HUNGARICA ACTA MATHEMATICA

AUCTORITATE

ACADEMIAE SCIENTIARUM NATURALIUM HUNGARICAE

EDIDIT

E. EGERVÁRY

VOL. I., NO. 1.

BUDAPESTINI

MCMXLVI

The HUNGARICA ACTA MATHEMATICA are being published by the *Hungarian Academy of Natural Sciences* in Budapest, edited by Prof. E. Egerváry (Budapest).

The HUNGARICA ACTA MATHEMATICA will be issued in fascicles not tied to any fixed dates; 6 fascicles will go to a volume. The HUNGARICA ACTA MATHEMATICA are obtainable through all booksellers.

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Budapest, VIII., Eszterházy-utca 9.

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HUNGARICA ACTA MATHEMATICA apparaîssent périodiquement; six fascicules forment un volume. HUNGARICA ACTA MATHEMATICA sont accessibles par chaque librairie.

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Budapest, VIII., Eszterházy-utca 9.

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Budapest, VIII., Eszterházy-utca 9.

ON A NEW FORM OF THE DIFFERENTIAL EQUATIONS OF THE PROBLEM OF THREE BODIES.

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MEMBER OF THE ACADEMY

(RECEVIED 30 DECEMBER 1945.)

The most celebrated of all classical dynamical problems are undoubtedly the problem of three bodies and the motion of a rigid body (top) under no forces. A great number of memoirs have been published on these subjects, but, as far as I am aware, none of the writers pointed out, that between these famous problems there is a remarkable analogy which can be advantageously used to the reduction of the problem of three bodies.

The object of the present memoir is to establish this analogy and to use it to derive a new form of the equations of the problem of three bodies as well as some new particular solutions of it.

According to a distinction due to J. J. Sylvester[1] there are two ways to arrive at the reduced form of the differential equations of a mechanical problem: by elimination or by ablimination. „Elimination is the act of extruding a variable from a system of equations in which it has appeared" while „the process whereby the space coordinates referring to absolute position are, so to say, avoided in a class of dynamical questions, is not one of elimination properly so called, the process to be applied in the cases before us is one not of extrusion, but of exclusion a b i n i t i o, or as it may be rendered in a single word, of ablimination". In other words: „The space element is not introduced and then expelled from but prevented from ever making its appearence at all in the resolving system of differential equations."

The motion of a rigid body under no forces is determined by a system of differential equations of 12-th order. It is well known that if the principal axes of inertia of the rigid body are introduced as moving axes, this system splits into three systems of the second order and two systems

[1] J. J. Sylvester, Collected Math. Papers, Vol. II. On the motion of a rigid body acted on by no external forces, pp. 577—601. On the motion of a rigid body moving freely about a fixed point, pp. 602—607.

of the third order. The systems of the second order determine the motion of the centre of gravity, one of the systems of third order constitutes the kinematical equations while the other is identical to the Eulerian equations of the top.

As the space coordinates referring to the absolute position of the top do not appear at all in the Eulerian equations, the use of the moving axes furnished the reduced form of the equations of motion by ablimination.

The question[2] arises now as to whether there are other dynamical systems whose equations of motion by the use of convenient moving axes split similarly into simpler systems of equations.

The present memoir has taken its rise from the plain observation that the kinetic and potential energy of the three bodies, as well as that of a rigid body, are obviously independent of the absolute position in space they depend only on the components of velocities of the principal axes of inertia of the system[3] and on the coordinates and velocities of the bodies relativ to these principal axes. Hence it may be expected that, as regards both problems, equations ot motion admit of beeing constructed, from which an element of absolute space is shut out.

Just this form of the equations of motion of the rigid body under no forces is supplied by the Eulerian equations.

I shall prove now that the system of the 18-th order of the problem of three bodies in the case, if the principal axes of inertia of the three bodies are introduced as moving axes, splits into three systems of the second order, one system of the third order and one system of the 9-th order. The systems of the second and third order have the same signification as in the case of the rigid body, while the system of the 9-th order may be regarded as a new form of the equations of the problem of three bodies, in which the coordinates referring to absolute position do not appear at all. The effectuation of the idea developed here will be sketched on the next pages.

Consider a system of n material particles m_k $(k = 1, 2, \dots n)$ whose coordinates referred to the principal axes of inertia of the system are x_k, y_k, z_k. Let $\omega_1, \omega_2, \omega_3$ denote the components of the angular velo-

[2] Klein—Sommerfeld, Theorie des Kreisels, Bd. II.

[3] The principal axes of inertia are undetermined, if two of the principal moments of inertia of the system of three particles are equal. But this may happen (apart from the trivial case of eqality of the masses and of the mutual distances) only for discret values of the time t, hence our method can be applied without restriction of the generality. Moreover it may be proved subsequently that our equations of motion retain their validity even in the case of equality of the moments of inertia.

city of the system of axes, resolved along the axes themselves and let $\dot{X}$, $\dot{Y}$, $\dot{Z}$ be the components of the velocity of the centre of gravity of the system referred to a system of axes fixed in space. The kinetic energy of the system is then

$$(1)\quad 2T = (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)\sum_k m_k + \sum\left\{\omega_1^2 \sum_k m_k (y_k^2 + z_k^2)\right\} + 2\sum\left\{\omega_1 \sum m_k (y_k \dot{z}_k - z_k \dot{y}_k)\right\} + \sum_k m_k (\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2)$$

