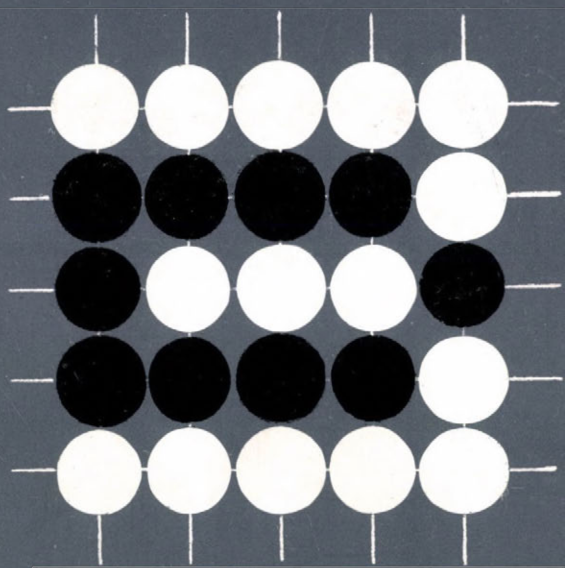
közlemények 26/1982

MTA Számítástechnikai és Automatizálási Kutató Intézet Budapest



MAGYAR TUDOMÁNYOS AKADÉMIA
SZÁMITÁSTECHNIKAI ÉS AUTOMATIZÁLÁSI KUTATÓ INTÉZETE

KÖZLEMÉNYEK

1982 ÁPRILIS

Felelős kiadó:

DR VÁMOS TIBOR

Szerkesztette:

BALLA KATALIN

ISBN 963 311 140 4
ISSN 0133-7459

CONTENT

Page

С О Д Е Р Ж А Н И Е

This special issue of *MTA SZTAKI Közlemények* contains the lectures presented by Hungarian participants of Equadiff–5 (the Conference on Differential Equations) held in Bratislava, Czechoslovakia, August, 1981. It attempts to give a review of problems in differential equations studied in Hungary.

The authors are indebted to the *Scientific Secretary of SZTAKI* for the opportunity to gather this survey of investigations. Special appreciation is due to Ms. Katalin Gabnai and Ms. Lidia Kertész for their excellent typing of the manuscript.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

ON SINGULAR BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ODE-S

KATALIN BALLA

Computer and Automation Institute
Hungarian Academy of Sciences

In the earlier paper [1] we discussed the numerical solution of a class of first order systems of nonlinear ordinary differential equations with a regular singularity. Since that a great number of problems arising in theoretical physics has been solved by that method. However, the most of problems led to systems of second order and for them the method was essentially simplified, i.e. the reduction to first order systems was avoided. Here we intend to describe this, namely we give an algorithm for transferring a certain manifold of solutions out of the point of singularity.

Problem and assumptions

For the vector-function $X = (x_1, \ldots, x_n)^T$ consider the system of n nonlinear second order ordinary differential equations of the form

$$t^2 X'' = A(t)X + tB(t)X' + f(t,X), \qquad 0 < t \leq t_0 \quad (1)$$

where the matrix-functions $A(t)$ and $B(t)$ are continuous on $[0, t_0]$. Let $\lim_{t \to 0} B(t) = B_0 = \mu E$ with μ real and assume that the eigenvalues of the matrix $\lim_{t \to 0} A(t) = A_0$ are either

i) disposed exterior of the parabola

$\Pi(\mu) = \{z: (1+\mu)^2 \, Re z + (Im\ z)^2 = 0\}$, i.e. $(1+\mu)^2 Re\ \lambda_{A_0} + (Im\ \lambda_{A_0})^2 > 0$

if $\mu \neq -1$ or

ii) do not lie on the nonpositive real axis $(-\infty, 0]$
i.e. either $Im\ \lambda_{A_0} \neq 0$ or $Re\ \lambda_{A_0} > 0$ if $\mu = -1$.

For the sake of simplicity, here we do not describe more complicated connections between the matrices A_0, B_0. In general case, the matrix equation $X^2 + (E+B_0)X + A_0 = 0$ should be considered.

Let the vector-function $f(t,X)$ be continuous with respect to (t,X) in $\Omega_{x;t}(c;t_0) = \{(t,X): 0 \leq t \leq t_0, |x_i| \leq c\}$ and

for any $t \in [0,t_0]$, let it be holomorphic in X, $|x_i| \leq c$.
Assume that $f(t,X) = \sum_{|p|\geq 2} f_p(t)X^p$, $X^p := x_1^{p_1} \cdot \ldots \cdot x_n^{p_n}$.

Find the manifold of solutions to (1) that satisfies the condition

$$|X(t)| = o(1), \qquad t \to 0. \tag{C1}$$

Theorem

For sufficiently small solutions to (1), C1 is equivalent to a condition

$$t\,X'(t) = \eta(t, X(t)), \tag{C2}$$

which holds for sufficiently small t. The vector-function $\eta(t,X)$ is continuous in (t,X), $t \leq t_1 \leq t_0$, $|x_i| \leq \omega \leq c$, holomorphic in X for any $t \in [0, t_1]$; $\eta(t,X) = \sum_{|p|\geq 1} \eta_p(t) X^p$

and it is the unique solution of the Cauchy problem

$$t \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} \eta = (E + B(t))\eta + A(t)X + f(t,X) \tag{2}$$

$\lim_{t \to 0} \eta(t,X) = \alpha_0 X + \gamma(X)$ uniformly in X for small $|X|$.

Here $\alpha_0 = \frac{E + B_0}{2} + \sqrt{A_0 + \frac{(E + B_0)^2}{4}}$, $\gamma(X)$ is the unique

solution of the Ljapunov problem:

$$\frac{\partial \gamma}{\partial X} (\alpha_0 X + \gamma) = (E + B_0 - \alpha_0)\gamma + f(0,X),$$

$$\gamma(X) = \sum_{|p|\geq 2} \gamma_p X^p.$$

Remark 1.

Let $A(t) \sim \sum_{i=0}^{\infty} A_i t^i$, $B(t) \sim \sum_{i=0}^{\infty} B_i t^i$,

$$f_p(t) \sim \sum_{i=0}^{\infty} f_p^{(i)} t^i, \quad t \to 0.$$

Then, $\eta_p(t) \sim \sum_{k=0}^{\infty} \eta_p^{(k)} t^k, \quad t \to 0.$ (3)

The coefficients $\eta_p^{(k)}$ can be obtained by formal substitution of the series $\sum_{|p|\geq 1} \sum_{k\geq o} \eta_p^{(k)} t^k X^p$ and the above asymptotic expansions into (2).

Remark 2.

The asymptotic expansion (3) is replaced by a convergent one, provided that $A(t)$, $B(t)$ and $f_p(t)$ have convergent power series expansions.

To prove the statement, the idea of [2] may be followed with slight changes. Let $tX' = \alpha(t)X$ be the equivalent condition (C2) for the truncated linear problem $t^2X'' = A(t)X + tB(t)X'$ (see [3]). For the function $\Theta(t,X) = \eta(t,X) - \alpha(t)X - \gamma(X)$ the Cauchy problem has the required unique solution [2]. Considering the vector $Z(t) = tX'(t) - \eta(t,X(t))$ and the equation that defines it by (1) and (2), one can show the equivalence of (C1) and (C2).

Remark 3.

The formal substitutions mentioned above yield chained systems of nonsingular linear equations for $\eta_p^{(k)}$ which can be solved recurrently. Therefore, the vector-function $\eta(t,X)$ can be determined with any required accuracy. When handling the numerical examples, we use this remark.

