)$ unseres Satzes zu gewinnen, müssen wir $F_1(\sigma)$ ein wenig verkleinern, was die guten Eigenschaften nicht beeinflußt. Lassen wir z.B. gewisse Blöcke $(x_{m-1}, x_m]$ weg, die mit Q bezeichnet werden. Da die Summe von $\frac{f_1(p)}{p}$ für einen Block $\log 2$ war, wird

$$\sum_{p \in Q} \frac{f_1(p)}{p} = +\infty, \qquad \sum_{p \in Q} \frac{f_1(p)}{p^\sigma} \to +\infty \qquad (\sigma \to 1+0),$$

wenn wir unendlich viele Blöcke weglassen, und mit der Definition

$$F_0(s) = \prod_{p \notin Q} \left(1 + \frac{1}{1!}\frac{f_1(p)}{p^s} + \frac{1}{2!}\frac{f_1^2(p)}{p^{2s}} + \ldots\right) =$$

$$= e^{\sum_{p \notin Q} \frac{f_1(p)}{p^s}} = e^{\sum_{p} \frac{f_1(p)}{p^s} - \sum_{p \in Q} \frac{f_1(p)}{p^s}} = F_1(s) e^{-\sum_{p \in Q} \frac{f_1(p)}{p^s}}$$

ist

$$F_0(\sigma) = F_1(\sigma) e^{-\sum_{p \in Q} \frac{f_1(p)}{p^\sigma}} = F_1(\sigma) o(1) = o\left(\frac{1}{\sigma - 1}\right).$$

Sorgen wir auf der anderen Seite dafür, daß die Anzahl der weggelassenen Blöcke $(x_{m-1}, x_m]$ unter den ersten m Blöcken $\log \sqrt[4]{N_m}$ nicht übersteigen soll. In diesem Fall ist

$$\sum_{p \in Q} \frac{f_1(p)}{p^{\sigma_m}} \leqq \sum_{\substack{p \leqq x_m \\ p \in Q}} \frac{f_1(p)}{p} + O(1) \leqq \log 2 \log \sqrt[4]{N_m} + O(1) \leqq \log \sqrt[4]{N_m}$$

und besteht die untere Abschätzung

$$|F_0(s)| = |F_1(s)| \left| e^{-\sum_{p \in Q} \frac{f_1(p)}{p^s}} \right| \geqq |F_1(s)| e^{-\sum_{p \in Q} \frac{f_1(p)}{p^{\sigma_m}}} \geqq |F_1(s)| e^{-\log \sqrt[4]{N_m}} = \frac{|F_1(s)|}{\sqrt[4]{N_m}}$$

für $\operatorname{Re} s = \sigma_m$. Daher folgt

$$\int\limits_{\substack{(\sigma_m) \\ |t| \leqq 2\pi}} |F_0(s)|^2 |ds| \geqq \frac{1}{\sqrt{N_m}} \int\limits_{\substack{(\sigma_m) \\ |t| \leqq 2\pi}} |F_1(s)|^2 |ds| \geqq \frac{1}{\sqrt{N_m}} N_m \frac{c_{70}}{\sigma_m - 1} = \sqrt{N_m} \frac{c_{70}}{\sigma_m - 1},$$

was ebenfalls $\neq O\left(\frac{1}{\sigma_m - 1}\right)$ ist und mit $M_0(x) = O(x)$ im Widerspruch steht. Qu.e.d.

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Literaturverzeichnis

[1] E. WIRSING, Das asymptotische Verhalten von Summen über multiplikative Funktionen. II, *Acta Math. Acad. Sci. Hung.*, **18** (1967), S. 411—467.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (3—4), (1968), pp. 405—412.

STANDARD-ULTRANTIFILTER IM VERBAND ALLER KOMPONENTEN EINER *l*-GRUPPE

Von

F. FIALA (Brno, Tschechoslowakei)

(*Vorgelegt von* L. RÉDEI)

Ist G eine l-Gruppe, $\Gamma(G)$ der Verband aller Komponenten in G, dann bezeichnet $\mathfrak{U}[\Gamma(G)]$ die Menge aller Ultraantifilter im Verband $\Gamma(G)$. Bezüglich der Terminologie verweisen wir auf 1. 1, ..., 1. 5. F. ŠIK hat in [4] in die Menge $\mathfrak{U}[\Gamma(G)]$ eine Topologie eingeführt und die Kompaktheit dieses Raumes bewiesen. Wir bezeichnen $\cup x = \bigcup_{K \in x} K$ für $x \in \mathfrak{U}[\Gamma(G)]$. Ist $\cup x \neq G$, nennen wir x den Standard-Ultraantifilter in $\Gamma(G)$. In [3] wurde eine Topologie in die Menge $\mathfrak{U}_s[\Gamma(G)]$ aller Standard-Ultraantifilter in $\Gamma(G)$ eingeführt und Eingenschaften dieses Raumes wurden untersucht.

In diesem Beitrag studiere ich ausführlicher einige topologische Eigenschaften des Raumes $\mathfrak{U}_s[\Gamma(G)]$. Der Absatz 1 umfasst Grundbegriffe und Hilfsbehauptungen. Der Satz 3. 3 bzw. 4. 1 zeigt einige notwendige und hinreichende Bedingungen für die Kompaktheit bzw. für die vollständige Regularität des Raumes $\mathfrak{U}_s[\Gamma(G)]$. Die Normalität wird im Absatz 5 untersucht. Im Absatz 2 studiere ich Beziehungen zwischen der durch die Topologie des Raumes $\mathfrak{U}[\Gamma(G)]$ induzierten Topologie in $\mathfrak{U}_s[\Gamma(G)]$ und der in [3] eingeführten Topologie in $\mathfrak{U}_s[\Gamma(G)]$.

1. Mit den Begriffen, die wir verwenden werden, kann man sich in den Arbeiten [1], [5], [7] ausführlicher bekannt machen. Wir führen hier nur die wichtigsten Definitionen an.

1. 1. Ist G eine l-Gruppe, dann nennen wir zwei Elemente $a, b \in G$ *disjunktiv*, wenn $|a| \wedge |b| = 0$ gilt (in Zeichen $a \delta b$). Nichtleere Untermengen $A, B \subseteq G$ nennen wir *disjunktiv*, wenn für beliebige Elemente $a \in A$, $b \in B$ die Beziehung $a \delta b$ gilt. Unter *dem disjunktiven Komplement* A' einer nichtleeren Untermenge $A \subseteq G$ verstehen wir die Menge aller Elemente aus G, die disjunktiv mit der Menge A sind. Für $A = \{a\}$ sei $\{a\}' = a'$, $(A')' = A''$.

1. 2. $\emptyset \neq A \subseteq G$ nennen wir *eine Komponente* in G, wenn eine solche Untermenge $\emptyset \neq B \subseteq G$ existiert, daß $B' = A$ gilt. Ist $a \in G$, dann nennen wir a'' bzw. a' *die Haupt-* bzw. *Dualhauptkomponente* in G. Gilt $A = B'$ und $B = A'$, so nennen wir A, B *komplementäre Komponenten*. Die Menge aller (Haupt-, Dualhaupt-) Komponenten in G bezeichnen wir $[\Pi(G), \Pi'(G)]$ $\Gamma(G)$ — kürzer $(\Pi, \Pi')\,\Gamma$.

Die Menge $\Gamma(G)$, *die durch die mengentheoretische Inklusion geordnet ist, bildet einen vollständigen Booleschen Verband, in dem* $\Pi(G)$ *und* $\Pi'(G)$ *Unterverbände sind* ([4], Teorema 1). *Infimum bestimmt man als mengentheoretischen Durchschnitt.*

1. 3. Unter *einem topologischen Raum* P verstehen wir einen Raum im Bourbakischen Sinne [2], d.h. für $A, B \subseteq P$ gilt: 1. $\overline{\emptyset} = \emptyset$, 2. $A \subseteq \overline{A}$, 3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$, 4. $\overline{\overline{A}} = \overline{A}$.

Wir werden uns auch mit *vollständig regulären* und *normalen Räumen* beschäftigen. Von diesen Räumen verlagen wir nicht, daß sie HAUSDORFFsch seien.

1. 4. Ist L ein oberer Halbverband, dann nennen wir eine nichtleere Untermenge $x \subseteqq L$ *Antifilter* (A. f.) in L, wenn sie folgende Eigenschaften hat:

1. Enthält L das größte Element 1, dann $1 \bar{\in} x$.
2. $a, b \in x \Rightarrow a \vee b \in x$.
3. $a \in x$, $c \leqq a \Rightarrow c \in x$.

Ist $a \in L$, $a \neq 1$, dann ist die Menge

$$x = \{b | b \in L,\ b \leqq a\}$$

ein A.f. in L, den wir *den durch das Element a erzeugten Haupt-A.f.* nennen. Ein maximaler A.f. in L heißt *Ultraantifilter* (U.a.f.). Die Menge aller U.a.f. in L bezeichnen wir mit $\mathfrak{A}(L)$. Ist L ein Verband von Untermengen einer Menge A und $x \in \mathfrak{A}(L)$, dann sei

$$\cup x = \bigcup_{K \in x} K.$$

Gilt $\cup x \neq A$, so sagen wir, daß x *ein Standard-Ultraantifilter* (Standard-U.a.f.) ist. Die Menge aller Standard-U.a.f. in L bezeichnen wir mit $\mathfrak{A}_s(L)$.

1. 5. In unseren Betrachtungen werden wir uns vor allem mit den Mengen $\mathfrak{A}(\Gamma)$ bzw. $\mathfrak{A}_s(\Gamma)$ aller U.a.f. bzw. aller Standard-U.a.f. im Verband Γ aller Komponenten einer l-Gruppe G und mit der Menge $\mathfrak{A}(\Pi')$ aller U.a.f. im Verband Π' aller Dualhauptkomponenten in G beschäftigen. In [4], IV, wurde eine Topologie in die Menge $\mathfrak{A}(\Gamma)$ derart eingeführt, daß alle Mengen der Form

$$\mathfrak{A}K = \{x | x \in \mathfrak{A}(\Gamma),\ K \in x\}$$

für $K \in \Gamma$ eine Basis Σ für offene Mengen bilden. Wir bezeichnen mit σ die so erhaltene Topologie und mit $(\mathfrak{A}(\Gamma), \sigma)$ den entsprechenden topologischen Raum.

1. 6. LEMMA. *Für $x \in \mathfrak{A}(\Gamma)$ bzw. für $x \in \mathfrak{A}(\Pi')$ gilt*

$$a \in \cup x \Leftrightarrow a' \bar{\in} x.$$

BEWEIS. [4], Lemma 1.

Aus dem vorhergehenden Lemma folgt die notwendige und hinreichende Bedingung dafür, daß $x \in \mathfrak{A}(\Gamma)$ standard ist ([5], 4. 12).

1. 7. LEMMA. *Für $x \in \mathfrak{A}(\Gamma)$ gilt*

$$x \in \mathfrak{A}_s(\Gamma) \Leftrightarrow x \cap \Pi' \neq \emptyset.$$

1. 8. Die Eigenschaft der Standard-U.a.f. aus 1. 7 wurde in [3], 1. 10, dazu benutzt, daß eine Topologie in die Menge $\mathfrak{A}_s(\Gamma)$ mit Hilfe der Basis Σ' für die offenen Mengen eingeführt wurde. Dabei bezeichnet Σ' das System aller Mengen in $\mathfrak{A}_s(\Gamma)$ der Form

$$\mathfrak{B}f' = \{x | x \in \mathfrak{A}_s(\Gamma),\ f' \in x\},$$

wobei $f' \in \Pi'$. Wir bezeichnen mit σ' die Topologie und mit $(\mathfrak{A}_s(\Gamma), \sigma')$ den topologischen Raum.

2. $G \neq 0$ sei eine l-Gruppe. Offenbar gilt $\mathfrak{A}_s(\Gamma) \subseteq \mathfrak{A}(\Gamma)$. Nach [6], 12. 6, gilt $\mathfrak{A}(\Gamma) = \mathfrak{A}_s(\Gamma)$ genau dann, wenn die l-Gruppe G eine schwache Einheit (d.h. ein solches Element f, daß $f' = 0$ gilt) enthält. Wir werden jetzt Eigenschaften der Untermenge $\mathfrak{A}_s(\Gamma)$ im topologischen Raum $(\mathfrak{A}(\Gamma), \sigma)$ untersuchen. Ferner vergleichen wir $\mathfrak{A}_s(\Gamma)$ als einen topologischen Unterraum im Raum $(\mathfrak{A}(\Gamma), \sigma)$ mit dem Raum $(\mathfrak{A}(\Gamma), \sigma')$.

2. 1. SATZ. $\mathfrak{A}_s(\Gamma)$ *ist eine offene Menge im Raum* $(\mathfrak{A}(\Gamma), \sigma)$.

BEWEIS. $x \in \mathfrak{A}_s(\Gamma)$ sei beliebig. Nach 1. 7 enthält x wenigstens eine Dualhauptkomponente f'. Dann gilt $x \in \mathfrak{A}f' \subseteq \mathfrak{A}_s(\Gamma)$.

2. 2. LEMMA. *Für* $K_1, K_2 \in \Gamma$ *sind folgende Bedingungen äquivalent:*
1. $K_1 \subseteq K_2$.
2. $\mathfrak{A}K_2 \subseteq \mathfrak{A}K_1$.

BEWEIS. [3], 3. 1.

2. 3. SATZ. $\mathfrak{A}_s(\Gamma)$ *ist eine dichte Untermenge im Raum* $(\mathfrak{A}(\Gamma), \sigma)$.

BEWEIS. $x \in \mathfrak{A}(\Gamma)$ sei beliebig. Wir zeigen, daß jede Umgebung des Punktes x ein $y \in \mathfrak{A}_s(\Gamma)$ enthält. $K \in x$ sei beliebig. Offenbar gilt $K \neq G$ und daraus $K' \neq 0$. Für $0 \neq f \in K$ gilt $G \neq f' \supseteq K$. Wir bezeichnen mit z den durch die Komponente f' erzeugten Haupt-A.f. Es sei y ein U.a.f. in Γ, der z enthält. Dann ist $y \in \mathfrak{A}_s(\Gamma)$, $y \in \mathfrak{A}f'$ und nach 2. 2 $\mathfrak{A}f' \subseteq \mathfrak{A}K$, d.h. $y \in \mathfrak{A}K$.

2. 4. FOLGERUNG. $\mathfrak{A}(\Gamma) \setminus \mathfrak{A}_s(\Gamma)$ *ist im Raum* $(\mathfrak{A}(\Gamma), \sigma)$ *eine abgeschlossene Grenzmenge.*

BEWEIS folgt aus 2. 1 und 2. 3.

2. 5. BEMERKUNG. Nach [5], 4. 12, ist $\mathfrak{A}(\Gamma) \setminus \mathfrak{A}_s(\Gamma) \neq \emptyset$ genau dann, wenn die l-Gruppe G keine schwache Einheit enthält. [3], 4. 10, gemäß ist der Verband Γ in diesem Falle unendlich. Eine Frage entsteht, wieviel U.a.f. in Γ existieren, die nicht standard sind. Es gibt im allgemeinen Fall mehr als einen solcher U.a.f., wie das Beispiel 2. 7 zeigt. Leider habe ich keine genaue Antwort gefunden.

2. 6. Ein duales Atom im Verband Γ heißt *maximale Komponente*. Eine l-Gruppe G heißt *vom Typ* α, wenn jede Komponente $\neq G$ in einer maximalen Komponente enthalten ist (d.h. wenn Γ ein dual atomarer Verband ist).

2. 7. BEISPIEL. G sei eine l-Gruppe vom Typ α, die keine schwache Einheit enthält. Der Verband $\Gamma(G)$ ist unendlich ([3], 4. 10), dual atomar und da er Boolescher Verband ist, ist er auch atomar mit unendlich vielen Atomen. Kein U.a.f. in $\Gamma(G)$, der alle Atome aus $\Gamma(G)$ enthält, ist standard ([3], 4. 10, Beweis $1 \Rightarrow 2$). Wir bezeichnen mit A die Menge aller Atome in $\Gamma(G)$. $K \in \Gamma(G)$ sei beliebig derart, daß die Beziehung $K_1 \subseteq K$ für unendlich viele Elemente $K_1 \in A$ gilt und $K_1 \subseteq K'$ für unendlich viele Elemente $K_1 \in A$ gilt. Es existiert ein U.a.f. x_K bzw. $x_{K'}$ in Γ, der die Menge $\{K\} \cup A$ bzw. $\{K'\} \cup A$ enthält. Offensichtlich gilt $x_K \neq x_{K'}$ und $x_K, x_{K'} \in \mathfrak{A}(\Gamma) \setminus \mathfrak{A}_s(\Gamma)$.

2. 8. Wir können eine Topologie in die Menge $\mathfrak{A}_s(\Gamma)$ auch so einführen, daß wir die Topologie σ aus $\mathfrak{A}(\Gamma)$ zur Menge $\mathfrak{A}_s(\Gamma)$ relativisieren. Wir bezeichnen den

topologischen Raum $(\mathfrak{A}_s(\Gamma), \sigma_s)$. $A \subseteqq \mathfrak{A}_s(\Gamma)$ sei eine beliebige Untermenge. Es folgt aus der Definition der Topologie σ in $\mathfrak{A}(\Gamma)$:

$$\sigma_s A = \{x | x \in \mathfrak{A}_s(\Gamma),\ K \in x \Rightarrow \text{ es existiert } y \in A \text{ so, daß } K \in y\},$$

d.h. $\sigma_s A = \{x | x \in \mathfrak{A}_s(\Gamma),\ x \subseteqq \bigcup_{y \in A} y\}$.

Aus der Definition der Topologie σ' in $\mathfrak{A}_s(\Gamma)$ folgt ([3], 1. 11)

$$\sigma' A = \{x | x \in \mathfrak{A}_s(\Gamma),\ f' \in x \Rightarrow \text{ es existiert } y \in A, \text{ so daß } f' \in y\},$$

d.h. $\sigma' A = \{x | x \in \mathfrak{A}_s(\Gamma),\ x \cap \Pi' \subseteqq \bigcup_{y \in A} y\}$

Durch den Vergleich bekommen wir

$$\sigma_s A \subseteqq \sigma' A,$$

d.h. die Topologie σ_s ist schwacher als die Topologie σ'. Wir bestimmen Bedingungen unter denen alle beide Topologien identisch sind. Zuerst aber notwendige Definitionen.

2. 9. L sei ein Verband. Ein A.f. in L heißt *einfach*, wenn für ihn

$$a, b \in L,\ a \wedge b \in x \Rightarrow a \in x \quad \text{oder} \quad b \in x$$

gilt. Nach [5], Corolario 4. 4, *ist jeder U.a.f. in Γ ein einfacher A.f. und umgekehrt.*

2. 10. LEMMA. *L sei ein distributiver Verband mit einem größten Element. Ist $a, b \in L$, $b \nleqq a$, dann existiert ein einfacher A.f. x in L, so daß $a \in x$, $b \bar{\in} x$.*

BEWEIS. [3], 2. 3.

2. 11. Wir sagen, daß für ein $x \in \mathfrak{A}(\Gamma)$ *die Bedingung* (p) *gilt*, wenn für jedes $K \in x$ ein $f' \in \Pi'$ mit den Eigenschaften $f' \in x$, $K \subseteqq f'$ existiert.

2. 12. SATZ. *Ist G eine l-Gruppe, dann sind folgende Bedingungen äquivalent:*
1. *Für jedes $x \in \mathfrak{A}_s(\Gamma)$ gilt die Bedingung* (p).
2. *Für $A \subseteqq \mathfrak{A}_s(\Gamma)$ gilt $\sigma_s A = \sigma' A$, d.h. $(\mathfrak{A}_s(\Gamma), \sigma_s) = (\mathfrak{A}_s(\Gamma), \sigma')$.*

BEWEIS. Vgl. [3], 3. 5.

$1 \Rightarrow 2$. $x \in \sigma' A$ sei beliebig. Daraus folgt, daß $\mathfrak{V} f' \cap A \neq \emptyset$ für jedes $f' \in x$ gilt. Der U.a.f. x ist standard. Wir zeigen, daß für jedes $\mathfrak{A}K \in \Sigma$, für welches $x \in \mathfrak{A}K$ gilt (d.h. $K \in x$), $\mathfrak{A}K \cap A \neq \emptyset$ ist, also $x \in \sigma A$. $K \in x$ sei beliebig. Der Voraussetzung gemäß gilt die Bedingung (p) für den U.a.f. x. Daraus folgt, daß eine Dualhauptkomponente g' mit den Eigenschaften $K \subseteqq g'$ und $g' \in x$ existiert. Aus der Definition der Basen Σ, Σ' und aus 2. 2 bekommen wir $\mathfrak{A}g' = \mathfrak{V}g' \subseteqq \mathfrak{A}K$. Daraus folgt: $\mathfrak{A}K \cap A \supseteqq \mathfrak{V}g' \cap A \neq \emptyset$, also $x \in \sigma_s A$. Es ist daher $\sigma' A \subseteqq \sigma_s A$. Nach 2. 8 ist immer $\sigma_s A \subseteqq \sigma' A$, also $\sigma_s A = \sigma' A$. Also gilt die Bedingung 2. [Ein anderer Beweis $1 \Rightarrow 2$. Wir setzen voraus, daß $\sigma_s A \subset \sigma' A$ gilt, d.h. daß ein $x \in \mathfrak{A}_s(\Gamma)$ so existiert, daß $x \in \sigma' A$, $x \bar{\in} \sigma_s A$ gilt. Daraus folgt, daß die Beziehung $\mathfrak{V} f' \cap A \neq \emptyset$ für jedes $f' \in x$ gilt und ein $K \in x$ derart existiert, daß $\mathfrak{A}K \cap A = \emptyset$ ist, also es existiert kein $g' \in x$, so daß $K \subseteqq g'$ gilt. Deshalb erfüllt x die Bedingung (p) nicht.]

$2\Rightarrow 1$. Wir setzen voraus, daß die Bedingung (p) für ein $x\in\mathfrak{A}_s(\Gamma)$ nicht gilt. Daraus folgt, daß ein $K\in x$ so existiert, daß für jedes $f'\supset K$ die Beziehung $f'\bar{\in}x$ gilt. $g'\in x$ sei beliebig. Der Voraussetzung gemäß ist $K\not\subseteq g'$. Nach 2. 10 existiert ein einfacher A.f. in $\Gamma(=\text{U.a.f.})$, wir bezeichnen ihn mit $y_{g'}$, für den $g'\in y_{g'}$, $\bar{\in}Ky_{g'}$ gilt. Wir setzen

$$A=\{y_{g'}|y_{g'}\in\mathfrak{A}_s(\Gamma),\ g'\in x\}.$$

Offensichtlich gilt $x\in\sigma' A$ und dabei ist $x\bar{\in}\sigma_s A$, weil $\mathfrak{A}K\cap A=\emptyset$ für $K\in x$ gilt. Daraus folgt $\sigma_s A\subset\sigma' A$, d.h. die Bedingung 2 gilt nicht.

3. Es wurde in [3], 3. 3, bewiesen, daß die Existenz einer schwachen Einheit in G eine hinreichende Bedingung für Kompaktheit des Raumes $(\mathfrak{A}_s(\Gamma),\sigma')$ ist. Wir zeigen jetzt, daß die Bedingung auch notwendig ist und wir fügen andere äquivalente Bedingungen hinzu.

3. 1. SATZ. *Es existiert eine l-Gruppe, für die der Raum* $(\mathfrak{A}_s(\Gamma),\sigma')$ *nicht kompakt ist.*

BEWEIS. Siehe das Beispiel 4. 14 in [3]. G sei die l-Gruppe aller reellen, auf dem Interval $\langle 0,1\rangle$ definierten Funktionen, so daß $f(x)\neq 0$ nur für endlich viele $x\in\langle 0,1\rangle$ besteht. G ist vom Typ α und enthält keine schwache Einheit. Dabei ist $\Gamma(G)$ unendlich und enthält unendlich viele duale Atome, daher ist die Menge $\mathfrak{A}_s(\Gamma(G))$ unendlich. Nach [3], 4. 12, ist jeder Standard-U.a.f. in $\Gamma(G)$ ein Haupt-A.f., also der Raum $(\mathfrak{A}_s(\Gamma(G)),\sigma')$ ist diskret. Aus der Überdeckung des Raumes $(\mathfrak{A}_s(\Gamma(G)),\sigma')$ mit allen isolierten Punkten kann man keine endliche Überdeckung auswählen, also ist der Raum $(\mathfrak{A}_s(\Gamma(G)),\sigma')$ nicht kompakt.

