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QUEUES WITH BATCH ARRIVALS. I

By

F. G. FOSTER (London)

(*Presented by* A. RÉNYI)

1. Introduction. The following single server queueing system is considered in this paper:

(i) Batches of exactly r units arrive at the sequence of instants, $\tau_1, \tau_2, \ldots, \tau_n, \ldots$, such that the inter-arrival times, $\tau_{n+1} - \tau_n > 0$ $(n = 1, 2, \ldots)$, are identically distributed independent random variables with common distribution function $F(x)$. Put

$$\varphi(s) = \int_0^\infty e^{-sx} dF(x), \quad \alpha = \int_0^\infty x dF(x) \quad \text{and} \quad \lambda = 1/\alpha.$$

We suppose $\alpha < \infty$.

(ii) Units are served individually by a single server. Since the units of a batch arrive simultaneously, we shall suppose that they are ordered for purposes of service. Batches are served in order of arrival. Denote by χ_n the service time of the n^{th} unit to be served. We suppose that $\{\chi_n\}$ $(n = 1, 2, \ldots)$ is a sequence of identically distributed independent positive random variables, independent also of the sequence $\{\tau_n\}$, and that their common distribution function, $H(x)$, is the exponential distribution:

$$H(x) = \mathbf{P}[\chi_n \leqq x] = 1 - e^{-\mu x} \qquad (x \geqq 0).$$

Put $\beta = \int_0^\infty x dH(x)$. Then $\mu = 1/\beta$. Define $\varrho = \lambda/\mu$.

In the terminology of [3], the system we consider has the 1-input (arrivals) untriggered with input quantity constantly r and a general distribution for the 1-input time. The 0-input (departures) is triggered with input quantity constantly unity and an exponential distribution for the 0-input time. The system has infinite capacity. On account of the characteristic property of the exponential distribution we have the alternative of supposing that the 0-input is untriggered also but with controlled input quantity: the input being virtual whenever the system contains no 1's. In other words, service begins rom time to time whether or not there are any units in the system, and if at the end of a service time a unit is present, it departs, otherwise nothing happens, and a fresh service begins.

Such batch-size queueing processes do not appear to have been treated explicitly in the literature, although they have obvious applications. They are, however, implicit in the work of ERLANG (see [1]) and WISHART [9]. These authors suppose that a service time devoted to one unit is composed of r consecutive phases. If, instead, we think of the unit as composed of r sub-units corresponding to the phases of service, we have the idea of batch arrivals. Justification for the explicit consideration of batch arrivals systems resides in the fact that the results one can obtain are elegant, and a natural generalization of the case of unit arrivals, as treated for example in [5] and [7]. This paper covers much the same ground as [9], but the analysis is different and the results obtained here are in fact new. (Cf. my remarks in the Discussion of [8].)

Denote by $\xi(t)$ the number of units in the system, including the one being served, at the instant t and put $\xi_n=\xi(\tau_n-0)$ $(n=1,2,\ldots)$. The main result of this paper is the determination of the limiting distribution,

$$p_j=\lim_{n\to\infty}\mathbf{P}[\xi_n=j].$$

I am indebted to Dr. L. TAKÁCS for suggesting a substantial improvement in my original method of proof. The distribution $\{p_j\}$ exists and is independent of the initial state of the system if and only if $r\varrho<1$. The proof of this statement follows the same lines as that for the case $r=1$, as given in [2].

The limiting distribution of the waiting time for an arbitrary unit will also be derived.

2. Let $\{\nu_n\}$ $(n=1,2,\ldots)$ be a sequence of identically distributed independent random variables with distribution

$$k_j=\mathbf{P}[\nu_n=j] \qquad (j=0,1,2,\ldots)$$

where

$$k_j=\int_0^\infty e^{-\mu x}\frac{(\mu x)^j}{j!}\,dF(x).$$

Then ν_n is thought of as the number of real or virtual departures during the n^{th} inter-arrival time.

Put $K(z)=\sum_{j=0}^{\infty}k_jz^j$. We note that $K(z)=\varphi\{\mu(1-z)\}$. We assume $r\varrho<1$. Then it follows from Rouché's theorem that the equation

$$K(z)=z^r \tag{1}$$

has exactly r roots (distinct or coincident) inside the circle $|z|=1$. For $K'(1)=-\mu\varphi'(0)=1/\varrho>r$, and so for some small positive δ, $K(1-\delta)<(1-\delta)^r$. Therefore, on the circle $|z|=1-\delta$, $|K(z)|\leq\Sigma k_j|z|^j<(1-\delta)^r=|z^r|$. Denote the roots by $z=\gamma_j$ $(j=1,2,\ldots,r)$. Clearly, one and only one of these roots is real and positive.

Define $P(z)=\sum_{j=0}^{\infty}p_jz^j$.

THEOREM 1.

$$P(z)=\prod_{j=1}^{r}\frac{1-\gamma_j}{1-\gamma_jz}, \tag{2}$$

the result being true whether or not some of the roots, γ_j, are coincident.

PROOF. By virtue of the characteristic property of the exponential distribution referred to in Section 1,

$$\xi_{n+1}=\max[\xi_n+r-\nu_n,0].$$

Letting $n\to\infty$ and taking generating functions, we have, for $|z|=1$,

$$P(z)=P(z)z^rK(z^{-1})+\sum_{j=0}^{\infty}c_j(1-z^{-j}),$$

where $\{c_j\}$ $(j=0,1,2,\ldots)$ is a sequence of real non-negative constants for which $\sum_{j=0}^{\infty}c_j=p_0$. Therefore

$$P(z)=\frac{\Sigma c_j(1-z^{-j})}{1-z^rK(z^{-1})}.$$

We now have to determine the constants c_j, and for this purpose we consider the zeros of the denominator. Clearly, there are exactly r zeros outside the unit circle, and these are $1/\gamma_j$ $(j=1,2,\ldots,r)$. Now consider the function

$$A(z)=P(z)\prod_{j=1}^{r}(1-\gamma_jz).$$

Since $P(z)$ is the generating function of a probability distribution, $A(z)$ must be regular for $|z|\leq1$. Now let $A(z)$ be defined for $|z|>1$ by

$$A(z)=\frac{\prod_{j=1}^{r}(1-\gamma_jz)\sum_{j=0}^{\infty}c_j(1-z^{-j})}{1-z^rK(z^{-1})}.$$

Since in this expression all the zeros of the denominator outside the circle $|z|=1$ are also zeros of the numerator, it follows that $A(z)$ is regular for $|z|>1$. Therefore, by analytic continuity, $A(z)$ is defined and regular for all z. Since, moreover, $A(z)=o(|z|)$ as $|z|\to\infty$, it follows that $A(z)=A$, a constant independent of z. Therefore

$$P(z)=\frac{A}{\prod_{j=1}^{r}(1-\gamma_j z)},$$

and from $P(1)=1$ we obtain finally (2).

We note that the distribution $\{p_j\}$ is thus formally the convolution of r geometric "distributions", if we allow complex probabilities. We have here a natural generalization of the known result for $r=1$ (see [5]).

EXAMPLE 1. *Inter-arrival times exponentially distributed.*

Suppose

$$F(x)=1-e^{-\lambda x} \qquad (x\geqq 0).$$

Then $\varphi(s)=\lambda/(\lambda+s)$. Therefore $K(z)=\varphi\{\mu(1-z)\}=\lambda/\{\lambda+\mu(1-z)\}=$ $=\varrho/(\varrho+1-z)$. Therefore the r roots γ_j are the roots of

$$\frac{\varrho}{\varrho+1-z}=z^r,$$

inside the circle $|z|=1$. This equation can be written,

$$(3) \qquad (1-z)z^r+\varrho z^r=\varrho,$$

which, being a polynomial equation of degree $r+1$, has exactly $r+1$ roots, and one of them is seen to be $z=1$.

Now consider the zeros of the expression

$$(1-z)\prod_{j=1}^{r}(1-\gamma_j z).$$

They are

$$1, \gamma_1^{-1}, \gamma_2^{-1}, \ldots, \gamma_r^{-1}.$$

But these are the reciprocals of the roots of equation (3). Therefore they are the roots of the equation

$$(1-z^{-1})z^{-r}+\varrho z^{-r}=\varrho,$$

i. e.

$$(4) \qquad 1-z\{1+\varrho(1-z^r)\}=0.$$

Thus we have the identity

$$(1-z)\prod_{j=1}^{r}(1-\gamma_j z)\equiv 1-z\{1+\varrho(1-z^r)\}.$$

It follows that, in this case, we have

$$P(z)=\frac{(1-z)\prod_{j=1}^{r}(1-\gamma_j)}{1-z\{1+\varrho(1-z^r)\}}.$$

From $P(1)=1$ we obtain $\prod_{j=1}^{r}(1-\gamma_j)=1-r\varrho$, so that finally

(5) $$P(z)=\frac{(1-r\varrho)(1-z)}{1-z\{1+\varrho(1-z^r)\}}.$$

Thus in the special case of exponential inter-arrival times, the generating function $P(z)$ can be expressed in a form not explicitly involving the roots γ_j.

EXAMPLE 2. *Inter-arrival times having an Erlang distribution.*

Suppose

$$F(x)=1-\sum_{j=0}^{k-1}\frac{(\lambda kx)^j}{j!}e^{-\lambda kx}.$$

Then $\varphi(s)=\{\lambda k/(\lambda k+s)\}^k$, and so $K(z)=[\varrho/\{\varrho+(1-z)k^{-1}\}]^k$. The roots γ_j are thus the roots of

(6) $$\varrho^k=z^r\{\varrho+(1-z)k^{-1}\}^k,$$

inside the circle $|z|=1$. This equation, being a polynomial equation of degree $r+k$, has $r+k$ roots. We know that $r+1$ of them are

$$1,\gamma_1,\gamma_2,\ldots,\gamma_r.$$

Let the other $k-1$ roots, which will be outside the circle $|z|=1$, be

$$\alpha_1,\alpha_2,\ldots,\alpha_{k-1}.$$

Now the equation whose roots are the reciprocals of the roots of (6) is

$$\varrho^k=z^{-r}\{\varrho+(1-z^{-1})k^{-1}\}^k,$$

i. e.

(7) $$(k\varrho)^k z^{r+k}-\{(1+k\varrho)z-1\}^k=0.$$

The left-hand side of this equation may therefore be identified with

$$(-1)^{k+1}(1-z)\prod_{j=1}^{r}(1-\gamma_j z)\prod_{j=1}^{k-1}(1-\alpha_j z).$$

Therefore we have, in this case,

$$P(z)=\frac{(-1)^{k+1}(1-z)\Pi(1-\alpha_j z)\Pi(1-\gamma_j)}{(k\varrho)^k z^{r+k}-\{(1+k\varrho)z-1\}^k}.$$

From $P(1)=1$ we obtain

$$\prod_{j=1}^{r}(1-\gamma_j)=\frac{(-1)^{k+1}(k\varrho)^{k-1}k(1-r\varrho)}{\prod_{j=1}^{k-1}(1-\alpha_j)},$$

so that finally

$$P(z)=\frac{(k\varrho)^{k-1}k(1-r\varrho)(1-z)}{(k\varrho)^k z^{r+k}-\{(1+k\varrho)z-1\}^k}\cdot\prod_{j=1}^{k-1}\frac{1-\alpha_j z}{1-\alpha_j}. \tag{8}$$

Thus in the case of Erlang inter-arrival times, the generating function $P(z)$ can be expressed in a form which involves explicitly only the roots α_j of (6), lying outside the unit circle, or equivalently the roots $1/\alpha_j$ of (7), lying inside the unit circle.

When k is odd, all the α_j are complex; when k is even, exactly one α_j is real and positive. This may be seen by a consideration of the pair of curves given by

$$y=(k\varrho)^k z^{r+k} \tag{9}$$

and

$$y=\{(1+k\varrho)z-1\}^k, \tag{10}$$

for real z. They intersect at $z=1$, and since $r\varrho<1$, the gradient at $z=1$ of (9) is less than that of (10). When k is even, the curves are seen to intersect at one other point lying between $z=0$ and $z=1$. But when k is odd, they do not intersect at any real value of z such that $|z|<1$.

For example, when $k=2$, there is only one α_j, say α, which must therefore be real and positive, and so we have, in this case,

$$P(z)=\frac{4\varrho(1-r\varrho)(1-z)}{4\varrho^2 z^{r+2}-\{(1+2\varrho)z-1\}^2}\,\frac{1-\alpha z}{1-\alpha}, \tag{11}$$

where α^{-1} is the unique real zero, lying within the interval $(0,1)$, of the denominator.

3. The waiting time distribution of an arbitrary arriving unit. We consider first the waiting time of the first unit in an arbitrary arriving batch. The waiting time is defined as the time which elapses between the instant at which the unit arrives and the instant at which its service begins. Denote by $\eta(t)$ the waiting time which the unit would have if it arrived at time t, and define $\eta_n=\eta(\tau_n-0)$. Thus η_n is the waiting time of the first unit in the n^{th} arriving batch. We consider the limiting distribution,

$$W(x)=\lim_{n\to\infty}\mathbf{P}\,[\eta_n\leq x].$$

Put

$$\Omega(s)=\int_0^\infty e^{-sx}\,dW(x).$$

THEOREM 2.

$$\Omega(s)=\prod_{j=1}^{r}\frac{1-\gamma_j}{1-\dfrac{\gamma_j\mu}{\mu+s}}. \tag{12}$$

PROOF. Each unit has a service time whose distribution is exponential with Laplace transform $\mu/(\mu+s)$. By the characteristic property of the exponential distribution, we can suppose that the service time of the unit at the head of the queue re-commences at the instant of the arrival of a batch. If an arriving batch finds j units in the system, the waiting time of the first unit in the batch will have Laplace transform $\{\mu/(\mu+s)\}^j$. The asymptotic probability of j units in the system is p_j. Therefore

$$\Omega(s)=\Sigma p_j\left(\frac{\mu}{\mu+s}\right)^j=P\left(\frac{\mu}{\mu+s}\right),$$

which gives the required result.

We note that, if we allow complex probabilities, the waiting time distribution is formally the convolution of r exponential "distributions", with concentrations $1-\gamma_j$ $(j=1,2,\ldots,r)$ at the origin.

COROLLARY. The waiting time distribution of a random unit in a batch has Laplace transform

$$\prod_{j=1}^{r}\frac{1-\gamma_j}{1-\dfrac{\gamma_j\mu}{\mu+s}}\cdot\frac{1}{r}\sum_{j=0}^{r-1}\left(\frac{\mu}{\mu+s}\right)^j. \tag{13}$$

4. Relationship with the unit arrivals system G/E$_r$/1. Let us now consider that the batches retain their identity in the queue: a batch is being served until its last member departs. Then if ξ is the number of units facing an arriving batch, the number of batches facing an arriving batch will be ζ where

$$\zeta=\begin{cases}0 & \text{if } \xi=0,\\ \left[\dfrac{\xi-1}{r}\right]+1 & \text{if } \xi\neq 0,\end{cases}$$

where $[x]$ denotes the greatest integer not greater than x. We may now interpret the random variable ζ as the number of units facing an arbitrary arriving unit in the unit arrivals system, G/E$_r$/1, which has mean inter-arrival time

$1/\lambda$, and an Erlang service time distribution with a mean of r/μ. The traffic intensity is thus $r\varrho$.

We consider the distribution of ζ. Define

$$q_j = \mathbf{P}[\zeta = j],$$

and put $Q(z) = \sum_{j=0}^{\infty} q_j z^j$. Now define

$$P_j = \sum_{i=0}^{j} p_i \quad \text{and} \quad Q_j = \sum_{i=0}^{j} q_i.$$

Then

$$\sum_{j=0}^{\infty} P_j z^j = \frac{P(z)}{1-z} \quad \text{and} \quad \sum_{j=0}^{\infty} Q_j z^j = \frac{Q(z)}{1-z}.$$

We have

$$Q_0 = P_0, \quad Q_1 = P_r, \quad Q_2 = P_{2r},$$

and generally,

$$Q_j = P_{jr}.$$

Therefore

$$\frac{Q(z)}{1-z} = \sum_{j=0}^{\infty} P_{jr} z^j.$$

But

$$P_j = \frac{1}{2\pi i} \int_C \frac{P(v)}{1-v} \frac{dv}{v^{j+1}},$$

where C is a contour around the origin excluding the poles of $P(z)/(1-z)$. Therefore

$$\frac{Q(z)}{1-z} = \sum_{j=0}^{\infty} \frac{z^j}{2\pi i} \int_C \frac{P(v)}{1-v} \frac{dv}{v^{j+1}},$$

so that

$$Q(z) = \frac{1-z}{2\pi i} \int_C \frac{P(v)}{v(1-v)} \frac{dv}{(1-v^{-r}z)}. \tag{14}$$

The poles of the integrand within C are at

$$v = \omega^j z^{1/r} \qquad (j = 1, 2, \ldots, r)$$

where ω^j is an r^{th} root of unity. The residue at $v = \omega^j z^{1/r}$ is

$$\frac{1}{r} \frac{P(\omega^j z^{1/r})}{1-\omega^j z^{1/r}}.$$

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Том XII — Вып. 1–2

РЕЗЮМЕ

ОЧЕРЕДИ ИЗ ГРУПП

Ф. Г. Фостер (Лондон)

Пусть прибывают в очередь группы из r единиц, которые обслуживаются по одному. Пусть промежутки времени между прибытием групп имеют любое данное распределение вероятностей, и времени обслуживания имеют показательное распределение. В работе определено предельное распределение числа ожидающих единиц во время прибытия новой пруппы и времени ожидания первого члена группы. Рассматриваются также некоторые частные случаи общей проблемы и связь рассмотренной проблемы с обычной теорией очереди, где единицы прибывают по одному.

О ПЕРИОДИЧЕСКИХ РЕШЕНИЯХ НЕКОТОРЫХ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА С ПЕРИОДИЧЕСКИМИ КОЭФФИЦИЕНТАМИ

И. Бихари (Будапешт)

Доказательство существования периодических решений уравнений с периодическими коэффициентами встречается с серьезными трудностями уже и в случае линейных уравнений. В случае нелинейных уравнений трудности лишь значительно возрастают. Для этой цели служат сложные методы, разработанные ввиду теоретической и практической важности вопроса. Поэтому кажется неожиданным, что в случае уравнения $y''+\varphi(x)f(y)h(y')=0$ (которое не менее специально чем изучавшиеся в литературе) при T-периодичном $\varphi(x)$ какими простыми средствами можно придти к результату. Настоящая работа дает условие существования T- и $2T$-периодичного решения, состоящего из $2n$ четвертьволн, т. е. получаем возможность ознакомиться и с более тонкой структурой периодических решений. Аналогичные результаты получаются для линейного уравнения $y''+[\alpha+\beta\varphi(x)]y=0$ и уравнения $y''+\varphi(x)f(y,y')=0$, где $f(u,v)$ однородная функция первой степени и $\operatorname{sg} f(u,v)=\operatorname{sg} u$. Работа объясняет, почему линейное уравнение $y''+\varphi(x)y=0$ в общем случае не имеет периодических решений.

СТАНДАРТНЫЕ ИДЕАЛЫ В СТРУКТУРАХ

Г. Гретцер и Э. Т. Шмидт (Будапешт)

Цель работы обобщением нейтрального идеала определить такой тип идеалов, который играл бы в теории структур роль, подобную роли нормальных делителей в теории групп. При этом требуется, чтобы этот тип идеалов обладал важнейшими свойствами нейтральных идеалов, что делает возможным обобщение теорем о нейтральных идеалах.

Элемент s (идеал S) структуры L называется стандартным, если для любых $x, y \in L$ (для любых идеалов X, Y структуры L) выполняется соотношение

$$x \cap (s \cup y) = (x \cap s) \cup (x \cap y) \quad ((X \cap (S \cup Y) = (X \cap S) \cup (X \cap Y)).$$

Теорема 1. Для элемента s структуры L следующие условия эквивалентны:

(α) элемент s — стандартный;

(β) для всех элементов u и t структуры L, для которых $u \leqq s \cup t$, имеет место соотношение

$$u = (u \cap s) \cup (u \cap t);$$

(γ) отношением конгруентности L является отношение Θ_s, для которого $x \equiv y\,(\Theta_s)$ выполняется в том и только в том случае, если $(x \cap y) \cup s_1 = x \cup y$ для некоторого $s_1 \leqq s$;

(δ) для любых элементов x и y структуры L

(i) $$s \cup (x \cap y) = (s \cup x) \cap (s \cup y),$$

(ii) из соотношений $s \cup x = s \cup y$ и $s \cap x = s \cap y$ следует равенство $x = y$.

Аналогичная теорема может быть получена для стандартных идеалов. Из этих двух теорем могут быть получены наиболее важные свойства стандартных элементов и идеалов (теоремы 3—6, леммы 1—9).

Доказывается, что понятие стандартного и нейтрального элемента совпадает в случае слабомодулярных структур, являющихся общим обобщением модулярных структур и структур с относительным дополнением (теоремы 7 и 8).

Авторы доказывают, что в структурах с отрезочным дополнением (где всякий отрезок $[0, a]$, как структура, дополнительный) ядро гомоморфизма и понятие стандартного идеала совпадают (теорема 11), что является обобщением одной теоремы Биркгофа [6]. Отсюда получается обобщение одного результата Дилуэрта [8] и одной теоремы Ванга [34].

Приводится ряд примеров того, как с помощью „словаря"

подгруппа → идеал
нормальный делитель → стандартный идеал
фактор-группа → фактор-структура
групповое действие → объединение

ряд теорем теории групп может быть переформулирован для структур. Так получаются теоремы изоморфизмы, лемма Цассенхауса, теорема Жордана—Гёльдера—Шрейера. Перефразировка проблемы Шрейера о расширении группы также приводит к разрешимой проблеме.

Из дальнейших результатов отметим две теоремы (теорема 21 и 23) о совпадении идеалов, удовлетворяющих первой теореме об изоморфизме, и нейтральных идеалов в специальном классе структур с отрезочным дополнением и в модулярных структурах с локально конечной длиной и нулевым элементом.

ОБ ОДНОМ СВОЙСТВЕ „СЕМЕЙСТВ" МНОЖЕСТВ

П. Эрдёш и А. Хайнал (Будапешт)

Система множеств $\mathfrak{F}$ называется системой со свойством **B**, если существует такое множество B, для которого $B \cap F = 0$ и $F \subseteq\!\!|\!\!\equiv B$ для всякого $F \in \mathfrak{F}$. $\mathfrak{F}$ называется системой со свойством $\mathbf{B}(s)$, если существвет множество B, для которого $0 < \overline{\overline{B \cap F}} < s$, где s любая мощность. $\mathfrak{F}$ обладает свойством $\mathbf{C}(q, r)$, если для всякого $\mathfrak{F}' \subseteq \mathfrak{F}$ из $\overline{\overline{\mathfrak{F}'}} \geqq q$ следует $\overline{\overline{\bigcap_{F \in \mathfrak{F}'} F}} < r$.

Если существует мощность p, для которой в случае $F \in \mathfrak{F}$ $\overline{\overline{F}} = p$, то это обозначается так: $p(\mathfrak{F}) = p$.

Вводятся следующие символы;

$\mathbf{M}(m, p, q, r) \to \mathbf{B}$ обозначает тот факт, что каждая система множеств $\mathfrak{F}$, для которой $\overline{\overline{\mathfrak{F}}} = m$, $p(\mathfrak{F}) = p$, и которая обладает свойством $\mathbf{C}(q, r)$, должна обладать свойством **B**;

$\mathbf{M}(m, p, r) \to \mathbf{B}(s)$ обозначает тот факт, что всякая система множеств, для которой $\overline{\overline{\mathfrak{F}}} = m$, $p(\mathfrak{F}) = p$ и которая обладает свойством $\mathbf{C}(2, r)$, должна обладать свойством $\mathbf{B}(s)$;

$\mathbf{M}(m, p, q, r) \not\to \mathbf{B}$, $\mathbf{M}(m, p, r) \to \mathbf{B}(s)$ обозначает отрицание соответствующих фактов.

Исходя из частных результатов Миллера, доказанных в [1], с помощью общей гипотезы континуума дается почти полная дискуссия выше упомянутых символов.

Отмечается, что с помощью полученных результатов можно сделать несколько выводов относительно возможных теоретико-множественных обобщений теоремы Тихонова, в частности, также с помощью гипотезы континуума доказывается, что топологическое произведение $\aleph_k$ 1-компактных дискретных топологических пространств не k-компактно, где k любое целое число (см. теорему 11).

Кроме того формулируется ряд нерешенных проблем.

О НЕКОТОРЫХ ЗАМЕЧАНИЯХ И ПРОБЛЕМАХ В СВЯЗИ С ОКРАШИВАНИЕМ ГРАФОВ

Ян Мыцельский (Вроцлав)

Первая часть работы рассматривает импликации девяти утверждений. Среди них наряду с утверждениями относительно топологического произведения бикомпактных пространств Хаусдорфа фигурирует и следующее: если каждый частичный граф некоторого графа может быть окрашен n цветами, то это же имеет место для полного графа.

Вторая часть работы доказывает эквиваленцию четырех утверждений. Эквиваленцию первых двух выражений высказывает теорема Куратовского, занимающаяся вложимостью в плоскость конечных графов.

ТЕОРЕМЫ О МАКСИМУМЕ И МИНИМУМЕ И ОБОБЩЕННЫЕ ФАКТОРЫ ГРАФОВ

Т. Галлаи (Будапешт)

Пусть каждой точке X конечного графа Γ без направления соответствуют неотрицательные целые числа $\varkappa(X)$ и $\varkappa'(X)$. Система дуг, состоящая из дуг, попарно не содержащих общих граней, называется совместимой (относительно $\varkappa$ и $\varkappa'$), если в любую точку X попадает не более $\varkappa(X)$ граничных точек и не более $\varkappa'(X)$ внутренних точек, относящихся к системе дуг. (Под дугой понимается путь или петля. Петля — такая окружность, одна из точек которой является двойной граничной точкой.) $\nu_{\max}$ максимум числа дуг совместимых систем дуг. $q = q(A, B, C)$ обозначает следующую систему весов: A, B и C любые такие множества точек графа Γ, что любая точка Γ встречается в одном и только одном из них. Точкам A, B и C соответствуют веса 0, 1 и 1/2. Граням, обе граничные точки которых относятся к A, сопоставляется вес 1; граням, одна из граничных точек которых принадлежит A, а другая C, сопоставляется вес 1/2. Система весов q обладает тем свойством, что сумма весов, относящихся к любой грани Γ и ее граничным точкам, $\geqq 1$. Всем таким системам весов q подходящим образом сопоставляется значение $S(q)$, зависящее от $\varkappa$ и $\varkappa'$. $S_{\min}$ минимум этих значений $S(q)$. Утверждение основной теоремы: $\nu_{\max} = S_{\min}$. Из основной теоремы выводятся теоремы, аналогичные „теореме n цепей" Менгера. Из основной теоремы получается также ряд теорем относительно существование обобщенных факторов. Под обобщенным фактором или $(\varkappa, \varkappa')$-фактором понимается такая совместимая система дуг, из граничных точек относящихся к которой дуг в любую точку X попадает точно $\varkappa(X)$ граничных точек. Эти теоремы в качестве специального случая содержат ряд известных теорем относительно обычных факторов.

ОБ ОБОБЩЕНИИ ТЕОРЕМЫ ПУЛЕН—ЭРМИТА, ОТНОСЯЩЕЙСЯ К ВЕЩЕСТВЕННЫМ КОРНЯМ МНОГОЧЛЕНОВ С ВЕЩЕСТВЕННЫМИ КОЭФФИЦИЕНТАМИ

Н. Обрешков (София)

Автор, обобщая хорошо известную теорему Пулен—Эрмита, доказывает, что если все корни многочлена

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \tag{1}$$

вещественны и комплексные корни многочлена

$$g(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m$$

с вещественными коэффициентами расположены в области

$$|\arg z| \leqq \frac{1}{\sqrt{n}}\,; \tag{2}$$

то все корни многочлена

$$b_0 f^{(m)}(x) + b_1 f^{(m-1)}(x) + \cdots + b_m f(x)$$

вещественны.

Из приложений этой теоремы упомянем следующее:

Если коэффициенты многочлена (1) вещественны и его корни расположены в области (2), то все корни многочлена

$$\frac{a_0}{n!}x^n + \frac{a_1}{(n-1)!}x^{n-1} + \cdots + \frac{a_{n-1}}{1!}x + a_n$$

вещественны.

О СУММЕ СТЕПЕНЕЙ КОМПЛЕКСНЫХ ЧИСЕЛ

Ф. В. Аткинсон (Торонто)

Цель работы доказать гипотезу П. Турана, согласно которой, если выполняется (1), то независимо от n

$$\max_{\nu=1,\ldots,n} \left| z_1^\nu + \cdots + z_n^\nu \right| > \frac{1}{6}. \tag{I}$$

Наилучшая постоянная в (I) не известна. Туран в книге[1] доказал лишь более слабое неравенство, когда вместо $\frac{1}{6}$ стоит $\log 2\left(\sum_{\nu=1}^{n}\frac{1}{\nu}\right)^{-1}$, которое позднее де Брёйн и Утияма улучшили до $(1-\varepsilon)\frac{\log\log n}{\log n}$, если $n > n_0(\varepsilon)$. В книге Турана было показано, что уже применяя его результат, можно дать очень простой метод приближенного решения алгебраических уравнений, применение (I) приводит к существенному упрощению метода.

ОБ ОБОБЩЕНИИ ОДНОГО НЕРАВЕНСТВА ПОЙА И СЕГЁ

Э. Макаи (Будапешт)

Неравенство, данное формулами (1) и (2) работы, и его обобщение для гильбертого пространства, данное формулой (6), дали В. Грёйб и В. Рейнболдт, опиравшиеся при доказательстве на теорию линейных операторов. Настоящая работа дает элементарное доказательство этих же неравенств.

[1] P. TURÁN, *Eine neue Methode in der Analysis und deren Anwendungen* (Budapest, 1953).

НЕКОТОРЫЕ ИНТЕРПОЛЯЦИОННЫЕ СВОЙСТВА МНОГОЧЛЕНОВ ЭРМИТА

К. К. Матур и А. Шарма (Лукноу, Индия)

Пусть $x_1, x_2, \ldots, x_n$ обозначают корни многочлена Эрмита $H_n(x)$. Авторы доказывают, что при четном n существует единственный (0, 2)-интерполяционный многочлен $R_n(x)$ не выше $2n-1$-ой степени, удовлетворяющий условиям

$$R_n(x_\nu) = \alpha_\nu, \quad \left\{\frac{d^2}{dx^2} R_n(x)\right\}_{x=x_\nu} = \beta_\nu \qquad (\nu = 1, 2, \ldots, n).$$

Здесь $\alpha_1, \alpha_2, \ldots, \alpha_n$ и $\beta_1, \beta_2, \ldots, \beta_n$ любые числа. При нечетном n такие многочлены, вообще говоря, не могут быть однозначно определены.

Аналогичную теорему авторы доказывают для так называемых (0, 1, 3)-интерполяционных многочленов.

В обоих случаях при четном n авторы приводят явный вид интерполяционных многочленов.

О КРУГОВЫХ И ШАРОВЫХ ОБЛАКАХ

А. Хеппеш (Будапешт)

Л. Фейеш Тот назвал k-слоевым шаровым облаком множество расположенных между двумя параллельными плоскостями не вклиняющихся друг в друга шаров, если каждая прямая, перпендикулярная к параллельным плоскостям, содержит внутреннюю или граничную точку по крайней мере k шаров, иначе говоря, если шары образуют относительно прямых, перпендикулярных к параллельным плоскостям, „k-кратно непроходимую стену". Аналогичным образом может быть определено на плоскости понятие k-слоевого кругового облака. Под толщиной кругового или шарового облака понимается расстояние между параллельными прямыми или плоскостями, между которыми находится облако. Фейеш Тот [1] доказал, что минимум толщины однослоевого шарового облака равен $2+\sqrt{2}$ (минимум толщины однослоевого кругового облака, очевидно, равен 2).

В настоящей работе доказывается, что минимум толщины k-слоевого кругового облака

$$d_k = (k-1)\sqrt{3} + 2,$$

а минимум толщины k-слоевого шарового облака удовлетворяет неравенству

$$D_k \leqq \left(k + \left[\frac{k-1}{2}\right]\right)\sqrt{3} + 2, \quad \text{если} \quad k \geqq 2.$$

ОБ АБСОЛЮТНОЙ СХОДИМОСТИ ТРИГОНОМЕТРИЧЕСКИХ РЯДОВ С ПРОБЕЛАМИ

П. Сюс (Будапешт)

В работе доказываются следующие теоремы:

Теорема 1. Пусть постоянная $K \geqq 2$. Тогда существует такая последовательность натуральных чисел

$$n_1, n_2, \ldots \qquad \left(\frac{n_{k+1}}{n_k} \geqq K\right),$$

что для любой монотонно убывающей последовательности $a_1, a_2, \ldots$

$$\sum_{k=1}^{\infty} a_k |\sin \pi n_k x| < \infty \tag{1}$$

может выполняться для нецелого x лишь в том случае, если

$$\sum_{k=1}^{\infty} a_k < \infty.$$

Теорема 2. Пусть $n_1, n_2, \ldots$ есть последовательность целых чисел, для которой

$$\lim_{k \to \infty} \frac{n_{k+1}}{n_k} = \infty.$$

Тогда существует такая монотонно убывающая последовательность $a_1, a_2, \ldots$, что, хотя $\sum_{k=1}^{\infty} a_k = \infty$, ряд (1) сходится на множестве мощности континуума.

Теорема 1 является распространением на некоторые ряды с пробелами одной классической теоремы Фату; теорема 2 утверждает, что такое распространение невозможно, если пробелы последовательности $\{n_k\}$ „слишком большие".

ОБ ОДНОЙ ЭКСТРЕМАЛЬНОЙ ЗАДАЧЕ ТЕОРИИ ИНТЕРПОЛИРОВАНИЯ

П. Эрдёш и П. Туран (Будапешт)

Если

$$1 \geqq x_1 > \cdots > x_n \geqq -1, \quad \omega(x) = \prod_{\nu=1}^{n} (x - x_\nu), \tag{1}$$

то фундаментальные многочлены относящегося к (1) интерполирования Лагранжа, как хорошо известно, имеют вид

$$l_k(x) = \frac{\omega(x)}{\omega'(x)(x - x_k)} \qquad (k = 1, 2, \ldots, n), \tag{2}$$

а так называемые фундаментальные многочлены второго рода интерполирования Эрмита—Фейера суть

$$\mathfrak{h}_k(x) = (x - x_k)\, l_k(x)^2 \qquad (k = 1, 2, \ldots, n). \tag{3}$$

Значение фундаментальных многочленов (3) освещено в работе Фейера [2], там же он показал, что при соответствующем выборе узлов (1)

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |\mathfrak{h}_k(x)| < \left(\frac{2}{\pi} + \varepsilon\right) \frac{\log n}{n}, \tag{4}$$

если только $n > n_0(\varepsilon)$. В настоящей работе авторы доказывают, что при любом выборе узлов (1)

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |\mathfrak{h}_k(x)| \geqq \left(\frac{2}{\pi} - c_1 \frac{\log \log n}{\log n}\right) \frac{\log n}{n},$$

где c_1 соответственно выбранная положительная постоянная, асимптотически определяя, таким образом, минимум

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} \mathfrak{h}_k(x).$$

Аналогичным образом авторы доказывают, что при любом выборе узлов (1)

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| \geqq \left(\frac{2}{\pi} - c_1 \frac{\log \log n}{\log n}\right) \log n. \tag{5}$$

Несколько менее слабую чем (5) оценку можно найти в работе С. Бернштейна. [1]; однако доказательство авторов основывается на совершенно других соображениях

ПРОБЛЕМЫ И РЕЗУЛЬТАТЫ ОТНОСИТЕЛЬНО ИНТЕРПОЛИРОВАНИЯ. II

П. Эрдёш (Будапешт)

Пусть $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$,

$$\omega(x) = \prod_{k=1}^{n} (x - x_k), \quad l_k(x) = \frac{\omega(x)}{\omega'(x)(x - x_k)}.$$

Доказывается, что

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > \frac{2}{\pi} \log n - c_1. \tag{1}$$

(1) обостряет теоремы Бернштейна и Эрдёша — Турана. Известно, что (1) не может быть улучшено, так как если x_k узлы многочленов Чебышева $T_n(x)$, то

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| < \frac{2}{\pi} \log n + c_2.$$

Неизвестна система точек, для которой

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)|$$

принимает свое наименьшее значение.

ОБ ОДНОЙ ПРОБЛЕМЕ БЭРА И ОДНОЙ ПРОБЛЕМЕ УАЙТХЕДА В ТЕОРИИ АБЕЛЕВЫХ ГРУПП

И. Ротмен (Урбана, США)

Автор исследует абелевы группы F, для которых $\mathrm{Ext}(F, T) = 0$ для всех периодических групп T, и те, для которых $\mathrm{Ext}(F, Z) = 0$, где Z группа целых, называя их B-группами и W-группами, соответственно. В счетном случае известно решение обеих проблем: и те и другие группы являются свободными группами.

Автор получает ряд частичных результатов в случае любой мощности. Типичные результаты:

1. Сепарабельная B-группа стройная.

2. Каждая W-группа сепарабельна, стройная и может быть вложена в полную прямую сумму бесконечных циклических групп как серрантная подгруппа.

УПОРЯДОЧЕННЫЕ ПОЛУГРУППЫ

Л. Фукс (Будапешт)

Работа рассматривает вопрос о вложимости в следующие упорядоченные полугруппы:

P: аддитивная полугруппа неотрицательных действительных чисел;

P_1: вещественный отрезок $[0, 1]$, действие $a \cdot b = \min(a + b, 1)$;

P_1^*: отрезок $[0, 1]$ и символ ∞, действие: $a \cdot b = a + b$ или ∞ в зависимости от того, будет ли $a + b \leqq 1$ или > 1.

Теорема 1. Положительно упорядоченная полугруппа тогда и только тогда может быть вложена в P, если она 1. архимедова, 2. не содержит аномальной пары, и 3. не имеет максимального элемента (если она содержит по крайней мере два элемента).

Теорема 3. Архимедова, естественно упорядоченная полугруппа изоморфна по упорядочению одной из подполугрупп от P, P_1 или P_1^*.

Последняя теорема является общим обобщением теорем Гёльдера [4] и Клиффорда [2].

О СИЛЕ СВЯЗАННОСТИ СЛУЧАЙНОГО ГРАФА

П. Эрдёш и А. Реньи (Будапешт)

Пусть $\Gamma_{n,N}$ случайный граф с n вершинами и N ребрами, без петель и без параллельных ребер. $\Gamma_{n,N}$ получается следующим образом: выбираем случайно N из всех возможных $\binom{n}{2}$ ребер, соединяющих n данные вершины $V_1, V_2, \ldots, V_n$; при этом по предположению все возможные $\binom{\binom{n}{2}}{N}$ выборы этих ребер одинакого вероятны.

Пусть для любого (неполного) графа G $c_p(G)$ обозначает минимальное число k, обладающее следующим свойством: можно вычеркнуть из графа G k подходящих вершин (вместе со всеми ребрами, которые инцидентны с этими вершинами) так, что получается несвязный граф. Пусть $c_l(G)$ обозначает минимальное число l, обладающее следующим свойством: можно вычеркнуть из графа G l ребер так, что получается несвязный граф. Наконец, пусть $d(V_k)$ $(k=1, 2, \ldots, n)$ число ребер, исходящих из вершины V_k (валентность вершины V_k) и положим $c(G) = \min\limits_{1 \leq k \leq n} d(V_k)$. Каждая из трех величин $c_p(G)$, $c_l(G)$ и $c(G)$ может быть рассмотрена как мера связанности графа G.

В работе доказывается (теорема 2), что если

$$N(n) = \frac{1}{2} n \log n + \frac{r}{2} n \log \log n + \alpha n + o(n), \tag{1}$$

где r данное целое неотрицательное число и α данное вещественное число, то (обозначая через $\mathbf{P}(\cdot)$ вероятность события в скобках) имеем

$$\lim_{n \to +\infty} \mathbf{P}(c_p(\Gamma_{n, N(n)}) = r) = 1 - e^{-\frac{e^{-2\alpha}}{r!}}, \tag{2}$$

и что (2) остается верным также, если в нем вместо $c_p(\Gamma_{n, N(n)})$ стоит или $c_l(\Gamma_{n, N(n)})$ или $c(\Gamma_{n, N(n)})$.

Далее доказывается (теорема 3), что число вершин случайного графа $\Gamma_{n, N(n)}$ (где $N(n)$ опять определено с помощью (1)), имеющих валентность r, в пределе при $n \to +\infty$ распределено по закону Пуассона с параметром $\lambda = \frac{e^{-2\alpha}}{r!}$.

Частные случаи этих теорем, когда $r = 0$, были доказаны авторами настоящей статьи уже раньше в их работе [4].

Therefore, summing the residues, we obtain

$$Q(z) = \frac{1-z}{r} \sum_{j=1}^{r} \frac{P(\omega^j z^{1/r})}{1-\omega^j z^{1/r}}. \tag{15}$$

We may note that the waiting time distribution for an arbitrary arriving unit in the unit arrivals system $G/E_r/1$ is identical with that for an arbitrary arriving batch in the batch arrivals system, and so is given by formula (12) above. Thus we have a solution in an interesting form to the Wiener—Hopf integral equation studied by LINDLEY [6].

EXAMPLE 3. *Inter-arrival times exponentially distributed.*

The generating function $P(z)$ for the batch arrivals system with exponential inter-arrival times is given by (5). Therefore if we consider the unit arrivals system $M/E_r/1$ with mean inter-arrival time $1/\lambda$ and mean service time r/μ, the generating function $Q(z)$ for the queue-size facing an arbitrary arriving unit is given by

$$\frac{Q(z)}{1-z} = \frac{1-r\varrho}{r} \sum_{j=1}^{r} \frac{1}{1-\omega^j z^{1/r}\{1+\varrho(1-z)\}}.$$

But the $\omega^j z^{1/r}\{1+\varrho(1-z)\}$ $(j=1,2,\ldots,r)$ are the r^{th} roots of $z\{1+\varrho(1-z)\}$. Therefore

$$Q(z) = \frac{(1-r\varrho)(1-z)}{1-z\{1+\varrho(1-z)\}^r}, \tag{16}$$

which is the known result for the system $M/E_r/1$ when the traffic intensity is $r\varrho$.

The waiting time distribution of an arbitrary arriving unit has in this case Laplace transform

$$\Omega(s) = P\left(\frac{\mu}{\mu+s}\right) = \frac{(1-r\varrho)\dfrac{s}{\mu+s}}{1-\dfrac{\mu}{\mu+s}\left\{1+\varrho-\varrho\left(\dfrac{\mu}{\mu+s}\right)^r\right\}} = \frac{1-r\varrho}{1-\dfrac{\lambda}{s}\left\{1-\left(\dfrac{\mu}{\mu+s}\right)^r\right\}},$$

which is the known Pollaczek formula for this case.

5. Further work. Let $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ denote the sequence of instants at which units depart from the batch arrivals system. In a sequel we shall consider the existence of the limiting distributions $\{p_j^*\}$ and $\{p_j^+\}$, defined, respectively, by

$$p_j^* = \lim_{t\to\infty} \mathbf{P}[\zeta(t)=j]$$

and

$$p_j^+ = \lim_{n\to\infty} \mathbf{P}[\xi(\sigma_n+)=j].$$

We shall also examine the relationships existing between the three distributions $\{p_j\}$, $\{p_j^*\}$ and $\{p_j^+\}$.

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ON PERIODIC SOLUTIONS OF CERTAIN SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

By

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(*Presented by* P. TURÁN)

As known, the investigation of the existence of periodic solutions of linear equations with periodic coefficients (e. g. the Hill and Mathieu equations) involves considerable amount of difficulties. The Floquet theory discussing this questions cannot be considered elementary at all. As to non-linear equations (e. g. the Liénard equation) the problem is still much more involved. N. LEVINSON, M. L. CARTWRIGHT, T. WAZEWSKI and other authors elaborated intricate analytical and analytic-topological methods to this end. On the other hand, the proof of the existence of such solutions is often very desirable for the practice too, for a solution like this is connected with some kind of stability (e. g. it forms a limit cycle in the sense of the Poincaré—Bendixson theory).

Even therefore it is surprising how simple tools lead to results concerning the equation

$$y'' + \varphi(x) f(y) h(y') = 0, \tag{1}$$

provided that $\varphi(x)$ is a *positive periodic* function, $f(y)$ and $h(u)$ are like those in [1] (p. 98, Theorems 5 and 6) where — as throughout [1] — only *positive monotone* $\varphi(x)$ was taken into account. The purpose of the present paper is to state results of this character, of course, also for linear equations.

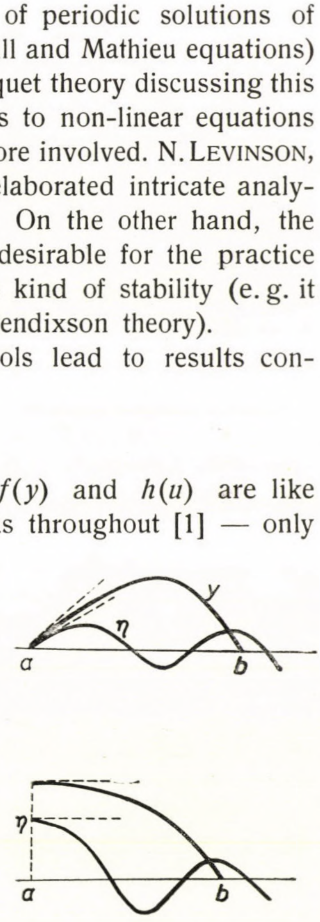

Fig. 1

On p. 102 of [1] a remark says that there is a solution $\eta(x)$ of (1) to any other one $y(x)$ of (1) with an arbitrary number of zeros between two adjacent zeros of $y(x)$ and $\eta(x)$ can be obtained by a convenient (sufficiently small) initial slope or (extreme) value η of $\eta(x)$ at $x = a$ (see Fig. 1). — This assertion must be corrected, for Corollary 8 on p. 87 in [1] does not imply really, as asserted, that the distance of two consecutive zeros tends to zero with the above initial values (see a counterexample below). In case of $\varphi(x) = k = \text{const} > 0$ there is given ([1], p. 88) a formula for this distance (i. e. for the period p, viz.

the corresponding solution is periodic) and it is doubtless that p is decreasing with $|\eta|$ (at linear equations unchanged) (see [1], Theorem 5), but, in general, does not tend to zero with $|\eta|$ or by the initial slope.

To throw light on the things take first the following example:

$$y''+\varphi(x)y\frac{1}{1+\varepsilon^2y'^2}=0. \tag{2}$$

Here $f(y)=y$, $h(u)=\dfrac{1}{1+\varepsilon^2u^2}$ satisfy the mentioned conditions and equation (2) turns to the linear equation $y''+\varphi(x)y=0$ as $\varepsilon\to 0$. If $\varphi(x)=k=\text{const}$, (2) may be solved by quadratures. According to $y(0)=\eta>0, y'(0)=0$, the solution is as follows (see [1], p. 89):

$$\varepsilon\int_y^\eta\frac{du}{\sqrt{-1+\sqrt{1+2\varepsilon^2k(\eta^2-u^2)}}}=x \qquad (\varepsilon>0) \tag{3}$$

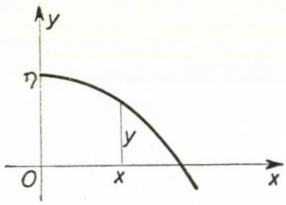

Fig. 2

and the period of $y(x)$ (see [1], p. 87) amounts to

$$p(\varepsilon,\eta,k)=4\varepsilon\int_0^\eta\frac{du}{\sqrt{-1+\sqrt{1+2\varepsilon^2k(\eta^2-u^2)}}}=$$

$$=\frac{2}{\sqrt{k}}\int_0^\eta\frac{\sqrt{1+\sqrt{1+2\varepsilon^2k(\eta^2-u^2)}}}{\sqrt{\eta^2-u^2}}\,du.$$

Replacing u by $\eta\sin z$ we get

$$p(\varepsilon,\eta,k)=\frac{4}{\sqrt{2k}}\int_0^{\frac{\pi}{2}}\sqrt{1+\sqrt{1+2\varepsilon^2k\eta^2\cos^2 z}}\,dz. \tag{4}$$

This value may be made as large as wanted by increasing of $|\eta|$, provided $\varepsilon\neq 0$. On the other hand, p is decreasing with $|\eta|$ and we have

$$p(\varepsilon,0,k)=\frac{2\pi}{\sqrt{k}}=p(0,\eta,k)=p(0,0,k),$$

i. e. p tends with $|\eta|\to 0$ to the period of the linear equation $y''+ky=0$.

We shall prove the following

THEOREM 1. *Let $\varphi(x)$ be a positive continuous periodic function having period T and monotone symmetrical half-periods, further let* $\min\varphi(x)$ *and*

$\max\varphi(x)$ *be denoted by* k *and* K, *respectively. If* $kT^2 \geqq n^2\pi^2$,[1] *then for* $n=1$ (2) *has a periodic solution of period* $2T$ *consisting of two half-waves, while for* $n=n_0>1$ (2) *has all the solutions as for* $n<n_0$ *and also a new one of period* $2T$ *consisting of* $2n_0$ *half-waves, provided that* $n_0=2l+1$ ($l=1,2,\ldots$), *or a solution of period* T *consisting of* n_0 *half-waves, provided that* $n_0=2l$ ($l=1,2,\ldots$).

PROOF. Let $\varphi(x)$ be, say, increasing for $0\leqq x\leqq \frac{T}{2}$, then the solutions y_1, y_2 of the equations

$$y''+ky\frac{1}{1+\varepsilon^2y'^2}=0,\quad y''+Ky\frac{1}{1+\varepsilon^2y'^2}=0$$

and the solution y of (2), all corresponding to the initial conditions $y(0)=\eta$, $y'(0)=0$, satisfy the inequalities (see [1], p. 102)

$$y_2<y<y_1 \qquad (0<x\leqq a),$$
$$y<y_1 \qquad (0<x\leqq b).$$

Here $a<b<c$ denote the first positive zeros of y_2, y, y_1, respectively (see Fig. 3) and $b\leqq\frac{T}{2}$ is assumed too. For the value of c we have

$$c(\varepsilon,\eta,k)=\frac{p(\varepsilon,\eta,k)}{4}$$

(see (4)). Obviously

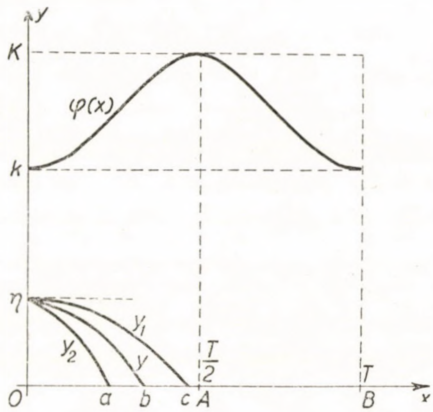

Fig. 3

$$c(\varepsilon,0,k)=\frac{\pi}{2\sqrt{k}} \qquad \text{and} \qquad c(\varepsilon,\infty,k)=+\infty$$

and c is a monotone increasing function of $|\eta|$. Therefore assuming

$$c(\varepsilon,0,k)=\frac{\pi}{2\sqrt{k}}\leqq\frac{T}{2} \quad \text{or} \quad kT^2\geqq\pi^2, \tag{5}$$

there is a value $\eta_0>0$ of η where $b(\varepsilon,\eta_0)=\frac{T}{2}$, because the interval $[a,c]$, with the point $x=b$ in its interior, passes wholly through the point $A\left(\frac{T}{2},0\right)$ as η covers $(0,+\infty)$. At the same time, exactly one quarter-wave of $y(x)$

[1] n denotes an integer.

gets into the interval $0 \leqq x \leqq \frac{T}{2}$. We state that $y(x)$ is periodic with period $2T$. Viz. the symmetrical $\eta(x)$ of $y(x)$ $\left(0 \leqq x \leqq \frac{T}{2}\right)$ with respect to the point A satisfies (2) and thus it is the continuation of $y(x)$ for $\frac{T}{2} \leqq x \leqq T$. Really, putting $T-x$ in (2) for x and regarding the relations

$$\eta(x) = -y(T-x), \quad \eta'(x) = y'(T-x),$$
$$\eta''(x) = -y''(T-x),$$
$$\varphi(T-x) = \varphi(-x) = \varphi(x),$$

we obtain

$$\eta''(x) + \varphi(x)\eta(x)\frac{1}{1+\varepsilon^2\eta'^2(x)} = 0.$$

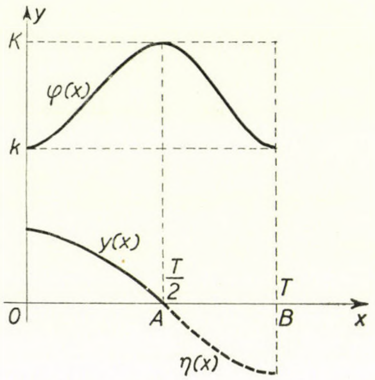

Fig. 4

It can be proved by the same argument that the continuation of $y(x)$ for $T \leqq x \leqq 2T$ is symmetrical with the branch $0 \leqq x \leqq T$ of $y(x)$ to the ordinate $x = T$, etc., thus the periodicity of $y(x)$ by the period $2T$ is clear.

If we assume instead of (5) the condition

$$(6) \qquad c(\varepsilon, 0, k) = \frac{\pi}{2\sqrt{k}} \leqq \frac{T}{2n}$$
$$\text{or} \quad kT^2 \geqq n^2\pi^2 \qquad (n \geqq 2),$$

then by a convenient choice of η we obtain also a new solution $y(x)$ of period T or $2T$, according as $n = 2l$ or $n = 2l+1$, respectively.[2] Viz. for a suitable η exactly n quarter-waves of $y(x)$ will lie in the interval $0 \leqq x \leqq \frac{T}{2}$ with $y'\left(\frac{T}{2}\right) = 0$ or $y\left(\frac{T}{2}\right) = 0$, respectively. This is really ensured by the condition (6), because the monotone increasing of $\varphi(x)$ involves the decreasing of the lengths, amplitudes, areas of the successive quarter-waves (see [1], Theorem 6). The branch $\frac{T}{2} \leqq x \leqq T$ of

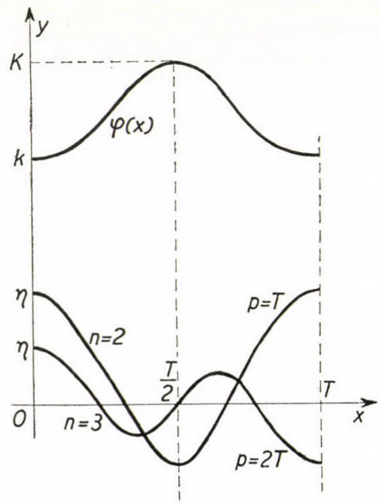

Fig. 5

[2] If (6) is satisfied for $n = n_0$, then it is satisfied for $n < n_0$, too, and the corresponding solutions exist also for $n = n_0$.

$y(x)$ is symmetrical with its own $0 \leqq x \leqq \frac{T}{2}$ branch to the *ordinate* $x = \frac{T}{2}$ or to the *point* $x = \frac{T}{2}$, respectively, etc.

The above statements are concerned with the non-linear case $\varepsilon \neq 0$. For every $\varepsilon \neq 0$ there are periodic solutions, provided that (6) is fulfilled. Let us denote the above periodic solution of period $2T$ for $n = 1$ by $y(x, \varepsilon, \eta)$. If $\varepsilon \to 0$ and (5) holds, then $\eta_0 \to +\infty$, i. e. there is no periodic solution of period $2T$ of the linear equation ($\varepsilon = 0$)

(7) $$y'' + \varphi(x) y = 0,$$

unless $b = \frac{T}{2}$ holds in advance. This is obvious by the fact, too, that the first positive zero of $y(x, 0, \eta)$ (viz. $x = b$) is independent of η.

At all events the condition

$$\frac{\pi}{\sqrt{K}} < T < \frac{\pi}{\sqrt{k}}$$

forms a necessary one for the existence of the above solution.

Although the limit passage $\varepsilon \to 0$ does not lead to periodic solutions, however, we obtain the following — only in part known — result concerning the linear equation

(8) $$y'' + (\alpha + \beta\varphi(x)) y = 0.$$

THEOREM 2. *Let $\varphi(x)$ be a continuous periodic function of period T with monotone symmetrical half-periods. Then α and β may be chosen so that* (8) *should have periodic solutions like those described in Theorem 1.*

PROOF. Let $\varphi(x)$ be e. g. even increasing for $0 \leqq x \leqq \frac{T}{2}$ and letting $k = \alpha + \beta\varphi(0)$, $K = \alpha + \beta\varphi\left(\frac{T}{2}\right)$ we have

$$k = \min(\alpha + \beta\varphi(x)), \quad K = \max(\alpha + \beta\varphi(x)) \qquad (\beta > 0).$$

Given $\eta \neq 0$ let y_2, y_1, y denote the solutions of the equations

$$y'' + Ky = 0, \quad y'' + ky = 0$$

and of (8) — all corresponding to the initial conditions $y(0) = \eta$, $y'(0) = 0$ — and let $a < b < c$ be the first positive zeros of y_2, y, y_1, respectively.

If $K = g(k) > k > 0$ where $g(k)$ is an arbitrary continuous monotone function with $g(k) \to 0$ as $k \to 0$,[3] then the interval $[a, c]$ (Fig. 3), with the

[3] Or more generally $g(0) \leqq \frac{\pi^2}{T^2}$.

point $x=b$ in its interior, passes through the place $x=\frac{T}{2}$ as $k\to 0$. Therefore there exists a value $k=k_0>0$ of k such that $b=\frac{T}{2}$ is satisfied. Then

$$\alpha=\frac{k_0\varphi\left(\frac{T}{2}\right)-K_0\varphi(0)}{\varphi\left(\frac{T}{2}\right)-\varphi(0)},\qquad \beta=\frac{K_0-k_0}{\varphi\left(\frac{T}{2}\right)-\varphi(0)}\qquad (K_0=g(k_0))$$

and $y(x)$ is periodic with period $2T$, etc.

If in the general equation (1) $\varphi(x)=k=\text{const}>0$, the solution is periodic with the period

$$p=4\int_0^\eta \frac{du}{H^{-1}(k[F(\eta)-F(u)])}\qquad \left(F(y)=\int_0^y f(u)\,du,\quad H(u)=\int_0^u \frac{z\,dz}{h(z)}\right).$$

This is increasing with $|\eta|$ (see [1], Theorem 5) and has a zero or positive limit as $\eta\to 0$ and a finite or infinite limit as $\eta\to\infty$. Denoting these limits by $p(0,k)$ and $p(\infty,k)$ we can state

THEOREM 3. *Assuming*

$$\frac{p(0,k)}{2}\leqq\frac{T}{n}\leqq\frac{p(\infty,K)}{2},$$

equation (1) *has for convenient* η *solutions of period* $2T$ *and* T *like above in Theorem 1. — The conditions imposed on* $\varphi(x), f(y), h(u)$ *are the same as in Theorem 5 in* [1].

The proof follows previous lines and may be omitted.

Equation (1) is that particular case of the general equation $y''=f(x,y,y')$ where $f(x,y,y')$ is factorized. However, the analogue of Theorem 2 may be extended to the solutions of the equation

$$y''+(\alpha+\beta\varphi(x))f(y,y')=0 \tag{9}$$

where $f(\lambda u,\lambda v)=\lambda f(u,v)$ and $\operatorname{sg} f(u,v)=\operatorname{sg} u$. This equation — discussed in a paper [2] of the author — shows common features with the linear equations.

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STANDARD IDEALS IN LATTICES

By

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(*Presented by* A. Rényi)

To Professor Ladislaus Fuchs *without whose constant help and encouragement we could never have achieved our humble results*

Contents

Introduction

The subject of this paper is to define a special class of lattice ideals, the class of standard ideals, and to examine its properties in detail. Before giving the definition of standard ideal we want the reader make acquainted with three tendencies of modern lattice theory which lead naturally to this notion.

The distributive lattices play a central role in lattice theory. This may be explained, on one hand, by the fact that lattices were abstracted from Boolean algebras through the distributive lattices. On the other hand, the distributive lattices have a lot of important properties that lattices in general do not have and, consequently, many of the researches were restricted to distributive lattices.

This fact gives the reason why some mathematicians have tried to define types of elements resp. ideals of lattices which preserve some properties of distributive lattices. It was of importance when in his paper [23] O. ORE has defined the notion of neutral element and ideal in modular lattices, and it was also of significance that in [4] G. BIRKHOFF succeeded in defining these notions in arbitrary lattices. The neutral elements play a central role, for instance, in the theory of direct factorizations of lattices (see [6]). Therefore the question how it is possible to generalize this notion to a wider class of lattice elements and ideals seems to be of interest.

Another trend of researches wants to elaborate the theory of lattice ideals similarly to the theory of ideals in rings or invariant subgroups in groups. Chiefly we are thinking of the fact that any ideal of a ring is the kernel of one and only one homomorphism, furthermore the ideals satisfy the well-known isomorphism theorems, the lemma of Zassenhaus and the Jordan—Hölder—Schreier refinement theorem. Such efforts have to overbridge many difficulties. Naturally, within the Boolean algebras — since the Boolean algebras are rings as well — the researcher does not meet any difficulty. It is also easy to settle this question in distributive lattices, only a good definition of the factor lattice is needed. (The simplest possible method is the following: we embed the distributive lattice in a Boolean algebra — for instance by the method of [13] — and so we get from the well-known notions and theorems of Boolean algebras the same in distributive lattices.)

The case of general lattices is not so simple. In general, the above mentioned theorems are not true. In his paper [31] K. SHODA avoided these difficulties by a suitable definition of the factor algebra; this definition of factor algebra, however, in case of lattices does not seem to be applicable. This was pointed out in [14] by J. HASHIMOTO, remarking that this definition of factor algebra in chains gives only the factor chain of two elements.

In [14], using an other definition of factor lattices, J. HASHIMOTO has proved interesting isomorphism theorems. HASHIMOTO made the very strong restriction: all the ideals occurring in the isomorphism theorems, are neutral. The question arises: is it possible to enlarge the class of neutral ideals, preserving the validity of the isomorphism theorems?

The third tendency of researches that we are going to sketch has started from the Birkhoff—Menger structure theorem of complemented modular lattices of finite length (see G. BIRKHOFF [2], [3] and K. MENGER [22]). This structure theorem asserts that the lattices of the above type coincide with the direct products of simple lattices. A theorem of R. P. DILWORTH [8] states that this structure theorem remains true without any alteration if we omit the supposition of modularity (of course, we must change the word "complemented" to "relatively complemented"). In fact, with this theorem began the investigation of the structure of relatively complemented lattices. The aim of these researches is to prove the results of the theory of relatively complemented modular lattices for relatively complemented lattices as well (some example from among these kinds of papers: J. E. MCLAUGHLIN [20], [21] and G. SZÁSZ [32]).

The following theorem of G. BIRKHOFF [6] is well known: in a complemented modular lattice if we let a congruence relation Θ correspond to the ideal of all x with $x \equiv 0$ (Θ), then we get a natural one-to-one correspondence between neutral ideals and congruence relations. A theorem of SHIH-CHIANG WANG [34], connected with this theorem of G. BIRKHOFF, asserts that the lattice of all congruence relations of a complemented modular lattice is a Boolean algebra if and only if all neutral ideals are principal. However, if we want to formulate these theorems for relatively complemented lattices or for section complemented lattices (i. e. in which the intervals $[0, a]$, as lattices, are complemented), then we do not get in general true assertions. So the question arises, how it is possible to get natural generalizations of these theorems for relatively complemented lattices, i. e. one may ask for the class of ideals, that plays, from the point of view of homomorphisms, a similar role in relatively complemented (section complemented) lattices, as the neutral ideal in complemented modular lattices.

We see that the developments of these there tendencies of lattice theory raise a common request, namely, that of finding appropriate generalizations of neutral ideals, of course, one generalization to each tendency! It was a great surprise to us, when it became clear that the *very same generalization* of neutral ideals answers all the questions raised above. This generalization is given by the notion of standard element and ideal.

An element s of the lattice L will be called *standard* if

$$x \cap (s \cup y) = (x \cap s) \cup (x \cap y)$$

for all pairs of elements x, y of L. A standard ideal of L is defined as a standard element of the lattice of all ideals of L.

The aim of the present paper is to study the most important properties of the standard elements and ideals and, as an application, to prove that the standard ideals make us possible to develop further the above listed three tendencies of lattice theory. We will prove that in some respects the notion of standard ideals is the best-possible one. Namely, the class of standard ideals is the widest one, satisfying the first isomorphism theorem, provided some natural conditions are assumed. Many other properties are also typical to the standard ideals, e. g. the existence of a "dictionary" — as given below. But, of course, if somebody will try to develop a theory of certain type of ideals, satisfying the requirements only of one of the above mentioned tendencies, then he will go further at the direction than we did.

It will appear from this paper that the notion of standard ideal corresponds to the notion of invariant subgroup of groups. Several theorems of group theory may be "translated" to lattice theory using the following "dictionary":[1]

subgroup → ideal

invariant subgroup → standard ideal

factor group → factor lattice[2]

group operation → join operation.[3]

We will use this "dictionary" for getting the appropriate forms of the isomorphism theorems, the Zassenhaus lemma, the solution of Schreier's extension problem and so on. We will see that the "dictionary" works well in all these cases. We get, of course, only the translations of the theorems but not those of the proofs!

The dictionary may be used also for translating negative assertions. An example: the invariant subgroup of an invariant subgroup is in general

[1] The "dictionary" may be used only in translating from group theory to lattice theory but not in the reversed direction! Therefore we used the sign → instead of equality.

[2] modulo a standard ideal!

[3] In the colloquium on Partially Ordered Sets (Oberwolfach, 26—30 October 1959) we have delivered a lecture in which a sketch of this theory was given. After the lecture Professor R. H. Bruck proposed an extension of the dictionary, that — after a short discussion — led to the correspondence

abelian group → distributive lattice.

Using this, one can define the solvability of a lattice, notions corresponding to the centralizator, and commutator subgroup and so on. It may be hoped that one can elaborate this part of the theory.

not invariant in the whole group and the same is true for standard ideals. (It is worth while mentioning that the neutral ideal of a neutral ideal is neutral in the whole lattice.)

Despite the fact that the notion of standard ideal is more general than that of neutral ideal, there appeared a lot of new properties of neutral ideals from the study of this generalization. Besides many not all too important properties, the best example is the result of Chapter VI (Theorem 23). This theorem characterizes neutral ideals in a special class of modular lattices. However, the proof shows clearly that the assertion is a typical one for standard ideals. Hence, we may say, that in this theorem we use the standard ideals as a method of proof.

The paper consists of six chapters.

The first chapter is of preliminary character. It contains notions which are not generally known, while for the fundamental notions of lattice theory and general algebra we refer to [6], [16] and [29]. The frequently used notions and theorems from the literature are enumerated.

In Chapter II, after the definition of standard element and ideal, we prove the two fundamental characterization theorems. In the remaining part we deduce some properties of the standard element and ideal which seems to be of importance.

In Chapter III we are interested in the connections between standard and neutral elements. In § 1 we verify the simplest connections, but already from these we deduce a new proof of a theorem concerning neutral ideals; a proof of this theorem within the theory of neutral ideals does not seem to be an easy task. In § 2 we prove the coincidence of standard and neutral elements in a rather wide class of lattices including modular as well as relatively complemented lattices. In § 3 we give a necessary and sufficient condition for a standard element to be neutral. In § 4 we deal with the lattice of all ideals of a weakly modular lattice. We prove that the lattice of all ideals is not necessarily weakly modular. In the remaining part of the section we discuss some properties of the ideal lattice.

In Chapter IV we prove that the class of standard ideals and that of the homomorphism kernels coincide in section complemented lattices. From this we infer the generalizations of the above mentioned theorems of G. Birkhoff and S. Wang. Then we prove the isomorphism theorems, the lemma of Zassenhaus and some of its consequences. In the last section we solve the lattice-theoretical equivalent of Schreier's extension problem.

In Chapter V we first prove that any distributive equality is capable of the characterization of the neutrality of an element of a modular lattice. Then in § 2 we prove that in modular lattices the uniquely relatively complemented elements are just the neutral ones, and thus we get a generalization of a well-known theorem of VON NEUMANN.

In Chapter VI we deal with ideals satisfying the first isomorphism theorem. In § 1 for a special class of section complemented lattices, while in § 3 for modular lattices with zero and of locally finite length we prove that this class of ideals coincides with the class of neutral ideals. In § 4 we show that under some natural conditions the standard ideals form the widest class of ideals satisfying the first isomorphism theorem.

There are 20 unsolved problems given at the end of the corresponding sections. We hope some of the readers will find it interesting to deal with them.

CHAPTER I
PRELIMINARIES

§ 1. Some notions and notations

The partial ordering relation will be denoted by $<$, in case of set lattices (that is lattices the elements of which are certain subsets of a given set) by $\subset$. In lattices the meet and the join will be designated by $\cap$ and $\cup$, and the complete meet and complete join by $\wedge$ and $\vee$. The least and greatest element of a partially ordered set (or of a lattice) we denote by 0 and 1. If a covers b (i. e. $a > b$, but $a > x > b$ for no x), then we write $a \succ b$.

If $\alpha(x)$ is a property defined on the set H, then we define $\{x; \alpha(x)\}$ as the set of all $x \in H$ for which $\alpha(x)$ is true. Hence in partially ordered sets $(a] = \{x; x \leqq a\}$ is the principal ideal generated by a, while $\{x; a \leqq x \leqq b\}$ is the interval $[a, b]$, provided that $a \leqq b$. If b covers a, then the interval $[a, b]$ is a prime interval. The dual principal ideal is denoted by $[a)$.

If any two elements a, b of L, satisfying $a < b$, may be connected by a finite maximal chain, then L is said to be semi-discrete. If the lengths of the maximal chains of the lattice L are finite and bounded, then L is called of finite length. If all intervals of the lattice L, as lattices, are of finite length, then L is of locally finite length. If L has a 0 and is of locally finite length, furthermore for all $a \in L$, in $[0, a]$ any two maximal chains are of the same length, then we say that in L the Jordan—Dedekind chain condition is satisfied. In this case the length of any maximal chain of the interval $[0, a]$ will be denoted by $d(a)$, and $d(x)$ is called the dimension function.

Let P and Q be partially ordered sets. The ordinal sum of P and Q is defined as the partially ordered set, which is the set union of P and Q, and the partial ordering remains unaltered in P and Q, while $x < y$ holds for all $x \in P$ and $y \in Q$; this partially ordered set will be denoted by $P \oplus Q$.

The set of all ideals of a lattice L, partially ordered under set inclusion, form a lattice, which will be denoted by $I(L)$.

LEMMA I. *$I(L)$ is a conditionally complete lattice. The meet of a set of ideals (if it exists) is the set-theoretical meet. The join of the ideals I_α ($\alpha \in A$) is the set of all x such that*

$$x \leqq i_{\alpha_1} \cup \cdots \cup i_{\alpha_n} \quad (i_{\alpha_j} \in I_{\alpha_j})$$

for some elements α_j of A.

If A is a general algebra and Θ is a congruence relation of A, then the congruence classes of A modulo Θ form a general algebra $A(\Theta)$. This is a homomorphic image of A.

We will use the two general isomorphism theorems (RÉDEI [29]):

THE FIRST GENERAL ISOMORPHISM THEOREM. *Let A be a general algebra and A' a subalgebra of A, further let Θ be an equivalence relation of A such that every equivalence class of A may be represented by an element of A'. Let Θ' denote the equivalence relation of A' induced by Θ. If Θ is a congruence relation, then so is Θ' and*

$$A(\Theta) \cong A'(\Theta').$$

The natural isomorphism makes a congruence class of A correspond to the contained congruence class of A'.

THE SECOND GENERAL ISOMORPHISM THEOREM. *Let A' be a homomorphic image of the general algebra A, let Θ be an equivalence relation of A, and denote Θ' the equivalence relation of A' under which the equivalence classes are the homomorphic images of those of A modulo Θ, and suppose that no two different equivalence classes of A modulo Θ have the same homomorphic image. Then Θ is a congruence relation if and only if Θ' is one and in this case*

$$A(\Theta) \cong A'(\Theta').$$

The natural isomorphism makes an equivalence class of A correspond to its homomorphic image.

§ 2. Congruence relations in lattices

Let Θ be a congruence relation of the lattice L, and denote by $L(\Theta)$ the homomorphic image of L induced by the congruence relation Θ, that is, the lattice of all congruence classes. If $L(\Theta)$ has a zero, then the complete inverse image of the zero is an ideal of L, called the kernel of the homomorphism $L \to L(\Theta)$.

A simple criterion for a binary relation η to be a congruence relation is formulated in

LEMMA II. (GRÄTZER and SCHMIDT [12].) *Let η be a binary relation defined on the lattice L. η is a congruence relation if and only if the following conditions hold for all $x, y, z \in L$:*

(a) $x \equiv x \ (\eta)$;

(b) $x \cup y \equiv x \cap y \ (\eta)$ *if and only if* $x \equiv y \ (\eta)$;

(c) $x \geqq y \geqq z$, $x \equiv y \ (\eta)$, $y \equiv z \ (\eta)$ *imply* $x \equiv z \ (\eta)$;

(d) $x \geqq y$ *and* $x \equiv y \ (\eta)$, *then* $x \cup z \equiv y \cup z \ (\eta)$ *and* $x \cap z \equiv y \cap z \ (\eta)$.

The congruence relations of L will be denoted by $\Theta, \Phi, \ldots$. The set of all congruence relations of L, partially ordered by "$\Theta \leqq \Phi$ if and only if $x \equiv y \ (\Theta)$ implies $x \equiv y \ (\Phi)$", will be designated by $\Theta(L)$.

LEMMA III. (BIRKHOFF [4] and KRISHNAN [18].)[4] *$\Theta(L)$ is a complete lattice. $x \equiv y \ (\bigwedge_{\alpha \in A} \Theta_\alpha)$ if and only if $x \equiv y \ (\Theta_\alpha)$ for all $\alpha \in A$; $x \equiv y \ (\bigvee_{\alpha \in A} \Theta_\alpha)$ if and only if there exists in L a sequence of elements $x \cup y = z_0 \geqq z_1 \geqq \cdots \geqq z_n = x \cap y$ such that $z_i \equiv z_{i-1} \ (\Theta_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ for suitable $\alpha_1, \ldots, \alpha_n \in A$.*

The least and greatest elements of the lattice $\Theta(L)$ will be designated by ω and ι, respectively.

Let H be a subset of L, $\Theta[H]$ will denote the least congruence relation under which any pair of elements of H is congruent. This we call the congruence relation induced by H. If H has just two elements, $H = \{a, b\}$, then $\Theta[H]$ will be written as Θ_{ab}. The congruence relation Θ_{ab} is called minimal.

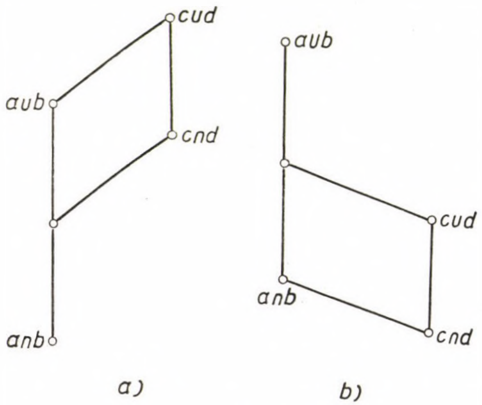

Fig. 1

First we describe — following R. P. DILWORTH — the minimal congruence relation Θ_{ab}. To this end we have to make some preparations.

Given two pairs of elements a, b and c, d of L, suppose that either

$$c \cap d \geqq a \cap b$$

and

$$(c \cap d) \cup (a \cup b) = c \cup d,$$

[4] See also GRÄTZER and SCHMIDT [12].

or

$$c \cup d \leqq a \cup b \quad \text{and} \quad (c \cup d) \cap (a \cap b) = c \cap d.$$

Then we say that a, b is weakly projective in one step to c, d, and write $\overline{a, b} \xrightarrow{1} \overline{c, d}$. The situation is given in Fig. 1. In other words, $\overline{a, b} \xrightarrow{1} c, d$ if and only if the intervals $[(a \cup b) \cap c \cap d, a \cup b]$, $[c \cap d, c \cup d]$ or $[a \cap b, (a \cap b) \cup c \cup d]$, $[c \cap d, c \cup d]$ are transposes in the sense of [6].

If there exist two finite sequences of elements $a = x_0, x_1, \ldots, x_n = c$ and $b = y_0, \ldots, y_n = d$ in L such that

$$(1) \qquad \overline{a, b} = \overline{x_0, y_0} \xrightarrow{1} \overline{x_1, y_1} \xrightarrow{1} \cdots \xrightarrow{1} \overline{x_n, y_n} = \overline{c, d},$$

then we say that a, b is weakly projective to c, d, in notation: $\overline{a, b} \to \overline{c, d}$, or if we are also interested in the number n, then we write $\overline{a, b} \xrightarrow{n} \overline{c, d}$.

If $\overline{a, b} \xrightarrow{1} \overline{c, d}$ and $\overline{c, d} \xrightarrow{1} \overline{a, b}$, then a, b and c, d are transposes, and we write $\overline{a, b} \xleftrightarrow{1} \overline{c, d}$. If the sequence (1) may be chosen in such a way that the neighbouring members are transposes, then a, b and c, d are called projective, and we write $\overline{a, b} \leftrightarrow \overline{c, d}$.

The notion of weak projectivity is due to R. P. DILWORTH [8] (see also MALCEV [19], GRÄTZER and SCHMIDT [12]). DILWORTH uses his terminology just reversed as we do.

The importance of this notion is shown by the fact that $\overline{a, b} \to \overline{c, d}$ and $a \equiv b\ (\Theta)$ imply $c \equiv d\ (\Theta)$ (applying this to $\Theta = \omega$, we get that $a = b$ implies $c = d$, a fact which will be used several times).

Now we are able to describe Θ_{ab}:

THEOREM I. (R. P. DILWORTH [8].) *Let a, b, c, d be elements of the lattice L. $c \equiv d\ (\Theta_{ab})$ holds if and only if there exist $y_i \in L$ with*

$$(2) \qquad c \cup d = y_0 \geqq y_1 \geqq \cdots \geqq y_k = c \cap d \quad \text{and} \quad \overline{a, b} \to \overline{y_{i-1}, y_i} \quad (i = 1, 2, \ldots, k).$$

It is easy to describe $\Theta[H]$, using Lemma III, Theorem I and the following trivial identity:

$$(3) \qquad \Theta[H] = \bigvee_{a, b \in H} \Theta_{ab}.$$

The symbol $\Theta[H]$ will be used mostly in case H is an ideal. Then one can prove the following important identity (see [14]):

$$(4) \qquad \Theta[\vee I_\alpha] = \vee \Theta[I_\alpha] \qquad (I_\alpha \in I(L)).$$

The following definition is of central importance in this paper. Let L be a lattice and I an ideal of L. By the factor lattice L/I of the lattice L modulo the ideal I is meant the homomorphic image of L induced by $\Theta[I]$, i. e.

$$L/I \cong L(\Theta[I]).$$

Finally, we mention the definition of permutability: the congruence relations Θ and Φ are called permutable if $a \equiv x\ (\Theta)$ and $x \equiv b\ (\Phi)$ imply the existence of a y such that $a \equiv y\ (\Phi)$ and $y \equiv b\ (\Theta)$.

§ 3. Lattices and elements with special properties

2 will denote the lattice of two elements.

Let U denote the non-modular lattice of five elements, generated by the elements p, q, r, that is, $p > q$, $p \cup r = q \cup r = i$, $p \cap r = o$. V will denote the modular, non-distributive lattice of five elements with the generators p, q, r, that is, $p \cup q = q \cup r = r \cup p = i$, $p \cap q = q \cap r = r \cap p = o$.

An element d of the lattice L is called distributive if

$$d \cup (x \cap y) = (d \cup x) \cap (d \cup y) \tag{5}$$

for all $x, y \in L$. In [25] O. Ore has proved that d is distributive if and only if $x \equiv y\ (\Theta[(d]])$ implies $x \cup y = [(x \cap y) \cup d] \cap (x \cup y)$.

An element n of L is said to be neutral if the sublattice $\{n, x, y\}$ is distributive, where x and y are arbitrary elements from L. The following theorem will be useful:

Theorem II. (Ore [24].) *The elements $x, y, z \in L$ generate a distributive sublattice of L if and only if for all permutations a, b, c of x, y, z the following equalities hold:*

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c), \tag{6}$$

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c), \tag{7}$$

$$(a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cup b) \cap (b \cup c) \cap (c \cup a). \tag{8}$$

Corollary. *An element n of L is neutral if and only if for all $x, y \in L$ the five equalities obtained from* (6)—(8) *by substituting permutations of x, y, n hold.*

Remark. It will follow from the theory of standard elements that this corollary may be sharpened, omitting three from the five conditions.

Theorem III. (Birkhoff [5].) *An element n of L is neutral if and only if*

(i) $n \cup (x \cap y) = (n \cup x) \cap (n \cup y)$ *for all* $x, y \in L$;

(i′) $n \cap (x \cup y) = (n \cap x) \cup (n \cap y)$ *for all* $x, y \in L$;

(ii) $n \cap x = n \cap y$ *and* $n \cup x = n \cup y$ $(x, y \in L)$

imply $x = y$, i. e. the relative complements of n are unique.

Theorem IV. (Ore [24].) *An element n of a modular lattice L is neutral if and only if condition* (i) *(or equivalently, condition* (i′)*) is satisfied.*

An ideal I of L is called distributive if it is a distributive element of $I(L)$. I is neutral if it is a neutral element of $I(L)$.

The lattice L is weakly modular (see GRÄTZER and SCHMIDT [12]) if from $\overline{a, b} \to \overline{c, d}$ $(a, b, c, d \in L; c \neq d)$ it follows the existence of $a_1, b_1 \in L$ satisfying $a \cap b \leqq a_1 < b_1 \leqq a \cup b$ and $\overline{c, d} \to \overline{a_1, b_1}$.

LEMMA IV. (GRÄTZER and SCHMIDT [12].) *Let the lattice L be*

A) *modular, or*

B) *relatively complemented, or*

C) *simple.*

Then L is weakly modular.

A lattice L with zero is called section complemented if all of its intervals of type $[0, a]$ are complemented as lattices. In general, the lattice L is section complemented if any element of L is contained in a suitable principal dual ideal which is section complemented as a lattice.[5]

The following assertion is trivial:

LEMMA V. *Any relatively complemented lattice is section complemented.*

Finally, we mention the $\vee$-distributive law:

$$x \cap \vee y_\alpha = \vee (x \cap y_\alpha).$$

A complete lattice L is called $\vee$-distributive if this law unrestrictedly holds in L.

Of importance is the theorem of FUNAYAMA and NAKAYAMA that asserts: $\Theta(L)$ is $\vee$-distributive.

The partition lattice $P(H)$ of the set H is defined as the partially ordered set of all partitions of H, where the partition p is said to be smaller than q if p is a refinement of q.

CHAPTER II
STANDARD ELEMENTS AND IDEALS

§ 1. Standard elements

We begin with repeating the definition of standard elements:

The element s of the lattice L is standard if the equality

$$x \cap (s \cup y) = (x \cap s) \cup (x \cap y) \tag{9}$$

holds for all $x, y \in L$.

[5] The section complemented lattices with zero are called by HERMES [16] „abschnittkomplementäre Verbände". The English name was suggested by Mr. LORENZ.

First of all, let us see some examples for standard elements. In the lat ice U (see Chapter I, § 3) p is a standard element. At the same time, it is clear that p is not neutral. (Furthermore, in the same lattice $(r]$ is a homomorphism kernel, but r is not standard.)

Obviously, any element of a distributive lattice is standard. Furthermore, in any lattice the elements 0 and 1 (if exist) are standard elements.

The simplest form for defining standard elements is the equality (9), however, it is not the most important property of a standard element. Some important characterizations of standard elements are given in

THEOREM 1. (The fundamental characterization theorem of standard elements.) *The following conditions upon an element s of the lattice L are equivalent:*

(α) *s is a standard element;*

(β) *the equality $u=(u\cap s)\cup(u\cap t)$ holds whenever $u\leqq s\cup t$ $(u, t\in L)$;*

(γ) *the relation Θ_s, defined by "$x\equiv y\ (\Theta_s)$ if and only if $(x\cap y)\cup s_1= x\cup y$ for some $s_1\leqq s$", is a congruence relation;*

(δ) *for all $x, y\in L$*

(i) $$s\cup(x\cap y)=(s\cup x)\cap(s\cup y),$$

(ii) $$s\cap x=s\cap y \quad \text{and} \quad s\cup x=s\cup y \quad \text{imply} \quad x=y.$$

PROOF. We will prove the equivalence of the four conditions cyclically

(α) *implies* (β). Indeed, if (α) holds and $u\leqq s\cup t$, then $u=u\cap(s\cup t)$. Owing to (9) we get $u=(u\cap s)\cup(u\cap t)$, which was to be proved.

(β) *implies* (γ). Using condition (β) and Lemma II we will prove that Θ_s as defined above is a congruence relation.

(a) $x\equiv x\ (\Theta_s)$. Indeed, for any $x\in L$, the equality $(x\cap x)\cup(x\cap s)=x$ trivially holds, so if we put $s_1=x\cap s$, we get the assertion.

(b) $x\cap y\equiv x\cup y\ (\Theta_s)$ if and only if $x\equiv y\ (\Theta_s)$. This is trivial from the definition of Θ_s.

(c) $x\geqq y\geqq z$, $x\equiv y\ (\Theta_s)$ and $y\equiv z\ (\Theta_s)$ imply $x\equiv z\ (\Theta_s)$. By hypothesis $x=y\cup s_1$ and $y=z\cup s_2$ for suitable elements $s_1, s_2\leqq s$. Consequently, $x=y\cup s_1=(z\cup s_2)\cup s_1=z\cup(s_1\cup s_2)$ for $s_1\cup s_2\leqq s$, that means, $x\equiv z\ (\Theta_s)$.

(d) In case $x\geqq y$ and $x\equiv y\ (\Theta_s)$ hold, $x\cup z\equiv y\cup z\ (\Theta_s)$ and $x\cap z\equiv y\cap z\ (\Theta_s)$. In fact, by assumption $x=y\cup s_1$ $(s_1\leqq s)$, and hence we get $x\cup z=(y\cup z)\cup s_1$, that is $x\cup z\equiv y\cup z\ (\Theta_s)$. To prove the second assertion, we start from the relations $x=y\cup s_1$ and $x\cap z\leqq y\cup s_1\leqq y\cup s$. Applying condition ($\beta$) to $u=x\cap z, t=y$ and using $x\cap y=y$, we get

$$x\cap z=(x\cap z\cap s)\cup(x\cap z\cap y)=(y\cap z)\cup s_2,$$

where $s_2=x\cap z\cap s\leqq s$, which means $x\cap z\equiv y\cap z\ (\Theta_s)$.

(γ) *implies* (δ). First we prove that (γ) implies (i). According to the definition of Θ_s, the congruences $x \equiv s \cup x$ (Θ_s) and $y \equiv s \cup y$ (Θ_s) hold for arbitrary $x, y \in L$. We get $x \cap y \equiv (s \cup x) \cap (s \cup y)$ (Θ_s). By monotonicity, $x \cap y \leqq (s \cup x) \cap (s \cup y)$, hence again by the definition of Θ_s it follows that $(s \cup x) \cap (s \cup y) = (x \cap y) \cup s_1$ with suitable $s_1 \leqq s$. Joining with s and keeping the inequalities $s_1 \leqq s$ and $s \leqq (s \cup x) \cap (s \cup y)$ in view, we derive $s \cup (x \cap y) = (s \cup x) \cap (s \cup y)$, which is nothing else than (i).

Secondly, we prove that (γ) implies (ii). Let the elements x and y be chosen as in (ii). We know that $s \cup y \equiv y$ (Θ_s), so meeting with x and using $x \cup s = y \cup s$ we get $x = (x \cup s) \cap x = (y \cup s) \cap x \equiv y \cap x$ (Θ_s), consequently, using (γ), $(x \cap y) \cup s_1 = x$ with suitable $s_1 \leqq s$. From the last equality $s_1 \leqq x$, accordingly, $s_1 \leqq s \cap x = s \cap y \leqq y$ (in the meantime we have used the supposition $s \cap x = s \cap y$ of (ii)), thus $x = (x \cap y) \cup s_1 \leqq (x \cap y) \cup y = y$. We may conclude similarly that $y \leqq x$, and thus $x = y$, which was to be proved.

(δ) *implies* (α). Let x and y be arbitrary elements of L and define $a = x \cap (s \cup y)$ and $b = (x \cap s) \cup (x \cap y)$. By (ii), it suffices to prove that $s \cap a = s \cap b$ and $s \cup a = s \cup b$.

To prove the first equality we start from $s \cap a$:

$$s \cap a = s \cap [x \cap (s \cup y)] = x \cap [s \cap (s \cup y)] = x \cap s.$$

It follows from the monotonicity that $x \cap s \leqq b = (x \cap s) \cup (x \cap y) \leqq [x \cap (s \cup y)] \cup [x \cap (s \cup y)] = a$. Meeting with s, we get $s \cap x \leqq s \cap b \leqq s \cap a$. But we have already proved that $s \cap x = s \cap a$, and so $s \cap a = s \cap b$. To prove $s \cup a = s \cup b$ we start from $s \cup a$ and use (i) several times:

$$s \cup a = s \cup [x \cap (s \cup y)] = (s \cup x) \cap [s \cup (s \cup y)] = (s \cup x) \cap (s \cup y) =$$
$$= s \cup (x \cap y) = s \cup (x \cap s) \cup (x \cap y) = s \cup b,$$

and so Theorem 1 is completely proved.

Rewriting (i) and weakening (ii), (δ) may be transformed to the following form:

LEMMA 1. *An element s of L is standard if and only if the following two conditions are satisfied:*

(i*) *the correspondence $x \to x \cup s$ is an endomorphism of L;*

(ii*) *if $x \geqq y$, $s \cup x = s \cup y$ and $s \cap x = s \cap y$, then $x = y$.*

It is easy to see that (i) is equivalent to (i*). Indeed, for any fixed s, the correspondence $x \to x \cup s$ is a join-endomorphism. That it is meet-endomorphism as well, is guaranteed just by (i). In the proof of Theorem I, at the step "(δ) implies (α)" we have used (ii) only for $x = a$ and $y = b$, and in this case $y \leqq x$ holds. Consequently, in the proof we have only used (ii*), and so one can replace (ii) by (ii*).

From condition (γ) of Theorem I we derive easily:

LEMMA 2. *Let s be a standard element of the lattice L. Then $(s]$ is a homomorphism kernel, namely $\Theta[(s]] = \Theta_s$. Conversely, if $x \equiv y$ $(\Theta[(s]])$ holds when and only when $(x \cap y) \cup s_1 = x \cup y$ with a suitable $s_1 \leqq s$, then s is a standard element.*

PROOF. The congruence relation Θ_s obviously satisfies $\Theta_s = \Theta[(s]]$, consequently $(s]$ is in the kernel of the homomorphism induced by Θ_s. We have to prove that $(s]$ is just the kernel. Otherwise there exists an $x > s$ with $x \equiv s$ (Θ_s). By definition, it follows $x = s \cup s_1$ $(s_1 \leqq s)$ which is obviously a contradiction. Conversely, if $\Theta[(s]] = \Theta_s$, then Θ_s is a congruence relation, since $\Theta[(s]]$ is one, and then from condition (γ) of Theorem 1 it follows that s is a standard element.

We have formulated Lemma 2 separately — despite the fact that it is an almost trivial variant of condition (γ) of Theorem 1 — because it points out that property of the standard elements which we think to be the most important one. It may be reformulated as follows: if $(s]$ is a principal ideal of L, then $x \equiv y$ $(\Theta[(s]])$ if and only if there exist a sequence of elements $x \cup y = z_0 \geqq z_1 \geqq \cdots \geqq z_m = x \cap y$ of L, an $s_1 \leqq s$, and a sequence of integers $n_1, \ldots, n_m$ such that $\overline{s_1, s} \xrightarrow{n_i} \overline{z_{i-1}, z_i}$ $(i = 1, \ldots, m)$. Now the definition of standardness is as follows: s is standard if and only if $n_i = 1$ may be chosen for all i. It follows then we may suppose $m = 1$ as well.

§ 2. Standard ideals

An ideal S of the lattice L is called standard if it is a standard element of the lattice $I(L)$, that is, if

$$I \cap (S \cup K) = (I \cap S) \cup (I \cap K) \tag{10}$$

holds for any pair of ideals I, K of L.

An example for standard ideals is given by the ideal $(p]$ of the lattice U. Further examples will be given at the end of this section.

Our chief aim in this section is to prove the analogue of Theorem 1 for standard ideals.

THEOREM 2. (The fundamental characterization theorem of standard ideals.) *The following seven conditions for an ideal S of the lattice L are equivalent:*

(α') *S is a standard ideal;*

(α'') *the equality*

$$I \cap (S \cup K) = (I \cap S) \cup (I \cap K)$$

holds if I and K are principal ideals;

(β') *for any ideal I, the elements of $S \cup I$ are of the form $s \cup x$ ($s \in S$, $x \in I$);*

(β'') *for any principal ideal I, the elements of $S \cup I$ are of the form $s \cup x$ ($s \in S$, $x \in I$);*

(γ') *the relation Θ_S of $I(L)$ defined by "$I \equiv K$ (Θ_S) if and only if $(I \cap K) \cup S_1 = I \cup K$ with a suitable $S_1 \subseteq S$" is a congruence relation of $I(L)$;*

(γ'') *the relation $\Theta[S]$ of L defined by "$x \equiv y$ ($\Theta[S]$) if and only if $(x \cap y) \cup s = x \cup y$ with a suitable $s \in S$" is a congruence relation;*

(δ') *for all I and $K \in I(L)$*

(i) $$S \cup (I \cap K) = (S \cup I) \cap (S \cup K),$$

(ii) $$S \cap I = S \cap K \quad \text{and} \quad S \cup I = S \cup K \quad \text{imply} \quad I = K.$$

PROOF. The conditions of this theorem are the analogues of those of Theorem 1. To make the similarity clear, first we show that (β') is equivalent to the following condition:

(β^*) if for the ideals I and J the inequality $J \subseteq S \cup I$ holds, then $J = (J \cap S) \cup (J \cap I)$.

It is, obviously, equivalent to (β^*) that any element of J may be written in the form $s \cup x (s \in S, x \in I)$. Since J is arbitrary, that means: any element of $S \cup I$ is of the form $s \cup x$, and this is condition (β'). So these two conditions are equivalent.

Another analogue of (β'') may also be formulated:

(β^{**}) if for the principal ideals I and J the inequality $J \subseteq S \cup I$ holds, then $J = (J \cap S) \cup (J \cap I)$.

Now, it is trivial that the equivalence of (α'), (β'), (γ') and (δ') is an immediate consequence of Theorem 1.

(α'') is a special case of (α'). The proofs that (α'') implies (β'') and (β'') implies (γ'') run on the similar lines as those of the corresponding implications in the proof of Theorem 1. Thus it is enough to prove that (γ'') implies (β'). Suppose (γ'') holds and let I be an arbitrary ideal of L, and $x \in S \cup I$. From Lemma I we get the existence of $s \in S$ and $i \in I$ with $x \leqq s \cup i$. Since $s \equiv s \cap i$ ($\Theta[S]$), therefore $s \cup i \equiv (s \cap i) \cup i = i$ ($\Theta[S]$), and so $x = x \cap (s \cup i) \equiv x \cap i$ ($\Theta[S]$). Accordingly, using (γ'') we get $x = (x \cap i) \cup s'$ where $s' \in S$. But $x \cap i \in I$, hence (β') is proved.

The proof of Theorem 2 is complete.

The analogues of Lemmas 1 and 2 are naturally true. We formulate only the analogue of the most important part of Lemma 2.

LEMMA 3. *Let S be a standard ideal of L. Then the congruence relation $\Theta[S]$ of L defined by condition (γ'') of Theorem 2 is the congruence relation induced by S and S is the kernel of the homomorphism induced by $\Theta[S]$.*

We may say that Lemma 3 gives an approval of the notation we have used in condition (γ'') of Theorem 2.

We get many examples of standard ideals from the following

LEMMA 4. *The principal ideal $(s]$ of L is standard if and only if s is a standard element of L.*

PROOF. The assertion is clear comparing Lemma 2 with condition (γ'') of Theorem 2, since $s_1 \in (s]$ and $s_1 \leqq s$ are equivalent statements.

It follows now from Lemma 4 that the existence of standard elements in a lattice implies the existence of standard ideals. The converse of this statement is not true. We construct a lattice L in which there exists a standard ideal, but has no standard element. Consider the direct product of the chain of the integers with **2**. The elements of this lattice are of the form $(n, 0)$ and $(n, 1)$ where n is an arbitrary integer and 0 and 1 are the elements of **2**. We define new elements x_n $(n = 0, \pm 1, \ldots)$, subject to the following relations:

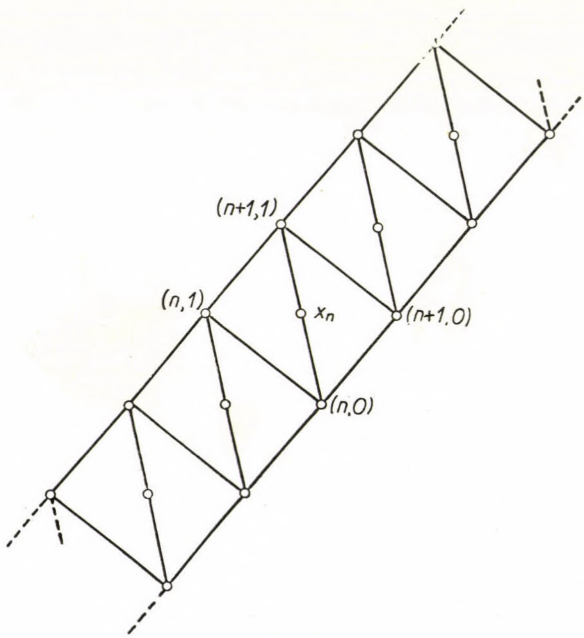

Fig. 2

$$x_n \cup (n, 1) = x_n \cup (n+1, 0) = (n+1, 1),$$
$$x_n \cap (n, 1) = x_n \cap (n+1, 0) = (n, 0).$$

The resulting partially ordered set L is shown in Fig. 2. One can easily prove that L is a lattice and L is simple, that is, $\Theta(L)$ consists of two elements. In L there is no standard element (if s were one, then $\omega < \Theta_s < \iota$ would be a contradiction), but the whole lattice is a standard ideal.

A proper standard ideal is obtained if we take two copies of this lattice, L_1 and L_2, and define $L \simeq L_1 \oplus L_2$. Then this lattice contains no standard element, but L_1 and L_2 are standard ideals.

It is natural to ask, why the following condition is not included in Theorem 2:

(δ'') if I and K are principal ideals, then

(i) $$S \cup (I \cap K) = (S \cup I) \cap (S \cup K);$$

(ii) $$S \cap I = S \cap K \quad \text{and} \quad S \cup I = S \cup K \text{ imply } I = K.$$

The reason is that we could not prove the equivalence of this condition to the others. Therefore we ask

PROBLEM 1. Does condition (δ'') characterize the standardness of the ideal S?

§ 3. Basic properties of standard elements and ideals

In this section and in the next one we shall deduce from the fundamental characterization theorems some important properties of standard elements and ideals.

If S is a standard ideal, then we call the congruence relation $\Theta[S]$ generated by S a standard congruence relation. If $S = (s]$, then $\Theta[S] = \Theta_s$, so Θ_s is a standard congruence relation that we may call principal standard congruence relation. First we see some results on the connection between standard ideals and standard congruence relations.

THEOREM 3. *The standard elements form a distributive sublattice of the lattice L. The principal standard congruence relations form a sublattice of* $\Theta(L)$. *Between these two lattices the correspondence* $s \to \Theta_s$ *is an isomorphism.*

Further, the standard ideals form a $\vee$*-distributive sublattice of* $I(L)$ *which is closed under forming complete join. The standard congruence relations form a sublattice of* $\Theta(L)$. *The correspondence* $S \to \Theta[S]$ *is an isomorphism between these two lattices.*

PROOF. First we verify the assertions concerning standard elements. Let s_1 and s_2 be standard. Then by an iterated use of (9) we get that for all $x, y \in L$

$$x \cap [(s_1 \cup s_2) \cup y] = x \cap [s_1 \cup (s_2 \cup y)] = (x \cap s_1) \cup [x \cap (s_2 \cup y)] =$$
$$= (x \cap s_1) \cup (x \cap s_2) \cup (x \cap y) = [x \cap (s_1 \cup s_2)] \cup (x \cap y),$$

that means, by definition, that $s_1 \cup s_2$ is standard. It is almost trivial that the correspondence $s \to \Theta_s$ is a join-endomorphism. Indeed, owing to Lemma 2 and the standardness of $s_1 \cup s_2$, the equality $\Theta_{s_1} \cup \Theta_{s_2} = \Theta_{s_1 \cup s_2}$ is equivalent to $\Theta[(s_1]] \cup \Theta[(s_2]] = \Theta[(s_1 \cup s_2]]$, and this is a special case of formula (4). Further, if $s_1 \neq s_2$, then $\Theta_{s_1} \neq \Theta_{s_2}$, for the kernels of the homomorphisms induced by Θ_{s_1}, resp. Θ_{s_2} are different (see Lemma 2).

Now we prove $\Theta_{s_1} \cap \Theta_{s_2} = \Theta_{s_1 \cap s_2}$. If $x \equiv y \; (\Theta_{s_1} \cap \Theta_{s_2})$, then $x \equiv y \; (\Theta_{s_1})$, and so $(x \cap y) \cup s_1' = x \cup y$ $(s_1' \leqq s_1)$, on the other hand $x \equiv y \; (\Theta_{s_2})$ holds as well, and from this $s_1' = (x \cup y) \cap s_1' \equiv (x \cap y) \cap s_1' \; (\Theta_{s_2})$, hence with a suitable $s \leqq s_2$ the relation $s_1' = [(x \cap y) \cap s_1'] \cup s$ holds. Consequently, $s \leqq s_1'$ is valid, therefore $s \leqq s_1 \cap s_2$ and $(x \cap y) \cup s = (x \cap y) \cup [(x \cap y) \cap s_1'] \cup s = x \cup y$. We have proved the following: $x \equiv y \; (\Theta_{s_1} \cap \Theta_{s_2})$ if and only if $(x \cap y) \cup s = x \cup y$ with a suitable $s \in (s_1 \cap s_2]$. According to Lemma 2, this means that $s_1 \cap s_2$ is standard and $\Theta_{s_1} \cap \Theta_{s_2} = \Theta_{s_1 \cap s_2}$. Thus we have shown that the standard elements form a sublattice of L, the principal standard congruence relations form a sublattice of $\Theta(L)$, and the correspondence $s \to \Theta_s$ is an isomorphism. It follows now, since $\Theta(L)$ is a distributive lattice, that the lattice of standard elements is distributive.

Applying the results proved so far to the lattice of all ideals of L, we get that the standard ideals form a sublattice of $I(L)$, the congruence relations Θ_S form a sublattice of $\Theta(I(L))$, and $S \to \Theta_S$ is an isomorphism. But we need the same assertions for $\Theta[S]$ instead of Θ_S. Therefore we prove a lemma from which the desired conclusion will follow.

First we need some notions. Let Θ be a congruence relation of L; Θ defines in the natural way a homomorphism of $I(L)$ under which $I \equiv J$ $(I, J \in I(L))$ if and only if to any $x \in I$ there exists a $y \in J$ such that $x \equiv y \; (\Theta)$, and conversely. That means: $I \equiv J$ if and only if the images of I and J under the homomorphism $L \to L(\Theta)$ are the same. This congruence relation of $I(L)$ we call the extension of Θ to $I(L)$. On the other hand, any congruence relation Φ of $I(L)$ induces a congruence relation of L under which $x \equiv y$ if and only if $(x] \equiv (y] \; (\Phi)$. This we call the restriction of Φ to L. Now we may state

LEMMA 5. *Let S be a standard ideal. Then Θ_S is the extension of $\Theta[S]$ to $I(L)$ and $\Theta[S]$ is the restriction of Θ_S to L.*

PROOF. Let $\overline{\Theta}[S]$ be the extension of $\Theta[S]$ to $I(L)$ and $I \equiv J \; (\overline{\Theta}[S])$; we suppose $I \subseteq J$. Choosing a $y \in J$ we can find an $x \in I$ $(y \geqq x)$ with $x \equiv y \; (\Theta[S])$, and so there exists an s_{xy} with $x \cup s_{xy} = y$. The ideal S' generated by the s_{xy} (x and y run over the elements of I and J) satisfies $S' \subseteq S$ and $I \cup S' = J$. On the other hand, if $I \cup S' = J$ with a suitable $S' \subseteq S$, then with $y \in J$ it follows that $y = s \cup x$ ($s \in S', x \in I, S$ is standard!), and so $x \equiv y$ ($\Theta[S]$). Thus $\overline{\Theta}[S] = \Theta_S$. To show the second assertion, suppose $(a] \equiv (b]$ (Θ_S). Then there exists an $S' \subseteq S$ with $(a \cap b] \cup S' = (a \cup b]$. It follows that $a \cup b \in (a \cap b] \cup S$, and since S is standard, we may find an $s \in S$ with $(a \cap b) \cup s = a \cup b$, which proves $a \equiv b$ ($\Theta[S]$).

COROLLARY. *The correspondence* $\Theta[S] \to \Theta_S$ *is an isomorphism between the lattice of all standard congruence relations of* L *and the lattice of all principal standard congruence relations of* $I(L)$.

Combining Corollary of Lemma 5 with the facts we have proved above, we get all the assertions of Theorem 3 with the exception of the statement that the lattice of all standard ideals of L is closed under forming complete join and is V-distributive.

Suppose the S_α are standard ideals, $S = \bigvee S_\alpha$, I is an arbitrary ideal and $x \in I \cup S$. Owing to Lemma I we may find $s_i \in S_{\alpha_i}$, $y \in I$ such that $x \leq \bigvee_{i=1}^{n} s_i \cup y$, consequently, $x \in \bigvee_{i=1}^{n} S_{\alpha_i} \cup I$. We know already that $\bigvee_{i=1}^{n} S_{\alpha_i}$ is a standard ideal, hence $x = u \cup v$, $u \in \bigvee_{i=1}^{n} S_{\alpha_i} \subseteq S$ and $v \in I$. Thus, by condition (β') of Theorem 2, S is a standard ideal.

Now we may apply formula (4) which compared with Lemma 3 gives $\Theta[\bigvee S_\alpha] = \bigvee \Theta[S_\alpha]$. Thus the standard congruence relations form a sublattice of $\Theta(L)$ which is closed under forming complete join. In $\Theta(L)$ there holds the V-distributive law and this is preserved under taking a sublattice which is closed under forming complete join, therefore the lattice of standard congruence relations is V-distributive, and then the same is true for the lattice of standard ideals. Thus the proof of Theorem 3 is completed.

Naturally arises the question: is the complete meet of standard ideals (if it exists) a standard ideal? We will show by a counterexample that this is not true in general. Let N be the chain of all negative integers, and denote by $0, a, b, 1$ the elements of a Boolean algebra of four elements and

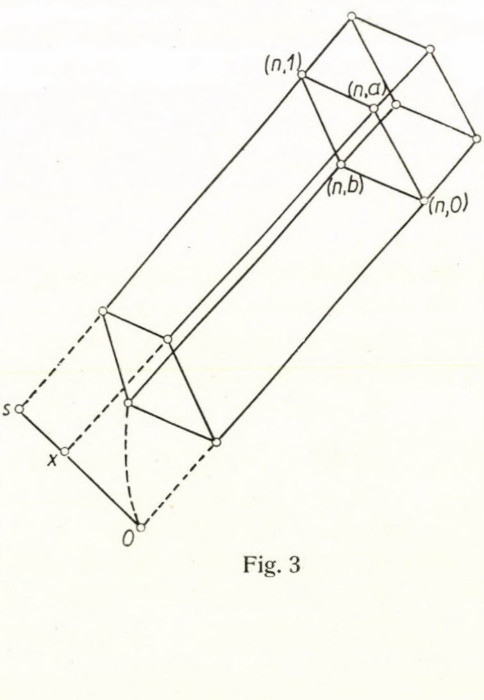

Fig. 3

form the direct product of N and this Boolean algebra. The lattice L is formed by adding three further elements $s, x, 0$ to this direct product subject to

$$\left.\begin{array}{l} x \cap (n, b) = 0, \quad x \cup (n, b) = (n, 1), \quad x \cup (n, a) = (n, a), \\ s \cup (n, a) = s \cup (n, b) = (n, 1), \quad s \cap (n, a) = x, \quad s \cap (n, b) = 0 \end{array}\right\} \quad (n \in N).$$

The resulting partially ordered set is given by Fig. 3. It is easy to prove that L is a lattice. Define $s_i = (-i, 1)$ $(i = 1, 2, \ldots)$. The principal ideals $(s_i]$ are standard, while their complete meet $(s]$ is not a standard ideal, for

$$(n, b) \cap [s \cup (n, a)] = (n, b),$$
$$[(n, b) \cap s] \cup [(n, b) \cap (n, a)] = 0 \cup (n, 0) = (n, 0),$$

and so (9) does not hold.

Owing to the definition, the following assertion is immediate:

LEMMA 6. *Let the correspondence $x \to \bar{x}$ be the homomorphism of a lattice L onto a lattice $\bar{L}$. If s is a standard element of L, then $\bar{s}$ is a standard element of $\bar{L}$.*

COROLLARY. *Let $x \to \bar{x}$ be a homomorphism of L onto $\bar{L}$, let S be an ideal of L, and denote by $\bar{S}$ the homomorphic image of S under this homomorphism. If S is standard in L, then $\bar{S}$ is standard in $\bar{L}$.*

PROOF OF THE COROLLARY. Let Θ be the congruence relation which induces the homomorphism $x \to \bar{x}$. Then the extension $\bar{\Theta}$ of Θ to $I(L)$ (defined before Lemma 5) is a congruence relation of $I(L)$ and $(x] \equiv (y]\ (\bar{\Theta})$ $(x, y \in L)$ if and only if $x \equiv y\ (\Theta)$. Hence the homomorphism $X \to \bar{X}$ $(X \in I(L))$ induced by $\bar{\Theta}$ is an extension of the homomorphism $x \to \bar{x}$ and carries S onto $\bar{S}$. Thus we may apply Lemma 6 to $I(L)$ and get the Corollary.

The converse of Lemma 6 is not true. One can find easily a lattice L, a homomorphism $x \to \bar{x}$ of L onto $\bar{L}$ and in L a standard element $\bar{s}$ such that in L there is no standard element s with $s \to \bar{s}$. As an example take the lattice U (see § 3 of Chapter I) and the homomorphism induced by Θ_{pq}. In the homomorphic image of U (which is the Boolean algebra of four elements) the image of r is standard, while r is not standard and is not congruent to any standard element (as a matter of fact, r forms alone a congruence class under Θ_{pq}).

From the point of view of later applications it is of importance the

LEMMA 7. *Any two standard congruence relations are permutable.*

PROOF. We have to prove that if S and T are standard ideals, x, y and z elements of the lattice with $x \equiv y\ (\Theta[S])$, $y \equiv z\ (\Theta[T])$, then for a suitable element u the relations $x \equiv u\ (\Theta[T])$, $u \equiv z\ (\Theta[S])$ hold.

First we consider the case $x \geqq y \geqq z$. Then by condition (γ'') of Theorem 2, we get elements $s \in S$ and $t \in T$ with $x = y \cup s$, $y = z \cup t$. We assert that $u = z \cup s$ fulfils the requirements. Indeed, $z \equiv z \cup s = u$ ($\Theta[S]$) and because of $u \cup t = z \cup s \cup t = y \cup s = x$ we have $u \equiv u \cup t = x$ ($\Theta[T]$).

In the general case consider the elements $x, x \cup y, x \cup y \cup z$. We have $x \equiv x \cup y$ ($\Theta[S]$) and $x \cup y \equiv x \cup y \cup z$ ($\Theta[T]$), therefore with a suitable element v, $x \equiv v$ ($\Theta[T]$) and $v \equiv x \cup y \cup z$ ($\Theta[S]$). We obtain in a similar way the existence of a w with $z \equiv w$ ($\Theta[S]$) and $w \equiv x \cup y \cup z$ ($\Theta[T]$). The element $u = v \cap w$ fulfils the requirements, for $u = v \cap w \equiv v \cap (x \cap y \cap z) = v$ ($\Theta[T]$) and this, together with $v \equiv x$ ($\Theta[T]$), gives $u \equiv x$ ($\Theta[T]$). Similarly, we can prove $u \equiv z$ ($\Theta[S]$), completing the proof of the lemma.

Let s be a standard element of the lattice L. Then from Lemma 2 it is clear that $L/(s] \simeq [s)$. Indeed, for all $x \in L$ we have $x \equiv s \cup x$ (Θ_s), and so any element of L is congruent to a suitable element of $[s)$, therefore $L/(s]$ is a homomorphic image of $[s)$. But this homomorphism is an isomorphism, for $x \geqq y \geqq s$ and $x \equiv y$ (Θ_s) implies $x \leqq y \cup s = y$, i. e. $x = y$.

To determine the factor lattice modulo a non-principal standard ideal is not so simple A solution of this problem is given in

THEOREM 4. *Let S be a standard ideal of the lattice L. Then the lattice of all ideals of L/S is isomorphic to the interval $[S, L]$ of $I(L)$, and, consequently, the interval $[S, L]$ of $I(L)$ determines L/S up to isomorphism.*

PROOF. We know from Lemma 5 that the extension of $\Theta[S]$ to $I(L)$ is Θ_S. Hence the homomorphism $L \to L/S$ induces in $I(L)$ a homomorphism $I(L) \to I(L)/(S]$. But, as we have remarked above, $I(L)/(S] \simeq [S, L]$ where the interval $[S, L]$ is taken in $I(L)$. This, together with a theorem of KOMATU [17], according to which every lattice is determined up to isomorphism by the lattice of its ideals, we get the theorem.

§ 4. Additional properties of standard elements and ideals

In our paper [10] there is a lemma that states: in a distributive lattice if the join and meet of two ideals are principal ideals, then the two ideals themselves are principal. We now generalize this to standard ideals of arbitrary lattices:

LEMMA 8. *Let I be an arbitrary and S a standard ideal of the lattice L. If $I \cup S$ and $I \cap S$ are principal, then I itself is principal.*

PROOF. Let $I \cup S = (a]$ and $I \cap S = (b]$. By condition (β') of Theorem 2 we have $a = s \cup x$ ($s \in S, x \in I$). We state that $I = (x \cup b]$. Indeed, suppose

$w \geqq x \cup b$ and $w \in I$. Then $(a] \supseteq S \cup (w] \supseteq S \cup (x \cup b] \supseteq S \cup (x] = (a]$, that is, $S \cup (a] = S \cup (x \cup b]$. Further, $(b] = S \cap I \supseteq S \cap (w] \supseteq S \cap (x \cup b] \supseteq S \cap (b] = (b]$, and so $S \cap (w] = S \cap (x \cup b]$. This two equalities imply (see condition (ii) of (δ') of Theorem 2) that $(w] = (x \cup b]$, and so $w = x \cup b$. Therefore, there are no elements in I greater than $x \cup b$, that is, $I = (x \cup b]$, completing the proof of the lemma.

By means of a simple example one can show that under the hypothesis of Lemma 8 S is not necessarily a principal ideal.

Since the ideals of a distributive lattice are standard, an exact analogue of the lemma of [10] is the

COROLLARY. *If the join and meet of two standard ideals are principal, then both standard ideals are principal.*

This corollary does not call for proof.

LEMMA 9. *Let s be a standard element of the lattice L and a an arbitrary element of L. Then $a \cap s$ is a standard element of the lattice $(a]$.*

PROOF. Any element of the ideal $(a]$ may be written in the form $a \cap x$ $(x \in L)$. Hence it is enough to prove that

$$(x \cap a) \cap [(s \cap a) \cup (y \cap a)] = [(x \cap a) \cap (s \cap a)] \cup [(x \cap a) \cap (y \cap a)].$$

Starting from the left member and applying (9) repeatedly, we get

$$(x \cap a) \cap [(s \cap a) \cup (y \cap a)] = (x \cap a) \cap [(s \cup y) \cap a] = (x \cap a) \cap (s \cup y) =$$
$$= (x \cap a \cap s) \cup (x \cap a \cap y) = [(x \cap a) \cap (s \cap a)] \cup [(x \cap a) \cap (y \cap a)]$$

which was to be proved.

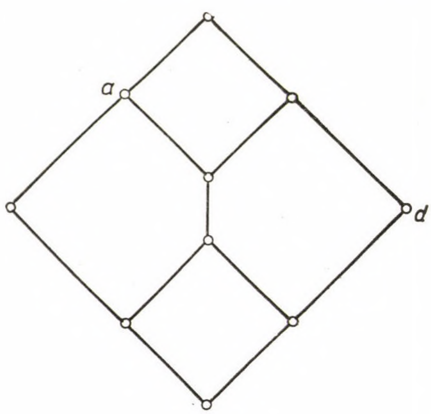

Fig. 4

COROLLARY. *Let S be a standard ideal and I an arbitrary ideal of the lattice L. Then $S \cap I$ is a standard ideal of the lattice I.*

Perhaps it is not worthless to note that the conclusions of this lemma are not valid for distributive elements. A counterexample is the lattice of Fig. 4, where d is a distributive element of the lattice, but the element $a \cap d$ is not a distributive element of the lattice $(a]$.

As we have seen, the neutrality of the element n was defined in such a way that for all $x, y \in L$ the sublattice $\{n, x, y\}$ is distributive. Though, in general, the notion of standard elements does not coincide with the notion of neutral elements, we may hope the validity of a weaker assertion for standard elements. Indeed, the following analogue of the definition of neutral elements is true:

THEOREM 5. *Let s_1 and s_2 be standard elements of the lattice L. Then the sublattice $\{s_1, s_2, x\}$ of L is distributive for all $x \in L$.*

PROOF. Our proof is based upon Theorem II. According to this, we have to prove the validity of (6), (7) and (8).

Condition (7) is valid, for it asserts the same as (9) since b or c is standard. As a consequence of condition (i) of (δ) of Theorem 1, (6) holds if a is standard; otherwise b and c are standard. In this case let us start with the right member of (6), apply (9) for the elements a, $a \cup c$ for the standard element b and then for a, b and the standard element c. We get

$$(a \cup c) \cap (a \cup b) = [(a \cup c) \cap a] \cup [(a \cup c) \cap b] = a \cup (a \cap b) \cup (c \cap b) = a \cup (c \cap b).$$

Finally, we prove (8). (8) is a symmetric function of its variables, therefore we have to prove it for one permutation of its variables only. Using the assertion of Theorem 3, according to which $s_1 \cup s_2$ and $s_1 \cap s_2$ are standard, further equality (9) and condition (i) of (δ) of Theorem 1, we get

$$(s_1 \cap s_2) \cup (s_1 \cap x) \cup (s_2 \cap x) = (s_1 \cap s_2) \cup [(s_1 \cup s_2) \cap x] =$$
$$= [(s_1 \cap s_2) \cup (s_1 \cup s_2)] \cap [(s_1 \cap s_2) \cup x] = (s_1 \cup s_2) \cap [(s_1 \cap s_2) \cup x] =$$
$$= (s_1 \cup s_2) \cap (s_1 \cup x) \cap (s_2 \cup x),$$

and this is just (8). Thus the proof of Theorem 5 is completed.

Applying Theorem 5 to $I(L)$ we get:

COROLLARY. *Let S_1 and S_2 be the standard ideals of the lattice L. Then S_1, S_2 and an arbitrary ideal X of L generate a distributive sublattice of $I(L)$.*

We have got Theorem 5 as an analogue of the definition of neutral elements (ideals). It is natural to ask, whether or not it is possible to get from Theorem 5 a new characterization of standard elements (ideals). That should mean that in a lattice the standard elements form a (unique) maximal subset for which the assertion of Theorem 5 is true. This is not true in general; not even in modular lattices. Consider the lattice V (see § 3 of Chapter I); there are in V only two standard elements: o and i. We may enlarge the set $\{o, i\}$ by the element p, and the assertion of Theorem 5 is true for this enlarged set as well. (See Problem 2.)

Now consider the lattice L and let us fix an element s of L. We call the mapping $x \to (x \cap s, x \cup s)$ of L into $L_s = (s] \times [s)$ the natural mapping of L into L_s. We can obtain by means of this notion a new characterization of standard elements, which is a direct generalization of a theorem of G. BIRKHOFF [5].

THEOREM 6. *The natural mapping of the lattice L into the lattice L_s is a meet isomorphism if and only if s is a standard element. It is an isomorphism if and only if s is neutral.*

PROOF. The first part of this theorem is essentially condition (δ) of Theorem 1. Namely, condition (i) assures that the mapping is a meet-homomorphism and (ii) that different elements have different images. The second part of the theorem is equivalent to Theorem III.

We remark that the idea of the proof of this theorem is due to BIRKHOFF [5]. This theorem seems to be a good tool for proving the standardness of an element.

COROLLARY. *Let L be a bounded lattice. $L_s \simeq L$ in the natural way (the direct components are supposed to be ideals of L and $(x, y) \to x \cup y$) if and only if s is a neutral element having a complement, that is s is an element of the center. In other words, L has a non-trivial direct decomposition if and only if its center has an element different from 0 and 1.*

In connection with Theorem 5 the following problem arises:

PROBLEM 2. Is it possible to characterize the set of standard elements as a maximal subset of the lattice L satisfying the condition of Theorem 5 and having some additional properties?

CHAPTER III

STANDARD AND NEUTRAL ELEMENTS

§ 1. Relations between standard and neutral elements

We have mentioned in the Introduction some causes which required the definition of standardness to be a generalization of neutrality and to coincide with the same in modular lattices. It is obvious that our definition fulfils this requirement, that is, the following assertions are valid:

LEMMA 10. *All the neutral elements of the lattice L are standard. Furthermore, in modular lattices the two notions coincide. A standard element s is neutral if and only if condition* (i′) *of Theorem III holds.*

PROOF. The first assertion is clear from the definitions. The second assertion is a consequence of Theorem IV and we get the third statement by a simple comparison of the conditions of Theorem 1 (δ) with those of Theorem III.

From these simple observations one can get a result of some interest for neutral ideals.

LEMMA 11. *Let n be a neutral element of the lattice L. Then $(n]$ is a neutral ideal of L and conversely.*

PROOF. Every neutral element is standard, and so by Lemma 4, $(n]$ is standard. Then by Lemma 10 it is enough to prove that

$$(n] \cap (I \cup J) = ((n] \cap I) \cup ((n] \cap J) \qquad (I, J \in I(L)).$$

Let $A = (n] \cap (I \cup J)$ and $B = ((n] \cap I) \cup ((n] \cap J)$. Since $A \supseteq B$ holds always, it is enough to prove that $a \in A$ implies $a \in B$. For some $i \in I$ and $j \in J$ we have $a \leqq i \cup j$, and so $a \leqq a \cap (i \cup j) \leqq n \cap (i \cup j)$. But $n \cap (i \cup j) = (n \cap i) \cup (n \cap j)$, because n is neutral and $(n \cap i) \cup (n \cap j)$ is an element of B, hence $a \in B$ holds as well. This completes the proof of the first part of the lemma. The converse statement is trivial.

It is of some interest that we could not find in the literature the assertion of this lemma (in [6] it is stated only for modular lattices). In § 3 of this chapter we shall derive this lemma from a more general theorem, using the deep theorem of ORE (or Lemma 12). We should like to point out that a direct proof of this lemma through Theorem III meets the same difficulty as that mentioned in this paper as Problem 1.

From Lemma 10 it is clear

LEMMA 12. *If an element s is standard in the lattice L as well as in its dual, then s is neutral.*

COROLLARY. *n is neutral if and only if*

$$x \cap (n \cup y) = (x \cap n) \cup (x \cap y),$$

$$x \cup (n \cap y) = (x \cup n) \cap (x \cup y)$$

for all $x, y \in L$.

Thus we see that from the five equalities which we have got from Theorem II to be characteristic for the neutrality of an element, three may be omitted.

LEMMA 13. *Let s and n be elements of L such that n is neutral, $s \leqq n$ and s is standard in $(n]$. Then s is a standard element of L.*

PROOF. From Theorem 6 we know that $x \to (x \cap n, x \cup n)$ is an isomorphism between L and a sublattice of $L_n = (n] \times [n)$. Under this isomorphism $s \to (s, n)$. Since s is standard in $(n]$ and n in $(n]$, therefore s is standard in

L_n. Since the property of being standard is preserved under taking a sublattice and under isomorphism, we get that s is standard in L.

If s is neutral in n, then — it is clear from the above proof — s is neutral in L too, i. e.

COROLLARY 1. *Let s be a neutral element of $(n]$ and n neutral in the lattice L. Then s is a neutral element of L.**

Applying Lemma 13 to $I(L)$ we get

COROLLARY 2. *Let S and N be ideals of the lattice L, $S \subseteq N$ such that N is neutral in L and S is standard in N. Then S is a standard ideal of L.*

Applying Corollary 1 to $I(L)$ we get a theorem of HASHIMOTO [14]:

COROLLARY 3. *A neutral ideal of a neutral ideal is neutral in the whole lattice.*

It is easy to see that the same assertion is not true for standard ideals or elements. As a counterexample take the lattice U and the elements $p > q$. p is standard in U, q is standard in $(p]$, but $(q]$ is not even a homomorphism kernel!

PROBLEM 3. We have seen in the Corollary of Lemma 12 that it is possible to define the neutral elements with the aid of two equalities. Is it possible to define neutrality by a single equality? (E. g.: is the neutrality of n equivalent to the condition that $(x \cap y) \cup (y \cap n) \cup (n \cap x) = (x \cup y) \cap (y \cup n) \cap \cap (n \cup x)$ holds for all $x, y \in L$?)

PROBLEM 4. Let G be a finite group and $L(G)$ the lattice of all subgroups of G. Characterize the standard elements of $L(G)$ (the same problem for neutral elements of $L(G)$ has been solved by G. ZAPPA [35]).

§ 2. Standard elements in weakly modular lattices

Our aim in this section is to prove the coincidence of (distributive and) standard and neutral elements in weakly modular lattices. This theorem contains a part of Lemma 10, that has asserted the same in modular lattices. There the proof was trivial, in consequence of the application of Theorem IV. But in weakly modular lattices we are in lack of a theorem of this kind, therefore the proof is not so simple.

* *Added in proof* (13 February 1961). The following assertion may be proved: Let a and b be neutral elements of the lattice L, $a \leq b$ and c a standard (neutral) element of , b . Then c is a standard (neutral) element of L.

THEOREM 7. *In a weakly modular lattice L, an element d is distributive if and only if it is neutral.*

PROOF. It follows easily from a theorem of ORE's paper [25] that d is distributive if and only if $x \equiv y\ (\Theta[(d]])$ is equivalent to $[(x \cap y) \cup d] \cap \cap (x \cup y) = x \cup y$. It follows that the kernel of the homomorphism induced by the congruence relation $\Theta[(d]]$ is $(d]$. Further, if $x, y \geqq d$ and $x \equiv y\ (\Theta[(d]])$, then $x = y$, because $x \cup y = [(x \cap y) \cup d] \cap (x \cup y) = x \cap y$. From these facts we will use only the following:

(*) If $a \leqq b \leqq d \leqq c \leqq e$ and d is a distributive element, then $\overline{a, b} \rightarrow \overline{c, e}$ implies $c = e$.

Indeed, under the stated conditions, $\overline{a, b} \rightarrow \overline{c, e}$ implies $c \equiv e\ (\Theta[(d]])$, and so $c = e$.

Now let d be a distributive element of the weakly modular lattice L. First we prove that d is standard, that is, we prove the validity of (9). Suppose (9) does not hold for a fixed couple $x, y \in L$. Then

$$x \cap (d \cup y) > (x \cap d) \cup (x \cap y).$$

Denote by a the left member of this inequality and by b the right member. We prove that

$$\overline{d, d \cap x} \rightarrow \overline{a, b}, \tag{11}$$

namely,

$$\overline{d, d \cap x} \xrightarrow{1} \overline{(d \cup x) \cap (d \cup y), b} \xrightarrow{1} \overline{a, b}.$$

Indeed, because of $d \cap x \leqq b$ we have to prove for the validity of

$$\overline{d, d \cap x} \xrightarrow{1} \overline{(d \cup x) \cap (d \cup y), b} \quad \text{only} \quad d \cup b = (d \cup x) \cap (d \cup y).$$

But $d \cup b = d \cup (x \cap d) \cup (x \cap y) = d \cup (x \cap y) = (d \cup x) \cap (d \cup y)$, for d is distributive. Now, using the inequalities $a \leqq (d \cup x) \cap (d \cup y)$ and $a > b$, we see that $b = b \cap a$ and $a = (d \cup x) \cap (d \cup y) \cap a$ are trivial. Thus

$$\overline{(d \cup x) \cap (d \cup y), b} \xrightarrow{1} \overline{a, b}$$

and (11) is proved.

Next we verify that

$$\overline{d, d \cup y} \rightarrow \overline{a, b}, \tag{12}$$

namely

$$\overline{d, d \cup y} \xrightarrow{1} \overline{d \cap x, a} \xrightarrow{1} \overline{a, b}.$$

To prove the first part of this statement, we have to show only $a \cap d = d \cap x$, but $a \cap d = d \cap x \cap (d \cup y) = d \cap x$. The second part of the assertion is clear.

Let us use the condition $a > b$ and the weak modularity of L: from these it follows the existence of elements u, v for which

$$\overline{a, b} \rightarrow \overline{u, v}, \quad d \leqq v < u \leqq d \cup y. \tag{13}$$

From (11) and (13) it follows $\overline{d, d \cap x} \rightarrow \overline{u, v}$, in contradiction to (*). Thus we have got a contradiction from $a > b$, so $a = b$, i. e. d is standard.

The second step of the proof is: using that d is standard, we prove that it is neutral.

If this statement is not true, then by Lemma 10 we conclude the existence of elements x, y of L such that

$$d \cap (x \cup y) > (d \cap x) \cup (d \cap y),$$

i. e. the condition (i′) of Theorem III does not hold. Putting $s_1 = d \cap (x \cup y)$ and $s_2 = (d \cap x) \cup (d \cap y)$ let us suppose $s_1 > s_2$. First we prove that

$$s_1 \cup x > s_2 \cup x \quad \text{and} \quad s_1 \cup y > s_2 \cup y.$$

Suppose that one of these does not hold, for instance, $s_1 \cup x \not> s_2 \cup x$; then from $s_1 > s_2$ we have $s_1 \cup x = s_2 \cup x$. We will see that it follows $\overline{d \cap x, x} \rightarrow \overline{s_1, s_2}$, namely

$$\overline{d \cap x, x} \xrightarrow{1} \overline{s_2 \cup (d \cap x), s_2 \cup x} \xrightarrow{1} \overline{s_1, s_2}.$$

To prove this it is enough to show that $s_1 \cap [s_2 \cup (d \cap x)] = s_2$ and $s_1 \cap (s_2 \cup x) = s_1$ Indeed, $s_1 \cap [s_2 \cup (d \cap x)] = s_1 \cap s_2 = s_2$ and $s_1 \cap (s_2 \cup x) = s_1 \cap (s_1 \cup x) = s_1$ (we have used $s_1 \cup x = s_2 \cup x$ in this step). Again from $s_1 > s_2$ and from the weak modularity it follows the existence of elements u, v with $d \cap x \leqq u < v \leqq x$ and $\overline{s_1, s_2} \rightarrow \overline{u, v}$. But $s_1, s_2 \leqq d$, and so $s_1 \equiv s_2$ (Θ_d), consequently $u \equiv v$ (Θ_d). Therefore (see condition (γ) of Theorem 1) $v = u \cup d_1$ with a suitable $d_1 \leqq d$. Then $v = u \cup d_1 \leqq u \cup (d \cap x) = u$, for we get from $v = u \cup d_1$ that $d_1 \leqq v \leqq x$, and hence $d_1 \leqq d \cap x$. The inequality we have just proved is in contradiction to the hypothesis $v > u$. Thus we have proved that $s_1 \cup x > s_2 \cup x$, and in a similar way one can prove $s_1 \cup y > s_2 \cup y$.

Now, using $s_1 \cup x > s_2 \cup x$ and $s_1 \cup y > s_2 \cup y$, we prove that

$$\overline{d \cap (s_2 \cup x), s_2 \cup x} \rightarrow \overline{s_1 \cap (s_2 \cup y), s_1},$$

namely,

$$\overline{d \cap (s_2 \cup x), s_2 \cup x} \xrightarrow{1} \overline{d \cap x, x} \xrightarrow{1} \overline{s_2 \cup y, s_2 \cup (x \cup y)} \xrightarrow{1} \overline{(s_2 \cup y) \cap s_1, s_1}.$$

From these $\overline{d \cap (s_2 \cup x), s_2 \cup x} \xrightarrow{1} \overline{d \cap x, x}$ is clear. To verify $\overline{d \cap x, x} \xrightarrow{1}$ $\xrightarrow{1} \overline{s_2 \cup y, s_2 \cup (x \cup y)}$ we use the inequality $d \cap x \leqq (d \cap x) \cup (d \cap y) = s_2 \leqq s_2 \cup y$, and so $(d \cap x) \cup (s_2 \cup y) = s_2 \cup y$, further $x \cup (s_2 \cup y) = s_2 \cup (x \cup y)$. To prove $\overline{s_2 \cup y, s_2 \cup (x \cup y)} \xrightarrow{1} \overline{(s_2 \cup y) \cap s_1, s_1}$ we have only to observe the inequality $s_1 = d \cap (x \cup y) \leqq s_2 \cup (x \cup y) = x \cup y$, and then $[s_2 \cup (x \cup y)] \cap s_1 = s_1$.

Before applying weak modularity we have to show that $s_1 \neq s_1 \cap (s_2 \cup y)$. Indeed, in case $s_1 = s_1 \cap (s_2 \cup y)$ it follows $s_1 \leqq s_2 \cup y$, and then $s_1 \cup y = s_2 \cup y$, which is a contradiction to $s_1 \cup y > s_2 \cup y$. From this we see that $d \cap (s_2 \cup x) = s_2 \cup x$ is also impossible, for $\overline{d \cap (s_2 \cup x), s_2 \cup x} \to \overline{s_1 \cap (s_2 \cup y), s_1}$, and so $d \cap (s_2 \cup x) = s_2 \cup x$ implies $s_1 \cap (s_2 \cup y) = s_1$. Now, using the weak modularity and $\overline{d \cap (s_2 \cup x), s_2 \cup x} \to \overline{s_1 \cap (s_2 \cup y), s_1}$, it follows the existence of u, v such that $d \cap (s_2 \cup x) \leqq u < v \leqq s_2 \cup x$ and $\overline{s_1 \cap (s_2 \cup y), s_1} \to \overline{u, v}$. It follows now $u \equiv v\ (\Theta_d)$ in a similar way as in the first step of the proof, thus $v = u \cup d'\ (d' \leqq d)$. But from $v \leqq s_2 \cup x$ we have $d' \leqq d \cap (s_2 \cup x)$ for $d \geqq s_1 > s_2$. Consequently, $v = u \cup d' \leqq u \cup [d \cap (s_2 \cup x)] = u$, a contradiction to $v > u$.

Thus we have verified the validity of the conditions of Theorem III, thus d is neutral. The proof of Theorem 7 is completed.

COROLLARY 1. *In a weakly modular lattice every standard element is neutral.*

The assertion is clear from condition (δ) of Theorem 1.

Apply this theorem to $I(L)$:

COROLLARY 2. *If $I(L)$ is weakly modular, then any standard ideal of L is neutral.*

COROLLARY 3. *In a relatively complemented lattice L any standard element is neutral.*

COROLLARY 4. *In a modular lattice any standard element and ideal is neutral.*

Corollaries 3 and 4 are immediate consequences of Lemma IV.

Unfortunately, we cannot establish Theorem 7 for distributive ideals, not even the more important Corollary 1 for standard ideals. A detailed discussion of the proof shows that the idea of the proof essentially uses that distributive, resp. standard elements are dealt with and not distributive, resp. standard ideals. It will be clear from § 4 of this chapter that we cannot get the results for ideals by a simple application of Theorem 7 to $I(L)$.

We shall now deal separately with (standard, i.e.) neutral elements of a special class of weakly modular lattices. We intend to show that in relatively complemented lattices the set of all neutral elements is again a relatively complemented lattice. First we prove

LEMMA 14. *Let a, b, c be neutral elements of a lattice L, and suppose $a < b < c$. If a relative complement d of b in the interval $[a, c]$ exists, then it is also neutral and uniquely determined.*

PROOF. We know from Theorem 6 that we can embed L in $L_b = (b] \times [b)$ under the correspondence $x \to (x \cap b, x \cup b)$. Under this $d \to (a, c)$, therefore d

is neutral (for both components of d are neutral) in L_b, and consequently it is neutral in L. The unicity assertion is trivial.

COROLLARY 1. (BIRKHOFF.) *Any complement of a neutral element is neutral.*

COROLLARY 2. *The neutral elements (if any) of a relatively complemented lattice form a relatively complemented distributive sublattice.*

We note that from Corollary 1 we do not get Lemma 14, only that d is neutral in $[a, c]$.

Lemma 14 is not true for standard elements. As an example take the lattice U where o, p, i are standard, while (the unique) relative complement of p in $[o, i]$ is r which is not standard.

PROBLEM 5. Is a distributive (or at least a standard) ideal of a weakly modular lattice neutral?

§ 3. A neutrality condition for standard elements

The last statement of Lemma 10 gives a necessary and sufficient condition for a standard element to be neutral. The condition is not trivial, for it is a conclusion of the comparison of the deep Theorem III with condition (δ) of Theorem 1. But in the previous paragraph, when we wanted to prove the coincidence of standard and neutral elements in weakly modular lattices, we have seen that this condition is not easy to apply. Therefore we set ourselves the aim of finding a sharper condition from which Corollary 1 of Theorem 7 may be easily derived. This is the content of

THEOREM 8. *A standard element s of the lattice L is neutral if and only if $a \geqq b \geqq s \geqq c \geqq e$ and $\overline{a, b} \rightarrow \overline{c, e}$ imply $c = e$.*

To prove the "only if" part of the theorem, suppose s is neutral. Then the dual ideal $[s)$ — as an ideal of the dual lattice $\tilde{L}$ — is standard. So it is impossible that a congruence of the form $c \equiv e$ (Θ_s) would hold in the dual lattice $\tilde{L}$, thus $c = e$.

Now we interrupt our proof to observe that the property (*) (stated in the previous section) is characteristic for distributive elements. Indeed, if d is not distributive, then there exist x, y with $d \cup (x \cap y) < (d \cup x) \cap (d \cup y)$. We prove that $d \cup (x \cap y) \equiv (d \cup x) \cap (d \cup y)$ $(\Theta[(d]])$. Indeed, $d \equiv d \cap x \cap y$ $(\Theta[(d]])$. Joining both sides of this relation first with x, then with y, we get

$$d \cup x \equiv x \quad (\Theta[(d]]) \quad \text{and} \quad d \cup y \equiv y \quad (\Theta[(d]]).$$

Meeting the corresponding sides, it results

$$(d \cup x) \cap (d \cup y) \equiv x \cap y \quad (\Theta[(d]])$$

Finally, joining both sides with d, we reach $(d \cup x) \cap (d \cup y) \equiv d \cup (x \cap y)$ $(\Theta[(d]])$, as desired.

Now we prove the "if" part of Theorem 8 If s is not neutral, then s is not distributive in the dual lattice $\tilde{L}$. We may apply the result just obtained to get the existence of $a \leqq b \leqq s \leqq c \leqq e$ in $\tilde{L}$ with $\overline{a, b} \to \overline{c, e}$. This is the same as the required relation in L, completing the proof of Theorem 8.

From the proof we see that the fact that standard elements and not ideals are dealt with, is again very essential.

Suppose that in the lattice L the following condition holds which is a weakened form of weak modularity:

(14) whenever $a > b \geqq c > d$ and $\overline{a, b} \to \overline{c, d}$, then $\overline{c, d} \to \overline{a_1, b_1}$ with suitable elements $a \geqq a_1 > b_1 \geqq b$.

COROLLARY 1. *If the lattice L satisfies* (14), *then every standard element in L is neutral.*

PROOF. Suppose L satisfies (14) and $s \in L$ is standard, but not neutral. Then, owing to Theorem 8 we can find elements $a > b \geqq s \geqq c > d$ such that $\overline{a, b} \to \overline{c, d}$. Now, applying (14), we infer the existence of a pair of elements a_1, b_1 such that $a \geqq a_1 > b_1 \geqq b$ and $\overline{c, d} \to \overline{a_1, b_1}$. Consequently, $a_1 \equiv b_1$ (Θ_s), which is impossible, since $a_1 > b_1 \geqq s$ and s is standard.

Since condition (14) is a generalization of weak modularity, it follows that the last corollary implies Corollary 1 of Theorem 8. We will prove by means of a simple example that this new corollary is stronger than the former one, that is, there exists a not weakly modular lattice L which satisfies (14).

Let L be the lattice defined in § 2 of Chapter II. We adjoin three new elements: $x, 0, 1$, subject to the following relations:

$$x \cup a = 1, \quad x \cap a = 0$$

for all $a \in L$. We get a lattice H whose diagram is given in Fig. 5. In this lattice $\overline{0, x} \to \overline{(2, 0), (1, 0)}$ and despite this fact $\overline{(2, 0), (1, 0)} \to \overline{u, v}$ holds for no $u, v \in L$ for which $x \geqq u > v \geqq 0$. This can be seen from the fact that the two different elements of the interval $[0, x]$ are not congruent modulo $\Theta_{(2,0)(1,0)}$. Consequently, H is not weakly modular. But condition (14) holds in H. Indeed, within L it holds, for L is simple (see Lemma IV). The only remaining case of interest is $1 > a \geqq b > c$ $(a, b, c \in L)$, when $\overline{1, a} \to \overline{b, c}$ always holds. But in this case $\overline{b, c} \to \overline{a, d}$ where d is an arbitrary element with $1 > d \succ a$.

In this counterexample (14) holds and so does the dual of (14). It is easy to show that any counterexample of this kind is infinite.

LEMMA 15. *Let L be a semi-discrete lattice in which* (14) *and its dual hold. Then L is weakly modular.*

PROOF. Let $a, b, c, d \in L$, $a > b$, $c > d$ and $d \cap a = b$, $d \cup a \geqq c$. Then $\overline{a, b} \to \overline{c, d}$. If $c = d \cup a$, then simply $\overline{c, d} \xrightarrow{1} \overline{a, b}$. If $c < d \cup a$, we choose an element x with $c \leqq x \prec d \cup a$. Then $\overline{x, d \cup a} \xrightarrow{1} \overline{a, b} \xrightarrow{2} \overline{c, d}$, i.e. $\overline{x, d \cup a} \to \overline{c, d}$ and $d \cup a > x \geqq c > d$. Using (14) we get $\overline{c, d} \to \overline{u, v}$ with suitable $d \cup a \geqq u > v \geqq x$. But $d \cup a \succ x$, therefore $u = d \cup a$, $v = x$, that is, $\overline{c, d} \to \overline{x, d \cup a}$. Trivially, $\overline{x, d \cup a} \to \overline{a, b}$, and so $\overline{c, d} \to \overline{a, b}$.

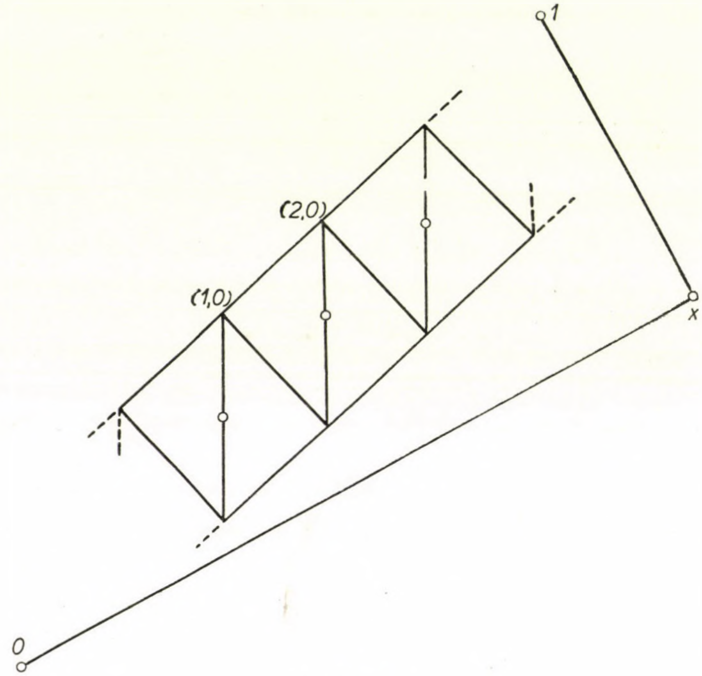

Fig. 5

In case $a > b, c > d, b \cup c = a$ and $b \cap c \leqq d$, the relation $\overline{a, b} \to \overline{c, d}$ holds, and then we can verify weak modularity by the dual of the above reasoning. The general case $\overline{a, b} \xrightarrow{n} \overline{c, d}$ may be deduced using a simple induction on n.

We see that in Lemma 15, instead of the semi-discreteness of the lattice L, we have used the following weaker property: if $a > b$, then there exist x and y with $a \succ x \geqq b$ and $a \geqq y \succ b$.

PROBLEM 6. Is (14) equivalent to weak modularity in finite lattices?

§ 4. On the lattice of all ideals of a weakly modular lattice

In § 2 the problem arose: why is it not possible to get the results of Theorem 7 for distributive and standard ideals by a simple application of the theorem to $I(L)$. In general, a way of getting a theorem for standard ideals is to prove the same first for standard elements. For instance, we got in this way the coincidence of standard and neutral ideals in modular lattices.

Whenever we make a step of this kind we have to ponder over the question: did we make a supposition on the lattice L which is not preserved if we pass from L to $I(L)$? In case of modular lattices there is no trouble, for if L is modular, then so is $I(L)$. But this is not the case in weakly modular lattices:

THEOREM 9. *The lattice of all ideals of a weakly modular lattice is not necessarily weakly modular.*

PROOF. We have to construct a weakly modular lattice K such that $I(K)$ is not weakly modular. Consider the chain of non-negative integers and take the direct product of this chain by the chain of two elements. The elements of this lattice are of the form $(n, 0)$ and $(n, 1)$, where 0 and 1 are the zero and unit elements of **2** and n is an arbitrary non-negative integer. Further, we define the elements x_n $(n = 1, 2, \ldots)$ satisfying the following relations:

$$x_n \cup (n-1, 1) = x_n \cup (n, 0) = (n, 1),$$
$$x_n \cap (n-1, 1) = x_n \cap (n, 0) = (n-1, 0).$$

Thus we have got a lattice L. Finally, we define three further elements $x, y, 1$ subject to

$$\left.\begin{aligned} x \cup y = x \cup z = y \cup z = 1, \\ x \cap y = x \cap z = y \cap z = (0,0) \end{aligned}\right\} \quad (z \neq 0,\ z \in L).$$

Denote the partially ordered set of all these elements by K. The elements of K are denoted by ∘ in Fig. 6.

It is easy to see that K is a lattice. K is simple and so, by Lemma IV, weakly modular. All but two ideals of K are principal ideals, these exceptional ones are denoted by ⊙ in the diagram, thus the diagram of K, completed by these two elements, gives the diagram of $I(K)$. Now, it is easy to see that K is not weakly modular. Indeed, under the congruence relation generated by the congruence of the two new elements, no two different elements of K are congruent. While from the congruence of any two different elements of K it follows the congruence of the two new elements, we have considered K to be imbedded in $I(K)$. The existence of the lattice K proves Theorem 9.

Some related unsolved problems are listed at the end of this section.

So far we could assure the weak modularity only of the lattice of all ideals of a modular lattice. Naturally, the same is true for every weakly modular lattice in which the ascending chain condition holds, because in this case the lattice of all ideals is identical with (more precisely isomorphic to) the original lattice. The following question arises: is it possible that the lattice of all ideals of a relatively complemented lattice is weakly modular if in the lattice the ascending chain condition does not hold? Is it possible

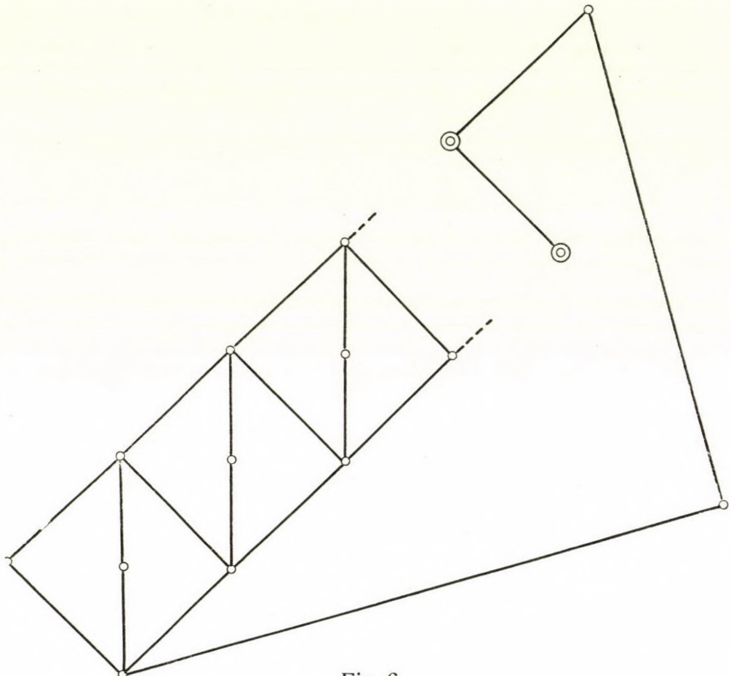

Fig. 6

that the ideal lattice of the same is relatively complemented? The interest of this latter question is that in modular lattices the answer is always negative as a consequence of a theorem of HASHIMOTO [15]. Despite this, the following assertion is true:

There exists a relatively complemented lattice L, not satisfying the ascending chain condition, such that $I(L)$ is relatively complemented. This lattice may be chosen to be semi-modular.

To construct L, consider an infinite set H. We say that the partition p of H, which devides the set H into the disjoint subsets H_α, is finite, if all but a finite number of the H_α consist of one element, and every H_α consists of a finite number of elements. We denote by $FP(H)$ the set of all finite and by $P(H)$ the set of all partitions of H.

It is clear that the join and meet of any two finite partitions are finite again, and if a partition is smaller than a finite partition, then it is also finite. It follows that $FP(H)$ is an ideal of the lattice $P(H)$. Now, it is easy to prove that just the finite partitions are the elements of the lattice $P(H)$ which are inaccessible from below. Indeed, if p is a finite partition, then the interval $[\omega, p]$ of the lattice $P(H)$ is finite, therefore p is inaccessible from below. Now suppose p is not finite, and let $\{H_\alpha\}$ be the corresponding partition of H (the H_α are pairwise disjoint). Either infinitely many H_α are containing more than one element, or at least one H_α contains an infinity of elements. In the first case, assume that $H_1, H_2, \ldots$ contain more than one element. We define the partition p_i to be the same as p on the set $H \setminus \bigvee_{j=i+1}^{\infty} H_j$, while on the set $\bigvee_{j=i+1}^{\infty} H_j$ let all the classes of p_i consist of one element. Obviously, $p_1 < p_2 < \cdots$ and $\bigvee p_i = p$, consequently, p is accessible from below. In the second case, let H_1 be a set which contains infinitely many elements $\{x_1, x_2, \ldots\}$. We define the partition p_i: upon the set $H \setminus H_1$ it is the same as p, $\{x_1, \ldots, x_i\}$ is one class, and all the x_n $(n > i)$ form separate classes. Again, $p_1 < p_2 < \cdots$ and $\bigvee p_i = p$, so p is accessible from below.

It is also clear that every partition is the complete join of finite partitions and, finally, it is well known (it follows trivially from Lemma III) that $P(H)$ is meet continuous. It follows from a theorem of KOMATU [17] that $P(H)$ is isomorphic to the lattice of all ideals of $FP(H)$.

Now we will prove that $FP(H)$ satisfies the requirements. We have to prove yet that in $FP(H)$ the ascending chain condition does not hold, that $FP(H)$ and $P(H)$ are relatively complemented, and finally that $FP(H)$ is semi-modular. The first of these assertions is trivial, since H is infinite. The second and the third assertions have been proved in [25] for $P(H)$, but these properties are preserved under taking an ideal of the lattice, therefore these hold in $FP(H)$.

We could assure the weak modularity of the ideal lattice of a modular lattice, for the modularity of a lattice may be defined by an equality. We now show that if the weak modularity of a lattice is a consequence of the fulfilment of a system of equalities, then the ideal lattice is also weakly modular. First we prove a general theorem which will serve for other purposes as well.

To formulate the theorem we need two notions. Following ORE [25] we call a subset $\bar{I}$ of the ideal I a *covering system* of I if $I=\{x; \exists y \in \bar{I}, x \leqq y\}$. Thus, for instance, $\bar{I}=I$ is always a covering system and if $I=(a]$, then $\{a\}$ is a covering system. If I is generated by the set $\{x_\alpha\}$, then the finite joins of the x_α form a covering system.

Let $f_\alpha(y, x_1, \ldots, x_n)$ and $g_\alpha(y, x_1, \ldots, x_n)$ be lattice polynomials, where n depends on α and α runs over an arbitrary set of indices A. (It is not a restriction that $f_\alpha(y, x_1, \ldots, x_n)$ and $g_\alpha(y, x_1, \ldots, x_n)$ depend on the same number of variables. Indeed, if $g_\alpha=g_\alpha(y, x_1, \ldots, x_r)$, $r<n$, then define $g'_\alpha(y, x_1, \ldots, x_n)=g_\alpha(y, x_1, \ldots, x_r) \cup (x_1 \cap x_2 \cap \cdots \cap x_r \cap \cdots \cap x_n \cap y)$. Independently of the values of the $x_1, \ldots, x_n$, the equality $g_\alpha(y, x_1, \ldots, x_r)=$ $=g'_\alpha(y, x_1, \ldots, x_n)$ always holds.) We say that the element s is of the type $f_\alpha=g_\alpha$ $(\alpha \in A)$, if for all $a_1, \ldots, a_n \in L$ and $\alpha \in A$ we have $f_\alpha(s, a_1, \ldots, a_n)=$ $=g_\alpha(s, a_1, \ldots, a_n)$. It is clear from (9) that the standard elements are of the type $f_\alpha=g_\alpha$ with the polynomials $f_1(y, x_1, x_2)=x_1 \cap (y \cup x_2)$ and $g_1(y, x_1, x_2)=$ $=(x_1 \cap y) \cup (x_1 \cap x_2)$ and $A=\{1\}$. Similarly, the neutral elements are also of the type $f_\alpha=g_\alpha$; we get a system of five polynomials from the Corollary of Theorem II and another system consisting of two polynomials from the Corollary of Lemma 12.

THEOREM 10. *Given the ideal I of the lattice L and a covering system $\bar{I}$ of I and the lattice polynomials f_α, g_α $(\alpha \in A)$. If every element of $\bar{I}$ is of the type $f_\alpha=g_\alpha$ $(\alpha \in A)$, then I as an element of $I(L)$ is of the type $f_\alpha=g_\alpha$ $(\alpha \in A)$.*

PROOF. It is enough to prove the theorem for one pair of polynomials $f_\alpha=g_\alpha$. For if the theorem failed to be true, then there would be a pair of polynomials $f=g$ such that I does not satisfy the corresponding equality.

Consider the polynomials f and g, and construct the following subsets of L:

$$F=\{t; t \leqq f(a, j_1, \ldots, j_n), a \in \bar{I}, j_1 \in J_1, \ldots, j_n \in J_n\},$$

$$G=\{t; t \leqq g(a, j_1, \ldots, j_n), a \in \bar{I}, j_1 \in J_1, \ldots, j_n \in J_n\}$$

where $J_1, \ldots, J_n$ are fixed ideals of L. We prove that F is an ideal. It is enough to prove that $t_1, t_2 \in F$ implies $t_1 \cup t_2 \in F$. Indeed, if $t_1, t_2 \in F$, then there exist $a_i \in \bar{I}$ and $j_{1,i} \in J_1, \ldots, j_{n,i} \in J_n$ $(i=1,2)$ with

$$t_i \leqq f(a_i, j_{1,i}, \ldots, j_{n,i}).$$

Now choose an element a of $\bar{I}$ for which $a_1 \cup a_2 \leqq a$. Then $f(a, j_{1,1} \cup$ $\cup j_{1,2}, \ldots, j_{n,1} \cup j_{n,2})$ is an element of F, and since the lattice polynomials are isotone functions of their variables, $t_1 \cup t_2 \leqq f(a, j_{1,1} \cup j_{1,2}, \ldots, j_{n,1} \cup j_{n,2})$ is clear, and so $t_1 \cup t_2 \in F$. Similarly, we can prove that G is also an ideal. If $t \in F$, then $t \leqq f(a, j_1, \ldots, j_n)$, but $f(a, j_1, \ldots, j_n)=g(a, j_1, \ldots, j_n)$, for a is an element of the type $f=g$, and so $t \leqq g(a, j_1, \ldots, j_n)$, that is, $t \in G$. We get $F \subseteq G$ and similarly $G \subseteq F$, that is, $F=G$. Owing to Lemma I, $F=f(I, J_1, \ldots, J_n)$ is clear. $G=g(I, J_1, \ldots, J_n)$ holds as well. Summing up, we got that $f(I, J_1, \ldots, J_n)=g(I, J_1, \ldots, J_n)$ and that was to be proved.

Now we turn our attention to corollaries of this theorem. We say that the lattice L is of the type $f_\alpha=g_\alpha$ if every element of L is of the same type, i. e. if the equalities $f_\alpha=g_\alpha$ $(\alpha \in A)$ identically hold.

COROLLARY 1. *Let f_α, g_α $(\alpha \in A)$ be lattice polynomials and suppose L is of the type $f_\alpha=g_\alpha$ $(\alpha \in A)$. Then this system of equalities holds in $I(L)$ too.*

Corollary 1 follows immediately from Theorem 10 taking $\bar{I}=I$ for all ideals $I \in I(L)$.

This corollary was known by BIRKHOFF (see Ex. 1 and Ex. 2 of pages 79 and 80 in [6], especially Ex. 2 (b*)). It answers in affirmative the first question of Ex. 2 (b*).

From Corollary 1 it follows immediately the following assertion, consisting of a theorem of STONE and one of DILWORTH:

COROLLARY 2. *The lattice of all ideals of a modular lattice is modular; the lattice of all ideals of a distributive lattice is distributive.*

Since both the standard and neutral elements are of the type $f_\alpha=g_\alpha$, we get from Theorem 10 the following

COROLLARY 3. *The principal ideal generated by a neutral element is neutral. A standard element generates a standard principal ideal.*

As a generalization of Lemma 13 we get

COROLLARY 4. *Let N be a neutral ideal of the lattice L. If F is an ideal of the type $f_\alpha=g_\alpha$ $(\alpha \in A)$ of the lattice N and the zero of any lattice is of the type $f_\alpha=g_\alpha$ $(\alpha \in A)$, then F is an ideal of the same type of the lattice L.*

PROOF. Owing to Theorem 10, we get that $(N]$ is a neutral ideal of $I(L)$ and $(F]$ is an ideal of the type $f_\alpha=g_\alpha$ of the lattice $(N]$. Therefore, it is enough to prove this assertion for a neutral element n and for an element f of type $f_\alpha=g_\alpha$. The proof may be carried out just in the same way as that of Lemma 13, we have to use only the assumption that the zero is of type $f_\alpha=g_\alpha$ (this has been satisfied trivially for standard elements).

We note that the supposition: the zero element of every lattice is of type $f_\alpha=g_\alpha$ $(\alpha \in A)$ is essential. If this does not hold, then the zero element is of type $f_\alpha=g_\alpha$ in the principal ideal generated by the zero element, the zero element is neutral and despite this fact the conclusion of Corollary 4 does not hold.

An interesting special case of Corollary 4 is

COROLLARY 4'. *An element of type $f_\alpha = g_\alpha$ ($\alpha \in A$) of a neutral ideal N of the lattice L is of type $f_\alpha = g_\alpha$ ($\alpha \in A$) in the whole lattice.*

Finally, we consider the question: is the converse of Theorem 10 or any of its corollaries true?

The converse of Theorem 10 is not true. It is not true even in the very special case when the defining system of the standard elements is in question. In fact, we have shown in § 2 of Chapter II the existence of standard ideals in lattices without standard elements.

The converses of Corollaries 1 and 2 are naturally true, for L is a sublattice of $I(L)$.

The converse of Corollary 4 states the following:

Assume that an ideal N of the lattice L has the property that whenever F is an ideal of the type $f_\alpha = g_\alpha$ ($\alpha \in A$) (provided the 0 of any lattice is of this type) in N, then it is of the same type in the whole L. Then N is a neutral ideal.

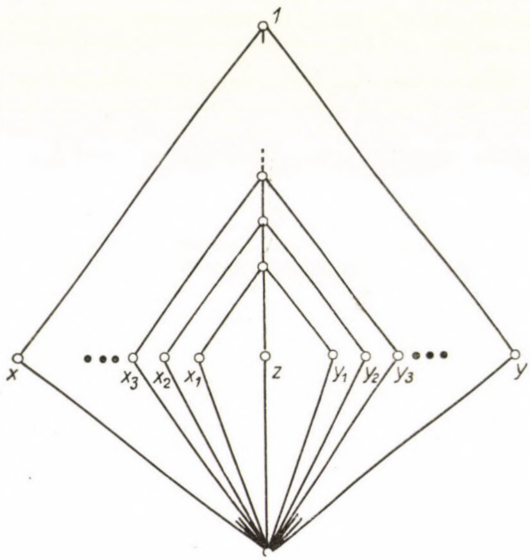

Fig. 7

This assertion is obviously true. Indeed, if an ideal N has the property required, then since N is a neutral ideal of N and since the property of being neutral is a property of type $f_\alpha = g_\alpha$ ($\alpha \in A$), it follows N is neutral in the whole lattice and this was to be proved.

We now prove that the converse of Corollary 4' is not true in general. Let L be a lattice generated by the elements $x, y, z, x_1, x_2, \ldots$ and $y_1, y_2, \ldots$. We require that

$$x \cap y = y \cap z = z \cap x = x_i \cap x_j = x_i \cap y_j = y_i \cap y_j = 0 \qquad (i > j)$$

and

$$z < x_1 \cup y_1 < x_2 \cup y_2 < \cdots < x \cup y, \quad z \cup y_i = z \cup x_i = x_i \cup y_i.$$

Let N be the ideal of L generated by the z, x_i and y_i ($i = 1, 2, \ldots$). The figure of L is shown in Fig. 7. We prove that any element n of the type $f_\alpha = g_\alpha$ of N is of the same type in L. This follows easily from the following assertion: if L_1 is a sublattice of L generated by n, x and $u_1, \ldots, u_k$, then one can find a sublattice L_2 of N such that $L_1 \cong L_2$ and under this isomorphism n corresponds to n. The validity of this assertion follows directly from the construction.

It remains to show that N is not neutral. Indeed, $N \cap (x] = (0]$ and $N \cap (y] = (0]$, but $N \cap [(x] \cup (y]] = N$.

PROBLEM 7. Is it possible to construct a relatively complemented lattice L such that $I(L)$ is not weakly modular? (This would be a sharpening of Theorem 9.)

PROBLEM 8. Is any homomorphism kernel of a relatively complemented lattice a neutral ideal?

REMARK. Using Theorem 11 we see that Problem 8 is a special case of Problem 5.

PROBLEM 9. Let the lattice polynomials f_α, g_α $(\alpha \in A)$ and f'_β, g'_β $(\beta \in B)$ be given. Give condition on the polynomials that an element be of the type $f_\alpha = g_\alpha$ $(\alpha \in A)$ if and only if it is of type $f'_\beta = g'_\beta$ $(\beta \in B)$.

PROBLEM 10. Give types of weakly modular lattice which are defined by identical relations and are different from the following three classes of lattices: a) the class consisting only of the lattice of one element; b) the class of distributive lattices; c) the class of modular lattices.

REMARK. BIRKHOFF states in [6] that among the modular lattices generated by three elements, one can define with the aid of identical relations only the above listed three classes of lattices.

PROBLEM 11. Find identities (in the variables s, x, y) ensuring that in the lattice generated by s, x and y the element s should be standard.

REMARK. The same problem for neutral elements is solved by Corollary of Theorem II.

PROBLEM 12. Determine the free standard lattice $FSL(3)$, that is, the free lattice generated by the elements s, x, y and we suppose s to be standard in $FSL(3)$.

REMARK. The same problem for neutral elements has been solved in [6], for the free neutral lattice with three generators is the free distributive lattice with three generators.

CHAPTER IV
HOMOMORPHISMS AND STANDARD IDEALS

§ 1. Homomorphism kernels and standard ideals

It is assured already by Lemma 3 that a standard ideal is a homomorphism kernel. The converse statement — as we have remarked — is not true in general, not even in modular lattices.[6] A simple example for that is shown in Fig. 8. The principal ideal $(a]$ of this lattice is a homomorphism kernel (obviously, because it is a prime ideal), but it is not standard for $x \cap (a \cup t) = x$ but $(x \cap a) \cup (x \cap t) = y$.

Now we prove:

THEOREM 11. *Let L be a section complemented lattice. Then every homomorphism kernel of L is a standard ideal and every standard ideal is the kernel of precisely one congruence relation.*

PROOF. Let the ideal I of the lattice L be the kernel of the homomorphism induced by the congruence relation Θ. Let $a \equiv b$ (Θ), $a \geqq b$, $a, b \in L$. We pick out an arbitrary element u of I. From the definition of section complementedness it follows the existence of an element $c \leqq u \cap a \cap b$ of L, for which the dual ideal $[c)$ as a lattice is section complemented, that is, any interval of type $[c, d]$ is complemented. From $u \in I$ it follows $c \in I$. Let b' be the relative complement of b in the interval $[c, a]$. From $a \equiv b$ (Θ) we conclude that $c = b \cap b' \equiv a \subset b' = b'$ (Θ), and, since I is a homomorphism kernel, $b' \in I$. Then $b \cup b' = a$ and $b' \in I$, thus by condition (γ'') of Theorem 2, I is a standard ideal.

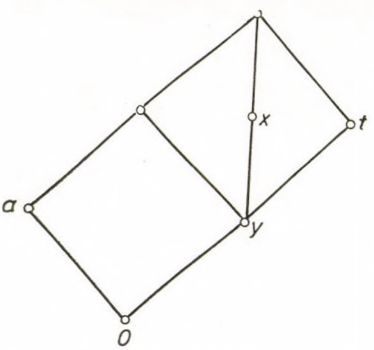

Fig. 8

At the same time we have proved that if I is the kernel of the homomorphism induced by Θ, then $\Theta = \Theta[I]$, and it follows that every standard ideal is the kernel of at most one homomorphism. It is known already from Lemma 3 that every standard ideal is the kernel of at least one homomorphism. Thus the proof of Theorem 11 is completed.

[6] It is included in a theorem of HASHIMOTO [14] that in a finite modular lattice every homomorphism kernel is standard and every congruence relation is a standard one if and only if the lattice is a direct product of simple lattices.

Theorem 11 is not true with "neutral ideal" instead of "standard ideal". A counterexample is shown in Fig. 9. That lattice is section complemented, the principal ideal $(a]$ is standard, but not neutral.

Since relatively complemented lattices form an important subclass of section complemented lattices (see Lemma V), therefore we formulate the assertions of Theorem 11 again for relatively complemented lattices.

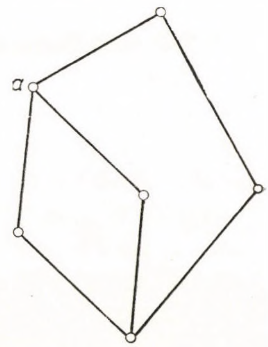

Fig. 9

COROLLARY 1. *In relatively complemented lattices there is a one-to-one correspondence between the standard ideals and the homomorphisms having kernel, letting a homomorphism correspond to its kernel.*

From Lemma 10 we know that in a modular lattice every standard ideal is neutral. Hence we get

COROLLARY 2. *In section complemented modular lattices there is a one-to-one correspondence between the neutral ideals and the homomorphisms having kernel.*

It is obvious that every homomorphism of the lattice L has a kernel if (and only if) L has a zero. Adding this trivial remark to Corollary 2 we get

COROLLARY 3. *In relatively complemented modular lattices with zero there is a one-to-one correspondence between the neutral ideals and homomorphisms, letting a homomorphism correspond to its kernel. This correspondence is an isomorphism between the lattice of all neutral ideals of L and $\Theta(L)$.*

This Corollary 3 is BIRKHOFF's theorem mentioned in the Introduction. We did not use in the formulation the expression "section complemented", for in modular lattices it means the same as "relatively complemented". This corollary shows that Theorem 11 is actually a generalization of BIRKHOFF's theorem.

Using Corollary 2 of Theorem 7, we can give another generalization of BIRKHOFF's theorem:

COROLLARY 4. *Let L be a lattice with zero, and suppose that $I(L)$ is weakly modular. Then there is a natural one-to-one correspondence between the neutral ideals and homomorphisms of L.*

From Theorem 11 and its Corollary 4 we shall get a generalization as well as a new proof of the structure theorem of relatively complemented lattices. An important tool in proving the structure theorem was the assertion that any two congruence relations of a relatively complemented lattice are permutable. First we generalize this assertion:

COROLLARY 5. *Any two congruence relations of a section complemented lattice L are permutable.*

PROOF. Suppose the assertion is not true. Then (as it is obvious from the proof of Lemma 7) there exist $a>b>c$ $(a, b, c \in L)$ and $\Theta, \Phi \in \Theta(L)$ such that $a \equiv b\ (\Theta)$, $b \equiv c\ (\Phi)$, while no element d of $[c, a]$ satisfies $a \equiv d\ (\Phi)$ and $d \equiv c\ (\Theta)$.

By the definition of section complementedness, there exists an $e \in L$ such that $a, b, c \in [e)$ and $[e)$ is section complemented. Any congruence relation Θ of L induces in $[e)$ a congruence relation $\bar{\Theta}$: $x \equiv y\ (\bar{\Theta})$ $(x, y \in [e))$ if and only if $x \equiv y\ (\Theta)$. $[e)$ is a section complemented lattice with zero, therefore every congruence relation of $[e)$ is a standard congruence relation. Owing to Lemma 7 we see that $\bar{\Theta}$ and Φ are permutable, that is, there exists $d \in [c, a]$ with $a \equiv d\ (\bar{\Phi})$ and $d \equiv c\ (\bar{\Theta})$. This implies $a \equiv d\ (\Phi)$ and $d \equiv c\ (\Theta)$, and this contradiction (the existence of such a d) proves the validity of Corollary 5.

REMARK. In applications it is enough to have Corollary 5 for section complemented lattices with zero. In this special case all the proof of Corollary 5 is the following: from the supposition and Theorem 11 it follows that every congruence relation of L is a standard congruence relation and by Lemma 7 any two standard congruence relations are permutable. — We remark that in proving Corollary 6 we shall not use Corollary 5.

DILWORTH has stated a structure theorem for relatively complemented lattices with ascending chain condition and with zero. We will suppose only that the lattice is section complemented and satisfies the ascending chain condition, and in this case we shall give a necessary and sufficient condition for the validity of the structure theorem.

COROLLARY 6. *Let L be a section complemented lattice with zero satisfying the ascending chain condition. A necessary and sufficient condition for L to be the direct product of simple lattices is that L be weakly modular.*

PROOF. First, suppose L is weakly modular. If L is not simple, there is a non-trivial congruence relation Θ of L, and let $I = \{x; x \equiv 0\ (\Theta)\}$. From the ascending chain condition it follows that $I(L) \simeq L$ and I is principal, $I = (n]$. Thus $I(L)$ is weakly modular, hence from Corollary 4 of Theorem 11 it follows that $(n]$ is neutral. From Corollary 3 of Theorem 10 we get the neutrality of n. From the ascending chain condition it follows the existence of maximal neutral elements n'_α $(\alpha \in A)$ different from 1. The complements n_α $(\alpha \in A)$ of the n'_α are minimal neutral elements of L. The index set A is finite. Otherwise, let $n_1, n_2, \ldots$ be distinct minimal neutral elements. Then putting $m_j = \bigcup_{i=1}^{j} n_i$, we have $0 < m_1 < m_2 < \cdots$ contradicting the ascending chain condition. Let $n_1, \ldots, n_k$ be the minimal neutral elements of L, and put $n = \bigvee_{i=1}^{k} n_i$. If $n \neq 1$, there is a neutral element n'_α with $n \leqq n'_\alpha < 1$. But $(n'_\alpha)' = n_i$ for some i, thus $(n'_\alpha)' \leqq n$, a contradiction. Thus $\bigvee n_i = 1$, that is (Corollary of Theorem 6), $L = (n_1] \times \cdots \times (n_k]$. We have to prove yet that the $(n_i]$ are simple. Indeed, if this does not hold, say $(n_1]$ is not simple, then by the same argument as above, we get the existence of a neutral element n of $(n_1]$ with $0 < n < n_1$. But then (Corollary 3 of Lemma 13) n is neutral in L, contradicting the minimality of n_1.

Conversely, suppose that L is the direct product of simple lattices. By Lemma IV, simple lattices are weakly modular, and it is obvious from the definition of weak modularity that the direct product of a finite number of weakly modular lattices is weakly modular again. Thus L is weakly modular, completing the proof.

As a further application of Theorem 11 we now prove a generalization of a theorem of SHIH-CHIAH WANG mentioned in the Introduction.

THEOREM 12. *Let L be a relatively complemented lattice with* 0 *and* 1. *$\Theta(L)$ is a Boolean algebra if and only if every standard ideal of L is principal.*

PROOF. Suppose every standard ideal of L is principal. It follows that every congruence relation of L is of the form Θ_s where s is a standard element. Indeed, every congruence relation Θ is of the form $\Theta = \Theta[S]$, where S is the kernel of the homomorphism induced by Θ, every homomorphism kernel is standard (these assertions are consequences of Theorem 11), and finally, every standard ideal is generated by one element $S = (s]$. From Corollary 1 of Theorem 7 it follows that s is a neutral element. L is section complemented, therefore s has a complement t. By Corollary 1 of Lemma 14, t is also neutral. Using Theorem 3 we get $\Theta_s \cap \Theta_t = \Theta_{s \cap t} = \Theta_0 = \omega$ and $\Theta_s \cup \Theta_t = \Theta_{s \cup t} = \Theta_1 = \iota$, that is, Θ_t is the complement of Θ_s. Therefore, every congruence relation of $\Theta(L)$ has a complement, that is, $\Theta(L)$ is a Boolean algebra.

Conversely, suppose that $\Theta(L)$ is a Boolean algebra. Owing to Theorem 11 we see that every congruence relation of L is of the form $\Theta[S]$ where S is a standard ideal. Let the congruence relation $\Theta[T]$ be the complement of $\Theta[S]$ (T is also a standard ideal). From Theorem 3 it follows $\Theta[S \cap T] = \omega$, $\Theta[S \cup T] = \iota$. Again from Theorem 3 we get $S \cap T = (0]$ and $S \cup T = L$. L has a unit element, therefore both $S \cap T$ and $S \cup T$ are principal ideals and, consequently, from the Corollary of Lemma 8 we obtain the result: S and T are principal ideals. We have proved that every standard ideal of L is principal, and thus the proof of Theorem 12 is completed.

We know (Lemma 10) that in modular lattices every standard ideal is neutral, consequently, from Theorem 12 we get as a special case the theorem of S. WANG:

COROLLARY. *The lattice of all congruence relations of a complemented modular lattice is a Boolean algebra if and only if every neutral ideal is principal.*

PROBLEM 13. Describe those finite lattices in which we get a one-to-one correspondence between the homomorphisms and standard ideals, letting a homomorphism correspond to its kernel.

REMARK. The lattice of Fig. 7 shows that such a lattice is not necessarily the direct product of simple lattices.

§ 2. Isomorphism theorems

In this section we will show that both isomorphism theorems are true for standard ideals.

THEOREM 13. (First isomorphism theorem for standard ideals.) *Let L be a lattice, S a standard ideal and I an arbitrary ideal of L. Then $S \cap I$ is a standard ideal of I and*

$$(I \cup S)/S \simeq I/(I \cap S).$$

PROOF. Corollary of Lemma 9 is just the first assertion of our theorem. The simplest mean to prove the isomorphism statement is the use of the first general isomorphism theorem of RÉDEI [29] (see Ch. I, § 2). We have only to prove that every congruence class of the lattice $I \cup S$ may be represented by an element of I. Indeed, any element x of $I \cup S$ is of the form $y \cup s$ where $s \in S$ and $y \in I$ (see condition (β') of Theorem 2). Further, $x = y \cup s \equiv y(\Theta\ [S])$, and so the congruence class that contains x may be represented by $y \in I$.

According to Theorem 4, the isomorphism statement of the first isomorphism theorem is equivalent to the isomorphism of the intervals $[S, I \cup S]$ and $[I \cap S, I]$ of $I(L)$. We can add to the isomorphism statement the following

SUPPLEMENT. *Let L be a lattice and S a standard ideal of L. Then*

$$[I \cap S, I] \simeq [S, I \cup S]$$

for all $I \in I(L)$. An isomorphism is given by the correspondence

$$X \to X \cup S \qquad (X \in [I \cap S, I]).$$

The inverse correspondence is

$$Y \to Y \cap I \qquad (Y \in [S, I \cup S]).$$

PROOF. From condition (δ') (i) of Theorem 2 we get that $X \to X \cup S$ $(X \in [I \cap S, I])$ is a homomorphism. If $X_1, X_2 \in [I \cap S, I]$, then $S \cap X_1 = S \cap X_2$ (for $X_1, X_2 \subseteq I$, and so $S \cap X_i = S \cap X_i \cap I = S \cap I$, $i = 1, 2$). Thus from condition (δ') (ii) of Theorem 2 we get that $S \cup X_1 \neq S \cup X_2$. Therefore, $X \to X \cup S$ is an isomorphism of $[I \cap S, I]$ *into* $[S, S \cup I]$. We prove that

$$(Y \cap I) \cup S = Y \qquad (Y \in [S, I \cup S]),$$

and this will prove that $Y \to Y \cap I$ $(Y \in [S, I \cup S])$ is the inverse of $X \to X \cup S$ $(X \in [I \cap S, I])$, and the latter correspondence maps $[I \cap S, I]$ *onto* $[S, I \cup S]$. Indeed, using condition (δ') (i) of Theorem 2, we get

$$(Y \cap I) \cup S = (Y \cup S) \cap (I \cup S) = Y \cap (I \cup S) = Y,$$

and this completes the proof of the Supplement.

In the last proof we have got a new proof of the isomorphism theorem. In case S is a principal ideal, the isomorphism given in the Supplement is essentially the isomorphism of the two factor lattices.

We remark that HASHIMOTO has got the first isomorphism theorem under the condition that both S and I are neutral ideals.

THEOREM 14. (Second isomorphism theorem for standard ideals.) *Let L be a lattice, S an ideal and T a standard ideal of L, $S \supseteq T$. Then S is standard if and only if S/T is standard in L/T, and in this case*

$$L/S \cong (L/T)/(S/T).$$

PROOF. If S is standard, then from Lemma 6 we get that S/T is standard in L/T. Conversely, suppose S/T is standard in L/T. We show condition (γ'') of Theorem 2 holds for S. We have seen in the proof of Theorem 1, in the step "(β) implies (γ)", that it is enough to prove that $x \equiv y$ $(\Theta[S])$ and $x \geqq y$ imply $x \cap u \equiv y \cap u$ $(\Theta[S])$ for all $u \in L$. (Here $\Theta[S]$ denotes the relation defined in condition (γ'') of Theorem 2.) We denote by $\bar{a}$ the image of the element a under the homomorphism $L \sim L/T$. Then we have $\bar{x} \equiv \bar{y} (\Theta[S/T])$, and since S/T is standard in L/T, therefore with a suitable $\bar{s} \in S/T$ we get $\bar{x} \cap \bar{u} = (\bar{y} \cap \bar{u}) \cup \bar{s}$. Further, since T is standard in L, we can find a $t \in T$ such that $x \cap u = [(y \cap u) \cup s] \cup t$. We put $s_1 = s \cup t$ and get $x \cap u = (y \cap u) \cup s_1$, $s_1 \in S$. This proves that S is standard.

We remark that during the proof we have made effective use of the fact that the congruence classes of L/T under $\Theta[S/T]$ are the homomorphic images of those of L under $\Theta[S]$.

The isomorphism is again proved simply by reference to the general second isomorphism theorem of RÉDEI [29].

COROLLARY. *There is a natural isomorphism*

$$X \to X/T$$

between the dual ideal $[T)$ *of* $I(L)$ *and* $I(L/T)$, *and this isomorphism makes the standard ideals of* L *containing* T *correspond to the standard ideals of* L/T.

The second isomorphism theorem of HASHIMOTO supposes that S and T are neutral and it is confined to the isomorphism statement.