Numerical example

A typical example comes from theory of fields [4]. Find the solution $w(x)$, $f(x)$ of the boundary value problem

$$x^2 w'' = 2w - 3w^2 + w^3 - x^2 f^2 + x^2 w f^2$$

$$x^2 f'' = (2 - \tfrac{1}{2}\beta x^2) f - 2x f' - 4 f w + 2 f w^2 + \tfrac{1}{2}\beta x^2 f^3 \qquad (1')$$

$$w \to 0, \quad f \to 0 \quad \text{when} \quad x \to 0, \qquad (C1')$$

$$w \to 1, \quad f \to 1 \quad \text{when} \quad x \to \infty. \qquad (D1)$$

In this case $B_o \neq \mu E$ but the assumptions i) and ii) can be naturally changed and the theorem remains valid.

The problem (1'), (C1') gives the following equivalent boundary condition:

$$x_o w' = \rho_1(x_o, w, f), \qquad x_o f' = \rho_2(x_o, w, f) \qquad (C2')$$

where x_o is small. Making use of the above consideration, we find $\rho_1(x,w,f) = a_1(x)w + a_2(x)f + b_1(x)w^2 + b_2(x)wf + b_3(x)f^2 + \ldots$,

and $\rho_2(x,w,f) = A_1(x)w + A_2(x)f + B_1(x)w^2 + B_2(x)wf + B_3(x)f^2 + \ldots$.

The linear terms yield a Ricatti-type system:

$$x a_1' + a_1^2 + a_2 A_1 - a_1 - 2 = 0$$

$$x a_2' + a_1 a_2 + a_2 A_2 - a_2 = 0$$

$$x A_1' + A_1 a_1 + A_2 A_1 + A_1 = 0$$

$$x A_2' + A_1 a_2 + A_2^2 + A_2 + (2 - \tfrac{1}{2}\beta x^2) = 0$$

Substituting the corresponding expansions, one gets

$$a_1 \equiv 2, \quad a_2 \equiv 0, \quad A_1 \equiv 0, \quad A_2 = 1 - \frac{1}{10}\beta x^2 - \frac{1}{700}\beta^2 x^4$$

(up to fourth order terms). For second order terms we give only the result of substitution: $b_1 \equiv -\frac{3}{5}$, $b_2 \equiv 0$,

$b_3 = -\frac{1}{5}x^2 - \frac{1}{175}x^4 + \dots$, $B_1 \equiv 0$,

$B_2 = -\frac{4}{5} - \frac{4\beta}{175}x^2 - \frac{2\beta^2}{2625}x^4 + \dots$, $B_3 \equiv 0$.

The higher order terms can be omitted.

In order to handle (D1), we notice that the substitution $v = 1 - w$, $g = 1 - f$ yields a similar to (C1') problem at the infinity. However the singularity at the infinity is of second kind. Here we have no room to describe how to replace (D1'), we refer only to [5]. The result is a boundary value

$$v' = \delta_1(x_\infty, v, g), \qquad g' = \delta_2(x_\infty, v, g), \qquad x_\infty \text{ is large.}$$

The approximative values of δ_1 and δ_2 accurate up to second order terms are as follows:

$$\delta_1 = \left(-1 + \frac{1}{2x^2} + \dots\right)v + \left[\frac{2}{2+\sqrt{\beta}} - \frac{2}{(2+\sqrt{\beta})^2 x^2} + \dots\right] vg + \dots$$

$$\delta_2 = \left(-\sqrt{\beta} - \frac{1}{x}\right)g + \left[\frac{2}{(2+\sqrt{\beta})x^2} + \dots\right]v^2 + \left[\frac{\sqrt{\beta}}{2} - \frac{1}{6x} + \frac{1}{9\sqrt{\beta}\,x^2} + \dots\right]g^2 + \dots$$

If we return to original functions f and w, we get a boundary value

$$\begin{aligned} f' &= \tilde{\delta}_1(x_\infty, f, w) \\ w' &= \tilde{\delta}_2(x_\infty, f, w) \end{aligned} \qquad \text{(D2')}$$

The "usual" boundary value problem for the equation (1') with boundary values (C2') and (D2') on the interval $[x_0, x_\infty]$ may be solved by the well-known Newton technique.

REFERENCES

[1] K. Balla, On singular boundary value problems for systems of nonlinear ODE-s, Colloquia Mathematica Societatás János Bolyai. 30. Qualitative Theory of Differential Equations. Szeged (Hungary), 1977. 63-70.

[2] K. Balla, On solution of singular boundary value problems for nonlinear systems of ODE-s (in Russian). Zhurnal Vichislitelnoj Matematiki i Matematicheskoj Fiziki 20(1980), 909-922.

[3] K. Balla, On the replacement of the condition of boundedness for certain systems of linear ODE-s with regular singularity. Colloquia Mathematica Societatis János Bolyai. 22. Numerical Methods, Keszthely (Hungary), 1977. 121-130.

[4] G.'t Hooft, Magnetic monopoles in unified gauge theories. Nuclear Physics B79(1974) 276-284.

[5] N.B. Konjukhova, On segregation of stable manifolds for some nonlinear systems of ODE-s with a singularity (in Russian). Zhurnal Vichislitelnoj Matematiki i Matematicheskoj Fiziki. 13(1973), 609-626.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

OPTIMIZATION OF PARAMETERS IN LINEAR SYSTEMS OF CERTAIN TYPE

ILONA BÁN

Computer and Automation Institute
Hungarian Academy of Sciences

INTRODUCTION

One of the classical problems in the examination of stabil systems characterisable with linear differential equation is to find the solution with the maximal degree of stability of the differential equation where the parameters $K_1, \ldots, K_n$ (on which the coefficients depend) are chosen from a given domain.

II. FORMULATION OF THE PROBLEM

Consider the linear differential equation

$$x^{(n)} + g_1(K_1, \ldots, K_n)x^{(n-1)} + \ldots + g_n(K_1, \ldots, K_n)x = 0. \qquad (1)$$

of which the characteristic equation is

$$\lambda^n + g_1(K_1, \ldots, K_n)\lambda^{n-1} + \ldots + g_n(K_1, \ldots, K_n) = 0 \qquad (2)$$

Obviously g_i $i=1, \ldots, n$ are positive in the domain of stability, since this is a necessary condition for the stability.

We suppose that there is a one-to-one correspondence between g_i $i=1,\dots,n$ and K_i $i=1,\dots,n$. We solve then the problem

$$\min_{(g_1,\dots,g_n)\in\Phi} \quad \max_{i=1,\dots n} \quad Re\,\lambda_i \tag{3}$$

where $\lambda_i = u_i + iv_i$, $i=1,\dots,n$. are the roots of the characteristic equation, a u_i are the real, and v_i the imaginary parts, $\Phi = \Gamma \cap U$ where U is the domain of stability, and Γ is the map of the domain Ω into the domain of coefficients, where we look for the solution, and $\Omega \subset R^n$ is the given domain in the space of parameters. We suppose that $\Gamma = \Gamma_1 \times \dots \times \Gamma_n$ and $\Gamma_i = [\underline{\Gamma}_i, \overline{\Gamma}_i]$, $\underline{\Gamma}_i \leq \overline{\Gamma}_i$. Consider a sequence of polinomials of increasing degree where each polinomial contains the roots of the previous one. Let K_i^j be the i-th coefficient of the polinomial of degree j. $(i=1,\dots,j)$. Using the relations between the coefficeints and the roots we get that the coefficients of the polinomials of the sequence can be expressed in terms of the coefficeints of the previous polinomials and the new root in the following way:

$$K_1^1 = -\lambda_1$$

$$K_1^2 = K_1^1 - \lambda_2$$

$$K_2^2 = -K_1^1 \lambda_2$$

$$\vdots$$

$$K_1^j = K_1^{j-1} - \lambda_j$$

$$K_2^j = K_2^{j-1} - K_1^{j-1} \lambda_j$$

$$\vdots$$

$$\vdots$$

$$K_i^j = K_i^{j-1} - K_{i-1}^{j-1}, \lambda_j$$

$$\vdots$$

$$K_j^j = -K_{j-1}^{-1} \lambda_j$$

Lemma 1.