If the system, referred to its own principal axes of inertia possesses s degrees of freedom, x_k, y_k, z_k may be expressed in terms of the generalised coordinates $q_1, q_2, \dots q_s$ and the kinetic energy of the system will take the form

$$(2)\quad 2T = (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)\sum_k m_k + \sum \omega_1^2 J_1(q) + 2\sum\sum_\sigma \omega_1 \dot{q}_\sigma F_{1\sigma}(q) + \sum_\sigma\sum_\varrho \dot{q}_\sigma \dot{q}_\varrho G_{\sigma\varrho}(q)$$

As far as the mechanical system is under the action of conservativ internal forces, the potential of these forces is a function of the generalised coordinates

$$(3)\quad V = V(q_1, q_2, \dots q_s)$$

If we wish now to find the form which is taken by the equations of motion when the coordinates q_σ and the quasi-velocities ω_1, ω_2, ω_3 are introduced as dependent variables then the method used by Lagrange to the derivation of the Eulerian equations seems to be the simplest way. The application of this method leads to the following equations of mixed type:

$$(4)\quad \frac{d}{dt}\frac{\partial T}{\partial \omega_j} = \frac{\partial T}{\partial \omega_k}\omega_l - \frac{\partial T}{\partial \omega_l}\omega_k \qquad (j, k, l, = 1, 2, 3)$$

$$(5)\quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} = -\frac{\partial V}{\partial q_\sigma} \qquad (\sigma = 1, 2, \dots s)$$

A rigid system of material particles has 0 degrees of freedom relativ to its own principal axes of inertia, consequently equations of the type (5) do not appear at all in this case, while the equations of type (4) are identical to the Eulerian equations.

A system of three particles has 3 degrees of freedom relativ to its principal axes of inertia and if we wish to ensure the analogy of the equations of motion to the Eulerian equations then evidently the three generalised coordinates must be chosen in such a way that the two principal moments of inertia of the system should occurre amongst them. Instead of them however it will be more convenient to introduce the radii of gyration q_1, q_2 (affected with proper signs) which are connected with the principal moments of inertia by the equations

$$(6) \qquad M = \sum m_j \;;\; M q_1^2 = \sum m_j x_j^2 \;;\; M q_2^2 = \sum m_j y_j^2$$

Applying as a third generalised coordinate an angular coordinate $\varkappa$, the relativ coordinates x_j, y_j, z_j of the three bodies are uniquely determined in terms of the generalised coordinates q_1, q_2, $\varkappa$ by means of the following equations

$$(7) \qquad \begin{aligned} x_j &= \sqrt{\frac{m_k + m_l}{m_j}}\, q_1 \cos(\varkappa + \delta_j) \\ y_j &= \sqrt{\frac{m_k + m_l}{m_j}}\, q_2 \sin(\varkappa + \delta_j) \\ z_j &= 0 \end{aligned} \qquad \begin{aligned} &(\delta_1, \delta_2, \delta_3 \text{ are constants} \\ &\text{satisfying the equations} \\ &tg\,(\delta_k - \delta_l) = \sqrt{\frac{M m_j}{m_k \cdot m_l}}\,) \end{aligned}$$

This choice of coordinates leads to the following expression of the kinetic energy of the system

$$(8) \quad 2\,T = M \{ \dot X^2 + \dot Y^2 + \dot Z^2 + q_2^2 \omega_1^2 + q_1^2 \omega_2^2 + (q_1^2 + q_2^2)(\omega_3^2 + \dot\varkappa^2) + \\ + 4\, q_1 q_2 \omega_3 \dot\varkappa + \dot q_1^2 + \dot q_2^2 \}$$

while the potential energy in the case of forces proportional to the ν—1-th power of the distances is given by

$$(9) \qquad V(q_1, q_2, \varkappa) = \\ = \sum m_k m_l \left\{ \left(\frac{M}{m_k} + \frac{M}{m_l} \right) \left[q_1^2 \sin^2(\varkappa + \delta_j) + q_2^2 \cos^2(\varkappa + \delta_j) \right] \right\}^{\frac{\nu}{2}}$$

Lastly, if we denote the direction-cosines of the set of axes fixed in space and of the set of the principal axes by α_j, β_j, γ_j $(j = 1, 2, 3)$, the equations of motion of the three bodies take the form

$$(10) \qquad \ddot{X} = 0 \qquad \ddot{Y} = 0 \qquad \ddot{Z} = 0$$

$$(11) \quad \left.\begin{aligned} &\frac{d}{dt}(q_2^2 \omega_1) + q_2^2 \omega_2 \omega_3 + 2 q_1 q_2 \omega_2 \dot{\varkappa} = 0 \\ &\frac{d}{dt}(q_1^2 \omega_2) - q_1^2 \omega_3 \omega_1 - 2 q_1 q_2 \omega_1 \dot{\varkappa} = 0 \\ &\frac{d}{dt}\left[(q_1^2 + q_2^2)\omega_3 + 2 q_1 q_2 \dot{\varkappa}\right] + (q_1^2 - q_2^2)\omega_1 \omega_2 = 0 \end{aligned}\right\}$$