The question naturally arises: what happens with the neutral ideals under the correspondence of the Corollary of Theorem 14? The answer is: neutral ideals of $[T)$ are mapped upon neutral ideals, but the converse is not true. Of course, T/T is always neutral in L/T, thus, if T is not neutral, then we have found a not neutral ideal carried into a neutral one.

And what happens if T is neutral? An answer is given by

THEOREM 15. (Second isomorphism theorem for neutral ideals.) *Let* L *be a lattice,* S *an ideal,* T *a neutral ideal of* $L, S \supseteq T$. *The ideal* S *is neutral if and only if* S/T *is neutral in* L/T. *In this case we have the isomorphism*

$$L/S \simeq (L/T)/(S/T).$$

SUPPLEMENT. *The ideal* S $(S \supseteq T)$ *is of type* $f_\alpha = g_\alpha$ $(\alpha \in A)$ *(it is presupposed that the zero of any lattice is of this type) if and only if* S/T *is of type* $f_\alpha = g_\alpha$ $(\alpha \in A)$ *in* L/T.

PROOF. Owing to Corollary 3 of Theorem 10 we can reduce the theorem to the case of neutral elements. Then, using Theorem 6, we trivially get (supposing n is neutral) that an element $s \geqq n$ of this lattice is neutral if and only if s is neutral in the lattice $[n)$. On the other hand, $[n)$ is isomorphic to the factor lattice $L/(n]$ and thus the assertion is proved. The assertions for the elements of type $f_\alpha = g_\alpha$ $(\alpha \in A)$ are to be proved in the same way.

We see that the second isomorphism theorem for standard ideals is characteristic for standard ideals. A slightly weaker form of Theorem 14 is already not characteristic for them. Namely:

There exists a lattice L *and a homomorphism kernel* S *of* L *such that* S *is not standard, but if* T *is a homomorphism kernel of* L *with* $T \subseteq S$, *then* S/T *is a standard ideal of* L/T *and*

$$L/S \simeq (L/T)/(S/T).$$

To construct such a lattice we take the chain of non-positive integers, and form the direct product of this chain with the chain of two elements.

The elements of this direct product are of the form $(n, 0)$ and $(n, 1)$ (n is a non-positive integer). We define one further element x subject to

$$\left.\begin{array}{l} x \cup (0,0) = x \cup (-n,1) = (0,1), \\ x \cap (0,0) = (-1,0), \\ x \cap (-n,1) = (-n,0) \end{array}\right\} \qquad (n = 1, 2, \ldots).$$

The diagram of this lattice is shown by Fig. 10. Let $S = (s]$. By an easy discussion one can show that the requirements are fulfilled.

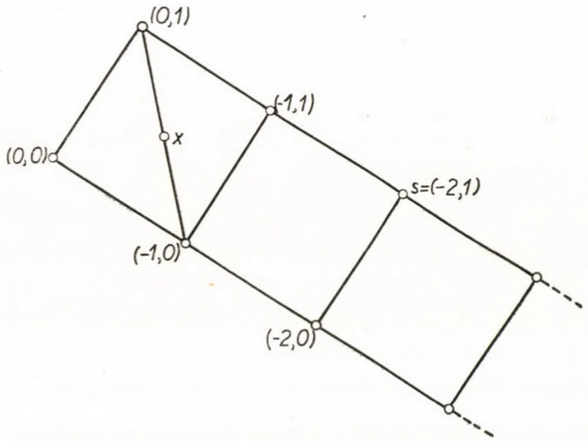

Fig. 10

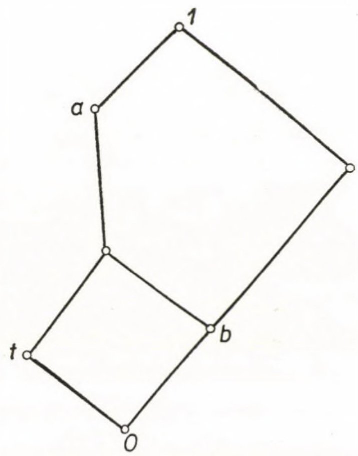

Fig. 11

A very difficult question is: which class of ideals is characterized by the first isomorphism theorem? It is easy to prove that this class is wider than the class of standard ideals. Before giving the example, we first give a precise meaning of the expression: an ideal satisfies the first isomorphism theorem.

We say that the ideal T of the lattice L satisfies the first isomorphism theorem if

$$T \cup I / T \cong I / I \cap T$$

for all $I \in I(L)$.

Now consider the following lattice of Fig. 11. The ideal $(t]$ of this lattice satisfies the first isomorphism theorem, but it is not standard. In this lattice the following ideals satisfy the first isomorphism theorem: $(0]$, $(t]$, $(b]$, $(a]$, $(1]$. These do not form a sublattice of $I(L)$. It is a bit more difficult, but still possible to give such a counterexample, where the ideals satisfying the first isomorphism theorem do form a sublattice of the lattice of ideals.

In some special class of lattices we shall examine the ideals satisfying the first isomorphism theorem in Chapter VI.

Now we are in the position to show how the dictionary, mentioned in the Introduction, works. Let us repeat first the dictionary:

subgroup → ideal

invariant subgroup → standard ideal

factorgroup → factor lattice modulo a standard ideal

group operation → join.

Consider the isomorphism theorems: we see that these are word by word translations of the corresponding statements of group theory. To get further examples, consider the following assertion of group theory:

The subgroup N of the group G is invariant if and only if NH is a subgroup for every subgroup H of G.

Here NH is the complex product of N with H. The corresponding notion is the "complex join" of two ideals I, K, let us denote it by $I \vee K$ (the notation $I \cup K$ would be ambiguous, it is used to denote the ideal-theoretical join of I and K), that is, $I \vee K = \{x; x = i \cup k; i \in I, k \in K\}$. Now the "translation" of the above theorem is:

The ideal S of the lattice L is standard if and only if $S \vee K$ is an ideal of L for every ideal K of L.

This theorem is actually true, for it is nothing else but a reformulation of condition (β') of Theorem 2.

The fact that one can use the dictionary so fruitfully seems to prove not only the applicability of the notion of standard ideals, but at the same time the usefulness of the definition of factor lattice.

The existence of the dictionary suggests the idea of possibility to define for general algebras a notion of "standard subalgebra", which is a common generalization of ideals in rings, of invariant subgroups in groups and standard ideals in lattices, and for which one can prove those theorems which can be translated from group theory to lattice theory. In fact, this can be done if we confine ourselves only to the isomorphism theorems and do not consider the result concerning complex product of subgroups and the theorem of § 4 of this chapter. So far we did not succeed in finding any notion which fulfils all the requirements.

§ 3. The Zassenhaus lemma

In groups as well as in rings one can prove the Zassenhaus lemma using the two isomorphism theorems. So it is not surprising that translating the Zassenhaus lemma to lattices, we can prove it without any difficulty.

THEOREM 16. (The Zassenhaus lemma.) *Let I and K be ideals of the lattice L. Further, let S and T be standard ideals of the lattices I and K, respectively. Then $S\cup(I\cap T)$ is a standard ideal of the lattice $S\cup(I\cap K)$ and $T\cup(S\cap K)$ is that of $T\cup(I\cap K)$. Finally, we have the isomorphism:*

$$[S\cup(I\cap K)]/[S\cup(I\cap T)]\simeq[T\cup(I\cap K)]/[T\cup(S\cap K)].$$

PROOF. (The situation is shown by Fig. 12.) Owing to the first isomorphism theorem we have

$$(*)\qquad [S, S\cup(I\cap K)]\simeq[S\cap K, I\cap K].$$

From Lemma 9 it follows that $S\cap K$ and $I\cap T$ are standard ideals of the lattice $I\cap K$, and so from Theorem 3 we get that their join $(S\cap K)\cup(I\cap T)$ is likewise a standard ideal of the same lattice. Consequently, by Lemma 6, $(S\cap K)\cup(I\cap T)/S\cap K$ is a standard ideal of $I\cap K/S\cap K$. From the Supplement of the first isomorphism theorem we know that the isomorphism $(*)$ may be set up by the correspondence $X\to X\cup S$ $(X\in[S\cap K, I\cap K])$. This carries the ideal $(S\cap K)\cup(I\cap T)$ into the ideal $S\cup(I\cap T)$ and $S\cap K$ into S. Thus we get from the above statement that $S\cup(I\cap T)/S$ is a standard ideal of $S\cup(I\cap K)/S$. From the first assertion of the second isomorphism theorem we get that $S\cup(I\cap T)$ is standard in $S\cup(I\cap K)$ and thus the first assertion of the Zassenhaus lemma is proved.

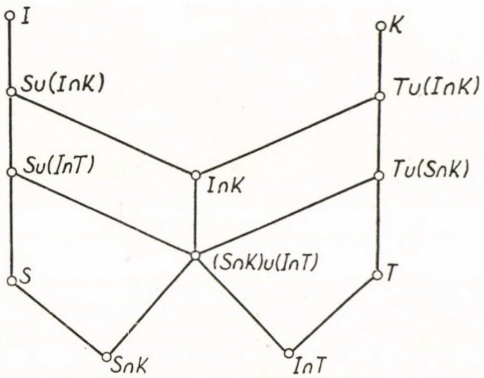

Fig. 12

It follows from the above statements that

$$[S\cup(I\cap K)/S]/[S\cup(I\cap T)/S]\simeq[I\cap K/S\cap K]/[(S\cap K)\cup(I\cap T)/S\cap K],$$

and from the second isomorphism theorem we get

$$[S\cup(I\cap K)]/[S\cup(I\cap T)]\simeq[I\cap K]/[(S\cap K)\cup(I\cap T)].$$

Similarly,

$$[T\cup(I\cap K)]/[T\cup(S\cap K)]\simeq[I\cap K]/[(S\cap K)\cup(I\cap T)],$$

and these two isomorphisms together prove the required isomorphism statement.

From Theorem 4 it follows that the isomorphism statement of the Zassenhaus lemma is equivalent to the isomorphism of the following two intervals of $I(L)$:

$$[S\cup(I\cap T), S\cup(I\cap K)]\simeq[T\cup(S\cap K), T\cup(I\cap K)].$$

SUPPLEMENT. *An isomorphism of these two intervals is given by the correspondence*

$$X \to (X \cap K) \cup T \qquad (X \in [S \cup (I \cap T), S \cup (I \cap K)])$$

whose inverse is

$$Y \to (Y \cap I) \cup S \qquad (Y \in [T \cup (S \cap K), T \cup (I \cap K)]).$$

A special case of some interest of the first assertion of the Zassenhaus lemma is the following:

COROLLARY. *Let A and B be elements of the lattice L and suppose $A > B$. If a is a standard element of the lattice $(A]$ and b is a standard element of the lattice $(B]$, then $a \cup b$ is a standard element of $(a \cup B]$.*

This corollary is a generalization of a part of Theorem 3, which asserts that the join of two standard elements is again a standard element. It would be interesting to give a direct proof of the Corollary.

The proof of the supplement may be got simply by applying twice the supplement of the first isomorphism theorem.

If both S and T are principal ideals, then the isomorphism of the supplement is the isomorphism of the two factor lattice.

The Jordan—Hölder—Schreier theorem follows from the Zassenhaus lemma as usually. To formulate the theorem we need the usual notions:

A chain

$$I = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n = J$$

of the ideals of the lattice L is called a standard series of length n of the interval $[J, I]$ if I_k is standard in I_{k-1} for $k = 1, 2, \ldots, n$. A standard series is called proper if we have always $\supset$ instead of $\supseteq$. A composition series is a proper standard series which has no proper refinement in the interval $[J, I]$.

THE JORDAN—HÖLDER—SCHREIER—ZASSENHAUS THEOREM. *Any two standard series of the interval $[J, I]$ have refinements such that the lengths of the two refined series are the same and the factor lattices of the two refined series are — disregarding of the order — pairwise isomorphic. Further, if in an interval there exists a composition series, then every standard series may be refined to a composition series. Consequently, any two composition series have the same length.*

Obviously, we may make the refinement in the same fashion as in groups and rings. Therefore we may omit the details.

We note that if we defined the standard series, requiring that every member of the series be a standard ideal in the whole lattice, then our Jordan—Hölder—Schreier—Zassenhaus theorem would be a consequence of the general Jordan—Hölder—Schreier—Zassenhaus theorem (see BIRKHOFF [6]). Indeed, in this case from Lemma 7 it follows that any two congruence relations in question are pairwise permutable. Furthermore, it is not a restriction to suppose that the lattice L has a zero, for we may pass from L to L/J where J is the least standard ideal which occurs.

In [14] HASHIMOTO has proved the above theorem, supposing all the ideals which occur are neutral. From the above remarks it follows that the results of HASHIMOTO are special cases of the theorem of [6].

§ 4. The Schreier extension problem for lattices

We will now examine a lattice-theoretical analogue of the well-known Schreier problem of the theory of groups.

The original problem is the following:

Let the groups G_1 and G_2 be given. Describe all groups G containing G_1 as an invariant subgroup such that $G/G_1 \simeq G_2$.

Translating this problem with the aid of the "dictionary" to lattice theory, we get the following problem:

Let the lattices L_1 and L_2 be given. Describe all lattices L containing L_1 as a standard ideal such that $L/L_1 \simeq L_2$.

Such a lattice will be called the Schreier extension of L_1 by L_2. Schreier extensions always exist, e. g. the direct product of the two given lattices.

SCHREIER's extension problem in groups and rings may be solved by means of certain functions. It is possible to define these functions as a consequence of the invertibility of the group operation. Although the problem is solved for semigroups too — with a suitable definition of invariant subsemigroup — but in this case the requirement is essentially the regularity of the operation with respect to the invariant subsemigroup. Because there is no possibility of defining regularity of the lattice operations, therefore we have to give up the hope of finding a solution similar to the method of the Schreier's functions. This is suggested already by a result of SZÁSZ [33], according to which the rather general notion "das schiefe Product" of RÉDEI ([26] and [27]), including in case of groups all Schreier's extensions, gives in lattices the direct product and nothing more.

Therefore, we had to recourse to other methods.

THEOREM 17. *Given the lattices L_1 and L_2, suppose L_1 has a unit and L_2 has a zero element. L is a Schreier extension of L_1 by L_2 if and only if L is isomorphic to a meet-sublattice*[7] *of $L_1 \times L_2$ which contains all the elements of the form $(x, 0)$ $(x \in L_1)$ and $(1, y)$ $(y \in L_2)$.*

[7] A subset H of the lattice L is called a meet-sublattice if it is closed under forming finite meet and under the partial order induced by that of L it is a lattice. This means that the join in H is not necessarily the same as in L.

Thus, while the group-theoretical Schreier extension problem is reduced to finding certain functions, now the same problem for lattices is reduced to finding certain meet-sublattices of a well-constructed lattice.

PROOF. Suppose L is a Schreier extension of L_1 by L_2. Let s be the greatest element of the ideal L_1 of L and embed L in L_s (see the definition of L_s before Theorem 6) in the natural way, that is, with the correspondence $x \to (x \cap s, x \cup s)$. It follows from Theorem 6 that it is a meet isomorphism. The image of L in L_s contains all the elements of the form $(x, 0)$ and $(1, y)$. Because of $L_1 \cong \{a; a \in L, a = (x, 0)\}$ and $L_2 \cong \{a; a \in L, a = (1, y)\}$ we get that L fulfils the requirements of Theorem 17.

Now we suppose that L is a meet-sublattice of $L_1 \times L_2$, and it contains the elements of the form $(x, 0)$ and $(1, y)$. We shall prove that $S = \{a; a \in L, a = (x, 0)\}$ is a standard ideal of L. Consider that partition of L which is induced by the congruence relation $\Theta[S]$ of $L_1 \times L_2$. If we prove that this partition is compatible, then it follows that S is a standard ideal. It is enough to show that if $y = x \vee s, s \in S$ ($\vee$ denotes the join in L), then in $L_1 \times L_2$ the relation $x \equiv y$ $(\Theta[S])$ holds. This does not hold trivially, for in general $x \vee s \geqq x \cup s$. But, from $(1, 0) \in L$ and from $x \cup (1, 0) \geqq x$ it follows that $x \leqq x \vee s \leqq x \cup (1, 0)$, and so $x \equiv x \cup (1, 0)$ $(\Theta[S])$, thus $x \equiv x \cup s$ $(\Theta[S])$ as we wished to prove.

Thus S is a standard ideal. $S \cong L_1$ is trivial, and $L/S \cong L_2$ is also immediate, thus the proof of Theorem 17 is completed.

It is clear from Theorem 6 that if the kernel were defined to be a neutral ideal, then we would not get the meet-sublattices, but the sublattices of $L_1 \times L_2$ as the solution of the extension problem.

A few words about the conditions: L_1 has a unit and L_2 has a zero element. The first hypothesis can easily be omitted, we have only to refer to Theorem 4, but obviously, without this condition the theorem would be more difficult.

The second condition is much more important. It assures that L_1 may be regarded as an ideal of $L_1 \times L_2$. Omitting this supposition, some new idea seems to be necessary to settle the question.

It seems that the standard ideals are the possible widest generalization of the class of neutral ideals, for which the Schreier problem has a solution. For instance, it is probable that there is no analogue of Theorem 17 for distributive ideals. To be precise we may say the following: it is natural to require in solving the Schreier problem that we get a finite general algebra as an extension of a finite general algebra by another finite one. This condition holds for groups, semigroups, rings and lattices (Theorem 17). It should not hold if we supposed the kernel to be only a distributive ideal. An example for this is given by the lattice of Fig. 4, which is the Schreier extension of **2** by **2**, and the kernel of this extension is a distributive ideal.

PROBLEM 14. Try to omit the hypothesis that L_2 has a zero in Theorem 17.

CHAPTER V

CHARACTERIZATIONS OF NEUTRAL ELEMENTS IN MODULAR LATTICES

§ 1. Characterizations by distributive equalities

The notion of standard elements has been defined by (9) which is a distributive equality. Furthermore, the condition (δ) (i) of Theorem 1 is also a distributive equality. Thus we see that there is a close connection between distributive equalities and standard elements.

This connection is deeper in modular lattices as a consequence of Theorem IV of ORE which asserts that in modular lattices both (6) and (7) are capable of the definition of neutral elements.

First the question arises what the situation is with the distributive equality (8). Is it also capable of definition of the neutral elements in modular lattices? Further: one can imagine that there are in the class of modular lattices other equalities, capable of the characterization of the distributivity. Are all of these identities able to define the neutrality of an element?

In general, in lattice theory, distributive equality means one of the equalities (6), (7) and (8). The cause why these are called distributive equalities is given by the following theorem (DEDEKIND [7], MENGER [22]): let $f(x, y, z) = g(x, y, z)$ be any one of the equalities (6), (7), (8); then a distributive lattice may be defined as a lattice of type $f = g$. Basing upon this theorem, we give the definition of distributive equality as follows:

Let $\mathfrak{L}$ be a class of lattices. $f(x, y, z) = g(x, y, z)$ is a *distributive equality* of $\mathfrak{L}$ if and only if within $\mathfrak{L}$ the distributive lattices are just the lattices of type $f = g$.

We remark that f and g are elements of $FL_{\underset{\sim}{\mathfrak{L}}}(3)$, that is, the free lattice with three generators over $\mathfrak{L}$. ($FL_{\underset{\sim}{\mathfrak{L}}}(3)$ consists of all polynomials of three indeterminates and $f(x, y, z) = g(x, y, z)$ in $FL_{\underset{\sim}{\mathfrak{L}}}(3)$ if and only if $f(a, b, c) = g(a, b, c)$ for all $a, b, c \in L \in \mathfrak{L}$.)

We say that the distributive equality $f = g$ is equivalent to the distributive equality $f' = g'$ over $\mathfrak{L}$ if for all $a, b, c \in L \in \mathfrak{L}$, $f(a, b, c) = g(a, b, c)$ is equivalent to $f'(a, b, c) = g'(a, b, c)$. This amounts to the coincidence of the congruence relations $\Theta_{f, g}$ and $\Theta_{f', g'}$ of $FL_{\underset{\sim}{\mathfrak{L}}}(3)$.

Now, our aim is to determine all non-equivalent distributive equalities of the class of modular lattices, and to decide whether it is possible or not to define by any one of them the neutrality of an element.

We reach both of our aims by proving the following theorem:

THEOREM 18. *Let $f(x, y, z) = g(x, y, z)$ be a distributive equality of modular lattices. The elements a, b, c of the modular lattice L generate a distributive sublattice of L if and only if $f(a, b, c) = g(a, b, c)$.*

Before proving the theorem, let us draw some consequences of it:

COROLLARY 1. *Let $f(x, y, z) = g(x, y, z)$ be an arbitrary distributive equality of modular lattices. An element n of the modular lattice L is neutral if and only if $f(n, a, b) = g(n, a, b)$ for all $a, b \in L$.*

The proof is obvious, comparing the definition of neutrality with the assertion of Theorem 17.

COROLLARY 2. *In the class of modular lattices all distributive equalities are equivalent.*

PROOF. Let $f(x, y, z) = g(x, y, z)$ and $f'(x, y, z) = g'(x, y, z)$ be distributive equalities. If $f(a, b, c) = g(a, b, c)$ for $a, b, c \in L$, then by Theorem 17 the sublattice generated by a, b, c is distributive, thus $f'(a, b, c) = g'(a, b, c)$, and conversely. That means the equivalence of any two distributive equalities.

The proof of the theorem is based on the following lemma which seems to have an interest of his own.

LEMMA 16. *Let L be a modular but not distributive lattice generated by three elements. L has a homomorphism onto V, that is, onto the modular but not distributive lattice of five elements.*

PROOF. Let a, b, c be generators of L and p, q, r those of V. The correspondence $a \to p$, $b \to q$, $c \to r$ may be extended to a homomorphism of L onto V. It is rather easy to show that this correspondence is a homomorphism, but one can spare the trouble of this proof by a simple reference to a theorem of BIRKHOFF, which effectively describes the free modular lattice with three generators (see Fig. 15).

So far the lattice V has been used as a sublattice to characterize the non-distributivity of a modular lattice. Lemma 16 shows that it may be used also as a homomorphic image to characterize the non-distributivity in case L is generated by three elements. In general this is not true, as shown by the modular lattice of six elements, having four atoms. This lattice is simple, therefore it has no homomorphism onto V.

PROOF OF THEOREM 18. Suppose the theorem is not true, that is, there exists a modular lattice L, three elements a, b, c of L and a distributive equality $f(x, y, z) = g(x, y, z)$ such that $f(a, b, c) = g(a, b, c)$ and the sublattice L_1 generated by a, b, c is not distributive. Then by Lemma 16 it has a homomorphism onto V such that the homomorphic images of a, b, c are p, q, r. Consequently, $f(p, q, r) = g(p, q, r)$. Let u, v, w be three elements of V distinct from p, q, r. Then these elements generate a distributive sublattice of V, thus $f(u, v, w) = g(u, v, w)$. We have shown that $f(u, v, w) = g(u, v, w)$ for all $u, v, w \in V$, that is, V is of type $f = g$. This is obviously a contradiction to the definition of distributive equalities. The proof is complete.

Finally, we make some general remarks. Let $\mathfrak{L}$ be a class of lattices, $f, g, f', g' \in FL_{\mathfrak{L}}(3)$. We say that $f = g$ is equivalent to $f' = g'$ if $\Theta_{f,g} = \Theta_{f',g'}$, although this is not the generally used definition. In [6], though not explicitly stated, another definition is used; this we shall now call the definition of weak equivalence. Denote by $\mathfrak{L}_{f=g}$ the class of lattices of type $f = g$ in the class $\mathfrak{L}$. We say that $f = g$ is weakly equivalent to $f' = g'$ if $\mathfrak{L}_{f=g} = \mathfrak{L}_{f'=g'}$.

An example of weakly equivalent but not equivalent equalities: $x \cup y \cup (x \cap y \cap z) = x \cup y \cup z$ and $x \cap y \cap (x \cup y \cup z) = x \cap y \cap z$. They are weakly equivalent, for they define the lattice of one element, but they are not equivalent, for in the lattice of Fig. 13 the elements a, b, c satisfy only one of them.

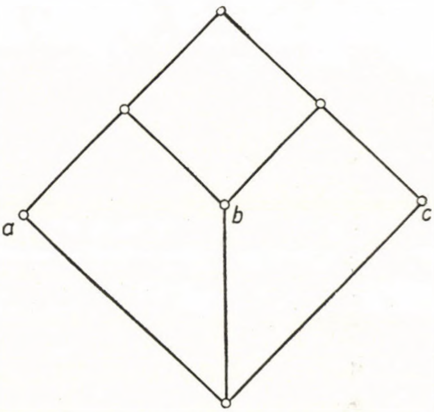

Fig. 13

It is trivial that per definitionem any two distributive equalities are weakly equivalent. Theorem 17 asserts that they are equivalent in modular lattices. The same assertion in the class of all lattices is not true, as it is shown again by the lattice of Fig. 13, where $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ but $a \cup (b \cap c) \neq (a \cup b) \cap (a \cup c)$.

PROBLEM 15. Let degree $\mathfrak{m}$ (an infinite or finite cardinal) of non-distributivity of the modular lattice L be defined as the power of a subset H of L maximal with respect to the property that any three elements of H generate a non-distributive sublattice of L. Is $\mathfrak{m}$ an invariant of the lattice? Is it true that L has the degree of non-distributivity 3 if and only if it has a homomorphic image isomorphic to V? If the degree of non-distributivity of L is $\mathfrak{m}$, has L homomorphism onto a simple lattice, having the same degree of non-distributivity?

§ 2. Neutral elements as elements with unique relative complements

A theorem of J. VON NEUMANN (see [6]) asserts that an element n of a complemented modular lattice L is neutral if and only if its complement is unique. In modular lattices the same assertion does not hold in general; the element a of the lattice of Fig. 14 is uniquely complemented, but it is not neutral. This observation is due to HALL (see [6]). Applying twice Ex. 2 of p. 115 of [6] we get[8] that an element n of a complemented modular lattice L is uniquely complemented if and only if it is uniquely relatively complemented, that is, if it has just one relative complement in any interval containing n. In this way it is possible to generalize NEUMANN's theorem to arbitrary modular lattices.

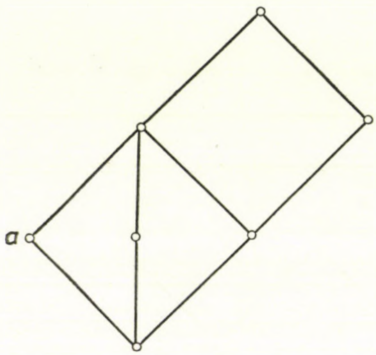

Fig. 14

THEOREM 19. *An element n of a modular lattice L is neutral if and only if it has at most one relative complement in any interval containing it.*

PROOF. If n is neutral, then by condition (ii) of Theorem III it obviously satisfies the stated condition.

Let n be an element of the modular lattice L, and x, y arbitrary elements of L. The free modular lattice FML 3), generated by n, x, y, according to a theorem of BIRKHOFF [6] is given in Fig. 15. If n satisfies the condition of Theorem 19, then $u \equiv v$, for u and v are the relative complements of a in the interval $[a \cap u, a \cup u]$. It follows that the lattice generated by a, x and y must be a homomorphic image of $FML(3)(\Theta_{uv})$. But $FML(3)(\Theta_{uv})$ is distributive and it follows that n is neutral, as asserted.

COROLLARY 1. (Theorem of NEUMANN.) *In a complemented modular lattice an element n is neutral if and only if it has precisely one complement.*

Indeed, as we have remarked above, in complemented modular lattices an element n is uniquely relatively complemented if and only if it is uniquely complemented. Hence the corollary.

COROLLARY 2. *An element n of the modular lattice L is neutral if it is neutral in every interval $[n \cap x, n \cup x]$ $(x \in L)$.*

Corollary 2 is an immediate consequence of Theorem 19.

In weakly modular lattices Theorem 19 fails to be valid. In the lattice of Fig. 16 the element a is uniquely relatively complemented, but not neutral.

[8] See also [32].

Corollary 2 is obviously true in relatively complemented and in complemented lattices. For if L is relatively complemented and $n \in L$ is neutral in any $[n \cap x, n \cup x]$ $(x, y \in L)$, then let a be a relative complement of n in $[n \cap x \cap y, n \cup x \cup y]$. Since n is neutral in $[n \cap a, n \cup a]$, $\{n, x, y\}$ is a distributive lattice, and so n is neutral. But Corollary 2 is not true in general. Consider the lattice of Fig. 17. In this lattice a is not neutral, but it is in every interval of the form $[a \cap x, a \cup x]$.

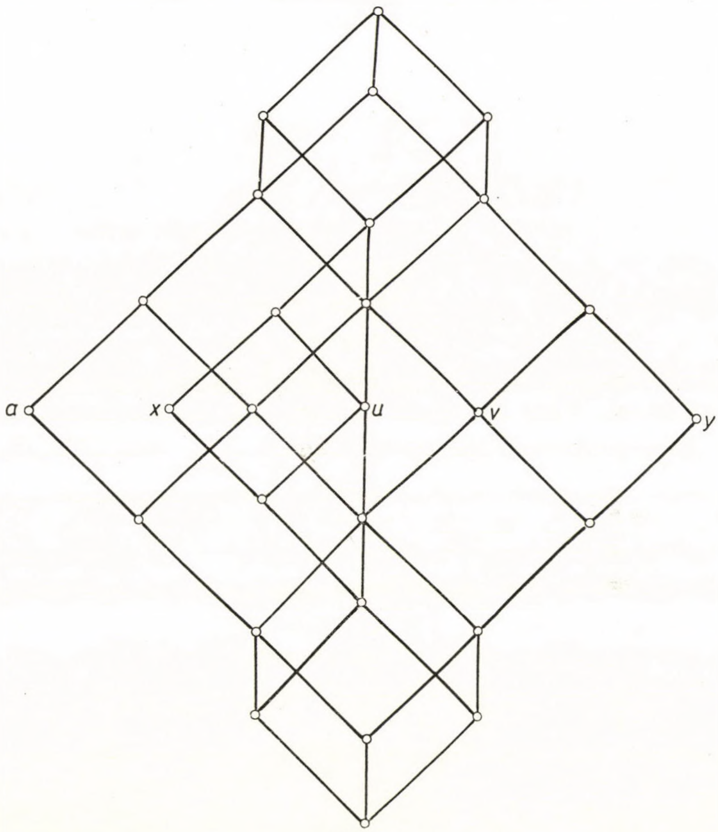

Fig. 15

PROBLEM 16. Is the assertion of Theorem 19 true in relatively complemented lattices?

PROBLEM 17. Is the assertion of Corollary 2 of Theorem 19 true in weakly modular lattices?

NOTE. The corresponding assertion for standard elements is not true. If $s \in L$ is locally standard (i. e. standard in every interval $[s \cap x, s \cup x]$) and

L is weakly modular, it does not follow that s is standard, as shown by the lattice of Fig. 18.

PROBLEM 18. Call an element s of L a *standard element of order two* if it is a standard element of a standard ideal of S. Do the standard elements of order two form a sublattice of L? Characterize the standard elements of order two. What can be said of the standard elements of higher order?

REMARK. The following theorem may be useful: if s is a standard element of order two of L, then there is a unique maximal standard ideal S such that s is standard in S.

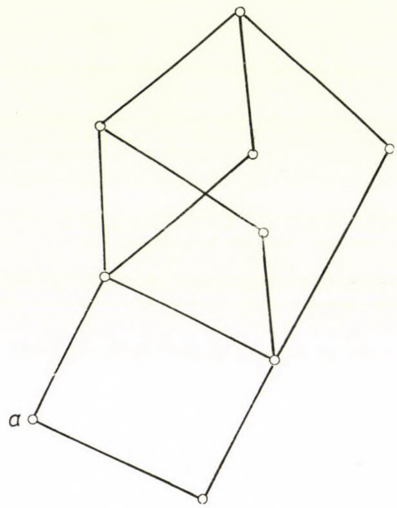

Fig. 16

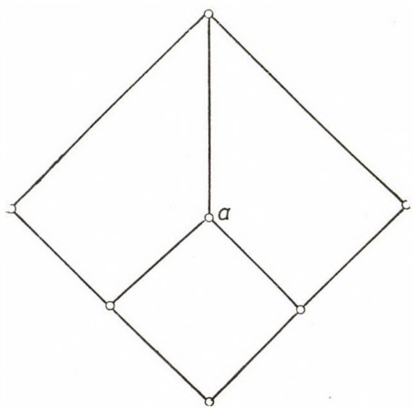

Fig. 17

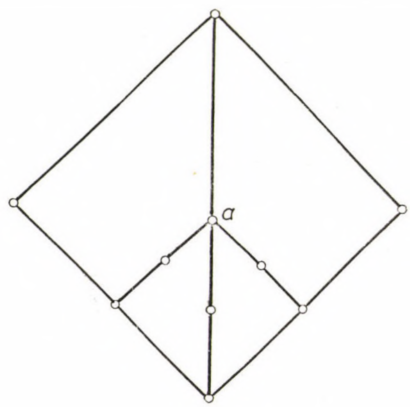

Fig. 18

CHAPTER VI

IDEALS SATISFYING THE FIRST ISOMORPHISM THEOREM

§ 1. The case of relatively complemented lattices with zero satisfying the ascending chain condition

In this section we want to prove the following theorem:

THEOREM 20. *Let L be a relatively complemented lattice with zero in which the ascending chain condition holds. An ideal I of the lattice L satisfies the first isomorphism theorem, i. e.*

$$I \cup K/I \simeq K/I \cap K$$

for all ideals K of L if and only if I is neutral.

PROOF. If I is neutral, then from Theorem 13 it follows that it satisfies the first isomorphism theorem. Now let us suppose that the ideal I of the relatively complemented lattice L with the ascending chain condition satisfies the first isomorphism theorem. By the structure theorem of DILWORTH, $L = L_1 \times \cdots \times L_k$ with simple lattices L_j, and consequently, $I = I_1 \times \cdots \times I_k$ with $I_j \subseteq L_j$ $(j = 1, \ldots, k)$. We prove that I satisfies the first isomorphism theorem if and only if every I_j satisfies in L_j the first isomorphism theorem. This is an immediate consequence of the following identity:

$$L/I \simeq L_1/I_1 \times \cdots \times L_k/I_k.$$

Hence we have reduced the question to the case of simple relatively complemented lattices. Instead of this we shall now consider a bit more general class of lattices, which will lead not only to the proof of Theorem 20, but at the same time to a generalization of it:

LEMMA 17. *Let L be a complemented simple lattice. No principal ideal of L except for* (0] *and* (1] *satisfies the first isomorphism theorem.*

PROOF. Let $a \in L$, and b the complement of a. Applying the first isomorphism theorem, we get

$$(a \cup b]/(a] \simeq (b]/(a \cap b].$$

The left member is isomorphic to the lattice of one element (except $a = 0$) and the right member is isomorphic to the principal ideal $(b]$. This is a contradiction, unless $a = 0$ or $b = 0$, as stated.

Lemma 17, compared to the arguments we have made above (using the structure theorem given in Corollary 4 of Theorem 11), leads to the following generalization of Theorem 20:

THEOREM 21. *Let L be a section complemented weakly modular lattice with the ascending chain condition. An ideal I of L satisfies the first isomorphism theorem if and only if it is neutral.*

COROLLARY. *Let I be a section complemented weakly modular lattice satisfying the ascending chain condition (for instance, a finite relatively complemented lattice) in which the first isomorphism theorem unrestrictedly holds. Then I is a finite Boolean algebra.*

In § 3 we shall show that the assertion of Theorem 20 holds in finite modular lattices as well. But it is not already true for finite weakly modular lattices. Fig. 17 shows a lattice L which is simple. The dual $\tilde{L}$ of L has an element $s \neq 0, 1$ which is locally standard[9] and thus it satisfies the first isomorphism theorem.

§ 2. A general theorem

In the following section we want to prove that the conclusion of Theorem 20 holds in modular lattices of locally finite length with zero. In this section we prove a general theorem which characterizes the ideals satisfying the first isomorphism theorem in certain classes of lattices which are a little more general than the class of finite lattices. The condition of this theorem is rather difficult, but starting from this, we shall be able to solve the problem in modular lattices of locally finite length with zero.

First we turn our attention to proving two lemmas of preliminary character.

LEMMA 18. *Let L be a lattice of finite length and let L satisfy the Jordan—Dedekind chain condition. Then $L(\Theta) \simeq L$ $(\Theta \in \Theta(L))$ implies $\Theta = \omega$, that is, no proper homomorphic image of L is isomorphic to L.*

PROOF. Suppose $\Theta \neq \omega$ and $L(\Theta) \simeq L$. Then there exist $a, b \in L$ such that $a > b$ and $a \equiv b$ (Θ). Let C be a maximal chain of length n such that $a, b \in C$. The image of C under Θ is a maximal chain of $L(\Theta)$, and its length is at most $n-1$, for the homomorphic images of a and b are the same. From the isomorphism $L \simeq L(\Theta)$ it follows that the Jordan—Dedekind chain condition holds in $L(\Theta)$, and consequently the length of $L(\Theta)$ is at most $n-1$. Since the length of the original lattice is n, therefore this contradicts the isomorphism of L and $L(\Theta)$.

COROLLARY. *Let L be a lattice of finite length and suppose that every homomorphic image of L satisfies the Jordan—Dedekind chain condition. Then $\Theta, \Phi \in \Theta(L)$ and $\Theta \geqq \Phi$, further $L(\Theta) \simeq L(\Phi)$ imply $\Theta = \Phi$.*

[9] See the definition in the Note after Problem 17.

We remark that the conclusion of Lemma 18 does not hold if the Jordan—Dedekind chain condition is not presupposed. An example is given by Fig. 19. Let Θ be the congruence relation Θ_{ab}. Then $\Theta_{ab} > \omega$ and $L(\Theta_{ab}) \simeq L$.

A generalization of the first isomorphism theorem is a homomorphism theorem which is always true:

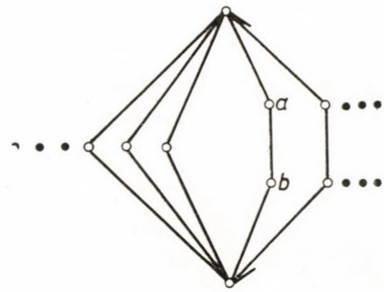

Fig. 19

LEMMA 19. *Let I and K be arbitrary ideals of L. Then we have*

$$K \sim K/I \cap K \sim I \cup K/I.$$

SUPPLEMENT 1. *A natural congruence relation Θ_1 of K under which $K(\Theta_1) \simeq K/I \cap K$ may be given in the following way: the elements a, b of K are congruent under Θ_1 if and only if there exist $u, v \in I \cap K$ and $a \cup b = x_0 \geqq x_1 \geqq \cdots \geqq x_n = a \cap b$ such that $\overline{u, v} \rightarrow \overline{x_{i-1}, x_i}$ $(i = 1, 2, \ldots, n)$ within K.*

SUPPLEMENT 2. *A natural congruence relation Θ_2 of K under which $K(\Theta_2) \simeq I \cup K/I$ may be given as follows: the elements a, b* of *K are congruent under Θ_2 if and only if there exist $u, v \in I$ and $a \cup b = y_0 \geqq y_1 \geqq \geqq \cdots \geqq y_n = a \cap b$ such that $\overline{u, v} \rightarrow \overline{y_{i-1}, y_i}$ $(i = 1, 2, \ldots, n)$ within $I \cup K$.*

Without loss of generality we may suppose $L = I \cup K$. Consider the congruence relation $\Theta[I]$. This makes a partition of K into congruence classes and, obviously, this partition of K is compatible. Consider the congruence relation Θ^* of K which induces the same partition on K. We assert that

$$I \cup K/I \simeq K(\Theta^*).$$

From the first general isomorphism theorem we get that it is enough to prove that every congruence class of $I \cup K$ modulo $\Theta[I]$ contains an element from K. Indeed, let $x \in I \cup K$, then with suitable $y \in I$ and $z \in K$ we have $x \leqq y \cup z$. We put $t = x \cap y \cap z$ $(\in I)$. Then $y \equiv t$ $(\Theta[I])$, hence $y \cup z \equiv t \cup z = z$ $(\Theta[I])$, and so $x = x \cap (y \cup z) \equiv x \cap z$ $(\Theta[I])$. Thus $x \cup z \in K$ is congruent to z modulo $\Theta[I]$.

On the other hand, denote Θ_0 the congruence relation of the lattice K generated by $I \cap K$ (i. e. $\Theta_0 = \Theta[I \cap K]$ on the lattice K). Obviously, $a \equiv b$ (Θ_0) implies $a \equiv b\,(\Theta^*)$, that is, $\Theta_0 \leqq \Theta^*$. Thus $K(\Theta_0) \sim K(\Theta^*)$. We have already seen that $I \cup K/I \simeq K(\Theta^*)$, hence we have $K \sim K/K \cap I \sim I \cup K/I$ which was to be proved.

The assertion of the supplements is immediate if we compare the definitions of Θ^* and Θ_0 with that of Θ_2 and Θ_1 and with Theorem I and formula (3).

Now it will be easy to give the required characterization of the ideals satisfying the first isomorphism theorem.

THEOREM 22. *Suppose that every element a of L satisfies one of the following conditions:*

(a) $[0, a]$ *is a finite lattice;*

(b) *every homomorphic image of the lattice* $[0, a]$ *satisfies the Jordan—Dedekind chain condition.*

Then an ideal I of the lattice L satisfies the first isomorphism theorem with any principal ideal $K = (k]$ if and only if whenever the weak projectivity $\overline{u, v} \to \overline{x, y}$ holds within $I \cup K$, where $u, v \in I$ and $x, y \in K$, $x \succ y$, then with suitable elements w, z of $I \cap K$ the weak projectivity $\overline{w, z} \to \overline{x, y}$ holds within K.

PROOF. From the conditions it follows that if $K = (k]$ is a principal ideal, then for the lattice $[0, k]$ the conclusion of Lemma 18 is true. Indeed, if K satisfies condition (b) of Theorem 22, then the assertion follows from Lemma 18. If K satisfies condition (a) of Theorem 22, then it is a finite lattice. Let Θ, $\Phi \in \Theta(K)$ and $\Theta < \Phi$. Then $K(\Phi)$ consists of fewer elements than $K(\Theta)$, thus $K(\Theta) \simeq K(\Phi)$ is impossible.

Now, in Lemma 19 we have seen that $I \cup K/I \simeq K(\Theta^*)$ and $K/I \cap K \simeq K(\Theta_0)$, where the congruences Θ^* and Θ_0 of K have been defined in the proof of Lemma 10. Thus, if K satisfies the first isomorphism theorem, then from the previous section and from $\Theta_0 \leqq \Theta^*$ it follows that $\Theta_0 = \Theta^*$. From conditions (a) and (b) we conclude that the lattice L is discrete, that is, any two elements $a > b$ of L may be connected by a finite maximal chain. It follows that two congruence relations, Θ_0 and Θ^*, are the same if and only if Θ_0 and Θ^* collapse the same prime intervals. But if a covers b, then in Supplement 1 of Lemma 19 we may take $n = 1$ and in Supplement 2 $m = 1$, and thus the coincidence of Θ_0 and Θ^* upon every prime interval is just assured by the condition of this theorem.

Conversely, if the conditions of Theorem 22 hold, then the congruence relations Θ_0 and Θ^* are the same, that is, $K(\Theta_0) \simeq K(\Theta^*)$. Consequently, by Lemma 19, we get $I \cup K/I \simeq K/I \cap K$, completing the proof of Theorem 22.

§ 3. Modular lattices of locally finite length with zero

The main result of this section is the following:

THEOREM 23. *Let L be a modular lattice of locally finite length with zero. An ideal I of L satisfies the first isomorphism theorem if and only if it is neutral.*

We prove a bit more than the assertion of the theorem, namely

COROLLARY. *Let L be a modular lattice with zero which is of locally finite length. The following four conditions on the ideal I of the lattice L are equivalent:*

(a) *I satisfies the first isomorphism theorem, that is, for an arbitrary ideal K of L*

$$I \cup K/I \simeq K/I \cap K;$$

(b) *I satisfies the first isomorphism theorem for an arbitrary principal ideal* $K=(k]$;

(c) *I is standard;*

(d) *I is neutral.*

We prepare the proof of this theorem with four lemmas. Among these the first is surely known, but we did not find in the literature. The most interesting of these lemmas is the third (Lemma 22) which gives the structure of an interesting free lattice.

LEMMA 20. *Let L be a locally finite modular lattice and I an ideal of L. A prime interval p of L collapses under* $\Theta[I]$ *if and only if it is projective to a prime interval q of I.*

PROOF. It is easy to derive this assertion from Theorem I and formula (3). A direct proof is the following: we define the relation Θ: $a \equiv b\ (\Theta)$ $(a, b \in L)$ if and only if there exists a sequence of elements $a \cup b = y_0 \succ y_1 \succ \cdots \succ y_n = a \cap b$ such that each $[y_{i+1}, y_i]$ $(i = 0, 1, \ldots, n-1)$ is projective to a prime interval of I. All the conditions of Lemma II are trivially satisfied for Θ, thus Θ is a congruence relation and $\Theta = \Theta[I]$ is also obvious. If $a \succ b$, then $n = 1$, hence the assertion.

Let L be a modular lattice of locally finite length. Let us fix an ideal I and a prime interval p of L. By the previous lemma we can find prime intervals q in I such that $q \overset{n}{\longleftrightarrow} p$. Choose a q such that n be as small as possible. For the proof of Theorem 22 it will be useful to call this smallest n the order of p relative to I. $n = 0$ means that $p \subseteq I$ and p is of infinite order if p does not collapse under $\Theta[I]$.

LEMMA 21. *Let L be a modular lattice of locally finite length. An ideal I of L is standard if and only if the orders of the prime intervals of L relative to I are* 0, 1 *or infinite.*

PROOF. If I is standard, then by condition (γ'') of Theorem 2 the assertion is obviously true.

Conversely, suppose the ideal I of L satisfies the condition. Let Θ be the relation defined in the proof of Lemma 20. If $a \equiv b\ (\Theta)$, then there exists a sequence of elements $a \cup b = y_0 \succ y_1 \succ \cdots \succ y_n = a \cap b$ such that each $[y_{i+1}, y_i]$ is projective to a prime interval $[a_{i+1}, a_i]$ of I. Because all the $y_{i+1}, y_i]$ are of the order 1 or 0 relative to I, it follows that we may suppose $y_{i+1} \cup a_i = y_i$. Let $x = \bigvee a_i$, then obviously $y_n \cup x = y_0$. This means that I satisfies condition (γ'') of Theorem 2 and, consequently, I is standard.

COROLLARY. *An ideal I of a modular lattice L of locally finite length is not standard if and only if there exists a prime interval p of L of order* 2 *relative to I.*

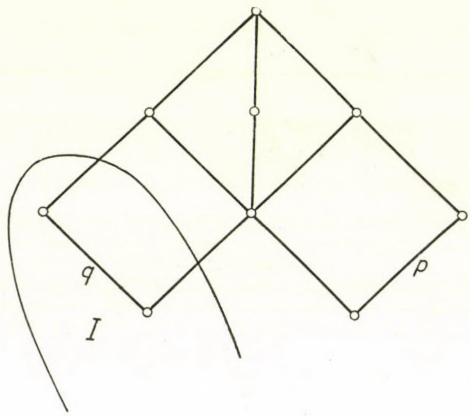

Fig. 20

The sublattice V (that is, the lattice of five elements which is modular but not distributive) is called *minimal* if its length in L is 2.

Now suppose that I is a non-standard ideal of the modular lattice L of locally finite length and consider a prime interval p of order 2, the existence of which is guaranteed by Lemma 21. It is possible that we can reach p from I in the way shown by Fig. 20. In this case the "turn" is through a minimal V. If this is the case, then we call the zero element of the minimal V a *turning element*. Consequently, if we can find to the ideal I a turning element, then I is surely not standard. The most important point of the proof of Theorem 22 is the converse of this statement. We cannot prove directly this assertion. First we have to find the most general situation which may occur in Fig. 20, that is, the corresponding free lattice.

LEMMA 22. *Let L be a modular lattice of locally finite length. I is a non-standard ideal, p a prime interval of order* 2 *relative to I, q a prime interval of I and $q \overset{2}{\leftrightarrow} p$, namely, $\overline{b, a} \overset{1}{\rightarrow} \overline{f, e} \overset{1}{\rightarrow} \overline{d, c}$, $p = [b, a]$ and $q = [d, c]$.*

The free modular lattice generated by the elements a, b, c, d and e, f is the following:

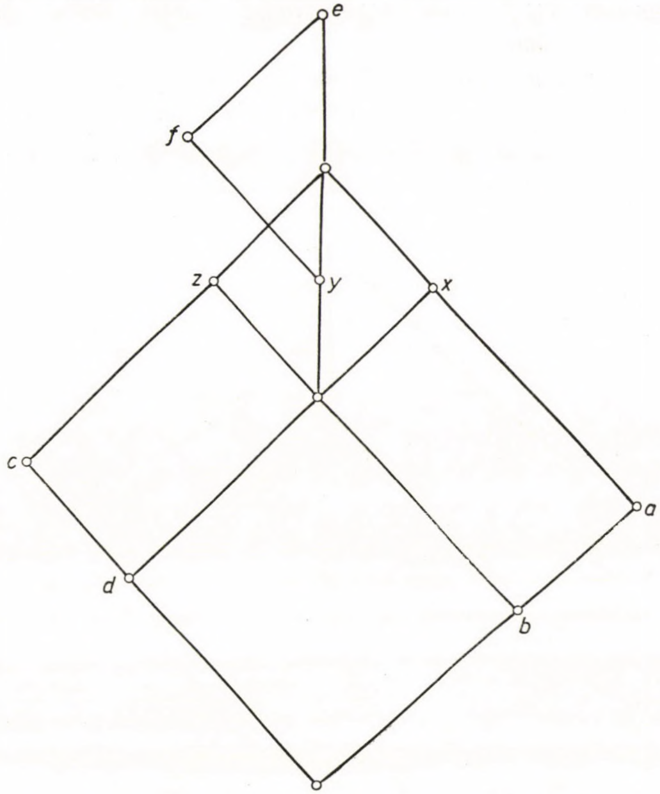

Fig. 21

REMARK. Simply to say, in Lemma 22 there is determined the free modular lattice generated by two covering pairs of elements. It is interesting the existence of this free lattice, for, in general, it is not allowed to prescribe covering relations in a free lattice.

PROOF. Consider the elements $x = a \cup d$, $y = f \cap (a \cup c)$, $z = b \cup c$, further the elements a, b, c, d, e, f, $b \cup d$, $a \cup c$, $b \cap d$. We prove that from the modularity and from the fact that the order of p relative to I is 2, finally from the covering relations it follows that these elements form a sublattice of L and all the joins and meets are the same as in Fig. 21 and these are consequences of the hypotheses.

First we show that x, y and z generate a minimal V, and

$$x \cup y = y \cup z = z \cup x = a \cup c, \quad x \cap y = y \cap z = z \cap x = b \cup d.$$

$x \cup z = a \cup c$ is clear from the definitions. We have $x \cup y = (a \cup d) \cup [f \cap (a \cup c)] =$ (from $a \cup d \leqq a \cup c$ and from the modularity) $= (a \cup d \cup f) \cap (a \cup c) =$ (because $a \cup f = e$) $= (d \cup e) \cap (a \cup c) = a \cup c$ for $c \leqq e$. We can get $y \cup z = a \cup c$ in a similar way.

From $x = a \cup d$ it follows that the interval $[b \cup d, x]$ is a transpose of $[b, a]$, and so it is of length 1 (we excluded the case $d \cup a = d \cup b$, for this implies $d \cap a > d \cap b$, thus $[d \cap b, d \cap a] \subseteq I$ is a transpose of $[b, a]$ the order of which relative to I is 1). Similarly, $[b \cup d, z]$ is also of length 1. Finally, $[b, a] \xrightarrow{1} [f, e] \xrightarrow{1} [y, a \cup c]$, and so $[y, a \cup c]$ is also a prime interval.

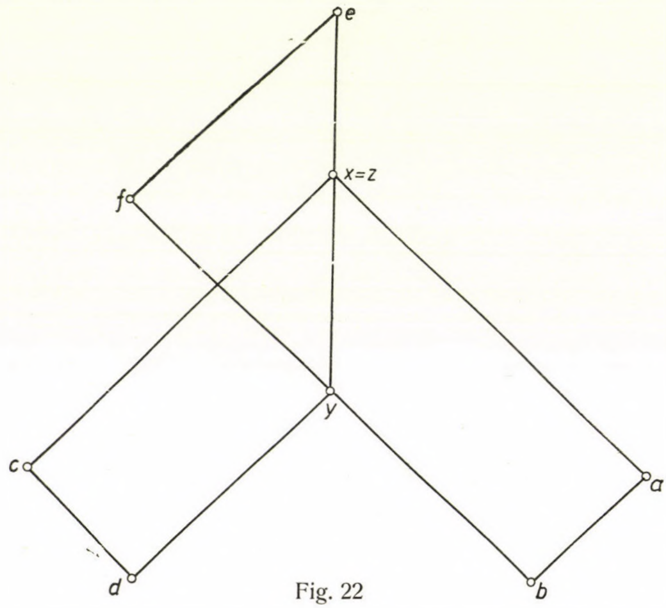

Fig. 22

We show that no two of x, y and z coincide. Suppose $x = z$. Because of $x \cup z = a \cup c$ we get $x = z = a \cup c$. Further, $[b \cup d, x]$ is a prime interval, and so $y = b \cup d$. In this case the diagram of the lattice (more precisely, a part of it) is shown by Fig. 22. We see that $a \cap c = a \cap d$ is impossible, for $a \cup c = a \cup d$ and $a \cup c = x$. Thus $[a \cap d, a \cap c]$ is a prime interval. Further, from $d \leqq y$ we get $a \cap d = a \cap y \cap d = b \cap d$. We prove that $[a \cap d, a \cap c]$ is a transpose of $[b, a]$. This will be a contradiction, for in this case the order of $[b, a]$ relative to I is 1 contrary to the hypotheses.

We have to prove that $b \cup (a \cap d) = b$ and $b \cup (a \cap c) = a$. We have $b \cup (a \cap d) =$ (from $a \cap d = b \cap d$) $= b$; further $b \cup (a \cap c) = (b \cup c) \cap a = z \cap a = a$.

The impossibility of $x = y$ is very easy to prove, for if $x = y$, then $f \geqq f \cap (a \cup c) = y = x = a \cup d \geqq a$, that is, $e = f \cup a = f$, a contradiction. We get a similar contradiction from $y = z$.

Of the remaining relations it is enough to prove $a \cap c = b \cap d$. Indeed, from $a \cap c = b \cap c$ (which is surely true, otherwise the order of $[b, a]$ relative to I were 1) and from $y \cap c = d$ we get $a \cap c = b \cap c = (y \cap b) \cap c = (y \cap c) \cap \cap b = d \cap b$. Thus the proof of Lemma 22 is completed.

Now we are able to prove the existence of turning elements.

LEMMA 23. *Let L be a modular lattice of locally finite length. An ideal I of L is non-standard if and only if there exists a turning element.*

PROOF. Using the notations of Lemma 21 and Lemma 22, consider the prime intervals $p = [b, a]$ and $q = [d, c]$ the existence of which is assured by Lemma 21. The sublattice of L generated by a, b, c, d and e, f is a homomorphic image of the free lattice of Lemma 22. Under this homomorphism, the minimal V of the free lattice does not collapse. (Indeed, if the minimal V collapsed, then both p and q would collapse.) Thus the minimal element of the minimal V may serve as a turning element.

We remark that the only congruences of the free lattice of Lemma 22, under which p does not collapse, are Θ_{fy}, Θ_{xa}, Θ_{cz} and their joins.

Now we are prepared for proving Theorem 23.

PROOF OF THEOREM 23. We prove the Corollary, for it is a stronger assertion than the theorem.

(c) implies (d) — this is stated in Lemma 10.

(d) implies (a) — this was proved in Theorem 13.

(a) implies (b) — this is trivial.

Thus we have to prove that (b) implies (c).

For this reason, let us suppose that L is a modular lattice with zero and of locally finite length, and I is an ideal satisfying

$$I \cup K/I \simeq K/K \cap I$$

for all $K = (k]$. If (b) does not imply (c), then I is not standard, that is, by Lemma 23 there exist turning elements in L. From the suppositions on L we obtain the existence of the dimension function $d(x)$. We denote from now on by u a turning element for which $d(u)$ is as small as possible. Finally, we denote by p and q the prime intervals (see Lemmas 22 and 23) from which the turning element u has been constructed.

Now we apply Theorem 21 to I and $(a]$. We may do so, for every interval $[0, a]$ of L satisfies the Jordan—Dedekind chain condition, and the same is true for any homomorphic image of $[0, a]$. $I \cup (a]$ contains the prime interval p, the order of which relative to I in $I \cup (a]$ is 2, for $I \cup (a]$ contains the whole minimal V because of $x \cup y \cup z = a \cup c \in (a] \cup I$. The order of p relative to $I \cap (a]$ in $(a]$ is at most 2, for if it were 1, then the order of p relative to I in $I \cup (a]$ would be also 1. Thus by Theorem 21 and Lemma 20 we can find a prime interval q_0 of $I \cap (a]$ and prime intervals $q_1, \ldots, q_n = p$ $(n \geqq 2)$ of $(a]$ such that $q_0 \xrightarrow{1} q_1 \xrightarrow{1} \cdots \xrightarrow{1} q_n = p$. Let the intervals be chosen in such a way that n be the order of p relative to $I \cap (a]$ in $(a]$. Then q_2 is of order 2 relative to $I \cap (a]$ in $(a]$. It is trivial that the order of q_2 relative to I in $I \cup (a]$ is also 2, otherwise it would be 1, and this would imply the same in $(a]$. We may now apply Lemma 23 to q_0, q_2 and $I \cap (a]$, to conclude the existence of a turning element v. This turning element of $I \cap (a]$ in $(a]$ is a turning element of I in the whole lattice, too, for q_2 is of order 2 relative to I. But the minimal V, the zero of which is the turning element v, is included in $(a]$, therefore $d(v) < d(a) - 1$. On the other hand, $d(a) \leqq d(u) + 1$, thus $d(v) < d(u)$. We have found a turning element of lower dimension than u, a contradiction to the minimality of $d(u)$. This contradiction proves the Corollary of Theorem 22 and at the same time Theorem 22.

We should point out that as a consequence of Theorem 22 we get that every ideal satisfying the first isomorphism theorem is a homomorphism kernel in modular lattices with zero and of locally finite length. We have obtained the same conclusion in section complemented weakly modular lattices with ascending chain condition in § 1 of this chapter. Thus the following problem arises:

PROBLEM 19. Give classes of lattices in which every ideal satisfying the first isomorphism theorem is a homomorphism kernel. (Does the class of weakly modular lattices serve for this purpose?)

REMARK. In general it is not true, see, for instance, the ideal $(q]$ of the lattice U.

PROBLEM 20. Does there exist a modular lattice L and an ideal I of L such that I satisfies the first isomorphism theorem and despite this

a) I is not a homomorphism kernel, or

b) I is not a neutral ideal?

§ 4. A characterization of standard ideals by the first isomorphism theorem

In the Introduction we alluded to the fact that the notion of standard ideals is the best-possible one from the point of view of the first isomorphism theorem.

To formulate precisely what this means we need some notions.

Let $\mathfrak{A}$ be a class of ideals, i. e. if we are given a lattice L and an ideal I of L, then we are able to determine whether $I \in \mathfrak{A}$ or not. We say that $\mathfrak{A}$ is of type $f_\alpha = g_\alpha$ ($\alpha \in A$) if $I \in \mathfrak{A}$ is equivalent to the fact that I is of type $f_\alpha = g_\alpha$ ($\alpha \in A$) in the sense of § 4 of Chapter III. We say that $\mathfrak{A}$ satisfies the first isomorphism theorem if from $I \in \mathfrak{A}$, I is an ideal of L it follows that I satisfies the first isomorphism theorem with any other ideal K of L. Finally, an ideal I of the lattice L is said to have the condition (**) if L_1/I is a sublattice of L/I under the natural mapping whenever $I \subseteq L_1 \subseteq L$, L_1 is a sublattice of L. Again, $\mathfrak{A}$ has property (**) if any of its ideals has it.

Only condition (**) needs a little explanation. It essentially requires that from the structure of L informations may be got about L/I.

For groups, (**) holds always (putting invariant subgroup instead of ideal and subgroup for sublattice).

Now we may state

THEOREM 24. *If the class $\mathfrak{A}$ of ideals*

1. *is of type $f_\alpha = g_\alpha$;*
2. *satisfies the first isomorphism theorem;*
3. *has the property* (**),

then $\mathfrak{A}$ contains only standard ideals.

The proof is easy, we have only to observe that it is an easy consequence of Theorem 10 that we may restrict ourselves to principal ideals. Now if $I = (d]$, then it may be easily proved that (*) of § 2 of Chapter III is equivalent to (**). As it was proved in § 3 of Chapter III, it follows that $\mathfrak{A}$ contains only distributive ideals. Now if d were distributive but not standard, then by Lemma 1 L would contain x, y with $x \geqq y$, $d \cup x = d \cup y$, $d \cap x = d \cap y$. By 1, $(d] \in \mathfrak{A}$ in $\{d, x, y\}$ contradicting 2.

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ON A PROPERTY OF FAMILIES OF SETS

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1. Introduction. In this paper we are going to generalize a problem solved by MILLER in his paper [1] and prove several results concerning this new problem and some related questions. We mention here that some of our theorems (Theorems 8 and 10) have the interesting consequence that the topological product of $\aleph_k$ 1-compact spaces (Lindelöf spaces) is not necessarily k-compact for any finite k.[1]

DEF. (1. 1) Let $\mathfrak{F}$ be a family of sets. $\mathfrak{F}$ is said by MILLER to possess property **B** if there exists a set B such that

$$F \cap B \neq 0 \qquad \text{for every} \qquad F \in \mathfrak{F},$$

$$F \subsetneqq B \qquad \text{for every} \qquad F \in \mathfrak{F}.$$

DEF. (1.2) Let $\mathfrak{F}$ be a family of sets. Let $p^*(\mathfrak{F})$ denote the least cardinal number p for which $\overline{\overline{F}} \leqq p$ for every $F \in \mathfrak{F}$. If $\overline{\overline{F}} \geqq p$ for every $F \in \mathfrak{F}$, we write $|\mathfrak{F}| \geqq p$. In what follows $p(\mathfrak{F}) = p$ denotes briefly that the family $\mathfrak{F}$ possesses the property

$$p^*(\mathfrak{F}) = p, \quad |\mathfrak{F}| \geqq p.$$

DEF. (1.3) Let $\mathfrak{F}$ be a family of sets and let $q \geqq 2$, $r \geqq 1$ be cardinal numbers. The family $\mathfrak{F}$ is said to possess property $\mathbf{C}(q, r)$ if $\overline{\overline{\bigcap_{F \in \mathfrak{F}'} F}} < r$ for every subfamily $\mathfrak{F}'$ of $\mathfrak{F}$, provided $\overline{\overline{\mathfrak{F}'}} \geqq q$.

NOTE. If for a family $\mathfrak{F}$ $|\mathfrak{F}| \geqq r$ and $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$, then $\mathfrak{F}$ consists of almost disjoint sets.

The result of MILLER which is our starting point can be stated as follows:

(1. 4) Let p be an infinite cardinal number, n an integer $(n > 0)$ and let $\mathfrak{F}$ be a family which possesses property $\mathbf{C}(p^+, n)$ such that $|\mathfrak{F}| \geqq p$. Then the family $\mathfrak{F}$ possesses property **B**.[2]

[1] In our example the spaces will be discrete ones. The generalized continuum hypothesis is used in the proof. As far as we know this result is new already for $k = 2$. This theorem should be compared with a theorem of J. Łos [3] (see Section 7).

[2] See [1], p. 35, Corollary.

To show that this result is best-possible MILLER proves the following:

(1.5) There exists a family $\mathfrak{F}$ $(p(\mathfrak{F})=\aleph_0, \overline{\overline{\mathfrak{F}}}=2^{\aleph_0})$ which possesses property $\mathbf{C}(2, \aleph_0)$ and fails to possess property **B**.[3]

However, one can ask what happens if $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_0)$ and $|\mathfrak{F}|$ is supposed to be greater than $\aleph_0$.

On the other hand, one can sharpen property **B** as follows:

DEF. (1.6) Let $\mathfrak{F}$ be a family, s a cardinal number, $s \geqq 2$. $\mathfrak{F}$ is said to possess property $\mathbf{B}(s)$ if there exists a set B such that $F \cap B \neq 0$ and $\overline{\overline{F \cap B}} < s$ for every $F \in \mathfrak{F}$.

Our problems will be of the following kind. Let $\mathfrak{F}$ be a family of sets, and let m, p, q, r, s be cardinal numbers such that $\overline{\overline{\mathfrak{F}}} = m$, $p(\mathfrak{F}) = p$ and suppose that $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$. Under what conditions for the cardinals m, p, q, r, s has $\mathfrak{F}$ to possess the properties **B** and $\mathbf{B}(s)$, respectively?

As the easy example (3.3) will show, nothing can be said about property $\mathbf{B}(s)$ if $q > 2$. The case $q = 2$ contains the essential difficulty in the researches concerning the property **B** too.

The problem just stated is clearly a generalization of the problem treated in (1.4) which is a corollary of [1], Theorem 1. We remark that it would be possible to generalize in a quite similar way the theorem itself not only its corollary, however, such a generalization does not seem to need new ideas and its formulation would be very complicated.

We restricted the formulation of the general problem with the assumption $p(\mathfrak{F}) = p$ instead of MILLER's original assumption $|\mathfrak{F}| \geqq p$. This has no importance in the problems concerning property **B**, however, in the problems for property $\mathbf{B}(s)$ it seems to be an essential restriction (see the remark at the end of Section 4).

2. Definitions. Notations. We use the usual notations of the set theory. We are going to list only those where there is a danger of misunderstanding.

In what follows $\mathfrak{F}, \mathfrak{G}, \ldots$ will denote families (sets of sets); capital letter will denote sets; $x, y, \ldots$ are the elements of the sets; m, t, p, q, r, s denote cardinals; $i, j, k, l, n, \ldots$ denote non-negative integers; $\alpha, \beta, \ldots$ denote ordinal numbers. Union and intersection of sets will be denoted by $\cup$ and $\cap$, respectively.

t^+ denotes the least cardinal greater than t (if t is finite, $t^+ = t+1$). t^- is the immediate predecessor of the cardinal t if it exists, if not, then $t^- = t$. (If t is finite, $t^- = t-1$ for $t > 0$ and $t^- = 0$ for $t = 0$.)

$\mathfrak{S}(S)$ denotes the set of all subsets of S.

[3] See [1], Theorem 3.

If $(x_\nu)_{\nu<\varphi}$ is an arbitrary sequence of type φ of not necessarily different elements x_ν, then $\{x_\nu\}_{\nu<\varphi}$ denotes the set of all x_ν's forthcoming in the sequence. This distinction will be sometimes omitted if there is no danger of misunderstanding.

Let $\varphi(x)$ be an arbitrary property of the elements of a set H. The set of all $x \in H$ which satisfy $\varphi(x)$ will be denoted by $\{x : \varphi(x)\}$. (We are going to use the logical signs $\wedge$ (and), $\vee$ (or) in the formulation of these formulas.)

The sets $\{X : X \subseteq S \wedge \overline{\overline{X}} = t\}$, $\{X : X \subseteq S \wedge \overline{\overline{X}} < t\}$ will be denoted by $[S]^t$, $[S]^{<t}$, respectively.

For an arbitrary family $\mathfrak{F}$ the set $\bigcup_{F \in \mathfrak{F}} F$ is denoted by $(\mathfrak{F})$. Other special notations concerning families will be introduced later.

For the study of the problem stated in the introduction we introduce the following symbols:

DEF. (2. 1) $\mathbf{M}(m, p, q, r) \rightarrow \mathbf{B}$ indicates the statement that every family $\mathfrak{F}$ which possesses property $\mathbf{C}(q, r)$ possesses property $\mathbf{B}$, provided $p(\mathfrak{F}) = p$ and $\overline{\overline{\mathfrak{F}}} = m$.

DEF. (2. 2) $\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(s)$ indicates the statement that every family $\mathfrak{F}$ which possesses property $\mathbf{C}(2, r)$ possesses property $\mathbf{B}(s)$, provided $p(\mathfrak{F}) = p$ and $\overline{\overline{\mathfrak{F}}} = m$. $\mathbf{M}(m, p, q, r) \not\rightarrow \mathbf{B}$ and $\mathbf{M}(m, p, r) \not\rightarrow \mathbf{B}(s)$ denote the negations of the corresponding statements, respectively.

To exclude trivial exceptions here we assume once for all $m > 0$, $p > 0$, $q > 1$, $r > 0$, $s > 1$.

We call briefly the symbols now introduced symbol-I and symbol-II, respectively.

The proof of some of our theorems makes use of the generalized continuum hypothesis or of the so-called measure hypothesis stated in [5]. These hypotheses will be cited as hypotheses (*), (**) and the corresponding theorems will be denoted by the same signs, respectively.

3. Preliminaries. A short summary of the content of the following sections. We briefly say that one of the symbols is monotone increasing (decreasing) in one of its variables, e. g. in m, if the fact that it is true for $m, p, q, r, (s)$ implies that it is true for $m', p, q, r, (s)$ for $m' \geqq m$ ($m' \leqq m$), respectively. The following monotonicity properties are immediate consequences of the definitions (2. 1) and (2. 2):

(3. 1) Both symbols are decreasing in m and r. Symbol-I is decreasing in q. Symbol-I is increasing in p, symbol-II is increasing in s.

We call attention that symbol-II is not increasing in p (see the end of Section 4).

It is also obvious that

(3. 2) $\mathbf{M}(m,p,r) \to \mathbf{B}(s)$ implies $\mathbf{M}(m,p,2,r) \to \mathbf{B}$ if $s \leqq p$. (For $s > p$ symbol-II is trivially true.)

Now we prove:

(3. 3) *Let* $p \geqq \aleph_0$, $s \leqq p$ *be cardinal numbers. There exists a family* $\mathfrak{F}$ *such that* $\overline{\overline{F}} = p$, $p(\mathfrak{F}) = p$, $\mathfrak{F}$ *possesses property* $\mathbf{C}(3, 1)$ *and it does not possess property* $\mathbf{B}(s)$.

PROOF. Let S be a set of power p. Let $\mathfrak{F}'$ be a system of subsets of S such that $\overline{\overline{\mathfrak{F}'}} = p$, $F_1 \cap F_2 = 0$ for $F_1, F_2 \in \mathfrak{F}'$, $F_1 \neq F_2$ and $\overline{\overline{F}} = p$ for every $F \in \mathfrak{F}'$. Put $\mathfrak{F} = \{S\} \cup \mathfrak{F}'$. It is obvious that $\mathfrak{F}$ satisfies the requirements of (3. 3) for $s = p$.

(3. 3) shows that in the investigations concerning property $\mathbf{B}(s)$ the assumption $q \leqq 2$, i. e. $q = 2$ is essential.

MILLER's result (1. 4) can be stated as follows:

THEOREM 1. *Suppose* $p \geqq \aleph_0$, $q \leqq p^+$. *Then for every* m *and for every* $r < \aleph_0$

$$\mathbf{M}(m, p, q, r) \to \mathbf{B}.$$

MILLER's theorem can be considered as a generalization of BERNSTEIN's theorem which states that if p is infinite, then every family $\mathfrak{F}$ ($\overline{\overline{\mathfrak{F}}} = p$, $p(\mathfrak{F}) = p$) (without any further assumption for property $\mathbf{C}(q, r)$) possesses property $\mathbf{B}$,[4] i. e.:

THEOREM 2. $\mathbf{M}(p, p, q, r) \to \mathbf{B}$ *if* $p \geqq \aleph_0$ *for every* q *and* r.

MILLER's counterexample (1. 5) can be stated generally as follows:

THEOREM 3. $\mathbf{M}(2^p, p, 2, p) \not\to \mathbf{B}$ *if* p *is infinite.*

Theorem 3 can be proved quite similarly as its special case for $p = \aleph_0$ cited in (1. 5) and therefore we omit the proof.

Theorem 2 shows that in the investigations concerning property $\mathbf{B}$ we may always suppose that $m \geqq p$, and Theorem 3 shows that if $m > p$, then to obtain positive results we have to suppose $r < p$.

We mention that without using (*) we can not decide the following

PROBLEM 1. $\mathbf{M}(\aleph_1, \aleph_0, 2, \aleph_0) \not\to \mathbf{B}$?

We can not prove without (*) that every $\mathfrak{F}$ ($p(\mathfrak{F}) = \aleph_0$, $\overline{\overline{\mathfrak{F}}} < 2^{\aleph_0}$) possesses property $\mathbf{B}$.

[4] See [4].

It is obvious that the property $\mathbf{C}(q', r)$ is weaker than the property $\mathbf{C}(q, r)$, provided $q' > q$.

Now we prove:

(3.4) $$\mathbf{M}(2^p, p, q, 1) \not\to \mathbf{B} \quad \textit{if} \quad q > 2^p, \quad p \geqq \aleph_0.$$

PROOF. Put $\mathfrak{F} = [S]^p$ where S is a set of power p. It is obvious that $\overline{\overline{\mathfrak{F}}} = 2^p$, $p(\mathfrak{F}) = p$, $\mathfrak{F}$ possesses property $\mathbf{C}(q, 1)$, but it does not possess property $\mathbf{B}$.

(3. 4) shows that we have to suppose $q \leqq 2^p$, and since here we do not want to discuss the difficulties caused by the continuum problem, we are going to suppose $q \leqq p^+$.

It results that the best-possible generalization of Theorem 1 would be the following:

(o) $\mathbf{M}(m, p, q, r) \to \mathbf{B}$ *for every* m, *provided* $q \leqq p^+$, $p \geqq \aleph_0$, $r < p$.

We can prove this only with the stronger assumption $r^+ < p$ (see Theorem 4) or with the restriction that m is not too large (see Theorem 5). Both proofs use (*).

The simplest unsolved problem here is

PROBLEM 2. $\mathbf{M}(\aleph_{\omega+1}, \aleph_1, 2, \aleph_0) \to \mathbf{B}$?

As to the symbol-II the problems are more ramified. First we have to discuss the case $m \leqq p$ which leads to some interesting result too. This will be done in Section 4. (3. 2) shows that we have to suppose $s \leqq p$. In Section 4 we are going to prove that at least in the case $p \geqq \aleph_0$, $m \geqq p$ we may suppose $r^+ \leqq s$.

So the best-possible refinement of the conjecture (o) would be the following:

(oo) $\mathbf{M}(m, p, r) \to \mathbf{B}(r^+)$ *for every* $m \geqq p \geqq \aleph_0$, *provided* $r < p$.

Now we have to distinguish two cases:

(i) If r is finite, then (oo) is false. However, it is always true for $\aleph_0$ instead of r^+ and using (*) corresponding to every m, p and r we can determine the least s (eventually finite) for which $\mathbf{M}(m, p, r) \to \mathbf{B}(s)$ is true. These results will be proved in Section 7. As a consequence of these results we prove the topological theorem mentioned in the introduction (in Section 8). There we state many conjectures which all would have been consequences of 2-compactness of the topological product of $\aleph_2$ Lindelöf spaces — now disproved — and which we can not disprove with our method.

(ii) If r is infinite, (oo) is very likely true, however, we can prove it — using (*) — only with similar restrictions as in the case of symbol-I, namely we can prove that

$$\mathbf{M}(m, p, r) \to \mathbf{B}(r^+) \quad \text{for} \quad m \geqq p > r > \aleph_0,$$

provided m is not too large (see Theorem 7), and we can prove that

$$\mathbf{M}(m, p, r) \to \mathbf{B}(r^{++}) \quad \text{for every} \quad m \geqq p > r^+ > \aleph_0$$

(see Theorem 6).

The simplest unsolved problems here are

PROBLEM 3.

a) $\mathbf{M}(\aleph_{\omega+1}, \aleph_{\omega+1}, \aleph_0) \to \mathbf{B}(\aleph_1)$?

b) $\mathbf{M}(\aleph_{\omega+1}, \aleph_1, \aleph_0) \to \mathbf{B}(\aleph_1)$?

c) $\mathbf{M}(\aleph_{\omega+1}, \aleph_2, \aleph_1) \to \mathbf{B}(\aleph_2)$?

The results on (o) and (oo) will be proved in Section 6. All the positive results concerning the case $m > p$ will be proved with the method of MILLER's theorem, and the proof runs always by induction on m. That is why we need a generalization of the induction process used in [1]. This will be done in Section 5 and as a corollary of it we obtain all the positive theorems (Theorems 4—9) already mentioned.

In Section 9 we deal with the case of finite sets ($p < \aleph_0$) and with some questions related to property **B**.

4. The symbol-II in the cases $m \leqq p$ $(p \geqq \aleph_0)$. The following theorems of A. TARSKI will play an important role in our investigations:

(*) LEMMA 1. *Let* S *be a set,* $\mathfrak{F}$ *a family such that* $(\mathfrak{F}) \subseteq S$, $\overline{\overline{S}} = \aleph_\alpha$, $|\mathfrak{F}| \geqq \aleph_\beta$. *Then*

a) $\overline{\overline{\mathfrak{F}}} \leqq \aleph_\alpha$, *provided* $\mathfrak{F}$ *possesses property* $\mathbf{C}(\aleph_{\alpha+1}, \aleph_\beta)$ *and* $cf(\alpha) \neq cf(\beta)$.

b) $\overline{\overline{\mathfrak{F}}} \leqq \aleph_\alpha$, *provided* $\mathfrak{F}$ *possesses property* $\mathbf{C}(\aleph_{\alpha+1}, r)$ *for an* $r < \aleph_\beta$.[5]

Note that in TARSKI's paper the theorems are proved under the stronger conditions that $\mathfrak{F}$ possesses the properties $\mathbf{C}(2, \aleph_\beta)$ and $\mathbf{C}(2, r)$, respectively, however, the proofs can be carried out in the same way for our case too.

LEMMA 2. *Let* S *be a set,* $\mathfrak{F}$ *a family such that* $(\mathfrak{F}) \subseteq S$, $\overline{\overline{S}} = \aleph_\alpha$, $|\mathfrak{F}| \geqq r$ *where* r *is finite. Then* $\overline{\overline{\mathfrak{F}}} \leqq \aleph_\alpha$, *provided* $\mathfrak{F}$ *possesses property* $\mathbf{C}(\aleph_{\alpha+1}, r)$.

Lemma 2 is a corollary of the fact that $\overline{\overline{[S]^r}} = \aleph_\alpha$ for every finite r. Note that the proof of Lemma 2 does not make use of (*).

(4. 1) $\mathbf{M}(m, p, r) \to \mathbf{B}(m^+)$ *for every* m, p, r.

$\mathbf{M}(m, p, r) \not\to \mathbf{B}(m)$ *if* $r > p$ *and* $p \geqq \aleph_0$ $(\aleph_0 \leqq m \leqq p)$.

[5] See [2], Theorem 5, I, p. 211 and Corollary 6, p. 213 for a) and b), respectively.

PROOF. The first statement is trivial, the second is to be seen quite similarly to (3.3).

REMARK. *If m is finite, $m \geqq 2$, then* $\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(m)$ *and* $\mathbf{M}(m, p, r) \not\rightarrow \mathbf{B}(m-1)$ *is true under the same conditions for p, r respectively.*

(4.1) shows that we may always suppose $r \leqq p$.

(4.2) $\mathbf{M}(m, \aleph_\alpha, r) \rightarrow \mathbf{B}(2)$ *for every α if $m < \aleph_\alpha$ and $r < \aleph_\alpha$. If $r = \aleph_\alpha$, then the same is true for $m < \aleph_{cf(\alpha)}$.*

PROOF. Let $\mathfrak{F}$ be a family $(p(\mathfrak{F}) = \aleph_\alpha, \overline{\overline{\mathfrak{F}}} = m)$ which possesses property $\mathbf{C}(2, r)$. It is obvious that the set $F - \bigcup_{F' \in \mathfrak{F},\, F' \neq F} F'$ is of power $\aleph_\alpha$ and so it is non-empty for an arbitrary $F \in \mathfrak{F}$. Let x_F be an element of this set and put $B = \{x_F\}_{F \in \mathfrak{F}}$. We have $\overline{\overline{B \cap F}} = 1$ for every $F \in \mathfrak{F}$, hence $\mathfrak{F}$ possesses property $\mathbf{B}(2)$.

(4.3) $\mathbf{M}(\aleph_{cf(\alpha)}, \aleph_\alpha, \aleph_\alpha) \rightarrow \mathbf{B}(\aleph_{cf(\alpha)})$ *for every α.*

PROOF. Let $\mathfrak{F}$ be a family such that $p(\mathfrak{F}) = \aleph_\alpha$, $\overline{\overline{\mathfrak{F}}} = \aleph_{cf(\alpha)}$, and suppose that $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$. Let $\mathfrak{F} = \{F_\nu\}_{\nu < \omega_{cf(\alpha)}}$ be a well-ordering of $\mathfrak{F}$.

The set $F_\nu - \bigcup_{\mu < \nu} F_\mu$ is of power $\aleph_\alpha$ for every $\nu < \omega_{cf(\alpha)}$. Let x_ν be an element of it.

Put $B = \{x_\nu\}_{\nu < \omega_{cf(\alpha)}}$. It is obvious that $B \cap F_\nu \neq 0$ for every $\nu < \omega_{cf(\alpha)}$ and $\overline{\overline{B \cap F_\nu}} < \aleph_{cf(\alpha)}$ for every $\nu < \omega_{cf(\alpha)}$, since if $\nu' > \nu$, then $x_{\nu'} \notin F_\nu$. It follows that $\mathfrak{F}$ possesses property $\mathbf{B}(\aleph_{cf(\alpha)})$.

Now we show that (4.2) is best-possible in "s", i. e.

(4.4) $\mathbf{M}(\aleph_{cf(\alpha)}, \aleph_\alpha, \aleph_\alpha) \not\rightarrow \mathbf{B}(s)$ *if* $s < \aleph_{cf(\alpha)}$.

PROOF. We are going to suppose that α is of the second kind. If α is of the first kind, the statement can be proved quite similarly. Let S be a set of power $\aleph_\alpha$ and let $S = \{x_\beta\}_{\beta < \omega_\alpha}$ be a well-ordering of type ω_α of S. Let $\{\alpha_\nu\}_{\nu < \omega_{cf(\alpha)}}$ be a monotone increasing sequence of type $\omega_{cf(\alpha)}$ of ordinal numbers less than α cofinal with α. Put $S_\nu = \{x_\beta\}_{\beta < \omega_{\alpha_\nu}}$ for every $\nu < \omega_{cf(\alpha)}$. Obviously, one can define the sequences $\{F_\nu^1\}_{\nu < \omega_{cf(\alpha)}}$, $\{F_\nu^2\}_{\nu < \omega_{cf(\alpha)}}$ of type $\omega_{cf(\alpha)}$ of subsets of S in such a way that — if we put $\mathfrak{F} = \{F_\nu^1\}_{\nu < \omega_{cf(\alpha)}} \cup \{F_\nu^2\}_{\nu < \omega_{cf(\alpha)}}$, then $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$ — and that the following conditions hold:

(1) $\overline{\overline{F_\nu^1}} = \overline{\overline{F_\nu^2}} = \aleph_\alpha$ for $\nu < \omega_{cf(\alpha)}$,

(2) $S_\nu \subseteq F_\nu^1$ for $\nu < \omega_{cf(\alpha)}$,

(3) $F_\nu^2 \cap F_{\nu'}^2 = 0$ for $\nu \neq \nu'$, $\nu, \nu' < \omega_{cf(\alpha)}$.

Then $p(\mathfrak{F}) = \aleph_\alpha$ by (1) and $\overline{\overline{\mathfrak{F}}} = \aleph_{cf(\alpha)}$. Suppose that the set B intersects every set F of the family $\mathfrak{F}$. Then $\overline{\overline{B}} \geqq \aleph_{cf(\alpha)}$ by (3), hence if $s < \aleph_{cf(\alpha)}$, there is a $B' \subseteq B$ such that $\overline{\overline{B'}} = s$ and there is a $\nu_0 < \omega_{cf(\alpha)}$ such that $B' \subseteq S_{\nu_0}$. But this means by (2) that $\overline{\overline{B \cap F^1_{\nu_0}}} \geqq s$ and thus $\mathfrak{F}$ does not possess property $\mathbf{B}(s)$.

It results from (4. 1), (4. 2), (4. 3) and (4. 4) that to complete the discussion of the case $m \leqq p$ $(p \geqq \aleph_0)$ we have to determine the value of symbol-II in the following cases:

A) $m = p$, $r < p$ $(p \geqq \aleph_0)$,

B) $m = \aleph_\beta$, $p = \aleph_\alpha$, $r = \aleph_\alpha$ where $cf(\alpha) < \beta \leqq \alpha$.

To obtain complete results we have to assume (*) in both cases. In the case A) there remains an unsolved problem even if we assume (*).

First we prove the following negative result concerning A):

(*) (4. 5) $\mathbf{M}(\aleph_\alpha, \aleph_\alpha, r) \not\rightarrow \mathbf{B}(r)$ *if* $r < \aleph_\alpha$.

(If r is finite, the assumption (*) can be omitted.)

PROOF. Let $\mathfrak{F}_1, \mathfrak{F}_2$ be families satisfying the following conditions:

(4. 5. 1) $$\overline{\overline{\mathfrak{F}_1}} = r^+, \quad \overline{\overline{\mathfrak{F}_2}} = \aleph_\alpha,$$

(4. 5. 2) $$p(\mathfrak{F}_1 \cup \mathfrak{F}_2) = \aleph_\alpha,$$

(4. 5. 3) $$F \cap F' = 0 \quad \text{for every pair } F, F' \in \mathfrak{F}_1 \cup \mathfrak{F}_2, \ F \neq F'.$$

Put $(\mathfrak{F}_1) = S_1$. By Zorn's lemma there exists a *maximal* system $\mathfrak{S}$ of subsets of S_1 satisfying the following conditions:

(4. 5. 4) $$\overline{\overline{X \cap F}} = 1 \quad \text{for every} \quad X \in \mathfrak{S}, F \in \mathfrak{F}_1,$$
$$\overline{\overline{X \cap Y}} < r \quad \text{for every pair} \quad X, Y \in \mathfrak{S}, \ X \neq Y.$$

From (4. 5. 1) and (4. 5. 4) we get

(4. 5. 5) $$\overline{\overline{X}} = r^+ \quad \text{for every} \quad X \in \mathfrak{S}$$

and using the maximality of $\mathfrak{S}$ we obtain:

(4. 5. 6) If the set B' intersects every set F of $\mathfrak{F}_1$, then there exists an element X_0 of $\mathfrak{S}$ such that $\overline{\overline{B' \cap X_0}} \geqq r$.

On the other hand, using Lemmas 1 and 2 we get from (4. 5. 1), (4. 5. 2) and (4. 5. 4) that

(4. 5. 7) $$\overline{\overline{\mathfrak{S}}} = \aleph_\alpha.$$

It follows that there exists a one-to-one mapping $h(X)$ which maps $\mathfrak{S}$ onto $\mathfrak{F}_2$. Put $\mathfrak{F}_2^* = \{h(X) \cup X\}_{X \in \mathfrak{S}}$ and define $\mathfrak{F}$ as follows:

$$\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2^*.$$

Since $r^+ \leqq \aleph_\alpha$ by the assumption, by (4. 5. 2) and (4. 5. 5) we have $p(\mathfrak{F}) = \aleph_\alpha$. By (4. 5. 3) and (4. 5. 4) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$ and by (4. 5. 1) and (4. 5. 7) $\overline{\overline{\mathfrak{F}}} = \aleph_\alpha$.

We have to prove that $\mathfrak{F}$ does not possess property $\mathbf{B}(r)$. But if the set B intersects every set F of $\mathfrak{F}$, then it has a subset B' satisfying the condition of (4.5.6), hence $\overline{\overline{B \cap X_0}} \geqq r$ for an $X_0 \in \mathbb{S}$ and therefore $\overline{\overline{B \cap F_0}} \geqq r$ for $F_0 = h(X_0) \cup X_0$, hence for an $F_0 \in \mathfrak{F}$. Thus $\mathfrak{F}$ does not possess property $\mathbf{B}(r)$.

REMARK. We have proved the following somewhat more general statement: *The family $\mathfrak{F}$ constructed above is such that each set which intersects every element of $\mathfrak{F}_1$ has to intersect an element of $\mathfrak{F}_2^*$ in at least r points.*

Now we need some preliminary definitions.

DEF. (4.6) Let $\mathfrak{F}$ be an arbitrary family and let S be a set such that $(\mathfrak{F}) \subseteq S$. For an arbitrary subset X of S and for an arbitrary cardinal number t we define the subfamily $\mathcal{G}(X, t, \mathfrak{F})$ as follows:

$$\mathcal{G}(X, t, \mathfrak{F}) = \{F : F \in \mathfrak{F} \wedge \overline{\overline{F \cap X}} \geqq t\}.$$

DEF. (4.7) Let $\mathfrak{F}$ and S have the same meaning as in (4.6). For an arbitrary $X \subseteq S$ we define the family $\mathfrak{F}|X$ as follows:

$$\mathfrak{F}|X = \{F \cap X\}_{F \in \mathfrak{F}}.$$

(Note that $\mathfrak{F}|X$ is not necessarily a subfamily of $\mathfrak{F}$.)

The following assertions are immediate consequences of the above definitions.

(4.8.1) *Let q, r be arbitrary. The families $\mathcal{G}(X, t, \mathfrak{F})$ and $\mathfrak{F}|X$ possess property $\mathbf{C}(q, r)$, provided the same holds for $\mathfrak{F}$.*

(4.8.2) $|\mathcal{G}(X, t, \mathfrak{F})|X| \geqq t.$

(4.8.3) *If the family $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$ and $t \geqq r$, then*

$$\overline{\overline{\mathcal{G}(X, t, \mathfrak{F})}} \leqq q^- \cdot \overline{\overline{\mathcal{G}(X, t, \mathfrak{F})|X}}.$$

Now we prove the following positive theorem concerning A):

(*) (4.9) *Suppose $r < \aleph_\alpha$. Then $\mathbf{M}(\aleph_\alpha, \aleph_\alpha, r) \to \mathbf{B}(r^+)$, provided the following condition does not hold:*

(v) *There exist ordinal numbers β, γ such that $\alpha = \beta + 1$, $r = \aleph_\gamma$, $cf(\beta) = cf(\gamma)$ and $\gamma < \beta$.*

(If r is finite, the assumption (*) can be omitted.)

PROOF. Let $\mathfrak{F}$ be a family ($p(\mathfrak{F}) = \aleph_\alpha$, $\overline{\overline{\mathfrak{F}}} = \aleph_\alpha$) which possesses property $\mathbf{C}(2, r)$. Put $S = (\mathfrak{F})$. Then $\overline{\overline{S}} = \aleph_\alpha$. Let $S = \{x_\nu\}_{\nu < \omega_\alpha}$ and $\mathfrak{F} = \{F_\mu\}_{\mu < \omega_\alpha}$ be well-orderings of type ω_α of the set S and of the family $\mathfrak{F}$, respectively. We may suppose that $r^+ < \aleph_\alpha$, for if not, then $r^+ = \aleph_\alpha$ is regular and the theorem follows from (4. 3), since the symbol-II is decreasing in r by (3. 1). Now we define a subsequence $\{x_{\nu_\varrho}\}_{\varrho<\omega_\alpha}$ of S by induction on ϱ as follows:

Let x_{ν_0} be an arbitrary element of F_0 and put $F_0=F_{\mu_0}$. Suppose that the elements x_{ν_σ} are already defined for every $\sigma<\varrho$, for a $\varrho<\omega_\alpha$. Put

(4. 9. 1) $$S_\varrho=\{x_{\nu_\sigma}\}_{\sigma<\varrho},\quad \mathcal{G}_\varrho=\mathcal{G}(S_\varrho, r, \mathcal{F}).$$

It is obvious that $\overline{\overline{S_\varrho}}\leqq\bar{\varrho}<\aleph_\alpha$ and by (4. 8. 3) $\overline{\overline{\mathcal{G}_\varrho}}\leqq\overline{\overline{\mathcal{G}_\varrho|S_\varrho}}$.

But $\mathcal{G}_\varrho|S_\varrho\subseteq\mathcal{F}(S_\varrho)$, thus using (*) we get

(4. 9. 2) $$\overline{\overline{\mathcal{G}_\varrho}}<\aleph_\alpha \quad \text{except if} \quad \overline{\overline{S_\varrho}}^{+}=\bar{\varrho}^{+}=\aleph_\alpha.$$

Put $\bar{\varrho}=\aleph_\beta$ and suppose $\beta+1=\alpha$. Then $\overline{\overline{S_\varrho}}=\aleph_\beta$, $|\mathcal{G}_\varrho|S_\varrho|\geqq r$ (by 4. 8. 2). Using Lemmas 1 and 2 for the family $\mathcal{G}_\varrho|S_\varrho$ we get

(4. 9. 3) $$\overline{\overline{\mathcal{G}_\varrho}}\leqq\overline{\overline{\mathcal{G}_\varrho|S_\varrho}}\leqq\aleph_\beta<\aleph_\alpha \quad \text{except if} \quad r=\aleph_\gamma,\ cf(\beta)=cf(\gamma).$$

Thus it results from (4. 9. 2) and (4. 9. 3) and from the assumption that v) does not hold that $\overline{\overline{\mathcal{G}_\varrho}}<\aleph_\alpha$. Let μ_ϱ be the least ordinal number μ for which

$$S_\varrho\cap F_\mu=0.^6$$

It is obvious that $F_{\mu_\varrho}\notin\mathcal{G}_\varrho$, and so the set

$$F_{\mu_\varrho}-(\mathcal{G}_\varrho)=F_{\mu_\varrho}-\bigcup_{F_\mu\in\mathcal{G}_\varrho}(F_{\mu_\varrho}\cap F_\mu)$$

is of power $\aleph_\alpha$, since $\overline{\overline{F_{\mu_\varrho}}}=\aleph_\alpha$ and $\overline{\overline{\bigcup_{F_\mu\in\mathcal{G}_\varrho}(F_{\mu_\varrho}\cap F_\mu)}}\leqq r\cdot\bar{\varrho}<\aleph_\alpha$.

Thus there exists a ν such that $x_\nu\in F_{\mu_\varrho}-((\mathcal{G}_\varrho)\cup S_\varrho))$.

Let ν_ϱ be the least ν of this kind. Thus x_{ν_ϱ}, F_{μ_ϱ} are defined for every $\varrho<\omega_\alpha$ and it follows by induction on ϱ that

(4. 9. 4) $$x_{\mu_\varrho}\in F_{\mu_\varrho},\ x_{\nu_\varrho}\notin(\mathcal{G}_\varrho) \quad \text{and} \quad x_{\nu_\sigma}\neq x_{\nu_\varrho},\ \mu_\sigma\neq\mu_\varrho \quad \text{for every} \quad \sigma<\varrho<\omega_\alpha.$$

Put $B=\{x_{\nu_\varrho}\}_{\varrho<\omega_\alpha}$. Now we prove

(4. 9. 5) $$B\cap F_\mu\neq 0 \quad \textit{for every} \quad \mu<\omega_\alpha.$$

For if not, then there exists a least μ^0 of this kind, and by (4. 9. 4) there is a $\mu_\varrho>\mu^0$ in contradiction to the definition of F_{μ_ϱ}.

(4. 9. 6) $$\overline{\overline{B\cap F_\mu}}<r^+ \quad \textit{for every} \quad \mu<\omega_\alpha.$$

[6] If such a μ does not exist, then we stop with the construction and obviously one can prove in the same way that S_ϱ assures property $\mathbf{B}(r^+)$ as we shall prove it later for B.

For if not, then there exists a $\mu^0 < \omega_\alpha$, a subset $B' \subset B$ and an $x_{\nu_{\varrho_0}} \in B$ such that $\overline{\overline{B'}} = r$, $B' + \{x_{\nu_{\varrho_0}}\} \subseteq F_{\mu_0}$ and $\sigma < \varrho_0$ for every $x_{\nu_\sigma} \in B'$, and this obviously contradicts (4.9.4), since then $F_{\mu_0} \in G_{\varrho_0}$.

(4.9.5) and (4.9.6) just mean that the family $\mathfrak{F}$ possesses property $\mathbf{B}(r^+)$.

REMARK. As we have already mentioned in Problem 3a) — for a special case — we do not know whether $\mathbf{M}(\aleph_{\beta+1}, \aleph_{\beta+1}, \aleph_\gamma) \to \mathbf{B}(\aleph_{\gamma+1})$ is true or not if β and γ satisfy the assumption (v), i. e. if $cf(\beta) = cf(\gamma)$ and $\gamma < \beta$.

Now we need the following

(*) LEMMA 3. *Let S be a set, $\overline{\overline{S}} = \aleph_{\alpha+1}$, and suppose that $\aleph_\alpha$ is regular. Then there exists a system $\mathfrak{S}$ of subsets of S satisfying the following conditions:*

$p(\mathfrak{S}) = \aleph_\alpha$, $\mathfrak{S}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$ and for an arbitrary $S' \subseteq S$ $(\overline{\overline{S'}} = \aleph_{\alpha+1})$ there exists an $X \in \mathfrak{S}$ such that $X \subseteq S'$.

Lemma 3 is a theorem of A. HAJNAL.[7]

Now turning to the case B) we are going to prove that *if $\aleph_\alpha$ is singular and $cf(\alpha) < \beta \leqq \alpha$, then the trivial result $\mathbf{M}(\aleph_\beta, \aleph_\alpha, \aleph_\alpha) \to \mathbf{B}(\aleph_{\beta+1})$ (see (4.1)) is best-possible, i. e.*

(*) (4.10) $$\mathbf{M}(\aleph_\beta, \aleph_\alpha, \aleph_\alpha) \not\to \mathbf{B}(\aleph_\beta) \quad \text{if} \quad cf(\alpha) < \beta \leqq \alpha.$$

We are going to prove this only for the case $\beta = \alpha$, the proof can be carried out similarly in the other cases too.[8]

Proof of the case $\beta = \alpha$. Let $\mathfrak{F}_1$ be a family satisfying the following conditions:

(4.10.1) $\overline{\overline{\mathfrak{F}_1}} = \aleph_{cf(\alpha)+1}$, $p(\mathfrak{F}_1) = \aleph_\alpha$ and $F \cap F' = 0$ for every $F, F' \in \mathfrak{F}_1$, $F \neq F'$.

$\aleph_{cf(\alpha)}$ being regular, we can apply Lemma 3 with $\mathfrak{F}_1$ instead of S and we obtain that there exists a system $\mathfrak{S}$ of subfamilies $\mathfrak{X}$ of $\mathfrak{F}_1$ satisfying the following conditions:

(4.10.2) $\overline{\overline{\mathfrak{S}}} = \aleph_{cf(\alpha)+1}$, $p(\mathfrak{S}) = \aleph_{cf(\alpha)}$, $\mathfrak{S}$ possesses property $\mathbf{C}(2, \aleph_{cf(\alpha)})$ and if $\mathfrak{F}'$ is a subfamily of $\mathfrak{F}_1$ such that $\overline{\overline{\mathfrak{F}'}} = \aleph_{cf(\alpha)+1}$, then there exists an $\mathfrak{X} \in \mathfrak{S}$ for which $\mathfrak{X} \subseteq \mathfrak{F}'$.

Let $\mathfrak{S} = \{\mathfrak{X}_\mu\}_{\mu < \omega_{cf(\alpha)+1}}$ and $\mathfrak{F}_1 = \{F_\tau\}_{\tau < \omega_{cf(\alpha)+1}}$ be well-orderings of type $\omega_{cf(\alpha)+1}$ of $\mathfrak{S}$ and $\mathfrak{F}_1$, respectively. Let further $\{\alpha_\nu\}_{\nu < \omega_{cf(\alpha)}}$ be a monotone increasing sequence of type $\omega_{cf(\alpha)}$ of ordinal numbers less than α cofinal with α.

[7] See [6], Theorem 9.

[8] We mention that if $cf(\alpha) < cf(\beta)$ (especially, if β is of the first kind), then the theorem is easy and can be proved without using (*). But for the cases $cf(\beta) \geqq cf(\alpha)$ we have to use the same complicated proof as for the case $\beta = \alpha$. It is possible that a simpler proof can be constructed in this case too, but we were unsuccessful in doing this.

By (4.10.2) $\mathfrak{X}_\mu \subseteq \mathfrak{F}_1$ and $\overline{\overline{\mathfrak{X}_\mu}} = \aleph_{cf(\alpha)}$ for every $\mu < \omega_{cf(\alpha)+1}$. Let $\mathfrak{X}_\mu = \{F_\nu^\mu\}_{\nu < \omega_{cf(\alpha)}}$ be a well-ordering of type $\omega_{cf(\alpha)}$ of $\mathfrak{X}_\mu$.

The set F_ν^μ — being an element of $\mathfrak{F}_1$ — is of power $\aleph_\alpha$, and so it can be split into the sum of $\aleph_\alpha$ disjoint subsets of power $\aleph_{\alpha_\nu}$, that means: there exists a sequence $\{F_\nu^\mu(\gamma)\}_{\gamma<\omega_\alpha}$ of type ω_α of subsets of F_ν^μ satisfying the following conditions:

(4.10.3) $\overline{\overline{F_\nu^\mu(\gamma)}} = \aleph_{\alpha_\nu}$ for every $\gamma < \omega_\alpha$, $F_\nu^\mu(\gamma_1) \cap F_\nu^\mu(\gamma_2) = 0$ for every $\gamma_1, \gamma_2 < \omega_\alpha$, $\gamma_1 \neq \gamma_2$, and $F_\nu^\mu = \bigcup_{\gamma<\omega_\alpha} F_\nu^\mu(\gamma)$ where $\mu < \omega_{cf(\alpha)+1}$, $\nu < \omega_{cf(\alpha)}$ are arbitrary.

Now, corresponding to every $\mu < \omega_{cf(\alpha)+1}$ we define a family $\mathfrak{F}_{2,\mu}$ as follows. First put $F^\mu(\gamma) = \bigcup_{\nu<\omega_{cf(\alpha)}} F_\nu^\mu(\gamma)$, and then put $\mathfrak{F}_{2,\mu} = \{F^\mu(\gamma)\}_{\gamma<\omega_\alpha}$.

We have, for every $\mu < \omega_{cf(\alpha)+1}$,

(4.10.4) $\overline{\overline{\mathfrak{F}_{2,\mu}}} = \aleph_\alpha$, $F^\mu(\gamma_1) \cap F^\mu(\gamma_2) = 0$ for $\gamma_1, \gamma_2 < \omega_\alpha, \gamma_1 \neq \gamma_2$, $p(\mathfrak{F}_{2,\mu}) = \aleph_\alpha$ and $(\mathfrak{F}_{2,\mu}) = (\mathfrak{X}_\mu)$.

In fact, the second statement follows from (4.10.1) and (4.10.3), the first one is a corollary of it, while the third and the fourth ones are consequences of (4.10.3), since

$$\overline{\overline{F^\mu(\gamma)}} = \sum_{\nu<\omega_{cf(\alpha)}} \overline{\overline{F_\nu^\mu(\gamma)}} = \sum_{\nu<\omega_{cf(\alpha)}} \aleph_{\alpha_\nu} = \aleph_\alpha \text{ for every } \mu < \omega_{cf(\alpha)+1}, \gamma < \omega_\alpha.$$

Now we put

$$\mathfrak{F} = \mathfrak{F}_1 \cup \bigcup_{\mu<\omega_{cf(\alpha)+1}} \mathfrak{F}_{2,\mu}.$$

We have

$$p(\mathfrak{F}) = \aleph_\alpha, \tag{4.10.5}$$

since $p(\mathfrak{F}_1) = p(\mathfrak{F}_{2,\mu}) = \aleph_\alpha$ for $\mu < \omega_{cf(\alpha)+1}$ by (4.10.1) and (4.10.4).

Taking into consideration that $\aleph_\alpha$ is singular and therefore $cf(\alpha)+1 < \alpha$, we get from (4.10.1) and (4.10.4)

$$\overline{\overline{\mathfrak{F}}} = \overline{\overline{\mathfrak{F}_1}} + \sum_{\mu<\omega_{cf(\alpha)+1}} \overline{\overline{\mathfrak{F}_{2,\mu}}} = \aleph_{cf(\alpha)+1} + \aleph_{cf(\alpha)+1} \cdot \aleph_\alpha = \aleph_\alpha. \tag{4.10.6}$$

We are going to prove that

(4.10.7) $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$.

Let F, F' be two elements of $\mathfrak{F}$ such that $F \neq F'$.

To see that $\overline{\overline{F \cap F'}} < \aleph_\alpha$, we distinguish four cases: (i) $F, F' \in \mathfrak{F}_1$, (ii) $F \in \mathfrak{F}_1$, $F' \in \mathfrak{F}_{2,\mu}$, (iii) $F \in \mathfrak{F}_{2,\mu}$, $F' \in \mathfrak{F}_{2,\mu}$, (iiii) $F \in \mathfrak{F}_{2,\mu}$, $F' \in \mathfrak{F}_{2,\mu'}$ for some $\mu \neq \mu'$.

In the cases (i) and (iii) F and F' are disjoint by (4.10.1) and (4.10.4), respectively. If (ii) holds, then — by (4.10.3) — either $F \cap F' = 0$ if $F \notin \mathfrak{X}_\mu$, or if $F \in \mathfrak{X}_\mu$, then $F = F^\mu_\nu$ for a $\nu < \omega_{cf(\alpha)}$ and $F' = F^\mu(\gamma)$ for a $\gamma < \omega_\alpha$ and $\overline{\overline{F \cap F'}} = \overline{\overline{F^\mu_\nu(\gamma)}} = \aleph_{\alpha_\nu} < \aleph_\alpha$.

Suppose now that (iiii) holds. By (4. 10. 2) we have $\overline{\overline{\mathfrak{X}_\mu \cap \mathfrak{X}_{\mu'}}} < \aleph_{cf(\alpha)}$, since $\mu \neq \mu'$. It is obvious that $(\mathfrak{F}) = \bigcup_{\tau < \omega_{cf(\alpha)+1}} F_\tau$, and so $F \cap F' = \bigcup_{\tau < \omega_{cf(\alpha)+1}} (F_\tau \cap (F \cap F'))$ but either $F_\tau \cap F$ or $F_\tau \cap F'$ is empty if $F_\tau \notin \mathfrak{X}_\mu \cap \mathfrak{X}_{\mu'}$, hence

$$F \cap F' = \bigcup_{F_\tau \in \mathfrak{X}_\mu \cap \mathfrak{X}_{\mu'}} (F_\tau \cap (F \cap F')).$$

Taking into consideration that by (4. 10. 3) $\overline{\overline{F_\tau \cap F}} < \aleph_\alpha$, it results that $\overline{\overline{F \cap F'}} < \aleph_\alpha$ in this case too.

It remains to prove that

(4. 10. 8) $\mathfrak{F}$ does not possess property $\mathbf{B}(\aleph_\alpha)$.

Let B be a set such that $\overline{\overline{B \cap F}} < \aleph_\alpha$ for every $F \in \mathfrak{F}$. Then, especially, corresponding to every $\tau < \omega_{cf(\alpha)+1}$ there exists a subscript $\nu(\tau) < \omega_{cf(\alpha)}$ such that

$$\overline{\overline{B \cap F_\tau}} < \aleph_{\alpha_{\nu(\tau)}}.$$

It results that there exists a subfamily $\mathfrak{F}'$ of $\mathfrak{F}_1$ and an ordinal number $\nu_0 < \omega_{cf(\alpha)}$ such that $\overline{\overline{\mathfrak{F}'}} = \aleph_{cf(\alpha)+1}$ and $\overline{\overline{B \cap F_\tau}} < \aleph_{\alpha_{\nu_0}}$ for every $F_\tau \in \mathfrak{F}'$. But, by (4. 10. 2), then there exists a $\mu_0 < \omega_{cf(\alpha)+1}$ such that $\mathfrak{X}_{\mu_0} \subseteq \mathfrak{F}'$. Thus we have $\overline{\overline{F^{\mu_0}_\nu \cap B}} < \aleph_{\alpha_{\nu_0}}$ for every $\nu < \omega_{cf(\alpha)}$, and so $\overline{\overline{B \cap (\mathfrak{X}_{\mu_0})}} \leqq \aleph_{cf(\alpha)} \cdot \aleph_{\alpha_{\nu_0}} < \aleph_\alpha$. But by (4. 10.4) $\mathfrak{F}_{2,\mu_0}$ consists of $\aleph_\alpha$ disjoint subsets of $(\mathfrak{X}_{\mu_0})$, consequently there is an $F \in \mathfrak{F}_{2,\mu_0} \subseteq \mathfrak{F}$ such that

$$B \cap F = 0.$$

Thus by (4. 10. 5)—(4. 10.8) the case $\beta = \alpha$ of (4. 10) is proved.

From (4. 3) and (4. 10) we obtain the following

(*) COROLLARY. *Suppose p is infinite. Then $\mathbf{M}(p, p, p) \to \mathbf{B}(p)$ holds if and only if p is a regular cardinal number.*

This should be compared with BERNSTEIN's theorem cited as Theorem 2 in Section 3.

REMARK. After (3. 1) we have stated without proof that $\mathbf{M}(m, p, r) \to \mathbf{B}(s)$ is not monotone increasing in p. This can be seen e. g. by the following examples:

$\mathbf{M}(\aleph_1, \aleph_0, \aleph_2) \to \mathbf{B}(\aleph_1)$ holds by (3. 2) but
$\mathbf{M}(\aleph_1, \aleph_1, \aleph_2) \not\to \mathbf{B}(\aleph_1)$ by (4. 1); or
$\mathbf{M}(\aleph_2, \aleph_0, \aleph_1) \to \mathbf{B}(\aleph_1)$ by (3. 2) but
(*) $\mathbf{M}(\aleph_2, \aleph_1, \aleph_1) \not\to \mathbf{B}(\aleph_1)$ by Theorem 3 and (3. 2) and
(*) $\mathbf{M}(\aleph_2, \aleph_2, \aleph_1) \not\to \mathbf{B}(\aleph_1)$ by (4. 5).

However, every example which disproves the monotonicity in question is such that $s>p$. Under the condition $s\leqq p$ — and these are the only genuine cases — the monotonicity seems to hold. Suppose namely that $\mathbf{M}(m,p,r)\to\mathbf{B}(s)$ is true, $s\leqq p$, and for the sake of simplicity suppose further that $m,p,r,s\geqq\aleph_0$, and suppose (*).

Distinguish three cases: (i) $p<r$, (ii) $p=r$, (iii) $p>r$. If (i) holds, then by (3. 1) and (4. 1) $m^+\leqq s$, hence again by (3. 1) and (4. 1) $\mathbf{M}(m,p',r)\to\mathbf{B}(s)$ is true for every $p'>p$.

If (ii) holds, then $m\leqq p$ by Theorem 3 and by (3. 2), and so $p'>m,r$ for every $p'>p$, hence $\mathbf{M}(m,p',r)\to\mathbf{B}(s)$ is true by (4. 2).

If (iii) holds, then the implication is again trivial if $m\leqq p$, and if $m>p$, then by Theorem 6 which will be proved in Section 6 $\mathbf{M}(m,p',r)\to\mathbf{B}(p^+)$ is true for every $p'>p$.

By a slight modification of the proof of Theorem 6 one can obtain the following theorem:

(*) *If $\mathfrak{F}$ is a family, $p(\mathfrak{F})=p'$, $\overline{\overline{\mathfrak{F}}}=m$ and $\mathfrak{F}$ possesses property* $\mathbf{C}(2,r)$, *then there exists a set B such that $\overline{\overline{B\cap F}}=p$ for every $F\in\mathfrak{F}$, provided that the above-mentioned inequalities hold for the cardinal numbers in question.*

Put $\mathfrak{F}'=\{B\cap F\}_{F\in\mathfrak{F}}$. It is obvious that $\mathfrak{F}$ possesses property $\mathbf{B}(s)$, provided the same holds for $\mathfrak{F}'$, but $p(\mathfrak{F}')=p$, $\overline{\overline{\mathfrak{F}'}}\leqq m$ and $\mathfrak{F}'$ possesses property $\mathbf{C}(2,r)$, hence $\mathbf{M}(m,p,r)\to\mathbf{B}(s)$ implies $\mathbf{M}(m,p',r)\to\mathbf{B}(s)$ in this case too.

It is possible that one can find a simpler proof for the monotonicity which does not use the hypothesis (*), but we were unsuccessful in doing this.

5. Generalization of Miller's inductive construction. Let $(\mathfrak{F})$ be a family and S a set, $(\mathfrak{F})\subseteq S$.

DEF. (5. 1) Let $\mathfrak{F}'$ be a subfamily of $\mathfrak{F}$ and put $S'=(\mathfrak{F}')$. $\mathfrak{F}'$ is said to be closed in $\mathfrak{F}$ with respect to the cardinal number t (or briefly t-closed in $\mathfrak{F}$) if $F\in\mathfrak{F}$ and $\overline{\overline{F\cap S'}}\geqq t$ implies that $F\in\mathfrak{F}'$.

It is obvious that if $\mathfrak{F}'$ is an arbitrary subfamily of $\mathfrak{F}$, then (the intersection of any number of t-closed subfamilies being t-closed) for every t there exists a minimal t-closed subfamily of $\mathfrak{F}$ containing $\mathfrak{F}'$. However, we need concrete constructions for t-closed families containing $\mathfrak{F}'$.

DEF. (5. 2) We define the t, ε closure of $\mathfrak{F}'$ in $\mathfrak{F}$: $\mathrm{Clos}(\mathfrak{F}',\mathfrak{F},t,\varepsilon)$ for every ε. First we define a sequence $\{\mathcal{G}_\nu\}_{\nu<\omega_\varepsilon}$ of type ω_ε of subfamilies of $\mathfrak{F}$ by induction on ν as follows:

Put $\mathcal{G}_0=\mathfrak{F}'$ and $S_0=(\mathcal{G}_0)$. Suppose that the families $\mathcal{G}_\mu$ as well as the sets S_μ are already defined for every $\mu<\nu$ for a $\nu<\omega_\varepsilon$.

Put $S_\nu^* = \bigcup_{\mu<\nu} S_\mu$, $\mathcal{G}_\nu = \mathcal{G}(S_\nu^*, t, \mathcal{F})$ (where $\mathcal{G}$ is the function defined in (4.6)), and $S_\nu = (\mathcal{G}_\nu)$.

Thus $\mathcal{G}_\nu$ is defined for every $\nu < \omega_\varepsilon$. Now we put

$$\mathrm{Clos}\,(\mathcal{F}', \mathcal{F}, t, \varepsilon) = \bigcup_{\nu<\omega_\varepsilon} \mathcal{G}_\nu.$$

As an immediate consequence of the definition we get that

$$(\mathrm{Clos}\,(\mathcal{F}', \mathcal{F}, t, \varepsilon)) = \bigcup_{\nu<\omega_\varepsilon} S_\nu \quad \text{and} \quad \mathcal{F}' \subseteq \mathrm{Clos}\,(\mathcal{F}', \mathcal{F}, t, \varepsilon).$$

We have:

(5.3) Clos $(\mathcal{F}', \mathcal{F}, t, \varepsilon)$ *is t-closed for every* $t < \aleph_{cf(\varepsilon)}$.[9]

PROOF. Let F be an element of $\mathcal{F}$ such that $\overline{\overline{F \cap (\mathrm{Clos}\,(\mathcal{F}', \mathcal{F}, t, \varepsilon))}} \geqq t$. Then $\overline{\overline{F \cap S_{\nu_0}^*}} \geqq t$ for a suitable $\nu_0 < \omega_\varepsilon$ and thus $F \in \mathcal{G}_{\nu_0} \subseteq \mathrm{Clos}\,(\mathcal{F}', \mathcal{F}, t, \varepsilon)$.

In what follows in this section let $\mathcal{F}$ be a fixed family, $(\mathcal{F}) = S$. Suppose that $p(\mathcal{F}) = p$, $\overline{\overline{\mathcal{F}}} = m$, $\mathcal{F}$ possesses property $\mathbf{C}(q, r)$, where the cardinal numbers m, p, q, r, s and t satisfy the following inequalities:

(°) $\qquad m > p,\ p \geqq \aleph_0,\ 2 \leqq q \leqq p^+,\ r < p,\ r^+ \leqq s \leqq p,\ r \leqq t < p.$

Every statement proved in this section depends on the assumption (°). We are going to use the notations $p = \aleph_\alpha$, $m = \aleph_\beta$, $r = \aleph_\gamma$, $s = \aleph_\delta$ alternatively (provided r and s are infinite).

DEF. (5.4) Let $\varepsilon(t)$ denote the index of the least $\aleph$ greater than t. ($\varepsilon(t) = 0$ if t is finite and $\aleph_{\varepsilon(t)} = t^+$ if t is infinite.) This means that $\aleph_{\varepsilon(t)}$ is always regular. Put briefly Clos $(\mathcal{F}', t)$ for Clos $(\mathcal{F}', \mathcal{F}, t, \varepsilon(t))$.

We need the following

(*) LEMMA 4. *Let* $\mathcal{F}'$ *be a subfamily of* $\mathcal{F}$, $\overline{\overline{\mathcal{F}'}} = m' \geqq p$. *Then* $\overline{\overline{\mathrm{Clos}(\mathcal{F}', t)}} = m'$, *provided one of the following conditions* (α) *and* ($\alpha\alpha$) *holds:*

(α) $r = t$ *and the following condition does not hold*:

(vv) *There exist ordinal numbers* β' *and* γ *such that* $m' = \aleph_{\beta'}$, $r = \aleph_\gamma$ *and* $cf(\beta') = cf(\gamma)$.

($\alpha\alpha$) $r < t$.

(Note that in case r is finite, the hypothesis (*) can be omitted.)

PROOF. Let $\mathcal{G}_\nu$ denote the families defined in (5.2) corresponding to the given $\mathcal{F}', t$ and $\varepsilon(t)$. First we are going to prove by induction on ν that

(1) $\overline{\overline{\mathcal{G}_\nu}} = m'$ and $\overline{\overline{S_\nu}} \leqq m'$ for every $\nu < \omega_{\varepsilon(t)}$.

[9] It would be easy to see that (5.3) holds under more general conditions too, but we do not need this. E. g., it is true that for every t either Clos $(\mathcal{F}', \mathcal{F}, t, 0)$ or Clos $(\mathcal{F}', \mathcal{F}, t, 1)$ is t-closed.

This is true for $\nu=0$, since $\overline{\overline{\mathcal{G}_0}}=\overline{\overline{\mathcal{F}'}}=m'$ by the assumption and $\overline{\overline{S_0}}=\overline{\overline{(\mathcal{G}_0)}}\leqq p\cdot m'=m'$. Suppose that the theorem is proved for every $\mu<\nu$ where $\nu<\omega_{\varepsilon(t)}$.

Then $\overline{\overline{S_\nu^*}}\leqq\sum_{\mu<\nu}m'=m'\overline{\nu}$. But by (5.4) $\overline{\nu}<\aleph_0\cdot t$ and therefore $\overline{\overline{S_\nu^*}}\leqq m'\cdot\aleph_0\cdot t=m'$, since $t<p\leqq m'$.

Now we obtain from (4.8.1), (4.8.2), (4.8.3) and (5.2) that $\overline{\overline{\mathcal{G}_\nu}}\leqq q^-\cdot\overline{\overline{\mathcal{G}_\nu|S_\nu^*}}$, $\mathcal{G}_\nu|S_\nu^*$ possesses property $\mathbf{C}(q,r)$ and $|\mathcal{G}_\nu|S_\nu^*|\geqq t$. On the other hand, we have $q\leqq(m')^+$, since $q\leqq p^+$ and $p\leqq m'$.

Hence by Lemmas 1 and 2 each of the conditions (α) and ($\alpha\alpha$) implies that $\overline{\overline{\mathcal{G}_\nu|S_\nu^*}}\leqq m'$. Consequently, we have $\overline{\overline{\mathcal{G}_\nu}}\leqq p\cdot m'=m'$, since $q^-\leqq p$ if $q\leqq p^+$. Thus $\overline{\overline{\mathcal{G}_\nu}}=m'$, since $\mathcal{G}_\nu$ contains $\mathcal{G}_0$, and similarly as for the case $\nu=0$ we obtain that $\overline{\overline{S_\nu}}\leqq m'$, and (1) is proved.

Using that $t<p$, $p\geqq\aleph_0$ implies $\aleph_{\varepsilon(t)}\leqq p$, we get from (1)

$$m'\leqq\overline{\overline{\mathrm{Clos}\,(\mathcal{F}',t)}}\leqq\sum_{\nu<\omega_{\varepsilon(t)}}m'\leqq m'\cdot p=m'$$

and Lemma 4 is proved.

Let now $\mathcal{F}=\{F_\varrho\}_{\varrho<\omega_\beta}$ be a well-ordering of type ω_β of the family $\mathcal{F}$.

Now we are going to define the sequences $\{\mathcal{F}'_\sigma(t)\}_{\sigma<\varphi}$, $\{\mathcal{F}_\sigma(t)\}_{\sigma<\varphi}$ of type φ of subfamilies of $\mathcal{F}$ as well as the sequence $\{S_\sigma(t)\}_{\sigma<\varphi}$ of subsets of S for a $\varphi\leqq\omega_\beta$ by induction on σ as follows:

DEF. (5.5) Put $\mathcal{F}'_0(t)=\{F_\varrho\}_{\varrho<\omega_\alpha}$, $\mathcal{F}_0(t)=\mathrm{Clos}\,(\mathcal{F}'_0(t),t)$, $S_0(t)=(\mathcal{F}_0(t))$. Suppose that the families $\mathcal{F}'_{\sigma'}(t)$, $\mathcal{F}_{\sigma'}(t)$ and the sets $S_{\sigma'}(t)$ are already defined for every $\sigma'<\sigma$. Put

$$\mathcal{F}^*_\sigma(t)=\bigcup_{\sigma'<\sigma}\mathcal{F}_{\sigma'}(t),\quad S^*_\sigma(t)=\bigcup_{\sigma'<\sigma}S_{\sigma'}(t).$$

If there exists an index $\varrho<\omega_\beta$ such that $F_\varrho\notin\mathcal{F}^*_\sigma(t)$, then put $\varrho_\sigma=\varrho$ for the least ϱ of this kind, if not, then put $\sigma=\varphi$.

If ϱ_σ exists, then put

$$\mathcal{F}'_\sigma(t)=\mathcal{F}^*_\sigma(t)\cup\{\mathcal{F}_{\varrho_\sigma}\},\quad \mathcal{F}_\sigma(t)=\mathrm{Clos}\,(\mathcal{F}'_\sigma(t),t),\quad S_\sigma(t)=(\mathcal{F}_\sigma(t)).$$

Finally, if ϱ_σ is defined for every $\sigma<\omega_\beta$, then put $\varphi=\omega_\beta$.

(5.6) As an immediate consequence of the definition we obtain the following results:

(5.6.1) $$\mathcal{F}=\bigcup_{\sigma<\varphi}\mathcal{F}_\sigma(t),$$

(5.6.2) $$\mathcal{F}^*_{\sigma'}(t)\subset\mathcal{F}'_{\sigma'}(t)\subseteq\mathcal{F}_{\sigma'}(t)\subseteq\mathcal{F}^*_\sigma(t)\quad \textit{for every}\quad \sigma'<\sigma<\varphi,$$

(5.6.3) $$S_{\sigma'}(t)\subseteq S^*_\sigma(t)\subseteq S_\sigma(t)\quad \textit{for every}\quad \sigma'<\sigma<\varphi,$$

(5.6.4) $\mathcal{F}_\sigma(t)$ *is t-closed in* $\mathcal{F}$ *for every* $\sigma<\varphi$ *by* (5.3) *and* (5.4).

DEF. (5. 7) Put $\mathcal{H}_\sigma(t) = \mathcal{F}_\sigma(t) - \mathcal{F}^*_\sigma(t)$ for every $\sigma < \varphi$.

By (5. 5) and (5. 6. 1) we have

(5. 7. 1) $$\mathcal{F} = \bigcup_{\sigma<\varphi} \mathcal{H}_\sigma(t),$$

and by (5. 6. 2)

(5. 7. 2) $$\mathcal{H}_\sigma(t) \cap \mathcal{H}_{\sigma'}(t) = 0 \quad \text{for every} \quad \sigma' < \sigma < \varphi.$$

Now we prove the following lemma:

(5. 8) *Suppose that* $F_\varrho \in \mathcal{H}_\sigma(t)$ *for some* $\varrho < \omega_\beta$, $\sigma < \varphi$. *Then*

(β) $\overline{\overline{F_\varrho \cap S^*_\sigma(t)}} \leqq t$, *and if* t *is finite, then*

($\beta\beta$) $\overline{\overline{F_\varrho \cap S^*_\sigma(t)}} < t$.

PROOF. First of all — $\mathcal{F}_{\sigma'}(t)$ being t-closed by (5. 6. 4) — we may suppose $\overline{\overline{F_\varrho \cap S_{\sigma'}(t)}} < t$ for every $\sigma' < \sigma$, for if not, then by the definition (5. 1) F_ϱ belongs to $\mathcal{F}_{\sigma'}(t)$ in contradiction to (5. 7).

We distinguish two cases: (i) $\sigma = \sigma_1 + 1$ for a $\sigma_1 < \sigma$, (ii) σ is of the second kind.

(i) By (5. 5) and (5. 6. 3) we have $S^*_\sigma(t) = S_{\sigma_1}(t)$, hence ($\beta\beta$) holds for every t.

(ii) Let ω_τ be the least ordinal number cofinal with σ and let $\{\sigma_\eta\}_{\eta<\omega_\tau}$ be a monotone increasing sequence of ordinal numbers less than σ of type ω_τ cofinal with σ. We distinguish again two cases: (j) $\aleph_\tau \leqq t$, (jj) $\aleph_\tau > t$.

(j) We have by (5. 5) and (5. 6. 3)

$$S^*_\sigma(t) = \bigcup_{\sigma'<\sigma} S_{\sigma'}(t) = \bigcup_{\eta<\omega_\tau} S_{\sigma_\eta}(t).$$

Hence $\overline{\overline{F_\varrho \cap S^*_\sigma(t)}} \leqq \sum_{\eta<\omega_\tau} \overline{\overline{S_{\sigma_\eta} \cap F_\varrho}} \leqq t \cdot \aleph_\tau = t$ and thus (β) holds.

(jj) Using again $S^*_\sigma(t) = \bigcup_{\eta<\omega_\tau} S_{\sigma_\eta}(t)$, we obtain that ($\beta\beta$) holds, for if not, then $F_\varrho \cap S^*_\sigma(t)$ contains a subset of power t which — $\aleph_\tau$ being regular — is contained already in a set $S_{\sigma_{\eta_0}}$ for an $\eta_0 < \tau$.

If t is finite, then either (i) or (jj) holds for it, and therefore if t is 'nite, then ($\beta\beta$) is true.

DEF. (5. 9) By (5. 7. 1) and (5. 7. 2) corresponding to every $\varrho < \omega_\beta$ there exists exactly one $\sigma < \varphi$ such that $F_\varrho \in \mathcal{H}_\sigma(t)$. Put $\tilde{F}_\varrho = F_\varrho - S^*_\sigma(t)$ for this σ and put further $\tilde{\mathcal{H}}_\sigma(t) = \{\tilde{F}_\varrho\}_{F_\varrho \in \mathcal{H}_\sigma(t)}$. Put finally $\tilde{S}_\sigma(t) = (\tilde{\mathcal{H}}_\sigma(t))$.

We need the following results:

It results from the assumption $p(\mathcal{F}) = p > t$ by (5.8) and (5. 9) that

(5. 10. 1) $p(\tilde{\mathcal{H}}_\sigma(t)) = p$ *for every* $\sigma < \varphi$, and it is obvious from (5. 9) that

(5. 10. 2) *the family* $\tilde{\mathcal{H}}_\sigma(t)$ *possesses property* $\mathbf{C}(q, r)$ *for every* $\sigma < \varphi$.

(5.10.3) *Suppose* $F_\varrho \in \mathcal{H}_\sigma(t)$. *Then*

(γ) $\overline{\overline{F_\varrho \cap \bigcup_{\sigma'<\sigma} \tilde{S}_{\sigma'}(t)}} \leqq t$ *and the equality is excluded if* t *is finite, and*

($\gamma\gamma$) $F_\varrho \cap \tilde{S}_{\sigma''}(t) = 0$ *for every* $\sigma'' > \sigma$.

PROOF. (γ) By the definitions (5.7) and (5.9) $\tilde{S}_{\sigma'}(t) \subseteq S_{\sigma'}(t) \subseteq S^*_\sigma(t)$ for every $\sigma' < \sigma$, hence by (5.8) we get $\overline{\overline{F_\varrho \cap \bigcup_{\sigma'<\sigma} \tilde{S}_{\sigma'}(t)}} \leqq \overline{\overline{F_\varrho \cap S^*_\sigma(t)}} \leqq t$ (or $< t$ if t is finite).

($\gamma\gamma$) It is enough to see that $F_\varrho \cap \tilde{F}_{\varrho''} = 0$ for every $\tilde{F}_{\varrho''} \in \tilde{\mathcal{H}}_{\sigma''}(t)$. But $F_\varrho \subseteq S_\sigma(t) \subseteq S^*_{\sigma''}(t)$, and so by (5.9) $\tilde{F}_{\varrho''} \cap F_\varrho \subseteq \tilde{F}_{\varrho''} \cap S^*_{\sigma''} = 0$.

Now we prove the following

LEMMA 5. *Suppose that the families* $\tilde{\mathcal{H}}_\sigma(t)$ *possess property* $\mathbf{B}(s)$ *for every* $\sigma < \varphi$. *Then the family* $\mathcal{F}$ *possesses property* $\mathbf{B}(t^+ + s)$, *and if* t *is finite, then it possesses property* $\mathbf{B}((t-1)+s)$ *too.*

PROOF. By the assumption for every $\sigma < \varphi$ there exists a set B_σ such that $B_\sigma \subseteq \tilde{S}_\sigma(t)$ and $1 \leqq \overline{\overline{B_\sigma \cap \tilde{F}_\varrho}} < s$ for every $\tilde{F}_\varrho \in \tilde{\mathcal{H}}_\sigma(t)$.

Put $B = \bigcup_{\sigma<\varphi} B_\sigma$. By (5.7.1) for every $\varrho < \omega_\beta$ there exists a $\sigma < \varphi$ such that $F_\varrho \in \mathcal{H}_\sigma(t)$. Then $\tilde{F}_\varrho \in \tilde{\mathcal{H}}_\sigma(t)$, $\tilde{F}_\varrho \subseteq F_\varrho$, by (5.9), and B_σ intersects $\tilde{F}_\varrho$ by the assumption, hence we get

$$(1) \qquad B \cap F_\varrho \neq 0 \quad \text{for every} \quad \varrho < \omega_\beta.$$

Now we are going to prove that

$$(2) \qquad \overline{\overline{B \cap F_\varrho}} < t^+ + s \quad \text{for every} \quad \varrho < \omega_\beta.$$

Let now σ_ϱ be the uniquely determined ordinal number for which $F_\varrho \in \mathcal{H}_{\sigma_\varrho}(t)$. By the definition of B we have

$$(\mathrm{x}) \qquad \overline{\overline{B \cap F_\varrho}} \leqq \overline{\overline{\bigcup_{\sigma<\sigma_\varrho} (B_\sigma \cap F_\varrho)}} + \overline{\overline{B_{\sigma_\varrho} \cap F_\varrho}} + \overline{\overline{\bigcup_{\sigma>\sigma_\varrho} (B_\sigma \cap F_\varrho)}}.$$

Taking into consideration that $B_\sigma \subseteq \tilde{S}_\sigma(t)$, we obtain from (5.10.3) that $\overline{\overline{\bigcup_{\sigma<\sigma_\varrho} (B_\sigma \cap F_\varrho)}} \leqq \overline{\overline{\bigcup_{\sigma<\sigma_\varrho} (\tilde{S}_\sigma(t) \cap F_\varrho)}} \leqq t$ and $\overline{\overline{\bigcup_{\sigma>\sigma_\varrho} (B_\sigma \cap F_\varrho)}} = 0$. On the other hand, it results from (5.9) that $\tilde{S}_{\sigma_\varrho}(t) \cap F_\varrho = \tilde{F}_\varrho$ for every $F_\varrho \in \mathcal{H}_{\sigma_\varrho}(t)$, hence $\overline{\overline{B \cap F_\varrho}} = \overline{\overline{B_{\sigma_\varrho} \cap \tilde{F}_\varrho}} < s$. It follows that $\overline{\overline{B \cap F_\varrho}} < t^+ + s$ for every $\varrho < \omega_\beta$. (1) and (2) mean that $\mathcal{F}$ possesses property $\mathbf{B}(t^+ + s)$. Suppose now that t is finite. The formula (x) holds in this case too. We get from (5.10.3) that the first cardinal number on the right-hand side is less than t and the third one is 0, while the second is by the assumption less than s in this case too. Now if s is infinite, then the sum is less than s, hence less than $(t-1)+s$. If s is finite, then the first summand being less than t is at most $t-1$, hence the sum is less than $(t-1)+s$ in this case too. It results from (1) that if t is finite, then $\mathfrak{F}$ possesses property $\mathbf{B}((t-1)+s)$.

LEMMA 6. *The family $\mathfrak{F}$ possesses property $\mathbf{B}$, provided the same holds for the families $\tilde{\mathfrak{H}}_\sigma(t)$ for every $\sigma<\varphi$.*

PROOF. Lemma 6 is to be seen quite similarly to Lemma 5. Let $B_\sigma \subseteq \tilde{S}_\sigma(t)$ denote the sets satisfying the condition $B_\sigma \cap \tilde{F}_\varrho \neq 0$, $\tilde{F}_\varrho \subsetneqq B_\sigma$ for every $\tilde{F}_\varrho \in \tilde{\mathfrak{H}}_\sigma(t)$. Put $B=\bigcup_{\sigma<\varphi} B_\sigma$. The proof of the fact that B intersects every F_ϱ is the same as in Lemma 5. Let σ_ϱ denote, as before, the uniquely determined σ for which $\tilde{F}_\varrho \in \tilde{\mathfrak{H}}_{\sigma_\varrho}(t)$. It results from the definition (5. 9) and from (5. 10. 3) that $B \cap \tilde{F}_\varrho = B_{\sigma_\varrho} \cap \tilde{F}_\varrho$, hence $\tilde{F}_\varrho \neq B \cap \tilde{F}_\varrho$, since $\tilde{F}_\varrho \subsetneqq B_{\sigma_\varrho}$, and thus $\tilde{F}_\varrho \subsetneqq B$, therefore $F_\varrho \subsetneqq B$ for every $\varrho<\varphi$.

For the sake of brevity we introduce the following notations:

DEF. (5. 11) The cardinal number m is said to possess property $\mathbf{T}(p, r)$ if there exists an m' $(p \leq m' < m)$ such that m' satisfies the formula (vv) of Lemma 4, i. e. if there exist ordinal numbers β' and γ such that

$$m'=\aleph_{\beta'}, \quad r=\aleph_\gamma \quad \text{and} \quad cf(\beta')=cf(\gamma).$$

Quite similarly, p is said to possess property $\mathbf{Q}(r)$ if p satisfies the formula (v) of (4. 9), i. e. if there exist ordinal numbers α_1 and γ such that

$$p=\aleph_\alpha=\aleph_{\alpha_1+1}, \quad r=\aleph_\gamma, \quad cf(\alpha_1)=cf(\gamma) \quad \text{and} \quad \gamma<\alpha_1 .$$

Now we are going to prove

(*) LEMMA 7. *$p(\tilde{\mathfrak{H}}_\sigma(t))=p$, the families $\mathfrak{H}_\sigma(t)$ possess property $\mathbf{C}(q, r)$ and $\overline{\overline{\tilde{\mathfrak{H}}_\sigma(t)}}<m$ for every $\sigma<\varphi$, provided one of the conditions (δ) and $(\delta\delta)$ holds:*

(δ) $r=t$ and m does not possess property $\mathbf{T}(p, r)$.

$(\delta\delta)$ $r<t$.

(If t is finite, the hypothesis (*) is not used.)

PROOF. The first two statements were proved in (5. 10. 1) and (5. 10. 2). We have to prove the third one. It is obvious from the definitions (5. 7) and (5. 9) that $\overline{\overline{\tilde{\mathfrak{H}}_\sigma(t)}} \leq \overline{\overline{\mathfrak{H}_\sigma(t)}} \leq \overline{\overline{\mathfrak{F}_\sigma(t)}}$. We prove by induction on σ that $\overline{\overline{\mathfrak{F}_\sigma(t)}} \leq \leq \overline{p \cdot \sigma+1}<m$ for every $\sigma<\varphi$.

By the definition (5. 5) $\overline{\overline{\mathfrak{F}_0'(t)}}=\aleph_\alpha=p$ and, since by the assumption either $r<t$ or m possesses property $\mathbf{T}(p, r)$, by Lemma 4 $\overline{\overline{\mathfrak{F}_0(t)}}= =\overline{\overline{\operatorname{Clos}(\mathfrak{F}_0'(t), t)}}=p$.

Suppose that we have $\overline{\overline{\mathfrak{F}_{\sigma'}(t)}} \leqq p \cdot \overline{\sigma' + 1}$ for every $\sigma' < \sigma$ for a $0 < \sigma < \varphi$. Then by (5.5)

$$\overline{\overline{\mathfrak{F}_\sigma^*}} \leqq \sum_{\sigma' < \sigma} \overline{\overline{\mathfrak{F}_{\sigma'}(t)}} \leqq \sum_{\sigma' < \sigma} p \cdot \overline{\sigma' + 1} = p \cdot \overline{\sigma}.$$

Now $\overline{\overline{\mathfrak{F}'_\sigma(t)}} = \overline{\overline{\mathfrak{F}^*_\sigma(t)}} + 1 \leqq p \cdot \overline{\sigma + 1}$.

We have $\varphi \leqq \omega_\beta$ from the definition (5.5), and therefore $p \cdot \overline{\sigma + 1} < m$, hence we may apply Lemma 4 again to $\mathfrak{F}_\sigma(t) = \text{Clos}\,(\mathfrak{F}'_\sigma(t), t)$ and we obtain $\overline{\overline{\mathfrak{F}_\sigma(t)}} \leqq p \cdot \overline{\sigma + 1}$, thus this statement is proved for every $\sigma < \varphi$ and Lemma 7 is proved.

Note that from the statement $\overline{\overline{\mathfrak{F}_\sigma(t)}} \leqq p \cdot \overline{\sigma + 1}$ $(\sigma < \varphi)$ it results that $\varphi = \omega_\beta$, but we do not use this fact.

Finally, to have a view of our results we need the following quite evident

LEMMA 8. *The least cardinal number which possesses property* $\mathbf{T}(\aleph_\alpha, \aleph_\gamma)$ $(\alpha > \gamma)$ *is* $\alpha + 1$ *if* $cf(\alpha) = cf(\gamma)$, *and it is* $\aleph_{\alpha + \omega_{cf(\gamma)} + 1}$ *if* $cf(\alpha) \neq cf(\gamma)$.

PROOF. By the definition (5.11) we have to find the least β_1 for which there exists a β' such that $\alpha \leqq \beta' < \beta_1$ and $cf(\beta') = cf(\gamma)$. It is obvious that $\beta_1 = \beta' + 1$ for the least ordinal number β' satisfying this condition, and $\beta' = \alpha$ if $cf(\alpha) = cf(\gamma)$.

Suppose now $cf(\alpha) \neq cf(\gamma)$. $\beta' > \alpha$ has the form $\beta' = \alpha + \beta''$ and $cf(\alpha + \beta'') = cf(\gamma)$ can hold only if β'' is of the second kind. But then $cf(\alpha + \beta'') = cf(\beta'')$ and the least ordinal number β'' of the second kind satisfying $cf(\beta'') = cf(\gamma)$ is $\omega_{cf(\gamma)}$.

Let for the sake of brevity $\tau(\alpha, \gamma)$ denote the index of the least cardinal number which possesses property $\mathbf{T}(\aleph_\alpha, \aleph_\gamma)$.

EXAMPLES.

$$\tau(n, 0) = \omega + 1, \quad \tau(\omega, 0) = \omega + 1, \quad \tau(\omega + 1, 0) = \omega \cdot 2 + 1;$$

or more generally

$\tau(\alpha + \mu, \gamma) = \alpha + \omega_\gamma + 1$ for $1 \leqq \mu \leqq \omega_\gamma$ if $\gamma \leqq \alpha$ and ω_γ is regular.

6. Proof of the results concerning the conjectures (o) and (oo).

(*) THEOREM 4. *Suppose* $p \geqq \aleph_0$, $2 \leqq q \leqq p^+$ *and* $r^+ < p$. *Then for every cardinal number* m.

$$\mathbf{M}(m, p, q, r) \to \mathbf{B}.$$

(Note that if r is finite, the hypothesis (*) is not used.)

PROOF. For $m \leqq p$ the theorem follows from Theorem 2 (BERNSTEIN's theorem) if we use that symbol-I is decreasing in m (by (3.1)). We prove it by induction on m for every $m > p$. Suppose that the theorem is true for every $m' < m$. Let now $\mathfrak{F}$ be a family $(p(\mathfrak{F}) = p,\ \overline{\overline{\mathfrak{F}}} = m)$ which possesses property $\mathbf{C}(q, r)$.

Put $t = r^+$. Then the conditions (°) are satisfied for the cardinal numbers in question and $r < t$. Hence we can carry out the construction described in Section 5 and we can apply Lemma 7. It results that the families $\tilde{\mathfrak{H}}_\sigma(t)$ possess property $\mathbf{C}(q, r)$, $p(\tilde{\mathfrak{H}}_\sigma(t)) = p$ and $\overline{\overline{\tilde{\mathfrak{H}}_\sigma(t)}} < m$ for every $\sigma < \varphi$. Using the induction hypothesis we obtain that the families $\tilde{\mathfrak{H}}_\sigma(t)$ possess property **B** and thus by Lemma 6 the same holds for the family $\mathfrak{F}$ too. Q. e. d.

REMARK. Theorem 4 is clearly a generalization of Theorem 1 (MILLER's theorem) for infinite r's, however, it is not best-possible in r as we have already mentioned. It is possible that under the conditions $p \geqq \aleph_0$, $q \leqq p^+$ the theorem holds for every $r < p$. We have to deal only with the case $p = r^+$.

Here we can prove the following

(*) THEOREM 5. *Suppose* $r = \aleph_\gamma$, $r^+ = p$ *(i. e.* $p = \aleph_\alpha = \aleph_{\gamma+1}$*),* $2 \leqq q \leqq p^+$. *Then* $\mathbf{M}(m, p, q, r) \to \mathbf{B}$ *holds for every* m *less than* $\aleph_{\gamma+\omega_{cf(\gamma)}+1}$.

PROOF. For $m \leqq p$ the theorem is true by Theorem 2. We prove it by induction on m for every $p < m < \aleph_{\gamma+\omega_{cf(\gamma)}+1}$. Suppose that it is true for every $m' < m$ for an m satisfying the above condition. Let $\mathfrak{F}$ be a family for which $p(\mathfrak{F}) = p$, $\overline{\overline{\mathfrak{F}}} = m$ and suppose that $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$. Put $t = r$. The conditions (°) hold for the cardinal numbers in question, and so we can consider the families $\tilde{\mathfrak{H}}_\sigma(r)$ $(\sigma < \varphi)$ defined in (5.9). Since by the assumption $cf(\alpha) = cf(\gamma + 1)$ $(cf(\alpha) \neq cf(\gamma))$, it follows from Lemma 8 that m does not possess property $\mathbf{T}(p, r)$. It results from Lemma 7 ($\delta\delta$) that $p(\tilde{\mathfrak{H}}_\sigma(r)) = p$, $\tilde{\mathfrak{H}}_\sigma(r)$ possesses property $\mathbf{C}(q, r)$ and $\overline{\overline{\tilde{\mathfrak{H}}_\sigma(r)}} < m$ for every $\sigma < \varphi$. Hence by the induction hypothesis the families $\tilde{\mathfrak{H}}_\sigma(r)$ possess property **B**. Consequently, by Lemma 6, the same is true for $\mathfrak{F}$.

REMARK. We do not know for any γ whether the assumption $m < \aleph_{\gamma+\omega_{cf(\gamma)}+1}$ can be omitted. We have formulated the simplest unsolved problem in Section 3 (see Problem 2).

(*) THEOREM 6. *Suppose* $p > r \geqq \aleph_0$, *then* $\mathbf{M}(m, p, r) \to \mathbf{B}(r^{++})$ *for every* m.

PROOF. If $p = r^+$, then the theorem is trivially true by (3.2). Thus we may suppose $r^+ < p$. In the cases $m < p$ by (4.2) we have $\mathbf{M}(m, p, r) \to \mathbf{B}(2)$.

If p does not possess property $\mathbf{Q}(r)$, then by (4.9) $\mathbf{M}(p, p, r) \to \mathbf{B}(r^+)$ holds. If p possesses property $\mathbf{Q}(r)$, then it obviously does not possess roperty $\mathbf{Q}(r^+)$ (since if $r = \aleph_\gamma$, then $r^+ = \aleph_{\gamma+1}$ and $cf(\gamma) \neq cf(\gamma + 1)$).

It follows again from (4.9) that $\mathbf{M}(p,p,r^+)\to\mathbf{B}(r^{++})$ holds. As a consequence of (3. 1) we get that $\mathbf{M}(p,p,r)\to\mathbf{B}(r^{++})$ holds in every cases. Now we prove the theorem for $m>p$ by induction on m as follows:

Suppose that it is true for every $m'<m$. Let $\mathfrak{F}$ be a family $(p(\mathfrak{F})=p,\ \overline{\overline{\mathfrak{F}}}=m)$ which possesses property $\mathbf{C}(2,r)$. Put $t=r^+$. Then the conditions (°) hold for the cardinal numbers in question and we can consider the families $\tilde{\mathfrak{H}}_\sigma(t)$ $(\sigma<\varphi)$. Since $r<t$, it results from Lemma 7 that $p(\tilde{\mathfrak{H}}_\sigma(t))=p$, the families $\tilde{\mathfrak{H}}_\sigma(t)$ possess property $\mathbf{C}(2,r)$ and $\overline{\overline{\tilde{\mathfrak{H}}_\sigma(t)}}<m$ for every $\sigma<\varphi$. Thus by the induction hypothesis the families $\tilde{\mathfrak{H}}_\sigma(t)$ possess property $\mathbf{B}(r^{++})$. Applying Lemma 5 we obtain that $\mathfrak{F}$ possesses property $\mathbf{B}(r^{++}+r^{++})$, i. e. it possesses property $\mathbf{B}(r^{++})$.

REMARK. It is obvious from (3. 1) that under the conditions of Theorem 6 $\mathbf{M}(m,p,r)\to\mathbf{B}(s)$ holds for every $s\geqq r^{++}$ too. In the case $q=2$ Theorem 4 is a corrollary of Theorem 6. Similarly as in the case of Theorem 4, it is possible that Theorem 6 holds with r^+ instead of r^{++}.

(*) THEOREM 7. *Suppose $p>r\geqq\aleph_0$. (Put $p=\aleph_\alpha$ $r=\aleph_\gamma$.) Suppose further that p does not possess property $\mathbf{Q}(r)$. Then*

$$\mathbf{M}(m,p,r)\to\mathbf{B}(r^+)$$

for every $m<\aleph_{\alpha+\omega_{cf(\gamma)}+1}$, provided $cf(\alpha)\neq cf(\gamma)$.

PROOF. For $m<p$ the theorem is a corollary of (4. 2). In the case $m=p$ we get from (4.9) that $\mathbf{M}(p,p,r)\to\mathbf{B}(r^+)$ holds, since the assumption of our theorem assures that p and r do not satisfy the formula (v) of (4. 9).

We are going to prove our theorem for $m>p$ by induction on m as follows: Suppose that the theorem is true for every $m'<m$, for an m satisfying the above condition. Let $\mathfrak{F}$ be a family $(p(\mathfrak{F})=p,\ \overline{\overline{\mathfrak{F}}}=m)$ which possesses property $\mathbf{C}(2,r)$. Put $t=r$. The conditions (°) are satisfied, and so we can consider the families $\tilde{\mathfrak{H}}_\sigma(r)$. The assumption $cf(\alpha)\neq cf(\gamma)$ assures by Lemma 8 that m does not possess property $\mathbf{T}(p,r)$. Thus from Lemma 7 we obtain that $p(\tilde{\mathfrak{H}}(r))=p$, the families $\tilde{\mathfrak{H}}_\sigma(r)$ possess property $\mathbf{C}(2,r)$ and $\overline{\overline{\tilde{\mathfrak{H}}_\sigma(r)}}<m$ for every $\sigma<\varphi$.

Thus, by the induction hypothesis, the families $\tilde{\mathfrak{H}}_\sigma(r)$ possess property $\mathbf{B}(r^+)$ and, consequently, by Lemma 5, the family $\mathfrak{F}$ possesses $\mathbf{B}(r^++r^+)$ Since r is supposed to be infinite, this means that $\mathfrak{F}$ possesses property $\mathbf{B}(r^+)$ too.

REMARKS. If p possesses property $\mathbf{Q}(r)$, we do not know whether the theorem is true for $m=p$. (See the remark after (4. 9) and Problem 3a).)

If $cf(\alpha)=cf(\gamma)$, then by (4.9) the theorem is true for $m=p$, but we do not know whether it is true for $m=p^+$ or not. The simplest unsolved problem here is $\mathbf{M}(\aleph_{\omega+1}, \aleph_\omega, \aleph_0) \to \mathbf{B}(\aleph_{\omega+1})$.

Here the difficulty is essentially the same as in Problem 3b). It is obvious from the remark made after (4.9) that a positive solution of Problem 3b) would imply the positive solution of the problem just stated as well as a positive solution of Problem 3a).

7. The discussion of symbol-II in the cases $r<\aleph_0$ $(p \geqq \aleph_0)$. Note that in the case $r<\aleph_0$ $(p\geqq\aleph_0)$ symbol-I is completely discussed by MILLER's theorem. The positive theorems concerning symbol-II will be proved by MILLER's method quite similarly as the theorems of Section 6.

THEOREM 8. a) $\mathbf{M}(\aleph_{\alpha+n}, \aleph_\alpha, r) \to \mathbf{B}((r-1)(n+1)+2)$ *if r is finite and α is arbitrary.*

b) $\mathbf{M}(m, \aleph_\alpha, r) \to \mathbf{B}(\aleph_0)$ *for every m and α, provided $r<\aleph_0$.*[10]

PROOF. a) We are going to prove the theorem by induction on n. For $n=0$ the theorem is proved in (4.9). Suppose that it is true for an n and let $\mathfrak{F}$ be a family such that $p(\mathfrak{F})=\aleph_\alpha$, $\overline{\overline{\mathfrak{F}}}=\aleph_{\alpha+n+1}$ and suppose that it possesses property $\mathbf{C}(2,r)$. It is obvious that the conditions (°) hold for the cardinal numbers in question and we can apply the construction of Section 5 with $t=r$ to our family $\mathfrak{F}$.

By Lemma 7, $p(\mathfrak{H}_\sigma(r))=p$, the families $\mathfrak{H}_\sigma(r)$ possess property $\mathbf{C}(2,r)$ and $\overline{\overline{\mathfrak{H}_\sigma(r)}}<\aleph_{\alpha+n+1}$ for every $\sigma<\varphi$. This means that $\overline{\overline{\mathfrak{H}_\sigma(r)}}\leqq\aleph_{\alpha+n}$ for every $\sigma<\varphi$ and — using (3.1) — we get from the induction hypothesis that the families $\mathfrak{H}_\sigma(r)$ possess property $\mathbf{B}((r-1)(n+1)+2)$ for every $\sigma<\varphi$.

It follows from Lemma 5 that the family $\mathfrak{F}$ possesses property $\mathbf{B}((r-1)+ +(r-1)(n+1)+2)$, i. e. it possesses property $\mathbf{B}((r-1)(n+2)+2)$.

b) The proof can be carried out by induction on m using Lemmas 5 and 7 quite similarly as in the previous cases, and so we omit the proof.

REMARK. The hypothesis (*) is not used in the proof, since it is not used in the proof of Lemma 7 for the case of finite r.

With a slight modification of our construction it would be easy to prove the following

THEOREM 9. *Let $\mathfrak{F}$ be a family, $p(\mathfrak{F})=\aleph_\alpha$, $\overline{\overline{\mathfrak{F}}}=\aleph_{\alpha+n}$, and suppose that it possesses property* $\mathbf{C}(2,r)$ *for a finite r where α is arbitrary.*

Let there be given a function $l(F)$ which correlates to every $F\in\mathfrak{F}$ an integer $l(F)$.

[10] Note that n denotes always a non-negative integer and r is supposed to be greater than 0.

Then there exists a set B such that

$$\overline{\overline{B \cap F}} = \max\big(l(F), (r-1)(n+1)+1\big) \quad \textit{for every} \quad F \in \mathfrak{F}.$$

In particular, if $l(F) \equiv (r-1)(n+1)+1$, *then the set B intersects every F in exactly* $(r-1)(n+1)+1$ *points.*

We omit the proof.

Now we are going to prove that Theorem 8 is best-possible in s.

(*) THEOREM 10. a) $\mathbf{M}(\aleph_{\alpha+n}, \aleph_\alpha, r) \nrightarrow \mathbf{B}((r-1)(n+1)+1)$ *if r is finite and* α *is arbitrary.*

b) $\mathbf{M}(m, \aleph_\alpha, r) \nrightarrow \mathbf{B}(l)$ *if* $r > 1$ *is finite,* α *is arbitrary,* $m \geqq \aleph_{\alpha+\omega}$ *and* l *is an integer.*

PROOF. a) We have to prove that there exists a family $\mathfrak{F}$ satisfying the following conditions:

(1) $p(\mathfrak{F}) = \aleph_\alpha$.

(2) $\overline{\overline{\mathfrak{F}}} = \aleph_{\alpha+n}$.

(3) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$.

(4) If for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}$, then there exists an $F_0 \in \mathfrak{F}$ such that $\overline{\overline{F_0 \cap B}} \geqq (r-1)(n+1)+1$.

We are going to prove instead of this the following more general statement: There exists a family $\mathfrak{F}$ satisfying the conditions (1), (2), (3) and the following condition:

(5) There exist subfamilies $\mathfrak{F}_1$, $\mathfrak{F}_2$ of $\mathfrak{F}$ such that $\mathfrak{F}_1 \cup \mathfrak{F}_2 = \mathfrak{F}$, $\mathfrak{F}_1 \cap \mathfrak{F}_2 = 0$ and if for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}_1$, then there exists an $F_0 \in \mathfrak{F}_2$ such that $\overline{\overline{F_0 \cap B}} \geqq (r-1)(n+1)+1$.

It is obvious that (5) implies (4).

Put $(\mathfrak{F}) = S$. Obviously (1) and (2) imply $\overline{\overline{S}} \leqq \aleph_{\alpha+n}$. Thus we have:

(6) If there exists a family $\mathfrak{F}$ satisfying the conditions (1), (2), (3) and (5), then for an arbitrary set S' $(\overline{\overline{S'}} = \aleph_{\alpha+n})$ there exists a family $\mathfrak{F}'$ such that $(\mathfrak{F}') \subseteq S'$ and $\mathfrak{F}'$ satisfies the conditions (1), (2), (3) and (5) too.

We prove the existence of such a family $\mathfrak{F}$ by induction on n. For $n = 0$ the theorem is proved in (4.5) (see the remark after (4.5)).[11] Suppose that for an n there exists a family $\mathfrak{F}$ satisfying the formulas (1), (2), (3) and (5). Let S be a set, $\overline{\overline{S}} = \aleph_{\alpha+n+1}$. Then $\overline{\overline{[S]^{\aleph_{\alpha+n}}}} = \aleph_{\alpha+n+1}$ by the hypothesis (*). Let $\{A_\varrho\}_{\varrho < \omega_{\alpha+n+1}} = [S]^{\aleph_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n+1}$ of the set $[S]^{\aleph_{\alpha+n}}$.

[11] In case of finite r the construction given in (4.5) can be simplified as follows: Suppose that $\overline{\overline{\mathfrak{F}_1}} = r$ instead of $\overline{\overline{\mathfrak{F}_1}} = r^+$ and take for $\mathfrak{S}$ the system of all subsets X of $(\mathfrak{F}_1)$ satisfying the condition $\overline{\overline{X \cap F}} = 1$ for every $F \in \mathfrak{F}_1$ instead of the system $\mathfrak{S}$ defined in (4.5.4).

We are going to define a sequence $\{\mathfrak{F}_\varrho\}_{\varrho<\omega_{\alpha+n+1}}$ of type $\omega_{\alpha+n+1}$ of families $(\mathfrak{F}_\varrho)\subseteq S$ by induction on ϱ as follows:

Suppose that the families $\mathfrak{F}_{\varrho'}$ are defined for a $\varrho<\omega_{\alpha+n+1}$ in such a way that $\overline{\overline{(\mathfrak{F}_{\varrho'})}}\leqq\aleph_{\alpha+n}$ for every $\varrho'<\varrho$. Then $\overline{\overline{A_\varrho\cup\bigcup_{\varrho'<\varrho}(\mathfrak{F}_{\varrho'})}}\leqq\aleph_{\alpha+n}$, hence we can define a subset S_ϱ of S such that

$$(7)\qquad S_\varrho\subseteq S-\big(A_\varrho\cup\bigcup_{\varrho'<\varrho}(\mathfrak{F}_{\varrho'})\big)\quad\text{and}\quad \overline{\overline{S_\varrho}}=\aleph_{\alpha+n}.$$

By the induction hypothesis and by (6) there exists a family $\mathfrak{F}^*_\varrho$ satisfying the formulas (1), (2), (3) and (5) such that

$$(8)\qquad (\mathfrak{F}^*_\varrho)\subseteq S_\varrho;$$

let $\mathfrak{F}^{1,*}_\varrho$ and $\mathfrak{F}^{2,*}_\varrho$ denote the families satisfying (5) instead of $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively.

Since $\mathfrak{F}^*_\varrho$ satisfies (2), we have $\overline{\overline{\mathfrak{F}^{2,*}_\varrho}}\leqq\aleph_{\alpha+n}$ and we may suppose that the equality holds. Let $\mathfrak{F}^{2,*}_\varrho=\{F^{\varrho,2,*}_\nu\}_{\nu<\omega_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n}$ of the family $\mathfrak{F}^{2,*}_\varrho$.

Since $\overline{\overline{A_\varrho}}=\aleph_{\alpha+n}$, it is obvious that there exists a system $\mathbb{S}_\varrho$ of subsets of A_ϱ satisfying the following conditions:

(9) $\overline{\overline{\mathbb{S}_\varrho}}=\aleph_{\alpha+n}$, $\overline{\overline{X}}=r-1$ for every $X\in\mathbb{S}$ and $X\cap Y=0$ for every $X, Y\in\mathbb{S}, X\neq Y$.

Let $\mathbb{S}_\varrho=\{X^\varrho_\nu\}_{\nu<\omega_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n}$ of the set $\mathbb{S}_\varrho$.

We define the families $\mathfrak{F}_\varrho$, $\mathfrak{F}^1_\varrho$, $\mathfrak{F}^2_\varrho$ by the following formulas:

$$(10)\qquad \mathfrak{F}^1_\varrho=\mathfrak{F}^{1,*}_\varrho,\quad \mathfrak{F}^2_\varrho=\{X^\varrho_\nu\cup F^{\varrho,2,*}_\nu\}_{\nu<\omega_{\alpha+n}}$$

and

$$\mathfrak{F}_\varrho=\mathfrak{F}^1_\varrho\cup\mathfrak{F}^2_\varrho.$$

It is obvious that $(\mathfrak{F}_\varrho)\subseteq S_\varrho+A_\varrho$, hence $\overline{\overline{(\mathfrak{F}_\varrho)}}\leqq\aleph_{\alpha+n}$, and so $\mathfrak{F}_\varrho$ is defined for every $\varrho<\omega_{\alpha+n+1}$ and the formulas (7)—(10) are satisfied for every $\varrho<\omega_{\alpha+\nu+1}$.

Put

$$(11)\qquad \mathfrak{F}=\bigcup_{\varrho<\omega_{\alpha+n+1}}\mathfrak{F}_\varrho,\quad \mathfrak{F}^1=\bigcup_{\varrho<\omega_{\alpha+n+1}}\mathfrak{F}^1_\varrho,\quad \mathfrak{F}^2=\bigcup_{\varrho<\omega_{\alpha+n+1}}\mathfrak{F}^2_\varrho.$$

Now we have to verify that $\mathfrak{F}$ satisfies (1), (2), (3) and (5) for $n+1$ instead of n.

$p(\mathfrak{F}^*_\varrho)=\aleph_\alpha$, since $\mathfrak{F}^*_\varrho$ satisfies (1), and thus it follows immediately from the definitions (9), (10) and (11) that

$$(12)\qquad p(\mathfrak{F})=\aleph_\alpha.$$

It results immediately from (7) and (8) that

$$(13) \qquad (\mathfrak{F}^1_{\varrho'}) \cap (\mathfrak{F}^1_{\varrho}) = 0 \quad \text{if} \quad \varrho' < \varrho < \omega_{\alpha+n+1}.$$

Thus, since the families $\mathfrak{F}^1_\varrho$ are non-empty, we have

$$(14) \qquad \overline{\overline{\mathfrak{F}}} = \aleph_{\alpha+n+1}.$$

Now we prove:

(15) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$.

Let F, F' be two distinct elements of $\mathfrak{F}$. Then $F \in \mathfrak{F}_\varrho$ and $F' \in \mathfrak{F}_{\varrho'}$ for suitable ϱ and ϱ', respectively. We distinguish two cases: (i) $\varrho = \varrho'$, (ii) $\varrho \neq \varrho'$.

(i) If $F \in \mathfrak{F}^1_\varrho$, $F' \in \mathfrak{F}^1_\varrho$, then $\overline{\overline{F \cup F'}} < r$, since by (10) $\mathfrak{F}^1_\varrho = \mathfrak{F}^{1,*}_\varrho$ and $\mathfrak{F}^*_\varrho$ satisfies (3).

If $F \in \mathfrak{F}^1_\varrho$, $F' \in \mathfrak{F}^2_\varrho$, then $F \subseteq S_\varrho$, $F' = X^\varrho_\nu \cup F^{\varrho,2,*}_\nu$ for a suitable $\nu < \omega_{\alpha+n}$, but by (9) $X^\varrho_\nu \subseteq A_\varrho$, hence by (7) and (8) $F \cap F' = F \cap F^{\varrho,2,*}_\nu$ and $\overline{\overline{F \cap F'}} < r$ follows again from the fact that $\mathfrak{F}^*_\varrho$ satisfies (3).

If $F \in \mathfrak{F}^2_\varrho$, $F' \in \mathfrak{F}^2_\varrho$, then $F = X^\varrho_\nu \cup F^{\varrho,2,*}_\nu$, $F' = X^\varrho_{\nu'} \cup F^{\varrho,2,*}_{\nu'}$ for suitable ν', ν $(\nu \neq \nu')$, respectively. Using again that A_ϱ and S_ϱ are disjoint, we get

$$F \cap F' = (X^\varrho_\nu \cap X^\varrho_{\nu'}) \cup (F^{\varrho,2,*}_\nu \cap F^{\varrho,2,*}_{\nu'}).$$

Thus, using that by (9) $X^\varrho_\nu \cap X^\varrho_{\nu'} = 0$, we get by the same argument as above that $\overline{\overline{F \cap F'}} < r$ in this case too.

(ii) We may suppose $\varrho' < \varrho$. If $F \in \mathfrak{F}^1_\varrho$, then by (7), (8) and (10) F and F' are disjoint. If $F \in \mathfrak{F}^2_\varrho$, then $F = X^\varrho_\nu \cup F^{\varrho,2,*}_\nu$ for a suitable ν and it results from (7) and (8) that $F' \cap F \subseteq X^\varrho_\nu$, hence by (9) $\overline{\overline{F' \cap F}} \leqq r-1 < r$.

We have

$$(16) \qquad \mathfrak{F}^1 \cap \mathfrak{F}^2 = 0.$$

In fact, $\mathfrak{F}^{1,*}_\varrho \cap \mathfrak{F}^{2,*}_\varrho = 0$ for every ϱ, because $\mathfrak{F}^*_\varrho$ satisfies (5), thus it results from the definition (10) and e. g. from the fact that $\mathfrak{F}^*_\varrho$ satisfies (3), that $\mathfrak{F}^1_\varrho \cap \mathfrak{F}^2_\varrho = 0$ and it is obvious from (7) and (8) that $\mathfrak{F}_{\varrho'} \cap \mathfrak{F}_\varrho = 0$ for $\varrho' \neq \varrho$, hence $\mathfrak{F}_1 \cap \mathfrak{F}_2 = 0$ is true.

Now we prove:

(17) Suppose that for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}^1$.

Then there exists an $F_0 \in \mathfrak{F}_2$ such that $\overline{\overline{F_0 \cap B}} \geqq (r-1)(n+2)+1$.

First of all it follows from (13) that $\overline{\overline{B}} = \aleph_{\alpha+n+1}$. As a corollary of this there exists a subscript ϱ_0 such that $A_{\varrho_0} \subseteq B$. Since by the assumption B has to intersect every $F \in \mathfrak{F}^1$, we have that $F \cap B \neq 0$ for every $F \in \mathfrak{F}^1_{\varrho_0}$. But by (10) $\mathfrak{F}^1_\varrho = \mathfrak{F}^{1,*}_\varrho$ and it follows from the fact that $\mathfrak{F}^*_{\varrho_0}$, $\mathfrak{F}^{1,*}_{\varrho_0}$, $\mathfrak{F}^{2,*}_{\varrho_0}$ satisfy (5), that there exists an index ν_0 such that $\overline{\overline{B \cap F^{\varrho_0,2,*}_{\nu_0}}} \geqq (r-1)(n+1)+1$. Put

$F^0 = X^{\varrho_0}_{\nu_0} \cup F^{\varrho_0,2,*}_{\nu_0}$. Then $F^0 \in \mathfrak{F}^2$. Taking into consideration that by (7) and (8) $X^{\varrho_0}_{\nu_0} \cap F^{\varrho_0,2,*}_{\nu_0} = 0$ and by (9) $X^{\varrho_0}_{\nu_0} \subseteq A_{\varrho_0} \subseteq B$, $\overline{\overline{X^{\varrho}_{\nu_0}}} = r-1$, we obtain

$$\overline{\overline{B \cap F^0}} \geqq \overline{\overline{X^{\varrho_0}_{\nu_0}}} + \overline{\overline{B \cap F^{\varrho_0,2,*}_{\nu_0}}} \geqq (r-1)(n+2)+1.$$

Thus the families $\mathfrak{F}$, $\mathfrak{F}^1$ and $\mathfrak{F}^2$ satisfy by (12), (14), (15), (16) and (17) the formulas (1), (2), (3) and (5) for $n+1$ instead of n, and so the existence of such a family is proved for every n.

b) By (3.1) it suffices to prove that $\mathbf{M}(\aleph_{\alpha+\omega}, \aleph_\alpha, r) \not\rightarrow \mathbf{B}(l)$.

Let $\{S_n\}_{n<\omega}$ be a sequence of disjoint sets such that $\overline{\overline{S_n}} = \aleph_{\alpha+n}$. By the theorem just proved and by the remark (6) there exists a sequence $\{\mathfrak{F}_n\}_{n<\omega}$ of families such that $(\mathfrak{F}_n) \subseteq S_n$ and $\mathfrak{F}_n$ satisfies for every n the conditions (1), (2), (3) and (5).

Put $\mathfrak{F} = \bigcup_{n<\omega} \mathfrak{F}_n$. Then $p(\mathfrak{F}) = \aleph_\alpha$ and $\overline{\overline{\mathfrak{F}}} = \aleph_{\alpha+\omega}$, since the $\mathfrak{F}_n$'s satisfy (1) and (2) for every n and the $\mathfrak{F}_n$'s are obviously disjoint.

Since the sets S_n are disjoint, $F \cap F' = 0$, provided $F \in \mathfrak{F}_n$, $F' \in \mathfrak{F}_{n'}$ for $n \neq n'$. Thus, taking into consideration that $\mathfrak{F}_n$ satisfies (3) for every n, it follows that $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$. But $\mathfrak{F}$ does not possess property $\mathbf{B}(l)$ for any l, since there exists an n_0 such that $(r-1)(n_0+1)+1 > l$ and the subfamily $\mathfrak{F}_{n_0}$ of $\mathfrak{F}$ does not possess property $\mathbf{B}((n-1)(n_0+1)+1)$, because it satisfies (5).

Thus part b) of Theorem 10 is also proved.

REMARK. As we have already mentioned in (4.5), in the case $n=0$ of the part a) of Theorem 10 the hypothesis (*) is not used. We do not even know whether one can prove Theorem 10a) for $n=1$ without using (*).

8. Results on the topological products. A topological space $\mathfrak{X}$ is said to be $\varkappa$-compact if every family $\mathfrak{M}$ of closed subsets of it with void intersection, $\bigcap_{X \in \mathfrak{M}} X = 0$, contains a subfamily $\mathfrak{M}' \subseteq \mathfrak{M}$ $(\overline{\overline{\mathfrak{M}'}} < \aleph_\varkappa)$ with void intersection.

0-compactness means ordinary compactness.

1-compact spaces are the Lindelöf spaces.

For the sake of brevity we introduce the symbol $\boldsymbol{T}(m, \lambda) \rightarrow \varkappa$ to indicate the following statement:

If $\mathfrak{F}$ is a family of λ-compact *discrete* topological spaces, $\overline{\overline{\mathfrak{F}}} = m$, then the topological product of the elements of $\mathfrak{F}$ is $\varkappa$-compact.

As usual, $\boldsymbol{T}(m, \lambda) \not\rightarrow \varkappa$ denotes the negation of this statement.

TYCHONOV's classical theorem can be stated as follows: $\boldsymbol{T}(m, 0) \rightarrow 0$ for every cardinal number m.

Let S be a set, $\overline{\overline{S}}=m$, and let $\mu(x)$ be a measure defined on all subsets of S such that the values of $\mu(x)$ are 0 and 1, $\mu(\{x\})=0$ for every $x\in S$.

The cardinal number m is said to be of measure 0 if every σ-measure satisfying the above condition vanishes identically.[12]

A well-known result of ULAM states that every cardinal number m less than the first strongly inaccessible aleph is of measure 0.[13]

The hypothesis (**) states that a strongly inaccessible $>\aleph_0$ aleph is not of measure 0 or more generally:

(**) If m is strongly inaccessible, $>\aleph_0$, then there exists an m-additive measure satisfying the above conditions such that $\mu(S)=1$.

If we use (*), then ŁOS's theorem (Theorem 4 of [3]) states that

$$\boldsymbol{T}(\aleph_{\varkappa+1}, 1)\not\rightarrow\varkappa \quad \text{for every} \quad \varkappa\geqq 1,$$

provided $\aleph_\varkappa$ is regular and of measure 0.[14]

Now we are going to prove the following

(*) THEOREM 11. *$\boldsymbol{T}(\aleph_{\alpha+n}, \alpha+1)\not\rightarrow\alpha+n$ for every ordinal number α and for every $1\leqq n<\omega$.*

Before proving this theorem[15] we compare it with ŁOS's theorem and state the simplest unsolved problems. Put $\alpha=0$, then our theorem states that $\boldsymbol{T}(\aleph_n, 1)\not\rightarrow n$ for every $n\geqq 1$, and so it is stronger than ŁOS's theorem for the cases $\varkappa<\omega$. Moreover it is best-possible, namely $\boldsymbol{T}(\aleph_n, 1)\rightarrow n+1$ is trivially true, since the topological product of $\aleph_\alpha$ Lindelöf spaces contains a base of power $\aleph_\alpha$ for every α. For the case of singular $\varkappa$'s, e. g. for $\varkappa=\omega$ the following problem remains open:

PROBLEM 4. $\boldsymbol{T}(\aleph_\omega, 1)\rightarrow\omega$?

($\boldsymbol{T}(\aleph_{\omega}, 1)\rightarrow\omega+1$ is trivially true and $\boldsymbol{T}(\aleph_\omega, 1)\not\rightarrow n$ for every finite n is a consequence of both theorems.)

For $\varkappa$'s greater than ω ŁOS's theorem is stronger, since our result states nothing about $\varkappa$-compactness of the product of Lindelöf spaces for $\varkappa>\omega$.

But we do not know whether ŁOS's theorem is best-possible e. g. for $\varkappa=\omega+1$, since it states $\boldsymbol{T}(\aleph_{\omega+2}, 1)\rightarrow\omega+1$ and the following problem remains open:

PROBLEM 5. $\boldsymbol{T}(\aleph_{\omega+2}, 1)\rightarrow\omega+2$?

(Our Theorem 11 gives only that $\boldsymbol{T}(\aleph_{\omega+2}, \omega+1)\not\rightarrow\omega+2$.)

[12] See [3], p. 14.
[13] See [7].
[14] See [3], Theorem 4, p. 17.
[15] The proof is given on p. 115.

For $\aleph_\varkappa$'s not less than the first inaccessible cardinal number ŁOS's theorem does not state anything. The reason for this is that if, at least, we assume the hypothesis (**), then $T(m_0, 1) \to \alpha_0$ is true where $m_0 = \aleph_{\alpha_0}$ denotes the first strongly inaccessible cardinal number $> \aleph_0$. More generally we have the following

(**) THEOREM 12. *If $\aleph_\alpha$ is strongly inaccessible, $> \aleph_0$, then*

$$T(\aleph_\alpha, \alpha) \to \alpha.$$ [16]

We mention here that even using (*) and (**) we can not decide whether $T(\aleph_\alpha, 1) \to \alpha_0$ is true if $\alpha > \alpha_0$ where $\aleph_{\alpha_0}$ is the first inaccessible cardinal number $> \aleph_0$.

Our theorem shows that $T(\aleph_{\alpha_0} + n, \alpha_0 + 1) \not\to \alpha_0 + n$ for every $1 \leqq n < \omega$, but neither ŁOS's theorem nor our theorem disproves that $T(m, \alpha_0 + 1) \to \alpha_0 + \omega$ holds for every cardinal number m if $\aleph_{\alpha_0}$ is strongly inaccessible $> \aleph_0$.

PROOF OF THEOREM 11. Let r_0 be an integer such that $(r_0 - 1)(n+1) + 1 \geqq (r_0 - 1)n + 2$ (e. g. $r_0 = 2$). By Theorem 10 corresponding to every n there exists a family $\mathfrak{F}$ $((\mathfrak{F}) = S)$ satisfying the following conditions:

(1) $p(\mathfrak{F}) = \aleph_\alpha$.

(2) $\overline{\overline{\mathfrak{F}}} = \aleph_{\alpha+n}$.

(3) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r_0)$.

(4) If for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}$, then there exists an $F_0 \in \mathfrak{F}$, such that

$$\overline{\overline{F_0 \cap B}} \geqq (r_0 - 1)(n+1) + 1.$$

Let $\mathfrak{F} = \{F_\varrho\}_{\varrho < \omega_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n}$ of $\mathfrak{F}$. Let $\mathfrak{X}$ denote the topological product of the discrete spaces F_ϱ. The elements of $\mathfrak{X}$ are the sequences $(x_\varrho)_{\varrho < \omega_{\alpha+n}}$ where $x_\varrho \in F_\varrho$.

Corresponding to every finite sequence $\varrho_1 < \cdots < \varrho_k < \omega_{\alpha+n}$ we define the subset $B_{\varrho_1 \ldots \varrho_k}((x_\varrho)_{\varrho < \omega_{\alpha+n}})$ of S as the set of ϱ_ith components of $(x_\varrho)_{\varrho < \omega_{\alpha+n}}$ for $i = 1, \ldots, k$, i. e. we put

$$B_{\varrho_1 \ldots \varrho_k}((x_\varrho)_{\varrho < \omega_{\alpha+n}}) = \{x : x \in S \wedge (x = x_{\varrho_1} \vee \cdots \vee x = x_{\varrho_k})\}. \tag{5}$$

Now we define the subset $X_{\varrho_1 \ldots \varrho_k}$ of $\mathfrak{X}$ as follows:

(6) $$(x_\varrho)_{\varrho < \omega_{\alpha+n}} \in X_{\varrho_1 \ldots \varrho_k} \quad \text{if and only if}$$

$$\overline{\overline{B_{\varrho_1 \ldots \varrho_k}((x_\varrho)_{\varrho < \omega_{\alpha+n}}) \cap F_{\varrho_i}}} < (r_0 - 1)n + 2 \quad \text{for every} \quad i = 1, \ldots, k.$$

[16] For the proof see p. 116.

Put

$$\mathfrak{M} = \{X_{\varrho_1 \ldots \varrho_k}\}_{(\varrho_1 < \cdots < \varrho_k < \omega_{\alpha+n})}.$$

It is obvious that $X_{\varrho_1 \ldots \varrho_k}$ is a closed subset of $\mathfrak{X}$ for every sequence $\varrho_1 < \cdots < \varrho_k < \omega_{\alpha+n}$ and it results from (1) that the discrete spaces F_ϱ are $\aleph_{\alpha+1}$-compact for every $\varrho < \omega_{\alpha+n}$. Hence it is enough to prove the following assertions:

(7) $$\bigcap_{X_{\varrho_1 \ldots \varrho_k} \in \mathfrak{M}} X_{\varrho_1 \ldots \varrho_k} = 0$$

and

(8) $$\bigcap_{X_{\varrho_1 \ldots \varrho_k} \in \mathfrak{M}'} x_{\varrho_1 \ldots \varrho_k} \neq 0 \quad \text{if} \quad \mathfrak{M}' \subseteq \mathfrak{M}, \quad \overline{\overline{\mathfrak{M}'}} < \aleph_{\alpha+n}.$$

Proof of (7). Let $(x^0_\varrho)_{\varrho < \omega_{\alpha+n}}$ be an arbitrary fixed element of $\mathfrak{X}$. Let B^0 be the set of those $x \in S$ for which there exists a $\varrho < \omega_{\alpha+n}$ such that $x = x^0_\varrho$. It is obvious that $B^0 \cap F_\varrho \neq 0$ for every $\varrho < \omega_{\alpha+n}$, hence by (4) we have for a $\varrho_0 < \omega_{\alpha+n}$

$$\overline{\overline{B^0 \cap F_{\varrho_0}}} \geqq (r_0 - 1)(n+1) + 1.$$

Put $(r_0 - 1)(n+1) + 1 = k_0$. Then there exists a sequence $\varrho^0_1 < \cdots < \varrho^0_{k_0}$ such that $\varrho_0 = \varrho^0_{i_0}$ for an i_0 $(1 \leqq i_0 \leqq k_0)$, $\overline{\overline{\{x^0_{\varrho^0_i}\}_{1 \leqq i \leqq \varkappa_0}}} = k_0$ and $\{x^0_{\varrho^0_i}\}_{1 \leqq i \leqq k_0} \subseteq F_{\varrho_0} = F_{\varrho^0_{i_0}}$.

But this means that $B_{\varrho^0_1 \ldots \varrho^0_{k_0}}((x^0_\varrho)_{\varrho < \omega_{\alpha+n}}) \cap F_{\varrho^0_{i_0}} = k_0 \nless (r_0 - 1)n + 2$ and thus by (6) $(x^0_\varrho)_{\varrho < \omega_{\alpha+n}} \notin X_{\varrho^0_1 \ldots \varrho^0_{k_0}}$ which proves that the product considered in (7) is empty.

Proof of (8). Let $I(\mathfrak{M}')$ denote the set of ordinal numbers ϱ appearing as a subscript ϱ_i $(i = 1, \ldots, k)$ of an $X_{\varrho_1 \ldots \varrho_k} \in \mathfrak{M}'$. It is obvious that $X_{\varrho_1 \ldots \varrho_k} \neq$ $\neq X_{\varrho'_1 \ldots \varrho'_k}$ if the sequences $\varrho_1, \ldots, \varrho_k$ and $\varrho'_1, \ldots, \varrho'_k$ are different. Hence $\mathfrak{M}' \subseteq \mathfrak{M}$, $\overline{\overline{\mathfrak{M}'}} < \aleph_{\alpha+n}$ implies $\overline{\overline{I(\mathfrak{M}')}} < \aleph_{\alpha+n}$. Thus it is sufficient to see that

$$\bigcap_{\varrho_i (i=1,\ldots,k),\, \varrho_i < \varrho_0} X_{\varrho_1 \ldots \varrho_k} \neq 0 \quad \text{holds for every} \quad \varrho_0 < \omega_{\alpha+n}.$$

Put $\mathfrak{F}_{\varrho_0} = \{F_{\varrho'}\}_{\varrho' < \varrho_0}$ for every $\varrho_0 < \omega_{\alpha+n}$. Then $p(\mathfrak{F}_{\varrho_0}) = \aleph_\alpha$ by (1). $\mathfrak{F}_{\varrho_0}$ possesses property $\mathbf{C}(2, r_0)$ by (3) and $\overline{\overline{\mathfrak{F}_{\varrho_0}}} \leqq \aleph_{\alpha+n-1}$ $(n-1 \geqq 0)$ for every $\varrho_0 < \omega_{\alpha+n}$. Thus by Theorem 8a) there exists a set B such that

$$1 \leqq \overline{\overline{B \cap F_{\varrho'}}} < (r_0 - 1)n + 2 \quad \text{for every} \quad \varrho' < \varrho_0.$$

It results that we can point out an element $x^0_{\varrho'}$ of $B \cap F_{\varrho'}$ for every $\varrho' < \varrho_0$ and let $x^0_{\varrho'}$ be an arbitrary element of $F_{\varrho'}$ for $\varrho' \geqq \varrho_0$. It is obvious from (6) that the sequence $(x^0_\varrho)_{\varrho < \omega_{\alpha+n}}$ so defined is an element of the product in question.