3. 2. FOLGERUNG. *Es existiert eine l-Gruppe, für die der Raum* $(\mathfrak{A}_s(\Gamma),\sigma_s)$ *nicht kompakt ist.*

BEWEIS. Auch im Raum $(\mathfrak{A}(\Gamma),\sigma)$ sind Haupt-U.a.f. isolierte Punkte ([3], 4. 4).

3. 3. SATZ. *In einer l-Gruppe G sind folgende Bedingungen äquivalent:*

1. *Der Verband* $\Pi'(G)$ *hat ein kleinstes Element.*
2. *Der Verband* $\Pi(G)$ *hat ein größtes Element.*
3. $\Pi(G)\cap\Pi'(G)\neq\emptyset$.
4. *Die l-Gruppe G enthält eine schwache Einheit.*
5. *Der Raum* $(\mathfrak{A}_s(\Gamma(G)),\sigma')$ *is kompakt.*

BEWEIS. $1\Rightarrow 2$. Hat der Verband $\Pi'(G)$ ein kleinstes Element, dann ist es die Null der l-Gruppe G ([4], Nota 3). $0'=G\in\Pi(G)$ ist offenbar das größte Element des Verbandes $\Pi(G)$.

$2\Rightarrow 3$. Hat $\Pi(G)$ ein größtes Element, dann ist es die ganze l-Gruppe G ([4], Nota 3). Offenbar gilt $0'=G\in\Pi'(G)$, also $G\in\Pi(G)\cap\Pi'(G)$, d.h. $\Pi(G)\cap\Pi'(G)\neq\emptyset$.

$3\Rightarrow 4$. Enthält die l-Gruppe G keine schwache Einheit, dann existiert wenigstens ein U.a.f. x in $\Gamma(G)$, der nicht standard ist ([6], 12. 6), also gilt $x\cap\Pi'(G)=\emptyset$ (1. 7). Jeder U.a.f. in $\Gamma(G)$ enthält genau eine von je zwei komplementären Komponenten ([5], Corolario 4. 4), deshalb $\Pi(G)\subseteq x$. Daraus folgt, daß $\Pi(G)\cap\Pi'(G)=\emptyset$. Folglich gilt die Bedingung 3 nicht.

$4\Rightarrow 5$. [3], 3. 3.

5⇒1. Wir werden Betrachtungen über die durch das System Σ' gebildete Überdeckung des Raumes $(\mathfrak{A}_s(\Gamma(G)), \sigma')$ anstellen. Der Voraussetzung gemäß existieren endlich viele Basenmengen $\mathfrak{B}f'_1, \mathfrak{B}f'_2, \ldots, \mathfrak{B}f'_n$, die den Raum $(\mathfrak{A}_s(\Gamma(G)), \sigma')$ überdecken. Mit anderen Worten: es existieren endlich viele Dualhauptkomponenten $f'_1, f'_2, \ldots, f'_n$, von denen wenigstens eine in einem beliebigen U.a.f. aus $\mathfrak{A}_s(\Gamma(G))$ enthalten ist. Wir bezeichnen

$$f' = \bigwedge_{i=1}^{n} f'_i.$$

Offenbar gilt $f' \in x$ für jedes $x \in \mathfrak{A}_s(\Gamma(G))$, d.h. $\mathfrak{B}f' = \mathfrak{A}_s(\Gamma(G))$. Wir zeigen, daß f' das kleinste Element in $\Pi'(G)$ ist. Es existiere umgekehrt $g' \in \Pi'(G)$, so daß $f' \not\equiv g'$. Nach 2. 10 existiert ein einfacher A.f. y in $\Gamma(G)$ derart, daß $g' \in y, f' \bar{\in} y$ gilt. Der Bemerkung in 2. 9 gemäß ist y ein U.a.f. in $\Gamma(G)$ und offensichtlich gilt $y \in \mathfrak{A}_s(\Gamma(G))$. Dabei ist $f' \bar{\in} y$, was im Widerspruch dazu steht, daß jeder Standard-U.a.f. in $\Gamma(G)$ f' enthält. f' ist also das kleinste Element in $\Pi'(G)$.

3. 4. Unter einer r-Gruppe verstehen wir eine l-Gruppe, die eine Realisierung hat. In [5], 4.15, wurde die *Γ-Realisierung* G einer r-Gruppe $\mathfrak{G}$ definiert. In [3], 2. 1, wurden bewiesen, daß *der durch die Γ-Realisierung induzierte topologische Raum* (M, G) ([5], I) *mit dem topologischen Raum* $(\mathfrak{A}_s(\Gamma), \sigma')$ *homöomorph ist.* Daraus folgt die folgende Behauptung.

3. 5. SATZ. *$\mathfrak{G}$ sei eine r-Gruppe, $G = (G_x | x \in M)$ ihre Γ-Realisierung. Dann sind die Bedingungen* 1—5 *des Satzes* 3. 3 *mit der folgenden Bedingung äquivalent:*
6. *Der Raum (M, G) ist kompakt.*

4. In [3], 2. 16, wurden hinreichende Bedingungen für die vollständige Regularität des Raumes $(\mathfrak{A}_s(\Gamma), \sigma')$ bestimmt. Der folgende Satz zeigt, daß sie auch notwendige Bedingungen sind.

4. 1. SATZ. *In einer l-Gruppe sind folgende Bedingungen äquivalent:*
1. *Für jedes $x \in \mathfrak{A}_s(\Gamma)$ ist $x \cap \Pi'$ ein U.a.f. in Π'.*
2. *Jede Menge $\mathfrak{B}f' \in \Sigma'$ ist offen und abgeschlossen.*
3. *Der Raum $(\mathfrak{A}_s(\Gamma), \sigma')$ ist vollständig regulär.*

BEWEIS. 1⇔2⇒3. [3], 2. 16.

3⇒1. Wir zeigen das indirekt. Gilt 1 nicht, dann existiert wenigstens ein U.a.f. $x \in \mathfrak{A}_s(\Gamma)$ derart, daß $x \cap \Pi'$ kein U.a.f. in Π' ist, d.h. es existiert $f' \in \Pi', f' \bar{\in} x$ derart, daß für ein beliebiges $g' \in x$ $f' \vee g' \neq G$ ist. Nach [3], 2. 16, ist die Menge $\mathfrak{B}f'$ nicht abgeschlossen. y sei ein beliebiger die Menge $\{f', x \cap \Pi'\}$ enthaltender U.a.f. in Γ. Offenbar gilt $y \in \mathfrak{B}f'$. Wir setzen

$$A = \mathfrak{A}_s(\Gamma) \setminus \mathfrak{B}f'.$$

A ist abgeschlossen und $y \bar{\in} A$. φ sei eine beliebige reelle auf der Menge $(\mathfrak{A}_s(\Gamma), \sigma')$ definierte Funktion derart, daß die folgenden Bedingungen gelten: $0 \leqq \varphi(t) \leqq 1$ für $t \in \mathfrak{A}_s(\Gamma)$, $\varphi(y) = 0$, $\varphi^1(A) = 1$. Weil $x \in A$ ist, gilt $\varphi(x) = 1$. Wir zeigen, daß φ nicht stetig im Punkt x ist. Wir wählen $\varepsilon = \frac{1}{2}$. Zur $\frac{1}{2}$-Umgebung der Zahl 1 (in Zeichen $0_{\frac{1}{2}}$) existiert keine Umgebung $\mathfrak{B}g'$ des Punktes x, so daß $\varphi^1(\mathfrak{B}g') \subseteqq 0_{\frac{1}{2}}$, denn aus der Definition des U.a.f. y folgt, daß $y \in \mathfrak{B}g'$ für jedes $g' \in x$ gilt. Daraus

folgt $\varphi(y)=0\in\varphi^1(\mathfrak{B}g')\not\subseteq 0_{\frac{1}{2}}$. Deshalb ist die Funktion φ im Punkt x nicht stetig und $(\mathfrak{A}_s(\Gamma), \sigma')$ ist kein vollständig regulärer Raum. Also gilt die Bedingung 3 nicht.

4. 2. FOLGERUNG. *$\mathfrak{G}$ sei eine r-Gruppe, $G=(G_x|x\in M)$ ihre Γ-Realisierung. Dann sind die Bedingungen* 1—3 *des Satzes* 4. 1 *mit der folgenden Bedingung äquivalent:*

4. *Der Raum (M, G) ist vollständig regulär.*

BEWEIS. Nach [3], 2. 1, sind die topologischen Räume (M, G) und $(\mathfrak{A}_s(\Gamma), \sigma')$ homöomorph.

5. Wir werden jetzt Bedingungen suchen, unter denen der Raum $(\mathfrak{A}_s(\Gamma), \sigma')$ normal ist.

5. 1. LEMMA. *$f'\,g'\in\Pi'$ seien beliebig. Folgende Bedingungen sind äquivalent:*
1. $\mathfrak{B}f'\cap\mathfrak{B}g'=\emptyset$.
2. $f'\vee g'=G$.

BEWEIS. 1⇒2. Wir setzen vor, daß 2 nicht gilt, d.h. daß $f'\vee g'\neq G$ ist. y sei ein U.a.f. in Γ, der die Komponente $f'\vee g'$ enthält. Offensichtlich ist $y\in\mathfrak{B}f'\cap\mathfrak{B}g'$, also gilt Bedingung 1 nicht. Damit ist die Implikation 1⇒2 bewiesen.

2⇒1. Aus der Voraussetzung folgt, daß jeder U.a.f. in Γ, der f' enthält, g' nicht enthält und umgekehrt, also $\mathfrak{B}f'\cap\mathfrak{B}g'=\emptyset$.

5. 2. SATZ. *G sei eine l-Gruppe. Dann sind folgende Bedingungen äquivalent:*
1. *Der Raum $(\mathfrak{A}_s(\Gamma), \sigma')$ ist normal.*
2. *Zu je zwei abgeschlossenen Mengen $A, B\subseteq(\mathfrak{A}_s(\Gamma), \sigma')$, $A\cap B=\emptyset$, existieren Untermengen $\Pi'_*(A)\subseteq\bigcup_{x\in A}x$, $\Pi'_*(B)\subseteq\bigcup_{y\in B}y$ in Π' mit den Eigenschaften:*

a) *Für jede $x\in A$, $y\in B$ gilt*

$$x\cap\Pi'_*(A)\neq\emptyset,\ y\cap\Pi'_*(B)\neq\emptyset.$$

b) *Für jede $f'\in\Pi'_*(A)$, $g'\in\Pi'_*(B)$ gilt $f'\vee g'=G$.*

BEWEIS. 1⇒2. Ist $(\mathfrak{A}_s(\Gamma), \sigma')$ normal, so existieren zu je zwei abgeschlossenen Mengen $A, B\subseteq(\mathfrak{A}_s(\Gamma), \sigma')$, $A\cap B=\emptyset$, solche offene Mengen $A_1, B_1\subseteq(\mathfrak{A}_s(\Gamma), \sigma')$, daß $A\subseteq A_1$, $B\subseteq B_1$, $A_1\cap B_1=\emptyset$ gilt. Zu jeder offenen Menge $C\subseteq(\mathfrak{A}_s(\Gamma), \sigma')$ existiert $\Pi'_c\subseteq\Pi'$ derart, daß $C=\bigcup_{f\in\Pi'_C}\mathfrak{B}f'$. Also ist

$$A_1=\bigcup_{f'\in\Pi'_{A_1}}\mathfrak{B}f',\qquad B_1=\bigcup_{g'\in\Pi'_{B_1}}\mathfrak{B}g'.$$

Wir bezeichnen $\Pi'_*(A)=\Pi'_{A_1}\cap(\bigcup_{x\in A}x)$, $\Pi'_*(B)=\Pi'_{B_1}\cap(\bigcup_{y\in B}y)$. $x\in A$ sei beliebig. Dann ist $x\in A_1$. Daraus folgt, daß ein $f'_1\in\Pi'_{A_1}$ so existiert, daß $x\in\mathfrak{B}f'_1$, d.h. daß $f'_1\in x$. Offenbar ist $f'_1\in\Pi'_*(A)$, also $f'_1\in\Pi'_*(A)\cap x$, d.h. $\Pi'_*(A)\cap x\neq\emptyset$. Analog zeigt man, daß für beliebiges $y\in B$ die Beziehung $\Pi'_*(B)\cap y\neq\emptyset$ gilt. Sind ferner $f'\in\Pi'_*(A)$, $g'\in\Pi'_*(B)$ beliebig, so gilt $\mathfrak{B}f'\subseteq A_1$, $\mathfrak{B}g'\subseteq B_1$, also $\mathfrak{B}f'\cap\mathfrak{B}g'=\emptyset$. Nach 5. 1 gilt $f'\vee g'=G$. Wir haben damit die Bedingung 2 bewiesen.

$2 \Rightarrow 1$. $A, B \subseteqq (\mathfrak{A}_s(\Gamma), \sigma')$ seien beliebige abgeschlossene Mengen derart, daß $A \cap B = \emptyset$. Wir bezeichnen

$$A^* = \bigcup_{f' \in \Pi'_*(A)} \mathfrak{B}f', \qquad B^* = \bigcup_{g' \in \Pi'_*(B)} \mathfrak{B}g'.$$

A^* und B^* sind offen und dabei gilt $A \subseteqq A^*$, $B \subseteqq B^*$ (das folgt aus der Eigenschaft a)) und $A^* \cap B^* = \emptyset$ (das folgt aus der Eigenschaft b) und aus 5.1). $(\mathfrak{A}_s(\Gamma), \sigma')$ ist also normal.

Wir geben noch einige hinreichende Bedingungen für Normalität des Raumes $(\mathfrak{A}_s(\Gamma), \sigma')$ an.

5. 3. LEMMA. *Folgenden Bedingungen sind in einer l-Gruppe G äquivalent:*

1. *$(\mathfrak{A}_s(\Gamma), \sigma')$ ist ein T_1-Raum.*
2. *Für $x, y \in \mathfrak{A}_s(\Gamma)$, $x \neq y$, sind die Mengen $\cup x$, $\cup y$ durch die mengentheoretische Inklusion unvergleichbar.*
3. *Für jedes $x \in \mathfrak{A}_s(\Gamma)$ gilt $x \cap \Pi' \in \mathfrak{A}(\Pi')$, und x ist der einzige U.a.f. in Γ, der $x \cap \Pi'$ enthält.*
4. *Der topologische Raum $(\mathfrak{A}_s(\Gamma), \sigma')$ ist Hausdorffsch.*

BEWEIS. [3], 2. 18.

5. 4. FOLGERUNG. *G sei eine l-Gruppe, die eine schwache Einheit enthält. Es gelte eine der Bedingungen* 1—4 *des Lemmas* 5. 3. *Dann ist der Raum $(\mathfrak{A}_s(\Gamma), \sigma')$ normal.*

BEWEIS. Enthält G eine schwache Einheit, so ist der Raum $(\mathfrak{A}_s(\Gamma), \sigma')$ kompakt (3. 3) und falls eine der Bedingungen des Lemmas 5. 3 gilt, dann ist der Raum Hausdorffsch, also normal.

5. 5. FOLGERUNG. *G sei eine l-Gruppe mit einer schwachen Einheit. Es gelte für jedes $x \in \mathfrak{A}_s(\Gamma)$ die Bedingung* (p). *Dann ist der Raum $(\mathfrak{A}_s(\Gamma), \sigma')$ normal.*

BEWEIS. Gilt die Bedingung (p) für jedes $x \in \mathfrak{A}_s(\Gamma)$, so gelten die Bedingungen 1—4 des Lemmas 5. 3 ([3], 2. 26). Die Behauptung folgt aus 5. 4.

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Literaturverzeichnis

[1] G. BIRKHOFF, *Lattice theory* (New York, rev. ed., 1948).
[2] N. BOURBAKI, *Topologie générale* (Actual. Sci. Ind. 858, Paris, 1940).
[3] F. FIALA, Über einen gewissen Ultraantifilterraum, *Math. Nachr.*, **33** (1967), S. 231—249.
[4] F. ŠIK, Compacidad de ciertos espacios de utlraantifiltros, *Mem. Fac. Cie. Univ. Habana 1, ser. mat.*, fasc. 1° (1963), S. 19—25.
[5] F. ŠIK, Estructura y realisaciones de grupos reticulados I, II, *Mem. Fac. Cie. Univ. Habana 1. ser. mat.*, fasc. 2° und 3° (1964), S. 1—29.
[6] F. ŠIK, Struktur und Realisierungen von Verbandsgruppen V, *Math. Nachr.*, **33** (1967), S. 221—229.
[7] F. ŠIK, Über subdirekte Summen geordneter Gruppen, *Czechoslovak Math. J.*, **10** (**85**) (1960), S. 400—424.

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Tomus 19 (3—4), (1968), pp. 413—435.

ON SOME PROBLEMS OF A STATISTICAL GROUP-THEORY. IV

By

P. ERDŐS and P. TURÁN (Budapest), members of the Academy

1. In this paper we shall continue the statistical investigation of S_n, the symmetric group of n letters. Let P be a generic element of S_n, and $\mathbf{O}(P)$ its order. In the first paper of this series (see [1]) we proved that the relation*

$$\log \mathbf{O}(P) = \left(\frac{1}{2}+o(1)\right)\log^2 n \tag{1.1}$$

holds in S_n, apart from $o(n!)$ P's. This was refined in [2] to a „logarithmic-Gaussian" distribution. The interest of these results is clearly shown by the theorem of E. LANDAU (see [3]) according to which

$$\lim_{n\to\infty} \frac{\max_{P\in S_n} \log \mathbf{O}(P)}{\sqrt{n\log n}} = 1. \tag{1.2}$$

These theorems can obviously be reformulated in terms of the orders of all cyclic subgroups of S_n. In this setting it is natural to raise the same question for the *pairwise nonisomorphic* cyclic subgroups of S_n. So we have two problems.

I. What is the maximum number of the pairwise nonisomorphic cyclic subgroups of S_n?

II. Does there exist a „sharp almost all theorem" for their order too?

These problems are in turn equivalent to the problems, how many *different* values can $\mathbf{O}(P)$ assume and whether or not these values show a behaviour analogous to (1.1). We shall answer these problems by the following two theorems.

THEOREM I. *The number $W(n)$ of different values of $\mathbf{O}(P)$ in S_n is*

$$\exp\left\{\frac{2\pi}{\sqrt{6}}\sqrt{\frac{n}{\log n}}+O\left(\frac{\sqrt{n}\log\log n}{\log n}\right)\right\}. \tag{1.3}$$

THEOREM II. *„Almost all" of the possible different $\mathbf{O}(P)$-values (with $o(W(n))$ exceptions at most) are of the form*

$$\exp\left\{(1+o(1))\frac{\sqrt{6}\log 2}{\pi}\sqrt{n\log n}\right\}. \tag{1.4}$$

* Here and in what follows the o and O-signs refer to $n\to+\infty$.

These theorems seem to us of interest for more than one reason. Comparing (1. 3) with the „exact" interval for $\mathbf{O}(P)$ furnished by (1. 2) it turned out that $W(n)$ is very small* and hence there must be „very big" intervals within

$$I: 1 \leqq x \leqq \exp\left\{(1+o(1))\sqrt{n \log n}\right\} \stackrel{\text{def}}{=} M, \tag{1. 5}$$

containing no $\mathbf{O}(P)$-values; this means the nonexistence of certain cyclic subgroups in S_n. The *explicit* determination of such intervals seems to us to be of interest. Since $\frac{\sqrt{6}}{\pi} \log 2 \sim 0{,}5404$ the value in (1. 4) is „essentially"

$$M^{0,5404}$$

which is very large compared to the value (1. 1) which is „essentially" only

$$\exp\{2(\log \log M)^2\}.$$

This fact gives obviously an interest to the problem for which m is the number of P's with $\mathbf{O}(P)=m$ maximal and what is the value of this maximum.

For the study of the distribution of the $\mathbf{O}(P)$-values it is of interest to study mean-values of $\mathbf{O}(P)$. We have found that

$$M_1(n) \stackrel{\text{def}}{=} \frac{1}{n!} \sum_{P \in S_n} \mathbf{O}(P) < \exp\left\{c_1 \sqrt{\frac{n}{\log n}}\right\} \tag{1. 6}$$

where c_1 — and later $c_2, c_3, \ldots$ — stand for positive numerical, explicitely calculable constants.** Owing to (1. 2) and the quick increase of the function $\exp\left(\sqrt{x \log x}\right)$ one had the guess for a much larger $M_1(n)$-value. Since we do not have at present an *asymptotically exact* formula for $M_1(n)$ we shall postpone the treatment of this problem to another occasion.

2. The statistical point of view leads to transparent laws in different sort of questions too. Such questions are e.g. the study of conjugacy-classes and commutability of elements of S_n. So — denoting the total number of conjugacy classes of S_n by $V(n)$ — we assert the

THEOREM III. *The elements P of S_n — with exception of the elements of $o(V(n))$ conjugacy-classes — are commutable exactly with*

$$\exp\left\{(1+o(1))\frac{\sqrt{6}}{4\pi}\sqrt{n}\log^2 n\right\} \tag{2. 1}$$

P's.

As an application of this theorem DR. J. DÉNES has communicated us orally the following remarkable theorem.

* A trivial upper bound for $W(n)$ is the number of divisors of $n!$. But this is $> \prod_{\frac{n}{2} < p \leqq n} 2 = 2^{\pi(n)-\pi\left(\frac{n}{2}\right)}$ which is much bigger than the expression in (1.3).

** If some constant depends on certain parameters, this will be always explicitely stated.

For all P's — with exception of the elements of $o(V(n))$ conjugacy classes — the „general commutator-equation"

$$XYX^{-1}Y^{-1}=P \quad (X,\ Y\in S_n) \tag{2.2}$$

has

$$\exp\left\{(1+o(1))\frac{\sqrt{6}}{2\pi}\sqrt{n}\log^2 n\right\} \tag{2.3}$$

solutions.

Dr. Dénes intends to publish his proof elsewhere.

For the „special commutator equation"

$$XYX^{-1}Y^{-1}=E \quad (E \text{ unitelement}) \tag{2.4}$$

the number of solutions* is the number of commutable pairs in S_n. As to this we found with Vera T. Sós that this number is

$$n!p(n), \tag{2.5}$$

$p(n)$ being the number of partitions of n. More generally we assert the

Theorem IV. *The number of commutable (a, b) pairs (with the convention of the footnote) from an arbitrary group G of order N is Nk where k stands for the number of conjugacy classes in G.*

We shall deduce two corollaries from it.

Corollary I. *In an arbitrary group of order N the number of commutable pairs (with the above convention) is at least** $N\log\log N$. By other words a finite group cannot be „too non-commutative".*

Taking into account the Hardy—Ramanujan asymptotical formula

$$p(n)\sim\frac{1}{4n\sqrt{3}}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right) \tag{2.6}$$

for the number of partition of n one can see from (2. 5) at once that Corollary I will no more be true if $N\log\log N$ is replaced by

$$N\exp\left(A\sqrt{\frac{\log N}{\log\log N}}\right), \qquad A>\frac{2\pi}{\sqrt{6}}.$$

Nevertheless it would be of interest to improve the lower bound in Corollary I or even determine the minimum number of conjugacy classes in groups of order N. The second corollary which is an immediate consequence of (2. 6) (and (2. 5)) is the

Corollary II. *The probability that a random pair (P_1, P_2) of S_n commute tends with $1/n$ rapidly to 0.*

* (X, Y) and (Y, X) are counted as different solutions if $X\neq Y$.

** The logarithm is meant here with the base 2.

Since Theorem IV and its Corollary I are not of statistical nature we postpone their proofs to Appendix I. For the same reason we postpone the solution of the following group theoretical extremal problem to Appendix II:

Determine all „least commutable" P's, i.e. the P's which commute with the minimal number of P's of S_n.

The solution is given by the

THEOREM V. *The P's with the required minimum property are exactly those whose canonical cycle-decomposition consists of two cycles with the length* 1 *and* $(n-1)$ *respectively, if only* $n \geqq 6$.