$$(12) \quad \left.\begin{aligned} &\frac{d}{dt}\left[(q_1^2 + q_2^2)\dot{\varkappa} + 2 q_1 q_2 \omega_3\right] = -\frac{1}{M}\frac{\partial V}{\partial \varkappa} \\ &\ddot{q}_1 - q_1(\omega_2^2 + \omega_3^2 + \dot{\varkappa}^2) - 2 q_2 \omega_3 \dot{\varkappa} = -\frac{1}{M}\frac{\partial V}{\partial q_1} \\ &\ddot{q}_2 - q_2(\omega_1^2 + \omega_3^2 + \dot{\varkappa}^2) - 2 q_1 \omega_3 \dot{\varkappa} = -\frac{1}{M}\frac{\partial V}{\partial q_2} \end{aligned}\right\}$$

$$(13) \quad \dot{\alpha}_j = \alpha_k \omega_l - \alpha_l \omega_k; \quad \dot{\beta}_j = \beta_k \omega_l - \beta_l \omega_k; \quad \dot{\gamma}_j = \gamma_k \omega_l - \gamma_l \omega_k$$
$$(j, k, l = 1, 2, 3)$$

The system of 18-th order of the problem of three bodies splits in this way into three equations of second order, into the system of third order of the kinematical equations and into a system of the 9-th order. This new form of the equations of the problem of three bodies, gained by ablimination, exhibits the following remarkable properties:

It was established in a really elementary way, by application of the Lagrangean method of generalised coordinates (contrary to the most part of the methods of reduction, which use contact-transformations).

The equations of motion are entirely symmetrical[4] with regard to the masses, the mutual distances as well as to the coordinates and velocities of the triangle formed by the bodies.

The equations of motion are obtained immediately in the form of a system of the 9-th order, contrary to the usual methods[5] which reduce the original system of 18-th order by means of the integrals of momentum

[4] A great part of the reduced equations gained by contact transformation is unsymmetrical with regard to the masses as well as to the geometrical data. Lately however several writers have given symmetrical reductions. See F. D. Murnaghan, A symmetric reduction of the planar three-body problem, Amer. Journ. of Math. Vol. 58, pp. 829—32. E. R. van Kampen and A. Wintner, On a symmetrical reduction of the problem of three bodies, Amer. Journ. of Math. Vol. 59. pp. 153—166.

[5] The straightforward derivation of a system of the 9-th order is endeavoured by E. Kähler Transformation der Differentialgleichungen des Dreikörperproblems, Math. Zeitschr. Bd. 24. S. 743—58. But his results are so much complicated that he omits to write explicitely the differential equations.

to the 12-th order and afterwards by means of the integrals of the angular momentum to the 9-th order.

The system (11, 12) of equations of the 9-th order of the problem of three bodies possesses two obvious integrals: the integral of energy:

$$(14)\quad \frac{M}{2}\left\{ q_2^2\omega_1^2 + q_1^2\omega_2^2 + (q_1^2+q_2^2)(\omega_3^2+\dot{\varkappa}^2) + 4\,q_1 q_2 \omega_3 \dot{\varkappa} + \dot{q}_1^2 + \dot{q}_2^2 \right\} + \\ + V(q_1, q_2, \varkappa) = C_1$$

and the integral of angular momentum:

$$(15)\quad q_2^4\omega_1^2 + q_1^4\omega_2^2 + \left\{ (q_1^2+q_2^2)\,\omega_3 + 2\,q_1 q_2 \dot{\varkappa} \right\}^2 = C_2$$

Consequently, by making use of these integrals, the equations of motion can be reduced to the 7-th order and, eliminating the time, to the 6-th order.

If the coordinates $q_1, q_2, \varkappa$ (or, what is the same thing, the mutual distances) are known, as functions of the time, then the system (11) of the third order may be regarded as the system of equations of motion of a *variable* top, which can be reduced by means of the integrals (14, 15) to the first order and solved by quadrature in consequence of the principle of the last multiplier.

The equations of the problem of three bodies established here put in evidence the proposition due to Lagrange that the complete solution of the problem requires only that we know at each instant the sides of the triangle formed by the bodies, the coordinates of each may then be determined by quadratures.

Indeed, the coordinates $q_1, q_2, \varkappa$ and the components $\omega_1, \omega_2, \omega_3$ of the angular velocity being known as functions of the time, it is not necessary to integrate the kinematical equations directly but using a contrivance well known from the theory of the top we can calculate the Eulerian angles ϑ, φ, ψ of the principal axes of inertia by quadratures in the following form

$$(16)\quad \begin{aligned} &\cos\vartheta = \frac{(q_1^2+q_2^2)\,\omega_3 + 2\,q_1 q_2 \dot{\varkappa}}{\sqrt{C_2}}\,; \quad \operatorname{tg}\psi = -\frac{q_1^2\omega_2}{q_2^2\omega_1} \\ &\varphi - \varphi_0 = \sqrt{C_2}\int \frac{q_2^2\omega_1^2 + q_1^2\omega_2^2}{q_2^4\omega_1^2 + q_1^4\omega_2^2}\,dt \end{aligned}$$

It has been proved[6] that the general three-body problem admits of no algebraical integral other than the ten classic ones and independent of the law of attraction. However in the case of special laws of attraction the problem may possess new integrals.[7] The existence of a new integral is suggested in the case of special laws of attraction by the structure of the expression (8) of the kinetic energy. Indeed the kinetic energy T does not contain the coordinate $\varkappa$ explicitely although it contains the corresponding velocity $\dot{\varkappa}$. Therefore, if the potential V is also independent of $\varkappa$, it is evident that $\varkappa$ is an ignorable (cyclic) coordinate and the integral corresponding to this ignorable coordinate is

$$(17) \qquad (q_1^2 + q_2^2)\,\dot{\varkappa} + 2\,q_1\,q_2\,\omega_3 = const.$$

It can be easily proved that the potential V is independent of $\varkappa$ if (and, disregarding the uninteresting case of elastic forces, only if) the masses are equal and the forces vary as the cube of the distances.[8]

In this case there exists a class of particular solutions of the three-body problem, corresponding to this integral, namely those steady motions in which q_1, q_2, $\dot{\varkappa}$, ω_1, ω_2, ω_3 have constant values. There are ∞^5 of these particular solutions, in which the bodies describe space-curves lying on a hyperboloid of revolution.