PROOF OF THEOREM 12. Let $\mathfrak{F}$ be a family, $\overline{\overline{\mathfrak{F}}} = \aleph_\alpha$ such that $\overline{\overline{F}} < \aleph_\alpha$ for every $F \in \mathfrak{F}$. Let $\mathfrak{F} = \{F_\nu\}_{\nu < \omega_\alpha}$ be a well-ordering of type ω_α of $\mathfrak{F}$. Put

$\mathfrak{F}_\nu = \{F_\mu\}_{\mu<\nu}$. Let $\mathfrak{X}$ and $\mathfrak{X}_\nu$ denote the topological product of the elements of $\mathfrak{F}$ and $\mathfrak{F}_\nu$, respectively. If $\Theta = (x_\nu^0)_{\nu<\omega_\alpha}$ is an element of $\mathfrak{X}$, then Θ/ν denotes the element $(x_\mu^0)_{\mu<\nu}$ of $\mathfrak{X}_\nu$.

Let there be given a family $\mathfrak{M}$ of closed subsets of $\mathfrak{X}$. Corresponding to every $X \in \mathfrak{M}$ and $\nu < \omega_\alpha$ we define a subset $Y(X, \nu)$ of $\mathfrak{X}_\nu$ as follows:

$$Y(X, \nu) = \{\Theta/\nu\}_{\Theta \in X}.$$

The set $\{Y(X, \nu)\}_{X \in \mathfrak{M}}$ is of power less than $\aleph_\alpha$ for every $\nu < \omega_\alpha$, since $\aleph_\alpha$ is strongly inaccessible and $\overline{\overline{\mathfrak{X}_\nu}} < \aleph_\alpha$ for every $\nu < \omega_\alpha$. As an easy consequence of this we obtain that $\bigcap_{X \in \mathfrak{M}} Y(X, \nu) \neq 0$ for every $\nu < \omega_\alpha$, provided $\bigcap_{X \in \mathfrak{M}} X \neq 0$ for every $\mathfrak{M}' \subseteq \mathfrak{M}$, $\overline{\overline{\mathfrak{M}'}} < \aleph_\alpha$.

Put $Z_\nu = \bigcap_{X \in \mathfrak{M}} Y(X, \nu)$. The Z_ν's form a ramification system. By a result of P. ERDŐS and A. TARSKI[17] it follows from the hypothesis (**) that there exists a $\Theta \in \mathfrak{X}$ such that $\Theta/\nu \in Z_\nu$ for every $\nu < \omega_\alpha$.

Let X be an arbitrary element of $\mathfrak{M}$. Then for an arbitrary $\nu < \omega_\alpha$ there exists a $\Theta_\nu \in X$ such that $\Theta_\nu/\nu = \Theta/\nu$. Since X is closed, it follows that $\Theta \in X$, and so $\Theta \in \bigcap_{X \in \mathfrak{M}} X$, i. e. $\mathfrak{X}$ is α-compact.

Now we state some unsolved problems which all would have been consequences of $\boldsymbol{T}(\aleph_2, 1) \rightarrow 2$. The answer to all these questions is very likely negative, but we can not disprove any of them. In the formulation of all these problems we consider (*) to be assumed.

PROBLEM 6. Let $\mathfrak{F}$ be a family ($\overline{\overline{\mathfrak{F}}} = \aleph_2$, $p(\mathfrak{F}) = \aleph_0$) such that every $\mathfrak{F}' \subseteq \mathfrak{F}$ ($\mathfrak{F}' \leqq \aleph_1$) possesses property **B**. Does then $\mathfrak{F}$ necessarily possess property **B** too?[18]

The family $\mathfrak{F}$ is said to possess property **G** if there exists a function $f(F)$ defined for every $F \in \mathfrak{F}$ such that $f(F)$ is an element of F and $f(F_1) \neq f(F_2)$ for $F_1 \neq F_2$.

PROBLEM 7. Let $\mathfrak{F}$ be a family ($\overline{\overline{\mathfrak{F}}} = \aleph_2$, $p(\mathfrak{F}) = \aleph_0$) such that every $\mathfrak{F}' \subseteq \mathfrak{F}$ ($\overline{\overline{\mathfrak{F}'}} \leqq \aleph_1$) possesses property **G**. Does then $\mathfrak{F}$ necessarily possess property **G** too?[19]

[17] See the footnote [4] on p. 328 of [8].

[18] The following theorem is an easy consequence of TYCHONOV's theorem: If $\mathfrak{F}$ is a family of finite sets such that every finite subfamily of $\mathfrak{F}$ possesses property **B**, then $\mathfrak{F}$ possesses property **B**.

[19] This problem is due to W. GUSTIN (oral communication). It is well known and an easy consequence of TYCHONOV's theorem that if for a family $\mathfrak{F}$ of finite sets every finite subfamily of it possesses property **G**, then the whole family possesses property **G** too. See e. g. [9].

PROBLEM 8. Let there be given a graph G of power $\aleph_2$. Suppose that every subgraph $\overline{\overline{G_1}} \leqq \aleph_1$ of G has chromatic number not greater than $\aleph_0$. Is it then true that the chromatic number of G is not greater than $\aleph_0$?[20]

Now we would like to formulate a problem which does not seem to follow directly from $\boldsymbol{T}(\aleph_2, 1) \rightarrow 2$, but which belongs to this class of problems too.

PROBLEM 9. Let there be given a graph G of power $\aleph_2$. Suppose that the edges of every subgraph G_1 of G can be directed so that the number of edges emanating from an arbitrary vertex is finite, provided $\overline{\overline{G_1}} \leqq \aleph_1$.

Is it true that the same holds for the graph G?[21]

A positive solution of Problem 9 would follow from the following generalization of TYCHONOV's theorem. (This generalization is probably false, but as far as we know has not yet been disproved.)

PROBLEM 10. Let $\mathfrak{F}$ be a family of *finite* sets, $\overline{\overline{\mathfrak{F}}} = \aleph_2$, and let $\mathfrak{F} = \{F_\nu\}_{\nu < \omega_2}$ be a well-ordering of type ω_2 of $\mathfrak{F}$. Let $\mathfrak{X}$ denote the Descartes product of the elements of $\mathfrak{F}$, i. e. $\mathfrak{X}$ is the set of all sequences $(x_\nu)_{\nu < \omega_2}$, $x_\nu \in F_\nu$. A subset X of $\mathfrak{X}$ is said to be $\aleph_0$-modified if there exists a set I of ordinal numbers less than ω_2, $\overline{\overline{I}} \leqq \aleph_0$ such that $\overset{1}{x_\nu} = \overset{2}{x_\nu}$ for every $\nu \in I$ implies that $(\overset{1}{x_\nu})_{\nu < \omega_2}$ belongs to $\mathfrak{X}$ if and only if $(\overset{2}{x_\nu})_{\nu < \omega_2}$ belongs to X.

Let $\mathfrak{M}$ be a family of $\aleph_0$-modified subsets of $\mathfrak{X}$ and suppose that the intersection of the elements of every subfamily $\mathfrak{M}'$ of $\mathfrak{M}$ is non-empty, provided $\overline{\overline{\mathfrak{M}'}} \leqq \aleph_1$. Is it true that for an arbitrary family $\mathfrak{M}$ satisfying these conditions $\bigcap_{X \in \mathfrak{M}} X \neq 0$?

9. Further problems. Suppose $p < \aleph_0$.[22] The theorem formulated in the footnote[18] on p. 117 or similar considerations show that to clear up all the problems it would be sufficient to determine the values of the symbols $\mathbf{M}(m, p, q, r) \rightarrow \mathbf{B}$, $\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(s)$ for finite m's, and so we now suppose that m, p, q, r, s are finite. Obviously, if $r = 1$, then the problems become trivial. So the simplest cases when one can find unsolved problems are $q = 2$, $r = 2$.

[20] It is well known that if every finite subgraph of G has chromatic number not exceeding n, then G has chromatic number not exceeding n. See [10].

[21] As an easy application of TYCHONOV's theorem P. ERDŐS and R. RADO proved the following theorem:

If the edges of every finite subgraph of a given graph G can be directed so that the number of edges emanating from an arbitrary vertex is less than a fixed integer n, then the same is true for the graph G.

[22] T. GALLAI pointed out that interesting and perhaps deep questions can be asked concerning the symbols for p less than $\aleph_0$.

One can ask whether $\mathbf{M}(m, p, 2, 2) \rightarrow \mathbf{B}$ is true for a $p > 2$ and for every m. The only non-trivial remark concerning this problem is that

$$\mathbf{M}(7, 3, 2, 2) \not\rightarrow \mathbf{B}. \tag{9.1}$$

This is shown by the Steiner triplets for $m = 7$.

The simplest unsolved problem here is

PROBLEM 11. Is it true that

$$\mathbf{M}(m, 4, 2, 2) \rightarrow \mathbf{B} \quad \text{holds for every} \quad m?$$

We can not even decide whether there exists an integer p_0 such that

$$\mathbf{M}(m, p_0, 2, 2) \rightarrow \mathbf{B} \quad \text{holds for every} \quad m.$$

REMARK. The example (9.1) is best-possible in m, i. e. $\mathbf{M}(6, 3, 2, 2) \rightarrow \mathbf{B}$ is true and it is interesting that for $m = 6$, $\mathbf{M}(6, 3, 2, 2) \rightarrow \mathbf{B}(2)$ is true too. There remain interesting unsolved problems even if we omit the assumption that $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$ for some q and r.

It is obvious that if m is sufficiently large, then a family $\mathfrak{F}$ with $p(\mathfrak{F}) = p$, $\overline{\overline{\mathfrak{F}}} = m$ has not to possess property $\mathbf{B}$. Let $m(p)$ denote the least integer m for which such a family exists.

We have

$$m(p) \leqq \binom{2p-1}{p}, \tag{9.2}$$

as it is shown by the subsets taken p at a time of a set having $2p-1$ elements.

More generally, one can ask for the least integers m for which there exists a family $\mathfrak{F}$ ($\overline{\overline{\mathfrak{F}}} = m$, $p(\mathfrak{F}) = p$) which does not possess property $\mathbf{B}(s)$ where $2 \leqq s \leqq p$. Let $m(p, s)$ denote this integer. (Obviously $m(p, p) = m(p)$.) Similarly as in (9. 2) we have

$$m(p, s) \leqq \binom{p+s-1}{p}. \tag{9.3}$$

(9. 1) shows that the estimations (9. 2) and (9. 3) are far to be best-possible already for $p = 3$. The following problem remains open:

PROBLEM 12. What is the order of magnitude of the functions $m(p)$, $m(p, s)$?

Let us now return to the infinite sets. We would like to raise several new problems, most of which are unsolved, which are all connected to a lesser or greater extent to the ones which we considered so far. To save space we will only outline the partial solutions which we have succeeded in obtaining up to the present.

The first of these problems is the following:

(9.4) Let there be given a family $\mathfrak{F}$ ($\overline{\overline{\mathfrak{F}}} = m$, $p(\mathfrak{F}) = p$) such that every subfamily $\mathfrak{F}'$ of $\mathfrak{F}$ possesses property $\mathbf{B}(r)$, provided that $\overline{\overline{\mathfrak{F}'}} < m$. Under what conditions for the cardinal numbers m, p, r and s does then $\mathfrak{F}$ necessarily possess property $\mathbf{B}(s)$ or property $\mathbf{B}$?

For the sake of brevity we introduce the symbols $\mathbf{S}(m, p, r) \to \mathbf{B}(s)$, $\mathbf{S}(m, p, r) \to \mathbf{B}$ ($\mathbf{S}(m, p, r) \not\to \mathbf{B}(s)$, $\mathbf{S}(m, p, r) \not\to \mathbf{B}$) to indicate the positive (negative) solutions of the problems, respectively. It is obvious that the problem stated in (9.4) is closely connected with the possible generalizations of TYCHONOV's theorem treated in Section 8. We point out only the simplest and typical problems. A general discussion of this symbol seems to be hopeless at present.

The example given by MILLER cited in Theorem 3 shows, if we assume (*), that

(*)(9.5) $$\mathbf{S}(\aleph_1, \aleph_0, 2) \not\to \mathbf{B}.$$

This follows from the fact that the system of almost disjoint sets of power $\aleph_1$ constructed by MILLER has the following property: if x is an element of the basic set and $S(x)$ is the union of the sets belonging to the system containing x and F is a set of the system not containing x, then $\overline{\overline{S(x) \cap F}} < \aleph_0$.

Comparing Theorems 8 and 10 we obtain as a corrollary that

(*)(9.6) $$\mathbf{S}(\aleph_2, \aleph_0, 4) \not\to \mathbf{B}(4).$$

The following problems remain open:

PROBLEM 13. a) $\mathbf{S}(\aleph_2, \aleph_0, 2) \to \mathbf{B}(2)$ or $\mathbf{S}(\aleph_2, \aleph_0, 2) \not\to \mathbf{B}$?

b) $\mathbf{S}(\aleph_2, \aleph_0, 4) \to \mathbf{B}(5)$ or $\mathbf{S}(\aleph_2, \aleph_0, 2) \not\to \mathbf{B}$?

The following problem concerning the symbol introduced in (9.4) is the simplest one for which our theorems proved so far do not give any information.

PROBLEM 14. Let r be an integer $r \geqq 2$. Is it true that $\mathbf{S}(\aleph_\omega, \aleph_0, r) \to \mathbf{B}(r)$ holds?

REMARK. It is easy to see that a negative solution of Problem 14 for any r would imply a negative solution of Problem 4.

The second question which arises concerning property $\mathbf{B}$ is the following: Theorem 3 (MILLER's example) assures that there exists a family $\mathfrak{F}$ ($\overline{\overline{\mathfrak{F}}} = 2^{\aleph_0}$, $p(\mathfrak{F}) = \aleph_0$) such that $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_0)$, but it does not possess property $\mathbf{B}$. However, his example is such that $(\mathfrak{F}) = \aleph_0$ and one can ask whether this is an essential restriction.

Concerning this question, using (*), we can prove the following theorem:

(*) (9.7) *There exists a family* $\mathcal{F}$ $(\overline{\overline{\mathcal{F}}}=\aleph_1,\ p(\mathcal{F})=\aleph_0)$ *which possesses property* $\mathbf{C}(2,\aleph_0)$ *such that it does not possess property* **B** *and satisfies the following condition:*

$(\triangle)$ $\overline{\overline{(\mathcal{F}')}}=\aleph_1$ *for every* $\mathcal{F}'\subseteq\mathcal{F},\ \overline{\overline{\mathcal{F}'}}=\aleph_1$.

We only outline the construction.

Let S be a set, $\overline{\overline{S}}=\aleph_1$. Applying Lemma 3 stated in Section 4 we obtain that there exists a system $\mathfrak{S}$ of subsets of S satisfying the following conditions:

(1) $p(\mathfrak{S})=\aleph_0,\ \overline{\overline{\mathfrak{S}}}=\aleph_1$.

(2) $\mathfrak{S}$ possesses property $C(2,\aleph_0)$.

(3) For an arbitrary $S'\subseteq S$ $(\overline{\overline{S'}}=\aleph_1)$ there exists an $A\in\mathfrak{S}$ such that $A\subseteq S'$.

Let $\mathfrak{S}=\{A_\nu\}_{\nu<\omega_1}$ and $S=\{x_\mu\}_{\mu<\omega_1}$ be well-orderings of type ω_1 of the sets $\mathfrak{S}$ and S, respectively.

Let $\mathfrak{S}_\nu$ be a system of subsets of A_ν for which $p(\mathfrak{S}_\nu)=\aleph_0,\ \overline{\overline{\mathfrak{S}_\nu}}=\aleph_1$, further let $\mathfrak{S}_\nu$ possess property $\mathbf{C}(2,\aleph_0)$. Let $\mathfrak{S}_\nu=\{B^\nu_\mu\}_{\mu<\omega_1}$ be a well-ordering of type ω_1 of the set $\mathfrak{S}_\nu$ for every $\nu<\omega_1$. It is obvious that one can define a monotone increasing sequence $\{\mu_\nu\}_{\nu<\omega_1}$ of type ω_1 of ordinal numbers less than ω_1 such that $\mu_\nu>\mu'$ for every $x_{\mu'}\in A_\nu$ (hence for every $x_{\mu'}\in B^\nu_\mu$ for every $\mu<\omega_1$).

Put $C^\nu_\mu=B^\nu_\mu\cup\{x_{\mu_\nu+\mu}\}$ and $\mathcal{F}=\{C^\nu_\mu\}_{\nu<\omega_1,\,\mu<\omega_1}$. It is obvious from (1) and (2) that $\overline{\overline{\mathcal{F}}}=\aleph_1$, $p(\mathcal{F})=\aleph_0$ and $\mathcal{F}$ possesses property $\mathbf{C}(2,\aleph_0)$. The fact that $\mathcal{F}$ does not possess property **B** follows from the property of $\mathfrak{S}$ stated in (3) (taking into consideration that a set which intersects every element of $\mathcal{F}$ has to be of power $\aleph_1$). Finally, it is easy to verify that if $\overline{\overline{\mathcal{F}'}}=\aleph_1$, then $\overline{\overline{(\mathcal{F}')}}=\aleph_1$ for every $\mathcal{F}'\subseteq\mathcal{F}$, since if $\overline{\overline{\mathcal{F}'}}=\aleph_1$, then $\mathcal{F}'$ either contains $\aleph_1$ C^ν_μ's with the same ν or $\aleph_1$ C^ν_μ's with pairwise different ν's.

The following refinement of the problem solved in (9.7) seems to be interesting. Let us say that the set X is almost contained in Y if $Y-X$ is finite.

PROBLEM 15. Let $\mathcal{F}$ be a family $(p(\mathcal{F})=\aleph_0,\ \overline{\overline{\mathcal{F}}}=\aleph_1)$ such that $\mathcal{F}$ possesses property $\mathbf{C}(2,\aleph_0)$ and suppose that (instead of $(\triangle)$) it possesses the following property:

At most $\aleph_0$ sets belonging to $\mathcal{F}$ are almost contained in a denumerable set.

Does such a family $\mathcal{F}$ necessarily possess property **B**?

The answer is probably negative to this question too, but we can not disprove it even if we omit the assumption that $\mathfrak{F}$ consists of almost disjoint sets.

The following question is connected with Problem 3 (namely a positive solution of it would imply a positive solution of Problem 3b)):

PROBLEM 16. Put $S = \{\nu\}_{\nu < \omega_{\omega+1}}$ $(\overline{\overline{S}} = \aleph_{\omega+1})$.

Let S_ν denote the set $\{\mu\}_{\mu<\nu}$ for every $\nu < \omega_{\omega+1}$. Then $\overline{\overline{S_\nu}} \leqq \aleph_\omega$, and so one can define a splitting of S_ν onto the sum of $\aleph_0$ disjoint sets such that

$$S_\nu = \bigcup_{n<\omega} S_n^\nu \quad \text{and} \quad \overline{\overline{S_n^\nu}} < \aleph_\omega \quad \text{for every} \quad \nu < \omega_{\omega+1}.$$

Is it possible to define the sets S_n^ν in such a way that for every $\nu < \omega_{\omega+1}$ of the second kind which is not cofinal with ω, there exists a monotone increasing sequence $\{\nu_\tau\}_{\tau<\varphi}$ of type φ of ordinal numbers less than ν cofinal with ν and such that $S_n^{\nu_\tau} \subseteq S_n^{\nu_{\tau'}}$ for every n and for every $\tau < \tau' < \varphi$?

A similar but simpler problem is the following one:

PROBLEM 17. Let S be the set of ordinal numbers less than ω_1. Is it possible to define a function $f(\nu)$ on S such that $f(\nu) \in S$, $f(\nu) < \nu$ for every $\nu < \omega_1$ which has the following property: If $\nu < \omega_1$ and ν is of the second kind, then there exists a sequence $\{\nu_n\}_{n<\omega}$ of type ω of ordinal numbers less than ν such that $\nu_n \to \nu$ and $f(\nu_{n+1}) = \nu_n$ for $n = 0, 1, 2, \ldots$. This problem is interesting in itself and seems to be very difficult.

The positive solution of the following problem would imply a negative solution of an immediate generalization of Problem 9, namely it would assure the existence of a graph G of power $\aleph_{\omega+1}$ the edges of every subgraph of power $\aleph_1$ of which can be directed so that the number of edges emanating from a vertex should be finite, but the whole graph can not be directed in such a way.

PROBLEM 18. Let S be a set of power $\aleph_\omega$. Does there exist a family $\mathfrak{F}$ such that $(\mathfrak{F}) \subseteq S$, $\overline{\overline{\mathfrak{F}}} = \aleph_{\omega+1}$, $p(\mathfrak{F}) = \aleph_0$, and which has the following property:

(1) If $S' \subseteq S$, $\overline{\overline{S'}} \leqq \aleph_0$, then there exist at most $\aleph_0$ sets F belonging to the family such that $\overline{\overline{F \cap S'}} = \aleph_0$.

REMARK. On the one hand, we can not disprove Problem 18 even if we require that $\mathfrak{F}$ should possess property $\mathbf{C}(2, \aleph_0)$, on the other hand we can not prove it if we require only that $\mathfrak{F}$ should possess the following weaker property instead of (1):

Every $S' \subseteq S$ $(\overline{\overline{S'}} = \aleph_0)$ contains at most $\aleph_0$ elements of the family.

We construct the graph G mentioned above as follows: Suppose that the family $\mathfrak{F}$ and the set S satisfy the requirement of Problem 18. Let the set of vertices of G be $\mathfrak{F} \cup S$. The edges are the pairs (F, x) where $F \in \mathfrak{F}$ and $x \in F$. It is easy to see that G has the property required.

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Added in proof (MARCH 3, 1961). The manuscript of this paper had been written before the authors knew that A. TARSKI has disproved the hypothesis (**). (See A. TARSKI, Some problems and results relevant to the foundations of set theory, *Proceedings of the International Congress for Logic, Methodology and Philosophy of Science* (Stanford, 1960).)

Thus we have no arguments to prove our Theorem 12 proved with the help of this hypothesis. It seems that the theorem is false at least for the inaccessible cardinals m which are strongly incompact.

It is obvious that the discussion of the unsolved problems concerning the symbol $\boldsymbol{T}(m, \lambda) \to \varkappa$ has to be changed in some places knowing the new result of A. TARSKI.

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SOME REMARKS AND PROBLEMS ON THE COLOURING OF INFINITE GRAPHS AND THE THEOREM OF KURATOWSKI[1]

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1. We consider the following propositions:

T. The topological product of any number of bicompact Hausdorff spaces is bicompact.[2]

T*. The topological product of any number of non-empty bicompact Hausdorff spaces is non-empty and bicompact.

I. In every Boolean algebra A there is a maximal ideal different from A.

R. Every Boolean algebra is isomorphic to a field of sets.

M. Every consistent elementary theory has a model.[3]

T_n. The topological product af any number of Hausdorff spaces, each having exactly n points, is non-empty and bicompact.[4]

S_n. Let M be any set of disjoint sets each having exactly n elements and $R(x, y)$ is a symmetric relation defined between the elements belonging to different elements of M. Suppose that for any finite set $F \subseteq M$ there exists an $f \in \mathsf{P}_{X \in F} X$ such that $R(f(X_1), f(X_2))$ holds for any $X_1, X_2 \in F$. Then there exists an $f \in \mathsf{P}_{X \in M} X$ such that $R(f(X_1), f(X_2))$ holds for any $X_1, X_2 \in M$.[5]

P_n. Every graph, each finite subgraph of which can be coloured with n colours, can be coloured with n colours.[6]

C_n. The Cartesian product of any number of sets, each having exactly n elements, is non-empty.

It is known that the axiom of choice implies each of the above propo-

[1] This is a lecture delivered on the Colloquium on the Theory of Graphs in Dobogókő, 22 October 1959.

[2] Here and farther *any number* means any positive finite or infinite cardinal number.

[3] We do not suppose that the number of symbols and statements of the theories is denumerable.

[4] Here and farther n is running over positive integers.

[5] P denotes the Cartesian product operator.

[6] A colouring of a graph with n colours is a partition of the set of vertices into n classes such that no two vertices in one class are joint by an edge.

sitions, but the following logical relations can be proved without the use o this axiom:

(1) $\mathrm{T} \longleftrightarrow \mathrm{T}^* \longleftrightarrow \mathrm{I} \longleftrightarrow \mathrm{R} \longleftrightarrow \mathrm{M} \longleftrightarrow \mathrm{T}_m \longleftrightarrow \mathrm{S}_n$ for $m = 2, 3, \ldots,\ n = 4, 5, \ldots;$

(2) $\mathrm{S}_4 \rightarrow \mathrm{S}_3 \rightarrow \mathrm{S}_2;$

(3) $\mathrm{S}_n \rightarrow \mathrm{P}_n \rightarrow \mathrm{C}_n$ for $n = 2, 3, \ldots;$

(4) $\mathrm{P}_{n+1} \rightarrow \mathrm{P}_n$ for $n = 2, 3, \ldots;$

(5) $\mathrm{P}_2 \longleftrightarrow \mathrm{C}_2.$

It would be interesting to know any further implication between these propositions. Some implications and independences between the propositions C_n are known, e. g. $\mathrm{C}_2 \longleftrightarrow \mathrm{C}_4$, $\mathrm{C}_{mn} \rightarrow \mathrm{C}_m$ and others (see [8], [10], [11]). The equivalences (1) and implications (2) are proved in the papers [4], [5], [6], [7]. Other interesting propositions which may be added to the equivalences (1) are given in [9].

Let us prove (3), (4) and (5):

$\mathrm{S}_n \rightarrow \mathrm{P}_n$ is obvious (compare the proof of P_n given in [2]).

$\mathrm{P}_n \rightarrow \mathrm{C}_n$. Let K be a set of disjoint n-element sets. We treat $\bigcup_{X \in K} X$ as a set of vertices of a graph, two vertices being joint if and only if they belong to the same X. By P_n it is easy to see that this graph can be coloured with n colours. Take all the vertices of one colour, this clearly defines a selection from K as required in C_n.

$\mathrm{P}_{n+1} \rightarrow \mathrm{P}_n$. Let G be a graph each finite subgraph of which can be coloured with n colours. We add a new vertex and join it to all vertices of G. Using P_{n+1} we easily see that the new graph can be coloured with $n+1$ colours. Removing the additional vertex we obtain n-colourings of G as needed in P_n.

$\mathrm{P}_2 \longleftrightarrow \mathrm{C}_2$. Owing to (3) it remains to prove $\mathrm{C}_2 \rightarrow \mathrm{P}_2$. If G is a connected graph, each finite subgraph of which can be coloured with 2 colours, then it is easy to see that, putting two vertices in the same class if and only if there exists a path from one to the other with an odd number of edges, we obtain a two-colouring of G. Now if G is not connected, using C_2 we select one of these classes for each component of G. We consider the partition of the vertices of G into 2 classes: the union of the selected classes and the remaining vertices. It is easy to see that it is a two-colouring of G.

REMARK (due to C. RYLL-NARDZEWSKI). The proposition P_n restricted to denumerable graphs can be proved without using the axiom of choice.

2. We consider the following properties of a graph G (by a graph we mean here a one-dimensional simplicial complex with the natural topology, we do not suppose that it is locally finite and the cardinality of the set of vertices of G is arbitrary):

(i) G does not contain topologically any one of KURATOWSKI's two graphs (Fig. 1).

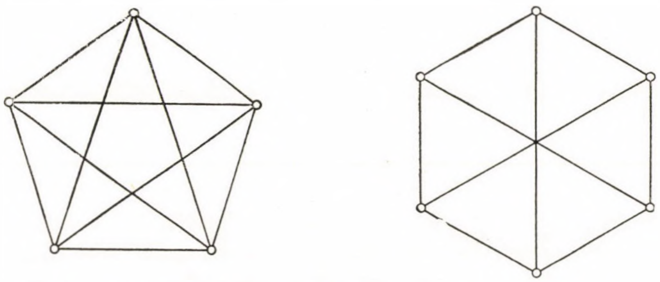

Fig. 1

(ii) Every finite subgraph of G is homeomorphically imbeddable in the plane R^2.

(iii) There exists a system of homeomorphisms $\{h_F(x)\}$ where F runs over all finite subgraphs of G such that h_F maps homeomorphically F into R^2 and for any F_1 and F_2

$$(*) \qquad h_{F_1}|F_1 \cap F_2 \text{ is homotopical to } h_{F_2}|F_1 \cap F_2.$$ [7]

(iv) One can define for every circuit C of G a partition of the set $|G| \setminus |C|$ [8] into two classes Int (C), Ext (C) such that two vertices belonging to different classes are not joint by an edge and

$$\text{if } |C_1| \subset |C_2| \cup \text{Int } (C_2), \text{ then Int } (C_1) \subset |C_2| \cup \text{Int } (C_2);$$
$$\text{if } |C_1| \subset |C_2| \cup \text{Ext } (C_2), \text{ then Ext } (C_1) \subset |C_2| \cup \text{Ext } (C_2).$$

THEOREM. *The properties* (i), (ii), (iii), (iv) *are equivalent.*

PROOF. (i) $\rightarrow$ (ii) by the well-known theorem of KURATOWSKI [3].

(ii) $\rightarrow$ (iii). We denote by S_F the set of homotopy types of homeomorphical applications of F into R^2 (F runs over the finite subgraphs of G). S_F is finite; we treat it as a discrete topological space. By the statement T (Section 1 of this paper) the topological product $\mathop{\mathsf{P}}\limits_F S_F$ is bicompact.

For any $t_1 \in S_{F_1}$ and $t_2 \in S_{F_2}$ we put $t_1 \sim t_2$ if and only if $(*)$ holds for some h_{F_1} of type t_1 and h_{F_2} of type t_2. Let $F_1, \ldots, F_m$ be any finite set of

[7] $f|X$ denotes the mapping f with domain restricted to X.

[8] $|H|$ denotes the set of vertices of the graph H. $\setminus$ denotes the set-theoretical difference.

finite subgraphs of G. We put $K_{F_1,\ldots,F_m}=\{f : f\in \mathop{\mathsf{P}}_F S_F,\ f(F_i)\sim f(F_j)$ for $i,j=1,\ldots,m\}$. Of course, the sets $K_{F_1,\ldots,F_m}$ are closed subsets of $\mathop{\mathsf{P}}_F S_F$. They are also non-empty, since if F is a finite subgraph of G such that $F_1,\ldots,F_m$ are subgraphs of F and h_F is a homeomorphism $h_F : F\to R^2$ (it exists by (ii)), then one can take for $f\in K_{F_1,\ldots,F_m}$ any function $f\in \mathop{\mathsf{P}}_F S_F$ such that $f(F_i)$ is the homotopy type of $h_F|F_i$. The finite intersections of the sets $K_{F_1,\ldots,F_m}$ are also non-empty, since

$$K_{F_1^{(1)},\ldots,F_m^{(1)}}\cap K_{F_1^{(2)},\ldots,F_n^{(2)}}\supset K_{F_1^{(1)},\ldots,F_m^{(1)},\,F_1^{(2)},\ldots,F_n^{(2)}}.$$

It follows that there exists an f_0 such that

$$f_0\in \bigcap_{m=1}^{\infty}\ \bigcap_{F_1,\ldots,F_m} K_{F_1,\ldots,F_m}$$

and clearly any system $\{h_F\}$, such that the homotopy type of h_F is $f_0(F)$ satisfies (iii); q. e. d.

(iii) $\to$ (iv). A system $\{h_F\}$ being given, for every circuit C and every vertex $v\in|G|\setminus|C|$ we put $v\in \mathrm{Int}(C)$ if the homeomorphism $h_{C\cup(v)}$ maps v inside the domain bounded by the image of C and $v\in \mathrm{Ext}(C)$ in the other case. It is easy to verify that our definition satisfies (iv).

(iv) $\to$ (i). Clearly a subgraph of a graph satisfying (iv) satisfies (iv) One can prove by a direct verification that no one of the Kuratowski graphs satisfies (iv); and our implication follows.

COROLLARY. (DIRAC and SCHUSTER [1].) *A denumerable graph satisfying* (i) *has a continuous* 1—1 *mapping into* R^2.

PROOF. By the theorem the graph satisfies (iii) and one can construct the mapping by an easy induction.

REMARK. The equivalence (ii) $\leftrightarrow$ (iii) remains valid if one replaces in these statements R^2 by any bicompact 2-manifold.

PROBLEM. Does there exist a finite set of finite graphs such that any finite graph G can not be homeomorphically imbedded in a given bicompact 2-manifold (e. g. the projective plane) if and only if G contains a subgraph homeomorphic to one of them?

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MAXIMUM-MINIMUM SÄTZE UND VERALLGEMEINERTE FAKTOREN VON GRAPHEN

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Einleitung

Von D. KÖNIG stammt der folgende Satz ([5], S. 233, Satz 14):[1]

SATZ (1) (KÖNIG). *In einem paaren Graphen*[2] *ist die maximale Anzahl der unabhängigen Kanten gleich der minimalen Anzahl der trennenden Punkte.*

Die Kanten $x_1, \dots, x_m$ heißen unabhängig, wenn entweder $m = 1$ ist, oder wenn je zwei von ihnen keinen gemeinsamen Punkt enthalten. Die Punkte $X_1, \dots, X_n$ heißen trennend, wenn jede Kante des Graphen einen dieser Punkte enthält. Die Behauptung des Satzes (1) ist für nichtpaare Graphen im allgemeinen nicht richtig. Es entsteht daher das Problem: durch geeignete Modifizierung der vorkommenden Größen eine auf beliebige Graphen geltende Verallgemeinerung zu finden. Wir wollen nun in dieser Arbeit zeigen, daß man die minimale Anzahl der trennenden Punkte (kurz: $p_{\min}$) durch eine solche Größe ersetzen kann, die im Falle paarer Graphen mit $p_{\min}$ zusammenfällt, und mit welcher die Behauptung des Satzes (1) für beliebige Graphen richtig ist. Diese Größe wird sich als das Minimum der geeignet bestimmten Werte von gewissen Gewichtssystemen ergeben. Weitergehend werden wir den Satz (1) auch in anderen Richtungen verallgemeinern. Im Satze (1) spielen nur Systeme von Kanten eine Rolle. Statt dieser werden wir aus Bogen, d. h. aus Wegen und Schlingen[3] zusammengesetzte Systeme in Betracht ziehen. Um den Begriff „unabhängig" auszudehnen, ordnen wir jedem Punkte X des Graphen zwei nichtnegative ganze Zahlen: eine Einlaufskapazität $\varkappa(X)$ und eine Durchgangskapazität $\varkappa'(X)$ zu. Ein aus Bogen bestehendes System H soll bezüglich $\varkappa$ und $\varkappa'$ aufnehmbar heißen, wenn je zwei Bogen von H keine gemeinsame Kante enthalten, in jedem Punkte X

[1] Wir sagen statt Knotenpunkte kurz nur Punkte.

[2] Ein Graph heißt paarer, wenn die Menge der Punkte des Graphen so in zwei Teilmengen zerlegt werden kann, daß jede Kante des Graphen zwei Punkte von verschiedenen Teilmengen verbindet.

[3] Eine Schlinge ist ein Kreis mit einem ausgezeichneten Punkte X des Kreises. Wir betrachten X als einen zweifachen Randpunkt der Schlinge.

höchstens $\varkappa(X)$ Randpunkte der Bogen von H fallen, und durch X höchstens $\varkappa'(X)$ Bogen von H gehen. Ersetzt man dann im Satze (1) die maximale Anzahl der unabhängigen Kanten durch die maximale Anzahl der aufnehmbaren Bogen, die Größe $p_{\min}$ wieder durch das Minimum der „Werte" von gewissen Gewichtssystemen, so entsteht ein allgemeiner Maximum-Minimum Satz, der Hauptsatz unserer Arbeit (Satz (3.3)). Aus diesem Satze ziehen wir mehrere Folgerungen. Unter anderem geben wir zwei, dem Mengerschen n-Kettensatz ([6], S. 222) ähnliche Trennungssätze (Sätze (13.2) und (13.4)). Wir leiten gleichfalls aus unserem Hauptsatze eine Reihe von Faktorisationssätzen ab. Ist jedem Punkte X des Graphen nur eine nichtnegative ganze Zahl $\varkappa(X)$ zugeordnet, so versteht man unter einem $\varkappa$-Faktor eine Menge von Kanten mit der Eigenschaft, daß zu jedem Punkte X genau $\varkappa(X)$ Kanten der Menge inzident sind (s. [10], S. 316). Sind zu jedem Punkte X zwei nichtnegative ganze Zahlen $\varkappa(X)$ und $\varkappa'(X)$ zugeordnet, so wollen wir unter einem verallgemeinerten Faktor ein solches, bezüglich $\varkappa$ und $\varkappa'$ aufnehmbares Bogensystem H verstehen, bei welchem in jeden Punkt X des Graphen genau $\varkappa(X)$ Randpunkte von H fallen. Es werden nun allgemeine notwendige und hinreichende — sowie im Falle spezieller Graphen und Kapazitäten einfache hinreichende — Bedingungen der Existenz von verallgemeinerten Faktoren angegeben (§ 14—16). Diese enthalten als Spezialfälle mehrere Sätze über $\varkappa$-Faktoren von PETERSEN, BAEBLER, TUTTE, BELCK und ORE.

Wir leiten unsere Sätze nur für solche Graphen ab, die keine Kantenschlingen, d. h. Kanten mit zusammenfallenden Endpunkten enthalten. Es bietet aber keine Schwierigkeit, unsere Ergebnisse auch auf Graphen mit Kantenschlingen auszubreiten.

Den Hauptsatz beweisen wir mit einer geeigneten Verallgemeinerung der Methode der alternierenden Züge (s. [3], [4], [10], [11]). Den Spezialfall, wo sämtliche $\varkappa'$-Werte verschwinden, kann man wesentlich kürzer behandeln. Es scheint wahrscheinlich zu sein, daß der allgemeine Fall auf diesen Spezialfall zurückführbar ist. Auf diese Weise könnte man zu einem einfacheren Beweis des Hauptsatzes gelangen.

Die vorliegende Arbeit ist in vier Abschnitte geteilt. Im ersten Abschnitt (§ 1—3) geben wir die Erklärung der nötigen Grundbegriffe und formulieren den Hauptsatz. Der zweite (§ 4—10) enthält den Beweis des Hauptsatzes. Im dritten (§ 11—13) untersuchen wir einige Spezialfälle des Hauptsatzes und leiten die erwähnten Trennungssätze ab. Der vierte Abschnitt (§ 14—16) enthält die Faktorisationssätze.

Um die weniger wichtigen Behauptungen von den eigentlichen Sätzen zu unterscheiden, lassen wir die Benennung „Satz" bei den ersteren weg.

I. FORMULIERUNG DES HAUPTSATZES

§ 1. Grundbegriffe

(1. 1) Der ungerichtete, endliche Graph Γ ist durch zwei elementenfremde endliche Mengen, durch die Menge Φ_Γ der „Punkte" und die Menge Ψ_Γ der „Kanten" sowie durch eine Inzidenzvorschrift I_Γ gegeben. I_Γ gibt an, ob ein beliebiger Punkt von Φ_Γ und eine beliebige Kante von Ψ_Γ *inzident* sind oder nicht, und diese Vorschrift genügt nur der einen Bedingung, daß zu jeder Kante genau zwei (verschiedene) Punkte inzident sind (s. [5]). Wir bezeichnen einen Punkt bzw. eine Kante immer mit X bzw. mit x, eventuell auch mit Indizes oder anderen Zeichen versehen. Ist nach I_Γ die Kante x zu den verschiedenen Punkten X_1 und X_2 inzident, so sagen wir: x *verbindet* die Punkte X_1 und X_2, X_1 und X_2 sind die Punkte oder *Randpunkte* von x, x ist eine *X_1X_2-Kante.* Von den Punkten und Kanten von Γ werden wir auch sagen, daß sie *in* oder *auf* Γ liegen.

Sind Φ_Γ und Ψ_Γ beide leer, so heißt der Graph Γ *leer*. $\Gamma' \subseteq \Gamma$ soll bedeuten: der Graph Γ' ist ein *Teilgraph* von Γ, d. h. es gilt $\Phi_{\Gamma'} \subseteq \Phi_\Gamma$, $\Psi_{\Gamma'} \subseteq \Psi_\Gamma$ und ein Punkt und eine Kante von Γ' sind in Γ' dann und nur dann inzident, wenn sie in Γ inzident sind. Ist $\Gamma' \subseteq \Gamma$ und $\Gamma' \neq \Gamma$, so heißt Γ' ein *echter* Teilgraph von Γ, und wir schreiben $\Gamma' \subset \Gamma$. Ein Teilgraph Γ' von Γ ist durch Angabe der Mengen $\Phi_{\Gamma'}$ und $\Psi_{\Gamma'}$ eindeutig bestimmt.

Ist $\Gamma' \subset \Gamma$ und liegt nur der eine Randpunkt der Kante x von Γ in Γ', so sagen wir, daß x den Graphen Γ' *berührt*, und der nicht zu Γ' gehörige Randpunkt von x heißt der *äußere* Punkt von x (bezüglich Γ').

Ist M eine beliebige endliche Menge, so bezeichnen wir mit $\nu(M)$ die Anzahl der Elemente von M. Die leere Menge bezeichnen wir mit $\emptyset$.

Ist E eine beliebige Teilmenge von Φ_Γ, so nennen wir die Punkte von E kurz *E-Punkte.* Ist $\varphi(X)$ eine beliebige in Φ_Γ definierte Funktion, so soll $\varphi(E)$ im Falle $E \neq \emptyset$ den Wert $\sum_{X \in E} \varphi(X)$, im Falle $E = \emptyset$ den Wert Null bedeuten.

Es sei $E \subseteq \Phi_\Gamma$ und $F \subseteq \Phi_\Gamma$. Wir nennen jene Kanten, bei denen ein Randpunkt ein E-Punkt und der andere ein F-Punkt ist, eine *EF-Kante.* Die Anzahl der EF-Kanten von Γ bezeichnen wir mit $\nu_\Gamma(E, F)$. Es gilt also z. B. $\nu_\Gamma(\Phi_\Gamma, \Phi_\Gamma) = \nu(\Psi_\Gamma)$. Ist eine der Mengen E und F leer, so sei $\nu_\Gamma(E, F) = 0$. Ist $E = \{X\}$, so schreiben wir statt EF-Kanten bzw. statt $\nu_\Gamma(E, F)$ auch XF-Kanten und $\nu_\Gamma(X, F)$. Die Zahl $\nu_\Gamma(X', X)$ gibt die Anzahl der XX'-Kanten von Γ an.

Das Zeichen $\varrho_\Gamma(X)$ soll den *Grad* von X in Γ, d. h. die Anzahl der zu X inzidenten Kanten von Γ bedeuten. Ist $\varrho_\Gamma(X)=0$, so heißt X ein *isolierter* Punkt von Γ.

Ist $E\subseteq\Phi_\Gamma$, so soll $[E]_\Gamma$ denjenigen Teilgraphen von Γ bezeichnen, der sämtliche E-Punkte und EE-Kanten von Γ und nur diese enthält. (Ist also $E=\emptyset$, so ist $[E]_\Gamma$ der leere Graph.)

Wir machen folgende Vereinbarungen: Γ soll immer einen endlichen ungerichteten Graphen bezeichnen, ferner wollen wir von den Zeichen bzw. Begriffen, die sich auf Γ beziehen, den Index Γ bzw. die Ausdrücke „von Γ", „in Γ" usw. im allgemeinen weglassen. Diese beziehen sich also — wenn anders nicht gesagt wird — immer auf den mit Γ bezeichneten Graphen. So bedeutet z. B. Φ die Menge Φ_Γ, und der Ausdruck „für jeden X" den Ausdruck „für jeden X von Γ".

Ist $E\subseteq\Phi$, so setzen wir $\bar{E}=\Phi-E$.

(1.2) Sind $X_0,\dots,X_n$ $(n\geqq 1)$[4] (nicht unbedingt verschiedene) Punkte und $x_1,\dots,x_n$ *verschiedene* Kanten von Γ, sind ferner X_{i-1} und X_i die beiden Randpunkte von x_i $(i=1,\dots,n)$, so heißt die Folge

$$p=(X_0x_1X_1\dots X_{n-1}x_nX_n)$$

ein *Zug* (von Γ). Kommt der Punkt X in der Folge $X_0,\dots,X_n$ genau m-mal vor $(m\geqq 1)$, so heißt X ein *m-facher* Punkt von p. Im Falle $m=1$ bzw. $m>1$ sprechen wir auch von *einfachen* bzw. *mehrfachen* Punkten von p (wir sagen auch, daß X in p einfach bzw. m-fach ist). X_0 ist der *Anfangspunkt*, X_n der *Endpunkt* von p, beide heißen die *Randpunkte* von p. Ähnlicherweise bezeichnen wir die Kanten x_1 und x_n. Die Anzahl der Kanten von p heißt die *Länge* von p.

Wir heben hervor, daß mit einem Zuge p eine bestimmte „Durchlaufsrichtung" verbunden ist, die jeder Kante von p eine *eindeutig* bestimmte Reihennummer zuteilt. (Das gleiche gilt für die Punkte im allgemeinen nicht!)

(1.3) Ist $p=(X_0x_1X_1\dots X_{n-1}x_nX_n)$ ein Zug von Γ, so bezeichne

$$[p]=[X_0x_1X_1\dots X_{n-1}x_nX_n]$$

denjenigen Teilgraphen von Γ, der sämtliche Punkte und Kanten von p, und nur diese, enthält. Sind $X_0,\dots,X_n$ verschieden, so heißt $[p]$ ein *Weg*. X_0 und X_n sind die Randpunkte des Weges. Sind $X_0,\dots,X_{n-1}$ $(n\geqq 2)$ verschieden und $X_n=X_1$, so heißt $[p]$ ein *Kreis*. Ein Kreis mit einem ausgezeichneten Punkt X' des Kreises heißt eine *Schlinge*. Wir nennen X' den Randpunkt der Schlinge, bzw. wir sagen, daß in X' *zwei* Randpunkte der

[4] Sind m und n ganze Zahlen und ist $m<n$, so bedeutet $(a_m,\dots,a_n)$ diejenige Folge, die dadurch zustande kommt, daß der Index i in a_i jede ganze Zahl von m bis n durchläuft. Im Falle $m=n$ bedeutet $(a_m,\dots,a_n)$ das einzige Element a_m.

Schlinge fallen. Wege und Schlingen heißen gemeinsam *Bogen*. Die von den Randpunkten verschiedenen Punkte eines Bogens heißen *innere* Punkte des Bogens. Ist h ein Bogen, so bedeutet die Gleichung

$$h = [X_0 x_1 X_1 \cdots X_{n-1} x_n X_n],$$

daß $p = (X_0 x_1 X_1 \cdots X_{n-1} x_n X_n)$ ein Zug ist und h und p dieselben Punkte, Kanten und Randpunkte besitzen.

Den Weg $[X_0 x_1 X_1]$ kann man mit der Kante x_1 selbst identifizieren. Wollen wir eine Kante gleich mit ihren Randpunkten angeben, so schreiben wir sie in der Form $[X_0 x_1 X_1]$.

(1.4) Das Zeichen $[h, X]$ bzw. $|h, X|$ soll angeben, wie viele Randpunkte bzw. inneren Punkte des Bogens h in X fallen. $[h, X]$ bzw. $|h, X|$ können nur die Werte 0, 1 oder 2 bzw. 0 oder 1 annehmen.

(1.5) Γ heißt *zusammenhängend*, wenn er höchstens einen Punkt enthält oder wenn zu je zwei Punkten von Γ ein Weg von Γ existiert, der beide Punkte enthält. Die maximalen zusammenhängenden Teilgraphen von Γ sind die *Komponenten* von Γ. Ist Γ nicht zusammenhängend, so besteht er aus mehreren nichtleeren Komponenten.

Ist $E \subseteq \Phi$, so nennen wir die Komponenten des Graphen $[E]$ die *E-Komponenten* von Γ und bezeichnen diese mit $[E_1], \ldots, [E_m]$ $(m \geqq 1)$. Ferner soll hier E_i $(1 \leqq i \leqq m)$ die Menge der Punkte von $[E_i]$ bedeuten. Ist $E = \emptyset$, so ist $m = 1$, $E_1 = \emptyset$ und $[E_1]$ der leere Graph.

§ 2. Bogen- und Gewichtssysteme

Von hier an bezeichne Γ durch die ganze Arbeit immer einen *nichtleeren* Graphen.

(2.1) Sind $h_1, \ldots, h_n$ Bogen von Γ, so heißt die Folge $H = (h_1, \ldots, h_n)$ ein *Bogensystem* von Γ. Wir betrachten zwei Systeme, die sich nur in der Reihenfolge ihrer Glieder unterscheiden, als identisch. Die leere Folge betrachten wir auch als ein Bogensystem. $H = 0$ soll ausdrücken, daß H leer ist.

$\nu(H)$ bezeichne die Anzahl der Glieder von H. Nach (1.4) gibt im Falle $H \neq 0$

$$[H, X] = \sum_{i=1}^{n} [h_i, X] \quad \text{bzw.} \quad |H, X| = \sum_{i=1}^{n} |h_i, X|$$

an, wie viele Randpunkte bzw. inneren Punkte der Bogen von H insgesamt in X fallen. Ist $H = 0$, so sei $[H, X] = |H, X| = 0$. Es gilt offensichtlich für jedes System H von Γ

$$\sum_{X \in \Phi} [H, X] = 2\nu(H). \tag{1}$$

(2.2) Ein zu Γ gehöriges *Gewichtssystem* q kommt dadurch zustande, daß wir zu jedem Punkt und zu jeder Kante von Γ eines der Gewichte 0, 1 oder 1/2 zuordnen. $q(X)$ bzw. $q(x)$ soll das Gewicht bezeichnen, welches in q zu X bzw. x gehört.

Wir sagen, daß der Bogen $h = [X_0 x_1 X_1 \ldots X_{n-1} x_n X_n]$ durch das Gewichtssystem q *gefüllt* ist, wenn einer der folgenden Fälle besteht: (1) Es gibt ein i $(0 \leqq i \leqq n)$ mit $q(X_i) = 1$. (2) Es gibt ein i $(1 \leqq i \leqq n)$ mit $q(x_i) = 1$. (3) Es gilt $q(X_0) = q(X_n) = 1/2$. (4) Es ist $q(X_0) = q(x_n) = 1/2$ oder $q(x_1) = q(X_n) = 1/2$. (5) Es gilt $q(x_1) = q(x_n) = 1/2$ $(n \geqq 2)$.

Wir wollen nur solche Systeme q betrachten, die jeden Bogen von Γ füllen. Man kann leicht einsehen: füllt q jede Kante, so füllt es auch jeden Bogen. Bezeichnet man mit A, B bzw. C die Menge der Punkte, die in q das Gewicht 0, 1 bzw. 1/2 bekommen, so muß in q, falls q jede Kante füllt, jede AA-Kante das Gewicht 1, jede AC-Kante das Gewicht 1/2 erhalten. Ferner ist es klar, daß diese Kantengewichte zusammen mit den Punktgewichten schon zu der Füllung sämtlicher Kanten genügen. Deshalb werden wir im folgenden nur solche Systeme betrachten, die außer den obenerwähnten keine von Null verschiedenen Gewichte enthalten. Diese Systeme sind durch die Mengen A, B und C eindeutig bestimmt. Zusammenfassend: Jedes von uns betrachtete sog. *füllende Gewichtssystem* q geben wir durch eine *geordnete Zerlegung* der Menge Φ in drei Teilmengen an, d. h. wir geben drei beliebige, paarweise fremde Teilmengen A, B und C von Φ mit $A \cup B \cup C = \Phi$ sowie eine bestimmte Reihenfolge A, B, C dieser Mengen an. Jeder A-, B- bzw. C-Punkt soll in q das Gewicht 0, 1 bzw. 1/2, jede AA- bzw. AC-Kante das Gewicht 1 bzw. 1/2, jede übrige Kante das Gewicht 0 erhalten. Diese Bestimmung von q werden wir kurz durch

$$q = q(A, B, C)$$

ausdrücken.

Wir bezeichnen die Menge der so definierten Gewichtssysteme mit Q. Q ist nichtleer.

§ 3. Aufnehmbare Bogensysteme. Der Wert eines Gewichtssystems

(3.1) Es seien $\varkappa(X)$ und $\varkappa'(X)$ zwei auf der Menge Φ der Punkte von Γ definierte Funktionen, die nur nichtnegative ganze Werte annehmen. Wir wollen diese „Kapazitätsfunktionen" festhalten und die Werte von $\varkappa(X)$ als Einlaufskapazitäten, diejenige von $\varkappa'(X)$ als Durchgangskapazitäten betrachten. Ein Bogensystem H von Γ nennen wir in bezug auf $\varkappa$ und $\varkappa'$ *aufnehmbar*, wenn H folgenden Bedingungen genügt:

(a) Enthält H mehr als einen Bogen, so haben diese Bogen paarweise keine gemeinsame Kante.

(b) Für jedes X gilt $[H, X] \leqq \varkappa(X)$ und $|H, X| \leqq \varkappa'(X)$.

Wir werden im folgenden den Ausdruck „in bezug auf $\varkappa$ und $\varkappa'$" im allgemeinen weglassen (auch bei anderen, von $\varkappa$ und $\varkappa'$ abhängigen Begriffen). Die Menge der aufnehmbaren Bogensysteme von Γ bezeichnen wir mit M. Da $H=0$ zu M gehört, ist M nie leer. Aus (a) folgt, daß jedes aufnehmbare System, falls es mehrere Bogen enthält, aus lauter verschiedenen Bogen besteht.

Man kann behaupten: Es existiert der Wert

$$\nu_{\max} = \max_{H \in M} \nu(H)$$

$$(\nu_{\max} \text{ hängt von } \Gamma, \varkappa \text{ und } \varkappa' \text{ ab}),$$

der „die maximale Anzahl der aufnehmbaren Bogen" angibt.

(3.2) Wir wollen jedem Gewichtssystem $q = q(A, B, C)$ als seinen *Wert* bezüglich $\varkappa$ und $\varkappa'$ eine *ganze* Zahl $S(q)$ zuordnen. Kann ein Gewicht maximalerweise zu der Füllung von μ Bogen eines aufnehmbaren Bogensystems beitragen, so werden wir dies mit der Multiplizität μ in Betracht nehmen. Ist jedoch $H \in M$, so kann ein Punkt X auf höchstens $\varkappa(X) + \varkappa'(X)$ Bogen von H liegen und es können in X höchstens $\varkappa(X)$ Bogen von H enden, ferner kann eine Kante in höchstens einem Bogen von H liegen. Die angegebenen Grenzen können im allgemeinen nicht durch kleinere ersetzt werden. Deshalb bekommt das zu einem Punkt X gehörige Gewicht 1 bzw. 1/2 die Multiplizität $\varkappa(X) + \varkappa'(X)$ bzw. $\varkappa(X)$. (Nach (2.2) kann das zu X gehörige halbe Gewicht nur zur Füllung der in X endenden Bogen beitragen!) Ferner enthalten alle zu den Kanten gehörenden Gewichte die Multiplizität 1.

Die mit Multiplizitäten betrachteten Gewichte sollen jedoch nicht einfach addiert werden. Statt dessen teile man zuerst die halben Gewichte so in Klassen ein, daß je zwei „benachbarte" immer zur selben Klasse gehören; dann berechne man einzeln die *ganzen Teile* der Summe der zur selben Klasse gehörigen Gewichte, und endlich addiere man diese zu der Summe der ganzen Gewichte. Genauer gesagt, betrachten wir die Komponenten $[C_i]$ $(i = 1, \ldots, m)$ von $[C]$, und reihen für ein jedes i die zu den C_i-Punkten und zu den AC_i-Kanten gehörigen halben Gewichte in eine Klasse ein. Wir definieren also $S(q)$ folgendermaßen:[5]

$$(1) \qquad S(q) = \varkappa(B) + \varkappa'(B) + \nu(A, A) + \sum_{i=1}^{m} \left[\frac{\varkappa(C_i) + \nu(A, C_i)}{2} \right].$$

[5] $[a]$ bedeutet den ganzen Teil der Zahl a.

Wir wollen diesen Ausdruck noch auf eine andere Form bringen. Wir nennen eine C-Komponente $[C_i]$ *gerade* oder *ungerade*, je nachdem ob die Zahl $\varkappa(C_i)+\nu(A, C_i)$ gerade oder ungerade ist, und bezeichnen die Anzahl der ungeraden C-Komponenten mit τ_q. Es gilt dann

$$S(q)=\varkappa(B)+\varkappa'(B)+\nu(A, A)+\frac{1}{2}(\varkappa(C)+\nu(A, C)-\tau_q). \tag{2}$$

Wir können behaupten: Es existiert der Wert

$$S_{\min}=\min_{q\in Q} S(q) \tag{3}$$

($S_{\min}$ hängt von Γ, $\varkappa$ und $\varkappa'$ ab).

Nun sind wir endlich in der Lage, unseren Hauptsatz formulieren zu können:

HAUPTSATZ (3. 3) *Es seien in den Punkten des nichtleeren Graphen Γ die Kapazitätsfunktionen $\varkappa$ und $\varkappa'$ definiert. Es gilt dann*

$$\nu_{\max}=S_{\min},$$

oder anders ausgedrückt: die maximale Anzahl der aufnehmbaren Bogen ist gleich dem Minimum der Werte der füllenden Gewichtssysteme.

Den Beweis des Hauptsatzes werden wir im II. Abschnitt durchführen.

BEMERKUNGEN. (1) Man kann unseren Hauptsatz in solcher Weise verallgemeinern, daß man auch den Kanten nichtnegative ganze Durchgangskapazitäten zuordnet. Bezeichnet $\varkappa'(x)$ die zur Kante x gehörige Kapazität, so muß man bei der Definition der aufnehmbaren Bogensysteme die Forderung (3. 1) (a) durch jene ersetzen, daß jede Kante x zu höchstens $\varkappa'(x)$ Bogen des Systems gehören darf, und bei der Berechnung von $S(q)$ das zu x gehörige Gewicht mit der Multiplizität $\varkappa'(x)$ in Betracht nehmen. Man kann einen Beweis des so entstehenden Satzes dadurch erhalten, daß man die Kanten mit $\varkappa'(x)=0$ wegläßt, zu jeder Kante x mit $\varkappa'(x)>1$ genau $\varkappa'(x)-1$ neue Kanten mit denselben Randpunkten hinzunimmt und auf den so entstehenden Graphen den Satz (3. 3) anwendet.

(2) Es entsteht ein interessantes Problem dadurch, daß man statt Bogensysteme nur Wegsysteme zuläßt. Wir vermuten, daß auch in diesem Falle ein ähnlicher Satz wie (3. 3) besteht.

II. BEWEIS DES HAUPTSATZES

§ 4. Eine Umformung des Problems

Wir nehmen im ganzen II. Abschnitt an, daß Γ ein (nichtleerer) Graph ist, in dessen Punkten die Kapazitätsfunktionen $\varkappa$ und $\varkappa'$ definiert sind.

Es gilt folgende Behauptung:

(4.1) *Ist* $H \in M$ *und* $q \in Q$, *so ist* $\nu(H) \leqq S(q)$.

BEWEIS. Es sei $H \in M$ und $q = q(A, B, C)$, und es seien $[C_i]$ $(i=1, \ldots, m)$ die C-Komponenten. Wir zerlegen H in drei Bogensysteme H_a, H_b und H_c. H_b bestehe aus sämtlichen solchen Bogen von H, die mindestens einen B-Punkt enthalten; H_a aus denen, die keinen B-Punkt, aber mindestens eine AA-Kante enthalten; endlich H_c aus denen, die weder B-Punkte, noch AA-Kanten enthalten. H_a, H_b und H_c können auch leer sein. Es gelten dann

$$(1)\qquad \begin{gathered} \nu(H_a) + \nu(H_b) + \nu(H_c) = \nu(H), \\ \nu(H_a) \leqq \nu(A, A) \quad \text{und} \quad \nu(H_b) \leqq \varkappa(B) + \varkappa'(B). \end{gathered}$$

Es sei H_i $(i=1, \ldots, m)$ das System sämtlicher Bogen von H_c, die C_i-Punkte enthalten. Im Falle $C = \varnothing$ sei $H_1 = 0$. Es gilt

$$(2)\qquad \sum_{i=1}^{m} \nu(H_i) \geqq \nu(H_c).$$

Ferner bezeichne λ_i, μ_i bzw. ν_i die Anzahl derjenigen Bogen von H_i, die genau 2, 1 bzw. 0 Randpunkte in C_i haben. Dann gelten

$$(3)\qquad \lambda_i + \mu_i + \nu_i = \nu(H_i),$$

$$(4)\qquad 2\lambda_i + \mu_i \leqq \varkappa(C_i) \qquad (i=1, \ldots, m).$$

Fällt nur ein bzw. kein Randpunkt des Bogens h von H_i in C_i, so enthält h mindestens eine bzw. zwei AC_i-Kanten. Es gilt daher $\mu_i + 2\nu_i \leqq \nu(A, C_i)$, woraus nach (4)

$$(5)\qquad \lambda_i + \mu_i + \nu_i \leqq \left[\frac{\varkappa(C_i) + \nu(A, C_i)}{2}\right] \qquad (i=1, \ldots, m)$$

folgt. Aus (1), (2), (3) und (5) ergibt sich $\nu(H) \leqq S(q)$.

(4.2) Es sei $H \in M$. Wir setzen

$$\delta_X(H) = \varkappa(X) - [H, X], \quad \delta(H) = \sum_{X \in \Phi} \delta_X(H), \quad \delta_{\min} = \min_{H \in M} \delta(H).$$

Der Wert $\delta_{\min}$ existiert und es gilt $\delta_X(H) \geqq 0$, $\delta(H) \geqq 0$, $\delta_{\min} \geqq 0$. Nach (2. 1) (1) gelten

(1) $$\delta(H) = \varkappa(\Phi) - 2\nu(H), \quad \delta_{\min} = \varkappa(\Phi) - 2\nu_{\max}.$$

Es ist ferner

(2) $$\nu_{\max} \leqq \frac{1}{2}\varkappa(\Phi).$$

Ist $q \in Q$, so folgt nach (4. 1) aus (1)

(3) $$S(q) \geqq \frac{1}{2}(\varkappa(\Phi) - \delta_{\min}).$$

Wir werden nun unseren Hauptsatz (3. 3) dadurch beweisen, daß wir ein solches q von Q konstruieren, welches die Gleichung

(4) $$S(q) = \frac{1}{2}(\varkappa(\Phi) - \delta_{\min})$$

befriedigt. Es folgt nämlich dann aus (3), das $S_{\min} = 1/2(\varkappa(\Phi) - \delta_{\min})$, was nach (1) mit $\nu_{\max} = S_{\min}$ gleichbedeutend ist.

§ 5. Eine weitere Umformung. Ketten

Um ein q zu bestimmen, welches (4. 2) (4) befriedigt, wollen wir eine weitere Umformung durchführen, die kurzgesagt darin besteht, daß man die Bogensysteme durch gewisse, zu diesen Systemen gehörige Teilgraphen von Γ ersetzt.

(5. 1) Wir wollen in dieser Arbeit unter einer *Kette* von Γ einen solchen Teilgraphen von Γ verstehen, der jeden Punkt von Γ enthält. Eine Kette von Γ ist durch die Angabe seiner Kanten eindeutig bestimmt. Deshalb sagen wir auch, daß eine Kette aus seinen Kanten „besteht". Diejenige Kette, die keine Kante enthält, heißt die *Nullkette* von Γ. Im folgenden sollen die vorkommenden Ketten, wenn anders nicht gesagt wird, immer zu Γ gehörige Ketten bedeuten.

Mit f (auch mit Indizes oder anderen zeichen versehen) bezeichnen wir immer Ketten. Ist $f' \subseteq f$, so soll $f - f'$ diejenige Kette bezeichnen, die aus sämtlichen solchen Kanten von f besteht, welche in f' nicht vorkommen.

Das Zeichen (f, X) soll die Anzahl derjenigen Kanten von f bedeuten, die zu X inzident sind (d. h. es ist $(f, X) = \varrho_f(X)$).

Ist h ein Bogen, H ein Bogensystem von Γ, so soll $\tilde{h}$ bzw. $\tilde{H}$ diejenige Kette bedeuten, die aus den Kanten von h bzw. aus den Kanten sämtlicher

Bogen von H besteht. (Ist $H=0$, so ist $\tilde{H}$ die Nullkette.) Nach (2. 1) und (3. 1) (a) gilt dann für jedes H von M und für jedes X

(1) $$(\tilde{H}, X) = [H, X] + 2|H, X|.$$

(5. 2) Ist $H \in M$ und gibt es ein H_1 in M mit $\tilde{H}_1 \subset \tilde{H}$ und $\delta(H_1) = \delta(H)$, so heißt H *reduzibel*. Ist $H \in M$ und ist es nicht reduzibel, so heißt es *irreduzibel*. Es bezeichne M_i die Menge der irreduziblen Systeme von M. Das System $H=0$ ist ein Element von M_i und wir können behaupten, daß zu jedem H von M ein H_1 in M_i mit $\tilde{H}_1 \subseteq \tilde{H}$ und $\delta(H_1) = \delta(H)$ existiert. Daraus folgt

(1) $$\min_{H \in M_i} \delta(H) = \delta_{\min}.$$

Es gilt ferner

(5. 3) *Ist $H \in M_i$ und besteht für den Punkt X die Ungleichung $|H, X| > 0$, so ist $[H, X] = \varkappa(X)$.*

Beweis. Gilt nämlich für X $|H, X| > 0$, so gibt es einen Bogen h von H, der X als inneren Punkt enthält. h enthält einen solchen Teilbogen h', daß der eine Randpunkt von h' der Punkt X ist, der andere jedoch mit einem Randpunkt von h zusammenfällt. Ersetzt man dann in H den Bogen h durch h', so entsteht ein solches System H_1, das im Falle $[H, X] < \varkappa(X)$ den Bedingungen $H_1 \in M$, $\tilde{H}_1 \subset \tilde{H}$ und $\nu(H_1) = \nu(H)$ genügt. Dies widerspricht jedoch nach (4. 2) (1) der Irreduzibilität von H.

Aus (5. 1) (1) und (5. 3) folgt

(5. 4) *Es sei $H \in M_i$. Dann gelten*

(1) *im Falle $(\tilde{H}, X) \geqq \varkappa(X)$ die Gleichungen $[H, X] = \varkappa(X)$ und $(\tilde{H}, X) - \varkappa(X) = 2|H, X|$;*

(2) *im Falle $(\tilde{H}, X) \leqq \varkappa(X)$ die Gleichung $|H, X| = 0$.*

Aus (5. 1) (1) und (5. 4) folgt weiter

(5. 5) *Ist $H \in M_i$, so besitzt $\tilde{H}$ die folgenden Eigenschaften: Für jedes X ist $(\tilde{H}, X) \leqq \varkappa(X) + 2\varkappa'(X)$. Ist $(\tilde{H}, X) > \varkappa(X)$, so ist die Zahl $(\tilde{H}, X) - \varkappa(X)$ gerade.*

(5. 6) Wir wollen unter Beachtung von (5. 5) eine beliebige Kette f (von Γ in bezug auf $\varkappa$ und $\varkappa'$) *aufnehmbar* nennen, wenn f den folgenden Bedingungen genügt:

(a) Für jedes X gilt $(f, X) \leqq \varkappa(X) + 2\varkappa'(X)$.

(b) Ist $(f, X) > \varkappa(X)$, so ist die Zahl $(f, X) - \varkappa(X)$ gerade.

Die Menge der aufnehmbaren Ketten (von Γ) bezeichnen wir mit $\tilde{M}$. Die Nullkette ist ein Element von $\tilde{M}$.

Es sei $f \in \tilde{M}$. Gilt dann für den Punkt X $(f, X) \geqq \varkappa(X)$ bzw. $(f, X) > \varkappa(X)$, so sagen wir, daß X durch f *gefüllt* bzw. *übergefüllt* ist.

Wir setzen

$$\delta_X(f) = \max(0, \varkappa(X) - (f, X)), \quad \delta(f) = \sum_{X \in \Phi} \delta_X(f), \quad \tilde{\delta}_{\min} = \min_{f \in \tilde{M}} \delta(f)$$

und nennen eine Kette von $\tilde{M}$ mit $\delta(f) = \tilde{\delta}_{\min}$ eine *extreme* Kette von $\tilde{M}$. Offensichtlich existieren extreme Ketten in $\tilde{M}$.

Man kann behaupten:

(5.7) *Ist $f' \subseteq f$, so gilt $\delta(f') \geqq \delta(f)$, und das Gleichheitszeichen gilt dann und nur dann, wenn für jedes X mit $(f', X) < (f, X)$ die Ungleichung $(f', X) \geqq \varkappa(X)$ besteht.*

Aus (5.4), (5.5) und (5.6) folgt

(5.8) *Ist $H \in M_i$, so ist $\tilde{H} \in \tilde{M}$ und es gilt für jedes X $\delta_X(\tilde{H}) = \delta_X(H)$, sowie $\delta(\tilde{H}) = \delta(H)$.*

(5.9) Ist $f \in \tilde{M}$ und gibt es ein f' in $\tilde{M}$ mit $f' \subset f$ und $\delta(f') = \delta(f)$, so heißt f *reduzibel*. Ist $f \in \tilde{M}$ und nicht reduzibel, so heißt es *irreduzibel*. Die Menge der irreduziblen Ketten von $\tilde{M}$ bezeichnen wir mit $\tilde{M}_i$. Die Nullkette ist ein Element von $\tilde{M}_i$, und zu jedem f von $\tilde{M}$ gibt es ein f' in $\tilde{M}_i$ mit $f' \subseteq f$ und $\delta(f') = \delta(f)$. Daraus folgt

(1) $$\min_{f \in \tilde{M}_i} \delta(f) = \tilde{\delta}_{\min}.$$

Ist $f \in \tilde{M}$ und ist k ein solcher Kreis von f, dessen sämtliche Punkte durch f übergefüllt sind, so heißt k ein *weglaßbarer* Kreis von f. Nach (5. 7) können wir behaupten:

(5.10) *Ist $f \in \tilde{M}$ und gibt es einen weglaßbaren Kreis von f, so ist f reduzibel.*

Es gilt auch die Umkehrung dieser Behauptung:

(5.11) *Ist f eine reduzible Kette von $\tilde{M}$, so enthält f einen weglaßbaren Kreis von f.*

Beweis. Existiert nämlicht ein f' in $\tilde{M}$ mit $f' \subset f$ und $\delta(f') = \delta(f)$, so sind nicht alle Punkte in $f'' = f - f'$ isoliert, und sämtliche nichtisolierten Punkte von f'' sind nach (5.7) durch f' gefüllt und durch f übergefüllt. Ferner sind alle diese Punkte nach (5.6) (b) geraden Grades in f''. Daraus folgt, daß f'' einen Kreis enthält. Dieser ist aber von f weglaßbar.

Ähnlicherweise kann man die Richtigkeit folgender Behauptung einsehen:

(5.12) *Ist $f \in \tilde{M}$ und ist f nicht die Nullkette, sind ferner sämtliche nicht-isolierten Punkte von f durch f übergefüllt, so enthält f einen weglaßbaren Kreis.*

Wir beweisen nun folgende Behauptung:

(5.13) *Zu jedem f von $\tilde{M}_i$ gibt es ein H in M_i mit $\tilde{H}=f$.*

Beweis. (I) Es sei $f \in \tilde{M}_i$. Ist f die Nullkette, so können wir $H=0$ setzen. Daher kann angenommen werden, daß f auch nichtisolierte Punkte enthält. Wir wollen in diesem Beweis einen solchen Bogen von f, dessen innere Punkte durch f übergefüllt sind, dessen Randpunkte jedoch nicht, einen *f-Bogen* nennen. Wir zeigen, daß f-Bogen existieren. Nach (5.12) und (5.10) gibt es ein X_0 mit $0<(f, X_0) \leqq \varkappa(X_0)$. f enthält eine Kante $[X_0 x_1 X_1]$. Ist $(f, X_1) \leqq \varkappa(X_1)$, so ist $[X_0 x_1 X_1]$ ein f-Bogen. Nehmen wir an, daß $(f, X_1) > \varkappa(X_1)$ ist. Es bezeichne L die Menge derjenigen Wege w von f, die folgende Eigenschaften besitzen: X_0 ist der eine Randpunkt von w, und sämtliche von X_0 verschiedenen Punkte von w sind durch f übergefüllt. L ist nichtleer. Es sei $w'=[X_0 x_1 X_1 \ldots X_{n-1} x_n X_n]$ ein Weg von L mit maximaler Anzahl von Kanten. Es gibt eine von x_n verschiedene Kante $[X_n x' X']$ von f. Es ist dann $h=[X_0 x_1 X_1 \ldots X_{n-1} x_n X_n x' X']$ ein f-Bogen. Ist nämlich h ein Weg, so muß $(f, X') \leqq \varkappa(X')$ sein. Im Falle $X'=X_0$ ist unsere Behauptung trivial. Der Fall $X'=X_i$ $(0<i \leqq n-1)$ kann nicht eintreten, da dann $[X_i x_{i+1} X_{i+1} \ldots \ldots X_n x' X']$ ein weglaßbarer Kreis von f wäre.

(II) Es bezeichne m eine positive ganze Zahl. Wir nehmen an, daß unser Satz für sämtliche solche Ketten von $\tilde{M}_i$ richtig ist, die weniger als m Kanten enthalten, und daß f $(f \in \tilde{M}_i)$ genau m Kanten besitzt. Es sei h_1 ein beliebiger f-Bogen. Für $f_1=f-\tilde{h}_1$ gilt $f_1 \in \tilde{M}$ und $\delta(f_1)=\delta(f)+2$. f_1 kann keinen weglaßbaren Kreis enthalten, denn ein solcher Kreis wäre auch von f weglaßbar. Nach (5.11) ist dann $f_1 \in \tilde{M}_i$. f_1 enthält weniger Kanten als f, und so gibt es laut unserer Annahme ein H_1 in M_i mit $\tilde{H}_1=f_1$. Nach (5.8) ist dann für jedes X $\delta_X(H_1)=\delta_X(f_1)$ und $\delta(H_1)=\delta(f_1)$. Fügen wir h_1 zu H_1, so entsteht ein System H mit $\tilde{H}=f$.

Wir zeigen, daß $H \in M$ ist. H genügt der Bedingung (3.1) (a). Um zu beweisen, daß H auch (3.1) (b) erfüllt, genügt es nur die Punkte von h_1 zu untersuchen. Ist X ein innerer Punkt von h_1, so ist $(f_1, X) \geqq \varkappa(X)$, daher gilt nach (5.4) $(f_1, X)=\varkappa(X)+2|H_1, X|$, was zusammen mit $(f_1, X)=(f, X)-2< <\varkappa(X)+2\varkappa'(X)$ die Ungleichung $|H_1, X|<\varkappa'(X)$ ergibt. Ist X ein Randpunkt von h_1, so ist $(f, X) \leqq \varkappa(X)$, und daher gilt $[H, X] \leqq (\tilde{H}, X) \leqq \varkappa(X)$. Daraus folgt, daß H die Bedingung (3.1) (b) erfüllt. Es gilt also $H \in M$.

Es gilt ferner $\delta(H)=\delta(H_1)-2=\delta(f)$.

Endlich beweisen wir, daß H irreduzibel ist. Nehmen wir nämlich das Gegenteil an, dann gibt es ein H' in M_i mit $\tilde{H}' \subset \tilde{H}$ und $\delta(H')=\delta(H)$. Nach (5.8) ist $\tilde{H}' \in \tilde{M}$ und $\delta(\tilde{H}')=\delta(H')$, woraus $\delta(\tilde{H}')=\delta(f)$ folgt. Das widerspricht jedoch der Irreduzibilität von f. Es ist damit der Beweis von (5.13) beendet.

(5.14) *Ist* $H \in M_i$, *so ist* $\tilde{H} \in \tilde{M}_i$.

Beweis. Laut (5.8) genügt es zu zeigen, daß $\tilde{H}$ irreduzibel ist. Nehmen wir an, daß es ein f' mit $f' \in \tilde{M}$, $f' \subset \tilde{H}$ und $\delta(f') = \delta(\tilde{H})$ gibt. Zu f' existiert ein f'' in $\tilde{M}_i$ mit $f'' \subseteq f'$ und $\delta(f'') = \delta(f')$. Nach (5.13) gibt es dann ein H' in M_i mit $\tilde{H}' = f''$. Laut (5.8) ist $\delta(H') = \delta(f'')$ und $\delta(\tilde{H}) = \delta(H)$. Es gilt also $\tilde{H}' \subset \tilde{H}$ und $\delta(H') = \delta(H)$. Das widerspricht aber der Irreduzibilität von H.

(5.15) Aus (5.8), (5.13) und (5.14) folgt

$$\min_{f \in \tilde{M}_i} \delta(f) = \min_{H \in M_i} \delta(H),$$

und das ergibt, zusammen mit (5.2) (1) und (5.9) (1) die Gleichung

$$\tilde{\delta}_{\min} = \delta_{\min}.$$

Um unseren Hauptsatz zu beweisen, genügt es daher nach (4.2) ein solches Gewichtssystem q von Q zu konstruieren, welches die Gleichung

$$S(q) = \frac{1}{2}(\varkappa(\Phi) - \tilde{\delta}_{\min}) \tag{1}$$

befriedigt. Zu dieser Konstruktion werden wir von einer extremen Kette f von $\tilde{M}$ ausgehen, und auf diese das Verfahren der alternierenden Züge anwenden.

§ 6. Alternierende Züge

(6.1) Um unsere Ausdrucksweise zu verkürzen, ist es vorteilhaft, einer jeden Kette f von Γ eine *charakteristische Funktion* $f(x)$ zuzuordnen. $f(x)$ sei auf sämtlichen Kanten von Γ definiert, und es bestehe $f(x) = 1$ oder $f(x) = 0$, je nachdem ob x eine Kante von f ist oder nicht.

Es sei f vorläufig ein beliebiges Element von $\tilde{M}$. Wir nennen den Zug $p = (X_0 x_1 X_1 \ldots X_{n-1} x_n X_n)$ einen *zu f gehörigen alternierenden Zug* (kurz: a-Zug), wenn entweder $n = 1$ ist, oder wenn $n > 1$ ist und p für jedes i $(1 \leqq i \leqq n-1)$ folgenden Bedingungen genügt:

(a) Ist $f(x_i) = f(x_{i+1})$, so ist X_i ein einfacher Punkt von p.

(b) Ist $f(x_i) = f(x_{i+1}) = 1$, so ist $(f, X) > \varkappa(X)$.

(c) Ist $f(x_i) = f(x_{i+1}) = 0$, so ist $(f, X) \leqq \varkappa(X) + 2\varkappa'(X) - 2$.

Wir nennen den Punkt X des a-Zuges p einen *Wechselpunkt* von p, wenn bei dem Durchlaufen von p der Wert von $f(x)$ sich bei *jedem* Durchgang über X ändert, oder wenn X ein Randpunkt von p ist. Nach (a) ist jeder mehrfache Punkt von p ein Wechselpunkt von p.

Ist $p = (X_0 x_1 X_1 \ldots X_{n-1} x_n X_n)$ ein zu f gehöriger a-Zug, so ist auch $\bar{p} = (X_n x_n X_{n-1} \ldots X_1 x_1 X_0)$, sowie

$$p_{ij} = (X_{i-1} x_i X_i \cdots X_{j-1} x_j X_j) \qquad (1 \leqq i \leqq j \leqq n)$$

ein solcher Zug. Die Züge p_{1i} bzw. p_{in} $(1 \leqq i \leqq n)$ heißen die *Anfangs-* bzw. *Endteile* von p.

Man kann in gewissen Fällen zwei zu f gehörige a-Züge zu einem dritten vereinigen. Wir wollen einige leicht ersichtliche hinreichende Bedingungen solcher Vereinigungen in dem folgenden Lemma zusammenfassen:

LEMMA (6.2) *Es seien* $p = (X_0 x_1 X_1 \ldots X_{m-1} x_m X_m)$ *und* $p' = (X_0' x_1' X_1' \ldots \ldots X_{n-1}' x_n' X_n')$ *zwei zu* f *gehörige* a*-Züge, die den folgenden vier Bedingungen genügen:* (1) $X_m = X_0'$; (2) p *und* p' *haben keine gemeinsamen Kanten;* (3) *ist* X *ein gemeinsamer Punkt von* p *und* p', *so ist* X *sowohl in* p *wie auch in* p' *ein Wechselpunkt;* (4) *Es besteht einer der folgenden drei Fälle:*

(a) $f(x_m) \neq f(x_1')$;

(b) $f(x_m) = f(x_1') = 1$, X_m *ist sowohl in* p *als auch in* p' *einfach und* $(f, X_m) > \varkappa(X_m)$;

(c) $f(x_m) = f(x_1') = 0$, X_m *ist in* p *und auch in* p' *einfach und* $(f, X_m) \leqq \leqq \varkappa(X_m) + 2\varkappa'(X_m) - 2$.

Dann ist auch

$$pp' = (X_0 x_1 X_1 \ldots X_{m-1} x_m X_m x_1' X_1' \ldots X_{n-1}' x_n' X_n')$$

ein zu f *gehöriger* a*-Zug.*

Der folgende Satz enthält die Grundidee der Anwendung der a-Züge.

SATZ (6.3) *Ist* f *eine extreme Kette von* $\tilde{M}$, *so gibt es keinen zu* f *gehörigen* a*-Zug* $p = (X_0 x_1 X_1 \ldots X_{n-1} x_n X_n)$ *mit den Eigenschaften:* $f(x_1) = = f(x_n) = 0$, $(f, X_0) < \varkappa(X_0)$, $(f, X_n) < \varkappa(X_n)$ *und im Falle* $X_0 = X_n$ $(f, X_0) \leqq \leqq \varkappa(X_0) - 2$.

BEWEIS. Es sei $f \in \tilde{M}$ und $\delta(f) = \tilde{\delta}_{\min}$. Nehmen wir an, daß solche zu f gehörigen a-Züge existieren, welche die in (6. 3) angeführten Eigenschaften besitzen, und es sei $p = (X_0 x_1 X_1 \ldots X_{n-1} x_n X_n)$ ein solcher Zug mit minimaler Anzahl von Kanten. Wir definieren eine Kette f' wie folgt: Ist $f(x_i) = 1$ $(1 \leqq i \leqq n)$, so sei $f'(x_i) = 0$. Ist $f(x_i) = 0$ $(1 \leqq i \leqq n)$, so sei $f'(x_i) = 1$. st $x \neq x_1, \ldots, x_n$, so sei $f'(x) = f(x)$. Wir werden zeigen, daß $f' \in \tilde{M}$ und $\delta(f') < \delta(f)$ ist, woraus die Richtigkeit unseres Satzes folgt.

(I) Zuerst sei X entweder ein Wechselpunkt von p, oder sei X kein Punkt von p. Fällt X mit einem der Punkte X_0 und X_n zusammen, so gilt $(f, X) < (f', X) \leqq \varkappa(X)$, und daraus folgt $\delta_X(f') < \delta_X(f)$. Ist $X \neq X_0, X_n$, so ist $(f', X) = (f, X)$. Es ist dann $\delta_X(f') = \delta_X(f)$, und wir sehen, daß f' bei X beide Bedingungen von (5.6) erfüllt.

(II) Es sei $X = X_i$ $(0 < i < n)$, und es gelte $f(x_{i-1}) = f(x_i) = 1$. Dann folgt nach (6.1)(a), (b): $X \neq X_0, X_n$, $(f', X) = (f, X) - 2$ und $(f, X) > \varkappa(X)$. Daraus ergibt sich $\delta_X(f') = \delta_X(f) = 0$ und man sieht, daß f' bei X beide Bedingungen unter (5.6) befriedigt.

(III) Es sei $X = X_i$ $(0 < i < n)$ und $f(x_{i-1}) = f(x_i) = 0$. Dann folgt nach (6.1)(a), (c): $X \neq X_0, X_n$, $(f', X) = (f, X) + 2$ und $(f, X) \leqq \varkappa(X) + 2\varkappa'(X) - 2$. Daraus kann man sehen, daß $\delta_X(f') \leqq \delta_X(f)$ gilt, sowie daß f' bei X der Bedingung (a) unter (5.6) genügt. Gilt für X noch $(f, X) \geqq \varkappa(X)$, so folgt aus den obigen, daß f' bei X auch (5.6)(b) erfüllt. Der andere Fall kann nicht eintreten. Wäre nämlich $(f, X) < \varkappa(X)$, so wäre — da X_i ein einfacher Punkt von p ist — $(X_0 x_1 X_1 \ldots X_{i-1} x_{i-1} X_i)$ ein zu f gehöriger a-Zug, mit den in (6.3) angeführten Eigenschaften, der weniger Kanten als p enthalten würde.

Aus (I), (II) und (III) folgt $f' \in \tilde{M}$ und $\delta(f') < \delta(f)$.

§ 7. Der Hilfspunkt X^*. Beweis eines Lemmas

(7.1) Es sei nun f eine beliebige *extreme* Kette von $\tilde{M}$. Im weiteren Teil des II. Abschnittes wollen wir diese Kette f festhalten. Um unsere Beweisführung zu vereinfachen, konstruieren wir nach BERGE ([3], S. 176) aus Γ, $\varkappa$ und f einen neuen Graphen Γ^* wie folgt: Wir nehmen zu Γ einen neuen Punkt X^* hinzu und verbinden X^* mit jedem durch f *nicht gefüllten* Punkt X von Γ durch $\delta_X(f)$ neue Kanten. Ferner erweitern wir die Funktionen $\varkappa$ und $\varkappa'$ auf Γ^* folgendermaßen: Es sei

(a) $\varkappa(X^*) = \tilde{\delta}_{\min}$; (b) $\varkappa'(X^*) = 0$.

Die zu Γ^* gehörige Kette f^* soll aus sämtlichen Kanten von f und aus sämtlichen neuen Kanten bestehen. (f^*, X) soll die Anzahl der zu X inzidenten Kanten von Γ^* bezeichnen. Wir stellen einige leicht ersichtliche Eigenschaften von f^* zusammen:

(1) Für jede zu X^* inzidente Kante x gilt $f^*(x) = 1$.

(2) $(f^*, X^*) = \varkappa(X^*)$.

(3) f^* ist eine (bezüglich $\varkappa$ und $\varkappa'$) aufnehmbare Kette von Γ^*.

(4) Jeder Punkt X von Γ^* ist durch f^* gefüllt.

(5) Ist X durch f^* übergefüllt, so gibt es keine XX^*-Kante.

Aus (6.3) leiten wir eine weitere Eigenschaft von f^* ab:

(7.2) *Es existiert in Γ^* kein zu f^* gehöriger a-Zug p^* von der Struktur $p^* = (X^* x_0 X_0 \ldots X_n x_{n+1} X^*)$ $(n \geqq 0)$.*

Beweis. Nehmen wir an, daß in Γ^* ein a-Zug p^* von der angegebenen Struktur existiert. Ist $n=0$, so ist $p^*=(X^*x_0X_0x_1X^*)$. Nach (7. 1)(1) und (6. 1)(b) gilt $(f^*, X_0)>\varkappa(X_0)$, was zu (7. 1)(5) in Widerspruch steht.

Es sei $n>0$. Dann ist nach (7. 1)(1), (2) und (6. 1)(b) der Teil $p=(X_0x_1X_1\ldots X_{n-1}x_nX_n)$ von p^* ein zu f gehöriger a-Zug von Γ. Aus der Existenz von x_0 und x_{n+1} folgt, daß X_0 und X_n durch f nicht gefüllt sind, und im Falle $X_0=X_n$ sogar $(f, X_0)\leqq\varkappa(X_0)-2$ besteht. Ferner gilt nach (7. 1)(4), (5) $(f^*, X_0)=\varkappa(X_0)$, $(f^*, X_n)=\varkappa(X_n)$, und so erhalten wir unter Beachtung von (7. 1)(1) und (6. 1)(b) die Gleichung $f(x_1)=f(x_n)=0$. Dies widerspricht jedoch dem Satz (6. 3).

(7. 3) Von hier an bis zum Ende des II. Abschnittes sollen sich unsere Begriffe und Bezeichnungen — wenn anderes nicht gesagt wird — nicht auf Γ, sondern auf Γ^* beziehen. Unter Punkten, Kanten und Zügen verstehen wir also immer beliebige Punkte, Kanten und Züge von Γ^*. Ferner haben $\nu(E, F)$ und $[E]$ die Bedeutung $\nu_{\Gamma^*}(E, F)$ bzw. $[E]_{\Gamma^*}$, und eine E-Komponente bedeutet eine E-Komponente von Γ^*.

Um unsere Ausdrucksweise weiter zu vereinfachen, wollen wir die Kanten von f^* *α-Kanten*, diejenigen Kanten, die nicht zu f^* gehören, *β-Kanten* nennen. (Für jede α-Kante ist also $f^*(x)=1$, und für jede β-Kante $f^*(x)=0$.) Die Zeichen $u, v, w, \bar{u}, \bar{v}, \bar{w}$ sollen einen beliebigen der Buchstaben α und β bedeuten, und zwar soll im selben Satz bzw. Beweis immer $\bar{u}\neq u$, $\bar{v}\neq v$, $\bar{w}\neq w$ gelten.

Sämtliche im weiteren Teil des II. Abschnittes vorkommenden a-Züge sind zu f^* gehörige a-Züge von Γ^*. Deshalb werden wir diese kurz nur als a-Züge bezeichnen. Vielmals wird es genügen, statt der Randkanten der a-Züge nur die Art dieser Kanten zu bezeichnen. Wir führen daher folgende kurze Schreibweise ein:

$$p=(X_1^u\ldots {}^vX_2)$$

soll bedeuten, daß X_1 der Anfangspunkt, X_2 der Endpunkt des a-Zuges p ist und die Anfangs- bzw. Endkante von p eine u- bzw. v-Kante ist. Kann kein Mißverständnis entstehen, so werden wir im allgemeinen in den a-Zügen neben den Randpunkten nur die in unseren Behauptungen explizit vorkommenden Elemente bzw. Kantenarten bezeichnen. Wir wiederholen nochmals die folgende Vereinbarung von (6. 1): p und $\bar{p}$ bezeichnen solche a-Züge, die sich nur in der Durchlaufsrichtung unterscheiden.

Im Beweis des folgenden Lemmas benützen wir keine speziellen Eigenschaften von Γ^*, $\varkappa$ und $\varkappa'$. Bezüglich f^* benützen wir nur (7. 1)(3).

Lemma (7. 4) *Es sollen die a-Züge $p=(X_1^u\ldots{}^vX_2)$ und $p_0=(X_0x_0\ldots X')$ die folgenden Bedingungen erfüllen:*

(1) *X' liegt auf p;*

(2) *liegt X_0 auf p, so ist er ein Wechselpunkt von p.*

Dann existiert ein a-Zug von der Struktur

$$p_1 = (X_0 x_0 \ldots {}^u X_1) \quad \textit{oder} \quad p_2 = (X_0 x_0 \ldots {}^v X_2),$$

der nur solche Kanten enthält, die in p oder in p_0 liegen.

Beweis. (I) Liegt x_0 in p, so bezeichne p' bzw. p'' denjenigen Anfangs- bzw. Endteil von p, dessen End- bzw. Anfangskante x_0 ist. X_0 ist entweder der Endpunkt von p' oder der Anfangspunkt von p''. Im ersten Falle kann man $p_1 = \bar{p}'$, im zweiten $p_2 = p''$ setzen.

(II) Wir nehmen an, daß x_0 nicht in p liegt. Es sei $p_3 = (X_0 x_0 \ldots x_3 X_3)$ der kürzeste Anfangsteil von p_0, dessen Endpunkt auf p liegt und es sei x_3 eine w-Kante. Da x_0 nicht auf p liegt, haben p_3 und p keine gemeinsame Kante. p_3 und p können nur die Punkte X_0 und X_3 gemeinsam haben, und im Falle $X_3 \neq X_0$ ist X_3 einfach in p_3.

(1) Ist $X_3 = X_1$ und $w = u$, so ist p_3 ein gesuchter Zug p_1. Ist $X_3 = X_1$ und $w \neq u$, so können wir nach (6.2)(a) $p_2 = p_3 p$ setzen. Im Falle $X_3 = X_2$ kann man ähnlich verfahren.

(2) Es sei $X_3 \neq X_1, X_2$. Es bezeichne p_4 einen Anfangsteil von p, dessen Endpunkt X_3 ist, und p_5 den zu p_4 komplementären Endteil von p. Es sei x_4 die Endkante von p_4, x_5 die Anfangskante von p_5. Wir können behaupten: Die gemeinsamen Punkte von p_3 und p_4 bzw. von p_3 und p_5 sind in den beiden Zügen Wechselpunkte.

Ist ferner X_3 ein Wechselpunkt von p, so besteht entweder $f^*(x_3) \neq f^*(x_4)$ oder $f^*(x_3) \neq f^*(x_5)$. Nach (6.2)(a) kann man im ersten Falle $p_1 = p_3 \bar{p}_4$, im zweiten $p_2 = p_3 p_5$ setzen.

Ist X_3 kein Wechselpunkt von p, so ist X_3 einfach in jedem der Züge p, p_4 und p_5. Ferner ist $X_3 \neq X_0$, und so ist X_3 auch in p_3 einfach. Man kann nun $p_2 = p_3 p_5$ (und auch $p_1 = p_3 \bar{p}_4$) setzen. Dies folgt im Falle $f^*(x_3) \neq f^*(x_5)$ wieder aus (6.2)(a). Im Falle $f^*(x_3) = f^*(x_5)$ haben $f^*(x_3), f^*(x_4)$ und $f^*(x_5)$ den gleichen Wert 1 bzw. 0, und so folgt laut (6.1)(b), (c) aus (6.2)(b), (c) unsere Behauptung.

§ 8. Die α-, β- und γ-Punkte

(8.1) In diesem Paragraphen nützen wir von Γ^*, $\varkappa$ und $\varkappa'$ nur die eine spezielle Eigenschaft aus, daß $\varkappa'(X^*) = 0$ ist, und von f^* nur die Eigenschaften (7.1)(1), (2), (3) und (7.2).

Wir führen einige neue Bezeichnungen ein. Ein a-Zug, dessen Anfangspunkt X^* ist, soll ein *X^*-Zug* heißen. Existiert zu dem Punkt X ein X^*-Zug, dessen Endpunkt X und dessen Endkante eine u-Kante ist, so heißt X *erreichbar*, genauer *u-erreichbar*. Unabhängig von dieser Definition wollen wir X^* immer erreichbar, und zwar β-erreichbar nennen. Nach (7.2) ist X^* nicht α-erreichbar. Es ist klar, daß jeder Punkt eines X^*-Zuges erreichbar ist.

(8.2) X soll ein *α-Punkt* heißen, wenn X α-erreichbar, jedoch nicht β-erreichbar ist und $(f^*, X) \leqq \varkappa(X)$ besteht. Ist X β-erreichbar, jedoch nicht α-erreichbar, und gilt $(f^*, X) > \varkappa(X) + 2\varkappa'(X) - 2$, so wollen wir X einen *β-Punkt* nennen. Nach unseren Annahmen gilt:

(8.3) *X^* ist ein β-Punkt.*

Wir nennen einen jeden solchen Punkt von Γ^*, der weder α- noch β-Punkt ist, einen *γ-Punkt*. Die unerreichbaren Punkte sind alle γ-Punkte. Ferner sind sämtliche Punkte, die sowohl α- als auch β-erreichbar sind, ebenfalls γ-Punkte. Ein erreichbarer γ-Punkt kann jedoch auch nur α- oder nur β-erreichbar sein. Diese wollen wir γ_α- bzw. *γ_β-Punkte* nennen.

Wir können behaupten:

(8.4) *Ist X ein γ_α-Punkt, so gilt $(f^*, X) > \varkappa(X)$; ist X ein γ_β-Punkt, so gilt $(f^*, X) \leqq \varkappa(X) + 2\varkappa'(X) - 2$.*

Es besteht ferner:

(8.5) *Liegt X auf dem X^*-Zuge p, und ist X ein u-Punkt, so ist X ein Wechselpunkt von p.*

Beweis. Ist X kein Wechselpunkt von p, so ist X kein Randpunkt von p, und er ist einfach in p. Es gibt daher einen und nur einen Anfangsteil p' von p mit dem Endpunkt X. Da auch p' ein X^*-Zug ist, muß nach (8.2) die Endkante von p' eine u-Kante sein. Laut (6.1)(b), (c) widerspricht das jedoch der Annahme, daß X ein u-Punkt ist.

Aus (7.1), (1), (2) und (6.1)(b) folgt:

(8.6) *X^* kann nur ein Randpunkt eines a-Zuges sein.*

Es gilt ferner:

(8.7) *Ist X_1 ein $\bar{u}$-Punkt und X_2 ein $\bar{v}$-Punkt, so gibt es keinen a-Zug p von der Struktur $p = (X_1^u \ldots {}^v X_2)$.*

Beweis. Nehmen wir an, daß ein solcher Zug p existiert. Nach (7.2) ist dann einer der Punkte X_1 und X_2, z. B. X_2 von X^* verschieden. Es gibt dann einen X^*-Zug $p_0' = (X^* \ldots X_2)$. Nach (8.6) kann man jetzt (7.4) anwenden, woraus die Existenz eines a-Zuges $p_1 = (X^* \ldots {}^u X_1)$ oder $p_2 = (X^* \ldots {}^v X_2)$ folgt. Das widerspricht jedoch unserer Annahme, daß X_1 ein $\bar{u}$-Punkt und X_2 ein $\bar{v}$-Punkt ist.

Aus (8.7) folgt:

(8.8) *Zwei u-Punkte können nur durch u-Kanten verbunden sein.*

(8.9) *Ein unerreichbarer Punkt kann von den erreichbaren Punkten nur mit den α- und β-Punkten durch Kanten verbunden sein, und zwar mit einem α-Punkt nur durch eine α-Kante, mit einem β-Punkt nur durch eine β-Kante.*

BEWEIS. Es sei X ein unerreichbarer, X' ein erreichbarer Punkt und $[XxX']$ eine u-Kante. Offensichtlich ist $X' \neq X^*$. Ferner ist X' nicht $\bar{u}$-erreichbar. Ist nämlich X' $\bar{u}$-erreichbar, so gibt es einen X^*-Zug $p = (X^* \ldots {}^{\bar{u}}X')$. p kann X und demnach auch x nicht enthalten. Nach (6.2)(a) ist dann $p(X'xX)$ ein a-Zug und X erreichbar.

Wir zeigen, daß X' kein γ_u-Punkt sein kann. Nehmen wir an, daß X' ein γ_u-Punkt ist. Es gibt einen X^*-Zug p_1 mit dem Endpunkt X'. Es bezeichne p_2 den kürzesten Anfangsteil von p_1, dessen Endpunkt X' ist. Dann ist X' in p_2 einfach, die Endkante von p_2 eine u-Kante, und es kann weder X noch x in p_2 liegen. Demzufolge ist aber nach (8.4) und (6.2)(b), (c) $p_2(X'xX)$ ein X^*-Zug, und so wäre X erreichbar.

Es kann also X' nur ein u-Punkt sein.

§ 9. Erreichbare und unerreichbare Komponenten

(9.1) Es bezeichne A, B^* und C die Menge der α-, β- und γ-Punkte, Φ^* die Menge sämtlicher Punkte von Γ^*. Laut (8.3) ist $X^* \in B^*$. Es sei $B = B^* - \{X^*\}$. Mit diesen Mengen bestimmen wir das zu Γ^* bzw. Γ gehörige füllende Gewichtssystem

$$q^* = q^*(A, B^*, C) \quad \text{bzw.} \quad q = q(A, B, C).$$

Unser Ziel ist zu zeigen, daß q ein gesuchtes, der Bedingung (5.15)(1) genügendes System ist. In § 9 werden wir zu diesem Zwecke die Eigenschaften der C-Komponenten von Γ^* untersuchen.

(9.2) Im übrigen Teil dieses Paragraphen benützen wir nur die unter (8.1) erwähnten Eigenschaften von Γ^*, $\varkappa$, $\varkappa'$ und f^*.

Wir bezeichnen die C-Komponenten von Γ^*, d. h. die Komponenten des Teilgraphen $[C]$ von Γ^* mit $[C_i]$ $(i = 1, \ldots, m)$. C_i $(i = 1, \ldots, m)$ soll die Menge der Punkte von $[C_i]$ bedeuten. Nach (8.9) besteht eine nichtleere C-Komponente entweder aus lauter erreichbaren oder aus lauter unerreichbaren Punkten. Dementsprechend nennen wir eine C-Komponente entweder *erreichbar* oder *unerreichbar*. Im Falle $C = C_1 = \emptyset$ nennen wir $[C_1]$ unerreichbar. Nach (8.9) können wir ferner behaupten:

(9. 3) *Berührt eine Kante eine C-Komponente, so ist der äußere Randpunkt der Kante (in Bezug der Komponente) entweder ein α- oder ein β-Punkt.*

Es gilt die Behauptung:

(9. 4) *Ist* $[C_i]$ *eine nichtleere C-Komponente und* $X \in C_i$, *bezeichnet ferner* $p = (X^* \ldots X'xX)$ *einen solchen* X^**-Zug, in dem* $X' \notin C_i$ *und* x *eine u-Kante ist, so ist* X' *ein* $\bar{u}$*-Punkt.*

BEWEIS. Nach (9. 3) genügt es zu zeigen, daß X' kein u-Punkt ist. Ist X' ein u-Punkt und $X' \neq X^*$, so existiert die vorletzte Kante von p, und diese muß eine u-Kante sein. Dies widerspricht jedoch dem Satze (8. 5). Ist X' ein u-Punkt und $X' = X^*$, so muß $u = \beta$ sein, was zu (7. 1)(1) in Widerspruch steht.

Unter Beachtung von (9. 4) wollen wir jede solche u-Kante, die die C-Komponente $[C_i]$ berührt und deren äußerer Randpunkt (in Bezug von $[C_i]$) ein $\bar{u}$-Punkt ist, eine *Eintrittskante* von $[C_i]$ nennen.

Aus (8. 9) folgt:

(9. 5) *Die unerreichbaren C-Komponenten besitzen keine Eintrittskante.*

Der folgende Satz bildet den Kernpunkt unserer ganzen Beweisführung:

SATZ (9. 6) *Jede erreichbare C-Komponente besitzt genau eine Eintrittskante.*

Wir zerlegen den Beweis dieses Satzes in mehrere Teile (von (9. 7) bis (9. 11)).

(9. 7) Es sei $[C_i]$ eine beliebige erreichbare C-Komponente. Wir wollen $[C_i]$ im folgenden festhalten. Erst zeigen wir, daß $[C_i]$ eine Eintrittskante besitzt. Es sei $X \in C_i$. Da $X^* \notin C_i$, gibt es einen X^*-Zug p mit dem Endpunkt X. Es bezeichne $p' = (X^* \ldots X'xX'')$ den kürzesten Anfangsteil von p, dessen Endpunkt zu C_i gehört. Dann ist $X' \notin C_i$. Ist x eine u-Kante, so ist nach (9. 4) X' ein $\bar{u}$-Punkt, und daher ist x eine Eintrittskante von $[C_i]$.

Es sei nun $[X_0 x_0 X_1]$ eine beliebige Eintrittskante von $[C_i]$ mit $X_0 \notin C_i$, $X_1 \in C_i$, es sei ferner x_0 eine u-Kante und X_0 ein $\bar{u}$-Punkt. Wir wollen im folgenden auch x_0 und u festhalten und zeigen, daß x_0 die einzige Eintrittskante von $[C_i]$ ist.

Wir wollen einen a-Zug p einen X_0*-Zug* nennen, wenn er folgende Bedingungen erfüllt: Es ist X_0 der Anfangspunkt, x_0 die Anfangskante von p, und mit Ausnahme von X_0 und x_0 gehören sämtliche Punkte und Kanten von p zu $[C_i]$. Ist $X \in C_i$ und gibt es einen X_0-Zug mit dem Endpunkt X, so sagen wir, daß X *in* $[C_i]$ *erreichbar* ist. Gibt es einen X_0-Zug mit dem Endpunkt X, dessen Endkante eine v-Kante ist, so heißt X in $[C_i]$ v*-erreichbar.*

Wir bemerken, daß jeder von X_0 verschiedene Punkt eines X_0-Zuges in $[C_i]$ erreichbar ist, und jeder nicht in $[C_i]$ liegende Punkt in $[C_i]$ unerreichbar ist.

(9. 8) *Ist X v-erreichbar und in $[C_i]$ erreichbar, so ist er auch in $[C_i]$ v-erreichbar.*

BEWEIS. Es sei $X \in C_i$, $p = (X^* \ldots {}^v X)$ ein X^*-Zug und p_0 ein X_0-Zug mit dem Endpunkt X. Ist die Endkante von p_0 eine v-Kante, so ist nichts zu beweisen. Es sei also $p_0 = (X_0 x_0 X_1 \ldots {}^{\bar{v}} X)$. Da $X^* \notin C_i$ ist, gibt es einen kürzesten Endteil von p, dessen Anfangskante nicht zu $[C_i]$ gehört. Es sei dieser Teil $p_2 = (X_2 x_2 X_3 \ldots {}^v X)$, und x_2 sei eine w-Kante. Außer X_2 und x_2 liegen sämtliche Punkte und Kanten von p_2 in $[C_i]$. Wendet man (9. 4) auf denjenigen Anfangsteil von p an, dessen Endkante x_2 ist, so sieht man, daß X_2 ein $\bar{w}$-Punkt ist. Ist $x_2 = x_0$, so ist p_2 ein X_0-Zug, und dann ist der Beweis fertig. Nehmen wir an, daß $x_2 \neq x_0$ ist. Liegt X_0 auf p_2, so kann nur $X_0 = X_2$ bestehen, also ist X_0 ein Wechselpunkt von p_2. Wir können daher in jedem Falle (7. 4) auf p_2 und p_0 anwenden. Demzufolge gibt es einen a-Zug $p_3 = (X_0 x_0 \ldots {}^w X_2)$ oder $p_4 = (X_0 x_0 \ldots {}^v X)$, und diese Züge enthalten nur solche Kanten, die entweder in p_2 oder in p_0 vorkommen. Nach (8. 7) kann aber p_3 nicht existieren. Es existiert also p_4. Wir zeigen nun, daß p ein X_0-Zug ist. Da von den Kanten von p_0 und p_2 nur x_0 und x_2 nicht zu $[C_i]$ gehören, genügt es zu beweisen, daß x_2 nicht in p_4 liegt. Nehmen wir an, daß x_2 in p_4 liegt. Von den Kanten von p_4 können nur x_0 und x_2 zu X_2 inzident sein. Ist $X_2 = X_0$, so sind wegen $x_0 \neq x_2$ genau zwei Kanten von p_4 mit X_2 inzident. Dies ist jedoch unmöglich, da X_0 der Anfangspunkt, jedoch nicht der Endpunkt von p_4 ist. Ist $X_2 \neq X_0$, so ist von den Kanten von p_4 genau eine Kante zu X_2 inzident. Dies ist aber wieder unmöglich, da jetzt X_2 kein Randpunkt von p_4 sein kann. Es ist also p_4 tatsächlich ein X_0-Zug, und daher ist X in $[C_i]$ v-erreichbar.

(9. 9) *Ist X' ein in $[C_i]$ erreichbarer, X ein in $[C_i]$ nicht erreichbarer Punkt und ist x eine von x_0 verschiedene $X'X$-Kante, so existiert ein a-Zug $p = (X_0 x_0 \ldots x X)$, dessen sämtliche, von x_0 und x verschiedene Kanten zu $[C_i]$ gehören.*

BEWEIS. Es sei x eine v-Kante. Liegt X auf einem X_0-Zug, so kann nur $X = X_0$ bestehen, und deshalb kann x zu keinem X_0-Zug gehören.

(I) Nehmen wir erst an, daß ein X_0-Zug $p_0 = (X_0 x_0 \ldots {}^{\bar{v}} X')$ existiert. Man kann dann (6. 2)(a) auf die a-Züge p_0 und $p' = (X' x X)$ anwenden, demzufolge ist $p = p_0 p'$ ein gesuchter a-Zug.

(II) Es sei jetzt X' in $[C_i]$ nicht $\bar{v}$-erreichbar. Es gibt einen X_0-Zug p_1 mit dem Endpunkt X'. Es bezeichne p_2 den kürzesten Anfangsteil von p_1, dessen Endpunkt X' ist. X' ist im X_0-Zug p_2 einfach, und die Endkante von p_2 ist eine v-Kante. Nach (9.8) kann jedoch X' jetzt nicht $\bar{v}$-erreichbar sein, daher muß X' ein γ_v-Punkt sein. Demzufolge ist nach (8.4) und (6.2)(b), (c) $p = p_2 p'$ ein gesuchter a-Zug.

(9.10) *Jeder Punkt von C_i ist in $[C_i]$ erreichbar.*

Beweis. Der Punkt X_1 der Kante x_0 ist offensichtlich in $[C_i]$ erreichbar. Nehmen wir an, daß in $[C_i]$ Punkte existieren, die in $[C_i]$ nicht erreichbar sind. Da $[C_i]$ zusammenhängend ist, gibt es dann in $[C_i]$ eine Kante x, die einen in $[C_i]$ erreichbaren Punkt X' mit einem in $[C_i]$ nicht erreichbaren Punkt X von C_i verbindet. Nach (9.9) existiert jedoch dann ein a-Zug $p = (X_0 x_0 \ldots xX)$, dessen sämtliche Kanten außer x_0 zu $[C_i]$ gehören. Es ist also X in $[C_i]$ erreichbar, was ein Widerspruch ist.

(9.11) Nehmen wir nun endlich an, daß eine von x_0 verschiedene Eintrittskante von $[C_i]$ existiert. Es sei $[XxX']$ eine solche Kante mit $X \notin C_i$, $X' \in C_i$ und x sei eine v-Kante, X' ein $\bar{v}$-Punkt. Nach (9.10) ist X' in $[C_i]$ erreichbar. Nach (9.9) gibt es dann einen a-Zug $p = (X_0 x_0 \ldots xX)$. Das widerspricht jedoch (8.7). Damit haben wir den Beweis von (9.6) beendet.

Nach (9.6) können wir nach der Art der Eintrittskanten die erreichbaren C-Komponenten in zwei Klassen teilen. Wir nennen eine erreichbare Komponente α- oder *β-erreichbar*, je nachdem ob die Eintrittskante eine α- oder β-Kante ist. Wir können dann nach (9.3), (8.9) und (9.6) zusammenfassend behaupten:

(9.12) *Ist $[C_i]$ eine beliebige C-Komponente, so sind die Kanten, die $[C_i]$ berühren, entweder AC_i- oder B^*C_i-Kanten, und zwar sind — mit Ausnahme der eventuell vorhandenen einzigen Eintrittskante — sämtliche AC_i-Kanten α-Kanten, sämtliche B^*C_i-Kanten β-Kanten.*

(1) *Ist $[C_i]$ unerreichbar, so gibt es keine Ausnahme.*

(2) *Ist $[C_i]$ α-erreichbar, so ist eine B^*C_i-Kante α-Kante.*

(3) *Ist $[C_i]$ β-erreichbar, so ist eine AC_i-Kante β-Kante.*

(9.13) *Bezeichnet man die Anzahl der α- bzw. β-erreichbaren C-Komponenten mit σ_α bzw. σ_β, so gibt es unter den AC-Kanten genau $\nu(A, C) - \sigma_\beta$, unter den B^*C-Kanten genau σ_α α-Kanten.*

Bezüglich q^* haben wir eine C-Komponente $[C_i]$ je nachdem gerade bzw. ungerade genannt, ob die Zahl $\varkappa(C_i) + \nu(A, C_i)$ gerade oder ungerade ist (s. (3.2)). Es gilt nun:

(9.14) *Es sind sämtliche erreichbaren C-Komponenten ungerade, sämtliche unerreichbaren gerade.*

BEWEIS. Betrachten wir eine beliebige C-Komponente $[C_i]$ $(1 \leqq i \leqq m)$. Es sei μ_i bzw. ν_i die Anzahl derjenigen α-Kanten, die $[C_i]$ berühren bzw. die in $[C_i]$ liegen. Da jede Kante zu genau zwei Punkten inzident ist, und die betrachteten α-Kanten außerhalb von $[C_i]$ genau μ_i, in den Punkten von $[C_i]$ genau $\sum_{X \in C_i} (f^*, X)$ Inzidenzen hervorrufen, gilt

(1) $$2(\mu_i + \nu_i) = \mu_i + \sum_{X \in C_i} (f^*, X).$$

Andernfalls ist nach (7.1)(3), (4) für jedes X die Zahl $(f^*, X) - \varkappa(X)$ gerade, und demzufolge ist auch

$$\sum_{X \in C_i} (f^*, X) - \varkappa(C_i) = \sum_{X \in C_i} ((f^*, X) - \varkappa(X))$$

gerade. Daraus und aus (1) folgt

(2) $$\mu_i + \varkappa(C_i) \equiv 0 \pmod 2.$$

Nach (9.12) ist aber

$$\mu_i = \nu(A, C_i) + \varepsilon_i,$$

wo $\varepsilon_i = 0$, 1 oder -1 ist, je nachdem ob $[C_i]$ unerreichbar, α- oder β-erreichbar ist. Dies ergibt zusammen mit (2) die Richtigkeit unserer Behauptung.

§ 10. Bestimmung von $S(q)$

Wir wollen bezüglich des Wertes $S(q^*)$ des unter (9.1) definierten Gewichtssystems q^* folgende Behauptung beweisen:

(10.1) *Es gilt* $2S(q^*) = \varkappa(\Phi^*)$.

BEWEIS. Es ist nach (3.2)(2)

$$S(q^*) = \varkappa(B^*) + \varkappa'(B^*) + \nu(A, A) + \frac{1}{2}(\varkappa(C) + \nu(A, C) - \tau),$$

wo τ die Anzahl der ungeraden C-Komponenten bedeutet.

Da $\varkappa(\Phi^*) = \varkappa(A) + \varkappa(B^*) + \varkappa(C)$ ist, ist (10.1) der folgenden Behauptung gleichwertig:

(1) $$\varkappa(A) = \varkappa(B^*) + 2\varkappa'(B^*) + 2\nu(A, A) + \nu(A, C) - \tau.$$

Um dies zu beweisen, werden wir die Anzahl ν_α derjenigen α-Kanten, die den Teilgraphen $[B^*]$ berühren, auf zweierlei Weisen ausdrücken.

(I) Ist X ein β-Punkt, so folgt im Falle $\varkappa'(X) > 0$ aus (8.2) und (7.1)(3), im Falle $\varkappa'(X) = 0$ aus (7.1)(3), (4) die Gleichung $(f^*, X) = \varkappa(X) + 2\varkappa'(X)$. Nach (8.8) können wir dann behaupten

(2) $$\nu_\alpha = \varkappa(B^*) + 2\varkappa'(B^*).$$

(II) Aus (7. 1)(4) und (8. 2) folgt, daß zu jedem α-Punkt X genau $\varkappa(X)$ α-Kanten inzident sind. Nach (8. 8) ist jede AA-Kante eine α-Kante. Wir können daher nach (9. 13) behaupten: unter den AB^*-Kanten kommen genau

$$\varkappa(A)-2\nu(A, A)-(\nu(A, C)-\sigma_\beta)$$

α-Kanten vor. Daraus erhalten wir nach (9. 13) für ν_α den Ausdruck

(3) $$\nu_\alpha=\varkappa(A)-2\nu(A, A)-\nu(A, C)+\sigma_\alpha+\sigma_\beta.$$

Es ist aber nach (9. 14) $\sigma_\alpha+\sigma_\beta=\tau$, und so ergibt sich aus (2) und (3) die Gleichung (1).

(10. 2) Um den Wert $S(q)$ zu erhalten, nehmen wir außer $B^*=B\cup\{X^*\}$ und (7. 1) (a), (b) auch in Betracht, daß $\nu_\Gamma(A, A)=\nu(A, A)$, $\nu_\Gamma(A, C)=\nu(A, C)$ ist und die C-Komponenten von Γ mit denen von Γ^* identisch sind, sowie daß diese Komponenten in Γ und in Γ^* gleichzeitig gerade oder ungerade sind. Wir erhalten so die Gleichungen

$$S(q^*)=S(q)+\tilde{\delta}_{\min} \quad \text{und} \quad \varkappa(\Phi^*)=\varkappa(\Phi)+\tilde{\delta}_{\min}.$$

Diese ergeben jedoch zusammen mit (10. 1) für $S(q)$ den Wert

$$S(q)=\frac{\varkappa(\Phi)-\tilde{\delta}_{\min}}{2}.$$

Nach (5. 15) haben wir damit den Beweis des Hauptsatzes (3. 3) beendet.

III. SPEZIELLE GEWICHTSSYSTEME UND KAPAZITÄTSFUNKTIONEN

§ 11. Gewichtssysteme mit besonderen Eigenschaften

In diesem Paragraphen wollen wir zeigen, daß man sich bei der Bestimmung des minimalen Wertes von $S(q)$ auf Gewichtssysteme mit besonderen Eigenschaften beschränken kann. Dies wird uns bei gewissen speziellen Kapazitätsfunktionen es ermöglichen, unserem Hauptsatz eine übersichtlichere Fassung zu geben.

(11. 1) Es seien in den Punkten des Graphen Γ die Kapazitätsfunktionen $\varkappa(X)$ und $\varkappa'(X)$ definiert,[6] und es sei $q=q(A, B, C)$ ein beliebiges Gewichtssystem mit $A\neq\emptyset$, ferner sei $X\in A$. Wir wollen untersuchen, welche Änderung $S(q)$ bei der Verlegung von X nach C erfährt. Es sei

$$A'=A-\{X\}, \quad C'=C\cup\{X\}, \quad q'=q'(A', B, C').$$

Es gilt folgende Behauptung:

[6] Von hier an beziehen sich unsere Begriffe und Bezeichnungen wieder auf den mit Γ bezeichneten Graphen.

(11.2) *Ist* $\nu(X, A) \geqq \varkappa(X)-1$, *so ist* $S(q') \leqq S(q)$.

Beweis. Es ist $\varkappa(C')=\varkappa(C)+\varkappa(X)$, $\nu(A', A')=\nu(A, A)-\nu(X, A)$ und $\nu(A', C')=\nu(A, C)+\nu(X, A)-\nu(X, C)$. Daraus folgt nach (3 2)(2)

$$S(q')=S(q)+\frac{1}{2}(\varkappa(X)-\nu(X, A)-\nu(X, C)+\tau_q-\tau_{q'}).$$

Ist $\nu(X, C)=0$, so sind sämtliche C-Komponenten auch C'-Komponenten, und außer diesen gibt es nur noch eine C'-Komponente: X. Mithin ist $\tau_{q'}=\tau_q$ oder $\tau_{q'}=\tau_q+1$, je nachdem ob $\varkappa(X)+\nu(X, A)$ gerade oder ungerade ist. Im ersten Falle folgt daraus, da $\nu(X, A)=\varkappa(X)-1$ nicht eintreten kann, die Behauptung $S(q') \leqq S(q)$. Im zweiten Falle ist die gleiche Behauptung offensichtlich richtig.

Nehmen wir nun an, daß $\nu(X, C)>0$ ist. Es seien $[C_i]$ $(i=1, \ldots, m)$ die C-Komponenten, und von diesen seien $[C_i]$ $(i=j+1, \ldots, m)$ diejenigen, für welche $\nu(C_i, A)>0$ gilt $(0 \leqq j<m)$. Dann sind die C'-Komponenten:

$[C'_{j+1}]$ mit $C'_{j+1}=(\bigcup_{i=j+1}^{m} C_i) \cup \{X\}$ und, falls $j>0$ ist, $[C'_i]=[C_i]$ $(i=1, \ldots, j)$.

Es bezeichne ferner σ die Anzahl der ungeraden Komponenten unter $[C_{j+1}], \ldots, [C_m]$, und es sei $\varepsilon=0$ oder 1, je nachdem ob $[C'_{j+1}]$ gerade oder ungerade ist. Es gilt dann

$$\tau_q-\tau_{q'}=\sigma-\varepsilon$$

und

$$S(q')=S(q)-\frac{1}{2}(\nu(X, A)-\varkappa(X)+\nu(X, C)-\sigma+\varepsilon).$$

Es besteht ferner $\nu(X, C) \geqq m-j \geqq \sigma$.

Gilt nun $\nu(X, A) \geqq \varkappa(X)$ oder $\nu(X, C)>\sigma$, so ist offensichtlich $S(q') \leqq S(q)$.

Ist $\nu(X, A)=\varkappa(X)-1$ und $\nu(X, C)=\sigma$, so zeigen wir, daß $\varepsilon=1$ ist, woraus wieder $S(q') \leqq S(q)$ folgen wird. In diesem Falle muß nämlich $\nu(X, C_i)=1$ und $\varkappa(C_i)+\nu(A, C_i) \equiv 1 \pmod 2$ für jedes $i=j+1, \ldots, m$ bestehen und deshalb ist die Zahl

$$\varkappa(C'_{j+1})+\nu(A', C'_{j+1})=\varkappa(X)+\nu(X, A)+\sum_{i=j+1}^{m}(\varkappa(C_i)+\nu(A, C_i)-1)$$

ungerade.

(11.3) Nehmen wir jetzt von $q=q(A, B, C)$ an, daß $C \neq \emptyset$ ist, und es sei $X \in C$. Das System q' definieren wir folgendermaßen: Es sei

$$B'=B \cup\{X\}, \quad C'=C-\{X\} \quad \text{und} \quad q'=q'(A, B', C').$$

Es gilt dann die Behauptung:

(11.4) *Ist* $\nu(X, A) \geqq \varkappa(X)+2\varkappa'(X)$, *so ist* $S(q') \leqq S(q)$.

BEWEIS. Eine einfache Rechnung zeigt, daß

$$S(q')=S(q)-\frac{1}{2}\left(\nu(A, X)-(\varkappa(X)+2\varkappa'(X))+\tau_{q'}-\tau_q\right)$$

ist. Es sei $[C_1]$ diejenige C-Komponente, die X enthält, und es sei $C^*= C_1-\{X\}$. Die Gesamtheit der C'-Komponenten besteht aus sämtlichen von C_1 verschiedenen C-Komponenten sowie (falls $C^* \neq \varnothing$) aus den Komponenten des Teilgraphen $[C^*]$. Mithin ist $\tau_{q'} \geqq \tau_q-1$, und die Gleichheit kann hier nur dann eintreten, wenn $[C_1]$ ungerade ist und sämtliche Komponenten von $[C^*]$ gerade sind oder $C^*=\varnothing$ ist. In diesen Fällen muß jedoch $\varkappa(X)+ \nu(A, X)$ ungerade sein, und so kann in $\nu(A, X) \geqq \varkappa(X)+2\varkappa'(X)$ das Gleichheitszeichen nicht gelten. Hieraus sehen wir, daß in jedem Falle $S(q') \leqq S(q)$ gilt.

(11.5) Es bezeichne Q_0 die Menge jener Gewichtssysteme $q=q(A, B, C)$, welche folgende Eigenschaften besitzen:

(a) A enthält, falls es nichtleer ist, nur solche Punkte X, die der Ungleichung $\nu(X, A)<\varkappa(X)-1$ genügen.

(b) C enthält, falls es nichtleer ist, nur solche Punkte X, die der Bedingung $\nu(X, A)<\varkappa(X)+2\varkappa'(X)$ genügen.

Nach (11.2) und (11.4) können wir dann behaupten:

(1) $$\min_{q \in Q_0} S(q)=S_{\min}.$$

Wir nennen die Punkte der Menge $E \subseteq \Phi$ (in Γ) *unabhängig,* wenn im Falle $\nu(E)>1$ je zwei von ihnen durch keine Kante (von Γ) verbunden sind. Dann kann man folgende Behauptungen machen:

(11.6) *Ist für jedes* X $\varkappa(X)+\varkappa'(X)=1$, *so ist in jedem System* $q=q(A, B, C)$ *von* Q_0 *die Menge* A *leer.*

(11.7) *Ist für jedes* X $\varkappa(X)+2\varkappa'(X)=2$, *so besitzt jedes System* $q=q(A, B, C)$ *von* Q_0 *die Eigenschaften:*

(1) *Die* A*-Punkte sind unabhängig, und für jedes* X *von* A *gilt* $\varkappa(X)=2$.

(2) *Zu jedem* C*-Punkt ist höchstens eine* AC*-Kante inzident.*

§ 12. Haupt- und Nebenpunkte. Hauptwege

(12.1) Wir wollen einige spezielle Kapazitätsfunktionen betrachten, bei denen unser Hauptsatz bzw. einige nachfolgende Sätze eine übersichtlichere Formulierung zulassen. Nehmen wir folgende Bedingungen:

(a) Für jedes X mit $\varkappa(X) > 0$ ist $\varkappa'(X) = 0$.

(b) Für jedes X mit $\varkappa(X) = 0$ ist $\varkappa'(X) = 1$.

Genügen die Funktionen $\varkappa$ und $\varkappa'$ diesen Bedingungen, so ist $\varkappa'$ durch $\varkappa$ eindeutig bestimmt, und wir nennen in diesem Falle $\varkappa$ und $\varkappa'$ zueinander *komplementär*. Ferner nennen wir jetzt die Punkte X mit $\varkappa(X) > 0$ *Hauptpunkte*, diejenigen mit $\varkappa(X) = 0$ *Nebenpunkte* und diejenigen Bogen, deren Randpunkte bzw. innere Punkte Haupt- bzw. Nebenpunkte sind, *Hauptbogen*. Die Bogen eines aufnehmbaren Bogensystems sind jetzt alle Hauptbogen, und diese können nur Randpunkte gemein haben.

Eine weitere beachtenswerte Spezialisierung erhalten wir durch die Annahme, daß bei komplementären Kapazitäten $\varkappa$ in jedem Hauptpunkt den gleichen Wert σ annimmt. Um ein solches Paar von Kapazitätsfunktionen anzugeben, genügt es, irgendwelche Punkte von Γ als Hauptpunkte auszeichnen, die übrigen als Nebenpunkte betrachten und den Wert σ vorschreiben. (Die Menge der Haupt- bzw. Nebenpunkte kann auch leer sein!)

Von hier an werden wir uns im II. Abschnitt nur mit den letzterwähnten Kapazitätsfunktionen beschäftigen, und zwar nur im Falle $\sigma = 1$.

(12.2) Es seien nun in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Die übrigen Punkte von Γ werden wir dann immer — ohne dies ausdrücklich zu betonen — Nebenpunkte nennen. Die Funktionen $\varkappa(X)$ und $\varkappa'(X)$ seien durch folgende Bedingungen definiert: Für jeden Hauptpunkt sei $\varkappa(X) = 1$ und $\varkappa'(X) = 0$, für jeden Nebenpunkt $\varkappa(X) = 0$ und $\varkappa'(X) = 1$. Wir bezeichnen die Anzahl der Haupt- bzw. Nebenpunkte in einer beliebigen Teilmenge E von Φ mit $\nu_h(E)$ bzw. $\nu_n(E)$.

Die aufnehmbaren Bogen fallen jetzt mit den *Hauptwegen* zusammen. Ein Hauptweg ist ein solcher Weg, dessen Randpunkte bzw. inneren Punkte Haupt- bzw. Nebenpunkte sind. Ein aufnehmbares Bogensystem besteht aus *unabhängigen* Hauptwegen, d. h. aus solchen Hauptwegen, die paarweise keinen gemeinsamen Punkt enthalten. $\nu_{\max}$ gibt jetzt *die maximale Anzahl der unabhängigen Hauptwege* an.

Betrachten wir nun den Wert $S_{\min}$. Nach (11.5)(1) und (11.6) genügt es bei der Bestimmung dieses Wertes, nur solche Systeme $q = q(A, B, C)$ zu betrachten, in denen A leer ist. Solche Systeme sind durch die Angabe der einzigen, beliebig wählbaren Teilmenge B von Φ bestimmt. Wir setzen daher für solche q

$$S(q) = S'(B) \quad \text{und} \quad \tau_q = \tau(\bar{B}) \quad (\bar{B} = \Phi - B).$$

$\tau(\bar{B})$ gibt die Anzahl der ungeraden, d. h. derjenigen Komponenten von $[\bar{B}]$, die eine ungerade Anzahl von Hauptpunkten enthalten. Bezeichnen $[\bar{B}_i]$

($i = 1, \ldots, m$) die Komponenten von $[\bar{B}]$, so gilt nach (3.2)(1), (2), (11.6) und (11.5)

$$(1) \qquad S'(B) = \nu(B) + \sum_{i=1}^{m} \left[\frac{1}{2} \nu_h(\bar{B}_i)\right] = \nu(B) + \frac{1}{2}(\nu_h(\bar{B}) - \tau(\bar{B}))$$

und

$$\min_{B \subseteq \Phi} S'(B) = S_{\min}.$$

Unseren Hauptsatz (3.3) kann man jetzt folgendermaßen formulieren:

SATZ (12.3) *Sind in Γ irgendwelche Punkte als Hauptpunkte ausgewählt, so ist die maximale Anzahl der unabhängigen Hauptwege dem Wert $\min_{B \subseteq \Phi} S'(B)$ gleich.*

Wegen späteren Anwendungen wollen wir noch folgende Behauptung beweisen:

(12.4) *Gilt für die Teilmenge B von Φ die Ungleichung $S'(B) < S'(\emptyset)$, so ist $\tau(\bar{B}) \geqq 2$.*

BEWEIS. Es ist $S'(\emptyset) = \frac{1}{2}(\nu_h(\Phi) - \tau(\Phi))$. Wir erhalten also aus $S'(B) < S'(\emptyset)$ die Ungleichung $\tau(\bar{B}) > \nu(B) + \nu_n(B) + \tau(\Phi)$. Da jetzt $B \neq \emptyset$ gilt, besteht $\tau(\bar{B}) \geqq 2$.

(12.5) Bisher haben wir von den Gewichten der füllenden Gewichtssysteme verlangt, daß sie jeden Bogen des Graphen füllen (s. (2.2)). Lassen wir diese Forderung fallen und verlangen nur, daß sämtliche Hauptwege gefüllt werden, so sind bei einem System $q = q(\emptyset, B, \bar{B})$ sämtliche in Nebenpunkten liegenden halben Gewichte zu der Füllung überflüssig. Es ist jetzt naheliegend, eine neue Art von Gewichtssystemen zu betrachten. Diese Systeme, die wir Systeme *zweiter Art* nennen und mit dem Buchstaben r bezeichnen wollen, definieren wir folgendermaßen: In r sollen nur die Punkte Gewichte erhalten, und zwar sollen wieder nur die Gewichte 0, 1 und 1/2 vorkommen. Einen Hauptweg w nennen wir jetzt durch r dann gefüllt, wenn w einen Punkt mit dem Gewicht 1 oder zwei Punkte (diese müssen nicht die Randpunkte von w sein) mit halben Gewichten enthält. r heißt *füllend,* wenn jeder Hauptweg von Γ durch r gefüllt ist, oder wenn in Γ kein Hauptweg existiert. Die Menge dieser füllenden Gewichtssysteme bezeichnen wir mit R. R ist offensichtlich nichtleer.

Den *Wert* von r definieren wir gleichfalls von neuem. Es bezeichne B die Menge derjenigen Punkte, die in r das Gewicht 1 erhalten und $\nu_{1/2}(E)$ die Anzahl derjenigen Punkte einer beliebigen Teilmenge E von Φ, die in r das Gewicht 1/2 haben. Unter Beachtung von (12.2) und der oben Gesagten sei der Wert von r

(1) $$S''(r) = \nu(B) + \sum_{i=1}^{m} \left[\frac{1}{2} \nu_{1/2}(\bar{B}_i) \right],$$

wo $[\bar{B}_i]$ $(i = 1, \ldots, m)$ die Komponenten von $[\bar{B}]$ bedeuten $(\bar{B} = \Phi - B)$.

Wir wollen nun zeigen, daß die Behauptung von (12. 3) auch mit den Gewichtssystemen zweiter Art gültig bleibt:

SATZ (12. 6) *Sind in Γ irgendwelche Punkte als Hauptpunkte ausgewählt, so ist die maximale Anzahl der unabhängigen Hauptwege dem Wert* $\min\limits_{r \in R} S''(r)$ *gleich.*

BEWEIS. (I) Es sei B eine beliebige Teilmenge von Φ, und $[\bar{B}_i]$ $(i = 1, \ldots, m)$ seien die Komponenten von $[\bar{B}]$. Ferner bezeichne r_1 jenes Gewichtssystem zweiter Art, in dem die Punkte von B das Gewicht 1, die Hauptpunkte von $\bar{B}$ das Gewicht 1/2, die Nebenpunkte von $\bar{B}$ das Gewicht 0 erhalten. Es ist dann $r_1 \in R$ und es gilt $\nu_{1/2}(\bar{B}_i) = \nu_h(\bar{B}_i)$ $(i = 1, \ldots, m)$. Nach (12. 2) (1) und (12. 5) (1) ist dann $S''(r_1) = S'(B)$, und so ist laut (12. 3) $\min\limits_{r \in R} S''(r) \leqq \nu_{\max}$.

(II) Es sei jetzt r ein beliebiges Element von R. Es bezeichne ferner B die Menge der Punkte, die in r das Gewicht 1 enthalten, und $[\bar{B}_i]$ $(i = 1, \ldots, m)$ seien die Komponenten von $[\bar{B}]$. Endlich sei W ein beliebiges System unabhängiger Hauptwege. Die Anzahl der Wege von W, die einen B-Punkt enthalten, ist $\leqq \nu(B)$. Ein Weg von W, der keinen B-Punkt enthält, muß ganz in einer der Komponenten $[\bar{B}_i]$ liegen. In einem $[\bar{B}_i]$ können aber höchstens $\left[\frac{1}{2} \nu_{1/2}(\bar{B}_i) \right]$ unabhängige Hauptwege liegen. Es folgt daraus nach (12. 5) (1), daß $\nu(W) \leqq S''(r)$ ist. Wir gelangen so zu $\nu_{\max} \leqq \min\limits_{r \in R} S''(r)$, und damit ist unser Beweis beendet.

§ 13. Trennungssätze

(13. 1) Es sei Γ wieder ein Graph, in dem irgendwelche Punkte als Hauptpunkte ausgezeichnet sind. Die einfachsten füllenden Gewichtssysteme zweiter Art sind offensichtlich jene, bei denen nur die Gewichte 1 und 0 vorkommen. Ein solches System ist durch die Menge D der Punkte mit positiven Gewichten vollständig bestimmt und so beschaffen, daß jeder Hauptweg von Γ einen D-Punkt enthält. Wir können hier statt der Füllung der Hauptwege auch von der Trennung der Hauptpunkte sprechen, da man von keinem Hauptpunkt zu einem anderen ohne Berührung eines D-Punktes gelangen kann. (Auch die Randpunkte der Hauptwege können an der Trennung teilnehmen!)

Wir nennen nun eine Teilmenge D von Φ *trennend*, wenn jeder Hauptweg von Γ mindestens einen D-Punkt enthält (oder wenn Γ keinen Hauptweg besitzt). T bezeichne die Menge der trennenden Teilmengen (T ist nichtleer). Es sei ferner

$$\pi_{\min} = \min_{D \in T} \nu(D).$$

Die Zahl $\pi_{\min}$ existiert und bedeutet *die minimale Anzahl der trennenden Punkte.*

Es stellt sich dann die Frage: Was für ein Zusammenhang besteht zwischen $\pi_{\min}$ und $\nu_{\max}$? Man kann als Antwort im allgemeinen keine genaue Gleichung angeben, sondern muß sich mit Ungleichungen begnügen.

Es besteht folgender

SATZ (13. 2) *Sind in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet, so gilt*

$$\pi_{\min} \leqq 2\nu_{\max}.$$

BEWEIS. Nach (12. 3) gibt es ein $B \subseteq \Phi$ mit

$$S'(B) = \nu(B) + \sum_{i=1}^{m} \left[\frac{1}{2} \nu_h(\bar{B}_i)\right] = \nu_{\max} \qquad (\bar{B} = \Phi - B), \tag{1}$$

wo $[\bar{B}_i]$ $(i = 1, \ldots, m)$ die Komponenten von $[\bar{B}]$ sind. Wir definieren eine Menge D wie folgt: Wir wählen in jedem $\bar{B}_i$, das Hauptpunkte enthält, je einen Hauptpunkt aus. Es soll nun D aus sämtlichen Punkten von B sowie aus sämtlichen Hauptpunkten von $\bar{B}$, mit Ausnahme der ausgewählten, bestehen. Es ist klar, daß dieses D trennend ist, und es gilt

$$\nu(D) \leqq \nu(B) + 2\sum_{i=1}^{m} \left[\frac{1}{2} \nu_h(\bar{B}_i)\right]. \tag{2}$$

Aus (1) und (2) folgt

$$\nu(D) \leqq 2\nu_{\max} - \nu(B), \tag{3}$$

und dies ergibt $\pi_{\min} \leqq 2\nu_{\max}$.

BEMERKUNGEN (1) Satz (13. 2) kann ohne weitere Voraussetzungen nicht verschärft werden. Dies zeigt folgendes Beispiel: Γ bestehe aus n isolierten „Dreiecken" und jeder Punkt von Γ sei Hauptpunkt. Dann ist $\nu_{\max} = n$, $\pi_{\min} = 2n$.

(2) Ist Γ ein paarer Graph und sind sämtliche Punkte von Γ Hauptpunkte, so kann man durch eine einfache Modifizierung des Beweises von (13. 2) zu der Ungleichung $\pi_{\min} \leqq \nu_{\max}$ gelangen. Da anderseits offensichtlich $\pi_{\min} \geqq \nu_{\max}$ besteht, erhalten wir $\pi_{\min} = \nu_{\max}$. Diese Gleichung ist mit der Behauptung des Königschen Satzes (1) unserer Einleitung identisch.

(13.3) Der Graph in der vorangehenden Bemerkung (1) ist nicht zusammenhängend. Nimmt man an, daß der Graph zusammenhängend ist, bzw. daß der „Zusammenhangsgrad" des Graphen einen gewissen Wert erreicht, so kann man den Satz (13.2) verschärfen.

Wir sagen: Die verschiedenen Punkte X' und X'' sind durch die von X' und X'' verschiedenen Punkte $X_1, \ldots, X_j$ $(j \geqq 1)$ *getrennt*, wenn jeder Weg von Γ, der X' mit X'' verbindet, mindestens einen der Punkte $X_1, \ldots, X_j$ enthält. Die Aussage *„Γ ist bezüglich der Hauptpunkte η-fach zusammenhängend"* soll folgendes bedeuten:

Im Falle $\eta = 1$: Höchstens eine der Komponenten von Γ enthält Hauptpunkte. In diesem Falle werden wir auch Γ bezüglich der Hauptpunkte zusammenhängend nennen.

Im Falle $\eta \geqq 2$: Γ ist bezüglich der Hauptpunkte zusammenhängend, und je zwei Hauptpunkte X' und X'' können durch weniger als η, von X' und X'' verschiedenen Punkten nicht getrennt werden.

Aus dieser Erklärung folgt: Ist Γ bezüglich der Hauptpunkte η-fach zusammenhängend $(\eta > 1)$, so ist er auch $(\eta - 1)$-fach zusammenhängend.

Nun wollen wir unter Beachtung von (4.2) (2) folgenden Satz aussprechen:

SATZ (13.4) *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Ist dann Γ bezüglich der Hauptpunkte η-fach zusammenhängend* $(\eta \geqq 1)$ *und ist* $\nu_{\max} < \left[\frac{1}{2}\nu_h(\Phi)\right]$, *so ist*

$$\pi_{\min} \leqq 2\nu_{\max} - \eta.$$

BEWEIS. Es sei B dieselbe Menge wie im Beweis von (13.2). Da Γ bezüglich der Hauptpunkte zusammenhängend ist, ist $S'(\emptyset) = \left[\frac{1}{2}\nu_h(\Phi)\right]$, und so gilt $S'(B) < S'(\emptyset)$. Dann ist aber nach (12.4) $\tau(\bar{B}) \geqq 2$. Jede ungerade $\bar{B}$-Komponente enthält jedoch einen Hauptpunkt, und so existieren in $\bar{B}$ zwei Hauptpunkte, die durch die Punkte von B getrennt werden. Es gilt daher $\nu(B) \geqq \eta$, und so folgt laut der Ungleichung (3) des Beweises von (13.2) die Behauptung $\pi_{\min} \leqq 2\nu_{\max} - \eta$.

BEMERKUNGEN. (1) Gilt nicht $\nu_{\max} < \left[\frac{1}{2}\nu_h(\Phi)\right]$, so ist nach (4.2) (2) $\nu_{\max} = \left[\frac{1}{2}\nu_h(\Phi)\right]$. In diesem Falle kann — wie das folgende Beispiel zeigt — $\pi_{\min} > 2\nu_{\max} - \eta$ eintreten.

Es sei $\Phi = \{X_1, \ldots, X_n, X'_1, \ldots, X'_\eta\}$ $(\eta \geqq 2,\ \eta < n < 2\eta)$ und Γ soll je eine $X_iX'_j$-Kante $(i = 1, \ldots, n;\ j = 1, \ldots, \eta)$ enthalten, jedoch keine andere

Kanten. $X_1, \ldots, X_n$ seien die Hauptpunkte, $X'_1, \ldots, X'_\eta$ die Nebenpunkte. Dann ist Γ bezüglich der Hauptpunkte η-fach zusammenhängend, $\nu_{\max} = \left[\frac{n}{2}\right] = \left[\frac{1}{2}\nu_h(\Phi)\right]$ und $\pi_{\min} = \eta$.

(2) Ohne weitere Bedingungen kann man auch (13. 4) nicht verschärfen. Das zeigt folgendes Beispiel: Es sei

$$A = \{X_1, \ldots, X_n\} \quad B = \{X'_1, \ldots, X'_\eta\} \qquad (2 \leqq 2\eta \leqq n),$$

$$C_i = \{X^i_1, X^i_2, X^i_3\} \quad (i = 1, \ldots, l;\ l \geqq 1),$$

und es seien A, B und sämtliche C_i paarweise elementenfremd. Φ bestehe aus sämtlichen Punkten der Mengen A, B und C_i $(i = 1, \ldots, l)$, die Menge der Kanten von Γ

aus je einer $X_i X'_j$-Kante $(i = 1, \ldots, n;\ j = 1, \ldots, \eta)$,
aus je einer $X^i_k X'_j$-Kante $(i = 1, \ldots, l;\ j = 1, \ldots, \eta;\ k = 1, 2, 3)$ und
aus je einer $X^i_j X^i_k$-Kante $(i = 1, \ldots, l;\ j, k = 1, 2, 3;\ j \neq k)$.

Die A-Punkte und die C_i-Punkte seien die Hauptpunkte, die B-Punkte die Nebenpunkte. Es ist dann Γ bezüglich der Hauptpunkte η-fach zusammenhängend, $\nu_{\max} = \eta + l$, $\pi_{\min} = \eta + 2l$. Es gilt daher $\pi_{\min} = 2\nu_{\max} - \eta$.

(3) Wir vermuten folgende Verallgemeinerung von (13. 2):

Sind in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet, und ist die maximale Anzahl solcher Hauptwege, die paarweise keinen Nebenpunkt gemeinsam haben, gleich $\nu'_{\max}$, so ist die minimale Anzahl der derart auswählbaren Nebenpunkte, daß jeder Hauptweg mindestens einen der ausgewählten Punkte enthält, nicht größer als $2\nu'_{\max}$.

Dieser Satz scheint tieferliegend zu sein als (13. 2). Man könnte natürlich einen Beweis dieser Vermutung erhalten, wenn man einen unserem Hauptsatz ähnlichen Satz über Hauptwege hätte. (S. Bemerkung (2) bei (3. 3).)

IV. VERALLGEMEINERTE FAKTOREN

§ 14. Allgemeine Existenzsätze

(14. 1) Es sei Γ ein nichtleerer Graph, $\varkappa(X)$ und $\varkappa'(X)$ seien auf der Menge der Punkte von Γ definierte Funktionen, die nur nichtnegative ganze Werte annehmen. Ein zu $\varkappa$ und $\varkappa'$ gehöriger *verallgemeinerter Faktor* von Γ oder kurz ein *$(\varkappa, \varkappa')$-Faktor* ist ein solches Bogensystem H von Γ, das für jeden Punkt X von Γ den Bedingungen

(1) $$[H, X] = \varkappa(X) \quad \text{und} \quad |H, X| \leqq \varkappa'(X)$$

genügt. Nach (4. 2) können wir auch sagen: Ein $(\varkappa, \varkappa')$-Faktor ist ein aufnehmbares Bogensystem H mit $\delta(H)=0$. Ist für jedes X $\varkappa'(X)=0$, so sind die $(\varkappa, \varkappa')$-Faktoren mit den gewöhnlichen $\varkappa$-Faktoren identisch (s. [10] und [7]).

Sind $\varkappa$ und $\varkappa'$ zueinander komplementär (s. (12. 1)), so sollen die $(\varkappa, \varkappa')$-Faktoren *topologische $\varkappa$-Faktoren* heißen.

Sind endlich zu dem Graphen Γ keine Funktionen $\varkappa$ und $\varkappa'$ angegeben, sondern sind nur in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet, so verstehen wir unter einem *topologischen σ-Faktor* — wo σ eine natürliche Zahl bedeutet — einen topologischen $\varkappa$-Faktor, bei dem die Funktion $\varkappa(X)$ in jedem Hauptpunkt den Wert σ, in jedem Nebenpunkt den Wert 0 annimmt.

Aus unserer Erklärung und aus (4. 2) folgt unmittelbar die Behauptung: Γ besitzt dann und nur dann einen $(\varkappa, \varkappa')$-Faktor, wenn

(2) $$\nu_{\max}=\frac{1}{2}\varkappa(\Phi)$$

besteht. Daraus beweisen wir den

SATZ (14. 2) *Es seien in den Punkten von Γ die Kapazitätsfunktionen $\varkappa(X)$ und $\varkappa'(X)$ erklärt. Γ enthält dann und nur dann einen $(\varkappa, \varkappa')$-Faktor, wenn für jedes Gewichtssystem $q=q(A, B, C)$*

(1) $$\frac{1}{2}\varkappa(\Phi)\leqq S(q)$$

besteht, oder anders formuliert, wenn bei jeder Zerlegung der Menge Φ in drei Mengen A, B und C die Ungleichung

(2) $$\varkappa(A)\leqq\varkappa(B)+2\varkappa'(B)+2\nu(A, A)+\nu(A, C)-\tau$$

besteht, wo τ die Anzahl der Komponenten $[C_i]$ von $[C]$ mit $\varkappa(C_i)+\nu(A, C_i)\equiv 1 \pmod 2$ bedeutet.

BEWEIS. Nach (4. 1) ist $\nu_{\max}\leqq S(q)$. Besteht also (14. 1) (2), so gilt für jedes q die Ungleichung (1). Gilt umgekehrt für jedes q die Ungleichung (1), so besteht laut unseres Hauptsatzes (3. 3) die Ungleichung $\nu_{\max}\geqq\frac{1}{2}\varkappa(\Phi)$, woraus nach (4. 2) (2) die Gleichung (14. 1) (2) folgt.

Die Ungleichung (1) ist unter Beachtung von $\varkappa(\Phi)=\varkappa(A)+\varkappa(B)+\varkappa(C)$ mit (2) gleichwertig.

BEMERKUNG. Ist für jedes X $\varkappa'(X)=0$, so gibt (14. 2) die Belck—Tuttesche Bedingung der Existenz von σ- bzw. $\varkappa$-Faktoren endlicher Graphen an ([2], Theorem IV; [10], Theorem XV).

Wir wollen uns mit den topologischen 1- und 2-Faktoren auch separiert beschäftigen.

(14. 3) Betrachten wir zuerst die topologischen 1-Faktoren. Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Die Definitionen von $\varkappa$ und $\varkappa'$ lauten dann: Es ist in jedem Hauptpunkt $\varkappa(X)=1$ und $\varkappa'(X)=0$, in jedem Nebenpunkt $\varkappa(X)=0$ und $\varkappa'(X)=1$. Nach (11. 5) und (11. 6) genügt es jetzt, in (14. 2) nur Systeme $q=q(A, B, C)$ mit $A=\varnothing$ zu betrachten. Wir bekommen so aus (14. 2) (2) mit den Bezeichnungen von (12. 2) den

SATZ (14. 4) *Es seien irgendwelche Punkte in Γ als Hauptpunkte ausgezeichnet. Γ enthält dann und nur dann einen topologischen 1-Faktor, d. h. $\frac{1}{2}\nu(\Phi)$ unabhängige Hauptwege, wenn für jede beliebige Teilmenge B von Φ die Ungleichung*

(1) $$\tau(\bar{B}) \leqq \nu_h(B) + 2\nu_n(B)$$

besteht.

Sind sämtliche Punkte von Γ Hauptpunkte, so gibt (14. 4) den bekannten Tutteschen Satz über die Existenz von 1-Faktoren an ([9], Theorem IV).

(14. 5) Jetzt wollen wir uns mit topologischen 2-Faktoren beschäftigen. Es seien daher wieder irgendwelche Punkte in Γ als Hauptpunkte ausgezeichnet, und $\varkappa$ und $\varkappa'$ sollen folgendermaßen definiert sein: In jedem Hauptpunkt ist $\varkappa(X)=2$ und $\varkappa'(X)=0$, in jedem Nebenpunkt $\varkappa(X)=0$ und $\varkappa'(X)=1$.

Existieren Wege in einem topologischen 2-Faktor, so setzen sich diese zu Kreisen zusammen. Diese Kreise bilden, zusammen mit den Schlingen des Faktors, ein *Kreissystem* mit folgenden Eigenschaften: Jeder Kreis des Systems enthält mindestens einen Hauptpunkt, jeder Hauptpunkt von Γ liegt auf einem Kreis des Systems und es haben je zwei Kreise des Systems (falls mehrere existieren) keinen gemeinsamen Punkt. Auf diese Weise gehört zu jedem topologischen 2-Faktor ein Kreissystem. (Zu dem leeren Faktor gehört das leere Kreissystem.) Umgekehrt ist es klar, daß die Hauptpunkte von Γ jedes Kreissystem mit den erwähnten Eigenschaften in solche Bogen zerlegen, die zusammen einen topologischen 2-Faktor bilden.

Nach (11. 5) und (11. 7) genügt es bei topologischen 2-Faktoren in (14. 2) nur Systeme q mit den Eigenschaften (11. 7) (1), (2) zu betrachten, und so erhält man aus (14. 2) (2) den

SATZ (14. 6) *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Γ besitzt dann und nur dann einen topologischen 2-Faktor, wenn für jede solche Zerlegung von Φ in drei Teilmengen A, B und C, bei welcher*

A aus unabhängigen Hauptpunkten besteht, und zu jedem C-Punkt höchstens eine AC-Kante inzident ist, die Ungleichung

$$\nu(A) \leqq \nu(B) + \sum_{i=1}^{m} \left[\frac{1}{2} \nu(A, C_i) \right] = \nu(B) + \frac{1}{2} (\nu(A, C) - \tau)$$

gültig ist. Hier bezeichnen $[C_i]$ $(i=1, \ldots, m)$ *die Komponenten von* $[C]$ *und* τ *die Anzahl der Komponenten* $[C_i]$ *mit ungeradem* $\nu(A, C_i)$.

Wir wollen aus (14. 6) zwei weitere Bedingungen der Existenz topologischer 2-Faktoren ableiten.

(14. 7) Es sei A eine beliebige Menge *unabhängiger* Punkte. Wir wollen jeden Bogen, dessen beide Randpunkte A-Punkte sind, die inneren Punkte jedoch nicht, *A-Bogen* nennen. Wir nennen das Bogensystem H ein *System unabhängiger A-Bogen,* wenn jeder Bogen von H (falls $H \neq \emptyset$ ist) ein A-Bogen ist und je zwei Bogen von H keinen gemeinsamen *inneren* Punkt enthalten. Ein Gewichtssystem r zweiter Art (s. (12. 5)) nennen wir ein *zu A gehöriges Gewichtssystem,* wenn in r alle A-Punkte, falls $A \neq \emptyset$, das Gewicht 0 erhalten und jeder A-Bogen, falls solche existieren, einen Punkt mit dem Gewicht 1 oder zwei Punkte mit halben Gewichten enthält. Es ist klar, daß zu A gehörige Systeme existieren. Wir wollen für ein jedes zu A gehöriges r einen *zu A gehörigen Wert* $S_A(r)$ definieren. Bezeichnet wieder B die Menge der Punkte, die in r das Gewicht 1 erhalten, und $\nu_{1/2}(E)$ die Anzahl der Punkte von E $(E \subseteq \Phi)$, die in r das Gewicht 1/2 bekommen, so sei

$$S_A(r) = \nu(B) + \sum_{i=1}^{m} \left[\frac{1}{2} \nu_{1/2}(C_i) \right],$$

wo $[C_i]$ $(i=1, \ldots, m)$ die Komponenten von $[C]$ mit $C = \Phi - (A \cup B)$ bedeuten.

Durch die selbe Schlußweise, die im Teile (II) des Beweises von (12. 6) benutzt wurde, kann man die Richtigkeit folgender Behauptung einsehen:

(14. 8) *Es sei A eine beliebige Menge unabhängiger Punkte von* Γ *und r ein zu A gehöriges Gewichtssystem.* Ist H ein System *unabhängiger A-Bogen, so gilt* $\nu(H) \leqq S_A(r)$.

Wir beweisen folgenden

SATZ (14. 9) *Es seien in* Γ *irgendwelche Punkte als Hauptpunkte ausgezeichnet.* Γ *enthält dann und nur dann einen topologischen 2-Faktor, wenn für jede beliebige Menge A unabhängiger Hauptpunkte und für jedes zu A gehöriges Gewichtssystem r die Ungleichung* $\nu(A) \leqq S_A(r)$ *besteht.*

BEWEIS. (I) Wir nehmen an, daß Γ einen topologischen 2-Faktor H besitzt. Es sei ferner A eine beliebige Menge unabhängiger Hauptpunkte und r ein beliebiges zu A gehöriges Gewichtssystem. Jene Kreise des zu H gehörigen

Kreissystems (*s.* (14. 5)), die A-Punkte enthalten, werden durch diese Punkte in genau $\nu(A)$ unabhängige A-Bogen zerlegt. Nach (14. 8) ist dann $\nu(A) \leqq S_A(r)$.

(II) Besitzt Γ keinen topologischen 2-Faktor, so kann man nach (14. 6) Φ so in die Teilmengen A, B und C zerlegen, daß A aus unabhängigen Hauptpunkten bestehe, zu jedem C-Punkt höchstens eine AC-Kante inzident sei und

$$(1) \qquad \nu(A) > \nu(B) + \sum_{i=1}^{m} \left[\frac{1}{2} \nu(A, C_i) \right]$$

gelte. ($[C_i]$ $(i = 1, \ldots, m)$ sind die Komponenten von $[C]$.) Aus (1) ergibt sich $A \neq \emptyset$. Wir definieren nun ein Gewichtssystem r wie folgt: Erhalte in r jeder B-Punkt das Gewicht 1, jeder C-Punkt, zu dem eine AC-Kante inzident ist, das Gewicht 1/2, und alle anderen Punkte das Gewicht 0. Dann ist es klar, daß r ein zu A gehöriges System ist und $S_A(r)$ der rechten Seite von (1) gleich ist. Damit haben wir unseren Satz bewiesen.

Es gilt der

SATZ (14. 10) *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Γ enthält dann und nur dann einen topologischen 2-Faktor, wenn zu jeder Menge A unabhängiger Hauptpunkte ein Bogensystem aus mindestens $\nu(A)$ unabhängigen A-Bogen existiert.*

BEWEIS. Die Notwendigkeit unserer Bedingung ist trivial. Nehmen wir nun an, daß kein topologischer 2-Faktor existiert. Dann gibt es nach (14. 9) eine Menge A unabhängiger Hauptpunkte und ein zu A gehöriges Gewichtssystem r mit $\nu(A) > S_A(r)$. Ist H ein beliebiges System unabhängiger A-Bogen, so besteht nach (14. 8) $\nu(H) \leqq S_A(r)$. Es gilt also $\nu(H) < \nu(A)$.

BEMERKUNG. Ist jeder Punkt in Γ Hauptpunkt, so geben die Sätze (14. 6), (14. 9) und (14. 10) Bedingungen der Existenz gewöhnlicher 2-Faktoren an. Die aus (14. 9) sich ergebende Bedingung ist einem Tutteschen Ergebnis über gerichtete Graphen ähnlich ([11], Satz (5. 1)).

§ 15. Topologische 1-Faktoren bei speziellen Graphen

In diesem Paragraphen wollen wir aus dem Satze (14. 4) für gewisse spezielle Graphen einige hinreichende Bedingungen der Existenz topologischer 1-Faktoren ableiten.

(15. 1) Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Wir wollen zu Γ eine Größe ξ' definieren, die eine Zusammenhangseigenschaft von Γ bezüglich der Hauptpunkte ausdrückt. Es bezeichne Λ die Menge derjenigen Teilmengen von Φ, die eine ungerade Anzahl von Haupt-

punkten enthalten. Gibt es in Φ mindestens einen Hauptpunkt, so ist Λ nichtleer, und in diesem Falle sei

$$\xi' = \min_{E \in \Lambda} \nu(E, \bar{E}) \qquad (\bar{E} = \Phi - E).$$

Enthält Φ keinen Hauptpunkt, so sei $\xi' = \infty$.

In den nachfolgenden Sätzen werden einige Bedingungen mehrmals vorkommen. Der Kürze halber wollen wir diese von den Sätzen getrennt formulieren.

BEDINGUNG (a). Der Grad eines jeden Hauptpunktes ist gleich μ ($\mu \geqq 1$), eines jeden Nebenpunktes kleiner oder gleich 2μ.

BEDINGUNG (b). Jede Komponente von Γ enthält eine gerade Anzahl von Hauptpunkten.

BEDINGUNG (c). Alle Nebenpunkte sind — falls solche existieren — geraden Grades.

Es gilt folgender Satz:

SATZ (15. 2) *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Bestehen dann die Bedingungen* (a), (b) *und $\xi' \geqq \mu$, so besitzt Γ einen topologischen* 1-*Faktor.*

BEWEIS. Es genüge Γ den angeführten Bedingungen und sei B eine beliebige Teilmenge von Φ. Wir wollen bezüglich B die Bezeichnungen unter (12. 2) anwenden. Ist B leer, so ist nach (b) $\tau(\bar{B}) = 0$, und so gilt (14. 4) (1). Es sei nun B nichtleer. Ist die Komponente $[\bar{B}_i]$ ungerade, so ist $\bar{B}_i \in \Lambda$, und demzufolge ist $\nu(\bar{B}_i, B) \geqq \xi' \geqq \mu$. Daraus folgt $\nu(\bar{B}, B) \geqq \mu\tau(\bar{B})$. (Dies ist auch im Falle $\xi' = \infty$ richtig!) Nach (a) ist aber $\nu(\bar{B}, B) \leqq \mu\nu_h(B) + 2\mu\nu_n(B)$ und wir können daher feststellen, daß B der Ungleichung (14. 4) (1) genügt. Nach (14. 4) enthält also Γ einen topologischen 1-Faktor.

BEMERKUNG. Im Falle $\mu = 1$ folgt das Bestehen von $\xi' \geqq \mu$ aus (b).

Genügt Γ neben (a) und (b) auch noch der Bedingung (c), so kann man statt $\xi' \geqq \mu$ eine schwächere Forderung stellen:

SATZ (15. 3) *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Bestehen dann die Bedingungen* (a), (b), (c) *und $\xi' \geqq \mu - 1$, so besitzt Γ einen topologischen* 1-*Faktor.*

BEWEIS. Ist jetzt $E \in \Lambda$, so ist ($\varrho(X)$ Bezeichnet den Grad von X)

$$\nu(E, \bar{E}) \equiv \sum_{X \in E} \varrho(X) - 2\nu(E, E) \equiv \mu\nu_h(E) \equiv \mu \pmod 2.$$

Ist also ξ' endlich, so gilt $\xi' \equiv \mu \pmod 2$, und dann folgt aus dem Bestehen von $\xi' \geqq \mu - 1$ auch $\xi' \geqq \mu$. Die letzte Behauptung ist auch im Falle $\xi' = \infty$ richtig. Dann folgt aber aus (15. 2) die Behauptung von (15. 3).

BEMERKUNG. Im Falle $\mu = 2$ folgt das Bestehen von $\xi' \geqq \mu - 1$ aus (b).

(15. 4) Für die Werte $\mu = 1, 2, 3$ und 4 wollen wir in den Sätzen (15. 2) und (15. 3) die Größe ξ' durch anschaulichere Begriffe ersetzen.

Vermehren sich die Komponenten des Graphen Γ durch Weglassen der Kante x bzw. der Kanten x_1 und x_2, so heißt x eine *Brücke* bzw. das Kantenpaar (x_1, x_2) eine *Doppelbrücke* (von Γ). Wir formulieren mit diesen Begriffen zwei weitere Bedingungen.

BEDINGUNG (d). Γ enthält keine Brücke.

BEDINGUNG (e). Γ enthält keine Doppelbrücke.

Die folgenden Behauptungen sind leicht ersichtlich:

Aus (b) und (d) folgt $\xi' \geqq 2$.

Aus (b) und (e) folgt $\xi' \geqq 3$, falls Γ mindestens zwei Kanten enthält.

Wir können demnach aus (15. 2) und (15. 3) zum folgenden Satz gelangen:

SATZ (15. 5). *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Das Bestehen der folgenden Bedingungen sichert dann die Existenz eines topologischen* 1*-Faktors von Γ:*

Im Falle $\mu = 1$ die Bedingungen (a), (b);
im Falle $\mu = 2$ die Bedingungen (a), (b), (d) *oder* (a), (b), (c);
im Falle $\mu = 3$ die Bedingungen (a), (b), (e) *oder* (a), (b), (c), (d);
im Falle $\mu = 4$ die Bedingungen (a), (b), (c), (e).

BEMERKUNGEN. (1) Es existieren Graphen, die der Bedingung (a) mit $\mu = 3$, sowie den Bedingungen (b) und (d) genügen, und die keinen topologischen 1-Faktor enthalten.

(2) Sind sämtliche Punkte Hauptpunkte, so geben (15. 2), (15. 3) und (15. 5) bekannte Sätze über 1-Faktoren an (s. [2], [3], [4], [9]).

§ 16. Topologische $\varkappa$-Faktoren bei speziellen Graphen

In diesem Paragraphen leiten wir aus (14. 2) einige hinreichende Bedingungen der Existenz topologischer $\varkappa$-Faktoren her (s. [2], S. 247 und [4], S. 144—146).

(16. 1) Es seien in den Punkten von Γ die Kapazitätsfunktionen $\varkappa(X)$ und $\varkappa'(X)$ vorläufig beliebig definiert.

Wir nehmen nun an, daß Γ *keinen* $(\varkappa, \varkappa')$*-Faktor besitzt* und wollen aus dieser Annahme, vorausgesetzt, daß Γ, $\varkappa$ und $\varkappa'$ gewisse Bedingungen erfüllen, eine Ungleichung (die Ungleichung (8)) ableiten. Dann werden wir solche Forderungen stellen, die dieser Ungleichung widersprechen. Diese Forderungen, zusammen mit den vorher erwähnten Bedingungen, werden dann die Existenz gewisser topologischer Faktoren sichern.

BEDINGUNG (a). Γ ist zusammenhängend.

BEDINGUNG (b) Die Zahl $\varkappa(\Phi)$ ist gerade.

Wir nehmen an, daß Γ den Bedingungen (a) und (b) genügt.

Da Γ keinen $(\varkappa, \varkappa')$-Faktor enthält, kann man nach (14. 2) die Menge Φ so in drei Teilmengen A, B und C zerlegen, daß

$$(1) \qquad \varkappa(A) > \varkappa(B) + 2\varkappa'(B) + 2\nu(A, A) + \nu(A, C) - \tau$$

besteht, wo τ die Anzahl der ungeraden Komponenten von $[C]$ bedeutet. (Die $[C_i]$ $(i = 1, \ldots, m)$ bezeichnen die Komponenten von $[C]$, und ein $[C_i]$ heißt ungerade, wenn $\varkappa(C_i) + \nu(A, C_i)$ ungerade ist.)

Es ist $A \cup B \neq \emptyset$. Ist nämlich $A = B = \emptyset$, so gilt $C = \Phi$, und so nach (a) und (b) auch $\tau = 0$, was zu (1) in Widerspruch steht. Aus $A \cup B \neq \emptyset$ und (a) folgt, daß zu jedem nichtleeren $[C_i]$ entweder eine AC_i-Kante oder eine BC_i-Kante existiert.

Wir nennen ein ungerades $[C_i]$ eine C_A- bzw. C_B-*Komponente*, wenn die Kanten, die den Teilgraphen $[C_i]$ berühren, alle AC_i- bzw. BC_i-Kanten sind. Es bezeichne τ_a bzw. τ_b die Anzahl der C_A- bzw. C_B-Komponenten, und J bedeute die Menge jener Indizes i, zu denen solche $[C_i]$ gehören, die C_A- oder C_B-Komponenten sind. Ist J nichtleer, so sei

$$\xi_\tau = \min_{i \in J} \nu(C_i, \bar{C}_i) \qquad (\bar{C}_i = \Phi - C_i).$$

Ist J leer, so sei $\xi_\tau = \nu(\Psi) + 3$. (Ψ bedeutet die Menge der Kanten von Γ.) In jedem Falle ist $\xi_\tau \geqq 1$ und es laufen aus jeder C_A- bzw. C_B-Komponente nach A bzw. B mindestens ξ_τ Kanten. Ferner gilt: Abgesehen von den C_B- bzw. C_A-Komponenten läuft aus jedem ungeraden $[C_i]$ mindestens eine Kante nach A bzw. nach B. Wir können nun folgende Ungleichungen feststellen:

$$(2) \qquad \nu(A, C) \geqq \tau_a \xi_\tau + \tau - \tau_a - \tau_b,$$

$$(3) \qquad \nu(B, C) \geqq \tau_b \xi_\tau + \tau - \tau_a - \tau_b.$$

Aus (1) und (2) folgt

$$(4) \qquad (\xi_\tau - 1)\tau_a - \tau_b + 2\varkappa'(B) < \varkappa(A) - \varkappa(B).$$

Für die Zahl $\nu(B, \bar{B})$ gilt offensichtlich ($\varrho(X)$ ist der Grad von X)

$$(5) \qquad \nu(B, \bar{B}) = \varrho(B) - 2\nu(B, B) \leqq \varrho(B). \qquad (\bar{B} = \Phi - B)$$

Wir wollen $\nu(B,\bar{B})$ auch von unten abschätzen. Es gilt

$$\nu(A,B)=\varrho(A)-2\nu(A,A)-\nu(A,C).$$

Es folgt daraus unter Beachtung von $\nu(B,\bar{B})=\nu(B,A)+\nu(B,C)$ sowie von (3) und (1)

$$\nu(B,\bar{B})>\varrho(A)-\varkappa(A)+\varkappa(B)+2\varkappa'(B)-\tau_a+(\xi_\tau-1)\tau_b.$$

Dies ergibt zusammen mit (5) die Ungleichung

(6) $$-\tau_a+(\xi_\tau-1)\tau_b+2\varkappa'(B)<\varrho(B)-\varrho(A)+\varkappa(A)-\varkappa(B).$$

Von hier an wollen wir uns auf topologische $\varkappa$-Faktoren beschränken, also nehmen an: $\varkappa$ und $\varkappa'$ sind komplementäre Funktionen (s. (12.1)). Ferner sollen Γ und $\varkappa(X)$ noch der folgenden Bedingung genügen (nach (12.1) können wir die Bezeichnungen Haupt- und Nebenpunkte benützen):

Bedingung (c). Es existiert eine Zahl λ mit $0<\lambda<1$, so daß für jeden Hauptpunkt $\varkappa(X)=\lambda\varrho(X)$ und für jeden Nebenpunkt $\varrho(X)\leqq 2/\lambda$ besteht.

Es folgt aus (c) (da $\varkappa$ und $\varkappa'$ komplementär sind) für jeden Punkt X von Γ die Ungleichung

$$\varkappa(X)\leqq\lambda\varrho(X)\leqq\varkappa(X)+2\varkappa'(X).$$

Daraus ergibt sich

(7) $$\varkappa(A)-\varkappa(B)\leqq\lambda(\varrho(A)-\varrho(B))+2\varkappa'(B).$$

(4) und (7) bzw. (6) und (7) ergeben die Ungleichungen

$$(\xi_\tau-1)\tau_a-\tau_b<\lambda(\varrho(A)-\varrho(B)),$$

$$-\tau_a+(\xi_\tau-1)\tau_b<(1-\lambda)(\varrho(B)-\varrho(A)).$$

Aus diesen erhalten wir endlich die gewünschte Ungleichung

(8) $$((1-\lambda)(\xi_\tau-1)-\lambda)\tau_a+(\lambda(\xi_\tau-1)-(1-\lambda))\tau_b<0.$$

Fordern wir nun, daß in (8) die Koeffizienten von τ_a und τ_b nichtnegativ seien, d. h. daß die beiden Ungleichungen

(9) $$\xi_\tau\geqq 1/\lambda \quad\text{und}\quad \xi_\tau\geqq 1/(1-\lambda)$$

bestehen, dann haben wir einen Widerspruch.

Wir definieren die Zahl ξ, die eine Zusammenhangseigenschaft von Γ charakterisiert, folgendermaßen: Ist $\nu(\Phi)>1$, so sei

$$\xi=\min_{\emptyset\subset E\subset\Phi}\nu(E,\bar{E})\qquad(\bar{E}=\Phi-E).$$

Ist $\nu(\Phi)=1$, so sei $\xi=0$.

Da in jedem Falle $\xi\leqq\nu(\Psi)$ ist, gilt offensichtlich $\xi_\tau\geqq\xi$.

Wir formulieren die

BEDINGUNG (d). Es gilt

$$\xi \geqq \max(1/\lambda, 1/(1-\lambda)).$$

Nach den obigen können wir dann folgenden Satz aussprechen:

SATZ (16.2) *Genügen Γ und $\varkappa$ den Bedingungen* (a), (b), (c) *und* (d), *so enthält Γ einen topologischen $\varkappa$-Faktor.*

BEMERKUNG. Sind sämtliche Punkte Hauptpunkte, so bekommt man aus (16.2) eine schwächere Form eines Oreschen Satzes über $\varkappa$-Faktoren ([7], Theorem 3.2.1).

Wir wollen den wichtigsten Spezialfall von (16.2) auch separiert formulieren:

SATZ (16.3) *Es seien im zusammenhängenden Graphen Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet und seien μ und $\sigma=\lambda\mu$ $(0<\lambda<1)$ natürliche Zahlen sowie $\sigma\nu_h(\Phi)$ eine gerade Zahl. Gilt für jeden Hauptpunkt $\varrho(X)=\mu$, für jeden Nebenpunkt $\varrho(X)\leqq 2\mu/\sigma$ und genügt Γ der Bedingung* (d), *so enthält Γ einen topologischen σ-Faktor.*

(16.4) Nehmen wir jetzt an, daß die in (16.1) gestellten Annahmen gelten, und daß $\varkappa(X)$ neben (a) und (c) noch der folgenden Bedingung genügt:

BEDINGUNG (e). $\varkappa(X)$ ist für jedes X gerade.

Es können dann C_B-Komponenten gar nicht existieren, also ist $\tau_b=0$. Ferner gilt für jede beliebige C_a-Komponente $[C_i]$

$$\nu(C_i, \bar{C}_i) \equiv \nu(C_i, A) \equiv \varkappa(C_i) + \nu(C_i, A) \equiv 1 \pmod 2.$$

Enthält Γ keine Brücke, so folgt daraus $\xi_\tau \geqq 3$, also steht die Forderung $1/(1-\lambda)\leqq 3$ zu (16.1) (8) in Widerspruch. Es gilt daher der

SATZ (16.5) *Genügen Γ und $\varkappa$ den Bedingungen* (a), (c), (e) *und enthält Γ keine Brücke, so besitzt Γ im Falle $\lambda\leqq 2/3$ einen topologischen $\varkappa$-Faktor.*

Wir wollen diesen Satz für topologische 2-Faktoren auch separiert formulieren.

SATZ (16.6) *Es seien im zusammenhängenden Graphen irgendwelche Punkte als Hauptpunkte ausgezeichnet und sei die natürliche Zahl $\mu\geqq 3$. Gilt für jeden Hauptpunkt $\varrho(X)=\mu$, für jeden Nebenpunkt $\varrho(X)\leqq\mu$, und enthält Γ keine Brücke, so besitzt Γ einen topologischen 2-Faktor, oder — was damit gleichwertig ist — Γ enthält ein solches Kreissystem, dessen Kreise (falls mehrere existieren) paarweise keinen gemeinsamen Punkt haben und zusammen alle Hauptpunkte enthalten.*

Sind sämtliche Punkte Hauptpunkte, so ergibt dieser Satz bekannte Sätze über 2-Faktoren (s. [1], [2], [4], [8]).

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SUR UNE GÉNÉRALISATION DU THÉORÈME DE POULAIN ET HERMITE POUR LES ZÉROS RÉELS DES POLYNÔMES RÉELS

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Soit

(1) $$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

un polynôme dont tous les zéros sont réels. D'après le théorème classique de POULAIN—HERMITE, si le polynôme

(2) $$g(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m$$

a seulement des zéros réels, le polynôme

(3) $$g(D) f(x) = b_0 f^{(m)}(x) + b_1 f^{(m-1)}(x) + \cdots + b_m f(x)$$

a aussi seulement des zéros réels. Chaque zéro multiple de (3) est aussi un zéro multiple de (1). Dans ce travail nous démontrons la généralisation suivante de ce théorème:

1. *Supposons que le polynôme* (1) *a seulement des zéros réels et que pour les arguments* φ *des zéros imaginaires du polynôme* (2), *dont tous les coefficients sont réels, on a l'inégalité*

(4) $$|\sin \varphi| \leqq \frac{1}{\sqrt{n}}.$$

Alors le polynôme (3) *a seulement des zéros réels. Si dans* (4) *on a le signe d'inégalité, chaque zéro multiple de* (3) *est aussi un zéro multiple de* (1).

La démonstration est basée sur la proposition nouvelle:

2. *Supposons que le polynôme* (1) *a seulement des zéros réels et soit* φ *un angle satisfaisant à la condition* (4). *Alors le polynôme*

(5) $$F(x) = f(x) - 2\varrho \cos \varphi f'(x) + \varrho^2 f''(x) \qquad (\varrho > 0)$$

a tous ses zéros réels et, si dans (4) *on a le signe d'inégalité, chaque zéro multiple de* (5) *est en même temps un zéro multiple de* (1).

Nous considérons d'abord le cas où (1) n'a pas des zéros multiples. D'après le théorème de POULAIN—HERMITE pour $\varphi = 0$ le polynôme (5) aura seulement des zéros réels et simples. Puisque les zéros de (5) sont des fonctions continues de φ pour $\varphi > 0$ assez petit, le polynôme (5) aura seulement des zéros réels et simples. Lorsque φ croît, aucun zéro réel de (5) ne peut devenir imaginaire sans se confondre avec un autre zéro. Désignons alors par φ_0 la plus petite valeur de φ, pour laquelle (5) a au moins un zéro multiple. Nous démontrerons que

$$\cos^2 \varphi_0 < 1 - \frac{1}{n}. \tag{6}$$

Désignons par λ un zéro multiple de (5) ($\varphi = \varphi_0$) et posons $\alpha = -\varrho \cos \varphi$. De

$$f(x) = c_0 + c_1(x-\lambda) + c_2(x-\lambda)^2 + \cdots + c_n(x-\lambda)^n$$

on obtient

$$F(x) = D_0 + D_1(x-\lambda) + D_2(x-\lambda)^2 + \cdots,$$

où

$$D_0 = c_0 + 2\alpha c_1 + 2\varrho^2 c_2, \quad D_1 = c_1 + 4\alpha c_2 + 6\varrho^2 c_3.$$

Comme λ est un zéro multiple de (5) il faut avoir

$$c_0 + 2\alpha c_1 + 2\varrho^2 c_2 = 0, \quad c_1 + 4\alpha c_2 + 6\varrho^2 c_3 = 0. \tag{7}$$

Soit $c_0 = 0$. Alors il suit que $c_1 \neq 0$, sinon (1) aura des zéros multiples. Pour $n = 2$ les équations (7) sont les suivantes: $\alpha c_1 + \varrho^2 c_2 = 0$, $c_1 + 4\alpha c_2 = 0$, d'où l'on obtient que $4\alpha^2 = \varrho^2$, c'est-à-dire $\cos^2 \varphi_0 = \frac{1}{4} < \frac{1}{2}$. Pour $n \geqq 3$, on obtient de (7)

$$\frac{\alpha^2}{\varrho^2} = \cos^2 \varphi_0 = \frac{c_2^2}{2(2c_2^2 - 3c_1c_3)}.$$

Nous allons démontrer que

$$\frac{c_2^2}{2(2c_2^2 - 3c_1c_3)} < 1 - \frac{1}{n}. \tag{8}$$

Le polynôme

$$c_1 + \binom{n-1}{1} \frac{c_2}{\binom{n-1}{1}} x + \binom{n-1}{2} \frac{c_3}{\binom{n-1}{2}} x^2 + \cdots$$

a tous ses zéros réels. D'après une inégalité connue d'Euler, on aura

$$\left(\frac{c_2}{n-1}\right)^2 \geqq \frac{c_1 c_3}{\binom{n-1}{2}}. \tag{9}$$

De cette inégalité, il suit d'abord que $c_2^2 > \frac{3}{2} c_1 c_3$. Donc dans (8) la division est admissible. Ensuite de (9) il découle facilement l'inégalité (8).