3. We shall also deal statistically with a different sort of questions. All groups G of order $\leqq n$ can be embedded into S_n; but it is an important longstanding question to determine for each G the minimal g such that G can be embedded into S_g. Here one can hope a simple law for g in a statistical sense only. What we can do at present is to present such a law for commutative groups with order $\leqq n$. Denoting the total number of such groups by $G(n)$ this is given by

THEOREM VI. *If* $\psi(n) \nearrow \infty$ *so that*

$$\lim_{x \to \infty} \frac{\log \psi(x)}{\log x} = 0 \tag{3.1}$$

then all commutative groups of order $\leqq n$, *with the exception of* $o(G(n))$ *such groups at most, can be embedded into* S_l *with*

$$l = \left[\frac{n}{\psi(n)}\right]. \tag{3.2}$$

We shall also show that the theorem is best-possible in the sense that choosing

$$L \overset{\text{def}}{=} [n^{1-\delta}]$$

with an arbitrary small $\delta > 0$, we show that more than $c_2(\delta)G(n)$ Abelian groups of order $\leqq n$ are not embeddable into S_L.

The theorems as well as the proofs of this paper are directly or indirectly connected with partition-problems, even with some which would be considered by nonarithmeticians in themselves rather weird and artificial. By this connection the partition-problems cannot be considered anymore to be an isolated playground for arithmeticians, since the search of analogous statistical laws for other „big" groups will *systematically* lead to such problems. Part of the partition problems revelant here are of the type that for what kind of partitions is it true that „almost all" of them consist of $\Phi(n)(1+o(1))$ summands, $\Phi(n)$ depending on the type of partitions under consideration. Such results for some special kind of partitions has been found by ERDŐS and LEHNER (see [4]); the method used here for another special type of partitions seems to be extendable to a *general* class of partition-problems. However, in this paper we shall confine ourselves to the case we actually need here.

4. Now we turn to the proofs of our assertions. For the proof of Theorem I we shall need some lemmata.

LEMMA I. *If for* $x>0$

$$f(x) = \prod_{q \text{ prime}} (1+e^{-qx})$$

then we have for* $x \to +0$

$$\log f(x) = \frac{\pi^2}{12x \log \frac{1}{x}} + O\left(\frac{1}{x} \frac{\log\log \frac{1}{x}}{\log^2 \frac{1}{x}}\right).$$

For the proof we write

$$\log f(x) = \int_0^\infty \log(1+e^{-xr})\, d\pi(r) = x \int_0^\infty \frac{\pi(r)}{1+e^{-xr}}\, dr$$

where $\pi(r)$ denotes the number of primes not exceeding r. Hence

$$\log f(x) = x \int_2^\infty \frac{\operatorname{Li} r}{1+e^{-xr}}\, dr + O(1) \int_2^\infty x \frac{re^{-\sqrt{\log r}}}{1+e^{-xr}}\, dr + O(1). \tag{4.1}$$

Splitting the second integral into

$$\int_2^{x^{-\frac{1}{2}}} + \int_{x^{-\frac{1}{2}}}^{x^{-1}\log^2 x^{-1}} + \int_{x^{-1}\log^2 x^{-1}}^\infty$$

one can easily see that this is

$$O\left(\frac{1}{x}\right) \exp\left(-\frac{1}{2}\sqrt{\log \frac{1}{x}}\right)$$

and also

$$\log f(x) = \int_2^\infty \frac{\log(1+e^{-xr})}{\log r}\, dr + O\left(\frac{1}{x}\right) \exp\left(-\frac{1}{2}\sqrt{\log \frac{1}{x}}\right). \tag{4.2}$$

Splitting the remaining integral into

$$\int_2^{x^{-1}\log^{-10} x^{-1}} + \int_{x^{-1}\log^{-10} x^{-1}}^{10x^{-1}\log x^{-1}} + \int_{10x^{-1}\log x^{-1}}^\infty \tag{4.3}$$

the first and third integrals in (4. 3) are evidently

$$O\left(\frac{1}{x \log^{10} \frac{1}{x}}\right);$$

* The asymptotic part of this lemma is implicitely contained in the paper of HARDY and RAMANUJAN [5], p. 130. For our aims however the remainder term is quite essential, whereas the paper of Hardy and Ramanujan contains no remainder term at all.

replacing in the second $\log r$ by $\log \frac{1}{x}$ the error is

$$O\left(\frac{\log\log\frac{1}{x}}{x\log^2\frac{1}{x}}\right).$$

Hence

$$\log f(x) = \frac{1}{\log\frac{1}{x}}\int_0^\infty \log(1+e^{-xr})\,dr + O\left(\frac{\log\log\frac{1}{x}}{x\log^2\frac{1}{x}}\right),$$

from which Lemma I follows at once.

5. Further we shall need the

LEMMA II. *If for* $x>0$

$$F(x) = \prod_{q \text{ prime}} (1+e^{-qx}+e^{-q^2x}+e^{-q^3x}+\ldots)$$

then we for $x \to +0$

$$\log F(x) = \frac{\pi^2}{12x\log\frac{1}{x}} + O\left(\frac{\log\log\frac{1}{x}}{x\log^2\frac{1}{x}}\right).$$

For the proof we write

$$\log F(x) = \log f(x) + \sum_q \log\left\{\frac{e^{-q^2x}+e^{-q^3x}+\ldots}{1+e^{-qx}}+1\right\}.$$

Hence

$$|\log F(x) - \log f(x)| \leqq \sum_q \log\{1+e^{-q^2x}+e^{-q^3x}+\ldots\} \leqq$$

$$\leqq \sum_q \sum_{\nu=2}^\infty \exp(-q^\nu x) = (1-e^{-x})\sum_{n=0}^\infty e^{-nx}\left(\sum_{q^\nu \leqq n,\ \nu\geqq 2}{}^{q,\nu} 1\right) =$$

$$= O(x)\sum_{n=0}^\infty \sqrt{n}\,e^{-nx} = O\left(\frac{1}{\sqrt{x}}\right)$$

which, together with Lemma I proves Lemma II.

6. The proof of Theorem I will easily be completed connecting Lemma II with the

LEMMA III. *Let for* $x>0$ *be*

$$h(x) = \sum_{n=0}^\infty a_n e^{-nx}, \qquad a_n \geqq 0$$

and for $x \to +0$ *with numerical positive* A

$$\log h(x) = \frac{A}{x \log \frac{1}{x}} \left\{ 1 + O\left(\frac{\log \log \frac{1}{x}}{\log \frac{1}{x}} \right) \right\}. \tag{6.1}$$

*Then**

$$S_N = \sum_{n \leqq N} a_n = \exp\left\{ 2 \sqrt{2A \frac{N}{\log N}} + O\left(\frac{\sqrt{N} \log \log N}{\log N} \right) \right\}.$$

For the proof let

$$x_0 = \frac{\lambda}{\sqrt{N \log N}} \tag{6.2}$$

λ positive constant, to be determined. Then

$$\log h(x_0) = \frac{2A}{\lambda} \sqrt{\frac{N}{\log N}} \left\{ 1 + O\left(\frac{\log \log N}{\log N} \right) \right\}. \tag{6.3}$$

Hence

$$\exp\left(-\lambda \sqrt{\frac{N}{\log N}} \right) S_N \leqq \sum_{n \leqq N} a_n e^{-n x_0} \leqq h(x_0) =$$

$$= \exp\left\{ \frac{2A}{\lambda} \sqrt{\frac{N}{\log N}} \left(1 + O\left(\frac{\log \log N}{\log N} \right) \right) \right\}$$

i. e.

$$S_N \leqq \exp\left\{ \left(\lambda + \frac{2A}{\lambda} \right) \sqrt{\frac{N}{\log N}} + O\left(\frac{\sqrt{N} \log \log N}{\log^{3/2} N} \right) \right\}.$$

Choosing

$$\lambda = \sqrt{2A} \tag{6.4}$$

we got already

$$S_N \leqq \exp\left\{ 2 \sqrt{2A \frac{N}{\log N}} + O\left(\frac{\sqrt{N} \log \log N}{\log^{3/2} N} \right) \right\} \tag{6.5}$$

which is the second part of Lemma III (slightly stronger). Further if $\delta = \delta(N) \to 0$ will be determined later, we write

$$\exp\left\{ \sqrt{2A \frac{N}{\log N}} \left(1 + O\left(\frac{\log \log N}{\log N} \right) \right) \right\} = h(x_0) = \tag{6.6}$$

$$= \sum_{n \leqq (1-\delta)N} + \sum_{(1-\delta)N \leqq n \leqq (1+\delta)N} + \sum_{(1+\delta)N < n \leqq 100N} +$$

$$+ \sum_{n > 100N} a_n e^{-n x_0} \stackrel{\text{def}}{=} S_1 + S_2 + S_3 + S_4.$$

* The Tauberian theorem differs again from that of HARDY—RAMANUJAN (l.c. [5]) containing in hypothesis as well as in assertion remainder terms, which is quite essential for our later aims. Besides in formula (5.281) 1. c. the factor $\frac{2\pi}{\sqrt{3}}$ seems to be replaced by $\frac{2\pi}{\sqrt{6}}$.

As to S_4 (6. 5) gives easily

$$(6.7)\quad S_4 < \int_{100N}^{\infty} \exp\left\{\sqrt{2A}\left(2\sqrt{\frac{y}{\log y}}-\frac{y}{\sqrt{N\log N}}\right)+c_2\frac{\sqrt{y}\log\log y}{\log^{3/2} y}\right\}dy = o(1).$$

For S_3 we get

$$(6.8)\quad S_3 \leqq 100N\exp\left\{c_3\frac{\sqrt{N}\log\log N}{\log^{3/2} N}\right\}\exp\left\{\sqrt{\frac{2A}{\log N}}\max_{y\leqq(1+\delta)N}\left(2\sqrt{y}-\frac{y}{\sqrt{N}}\right)\right\}.$$

Since for all sufficiently large N's

$$\max_{y\leqq(1+\delta)N}\left(2\sqrt{y}-\frac{y}{\sqrt{N}}\right) = \sqrt{N}\left\{2\sqrt{1+\delta}-(1+\delta)\right\} =$$

$$= \sqrt{N}\left\{1-\frac{\delta^2}{4}+O(\delta^3)\right\} < \sqrt{N}\left(1-\frac{\delta^2}{5}\right)$$

(6. 8) gives

$$(6.9)\quad S_3 < \exp\left\{\sqrt{2A\frac{N}{\log N}}\left(1-\frac{\delta^2}{5}\right)+O\left(\frac{\sqrt{N}\log\log N}{\log^{3/2} N}\right)\right\}.$$

As to S_1 (6. 5) gives

$$S_1 < N\exp\left\{c_3\frac{\sqrt{N}\log\log N}{\log^{3/2} N}\right\}\exp\left\{\sqrt{2A}\max_{y\leqq(1-\delta)N}\left(2\sqrt{\frac{y}{\log y}}-\frac{y}{\sqrt{N\log N}}\right)\right\}.$$

One can easily see that the last maximum is

$$(6.10)\quad \sqrt{\frac{N}{\log N}}\left\{2\sqrt{1-\delta}-(1-\delta)\right\}+O\left(\frac{\sqrt{N}}{\log^{3/2} N}\right) =$$

$$= \sqrt{\frac{N}{\log N}}\left\{1-\frac{\delta^2}{4}+O(\delta^3)\right\}+O\left(\frac{\sqrt{N}}{\log^{3/2} N}\right) < \left(1-\frac{\delta^2}{5}\right)\sqrt{\frac{N}{\log N}}+c_4\frac{\sqrt{N}}{\log^{3/2} N}.$$

Hence choosing

$$(6.11)\quad \delta = \sqrt{\frac{\log\log N}{\log N}}$$

(6. 6), (6. 7), (6. 9), (6. 10) and (6. 11) give for all sufficiently large N's

$$S_2 > \exp\left\{\sqrt{2A\frac{N}{\log N}}\left(1-c_5\frac{\log\log N}{\log N}\right)\right\}$$

and a fortiori

$$\exp\left\{-\sqrt{2A\frac{(1-\delta)N}{\log N}}\right\}\left(\sum_{(1-\delta)N<m\leqq(1+\delta)N} a_n\right) >$$

$$> \exp\left\{\sqrt{2A\frac{N}{\log N}} - c_6\frac{\sqrt{N}\log\log N}{\log^{3/2} N}\right\}$$

and — replacing N by $\dfrac{N}{1+\delta}$ —

$$S_N > \exp\left\{2\sqrt{2A\frac{N}{\log N}} - c_7\sqrt{\frac{N}{\log N}}\,\delta - c_6\frac{\sqrt{N}\log\log N}{\log^{3/2} N}\right\}$$

which completes the proof of Lemma III.

7. Now we can turn to the proof of Theorem I. Let

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}, \qquad 2 \leqq p_1 < p_2 < \dots < p_r \tag{7.1}$$

be an $\mathbf{O}(P)$-value; the canonical cycle-decomposition of this P should consist of m_ν cycles of length n_ν $(\nu = 1, 2, \dots, k)$ i.e.

$$1 \leqq n_1 < n_2 \dots < n_k, \tag{7.2}$$

$$m_\nu \geqq 1, \qquad \nu = 1, 2, \dots, k, \tag{7.3}$$

$$\sum_{\nu=1}^{k} m_\nu n_\nu = n \tag{7.4}$$

so that

$$\mathbf{O}(P) = [n_1, n_2, \dots, n_k] = N. \tag{7.5}$$

Since each $p_\nu^{\alpha_\nu}$ is a factor of some n_j and

$$n_1 + n_2 \leqq n_1 n_2 \quad \text{for integer} \quad n_1 \geqq 2,\ n_2 \geqq 2 \tag{7.6}$$

we have

$$p_1^{\alpha_1} + p_2^{\alpha_2} + \dots + p_r^{\alpha_r} \leqq n_1 + n_2 + \dots + n_k \leqq \sum_{j=1}^{k} m_j n_j = n, \tag{7.7}$$

$$p_1 < p_2 < \dots < p_r.$$

Hence to each such $\mathbf{O}(P)$-value we make correspond uniquely a set of powers of distinct primes with the sum $\leqq n$.

But conversely, having any sum of prime-powers

$$q_1^{\beta_1} + q_2^{\beta_2} + \dots + q_l^{\beta_l} \leqq n \tag{7.8}$$

with

$$2 \leqq q_1 < q_2 < \dots < q_l, \ q_l \text{ primes} \tag{7.9}$$

the number $q_1^{\beta_1}q_2^{\beta_2}\dots q_l^{\beta_l}$ is the order of a P_1; such a P_1 is furnished by any permutations of S_n consisting of l cycles with cycle-lengths $q_1^{\beta_1}, q_2^{\beta_2}, \dots, q_l^{\beta_l}$ respectively and

$$n-\sum_{j=1}^{l} q_j^{\beta_j}$$

further cycles of length 1. Hence the different $\mathbf{O}(P)$-values are the $q_1^{\beta_1}q_2^{\beta_2}\dots q_l^{\beta_l}$-numbers extended to all systems satisfying (7. 8)—(7. 9). Hence $W(n)$ is identical with the number of solutions of (7. 8)—(7. 9).

Now the number of solutions of (7. 8)—(7. 9) is evidently $\sum_{\nu \leqq n} a_\nu$ where for $x>0$

$$\sum_{\nu=0}^{\infty} a_\nu e^{-\nu x} = F(x) = \prod_{q \text{ prime}} (1+e^{-qx}+e^{-q^2x}+e^{-q^3x}+\dots).$$

Application of Lemma II and III completes the proof of Theorem I.

Owing to the grouptheoretical connection it would be desirable to obtain better approximations to $W(n)$ than the one given by our Theorem I. The Tauberian theorems of Ingham (see INGHAM [6]) would certainly lead to a better result but the verification of its assumptions would be much more laborious than the way we have chosen. The same holds on the saddle point technique of ROTH—SZEKERES (see [7]).

8. For the proof of Theorem II we shall need two further lemmata.

LEMMA IV. *For* $x \to +0$ *we have*

$$\sum_{q \text{ prime}} \frac{1}{e^{qx}+1} = \frac{\log 2}{x \log \frac{1}{x}} + O\left(\frac{\log\log\frac{1}{x}}{x \log^2 \frac{1}{x}}\right).$$

For the proof we write

$$\sum_{q} \frac{1}{e^{qx}+1} = \int_0^\infty \frac{d\pi(r)}{e^{xr}+1}.$$

Analogously as in Lemma I this is

$$\frac{1}{\log\frac{1}{x}} \int_0^\infty \frac{dr}{e^{xr}+1} + O\left(\frac{\log\log\frac{1}{x}}{x \log^2 \frac{1}{x}}\right) = \frac{\log 2}{x \log \frac{1}{x}} + O\left(\frac{\log\log\frac{1}{x}}{x \log^2 \frac{1}{x}}\right)$$

indeed.

9. The next lemma is the crucial one. Let us consider all solutions of the system (7. 8)—(7. 9). Then we assert that the number of summands follows a rather strong statistical law. More exactly we assert the

LEMMA V. *For almost all* solutions of the system* (7. 8)—(7. 9) *the inequality*

$$l = \frac{2\sqrt{6}}{\pi} \log 2 \sqrt{\frac{n}{\log n}} + O\left(\sqrt{n} \log^{-0,73} n\right) \tag{9. 1}$$

holds.

For the proof we define

$$H_{mn} = \sum_{\substack{p_1^{\alpha_1} + p_2^{\alpha_2} + \ldots + p_m^{\alpha_m} \leqq n \\ p_1 < p_2 < \ldots < p_m}} 1 \tag{9. 2}$$

Then we have for $x > 0$, y real

$$G(y, x) \overset{\text{def}}{=} \sum_{\mu, \nu} A_{\mu\nu} e^{-\mu y - \nu x} = \prod_{q \text{ prime}} \{1 + e^{-y-qx} + e^{-y-q^2x} + \ldots\} \tag{9. 3}$$

and we have to investigate

$$S_{mn} = \sum_{\substack{\mu \leqq m \\ \nu \leqq n}} A_{\mu\nu} = \sum_{\mu \leqq m} H_{\mu n} \tag{9. 4}$$

where $m = m(n)$ will be determined later. Writing $G(y, x)$ in the form

$$G(y, x) = F(x) \prod_{q \text{ prime}} \left\{1 + \frac{(e^{-y} - 1)(e^{-qx} + e^{-q^2x} + \ldots)}{1 + e^{-qx} + e^{-q^2x} + \ldots)}\right\} \overset{\text{def}}{=} F(x) G_0(y, x); \tag{9. 5}$$

putting

$$x_1 = \frac{\pi}{\sqrt{6n \log n}}, \qquad y_1 = y_1(n) \to 0 \tag{9. 6}$$

to be determined, we get

$$S_{mn} \leqq \left(F(x_1) e^{nx_1}\right) \left(G_0(y_1, x_1) e^{my_1}\right). \tag{9. 7}$$

For the first factor on the right Lemma II gives the upper bound

$$\exp\left\{\frac{2\pi}{\sqrt{6}} \sqrt{\frac{n}{\log n}} + O\left(\frac{\sqrt{n} \log\log n}{\log^{3/2} n}\right)\right\}. \tag{9. 8}$$

The second factor in (9. 7) is owing to (9. 5)

$$\leqq \exp\left\{my_1 - \sum_{q \text{ prime}} \frac{(1 - e^{-y_1}) e^{-qx_1}}{1 + e^{-qx_1} + e^{-q^2x_1} + \ldots}\right\} = \exp y_1 \left\{m - \left(1 - \frac{y_1}{2}\right) \sum_{q \text{ prime}} \frac{1}{e^{qx_1} + 1} + \right.$$

$$\left. + \left(1 - \frac{y_1}{2}\right) \sum_{q \text{ prime}} \left(\frac{1}{e^{qx_1} + 1} - \frac{1}{e^{qx_1} + 1 + e^{(q-q^2)x_1} + e^{(q-q^3)x_1} + \ldots}\right)\right\}.$$

* i.e. with $o(W(n))$ exceptions at most, with the notation of Theorem I.

The last sum being

$$< \sum_{q \text{ prime}} \frac{e^{(q-q^2)x_1} + e^{(q-q^3)x_1} + \dots}{(e^{qx_1}+1)^2} < \sum_{q \text{ prime}} \sum_{\nu=2}^{\infty} \exp\left(-q^\nu \frac{x_1}{2}\right) = O\left(\frac{1}{\sqrt{x_1}}\right) = O(n \log n)^{1/4}$$

the last factor in (9. 7) has the upper bound

$$\exp y_1 \left\{ m - \left(1 - \frac{y_1}{2}\right) \sum_{q \text{ prime}} \frac{1}{e^{qx_1}+1} + O(n \log n)^{1/4} \right\}.$$

Using Lemma IV this is in turn

$$\leqq \exp y_1 \left\{ m - \left(1 - \frac{y_1}{2}\right) \frac{2\sqrt{6}\,\log 2}{\pi} \sqrt{\frac{n}{\log n}} + O\left(\frac{\sqrt{n}\,\log\log n}{\log^{3/2} n}\right) \right\}. \tag{9.9}$$

This together with (9. 7) and (9. 8) gives

$$S_{mn} \leqq \exp\left\{ \frac{2\pi}{\sqrt{6}} \sqrt{\frac{n}{\log n}} + c_7 \frac{\sqrt{n}\,\log\log n}{\log^{3/2} n} + \right.$$

$$\left. + y_1 \left(m - \frac{2\sqrt{6}\log 2}{\pi} \sqrt{\frac{n}{\log n}} \right) + \frac{y_1^2}{2} \frac{2\sqrt{6}\log 2}{\pi} \sqrt{\frac{n}{\log n}} \right\}.$$

Now choosing

$$m = \frac{2\sqrt{6}\log 2}{\pi} \sqrt{\frac{n}{\log n}} - \frac{\sqrt{n}}{\log^{0,73} n}, \qquad y_1 = \log^{-0,26} n \tag{9.10}$$

this gives for all sufficiently large n's

$$S_{mn} \leqq \exp\left\{ \frac{2\pi}{\sqrt{6}} \sqrt{\frac{n}{\log n}} - \frac{1}{2} \frac{\sqrt{n}}{\log^{0,99} n} \right\}. \tag{9.11}$$

10. We shall need also an upper bound for

$$S^*_{M,n} \overset{\text{def}}{=} \sum_{\substack{\mu \leqq M \\ \nu \leqq n}} A_{\mu\nu} = \sum_{\mu \leqq M} H_{\mu n} \tag{10.1}$$

$M = M(n)$ to be determined later. First let us observe that for $\nu \leqq n$ and $\mu > 3\sqrt{n}$

$$p_1^{\alpha_1} + p_2^{\alpha_2} + \dots + p_\mu^{\alpha_\mu} > \sum_{1 \leqq l \leqq 3\sqrt{n}} l > n$$

i. e.

$$S^*_{M,n} = \sum_{\substack{M \leqq \mu \leqq 3\sqrt{n} \\ \nu \leqq n}} A_{\mu\nu}. \tag{10.2}$$

We apply Cauchy's coefficient-estimation with

$$x = x_1 = \frac{\pi}{\sqrt{6n \log n}}, \qquad y = y_1 = -|y_1| \to 0$$

to be determined for $A_{\nu\mu}$; this gives, using (9. 5), (9. 7) and (9. 8) again

$$A_{\mu\nu} \leqq \exp\left\{\frac{2\pi}{\sqrt{6}}\sqrt{\frac{n}{\log n}}+O\left(\frac{\sqrt{n}\log\log n}{\log^{3/2} n}\right)\right\}\cdot \tag{10.3}$$

$$\cdot\exp\left\{-\mu|y_1|+(e^{|y_1|}-1)\frac{e^{-qx_1}+e^{-q^2x_1}+\ldots}{1+e^{-qx_1}}\right\}.$$

The second factor in (10.3) cannot exceed

$$\exp|y_1|\left\{-\mu+(1+|y_1|)\sum_{q\,\text{prime}}\frac{1}{e^{qx_1}+1}+O(n\log n)^{1/4}\right\}.$$

Using again Lemma IV this is

$$\leqq \exp\left\{|y_1|\left(-\mu+\frac{2\sqrt{6}\log 2}{\pi}\sqrt{\frac{n}{\log n}}\right)+c_8 y_1^2\sqrt{\frac{n}{\log n}}\right\}$$

and thus from (10. 2) and (10. 3)

$$S^*_{M,n} \leqq \exp\left\{\frac{2\pi}{\sqrt{6}}\sqrt{\frac{n}{\log n}}+c_9\frac{\sqrt{n}\log\log n}{\log^{3/2} n}+\right. \tag{10.4}$$

$$\left.+|y_1|\left(-M+\frac{2\sqrt{6}\log 2}{\pi}\sqrt{\frac{n}{\log n}}\right)+c_8 y_1^2\sqrt{\frac{n}{\log n}}\right\}.$$

Choosing now

$$M=\frac{2\sqrt{6}\log 2}{\pi}\sqrt{\frac{n}{\log n}}+\frac{\sqrt{n}}{\log^{0,73} n}, \tag{10.5}$$

$$y_1=-\log^{-0,26} n \tag{10.6}$$

we get

$$S^*_{M,n} \leqq \exp\left\{\frac{2\pi}{\sqrt{6}}\sqrt{\frac{n}{\log n}}-\frac{1}{2}\sqrt{n}\log^{-0,99} n\right\}. \tag{10.7}$$

Since from Theorem I we have

$$W(n)=\sum_{\substack{\mu,\nu\\ \nu\leqq n}} A_{\mu\nu} > \exp\left\{\frac{2\pi}{\sqrt{6}}\sqrt{\frac{n}{\log n}}-c_{10}\frac{\sqrt{n}\log\log n}{\log n}\right\}$$

this, (9. 11) and (10. 7) prove Lemma V.