Particular cases of our equations may be found in the works of several authors. In order to determine all the solutions in which the ratios of the mutual distances of the bodies remain constant, O. Pylarinos[9] associated a rigid triangle to the variable triangle of the bodies and he in this way succeeded to establish a system of equations from which all the particular solutions in question may be uniformly derived. We shall prove, that our equations of motion contain the equations of Pylarinos, as a particular case.

[6] Gravé, Nouvelles Annales de Math. Ser. 3. Vol. XV, pp. 537—47.

[7] It is known that in the case of forces inversely proportional to the cube of the distances Jacobi's equation may be integrated and (using the notation of this paper) leads to the integral

$$q_1^2 + q_2^2 = C_1(t - t_0)^2 + C_0$$

Using this, all particular solutions may be obtained by quadrature in which the bodies form an isoscele triangle.

[8] The existence of the new particular solution discoverd by D. Sokolov (Dokladi Akademii Nauk CCCR. 1945. Tom. XLVI, pp. 99—102) is also a consequence of the integral (17).

[9] O. Pylarinos, Über die Lagrangeschen Fälle im verallgemeinerten Dreikörperproblem, Math. Zeitschr. 1941. Bd. 47. S. 357—72. See also Th. Banachiewitz, Cas particulier du problem des n corps, Comptes Rendus, 1906. Tome 142, pp. 510—12.

Investigating the motion of three bodies along a straight line Euler and Jacobi[10] introduced such coordinates which ensure the symmetry of the reduced equations. These equations can be immediately obtained from our general equations by putting in them $q_2 = 0$, $\omega_1 = \omega_2 = \omega_3 = 0$.

The expression (8) of the kinetic energy just as much as the equations (11, 12) derived from it are unsymmetrical with respect to the coordinates q_1, q_2 as well as to the velocities $\omega_1, \omega_2, \omega_3$ (though they are symmetrically related to the bodies). This circumstance is explained by the fact that the system of three bodies forms a degenerating figure in space, which exhibits a strict analogy to a rigid plane figur.

This is verified by the integral

$$\omega_1^2 + \omega_2^2 = f\,(q_1, q_2, \varkappa, \dot q_1, \dot q_2, \dot\varkappa) \tag{18}$$

which may be obtained by linear combination of the integrals (14, 15) and which is a strict analogon of the integral

$$\omega_1^2 + \omega_2^2 = const. \tag{19}$$

well known from the theory of motion of a plane lamina.

However the method developed in the present memoir can be applied without any principial difficulty to the problem of four bodies and the equations of motion obtained in this case are entirely symmetrical.

I.

THE EQUATIONS OF MOTION OF A SYSTEM OF MATERIAL PARTICLES REFERRED TO ITS OWN PRINCIPAL AXES OF INERTIA

Suppose that the position of a point is specified by its coordinates x, y, z at any instant t with reference to the instantaneous position of a right-handed system of axes $oxyz$ which are themselves in motion. Let $\omega_1, \omega_2, \omega_3$ denote the components of the angular velocity of the system $oxyz$, resolved along the instantaneous position of the axes. Then the components of the velocity of the point are

$$\begin{aligned} v_x &= \dot x - y\,\omega_3 + z\,\omega_2 \\ v_y &= \dot y - z\,\omega_1 + x\,\omega_3 \\ v_z &= \dot z - x\,\omega_2 + y\,\omega_1 \end{aligned} \tag{20}$$

[10] L. Jacobi, Gesammelte Werke, Bd. III.

If the origin of the axes coincides with the centre of gravity of the system of particles m_k $(x_k\ y_k\ z_k)$ $(k = 1, 2, \dots n)$ and if the coordinates of the centre of gravity with reference to a system of axes fixed in space are X, Y, Z, the kinetic energy of the system of particles is (using the abbreviation $M = \sum m_k$)

$$(21) \qquad 2T = M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \sum m_k (v_{kx}^2 + v_{ky}^2 + v_{kz}^2)$$

If furthermore the moving axes coincide with the principal axes of inertia of the system of particles, then the coordinates x_k, y_k, z_k verify the equations

$$(22) \qquad \begin{aligned} &\sum m_k x_k = \sum m_k y_k = \sum m_k z_k = \sum m_k y_k z_k = \sum m_k z_k x_k = \\ &= \sum m_k x_k y_k = 0 \end{aligned}$$

Applying these equations and the formulae (20) of the components of velocity, we obtain the kinetic energy of the system of particles in the form

$$(1) \qquad \begin{aligned} 2T &= M \sum \dot{X}^2 + \sum \left\{ \omega_1^2 \sum m_k (y_k^2 + z_k^2) \right\} + \\ &+ 2 \sum \left\{ \omega_1 \sum m_k (y_k \dot{z}_k - z_k \dot{y}_k) \right\} + \sum m_k (\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2) \end{aligned}$$