Considérons maintenant le cas où $c_0 \neq 0$. Des équations (7), on a

$$\frac{\alpha^2}{\varrho^2} = \cos^2 \varphi_0 = \frac{(3c_0c_3 - c_1c_2)^2}{2(c_1^2 - 2c_0c_2)(2c_2^2 - 3c_1c_3)}.$$

Il est bien connu que le nombre $c_1^2 - 2c_0c_2$ est positif. Le nombre $2c_2^2 - 3c_1c_3$ est aussi positif. En effet, pour $n = 2$ c'est évident. Pour $n \geqq 3$ l'inégalité $2c_2^2 - 3c_1c_3 > 0$ découle de l'inégalité d'Euler

$$2c_2^2 \geqq 3\frac{n-1}{n-2}c_1c_3$$

concernant le polynôme $c_0 + c_1x + c_2x^2 + \cdots$. Nous démontrerons l'inégalité générale

$$\text{(10)} \qquad \frac{(3c_0c_3 - c_1c_2)^2}{2(c_1^2 - 2c_0c_2)(2c_2^2 - 3c_1c_3)} < 1 - \frac{1}{n}.$$

Pour $n = 2$ $(c_3 = 0)$ cette inégalité devient

$$\frac{c_1^2}{2(c_1^2 - 2c_0c_2)} < 1,$$

ce qui est équivalent à l'inégalité connue $c_1^2 > 4c_0c_2$. Donc on peut supposer que $n \geqq 3$. D'après les conditions on a

$$\text{(11)} \qquad c_0 + c_1x + \cdots + c_nx^n = c_0(1 + x_1x)(1 + x_2x)\cdots(1 + x_nx),$$

où les nombres $x_1, x_2, \ldots, x_n$ sont réels. De (11) on obtient que

$$\frac{c_1}{c_0} = \Sigma x_1, \quad \frac{c_2}{c_0} = \Sigma x_1x_2, \quad \frac{c_3}{c_0} = \Sigma x_1x_2x_3,$$

d'où l'on a

$$\frac{1}{c_0^2}(c_1^2 - 2c_0c_2) = \Sigma x_1^2,$$

$$\frac{1}{c_0^2}(c_1c_2 - 3c_0c_3) = \Sigma x_1^2x_2,$$

$$\frac{1}{c_0^2}(2c_2^2 - 3c_1c_3) = 2(\Sigma x_1^2x_2^2 + 2\Sigma x_1^2x_2x_3 + 6\Sigma x_1x_2x_3x_4),$$

$$-3(\Sigma x_1^2x_2x_3 + 4\Sigma x_1x_2x_3x_4) = 2\Sigma x_1^2x_2^2 + \Sigma x_1^2x_2x_3.$$

Donc l'inégalité (10) prend la forme

$$\frac{(\Sigma x_1^2x_2)^2}{2\Sigma x_1^2(2\Sigma x_1^2x_2^2 + \Sigma x_1^2x_2x_3)} < 1 - \frac{1}{n},$$

où

$$\text{(12)} \qquad U = 2(n-1)\Sigma x_1^2(2\Sigma x_1^2x_2^2 + \Sigma x_1^2x_2x_3) - n(\Sigma x_1^2x_2)^2 > 0.$$

Nous démontrerons maintenant l'identité suivante:

$$U=\sum_{i<j}^{1\dots n} L_{ij}(x_i-x_j)^2, \tag{13}$$

où L_{ij} sont les nombres

$$L_{ij}=\left(\sum_{s=1}^{n} x_s^2-x_i^2-x_j^2\right)\sum_{s=1}^{n} x_s^2+n x_i^2 x_j^2.$$

En effet, pour la forme $(\Sigma x_1^2 x_2)^2$ nous avons

$$(\Sigma x_1^2x_2)^2=\Sigma x_1^4x_2^2+2\Sigma x_1^3x_2^3+2\Sigma x_1^3x_2^2x_3+4\Sigma x_1^2x_2^2x_3x_4+6\Sigma x_1^2x_2^2x_3^2+2\Sigma x_1^4x_2x_3.$$

Désignons l'expression $\sum_{s=1}^{n} x_s^2$ par V. Nous allons trouver dans (12) tous les termes qui contiennent le produit x_1x_2, dont les multiplicateurs sont du degré pair relativement les variables $x_1, x_2, \dots, x_n$. Il est évident que nous nous devrons borner à la partie

$$2(n-1)\Sigma x_1^2\Sigma x_1^2x_2x_3-n(\Sigma x_1^2x_2)^2$$

de la forme U. On voit facilement que le multiplicateur cherché de x_1x_2 dans U est égal à

$$2(n-1)V(V-x_1^2-x_2^2)-n\left[2x_1^2x_2^2+2(x_1^2+x_2^2)\sum_{p=3}^{n}x_p^2\right]-4n\sum_{i<j}^{3\dots n}x_i^2x_j^2-2n\sum_{p=3}^{n}x_p^2=$$

$$=2n(V-x_1^2-x_2^2)^2-2V(V-x_1^2-x_2^2)-4n\sum_{i<j}^{3\dots n}x_i^2x_j^2-2n\sum_{p=3}^{n}x_p^4-2nx_1^2x_2^2=$$

$$=-2V(V-x_1^2-x_2^2)-2nx_1^2x_2^2.$$

Donc en désignant par L_{ij} le coefficient de $-2x_1x_2$ on aura

$$L_{12}=V(V-x_1^2-x_2^2)+nx_1^2x_2^2.$$

En désignant alors par L_{ij}, $i\neq j$, l'expression

$$L_{ij}=V(V-x_i^2-x_j^2)+nx_i^2x_j^2,$$

il suit qu'on aura l'identité

$$-2\sum_{i<j}^{1\dots n}L_{ij}x_ix_j=2(n-1)\Sigma x_1^2\Sigma x_1^2x_2x_3-$$

$$-n(2\Sigma x_1^3x_2^3+2\Sigma x_1^3x_2^2x_3+4\Sigma x_1^2x_2^2x_3x_4+2\Sigma x_1^4x_2x_3).$$

L'identité (13) sera démontrée si nous démontrons l'identité suivante:

$$4(n-1)\Sigma x_1^2\Sigma x_1^2x_2^2-n(\Sigma x_1^3x_2^2+6\Sigma x_1^2x_2^2x_3^2)=\sum_{i<j}^{1\dots n}L_{ij}(x_i^2+x_j^2). \tag{14}$$

Puisque

$$\Sigma x_1^2 \Sigma x_1^2 x_2^2 = \Sigma x_1^4 x_2^2 + 3\Sigma x_1^2 x_2^2 x_3^2,$$

l'égalité (14) prend la forme

$$(15)\qquad (3n-4)\Sigma x_1^4 x_2^2 + 6(n-2)\Sigma x_1^2 x_2^2 x_3^2 = \sum_{i<j}^{1\ldots n} L_{ij}(x_i^2 + x_j^2).$$

Le terme $L_{12}(x_1^2+x_2^2)$ dans le second membre de (15) est égal à

$$V(V-x_1^2-x_2^2)(x_1^2+x_2^2)+nx_1^2x_2^2(x_1^2+x_2^2)-$$
$$-(x_1^2+x_2^2)^2(V-x_1^2-x_2^2)+(x_1^2+x_2^2)(V-x_1^2-x_2^2)^2+nx_1^2x_2^2(x_1^2+x_2^2)=A+B,$$

où

$$A = 2x_1^2x_2^2(x_3^2+\cdots+x_n^2)+2(x_1^2+x_2^2)\sum_{i<j}^{3\ldots n} x_i^2x_j^2,$$

$$B=(x_1^4+x_2^4)(x_3^2+\cdots+x_n^2)+(x_1^2+x_2^2)(x_3^4+\cdots+x_n^4)+n(x_1^4x_2^2+x_2^4x_1^2).$$

Il est évident que tous les termes semblables à A dans le second membre de (15) auront pour somme la fonction symétrique $S=\Sigma x_1^2x_2^2x_3^2$, multipliée par un nombre entier et positif K. Mais dans chaque expression A il y a $2(n-2)+4\binom{n-2}{2}=2(n-2)^2$ termes de la fonction S et puisque le nombre des expressions A est égal à $\binom{n}{2}$, le nombre K sera égal à

$$K=\frac{2(n-2)^2\binom{n}{2}}{\binom{n}{3}}=6(n-2).$$

Ainsi dans chaque expression de la forme B dans (15) il y a $2n-4+2n-4+2n=6n-8$ termes de la fonction symétrique $S_1=\Sigma x_1^4x_2^2$. Mais le nombre des termes de cette fonction est égal à $n(n-1)$ et dans (15) on a $\binom{n}{2}$ termes de la forme B. Donc la somme des termes de type B dans le second membre de (15) sera égale à L, où le nombre L est déterminé par

$$L=\frac{(6n-8)\binom{n}{2}}{n(n-1)}=3n-4.$$

Ainsi l'identité (15) est démontrée, de même que l'identité (13).

De (13) il découle que, quels que soient les nombres réels $x_1, x_2, \ldots, x_n$, on a toujours

$$(16)\qquad U\geqq 0.$$

On a dans (16) le signe d'égalité seulement dans le cas où tous les nombres $x_1, x_2, \ldots, x_n$ sont égaux.

Par là c'est démontré qu'on aura toujours l'inégalité $U>0$ à l'exception du cas où $x_1=x_2=\cdots=x_n$. Dans le dernier cas le polynôme (11) prend la forme

$$(17) \qquad c(x-x_1)^n$$

et le polynôme (5) devient

$$(18) \qquad c(x-x_1)^{n-2}[(x-x_1)^2-2\varrho n\cos\varphi(x-x_1)+n(n-1)\varrho^2].$$

Les zéros du polynôme (18) sont x_1 (de multiplicité $n-2$) et les nombres

$$n\varrho\left(\cos\varphi\pm\sqrt{\cos^2\varphi-\frac{n\cdot 1}{n}}\right).$$

Pour $\cos^2\varphi\geqq\frac{n-1}{n}$ tous les zéros de ce polynôme sont réels.

Supposons pour le moment que le polynôme (1) n'a pas la forme (17). De (1) nous obtenons un polynôme qui aura seulement des zéros réels et simples, en remplaçant chaque zéro z_k de multiplicité m par les m zéros simples $z_k, z_k(1+\varepsilon), z_k(1+2\varepsilon), \ldots, z_k(1+(m-1)\varepsilon)$, où $\varepsilon>0$ est suffisamment petit. Pour ε convenable on obtient ainsi un polynôme $f(x,\varepsilon)$, dont les zéros sont tous réels et simples. D'après les considérations ci-dessus il y aura un angle $\varphi_0'>0$, $\cos^2\varphi_0'<\frac{n-1}{n}$, tel que pour $0\leqq\varphi\leqq\varphi_0'$ l'équation $F(x,\varepsilon)=0$ pour $f(x,\varepsilon)$ aura seulement des racines réelles. Puisque le polynôme $f(x,\varepsilon)$ tend vers $f(x)$, le polynôme $F(x,\varepsilon)$ tendra vers $F(x)$, lorsque $\varepsilon\to 0$ et les zéros du polynôme $F(x)$ seront tous réels pour $0\leqq\varphi\leqq\varphi_0$, où φ_0 est déterminé par

$$\cos^2\varphi_0=\frac{n-1}{n}.$$

On voit aussi que pour $\varphi<\varphi_0$ chaque zéro multiple du polynôme (5) doit être un zéro multiple du polynôme (1).

On voit tout de suite que la conclusion est vraie pour le polynôme (17) aussi et le théorème est complètement démontré.

Le théorème **1** se démontre par une voie inductive. Soient

$$z_k=\varrho_k(\cos\varphi_k+i\sin\varphi_k), \quad \bar{z}_k=\varrho_k(\cos\varphi_k-i\sin\varphi_k) \qquad (k=1,2,\ldots,p)$$

les zéros imaginaires du polynôme (1). Pour les arguments on suppose donc que

$$|\sin\varphi_k|<\frac{1}{\sqrt{n}}.$$

Le polynôme (2) a alors la forme

$$g(x)=g_1(x)(x-z_1)(x-\bar{z}_1)\cdots(x-z_p)(x-\bar{z}_p),$$

où $g_1(x)=b_0(x-\gamma_1)(x-\gamma_2)\cdots(x-\gamma_q)$ et les zéros $\gamma_1, \gamma_2, \ldots, \gamma_q$ du polynôme $g_1(x)$ sont les zéros réels de $g(x)$. D'après le théorème de HERMITE—POULAIN les zéros du polynôme $h(x)=g_1(D)f(x)$ sont tous réels et chaque zéro multiple de ce polynôme est un zéro multiple de $f(x)$. D'après la proposition **2** les zéros du polynôme

$$h_1(x)=z_1\bar{z}_1h(x)-(z_1+\bar{z}_1)h'(x)+h''(x)$$

seront tous réels et chaque zéro multiple de $h_1(x)$ sera un zéro multiple de $f(x)$. On aura la même proposition pour les zéros du polynôme

$$h_2(x)=z_2\bar{z}_2h_1(x)-(z_2+\bar{z}_2)h_1'(x)+h_1''(x)$$

etc. En suivant ce raisonnement on parviendra au polynôme (3) et le théorème **1** est ainsi démontré. On ne peut pas améliorer l'inégalité (4).

Considérons maintenant quelques conséquences du théorème **1**. Supposons que pour les arguments φ des zéros imaginaires du polynôme

$$f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n, \tag{19}$$

dont les coefficients sont réels, on a l'inégalité

$$|\sin\varphi|\leqq\frac{1}{\sqrt{n}}. \tag{20}$$

Il est évident que pour les zéros du polynôme

$$g(x)=x^nf\left(\frac{1}{x}\right)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n$$

on aura la même inégalité (20). D'après le théorème **1** le polynôme

$$g(D)x^n=n!\,a_0+(n-1)!\,a_1x+n(n-1)\cdots3a_2x^2+\cdots+a_nx^n$$

a seulement des zéros réels. Donc on a la proposition:

3. *Si les arguments des zéros du polynôme* (19), *dont les coefficients sont réels, satisfont à l'inégalité* (20), *la polynôme*

$$a_0+\frac{a_1}{1!}x+\frac{a_2}{2!}x^2+\cdots+\frac{a_n}{n!}x^n$$

a seulement des zéros réels.

De la même manière on démontre la proposition:

4. *Supposons que les arguments des zéros imaginaires du polynôme réel* (19) *satisfont à l'inégalité*

$$|\sin\varphi|\leqq\frac{1}{\sqrt{p}},$$

où p est un nombre entier et positif. Alors, si $p \geqq n$, le polynôme

$$\frac{a_0}{(p-n)!}+\frac{a_1}{(p+1-n)!}x+\frac{a_2}{(p+2-n)!}x^2+\cdots+\frac{a_n}{p!}x^n$$

a tous ses zéros réels. Si $p < n$, le polynôme

$$a_{n-p}+\frac{a_{n-p+1}}{1!}x+\frac{a_{n-p+2}}{2!}x^2+\cdots+\frac{a_n}{p!}x^p$$

a tous ses zéros réels.

On doit à SCHUR [1] le théorème suivant:

Si les zéros du polynôme

(21) $$f(x)=a_0+a_1x+\cdots+a_mx^m \qquad (a_m \neq 0)$$

sont réels et les zéros du polynôme

(22) $$\varphi(x)=b_0+b_1x+\cdots+b_nx^n \qquad (b_n \neq 0)$$

sont réels et du même signe, les zéros du polynôme

(23) $$a_0b_0+1!\,a_1b_1x+2!\,a_2b_2x^2+\cdots+k!\,a_kb_kx^k \qquad (k=\min(m,n))$$

sont aussi réels.

MALO [2] a démontré plus tôt le théorème:

Si les zéros du polynôme (21) $(a_m \neq 0)$ sont réels et les zéros du polynôme (22) $(b_n \neq 0)$ sont réels et du même signe, les zéros du polynôme $a_0b_0+a_1b_1x+\cdots+a_kb_kx^k$ $(k=\min(m,n))$ sont tous réels.

En se basant sur le théorème **1**, nous démontrons la généralisation suivante du théorème de SCHUR:

5. *Supposons que les zéros du polynôme (21) $(a_m \neq 0)$ sont réels et que les arguments des zéros du polynôme réel (22) $(b_n \neq 0)$ satisfont tous à l'inégalité $-\alpha \leqq \varphi \leqq \alpha$, ou à l'inégalite $\pi-\alpha \leqq \varphi \leqq \pi+\alpha$, où $\alpha > 0$ est l'angle déterminé par $\sin\alpha=\frac{1}{\sqrt{m}}$. Alors le polynôme (23) à seulement des zéros réels.*

Considérons d'abord le cas où $a_0b_0 \neq 0$. On peut se borner au cas de l'inégalité $\pi-\alpha \leqq \varphi \leqq \pi+\alpha$. Alors les coefficients du polynôme (22) seront du même signe qu'on peut supposer positif. Nous suivrons la marche de la démonstration de SCHUR. Soit z un nombre réel et considérons d'abord le cas où $m \leqq n$. Le polynôme

(24) $$F(x)=b_0f(x)+b_1zf'(x)+\cdots+b_mz^mf^{(m)}(x)$$

a la forme

$$F(x)=P_0(z)+P_1(z)\frac{x}{1!}+P_2(z)\frac{x^2}{2!}+\cdots+P_m(z)\frac{x^m}{m!},$$

où

$$P_0(z) = a_0 b_0 + 1!\, a_1 b_1 z + 2!\, a_2 b_2 z^2 + \cdots + m!\, a_m b_m z^m$$

et

$$P_\mu(z) = \mu!\, a_\mu b_0 + (\mu+1)!\, a_{\mu+1} b_1 z + \cdots + m!\, a_m b_{m-\mu} z^{m-\mu} \qquad (1 \leqq \mu \leqq m).$$

Si pour les arguments des zéros du polynôme (22) on a $|\sin\varphi| < \frac{1}{\sqrt{m}}$, la même proposition sera valable pour les zéros du polynôme

$$b_0 + b_1 z x + b_2 z^2 x^2 + \cdots + b_n z^n x^n.$$

Donc d'après le théorème **1** le polynôme (24) aura seulement des zéros réels, quel que soit le nombre réel z. Les polynômes $P_0(z)$ et $P_1(z)$ ne peuvent pas avoir des zéros communs puisque dans le cas contraire $x = 0$ sera un zéro multiple de (24) et donc un zéro pareil du polynôme (21), ce qui est impossible à cause de $a_0 \neq 0$.

Des conséquences connues du théorème de Descartes il suit que les polynômes

$$P_0(z), P_1(z), P_2(z), \ldots, P_m(z) \tag{25}$$

forment une suite de Sturm. Pour $z = -\infty$ dans suite (25) il n'y a pas des variations et pour $z = \infty$ cette suite a m variations. Donc d'après le théorème généralisé de Sturm le polynôme $P_0(z)$ doit avoir seulement des zéros réels et le rapport $P_1(z)/P_0(z)$ doit passer de positif en négatif m fois en s'annulant lorsque z croît de $-\infty$ à ∞. Donc il suit encore que les zéros du polynôme $P_0(z)$ sont tous simples. Le cas $|\sin\varphi| \leqq \frac{1}{\sqrt{m}}$ est un cas limite de $|\sin\varphi| < \frac{1}{\sqrt{m}}$ et la réalité des zéros du polynôme est garantie par le théorème classique de Hurwitz.

Considérons maintenant le cas $m > n$. Au lieu de polynôme $\varphi(x)$ nous prendrons le polynôme

$$(1+\varepsilon x)^{m-n} \varphi(x) = b_0(\varepsilon) + b_1(\varepsilon) x + \cdots + b_m(\varepsilon) x^m,$$

où ε est un nombre positif. Le polynôme

$$a_0 b_0(\varepsilon) + 1!\, a_1 b_1(\varepsilon) x + \cdots + n!\, a_n b_n(\varepsilon) x^n + \cdots + m!\, a_m b_m(\varepsilon) x^m$$

aura tous ses zéros réels. En faisant ici ε tendre vers zéro, on obtient le polynôme

$$a_0 b_0 + 1!\, a_1 b_1 x + \cdots + n!\, a_n b_n x^n,$$

qui doit avoir, grâce au théorème de Hurwitz, seulement des zéros réels. Si $a_0 = b_0 = 0$, nous pouvons écrire

$$f(x) = x^p f_1(x), \quad \varphi(x) = x^q \varphi_1(x),$$

où $f_1(x)$ et $\varphi_1(x)$ sont des polynômes différents de zéro pour $x=0$. Comme ci-dessus, nous introduisons les polynômes

$$f_\delta(x)=(x+\delta)^p f_1(x), \quad \varphi_\delta(x)=(x+\delta)^q \varphi_1(x)$$

et le polynôme composé de ces deux polynômes tendra vers le polynôme composé (23).

6. (Généralisation du théorème de MALO.) *Supposons que les arguments des zéros imaginaires du polynôme réel* (21) *satisfont à l'inégalité* $|\sin\varphi| \leqq \dfrac{1}{\sqrt{m}}$ *et que les arguments des zéros imaginaires du polynôme réel* (22), *dont tous les coefficients sont du même signe, satisfont à la même inégalité. Alors le polynôme*

$$a_0b_0+a_1b_1x+a_2b_2x^2+\cdots+a_kb_kx^k \qquad (k=\min(m,n))$$

a seulement des zéros réels.

Pour la démonstration on applique les théorèmes **3** et **5**.

(Reçu le 1 mars 1960.)

Ouvrages cités

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[2] E. MALO, Note sur les équations algébriques dont toutes les racines sont réelles, *Journal de Math. spéciales* (4), **4** (1895), p. 7.

ON SUMS OF POWERS OF COMPLEX NUMBERS

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(*Presented by* P. TURÁN)

1. Let $z_1, \ldots, z_n$ be complex numbers such that

(1) $$1 = z_1 \geqq |z_2| \geqq \cdots \geqq |z_n|,$$

and write

(2) $$s_k = \sum_{m=1}^{n} z_m^k, \quad s = \max_{1 \leqq k \leqq n} |s_k|.$$

The problem has been posed by TURÁN[1] of finding a positive lower bound for s, valid for all choices of $z_1, \ldots, z_n$ subject to (1). The lower bound $(\log 2) \Big/ \sum_{1}^{n} \frac{1}{m}$, due to TURÁN, was improved by N. G. DE BRUIJN to $C \log \log n / \log n$, for some $C > 0$, and sufficiently large n. It was subsequently shown by S. UCHIYAMA[2] that C could be taken arbitrarily close to 1. The aim of the present note is to verify the conjecture that s has a positive lower bound independent of n. Without any suggestion that this is a precise value, we show that

(3) $$s > 1/6.$$

2. As in [2], we use the result that

(4) $$\exp \left\{ -\sum_{1}^{n} m^{-1} s_m y^m \right\} = \prod_{r=1}^{n} (1 - z_r y) + \sum_{m=n+1}^{\infty} c_m y^m,$$

for all y; here the c_m are functions of $z_1, \ldots, z_n$. In particular, writing

$$g(\theta) = -\sum_{1}^{n} m^{-1} s_m e^{mi\theta}$$

we have

(5) $$e^{g(\theta)} = \prod_{r=1}^{n} (1 - z_r e^{i\theta}) + \sum_{n+1}^{\infty} c_m e^{mi\theta},$$

[1] P. TURÁN, *Eine neue Methode in der Analysis und deren Anwendungen* (Budapest, 1953, and Peking, 1956). — P. TURÁN, Über die Potenzsummen komplexer Zahlen, *Archiv der Math.*, **9** (1958), pp. 59—64.

[2] S. UCHIYAMA, Sur les sommes de puissances des nombres complexes, *Acta Math. Acad. Sci. Hung.*, **9** (1958), pp. 275—278.

and the special case $\theta=0$ gives, since $z_1=1$,

$$e^{g(0)}=\sum_{n+1}^{\infty} c_m. \tag{6}$$

We now evaluate the c_m as Fourier coefficients of $e^{g(\theta)}$, according to (5). We get

$$c_m=(2\pi)^{-1}\int_{-\pi}^{\pi} e^{-mi\theta}e^{g(\theta)}d\theta.$$

Integration by parts gives

$$c_m=[-(2\pi mi)^{-1}e^{-mi\theta+g(\theta)}]_{-\pi}^{\pi}+(2\pi mi)^{-1}\int_{-\pi}^{\pi} g'(\theta)e^{-mi\theta+g(\theta)}d\theta,$$

and here the integrated term plainly vanishes. Substituting in (6) we get

$$e^{g(0)}=(2\pi i)^{-1}\sum_{n+1}^{\infty} m^{-1}\int_{-\pi}^{\pi} g'(\theta)e^{-mi\theta+g(\theta)}d\theta=(2\pi i)^{-1}\int_{-\pi}^{\pi} g'(\theta)e^{g(\theta)}h(\theta)d\theta \tag{7}$$

where

$$h(\theta)=\sum_{n+1}^{\infty} m^{-1}e^{-mi\theta}$$

or

$$1=(2\pi i)^{-1}\int_{-\pi}^{\pi} g'(\theta)e^{g(\theta)-g(0)}h(\theta)d\theta. \tag{8}$$

In deriving (7) from the preceding equation we have inverted the order of summation and integration. This may be justified by the theory of mean-square convergence. Alternatively, it would be possible to avoid this difficulty by replacing $h(\theta)$ by a sufficiently long partial sum of the series which defines it; this, however, would complicate the working.

3. Taking absolute values in (8) and using Schwarz's inequality, we have

$$1\leqq(2\pi)^{-2}\int_{-\pi}^{\pi}|g'(\theta)|^2d\theta\int_{-\pi}^{\pi}|e^{g(\theta)-g(0)}|^2|h(\theta)|^2d\theta. \tag{9}$$

The first integral on the right is readily estimated. Since

$$g'(\theta)=-i\sum_{1}^{n} s_m e^{mi\theta},$$

the Parseval equality shows that

$$\int_{-\pi}^{\pi}|g'(\theta)|^2d\theta=2\pi\sum_{1}^{n}|s_m|^2\leqq 2\pi n s^2. \tag{10}$$

For the second integral in (9), we consider first the interval $-\pi/n \leqq \theta \leqq \pi/n$. Here

$$|g(\theta)-g(0)| \leqq \sum_1^n m^{-1}|s_m|\,|e^{mi\theta}-1| \leqq \sum_1^n m^{-1}|s_m|\,|m\theta| \leqq |\theta n s| \leqq \pi s$$

in this interval. Hence

$$\int_{-\pi/n}^{\pi/n} |e^{g(\theta)-g(0)}|^2 |h(\theta)|^2 d\theta \leqq e^{2\pi s} \int_{-\pi/n}^{\pi/n} |h(\theta)|^2 d\theta < e^{2\pi s} \int_{-\pi}^{\pi} |h(\theta)|^2 d\theta = \tag{11}$$
$$= 2\pi e^{2\pi s} \sum_{n+1}^{\infty} m^{-2} < 2\pi e^{2\pi s} n^{-1}.$$

For $\pi/n \leqq \theta \leqq \pi$ we estimate $g(\theta)-g(0)$ as follows:

$$|g(\theta)-g(0)| \leqq \sum_{m \leqq \pi/\theta} m^{-1}|s_m|\,|m\theta| + \sum_{\pi/\theta < m \leqq n} m^{-1}|s_m|\,2 \leqq \pi s + 2s(\log(n\theta/\pi)+1).$$

Hence

$$|e^{g(\theta)-g(0)}| \leqq e^{s(\pi+2)}(n\theta/\pi)^{2s}.$$

We also have to estimate $h(\theta)$. We have

$$(1-e^{-i\theta})h(\theta) = (n+1)^{-1}e^{-(n+1)i\theta} - \sum_{r=2}^{\infty} e^{-(n+r)i\theta}\{(n+r-1)^{-1}-(n+r)^{-1}\}$$

whence

$$|(1-e^{-i\theta})h(\theta)| \leqq 2/(n+1).$$

Hence

$$|h(\theta)| \leqq (n+1)^{-1}\left|\operatorname{cosec}\frac{1}{2}\theta\right| < \pi/(n\theta)$$

if $\pi/n \leqq \theta \leqq \pi$.

Hence

$$\int_{\pi/n}^{\pi} |e^{g(\theta)-g(0)}|^2 |h(\theta)|^2 d\theta \leqq e^{2s(\pi+2)} \int_{\pi/n}^{\pi} (n\theta/\pi)^{4s-2} d\theta = (\pi/n) e^{2s(\pi+2)} \int_1^n \varphi^{4s-2} d\varphi <$$
$$< \pi e^{2s(\pi+2)} n^{-1} (1-4s)^{-1}, \tag{12}$$

assuming that $s<1/4$; this is enough since we aim to prove only that $s>1/6$. Since a similar bound holds for the integral over $(-\pi, -\pi/n)$, it follows from (11) and (12) that

$$\int_{-\pi}^{\pi} |e^{g(\theta)-g(0)}|^2 |h(\theta)|^2 d\theta < 2\pi n^{-1} e^{2\pi s}\{1+e^{4s}(1-4s)^{-1}\}.$$

Inserting this and (10) in (9) we get

$$1 < s^2 e^{2\pi s}\{1+e^{4s}(1-4s)^{-1}\}. \tag{13}$$

Since the expression on the right tends to zero with s, and is independent of n, we deduce that s has a positive lower bound which is independent of n. That (13) implies (3) follows from the fact that the inequality (13) is false for $s = 1/6$, together with the fact that the function on the right is monotonic increasing in $0 < s < 1/4$.

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ON GENERALIZATIONS OF AN INEQUALITY DUE TO PÓLYA AND SZEGŐ

By

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(*Presented by* P. TURÁN)

1. In a recent paper W. GREUB and W. RHEINBOLDT [1] prove the inequality

$$\sum_{k=1}^{\infty} a_k^2 \xi_k^2 \sum_{k=1}^{\infty} b_k^2 \xi_k^2 \leqq \frac{(M_1 M_2 + m_1 m_2)^2}{4 M_1 M_2 m_1 m_2} \left[\sum_{k=1}^{\infty} a_k b_k \xi_k^2\right]^2 \tag{1}$$

where $\xi_1^2 + \xi_2^2 + \cdots + \xi_k^2 + \cdots < \infty$ and

$$0 < m_1 \leqq a_k \leqq M_1, \quad 0 < m_2 \leqq b_k \leqq M_2. \tag{2}$$

They show that it is equivalent to the inequality

$$\sum_{k=1}^{\infty} \gamma_k \xi_k^2 \sum_{k=1}^{\infty} \frac{1}{\gamma_k} \xi_k^2 \leqq \frac{(M+m)^2}{4Mm} \left[\sum_{k=1}^{\infty} \xi_k^2\right]^2 \qquad (\Sigma \gamma_k < \infty, \ 0 < m \leqq \gamma_k \leqq M) \tag{2'}$$

which the authors attribute to L. V. KANTOROVICH[1] and is a generalization of an inequality due to PÓLYA and SZEGŐ [2]. Their proof is done *via* the theory of linear operators. However, owing to the elementary character of the inequality (1) it is of some interest to have a proof involving only the elements of analysis. (In part 3 of this paper it will be seen that the same argument suffices to prove that general inequality in Hilbert space from which GREUB and RHEINBOLDT derived (1).)

The proof given here will enable us to discuss the case of equality in (1). By putting

$$\alpha_k = (m_1 M_1)^{-\frac{1}{2}} a_k, \quad \beta_k = (m_2 M_2)^{-\frac{1}{2}} b_k, \quad a = (M_1/m_1)^{\frac{1}{2}}, \quad b = (M_2/m_2)^{\frac{1}{2}}$$

it is seen that the above statement is equivalent to the following assertion:

[1] The inequality (2') is contained in an integral inequality obtained by P. SCHWEITZER [3] who has shown that if $0 < m \leqq F(x) \leqq M$ in $p \leqq x \leqq q$, then

$$\int_p^q F(x)\,dx \int_p^q \frac{1}{F(x)}\,dx \leqq \frac{(M+m)^2}{4Mm}.$$

By taking $p = 0$, $q = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 + \cdots$, $F(x) = \gamma_1$ if $0 \leqq x < \xi_1^2$, $F(x) = \gamma_k$ if $\xi_1^2 + \xi_2^2 + \cdots + \xi_{k-1}^2 \leqq x < \xi_1^2 + \xi_2^2 + \cdots + \xi_k^2$ $(k = 2, 3, \ldots)$, we have (2'), provided that $F(x)$ and $[F(x)]^{-1}$ are Riemann integrable which is true if f. i. the γ_k's form an increasing sequence of numbers.

If $a^{-1} \leqq \alpha_k \leqq a$, $b^{-1} \leqq \beta_k \leqq b$ *and* $\xi_1^2 + \xi_2^2 + \cdots + \xi_k^2 \cdots < \infty$, *then*

$$[(ab)+(ab)^{-1}]^2 \left(\sum_{k=1}^{\infty} \alpha_k \beta_k \xi_k^2 \right)^2 - 4 \sum_{k=1}^{\infty} \alpha_k^2 \xi_k^2 \sum_{k=1}^{\infty} \beta_k^2 \xi_k^2 \geqq 0. \tag{3}$$

Put now

$$l_{1k}(x) = \alpha_k x - ab\beta_k \quad \text{and} \quad l_{2k}(x) = ab\alpha_k x - \beta_k.$$

Then

$$l_{1k}(1) = \alpha_k - ab\beta_k \leqq a - abb^{-1} = 0 \tag{4$_1$}$$

and

$$l_{2k}(1) = ab\alpha_k - \beta_k \geqq aba^{-1} - b = 0 \tag{4$_2$}$$

with signs of equality only if

(5₁) $\alpha_k = a, \ \beta_k = b^{-1}$ and (5₂) $\alpha_k = a^{-1}, \beta_k = b,$

respectively.

Further if

$$f(x) = \sum_{k=1}^{\infty} \frac{\xi_k^2}{ab} l_{1k}(x) l_{2k}(x) = \sum_{k=1}^{\infty} \alpha_k^2 \xi_k^2 x^2 - [ab + (ab)^{-1}] \sum_{k=1}^{\infty} \alpha_k \beta_k \xi_k^2 x + \sum_{k=1}^{\infty} \beta_k^2 \xi_k^2,$$

then $f(x) \to +\infty$ as $x \to \infty$ and $f(1) \leqq 0$. Hence the quadratic equation $f(x) = 0$ has at least one real root, and therefore its discriminant, the left-hand side of (3) is non-negative.

2. For finding the conditions of equality in (3) we define two sets of natural numbers K_1 and K_2 as follows: if $k \in K_i$, then (5_i) holds ($i = 1, 2$). The left-hand side of (3) vanishes if and only if $f(1)$ and $f'(1)$ vanish simultaneously. $f(1) = 0$ if and only if $K_1 \cup K_2$ is the set K of natural numbers. In this case an easy computation shows that

$$f'(1) = \frac{a^2b^2 - 1}{a^2b^2} \left(a^2 \sum_{k \in K_1} \xi_k^2 - b^2 \sum_{k \in K_2} \xi_k^2 \right)$$

and $f'(1)$ (consequently the left of (3)) vanishes if either $a^2b^2 = 1$ (or what amounts to the same $a = b = 1$) or

$$a^2 \sum_{k \in K_1} \xi_k^2 = b^2 \sum_{k \in K_2} \xi_k^2 \qquad (K_1 \cup K_2 = K).$$

3. GREUB and RHEINBOLDT [1] found a generalization of inequality (1) in Hilbert space which they call the *generalized Pólya—Szegő inequality:*

Given two permutable, linear and self-adjoint operators A *and* B *of the Hilbert space* H *which fulfil the conditions*[2]

$$0 < m_1 I \leqq A \leqq M_1 I, \quad 0 < m_2 I \leqq B \leqq M_2 I,$$

[2] I denotes the identity operator in H. $C \leqq D$ means that $(Cx, x) \leqq (Dx, x)$ for all $x \in H$. C is positive if $(Cx, x) \geqq 0$ for all $x \in H$.

then

$$(6)\qquad (Ax, Ax)(Bx, Bx) \leqq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2} (Ax, Bx)^2$$

for all $x \in H$.

The device used in showing (1) can be applied to yield a short proof of this inequality, the main result of the paper of GREUB and RHEINBOLDT. We will use the fact that the product of two permutable positive bounded linear self-adjoint operators is positive, too.

There is no loss of generality in taking $M_1 = a$, $m_1 = a^{-1}$, $M_2 = b$, $m_2 = b^{-1}$, so that it is enough to show that in this case

$$(7)\qquad [ab + (ab)^{-1}]^2 (Ax, Bx)^2 - 4(Ax, Ax)(Bx, Bx) \geqq 0.$$

PROOF. Consider the quadratic form

$$f(\lambda) = -(Ax, Ax)\lambda^2 + [ab + (ab)^{-1}](Ax, Bx)\lambda - (Bx, Bx)$$

for fixed x. We have $f(1) = (Mx, Nx)$ with $M = abB - A$ and $N = A - (ab)^{-1}B$. Both M and N are positive and bounded, for

$$(Mx, x) = ab(Bx, x) - (Ax, x) \geqq (abb^{-1} - a)(x, x) = 0$$

and

$$(Nx, x) = (Ax, x) - (ab)^{-1}(Bx, x) \geqq (a^{-1} - (ab)^{-1} b)(x, x) = 0,$$

and so the operator NM is positive. Hence $f(1) = (NMx, x) \geqq 0$.

On the other hand, $f(\lambda) \to -\infty$ if $\lambda \to \infty$ as (Ax, Ax) is positive. Hence the quadratic equation $f(\lambda) = 0$ has at least one real root and (7) holds.

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SOME INTERPOLATORY PROPERTIES OF HERMITE POLYNOMIALS

By

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(*Presented by* P. TURÁN)

1. Professor TURÁN and his colleagues in a series of papers on interpolation have discussed i) the problem of existence and uniqueness, ii) the problem of explicit representation and iii) the problem of convergence for (0, 2)-interpolation by taking as abscissae the zeros of $\Pi_n(x) = (1-x^2)P'_{n-1}(x)$ where $P_{n-1}(x)$ is the Legendre polynomial of degree $\leqq n-1$. By (0, 2)-interpolation we mean the construction of a polynomial of degree $\leqq 2n-1$, when the value of the function and its second derivative at the zeros of $\Pi_n(x)$ are prescribed.

Later on SAXENA and SHARMA [3] have studied the aforesaid problems for (0, 1, 3)-interpolation taking the same abscissae as those used by P. TURÁN. Later SAXENA [4] has extended the results to (0, 1, 2, 4)-interpolation.

The object of this note is to consider the problem of existence and uniqueness and of explicit representation for (0, 2)- and (0, 1, 3)-interpolation, respectively, choosing the abscissae as the zeros of $H_n(x)$, the Hermite polynomial of degree n, which are given by

$$\infty > x_{1,n} > x_{2,n} > \cdots > x_{n,n} > -\infty.$$

2. We shall prove the following theorems:

THEOREM I. (Case (0, 2).) *If $n = 2k$, then to prescribed values y_ν and y_ν^* there is a uniquely determined polynomial $g(x)$ of degree $\leqq 2n-1$ such that*

$$(2.1)\qquad g(x_{\nu,n}) = y_\nu \quad \textit{and} \quad g''(x_{\nu,n}) = y_\nu^* \qquad (\nu = 1, 2, \ldots, n)$$

if $x_{\nu,n}$'s stand for the zeros of $H_n(x)$.

This means, of course, that in case

$$(2.2)\qquad y_\nu = y_\nu^* = 0 \qquad (\nu = 1, 2, \ldots, n;\ n \text{ even})$$

the only solution of (2.1) is $g(x) \equiv 0$.

THEOREM II. (Case (0, 1, 3).) *If $n = 2k$, then to prescribed values $y_\nu, y_\nu^*, y_\nu^{**}$ there is a uniquely determined polynomial $f(x)$ of degree $\leqq 3n-1$ such that*

$$(2.3)\qquad f(x_{\nu,n}) = y_\nu, \quad f'(x_{\nu,n}) = y_\nu^* \quad \textit{and} \quad f'''(x_{\nu,n}) = y_\nu^{**} \qquad (\nu = 1, 2, \ldots, n)$$

(if $x_{\nu,n}$'s stand for the zeros of $H_n(x)$.

This means, of course, that in case

(2. 4) $$y_\nu = y_\nu^* = y_\nu^{**} = 0 \qquad (\nu = 1, 2, \ldots, n;\ n \text{ even})$$

the only solution of (2. 3) is $f(x) \equiv 0$.

3. Preliminaries. In this section we shall give certain well-known formulae which we shall use later on.

(3. 1) $$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

is the well-known differential equation satisfied by $H_n(x)$. At $x = x_j$,

(3. 2) $$H_n''(x_j) = 2x_j H_n'(x_j) \qquad (j = 1, 2, \ldots, n).$$

We shall denote by $l_\nu(x)$ the fundamental polynomials of the Lagrange interpolation based on the x_ν's, i. e.

(3. 3) $$l_\nu(x) = \frac{H_n(x)}{(x - x_\nu)H_n'(x_\nu)}.$$

From this it is easy to see that

(3. 4) $$l_\nu(x_\nu) = 1, \quad l_\nu(x_j) = 0,$$

(3. 5) $$l_\nu'(x_\nu) = x_\nu, \quad l_\nu'(x_j) = \frac{H_n'(x_j)}{H_n'(x_\nu)(x_j - x_\nu)},$$

(3. 6) $$\begin{cases} l_\nu''(x_\nu) = \dfrac{4x_\nu^2 + 2(1-n)}{3}, \\ l_\nu''(x_j) = \dfrac{2H_n'(x_j)}{(x_j - x_\nu)H_n'(x_\nu)} \left\{ x_j - \dfrac{1}{x_j - x_\nu} \right\}. \end{cases}$$

Besides this we shall also make use of the fact that

(3. 7) $$\sum_{r=0}^{n} \frac{H_r(x)H_r(y)}{2^r r!} = \frac{1}{2^{n+1} n!} \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{x - y}.$$

Taking $y = x_\nu$ and replacing $n+1$ by n in (3. 7), we get

(3. 8) $$\sum_{r=0}^{n-1} \frac{H_r(x)H_r(x_\nu)}{2^r r!} = \frac{1}{2^n (n-1)!} \frac{H_n(x)H_{n-1}(x_\nu)}{(x - x_\nu)}.$$

We also require the following well-known properties:

(3. 9) $$H_n'(x) = 2nH_{n-1}(x),$$

(3. 10) $$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x) \qquad (n = 2, 3, 4, \ldots),$$

(3. 11) $$H_{2m}(0) = \frac{(-1)^m (2m)!}{m!}, \qquad H_{2m+1}'(0) = (-1)^m \frac{(2m+2)!}{(m+1)!}.$$

Also $H_0(x) = 1$, and $H_1(x) = 2x$.

4. Proof of Theorem I. Since $g(x)$ is a polynomial of degree $\leqq 2n-1$, we have

$$g(x) = H_n(x) r_{n-1}(x) \tag{4.1}$$

where $r_{n-1}(x)$ is a polynomial of degree $\leqq n-1$, so that the first part of condition (2.1) is obviously satisfied and from the second we have

$$g''(x_j) = H_n''(x_j) r_{n-1}(x_j) + 2H_n'(x_j) r_{n-1}'(x_j) = 0.$$

Since x_j's are the simple zeros of $H_n(x)$, we get

$$r_{n-1}'(x_j) + x_j r_{n-1}(x_j) = 0,$$

with the help of (3.2). Hence

$$r_{n-1}'(x) + x r_{n-1}(x) = c H_n(x) \tag{4.2}$$

where c is a constant.

Let the solution of this differential equation be

$$r_{n-1}(x) = \sum_{\nu=0}^{n-1} c_\nu H_\nu(x).$$

Substituting this in (4.2), we get with the help of (3.9) and (3.10),

$$3\sum_{\nu=0}^{n-2} (\nu+1) c_{\nu+1} H_\nu(x) + \frac{1}{2} \sum_{\nu=1}^{n} c_{\nu-1} H_\nu(x) = c H_n(x). \tag{4.3}$$

Now equating the coefficients of $H_\nu(x)$ on both sides, we get

$$3c_1 = 0,$$

$$3(\nu+1)c_{\nu+1} + \frac{1}{2} c_{\nu-1} = 0 \qquad (\nu = 1, 2, \ldots, n-2),$$

$$\frac{1}{2} c_{n-2} = 0 \quad \text{and} \quad \frac{1}{2} c_{n-1} = c.$$

Since $c_1 = 0$ and n is even, we have

$$c_1 = c_3 = c_5 = \cdots = c_{n-1} = 0.$$

Also

$$c_{n-2} = c_{n-4} = \cdots = c_2 = c_0 = 0.$$

Hence all the c_ν's are zero. So the solution is $g(x) \equiv 0$.

When n is odd, we get from (4.3)

$$3c_1 = 0,$$

$$3(\nu+1)c_{\nu+1} + \frac{1}{2} c_{\nu-1} = 0 \quad \text{for} \quad \nu = 1, 2, \ldots, n-2.$$

Therefore

$$c_1 = c_3 = c_5 = \cdots = c_{n-2} = 0$$

and $c_0, c_2, c_4, \ldots, c_{n-1}$ can be determined and are non-zero. Therefore, when n is odd, there are an infinity of solutions if they exist.

Theorem II can be proved on the same lines.

5. Problem of explicit representation. (Case (0, 2).) Given distinct points

$$\infty > x_1 > x_2 \cdots > x_n > -\infty$$

and arbitrary numbers

$$\alpha_1, \alpha_2, \ldots, \alpha_n;\ \beta_1, \beta_2, \ldots, \beta_n;$$

it is to be decided whether or not there is a polynomial $R_n(x)$ of degree $\leqq 2n-1$ such that

$$R_n(x_\nu) = \alpha_\nu, \qquad R_n''(x_\nu) = \beta_\nu \qquad (\nu = 1, 2, \ldots, n).$$

Throughout this paper we shall take n to be even. So for $n = 2k$ we have

$$R_{2k}(x) = \sum_{\nu=1}^{2k} \alpha_\nu r_\nu(x) + \sum_{\nu=1}^{2k} \beta_\nu \varrho_\nu(x) \tag{5.1}$$

where $r_\nu(x)$ and $\varrho_\nu(x)$ are the fundamental polynomials of the first and second kind of (0, 2)-interpolation, of degree $\leqq 2n-1 = 4k-1$, uniquely determined by the following conditions:

$$r_\nu(x_j) = \begin{cases} 1 & \text{for } j = \nu \\ 0 & \text{for } j \neq \nu \end{cases} \qquad (j = 1, 2, \ldots, n), \tag{5.2}$$

$$r_\nu''(x_j) = 0 \qquad (j = 1, 2, \ldots, n) \tag{5.3}$$

and

$$\varrho_\nu(x_j) = 0 \qquad (j = 1, 2, \ldots, n), \tag{5.4}$$

$$\varrho_\nu''(x_j) = \begin{cases} 1 & \text{for } j = \nu \\ 0 & \text{for } j \neq \nu \end{cases} \qquad (j = 1, 2, \ldots, n). \tag{5.5}$$

We then have the following

THEOREM III. *For the fundamental polynomials $r_\nu(x)$ and $\varrho_\nu(x)$ the following explicit forms hold true:*

$$\begin{aligned} r_\nu(x) = l_\nu^2(x) + \frac{e^{-\frac{x^2}{2}} H_n(x)}{H_n'(x_\nu)} \Bigg[A \int_0^x e^{\frac{t^2}{2}} H_n(t)\,dt + \\ + B \int_0^x \frac{e^{\frac{t^2}{2}} H_n(t)}{t - x_\nu}\,dt - \frac{1}{H_n'(x_\nu)} \int_0^x \frac{H_n'(t) - (Ct + D) l_\nu(t)}{(t - x_\nu)^2} e^{\frac{t^2}{2}}\,dt + K \Bigg] \end{aligned} \tag{5.6}$$

$$(\nu = 1, 2, \ldots, n),$$

where A, B, C, D and K are constants given by

$$A=-\frac{\left(-\frac{4}{3}\right)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{2H_n'(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{\left(-\frac{3}{4}\right)^r(n-2r-1-2x_\nu^2)}{r!}H_{2r}(x_\nu), \tag{5.7}$$

$$B=-\frac{2x_\nu^2}{H_n'(x_\nu)}, \tag{5.8}$$

$$C=x_\nu H_n'(x_\nu), \tag{5.9}$$

$$D=(1-x_\nu^2)H_n'(x_\nu), \tag{5.10}$$

$$K=\frac{2^n(n-1)!}{H_n'(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{(-1)^r(n-2r-2-2x_\nu^2)}{2^{2r+1}\Gamma\left(r+\frac{3}{2}\right)}H_{2r+1}(x_\nu)\sum_{i=1}^{r+1}3^{i-1}\frac{\Gamma\left(r+\frac{3}{2}-i\right)}{\Gamma(r+2-i)}, \tag{5.11}$$

$$\varrho_\nu(x)=\frac{e^{-\frac{x^2}{2}}H_n(x)}{2H_n'(x_\nu)}\left[a\int_0^x e^{\frac{t^2}{2}}H_n(t)dt+\frac{1}{H_n'(x_\nu)}\int_0^x\frac{e^{\frac{t^2}{2}}H_n(t)}{t-x_\nu}dt+b\right], \tag{5.12}$$

where a and b are constants given by

$$a=-\frac{\left(-\frac{4}{3}\right)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{2H_n'(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\left(-\frac{3}{4}\right)^r\frac{H_{2r}(x_\nu)}{r!} \tag{5.13}$$

and

$$b=\frac{2^n(n-1)!}{H_n'(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{(-1)^rH_{2r+1}(x_\nu)}{2^{2r+1}\Gamma\left(r+\frac{3}{2}\right)}\sum_{i=1}^{r+1}3^{i-1}\frac{\Gamma\left(r+\frac{3}{2}-i\right)}{\Gamma(r+2-i)}. \tag{5.14}$$

Other equivalent forms for (5. 13), (5. 14), (5. 7) and (5. 11) are the following:

$$a=-\frac{\left(-\frac{1}{12}\right)^n}{\sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right)H_n'(x_\nu)}\int_{-\infty}^{\infty}\frac{H_n(it)}{it-x_\nu}e^{-\frac{t^2}{2}}dt, \tag{5.13a}$$

$$b=-\frac{i}{\sqrt{2\pi}H_n'(x_\nu)}\int_0^1\frac{du}{\sqrt{1-u^2}}\int_{-\infty}^{\infty}te^{-\frac{t^2}{2}}\frac{H_n(iut)}{iut-x_\nu}dt, \tag{5.14a}$$

(5. 7a)
$$A=\frac{\left(-\frac{1}{12}\right)^{\frac{n}{2}}}{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)H_n'(x_\nu)}\left[2x_\nu^2\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}\frac{H_n(it)}{it-x_\nu}dt+\right.$$
$$\left.+\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}\frac{H_n'(it)-(itx_\nu+1-x_\nu^2)l_\nu(it)H_n'(ix_\nu)}{(it-x_\nu)^2}dt\right],$$

(5. 11a)
$$K=\frac{i}{\sqrt{2\pi}H_n'(x_\nu)}\int_0^1\frac{du}{\sqrt{1-u^2}}\left[\int_{-\infty}^{\infty}te^{-\frac{t^2}{2}}\left\{2x_\nu^2\frac{H_n(iut)}{iut-x_\nu}+\right.\right.$$
$$\left.\left.+\frac{H_n'(iut)-(iutx_\nu+1-x_\nu^2)l_\nu(iut)H_n'(iux_\nu)}{(iut-x_\nu)^2}\right\}dt\right].$$

6. Proof of Theorem III. For the determination of $\varrho_\nu(x)$ we need the following

LEMMA 6. 1. *We have*

(6. 1)
$$\int_0^x e^{\frac{t^2}{2}}H_{2k}(t)\,dt=2\sum_{r=1}^{k}(-12)^{r-1}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+\frac{3}{2}-r\right)}e^{\frac{x^2}{2}}H_{2k+1-2r}(x)+$$
$$+(-12)^k\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}}\int_0^x e^{\frac{t^2}{2}}dt$$

and

$$(6.\,2)\quad \int_0^x e^{\frac{t^2}{2}}H_{2k+1}(t)dt=2\sum_{r=1}^{k+1}(-12)^{r-1}\frac{k!}{(k+1-r)!}\left\{e^{\frac{x^2}{2}}H_{2k+2-2r}(x)-H_{2k+2-2r}(0)\right\}.$$

PROOF. The proof follows at once from the formulae

$$\int_0^x e^{\frac{t^2}{2}}H_{2k}(t)dt=2e^{\frac{x^2}{2}}H_{2k-1}(x)-6(2k-1)\int_0^x e^{\frac{t^2}{2}}H_{2k-2}(t)dt$$

and

$$\int_0^x e^{\frac{t^2}{2}}H_{2k+1}(t)dt=2\left\{e^{\frac{x^2}{2}}H_{2k}(x)-H_{2k}(0)\right\}-6\cdot 2k\int_0^x e^{\frac{t^2}{2}}H_{2k-1}(t)dt.$$

LEMMA 6. 2. *When n is even,*

$$\int_0^x e^{\frac{t^2}{2}} \frac{H_n(t)}{t-x_\nu}\, dt = \frac{2^{n+1}(n-1)!}{H_{n-1}(x_\nu)} \left[\sum_{r=0}^{\frac{n}{2}-1} \frac{H_{2r}(x_\nu)}{2^{2r}(2r)!} \cdot \right.$$

$$(6.3)\quad \cdot \left\{ \sum_{i=1}^{r} (-12)^{i-1} \frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(r+\frac{3}{2}-i\right)} e^{\frac{x^2}{2}} H_{2r+1-2i}(x) + \frac{(-12)^r \Gamma\left(r+\frac{1}{2}\right)}{2\sqrt{\pi}} \int_0^x e^{\frac{t^2}{2}} dt \right\} +$$

$$\left. + \sum_{r=0}^{\frac{n}{2}-1} \frac{H_{2r+1}(x_\nu) r!}{2^{2r+1}(2r+1)!} \sum_{i=1}^{r+1} \frac{(-12)^{i-1}}{(r+1-i)!} \left\{ e^{\frac{x^2}{2}} H_{2r+2-2i}(x) - H_{2r+2-2i}(0) \right\} \right].$$

PROOF. From (3. 8) we have

$$\int_0^x e^{\frac{t^2}{2}} \frac{H_n(t)}{t-x_\nu}\, dt = \frac{2^n(n-1)!}{H_{n-1}(x_\nu)} \sum_{r=0}^{n-1} \frac{H_r(x_\nu)}{2^r r!} \int_0^x e^{\frac{t^2}{2}} H_r(t)\, dt.$$

Now breaking the right-hand sum according to even and odd values of r and applying Lemma 6. 1, we get the required result.

Determination of $\varrho_\nu(x)$ when n is even. Let

$$(6.4)\qquad \mu_\nu(x) = \frac{H_n(x) e^{-\frac{x^2}{2}}}{2H_n'(x_\nu)} \left[a \int_0^x e^{\frac{t^2}{2}} H_n(t)\, dt + \frac{1}{H_n'(x_\nu)} \int_0^x e^{\frac{t^2}{2}} \frac{H_n(t)}{t-x_\nu}\, dt + b \right].$$

It is easy to see with the help of (3. 2) that conditions (5. 4) and (5. 5) are satisfied for all values of a and b. So

$$\mu_\nu(x) \equiv \varrho_\nu(x).$$

For the determination of a and b we shall make use of the fact that the right-hand side of (6. 4) is a polynomial of degree $\leqq 2n-1$, so that the coefficient of

$$\int_0^x e^{\frac{t^2}{2}} dt \quad \text{and} \quad e^{-\frac{x^2}{2}}$$

must vanish separately. From these we get a and b as given in (5. 13) and (5. 14), respectively.

Hence $\varrho_\nu(x)$ is uniquely determined.

7. For the determination of $r_\nu(x)$ we require the following

LEMMA 7. 1. *We have*

$$-\int_0^x e^{\frac{1}{2}t^2}\,\frac{H_n'(t)-(tx_\nu+1-x_\nu^2)l_\nu(t)H_n'(x_\nu)}{(t-x_\nu)^2}\,dt=$$

$$=\frac{2^{n+1}(n-1)!}{H_{n-1}(x_\nu)}\left[\sum_{r=0}^{\frac{n}{2}-1}\frac{(n-2r-1)}{2^{2r}(2r)!}H_{2r}(x_\nu)\left\{\sum_{i=1}^{r}(-12)^{i-1}\cdot\right.\right.$$

(7. 1)
$$\left.\cdot\frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(r+\frac{3}{2}-i\right)}e^{\frac{x^2}{2}}H_{2r+1-2i}(x)+(-12)^r\frac{\Gamma\left(r+\frac{1}{2}\right)}{2\sqrt{\pi}}\int_0^x e^{\frac{t^2}{2}}dt\right\}+$$

$$\left.+\sum_{r=0}^{\frac{n}{2}-1}\frac{(n-2r-2)H_{2r+1}(x_\nu)}{2^{2r+1}(2r+1)!}\sum_{i=1}^{r+1}(-12)^{i-1}\frac{r!}{(r+1-i)!}\left\{e^{\frac{x^2}{2}}H_{2r+2-2i}(x)-H_{2r+2-2i}(0)\right\}\right].$$

PROOF. From (3. 8) when $x=t$, we get

$$\frac{H_n(t)}{t-x_\nu}=\frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\sum_{r=0}^{n-1}\frac{H_r(x_\nu)H_r(t)}{2^r r!}.$$

Differentiating this twice, we have

$$\frac{H_n'(t)}{t-x_\nu}-\frac{H_n(t)}{(t-x_\nu)^2}=\frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\sum_{r=0}^{n-1}\frac{H_r(x_\nu)H_r'(t)}{2^r r!}$$

and

$$\frac{H_n''(t)}{t-x_\nu}-\frac{2H_n'(t)}{(t-x_\nu)^2}+\frac{2H_n(t)}{(t-x_\nu)^3}=\frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\sum_{r=0}^{n-1}\frac{H_r(x_\nu)H_r''(t)}{2^r r!}.$$

From these it can easily be seen that

$$-\frac{H_n'(t)-(tx_\nu+1-x_\nu^2)l_\nu(t)H_n'(x_\nu)}{(t-x_\nu)^2}=$$

$$=-\frac{H_n'(t)}{(t-x_\nu)^2}+\frac{x_\nu H_n(t)}{(t-x_\nu)^2}+\frac{H_n(t)}{(t-x_\nu)^3}=\frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\sum_{r=0}^{n-2}\frac{(n-r-1)}{2^r r!}H_r(x_\nu)H_r(t).$$

Therefore

$$-\int_0^x e^{\frac{t^2}{2}}\,\frac{H_n'(t)-(tx_\nu+1-x_\nu^2)l_\nu(t)H_n'(x_\nu)}{(t-x_\nu)^2}\,dt=$$

$$=\frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\sum_{r=0}^{n-2}\frac{(n-r-1)}{2^r r!}H_r(x_\nu)\int_0^x e^{\frac{t^2}{2}}H_r(t)dt.$$

Now breaking the right-hand sum in even and odd sums and applying Lemma 6. 2 we get the required lemma.

Determination of $r_\nu(x)$ when n is even. Let

$$\lambda_\nu(x) = l_\nu^2(x) + \frac{H_n(x)}{H_n'(x_\nu)}\Bigg[A e^{-\frac{x^2}{2}} \int_0^x e^{\frac{1}{2}t^2} H_n(t)\,dt + \tag{7.2}$$

$$+ e^{-\frac{x^2}{2}} B \int_0^x e^{\frac{t^2}{2}} \frac{H_n(t)}{t-x_\nu}\,dt - \frac{e^{-\frac{1}{2}x^2}}{H_n'(x_\nu)} \int_0^x e^{\frac{t^2}{2}} \frac{H_n'(t)-(Ct+D)\,l_\nu(t)}{(t-x_\nu)^2}\,dt + K e^{-\frac{x^2}{2}}\Bigg].$$

Condition (5. 2) is obviously satisfied for all values of A, B, C, D and K, i. e.

$$\lambda_\nu(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \text{for } j = \nu, \end{cases}$$

and for condition (5. 3), $j \neq \nu$, it is easy to see with the help of (3. 2) and (3. 3) that

$$\lambda_\nu''(x_j) = 0.$$

But for $j = \nu$, B is chosen so that

$$\lambda_\nu''(x_\nu) = 0,$$

i. e.

$$2BH_n'(x_\nu) - 2 \lim_{x \to x_\nu} \frac{H_n'(x) - (Cx+D)\,l_\nu(x)}{H_n'(x_\nu)\,(x-x_\nu)^2} + 2[l_\nu'^2(x_\nu) + l_\nu(x_\nu)\,l_\nu''(x_\nu)] = 0. \tag{7.3}$$

Now since $\dfrac{H_n'(x) - (Cx+D)\,l_\nu(x)}{(x-x_\nu)^2}$ is a polynomial, we have

$$H_n'(x_\nu) - (Cx_\nu + D)\,l_\nu(x_\nu) = 0,$$

$$H_n''(x_\nu) - C\,l_\nu(x_\nu) - (Cx_\nu + D)\,l_\nu'(x_\nu) = 0.$$

From these it is easy to see with the help of (3. 4) and (3. 5) that

$$C = x_\nu H_n'(x_\nu) \quad \text{and} \quad D = (1 - x_\nu^2) H_n'(x_\nu).$$

From (7. 2), (3. 6), (3. 5) and (3. 4) we get

$$B = -\frac{2x_\nu^2}{H_n'(x_\nu)}.$$

So $\lambda_\nu(x)$ satisfies both conditions (5. 2) and (5. 3). Therefore

$$\lambda_\nu(x) \equiv r_\nu(x).$$

For the determination of A and K we shall make use of the fact that the right-hand side of (7. 2) is a polynomial of degree $\leqq 2n-1$ so that

(i) the coefficient of $\int_0^x e^{+\frac{t^2}{2}} dt$ is 0

and

(ii) the coefficient of $e^{-\frac{x^2}{2}}$ is 0.

Now applying (6. 2) and (6. 3) in (7. 2) we get (5. 7) from condition (i). From (ii) we get (5. 11).

Hence $r_\nu(x)$ is uniquely determined as given in (5. 6).

8. Problem of explicit representation. (Case (0, 1, 3).) Given the n distinct zeros

$$+\infty > x_1 > x_2 > \cdots > x_n > -\infty$$

of $H_n(x)$ and arbitrary numbers

$$a_1, a_2, a_3, \ldots, a_n;\ b_1, b_2, b_3, \ldots, b_n;\ c_1, c_2, c_3, \ldots, c_n,$$

we know from Theorem II that there exists a polynomial $R_n(x)$ of degree $\leqq 3n-1$ such that

$$R_n(x_\nu) = a_\nu, \quad R_n'(x_\nu) = b_\nu, \quad R'''(x_\nu) = c_\nu \qquad (\nu = 1, 2, \ldots, n).$$

So for $n = 2k$ we have

$$R_{2k}(x) = \sum_{\nu=1}^{2k} a_\nu u_\nu(x) + \sum_{\nu=1}^{2k} b_\nu v_\nu(x) + \sum_{\nu=1}^{2k} c_\nu w_\nu(x)$$

where $u_\nu(x)$, $v_\nu(x)$ and $w_\nu(x)$ are the fundamental polynomials of the first, second and third kind of (0, 1, 3)-interpolation, belonging to the x_j-points, are polynomials of degree $\leqq 3n-1 = 6k-1$ uniquely determined by the given expressions:

$$(8.1) \qquad u_\nu(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \text{for } j = \nu, \end{cases} \quad u_\nu'(x_j) = u_\nu'''(x_j) = 0 \qquad (j = 1, 2, \ldots, n),$$

$$(8.2) \quad v_\nu(x_j) = 0, \quad u_\nu'(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \text{for } j = \nu, \end{cases} \quad v_\nu'''(x_j) = 0 \qquad (j = 1, 2, \ldots, n),$$

$$(8.3) \quad w_\nu(x_j) = 0, \quad w_\nu'(x_j) = 0, \quad w_\nu'''(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \text{for } j = \nu \end{cases} \qquad (j = 1, 2, \ldots, n).$$

THEOREM IV. (Case (0, 1, 3).) *For the fundamental polynomials $u_\nu(x)$, $v_\nu(x)$ and $w_\nu(x)$ the following explicit forms can be given:*

$$u_\nu(x)=l_\nu^3(x)+c_\nu v_\nu(x)+\frac{H_n^2(x)}{H_n'^2(x)}\Big\{Ae^{-x^2}\int_0^x e^{t^2}H_n(t)\,dt+ \tag{8.4}$$

$$+Be^{-x^2}\int_0^x e^{t^2}\frac{H_n(t)}{t-x_\nu}dt-\frac{e^{-x^2}}{H_n'(x_\nu)}\int_0^x e^{t^2}\frac{H_n'(t)-(ct^2+dt+a)l_\nu(t)}{(t-x_\nu)^3}dt+ke^{-x^2}\Big\},$$

where c_ν, A, B, c, d, a and k are constants given by

$$c_\nu=-3x_\nu, \tag{8.5}$$

$$A=\frac{(-2)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{6\sqrt{\pi}H_n'(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}[(n-2r)H_{2r+1}(x_\nu)+ \tag{8.6}$$

$$+\{-10x_\nu(n-1)+2x_\nu(r+9x_\nu^2)H_{2r}(x_\nu)\}]\frac{\Gamma\left(r+\frac{1}{2}\right)}{(2r)!}(-2)^r,$$

$$B=-\frac{3x_\nu}{H_n'(x_\nu)}[2x_\nu^2+(1-n)], \tag{8.7}$$

$$c=\frac{x_\nu^2+2(1-n)}{3}H_n'(x_\nu), \tag{8.8}$$

$$d=x_\nu\left[1-\frac{2}{3}\{x_\nu^2+2(1-n)\}\right]H_n'(x), \tag{8.9}$$

$$a=\left[1-x_\nu^2\Big\{1-\frac{1}{3}(x_\nu^2+2(1-n))\Big\}\right]H_n'(x) \tag{8.10}$$

and

$$k=-\frac{2^{n-1}(n-1)!}{3H_n'(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{(-1)^r r!}{(2r+1)!}(n-2r-1)H_{2r+2}(x_\nu)+ \tag{8.11}$$

$$+x_\nu(2r-10n+11+18x_\nu^2)H_{2r+1}(x_\nu)]\sum_{i=1}^{r+1}\frac{2^{i-1}}{\sqrt{\pi}}\frac{\Gamma\left(r+\frac{3}{2}-i\right)}{\Gamma(r+2-i)},$$

(8.12)
$$v_\nu(x) = \frac{H_n(x)}{H'_n(x_\nu)}\left[l_\nu^2(x) + H_n(x)e^{-x^2}\left\{A'\int_0^x e^{t^2}H_n(t)\,dt + \right.\right.$$
$$\left.\left. + B'\int_0^x e^{t^2}\frac{H_n(t)}{t-x_\nu}\,dt - \frac{1}{H_n'^2(x_\nu)}\int_0^x e^{t^2}\frac{H'_n(t)-(c't+d')l_\nu(t)}{(t-x_\nu)^2}\,dt + k'\right\}\right],$$

where A', B', c', d' *and* k' *are constants given by*

(8.13)
$$A' = -\frac{(-2)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}H_n'^2(x_\nu)\,H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{n-2r-1-\dfrac{14x_\nu^2+1-n}{3}}{(2r)!}(-2)^r\cdot$$
$$\cdot\Gamma\left(r+\frac{1}{2}\right)H_{2r}(x_\nu),$$

(8.14)
$$B' = -\frac{14x_\nu^2+1-n}{3H_n'^2(x_\nu)},$$

(8.15)
$$c' = x_\nu H'_n(x_\nu),$$

(8.16)
$$d' = (1-x_\nu^2)H'_n(x_\nu),$$

(8.17)
$$k' = \frac{2^{n-1}(n-1)!}{\sqrt{\pi}H_n'^2(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{n-2r-2-\dfrac{14x_\nu^2+1-n}{3}}{(2r+1)!}\cdot$$
$$\cdot(-1)^r r!\,H_{2r+1}(x_\nu)\sum_{i=1}^{r+1}2^{i-1}\frac{\Gamma\left(r+\frac{3}{2}-i\right)}{\Gamma(r+2-i)},$$

(8.18)
$$w_\nu(x) = \frac{H_n^2(x)e^{-x^2}}{6H_n'^2(x_\nu)}\left[c''\int_0^x e^{t^2}H_n(t)\,dt + \frac{1}{H'_n(x_\nu)}\int_0^x e^{t^2}\frac{H_n(t)}{t-x_\nu}\,dt + d''\right],$$

where

(8.19)
$$c'' = +\frac{(-2)^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)}{H'_n(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{(-1)^r H_{2r}(x_\nu)}{2^r r!}$$

and

$$(8.20)\quad d''=\frac{2^{n-1}(n-1)!}{H_n'(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{(-1)^r r!}{(2r+1)!}H_{2r+1}(x_\nu)\sum_{i=1}^{r+1}2^{i-1}\frac{\Gamma\left(r+\frac{3}{2}-i\right)}{\Gamma(r+2-i)}.$$

Equivalent expressions for (8.6), (8.11), (8.13), (8.17), (8.19) and (8.20) are as follows:

$$(8.6a)\quad A=\frac{1}{(-8)^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right)H_n'(x_\nu)}\left[\int_{-\infty}^{\infty}e^{-t^2}\frac{H_n'(it)-(-ct^2+idt+a)l_\nu(it)}{(it-x_\nu)^3}dt+\right.$$
$$\left.+3x_\nu(2x^\nu+1-n)\int_{-\infty}^{\infty}e^{-t^2}\frac{H_n(it)}{it-x_\nu}dt\right],$$

$$(8.11a)\quad k=\frac{i}{\sqrt{\pi}H_n'(x_\nu)}\left[\int_0^1\frac{du}{\sqrt{1-u^2}}\,3x_\nu(2x_\nu^2+1-n)\int_{-\infty}^{\infty}te^{-t^2}\frac{H_n(iut)}{(iut-x_\nu)}dt+\right.$$
$$\left.+\int_0^1\frac{du}{\sqrt{1-u^2}}\int_{-\infty}^{\infty}te^{-t^2}\frac{H_n'(iut)-\{-cu^2t^2+idut+a\}l_\nu(iut)}{(iut-x_\nu)^3}dt\right],$$

$$(8.13a)\quad A'=\frac{1}{(-8)^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right)H_n^2(x_\nu)}\left[\frac{14x_\nu^2+1-n}{3}\int_{-\infty}^{\infty}e^{-t^2}\frac{H_n(it)}{it-x_\nu}dt+\right.$$
$$\left.+\int_{-\infty}^{\infty}e^{-t^2}\frac{H_n'(it)-(itx_\nu+1-x_\nu^2)l_\nu(it)H_n'(x_\nu)}{(it-x_\nu)^2}dt\right],$$

$$(8.17a)\quad k'=\frac{i}{\sqrt{\pi}H_n'^2(x_\nu)}\left[\frac{14x_\nu+1-n}{3}\int_0^1\frac{du}{\sqrt{1-u^2}}\int_{-\infty}^{\infty}te^{-t^2}\frac{H_n(iut)}{iut-x_\nu}dt+\right.$$
$$\left.+\int_0^1\frac{du}{\sqrt{1-u^2}}\int_{-\infty}^{\infty}te^{-t^2}\frac{H_n'(iut)-(iutx_\nu+1-x_\nu^2)l_\nu(iut)H_n'(x_\nu)}{(iut-x_\nu)^2}dt\right],$$

(8.19a) $$c'' = -\frac{1}{(-8)^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right)H_n'(x_\nu)}\int_{-\infty}^{\infty} e^{-t^2}\frac{H_n(it)}{it-x_\nu}\,dt,$$

(8.20a) $$d'' = \frac{-i}{\sqrt{\pi}H_n'(x_\nu)}\int_0^1 \frac{du}{\sqrt{1-u^2}}\int_{-\infty}^{\infty} te^{-t^2}\frac{H_n(iut)}{iut-x_\nu}\,dt.$$

The proof of this theorem can be given on the same lines as given in Theorem III.

We shall require the following formulae which can be derived on the same lines as in Lemmas 6.2, 6.3 and 7.1.

When $n=2k$,

(8.21) $$\int_0^x e^{t^2}H_{2k}(t)\,dt = \sum_{r=1}^{k}(-8)^{r-1}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+\frac{3}{2}-r\right)}e^{x^2}H_{2k+1-2r}(x) + (-8)^k\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}}\int_0^x e^{t^2}dt.$$

When $n=2k+1$,

(8.22) $$\int_0^x e^{t^2}H_{2k+1}(t)\,dt = \sum_{r=1}^{k}(-8)^{r-1}\frac{k!}{(k+1-r)!}\left[e^{x^2}H_{2k-2r+2}(x) - H_{2k-2r+2}(0)\right].$$

(8.23) $$\int_0^x e^{t^2}\frac{H_n(t)}{t-x_\nu}\,dt = \frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\left[\sum_{r=0}^{\frac{n}{2}-1}\frac{H_{2r}(x_\nu)}{2^{2r}(2r)!}\left\{\sum_{i=1}^{r}(-8)^{i-1}\frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(r+\frac{3}{2}-i\right)}\cdot e^{x^2}H_{2r+1-2i}(x) + (-8)^r\frac{\Gamma\left(r+\frac{1}{2}\right)}{\sqrt{\pi}}\int_0^x e^{t^2}dt\right\} + \sum_{r=0}^{\frac{n}{2}-1}\frac{H_{2r+1}(x_\nu)}{2^{2r+1}(2r+1)}\sum_{i=1}^{r+1}\frac{(-8)^{i-1}r!}{(r+1-i)!}\left\{e^{x^2}H_{2r+2-2i}(x) - H_{2r+2-2i}(0)\right\}\right].$$

LEMMA 8. 1.

$$\int_0^x \frac{H_n'(t)-(ct^2+dt+a)l_\nu(t)}{(t-x_\nu)^3} e^{t^2} dt =$$

$$= \frac{2^n(n-1)!}{3H_{n-1}(x_\nu)} \Bigg[\sum_{r=0}^{\frac{n}{2}-1} \frac{(n-2r)H_{2r+1}(x_\nu) - x_\nu(n-2r-1)H_{2r}(x_\nu)}{2^{2r}(2r)!} \cdot$$

$$(8.24) \quad \cdot \left\{ \sum_{i=1}^{r} (-8)^{i-1} \frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(r+\frac{3}{2}-i\right)} e^{x^2} H_{2r+1-2i}(x) + \frac{(-8)^r \Gamma\left(r+\frac{1}{2}\right)}{\sqrt{\pi}} \int_0^x e^{t^2} dt \right\} +$$

$$+ \sum_{r=0}^{\frac{n}{2}-1} \frac{(n-2r-1)H_{2r+2}(x_\nu) - x_\nu(n-2r-2)H_{2r+1}(x_\nu)}{2^{2r+1}(2r+1)!} \cdot$$

$$\cdot \sum_{i=1}^{r+1} (-8)^{i-1} \frac{r!}{(r+1-i)!} \left\{ e^{x^2} H_{2r-2i+2}(x) - H_{2r-2i+2}(0) \right\} \Bigg]$$

where c, d and a are as given in (8. 8), (8. 9) *and* (8. 10).

This lemma can be proved easily on the same lines as Lemma 7. 1.

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ÜBER KREIS- UND KUGELWOLKEN

Von

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(*Vorgelegt von* G. HAJÓS)

Der Begriff „Wolke" wurde in einer kürzlich erschienenen Arbeit von L. FEJES TÓTH [1] eingeführt. Wir betrachten in der Ebene eine Menge von nicht übereinandergreifenden Einheitskreisen, die zwischen zwei parallelen Geraden liegen. Diese Menge wird Kreiswolke genannt, wenn jede solche Gerade, welche die obigen zwei parallelen Geraden senkrecht schneidet, mindestens einen der Kreise trifft. Man kann es so interpretieren, daß eine Kreiswolke gegen die senkrechten Strahlen eine undurchdringliche Wand bildet. Ganz analog nennen wir im Raum eine Menge von nicht übereinandergreifenden Einheitskugeln eine Kugelwolke, wenn sie zwischen zwei parallelen Ebenen liegt, und wenn jede zu diesen Ebenen senkrechte Gerade mindestens eine der Kugeln trifft.

In der erwähnten Arbeit wurde das Problem aufgeworfen und gelöst, wie groß die minimale Dicke[1] einer Kugelwolke ist. Der Satz von FEJES TÓTH lautet folgendermaßen:

Jede Kugelwolke hat eine Dicke $\geqq \sqrt{2}+2$, *und Gleichheit gilt nur dann, wenn die Wolke aus zwei quadratischen, einander berührenden Kugelschichten besteht.*

Die minimale Dicke einer Kreiswolke ist offenkundig 2 (Fig. 1).

FEJES TÓTH hat meine Aufmerksamkeit auf das Problem der Bestimmung der minimalen Dicke d_k bzw. D_k einer k-fachen Kreis- bzw. Kugelwolke gerichtet, wo eine Wolke k-fach genannt wird, wenn jeder der genannten parallelen Strahlen mindestens k Kreise bzw. Kugeln trifft. (Die früher definierten Wolken sind also in diesem Sinne einfache Wolken.)

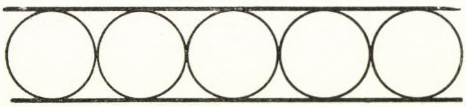

Fig. 1

Legen wir k einfache extremale Kreis- bzw. Kugelwolken in geeigneter Weise aufeinander, so erhalten wir eine k-fache Kreiswolke von der Dicke $(k-1)\sqrt{3}+2$ (Fig. 2) bzw. eine k-fache Kugelwolke von der Dicke

[1] Dicke nennen wir — wie gewöhnlich — den Mindestabstand von zwei parallelen Geraden bzw. Ebenen, die die vorliegende ebene bzw. räumliche Menge umfassen.

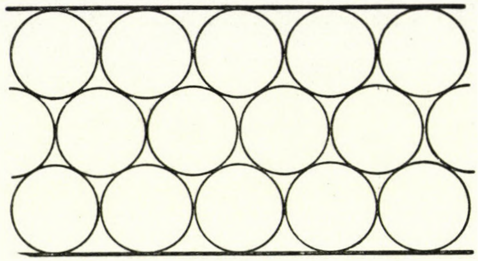

Fig. 2

$(2k-1)\sqrt{2}+2$. Folglich ist

(1) $$d_k \leqq (k-1)\sqrt{3}+2$$

bzw.

(2) $$D_k \leqq (2k-1)\sqrt{2}+2.$$

Beide Wolken liefern einen Teil der dichtesten gitterförmigen Kreis- bzw. Kugellagerung ([2]). Man könnte daher im ersten Augenblick vermuten, daß in (1) und (2) für jedes k das Gleichheitszeichen gilt. Wir werden zeigen, daß dies in der Ebene tatsächlich zutrifft, im Raum aber nicht.

Wir beweisen also folgenden

SATZ. *Bedeutet d_k bzw. D_k die kleinstmögliche Dicke einer k-fachen Wolke von Einheitskreisen bzw. Einheitskugeln, so gilt*

(3) $$d_k = (k-1)\sqrt{3}+2 \quad \textit{für} \quad k>1$$

und

(4) $$D_k < (2k-1)\sqrt{2}+2 \quad \textit{für} \quad k>1.$$

Wir wenden uns zunächst dem Beweis von (3) zu.

Wir betrachten eine beliebige k-fache Kreiswolke in der Ebene, welche in einem Parallelstreifen S von der Breite b enthalten ist. Es seien g_i $(i=1,2)$ die den Streifen begrenzenden Geraden, g_i' eine in S liegende Gerade, die mit g_i den Parallelstreifen S_i von der Breite 1 bestimmt $(i=1,2)$, und schließlich S_3 der von g_1' und g_2' begrenzte mittlere Teilstreifen von S (Fig. 3).

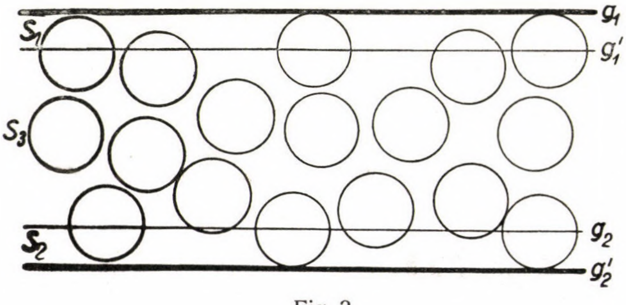

Fig. 3

Ohne Beschränkung der Allgemeinheit können wir voraussetzen, daß die Wolke gesättigt ist, d. h. daß sich zu den Kreisen kein weiterer Kreis hinfügen läßt, so daß eine Wolke von derselben Dicke entsteht.

Wir definieren zunächst die zu einem Kreis gehörige Zelle. Diese besteht aus der Gesamtheit derjenigen Punkte von S, deren Abstand von dem

Mittelpunkt des ausgewählten Kreises kleiner als ihr Abstand von den übrigen Kreismittelpunkten ist. Offenkundig sind die Zellen konvexe Vielecke, die den Streifen S (abgesehen von den Randpunkten der Zellen) schlicht und lückenlos überdecken. Jede Zelle enthält das Innere des zugehörigen Kreises, und wegen der Gesättigtheit der Wolke hat sie bei jeder in S_3 liegenden Ecke einen Winkel $> \frac{\pi}{3}$. Um einen solchen Eckpunkt geschlagener Einheitskreis muß nämlich mindestens den nächsten Kreis der Wolke treffen.

Wir geben jetzt eine andere Zerlegung von S an. Wir verbinden die Mittelpunkte von zwei Kreisen durch eine Strecke, wenn ihre Zellen gemeinsame Randstrecken in S_3 (also im inneren Teilstreifen) haben. Die Verbindungsstrecke und die zugehörige gemeinsame Seite der Zellen sind senkrecht. Diese Strecken bilden ein zusammenhängendes Netz, welches völlig in S_3 liegt (Fig. 4). So ergeben sich zwei äußere, unendliche Bereiche B_1 und B_2, die sich von g_1 bzw. von g_2 bis zum Rande des Netzes erstrecken. Der von S übriggebliebene Teil, der durch das Netz in Vielecke zerlegt wird, sei mit B_3 bezeichnet. Diese Vielecke und die im Inneren von S_3 liegenden Zellenecken sind einander ein-eindeutig zugeordnet. Jedes Vieleck hat soviel Seiten, wieviel Zellen in der zugehörigen Ecke zusammenstoßen. Die Zellenwinkel in dieser Ecke sind komplementäre Winkel des Vielecks. Da aber die Zellenwinkel $> \frac{\pi}{3}$ sind, sind die Winkel des Vielecks $< \frac{2\pi}{3}$. Teilen wir die Vielecke durch einander nicht kreuzende Diagonale weiter, so bekommen wir zum Schluß lauter Dreiecke, deren Eckpunkte in Kreismittelpunkten liegen, deren Seiten $\geqq 2$ und deren Winkel $< \frac{2\pi}{3}$ ausfallen. Daraus folgt, daß in jedem Dreieck die größte Höhe $\geqq \sqrt{3}$ und sämtliche Höhen $\geqq 1$ sind. Folglich is der Inhalt eines Dreiecks stets $\geqq \sqrt{3}$ bzw. der Gesamtinhalt der Wolke innerhalb eines jeden Dreiecks genau $\pi/2$. Hieraus ergibt sich, daß die Dichte[2] der Wolke in B_3 höchstens $\pi/\sqrt{12}$ ist.

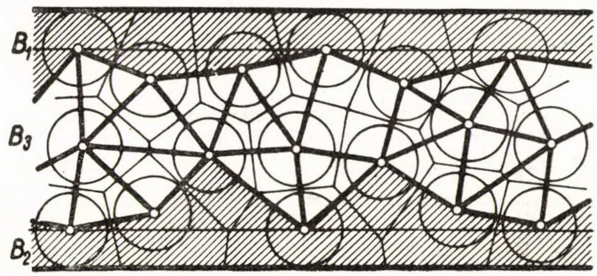

Fig. 4

[2] Unter der Dichte der Wolke in einem im Endlichen liegenden Gebiet G verstehen wir das Inhaltsverhältnis der in G liegenden Kreisteile und von G. Die Dichte in einem unendlichen Gebiet läßt sich dann durch einen Grenzübergang definieren.

Im folgenden Schritt zeigen wir, daß es in B_i einen den Sreifen S_i enthaltenden Teilbereich B_i' gibt, in welchem die Dichte der Wolke den Wert $\pi/4$ nicht übertritt. (Hier und im folgenden bedeutet i stets 1 oder 2.)

Der Bereich B_i ist einerseits von g_i anderseits von einem Streckenzug (vom Rand des Netzes) begrenzt. Diese Strecken verbinden die Mittelpunkte solcher Kreispaare, deren Zellen eine gemeinsame Randstrecke in S_3 und einen gemeinsamen Randpunkt an der Gerade g_i' haben. Es ist leicht einzusehen, daß der Winkel von einer solchen Zellenseite und von g_i' wegen der Gesättigtheit der Wolke größer als $\pi/6$ ist. Deshalb ist der Neigungswinkel zwischen g_i' und einer beliebigen Strecke des Streckenzuges kleiner als $\pi/3$ (Fig. 5).

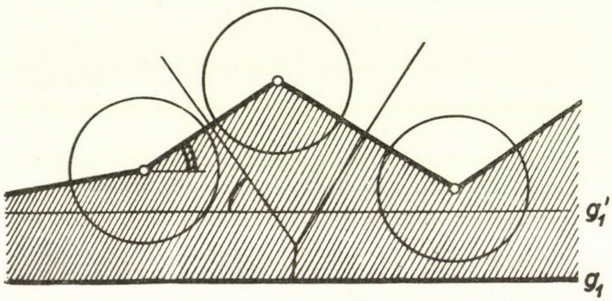

Fig. 5

Die von den Ecken des Streckenzuges auf g_i gefällten Lote zerlegen B_i in rechtwinklige Trapeze (dabei werden als Ecken sämtliche Kreismittelpunkte angesehen, die auf diesem Streckenzug liegen). Zufolge des obigen Resultates ist der Neigungswinkel der Schenkel eines jeden Trapezes kleiner als $\pi/3$. Die Höhe des Trapezes, d. h. der Abstand der parallelen Seiten ist also mindestens 1, deshalb enthält jedes Trapez insgesamt einen halben Kreis aus der Wolke.

Im folgenden wird ein Trapez steil genannt, wenn der Neigungswinkel seiner Schenkel größer als $\pi/6$ ist. Zerlegen wir ein steiles Trapez durch eine Strecke in ein Rechteck und ein rechtwinkliges Dreieck weiter, so enthält sowohl das Rechteck, wie das Dreieck genau einen Viertelkreis der Wolke. Die Dichte der Wolke ist also im Rechteck trivialerweise $\leqq \frac{\pi}{4}$ und im abgeschnittenen rechtwinkligen Dreiecks $\leqq \frac{\pi}{\sqrt{12}}$. Dieses Dreieck ist nämlich die Hälfte eines solchen Dreiecks, welches nur Seiten $\geqq 2$ und Winkel $\leqq \frac{2\pi}{3}$, also einen Inhalt $\geqq \sqrt{3}$ hat. Elementares Rechnen zeigt, daß der Inhalt jedes anderen (nicht steilen) Trapezes wenigstens 2 beträgt, infolgedessen die Dichte der Wolke in diesen Trapezen höchstens $\frac{\pi}{4}$ ist.

Es bedeute B_i' die Gesamtheit der nichtsteilen Trapeze und der Rechtecke der steilen Trapeze, und B_3' die Vereinigung der aus B_1 und B_2 abgeschnittenen rechtwinkligen Dreiecke und der Dreiecke von B_3. Mit Hilfe dieser Bezeichnungen können wir die obigen Ergebnisse folgendermaßen zusammenfassen: Wir haben den Streifen S derart in die Teile B_j' $(j=1,2,3)$ zerlegt, daß die Dichte der Wolke in B_1' und B_2' $\leqq \frac{\pi}{4}$ und in $B_3' \leqq \frac{\pi}{\sqrt{12}}$ ist. Im Hinblick darauf, daß B_i' den Streifen S_i enthält und $\frac{\pi}{\sqrt{12}} > \frac{\pi}{4}$ ist, können wir die folgende Abschätzung für die Dichte D der Wolke in S angeben:

$$D \leqq \frac{1}{b}\left(2\frac{\pi}{4}+(b-2)\frac{\pi}{\sqrt{12}}\right). \tag{5}$$

Projizieren wir die Wolke senkrecht auf g_1, so wirft jeder Kreis eine Schattenstrecke von der Länge 2. Da aber die Schattenstrecken g_1 k-fach überdecken, gehören zu einer Einheitsstrecke von g_1 durchschnittlich mindestens $\frac{k}{2}$ Kreise der Wolke. Daraus ergibt sich für D eine Abschätzung von unten:

$$D \geqq \frac{\frac{k}{2}\pi}{b},$$

woraus sich, mit Rücksicht auf $b \geqq d_k$, (1) und (5), die behauptete Gleichung (3) ergibt.

Um die Gültigkeit der Ungleichung (4) zu zeigen, werden wir eine k-fache Wolke von der Dicke $\left(k+\left[\frac{k-1}{3}\right]\right)\sqrt{3}+2$ konstruieren.[3]

Es seien $\mathbf{a}_1$ und $\mathbf{a}_2$ vom Anfangspunkt O ausgehende Vektoren, die mit O das gleichseitige Dreieck OA_1A_2 von Seitenlänge 2 bestimmen, und $\mathbf{c}$ ein zu $\mathbf{a}_1$ und $\mathbf{a}_2$ senkrechter Vektor von der Länge $\sqrt{3}$. Wir betrachten das von $\mathbf{a}_1$ und $\mathbf{a}_2$ erzeugte Punktgitter und die gitterförmige Kugelschicht G_0, die aus den um diese Gitterpunkte geschlagenen Einheitskugeln besteht. Wir bezeichnen mit G_i die Kugelschicht, welche von G_{i-1} durch die Translation $\frac{\mathbf{a}_i}{2}+\mathbf{c}$ entsteht $(i>0)$, wo $\mathbf{a}_i \equiv \mathbf{a}_1$ bzw. $\mathbf{a}_i \equiv \mathbf{a}_2$, je nachdem i ungerade bzw. gerade ist. Die Kugeln der Schichten greifen nicht übereinander.

Die senkrechte Projektion G_i' der Schicht G_i auf die Ebene des Dreiecks OA_1A_2 liefert eine gitterförmige Lagerung von Einheitskreisen, und es gilt offensichtlich $G_i' = G_{i-1}' + \frac{\mathbf{a}_i}{2}$. Wir greifen aus den Kugelprojektionen die um

[3] Das Symbol $[x]$ bedeutet die größte ganze Zahl, die den Wert x nicht übertrifft.

$O+\frac{\mathbf{a}_1}{2}, O+\frac{\mathbf{a}_1}{2}+\frac{\mathbf{a}_2}{2}$ bzw. $O+\frac{\mathbf{a}_2}{2}$ geschlagenen Einheitskreise K_1, K_2 bzw. K_3 heraus. Es ist leicht einzusehen, daß K_j zum Kreisgitter G'_j gehört ($j=1, 2, 3$), und daß das Dreieck OA_1A_2 durch K_1 und K_2 einfach, durch K_1, K_2 und K_3 zweifach, und durch K_1, K_2, K_3 und G'_4 dreifach überdeckt ist. Folglich liefert wegen der Gitterförmigkeit G_1 und G_2 eine einfache, G_1, G_2 und G_3 eine zweifache, schließlich G_1, G_2, G_3 und G_4 eine dreifache Wolke. Da $G'_i=G'_{i+4}$ ist, ist die Vereinigung der Schichten G_i von $i=1$ bis $i=k+1+\left[\frac{k-1}{3}\right]$ eine k-fache Wolke, welche eine Dicke $\left(k+\left[\frac{k-1}{3}\right]\right)\sqrt{3}+2$ hat. Da aber für $k>1$ $\left(k+\left[\frac{k-1}{3}\right]\right)\sqrt{3}<(2k-1)\sqrt{2}$ ist, ist damit der Beweis von (4) erbracht.

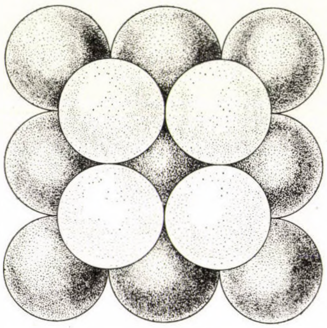

Fig. 6

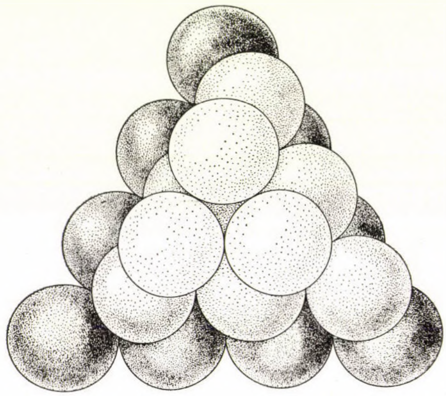

Fig. 7

Fig. 6 bzw. Fig. 7 zeigt die extremale einfache Wolke von FEJES TÓTH bzw. unsere Konstruktion für $k=2$.

(Eingegangen am 5. April 1960.)

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ÜBER DIE ABSOLUTE KONVERGENZ LAKUNAERER TRIGONOMETRISCHER REIHEN

Von

P. SZÜSZ (Budapest)

(*Vorgelegt von* P. TURÁN)

In der vorliegenden Note werden die folgenden beiden Sätze bewiesen:

SATZ 1. *Ist K eine beliebig große positive Zahl, so existiert eine Folge* $\{n_k\}$ $(k=1, 2, \ldots)$ *natürlicher Zahlen mit den folgenden Eigenschaften:*

a) *Es ist*

$$\frac{n_{k+1}}{n_k} > K \qquad (k=1, 2, \ldots).$$

b) *Ist* $a_1, a_2, \ldots$ *eine monoton abnehmende Folge positiver Zahlen, so folgt aus*

$$\sum_{k=1}^{\infty} |a_k \sin \pi n_k x| < \infty \tag{1}$$

für irgendein $x \neq 0, \pm 1, \ldots$ *die Relation*

$$\sum_{k=1}^{\infty} a_k < \infty. \tag{2}$$

SATZ 2. *Es sei* $\{n_k\}$ $(k=1, 2, \ldots)$ *eine monoton zunehmende Folge natürlicher Zahlen, für die nur die Relation*

$$\frac{n_{k+1}}{n_k} \to \infty \quad \text{für} \quad k \to \infty \tag{3}$$

besteht. Dann gibt es eine monoton abnehmende Folge $\{a_k\}$ *positiver Zahlen, für die zwar* $\sum_{k=1}^{\infty} a_k = \infty$ *gilt, aber trotzdem gibt es eine Menge der Mächtigkeit des Kontinuums der Zahlen x, für die* $\sum_{k=1}^{\infty} |a_k \sin \pi n_k x| < \infty$ *gilt.*

Satz 1 ist eine Übertragung eines klassischen Satzes von FATOU[1] auf Lückenfolgen. Satz 2 besagt, daß eine dem Fatouschen Satz analoge Behauptung nicht mehr bestehen kann, wenn die Lücken der Folge $\{n_k\}$ „zu groß" sind.

[1] Vgl. z. B. ZYGMUND [1], S. 134.

Beweis des Satzes 1. Ohne Beschränkung der Allgemeinheit sei angenommen, daß K (von a)) eine natürliche Zahl ist ($K \geqq 2$).

Man setze

$$n_1 = 1, \quad n_{k+1} = K n_k + 1 \qquad (k = 1, 2, \ldots). \tag{4}$$

Nun zeige ich, daß falls $\{n_k\}$ durch (4) definiert ist, so kann für eine beliebige, monoton abnehmende Folge $\{a_k\}$ positiver Zahlen die Relation $\sum_{k=1}^{\infty} |a_k \sin \pi n_k x| < \infty$ mit einem nichtganzen x nur dann stattfinden, wenn $\sum_{k=1}^{\infty} a_k < \infty$ ist. Es darf ohne Beschränkung der Allgemeinheit

$$0 \leqq x < 1$$

angenommen werden; da x voraussetzungsgemäß nichtganz ist, gibt es eine Zahl δ mit $0 < \delta \leqq \frac{1}{2}$ und

$$\delta \leqq x \leqq 1 - \delta. \tag{5}$$

Nun gilt bekanntlich für jedes reelle t [2]

$$|\sin \pi t| \geqq 2 \|t\|; \tag{6}$$

daher genügt es für jede monoton abnehmende Folge $\{a_k\}$ mit $\sum_{k=1}^{\infty} a_k = \infty$ und für jedes nichtganze x die Relation

$$\sum_{k=1}^{\infty} a_k \|n_k x\| = \infty \tag{7}$$

zu beweisen.

Man setze

$$\gamma = \frac{\delta}{8K}, \tag{8}$$

wobei δ durch (5) definiert ist.

Nun zeige ich, daß aus

$$\|n_k x\| < \gamma \tag{9}$$

und

$$\|n_{k+1} x\| < \gamma \tag{10}$$

ein Widerspruch zu (5) folgt. (9) bzw. (10) ist gleichbedeutend mit

$$x = \frac{l_k + \vartheta_k}{n_k} \tag{11}$$

[2] $\|t\|$ bedeutet den Abstand von t von der nächstbenachbarten ganzen Zahl.

bzw.

(12)
$$x = \frac{l_{k+1} + \vartheta_{k+1}}{n_{k+1}}$$

mit ganzen l und $|\vartheta_k| < \gamma$, $|\vartheta_{k+1}| < \gamma$. Hieraus folgt weiter mit Rücksicht auf (4)

(13)
$$|n_k(Kl_k - l_{k+1}) + l_k| < \frac{K+2}{8K} \delta n_k.$$

Aus (5) folgt bei festgelegtem δ für genügend großes n_k

(14)
$$\frac{\delta}{2} n_k < l_k < \left(1 - \frac{\delta}{2}\right) n_k.$$

(13) ist gleichbedeutend mit

(13')
$$\left| Kl_k - l_{k+1} - \frac{l_k}{n_k} \right| < \frac{\delta}{4};$$

da aber K, also auch $Kl_k - l_{k+1}$ ganz ist, ist (13') mit (14) nicht verträglich. Daher können (für genügend großes n_k) (5), (9) und (10) nicht gleichzeitig bestehen, also falls $0 < x < 1$ ist, muß von irgendeinem k an stets entweder

$$\|n_k x\| > \gamma$$

oder

$$\|n_{k+1} x\| > \gamma$$

sein (oder auch beide). Damit ist (7), also auch unser Satz 1 bewiesen.

Beweis des Satzes 2. Man setze

(15)
$$\frac{n_{k+1}}{n_k} = d_k;$$

voraussetzungsgemäß gilt

(16)
$$\lim_{k \to \infty} d_k = \infty.$$

Nun sei $\{a_k\}$ eine monoton gegen Null strebende Folge positiver Zahlen, die den Relationen

(17)
$$\sum_{k=1}^{\infty} a_k = \infty$$

und

(18)
$$\sum_{k=1}^{\infty} \frac{a_k}{d_k} < \infty$$

genügt. Eine solche existiert immer (vgl. Knopp [2], S. 311).

Es gilt für jedes reelle t

(19) $$|\sin \pi t| \leqq \pi \|t\|;$$

daher genügt es zu zeigen, daß für eine Menge der Zahlen x der Mächtigkeit des Kontinuums die Relation

(20) $$\sum_{k=1}^{\infty} a_k \|n_k x\| < \infty$$

besteht.

Die natürliche Zahlenfolge sei in zwei fremde Teilfolgen $k_1, k_2, \ldots$ und $l_1, l_2, \ldots$ eingeteilt. Es bezeichne

(21) $$E(\{k_j\}, \{l_j\})$$

die Menge der Zahlen x, für die die Relationen

(22) $$\|n_{k_j} x\| \leqq d_{k_j}^{-1} \qquad (j = 1, 2, \ldots)$$

und

(23) $$\|n_{l_j} x\| > d_{l_j}^{-1} \qquad (j = 1, 2, \ldots)$$

gelten.

Kann ich zeigen, daß für jede Einteilung $\{k_j\}, \{l_j\}$ die Menge $E(\{k_j\}, \{l_j\})$ nichtleer ist, so bin ich fertig. In diesem Falle wähle ich nämlich eine Teilfolge $l_1, l_2, \ldots$ der natürlichen Zahlenfolge mit $\sum_{j=1}^{\infty} a_{l_j} < \infty$ und verlange, daß für sämtliche k mit $k \neq l_j$ $(j = 1, 2, \ldots)$ die Relation (22) gelten soll.

Für die Zahlen x mit $x \in E(\{k_j\}, \{l_j\})$ ist wegen $\|n_{k_j} x\| \leqq d_{k_j}^{-1}$, wegen $\sum_{j=1}^{\infty} a_{l_j} < \infty$ und wegen (18)

(24) $$\sum_{k=1}^{\infty} a_k \|n_k x\| < \infty.$$

Falls $E(\{k_j\}, \{l_j\})$ bei jeder Einteilung nichtleer ist, so ist die Menge von x mit (24) der Mächtigkeit des Kontinuums, weil ja für l_j bei jedem j die Wahl

$$\|n_{l_j} x\| \leqq d_{l_j}^{-1} \quad \text{oder} \quad \|n_{l_j} x\| > d_{l_j}^{-1}$$

möglich ist.

Daher habe ich nur den folgenden Hilfssatz zu beweisen:

HILFSSATZ. *$E(\{k_j\}, \{l_j\})$ ist bei einer beliebigen Zerlegung der natürlichen Zahlenfolge in fremde Teilfolgen $\{k_j\}$ und $\{l_j\}$ nichtleer.*

BEWEIS. Es bezeichne

$$E_m(\{k_j\}, \{l_j\})$$

die Menge von x, für die die Relationen (22) und (23) gelten, jedoch nicht notwendigerweise für alle k und l, sondern nur für die mit

$$k_j \leqq m, \quad l_j \leqq m.$$

Offenbar ist

$$E_1 \supseteq E_2 \supseteq \cdots.$$

Nun zeige ich die Existenz einer Folge $I_1, I_2, \ldots$ von Intervallen mit den folgenden Eigenschaften:

a) $I_{n+1} \subset I_n \quad (n = 1, 2, \ldots)$.

b) $I_n \subset E_n \quad (n = 1, 2, \ldots)$.

c) Ist I_n offen, so ist auch die abgeschlossene Hülle von I_{n+1} in I_n enthalten. Da durch das Intervallensystem $I_1, I_2, \ldots$ eine Intervallschachtelung definiert ist, die nicht zu einer nicht in $E(\{k_j\}, \{l_j\})$ enthaltenen Zahl führen kann, wird durch den Nachweis der Existenz einer Intervallenfolge $I_1, I_2, \ldots$ mit a), b) und c) auch unser Hilfssatz bewiesen sein.

Der Beweis der letzten Behauptung wird durch vollständige Induktion geführt.

Für $m = 1$ besteht $E_m(\{k_j\}, \{l_j\})$ aus offenen Intervallen, falls $l_1 = 1$, und aus abgeschlossenen Intervallen, falls $k_1 = 1$, deren Längen gleich $\left(1 - \frac{2}{d_1}\right)\frac{1}{n_1}$ sind, falls $l_1 = 1$ ist, und gleich $\frac{2}{d_1 n_1}$ sind, falls $k_1 = 1$ ist. In $E_1(\{k_j\}, \{l_j\})$ ist also wegen

$$\left(1 - \frac{2}{d_1}\right) > \frac{4}{d_1}, \quad \text{falls } d_1 > 2,$$

stets ein Intervall der Mindestlänge $\frac{1}{n_2}$ enthalten, wenn $k_1 = 1$ ist, und ist ein Intervall der Mindestlänge $\frac{2}{n_2}$ enthalten, wenn $l_1 = 1$ ist (bei der Induktion wird nur so viel ausgenützt werden). Ferner ist dieses Intervall offen für $l_1 = 1$ und abgeschlossen für $k_1 = 1$. Nehmen wir an, dies gilt bis I_m, d. h. I_m ist offen, falls m unter den l vorkommt, und besitzt dann eine Mindestlänge $\frac{2}{n_{m+1}}$, und ist abgeschlossen, falls m unter den k vorkommt, dann besitzt sie eine Mindestlänge $\frac{1}{n_{m+1}}$.

Zunächst sei angenommen, daß m den k angehört. Dann existiert nach der Induktionsvoraussetzung ein abgeschlossenes Intervall I_m von der Mindestlänge $\frac{1}{n_{m+1}}$, die $E_m(\{k_j\}, \{l_j\})$ angehört; dieses enthält (im Inneren oder an den Endpunkten) mindestens einen Punkt der Gestalt $\frac{r}{n_{m+1}}$; für eine Umgebung von der Mindestlänge $\frac{1}{d_{m+1}n_{m+1}}=\frac{1}{n_{m+2}}$ gilt dann (22), und diese Umgebung bildet wieder ein abgeschlossenes Intervall. Für den übrigen Teil des Intervalles I_m gilt (23). Der kann in höchstens zwei Teilintervalle zerfallen, die Länge von mindestens einem ist also größer als

$$\frac{1}{2}\left(1-\frac{2}{d_{m+1}}\right)\frac{1}{n_{m+1}}>\frac{2}{n_{m+2}},$$

dieses Intervall enthält also ein offenes Teilintervall mit einer größeren Länge als $\frac{2}{n_{m+2}}$. Damit ist die Induktion durchgeführt, falls m den k angehört; der entgegengesetzte Fall kann auf völlig analoge Weise erledigt werden.

Damit ist der Hilfssatz, also auch unser Satz 2 bewiesen.

BUDAPEST, FORSCHUNGSINSTITUT FÜR MATHEMATIK DER
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AN EXTREMAL PROBLEM IN THE THEORY OF INTERPOLATION

By

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1. Let the infinite triangular matrix

$$A = \begin{pmatrix} x_{11} & & & \\ x_{12} & x_{22} & & \\ \vdots & & & \\ x_{1n} & x_{2n} & \cdots & x_{nn} \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

be given, where for $n=1, 2, \ldots$ the inequality

$$1 \geqq x_{1n} > x_{2n} > \cdots > x_{nn} \geqq -1 \tag{1.1}$$

holds. Putting

$$\omega_n(x, A) = \prod_{j=1}^{n} (x - x_{jn}), \tag{1.2}$$

$$l_{jn}(x, A) = \frac{\omega_n(x, A)}{\omega_n'(x_{jn}, A)(x - x_{jn})}, \tag{1.3}$$

the polynomial

$$L_n(x, y_{1n}, \ldots, y_{nn}, A) = \sum_{j=1}^{n} y_{jn} l_{jn}(x, A), \tag{1.4}$$

the so-called n^{th} Lagrange interpolation polynomial belonging to A, is the only polynomial of degree $\leqq n-1$ having the value y_{jn} at $x = x_{jn}$ for $j = 1, 2, \ldots, n$. Particularly important is the case when the values y_{jn} are given by

$$y_{jn} = f(x_{jn}) \qquad (j = 1, 2, \ldots, n)$$

where $f(x)$ is a prescribed function continuous in $[-1, +1]$; in this case we shall denote the polynomial in (1.4) more simply by $L_n(x, f, A)$. From the classical investigations of G. FABER[1] and S. BERNSTEIN[2] it follows that *no* matrix A is "effective for the whole class C of functions continuous in

[1] G. FABER [5]. The numbers in brackets refer to the literature quoted at the end of the paper.

[2] S. BERNSTEIN [1].

$[-1, +1]$"; the latter even proved that for every A with (1. 1) there is an $f_0(x) \in C$ and a $-1 \leqq \xi_0 \leqq +1$ such that

$$\overline{\lim_{n\to\infty}} |L_n(\xi_0, f_0, A)| = +\infty,$$

in contrary to everything what was expected since NEWTON.

2. As FEJÉR discovered essentially in 1913, the situation changes completely if instead of the sequence of the Lagrange polynomials $L_n(x, f, A)$ one considers an appropriate special case of the general Hermite interpolation[3] (which HERMITE himself considered only from formal point of view). FEJÉR considered the polynomials $H_n(x, f, A)$ of degree $\leqq 2n-1$ uniquely determined by the requirements

$$\left.\begin{aligned} &H_n(x_{jn}, f, A) = f(x_{jn}), && (2.1)\\ &\left(\frac{dH_n(x, f, A)}{dx}\right)_{x=x_{jn}} = 0 && (2.2)\end{aligned}\right\} \quad (j = 1, 2, \ldots, n).$$

He proved that choosing e. g. for A the matrix P, the n^{th} row of which consists of the roots α_{jn} of the n^{th} Legendre polynomial

$$\{(x^2-1)^n\}^{(n)},$$

one has, whenever $f \in C$, the relation

$$\lim_{n\to\infty} H_n(x, f, P) = f(x)$$

for $-1 < x < +1$, but not necessarily[4] for $x = \pm 1$. Later he proved[5] that choosing as A the matrix T, the n^{th} row of which consists of the roots β_{jn} of the n^{th} Chebyshev polynomial $T_n(x)$ defined by

$$T_n(\cos\vartheta) = \cos n\vartheta, \tag{2.3}$$

the relation

$$\lim_{n\to\infty} H_n(x, f, T) = f(x) \tag{2.4}$$

[3] L. FEJÉR [6].

[4] As it was shown recently by E. EGERVÁRY and P. TURÁN [2] for the sequence of polynomials $H_n^*(x, f)$ of degree $\leqq 2n-3$, defined by

$$H_n^*(\alpha_{j,n-2}, f) = f(\alpha_{j,n-2}), \quad H_n^*(\pm 1, f) = f(\pm 1),$$

$$\left(\frac{dH_n^*(x, f)}{dx}\right)_{x=\alpha_{j,n-2}} = 0 \qquad (j = 1, 2, \ldots, n-2),$$

the relation

$$\lim_{n\to\infty} H_n^*(x, f) = f(x)$$

holds *uniformly for* $[-1, +1]$.

[5] L. FEJÉR [7].

holds uniformly for $[-1, +1]$. Here, generally, $H_n(x, f, A)$ stands for the polynomial of degree $\leqq 2n-1$ defined by

$$\left.\begin{aligned} &(2.5) \qquad H_n(x_{jn}, f, A) = f(x_{jn}), \\ &(2.6) \qquad \left(\frac{dH_n(x, f, A)}{dx}\right)_{x=x_{jn}} = y'_{jn} \end{aligned}\right\} \quad (j=1, 2, \ldots, n)$$

where the real numbers y'_{jn} are subject only to the restriction

$$\lim_{n\to\infty} \max_{j=1,\ldots,n} \frac{|y'_{jn}| \log n}{n} = 0. \tag{2.7}$$

3. The relation (2.4) is surprising owing to the great arbitrariness of the slopes y'_{jn}. This raises naturally the question that perhaps choosing another matrix A instead of T this arbitrariness of the slopes can be increased. To give a more exact form to this question we remark that, as easy to see,[6] everything depends upon the expression

$$M_n(A) \overset{\text{def}}{=} \max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| \tag{3.1}$$

where

$$\mathfrak{h}_{jn}(x, A) = \frac{\omega_n(x, A)^2}{\omega'_n(x_{jn}, A)^2 (x - x_{jn})}. \tag{3.2}$$

Hence it is natural to ask for the "optimal" matrix $A = A^*$ (which is not necessarily unique), i. e. for which

$$M_n(A) = \text{minimal} \tag{3.3}$$

for $n = 1, 2, \ldots$. Since, according to FEJÉR,[7] for arbitrarily small $\varepsilon > 0$ for $n > n_0(\varepsilon)$ the inequality

$$M_n(T) < \left(\frac{2}{\pi} + \varepsilon\right) \frac{\log n}{n} \tag{3.4}$$

holds, we certainly have, denoting[8]

$$\min_A M_n(A) = M_n(A^*) \overset{\text{def}}{=} g(n), \tag{3.5}$$

the inequality

$$\overline{\lim_{n\to\infty}} \frac{n}{\log n} g(n) \leqq \frac{2}{\pi}. \tag{3.6}$$

[6] L. FEJÉR [7].
[7] See L. FEJÉR [7] with a slightly different notation.
[8] It is easy to see that for fixed n the minimum exists.

Now we are going to prove

$$\varliminf_{n\to\infty} \frac{n}{\log n} g(n) \geqq \frac{2}{\pi}, \tag{3.7}$$

i. e.

$$\lim_{n\to\infty} \frac{n}{\log n} g(n) = \frac{2}{\pi}. \tag{3.8}$$

By (3.7) *our extremal problem is at least asymptotically solved and shown that the choice* $A = T$ *gives essentially the greatest freedom for the choice of the slopes* y'_{jn}. More exactly, we are going to prove the following theorem where c_1 (and later $c_2, c_3, \ldots$) denote positive numerical constants.

THEOREM I. *By whatever choice of the matrix A we have the inequality*

$$(M_n(A) \overset{\text{def}}{=}) \max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| \geqq \frac{2}{\pi n} (\log n - c_1 \log\log n).$$

It would be of interest to determine the exact value of $g(n)$, at least for small n's. A proof of the weaker inequality

$$g(n) \geqq c_2 \frac{\log n}{n} \tag{3.9}$$

could have been proved more briefly; we shall, however, omit this version. Probably also the inequality

$$\int_{-1}^{1} \left\{ \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| \right\} dx > c_3 \frac{\log n}{n} \tag{3.10}$$

holds or even the inequality

$$\sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| > c_4 \frac{\log n}{n} \tag{3.11}$$

in $[-1, +1]$ with the exception of a set with measure tending to 0 with $\frac{1}{n}$; we could not prove so far whether or not for all $-1 \leqq a < b \leqq 1$

$$\max_{a\leqq x\leqq b} \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| > \left(\frac{2}{\pi} - \varepsilon\right) \frac{\log n}{n} \tag{3.12}$$

holds for all $n > n_0(\varepsilon, a, b)$ (or even for $n > n_1(\varepsilon)$).

In our theorem the factor $\log\log n$ can perhaps be replaced by 1; a further refinement, enabling to prove that $g(n)$ is a *convex* function of n, seems to be very difficult.

Our method furnishes mutatis mutandis a proof for the inequality

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |l_{jn}(x, A)| \geqq \frac{2}{\pi} \log n - c_5 \log\log n \tag{3.13}$$

for all matrices A; a somewhat weaker inequality was proved in S. BERNSTEIN's paper [1]. The significance of (3. 13) is given, of course, by the fact that, in conjunction with the fact that for $n > n_1(\varepsilon)$

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |l_{jn}(x, T)| \leqq \left(\frac{2}{\pi} + \varepsilon\right) \log n,$$

it solves asymptotically the extremal problem to find the minimum of $\max\limits_{-1\leqq x\leqq +1} \sum\limits_{j=1}^{n} |l_{jn}(x, A)|$ when A varies. We shall sketch our proof for (3. 13) (Theorem II) and drop the formulation of problems analogous to (3. 10), (3. 11) and (3. 12) with $l_{jn}(x, A)$ instead of $\mathfrak{h}_{jn}(x, A)$.

Since in the proof of our theorem we are always dealing with a large but fixed n, for simplifying the notation we omit n from the indices. Hence for

$$1 \geqq x_1 > x_2 > \cdots > x_n \geqq -1,$$

$$\omega(x) = \prod_{j=1}^{n} (x - x_j), \quad l_j(x) = \frac{\omega(x)}{\omega'(x_j)(x - x_j)}$$

we have to prove that

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} \frac{\omega(x)^2}{\omega'(x_j)^2 |x - x_j|} = \max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| = \tag{3.14}$$

$$= \max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |x - x_j| \, l_j(x)^2 \geqq \frac{2}{\pi n} (\log n - c_1 \log\log n).$$

4. We shall need two lemmas.

LEMMA I. *If for a* $0 < b < \frac{1}{2}$ *and* $0 < \eta_1 < 1$ *and a rational polynomial* $J(x)$ *of degree* n *the inequalities*

$$|J(x)| \leqq M \quad \textit{for} \quad -1 \leqq x \leqq +1,$$
$$|J(x)| \leqq \eta_1 M \quad \textit{for} \quad -b \leqq x \leqq +b$$

hold, then for $0 < \eta_2 < \frac{1}{4}$ *and*

$$-(1-\eta_2) b \leqq x \leqq (1-\eta_2) b$$

the inequality

$$\left|\frac{dJ}{dx}\right| \leqq M \left\{ (1 + b^2) \eta_1 n + \frac{4}{\eta_2^2 b^2} \right\}$$

holds.

For the proof of this lemma we may suppose $M=1$, and consider the pure cosine polynomial

$$J(\cos\vartheta)=J_1(\vartheta). \tag{4.1}$$

We apply the well-known interpolation formula of M. RIESZ[9] which gives

$$\frac{dJ_1}{d\vartheta}=\frac{1}{2n}\sum_{j=1}^{n}J_1(\vartheta+\vartheta_j)\frac{(-1)^{j+1}}{1-\cos\vartheta_j}$$

where

$$\vartheta_j=\frac{(2j-1)\pi}{2n}.$$

Since our hypothesis amounts to

$$|J_1(\vartheta)|\leqq 1 \quad\text{for}\quad 0\leqq\vartheta\leqq\pi,$$
$$|J_1(\vartheta)|\leqq\eta_1 \quad\text{for}\quad \arccos b\leqq\vartheta\leqq\pi-\arccos b,$$

we get for

$$\arccos(1-\eta_2)b\leqq\vartheta\leqq\pi-\arccos(1-\eta_2)b$$

the estimation

$$\left|\frac{dJ_1}{d\vartheta}\right|\leqq\frac{\eta_1}{2n}\sum_{\arccos b\leqq\vartheta+\vartheta_j\leqq\pi-\arccos b}\frac{1}{1-\cos\vartheta_j}+ \tag{4.2}$$
$$+\frac{\eta_1}{2n}\sum_{\pi+\arccos b\leqq\vartheta+\vartheta_j\leqq 2\pi-\arccos b}\frac{1}{1-\cos\vartheta_j}+\frac{1}{2n}\sum_j{}'\frac{1}{1-\cos\vartheta_j}$$

where the last summation is extended to the ϑ_j's not contained in the previous two. Since

$$\frac{1}{2n}\sum_{j=1}^{n}\frac{1}{1-\cos\vartheta_j}=n,$$

we get

$$\left|\frac{dJ_1}{d\vartheta}\right|\leqq\eta_1 n+\frac{1}{1-\cos(\arccos(1-\eta_2)b-\arccos b)}=\eta_1 n+$$
$$+\frac{1}{1-(1-\eta_2)b^2-\sqrt{1-(1-\eta_2)^2b^2}\cdot\sqrt{1-b^2}}=$$
$$=\eta_1 n+\frac{\{1-(1-\eta_2)b^2\}+\sqrt{1-(1-\eta_2)^2b^2}\cdot\sqrt{1-b^2}}{\{1-(1-\eta_2)b^2\}^2-\{1-(1-\eta_2)^2b^2\}(1-b^2)}<$$
$$<\eta_1 n+\frac{2}{1+(1-\eta_2)^2-2(1-\eta_2)}\frac{1}{b^2}=\eta_1 n+\frac{2}{\eta_2^2}\frac{1}{b^2}.$$

[9] M. RIESZ [9].

Hence for $-(1-\eta_2)b \leqq x \leqq (1-\eta_2)b$

$$\left|\frac{dJ(x)}{dx}\right| = \left|\frac{dJ_1(\vartheta)}{d\vartheta}\right| \frac{1}{\sqrt{1-x^2}} \leqq \left(\eta_1 n + \frac{2}{\eta_2^2 b^2}\right)\frac{1}{\sqrt{1-b^2}} < \eta_1(1+b^2)n + \frac{4}{\eta_2^2 b^2},$$

indeed.

LEMMA II. *Let $J_2(x)$ be a rational polynomial of degree $\leqq m$ which assumes its absolute maximum μ with respect to $[-1, +1]$ at $x=\bar{\xi}$. Then there is an interval I in $[-1, +1]$ of length $\frac{1}{2m^2}$ such that one of its endpoints is $\bar{\xi}$ and in which the inequality*

$$|J_2(x)| \geqq \frac{1}{2}\mu$$

holds.

We choose, namely, as I that one among the intervals

$$\left[\bar{\xi}, \bar{\xi}+\frac{1}{2m^2}\right], \quad \left[\bar{\xi}-\frac{1}{2m^2}, \bar{\xi}\right]$$

which lies in $[-1, +1]$. We may suppose the first. Then using MARKOV's classical theorem[10] we get in I

$$|J_2(x)| = \left|J_2(\bar{\xi}) + \int_{\bar{\xi}}^{x} J_2'(t)\,dt\right| \geqq |J_2(\bar{\xi})| - \int_{\bar{\xi}}^{\bar{\xi}+\frac{1}{2m^2}} \mu m^2\,dt = \mu - \frac{\mu}{2} = \frac{\mu}{2},$$

indeed.

5. We shall employ the following notations. Let

$$M \overset{\text{def}}{=} \max_{-1\leqq x\leqq +1} |\omega(x)|, \tag{5.1}$$

and this should be attained here for $x=\xi$, say. We shall consider the intervals

$$d_\nu:\ -\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^\nu \leqq x \leqq \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^\nu \tag{5.2}$$

and

$$d'_\nu:\ -\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^\nu\left(1-\frac{1}{\log^3 n}\right) \leqq \tag{5.3}$$

$$\leqq x \leqq \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^\nu\left(1-\frac{1}{\log^3 n}\right)$$

for

$$\nu = 0, 1, \ldots, [\log^2 n] \overset{\text{def}}{=} R. \tag{5.4}$$

[10] See MARKOV [8].

We shall use $d'_{\nu+1}-d'_\nu$ and $\bar{d}_\nu$ (the complementary of d_ν with respect to $[-1, +1]$) in the usual sense. We shall denote by ξ_ν one of the values x in d_ν with

$$|\omega(\xi_\nu)| \overset{\text{def}}{=} \max_{x\in d_\nu} |\omega(x)| \overset{\text{def}}{=} M_\nu. \tag{5.5}$$

The intervals d_ν are for $n>c_6$ in $[-1, +1]$ and thus

$$M_0 \leqq M_1 \leqq \cdots \leqq M_R \leqq M. \tag{5.6}$$

6. The proof of our Theorem I is split into three cases.

Case I. There is an index $1\leqq k_0 \leqq n$ and a $-1\leqq \xi^* \leqq +1$ such that

$$\max_{-1\leqq x\leqq +1} |l_{k_0}(x)| = |l_{k_0}(\xi^*)| \geqq n^3. \tag{6.1}$$

Applying Lemma II to $l_{k_0}(x)$ we obtain the existence of an interval I in $[-1, +1]$ of length $>\frac{1}{2n^2}$ such that in I the inequality

$$|l_{k_0}(x)| \geqq \frac{1}{2} n^3 \tag{6.2}$$

holds. We choose in I a ξ^{**} as follows. If x_{k_0} is not in I, then let ξ^{**} be the middle-point of I, say; then

$$|\xi^{**}-x_{k_0}| \geqq \frac{1}{4n^2}. \tag{6.3}$$

If x_{k_0} is in I, then ξ^{**} can be chosen in I so that (6.3) holds again. Then we have

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_k(x)| \geqq \sum_{j=1}^{n} |\mathfrak{h}_j(\xi^{**})| \geqq |\mathfrak{h}_{k_0}(\xi^{**})| =$$

$$= |\xi^{**}-x_{k_0}|\, l_{k_0}(\xi^{**})^2 \geqq \frac{1}{4n^2}\frac{1}{4} n^6 > \frac{2}{\pi}\frac{\log n}{n}$$

for $n>c_7$. Hence in this case our theorem is proved and we may suppose in the sequel the inequality

$$\max_{-1\leqq x\leqq +1} |l_k(x)| < n^3 \tag{6.4}$$

for $k=1, 2, \ldots, n$. This last inequality will be used only in the form that it implies[11] upon the x_j's that writing them in the form

$$x_j = \cos \vartheta_j \qquad (0\leqq \vartheta_j \leqq \pi;\ j=1, 2, \ldots, n)$$

[11] See ERDŐS [3]. His proof is an improvement of that contained in ERDŐS—TURÁN [4], esp. p. 548—552.

the ϑ_j's are uniformly distributed in the sense that for $0 \leqq \alpha < \beta < \pi$

$$\left| \sum_{\alpha \leqq \vartheta_j \leqq \beta}{}_j 1 - \frac{\beta - \alpha}{\pi} n \right| < c_8 \log^2 n. \tag{6. 5}$$

7. *Case II.* With the notation of **5** we suppose the inequality

$$M_0 < \frac{M}{\log^2 n} \tag{7. 1}$$

holds.

We apply Lemma I with

$$J(x) = \omega(x), \quad b = \frac{1}{\log n},$$

$$\eta_1 = \frac{1}{\log^2 n}, \quad \eta_2 = \frac{1}{\log^3 n};$$

the assumption (7. 1) assures the applicability of this lemma. This gives for $x \in d_0'$ the estimation

$$|\omega'(x)| \leqq M \left\{ \left(1 + \frac{1}{\log^2 n}\right) \frac{n}{\log^2 n} + 4 \log^8 n \right\} < M \frac{2n}{\log^2 n}$$

roughly, for $n > c_9$. Hence we obtain

$$\max_{-1 \leqq x \leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| \geqq \sum_{j=1}^{n} |\mathfrak{h}_j(\xi)| \geqq \frac{1}{2} \sum_{j=1}^{n} \frac{\omega(\xi)^2}{\omega'(x_j)^2} \geqq \frac{M^2}{2} \sum_{x_j \in d_0'}{}_j \frac{1}{\omega'(x_j)^2} \geqq \frac{\log^4 n}{8n^2} \sum_{x_j \in d_0'}{}_j 1.$$

Applying (6. 5), the last sum is (roughly) for $n > c_{10}$

$$> \frac{1}{4} \frac{n}{\log n},$$

i. e.

$$\max_{-1 \leqq x \leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| \geqq \frac{\log^3 n}{32 n} > \frac{2}{\pi} \frac{\log n}{n}$$

for $n > c_{11}$. Hence also in this case our theorem is proved and in the sequel we may suppose (Case III)

a) the uniformly dense distribution in (6. 5),

b) the inequality

$$M_0 \geqq \frac{M}{\log^2 n}. \tag{7. 2}$$

8. *Case III (and the last).* First we assert that there is an index ν_0 with $0 \leqq \nu_0 \leqq [\log^2 n] = R$ and

$$M_{\nu_0 + 1} \leqq M_{\nu_0} \left(1 + \frac{1}{\log n}\right). \tag{8. 1}$$

For if not, then we should have for all these ν's

$$M_{\nu+1} > M_\nu\left(1+\frac{1}{\log n}\right),$$

i. e. from (5. 6), (7. 2) for $n > c_{12}$ by multiplying we get

$$M \geqq M_R > M_0\left(1+\frac{1}{\log n}\right)^R > M_0\sqrt{n} > \frac{\sqrt{n}}{\log^2 n} M > 2M$$

which is false. Hence (8. 1) is true. With this ν_0 we have, with the notations of **5**,

$$\text{(8. 2)} \qquad \max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| \geqq \sum_{j=1}^{n} |\mathfrak{h}_j(\xi_{\nu_0})| =$$

$$= \sum_{x_j\in d'_{\nu_0}} {}_j + \sum_{x_j\in d'_{\nu_0+1}-d'_{\nu_0}} {}_j + \sum_{x_j\in \overline{d'_{\nu_0+1}}} {}_j \overset{\text{def}}{=} S_1+S_2+S_3 \geqq S_1+S_2.$$

To obtain a lower bound for S_1 we use Lemma I with $n > c_{13}$ and

$$\eta_1 = \frac{M_{\nu_0}}{M}, \quad \eta_2 = \frac{1}{\log^3 n},$$

$$b = \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\left(>\frac{1}{\log n}\right).$$

This gives for $x_j \in d'_{\nu_0}$ owing to (7. 2) and (5. 6) for $n > c_{14}$

$$|\omega'(x_j)| \leqq M\left\{\left(1+\frac{25}{\log^2 n}\right)\frac{M_{\nu_0}}{M} n + 4\log^8 n\right\} <$$

$$< M\left\{\left(1+\frac{25}{\log^2 n}\right)\frac{M_{\nu_0}}{M} n + \left(\frac{M_{\nu_0}}{M}\log^2 n\right) 4\log^8 n\right\} =$$

$$= M_{\nu_0}\left\{\left(1+\frac{25}{\log^2 n}\right) n + 4\log^{10} n\right\} < M_{\nu_0}\left(1+\frac{30}{\log^2 n}\right) n,$$

and hence

$$\text{(8. 3)} \qquad S_1 = \sum_{x_j\in d'_{\nu_0}} {}_j \frac{M_{\nu_0}^2}{\omega'(x_j)^2\,|\xi_{\nu_0}-x_j|} > \frac{1}{\left(1+\frac{30}{\log^2 n}\right)^2 n^2} \sum_{x_j\in d'_{\nu_0}} {}_j \frac{1}{|\xi_{\nu_0}-x_j|}.$$

In order to obtain a lower bound for S_2 we apply again Lemma I with

$$\eta_1 = \frac{M_{\nu_0+1}}{M}, \quad \eta_2 = \frac{1}{\log^3 n},$$

$$b = \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0+1}\left(>\frac{1}{\log n}\right).$$

This gives for $x_j \in d'_{\nu_0+1}$, as before,

$$|\omega'(x_j)| \leqq M_{\nu_0+1}\left(1+\frac{30}{\log^2 n}\right)n,$$

i. e. by using (8. 1)

$$S_2 = \sum_{x_j \in d'_{\nu_0+1}-d'_{\nu_0}}^{j} \frac{M_{\nu_0}^2}{\omega'(x_j)^2\,|\xi_{\nu_0}-x_j|} >$$

$$> \frac{M_{\nu_0}^2}{M_{\nu_0+1}^2} \frac{1}{\left(1+\frac{30}{\log^2 n}\right)^2} \frac{1}{n^2} \sum_{x_j \in d'_{\nu_0+1}-d'_{\nu_0}}^{j} \frac{1}{|\xi_{\nu_0}-x_j|} >$$

$$> \frac{1}{\left(1+\frac{30}{\log^2 n}\right)^4} \frac{1}{n^2} \sum_{x_j \in d'_{\nu_0+1}-d'_{\nu_0}}^{j} \frac{1}{|\xi_{\nu_0}-x_j|}.$$

This and (8. 3) give together for $n > c_{15}$

$$S_1 + S_2 > \frac{1}{\left(1+\frac{30}{\log^2 n}\right)^4 n^2} \sum_{x_j \in d'_{\nu_0+1}}^{j} \frac{1}{|\xi_{\nu_0}-x_j|}. \tag{8. 4}$$

9. Now we use the full force of the uniform distribution in (6. 5). To do so we write first

$$\xi_{\nu_0} = \cos \Theta_{\nu_0}$$

and have

$$-\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0} \leqq \cos \Theta_{\nu_0} \leqq \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0},$$

i. e.

$$\left|\frac{\pi}{2} - \Theta_{\nu_0}\right| < \arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\right\}; \tag{9. 1}$$

we remark further that the x_j's in (8. 4) are exactly the ϑ_j's with

$$\left|\frac{\pi}{2} - \vartheta_j\right| \leqq \arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0+1}\left(1-\frac{1}{\log^3 n}\right)\right\} \stackrel{\text{def}}{=} \alpha. \tag{9. 2}$$

Since

$$\frac{1}{|\xi_{\nu_0}-x_j|} = \frac{1}{|\cos \Theta_{\nu_0} - \cos \vartheta_j|} \geqq \frac{1}{|\Theta_{\nu_0}-\vartheta_j|},$$

we have in the remaining Case III

$$\max_{-1 \leqq x \leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| > \left(1-\frac{30}{\log^2 n}\right)^4 \frac{1}{n^2} \sum_{\left|\frac{\pi}{2}-\vartheta_j\right| \leqq \alpha}^{j} \frac{1}{|\Theta_{\nu_0}-\vartheta_j|}. \tag{9. 3}$$

Since from (9. 1) we have

$$\left|\Theta_{\nu_0}-\left(\frac{\pi}{2}\pm\alpha\right)\right|\geqq \arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0+1}\left(1-\frac{1}{\log^3 n}\right)\right\}-$$

$$-\arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^3 n}\right)^{\nu_0}\right\}>\arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\left(1+\frac{1}{2\log^2 n}\right)\right\}-$$

$$-\arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\right\}>\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\frac{1}{2\log^2 n}>\frac{1}{2\log^3 n},$$

the range of summation in (9. 3) is not increased by replacing the original one by

$$|\Theta_{\nu_0}-\vartheta_j|\leqq\frac{1}{2\log^3 n}.$$

Denoting the arcs

$$\Theta_{\nu_0}-(\varkappa+1)\frac{\log^5 n}{n}\leqq\vartheta<\Theta_{\nu_0}-\varkappa\frac{\log^5 n}{n}\qquad\left(\varkappa=0,1,\ldots,\left[\frac{3}{2}\frac{n}{\log^8 n}\right]\right)$$

and

$$\Theta_{\nu_0}+\lambda\frac{\log^5 n}{n}<\vartheta\leqq\Theta_{\nu_0}+(\lambda+1)\frac{\log^5 n}{n}\qquad\left(\lambda=0,1,\ldots,\left[\frac{3}{2}\frac{n}{\log^8 n}\right]\right)$$

by $U_\varkappa$ and V_λ, respectively, (6. 5) results

$$\sum_{\vartheta_j\in V_\lambda}{}^{j}\frac{1}{|\Theta_{\nu_0}-\vartheta_j|}\geqq\frac{n}{\log^5 n}\frac{1}{\lambda+1}\sum_{\vartheta_j\in V_\lambda}{}^{j}1>$$

$$>\frac{n}{\log^5 n}\frac{1}{(\lambda+1)}\left\{\frac{1}{\pi}\log^5 n-c_8\log^2 n\right\}=\frac{n}{\pi}\frac{1}{\lambda+1}\left\{1-\frac{\pi c_8}{\log^3 n}\right\},$$

and similarly for

$$\sum_{\vartheta_j\in U_\varkappa}{}^{j}\frac{1}{|\Theta_{\nu_0}-\vartheta_i|}.$$

Hence from (9. 3) in Case III

$$\max_{-1\leqq x\leqq+1}\sum_{j=1}^{n}|\mathfrak{h}_j(x)|>\left(1-\frac{30}{\log^2 n}\right)^4\frac{2}{\pi n}\left(1-\frac{\pi c_8}{\log^3 n}\right)\sum_{0\leqq\lambda\leqq\left[\frac{n}{\log^8 n}\right]}{}^{\lambda}\frac{1}{\lambda+1}>$$

$$>\frac{2}{\pi}\frac{\log n}{n}-c_{16}\frac{\log\log n}{n}$$

for $n>c_{17}$. Q. e. d.

10. As told we shall sketch the proof of

THEOREM II. *For* $n > c_{18}$ *we have*

$$\max_{-1 \leqq x \leqq +1} \sum_{\nu=1}^{n} |l_\nu(x)| > \frac{2}{\pi} \log n - c_{19} \log \log n.$$

PROOF. Without loss of generality we may suppose the inequality

$$|l_\nu(x)| \leqq \log n \tag{10. 1}$$

for $-1 \leqq x \leqq +1$ and $\nu = 1, 2, \ldots, n$, from which the equidistribution (6. 5) follows at once. So we shall have only two cases (keeping the previous notations).

Case I.

$$M_0 < \frac{1}{20 \log^2 n} M. \tag{10. 2}$$

We apply Lemma I with

$$J(x) = \omega(x), \quad b = \frac{1}{\log n},$$

$$\eta_1 = \frac{1}{20 \log^2 n}, \quad \eta_2 = \frac{1}{\log^3 n};$$

again (10. 2) assures the applicability of this lemma. This gives for $x \in d_0'$ as in **7** for $n > c_{20}$

$$|\omega'(x)| < \frac{M}{10} \frac{n}{\log^2 n}$$

and

$$\max_{-1 \leqq x \leqq +1} \sum_{\nu=1}^{n} |l_\nu(x)| \geqq \sum_{\nu=1}^{n} |l_\nu(\xi)| \geqq \frac{M}{2} \sum_{x_j \in d_0'}^{j} \frac{1}{|\omega'(x_j)|} > 5 \frac{\log^2 n}{n} \sum_{x_j \in d_0'}^{j} 1 > \frac{5}{4} \log n$$

using (6. 5) roughly.

Case II. We may suppose

$$M_0 \geqq \frac{M}{20 \log^2 n}. \tag{10. 3}$$

Again we have for $n > c_{21}$ an index ν_1 with $0 \leqq \nu_1 \leqq R$ and

$$M_{\nu_1+1} \leqq M_{\nu_1} \left(1 + \frac{1}{\log n}\right); \tag{10. 4}$$

for if not, we should have

$$M > M_0 \sqrt{n} > M \frac{\sqrt{n}}{20 \log^2 n} > 2M$$

which is false. Again

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |l_j(x)| \geqq \sum_{j=1}^{n} |l_j(\xi_{\nu_1})| \geqq \sum_{x_j\in d'_{\nu_1}} + \sum_{x_j\in d'_{\nu_1+1}-d'_{\nu_1}} \stackrel{\text{def}}{=} S'_1+S'_2.$$

To obtain a lower bound for S'_1 we use Lemma I for $n>c_{22}$ with

$$\eta_1=\frac{M_{\nu_1}}{M}, \quad \eta_2=\frac{1}{\log^3 n},$$

$$b=\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_1}\left(>\frac{1}{\log n}\right).$$

This gives for $x_j\in d'_{\nu_1}$, using also (10. 3), for $n>c_{23}$

$$|\omega'(x_j)|\leqq M\left\{\left(1+\frac{25}{\log^2 n}\right)\frac{M_{\nu_1}}{M}n+4\log^8 n\right\}<$$

$$<M\left\{\left(1+\frac{25}{\log^2 n}\right)\frac{M_{\nu_1}}{M}n+\left(\frac{M_{\nu_1}}{M}20\log^2 n\right)4\log^8 n\right\}<M_{\nu_1}\left(1+\frac{30}{\log^2 n}\right)n.$$

The further part of the proof runs exactly after the pattern of Theorem I and can be dropped.

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PROBLEMS AND RESULTS ON THE THEORY OF INTERPOLATION. II

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Let $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$ be n arbitrary points in the interval $(-1, +1)$. $\omega_n(x) = \prod_{i=1}^{n} (x - x_i)$, $l_k(x) = \omega_n(x)/\omega_n'(x_k)(x - x_k)$. It is well known that the sum $\sum_{k=1}^{n} |l_k(x)|$ plays a decisive role in the convergence and divergence properties of the Lagrange interpolation polynomials. FABER [1] proved that $\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)|$ tends to infinity with n, in fact he proved that

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > \frac{1}{12} \log n. \tag{1}$$

Later FEJÉR [2] obtained a very simple proof for (1). The problem of determining the n points for which $\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)|$ is minimal is unsolved up to the present. BERNSTEIN [3] asserts that for every $\varepsilon > 0$, if $n > n_0$,

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > (1 - \varepsilon) \frac{2}{\pi} \log n. \tag{2}$$

BERNSTEIN in his important paper proved (2) in full detail for trigonometric interpolation. He states that (2) for interpolation in $(-1, +1)$ is a simple consequence of this result. I was not able to reconstruct the proof. However, we proved with TURÁN [4] that (2) is true, even if the right side is replaced by $\frac{2}{\pi} \log n - c \log\log n$; here and throughout this paper $c, c_1, c_2, \ldots$ will denote positive absolute constants.

The main task of the present paper is the proof of the following

THEOREM 1. *Let* $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$. *Then*

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > \frac{2}{\pi} \log n - c_1.$$

This result can not be improved very much, since it is known that for the roots of the n^{th} Chebyshev polynomial $T_n(x)$

$$\max_{-1\leqq x\leqq 1} \sum_{k=1}^{n} |l_k(x)| < \frac{2}{\pi} \log n + c_2.$$

In fact, it is known and can be shown by a simple calculation that if $y_1 < y_2 < \cdots < y_n$ are the roots of $T_n(x)$, then

$$\frac{2}{\pi}\log n - c_2 < \max_{y_i < x < y_{i+1}} \sum_{k=1}^{n} |l_k(x)| < \frac{2}{\pi}\log n + c_2.$$

Let $\begin{matrix} x_1^{(1)} & \\ x_1^{(2)} & x_2^{(2)} \\ \cdots & \cdots \end{matrix}$ be a triangular matrix called point group in the theory of interpolation, $-1 \leqq x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} \leqq 1$. BERNSTEIN [3] proved that there exists an x_0 $(-1 < x_0 < 1)$ so that

$$\overline{\lim} \sum_{k=1}^{n} |l_k(x_0)| = \infty.$$

More precisely, he proved that for every fixed $-1 \leqq a < b \leqq 1$

$$\max_{a<x<b} \sum_{k=1}^{n} |l_k(x)| > \left(\frac{1}{4} - \varepsilon\right) \log n \tag{3}$$

for $n > n_0(\varepsilon, a, b)$. I think that in (3) $\frac{1}{4}$ can be replaced by $\frac{2}{\pi}$, but I have not been able to prove this.

In my paper [5] I stated that I can prove that there exists an x_0 so that for infinitely many n

$$\sum_{k=1}^{n} |l_k(x_0)| > \frac{2}{\pi} \log n - c. \tag{4}$$

(4) is quite possibly true, but unfortunately I am very far from being able to prove it.

To prove our Theorem we first need some lemmas.

LEMMA 1. *Let* $\cos\theta_i = y_i$ $(1 \leqq i \leqq n)$ *be the roots of the* n^{th} *Chebyshev polynomial* $T_n(x)$. *Then for every* $-1 \leqq x \leqq 1$ *and* $t > c_3$

$$\frac{1}{n} \sum\nolimits_t \left| \frac{(1-y_i^2)^{\frac{1}{2}}}{x - y_i} \right| > \frac{2}{\pi} \log n - c_4 \log t,$$

where Σ_t *denotes that the summation is extended only over those* y_i*'s for which* $|\theta - \theta_i| > t\pi/n$, $\cos\theta = x$.

The proof of Lemma 1 is by simple computation and is left to the reader.

$\cos\vartheta_0 = x_0$ will denote the point in $(-1, +1)$ where $|\omega_n(x)|$ assumes its absolute maximum. $\bar{I}_t$ will denote the intersection with $(0, \pi)$ of an interval of length $t\pi/n$, one endpoint of which is ϑ_0, I_t will be the interval in $(-1, +1)$ obtained from $\bar{I}_t$ by the mapping $\cos\vartheta = x$. There are two intervals I_t, one to the right, the other to the left of x_0.

LEMMA 2. *Assume that there exists a $t > c_3$ so that for every $t' \geqq t$ every interval $I_{t'}$ contains more than $t'\left(1-\frac{1}{(\log t')^2}\right)$ x_i's. Then*

$$\frac{1}{n}\sum_{i=1}^{n}\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0-x_i}\right| > \frac{2}{\pi}\log n - c_5\log t.$$

The term $|(1-x_i^2)^{\frac{1}{2}}|$ is really understood to mean $\max\left(|(1-x_i^2)^{\frac{1}{2}}|, \frac{1}{n}\right)$, to save space I will always replace this by $|(1-x_i^2)^{\frac{1}{2}}|$.

Let y_i be such that there are k y's in the interval (x_0, y_i), and let $x_{i'}$ be such that there are k x's in $(x_0, x_{i'})$. Clearly $\theta_i - \theta_0 = \dfrac{k\pi + O(1)}{n}$ and by our condition on the x's

$$\vartheta_{i'} - \vartheta_0 < \frac{k\pi}{n} + \frac{c_6 k\pi}{n(\log k)^2} + \frac{t\pi}{n} < \frac{k\pi}{n} + \frac{c_7 k\pi}{n(\log k)^2} \tag{5}$$

for $k > t^2$. From (5) we obtain by a simple trigonometrical calculation for $k > t^2$

$$\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0-x_i}\right| - \left|\frac{(1-y_i^2)^{\frac{1}{2}}}{y_0-y_i}\right| > -\frac{c_8}{k(\log k)^2}. \tag{6}$$

Lemma 2 immediately follows from (6) and Lemma 1.

LEMMA 3. *Assume that the x_i's and x_0 have the same properties as in Lemma 2 and the further property that for some $t' > t$ there is an $I_{t'}$ which contains more than t'^3 x_i's. Then if $t > c_3$,*

$$\sum = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0-x_i}\right| > \frac{2}{\pi}\log n.$$

Let t^* be the greatest t' for which an interval I_{t^*} contains t^{*3} x's. Write

$$\sum = \sum{}' + \sum{}_{t^*}$$

where in Σ' $|\vartheta_0 - \vartheta_i| \leqq \dfrac{t^*\pi}{n}$ and in Σ_{t^*} $|\vartheta_i - \vartheta_0| > \dfrac{t^*\pi}{n}$.

As in the proof of Lemma 2 we can show that

$$\sum_{t^*} > \frac{2}{\pi} \log n - c_9 \log t^*. \tag{7}$$

A simple trigonometrical computation shows that for the x_i's in Σ' $\left(\text{here } |\vartheta_i - \vartheta_0| \leqq \frac{t^* \pi}{n} \text{ and by our remark } |(1-x_i^2)^{\frac{1}{2}}| \geqq \frac{1}{n}\right)$

$$\frac{1}{n}\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0 - x_i}\right| > \frac{c_{10}}{t^{*2}}.$$

Thus, since there are at least t^{*3} summands in Σ', we have

$$\Sigma' > ct'. \tag{8}$$

(7) and (8) imply Lemma 3 for sufficiently large $t > c_3$.

LEMMA 4. *Let* $\cos\lambda_0 = x_0$ *be any point in* $(-1, +1)$. *There exists a polynomial* $F_r(x)$ *of degree* r *for which* $F_r(z_0) = 1$ *and*

$$\left| F_r\left[\cos\left(\lambda_0 + s\frac{\pi}{n}\right)\right]\right| < \frac{c_{11}}{|s|}$$

if $\lambda_0 + \frac{s\pi}{n}$ *is in* $(0, \pi)$.

Lemma 4 is well known [6].

LEMMA 5. *Let* $g_m(x)$ *be any polynomial of degree* m, *assume that it assumes its absolute maximum in* $(-1, +1)$ *at* $\cos\lambda_0 = z_0$. *Then if* $\cos\lambda_i = z_i$ *is any root of* $g_m(x)$, *we have*

$$|\lambda_0 - \lambda_i| \geqq \frac{\pi}{2m},$$

equality only holds if $g_m(x) = T_m(x)$.

This is a theorem of M. RIESZ [7].

LEMMA 6. *Assume that the* x_i*'s are such that there is a* $t > c_{12}$ *so that at least one of the intervals* I_t *contains fewer than* $t\left(1 - \frac{1}{(\log t)^2}\right)$ x_i*'s, and that for* $t' \geqq t$ *the intervals* $I_{t'}$ *contain not more than* t'^3 x_i*'s. Then*

$$\max_{x_k \subset I_t} \max_{x \text{ in } J_t} |l_k(x)| > t$$

where by J_t $(J_t \subset I_t)$ *we denote the interval*

$$J_t = \left\{ \cos\left(\vartheta_0 + \frac{t\pi}{n(\log t)^3}\right), \cos\left(\vartheta_0 + \frac{t\pi}{n} - \frac{t\pi}{n(\log t)^3}\right)\right\}.$$

Lemma 6 is very far from being best-possible, the conditions could be weakened and the conclusions strengthened, but it will suffice for our purpose in its present form. The proof of Lemma 6 is the most difficult part of the paper [8].

Let $g(x)$ be a polynomial whose roots in I_t coincide with those of $\omega_n(x)=\prod_{i=1}^{n}(x-x_i)$ and outside of J_t they coincide with the roots of the m^{th} Chebyshev polynomial $T_m(x)$, $m=\left[n\left(1-\frac{1}{(\log t)^3}\right)\right]$. By our assumptions the degree of $g(x)$ is less than

$$t-\frac{t}{(\log t)^2}+m-t\left(1-\frac{2}{(\log t)^3}\right)<m \tag{9}$$

for $t>c_{12}$ (i.e. the degree of $g_m(x)$ equals the number of x_i in I_t plus m minus the number of roots of $T_m(x)$ in J_t).

From Lemma 5 and (9) it follows that $g(x)$ must assume its absolute maximum for $(-1,+1)$ in J_t at the point $\cos\lambda_0=z_0$, say.

Denote by $I_t^{(l)}$ $(l=1,2,\ldots)$ the intersection with $(-1,+1)$ of the intervals

$$\left\{\cos\left(\vartheta_0+\frac{2^{l-1}t\pi}{n}\right),\ \cos\left(\vartheta_0+\frac{2^{l}t\pi}{n}\right)\right\} \tag{10}$$

and

$$\left\{\cos\left(\vartheta_0-\frac{(2^{l}-1)t\pi}{n}\right),\ \cos\left(\vartheta_0-\frac{(2^{l-1}-1)t\pi}{n}\right)\right\}.$$

We now apply Lemma 4 with $r=\left[\frac{n(\log t)^4}{t}\right]$. Since $\cos\lambda_0=z_0$ is in J_t and the distance of the endpoints of $\bar{J}_t$ from the endpoints of $\bar{I}_t$ (in ϑ) is $\frac{t\pi}{n(\log t)^3}$, we obtain from Lemma 4 by a simple computation that for the x's in $I_t^{(l)}$

$$|F_r(x)|<\frac{1}{2^l} \tag{11}$$

for sufficiently large t (i.e. the s in Lemma 4 is for $l=1$ not less than $\log t$ [z_0 is in J_t] and for $l>1$ it is not less than $2^{l-1}\log t$).

Consider now

$$G(x)=Ag(x)\,(F_r(x))^{[t/(\log t)^8]} \tag{12}$$

where A is chosen so that $G(z_0)=1$. The degree of $G(x)$ is not greater than $\left(m=\left[n\left(1-\frac{t}{(\log t)^3}\right)\right]\right)$

$$n-\frac{n}{(\log t)^3}+\frac{t}{(\log t)^8}\frac{n(\log t)^4}{t}<n.$$

Thus by the Lagrange interpolation formula (taken on $x_1, x_2, \ldots, x_n$) we have by (12)

$$1=G(z_0)=\sum_{i=1}^{n} G(x_i)\, l_i(z_0). \tag{13}$$

For the x_i's in I_t $G(x_i)=0$. Thus we can write (13) as

$$1=\sum_{l=1}^{\infty}\sum{}^{(l)} G(x_i)\, l_i(z_0) \tag{14}$$

where in $\Sigma^{(l)}$ the summation is extended over the x_i's in $I_t^{(l)}$. The summation in (14) clearly has to be extended only over a finite number of l's.

Since $|g(z_0)|\geqq|g(x)|$ for $-1\leqq x\leqq 1$ and $F_r(z_0)=1$, we obtain from (11) and (12) that

$$|G(x_i)|<\left(\frac{1}{2^l}\right)^{[t/\log t)^8]} \quad \text{for the } x_i\text{'s in } I_t^{(l)}. \tag{15}$$

Assume now that our Lemma is false. Then for all $i\not\subset I_t$

$$|l_i(z_0)|\leqq t. \tag{16}$$

Further by the assumptions of our Lemma the number of the x_i's in $I_t^{(l)}$ is not greater than $2^{3l+1}t^3$ (since $I_t^{(l)}$ is contained in the union of the two intervals $I_{2^l t}$). Thus, finally, we obtain from (14), (15) and (16) that

$$1<t^4\sum_{l=1}^{\infty}2^{3l+1}\left(\frac{1}{2^l}\right)^{[t/(\log t)^8]}. \tag{17}$$

The terms of the series (17) drop faster than a geometric series of quotient $\frac{1}{2}$, thus (17) implies

$$1<32t^4\left(\frac{1}{2}\right)^{[t/(\log t)^8]}$$

which is clearly false for $t>c_{12}$. This contradiction proves the Lemma.

Now we are ready to prove our Theorem. In fact, we shall show that if x_0 is the place in $(-1,+1)$ where $\omega_n(x)$ assumes its absolute maximum, then

$$\sum_{k=1}^{n}|l_k(x_0)|>\frac{2}{\pi}\log n-c_1 \tag{18}$$

for sufficiently large c_1. We can clearly assume $\omega_n(x_0) = 1$ (replacing $\omega_n(x)$ by $c\omega_n(x)$), and thus by the classical theorem of Bernstein

(19) $$|\omega_n'(x_k)| \leqq \min\left(n^2, \frac{n}{|1-x_k^2|^{\frac{1}{2}}}\right).$$

Thus from (19)

(20) $$\sum_{k=1}^{n} |l_k(x_0)| \geqq \frac{1}{n} \sum_{k=1}^{n} \left| \frac{(1-x_k^2)^{\frac{1}{2}}}{x_0 - x_k} \right|.$$

Let the constant c_{12} be sufficiently large. If for every $t > c_{12}$ every I_t contains more than $t\left(1 - \frac{1}{(\log t)^2}\right)$ x's, then our Theorem follows from (20) and Lemma 2. Assume next that there exists a $t > c_{12}$ for which I_t contains not more than $t\left(1 - \frac{1}{(\log t)^2}\right)$ x's, and let t_0 be the largest such t. Assume first that there exists a $t' \geqq t_0$ for which $I_{t'}$ contains more than t'^3 x's, then our Theorem follows from (20) and Lemma 3. If no such t' exists, consider the largest interval I_{t_0} which contains not more than $t_0\left(1 - \frac{1}{(\log t_0)^2}\right)$ x_k's. By Lemma 6 there is an x_i not in I_{t_0} so that for a certain z_0 in J_{t_0}

(21) $$|l_i(z_0)| > t_0.$$

Now since z_0 is in J_{t_0} ($\cos \lambda_0 = z_0$, $\cos \vartheta_0 = x_0$, $\cos \vartheta_i = x_i$, $x_i \not\subset I_{t_0}$),

(22) $$|\vartheta_i - \vartheta_0| \leqq (\log t_0)^3 |\vartheta_i - \lambda_0|.$$

Thus from (22) by a simple computation

(23) $$|x_i - x_0| < (\log t_0)^6 |x_i - z_0|.$$

From (23), (21) and $|\omega_n(x_0)| \geqq |\omega_n(z_0)|$ we have

(24) $$|l_i(x_0)| > \frac{t_0}{(\log t_0)^6}.$$

From Lemma 2 we have

(25) $$\frac{1}{n} \sum_{k=1}^{n}{}' \left| \frac{(1-x_k^2)^{\frac{1}{2}}}{x_0 - x_k} \right| > \frac{2}{\pi} \log n - c_{13} \log t_0$$

where the dash indicates that $k = i$ is omitted. (25) holds, since a simple computation shows from Lemma 5 that

$$\left| \frac{(1-x_i^2)^{\frac{1}{2}}}{x_0 - x_i} \right| < c_{14} n.$$

Thus, finally, from (20), (24) and (25) we have

$$\sum_{k=1}^{n}|l_k(x_0)| \geqq \frac{1}{n}\sum_{k=1}^{n}{}' \left|\frac{(1-x_k^2)^{\frac{1}{2}}}{x_0-x_k}\right| + \tag{26}$$
$$+|l_i(x_0)| > \frac{2}{\pi}\log n - c_{13}\log t_0 + \frac{t_0}{(\log t_0)^6} > \frac{2}{\pi}\log n$$

if t is sufficiently large ($t > c_{13}$, say). Thus the proof of Theorem 1 is complete.

It would have been possible to organize the proof differently, since it can be shown that I_t can never contain more than t^3 x_i's. In fact, we have the following

THEOREM 2. *Let* $\omega_n(x) = \prod_{i=1}^{n}(x-x_i)$ *(we do not assume that the* x_i*'s are in* $(-1, +1)$*). Assume that* $\omega_n(x)$ *assumes its absolute maximum in* $(-1, +1)$ *at* $\cos\vartheta_0 = x_0$. *Then every interval* I_t *contains at most* $c_{14}t$ *of the* x_i*'s.*

We do not give the proof of Theorem 2. The best value of c_{14} is not known. Perhaps $c_{14} = 2$.

The problem of determining the points $-1 \leqq x_1 < \cdots < x_n \leqq 1$ for which

$$\int_{-1}^{+1}\sum_{k=1}^{n}|l_k(x)|\,dx$$

is a minimum is unsolved, and so far as I know has not yet been considered. I believe that to every $\varepsilon > 0$ there exists an n_0 so that for $n > n_0$

$$\int_{-1}^{+1}\sum_{k=1}^{n}|l_k(x)|\,dx > (1-\varepsilon)\int_{-1}^{+1}\sum_{k=1}^{n}|L_k(x)|\,dx \tag{27}$$

where $L_k(x) = \dfrac{T_n(x)}{T_n'(y_k)\,(x-y_k)}$ are the fundamental functions of the Lagrange interpolation taken at the roots $y_1, y_2, \ldots, y_n$ of the n^{th} Chebyshev polynomial. I have not been able to prove (27), but I can prove the following weaker

THEOREM 3. *There exists a constant* c_{15} *so that for every* $-1 \leqq x_1 < < x_2 < \cdots < x_n \leqq 1$ *we have*

$$\int_{-1}^{+1}\sum_{k=1}^{n}|l_k(x)|\,dx > c_{15}\log n. \tag{28}$$

In fact, to every ε there exists a δ so that the number of indices $1 \leqq k \leqq n$, for which

$$\int_{-1}^{+1} |l_k(x)|\, dx < \frac{\delta \log n}{n}, \tag{29}$$

is less than εn, and the number of k's, for which $\int_{-1}^{+1} |l_k(x)|\, dx > \frac{c_{16}}{n}$ is less than $c_{17} \frac{n}{\log n}$.

We do not give the proof of Theorem 3, it can be obtained by using the methods of my paper [5].

As far as I know the problem of determining the sequence $-1 \leqq x_1 < < x_2 < \cdots < x_n \leqq 1$ for which

$$\int_{-1}^{+1} \sum_{k=1}^{n} l_k^2(x)\, dx \tag{30}$$

is minimal has not been considered. It is possible that the integral (30) is minimal if the x_i's are the roots of the integral of the Legendre polynomial. FEJÉR [9] proved that these are the only points for which

$$\sum_{k=1}^{n} l_k^2(x) \leqq 1 \quad \text{for} \quad -1 \leqq x \leqq 1.$$

THEOREM 4. *To every ε there exists an n_0 so that for every $n > n_0$ the integral* (30) *is greater than $2-\varepsilon$.*

We only outline the idea of the proof. If the projections of the points $x_1, x_2, \ldots, x_n$ on the unit circle are not asymptotically uniformly distributed, then there exists a k so that [10]

$$\max_{-1 \leqq x \leqq 1} |l_k(x)| > (1+\delta)^n, \tag{31}$$

and from (31) by Markov's theorem

$$\int_{-1}^{+1} l_k^2(x)\, dx > \frac{(1+\delta)^{2n}}{8n^2} > 2$$

for $n > n_0$. Thus we can assume that the projections of the x_k's on the unit circle are asymptotically uniformly distributed. In this case we obtain our Theorem by showing that

$$\int_{-1}^{+1} \sum_{k=1}^{n} l_k^2(x)\, dx > (1-\varepsilon) \int_{-1}^{+1} \sum_{k=1}^{n} L_k^2(x)\, dx \tag{32}$$

where $L_k(x) = \frac{P_n(x)}{P'_n(z_k)(x-z_k)}$ $\left(P_n(x) = \prod_{k=1}^{n}(x-z_k)\right)$ is the n^{th} Legendre polynomial. The proof of (32) follows easily from the fact that

$$\int_{-1}^{+1} L_k^2(x)\,dx \leqq \int_{-1}^{+1} f_{n-1}^2(x)\,dx$$

where $f_{n-1}(x)$ is any polynomial of degree $\leqq n-1$ for which $f_{n-1}(z_k)=1$, and by a simple computation. We suppress the details.

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ON A PROBLEM OF BAER AND A PROBLEM OF WHITEHEAD IN ABELIAN GROUPS

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(*Presented by* L. RÉDEI)

1. Introduction

The first problem of the title, proposed by BAER [1] in 1936, asks for a characterization of those groups F such that $\text{Ext}(F, T) = 0$ for all torsion groups T. (Call such groups *B-groups.*) The second, proposed by J. H. C. WHITEHEAD in 1952, asks for a characterization of those groups F such that $\text{Ext}(F, Z) = 0$, where Z is the integers. (Call such groups *W-groups.*)[1] Both of these problems have been partially solved: BAER [1] proved that any countable B-group is free; K. STEIN [6][2] proved that any countable W-group is free. (We shall give simple homological algebraic proofs of these theorems. The notation and terminology is that of [3].) Since subgroups of B-groups (W-groups) are again B-groups (W-groups), these groups are $\aleph_1$-free. The simplest example of an $\aleph_1$-free group which is not free is Π, the direct product of countably many copies of Z. Only recently, BAER [2], J. ERDŐS [4], and SĄSIADA [5] independently gave proofs that Π is not a B-group. In this paper we generalize their result by showing that separable B-groups are slender. Further, we show that any W-group is slender and separable. In addition, if all B-groups are separable, then any B-group is a W-group. In the last section we show that certain subgroups of Π are not W-groups.

2. Homological algebra

Let R be a commutative ring with unit. A sequence of R-modules and R-homomorphisms $\cdots \to A_n \to A_{n+1} \to A_{n+2} \to \cdots$ is *exact* in case the image of any homomorphism equals the kernel of the next one. In particular, $0 \to A \xrightarrow{f} B$ exact implies f is a monomorphism, and $B \xrightarrow{g} C \to 0$ exact implies g is an epimorphism. An *extension of C by A* is an exact sequence $0 \to A \to B \to C \to 0$. Thus A may be identified with a submodule of B, and $B/A \approx C$. An equiva-

[1] This problem is of homological interest. It asks whether Z is a universal test group for freedom. The dual question — Is there a universal test group for divisibility? — is an easy exercise; Q/Z is such a group, where Q is the rationals.

[2] I am indebted to R. SWAN for this reference.

lence relation is imposed on extensions of C by A, and one may add two equivalence classes by the "Baer sum." Under this operation, the classes of extensions of C by A form an R-module, which is denoted $\text{Ext}_R(C, A)$ or simply $\text{Ext}(C, A)$. The zero element of this module is the class of the *split sequence:* $0 \to A \to A \oplus C \to C \to 0$. $\text{Ext}(C, A) = 0$ if and only if every extension of C by A is split. If $R = Z$, F is a free group if and only if $\text{Ext}(F, A) = 0$ for any group A; D is a divisible group if and only if $\text{Ext}(C, D) = 0$ for any group C.

There is another R-module which can be assigned to a pair of R-modules C and A: $\text{Hom}_R(C, A)$, the R-homomorphisms of C into A. Any homomorphism $f: A \to B$ induces a homomorphism $f^*: \text{Hom}(C, A) \to \text{Hom}(C, B)$ by $f^*(g) = fg$. Also f induces a homomorphism $f_*: \text{Hom}(B, C) \to \text{Hom}(A, C)$ (note the change in direction) by $f_*(h) = hf$. In particular, if $f: A \to A$ is multiplication by $r \in R$, i. e., $f(a) = ra$, then the induced maps f^* and f_* are also multiplication by r. What we have just said remains true if we replace "Hom" by "Ext" with the exception, of course, that the induced homomorphisms f^* and f_* are defined differently.

Of great importance are the two *induced exact sequences.* Suppose $0 \to A \to B \to C \to 0$ is exact, and M is a fixed module. Then the following sequences are exact:

$$(*) \qquad \begin{aligned} 0 \to \text{Hom}(M, A) \to \text{Hom}(M, B) \to \text{Hom}(M, C) \to \\ \to \text{Ext}(M, A) \to \text{Ext}(M, B) \to \text{Ext}(M, C); \end{aligned}$$

$$(**) \qquad \begin{aligned} 0 \to \text{Hom}(C, M) \to \text{Hom}(B, M) \to \text{Hom}(A, M) \to \\ \to \text{Ext}(C, M) \to \text{Ext}(B, M) \to \text{Ext}(A, M). \end{aligned}$$

(A *connecting homomorphism* to pass from the last Hom to the first Ext can be defined.) If R is a principal ideal domain, one can insert "$\to 0$" at the end of $(*)$ and $(**)$.

We shall also need the following two formulas:

$$\text{Ext}(\Sigma A_\alpha, B) \approx \Pi\, \text{Ext}(A_\alpha, B) \quad \text{and} \quad \text{Ext}(A, \Pi B_\beta) \approx \Pi\, \text{Ext}(A, B_\beta).$$

In particular, Ext commutes with finite direct sums in either variable.

It is assumed the reader is familiar with the requisite abelian group theory.

3. New proofs of known results

We adopt the following notation: $Q =$ the rationals; $C(n) =$ the cyclic group of order n; $C(p^\infty) =$ the torsion divisible group of type p^∞; $\Sigma G_i =$ $=$ the direct sum of groups G_i; $\Pi G_i =$ the direct product of the G_i; $I_p^* =$ $=$ the p-adic integers. If $x \in G$, $[x]$ is the cyclic subgroup generated by x.

When the meaning is clear from the context, we shall abbreviate ΣG_i by Σ, ΠG_i by Π.

LEMMA 0. *Every subgroup of a B-group (W-group) is a B-group (W-group). Every B-group (W-group) is torsion-free.*

PROOF. Let H be a subgroup of the B-group F. Then exactness of $0 \to H \to F$ induces exactness of $\operatorname{Ext}(F, T) \to \operatorname{Ext}(H, T) \to 0$ for any torsion T. Since $\operatorname{Ext}(F, T) = 0$, $\operatorname{Ext}(H, T) = 0$. (A similar argument works for W-groups, where T is replaced by Z.)

Suppose F is a B-group. If F is not torsion-free, it has a cyclic subgroup $C(n)$, which is also a B-group. But $0 \to C(n) \to C(n^2) \to C(n) \to 0$ is a non-split sequence showing $\operatorname{Ext}(C(n), C(n)) \neq 0$, a contradiction.

Suppose F is a W-group. If F is not torsion-free, it has a cyclic subgroup $C(n)$, which is also a W-group. But $0 \to Z \xrightarrow{f} Z \to C(n) \to 0$ is exact, where f is multiplication by n, and does not split. Therefore $\operatorname{Ext}(C(n), Z) \neq 0$, a contradiction.

LEMMA 1. $\operatorname{Ext}\left(C(p^\infty), \sum_{i=1}^{\infty} C(p^i)\right)$ *is uncountable.*

PROOF. Exactness of $0 \to \Sigma C(p^i) \to \Pi C(p^i) \to \Pi/\Sigma \to 0$ induces exactness of $0 \to \operatorname{Hom}(C(p^\infty), \Pi/\Sigma) \to \operatorname{Ext}(C(p^\infty), \Sigma)$. But $C(p^\infty)$ is a summand of Π/Σ so that $\operatorname{Hom}(C(p^\infty), C(p^\infty))$ is a summand of $\operatorname{Hom}(C(p^\infty), \Pi/\Sigma)$. Since $\operatorname{Hom}(C(p^\infty), C(p^\infty)) \approx I_p^*$, it is uncountable.

LEMMA 2. *If L is torsion-free of rank 1, L not cyclic, then there exists a countable torsion group T (depending on L) such that* $\operatorname{Ext}(L, T)$ *is uncountable.*

PROOF. *Case 1.* There exists an infinite set of primes P such that L contains an element x divisible by each $p \in P$. Let $Z = [x]$. Then $L/Z = \sum_{p \in P} A_p$, A_p a non-zero p-primary group. Exactness of $0 \to Z \to L \to L/Z \to 0$ induces exactness of $T \to \operatorname{Ext}(L/Z, T) \to \operatorname{Ext}(L, T) \to 0$, where $T = \sum_{p \in P} C(p)$. Since T is countable, it suffices to prove $\operatorname{Ext}(L/Z, T)$ is uncountable. But $\operatorname{Ext}(L/Z, T) = \operatorname{Ext}(\Sigma A_p, T) \approx \Pi \operatorname{Ext}(A_p, T)$ which is uncountable since each $\operatorname{Ext}(A_p, T) \neq 0$; (see proof of Lemma 0).

Case 2. L contains an element x of infinite p-height. If $Z = [x]$, $C(p^\infty)$ is a summand of L/Z. Let $T = \sum_{i=1}^{\infty} C(p^i)$. Exactness of $0 \to Z \to L \to L/Z \to 0$ induces exactness of $T \to \operatorname{Ext}(L/Z, T) \to \operatorname{Ext}(L, T) \to 0$. Thus it suffices to show $\operatorname{Ext}(L/Z, T)$ or its summand $\operatorname{Ext}(C(p^\infty), T)$ is uncountable. This has been done in Lemma 1.

If L does not satisfy Case 1, its characteristic has only finitely many non-zero entries; if L does not satisfy Case 2, its characteristic has no ∞'s as entries. Thus, if L satisfies neither case, L is cyclic.

THEOREM 1. *A countable B-group F is free.*

PROOF. We first assume F has finite rank n; we perform an induction on n. If $n=1$, Lemma 2 gives the desired result. For the general case, let $0 \to H \to F \to L \to 0$ be exact, where L is torsion-free of rank 1. H is free, since its rank is $n-1$. We must show L is cyclic to complete the argument. If L is not cyclic, there exists a countable group T with $\mathrm{Ext}(L, T)$ uncountable. We have exactness of $\mathrm{Hom}(H, T) \to \mathrm{Ext}(L, T) \to \mathrm{Ext}(F, T) = 0$. Since H is free of finite rank, $\mathrm{Hom}(H, T) \approx \Sigma T$. But now we have a countable group with an uncountable quotient, a contradiction. Hence L is cyclic. PONTRJAGIN's Lemma extends the theorem to arbitrary countable B-groups.

REMARK. For this result, it is only necessary that $\mathrm{Ext}(F, T) = 0$ where $T \approx \sum_p \sum_i C(p^i)$.

LEMMA 3. *A W-group F of rank* 1 *is cyclic.*

PROOF. Suppose F is not cyclic. Exactness of $0 \to Z \to F \to F/Z \to 0$ induces exactness of $\mathrm{Hom}(F, Z) \to \mathrm{Hom}(Z, Z) \to \mathrm{Ext}(F/Z, Z) \to \mathrm{Ext}(F, Z) = 0$. Since F is not cyclic, $\mathrm{Hom}(F, Z) = 0$, and $Z \approx \mathrm{Ext}(F/Z, Z)$. Since Z is indecomposable, F/Z is p-primary; since F is not cyclic, $F/Z \approx C(p^\infty)$. But if $q \neq p$ is a prime, $\mathrm{Ext}(C(p^\infty), Z)$ is q-divisible, since multiplication by q is an automorphism of $C(p^\infty)$ which induces a similar automorphism of $\mathrm{Ext}(C(p^\infty), Z)$. This contradiction completes the proof.

THEOREM 2. *A countable W-group F is free.*

PROOF. By PONTRJAGIN's Lemma, we may assume F has finite rank n; we perform an induction on n. If $n=1$, we use Lemma 3. Suppose $0 \to H \to F \to L \to 0$ is exact, L torsion-free of rank 1. By induction, H is free. This sequence induces exactness of $\mathrm{Hom}(H, Z) \to \mathrm{Ext}(L, Z) \to \mathrm{Ext}(F, Z) = 0$. Hence $\mathrm{Ext}(L, Z)$ is finitely generated. But since L is torsion-free, $\mathrm{Ext}(L, Z)$ is divisible ([3], p. 135). Therefore $\mathrm{Ext}(L, Z) = 0$ and so L is cyclic, by Lemma 3. Hence F is free.

4. New properties of B-groups and W-groups

At this point, an economy of ideas is required. Both the BAER and WHITEHEAD problems are particular cases of a more general problem. Let $\mathfrak{S}$ be a class of abelian groups; find all groups F such that $\mathrm{Ext}(F, S) = 0$ for all S in $\mathfrak{S}$. Call such a group an *$\mathfrak{S}$-group*. In BAER's problem, $\mathfrak{S}$ is the class of all torsion groups; in WHITEHEAD's problem, $\mathfrak{S}$ has the unique element Z. Observe that the problem may be further generalized by replacing groups by modules. If G is a group, $|G|$ shall denote its cardinality.

LEMMA 4. (The Density Lemma.) *Let F be a torsion-free group, H a pure subgroup such that F/H is divisible. Suppose there is a countable S in $\mathfrak{S}$ such that* $\text{Ext}(Q, S) \neq 0$. *Then if $2^{|H|} < 2^{|F|}$, F is not an $\mathfrak{S}$-group.*

PROOF. Since $2^{|H|} < 2^{|F|}$, $|H| < |F|$, so that $F/H \approx \sum_{|F|} Q$. Now exactness of $0 \to H \to F \to F/H \to 0$ induces exactness of $\text{Hom}(H, S) \xrightarrow{\delta} \text{Ext}(F/H, S) \to$ $\to \text{Ext}(F, S)$. Since S is countable, $|\text{Hom}(H, S)| \leqq \aleph_0^{|H|} = 2^{|H|}$. On the other hand, $\text{Ext}(F/H, S) \approx \text{Ext}(\sum_{|F|} Q, S) \approx \prod_{|F|} \text{Ext}(Q, S)$. Since $\text{Ext}(Q, S) \neq 0$, $|\text{Ext}(F/H, S)| \geqq 2^{|F|}$. Thus δ cannot be an epimorphism and so $\text{Ext}(F, S) \neq 0$. Thus F is not an $\mathfrak{S}$-group.

We make two remarks here. Suppose we were considering the more general problem of determining $\mathfrak{S}$-modules, where *module* means R-module, R a principal ideal domain. Lemma 4 is still true if we replace Q by the quotient field of R, and if we further assume this quotient field is countable. In particular, the lemma is true if $R = I_p$, the p-adic rationals.

Let us return to groups. A group F is *Hausdorff* in case $\bigcap_n n!F = 0$. If F is Hausdorff, the subgroups $n!F$ define a metric topology on F, *the n-adic topology*, which makes F a topological group. A pure subgroup H of F is a subspace (i. e., the n-adic topology on H is the same topology as that induced on H in virtue of its being contained in F). Also a subgroup H is dense in F if and only if F/H is divisible. Thus we may paraphrase Lemma 4 in saying that if F is an $\mathfrak{S}$-group, no subgroup of smaller cardinality can be dense in F.[3] But even more is true. Since any subgroup of an $\mathfrak{S}$-group is again an $\mathfrak{S}$-group, we know that the closure $\bar{H}$ of a subgroup H has the same cardinality as H.[4]

DEFINITION. Let Z_i be an infinite cyclic group with generator e_i $(i = 1, 2, \ldots)$, and let $\Pi = \Pi Z_i$. A group F is *slender* in case $f(e_i) = 0$ for almost all i, f any homomorphism from Π to F.

ŁOS [5] has shown that any free group is slender. Since Π is Hausdorf, we consider it topologized. Let Σ denote the subgroup ΣZ_i, and let $\bar{\Sigma}$ be the closure of Σ in Π; $\bar{\Sigma}$ is a subgroup. I conjecture that slenderness is really a reflection of the topological relation between Σ and $\bar{\Sigma}$; in all slen-

[3] We assume $|H|$ and $|F|$ are such that $|H| < |F|$ and $2^{|H|} < 2^{|F|}$.

[4] Unless $|H| < |\bar{H}|$ and $2^{|H|} = 2^{|\bar{H}|}$.

derness arguments one should replace Π by $\bar{\Sigma}$. The following proof (essentially due to SĄSIADA) illustrates this conjecture.

LEMMA 5. *$\bar{\Sigma}$ has no direct summand isomorphic to Σ.*

PROOF. Suppose $\bar{\Sigma} = A \oplus B$, where $A \approx \Sigma$; let $f\colon \bar{\Sigma} \to A$ be the projection. Suppose $f(\Sigma)$ has infinite rank in A. We may assume $f(e_i) \neq 0$ for all i. Since A is reduced and torsion-free, there exists a sequence of positive integers m_i such that $f(m_i!\,e_i) \notin m_{i+1}!\,A$. Set $\mathbf{X} =$ all $x \in \Pi$ such that $(x_i) = 0$ or $\pm m_i!$ ($(x)_i =$ the i^{th} co-ordinate of x); set $\mathbf{Y} =$ all $x \in \Pi$ such that $(x)_i = 0$ or $m_i!$. Note that $\mathbf{Y} \subset \mathbf{X} \subset \bar{\Sigma}$. Further, if $x, y \in \mathbf{Y}$, then $x - y \in \mathbf{X}$. Suppose $x \neq 0$, $x \in \mathbf{X} \cap \ker f$; let the first non-zero co-ordinate of x be the j^{th}. Then $0 = f(x) = f(m_j!\,e_j) + f(\{0, \ldots, 0, \pm m_{j+1}!\,e_{j+1}, \ldots\})$, so that $f(m_j!\,e_j) \in m_{j+1}!\,A$, a contradiction. Now $\mathbf{Y}$ is uncountable while A is countable. Hence there exist x and $y \in \mathbf{Y}$, $x \neq y$, such that $f(x) = f(y)$. Thus $f(x-y) = 0$ contradicting $\mathbf{X} \cap \ker f = 0$. We must conclude that $f(\Sigma)$ has finite rank in A. But $f(\Sigma)$ is dense in $f(\bar{\Sigma}) = A$, so that A has finite rank, another contradiction.

DEFINITION. A torsion-free group G is *separable* in case any pure subgroup of finite rank is a direct summand of G.[5]

LEMMA 6. *Let $\mathfrak{S}$ be a class of groups containing a countable S such that* $\operatorname{Ext}(Q, S) \neq 0$. *Then a separable $\aleph_1$-free $\mathfrak{S}$-group F is slender.*

PROOF. Suppose $f\colon \Pi \to F$ with $f(e_i) \neq 0$ for infinitely many i; let H be the pure subgroup generated by $f(\Sigma)$. If H has finite rank, it is a direct summand of F, by separability. Let $\pi\colon F \to H$ be the projection. Then $\pi f\colon \Pi \to H$ has $\pi f(e_i) \neq 0$ for infinitely many i, contradicting the fact that H is slender (H is free). Hence $f(\Sigma)$ has infinite rank. Now $f(\Sigma)$ is pure and dense in $f(\bar{\Sigma})$; also $f(\bar{\Sigma})$ is an $\mathfrak{S}$-group, so that we may apply the density lemma. Thus countability of $f(\Sigma)$ implies countability of $f(\bar{\Sigma})$. Since $f(\Sigma) \subset f(\bar{\Sigma})$ and F is $\aleph_1$-free, $f(\bar{\Sigma})$ is free of countable rank, i. e., $f(\bar{\Sigma}) \approx \Sigma$. But then $\bar{\Sigma}$ has a summand isomorphic to Σ, a contradiction.

THEOREM 3. *A separable B-group is slender.*

PROOF. An immediate consequence of Theorem 1 and Lemma 6.

We now turn our attention to W-groups.

LEMMA 7. *Let F be a W-group with pure subgroup H of finite rank. Then F/H is a W-group.*

PROOF. Exactness of $0 \to H \to F \to F/H \to 0$ induces exactness of $\operatorname{Hom}(H, Z) \to \operatorname{Ext}(F/H, Z) \to \operatorname{Ext}(F, Z) = 0$. Since H has finite rank, it is

[5] This notion of separability is not the usual one, but the two notions coincide in the case of homogeneous groups, and hence in the present problem.

free. Hence Hom (H, Z) is free of finite rank, so that Ext $(F/H, Z)$ is finitely generated. Since F/H is torsion-free, Ext $(F/H, Z)$ is divisible. Therefore Ext $(F/H, Z)=0$.

COROLLARY 1. *Any W-group F is separable.*

PROOF. Let H be a pure subgroup of finite rank in F; H is free. Hence Ext $(F/H, H) \approx \Sigma$ Ext $(F/H, Z)=0$, by Lemma 7. Hence Ext $(F/H, H)=0$, i. e., H is a direct summand of F.

THEOREM 4. *Any W-group is slender.*

THEOREM 5. *Any W-group F can be imbedded as a pure subgroup in a direct product of Z's.*

PROOF. By Corollary 1, for each $x \in F$, there is a δ_x: $F \to Z$ such that $\delta_x(x)=h(x)$, where $h(x)$ is the height of x. Define D: $F \to \prod_{x \in F} Z$ by $D(y) = \{\delta_x(y)\}$. D is a monomorphism, by our initial remark. $D(F)$ is a pure subgroup, since: (1) the height of an element in ΠZ is the minimum of the heights of its co-ordinates; (2) any homomorphism, e. g., δ_x, cannot lower heights.

We have observed that B-groups and W-groups share many properties. It is a plausible conjecture that these two classes of groups are identical. The following two theorems shed some light on this conjecture:

THEOREM 6. *Suppose every B-group is separable. Then any B-group F is a W-group.*

PROOF. Let H be a pure subgroup of F of finite rank. Since F is separable, F/H is a B-group (being a summand of F). If (∗) $0 \to H \to G \to F/H \to 0$ is exact, then Ext $(F/H, T) \to$ Ext $(G, T) \to$ Ext (H, T) is exact for any torsion group T. Hence G is a B-group. Since F/H is torsion-free, H is a pure subgroup of G. Therefore H is a summand of G, because G is separable. Thus the sequence (∗) always splits, i. e., Ext $(F/H, H)=0$. Since H is free of finite rank, Ext $(F/H, Z)=0$. But $F \approx H \oplus F/H$. Therefore Ext $(F, Z) \approx$ Ext $(H \oplus F/H, Z) \approx$ Ext $(H, Z) \oplus$ Ext $(F/H, Z)=0$. Hence F is a W-group.

I cannot prove that every B-group is separable, but I can prove one result in this direction.

THEOREM 7. *Let F be a B-group, H a pure subgroup of finite rank. Then F/H is* $\aleph_1$*-free.*

We first prove

LEMMA 8. *Let L be a torsion-free group of finite rank which is not free. Then there exists a countable torsion group T such that* Ext (L, T) *is uncountable.*

(We remark that this lemma gives another proof of Theorem 1.)

PROOF. Assume rank $L=n$. If $n=1$, this is Lemma 2. Let $0\to M\to\to L\to L/M\to 0$ be exact, where M is pure of rank $n-1$. This induces exactness of $\mathrm{Hom}(M, T)\to \mathrm{Ext}(L/M, T)\to \mathrm{Ext}(L, T)\to \mathrm{Ext}(M, T)\to 0$, for any torsion T. If M is not free, choose a countable T, by induction, such that $\mathrm{Ext}(M, T)$ is uncountable; then $\mathrm{Ext}(L, T)$ is uncountable. If M is free, then L/M is not free, so that we can choose a countable torsion T so that $\mathrm{Ext}(L/M, T)$ is uncountable. But $\mathrm{Hom}(M, T)\approx\Sigma T$ is countable so that $\mathrm{Ext}(L, T)$ is uncountable.

We return to the proof of Theorem 7.

Exactness of $0\to H\to F\to F/H\to 0$ induces exactness of $\mathrm{Hom}(H, T)\to\to \mathrm{Ext}(F/H, T)\to \mathrm{Ext}(F, T)=0$, T any torsion group. Since H is free of finite rank, $\mathrm{Hom}(H, T)\approx\Sigma T$. Hence $|\mathrm{Ext}(F/H, T)|\leqq|T|$. Let G be a subgroup of F/H of finite rank. Exactness of $0\to G\to F/H$ induces exactness of $\mathrm{Ext}(F/H, T)\to \mathrm{Ext}(G, T)\to 0$. By Lemma 8, if G is not free, we can choose a countable T such that $\mathrm{Ext}(G, T)$ is uncountable, contradicting the inequality above. Hence G must be free. PONTRJAGIN's Lemma completes the proof.

The following questions remain open: If F is a B-group, is F separable? An easier question is: if F is a B-group, H a pure subgroup of finite rank, is F/H a B-group? (An affirmative answer to this question would imply Theorem 7.) Which pure subgroups of a product of copies of Z are slender?

5. Further investigations

We have seen that any W-group can be imbedded as a pure subgroup in a direct product of Z's. Thus one approach to solving WHITEHEAD's problem is to eliminate all non-free subgroups (assuming that the conjecture — all W-groups are free — is correct). I propose the following plan of attack. Let us find a family of subgroups $\{A_i\}$ with two properties: 1. each A_i is not a W-group; 2. if a subgroup S of a direct product of Z's is not free, then it contains a copy of some A_i. Since any subgroup of a W-group is a W-group, such a method could solve the problem.

Let us only look at Π and its subgroups, where Π is a direct product of countably many Z's. As candidates for the A_i, I suggest the following subgroups. G is of *type* $n!$ if it is isomorphic to π^{-1} (the divisible subgroup of Π/Σ), where π: $\Pi\to\Pi/\Sigma$ is the natural map. G is of type p^n if it is isomorphic to π^{-1} (the p-divisible subgroup of Π/Σ). The following question remains open: If a pure subgroup S of Π contains no subgroup of type $n!$ or of type p^n, is S free?

LEMMA 9. *The group G of type $n!$ is not a W-group.*

Proof. G is uncountable with pure dense subgroup Σ. Since Σ is countable, we apply the density lemma.

In order to show groups of type p^n are not W-groups, we must examine I_p-modules.