In the stability domain the coefficients K_i^j of a polinomial with real coefficients of degree j are polinomials with positive real coefficients of v_k, $k=1,\dots,n$. (the coefficients are functions of u_k, $k=1,\dots,n$ of course)

Proof: We use mathematical induction with respect to the number of roots. For $j=1,2$ the statement is trivial. Let us suppose that lemma 1 is proved for $j=n-1$ i.e.

$$K_i^{n-1} = f_i^{n-1} + p_i^{n-1}(v_k^2) \qquad k=1,\dots,n \tag{4}$$

where f_i^{n-1} does not depend on v_k and $p_i^{n-1}(v_k^2)$ is always positive.

Let us first suppose that n is odd. In that case λ_n is real, as K_i^n are real. We have $u_n < 0$, since we are in the domain of stability as we supposed it. In this case our statement trivially follows from the statement for $n-1$, as

$$K_i^n = K_i^{n-1} - K_{i-1}^{n-1} \lambda_n$$

and so

$$K_i^n = f_i^{n-1} + p_i^{n-1}(v_k^2) - (f_{i-1}^{n-1} + p_{i-1}^{n-1}(v_k^2))u_n = f_i^n - f_{i-1}^{n-1}u_n + p_i^{n-1}(v_k^2) + u_n\, p_{i-1}^{n-1}(v_k^2) =$$

$$= f_i^n + p_i^n$$

If n is even, then we rewrite the equality (4) so, that we use the coefficients with upper index $n-2$. As

$$K_i^{n-1} = K_i^{n-2} - K_{i-1}^{n-2}\,\lambda_{n-1}$$

$$K_{i-1}^{n-1} = K_{i-1}^{n-2} - K_{i-2}^{n-2}\,\lambda_{n-1}$$

so we have

$$K_i^n = -K_{i-1}^{n-2}\lambda_{n-1} + K_i^{n-2} - (K_{i-1}^{n-2} - K_{i-2}^{n-2}\,\lambda_{n-1})\lambda_n = K_i^{n-2} - K_{i-1}^{n-2}\,\lambda_n - K_{i-1}^{n-2}\,\lambda_{n-1} +$$

$$+ K_{i-1}^{n-2}\,\lambda_{n-1}\lambda_i$$

As n is even, and the polinomial has real coefficients, it follows that

$$\lambda_{n-1} = u_{n-1} + i\,v_{n-1}, \quad \lambda_n = u_n + iv_n$$

and $\quad u_n = u_{n-1}, \quad v_n = -v_{n-1}$

In this way

$$K_i^n = K_i^{n-2} - K_{i-1}^{n-2}\,2u_n + K_{i-1}^{n-2}(u_n^2 + v_n^2)$$

from which our statement follows, as $u_n < 0$.

Lemma 2.

The domain of stability is not bounded in the space of coefficients.

Proof: Let us define $g_i = (-1)^j \sum_{i_s \subset N} \prod_{i_\ell \in I_s} \lambda_{i_\ell} \qquad j=1,\dots,n$

and I_s runs through the subsets with j elements of $N = \{1,\dots,n\}$,

Let us suppose that $g = \{g_1,\dots,g_n\}$ is in the domain of stability, that is $Re\ \lambda_i < 0 \quad i=1,\dots,n$. Then for $c > 0$ we have $Re\ c\ \lambda_i < 0$. So the corresponding point in the space of coefficients

$$g^* = (g_1^*,\dots,g_n^*) \qquad \text{where} \quad g_j^* = c^j\ g_j \quad j=1,\dots,n \ .$$

is in the domain of stability too, so it is unbounded.

Definition: The upper boundary of the n dimensional rectangle Γ is the set

$$\{x \mid x \in \Gamma,\ \exists\, i \quad x_i = \overline{\Gamma}_i\}$$

The lower boundary is defined in an analogous way.

Lemma 3.

The optimum is on the common part of the upper boundary of Γ and the domain of stability.

Proof: As the domain of stability is open, the optimum, if it exists, is in the interior of this domain. Let us suppose, that the optimum is not on the upper boundary of Γ. It is obvious from the relations between the coefficients and the roots that in the expression $g_i = f_i + p_i(v_k^2)$

$$f_i = (-1)^i \sum_{i_s \subset N} \prod_{i_\ell \in I_s} u_{i_\ell} \qquad i=1,\dots,n$$

are positive too. It follows that when increasing u_i the upper boundary may be reached. As the coefficients are continuous monotonically increasing functions of $u_i \quad i=1,\dots,n$ if we increase that u_i which is according to its module, the least we can reach the upper boundary, which contradicts to the hypothesis.

Lemma 4.

If $\underline{\Gamma}_i = 0$ $i=1,\ldots,n$ then in the optimal case $v_i = 0$, $i = 1,\ldots,n$.

Proof: Suppose that in the optimal point $v_i \neq 0$. Then the coefficients, corresponding to the roots u_i, $v_i = 0$ are in the interiour of the domain Φ. As the real parts did not change, these new roots are optimal too, which contradicts lemma 3.

Remark: In the case $\underline{\Gamma}_i \neq 0$ the point $v_i = 0$ may fall in the exterior of the domain.

Lemma 5.

In the case $\underline{\Gamma}_i = 0$, $i = 1,\ldots,n$ there exists a global optimum. In this case the real parts are equal.
Proof: Let us suppose that they are not equal. Let u_1 be the greatest and u_2 the least with respect to their module of real parts. As

$$K_j = (-1)^j \Sigma\, u_{i_1} \ldots u_{i_j} = u_1(-1)^j \Sigma\, u_{i_1} \ldots u_{i_{j-1}} +$$

$$+ u_2(-1)^j \sum_{i_\ell \neq 1,2} u_{i_1} \ldots u_{i_{j-1}} + u_1 u_2 (-1)^j \sum_{i_\ell \neq 1,2} u_{i_1}, \ldots, u_{i_{j-2}} =$$

$$= -(u_1+u_2)(-1)^{j-1} \Sigma\, u_{i_1} \ldots u_{i_{j-1}} + u_1 u_2 (-1)^{j-2} \Sigma\, u_{i_1} \ldots u_{i_{j-2}}$$

it is obvious that $u_1^* = u_2^* = -\sqrt{u_1 u_2}$ gives a value g which is better, then the hypothetically "optimal" one. We have proved the following:

Theorem

Let us consider a control system, described by differential equation with real coefficients

$$x^{(n)} + g_1 x^{(n-1)} + \ldots + g_n x = 0$$

Then the solution of the equation with the greatest degree of stability according to the conditions $g_i < \Gamma_i$ is the following

$$x(t) = (c_1 + c_2 t + \ldots + c_n t^{n-1}) e^{ut}$$

where u is the least real root of which the corresponding coefficients are in Γ and c_i are constants, depending on the boundary conditions.