If the system of particles possesses with reference to its own principal axes of inertia s degrees of freedom, x_k, y_k, z_k can be expressed in terms of the generalised coordinates $q_1, q_2, \dots q_s$ and the expression (1) of the kinetic energy becomes

$$(2) \qquad \begin{aligned} 2T &= M \sum \dot{X}^2 + \sum \omega_j^2 J_j(q) + 2 \sum \sum \omega_j \dot{q}_\sigma F_{j\sigma}(q) + \\ &+ \sum \sum \dot{q}_\sigma \dot{q}_\varrho G_{\sigma\varrho}(q) \end{aligned}$$

As far as only conservative, internal forces are acting on the system, those potential can be expressed as a function of the generalised coordinates

$$(3) \qquad V(q_1, q_2, \dots q_s)$$

In order to establish the equations of motion suppose that the quasi-velocities ω_1, ω_2, ω_3 are expressed in terms of the Eulerian angles ϑ, φ, ψ by means of the well-known relations

$$(23)\qquad \begin{aligned} \omega_1 &= \dot{\vartheta} \sin\psi - \dot{\varphi} \sin\vartheta \cos\psi \\ \omega_2 &= \dot{\vartheta} \cos\psi + \dot{\varphi} \sin\vartheta \sin\psi \\ \omega_3 &= \dot{\psi} \qquad\quad + \dot{\varphi} \cos\vartheta \end{aligned}$$

Then the kinetic energy involves only true coordinates and their derivatives and the equations of motion can be immediately written. Lagrange's equation referring to the coordinate ψ is

$$(24)\qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} = - \frac{\partial V}{\partial \psi}$$

but

$$\frac{\partial T}{\partial \dot{\psi}} = \frac{\partial T}{\partial \omega_3} \cdot \frac{\partial \omega_3}{\partial \dot{\psi}} = \frac{\partial T}{\partial \omega_3}$$

$$\frac{\partial T}{\partial \psi} = \frac{\partial T}{\partial \omega_1} \cdot \frac{\partial \omega_1}{\partial \psi} + \frac{\partial T}{\partial \omega_2} \cdot \frac{\partial \omega_2}{\partial \psi} = \frac{\partial T}{\partial \omega_1}\omega_2 - \frac{\partial T}{\partial \omega_2}\omega_1$$

and

$$\frac{\partial V}{\partial \psi} = 0,$$

consequently the equation (24) is identical to the equation

$$\frac{d}{dt}\frac{\partial T}{\partial \omega_3} = \frac{\partial T}{\partial \omega_1}\omega_2 - \frac{\partial T}{\partial \omega_2}\omega_1$$

The two other equations follow by symmetry.

The coordinates q_σ being independent of the Eulerian angles and their derivatives, Lagrange's equations referring to these coordinates retain their original form

$$(5)\qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} = \frac{\partial V}{\partial q_\sigma}$$

Lastly the motion of the centre of gravity is determined by the well-known equations

$$(10)\qquad \ddot{X} = \ddot{Y} = \ddot{Z} = 0$$

Thus we have the result that the system of equations of the system of particles referred to its own principal axes of inertia contains 3 Eulerian equations of the first order and s Lagrangean equations of the second order.

The system of the $2s+3$-th order possesses two obvious integrals

$$T + V = C_1 \qquad \textit{(integral of energy)} \tag{25}$$

$$\left(\frac{\partial T}{\partial \omega_1}\right)^2 + \left(\frac{\partial T}{\partial \omega_2}\right)^2 + \left(\frac{\partial T}{\partial \omega_3}\right)^2 = C_2 \qquad \textit{(integral of the) angular momentum)} \tag{26}$$

If, after having integrated the system, the components of the angular velocity are known as functions of the time, the direction-cosines α_j, β_j, γ_j, $(j = 1, 2, 3)$ of the principal axes of inertia can be obtained by integration of the kinematical equations

$$\dot{\alpha}_j = \alpha_k \omega_l - \alpha_l \omega_k ; \quad \dot{\beta}_j = \beta_k \omega_l - \beta_l \omega_k ; \quad \dot{\gamma}_j = \gamma_k \omega_l - \gamma_l \omega_k \tag{13}$$

But in order to determine the position in space it seems to be more advantageous to calculate the Eulerian angles ϑ, φ, ψ by a method which is well known from the theory of the top. If the invariable vector of length $\sqrt{C_2}$ representing the angular momentum of the system coincides with the axis of Z fixed in space, its components along the moving axes verify the equations

$$\frac{\partial T}{\partial \omega_j} = \sqrt{C_2}\, \gamma_j \qquad (j = 1, 2, 3)$$

and applying the well-known relations between the direction-cosines and the Eulerian angles we get immediately

$$\cos \vartheta = \gamma_3 = \frac{1}{\sqrt{C_2}} \frac{\partial T}{\partial \omega_3} ; \quad tg\, \psi = - \frac{\gamma_2}{\gamma_1} = - \frac{\partial T}{\partial \omega_2} : \frac{\partial T}{\partial \omega_1} \tag{27}$$

Lastly the third Eulerian angle can be obtained by use of the equations (23) in the form

$$\varphi = \varphi_0 + \sqrt{C_2} \int \frac{\omega_1 \dfrac{\partial T}{\partial \omega_1} + \omega_2 \dfrac{\partial T}{\partial \omega_2}}{\left(\dfrac{\partial T}{\partial \omega_1}\right)^2 + \left(\dfrac{\partial T}{\partial \omega_2}\right)^2} dt \tag{28}$$

II.