Lemma 10. *For any group G,* $\operatorname{Ext}_Z(I_p \otimes G, I_p) \approx \operatorname{Ext}_{I_p}(I_p \otimes G, I_p)$.

Proof. Observe that $\operatorname{Hom}_Z(I_p \otimes G, I_p) = \operatorname{Hom}_{I_p}(I_p \otimes G, I_p)$. Exactness of $0 \to I_p \to Q \to C(p^\infty) \to 0$ induces exactness of the rows of the commutative diagram:

$$\begin{array}{ccccccc} \operatorname{Hom}_Z(I_p \otimes G, I_p) & \to \operatorname{Hom}_Z(I_p \otimes G, Q) & \to \operatorname{Hom}_Z(I_p \otimes G, C(p^\infty)) & \to \operatorname{Ext}_Z(I_p \otimes G, I_p) \to 0, \\ \| & \| & \| & \\ \operatorname{Hom}_{I_p}(I_p \otimes G, I_p) & \to \operatorname{Hom}_{I_p}(I_p \otimes G, Q) & \to \operatorname{Hom}_{I_p}(I_p \otimes G, C(p^\infty)) & \to \operatorname{Ext}_{I_p}(I_p \otimes G, I_p) \to 0. \end{array}$$

Hence the cokernels are isomorphic.

Lemma 11. *If* $\operatorname{Ext}_Z(G, Z) = 0$, *then* $\operatorname{Ext}_{I_p}(I_p \otimes G, I_p) = 0$.

Proof. We shall show that $\operatorname{Ext}_Z(I_p \otimes G, I_p) = 0$ and the result will follow from Lemma 10. All Ext's appearing in this proof shall be Ext_Z's.

Exactness of $0 \to Z \to I_p \to I_p/Z \to 0$ induces exactness of $\operatorname{Ext}(G, Z) \to$ $\to \operatorname{Ext}(G, I_p) \to \operatorname{Ext}(G, I_p/Z)$. But $I_p/Z \approx \sum_{q \neq p} C(q^\infty)$ which is divisible; hence $\operatorname{Ext}(G, I_p/Z) = 0$. Since $\operatorname{Ext}(G, Z)$ is also 0, we have $\operatorname{Ext}(G, I_p) = 0$.

Exactness of $0 \to G \to I_p \otimes G \to (I_p/Z) \otimes G \to 0$ (G is torsion-free) induces exactness of $\operatorname{Ext}(I_p/Z \otimes G, I_p) \to \operatorname{Ext}(I_p \otimes G, I_p) \to \operatorname{Ext}(G, I_p) = 0$. Now $\operatorname{Ext}(I_p/Z \otimes G, I_p) \approx \operatorname{Ext}(\sum_{q \neq p} C(q^\infty) \otimes G, I_p) \approx \prod_{q \neq p} \operatorname{Ext}(C(q^\infty) \otimes G, I_p)$. Exactness of $0 \to I_p \to Q \to C(p^\infty) \to 0$ induces exactness of $0 = \operatorname{Hom}(C(q^\infty) \otimes G, C(p^\infty)) \to$ $\to \operatorname{Ext}(C(q^\infty) \otimes G, I_p) \to \operatorname{Ext}(C(q^\infty) \otimes G, Q) = 0$, since $C(q^\infty) \otimes G$ is q-primary. Hence $\operatorname{Ext}(C(q^\infty) \otimes G, I_p) = 0$, $\operatorname{Ext}(I_l/Z \otimes G, I_p) = 0$, and finally $\operatorname{Ext}(I_p \otimes G, I_p) = 0$.

Corollary 2. *If G is a W-group, $I_p \otimes G$ is a W-module (over the ring I_p).*

Lemma 12. *A group G of type p^n is not a W-group.*

Proof. Otherwise $I_p \otimes G$ would be a W-module, and so would satisfy the module version of the density lemma. But $I_p \otimes G$ is uncountable, while $I_p \otimes \Sigma$ is a pure dense submodule which is countable. This contradiction completes the proof.

Note (added 15 October 1960). Ti Yen succeeded in solving a problem raised at the end of **4.**

Theorem. *Let F be a B-group with pure subgroup H of finite rank. Then F/H is a B-group.*

Exactness of $0 \to H \to F \to F/H \to 0$ induces exactness of $\mathrm{Hom}(F, T) \xrightarrow{\alpha} \mathrm{Hom}(H, T) \to \mathrm{Ext}(F/H, T) \to \mathrm{Ext}(F, T) = 0$ where T is any torsion group. In order to show $\mathrm{Ext}(F/H, T) = 0$, it suffices to prove α is an epimorphism. Let $f\colon H \to T$. Since H is free of finite rank, $f(H)$ is finite. Replacing T by $f(H)$ in the above exact sequence yields $\mathrm{Hom}(F, f(H)) \to \mathrm{Hom}(H, f(H)) \to \mathrm{Ext}(F/H, f(H)) = 0$. Hence f can be extended over F to a map into $f(H)$; a fortiori, f can be extended to a map from F to T. Hence α is an epimorphism, $\mathrm{Ext}(F/H, T) = 0$, and F/H is a B-group.

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NOTE ON FULLY ORDERED SEMIGROUPS

By

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(*Presented by* L. RÉDEI)

Dedicated to P. TURÁN *on his 50th birthday*

By a *fully ordered* (briefly: f. o.) *semigroup*[1] S is meant a semigroup which is at the same time a f. o. set under an ordering relation $\leqq$ such that $a \leqq b$ $(a, b \in S)$ implies $ac \leqq bc$ and $ca \leqq cb$ for all $c \in S$. (If S is cancellative, then this is equivalent to the fact that $a < b$ implies $ac < bc$ and $ca < cb$, but we do not assume the cancellation laws.) We say that S is *positively ordered* if $ab \geqq a$ and $ab \geqq b$ for all $a, b \in S$, and *naturally ordered* if it is positively ordered and $a < b$ implies the existence of elements $c, d \in S$ such that $b = ca = ad$. A positively ordered semigroup S is called *archimedean* if $a^n < b$ for all positive integers n implies $a = e$ ($=$the identity of S).[2] Finally, $a, b (\in S)$ are said to form an *anomalous pair* if $a \neq b$, $a^n < b^{n+1}$ and $b^n < a^{n+1}$ for all natural integers n.

Fully ordered semigroups have received some attention recently. The classical result of O. HÖLDER [4] which gives a sufficient condition that a f. o. semigroup S be embeddable in the additive semigroup P of all non-negative real numbers (with preservation of ordering) has been generalized in various ways. HÖLDER's conditions were:

(a) S is cancellative;

(b) S is naturally ordered;

(c) S is archimedean.

ALIMOV [1] replaced conditions (b) and (c) by the single one:

(d) S contains no anomalous pair,

thereby giving a necessary and sufficient condition for the embeddability of S in the real group. In § 1 we prove for positively ordered semigroups another necessary and sufficient condition in which the rather restrictive condition (a) is replaced by (c) and the rather weak condition (e) which relate more closely to the ordering relation:

(e) S contains no maximal element unless it consists of a single element.

[1] For the notions and basic facts related to f. o. semigroups we refer to CLIFFORD [3].

[2] We do not assume the existence of the identity e in S. If in a statement e occurs, then this means "the eventually existing identity".

A remarkable analogue of HÖLDER's theorem has been found by CLIFFORD [2]. He proved that if a f. o. semigroup S satisfies the conditions (b), (c) and the following two:

(a*) S is not cancellative;

(f) S is commutative,

then it is o-isomorphic (order-isomorphic) to a subsemigroup of one of the following two f. o. semigroups:

P_1: the real interval [0, 1] with the operation: $a \circ b = \min(a+b, 1)$,

P_1^*: the interval [0, 1] and the symbol ∞ with $a \circ b = a+b$ or ∞ according as $a+b \leqq 1$ or >1.

In § 2 we shall show that conditions (b) and (c) imply (f), i. e. in CLIFFORD's theorem the hypothesis of commutativity can be omitted just as in HÖLDER's theorem. In the final § 3 we shall give a new proof of this theorem of CLIFFORD, one which seems to be more direct and simpler than CLIFFORD's original proof and which establishes at the same time HÖLDER's theorem too.

§ 1. Subsemigroups of the positive reals

This section is devoted to the proof of the following theorem:

THEOREM 1. *A necessary and sufficient condition that a positively f. o. semigroup be o-isomorphic to a subsemigroup of the additive semigroup of all non-negative real numbers is that it satisfy conditions* (c), (d) *and* (e).[3]

The necessity of these conditions being obvious, we may turn immediately to the proof of their sufficiency. Suppose therefore that (c), (d) and (e) are satisfied. We may assume that S contains more than one element.

Let $a, b \in S$ and $a \neq e$. We prove that $ab > b$ and $ba > b$. For, by (e), S contains an element c such that $c > b$, and because of (c) we can choose n so large that $a^n \geqq c$. If $ab = b$ held, then also $a^n b = b$ whence $a^n b \geqq a^n \geqq c > b$ would be a contradiction. Thus $ab > b$, and similarly $ba > b$.

To prove commutativity, assume on the contrary that $ab \neq ba$. Then neither $a = e$ nor $b = e$, and what has been proved implies $(ab)^n < b(ab)^n a = (ba)^{n+1}$ and $(ba)^n < (ab)^{n+1}$, i. e. ab and ba form an anomalous pair, contrary to (d). Thus S is commutative.

Next assume that $ab = ac$ where $b < c$. By making use of commutativity, a simple induction shows that $ab^n = ac^n$ for all n. Surely, $c \neq e$, therefore $ac^n < ac^{n+1} = ab^{n+1}$, whence $c^n < b^{n+1}$ for all n. Since obviously $b^n \leqq c^n < c^{n+1}$

[3] Mr. G. GRÄTZER noted that (c) and (e) may be united into the single condition: $a^n \leqq b$ for all positive n implies $a = e$.

for all n, b and c form an anomalous pair. This contradiction proves that S is cancellative.

By ALIMOV's theorem [1], (a) and (d) imply what we wished to prove.

§ 2. The commutativity statement

We shall need the following

LEMMA. (CLIFFORD.) *Let S be a f. o. semigroup satisfying* (a*), (b) *and* (c). *Then*

1. *S contains a maximal element u,*
2. *for every $a \neq e$ there exists a natural integer k with $a^k = u$,*
3. *$ab = ac \neq u$ (or $ba = ca \neq u$) implies $b = c$.*

By hypothesis, three elements $a, b, c \in S$ exist such that $ab = ac$ and $b < c$ (or $ba = ca$ and $b < c$). By (b), $c = bx$ for some $x \in S$, $x \neq e$, therefore $y = yx$ holds for $y = ab$. If there existed an element $z > y$, $z \in S$, then choosing n so as to satisfy $x^n \geqq z$, we should have $y = yx^n \geqq yz \geqq z > y$, a contradiction. This establishes 1. and 3. at once. By (c), $a \neq e$ implies $a^k \geqq u$ for some k, and so $a^k = u$, completing the proof.

Now we are ready to prove:

THEOREM 2. *An archimedean, naturally f. o. semigroup is commutative.*

For the sake of simplicity we omit the identity from S if S has one; it is evident that this does not affect generality, because it must be the least element of S.

First assume that S possesses a minimal element a. Then to any $b \in S$, different from the eventually existing maximal element u, there exists an integer $k \geqq 1$ satisfying $a^k \leqq b < a^{k+1}$. Supposing $a^k < b$, there exists a $c \in S$ such that $b = a^k c$. But by the choice of a, $c \geqq a$, thus $b \geqq a^k a > b$, which is absurd. Thus $b = a^k$ and S is a cyclic semigroup (generated by a).

Secondly assume that S has no minimal element. Then to any $x \in S$ there exists a $z \in S$ such that $z^2 \leqq x$; indeed, if $y < x$ and $x = yy_0$, then $z = \min(y, y_0)$ is a desired element. By way of contradiction, suppose that $ab > ba$ for some a, b in S. At first let[4] $ab < u$; then also $a < u, b < u$. If $ab = bax$ and $z^2 \leqq x$, then by (c) we can determine integers m, n satisfying $z^m \leqq a < z^{m+1}$ and $z^n \leqq b < z^{n+1}$. But these lead to the inequalities $ab = bax \geqq$ $\geqq z^{n+m+2} > ab$ (in the non-cancellative case strict inequality because of the Lemma), a contradiction. Next let $ab = u$, and say $a < b$. Then $b < u$ (otherwise $ba = u = ab$) and $a^k < b < a^{k+1}$ (equality would imply that a and b

[4] If no maximal element u exists in S, then we may think of $ab < u$ to hold generally.

commute), for some $k \geqq 1$. Therefore $b = a^k c$ for a certain $c < a$, and since $ac \leqq b$ and $ca \leqq ba < ab = u$, we can apply what has been proved to conclude that a and c commute. This is again a contradiction, for then a and b also commute. Consequently, S must be commutative.

§ 3. Archimedean, naturally fully ordered semigroups

The next result is a generalization of HÖLDER's theorem, containing also CLIFFORD's theorem freed of the commutativity hypothesis.

THEOREM 3.[5] *Let S be an archimedean, naturally f. o. semigroup. Then S is o-isomorphic to a subsemigroup of P, P_1 or P_1^*.*

From the preceding theorem we know that S is necessarily commutative. If S is an infinite cyclic semigroup, generated by a, then $a^k \to k$ is an o-isomorphism of S into P. If S is a finite cyclic semigroup with the elements $(e<)a < a^2 < \cdots < a^n = a^{n+1}$, then the mapping $a^k \to \dfrac{k}{n}$ embeds S in P_1.

If S is not cyclic, then we again omit the eventually existing identity of S. By the proof of Theorem 2, to any $x \in S$ we can find a $z \in S$ satisfying $z^2 \leqq x$, and hence also one satisfying $z^t \leqq x$ for any preassigned integer $t > 0$. Now choose and fix an arbitrary $a \in S$ with $a < u$, and put $f(a) = 1$. To any $b \in S, b \neq u$, we define two sets of rational numbers: let L consist of all fractions m/n with $a \leqq x^n$ and $x^m \leqq b$ for some $x \in S$, and let k/l belong to the set U if $b \leqq y^k$ and $y^l \leqq a$ for some $y \in S$. The archimedean character of S guarantees that neither of L and U is empty. We show that $m/n \leqq k/l$. To any large t we can find a $z \in S$ with $z^t \leqq \min(x, y)$, hence $r \geqq t$ and $s \geqq t$ hold for the integers r, s defined by $z^r \leqq x < z^{r+1}$, $z^s \leqq y < z^{s+1}$. Therefore $z^{rm} \leqq b < z^{(s+1)k}$ and $z^{sl} \leqq a < z^{(r+1)n}$ whence $rm < (s+1)k$ and $sl < (r+1)n$. We infer that $\dfrac{m}{n} < \left(1 + \dfrac{1}{r}\right)\left(1 + \dfrac{1}{s}\right)\dfrac{k}{l}$ for arbitrarily large r, s, thus $m/n \leqq k/l$, in fact. On the other hand, the same argument shows that to any large $t > 0$ there exist $r, s \geqq t$ such that $s/(r+1) \in L$ and $(s+1)/r \in U$. Since the difference of these fractions tends to 0 with increasing t, it follows that there exists one and only one real number β such that $p \leqq \beta \leqq q$ for all $p \in L$ and $q \in U$. We put $f(b) = \beta$.

There is no difficulty in proving that the function f from $S \setminus u$ to the real axis is monotone and satisfies $f(b) = f(c)$ only if $b = c$. Moreover, we

[5] The proof of this theorem differs from the usual proofs of HÖLDER's theorem in that we must argue with 'small' elements rather than 'large' elements, due to the singular behaviour of the maximal element u.

have $f(bc)=f(b)+f(c)$ whenever $bc<u$ — which can again be proved by using sufficiently small elements $z\in S$.

If S is cancellative, then it contains no maximal element, and f is an o-isomorphism of S into P.

If S is not cancellative, then — in view of the Lemma — it contains a maximal element u. The set of values of $f(b)$ for all $b\in S\setminus u$ is bounded: if $a^i=u$, then i is an upper bound. Thus there exists a smallest real number α such that $f(b)\leqq\alpha$ for all $b\in S\setminus u$. If no $c\in S\setminus u$ exists with $f(c)=\alpha$, then set $f(u)=\alpha$, and if such a c exists, then let $f(u)=\infty$. The function $g(b)=\frac{1}{\alpha}f(b)$ is obviously an o-isomorphism of S into P_1 or P_1^*. This completes the proof.

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ON THE STRENGTH OF CONNECTEDNESS OF A RANDOM GRAPH

By

P. ERDŐS (Budapest), corresponding member of the Academy,
and A. RÉNYI (Budapest), member of the Academy

Let G be a non-oriented graph without parallel edges and without slings, with vertices $V_1, V_2, \ldots, V_n$. Let us denote by $d(V_k)$ the *valency* (or degree) of a point V_k in G, i. e. the number of edges starting from V_k. Let us put

(1) $$c(G) = \min_{1 \leqq k \leqq n} d(V_k).$$

If G is an arbitrary non-complete graph, let $c_p(G)$ denote the least number k such that by deleting k appropriately chosen vertices from G (i. e. deleting the k points in question and all edges starting from these points) the resulting graph is not connected. If G is a complete graph of order n, we put $c_p(G) = n-1$. Let $c_e(G)$ denote the least number l such that by deleting l appropriately chosen edges from G the resulting graph is not connected. We may measure the strength of connectedness of G by any of the numbers $c_p(G)$, $c_e(G)$ and in a certain sense (if G is known to be connected) also by $c(G)$. Evidently one has

(2) $$c(G) \geqq c_e(G) \geqq c_p(G).$$

It is known further that any two points of G are connected by at least $c_p(G)$ paths having no point in common, except the two endpoints (theorem of MENGER—WHITNEY, see [1] and [2]) and by at least $c_e(G)$ paths having no edge in common (theorem of FORD and FULKERSON, see [3]).

We shall denote by $\nu_r(G)$ the number of vertices of G which have the valency r $(r = 0, 1, 2, \ldots)$.

As in two previous papers ([4], [5]) we consider the random graph $\Gamma_{n,N}$ defined as follows: Let there be given n labelled points $V_1, V_2, \ldots, V_n$. Let us choose at random N edges among the $\binom{n}{2}$ possible edges connecting these n points, so that each of the $\binom{\binom{n}{2}}{N}$ possible choices of these edges should be equiprobable. We denote by $\Gamma_{n,N}$ the random graph thus obtained. We shall denote by $\mathbf{P}(\cdot)$ the probability of the event in the brackets.

The aim of this note is to investigate the strength of connectedness of the random graph $\Gamma_{n,N}$ when n and N both tend to $+\infty$, $N=N(n)$ being a function of n. As it has been shown in [4], the following theorem holds:

THEOREM 1. *If we have* $N(n)=\frac{1}{2}n\log n+\alpha n+o(n)$ *where* α *is a real constant, then the probability of* $\Gamma_{n,N(n)}$ *being connected tends to* $\exp(-e^{-2\alpha})$ *for* $n\to+\infty$.

In this paper we shall prove the following theorem:

THEOREM 2. *If we have* $N(n)=\frac{1}{2}n\log n+\frac{r}{2}n\log\log n+\alpha n+o(n)$ *where* α *is a real constant and* r *a non-negative integer, then*

$$\lim_{n\to+\infty}\mathbf{P}(c_p(\Gamma_{n,N(n)})=r)=1-\exp\left(-\frac{e^{-2\alpha}}{r!}\right), \tag{3}$$

further

$$\lim_{n\to+\infty}\mathbf{P}(c_e(\Gamma_{n,N(n)})=r)=1-\exp\left(-\frac{e^{-2\alpha}}{r!}\right) \tag{4}$$

and

$$\lim_{n\to+\infty}\mathbf{P}(c(\Gamma_{n,N(n)})=r)=1-\exp\left(-\frac{e^{-2\alpha}}{r!}\right). \tag{5}$$

REMARK. Clearly Theorem 2 can be considered as a generalization of Theorem 1. As a matter of fact, any of the statements $c_p(G)=0$ or $c_e(G)=0$ is equivalent to G not being connected and thus for $r=0$ (3) and (4) reduce to the statement of Theorem 1. It has been shown further in [4] that if $N(n)=\frac{n}{2}\log n+\alpha n+o(n)$ and $\Gamma_{n,N(n)}$ is not connected, then it consists almost surely of a connected component and of a few isolated points. Therefore (5) is for $r=0$ also equivalent to the statement of Theorem 1. Thus in proving Theorem 2 we may restrict ourselves to the case $r\geqq 1$.

The statement (5) of Theorem 2 gives information about the *minimal* valency of points of $\Gamma_{n,N}$. In a forthcoming note we shall deal with the same question for larger ranges of N (when $c(\Gamma_{n,N})$ tends to infinity with n), further with the related question about the *maximal* valency of points of $\Gamma_{n,N}$.

We shall prove further the following

THEOREM 3. *If we have* $N(n)=\frac{1}{2}n\log n+\frac{r}{2}n\log\log n+\alpha n+o(n)$ *where* α *is a real constant and* r *a non-negative integer, then we have*

$$\lim_{n\to+\infty}\mathbf{P}(\nu_r(\Gamma_{n,N(n)})=k)=\frac{\lambda^k e^{-\lambda}}{k!} \quad \textit{for} \quad k=0,1,\ldots \tag{6}$$

where $\lambda = \frac{e^{-2\alpha}}{r!}$; *in other words, the distribution of* $\nu_r(\Gamma_{n, N(n)})$ *tends to a Poisson distribution.*

PROOF OF THEOREMS 2 AND 3. Let $r \geqq 1$ be an integer and $-\infty < \alpha < +\infty$. Let us suppose that

$$N(n) = \frac{1}{2} n \log n + \frac{r}{2} n \log\log n + \alpha n + o(n). \tag{7}$$

Let $\Gamma_{n,N}$ be a random graph with the n vertices $V_1, V_2, \ldots, V_n$ and having N edges. Let $P_k(n, N, r)$ denote the probability that by removing r suitably chosen points from $\Gamma_{n,N}$ there remain two disjoint graphs, consisting of k and $n-k-r$ points, respectively. We may suppose $k < \left[\frac{n-r}{2}\right]$. First we have clearly

$$P_k(n, N, r) \leqq \binom{n}{r}\binom{n-r}{k} \frac{\binom{\binom{n}{2} - k(n-k-r)}{N}}{\binom{\binom{n}{2}}{N}}.$$

It follows by some obvious estimations that

$$\sum_{(r+3)\frac{\log n}{\log\log n} < k \leqq \left[\frac{n-r}{2}\right]} P_k(n, N(n), r) = O\left(\frac{1}{n}\right). \tag{8}$$

Now we consider the case $k \leqq (r+3)\frac{\log n}{\log\log n}$. Let $P_k^*(n, N, r)$ denote the probability that by removing r suitably chosen points (the set of which will be denoted by Δ) $\Gamma_{n,N}$ can be split into two disjoint subgraphs Γ' and Γ'' consisting of k and $n-k-r$ points, respectively, but that $\Gamma_{n,N}$ can not be made disconnected by removing only $r-1$ points. If $\Gamma_{n,N}$ has these properties and if s denotes the number of edges of $\Gamma_{n,N}$ connecting a point of Δ with a point of Γ', then we have clearly $s \geqq r$. Otherwise, by definition, $s \leqq rk$. Thus we have

$$P_k^*(n, N, r) \leqq \sum_{s=r}^{rk} \binom{n}{r}\binom{n-r}{k}\binom{rk}{s} \frac{\binom{\binom{n}{2} - k(n-k)}{N-s}}{\binom{\binom{n}{2}}{N}}. \tag{9}$$

It follows that

$$\sum_{k=2}^{\left[(r+3)\frac{\log n}{\log\log n}\right]} P_k^*(n, N(n), r) = O\left(\frac{1}{\log n}\right). \tag{10}$$

From (8) and (10) it follows that for $n \to +\infty$

$$\mathbf{P}(c_p(\Gamma_{n, N(n)}) = r) \sim \mathbf{P}(c(\Gamma_{n, N(n)}) = r). \tag{11}$$

As a matter of fact, (8) and (10) imply that if by removing r suitably chosen points (but not by removing less than r points) $\Gamma_{n, N(n)}$ can be split into two disjoint subgraphs Γ' and Γ'' consisting of k and $n-k-r$ points, respectively, where $k \leqq \left[\frac{n-r}{2}\right]$, then only the case $k=1$ has to be considered, the probability of $k>1$ being negligibly small. It remains to prove (5). This can be done as follows. First we prove that

$$\lim_{n \to +\infty} \mathbf{P}(c(\Gamma_{n, N(n)}) \leqq r-1) = 0. \tag{12}$$

For $r=1$ this follows already from Theorem 1. Thus we may suppose here $r \geqq 2$. We have

$$\mathbf{P}(c(\Gamma_{n, N}) \leqq r-1) \leqq \sum_{h=1}^{r-1} n \binom{n-1}{h} \frac{\binom{\binom{n}{2}-(n-1)}{N-h}}{\binom{\binom{n}{2}}{N}},$$

and thus

$$\mathbf{P}(c(\Gamma_{n, N(n)}) \leqq r-1) = O\left(\frac{1}{\log n}\right) \tag{13}$$

which proves (12).

Now let $\nu_r(\Gamma_{n, N})$ denote the number of vertices of $\Gamma_{n, N}$ which have the valency r. Then we have clearly by (12)

$$\mathbf{P}(c(\Gamma_{n, N(n)}) = r) \sim \mathbf{P}(\nu_r(\Gamma_{n, N(n)}) \neq 0). \tag{14}$$

Now evidently

$$\mathbf{P}(\nu_r(\Gamma_{n, N(n)}) \neq 0) = \sum_{j=1}^{n} (-1)^{j-1} S_j \tag{15}$$

where

$$S_j = \sum\sum \cdots \sum_{1 \leqq k_1 < k_2 < \cdots < k_j \leqq n} \mathbf{P}(d(V_{k_1}) = r, d(V_{k_2}) = r, \ldots, d(V_{k_j}) = r). \tag{16}$$

Evidently, if we stop after taking an even or odd number of terms of the sum on the right-hand side of (15), we obtain a quantity which is greater or smaller, respectively, than the left-hand side of (15). Now clearly

$$\mathbf{P}(d(V_k)=r)=\binom{n-1}{r}\frac{\binom{\binom{n}{2}-(n-1)}{N(n)-r}}{\binom{\binom{n}{2}}{N(n)}}\sim\frac{e^{-2\alpha}}{nr!},$$

and thus

$$\lim_{n\to+\infty} S_1=\frac{e^{-2\alpha}}{r!}. \tag{17}$$

Now let us consider $\mathbf{P}(d(V_{k_1})=r,\ d(V_{k_2})=r)$ where $k_1\neq k_2$. If both V_{k_1} and V_{k_2} have valency r, three cases have to be considered: a) either V_{k_1} and V_{k_2} are not connected, and there is no point which is connected with both V_{k_1} and V_{k_2}; b) or V_{k_1} and V_{k_2} are not connected, but there is a point connected with both; c) V_{k_1} and V_{k_2} are connected. We denote the probabilities of the corresponding subcases by $\mathbf{P}_a(d(V_{k_1})=r,\ d(V_{k_2})=r)$, $\mathbf{P}_b(d(V_{k_1})=r,\ d(V_{k_2})=r)$ and $\mathbf{P}_c(d(V_{k_1})=r,\ d(V_{k_2})=r)$, respectively. We evidently have

$$\mathbf{P}_a(d(V_{k_1})=r,\ d(V_{k_2})=r)=\frac{(n-2)!}{r!^2(n-2r-2)!}\frac{\binom{\binom{n}{2}-(2n-3)}{N(n)-2r}}{\binom{\binom{n}{2}}{N(n)}}\sim\left(\frac{e^{-2\alpha}}{n\cdot r!}\right)^2,$$

and thus

$$\sum\sum_{1\leq k_1<k_2\leq n}\mathbf{P}_a(d(V_{k_1})=r,\ d(V_{k_2})=r)\sim\frac{1}{2}\left(\frac{e^{-2\alpha}}{r!}\right)^2. \tag{18}$$

On the other hand (denoting by l the number of points which are connected with both V_{k_1} and V_{k_2}), we have

$$\mathbf{P}_b(d(V_{k_1})=r,\ d(V_{k_2})=r)=$$

$$=\sum_{l=1}^{r}\frac{(n-2)!}{l!(r-l)!(n-2r+l-2)!}\frac{\binom{\binom{n}{2}-(2n-3)}{N(n)-2r}}{\binom{\binom{n}{2}}{N(n)}}=O\left(\frac{1}{n^3}\right). \tag{19}$$

Similarly one has

$$\mathbf{P}_c(d(V_{k_1})=r,\ d(V_{k_2})=r)=$$

(20)
$$=\sum_{l=0}^{r-1}\frac{(n-2)!}{l!(r-l-1)!^2(n-2r+l)!}\frac{\binom{\binom{n}{2}-(2n-3)}{N(n)-2r}}{\binom{\binom{n}{2}}{N(n)}}=O\left(\frac{1}{n^4}\right).$$

Thus we obtain

$$\lim_{n\to+\infty} S_2=\frac{1}{2}\left(\frac{e^{-2\alpha}}{r!}\right)^2.$$

The cases $j>2$ can be dealt with similarly. Thus we obtain

(21)
$$\lim_{n\to+\infty} S_j=\frac{1}{j!}\left(\frac{e^{-2\alpha}}{r!}\right)^j \qquad (j=1,2,3,4,\dots).$$

It follows from (16) and (21) that

(22)
$$\lim_{n\to+\infty}\mathbf{P}(\nu_r(\Gamma_{n,N(n)})\neq 0)=1-\exp\left(-\frac{e^{-2\alpha}}{r!}\right).$$

In view of (2), (11) and (14) Theorem 2 follows.

To prove Theorem 3 it is sufficient to remark that by the well-known formula of CH. JORDAN

(23)
$$\mathbf{P}(\nu_r(\Gamma_{n,N(n)})=k)=\sum_{j=0}^{n-k}(-1)^j\binom{j+k}{j}S_{j+k},$$

and thus by (21), putting $\lambda=\dfrac{e^{-2\alpha}}{r!}$, we obtain for $k=0,1,\dots$

(24)
$$\lim_{n\to+\infty}\mathbf{P}(\nu_r(\Gamma_{n,N(n)})=k)=\frac{\lambda^k}{k!}\sum_{j=0}^{\infty}\frac{(-1)^j\lambda^j}{j!}=\frac{\lambda^k e^{-\lambda}}{k!}.$$

Thus Theorem 3 is proved.

Our thanks are due to T. GALLAI for his valuable remarks.

(Received 12 October 1960)

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ON W. FENCHEL'S SOLUTION OF THE PLANK PROBLEM

By

M. BOGNÁR (Budapest)

(*Presented by* G. HAJÓS)

We denote by E^n the n-dimensional Euclidean space. A *strip* $S \subset E^n$ is a closed convex set bounded by two parallel hyperplanes. Their distance is the *width* of S. The vector $\mathbf{v}$ is said to be the *width vector* of S, if S is bounded by hyperplanes normal to $\mathbf{v}$ and the width of S is $2|\mathbf{v}|$. The *width* of a convex body $K \subset E^n$ with inner points is the minimal width of a strip covering K. We denote points also by vectors leading to them from the origin.

A. TARSKI stated in 1932 the following conjecture, known as plank problem:

If a convex body K is covered by a finite number of strips, then the sum of their widths is not less than the width of K.

TH. BANG's [1] proof of this conjecture has been simplified by W. FENCHEL [2]. This simplified proof is based on three simple lemmas, which reduce the conjecture to the following statement:

The union of the strips $S_1, \ldots, S_r$ *of width vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_r$ *does not contain each of the* 2^r *points* $\lambda(\pm \mathbf{v}_1 \pm \cdots \pm \mathbf{v}_r)$, *where* $\lambda > 1$ *and the origin is an arbitrary point.*

Present note gives an alternative proof of this statement. We denote by Π the set of the 2^r points $\lambda(\pm \mathbf{v}_1 \pm \cdots \pm \mathbf{v}_r)$, further by (P, H) and (P, S) the distance of the point P from the hyperplane H and from the strip S, respectively.

We start the proof with the special case in which the middle hyperplanes $H_1, \ldots, H_r$ of the strips $S_1, \ldots, S_r$ have a common point C. Let P be an element of Π of maximal distance CP. We show that P is not covered by $S_1, \ldots, S_r$, i. e. P proves our statement.

If the sign of $\mathbf{v}_i$ $(i = 1, \ldots, r)$ is altered in the vector defining P, we get a vector which defines $P_i \in \Pi$. By definition of P we have $CP \geqq CP_i$. Consequently, since $C \in H_i$ and $PP_i \perp H_i$, we obtain $(P, H_i) \geqq (P_i, H_i)$, $(P, H_i) \geqq \frac{1}{2} PP_i = \lambda |\mathbf{v}_i|$ and $(P, S_i) \geqq (\lambda - 1) |\mathbf{v}_i|$. This proves that $S_1, \ldots, S_r$ contain neither P nor any point Q for which $PQ < (\lambda - 1) d$, where $d = \min(|\mathbf{v}_1|, \ldots, |\mathbf{v}_r|)$.

We reduce now the general case to the just settled special one. We imbed our E^n in an E^{n+1}, leave the origin unaltered in E_n, denote by $\mathbf{e}$ a unit vector normal to E^n and by C' the point $t\mathbf{e}$. Let us define the hyperplane H_i' $(i=1,\dots,r)$ by $C'\in H_i'$ and $H_i\subset H_i'$, further the strip S_i' by his middle hyperplane H_i' and by $S_i'\cap E^n=S_i$. We denote by $\mathbf{v}_i'$ the width vector of S_i' for which $\mathbf{v}_i'\mathbf{v}_i>0$ and by Π' the set of the 2^r points $\lambda(\pm\mathbf{v}_i'\pm\pm\cdots\pm\mathbf{v}_r')$. We apply the special case of our statement to the strips $S_1',\dots,S_r'$, and infer that the strips $S_1',\dots,S_r'$ cover neither P' nor any point Q' for which $P'Q'<(\lambda-1)d'$ where $d'=\min(|\mathbf{v}_1'|,\dots,|\mathbf{v}_r'|)$.

If $t\to\infty$, then clearly $\mathbf{v}_i'\to\mathbf{v}_i$ $(i=1,\dots,r)$, further $\Pi'\to\Pi$ and $d'\to d$. Consequently, if t is sufficiently large, we have $d'>\frac{d}{2}$ and P' defines $P\in\Pi$ for which $P'P<(\lambda-1)\frac{d}{2}$, i. e. P satisfies our conditions for Q'. Hence P is not covered by $S_1',\dots,S_r'$, nor by the strips $S_1,\dots,S_r$ contained in them. This completes the proof of the statement.

We remark, in order to facilitate comparison with FENCHEL's paper, that the expression

$$\lambda\left(\sum_{i=1}^{r}\varepsilon_i\mathbf{v}_i\right)^2+2\sum_{i=1}^{r}\varepsilon_i c_i$$

dealt with by FENCHEL is equal to

$$\lim_{t\to\infty}\frac{1}{\lambda}\left[\left(\lambda\sum_{i=1}^{r}\varepsilon_i\mathbf{v}_i'-t\mathbf{e}\right)^2-(t\mathbf{e})^2\right],$$

where ε_i is $+1$ or -1 $(i=1,\dots,r)$. This shows that our maximum method leads to the same point P as FENCHEL's maximum method.

(Received 1 September 1959)

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ÜBER DIE VERALLGEMEINERUNG DER THEORIE DER REKURSIVEN FUNKTIONEN FÜR ABSTRAKTE MENGEN GEEIGNETER STRUKTUR ALS DEFINITIONSBEREICHE[1]

Von

RÓZSA PÉTER (Budapest)

(*Vorgelegt von* L. KALMÁR)

1. Wie ich von L. KALMÁR erfahren habe, hat HU SCHI-HUA den Gedanken aufgeworfen, daß die Äquivalenz der Theorie der partiell-rekursiven Funktionen mit der Theorie der Markovschen Algorithmen durch Vermeidung der Gödelisierung der „Worte" vielleicht einfacher bewiesen werden kann, wenn als vermittelnder Zwischenbegriff der Begriff der auf der Menge der Worte definierten partiell-rekursiven Funktion eingeführt wird. Dabei hätte man zu benutzen, daß die Worte ähnlich wie die natürlichen Zahlen aufgebaut werden, nur statt aus 0 ausgehend die einzige Funktion x' zu benutzen, geht man hier aus dem leeren Wort $\wedge$ aus, und benutzt zur Bildung der Worte das Anknüpfen der Buchstaben eines Alphabets $\{\ldots, a_i, \ldots\}$.

Eine ähnliche Situation kommt in der Praxis oft vor, wie darauf auch KLEENE's Buch[2] hingewiesen hat, worin die verwendeten „entities'' von mehreren 0-Elementen ausgehend aufgebaut werden, aber durch eine einzige Nachfolgerfunktion. Ähnlicherweise werden zum Beispiel auch Terme oder Formeln aus gewissen Anfangselementen ausgehend durch gewisse Operatio-

[1] Über die Ergebnisse dieser Arbeit habe ich (mit geringer Abweichung) in Warszawa im „Internationalen Symposium der Grundlagen der Mathematik: Infinitistische Methoden" am 3-ten September 1959 einen Vortrag gehalten. — *Zusatz bei der Korrektur (am 25. August 1961):* Nach diesem Symposium und nach Einreichung noch im selben Monat vorliegender Arbeit haben verschiedene Verfasser gewisse Ergebnisse über den Spezialfall der Wortemengen publiziert. Ich habe von L. KALMÁR erfahren, daß ihm HU SCHI-HUA am 27. Dezember 1959 brieflich mitgeteilt hat: es werden von ihm über dieses Thema (genauer: über die Verallgemeinerung der Theorie der partiell-rekursiven Funktionen für Wortemengen mit endlichem Alphabet) drei Arbeiten im Acta Math. Sinica bzw. im Scientifica Sinica erscheinen (die Eingegangen-Daten dieser Arbeiten sind mir nicht bekannt). G. ASSER hat über „Rekursive Wortfunktionen" auf der Tagung der Deutschen Math. Vereinigung (19—23. Oktober 1959) in Münster einen Vortrag gehalten; das wird auch in der (demnach offenbar später eingegangenen) Arbeit „Rekursive Wortarithmetik" von V. VUČKOVIĆ (*Ac. Serbe des Sc., Publ. de l'Inst. Math.,* **14** (1960), S. 9—60) zitiert. Die Ergebnisse von G. ASSER wurden in seiner Arbeit „Rekursive Wortfunktionen" in der *Zeitschr. f. math. Logik und Grundlagen d. Math.,* **6** (1960), S. 258—278, veröffentlicht. Sowohl G. ASSER als auch V. VUČKOVIĆ beschränken sich auf Wortemengen, sogar mit endlichem Alphabet.

[2] S. C. KLEENE, *Introduction to metamathematics* (Amsterdam—Groningen, 1952).

nen gebildet, und auf den so erhaltenen „zahlenartigen" Mengen werden weitere Begriffe rekursiv definiert. Um auch ein Beispiel von den vielen zu nennen: in dem Gentzenschen Beweis der Widerspruchsfreiheit der Zahlentheorie werden die Reduzierten einer Formel rekursiv definiert auf der Formelnmenge. In diesen Fällen wird wesentlich benutzt, daß die betreffenden zahlenartigen Gebilde, zum Beispiel Formeln, aus den Ausgangsgebilden eindeutig aufgebaut werden.

Man könnte noch zahlreiche weitere Beispiele aufzählen.

Es scheint demnach lohnend, die zahlenartig aufbaubaren Mengen und die auf diesen definierten rekursiven Funktionen allgemein zu untersuchen. Die allgemeine Theorie wird natürlich nicht konstruktiv sein, aber es können sich daraus nützliche Anwendungen für die konstruktiven Spezialfälle ergeben.

Betreffend der allgemeinen Kentnisse über rekursive Funktionen berufe ich mich auf mein Buch.[3] Es wird zum Beispiel oft benutzt, daß die in $\frac{m}{n}$ enthaltene größte ganze Zahl $\left[\frac{m}{n}\right]$, der Binomialkoeffizient $\binom{m}{n}$ und

$$m \dot{-} n = \begin{cases} m - n, & \text{falls } m \geqq n, \\ 0 & \text{sonst} \end{cases}$$

primitiv-rekursive zahlentheoretische Funktionen sind, und m/n eine primitiv-rekursive zahlentheoretische Beziehung ist.

In Kapitel I gebe ich die Definition einer zahlenartig aufbaubaren Menge an; zeige, wie verschiedene Rekursionsarten auf einer solchen Menge definiert werden können, und mit welchen Bedingungen die Ergebnisse über die entsprechenden zahlentheoretischen Rekursionsbegriffe auf diese übertragen werden können. In den Weiteren betrachte ich als ein Beispiel den Fall der Menge der aus einem Alphabet gebildeten Worte. Das wird auch zur Darstellung von partiell- bzw. allgemein-rekursiven Funktionen auf zahlenartig aufbaubaren Mengen in einer Art Kleenesche explizite Form benutzt.

I

2. Sei H eine beliebige nicht leere Menge, H_0 eine nicht leere Teilmenge von H (die Elemente von H_0 werden die Rolle von 0 übernehmen) und F eine Teilmenge der auf H definierten Funktionen mit einer beliebigen Anzahl von Argumenten, die als Werte ebenfalls Elemente von H annehmen (die Elemente von F übernehmen die Rolle von x').[4] Sei H_1 die Menge derjenigen

[3] R. Péter, *Rekursive Funktionen* (Budapest, 1957), 2-te Auflage.

[4] In den Weiteren wird Funktion — falls nicht explizit etwas anderes gesagt wird — immer eine derartige Funktion bedeuten.

Elemente von H, die sich als Werte ergeben, wenn für alle Argumente der zu F gehörigen Funktionen Elemente aus H_0 eingesetzt werden; ferner sei H_2 die Menge derjenigen Elemente von H, die sich als Werte ergeben, wenn für alle Argumente der zu F gehörigen Funktionen Elemente aus H_0+H_1 eingesetzt werden, aber mindestens für ein Argument ein Element aus H_1, usw.; sind $H_0, \ldots, H_n$ bereits definiert, so sei H_{n+1} die Menge derjenigen Elemente von H, die sich als Werte ergeben, wenn für alle Argumente der zu F gehörigen Funktionen Elemente aus $H_0+\cdots+H_n$ eingesetzt werden, aber mindestens für ein Argument ein Element von H_n. Wir nehmen an, daß die Vereinigungsmenge von $H_0, H_1, H_2, \ldots$ die Menge H erschöpft. Eine Menge von dieser Struktur nenne ich wegen der ganzzahlenartigen Aufbaubarkeit *holomorph.*

Eine für die Möglichkeit der Definition von zahlentheoretischen Funktionen durch Rekursion wesentliche Eigenschaft der natürlichen Zahlen ist ihre *eindeutige* Aufbaubarkeit aus 0 mit Hilfe der Funktion x'. Etwas ähnliches werde ich auch im allgemeinen Fall der rekursiven Definition von Funktionen auf einer holomorphen Menge H benötigen, nämlich, daß die Mengen $H_0, H_1, H_2, \ldots$ paarweise fremd sind, daß sogar jedes ihrer Elemente nur auf eine Weise durch Einsetzen von Elementen vorangehender Mengen für Argumente einer Funktion aus F entsteht, daß also eindeutig bestimmt ist, welche diese Funktion aus F ist, und welche Elemente für ihre Argumente stehen. Eine holomorphe Menge von dieser Eigenschaft nenne ich mit einem aus der Algebra entliehenen Ausdruck eine *freie holomorphe Menge.* Eine holomorphe Menge H heißt also frei, falls aus $f(x_1, \ldots, x_r)=g(y_1, \ldots, y_s)$ mit $f, g \in F$ und $x_1, \ldots, x_r, y_1, \ldots, y_s \in H$ stets $f=g$ (also $r=s$) und $x_1=y_1, \ldots, x_r=y_s$ folgt.

In den Folgenden soll H immer eine freie holomorphe Menge bezeichnen; auch die Bezeichnungen $F, H_0, H_1, \ldots$ sollen festgehalten werden. Ist $h \in H_i$, dann sage ich, daß $i=o(h)$ die Ordnung von h ist.

Als Beispiele von freien holomorphen Mengen seien die folgenden angeführt:

Beispiel 1. Es sei $H=\{0, 1, 2, \ldots\}$ mit $H_0=\{0\}$, $F=\{x'\}$. Hier ist $o(x)=x$ für jedes $x \in H$. Auf dieses Beispiel werde ich mich mit den Worten „zahlentheoretischer Fall" beziehen.

Beispiel 2. H sei die Menge der Worte über einem beliebigen „Alphabet" $A=\{\ldots, a_i, \ldots\}$, d. h. die Menge der endlichen Folgen aus Elementen von A; H_0 sei $\{\wedge\}$, wobei $\wedge$ das „leere Wort" bezeichnet, F bestehe aus den Funktionen einer Variablen, die jedes Wort $x \in H$ in das durch Anknüpfen einer „Buchstaben" (d. h. eines Elementes a_i von A) entstehende Wort xa_i überführt (i soll eine Indexmenge von beliebiger Mächtigkeit durchlaufen). Hier ist die Ordnung eines beliebigen Wortes $a_{i_1}a_{i_2}\ldots a_{i_r}$ gleich seiner Länge r.

BEISPIEL 3. H sei die Menge der Formeln des Aussagenkalküls; H_0 sei die Menge der Aussagenvariablen, F bestehe aus den für $x \in H$, $y \in H$ durch

$$\nu(x) = \bar{x}, \quad \varkappa(x, y) = (x \& y), \quad \delta(x, y) = (x \vee y),$$
$$\iota(x, y) = (x \rightarrow y), \quad \varepsilon(x, y) = (x \leftrightarrow y)$$

definierten Funktionen $\nu, \varkappa, \delta, \iota$ und ε. In diesem Fall wird die Freiheit der holomorphen Menge H durch die angedeutete Art der Verwendung von Klammern gesichert.

3. Im zahlentheoretischen Fall ist für die primitive Rekursion der Begriff des *unmittelbaren Vorgängers* x, und für die Wertverlaufsrekursion der Begriff der *Vorgänger* $0, 1, \ldots, x$ einer von 0 verschiedenen Zahl x' wesentlich. Offenbar benötigt man auch im allgemeinen Fall entsprechende Begriffe. Zunächst könnte man daran denken, diese Begriffe auf Grund der Struktur von H fest zu definieren, etwa im Fall $x \notin H_0$ die *unmittelbaren Konstituenten* von x, d. h. die eindeutig bestimmten Elemente $x_1^*, \ldots, x_m^*$ von H, für welche $x = f^*(x_1^*, \ldots, x_m^*)$ mit einer Funktion $f^* \in F$ gilt, als die unmittelbaren Vorgänger von x zu betrachten, und den Begriff der Vorgänger entsprechend zu definieren (siehe Beispiel 4). Dann wäre aber $o(x_i^*) = o(x) - 1$ nicht für alle $1 \leqq i \leqq m$ gesichert, obwohl wichtige Spezialfälle zeigen, daß es zweckmäßig ist darauf zu bestehen. Es ergibt sich sogar, daß es überhaupt nicht zweckmäßig ist, den Vorgängerbegriff für jede freie holomorphe Menge ein für allemal festzulegen, da bei verschiedenen Verallgemeinerungen der Sätze der Theorie der rekursiven Funktionen verschiedene Vorgängerbegriffe als zweckmäßig erscheinen. Ich setze daher voraus, es sei jedem $x \in H$ eine Untermenge $V(x)$ von H zugeordnet, deren Elemente die *Vorgänger* von x heißen. Sei auch $x \in V(x)$; die von x verschiedenen Elemente von $V(x)$ nenne ich die *echten Vorgänger* von x. Als kurze Bezeichnung verwende ich $y \prec x$ dafür, daß y ein echter Vorgänger von x ist, und somit $y \preceq x$ dafür, daß y ein Vorgänger von x ist. Jedenfalls setze ich die Gültigkeit der folgenden „Vorgängeraxiome" voraus:

V1. Für jedes $x \in H$ ist $x \in V(x)$ (wie gesagt); d. h. $x \preceq x$.

V2. Für jedes $x \in H$, $x \notin H_0$ gehören die unmittelbaren Konstituenten von x zu $V(x)$; d. h. für $x = f^*(x_1^*, \ldots, x_m^*)$, $f^* \in F$, $x_1^*, \ldots, x_m^* \in H$ gilt $x_1^* \preceq x, \ldots$ $\ldots, x_m^* \preceq x$. (Offenbar gilt dann sogar $x_1^* \prec x, \ldots, x_m^* \prec x$.)

V3. Aus $y \prec x$ und $z \prec y$ folgt $z \prec x$.

V4. Aus $y \prec x$ folgt $o(y) < o(x)$.

Ein Vorgänger y von $x \in H$, $x \notin H_0$ mit $o(y) = o(x) - 1$ heiße ein *unmittelbarer Vorgänger* von x.

V5. Jeder echte Vorgänger von $x \in H$ ist ein Vorgänger eines unmittelbaren Vorgängers von x; d. h. für $y \prec x$ gibt es ein z mit $y \preceq z$, $z \prec x$ und $o(z) = o(x) - 1$.

V6. Die Menge der unmittelbaren Vorgänger von $x \in H$ ist endlich, und zwar liegt bei festem $f^* \in F$ die Anzahl der unmittelbaren Vorgänger von $x = f^*(x_1^*, \ldots, x_m^*)$ unter einer (eventuell von f^* abhängiger, aber) von $x_1^*, \ldots, x_m^*$ unabhängiger Schranke.

V1—V5 drücken plausible Forderungen aus; V6 wird dadurch motiviert, daß sonst bereits in eine primitive Rekursion (d. h. Definition des Wertes einer Funktion f an einer beliebigen Stelle $x = f^*(x_1^*, \ldots, x_m^*) \in H$, in einer im allgemeinen von $f^* \in F$ abhängigen Weise, mit Hilfe der Werte von f für die unmittelbaren Vorgänger von x, wobei die Funktionswerte an den Stellen $x \in H_0$ direkt angegeben werden) Funktionen von einer veränderlichen Anzahl von Argumenten eingehen könnten.

Eine freie holomorphe Menge H mit einem den Axiomen V1—V6 genügenden Vorgängerbegriff nenne ich eine (partiell) *angeordnete freie holomorphe Menge.* (Die Beziehung $y \prec x$ definiert wegen V1, V3 und V4 offenbar eine partielle Anordnung von H, die freilich mit der Struktur von H als freie holomorphe Menge verknüpft ist, ebenso wie etwa die Anordnung eines angeordneten Körpers mit der Körperstruktur.)

Als Beispiele von angeordneten freien holomorphen Mengen seien die folgenden angeführt. Der Leser möge V1—V6 bei jedem Beispiel verizifieren.

Beispiel 1′. Im zahlentheoretischen Fall sei $V(x) = \{0, 1, \ldots, x\}$. Offenbar ist dann der einzige unmittelbare Vorgänger von x' seine einzige unmittelbare Konstituente x.

Beispiel 2′. Aus der freien holomorphen Menge H des Beispiels 2 (der Menge der Worte über einem beliebigen Alphabet) gewinnt man eine angeordnete freie holomorphe Menge, indem man unter Vorgänger eines Wortes $a_{i_1} a_{i_2} \ldots a_{i_r}$ seine „Anfangstücke"

$$\wedge, \; a_{i_1}, \; a_{i_1} a_{i_2}, \ldots, a_{i_1} a_{i_2} \ldots a_{i_{r-1}}, \; a_{i_1} a_{i_2} \ldots a_{i_r}$$

versteht. Dann ist wiederum der einzige unmittelbare Vorgänger eines Wortes $x \neq \wedge$ seine einzige unmittelbare Konstituente $x^{-1} = a_{i_1} a_{i_2} \ldots a_{i_{r-1}}$ (im Fall $r = 1$ $x^{-1} = \wedge$). Um x^{-1} für jedes $x \in H$ zu definieren, setze ich $\wedge^{-1} = \wedge$; dies ist selbstverständlich kein unmittelbarer Vorgänger von $\wedge$.

Beispiel 2″. Aus derselben freien holomorphen Menge H gewinnt man eine andere angeordnete freie holomorphe Menge, indem man unter Vorgänger eines Wortes $x = a_{i_1} a_{i_2} \ldots a_{i_r}$ das leere Wort $\wedge$, ferner sämtliche „Abschnitte"

$a_{i_k}a_{i_{k+1}}\dots a_{i_l}$ $(1 \leqq k \leqq l \leqq r)$ von x versteht. Z. B. sind dann die Vorgänger des Wortes $a_1a_2a_3a_4$ die folgenden Worte:

$$\wedge,\ a_1,\ a_2,\ a_1a_2,\ a_3,\ a_2a_3,\ a_1a_2a_3,\ a_4,\ a_3a_4,\ a_2a_3a_4,\ a_1a_2a_3a_4.$$

Unter diesen gibt es zwei unmittelbare Vorgänger von x (hier $a_1a_2a_3$ und $a_2a_3a_4$); und offenbar gilt dasselbe für jedes Wort x, das mindestens aus zwei Buchstaben besteht. Der eine entsteht durch Weglassen des letzten Buchstaben, der andere durch Weglassen des ersten Buchstaben des Wortes. Der erste wurde bereits mit x^{-1} bezeichnet, der zweite soll mit ${}^{-1}x$ bezeichnet werden. In diesem Fall ist nur eine der unmittelbaren Vorgänger, nämlich x^{-1}, eine unmittelbare Konstituente von x. Für ein Wort $x = a_i$ von der Länge 1, das $a_i^{-1} = \wedge$ als einzigen (unmittelbaren) Vorgänger (und zugleich einzige unmittelbare Konstituente) besitzt, soll ${}^{-1}a_i = \wedge$ gesetzt werden; und für $x = \wedge$ sei ebenfalls $\wedge^{-1} = {}^{-1}\wedge = \wedge$. Später werden wir sehen, daß dieser Vorgängerbegriff von einem wichtigen Gesichtspunkt aus vorteilhafter ist, als der in 2′ verwendete, auf erstem Blick natürlicher erscheinende Vorgängerbegriff.

Beispiel 3′. Aus der freien holomorphen Menge H des Beispiels 3 (der Menge der Formeln des Aussagenkalküls) gewinnt man eine angeordnete freie holomorphe Menge, indem man unter Vorgänger einer Formel ihre Teilformeln (im üblichen Sinn) versteht. Man sieht leicht, daß in diesem Fall eine Formel von der Form $\overline{x}$ die Formel x als einzigen unmittelbaren Vorgänger (und zugleich einzige unmittelbare Konstituente), eine Formel von der Form $(x \& y)$, $(x \vee y)$, $(x \rightarrow y)$ oder $(x \leftrightarrow y)$ aber entweder beide, oder nur eine seiner unmittelbaren Konstituenten x und y als unmittelbare Vorgänger besitzt. Z. B. hat die Formel $\big((A \& B) \rightarrow (B \vee A)\big)$ zwei unmittelbare Vorgänger, nämlich $(A \& B)$ und $(B \vee A)$; die Formel $\big((A \& (A \rightarrow B)) \rightarrow B\big)$ aber nur einen, nämlich $\big(A \& (A \rightarrow B)\big)$.

Beispiel 4. Auch im Fall einer beliebigen freien holomorphen Menge kann man einen den Axiomen V1—V6 genügenden Vorgängerbegriff definieren. Ich nenne für jedes $x \in H$ das Element x selbst, seine unmittelbaren Konstituenten, die unmittelbaren Konstituenten der unmittelbaren Konstituenten von x usw. die *Konstituenten* von x. Genauer definiert man die Menge der Konstituenten von x durch (Wertverlaufs-) Rekursion in Bezug auf die Ordnung von x wie folgt. Für $o(x) = 0$, d. h. $x \in H_0$, ist x die einzige Konstituente von x. Für $o(x) = n+1$, also $x = f^*(x_1^*, \dots, x_m^*)$ mit $o(x_1^*) \leqq n, \dots, o(x_m^*) \leqq n$ sind die Konstituenten von x die Folgenden: x selbst, ferner die Konstituenten von $x_1^*, \dots, x_m^*$. Versteht man unter Vorgänger von x seine Konstituenten, so gewinnt man aus H eine angeordnete freie holomorphe Menge. In diesem Fall kommen die unmittelbaren Vorgänger von x unter seinen unmittelbaren Konstituenten vor, aber nicht notwendig jede unmittelbare Konstituente von x

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РЕЗЮМЕ

ЗАМЕЧАНИЕ К ДОКАЗАТЕЛЬСТВУ ФЕНХЕЛА „ПРОБЛЕМЫ ДОЩЕЧЕК"

М. Богнар (Будапешт)

Доказательство В. Фенхела [2] поставленной А. Тарски „Проблемы дощечек" опирается на следующую лемму:

Лемма. Пусть $S_1, \ldots, S_r$ суть параллельные полосы в n-мерном евклидовом пространстве. Пусть при $\varrho = 1, \ldots, r$ толщина S_ϱ равна d_ϱ, а ее нормальный вектор $\mathbf{v}_\varrho$, где $|\mathbf{v}_\varrho| = \frac{1}{2} d_\varrho$. Пусть $\lambda > 1$. Тогда среди 2^r точек

$$\lambda(\pm \mathbf{v}_1 \pm \cdots \pm \mathbf{v}_r)$$

будет такая, которая не покрывается ни одной из полос $S_1, \ldots, S_r$.

В настоящей работе дается новое, чисто геометрическое доказательство леммы.

ОБОБЩЕНИЕ ТЕОРИИ РЕКУРСИВНЫХ ФУНКЦИЙ НА АБСТРАКТНЫХ МНОЖЕСТВАХ ПОДХОДЯЩЕЙ КОНСТРУКЦИИ КАК НА ОБЛАСТЯХ ОПРЕДЕЛЕНИЯ

Р. Петер (Будапешт)

Согласно устному сообщению Л. Калмара, Ху Ши-хуа высказал предположение, согласно которому эквивалентность теории частных-рекурсивных функций с теорией алгорифмов Маркова была бы более проще доказываема, если бы вместо гёделизации слов в качестве посреднического понятия было бы введено понятие частной-рекурсивной функции, определенной на множестве слов. Для этого можно было бы воспользоваться тем фактом, что, исходя из пустого слова, слова строятся аналогично числам, применяя вместо образования единственного следующего x' присоединение к слову x различных букв некоторого алфавита.

В практике встречается и ряд других примеров того, что на множествах, строящихся аналогично числам, например, на множествах формул, рекурсией определяются дальнейшие понятия. Поэтому кажется полезным множества, строящиеся аналогично числам, и определенные на них рекурсивные функции сделать объектом общего исследования. Общая теория не конструктивная, но из нее могут получаться полезные приложения к конструктивным специальным случаям.

Мнжество H, которое однозначно генерируется из элементов некоторого его подмножества H_0 некоторыми определенными на нем функциями, может быть названо

„свободным голоморфным множеством". В главе I работы даются определения различных видов рекурсий на таких множествах, предпологая, что понятие „предыдущих элементов" определено на них подходящим образом, и примеры того, как можно избежать применения методов теории чисел в доказательствах теорем касающихся их. С другой стороны, доказывается, что всякая примитивная-рекурсивная функция в смысле теории чисел может быть распространена до примитивной-рекурсивной функции на множестве H. О функциях, определенных на множестве H рекурсией множества значений, симультанной рекурсией, вложенной рекурсией, показывается, что если на некотором расширении H' множества H (возможно, на самом H) можно определить вспомогательные функции, удовлетворяющие некоторым условиям, то они могут быть расширены до примитивных-рекурсивных функций на множестве H. Примитивные рекурсии H сводятся к более простому виду.

В главе II, специализируя общие результаты на множества слов, показывается, что они справедливы здесь без добавления новых основных функций.

Эти результаты в дальнейшем используются для приведения определенных на свободных упорядоченных голоморфных множествах частных и общих рекурсивных функций к явной форме Клини.*

ПРОЕКТИВНЫЕ n-МЕРНЫЕ СИМПЛЕКСЫ В $2n$—2-МЕРНОМ ПРОЕКТИВНОМ ПРОСТРАНСТВЕ

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n-мерные симплексы (P) и (Q), расположенные в $2n-1$- или $2n-2$-мерном проективном пространстве, называются проективным относительно $n-2$-мерного проективного пространства t, если t есть общая трансверсаль прямых, связывающих соответствующие вершины симплексов. Пусть R_i $(i=1,2,\ldots,n+1)$ есть общая точка соответствующей пары $n-1$-мерных граней симплексов. Автор дает синтетическое доказательство следующего обобщения теоремы Дезарга:

(P) и (Q) проективны относительно t в том и только в том случае, если точки R_i $(i=1,2,\ldots,n+1)$ коллинейны.

ОБ ОДНОЙ ТЕОРЕМЕ ОБ ОТНОСИТЕЛЬНОЙ НЕПРОТИВОРЕЧИВОСТИ, СВЯЗАННОЙ С ОБОБЩЕННОЙ ТЕОРЕМОЙ КОНТИНУУМА

А. Хайнал (Будапешт)

Настоящая работа содержит подробное доказательство теоремы автора, опубликованной в предварительном сообщении в 1956-ом году. (См. *Zeitschrift f. Math. Logik und Grundlagen d. Math.*, **2** (1956), стр. 131—136.)

Пусть Λ, N обозначают порядковые числа, которые могут быть определены некоторой формулой в системе аксиом теории множеств Гёделя, данной в [1], — в системе аксиом Σ^*.

* Замечание редакции. Окончание работы по техническим причинам будет опубликованно в следующем выпуске журнала.

Пусть $F_{\Lambda, N}$ означает аксиому $2^{\aleph} \leqq \aleph_{\Lambda+N+1}$, $F^*_{\Lambda, N}$ аксиому

$$(\varrho)(\Lambda \leqq \varrho < \Lambda \dotplus N \dotplus 1 \supset 2^{\aleph_\varrho} = \aleph_{\Lambda \dotplus N \dotplus 1}) \cdot (\mu)(\mu \geqq \Lambda \dotplus N \supset 2^{\aleph_\mu} = \aleph_{\mu \dotplus 1}).$$

Основная теорема работы — теорема 2 — утверждает, что для широкого класса порядковых чисел — для так называемых абсолютно определимых порядковых чисел — имеет место следующее:

Если система аксиом Σ^*, $F_{\Lambda, N}$ непротиворечива, то непротиворечива и система аксиом Σ^*, $F^*_{\Lambda, N}$. (Значение абсолютно определяемых порядковых чисел см. в § 6.)

Из ряда интересных следствий теоремы 2 упомянем следующее:

Утверждение $2^{\aleph_0} = \aleph_1$ — гипотеза континуума — в том и только в том случае может быть доказана в системе аксиом Σ^*, если может быть доказана одна из формул $2^{\aleph_0} \neq \aleph_2$, $2^{\aleph_0} \neq 2^{\aleph_1}$.

Упомянем, что доказательства конструктивны, так, например, из любого опровержения в Σ^* формулы $2^{\aleph_0} = 2^{\aleph_1}$ — так называемой гипотезы Лузина — может быть построено доказательство формулы $2^{\aleph_0} = \aleph_1$.

О ГРАФАХ ИМЕЮЩИХ ДВЕ ОТМЕЧЕННЫЕ ТОЧКИ

А. Адам (Сегед)

В настоящей работе исследованы сильно связные графы с двумя отмеченными точками. Работа состоит из трех глав. Глава I дает определение самых основных понятий, глава II занимается вопросами последовательного и параллельного разложения.

Глава III содержит структурные результаты, относящиеся к одному классу графов, неразложимых в смысле, указанном в работе [12]. Этот класс можно охарактеризовать как класс двухполюсных неразложимых графов, содержащих цепь без двойного ребра. В этих графах определены мосты и результаты освещают возможные помещения мостов.

Содержание работы в основном совпадает с содержанием работ [1], [2], [4].

НЕСКОЛЬКО ЗАМЕЧАНИЙ О РАЗМЕРНОСТИ И ЭНТРОПИИ СЛУЧАЙНЫХ ВЕЛИЧИН

И. Чисар (Будапешт)

А. Реньи ([1], [2]) ввел понятие размерности и d-мерной энтропии случайных величин. Он доказал [2], что если ξ случайная величина с абсолютно непрерывным распределением, для которой $\mathbf{H}_0([\xi]) < +\infty$, то размерность ξ

$$\mathbf{d}(\xi) = \lim_{n\to\infty} \frac{\mathbf{H}_0(\xi_n)}{\log n} = 1,$$

если далее

$$\mathfrak{I} = \int_{-\infty}^{+\infty} f(x) \log \frac{1}{f(x)}\, dx$$

конечен, то одномерная энтропия ξ существует, а именно

$$\mathbf{H}_1(\xi) = \lim_{n\to\infty} (\mathbf{H}_0(\xi_n) - \log n) = \mathfrak{I},$$

где $\mathbf{H}_0(\xi_n)$ энтропия Шэннона дискретной случайной величины $\xi_n = \frac{1}{n}[n\xi]$, а $f(x)$ функция плотности распределении от ξ.

В настоящей работе, изучая вопрос об обратимости теоремы Реньи, доказывается, что если распределение ξ не абсолютно непрерывно, то одномерной энтропии не существует, а $\lim_{n\to\infty}(\mathbf{H}_0(\xi_n)-\log n) = -\infty$ (теорема 1). Доказательство элементарно, оно основывается на оценке величины $\mathbf{H}_0(\xi_n)-\log n$ с помощью неравенства Иенсена.

Также элементарно доказывается, что $\lim_{n\to\infty}(\mathbf{H}_0(\xi_n)-\log n)$ существует при любом распределении ξ (теорема 2). Дается также более простое, использующее основные теоремы относительно интеграла Лебега, доказательство того, что в абсолютно непрерывном случае

$$\lim_{n\to\infty}(\mathbf{H}_0(\xi_n)-\log n) = \int_{-\infty}^{+\infty} f(x)\log\frac{1}{f(x)}\,dx = \mathfrak{I},$$

не требуя даже конечности $\mathfrak{I}$ (теорема 3). Теорема 4 утверждает, что в случае не абсолютно непрерывного распределения ξ (даже если ξ ограничена) $\mathbf{H}_0(\xi_n)-\log n$ может стемиться к $-\infty$ как угодно медленно, следовательно утверждение теоремы 1 не может быть усилено.

Полученные результаты могут быть обобщены и для r-мерных случайных векторных величин.

НОРМАЛЬНЫЙ ПОРЯДОК АДДИТИВНЫХ АРИФМЕТИЧЕСКИХ ФУНКЦИЙ НА МНОЖЕСТВЕ „СДВИНУТЫХ" ПРОСТЫХ ЧИСЕЛ

М. Б. Барбан (Ташкент, СССР)

Цель настоящей работы доказать следующую теорему:

Пусть $f(m)$ — неотрицательная сильно аддитивная $(f(m) = \sum_{p|m} f(p))$ арифметическая функция,

$$A_n = \sum_{p<n}\frac{f(p)}{p}, \quad \Lambda = \max_{p<n} f(p), \quad A_n \to \infty \quad \text{и} \quad \Lambda_n = o(A_n).$$

Тогда A_n — нормальный порядок $f(m)$.

Метод, используемый работой, аналогичен методу Турана [3] и Линника [5].

О КОЛЬЦАХ С УСЛОВИЕМ МИНИМАЛЬНОСТИ ДЛЯ ГЛАВНЫХ ПРАВЫХ ИДЕАЛОВ. II

Ф. Сас (Будапешт)

Теоретико-кольцевые понятия употребительны в смысле Джекобсона [13], причем кольца всегда ассоциативны. Алгебраические понятия обнаруживаются еще в [1], [4], [7] и [21].

MHR-кольцо означает кольцо с условием минимальности для главных идеалов.

MH_1R-кольцо является кольцом с условием минимальности для правых идеалов, лежащих в одном главном правом идеале.

MHI-кольцо (или MH_1I-кольцо) является двусторонним аналогом MHR-кольца (или MH_1R-кольца), т. е. кольцом с условием минимальности для главных идеалов (или для идеалов, лежащих в одном главном идеале).

$MMHR$-кольцо является MHR-кольцом с условием максимальности для главных правых идеалов.

MHU-кольцо является кольцом с условием минимальности для подколец, порожденных одним элементом.

Правый идеал eA ($e^2 = e \in A$) кольца A называется „соответствующим”, если подкольцо $(1-e)AeA(1-e)$ нильпотентно, где $1 \in I$ (I кольцо целых чисел).

Пусть E кольцо всех эндоморфизмов одной абелевой группы M и $A = A^{(0)}$ некоторое подкольцо в $A^{(0)}$, причем $A^{(n+1)}$ централизатор подкольца $A^{(n)}$ в кольце A. Тривиально имеют место $A^{(2k)} = A^{(2)}$ и $A^{(2k-1)} = A^{(1)}$, если $k \geqq 1$, а случай $k = 0$ является критическим.

A-модуль M с условием $MA = M$ называется перфектным. Вполне редуцибельный гомоморфный образ M/K модуля M является „отмеченным”, если из существования A-гомоморфизма φ произвольного вполне редуцибельного модуля M/L на M/K следует, что φ является изоморфизмом.

В этой статье доказываются следующие результаты:

Кольцо A является тогда и только тогда полупростым MHR-кольцом, когда оно есть прямая сумма своих идемпотентных минимальных правых идеалов. В этом случае A является и теоретико-кольцевой прямой суммой простых MHR-колец, где прямое разложение в сущности единственно. MHI-кольцо A, которое является подпрямой суммой простых колец, есть прямая сумма простых колец, и поэтому оно есть MH_1I-кольцо. Мы построили коммутативное MHI-кольцо (т. е. и MHR-кольцо), которое не является MH_1I-кольцом (или MH_1R-кольцом) и радикал Джекобсона которого ненильпотентен. Исследовались и специальные полупростые MHR-кольца. Мы получили одно необходимое и достаточное условие того, чтобы кольцо было прямой суммой тел, и. т. д. Всякое полупростое MHR-кольцо регулярно в смысле Неймана, а конкретный пример показывает, что обратное утверждение неверно. Обобщается также один критерый Харада—Ковача или Кертеса.

Аддитивная группа MHR-колец или MHR-колец с единицей, примитивных MHR-колец, MHR-нильколец была полностью описана. В нильпотентных MH_1R-кольцах имеет место и условие минимальности для аддитивных подгрупп, лежащих в одном главном правом идеале.

Изучено строение абелевой группы G, кольцо A всех эндоморфизмов которой является MHR-кольцом. В этом случае A есть кольцо с условием минимальности и для главных левых идеалов, для всех правых идеалов, для всех левых идеалов. Мы построили абелеву группу M, кольцо E всех эндоморфизмов которой имеет MHR-подкольцо $A = A^{(0)}$ с условием $A^{(2)} \neq A^{(0)}$. Исследовались и системы Лоевия над MHR-кольцами, используя и понятие „отмеченных” вполне редуцибельных фактормодулей. Мы изучили и другие проблемы для модулей и перфектных модулей над MHR-кольцами.

Некоторая трансфинитная степень радикала Джекобсона J произвольного MHR-кольца A есть 0. Нижний нильрадикал Бэра—Мэккоя B, радикал L Левицкого, верхний нильрадикал N и радикал J Джекобсона равны между собой у MHR-колец. Пусть Z зероидрадикал Л. Фукса и G радикал Брауна—Мэккоя. Тогда в MHR-кольцах $J \subseteq Z$, и при существовании правой единицы $J = Z$ и $J = G$.

Мы изучили и $MMHR$-кольца, „соответствующие” главные правые идеалы цоколя $MMHR$-колец и артиновских колец, и некоторые свойства MHU-колец.

Приведены и некоторые открытые проблемы у всех §.

О РАЗЛОЖЕНИИ ОПЕРАТОРОВ В ГИЛЬБЕРТОВОМ ПРОСТРАНСТВЕ

Г. Лангер (Дрезден)

Простой конструкцией доказывается, что нерастягивающий оператор T (т. е. оператор, обладающий свойством $\|T\| \leqq 1$) в гильбертовом пространстве $\mathfrak{H}$ может быть представлен ортогональной суммой двух операторов T_0 и T_1 так, что T_0 в соответствующем гильбертовом пространстве $\mathfrak{H}_0$ унитарен, а T_1 в пространстве $\mathfrak{H}_1 = \mathfrak{H} \ominus \mathfrak{H}_0$ таков, что для каждого элемента $x \in \mathfrak{H}_1$ $(x \neq 0)$, существует целое число n такое, что $\|T^{(n)}x\| < \|x\|$. Из этого следует, что все комплексные числа λ $(|\lambda| = 1)$ принадлежат к множеству $\varrho(T_1) \cup \sigma_c(T_1)$ ([5]). С помощью преобразования Кэли соответствующие предложения доказываются для замкнутого оператора A, обладающего свойством $\partial(A) \subset \subset \partial(A^*)$ и неотрицательной мнимой частю (т. е. для которого $\left(\frac{A - A^*}{2i} x, x\right) \geqq 0$ при $x \in \partial(A)$).

ОБ ОДНОЙ ТЕОРЕМЕ КУЗЬМИНА

П. Сюс (Будапешт)

Кузьмин в 1928-ом году доказал гипотезу Гаусса, согласно которой

$$\lim_{n \to \infty} m_n(x) = \frac{\log(1+x)}{\log 2}, \tag{1}$$

где $m_n(x)$ мера тех вещественных чисел, лежащих на отрезке $(0, 1)$, для которых

$$\frac{1}{\zeta_{n+1}(\alpha)} \leqq x;$$

здесь $\zeta_{n+1}(\alpha)$ определено следующим образом: если $\alpha = [0; a_1, a_2, \ldots]$ правильное представление α в виде цепной дроби, то

$$\zeta_{n+1}(\alpha) = [a_n; a_{n+1}, \ldots].$$

Вместо (1) Кузьмин доказал более сильное соотношение

$$m_n(x) = \frac{\log(1+x)}{\log 2} + O(q^{\sqrt{n}}), \tag{2}$$

где $q < 1$.

Несколько позже (в 1929-ом году), не зная работы Кузьмина, П. Леви также доказал соотношение (2), с лучшим остаточным членом $O(q^n)$ вместо $O(q^{\sqrt{n}})$. Его метод значительно отличается от метода Кузьмина.

Настоящая работа из исходной точки Кузьмина просто выводит соотношение

$$m_n(x) = \frac{\log(1+x)}{\log 2} + O(q^n);$$

полученная постоянная q значительно меньше чем у П. Леви.

ДАЛЬНЕЙШИЕ „ОДНОСТОРОННИЕ" ТЕОРЕМЫ НОВОГО ТИПА В ТЕОРИИ ДИОФАНТИЧЕСКОГО ПРИБЛИЖЕНИЯ

П. Туран (Будапешт)

Автор в книге [1], значительно переработанное издание которой готовится к печати на английском языке в серии *Interscience Tracts*, показал, что систематические экстремальные задачи относительно суммы степеней комплексных чисел, с одной стороны, могут рассматриваться как распространение классичечких теорем теории диофантического приближения, с другой стороны, имеют приложения в различных проблемах анализа и аналитической теории чисел. Различные вопросы относительно знака остаточного члена теоремы о простых числах привели к необходимости изучения „односторонних" задач. В этом направлении в настоящей работе доказывается следующая теорема:

Если $1=|z_1|\geqq|z_2|\geqq\cdots\geqq|z_n|$ и с некоторым $0<\varkappa\leqq\frac{\pi}{2}$ $\varkappa\leqq|\operatorname{arc} z_j|\leqq\pi$ $(j=1,2,\ldots,n)$, то для любых комплексных $b_1,\ldots,b_n$, удовлетворяющих условию $\min\limits_{j=1,\ldots,n}\operatorname{Re}(b_1+\cdots+b_j)>0$, и неотрицательного целого m существуют такие целые ν_1 и ν_2, что

$$m+1\leqq\nu_j\leqq m+n\left(3+\frac{\pi}{\varkappa}\right)\qquad(j=1,2),$$

$$\operatorname{Re}\left(\sum_{j=1}^{n}b_j z_j^{\nu_1}\right)\geqq\left(\frac{n}{24e^3\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)}\right)^{2n}\min_{j=1,\ldots,n}\operatorname{Re}(b_1+\cdots+b_j)$$

и

$$\operatorname{Re}\left(\sum_{j=1}^{n}b_j z_j^{\nu_2}\right)\leqq-\left(\frac{n}{24e^3\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)}\right)^{2n}\min_{j=1,\ldots,n}\operatorname{Re}(b_1+\cdots+b_j).$$

Первое приложение этой теоремы к остаточному члену теоремы о простых числах нашел С. Кнаповски, его результат скоро будет опубликован. В качестве дальнейшего приложения может быть получена следующая теорема, принадлежащая к направлению, инициатором которого был Чебышев:

Если $\Lambda(n)$ функция Манголдта

$$\Lambda(n)=\begin{cases}\log p, \text{ если } n=p^{\alpha} \ (p \text{ простое число}),\\ 0, \text{ в противном случае},\end{cases}$$

то существует такая положительная постоянная c, которая может быть задана в явном виде, что при $T>c$

$$g(x)=\sum_{\substack{n\leqq x\\ n\equiv 1 \bmod 4}}\Lambda(n)-\sum_{\substack{n\leqq x\\ n\equiv 3 \bmod 4}}\Lambda(n)$$

меняет знак на интервале $T^{1/3}\leqq x\leqq T$ (и даже принимает значение, превосходящее $\frac{\sqrt{T}}{\log^{10}T}$ и меньшее чем $-\frac{\sqrt{T}}{\log^{10}T}\Big)$.

ЗАМЕЧАНИЯ ОБ ИНТЕРПОЛИРОВАНИИ (СХОДИМОСТЬ В СРЕДНЕМ НА БЕСКОНЕЧНОМ ПРОМЕЖУТКЕ)

Я. Балаж и П. Туран (Будапешт)

Пусть дана бесконечная треугольная матрица

$$x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)} \qquad (n = 1, 2, \ldots), \tag{1}$$

причем

$$x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)}.$$

Пусть $L_n(f; x)$ обозначает n-ый интерполяционный многочлен Лагранжа, относящийся к системе узлов (1) и к данной функции $f(x)$.

Если $g(x)$ такая целая функция, что

$$g(x) = \sum_{\nu=0}^{\infty} a_\nu x^{2\nu},$$

все коэффициенты a_ν положительны и

$$\int^{+\infty} \frac{\log g(x)}{x^2}\, dx = +\infty,$$

функция $p(x)$ такова, что

$$\int_{-\infty}^{+\infty} g(x) p(x)\, dx < +\infty,$$

систему узлов (1) образуют корни многочленов, ортогональных на $(-\infty, +\infty)$ по весу $p(x)$, а определенная на $(-\infty, +\infty)$ непрерывная функция $f(x)$ такова, что

$$\lim_{n \to \pm\infty} \frac{f(x)}{\sqrt{g(x)}} = 0,$$

то

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} \{f(x) - L_n(f; x)\}^2 p(x)\, dx = 0.$$

Насколько авторам известно, это первый общий результат относительно интервала $(-\infty, +\infty)$. Более того, этот результат не был известен даже для случая многочленов Эрмита, ортогональных на $(-\infty, +\infty)$ по весу e^{-x^2}.

Теорему можно применить к общей теории многочленов, ортогональных на $(-\infty, +\infty)$ по весу $p(x)$. Авторы займутся этим вопросов в следующей работе.

ist sein unmittelbarer Vorgänger (im Fall $o(x_i^*) < n$ ist nämlich x_i^* kein unmittelbarer Vorgänger von x).

Dieses Beispiel dient nur um zu zeigen, daß aus jeder freien holomorphen Menge durch einen geeigneten Vorgängerbegriff eine angeordnete freie holomorphe Menge gewonnen werden kann; in Spezialfällen kann sich aber (wie beim Beispiel 2″ schon gesagt wurde) ein von dem hier angeführten verschiedener Vorgängerbegriff als geeigneter erweisen. Jedenfalls folgt aus V1, V2 und V3, daß die Konstituenten von $x \in H$ stets unter seinen Vorgängern auftreten; also ist der hier angeführte Vorgängerbegriff der engste, der diesen Axiomen genügt.

4. Von nun an soll H stets eine angeordnete freie holomorphe Menge bezeichnen. Für jedes $f^* \in F$ bedeute $t = t(f^*)$ das Maximum der (wegen V6 beschränkten) Anzahl der unmittelbaren Vorgänger von $x = f^*(x_1^*, \ldots, x_m^*)$, wobei $x_1^*, \ldots, x_m^*$ die Menge H durchlaufen. Indem wir die unmittelbaren Vorgänger von x in einer beliebigen, aber festen Reihenfolge, nötigenfalls (wenn nämlich ihre Anzahl weniger als t beträgt) mit Wiederholung, aufzählen, werden die unmittelbaren Vorgänger von x dauernd mit $x_1, \ldots, x_t$ bezeichnet. (Die gewählte Reihenfolge der unmittelbaren Vorgänger soll für Elemente, welche dieselben unmittelbaren Vorgänger besitzen, die selbe sein.)

Es sei nun gegeben eine angeordnete freie holomorphe Menge H. Jedem $h \in H_0$ bzw. $f^* \in F$ sei je eine Funktion[4] g_h von r bzw. g_{f^*} von $2t + r (= 2t(f^*) + r)$ Argumenten zugeordnet (wo r eine beliebige natürliche Zahl, einschließlich Null, bedeutet). Dann sagen wir, daß die Funktion f von $r+1$ Argumenten durch *primitive Rekursion* mit Hilfe dieser Funktionen definiert wird, wenn

$$f(h, u_1, \ldots, u_r) = g_h(u_1, \ldots, u_r), \quad \text{falls} \quad h \in H_0,$$

und falls x der Form $x = f^*(\ldots)$ ist, wo $f^* \in F$,

$$f(x, u_1, \ldots, u_r) = g_{f^*}(x_1, \ldots, x_t, u_1, \ldots, u_r, f(x_1, u_1, \ldots, u_r), \ldots, f(x_t, u_1, \ldots, u_r)),$$

wobei $u_1, \ldots, u_r \in H$. Im allgemeinen kommen hier unendlich viele Definitionsgleichungen vor, ihre Mächtigkeit ist $\overline{\overline{H_0 + F}}$.

Es seien nun gewisse Ausgangsfunktionen angegeben. Die Elemente von H_0 (als Konstanten) und die Funktionen von F sollen jedenfalls Ausgangsfunktionen sein, es können aber auch andere Ausgangsfunktionen vorkommen. Dann wird eine Funktion (in den Ausgangsfunktionen) primitiv-rekursiv genannt, falls sie aus den Ausgangsfunktionen durch endlich viele Substitutionen und primitive Rekursionen entsteht. Dabei darf jede Funktion so betrachtet werden, als eine Funktion ihrer Argumente und beliebig (endlich) vieler „hinzugenommener" Argumente, von denen sie nicht tatsächlich abhängt. Es soll

$b(y_1, \ldots, y_r)$ die charakteristische Funktion einer Beziehung $B(y_1, \ldots, y_r)$ heißen, wenn

$$b(y_1, \ldots, y_r) = \begin{cases} h_0, & \text{falls } B(y_1, \ldots, y_r) \text{ gilt,} \\ h_1 & \text{sonst,} \end{cases}$$

und eine Beziehung heißt primitiv-rekursiv, falls ihre charakteristische Funktion primitiv-rekursiv ist; hierbei bedeuten h_0 und h_1 zwei feste, sogleich zu bestimmende Elemente von H. (In den Weiteren werde ich die charakteristische Funktion einer mit B oder B_i bezeichneten Beziehung immer mit b bzw. mit b_i bezeichnen.)

Die Funktion $f(x) = x$ und die charakteristische Funktion der Gleichheit $x = y$ sollen als primitiv-rekursiv gelten, also notwendigerfalls zu den Ausgangsfunktionen hinzugenommen werden.

Man beweist leicht, daß die Klasse der (in gegebenen Ausgangsfunktionen) primitiv-rekursiven Funktionen davon unabhängig ist, in welcher Reihenfolge (und mit welcher Wiederholungen) die unmittelbaren Vorgänger von x durch $x_1, \ldots, x_t$ bezeichnet worden sind.

5. Weder H_0 noch F ist leer; sei h_0 ein bestimmtes Element von H_0 und f_0 ein bestimmtes Element von F. Setzen wir ferner

$$h_1 = f_0(h_0, \ldots, h_0), \quad h_2 = f_0(h_1, \ldots, h_1), \ldots,$$

dann ist 0 die Ordnung von h_0, dann 1 die Ordnung von h_1, ferner 2 die Ordnung von h_2, usw.; so kann zu jeder natürlichen Zahl n ein Element h_n der Ordnung n angegeben werden.

Ich werde in den Folgenden $h_0, h_1, h_2, \ldots$ mit den natürlichen Zahlen identifizieren (zum Beispiel verstehe ich unter dem „h_n-ten" Glied einer Folge das n-te Glied). So kann die Ordnung eines Elementes von H durch die primitive Rekursion

$$o(h) = h_0, \quad \text{falls} \quad h \in H_0,$$

und falls $x = f^*(\ldots)$, wo $f^* \in F$ ist,

$$o(x) = f_0(o(\bar{x}), \ldots, o(\bar{x}))$$

definiert werden, wobei $\bar{x}$ ein beliebiger unmittelbarer Vorgänger von x ist. Einen bestimmten solchen Vorgänger hat man oft zu verwenden; sei x^{-1} ein solcher, wobei $h^{-1} = h$ ist, falls $h \in H_0$ (in diesem Fall ist h^{-1} ein nicht echter Vorgänger), und falls $x = f^*(x_1^*, \ldots, x_m^*) \in H_{n+1}$, wobei $f^* \in F$ ist, dann sei $x^{-1} = x_i^*$, wobei x_i^* von $x_1^*, \ldots, x_m^*$ dasjenige mit kleinstem Index ist, für welches $x_i^* \in H_n$ gilt.

x^{-1} soll als primitiv-rekursiv gelten, also notwendigerfalls zu den Ausgangsfunktionen hinzugenommen werden.[5]

6. Nun gilt der folgende

SATZ. *Jede zahlentheoretische primitiv-rekursive Funktion kann zu einer primitiv-rekursiven Funktion in H erweitert werden.*

BEWEIS. Offenbar genügt es zu beweisen, daß falls $\varphi(n_1, \ldots, n_r)$ eine zahlentheoretische primitiv-rekursive Funktion ist, und, wie gesagt,

$$0 = h_0, \ 1 = h_1, \ 2 = h_2, \ldots$$

gesetzt wird (also h_{n+1} auch als $h_n + 1$ geschrieben werden kann, und allgemein eine zahlentheoretische Funktion $\varphi(n)$ in jeder der Formen

$$\varphi(n), \ h_{\varphi(n)}, \ \varphi(h_n), \ o(\varphi(n))),$$

dann eine in H primitiv-rekursive Funktion $f(y_1, \ldots, y_r)$ mit

$$o(f(y_1, \ldots, y_r)) = \varphi(o(y_1), \ldots, o(y_r))$$

vorhanden ist. Die zahlentheoretischen primitiv-rekursiven Funktionen werden aus 0 und $n+1$ ausgehend durch endlich viele Substitutionen und primitive Rekursionen aufgebaut. Man hat also zu zeigen, daß die Behauptung 1. für 0 und $n+1$ gilt, und sich 2. bei Substitutionen und 3. bei primitiven Rekursionen vererbt.

1. Zu 0 kann in H die Konstante h_0 gehören, da

$$o(h_0) = h_0 = 0$$

ist.

Zu $n+1$ kann in H die Funktion $f_0(x, \ldots, x)$ gewählt werden, da

$$o(f_0(x, \ldots, x)) = o(x) + 1$$

ist.

2. Falls bereits

$$o(g(y_1, \ldots, y_k)) = \psi(o(y_1), \ldots, o(y_k))$$

und für $i = 1, \ldots, k$

$$o(f_i(y_1, \ldots, y_r)) = \varphi_i(o(y_1), \ldots, o(y_r))$$

[5] (a) Im Fall einer Wortemenge stimmt das mit dem in den Beispielen 2′ und 2″ der Nr. 3 angegebenen x^{-1} überein.

(b) Wird im allgemeinen Fall die Reihenfolge $x_1, \ldots, x_t$ der unmittelbaren Vorgänger von x etwa so gewählt, daß $x_1 = x_i^*$ gilt, so läßt sich x^{-1} durch die primitive Rekursion

$$h^{-1} = h, \quad \text{falls} \quad h \in H_0,$$

und falls $x = f^*(\ldots)$, wo $f^* \in F$,

$$x^{-1} = x_1$$

definieren.

gilt, wo ψ, φ_i zahlentheoretische Funktionen sind, so kann zur aus ihnen durch Substitution entstehenden Funktion

$$\psi(\varphi_1(n_1, \ldots, n_r), \ldots, \varphi_k(n_1, \ldots, n_r))$$

in H die Funktion

$$g(f_1(y_1, \ldots, y_r), \ldots, f_k(y_1, \ldots, y_r))$$

gewählt werden, da

$$o(g(f_1(y_1, \ldots, y_r), \ldots, f_k(y_1, \ldots, y_r))) =$$
$$= \psi(o(f_1(y_1, \ldots, y_r)), \ldots, o(f_k(y_1, \ldots, y_r))) =$$
$$= \psi(\varphi_1(o(y_1), \ldots, o(y_r)), \ldots, \varphi_k(o(y_1), \ldots, o(y_r)))$$

gilt.

3. Falls bereits

$$o(a(u_1, \ldots, u_r)) = \alpha(o(u_1), \ldots, o(u_r))$$

und

$$o(b(x, u_1, \ldots, u_r, y)) = \beta(o(x), o(u_1), \ldots, o(u_r), o(y))$$

gilt, wo α und β zahlentheoretische Funktionen sind, so kann zur Funktion $\varphi(n, p_1, \ldots, p_r)$, welche durch die primitive Rekursion

$$\varphi(0, p_1, \ldots, p_r) = \alpha(p_1, \ldots, p_r)$$
$$\varphi(n+1, p_1, \ldots, p_r) = \beta(n, p_1, \ldots, p_r, \varphi(n, p_1, \ldots, p_r))$$

definiert wird, in H die durch die primitive Rekursion

$$f(h, u_1, \ldots, u_r) = a(u_1, \ldots, u_r), \quad \text{falls} \quad h \in H_0,$$

und für $x = f^*(\ldots)$, wo $f^* \in F$,

$$f(x, u_1, \ldots, u_r) = b(x^{-1}, u_1, \ldots, u_r, f(x^{-1}, u_1, \ldots, u_r))$$

definierte Funktion $f(x, u_1, \ldots, u_r)$ gewählt werden, da für $h \in H_0$

$$o(f(h, u_1, \ldots, u_r)) = o(a(u_1, \ldots, u_r)) =$$
$$= \alpha(o(u_1), \ldots, o(u_r)) = \varphi(0, o(u_1), \ldots, o(u_r)) = \varphi(o(h), o(u_1), \ldots, o(u_r))$$

gilt; ferner falls für alle unmittelbaren Vorgänger $\overline{x}$ von $x \notin H_0$ bereits

$$o(f(\overline{x}, u_1, \ldots, u_r)) = \varphi(o(\overline{x}), o(u_1), \ldots, o(u_r))$$

gilt, so gilt auch

$$o(f(x, u_1, \ldots, u_r)) = o(b(x^{-1}, u_1, \ldots, u_r, f(x^{-1}, u_1, \ldots, u_r))) =$$
$$= \beta(o(x^{-1}), o(u_1), \ldots, o(u_r), o(f(x^{-1}, u_1, \ldots, u_r)) =$$
$$= \beta(o(x^{-1}), o(u_1), \ldots, o(u_r), \varphi(o(x^{-1}), o(u_1), \ldots, o(u_r))) =$$
$$= \varphi(o(x^{-1}) + 1, o(u_1), \ldots, o(u_r)) = \varphi(o(x), o(u_1), \ldots, o(u_r)).$$

7. Beispiele für Definitionen primitiv-rekursiver Funktionen in H.

1. $h_{o(x)} = o(x)$ wurde bereits als primitiv-rekursive Funktion definiert.

2. Die Definition von $x^{-o(y)}$:

$$x^{-o(h)} = x, \quad \text{falls} \quad h \in H_0,$$

und falls $y = f^*(\ldots)$, wo $f^* \in F$,

$$x^{-o(y)} = (x^{-o(y^{-1})})^{-1}.$$

Das ergibt einen (nicht notwendig echten) Vorgänger von x, dessen Ordnung — falls die Ordnung von y nicht größer als die Ordnung von x ist — um die Ordnung von y kleiner als die Ordnung von x ist.[6]

3. Definition eines bestimmten (nicht notwendig echten) Vorgängers 0-ter Ordnung von x, mit $e(x)$ bezeichnet:

$$e(h) = h, \quad \text{falls} \quad h \in H_0,$$

und falls $x = f^*(\ldots)$, wo $f^* \in F$,

$$e(x) = e(x^{-1}).$$

4. Definition der charakteristischen Funktion der Beziehung, daß x der Form $f_0^*(\ldots)$ ist, wobei f_0^* ein bestimmtes Element von F ist (falls sie provisorisch mit f bezeichnet wird):

$$f(h) = h_1, \quad \text{falls} \quad h \in H_0,$$

falls $x = f_0^*(\ldots)$,

$$f(x) = h_0,$$

und falls $x = f^*(\ldots)$, wo $f^* \in F$ von f_0^* verschieden ist,

$$f(x) = h_1.$$

5. Definition von $\mathrm{sg}(x)$ (d. h. der charakteristischen Funktion von $x \in H_0$):

$$\mathrm{sg}(h) = h_0, \quad \text{falls} \quad h \in H_0,$$

und falls $x = f^*(\ldots)$, wo $f^* \in F$,

$$\mathrm{sg}(x) = h_1.$$

6. Definition von $\overline{\mathrm{sg}}(x)$ (d. h. der charakteristischen Funktion von $x \notin H_0$):

$$\overline{\mathrm{sg}}(h) = h_1, \quad \text{falls} \quad h \in H_0,$$

und falls $x = f^*(\ldots)$, wo $f^* \in F$,

$$\overline{\mathrm{sg}}(x) = h_0.$$

[6] Man sieht leicht, daß diese Funktion nicht von y, nur von $o(y)$ abhängig ist (dasselbe gilt auch in den Weiteren immer, wenn eine Funktion in Abhängigkeit von $o(\ldots)$ definiert wird), ferner, daß sie für $o(y) = 1$ in x^{-1} übergeht.

Damit erhält man die charakteristische Funktion der Negation $\overline{B}(y_1, \ldots, y_r)$ einer Beziehung $B(y_1, \ldots, y_r)$ als $\overline{\mathrm{sg}}(b(y_1, \ldots, y_r))$.

7. Wird noch $d(x, y)$ etwa durch

$$d(h_0, y) = h_0,$$

$$d(h, y) = \mathrm{sg}(y), \quad \text{falls} \quad h \in H_0 \quad \text{und} \quad h \neq h_0,$$

und für $x = f^*(\ldots)$, wo $f^* \in F$,

$$d(x, y) = \mathrm{sg}(y)$$

definiert, so erhält man die charakteristische Funktion einer Disjunktion

$$B_1(y_1, \ldots, y_r) \vee B_2(y_1, \ldots, y_r)$$

als

$$d(b_1(y_1, \ldots, y_r),\ b_2(y_1, \ldots, y_r)).$$

Aus 6. und 7. sieht man, daß sämtliche aussagenlogischen Verknüpfungen von primitiv-rekursiven Beziehungen in H primitiv-rekursiv sind.

8. Samt den Funktionen $g_i(y_1, \ldots, y_r)$ $(i = 1, 2, \ldots, n+1)$ und den Beziehungen $B_i(y_1, \ldots, y_r)$ $(i = 1, 2, \ldots, n)$, wobei an jeder Stelle $(y_1, \ldots, y_r)$ höchstens eine der Beziehungen gilt, ist auch die aus ihnen folgenderweise „zusammengeflickte" Funktion $f(y_1, \ldots, y_r)$ in H primitiv-rekursiv:

$$f(y_1, \ldots, y_r) = \begin{cases} g_1(y_1, \ldots, y_r), & \text{falls} \quad B_1(y_1, \ldots, y_r), \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ g_n(y_1, \ldots, y_r), & \text{falls} \quad B_n(y_1, \ldots, y_r), \\ g_{n+1}(y_1, \ldots, y_r) & \text{sonst.} \end{cases}$$

Wenn die Konjunktion der Negationen der Beziehungen $B_i(y_1, \ldots, y_r)$ $(i = 1, 2, \ldots, n)$ mit $B_{n+1}(y_1, \ldots, y_r)$ bezeichnet wird, so kann mit Verwendung dieser ebenfalls primitiv-rekursiven Beziehung die letzte Zeile der Definition in der Form

$$f(y_1, \ldots, y_r) = g_{n+1}(y_1, \ldots, y_r), \quad \text{falls} \quad B_{n+1}(y_1, \ldots, y_r)$$

geschrieben werden.

Zum Beweis definiere ich zuerst eine Funktion $s_2(z_1, z_2)$ durch die primitive Rekursion:

$$s_2(h_0, z_2) = z_2, \quad s_2(h, z_2) = h, \quad \text{falls} \quad h \in H_0 \quad \text{und} \quad h \neq h_0,$$

und für $z_1 = f^*(\ldots)$, wo $f^* \in F$,

$$s_2(z_1, z_2) = z_1;$$

dann sukzessiv die Funktionen

$$s_3(z_1, z_2, z_3) = s_2(s_2(z_1, z_2), z_3),$$
$$s_4(z_1, z_2, z_3, z_4) = s_2(s_3(z_1, z_2, z_3), z_4),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$s_{n+1}(z_1, \ldots, z_{n+1}) = s_2(s_n(z_1, \ldots, z_n), z_{n+1}).$$

Es gilt für $i = 2, 3, \ldots, n+1$ bei jedem $j = 1, 2, \ldots, i$

$$s_i(h_0, \ldots, h_0, z_j, h_0, \ldots, h_0) = z_j,$$

und so speziell

$$s_i(h_0, \ldots, h_0) = h_0;$$

denn man entnimmt aus der Definition von $s_2(z_1, z_2)$ unmittelbar

$$s_2(z_1, h_0) = z_1, \quad s_2(h_0, z_2) = z_2,$$

und auch das benutzend geht diese Eigenschaft für $i = 2, 3, \ldots, n$ sukzessiv von $s_i(z_1, \ldots, z_i)$ auf

$$s_{i+1}(z_1, \ldots, z_{i+1}) = s_2(s_i(z_1, \ldots, z_i), z_{i+1})$$

über.

Daraus ergibt sich, wenn eine Funktion $s_1(x, y)$ durch die primitive Rekursion

$$s_1(h_0, y) = y, \quad s_1(h, y) = h_0, \quad \text{falls} \quad h \in H_0 \quad \text{und} \quad h \neq h_0,$$

und für $x = f^*(\ldots)$, wo $f^* \in F$,

$$s_1(x, y) = h_0$$

definiert wird, unsere „zusammengeflickte" Funktion als

$$f(y_1, \ldots, y_r) = s_{n+1}(s_1(b_1(y_1, \ldots, y_r), g_1(y_1, \ldots, y_r)), \ldots$$
$$\ldots, s_1(b_{n+1}(y_1, \ldots, y_r), (g_{n+1}(y_1, \ldots, y_r)))).$$

9. Definition der charakteristischen Funktion der Beziehung $y \in H_{o(x)}$ (provisorisch mit $g(x, y)$ bezeichnet):

$$g(h, y) = \operatorname{sg}(y), \quad \text{falls} \quad h \in H_0,$$

und für $x = f^*(\ldots)$, wo $f^* \in F$,

$$g(x, y) = \begin{cases} h_0, & \text{falls} \quad \operatorname{sg}(x^{-o(y)}) = h_0 \ \& \ \operatorname{sg}((f_0(x, \ldots, x))^{-o(y)}) \neq h_0, \\ h_1 & \text{sonst.} \end{cases}$$

(Einfacher ergibt sich diese Funktion durch Einsetzen von $o(x)$ für x und $o(y)$ für y in der charakteristischen Funktion der Gleichheit $x = y$.)

10. Definition der charakteristischen Funktion der Beziehung, daß die Ordnung von x größer als die Ordnung von y ist (provisorisch mit $f(x, y)$

bezeichnet):

$$f(x, y) = \begin{cases} h_0, & \text{falls} \quad \text{sg}(x^{-o(y)}) \neq h_0, \\ h_1 & \text{sonst.} \end{cases}$$

11. Definition der charakteristischen Funktion der Beziehung

$$Ey)[y \preceq x \ \& \ B(y, u_1, \ldots, u_r)]$$

(provisorisch mit $f(x, u_1, \ldots, u_r)$ bezeichnet):

$$f(h, u_1, \ldots, u_r) = \left. \begin{cases} h_0, & \text{falls} \quad B(h, u_1, \ldots, u_r) \\ h_1 & \text{sonst} \end{cases} \right\}, \quad \text{falls} \quad h \in H_0,$$

und falls $x = f^*(\ldots)$, wo $f^* \in F$,

$$f(x, u_1, \ldots, u_r) = \begin{cases} h_0, & \text{falls} \quad f(x_1, u_1, \ldots, u_r) = h_0 \vee \cdots \\ & \cdots \vee f(x_t, u_1, \ldots, u_r) = h_0 \vee B(x, u_1, \ldots, u_r), \\ h_1 & \text{sonst.} \end{cases}$$

Ganz ähnlich, nur Konjunktion statt Disjunktion verwendend, definiert man die charakteristische Funktion der Beziehung

$$(y)[y \preceq x \to B(y, u_1, \ldots, u_r)].$$

12. Definition der — provisorisch mit f bezeichneten — Funktion

$$f(x, u_1, \ldots, u_r) = \mu_y[y \leqq x \ \& \ B(y, u_1, \ldots, u_r)],$$

welche einen bestimmten, nicht notwendig echten, Vorgänger y von x bedeutet, für welches $B(y, u_1, \ldots, u_r)$ gilt, und den Wert h_0 annimmt, falls kein solches y existiert:

$$f(h, u_1, \ldots, u_r) = \begin{cases} h, & \text{falls} \quad h \in H_0 \quad \text{und} \quad B(h, u_1, \ldots, u_r), \\ h_0, & \text{falls} \quad h \in H_0 \quad \text{und} \quad \overline{B}(h, u_1, \ldots, u_r), \end{cases}$$

und falls $x = f^*(\ldots)$, wo $f^* \in F$,

$$f(x, u_1, \ldots, u_r) = \begin{cases} f(x_1, u_1, \ldots, u_r), & \text{falls} \quad (Ey)[y \leqq x_1 \ \& \ B(y, u_1, \ldots, u_r)], \\ f(x_2, u_1, \ldots, u_r), & \text{falls} \quad (y)[y \leqq x_1 \to \overline{B}(y, u_1, \ldots, u_r)] \ \& \\ & \qquad \& \ (Ey)[y \leqq x_2 \ \& \ B(y, u_1, \ldots, u_r)], \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f(x_t, u_1, \ldots, u_r) & \text{falls} \quad (y)[y \leqq x_1 \to \overline{B}(y, u_1, \ldots, u_r) \ \& \\ & \qquad \cdots \& \ (y)[y \leqq x_{t-1} \to \overline{B}(y, u_1, \ldots, u_r)] \ \& \\ & \qquad \& \ (Ey)[y \leqq x_t \ \& \ B(y, u_1, \ldots, u_r)], \\ x, & \text{falls} \quad (y)[y \leqq x_1 \to \overline{B}(y, u_1, \ldots, u_r)] \ \& \\ & \qquad \cdots \& \ (y)[y \leqq x_t \to \overline{B}(y, u_1, \ldots, u_r)] \ \& \\ & \qquad \& \ B(x, u_1, \ldots, u_r), \\ h_0 & \text{sonst.} \end{cases}$$

8. Zur Untersuchung der Zusammenhang der verschiedenen Rekursionsarten in H benötigt man als Hilfsmittel eine eineindeutige Abbildung der Menge K der aus Elementen von H gebildeten endlichen (nicht leeren) Folgen auf eine Teilmenge von H. (Im zahlentheoretischen Fall kann z. B. einer Zahlenfolge $a_0, a_1, \ldots, a_n$ die Zahl $p_0^{a_0} p_1^{a_1} \ldots p_n^{a_n}$ zugeordnet werden, wobei $p_0, p_1, \ldots$ die wachsende Folge der Primzahlen ist.) Da $h_0, h_1, h_2, \ldots$ alle verschieden sind, ist H eine unendliche Menge, daher existieren nach bekannten mengentheoretischen Sätzen solche Abbildungen. (Zum allgemeinen Existenzbeweis wird auch der Wohlordnungssatz verwendet, aber die vorliegende allgemeine Theorie ist ohnehin nicht konstruktiv; außerdem können in den wichtigsten Spezialfällen — für abzählbares H, für H von der Mächtigkeit des Kontinuum, oder der Mächtigkeit $2^{2^{\aleph_0}}$, usw. — Abbildungen der gewünschten Art auch ohne Verwendung des Wohlordnungssatzes effektiv angegeben werden.)

Das Element von H, welches bei einer solchen Abbildung der Elementenfolge $(y_0, y_1, \ldots, y_n)$ entspricht, soll mit $c_n(y_0, y_1, \ldots, y_n)$ bezeichnet werden; und falls ein $x \in H$ einer endlichen Elementenfolge entspricht, sei $\operatorname{long}(x)$ die „Länge" dieser Folge (genauer: n, falls die Folge $(y_0, y_1, \ldots, y_n)$ ist, also um 1 weniger als die Gliederzahl der Folge); und für $j = 0, 1, \ldots, \operatorname{long}(x)$ sei $k_j(x)$ das j-te Glied der Folge. Die Definitionen von $\operatorname{long}(x)$ und $k_j(x)$ können verschiedenartig auch auf solche $x \in H$ ausgedehnt werden, welche keiner Elementenfolge entsprechen, und auch auf Werte $j > \operatorname{long}(x)$; sogar auf beliebige von $h_0, h_1, h_2, \ldots$ (die mit den natürlichen Zahlen identifiziert wurden) verschiedene $j \in H$; so wird $\operatorname{long}(x)$ zu einer einstelligen, $k_y(x)$ zu einer zweistelligen Funktion in H, im bisher gebrauchten Sinne. Nach geeigneter Fixierung der Abbildung und der Ausdehnung der Definitionen sollen $\operatorname{long}(x)$ und $k_y(x)$ — falls sie in H nicht als primitiv-rekursive Funktionen definiert werden können — zu den Ausgangsfunktionen von H hinzugenommen werden.

Nach den Definitionen gilt

$$\operatorname{long}(c_n(y_0, y_1, \ldots, y_n)) = n$$

und für $j = 0, 1, \ldots, n$

$$k_j(c_n(y_0, y_1, \ldots, y_n)) = y_j.$$

9. Die Wertverlaufsrekursion. Wir wollen nun Folgen aus den Vorgängern eines beliebigen $x \in H$ bilden. Für diese kann verschiedenartig eine vollständige Anordnung (etwa auch mit Wiederholungen) angegeben werden. Wir wählen ein für allemal die durch folgende Rekursion nach der Ordnung von x erklärte Anordnung. Für $o(x) = 0$ ist x selbst sein einziger Vorgänger. Nehmen wir an, daß für alle Elemente von x der Ordnung n die Anordnung ihrer Vorgänger schon definiert ist, und sei nun $o(x) = n + 1$. Dann haben die unmittelbaren Vorgänger $x_1, \ldots, x_t$ von x die Ordnung n, und sämtliche echten Vorgänger von x kommen nach V5 der Nr. 3 unter den — nach Annahme bereits geordneten — Vorgängern von $x_1, \ldots, x_t$ vor. Man nehme nun zuerst die Vorgänger von x_1 in ihrer Reihenfolge, dann die von diesen verschiedenen Vorgänger von x_2 in ihrer Reihenfolge, usw., endlich die von den Vorgängern der Elemente $x_1, \ldots, x_{t-1}$ verschiedenen Vorgänger von x_t in ihrer Reihenfolge, und zuletzt noch x. In den Weiteren werde ich die Vorgänger von x in dieser Reihenfolge mit

$$\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_s$$

bezeichnen.

Der Wertverlauf einer Funktion $f(x, u_1, \ldots, u_r)$ über die Vorgänger von x wird durch die Folge

$$f(\bar{x}_0, u_1, \ldots, u_r), f(\bar{x}_1, u_1, \ldots, u_r), \ldots, f(\bar{x}_s, u_1, \ldots, u_r),$$

also mit Benutzung der erwähnten Abbildung durch

$$\varphi(x, u_1, \ldots, u_r) = c_s\big(f(\bar{x}_0, u_1, \ldots, u_r), \ldots, f(\bar{x}_s, u_1, \ldots, u_r)\big)$$

charakterisiert; diese Funktion soll die *Wertverlaufsfunktion* von f heißen. Daraus erhält man, falls $\bar{x} \leq x$ und so $\bar{x} = \bar{x}_j$ mit einem $0 \leqq j \leqq s$ ist,

$$f(\bar{x}, u_1, \ldots, u_r) = k_j\big(\varphi(x, u_1, \ldots, u_r)\big);$$

speziell für $j = s = \mathrm{long}\big(\varphi(x, u_1, \ldots, u_r)\big)$

$$f(x, u_1, \ldots, u_r) = k_{\mathrm{long}(\varphi(x, u_1, \ldots, u_r))}\big(\varphi(x, u_1, \ldots, u_r)\big).$$

Damit sind wir in der Lage, die Wertverlaufsrekursion in H definieren zu können. Da handelt es sich um solche Rekursionen, wobei zur Definition des Funktionswertes an einer Stelle x Funktionswerte an echten, aber nicht notwendig *unmittelbaren* Vorgängern verwendet werden. Da sämtliche echte Vorgänger Vorgänger der unmittelbaren Vorgänger $x_1, \ldots, x_t$ sind, und nach den Obigen der Wert von f an jedem Vorgänger eines x_i $(1 \leqq i \leqq t)$ durch Anwendung von k_j bei geeignetem j auf $\varphi(x_i, u_1, \ldots, u_r)$ angegeben werden kann, so kann als allgemeine Wertverlaufsrekursion die folgende Definition einer Funktion $f(x, u_1, \ldots, u_r)$ angenommen werden:

$$f(h, u_1, \ldots, u_r) = g_h(u_1, \ldots, u_r), \quad \text{falls} \quad h \in H_0,$$

und für $x = f^*(\ldots)$, wo $f^* \in F$,

$$\text{(W)} \qquad f(x, u_1, \ldots, u_r) = g_{f^*}\big(x_1, \ldots, x_t, u_1, \ldots, u_r, \varphi(x_1, u_1, \ldots, u_r), \ldots \\ \ldots, \varphi(x_t, u_1, \ldots, u_r)\big),$$

wobei $\varphi(x, u_1, \ldots, u_r)$ die Wertverlaufsfunktion von $f(x, u_1, \ldots, u_r)$ ist.

10. Um ähnlich wie im zahlentheoretischen Fall, auch allgemein die Wertverlaufsrekursion auf primitive Rekursion zurückzuführen, untersuchen wir zunächst eine „rekursionsartige Darstellbarkeit" der Funktion $c_n(y_0, y_1, \ldots, y_n)$ der Nr. 8. Sei $(y_0, y_1, \ldots, y_n)$ für $n \geqq 1$ eine beliebige endliche Folge von Elementen aus H, und sei

$$x = c_{n-1}(y_0, y_1, \ldots, y_{n-1}), \quad y = c(y_n) = c_0(y_n),$$

dann gilt für $j = 0, 1, \ldots, n-1$

$$y_j = k_j(x), \quad y_n = k_0(y), \quad n = \operatorname{long}(x) + 1,$$

also

$$c_n(y_0, y_1, \ldots, y_{n-1}, y_n) = c_{\operatorname{long}(x)+1}\big(k_0(x), k_1(x), \ldots, k_{\operatorname{long}(x)}(x), k_0(y)\big).$$

Setzen wir für beliebige $x, y \in H$

$$p(x, y) = c_{\operatorname{long}(x)+1}\big(k_0(x), k_1(x), \ldots, k_{\operatorname{long}(x)}(x), k_0(y)\big),$$

und nehmen wir $c(x)$ und $p(x, y)$ zu den Ausgangsfunktionen hinzu, falls sie in H nicht als primitiv-rekursive Funktionen definiert werden können. Dann gilt nach den Obigen für $n \geqq 1$ die „rekursionsartige Darstellung" von c_n:

$$c_n(y_0, y_1, \ldots, y_n) = p\big(c_{n-1}(y_0, y_1, \ldots, y_{n-1}), c(y_n)\big).$$

Daraus ergibt sich, daß falls für die Folge $y_0, y_1, \ldots, y_n$ die für $y = 0, 1, \ldots, o(x)$ angenommenen Werte einer primitiv-rekursiven Funktion $\bar{f}(y, z_1, \ldots, z_r)$ eingesetzt werden, dann die so entstehende Funktion

$$c^{\bar{f}}(x, z_1, \ldots, z_r) = c_{o(x)}\big(\bar{f}(0, z_1, \ldots, z_r), \bar{f}(1, z_1, \ldots, z_r), \ldots, \bar{f}(o(x), z_1, \ldots, z_r)\big)$$

primitiv-rekursiv ist, da sie durch folgende primitive Rekursion definiert werden kann:

$$c^{\bar{f}}(h, z_1, \ldots, z_r) = c\big(\bar{f}(0, z_1, \ldots, z_r)\big), \quad \text{falls} \quad h \in H_0,$$

und falls $x = f^*(\ldots)$, wo $f^* \in F$,

$$c^{\bar{f}}(x, z_1, \ldots, z_r) = p\big(c^{\bar{f}}(\overset{-1}{x}, z_1, \ldots, z_r), c(\bar{f}(o(x), z_1, \ldots, z_r))\big).$$

Für die Anwendungen ist nur die Existenz einer primitiv-rekursiven Funktion $p(x, y)$ mit dieser Eigenschaft wichtig.

11. Das werde ich nun zur rekursiven Definition der Wertverlaufsfunktion $\varphi(x, u_1, \ldots, u_r)$ der durch (W) in Nr. 9 definierten Funktion $f(x, u_1, \ldots, u_r)$ benutzen.

Nach V5 in Nr. 3 kommen sämtliche echten Vorgänger $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{s-1}$ eines Elementes $x (= \bar{x}_s)$ der Form $x = f^*(\ldots)$, wo $f^* \in F$, unter den Vorgängern seiner unmittelbaren Vorgänger $x_1, \ldots, x_t$ vor. Das kleinste Index i, für welches $\bar{x}_j$ bei einem $j < s$ ein Vorgänger von x_i ist, ferner das l, für welches

$\overline{x}_j$ der Vorgänger mit dem Index l in der festgelegten Reihenfolge der Vorgänger von x_i ist, endlich auch s sind von x abhängig, dies bedeutet aber offenbar nur eine Abhängigkeit von $x_1, \ldots, x_t$: für Elemente mit denselben unmittelbaren Vorgängern stimmen sie überein. Es sind also bei jedem $f^* \in F$ diese i und l für Zahlenwerte $j < s$ Funktionen von $j, x_1, \ldots, x_t$, und s ist eine Funktion von $x_1, \ldots, x_t$, falls x der Form $x = f^*(\ldots)$ ist. Jede dieser Funktionen nimmt nur Zahlenwerte an; i höchstens den Wert $t(f^*)$. Offenbar kann die Definition dieser Funktionen auf alle Elemente von H mit Beibehaltung dieser Eigenschaft ausgedehnt werden (an Stellen, wo sie bisher nicht definiert sind, kann man ihnen etwa den Wert h_0 zuordnen).

Seien also zu jedem $f^* \in F$

$$\sigma_{f^*}(z_1, \ldots, z_{t(f^*)}), \quad \iota_{f^*}(y, z_1, \ldots, z_{t(f^*)}), \quad \lambda_{f^*}(y, z_1, \ldots, z_{t(f^*)})$$

in H definierten Funktionen, die nur Zahlenwerte annehmen (also mit ihrer eigenen Ordnung übereinstimmen), dabei ι_{f^*} höchstens den Wert $t(f^*)$; und so beschaffen sind, daß falls $x = f^*(\ldots) \in H$ als unmittelbare Vorgänger $x_1, \ldots, x_t$ besitzt, und j eine kleinere Zahl als $\sigma_{f^*}(x_1, \ldots, x_t)$ ist, dann

$$\sigma_{f^*}(x_1, \ldots, x_t), \quad \iota_{f^*}(j, x_1, \ldots, x_t), \quad \lambda_{f^*}(j, x_1, \ldots, x_t)$$

der Reihe nach die oben definierten Zahlen

$$s, \quad i, \quad l$$

ergeben. Diese Funktionen sollen — falls sie in H nicht als primitiv-rekursive Funktionen definiert werden können — zu den Ausgangsfunktionen hinzugenommen werden.

Nach der Bedeutung von s, i und l und nach Nr. 9 gilt nun für die Funktion $f(x, u_1, \ldots, u_r)$ mit der Wertverlaufsfunktion $\varphi(x, u_1, \ldots, u_r)$ für ein $x = f^*(\ldots) \in H$ mit den Vorgängern $\overline{x}_0, \ldots, \overline{x}_{s-1}, \overline{x}_s = x$ und mit den unmittelbaren Vorgängern $x_1, \ldots, x_t$ für $j = 0, 1, \ldots, s-1$

$$f(\overline{x}_j, u_1, \ldots, u_r) = k_{\lambda_{f^*}(j, x_1, \ldots, x_t)}(\varphi(x_{\iota_{f^*}(j, x_1, \ldots, x_t)}, u_1, \ldots, u_r)),$$

und daher

$$\varphi(x, u_1 \ldots, u_r) = c_s(f(\overline{x}_0, u_1, \ldots, u_r), \ldots, f(\overline{x}_{s-1}, u_1, \ldots, u_r), f(\overline{x}_s, u_1, \ldots, u_r)) =$$
$$= c_{\sigma_{f^*}(x_1, \ldots, x_t)}(k_{\lambda_{f^*}(0, x_1, \ldots, x_t)}(\varphi(x_{\iota_{f^*}(0, x_1, \ldots, x_t)}, u_1, \ldots, u_r)), \ldots$$
$$\ldots, k_{\lambda_{f^*}(\sigma_{f^*}(x_1, \ldots, x_t)-1, x_1, \ldots, x_t)}(\varphi(x_{\iota_{f^*}(\sigma_{f^*}(x_1, \ldots, x_t)-1, x_1, \ldots, x_t)}, u_1, \ldots, u_r)), f(x, u_1, \ldots, u_r)).$$

Wird also

$$\overline{g}_{f^*}(z_1, \ldots, z_t, v_0, v_1, \ldots, v_t, w) = c_{\sigma_{f^*}(z_1, \ldots, z_t)}(k_{\lambda_{f^*}(0, z_1, \ldots, z_t)}(v_{\iota_{f^*}(0, z_1, \ldots, z_t)}), \ldots$$
$$\ldots, k_{\lambda_{f^*}(\sigma_{f^*}(z_1, \ldots, z_t)-1, z_1, \ldots, z_t)}(v_{\iota_{f^*}(\sigma_{f^*}(z_1, \ldots, z_t)-1, z_1, \ldots, z_t)}), w)$$

gesetzt, dann gilt (da ι_{f^*} zwar auch den Wert $h_0=0$ annehmen kann, aber in den uns interessierenden Fällen nie) etwa mit

$$\overline{\overline{g}}_{f^*}(z_1,\ldots,z_t,v_1,\ldots,v_t,w)=\overline{g}_{f^*}(z_1,\ldots,z_t,h_0,v_1,\ldots,v_t,w)$$

der Zusammenhang

$$(\overline{\mathrm{W}})\qquad \varphi(x,u_1,\ldots,u_r)=\overline{\overline{g}}_{f^*}(x_1,\ldots,x_t,\varphi(x_1,u_1,\ldots,u_r),\ldots$$
$$\ldots,\varphi(x_t,u_1,\ldots,u_r),f(x,u_1,\ldots,u_r)).$$

12. Nach dem Schlußergebnis der Nr. 10 ist hier die Funktion $\overline{g}_{f^*}$ primitiv-rekursiv. Denn erstens ist für $t=t(f^*)$

$$v_{\iota_{f^*}(y,z_1,\ldots,z_t)}=v^{f^*}(y,z_1,\ldots,z_t,v_0,v_1,\ldots,v_t)=$$
$$=\begin{cases} v_1, & \text{falls}\quad \iota_{f^*}(y,z_1,\ldots,z_t)=1,\\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\\ v_t, & \text{falls}\quad \iota_{f^*}(y,z_1,\ldots,z_t)=t,\\ v_0 & \text{sonst}\end{cases}$$

nach 8 der Nr. 7 eine primitiv-rekursive Funktion. Daher ist auch die durch

$$\overline{f}(y,z_1,\ldots,z_t,v_0,v_1,\ldots,v_t,w)=\begin{cases} k_{\lambda_{f^*}(y,z_1,\ldots,z_t)}(v_{\iota_{f^*}(y,z_1,\ldots,z_t)}),\\ \quad\text{falls}\quad o(\iota_{f^*}(y,z_1,\ldots,z_t))<o(\sigma_{f^*}(z_1,\ldots,z_t)),\\ w\quad \text{sonst}\end{cases}$$

definierte Funktion $\overline{f}$ primitiv-rekursiv, und mit ihr nach Nr. 10 auch die Funktion

$$c^{\overline{f}}(x,z_1,\ldots,z_t,v_0,v_1,\ldots,v_t,w)=$$
$$=c_{o(x)}(\overline{f}(0,z_1,\ldots,z_t,v_0,v_1,\ldots,v_t,w),\ldots,\overline{f}(o(x),z_1,\ldots,z_t,v_0,v_1,\ldots,v_t,w)).$$

Man sieht aber leicht, daß

$$\overline{g}_{f^*}(z_1,\ldots,z_t,v_0,v_1,\ldots,v_t,w)=c^{\overline{f}}(\sigma_{f^*}(z_1,\ldots,z_t),z_1,\ldots,z_t,v_0,v_1,\ldots,v_t,w)$$

gilt; so ist auch $\overline{g}_{f^*}$, und daher auch die in $(\overline{\mathrm{W}})$ der Nr. 11 auftretende Funktion $\overline{\overline{g}}_{f^*}$ primitiv-rekursiv in H.

Sind auch die in der Definition (W) von Nr. 9 auftretenden Funktionen g_h und g_{f^*} primitiv-rekursiv in H, so gilt das auch für die Wertverlaufsfunktion

$$\varphi(x,u_1,\ldots,u_r)=c_s(f(\overline{x}_0,u_1,\ldots,u_r),\ldots,f(\overline{x}_s,u_1,\ldots,u_r)),$$

denn diese kann (in Betracht genommen, daß für $h\in H_0$ der einzige Vorgänger von h selbst h, und $c_0=c$ ist) nach (W) und $(\overline{\mathrm{W}})$ durch folgende primitive Rekursion definiert werden:

$$\varphi(h,u_1,\ldots,u_r)=c(g_h(u_1,\ldots,u_r)),\quad \text{falls}\quad h\in H_0,$$

und falls $x=f^*(\ldots)$, wo $f^* \in F$,

$$\varphi(x, u_1, \ldots, u_r)=\overline{\overline{g}}_{f^*}(x_1, \ldots, x_t, \varphi(x_1, u_1, \ldots, u_r), \ldots, \varphi(x_t, u_1, \ldots, u_r),$$
$$g_{f^*}(x_1, \ldots, x_t, u_1, \ldots, u_t, \varphi(x_1, u_1, \ldots, u_r), \ldots, \varphi(x_t, u_1, \ldots, u_r))).$$

Endlich ergibt sich $f(x, u_1, \ldots, u_r)$ durch die Substitution

$$f(x, u_1, \ldots, u_r)=k_{\mathrm{long}(\varphi(x, u_1, \ldots, u_r))}(\varphi(x, u_1, \ldots, u_r))$$

als primitiv-rekursive Funktion in H. *Demnach führt die Wertverlaufsrekursion nicht von der Klasse der in H primitiv-rekursiven Funktionen heraus.*

13. Reduktionen. 1. Bisher war auch schon mühsam, die vielen Parameter mitzuschleppen; und das ist auch nicht notwendig. In H lassen sich die Parameter zu einem einzigen zusammenziehen.

Wird $f(x, u_1, \ldots, u_r)$ durch die primitive Rekursion

$$f(h, u_0, \ldots, u_r)=g_h(u_0, \ldots, u_r), \quad \text{falls} \quad h \in H_0,$$

und für $x=f^*(\ldots)$, wo $f^* \in F$,

$$f(x, u_0, \ldots, u_r)=g_{f^*}(x_1, \ldots, x_t, u_0, \ldots, u_r, f(x_1, u_0, \ldots, u_r), \ldots, f(x_t, u_0, \ldots, u_r))$$

definiert, so sei

$$f'(x, u)=f(x, k_0(u), \ldots, k_r(u)),$$

woraus sich

$$f(x, u_1, \ldots, u_r)=f'(x, c_r(u_0, \ldots, u_r))$$

ergibt.

f' kann durch folgende primitive Rekursion definiert werden:

$$f'(h, u)=g_h(k_0(u), \ldots, k_r(u))=g'_h(u), \quad \text{falls} \quad h \in H_0,$$

und für $x=f^*(\ldots)$, wo $f^* \in F$,

$$f'(x, u)=g_{f^*}(x_1, \ldots, x_t, k_0(u), \ldots, k_r(u), f(x_1, k_0(u), \ldots, k_r(u)), \ldots$$
$$\ldots, f(x_t, k_0(u), \ldots, k_r(u)))=$$
$$=g_{f^*}(x_1, \ldots, x_t, k_0(u), \ldots, k_r(u), f'(x_1, u), \ldots, f'(x_t, u))=$$
$$=g'_{f^*}(x_1, \ldots, x_t, u, f'(x_1, u), \ldots, f'(x_t, u)),$$

wobei g'_h, g'_{f^*} mit g_h bzw. g_{f^*} primitiv-rekursiv sind.

Aus $f'(x, u)$ erhält man $f(x, u_0, \ldots, u_r)$ durch die obige Substitution.

2. In H kann eine Rekursion der letzten Form noch weiter reduziert werden: auf eine Form, wobei die Abhängigkeit vom einzigen Parameter u nur im Fall $x \in H_0$ zum Ausdruck kommt.

Sei nämlich

$$f''(x, u) = c_1(u, f'(x, u)),$$

woraus sich

$$f'(x, u) = k_1(f''(x, u)),$$

und für beliebiges x (also zum Beispiel auch für x^{-1} statt x)

$$u = k_0(f''(x, u))$$

ergibt.

So gewinnt man folgende primitiv-rekursive Definition für $f''(x, u)$:

$$f''(h, u) = c_1(u, g'_h(u)), \quad \text{falls} \quad h \in H_0,$$

und für $x = f^*(\ldots)$, wo $f^* \in F$,

$$f''(x, u) = c_1(k_0(f''(x^{-1}, u)), g'_{f^*}(x_1, \ldots, x_t, k_0(f''(x^{-1}, u)),$$
$$k_1(f''(x_1, u)), \ldots, k_1(f''(x_t, u)))).$$

Aus $f''(x, u)$ erhält man $f'(x, u)$ durch die obige Substitution.

Demnach kann man sich in H, von den bereits angenommenen Ausgangsfunktionen ausgehend, nebst Substitutionen auf primitive Rekursionen der Form

$$f(h, u) = g_h(u), \quad \textit{falls} \quad h \in H_0,$$

und falls $x = f^*(\ldots)$, wo $f^* \in F$,

$$f(x, u) = g_{f^*}(x_1, \ldots, x_t, f(x_1, u), \ldots, f(x_t, u))$$

beschränken.

14. Die simultane Rekursion für mehrere Funktionen. *Diese läßt sich auf eine primitive Rekursion in H nebst Substitutionen zurückführen.* Das zeige ich an einem Beispiel; der allgemeine Fall könnte aber genau so behandelt werden.

Sei

$$f'(h, u) = g'_h(u), \quad f''(h, u) = g''_h(u), \quad \text{falls} \quad h \in H_0,$$

und für $x = f^*(\ldots)$, wo $f^* \in F$,

$$f'(x, u) = g'_{f^*}(x_1, f'(x_1, u), f''(x_2, u)), \quad f''(x, u) = g''_{f^*}(x_2, f'(x_3, u)),$$

wo x_1, x_2, x_3 unmittelbare Vorgänger von x sind.

Setzen wir

$$\bar{f}(x, u) = c_1(f'(x, u), f''(x, u)),$$

woraus sich

$$f'(x, u) = k_0(\bar{f}(x, u)) \quad \text{und} \quad f''(x, u) = k_1(\bar{f}(x, u))$$

ergibt.

$\bar{f}(x, u)$ kann dabei durch folgende primitive Rekursion definiert werden:

$$\bar{f}(h, u) = c_1(g'_h(u), g''_h(u)), \quad \text{falls} \quad h \in H_0,$$

und für $x = f^*(\ldots)$, wo $f^* \in F$,

$$\bar{f}(x, u) = c_1(g'_{f^*}(x_1, k_0(\bar{f}(x_1, u)), k_1(\bar{f}(x_2, u))), g''_{f^*}(x_2, k_0(\bar{f}(x_3, u)))).$$

Damit ist die Behauptung für dieses Beispiel bewiesen.

15. Die eingeschachtelte Rekursion. Hier handelt es sich um Rekursionen, wobei für die Parameter Einsetzungen erfolgen; sogar Werte der zu definierenden Funktion (an unmittelbare-Vorgänger-Stellen) können für die Parameter eingesetzt werden. Ich behandle hier ebenfalls ein Beispiel, aber mit allgemein ebenso verwendbaren Methoden.

Sei das betrachtete Beispiel die Definition:

$$f(h, u) = g_h(u), \quad \text{falls} \quad h \in H_0,$$

und für $x = f^*(\ldots)$, wo $f^* \in F$,

$$f(x, u) = g_{f^*}(x_1, x_2, u, f(x_1, f(x_2, l_{f^*}(x_1, x_2, u)))),$$

wo x_1, x_2 unmittelbare Vorgänger von x, und die Funktionen g_h, g_{f^*}, l_{f^*} in H primitiv-rekursiv sind.

Bei der Berechnung der Werte von $f(x, u)$ sieht man, daß dabei „Ineinanderschachtelungen" der Funktionen g_h, g_{f^*}, l_{f^*} als „Bausteine" auftreten; dabei erfolgt in die beiden ersten Leerstellen der Funktionen g_{f^*} und l_{f^*} niemals eine Einsetzung einer Funktion. Eine Funktion $\bar{f}(x, u)$, welche die folgenden Bedingungen erfüllt, würde sämtliche „Bausteine" als Werte annehmen:

1. $\bar{f}(x, u)$ nimmt für ein x den Wert u an,
2. es gibt zu jedem $h \in H_0$ und $y \in H$ ein $x \in H$ mit
$$\bar{f}(x, u) = g_h(\bar{f}(y, u)),$$
3. es gibt zu jedem $f^* \in F$ und $y_1, y_2, y_3, y_4 \in H$ ein $x \in H$ mit
$$\bar{f}(x, u) = g_{f^*}(y_1, y_2, \bar{f}(y_3, u), \bar{f}(y_4, u)),$$
4. es gibt zu jedem $f^* \in F$ und $y_1, y_2, y_3 \in H$ ein $x \in H$ mit
$$\bar{f}(x, u) = l_{f^*}(y_1, y_2, \bar{f}(y_3, u)).$$

Falls dabei y, y_1, y_2, y_3, y_4 immer echte Vorgänger von x sind, erhält man aus diesen Forderungen eine rekursive Definition für eine geeignete Funktion $\bar{f}(x, u)$.

16. Im zahlentheoretischen Fall könnte z. B. als für 3. geeignetes x die der Folge (y_1, y_2, y_3, y_4) zugeordnete Zahl $p_1^{y_1} p_2^{y_2} p_3^{y_3} p_4^{y_4}$ gewählt werden; umgekehrt, bei gegebenem x wären die Werte y_1, y_2, y_3, y_4 die Exponenten von p_1, p_2, p_3, p_4 in der Primfaktorenzerlegung von x, und diese wären (als kleinere

Zahlen) sicher echte Vorgänger von x. Diesen Exponenten entsprechen im allgemeinen Fall die Werte $k_1(x), k_2(x), k_3(x), k_4(x)$. In der (unter anderem) durch $k_{o(y)}(x)$ charakterisierten Abbildung kann aber vorkommen, daß eine eingliedrige Folge auf sich selbst abgebildet wird; deshalb kann für $k_{o(y)}(x)$ allgemein nicht die Forderung gestellt werden, daß es ein echter Vorgänger von x sei. Es hilft aber auch die schwächere Forderung, daß $k_{o(y)}(x)$ stets Vorgänger von x sei.

Die folgenden Betrachtungen gelten also nur dann, wenn die verwendete Abbildung der endlichen Elementenfolgen in H so beschaffen ist, daß stets

(K) $$k_{o(y)}(x) \preceq x$$

gilt, woraus sich für ein beliebiges $\bar{x} \prec x$

$$k_{o(y)}(\bar{x}) \prec x$$

ergibt. D. h. die folgenden Betrachtungen gelten nur dann für eine angeordnete freie holomorphe Menge H, wenn es eine solche Abbildung der endlichen Elementenfolgen in H gibt, daß bei beliebigen $y_0, y_1, \ldots, y_n \in H$ für das ihrer Folge zugeordnete Element $c_n(y_0, y_1, \ldots, y_n)$

$$y_0 \preceq c_n(y_0, \ldots, y_n), y_1 \preceq c_n(y_0, \ldots, y_n), \ldots, y_n \preceq c_n(y_0, \ldots, y_n)$$

gilt. Auch bei der Ausdehnung des Definitionsbereiches von $k_j(x)$ (dessen Wert, falls $x \in H$ einer endlichen Elementenfolge von H entspricht, und $j < \mathrm{long}(x)$ ist, das j-te Glied dieser Folge ist) auf alle Elemente von H, hat man auf die Bedingung (K) zu achten; das kann so geschehen, daß dieser Funktion an allen neu hinzugenommenen Stellen der Wert $e(x)$ zugeordnet wird.

Bei geeigneter Definition des Vorgängerbegriffes kann die Bedingung (K) für die in den Anwendungen wichtige Rolle erhaltenden speziellen freien holomorphen Mengen erfüllt werden. Die geeignete Definition des Vorgängerbegriffes ist aber wesentlich. Z. B. im Fall einer Wortemenge über einem Alphabet liegt es an der Hand einer Wortefolge jenes Wort zuzuordnen, welches durch das Nacheinandersetzen der mit durch geeignete „Trennzeichen" getrennten Worte der Folge entsteht. Nun sind die einzelnen Worte der Folge keine „Anfangstücke", nur „Abschnitte" des so entstehenden Wortes; daher wird die Bedingung (K) im Fall des Beispiels 2′ in Nr. 3 nicht erfüllt; im Fall des Beispiels 2″ dagegen kann sie erfüllt werden.

17. Von nun an nehmen wir an, daß in H die Bedingung (K) erfüllt ist. Dann kann eine Funktion $\bar{f}(x, u)$ wie folgt rekursiv definiert werden (da man die Funktion $k_{o(y)}(x)$ auch zu Fallunterscheidungen gebrauchen kann):

$$\bar{f}(h, u) = u, \quad \text{falls} \quad h \in H_0,$$

und für $x = f^*(\ldots)$, wo $f^* \in F$,

$$\bar{f}(x,u) = \begin{cases} g_h(\bar{f}(k_1(x_1),u)), & \text{falls} \quad k_0(x_1) = h \in H_0, \\ g_{f^*}(k_1(x_1), k_2(x_1), \bar{f}(k_3(x_1),u), \bar{f}(k_4(x_1),u)), & \text{falls} \quad k_0(x_1) \in H_1, \\ l_{f^*}(k_1(x_1), k_2(x_1), \bar{f}(k_3(x_1),u)) & \text{sonst.} \end{cases}$$

Man sieht, daß diese Funktion $\bar{f}(x,u)$ den Wert u annimmt, und auch die Forderungen 2., 3., 4. in Nr. 15 erfüllt; genauer:

(2) Für jedes $h \in H_0$, $y \in H$ und

$$x = f_0(c_1(h,y), \ldots, c_1(h,y))$$

gilt

$$\bar{f}(x,u) = g_h(\bar{f}(y,u)),$$

(3) für jedes $f^* \in F$; $y_1, y_2, y_3, y_4 \in H$ und

$$x = f^*(c_4(h_1, y_1, y_2, y_3, y_4), \ldots, c_4(h_1, y_1, y_2, y_3, y_4))$$

gilt

$$\bar{f}(x,u) = g_{f^*}(y_1, y_2, \bar{f}(y_3,u), \bar{f}(y_4,u)),$$

(4) für jedes $f^* \in F$; $y_1, y_2, y_3 \in H$ und

$$x = f^*(c_3(h_2, y_1, y_2, y_3), \ldots, c_3(h_2, y_1, y_2, y_3))$$

gilt

$$\bar{f}(x,u) = l_{f^*}(y_1, y_2, f(y_3, u)).$$

18. Prüfen wir nach, ob die Definition von $\bar{f}(x,u)$ ein Spezialfall der allgemeinen Wertverlaufsrekursion ist. Es treten darin für gewisse i Funktionswerte $\bar{f}(k_i(x_1),u)$ auf. Nach (K) gilt

$$k_i(x_1) \prec x_1.$$

Sei $k_i(x_1)$ der Vorgänger mit dem Index v von x_1. Dann gilt nach Nr. 9, wenn die Wertverlaufsfunktion von $\bar{f}(x,u)$ mit $\bar{\varphi}(x,u)$ bezeichnet wird,

$$\bar{f}(k_i(x_1),u) = k_v(\bar{\varphi}(x_1,u)).$$

Um die Definition von $\bar{f}(x,u)$ und ähnliche Definitionen auf primitive Rekursion zurückführen zu können, hätte man (bei jedem festen f^*, falls x der Form $x = f^*(\ldots)$ ist) dieses v als Funktion von $i, x_1, \ldots, x_t$ zu bestimmen (und zwar als primitiv-rekursive Funktion). Ihr Definitionsbereich kann wie üblich auf alle Elemente von H ausgedehnt werden; und die so erhaltene Funktion soll — falls sie nicht als primitiv-rekursive Funktion in H definiert werden kann — zu den Ausgangsfunktionen hinzugenommen werden.

Dann sieht man leicht, daß die Definition von $\bar{f}(x,u)$ ein Spezialfall der allgemeinen Wertverlaufsrekursion ist, und daher ist $\bar{f}(x,u)$ eine primitiv-rekursive Funktion.

19. Die eingeschachtelten $\bar{f}$-Ausdrücke können aufgelöst werden, im folgenden Sinne. Es gilt der

HILFSSATZ. *Durch eine Wertverlaufsrekursion kann eine Funktion $r(x, y)$ definiert werden, für die stets*

$$\bar{f}(x, \bar{f}(y, u)) = \bar{f}(r(x, y), u)$$

gilt.

BEWEIS DES HILFSSATZES. Erstens gilt für $h \in H_0$ stets

$$\bar{f}(h, \bar{f}(y, u)) = \bar{f}(y, u),$$

so gilt die Behauptung für $h \in H_0$, wenn

$$r(h, y) = y \quad \text{für} \quad h \in H_0$$

gesetzt wird.

Angenommen, daß für alle x-Werte höchstens n-ter Ordnung $r(x, y)$ bereits im Sinne des Hilfssatzes definiert wurde, dann gilt für ein beliebiges $x = f^*(\ldots)$ von $n+1$-ter Ordnung, wobei $f^* \in F$,

$$\bar{f}(x, \bar{f}(y, u)) = \left\{\begin{array}{l} g_h(\bar{f}(r(k_1(x_1), y), u)), \quad \text{falls} \quad k_0(x_1) = h \in H_0, \\ g_{f^*}(k_1(x_1), k_2(x_1), \bar{f}(r(k_3(x_1), y), u), \\ \qquad \bar{f}(r(k_4(x_1), y), u)), \quad \text{falls} \quad k_0(x_1) \in H_1, \\ l_{f^*}(k_1(x_1), k_2(x_1), \bar{f}(r(k_3(x_1), y), u)) \quad \text{sonst} \end{array}\right\} =$$

$$= \begin{cases} \bar{f}(f_0(c_1(h, r(k_1(x_1), y)), \ldots, c_1(h, r(k_1(x_1), y))), u), \quad \text{falls} \quad k_0(x_1) = h \in H_0, \\ \bar{f}(f^*(c_4(h_1, k_1(x_1), k_2(x_1), r(k_3(x_1), y), r(k_4(x_1), y)), \ldots \\ \quad \ldots, c_4(h_1, k_1(x_1), k_2(x_1), r(k_3(x_1), y), r(k_4(x_1), y))), u), \quad \text{falls} \quad k_0(x_1) \in H_1, \\ \bar{f}(f^*(c_3(h_2, k_1(x_1), k_2(x_1), r(k_3(x_1), y)), \ldots, c_3(h_2, k_1(x_1), k_2(x_1), \\ \quad r(k_3(x_1), y))), u) \quad \text{sonst}, \end{cases}$$

also kann für $x = f^*(\ldots)$, wo $f^* \in F$,

$$r(x, y) = \begin{cases} f_0(c_1(h, r(k_1(x_1), y)), \ldots, c_1(h, r(k_1(x_1), y))), \quad \text{falls} \quad k_0(x_1) = h \in H_0, \\ f^*(c_4(h_1, k_1(x_1), k_2(x_1), r(k_3(x_1), y), r(k_4(x_1), y)), \ldots, c_4(h_1, k_1(x_1), \\ \quad k_2(x_1), r(k_3(x_1), y), r(k_4(x_1), y))), \quad \text{falls} \quad k_0(x_1) \in H_1, \\ f^*(c_3(h_2, k_1(x_1), k_2(x_1), r(k_3(x_1), y)), \ldots, c_3(h_2, k_1(x_1), k_2(x_1), \\ \quad r(k_3(x_1), y))) \quad \text{sonst} \end{cases}$$

dem Hilfssatz entsprechend definiert werden. Damit erhält man mit der bereits angegebenen Definition von $r(h, y)$ für $h \in H_0$ nach (K) ähnlich wie in Nr. 17—18 für $\bar{f}(x, u)$ eine Wertverlaufsrekursion für $r(x, y)$, und so ist $r(x, y)$ primitiv-rekursiv.

20. Der Hilfssatz ermöglicht die Werte von $f(x, u)$ aus den Werten von $\overline{f}(x, u)$ herauszuschöpfen. Es gilt nämlich die

BEHAUPTUNG: *Es gibt eine primitiv-rekursive Funktion $s(x)$, so daß für jedes $x \in H$*

$$f(x, u) = \overline{f}(s(x), u)$$

gilt.

BEWEIS DER BEHAUPTUNG. Für $h \in H_0$ gilt

$$f(h, u) = g_h(u) = g_h(\overline{f}(h_0, u)) = \overline{f}(f_0(c_1(h, h_0), \ldots, c_1(h, h_0)), u),$$

also gilt die Behauptung für $x = h \in H_0$, wenn

$$s(h) = f_0(c_1(h, h_0), \ldots, c_1(h, h_0)), \quad \text{falls} \quad h \in H_0$$

definiert wird.

Angenommen, daß für die Stellen x der Ordnung n die Funktion $s(x)$ der Behauptung entsprechend definiert ist, gilt bei $x = f^*(\ldots)$ der Ordnung $n+1$, wo $f^* \in F$,

$$\begin{aligned} f(x, u) &= g_{f^*}(x_1, x_2, \overline{f}(h_0, u), \overline{f}(s(x_1), \overline{f}(s(x_2), l_{f^*}(x_1, x_2, \overline{f}(h_0, u))))) = \\ &= g_{f^*}(x_1, x_2, \overline{f}(h_0, u), \overline{f}(s(x_1), \overline{f}(s(x_2), \overline{f}(f^*(c_3(h_2, x_1, x_2, h_0), \ldots \\ &\qquad \ldots, c_3(h_2, x_1, x_2, h_0)), u)))) = \\ &= g_{f^*}(x_1, x_2, \overline{f}(h_0, u), \overline{f}(s(x_1), \overline{f}(r(s(x_2), f^*(c_3(h_2, x_1, x_2, h_0), \ldots \\ &\qquad \ldots, c_3(h_2, x_1, x_2, h_0))), u))) = \\ &= g_{f^*}(x_1, x_2, \overline{f}(h_0, u), \overline{f}(r(s(x_1), r(s(x_2), f^*(c_3(h_2, x_1, x_2, h_0), \ldots \\ &\qquad \ldots, c_3(h_2, x_1, x_2, h_0)))), u)) = \\ &= \overline{f}(f^*(c_4(h_1, x_1, x_2, h_0, r(s(x_1), r(s(x_2), f^*(c_3(h_2, x_1, x_2, h_0), \ldots \\ &\ldots, c_3(h_2, x_1, x_2, h_0))))), \ldots, c_4(h_1, x_1, x_2, h_0, r(s(x_1), r(s(x_2), f^*(c_3(h_2, x_1, x_2, h_0), \ldots \\ &\qquad \ldots, c_3(h_2, x_1, x_2, h_0)))))), u), \end{aligned}$$

und daher wird $s(x)$ für $x = f^*(\ldots)$, wo $f^* \in F$, durch

$$\begin{aligned} s(x) = f^*(c_4(h_1, x_1, x_2, h_0, r(s(x_1), r(s(x_2), f^*(c_3(h_2, x_1, x_2, h_0), \ldots \\ \ldots, c_3(h_2, x_1, x_2, h_0))))), \ldots \\ \ldots, c_4(h_1, x_1, x_2, h_0, r(s(x_1), r(s(x_2), f^*(c_3(h_2, x_1, x_2, h_0), \ldots, c_3(h_2, x_1, x_2, h_0)))))) \end{aligned}$$

der Behauptung entsprechend definiert. Das ergibt mit der obigen Definition von $s(h)$ für $h \in H_0$ eine primitive Rekursion für $s(x)$.

Samt $s(x)$ ist aber wegen

$$f(x, u) = \overline{f}(s(x), u)$$

auch $f(x, u)$ primitiv-rekursiv.

Falls die Bedingung (K) *erfüllt ist, können sämtliche eingeschachtelten Rekursionen in H genau so auf primitive Rekursionen aufgelöst werden.*

21. Ähnlich könnte man die Analogien mit den anderen, für den zahlentheoretischen Fall bekannten Rekursionsarten verfolgen; nun gehe ich aber auf die *Definition der allgemein-rekursiven (und der partiell-rekursiven) Funktion* in H über.

Ich setze voraus, für jedes Element h von H_0 sei ein entsprechendes Zeichen für h (für verschiedene Elemente von H_0 verschiedene Zeichen) vorhanden; ich nenne diese Zeichen die *Konstanten.* Ferner sei für jede der (nichtkonstanten) Ausgangsfunktionen je ein entsprechendes von den Konstanten verschiedenes Zeichen (für verschiedene Ausgangsfunktionen verschiedene Zeichen) vorhanden; ich nenne sie *Funktionszeichen.* Für jedes Funktionszeichen ist dann die Anzahl der Argumente der durch ihm bezeichneten Funktion, kurz die Argumentenzahl des fraglichen Funktionszeichens, gegeben. Es sei auch eine abzählbar unendliche Folge von — von den Konstanten und Funktionszeichen verschiedenen — Zeichen, *Variable* genannt, ferner eine abzählbar unendliche Folge von — von den Konstanten, Variablen und Funktionszeichen verschiedenen — Zeichen, *Funktionsvariable* genannt, vorhanden. Zu jeder Funktionsvariablen sei eine positive ganze Zahl, die Argumentenzahl dieser Funktionsvariablen, zugeordnet derart, daß jede positive ganze Zahl die Argumentenzahl von unendlich vielen Funktionsvariablen ist. (Die Variablen werden die Elemente von H durchlaufen, die Funktionsvariablen werden auf H oder auf einer Teilmenge von H definierte, Elemente von H als Werte annehmende Funktionen mit der entsprechenden Anzahl von Argumenten bezeichnen, nämlich zu definierende Funktionen oder in die Definition eingehende Hilfsfunktionen.) Gewisse, aus Konstanten, Variablen, Funktionszeichen, Funktionsvariablen, Klammern und Kommata bestehende endliche Zeichenreihen werden *Terme* genannt, nämlich: jede Konstante und jede Variable ist ein Term, ferner, falls $\mathfrak{t}_1, \ldots, \mathfrak{t}_r$ bereits Terme sind, so sei jede Zeichenreihe $\mathfrak{f}(\mathfrak{t}_1, \ldots, \mathfrak{t}_r)$, wo $\mathfrak{f}$ ein Funktionszeichen oder eine Funktionsvariable mit der Argumentenzahl r ist, ebenfalls ein Term (und keine sonstige Zeichenreihe soll als Term gelten). Zu den bereits auf gezählten Zeichen nehmen wir noch das Gleichheitszeichen „=" hinzu; eine Zeichenreihe $\mathfrak{t}_1 = \mathfrak{t}_2$, wobei $\mathfrak{t}_1$ und $\mathfrak{t}_2$ Terme sind, wird eine Gleichung, und eine beliebige (eventuell unendliche) Menge von Gleichungen ein Gleichungssystem genannt.

Jedes Element x von H entsteht in eindeutiger Weise aus endlich vielen Elementen von H_0 durch endlichvielmalige Anwendung von Funktionen aus F. Werden die diese Funktionen (die ja unter den Ausgangsfunktionen vorkommen) bezeichnenden Funktionszeichen entsprechend auf die jene Elemente von H_0 bezeichnenden Konstanten angewandt, so ensteht ein Term, den ich als das dem Element x entsprechende (oder das Element x bezeichnende) *Elemententerm* bezeichne. Für $h \in H_0$ ist unter dem das Element h bezeichnenden Elemententerm die entsprechende Konstante zu verstehen.

Ein Gleichungssystem $\mathfrak{G}$ wird das *definierende Gleichungssystem* einer auf einer Teilmenge von H (eventuell auf H) definierten und Elemente von H als Werte annehmenden Funktion $f(y_1, \ldots, y_r)$ genannt, falls es für eine geeignete Funktionsvariable $\mathfrak{f}$ mit der Argumentenzahl r und für beliebige $y_1, \ldots, y_r, y \in H$ dann und nur dann eine Ableitung aus $\mathfrak{G}$ mit einem Endgleichung der Form

$$\mathfrak{f}(\mathfrak{y}_1, \ldots, \mathfrak{y}_r) = \mathfrak{y}$$

gibt, wo $\mathfrak{y}_1, \ldots, \mathfrak{y}_r$ und $\mathfrak{y}$ die den Elementen $y_1, \ldots, y_r, y$ von H entsprechenden Elemententerme sind, falls $f(y_1, \ldots, y_r)$ definiert ist und gleich dem Element y ist. Dabei bedeutet eine Ableitung aus einem Gleichungssystem $\mathfrak{G}$ die endlichmalige Anwendung der folgenden Schritte, von den Gleichungen des gegebenen Systems $\mathfrak{G}$ ausgehend:

1. Das Einsetzen eines Elemententermes für eine Variable in einer Gleichung, überall, wo diese Variable auftritt.

2. Falls die Gleichungen $\mathfrak{a} = \mathfrak{b}$ und $\mathfrak{c} = \mathfrak{d}$ bereits in der Ableitung vorgekommen sind (eventuell zum Gleichungssystem $\mathfrak{G}$ gehören), dann die Ersetzung in $\mathfrak{a} = \mathfrak{b}$ eines Teiltermes $\mathfrak{c}$ durch $\mathfrak{d}$ (auf einer oder mehreren Stellen).

Eine auf einer Teilmenge von H bzw. auf H definierte Funktion, wofür ein definierendes Gleichungssystem existiert, heißt eine partiell-rekursive bzw. eine allgemein-rekursive Funktion. In beiden Fällen wird also verlangt, daß für zwei Gleichungen der Form

$$\mathfrak{f}(\mathfrak{y}_1, \ldots, \mathfrak{y}_r) = \mathfrak{y}, \quad \mathfrak{f}(\mathfrak{y}_1, \ldots, \mathfrak{y}_r) = \mathfrak{y}'$$

mit dem obigen (die zu definierende partiell-rekursive bzw. allgemein-rekursive Funktion f bezeichnenden) Funktionsvariablen $\mathfrak{f}$ und mit Elemententermen $\mathfrak{y}_1, \ldots, \mathfrak{y}_r$, $\mathfrak{y}$ und $\mathfrak{y}'$, die Endgleichungen von Ableitungen aus dem definierenden Gleichungssystem von f sind, $\mathfrak{y}$ mit $\mathfrak{y}'$ übereinstimmen soll; und im Fall einer allgemein-rekursiven Funktion f wird auch verlangt, daß es für beliebige Elemententerme $\mathfrak{y}_1, \ldots, \mathfrak{y}_r$ ein Elemententerm $\mathfrak{y}$ geben soll, wofür die Gleichung $\mathfrak{f}(\mathfrak{y}_1, \ldots, \mathfrak{y}_r) = \mathfrak{y}$ die Endgleichung einer Ableitung aus dem definierenden Gleichungssystem von f ist.

Im folgenden werde ich einfachheitshalber die Konstanten, Elemententerme, Funktionszeichen und Funktionsvariablen manchmal nicht von den durch sie bezeichneten Elementen von H_0 und H, Ausgangsfunktionen bzw. zu definierenden Funktionen oder Hilfsfunktionen unterscheiden, was wohl nie zu einem Mißverständnis führen kann.

22. Bemerkungen. (1) Enthält F wenigstens eine mindestens zweistellige Funktion (was bei einer Wortemenge nicht der Fall ist), dann kann einer beliebigen endlichen Folge von Elementen aus H in einfacher Weise ein Element von H derart zugeordnet werden, daß daraus die Glieder der Folge wiederzuerkennen sind.

Sei nämlich $f_1(x_1, \ldots, x_m) \in F$ mit $m \geqq 2$, und setzen wir

$$f_1(x, y) = f_1(x, y, \ldots, y).$$

Nach Annahme sind für ein $x = f^*(x_1^*, \ldots, x_m^*)$, wo $f^* \in F$, die Funktion f^* und die Argumente $x_1^*, \ldots, x_m^*$ eindeutig bestimmt. Die Funktionen $g_i(x)$ für $i = 1, 2, \ldots, m$, wobei $g_i(x)$ für $x = f_1(y_1, \ldots, y_i, \ldots, y_m)$ den Wert y_i und sonst den Wert h_0 annimmt, können leicht als primitiv-rekursive Funktionen in H definiert werden.

Nun kann einem geordneten Paar (x, y) das Element $f_1(x, y)$ zugeordnet werden, woraus die Glieder des Paars als

$$g_1(f_1(x, y)) = x, \quad g_2(f_1(x, y)) = y$$

wiedererhalten werden können. Durch wiederholte Paarbildung kann einer beliebigen endlichen Folge ebenfalls ein einziges Element zugeordnet werden, woraus die Glieder der Folge wiederzuerkennen sind. Es ist aber zweckmäßiger ebenfalls durch Paarbildung zu bezeichnen, um wievieltes Glied es sich handelt, denn so erkennt man aus dem einer endlichen Folge zugeordneten Element sofort, wievielgliedrig die betreffende Folge ist.

Seien also den Folgen

$$x_0$$
$$x_0, x_1$$
$$x_0, x_1, x_2$$
$$\cdots\cdots\cdots$$

der Reihe nach die Elemente

$$f_1(x_0, h_0)$$
$$f_1(f_1(x_0, h_0), f_1(x_1, h_1))$$
$$f_1(f_1(f_1(x_0, h_0), f_1(x_1, h_1)), f_1(x_2, h_2))$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

zugeordnet.

Dann sieht man gleich, daß falls x einer endlichen Folge zugeordnet ist, so der Höchstindex der Glieder $x_0, x_1, \ldots$ dieser Folge gleich

$$\overline{\operatorname{long}}(x) = g_2(g_2(x))$$

ist. Für andere x kann $\overline{\operatorname{long}}(x) = h_0$ definiert werden. Man hat aber noch diese Fallunterscheidung zu charakterisieren.

Erst sei $\mathrm{ch}_{o(y)}(x)$ die charakteristische Funktion der Beziehung, daß x einer Folge

$$x_0, x_1, \ldots, x_{o(y)}$$

zugeordnet ist. Diese kann (nach 4. von Nr. 7) als primitiv-rekursive Funktion von x und y definiert werden: für $h \in H_0$ sei

$$\mathrm{ch}_{o(h)}(x) = \begin{cases} h_0, & \text{falls } x \text{ von der Form } x = f_1(\ldots) \text{ ist} \\ & \text{und } g_2(x) = g_3(x) = \cdots = g_m(x) = h_0, \\ h_1 & \text{sonst,} \end{cases}$$

und wenn $y = f^*(\ldots)$, wo $f^* \in F$ ist, so

$$\mathrm{ch}_{o(y)}(x) = \begin{cases} h_0, & \text{falls } x, g_2(x), g_3(x), \ldots, g_m(x) \text{ von der Form } f_1(\ldots) \text{ sind,} \\ & g_2(x) = g_3(x) = \cdots = g_m(x), \\ & \mathrm{ch}_{o(y-1)}(g_1(x)) = h_0 \quad \text{und} \quad g_2(g_2(x)) = h_{o(y)}, \\ h_1 & \text{sonst.} \end{cases}$$

Dann ist die charakteristische Funktion $\overline{\mathrm{ch}}(x)$ der Beziehung, daß x überhaupt einer endlichen Folge $x_0, x_1, \ldots, x_i$ zugeordnet ist (da dann $g_2(g_2(x)) = h_i$ ist):

$$\overline{\mathrm{ch}}(x) = \mathrm{ch}_{o(g_2(g_2(x)))}(x),$$

und damit ist auch

$$\overline{\mathrm{long}}(x) = \begin{cases} g_2(g_2(x)), & \text{falls } \overline{\mathrm{ch}}(x) = h_0, \\ h_0 & \text{sonst} \end{cases}$$

primitiv-rekursiv.

Aus dem der Folge x_0, x_1, x_2 zugeordneten Element

$$x = f_1(f_1(f_1(x_0, h_0), f_1(x_1, h_1)), f_1(x_2, h_2))$$

ergibt sich

$$x_2 = g_1(g_2(x)), \quad x_1 = g_1(g_2(g_1(x))), \quad x_0 = g_1(g_1(g_1(x))),$$

und man sieht leicht, daß allgemein für $\overline{\mathrm{ch}}(x) = h_0$, mit Verwendung der leicht als primitiv-rekursiv definierbaren Funktion

$$g_1^{o(y)}(x) = \underbrace{g_1(g_1(\ldots(g_1}_{o(y)\text{-mal}}(x))\ldots)),$$

das Glied mit dem Index $\overline{\mathrm{long}}(x) \dot{-} o(y)$ der zu x gehörigen Folge für $o(y) < \overline{\mathrm{long}}(x)$ gleich

$$g_1(g_2(g_1^{o(y)}(x)))$$

ist, für $o(y) \geqq \overline{\mathrm{long}}(x)$ aber gleich

$$g_1^{\overline{\mathrm{long}}(x)+1}(x)$$

ist (man denke daran, daß $\overline{\text{long}}(x)$ irgendein h_i ist). Für $\overline{\text{ch}}(x)=h_1$ kann x als sein einziges Glied gelten. Wird also $\bar{k}_{\overline{\text{long}}(x)\dot{-}o(y)}(x)$ durch

$$\bar{k}_{\overline{\text{long}}(x)\dot{-}o(y)}(x)=\begin{cases} x, & \text{falls } \overline{\text{ch}}(x)=h_1, \\ g_1(g_2(g_1^{o(y)}(x))), & \text{falls } \overline{\text{ch}}(x)=h_0 \text{ und } o(y)<\overline{\text{long}}(x), \\ g_1^{\overline{\text{long}}(x)+1}(x), & \text{falls } \overline{\text{ch}}(x)=h_0 \text{ und } o(y)\geqq\overline{\text{long}}(x) \end{cases}$$

definiert, so ist damit auch die (die Rolle der in Nr. 8 betrachteten Funktion $k_{o(y)}(x)$ übernehmende) Funktion

$$\bar{k}_{o(y)}(x)=\begin{cases} \bar{k}_{\overline{\text{long}}(x)\dot{-}(\overline{\text{long}}(x)\dot{-}o(y))}(x), & \text{falls } o(y)\leqq\text{long}(x), \\ e(x) & \text{sonst} \end{cases}$$

primitiv-rekursiv.

Aus der Definition sieht man, daß die der Bedingung (K) entsprechende Beziehung

$$\bar{k}_{o(y)}(x)\leqq x$$

immer gilt.

(2) Wenn F nur einstellige Funktionen enthält, dann hat ein beliebiges Element x von H die Form:

$$x=f_i(f_{i-1}(\ldots(f_2(f_1(h)))\ldots)),$$

wo $f_l\in F$ für $l=1,2,\ldots,i$ und h ein Element von H_0 ist. Das kann unmißverständlich auch ohne Klammern geschrieben werden, etwa die Zeichen in der Reihenfolge ihrer Verwendung aufgereiht:

$$x=hf_1f_2\ldots f_i.$$

Wenn als Vorgänger des Elementes x die Konstituenten von x, also

$$h,\quad hf_1,\quad hf_1f_2,\ldots,hf_1\ldots f_{i-1},\quad hf_1\ldots f_i$$

betrachtet werden, so besitzt x den einzigen unmittelbaren Vorgänger $x^{-1}=hf_1\ldots f_{i-1}$, und alle seine echten Vorgänger sind die Vorgänger von x^{-1}. So lautet hier eine primitive Rekursion:

$$f(h,u_1,\ldots,u_r)=g_h(u_1,\ldots,u_r),\quad \text{falls}\quad h\in H_0,$$

und für $f_i\in F$ (D

$$f(xf_i,u_1,\ldots,u_r)=g_{f_i}(x,u_1,\ldots,u_r,f(x,u_1,\ldots,u_r)).$$

Nun kann die angeordnete freie holomorphe Menge H umkehrbar eindeutig (und sogar in einem plausiblen Sinne isomorph) auf eine andere angeordnete freie holomorphe Menge $\mathfrak{H}$ abgebildet werden, indem jedem Element $x=hf_1\ldots,f_i$ von H die Zeichenreihe $\mathfrak{x}=\mathfrak{h}\mathfrak{f}_1\ldots\mathfrak{f}_i$ zugeordnet wird, wobei $\mathfrak{h}$ die Konstante und $\mathfrak{f}_1,\ldots,\mathfrak{f}_i$ die Funktionszeichen sind, die der Reihe nach das

Element h von H_0 und die Funktionen $f_1, \ldots, f_i$ (die ja unter den Ausgangsfunktionen vorkommen) bezeichnen. $\mathfrak{H}$ besteht also aus allen Zeichenreihen, deren erstes Glied eine Konstante und die etwaigen übrigen Glieder solche Funktionszeichen sind, die Elemente von F bezeichnen. Offenbar ist $\mathfrak{H}$ ebenfalls eine angeordnete freie holomorphe Menge, mit der Menge $\mathfrak{H}_0$ der Konstanten in der Rolle von H_0, und in der Rolle von F mit der Menge $\mathfrak{F}$ der Funktionen, die jedem zu $\mathfrak{H}$ gehörigen Zeichenreihe $\mathfrak{x}$ die durch Anknüpfen eines (ein Element von F bezeichnenden) Funktionszeichens $\mathfrak{f}_i$ entstehende Zeichenreihe $\mathfrak{x}\mathfrak{f}_i$ zuordnet (diese Funktion werde ich ebenfalls mit $\mathfrak{f}_i$ bezeichnen, so daß $\mathfrak{F}$ zugleich die Menge der die Elemente von F bezeichnenden Funktionszeichen bezeichnen kann), endlich mit dem Vorgängerbegriff, welcher einer zu $\mathfrak{H}$ gehörigen Zeichenreihe $\mathfrak{x} = \mathfrak{h}\mathfrak{f}_1 \ldots \mathfrak{f}_i$ die Zeichenreihen

$$\mathfrak{h}, \quad \mathfrak{h}\mathfrak{f}_1, \quad \mathfrak{h}\mathfrak{f}_1\mathfrak{f}_2, \; \ldots, \mathfrak{h}\mathfrak{f}_1 \ldots \mathfrak{f}_{i-1}, \quad \mathfrak{h}\mathfrak{f}_1 \ldots \mathfrak{f}_i$$

und nur diese als Vorgänger (also $\mathfrak{x}^{-1} = \mathfrak{h}\mathfrak{f}_1 \ldots \mathfrak{f}_{i-1}$ als einzigen unmittelbaren Vorgänger) zuordnet. Für jedes $x \in H$ bezeichne ich die derart zugeordnete Zeichenreihe, d. h. „Wort" (des Alphabets

$$A' = \{\ldots, \mathfrak{h}, \ldots; \ldots, \mathfrak{f}, \ldots\} = \mathfrak{H}_0 + \mathfrak{F},$$

wo $\mathfrak{h}$ die zu $\mathfrak{H}_0$ gehörigen Konstanten und $\mathfrak{f}$ die zu $\mathfrak{F}$ gehörigen Funktionszeichen durchläuft) als das „Gödel-Wort" von x.

Einer primitiven Rekursion (D) in H entspricht eine primitive Rekursion in $\mathfrak{H}$, welche formal ähnlich lautet, nämlich

$$f(\mathfrak{h}, u_1, \ldots, u_r) = g_{\mathfrak{h}}(u_1, \ldots, u_r), \quad \text{falls} \quad \mathfrak{h} \in \mathfrak{H}_0,$$

und für $\mathfrak{f}_i \in \mathfrak{F}$ $(\mathfrak{D})$

$$f(\mathfrak{x}\mathfrak{f}_i, u_1, \ldots, u_r) = g_{\mathfrak{f}_i}(\mathfrak{x}, u_1, \ldots, u_r, f(\mathfrak{x}, u_1, \ldots, u_r));$$

und dasselbe gilt auch für andere Rekursionsarten.

Nun ist $\mathfrak{H}$ eine Teilmenge der Wortemenge H' mit dem Alphabet A'. Zu $\mathfrak{H}$ gehören, wie gesagt, nur diejenigen Worte von H' (und genau diese sind „sinnvoll" in H), welche mit einem $\mathfrak{h} \in \mathfrak{H}_0$ beginnen, und weiter kein Element von $\mathfrak{H}_0$ enthalten. Die charakteristische Funktion $\mathrm{sv}(\mathfrak{x})$ der Beziehung: „Ein Wort obiger Art zu sein", d. h. „Sinnvoll in H zu sein" (welche den Wert $\wedge$, d. h. das leere Wort, oder ein bestimmtes $\mathfrak{h}_0 \in \mathfrak{H}_0$, etwa die dem Element h_0 von H_0 entsprechende Konstante als Wert annimmt, je nachdem diese Beziehung gilt oder nicht gilt) kann wie folgt als primitiv-rekursive Funktion in H' definiert werden:

$$\mathrm{sv}(\wedge) = \mathfrak{h}_0,$$

$$\mathrm{sv}(x\mathfrak{h}) = \begin{cases} \wedge, & \text{falls} \quad x = \wedge, \\ \mathfrak{h}_0 & \text{sonst,} \end{cases}$$

falls $\mathfrak{h} \in \mathfrak{H}_0$; und für $\mathfrak{f}_i \in \mathfrak{F}$

$$\mathrm{sv}(x\mathfrak{f}_i) = \begin{cases} \wedge, & \text{falls} \quad \mathrm{sv}(x) = \wedge, \\ \mathfrak{h}_0 & \text{sonst.} \end{cases}$$

Dann kann in H' der primitiv-rekursiven Definition ($\mathfrak{D}$) in $\mathfrak{H}$ von $f(x, u_1, \ldots, u_r)$ die primitiv-rekursive Definition einer Funktion $f'(x, u_1, \ldots, u_r)$ zugeordnet werden, welche an „sinnvolle" Stellen dieselben Werte wie $f(x, u_1, \ldots, u_r)$ annimmt, und an anderen Stellen den Wert $\wedge$: Sei

$$f'(\wedge, u_1, \ldots, u_r) = \wedge,$$

für $\mathfrak{h} \in \mathfrak{H}_0$

$$f'(x\mathfrak{h}, u_1, \ldots, u_r) = \begin{cases} g_{\mathfrak{h}}(u_1, \ldots, u_r), & \text{falls} \quad x = \wedge \text{ und} \\ & \mathrm{sv}(u_1) = \cdots = \mathrm{sv}(u_r) = \wedge, \\ \wedge & \text{sonst,} \end{cases}$$

und für $\mathfrak{f}_i \in \mathfrak{F}$

$$f'(x\mathfrak{f}_i, u_1, \ldots, u_r) = \begin{cases} g_{\mathfrak{f}_i}(x, u_1, \ldots, u_r, f'(x, u_1, \ldots, u_r)), \\ \quad \text{falls} \quad \mathrm{sv}(x) = \mathrm{sv}(u_1) = \cdots = \mathrm{sv}(u_r) = \wedge, \\ \wedge \quad \text{sonst;} \end{cases}$$

dann sieht man durch vollständige Induktion nach der Ordnung der sinnvollen Argumente x, daß

$$f'(x, u_1, \ldots, u_r) = \begin{cases} f(x, u_1, \ldots, u_r), \\ \quad \text{falls} \quad \mathrm{sv}(x) = \mathrm{sv}(u_1) = \cdots = \mathrm{sv}(u_r) = \wedge, \\ \wedge \quad \text{sonst.} \end{cases}$$

Ähnlich findet man entsprechende Rekursionen auf der Wortemenge H' zu den anderen Rekursionsarten in H. (Auf die Frage der Umkehrung komme ich auf anderer Stelle zurück.)

Daher beschränke ich mich in den Folgenden auf Wortemengen.

II

23. Von nun an soll H immer eine Wortemenge im Sinne des Beispiels 2 in Nr. 2 bezeichnen, wobei $H_0 = \{\wedge\}$ und $F = \{\ldots, xa_i, \ldots\}$ ist; die a_i sind dabei die „Buchstaben" eines beliebigen „Alphabets" $A = \{\ldots, a_i, \ldots\}$.

Als Vorgänger eines Wortes sollen $\wedge$ und sämtliche „Abschnitte" des Wortes betrachtet werden; H soll also immer eine angeordnete freie holomorphe Menge der Art des Beispiels 2'' in Nr. 3 sein. Mit den dort eingeführten Bezeichnungen sei die festgewählte Reihenfolge der unmittelbaren Vorgänger eines Wortes x

$$x_1 = x^{-1}, \quad x_2 = {}^{-1}x$$

(für $o(x)=1$ gilt dann $x_1=x_2=\wedge$). Statt „x ist der Form $x=f^*(\ldots)$, wo $f^*\in F$" kann hier „x ist der Form $(\ldots)a_i$, wo $a_i\in A$" gesagt werden. Also lautet hier eine primitiv-rekursive Definition wie folgt:

$$f(\wedge, u_1,\ldots,u_r)=g(u_1,\ldots,u_r),$$

und falls x der Form $x=(\ldots)a_i$ ist, wo $a_i\in A$,

$$f(x,u_1,\ldots,u_r)=g_{a_i}(x^{-1},\,{}^{-1}x,u_1,\ldots,u_r,f(x^{-1},u_1,\ldots,u_r),f({}^{-1}x,u_1,\ldots,u_r)),$$

oder aber

$$f(\wedge,u_1,\ldots,u_r)=g(u_1,\ldots,u_r)$$

und für $a_i\in A$

$$f(xa_i,u_1,\ldots,u_r)=$$
$$=g_{a_i}(x,\,{}^{-1}(xa_i),u_1,\ldots,u_r,f(x,u_1,\ldots,u_r),f({}^{-1}(xa_i),u_1,\ldots,u_r));$$

wie man leicht sieht, ist das auch mit der folgenden Definition äquivalent:

$$f(\wedge,u_1,\ldots;u_r)=g(u_1,\ldots,u_r)$$

und für $a_i\in A$

$$f(a_i,u_1,\ldots,u_r)=\bar{g}_{a_i}(u_1,\ldots,u_r),$$

ferner für $x\neq\wedge$, $a_i\in A$

$$f(xa_i,u_1,\ldots,u_r)=$$
$$=g_{a_i}(x,u_1,\ldots,u_r,f(x,u_1,\ldots,u_r),f({}^{-1}(xa_i),u_1,\ldots,u_r)).$$

Die Funktion x^{-1} kann hier durch folgende primitive Rekursion definiert werden:

$$\wedge^{-1}=\wedge$$

und für $a_i\in A$

$$(xa_i)^{-1}=x,$$

wobei aber auch die Identitätsfunktion $f(x)=x$ verwendet wird. Hier kann man auch diese durch eine primitive Rekursion in den ursprünglichen Ausgangsfunktionen $\wedge$ und xa_i definieren:

$$f(\wedge)=\wedge,$$

und falls $a_i\in A$,

$$f(xa_i)=xa_i.$$

Die Definition von ${}^{-1}x$ lautet:

$${}^{-1}\wedge=\wedge$$

und für $a_i\in A$

$${}^{-1}(xa_i)=\begin{cases}\wedge, & \text{falls } x=\wedge,\\ ({}^{-1}x)a_i & \text{sonst,}\end{cases}$$

und das ist auch eine primitive Rekursion in H (mit Benutzung einer „zusammengeflickten" Funktion; siehe 8. in Nr. 7).

Mit Hilfe der Identitätsfunktion lautet die Definition der Aneinanderknüpfung zweier Worte xy (provisorisch mit $f(x, y)$ bezeichnet):

$$f(x, \wedge) = x$$

und für $a_i \in A$

$$f(x, ya_i) = f(x, y)a_i.$$

24. Betrachten wir noch einige allgemeine Definitionen aus Nr. 7, um zu klären, was sich aus diesen für unseren Spezialfall ergibt.

1. h_0, als ein Element aus H_0, kann hier nur $\wedge$ sein. f_0, als ein ausgewähltes Element aus F, sei hier xa_0, wo $a_0 \in A$. Dann ist hier

$$h_1 = \wedge a_0 = a_0,$$
$$h_2 = a_0 a_0,$$
$$h_3 = a_0 a_0 a_0,$$
$$\ldots\ldots\ldots\ldots\ldots$$

Diese sollen der kürzeren Schreibweise halber mit $a_0^1, a_0^2, a_0^3, \ldots$ bezeichnet werden; a_0^0 soll $\wedge$ bedeuten. So ist hier

$$h_{o(x)} = a_0^{o(x)}.$$

Hier identifizieren wir also die natürlichen Zahlen $0, 1, 2, \ldots$ der Reihe nach mit $\wedge, a_0, a_0^2, \ldots$. Die Ordnung r eines Wortes $a_1 a_2 \ldots a_r$ wird dadurch mit $a_0 a_0 \ldots a_0 = a_0^r$ identifiziert.

1a) Die charakteristische Funktion $\zeta(x)$ der Beziehung, daß x eine natürliche Zahl ist, kann folgenderweise als primitiv-rekursive Funktion definiert werden:

$$\zeta(\wedge) = \wedge$$

und für $a_0 \in A$

$$\zeta(xa_i) = \begin{cases} \zeta(x), & \text{falls} \quad a_i = a_0, \\ a_0 & \text{sonst.} \end{cases}$$

2. $x^{-o(y)}$ ergibt hier für $o(y) \leqq o(x)$ den Rest, der vom Wort x übrigbleibt, wenn seine letzten Buchstaben der Anzahl $o(y)$ weggelassen werden. Für $o(y) > o(x)$ ist $x^{o(y)} = \wedge$.

2a) Hier kann auch ${}^{-o(y)}x$ als primitiv-rekursive Funktion definiert werden, worunter für $o(y) \leqq o(x)$ der Rest von x bei Weglassung der ersten Buchstaben der Anzahl $o(y)$ verstanden werden soll, und $\wedge$, wenn $o(y) > o(x)$ ist:

$${}^{-o(\wedge)}x = x$$

und für $a_i \in A$

$${}^{-o(ya_i)}x = {}^{-1}({}^{-o(y)}x).$$

3. $e(x)$ ist hier stets gleich $\wedge$, als einziger Vorgänger 0-ter Ordnung von x. Hätte man die Buchstaben von A in H_0 aufgenommen, so würde $e(x)$ den ersten Buchstaben des Wortes x bedeuten. Diese Funktion werden wir hier brauchen; und auch allgemeiner das aus den ersten Buchstaben der Anzahl $o(y)$ von x bestehende Wort. Das soll mit $e_{o(y)}(x)$ bezeichnet werden; dabei sei $e_{o(y)}(x)=\wedge$ für $o(y)>o(x)$. Die Definition von $e_{o(y)}(x)$ lautet (mit Bezugnahme auf 10. von Nr. 7):

3a)

$$e_{o(y)}(\wedge)=\wedge$$

und für $a_i \in A$

$$e_{o(y)}(xa_i)=\begin{cases} e_{o(y)}(x), & \text{falls } o(xa_i)>o(y), \\ xa_i, & \text{falls } o(xa_i)=o(y), \\ \wedge & \text{sonst.} \end{cases}$$

3b) Auch $l_{o(y)}(x)$, welches für $o(y)>o(x)$ den Wert $\wedge$ annimmt, und sonst das aus den letzten Buchstaben der Anzahl $o(y)$ von x bestehende Wort bedeutet, kann durch eine primitive Rekursion definiert werden:

$$l_{o(y)}(\wedge)=\wedge$$

und für $a_i \in A$

$$l_{o(y)}(xa_i)=\begin{cases} {}^{-1}l_{o(y)}(x)a_i, & \text{falls } o(xa_i)>o(y)>0, \\ xa_i, & \text{falls } o(xa_i)=o(y), \\ \wedge & \text{sonst.} \end{cases}$$

4. Dies ergibt hier die charakteristische Funktion der Beziehung, daß das Wort x mit einem bestimmten Buchstaben a_i endet.

Alles andere von Nr. 7 überträgt sich wörtlich auf unseren Spezialfall.

Die charakteristische Funktion der Gleichheit $x=y$ kann auch hier nicht allgemein durch primitive Rekursionen und Substitutionen definiert werden, muß also zu den Ausgangsfunktionen hinzugenommen werden.

25. Hier können aber auch weitere wichtige primitiv-rekursive Funktionen definiert werden. Zum Beispiel:

(1) Definition der charakteristischen Funktion der Beziehung, daß x ein Abschnitt von y ist, das heißt, nach der Definition der Vorgänger eines Wortes, daß $y \preceq x$ gilt (provisorisch mit $f(x,y)$ bezeichnet):

$$f(\wedge, y)=\begin{cases} \wedge, & \text{falls } y=\wedge, \\ a_0 & \text{sonst,} \end{cases}$$

und für $a_i \in A$

$$f(xa_i, y)=\begin{cases} \wedge, & \text{falls } xa_i=y \vee f(x,y)=\wedge \vee f({}^{-1}(xa_i), y)=\wedge, \\ a_0 & \text{sonst.} \end{cases}$$

(2) Definition des Wortes $\mathrm{subst}(x, y, z)$, welches aus x entsteht, wenn darin vom Ende nach dem Anfang gehend jedes Vorkommen von y durch z ersetzt wird (wobei natürlich $\mathrm{subst}(x, y, z) = x$ ist, wenn y nicht unter den Abschnitten von x vorkommt; und für $y = \wedge$ sei $\mathrm{subst}(x, y, z) = \wedge$):

$$\mathrm{subst}(\wedge, y, z) = \wedge$$

und für $a_i \in A$

$$\mathrm{subst}(xa_i, y, z) = \begin{cases} \wedge, & \text{falls } y = \wedge, \\ z({}^{-(o(y) \dot{-} 1)}\mathrm{subst}({}^{-1}(xa_i), y, z)), \\ \quad \text{falls } e_1(xa_i)e_{o(y) \dot{-} 1}(\mathrm{subst}({}^{-1}(xa_i), y, z)) = y, \\ e_1(xa_i)\,\mathrm{subst}({}^{-1}(xa_i), y, z) & \text{sonst.} \end{cases}$$

Hier kann $o(y) \dot{-} 1$ überall durch $o(y^{-1})$ ersetzt werden; also ist die Funktion $\mathrm{subst}(x, y, z)$ primitiv-rekursiv.

Für $z = \wedge$ bedeutet $\mathrm{subst}(x, y, z)$ das Wort, das aus x übrigbleibt, wenn davon vom Ende nach dem Anfang gehend jedes Vorkommen von y gestrichen wird.

(3) Eine andere Art Ersetzung ist, wenn in x ein mit dem $o(i)+1$-ten Buchstaben beginnender Abschnitt der Ordnung $o(y)$ durch z ersetzt wird, falls er gleich y ist, und unverändert gelassen wird, wenn nicht. Das so entstehende Wort, mit $\mathrm{ers}(i, x, y, z)$ bezeichnet, wobei wieder $\mathrm{ers}(i, x, y, z) = \wedge$ für $y = \wedge$ sei, kann durch folgende „zusammengeflickte" Definition angegeben werden:

$$\mathrm{ers}(i, x, y, z) = \begin{cases} \wedge, & \text{falls } y = \wedge, \\ e_{o(i)}(x)\,z\,l_{o(x) \dot{-} o(iy)}(x), & \text{falls } {}^{-o(i)}e_{o(iy)}(x) = y \neq \wedge, \\ x & \text{sonst,} \end{cases}$$

woraus sich $\mathrm{ers}(i, x, y, z)$ wegen $o(x) \dot{-} o(iy) = o(x^{-o(iy)})$ als primitiv-rekursive Funktion ergibt.