Algorithm

As in the case of optimal solution for some value

$$g_i = (-1)^i \binom{n}{i} u^i$$

the algorithm is working in the following way: for each j we determine

$$u_j = \sqrt[j]{\frac{g_j}{\binom{n}{j}(-1)^j}}$$

Then we determine the quantities g_j using the relations between the roots and the coefficients, and we look whether these values are in Φ . If not, then we reject them. If yes then we take this solution in the set of the possible solutions. If we have already examined it for $j = 1,\ldots,n$ the quantities g_j, then then we choose the best from possible solutions (max.n)

Example:

Let us consider the control system,

The characteristic equation of this system is

$$x^3 + K_1 x^2 + (1+K_2)x + K_3 = 0$$

In some cases the following table shows the solution:

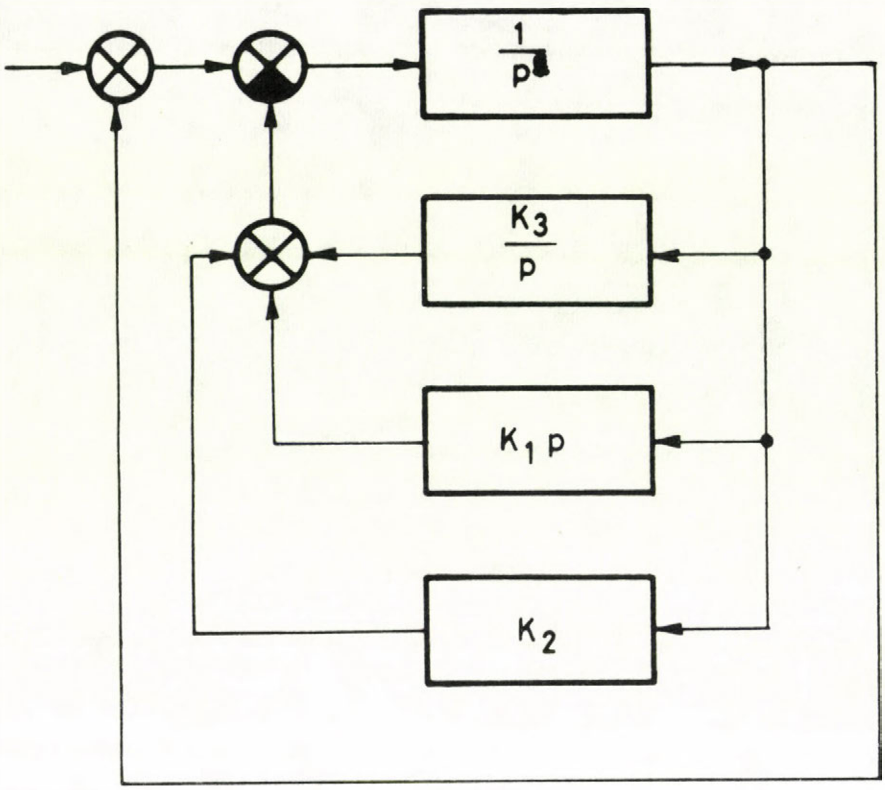
$\bar{\Gamma}_1$	$\bar{\Gamma}_2$	$\bar{\Gamma}_3$	root	opt.	coefficients		
3	3	1	-1	1	3	3	1
6	11	6	-1.8571	1	5.4514	9.9058	6
3	6	3	-1	1	3	3	1

REFERENCES

1. Morris Marden: The geometry of zeros of a polinomial in a complex variable 1949.

2. Zajac E.E.: Bounds of the Decay rate of damped linear systems "Quart. of Appl. Math." 1963. v.20 No.4.

3. Jindrich Spal: Algebraic Approach of the Root-Loci Method Kybernetika Čislo 5. Ročnik 6/1970


1
p
K3
p
K1 p
K2

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

QUALITATIVE PROPERTIES OF THE HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

ÁRPÁD ELBERT

Mathematical Institute of
the Hungarian Academy of Sciences

The subject of our lecture is the second order differential equation of the form

(1) $$y'' + p\,f(y,y') = 0$$

for $y = y(t)$ where the function $p = p(t)$ is piecewise continuous on the interval $I \subset (-\infty, \infty)$ and $f(u,v)$ satisfies the following relations

(2) $F(u) = u\,f(u,1) > 0$ and continuous for $u \in R \setminus \{0\}$,

(3) $f(cu,\ cv) = c\,f(u,v)$, $\quad c \in R, (u,v) \in R \times R$.

The existence and the uniqueness of the solutions of the initial value problem for (1) is also assumed. The property (3) ensures that if $y = y(t)$ is a solution of (1) and c is some constant then the function $c\,y(t)$ is also a solution of (1). Thus the differential equation (1) resembles the linear differential equation indicating the name of half-linear one (see [1]). The special case $f(u,v) = u^{\overset{*}{n}}\,|v|^{1-n}$ with $n > 0$ was studied in [2] and [3]. Here we make an attempt to deal the oscillatory case (in view of the inequality (2)) as generally as possible.

Let $S = S(\varphi)$ be the solution of the following differential equation

(4) $$\ddot{S} + f(S,\dot{S}) = 0$$

with the initial conditions $S(0) = 0$, $\dot{S}(0) = 1$. If

$$\hat{\pi} = \int_{-\infty}^{\infty} \frac{dt}{1+F(t)} < \infty$$

then $S(\varphi)$ is an oscillatory function with the period $2\hat{\pi}$, and there are quantities π_+, π_-, K depending only of f with the properties

$$\pi_+ + \pi_- = \hat{\pi}$$

(5) $$\dot{S}(\pi_+) = \dot{S}(-\pi_-) = 0, \quad \dot{S}(\varphi)>0 \quad \text{for} \quad -\pi_- <\varphi<\pi_+,$$

$$S(\varphi) = -K\, S(\varphi-\hat{\pi}), \qquad K > 0.$$

If $\varphi \neq \varphi_i = \pi_+ + i\hat{\pi}$, $i=0,\pm 1,\ldots,$ then we define the function $T = T(\varphi)$ by

$$T(\varphi) = \frac{S(\varphi)}{\dot{S}(\varphi)} .$$

Owing to (5) the function T is periodic with the period $\hat{\pi}$ and it has discontinuities at $\varphi = \varphi_i$ and it varies from $-\infty$ to ∞ as φ varies from φ_i to φ_{i+1}. Moreover T fulfils the differential equation

(6) $$\dot{T} = 1 + F(T),$$

where the function F is nonnegative due to (2).

Let us consider the curve i on the plane (x_1,x_2) given in parametric from by $(\dot{S}(\varphi),\ S(\varphi))$ for $-\infty < \varphi < \infty$. Since K is in general not equal to 1, the curve i is winding round the origin $(0,0)$. Let $\varphi = \varphi_o$ be fixed. Then the points of the halfray starting from the origin and crossing the point $P_o(\dot{S}(\varphi_o),S(\varphi_o))$ are $x_1 = \rho\ \dot{S}(\varphi_o)$, $x_2 = \rho\ S(\varphi_o)$ with $\rho > 0$. Hence for the point P_o of i we have $\rho = 1$. Thus we have a generalized polar transformation on the plane (x_1,x_2)

$$x_1 = \rho\ S(\varphi)$$

(7)

$$x_2 = \rho\ S(\varphi)$$

which depends on the equation (1) in the above sense.