ON THE COORDINATES WHICH DETERMINE THE POSITION OF THREE PARTICLES RELATIV TO THEIR OWN PRINCIPAL AXES OF INERTIA

If the principal axes of inertia of a system of three material particles $m_j\ (x_j,\ y_j,\ o)\ (j = 1, 2, 3)$ coincide with the axes $oxyz$, their coordinates verify the equations

$$\sum m_j x_j = \sum m_j y_j = \sum m_j x_j y_j = 0 \tag{29}$$

Introduce now the radii of gyration q_1, q_2 defined by the equations

$$M = \sum m_j;\ M q_1^2 = \sum m_j x_j^2;\ M q_2^2 = \sum m_j y_j^2; \qquad sgn\ q_1 q_2 = sgn \left| 1\ x_j\ y_j \right| \tag{30}$$

and consider the system of points $\zeta_j = \xi_j + i\,\eta_j = \varrho_j \cdot e^{i\delta_j}$ which is obtained from the original one by the affin transformation

$$\xi_j = \frac{x_j}{q_1} \quad ; \quad \eta_j = \frac{y_j}{q_2} \tag{31}$$

Then in consequence of the equations (29) and of the equation $\sum m_j \xi_j^2 = \sum m_j \eta_j^2$ the complex coordinates ζ_j verify the equations

$$\begin{aligned} m_1 \zeta_1 + m_2 \zeta_2 + m_3 \zeta_3 &= 0 \\ m_1 \zeta_1^2 + m_2 \zeta_2^2 + m_3 \zeta_3^2 &= 0 \end{aligned} \tag{32}$$

Eliminating for instance ζ_1 we get

$$\frac{m_1 + m_2}{m_3} \zeta_2^2 + 2 \zeta_2 \zeta_3 + \frac{m_1 + m_3}{m_2} \zeta_3^2 =$$

$$= \frac{m_1}{m_2 + m_3} \left\{ (\zeta_2 - \zeta_3)^2 + \frac{M}{m_1 m_2 m_3} (m_2 \zeta_2 + m_3 \zeta_3)^2 \right\} = 0$$

consequently

$$\frac{\zeta_2 - \zeta_3}{m_2 \zeta_2 + m_3 \zeta_3} = \frac{\zeta_2 - \zeta_3}{- m_1 \zeta_1} = \pm i \sqrt{\frac{M}{m_1 m_2 m_3}} ;$$

$$\frac{\zeta_2}{\zeta_3} = \frac{\varrho_2}{\varrho_3} e^{i(\delta_2 - \delta_3)} = \frac{1 \pm i \sqrt{\dfrac{M m_3}{m_1 m_2}}}{1 \mp i \sqrt{\dfrac{M m_2}{m_1 m_3}}} \tag{33}$$

the values of ζ_3/ζ_1 and ζ_1/ζ_2 can be obtained from here by cyclic permutation.

From the equation (33) we get

$$\varrho_2^2 : \varrho_3^2 = \left(1 + \frac{M m_3}{m_1 m_2}\right) : \left(1 + \frac{M m_2}{m_1 m_3}\right)$$

and in consequence of

$$\sum m_j (\xi_j^2 + \eta_j^2) = \sum m_j \varrho_j^2 = 2M \tag{34}$$

$$\varrho_j = \sqrt{\frac{m_k + m_l}{m_j}} \tag{35}$$

The arguments δ_j are determined, apart from an additive constant, by the equations (33). Hence if δ_1, δ_2, δ_3 denote constants such that they verify the equations

$$tg\,(\delta_k - \delta_l) = \pm \sqrt{\frac{M m_j}{m_k m_l}}, \tag{36}$$

the general solution of the equations (32) is given by

$$\zeta_j = \xi_j + i \eta_j = \sqrt{\frac{m_k + m_l}{m_j}}\, e^{i(\delta_j + \varkappa)} \tag{37}$$

where $\varkappa$ denotes an arbitrary real constant.

Consequently the relativ coordinates of the original system of points m_j $(x_j\, y_j\, o)$ are expressed in terms of the generalised coordinates q_1, q_2, $\varkappa$ by means of the equations[11]

[11] These relative coordinates have been stated without proof by R. Radau (Sur une transformation des coordonnées de trois corps. dans laquelle figurent les moments d'inertie, Comptes Rendus, 1869. pp. 1465—68.

$$x_j = \sqrt{\frac{m_k + m_l}{m_j}}\, q_1 \cos(\varkappa + \delta_j)$$

(38)
$$y_j = \sqrt{\frac{m_k + m_l}{m_j}}\, q_2 \sin(\varkappa + \delta_j)$$

$$z_j = 0$$

The kinetic energy relativ to the principal axes of inertia is (with regard to $z_j = 0$)

$$2\, T_{rel} = \sum m_j (\dot{x}_j^2 + \dot{y}_j^2)$$

But the equations (38) give

$$\dot{x}_j = x_j \frac{\dot{q}_1}{q_1} - y_j \frac{q_1}{q_2}\dot{\varkappa} \;;\; \dot{y}_j = y_j \frac{\dot{q}_2}{q_2} + x_j \frac{q_2}{q_1}\dot{\varkappa}$$

consequently with regards to (29, 30)

(39)
$$2\, T_{rel} = M \left\{ \dot{q}_1^2 + \dot{q}_2^2 + (q_1^2 + q_2^2)\, \dot{\varkappa}^2 \right\}$$

One of the components of the relativ angular momentum is

(40)
$$\sum m_j (x_j \dot{y}_j - y_j \dot{x}_j) = 2\, M q_1 q_2 \dot{\varkappa}\,,$$

the others vanish.