11. Now we can turn to the proof of Theorem II.
As told all $\mathbf{O}(P)$-values are the numbers

$$q_1^{\beta_1} q_2^{\beta_2}\ldots q_l^{\beta_l} \tag{11.1}$$

satisfying (7. 8)—(7. 9). Owing to Lemma V with $o(W(n))$ exceptions also*

$$l = \frac{2\sqrt{6}\log 2}{\pi}\sqrt{\frac{n}{\log n}} + O\left(\sqrt{n}\log^{-0,73} n\right) \tag{11.2}$$

is satisfied. But we have always

$$l! \leqq q_1^{\beta_1} q_2^{\beta_2} \dots q_l^{\beta_l} < \left(\frac{q_1^{\beta_1} + \dots + q_l^{\beta_l}}{l}\right)^l \leqq \left(\frac{n}{l}\right)^l. \tag{11.3}$$

Now using (11. 2) we have on one hand

$$l! > \exp(l\log l - l) \geqq \exp\left\{\frac{\sqrt{6}\log 2}{\pi}(1-o(1))\sqrt{n\log n}\right\} \tag{11.4}$$

and on the other hand

$$\left(\frac{n}{l}\right)^l = \exp\left(l\log\frac{n}{l}\right) \leqq \exp\left\{\frac{\sqrt{6}\log 2}{\pi}(1+o(1))\sqrt{n\log n}\right\}$$

which prove Theorem II.

12. Next we turn to Theorem III. Let — as in Theorem IV — $p(n)$ stand for the number of the partitions of n; we characterise the partitions with the numbers m_ν, n_ν in section 7. According to the classical asymptotic formula of HARDY and RAMANUJAN (see [5]) we have

$$p(n) = (1+o(1))\frac{1}{4n\sqrt{3}}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right). \tag{12.1}$$

Let us denote by

$$\pi_1, \pi_2, \dots, \pi_{p(n)} \tag{12.2}$$

the partitions of n and define the „partition-function" $h(\pi_\nu)$ by

$$h(\pi_\nu) = \frac{m_1^2 n_1 + \dots + m_k^2 n_k}{m_1 + \dots + m_k}. \tag{12.3}$$

Then we need the

LEMMA VI. *For almost all partitions, i.e. with $o(p(n))$ exceptions at most, for $n > n_0$ the inequality*

$$h(\pi_\nu) \leqq \sqrt{n}\log^2 n \tag{12.4}$$

holds (the right side could be replaced by $c_{11}\sqrt{n}\log n$).

For the proof of this lemma we consider first the set Π of the π_ν-partitions with

$$\max_{j=1,2,\dots,k} m_j n_j > \sqrt{n}\log^2 n. \tag{12.5}$$

* The remainder term was of course only for Lemma V of vital importance.

Fixing m_j and n_j let $\Pi^{(m_j, n_j)}$ be the corresponding subset of Π and $|\Pi|$ the cardinality of Π. Then

$$|\Pi^{(m_j, n_j)}| \leqq p(n - m_j n_j) \leqq p\left(n - \sqrt{n}\log^2 n\right)$$

and from (12. 1)

$$< (1+o(1))\frac{1}{4\left(n-\sqrt{n}\log^2 n\right)\sqrt{3}} \exp\left\{\frac{2\pi}{\sqrt{6}}\sqrt{n-\sqrt{n}\log^2 n}\right\} <$$

$$< (1+o(1))\frac{1}{4n\sqrt{3}} \exp\left\{\frac{2\pi}{\sqrt{6}}\sqrt{n}\left(1-\frac{\log^2 n}{2\sqrt{n}}\right)\right\} = p(n)\,(1+o(1)) \exp\left\{-\frac{\pi}{\sqrt{6}}\log^2 n\right\}.$$

Since $\Pi = \bigcup_{m_j, n_j} \Pi^{(m_j, n_j)}$ and m_j, n_j has at most n values, we get

$$|\Pi| < p(n) 2n^3 \exp\left\{-\frac{\pi}{\sqrt{6}}\log^2 n\right\} = o\left(p(n)\right).$$

Hence with exception of $o\left(p(n)\right)$ π_ν's at most the inequality

$$\max_j m_j n_j \leqq \sqrt{n}\log^2 n \tag{12.6}$$

holds. For these partitions we have

$$h(\pi_\nu) \leqq \max_j m_j n_j \frac{\sum_j m_j}{\sum_j m_j} \leqq \sqrt{n}\log^2 n$$

indeed.

We shall need also the following theorem of ERDŐS—LEHNER (see [4]).

With the notation of section 7 for all but $o\left(p(n)\right)$ partitions the inequality

$$\left|(m_1 + \ldots + m_k) - \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n\right| \leqq \omega(n)\sqrt{n} \tag{12.7}$$

holds, if only $\omega(x) \nearrow \infty$ for $x \to \infty$.

I.e. almost all partitions consist of $(1+o(1))\dfrac{\sqrt{6}}{2\pi}\sqrt{n}\log n$ summands.

13. Now we can prove Theorem III as follows. Let us consider the conjugacy-classes of S_n; as well known P_1 and P_2 belong to the same class if and only if the cycles of their canonical cycle decomposition are pairwise identical in their number as well to their respective length. Hence a conjugacy-class $\mathfrak{A}$ is determined by the common

$$(n_1, n_2, \ldots, n_k;\ m_1, m_2, \ldots, m_k) \tag{13.1}$$

numbers of their P's. Thus first of all

$$V(n) = p(n). \tag{13.2}$$

i.e. the number of the conjugacy-classes of S_n equals to the number of partitions of n. Secondly the number of P's in the class $\mathfrak{A}$ is

$$\frac{n!}{m_1!\, m_2! \dots m_k!\, n_1^{m_1}\, n_2^{m_2} \dots n_k^{m_k}}, \tag{13.3}$$

owing to Cauchy's formula (see e.g. RIORDAN [8]). Fixing P_0 in the class $\mathfrak{A}$ let the centraliser of P_0 be $C(P_0)$. As well-known there is a one-to-one correspondence between the conjugates of P_0 and the cosets of $C(P_0)$ in S_n. Hence, with the notation of section 12 we have

$$|\mathfrak{A}| = \frac{|S_n|}{|C(P_0)|} \tag{13.4}$$

and thus the number of P's commutable with P_0 is owing to (13. 3)

$$|C(P_0)| = m_1!\, m_2! \dots m_k!\, n_1^{m_1}\, n_2^{m_2} \dots n_k^{m_k} \tag{13.5}$$

for all P_0's of the class $\mathfrak{A}$.

Now we remark that

$$\frac{(m_1+m_2+\dots+m_k)!}{|C(P_0)|} = \frac{(m_1+m_2+\dots+m_k)!}{m_1!\, m_2! \dots m_k!}\left(\frac{1}{n_1}\right)^{m_1}\left(\frac{1}{n_2}\right)^{m_2}\dots\left(\frac{1}{n_k}\right)^{m_k} < \tag{13.6}$$

$$< \left(\frac{1}{n_1}+\frac{1}{n_2}+\dots+\frac{1}{n_k}\right)^{m_1+\dots+m_k} \leqq \left(\sum_{\nu=1}^{k}\frac{1}{\nu}\right)^{m_1+\dots+m_k} < (1+\log k)^{m_1+\dots+m_k}.$$

Hence

$$|C(P_0)| > \frac{(m_1+\dots+m_k)!}{(1+\log n)^{m_1+\dots+m_k}} > \left(\frac{m_1+\dots+m_k}{2e\log n}\right)^{m_1+\dots+m_k}. \tag{13.7}$$

So far we made no restrictions on the conjugacy class $\mathfrak{A}$. Now we take into consideration Lemma VI, (12. 7) and (13. 2); these give that with exception of the P's of

$$o(V(n)) = o(p(n))$$

conjugacy-classes in the remaining set Π_1 of conjugacy-classes, both inequalities (12. 4) and (12. 7) hold. Hence for $P_0 \in \Pi_1$ (12. 7) and (13. 7) give (using also the inequality $l! > \left(\frac{l}{e}\right)^l$)

$$|C(P_0)| > \left(\frac{\sqrt{n}}{50}\right)^{(1-o(1))\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n} > \exp\left\{(1-o(1))\frac{\sqrt{6}}{4\pi}\sqrt{n}\log^2 n\right\},$$

which gives the first half of Theorem III.

On the other hand (13. 5) gives (using the inequality $l! \leqq l^l$ for all P_0's)

$$|C(P_0)| \leqq (m_1 n_1)^{m_1}(m_2 n_2)^{m_2} \dots (m_k n_k)^{m_k} \leqq \left(\frac{m_1^2 n_1 + \dots + m_k^2 n_k}{m_1+\dots+m_k}\right)^{m_1+\dots+m_k}.$$

Now restricting P_0 to Π_1 we may apply (12. 4) and (12. 7); thus

$$|C(P_0)| \leqq \left(\sqrt{n}\log^2 n\right)^{(1+o(1))\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n} = \exp\left\{(1+o(1))\frac{\sqrt{6}}{4\pi}\sqrt{n}\log^2 n\right\},$$

which completes the proof of Theorem III.

As a by-product we notice the following

COROLLARY. *All conjugacy-classes of* S_n, *with* $o(p(n))$ *exceptions at most, contain*

$$n!\exp\left\{-(1+o(1))\frac{\sqrt{6}}{4\pi}\sqrt{n}\log^2 n\right\}$$

P's.

14. Next we turn to theorem VI. We shall need the well-known theorem of ERDŐS—SZEKERES* (see [9]).

Denoting the number of pairwise nonisomorphic Abelian groups of order m by $k(m)$ we have

$$\sum_{m\leqq n} k(m) = A_0 n(1+o(1)), \qquad A_0 = \zeta(2)\zeta(3)\ldots, \tag{14. 1}$$

$\zeta(s)$ is the Riemann zeta function.

Taking this in account we have only to prove that (3. 2) is satisfied for all but $o(n)$ Abelian groups of order $\leqq n$.

We shall need the remark (which follows at once from the fundamental theorem of finite Abelian groups) that

$$k(m_1m_2)=k(m_1)k(m_2) \quad \text{if} \quad (m_1, m_2)=1. \tag{14. 2}$$

We shall denote by $z(m)$ the maximal prime-power divisor of m and let us fix a $\psi(n)$ with property (3. 1). Let M be the set of integers m not exceeding n with the property

$$z(m) > \frac{n}{2\psi(n)}. \tag{14. 3}$$

We shall need the

LEMMA VIII. *The inequality*

$$U = \sum_{m\in M} k(m) = o(n)$$

holds.

Let

$$z(m) = q^\alpha > \frac{n}{2\psi(n)}.$$

Then we have

$$m = q^\alpha m_1, \qquad (q^\alpha, m_1) = 1$$

* Their theorem furnishes also a remainder-term. However here — in contrary to the previous discussions — it is immaterial. Even $>O(n)$ is enough.

and hence — using also that $k(q^\alpha)=p(\alpha)$ and (14. 1) —

$$U=\sum_{\frac{n}{2\psi(n)}\leqq q^\alpha\leqq n} k(q^\alpha) \sum_{\substack{m_1\leqq \frac{n}{q^\alpha},\,(m_1,\,q)=1\\ z(m_1)<q^\alpha}} k(m_1)\leqq \sum_{\frac{n}{2\psi(n)}\leqq q^\alpha\leqq n} p(\alpha) \sum_{m_1\leqq \frac{n}{q^\alpha}} k(m_1)=O(n) \sum_{\frac{n}{2\psi(n)}\leqq q^\alpha\leqq n} \frac{p(\alpha)}{q^\alpha}.$$

(14. 4)

The contribution of $\alpha=1$ is $o(n)$ owing to

$$\sum_{\frac{n}{2\psi(n)}\leqq q\leqq n} \frac{1}{q}=o(1). \tag{14. 5}$$

Since roughly

$$\alpha<2\log n$$

and

$$p(\alpha)<ce^{2\sqrt{\alpha}}<ce^{4\sqrt{\log n}},$$

the contribution of $\alpha\geqq 2$ is

$$<O\left(ne^{4\sqrt{\log n}}\right)\sum_{2\leqq\alpha\leqq 2\log n}\ \sum_{m>\left(\frac{n}{2\psi(n)}\right)^{1/\alpha}} m^{-\alpha}< \tag{14. 6}$$

$$<O\left(ne^{4\sqrt{\log n}}\right)\left(\frac{n}{\psi(n)}\right)^{-\frac{1}{2}}\log n=o(n),$$

which completes the proof of lemma VIII.

Owing to this lemma it is enough to restrict ourselves to Abelian groups of order $m\leqq n$ with the property

$$z(m)\leqq\frac{n}{2\psi(n)}. \tag{14. 7}$$

15. Let m be such an integer and let

$$m=a_1a_2\ldots a_k\qquad(=q_1^{\beta_1}q_2^{\beta_2}\ldots q_s^{\beta_s}) \tag{15. 1}$$

be any decomposition of m into prime-powers. Let us consider

$$D(m)=a_1+a_2+\ldots+a_k. \tag{15. 2}$$

Since an easy induction gives for integers $b_j\geqq 2$

$$b_1+b_2+\ldots+b_l\leqq b_1b_2\ldots b_l,$$

we get

$$D(m)\leqq q_1^{\beta_1}+q_2^{\beta_2}+\ldots+q_s^{\beta_s}. \tag{15. 3}$$

Since only at most one of the $q_j^{\beta_j}$ can exceed $\sqrt{n}$, we get from (14. 7)

$$D(m)\leqq\frac{n}{2\psi(n)}+s\sqrt{n}\leqq\frac{n}{2\psi(n)}+\sqrt{n}\log n\leqq\left[\frac{n}{\psi(n)}\right]=l. \tag{15. 4}$$

Now take for $n > c_{12}$ any commutative group G of order $m \leqq n$ with (14. 7) of type $(a_1, a_2, \ldots, a_k)$. But then forming the permutations of $a_1 + \ldots + a_k \leqq l$ elements

$$(12\ldots a_1)^{v_1}(a_1+1, a_1+2, \ldots, a_1+a_2)^{v_2}\ldots(a_1+\ldots+a_{k-1}+1, \ldots, a_1+a_2+\ldots+a_k)^{v_k}$$

$$1 \leqq v_1 \leqq a_1,\ 1 \leqq v_2 \leqq a_2, \ldots, 1 \leqq v_k \leqq a_k$$

these form a group G^* isomorphic to G which is a subgroup of S_l indeed.

In order to prove that the theorem is no more true for $S_{[n^{1-\delta}]}$ with an arbitrarily small numerical positive δ we have only to remark that for a prime q the group G, the order of which is divisible by q, cannot be embedded into S_d with $d < q$ further that the number of integers $\leqq n$ divisible by a $q > n^{1-\delta}$ is

$$\sum_{n^{1-\delta} \leqq q \leqq n} \left[\frac{n}{q}\right] > n \log \frac{1}{1-\delta} - o(n)$$

and finally (14. 1).

APPENDIX I

As told we are going to give the proof of Theorem IV and its corollaries in this Appendix. Let us consider a fixed conjugacy-class Q_j of G containing $|Q_j|$ elements; let α be one of its elements and $C(\alpha)$ its centraliser, containing $|C(\alpha)|$ elements. Hence α commutes in G with $\left(\frac{N}{|Q_j|} - 1\right)$ elements, different from α. Since the same holds for all elements of Q_j, the total number of commutable (α, β) pairs with $\alpha \in Q_j$, $\alpha \neq \beta$ is

$$|Q_j|\left(\frac{N}{|Q_j|} - 1\right) = N - |Q_j|.$$

Summation for $j = 1, 2, \ldots, k$ gives

$$kN - \sum_{j=1}^{k} |Q_j| = kN - N.$$

This gives the total number of commutable (α, β)-pairs with $\alpha \neq \beta$; since we have N further commutable pairs (α, α), the proof of Theorem IV is finished.

In order to prove Corollary I we appeal to the following well-known theorem (see [10]).

Let $2 = \alpha_1 < \alpha_2 < \ldots$ be defined by the recursion

$$\alpha_{v+1} = \alpha_1 \alpha_2 \ldots \alpha_v + 1. \tag{I. 1}$$

If for a fixed v and positive integers $x_1, x_2, \ldots, x_v$

$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_v} < 1 \tag{I. 2}$$

then

$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_v} \leqq 1 - \frac{1}{\alpha_{v+1} - 1}. \tag{I. 3}$$

Next we apply the following reasoning of LANDAU (see [11]). With the above Q_j's we have

$$|Q_1| = 1 \quad \text{and for} \quad 2 \leqq j \leqq k \qquad |Q_j| = \frac{N}{y_{j-1}}, \qquad (y_l \text{ positive integers})$$

and hence

$$\frac{1}{y_1} + \frac{1}{y_2} + \ldots + \frac{1}{y_{k-1}} = 1 - \frac{1}{N}.$$

But then (I. 2) is fulfilled and (I. 3) is applicable with $v = k-1$. This gives

$$1 - \frac{1}{N} \leqq 1 - \frac{1}{\alpha_k - 1}, \quad \text{i. e.} \quad \alpha_k \geqq N+1. \tag{I. 4}$$

Now from (I. 1)

$$\frac{\alpha_{v+1} - 1}{\alpha_v - 1} = \alpha_v, \qquad \alpha_{v+1} = \alpha_v^2 - \alpha_v + 1 < \alpha_v^2$$

i. e.

$$\alpha_k \leqq 2^{2^k}.$$

Hence (I. 4) gives

$$2^{2^k} > N \quad \text{or} \quad k > \log\log N$$

which together with Theorem IV proves Corollary I indeed.

APPENDIX II

1. Now we prove Theorem V. According to (13. 5) the number of elements with which a fixed $P_0 \in S_n$ is commutable depends only upon the conjugacy class to which P_0 belongs; the extremal class is a class $Q = Q(n_1, n_2, \ldots, n_k; m_1, m_2, \ldots, m_k)$ for which

$$f(Q) \overset{\text{def}}{=} m_1!\, m_2! \ldots m_k!\, n_1^{m_1} n_2^{m_2} \ldots n_k^{m_k} \tag{II. 1}$$

is minimal under the restrictions (7. 2)—(7. 3)—(7. 4). Hence we have to show that the minimum is $(n-1)$ and the only extremal class Q_0 corresponds to

$$k = 2, \quad m_1 = m_2 = 1, \quad n_1 = 1, \quad n_2 = (n-1). \tag{II. 2}$$

First we remark that for

$$k \geqq 4, \qquad 1 \leqq a_1 < a_2 < \ldots < a_k \tag{II. 3}$$

the inequality

$$a_1 a_2 a_3 \ldots a_k \geqq a_1 + a_2 + \ldots + a_k + 14 \tag{II. 4}$$

holds. Namely the expression

$$\frac{1}{a_1 a_2 a_3} + \frac{1}{a_1 a_2 a_4} + \frac{1}{a_1 a_3 a_4} + \frac{1}{a_2 a_3 a_4}$$

attains its maximum for $a_1=1$, $a_2=2$, $a_3=3$, $a_4=4$, which is 5/12. Hence

$$1 \geqq \frac{1}{a_1 a_2 a_3} + \ldots + \frac{1}{a_2 a_3 a_4} + \frac{7}{12}.$$

Thus

$$a_1 a_2 a_3 a_4 \geqq a_1 + a_2 + a_3 + a_4 + 14\left(\frac{a_1}{1}\right)\left(\frac{a_2}{2}\right)\left(\frac{a_3}{3}\right)\left(\frac{a_4}{4}\right) \geqq a_1 + a_2 + a_3 + a_4 + 14$$

i.e. (II. 4) holds for $k=4$. If it holds for $k \leqq k_0$

$$a_1 a_2 \ldots a_{k_0} \geqq a_1 + a_2 + \ldots + a_{k_0} + 14$$

then multiplying by $a_{k_0+1}(>2)$

$$a_1 a_2 \ldots a_{k_0+1} \geqq a_1 + \ldots + a_{k_0} + 14 a_{k_0+1} =$$

$$= (a_1 + a_2 + \ldots + a_{k_0+1}) + 13 a_{k_0+1} > a_1 + a_2 + \ldots + a_{k_0+1} + 14$$

which proves (II. 4). One can see quite analogously that for

(II. 5) $$k \geqq 3, \quad 2 \leqq a_1 < a_2 < \ldots < a_k$$

the inequality

(II. 6) $$a_1 a_2 \ldots a_k \geqq a_1 + a_2 + \ldots + a_k + 14$$

holds.

For $k=1$ we have

$$f(Q) = n > n-1 = f(Q_0)$$

i.e. this class cannot be extremal. Putting aside temporarily the case $k=2$ we suppose

(II. 7) $$k \geqq 3$$

and hence

(II. 8) $$n_k \geqq 3.$$

Let now $Q^*(n_1^*, \ldots, n_k^*;\ m_1^*, \ldots, m_k^*)$ be an extremal-class. We assert that

(II. 9) $$n_\nu^* \geqq 3 \to m_\nu^* = 1,$$

i.e. a cycle-length $\geqq 3$ can occur at most once in an extremal-class. For if not, let n_j^* the longest cycle-length with $m_j^* \geqq 2$. This implies that if the cycle-length $m_j^* n_j^*$ occurs at all in Q^* and $= n_l^*$ then

(II. 10) $$m_l^* = 1.$$

Then we consider the class Q_1 which comes from Q^* contracting all cycles with length n_j^* into a single cycle (of length $m_j^* n_j^*$). In the case (II. 10)

(II. 11) $$\frac{f(Q_1)}{f(Q^*)} = \frac{2!(m_j^* n_j^*)^2}{1!(m_j^* n_j^*) m_j^*! (n_j^*)^{m_j^*}} = \frac{2}{(m_j^*-1)!(n_j^*)^{m_j^*-1}} < 1$$

and in the case when the cycle-length $m_j^* n_j^*$ does not occur in Q^*

$$\frac{f(Q_1)}{f(Q^*)} = \frac{1!(m_j^* n_j^*)}{m_j^*!(n_j^*)^{m_j^*}} = \frac{1}{(m_j^*-1)!(n_j^*)^{m_j^*-1}} < 1 \tag{II. 12}$$

i.e. Q^* could not be an extremal class. Hence (II. 9) is proved.

Next we assert that

$$n_j^* = 1 \to m_v^* = 1 \tag{II. 13}$$

i.e. the cycle-length $n_1 = 1$ can occur in an extremal-class at most once. Obviously we may suppose

$$(3 \leqq) n_k^* \leqq n-1. \tag{II. 14}$$

Supposing

$$m_1^* \geqq 2$$

we consider the class Q_2 which arises from Q^* taking away one cycle of length 1 and replacing the cycle with length n_k^* by one of length (n_k^*+1). Then owing to (II. 14) and (II. 9) we have

$$\frac{f(Q_2)}{f(Q^*)} = \frac{(m_1^*-1)!\, 1^{m_1^*-1} 1!(n_k^*+1)}{m_1^*!\, 1^{m_1^*} \cdot 1!\, n_k^*} = \frac{n_k^*+1}{n_k^* \cdot m_1^*} \leqq \frac{4}{3} \cdot \frac{1}{2} < 1$$

and thus Q^* could be an extremal-class. Hence (II. 13) is proved.