26. In wichtigen Anwendungen, z. B. in einer exakten Fassung der (in der Programmierung wichtige Rolle erhaltenden) algorithmischen Sprache „Algol",[7] treten aber meistens nicht primitive Rekursionen, sondern simultane

[7] A. J. Perlis and K. Samelson, Report on the algorithmic language ALGOL by the ACM Committee on Programming Languages and the GAMM Committee on Programming, *Numerische Mathematik*, **1** (1959), S. 41–60. — *Zusatz bei der Korrektur (am 25. August 1961):* In der Zeitschrift *Numerische Mathematik* ist in 1960 ein Referat über die algorithmische Sprache „Algol 60" erschienen. Ich habe bewiesen, daß die darin für den ersten Augenblick als zirkelhaft definiert erscheinenden Prädikate bei exakter Fassung primitivrekursive Beziehungen auf einer Wortemenge sind. Darüber habe ich am II. Ungarischen Mathematikerkongress (1960) einen Vortrag gehalten; eine ausführlichere Mitteilung ist im Erscheinen in den *Publ. Math. Inst. Hung. Acad. Sci.*

und Wertverlaufsrekursionen auf (in den letzteren werden meistens nicht nur Anfangsstücke, sondern beliebige Abschnitte der Worte als Vorgänger verwendet; letzten Endes war gerade das entscheidend in der Wahl unserer Anordnung der Wortemenge H). Um auch diese auf primitive Rekursionen zurückführen zu können, haben wir nach Nr. 8—20 eine geeignete Abbildung der endlichen Elementenfolgen von H auf H zu wählen.

Besteht das Alphabet aus einem einzigen Buchstaben a_0, dann sind sämtliche Elemente der Wortemenge:

$$\wedge,\ a_0,\ a_0^2, \ldots,$$

die wir mit den natürlichen Zahlen

$$0, 1, 2, \ldots$$

identifiziert haben. Hier hat jedes Wort $x \neq \wedge$ den einzigen unmittelbaren Vorgänger

$$x^{-1} = {}^{-1}x = x-1.$$

So kann diese Wortemenge einfach als identisch der Menge der natürlichen Zahlen betrachtet werden. Hier erhält man die Zusammenhänge der verschiedenen Rekursionsarten mit Benutzung der zahlentheoretischen Eigenschaften der Elemente, z. B. der eindeutigen Primfaktorenzerlegung. Auch die charakteristische Funktion der Gleichheit $x = y$ kann hier bekanntlich primitivrekursiv definiert werden. Als einzige Ausgangsfunktion bleibt hier neben der Konstanten 0, d. h. $\wedge$ die Funktion xa_0, das mit $x+1$ identifiziert wurde.

In den Weiteren nehme ich daher an, daß das Alphabet A wenigstens zwei Buchstaben a_0, a_1 enthält; für die Mächtigkeit von A stelle ich vorläufig keine andere Bedingungen.

27. Einer endlichen Folge

$$y_0 \in H, \quad y_1 \in H, \ldots, y_n \in H$$

könnte man mit einem geeigneten „Trennzeichen" $\alpha \in H$ das Wort

$$y_0 \alpha y_1 \alpha \ldots \alpha y_n \alpha$$

zuordnen; man hat aber bei der Wahl von α darauf zu achten, daß aus diesem Wort die Glieder der Folge eindeutig wiederzuerkennen seien. Ich wähle als Trennzeichen die Worte α_i, wo

$$\alpha_i = a_1 a_0^i a_1 \quad \text{für} \quad i = 1, 2, \ldots$$

ist. Genauer sei j die größte Zahl, für welche a_0^j Vorgänger von mindestens einem Glied der Folge $(y_0, \ldots, y_n)$ ist, dann ordne ich dieser Folge das Wort

$$c_n(y_0, \ldots, y_n) = y_0 \alpha_{j+1} \ldots \alpha_{j+1} y_n \alpha_{j+1}$$

zu.

So ist x dann und nur dann einer endlichen Wortefolge aus H zugeordnet, wenn es mit einem α_{j+1} endet, und dabei weder

$$a_1 a_0^{j+1} a_1 a_0^{j+1} a_1$$

noch ein

$$a_{i_1} a_0^{j+1} a_{i_2}$$

Vorgänger von x ist, wenn nicht $a_{i_1} = a_{i_2} = a_1$ gilt.

Ist x einer Folge $(y_0, \ldots, y_n)$ zugeordnet, also der Form

$$x = y_0 \alpha_{j+1} \ldots \alpha_{j+1} y_n \alpha_{j+1},$$

so wird in Abhängigkeit von x die „Länge" n dieser Folge durch $\operatorname{long}(x)$ und das Glied y_i durch $k_i(x)$ bezeichnet. Wie man sieht, ist in diesem Fall $k_i(x)$ für jede Zahl $i \leqq n$ ein Vorgänger von x; wird es für andere Werte x und i als $\wedge$ definiert, so gilt stets

$$k_i(x) \preceq x,$$

in Übereinstimmung mit der Forderung (K) der Nr. 16.

28. Ich zeige, daß $\operatorname{long}(x)$ und $k_i(x)$ als primitiv-rekursive Funktionen in H definiert werden können; dann auch die Funktion $p(x, y)$ der Nr. 10.

Ich schicke die primitiv-rekursive Definition einiger Hilfsfunktionen voraus.

1. Sei $b_\alpha(x)$ die charakteristische Funktion der Beziehung, daß x der Form α_i ist $(i = 1, 2, \ldots)$. Dann ist

$$b_\alpha(x) = \begin{cases} \wedge, & \text{falls } e_1(x) = a_1 \,\&\, l_1(x) = a_1 \,\&\, {}^{-1}(x^{-1}) \neq \wedge \,\&\, \zeta({}^{-1}(x^{-1})) = \wedge, \\ a_0 & \text{sonst.} \end{cases}$$

2. Sei nun $f_\alpha(x)$ gleich α_{j+1}, falls x einer endlichen Folge aus H zugeordnet ist, und mit α_{j+1} endet; sonst sei $f_\alpha(x) = \wedge$. Dann gilt

$$f_\alpha(x) = \mu_y [y \preceq x \,\&\, b_\alpha(y) \,\&\, l_{o(y)}(x) = y \,\&\, a_1 a_0^{o(y^{-2})} a_1 a_0^{o(y^{-2})} a_1 \vdots\!\leqq x \,\&$$
$$\& (u)(v)[u \preceq x \rightarrow (v \preceq x \rightarrow (x = u a_0^{o(y^{-2})} v \rightarrow (l_1(u) = a_1 \,\&\, e_1(v) = a_1)))]].$$

Zur Definition von $\operatorname{long}(x)$ und $k_i(x)$ hat man vor Augen zu halten, daß für $f_\alpha(x) \neq \wedge$

$$x = k_0(x) f_\alpha(x) k_1(x) f_\alpha(x) \ldots f_\alpha(x) k_{\operatorname{long}(x)}(x) f_\alpha(x)$$

gilt.

3. Zuerst ergibt sich daraus die Definition von $k_0(x)$:

$$k_0(x) = \mu_y [y \preceq x \,\&\, f_\alpha(x) \vdots\!\leqq y \,\&\, (Ez)[z \preceq x \,\&\, x = y f_\alpha(x) z]].$$

(Für $f_\alpha(x) = \wedge$ folgt daraus $k_0(x) = \wedge$.)

4. Wird für $f_\alpha(x) \neq \wedge$ vom Anfang von x dieses $k_0(x)$ und das danach stehende $f_\alpha(x)$ weggelassen, so soll der Rest mit $g(x)$ bezeichnet werden:

$$g(x) = {}^{-o(k_0(x) f_\alpha(x))}x.$$

(Daraus folgt $g(x) = x$ für $f_\alpha(x) = \wedge$.)

5. Das hat man nun zu iterieren. Sei

$$\bar{g}_{o(\wedge)}(x) = x$$

und für $a_i \in A$

$$\bar{g}_{o(ya_i)}(x) = g(\bar{g}_{o(y)}(x)).$$

(Für $f_\alpha(x) = \wedge$ gilt immer $\bar{g}_{o(y)}(x) = x$.)

6. Nun kann $\operatorname{long}(x)$ durch

$$\operatorname{long}(x) = o(\mu_y[y \preceq x \;\&\; \bar{g}_{o(ya_0)}(x) = \wedge])$$

definiert werden (für $f_\alpha(x) = \wedge$ ergibt sich daraus $\operatorname{long}(x) = \wedge$).

7. Endlich erhält man

$$k_{o(y)}(x) = \begin{cases} k_0(\bar{g}_{o(y)}(x)), & \text{falls} \quad o(y) \leq \operatorname{long}(x), \\ \wedge & \text{sonst.} \end{cases}$$

8. Zur Definition von $p(x, y)$ der Nr. 10 hat man zu untersuchen, wie das der Folge $(y_0, \ldots, y_n, y_{n+1})$ zugeordnete Wort

$$c_{n+1}(y_0, \ldots, y_n, y_{n+1}) = y_0 \alpha_{j+1} \ldots \alpha_{j+1} y_n \alpha_{j+1} y_{n+1} \alpha_{j+1}$$

aus

$$x = c_n(y_0, \ldots, y_n) \quad \text{und} \quad y = c_0(y_{n+1}) = c(y_{n+1})$$

gebildet werden kann. Dabei ist j die größte der Zahlen i, für welche a_0^i Vorgänger von mindestens einem der $y_0, \ldots, y_n, y_{n+1}$ ist. So ist α_{j+1} dasjenige von

$$f_\alpha(c_n(y_0, \ldots, y_n)) \quad \text{und} \quad f_\alpha(c(y_{n+1})),$$

welches von größerer Ordnung ist.

9. Wird daher

$$m(x, y) = \begin{cases} f_\alpha(x), & \text{falls} \quad o(f_\alpha(y)) \leq o(f_\alpha(x)), \\ f_\alpha(y) & \text{sonst} \end{cases}$$

gesetzt, dann gilt

10.

$$c_{\operatorname{long}(x)+1}(k_0(x), k_1(x), \ldots, k_{\operatorname{long}(x)}(x), k_0(y)) =$$
$$= \operatorname{subst}(x, f_\alpha(x), m(x, y)) \operatorname{subst}(y, f_\alpha(y), m(x, y)).$$

Demnach läßt sich in einer Wortemenge $p(x, y)$ — und natürlich auch $c(x) = x f_\alpha(x)$ — als primitiv-rekursive Funktion definieren, ohne neue Ausgangsfunktionen heranzuziehen.

29. Es wurde ferner in Nr. 11 gefordert, daß bei der gewählten Anordnung

$$\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_s$$

sämtlicher Vorgänger von einem $x \in H$ der Form $x = f^*(\ldots)$, bei festem $f^* \in F$, falls i die kleinste Zahl ist, für welche $\bar{x}_j$ bei einem $j < s$ ein Vorgänger, und zwar mit dem Index l, des i-ten unmittelbaren Vorgängers x_i von x ist, dann

$$s, \quad i, \quad l$$

von den unmittelbaren Vorgängern von x bzw. von diesen und von j primitiv-rekursiv abhängen sollen. Im Fall der Wortemenge sollen also s, i, l bei festem a_i primitiv-rekursive Funktionen der Vorgänger $x_1 = x^{-1}$, $x_2 = {}^{-1}x$ von $x = (\ldots)a_i$ bzw. von diesen und j sein.

Die Vorgänger eines nicht leeren Wortes $x = d_1 d_2 \ldots d_n$, wo $d_r \in A$ für $r = 1, 2, \ldots, n$, und zwar $d_n = a_i$, sind in der gewählten Reihenfolge:

$$\text{(V)} \qquad \begin{array}{l} \wedge, \\ d_1, \\ d_2, d_1 d_2, \\ d_3, d_2 d_3, d_1 d_2 d_3, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ d_{n-1}, d_{n-2} d_{n-1}, \ldots, d_1 d_2 \ldots d_{n-1}, \\ d_n, d_{n-1} d_n, \ldots, d_2 d_3 \ldots d_n, d_1 d_2 \ldots d_n. \end{array}$$

Ihre Anzahl ist

$$1 + \binom{n+1}{2};$$

die Anzahl derjenigen darunter, die d_n noch nicht enthalten, also Vorgänger von x^{-1} sind, ist $1 + \binom{n}{2}$. Selbstverständlich ist auch die Anzahl der Vorgänger von ${}^{-1}x$ (welche d_1 nicht enthalten) ebenfalls $1 + \binom{n}{2}$; von diesen kommen $1 + \binom{n-1}{2}$ bereits unter den Vorgängern von x^{-1} vor. Die Vorgänger von ${}^{-1}x$, welche keine Vorgänger von x^{-1} sind, stehen alle in der letzten Reihe von (V):

$$d_n, d_{n-1} d_n, \ldots, d_2 d_3 \ldots d_n.$$

Wenn $\bar{x}_j$ das r-te Glied dieser Reihe ist, dann ist

$$j = \binom{n}{2} + r,$$

und unter den Vorgängern von ${}^{-1}x = d_2 d_3 \dots d_n$ hat $\overline{x}_j$ den Index

$$i = \binom{n-1}{2} + r;$$

so gilt in diesem Fall

$$i = j - (n-1).$$

Daher erhält man mit

$$\sigma(z) = \binom{o(z)+1}{2},$$

$$\iota(y,z) = \begin{cases} 1, & \text{falls } \zeta(y) = \wedge \ \& \ o(y) \leqq \binom{o(z)}{2}, \\ 2, & \text{falls } \zeta(y) = \wedge \ \& \ 1 + \binom{o(z)}{2} \leqq o(y) < \binom{o(z)+1}{2}, \\ \text{und etwa } \wedge \text{ sonst,} \end{cases}$$

$$\lambda(y,z) = \begin{cases} o(y), \quad \text{falls } \zeta(y) = \wedge \ \& \ o(y) \leqq \binom{o(z)}{2}, \\ o(y) \dot{-} (o(z) \dot{-} 1), \quad \text{falls } \zeta(y) = \wedge \ \& \\ \qquad 1 + \binom{o(z)}{2} \leqq o(y) < \binom{o(z)+1}{2}, \\ \text{und etwa } \wedge \text{ sonst,} \end{cases}$$

die nach dem Satz von Nr. 6 primitiv-rekursive Funktionen sind,

$$s = \sigma(x) = \sigma(x^{-1}a_i), \quad i = \iota(j,x) = \iota(j, x^{-1}a_i), \quad l = \lambda(j,x) = \lambda(j, x^{-1}a_i),$$

also, wie gefordert war, bei jedem a_i als primitiv-rekursive Funktionen von x^{-1} und ${}^{-1}x$, bzw. von diesen und j, ohne Hinzunahme neuer Ausgangsfunktionen.

30. Endlich wurde in Nr. 18 noch gefordert, daß falls $k_i(x_1)$ der Vorgänger mit dem Index v von x_1 ist, dann v eine primitiv-rekursive Funktion von i und den unmittelbaren Vorgängern von x sei.

Im Fall der Wortemenge ist

$$x_1 = x^{-1},$$

und der Vorgänger mit dem Index v von x^{-1} ist zugleich der Vorgänger $\overline{x}_v$ mit dem Index v von x. Wir haben also jenes v zu bestimmen, für das

$$k_i(x_1) = \overline{x}_v$$

gilt.

Für $f_\alpha(x_1) = \wedge$ oder $i > \operatorname{long}(x_1)$ ist $k_i(x_1) = \wedge$, also $v = 0$. Sonst ist nach 4. und 5. der Nr. 28

$$x_1 = d_1 d_2 \ldots d_{o(x_1)} = k_0(x_1) f_\alpha(x_1) k_1(x_1) f_\alpha(x_1) \ldots f_\alpha(x_1) k_{\operatorname{long}(x_1)}(x_1) f_\alpha(x_1) =$$

$$= e_{o(x_1) \dot{-} o(\bar{g}_i(x_1))}(x_1) k_i(x_1) l_{o(\bar{g}_i(x_1)) \dot{-} o(k_i(x_1))}(x_1),$$

also

$$k_i(x_1) = d_{i_1+1} d_{i_1+2} \ldots d_{i_1+i_2},$$

wo

$$i_1 = o(x_1) \dot{-} o(\bar{g}_i(x_1)) = o(x_1^{-o(\bar{g}_i(x_1))})$$

und

$$i_2 = o(k_i(x_1))$$

primitiv-rekursive Funktionen von i und x_1 sind.

Man entnimmt leicht aus (V) in Nr. 29, daß

$$v = \binom{i_1 + i_2}{2} + i_2$$

ist, also ergibt sich mit der Funktion

$$\varphi(y, z) = \begin{cases} \dbinom{o(z^{-o(\bar{g}_{o(y)}(z))}) + o(k_{o(y)}(z))}{2} + o(k_{o(y)}(z)), \\ \qquad \text{falls } f_\alpha(z) \neq \wedge \;\&\; \zeta(y) = \wedge \;\&\; o(y) \leqq \operatorname{long}(z), \\ \wedge \quad \text{sonst,} \end{cases}$$

die (wegen $\operatorname{long}(z) = o(\operatorname{long}(z))$) nach dem Satz der Nr. 6 primitiv-rekursiv ist,

$$v = \varphi(i, x_1)$$

als primitiv-rekursive Funktion von i und von den unmittelbaren Vorgängern x_1 und x_2 von x, ohne Hinzunahme neuer Ausgangsfunktionen.

Man zeigt ähnlicherweise, daß auch jenes kleinste u und jenes w, für welche $k_i(\bar{x}_j)$ bei $j < s$ der Vorgänger mit dem Index w von x_u ist, primitiv-rekursive Funktionen von i, j, x_1 und x_2 sind (bei jedem festen a_i, falls x der Form $x = (\ldots)a_i$ ist). Dies kann man dazu verwenden, um von gewissen speziellen Wertverlaufsrekursionen zu zeigen, daß sie wirklich unter die in Nr. 9 behandelten allgemeinen Wertverlaufsrekursion fallen; nämlich von solchen, bei welchen der Wert der zu definierenden Funktion an einer Stelle x mit Hilfe von Werten derselben Funktion an Stellen der Form $k_i(\bar{x}_j)$ gegeben wird.

31. Es hat sich herausgestellt, daß in einer Wortemenge nicht notwendig ist neue Ausgangsfunktionen aufzunehmen. *Allein von* $\wedge$, *von den Funktionen der Menge F und von der charakteristischen Funktion der Gleichheit* $x = y$

ausgehend können in jeder Wortemenge die simultanen Rekursionen, die Wertverlaufsrekursionen und die eingeschachtelten Rekursionen auf primitive Rekursionen, und diese auf das dem in Nr. 13 angegebenen reduzierten Form entsprechende Schema

$$f(\wedge, u) = g(u),$$

und falls $a_i \in A$,

$$f(xa_i, u) = g_{a_i}(x, f(x, u), f({}^{-1}(xa_i), u))$$

*nebst Substitutionen zurückgeführt werden.**

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* *Anmerkung der Redaktion.* Der Schluß der Arbeit wird aus technischen Gründen in unserem nächsten Heft erscheinen.

PROJECTIVE n-SIMPLEXES IN A $[2n-2]$

By

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(*Presented by* G. HAJÓS)

1. Introduction. Let (P) be an n-dimensional simplex in a projective space p of n dimensions $(n \geqq 2)$, or briefly an n-simplex in an $[n]$ p, and let (Q) be another in a different $[n]$ q. If the $n+1$ joins of their corresponding vertices P_i, Q_i $(i=0, \ldots, n)$ have an $[n-2]$ as a common transversal t, they are then said to be *projective from* t ([6]—[8]) and determine a $[2n-1]$ or lie in a $[2n-2]$, *assuming that they don't lie in a lower space.* If p_i is the $[n-1]$ of (P) opposite P_i and q_i that of (Q) opposite Q_i, in either case then they meet in a point $R_i = p_i \cdot q_i$.

The purpose of this paper is to prove *synthetically* that

(P), (Q) *are projective from an* $[n-2]$, *if and only if the* $n+1$ *points* R_i *are collinear.*

We follow here the line of argument adopted by BAKER ([1]) and COXETER ([4]) to prove the Desargues' two-triangle theorem and its converse ([3]). In fact, for $n=2$, our proposition reduces to this theorem. Hence we too have named ([7]) it after Desargues.

But the treatment there ([7]) and for a particular case ([5]) of projective tetrahedra $(n=3)$ in a $[4]$ is *analytic*. The synthetic proof of the proposition, as based on the axioms and the corresponding propositions of incidence in $[2n-1]$ or on the Desargues' theorem in a plane and the axioms of $[2n-2]$ only for $n>2$, has its own interest and thus deserves a special study.

The knowledge of the axioms of incidence in an $[m]$ for all integral values of m is assumed.

2. (P), (Q) **in a** $[2n-1]$. a) When (P), (Q) are in a $[2n-1]$, the proposition is just an obvious one of incidence alone bordering almost on triviality like the Desargues' theorem in a solid ([3]). For, in a $[2n-1]$, if (P), (Q) are projective from an $[n-2]$, they are just projections of each other from the $[n-2]$ in their respective $[n]$'s, thus justifying their adjective projective, such that the $n+1$ points R_i lie in the line $r = p \cdot q$ common to their 2 $[n]$'s p, q.

b) Conversely, if p_i, q_i meet in a point R_i, the $n+1$ $[2n-2]$'s $p_i q_i$ meet in an $[n-2]$, say t, lying in the $[n-1]$ common to n of them through a particular join P_jQ_j which then meets t. Thus all the $n+1$ joins P_iQ_i are met by t showing (P), (Q) to be projective from t.

3. (P), (Q) **in a** $[2n-2]$. a) Let (P), (Q) lie in a $[2n-2]$ e and projective from an $[n-2]$ t; let A and B be 2 points on a line, meeting t, outside e; further let $p_{ij}=p_i\cdot p_j$ be the $[n-2]$ of (P) opposite P_iP_j and similarly $q_{ij}=q_i\cdot q_j$. A, p_{ij}, t then determine a $[2n-2]$ which contains B, q_{ij} too. Therefore the 2 $[n-1]$'s Ap_{ij}, Bq_{ij} meet in a point R_{ij}.

Again AB, e determine a $[2n-1]$ f, and the 2 $[n]$'s Ap_i, Bq_i, lying therein, meet in a line r_i which then contains the n points R_{ij} $(i\neq j)$ and the $(n+1)^{\text{th}}$ point $R_i=p_i\cdot q_i$ too. The $n+1$ lines r_i lie in a plane g, for every two of them meet in a point like R_{ij}. The 2 $[n+1]$'s Ap, Bq, lying in f, meet in a solid which then contains g as well as the plane $h=p\cdot q$ common to the 2 $[n]$'s p, q of (P), (Q) lying in e. Thus the $n+1$ points R_i, lying in g and h both, lie on their common line $r=g\cdot h$.

b) Conversely, suppose that the $n+1$ points R_i lie in a line r which lies in the plane h, and let t be the definite $[n-2]$ transversal of the n joins P_jQ_j $(i\neq j)$ of general position in e ([1], p. 39, ex. 7). Then we have to prove that t meets P_iQ_i too.

Let g be a plane meeting e in r; let r_i be a line in g, through every R_i; the $n+1$ $[n]$'s r_ip_i, lying in the $[n+1]$ gp, meet in a point A, and the $n+1$ $[n]$'s r_iq_i similarly meet in B.

Now following Baker ([1], p. 8), it follows from what precedes that AB meets t. For, (Ap_i) is an n-simplex formed of A and the n vertices of (P) in p_i, and (Bq_i) is another formed similarly, both lying in the $[2n-1]$ $f=ge$; their corresponding $[n-1]$'s meet on the line r_i common to their $[n]$'s Ap_i, Bq_i, and therefore, by § 2b), they are projective from an $[n-2]$ which is no other than t. Similarly now are projective the 2 n-simplexes (Ap_j), (Bq_j) $(i\neq j)$, similarly formed, from the same $[n-2]$, viz., t. Thus t meets the join of their corresponding vertices P_i, Q_i.

4. Alternative proof of the preceding proposition. But to prove the theorem in a $[2n-2]$ e $(n>2)$, we may not like to peep out of this space into a $[2n-1]$ and beg for help there, as if the axioms of e are not enough for our purpose. To justify our preference for an approach independent of an outside space we may proceed as follows:

a) Let (P), (Q) be as in § 3a). $R_i=p_i\cdot q_i$ is seen to be the point of intersection $(h\cdot p_i)\cdot(h\cdot q_i)$ of the 2 lines, in the plane h, where it meets p_i, q_i. The 3 points R_i, R_j, R_k are collinear, if and only if the 2 trilaterals $u\equiv(h\cdot p_i)(h\cdot p_j)(h\cdot p_k)$, $v\equiv(h\cdot q_i)(h\cdot q_j)(h\cdot q_k)$ are Desargues trilaterals. To prove that they are we may observe that the 3 joins of their corresponding vertices concur at the point of intersection of h with the $[2n-4]$ c determined by the $[n-2]$ t and the $[n-3]$ $p_{ijk}=p_{ij}\cdot p_k$ or $q_{ijk}=q_{ij}\cdot q_k$. For, the join of the vertex $(h\cdot p_i)\cdot(h\cdot p_j)=h\cdot p_{ij}=h\cdot p_{ijk}P_k$ of u to the corresponding vertex $h\cdot q_{ijk}Q_k$ of v is the line of intersection of h with the $[2n-3]$ $P_k c Q_k$, and similarly behave the other 2 joins meeting c in the same point.

Thus any 3 of the $n+1$ points R_i and therefore all are collinear.

b) Conversely, let the $n+1$ points R_i lie in a line r and let t be as in § 3b). Following the argument and notation of the preceding section, we find that the 3 $[2n-3]$'s $p_{ij}q_{ij}, p_{jk}q_{jk}, p_{ki}q_{ki}$ have a $[2n-4]$ c in common, through the $[2n-5]$ $p_{ijk}q_{ijk}$, meeting therefore the 3 joins P_iQ_i, P_jQ_j, P_kQ_k. Thus $c=tp_{ijk}$, and there are $\binom{n}{2}$ such spaces, through t, all meeting the join P_iQ_i, one for each pair of values of j, k other than i. Hence P_iQ_i meets t, for the said spaces evidently just happen to have t in common and no other point.

5. Applications. a) We have seen in [8] that: *If $n-1$ consecutive edges of the $(n+1)$-gon formed of $n+1$ vertices of a simplex S in a $[2n-2]$ be "conjugate" for a quadric Q therein to their respectively opposite $[n-2]$' s, their n-simplex is projective to the corresponding one of the polar simplex S' of S for Q from an $[n-2]$ through the polar $[n-3]$ of their $[n]$ for Q.*

DEFINITION. A line l is said to be *conjugate* to an $[n-2]$ m for Q, if the polar $[n-1]$ m' of m for Q meets l in a point which therefore is the pole, for Q, of the hyperplane determined then by m and the polar $[2n-4]$ l' of l for Q.

When $n=3$, this proposition is used ([8]) to construct a *self-conjugate heptad* (h) *of points* for Q in a [4] such that the plane containing any 3 of them is conjugate for Q to the solid containing the other four.

In fact, it is the construction of this (h) that led to the theory of projective tetrahedra ([6]) from a line and, consequently, to that of projective n-simplexes from an $[n-2]$ in a $[2n-2]$.

b) The pair of n-simplexes (P), (Q), projective from an $[n-2]$ t, gives rise to 2^n-1 more such pairs obviously projective from the same $[n-2]$ t. For, there are 2 choices for every point, P_i or Q_i, to belong to an n-simplex of a pair independent of each other. For example, $n+1$ pairs are of the type $Q_0P_1\ldots P_n$, $P_0Q_1\ldots Q_n$; $\binom{n+1}{2}$ pairs of the type $Q_0Q_1P_2\ldots P_n$, $P_0P_1Q_2\ldots Q_n$, and so on.

If R_i (§1) is called the *arguesian point* ([7]) and their line r as the *arguesian line* of (P), (Q), we find that: *There are, in all, 2^n arguesian lines,* *one for each pair of projective n-simplexes, and* $2^n(n+1)$ *arguesian points,* $n+1$ *on each line and each common to 2 lines, such that every line meets* $n+1$ *other lines, skew to each other.*

It is further shown in [7] that: *The arguesian points distribute into* $n+1$ *groups of* 2^{n-1} *each such that the points of a group form the vertices of a dual of an* $(n-1)$*-dimensional S-configuration* ([5]) *whose diagonal* $(n-1)$*-simplex forms a prime face of the n-simplex with vertices at the* $n+1$ *points which are the harmonic conjugates of the* $n+1$ *points of intersection of the transversal* $[n-2]$ t *with the* $n+1$ *joins* P_iQ_i *w. r. t.* P_i, Q_i*, respectively.*

When $n=3$, the 8 arguesian lines lie on a quadric, and when $n=4$, the 16 arguesian lines form the general 16_5 figure on the Segre quartic surface ([2], pp. 166—172). For these cases of evident interest, we may refer further to [6] and [7], respectively.

c) We may introduce a pair of $(n+1)$-simplexes projective from an $[n-2]$ t in a $[2n-2]$, $[2n-1]$ or $[2n]$ such that t meets all the $n+2$ joins of their corresponding vertices. They give rise to $n+2$ arguesian lines, one for each pair of their corresponding n-simplexes, which then evidently lie in a plane, referred to as their *arguesian plane*. In the $[2n]$, it is obviously a proposition of incidence alone.

Again one such pair gives rise to $2^{n+1}-1$ more pairs projective from t. Thus: *There arise* 2^{n+1} *arguesian planes, one for each pair, and* $2^n(n+2)$ *arguesian lines,* $n+2$ *in each plane and each common to 2 planes which then lie in a solid such that there are* $2^n(n+2)$ *such solids, each containing* $2n+3$ *lines. There are* $2^{n-2}(n+2)(n+1)$ *arguesian points,* $(n+1)^2$ *in each solid, each lying on 4 lines and 4 planes which lie in a* [4], *for* $n>3$, *such that there are* $2^{n-2}(n+2)(n+1)$ *such* [4]'*s in all, each containing 4 solids, 4 planes,* $4(n+1)$ *lines and* $2n^2+2n+1$ *points.*

For further details of this interesting configuration, when $n=3$, we refer to [6] and for $n>3$ to [7].

d) Similarly we may consider, in general, a pair of $(n+m)$-simplexes $(m>n-1)$ projective from an $[n-2]$ t in a $[2n-2], \ldots,$ or $[2n+m-1]$ such that t meets all the $n+m+1$ joins of their corresponding vertices. They give rise to $\binom{n+m+1}{m}$ arguesian lines and $\binom{n+m+1}{n}$ arguesian points, $n+1$ on each line, all lying in an $[m+1]$, called their *arguesian* $[m+1]$. Obviously, it is a proposition of incidence in the $[2n+m-1]$.

One such pair gives rise to $2^{n+m}-1$ more pairs projective from t, and thus there arise 2^{n+m} arguesian $[m+1]$'s in all. Given m, we can build up the complete picture of the configuration of arguesian points, lines, planes, ..., and $[m+1]$'s.

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ON A CONSISTENCY THEOREM CONNECTED WITH THE GENERALIZED CONTINUUM PROBLEM[1]

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§ 1. Introduction. (Summary, notations)

It is known that GÖDEL [1] has proved the consistency of the generalized continuum hypothesis $(\alpha)(2^{\aleph_\alpha}=\aleph_{\alpha+1})$ with the axiom system Σ^*.[2] Presumably the generalized continuum hypothesis is independent of the axiom system Σ^*, however, the consistency of the negation of this hypothesis with Σ^* has not yet been proved.

Moreover, presumably each of the particular cases $2^{\aleph_0}=\aleph_1$, $2^{\aleph_1}=\aleph_2, \ldots$ $\ldots, 2^{\aleph_\omega}=\aleph_{\omega+1}, \ldots, 2^{\aleph_\lambda}=\aleph_{\lambda+1}, \ldots$ is independent of the axiom system Σ^*. However, for a given ordinal λ the question of independence of the statement $2^{\aleph_\lambda}=\aleph_{\lambda+1}$ has a precise meaning only in the case when the ordinal number λ can be defined by means of the notions of the set theory, i. e. if it is a particular class which is at the same time an ordinal number as, for example, 0, 1 or ω (or $\omega+1$ or ω_1). Let Λ, $\mathrm{N}, \ldots$ denote such *particular ordinal numbers.*

Now, since no theorem of the set theory is known which would allow to infer from the assumption $2^{\aleph_\mu}=\aleph_{\mu+1}$ for certain ordinals μ the equality $2^{\aleph_\Lambda}=\aleph_{\Lambda+1}$ for a Λ different from these μ, we can formulate the conjecture that the equality $2^{\aleph_\Lambda}=\aleph_{\Lambda+1}$ cannot be proved for any Λ from the axioms of Σ^* even if we assume that $2^{\aleph_\mu}=\aleph_{\mu+1}$ is fulfilled for every μ different from Λ.

It is obvious that until the independence of the generalized continuum hypothesis is proved, the question whether an equality of the form $2^{\aleph_\Lambda}=\aleph_{\Lambda+1}$ can be proved from the axioms of Σ^* and certain additional assumptions may be investigated only if we assume that it cannot be proved from the axioms of Σ^* *alone.*

So we hope that some interest may be attached to theorems which

[1] This paper contains the detailed proofs of the results of the author's dissertation submitted in fulfilment of the requirements for the degree of Candidate of Mathematical Sciences. A preliminary report containing these results has been published in the *Zeitschrift f. Math. Logik und Grundlagen d. Math.*, **2** (1956), pp. 131—136.

[2] For the axiom system Σ^* see in [1].

investigate the connections — or more precisely the absence of connections — of these special cases assuming the independence of some special equalities.

The assumption that the equality $2^{\aleph_\Lambda} = \aleph_{\Lambda \dotplus 1}$ cannot be proved from the axioms of Σ^* can be stated in the form that there exists a particular ordinal number $\mathrm{N} \neq 0$ (e. g. $\mathrm{N} = 1$), such that the assumption $2^{\aleph_\Lambda} \geqq \aleph_{\Lambda \dotplus \mathrm{N} \dotplus 1}$ is consistent with the axioms of Σ^*.

The main result of our paper, Theorem 2, asserts that if for a certain kind of particular ordinal numbers Λ, N the inequality $2^{\aleph_\Lambda} \geqq \aleph_{\Lambda \dotplus \mathrm{N} \dotplus 1}$ is consistent with the axioms of Σ^*, then the same is true for the equality $2^{\aleph_\Lambda} = \aleph_{\Lambda \dotplus \mathrm{N} \dotplus 1}$ even if we assume $2^{\aleph_\mu} = \aleph_{\mu \dotplus 1}$ for every $\mu \geqq \Lambda \dotplus \mathrm{N}$.

Choosing specially $\mathrm{N} = 1$, Theorem 2 yields the following: If $2^{\aleph_\Lambda} = \aleph_{\Lambda \dotplus 1}$ cannot be proved from the axioms of Σ^*, then it cannot be proved even if we assume $2^{\aleph_\mu} = \aleph_{\mu \dotplus 1}$ for every μ greater than Λ. This is a partial solution of the formerly stated problem. For other values of the parameter N Theorem 2 has other interesting corollaries. We remark that the independence of the proposition $2^{\aleph_\Lambda} = \aleph_{\Lambda \dotplus 1}$ of the hypothesis that for every μ less than Λ $2^{\aleph_\mu} = \aleph_{\mu \dotplus 1}$, seems to be a more difficult problem which we cannot solve.

Before stating Theorem 2 in a precise form (viz. specifying the suitable kind of the particular ordinal numbers Λ, N for which we shall prove the theorem), we shall indicate the way leading to the proof, especially the intermediate theorems.

In what follows we shall use the notations and concepts introduced in [1] as well as the theorems proved there.[3] (Note that the letters C, F, R, S also occur as variables in § 2 and § 3 in our paper too.) In addition, in the definition of some metaconcepts as well as in the formulation of some metatheorems we shall use the symbol $\vdash \varphi$ to signify that the formula φ of the axiom system Σ^* is a theorem in this axiom system.

For the sake of brevity for any particular ordinal numbers Λ, N the formulas $2^{\aleph_\Lambda} \geqq \aleph_{\Lambda \dotplus \mathrm{N} \dotplus 1}$ and $2^{\aleph_\Lambda} = \aleph_{\Lambda \dotplus \mathrm{N} \dotplus 1} \cdot (\mu)(\Lambda \dotplus \mathrm{N} \leqq \mu \supset 2^{\aleph_\mu} = \aleph_{\mu \dotplus 1})$ will be called axioms $\mathrm{F}_{\Lambda, \mathrm{N}}$ and $\mathrm{F}^*_{\Lambda, \mathrm{N}}$, respectively.[4]

The proof of Theorem 2 consists in giving an inner model[5] in the axiom system Σ^*, $\mathrm{F}_{\Lambda, \mathrm{N}}$ for the axiom system Σ^*, $\mathrm{F}^*_{\Lambda, \mathrm{N}}$.

[3] For the logical notions we also use the symbols $\equiv, \supset, \vee, \cdot$ (x), $(\exists x)$, $(\exists ! x)$, but for the use of the parantheses in the formulas we shall make use of the following convention. We shall say that "$\equiv$" ranks ahead of "$\supset$", and rank our operators in the order in which we have listed them above.

[4] The operation $X \dotplus Y$ appearing in the formulas is not defined in [1]. However, it is a straightforward generalization of the operation $X \dotplus 1$ defined in [1], associating to any two ordinals α, β their sum $\alpha \dotplus \beta$ in the usual sense. For the definition of $X \dotplus Y$ see 2.1.4, p. 328.

[5] For the concept of inner model see [2], p. 165.

In order to define this model and to prove the properties of it from the axioms of $\Sigma^*, F_{\Delta, N}$, it is necessary to develop the abstract set theory, more detailed from the axioms of set theory as it is done in Chapters II—IV of [1]. We are going to do it in § 2. If a theorem is stated without further specifications, this means that it follows from the axioms of Σ. If the axiom E is needed for a theorem or a definition, its number is marked by *, just like in [1].

In the method of the proofs we also follow [1]. In connection with this we cite the following part of the introduction of [1]:

"Although the definitions and theorems are mostly stated in logistic symbols, the theory developed is not to be considered as a formal system but as an axiomatic theory in which the meaning and the properties of the logical symbols are presupposed to be known. However, to everyone familiar with mathematical logic it will be clear that the proofs could be formalized, using only the rules of HILBERT's "Engeren Funktionenkalkul". In several places, we are concerned with metamathematical considerations about the notions and propositions of the system Σ^*. However, the only purpose of these metamathematical considerations is to show how the proofs of theorems of a certain kind can be accomplished by a general method."

We have to remark here that Theorem 2 and its corollaries are, as a matter of course, not theorems of set theory but they are also metamathematical theorems about set theory in a somewhat more general sense as metatheorems mentioned above. In the supplementary part of our paper in § 6 we shall prove theorems of similar character, and so in § 6 metatheorem, metaconcept are used in this more general sense. Anyway we wish to note that after the complete formalisation of the axiom system all these theorems proved in this paper might be considered as arithmetical ones of constructive nature.

By the *model determined by the class M* we understand the inner model, the fundamental notions of which are defined by the formulas

$$\mathfrak{Cls}_M(X) \equiv X \subseteq M . (u)(u \in M \supset X . u \in M),$$

$$\mathfrak{M}\mathfrak{s}_M(X) \equiv X \in M,$$

$$X \in_M Y \equiv \mathfrak{M}\mathfrak{s}_M(X) . \mathfrak{Cls}_M(Y) . X \in Y.$$

The model determined by the class M will be denoted by Δ_M.

We say that a class M is *almost universal*,[6] if for any of its subsets u it contains an element v such that $u \subseteq v$.

[6] The universal class (and by Axiom D only this class) has the property of containing any of its subsets as element. Hence the expression "almost universal" for the above property.

In § 3 we shall give a sketch of the proof of the following

LEMMA.* *If the class M is complete, closed with respect to the fundamental operations and almost universal, then the model determined by the class M satisfies the axiom system* Σ.[7]

Using this lemma we can prove the following

THEOREM 1*.[8] *For any pair* λ, ν *of ordinals, if* $2^{\aleph_\lambda} \geqq \aleph_{\lambda \dotplus \nu}$, *then there exists a complete and almost universal class M such that* Δ_M *satisfies the axiom system* Σ^* *and, in addition, the following formulas are fulfilled:*

(1) $$\alpha \leqq \lambda \dotplus \nu \dotplus 1 \supset \aleph_M{}^{\backprime}\alpha = \aleph^{\backprime}\alpha,$$

(2) $$\left(2^{\aleph_M{}^{\backprime}\lambda}\right)_M = \aleph_M{}^{\backprime}\lambda \dotplus \nu \dotplus 1,$$

(3) $$\lambda \dotplus \nu \leqq \mu \supset \left(2^{\aleph_M{}^{\backprime}\mu}\right)_M = \aleph_M{}^{\backprime}\mu \dotplus 1.$$

If the particular ordinal number Λ is absolute with respect to every model determined by a complete and almost universal class M satisfying the axioms of Σ^* for which $(\alpha)(\alpha \leqq \Lambda \dotplus 1 \supset \aleph_M{}^{\backprime}\alpha = \aleph^{\backprime}\alpha)$ is fulfilled, then we call Λ an *absolutely definable (particular) ordinal number in the weaker sense* (see Definition 5. 4 on p. 365) it can be proved e. g. that $0, 1, 2, \ldots, n, \ldots, \omega, \omega \dotplus 1, \ldots, \omega \dotplus \omega, \ldots, \omega_1, \ldots, \omega_{\omega_1}, \ldots$ are absolutely definable ordinal numbers.

If Λ, N are absolutely definable ordinal numbers satisfying axiom $\mathrm{F}_{\Lambda,\mathrm{N}}$, then by Theorem 1 there exists a class M such that Λ and N are absolute with respect to Δ_M by (1), and this model satisfies beyond the axiom system Σ^* axiom $\mathrm{F}^*_{\Lambda,\mathrm{N}}$ as well, for the absoluteness of Λ and N and of the other concepts, occurring without the subscript M, implies that the conjunction of (2) and (3) is just the relativized of $\mathrm{F}^*_{\Lambda,\mathrm{N}}$ with respect to the model Δ_M.[9] Hence we get Theorem 2 in the following precise form:

THEOREM 2. *If* Λ *and* N *are absolutely definable ordinal numbers and the axiom system* $\Sigma^*, \mathrm{F}_{\Lambda,\mathrm{N}}$ *is consistent, then the same is true for the axiom system* $\Sigma^*, \mathrm{F}^*_{\Lambda,\mathrm{N}}$.

[7] The lemma in a somewhat weaker form is stated without proof in [3]. The concept of the model determined by a class, as well as the lemma, are, respectively, immediate generalizations of the model Δ determined by the class L (the class of the constructible sets) and of the theorem, proved in [1], that the model Δ satisfies the axiom system Σ. The lemma is a theorem formalizable and provable in the axiom system Σ^*.

[8] For Theorem 1 see § 4. Theorem 1 is also a formalizable and provable one in the axiom system Σ^*. The subscript M, as usual, indicates the relativization of the corresponding concept to Δ_M.

[9] This sketch of the proof does not show quite clearly the constructive character of Theorem 2 because of the expression "there exists a class M" taking place in it, but this might be obvious by the proof given in § 5, p. 366.

The most essential and, of course, most extensive part of the proof of our result is the proof of Theorem 1. This will be given in § 4.[10]

In § 5 we shall give the proof of Theorem 2 and of the corollaries. As to the corollaries of Theorem 2 see pp. 366—368. Here we mention only $\mathbf{K}_3$ which seems to be the most interesting from set-theoretical point of view and which asserts that from any disproof of the so-called Lusin hypothesis $2^{\aleph_0} = 2^{\aleph_1}$ one can construct a proof of the continuum hypothesis $2^{\aleph_0} = \aleph_1$.

Finally, in § 6 we deal with the range of the absolutely definable particular ordinal numbers, i. e. the range of the validity of our results. We shall prove that for the notion of Church—Kleene's recursive ordinal number a corresponding metaconcept for the axiom system Σ^* can be defined, and that every particular ordinal number in the range of this metaconcept is absolutely definable.[11]

Moreover, we prove the more general theorem that every particular ordinal number which can be defined by a formula of the model Δ is absolutely definable.[12] The construction of non-denumerable absolutely definable ordinal numbers is made possible by a theorem which asserts that the particular ordinal number ω_A is absolutely definable if the same is true for A.

§ 2. Preliminary results

2. 1. In this section we prove some simple theorems of Σ (or of Σ^*, respectively) and we give the definition of some set-theoretical concepts not defined in [1].

We may remark that if after the definition of a concept nothing is said about its normality, it is normal and its normality will be proved by the method explained on p. 13 of [1] in the index to be found at the end of our paper.

In what follows we shall often refer to the following simple theorems not stated explicitly in [1]:

2. 1. 1. $u \in B^{\text{``}}X \equiv (\exists v)(v \in X . \langle u v \rangle \in B)$.

2. 1. 2. $\mathfrak{Fnc}(A) \supset (u \in A^{\text{``}}Y \equiv (\exists v)(v \in Y . u = A^{\text{`}}v))$.

The first one is a direct consequence of the definition [1], 4. 52, while the second follows from this by [1], 4. 61, 4. 63, 4. 65.

[10] In the first part of § 4 we are going to give a detailed description of the ideas used in the proof and point out where the proof runs parallel to the proof of [1] of GÖDEL and where we need an essentially new idea for carrying out the proof.

[11] See 6.2.

[12] See 6. 1.

In [1], 7.5 the following theorem is proved concerning the definition by means of transfinite induction:

$$(G)(\exists! F)(F\,\mathfrak{Fn}\,On\,.(\alpha)(F`\alpha = G`(F\upharpoonright\alpha))).$$

It follows that if the class G depends on the parameters $X_1, \ldots, X_n$, i. e. $\mathfrak{G}(X_1, \ldots, X_n)$ is a given operation, there exists an operation $\mathfrak{F}(X_1, \ldots, X_n)$ such that

$$(1)\quad \mathfrak{F}(X_1, \ldots, X_n)\,\mathfrak{Fn}\,On.(\alpha)(\mathfrak{F}(X_1, \ldots, X_n)`\alpha = \mathfrak{G}(X_1, \ldots, X_n)`(\mathfrak{F}(X_1, \ldots, X_n)\upharpoonright\alpha))$$

for every $X_1, \ldots, X_n$.

The following metatheorem is needed:

M_7. *If the operation $\mathfrak{G}(X_1, \ldots, X_n)$ is normal, there exists a normal operation $\mathfrak{F}(X_1, \ldots, X_n)$ defined by the formula* (1).

PROOF. By M_2 [1] there exists a uniquely determined class K such that

$$(2)\qquad f \in K \equiv (\exists\beta)(f\,\mathfrak{Fn}\,\beta.(\alpha)(\alpha \in \beta \supset f`\alpha = \mathfrak{G}(X_1, \ldots, X_n)`(f\upharpoonright\alpha)))$$

for every $X_1, \ldots, X_n$. Therefore an operation $\mathfrak{K}(X_1, \ldots, X_n)$ is defined by (2) and the normality of $\mathfrak{G}(X_1, \ldots, X_n)$ implies that $\mathfrak{K}(X_1, \ldots, X_n)$ is normal. It is proved in [1], 7.5 that if the class K satisfies (2), then $F = \mathfrak{S}(K)$ satisfies the formula

$$F\,\mathfrak{Fn}\,On.(\alpha)(F'\alpha = \mathfrak{G}(X_1, \ldots, X_n)`(F\upharpoonright\alpha)).$$

Hence the operation $\mathfrak{F}(X_1, \ldots, X_n) = \mathfrak{S}(\mathfrak{K}(X_1, \ldots, X_n))$ satisfies the formula (1), and $\mathfrak{K}(X_1, \ldots, X_n)$ beeing normal, it is normal.

Concerning the inductive definition of functions over the set of integers the following theorem is proved in [1], 8.45:

$$(a)(G)(\exists! F)(F\,\mathfrak{Fn}\,\omega.F`0 = a.(k)(F`k\dot{+}1 = G`(F`k))).$$

Analogously to M_7 we need a corresponding theorem for normal operations.

M_8. *Let $\mathfrak{A}(X_1, \ldots, X_n)$ and $\mathfrak{G}(Y, Z, X_1, \ldots, X_n)$ be normal operations. Suppose that $\mathfrak{M}(\mathfrak{A}(X_1, \ldots, X_n))$ and $\mathfrak{M}(\mathfrak{G}(y, z, X_1, \ldots, X_n))$. Then a normal operation $\mathfrak{F}(X_1, \ldots, X_n)$ can be defined by the following formulas:*

$$(3)\qquad \begin{aligned} &\mathfrak{F}(X_1, \ldots, X_n)\,\mathfrak{Fn}\,\omega, \\ &\mathfrak{F}(X_1, \ldots, X_n)`0 = \mathfrak{A}(X_1, \ldots, X_n), \\ &\mathfrak{F}(X_1, \ldots, X_n)`k\dot{+}1 = \mathfrak{G}(k, \mathfrak{F}(X_1, \ldots, X_n)`k, X_1, \ldots, X_n). \end{aligned}$$

PROOF. First we define an operation $\mathfrak{G}_1(X_1, \ldots, X_n)$ by the stipulations

$$(4)\qquad \begin{aligned} &\mathfrak{G}_1(X_1, \ldots, X_n)\,\mathfrak{Fn}\,V, \\ &\mathfrak{G}_1(X_1, \ldots, X_n)`0 = \mathfrak{A}(X_1, \ldots, X_n). \end{aligned}$$

If there exists a k such that $\mathfrak{D}(x)=k\dot{+}1$, then $\mathfrak{G}_1(X_1,\dots,X_n)^{\text{`}}x=$ $=\mathfrak{G}(k, x^{\text{`}}k, X_1,\dots,X_n)$, and $\mathfrak{G}_1(X_1,\dots,X_n)^{\text{`}}x=0$ in the other cases.

It is easy to verify that the assumptions of M_8 imply the existence and the normality of $\mathfrak{G}_1(X_1,\dots,X_n)$. By M_7 there exists a normal operation $\mathfrak{F}_1(X_1,\dots,X_n)$ such that

$$(5)\qquad \begin{aligned}&\mathfrak{F}_1(X_1,\dots,X_n)\,\mathfrak{Fn}\,On,\\ &\mathfrak{F}_1(X_1,\dots,X_n)^{\text{`}}\alpha=\mathfrak{G}_1(X_1,\dots,X_n)^{\text{`}}(\mathfrak{F}_1(X_1,\dots,X_n)\restriction\alpha).\end{aligned}$$

It follows from (4) and (5) that $\mathfrak{F}_1(X_1,\dots,X_n)$ satisfies the second and the third formulas in (3). Put $\mathfrak{F}(X_1,\dots,X_n)=\mathfrak{F}_1(X_1,\dots,X_n)\restriction\omega$. Then $\mathfrak{F}$ is normal and satisfies the formulas (3). Using once more the normality of the operations $\mathfrak{A}$ and $\mathfrak{G}$ one can prove by induction on k that if the formulas (3) are fulfilled by F_1 and F_2 instead of $\mathfrak{F}(X_1,\dots,X_n)$, then $F_1{}^{\text{`}}k=F_2{}^{\text{`}}k$ for every k, hence $F_1=F_2$ by [1], 4.67. It follows that the normal operation $\mathfrak{F}(X_1,\dots,X_n)$ constructed above is defined by the formulas (3).

Now we are going to define the operation $X\dot{+}Y$ mentioned in the Introduction.[13]

First we define an operation $\mathfrak{R}(X)$ as follows:

2.1.3. DEF. $\mathfrak{R}(X)\,\mathfrak{Fn}\,On.(\mathfrak{O}(X)\supset(\beta)(\mathfrak{R}(X)^{\text{`}}\beta=X+\mathfrak{R}(X)^{\text{``}}\beta)).$ $.(\sim\mathfrak{O}(X)\supset(\beta)(\mathfrak{R}(X)^{\text{`}}\beta=0)).$

We have to prove the existence of $\mathfrak{R}(X)$. Let us construct $\mathfrak{R}(X)$. The operation $\mathfrak{G}(X)$ is defined as follows:

$$\text{o)}\qquad \begin{aligned}&\mathfrak{G}(X)\,\mathfrak{Fn}\,V.\\ &(\mathfrak{O}(X)\supset(y)(\mathfrak{G}(X)^{\text{`}}y=X+\mathfrak{W}(y))).\\ &(\sim\mathfrak{O}(X)\supset(y)(\mathfrak{G}(X)^{\text{`}}y=0)).\end{aligned}$$

If X is an ordinal number, then it is a set, therefore in this case $X+\mathfrak{W}(y)$ is a set, too, by [1], 5.13, 5.16. Thus to every class X there exists exactly one class $\mathfrak{G}(X)$ satisfying the formula (o) by M_5 [1], and it is easy to verify that the operation $\mathfrak{G}(X)$ defined by the formula (o) is normal.[14]

[13] We have to remark that, as to the simplicity of the problem in question, the following discussion seems to be rather complicated. The reason for this is that the axiomatical theory of the "operations" of ordinal numbers is not elaborated, consequently there are no general methods to our disposal.

[14] All the concepts appearing in (o) are normal. But the definition of $\mathfrak{G}(X)$ has to have the form $u\in\mathfrak{G}(X)\equiv\psi(u,X)$ where ψ is normal. But this modification may be carried out without difficulties.

Now by M_7 there exists a normal operation $\mathfrak{N}(X)$ satisfying the formula

(oo) $$\mathfrak{N}(X)\mathfrak{Fn}\, On.(\beta)(\mathfrak{N}(X)`\beta = \mathfrak{G}(X)`(\mathfrak{N}(X)\restriction\beta)).$$

This operation $\mathfrak{N}(X)$ satisfies the definition 2. 1. 3 for every X. In fact, $\mathfrak{N}(X)\mathfrak{Fn}\, On$ for every X, if X is an ordinal number, then $\mathfrak{N}(X)`\beta = X + \mathfrak{W}(\mathfrak{N}(X)\restriction\beta) = X + \mathfrak{N}(X)``\beta$, if X is not an ordinal number, then $\mathfrak{N}(X)`\beta = \mathfrak{G}(X)`(\mathfrak{N}(X)\restriction\beta) = 0$ by (o) and (oo). On the other hand, from [1], 7.5 it follows that for every class X there exists only one class Y which satisfies the definition 2. 1. 3 (with Y instead of $\mathfrak{N}(X)$), and therefore the operation $\mathfrak{N}(X)$ constructed above is defined by 2. 1. 3 and it is normal.

We define the operation $X \dotplus Y$.[15]

2.1.4. Def. $(\mathfrak{O}(X).\mathfrak{O}(Y) \supset X \dotplus Y = \mathfrak{N}(X)`Y).(\sim(\mathfrak{O}(X).\mathfrak{O}(Y)) \supset X \dotplus Y = X + \{X\})$.

2.1.5. $u \in \alpha \dotplus \beta \equiv u \in \alpha \vee (\exists\xi)(\xi < \beta . \alpha \dotplus \xi = u)$.

Proof. $\alpha \dotplus \beta = \mathfrak{N}(\alpha)`\beta$ by 2.1.4 and $\mathfrak{N}(\alpha)`\beta = \alpha + \mathfrak{N}(\alpha)``\beta$ by 2.1.3. Hence $u \in \alpha \dotplus \beta \equiv u \in \alpha \vee u \in \mathfrak{N}(\alpha)``\beta$. But, $\mathfrak{N}(\alpha)$ beeing a function, $u \in \mathfrak{N}(\alpha)``\beta \equiv (\exists v)(v \in \beta . \mathfrak{N}(\alpha)`v = u)$ by 2. 1. 2. Taking into account that all the elements of β are the ordinal numbers ξ less than β, 2. 1. 5 is proved.

2. 1. 5 is the usual inductive definition of the "sum of two ordinal numbers" in the non-axiomatic set theory. The following properties of the operation $X \dotplus Y$ are to be deduced from 2. 1. 5 without difficulties:

2. 1. 6. $\mathfrak{O}(\alpha \dotplus \beta)$; $\alpha \dotplus 0 = \alpha$; $\alpha \dotplus (\beta \dotplus 1) = (\alpha \dotplus \beta) \dotplus 1$;
$\alpha < \beta \equiv (\exists\xi)(\xi \neq 0 . \alpha \dotplus \xi = \beta)$.

The proofs are left to the reader.

The following corollaries of [1], 7. 62 are needed:

2. 1. 7. $A\,\mathfrak{We}\,R . \mathfrak{Ord}(X) . \mathfrak{Ord}(Y) . G\,\mathfrak{Isom}_{R,E}(A, X) . H\,\mathfrak{Isom}_{R,E}(A, Y) \supset X = Y . G = H$, i. e. a well-ordered class can be isomorphic to at most one ordinal.

2. 1. 8. $A \subseteq On . B \subseteq On . G\,\mathfrak{Isom}_{E,E}(A, B) : H\,\mathfrak{Isom}_{E,E}(A, B) \supset G = H$.

As it is mentioned in [1], p. 6, axiom E is a very strong form of the axiom of choice. One can prove from axiom E that the whole universe of sets can be well-ordered.

2. 1. 9.* $(\exists W)(V\,\mathfrak{We}\,W)$.

[15] The operation $X \dotplus Y$ is the generalization of the operation $X \dotplus 1$ defined in [1], 7.4. The extension of the operation $\alpha \dotplus \beta$ for arbitrary classes X, Y is to be executed in such a way that for every X and for $Y = 1$ the operation $X \dotplus Y$ shall be equal to the operation $X \dotplus 1$.

We shall make use of this theorem in the proof of the Lemma. However, we believe that it is not necessary to give here a detailed proof of it, we shall only give a sketch of proving it.

Using axiom D one can define in Σ a function Ra which associates to every set x an ordinal number, called the rank of the set x, in such a manner that $Ra'x = 0$ and every set x of the rank α is a subset of the class containing the sets of rank less than α as elements. Moreover, one can prove that for any α the class of the sets x of rank α is a set.[16] Let v_α denote this set. According to the well-ordering theorem proved in [1], 7.71, each of the sets v_α has a well-ordering and it is easy to see that the class of all well-orderings of a set is a set. Using once more axiom E one can define a function which associates to every set v_α one of its well-orderings w_α. Now the well-ordering W of V may be defined as follows:

$$x\,W y \equiv Ra'x < Ra'y \vee Ra'x = Ra'y \,.\, x w_\alpha y.$$

It is obvious that V is well-ordered by W. We remark that every proper W-section of V is a set, and therefore V is isomorphic to On with respect to W and E by [1], 7.7.1.

2.1.10.* $m \subseteq \omega_{\alpha \dot{+} 1} . \overline{\overline{m}} < \omega_{\alpha \dot{+} 1} \supset (\exists \delta)(\delta < \omega_{\alpha \dot{+} 1} . m \subset \delta)$, i.e. to every subset m of $\omega_{\alpha \dot{+} 1}$, if m is of power less than $\omega_{\alpha \dot{+} 1}$, there exists a $\delta < \omega_{\alpha \dot{+} 1}$ such that all the elements of m are less than δ.

PROOF. Put $\delta = \mathfrak{S}(m) \dot{+} 1$. $m \subset \delta$ by [1], 4.451. If $\gamma \in m$, then $\gamma < \omega_{\alpha \dot{+} 1}$, and therefore $\overline{\overline{\gamma}} < \omega_{\alpha \dot{+} 1}$ by [1], 8.26. It follows — applying [1], 8.64 with I instead of F — that $\overline{\overline{\mathfrak{S}(m)}} < \omega_{\alpha \dot{+} 1}$, hence $\overline{\overline{\delta}} < \omega_{\alpha \dot{+} 1}$ by [1], 8.63. But then $\delta < \omega_{\alpha \dot{+} 1}$, since $\omega_{\alpha \dot{+} 1} \leqq \delta$ would imply $\omega_{\alpha \dot{+} 1} \leqq \overline{\overline{\delta}}$ by [1], 8.28.

2.1.11. DEF. We introduce the symbols $\mathfrak{Clos}_2(A, R)$ and $\mathfrak{Clos}_3(A, R)$ to denote that A is closed with respect to R as diadic relation and A is closed with respect to R as triadic relation, respectively.[17]

2.1.12. $\mathfrak{Clos}_n(A, R) . \mathfrak{Clos}_n(B, R) \supset \mathfrak{Clos}_n(A.B, R)$ for $n = 2, 3$.

PROOF. $R``A^{n-1} \subseteq A$ and $R``B^{n-1} \subseteq B$ by the assumption. $(A.B)^{n-1} \subseteq A^{n-1}$, $(A.B)^{n-1} \subseteq B^{n-1}$, and therefore $R``(A.B)^{n-1} \subseteq R``A^{n-1} \subseteq A$, $R``(A.B)^{n-1} \subseteq$ $\subseteq R``B^{n-1} \subseteq B$ by the monotony property of the operation $X``Y$ stated in [1], 4.86.

In [1], 8.72 the concept of the closure of a class X with respect to $R_1, \ldots, R_i$ and with respect to $S_1, \ldots, S_j$ as triadic relations is introduced. Let $[X]_{(R_1 \ldots R_i)(S_1 \ldots S_j)}$ denote the closure of X.

16 The definition of such a function Ra in Σ is elaborated in details in [2].

17 For the definition of this concepts see [1], 8.7 and 8.71.

2. 1. 13.* DEF. For every $X, R_1, \dots, R_i, S_1, \dots, S_j$, $[X]_{(R_1,\dots,R_i)(S_1\dots S_j)}$ is equal to the uniquely determined class Y satisfying the formulas

$$\mathfrak{M}(X).\mathfrak{Un}(R_1).\dots.\mathfrak{Un}(R_i).\mathfrak{Un}(S_1).\dots.\mathfrak{U}_n(S_j).\mathfrak{Clos}_2(Y.R_1).\dots.\mathfrak{Clos}_2(Y.R_i).$$
$$.\mathfrak{Clos}_3(Y, S_1).\dots.\mathfrak{Clos}_3(Y, S_j).(Z)(X \subseteq Z.\mathfrak{Clos}_2(Z, R_1).\dots.\mathfrak{Clos}_2(Z, R_i).$$
$$.\mathfrak{Clos}_3(Z, B_1).\dots.\mathfrak{Clos}_3(Z, S_j) \supset Y \subseteq Z) \vee \sim(\mathfrak{M}(X).\mathfrak{Un}(R_1).\dots.$$
$$.\mathfrak{Un}(R_i).\mathfrak{Un}(S_1)\dots\mathfrak{Un}(S_j)).Y = V.$$

The existence of the operation $[X]_{(R_1,\dots R_i)(S_1,\dots,S_j)}$ is assured by [1], 8.73 and its normality may be seen from the inductive definition of the closure of a set given there, referring to metatheorem M_8.

2. 1. 14.* DEF. $2^X = \overline{\overline{\mathfrak{P}(X)}}$.

2. 1. 15.* $2^{\aleph_\alpha} > \aleph_\alpha$, $2^{\aleph_\alpha} \geqq \aleph_{\alpha+1}$.

By [1], 8.32

2. 1. 16.* $\overline{\overline{x}} = \overline{\overline{y}} \supset 2^x = 2^y$.

PROOF. If $\overline{\overline{x}} = \overline{\overline{y}}$, then $x \simeq y$ by [1], 8.25. Let h be a function such that $\mathfrak{Un}_2(h)$, $\mathfrak{D}(h) = x$ and $\mathfrak{W}(h) = y$. By M_5 [1] there exists a function H defined on $\mathfrak{P}(x)$ such that $H`u = h``u$ for every $u \in \mathfrak{P}(x)$. It is obvious that $\mathfrak{Un}_2(H)$, $\mathfrak{D}(H) = \mathfrak{P}(x)$ and $\mathfrak{W}(H) = \mathfrak{P}(y)$. Consequently $\overline{\overline{\mathfrak{P}(x)}} = \overline{\overline{\mathfrak{P}(y)}}$.

2. 1. 17.* $\overline{\overline{x}} \leqq \overline{\overline{y}} \supset 2^x \leqq 2^y$.

PROOF. $2^x = 2^{\overline{\overline{x}}}$ and $2^y = 2^{\overline{\overline{y}}}$ by 2. 1. 16. But $\overline{\overline{x}} \leqq \overline{\overline{y}}$ implies that $\overline{\overline{x}} \subseteq \overline{\overline{y}}$, and therefore $\mathfrak{P}(\overline{\overline{x}}) \subseteq \mathfrak{P}(\overline{\overline{y}})$, hence $2^x \leqq 2^y$ by [1], 8. 28.

2. 2. We have already defined the model determined by the class M in the Introduction. Let us consider here this concept more detailed. The notions $\mathfrak{Cls}_M(X)$, $\mathfrak{M}_M(X)$, $X \in_M Y$ depend on the class M. We introduce the symbols $\mathfrak{Cls}(X, M)$, $\mathfrak{M}(X, M)$, $\mathfrak{E}(X, Y, M)$ to denote them.

2. 2. 1. DEF. $\mathfrak{Cls}(X, M) \equiv X \subseteq M.(u)(u \in M \supset u.X \in M)$.

$\mathfrak{Cls}(X, M)$ is denoted briefly by $\mathfrak{Cls}_M(X)$, and if $\mathfrak{Cls}(X, M)$ holds, then X is said to be an M-class.

2. 2. 2. DEF. $\mathfrak{M}(X, M) \equiv X \in M$.

$\mathfrak{M}(X, M)$ is denoted briefly by $\mathfrak{M}_M(X)$, and if $\mathfrak{M}_M(X)$ holds, then X is said to be an M-set.

2. 2. 3. DEF. $\mathfrak{E}(X, Y, M) \equiv \mathfrak{M}(X, M), \mathfrak{Cls}(Y, M).X \in Y$.

$\mathfrak{E}(X, Y, M)$ is denoted briefly by $X \in_M Y$, and if $X \in_M Y$ holds, X is said to be an M-element of Y.

The relativized and the absoluteness of a concept is defined only with respect to the model Δ in [1].

The definition of the relativization and of the absoluteness is given for inner models generally e. g. in [2]. However, it is obvious that some general remarks about relativization and absoluteness announced in [1] (for example [1], 10. 18) may be used for the relativization and absoluteness with respect to every inner model.

If the relativized of a particular class A, operation $\mathfrak{A}$, notion $\mathfrak{B}$, variable Γ exists with respect to the model Δ_M, we denote it by $A_M, \mathfrak{A}_M, \mathfrak{B}_M, \Gamma_M$, respectively. If there is no possibility of misunderstanding, the relativized of the class and set variables $X, Y, \dots, x, y, \dots$ will be denoted by $\bar{X}, \bar{Y}, \dots, \bar{x}, \bar{y}, \dots$ instead of $X_M, Y_M, \dots, x_M, y_M, \dots$, respectively.

It is obvious that Δ_M is not a set-theoretical concept, and so, for example, the statement "there exists a model Δ_M such that in the model Δ_M all the axioms of Σ^* hold" can not be formalized in our system. However, the property of the class M that the axioms of Σ or of Σ^* hold in the model determined by the class M is easy to formalize satisfactory.

Let us write all the axioms of Σ^* in a form not containing free variables and concepts other than $\mathfrak{Cls}(X)$, $\mathfrak{M}(X)$ and $X \in M$. The relativized of this form of the axioms always exists (i. e. we may write in this formulas $\mathfrak{Cls}(X, M)$, $\mathfrak{M}(X, M)$, $\mathfrak{E}(X, Y, M)$ instead of $\mathfrak{Cls}(X)$, $\mathfrak{M}(X)$, $X \in Y$, respectively) and each of these formulas contains at most the free variable M. Let us denote the conjunction of the relativized axioms of Σ and Σ^* by $\Psi_0(M)$ and $\Psi_0^*(M)$, respectively.[18] $\Psi_0(M)$ and $\Psi_0^*(M)$ formalize that the axioms of Σ or Σ^* hold in Δ_M.

From general logical considerations follows

M_9. If $\Psi_0(M)$ (or $\Psi_0^*(M)$) and the formula φ of Σ (or of Σ^*) is provable in Σ (in Σ^*), then φ holds in the model (i. e. if φ is provable e. g. in Σ, then $(M)(\Psi_0(M) \supset \varphi_M)$ is provable in Σ too).

Another special kind of inner models has been introduced by SHEPHERDSON in [2]. The inner model $\mathfrak{M}$ satisfying the axioms of group A, B, C of Σ is said to be a complete model,[19] provided the following conditions hold:

(i) $(X)(Y)(X \in_{\mathfrak{M}} Y \equiv \mathfrak{M}_{\mathfrak{M}}(X) . \mathfrak{Cls}_{\mathfrak{M}}(Y) . X \in Y)$,

(ii) $(A_{\mathfrak{M}})(X)(X \in A_{\mathfrak{M}} \supset \mathfrak{M}_{\mathfrak{M}}(X))$

where the subscript $\mathfrak{M}$ indicates the relativization with respect to the model $\mathfrak{M}$.

SHEPHERDSON has proved in [2] that the concepts defined in [1] absolute with respect to Δ are mostly absolute with respect to any complete model.

[18] We need not the concrete form of these formulas. We are going to use them only to formalize some of our theorems and to make clear the proof of Theorem 2.

[19] See [2].

We are going to make use of these results, and therefore we need the following theorem:

2.2.4. If for a class M the axioms A, B, C hold in Δ_M, then Δ_M is a complete model.

PROOF. The formula (i) is fulfilled by the definition 2.2.3 of $\mathfrak{E}(X, Y, M)$. The variable A_M runs over the M-classes. But $A_M \subseteq M$ for every M-class by 2.2.1, therefore $X \in A_M$ implies that $X \in M$, but then X is an M-set by 2.2.2 and (ii) is fulfilled too. Now for the convenience of the reader we are going to recapitulate the results of SHEPHERDSON [2] which will be used in this paper.

2.2.5. If for a class M the axioms of A, B, C hold in the model Δ_M, then, Δ_M being a complete model by 2.2.4, the following concepts are absolute with respect to Δ_M, by [2], 2.10, 2.12, 2.317, 2.320, 2.322, 2.312(b):

$$X \in Y,\ X \subseteq Y,\ \mathfrak{Ex}(X, Y),\ \mathfrak{Em}(X, Y),\ \mathfrak{Un}(X),\ \mathfrak{Un}_2(X),$$

$$X \times Y,\ X^2, X^3, \ldots,\ \mathfrak{Rel}'X),\ \mathfrak{Rel}_3(X),\ \mathfrak{D}(X),\ \mathfrak{W}(X),\ X.Y,$$

$$\mathfrak{Cnv}_i(X)\ (i = 1, 2, 3),\ X \upharpoonright Y,\ A^{\text{``}}X,\ X - Y,\ X + Y,\ A^{\text{`}}X,$$

$$\mathfrak{S}(X),\ \mathfrak{Lim}(X),\ \mathfrak{Max}(X),\ \mathfrak{Fnc}(X),\ X \mathfrak{Fn}\, Y,$$

$$\mathfrak{F}_i(X, Y) \text{ for } i = 2, \ldots 8,\ \mathfrak{Comp}(X),\ X \mathfrak{Con}\, Y,\ 0, 1, \ldots, \omega,$$

$$\mathfrak{Fin}(X),\ X < Y,\ X \leqq Y,\ \mathfrak{Ord}(X).$$

Taking into account that $V_M = M$ for Δ_M under the conditions of 2.2.5 the following equalities hold:

2.2.6. $-_M \bar{X} = M - \bar{X}$, $E_M = E.M$, $\mathfrak{P}_M(\bar{X}) = M.\mathfrak{P}(\bar{X})$, $On_M = On.M$, $On_M \in On \vee On_M = On$ by [2], 2.213, 2.315, 2.316.

If, in addition, $\mathfrak{M}(X)$ is absolute with respect to Δ_M, the following concepts are absolute with respect to Δ_M:

2.2.7. $\{X, Y\}$, $\langle X, Y \rangle$, $\langle X, Y, Z \rangle, \ldots, X \dotplus 1$, On, $\alpha, \beta, \gamma, \ldots, L$ (the class of constructible sets).

Now for the convenience of the reader we collect here some meta-theorems used in this paper to prove the absoluteness of a concept:[20]

AB$_1$. If all the concepts except $X_1, \ldots, X_n$ appearing in the formula $\varphi(X_1, \ldots, X_n)$ are absolute, then φ is absolute.

AB$_2$. If the relativized of the operation $\mathfrak{A}(X_1, \ldots, X_n)$ (particular class A) exists, then $\mathfrak{A}(X_1, \ldots, X_n)$, (A) is an M-class for every $\bar{X}_1, \ldots, \bar{X}_n$ (see [1], 10.1).

[20] In these theorems the subscript M always denotes the relativization with respect to the model Δ_M.

AB_3. If the relativized of an operation $\mathfrak{A}(X_1, \dots, X_n)$ exists and $\mathfrak{A}(\bar{X}_1, \dots, \bar{X}_n)$ satisfies the relativized of the defining postulate of $\mathfrak{A}$ for every $\bar{X}_1, \dots, \bar{X}_n$, then $\mathfrak{A}$ is absolute. Similarly for a particular class A (see [1], p. 47).

AB_4. If $\mathfrak{A}_M$ and A_M exist and the defining postulates of $\mathfrak{A}$ and A are absolute, then $\mathfrak{A}$ and A are absolute (see [1], p. 51).

AB_5. If the operations $\mathfrak{A}(X_1, \dots, X_n)$, $\mathfrak{B}_1(X_1, \dots, X_m), \dots, \mathfrak{B}_n(X_1, \dots, X_m)$ are absolute, then the same is true for the operation $\mathfrak{C}(X_1, \dots, X_m)$ defined by the equality

$$\mathfrak{C}(X_1, \dots, X_m) = \mathfrak{A}(\mathfrak{B}_1(X_1, \dots, X_m), \dots, \mathfrak{B}_n(X_1, \dots, X_m))$$

(see [1], 10.18).

AB_6. Let $\mathfrak{A}(X_1, \dots, X_n, U)$ be an operation, Γ a variable of the range $\mathfrak{B}(U)$. Suppose that $\mathfrak{A}(X_1, \dots, X_n, \Gamma)$ satisfies the formula $\varphi(\mathfrak{A}(X_1, \dots, X_n), X_1, \dots, X_n, \Gamma)$, $(X_1) \cdots (X_n)(\Gamma)(\exists ! Y)\varphi(Y, X_1, \dots, X_n, \Gamma)$ and for the classes U for which $\sim\mathfrak{B}(U)$ holds, $\mathfrak{A}(X_1, \dots, X_n, U) = 0$. Then $\mathfrak{A}$ is absolute under the conditions that $\mathfrak{B}(U)$ and 0 are absolute, $\mathfrak{A}_M$ exists and one of the following formulas holds:

$$\varphi(\bar{Y}, \bar{X}_1, \dots, \bar{X}_n, \Gamma) \equiv \varphi_M(\bar{Y}, \bar{X}_1, \dots, \bar{X}_n, \Gamma), \tag{1}$$

$$\mathfrak{A}(\bar{X}_1, \dots, \bar{X}_n, \Gamma) = \mathfrak{A}_M(\bar{X}_1, \dots, \bar{X}_n, \Gamma). \tag{2}$$

PROOF. Let $\psi(Y, X_1, \dots, X_n, U)$ be the defining postulate of $\mathfrak{A}(X_1, \dots, X_n, U)$. By the assumption ψ has the following form:

$$\psi(Y, X_1, \dots, X_n, U) \equiv \mathfrak{B}(U).\varphi(Y, X_1, \dots, X_n, U) \vee \sim\mathfrak{B}(U).(Y = 0).$$

By AB_4 it is sufficient to prove that ψ is absolute.

$$\psi_M(\bar{Y}, \bar{X}_1, \dots, \bar{X}_n, \bar{U}) \equiv \mathfrak{B}(\bar{U}).\varphi_M(\bar{Y}, \bar{X}_1, \dots, \bar{X}_n, \bar{U}) \vee \sim\mathfrak{B}(\bar{U}).(\bar{Y} = 0),$$

since $\mathfrak{B}$ and 0 are absolute. But then $\psi(\bar{Y}, \bar{X}_1, \dots, \bar{X}_n, \bar{U}) \equiv \psi_M(\bar{Y}, \bar{X}_1, \dots, \bar{X}_n, \bar{U})$ follows from (1) and thus $\mathfrak{A}$ is absolute if (1) holds.

On the other hand, if $\bar{U}$ is an M-class for which $\sim\mathfrak{B}_M(\bar{U})$, then by the absoluteness of $\mathfrak{B}$, $\sim\mathfrak{B}(\bar{U})$, hence $\mathfrak{A}_M(\bar{X}_1, \dots, \bar{X}_n, \bar{U}) \equiv \mathfrak{A}(\bar{X}_1, \dots, \bar{X}_n, \bar{U}) = 0$ for this $\bar{U}$. Consequently, if (2) holds, then $\mathfrak{A}_M(\bar{X}_1, \dots, \bar{X}_n, \bar{U}) \equiv \mathfrak{A}(\bar{X}_1, \dots, \bar{X}_n, \bar{U})$ is fulfilled and $\mathfrak{A}$ is absolute in this case too.

We need the following theorems concerning the absoluteness of the concepts $X \mathfrak{Isom}_{R,S}(Y, Z)$ and $X \dotplus Y$:

2.2.8. Under the conditions of 2.2.5 $X \mathfrak{Isom}_{R,S}(Y, Z)$ is absolute with respect to Δ_M.

PROOF.

$$\bar{X}\,\mathfrak{Isom}_{\bar{R},\bar{S}}(\bar{Y},\bar{Z}) \equiv \mathfrak{D}(\bar{X}) = \bar{Y}.\mathfrak{W}(\bar{X}) = \bar{Z}.\mathfrak{Un}_2(\bar{X}).$$

$$.\mathfrak{Rel}(\bar{X}).(u)(v)\big(u \in \bar{Y}.v \in \bar{Y} \supset (\langle uv\rangle \in \bar{R} \equiv \langle \bar{X}`u, \bar{X}`v\rangle \in \bar{S})\big)$$

by the definition [1], 6. 4.

$$(\bar{X}\,\mathfrak{Isom}_{\bar{R},\bar{S}}(\bar{Y},\bar{Z}))_M \equiv \mathfrak{D}(\bar{X}) = \bar{Y}.\mathfrak{W}(\bar{X}) = \bar{Z}.\mathfrak{Un}_2(\bar{X}).$$

$$.\mathfrak{Rel}(\bar{X}).(\bar{u})(\bar{v})\big(\bar{u} \in \bar{Y}.\bar{v} \in \bar{Y} \supset (\langle \bar{u}\bar{v}\rangle \in \bar{R} \equiv \langle \bar{X}`\bar{u}, \bar{X}`\bar{v}\rangle \in \bar{S})\big)$$

by 2. 2. 5. Hence it is sufficient to prove that

$$(u)(v)\big(u \in \bar{Y}.v \in \bar{Y} \supset (\langle uv\rangle \in \bar{R} \equiv \langle \bar{X}`u\bar{X}`v\rangle \in \bar{S})\big) \equiv$$

$$\equiv (\bar{u})(\bar{v})\big(\bar{u} \in \bar{Y}.\bar{v} \in \bar{Y} \supset (\langle \bar{u}\bar{v}\rangle \in \bar{R} \equiv \langle \bar{X}`\bar{u}\bar{X}`\bar{v}\rangle \in \bar{S})\big).$$

From the left side follows immediately the right side, since every M-set is a set by 2. 2. 2. The reverse implication holds, since if one of the sets u, v is not an M-set, the implication holds vacuously, because by 2. 2. 1 the hypothesis $u \in \bar{Y}.v \in \bar{Y}$ is false. Therefore $X\,\mathfrak{Isom}_{R,S}(Y,Z)$ is absolute.

2. 2. 9. Under the conditions of 2. 2. 7 $X \dotplus Y$ is absolute with respect to Δ_M.

PROOF.

$$u \in \alpha \dotplus \beta \equiv u \in \alpha \vee (\exists \xi)(\xi < \beta . u = \alpha \dotplus \xi)$$

by 2. 1. 5. But 2. 1. 5 may be proved without using axiom D. Therefore from the assumptions it follows that the relativized of this theorem holds, i. e.

$$\bar{u} \in \alpha \dotplus_M \beta \equiv \bar{u} \in \alpha \vee (\exists \xi)(\xi < \beta . \bar{u} = \alpha \dotplus \xi)$$

by the absoluteness of the concepts appearing without the subscript M in this formula (2. 2. 5, 2. 2. 7). But this equivalence holds with u instead of $\bar{u}$, since if u is not an M-set, both sides are false. It follows by transfinite induction on β that $\alpha \dotplus \beta = \alpha \dotplus_M \beta$ for every α, β. If one of the classes $\bar{X}, \bar{Y}$ is not an ordinal number, then this one is not an ordinal number in the model either by the absoluteness of $\mathfrak{O}(X)$ (2. 2. 7). Therefore in this case we have $\bar{X} \dotplus \bar{Y} = \bar{X} \dotplus 1$ and $\bar{X} \dotplus_M \bar{Y} = \bar{X} \dotplus_M 1$, hence $\bar{X} \dotplus \bar{Y} = \bar{X} \dotplus_M \bar{Y}$ by the absoluteness of the operation $X \dotplus 1$ (2. 2. 7). So $\bar{X} \dotplus \bar{Y} = \bar{X} \dotplus_M \bar{Y}$ holds for every $\bar{X}, \bar{Y}$, and therefore $\bar{X} \dotplus \bar{Y}$ is absolute.

§ 3. Lemma

In the Introduction we have defined the notion to be *almost universal*. Let $\mathfrak{Au}(X)$ denote that the class X is almost universal.

3. 1. DEF. $\mathfrak{Au}(X) \equiv (u)\big(u \subseteq X \supset (\exists v)(v \in X . u \subseteq v)\big).$

The class X is said to be *closed with respect to the fundamental operations* if $u, v \in X$ implies $\mathfrak{F}_i(u, v) \in X$ for every u, v $(i = 1, \ldots, 8)$.

3.2. DEF. $\mathfrak{Fcl}(X) \equiv (u)(v)(u \in X . v \in X \supset \mathfrak{F}_1(u, v) \in X . \ldots . \mathfrak{F}_8(u, v) \in X)$.

We need the following

LEMMA.* *If the class M is complete, closed with respect to the fundamental operations and almost universal, then the model determined by the class Δ_M satisfies the axiom system Σ.*

This lemma is formalizable and provable in the axiom system Σ^*:

3.3.* $\mathfrak{Comp}(M) . \mathfrak{Fcl}(M) . \mathfrak{Au}(M) \supset \psi_0(M)$.

We have already mentioned that the Lemma is an immediate generalization of the theorem proved in [1], VI which states that the model determined by the class L of the constructible sets satisfies the axiom system Σ. In [1], 9.51, 9.6, 9.63 it is proved that the assumptions of our Lemma are satisfied by the class L. However, in the proof of the theorem mentioned above GÖDEL makes use of two other properties of the class L as well. In [1], 9.87 must be used that the class L has a well-ordering by the construction and in the proof of $(C_1)_L$ GÖDEL makes use of the fact that L, by the construction, has an element $F^{\prime\prime}\omega$ satisfying $(C_1)_L$.

In the proof of Theorem 1 we are going to construct classes M in a similar way to GÖDEL's L and we shall make use of the Lemma to prove that Δ_M satisfies the axiom system Σ from these classes M too. These classes M will be well-ordered and $(C_1)_M$ may be verified for these classes without difficulties. However, we think that for the sake of overlooking the discussion it is more suitable to state this Lemma interesting by itself[21] in this general form.

We shall omit the detailed proof of the Lemma for it may be carried out in the same manner as in [1], VI using the previously proved results in [1], pp. 38—44. Only in the proof of the result $\mathfrak{Cls}_M(M . Q_4^{\prime\prime} A)$ corresponding to [1], 9.87 must be used axiom E, and we must give a proof of $(C_1)_M$. But first we sketch the proof of the Lemma. Using the completeness of M it is easy to verify that $X \in Y$ is absolute and $(A_1)_M$ is fulfilled. Then, step by step, every notions and operations appearing in the axioms may be proved to be absolute with respect to Δ_M. From these it follows that $(A_2)_M$ and $(A_3)_M$ are fulfilled. The conditional existence axioms $(A_4)_M, (B_1)_M, \ldots, (B_8)_M$ are fulfilled for sets M being closed with respect to the fundamental operations. From this it may be deduced that $E . M$, $\overline{A} . \overline{B}$, $M - \overline{A}$, $M . (V \times \overline{A})$, $\mathfrak{D}(\overline{A})$, $\mathfrak{Conv}_k(\overline{A})$ $(k = 1, 2, 3)$ are M-classes and they satisfy the corresponding relativized axioms of group

[21] The remark that the Lemma is provable in Σ if we assume that M might be well-ordered has no importance from our point of view, because we have to make use essentially of axiom E in another point of the proof of Theorem 1 too.

B. $(D)_M$ is an immediate consequence of D using the absoluteness of the notions occurring in D.

Now the relativized of the axioms of the group C are fulfilled, because the relativized of the metatheorem M_2 [1] is true for the model (M_2 being proved from the axioms A, B, D), and therefore M-classes may be constructed satisfying these axioms such that these M-classes are sets by the corresponding axiom of group C, and consequently they are M-sets by the absoluteness of $\mathfrak{M}(X)$. However, in the case of the axioms C_2, C_3, C_4 the proof may be carried out in a more direct way.

In the case of axioms C_1 we must argue as follows. By the absoluteness of the notions appearing in C_1, $(C_1)_M$ is the following formula:

$$(\exists \bar{a})\big(\sim \mathfrak{Em}(\bar{a}).(\bar{x})(x \in \bar{a} \supset (\exists \bar{y})(\bar{y} \in \bar{a}.\bar{x} \subset \bar{y}))\big).$$

Let $\varphi(X)$ be the following formula:

$$\varphi(X) \equiv X = 0 \vee (\exists u)(X = u \dotplus 1).(v)\big(v \in X \supset (\exists w)(w \in X.v = w \dotplus 1)\big).$$

All the concepts appearing in this formula can be defined using only axioms A, B, D, and their normality follows from these axioms as well. Therefore if we define the particular class A by the stipulation $X \in A \equiv \varphi(X)$, then φ_M and A_M exist.

It is easy to verify that $\varphi(X)$ is absolute, and consequently A is absolute by AB_2. Hence A is an M-class.

But A is ω. In fact, from the definition of A it follows that $0 \in A$, $x \in A \supset x \dotplus 1 \in A$ and $x \dotplus 1 \in A \supset x \in A$. $0 \in A$ and $x \in A \supset x \dotplus 1 \in A$ implies $\omega \subseteq A$. By axiom D, if $A - \omega$ were not empty, then it would have an element x such that $x.(A-\omega) = 0$. But in this case $\varphi(x)$, and consequently $x = u \dotplus 1$ for an u where $u \in A$ and $u \in x$, and therefore $u \in \omega$, but then $x \in \omega$. But this contradicts $x \in A - \omega$, thus we have $A - \omega = 0$, $A = \omega$. ω is a set, and being an M-class, it is an M-set by the absoluteness of $\mathfrak{M}(X)$. $\omega = \bar{a}$ satisfies axiom $(C_1)_M$.

§ 4. Proof of Theorem 1

THEOREM 1.* *For any pair λ, ν of ordinals, if $2^{\aleph_\lambda} \geqq \aleph_{\lambda \dotplus \nu \dotplus 1}$, then there exists a complete and almost universal class M such that Δ_M satisfies the axiom system Σ^* and, in addition, the following formulas are fulfilled:*

$(+)$ $\quad \aleph_M{}^{\prime}\alpha = \aleph^{\prime}\alpha$ *for* $\alpha \leqq \lambda \dotplus \nu \dotplus 1$,

$(++)$ $\quad (2^{\aleph_M{}^{\prime}\varrho})_M = \aleph_M{}^{\prime}\lambda \dotplus \nu \dotplus 1$ *for* $\lambda \leqq \varrho < \lambda \dotplus \nu \dotplus 1$,

$(+++)$ $\quad (2^{\aleph_M{}^{\prime}\mu})_M = \aleph_M{}^{\prime}\mu \dotplus 1$ *for* $\lambda \dotplus \nu \leqq \mu$.

Theorem 1 is formalizable and provable in Σ^*:

$$(\lambda)(\nu)(2^{\aleph_\lambda} \geqq \aleph_{\lambda\dotplus\nu\dotplus1} \supset (\exists M)(\mathfrak{Comp}(M).\mathfrak{Au}(M).\psi_0^*(M).$$

$$(\alpha)(\alpha \leqq \lambda\dotplus\nu\dotplus1 \supset \aleph_M{}^{\backprime}\alpha = \aleph^{\backprime}\alpha).(\varrho)(\lambda \leqq \varrho < \lambda\dotplus\nu\dotplus1 \supset (2^{\aleph_M{}^{\backprime}\varrho})_M = \aleph_M{}^{\backprime}\lambda\dotplus\nu\dotplus1).$$

$$.(\mu)(\lambda\dotplus\nu \leqq \mu \supset 2^{\aleph_M{}^{\backprime}\mu} = \aleph_M{}^{\backprime}(\mu\dotplus1)).$$

The idea of the proof is the following: The assumption $2^{\aleph_\lambda} \geqq \aleph_{\lambda\dotplus\nu\dotplus1}$ implies that the set ω_λ of power $\aleph_\lambda$ (which is constructible as all ordinals are) has at least $\aleph_{\lambda\dotplus\nu\dotplus1}$ subsets. If we add $\aleph_{\lambda\dotplus\nu\dotplus1}$ such subsets to the class L, then we have to add not too much further sets to L in order that the model determined by the wider class arising thus should satisfy again the axiom system Σ^*, so that ω_λ will not possess more than $\aleph_{\lambda\dotplus\nu\dotplus1}$ subsets in that model and the cardinal of the power sets of sets of a cardinal not less than $\aleph_{\lambda\dotplus\nu}$ will not increase.

The technical execution of the proof is not so simple. This runs on the following lines. Making use of the assumption $2^{\aleph_\lambda} \geqq \aleph_{\lambda\dotplus\nu\dotplus1}$ we define a function h which establishes a one-to-one mapping of the ordinals less than $\omega_{\lambda\dotplus\nu\dotplus1}$ onto a subset of $\mathfrak{P}(\omega_{\lambda\dotplus\nu\dotplus1})$ as well as a function k which associates to every ordinal number less than $\omega_{\lambda\dotplus\nu\dotplus1}$ a one-to-one mapping of this ordinal number onto its cardinal number.

Writing 12 instead of 9 in the definitions [1] of the particular classes S, J, K_1, K_2 we denote the particular classes thus obtained by S, J, K_1, K_2 again and we define the functions $J_0, \dots, J_{11}$ by $J_i{}^{\backprime}\langle\alpha\beta\rangle = J^{\backprime}\langle i\alpha\beta\rangle$ for $i<12$ (instead of $i<9$).

Analogously to the function F defined in [1], 9.5 we define a function G on the class of all ordinal numbers in the following way:

$$G^{\backprime}\alpha = G^{\backprime\backprime}\alpha \quad \text{for} \quad \alpha \in \mathfrak{W}(J_0),$$

$$G^{\backprime}\alpha = \mathfrak{F}_i(G^{\backprime}\beta, G^{\backprime}\gamma) \quad \text{for} \quad \alpha \in \mathfrak{W}(J_i) \qquad (i = 1, \dots, 8);$$

if $\alpha < \omega_{\lambda\dotplus\nu\dotplus1}$,

$$G^{\backprime}\alpha = G^{\backprime}\beta . On \quad \text{for} \quad \alpha \in \mathfrak{W}(J_9),$$

$$G^{\backprime}\alpha = G^{\backprime}\beta . h^{\backprime}G^{\backprime}\gamma \quad \text{for} \quad \alpha \in \mathfrak{W}(J_{10}),$$

$$G^{\backprime}\alpha = G^{\backprime}\beta . k^{\backprime}G^{\backprime}\gamma \quad \text{for} \quad \alpha \in \mathfrak{W}(J_{11});$$

if $\alpha \geqq \omega_{\lambda\dotplus\nu\dotplus1}$,

$$G^{\backprime}\alpha = G^{\backprime}\beta . 0 = 0 \quad \text{for} \quad \alpha \in \mathfrak{W}(J_9),$$

$$G^{\backprime}\alpha = G^{\backprime}\beta . h . G^{\backprime\backprime}\omega_{\lambda\dotplus\nu\dotplus1} \quad \text{for} \quad \alpha \in \mathfrak{W}(J_{10}),$$

$$G^{\backprime}\alpha = G^{\backprime}\beta . k . G^{\backprime\backprime}\omega_{\lambda\dotplus\nu\dotplus1} \quad \text{for} \quad \alpha \in \mathfrak{W}(J_{11})$$

with $\beta = K_1{}^{\backprime}\alpha$, $\gamma = K_2{}^{\backprime}\alpha$.

On the analogy of the definition of L we define M as the class of the values of the function G: $M = \mathfrak{W}(G)$. Then M satisfies the requirements of Theorem 1.

The fact that the model determined by M satisfies the axioms of Σ follows from the Lemma without difficulties.

The truth of the axiom of choice in the model follows from the fact that the function G inducing the well-ordering of the class M can be proved to be an M-class after we have verified that h, k are M-sets. However, to prove this analogously to the proof of the theorem that $V = L$ holds in Δ, absoluteness proofs are needed. But it has no sense to speak of the absoluteness of G, for the functions h, k are not particular classes. Therefore we have to define operations $\mathfrak{G}(u_1, u_2, \xi)$, $\mathfrak{M}(u_1, u_2, \xi)$ writing in the definition of G the variables u_1, u_2, ξ instead of $h, k, \omega_{\lambda \dot{+} \nu \dot{+} 1}$, respectively.

In 4.1 we are going to define the particular classes S, J, K_1, K_2, J_i and we prove some results concerning these classes. In 4.2 we shall prove some emmas needed to the construction and to the examination of the above mentioned operations. In 4.3 we are going to define the operations $\mathfrak{G}(u_1, u_2, \xi)$, $\mathfrak{M}(u_1, u_2, \xi)$ and other operations analogous to GÖDEL's particular classes Od, As, C, and we shall prove in this section that the class $\mathfrak{M}(u_1, u_2, \xi)$ satisfies the assumptions of the Lemma, and therefore in the model $\Delta_{\mathfrak{M}(u_1, u_2, \xi)}$ all the axioms of Σ hold for every u_1, u_2, ξ. In 4.4 we shall prove that all the operations defined in 4.3 are absolute with respect to Δ_M for every M satisfying the assumptions of the Lemma. In 4.5 we prove some general theorems for the classes $\mathfrak{M}(u_1, u_2, \xi)$. The functions h and k will be defined only in Section 4.6. We prove in this section that $h, k \in \mathfrak{M}(h, k, \omega_{\lambda \dot{+} \nu \dot{+} 1}) = M$, and therefore by the absoluteness of $\mathfrak{G}(u_1, u_2, \xi)$ $\mathfrak{G}_M(h, k, \omega_{\lambda \dot{+} \nu \dot{+} 1}) = \mathfrak{G}(h, k, \omega_{\lambda \dot{+} \nu \dot{+} 1})$ in consequence of which axiom E holds in the model Δ_M. In 4.7 we prove the formula $(+)$ using the fact that k belongs to the model and we prove that $(2^{\aleph_M{}^{(}\lambda})_M \geqq \aleph_M{}^{(}\lambda \dot{+} \nu \dot{+} 1$ using that h belongs to the model. In addition, in this section we prove that $V_M = \mathfrak{M}_M(h, k, \aleph_M{}^{(}\lambda \dot{+} \nu \dot{+} 1)$ holds, and therefore to prove Theorem 1 it is sufficient to prove the following theorem in Σ^*: If for a pair of ordinal numbers α, β $(\exists u_1)(\exists u_2)\big(V = \mathfrak{M}(u_1, u_2, \omega_{\alpha \dot{+} \beta \dot{+} 1})\big)$, then the following formulas are fulfilled:

(1) $2^{\aleph_{\alpha \dot{+} \beta}} = \aleph_{\alpha \dot{+} \beta \dot{+} 1}$,

(2) $2^{\aleph_\mu} = \aleph_{\mu \dot{+} 1}$ for every $\mu > \alpha \dot{+} \beta$.

This theorem is proved in Sections 4.8—4.10 analogously to the theorem proved in [1], Chapter VIII, that $V = L$ implies the generalized continuum hypothesis. It is obvious that to carry out the proof a generalization of the theorem [1], 12.6 is needed and this generalization is made possible by the fact that $G^{\prime}\alpha$ is defined for $\alpha \geqq \omega_{\lambda \dot{+} \nu \dot{+} 1}$ with the help of the fundamental operations and of the operations $x.0$, $x.h$, $x.k$ of very simple character. It is easy to find the generalization of [1], 12.6 from which (2) may be deduced. However, to find the generalization from which (1) may be deduced, too, a new idea is needed. In 4.8 we shall give a detailed sketch of the proofs elaborated in Sections 4.8—4.10.

If in the proof of a theorem the assumption $2^{\aleph_\lambda} \geqq \aleph_{\lambda \dot{+} \nu \dot{+} 1}$ is used, its number is marked by $**$.

4.1. As we have already mentioned we need a new definition of the particular classes S, J, J_i, K_1, K_2 defined in [1], 9.2, 9.21, 9.22, 9.24.

4.1.1. Def.

$$(\mu < 12 . \nu < 12 \supset (\langle \mu \alpha \beta \rangle S \langle \nu \gamma \delta \rangle \equiv \langle \alpha \beta \rangle R \langle \gamma \delta \rangle \vee \langle \alpha \beta \rangle = \langle \gamma \delta \rangle . \mu < \nu)).$$
$$. S \subseteq 12 \times On^2$$

where R is the relation defined by [1], 7.81.

In the same manner as in [1], 9.2 it is easy to see that S exists, $12 \times On^2$ is well-ordered by S, $12 \times On^2$ is a proper class and any proper S-section of $12 \times On^2$ is a set. Hence $12 \times On^2$ is isomorphic to On with respect to S and E by [1], 7.7. Therefore there exists a function J satisfying the following defining postulate:

4.1.2. Def.

$$J \,\mathfrak{Fn}\, 12 \times On^2 . \mathfrak{W}(J) = On . (\mu < 12 . \nu < 12 \supset (\langle \mu \alpha \beta \rangle S \langle \nu \gamma \delta \rangle) \supset$$
$$\supset (J^{\prime}\langle \mu \alpha \beta \rangle < J^{\prime}\langle \nu \gamma \delta \rangle)).$$

We define the functions $J_0, \ldots, J_{11}$ as follows:

4.1.3. Def.

$$J_0 \,\mathfrak{Fn}\, On^2 . J_0^{\prime}\langle \alpha \beta \rangle = J^{\prime}\langle 0 \alpha \beta \rangle,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$J_{11} \,\mathfrak{Fn}\, On^2 . J_{11}^{\prime}\langle \alpha \beta \rangle = J^{\prime}\langle 11 \alpha \beta \rangle.$$

Analogously to [1], 9.23 we have

4.1.4. The classes $\mathfrak{W}(J_i)$ $(i = 0, \ldots, 11)$ are mutually exclusive and their sum is On.

We define the classes K_1, K_2:

4.1.5. Def.

$$\langle \alpha \gamma \rangle \in K_1 \equiv (\exists \mu)(\exists \beta)(\mu < 12 . \gamma = J^{\prime}\langle \mu \alpha \beta \rangle) . K_1 \subseteq On^2,$$
$$\langle \beta \gamma \rangle \in K_2 \equiv (\exists \mu)(\exists \alpha)(\mu < 12 . \gamma = I^{\prime}\langle \mu \alpha \beta \rangle) . K_2 \subseteq On^2.$$

K_1, K_2 are functions defined on On. The following theorems may be proved literally as the theorems [1], 9. 25, 9. 26, 9. 27:

4. 1. 6. $J_i^{\prime}\langle\alpha\beta\rangle \geqq \mathfrak{Max}(\{\alpha\beta\})$ for $i<12$,

$J_i^{\prime}\langle\alpha\beta\rangle > \mathfrak{Max}(\{\alpha\beta\})$ for $i<12,\ i\neq 0$.

$K_1^{\prime}\alpha \leqq \alpha,\quad K_2^{\prime}\alpha \leqq \alpha,$

$K_1^{\prime}\alpha < \alpha,\quad K_2^{\prime}\alpha < \alpha$ for $\alpha \notin \mathfrak{W}(J_0)$.

4. 1. 7.* $\alpha<\omega_\gamma . \beta<\omega_\gamma \supset J_i^{\prime}\langle\alpha\beta\rangle<\omega_\gamma,\quad \omega_\gamma \in \mathfrak{W}(I_0)$.

We need the following simple result:

4. 1. 8.* If $\beta<\omega_\alpha$, then there exists a $\gamma<\omega_\alpha$ such that $\beta<\gamma$ and $\gamma \in \mathfrak{W}(J_0)$.

Proof. $\beta = J^{\prime}\langle i\alpha_1\beta_1\rangle$. Let $\langle\alpha_2\beta_2\rangle$ be the successor of $\langle\alpha_1\beta_1\rangle$ in the well-ordering R of On^2. Let j be the integer for which $i+j=12$. Then $\gamma=\beta\dotplus j = J^{\prime}\langle 0\alpha_2\beta_2\rangle$ by 4. 1. 1 and 4. 1. 2 and $\beta<\gamma$ by 2. 1. 6. If $\overline{\overline{\beta}}<\omega_\alpha$, then $\overline{\overline{\beta}}<\omega_\alpha$, and therefore $\overline{\overline{\beta\dotplus 1}} = \overline{\overline{\beta+\{\beta\}}}<\omega_\alpha$ by [1], 8. 63, hence $\overline{\overline{\beta\dotplus j}}<\omega_\alpha$ for every $j<12$. So we have $\overline{\overline{\gamma}}<\omega_\alpha$, but then $\gamma<\omega_\alpha$ by [1], 8. 28 and $\gamma\in\mathfrak{W}(J_0)$ by the definition of J_0.

4. 2.

ML_1. *Are the operations $\mathfrak{H}_0(U_1, \ldots, U_l)$, $\mathfrak{H}_i(X, Y, U_1, \ldots, U_l)$ $(i=1, \ldots, 11)$ normal, and if, in addition, $\mathfrak{M}(\mathfrak{H}_i(x, y, U_1, \ldots, U_l))$ for every $x, y, U_1, \ldots, U_l$ $(i=1, \ldots, 11)$, then there exists a normal operation $\mathfrak{H}(U_1, \ldots, U_l, Z)$ such that $\mathfrak{H}(U_1, \ldots, U_l, Z)=0$ if Z is not an ordinal number, and for every $U_1, \ldots, U_l, \xi$ $\mathfrak{H}(U_1, \ldots, U_l, \xi)$ is equal to the class H satisfying the following formulas:*

$$H \,\mathfrak{Fn}\, On,$$
$$\alpha<\xi \supset H^{\prime}\alpha = \mathfrak{H}_0(U_1, \ldots, U_l)^{\prime}\alpha,$$
$$(1)\qquad \alpha\in\mathfrak{W}(J_0)-\xi \supset H^{\prime}\alpha = H^{\prime\prime}\alpha,$$
$$\alpha\in\mathfrak{W}(J_i)-\xi \supset H^{\prime}\alpha = \mathfrak{H}_i(H^{\prime}K_1^{\prime}\alpha, H^{\prime}K_2^{\prime}\alpha, U_1, \ldots, U_l) \quad \text{for} \quad i=1, \ldots, 11.$$

Proof.[22] In order to prove the existence of the operation $\mathfrak{H}$ first we define an operation $\mathfrak{A}(U_1, \ldots, U_l, Z)$ as follows: $\mathfrak{A}(U_1, \ldots, U_l, Z)=0$ if Z is not an ordinal number and for every $U_1, \ldots, U_l, \xi$ $\mathfrak{A}(U_1, \ldots, U_l, \xi)=A$ is defined by the formulas

$$A \,\mathfrak{Fn}\, V,$$
$$\mathfrak{D}(x)\in\xi \supset A^{\prime}x = \mathfrak{H}_0(U_1, \ldots, U_l)^{\prime}\mathfrak{D}(x),$$
$$(2)\qquad \mathfrak{D}(x)\in\mathfrak{W}(J_0)-\xi \supset A^{\prime}x = \mathfrak{W}(x),$$
$$\mathfrak{D}(x)\in\mathfrak{W}(J_i)-\xi \supset A^{\prime}x = \mathfrak{H}_i(x^{\prime}K_1^{\prime}\mathfrak{D}(x), x^{\prime}K_2^{\prime}\mathfrak{D}(x), U_1, \ldots, U_l)$$
$$\text{for} \quad i=1, \ldots, 11,$$

and $A^{\prime}x=0$ everywhere else.

[22] The proof is similar to the existence proof of F given in [1], 9. 3.

The existence and unicity of A is assured by M_6 [1] for every $U_1, \dots, U_l, \xi$ taking into consideration that $\mathfrak{M}(\mathfrak{H}_i(x, y, U_1, \dots, U_l))$ and the concepts appearing in the formulas are normal by the assumptions. Thus the operation $\mathfrak{A}(U_1, \dots, U_l, Z)$ exists and it is easy to verify that it is normal. By M_7 there exists a normal operation $\mathfrak{H}^+(U_1, \dots, U_l, Z)$ such that $\mathfrak{H}^+(U_1, \dots, U_l, Z) = H$ satisfies

$$H \,\mathfrak{Fn}\, On, \quad H`\alpha = A`(H \upharpoonright \alpha) \tag{3}$$

where A satisfies (2). But if for $U_1, \dots, U_l, \xi$ H satisfies (3) and A satisfies (2), then H satisfies (1) as seen by the following proof. $H \,\mathfrak{Fn}\, On$. If $\alpha < \xi$, then $\mathfrak{D}(H \upharpoonright \alpha) = \alpha \in \xi$, and therefore $H`\alpha = \mathfrak{H}_0(U_1, \dots, U_l)`\alpha$. If $\alpha \in \mathfrak{W}(J_0) - \xi$, then $\mathfrak{D}(H \upharpoonright \alpha) = \alpha \in \mathfrak{W}(J_0) - \xi$ and $H`\alpha = \mathfrak{W}(H \upharpoonright \alpha) = H``\alpha$. If $\alpha \in \mathfrak{W}(J_i) - \xi$ $(i = 1, \dots, 11)$, then $\mathfrak{D}(H \upharpoonright \alpha) = \alpha \in \mathfrak{W}(J_i) - \xi$ and $H`\alpha = A`(H \upharpoonright A) = \mathfrak{H}_i((H \upharpoonright \alpha)`K_1`\alpha, (H \upharpoonright \alpha)`K_2`\alpha, U_1, \dots, U_l)$. But $K_1`\alpha, K_2`\alpha < \alpha$ by 4.1.6 and $(H \upharpoonright \alpha)`\beta = H`\beta$ if $\beta < \alpha$, therefore $H`\alpha = \mathfrak{H}_i(H`K_1`\alpha, H`K_2`\alpha, U_1, \dots, U_l)$.

Hence if we define the operation $\mathfrak{H}(U_1, \dots, U_l, Z)$ by the stipulations

$$\mathfrak{O}(Z) \supset \mathfrak{H}(U_1, \dots, U_l, Z) = \mathfrak{H}^+(U_1, \dots, U_l, Z),$$
$$\sim \mathfrak{O}(Z) \supset \mathfrak{H}(U_1, \dots, U_l, Z) = 0,$$

$\mathfrak{H}$ satisfies the requirements of ML_1. By transfinite induction it is easy to see that there exists only one class H satisfying (1) for every $U_1, \dots, U_l, \xi$, and so the normal operation $\mathfrak{H}$ constructed above is defined by the stipulations given in ML_1.

ML_2. *Are the operations $\mathfrak{H}_i(X, Y, U_1, \dots, U_l)$ $(i = 1, \dots, 11)$ normal and if, in addition, $\mathfrak{M}(\mathfrak{H}_i(x, y, U_1, \dots, U_l))$ for every $x, y, U_1, \dots, U_l$, then there exists a normal operation $\mathfrak{H}(U_1, \dots, U_l, Z)$ such that $\mathfrak{H}(U_1, \dots, U_l, Z) = 0$ if Z is not an ordinal number and for every ordinal number ξ $\mathfrak{H}(U_1, \dots, U_l, \xi)$ is equal to the class H satisfying the following formulas:*

$$\begin{gathered} H \,\mathfrak{Fn}\, \xi, \\ \alpha \in \mathfrak{W}(J_0).\xi \supset H`\alpha = H``\alpha, \\ \alpha \in \mathfrak{W}(J_i).\xi \supset H`\alpha = \mathfrak{H}_i(H`K_1`\alpha, H`K_2`\alpha, U_1, \dots, U_l) \quad \textit{for} \quad i = 1, \dots, 11. \end{gathered} \tag{4}$$

PROOF. It follows from ML_1 that there exists a normal operation $\mathfrak{H}^+(U_1, \dots, U_l)$ such that $\mathfrak{H}^+(U_1, \dots, U_l) = H^+$ where the following formulas are satisfied by H^+:

$$\begin{gathered} H^+ \,\mathfrak{Fn}\, On, \\ \alpha \in \mathfrak{W}(J_0) \supset H^+`\alpha = H^+``\alpha, \\ \alpha \in \mathfrak{W}(J_i) \supset H^+`\alpha = \mathfrak{H}_i(H^+`K_1`\alpha, H^+`K_2`\alpha, U_1, \dots, U_l) \quad \text{for} \quad i = 1, \dots. 11. \end{gathered} \tag{5}$$

If H^+ satisfies (5), then $H = H^+ \upharpoonright \xi$ satisfies (4). This is to be seen as follows. $\mathfrak{D}(H^+ \upharpoonright \xi) = \xi$, and consequently the first of the formulas (4) is fulfilled. If $\alpha \in \xi$, then $H`\alpha = H^+`\alpha$. Suppose $\alpha \in \mathfrak{W}(J_0).\xi$. Then $H^+`\alpha = H^+``\alpha$, hence $H`\alpha = H^+``\alpha$. But by 2.2.2 $H^+``\alpha$ and $H``\alpha$ are the sets of the elements $H^+`\beta$ and $H`\beta$, respectively, where $\beta < \alpha$. If $\beta < \alpha$, then $\beta \in \xi$, and therefore $H^+`\beta = H`\beta$, hence $H`\alpha = H``\alpha$. Now suppose $\alpha \in \mathfrak{W}(J_i).\xi$ $(i = 1, \ldots, 11)$. $H`\alpha = H^+`\alpha = \mathfrak{H}_i(H^+`K_1`\alpha, H^+`K_2`\alpha, U_1, \ldots, U_l)$. $K_1`\alpha, K_2`\alpha < \alpha$ by 4.1.6, and therefore $K_1`\alpha, K_2`\alpha \in \xi$. Hence $H`\alpha = \mathfrak{H}_i(H`K_1`\alpha, H`K_2`\alpha, U_1, \ldots, U_l)$. Let $\mathfrak{H}(U_1, \ldots, U_l, Z)$ be the operation defined by the following stipulations:

$$\mathfrak{O}(Z) \supset \mathfrak{H}(U_1, \ldots, U_l, Z) = \mathfrak{H}^+(U_1, \ldots, U_l) \upharpoonright Z,$$
$$\sim \mathfrak{O}(Z) \supset \mathfrak{H}(U_1, \ldots, U_l, Z) = 0.$$

It follows from the facts proved above that $\mathfrak{H}$ is normal and satisfies the definition (4). It is easy to see by transfinite induction that there exists only one class H satisfying the formulas (4) for every $U_1, \ldots, U_l, \xi$. Hence under the conditions of ML_2 a normal operation is defined by the stipulations given in ML_2.

ML_3. *Let $\mathfrak{H}_i(X, Y)$ $(i = 1, \ldots, 11)$ be operations of the form $\mathfrak{H}_1(X, Y) = \{X, Y\}$ and $\mathfrak{H}_i(X, Y) = X.\mathfrak{K}_i(Y)$ for $i = 2, \ldots, 11$ where $\mathfrak{K}_i(Y)$ are arbitrary operations.*

Let ξ be an ordinal number and H_0 a class such that $H_0`\alpha \subseteq H_0``\alpha$ for every α less than ξ. Suppose that the following formulas are fulfilled by the class H:

$$(\exists A)(H \,\mathfrak{Fn}\, A \,.\, \mathfrak{Ord}(A)),$$
$$\alpha \in \xi . A \supset H`\alpha = H_0`\alpha, \tag{6}$$
$$\alpha \in (\mathfrak{W}(J_0) - \xi) . A \supset H`\alpha = H``\alpha,$$
$$\alpha \in (\mathfrak{W}(J_i) - \xi) . A \supset \mathfrak{H}_i(H`K_1`\alpha, H`K_2`\alpha) \quad \textit{for} \quad i = 1, \ldots, 11.$$

Then $H`\alpha \subseteq H``\alpha$ and $H``\alpha$ is complete for every $\alpha \in A$.

PROOF.[23] Let A be the class for which $H \,\mathfrak{Fn}\, A$. Let α be the first element of A for which $H`\alpha \subseteq H``\alpha$ is false. Then we have $H`\beta \subseteq H``\beta$ for every $\beta < \alpha$ (A is either an ordinal number η or On thus $\alpha \in A$ implies $\beta \in A$ for every $\beta < \alpha$.) If $\alpha < \xi$, then $H`\alpha = H_0`\alpha$ and $H`\beta = H_0`\beta$ for every $\beta < \alpha$, i.e. $H``\alpha = H_0``\alpha$, hence $H`\alpha \subseteq H``\alpha$ by the assumption. If $\alpha \in \mathfrak{W}(J_0) - \xi$, then $H`\alpha = H``\alpha$, hence $H`\alpha \subseteq H``\alpha$. If $\alpha \in \mathfrak{W}(J_1) - \xi$, then $H`\alpha = \mathfrak{H}_1(H`K_1`\alpha, H`K_2`\alpha) = \{H`K_1`\alpha, H`K_2`\alpha\}$. $K_1`\alpha, K_2`\alpha < \alpha$ by 4.1.6, and therefore $\{H`K_1`\alpha, H`K_2`\alpha\} \subseteq H``\alpha$ by 2.2.2. If $\alpha \in \mathfrak{W}(J_i) - \xi$ $(i = 2, \ldots, 11)$

[23] Similarly to [1], 9.5.

then $H^{\prime}\alpha = \mathfrak{H}_i(H^{\prime}K_1^{\prime}\alpha, H^{\prime}K_2^{\prime}\alpha) = H^{\prime}K_1^{\prime}\alpha . \mathfrak{K}_i(H^{\prime}K_2^{\prime}\alpha)$. Put $K_1^{\prime}\alpha = \beta$. $H^{\prime}\alpha \subseteq \subseteq H^{\prime}\beta$, and since by 4.1.6 $\beta < \alpha$, $H^{\prime}\beta \subseteq H^{\prime\prime}\beta$ and $H^{\prime\prime}\beta \subseteq H^{\prime\prime}\alpha$. Hence $H^{\prime}\alpha \subseteq H^{\prime\prime}\alpha$. But this is a contradiction, therefore we have $H^{\prime}\alpha \subseteq H^{\prime\prime}\alpha$ for every $\alpha \in A$. The completeness of $H^{\prime\prime}\alpha$ is a corollary of the statement just proved. In fact, if $u \in H^{\prime\prime}\alpha$, then $u = H^{\prime}\beta$ for a certain $\beta < \alpha$, but then $H^{\prime}\beta \subseteq H^{\prime\prime}\beta \subseteq H^{\prime\prime}\alpha$, hence $u \subseteq H^{\prime\prime}\alpha$.

4.3. In this section we define the operations $\mathfrak{G}(U_1, U_2, \xi)$, $\mathfrak{M}(U_1, U_2, \xi)$ etc. already mentioned. We make the following convention: Let $\mathfrak{A}(\ldots, Z) = 0$ for the classes Z not being ordinal numbers, for any operation $\mathfrak{A}(\ldots, Z)$ defined in this section.

First we define the operations $\mathfrak{g}_i(X, Y, U_1, U_2, Z)$ $(i = 1, \ldots, 11)$ as follows:

4.3.1. Def.

$$\mathfrak{g}_i(X, Z, U_1, U_2, \xi) = \mathfrak{F}_i(X, Y) \quad \text{for} \quad i = 1, \ldots, 8,$$

$$\mathfrak{g}_9(X, Y, U_1, U_2, \xi) = X . On,$$

$$\mathfrak{g}_{10}(X, Y, U_1, U_2, \xi) = X . U_1^{\prime}Y,$$

$$\mathfrak{g}_{11}(X, Y, U_1, U_2, \xi) = X . U_2^{\prime}Y.$$

Now we define the operation $\mathfrak{g}(U_1, U_2, Z)$ in the following way:

4.3.2. Def. Let for every U_1, U_2, ξ $\mathfrak{g}(U_1, U_2, \xi)$ be equal to the uniquely determined set g satisfying the following formulas:

$$g \,\mathfrak{Fn}\, \xi,$$

$$\alpha \in \mathfrak{W}(J_0) . \xi \supset g^{\prime}\alpha = g^{\prime\prime}\alpha,$$

$$\alpha \in \mathfrak{W}(J_i) . \xi \supset g^{\prime}\alpha = \mathfrak{g}_i(g^{\prime}K_1^{\prime}\alpha, g^{\prime}K_2^{\prime}\alpha, U_1, U_2, \xi) \quad \text{for} \quad i = 1, \ldots, 11.$$

We have to prove the existence and the normality of $\mathfrak{g}(U_1, U_2, Z)$.

The operations $\mathfrak{g}_i(X, Y, U_1, U_2, Z)$ $(i = 1, \ldots, 11)$ being normal and $\mathfrak{g}_i(x, y, U_1, U_2, Z)$ being a set for every x, y, U_1, U_2, Z by [1], 9.1 and 4.3.1, it follows from ML_2 the existence of a normal operation $\mathfrak{g}^{+}(U_1, U_2, Z, Z_1) = g^{+}$ satisfying the following formulas:

$$g^{+} \,\mathfrak{Fn}\, \xi,$$

$$\alpha \in \mathfrak{W}(J_0) . \xi \supset g^{+\prime}\alpha = g^{+\prime\prime}\alpha,$$

$$\alpha \in \mathfrak{W}(I_i) . \xi \supset g^{+\prime}\alpha = \mathfrak{g}_i(g^{+\prime}K_1^{\prime}\alpha, g^{+\prime}K_2^{\prime}\alpha, U_1, U_2 Z) \quad \text{for} \quad i = 1, \ldots, 11.$$

Consequently, the operation $\mathfrak{g}(U_1, U_2, Z) = \mathfrak{g}^{+}(U_1, U_2, Z, Z)$ is normal and satisfies the requirements of 4.3.2.

The operations $\mathfrak{G}_i(X, Y, U_1, U_2, Z)$ are defined as follows:

4. 3. 3. DEF.

$$\mathfrak{G}_i(X, Y, U_1, U_2, \xi) = \mathfrak{F}_i(X, Y) \quad \text{for} \quad i = 1, \ldots, 8,$$
$$\mathfrak{G}_9(X, Y, U_1, U_2, \xi) = X.0 = 0,$$
$$\mathfrak{G}_{10}(X, Y, U_1, U_2, \xi) = X.U_1.\mathfrak{W}(\mathfrak{g}(U_1, U_2, \xi)),$$
$$\mathfrak{G}_{11}(X, Y, U_1, U_2, \xi) = X.U_2.\mathfrak{W}(\mathfrak{g}(U_1, U_2, \xi)).$$

Now we define the operation $\mathfrak{G}(U_1, U_2, Z)$.

4. 3. 4. DEF. For every U_1, U_2, ξ let $\mathfrak{G}(U_1, U_2, \xi)$ be equal to the class G determined by the formulas:

$$G \,\mathfrak{Fn}\, On,$$
$$\alpha < \xi \supset G^{\prime}\alpha = \mathfrak{g}(U_1, U_2, \xi)^{\prime}\alpha,$$
$$\alpha \in \mathfrak{W}(J_0) - \xi \supset G^{\prime}\alpha = G^{\prime\prime}\alpha,$$
$$\alpha \in \mathfrak{W}(J_i) - \xi \supset G^{\prime}\alpha = \mathfrak{G}_i(G^{\prime}K_1^{\prime}\alpha, G^{\prime}K_2^{\prime}\alpha, U_1, U_2, \xi) \quad \text{for} \quad i = 1, \ldots, 11.$$

The operations $\mathfrak{G}_i(X, Y, U_1, U_2, Z)$, $\mathfrak{g}(U_1, U_2, Z)$ being normal and $\mathfrak{G}_i(x, y, U_1, U_2, Z)$ being a set for every x, y, U_1, U_2, Z $(i = 1, \ldots, 11)$ by [1], 9. 4 and 4. 3. 3, there exists by ML_1 a normal operation $\mathfrak{G}^+(U_1, U_2, Z, Z_1) = G^+$ such that for every U_1, U_2, Z, ξ

$$G^+ \,\mathfrak{Fn}\, On,$$
$$\alpha < \xi \supset G^{+\prime}\alpha = \mathfrak{g}(U_1, U_2, Z)^{\prime}\alpha$$
$$\alpha \in \mathfrak{W}(J_0) - \xi \supset G^{+\prime}\alpha = G^{+\prime\prime}\alpha,$$
$$\alpha \in \mathfrak{W}(J_i) - \xi \supset G^{+\prime}\alpha = \mathfrak{G}_i(G^{+\prime}K_1^{\prime}\alpha, G^{+\prime}K_2^{\prime}\alpha, U_1, U_2, Z).$$

It results that the operation $\mathfrak{G}(U_1, U_2, Z) = \mathfrak{G}^+(U_1, U_2, Z, Z)$ is normal and satisfies the requirements of 4. 3. 4.

The operation $\mathfrak{M}(U_1, U_2, Z)$ is defined as follows:

4. 3. 5. DEF. $\mathfrak{M}(U_1, U_2, \xi) = \mathfrak{W}(\mathfrak{G}(U_1, U_2, \xi))$.

The classes $\mathfrak{M}(U_1, U_2, \xi)$ are constructed analogously to GÖDEL's L. Analogously to the "order of a constructible set" *the* U_1, U_2, ξ*-order of a set* $x \in \mathfrak{M}(U_1, U_2, \xi)$ is defined as follows: the smallest α for which $x = \mathfrak{G}(U_1, U_2, \xi)^{\prime}\alpha$ is called the U_1, U_2, ξ*-order of the set* $x \in \mathfrak{M}(U_1, U_2, \xi)$. We define the operation $\mathfrak{Od}(U_1, U_2, Z)$ in such a way that for every U_1, U_2, ξ the function $\mathfrak{Od}(U_1, U_2, \xi)$ associates the U_1, U_2, ξ-order of x to each element x of $\mathfrak{M}(U_1, U_2, \xi)$.

4. 3. 6. DEF.

$$(\langle yx\rangle \in \mathfrak{Od}(U_1, U_2, \xi) \equiv \langle xy\rangle \in \mathfrak{G}(U_1, U_2, \xi).$$
$$.(z)(z \in y \supset \sim \langle xz\rangle \in \mathfrak{G}(U_1, U_2, \xi))).\mathfrak{Rel}(\mathfrak{Od}(U_1, U_2, \xi)).$$

For every U_1, U_2, ξ there exists by M_4 [1] exactly one class $Y = \mathfrak{Od}(U_1, U_2, \xi)$ satisfying this formula, the concepts occurring on the right-hand side of the equivalence being normal. Thus the operation $\mathfrak{Od}(U_1, U_2, Z)$ exists.

It follows immediately from the definitions that

4.3.7. $\mathfrak{Od}\,\mathfrak{Fn}\,\mathfrak{M}$; $x \in \mathfrak{M} . \alpha = \mathfrak{Od}^{\prime}x \supset \mathfrak{G}^{\prime}\alpha = x.$[24]

4.3.8. $\alpha < \xi \supset \mathfrak{Comp}(\mathfrak{g}^{\prime\prime}\alpha) . \mathfrak{g}^{\prime}\alpha \subseteq \mathfrak{g}^{\prime\prime}\alpha.$

PROOF. 4.3.8 follows from ML_3. In fact, if we put $\mathfrak{H}_0 = 0$, $\xi = 0$, $A = \xi$, $\mathfrak{H}_i(X, Y) = \mathfrak{g}_i(X, Y, U_1, U_2, \xi)$ $(i = 1, \ldots, 11)$ in ML_3, then the class H defined by the formulas (6) is $\mathfrak{g}$ and the assumptions of ML_3 are fulfilled by [1], 9.1 and 4.3.1.

4.3.9. $\mathfrak{G}^{\prime}\alpha \subseteq \mathfrak{G}^{\prime\prime}\alpha$; $\mathfrak{Comp}(\mathfrak{G}^{\prime\prime}\alpha)$.

PROOF. Put

$$\mathfrak{H}_0 = \mathfrak{g}(U_1, U_2, \xi),\ \xi = \xi,\ A = On,\ \mathfrak{H}_i(X, Y) = \mathfrak{G}_i(X, Y, U_1, U_2, \xi)\ (i = 1, \ldots, 11)$$

in ML_3. Then the class H defined by the formulas (6) is $\mathfrak{G}(U_1, U_2, \xi)$ for every U_1, U_2, ξ. By [1], 9.1, 4.3.3 and 4.3.8 the assumptions of ML_3 are fulfilled. Consequently 4.3.9 follows from ML_3.

4.3.10. $y \in \mathfrak{M} . x \in y \supset x \in \mathfrak{M} .\ \mathfrak{Od}^{\prime}x < \mathfrak{Od}^{\prime}y.$

PROOF. If $y \in \mathfrak{M}$, then by 4.3.7 there is an α such that $\alpha = \mathfrak{Od}^{\prime}y$ and $\mathfrak{G}^{\prime}\alpha = y$. But $\mathfrak{G}^{\prime}\alpha \subseteq \mathfrak{G}^{\prime\prime}\alpha$ by 4.3.9 and x being an element of y we have $x \in \mathfrak{G}^{\prime\prime}\alpha$. Then $x = \mathfrak{G}^{\prime}\beta$ for a β $(\beta < \alpha)$ thus $x \in \mathfrak{M}$ and $\mathfrak{Od}^{\prime}x \leqq \beta < \alpha$.

As a corollary of 4.3.10 we have

4.3.11. $\mathfrak{Comp}(\mathfrak{M})$.[25]

On the analogy of the particular class As defined in [1], 11.8 an operation $\mathfrak{As}(U_1, U_2, Z)$ will be defined. The function $\mathfrak{As}$ associates to every non-empty set x $(x \in \mathfrak{M})$ the element y of the least order of x.

4.3.12. DEF.

$$(\langle yx \rangle \in \mathfrak{As}(U_1, U_2, \xi) \equiv y \in x . x \in \mathfrak{M}(U_1, U_2, \xi).$$
$$.(z)(\mathfrak{Od}(U_1, U_2, \xi)^{\prime}z < \mathfrak{Od}(U_1, U_2, \xi)^{\prime}y \supset \sim (z \in x))) . \mathfrak{Rel}(\mathfrak{As}(U_1, U_2, \xi)).$$

The existence of the operation $\mathfrak{As}$ similarly as in the case of the operation $\mathfrak{Od}$ is assured by M_4 [1].

4.3.13. $\mathfrak{As}\,\mathfrak{Fn}(\mathfrak{M} - \{0\})$; $\mathfrak{As}^{\prime}x \in x$ if $x \in \mathfrak{M}$ and $x \neq 0$.

[24] Here and often in the sequel the parameters U_1, U_2, ξ are omitted. If the sign of such an operation appears in a theorem without the parameters, then the theorem is true for every U_1, U_2, ξ.

[25] Similarly to [1], 9.51.

PROOF. If $\langle yx\rangle$ and $\langle zx\rangle$ are elements of $\mathfrak{As}$, then $y\in x$ and $\mathfrak{Od}'y=\mathfrak{Od}'z$ by the definition 4. 3. 12, hence $y=z$ by 4. 3. 7. It follows that $\mathfrak{As}$ is a function. $(\exists y)(\langle yx\rangle\in\mathfrak{As})$ implies that $x\in\mathfrak{M}$ and $y\in x$, hence $\mathfrak{D}(\mathfrak{As})\subseteq\mathfrak{M}-\{0\}$. On the other hand, if $x\in\mathfrak{M}-\{0\}$, then x being a non-empty subset of $\mathfrak{M}$ the set $\mathfrak{Od}''x$ is non-empty by 4. 3. 7. $\mathfrak{Od}''x$ is a set of ordinal numbers. Let α be the smallest element of it, and let y be the element of order α. Then $\langle yx\rangle\in\mathfrak{As}$, hence $x\in\mathfrak{D}(\mathfrak{As})$. It follows that $\mathfrak{D}(\mathfrak{As})=\mathfrak{M}-\{0\}$. The second part of 4. 3. 13 is a corollary of the statement just proved, for if $x\in\mathfrak{M}$ and $x\neq 0$, then $x\in\mathfrak{D}(\mathfrak{As})$, hence $\langle\mathfrak{As}'x\,x\rangle\in\mathfrak{As}$, and therefore $\mathfrak{As}'x\in x$.

4. 3. 14.

$$\alpha\in\mathfrak{W}(J_0)\supset\mathfrak{G}'\alpha=\mathfrak{G}''\alpha.$$

$$\alpha\in\mathfrak{W}(J_i)\supset\mathfrak{G}'\alpha=\mathfrak{Fi}\,(\mathfrak{G}'\beta,\mathfrak{G}'\gamma)\quad\text{for}\quad i=1,\dots,8$$

where $\beta=K_1'\alpha$, $\gamma=K_2'\alpha$, $\alpha=J_i'\langle\beta\gamma\rangle$.

PROOF. For $\alpha\geqq\xi$ the statement follows directly from the definitions 4. 3. 3, 4. 3. 4 of the operations $\mathfrak{G}_i$ and $\mathfrak{G}$.

If $\alpha<\xi$, then $\mathfrak{G}'\alpha=\mathfrak{g}'\alpha$ by 4. 3. 4 and $\beta,\gamma\leqq\alpha<\xi$ by 4. 1. 6, hence $\mathfrak{G}'\alpha=\mathfrak{g}'\alpha=\mathfrak{g}_i(\mathfrak{g}'\beta,\mathfrak{g}'\gamma)=\mathfrak{Fi}\,(\mathfrak{G}'\beta,\mathfrak{G}'\alpha)$ for $i=1,\dots,8$ by 4. 3. 1 and 4. 3. 2. $\mathfrak{G}'\beta=\mathfrak{g}'\beta$ for every $\beta<\alpha$, hence if $\alpha\in\mathfrak{W}(J_0)$, then $\mathfrak{G}'\alpha=\mathfrak{g}'\alpha=\mathfrak{g}''\alpha=\mathfrak{G}''\alpha$ by 4. 3. 2.

4. 3. 15. $\mathfrak{Fcl}(\mathfrak{M})$, i. e. $\mathfrak{M}$ is closed with respect to the fundamental operations.[26]

PROOF. Suppose $x\in\mathfrak{M}$, $y\in\mathfrak{M}$. Put $\mathfrak{Od}'x=\beta$, $\mathfrak{Od}'y=\gamma$. $\mathfrak{G}'\beta=x$, $\mathfrak{G}'\gamma=y$ by 4. 3. 7. $\mathfrak{G}'J_i'\langle\beta\gamma\rangle=\mathfrak{F}_i(x,y)$ for $i=1,\dots,8$ by 4. 3. 14, hence $\mathfrak{F}_i(x,y)\in\mathfrak{M}$ for $i=1,\dots,8$.

4. 3. 16. $\mathfrak{Au}(\mathfrak{M})$, i. e. $\mathfrak{M}$ is almost universal.[27]

PROOF. Suppose $x\subseteq\mathfrak{M}$. $\mathfrak{Od}''x$ is a set, $\mathfrak{Od}$ being a function by 4. 3. 7. $\mathfrak{Od}''x$ is a set of ordinal numbers. Therefore by [1], 7. 451 there exists an ordinal number α such that every element β of $\mathfrak{Od}''x$ is less than α and we may suppose $\alpha\in\mathfrak{W}(J_0)$. $x\subseteq\mathfrak{G}''\mathfrak{Od}''x$ by 4. 3. 7, and — $\mathfrak{Od}''x$ being a subset of α — $\mathfrak{G}''\mathfrak{Od}''x\subseteq\mathfrak{G}''\alpha$, hence $x\subseteq\mathfrak{G}''\alpha=\mathfrak{G}'\alpha$. Consequently x is a subset of an element $\mathfrak{M}$, $\mathfrak{M}$ is almost universal.

4. 3. 17.* $\psi_0(\mathfrak{M}(U_1,U_2,\xi))$. The model determined by the class $\mathfrak{M}$ satisfies the axiom system Σ for every U_1, U_2, ξ.

[26] Similarly to [1], 9. 6.
[27] Similarly to [1], 9. 63.

PROOF. $\mathfrak{M}$ is complete, closed with respect to the fundamental operations, and almost universal by 4.3.11, 4.3.15, 4.3.16. Consequently $\psi_0(\mathfrak{M})$ holds by the Lemma 3.3.

On the analogy of the particular class C defined in [1], 11.81 an operation $\mathfrak{C}(U_1, U_2, Z)$ will be defined. For every U_1, U_2, ξ the function $\mathfrak{C}(U_1, U_2, \xi)$ associates to every ordinal number α the smallest ordinal number β, which is the order of an element of $\mathfrak{G}`\alpha$, whenever the set $\mathfrak{G}`\alpha$ is non-empty.

4.3.18. DEF.

$$\mathfrak{C}(U_1, U_2, \xi)\ \mathfrak{Fn}\ On.$$

$$\mathfrak{C}(U_1, U_2, \xi)`\alpha = \mathfrak{Od}(U_1, U_2, \xi)`\mathfrak{As}(U_1, U_2, \xi)`\mathfrak{G}(U_1, U_2, \xi)`\alpha.$$

$\mathfrak{Od}`\mathfrak{As}`\mathfrak{G}`x$ is a normal term, the operations $\mathfrak{Od}, \mathfrak{As}, \mathfrak{G}$ being normal, and $\mathfrak{M}(\mathfrak{Od}`\mathfrak{As}`\mathfrak{G}`x)$ for every x by the definition of the operation $`$. Therefore, by M_5 [1] for every U_1, U_2, ξ there exists exactly one class $\mathfrak{C}$ satisfying this definition. Hence the operation $\mathfrak{C}(U_1, U_2, Z)$ exists.

4.3.19. $\mathfrak{G}`\mathfrak{C}`\alpha \in \mathfrak{G}`\alpha$ if $\mathfrak{G}`\alpha$ is non-empty.

PROOF. It follows by 4.3.13 from the assumption that $\mathfrak{As}`\mathfrak{G}`\alpha \in \mathfrak{G}`\alpha$, and therefore $\mathfrak{As}`\mathfrak{G}`\alpha \in \mathfrak{M}$ by 4.3.10. Put $x = \mathfrak{As}`\mathfrak{G}`\alpha$ and $\mathfrak{Od}`x = \beta$. $x = \mathfrak{G}`\beta$ by 4.3.7, and therefore, $\beta$ being equal to $C`\alpha$, we have $x = \mathfrak{G}`\mathfrak{C}`\alpha = \mathfrak{As}`\mathfrak{G}`\alpha$, hence $\mathfrak{G}`\mathfrak{C}`\alpha \in \mathfrak{G}`\alpha$.

4.3.20. $\mathfrak{C}`\alpha \leqq \alpha$, and if $\alpha \neq 0$, $\mathfrak{C}`\alpha < \alpha$.

PROOF. If $\mathfrak{G}`\alpha$ is non-empty, then $\mathfrak{C}`\alpha$ is the order of one of its elements, hence $\mathfrak{C}`\alpha \leqq \mathfrak{Od}`\mathfrak{G}`\alpha \leqq \alpha$ by 4.3.10. If $G`\alpha$ is empty, then $\mathfrak{G}`\alpha \notin \mathfrak{D}(\mathfrak{As})$ by 4.3.13 and thus $\mathfrak{As}`\mathfrak{G}`\alpha = 0$. Consequently, in this case $\mathfrak{C}`\alpha = \mathfrak{Od}`0 = 0$. Hence $\mathfrak{C}`\alpha \leqq \alpha$ and $\mathfrak{C}`\alpha = \alpha$ holds if and only if $\alpha = 0$.

4.4. In this section we are going to prove that all the operations defined in Section 4.3 are absolute with respect to any model Δ_M, provided that M is complete, almost universal and $\Psi_0(M)$ fulfilled.

In this section a concept is briefly called absolute if it is absolute with respect to any model determined by a complete and almost universal class M satisfying $\Psi_0(M)$. M always denotes a complete and almost universal class for which $\Psi_0(M)$.

4.4.1. $\mathfrak{M}(X)$ is absolute.

PROOF. $\mathfrak{M}_M(X) \equiv X \in M$. We have to prove $\mathfrak{M}_M(\overline{X}) \equiv \mathfrak{M}(\overline{X})$. If $\mathfrak{M}_M(\overline{X})$, then $\overline{X} \in M$, hence $\overline{X}$ is a set. Suppose, on the other hand, that $\overline{X}$ is a set. Since $\overline{X} \subseteq M$ by 2.2.1 and M is almost universal, there exists a $\overline{v}$ such that

$\bar{X} \subseteq \bar{v}$ and $\bar{v} \in M$. Thus $\bar{X}.\bar{v} = \bar{X}$ and $\bar{X}.\bar{v} \in M$ by 2.2.1. Hence $\bar{X} \in \mathfrak{M}$, i. e. $\mathfrak{M}_M(\bar{X})$.

The assumption that M is almost universal is used only in this proof. Without referring occasionally we shall make use of the fact that by the assumptions and by 4.4.1 all the concepts listed in 2.5 and 2.7 are absolute.

All the concepts in question are defined with the help of the axioms of Σ, and therefore the relativized of the concepts exist. Therefore in the case of the particular classes by AB_4 it is enough to prove the absoluteness of the defining postulates, and if $\mathfrak{A}(\ldots Z)$ is any of the operations defined in 4.3 and $\varphi(\ldots, \xi)$ is the defining postulate of $\mathfrak{A}$ for $\ldots \xi$, then the range $\mathfrak{D}(X)$ of the variable ξ being absolute, it is enough to prove by AB_6 the equivalence

$$\varphi(\ldots \bar{U}_1, \bar{U}_2, \xi) \equiv \varphi_M(\ldots \bar{U}_1, \bar{U}_2, \xi) \quad \text{for every} \quad \ldots \bar{U}_1, \bar{U}_2, \xi.$$

4.4.2. The particular classes Le, R, S, J, J_i $(i=1, \ldots, 11)$, K_1, K_2 are absolute.

4.4.2 may be proved showing step by step that the concepts appearing in the defining postulates are absolute. Although we have changed the definition of GÖDEL's S, J, J_i, K_1, K_2, the proofs are literally the same as in [1], 11.5, ..., 11.54, and so they may be omitted.

4.4.3. The operations $\mathfrak{g}_i(X, Y, U_1, U_2, Z)$ $(i=1, \ldots, 11)$ are absolute.

PROOF. By the remark made above it is enough to see that the operations $\mathfrak{F}_i(X, Y)$ $(i=1, \ldots, 8)$, $U_1{}^{\text{‘}}Y$, $U_2{}^{\text{‘}}Y$ and the particular class On are absolute.

4.4.4. The operation $\mathfrak{g}(U_1, U_2, Z)$ is absolute.

PROOF. Similarly as in 4.4.3 it is enough to see that the notions $X \in Y$, $X \mathfrak{Fn} Y$, the operations $\mathfrak{W}(X)$, $X.Y$, $X^{\text{‘}}Y$ and the bounded variable α are absolute and the same is true for the particular classes J_i, K_1, K_2 and for the operations $\mathfrak{g}_i(X, Y, U_1, U_2, Z)$ by 4.4.2 and 4.4.3.

4.4.5. The operations $\mathfrak{G}_i(X, Y, U_1, U_2, Z)$ $(i=1, \ldots, 11)$ are absolute.

PROOF. The operations $\mathfrak{F}_i(X, Y)$ $(i=1, \ldots, 8)$, $\mathfrak{W}(X)$ and the particular class 0 are absolute and the same is true for the operation $\mathfrak{g}(U_1, U_2, Z)$ by 4.4.4.

4.4.6. The operation $\mathfrak{G}(U_1, U_2, Z)$ is absolute.

PROOF. The notions $X \in Y$, $X < Y$, $X \mathfrak{Fn} Y$, the operations $\mathfrak{W}(X)$, $X - Y$, $X^{\text{‘}}Y$, $X^{\text{‘‘}}Y$, the particular class On, the variable α are absolute and the same is true for the particular classes J_i, K_1, K_2 and for the operations $\mathfrak{G}_i(X, Y, U_1, U_2, Z$ and $\mathfrak{g}(U_1, U_2, Z)$ by 4.4.2, 4.4.4 and 4.4.5.

4.4.7. The operation $\mathfrak{Mt}(U_1, U_2, Z)$ is absolute.

PROOF. $\mathfrak{W}(X)$ and $\mathfrak{G}(U_1, U_2, Z)$ are absolute by 4.4.6.

4.4.8. The operation $\mathfrak{Od}(U_1, U_2, Z)$ is absolute.

PROOF. $\mathfrak{Od}(U_1, U_2, \xi) = T$ is defined by the following formula:

(1) $(\langle yx\rangle \in T \equiv \langle xy\rangle \in \mathfrak{G}(U_1, U_2, \xi).(z)(z \in y \supset \sim \langle xz\rangle \in \mathfrak{G}(U_1, U_2, \xi))).\mathfrak{Rel}(T)$.

Let us consider the following formulas

(2) $(\langle yx\rangle \in \bar{T} \equiv \langle xy\rangle \in \mathfrak{G}(\bar{U}_1, \bar{U}_2, \xi).(z)(z \in y \supset \sim \langle xz\rangle \in \mathfrak{G}(\bar{U}_1, \bar{U}_2, \xi))).\mathfrak{Rel}(\bar{T})$,

(3) $(\langle \bar{y}\bar{x}\rangle \in \bar{T} \equiv \langle \bar{x}\bar{y}\rangle \in \mathfrak{G}(\bar{U}_1, \bar{U}_2, \xi).(\bar{z})(\bar{z} \in \bar{y} \supset \sim \langle \bar{x}\bar{z}\rangle \in \mathfrak{G}(\bar{U}_1, \bar{U}_2, \xi))).\mathfrak{Rel}(\bar{T})$.

$X \in Y$, $\mathfrak{Rel}(X)$, the operations $\langle XY\rangle$, $\mathfrak{G}(U_1 U_2, Z)$ being absolute (the last one by 4.4.6), (3) is just the relativized of (1). Therefore it is enough to prove that for every $\bar{U}_1, \bar{U}_2, \xi$ (2) holds if and only if (3) holds.

$\mathfrak{G}(\bar{U}_1, \bar{U}_2, \xi)$ is an M-class for every $\bar{U}_1, \bar{U}_2, \xi$ by AB_2. Hence if any side of the equivalence in (2) is true, either $\langle yx\rangle$ or $\langle xy\rangle$ is an M-set. But in both cases we have, by the completeness of M, that $x \in M$ and $y \in M$. Hence (2) holds if and only if the following formula holds:

$$(\langle \bar{y}\bar{x}\rangle \in \bar{T} \equiv \langle \bar{x}\bar{y}\rangle \in \mathfrak{G}(\bar{U}_1, \bar{U}_2, \xi).(z)(z \in \bar{y} \supset \sim \langle \bar{x}z\rangle \in \mathfrak{G}(\bar{U}_1, \bar{U}_2, \xi))).\mathfrak{Rel}(\bar{T}).$$

Therefore it is sufficient to prove the following formula:

$$(z)(z \in \bar{y} \supset \sim \langle \bar{x}z\rangle \in \mathfrak{G}(\bar{U}_1, \bar{U}_2, \xi)) \equiv (\bar{z})(\bar{z} \in \bar{y} \supset \sim \langle \bar{x}\bar{z}\rangle \in \mathfrak{G}(\bar{U}_1, \bar{U}_2, \xi)).$$

It is obvious that the left-hand side implies the right-hand side. The reverse implication is true too, because if z is not an M-set, $z \in \bar{y}$ is false and the implications hold vacuously.

4.4.9. The operation $\mathfrak{As}(U_1, U_2, Z)$ is absolute.

4.4.9 is to be proved by a quite similar discussion as 4.4.8 using the absoluteness of $\mathfrak{Od}$ just proved, and so we may omit the proof.

4.5. In this section we prove some results concerning the classes $\mathfrak{Mt}(U_1, U_2, \xi)$. The variables $\bar{x}, \bar{y}, \ldots$ run over $\mathfrak{Mt}$-sets, i. e. their range is $\mathfrak{Mt}(U_1, U_2, \xi)$. There is no danger of misunderstanding, for in all the theorems announced in this section U_1, U_2, ξ are arbitrary but fixed values of the parameters.

4.5.1.* $\mathfrak{Od}{}^{\backprime}\mathfrak{F}_i(\bar{x}, \bar{y}) < \omega_\alpha$ if $\mathfrak{Od}{}^{\backprime}\bar{x} < \omega_\alpha$ and $\mathfrak{Od}{}^{\backprime}\bar{y} < \omega_\alpha$ $(i = 1, \ldots, 8)$.

PROOF. Put $\beta = \mathfrak{Od}{}^{\backprime}\bar{x}$, $\gamma = \mathfrak{Od}{}^{\backprime}\bar{y}$. Then $\bar{x} = \mathfrak{G}{}^{\backprime}\beta$ and $\bar{y} = \mathfrak{G}{}^{\backprime}\gamma$ by 4.3.7. $\mathfrak{G}{}^{\backprime}J_i{}^{\backprime}\langle\beta\gamma\rangle = \mathfrak{F}_i(\bar{x}, \bar{y})$ by 4.3.14, therefore $\mathfrak{Od}{}^{\backprime}\mathfrak{F}_i(\bar{x}, \bar{y}) \leqq J_i{}^{\backprime}\langle\beta\gamma\rangle$. But $\beta < \omega_\alpha$ and $\gamma < \omega_\alpha$ by the assumption, hence $J_i{}^{\backprime}\langle\beta\gamma\rangle < \omega_\alpha$ by 4.1.7. Consequently $\mathfrak{Od}{}^{\backprime}\mathfrak{F}_i(\bar{x}, \bar{y}) < \omega_\alpha$.

4.5.2.* $\langle \bar{x}, \bar{y}\rangle \in \mathfrak{M}$, $\bar{x}.\bar{y} \in \mathfrak{M}$, $\mathfrak{Od}'\langle \bar{x}\bar{y}\rangle < \omega_\alpha$, $\mathfrak{Od}'(\bar{x}.\bar{y}) < \omega_\alpha$ if $\mathfrak{Od}'\bar{x} < \omega_\alpha$ and $\mathfrak{Od}'\bar{y} < \omega_\alpha$.

PROOF. $\langle \bar{x}\bar{y}\rangle = \{\{\bar{x}\}\{\bar{x}\bar{y}\}\}$, $\bar{x}.\bar{y} = \bar{x} - (\bar{x} - \bar{y})$ and 4.5.2 follows by the repeated applications of the cases $i = 1$, $i = 3$ of 4.5.1, respectively.

4.5.3.* $\bar{x} + \bar{y} \in \mathfrak{M}$ and $\mathfrak{Od}'\bar{x} + \bar{y} < \omega_\alpha$ if $\mathfrak{Od}'\bar{x} < \omega_\alpha$ and $\mathfrak{Od}'\bar{y} < \omega_\alpha$.

PROOF. Put $\beta = \mathfrak{Od}'\bar{x}$, $\gamma = \mathfrak{Od}'\bar{y}.\mathfrak{Max}(\{\beta, \gamma\}) < \omega_\alpha$ by the assumption, therefore by 4.1.8 there exists a δ such that $\delta \in \mathfrak{W}(J_0)$, $\mathfrak{Max}(\{\beta, \gamma\}) < \delta < \omega_\alpha$. Put $\bar{z} = \mathfrak{G}'\delta$. $\mathfrak{G}'\delta = \mathfrak{G}''\delta$ by 4.3.14. $\bar{x} = \mathfrak{G}'\beta$ and $\bar{y} = \mathfrak{G}'\gamma$ by 4.3.7, therefore $\bar{x} \in \bar{z}$ and $\bar{y} \in \bar{z}$ and $\bar{z}$ being complete by 4.3.9 $\bar{x} + \bar{y} \subseteq \bar{z}$. It follows that $\bar{x} + \bar{y} = z - ((z - \bar{x}) - \bar{y})$. $\delta < \omega_\alpha$ implies $\mathfrak{Od}'\bar{z} < \omega_\alpha$, and from the case $i = 3$ of 4.5.1 it follows that $\mathfrak{Od}'((\bar{z} - \bar{x}) < \omega_\alpha$, $\mathfrak{Od}'((z - \bar{x} - \bar{y}) < \omega_\alpha$ and, finally, $\mathfrak{Od}'\bar{x} + \bar{y} < \omega_\alpha$.

4.5.4.* The power of the set $\bar{x}$ is less than ω_α, provided $\mathfrak{Od}'\bar{x} < \omega_\alpha$.

PROOF. Put $\beta = \mathfrak{Od}'\bar{x}$. $\bar{x} = \mathfrak{G}'\beta$ by 4.3.7 and $\bar{x} \subseteq \mathfrak{G}''\beta$ by 4.3.9. $\beta < \omega_\alpha$ by the assumption, hence $\overline{\overline{\beta}} < \omega_\alpha$ by [1], 8.26. Thus $\overline{\overline{\mathfrak{G}''\beta}} < \omega_\alpha$ by [1], 8.31. Consequently $\overline{\overline{x}} < \omega_\alpha$ by [1], 8.28.

4.5.5.* If $\mathfrak{Od}'\bar{x} < \omega_{\alpha \dotplus 1}$, $\mathfrak{Od}'\bar{y} < \omega_{\alpha \dotplus 1}$, then there exists a $\delta \in \mathfrak{W}(J_0)$, $\delta < \omega_{\alpha \dotplus 1}$ such that $\bar{x} \times \bar{y} \subseteq \mathfrak{G}'\delta$.[28]

PROOF. $\overline{\overline{x}} < \omega_{\alpha \dotplus 1}$ and $\overline{\overline{y}} < \omega_{\alpha \dotplus 1}$ by 4.5.4, hence $\overline{\overline{\bar{x} \times \bar{y}}} < \omega_{\alpha \dotplus 1}$ by [1], 8.6.3, and therefore $\overline{\overline{\mathfrak{Od}''(\bar{x} \times \bar{y})}} < \omega_{\alpha \dotplus 1}$ by [1], 8.31. Suppose $z \in \bar{x} \times \bar{y}$. Then $z = \langle uv\rangle$, $u \in \bar{x}$, $v \in \bar{y}$, therefore $u \in \mathfrak{M}$, $v \in \mathfrak{M}$, $\mathfrak{Od}'u < \omega_{\alpha \dotplus 1}$ and $\mathfrak{Od}'v < \omega_{\alpha \dotplus 1}$ by 4.3.10. It follows from 4.5.2 that $z \in \mathfrak{M}$ and $\mathfrak{Od}'z < \omega_{\alpha \dotplus 1}$. Hence we have $\mathfrak{Od}''(\bar{x} \times \bar{y}) \subseteq \omega_{\alpha \dotplus 1}$. It results by 2.1.10 that there exists a $\delta < \omega_{\alpha \dotplus 1}$ such that $\mathfrak{Od}''(\bar{x} \times \bar{y}) \subseteq \delta$ and by 4.1.8 we may suppose $\delta \in \mathfrak{W}(J_0)$. $\bar{x} \times \bar{y} \subseteq$ $\subseteq \mathfrak{G}''\mathfrak{Od}''(\bar{x} \times \bar{y})$ by 4.3.7, consequently $\bar{x} \times \bar{y} \subset \mathfrak{G}''\delta = \mathfrak{G}'\delta$.

4.5.6.* $\overline{\overline{\mathfrak{G}''\omega_\alpha}} = \omega_\alpha$.

PROOF. $\overline{\overline{\mathfrak{G}''\omega_\alpha}} \leqq \omega_\alpha$ by 4.5.4. On the other hand, the function $\mathfrak{G}$ is one-to-one on the set $\mathfrak{W}(J_0).\omega_\alpha$, for if $\beta, \gamma \in \mathfrak{W}(J_0).\omega_\alpha$ and e. g. $\beta < \gamma$, then $\mathfrak{G}'\beta \in \mathfrak{G}'\gamma$, hence $\mathfrak{G}'\beta \neq \mathfrak{G}'\gamma$. Consequently $\overline{\overline{\mathfrak{G}''\omega_\alpha}} \geqq \overline{\overline{\mathfrak{G}''(\omega_\alpha.\mathfrak{W}(J_0))}} = \overline{\overline{\omega_\alpha.\mathfrak{W}(J_0)}}$. But $\omega_\alpha.\mathfrak{W}(J_0) = J_0''\omega_\alpha^2$ and the function J_0 is one-to-one too, hence $\overline{\overline{\mathfrak{G}''\omega_\alpha}} \geqq \overline{\overline{\omega_\alpha^2}} = \omega_\alpha$. It follows that $\overline{\overline{\mathfrak{G}''\omega_\alpha}} = \omega_\alpha$.

[28] 4.5.5 remains true even if we write α instead of $\alpha \dotplus 1$ and it can be proved that $\mathfrak{Od}'\bar{x} \times \bar{y} < \omega_\alpha$. It is obvious that a more general theorem may be formulated and proved instead of the special cases proved here. But we are not in the need of such a general theorem.

4.5.7.*

$$\text{If } \alpha<\xi, \begin{cases} \mathfrak{G}{}^{\backprime}\alpha=\mathfrak{G}{}^{\backprime}\beta.On & \text{for } \alpha\in\mathfrak{W}(J_9),\\ \mathfrak{G}{}^{\backprime}\alpha=\mathfrak{G}{}^{\backprime}\beta.U_1{}^{\backprime}\mathfrak{G}{}^{\backprime}\gamma & \text{for } \alpha\in\mathfrak{W}(J_{10}),\\ \mathfrak{G}{}^{\backprime}\alpha=\mathfrak{G}{}^{\backprime}\beta.U_2{}^{\backprime}\mathfrak{G}{}^{\backprime}\gamma & \text{for } \alpha\in\mathfrak{W}(J_{11}).\end{cases}$$

$$\text{If } \alpha\geqq\xi, \begin{cases} \mathfrak{G}{}^{\backprime}\alpha=\mathfrak{G}{}^{\backprime}\beta.0=0 & \text{for } \alpha\in\mathfrak{W}(J_9),\\ \mathfrak{G}{}^{\backprime}\alpha=\mathfrak{G}{}^{\backprime}\beta.U_1.\mathfrak{G}{}^{\backprime\backprime}\xi & \text{for } \alpha\in\mathfrak{W}(J_{10}),\\ \mathfrak{G}{}^{\backprime}\alpha=\mathfrak{G}{}^{\backprime}\beta.U_2.\mathfrak{G}{}^{\backprime\backprime}\xi & \text{for } \alpha\in\mathfrak{W}(J_{11})\end{cases}$$

where $\beta=K_1{}^{\backprime}\alpha$, $\gamma=K_2{}^{\backprime}\alpha$, $\alpha=J_i{}^{\backprime}\langle\beta,\gamma\rangle$ $(i=9, 10, 11)$.

PROOF. If $\alpha<\xi$, then $\mathfrak{G}{}^{\backprime}\alpha=g{}^{\backprime}\alpha$ by 4.3.4 and $\beta,\gamma\leqq\alpha<\xi$ by 4.1.6 and the statement follows from the definitions 4.3.1, 4.3.2. If $\alpha\geqq\xi$, the theorem is true by 4.3.3 and 4.3.4, taking into consideration that $\mathfrak{g}{}^{\backprime\backprime}\xi=\mathfrak{G}{}^{\backprime\backprime}\xi$.

4.5.8.* If, in particular, the parameter ξ is equal to a cardinal number ω_δ, then

$$\text{if } \mathfrak{Max}(\{\beta,\gamma\})<\omega_\delta=\xi, \begin{cases} \mathfrak{G}{}^{\backprime}J_9{}^{\backprime}\langle\beta\gamma\rangle=\mathfrak{G}{}^{\backprime}\beta.On,\\ \mathfrak{G}{}^{\backprime}J_{10}{}^{\backprime}\langle\beta\gamma\rangle=\mathfrak{G}{}^{\backprime}\beta.U_1\mathfrak{G}{}^{\backprime}\gamma,\\ \mathfrak{G}J_{11}{}^{\backprime}\langle\beta\gamma\rangle=\mathfrak{G}{}^{\backprime}\beta.U_2\mathfrak{G}{}^{\backprime}\gamma;\end{cases}$$

$$\text{if } \mathfrak{Max}(\{\beta\gamma\})\geqq\omega_\delta=\xi, \begin{cases} \mathfrak{G}{}^{\backprime}J_9{}^{\backprime}\langle\beta\gamma\rangle=\mathfrak{G}{}^{\backprime}\beta.0=0,\\ \mathfrak{G}{}^{\backprime}J_{10}{}^{\backprime}\langle\beta\gamma\rangle=\mathfrak{G}{}^{\backprime}\beta.U_1.\mathfrak{G}{}^{\backprime\backprime}\omega_\delta,\\ \mathfrak{G}{}^{\backprime}J_{11}{}^{\backprime}\langle\beta\gamma\rangle=\mathfrak{G}{}^{\backprime}\beta.U_2.\mathfrak{G}{}^{\backprime\backprime}\omega_\delta.\end{cases}$$

4.5.8 follows immediately from 4.5.7, taking into consideration that $\mathfrak{Max}(\{\beta\gamma\})<\omega_\alpha\equiv J_i{}^{\backprime}\langle\beta\gamma\rangle<\omega_\alpha$ for $i<12$ by 4.1.6 and 4.1.7.

4.5.9.* If $\xi=\omega_{\delta\dot{+}1}$ and $\alpha<\omega_{\delta\dot{+}1}$, then $\mathfrak{Od}{}^{\backprime}\alpha<\omega_{\delta\dot{+}1}$.[29]

PROOF. The property of α, $\alpha<\omega_{\delta\dot{+}1}\supset\mathfrak{Od}{}^{\backprime}\alpha<\omega_{\delta\dot{+}1}$ is to be proved by induction on α. Suppose that the statement is true for every $\beta<\alpha$ where $\alpha<\omega_{\delta\dot{+}1}$. From the assumption $\beta<\alpha$ it follows that $\beta<\omega_{\delta\dot{+}1}$, hence $\mathfrak{Od}{}^{\backprime}\beta<$ $<\omega_{\delta\dot{+}1}$. Consequently, $\mathfrak{Od}{}^{\backprime\backprime}\alpha\subseteq\omega_{\delta\dot{+}1}$ and $\overline{\overline{\mathfrak{Od}{}^{\backprime\backprime}\alpha}}\leqq\overline{\overline{\alpha}}<\omega_{\delta\dot{+}1}$ by [1], 8.26 and [1], 8.31. Then by 2.1.10 there exists an ordinal number γ $(\gamma<\omega_{\delta\dot{+}1})$ such that $\mathfrak{Od}{}^{\backprime\backprime}\alpha\subset\gamma$ and by 4.1.8 we may suppose $\gamma\in\mathfrak{W}(I_0)$. Hence $\alpha\subseteq\mathfrak{G}{}^{\backprime\backprime}\mathfrak{Od}{}^{\backprime\backprime}\alpha\subseteq$ $\subseteq\mathfrak{G}{}^{\backprime\backprime}\gamma=\mathfrak{G}{}^{\backprime}\gamma$. $\mathfrak{Max}(\{\gamma 0\})$ being less than $\omega_{\delta\dot{+}1}$, $\mathfrak{G}{}^{\backprime}I_9{}^{\backprime}\langle\gamma 0\rangle=\mathfrak{G}{}^{\backprime}\gamma.On$ by 4.5.8. Put $\bar{x}=\mathfrak{G}{}^{\backprime}\gamma.On$. $\mathfrak{Od}{}^{\backprime}\bar{x}\leqq I_9{}^{\backprime}\langle\gamma 0\rangle<\omega_{\delta\dot{+}1}$ by 4.1.7. It is obvious that $\alpha\subseteq\bar{x}$. But $\bar{x}$ is the intersection of two complete sets, is complete by [1],

[29] 4.5.9 remains true if we write δ instead of $\delta\dot{+}1$. More generally, it remains true for arbitrary ξ and ω_δ (not depending on ξ) even if the operation $X.On$ is omitted from the construction of the function $\mathfrak{G}$. However, we do not need this general theorem and the proof of 4.5.9 is more simple with the help of the operation $X.On$.

6. 6. 5, and therefore $\bar{x}$, as a set of ordinal numbers, is an ordinal number, too, by [1], 7. 1. It follows that either $\alpha \in \bar{x}$ or $\alpha = \bar{x}$. But $\mathfrak{Od}`\bar{x} < \omega_{\delta \dotplus 1}$ in both cases implies by 4. 3. 10 that $\mathfrak{Od}`\alpha < \omega_{\delta \dotplus 1}$.

4. 6. In this section we are going to define the functions h, k already mentioned.

First we define a notion $\mathfrak{h}(U_1, X, Y)$ as follows:

4. 6. 1.* DEF. $\mathfrak{h}(U_1, X, Y) \equiv U_1 \, \mathfrak{Fn} \, \omega_{X \dotplus Y \dotplus 1} . \mathfrak{Un}_2(U_1) . \mathfrak{W}(U_1) \subseteq \mathfrak{P}(\omega_X)$.

4. 6. 2.** $(\exists u_1)\mathfrak{h}(u_1, \lambda, \nu)$.

PROOF. $\omega_\lambda = \aleph_\lambda$ and $2^{\aleph_\lambda} = \overline{\overline{\mathfrak{P}(\omega_\lambda)}}$. By the assumption of the Theorem 1, $2^{\aleph_\lambda} \geqq \aleph_{\lambda \dotplus \nu \dotplus 1} = \omega_{\lambda \dotplus \nu \dotplus 1}$. But $2^{\aleph_\lambda}$, being a cardinal number, is an ordinal number, hence $\omega_{\lambda \dotplus \nu \dotplus 1} \subseteq 2^{\aleph_\lambda}$. By [1], 8. 24 and 8. 1 there exists an u such that u is one-to-one, $\mathfrak{D}(u_1) = 2^{\aleph_\lambda}$, and $\mathfrak{W}(u_1) = \mathfrak{P}(\omega_\lambda)$. Put $u_1 = u \upharpoonright \omega_{\lambda \dotplus \nu \dotplus 1}$. It is obvious that $\mathfrak{h}(u_1, \lambda, \nu)$ holds. On the other hand, $(\exists u_1)\mathfrak{h}(u_1, \lambda, \nu)$ implies that $\omega_{\lambda \dotplus \nu \dotplus 1} \leqq \overline{\overline{\mathfrak{P}(\omega_\lambda)}} = 2^{\aleph_\lambda}$, hence we have

4. 6. 3.* $(\exists u_1)\mathfrak{h}(u_1, \lambda, \nu) \equiv 2^{\aleph_\lambda} \geqq \aleph_{\lambda \dotplus \nu \dotplus 1}$.

Let h denote a set for which $\mathfrak{h}(h, \lambda, \nu)$ holds. Such a h exists by 4. 6. 2 and h is a one-to-one mapping of the set $\omega_{\lambda \dotplus \nu \dotplus 1}$ onto a subset of $\mathfrak{P}(\omega_\lambda)$.

Now we define $\mathfrak{k}(U_2, X, Y)$ as follows:

4. 6. 4.* DEF. $\mathfrak{k}(U_2, X, Y) \equiv U_2 \, \mathfrak{Fn} \, \omega_{X \dotplus Y \dotplus 1} . (\alpha)(\alpha \in \omega_{X \dotplus Y \dotplus 1} \supset \mathfrak{D}(U_2`\alpha) =$
$= \alpha . \mathfrak{W}(U_2`\alpha) = \bar{\alpha} . \mathfrak{Un}_2(U_2`\alpha) . \mathfrak{Rel}(U_2`\alpha))$.

4. 6. 4. 1* $(\exists u_2)\mathfrak{k}(u_2, \lambda, \nu)$.

PROOF. Let A be a function defined on the class $V - \{0\}$ which associates to every non-empty set x one of its element. Such an A exists by axiom E. Let B be the class defined by the formula

$$(\langle xy \rangle \in B \equiv (z)(z \in x \equiv \mathfrak{D}(z) = y . \mathfrak{W}(z) = \bar{y} . \mathfrak{Un}_2(z) . \mathfrak{Rel}(z))) . \mathfrak{Rel}(B).$$

The existence of B is assured by M_4 [1]. But if y is arbitrary, there exists, by M_4 [1], exactly one class X such that

$$(1) \qquad (z)(z \in X \equiv \mathfrak{D}(z) = y . \mathfrak{W}(z) = \bar{y} . \mathfrak{Un}_2(z) . \mathfrak{Rel}(z)).$$

If X satisfies (1), then $z \subseteq y \times \bar{y}$ for every $z \in X$, hence $X \subseteq \mathfrak{P}(y \times \bar{y})$, and therefore X is a set by [1], 5. 12 and 5. 18.

It follows that B is a function which associates to every set y the set of all one-to-one mappings of y and $\bar{y}$. Put $C = A | B$. The set $B`y$ is nonempty for every $y$ by [1], 8. 24, hence $\mathfrak{W}(B) \subseteq \mathfrak{D}(A)$. Therefore $C \, \mathfrak{Fn} \, V$ and $C`y \in B`y$, hence $\mathfrak{D}(C`y) = y$, $\mathfrak{W}(C`y) = \bar{y}$, $\mathfrak{Un}_2(C`y)$, $\mathfrak{Rel}(C`y)$ for every y. $u_2 = C \upharpoonright \omega_{\lambda \dotplus \nu \dotplus 1}$ is a set and $\mathfrak{k}(u_2, \lambda, \nu)$ holds.

Let k denote a set satisfying $\mathfrak{k}(k, \lambda, \nu)$. Now we make the convention different from that used in Section 3. If the sign of an operation depending on the parameters U_1, U_2, ξ appears anywhere without writing out its arguments, then it denotes the value of this operation for $U_1 = h$, $U_2 = k$, $\xi = \omega_{\lambda \dotplus \nu \dotplus 1}$, e. g.: $\mathfrak{G} = \mathfrak{G}(h, k, \omega_{\lambda \dotplus \nu \dotplus 1})$, $\mathfrak{M} = \mathfrak{M}(h, k, \omega_{\lambda \dotplus \nu \dotplus 1})$. Corresponding to this convention the subscript $\mathfrak{M}$ indicates the relativization with respect to the model $\Delta_{\mathfrak{M}}$ and $\bar{x}, \bar{y}, \ldots, X, Y, \ldots$ denote the set — and class — variables of this model.

Now we are going to prove that h, k are $\mathfrak{M}$-sets. We need some preliminary results.

4. 6. 5.** If $\alpha < \omega_{\lambda \dotplus \nu \dotplus 1}$, then $h^{\backprime}\alpha \in \mathfrak{M}$ and $\mathfrak{Od}^{\backprime}h^{\backprime}\alpha < \omega_{\lambda \dotplus \nu \dotplus 1}$.

PROOF. $h^{\backprime}\alpha \in \mathfrak{W}(h)$ and thus $h^{\backprime}\alpha \in \mathfrak{P}(\omega_\lambda)$ by 4. 6. 1, hence $h^{\backprime}\alpha \subseteq \omega_\lambda$. $\omega_\lambda < \omega_{\lambda \dotplus \nu \dotplus 1}$, and therefore $\mathfrak{Od}^{\backprime}\omega_\lambda < \omega_{\lambda \dotplus \nu \dotplus 1}$ and $\mathfrak{Od}^{\backprime}\alpha < \omega_{\lambda \dotplus \nu \dotplus 1}$ by 4. 5. 9. Put $\beta = \mathfrak{Od}^{\backprime}\omega_\lambda$, $\gamma = \mathfrak{Od}^{\backprime}\alpha$. Then $\mathfrak{G}^{\backprime}\beta = \omega_\lambda$, $\mathfrak{G}^{\backprime}\gamma = \alpha$ by 4.3.7 and $\mathfrak{G}^{\backprime}J_{10}^{\backprime}\langle\beta\gamma\rangle = \mathfrak{G}^{\backprime}\beta . h^{\backprime}\mathfrak{G}^{\backprime}\gamma = \omega_\lambda . h^{\backprime}\alpha = h^{\backprime}\alpha$ by 4. 5. 8 and thus $h^{\backprime}\alpha \in \mathfrak{M}$. But $J_{10}^{\backprime}\langle\beta\gamma\rangle < \omega_{\lambda \dotplus \nu \dotplus 1}$ by 4. 1. 7, therefore $\mathfrak{Od}^{\backprime}h^{\backprime}\alpha < \omega_{\lambda \dotplus \nu \dotplus 1}$.

4. 6. 6.* If $\alpha < \omega_{\lambda \dotplus \nu \dotplus 1}$, then $k^{\backprime}\alpha \in \mathfrak{M}$ and $\mathfrak{Od}^{\backprime}k^{\backprime}\alpha < \omega_{\lambda \dotplus \nu \dotplus 1}$.

PROOF. $k^{\backprime}\alpha \subseteq \alpha \times \overline{\overline{\alpha}}$ by 4. 6. 4. $\overline{\overline{\alpha}} \leqq \alpha$ by [1], 8. 26, hence $\overline{\overline{\alpha}}$ being an ordinal number we have $\mathfrak{Od}^{\backprime}\overline{\overline{\alpha}} < \omega_{\lambda \dotplus \nu \dotplus 1}$ and $\mathfrak{Od}^{\backprime}\alpha < \omega_{\lambda \dotplus \nu \dotplus 1}$ by 4. 5. 9. Then by 4. 5. 5 there exists a $\beta < \omega_{\lambda \dotplus \nu \dotplus 1}$ such that $\alpha \times \overline{\overline{\alpha}} \subseteq \mathfrak{G}^{\backprime}\beta$. Put $\gamma = \mathfrak{Od}^{\backprime}\alpha$. Then $\gamma < \omega_{\lambda \dotplus \nu \dotplus 1}$ and $\mathfrak{G}^{\backprime}\gamma = \alpha$ by 4. 3. 7. Hence $\mathfrak{G}^{\backprime}I_{11}^{\backprime}\langle\beta\gamma\rangle = \mathfrak{G}^{\backprime}\beta . k^{\backprime}\mathfrak{G}^{\backprime}\gamma = k^{\backprime}\alpha$ by 4. 5. 8. It follows that $k^{\backprime}\alpha \in \mathfrak{M}$ and using 4. 1. 7 $\beta, \gamma < \omega_{\lambda \dotplus \nu \dotplus 1}$ implies $\mathfrak{Od}^{\backprime}k^{\backprime}\alpha < \omega_{\lambda \dotplus \nu \dotplus 1}$.

4. 6. 7.** $h \subseteq \mathfrak{G}^{\backprime\backprime}\omega_{\lambda \dotplus \nu \dotplus 1}$, $k \subseteq \mathfrak{G}^{\backprime\backprime}\omega_{\lambda \dotplus \nu \dotplus 1}$.

PROOF. h and k are functions defined on $\omega_{\lambda \dotplus \nu \dotplus 1}$ by 4. 6. 1 and 4. 6. 4, thus the elements of h and k are the ordered pairs $\langle h^{\backprime}\alpha\alpha\rangle$ and $\langle k^{\backprime}\alpha\alpha\rangle$ for $\alpha < \omega_{\lambda \dotplus \nu \dotplus 1}$, respectively. But $h^{\backprime}\alpha$, $k^{\backprime}\alpha$ are elements of $\mathfrak{M}$ and $\mathfrak{Od}^{\backprime}\alpha$, $\mathfrak{Od}^{\backprime}h^{\backprime}\alpha$, $\mathfrak{Od}^{\backprime}k^{\backprime}\alpha < \omega_{\lambda \dotplus \nu \dotplus 1}$ by 4. 5. 9, 4. 6. 5 and 4. 6. 6, hence $\langle h^{\backprime}\alpha\alpha\rangle \in M$, $\langle k^{\backprime}\alpha\alpha\rangle \in \mathfrak{M}$ and $\mathfrak{Od}^{\backprime}\langle h^{\backprime}\alpha\alpha\rangle < \omega_{\lambda \dotplus \nu \dotplus 1}$, $\mathfrak{Od}^{\backprime}\langle k^{\backprime}\alpha\alpha\rangle < \omega_{\lambda \dotplus \nu \dotplus 1}$ by 4. 5. 2. This means that $\langle h^{\backprime}\alpha\alpha\rangle \in \mathfrak{G}^{\backprime\backprime}\omega_{\lambda \dotplus \nu \dotplus 1}$, $\langle k^{\backprime}\alpha\alpha\rangle \in \mathfrak{G}^{\backprime\backprime}\omega_{\lambda \dotplus \nu \dotplus 1}$, hence $h, k \subseteq \mathfrak{G}^{\backprime\backprime}\omega_{\lambda \dotplus \nu \dotplus 1}$.

4. 6. 8.** $h, k \in \mathfrak{M}$.

PROOF. $\mathfrak{Max}(\{\omega_{\lambda \dotplus \nu \dotplus 1}0\}) = \omega_{\lambda \dotplus \nu \dotplus 1}$, $\omega_{\lambda \dotplus \nu \dotplus 1} \in \mathfrak{M}(J_0)$ by 4. 1. 7, and therefore $\mathfrak{G}^{\backprime}\omega_{\lambda \dotplus \nu \dotplus 1} = \mathfrak{G}^{\backprime\backprime}\omega_{\lambda \dotplus \nu \dotplus 1}$ by 4. 3. 14. We get from 4. 5. 8

$$\mathfrak{G}^{\backprime}J_{10}^{\backprime}\langle\omega_{\lambda \dotplus \nu \dotplus 1}0\rangle = \mathfrak{G}^{\backprime\backprime}\omega_{\lambda \dotplus \nu \dotplus 1} . h . \mathfrak{G}^{\backprime\backprime}\omega_{\lambda \dotplus \nu \dotplus 1},$$

$$\mathfrak{G}^{\backprime}J_{11}^{\backprime}\langle\omega_{\lambda \dotplus \nu \dotplus 1}0\rangle = \mathfrak{G}^{\backprime}\omega_{\lambda \dotplus \nu \dotplus 1} . k . \mathfrak{G}^{\backprime\backprime}\omega_{\lambda \dotplus \nu \dotplus 1}.$$

It follows from 4.6.7 that $\mathfrak{G}^{\backprime}J_{10}^{\backprime}\langle\omega_{\lambda \dotplus \nu \dotplus 1}0\rangle = h$, $\mathfrak{G}^{\backprime}J_{11}^{\backprime}\langle\omega_{\lambda \dotplus \nu \dotplus 1}0\rangle = k$, and therefore $h, k \in \mathfrak{M}$.

Now we are going to prove that the model determined by the class $\mathfrak{M}$ satisfies axiom E. The notions $X \in Y$, $\mathfrak{Em}(X)$, $\mathfrak{Un}(X)$ and the operation $\langle XY \rangle$ being absolute with respect to $\Delta_{\mathfrak{M}}$ by 2.2.5, 2.2.7, 4.3.11, 4.3.16 and 4.3.17, we have to prove

4.6.9.** $(E)_{\mathfrak{M}}$; $(\exists \bar{A})(\mathfrak{Un}(\bar{A}).(\bar{x})(\sim \mathfrak{Em}(\bar{x}) \supset (\exists \bar{y})(\langle \bar{y}\bar{x} \rangle \in \bar{A}.\bar{y} \in \bar{x})))$.

PROOF. The operation $\mathfrak{As}(U_1, U_2, Z)$ is absolute with respect to $\Delta_{\mathfrak{M}}$ by by 4.3.11, 4.3.16, 4.3.17 and 4.4.9. But $h, k, \omega_{\lambda \dot{+} \nu \dot{+} 1}$ being $\mathfrak{M}$-sets (h and k by 4.6.8), the class $\mathfrak{As} = \mathfrak{As}(h, k, \omega_{\lambda \dot{+} \nu \dot{+} 1})$ is an $\mathfrak{M}$-class by the metatheorem AB_2.

The $\mathfrak{M}$-class $\bar{A} = \mathfrak{As}$ satisfies $(E)_{\mathfrak{M}}$. In fact, $\mathfrak{As}\,\mathfrak{Fn}(\mathfrak{M} - \{0\})$ by 4.3.13 and $\mathfrak{As}`\bar{x} \in \bar{x}$ for every $\bar{x} \in \mathfrak{M} - \{0\}$, i. e. for every $\bar{x}$, provided $\sim \mathfrak{Em}(\bar{x})$. It follows that $\mathfrak{Un}(\mathfrak{As})$ and that for every non-empty $\bar{x}$ there exists and $\bar{y}$, e. g. $= \mathfrak{As}`\bar{x}$, such that $\langle \bar{y}\bar{x} \rangle \in \mathfrak{As}$ and $\bar{y} \in \bar{x}$.

4.6.10.** $\Psi_0^*(\mathfrak{M})$, i. e. the axioms of Σ^* hold in $\Delta_{\mathfrak{M}}$.

4.6.10 follows from 4.3.17 and 4.6.9.

4.7. In this section we are going to prove that the class $\mathfrak{M}$ satisfies the formula $(+)$ and we are going to reduce the proof of Theorem 1 to the proof of a theorem analogous to GÖDEL's theorem which asserts that $V = L$ implies $(\alpha)(2^{\aleph_\alpha} = \aleph_{\alpha \dot{+} 1})$. The proof of this theorem will be given in Sections 4.8, 4.9, 4.10.

We have proved in 4.6.10 $\psi_0^*(\mathfrak{M})$, and therefore the relativized of the concepts defined in Σ^* exist and the relativized of the theorems proved in Σ^* hold. $\mathfrak{M}$ is complete and almost universal by 4.3.11 and 4.3.16, consequently the concepts listed in 2.2.5, 2.2.7 are absolute with respect to $\Delta_{\mathfrak{M}}$ and the equalities 2.2.6 hold for $\mathfrak{M}$. We shall make use of these facts without referring to them. In what follows instead of "absolute with respect to $\Delta_{\mathfrak{M}}$" we briefly say absolute. First we need some preliminary results.

4.7.1.** $\bar{u} \simeq \bar{v}$ if $\bar{u} \simeq_{\mathfrak{M}} \bar{v}$.

PROOF. $\bar{u} \simeq_{\mathfrak{M}} \bar{v} \equiv (\exists \bar{w})(\mathfrak{D}(\bar{w}) = \bar{u}.\mathfrak{W}(\bar{w}) = \bar{v}.\mathfrak{Un}_2(\bar{w}).\mathfrak{Rel}(\bar{w}))$, since $\bar{u} \simeq v \equiv$ $\equiv (\exists w)(\mathfrak{D}(w) = \bar{u}.\mathfrak{W}(w) = \bar{v}.\mathfrak{Un}_2(w).\mathfrak{Rel}(w))$ by [1], 8.12 and $\mathfrak{Un}_2(X)$, $\mathfrak{Rel}(X)$, $\mathfrak{D}(X)$, $\mathfrak{W}(X)$ are absolute.

Hence $\bar{u} \simeq_{\mathfrak{M}} \bar{v}$ implies $\bar{u} \simeq \bar{v}$.

4.7.2.** $N \subseteq N_{\mathfrak{M}}$.

PROOF. $N \subseteq On$ by [1], 8.22, therefore, On being absolute, $N \subseteq \mathfrak{M}$. Hence we have to prove that $\bar{x} \in N$ implies $\bar{x} \in N_{\mathfrak{M}}$.

$\bar{x} \in N \equiv \mathfrak{Ord}(\bar{x}).(u)(u \in \bar{x} \supset \sim u \simeq \bar{x})$ (see [1], p. 31). It follows that $\bar{x} \in N_{\mathfrak{M}} \equiv \mathfrak{Ord}(\bar{x}).(\bar{u})(\bar{u} \in \bar{x} \supset \sim \bar{u} \simeq_{\mathfrak{M}} \bar{x})$. Suppose $\sim \bar{x} \in N_{\mathfrak{M}}$. Then either

$\sim \mathfrak{Ord}(\bar{x})$, hence $\sim \bar{x} \in N$), or there exists a $\bar{u}$ such that $\bar{u} \simeq_{\mathfrak{M}} \bar{x}$ and $\bar{u} \in \bar{x}$. Hence $\bar{u} \simeq \bar{x}$ by 4.7.1, and therefore $\sim \bar{x} \in N$.

4.7.3.** $\aleph_{\mathfrak{M}}\, \mathfrak{Fn}\, On$; $\aleph_{\mathfrak{M}}{}^{\backprime}\alpha \leqq \aleph^{\backprime}\alpha$.

PROOF. $\aleph\, \mathfrak{Fn}\, On$ by [1], 8.57, and therefore $\aleph_{\mathfrak{M}}\, \mathfrak{Fn}\, On$, since $X\, \mathfrak{Fn}\, Y$ and On are absolute. Further $\aleph_{\mathfrak{M}}\, \mathfrak{Isom}_{E.\mathfrak{M}, E.\mathfrak{M}}(On, N'_{\mathfrak{M}})$ by [1], 8.57, $X\, \mathfrak{Isom}_{R,S}(Y, Z)$ being absolute by 2.2.8. It follows that $G = \aleph \mid \aleph_{\mathfrak{M}}^{-1}$ is a strictly monotonic mapping of the class $N'_{\mathfrak{M}}$ onto N, i. e. $G^{\backprime}\beta < G^{\backprime}\gamma$ if $\beta < \gamma$, $\beta, \gamma \in N'_{\mathfrak{M}}$.