We have from the equations (31) (33)

$$\frac{x_j - x_k}{q_1} = \xi_j - \xi_k = \sqrt{\frac{M\, m_l}{m_j\, m_k}}\, \eta_l = \sqrt{\frac{M\, m_l}{m_j\, m_k}}\, \frac{y_l}{q_2}\;;$$

$$\frac{y_j - y_k}{q_2} = \sqrt{\frac{M\, m_l}{m_j\, m_k}}\, \frac{x_l}{q_1}\,,$$

thus the potential of central forces proportional to the ν—1-th power of the mutual distances has the expression

(41)
$$V(q_1, q_2, \varkappa) = \\ = \sum m_j m_k \left\{ \left(\frac{M}{m_j} + \frac{M}{m_k} \right) \left[q_1^2 \sin^2(\varkappa + \delta_j) + q_2^2 \cos^2(\varkappa + \delta_j) \right] \right\}^{\frac{\nu}{2}}$$

III.

THE SYSTEM OF DIFFERENTIAL EQUATIONS OF THE PROBLEM OF THREE BODIES AND SOME PARTICULAR CASES OF IT

If the system of particles treated in I. consists of three masses, one of the coordinates, e.g. z_j vanishes and the coordinates x_j, y_j can be expressed in terms of the generalised coordinates q_1, q_2 and $\varkappa$ by means of the equations (38).

Substituting the values of the coordinates x_j, y_j and their derivativs (and using (39, 40)) in the general expression of the kinetic energy, we get the expression of the kinetic energy of three particles in terms of the variables $\dot{X}\ \dot{Y}\ \dot{Z}\ q_1\ q_2\ \dot{q}_1\ \dot{q}_2\ \dot{\varkappa}\ \omega_1\ \omega_2\ \omega_3$ in the following form

$$(8)\qquad \begin{aligned} 2T = M \Big\{ & \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 + q_2^2 \omega_1^2 + q_1^2 \omega_2^2 + (q_1^2 + q_2^2)(\omega_3^2 + \dot{\varkappa}^2) + \\ & + 4 q_1 q_2 \omega_3 \dot{\varkappa} + \dot{q}_1^2 + \dot{q}_2^2 \Big\} \end{aligned}$$

Substituting the expression (8) of T and the expression (9) of V in the equations (4, 5, 25, 26) of I., we get immediately the differential equations (10, 11, 12) of the problem of three bodies as well as their integrals (14, 15).

After having integrated the system, the components ω_1, ω_2, ω_3 are known as functions of the time and the Eulerian angles determining the position in space will be obtained from the general equations (27, 28) in the form (16).

The triangle formed by the bodies remaines similar to itself if and only if $\varkappa$ and the ratio of q_1 and q_2 remains constant throughout the motion. Hence if we denote the initial values of these quantities by $\varkappa_0$, q_{10}, q_{20}, the homothetic motions will be represented by particular integrals of the form

$$(43)\qquad \varkappa = \varkappa_0\,;\quad q_1(t) = q_{10} \cdot \lambda(t)\,;\quad q_2(t) = q_{20} \cdot \lambda(t)$$

Substituting these expressions in the differential equations (11, 12) we get the following equations which must be verified by the variables λ, ω_1, ω_2, ω_3.

$$
(44) \quad
\begin{aligned}
&\dot{\omega}_1 \qquad\qquad + \omega_2 \omega_3 + 2 \frac{\dot{\lambda}}{\lambda} \omega_1 = 0 \\
&\dot{\omega}_2 \qquad\qquad - \omega_3 \omega_1 + 2 \frac{\dot{\lambda}}{\lambda} \omega_2 = 0 \\
&\dot{\omega}_3 + \frac{q_{10}^2 - q_{20}^2}{q_{10}^2 + q_{20}^2} \omega_1 \omega_2 + 2 \frac{\dot{\lambda}}{\lambda} \omega_3 = 0
\end{aligned}
$$

$$
(45) \quad
\begin{aligned}
2\, q_{10}\, q_{20} \frac{d}{dt} (\lambda^2 \omega_3) &= - \frac{1}{M} \frac{\partial V}{\partial \varkappa} \\
\ddot{\lambda} - \lambda (\omega_2^2 + \omega_3^2) &= - \frac{1}{M} \frac{\partial V}{\partial q_1} \\
\ddot{\lambda} - \lambda (\omega_1^2 + \omega_3^2) &= - \frac{1}{M} \frac{\partial V}{\partial q_2}
\end{aligned}
$$

The first three of these equations agree with those which have been established by O. Pylarinos in order to discusse the homothetic motions.

If the motion of the three bodies takes place along a straight line, one of the radii of gyration, for instance q_2 vanishes and each component of the angular velocity may be supposed to be equal to 0. Consequently the motion of three bodies along a straight line will be determined by the following system of two differential equations

$$
(46) \quad
\begin{aligned}
\ddot{q}_1 - q_1 \dot{\varkappa}^2 &= - \frac{1}{M} \frac{\partial V}{\partial q_1} \\
\frac{d}{dt} (q_1^2 \dot{\varkappa}) &= - \frac{1}{M} \frac{\partial V}{\partial \varkappa}
\end{aligned}
$$

These equations are·apart from some differences in the notation identical to those which have been used by Euler and Jacobi on the occasion of the study of the problem under consideration.