Let us consider a solution $y = y(t)$ of (1). This solution defines a curve c on the plane (x_1,x_2) given by $(y'(t),y(t))$ for $t \in I$. We fix the values φ_o and ρ_o by the initial conditions $y'(t^o) = \rho_o\ \dot{S}(\varphi_o)$, $y(t^o) = \rho_o\ S(\varphi_o)$. Then by (7) we have two welldefined functions $\varphi(t)$ and $\rho(t)$ as polar coordinates of the curve c

$$y'(t) = \rho(t)\ \hat{S}(\varphi(t))$$

(8)

$$y(t) = \rho(t)\ S(\varphi(t)\ .$$

which is a generalization of the Prüfer transformation.
The functions $\varphi(t)$ and $\rho(t)$ satisfy first order differential equation system

$$\varphi' = \frac{1}{1+F(T)} + p(t)\ \frac{F(T)}{1+F(T)}$$

(9)

$$\rho' = \rho\ \frac{f(T,1)}{1+F(T)} \quad (1-p).$$

It is interesting to remark that the system (9) is of triangular form because the right hand side of the first equation depends only on the unknown φ but not on ρ, while in the second equation both the unknowns occur. This property of the system (9) enables us to generalize several properties known for linear second order differential equations (see e.g. in [4]).

We start with a simple lemma.

LEMMA. Let the functions $p_1(t)$, $p_2(t)$ be piecewise continuous and let the relation

(10) $$p_1(t) \leq p_2(t) \quad \text{for} \quad t \in I$$

be valid. If $\varphi_1(t), \varphi_2(t)$ are the solutions of the differential equations

(11) $$\varphi_j' = \frac{1}{1+F(T(\varphi_j))} + \varphi_j(t) \frac{F(T(\varphi_j))}{1+F(T(\varphi_j))}$$

with the initial condition $\varphi_j(t^o) = \varphi_{jo}$ $(j=1,2)$. Then the inequality

$$\varphi_1(t) \leqq \varphi_2(t) \quad \text{for} \quad [t_o, \infty] \cap I$$

holds. In the case $\varphi_{1o} = \varphi_{2o}$ we have also

$$\varphi_1(t) \geqq \varphi_2(t) \quad \text{for} \quad (-\infty, t_o] \cap I.$$

Let $y = y(t)$ be a solution of (1), and $t_1, t_2, \ldots$ be consecutive zeros of $y(t) = 0$. Then by (8) we have $\hat{S}(\varphi(t_i)) = 0$, hence $\varphi(t_i) \equiv 0(\mathrm{mod}\,\hat{\pi})$. By (9) $\varphi'(t_i) = 1 > 0$ thus $\varphi(t)$ is strictly increasing in the neighbourhood of $t = t_i$. Therefore $\varphi(t_{i+1}) = \varphi(t_i) + \hat{\pi}$ and $\varphi(t_i) < \varphi(t) < \varphi(t_{i+1})$

for $t_i < t < t_{i+1}$. Similarly we have for the zeros $t'_1, t'_2, \ldots$ of $y'(t) = 0$ that $\varphi(t'_i) \equiv \pi_+ (mod\ \hat{\pi})$. If $p(t)$ is positive on the interval (t_i, t_{i+1}) then by (9) the function φ is strictly increasing hence there is exactly one value $t'_i \in (t_i, t_{i+1})$ where $y(t)$ has local extremal value and

$\varphi(t'_i) = \varphi(t_i) + \pi_+$.

This observation and the Lemma has many applications.

Theorem 1. Let t_1 and t_2 be two consecutive zeros of a nontrivial solution $y(t)$ of (1). Then every solution $\bar{y}(t)$ different from $cy(t)$, where c is any constant, vanishes once and only once in (t_1, t_2).

Usually we recall this theorem saying that the zeros of linearly independent solutions of (1) are interlacing. In the next theorem we formulate a stronger version of this interlacing property.

Theorem 2. Let $y(t)$ be a nontrivial solution of (1) and $J = (\tau_1, \tau_2)$ be an interval such that $J \subset I$ and $y(t)\, y'(t) \neq 0$ on J, $y(\tau_j)\, y'(\tau_j) = 0$ for $j = 1,2$. Let $\bar{y}(t)$ be other linearly independent solution of (1) satisfying either $\bar{y}(\tau_1) y(\tau_1) > 0$, $\bar{y}'(\tau_1) y'(\tau_2) > 0$ if $yy' < 0$ or $\bar{y}(\tau_1) y(\tau_2) > 0$, $\bar{y}'(\tau_1)\, y'(\tau_1) > 0$ if $yy' > 0$. Then there is a value $t^* \in J$ with $\bar{y}(t^*) = 0$ in the first case or $\bar{y}'(t^*) = 0$ in the second case.

Concerning the strong interlacing property we conjecture that if the solutions of differential equation

$$y'' + p(t)\, g(y, y') = 0$$

has the property given in Theorem 2 then the equation is half-linear, i.e. the function $g(y, y')$ satisfies the homogenity relation of type (3). In fact we could prove this conjecture only in the case when $g(y,y') = h(y)\, k(y')$. We have found that $g = y^{\overset{*}{n}}\, |y'|^{1-n}$, which was treated already in [3]. On the other hand similar statement is not true in general if we assume only the common interlacing property formulated by Theorem 1.

Now we want to compare the solutions of the differential equations

$$(12_j) \qquad y_j'' + p_j\, f(y_j, y_j') = 0, \quad j=1,2.$$

We say that the equation (12_2) is a Sturmian majorant to (12_1) if the inequality (10) holds.

Theorem 3. Let $t^o \in I$ and $j \in \{1,2\}$, and let the solution $y_j = y_j(t)$ of (12_j) satisfy the initial conditions $y_j(t^o) = y_j^o \neq 0$, $y_j'(t^o) = y_j'^o$ with

$$\frac{y_1'^o}{y_1^o} = \frac{y_2'^o}{y_2^o} .$$

If t_o, t_1 are two consecutive zeros of $y_2(t) = 0$ such that $t_o < t^o < t_1$, and if (12_2) is a Sturmian majorant to (12_1), then

$$\frac{y_1(t)}{y_1^o} \geqq \frac{y_2(t)}{y_2^o} \quad \text{for} \quad t_o \leq t \leq t_1 .$$

Theorem 4. Let the conditions be the same as in Theorem 3 with the only exception that we assume now

$$\frac{y_1'^o}{y_1^o} \geqq \frac{y_2'^o}{y_2^o} .$$

Denote $t_{j_1}, t_{j_2}, \ldots$ the zeros of $y_j(t) = 0$ on $(t^o, \infty) \cap I$. Then $t_{1k} \geqq t_{2k}$ for $k = 1, 2, \ldots$. If the coefficient $p_1(t) > 0$ then similar statement is true for the zeros of $y_j'(t) = 0$, too.

REFERENCES

[1] I. Bihari, On the second order half-linear differential equation, Studia Sci. Math. Hung. 3 (1968)411-437.

[2] J.D. Mirzov, On some analogs of Sturm's and Kneser's theorems for nonlinear systems, J. Math. Anal. Appl. 53(1976)418-426.

[3] Á. Elbert, A half-linear second order differential equation, Colloquia Math. Soc. J. Bolyai, 30. Qualitative Theory of Ordinary Differential Equations, Szeged (Hungary) 1979, 153-180.