On the other hand $N' = N - \omega$, hence $N'_{\mathfrak{M}} = N_{\mathfrak{M}} - \omega$ and thus $N' \subseteq N'_{\mathfrak{M}}$ by 4.7.2. Therefore by [1], 7.611 $G^{\backprime}\beta \geqq \beta$ for every $\beta \in N'_{\mathfrak{M}}$. Put $\beta = \aleph_{\mathfrak{M}}{}^{\backprime}\alpha$. Then $\beta \in N'_{\mathfrak{M}}$ and $G^{\backprime}\beta = G^{\backprime}\aleph_{\mathfrak{M}}{}^{\backprime}\alpha = \aleph^{\backprime}(\aleph_{\mathfrak{M}}^{-1}{}^{\backprime}(\aleph_{\mathfrak{M}}{}^{\backprime}\alpha)) = \aleph^{\backprime}\alpha \geqq \beta = \aleph_{\mathfrak{M}}{}^{\backprime}\alpha$.

4.7.4.** $\gamma \simeq_{\mathfrak{M}} \bar{\bar{\gamma}}$ if $\gamma < \omega_{\lambda \dotplus \nu \dotplus 1}$.

PROOF. $\gamma \simeq_{\mathfrak{M}} \bar{\bar{\gamma}} \equiv (\exists \bar{w})(\mathfrak{D}(\bar{w}) = \gamma . \mathfrak{W}(\bar{w}) = \bar{\bar{\gamma}} . \mathfrak{Un}_2(\bar{w}) . \mathfrak{Rel}(\bar{w}))$ by [1], 8.12. $k^{\backprime}\gamma \in \mathfrak{M}$ by 4.6.6. Put $k^{\backprime}\gamma = \bar{w}$. Then $\mathfrak{D}(\bar{w}) = \gamma$, $\mathfrak{W}(\bar{w}) = \bar{\bar{\gamma}}$, $\mathfrak{Un}_2(\bar{w})$, $\mathfrak{Rel}(\bar{w})$ by 4.6.4.

Now we can prove that $\mathfrak{M}$ satisfies $(+)$, i. e.

4.7.5.** $\alpha \leqq \lambda \dotplus \nu + 1 \supset \aleph_{\mathfrak{M}}{}^{\backprime}\alpha = \aleph^{\backprime}\alpha$.

PROOF. The formula appearing in 4.7.5 being normal we may prove this by transfinite induction on α. $\aleph^{\backprime}0 = \omega$, $\aleph_{\mathfrak{M}}{}^{\backprime}0 = \omega_{\mathfrak{M}}$, hence ω being absolute we have $\aleph^{\backprime}0 = \aleph_{\mathfrak{M}}{}^{\backprime}0$. $X \leqq Y$ being absolute, $\omega = \aleph_{\mathfrak{M}}{}^{\backprime}0 \leqq \aleph_{\mathfrak{M}}{}^{\backprime}\alpha$, for every α, hence $\aleph_{\mathfrak{M}}{}^{\backprime}\alpha$ is infinite. Therefore $\overline{\overline{\aleph_{\mathfrak{M}}{}^{\backprime}\alpha}} = \aleph^{\backprime}\delta$ for a δ. Suppose that α is the least ordinal number for which $\aleph_{\mathfrak{M}}{}^{\backprime}\alpha \neq \aleph^{\backprime}\alpha$. We have to prove that $\alpha \leqq \lambda \dotplus \nu \dotplus 1$ leads to a contradiction.

Put $\gamma = \aleph_{\mathfrak{M}}{}^{\backprime}\alpha$. $\gamma \leqq \aleph^{\backprime}\alpha$ by 4.7.3, hence $\gamma < \aleph^{\backprime}\alpha \leqq \aleph^{\backprime}\lambda \dotplus \nu \dotplus 1 = \omega_{\lambda \dotplus \nu \dotplus 1}$. It follows from 4.7.4 that $\gamma \simeq_{\mathfrak{M}} \bar{\bar{\gamma}}$, i. e. $\aleph_{\mathfrak{M}}{}^{\backprime}\alpha \simeq_{\mathfrak{M}} \aleph^{\backprime}\delta$ where $\delta < \alpha$, since $\gamma < \aleph^{\backprime}\alpha$. On the other hand, δ being less than α, $\aleph^{\backprime}\delta = \aleph_{\mathfrak{M}}{}^{\backprime}\delta$, hence $\aleph_{\mathfrak{M}}{}^{\backprime}\alpha \simeq_{\mathfrak{M}} \aleph_{\mathfrak{M}}{}^{\backprime}\delta$ in contradiction to the fact that the function $\aleph$ is strictly monotone.

4.7.6.** $\mathfrak{h}_{\mathfrak{M}}(h, \lambda, \nu)$.

PROOF. $\mathfrak{h}_{\mathfrak{M}}(\bar{U}_1, \bar{X}, \bar{Y}) \equiv \bar{U}_1\, \mathfrak{Fn}\, \aleph_{\mathfrak{M}}{}^{\backprime}\bar{X} \dotplus \bar{Y} \dotplus 1 . \mathfrak{Un}_2(\bar{U}_1) . \mathfrak{W}(\bar{U}_1) \subseteq \mathfrak{P}(\aleph_{\mathfrak{M}}{}^{\backprime}X) . \mathfrak{M}$ by 4.6.1, and thus we have to prove that the formulas $h\, \mathfrak{Fn}\, \aleph_{\mathfrak{M}}{}^{\backprime}\lambda + \nu \dotplus 1$, $\mathfrak{Un}_2(h)$, $\mathfrak{W}(h) \subseteq \mathfrak{P}(\aleph_{\mathfrak{M}}{}^{\backprime}\lambda) . \mathfrak{M}$ hold.

The first formula follows from $h\, \mathfrak{Fn}\, \omega_{\lambda \dotplus \nu \dotplus 1}$ using 4.7.5 for $\alpha = \lambda \dotplus \nu \dotplus 1$, while the second follows directly from $\mathfrak{h}(h, \lambda, \nu)$ too. As to the third one it also follows from $\mathfrak{h}(h, \lambda, \nu)$ using 4.7.5 for $\alpha = \lambda$ — taking into consideration the equality $\mathfrak{P}_{\mathfrak{M}}(X) = \mathfrak{P}(X) . \mathfrak{M}$ and the inclusion $\mathfrak{W}(h) \subseteq \mathfrak{M}$ (4.6.5).

4.7.7.** $(2^{\aleph_{\mathfrak{M}}{}^{\backprime}\lambda})_{\mathfrak{M}} \geqq \aleph_{\mathfrak{M}}{}^{\backprime}\lambda + \nu + 1$.

PROOF. 4.7.7 follows from 4.6.3, 4.6.8 and 4.7.6.

4.7.8.** $(\exists \bar{u}_1)(\exists \bar{u}_2)(V_{\mathfrak{M}} = \mathfrak{M}_{\mathfrak{M}}(\bar{u}_1, \bar{u}_2, \aleph_{\mathfrak{M}}{}^{\backprime}\lambda \dotplus \nu \dotplus 1))$.

PROOF. $h, k \in \mathfrak{M}$ by 4.6.8. Put $\bar{u}_1 = h$, $\bar{u}_2 = k$. We know that $V_{\mathfrak{M}} = \mathfrak{M} = \mathfrak{M}(h, k, \omega_{\lambda \dotplus \nu \dotplus 1})$. On the other hand, $\mathfrak{M}(U_1, U_2, Z)$ being absolute with respect to $\Delta_{\mathfrak{M}}$ by 4.4.7, we have $\mathfrak{M}(h, k, \omega_{\lambda \dotplus \nu \dotplus 1}) = \mathfrak{M}_{\mathfrak{M}}(h, k, \omega_{\lambda \dotplus \nu \dotplus 1})$ and taking into consideration that $\aleph_{\mathfrak{M}}{}^{\backprime}\lambda \dotplus \nu \dotplus 1 = \omega_{\lambda \dotplus \nu \dotplus 1}$ by 4.7.5, we get that $V_{\mathfrak{M}} = \mathfrak{M}_{\mathfrak{M}}(h, k, \aleph_{\mathfrak{M}}{}^{\backprime}\lambda \dotplus \nu \dotplus 1)$ and thus 4.7.8 is proved.

4.7.9.* $(\alpha)(\beta)((\exists u_1)(\exists u_2)(V = \mathfrak{M}(u_1, u_2, \aleph_{\alpha \dotplus \beta \dotplus 1})) \supset$

$$\supset (\varrho)(\varrho < \alpha \dotplus \beta \dotplus 1 \supset 2^{\aleph_\varrho} \leqq \aleph_{\alpha \dotplus \beta \dotplus 1}) . (\mu)(\alpha \dotplus \beta < \mu \supset 2^{\aleph_\mu} = \aleph_{\mu \dotplus 1})).$$

We are going to prove 4.7.9 in Sections 4.8, 4.9, 4.10. Now using 4.7.9 we finish the proof of Theorem 1.

4.7.10.** $(\varrho)(\lambda \leqq \varrho < \lambda \dotplus \nu \dotplus 1 \supset (2^{\aleph_{\mathfrak{M}}{}^{\backprime}\varrho})_{\mathfrak{M}} = \aleph_{\mathfrak{M}}{}^{\backprime}\lambda \dotplus \nu \dotplus 1)$.

$$(\mu)(\lambda \dotplus \nu \leqq \mu \supset (2^{\aleph_{\mathfrak{M}}{}^{\backprime}\mu})_{\mathfrak{M}} = \aleph_{\mathfrak{M}}{}^{\backprime}\mu \dotplus 1).$$

PROOF. The axioms of Σ^* hold in the model $\Delta_{\mathfrak{M}}$, hence the relativized of 4.7.9 holds too. Therefore we have

$$(\alpha)(\beta)[(\exists \bar{u}_1)(\exists \bar{u}_2)(V_{\mathfrak{M}} = \mathfrak{M}_{\mathfrak{M}}(\bar{u}_1, \bar{u}_2, \aleph_{\mathfrak{M}}{}^{\backprime}\alpha \dotplus \beta \dotplus 1)) \supset$$
$$\supset (\varrho)(\varrho < \alpha \dotplus \beta \dotplus 1 \supset (2^{\aleph_{\mathfrak{M}}{}^{\backprime}\varrho})_{\mathfrak{M}} \leqq \aleph_{\mathfrak{M}}{}^{\backprime}\alpha \dotplus \beta \dotplus 1).$$
$$. (\mu)(\alpha \dotplus \beta < \mu \supset (2^{\aleph_{\mathfrak{M}}{}^{\backprime}\mu})_{\mathfrak{M}} = \aleph_{\mathfrak{M}}{}^{\backprime}\mu \dotplus 1)]$$

by the absoluteness of the concepts appearing without the subscript $\mathfrak{M}$. It results that

$$(\exists \bar{u}_1)(\exists \bar{u}_2)(V_{\mathfrak{M}} = \mathfrak{M}_{\mathfrak{M}}(\bar{u}_1, \bar{u}_2, \aleph_{\mathfrak{M}}{}^{\backprime}\lambda \dotplus \nu \dotplus 1)) \supset$$
$$\supset (\varrho)(\varrho < \lambda \dotplus \nu \dotplus 1 \supset (2^{\aleph_{\mathfrak{M}}{}^{\backprime}\varrho})_{\mathfrak{M}} \leqq \aleph_{\mathfrak{M}}{}^{\backprime}\lambda \dotplus \nu \dotplus 1).$$
$$. (\mu)(\lambda \dotplus \nu < \mu \supset (2^{\aleph_{\mathfrak{M}}{}^{\backprime}\mu})_{\mathfrak{M}} = \aleph_{\mathfrak{M}}{}^{\backprime}\mu \dotplus 1)$$

holds too. On the other hand,

$$(\exists \bar{u}_1)(\exists \bar{u}_2)(V_{\mathfrak{M}} = \mathfrak{M}_{\mathfrak{M}}(\bar{u}_1, \bar{u}_2, \aleph_{\mathfrak{M}}{}^{\backprime}\lambda \dotplus \nu \dotplus 1))$$

by 4.7.8, hence we have

$$(\varrho)(\varrho < \lambda \dotplus \nu \dotplus 1 \supset (2^{\aleph_{\mathfrak{M}}{}^{\backprime}\varrho})_{\mathfrak{M}} \leqq \aleph_{\mathfrak{M}}{}^{\backprime}\lambda \dotplus \nu \dotplus 1).$$
$$. (\mu)(\lambda \dotplus \nu < \mu \supset (2^{\aleph_{\mathfrak{M}}{}^{\backprime}\mu})_{\mathfrak{M}} = \aleph_{\mathfrak{M}}{}^{\backprime}\mu \dotplus 1).$$

Comparing this with 4.7.7 we obtain 4.7.10. But 4.7.10 means that $\mathfrak{M}$ satisfies $(++)$ and $(+++)$. Now by 4.3.11, 4.3.17, 4.6.10, 4.7.5 and 4.7.10 the class $\mathfrak{M} = \mathfrak{M}(h, k, \omega_{\lambda \dotplus \nu \dotplus 1})$ satisfies all the requirements of Theorem 1 and thus Theorem 1 is proved, provided that the proof of 4.7.9 will be carried out in Σ^*.

4.8. PROOF OF 4. 7. 9.

Let λ, ν be two ordinal numbers for which $(\exists u_1)(\exists u_2)(V=\mathfrak{M}(u_1, u_2, \omega_{\lambda\dot{+}\nu\dot{+}1}))$ and let h and k denote such sets for which

(—) $V=\mathfrak{M}(h, k, \omega_{\lambda\dot{+}\nu\dot{+}1})$.

We have to prove, using (—), the formulas

(— —) $(\varrho)(\varrho<\lambda\dot{+}\nu\dot{+}1\supset 2^{\aleph_\varrho}\leqq\aleph_{\lambda\dot{+}\nu\dot{+}1})$,

(— — —) $(\mu)(\lambda\dot{+}\nu\leqq\mu\supset 2^{\aleph_\mu}=\aleph_{\mu\dot{+}1})$.

We need several preliminary results,[30] but before proving them we sketch the idea of the proof. As we have already mentioned, theorem 4. 7. 9 is the analogue of the theorem proved in [1], VIII, that $V=L$ implies the generalized continuum hypothesis. The proof of this theorem in [1] is based upon the auxiliary theorem proved in [1], 12. 6. It is obvious that we have to prove an auxiliary theorem similar to [1], 12. 6. But, in [1], 12. 6 it is essentially used that the fundamental operations $\mathfrak{F}_i(X, Y)$ $(i=1,\ldots,8)$ used at the construction of the function F are very simple and of constructive character. Our operations used at the construction of the function $\mathfrak{G}$ (especially the operations $\mathfrak{g}_{10}$ and $\mathfrak{g}_{11}$) have not all the properties of the $\mathfrak{F}_i$'s needed to the proof of [1], 12. 6. However, for the α's greater than $\omega_{\lambda\dot{+}\nu\dot{+}1}$ our function $\mathfrak{G}$ is constructed quite similarly to GÖDEL's F, and therefore we can prove a theorem similar to [1], 12. 6 formulating some further restrictions for the sets of ordinal numbers m and m' appearing in [1], 12. 6.

However, this would only enable us to prove the formula (— — —) for $\mu>\lambda\dot{+}\nu$, while for the proof of (— —) we need a further generalization of [1], 12. 6 which deals with the "isomorphism" of the functions $\mathfrak{G}(h, k, \xi_1)$, $\mathfrak{G}(h, k, \xi_2)$ where ξ_1, ξ_2 may be different ordinal numbers. This theorem will be given in 4. 9. We mention that using the assumption (—) we may always use the theorems proved in Sections 4. 1—4. 5 for every $\mathfrak{M}$-set as theorems valid for every set.

We introduce the following notations. If $\mathfrak{A}(U_1, U_2, Z)$ is an arbitrary operation defined in Section 4. 3, $\mathfrak{A}(h, k, \xi)$ is denoted by $\mathfrak{A}_\xi$. If, especially, $\xi=\omega_{\lambda\dot{+}\nu\dot{+}1}$, then $\mathfrak{A}_{\lambda\dot{+}\nu\dot{+}1}=\mathfrak{A}$.

We need the following definitions. The class X is said to be ξ-closed if it is closed with respect to $\mathfrak{E}_\xi$, K_1, K_2 and with respect to $J_0,\ldots,J_{11}$ as triadic relations. $\mathfrak{Clos}(X)_\xi$ denotes that X is ξ-closed.[31]

30 Every theorem proved in 4. 8, 4. 9, 4. 10 may depend on the assumption $V=\mathfrak{M}(h, k, \omega_{\lambda\dot{+}\nu\dot{+}1})$.

31 In accordance with the above made convention $\mathfrak{Clos}(X)\omega_{\lambda\dot{+}\nu\dot{+}1}$ is denoted by $\mathfrak{Clos}(X)$ and we say briefly closed instead of $\omega_{\lambda\dot{+}\nu\dot{+}1}$-closed.

4.8.1. DEF. $\mathfrak{Clos}(X)_\xi \equiv \mathfrak{Clos}_2(X, \mathfrak{C}_\xi).\mathfrak{Clos}_2(X, K_1).\mathfrak{Clos}_2(X_2, K_2).$
$.\mathfrak{Clos}_3(X, J_0) \ldots \mathfrak{Clos}_3(X, J_{11}).$

The closure of a class X with respect to $\mathfrak{C}_\xi, K_1, K_2$ and with respect to $J_0, \ldots, J_{11}$ as triadic relations is denoted by $[X]_\xi$, and is said to be the ξ-closure of X ($[X]_{\omega_{\lambda \dot{+} \nu \dot{+} 1}} = [X]$).

4.8.2. DEF. $[X]_\xi = [X]_{(\mathfrak{C}_\xi, K_1, K_2)(J_0, \ldots, J_{11})}$.

4.8.3. $\mathfrak{G}_{\xi_1}{}^{\prime}\alpha = \mathfrak{G}_{\xi_2}{}^{\prime}\alpha$ if $\alpha < \xi_1$ and $\alpha < \xi_2$.

PROOF. $\mathfrak{G}_{\xi_1}{}^{\prime}\alpha = \mathfrak{g}_{\xi_1}{}^{\prime}\alpha$, $\mathfrak{G}_{\xi_2}{}^{\prime}\alpha = \mathfrak{g}_{\xi_2}{}^{\prime}\alpha$ by 4.3.4. But if e.g. $\xi_1 < \xi_2$, then $\mathfrak{g}_{\xi_1} = \mathfrak{g}_{\xi_2} \upharpoonright \xi_1$ by 4.3.2.

4.8.4. $\mathfrak{Clos}(X)_\xi.\mathfrak{Clos}(Y)_\xi \supset \mathfrak{Clos}(X.Y)_\xi$.

PROOF. 4.8.4 follows from 2.1.12.

4.8.5. The ξ-closure of a set x is ξ-closed, $x \subseteq [x]_\xi$ and $\overline{\overline{[x]_\xi}} = \overline{\overline{x}}$, provided x is infinite.

PROOF. 4.8.5 follows from the definition 2.1.13 by [1], 8.7.3 using that $\mathfrak{C}_\xi, K_1, K_2$ and $J_0, \ldots, J_{11}$ are functions.

4.8.6. If $m \subseteq On$ and $m' \subseteq On$; m, m' both closed with respect to K_1, K_2 and J_i $(i = 0, \ldots, 11)$ as triadic relations, and if $G\, \mathfrak{Isom}_{E,E}(m, m')$, then G is an isomorphism for the triadic relations J_i.

PROOF. Similarly to [1], 12.5.

4.8.7. If $m \subseteq On$ and m is closed with respect to K_1, K_2 and J_i $(i = 0, \ldots, 11)$ as triadic relations and $G\, \mathfrak{Isom}_{E,E}(m, \vartheta)$, then ϑ is ξ-closed for every ξ.

PROOF. We have $\mathfrak{C}_\xi{}^{\prime}\alpha \leqq \alpha$ for every α by 4.3.20, hence ϑ is closed with respect to $\mathfrak{C}_\xi$. The fact that ϑ is closed with respect to K_1, K_2 and J_i $(i = 0, \ldots, 11)$ may be proved as the analogous theorem in [1], 12.4. The details of the proof are left to the reader.

4.8.8. Suppose $n \subseteq On$, $\overline{\overline{n}} < \omega_{\lambda \dot{+} \nu \dot{+} 1}$, then there exists an m $(m \subseteq On)$ and a ξ such that

a) $n \subseteq m$,

b) m is closed,

c) $\overline{\overline{m}} < \omega_{\lambda \dot{+} \nu \dot{+} 1}$,

d) $m \cdot \omega_{\lambda \dot{+} \nu \dot{+} 1} = \xi$,

and as a corollary of a), b), c) and d) ξ is closed, $\xi \in \mathfrak{W}(J_0)$ and $\xi < \omega_{\lambda \dot{+} \nu \dot{+} 1}$.

PROOF. We consider a function f defined on ω as follows:

$f\, \mathfrak{Fn}\, \omega$,

$f'0 = n,$

$f'l \dotplus 1 = [\mathfrak{S}(f'l.\omega_{\lambda\dotplus\nu\dotplus 1}) + f'l].$

Such an f exists by [1], 8.45 because by M_5 [1] there exists a function G for which $G'x = [\mathfrak{S}(x.\omega_{\lambda\dotplus\nu\dotplus 1}) + x]$ for every x, since the operations $X+Y$, $X.Y$, $\mathfrak{S}(X)$ and $[X]$ are normal and $x+y$, $x.y$, $\mathfrak{S}(x)$ and $[x]$ are sets for every set x and y by [1], 5.11, 5.12 and by 4.8.5.

Put $m = \mathfrak{S}(\mathfrak{W}(f))$.

Put $m = \mathfrak{S}(\mathfrak{W}(f))$. We have to prove that m satisfies the requirements of the theorem. $m \subseteq On$, since $\mathfrak{W}(\mathfrak{C}) \subseteq On$, $\mathfrak{W}(K_1) \subseteq On$, $\mathfrak{W}(K_2) \subseteq On$, $\mathfrak{W}(J_i) \subseteq On$.

Ad a) $f'l \subseteq m$ for every l, hence $f'0 = n \subseteq m$.

Ad b) $f'l \subseteq f'l \dotplus 1$ by 4.8.5. It follows by induction on s that $l < s$ implies $f'l \subseteq f's$. Now m is closed with respect to $\mathfrak{C}, K_1, K_2$, because if $u \in m$, then there exists an l such that $u \in f'l$, and therefore $\mathfrak{C}'u, K_1'u, K_2'u \in f'l \dotplus 1$, hence $\mathfrak{C}'u, K_1'u, K_2'u \in m$. On the other hand, if $u \in m$ and $v \in m$, then there exists an l and an s such that $u \in f'l$, $v \in f's$. We may suppose $l \leqq s$. Then $u \in f's$ holds too, and therefore $J_i'\langle u, v\rangle \in f's \dotplus 1$, hence $J_i'\langle uv\rangle \in m$ for $i = 0, \ldots, 11$. Thus m is closed.

Ad c) First we prove by induction on l that $\overline{\overline{f'l}} < \omega_{\lambda\dotplus\nu\dotplus 1}$. $f'0 = n$ and $\overline{\overline{n}} < \omega_{\lambda\dotplus\nu\dotplus 1}$ by the assumption. Suppose $\overline{\overline{f'l}} < \omega_{\lambda\dotplus\nu\dotplus 1}$. Then $\overline{\overline{f'l.\omega_{\lambda\dotplus\nu\dotplus 1}}} < \omega_{\lambda\dotplus\nu\dotplus 1}$ by [1], 8.28. But $f'l.\omega_{\lambda\dotplus\nu\dotplus 1} \subseteq \omega_{\lambda\dotplus\nu\dotplus 1}$, hence by 2.1.10 there exists a $\delta < \omega_{\lambda\dotplus\nu\dotplus 1}$ such that $f'l.\omega_{\lambda\dotplus\nu\dotplus 1} \subset \delta$.[32] It follows that $\mathfrak{S}(f'l.\omega_{\lambda\dotplus\nu\dotplus 1}) \subseteq \delta$, hence $\overline{\overline{\mathfrak{S}(f'l.\omega_{\lambda\dotplus\nu\dotplus 1})}} \leqq \delta < \omega_{\lambda\dotplus\nu\dotplus 1}$ by [1], 8.26. Therefore $\overline{\overline{\mathfrak{S}(f'l.\omega_{\lambda\dotplus\nu\dotplus 1}) + f'l}} < \omega_{\lambda\dotplus\nu\dotplus 1}$ by [1], 8.63. Finally, it follows from 4.8.5 that $\overline{\overline{f'l \dotplus 1}} < \omega_{\lambda\dotplus\nu\dotplus 1}$,[33] and therefore $\overline{\overline{f'l}} < \omega_{\lambda\dotplus\nu\dotplus 1}$ for every l. It follows by [1], 8.64 that $\overline{\overline{m}} \leqq \overline{\overline{\omega_{\lambda\dotplus\nu} \times \omega}} = \omega_{\lambda\dotplus\nu} < \omega_{\lambda\dotplus\nu\dotplus 1}$.

Ad d) We have to prove that $m.\omega_{\lambda\dotplus\nu\dotplus 1}$ is an ordinal number. By [1], 7.1 it is enough to prove that $m.\omega_{\lambda\dotplus\nu\dotplus 1}$ is complete. Suppose $u \in m.\omega_{\lambda\dotplus\nu\dotplus 1}$. Then, $\omega_{\lambda\dotplus\nu\dotplus 1}$ being an ordinal number, $u \subseteq \omega_{\lambda\dotplus\nu\dotplus 1}$, hence we have to prove that $u \subseteq m$ holds too. But if $u \in m.\omega_{\lambda\dotplus\nu\dotplus 1}$, then $u \in f'l$ for some l, and therefore $u \subseteq \mathfrak{S}(f'l.\omega_{\lambda\dotplus\nu\dotplus 1})$, hence $u \subseteq f'l \dotplus 1$ by 4.8.5, and thus $u \subseteq m$.

Put $m.\omega_{\lambda\dotplus\nu\dotplus 1} = \xi$. Then $\xi \in \mathfrak{W}(I_0)$ by 4.1.6, ξ is closed by 4.8.4, $\omega_{\lambda\dotplus\nu\dotplus 1}$ being closed by 4.1.7, and $\overline{\overline{\xi}} < \omega_{\lambda\dotplus\nu\dotplus 1}$ implies $\xi < \omega_{\lambda\dotplus\nu\dotplus 1}$. 4.8.8 is needed to the proof of (— —). It is obvious that 4.8.8 remains valid if we write any "regular" cardinal number instead of $\omega_{\lambda\dotplus\nu\dotplus 1}$.

4.9. In this section we are going to prove the generalization of [1], 12.6 mentioned in 4.8. First we state it in the following form which corresponds to [1], 12.3:

[32] At this point the regularity of the cardinal number $\omega_{\lambda\dotplus\nu\dotplus 1}$ is essentially used.

[33] It is obvious that we may suppose n to be infinite.

4.9.1. Is the set $m \subseteq On$ closed, and is ϑ the ordinal number for which there exists a G such that $G\,\mathfrak{Isom}_{E,E}(m, \vartheta)$, then if there exists an ordinal number ξ such that $m.\omega_{\lambda \dotplus \nu \dotplus 1} = \xi$, then (abbreviating $G{}^{\backprime}\alpha$ by α')

$$(\mathfrak{G}{}^{\backprime}\alpha \in \mathfrak{G}{}^{\backprime}\beta \equiv \mathfrak{G}_{\xi}{}^{\backprime}\alpha' \in \mathfrak{G}_{\xi}{}^{\backprime}\beta').(\mathfrak{G}{}^{\backprime}\alpha = \mathfrak{G}{}^{\backprime}\beta \equiv \mathfrak{G}_{\xi}{}^{\backprime}\alpha' = \mathfrak{G}_{\xi}{}^{\backprime}\beta')$$

for every $\alpha, \beta \in m$.

But it is more convenient to prove the theorem in the following symmetric form which corresponds to [1], 12.6:

4.9.2. If $m_1 \subseteq On$, $m_2 \subseteq On$ and m_1 is ξ_1-closed, m_2 is ξ_2-closed and there exists a G such that $G\,\mathfrak{Isom}_{E,E}(m_1, m_2)$ and $m_1.\xi_1 = m_2.\xi_2$, then (if $G{}^{\backprime}\alpha$ is abbreviated by α')

$$(\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha \in \mathfrak{G}_{\xi_1}{}^{\backprime}\beta \equiv \mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' \in \mathfrak{G}_{\xi_2}{}^{\backprime}\beta').(\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha = \mathfrak{G}_{\xi_1}{}^{\backprime}\beta \equiv \mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' = \mathfrak{G}_{\xi_2}{}^{\backprime}\beta')$$

for every $\alpha, \beta \in m_1$.

4.9.2 is the theorem corresponding to [1], 12.6. The assumption $m_1.\xi_1 = m_2.\xi_2$ assures that the parts of m_1 and m_2 for which $\mathfrak{G}_{\xi_1}$ and $\mathfrak{G}_{\xi_2}$ are defined with the help of operations of different character from the fundamental operations are left fixed by the mapping G.

First we prove that 4.9.2 implies 4.9.1. Suppose that the conditions of 4.9.1 hold for $m, \vartheta, \omega_{\lambda \dotplus \nu \dotplus 1}$ and ξ. Put $m_1 = m$, $m_2 = \vartheta$, $\xi_1 = \omega_{\lambda \dotplus \nu \dotplus 1}$, $\xi_2 = \xi$. Then m_2 is ξ_2-closed by 4.8.7 and $m_1.\xi_1 = m.\omega_{\lambda \dotplus \nu \dotplus 1} = \xi = \xi_2$. But ξ, being a subset of m, is a subset of ϑ by 2.1.8, hence $\xi_2 = \xi_2.\vartheta = \xi_2.m_2$, i. e. $m_1.\xi_1 = m_2.\xi_2$. It follows that the conditions of 4.9.2 hold for m_1, m_2, ξ_1, ξ_2 and thus 4.9.2 implies 4.9.1.

The proof of 4.9.2 is quite similar to the proof of [1], 12.6. The theorem is to be proved by induction on $\eta = \mathfrak{Max}(\{\alpha\beta\})$, i. e. we have to prove

$$(\alpha)(\beta)(\mathfrak{Max}(\{\alpha\beta\}) = \eta.\alpha \in m.\beta \in m \supset (\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha \in \mathfrak{G}_{\xi_1}{}^{\backprime}\beta \equiv \mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' \in \mathfrak{G}_{\xi_2}{}^{\backprime}\beta').$$
$$.(\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha = \mathfrak{G}_{\xi_1}{}^{\backprime}\beta \equiv \mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' = \mathfrak{G}_{\xi_2}{}^{\backprime}\beta')).$$

This expression is normal, therefore we can apply induction by [1], 7.161.

Now for the same reasons as in [1] we have to prove that

1. $\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha \in \mathfrak{G}_{\xi_1}{}^{\backprime}\eta \equiv \mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' \in \mathfrak{G}_{\xi_2}{}^{\backprime}\eta'$,
2. $\mathfrak{G}_{\xi_1}{}^{\backprime}\eta \in \mathfrak{G}_{\xi_1}{}^{\backprime}\beta \equiv \mathfrak{G}_{\xi_2}{}^{\backprime}\eta' \in \mathfrak{G}_{\xi_2}{}^{\backprime}\beta'$,
3. $\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha = \mathfrak{G}_{\xi_1}{}^{\backprime}\eta \equiv \mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' = \mathfrak{G}_{\xi_2}{}^{\backprime}\eta'$

under the hypotheses that $\eta \in m_1$ and $\alpha, \beta \in m_1.\eta$ and

I. $\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha \in \mathfrak{G}_{\xi_1}{}^{\backprime}\beta \equiv \mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' \in \mathfrak{G}_{\xi_2}{}^{\backprime}\beta'$,

II. $\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha = \mathfrak{G}_{\xi_1}{}^{\backprime}\beta \equiv \mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' = \mathfrak{G}_{\xi_2}{}^{\backprime}\beta'$ for $\alpha, \beta \in m_1.\eta$.

In what follows in Section 4.9 everything depends on these induction hypotheses.

We introduce the following abbreviations:

$$\mathfrak{S}_{\xi_1}{}^{\backprime\backprime} m_1 = r_1, \quad \mathfrak{S}_{\xi_2}{}^{\backprime\backprime} m_2 = r_2, \quad \mathfrak{S}_{\xi_1}{}^{\backprime\backprime}(m_1 . \eta) = r_{1\eta}, \quad \mathfrak{S}_{\xi_2}{}^{\backprime\backprime}(m_2 . \eta') = r_{2\eta'} .$$

Now we can define a one-to-one mapping H of $r_{1\eta}$ onto $r_{2\eta'}$ by $H = \mathfrak{S}_{\xi_2} | G | \mathfrak{S}_{\xi_1}^{-1}$ i. e. $H{}^{\backprime}x = \mathfrak{S}_{\xi_2}{}^{\backprime}\alpha'$ if $x = \mathfrak{S}_{\xi_1}{}^{\backprime}\alpha$, $\alpha \in m_1 \eta$. Because of the induction hypothesis II, H is one-to-one. Because of the induction hypothesis, H is an isomorphism with respect to E. Note that the hypothesis of theorem 4.9.2 and the induction hypotheses are symmetric in m_1, m_2; ξ_1, ξ_2 and η, η', so that whatever is proved from them will also hold if $m_1, \eta, r_1, r_{1\eta}, G, H$ are interchanged with $m_2, \eta', r_2, r_{2\eta'}, G^{-1}, H^{-1}$, respectively.

Now the following theorems are to be proved literally in the same way as the corresponding theorems in [1], therefore we omit the proofs:

(1) r_1 is closed with respect to the fundamental operations.

(2) $x \in r_1 \supset \mathfrak{Od}_{\xi_1}{}^{\backprime}x \in m_1$.

(3) $x \in r_1 . x \neq 0 \supset x . r_1 \neq 0$.

(3.1) $\{xy\} \in r_1 \supset x, y \in r_1$; $\langle xy \rangle \in r_1 \supset x, y \in r_1$, $\langle x, y, z \rangle \in r_1 \supset x, y, z \in r_1$.

(4) $y \in r_1 . \langle yx \rangle \in Q_i \supset x \in r_1$ if $i \neq 5$.

(5) $x \in r_{1\eta} . y \in r_1 . x \supset y \in r_{1\eta}$.

(6) $y \in \mathfrak{S}_{\xi_1}{}^{\backprime}\eta . y \in r_1 \supset y \in r_{1\eta}$.

(7) $\{xy\} \in r_{1\eta} \supset x, y \in r_{1\eta}$; $\langle xy \rangle \in r_{1\eta} \supset x, y \in r_{1\eta}$, $\langle xyz \rangle \in r_{1\eta} \supset x, y, z \in r_{1\eta}$.

(8) H is an isomorphism with respect to the relations

$z = \{xy\}, z = \langle xy \rangle, z = \langle xyt \rangle$ and the Q_i $\quad (i = 4, \dots, 8)$.

Now with the same considerations as in [1] it is to be seen that it is sufficient to prove 1. $\mathfrak{S}_{\xi_1}{}^{\backprime}\alpha \in \mathfrak{S}_{\xi_1}{}^{\backprime}\eta \equiv \mathfrak{S}_{\xi_2}{}^{\backprime}\alpha' \in \mathfrak{S}_{\xi_2}{}^{\backprime}\eta'$ for $\alpha \in m . \eta$ and by symmetry reasons it is sufficient to show that

$$\mathfrak{S}_{\xi_1}{}^{\backprime}\alpha \in \mathfrak{S}_{\xi_1}{}^{\backprime}\eta \supset \mathfrak{S}_{\xi_2}{}^{\backprime}\alpha' \in \mathfrak{S}_{\xi_2}{}^{\backprime}\eta' .$$

Thus we assume $\mathfrak{S}_{\xi_1}{}^{\backprime}\alpha \in \mathfrak{S}_{\xi_1}{}^{\backprime}\eta$ and consider the following separate cases:

1. $\eta \in \xi_1$, $\quad i+2.$ $\eta \in \mathfrak{W}(J_i) - \xi_1 \qquad (i = 0, \dots, 11)$.

Note that $\eta \in \xi_1 \equiv \eta' \in \xi_2$ and $\eta \in \xi_1 \supset \eta = \eta'$ by the assumption $m_1 . \xi_1 = m_2 . \xi_2$. In fact, $m_1 . \xi_1$ and $m_2 . \xi_2$ are E sections of m_1 and m_2, respectively, therefore we have $G \upharpoonright m_1 . \xi_1 = I \upharpoonright m_1 \xi_1$ by 2.1.8.

On the other hand, $\eta \in \mathfrak{W}(J_i)$ implies $\eta' \in \mathfrak{W}(J_i)$ by 4.8.6, hence $\eta \in \mathfrak{W}(J_i) - \xi_1 \equiv \eta' \in \mathfrak{W}(J_i) - \xi_2$ for $i = 0, \dots, 11$.

Ad 1. $\eta \in \xi_1$. Then $\eta' \in \xi_2$ and $\alpha < \xi_1$, since $\alpha < \eta$. Thus $\eta = \eta'$, $\alpha = \alpha'$ and $\alpha < \xi_2$, therefore $\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha = \mathfrak{G}_{\xi_2}{}^{\backprime}\alpha'$ and $\mathfrak{G}_{\xi_1}{}^{\backprime}\eta = \mathfrak{G}_{\xi_2}{}^{\backprime}\eta'$ by 4.8.3. It follows that $\mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' \in \mathfrak{G}_{\xi_2}{}^{\backprime}\eta'$.

Ad 2—10. The proof is the same as in [1] for the cases 1, 2, 3, 4, 5, 6 (see [1], p. 60), using the results (1)—(8) listed on p. 361, taking into consideration that $\mathfrak{G}_i(X, Y)_\xi = \mathfrak{F}_i(X, Y)$ for $i = 1, \dots, 8$ by 4.3.1 and 4.3.3.

Ad 11. $\eta \in \mathfrak{W}(J_9) - \xi_1$. In this case our statement is obviously true, since $\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha \in \mathfrak{G}_{\xi_1}{}^{\backprime}\eta$ is false for every α by 4.5.7.

Ad 12—13. $\eta \in \mathfrak{W}(J_i) - \xi_1$ $(i = 10, 11)$. Then $\eta' \in \mathfrak{W}(J_i) - \xi_2$, $\eta = J_i{}^{\backprime}\langle\beta\gamma\rangle$, $\eta' = J_i{}^{\backprime}\langle\beta'\gamma'\rangle$, $\beta, \gamma \in m.\eta$ by 4.1.6 and 4.8.6 and

$$\mathfrak{G}_{\xi_1}{}^{\backprime}\eta = \mathfrak{G}_{\xi_1}{}^{\backprime}\beta . s . \mathfrak{G}_{\xi_1}{}^{\backprime\backprime}\xi_1, \quad \mathfrak{G}_{\xi_2}{}^{\backprime}\eta' = \mathfrak{G}_{\xi_2}{}^{\backprime}\beta' . s . \mathfrak{G}_{\xi_2}{}^{\backprime\backprime}\xi_2$$

by 4.5.7, where $s = h$ or $s = k$ if $i = 10$ or $i = 11$, respectively. $\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha \in \mathfrak{G}_{\xi_1}{}^{\backprime}\eta$ implies that

$$(0)\ \ \mathfrak{G}_{\xi_1}{}^{\backprime}\alpha \in \mathfrak{G}_{\xi_1}{}^{\backprime}\beta, \qquad (00)\ \ \mathfrak{G}_{\xi_1}{}^{\backprime}\alpha \in s, \qquad (000)\ \ \mathfrak{G}_{\xi_1}{}^{\backprime}\alpha \in \mathfrak{G}_{\xi_1}{}^{\backprime\backprime}\xi_1.$$

But then there exists, by (000), a $\delta_1 < \xi_1$ for which $\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha = \mathfrak{G}_{\xi_1}{}^{\backprime}\delta_1$. Put $\delta = \mathfrak{Od}_{\xi_1}{}^{\backprime}\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha$. It follows by 4.3.10 that $\delta < \xi_1$. $\delta \in m_1$ by (2). We get $\delta \in m_1.\eta$ and $\delta \in m_1.\xi_1$. Hence $\delta = \delta'$, $\delta' \in \xi_2$, and therefore $\mathfrak{G}_{\xi_1}{}^{\backprime}\delta = \mathfrak{G}_{\xi_2}{}^{\backprime}\delta'$ by 4.8.3. $\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha = \mathfrak{G}_{\xi_1}{}^{\backprime}\delta$ by 4.3.7, hence $\mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' = \mathfrak{G}_{\xi_2}{}^{\backprime}\delta'$ by II.

It follows that

(0′) $\mathfrak{G}_{\xi_2}{}^{\backprime}\alpha \in \mathfrak{G}_{\xi_2}{}^{\backprime}\beta'$ by (0) and I;

(00′) $\mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' \in s$ by (00), since it follows from the above proved statements that $\mathfrak{G}_{\xi_1}{}^{\backprime}\alpha = \mathfrak{G}_{\xi_2}{}^{\backprime}\alpha'$; finally

(000′) $\mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' \in \mathfrak{G}_{\xi_2}{}^{\backprime\backprime}\xi_2$, since $\mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' = \mathfrak{G}_{\xi_2}{}^{\backprime}\delta'$ where $\delta' \in \xi_2$.

But (0′), (00′) and (000′) imply $\mathfrak{G}_{\xi_2}{}^{\backprime}\alpha' \in \mathfrak{G}_{\xi_2}{}^{\backprime}\eta'$.

4.10. In this section we are going to prove that (— —) and (— — —) follow from the assumptions of 4.7.9.

4.10.1. If $\mu \geqq \lambda \dotplus \nu + 1$, $\gamma < \omega_{\mu \dotplus 1}$, $\gamma \in \mathfrak{W}(J_0)$ and $u \subseteq G^{\backprime\backprime}\gamma$, then $\mathfrak{Od}^{\backprime}u < \omega_{\mu \dotplus 1}$.[34]

PROOF. There exists a δ such that $u = \mathfrak{G}^{\backprime}\delta$ (by (—)). Put $a = \mathfrak{Max}(\{\gamma, \omega_{\lambda \dotplus \nu \dotplus 1}\}) + \{\delta\}$ and $m = [a]$. Then $\overline{\overline{a}} = \overline{\overline{\mathfrak{Max}(\{\gamma, \omega_{\lambda \dotplus \nu \dotplus 1}\})}} = \mathfrak{Max}(\{\overline{\overline{\gamma}}, \omega_{\lambda \dotplus \nu \dotplus 1}\})$ by [1], 8.63, hence $\overline{\overline{a}} < \omega_{\mu \dotplus 1}$ by the assumptions. Therefore $\overline{\overline{m}} < \omega_{\mu \dotplus 1}$ by 4.8.5.

By [1], 7.72 there exists an ordinal number ϑ and a one-to-one mapping G such that $G\,\mathfrak{Isom}_{E,E}(m, \vartheta)$. Put $\xi = \omega_{\lambda \dotplus \nu \dotplus 1}$. Then $m.\omega_{\lambda \dotplus \nu \dotplus 1} = \xi$ holds, since $\omega_{\lambda \dotplus \nu \dotplus 1} \subseteq m$ by 4.8.5. m being closed by 4.8.5 we can apply 4.9.1 to $\xi = \omega_{\lambda \dotplus \nu \dotplus 1}$.

[34] Theorem 4.10.1 corresponds to [1], 12.2.

We have $\mathfrak{G}`\alpha \in \mathfrak{G}`\beta \equiv \mathfrak{G}`\alpha' \in \mathfrak{G}`\beta'$ for every $\alpha, \beta \in m$ where $G`\alpha$ is abbreviated by $\alpha'$. On the other hand, $\gamma \subseteq m$, and therefore $\gamma$ is an $E$-section of $m$. It follows that $\gamma \leqq \vartheta$ and $\alpha = \alpha'$ for $\alpha < \gamma$. It results that $\mathfrak{G}`\alpha \in \in \mathfrak{G}`\delta \equiv \mathfrak{G}`\alpha \in \mathfrak{G}`\delta'$ for every $\alpha < \gamma$, i. e. $\mathfrak{G}`\delta . \mathfrak{G}``\gamma = \mathfrak{G}`\delta' . \mathfrak{G}``\gamma$. But $\gamma \in \mathfrak{W}(J_0)$, and so we have $\mathfrak{G}`\delta` . \mathfrak{G}`\gamma = u$. $\overline{\overline{\vartheta}} = \overline{\overline{m}} < \omega_{\mu+1}$ and $\delta` \in \vartheta$, hence $\mathfrak{Od}`u < \omega_{\mu+1}$ by 4. 5. 2.

4. 10. 2. $(\mu)(\lambda + \nu + 1 \leqq \mu \supset 2^{\aleph_\mu} = \aleph_{\mu+1})$.

PROOF. $2^{\aleph_\mu} \geqq \aleph_{\mu+1}$ by 2. 1. 15, and so we have to prove that $2^{\aleph_\mu} \leqq \aleph_{\mu+1}$ for $\mu \geqq \lambda + \nu + 1$. $\overline{\overline{\mathfrak{G}``\omega_\mu}} = \aleph_\mu$ by 4. 5. 6 and thus $2^{\aleph_\mu} = \overline{\overline{\mathfrak{P}(G``\omega_\mu)}}$ by 2. 1. 16. On the other hand, $\overline{\overline{\mathfrak{G}``\omega_{\mu+1}}} = \aleph_{\mu+1}$, therefore by [1], 8. 28 it is enough to see that $\mathfrak{P}(\mathfrak{G}``\omega_\mu) \subseteq \mathfrak{G}``\omega_{\mu+1}$. Suppose $u \in \mathfrak{P}(\mathfrak{G}``\omega_\mu)$ and put $\gamma = \omega_\mu$. Then $\mu \geqq \lambda + \nu + 1$, $u \subseteq \mathfrak{G}``\gamma$, $\gamma \in \mathfrak{W}(J_0)$ (4. 1. 6) and $\gamma < \omega_{\mu+1}$, hence $\mathfrak{Od}`u < \omega_{\mu+1}$ by 4. 10. 1, i. e. $u \in \mathfrak{G}``\omega_{\mu+1}$.

Now to prove (— — —) it would be enough to prove $2^{\aleph_{\lambda+\nu}} = \aleph_{\lambda+\nu+1}$. But the proof of this equality is somewhat different from the proof of 4. 10. 2. We shall prove it proving (— —), and just the proof of this equality makes the main difficulty in our proof. The generalization of the mapping theorem [1], 12. 6 formulated in 4. 9. 2 which deals with the "isomorphism" of $\mathfrak{G}_{\xi_1}$, $\mathfrak{G}_{\xi_2}$ for different ξ_1 and ξ_2 is essentially needed only in this part of the proof.

First we need the following preliminary result:

4. 10. 3. If $u \subseteq \mathfrak{G}``\gamma, \gamma < \omega_{\lambda+\nu+1}$, then there exists a $\xi < \omega_{\lambda+\nu+1}$ such that $\mathfrak{Od}_\xi`u < \omega_{\lambda+\nu+1}$.

PROOF. There exists a δ such that $u = \mathfrak{G}`\delta$ (by (—)). Put $n = \gamma + \{\delta\}$. Then $\overline{\overline{n}} = \overline{\overline{\gamma}} < \omega_{\lambda+\nu+1}$ by [1], 8. 26 and 8. 63. Therefore by 4. 8. 8 there exists an $m \subseteq On$ such that $n \subseteq m$, and consequently $\gamma \subseteq m$, $\delta \in m$; m is closed, $\overline{\overline{m}} < \omega_{\lambda+\nu+1}$, furthermore there exists a ξ for which $m . \omega_{\lambda+\nu+1} = \xi$, $\xi < \omega_{\lambda+\nu+1}$, $\xi \in \mathfrak{W}(I_0)$.

By [1], 7. 72 there exists a ϑ and a G such that $G\,\mathfrak{Isom}_{E,E}(m, \vartheta)$. We can apply 4. 9. 1, thus we have

$$\mathfrak{G}`\alpha \in \mathfrak{G}`\beta \equiv \mathfrak{G}_\xi`\alpha' \in \mathfrak{G}_\xi`\beta' \quad \text{for every } \alpha, \beta \in m.$$

ξ being a common E-section of m and ϑ similarly as in 4. 10. 1, we get $\alpha < \xi \equiv \alpha' < \xi$ and $\alpha < \xi \supset \alpha = \alpha'$.

But $\gamma \subseteq m$ and $\gamma \subseteq \omega_{\lambda+\nu+1}$, hence $\gamma \subseteq \xi$. It follows that

$$\mathfrak{G}`\alpha \in \mathfrak{G}`\delta \equiv \mathfrak{G}_\xi`\alpha \in \mathfrak{G}_\xi`\delta' \quad \text{for every } \alpha \in \gamma.$$

But $\mathfrak{G}`\alpha = \mathfrak{G}_\xi`\alpha$ for $\alpha < \xi < \omega_{\lambda+\nu+1}$ by 4. 8. 3, and therefore

$$\mathfrak{G}`\alpha \in \mathfrak{G}`\delta \equiv \mathfrak{G}`\alpha \in \mathfrak{G}_\xi`\delta' \quad \text{for every } \alpha < \gamma.$$

This means that $\mathfrak{G}'\delta.\mathfrak{G}''\gamma = \mathfrak{G}_\xi'\delta'.\mathfrak{G}''\gamma$ and using $u \subseteq \mathfrak{G}''\gamma$ it follows that $u = \mathfrak{G}''\gamma.\mathfrak{G}_\xi'\delta'$. On the other hand, using again that $\mathfrak{G}'\alpha = \mathfrak{G}_\xi'\alpha$ for every $\alpha < \xi$, we have $\mathfrak{G}''\gamma = \mathfrak{G}_\xi''\gamma \subseteq \mathfrak{G}_\xi''\xi$ and $\xi \in \mathfrak{W}(J_0)$ implies $\mathfrak{G}_\xi''\xi = \mathfrak{G}_\xi'\xi$.

Thus $u = \mathfrak{G}_\xi'\xi.\mathfrak{G}_\xi'\delta'$.

But $\delta' \in \vartheta$ and $\overline{\overline{\vartheta}} = \overline{\overline{m}} < \omega_{\lambda+\nu+1}$, hence $\delta' < \omega_{\lambda+\nu+1}$ and $\xi < \omega_{\lambda+\nu+1}$. It follows from 4.5.2 that $\mathfrak{Od}_\xi'u < \omega_{\lambda+\nu+1}$.

4.10.4. $\mathfrak{Od}'\mathfrak{G}_\xi'\alpha < \omega_{\lambda+\nu+2}$ for $\alpha, \xi < \omega_{\lambda+\nu+1}$.[35]

PROOF. If for an α in question there exists a $\gamma < \omega_{\lambda+\nu+2}$ $(\gamma \in \mathfrak{W}(I_0))$ such that $\mathfrak{G}_\xi'\alpha \in \mathfrak{G}''\gamma$, then $\mathfrak{Od}'\mathfrak{G}_\xi'\alpha < \omega_{\lambda+\nu+2}$ by 4.10.1. Hence we have to prove that

$$(\xi)\left(\alpha < \omega_{\lambda+\nu+1}.\xi < \omega_{\lambda+\nu+1} \supset (\exists\gamma)(\gamma < \omega_{\lambda+\nu+2}.\gamma \in \mathfrak{W}(J_0).\mathfrak{G}_\xi'\alpha \subseteq \mathfrak{G}''\gamma)\right)$$

holds.

This expression being normal we can apply induction on α.

Suppose that this is true for every $\beta < \alpha$ for an $\alpha < \omega_{\lambda+\nu+1}$. By 4.10.1 it follows that $\mathfrak{G}_\xi''\alpha \subseteq \mathfrak{G}''\omega_{\lambda+\nu+2}$. But then $\mathfrak{G}_\xi'\alpha \subseteq \mathfrak{G}''\omega_{\lambda+\nu+2}$ by 4.3.9.

Put $a = \mathfrak{G}_\xi'\alpha$. Then $\mathfrak{Od}''a \subseteq \omega_{\lambda+\nu+2}$. But $\overline{\overline{a}} \leqq \omega_{\lambda+\nu+1}$ by 4.5.4, hence, $\mathfrak{Od}$ being a function, $\overline{\overline{\mathfrak{Od}''a}} \leqq \omega_{\lambda+\nu+1}$. It follows from 2.1.10 and 4.1.8 that there exists a $\gamma < \omega_{\lambda+\nu+2}$ $(\gamma \in \mathfrak{W}(J_0))$ such that $\mathfrak{Od}''a \subseteq \gamma$. But then $G_\xi'\alpha \subseteq G''\gamma$ for this γ.

4.10.5. There exists an ordinal number π $(\pi < \omega_{\lambda+\nu+2})$ such that $\mathfrak{Od}'u < \pi$ for every u for which there exists a $\gamma < \omega_{\lambda+\nu+1}$, $u \subseteq \mathfrak{G}''\gamma$.

PROOF. We define a function T as follows:

$$T\,\mathfrak{Fn}\,\omega^2_{\lambda+\nu+1}.(\langle\xi\alpha\rangle \in \omega^2_{\lambda+\nu+1} \supset T'\langle\xi\alpha\rangle = \mathfrak{Od}'(\mathfrak{G}_\xi'\alpha).$$

The operations $X'Y$, $\mathfrak{Od}(U_1, U_2, Z)$, $\mathfrak{G}(U_1, U_2, Z)$[36] being normal, such a function exists by M_5 [1], since $\mathfrak{Od}'\mathfrak{G}_\xi'\alpha$ is a set for every ξ and α.

We define p as follows:

$u \in p \equiv (\exists\gamma)(\gamma < \omega_{\lambda+\nu+1}.u \subseteq \mathfrak{G}''\gamma)$. p exists by M_4 [1] and $p \subseteq \mathfrak{P}(\mathfrak{G}''\omega_{\lambda+\nu+1})$ implies that p is a set.

Suppose $U \in p$. Then by 4.10.3 there exists a $\xi < \omega_{\lambda+\nu+1}$ such that $\mathfrak{Od}_\xi'u < \omega_{\lambda+\nu+1}$.

Put $\alpha = \mathfrak{Od}_\xi'u$. Then $u = \mathfrak{G}_\xi'\alpha$ where $\alpha, \xi < \omega_{\lambda+\nu+1}$, i. e. $\mathfrak{Od}'u \in \mathfrak{W}(T)$. But $\mathfrak{W}(T) \subseteq \omega_{\lambda+\nu+2}$ by 4.10.4 and $\overline{\overline{\mathfrak{W}(T)}} \leqq \overline{\overline{\mathfrak{D}(T)}} = \overline{\overline{\omega^2_{\lambda+\nu+1}}} = \omega_{\lambda+\nu+1} < \omega_{\lambda+\nu+2}$ by [1], 8.62. Hence by 2.1.10 there exists a $\pi < \omega_{\lambda+\nu+2}$ such that $\mathfrak{W}(T) \subseteq \pi$. It follows that $\mathfrak{Od}'u < \pi$ for every $u \in p$.

[35] It is easy to see that 4.10.3 implies directly (— —), and therefore Lemma 4.10.4 is not essential. However, it seems to be more convenient to carry out the proof as we do it in the text.

[36] At this point we have to use essentially the normality of the operations $\mathfrak{Od}$ and $\mathfrak{G}$.

4.10.6. $(\varrho)(\varrho \leqq \lambda \dotplus \nu \supset 2^{\aleph_\varrho} \leqq \aleph_{\lambda \dotplus \nu \dotplus 1})$.

PROOF. Let π be the ordinal number which satisfies the requirements of 4.10.5. Then $\overline{\overline{\pi}} < \omega_{\lambda \dotplus \nu \dotplus 2}$, hence $\overline{\overline{\pi}} \leqq \omega_{\lambda \dotplus \nu \dotplus 1}$. $\overline{\overline{G^{((}\omega}} = \aleph_\varrho$ by 4.5.6 and $2^{\aleph_\varrho} = \overline{\overline{\mathfrak{P}(\mathfrak{G}^{((}\omega_\varrho)}}$ by 2.1.16. $\mathfrak{G}$ being a function, $\overline{\overline{G^{((}\pi}} \leqq \aleph_{\lambda \dotplus \nu \dotplus 1}$, therefore it is sufficient to see that $\mathfrak{P}(\mathfrak{G}^{((}\omega_\varrho) \subseteq G^{((}\pi$, but ϱ being less than $\lambda \dotplus \nu \dotplus 1$, $\omega_\varrho = \gamma < \omega_{\lambda \dotplus \nu \dotplus 1}$ and the inclusion follows from 4.10.5.

4.10.7. $2^{\aleph_{\lambda \dotplus \nu}} = \aleph_{\lambda \dotplus \nu \dotplus 1}$.

PROOF. $2^{\aleph_{\lambda \dotplus \nu}} \geqq \aleph_{\lambda \dotplus \nu \dotplus 1}$ by 2.1.15 and $2^{\aleph_{\lambda \dotplus \nu}} \leqq \aleph_{\lambda \dotplus \nu \dotplus 1}$ by 4.10.6.

(— —) and (— — —) hold by 4.10.2, 4.10.6 and 4.10.7. Thus we have proved 4.7.9, consequently the proof of Theorem 1 is finished as well.

§ 5. Proof of Theorem 2. Corollaries

We are going to define some metaconcepts.

DEFINITION 5.1. The formula $\varphi(X_1, \ldots, X_n)$ of Σ^* is called to be *constructive* if we can prove in Σ^* that φ is absolute with respect to any model Δ_M satisfying Σ^* and determined by a complete and almost universal class M, i. e. φ is constructive if

$$\vdash (M)(\mathfrak{Comp}(M).\mathfrak{Au}(M).\psi_0^*(M) \supset (X_{1,M}) \ldots (X_{n,M})(\varphi(X_{1,M}, \ldots, X_{n,M}) \equiv \\ \equiv \varphi_M(X_{1,M} \ldots X_{n,M}))).$$

DEFINITION 5.2. The particular class $\mathfrak{A}$ (of Σ^*) is called to be *absolutely definable* if

$$\vdash (M)(\mathfrak{Comp}(M).\mathfrak{Au}(M).\psi_0^*(M) \supset \mathfrak{A}_M = \mathfrak{A}).$$

DEFINITION 5.3. The particular ordinal number Λ is called to be *absolutely definable in the stronger sense if it is absolutely definable* (as a particular class)

DEFINITION 5.4. The particular ordinal number Λ is called to be *absolutely definable in the weaker sense* if

$$\vdash (M)(\mathfrak{Comp}(M).\mathfrak{Au}(M).\psi_0^*(M).(\alpha)(\alpha \leqq \Lambda \dotplus 1 \supset \aleph_M{}^{\backprime}\alpha = \aleph^{\backprime}\alpha). \supset \Lambda_M = \Lambda).$$

It is obvious that every particular ordinal number absolutely definable in the stronger sense is absolutely definable in the weaker sense. In § 6 we are going to prove theorems concerning absolutely definable ordinal numbers.

THEOREM 2. *Are the particular ordinal numbers Λ and* N *absolutely definable in the weaker sense, then the axiom system Σ^*, $F^*_{\Lambda,N}$ is consistent, provided that the same is true for the axiom system Σ^*, $F_{\Lambda,N}$.*

PROOF. In the axiom system Σ^*, $F_{\Lambda,N}$, $2^{\aleph_\Lambda} \geqq \aleph_{\Lambda \dotplus N \dotplus 1}$ is an axiom. Hence by Theorem 1 we can prove in $\Sigma^*, F_{\Lambda,N}$ the formula

$$
(1) \qquad
\begin{aligned}
&(\exists M)(\mathfrak{Comp}(M).\mathfrak{Au}(M).\psi_0^*(M).\\
&.(\alpha)(\alpha \leqq \Lambda \dotplus N \dotplus 1 \supset \aleph_M{}^{(}\alpha = \aleph^{(}\alpha).\\
&.(\varrho)(\Lambda \leqq \varrho < \Lambda \dotplus N \dotplus 1 \supset (2^{\aleph_M{}^{(}\varrho})_M = \aleph_M{}^{(}\Lambda \dotplus N \dotplus 1).\\
&.(\mu)(\Lambda \dotplus N \leqq \mu \supset (2^{\aleph_M{}^{(}\mu})_M = \aleph_M{}^{(}\mu \dotplus 1)).
\end{aligned}
$$

Let now $\psi_0^{**}(M)$ denote the conjunction of the relativized of the axioms of $\Sigma^*, F^*_{\Lambda N}$ with respect to Δ_M. Λ and N being absolutely definable in the weaker sense, Λ and N are absolute with respect to any Δ_M for which $\mathfrak{Comp}(M)$, $\mathfrak{Au}(M)$, $\psi_0^*(M)$, $(\alpha)(\alpha \leqq \Lambda \dotplus N \dotplus 1 \supset \aleph_M{}^{(}\alpha = \aleph^{(}.\alpha)$, since $\Lambda \dotplus 1 \leqq \leqq \Lambda \dotplus N \dotplus 1$, $N \dotplus 1 \leqq \Lambda \dotplus N \dotplus 1$, and thus, taking into consideration the absoluteness of the concepts appaering without the subscript M, $(\varrho)(\Lambda \leqq \varrho < < \Lambda \dotplus N \dotplus 1 \supset (2^{\aleph_M{}^{(}\varrho})_M = \aleph_M{}^{(}\Lambda \dotplus N \dotplus 1)$. $(\mu)(\Lambda \dotplus N \leqq \mu \supset (2^{\aleph_M{}^{(}\mu})_M = \aleph_M{}^{(}\mu \dotplus 1)$ is just the relativized of $F^*_{\Lambda,N}$. It follows that the theorem $(\exists M)\psi_0^{**}(M)$ is provable in Σ^*, $F_{\Lambda,N}$.

Suppose now that a contradiction is derived in $\Sigma^*, F^*_{\Lambda,N}$, i. e. we have in $\Sigma^*, F^*_{\Lambda,N}$ a proof for a closed formula Θ and for its negation $\sim\Theta$ as well. Then we can construct by M_9 from these proofs the proofs of the formulas

$$(M)(\psi_0^{**}(M) \supset \Theta_M) \quad \text{and} \quad (M)(\psi_0^{**}(M) \supset \sim\Theta_M)$$

in $\Sigma^*, F_{\Lambda,N}$, respectively, i. e. we can construct from these proofs the proof of $(M)(\sim\psi_0^{**}(M))$ in $\Sigma^*, F_{\Lambda,N}$ which is just the negation of $(\exists M)\psi_0^{**}(M)$. Q. e. d.

Applying Theorem 2 to $N = 1$ we obtain the following corollary:

K_1. *If Λ is absolutely definable in the weaker sense and $2^{\aleph_\Lambda} = \aleph_{\Lambda \dotplus 1}$ cannot be proved in Σ^*, then it cannot be proved in Σ^* even if we assume that $2^{\aleph_\mu} = \aleph_{\mu \dotplus 1}$ for every μ greater than Λ.*

PROOF. $2^{\aleph_\Lambda} \neq \aleph_{\Lambda \dotplus 1} \equiv 2^{\aleph_\Lambda} \geqq \aleph_{\Lambda \dotplus 1 \dotplus 1} = \aleph_{\Lambda \dotplus 2}$.

Hence if $2^{\aleph_\Lambda} = \aleph_{\Lambda \dotplus 1}$ cannot be proved in Σ^*, then the axiom system $\Sigma^*, F_{\Lambda,1}$ must be consistent. It follows by Theorem 2 that the axiom system $\Sigma^*, F^*_{\Lambda,1}$ is consistent as well. But this means that the axioms Σ^*, $2^{\aleph_\Lambda} = \aleph_{\Lambda \dotplus 2}$, $(\mu)(\Lambda < \mu \supset 2^{\aleph_\mu} = \aleph_{\mu \dotplus 1})$ are consistent, hence one cannot derive $2^{\aleph_\Lambda} \neq \aleph_{\Lambda \dotplus 2}$ from the axioms of Σ^* and $(\mu)(\Lambda < \mu \supset 2^{\aleph_\mu} = \aleph_{\mu \dotplus 1})$.

It is obvious from the proof of Theorem 2 that from any proof of the theorem $2^{\aleph_\Lambda} = \aleph_{\Lambda \dotplus 1}$ given in Σ^*, $(\mu)(\Lambda < \mu \supset 2^{\aleph_\mu} = \aleph_{\mu \dotplus 1})$ we can construct a proof of the formula $2^{\aleph_\Lambda} = \aleph_{\Lambda \dotplus 1}$ in Σ^*.

A similar remark is valid for the other corollaries the proof of which we are going only to sketch.

$\mathbf{K}_2$. $\vdash 2^{\aleph_\Delta} = \aleph_{\Delta \dotplus 1}$ *if (and only if)* $\vdash 2^{\aleph_\Delta} \neq \aleph_{\Delta \dotplus 2}$.

$\mathbf{K}_3$. $\vdash 2^{\aleph_\Delta} = \aleph_{\Delta \dotplus 1}$ *if (and only if)* $\vdash 2^{\aleph_\Delta} \neq 2^{\aleph_{\Delta \dotplus 1}}$, *provided that* Δ *is absolutely definable in the weaker sense.*

The corrollary $\mathbf{K}_3$ has the following interesting consequence:

It was well known that the hypothesis $2^{\aleph_0} = 2^{\aleph_1}$, called Lusin hypothesis, contradicts Cantor's continuum hypothesis. But $\mathbf{K}_3$ says for $\Delta = 0$ that a disproof of the Lusin hypothesis would give a proof of the continuum hypothesis.

The corollaries $\mathbf{K}_2$, $\mathbf{K}_3$ are immediate consequences of the following more general theorems $\mathbf{K}_4$, $\mathbf{K}_5$ if we take into consideration that

$$2^{\aleph_\Delta} < \aleph_{\Delta \dotplus 2} \equiv 2^{\aleph_\Delta} = \aleph_{\Delta \dotplus 1}:$$

$\mathbf{K}_4$. $\vdash 2^{\aleph_\Delta} < \aleph_{\Delta \dotplus \mathrm{N} \dotplus 1}$ *if (and only if)* $\vdash 2^{\aleph_\Delta} \neq \aleph_{\Delta \dotplus \mathrm{N} \dotplus 1}$.

$\mathbf{K}_5$. $\vdash 2^{\aleph_\Delta} < \aleph_{\Delta \dotplus \mathrm{N} \dotplus 1}$ *if (and only if)* $\vdash 2^{\aleph_\Delta} \neq 2^{\aleph_{\Delta \dotplus \mathrm{N}}}$, *provided that* Δ *and* N *are absolutely definable in the weaker sense.*

PROOF. The "only if" is trivial in both cases.

Suppose that one of the formulas $2^{\aleph_\Delta} \neq \aleph_{\Delta \dotplus \mathrm{N} \dotplus 1}$, $2^{\aleph_\Delta} \neq 2^{\aleph_{\Delta \dotplus \mathrm{N}}}$ is proved in Σ^*. This means that the axiom system Σ^*, $\mathrm{F}^*_{\Delta,\mathrm{N}}$ is proved to be inconsistent. Then by Theorem 2 we can derive a contradiction in Σ^*, $\mathrm{F}_{\Delta,\mathrm{N}}$, too, i. e. we have a proof of $2^{\aleph_\Delta} < \aleph_{\Delta \dotplus \mathrm{N} \dotplus 1}$ in Σ^*.

REMARK.[37] There is a well-known theorem of the set theory which asserts that $2^{\aleph_\Delta} \neq \aleph_{\Delta \dotplus \mathrm{M}}$ if $cf(\Delta + \mathrm{M}) \leqq cf(\Delta)$, since in this case $\aleph_{\Delta \dotplus \mathrm{M}}^{\aleph_\Delta} > \aleph_{\Delta \dotplus \mathrm{M}}$.[38] For example, $2^{\aleph_0} \neq \aleph_\omega$. If our theorems were valid for arbitrary $\mathrm{M} \geqq 1$ instead of $\mathrm{N} + 1$, corollary $\mathbf{K}_4$ would imply, for example, $2^{\aleph_0} < \aleph_\omega$. Naturally this is not the case. However, if M is of the second kind but $cf(\Delta + \mathrm{M}) > cf(\Delta)$, then the equality $2^{\aleph_\Delta} = \aleph_{\Delta \dotplus \mathrm{M}}$ is not disproved. For example, $2^{\aleph_0} = \aleph_{\omega_1}$ can be consistent with the axioms of Σ^*, but we can prove neither Theorem 2 nor corollary $\mathbf{K}_4$ for this case, for we can not prove the theorem corresponding to 4. 8. 8 in this case.

We mention here that there is a conjecture that the continuum hypothesis is independent of the axiom system Σ^* in such a manner that $2^{\aleph_\Delta}$ cannot be limited from above in a non-trivial way. Anyway our result shows in accordance with this conjecture why non-trivial inequalities of the form $2^{\aleph_\Delta} \neq 2^{\aleph_{\Delta \dotplus \mathrm{N}}}$, $2^{\aleph_\Delta} \neq \aleph_{\Delta \dotplus \mathrm{N} \dotplus 1}$ are not proved in the set theory.

We mention another group of corollaries of Theorem 1.

[37] In this remark we use set-theoretical concepts not defined in this paper. For the definition of $cf(\alpha)$ see e. g. [4].

[38] See e. g. TARSKI's paper [4].

Let $\varphi(X)$ and $\psi(X)$ be constructive formulas. Then

K$_6$. $\vdash(\alpha)(\beta)(\varphi(\alpha).\psi(\beta)\supset 2^{\aleph_\alpha}<\aleph_{\alpha+\beta+1})$ *if (and only if)* $\vdash(\alpha)(\beta)(\varphi(\alpha).\psi(\beta)\supset 2^{\aleph_\alpha}\neq\aleph_{\alpha+\beta+1})$.

K$_7$. $\vdash(\alpha)(\beta)(\varphi(\alpha).\psi(\beta)\supset 2^{\aleph_\alpha}<\aleph_{\alpha+\beta+1})$ *if (and only if)* $\vdash(\alpha)(\beta)(\varphi(\alpha).\psi(\beta)\supset 2^{\aleph_\alpha}\neq 2^{\aleph_{\alpha+\beta}})$.

The proof of these theorems is to be obtained from Theorem 1 quite similarly to the proof of Theorem 2.

The identically true formulas (e. g. $X=X$) being constructive we obtain immediately the following consequences (writing $\beta=1$):

K$_8$. $\vdash(\alpha)(2^{\aleph_\alpha}=\aleph_{\alpha+1})$ *if (and only if)* $\vdash(\alpha)(2^{\aleph_\alpha}\neq\aleph_{\alpha+2})$.

K$_9$. $\vdash(\alpha)(2^{\aleph_\alpha}=\aleph_{\alpha+1})$ *if (and only if)* $\vdash(\alpha)(2^{\aleph_\alpha}\neq 2^{\aleph_{\alpha+1}})$.

Finally, we mention that applying Theorem 2 to $A=0$, $N=0$ we obtain the result that if the axiom system Σ^* is consistent, then the axiom system Σ^*, $(\alpha)(2^{\aleph_\alpha}=\aleph_{\alpha+1})$ is consistent as well, since axiom $F_{0,0}=2^{\aleph_0}\geqq\aleph_1$ is a theorem of Σ^*. This special case is weaker than GÖDEL's result for the continuum hypothesis, since he reduces the consistency of Σ^* $(\alpha)(2^{\aleph_\alpha}=\aleph_{\alpha+1})$ to the consistency of Σ, but the model, constructed in Σ^* for our pourposes, has the interesting property that $\aleph_1$ is absolute with respect to this model.

§ 6. Supplement

We make the following conventions. Every metaconcept the definition of which depends on the axiom system is to be regarded as a metaconcept of Σ^*.

$\mathfrak{A}, \mathfrak{B}, \ldots$ denote particular classes. An arbitrary concept is briefly said absolute if it is absolute with respect to any model Δ_M satisfying the axiom system Σ^* and determined by a complete and almost universal class M.

6.1. In this section we deal with a characterization of the absolutely definable particular classes.

Let Δ_M be a model determined by a class M.

We are going to define the metaconcepts

"*to be a formula of* Δ_M" and

"*to be a concept of* Δ_M".

These concepts are to be defined "relativizing" the common recursive definition of the formulas and concepts as follows:

$$\mathfrak{Cls}(X, M), \quad \mathfrak{M}(X, M), \quad \mathfrak{E}(X, Y, M)$$

are notions of the model Δ_M (for fixed M).

Let $\overline{X}, \overline{Y}, \ldots$ and $\overline{x}, \overline{y}, \ldots$ denote the M-class and M-set variables, respectively. $\overline{X}, \overline{Y}, \ldots$ and $\overline{x}, \overline{y}, \ldots$ *are variables of the model* Δ_M.

Generally, if $\mathfrak{B}(X, M)$ is a notion of Δ_M, *the variable* Γ *defined by the stipulations*

$$(\Gamma)\varphi(\Gamma) \equiv (X)(\mathfrak{B}(X, M) \supset \varphi(X)),$$
$$(\exists \Gamma)\varphi(\Gamma) \equiv (\exists X)(\mathfrak{B}(X, M).\varphi(X))$$

is a variable of the model Δ_M.

The formula $\varphi(\Gamma_1, \ldots, \Gamma_n, M)$ is called to be *a formula of* Δ_M if every concept contained in it is a concept of Δ_M.

The *notion* $\mathfrak{B}(X_1, \ldots, X_n, M)$ *is called to be a notion of* Δ_M if it is introduced by the stipulations:

$$\mathfrak{B}(\overline{X}_1, \ldots, \overline{X}_n, M) \equiv \varphi(\overline{X}_1, \ldots, \overline{X}_n, M)$$

where $\varphi(\overline{X}_1, \ldots, \overline{X}_n, M)$ is a formula of Δ_M and $\mathfrak{B}(X_1, \ldots, X_n, M)$ is false for every $X_1, \ldots, X_n$ if one of the classes $X_1, \ldots, X_n$ is not an M class.

The *operation* $\mathfrak{A}(X_1, \ldots, X_n, M)$ *is called to be an operation of* Δ_M if it is defined by the following stipulations:

$\varphi(\overline{Y}, \overline{X}_1, \ldots, \overline{X}_n, M)$ is a formula of Δ_M for which

$$\vdash (\overline{X}_1)\ldots(\overline{X}_n)(\exists!\, \overline{Y})\, \varphi(\overline{Y}, \overline{X}_1, \ldots, \overline{X}_n, M) \text{ and } \varphi(\mathfrak{A}(\overline{X}_1, \ldots, \overline{X}_n, M), \overline{X}_1, \ldots, \overline{X}_n, M)$$

and $\mathfrak{A}(X_1, \ldots, X_n, M) = 0$ if one of the classes $X_1, \ldots, X_n$ is not an M-class.

Finally, the *particular class* $\mathfrak{A}$ *is called to be a particular class of* Δ_M if it is defined by $\varphi(\mathfrak{A}, M)$ where $\varphi(\overline{X}, M)$ is a formula of Δ_M and $\vdash (\exists!\, \overline{X})\, \varphi(\overline{X}, M)$.

The following facts are easy to see:

(1) The relativized of a formula or a concept with respect to Δ_M is a formula or a concept of Δ_M, respectively.

(2) If Δ_M satisfies the axiom system Σ^* and a metatheorem like M_1 [1], for example, is proved from the axioms of Σ^*, for a certain kind of formulas, then the relativized metatheorem is valid for the "relativized kind" of the formulas of Δ_M.

In what follows L, F denote the classes defined in [1], 9.3, 9.4, respectively, and Δ_L is denoted by Δ as in [1]. We prove the following metatheorem:

C$_1$. *The special class* $\mathfrak{A}$ *is absolutely definable in the stronger sense if and only if it is a special class of the model* Δ.

PROOF. The class L being complete and almost universal by [1], 9.51, 9.63, the class $\mathfrak{A}$ (if absolutely definable) is absolute with respect to Δ. This means that $\mathfrak{A}_L = \mathfrak{A}$, hence $\mathfrak{A}$ is a special class of Δ by (1).

Suppose now that $\mathfrak{A}$ is a special class of Δ.

This means that the defining postulate $\varphi(X)$ of $\mathfrak{A}$ is such that $\varphi(\overline{X})$ is a formula of Δ. On the other hand, it is easy to see that every formula of Σ^* is equivalent to a formula not containing other concepts than class variables and the notion $X \in Y$. Hence by (2) we may suppose that $\varphi(\overline{X})$ contains only L-class variables and the only notion $\mathfrak{E}(X, Y, L)$.

Therefore it is enough to prove that the notion $\mathfrak{E}(X, Y, L)$ and the L-class variables $\overline{X}, \overline{Y}, \ldots$ are absolute.

Let now M be an arbitrary complete and almost universal class for which $\psi_0^*(M)$ holds. Let $X_M, Y_M, \ldots$ and $x_M, y_M, \ldots$ denote the class and set variables of the model Δ_M, respectively.

First we prove that $\mathfrak{Cls}(X, L)$ is absolute with respect to Δ_M.

$$\mathfrak{Cls}_L(X_M) \equiv X_M \subseteq L.(u)(u \in L \supset u.X_M \in L)$$

by the definition 2.2.1 of $\mathfrak{Cls}_L(X)$. Using that the notions $X \in Y$, $X \subseteq Y$, the operation $X.Y$ and the special class L are absolute by 2.2.5 and 2.2.7, the relativized of $\mathfrak{Cls}_L(X_M)$ is the following:

$$(\mathfrak{Cls}_L(X_M))_M \equiv X_M \subseteq L.(u_M)(u_M \in L \supset u_M.X_M \in L).$$

We have to prove that $\mathfrak{Cls}_L(X_M) \equiv (\mathfrak{Cls}_L(X_M))_M$ for every X_M.

u_M being always a set, $\mathfrak{Cls}_L(X_M)$ implies obviously $(\mathfrak{Cls}_L(X_M))_M$. Suppose now that $(\mathfrak{Cls}_L(X_M))_M$ holds. Then $X_M \subseteq L$, and therefore it is enough to see that the implication $u \in L \supset u.X_M \in L$ holds for every set u. It holds for every $u \in M$ by the assumption $(\mathfrak{Cls}_L(X_M))_M$. On the other hand, L is absolute with respect to Δ_M by 2.2.7, hence, M being complete, $L \subseteq M$. It follows that if $u \notin M$, then $u \notin L$ and in this case the implication holds vacuously.

Thus $\mathfrak{Cls}_L(X)$ is proved to be absolute.

Now we can prove that $X \in_L Y$ is absolute. By 2.2.3 $X \in_L Y \equiv X \in Y . \mathfrak{Cls}_L(Y).X \in L$. But $X \in Y$ and L are absolute by 2.2.7, and we have just proved the absoluteness of $\mathfrak{Cls}_L(X)$. Hence $X \in_L Y$ is absolute. It remains to prove that the L-class variables $\overline{X}, \overline{Y}$ are absolute, i. e. $\mathfrak{Cls}_L(X) \equiv (\mathfrak{Cls}_L(X))_M$ *for every class* X. $\mathfrak{Cls}_L(X)$ being absolute, this holds for every X which is an M-class. If X is not an M-class, then the right-hand side is obviously false. Therefore we have to prove that if X is not an M-class, then it can not be an L-class, i. e.

$$\mathfrak{Cls}_L(X) \supset \mathfrak{Cls}_M(X),$$
$$\mathfrak{Cls}_L(X) \equiv X \subseteq L.(u)(u \in L \supset u.X \in L),$$
$$\mathfrak{Cls}_M(X) \equiv X \subseteq M.(u)(u \in M \supset u.X \in M).$$

Suppose now that $\mathfrak{Cls}_L(X)$ holds. As we have already seen, $L \subseteq M$, hence $X \subseteq M$. It remains to prove that $u.X \in M$ holds for every $u \in M$. Suppose $u \in M$. Then $u.X \subseteq X$, hence, X being an L-class, $u.X \subseteq L$. $u.X$ is a set, and therefore, L being almost universal, there exists a $v \in L$ such that $u.X \subseteq v$. We have $u.X = u.(X.v)$. X being an L-class, $X.v$ is an L-set, hence it is an M-set too. It follows that $u.X$, being the intersection of two M-sets, is an M-set too.

Thus the absoluteness of $\overline{X}, \overline{Y}, \ldots$ is proved.

As an immediate corollary of $\mathbf{C}_1$ we get the following characterization of the particular ordinal numbers absolutely definable in the stronger sense:

$\mathbf{C}_2$. *The particular ordinal number A is absolutely definable in the stronger sense if and only if it is a particular ordinal number of Δ.*

$\mathbf{C}_2$ follows from $\mathbf{C}_1$ taking into consideration that $\mathfrak{D}(X)$ is absolute by 2.2.7.

6.2. In this section we outline the proof of the fact that the constructive ordinal numbers of CHURCH—KLEENE are absolutely definable in the stronger sense. We have to define the axiomatical notion of a recursive ordinal number. It is obvious that for our purposes the definition of a metaconcept will be necessary.

We are going to define the metaconcept of a constructive particular ordinal number.

First we need some preliminary results.

We make the following convention: $\mathfrak{f}(n_1 \ldots n_k)$, $\mathfrak{g}(n_1 \ldots n_k)$ denote such operations for which $\mathfrak{f}(n_1, \ldots, n_k)$, $\mathfrak{g}(n_1 \ldots n_k), \ldots$ are integers for every $n_1, \ldots, n_k$ if $n_1, \ldots, n_k$ are integers and $\mathfrak{f}(X_1, \ldots, X_n) = 0$, $\mathfrak{g}(X_1, \ldots, X_n) = 0$ if some of the classes $X_1, \ldots, X_n$ is not an integer.

6.2.1. Are the operations $\mathfrak{g}(n_1, \ldots, n_m)$, $\mathfrak{h}_1(n_1, \ldots, n_k), \ldots, \mathfrak{h}_m(n_1, \ldots, n_k)$ normal, then there exists exactly one normal operation $\mathfrak{f}(n_1, \ldots, n_k)$ satisfying

$$\text{(I)} \qquad \mathfrak{f}(n_1, \ldots, n_k) = \mathfrak{g}(\mathfrak{h}_1(n_1, \ldots, n_k), \ldots, \mathfrak{h}_m(n_1, \ldots, n_k)).$$

6.2.2. Are the operations $\mathfrak{g}(n_2, \ldots, n_k)$, $\mathfrak{h}(n_1, n_1', n_2, \ldots, n_k)$ normal, then there exists exactly one normal operation $\mathfrak{f}(n_1, \ldots, n_k)$ satisfying

$$\text{(II)} \qquad \begin{aligned} \mathfrak{f}(0, n_2, \ldots, n_k) &= \mathfrak{g}(n_2, \ldots, n_k), \\ \mathfrak{f}(n_1 + 1, n_2, \ldots, n_k) &= \mathfrak{h}(n_1, \mathfrak{f}(n_1, n_2, \ldots, n_k), n_2, \ldots, n_k). \end{aligned}$$

The proofs can be carried out without difficulty (in case of 6.2.2 using M_8) thus we omit them.

The following formulas define normal operations:

6.2.3. DEF. $s(n) = n + 1$, $c(n) = 0$ $u_k^i(n_1, \ldots, n_k) = n_i$ $(1 \leq i \leq k)$.

6.2.4. DEF. The operation $\mathfrak{f}(n_1, \ldots, n_k)$ is called to be a *primitive re-*

cursive operation if it can be obtained from the operations defined in 6. 2. 3 by the successive applications of (I) and (II).

It follows easily from the definition:

6. 2. 5. Every primitive recursive operation is a normal operation.

We define the metaconcept of a general recursive operation using the characterization of general recursive functions given by KLEENE.

6. 2. 6. DEF. The operation $\mathfrak{f}(n_1, \ldots, n_k)$ is called to be a *general recursive operation* if there exist primitive recursive operations $\mathfrak{g}(m)$, $\mathfrak{h}(n_1, \ldots, n_k, n)$ for which

$$(1) \qquad (\exists n)(\mathfrak{h}(n_1, \ldots, n_k, n) = 0),$$

$$(2) \qquad (\mathfrak{h}(n_1, \ldots, n_k, n) = 0.(m)(m < n \supset \mathfrak{h}(n_1, \ldots, n_k, n) \neq 0)) \supset$$
$$\supset \mathfrak{f}(n_1, \ldots, n_k) = \mathfrak{g}(n).$$

We have

6. 2. 7. If the operations $\mathfrak{g}(m)$ and $\mathfrak{h}(n_1, \ldots, n_k, n)$ are primitive recursive and $\mathfrak{h}(n_1, \ldots, n_k, n)$ satisfies (1), then there exists exactly one normal operation which satisfies (2).

PROOF. Taking into consideration the convention made on p. 371. the formula

$$u \in \mathfrak{f}(n_1, \ldots, n_k) \equiv (\exists n)(u \in \mathfrak{g}(n).\mathfrak{h}(n_1, \ldots, n_k, n) = 0.$$
$$.(m)(m < n \supset \mathfrak{h}(n_1, \ldots, n_k, n) \neq 0))$$

defines a normal operation which satisfies (2).

As a consequence of 6. 2. 6 and 6. 2. 7 we have

6. 2. 8. Every general recursive operation is normal.

6. 2. 9. DEF. The particular class $\mathfrak{A}$ is said to be a *recursively enumerable relation* if there exists a general recursive operation $\mathfrak{f}(k, m, n)$ for which

$$(3) \qquad (k)(m)(\langle km\rangle \in \mathfrak{A} \equiv (\exists n)(\mathfrak{f}(k, m, n) = 0)).\mathfrak{A} \subseteq \omega^2.$$

It results from 6. 2. 8 by M_4 [1]:

6. 2. 10. Corresponding to every general recursive operation $\mathfrak{f}(k, m, n)$ there exists exactly one special class $\mathfrak{A}$ satisfying (3).

We define the metaconcept of a recursive ordinal number using the characterization of CHURCH—KLEENE's recursive ordinal numbers given by MARKVALD [5].

6. 2. 11. DEF. The particular ordinal number Λ is said to be a *recursive ordinal number* if there exists a recursively enumerable relation $\mathfrak{A}$ for which

$$\vdash (\exists X)(X\, \mathfrak{Isom}_{\mathfrak{A}, E}(\omega, \Lambda)).$$

Now using the metatheorems $AB_1, \ldots, AB_6$ and theorems 2.2.5—2.2.7 it offers no difficulty to prove step by step that every primitive recursive operation, general recursive operation, recursively enumerable relation are absolute, so we get the metatheorem:

C$_3$. *Every recursive ordinal number is absolutely definable in the stronger sense.*

It is obvious that in the same way one can prove for more general kind of special ordinal numbers Λ that Λ is absolutely definable. However, the theorems proved so far assure only the existence of denumerable absolutely definable ordinal numbers (in the stronger sense).

6.3. In this section we are going to prove a metatheorem which assures the existence of non-denumerable ordinal numbers absolutely definable in the weaker sense.

C$_4$. *If* Λ_1 *is an absolutely definable ordinal number in the weaker sense, then the same is true for* $\aleph_{\Lambda_1} = \Lambda$.

PROOF. We have to prove that Λ is absolute with respect to every model Δ_M determined by a complete and almost universal class M satisfying $\psi_0^*(M)$, and

(x) $(\alpha)(\alpha \leqq \Lambda \dotplus 1 \supset \aleph_M{}^{\text{‘}}\alpha = \aleph^{\text{‘}}\alpha)$.

Let M be a class satisfying the above conditions. $\Lambda_M = \aleph_M{}^{\text{‘}}\Lambda_{1M}$, since $X^{\text{‘}}Y$ is absolute. (x) implies $(\alpha)(\alpha \leqq \Lambda_1 \dotplus 1 \supset \aleph_M{}^{\text{‘}}\alpha = \aleph^{\text{‘}}\alpha)$, since $\Lambda_1 \leqq \Lambda$. Thus Λ_1 is absolute with respect to Δ_M — since it is absolutely definable in the weaker sense — that means $\Lambda_{1M} = \Lambda_1$.

Applying (x) to $\Lambda_1 = \alpha$ we get $\Lambda_M = \aleph_M{}^{\text{‘}}\Lambda_1 = \aleph^{\text{‘}}\Lambda_1 = \Lambda$. Q. e. d.

*

The following indices are to be considered as the continuations of the corresponding ones of [1]:

Normal concepts

2.1.4 $Z \in X \dotplus Y \equiv (\exists\alpha)(\exists\beta)(X = \alpha . Y = \beta . Z \in \mathfrak{N}(\alpha)^{\text{‘}}\beta) \vee$
$\vee \sim (\exists\alpha)(\exists\beta)(X = \alpha . Y = \beta) . Z \in X \dotplus 1$

2.1.11 $\mathfrak{Clos}_2(X, Y) \equiv X^{\text{‘‘}}Y \subseteq Y$; $\mathfrak{Clos}_3(X, Y) \equiv X^{\text{‘‘}}Y^2 \subseteq Y$

2.1.14 $Z \in 2^X \equiv Z \in \overline{\overline{\mathfrak{P}(X)}}$

2.2.1 $\mathfrak{Cls}(X, M) \equiv X \subseteq M . (u)(u \in M \supset u . X \in M)$

2.2.2 $\mathfrak{M}(X, M) \equiv X \in M$

2.2.3 $\mathfrak{E}(X, Y, M) \equiv \mathfrak{M}(X, M) . \mathfrak{Cls}(Y, M) . X \in Y$

3.1 $\mathfrak{Au}(X) \equiv (u)(u \subseteq X \supset (\exists v)(v \in X . u \subseteq v))$

3.2 $\mathfrak{Fcl}(X) \equiv (u)(v)(u \in X . v \in X \supset \mathfrak{F}_1(u, v) \in X . \dots . \mathfrak{F}_8(u, v) \in X)$

4.3.1 $U \in \mathfrak{g}_i(X, Y, U_1, U_2, Z) \equiv \mathfrak{D}(Z) . U \in \mathfrak{F}_i(X, Y)$ for $i = 1, \dots, 8$

$U \in \mathfrak{g}_9(X, Y, U_1, U_2, Z) \equiv \mathfrak{D}(Z) . U \in X . On$

$U \in \mathfrak{g}_{10}(X, Y, U_1, U_2, Z) \equiv \mathfrak{D}(Z) . U \in (X . U_1{}^{\backprime} Y)$

$U \in \mathfrak{g}_{11}(X, Y, U_1, U_2, Z) \equiv \mathfrak{D}(Z) . U \in (X . U_2{}^{\backprime} Y)$

4.3.3 $U \in \mathfrak{G}_i(X, Y, U_1, U_2, Z) \equiv \mathfrak{D}(Z) . U \in \mathfrak{F}_i(X, Y)$ for $i = 1, \dots, 8$

$U \in \mathfrak{G}_9(X, Y, U_1, U_2, Z) \equiv U \in O$

$U \in \mathfrak{G}_{10}(X, Y, U_1, U_2, Z) \equiv \mathfrak{D}(Z) . U \in X . U_1 . \mathfrak{W}(\mathfrak{g}(U_1, U_2, Z))$

$U \in \mathfrak{G}_{11}(X, Y, U_1, U_2, Z) \equiv \mathfrak{D}(Z) . U \in X . U_2 . \mathfrak{W}(\mathfrak{g}(U_1, U_2, Z))$

4.3.5 $U \in \mathfrak{M}(U_1, U_2, Z) \equiv \mathfrak{D}(Z) . U \in \mathfrak{W}(\mathfrak{G}(U_1, U_2, Z))$

4.3.6 $U \in \mathfrak{Od}(U_1, U_2, Z) \equiv \mathfrak{D}(Z) . ((\exists x)(\exists y)(U = \langle yx \rangle . \langle xy \rangle \in$
$\in \mathfrak{G}(U_1, U_2, Z) . (z)(z \in y \supset \sim \langle xz \rangle \in \mathfrak{G}(U_1, U_2, Z)))$

4.3.12 $U \in \mathfrak{As}(U_1, U_2, Z) \equiv \mathfrak{D}(Z) . ((\exists x)(\exists y)(U = \langle yx \rangle . y \in x . x \in$
$\in \mathfrak{M}(U_1, U_2, Z) . (z)(\mathfrak{Od}(U_1, U_2, Z){}^{\backprime} z < \mathfrak{Od}(U_1, U_2, Z){}^{\backprime} y \supset \sim z \in x)))$

4.3.18 $U \in \mathfrak{C}(U_1, U_2, Z) \equiv \mathfrak{D}(Z) .$
$. (U \in \mathfrak{Od}(U_1, U_2 . Z) | (\mathfrak{As}(U_1, U_2, Z) | \mathfrak{G}(U_1, U_2, Z)))$

4.6.1 $\mathfrak{h}(U_1, X, Y) \equiv U_1 \mathfrak{Fn}\, \omega_{X \dot{+} Y \dot{+} 1} . \mathfrak{Un}_2(U_1) . \mathfrak{W}(U_1) \subseteq \mathfrak{P}(\omega_X)$

4.6.4 $\mathfrak{k}(U_2, X, Y) \equiv U_2 \mathfrak{Fn}\, \omega_{X \dot{+} Y \dot{+} 1} . ((\alpha)(\alpha \in \omega_{X \dot{+} Y \dot{+} 1} \supset$
$\supset \mathfrak{D}(U_2{}^{\backprime}\alpha) = \alpha . \mathfrak{W}(U_2{}^{\backprime}\alpha) = \overline{\overline{\alpha}} . \mathfrak{Un}_2(U_2{}^{\backprime}\alpha) . \mathfrak{Rel}(U_2{}^{\backprime}\alpha)))$

4.8.1 $\mathfrak{Clos}(X)_\xi \equiv \mathfrak{Clos}_2(X, \mathfrak{C}_\xi) . \mathfrak{Clos}_2(X, K_1) . \mathfrak{Clos}_2(X, K_2) .$
$. \mathfrak{Clos}_3(X, I_0) . \dots . \mathfrak{Clos}_3(X, I_{11})$

4.8.2 $Z \in [X]_\xi \equiv Z \in [X]_{(\mathfrak{C}_\xi K_1, K_2)(I_0, \dots, I_{11})}$

Special symbols

2.1.4 $X \dot{+} Y$

2.1.13 $[X]_{(R_1 \dots R_i)(S_1 \dots S_j)}$

2.1.14 2^X

4.8.2 $[X]_\xi$

Letters and combinations of letters

p. 323	Δ_M		
2.1.3	$\mathfrak{N}(X)$		
2.1.11	$\mathfrak{Clos}_2(A, R)$, $\mathfrak{Clos}_3(A, R)$		
2.2.1	$\mathfrak{Cls}(X, M)$		
2.2.2	$\mathfrak{M}(X, M)$		
2.2.3	$\mathfrak{E}(X, Y, M)$		
3.1	$\mathfrak{Au}(X)$		
3.2	$\mathfrak{Fcl}(X)$		
p. 331	$\psi_0(X)$		
p. 331	$\psi_0^*(X)$		

4.1.1	S	[1], 9.2	Note that these letters have somewhat different meaning as in [1].
4.1.2	J	[1], 9.21	
4.1.3	$J_0, \ldots, J_{11}$	[1], 9.22	
4.1.5	K_1, K_2	[1], 9.24	

4.3.1	$\mathfrak{g}_i(X, Y, U_1, U_2, Z)$ $(i=1, \ldots, 11)$	Note that for the abbreviations of these operations different conventions are introduced. The first abbreviation is introduced on p. 345, in footnote 24, is valid in Sections 4.3, 4.4, 4.5. The second one is introduced on p. 353 and is used in Sections 4.6 and 4.7. Finally, the third one is introduced on p. 357 and is used in Sections 4.8, 4.9, 4.10.
4.3.2	$\mathfrak{g}(U_1, U_2, Z)$	
4.3.3	$\mathfrak{G}_i(X, Y, U_1, U_2, Z)$ $(i=1, \ldots, 11)$	
4.3.4	$\mathfrak{G}(U_1, U_2, Z)$	
4.3.5	$\mathfrak{M}(U_1, U_2, Z)$	
4.3.6	$\mathfrak{Od}(U_1, U_2, Z)$	
4.3.12	$\mathfrak{As}(U_1, U_2, Z)$	
4.3.18	$\mathfrak{C}(U_1, U_2, Z)$	

4.6.1	$\mathfrak{h}(U_1, X, Y)$	Note that the letters h, k are used in 4.8, 4.9, 4.10 in another sense, and if there is no danger of misunderstanding, k appears as an integer variable too.
p. 352	h	
4.6.4	$\mathfrak{k}(U_2, X, Y)$	
p. 353	k	

Technical terms

p. 321	particular ordinal number
p. 323	model determined by the class M
3. 1	almost universal
4. 3. 6	U_1, U_2, ξ-order of a set
4. 8. 1	ξ-closed (closed)
4. 8. 2	ξ-closure
5. 1	constructive formula
5. 2	absolutely definable particular class
5. 3	absolutely definable particular ordinal number in the stronger sense
5. 4	absolutely definable particular ordinal number in the weaker sense
6. 2. 4	primitive recursive operation
6. 2. 6	general recursive operation
6. 2. 9	recursively enumerable relation
6. 2. 11	recursive ordinal number

(Note that the word absolute is used for the abbreviation of different expressions of the form "absolute with respect to..." by the conventions made in the text.)

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ON GRAPHS IN WHICH TWO VERTICES ARE DISTINGUISHED

By

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(*Presented by* L. KALMÁR)

Introduction

At this moment it is elaborated no satisfactorily comprehensive mathematical theory in order to elucidate the questions of the analysis and synthesis of relay-contact networks. Although there appear papers frequently which investigate problems belonging to this field, it seems that the development leading to a complete theory requires yet much of labor. (The various minimization questions are accessible very difficultly by exact methods which are of sufficiently constructive nature.)

The problems of this character fall to the boundaries of the mathematical logic (especially the theory of truth functions) and the theory of graphs.[1] The publications mostly omit the application of the graph theory and consider only certain truth-functional formulae. This one-sidedness seems to be disadvantageous, we remind the reader of the similar opinion of Z. PAWLAK (see [7]). In order to demonstrate the success of this view we mention the paper [12] of B. TRACHTENBROT in which the connection of the graph-theoretical and truth-functional aspects leads to important results.[2]

The problems of this field require to use jointly the logic and the theory of graphs. However, there seem to be useful the purely graph-theoretical preparatory investigations. The present article gives some theorems of this nature: it contains certain structural results on graphs considered from the point of view how they connect two distinguished vertices. The investigations described here were published in Hungarian language in the papers [1], [2], [4]. Although the subject is essentially the same, the presentation of the matter has many differences from the Hungarian publications. These differences follow from two motives. Firstly, the relative perfectness of the matter makes possible to give a more conscious ordering of the results contrarily to the particular publications. Secondly, I rely here rather on the results of TRACHTENBROT [12].

[1] See e. g. [12] or [3] concerning boundary problems of what character can arise.

[2] His former paper [11] contains a part of the results of [12] in more concise form and without proofs.

The paper consists of three chapters. Chapter I gives the introduction of the most general concepts. Chapter II discusses especially the questions of the series and parallel decomposition, its main result (Theorem 3) gives equivalent conditions for a 2-graph not to be decomposable into single edges.[3]

Chapter III is devoted to the structural analysis of a class of 2-graphs. Since the paper [12] of TRACHTENBROT describes the decomposition of the 2-graphs into irreducible ones, the question of studying the 2-graphs structurally can be restricted by requiring the irreducibility. ("Irreducible" is meant in the sense of TRACHTENBROT, cf. Footnote [9].) § 5 characterizes the class of 2-graphs investigated and defines certain subgraphs in a graph belonging to this class. These subgraphs are: the 2-graph denoted by $\mathfrak{G}'$ and the n-graphs named bridges. The results of Chapter III determine which 2-graphs can occur as $\mathfrak{G}'$ and describe the possibilities of joining the bridges to $\mathfrak{G}'$. Only the inner structure of the bridges remains open to the complete description of these graphs.

I call the attention of the reader to the most immediate further problems: the question of investigating the complementary class of the irreducible 2-graphs (cf. Footnote [13]) and the question of considering the bridges (cf. Footnotes [17], [20]).

I wish to express my deep gratitude to Prof. T. GALLAI, Prof. L. KALMÁR and Dr. G. POLLÁK for their suggestions and assistance of various nature.

I. GENERAL PRELIMINARIES

§ 1

A *graph* is a finite collection of *vertices* and a finite collection of *edges*. Each edge connects two distinct vertices, named the *terminals* of the edge. The same pair of vertices can be joined by more than one edge. Each vertex is incident to at least one edge.

The edges of a graph are not oriented a priori. However, if we denote an edge by a letter (e. g. by k), then this edge is considered in a fixed orientation, correspondingly one of its terminals is called its *beginning vertex*, and the other one is called its *end vertex*.

[3] The equivalence of the properties α), γ) of Theorem 3 was exposed without proof both in the paper [9] of RIORDAN and SHANNON (p. 84, Footnote [5]) and in the abstract [10] of LUNTZ. Essentially the same result was arrived in the paper [5] of ELGOT and WRIGHT, I remark that the publication of this theorem in [1] has preceded their article.

A (finite) sequence

$$A_0, k_1, A_1, k_2, A_2, \ldots, A_{n-1}, k_n, A_n \qquad (n \geqq 0)$$

of vertices and edges is called an *arc* if the vertices $A_0, A_1, \ldots, A_{n-1}$ are pairwise distinct, and A_n is distinct from each of $A_1, A_2, \ldots, A_{n-1}$, and the beginning vertex of the edge k_i is A_{i-1}, and the end vertex of k_i is A_i (for any i, $1 \leqq i \leqq n$). An arc is called *chain* or *circuit,* according as $A_0 \neq A_n$ or $A_0 = A_n$, respectively. A_0 is the *beginning vertex* of the chain, A_n is the *end vertex* of the chain; "*terminals* of the chain" will be a common term for them.[4] $A_0 = A_n$ is called the *terminal* of the circuit. The vertices $A_1, A_2, \ldots, A_{n-1}$ are the *inner vertices* of the arc. If $n = 0$, then we speak of a *degenerated chain.* We make no distinction between the edge k and the chain A, k, B. Two chains are said to be *disjoint* if they have no edge in common and any common vertex is a terminal of both of them.[5]

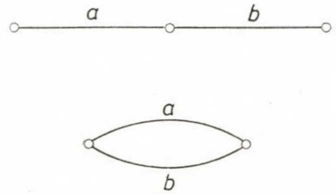

Fig. 1

The arcs will be denoted by Latin small letters. The terminals of a chain are eventually in round brackets, e. g. $a(PQ)$. If A and B are vertices of a chain and A precedes B in the chain, then $a[AB]$ denotes the subchain of a a between A and B. The converse direction is denoted as $(-1)^{\text{th}}$ power, e. g. $a^{-1}(QP)$, $a^{-1}[BA]$. If one of the equalities $k = a[AB]$ and $k^{-1} = a[BA]$ is true for the edge $k(AB)$ and the chain a, then we can express this fact by the following three manners, equivalently:

a contains *k*,

k is an edge of *a*,

k occurs in the chain *a*.

If the chains $a(AB)$ and $b(BC)$ have no vertex in common but B, then the product ab means the chain containing firstly the vertices and edges of a, after these the vertices and edges of b (in the original ordering), but B occurs only once in ab.

Let there be given a subset η of the set of the edges of a graph. The graph consisting of the elements of η and the vertices incident to them is called a *subgraph* of the graph.

[4] We shall use the word "terminal" in an analogous sense also without explicit definition.

[5] Therefore it is allowed that two disjoint chains join to each other as it can be seen on Fig. 1.

LEMMA 1. *Let α be an arbitrary subset of the set of the vertices of the graph $\mathfrak{G}$. We introduce a binary relation ϱ_α in the set of the edges of $\mathfrak{G}$ by what follows. ϱ_α holds for the edges k_1, k_2 if and only if there exists an arc whose first edge is k_1 and whose last edge is k_2 such that this arc contains no element of α as its inner vertex. Then ϱ_α is an equivalence relation.*

PROOF. ϱ_α is trivially reflexive and symmetrical. We must verify that it is transitive too. Let $a(AB)$ be an arc whose first and last edges are k_1 and k_2, respectively, and which contains no element of α innerly. Let $b(CD)$ be an arc having similar properties for k_2 and k_3. Let the first inner vertex of a which occurs also in b denoted by E. Then one of the arcs $a[AE]\cdot b[ED]$ and $a[AD]\cdot k_3^{-1}$ exists (according as $D \neq E$ or $D = E$, respectively) and has no element of α innerly; so ϱ_α holds also for k_1 and k_3.

§ 2

A connected graph is called a *graph with n terminals* (or briefly *n-graph*) if there are distinguished n vertices in it ($n \geqq 2$). We shall demand later some connection properties to the n-graphs. A vertex of an n-graph distinct from the terminals is called an *inner vertex* of them. A chain is called an *inner chain* if all inner vertices in it are inner vertices of the graph. An n-graph is *strictly connected* if

(1) it consists of a single edge (and the terminals of the graph are the terminals of the edge), or

(2a) its any edge is contained in an inner chain the terminals of which are terminals of the graph, and

(2b) any two vertices can be joined by an inner chain, and

(2c) there is no edge between any two terminals of the graph.

The present paper will investigate the 2-graphs especially. In a 2-graph the terminals are called *beginning vertex* and *end vertex* of the graph (denoted by P and Q, respectively). The 2-graph consisting of a single edge between its terminals is called the *trivial 2-graph*. If the beginning vertex of a chain is the beginning vertex of the graph and the end vertex of the chain is the end vertex of the graph, then this chain is called a *path*. A 2-graph is said to be *strongly connected* if its any edge is contained in a path.[6] A strictly connected 2-graph is also strongly connected.[7]

[6] The following equivalent conditions are known for a graph to be strongly connected (we use also concepts not yet introduced):

(a) any subgraph of the graph has at least two terminals (see [12], Lemma 2, p. 229),

(b) any vertex of the graph is contained in a path (see [1], Theorem 1, p. 213).

[7] Furthermore: a 2-graph is strictly connected if and only if it is an indecomposable or series decomposable strongly connected 2-graph.

Let a subgraph $\mathfrak{H}$ of the 2-graph $\mathfrak{G}$ be considered according to the subset η of the set ε of the edges of $\mathfrak{G}$. If we do not agree otherwise, then a vertex is called a *terminal* of $\mathfrak{H}$ if

it is a terminal of $\mathfrak{G}$ and belongs to $\mathfrak{H}$, or

there exists an element of η incident to it and there exists an element of $\varepsilon-\eta$ incident to it.[8] The whole graph and the one-edge subgraphs form obviously 2-subgraphs (i. e. subgraphs with 2 terminals) of a 2-graph. The 2-subgraphs different from these will be called *proper* ones. A 2-graph is *irreducible* if it differs from the graphs on Fig. 2 and has no proper 2-subgraph.[9] In [12] it is studied the decomposition of the 2-graphs into irreducible ones (§ 3).

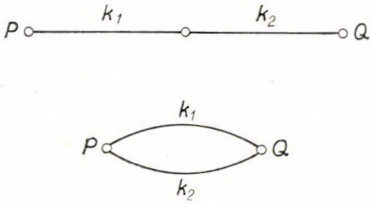

Fig. 2

A non-degenerated chain $a(AB)$ of a 2-graph is a *double chain* if there exist two paths b and c such that $b[AB]=a$ and $c[BA]=a^{-1}$. Thus there are defined also the *double edges*. Evidently, a 2-graph contains a double edge if and only if it contains a double chain.

Seven chains $a(AB)$, $b(PC)$, $c(CA)$, $d(CB)$, $e(AD)$, $f(BD)$, $g(DQ)$ of a 2-graph are said *to join to each other as the Wheatstone bridge* if

they are pairwise disjoint, and

a, c, d, e, f are non-degenerated, and

the vertices P, A, B, C, D, Q are pairwise distinct, only the coincidences $P=C$ or $Q=D$ are allowed (if b or g are degenerated, respectively).

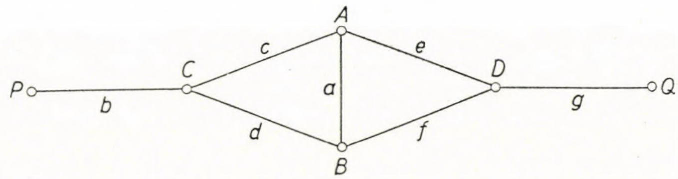

Fig. 3

(See Fig. 3.) If both of b and g are degenerated, then we speak of a *simple Wheatstone bridge*.

In Chapter II we shall consider strongly connected 2-graphs. Throughout Chapter III (except § 9) $\mathfrak{G}$ will denote a strongly connected irreducible 2-graph.

[8] It will occur one exceptional agreement in § 5 at defining the terminals of $\mathfrak{G}'$.

[9] We mention that we use the word "irreducible" in the same sense as TRACHTENBROT. This terminology differs from the one of the Hungarian publications: there the term "irreducible" has the same meaning as "indecomposable" in the present paper.

II. SERIES-PARALLEL DECOMPOSITION

§ 3

Let there be given a 2-graph $\mathfrak{G}$. Let two edges k_1, k_2 be equivalent if and only if there exists an arc whose first edge is k_1 and whose last edge is k_2 such that it contains neither P nor Q as its inner vertex (see Lemma 1). It is clear that each of the subgraphs belonging to the equivalence classes has exactly two terminals, namely the terminals of $\mathfrak{G}$. Let P be considered as the beginning vertex of each subgraph and Q as the end vertex of each subgraph; thus we have defined 2-graphs in $\mathfrak{G}$ such that any edge of $\mathfrak{G}$ is contained in exactly one of these 2-graphs. The 2-graphs now defined are called the *parallel components* of $\mathfrak{G}$. Any parallel component of $\mathfrak{G}$ is considered as a 2-graph having P as its beginning vertex and Q as its end vertex. If $\mathfrak{G}$ has at least two parallel components, then it is *parallel decomposable,* in the contrary case (i. e. if it has only one parallel component, namely itself) it is *parallel indecomposable.*

A vertex A is called a *knot vertex* of $\mathfrak{G}$ if every path of $\mathfrak{G}$ contains A as its inner vertex.

LEMMA 2. *Each path of $\mathfrak{G}$ contains the knot vertices in the same order.*

PROOF. Suppose that the lemma is false. This means that there are two knot vertices A, B and two paths a, b such that A precedes B in the path a and B precedes A in the path b. Let C be the first vertex of b which occurs also in the chain $a[BQ]$. Then C precedes A in b (because it must be B or a vertex preceding B in b), and there exists the path $b[PC]\cdot a[CQ]$. This path does not contain A in contradiction to the definition of the knot vertices.

Lemma 2 gives the possibility of introducing an ordering (denoted by $\prec$) in the set formed by the knot vertices and the terminals.

LEMMA 3. *Let the knot vertices of $\mathfrak{G}$ be denoted by $A_1, A_2, \dots, A_{n-1}$ (so that $A_1 \prec A_2 \prec \dots \prec A_{n-1}$), let an edge k and a path a be given. If k occurs in the chain $a[A_{i-1}A_i]$ and b is a path containing k, then $b[A_{i-1}A_i]$ contains k ($1 \leqq i \leqq n$, P is denoted also by A_0, Q is denoted also by A_n).*

PROOF. Suppose that k is contained in $b[A_{j-1}A_j]$ where $j \neq i$.

Case 1: $j<i$. Let B be the first vertex of b which occurs also in $a[A_{i-1}Q]$. Then $b[PB]\cdot a[BQ]$ is a path which does not contain A_{i-1}, what is a contradiction.

Case 2: $j>i$. The proof is similar, the roles of a and b, i and j are interchanged.

Now we define classes in the set of the edges of $\mathfrak{G}$: k belongs to the i^{th} class $(1 \leqq i \leqq n)$ if there exists a path a such that k is contained in $a[A_{i-1}A_i]$. Lemma 3 ensures that any edge is the element of exactly one class The subgraphs corresponding to the equivalence classes are called the *series components* of $\mathfrak{G}$. (The i^{th} series component is the series component which belongs to the i^{th} class.) We shall consider each series component after the following lemma as a 2-graph:

LEMMA 4. *Let the i^{th} and j^{th} series components be given* $(1 \leqq i < j \leqq n)$. *The following statements are true:*

if $i+1=j$, *then the two components have exactly one common vertex, namely* A_i,

if $i+1<j$, *then the two components have no vertex in common.*

PROOF. Assume that the lemma is not fulfilled.

Case 1: let the i^{th} component be the first one which has a vertex (distinct from A_i) common to some of the further components. Let A be common vertex in the i^{th} component such that A is the first vertex of similar nature on a suitable path a. The property supposed above of A means that there exists a path b such that A occurs in the chain $b[A_{j-1}A_j]$ where $j>i$. Then the path $a[PA]\cdot b[AQ]$ exists and cannot contain the vertex A_i, so A_i is no knot vertex. This contradiction disproves the hypothesis.

Case 2: let the i^{th} component be the last one which has a vertex (distinct from A_{i-1}) common to some of the preceding components. The inference symmetrical to Case 1 leads to a contradiction.

For any i $(1 \leqq i \leqq n)$ let A_{i-1} be the beginning vertex and let A_i be the end vertex of the i^{th} series component. If $\mathfrak{G}$ has at least two series components, then it is *series decomposable,* in the contrary case it is *series indecomposable.* $\mathfrak{G}$ is series indecomposable if and only if it has no knot vertex, i. e. for any inner vertex A of $\mathfrak{G}$ there exists a path which does not contain A. It is clear that a parallel decomposable 2-graph cannot have a knot vertex, so there holds

LEMMA 5. *Any parallel decomposable 2-graph is series indecomposable.*

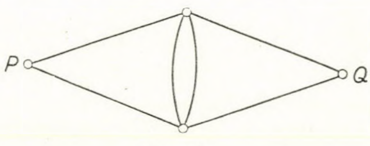

Fig. 4

If a 2-graph is parallel indecomposable and series indecomposable, then we call it to be *indecomposable.* Any irreducible 2-graph is indecomposable.[10]

We can introduce the concept of the *constituent* of a 2-graph $\mathfrak{G}$ inductively. $\mathfrak{G}$ itself is a constituent. If $\mathfrak{H}$ is a constituent of $\mathfrak{G}$, then any series

[10] The converse statement does not hold, let us investigate e. g. the graph on Fig. 4.

or parallel component of $\mathfrak{H}$ is also a constituent of $\mathfrak{G}$. (Evidently, if $\mathfrak{H}$ is a series component of the constituent $\mathfrak{H}'$, then $\mathfrak{H}$ is indecomposable or parallel decomposable; and vice versa.) It is clear that any 2-graph admits a unique decomposition into indecomposable constituents (the uniqueness is meant up to the ordering and association of the parallel components, up to the association of the series components.) It is evident that any constituent is strongly connected.

§ 4

THEOREM 1. *If $\mathfrak{G}$ is a non-trivial indecomposable 2-graph, then $\mathfrak{G}$ contains a pair of disjoint paths.*[11, 12]

PROOF. Let c be a path of $\mathfrak{G}$. A denotes the first vertex of c such that there exists a chain $a(AQ)$ disjoint to both of $c[PA]$ and $c[AQ]$. (There exists always a vertex $\neq Q$ of this property. This fact becomes evident if one considers a path such that its last edge differs from the last edge of c.) B denotes the last vertex of c such that there exists a chain $b(PB)$ disjoint to both of $c[PB]$ and $c[BQ]$.

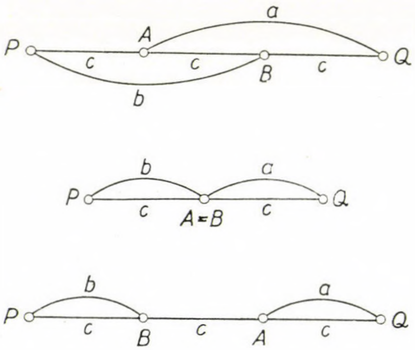

Fig. 5

Case 1: a and b are not disjoint. Let D be the first vertex of b which is contained also in a. Then $b[PD]\cdot a[DQ]$ is a path disjoint to c.

Case 2: a and b are disjoint. If $c[AB]$ exists, then the number of its edges is called the *index* of c. If $c[BA]$ exists and has n edges, then $-n$ is called the index of c. (In the case $A=B$ the index is 0.) Let now c be a path having maximal index. The further proof splits into three subcases (in each subcase see the corresponding part of Fig. 5), we remark that the second and third ones cannot occur really.

Case 2a: the index of c is positive. Then $b[PB]\cdot c[BQ]$ and $c[PA]\cdot a[AQ]$ are disjoint paths.

[11] This theorem was firstly formulated in [12] (p. 244). It can be deduced from the well-known theorem of MENGER (see e. g. [6], p. 244). The present proof differs from the one of TRACHTENBROT, and it is a modified version of the inference which proves in [1] immediately Theorem 2 (Corollary 1 of the present article).

[12] The statement is obvious for parallel decomposable graphs. The theorem can be exposed also as follows: if any two paths of a 2-graph have an inner vertex in common, then all the paths have an inner vertex in common.

Case 2b: the index of c is 0. Let us consider a path d which does not contain $A = B$. There exists a chain having the following properties: it is a part of d, its beginning vertex is on b or on $c[PA]$, its end vertex is on a or on $c[AQ]$, and it includes no proper subchain having these properties. In any possible cases one can find two disjoint paths.

Case 2c: the index of c is negative. Let us consider a path d which does not contain B. Its suitable subchain has its beginning vertex on b or on $c[PB]$ and its end vertex on a or on $c[BQ]$, and it is minimal among the subchains of this property. If the end vertex of this subchain lies on $c[BA]$, then we get a contradiction to the maximality of the index of c. In the other case the proof can be completed as in the case 2b.

THEOREM 2. *Any non-trivial indecomposable 2-graph* $\mathfrak{G}$ *contains five chains joining to each other as the simple Wheatstone bridge.*

PROOF. Let h_1 and h_2 be two paths of $\mathfrak{G}$ disjoint to each other. Let A and B be arbitrary inner vertices of h_1 and h_2, respectively. Let $l(AB)$ be a chain connecting A and B, containing neither P nor Q. Let the first vertex of l which lies on h_2 be denoted by C; let the last vertex of $l[AC]$ which lies on h_1 be denoted by D. Then the chains $a = l[DC]$, $c = h_1[PD]$, $d = h_2[PC]$, $e = h_1[DQ]$, $f = h_2[CQ]$ form a simple Wheatstone bridge.

THEOREM 3. *The following three properties are equivalent for a 2-graph* $\mathfrak{G}$:

α) $\mathfrak{G}$ *has a non-trivial indecomposable constituent,*

β) $\mathfrak{G}$ *contains chains joining to each other as the Wheatstone bridge,*

γ) $\mathfrak{G}$ *contains a double chain.*

PROOF. $\alpha) \to \beta)$. The non-trivial indecomposable constituent of $\mathfrak{G}$ has a simple Wheatstone bridge by Theorem 2. One can see that if some (series or parallel) component of a graph contains a (simple or non-simple) Wheatstone bridge, then the graph itself has a Wheatstone bridge. Hence the statement follows inductively.

$\beta) \to \gamma)$. The chain a of a Wheatstone bridge is evidently a double one.

$\gamma) \to \alpha)$. If no (series or parallel) component of $\mathfrak{G}$ contains a double chain, then $\mathfrak{G}$ cannot have such a one. So the falsity of α) implies the falsity of γ) inductively.

COROLLARY 1. *Any non-trivial indecomposable 2-graph contains a double chain.*

THEOREM 4. *If* $a_1(A_1B_1)$ *is a double chain of the 2-graph* $\mathfrak{G}$, *then* $\mathfrak{G}$ *contains chains joining to each other as the Wheatstone bridge such that* a_1 *or* a_1^{-1} *is included in* a.

Proof. There exists a double chain $a(AB)$ including (eventually non-properly) a_1 such that a is not included in any double chain properly. Let us consider two paths h_1, h_2 such that $h_1[AB] = a$ and $h_2[BA] = a^{-1}$. Let C be the last vertex of $h_2[PB]$ such that it occurs also in $h_1[PA]$. Let D be the first vertex of $h_2[AQ]$ which occurs also in $h_1[BQ]$. Then the chains a, $b = h_1[PC]$, $c = h_1[CA]$, $d = h_2[CB]$, $e = h_2[AD]$, $f = h_1[BD]$, $g = h_1[DQ]$ fulfil almost all the statements of the theorem immediately by their definition. We must yet prove only two statements. $h_2[AD]$ and $h_1[PA]$ are disjoint: if we suppose the contrary, then let the first inner vertex of $h_2[AD]$ which occurs in $h_1[PA]$ be denoted by F: there exists the path

$$h_1[PF] \cdot h_2^{-1}[FB] \cdot h_1[BQ],$$

so $h_2[BF]$ is a double chain including a properly; this contradicts the definition of a. We can prove in a similar way that $h_2[CB]$ and $h_1[BQ]$ are disjoint (if the last inner vertex of $h_2[CB]$ common to $h_1[BQ]$ is denoted by G, then there exists $h_1[PA] \cdot h_2^{-1}[AG] \cdot h_1[GQ]$).

III. BRIDGES AND THEIR SITUATION

§ 5

Throughout §§ 5—8 let $\mathfrak{G}$ be an irreducible 2-graph which satisfies the following condition: there exists a path of $\mathfrak{G}$ in which no double edge occurs.[13, 14] We are going to define a 2-graph $\mathfrak{G}'$ and some n-graphs (named bridges) in $\mathfrak{G}$, so that each edge of $\mathfrak{G}$ is contained in exactly one of $\mathfrak{G}'$ and the bridges.

Let $\mathfrak{G}'$ be the subgraph of $\mathfrak{G}$ determined by the set of edges of the paths without double edge. $\mathfrak{G}'$ contains the terminals of $\mathfrak{G}$. We agree that only they are considered to be the terminals of $\mathfrak{G}'$. So $\mathfrak{G}'$ becomes a 2-graph

[13] The complementary class of the irreducible 2-graphs can be characterized as the collection of the graphs whose each path contains a double edge. It is desirable to get structural results on these graphs, but it seems to be a difficult problem to arrive deeper ones.

[14] In [2] I have introduced a more complicated classification of the indecomposable 2-graphs into seven classes. Four of these classes were shown to be empty by the following theorem of G. Pollák (see [8]): to any edge k of a non-trivial indecomposable graph there exists a double edge k^* such that a suitable path contains both of k and k^*. The remaining three classes correspond to the two classes now introduced, it is made there a distinction according as every edge beside $\mathfrak{G}'$ is a double one or not. So the situation of bridges is studied separately in [2] and [4] corresponding to this distinction.

with the terminals P and Q. It is evidently strongly connected and contains no double edge (a double edge of $\mathfrak{G}'$ would be a double edge of $\mathfrak{G}$). So Theorem 3 ensures that every indecomposable constituent of $\mathfrak{G}'$ is a trivial 2-graph.

The edges being beside $\mathfrak{G}'$ form a subgraph $\mathfrak{G}^0$ of $\mathfrak{G}$ (non-connected in general). Let us introduce a classification of these edges in accordance with Lemma 1, being α the set of the vertices occurring in both of $\mathfrak{G}^0$ and $\mathfrak{G}'$. The subgraphs of $\mathfrak{G}$, each of which is determined by the edges of an equivalence class, are named *bridges*. We define the terminals of the bridges in the sense of § 2. Each bridge is a strictly connected n-subgraph of $\mathfrak{G}$.

Fig. 6 shows some graphs on which one can study the former definitions. (The last graph does not belong to the class introduced here.)

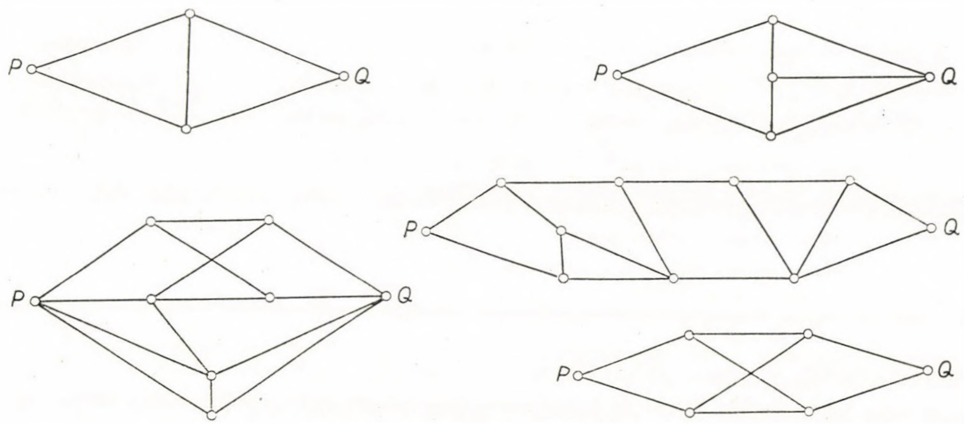

Fig. 6

THEOREM 5. *Let $\mathfrak{G}$ be an irreducible 2-graph having at least one path without double edge. Then either*

(a) *$\mathfrak{G}'$ is series decomposable and its each component is a trivial 2-graph, or*

(b) *$\mathfrak{G}'$ is parallel decomposable and its each component is a graph described in* (a).

PROOF. Suppose that the theorem is false for some $\mathfrak{G}$. This means that one of the following four statements holds for the structure of $\mathfrak{G}'$:

α) $\mathfrak{G}'$ is a trivial 2-graph,

β) $\mathfrak{G}'$ is series decomposable and its some component is parallel decomposable,

γ) $\mathfrak{G}'$ is parallel decomposable and its some component is a trivial 2-graph,

δ) $\mathfrak{G}'$ is parallel decomposable and has a component whose some series component is parallel decomposable.

The impossibility of α) and γ) follows from the irreducibility (especially: from the parallel indecomposability) of $\mathfrak{G}$.

The impossibility of β). Let the parallel decomposable component of $\mathfrak{G}'$ be denoted by $\mathfrak{H}$.

Case 1: there is no bridge having a terminal (innerly) in $\mathfrak{H}$ and another terminal beside $\mathfrak{H}$. Then $\mathfrak{H}$ and the bridges whose terminals are in $\mathfrak{H}$ form a 2-subgraph of $\mathfrak{G}$. This contradicts the irreducibility of $\mathfrak{G}$.

Case 2: there exists a bridge $\mathfrak{D}$ such that its terminal A is an inner vertex of $\mathfrak{H}$ and its terminal B is not contained in $\mathfrak{H}$. Our purpose is to show that some edges of $\mathfrak{H}$ are double edges, this will be a contradiction. Indeed, if B is contained in a series component which precedes $\mathfrak{H}$, then let $a(BA)$ be an inner chain of $\mathfrak{D}$, let b be a path of $\mathfrak{G}'$ containing A, let c be a path of $\mathfrak{G}'$ which does not contain A, let C be the last vertex of $b[PA]$ which occurs also in c. Then there exists the path

$$d[PB]\cdot a\cdot b^{-1}[AC]\cdot c[CQ]$$

where d is an arbitrary path of $\mathfrak{G}'$ containing b. So $\mathfrak{H}$ contains a double chain (Fig. 7). If the series component containing B is preceded by $\mathfrak{H}$, then the proof is analogous.

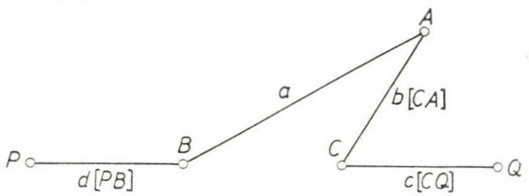

Fig. 7

The impossibility of δ) can be verified similarly to the impossibility of β). Let $\mathfrak{H}$ be a parallel decomposable component of the component $\mathfrak{H}'$ of $\mathfrak{G}'$. The former inference can be applied without any essential modification ($\mathfrak{G}'$ must be substituted by $\mathfrak{H}'$); it remains a further case: the case if $\mathfrak{H}'$ does not contain B (but B is contained in another parallel component of $\mathfrak{G}'$). One can apply the former inference also in this case.

§ 6

In what follows the preceding relation $\prec$ will be applied always in the set of the vertices of $\mathfrak{G}'$. If the first statement of Theorem 5 holds for $\mathfrak{G}'$, then it is not necessary to elucidate nearerly the meaning of the sign $\prec$; if the second statement holds for $\mathfrak{G}'$, then $\prec$ is meant in some component of $\mathfrak{G}'$.

We shall introduce a distinction among the bridges. A bridge $\mathfrak{D}$ of the 2-graph $\mathfrak{G}$ is called a *pseudo-bridge* if either

(a) $\mathfrak{G}'$ is series decomposable, or

(b) $\mathfrak{G}'$ is parallel decomposable and there exists a component $\mathfrak{H}$ of $\mathfrak{G}'$ which contains all the terminals of $\mathfrak{D}$.

The other bridges are called *real bridges.* (Evidently, a bridge is real if and only if there exist two parallel components $\mathfrak{H}_1, \mathfrak{H}_2$ of $\mathfrak{G}'$ such that two suitable terminals A and B of $\mathfrak{D}$ are inner vertices of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively.)

The terminal A of the bridge $\mathfrak{D}$ is called an *inner terminal* of $\mathfrak{D}$ if it differs from the terminals of $\mathfrak{G}$. The real bridges $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are called to be *equivalent* if their all inner terminals coincide (i. e. each inner terminal of $\mathfrak{D}_1$ is an inner terminal of $\mathfrak{D}_2$, and conversely).

Let us consider the bridge $\mathfrak{D}$, let A and B be two terminals of $\mathfrak{D}$ such that $A \prec B$. Then an inner chain $a(AB)$ of $\mathfrak{D}$ is called a *shunt*. Now let there be given two terminals C, D of $\mathfrak{D}$ being (innerly) in distinct parallel components of $\mathfrak{G}'$. Then an inner chain $c(CD)$ of $\mathfrak{D}$ is called a *cross-chain*. "*Direct chain*" will be a common term for shunts and cross-chains.[15]

Let $\mathfrak{G}'$ be parallel decomposable. Two distinct components $\mathfrak{H}_\alpha, \mathfrak{H}_\beta$ of $\mathfrak{G}'$ are called to be *associated* if there exists a sequence

$$\mathfrak{H}_\alpha = \mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n = \mathfrak{H}_\beta \qquad (n \geqq 2)$$

such that for any i $(2 \leqq i \leqq n)$ there exists a (real) bridge $\mathfrak{D}$ two suitable terminals of which are inner vertices of $\mathfrak{H}_{i-1}$ and $\mathfrak{H}_i$, respectively. Each component is regarded to be associated with itself. So we have defined an equivalence relation.

LEMMA 6. *The following three statements are equivalent for an inner chain $a(AB)$ of a bridge $\mathfrak{D}$ where A and B are inner terminals of $\mathfrak{D}$:*

α) $a(AB)$ is a direct chain of $\mathfrak{D}$,

β) there exists a path b of $\mathfrak{G}$ such that $b[AB] = a$ and each edge of the chains $b[PA]$, $b[BQ]$ belongs to $\mathfrak{G}'$,

γ) there exists a path b of $\mathfrak{G}$ such that $b[AB] = a$.

PROOF. $\alpha) \rightarrow \beta)$. Let c and d be two paths of $\mathfrak{G}'$ containing A and B, respectively. Then $c[PA] \cdot a \cdot d[BQ]$ exists and satisfies β).

$\beta) \rightarrow \gamma)$ is trivial.

$\gamma) \rightarrow \alpha)$. Suppose that γ) is fulfilled but α) is not true. The falsity of α) means that A and B cannot lie in distinct parallel components of $\mathfrak{G}'$ and $B \prec A$ holds. So the pair of vertices A, B has the two properties:

(a′) A precedes B in the path b,

(b′) $B \prec A$.

[15] If a is a shunt, then a^{-1} is no direct chain. If a is a cross-chain, then also a^{-1} is a cross-chain.

Let us consider all the pairs of vertices C, D of $\mathfrak{G}'$ having the properties:

(a) D precedes C in the path b, and

(b) $C \prec D$

(the chain b is fixed). One can show that there exist two vertices C', D' satisfying (a), (b), and the property:

(c) if $C \prec X \prec D$ holds for a vertex X of $\mathfrak{G}'$, then X does not occur in b.[16]

Let now e be a path of $\mathfrak{G}'$ containing C and D. Then the path

$$b[PD] \cdot e^{-1}[DC] \cdot b[CQ]$$

exists, so $e[CD]$ is a double chain. This contradiction proves the lemma.

LEMMA 7. *An edge $k(AB)$ of a bridge $\mathfrak{D}$ is double if and only if there exist two direct chains a, b of $\mathfrak{D}$ such that $a[AB]=k$ and $b[BA]=k^{-1}$.*

This lemma follows immediately from the equivalence of the properties α) and β) in the preceding lemma.

THEOREM 6. *Any shunt of a bridge $\mathfrak{D}$ of $\mathfrak{G}$ contains a double edge.*[17]

PROOF. Let there be given a shunt a of $\mathfrak{D}$. The statement β) of Lemma 6 holds for a; if a contains no double edge, then each edge of b belongs to $\mathfrak{G}'$ (by the definition of $\mathfrak{G}'$), what is a contradiction.

THEOREM 7. *To each inner vertex A of $\mathfrak{G}'$ there exists a bridge $\mathfrak{D}$ such that A is a terminal of $\mathfrak{D}$. If $\mathfrak{G}'$ is parallel decomposable, then its any two components are associated. A bridge cannot include a 2-subgraph consisting of more than one edge. Let the equivalent (real) bridges $\mathfrak{D}_\alpha$, $\mathfrak{D}_\beta$ be considered; one of them has at least three terminals.*[18]

PROOF. The propositions of the theorem are implied obviously by the irreducibility of $\mathfrak{G}$. The second statement is perhaps not so evident as the other ones; if it were false, then let the equivalence classes of the components

[16] This fact can be verified in the following manner. We start with a pair of vertices C, D satisfying (a) and (b). If there exists a vertex X occurring in b which satisfies $C \prec X \prec D$, then we have three possibilities: X can occur in $b[PD]$ or in $b[DC]$ or in $b[CQ]$. In each of these cases the statements (a) and (b) are fulfilled either by C and X or by X and D. This inference leads in a finite number of steps to a pair of vertices which satisfies (c).

[17] The analogous statement is evident for cross-chains. Lemma 7 makes clear what this restriction means for the inner structure of $\mathfrak{D}$ ("inner structure" is thought together with the partial ordering $\prec$ of the terminals).

[18] I. e. if they have two inner terminals, then either P or Q occurs as a terminal of one of them.

of $\mathfrak{G}'$ be formed according to the relation of being associated. If we regard these classes and to any class the bridges which have the terminals in it, then we get that $\mathfrak{G}$ is parallel decomposable.

Theorem 6 and the third statement of Theorem 7 imply

COROLLARY 2. *If a bridge has only two terminals, then it consists of one edge and it is a real bridge.*

THEOREM 8. *Let us consider two inner chains $f(AB)$ and $g(CD)$ of a bridge $\mathfrak{D}$. If one of the following assumptions α), β), γ), δ) is fulfilled, then f and g are not disjoint:*

α) $P \preceq A \prec C \prec B \prec D \preceq Q$.

β) $P \prec A \prec D \prec Q$ and $C \prec B$ are valid in two distinct parallel components of $\mathfrak{G}'$, there holds at most one of the coincidences $P = C$ and $Q = B$.[19]

γ) $P \prec A \prec C \prec B \prec Q$ and $P \prec D \prec Q$ are valid in distinct parallel components of $\mathfrak{G}'$.

δ) $P \prec A \prec C \prec Q$, $P \prec B \prec Q$, $P \prec D \prec Q$ are valid in three distinct parallel components of $\mathfrak{G}'$.

PROOF. Suppose that f and g are disjoint. If a, b, c denote some suitable paths of $\mathfrak{G}'$, then the paths

$$a[PA]\cdot f\cdot a^{-1}[BC]\cdot g\cdot a[DQ],$$
$$b[PC]\cdot g\cdot a^{-1}[DA]\cdot f\cdot b[BQ],$$
$$a[PA]\cdot f\cdot a^{-1}[BC]\cdot g\cdot b[DQ],$$
$$c[PD]\cdot g^{-1}\cdot a^{-1}[CA]\cdot f\cdot b[BQ]$$

exist in the cases α), β), γ), δ), respectively. So some subchain of a is shown to be a double chain, what is a contradiction.

LEMMA 8. *Let A, B be terminals of a bridge $\mathfrak{D}_\alpha$ and let C, D be terminals of a bridge $\mathfrak{D}_\beta$ (distinct to $\mathfrak{D}_\alpha$). Then A, B, C, D fulfil none of the statements α), β), γ), δ) occurring in Theorem 8.*[20]

PROOF. If we suppose that A, B, C, D satisfy some of α), β), γ), δ) and consider two inner chains $f(AB)$, $g(CD)$ of $\mathfrak{D}_\alpha$, $\mathfrak{D}_\beta$, respectively, then the inference seen in the proof of Theorem 8 leads to a contradiction.

[19] The case if both of them hold is included in α).

[20] Theorem 8 and Lemma 8 could be condensed into one theorem. (Indeed, in [2] and [4] I have chosen this way for exposing them.) In the present paper the method of separating them is followed for methodical reason: the statement of Lemma 8 will be contained in further results more explicitly, however, the present paper does not investigate the possibility of improving Theorem 8.

§ 7

THEOREM 9. *Let $\mathfrak{G}'$ be parallel decomposable. Then the real bridges of $\mathfrak{G}$ satisfy the following three statements:*

(1) *If the equivalent bridges $\mathfrak{D}_\alpha, \mathfrak{D}_\beta$ have at least four inner terminals, then any two of these terminals are in distinct components of $\mathfrak{G}'$. If the equivalent bridges $\mathfrak{D}_\alpha, \mathfrak{D}_\beta$ have exactly three inner terminals, and these terminals lie in two components of $\mathfrak{G}'$, then all terminals of $\mathfrak{D}_\alpha$ and all terminals of $\mathfrak{D}_\beta$ are inner ones.*

(2) *Let $A_\alpha \prec A_\beta$ hold for two suitable inner terminals A_α, A_β of the non-equivalent bridges $\mathfrak{D}_\alpha, \mathfrak{D}_\beta$, respectively. Then P does not occur among the terminals of $\mathfrak{D}_\beta$, Q does not occur among the terminals of $\mathfrak{D}_\alpha$; and $A'_\alpha \preceq A'_\beta$ holds for any two inner terminals A'_α, A'_β of $\mathfrak{D}_\alpha, \mathfrak{D}_\beta$, respectively, if they are in a common component of $\mathfrak{G}'$.*

(3) *Suppose that $\mathfrak{G}'$ has at least three components and each of the distinct bridges $\mathfrak{D}_\alpha, \mathfrak{D}_\beta$ has an inner terminal in $\mathfrak{H}_1$ where $\mathfrak{H}_1$ is a component of $\mathfrak{G}'$ and $\mathfrak{H}_1$ has at least two inner vertices which are terminals of real bridges. Then there exists a component $\mathfrak{H}_2$ such that if a bridge has an inner terminal in $\mathfrak{H}_1$, then it has inner terminals only in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, and $\mathfrak{H}_2$ has only one inner vertex which is a terminal of some real bridge.*

The theorem follows from the following lemma and the fact that Lemma 8 is absolutely true.

LEMMA 9. *The following propositions are equivalent for the real bridges of $\mathfrak{G}$:*

(a) *Lemma 8 is valid for any two bridges.*

(b) *Theorem 9 is valid for the bridges.*

(c) *Let $\mathfrak{D}_\alpha, \mathfrak{D}_\beta$ be distinct real bridges. Let A_α, A_β be inner terminals of $\mathfrak{D}_\alpha, \mathfrak{D}_\beta$, respectively, satisfying $A_\alpha \prec A_\beta$ in some component $\mathfrak{H}_1$ of $\mathfrak{G}'$. Let A'_α be an inner terminal of $\mathfrak{D}_\alpha$ being in a component $\mathfrak{H}_2 (\neq \mathfrak{H}_1)$ of $\mathfrak{G}'$. Then one of the relations $A_\alpha \preceq A'_\beta$ and $A'_\alpha \preceq A'_\beta$ is true for any inner terminal A'_β of $\mathfrak{D}_\beta$, further P does not occur among the terminals of $\mathfrak{D}_\beta$ and Q does not occur among the terminals of $\mathfrak{D}_\alpha$. — It is valid the statement which can be got from the preceding part of* (c) *by converting the signs $\prec$ and $\preceq$, by interchanging P and Q.*

PROOF. (a)→(c). Let the supposition of (c) be fulfilled. If the conclusion were false for A'_β, then either A'_β would be an inner vertex of a component of $\mathfrak{G}'$ distinct from both of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, or it would be true one of $A'_\beta \prec A_\alpha$ and $A'_\beta \prec A'_\alpha$. Consider the cross-chains $f(A_\alpha A'_\alpha)$ and $g(A_\beta A'_\beta)$ of $\mathfrak{D}_\alpha$ and $\mathfrak{D}_\beta$, respectively. The existence of f and g contradicts

Lemma 8. We get a contradiction to Lemma 8 also in the case when the statement of (c) for P and Q is false.

(c)→(b). First we prove (1). Let A, B, C be inner terminals of the equivalent bridges $\mathfrak{D}_\alpha$ and $\mathfrak{D}_\beta$ satisfying $A \prec B$, do not let C be in the same component as A and B. We can suppose that $A \prec X \prec B$ implies that X is not a terminal of $\mathfrak{D}_\alpha$ and $\mathfrak{D}_\beta$. Now we investigate an arbitrary terminal D of $\mathfrak{D}_\alpha$ and $\mathfrak{D}_\beta$; we are going to apply (c). By the substitution of A, B, C, D for $A_\alpha, A_\beta, A'_\alpha, A'_\beta$, respectively, (c) ensures that either $C \preceq D$ or $A \preceq D$ holds. If we apply the symmetrical part of (c), then a similar inference leads to the validity of $D \preceq C$ or $D \preceq B$. These preceding relations mean that D is equal to one of A, B, C.

Let the assumption of (2) be true, and let us suppose that $A_\alpha \prec X \prec A_\beta$ implies that X is not an inner terminal of $\mathfrak{D}_\alpha$ or $\mathfrak{D}_\beta$. The statements for P and Q follow trivially from (c). If A_α and A'_α lie in distinct components of $\mathfrak{G}'$, then (c) implies evidently the statement to be proved. In the other case $A_\alpha, A_\beta, A'_\alpha, A'_\beta$ are knot vertices of the same component $\mathfrak{H}$ of $\mathfrak{G}'$; our purpose is to show that $A'_\beta \prec A'_\alpha$ implies that $\mathfrak{D}_\alpha$ and $\mathfrak{D}_\beta$ are equivalent bridges. Indeed, we have only one possibility for avoiding the contradiction to (c): there holds $A'_\beta = A_\alpha \prec A_\beta = A'_\alpha$, and $\mathfrak{H}_1$ contains no other inner terminal of $\mathfrak{D}_\alpha$ or $\mathfrak{D}_\beta$. Let us now consider arbitrary further inner terminals A^*_α, A^*_β of $\mathfrak{D}_\alpha, \mathfrak{D}_\beta$, respectively, (c) ensures both of $A^*_\alpha \preceq A^*_\beta$ and $A^*_\beta \preceq A^*_\alpha$. This means that $A^*_\alpha = A^*_\beta$, hence $\mathfrak{D}_\alpha$ and $\mathfrak{D}_\beta$ are equivalent bridges.

Let the supposition of (3) be fulfilled. Then there exist two suitable different bridges $\mathfrak{D}_\alpha, \mathfrak{D}_\beta$ such that A_α and A_β are distinct knot vertices of $\mathfrak{H}_1$ where A_α, A_β are terminals of $\mathfrak{D}_\alpha, \mathfrak{D}_\beta$, respectively. Let $\mathfrak{H}_2 (\neq \mathfrak{H}_1)$ be a component of $\mathfrak{G}'$ containing some inner terminal of $\mathfrak{D}_\alpha$. Then (c) ensures that each terminal of $\mathfrak{D}_\beta$ lies in $\mathfrak{H}_1$ or $\mathfrak{H}_2$. Furthermore, a similar inference shows that every terminal of $\mathfrak{D}_\alpha$ is contained in $\mathfrak{H}_1$ or $\mathfrak{H}_2$. If $\mathfrak{D}_\gamma$ is an arbitrary bridge having an inner terminal in $\mathfrak{H}_1$, then the same inference can be carried out for $\mathfrak{D}_\gamma$ and for one of $\mathfrak{D}_\alpha$ and $\mathfrak{D}_\beta$, consequently also $\mathfrak{D}_\gamma$ has all its terminals in $\mathfrak{H}_1$ and $\mathfrak{H}_2$. We must yet prove the last statement of (3). Since $\mathfrak{G}'$ consists of at least three components and its any two components are associated by Theorem 7, there exists a bridge $\mathfrak{D}_\delta$ having some inner terminals in $\mathfrak{H}_2$ and in a component of $\mathfrak{G}'$ different from $\mathfrak{H}_1$ and $\mathfrak{H}_2$. ($\mathfrak{D}_\delta$ cannot have an inner terminal in $\mathfrak{H}_1$.) So there exist two non-void classes of the bridges having inner terminals in $\mathfrak{H}_2$: the bridges having an inner terminal in $\mathfrak{H}_1$ and the bridges having no inner terminal in $\mathfrak{H}_1$. If we choose one representative from both classes, then (c) ensures that there exists exactly one knot vertex of $\mathfrak{H}_2$ occurring as a terminal of (at least) one of the representatives. Since the choice can happen arbitrarily, $\mathfrak{H}_2$ has only one inner vertex which occurs as a terminal of a bridge.

(b)→(a). If (a) were not fulfilled, then there would exist two (different) bridges $\mathfrak{D}_\alpha$, $\mathfrak{D}_\beta$ such that suitable terminals A, B of $\mathfrak{D}_\alpha$ and suitable terminals C, D of $\mathfrak{D}_\beta$ satisfy one of α), β), γ), δ) in Theorem 8. In each of these cases one can see that some of the statements (1), (2), (3) occurring in Theorem 9 is not fulfilled.

THEOREM 10. *Let $\mathfrak{G}'$ be parallel decomposable. If $\mathfrak{D}$ is a pseudo-bridge of $\mathfrak{G}$, then $\mathfrak{D}$ has exactly three terminals, namely P, Q and the inner vertex A of some component of $\mathfrak{G}'$ which consists of two edges.*

PROOF. Suppose that the theorem is not satisfied by a pseudo-bridge $\mathfrak{D}$. Corollary 2 assures that $\mathfrak{D}$ has at least three terminals, this means that the component $\mathfrak{H}_1$ of $\mathfrak{G}'$ containing the inner terminals of $\mathfrak{D}$ has (at least) two inner vertices. Since any two components of $\mathfrak{G}'$ must be associated by Theorem 7, $\mathfrak{H}_1$ contains an inner vertex A which is a terminal of some real bridge. Taking into account the fact that any inner vertex of $\mathfrak{H}_1$ occurs as a terminal of a bridge (Theorem 7), we can separate two cases.

Case 1: $\mathfrak{H}_1$ has three vertices A, B, C such that $P \preceq A \prec C \prec B \preceq Q$ is valid, there holds at most one of the equalities, A and B are terminals of a pseudo-bridge, C is a terminal of a real bridge. This contradicts either the case γ) or (if one of the equalities holds) the case β) of Lemma 8.

Case 2: if three vertices A, B, C of $\mathfrak{H}_1$ satisfy $P \preceq A \prec C \prec B \preceq Q$ and A, B are terminals of the same pseudo-bridge, then C does not occur as a terminal of a real one. This statement can be expressed in the following manner. Let $A_1, A_2, \ldots, A_n$ be the inner vertices of $\mathfrak{H}_1$ which occur as terminals of real bridges (according to their ordering on the unique path h of $\mathfrak{H}_1$), let P and Q be denoted by A_0 and A_{n+1}, respectively, too. So there must exist to any pseudo-bridge $\mathfrak{D}$ a chain $h[A_{i-1}A_i]$ $(i = 1, 2, \ldots, n+1)$ such that all terminals of $\mathfrak{D}$ are in $h[A_{i-1}A_i]$. If $\mathfrak{D}_\alpha$ denotes a fixed pseudo-bridge, then it is clear that A_{i-1} and A_i terminate a 2-subgraph which contains (at least) the edges of $h[A_{i-1}A_i]$ and $\mathfrak{D}_\alpha$. This contradicts the irreducibility of $\mathfrak{G}$.

COROLLARY 3. *If $\mathfrak{G}'$ is parallel decomposable and A is an inner vertex of $\mathfrak{G}'$, then A is a terminal of some real bridge.*

Having this corollary, the reader can formulate the case (3) of Theorem 9 in a somewhat simpler form.

THEOREM 11. *Let $\mathfrak{G}'$ be series decomposable. Then one of the following two statements holds:*

α) $\mathfrak{G}'$ consists of two edges, $\mathfrak{G}'$ has at least one (pseudo-)bridge, each vertex of $\mathfrak{G}'$ is a terminal of any bridge.

β) $\mathfrak{G}'$ consists of at least three edges, $\mathfrak{G}$ has exactly one (pseudo-)bridge, each vertex of $\mathfrak{G}'$ is a terminal of the bridge.

PROOF. If $\mathfrak{G}'$ has two edges, then the further part of α) is implied by the irreducibility of $\mathfrak{G}$. If $\mathfrak{G}'$ has three or more edges, then it is to be proved that $\mathfrak{G}$ has only one bridge. Assume that there are at least two bridges; we distinguish two cases.

Case 1: there are two distinct bridges $\mathfrak{D}_\alpha, \mathfrak{D}_\beta$ such that $P \preceq A \prec C \prec \prec B \prec D \preceq Q$ is valid for suitable terminals A, B of $\mathfrak{D}_\alpha$ and for suitable ones C, D of $\mathfrak{D}_\beta$. This contradicts Lemma 8.

Case 2: the condition of the preceding case is not fulfilled. We are showing firstly the following fact: there exists a bridge $\mathfrak{D}_\alpha$ and a vertex A of $\mathfrak{G}$ such that A is not a terminal of $\mathfrak{D}_\alpha$. If this statement were false, then each vertex of $\mathfrak{G}'$ would be a terminal of any bridge; since $\mathfrak{G}'$ contains a least four vertices, it would hold the condition of Case 1.

We want to get a contradiction to the irreducibility of $\mathfrak{G}$. Let $\mathfrak{H}$ be the (unique) path of $\mathfrak{G}'$, let B be the last vertex of $h[PA]$ which is a terminal of $\mathfrak{D}_\alpha$, let C be the first vertex of $h[AQ]$ which is a terminal of $\mathfrak{D}_\alpha$. If we regard that the condition defining Case 1 does not hold, it becomes clear that any bridge (different from $\mathfrak{D}_\alpha$) of $\mathfrak{G}$ has all its terminals either beside $h[BC]$ or in $h[BC]$. This means that $\mathfrak{G}$ contains a proper 2-subgraph with the terminals B, C: this subgraph consists of the edges of $h[BC]$ and of the edges of some bridges.

§ 8

Theorems 5—11 have detailed some necessary conditions for the situation of the bridges. The following theorem will express that together these conditions are sufficient too.

THEOREM 12. *Let an arbitrary 2-graph $\mathfrak{G}$ be given. Let $\mathfrak{G}'$ denote a subgraph of $\mathfrak{G}$ which contains the terminals of $\mathfrak{G}$ and has the structure described in the statements* (a), (b) *of Theorem 5. Let the edges of $\mathfrak{G}$ which are not contained in $\mathfrak{G}'$ be classified as it is exposed in § 5. Let the n-subgraphs according to this classification be called H-graphs. If the H-graphs are strictly connected n-graphs and (after introducing the concepts occurring) Theorems 6—11 preserve the validity for them, then $\mathfrak{G}$ is a strongly connected irreducible 2-graph having the property that a path b of $\mathfrak{G}$ contains a double edge if and only if b has at least one edge which does not belong to $\mathfrak{G}'$.*

PROOF. A) $\mathfrak{G}$ *is strongly connected.* Any edge of $\mathfrak{G}'$ is evidently contained in a path. Let k be an edge of a bridge $\mathfrak{D}$. It is contained in an inner chain a whose terminals are terminals of $\mathfrak{D}$ (cf. the definition of the strictly connected n-graphs). At least one of a and a^{-1} is a direct chain, the further proof coincides with proving the implication $\alpha)\rightarrow\beta)$ in Lemma 6.

B) $\mathfrak{G}$ *is irreducible.* Let A, B be arbitrary distinct vertices of $\mathfrak{G}$. (There are the following possibilities in order to choose them: they are vertices of $\mathfrak{G}'$, — inner vertices of some bridge, — inner vertices of two bridges, — one of them belongs to $\mathfrak{G}'$ and the other is an inner vertex of a bridge.) Let us form the subgraphs of G in accordance with Lemma 1, taking $\{A, B\}$ as α. Theorems 7, 10 and 11 ensure that any formed subgraph either consists of one edge or has at least three terminals. (If there exists an edge between A and B, then this edge itself is a subgraph. The other edges form one or two subgraphs; the second of these cases can occur only if A and B belong to $\mathfrak{G}'$, $\mathfrak{G}'$ consists of two parallel components and A, B are inner vertices of distinct components, in this case P and Q are in different subgraphs.)

C) *If a path contains an edge beside* $\mathfrak{G}'$, *then it contains a double edge* by Theorem 6. (We must consider a subchain of the path which is a direct chain of a bridge.)

D) *No edge of* $\mathfrak{G}'$ *is a double one.* Suppose the contrary. Let $k(AB)$ be an edge of $\mathfrak{G}'$ such that $k=a[AB]=b^{-1}[AB]$ where a is the path of $\mathfrak{G}'$ containing k and b is a suitable path of $\mathfrak{G}$. Let the ordering relation $\prec$ be defined for the vertices of $\mathfrak{G}'$ occurring in b; let $X \prec Y$ be valid if X precedes Y in b. Let C be the first vertex of b such that there exists a vertex X satisfying both of $X \prec C$ and $C \prec X$. Let D be the last vertex of b which lies in the chain $g[PC]$ where g is the unique path of $\mathfrak{G}'$ containing C. Evidently, $C \prec D$. The edge of b immediately preceding C is an edge of some bridge, the same is true for the edge of b following D immediately. Let us consider the chains $b[EC]$ and $b[DF]$ which are uniquely determined by the assumption that E and F belong to $\mathfrak{G}'$ and their no inner vertex is contained in $\mathfrak{G}'$. The definitions of C and D assure that each of $F \prec C$, $D \prec E$ and $F \prec E$ is false. So the existence of $b[EC]$ and $b[DF]$ contradicts either Theorem 8 or Lemma 8. The proof becomes complete by the remark that Lemma 8 follows from Theorem 9 by Lemma 9.

§ 9

In this § we consider (in general) reducible 2-graphs.

It can arise the question what is the criterion in order that an edge of an arbitrary 2-graph $\mathfrak{G}$ should be a double one, if we know the irreducible 2-graphs occurring in the decomposition of $\mathfrak{G}$ due to TRACHTENBROT. This problem can be elucidated successively by the evident

THEOREM 13. *Let us consider the 2-graph* $\mathfrak{G}^*$, *let the edges of* $\mathfrak{G}^*$ *be denoted by* $k_1, \dots, k_n$. *Let the 2-graph* $\mathfrak{G}_i$ *be substituted for the edge* k_i *(for any* i, $1 \leqq i \leqq n$*), let the 2-graph resulting be denoted by* $\mathfrak{G}$. *An edge* k *of* $\mathfrak{G}_i$ *becomes a double edge of* $\mathfrak{G}$ *if and only if either*

k is a double edge of $\mathfrak{G}_i$, or

k_i is a double edge of $\mathfrak{G}^*$.

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SOME REMARKS ON THE DIMENSION AND ENTROPY OF RANDOM VARIABLES

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Let ξ be a real random variable. Let us put

$$\xi_n = \frac{1}{n}[n\xi], \quad \mathbf{P}\left(\xi_n = \frac{k}{n}\right) = \mathbf{P}\left(\frac{k}{n} \leqq \xi < \frac{k+1}{n}\right) = p_{nk}. \tag{1}$$

The dimension of ξ is defined as the limit

$$\mathbf{d}(\xi) = \lim_{n\to\infty} \frac{\mathbf{H}_0(\xi_n)}{\log n} \tag{2}$$

(provided this exists) where $\mathbf{H}_0(\xi_n)$ denotes the entropy of the discrete random variable $\xi_n = \frac{1}{n}[n\xi]$, as defined by SHANNON:

$$\mathbf{H}_0(\xi_n) = \sum_{k=-\infty}^{+\infty} p_{nk} \log \frac{1}{p_{nk}} \tag{3}$$

(the log is to be understood with respect to the base 2).

If the finite limit

$$\mathbf{H}_d(\xi) = \lim_{n\to\infty} (\mathbf{H}_0(\xi_n) - d \log n) \tag{4}$$

exists, we call it the d-dimensional entropy of ξ.

These notions were introduced by A. RÉNYI (see [1], [2]). He proved [2] that if the distribution of ξ is absolutely continuous and $\mathbf{H}_0(\xi_1) = \mathbf{H}_0([\xi]) < +\infty$, then $\mathbf{d}(\xi) = 1$ and

$$\mathbf{H}_1(\xi) = \int_{-\infty}^{+\infty} f(x) \log \frac{1}{f(x)} dx,$$

provided that this integral exists. ($f(x)$ denotes the density function of ξ.)

In this paper we investigate the problem whether this theorem of RÉNYI is reversible or not. We shall prove that the distribution of ξ is necessarily absolutely continuous if $\mathbf{H}_0([\xi]) < +\infty$ and $\mathbf{H}_0(\xi_n) - \log n$ does not tend to $-\infty$, and then (and only then) $\mathbf{H}_1(\xi)$ exists (Theorems 1 and 2). But a necessary and sufficient condition for the absolute continuity of the distribution of ξ cannot be given in terms of the asymptotic behaviour of $\mathbf{H}_0(\xi_n)$ (Theorem 4). The results obtained can be generalized also for random vectors (Theorem 5).

THEOREM 1. *If the distribution of a real random variable* ξ *is not absolutely continuous and* $\mathbf{H}_0([\xi])<+\infty$, *then*

$$\mathbf{H}_0(\xi_n)-\log n \to -\infty \quad \textit{for} \quad n\to+\infty.$$

PROOF. Since the distribution of ξ is not absolutely continuous, there exists a positive number $\varepsilon>0$ such that for every $\delta>0$ there is a finite number of disjoint intervals $I_1', I_2', \ldots, I_{r(\delta)}'$ for which $\sum_{i=1}^{r(\delta)} \mu(I_i')<\frac{\delta}{2}$ (μ denotes the Lebesgue measure) but $\sum_{i=1}^{r(\delta)} \mathbf{P}(\xi\in I_i')>\varepsilon$. Let $n>\frac{4r(\delta)}{\delta}$ be an arbitrary fixed integer. Let us consider all the intervals $I_j=\left[\frac{k_j}{n}, \frac{k_j+1}{n}\right)$ (the k_j's are integers) having at least one point common to $\sum_{i=1}^{r(\delta)} I_i'$. Their number will be denoted by $s=s(n,\delta)$. Then

$$\sum_{j=1}^{s} p_{nk_j}=\sum_{j=1}^{s}\mathbf{P}(\xi\in I_j)\geqq\sum_{i=1}^{r(\delta)}\mathbf{P}(\xi\in I_i')>\varepsilon, \tag{5}$$

further, as for each I_i' there are at most two I_j's having a point common to I_i' but not contained in it $(i=1,2,\ldots,r(\delta))$, and therefore there are at most $2r(\delta)$ I_j's, not contained in $\sum_{i=1}^{r(\delta)} I_i'$, we have

$$\frac{s}{n}=\sum_{j=1}^{s}\mu(I_j)\leqq\sum_{i=1}^{r(\delta)}\mu(I_i')+2r(\delta)\frac{1}{n}<\frac{\delta}{2}+2r(\delta)\frac{\delta}{4r(\delta)}=\delta. \tag{6}$$

Let us put

$$\sum_{\substack{k=ln\\ k\neq k_1,\ldots,k_s}}^{(l+1)n-1} p_{nk}=p_{1l}' \quad (p_{1l}'\leqq p_{1l}); \qquad \sum_{\substack{k=ln\\ k\neq k_1,\ldots,k_s}}^{(l+1)n-1} 1=\nu_l \quad (\nu_l\leqq n).$$

Then, applying Jensen's inequality to the convex function $-\log x$, we obtain

$$\mathbf{H}_0(\xi_n)=\sum_{k=-\infty}^{+\infty} p_{nk}\log\frac{1}{p_{nk}}=\sum_{l=-\infty}^{+\infty}\sum_{\substack{k=ln\\ k\neq k_1,\ldots,k_s}}^{(l+1)n-1} p_{nk}\log\frac{1}{p_{nk}}+\sum_{j=1}^{s}p_{nk_j}\log\frac{1}{p_{nk_j}}\leqq$$

$$\leqq\sum_{l=-\infty}^{+\infty}p_{1l}'\log\frac{\nu_l}{p_{1l}'}+\left(\sum_{j=1}^{s}p_{nk_j}\right)\log\frac{s}{\sum_{j=1}^{s}p_{nk_j}}\leqq \tag{7}$$

$$\leqq\sum_{l=-\infty}^{+\infty}p_{1l}'\log\frac{1}{p_{1l}'}+\sum_{l=-\infty}^{\infty}p_{1l}'\log n+\left(\sum_{j=1}^{s}p_{nk_j}\right)\log\frac{1}{\sum_{j=1}^{s}p_{nk_j}}+\left(\sum_{j=1}^{s}p_{nk_j}\right)\log s.$$

The function $x \log \frac{1}{x}$ is monotonically increasing for $x \leqq \frac{1}{e}$, therefore we have, except for at most two values of l, $p'_{1l} \log \frac{1}{p'_{1l}} \leqq p_{1l} \log \frac{1}{p_{1l}}$. Since, further, the maximum of $x \log \frac{1}{x}$ is $\frac{\log e}{e}$ and by (6) $s < \delta n$, we obtain from (7)

$$\text{(8)} \quad \begin{aligned} \mathbf{H}_0(\xi_n) &\leqq \sum_{l=-\infty}^{+\infty} p_{1l} \log \frac{1}{p_{1l}} + 2\frac{\log e}{e} + \log n \sum_{l=-\infty}^{+\infty} p'_{1l} + \frac{\log e}{e} + \left(\sum_{j=1}^{s} p_{nk_j}\right) \log \delta n = \\ &= \mathbf{H}_0(\xi_1) + 3\frac{\log e}{e} + \log n \left(\sum_{l=-\infty}^{+\infty} p_{1l} + \sum_{j=1}^{s} p_{nk_j}\right) + \left(\sum_{j=1}^{s} p_{nk_j}\right) \log \delta. \end{aligned}$$

As we have $\sum_{l=-\infty}^{+\infty} p'_{1l} + \sum_{j=1}^{s} p_{nk_j} = \sum_{k=-\infty}^{+\infty} p_{nk} = 1$ and by (5) $\sum_{j=1}^{s} p_{nk_j} > \varepsilon$, thus we obtain from (8), if $\delta < 1$:

$$\text{(9)} \qquad \mathbf{H}_0(\xi_n) \leqq \mathbf{H}_0(\xi_1) + 3\frac{\log e}{e} + \log n + \varepsilon \log \delta.$$

As (9) is true for every $n > \frac{4r(\delta)}{\delta}$, we have

$$\text{(10)} \qquad \overline{\lim_{n \to \infty}} (\mathbf{H}_0(\xi_n) - \log n) \leqq \mathbf{H}_0(\xi_1) + 3\frac{\log e}{e} + \varepsilon \log \delta.$$

Here $\mathbf{H}_0(\xi_1) = \mathbf{H}_0([\xi])$ is a finite constant, $\varepsilon > 0$ is fixed and $\delta > 0$ can be chosen arbitrarily small, therefore we have $\lim_{n \to \infty} (\mathbf{H}_0(\xi_n) - \log n) = -\infty$. Thus Theorem 1 is proved.

THEOREM 2. *For all real random variables* ξ *with* $\mathbf{H}_0([\xi]) < +\infty$ *there exists the limit* $\lim_{n \to \infty} (\mathbf{H}_0(\xi_n) - \log n)$; *it is either finite or equal to* $-\infty$.

PROOF. Using the well-known formula $\mathbf{H}_0(\xi, \eta) = \mathbf{H}_0(\eta) + \mathbf{H}_0(\xi \mid \eta)$ (where $\mathbf{H}_0(\xi, \eta)$ denotes the entropy of the joint distribution of ξ and η and $\mathbf{H}_0(\xi \mid \eta)$ denotes the conditional entropy of ξ under the condition that the value of η is given) and the inequality $\mathbf{H}_0(\eta) \leqq \log N$ (if N is the number of the possible values of η) (see e. g. [1]), we obtain for any two integers k and l

$$\text{(11)} \qquad \mathbf{H}_0(\xi_k) = \mathbf{H}_0(\xi_l, \xi_k) - \mathbf{H}_0(\xi_l \mid \xi_k) \geqq \mathbf{H}_0(\xi_l) - \log \left(\left[\frac{l}{k}\right] + 2\right).$$

In fact, if the value of ξ_k is fixed, ξ_l can take on at most $\left[\frac{l}{k}\right] + 2$ different values, thus $\mathbf{H}_0(\xi_l\,|\,\xi_k) \leqq \log\left(\left[\frac{l}{k}\right]+2\right)$ and, on the other hand, $\mathbf{H}_0(\xi_l, \xi_k) \geqq \mathbf{H}_0(\xi_l)$.

Now for any fixed $\varepsilon>0$ we have $\left[\frac{l}{k}\right]+2 \leqq (1+\varepsilon)\frac{l}{k}$ if $\varepsilon\frac{l}{k}\geqq 2$; thus from (11) we obtain for $l \geqq \frac{2}{\varepsilon}k$

$$\mathbf{H}_0(\xi_k) \geqq \mathbf{H}_0(\xi_l) - \log\left((1+\varepsilon)\frac{l}{k}\right) = \mathbf{H}_0(\xi_l) - \log l + \log k - \log(1+\varepsilon),$$

i. e.

$$\mathbf{H}_0(\xi_k) - \log k \geqq \mathbf{H}_0(\xi_l) - \log l - \log(1+\varepsilon) \quad \text{if} \quad l \geqq \frac{2}{\varepsilon}k. \tag{12}$$

From (12) we obtain

$$\mathbf{H}_0(\xi_k) - \log k \geqq \overline{\lim_{l\to\infty}}\,(\mathbf{H}_0(\xi_l) - \log l) - \log(1+\varepsilon), \tag{13}$$

whence

$$\underline{\lim_{k\to\infty}}\,(\mathbf{H}_0(\xi_k) - \log k) \geqq \overline{\lim_{l\to\infty}}\,(\mathbf{H}_0(\xi_l) - \log l) - \log(1+\varepsilon). \tag{14}$$

As $\varepsilon>0$ can be chosen arbitrarily small, the existence of $\lim\limits_{n\to\infty}(\mathbf{H}_0(\xi_n) - \log n)$ follows from (14). From (13) we obtain that this limit cannot be greater than $\mathbf{H}_0(\xi_1) - \log 1 = \mathbf{H}_0([\xi]) < +\infty$. Thus Theorem 2 is proved.

REMARK. For the existence of the one-dimensional entropy of ξ by Theorem 2 it is sufficient (and necessary) that $\overline{\lim\limits_{n\to\infty}}\,(\mathbf{H}_0(\xi_n) - \log n) > -\infty$. In this case the distribution of ξ is by Theorem 1 absolutely continuous.

THEOREM 3. *If the distribution of ξ is absolutely continuous, and $\mathbf{H}_0([\xi]) < +\infty$, then, denoting by $f(x)$ the density function of ξ, we have*

$$\lim_{n\to\infty}(\mathbf{H}_0(\xi_n) - \log n) = \int_{-\infty}^{+\infty} f(x)\log\frac{1}{f(x)}\,dx = \mathfrak{I}. \tag{15}$$

REMARK. In the case when $\mathfrak{I}$ is finite (15) was proved by A. RÉNYI [2]. His proof could be applied, with some changes, also in the case $\mathfrak{I} = -\infty$. Here we modify somewhat this proof, so that the separation of the cases $\mathfrak{I} > -\infty$ and $\mathfrak{I} = -\infty$ becomes unnecessary and the proof is shorter also in the case when $\mathfrak{I}$ is finite.

PROOF. The integral $\mathfrak{I}$ cannot be meaningless or equal to $+\infty$. In fact, let us put $E = \{x : f(x) < 1\}$, $I_l = \{x : l \leqq x < l+1\}$. Then by Jensen's inequality, generalized for integrals, we obtain

$$\int_{-\infty}^{+\infty} \left| f(x) \log \frac{1}{f(x)} \right|_+ dx = \int_E f(x) \log \frac{1}{f(x)} dx = \sum_{l=-\infty}^{+\infty} \int_{EI_l} f(x) \log \frac{1}{f(x)} dx \leqq$$

$$\leqq \sum_{l=-\infty}^{+\infty} \left(\int_{EI_l} f(x) dx \right) \log \frac{\mu(EI_l)}{\int_{EI_l} f(x) dx} \leqq \sum_{l=-\infty}^{+\infty} \left(\int_{EI_l} f(x) dx \right) \log \frac{1}{\int_{EI_l} f(x) dx} \tag{16}$$

($|x|_+$ and $|x|_-$ denote the positive and negative parts of x, respectively, $x = |x|_+ - |x|_-$).

Here is $\int_{EI_l} f(x) dx \leqq \int_{I_l} f(x) dx = p_{1l}$. Using that the function $x \log \frac{1}{x}$ is increasing for $x \leqq \frac{1}{e}$ and $p_{1l} \leqq \frac{1}{e}$ except at most for two values of l and the maximum of $x \log \frac{1}{x}$ is $\frac{\log e}{e}$, we obtain from (16)

$$\int_{-\infty}^{+\infty} \left| f(x) \log \frac{1}{f(x)} \right|_+ dx \leqq \sum_{l=-\infty}^{+\infty} p_{1l} \log \frac{1}{p_{1l}} + 2 \frac{\log e}{e} = \mathbf{H}_0([\xi]) + 2 \frac{\log e}{e} < +\infty. \tag{17}$$

This shows that $\mathfrak{I}$ cannot be meaningless or equal to $+\infty$. Now let us put

$$g_n(x) = n p_{nk} \log \frac{1}{n p_{nk}} = \left(\left(\frac{1}{n} \right)^{-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) dx \right) \log \frac{1}{\left(\frac{1}{n} \right)^{-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) dx} \tag{18}$$

$$\text{for} \quad \frac{k}{n} \leqq x < \frac{k+1}{n},$$

then

$$\lim_{n \to \infty} g_n(x) = f(x) \log \frac{1}{f(x)} \qquad \text{almost everywhere.} \tag{19}$$

Therefore we have by the theorem of Lebesque $\left(\text{using that } g_n(x) \leqq \frac{\log e}{e} \right)$

$$\lim_{n \to \infty} \int_{-L}^{+L} |g_n(x)|_+ dx = \int_{-L}^{+L} \left| f(x) \log \frac{1}{f(x)} \right|_+ dx, \tag{20}$$

and by the lemma of Fatou

$$(21)\qquad \underline{\lim}_{n\to\infty}\int_{-L}^{+L}|g_n(x)|_-dx \geqq \int_{-L}^{+L}\left|f(x)\log\frac{1}{f(x)}\right|_-dx$$

for any positive integer L.

Subtracting (21) from (20) we get

$$(22)\qquad \overline{\lim}_{n\to\infty}\int_{-L}^{+L}g_n(x)dx \leqq \int_{-L}^{+L}f(x)\log\frac{1}{f(x)}\,dx.$$

Let us choose $L_0=L_0(\varepsilon)$ so that $\sum_{l<-L_0 \text{ or } l\geqq L_0} p_{1l}\log\frac{1}{p_{1l}}<\varepsilon$ (this is possible for any $\varepsilon>0$, since $\sum_{l=-\infty}^{+\infty}p_{1l}\log\frac{1}{p_{1l}}=\mathbf{H}_0([\xi])<+\infty$).

Then, using that by Jensen's inequality $\sum_{k=ln}^{(l+1)n-1}p_{nk}\log\frac{1}{np_{nk}}\leqq p_{1l}\log\frac{1}{p_{1l}}$, we have for $L\geqq L_0$

$$(23)\qquad \int_{-L}^{+L}g_n(x)dx=\sum_{k=-Ln}^{Ln-1}p_{nk}\log\frac{1}{np_{nk}}\geqq\sum_{k=-\infty}^{+\infty}p_{nk}\log\frac{1}{np_{nk}}-\varepsilon=\mathbf{H}_0(\xi_n)-\log n-\varepsilon.$$

From (22) and (23) we obtain

$$\overline{\lim}_{n\to\infty}(\mathbf{H}_0(\xi_n)-\log n)-\varepsilon\leqq\int_{-L}^{+L}f(x)\log\frac{1}{f(x)}dx \quad \text{for every} \quad L\geqq L_0$$

thus

$$(24)\qquad \overline{\lim}_{n\to\infty}(\mathbf{H}_0(\xi_n)-\log n)\leqq\int_{-\infty}^{+\infty}f(x)\log\frac{1}{f(x)}\,dx+\varepsilon.$$

Here $\varepsilon>0$ is arbitrary, therefore we have

$$(25)\qquad \overline{\lim}_{n\to\infty}(\mathbf{H}_0(\xi_n)-\log n)\leqq\int_{-\infty}^{+\infty}f(x)\log\frac{1}{f(x)}\,dx=\mathfrak{J}.$$

On the other hand, by Jensen's inequality for integrals,

$$\mathfrak{J}=\int_{-\infty}^{+\infty} f(x)\log\frac{1}{f(x)}\,dx=\sum_{k=-\infty}^{+\infty}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\log\frac{1}{f(x)}\,dx\leqq$$

$$\leqq\sum_{k=-\infty}^{+\infty}\left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,dx\right)\log\frac{\frac{1}{n}}{\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,dx}=\sum_{k=-\infty}^{+\infty} p_{nk}\log\frac{1}{np_{nk}}=\mathbf{H}_0(\xi_n)-\log n. \tag{26}$$

(25) and (26) prove (15).

REMARK. Theorem 3 combined with Theorem 1 gives a new proof for Theorem 2. Compared with the proof thus obtained the original proof of Theorem 2 has the advantage of being completely elementary.

Now we turn to prove that Theorem 1 cannot be sharpened in the sense that $\mathbf{H}_0(\xi_n)-\log n$ can tend to $-\infty$ arbitrarily slowly for a suitable ξ with a not absolutely continuous distribution. Hence, as by Theorem 3 $\mathbf{H}_0(\xi_n)-\log n\to-\infty$ is possible also in the absolutely continuous case (namely when $\int_{-\infty}^{+\infty} f(x)\log\frac{1}{f(x)}\,dx=-\infty$), it will follow that in terms of the asymptotic behaviour of $\mathbf{H}_0(\xi_n)$ no necessary and sufficient condition can be given for the absolutely continuity of the distribution of ξ.

THEOREM 4. *If c_n $(n=1,2,\ldots)$ is an arbitrary sequence of real numbers for which $\lim_{n\to\infty} c_n=+\infty$, then there exists a real random variable ξ, with values in the interval $[0,1)$ having a singular distribution, such that $\mathbf{H}_0(\xi_n)-\log n>-c_n$ for any sufficiently large n.*

PROOF. We shall use a construction used by A. RÉNYI for a similar purpose (see [2], p. 195). Let $a_1, a_2, \ldots$ be a sequence of positive integers which will be specified later and $s_n=\sum_{i=1}^{n} a_i$. We define a measure ν corresponding to the sequence $\{a_i\}$ on the subintervals of $I=[0,1)$, having the form

$\left[\frac{k}{2^{s_n}}, \frac{k+1}{2^{s_n}}\right)$ $(n=1, 2, \ldots; k=0, 1, \ldots, 2^{s_n}-1)$ by putting

$$\nu\left[\frac{k}{2^{s_1}}, \frac{k+1}{2^{s_1}}\right)=\begin{cases}\frac{1}{2} & \text{if } k=0 \text{ or } k=2^{a_1}-1,\\ 0 & \text{otherwise,}\end{cases} \tag{27}$$

$$\nu\left[\frac{k}{2^{s_n}}, \frac{k+1}{2^{s_n}}\right)=\begin{cases}\frac{1}{2^n} & \text{if } k=l\cdot 2^{a_n} \text{ or } (l+1)2^{a_n}-1 \\ & \qquad \text{where } \nu\left[\frac{l}{2^{s_{n-1}}}, \frac{l+1}{2^{s_{n-1}}}\right)=\frac{1}{2^{n-1}},\\ 0 & \text{otherwise.}\end{cases}$$

Let us extend this measure in the usual way to all Borel subsets of $I=[0, 1)$ and denote by $\xi^{(\nu)}$ a random variable with the distribution function $F(x)=\nu[0, x)$. It is easy to show that if there are infinitely many a_i's different from 1, the distribution function $F(x)$ of ξ is singular.

As $\xi^{(\nu)}_{2^{s_n}}=\frac{1}{2^{s_n}}[2^{s_n}\xi^{(\nu)}]$ takes on 2^n different values, each with the probability $\frac{1}{2^n}$, we have by (3)

$$\mathbf{H}_0(\xi^{(\nu)}_{2^{s_n}})=2^n\frac{1}{2^n}\log 2^n=n. \tag{28}$$

Further, applying (12) with $\varepsilon=2$ we obtain that

$$\mathbf{H}_0(\xi^{(\nu)}_k)-\log k\geqq \mathbf{H}_0(\xi^{(\nu)}_l)-\log l-\log 3 \quad \text{if} \quad l\geqq k. \tag{29}$$

Let us choose the sequence $\{a_i\}$ to contain infinitely many $a_i>1$, so that for sufficiently large n $(n\geqq n_0)$

$$s_n-n=\sum_{i=1}^{n}(a_i-1)<\min_{i>2^{s_{n-1}}} c_i-\log 3 \tag{30}$$

(this is always possible, since $\lim\limits_{n\to\infty} c_n=+\infty$).

Then, if $2^{s_{n-1}}<k\leqq 2^{s_n}$ where $n\geqq n_0$, we have by (29), (28) and (30)

$$\mathbf{H}_0(\xi^{(\nu)}_k)-\log k\geqq \mathbf{H}_0(\xi^{(\nu)}_{2^{s_n}})-\log 2^{s_n}-\log 3=n-s_n-\log 3> \tag{31}$$

$$>-\min_{i>2^{s_{n-1}}} c_i\geqq -c_k.$$

Thus Theorem 4 is proved.

The results obtained can be generalized to random vectors. Let $\vec{\xi}=(\xi^1, \xi^2, \dots, \xi^r)$ be a random r-dimensional vector. Let us put

$$\vec{\xi}_n=\left(\frac{1}{n}[n\xi^1], \frac{1}{n}[n\xi^2], \dots, \frac{1}{n}[n\xi^r]\right), \tag{32}$$

$$\mathbf{P}\left(\vec{\xi}_n=\left(\frac{k^{(1)}}{n}, \frac{k^{(2)}}{n}, \dots, \frac{k^{(r)}}{n}\right)\right)=\mathbf{P}\left(\frac{k^{(1)}}{n}\leqq\xi^1<\frac{k^{(1)}+1}{n}, \dots, \frac{k^{(r)}}{n}\leqq\xi^r<\frac{k^{(r)}+1}{n}\right)=$$

$$=p_{n;k^{(1)},\dots,k^{(r)}}.$$

Suppose that $\mathbf{H}_0(\vec{\xi}_1)<+\infty$.

If the distribution of $\vec{\xi}$ is not absolutely continuous, then there exists an $\varepsilon>0$ such that for every $\delta>0$, $n>n_0(\delta)$ there can be found intervals of the form

$$I_j=\left\{(x^1, \dots, x^r)\colon \frac{k_j^{(1)}}{n}\leqq x^1\leqq\frac{k_{j+1}^{(1)}}{n}, \dots, \frac{k_j^{(r)}}{n}\leqq x^r\leqq\frac{k_{j+1}^{(r)}}{n}\right\}$$

(n and the $k_j^{(i)}$'s are integers, $j=1, 2, \dots, s=s(\delta, n)$) for which

$$\sum_{j=1}^{s}\mu(I_j)=\frac{s}{n^r}<\delta \quad\text{and}\quad \sum_{j=1}^{s}\mathbf{P}(\vec{\xi}\in I_j)=\sum_{j=1}^{s}p_{n;k_j^{(1)},\dots,k_j^{(r)}}>\varepsilon. \tag{33}$$

(In fact, by definition there exist such disjoint intervals $I_1', I_2', \dots, I_{r(\delta)}'$ that $\sum_{i=1}^{r(\delta)}\mu(I_i')<\frac{\delta}{2}$ and $\sum_{i=1}^{r(\delta)}\mathbf{P}(\vec{\xi}\in I_i')>\varepsilon$. For sufficiently large n the set $\sum_{i=1}^{r(\delta)}I_i'$ can be covered with intervals I_j of the desired form ($j=1, 2, \dots, s$) so that $\sum_{j=1}^{s}\mu(I_j)<2\sum_{i=1}^{r(\delta)}\mu(I_i')$ and for this I_j's (33) holds.)

Using (33) we obtain similarly to Theorem 1 that $\lim_{n\to\infty}(\mathbf{H}_0(\vec{\xi}_n)-r\log n)=-\infty$. Theorems 2 and 3 and their proofs can also be carried over to the r-dimensional case with obvious changes.