IV.

PARTICULAR SOLUTIONS IN THE CASE OF EQUAL MASSES AND OF FORCES PROPORTIONAL TO THE CUBE OF THE MUTUAL DISTANCES

If we suppose the units conveniently chosen, the potential of three equal masses in the case of forces proportional to the cube of the distances is given by

$$V = m^2 \sum r_{kl}^4 = \frac{1}{4}\left(3\, q_1^4 + 2\, q_1^2\, q_2^2 + 3\, q_2^4\right) \tag{47}$$

Thus $\dfrac{\partial V}{\partial \varkappa} = 0$ and the equations (12) take the form

$$\frac{d}{dt}\left[\left(q_1^2 + q_2^2\right) \dot{\varkappa} + 2\, q_1\, q_2\, \omega_3\right] = 0$$

$$\ddot{q}_1 - q_1 \left(\omega_2^2 + \omega_3^2 + \dot{\varkappa}^2\right) - 2\, q_2\, \omega_3\, \dot{\varkappa} = -\, q_1 \left(3\, q_1^2 + q_2^2\right) \tag{48}$$

$$\ddot{q}_2 - q_2 \left(\omega_1^2 + \omega_3^2 + \dot{\varkappa}^2\right) - 2\, q_1\, \omega_3\, \dot{\varkappa} = -\, q_2 \left(3 q_2^2 + q_1^2\right)$$

Attempting to satisfy the equations of motions by constant values of q_1, q_2, $\dot{\varkappa}$, ω_1, ω_2, ω_3 we get immediately from the third equation of (11) $\omega_1 \omega_2 = 0$. In order to have a definit case, suppose $\omega_2 = 0$. Then the first and second of the equations (11) reduce to

$$q_2^2\, \omega_1 = const. \;,\quad \omega_1\, q_1 \left(q_1\, \omega_3 + 2\, q_2\, \dot{\varkappa}\right) = 0 \tag{49}$$

The assumption $\omega_1 = 0$ resp. $q_1 = 0$ leads to a motion in a plane resp. to a collinear solution, therefore suppose

$$q_1\, \omega_3 + 2\, q_2\, \dot{\varkappa} = 0 \tag{50}$$

This equation together with the equations (48) suffice to express $\dot{\varkappa}$, ω_1, ω_3 in terms of q_1 and q_2, while the equations (48^1, 49^1) are satisfied by any constant values of the variables.

We get in this way

$$\omega_1 = \frac{\sqrt{(q_1^2 - q_2^2)(10\, q_1^2 + 4\, q_2^2)}}{q_1}$$

$$(51) \qquad \omega_2 = 0 \qquad\qquad \varkappa = \varkappa_0 - \sqrt{3\, q_1^2 + q_2^2}\,(t - t_0)$$

$$\omega_3 = \frac{2\, q_2}{q_1} \sqrt{3\, q_1^2 + q_2^2}$$

and the constant values $q_1 \geqq q_2$ can be chosen arbitrarily.

For the Eulerian angles we obtain from (16) the values

$$(52) \qquad \cos \vartheta = \frac{q_2}{q_1} \sqrt{\frac{6\, q_1^2 + 2\, q_2^2}{5\, q_1^2 + 3\, q_2^2}}\;;\quad \psi = 0\;;\quad \varphi - \varphi_0 = \sqrt{10\, q_1^2 + 6\, q_2^2}\,(t - t_0)$$

Finally the position of the bodies in space will be determined by the equations

$$X_j = q_2 \left[\sqrt{\frac{6\, q_1^2 + 2\, q_2^2}{5\, q_1^2 + 3\, q_2^2}} \cos \varphi \cos \left(\varkappa + \frac{2 j \pi}{3} \right) - \sin \varphi \sin \left(\varkappa + \frac{2 j \pi}{3} \right) \right]$$

$$(53) \qquad Y_j = q_2 \left[\sqrt{\frac{6\, q_1^2 + 2\, q_2^2}{5\, q_1^2 + 3\, q_2^2}} \sin \varphi \cos \left(\varkappa + \frac{2 j \pi}{3} \right) + \cos \varphi \sin \left(\varkappa + \frac{2 j \pi}{3} \right) \right]$$

$$Z_j = - \sqrt{\frac{(q_1^2 - q_2^2)(6\, q_1^2 + 2\, q_2^2)}{5\, q_1^2 + 3\, q_2^2}} \cos \left(\varkappa + \frac{2 j \pi}{3} \right)$$

where φ and $\varkappa$ are to be replaced by their values given in (51, 52).

The coordinates of each of the bodies satisfy the equations

$$(54) \qquad \frac{X^2 + Y^2}{q_2^2} - \frac{Z^2}{6\, q_1^2 + 2\, q_2^2} - 1 = 0$$

consequently each body describes a curve on the hyperboloid of revolution (54) which projects into a hypocycloid on the $X\ Y$ plane.

As the values of $q_1, q_2, \varkappa_0, \varphi_0, t_0$ are arbitrary, we have found a family of ∞^5 particular solutions and all these solutions correspond to permanent rotations of the system of principal axes round a line fixed in space.

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46.745. — Egyetemi Nyomda, Budapest. (F.: Tirai Richárd.)

CONTENTS