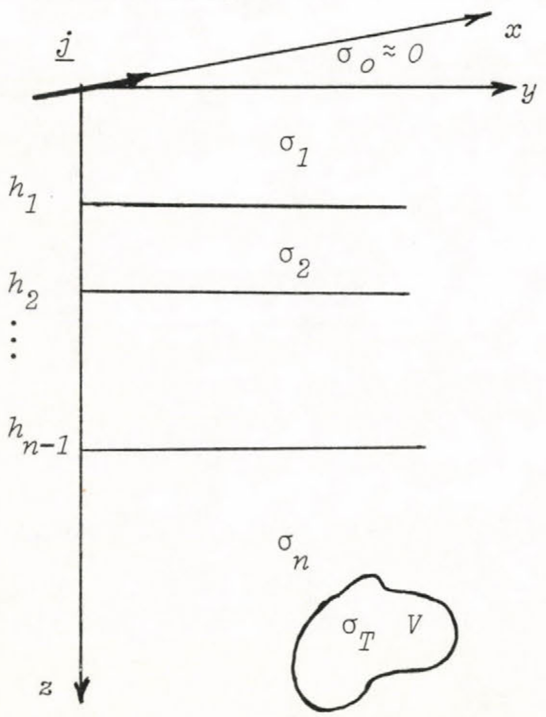
[4] Ph. Hartman, Ordinary differential equations, John-Willey et Sons, Inc., N.Y.-London-Sidney, 1964.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

МЕТОД РАСЧЕТА АНОМАЛЬНОГО ЭЛЕКТРОМАГНИТНОГО ПОЛЯ ОТ ЛОКАЛЬНОЙ НЕОДНОРОДНОСТИ

Р.Х. Фарзан

Рассматривается математическая модель распространения электромагнитного поля, возбуждаемого горизонтальным электрическим диполем, лежащим на поверхности Земли, в слоистой среде с неоднородностью конечных размеров V . Если поле изменяется по времени по закону $exp(-i\omega t)$, где $\omega = const$, с помощью преобразования Фурье система уравнений Максвелла сводится к системе векторных линейных дифференциальных эллептических уравнений для векторов электрического и магнитного полей в трехмерной области с разрывными коэффициентами и с правой частью, представленной δ -функцией Дирака



$$\text{/1/} \quad \begin{aligned} rot\ \underline{E} &= i\omega\mu_o \underline{M}, \\ rot\ \underline{M} &= \sigma \underline{E} + \underline{j}, \end{aligned}$$

где μ_o - магнитная проницаемость, σ - кусочно-постоянная проводимость тока, плотность распределения источников поля $\underline{j} = \{I\ell\ \delta(\underline{R}), 0, 0\}$. На границах сред касательные составляющие векторов поля непрерывны.

Введем обозначения:

$$\text{/2/} \quad \sigma_c(z) = \begin{cases} \sigma_o, & z < 0, \\ \sigma_1 & 0 < z < h_1, \\ \sigma_2 & h_1 < z < h_2, \\ \vdots & \\ \sigma_n & h_{n-1} < z; \end{cases} \qquad \sigma(M) = \begin{cases} \sigma_c(z), & M \notin V, \\ \sigma_T\ , & M \in V. \end{cases}$$

Обозначим через $\underline{E}^{(n)}, \underline{H}^{(n)}$ /вектора "нормального" поля/ решение системы уравнений

/3/
$$rot\ \underline{E}^{(n)} = i\omega\mu_0 \underline{H}^{(n)},$$
$$rot\ \underline{H}^{(n)} = \sigma_c(z)\underline{E}^{(n)} + \underline{j}.$$

Тогда остающаяся "аномальная" часть полного поля $\underline{E}^{(a)} = \underline{E} - \underline{E}^{(n)}$ и $\underline{H}^{(a)} = \underline{H} - \underline{H}^{(n)}$ есть решение системы

/4/
$$rot\ \underline{E}^{(a)} = i\omega\mu_0 \underline{H}^{(n)},$$
$$rot\ \underline{H}^{(n)} = \sigma_c(z)\,\underline{E}^{(a)} + (\sigma(M) - \sigma_c(z))\underline{E}.$$

Система /3/ есть система уравнений для слоистой среды без аномалий. Ее решение считаем известным [1, 2] . Из /4/ видно, что эту систему тоже можно рассматривать как систему уравнений для слоистой среды с плотностью распределения "аномальных" электрических источников

/5/
$$\underline{j}^{(a)} = (\sigma(M) - \sigma_c(z))\underline{E}\ .$$

Видно, что $\underline{j}^{a}$ для $M \notin V$. Следовательно, для решения этой системы нужно знать полное электрическое поле в точках тела V.

Пусть $\underline{j}^{(x)}$ - источник электрического поля единичной мощности - диполь, параллельный оси Ox и расположенный в точке M_o ($M_o \in V$):
$$\underline{j}^{x} = \{\delta(\underline{R}_{MM_o}, 0, 0\}^T, \quad \underline{R}_{MM_o} = \overrightarrow{M_oM}.$$

Пусть $\underline{E}^{(x)}$ - поле, возбуждаемое этим источником в точке M:
$$\underline{E}^{(x)}(M, M_o) = \{E_x^{(x)}, E_y^{(x)}, E_z^{(x)}\}^T.$$

Заметим, что это - решение системы уравнений для слоистой среды с прогруженным источником, и эту проблему также считаем решенной [3] .

Аналогично получаются поля $\underline{E}^{(y)}$ и $\underline{E}^{(z)}$. Компоненты этих векторов составляют фундаментальную матрицу электрического поля

$$/6/ \qquad \hat{E}(M,M_o) = \left\{ \begin{matrix} E_x^{(x)} & E_x^{(y)} & E_x^{(z)} \\ E_y^{(x)} & E_y^{(y)} & E_y^{(z)} \\ E_z^{(x)} & E_z^{(y)} & E_z^{(z)} \end{matrix} \right\} .$$

Электрическое аномальное поле, возбуждаемое источником $\underline{j}^{(a)}$ выразится тогда интегралом

$$/7/ \qquad \underline{E}^{(a)}(M) = \iiint\limits_{M_o \in V} \hat{E}(M,M_o)\underline{j}^{(a)}(M_o)\, d\,V_{M_o} .$$

Выражение /7/, после подстановки /5/ можно переписать в виде:

$$/8/ \qquad \underline{E}(M) + (\sigma_c - \sigma_T) \iiint\limits_{M_o \in V} \hat{E}(M,M_o)\underline{E}(M_o) d\,V_{M_o} = \underline{E}^{(n)}(M_o)$$

где первая часть известна. Если $M \in V$, то мы получаем интегральное векторное уравнение, причем под интегралом - сингулярная при $M_o = M$ функция. Для численного решения этого уравнения область V разбиваем на элементы V_i, $i=1,\dots,N$, внутри которых электрическое поле $\underline{E}$ можно считать приближенно постоянным и равным его значению в некоторой точке $M_i \in V_i$. Тогда интегральное уравнение /8/ в точках M_i перепишется в виде:

$$\underline{E}(M_i) + (\sigma_c - \sigma_T) \sum_{j=1}^{N} \iiint\limits_{M_o \in V_j} \hat{E}(M_i,M_o) d\,V_{M_o} \cdot \underline{E}(M_j) = \underline{E}^{(n)}(M_i), \quad i=1,\dots,N;$$

или, вводя обозначение

$$/9/ \qquad \hat{\alpha}_{ij} = (\sigma_c - \sigma_T) \iiint\limits_{M_o \in V_j} \hat{E}(M_i,M_o) d\,V_{M_o}$$

получим

$$/10/ \qquad \underline{E}(M_i) = \sum_{j=1}^{N} \hat{\alpha}_{ij}\, \underline{E}(M_j) = \underline{E}^{(n)}(M_i), \quad i=1,\dots,N.$$

Поскольку мы считаем известной матрицу $\hat{E}$, матрица $\hat{\alpha}_{ij}$ также известна. При известных $\hat{\alpha}_{ij}$ система /10/ представляет собой систему линейных алгебраических уравнений, решив которую можно определить электрическое поле в точках тела V. Затем, по формуле /5/ определив распределение аномальных источников, решаем систему /4/ для аномального поля. Вместе с нормальным полем это дает возможность определить полное электрическое и магнитное поле во всех точках пространства.