(II. 11) and (II. 13) give at the same time that the cycle-length 2 can occur at most twice in an extremal-class and it could occur twice *only* if Q^* has (exactly) one cycle of length 4. If Q^* has no cycle with length 8, then replacing Q^* by Q_3 which contains a cycle of length 8, taking off the two cycles of length 2 and the one with length 4 we get

$$\frac{f(Q_3)}{f(Q^*)} = \frac{1!\,8^1}{2!\,2^2 \cdot 1!\,4^1} < 1; \tag{II. 15}$$

if Q^* has (exactly) one cycle of length 8 then

$$\frac{f(Q_3)}{f(Q^*)} = \frac{2!\,8^2}{2!\,2^2 \cdot 1!\,4^1 \cdot 1!\,8^1} < 1. \tag{II. 16}$$

Thus the cycle-length 2 can occur at most once too, i.e.

$$m_1^* = m_2^* = \ldots = m_k^* = 1$$

and

$$f(Q^*) = n_1^* n_2^* \ldots n_k^* \tag{II. 17}$$

with

$$n_1^* + n_2^* + \ldots + n_k^* = n, \qquad k \geqq 3$$

$$1 \leqq n_1^* < n_2^* < \ldots < n_k^*. \tag{II. 18}$$

If $k \geqq 4$ then (II. 4), (II. 17) and (II. 18) give

$$f(Q^*) \geqq n+14 > f(Q_0)$$

i.e. we may suppose

(II. 19) $$k=3.$$

(II. 5) gives at once that in this case

$$n_1=1$$

and hence

$$f(Q^*)=n_2n_3$$

$$2\leqq n_2<n_3, \quad n_2+n_3=n-1$$

for which for $n\geqq 6$

$$f(Q^*)\geqq 2(n-3)>(n-1),$$

i.e. Q^* cannot be extremal.

For the case $k=2$ the above reasoning can be repeated and gives that for $Q^*\neq Q_0$ $f(Q^*)>f(Q_0)$ which completes the proof.

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References

[1] P. ERDŐS and P. TURÁN, On some problems of a statistical group theory, I, *Zeitschr. für Wahrscheinlichkeitstheorie und verw. Gebiete*, **4** (1965) pp. 175—186.

[2] P. ERDŐS and P. TURÁN, On some problems of a statistical group theory. III, *Acta Math. Acad. Sci. Hung.*, **18** (1967), pp. 309—320.

[3] E. LANDAU, *Handbuch der Lehre von der Verteilung der Primzahlen*, I. (1909), p. 222.

[4] P. ERDŐS and J. LEHNER, The distribution of the number of summands in the partitions of a positive integer, *Duke Math. Journ.*, **8** (**2**) (1941), pp. 335—345.

[5] G. H. HARDY and S. RAMANUJAN, Asymptotic formulae for the distribution of integers of various types, *Proc. of Lond. Math. Soc.* (**2**) **16** (1917), pp. 117—132.

[6] A. E. INGHAM, A Tauberian theorem for partitions, *Annals of Math.*, **42** (1941), pp. 1075—1090.

[7] K. F. ROTH and G. SZEKERES, Some asymptotic formulae in the theory of partitions, *The Quart. Journ. of Math.*, Oxford Second Series, **5** (1954), pp. 241—259.

[8] J. RIORDAN, *An introduction to combinatorial analysis* (New York, 1958).

[9] P. ERDŐS and G. SZEKERES, Über die Anzahl der Abelschen Gruppen gegebener Ordnung, *Acta Litt. ac. Scient. Szeged*, **7** (1934), pp. 95—102.

[10] The series $\sum\frac{1}{\alpha_\nu}$ is ussually called Engel-series of second kind; for its properties and for a proof of (I. 2)—(I. 3) see e.g. P. ERDŐS, On the integer solutions of the equation $\frac{1}{x_1}+\ldots+\frac{1}{x_n}=\frac{a}{b}$ (in Hungarian), *Matematikai Lapok*, **I. 3** (1950), pp. 192—210.

[11] E. LANDAU, *Math. Annalen*, **56** (1903), pp. 674—678.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (3—4), (1968), pp. 437—440.

AN EXTREMAL PROBLEM

By

R. E. L. TURNER (Madison, Wisconsin, U.S.A.)

(*Presented by* P. TURÁN)

In the course of estimating the size of solutions of commutator equations in perturbation theory (cf. [1], p. 533) the following problem arises: consider a denumerable sequence of points $0=p_0<p_1<p_2\ldots<\infty$ on the real axis; let δ_i be the distance from p_i to its nearest neighbour and form the sum

$$M(P)=\sum_{k=1}^{\infty}\frac{\delta_0\delta_k}{p_k^2}, \tag{1}$$

P standing for the collection $\{p_1, p_2, \ldots\}$. How large can $M(P)$ be? Comparing the sum with an integral gives a bound of 2. We show here that $M(P)\leqq\pi^2/6$. If P consists of the positive integers, then $M(P)=\pi^2/6$, so the bound cannot be improved. We obtain the bound for infinite sets P by examining a sum $M_n(P)$ corresponding to (1) for finite sets $P=\{p_1, p_2, \ldots, p_n\}$. It is shown that for a fixed n, the configuration P for which $M_n(P)$ is a maximum consists of points which, starting with p_1, are equally spaced up to some point $p_{k(n)}$, and for points to the right of $p_{k(n)}$, have strictly increasing separation. Thus while the set P yielding the maximum of $M(P)$ exhibits the regularity one comes to expect of extremal configurations, the extrema for finite sets do not.

The sum $M_n(P)$ is homogeneous of degree zero in the p_i and hence we may assume $p_1=1$ (or $p_k-p_{k-1}=1$ for some k). It is clear that if $M_n(P)\leqq\pi^2/6$ for all n and P, then $M(P)\leqq\pi^2/6$. Allowing $p_i\leqq p_{i+1}$ one sees that

$$\mu_n=\text{l. u. b.}_P\, M_n(P)$$

is assumed by a configuration $\tilde{P}_n$ and that μ_n is strictly increasing with n. Since coincident points contribute zero to $M_n(P)$, however, $\tilde{P}_n$ must consist of n distinct points.

LEMMA 1. *Let* $\tilde{P}_n=\{p_1, p_2, \ldots, p_n\}$ *be a configuration for which* $M_n(P)$ *is a maximum. Then*

(i) $p_n-p_{n-1}=p_{n-1}$ *and if* $p_{j-1}-p_{j-2}<p_j-p_{j-1}<\ldots<p_n-p_{n-1}$ *for some* $j\geqq 3$, *we have* $p_{n-k}-p_{n-k-1}=r_k p_{n-k-1}$ *where* $r_0=1$ *and*

$$r_{k+1}=\frac{(r_k+1)^2-1}{(r_k+1)^2+1}$$

for $0\leqq k\leqq n-j$.

(ii)

$$\sum_{l=n-k}^{n} \frac{p_l p_{l-1}}{p_l^2} = \eta_k \frac{1}{p_{n-k-1}}$$

for $0 \leqq k \leqq n-j$, *where* $\eta_k = (r_k+1)^{-2}$.

PROOF. If $p_n - p_{n-1} < p_{n-1} - p_{n-2}$ and we change $\tilde{P}_n$ by moving p_n a distance x to the right, $M_n(P)$ will be a constant plus

$$(p_n + x - p_{n-1}) \left(\frac{1}{p_{n-1}^2} + \frac{1}{(p_n+x)^2} \right)$$

which has a derivative

$$\frac{1}{p_{n-1}^2} + \frac{2p_{n-1}}{p_n^3} - \frac{1}{p_n^2} > 0$$

at $x=0$, contradicting the maximality.

Thus, $p_n - p_{n-1} \geqq p_{n-1} - p_{n-2}$ and if $x \geqq 0$, replacing p_n by $p_n + x$ yields a derivative $(2p_{n-1} - p_n)p_n^{-3}$ for M_n at $x=0$, showing that $p_n - p_{n-1} = p_{n-1}$ for a maximum. In that case $(p_n - p_{n-1})p_n^{-2} = \eta_0 p_n^{-1}$.

Proceeding by induction, we assume the conclusions (i) and (ii) are true for $\{p_{n-k}, \ldots, p_n\}$ with $k < n-j$. Then

$$M_n = \sum_{i=1}^{n-k-2} \frac{\delta_i}{p_i^2} + \frac{p_{n-k-1} - p_{n-k-2}}{p_{n-k-1}^2} + \frac{1}{(r_k+1)^2} \cdot \frac{1}{p_{n-k-1}}$$

and its derivative with respect to p_{n-k-1} is

$$\frac{2p_{n-k-2}}{p_{n-k-1}^3} - \frac{1}{p_{n-k-1}^2} \left(1 + \frac{1}{(r_k+1)^2} \right).$$

For a maximum the derivative must be zero, or

$$p_{n-k-1} - p_{n-k-2} = \frac{(r_k+1)^2 - 1}{(r_k+1)^2 + 1} p_{n-k-2}$$

while

$$\sum_{l=n-k-1}^{n} \frac{p_l - p_{l-1}}{p_l^2} = \frac{r_{k+1} \cdot p_{n-k-2}}{(r_{k+1}+1)^2 p_{n-k-2}^2} + \eta_k \frac{1}{p_{n-k-1}} =$$

$$= \frac{1}{p_{n-k-2}} \left\{ \frac{r_{k+1}}{(r_{k+1}+1)^2} + \frac{1 - r_{k+1}}{1 + r_{k+1}} \cdot \frac{1}{r_{k+1}+1} \right\} = \eta_{k+1} \frac{1}{p_{n-k-2}}.$$

DEFINITION. We say that p_j has property L if δ_j is equal to the max $(r_{n-j} p_{j-1}, p_{j-1} - p_{j-2})$.

LEMMA 2. *Suppose the points* $\{p_j, p_{j+1}, \ldots p_n\}$ *have property* L.

(i) *If* $r_{n-m} p_{m-1} > p_{m-1} - p_{m-2}$ *for some* m *with* $j \leqq m < n$, *then* $r_{n-\tilde{m}} p_{\tilde{m}-1} > p_{\tilde{m}-1} - p_{\tilde{m}-2}$ *for* $\tilde{m} > m$.

(ii) *If* $\delta_m = \delta_{m+1}$ *for some* m *with* $j < m < n$, *then* $\delta_{\tilde{m}} = \delta_{\tilde{m}+1}$ *for* $j \leqq \tilde{m} < m$.

PROOF. (i) Since r_k decreases as k increases, we have $r_{n-m-1} p_m > r_{n-m} p_{m-1} = p_m - p_{m-1}$, using Lemma 1, property L and the hypothesis of (i). Thus the conclusion follows by induction.

(ii) If $\delta_{\tilde{m}} < \delta_{\tilde{m}+1}$ and $j \leqq \tilde{m} < m$, then $r_{n-\tilde{m}-1} p_{\tilde{m}} > p_{\tilde{m}} - p_{\tilde{m}-1}$ and from (i), $r_{n-m-1} p_m > p_m - p_{m-1} = \delta_m$. Then $\delta_{m+1} = r_{n-m-1} p_m > \delta_m$, contrary to assumption.

LEMMA 3. *Suppose* $\tilde{P}_n = \{p_1, p_2, \ldots, p_n\}$ *maximizes* $M_n(P)$. *Then the points* p_i $(i = 2, 3, \ldots, n)$ *have property* L.

PROOF. We know that p_n has property L. If there is a point not having the property then there is a first one, p_k, starting from the end p_n. We let $r_{n-k} p_{k-1} = R$ and $p_{k-1} - p_{k-2} = D$. Then one of the following three situations prevails:

(i) $\delta_k < D$.
(ii) $\delta_k > D$ and $\delta_k > R$.
(iii) $D \leqq \delta_k < R$.

We will show that in each case M_n can be increased, contradicting the hypothesis.

Case (i) $(\delta_k < D)$. Since p_{k+1} has property L, $\delta_{k+1} \geqq p_k - p_{k-1}$. If there is strict inequality, then by Lemmas 1 and 2, $\sum_{k+1}^{n} p_l^{-2} (p_l - p_{l-1}) = \eta_{n-k-1} p_k^{-1}$. Consider shifting p_k to the right a distance x and shifting $p_{k+1}, p_{k+2}, \ldots, p_n$ so that they retain the maximizing ratios. Then M_n will be a constant plus

$$(p_k + x - p_{k-1}) \left(\frac{1}{p_{k-1}^2} + \frac{1}{(p_k + x)^2} \right) + \eta_{n-k-1} \frac{1}{p_k + x}.$$

Using homogeneity, we assume $p_k - p_{k-1} = 1$, and then see that the derivative of the last expression at $x = 0$ is

$$\frac{1}{p_{k-1}^2} + \frac{1 - \eta_{n-k-1}}{(p_{k-1} + 1)^2} - \frac{2}{(p_{k-1} + 1)^3} > 0$$

which is impossible for a maximum.

If $\delta_{k+1} = p_k - p_{k-1} = 1$, then from property L it follows that the p_i $(i > k)$ will be equally spaced up to a point p_q, with distances to the right of p_q being determined by $\{r_k\}$. If the points $p_k, p_{k+1}, \ldots p_q$ are moved a distance x to the right and the points to the right of p_q are moved so as to preserve the maximizing ratios, then M_n will be a fixed term plus

$$S(x) = \frac{1 + x}{p_{k-1}^2} + \sum_{i=k}^{q} \frac{1}{(p_i + x)^2} + \eta_{n-q-1} \frac{1}{p_q + x}.$$

Using an integral to estimate the sum in the expression for the derivative $S'(0)$ one sees that $S'(0) > (1 - \eta_{n-q-1}) p_q^{-2} > 0$ and thus $\delta_k < D$ cannot occur.

Case (ii) $(\delta_k > D, \delta_k > R)$. We can proceed as in case (i), now moving p_k to $p_k + x$ with $x \leqq 0$, and shifting the points $p_{k+1}, \ldots, p_n$ to the left, maintaining the appropriate distances. If $p_{k+1} - p_k > p_k - p_{k-1}$, the change in M_n has a derivative

$$\frac{2p_{k-1} - p_k(1 + \eta_{n-k-1})}{p_k^3}$$

at $x = 0$. Then, since $r_{n-k} p_{k-1} = R < p_k - p_{k-1}$ and $r_{n-k} = (1 - \eta_{n-k-1})(1 + \eta_{n-k-1})^{-1}$ it follows that $2p_{k-1} < p_k(1 + \eta_{n-k-1})$, yielding a negative derivative and the possibility of increasing M_n.

If $p_{k+1}-p_k = p_k-p_{k-1}=1$, then as before we obtain a derivative

$$\frac{1}{p_k^2}-\sum_{i=k}^{q}\frac{2}{p_i^3}-\eta_{n-q-1}\frac{1}{p_q^2}$$

or, with $p_k=p$,

$$\frac{1}{p^2}-\sum_{i=k}^{q}\frac{2}{(p+i-k)^3}-\frac{\eta_{n-q-1}}{(p+q-k)^2}.$$

Since $r_{n-q}p_{q-1}\leqq p_{q-1}-p_{q-2}=1$, $r_{n-q}\leqq p_{q-1}^{-1}=(p_q-1)^{-1}$ and

$$\eta_{n-q-1}=\frac{1-r_{n-q}}{1+r_{n-q}}\geqq\frac{1-(p_q-1)^{-1}}{1+(p_q-1)^{-1}}=\frac{p+q-k-2}{p+q-k}.$$

Then the derivative is strictly less than

$$\frac{1}{p^2}-\int_{p}^{p+q-k}\frac{2dx}{x^3}-\frac{2}{(p+q-k)^3}-\frac{p+q-k-2}{(p+q-k)^3}=0,$$

ruling out case (ii).

Case (iii) $(D\leqq\delta_k<R)$. First, $p_{k+1}-p_k>p_k-p_{k-1}$; for if not, then $r_{n-k-1}p_k\leqq$ $\leqq p_k-p_{k-1}<R=r_{n-k}p_{k-1}$. But as $r_{i+1}<r_i$, the resulting inequality is patently false. The inequality $p_{k+1}-p_k>p_k-p_{k-1}$, combined with Lemma 2 shows that p_i-p_{i-1} is strictly increasing for $i\geqq k$. If p_k is moved to $p_k+x(x\geqq 0)$ and the points $p_{k+1}, \ldots, p_n$ are shifted, maintaining the proper ratios, then since $\delta_k\geqq D$, the rate of change of M_n at $x=0$ is

$$\frac{1}{p_k^2}-\frac{2(p_k-p_{k-1})}{p_k^3}-\eta_{n-k-1}\frac{1}{p_k^2}$$

which reduces to

$$\frac{2(R-\delta_k)}{p_k^3(1+r_{n-k})}>0.$$

This shows case (ii) cannot occur and completes the proof of the lemma.

From the three lemmas we obtain the following result

COROLLARY. *Let* $\tilde{P}_n=\{p_1, \ldots, p_n\}$. *Then if* $p_1=1$, $p_k=k$ *as long as* $kr_{n-k-1}\leqq 1$. *Beyond that point* $p_{k+1}=p_k(1+r_{n-k-1})$.

THEOREM. $M(P)\leqq\pi^2/6$.

PROOF. It is easy to verify that r_k decreases to zero as k increases. From the Corollary one sees that the number of equidistant points can be made arbitrarily large by making n large. Then μ_n will approach $\pi^2/6$ as $n\to\infty$, and since μ_n is increasing, it must remain less than $\pi^2/6$. It follows that $M(P)\leqq\pi^2/6$.

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Reference

[1] R. E. L. TURNER, Perturbation of compact spectral operators, *Comm. Pure Appl. Math.*, **18** (1965), pp. 519—541.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 19 (3—4), (1968), pp. 441—450.

SOME PROBLEMS ON THE ITERATION OF MULTIPLICATIVE NUMBER-THEORETICAL FUNCTIONS

By

I. KÁTAI (Budapest)

(*Presented by* P. TURÁN)

1. Let $\vartheta(n)$ denote a totally multiplicative number-theoretical function — i.e. satisfying the relation $\vartheta(mn)=\vartheta(m)\vartheta(n)$ for every pairs m, n of natural numbers — which assumes positive integer values only.

The letters p and q stand for prime numbers. Further p_i $(i=1, 2, \ldots)$ denote the ith prime in the natural order.

Let

$$\vartheta_0(n)=n, \quad \vartheta_1(n)=\vartheta(n), \quad \vartheta_k(n)=\vartheta(\vartheta_{k-1}(n)) \tag{1.1}$$

for $k=2, 3, \ldots$; $n=1, 2, \ldots$.

It is evident that $\vartheta_k(n)$ is a totally multiplicative function for any constant k.

For every n and k let $H_k(n)$ denote the set of all prime divisors of $\vartheta_k(n)$.

We shall say: the sequence of the sets

$$H_0(n);\ H_1(n), \ldots, H_k(n), \ldots \tag{1.2}$$

is the trajectory of n generated by ϑ.

Let $E(n)$ denote the set of those primes which belong to $H_k(n)$ for infinitely many k.

Let $E(=E_\vartheta)$ be the union of the sets $E(n)$ for $n=2, 3, \ldots$.

We say, that the elements of $E(n)$ are the asymptotic prime divisors of n. Further we call E the kernel of the iteration.

The most plausible question for a concrete $\vartheta(n)$ is the following. E has finitely or infinitely many elements.

We say, that n tends to zero p-adically by the iteration, if for every positive integer v there exists a constant $k_0=k_0(n, v, p)$ such that p^v is a divisor of $\vartheta_k(n)$, when $k\geqq k_0$. Let $P(n)$ denote the set of all p, for which n tends to zero p-adically.

It is evident, that

$$P(n)\subset E(n). \tag{1.3}$$

Let

$$P=\bigcup_{n=2}^{\infty} P(n).$$

Let $\varkappa(n)$ be the minimal index k for which $\vartheta_k(n)$ has only such prime divisors which occur in $E(n)$. If such k's do not exist we put $\varkappa(n)=+\infty$. The following relations evidently hold:

$$\varkappa(n)=\max_{q|n} \varkappa(q), \tag{1.4}$$

$$\varkappa(n)=1+\varkappa(\vartheta(n)), \quad \text{if} \quad \varkappa(n)>0. \tag{1.5}$$

2. First we investigate the class of functions defined by the relations:

$$\vartheta(p)=p+a \tag{2. 1}$$

for every prime p, where a is a fixed natural number. Let $b>1$ denote a natural number coprime to a.

The following theorem is true.

THEOREM 1. *For a ϑ defined by* (2. 1) *the set E has finitely many elements. The elements of E are smaller than $(2b-1)a$.*

PROOF. Let $p>(b-1)a$. Since

$$p, p+a, \ldots, p+(b-1)a \tag{2. 2}$$

is a complete system of residues mod b, so there exists at least one composite number in (2. 2). Let $p+la$ denote the smallest composite number in (2. 2).

Taking into account that

$$E(n)=E(\vartheta_k(n)) \quad \text{for} \quad k=1, 2, \ldots, \tag{2. 3}$$

hence

$$E(p)=E(\vartheta_{l-1}(p))=E(p+la) \tag{2. 4}$$

follows. Further

$$E(n)=\bigcup_{q|n} E(q). \tag{2. 5}$$

From

$$p+la\leqq p+(b-1)a<2p$$

it follows that the greatest primefactor of $p+la$ is smaller than p, whence

$$E(p)\subset \bigcup_{q<p} E(q). \tag{2. 6}$$

follows. From (2. 6) it follows that

$$E(p)\subset \bigcup_{q\leqq K} E(q) \tag{2. 7}$$

where $K=(b-1)a$.

We shall now prove, that the set

$$F\overset{\text{def}}{=}\bigcup_{q\leqq K}^{!} E(q)$$

has no elements greater than $2K+a$.

Let p be any element of F, greater than K. Let p_1 defined as follows: $p, p-a, \ldots, p-va=p_1$ are all primes and $p-(v+1)a$ is a composite number. Then p_1 is a divisor of $q+a$, where $q\leqq K$. Hence

$$p\leqq p_1+va\leqq p_1+K\leqq 2K+a=(2b-1)a$$

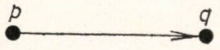
and the assertion is proved.

REMARKS 1. Since we obtained an effective upper bound for the order of primes belonging to E, so for every concrete a we may calculate the elements of E.

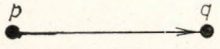
2. Let E_1 denote the set of all prime divisor of a. It is evident, that its elements belong to the kernel of iteration, i.e. $E_1 \subset E$. Let

$$E_2 = E - E_1 .$$

We can illustrate the kernel of the iteration and the motion by the iteration as the vertices and the edges of an oriented graph. In other words: if $p, q \in E$ and $q | \vartheta(p)$, then we write



Further we eventually write a wider arrow



if $q^2 | \vartheta(p)$.

Using simple numerical calculations we determine the graph of (2. 1) for $a = 1, \ldots, 40$. (See the figures below.)

3. Let $\varkappa(n)$ as defined in (1. 4) and let $\vartheta(n)$ be generated by the relation (2. 1).

THEOREM 2. *We have*

$$\overline{\lim_{p \to \infty}} \frac{\varkappa(p)}{\log \log p} > c \qquad (>0). \tag{3. 1}$$

PROOF. The assertion is an immediate consequence of a theorem of YU. V. LINNIK stating that the smallest prime number in the arithmetical progression $\equiv l \pmod k$ is smaller than k^{c_1}, where c_1 is an absolute constant.

The determination of the mean order of magnitude of $\varkappa(n)$ is a hard problem and yet not answered.

4. The investigation of $P(n)$ is a difficult problem in the general case.

It is evident, that $P(n)$ and E_1 are disjoint if and only if $(n, a) = 1$. Further, if $q | a$, then the condition $q^2 | q + a$ is a necessary and sufficient one to $q \in P$.