The construction in the proof of Theorem 4 must be modified as follows: Let us choose a sequence of positive integers $\{a_n\}$ containing infinitely many $a_i>1$ so that for every $n\geqq n_0$

$$s_n-n<\frac{1}{r}\min_{i>2^{s_n-1}}c_i-\log 3.$$

We denote by ν^i the measure on the Borel subsets of the interval $[0, 1)$ of the axis x^i, corresponding to the sequence $\{a_n\}$ by the construction in the proof of Theorem 4. Then $\nu=\nu^1\times\nu^2\times\dots\times\nu^r$ is a measure defined on the Borel subsets of the unit cube and if $\vec{\xi}$ is a random vector with $\mathbf{P}(\vec{\xi}\in A)=\nu(A)$ (for any

Borel subset A of the unit cube), then $\vec{\xi}$ has a singular distribution and we obtain similarly to the proof of Theorem 4 that

$$\mathbf{H}_0(\vec{\xi}_n) - r\log n > -c_n \quad \text{for every} \quad n > 2^{s_{n_0}-1}.$$

Thus we have

THEOREM 5. *For any random r-dimensional vector* $\vec{\xi}$ *with* $\mathbf{H}_0(\vec{\xi}_1) < +\infty$ *there exists* $\lim_{n\to\infty}(\mathbf{H}_0(\vec{\xi}_n) - r\log n)$. *This limit is equal to* $-\infty$ *if the distribution of* $\vec{\xi}$ *is not absolutely continuous and equal to*

$$\mathfrak{I} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x^1, x^2, \ldots, x^n) \log \frac{1}{f(x^1, x^2, \ldots, x^n)} dx^1 dx^2 \cdots dx^r$$

if the distribution of $\vec{\xi}$ *is absolutely continuous with the density function* $f(x^1, x^2, \ldots, x^r)$ $(-\infty \leqq \mathfrak{I} < +\infty)$. *Further, for any sequence of real numbers* $c_n \to +\infty$ *there exists a bounded random vector* $\vec{\xi}$ *with singular distribution for which* $\mathbf{H}_0(\vec{\xi}_n) - r\log n > -c_n$ *for every sufficiently large* n.

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НОРМАЛЬНЫЙ ПОРЯДОК АДДИТИВНЫХ АРИФМЕТИЧЕСКИХ ФУНКЦИЙ НА МНОЖЕСТВЕ "СДВИНУТЫХ" ПРОСТЫХ ЧИСЕЛ

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(Представлено П. Тураном)

Харди и Рамануджан [1] ввели понятие о нормальном порядке арифметической функции, заданной на некотором множестве чисел.

Пусть $f(m)$ — функция, заданная на некотором множестве M целых положительных чисел, $Q(n)$ — количество чисел этого множества, не превосходящих n, $g(n)$ — возрастающая функция. Обозначим через $N(n)$ количество чисел множества M, не превосходящих n и обладающих свойством

$$(1-\varepsilon)g(n)<f(m)<(1+\varepsilon)g(n).$$

Мы говорим, что $g(n)$ есть нормальный порядок функции $f(m)$, если $\lim\limits_{n\to\infty}\dfrac{N(n)}{Q(n)}=1$ для любых достаточно малых положительных ε.

Приведенное определение, хотя и несколько отличается от определения Харди и Рамануджана, по существу совпадает с последним, когда $g(n)$ — медленно растущая функция, а множество M — не слишком редкое (например, если $Q(n)>n^{\varepsilon_1}$).

В упомянутой статье Харди и Рамануджан доказали, что $\ln\ln n$ является нормальным порядком функции $\nu(m)=\sum\limits_{p|m}1$ на множестве всех натуральных чисел и на множестве бесквадратных чисел.

С тех пор этот результат был обобщен различными авторами в различных направлениях.

Эрдёш [2], используя метод Харди и Рамануджана и остроумным образом полученную им оценку сверху для количества простых чисел, не превосходящих n и удовлетворяющих условию $\nu(p-1)=K$, доказал, что $\ln\ln n$ является нормальным порядком функции $\nu(m)$ на множестве чисел вида $p-1$.

Туран [3] дал новое доказательство теоремы Харди и Рамануджана и обобщил ее на широкий класс других арифметических функций, заданных на множестве всех натуральных чисел, а для функции $\nu(m)$ доказал аналогичный результат для множества чисел, являющихся значениями целочисленного неприводимого полинома в области всех натуральных чисел

Кубилюс [4], применяя метод Турана, доказал аналогичную теорему для всех сильно аддитивных $\left(f(m)=\sum_{p|m} f(p)\right)$ арифметических функций, заданных в области всех натуральных чисел.

Метод Турана основывался на оценке средне-квадратического уклонения арифметической функции от своего среднего значения (т. е. дисперсии, если пользоваться терминологией теории вероятностей).

Однако непосредственное применение метода Турана к задаче Эрдёша и к ее обобщению на другие арифметические функции очевидным образом упирается в серьезные, подчас проблематичные сведения о распределении простых чисел в прогрессиях с растущим модулем.

Если же использовать простое свойство дисперсии не уменьшаться, когда множество чисел, на котором задана функция, заменяется на объемлющее (ср. Линник [5]), то для наших целей было бы достаточно найти такую "оболочку" над множеством простых чисел, чтобы для количества ее чисел в интервале $(1, n)$ выполнялось неравенство $O\left(\frac{n}{\ln n}\right)$, и чтобы вопросы распределения чисел этой "оболочки" в прогрессиях с растущим модулем уже не были проблематичными.

Но такой "оболочкой" как раз и являются числа $m \leqq n$, не делящиеся на простые, не превосходящие $n^{\frac{1}{\alpha}}$ для растущих α, что доказывается методом Бруна. То, что решето Бруна дает асимптотическое равенство для количества чисел $m \leqq n$, которые остаются после отсеивания из них чисел, принадлежащих некоторым фиксированным классам вычетов по простым модулям, не превосходящим $n^{\frac{1}{\alpha}}$ $(\alpha=\alpha(n)\rightarrow\infty)$, факт — широко используемый в вероятностной теории чисел (см [4], интересно сравнить с [6]).

Таким образом, на основе лишь элементарного метода появляется возможность обобщения теоремы Эрдёша, а именно, имеет место следующая

Теорема. Пусть $f(m)$ — неотрицательная сильно аддитивная арифметическая функция,

$$A_n=\sum_{p<n}\frac{f(p)}{p}, \quad \Lambda_n=\max_{p<n}|f(p)|, \quad A_n\rightarrow\infty \quad \text{и} \quad \Lambda_n=o(A_n) \quad \text{при} \quad n\rightarrow\infty.$$

Тогда A_n — нормальный порядок $f(m)$.

Нам понадобится несколько лемм.

Лемма 1. Пусть $(k, l)=1$. Обозначим через $\pi_l(k, n, n^{\frac{1}{\alpha}})$ количество чисел прогрессии $km+l$, не превосходящих n и не делящихся на простые, меньшие или равные $n^{\frac{1}{\alpha}}$. Тогда для достаточно больших, но фиксированных α, и всех $k \leqq n^{\delta}$ $(\delta = \delta(\alpha))$, при $n \geqq n_0(\alpha)$

$$\pi_l(k, n, n^{\frac{1}{\alpha}}) = \frac{n}{\varphi(k)} \prod_{p \leqq n^{1/\alpha}} \left(1 - \frac{1}{p}\right) + \theta \frac{e^{-\alpha \ln \alpha} n}{\varphi(k) \ln n},$$

где через θ здесь и в дальнейшем будет обозначена величина, по абсолютному значению меньшая 1, не обязательно одна и та же на протяжении всего изложения.

Доказательство проводится методом решета Бруна (см., например, [7]).

Лемма 2. В условиях и обозначениях предыдущей леммы

$$\pi_l(k, n, n^{\frac{1}{\alpha}}) < c_1(\alpha) \frac{n}{\varphi(k) \ln n}.$$

Лемма является очевидным следствием предыдущей.

Лемма 3. Обозначим через $M_l(a, n, n^{\frac{1}{\alpha}})$ число решений неопределенного уравнения $y_1 - l = a y_2$ в натуральных числах y_1, y_2, не превосходящих n и не делящихся на простые, меньшие или равные $n^{\frac{1}{\alpha}}$. Тогда, если $a \leqq n^{\sigma}$, $\sigma = \sigma(\alpha) < 1$, то

$$M_l(a, n, n^{\frac{1}{\alpha}}) < c_2(\alpha) \frac{n}{\varphi(a) \ln^2 n} \prod_{p|a} \left(1 + \frac{2}{p}\right).$$

Это неравенство, очевидно, также доступно методу решета в форме Бруна или Сельберга.

Лемма 4.

$$\sum_{a \leqq n} \frac{\prod_{p|a} \left(1 + \frac{2}{p}\right)}{\varphi(a)} = O(\ln n).$$

Доказательство.

$$\sum_{a \leqq n} \frac{\prod_{p|a} \left(1 + \frac{2}{p}\right)}{\varphi(a)} = O\left(\sum_{a \leqq n} \frac{\prod_{p|a} \left(1 + \frac{3}{p}\right)}{a}\right) =$$

$$= O\left(\sum_{a \leqq n} \frac{1}{a} \sum_{d|a} \mu^2(d) \frac{3^{\nu(d)}}{d}\right) = O\left(\sum_{d \leqq n} \frac{\mu^2(d)}{d} 3^{\nu(d)} \sum_{\substack{a \equiv 0(d) \\ a \leqq n}} \frac{1}{a}\right) =$$

$$= O\left(\ln n \sum_{d \leqq n} 3^{\nu(d)} \frac{\mu^2(d)}{d^2}\right) = O\left(\ln n \prod_{p} \left(1 + \frac{3}{p^2}\right)\right) = O(\ln n).$$

Теперь мы можем приступить к доказательству теоремы.

Очевидно, теорема является следствием следующей оценки дисперсии:

$$D=\sum_{p-l\leqq n}(f(p-l)-A_n)^2=o\left(\frac{n}{\ln n}A_n^2\right). \tag{1}$$

Но

$$D\leqq\sum_{\nu-l\leqq n}(f(\nu-l)-A_n)^2=\sum_{\nu-l\leqq n}f^2(\nu-l)-2A_n\sum_{\nu-l\leqq n}f(\nu-l)+A_n^2\sum_{\nu-l\leqq n}1, \tag{2}$$

где ν пробегает числа, не делящиеся на простые, не превосходящие $n^{\frac{1}{\alpha}}$.

Оценим суммы в правой части (2).

$$\sum_{\nu-l\leqq n}f^2(\nu-l)=\sum_{p\leqq n}f^2(p)\sum_{\substack{\nu-l\equiv 0(p)\\ \nu-l\leqq n}}1+\sum_{\substack{pq\leqq n\\ p\neq q}}f(p)f(q)\sum_{\substack{\nu-l\equiv 0(pq)\\ \nu-l\leqq n}}1=\sum{}'+\sum{}''.$$

Далее

$$\sum{}'=\sum_{p\leqq n^\delta}f^2(p)\sum_{\substack{\nu-l\equiv 0(p)\\ \nu-l\leqq n}}1+\sum_{n^\delta<p\leqq n}f^2(p)\sum_{\substack{\nu-l\equiv 0(p)\\ \nu-l\leqq n}}1=\sum{}_1'+\sum{}_2'.$$

Применяя к $\sum_1'$ лемму 2, получим

$$\sum{}_1'=O\left(\frac{n}{\ln n}\sum_{p\leqq n^\delta}\frac{f^2(p)}{p-1}\right)=O\left(\frac{n}{\ln n}A_nA_n\right)=o\left(A_n^2\frac{n}{\ln n}\right).$$

К $\sum_2'$ применим леммы 3 и 4:

$$\sum{}_2'=O\Big(A_n^2\sum_{n^\delta<p\leqq n}\sum_{\substack{\nu-l\equiv 0(p)\\ \nu-l\leqq n}}1\Big)=O\Big(A_n^2\sum_{\substack{\nu-l=a\nu_1\\ \nu-l\leqq n\\ n^\delta<\nu_1\leqq n}}1\Big)=$$

$$=O\Big(A_n^2\sum_{a\leqq n^{1-\delta}}M_l(a,n,n^{\frac{1}{\alpha}})\Big)=O\left(A_n^2\frac{n}{\ln^2 n}\sum_{a\leqq n^{1-\delta}}\frac{\prod\limits_{p|a}\left(1+\frac{2}{p}\right)}{\varphi(a)}\right)=o\left(A_n^2\frac{n}{\ln n}\right).$$

Таким образом,

$$\sum{}'=o\left(\frac{n}{\ln n}A_n^2\right).$$

Теперь оценим $\sum''$:

$$\sum{}''=\sum_{\substack{pq\leqq n^\delta\\ p\neq q}}f(p)f(q)\sum_{\substack{\nu-l\equiv 0(pq)\\ \nu-l<n}}1+\sum_{n^\delta<pq\leqq n}f(p)f(q)\sum_{\substack{\nu-l\equiv 0(pq)\\ \nu-l\leqq n}}1=\sum{}_1''+\sum{}_2''.$$

$\sum_2''$ оценивается точно также, как $\sum_2'$. Таким образом,

$$\sum{}_2''=o\left(\frac{n}{\ln n}A_n^2\right).$$

К $\sum_1''$ применим лемму 1:

$$\sum_1'' = n \prod_{p \leqq n^{1/\alpha}} \left(1 - \frac{1}{p}\right) \sum_{\substack{pq \leqq n^\delta \\ p \neq q}} \frac{f(p)f(q)}{\varphi(pq)} + \theta \frac{n}{\ln n} e^{-\alpha \ln \alpha} \sum_{\substack{pq \leqq n^\delta \\ p \neq q}} \frac{f(p)f(q)}{\varphi(pq)} + O(1)$$

(последнее за счет pq, не взаимно-простых с l).

Так как

$$\sum_{\substack{pq \leqq n^\delta \\ p \neq q}} \frac{f(p)f(q)}{\varphi(pq)} \leqq 4 \left(\sum_{p < n} \frac{f(p)}{p}\right)^2,$$

то

$$\sum_1'' = n \prod_{p \leqq n^{1/\alpha}} \left(1 - \frac{1}{p}\right) \sum_{\substack{pq \leqq n^\delta \\ p \neq q}} \frac{f(p)f(q)}{\varphi(pq)} + 4\theta \frac{n}{\ln n} e^{-\alpha \ln \alpha} A_n^2 + O(1).$$

Теперь, собирая все оценки, получим

$$(3) \qquad \sum_{\nu - l \leqq n} f^2(\nu - l) = n \prod_{p \leqq n^{1/\alpha}} \left(1 - \frac{1}{p}\right) \sum_{\substack{pq \leqq n^\delta \\ p \neq q}} \frac{f(p)f(q)}{\varphi(pq)} + 4\theta \frac{n}{\ln n} e^{-\alpha \ln \alpha} A_n^2 + o\left(\frac{n}{\ln n} A_n^2\right).$$

Точно также мы можем оценить $\sum_{\nu - l \leqq n} f(\nu - l)$.

$$(4) \qquad \sum_{\nu - l \leqq n} f(\nu - l) = n \prod_{p \leqq n^{1/\alpha}} \left(1 - \frac{1}{p}\right) \sum_{p \leqq n^\delta} \frac{f(p)}{\varphi(p)} + 2\theta \frac{n}{\ln n} e^{-\alpha \ln \alpha} A_n + o\left(\frac{n}{\ln n} A_n\right).$$

Подставляя (3), (4) в (2) и применяя лемму 1 при $k = 1$ для оценки $\sum_{\nu - l \leqq n} 1$, получим

$$D \leqq n \prod_{p \leqq n^{1/\alpha}} \left(1 - \frac{1}{p}\right) \left[\sum_{\substack{pq \leqq n^\delta \\ p \neq q}} \frac{f(p)f(q)}{\varphi(pq)} - 2A_n \sum_{p \leqq n^\delta} \frac{f(p)}{\varphi(p)} + A_n^2\right] + 10\theta \frac{n}{\ln n} e^{-\alpha \ln \alpha} A_n^2 + o\left(\frac{n}{\ln n} A_n^2\right).$$

Так как $\prod_{p \leqq n^{1/\alpha}} \left(1 - \frac{1}{p}\right) < \frac{c_3(\alpha)}{\ln n}$, а выбирая α достаточно большим можно выражение $e^{-\alpha \ln \alpha}$ сделать произвольно малым, то для завершения доказательства достаточно показать, что

$$W=\sum_{\substack{pq\leqq n^\delta\\ p\neq q}}\frac{f(p)f(q)}{\varphi(pq)}-2A_n\sum_{p\leqq n^\delta}\frac{f(p)}{\varphi(p)}+A_n^2=o(A_n^2). \tag{5}$$

Но, во-первых

$$\sum_{\substack{pq\leqq n^\delta\\ p\neq q}}\frac{f(p)f(q)}{\varphi(pq)}\leqq\Big(\sum_{p\leqq n^\delta}\frac{f(p)}{\varphi(q)}\Big)^2. \tag{6}$$

Во-вторых, применяя хорошо известную оценку $\sum_{p<y}\frac{1}{p}=\ln\ln y+O(1)$, имеем

$$\sum_{p\leqq n^\delta}\frac{f(p)}{\varphi(p)}=\sum_{p\leqq n^\delta}\frac{f(p)}{p}+\sum_{p\leqq n^\delta}f(p)\Big(\frac{1}{p-1}-\frac{1}{p}\Big)=\sum_{p\leqq n^\delta}\frac{f(p)}{p}+O(A_n)= \tag{7}$$

$$=A_n+O\Big(A_n\sum_{n^\delta<p<n}\frac{1}{p}\Big)+O(A_n)=A_n+o(A_n).$$

Теперь подстановка (6) и (7) в выражение для W (5) завершает доказательство теоремы.

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ON THE NORMAL ORDER OF ADDITIVE ARITHMETIC FUNCTIONS

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(Summary)

The aim of the present paper is the proof of the following theorem:
If $f(m)$ is a non-negative strongly additive $\left(f(m)=\sum_{p\mid m} f(p)\right)$ *arithmetic function,*

$$A_n=\sum_{p<n}\frac{f(p)}{p},\quad \Lambda_n=\max_{p<n} f(p),\quad A_n\to\infty \quad \textit{and} \quad \Lambda_n=o(A_n),$$

then A_n is the normal order of $f(n)$.

The method used in this paper is similar to those of Turán [3] and Linnik [5].

ÜBER RINGE MIT MINIMALBEDINGUNG FÜR HAUPTRECHTSIDEALE. II

Von
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(*Vorgelegt von* L. RÉDEI)

Professor L. RÉDEI *zu seinem 60. Geburtstag gewidmet*

§ 1. Einleitung[1]

Die im Titel erwähnten Ringe werden kurz MHR-Ringe genannt, die schon in unserer Arbeit [26] untersucht worden sind. Unter einem Ring verstehen wir immer einen assoziativen Ring, unter dem Radikal das Jacobsonsche Radikal, unter einem halbeinfachen Ring einen Ring mit dem Radikal (0). Für die Grundbegriffe verweisen wir auf [13], ferner auf [1], [4], [7] und [21]. Unter einem MH_1R-Ring verstehen wir einen Ring mit Minimalbedingung für die in Hauptrechtsidealen liegenden Rechtsideale. Hiernach ist jeder Artinsche Ring (d. h. ein Ring mit Minimalbedingung für Rechtsideale) ein MH_1R-Ring und jeder MH_1R-Ring ist ein MHR-Ring, aber die Umkehrungen sind falsch. Ferner wird ein Ring mit Minimalbedingung für Hauptideale (bzw. für die in Hauptidealen liegenden Ideale) kurz MHI-Ring (bzw. MH_1I-Ring) genannt. Die MHR-Ringe mit Maximalbedingung für Hauptrechtsideale nennen wir $MMHR$-Ringe. Ein MHU-Ring bedeutet aber stets einen Ring mit Minimalbedingung für die durch ein Element erzeugten Unterringe.

Bekanntlich wird ein Ring A im von Neumannschen Sinne regulär genannt, wenn es zu jedem Element $a \in A$ ein $x \in A$ mit $a = axa$ gibt. Ist ferner A ein beliebiger Ring mit einem idempotenten Element e, so wird das Hauptrechtsideal $(e)_r = eA$ von A „regelmäßig" genannt, wenn der Unterring $Q_e = (1-e)AeA(1-e)$ nilpotent ist. Bezeichnet I den Ring der ganzen rationalen Zahlen, so ist $1 \in I$ im Falle $1 \notin A$ ein Operator mit $(1-e)AeA(1-e) = \{aeb - eaeb - aebe + eaebe \mid a, b \in A\}$. Der Unterring Q_e ist ein Quasiideal im Sinne von O. STEINFELD [23]. Bezeichnet e_{ij} die Basiselemente des vollen Matrizenringes $A = (I/(2))_2$ $(i, j = 1, 2)$, so ist $e_{11}A$ wegen $e_{22} \in Q_{e_{11}}$ kein regelmäßiges Hauptrechtsideal. Ist aber A kein Nilring und ist A entweder

[1] Diese Arbeit ist ein Teil der Dissertation (1959) des Verfassers. Wir verweisen noch auf [26] und auf die Note [6], die voneinander unabhängig zustande gekommen sind. Vgl. noch F. SZÁSZ, A főjobbideálokra nézve minimum-feltételű gyűrűk, *MTA Mat. és Fiz. Oszt. Közl.*, **11** (1961), S. 135—177, was eine abgekürzte Version der Kandidatsdissertation ist.

ein $MMHR$-Ring oder ein Artinscher Ring, so ist eA mit jedem Hauptidempotent e regelmäßig. Wiederum ist eA in einem beliebigen Ring A regelmäßig, wenn $e(=e^2)$ zum Zentrum von A gehört. Ist nun φ ein Ringhomomorphismus von A in B, so ist mit eA auch $e\varphi \cdot A\varphi$ ebenfalls regelmäßig.

Ist E der volle Endomorphismenring einer beliebigen Abelschen Gruppe M, und $A = A^{(0)}$ ein Unterring von E, so sei $A^{(n+1)}$ der Zentralisator von $A^{(n)}$ bezüglich E für jede natürliche Zahl n. Es gelten immer $A^{(2k-1)} = A^{(1)}$ und $A^{(2k)} = A^{(2)}$ für alle $k \geqq 1$. Der kritische Fall ist im allgemeinen $k=0$. Ist nun A die ringtheoretische direkte Summe beliebig vieler einfacher Artinscher Ringe, so gilt auch $A^{(0)} = A^{(2)}$.

Ein A-Rechtsmodul M mit $MA = M$ ist perfekt. Ein vollständig reduzibler A-Faktormodul M/K von M wird „ausgezeichnet" genannt, wenn jeder A-Homomorphismus eines beliebigen vollständig reduziblen A-Faktormoduls M/L auf M/K ein Isomorphismus ist. Wir bezeichnen die Menge $\{x \mid x \in M, x \cdot J^{\alpha} = 0\}$ mit $K_{\alpha}(M)$ für einen A-Rechtsmodul M und für jede Ordnungszahl α, wobei J das Radikal von A ist.

Nun werden wir die Ergebnisse der vorliegenden Arbeit zusammenfassen.

Im § 2 werden das Noethersche und das Wedderburn—Artinsche Kriterium verallgemeinert, ferner Eindeutigkeitsfragen erörtert, bzw. spezielle halbeinfache MHR-Ringe charakterisiert. Es werden auch die subdirekten Darstellungen von MHI-Ringen, und eine Verallgemeinerung eines Kriteriums von A. KERTÉSZ betrachtet, das die halbeinfachen Artinschen Ringe durch eine Folge von äquivalenten Bedingungen bestimmt hat. Wir verallgemeinern auch ein Harada—Kovácssches Kriterium über die regulären Ringe. Hiernach und nach [26] ist jeder halbeinfache MHR-Ring regulär. Wir haben aber einen regulären Ring explizit gegeben, der kein (halbeinfacher) MHR-Ring ist

Im § 3 werden die additiven Gruppen der MHR-Ringe, insbesondere der MHR-Ringe mit Einselement, der MHR-Radikalringe bzw. der primitiven MHR-Ringe betrachtet. In einem nilpotenten MH_1R-Ring gilt die Minimalbedingung auch für die in Hauptrechtsidealen liegenden additiven Untergruppen, und jeder Unterring ist ebenfalls ein MH_1R-Ring.

Im § 4 untersuchen wir die Struktur einer Abelschen Gruppe M mit (vollem) MHR-Endomorphismenring E. Ist nun der volle Endomorphismenring E von M beliebig, und A ein einfacher MHR-Unterring von E, so kann nach einem expliziten Beispiel auch $(A=)A^{(0)} \neq A^{(2)}$ gelten, d. h. der Zentralisator $A^{(2)}$ des Zentralisators $A^{(1)}$ von A bezüglich E ist von A verschieden. Wir werden auch die Operatorenmoduln und die Loewyschen Systeme über MHR-Ringe erörtern.

§ 5 untersucht die Zusammenhänge von verschiedenen Typen der Radikale von MHR-Ringen. In beliebigen MHR-Ringen stimmen das Baer—

McCoysche Radikal B, das Levitzkische Radikal K, das obere Nilradikal N und das Jacobsonsche Radikal J überein. Hat der MHR-Ring ein Rechtseinselement, so stimmen auch das Fuchssche Radikal Z und das Brown—McCoysche Radikal G mit dem Jacobsonschen Radikal J überein. Im allgemeinen gelten in MHR-Ringen $Z \supseteq J$ und $G \supseteq J$, und es können durch spezielle Beispiele auch $Z \neq J$ und $G \neq J$ bestätigt werden.

Im § 6 betrachten wir die $MMHR$-Ringe und die regelmäßigen Hauptrechtsideale des Sockels eines Ringes. Hiernach ist jedes Rechtsideal R eines regelmäßigen Hauptrechtsideales, als Ringes, des Sockels A_1 von A ebenfalls ein Rechtsideal in A, und es gilt noch eine Folge der Behauptungen, insbesondere eine relative Charakterisierung bezüglich der regelmäßigen Hauptrechtsideale des Sockels eines Ringes.

Im § 7 werden die MHU-Ringe erörtert. Es gibt einen MHU-Ring, der kein MHR-Ring ist, und einen MHR-Ring, der kein MHU-Ring ist. Ist nun A ein MHU-Nilring, der auch ein MHR-Ring ist, so bilden die Elemente von A bezüglich der Verknüpfung $x \cdot y = x + y - xy$ eine periodische verallgemeinerte auflösbare Gruppe G, die das direkte Produkt ihrer Sylowschen p-Untergruppen ist. Die Bedingung, daß jeder endlich erzeugbare echte Unterring von A ein Hauptrechtsideal von A ist, ist hinreichend aber nicht notwendig dafür, daß ein MHR-Ring auch ein MHU-Ring sei, bzw., daß ein MHU-Ring auch ein MHR-Ring sei (vgl. Satz 3 von [25]).[2]

§ 2. Über halbeinfache MHR-Ringe

Wir haben schon in unserer Arbeit [26] gezeigt, daß jeder halbeinfache MHR-Ring A, als ein A-Rechtsmodul, vollständig reduzibel ist. Diese Ringe A sind auch als zweiseitige (A, A)-Doppelmoduln vollständig reduzibel. Es gilt nun die folgende Behauptung als eine Umkehrung und Ergänzung dieser Tatsachen:

2. 1. *Ist ein Ring A die direkte Summe seiner idempotenten minimalen Rechtsideale, so ist A ein halbeinfacher MHR-Ring. Ein Ring A ist genau dann ein halbeinfacher MHR-Ring, wenn A in die ringtheoretische direkte Summe von einfachen MHR-Ringen zerlegt werden kann. Diese letztere Zerlegung ist bis auf Isomorphie und Permutation eindeutig.*

Nach [26] ist nämlich A, als die direkte Summe idempotenter minimaler Rechtsideale, auch als ein (A, A)-Doppelmodul vollständig reduzibel. Daher ist das Radikal J ein direkter Summand von A, folglich $J = 0$, denn

[2] Vgl. F. Szász, Die Ringe, deren endlich erzeugbare echte Unterringe Hauptrechtsideale sind, *Acta Math. Acad. Sci. Hung.*, **13** (1962) (im Erscheinen), und die Kandidatsdissertation des Verfassers.

aus $e^2 = e \in J$ erhält man $e = 0$. Die weitere Behauptungen folgen aus dem Kapitel III von [13].

SATZ 2. 2. *Jeder MHR-Ring A ohne nilpotente Elemente* ($\neq 0$) *ist die ringtheoretische direkte Summe von Schiefkörpern (*GERTSCHIKOFF [11]*). Jeder nullteilerfreie MHR-Ring ist ein Schiefkörper. Jeder kommutative halbeinfache MHR-Ring ist die ringtheoretische direkte Summe von Körpern. Jeder bireguläre* [13] *MHR-Ring ist die ringtheoretische direkte Summe von vollen Matrizenringen über Schiefkörpern.*

BEWEIS. Jeder *MHR*-Ring A ohne nilpotente Elemente ($\neq 0$) ist nämlich halbeinfach, denn das Radikal ist ein Nilideal. Nach dem Satz von LITOFF (Kapitel IV von [13]) läßt sich aber jeder endlich erzeugbare Unterring von A in einen solchen Unterring von A einbetten, der einem vollen Matrizenring über einem Schiefkörper isomorph ist, und der im allgemeinen nilpotente Elemente ($\neq 0$) hat. Hiernach ist jede zweiseitige einfache direkte Komponente von A ein Schiefkörper.

Ist nun insbesondere A nullteilerfrei, so ist A selbst ein Schiefkörper.

Da jeder kommutative einfache Ring ein Körper ist, und jedes (zweiseitige) Hauptideal eines biregulären Ringes ein Einselement hat, gelten auch die weiteren Behauptungen im Satz 2. 2.

BEMERKUNG. Der Satz von SZELE [27] ist eine Folgerung sowohl des Wedderburn—Artinschen Struktursatzes als auch unserer Resultate über *MHR*-Ringe. Dagegen kann auch ein kürzer elementarer Beweis des Szeleschen Satzes über die Ringe ohne echte Rechtsideale gegeben werden (vgl. Lemma 1 von [24]).

SATZ 2. 3. *Jeder halbeinfache MHR-Ring ist ein MH_1I-Ring. Ein MHI-Ring A ist dann und nur dann eine ringtheoretische subdirekte Summe einfacher Ringe E'_α ($\alpha \in \Omega$), wenn A die ringtheoretische direkte Summe der Ringe $E_\alpha (\cong E'_\alpha)$ ist. Dann ist A auch ein MH_1I-Ring.*

BEWEIS. Ein halbeinfacher *MHR*-Ring ist zweiseitig vollständig reduzibel, also ist er ein MH_1I-Ring.

Es sei nun A ein *MHI*-Ring, der eine subdirekte Summe einfacher Ringe E'_α ($\alpha \in \Omega$) ist. Dann existieren maximale Ideale B_α mit $A/B_\alpha \cong E'_\alpha$, $\bigcap_{\alpha \in \Omega} B_\alpha = 0$, $A^2 = B_\alpha + A$. Wir zeigen, daß jedes Hauptideal (a) von A, als ein (A, A)-Doppelmodul, vollständig reduzibel ist. Es sei nämlich $C_\alpha = (a) \cap B_\alpha$. Dann können die Ideale C_α mit $C_\alpha = (a)$ weggelassen werden, und (a) ist eine subdirekte Summe der übrigbleibenden einfachen Ringe $(a)/C_\alpha (\cong A/B_\alpha;$ $a \notin B_\alpha)$. Ist nun (m) ein beliebiges minimales Ideal von A in (a), so existiert ein Ideal C_α mit $(m) \not\subseteq C_\alpha$, also mit $(a) = (m) \oplus C_\alpha$. Hiernach ist das endomorphe Bild C_α von (a) ebenfalls ein Hauptideal. Ist nun (m_1) ein minimales Ideal von A in C_α, so gilt $C_\alpha = (m_1) \oplus C_\beta$ mit einem Hauptideal C_β von A. Da aber A ein MHI-Ring ist, besteht $(a) = (m) \oplus (m_1) \oplus \cdots \oplus (m_k)$ mit einem geeigneten Index k, denn $(a) \supset C_\alpha \supset C_\beta \supset \cdots$ muß abbrechen. Deshalb ist aber auch $A = \bigcup_{a \in A} (a)$, als ein (A, A)-Doppelmodul, vollständig reduzibel, w. z. b. w.

Herr Dr. A. KERTÉSZ hat in [15] und [16] bewiesen, daß eine Folge untereinander äquivalenter Bedingungen unter den Ringen mit Linkseinselement genau die Artinschen halbeinfachen Ringe charakterisiert. Diese Bedingungen bestimmen aber genau die halbeinfachen MHR-Ringe in der größeren Klasse der Ringe ohne Rechtsannullatoren $(\neq 0)$. Ist nun $A' = A$, wenn A ein Einselement hat, und sonst A' die Dorrohsche Erweiterung von A mit Einselement, so gilt der

SATZ 2. 4. *Für einen Ring A ohne Rechtsannullatoren $(\neq 0)$ sind die folgenden Bedingungen untereinander äquivalent:*

a) *A ist ein halbeinfacher MHR-Ring;*

b) *A ist die direkte Summe idempotenter minimaler Rechtsideale;*

c) *A ist ein MHR-Ring und eine subdirekte Summe von Ringen, die zu idempotenten minimalen Rechtsidealen von A A-isomorph sind;*

d) *A ist ein MHR-Ring, in dem der Durchschnitt gewisser modularer* ([13]) *maximaler Rechtsideale $(0)_r$ ist;*

e) *A stimmt mit seinem Sockel A_1 überein;*

f) *der Rechtsannullator in A' jedes Elementes von A ist der Durchschnitt endlich vieler modularer maximaler Rechtsideale von A';*

g) *es gibt zu jedem Rechtsideal R von A ein Rechtsideal S von A mit $A = R + S$, $R \cap S = 0$;*

h) *existiert die direkte Summe $R^* = \sum_{\alpha \in \Omega} \oplus R_\alpha$ für eine möglichst maximale Menge der minimalen Rechtsideale R_α $(\alpha \in \Omega)$, so besteht $A = R^*$.*

BEWEIS. Gilt $A = A_1$ für den Sockel (d. h. Rechtssockel) A_1 von A, so ist A wegen $A_1 \cdot J = 0$ genau dann halbeinfach, wenn A keinen Rechtsannullator $(\neq 0)$ besitzt. Betrachten wir nun A^+ als einen A'-Rechtsmodul. Der Beweis folgt nun aus dem Kertészschen Hauptsatze in [16].

BEMERKUNGEN. Ist jedes Rechtsideal R eines Ringes A, als eines A-Rechtsmoduls, ein rechtsseitiger direkter Summand, so gilt $A = B \oplus C$, wobei $\oplus$ eine ringtheoretische direkte Summe, B ein halbeinfacher MHR-Ring und C ein Zeroring mit additiver elementarer Abelscher Gruppe ist. (Die additive Ordnung $O^+(c)$ jedes Elementes c von C ist eine quadratfreie Zahl.) — Ist insbesondere jedes Rechtsideal R eines Ringes A, als eines (A, A)-Doppelmoduls, ein zweiseitiger direkter Summand, so gilt $A \dot{=} B \oplus C$, wobei B die ringtheoretische direkte Summe von Schiefkörpern, und C ein Zeroring mit elementarer Abelscher additiver Gruppe ist.

SATZ 2. 5. *Ein Ring A ist dann und nur dann die ringtheoretische direkte Summe von Schiefkörpern, wenn jedes in einem Hauptrechtsideale liegende Rechtsideal R von A ein Rechtseinselement e besitzt.*

BEWEIS. Es seien also e und e_1 Rechtseinselemente von $R \subseteq (a)_r$ bzw. von $R_1 = (1-e)R(\subseteq (a)_r)$. Da $R_1^2 = (1-e)R(1-e)R = 0$ ist, folgt aus der Halbeinfachheit von A auch $R_1 = 0$. Dies bedeutet aber, daß e gleichzeitig auch ein Linkseinselement von R ist. Nach unserem Satz 4 in [26] ist aber ein beliebiger Ring B dann und nur dann ein halbeinfacher MHR-Ring, wenn jedes in einem Hauptrechtsideale liegende Rechtsideal R von B ein Linkseinselement besitzt. Hiernach ist also der Ring A ein halbeinfacher MHR-Ring. Gilt nun $e_1A \cap e_2A = 0$ $(e_i^2 = e_i)$, so hat $e_1A + e_2A$ ebenfalls ein (zweiseitiges) Einselement, und die Komponenten dieses Einselementes können als e_1 und e_2 mit $e_1e_2 = e_2e_1 = 0$ gewählt werden. Dann ist jedes Rechtsideal von A ein Ideal, und somit ist A die ringtheoretische direkte Summe von Schiefkörpern. Die Umkehrung ist trivial.

SATZ 2. 6. *Ein beliebiger Ring A ist dann und nur dann regulär im von Neumannschen Sinne, wenn jedes Hauptrechtsideal von A ein Linkseinselement hat. Ein beliebiger Ring A ist dann und nur dann regulär, wenn $(a)_r(a)_l = (a)_r \cap (a)_l$ für jedes Element $a \in A$ gilt.* (Vgl. S. LAJOS und K. ISÉKI bezüglich Halbgruppen.) *Jeder halbeinfache MHR-Ring ist regulär.*

BEWEIS. Der Ring A besitzt nicht notwendig ein Einselement. (Wir verweisen bezüglich regulärer Ringe ohne Einselement auf MCCOY's Buch.) Hiernach sei $e = na + ab$ $(n \in I;\ a, b \in A)$ ein Linkseinselement von $(a)_r$. Dann gilt aber $a = ac$ mit $c = na + ba$, also $a = eac = a(n+b)a(n+b)a$. Dies bedeutet die Regularität von A. Umgekehrt folgt nun aus $a = ada$ $(d \in A)$ auch $ad(na + ax) = na + ax$, also die Existenz eines Linkseinselementes $e = ad$ in $(a)_r$.

Gilt $(a)_r \cdot (a)_l = (a)_r \cap (a)_l$ für jedes $a \in A$, so ergibt sich aus $a = (n_1a + ab_1) \cdot (b_2a + n_2a)$ $(n_i \in I;\ b_i \in A)$ durch eine wiederholte Einsetzung $a = a[(n_1 + b_1) \cdot (b_2a + n_2a)(n_1 + b_1)(n_2 + b_2)]a$, also die Regularität von A. Umgekehrt genügt es, in regulären Ringen $(a)_r \cap (a)_l \subseteq (a)_r \cdot (a)_l$ zu beweisen. Jedes Element des Durchschnittes hat die Form $b = ay = za$ $(y, z \in A)$. Gilt nun $a = axa$, so ist auch $b = za = zaxa = ayxa \in (a)_r \cdot (a)_l$, w. z. b. w.

Die letzte Behauptung des Satzes 2. 6 ist eine Folgerung der obigen Verallgemeinerung des von Neumannschen Kriteriums bezüglich regulärer Ringe, und die unserer Arbeit [26].

Also ist jeder halbeinfache MHR-Ring regulär, und bekanntlich ist jeder reguläre Ring halbeinfach (im Jacobsonschen Sinne). Das folgende Beispiel zeigt aber die Existenz eines regulären Ringes, der kein MHR-Ring ist.

BEISPIEL 2.7. Es sei A die komplette direkte Summe unendlich vieler regulärer Ringe $A_1, A_2, \ldots, A_n, \ldots$ usw. Es seien $b_1 = \langle a_{11}, a_{12}, \ldots, a_{1n}, \ldots \rangle$ bzw. $b_k = \langle 0, 0, \ldots, 0, a_{kk}, a_{k,k+1}, \ldots \rangle$ die Elemente von A mit beliebigen $a_{1n} (\neq 0, \in A_1)$ und mit $a_{k-1,n} \cdot a_{kn} = a_{k-1,n}$ $(\in A_n)$. Dann ist $(b_1)_r \supset (b_1 b_2)_r \supset \supset (b_1 b_2 b_3)_r \supset \ldots$ eine unendliche absteigende Kette der Hauptrechtsideale, obwohl A offenbar regulär ist. Es kann in A auch ein in einem Hauptrechtsideale liegendes Rechtsideal ohne Linkseinselemente explizit gegeben werden: $R^* = \sum_{j=1}^{\infty} \oplus (e_j - e_{j+1}) e_j A$, wobei $(b_1 \cdot b_2 \ldots b_j)_r = e_j A$ $(e_j^2 = e_j)$ ist.

Zum Schluß erwähnen wir eine Charakterisierung der MH_1R-Ringe.

SATZ 2.8. *Ein Ring A ist dann und nur dann ein MH_1R-Ring, wenn die Minimalbedingung für die in Hauptrechtsidealen liegenden Nilrechtsideale von A gilt, und jedes in einem Hauptrechtsideale liegende Rechtsideal R von A die Gestalt $R = eA \oplus R_1$ hat, wobei $e^2 = e$ und R_1 ein Nilrechtsideal von A ist.*

BEMERKUNG 2.9. Da jeder nullteilerfreie einfache MHR-Ring ein Körper ist, scheint es uns sehr merkwürdig, alle nullteilerfreien einfachen Ringe zu bestimmen. Ist ein nullteilerfreier einfacher Ring kein Schiefkörper, so ist er aber deswegen kein MHR-Ring. Es kann aber auch ein solcher Ring ergeben werden. Man nimmt nämlich einen nullteilerfreien Ring A, der sich in keinen Schiefkörper aber in einen einfachen nullteilerfreien Ring E einbetten läßt (MALCZEW; COHN). Dann hat E sicher die obigen Eigenschaften. Wir wissen bisher noch nicht, ob was ein Kriterium dafür ist, daß ein MHR-Ring in einen einfachen MHR-Ring eingebettet werden kann. Man wünscht ebenfalls eine notwendige und hinreichende Bedingung dafür zu suchen und finden, daß in einer subdirekten Summe A beliebiger einfacher Ringe E_α die Ringe E_α bis auf Isomorphie und Permutation durch den ursprünglichen Ring A eindeutig bestimmt seien. Man möchte auch die solchen Ringe explizit bestimmen, deren jedes in einem Hauptrechtsideale liegende Linksideal ein Rechtseinselement (bzw. Linkseinselement) hat. Die ringtheoretischen direkten Summen von Schiefkörpern sind nämlich sicher solche Ringe. Wir wissen bisher ebenfalls auch noch nicht, ob es einen Ring gibt, der die direkte Summe von $\mathfrak{m}$ idempotenten minimalen Rechtsidealen und die von $\mathfrak{n}$ idempotenten minimalen Linksidealen ist, wobei $\mathfrak{m}$ und $\mathfrak{n}$ zwei verschiedene unendliche Mächtigkeiten sind.

§ 3. Die additive Gruppe der *MHR*-Ringe

Die Methoden von L. FUCHS—T. SZELE über die additive Gruppe der Artinschen Ringe, die später von L. FUCHS weiterentwickelt wurden [7], können auch auf die Untersuchung der additiven Gruppe der *MHR*-Ringe angewandt und verallgemeinert werden.

SATZ 3. 1. *Eine Abelsche Gruppe A^+ ist dann und nur dann die additive Gruppe eines MHR-Ringes, wenn $A^+ = B \oplus C$ gilt, wobei B vollständig und C reduziert periodisch sind. Existiert eine Prüfersche Gruppe $Z(p^\infty)$ in A^+, so liegt $Z(p^\infty)$ im zweiseitigen Annulator des MHR-Ringes A.*

BEWEIS. Ist A ein *MHR*-Ring und B die maximale vollständige Untergruppe von A^+, so gilt $A^+ = B \oplus C$ mit reduzierter C. Ist nun $(ma)_r$ $(= m(a)_r)$ ein minimales Hauptrechtsideal unter den Hauptrechtsidealen $(na)_r$ $(n \in I)$, so gilt $ma \in B$. Daher ist C periodisch wegen $B \cap C = 0$. Umgekehrt habe A^+ die erwähnte Struktur, und es sei $B = B_1 \oplus B_2$ mit torsionfreier B_1 und mit periodischer B_2. Definiert man nun über B_1 einen kommutativen Körper [7] und sei $B_2 A = A B_2 = CA = AC = 0$, so entsteht über A^+ ein spezieller *MHR*-Ring.

Ist ferner $Z(p^\infty) \subseteq A^+$, $a \in Z(p^\infty)$, $O(a) = p^k$, $A^+ = B \oplus C$ mit den vorigen Bezeichnungen bezüglich des *MHR*-Ringes A, so gilt $x = b + c (x \in A)$ mit $b \in B$, $c \in C$, $O(c) = l$. Die Gleichungen $lu = a$ und $p^k v = b$ sind in A^+ lösbar, und es gilt $ax = ab + ac = a(p^k v) + (lu)c = (p^k a)v + u(lc) = 0$, also $Z(p^\infty) \cdot A = 0$. Ganz ähnlich ergibt sich auch $A \cdot Z(p^\infty) = 0$.

SATZ 3. 2. *A^+ ist dann und nur dann die additive Gruppe eines MHR-Ringes mit Einselement, wenn $A^+ = B \oplus C$ gilt, wobei B torsionsfrei vollständig und C beschränkt periodisch sind. A^+ ist genau dann die additive Gruppe eines primitiven MHR-Ringes, wenn A^+ entweder eine torsionsfreie vollständige Gruppe oder eine elementare Abelsche p-Gruppe ist.*

BEWEIS. Ist A ein *MHR*-Ring mit Einselement, so gilt $Z(p^\infty) \nsubseteq A^+$. Die maximale vollständige Untergruppe B von A^+ ist also torsionsfrei, und der maximale periodische Unterring C von A, als ein ringtheoretischer direkter Summand von A, ist ein endomorphes Bild des Ringes A. Also hat der Ring C ein Einselement e mit $O(e) = m$. Hiernach gilt aber $mC = 0$. Besteht nun umgekehrt $A^+ = B \oplus C$ mit torsionsfreier vollständiger B und mit beschränkter periodischer C, so konstruieren wir auf B einen kommutativen Körper [7]. Ferner ist C die direkte Summe von Gruppen $Z(p_m^{k_i})$ für gewisse Primzahlen p_m und Exponenten k_i. Wir sammeln nun die Summanden $Z(p_m^{k_i})$, die zu einem festen Paar (p_m, k_i) gehören, in eine Summe $D_{m,k}$ zusammen. Nach einem wichtigen Lemma von L. FUCHS ([7], S. 281) kann auf jede Gruppe

$D_{m,k}$ ein kommutativer Ring mit Einselement aufgebaut werden, dessen sämtliche verschiedene Ideale $0, p_m D_{m,k}, p_m^2 D_{m,k}, \ldots, p_m^{k-1} D_{m,k} \neq 0$ sind. Da nur endlich viele verschiedene $D_{m,k}$ existieren, entsteht mit den Definitionen $BD_{m,k} = D_{m,k}B = 0$ und $D_{m,k} \cdot D_{n,l} = D_{n,l}D_{m,k} = \delta_{mn}\delta_{kl}D_{m,k}$ offenbar ein MHR-Ring A über A^+.

Jeder primitive MHR-Ring ist nach [26] einfach. Gilt nun $pA = A$ für jede Primzahl p, so ist A^+ vollständig, und zwar torsionsfrei nach der letzten Behauptung des Satzes 3.1. Im Falle $pA \neq A$ ist aber $pA = 0$, und hiernach ist A^+ eine elementare Abelsche p-Gruppe. Umgekehrt können auf die so bekommenen Gruppen kommutative Körpern [7] aufgebaut werden, w. z. b. w.

SATZ 3. 3. *Die Abelsche Gruppe A^+ ist dann und nur dann die additive Gruppe eines MHR-Radikalringes, wenn A^+ periodisch ist. Ist insbesondere A ein nilpotenter MH_1R-Ring, so gilt in A die Minimalbedingung auch für die in Hauptrechtsidealen liegenden additiven Untergruppen von A^+,*

BEWEIS. Es sei A^+ die additive Gruppe eines MHR-Radikalringes A. Dann ist A ein Nilring nach dem Satz 1 unserer Arbeit [26]. Wäre nun $x(\neq 0, \in A)$ ein Element der Ordnung $O^+(x) = 0$, so würde $(2^k x)_r = (2^{k+1}x)_r$ mit einem Exponenten k bestehen. Dann gilt aber mit der Bezeichnung $y = 2^k x$ eine Gleichung $y = n(2y) + (2y)z$, folglich auch $(2n-1)^m y = (2n-1)^{m-1}(-2z)y = \cdots = (-2z)^m y = 0$ $(n \in I, z \in A, z^m = 0)$. Da aber $O(y) = 0$ gilt, und der Ring I der ganzen Zahlen nullteilerfrei ist, gelten $2n-1 = 0$ und $n = 1/2 \in I$, was unmöglich ist. Hiernach ist A^+ tatsächlich periodisch. Umgekehrt kann auf jede periodische additive Gruppe ein Zeroring als ein spezieller MHR-Radikalring aufgebaut werden.

Es sei nun A ein nilpotenter MH_1R-Ring, $A^n = 0$ $(A^{n-1} \neq 0)$ und L_k $(k = 1, 2, 3, \ldots, n-1)$ der Linksannullator von A^k in A. Dann ist $0 \subset L_1 \subset L_2 \subset \cdots \subset L_{n-1} = A$. Es sei ferner $G_1 \supset G_2 \supset G_3 \supset \ldots$ eine unendliche absteigende Kette von in einem Hauptrechtsideale $(a)_r$ von A liegenden additiven Untergruppen von A^+, und $a \in L_t$ mit $t \in I$, $1 \leqq t \leqq n-1$. Ist nun $R_i = G_i + G_i A$, so gelten $R_i \subseteq (a)_r$ und $R_i A \subseteq R_i$. Da aber A ein MH_1R-Ring ist, so existiert ein Index i_1 mit $R_{i_1+1} = R_{i_1+2} = \ldots$, also auch mit $G_{i_1+1} + L_{t-1} = G_{i_1+2} + L_{t-1} = \ldots$ wegen $G_i A \subseteq L_t A \subseteq L_{t-1}$. Aus $x \in L_t A$ erhält man nämlich xA_{t-1}, folglich auch $x \in L_{t-1}$ und $L_{t-1} \supseteq L_t A$. Da aber der Verband der Untergruppen von A^+ modular ist, gilt $G_{i_1+1} \cap L_{t-1} \supset G_{i_1+2} \cap L_{t-1} \supset \ldots$. Nach endlich vielen Schritten gewinnen wir die Existenz eines Indexes i_{t-1} mit $G_{i_{t-1}+1} + L_1 = G_{i_{t-1}+2} + L_1 = \ldots$, folglich auch mit $G_{i_{t-1}+1} \cap L_1 \supset G_{i_{t-1}+2} \cap L_1 \supset \ldots$. Diese unendliche Kette besteht wegen $L_1 A = 0$ aus in $(a)_r$ liegenden Rechtsidealen von A, und da A ein MH_1R-Ring ist, muß auch die Kette $G_1 \supset G_2 \supset G_3 \supset \ldots$ nach endlich vielen Schritten abbrechen, w. z. b. w.

BEMERKUNG 3.4. Es kann mit einer ähnlichen Methode gezeigt werden, daß jeder Unterring eines nilpotenten MH_1R-Ringes ebenfalls ein MH_1R-Ring ist. Gilt ferner die Maximalbedingung für die in einem Hauptrechtsideale liegenden Rechtsideale eines nilpotenten MH_1R-Ringes A, so gilt die Maximalbedingung auch für die in einem Hauptrechtsideale liegenden additiven Untergruppen von A^+. Wir wissen aber noch nicht, ob es einen MHR-Radikalring (bzw. einen nilpotenten MHR-Ring bzw. einen MH_1R-Nilring) gibt, dessen ein Unterring S die entsprechende Kettenbedingung nicht erbt. Auch das Problem der Existenz eines MHR-Nilringes ohne die Minimalbedingung für Hauptlinksideale bildet für uns bisher eine offene Frage. Wir wissen ferner ebenfalls noch nicht die Lösung von zwei wichtigen Problemen bezüglich Abelscher Gruppen: 1. Was ist eine notwendige und hinreichende Bedingung dafür, daß die Abelsche Gruppe G einen vollen Endomorphismenring mit Minimalbedingung für zweiseitige Hauptideale besitze? 2. Was ist ein Kriterium dafür, daß auf eine Abelsche Gruppe G wenigstens ein MHR-Ring mit Maximalbedingung für Hauptrechtsideale (kurz: $MMHR$-Ring) aufgebaut werden kann?

§ 4. Die MHR-Ringe als Operatorenbereiche

Wir werden die Operatoren und die Endomorphismen einer Abelschen Gruppe M stets rechtsseitig schreiben. Offensichtlich erhält man nun zwei verschiedene Problemkreise, wenn man eine Eigenschaft erstens für den Verband der Linksideale von E, zweitens für den Verband der Rechtsideale von E erfordert, wobei E den vollen Endomorphismenring von M bezeichnet.

SATZ 4.1. *Ist E der volle Endomorphismenring einer Abelschen Gruppe M, so sind die folgenden Bedingungen untereinander äquivalent:*

I. *in E gilt die Minimalbedingung für Hauptrechtsideale;*

II. *in E gilt die Minimalbedingung für Hauptlinksideale;*

III. *in E gilt die Minimalbedingung für Rechtsideale;*

IV. *in E gilt die Minimalbedingung für Linksideale;*

V. *M ist die direkte Summe einer endlichen Abelschen Gruppe und endlich vieler Exemplare $\mathfrak{R}$ der additiven Gruppe aller rationalen Zahlen.*

BEWEIS. Aus I folgt V. Nehmen wir I an. Da E ein Einselement e hat, ist nE für jede ganze Zahl n ein Hauptrechtsideal in E. Ist nun mE ein minimales Hauptrechtsideal unter den Rechtsidealen nE, so ist mE^+ eine vollständige Gruppe, also gilt $k(mE)=mE$ für jedes $k(\in I, \neq 0)$. Dann ist aber mM wegen $mM=m(ME)=M(mE)=M(kmE)=km(ME)=kmM$ ebenfalls vollständig. Es gibt also eine Untergruppe M_0 von M mit

$M = mM \oplus M_0$ nach dem Satz von BAER, wobei M_0 offenbar eine m-beschränkte periodische Gruppe ist. Da $me = kmx_k$ in E für jedes $k(\neq 0, \in I)$ lösbar ist, gilt $Z(p^\infty) \nsubseteq M$. Folglich ist mM torsionsfrei. Ferner ist M_0 die direkte Summe von zyklischen Gruppen wegen $mM_0 = 0$ und nach dem Satz von BAER. Ist nun $M = \sum_{i \geqq 1} \oplus M_i$ $(M_i \neq 0)$ eine direkte Zerlegung von M, so bilden die direkten Summanden $S_n = \sum_{j \geqq n+1} \oplus M_j$ eine absteigende Kette, und es gilt $M = M_1 \oplus \cdots \oplus M_k \oplus S_k$ für jedes $k(\in I, \geqq 1)$. Ist nun die Abbildung $a_n \in E$ die Projektion von M auf S_n, so gilt offenbar $(a_n)_r \subseteq (a_{n-1})_r$ wegen $a_{n-1}a_n = a_n$. Besteht aber $(a_n)_r = (a_{n-1})_r$ im MHR-Ring E, so ist die Gleichung $a_n y_n = a_{n-1}$ in E lösbar, und so ergibt sich $M_n = M_n a_{n-1} = M_n a_n y_n = 0$ wegen $M_n a_n = 0$. Dies bedeutet nun, daß M die direkte Summe nur endlich vieler ihrer Untegruppen ist. Hiernach folgt aus I tatsächlich V.

Aus II folgt V. Der Beweis ist dem „aus I folgt V'' ähnlich. Man bestätigt aber $(a_n)_l \subseteq (a_{n-1})_l$ wegen $a_n a_{n-1} = a_n$. Ist E ein Ring mit Minimalbedingung für Hauptlinksideale (d. h. ein MHL-Ring), so hat $z_n a_n = a_{n-1}$ in E Lösungen, wenn $n \geqq n_0$ ist. Aus $M_n = M_n a_{n-1} = M_n z_n a_n \subseteq M a_n = S_n$ folgt aber $M_n = 0$ wegen $M_n \cap S_n = 0$, und dies bedeutet wirklich das Bestehen von V.

Aus V folgen III und IV. Im Falle V ist nämlich der volle Endomorphismenring E_1 der torsionsfreien vollständigen Gruppe mM ein voller Matrizenring vom Typ $n \times n$ über dem rationalen Zahlkörper K_0. Der volle Endomorphismenring E_2 der endlichen Gruppe M_0 ist aber offenbar endlich. Da ferner die additive Gruppe $\mathfrak{R}$ aller rationalen Zahlen keine echte Untergruppe von endlichem (gruppentheoretischem) Index hat, so ist $\mathrm{Hom}(mM, M_0) = 0$. Aus der Torsionsfreiheit von mM folgt nun auch $\mathrm{Hom}(M_0, mM) = 0$. Hiernach gewinnen wir eine ringtheoretische direkte Zerlegung $E = E_1 \oplus E_2$. Also gilt in E die Minimalbedingung sowohl für die Rechtsideale, als auch für die Linksideale.

Aus III folgt I. Dies ist trivial.

Aus IV folgt II. Dies ist ebenfalls trivial.

Das folgende Beispiel zeigt uns die Existenz einer Abelschen Gruppe M derart, daß der Zentralisator $A^{(2)}$ des Zentralisators $A^{(1)}$ von einem einfachen MHR-Unterringe A des vollen Endomorphismenringes E von M bezüglich E von $A (= A^{(0)})$ verschieden ist.

BEISPIEL 4.2. Es sei nämlich A der Ring aller $\aleph_0 \times \aleph_0$ Matrizen mit endlich vielen Spalten- und Zeilenvektoren $(\neq 0)$ über dem rationalen Zahlkörper K_0. Bezeichne e_{ij} die Matrizeneinheiten $(i, j = 1, 2, \ldots, n, \ldots)$ und h_{kl} einen Homomorphismus mit $e_{ij} h_{kl} = \delta_{ik} e_{lj}$. Dann ist dieser h_{mk} ein

A-Homomorphismus wegen $(e_{ij}e_{kl})h_{mn}=(e_{ij}h_{mn})e_{kl}$ im vollständig reduziblen A-Rechtsmodul A, und es gelten $h_{mn}\in A^{(1)}$, $\sum_{i=1}^{\infty} h_{ii}=1$, $h_{ij}h_{kl}=\delta_{jk}h_{il}$, wenn wir den Ring A mit dem Ring $\bar{A}$ aller Rechtsmultiplikationen $\bar{a}$ $(a\in A)$ und K_0 mit dem Körper aller A-Endomorphismen von $(e_{11})_r$ identifizieren. Es sei ferner $r_{ij}=\sum_{n=1}^{\infty} h_{n1}r_{ij}h_{1n}$ $(r_{ij}\in K_0)$, $b\in A^{(1)}$ ein beliebiges Element, und $b_{ij}=h_{ii}bh_{jj}$. Dann gilt $b=\sum_{i,j}^{\infty} b_{ij}=\sum_{i,j}^{\infty} r_{ij}h_{ij}=\sum_{i,j}^{\infty} h_{ij}r_i$, wegen $h_{ij}r_{ij}=h_{i1}r_{ij}h_{1j}=r_{ij}h_{ij}$. Ist nun $a_1=e_{1i}h_{1i}b_{ij}h_{j1}$ $(\in(e_{11})_r)$, so ist $e_{1i}x\to a_1x$ $(x\in A)$ ein A-Endomorphismus von $(e_{11})_r$, also gilt $h_{1i}b_{ij}h_{j1}=s_{ij}\in K_0$. Hiernach erhält man $b_{ij}=h_{i1}s_{ij}h_{1j}=s_{ij}h_{ij}=h_{ij}s_{ij}$ und $b=\sum_{i,j}^{\infty} s_{ij}h_{ij}$, wobei s_{ij} nur für endlich viele j von Null verschieden ist, denn $e_{kl}b$ muß nur aus endlich vielen Gliedern $(\neq 0)$ bestehen. Bezeichne nun A^* den Ring aller formalen unendlichen Summen $a^*=\sum_j{}' \sum_{i=1}^{\infty} r_{ij}e_{ij}$ mit r_{ij} mit $r_{ij}\neq 0$ nur für endlich viele j. Ist ferner $M=A$, $m\in M$, $a^*\in A^*$, $b\in A^{(1)}$, so gilt $(ma^*)b=(mb)a^*$, folglich auch $A^*\subseteq A^{(2)}$. Also erhält man $A^{(0)}\neq A^{(2)}$ wegen $A^{(0)}=A=M=\bar{A}\subset A^*$, w. z. b. w.

SATZ 4.3. *Ist A ein halbeinfacher MHR-Ring, so ist jeder perfekte A-Modul M vollständig reduzibel, und die Mächtigkeit der Menge aller A-nichtisomorphen irreduziblen A-Moduln ist gleich der Mächtigkeit von Menge aller einfachen Ideale von A.*

BEWEIS. Bekantlich ist jeder treue minimale A-Modul über einem primitiven Ring A mit $\neq 0$ Sockel einem minimalen Rechtsideale R von A notwendig isomorph. Ähnlich ist jeder perfekte minimale A-Modul über einem halbeinfachen MHR-Ring A einem minimalen Rechtsideale R des Operatorenbereiches A-isomorph. — Ist nun $M(=MA)$ ein beliebiger A-Modul über einem halbeinfachen MHR-Ring A, so besteht $m=\Sigma m_ia_i$ mit $m_ia_i\neq 0$ $(m_i\in M; a_i\in A)$ und mit $a_i=\Sigma b_{ij}$, wobei b_{ij} in einem minimalen Rechtsideale R_{ij} von A liegt. Ist nun $m_ib_{ij}\neq 0$, so ist $m_iR_{ij}(\neq 0)$ ein minimaler A-Untermodul von M, und somit ist M vollständig reduzibel. Die letzte Behauptung folgt durch klassische Methoden.

BEMERKUNG. Das folgende Beispiel zeigt, daß es einen einfachen MHR-Ring A und einen A-Modul M derart gibt, daß keine echte direkte Zerlegung $M=M_0\oplus M_1$ von M mit $M_0A=0$ und mit $M_1A=M_1$ existiert. (Ist A nämlich ein Artinscher halbeinfacher Ring, so gilt für jeden A-Modul M eine direkte Zerlegung $M=M_0\oplus M_1$ mit $M_0A=0$, $M_1A=M_1$, wegen einer Peirceschen Zerlegung von M.) Es sei also A der im Beispiel 4.2 betrachtete Ring A und M die Menge aller Paare (a, n) mit $a\in A$, $n\in I$, wobei wir die Gleichheit und Addition der Paare gewöhnlich definieren. Definiert man auch $(a, n)b = (ab + nb, 0)$, so ist M ein A-Modul. Dann folgt aus $M = M_0 \oplus M_1$, aus $M_0 A = 0$ und aus der Halbeinfachheit von A sicher $M_0 = 0$.

SATZ 4. 4. *Ist A ein beliebiger Ring, und ist xR ein direkter Summand von M für jedes in einem Hauptrechtsideale von A liegende Rechtsideal R von A und für jedes Element x von jedem A-Modul M, so ist A notwendig ein halbeinfacher MHR-Ring.*

BEWEIS. Es sei A ein Ring mit der obigen Eigenschaft und M der spezielle Modul aller Paare (a, n) $(a \in A, n \in I)$ mit $(a, n)b = (ab + nb, 0)$ und mit gewöhnlich definierter Gleichheit und Addition bzw. mit der obigen Operatormultiplikation. Es sei $x = (0, 1)$ und $(e, 0)$ die Komponente von x in der Zerlegung $M = xR \oplus K$ für ein Rechtsideal R, das in einem Hauptrechtsideale $(a)_r (\subseteq A)$ liegt. Dann ist $e (\in A)$ notwendig ein Linkseinselement in R, und somit ist A nach unserem Satz 4 in [26] ein halbeinfacher MHR-Ring, w. z. b. w.

4. 5. *Es sei A ein MHR-Ring mit dem Radikal J, und M ein perfekter A-Rechstmodul, K ein Untermodul von M. Der Faktormodul M/K von M ist dann und nur dann vollständig reduzibel, wenn $MJ \subseteq K$ gilt.*

BEWEIS. Ist M/K vollständig reduzibel, so besteht $(M/K)J = K/K$, also $MJ \subseteq K$. Umgekehrt folgt aus $MJ \subseteq K$, daß M/K ein (perfekter) A/J-Modul ist, der nach 4. 3 vollständig reduzibel ist.

4. 6. *Sind sowohl M/K_1 als auch M/K_2 vollständig reduzible „ausgezeichnete" Faktormoduln eines perfekten A-Moduls M, so gilt $K_1 = K_2$* (Die Definition s. im § 1.)

BEWEIS. Unter den Voraussetzungen erhält man nach 4. 5 sofort $MJ \subseteq K_1$ und $MJ \subseteq K_2$, also $MJ \subseteq K_1 \cap K_2$. Dies bedeutet aber nach 4. 5, daß $M/(K_1 \cap K_2)$ ebenfalls vollständig reduzibel ist. Da auch die A-Homomorphismen

$$m + (K_1 \cap K_2) \to m + K_i \qquad (i = 1, 2; m \in M)$$

nach der Definition des ausgezeiechneten Faktormoduls Isomorphismen sind, besteht $K_1 = K_1 \cap K_2 = K_2$, w. z. b. w.

4. 7. *Es sei A ein MHR-Ring mit nilpotentem Radikal J und mit der Eigenschaft $a \in aA$ für jedes $a \in A$. Es sei ferner $M = M_0$ ein perfekter A-Modul und M_k/M_{k+1} ein vollständig reduzibler ausgezeichneter Modul für $k = 0, 1, 2, 3, \ldots$. Dann kann das obere Loewysche System $M_0 \supset M_1 \supset M_2 \supset \ldots$ in der kanonischen Form $M \supset MJ \supset MJ^2 \supset \ldots$ eindeutig dargestellt werden. Das aus den „Hypersockeln" von M bestehende untere Loewysche System läßt sich aber unter den Voraussetzungen in der kanonischen Gestalt*

$K_0(M) \subset K_1(M) \subset K_2(M) \subset \ldots$ *eindeutig darstellen* $(K_\gamma(M) = \{m \mid m \in M,$ $mJ^\gamma = 0\})$.[3]

BEMERKUNG 4.8. Wir wissen bisher noch nicht, wie weit die Voraussetzungen $MA = M$, $J^n = 0$ und $a \in aA$ (für jedes $a \in A$) dafür abgeschwächt werden können, daß es eine ausführliche Bekanntmachung über eine explizite Darstellung der oberen bzw. unteren Loewyschen Systeme über MHR-Ringe möglich sei. Es ist aber ebenfalls eine offene Frage, ob genau wann die Bedingung $A^{(0)} = A^{(2)}$ für einen halbeinfachen MHR-Unterring A des vollen Endomorphismenringes E von einer Abelschen Gruppe M gilt.

§ 5. Die verschiedenen Typen der Radikale von MHR-Ringen

Bezeichne J das Jacobsonsche Radikal [13], B das Baer—McCoysche (untere) Nilradikal [2], [19], N das obere Nilradikal, L das Levitzkische Radikal [18], Z das Fuchssche (Zeroid-) Radikal [9] und G das Brown—McCoysche Radikal [3]. In beliebigen Ringen bestehen immer $B \subseteq L \subseteq N \subseteq J \subseteq G$ und $N \subseteq Z$. Es gibt Ringe sowohl mit $Z \subset J$ als auch mit $G \subset Z$. Wir sahen schon bei Satz 1 von [26], daß J in MHR-Ringen stets ein Nilideal ist. Vielmehr hat J in MHR-Ringen die stärkere Eigenschaft, daß jedes „unendliche Produkt" $j_1 \cdot j_2 \cdot j_3 \cdots$ $(j_i \in J)$ gegen Null konvergiert. (Ist aber $A = \{a\}$, $O(a) = 0$, $a^n = 0$ $(n \geqq 2)$, so streben alle „unendlichen Produkte" in A ebenfalls gegen Null, obwohl der Radikalring $A = \{a\}$ kein MHR-Ring ist.)

Es gilt nun auch

SATZ 5.1. *Ist A ein MHR-Ring, so gilt in A immer* $B = L = N = J$.

BEWEIS. A/B ist ein Ring ohne nilpotente Ideale $(\neq 0)$ bzw. nilpotente Rechtsideale $(\neq 0)$. Hiernach ist ein in J/B liegendes minimales Rechtsideal weder nilpotent noch idempotent. Dies bedeutet aber $J = B$, also $B = L = N = J$.

SATZ 5.2. *Es gilt in MHR-Ringen A stets $J \subseteq Z$. Hat insbesondere der MHR-Ring A ein Rechtseinselement e, so ist $J = Z = G$.*

BEWEIS. Es sei erstens A ein beliebiger MHR-Ring. Nach [9] ist $Z = Z_l \cap Z_r$, wobei Z_l (bzw. Z_r) die Summe aller l-Zeroideale (bzw. r-Zeroideale) von A bezeichnet, die mit dem Durchschnitt aller maximalen l-Zerofaktorideale (bzw. r-Zerofaktorideale) übereinstimmt. Diese maximalen l- (bzw. r-) Zerofaktorideale sind nach [9] Primideale in A. Daher ergibt sich

[3] Vgl. F. SZÁSZ, Beziehungen zwischen den Abelschen Gruppen und den assoziativen Ringen mit Minimalbedingung für Hauptrechtsideale, *II. Magyar Matematikai Kongresszus (Budapest, 1960), Előadáskivonatok,* Ia, S. 60—62.

auch $Z \supseteq J$, denn B ist der Durchschnitt aller Primideale, und nach Satz 5.1 gilt $J = B$.

Es sei nun A ein MHR-Ring mit Rechtseinselement e. Wir werden beweisen, daß jedes Element $c(\neq 0)$ von jedem echten zweiseitigen Ideal $C(\neq A)$ ein Linksnullteiler ist. Im entgegengesetzten Falle existiert nämlich ein solches minimales Hauptrechtsideal $(d)_r \subseteq C$, daß d kein Linksnullteiler von A ist. Hiernach ergibt sich $(d)_r = (d^2)_r$, also auch $d = d^2 a$ $(a \in A)$ wegen $e \in A$, weil dann mit d auch d^2 kein Linksnullteiler in A ist. Da offenbar $e - da \notin C$ besteht, gilt es auch $e - da \neq 0$. Dies ist aber unmöglich wegen $d(e - da) = 0$ und wegen der obigen Wahl von d. Daher ist jedes echte Ideal C von A ein l-Zerofaktorideal, und somit besteht $Z_l = J$, denn in A stimmen die primitiven Ideale und die maximalen Ideale untereinander überein (vgl. „*Nachträgliche Bemerkungen*" unserer Arbeit [26]). Dann gilt aber $Z \subseteq Z_l = J \subseteq Z$, also $J = Z (= Z_l \subseteq Z_r)$.

Besitzt nun der MHR-Ring A ein Rechtseinselement e, so ist A/J ein Artinscher halbeinfacher Ring, also halbeinfach auch im Brown—McCoyschen Sinne. Hiernach gilt aber $J = G$, w. z. b. w.

BEISPIELE.

5.3. Ist $A = \{a, b\}$ mit $a + a = b + b = a^2 = ab = ba = b^2 + b = 0$, so ist A endlich, und zwar $|A| = 4$. Ferner gelten $Z = A$, $B = J = G = \{a\} \neq A$, also $G \subset Z$. A hat kein Rechtseinselement.

5.4. Es sei A der im Beispiel 4.2 betrachtete einfache MHR-Ring ohne Rechtseinselement. Dann gelten $B = J = 0$ und $G = Z = A$.

5.5. Ist $A = \{a, b\}$ mit $a + a = b + b = a^2 + a = b^2 = ba = ab + b = 0$, so gilt im endlichen Ring A ohne Rechtseinselement: $B = J = G = Z = \{b\} \neq A$.

5.6. Ist A der Ring ohne Rechtseinselement aller rationalen Zahlen mit geraden Zählern und mit ungeraden Nennern, so gelten in A, der kein MHR-Ring ist, $B = N = Z = 0$ und $J = G = A$. Ferner ist (0) in A offenbar ein Primideal, aber weder ein maximales Ideal noch ein primitives Ideal. (Vgl. „*Nachträgliche Bemerkungen*" unserer Arbeit [26].)

BEMERKUNGEN. Nach den obigen Beispielen kann auch die allgemeine Frage gestellt werden, ob in allen MHR-Ringen das Brown—McCoysche Radikal notwendig ein Teil des Fuchsschen Radikales ist. Man kennt noch ebenfalls kein Kriterium dafür, daß der Faktorring A/Z eines MHR-Ringes A nach seinem Fuchsschen Radikal Z halbeinfach im Sinne von Fuchs sei. Da eine transfinite Potenz des Jacobsonschen Radikales J von jedem MHR-Ringe A verschwindet, d. h. $J^\gamma = 0$ mit einer Ordnungszahl γ gilt (vgl. [26]), kann es auch folgendes gefragt werden: zu welchen Ordnungszahlen γ gibt es wenigstens einen MHR-Ring A mit Radikal J, für das die Bedingungen

$J^\gamma = 0$, $J^\beta \neq 0$, $\beta < \gamma$ gelten? Dies steht in Beziehung mit dem Problem, ob es zu welchen Ordnungszahlen γ wenigstens einen *MHR*-Ring A mit Rechtseinselement e und mit Radikal J derart gibt, daß der Linksannullator von J^β ($\beta \leqq \gamma$) in A genau der β-te transfinite Rechtssockel von A ist. Es wäre merkwürdig, auch einen *MHR*-Radikalring ohne die Minimalbedingung für Hauptlinksideale zu konstruieren, obwohl wir vermuten, daß ein solcher Radikalring vielleicht überhaupt nicht existieren kann. Bezüglich hierher gehörender eventueller Verallgemeinerungen verweisen wir noch auf eine Arbeit von Herrn H. J. HOEHNKE (Nilpotenzkriterien, *Math. Annalen*, **132** (1957), S. 404—411).

§ 6. *MMHR*-Ringe und regelmäßige Hauptrechtsideale

Nach der im § 1 ergebenen Definition ist jeder halbeinfache Artinsche Ring ein *MMHR*-Ring. Sowohl die ringtheoretische direkte Summe von einfachen Artinschen Ringen (die im allgemeinen kein Artinscher Ring ist), als auch jeder Zeroring $Z(p^\infty)$ ist ein *MHR*-Ring, aber im allgemeinen kein *MMHR*-Ring. Dagegen sind die Artinschen Ringe A ohne $Z(p^\infty)$ in A^+ immer *MMHR*-Ringe (vgl. FUCHS [7]).

SATZ 6. 1. *Ist A ein MMHR-Ring mit Radikal J, so ist A/J stets ein Artinscher halbeinfacher Ring. Dann ist die additive Gruppe A^+ von A notwendig die direkte Summe einer torsionsfreien vollständigen Gruppe und endlich vieler ihrer reduzierten p-Komponenten.*

BEWEIS. Es genügt zu zeigen, daß jeder halbeinfache *MMHR*-Ring $B(=A/J)$ ein Artinscher Ring ist. Ist $B = \Sigma \oplus e_\alpha B$ ($e_\alpha^2 = e_\alpha$), so gilt $(e_1)_r \subset \subset (e_1 + e_2)_r \subset (e_1 + e_2 + e_3)_r \subset \ldots$ durch geeignete Wahl der idempotenten Elemente $e_1, e_2, \ldots,$ deren jedes endliche System als ein System paarweise orthogonaler idempotenter Elemente gewählt werden kann. Da A ein *MMHR*-Ring ist, gilt $B = (e_1 + e_2 + \cdots + e_s)_r$ mit einem Index s. Hiernach hat B ein Linkseinselement $e_1 + e_2 + \cdots + e_s$, und deshalb ist B ein halbeinfacher Artinscher Ring.

Ist nun A ein *MHR*-Ring mit $Z(p^\infty) \subseteq A^+$, so ist A kein *MMHR*-Ring wegen $Z(p^\infty) \cdot A = 0$ nach dem Satz 3. 1. Also ist die maximale vollständige Untergruppe eines *MMHR*-Ringes A stets torsionsfrei. Ist ferner $a_p \neq 0$ mit $O(a_p) = p$, so gilt $(a_{p_1})_r \subset (a_{p_1} + a_{p_2})_r \subset (a_{p_1} + a_{p_2} + a_{p_3})_r \subset \ldots.$ Da A ein *MMHR*-Ring ist, so hat A tatsächlich nur endlich viele verschiedene p-Komponenten. Die Anwendung der Resultate von § 3 beendigt nun den Beweis.

Bemerkungen und Beispiele.

1. Die erwähnte Gestalt von A^+ eines $MMHR$-Ringes A ist nur eine notwendige Bedingung bezüglich A^+ über A^+ einen $MMHR$-Ring A zu konstruieren. Sind insbesondere alle p-Komponenten A_p^+ beschränkt, so ist diese stärkere Bedingung schon sicher hinreichend. Ein genaueres Kriterium ist noch unbekannt.

2. Ist A die direkte Summe aller Zeroringe $Z(p^k)$ $(k=1, 2, \ldots$ usw.), so ist A ein $MMHR$-Ring mit unbeschränkter periodischer additiver Gruppe. (Hierbei ist p festgewählt.)

3. Die ringtheoretische direkte Summe A von unendlich vielen Exemplaren der untereinander isomorphen endlichen Primkörper K_p ist sicher ein MHR-Ring, aber kein $MMHR$-Ring, obwohl $pA=0$ gilt. Hier können die Exemplare K_p durch die vollen Matrizenringe $(K_p)_n$ $(n=1, 2, 3, \ldots$ usw.) derart ersetzt werden, daß A ebenfalls ein MHR-Ring mit $pA=0$ aber kein $MMHR$-Ring bleibt.

Nun möchten wir die Untersuchungen von Dieudonné und Hopkins ([5], [12]) über den Sockel A_1 von A ergänzen und fortentwickeln. Wir benützen dafür den Begriff der regelmäßigen Hauptrechtsideale (s. § 1) und die folgenden Bezeichnungen. Ist A ein beliebiger Ring mit Rechtssockel (kurz nur Sockel) A_1 und mit den homogenen Komponenten H_μ [13], so gilt $A_1=\sum_{\mu\in\Omega}\oplus H_\mu$. ($H_\mu$ ist die Summe aller zu einem festen minimalen Rechtsideale R von A A-isomorphen Rechtsideale von A.) Es sei ferner N_μ die Summe aller nilpotenten Rechtsideale von A in H_μ, und M_μ ein Rechtsideal von A in H_μ mit $H_\mu=M_\mu\oplus N_\mu$. Bekanntlich ist jedes Rechtsideal sowohl von M_μ als auch von H_μ sicher ein Rechtsideal ebenfalls von A [5]. Es sei nun N die Summe aller N_μ, N^* die Summe einiger, endlich vieler, festgewählter N_μ; H^* die Summe der entsprechenden (d. h. zu den in N^* liegenden N_μ gehörenden) H_μ und M^* die Summe der entsprechenden M_μ. Es gilt der folgende

Satz 6.2. I. *Ist eA $(e^2=e)$ ein regelmäßiges Hauptrechtsideal von A im Sockel A_1, so gilt $eA=eAe$, und jedes Rechtsideal von eA ist ein Rechtsideal ebenfalls von A. Ferner ist eA ein Artinscher halbeinfacher Ring, und es gilt auch $eN=0$.*

II. *Ist nun $N_0=Ne$ und $n\in N_0$, so bestehen $(e+n)^2=e+n$ und $eA\oplus N=(e+n)A\oplus N$. Ferner ist $(e+n)A$ mit eA $(e^2=e,\ n\in Ne)$ ebenfalls ein regelmäßiges Hauptrechtsideal. Die Abbildung $ex\to(e+n)x$ $(x\in A)$ ist ein A-Isomorphismus von eA auf $(e+n)A$ und aus $(e+n_1)A=(e+n_2)A$ folgt notwendig $n_1=n_2$ $(n_i\in Ne)$. Es gelten auch die Beziehungen $A(1-e)\subseteq$ $\subseteq(1-e-n)A$ und $A(1-e-n)\subseteq(1-e)A$.*

III. *Besteht umgekehrt* $R \oplus N = eA \oplus N$, *wobei* R *ein beliebiges Rechtsideal,* eA *ein regelmäßiges Hauptrechtsideal von* A *und* $e^2 = e$ *ist, so ist* R *notwendig ein regelmäßiges Hauptrechtsideal von* A, *und es existiert ein* $m \in Ne$ *mit* $R = (e+m)A$.

BEWEIS. Es seien $Q_e = (1-e)AeA(1-e)$ und $B_e = eA(1-e) + AeA(1-e)$, wobei eA $(e^2 = e)$ regelmäßig ist, und in A_1 liegt. Dann ist B_e ein Linksideal von A, und es gelten $B_e = Q_e + eA(1-e)$ und $B_e^k = Q_e^k + eA(1-e)Q_e^{k-1}$ $(k = 2, 3, 4, \ldots)$. Da eA regelmäßig ist, existiert ein Exponent l mit $Q_e^l = 0$, also mit $B_e^{l+1} = 0$. Dann ist aber $D_e = eA \cap (B_e + B_e A)$ sowohl idempotent wegen $e^2 = e$ und $D_e \subseteq eA \subseteq A_1$ als auch nilpotent wegen $B_e^{l+1} = 0$. Also gelten $D_e = 0$ und $eA = eAe$ wegen $eA(1-e) \subseteq D_e (= 0; = eA \cap (B_e + B_e A))$. $A = eA \oplus (1-e)A$ und $eA = eAe$ $(e^2 = e)$ bedeuten aber, daß jedes Rechtsideal von eA ebenfalls ein Rechtsideal von A ist. Hiernach ist eA ein halbeinfacher Artinscher Ring.

Aus $eA \cap N = 0$ folgt offenbar $eN = 0$.

Ist nun $N_0 = Ne$ und $n \in N_0$, so gilt $(e+n)^2 = e+n$ wegen $ne = n$, $en = 0$ und $n^2 = 0$. Da aber $ea = (e+n)a - na$ und $(e+n)b = eb + nb$ bestehen, ist $eA \oplus N = (e+n)A \oplus N$. Ein elementares Rechnen zeigt nun, daß die Kerne K_1 bzw. K_2 der A-Homomorphismen $\varphi_1 : x \to ex$ $(x \in A)$ bzw. $\varphi_2 : x \to (e+n)x$ $(x \in A)$ übereinstimmen, und deshalb ist die Abbildung $\varphi : ex \to (e+n)x$ ein A-Isomorphismus von eA auf $(e+n)A$. (Es gelten nämlich $K_1 \subseteq K_2$ und $K_2 \subseteq K_1$.)

Es seien nun $Q_e^l = 0$ und $Q_{e+n} = (1-e-n)A(e+n)A(1-e-n)$. Wir möchten zeigen, daß Q_{e+n} nilpotent, d. h. $(e+n)A$ regelmäßig in A ist. Dies kann wegen $Q_{e+n} \subseteq \{Q_e, N\}$ folgenderweise eingesehen werden. Jedes Element von $\{Q_e, N\}^{2l}$ ist die Summe gewisser Produkte P, deren Faktoren zur mengentheoretischen Vereinigung $Q_e \cup N$ gehören. Dann gilt aber sicher $\{Q_e, N\}^{2l} = 0$. Hat nämlich ein Produkt P keinen Faktor aus N, so gilt $P \in Q_e^{2l}$, also $P = 0$ wegen $Q_e^l = 0$. Besitzt nun ein Produkt P genau einen Faktor n aus N, so ist $P = q_1 q_2 \ldots q_s n q_1' q_2' \ldots q_t'$ mit $q_i, q_j' \in Q_e$, $s \geqq 0, t \geqq 0$. (Das leere Produkt sei nach Definition $1 \in I$.) In diesem Ausdruck ist wegen $(l-1) + 1 + (l-1) < 2l$ entweder $s \geqq l$ oder $t \geqq l$, und somit gilt auch $P = 0$ wegen $Q_e^l = 0$. Enthält drittens ein Produkt P wenigstens zwei Faktoren aus N, so ist ebenfalls $P = 0$ wegen $N^2 = 0$, $AN \subseteq N$ und $NA \subseteq N$. Deshalb ist $(e+n)A$ wirklich regelmäßig in A, wenn eA das gleiche ist.

Besteht ferner $(e+n_1)A = (e+n_2)A$ $(n_i \in N_0 = Ne)$, wobei $eA \subseteq A_1$ regelmäßig ist, so existiert ein $a \in A$ mit $(e+n_1)e = (e+n_2)a$. Folglich ergibt sich $n_2 - n_1 = (n_2 - n_1)e = n_2 e - (ea + n_2 a - e) = (e+n_2)(e-a)$. Hiernach gilt aber $n_2 - n_1 \in N \cap (e+n_2)A$, also auch $n_2 - n_1 = 0$ und $n_2 = n_1$.

Da die Abbildung $ex \to (l+u)x$ nach den Vorigen ein A-Isomorphismus ist, und $eA = eAe$, $(e+n)A = (e+n)A(e+n)$ gelten, bestehen auch

$$A(1-e) = eA(1-e) + (1-e)A(1-e) = (1-e)A(1-e) \subseteq (1-e)A = \\ = (1-e-n)A,$$

und wegen Symmetrie auch $A(1-e-n) \subseteq (1-e)A$. Dies bedeutet aber, daß jeder Linksannullator von e ebenfalls ein Rechtsannullator von $e+n$ ist, bzw. daß jeder Linksannullator von $e+n$ ebenfalls ein Rechtsannullator von e ist.

Gilt zum Schluß $R \oplus N = eA \oplus N$ mit einem beliebigen Rechtsideal R und mit einem regelmäßigen Hauptrechtsideal eA ($e^2 = e$) von A in A_1, so gilt $e = f + n$ mit $f \in R$, $n \in N$. Aus $e^2 = e$, $N^2 = 0$, $RN = 0$ und $R \cap N = 0$ erhält man aber $f^2 + fn + nf + n^2 = f + n$, also $f^2 = f$ und $nf = n$. Da $e = f^2 + n, f \in R$ und $fA \oplus N \supseteq eA \oplus N = R \oplus N \supseteq fA \oplus N$ bestehen, und der Verband der Untergruppen von A^+ modular ist, ergibt sich $R = fA$ mit $f = e - n$. (Offensichtlich gelten nämlich $fA \subseteq R$, $fA + N = R + N$ und $fA \cap N = = R \cap N (= 0)$, und daher folgt unsere Behauptung.)

SATZ 6. 3. *Ist insbesondere A entweder ein MMHR-Ring, oder ein Artinscher Ring, so ist M^* in A stets ein regelmäßiges Hauptrechtsideal.*

BEWEIS. Da M^* die Summe endlich vieler idempotenter minimaler Rechtsideale $e_1A, \ldots, e_nA$ ($e_i^2 = e_i$) von A in H^* ist, kann es $e_{i+1} \in (1-e_i) \ldots \ldots (1-e_2)(1-e_1)A$ wegen der Anwendung klassischer Methoden vorausgesetzt werden. Hiernach hat aber M^* ein Linkseinselement e, d. h. $M^* = eA$ ($e^2 = e$). Es genügt zu zeigen, daß $Q_e = (1-e)AeA(1-e)$ nilpotent ist. Im Falle $d^* \in D^* = H^* \cap (1-e)A$ existieren solche Elemente $a_1, a_2 \in A$ und $n \in N$, daß $d^* = ea_1 + n = (1-e)a_2$ ist, denn es gilt $H^* = M^* \oplus N^*$. Nach einer Linksmultiplikation mit e erhält man $e^2a_1 + en = 0$, folglich auch $ea_1 = 0$ wegen $eA \cap N = 0$. Dies bedeutet aber, daß $d^* = n$, also $D^* \subseteq N^*$. Hiernach gilt nun $Q_e^2 = 0$ wegen $Q_e \subseteq D^* \subseteq N^* \subseteq N$ und wegen $N^2 = 0$, und somit ist M^* tatsächlich regelmäßig in A.

BEMERKUNG 6. 4. Natürlich kann auch die Frage gestellt werden, ob sich die Voraussetzungen des Satzes 6. 2 bzw. des Satzes 6. 3 noch wie weit abschwächen lassen. Ferner kennen wir bisher ebenfalls nicht die Existenz eines beliebigen Ringes (bzw. MHR-Ringes) A derart, daß $(1-e)A(1-e)$ kein nilpotenter Ring, aber $(1-e)AeA(1-e)$ ein nilpotenter Ring ist. Dies steht in Zusammenhang mit der Möglichkeit, daß sich die „regelmäßigen" Hauptrechtsideale vielleicht auch anders definieren lassen. Es wäre also merkwürdig, diese Beziehungen ausführlicher zu untersuchen.

§ 7. Über die *MHU*-Ringe

V. I. ŠNEĬDMYULLER [22] betrachtete die Klasse K der Ringe mit Minimalbedingung für alle Unterringe. Jeder Ring aus dieser Klasse K ist ein MHU-Ring im Sinne der Definition im § 1. Ein unendlicher Zeroring A mit $pA=0$ gehört nicht zu dieser Klasse K, obwohl dieser Ring A ein MHU-Ring ist. Ferner haben wir im unseren Satz 3. 3 gesehen, daß jeder nilpotente MH_1R-Ring auch ein MHU-Ring ist. Die endlichen Ringe sind natürlich MHR-Ringe und auch MHU-Ringe. $Z(p^\infty)$ ist ferner ebenfalls sowohl ein MHR-Ring als auch ein MHU-Ring, obwohl $Z(p^\infty)$ unendlich ist. Dagegen ist der Ring I der ganzen Zahlen weder ein MHR-Ring noch ein MHU-Ring. Der rationale Zahlkörper K_0 ist offenbar ein MHR-Ring, aber kein MHU-Ring. Ist nun $A=\{a_1, a_2, a_3, \ldots\}$ ein Ring mit $a_n+a_n=$ $=a_n^{n+1}=a_na_m+a_ma_n=0$ $(m\neq n)$, so ist A ein MHU-Ring, aber kein MHR-Ring. Gehört nun der Ring A zur Klasse K^* aller solchen Ringe, deren jeder endlich erzeugbare echte Unterring ein Hauptrechtsideal in A ist, so ist A genau dann ein MHU-Ring, wenn er auch ein MHR-Ring ist. Neulich haben wir diese Klasse K^* ganz explizit bestimmt (vgl. in speziellem Falle auch Satz 3 von [25]).[4] Die endlichen Körper, die im allgemeinen nicht zur Klasse K^* gehören müssen, zeigen uns die Existenz solcher Ringe, die gleichzeitig MHR-Ringe und auch MHU-Ringe sind. Jeder unendliche MHR-Ring A (MHU-Ring A) aus der Klasse K^* hat eine periodische additive Gruppe A^+ mit $A_p \simeq Z(p^\infty)$ bzw. $\simeq Z(p^\infty)\oplus I|(p)$ oder $|A_p|<\infty$ für jede Primzahl p. Dies ist ein tieferes Korollar von [25].

Die folgenden Tatsachen bezüglich der MHU-Ringe können aber viel leichter eingesehen werden. Sowohl jedes homomorphe Bild als auch jeder Unterring eines MHU-Ringes ist ebenfalls ein MHU-Ring. Ferner hat jeder MHU-Ring $(\neq 0)$ in jedem seiner Unterringe $(\neq 0)$ minimale Unterringe $(\neq 0)$, die Ringe von Primzahlordnung (also Zeroringe oder Primkörper) sind (vgl. Lemma 1 von [24]).

SATZ 7. 1. *Die additive Gruppe eines MHU-Ringes ist periodisch. Jeder MHU-Schiefkörper ist ein kommutativer, absolut algebraischer Körper der Charakteristik* $p(\neq 0)$. *Ist ferner A ein Nilring, der gleichzeitig sowohl ein MHU-Ring als auch ein MHR-Ring ist, so bilden die Elemente von A bezüglich der Verknüpfung* $r_1 \circ r_2 = r_1 + r_2 - r_1 r_2$ *stets eine (nicht notwendig auflösbare) periodische Gruppe G, die das direkte Produkt ihrer Sylowschen p-Untergruppen ist. Die Gruppe G besitzt dann ein transfinites auflösbares invariantes System.*

[4] Wir verweisen noch auf die in [2] zitierte Arbeit.

BEWEIS. Ist P das maximale periodische Ideal des MHU-Ringes A, so ist $(A/P)^+$ torsionsfrei. Da A/P keinen Unterring einer Ordnung p hat, gilt notwendig $A = P$.

Da der Ring I der ganzen rationalen Zahlen kein MHU-Ring ist, hat jeder MHU-Schiefkörper A eine Charakteristik $p(\neq 0)$, und besitzt A lauter absolut algebraische Elemente wegen des Abbrechen jeder absteigenden Kette $\{x\} \supset \{x^2\} \supset \cdots \supset \{x^{2^k}\} \supset \ldots$ in A. Also ist A nach einem Satz von JACOBSON [13] kommutativ, denn es gibt zu jedem $x \in A$ einen Exponenten n mit der Bedingung $x^n = x$.

Ist nun A ein MHU-Nilring, der gleichzeitig auch ein MHR-Ring ist, so bilden die Elemente von A bekanntlich eine Gruppe G bezüglich der Operation $a_1 \circ a_2 = a_1 + a_2 - a_1 a_2$, denn diese Verknüpfung ist assoziativ. Ferner ist 0 das Einselement dieser Gruppe, und im Falle $a^n = 0$ ist $b = -\{a + a^2 + \cdots \cdots + a^{n-1})$ das Inverselement von a in G. Da im Falle $O^+(a) = m(\neq 0, \in I)$ der Unterring $\{a\}$ von A endlich ist, so ist die Ordnung von a in G ebenfalls endlich. Also ist G periodisch. Jede p-Komponente A_p des Ringes A ist eine Untergruppe in der Gruppe G, und zwar kann es gezeigt werden, daß G das direkte Produkt der Sylowschen p-Untergruppen A_p ist. (Für den Beweis verweisen wir auf [22].) Jedes Ideal von A ist ferner ein Normalteiler in G. Da A ein MHR-Nilring ist, so existiert eine Ordnungszahl γ mit $A^\gamma = 0$, $A^\beta = 0$, $\beta < \gamma$ (vgl. [26]). Betrachten wir nun das transfinite invariante System $A \supset A^2 \supset \cdots \supset A^\omega \supset \cdots \supset A^\gamma = 0$, das aus Normalteilern A^β von G besteht. Dann sind die Faktorgruppen $A^\beta/A^{\beta+1}$ bei jedem „Sprung" $(\beta, \beta+1)$ kommutativ wegen $(a_1 \circ A^{\beta+1}) \circ (a_2 \circ A^{\beta+1}) = a_1 + a_2 + A^{\beta+1} = (a_2 \circ A^{\beta+1}) \circ (a_1 \circ A^{\beta+1})$, w. z. b. w.

BEMERKUNG 7.2. Wir sahen schon, daß die Klassen von MHR-Ringen bzw. von MHU-Ringen untereinander verschieden sind. Nun stellt sich eine Frage, ob was ein Kriterium dafür ist, daß ein MHR-Ring auch ein MHU-Ring sei. Es kann auch die umgekehrte Frage betrachtet werden, d. h. genau wann folgt aus der MHU-Eigenschaft die MHR-Eigenschaft? Diese scheinen uns im allgemeinen schwer.

Anhang

Zum Schluß möchten wir einige allgemeinere offene Grundfragen bezüglich der MHR-Ringe erwähnen, die mit dem Gegenstand unserer Arbeit [26] und der soeben betrachteten Theorie eng zusammenhängen.

PROBLEM 1. Man untersuche die Minimalbedingungen in vollen Matrizenringen A_n $(n = 2, 3, 4, \ldots)$ über einem MHR-Ring A!

PROBLEM 2. Gibt es einen MHR-Ring ohne die Minimalbedingung für endlich erzeugte Rechtsideale?

PROBLEM 3. Gibt es einen MHR-Ring mit endlich vielen Hauptrechtsidealen und mit unendlich vielen Hauptlinksidealen?

PROBLEM 4. Was ist ein Kriterium dafür, daß ein MHR-Ring auch ein Ring mit Minimalbedingung für Hauptlinksideale sei?

PROBLEM 5. Gibt es einen MHR-Ring ohne die Minimalbedingung für die Potenzen eines festen Hauptrechtsideales?

PROBLEM 6. Genau wann ist ein MHR-Ring auch ein $MMHR$-Ring?

PROBLEM 7. Welche MHR-Ringe können in einen MHR-Ring mit Einselement eingebettet werden?

PROBLEM 8. Man ergebe gewisse Verallgemeinerungen der Theorie von gewöhnlichen primären bzw. vollständig primären Ringe durch [13], durch [26], und durch die Ergebnisse dieser Arbeit!

PROBLEM 9. Man untersuche im Zusammenhang mit Problem 8 die eventuellen verschiedenen direkten Zerlegungen von MHR-Ringen A, deren Radikal J von Null verschieden ist!

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EIN ZERSPALTUNGSSATZ FÜR OPERATOREN IM HILBERTRAUM

Von

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1. Wir beweisen in dieser Note durch eine einfache Konstruktion, daß eine Kontraktion T in einem Hilbertraum $\mathfrak{H}$ von einem Teilraum $\mathfrak{H}_0$ reduziert wird, in dem sie eine unitäre Transformation induziert, und daß T im orthogonalen Komplement $\mathfrak{H}_1$ von $\mathfrak{H}_0$ die Eigenschaft hat, daß zu jedem $x \in \mathfrak{H}_1$ $(x \neq 0)$ eine ganze Zahl n existiert, so daß $\|T^{(n)}(x)\| < \|x\|$ gilt.[1] Dieses Ergebnis und damit zusammenhängende einfache Spektralaussagen werden anschließend mit Hilfe der Cayleytransformation auf abgeschlossene Operatoren A übertragen, deren Definitionsbereich $\partial(A)$ die Beziehung $\partial(A) \subset \partial(A^*)$ erfüllt und die einen nichtnegativen Imaginärteil haben. Spezielle Operatoren, die diesen Voraussetzungen genügen, sind z. B. die dissipativen Operatoren in [2].

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2. SATZ 1. *Ist T eine Kontraktion in einem Hilbertraum $\mathfrak{H}$, dann ist T die orthogonale Summe zweier Transformationen T_0 und T_1. Dabei ist T_0 in dem zugehörigen Hilbertraum $\mathfrak{H}_0$ unitär, und T_1 in $\mathfrak{H}_1 = \mathfrak{H} \ominus \mathfrak{H}_0$ hat die Eigenschaft, daß zu jedem $x \in \mathfrak{H}_1$ $(x \neq 0)$ eine ganze Zahl n existiert, so daß $\|T_1^{(n)}x\| < \|x\|$ gilt.*

BEWEIS. Es sei

$$\mathfrak{H}_0 = \{x: \|T^{(n)}x\| = \|x\| \quad \text{für} \quad n = 0, \pm 1, \pm 2, \ldots\}. \tag{1}$$

Aus $x \in \mathfrak{H}_0$ folgt für $n = 0, \pm 1, \pm 2, \ldots$ wegen $\|T^*\| = \|T\| \leq 1$

$$\|T^{*(n)}T^{(n)}x - x\|^2 = \|T^{*(n)}T^{(n)}x\|^2 - 2\|T^{(n)}x\|^2 + \|x\|^2 \leq$$
$$\leq \|x\|^2 - \|T^{(n)}x\|^2 = 0,$$

d. h.

$$T^{*n}T^n x = T^n T^{*n} x = x \quad \text{für} \quad n = 0, 1, 2, \ldots. \tag{2}$$

[1] Für eine lineare, beschränkte Transformation B sei $B^{(n)} = B^n$ für $n = 0, 1, 2, \ldots$ und $B^{(n)} = B^{*|n|}$ für $n = -1, -2, \ldots$, vgl. [1].

Umgekehrt folgt aus $T^{*(n)}T^{(n)}x = x$ auch

$$\|T^{(n)}x\|^2 = (T^{*(n)}T^{(n)}x, x) = \|x\|^2 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Es ist also

$$\mathfrak{H}_0 = \{x\colon\ T^{*(n)}T^{(n)}x = x \quad \text{für} \quad n = 0, \pm 1, \pm 2, \ldots\}. \tag{3}$$

Aus der Darstellung (3) von $\mathfrak{H}_0$ folgen die Linearität und die Abgeschlossenheit von $\mathfrak{H}_0$, also ist $\mathfrak{H}_0$ ein Teilraum von $\mathfrak{H}$. Ist $x \in \mathfrak{H}_0$, dann folgt für $y = Tx$ aus (1)

$$\|T^n y\| = \|T^{n+1}x\| = \|Tx\| = \|y\| \quad \text{für} \quad n = 1, 2, \ldots$$

und wegen $T^*Tx = x$

$$\|T^{*n}y\| = \|T^{*n}Tx\| = \|T^{*n-1}x\| = \|Tx\| = \|y\| \quad \text{für} \quad n = 1, 2, \ldots,$$

d. h. $Tx \in \mathfrak{H}_0$.

Entsprechend zeigt man $T^*x \in \mathfrak{H}_0$ für $x \in \mathfrak{H}_0$. Aus $T\mathfrak{H}_0 \subset \mathfrak{H}_0$ und $T^*\mathfrak{H}_0 \subset \mathfrak{H}_0$ folgt aber für $\mathfrak{H}_1 = \mathfrak{H} \ominus \mathfrak{H}_0$ auch $T\mathfrak{H}_1 \subset \mathfrak{H}_1$ und $T^*\mathfrak{H}_1 \subset \mathfrak{H}_1$.

Bezeichnet schließlich T_0 bzw. T_1 den von T in $\mathfrak{H}_0$ bzw. $\mathfrak{H}_1$ induzierten Operator, dann folgt aus (2), daß T_0 unitär ist. Die Aussage über T_1 ergibt sich aus (1) und der Beziehung $(T_1)^* = (T^*)_1$, wobei $(T^*)_1$ den von T^* in $\mathfrak{H}_1$ induzierten Operator bezeichnet. Damit ist Satz 1 bewiesen.

Der Teilraum $\mathfrak{H}_0$ läßt sich offensichtlich auch folgendermaßen charakterisieren:

$\mathfrak{H}_0$ ist maximaler Teilraum von $\mathfrak{H}$ in dem Sinne, daß es keinen ihn echt enthaltenden Teilraum $\mathfrak{H}_0'$ gibt, so daß $T\mathfrak{H}_0' \subset \mathfrak{H}_0'$ und $T^*\mathfrak{H}_0' \subset \mathfrak{H}_0'$ gilt und der von T in $\mathfrak{H}_0'$ induzierte Operator unitär ist.

Ist λ ein Eigenwert einer Kontraktion T mit $|\lambda| = 1$ und zugehörigem Eigenelement x_0, dann ist bekanntlich $\bar{\lambda}$ ein Eigenwert von T^* mit dem gleichen Eigenelement x_0 ([3], S. 440; vgl. auch [4]). Die in Satz 1 erklärte Transformation T_1 hat folglich keine Eigenwerte vom Betrag 1. Weiterhin liegen auf der Einheitskreislinie keine Punkte des Residualspektrums $\sigma_r(T)$ [2] einer Kontraktion T. Andernfalls folgte aus $\lambda \in \sigma_r(T)$, $|\lambda| = 1$ nach [5], S. 581, $\bar{\lambda} \in \sigma_p(T^*)$, also $\lambda \in \sigma_p(T)$. Für die Transformation T_1 ist folglich die Einheitskreislinie ganz in $\sigma_c(T_1) \cup \varrho(T_1)$ enthalten.

3. A sei ein abgeschlossener Operator in $\mathfrak{H}$ mit in $\mathfrak{H}$ dichtem Definitionsbereich $\partial(A)$ und es gelte $\partial(A) \subset \partial(A^*)$. Dann sind die Operatoren $\mathrm{Re}(A) = \dfrac{A + A^*}{2}$ und $\mathrm{Im}(A) = \dfrac{A - A^*}{2i}$ auf der dichten Menge $\partial(A)$ definiert

[2] Für einen linearen abgeschlossenen Operator B seien das Spektrum $\sigma(B)$ und seine Teile $\sigma_p(B), \sigma_c(B), \sigma_r(B)$ sowie die Resolventenmenge $\varrho(B)$ definiert wie in [5], S. 599.

und symmetrisch, denn es ist (vgl. [3], S. 285, 289)

$$\left(\frac{A+A^*}{2}\right)^* \supseteq \frac{A+A^*}{2} \quad \text{und} \quad \left(\frac{A-A^*}{2i}\right)^* \supseteq \frac{A-A^*}{2i}.$$

Für $x \in \partial(A)$ gilt

$$Ax = \mathrm{Re}(A)x + i\,\mathrm{Im}(A)x.$$

Wir sagen, ein abgeschlossener Operator A mit $\partial(A) \subset \partial(A^*)$ habe einen nichtnegativen Imaginärteil, wenn

$$(\mathrm{Im}(A)x, x) \geqq 0 \quad \text{für alle} \quad x \in \partial(A)$$

gilt.

Es existiere für den abgeschlossenen Operator A mit nichtnegativem Imaginärteil ein Punkt ζ mit $\zeta \in \varrho(A)$ und $\mathrm{Im}(\zeta) < 0$. Als Cayleytransformierte von A (vgl. [6], S. 398) bezeichnen wir den auf ganz $\mathfrak{H}$ durch die Gleichung

$$(4) \qquad T = (A - \bar{\zeta}I)(A - \zeta I)^{-1} = I + (\zeta - \bar{\zeta})(A - \zeta)^{-1}$$

definierten linearen Operator. T ist dann eine Kontraktion (vgl. [3], S. 424), denn es wird mit $y = (A - \zeta I)x$, $x \in \partial(A)$,

$$\|y\|^2 - \|Ty\|^2 = -4\,\mathrm{Im}(\zeta)(\mathrm{Im}(A)x, x) \geqq 0.$$

Aus (4) folgt noch $1 \notin \sigma_p(T)$, $\mathfrak{R}(T - I) = \partial(A)$ [3] und $\mathfrak{R}(T^* - I) = \partial(A^*)$.

Ist umgekehrt T eine Kontraktion mit $1 \notin \sigma_p(T)$, dann existieren nach den Bemerkungen am Ende von Abschnitt 2 die (nicht notwendig beschränkten) Operatoren $(T-I)^{-1}$ und $(T^*-I)^{-1}$ und haben beide in $\mathfrak{H}$ dichte Definitionsbereiche. Wir setzen voraus, daß $\mathfrak{R}(T-I) \subset \mathfrak{R}(T^*-I)$ gilt.

Es sei ζ eine komplexe Zahl mit $\mathrm{Im}(\zeta) < 0$. Dann ist der Operator

$$(5) \qquad A = (\zeta T - \bar{\zeta}I)(T - I)^{-1} = \zeta I + (\zeta - \bar{\zeta})(T - I)^{-1}$$

ein abgeschlossener Operator mit dichtem Definitionsbereich $\partial(A) = \mathfrak{R}(T-I)$. Es ist $\partial(A) \subset \partial(A^*)$, und A hat einen nichtnegativen Imaginärteil, denn für $y \in \partial(A)$ ist mit $y = (T - I)x$

$$(\mathrm{Im}(A)y, y) = \mathrm{Im}(\zeta)(\|Tx\|^2 - \|x\|^2) \geqq 0.$$

Die Transformationen (4) und (5) sind zueinander invers, d. h., ist A ein abgeschlossener Operator mit $\partial(A) \subset \partial(A^*)$ und nichtnegativem Imaginärteil und T seine Cayleytransformierte, dann besteht zwischen T und A auch die Beziehung (5).

Wir untersuchen die Beziehungen zwischen den Spektren von T und A. Den Formeln (4) und (5) entsprechen die Abbildungen

$$(6) \qquad \lambda \to \nu(\lambda) = \frac{\lambda - \bar{\zeta}}{\lambda - \zeta} \quad \text{und} \quad \nu \to \lambda(\nu) = \frac{\zeta\nu - \bar{\zeta}}{\nu - 1}$$

[3] $\mathfrak{R}(B)$ bezeichnet den Wertebereich des Operators B.

der geschlossenen komplexen Ebene auf sich. Aus (4) und (5) folgt (vgl. [6], S. 401), daß einem Eigenwert λ von A eineindeutig ein Eigenwert ν von T entspricht, wobei zwischen λ und ν die Beziehungen (6) bestehen und die zugehörigen Eigenelemente übereinstimmen. Aus (4), (5) und (6) folgt für $\lambda \notin \sigma_p(A)$, d. h. $\nu \notin \sigma_p(T)$

$$(T-\nu I)y=(1-\nu)(A-\lambda I)x$$

mit $y \in \mathfrak{H}$, $x \in \partial(A)$ und $y=(A-I)x$. $A-\lambda I$ bildet also $\partial(A)$ genau dann eineindeutig auf $\mathfrak{H}$ (bzw. auf einen nicht dichten Teil von $\mathfrak{H}$) ab, wenn $T-\nu I$ den Raum $\mathfrak{H}$ eineindeutig auf sich (bzw. auf einen nicht dichten Teil von $\mathfrak{H}$) abbildet, d. h., es ist $\lambda \in \varrho(A)$ (bzw. $\lambda \in \sigma_r(A)$) genau dann, wenn $\nu \in \varrho(T)$ (bzw. $\nu \in \sigma_r(T)$) gilt. Die Punkte $\lambda \in \sigma(A)$ (bzw. $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_r(A)$) entsprechen also eineindeutig den Punkten $\nu \in \sigma(T)$ (bzw. $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$), wenn zwischen ν und λ die Relationen (6) bestehen.

SATZ 2. *A sei ein abgeschlossener Operator mit $\partial(A) \subset \partial(A^*)$ und nichtnegativem*[4] *Imaginärteil, für den eine komplexe Zahl ζ mit $\mathrm{Im}(\zeta)<0$ und $\zeta \in \varrho(A)$ existiert. Dann ist A die orthogonale Summe eines selbstadjungierten Operators A_0 in $\mathfrak{H}_0$ und eines abgeschlossenen Operators A_1 in $\mathfrak{H}_1$, $\mathfrak{H}=\mathfrak{H}_0 \oplus \mathfrak{H}_1$. Die untere Halbebene $\mathrm{Im}(z)<0$ gehört zu $\varrho(A)$; die reelle Achse ist in $\sigma_c(A_1) \cup \varrho(A_1)$ enthalten.*

BEWEIS. T sei die Cayleytransformierte von A. Nach Satz 1 besitzt T eine Darstellung $T=T_0 \oplus T_1$, wobei T_0 unitär ist in $\mathfrak{H}_0$ und T_1 eine Kontraktion ist in $\mathfrak{H}_1$, für welche die Einheitskreislinie ganz in $\varrho(T_1) \cup \sigma_c(T_1)$ enthalten ist. Es ist $1 \notin \sigma_p(T)$ und $\Re(T_1-I) \subset \Re(T_1^*-I)$. A_0 (bzw. A_1) sei die nach (5) aus T_0 (bzw. T_1) gebildete Transformation mit nichtnegativem Imaginärteil. Dann ist

$$A=A_0 \oplus A_1,$$

denn es ist mit $x \in \partial(A)$ auch $P_0x=x_0 \in \partial(A_0)$, $P_1x=x_1 \in \partial(A_1)$ und $Ax_0=A_0x_0$, $Ax_1=A_1x_1$, wenn P_0 bzw. P_1 die orthogonale Projektion auf $\mathfrak{H}_0$ bzw. $\mathfrak{H}_1$ bezeichnet. Die Aussagen über das Spektrum von A und A_1 folgen aus den entsprechenden Aussagen über das Spektrum von T und T_1 im 2. Abschnitt.

FOLGERUNG 1. *Genügt A den Voraussetzungen von Satz 2, dann sind die zu zwei verschiedenen reellen Eigenwerten von A gehörenden Eigenelemente orthogonal, und kein reeller Punkt gehört zu $\sigma_r(A)$.*

FOLGERUNG 2. *Ein Operator A in einem n-dimensionalen unitären Raum mit positivem Imaginärteil (d. h. der Imaginärteil von A ist nichtnegativ, und*

[4] Die folgenden Aussagen gelten mutatis mutandis auch für Operatoren mit nichtpositivem Imaginärteil.

für mindestens ein Element x des Raumes ist $\left(\frac{A-A^*}{2i}x, x\right) > 0$) *hat mindestens einen nichtreellen Eigenwert.*

Andernfalls wäre nämlich der Operator A_1 in der Zerlegung von Satz 2 eigenwertfrei, also $\mathfrak{H}_1 = \{0\}$, d. h. A selbstadjungiert. Die Aussage von Folgerung 2 gilt nicht mehr in einem unendlichdimensionalen Raum. Das zeigt ein Beispiel in [2], S. 57.

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ÜBER EINEN KUSMINSCHEN SATZ

Von

P. SZÜSZ (Budapest)

(*Vorgelegt von* A. RÉNYI)

Meinem verehrten Lehrer, Herrn Prof. P. TURÁN *zum 50. Geburtstag gewidmet*

In der vorliegenden Note werden die folgenden Bezeichnungen benutzt:[1] α bedeutet stets eine zwischen Null und Eins gelegene reelle Zahl; $a_1, a_2, \ldots$ bedeuten die Teilnenner der regelmäßigen Kettenbruchentwicklung von α, d. h.

$$\alpha = [0; a_1, a_2, \ldots]. \tag{1}$$

(Im Falle eines rationalen α bricht die Entwicklung (1) nach endlich vielen Schritten ab.) Ferner wird $a_0 = 0$ und

$$\zeta_0 = \alpha, \quad \zeta_n = a_n + \frac{1}{\zeta_{n+1}} \qquad (n = 0, 1, \ldots), \tag{2}$$

$$z_n = \frac{1}{\zeta_{n+1}} \qquad (n = 0, 1, \ldots) \tag{3}$$

gesetzt. Offenbar gilt $0 < z_n \leqq 1$. Es bezeichne $m_n(x)$ das Lebesguesche Maß der Menge der Zahlen α, für die $z_n \leqq x$ gilt $(n = 0, 1, \ldots)$.

Noch GAUSS hat die Frage aufgeworfen, den Grenzwert $\lim\limits_{n\to\infty} m_n(x)$ zu bestimmen. In einem Brief an LAPLACE behauptete er, es sei ihm gelungen, die Limesrelation

$$\lim_{n\to\infty} m_n(x) = \frac{\log(1+x)}{\log 2} \tag{4}$$

zu beweisen. Sein Beweis wurde jedoch nicht veröffentlicht. Der erste bekannte Beweis von (4) rührt von KUSMIN [1] her.[2] Er bewies statt (4) sogar schärfer

$$m_n(x) = \frac{\log(1+x)}{\log 2} + O(q^{\sqrt{n}}), \tag{5}$$

wobei q eine Konstante bedeutet, die kleiner ist als 1. Aus (4) bzw. (5) haben KHINTCHINE [2], [3] und P. LÉVY [5] verschiedene Schlüsse gezogen, u. a. die Existenz der Dichte der k mit $a_k = r$ (r ist eine natürliche Zahl),

[1] Die Bezeichnungen stimmen mit denen von KHINTCHINE [4] überein.

[2] Der Kusminsche Beweis wurde auch im Buch [4] von KHINTCHINE wiedergegeben.

die Existenz von $\lim\limits_{n\to\infty} \sqrt[n]{a_1 a_2 \ldots a_n}$, die von $\lim\limits_{n\to\infty} \sqrt[n]{q_n}$ für fast α bewiesen, wobei q_n den n-ten Näherungsnenner von α bedeutet.

Ohne Kenntnis der Kusminschen Arbeit hat P. LÉVY[3] die Relation (5) wiederentdeckt, sogar mit dem noch schärferen Restglied $O(q^n)$. Die beiden Beweise sind voneinander gründlich verschieden. KUSMIN macht von der schon von GAUSS bekannten rekurrenten Formel

$$m_{n+1}(x) = \sum_{k=1}^{\infty} \left\{ m_n\left(\frac{1}{k}\right) - m_n\left(\frac{1}{k+x}\right) \right\} \tag{6}$$

Gebrauch, während P. LÉVY eine rekurrente Formel benutzt, in der die Verteilungsfunktionen $m_n(x)$ und auch $m_n^*(x)$ auftreten, wobei $m_n^*(x)$ das Maß der Zahlen α ist, für die $z^* < x$ ist; hier bedeutet z_n den (endlichen) Kettenbruch $[0; a_n, a_{n-1}, \ldots, a_1]$. Jedenfalls scheinen mir beide Beweise unnötig kompliziert zu sein.

In der vorliegenden Note zeige ich, wie man aus dem Kusminschen Ansatz (6) sehr einfach und rasch zum schärferen P. Lévyschen Resultat

$$m_n(x) = \frac{\log(1+x)}{\log 2} + O(q^n) \tag{7}$$

gelangen kann, wobei für q die Zahl $2\zeta(3) - \zeta(2)$ gesetzt werden darf. ($\zeta(s)$ ist die Riemannsche Zeta-Funktion.) P. LÉVY's q ist kleiner, nämlich 0,7. Durch eine Verfeinerung der Rechnung kann ich die Zahl q auf

$$\sqrt{(2\zeta(3) - \zeta(2))(7 + 2\zeta(4) - 6\zeta(3) - \zeta(2))} < 0{,}4$$

herabdrücken.

§ 1 enthält den Beweis von (7) mit $q = 2\zeta(3) - \zeta(2)$; im § 2 wird gezeigt, daß die Zahl $2\zeta(3) - \zeta(2)$ durch eine wesentlich kleinere ersetzt werden kann.

§ 1

Es sei $f_0(x)$ eine für $0 \leqq x \leqq 1$ definierte, zweimal stetig differenzierbare Funktion mit $f_0(0) = 0, f_0(1) = 1$; ferner seien $f_1(x), f_2(x), \ldots$ durch die rekurrente Formel (6) definiert. (Setzt man insbesondere $f_0(x) = m_0(x) = x$, so erhält man die Funktionen $m_n(x)$, die im Gaußschen Problem auftreten.) Da nach Differentiation von (6) eine gleichmäßig konvergente Reihe entsteht, ist die gliedweise Differentiation von (6) statthaft. Ersetzt man $m_n(x)$ durch $f_n(x)$ und führt man diese Differentiation durch, so erhält man

$$f'_{n+1}(x) = \sum_{k=1}^{\infty} f'_n\left(\frac{1}{k+x}\right) \frac{1}{(k+x)^2} \qquad (n = 0, 1, \ldots). \tag{1.1}$$

[3] P. LÉVY [5], wiedergegeben auch im Buch [6] von P. LÉVY (S. 301—306).

Um (7) zu beweisen, genügt es zu zeigen, daß für jede Funktionenfolge $f_0(x), f_1(x), \ldots$, für die $f_0(x)$ zweimal stetig differenzierbar ist und $f_1(x)$, $f_2(x), \ldots$ durch (6) definiert sind, die Relation

$$(1.2) \qquad f_n'(x) = \frac{1}{\log 2} \cdot \frac{1}{1+x} + O(q^n)$$

gilt. Hieraus folgt nämlich (7) sofort durch Integration.

Man setze

$$(1.3) \qquad f_n'(x) = \frac{g_n(x)}{1+x} \qquad (n = 0, 1, \ldots).$$

Dann transformiert sich die rekurrente Formel (1. 1) in

$$(1.4) \qquad g_{n+1}(x) = \sum_{k=1}^{\infty} g_n\left(\frac{1}{k+x}\right) \frac{1+x}{(k+x)(k+1+x)}.$$

Um (1. 2) zu zeigen, genügt es, die Relation

$$(1.5) \qquad g_n(x) = \frac{1}{\log 2} + O(q^n)$$

zu beweisen. Statt (1. 5) würde die Relation

$$g_n(x) = C_n + O(q^n)$$

mit einer nur von n abhängigen Konstanten C_n aus

$$(1.6) \qquad g_n'(x) = O(q^n)$$

folgen; die Relation $C_n = \frac{1}{\log 2} + O(q^n)$ folgt dann aus der Normierungsvoraussetzung $f_n(0) = 0$, $f_n(1) = 1$. Daher genügt es (1. 6) zu beweisen.

SATZ 1. *Es sei* $g_0(x)$ *eine im Intervall* (0, 1) *einmal stetig differenzierbare Funktion, ferner seien* $g_1(x), g_2(x), \ldots$ *durch die rekurrente Formel* (1. 4) *definiert. Es sei vorausgesetzt, daß entweder*

$$(1.7) \qquad -M_0 \leqq g_0'(x) \leqq 0$$

oder

$$(1.8) \qquad 0 \leqq g_0'(x) \leqq M_0$$

gilt. Bezeichnet dann M_n *das Maximum des absoluten Betrages von* $g_n'(x)$ *in* (0, 1), *so gilt*

$$(1.9) \qquad |g_{n+1}'(x)| \leqq (2\zeta(3) - \zeta(2)) \frac{M_n}{(1+x)^2},$$

wobei $\zeta(s)$ *die Riemannsche Zeta-Funktion bedeutet.*

Dem Beweis muß ein Hilfssatz vorangeschickt werden, der auch nicht ganz ohne Interesse sein dürfte.

HILFSSATZ. *$g_0(x), g_1(x), \ldots$ bedeuten dasselbe wie im Satz 1. Ist dann $g_0(x)$ in $(0, 1)$ monoton zunehmend, so sind $g_1(x), g_3(x), g_5(x), \ldots$ monoton abnehmend, $g_2(x), g_4(x), \ldots$ monoton zunehmend; ist $g_0(x)$ monoton abnehmend, so sind $g_1(x), g_3(x), \ldots$ monoton zunehmend, $g_2(x), g_4(x), \ldots$ monoton abnehmend.*

BEWEIS DES HILFSSATZES. Nehmen wir an, $g_0(x)$ sei monoton zunehmend, d. h. es gilt (1. 8). (Der Fall, wenn (1. 7) gilt, kann völlig analog behandelt werden.) Dann gilt wegen (1. 4)

$$g_1'(x) = -\sum_{k=1}^{\infty} g_0'\left(\frac{1}{k+x}\right) \frac{1+x}{(k+x)^3(k+1+x)} + \\ + \sum_{k=1}^{\infty} g_0\left(\frac{1}{k+x}\right) \frac{k(k-1)-(1+x)^2}{(k+x)^2(k+1+x)^2}, \tag{1. 10}$$

oder, indem man von der rechten Seite den durch Differentiation der Identität $\sum_{k=1}^{\infty} \frac{1+x}{(k+x)(k+1+x)} = 1$ beweisbaren Ausdruck

$$\sum_{k=1}^{\infty} g_0\left(\frac{1}{1+x}\right) \frac{k(k-1)-(1+x)^2}{(k+x)^2(k+1+x)^2} = 0 \tag{1. 11}$$

subtrahiert,

$$g_1'(x) = -\sum_{k=1}^{\infty} g_0'\left(\frac{1}{k+x}\right) \frac{1+x}{(k+x)^3(k+1+x)} - \\ - \sum_{k=2}^{\infty} \left\{ g_0\left(\frac{1}{1+x}\right) - g_0\left(\frac{1}{k+x}\right) \right\} \frac{k(k-1)-(1+x)^2}{(k+x)^2(k+1+x)^2}. \tag{1. 12}$$

Nun ist $g_0(x)$ in $(0, 1)$ voraussetzungsgemäß monoton zunehmend, $g_0'\left(\frac{1}{k+x}\right)$ ist also positiv. Die erste Summe auf der rechten Seite von (1. 12) ist also negativ. Um einzusehen, daß auch die zweite Summe negativ ist, beachte man, daß $k(k-1)-(1+x)^2$ für $k \geqq 3$ für jedes x $(0 \leqq x \leqq 1)$ positiv ist. Hieraus folgt wegen des monotonen Zunehmens von $g_0(x)$

$$\sum_{k=2}^{\infty} \left\{ g_0\left(\frac{1}{1+x}\right) - g_0\left(\frac{1}{k+x}\right) \right\} \frac{k(k-1)-(1+x)^2}{(k+x)^2(k+1+x)^2} \geqq \\ \geqq \left\{ g_0\left(\frac{1}{1+x}\right) - g_0\left(\frac{1}{2+x}\right) \right\} \sum_{k=2}^{\infty} \frac{k(k-1)-(1+x)^2}{(k+x)^2(k+1+x)^2},$$

also durch abermalige Anwendung von (1.11)

$$\sum_{k=2}^{\infty}\left\{g_0\left(\frac{1}{1+x}\right)-g_0\left(\frac{1}{k+x}\right)\right\}\frac{k(k-1)-(1+x)^2}{(k+x)^2(k+1+x)^2}\geqq$$

$$\geqq\left\{g_0\left(\frac{1}{1+x}\right)-g_0\left(\frac{1}{2+x}\right)\right\}\frac{1}{(2+x)^2}\geqq 0.$$

Daher ist in (1.12) auch die zweite Summe negativ; aus $g_0'(x)\geqq 0$ folgt also $g_1'(x)\leqq 0$, und durch Wiederholung der obigen Überlegung aus $g_n'(x)\geqq 0$ folgt $g_{n+1}'(x)\leqq 0$. Damit ist unser Hilfssatz bewiesen.

Beweis des Satzes 1. Differenziert man die rekurrente Formel (1.4) (wir wissen schon, daß das gliedweise Differenzieren statthaft ist), so gilt, analog zu (1.12),

$$(1.13)\qquad g_{n+1}'(x)=-\sum_{k=1}^{\infty}g_n\left(\frac{1}{k+x}\right)\frac{1+x}{(k+x)^3(k+1+x)}-$$

$$-\sum_{k=2}^{\infty}\left\{g_n\left(\frac{1}{1+x}\right)-g_n\left(\frac{1}{k+x}\right)\right\}\frac{k(k-1)-(1+x)^2}{(k+x)^2(k+1+x)^2},$$

oder durch Anwendung des Mittelwertsatzes der Differentialrechnung

$$(1.14)\qquad g_{n+1}'(x)=-\sum_{k=1}^{\infty}g_n'\left(\frac{1}{k+x}\right)\frac{1+x}{(k+x)^3(k+1+x)}-$$

$$-\sum_{k=2}^{\infty}g_n'\left(\frac{1}{\vartheta_k+x}\right)\frac{(k-1)(k(k-1)-(1+x)^2)}{(1+x)(k+x)^3(k+1+x)^2},$$

wo $1<\vartheta<k$.

Nun sind sämtliche Faktoren

$$\frac{(k-1)(k(k-1)-(1+x)^2)}{(1+x)(k+x)^3(k+1+x)^2}$$

der zweiten Summe höchstens bis auf den dem Fall $k=2$ entsprechenden positiv; ferner gilt

$$\frac{2-(1+x)^2}{(1+x)(2+x)^3(3+x)^2}\begin{cases}\geqq 0, & \text{falls } x\leqq\sqrt{2}-1,\\ <0, & \text{falls } x>\sqrt{2}-1.\end{cases}$$

Daher folgt aus (1.14) und aus dem Hilfssatz

$$|g_{n+1}'(x)|\begin{cases}\leqq\dfrac{M_n}{1+x}\left(\dfrac{1}{(1+x)(2+x)}+\displaystyle\sum_{k=2}^{\infty}\frac{k(k-1)^2+(2+x)(1+x)^2}{(k+x)^3(k+1+x)^2}\right), & \text{falls } x\leqq\sqrt{2}-1,\\[2ex] <\dfrac{M_n}{1+x}\left(\dfrac{1}{(1+x)(2+x)}+\dfrac{1+x}{(2+x)^3(3+x)}+\right.\\ \qquad\left.+\displaystyle\sum_{k=3}^{\infty}\frac{k(k-1)^2+(2+x)(1+x)^2}{(k+x)^3(k+1+x)^2}\right) & \text{sonst,}\end{cases}$$

oder

$$
(1.15)\qquad |g'_{n+1}(x)| \begin{cases} \leqq \dfrac{M_n}{(1+x)^2}\left(\dfrac{1}{2+x}+\displaystyle\sum_{k=2}^{\infty}\dfrac{(1+x)\big(k(k-1)^2+(2+x)(1+x)^2\big)}{(k+x)\;(k+1+x)^2}\right), \\ \hfill \text{falls}\quad x\leqq\sqrt{2}-1, \\ < \dfrac{M_n}{(1+x)^2}\left(\dfrac{1}{2+x}+\dfrac{(1+x)^2}{(2+x)^3(3+x)}+\right. \\ \qquad\left.+\displaystyle\sum_{k=3}^{\infty}\dfrac{(1+x)\big(k(k-1)^2+(2+x)(1+x)^2\big)}{(k+x)^3\;(k+1+x)^2}\right) \quad \text{sonst.} \end{cases}
$$

Eine elementare Diskussion, die auf der Descartesschen Regel beruht, zeigt, daß der Klammerausdruck auf der rechten Seite von (1. 15) für $x=0$ maximal ist. Daher gilt

$$
|g'_{n+1}(x)| \leqq \frac{M_n}{(1+x)^2}\left(\frac{1}{2}+\sum_{k=2}^{\infty}\frac{k(k-1)^2+2}{k^3(k+1)^2}\right),
$$

oder, indem man den Wert des Klammerausdruckes durch die Werte der Riemannschen Zeta-Funktion ausdrückt,

$$
|g'_{n+1}(x)| \leqq \frac{M_n}{(1+x)^2}\,(2\zeta(3)-\zeta(2)),
$$

womit (1. 9) bewiesen ist.

BEMERKUNG. Begnügt man sich in (1. 9) statt $2\zeta(3)-\zeta(2)$ mit einer um etwa 0,1 größeren Konstanten (die noch immer kleiner als 1 ist), so wird der Hilfssatz entbehrlich. So kann der Beweis noch wesentlich kürzer geführt werden.

§ 2

Aus (1. 9) folgt, daß in (7) für q die Zahl

$$
(2.1)\qquad C_1 = 2\zeta(3)-\zeta(2)
$$

gesetzt werden darf. Nun zeige ich, daß C_1 durch eine wesentlich kleinere Zahl ersetzt werden darf.

SATZ 2. *In* (7) *darf für* q *die Zahl*

$$
(2.2)\qquad C_3 = \sqrt{C_1 C_2}
$$

gesetzt werden, wobei

$$
(2.3)\qquad C_2 = 7+2\zeta(4)-6\zeta(3)-\zeta(2)
$$

bezeichnet.

Beweis. Setzt man (1.9) in (1.14) ein, so folgt aus der Positivität von $2-(1+x)^2$ für $x \leqq \sqrt{2}-1$ und daraus, daß $g_n'(t)$ ihr Vorzeichen im Intervall (0, 1) nicht ändert,

$$
(2.4) \qquad |g_{n+1}'(x)| \leqq C_1 M_{n+1} \left(\sum_{k=2}^{\infty} \frac{1+x}{(k+x)(k+1+x)^3} + \right.
$$

$$
\left. + \sum_{k=1}^{\infty} \frac{(k-1)(k(k-1)-(1+x)^2)}{(1+x)(k+x)(k+1+x)^4} \right),
$$

falls $x \leqq \sqrt{2}-1$; für $x > \sqrt{2}-1$ gilt dieselbe Abschätzung mit dem einzigen Unterschied, daß dann auch das dem Fall $k=2$ entsprechende Glied in der zweiten Summe fallen gelassen werden muß. Man erhält, ebenso wie bei dem Beweis von (1.9), daß die Klammer auf der rechten Seite von (2.4) für $x=0$ maximal ist, also

$$
|g_{n+1}'(x)| \leqq C_1 M_{n-1} \left(\sum_{k=1}^{\infty} \frac{1}{k(k+1)^3} + \sum_{k=2}^{\infty} \frac{(k-1)(k(k-1)-1)}{k(k+1)^4} \right),
$$

oder, indem man den Klammerausdruck wieder durch die Werte der Riemannschen Zeta-Funktion ausdrückt,

$$
(2.5) \qquad |g_{n+1}'(x)| \leqq C_1 C_2 M_{n-1},
$$

wobei C_2 die Bedeutung (2.3) hat. (2.5) ist gleichbedeutend mit

$$
M_{n+1} \leqq C_1 C_2 M_{n-1},
$$

womit auch unser Satz 2 bewiesen ist.

BUDAPEST, FORSCHUNGSINSTITUT FÜR MATHEMATIK
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ON SOME FURTHER ONE-SIDED THEOREMS OF NEW TYPE IN THE THEORY OF DIOPHANTINE APPROXIMATIONS

By

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1. A natural generalization of some classical theorems in the theory of diophantine approximations the present author has found in certain systematical estimations concerning generalized power-sums of complex numbers. The fruitfulness of this new trend was shown by various applications in the analysis and analytical number theory.[1] As a typical one of these estimations which can be compared with the results of the present paper we quote the following theorem:[2]

If m is a non-negative integer, further

$$1=|z_1|\geqq|z_2|\geqq\cdots\geqq|z_n| \tag{1.1}$$

and $b_1, b_2, \ldots, b_n$ *are complex numbers with*

$$\min_{\mu=1,\ldots,n}|b_1+\cdots+b_\mu|>0,$$

then there is an integer ν *with*

$$m+1\leqq\nu\leqq m+n \tag{1.2}$$

and

$$\left|\sum_{j=1}^{n} b_j z_j^{\nu}\right|\geqq\left(\frac{n}{8e(m+n)}\right)^{n}\min_{\mu=1,\ldots,n}|b_1+\cdots+b_\mu|. \tag{1.3}$$

As E. Makai proved[3] replacing on the right side the constant 8 by

[1] See my book *Eine neue Methode in der Analysis und deren Anwendungen* (Budapest, 1953). Here the emphasis was led on the applications; for a first account of a systematization of the underlying theory see my paper "Über eine neue Methode der Analysis" (*Wiss. Zeitschr. der Humboldt Univ. zu Berlin Jg.*, (1955/56), pp. 275—279) as well as the Chinese edition of my book. A completely rewritten and enlarged English version will appear in the *Interscience Tracts* series.

[2] Originally I stated this theorem with $24e^2$ instead of $8e$; this stronger form is contained in our paper with Vera T. Sós (On some new theorems in the theory of diophantine approximations, *Acta Math. Acad. Sci. Hung.*, **6** (1955), pp. 241—255), apart from a remark of S. Uchiyama, made in his paper "A note on the second main theorem of P. Turán", *ibid.*, **9** (1958), 379—380.

[3] See his paper "The first main theorem of P. Turán", *Acta Math. Acad. Sci. Hung.*, **10** (1959), pp. 405—412.

$\frac{2}{\log 2} \sim 2{,}89$, the estimation (1.3) becomes for a suitable system of $(b_1^*, b_2^*, \dots, b_n^*)$, $(z_1^*, \dots, z_n)$ with (1.1) false for each $n \geqq 2$ and sufficiently large m's.

2. The number of applications of this theorem in the analytical number theory became in the meantime still larger,[4] but some questions concerning the sign of the prime number formula or those concerning the distribution of primes in different progressions mod Δ suggest the search for theorems asserting that

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu}$$

assumes in every "not too large" intervals "not too small" positive *and* "not too large" negative values. Easy counterexamples show, however, that such "one-sided" theorems cannot hold *generally*, i. e. for variable z_j's satisfying *only* (1.1) (keeping the b_j's fixed) and all we can hope is to obtain "restricted" one-sided theorems, i. e. such ones where the z_j-variables are subject to a geometrical restriction (beside (1.1)). The following restriction seems to be the most suitable for our purposes:

There is a $\varkappa$ with $0 < \varkappa \leqq \frac{\pi}{2}$ such that on the complex z-plane no z_j-numbers are in the angle

$$|\operatorname{arc} z| \leqq \varkappa. \tag{2.1}$$

Under this restriction, which might be called $R(\varkappa)$-restriction in the sequel, I proved the following theorem:

If m is a non-negative integer, further

$$1 = |z_1| \geqq |z_2| \geqq \cdots \geqq |z_n|$$

and the z_j's satisfy an $R(\varkappa)$-restriction, then there are positive integers ν_1 and ν_2 with

$$m+1 \leqq \nu_j \leqq m + n\left(3 + \frac{\pi}{\varkappa}\right) \qquad (j = 1, 2), \tag{2.2}$$

further

$$\operatorname{Re} \sum_{j=1}^{n} z_j^{\nu_1} \geqq \left(\frac{1}{81(m+n)}\right)^{2\left(3+\frac{\pi}{\varkappa}\right)n^3} \tag{2.3}$$

[4] See among others my paper "On the so-called density hypothesis in the theory of the zeta-function of Riemann", *Acta Arith.*, **4** (1958), pp. 31—56; or S. Knapowski, On the mean values of certain functions in prime number theory, *Acta Math. Acad. Sci. Hung.*, **10** (1959), pp. 375—390; or W. Stas, Über eine Anwendung der Methode von Turán auf die Theorie des Restgliedes im Primidealsatz, *Acta Arith.*, **5** (1959), pp. 179—196 etc.

and

$$\text{(2.4)}\qquad \operatorname{Re}\sum_{j=1}^{n} z_j^{\nu_2} \leqq -\left(\frac{1}{81(m+n)}\right)^{2\left(3+\frac{\pi}{\varkappa}\right)n^3}.$$

My proof of this theorem together with another one-sided theorems suggested by the theory, appeared in Vol. XI of this periodical. As I learned from a letter of Mr. S. KNAPOWSKI he succeeded in proving by an ingenious use of theorem (2.2)—(2.3)—(2.4) the following theorem which was a long-standing desideratum in the literature[5] of the prime number theory:

Denoting, as usual, by $\Lambda(n)$ the von Mangoldt symbol

$$\Lambda(n)=\begin{cases}\log p & \text{for}\quad n=p^\alpha \quad (p \text{ prime}),\\ 0 & \text{otherwise},\end{cases}$$

and by $\varrho_0=\beta_0+i\gamma_0$ an arbitrary zero of the Riemann zeta-function with $\beta_0 \geqq \frac{1}{2}$, $\gamma_0>0$, there is a positive, explicitly calculable constant C such that for

$$T>\max\left(C, e^{e^{\log^2\gamma_0}}\right)$$

we have

$$\max_{1\leqq x\leqq T}\left(\sum_{n\leqq x}\Lambda(n)-x\right)>T^{\beta_0}e^{-15\frac{\log T}{\sqrt{\log\log T}}}$$

and

$$\min_{1\leqq x\leqq T}\left(\sum_{n\leqq x}\Lambda(n)-x\right)<-T^{\beta_0}e^{-15\frac{\log T}{\sqrt{\log\log T}}}.$$

3. Hence the significance of the theorem (2.2)—(2.3)—(2.4) is clear. But it is still unsatisfactory for two reasons. The theorem (1.1)—(1.2)—(1.3) can be applied in some cases[6] when not all b_j's are 1; hence in the theorem (2.2)—(2.3)—(2.4) the lack of the coefficients b_j ought to be removed. Secondly, a comparison of theorems (1.1)—(1.2)—(1.3) and (2.2)—(2.3)—(2.4) shows that the right sides of (2.3) and (2.4) are much weaker than that in (1.3) (the occurring exponents being of order n^3 and n, resp.). To meet these requirements we shall prove the following one-sided

THEOREM. *If m is a non-negative integer, further*

$$\text{(3.1)}\qquad 1=|z_1|\geqq|z_2|\geqq\cdots\geqq|z_n|$$

and the z_j's satisfy an $R(\varkappa)$-restriction in (2.1), further the b_j's are complex numbers with

$$\min_{\mu=1,2,\ldots,n}\operatorname{Re}\sum_{j=1}^{\mu}b_j>0,$$

[5] See J. E. LITTLEWOOD, Mathematical notes (12). An inequality for a sum of cosines, *Journal London Math. Soc.,* **12** (1937), pp. 217—222.

[6] See e. g. S. KNAPOWSKI, On an explicit estimation in the prime number theory, *Journal London Math. Soc.,* **34** (1959), pp. 437—441.

then there exist positive integers ν_1 *and* ν_2 *such that*

$$m+1 \leqq \nu_j \leqq m+n\left(3+\frac{\pi}{\varkappa}\right) \qquad (j=1,2), \tag{3.2}$$

further

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqq \frac{1}{2n+1}\left(\frac{n}{24e^3\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)}\right)^{2n} \min_{\mu=1,\dots,n}\left(\operatorname{Re}\sum_{j=1}^{\mu} b_j\right) \tag{3.3}$$

and

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_2} \leqq -\frac{1}{2n+1}\left(\frac{n}{24e^3\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)}\right)^{2n} \min_{\mu=1,\dots,n}\left(\operatorname{Re}\sum_{j=1}^{\mu} b_j\right). \tag{3.4}$$

Now the analogy between (1. 3) and (3. 3)—(3. 4) is much more satisfactory; but in the two-sided case we had no $R(\varkappa)$-restriction. Since the $R(\varkappa)$-restriction showed itself manageable in the applications, the problem became *important*, in which way the estimation (1. 3) can be improved when the z's satisfy an $R(\varkappa)$-restriction? To this problem and to the applications of the above theorem I shall return elsewhere.

4. To the proof of our theorem we shall need the

LEMMA. *The polynomial*

$$F(z)=z^N+a_1 z^{N-1}+\cdots+a_N \tag{4.1}$$

with real coefficients and all zeros in the circle $|z| \leqq 1$ *but outside the angle*

$$|\operatorname{arc} z| \leqq \varkappa \qquad \left(0<\varkappa \leqq \frac{\pi}{2}\right) \tag{4.2}$$

can be multiplied by a polynomial $\varphi(z)$ *with real coefficients (but not necessarily with leading coefficient* 1*) so that writing*

$$F(z)\varphi(z)=\Sigma_\nu e_\nu z^\nu \tag{4.3}$$

a) *the coefficients* e_ν *are non-negative,*

b) *the degree of* $F(z)\varphi(z)$ *is*

$$\leqq \frac{N}{2}\left(1+\frac{\pi}{\varkappa}\right),$$

c) *the inequality*

$$\Sigma_\nu e_\nu \leqq 2^N$$

holds and, finally,

d) *the coefficient of the highest power of* z *in* $F(z)\varphi(z)$ *is*

$$\geqq 3^{-N}.$$

5. For the proof of this lemma we shall need some preliminary remarks. First of all we notice the identity

$$(5.1)\quad (1-2\cos\gamma\cdot w+w^2)\sum_{j=0}^{k}\frac{\sin(j+1)\gamma}{\sin\gamma}w^j=1-\frac{\sin(k+2)\gamma}{\sin\gamma}w^{k+1}+\frac{\sin(k+1)\gamma}{\sin\gamma}w^{k+2}$$

for all complex w's and real γ; this is an easy consequence of the the relation

$$\frac{1}{1-2\cos\gamma\cdot w+w^2}=\sum_{\mu=0}^{\infty}\frac{\sin(\mu+1)\gamma}{\sin\gamma}w^{\mu},$$

valid for $|w|<1$. Let

$$(5.2)\qquad \frac{\pi}{2}\geqq\gamma>0$$

and we apply (5.1) with

$$k=\left[\frac{\pi}{\gamma}\right]-1;$$

this gives

$$(5.3)\quad (1-2\cos\gamma\, w+w^2)\sum_{\mu=0}^{\left[\frac{\pi}{\gamma}\right]-1}\frac{\sin(\mu+1)\gamma}{\sin\gamma}w^{\mu}=1-\frac{\sin\left(1+\left[\frac{\pi}{\gamma}\right]\right)\gamma}{\sin\gamma}w^{\left[\frac{\pi}{\gamma}\right]}+$$

$$+\frac{\sin\left[\frac{\pi}{\gamma}\right]\gamma}{\sin\gamma}w^{\left[\frac{\pi}{\gamma}\right]+1}=1+\frac{\left|\sin\left(1+\left[\frac{\pi}{\gamma}\right]\right)\gamma\right|}{\sin\gamma}w^{\left[\frac{\pi}{\gamma}\right]}+\frac{\left|\sin\left[\frac{\pi}{\gamma}\right]\gamma\right|}{\sin\gamma}w^{\left[\frac{\pi}{\gamma}\right]+1}.$$

Putting

$$a_\mu(\gamma)=\frac{\sin(\mu+1)\gamma}{\sin\gamma}\quad\text{for}\quad \mu\leqq\left[\frac{\pi}{\gamma}\right]-2$$

and

$$(5.4)\quad a_\mu(\gamma)=\frac{\sin(\mu+1)\gamma}{\sin\gamma}+\frac{|\sin(\mu+2)\gamma|}{2\sin\gamma}\quad\text{for}\quad \mu=\left[\frac{\pi}{\gamma}\right]-1$$

(5.3) gives the following modified identity:

$$(1-2\cos\gamma\cdot w+w^2)\sum_{\mu=0}^{\left[\frac{\pi}{\gamma}\right]-1}a_\mu(\gamma)w^{\mu}=1+\frac{\left|\sin\left(\left[\frac{\pi}{\gamma}\right]+1\right)\gamma\right|}{2\sin\gamma}w^{\left[\frac{\pi}{\gamma}\right]-1}+$$

$$(5.5)\quad +\left(\frac{\left|\sin\left(\left[\frac{\pi}{\gamma}\right]+1\right)\gamma\right|}{\sin\gamma}-2\cos\gamma\frac{\left|\sin\left(\left[\frac{\pi}{\gamma}\right]+1\right)\gamma\right|}{2\sin\gamma}\right)w^{\left[\frac{\pi}{\gamma}\right]}+$$

$$+\left(\frac{\left|\sin\left[\frac{\pi}{\gamma}\right]\gamma\right|}{\sin\gamma}+\frac{\left|\sin\left(\left[\frac{\pi}{\gamma}\right]+1\right)\gamma\right|}{2\sin\gamma}\right)w^{\left[\frac{\pi}{\gamma}\right]+1}\overset{\text{def}}{=}\sum_{\mu=0}^{\left[\frac{\pi}{\gamma}\right]+1}d_\mu(\gamma)w^{\mu}.$$

We evidently have for all μ's

$$d_\mu(\gamma) \geqq 0. \tag{5.6}$$

Further obviously

$$d_{\left[\frac{\pi}{\gamma}\right]+1}(\gamma) = \frac{\sin\left[\frac{\pi}{\gamma}\right]\gamma}{\sin\gamma} - \frac{\sin\left(1+\left[\frac{\pi}{\gamma}\right]\right)\gamma}{2\sin\gamma}. \tag{5.7}$$

Let

$$\frac{\pi}{N^*+1} < \gamma \leqq \frac{\pi}{N^*} \qquad (N^* \text{ integer } \geqq 2)$$

or rather

$$\gamma = \frac{\pi}{N^*} - \vartheta\frac{\pi}{N^*(N^*+1)} \qquad (0 \leqq \vartheta < 1); \tag{5.8}$$

then the expression in (5.7) goes into

$$\frac{\sin N^*\gamma - \frac{1}{2}\sin(N^*+1)\gamma}{\sin\gamma} \geqq \frac{\sin\frac{\vartheta\pi}{N^*+1} + \frac{1}{2}\sin\frac{(1-\vartheta)\pi}{N^*}}{\sin\frac{\pi}{N^*}} \overset{\text{def}}{=} U(\vartheta).$$

For $0 \leqq \vartheta \leqq \frac{1}{2}$ we have

$$U(\vartheta) \geqq \frac{1}{2}\frac{\sin\frac{\pi}{2N^*}}{\sin\frac{\pi}{N^*}} = \frac{1}{4\cos\frac{\pi}{2N^*}} \geqq \frac{1}{2\sqrt{2}} > \frac{2}{3\pi}$$

and for $\frac{1}{2} \leqq \vartheta \leqq 1$

$$U(\vartheta) \geqq \frac{\sin\frac{\pi}{2N^*+2}}{\sin\frac{\pi}{N^*}} > \frac{N^*}{N^*+1}\frac{1}{\pi} \geqq \frac{2}{3\pi}.$$

Hence we have

$$d_{\left[\frac{\pi}{\gamma}\right]+1}(\gamma) \geqq \frac{2}{3\pi}, \tag{5.9}$$

independently of γ.

6. Now we can turn to the proof of our lemma. Write

$$-\lambda_j \qquad (j = 1, 2, \ldots, \gamma_1) \tag{6.1}$$

for the negative zeros of $F(z)$, further

$$r_j e^{\pm i\alpha_j} \qquad (j = 1, 2, \ldots, \gamma_2) \tag{6.2}$$

for the complex-conjugate zeros of $F(z)$ with

$$\frac{\pi}{2} \leqq \alpha_j < \pi \tag{6.3}$$

and, finally,

$$\varrho_j e^{\pm i\beta_j} \qquad (j = 1, 2, \ldots, \gamma_3) \tag{6.4}$$

for the complex-conjugate zeros of $F(z)$ with

$$\varkappa < \beta_j < \frac{\pi}{2}; \tag{6.5}$$

some of these categories can be empty. Then we have

$$j_1 + 2j_2 + 2j_3 = N \tag{6.6}$$

and[7]

$$F(z) = \prod_{j=1}^{j_1} (z + \lambda_j) \cdot \prod_{j=1}^{j_2} (z^2 + 2r_j |\cos \alpha_j| z + r_j^2) \cdot \prod_{j=1}^{j_3} (z^2 - 2\varrho_j \cos \beta_j \cdot z + \varrho_j^2). \tag{6.7}$$

We apply the identity (5.5) with

$$w = \frac{z}{\varrho_j}, \quad \gamma = \beta_j;$$

multiplying by $\varrho_j^{\left[\frac{\pi}{\beta_j}\right]+1}$ this gives the identity

$$(z^2 - 2\varrho_j \cos \beta_j \cdot z + \varrho_j^2) \left\{ \sum_{\mu=0}^{\left[\frac{\pi}{\beta_j}\right]-1} a_\mu(\beta_j) \varrho_j^{\left[\frac{\pi}{\beta_j}\right]-1-\mu} z^\mu \right\} = \sum_{\mu=0}^{\left[\frac{\pi}{\beta_j}\right]+1} d_\mu(\beta_j) \varrho_j^{\left[\frac{\pi}{\beta_j}\right]+1-\mu} z^\mu. \tag{6.8}$$

We assert that we may choose as $\varphi(z)$ of our lemma

$$\varphi(z) = \prod_{j=1}^{j_3} \left\{ \sum_{\mu=0}^{\left[\frac{\pi}{\beta_j}\right]-1} a_\mu(\beta_j) \varrho_j^{\left[\frac{\pi}{\beta_j}\right]-1-\mu} z^\mu \right\}, \tag{6.9}$$

i. e.

$$F(z)\varphi(z) = \prod_{j=1}^{j_1} (z + \lambda_j) \cdot \prod_{j=1}^{j_2} (z^2 + 2r_j |\cos \alpha_j| z + r_j^2) \cdot \prod_{j=1}^{j_3} \left\{ \sum_{\mu=0}^{\left[\frac{\pi}{\beta_j}\right]+1} d_\mu(\beta_j) \varrho_j^{\left[\frac{\pi}{\beta_j}\right]+1-\mu} z^\mu \right\}. \tag{6.10}$$

This and (5.6) put assertion a) into evidence. The degree of $F(z)\varphi(z)$ is

$$j_1 + 2j_2 + \sum_{j=1}^{j_3} \left(1 + \left[\frac{\pi}{\beta_j}\right]\right),$$

[7] Empty product means 1.

i. e. owing to (6.5)

$$\leqq j_1+2j_2+j_3\left(1+\frac{\pi}{\varkappa}\right)\leqq \frac{1+\frac{\pi}{\varkappa}}{2}(j_1+2j_2+2j_3)=\frac{N}{2}\left(1+\frac{\pi}{\varkappa}\right)$$

which is precisely assertion b). Further, since we have now

$$0\leqq\lambda_j\leqq 1,\quad 0\leqq r_j\leqq 1,\quad 0\leqq\varrho_j\leqq 1,$$

the formula (6.10) leads to

$$\Sigma_\nu e_\nu = F(1)\varphi(1)\leqq 2^{j_1+2j_2}\prod_{j=1}^{j_3}\left\{\sum_{\mu=0}^{\left[\frac{\pi}{\beta_j}\right]+1} d_\mu(\beta_j)\right\}. \tag{6.11}$$

Using the explicit forms of the coefficients $d_\mu(\beta_j)$ from (5.5) we have

$$\begin{aligned}\sum_{\mu=0}^{\left[\frac{\pi}{\beta_j}\right]+1} d_\mu(\beta_j) &\leqq 1+2\frac{\left|\sin\left(\left[\frac{\pi}{\beta_j}\right]+1\right)\beta_j\right|}{\sin\beta_j}+\frac{\left|\sin\left[\frac{\pi}{\beta_j}\right]\beta_j\right|}{\sin\beta_j}=\\ &=1+\frac{\sin\left[\frac{\pi}{\beta_j}\right]\beta_j}{\sin\beta_j}-2\frac{\sin\left(\left[\frac{\pi}{\beta_j}\right]+1\right)\beta_j}{\sin\beta_j}.\end{aligned} \tag{6.12}$$

Putting

$$\left[\frac{\pi}{\beta_j}\right]=\frac{\pi}{\beta_j}-\vartheta_j \qquad (0\leqq\vartheta_j<1)$$

we get owing to $(0<)\beta_j<\frac{\pi}{2}$ the inequalities

$$0\leqq\frac{\sin\left[\frac{\pi}{\beta_j}\right]\beta_j}{\sin\beta_j}=\frac{\sin\vartheta_j\beta_j}{\sin\beta_j}<1$$

and

$$0\leqq-\frac{\sin\left(1+\left[\frac{\pi}{\beta_j}\right]\right)\beta_j}{\sin\beta_j}=\frac{\sin(1-\vartheta_j)\beta_j}{\sin\beta_j}\leqq 1;$$

thus (6.11) and (6.12) give

$$\Sigma_\nu e_\nu\leqq 2^{j_1+2j_2+2j_3}=2^N,$$

i. e. assertion c). In order to prove assertion d) of our lemma the coefficient of the highest power of z in $F(z)\varphi(z)$ is owing to (6.10)

$$\prod_{j=1}^{j_3} d_{\left[\frac{\pi}{\beta_j}\right]+1}(\beta_j),$$

and hence from (5.8) it is

$$\geqq \left(\frac{2}{3\pi}\right)^{j_3} \geqq \left(\frac{2}{3\pi}\right)^{\frac{N}{2}} > 3^{-N},$$

indeed. Hence our lemma is proved.

7. Now we can turn to the proof of our theorem; we may suppose all z_j's are different. We introduce the numbers ζ_j and B_j $(j=1, 2, \ldots, 2n)$ by

(7. 1) $$\left.\begin{matrix}\zeta_{2j-1}=z_j\\ \zeta_{2j}=\bar{z}_j\end{matrix}\right\} \qquad (j=1,2,\ldots,n)$$

and

(7. 2) $$\left.\begin{matrix}B_{2j-1}=b_j\\ B_{2j}=\bar{b}_j\end{matrix}\right\} \qquad (j=1,2,\ldots,n),$$

respectively, then we have

(7. 3) $$1=|\zeta_1|=|\zeta_2|\geqq|\zeta_3|=|\zeta_4|\geqq\cdots \qquad (\varkappa<|\operatorname{arc}\zeta_j|\leqq\pi).$$

Let $\eta_1, \eta_2, \ldots, \eta_l$ be the maximal number of different ζ_j's and let

(7. 4) $$1=|\eta_1|\geqq|\eta_2|\geqq\cdots\geqq\eta_l \qquad (\varkappa<\operatorname{arc}\eta_j\leqq\pi).$$

Evidently

(7. 5) $$n\leqq l\leqq 2n;$$

the η_j's have obviously the property that with an η also $\bar{\eta}$ occurs among the η_j's. Hence the coefficients of

(7. 6) $$\Phi(z)\overset{\text{def}}{=}\prod_{j=1}^{l}(z-\eta_j)$$

are real and we may apply our lemma to $\Phi(z)$ as $F(z)$; let the corresponding polynomial $\varphi(z)$ be $\varphi^*(z)$. Hence $N=l$ and

(7. 7) $$\Phi(z)\varphi^*(z)\overset{\text{def}}{=}\sum_{\nu\leqq\frac{l}{2}\left(1+\frac{\pi}{\varkappa}\right)} e'_\nu z^\nu$$

with non-negative coefficients e'_ν for which

(7. 8) $$\Sigma_\nu e'_\nu\leqq 2^l$$

and with the leading coefficient

(7. 9) $$\geqq 3^{-l}.$$

8. Let

$$\delta = 1 - \frac{2n}{m + n\left(3 + \frac{\pi}{\varkappa}\right)} \tag{8.1}$$

and

$$\psi(z) = \left(\frac{24}{1-\delta}\right)^{2n} \frac{1 + z + z^2 + \cdots + z^{l-1}}{\delta^{m+n\left(3+\frac{\pi}{\varkappa}\right)}} \Phi(z)\varphi^*(z) \overset{\text{def}}{=} \Sigma_\nu e_\nu'' z^\nu. \tag{8.2}$$

$\psi(z)$ is obviously a polynomial of degree

$$\leq \frac{l}{2}\left(3 + \frac{\pi}{\varkappa}\right) - 1 \leq n\left(3 + \frac{\pi}{\varkappa}\right) - 1 \tag{8.3}$$

with non-negative coefficients, and if the exact degree of $\Phi(z)\varphi^*(z)$ is N_0 then owing to (7.9)

$$\text{coeffs. } z^\nu \text{ in } \psi(z) \geq \left(\frac{24}{1-\delta}\right)^{2n} \frac{1}{3^l} \delta^{-m-n\left(3+\frac{\pi}{\varkappa}\right)} \geq \left(\frac{8}{1-\delta}\right)^{2n} \delta^{-m-n\left(3+\frac{\pi}{\varkappa}\right)} \tag{8.4}$$

for $\nu = N_0, N_0 + 1, \ldots, N_0 + l - 1$.

9. According to a lemma in our above quoted paper with VERA T. SÓS there is an R with

$$\delta \leq R \leq 1 \tag{9.1}$$

so that on the whole periphery of $|z| = R$ the lower estimation

$$|\Phi(z)| \geq 2\left(\frac{1-\delta}{4}\right)^l \tag{9.2}$$

holds, and the same for all partial products. Then the index h is uniquely defined by

$$1 = |\eta_1| \geq |\eta_2| \geq \cdots \geq |\eta_h| > R > |\eta_{h+1}| \geq \cdots. \tag{9.3}$$

Let similarly as in the proof of theorem (1.1)—(1.2)—(1.3)

$$H_1(z) \overset{\text{def}}{=} \prod_{j=h+1}^{l} (z - \eta_j) \overset{\text{def}}{=} \sum_{j=0}^{l-h} c_j^{(1)} z^j,[8] \tag{9.4}$$

where the coefficients $c_j^{(1)}$ are obviously *real*. As easy to see, we have

$$|c_j^{(1)}| \leq \binom{l-h}{j} \qquad (j = 0, 1, \ldots, l-h). \tag{9.5}$$

Further we consider the auxiliary polynomial $H_2(z)$ of degree $\leq h-1$,

[8] If the product is empty, it means 1.

defined by

$$H_2(\eta_j)=\frac{1}{\eta_j^{m+1+N_0}H_1(\eta_j)} \qquad (j=1,2,\dots,h) \tag{9.6}$$

(N_0 in (8.4)). Owing to the definition of the η_j's, the Lagrange interpolation formula gives at once that $H_2(z)$ is real on the real axis. Writing $H_2(z)$ first in the form of Newton interpolation polynomial

$$\begin{aligned} H_2(z)=c_0^{(2)}+c_1^{(2)}(z-\eta_1)+c_2^{(2)}(z-\eta_1)(z-\eta_2)+\cdots \\ \cdots+c_{h-1}^{(2)}(z-\eta_1)(z-\eta_2)\cdots(z-\eta_{h-1}), \end{aligned} \tag{9.7}$$

we have owing to Nörlund's integral representation

$$c_d^{(2)}=\frac{1}{2\pi i}\int\limits_{|s|=R}\frac{ds}{s^{m+1+N_0}H_1(s)(s-\eta_1)(s-\eta_2)\cdots(s-\eta_{d+1})} \qquad (d=0,1,\dots,h-1)$$

and thus using (9.2) the estimation for all d's

$$|c_d^{(2)}|\leqq\frac{1}{2R^{m+N_0}}\left(\frac{4}{1-\delta}\right)^l. \tag{9.8}$$

Writing $H_2(z)$ in the form

$$H_2(z)=\sum_{d=0}^{h-1}c_d^{(3)}z^d \tag{9.9}$$

the coefficients $c_d^{(3)}$ are real and given by

$$c_{h-1}^{(3)}=c_{h-1}^{(2)}$$

and for $0\leqq d\leqq h-2$

$$c_d^{(3)}=c_d^{(2)}-c_{d+1}^{(2)}\sum_{1\leqq j_1\leqq d+1}z_{j_1}+c_{d+2}^{(2)}\sum_{1\leqq j_1<j_2\leqq d+2}z_{j_1}z_{j_2}-\cdots,$$

thus using (9.8) we have for $d=0,1,\dots,h-1$ the inequality

$$\begin{aligned} |c_d^{(3)}|\leqq\frac{1}{2R^{m+N_0}}\left(\frac{4}{1-\delta}\right)^l\Bigg\{1+\binom{d+1}{1}+\binom{d+2}{2}+\cdots \\ +\binom{h-1}{h-1-d}\Bigg\}=\frac{1}{2R^{m+N_0}}\left(\frac{4}{1-\delta}\right)^l\binom{h}{d+1}. \end{aligned} \tag{9.10}$$

We need further the polynomial $H_3(z)$ defined by

$$H_3(z)\overset{\text{def}}{=}H_1(z)H_2(z)\overset{\text{def}}{=}\sum_{\nu=0}^{l-1}c_\nu^{(4)}z^\nu \tag{9.11}$$

with *real* coefficients and of degree $\leqq l-1$. For its coefficients we have

$$\sum_{\nu=0}^{l-1} |c_\nu^{(4)}| \leqq \left(\sum_{\nu=0}^{l-h} |c_\nu^{(1)}|\right)\left(\sum_{d=0}^{h-1} |c_d^{(3)}|\right) \leqq \tag{9.12}$$
$$\leqq 2^{l-h} 2^{h-1} \frac{1}{R^{m+N_0}} \left(\frac{4}{1-\delta}\right)^l = \frac{1}{2} \frac{1}{R^{m+N_0}} \left(\frac{8}{1-\delta}\right)^l$$

owing to (9.5) and (9.10).

10. Finally, we consider the polynomials

$$\psi(z) + z^{N_0} H_3(z) \overset{\text{def}}{=} \Sigma_\nu c_\nu^{(5)} z^\nu, \tag{10.1}$$

$$\psi(z) - z^{N_0} H_3(z) \overset{\text{def}}{=} \Sigma_\nu c_\nu^{(6)} z^\nu; \tag{10.2}$$

they are first of all polynomials with *real* coefficients, since $H_3(z)$ and $\psi(z)$ are such ones. But we assert, moreover, that all coefficients are *non-negative.* For $\nu < N_0$ this follows simply from the non-negativity of the coefficients e_ν''. For $N_0 \leqq \nu \leqq N_0 + l - 1$ we have

$$c_\nu^{(5)} = e_\nu'' + c_{\nu-N_0}^{(4)},$$

i. e. owing to (8.4) and (9.12)

$$c_\nu^{(5)} \geqq \left(\frac{8}{1-\delta}\right)^{2n} \left(\frac{1}{\delta}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)} - \frac{1}{2} \frac{1}{R^{m+N_0}} \left(\frac{8}{1-\delta}\right)^l.$$

Taking into account (8.3), further $l \leqq 2n$ and $R \geqq \delta$, the non-negativity of the coefficients $c_\nu^{(5)}$ follows indeed; the assertion for the $c_\nu^{(6)}$'s follows analogously. Further, owing to the definitions of $\psi(z)$ and $\Phi(z)$, we have from (10.1) and (10.2)

$$\Sigma_\nu c_\nu^{(5)} \eta_j^\nu = \eta_j^{N_0} H_3(\eta_j), \quad \Sigma_\nu c_\nu^{(6)} \eta_j^\nu = -\eta_j^{N_0} H_3(\eta_j) \tag{10.3}$$

for $j = 1, 2, \ldots, l$. But owing to the definitions of $H_3(z)$, $H_1(z)$ and $H_2(z)$ we have[9]

$$\Sigma_\nu c_\nu^{(5)} \eta_j^\nu = \begin{cases} \dfrac{1}{\eta_j^{m+1}} & \text{for} \quad j = 1, 2, \ldots, h \\ 0 & \phantom{\text{for}} \quad j = h+1, \ldots, l \end{cases}$$

and

$$\Sigma_\nu c_\nu^{(6)} \eta_j^\nu = \begin{cases} -\dfrac{1}{\eta_j^{m+1}} & \text{for} \quad j = 1, 2, \ldots, h \\ 0 & \phantom{\text{for}} \quad j = h+1, \ldots, l \end{cases}$$

[9] The second category here and later can be empty.

or

$$(10.4)\qquad \Sigma_\nu c_\nu^{(5)} \eta_j^{m+\nu+1} = \begin{cases} 1 & \text{for } j=1,2,\ldots,h \\ 0 & \text{for } j=h+1,\ldots,l \end{cases}$$

and

$$(10.5)\qquad \Sigma_\nu c_\nu^{(6)} \eta_j^{m+1+\nu} = \begin{cases} -1 & \text{for } j=1,2,\ldots,h \\ 0 & \text{for } j=h+1,\ldots,l. \end{cases}$$

Owing to the definition of the ζ_j's (10.4) and (10.5) hold also replacing the η_j's by the ζ_j's, only the role of h is taken perhaps by another $1 \leqq h_1 \leqq 2n$. Multiplying (10.4) (with ζ_j instead of η_j) by B_j and summing for $j=1,2,\ldots,2n$ we get

$$\Sigma_\nu c_\nu^{(5)} \left\{ \sum_{j=1}^{2n} B_j \zeta_j^{m+\nu+1} \right\} = \sum_{j=1}^{h_1} B_j$$

and similarly

$$\Sigma_\nu c_\nu^{(6)} \left\{ \sum_{j=1}^{2n} B_j \zeta_j^{m+\nu+1} \right\} = -\sum_{j=1}^{h_1} B_j$$

or, taking (7.1) and (7.2) into account,

$$(10.6)\qquad \Sigma_\nu c_\nu^{(5)} \left(\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{m+\nu+1} \right) = \operatorname{Re} \sum_{j=1}^{h_1} b_j$$

and

$$(10.7)\qquad \Sigma_\nu c_\nu^{(6)} \left(\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{m+\nu+1} \right) = -\operatorname{Re} \sum_{j=1}^{h_1} b_j .$$

Hence from the non-negativity of the coefficients $c_\nu^{(5)}$ and $c_\nu^{(6)}$ and taking into account the fact that the degree of the polynomials in (10.1) and (10.2) is owing to (8.3) and $l \leqq 2n$ at most $n\left(3+\frac{\pi}{\varkappa}\right)-1$ we obtained the existence of integers ν_1 and ν_2 with

$$m+1 \leqq \nu_1, \quad \nu_2 \leqq m+n\left(3+\frac{\pi}{\varkappa}\right)$$

and

$$(10.8)\qquad \operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqq \frac{\min\limits_{\mu=1,\ldots,n} \operatorname{Re} \sum\limits_{j=1}^{\mu} b_j}{\Sigma_\nu c_\nu^{(5)}}$$

and

$$(10.9)\qquad -\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_2} \geqq \frac{\min\limits_{\mu=1,\ldots,n} \operatorname{Re} \sum\limits_{j=1}^{\mu} b_j}{\Sigma_\nu c_\nu^{(6)}} .$$

11. In order to complete the proof of our theorem we have to find upper bounds for $\Sigma_\nu c_\nu^{(5)}$ and $\Sigma_\nu c_\nu^{(6)}$. Since they can be done analogously, it suffices to investigate $\Sigma_\nu c_\nu^{(6)}$, say. From (10. 2)

$$\Sigma_\nu c_\nu^{(6)} = \psi(1) - H_3(1) \leqq \psi(1) + \Sigma_\nu |c_\nu^{(4)}|$$

and using (8. 2) and (9. 12) it is

$$\leqq \left(\frac{24}{1-\delta}\right)^{2n} \frac{2n}{\delta^{m+n\left(3+\frac{\pi}{\varkappa}\right)}} \Phi(1)\varphi^*(1) + \frac{1}{2} \frac{1}{\delta^{m+n\left(3+\frac{\pi}{\varkappa}\right)}} \left(\frac{8}{1-\delta}\right)^{2n}.$$

Using also (7. 8) we get

$$\Sigma_\nu c_\nu^{(6)} < (2n+1) \left(\frac{48}{1-\delta}\right)^{2n} \left(\frac{1}{\delta}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)}. \tag{11.1}$$

Since we have

$$\left(\frac{1}{1-\dfrac{2n}{m+n\left(3+\dfrac{\pi}{\varkappa}\right)}}\right)^{m+n\left(3+\frac{\pi}{\varkappa}\right)} \leqq e^{6n},$$

(11. 1) completes the proof.

(Received 28 June 1960)

NOTES ON INTERPOLATION. VIII

(MEAN CONVERGENCE IN INFINITE INTERVALS)

By

J. BALÁZS (Budapest) and P. TURÁN (Budapest), member of the Academy[1]

1. Let be given the triangular infinite matrix A the n^{th} row of which being

$$x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}$$

with

$$x_1^{(n)} > x_2^{(n)} > \dots > x_n^{(n)} \tag{1.1}$$

for $n=1,2,\dots$; if there is no danger of misunderstanding, we shall drop the upper index. The n^{th} Lagrange interpolation polynomial $L_n(f, A)$ of a function f belonging to A is the polynomial

$$\sum_{\nu=1}^{n} f(x_\nu^{(n)})\, l_{\nu n}(x, A) = \sum_{\nu=1}^{n} f(x_\nu)\, l_\nu(x) \tag{1.2}$$

of degree $\leq n-1$ which coincides with $f(x)$ at $x=x_\nu^{(n)}$ $(\nu=1,2,\dots,n)$. Here

$$\omega(x) = \omega_n(x) = \omega_n(x, A) = \prod_{\nu=1}^{n} (x - x_\nu^{(n)}) = \prod_{\nu=1}^{n} (x - x_\nu) \tag{1.3}$$

and the "fundamental functions" $l_{\nu n}(x)$ are given by

$$l_{\nu n}(x, A) = l_\nu(x) = \frac{\omega(x)}{\omega'(x_\nu)(x-x_\nu)}. \tag{1.4}$$

2. Most of the investigations concerning these polynomials $L_n(f)$ refer to the "finite" case, i. e. when all points of A are e. g. in the interval $[-1, +1]$. The infinite case, i. e. when the points of A are in $[-\infty, +\infty]$, received much less attention; as a matter of fact, we are not aware of any results in this direction apart from those contained in the book of SHOHAT[2]

[1] The content of this paper was the subject of a lecture we gave at the mathematical symposium of the Humboldt University, Berlin on 11 November 1960.

[2] *Added in proof* (2 October 1961). Prof J. GERONIMUS kindly called our attention to the paper of J. SHOHAT entitled "Application of orthogonal Tchebyscheff polynomials etc.", *Annali di Math.*, Ser. IV, **18** (1939), pp. 201—238. Though this paper deals with problem of mean convergence, SHOHAT and TAMARKIN did not find it worth even quote in their book on Moment problem in 1943, though the book contains a detailed chapter on mechanical quadrature.

and TAMARKIN, *The problem of moments.* They deal with the sequence

$$\int_{-\infty}^{\infty} L_n(f, A)\,d\psi(x),$$

i. e. with the mechanical quadrature when the n^{th} row of A is given by the zeros of $q_n(x)+c_{n-1}q_{n-1}(x)$; $q_n(x)$ being the n^{th} orthogonal polynomial belonging to the weight $d\psi(x)$ in $[-\infty, +\infty]$. In what follows we shall confine ourselves to a special class B of weights of the form

$$p(x)\,dx = \frac{h(x)}{g(x)}\,dx; \tag{2. 1}$$

our considerations could be extended to a more general class C, but this would not be *effective*, i. e. the decision, whether or not a given weight belongs to the class C is difficult in general (polynomial approximability along the whole real axis with respect to a given weight is concerned). As to the class B of weights, let $g(x)$ be even and infinitely often derivable with

$$g^{(2\nu)}(x) > 0 \qquad (\nu = 0, 1, 2, \ldots;\ x \text{ real}), \tag{2. 2}$$

further for $x>0$ let $\log g(x)$ be a convex function of $\log x$ and

$$\int^{\infty} \frac{\log g(x)}{x^2}\,dx = +\infty; \tag{2. 3}$$

as to $h(x)$ we require only non-negativity on the real axis and

$$\int_{-\infty}^{\infty} h(x)\,dx < \infty. \tag{2. 4}$$

One sees at once that $g(x)$ can be chosen e. g. as $\exp(x^{2k})$, k positive integer, or more generally if $g(x)$ is an entire function of the form

$$g_0(x) = \sum_{\nu=0}^{\infty} a_\nu x^{2\nu},$$

the a_ν's being positive and tending to 0 "not too rapidly". Fixing a $p(x)$ from the class B we choose the matrix of interpolation from the matrix class A_0 the n^{th} row of which consists of the zeros of the equation

$$q_n(x) = 0. \tag{2. 5}$$

Now we assert the

THEOREM. *If $A \in A_0$, further $f(x)$ is continuous on the real axis satisfying*

$$\lim_{x\to\pm\infty} \frac{f(x)}{\sqrt{g(x)}} = 0, \tag{2. 6}$$

then we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\{f(x)-L_n(f, A)\}^2 p(x)dx=0. \tag{2.7}$$

Hence the Lagrange interpolation polynomials converge "in mean" to $f(x)$. As far as we know this is the first general theorem in this direction when the interval is infinite. For finite intervals ERDŐS and the second of us proved the corresponding theorem[3] in 1934.

With an arbitrarily small $0<\varepsilon<\frac{1}{3}$ choosing

$$h(x)=e^{-2\varepsilon x^2}, \quad g(x)=e^{(1-2\varepsilon)x^2}$$

$\frac{h}{g}$ obviously belongs to the class B; hence we got the

COROLLARY. For the Lagrange interpolation polynomial $L_n(f)$ taken at the zeros of $H_n(x)$, the n^{th} Hermite polynomial, we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\{f(x)-L_n(f)\}^2 e^{-x^2}dx=0$$

whenever $f(x)$ is continuous and

$$\lim_{x\to\pm\infty} f(x)e^{-\left(\frac{1}{2}-\varepsilon\right)x^2}=0$$

holds.

This corollary seems not to be noted before. Schwarz's inequality gives further

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|f(x)-L_n(f, A)|\frac{h(x)}{\sqrt{g(x)}}dx=0 \tag{2.8}$$

for the matrices A_0 whenever $f(x)$ is continuous and (2.6) is fulfilled. It would not be difficult to replace in the theorem the continuity by R-integrability of f and f^2, but in the interpolation theory continuity is the natural requirement.

It would be easy to deduce by a reasoning applied first by FEJÉR the facts

$$\lim_{n\to\infty}\max_{j=1,2,\dots,n}|x_j^{(n)}|=\infty \tag{2.9}$$

and, for arbitrarily large $d>0$,

$$\lim_{n\to\infty}\max_{\nu}(x_\nu^{(n)}-x_{\nu+1}^{(n)})=0 \tag{2.10}$$

[3] P. ERDŐS and P. TURÁN, On Interpolation. I, *Annals of Math.*, **38** (1937), pp. 142—155. The essential relation (13) is stated there if only $\int_{-1}^{1}p(x)dx$ exists and $p(x)\geqq M$ owing to misprint: the letter M is to be replaced by 0.

where the max refers to those $x_\nu^{(n)}$'s for which

$$-d \leqq x_{\nu+1}^{(n)} < x_\nu^{(n)} \leqq d.$$

But it is easy to prove by a direct reasoning that if $p(x) \geqq 0$ and all integrals

(2.11) $$\int_{-\infty}^{\infty} x^n p(x)\,dx$$

exist, then (2.9) holds, and to show by an example that

$$x_\nu^{(n)} \geqq -K \qquad (\nu = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$$

(K is independent of n) can occur.

3. For the proof we remark first that from (2.2) and (2.3) it follows that $\sqrt{g(x)}$ is even, $\log\sqrt{g(x)}$ is for $x>0$ a convex function of $\log x$ and

$$\int^{\infty} \frac{\log\sqrt{g(x)}}{x^2}\,dx = +\infty.$$

But then the theorem of S. BERNSTEIN—MANDELBROJT[4] gives that for the functions satisfying (2.6) and for an arbitrarily small $\varepsilon > 0$ there is a polynomial $\pi_k(x)$ of degree k such that on the whole real axis

(3.1) $$\frac{|f(x) - \pi_k(x)|}{\sqrt{g(x)}} \leqq \varepsilon.$$

Further we shall need the

LEMMA. *We have for* $n = 1, 2, \ldots$ *the inequality*

$$\sum_{\nu=1}^{n} g(x_\nu) \int_{-\infty}^{\infty} l_\nu(x)^2 p(x)\,dx \leqq \int_{-\infty}^{\infty} h(x)\,dx.$$

For the proof we consider the polynomial $W(x)$ of degree $\leqq 2n-1$ uniquely determined by the conditions

$$W(x_\nu) = g(x_\nu), \quad W'(x_\nu) = g'(x_\nu) \qquad (\nu = 1, 2, \ldots, n).$$

This polynomial, as easy to verify, can be written in the form

(3.2) $$W(x) = \sum_{\nu=1}^{n} g(x_\nu) \left\{ 1 - \frac{q''(x_\nu)}{q'(x_\nu)}(x - x_\nu) \right\} l_\nu(x)^2 + \sum_{\nu=1}^{n} g'(x_\nu)(x - x_\nu) l_\nu(x)^2.$$

Let

(3.3) $$\Delta(x) \stackrel{\text{def}}{=} g(x) - W(x)$$

which is again infinitely often derivable: then we have for $\nu = 1, 2, \ldots, n$

(3.4) $$\Delta(x_\nu) = \Delta'(x_\nu) = 0.$$

[4] S. BERNSTEIN, Leçons sur les propriétés extrémales etc., and S. MANDELBROJT, Séries adhérentes, régularisation des suites, applications, in part, p. 182.

Owing to the divergence of the integral in (2. 3) and the logarithmic convexity of $g(x)$ we easily get that for all fixed positive integer k's

$$\lim_{x\to\pm\infty}\frac{|x|^k}{g(x)}=0,$$

i. e. from (3. 3)

$$\operatorname{sg}\Delta(\pm\infty)=+1. \tag{3.5}$$

We now assert that on the real axis we have

$$\Delta(x)\geqq 0, \quad \text{i. e.} \quad W(x)\leqq g(x). \tag{3.6}$$

For if we had with a real ξ

$$\Delta(\xi)<0,$$

then owing to (3. 4) and (3. 5) $\Delta(x)$ would have at least $2n+1$ real roots (counted by multiplicity), i. e. after repeated applications of Rolle's theorem with a suitable real η

$$\Delta^{(2n)}(\eta)=0.$$

But since the degree of $W(x)$ is $\leqq 2n-1$, (3. 3) would give

$$g^{(2n)}(\eta)=0,$$

in contradiction to (2.2); hence (3. 6) is true. Multiplying in (3. 6) by $p(x)$ and integrating for $(-\infty, +\infty)$ we get

$$\int_{-\infty}^{\infty} W(x)p(x)dx\leqq\int_{-\infty}^{\infty} h(x)dx. \tag{3.7}$$

But owing to the orthogonality property of $q_n(x)$ we have

$$\int_{-\infty}^{\infty}(x-x_\nu)l_\nu(x)^2p(x)dx=\frac{1}{q_n'(x_\nu)^2}\int_{-\infty}^{\infty}q_n(x)\frac{q_n(x)}{x-x_\nu}p(x)dx=0;$$

hence inserting in (3.7) the explicit form of $W(x)$ in (3.2) the lemma follows.

4. Now we can finish the proof of the theorem as follows. We fix an arbitrarily small $\varepsilon>0$ and consider the polynomial $\pi_k(x)$ from (3. 1). Fixing an integer n with $n\geqq k+1$, denoting

$$f(x)-\pi_k(x)\overset{\text{def}}{=}D(x) \tag{4.1}$$

and taking into account the linearity of the operation $L_n(f; A)$ and the relation

$$L_n(\pi_k, A)\equiv\pi_k(x)$$

we get at once

$$J_n \overset{\text{def}}{=} \int_{-\infty}^{\infty} \{f(x) - L_n(f, A)\}^2 p(x)\,dx = \int_{-\infty}^{\infty} \{D(x) - L_n(D, A)\}^2 p(x)\,dx \leqq$$

(4. 2)

$$\leqq 2 \int_{-\infty}^{\infty} D^2(x) p(x)\,dx + 2 \int_{-\infty}^{\infty} L_n(D, A)^2 p(x)\,dx \overset{\text{def}}{=} J_n' + J_n''.$$

As to J_n', (3. 1), (4. 1) and (2. 1) give at once

$$J_n' \leqq \varepsilon^2 \int_{-\infty}^{\infty} h(x)\,dx. \tag{4. 3}$$

As to J_n'' we have

$$J_n'' = \sum_{\mu} \sum_{\nu} D(x_\mu)\, D(x_\nu) \int_{-\infty}^{\infty} l_\mu(x) l_\nu(x) p(x)\,dx. \tag{4. 4}$$

Observing as we did in our paper with ERDŐS that for $\mu \neq \nu$ again owing to the orthogonality

$$\int_{-\infty}^{\infty} l_\mu(x) l_\nu(x) p(x)\,dx = \frac{1}{q_n'(x_\mu) q_n'(x_\nu)} \int_{-\infty}^{\infty} \frac{q_n(x)}{(x - x_\mu)(x - x_\nu)} q_n(x) p(x)\,dx = 0$$

we get from (4. 4)

$$J_n'' = \sum_{\mu=1}^{n} D(x_\mu)^2 \int_{-\infty}^{\infty} l_\mu(x)^2 p(x)\,dx,$$

and using (3. 1), (4. 1)

$$J_n'' \leqq \varepsilon^2 \sum_{\mu=1}^{n} g(x_\mu) \int_{-\infty}^{\infty} l_\mu(x)^2 p(x)\,dx.$$

Hence by the lemma, (4. 3) and (4. 2) we get

$$J \leqq 2\varepsilon^2 \int_{-\infty}^{\infty} h(x)\,dx$$

which proves the theorem.

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