Элементы матрицы $\hat{E}$ /6/, как известно /3/ , выражаются через интегралы по бесконечному пределу от выражения, содержащего функцию Бесселя. Таким образом, элементы матрицы $\hat{\alpha}_{ij}$ /9/ есть четверные интегралы. Общее число скалярных уравнений - $3N$, число элементов всех матриц $\hat{\alpha}_{ij}$ - $9N^2$. Таким образом, при численном решении машинное время при увеличении N возрастает очень быстро.

В случае $i = j$ это усложняется тем, что под интегралом появляется неинтегрируемая сингулярность. Для вычисления этого интеграла предлагается следующий прием [4]. Из матрицы $\hat{E}$ выделяется часть $\hat{E}^{(o)}$, соответствующая электрическому полю в однородном пространстве с электрической проводимостью $\sigma = \sigma_n$. Оказывается, что именно эта часть содержит в себе сингулярность полной матрицы $\hat{E}$. Оставшаяся после выделения часть $\hat{E}^{(1)}$ выражает влияние неоднородности пространства и уже не содержит особенностей, и соответствующая ей матрица $\hat{\alpha}_{ii}^{(1)}$ вычисляется так же, как $\hat{\alpha}_{ij}$ в случае $i \neq j$.

Электрическое поле в однородном пространстве выражается в явном виде, и матрицу $\hat{E}^{(o)}$ можно записать

$$/11/ \qquad \hat{E}^{(o)} = \begin{Bmatrix} U + \frac{1}{k_n^2} U''_{xx} & \frac{1}{k_n^2} U''_{xy} & \frac{1}{k_n^2} U''_{xz} \\ \frac{1}{k_n^2} U''_{xy} & U - \frac{1}{k_n^2} U''_{yy} & \frac{1}{k_n^2} U''_{yz} \\ \frac{1}{k_n^2} U''_{xz} & \frac{1}{k_n^2} U''_{yz} & U + \frac{1}{k_n^2} U''_{zz} \end{Bmatrix}$$

$$U = \frac{i\omega\mu_o}{4\pi} \frac{e^{ik_nR}}{R}, \quad k_n^2 = i\omega\mu_o\sigma_n, \quad R = |\overrightarrow{M_oM}|$$

Поскольку, например,

$$\frac{1}{k_n^2} U''_{xx} = \frac{e^{ik_nR}}{4\pi R \cdot \sigma_n} \left\{ \frac{-k_n^2 R^2 - 3ik_nR + 3}{R^2} \cdot \frac{x^2}{R^2} + \frac{ik_nR - 1}{R^2} \right\},$$

то видно, что при $R \to O$ получается неинтегрируемая особенность. Запишем $\hat{\alpha}_{ii}^{(o)}$ в виде

/12/ $$\hat{\alpha}_{ii}^{(o)} = (\sigma_n - \sigma_T)\{\lim_{r\to o} \iiint_{V_r} \hat{E}^{(o)} dV + \lim_{r\to o} \iiint_{V_i - V_r} \hat{E}^{(o)} dV\}$$

где V_r - шар радиуса r с центром в M_i. Можно показать [4], используя теорию потенциала [5], что первый интеграл

/13/ $$\hat{q}_1 = \sigma_n \lim_{r\to o} \iiint_{V_r} \hat{E}^{(o)} dV = -\frac{1}{3}\hat{I}$$

где $\hat{I}$ - единичная матрица. Второй интеграл

/14/ $$\hat{Q}_1 = \sigma_n \lim_{r\to o} \iiint_{V_i - V_r} \hat{E}^{(o)} dV$$

особенностей содержать не будет. В случае тел с центральной симметрией /шар, куб, эллипсоид, прямоугольный параллепипед и т.д./ эта матрица также будет диагональной, а для шара и куба будет иметь вид:

/15/ $$\hat{Q}_1 = Q_1 \cdot \hat{I}.$$

В частности, для шара радиуса R_o получим в явном виде:

$$Q_1 = \frac{2}{3} [(1 - ik_n R_o) \exp(ik_n R_o) - 1].$$

Заметим также, что для симметричных тел матрица $\hat{\alpha}_{ii}^{(1)}$ также является диагональной:

/16/ $$\hat{\alpha}_{ii}^{(1)} = \frac{\sigma_n - \sigma_T}{\sigma_n} \begin{Bmatrix} Q_2 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{Bmatrix}.$$

Были проведены расчеты для двухслойной среды ($n=2$) для шара и для куба одинакового объема и вычислены Q_1, Q_2 при различных ξ :

$$\xi^2 = \frac{8\pi^2}{h^2 \omega \mu_0 \sigma_1}, \qquad h = h_1.$$

Расчеты проводились для следующих значений параметров:

объем тела $h^3 \cdot 0.064$:

глубина залегания центра тела $2h$:

отношение электрических проводимостей: $\frac{\sigma_2}{\sigma_1} = 100$, $\frac{\sigma_T}{\sigma_1} = 10$.

Результаты расчетов приведены на Табл. 1. и 2.

ТАБЛИЦА 1: Значения Q_1 для шара и куба

	шар	куб
5	-0,00039593 + 0,0060568	-0,0003948 + 0,0059552
10	-0,000051498 + 0,0015673	-0,000051607 + 0,0015419
30	-0,0000024637 + 0,00017807	-0,0000022865 + 0,00017526
100	-0,0000006358 + 0,00001615	-0,00000040412 + 0,00001589
300	-0,0000001987 + 0,000001798	-0,00000034172 + 0,000001770
1000	-0,0000006755 + 0,0000001616	-0,00000033236 + 0,0000001591

ТАБЛИЦА 2: Значения Q_2 для шара и куба

	шар	куб
5	-0,025794 + 0.0050821	-0,023618 + 0,0043654
10	-0,0076815 + 0,015885	-0,0073779 + 0,014328
30	0,0095096 + 0,004545	0,0082985 + 0,004145
100	0,010316 + 0,0004383	0,0090272 + 0,0004012
300	0,010323 + 0,0004383	0,0090272 + 0,0004012
1000	0,010323 + 0,000048222	0,0090339 + 0,000043995

Результаты показывают, что во-первых, при выделении $\hat{\alpha}_{ii}^{(o)}$ мы действительно выделяем главную часть, т.к. оставшаяся часть по абсолютной величине намного меньше главной части. Более того, в $\hat{\alpha}_{ii}^{(o)}$ основную роль играет вклад поля вблизи источника ($|q_1| >> |Q_1|$).

Далее, если q_1 определяет само наличие аномального источника данного объема, то Q_1 характеризуется конкретной конфигурацией локального тела, служащего источником. Оказывается, конкретная конфигурация не играет существенной роли. Различие между значениями полных матриц $\hat{\alpha}_{ii}$ для шара и куба очень мало. Поэтому, если иметь в виду конечную цель - решение обратной задачи, то очевидно, что на основании измерений компонентов поля на поверхности невозможно различить эти два тела. Следовательно, возможно выделять тела определенного объема и формы /компактные, вытянутые и т.д./, но не конкретной конфигурации.

Из этого же незначительного различия между шаром и кубом одинакового объема следует, что при численном интегрировании при разбиение тела V на элементы V_i не важно, какую конкретную конфигурацию придать элементам V_i, и можно выбирать ее исходя из соображения упрощения вычисления элементов $\hat{\alpha}_{ii}$ - четверных интегралов.

Изложенные результаты получены совместно с проф. В.И. Дмитриевым /Москва/.

ЛИТЕРАТУРА

1. В.И. Дмитриев: Электромагнитное поле в неоднородных средах. МГУ, Труды Вычислительного центра . Москва, 1969.

2. Э.А. Федорова, Р.Х. Фарзан: Методы и алгоритмы численного исследования электромагнитных полей в слоистых средах. Сб. "Математические модели электромагнитной разведки в геофизике". Изд. Будапештского Университета им. Этвеша Лоранда. Будапешт, 1980.