Every n can be represented in the form $n = n_1 n_2$ exactly once, where $(n_2, a) = 1$, and n_1 has only such primedivisors which divide a. Let

$$P_1(n) = P(n_1); \qquad P_2(n) = P(n_2),$$

$$P_1 = \bigcup_{n=2}^{\infty} P(n_1); \qquad P_2 = \bigcup_{n=2}^{\infty} P(n_2).$$

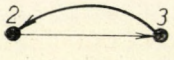
By numerical computations we obtain

THEOREM 3.

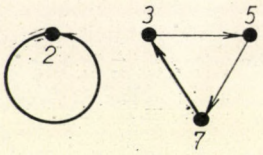
(1) $P_2 = E_2$ *for every* a *in the interval* $2 \leqq a \leqq 40$.

(2) $P_2(n) = E_2$ *for every* $n \geqq 2$, *if* a *is odd in the interval* $2 \leqq a \leqq 40$, *except the values* $a = 3, 15, 21, 39$.

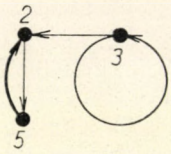
PROOF. (1) The proof of (1) goes by simple investigation of the graphs. (2) In our cases the graphs are connected. Further the graphs contain at least three elements. Further, when $p \in E_2$ and $p > 2$, then $2 | \vartheta(p)$. From these facts it follows that $2 \in H_k(2)$



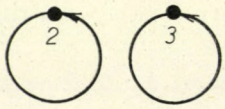
$a=1$

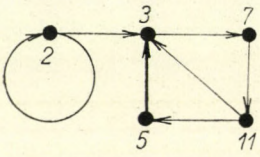


$a=2$

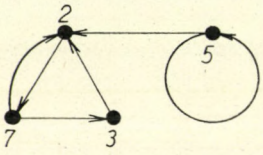


$a=3$





$a=4$



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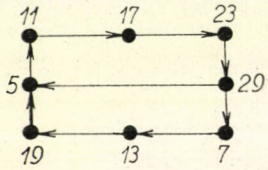

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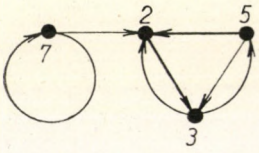

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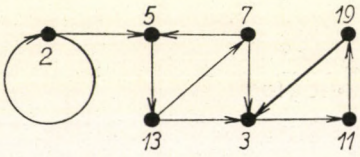

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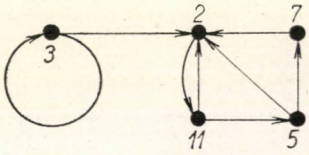

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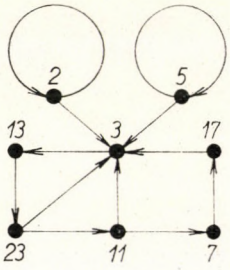

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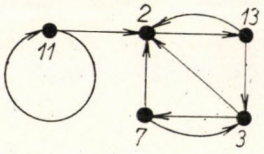

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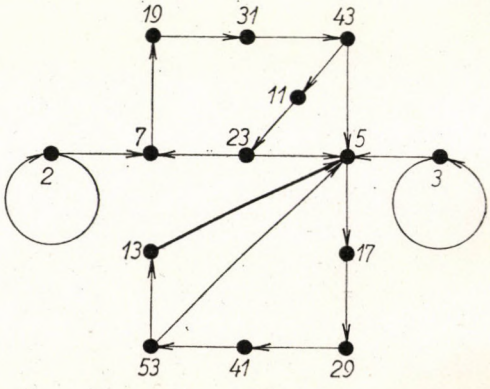
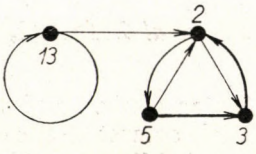


$a=12$



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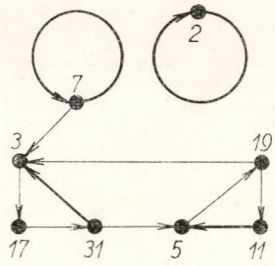

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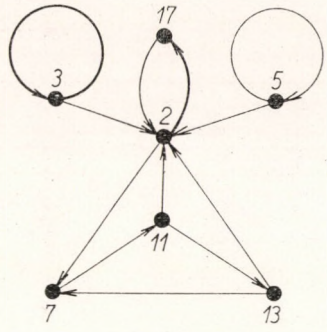

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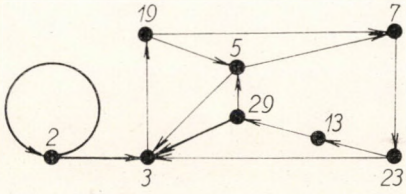

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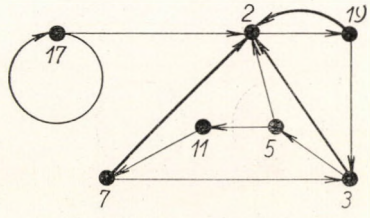

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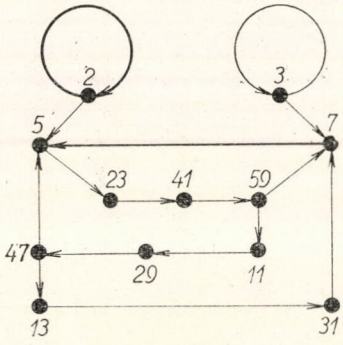

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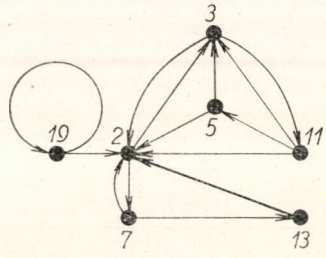

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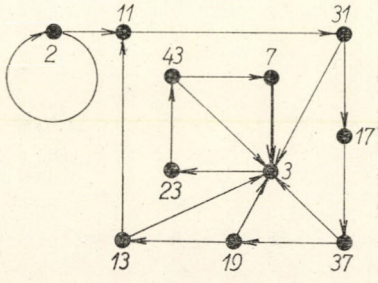

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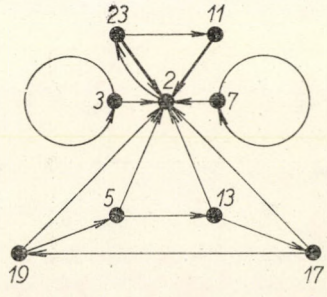

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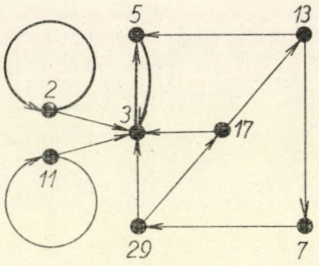

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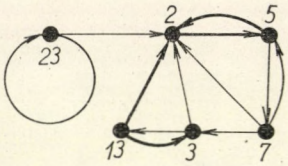

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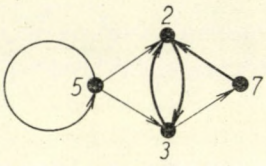

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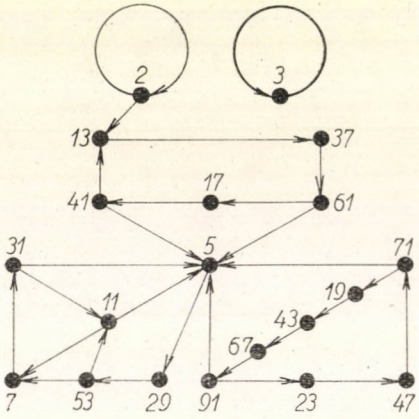

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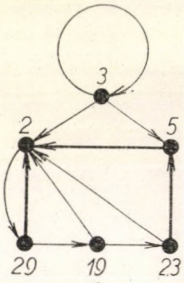

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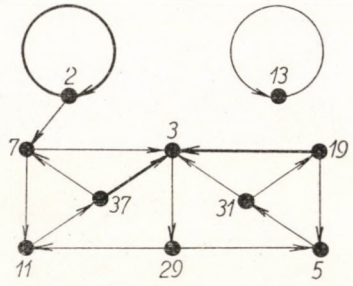

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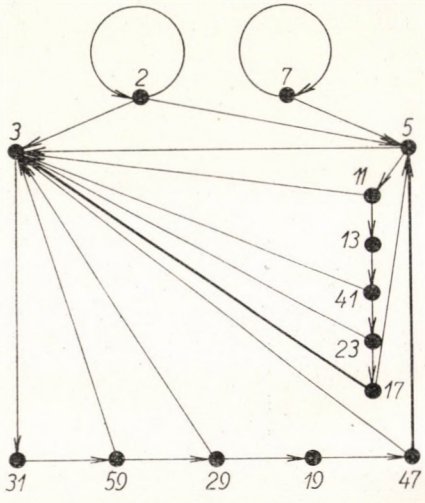

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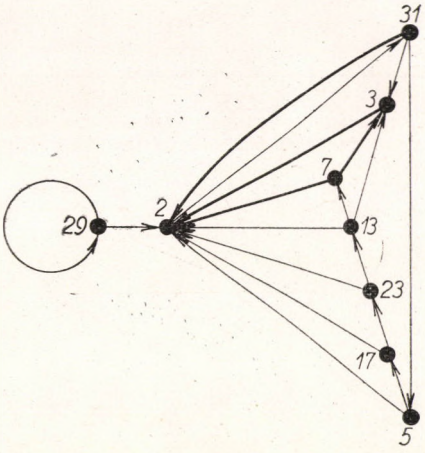

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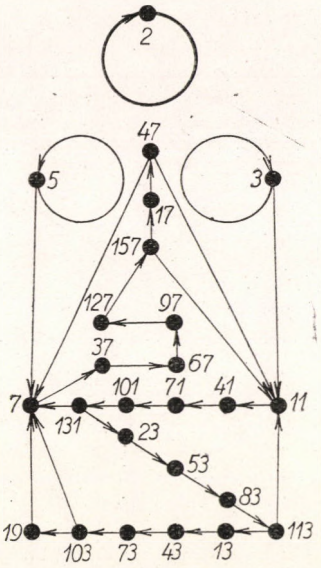

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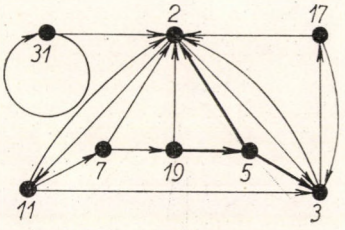

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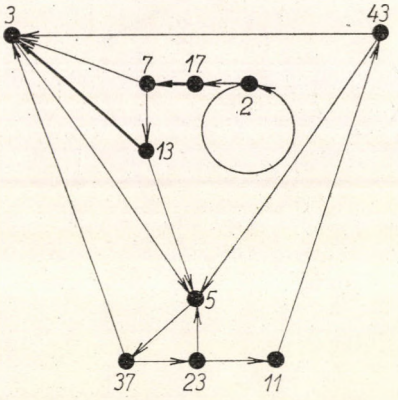

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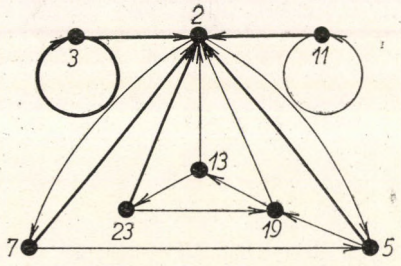

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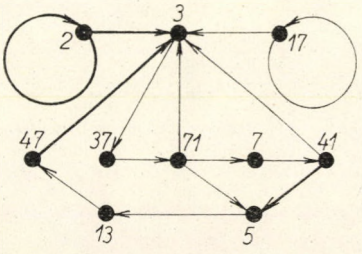

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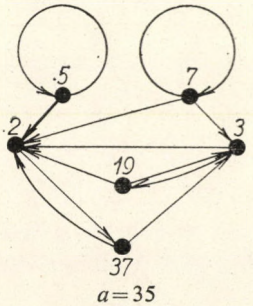

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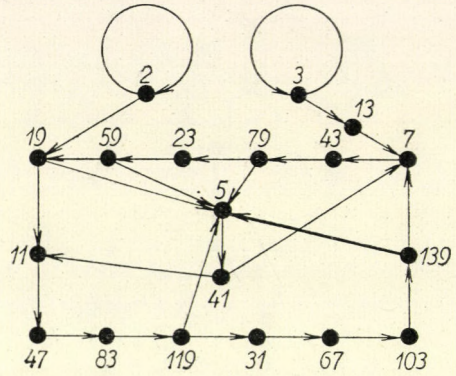

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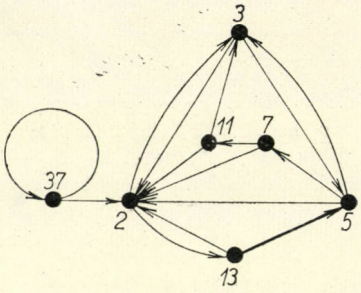

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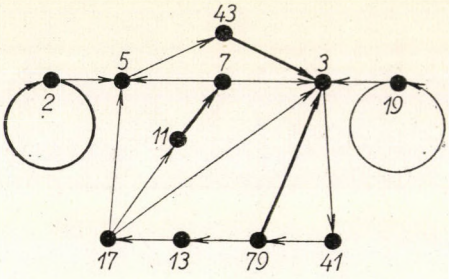

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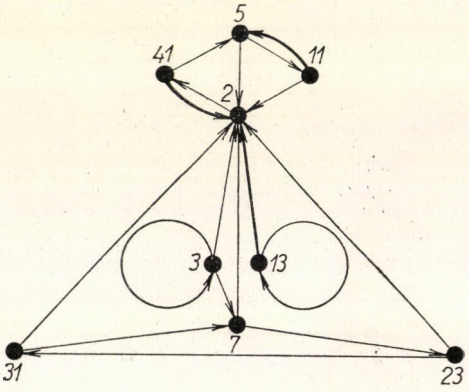

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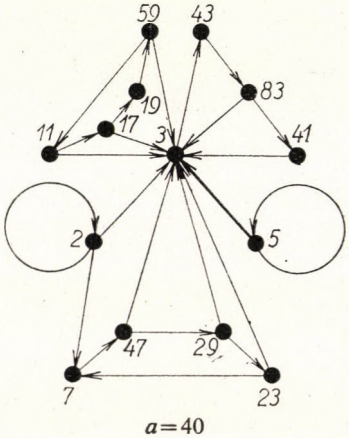


$a=40$

for every sufficiently large k, and from the connectedness of the graphs $H_k(2)=E_2$ follows for sufficiently large k. Let $2<p_1<\ldots<p_s$ all the elements of E_2.

Let

$$\vartheta_k(2) = 2^{\alpha_0(k)} p_1^{\alpha_1(k)} \ldots p_s^{\alpha_s(k)},$$

where $s\geqq 2$, and k so large that $\alpha_i(k)\geqq 1$ for $i=1, \ldots, s$. Then

$$\vartheta_{k+r}(2) = \vartheta_r^{\alpha_0(k)}(2)\vartheta_r^{\alpha_1(k)}(p_1) \ldots \vartheta_r^{\alpha_s(k)}(p_s)$$

whence it follows, that

$$\alpha_0(k+r)\geqq\alpha_0(r)+\alpha_1(k)+ \ldots + \alpha_s(k).$$

Hence immediately follows, that $\alpha_0(k)\to\infty$ for $k\to\infty$. Since $p_i|\vartheta_k(2)$ for $k\geqq k_0$ and $2^N|\vartheta(2)$ for $r\geqq k_1(N)$, hence $\vartheta_k(2^N)|\vartheta_{k+r}(2)$, and so $p_i^N|\vartheta_{k+r}(2)$ follows.

REMARKS. 1. For even a we can prove similar theorems, but the proofs are complicated.

2. I think, that for every sufficiently large a the graph with vertices in E_2 is not connected.

3. We cannot characterize all of those a's for which the prime p belongs to E.

4. Let $L=L_\vartheta$ be the set of all primes p, such that $\vartheta_k(n)$ converges to zero p-adically for every prime p, belonging to L, and for all $n\leqq x$, except perhaps $o(x)$ elements.

It is evident, that $L\subset E_2$. We can prove the following theorem, which we state without proof:

THEOREM 4.

(1) $L=E_2$, *if* $a=2, 4, 6, 8, 10, 14, 16, 18, 26, 30, 32, 34, 36, 38.$

(2) *If* $a=8$, *then* $L=\{5, 17, 29, 41, 53, 13\}$,

if $a=15$, *then* $L=\{2, 7\}$,

if $a=20$, *then* $L=\{3, 23, 43, 7\}$,

if $a=21$, *then* $L=\{5, 13, 17, 19\}$,

if $a=22$, *then* $L=\{3, 5\}$,

if $a=24$, *then* $L=\{5, 29, 53, 7, 31, 11\}$,

if $a=28$, *then* $L=\{3, 31, 19, 47\}$,

if $a=40$, *then* $L=\{3, 11, 17, 19, 59, 43, 83, 41\}$.

5. We can generalize our Theorem 1 to a wider class of multiplicative functions. Let $K(x)$ be a monotonically non-decreasing function defined for $x\geqq 2$, such that

$$\max_{x\leqq p_i\leqq 2x}(p_{i+1}-p_i) > K(x). \tag{5.1}$$

Let

$$p < \vartheta(p) < p+\frac{K(p)}{2} \tag{5.2}$$

for every sufficiently large prime p.

THEOREM 5. *For ϑ, satisfying the relation* (5. 2) *with* (5. 1) *the kernel of iteration E_ϑ has only finitely many elements.*

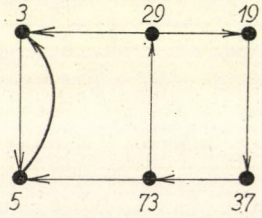
The proof is very similar to the proof of Theorem 1 and so can be omitted.

6. The investigation of the iterations-kernel of the functions generated by

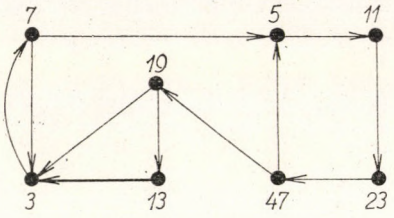
$$\vartheta(p)=ap+b \tag{6. 1}$$

seems to be very hard in the cases $a\geqq 2$.

Using numerical computations it is easy to obtain for the function $\vartheta(p)=2p-1$, that the kernel of the iteration contains the primes 2, 3, 5, 19, 37, 73, 29 among the primes smaller than 100, and the motion by the iteration can be illustrated by the following graph



Similarly, for $\vartheta(p)=2p+1$ among the primes smaller than 100 we obtain the following stable states and graph:



I think, that the following conjecture is true.

Let

$$\vartheta(p)=P(p), \tag{6. 2}$$

where $P(x)$ is an irreducible polynomial with integer coefficients, and with values $\geqq 2$ for $x\geqq 2$.

Then the kernel of the iteration generated by ϑ contains finitely or infinitely many elements as the degree of $P(x)$ is one or greater than one, respectively.

Acknowledgement. I am grateful to Mr. D. KÓLYA for drawing the figures.

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MODULES FOR PSEUDO-RINGS

By

E. M. PATTERSON (Aberdeen, Scotland)

(*Presented by* L. RÉDEI)

1. Introduction

A pseudo-ring $\mathfrak{A} = (A^*, A)$ is an algebraic system in which the elements form an additive abelian group A, containing a distinguished subgroup A^*, such that for each $a^* \in A^*$ and $a \in A$ a product $a^*a \in A$ is defined; with respect to this operation, A^* is a ring and A is a left A^*-module. If $A^* = A$, then we identify $\mathfrak{A}$ with the ring A^*.

The structure of pseudo-rings has been discussed in two recent papers [3, 4], in the second of which a generalisation of the Wedderburn—Artin theorem for rings was obtained. It is natural to ask whether the results in ring theory which stem from the Wedderburn—Artin theorem can also be generalised to pseudo-rings. In the present paper we focus attention on the theorem that any unital module over a semi-simple Artinian ring is completely reducible. In our main theorem (Theorem 1) we obtain a generalisation of this result for pseudo-rings.

Our investigations involve a detailed study of the concept of a pseudo-ring module (see [3], §§ 3, 4). If $\mathfrak{X} = (X^*, X)$ is an $\mathfrak{A}$-module, then X is an additive abelian group and X^* is a subgroup of X; for each $x^* \in X^*$ and $a \in A$ a product $x^*a \subset X$ is defined and satisfies the usual distributive laws as well as the condition $X^*A^* \subseteq X^*$. The group X can contain elements which are not directly involved in this product operation and because of this it can happen that $\mathfrak{X}$ provides a „bad" representation of $\mathfrak{A}$. An example of this type was given in a previous paper ([4], § 3; the definition of multiplication should read $(x_1, \overline{0})(a_1, a_2) = (x_1a_1, \overline{x_1a_2})$). In § 2 of the present paper, we give another example, which shows that not every unital module over a semi-simple Artinian pseudo-ring is completely reducible, so that we cannot expect the theorem from ring theory on complete reducibility to extend directly to pseudo-rings.

Examples of the type given in § 2 indicate the need for some criteria by which a module can be judged to be „good". In the case in which $\mathfrak{A}$ is a semi-simple Artinian pseudo-ring, it seems reasonable to regard the regular representation as setting a useful standard of goodness, so we first investigate the properties of $\mathfrak{A}$ as an $\mathfrak{A}$-module. This leads to a definition of good modules for pseudo-rings of this type. Our main theorem is that when $\mathfrak{A}$ is a semi-simple Artinian pseudo-ring a unital $\mathfrak{A}$-module is good if and only if it is completely reducible.

This result suggests that a study of pseudo-rings through the methods of homological algebra might be possible. However it is not clear what would be the most suitable definition of good module in the general case. Based on the ideas developed in our study of the semi-simple Artinian case, we show that for each pseudo-ring $\mathfrak{A}$ there exists a certain exact category $\mathfrak{C}(\mathfrak{A})$, which seems worthy of further investigation. When $\mathfrak{A}$ is a ring, $\mathfrak{C}(\mathfrak{A})$ is the usual abelian category of modules

over the ring. When $\mathfrak{A}$ is a semi-simple Artinian pseudo-ring, the objects in $\mathfrak{C}(\mathfrak{A})$ are the good $\mathfrak{A}$-modules, and $\mathfrak{C}(\mathfrak{A})$ is again abelian.

The notations and definitions are as in the previous papers [3, 4]. If $\mathfrak{X}=(X^*, X)$ is an $\mathfrak{A}$-module and $\{\mathfrak{X}_\lambda=(X_\lambda^*, X_\lambda): \lambda\in\Lambda\}$ is a family of $\mathfrak{A}$-submodules of $\mathfrak{X}$, indexed by a set Λ, such that

$$X=\sum_{\lambda\in\Lambda}\oplus X_\lambda, \qquad X^*=\sum_{\lambda\in\Lambda} X_\lambda^*,$$

then we say that $\mathfrak{X}$ is the direct sum of the $\mathfrak{X}_\lambda$ and write

$$\mathfrak{X}=\sum_{\lambda\in\Lambda}\oplus\mathfrak{X}_\lambda.$$

2. A bad module

Let A be the set of ordered triples (a_1, a_2, a_3) of real numbers, let A^* be the subset for which $a_2=0=a_3$, define addition in A in the obvious way and define multiplication by

$$(b_1, 0, 0)(a_1, a_2, a_3) = (b_1a_1, b_1a_2, b_1a_3).$$

Let X be the set of ordered triples (x_1, x_2, x_3) of real numbers, let X^* be the subset for which $x_3=0$, define addition in X in the obvious way and define multiplication by

$$(x_1, x_2, 0)(a_1, a_2, a_3) = (x_1a_1, x_2a_1, x_1a_2+x_2a_3).$$

Then $\mathfrak{A}=(A^*, A)$ is a primitive Artinian pseudo-ring and $\mathfrak{X}=(X^*, X)$ is an $\mathfrak{A}$-module. If $Y=\{(x_1, 0, x_3)\}$ and $Y^*=\{(x_1, 0, 0)\}$, then (Y^*, Y) is a proper submodule of $\mathfrak{X}$ and is not quasi-zero, so that $\mathfrak{X}$ is not irreducible (see [3], §§ 4, 7). On the other hand, every submodule which is not quasi-zero contains every element of X of the form $(0, 0, x_3)$ and so $\mathfrak{X}$ cannot be a direct sum of irreducible submodules.

This example shows that not every unital module over a primitive Artinian pseudo-ring is completely reducible in the sense of being a direct sum of irreducible submodules. Thus we cannot extend the theorem on complete reducibility of modules over semi-simple Artinian rings to arbitrary modules over semi-simple Artinian pseudo-rings.

3. Accessible modules

Let $\mathfrak{X}=(X^*, X)$ be a module over a pseudo-ring $\mathfrak{A}=(A^*, A)$. Write $X_1=X^*+X^*A$. Then (X^*, X_1) is a submodule of $\mathfrak{X}$ which gives as much information about $\mathfrak{A}$ as $\mathfrak{X}$ itself, since elements of X which are not in X_1 are in no way involved in the operation of multiplication. Thus we can focus our attention on modules for which $X=X_1$.

DEFINITION 1. A module $\mathfrak{X}=(X^*, X)$ over a pseudo-ring $\mathfrak{A}=(A^*, A)$ will be called *accessible* if

$$X=X^*+X^*A. \tag{3. 1}$$

It should be observed that the „bad" module discussed in § 2 is accessible. Thus the property of accessibility alone will not be sufficient for our purposes. However it plays an important part in our definition of good module.

Every module over a ring is clearly accessible. More generally, any pseudo-ring module for which $X^* = X$ is accessible.

If A^* contains a unity 1 and $\mathfrak{X}$ is unital in the sense that $x^*1 = x^*$ for all $x^* \in X^*$, condition (3. 1) can be written

$$X = X^*A. \tag{3. 2}$$

We shall require the following lemmas, the proofs of which are trivial.

LEMMA 1. *Let* $\{\mathfrak{X}_\lambda = (X_\lambda^*, X_\lambda) : \lambda \in \Lambda\}$ *be a family of accessible submodules of a module* $\mathfrak{X}$. *Then their sum* $\mathfrak{Y} = (Y^*, Y)$, *defined by*

$$Y = \sum_{\lambda \in \Lambda} X_\lambda, \qquad Y^* = \sum_{\lambda \in \Lambda} X_\lambda^*,$$

is also accessible.

LEMMA 2. *A quasi-zero accessible module is zero.*

Further general properties of accessibility will be discussed in § 6.

4. The regular representation of a semi-simple Artinian pseudo-ring

Throughout this section and § 5, $\mathfrak{A}$ denotes a semi-simple Artinian pseudo-ring which is accessible in the sense of Definition 1, so that

$$A = A^* + A^*A.$$

In a previous paper [4], we established the Wedderburn—Artin structure theorem for pseudo-rings of this type. This shows that $\mathfrak{A}$ is the direct sum of a finite number of primitive matrix pseudo-rings of the form $\mathfrak{M}(I_n, I, D) = (M^*, M)$. Here I is any set, $I_n = \{i_1, \ldots, i_n\}$ is a finite subset consisting of n elements of I, D is a division ring, M is the set of all row-finite matrices of type I whose i^{th} row is zero whenever $i \notin I_n$ and M^* is the subset of M consisting of those matrices whose j^{th} column is also zero whenever $j \notin I_n$.

LEMMA 3. *The* $\mathfrak{A}$*-module* $\mathfrak{A}$ *can be expressed as the direct sum*

$$\sum_{i=1}^{k} \oplus \mathfrak{B}_i$$

of a finite number of accessible irreducible right ideals $\mathfrak{B}_1, \ldots, \mathfrak{B}_k$.

PROOF. Let $M_r (r \in \{1, 2, \ldots, n\})$ be the subset of M consisting of those matrices whose i^{th} row is zero whenever $i \neq i_r$ and write $M_r^* = M_r \cap M^*$. Then $\mathfrak{M}_r = (M_r^*, M_r)$ is an irreducible right ideal of $\mathfrak{M}(I_n, I, D)$ satisfying $M_r = M_r^* M$, and

$$M = \sum_{r=1}^{n} \oplus M_r, \qquad M^* = \sum_{r=1}^{n} M_r^*.$$

Thus $\mathfrak{M}(I_n, I, D)$ is the direct sum $\sum_{r=1}^{n} \oplus \mathfrak{M}_r$ of accessible irreducible right ideals. Applying this result to each component in the decomposition of $\mathfrak{A}$ as the direct sum of primitive pseudo-rings, we obtain our desired result.

LEMMA 4. *There exist elements e_γ ($\gamma \in \Gamma$, some indexing set) of A such that*

$$A = A^* \oplus \sum_{\gamma \in \Gamma} \oplus A_\gamma^* e_\gamma, \tag{4. 1}$$

where each A_γ^* is a minimal left ideal of A^* and

$$a_\gamma^* \in A_\gamma^*, \quad a_\gamma^* e_\gamma = 0 \quad \Rightarrow \quad a_\gamma^* = 0. \tag{4. 2}$$

Moreover if $\mathfrak{B} = (B^*, B)$ is any accessible right ideal of $\mathfrak{A}$, then

$$B = B^* \oplus \sum_{\gamma \in \Gamma} \oplus B_\gamma^* e_\gamma, \tag{4. 3}$$

where $B_\gamma^* = A_\gamma^* \cap B^* = B^* A_\gamma^*$.

PROOF. By definition, A is a left A^*-module. Since $\mathfrak{A}$ is semi-simple and Artinian, so is A^* (this follows from the Wedderburn—Artin structure theorem: see [4]). Hence A^* is a semi-simple left Artinian ring. Since A^* is an A^*-submodule of the left module A, we have

$$A = A^* \oplus \sum_{\gamma \in \Gamma} \oplus A_\gamma,$$

where Γ is some indexing set and the A_γ are irreducible left A^*-modules. We can write $A_\gamma = A_\gamma^* e_\gamma$, where A_γ^* is some minimal left ideal of A^* and $e_\gamma \in A_\gamma$ ([1], p. 47). Thus A is given by (4. 1); condition (4. 2) follows from the minimality of A_γ^*.

Now suppose that $\mathfrak{B} = (B^*, B)$ is an accessible right ideal of $\mathfrak{A}$. Then B^* is a right ideal of the Artinian ring A^* and so there exists a left unity e^* for B^* such that $B^* = e^* A^*$. Since $\mathfrak{B}$ is accessible, we have

$$B = B^* A = e^* A^* A = e^* A.$$

Hence, by (4. 1), if $b \in B$ there exist $a^* \in A^*$, $a_\gamma^* \in A_\gamma^*$ such that

$$b = e^* a^* + \sum_{\gamma \in \Gamma} e^* a_\gamma^* e_\gamma.$$

Since $e^* a^* \in B^*$, it follows that

$$B \subseteqq B^* + \sum_{\gamma \in \Gamma} B_\gamma^* e_\gamma$$

where $B_\gamma^* = e^* A_\gamma^*$. The opposite inclusion holds since $B_\gamma^* \subseteqq B^*$, and so

$$B = B^* + \sum_{\gamma \in \Gamma} B_\gamma^* e_\gamma.$$

Since A_γ^* is a left ideal of A^*, $B_\gamma^* \subseteqq A_\gamma^*$. Hence, since the sum in (4. 1) is direct, we have (4. 3).

If $b^* \in A_\gamma^* \cap B^*$, then $b^* = e^* b^* \in e^* A_\gamma^*$ and so $A_\gamma^* \cap B^* \subseteqq B_\gamma^*$. But, from the above, $B_\gamma^* \subseteqq A_\gamma^* \cap B^*$ and so $B_\gamma^* = A_\gamma^* \cap B^*$. Since $e^* \in B^*$, $B_\gamma^* \subseteqq B^* A_\gamma^*$; but clearly $B^* A_\gamma^* \subseteqq B^* \cap A_\gamma^*$ since B^* is a right ideal of A^* and A_γ^* is a left ideal of A^*. Hence we have $B_\gamma^* = B^* A_\gamma^*$.

LEMMA 5. *Every accessible submodule of $\mathfrak{A}$ is normal.*

PROOF. Let $\mathfrak{B}=(B^*, B)$ be an accessible right ideal of $\mathfrak{A}$. Suppose that $b\in B\cap A^*$. By Lemma 4, there exist $b^*\in B^*$, $b_\gamma^*\in B_\gamma^*$ such that

$$b = b^* + \sum_{\gamma\in\Gamma} b_\gamma^* e_\gamma$$

and since $b\in A^*$ we have $\sum_{\gamma\in\Gamma} b_\gamma^* e_\gamma \in A^*$. But the sum in (4. 1) is direct, and so $\sum_{\gamma\in\Gamma} b_\gamma^* e_\gamma = 0$ and $b=b^*$. It follows that $B\cap A^*\subseteqq B^*$, which proves that $\mathfrak{B}$ is normal.

LEMMA 6. *Let $\mathfrak{B}=(B^*, B)$ be an accessible submodule of $\mathfrak{A}$ and suppose that a^* is any element of A^*. Then the quotient right ideal $(\mathfrak{B}: a^*)$ is accessible.*

PROOF. Write $(Q^*, Q)=(\mathfrak{B}: a^*)$. Then if $q\in Q$, $a^*q\in B$. By Lemma 4, there exist $q^*\in A^*$, $q_\gamma^*\in A_\gamma^*$ such that

$$q = q^* + \sum_{\gamma\in\Gamma} q_\gamma^* e_\gamma$$

and similarly there exist $b^*\in B^*$ and $b_\gamma^*\in B_\gamma^*$ such that

$$a^*q = b^* + \sum_{\gamma\in\Gamma} b_\gamma^* e_\gamma.$$

It follows that

$$a^*q^* + \sum_{\gamma\in\Gamma} a^* q_\gamma^* e_\gamma = b^* + \sum_{\gamma\in\Gamma} b_\gamma^* e_\gamma$$

and, since the sum in (4. 1) is direct, we get

$$a^*q^* = b^*, \quad a^*q_\gamma^* e_\gamma = b_\gamma^* e_\gamma.$$

Using (4. 2), we deduce that

$$a^*q_\gamma^* = b_\gamma^*$$

and so we have $q^*, q_\gamma^*\in Q^*$. It follows that $q\in Q^*+Q^*A$, which is sufficient to show that (Q^*, Q) is accessible.

LEMMA 7. *Let $\mathfrak{X}=(X^*, X)$ be an $\mathfrak{A}$-module which is expressible as the direct sum*

$$\sum_{\lambda\in\Lambda}\oplus \mathfrak{X}_\lambda$$

where Λ is some indexing set and $\mathfrak{X}_\lambda$ is a submodule of $\mathfrak{X}$, isomorphic with some accessible submodule of $\mathfrak{A}$. Then, in the same notation as used in Lemma 4,

$$X = X^* \oplus \sum_{\gamma\in\Gamma} \oplus X_\gamma^* e_\gamma, \tag{4. 4}$$

where $X_\gamma^* = X^* A_\gamma^*$ and

$$x_\gamma^*\in X_\gamma^*, \quad x_\gamma^* e_\gamma = 0 \quad \Rightarrow \quad x_\gamma^* = 0. \tag{4. 5}$$

Moreover, if $\mathfrak{Y}=(Y^*, Y)$ is an accessible submodule of $\mathfrak{X}$, then

$$Y = Y^* \oplus \sum_{\gamma\in\Gamma} \oplus Y_\gamma^* e_\gamma, \tag{4. 6}$$

where $Y_\gamma^* = Y^* A_\gamma^*$.

PROOF. Since $\mathfrak{X}_\lambda = (X_\lambda^*, X_\lambda)$ is isomorphic with an accessible submodule of $\mathfrak{A}$, it follows from Lemma 4 that

$$X_\lambda = X_\lambda^* \oplus \sum_{\gamma \in \Gamma} \oplus X_{\lambda\gamma}^* e_\gamma,$$

where $X_{\lambda\gamma}^* = X_\lambda^* A_\gamma^*$ and

(4. 7) $$x_{\lambda\gamma}^* \in X_{\lambda\gamma}^*, \quad x_{\lambda\gamma}^* e_\gamma = 0 \quad \Rightarrow \quad x_{\lambda\gamma}^* = 0.$$

Hence

$$X = \sum_{\lambda \in \Lambda} \oplus X_\lambda = X^* \oplus \sum_{\gamma \in \Gamma} \oplus X_\gamma^* e_\gamma,$$

where

(4. 8) $$X_\gamma^* = \sum_{\lambda \in \Lambda} \oplus X_{\lambda\gamma}^*.$$

Clearly

$$X_\gamma^* = \sum_{\lambda \in \Lambda} \oplus X_\lambda^* A_\gamma^* = X^* A_\gamma^*$$

and, from (4. 7),

$$x_\gamma^* \in X_\gamma^*, \quad x_\gamma^* e_\gamma = 0 \quad \Rightarrow \quad x_\gamma^* = 0,$$

since the sum in (4. 4) is direct.

Suppose now that $\mathfrak{Y} = (Y^*, Y)$ is an accessible submodule of $\mathfrak{X}$ and that $y \in Y$. Since $Y = Y^* A$, there exist a positive integer r and elements $y_1^*, \ldots, y_r^* \in Y^*$, $a_1, \ldots, a_r \in A$ such that

$$y = \sum_{j=1}^{r} y_j^* a_j.$$

By Lemma 4, there exist $a_j^* \in A^*$ and $a_{j\gamma}^* \in A_\gamma^*$ such that

$$a_j = a_j^* + \sum_{\gamma \in \Gamma} a_{j\gamma}^* e_\gamma,$$

so that

$$y = \sum_{j=1}^{r} y_j^* a_j^* + \sum_{\gamma \in \Gamma} \left(\sum_{j=1}^{r} y_j^* a_{j\gamma}^* \right) e_\gamma.$$

Hence

$$y \in Y^* + \sum_{\gamma \in \Gamma} Y_\gamma^* e_\gamma,$$

where $Y_\gamma^* = Y^* A_\gamma^*$. Since the sum in (4. 4) is direct and $Y_\gamma^* \subseteq X^* A_\gamma^* = X_\gamma^*$, it follows that

$$Y = Y^* \oplus \sum_{\gamma \in \Gamma} \oplus Y_\gamma^* e_\gamma.$$

Further properties of the $\mathfrak{A}$-module $\mathfrak{A}$ will be discussed later. However, we are now in a position to prove our main theorem.

5. Completely reducible modules over a semi-simple Artinian pseudo-ring

We again suppose that $\mathfrak{A} = (A^*, A)$ is an accessible semi-simple Artinian pseudo-ring. In this section we introduce the concept of a good module for such a pseudo-ring and prove our main result: a unital $\mathfrak{A}$-module is good if and only if it is completely reducible.

DEFINITION 2. An $\mathfrak{A}$-module $\mathfrak{X}=(X^*, X)$ will be called a *good* module if it satisfies the following:

(G1) $\mathfrak{X}$ is accessible.

(G2) Every accessible submodule of $\mathfrak{X}$ is normal.

(G3) For any accessible submodule $\mathfrak{Y}$ of $\mathfrak{X}$ and any element $x^* \in X^*$, the quotient $(\mathfrak{Y}: x^*)$ is an accessible right ideal of $\mathfrak{A}$.

We observe that the bad module described in § 2 satisfies (G2) but not (G3).

THEOREM 1. *Let $\mathfrak{X}$ be a non-zero unital $\mathfrak{A}$-module. Then $\mathfrak{X}$ is good if and only if there exists a family*

$$\{\mathfrak{X}_\lambda : \lambda \in \Lambda\},$$

where Λ is some indexing set and the $\mathfrak{X}_\lambda$ are submodules of $\mathfrak{X}$ such that

(i) $$\mathfrak{X} = \sum_{\lambda \in \Lambda} \oplus \mathfrak{X}_\lambda$$

(ii) *each $\mathfrak{X}_\lambda$ is $\mathfrak{A}$-isomorphic with an accessible irreducible right ideal of $\mathfrak{A}$.*

PROOF. Suppose first that $\mathfrak{X}=(X^*, X)$ satisfies the conditions stated in the theorem. By Lemma 7, X is given by (4. 4), where (4. 5) is satisfied. Condition (G1) follows at once from Lemma 1. To prove (G2), let $\mathfrak{Y}=(Y^*, Y)$ be an accessible submodule of $\mathfrak{X}$. From Lemma 7, it follows that if $y \in Y \cap X^*$, then $y \in Y^*$ and so $\mathfrak{Y}$ is normal.

Now let x^* be any element of X^* and consider the quotient $(Q^*, Q)=(\mathfrak{Y}: x^*)$. If $q \in Q$, then $x^*q \in Y$. By Lemma 4, there exist $q^* \in A^*$, $q_\gamma^* \in A_\gamma^*$ such that

$$q = q^* + \sum_{\gamma \in \Gamma} q_\gamma^* e_\gamma$$

and by Lemma 7, there exist $y^* \in Y^*$, $y_\gamma^* \in Y_\gamma^*$ such that

$$x^*q = y^* + \sum_{\gamma \in \Gamma} y_\gamma^* e_\gamma.$$

Hence

$$x^*q^* + \sum_{\gamma \in \Gamma} x^* q_\gamma^* e_\gamma = y^* + \sum_{\gamma \in \Gamma} y_\gamma^* e_\gamma,$$

so that, since the sum in (4. 4) is direct and (4. 5) is satisfied,

$$x^*q^* = y^*, \quad x^*q_\gamma^* = y_\gamma^*.$$

It follows that $q^*, q_\gamma^* \in Q^*$ and so $q \in Q^* + Q^*A$. Therefore $(\mathfrak{Y}: x^*)$ is accessible and so (G3) is satisfied. Hence $\mathfrak{X}$ is a good module.

Suppose conversely that $\mathfrak{X}$ is good. Let $x^* \in X^*$ and let $\mathfrak{B}_i=(B_i^*, B_i)$ $(i=1, \ldots, k)$ be as in Lemma 3. Write $x^*\mathfrak{B}_i$ for the submodule $(x^*B_i^*, x^*B_i)$ of $\mathfrak{X}$. The mapping $\beta: \mathfrak{B}_i \to x^*\mathfrak{B}_i$ defined by

$$b_i\beta = x^*b_i$$

is an $\mathfrak{A}$-epimorphism. Its kernel $\operatorname{Ker}\beta=(K^*, K)$ is an $\mathfrak{A}$-submodule of $\mathfrak{B}_i$. Since $\mathfrak{B}_i$ is irreducible, either $\operatorname{Ker}\beta=\mathfrak{B}_i$ or $K^*=\{0\}$. In the former case, $x^*\mathfrak{B}_i$ is zero. In the latter, $\operatorname{Ker}\beta$ is quasi-zero. But, if (Q^*, Q) is the quotient $(0: x^*)$ we have

$$b_i \in K \Rightarrow b_i \in Q.$$

By (G3) applied to the zero submodule of $\mathfrak{X}$, (Q^*, Q) is accessible. Hence, by Lemma 4, if $b_i \in K$ there exist $q^* \in Q^*$, $q_\gamma^* \in A_\gamma^* \cap Q^*$ such that

$$b_i = q^* + \sum_{\gamma \in \Gamma} q_\gamma^* e_\gamma . \tag{5.1}$$

But, again by Lemma 4, there exist $b_i^* \in B_i^*$, $b_{i\gamma}^* \in A_\gamma^* \cap B_i^*$ such that

$$b_i = b_i^* + \sum_{\gamma \in \Gamma} b_{i\gamma}^* e_\gamma$$

and so we have

$$q^* = b_i^* \in B_i^*, \quad q_\gamma^* = b_{i\gamma}^* \in B_i^* .$$

It follows that $q^*, q_\gamma^* \in K^*$ and so, using (5.1) we have

$$K \subseteqq K^* A .$$

Hence Ker β is accessible. Therefore, by Lemma 2, Ker β is zero if it is quasi-zero.

Thus either $x^*\mathfrak{B}_i$ is zero or $x^*\mathfrak{B}_i$ is $\mathfrak{A}$-isomorphic with $\mathfrak{B}_i$. If each $x^*\mathfrak{B}_i$ is zero $(i = 1, \ldots, k)$ then $x^*\mathfrak{A}$ is zero, which implies that $x^* = 0$, since $\mathfrak{X}$ is unital.

Let $\mathfrak{S}$ be the set of all non-zero submodules of $\mathfrak{X}$ of the form $x^*\mathfrak{B}_i$, where $x^* \in X^*$. Then $\mathfrak{S}$ is not empty, since $X^* \neq \{0\}$ (by hypothesis, $\mathfrak{X}$ is not zero and is accessible). Let

$$\mathfrak{T} = \{\mathfrak{X}_\delta = (X_\delta^*, X_\delta) : \delta \in \Delta\},$$

where Δ is some indexing set, be a subset of $\mathfrak{S}$ such that for each $\delta_1 \in \Delta$, we have

$$X_{\delta_1} \cap \sum_{\substack{\delta \in \Delta \\ \delta \neq \delta_1}} X_\delta = \{0\}.$$

Then $\mathfrak{T}$ will be called independent. Let $\mathfrak{C}$ be the collection of all independent subsets of $\mathfrak{S}$; $\mathfrak{C}$ is not empty, since any non-zero module $x^*\mathfrak{B}_i$ determines an independent subset. Let $\mathfrak{C}_0$ be a chain in $\mathfrak{C}$. Then the union of all the subsets in $\mathfrak{C}_0$ is independent and so, by Zorn's lemma, $\mathfrak{C}$ has a maximal element. Let $\{\mathfrak{X}_\lambda = (X_\lambda^*, X_\lambda) : \lambda \in \Lambda\}$, where Λ is some indexing set, be such a maximal element. We write

$$\mathfrak{W} = \sum_{\lambda \in \Lambda} \mathfrak{X}_\lambda$$

and put $\mathfrak{W} = (W^*, W)$. Since each $\mathfrak{X}_\lambda$ is isomorphic with an accessible right ideal of $\mathfrak{A}$, $\mathfrak{W}$ is accessible by Lemma 1.

Now suppose that $x^*\mathfrak{B}_i$ is any element of $\mathfrak{S}$. Write

$$\mathfrak{Y} = x^*\mathfrak{B}_i \cap \mathfrak{W}$$

and $\mathfrak{Y} = (Y^*, Y)$. Then $\mathfrak{Y}$ is a submodule of the irreducible module $x^*\mathfrak{B}_i$ and so either $\mathfrak{Y} = x^*\mathfrak{B}_i$ or $Y^* = \{0\}$.

Since $\mathfrak{W}$ is accessible, it follows from (G3) that $(Q^*, Q) = (\mathfrak{W} : x^*)$ is accessible. If $x^*b_i \in Y$ (where $b_i \in B_i$), then $b_i \in Q$ and so there exist $q_i^* \in Q^*$, $q_{i\gamma}^* \in A_\gamma^* \cap Q^*$ such that

$$b_i = q_i^* + \sum_{\gamma \in \Gamma} q_{i\gamma}^* e_\gamma . \tag{5.2}$$

Using Lemma 4, we quickly find that $q_i^* \in B_i^*$, $q_{i\gamma}^* \in A_\gamma^* \cap B_i^*$, so that $x^*q_i^* \in x^*B_i^*$ and $x^*q_{i\gamma}^* \in x^*B_i^*$. But also $x^*q_i^* \in W \cap X^*$; since $\mathfrak{W}$ is accessible, it is normal by

(G2) and so $W \cap X^* = W^*$. Therefore $x^* q_i^* \in W^*$ and similarly $x^* q_{i\gamma}^* \in W^*$. Hence we have $x^* q_i^* \in Y^*$ and $x^* q_{i\gamma}^* \in Y^*$, so that, using (5. 2), $x^* b_i \in Y^* A$, whence

$$Y \subseteqq Y^* A,$$

which proves that $\mathfrak{Y}$ is accessible. Hence, from Lemma 2, $\mathfrak{Y}$ is zero if it is quasi-zero.

It follows that either $\mathfrak{Y} = x^* \mathfrak{B}_i$ or $\mathfrak{Y}$ is zero. In the latter case,

$$\{\mathfrak{X}_\lambda : \lambda \in \Lambda\} \cup \{x^* \mathfrak{B}_i\}$$

is an independent set, contradicting the maximality of $\{\mathfrak{X}_\lambda : \lambda \in \Lambda\}$. Thus $\mathfrak{Y} = x^* \mathfrak{B}_i$, whence

$$x^* \mathfrak{B}_i \subseteqq \sum_{\lambda \in \Lambda} \mathfrak{X}_\lambda .$$

It is now easy to show that

$$\mathfrak{X} = \sum_{\lambda \in \Lambda} \oplus \mathfrak{X}_\lambda ,$$

which verifies condition (i) of Theorem 1. Condition (ii) follows from our definition of $\mathfrak{S}$; each element of $\mathfrak{S}$ is a non-zero submodule of $\mathfrak{X}$ of the form $x^* \mathfrak{B}_i$, which is isomorphic with the accessible irreducible right ideal $\mathfrak{B}_i$ of $\mathfrak{A}$.

This completes the proof of Theorem 1. Since any module over a ring is good, Theorem 1 generalises the theorem that any unital module over a semi-simple Artinian ring is completely reducible. The next result also generalises a theorem of ring theory.

THEOREM 2. *Every accessible submodule of a unital good* $\mathfrak{A}$*-module is a direct summand.*

PROOF. Let $\mathfrak{X} = (X^*, X)$ be a unital good $\mathfrak{A}$-module and let $\mathfrak{Y} = (Y^*, Y)$ be an accessible submodule. Since Y^* is an A^*-submodule of the A^*-module X^*, and A^* is a semi-simple Artinian ring, Y^* is a direct summand of X^*. Let Z^* be an A^*-submodule of X^* such that $X^* = Y^* \oplus Z^*$.

Using the same notation as in Lemma 4, we have $X^* A_\gamma^* = Y^* A_\gamma^* + Z^* A_\gamma^*$. Since Y^* and Z^* are right ideals of X^* we have $Y^* A_\gamma^* \subseteqq Y^*$ and $Z^* A_\gamma^* \subseteqq Z^*$. Hence

$$X^* A_\gamma^* = Y^* A_\gamma^* \oplus Z^* A_\gamma^* .$$

By Theorem 1, $\mathfrak{X}$ satisfies the conditions of Lemma 7. Hence, from that Lemma,

$$Y = Y^* \oplus \sum_{\gamma \in \Gamma} \oplus Y_\gamma^* e_\gamma .$$

Write $\mathfrak{Z}$ for the accessible submodule (Z^*, Z), where

$$Z = Z^* + \sum_{\gamma \in \Gamma} Z^* A_\gamma^* e_\gamma .$$

Since the sum in (4. 4) is direct, we have

$$Z = Z^* \oplus \sum_{\gamma \in \Gamma} \oplus Z^* A_\gamma^* e_\gamma .$$

Therefore, by Lemma 7,

$$X = X^* \oplus \sum_{\gamma \in \Gamma} \oplus X_\gamma^* e_\gamma = (Y^* \oplus Z^*) \oplus \sum_{\gamma \in \Gamma} \oplus (Y^* A_\gamma^* \oplus Z^* A_\gamma^*) e_\gamma = Y \oplus Z$$

and so $\mathfrak{Y}$ is a direct summand of $\mathfrak{X}$.

6. Accessibility for general pseudo-rings

We now suppose that $\mathfrak{A} = (A^*, A)$ is a general pseudo-ring. There is no guarantee that conditions (G1), (G2) and (G3) of Definition 2 will provide an adequate definition of good module when $\mathfrak{A}$ is not a semi-simple Artinian pseudo-ring, so we refrain from using the term in the general case. However, we shall examine some of the basic properties of modules satisfying (G1), (G2) and (G3). To do this, we require some general properties relating to accessibility, and we introduce the idea of an accessible homomorphism.

LEMMA 8. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be $\mathfrak{A}$-modules and let $\alpha: \mathfrak{X} \to \mathfrak{Y}$ be an $\mathfrak{A}$-homomorphism. If $\mathfrak{U}$ is an accessible submodule of $\mathfrak{X}$, then $\mathfrak{U}\alpha$ is an accessible submodule of $\mathfrak{Y}$.*

PROOF. If $\mathfrak{U} = (U^*, U)$ then, by definition, $\mathfrak{U}\alpha = (U^*\alpha, U\alpha)$. Hence, if $\mathfrak{U}$ is accesssible,

$$U\alpha = (U^* + U^*A)\alpha = U^*\alpha + (U^*\alpha)A$$

so that $\mathfrak{U}\alpha$ is accessible.

LEMMA 9. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be $\mathfrak{A}$-modules and let $\varepsilon: \mathfrak{X} \to \mathfrak{Y}$ be an $\mathfrak{A}$-epimorphism such that* Ker ε *is accessible. Then if $\mathfrak{B}$ is an accessible submodule of $\mathfrak{Y}$, the inverse image $\mathfrak{B}\varepsilon^{-1}$ is an accessible submodule of $\mathfrak{X}$.*

PROOF. Let $\mathfrak{W} = (W^*, W)$ be the inverse image $\mathfrak{B}\varepsilon^{-1}$, so that

$$W = \{x | x \in X,\ x\varepsilon \in V\},\quad W^* = \{x^* | x^* \in X^*,\ x^*\varepsilon \in V^*\}.$$

Suppose that $w \in W$. Then $w\varepsilon \in V$. If $\mathfrak{B}$ is accessible, there exist a positive integer r and elements $v^*, v_1^*, \ldots, v_r^* \in V^*$, $a_1, \ldots, a_r \in A$ such that

$$w\varepsilon = v^* + \sum_{j=1}^{r} v_j^* a_j.$$

Since ε is an $\mathfrak{A}$-epimorphism, there exist $w^*, w_j^* \in W^*$ such that $w^*\varepsilon = v^*$, $w_j^*\varepsilon = v_j^*$. Hence

$$w\varepsilon = w^*\varepsilon + \sum_{j=1}^{r} (w_j^*\varepsilon) a_j,$$

so that

$$\left(w - w^* - \sum_{j=1}^{r} w_j^* a_j\right)\varepsilon = 0.$$

Thus if Ker $\varepsilon = (K^*, K)$, we have

$$w - w^* - \sum_{j=1}^{r} w_j^* a_j \in K$$

and so, since Ker ε is accessible, there exist a positive integer s and elements $k^*, k_1^*, \ldots, k_s^* \in K^*, b_1, \ldots, b_s \in A$ such that

$$w - w^* - \sum_{j=1}^{r} w_j^* a_j = k^* + \sum_{i=1}^{s} k_i^* b_i.$$

Clearly $K^* \subseteqq W^*$ and so we have

$$w \in W^* + W^* A.$$

Thus $W \subseteqq W^* + W^* A$, which shows that $\mathfrak{W}$ is accessible.

LEMMA 10. *Let $\mathfrak{X}$ be an $\mathfrak{A}$-module and let $\mathfrak{Y}$ be an accessible submodule of $\mathfrak{X}$. Then if $\mathfrak{X}$ satisfies* (G2), *so does $\mathfrak{Y}$. Similarly if $\mathfrak{X}$ satisfies* (G3), *so does $\mathfrak{Y}$.*

PROOF. If $\mathfrak{U}$ is an accessible submodule of $\mathfrak{Y}$, then $\mathfrak{U}$ is an accessible submodule of $\mathfrak{X}$. Thus if $\mathfrak{X}$ satisfies (G2), $\mathfrak{Y}$ and $\mathfrak{U}$ are both normal in $\mathfrak{X}$, from which it is easily deduced that $\mathfrak{U}$ is normal in $\mathfrak{Y}$. Thus $\mathfrak{Y}$ satisfies (G2).

The other case is trivial, since accessible submodules of $\mathfrak{Y}$ are accessible submodules of $\mathfrak{X}$.

LEMMA 11. *Let $\mathfrak{X}$ be an $\mathfrak{A}$-module and let $\mathfrak{Y}$ be the image of $\mathfrak{X}$ under an $\mathfrak{A}$-epimorphism ε such that* Ker ε *is accessible. Then if $\mathfrak{X}$ satisfies* (G2), *so does $\mathfrak{Y}$. Similarly if $\mathfrak{X}$ satisfies* (G3), *so does $\mathfrak{Y}$.*

PROOF. Let $\mathfrak{V} = (V^*, V)$ be an accessible submodule of $\mathfrak{Y}$. By Lemma 9, the inverse image $\mathfrak{V}\varepsilon^{-1}$ is an accessible submodule of $\mathfrak{X}$ and hence, if $\mathfrak{X}$ satisfies (G2), $\mathfrak{V}\varepsilon^{-1}$ is normal in $\mathfrak{X}$. Thus if $\mathfrak{V}\varepsilon^{-1} = (W^*, W)$, we have $W \cap X^* = W^*$. Suppose that $v \in V \cap Y^*$. Then $v = w\varepsilon$ for some $w \in W$ and also $v = x^*\varepsilon$ for some $x^* \in X^*$. Thus $(w - x^*)\varepsilon = 0$, so that $w - x^* \in W$, showing that $x^* \in W$. But then $x^* \in W \cap X^* = W^*$, so that $v = x^*\varepsilon \in V^*$. Therefore $V \cap Y^* \subseteqq V^*$ and hence $\mathfrak{V}$ is normal, so that $\mathfrak{Y}$ satisfies (G2).

Suppose again that $\mathfrak{V}$ is an accessible submodule of $\mathfrak{Y}$ and that $y^* \in Y^*$. Then there exists $x^* \in X^*$ such that $y^* = x^*\varepsilon$ and we have

$$(\mathfrak{V} : y^*) = (\mathfrak{W} : x^*),$$

where $\mathfrak{W} = \mathfrak{V}\varepsilon^{-1}$. By Lemma 9, $\mathfrak{W}$ is accessible and so, if $\mathfrak{X}$ satisfies (G3), $(\mathfrak{V} : y^*)$ is accessible; therefore $\mathfrak{Y}$ satisfies (G3).

The above lemmas can be expressed in terms of homomorphisms. For this purpose, it is convenient to introduce the following definition.

DEFINITION 3. An $\mathfrak{A}$-homomorphism will be called *accessible* if its image and kernel are accessible modules.

Lemma 10 is equivalent to the following:

Let $\mu: \mathfrak{Y} \to \mathfrak{X}$ be an accessible $\mathfrak{A}$-monomorphism. Then if $\mathfrak{X}$ satisfies (G2), so does $\mathfrak{Y}$; if $\mathfrak{X}$ satisfies (G3), so does $\mathfrak{Y}$.

Similarly Lemma 11 is equivalent to:

Let $\varepsilon: \mathfrak{X} \to \mathfrak{Y}$ be an accessible $\mathfrak{A}$-epimorphism. Then if $\mathfrak{X}$ satisfies (G2), so does $\mathfrak{Y}$; if $\mathfrak{X}$ satisfies (G3), so does $\mathfrak{Y}$.

7. Modules for general pseudo-rings

The above considerations suggest that the modules satisfying (G1), (G2) and (G3), together with the accessible homomorphisms of these modules, might play an important part in the study of the structure of pseudo-rings in general. If this is to be so, we should expect to be able to construct a category whose objects are these modules and whose morphisms are the accessible homomorphisms. However the composition of two accessible homomorphisms is not necessarily accessible. To rectify this, we consider the following additional condition.

(G4). The intersection of two accessible submodules is accessible.

Then we have the following result.

LEMMA 12. *Let* $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ *be* $\mathfrak{A}$*-modules and suppose that* $\mathfrak{Y}$ *satisfies* (G4). *If* $\alpha: \mathfrak{X} \to \mathfrak{Y}$ *and* $\beta: \mathfrak{Y} \to \mathfrak{Z}$ *are accessible homomorphisms, then their composition* $\alpha \circ \beta: \mathfrak{X} \to \mathfrak{Z}$ *is accessible.*

PROOF. Since α is accessible, Im α is an accessible submodule of $\mathfrak{Y}$. By Lemma 8, Im $(\alpha \circ \beta) = (\text{Im } \alpha)\beta$ is accessible.

Since β is accessible, Ker β is an accessible submodule of $\mathfrak{Y}$. Hence, since $\mathfrak{Y}$ satisfies (G4), Im $\alpha \cap$ Ker β is accessible. But

$$\text{Ker } (\alpha \circ \beta) = (\text{Im } \alpha \cap \text{Ker } \beta)\alpha^{-1}$$

and so, by Lemma 9, Ker $(\alpha \circ \beta)$ is accessible.

The next result shows that (G4) imposes no further restrictions in the semi-simple Artinian case.

LEMMA 13. *If* $\mathfrak{A}$ *is a semi-simple Artinian pseudo-ring, then any unital good* $\mathfrak{A}$*-module* $\mathfrak{X}$ *satisfies* (G4).

PROOF. By Theorem 1, $\mathfrak{X}$ satisfies the conditions of Lemma 7. Hence, if $\mathfrak{Y} = (Y^*, Y)$ and $\mathfrak{Z} = (Z^*, Z)$ are accessible submodules of $\mathfrak{X}$, we have

$$Y = Y^* \oplus \sum_{\gamma \in \Gamma} \oplus Y_\gamma^* e_\gamma, \qquad Z = Z^* \oplus \sum_{\gamma \in \Gamma} \oplus Z_\gamma^* e_\gamma.$$

Write $\mathfrak{U} = \mathfrak{Y} \cap \mathfrak{Z} = (Y^* \cap Z^*, Y \cap Z)$. If $u \in Y \cap Z$, there exist $y^* \in Y^*$, $y_\gamma^* \in Y^*$, $z^* \in Z^*$, $z_\gamma^* \in Z_\gamma^*$ such that

$$u = y^* + \sum_{\gamma \in \Gamma} y_\gamma^* e_\gamma = z^* + \sum_{\gamma \in \Gamma} z_\gamma^* e_\gamma$$

from which it follows that $y^* = z^* \in Y^* \cap Z^*$ and $y_\gamma^* = z_\gamma^* \in Y^* \cap Z^*$, so that

$$u \in Y^* \cap Z^* + (Y^* \cap Z^*)A.$$

Therefore $\mathfrak{U}$ is accessible.

We now prove the main result of this section.

THEOREM 3. *For any pseudo-ring* $\mathfrak{A}$, *there exists an exact category* $\mathfrak{C}(\mathfrak{A})$, *whose objects are the* $\mathfrak{A}$*-modules satisfying* (G1), (G2), (G3) *and* (G4), *and whose morphisms are the accessible homomorphisms of these modules.*

When $\mathfrak{A}$ *is a semi-simple Artinian pseudo-ring,* $\mathfrak{C}(\mathfrak{A})$ *is abelian.*

PROOF. By Lemma 12, the composition of two morphisms is again a morphism. By (G1), the identity mapping of each object is a morphism. Hence $\mathfrak{C}(\mathfrak{A})$ is a category.

Given any morphism $\alpha\colon \mathfrak{X}\to\mathfrak{Y}$ in $\mathfrak{C}(\mathfrak{A})$, $\operatorname{Ker}\alpha$ is an accessible submodule of $\mathfrak{X}$ and $\operatorname{Im}\alpha$ is an accessible submodule of $\mathfrak{Y}$. By Lemmas 10 and 11, $\operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha$ are objects in $\mathfrak{C}(\mathfrak{A})$. Since $\mathfrak{Y}$ satisfies (G2), $\operatorname{Im}\alpha$ is normal in $\mathfrak{Y}$ and so $\operatorname{Coker}\alpha = \mathfrak{Y}-\operatorname{Im}\alpha$ is defined. If $\mu\colon \mathfrak{U}\to\mathfrak{B}$ is a monomorphism in $\mathfrak{C}(\mathfrak{A})$, then $\mathfrak{U}$ is isomorphic with the kernel of the natural epimorphism $\mathfrak{B}\to\mathfrak{B}-\operatorname{Im}\mu$. If $\varepsilon\colon \mathfrak{U}\to\mathfrak{B}$ is an epimorphism in $\mathfrak{C}(\mathfrak{A})$, then $\mathfrak{B}$ is isomorphic with the cokernel of the natural monomorphism $\operatorname{Ker}\varepsilon\to\mathfrak{U}$. Moreover, given any morphism $\alpha\colon \mathfrak{X}\to\mathfrak{Y}$ in $\mathfrak{C}(\mathfrak{A})$, we can write $\alpha=\varepsilon\mu$, where $\varepsilon\colon\mathfrak{X}\to\operatorname{Im}\alpha$ is an epimorphism and $\mu\colon \operatorname{Im}\alpha\to\mathfrak{Y}$ is a monomorphism. Thus it is easily seen that the category $\mathfrak{C}(\mathfrak{A})$ is exact (see [2], p. 18).

Suppose now that $\mathfrak{A}$ is a semi-simple Artinian pseudo-ring. By Theorem 1, each object in $\mathfrak{C}(\mathfrak{A})$ is isomorphic with a direct sum of accessible irreducible right ideals of $\mathfrak{A}$.

Suppose that $\alpha\colon \mathfrak{X}\to\mathfrak{Y}$ is any $\mathfrak{A}$-homomorphism, where $\mathfrak{X}, \mathfrak{Y}\in\mathfrak{C}(\mathfrak{A})$. Put $\mathfrak{X}=(X^*, X)$ and $\mathfrak{Y}=(Y^*, Y)$. By Lemma 7, we have

$$X = X^*\oplus\sum_{\gamma\in\Gamma}\oplus X^*A_\gamma^*e_\gamma, \qquad Y = Y^*\oplus\sum_{\gamma\in\Gamma}\oplus Y^*A_\gamma^*e_\gamma$$

where

$$x_\gamma^*\in X^*A_\gamma^*,\quad x_\gamma^*e_\gamma = 0 \;\Rightarrow\; x_\gamma^* = 0,$$

$$y_\gamma^*\in Y^*A_\gamma^*,\quad y_\gamma^*e_\gamma = 0 \;\Rightarrow\; y_\gamma^* = 0.$$

Since $\mathfrak{X}$ is accessible by (G1), $\operatorname{Im}\alpha$ is accessible by Lemma 8. Suppose that $\operatorname{Ker}\alpha = (K^*, K)$ and that $x\in K$. There exist $x^*\in X^*$, $x_\gamma^*\in X^*A_\gamma^*$ such that

$$x = x^* + \sum_{\gamma\in\Gamma} x_\gamma^*e_\gamma$$

and so

$$0 = x\alpha = x^*\alpha + \sum_{\gamma\in\Gamma}(x_\gamma^*\alpha)e_\gamma.$$

But clearly $x_\gamma^*\alpha\in(X^*A_\gamma^*)\alpha=(X^*\alpha)A_\gamma^*$, so that $x_\gamma^*\alpha\in Y^*A_\gamma^*$. Hence, since the sum for Y is direct, we have $x^*\alpha=0$, $x_\gamma^*\alpha=0$. It follows that $x^*, x_\gamma^*\in K^*$, whence $K\subseteqq K^*+K^*A$, showing that $\operatorname{Ker}\alpha$ is accessible. Thus we have proved that any $\mathfrak{A}$-homomorphism $\alpha\colon \mathfrak{X}\to\mathfrak{Y}$, where $\mathfrak{X}, \mathfrak{Y}\in\mathfrak{C}(\mathfrak{A})$, is accessible.

It follows that the sum of two morphisms of $\mathfrak{X}$ into $\mathfrak{Y}$ is again a morphism, from which it is easily verified that $\mathfrak{C}(\mathfrak{A})$ is additive. Finally $\mathfrak{C}(\mathfrak{A})$ is abelian, since by Theorem 1, the direct sum of a finite number of objects in $\mathfrak{C}(\mathfrak{A})$ is again in $\mathfrak{C}(\mathfrak{A})$.

Whether the category $\mathfrak{C}(\mathfrak{A})$ will prove to be of significance in a study of the structure theory of general pseudo-rings remains to be seen. It may be that $\mathfrak{C}(\mathfrak{A})$ is unduly restrictive in the sense of cutting out some useful $\mathfrak{A}$-modules. Indeed in extreme cases, $\mathfrak{C}(\mathfrak{A})$ can be trivial; for example if $\mathfrak{A}$ is the psendo-ring $(\mathbf{Z}, \mathbf{Q})$, where $\mathbf{Z}$ is the ring of integers and $\mathbf{Q}$ is the set of rational numbers regarded as a $\mathbf{Z}$-module, then no non-trivial $\mathfrak{A}$-module satisfies (G2). On the other hand, we have the following theorem.

THEOREM 4. *Let $\mathfrak{A}$ be a pseudo-ring which satisfies* (G1) *and* (G3). *Then the Jacobson radical of $\mathfrak{A}$ is the intersection of the kernels of the irreducible $\mathfrak{A}$-modules which belong to the category $\mathfrak{C}(\mathfrak{A})$.*

PROOF. Let $\mathfrak{Y}$ be an irreducible $\mathfrak{A}$-module. Then there exists a maximal modular normal right ideal $\mathfrak{B}=(B^*, B)$ of $\mathfrak{A}$ such that $\mathfrak{Y}$ is $\mathfrak{A}$-isomorphic with $\mathfrak{A}-\mathfrak{B}$. Defining $B_1=B^*+B^*A$, we obtain an accessible normal right ideal $\mathfrak{B}_1=(B^*, B_1)$ of $\mathfrak{A}$. As shown in a previous paper ([4], Theorem 3), $\mathfrak{B}_1$ is also a maximal modular right ideal of $\mathfrak{A}$ and $\mathfrak{A}-\mathfrak{B}_1$ is irreducible. Since $B_1 \subseteqq B$, Ker $(\mathfrak{A}-\mathfrak{B}_1) \subseteqq$ Ker $(\mathfrak{A}-\mathfrak{B})$. Hence the Jacobson radical of $\mathfrak{A}$, which is the intersection of the kernels of the irreducible $\mathfrak{A}$-modules, is the intersection of the kernels of those of the form $\mathfrak{A}-\mathfrak{B}_1$, where $\mathfrak{B}_1$ is accessible.

We complete the proof by showing that $\mathfrak{A}-\mathfrak{B}_1 \in \mathfrak{C}(\mathfrak{A})$. Since $\mathfrak{A}-\mathfrak{B}_1$ is irreducible it is accessible and has no non-trivial accessible submodules, so that conditions (G1), (G2) and (G4) are satisfied. To verify (G3), it is sufficient to consider the quotients of the form $(0: x^*)$, where x^* is of the form a^*+B_1 for some $a^* \in A^*$, since the only non-zero accessible submodule of $\mathfrak{A}-\mathfrak{B}_1$ is $\mathfrak{A}-\mathfrak{B}_1$ itself. But $(0: x^*)=$ $=(\mathfrak{B}_1: a^*)$. Since $\mathfrak{B}_1$ is an accessible submodule of $\mathfrak{A}$, and $\mathfrak{A}$ satisfies (G3), $(\mathfrak{B}_1: a^*)$ is accessible. Hence $(0: x^*)$ is accessible, which shows that $\mathfrak{A}-\mathfrak{B}_1$ satisfies (G3).

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References

[1] N. JACOBSON, *Structure of rings*, Amer. Math. Soc. Coll. Publ. Vol. XXXVII (Providence, R. I., 1956).
[2] B. MITCHELL, *Theory of categories* (New York and London: Academic Press, 1965).
[3] E. M. PATTERSON, The Jacobson radical of a pseudo-ring, *Math. Zeitschr.*, **89** (1965), pp. 348—364.
[4] E. M. PATTERSON, The Wedderburn-Artin theorem for pseudo-rings, *Math. Zeitschr.*, **98** (1967), pp. 31—41.

INDEX

Clay, J. R., The group of left distributive multiplications on an Abelian group 221
Krauss, P. H., Representation of conditional probability measures on Boolean algebras 229
Steinfeld, O., Primelemente und Primradikale in gewissen verbandsgeordneten algebraischen Strukturen 243
Harary, F. and *Palmer, E. M.*, On the point-group and line-group of a graph 263
Tandori, K., Bemerkung zur starken Summation der Fourierreihen 271
Ruzsa, I., Ein neues formales System der deontischen Logik 287
Rotman, B., On the comparison of order types 311
Ennola, V., On the greatest common divisor of two values of a polynomial 329
Bambah, R. P. and *Woods, A. C.*, On minimal density of plane coverings by circles 337
Wiegandt, R., Radical and semi-simplicity in categories 345
Halász, G., Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen 365
Fiala, F., Standard-Ultraantifilter im Verband aller Komponenten einer *l*-Gruppe 405
Erdős, P. and *Turán, P.*, On some problems of a statistical group-theory. IV 413
Turner, R. E. L., An extremal problem 437
Kátai, I., Some problems on the iteration of multiplicative number-theoretical functions 441
Patterson, E. M., Modules for pseudo-rings 451

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