3. Д. Молнарка, Т.Г. Павлова: Методы расчета электомагнитного поля источника, погруженного в слоистую среду. Там же.

4. В.И. Дмитриев, Р.Х. Фарзан: Методы расчета аномального электромагнитного поля от локальной неоднородности. Там же.

5. А.Н. Тихонов, А.А. Самарский.: Уравнения математической физики. Москва, "Наука", 1972.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények, 26/1982

UNTERSUCHUNG DER SICH BEI FERMENTATIONSPROZESSEN AUSGESTALTENDEN TEMPERATURVERTEILUNG

DR. LÁSZLÓ FENYVESI

TIVADAR KŐFALVI

National Institute of Agricultural Engineering
Gödöllő, Hungary

EINLEITUNG

Mehrere wichtigen agrotechnischen Technologien gründen sich auf den Fermentationsprozess (Silierungs-, Düngerbehandlungsverfahren, Methoden der Lebensmittelindustrie). Die die Fermentation auslösenden Mikroorganismen arbeiten im bestimmten Temperaturbereich. Eben deshalb ist die Charakterisierung des thermishcen Zustandes des Systems von entschneidender Wichtigkeit.

Daher sollte so ein Modell aufgebaut werden, mit dessen Hilfe die im Material von Einheitsmasse während einer Einheitszeit entstehende Wärmemenge bestimmt werden kann. Für diesen Zweck ist eine Temperaturmessung im fermentierenden Material, das in einem zylinderförmigen Behälter gelegt wurde, vorzunehmen.

In Kenntnis der Wärmeentwicklung müssen wir die konkreten Realisierungsfälle modellieren: den zylinderförmigen, koaxialhohlen und kompakten Körper, sowie das in einem prismatischen Behälter gestelltes, fermentierendes Materialaggregat.

Wir haben konkrete Rechnen bei der thermischen Modellierung des Silierung durchgeführt.

BESTIMMUNG DER IM LAUFE DER FERMENTATION ENTSTEHENDEN WÄRMEMENGE

Wir haben das fermentierende Material in einem zylinderförmigen Behälter eingesetzt und sicherten die Anaerobverhältnisse durch entsprechenden Abschluss. Wir messen während der Fermentation die Temperatur - bei mehreren Punkten des Behälters - sowie die seit der Einlagerung vergangene Zeit. Wir nehmen an: das Temperaturfeld ist zylindersymmetrisch, verändert sich nicht in der Richtung der Längsachse des Behälters, sowie die thermischen Charakteristika des Materials sind bekannt und konstant.

Durch Verwendung der Transportgleichungen in Zylinderkoordinaten haben wir für die Temperaturverteilung die nachstehende Differenzialgleichung aufgestellt:

$$\rho c \frac{\partial U}{\partial t} - \lambda \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r}\right) = A(t, r), \qquad (1)$$

hier:

- U = Innentemperatur U= f(r,t) [°K]
- t = die seit Untersuchungsbeginn verlaufene Zeit [h]
- r = im Behälter ausgelegter Radius [m]
- λ = Wärmeleitungskoeffizient des eingelagerten Materials [K J/m h°K]
- c = speziphische Wärme des Materials [K J/kg °K]
- ρ = Dichte des eingelagerten Materials [kg/m^3]

Bei der Aussenwand ist das Newton'sche Abkühlungsgesetz in Kraft:

$$\lambda \left| \frac{\lambda U}{\partial r} \right|_{r=r_1} + \alpha_1 (U - U_f) = 0, \qquad (2)$$

und im Inneraum:

$$\frac{\partial U}{\partial r} = 0. \tag{3}$$

Wir müssen auch die Anfangstemperaturverteilung des eingelagerten Materials angeben:

$$U_o = f(r) = U|_{t=o} . \tag{4}$$

Bei den obigen Zusammenhängen:

U_f = Aussentemperatur (^oK)

α_1 - Flächen-Wärmeleitungskoeffizient bei der Innenwand [K J/m2 h ^oK].

Bei der Lösung ist es zweckmässig die Rechnen mit einem der hohligen Lagerung entsprechenden Modell durchzuführen. Somit ist die geometrische Form der Untersuchungsbehälter (*Abb.1.*)

Nun, ist die Randbedingung des Innenhohlraumes:

$$\lambda \left|\frac{\partial U}{\partial r}\right|_{r=r_o} + \alpha_2(U - U_f) = 0, \qquad (5)$$

hier: α_2: der Flächen-Wärmeleitungskoeffizient bei der Innenwand. Die Lösung der Gleichung (1) haben wir durch Anwendung der Fourier'sche Methode in der nachstehenden Form ausgesucht:

$$U^*(t,r) = \nu^*(t,r) + \omega(t,r), \qquad (6)$$

wie die Summe der Lösung der homogenen (ν^*) und inhomogenen Gleichungen (ω). Unter Beachtung der Aussentemperatur haben wir bei der Lösung die nachstehende Transformation eingeführt:

$$\nu^* = \nu + U_f. \qquad (7)$$

Unter Berücksichtigung von (6) und (7) wird die neue Form von (1)

$$\rho c\frac{\partial \nu}{\partial t} - \lambda \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \nu}{\partial r}\right) + \rho c \frac{dU_f}{dt} = 0. \qquad (8)$$

Hier wird das inhomogene Glied explizite erscheinen. Durch Lösung der homogenen (ν_o) und der inhomogenen Gleichung (ν_1)

$$\nu = \nu_o + \nu_1. \qquad (9)$$

Wird die Separation bei dem homogenen Teil

$$\nu_o = \tau_o(t)Q_o(r), \qquad (10)$$

entsprechend durchgeführt, erreichen wir die nachstehenden Ergebnisse:

$$Q_o^n - B_n[I_o(m_n r) + \frac{\lambda_o m_n I_1(m_n r_1) - \alpha_1 I_o(m_n r_1)}{-\lambda_o m_n Y_1(m_n r_1) + \alpha_1 Y_o(m_n r_1)} Y_o(m_n r),$$

(11)

und $\tau_0 = D_n \, exp(a \, m_n^2 \, t).$ (12)

Die Eigenwerte können von der Gleichung

$$[-S_n I_1(S_n) + \gamma I_o(S_n)][-S_n Y_1(S_n \frac{r_o}{r}) + \alpha_2 \gamma Y_o(S_n \frac{r_o}{r_1})] -$$

$$- [-S_n Y_1(S_n) + \gamma Y_o(S_n)][-S_n I_1(S_n \frac{r_o}{r_1}) + \alpha_2 \gamma I_o(S_n \frac{r_o}{r_1})] = 0,$$

(13)

bestimmt werden, wo wir die Formelzeichen

$$S_n = m_n r_1, \qquad \gamma = \frac{\alpha_1 r_o}{\lambda}$$

eingeführt haben.

Die Werte von S_n haben wir mittels eines Rechners bestimmt, von denen geben wir einige bekannt:

n	S_n	n	S_n
1	3,098453	7	23,20777
2	6,479410	8	26,54226
3	9,839644	9	29,875338
4	13,18906	10	33,20740
5	16,53220	11	36,53860
6	19,87133	12	39,86923

Weiterhin unter Berücksichtigung von

$$\nu_1 = \tau_1(t)\, Q_o(r),$$

sind die Koeffizienten Q_o, und

$$\tau_1 = -\, B_n^{-1}\, exp(a\, m_n^2\, t) \int_{t_o}^{t_1} C_n(t)\, exp(-a\, m_n^2\, t)dt. \qquad (14)$$

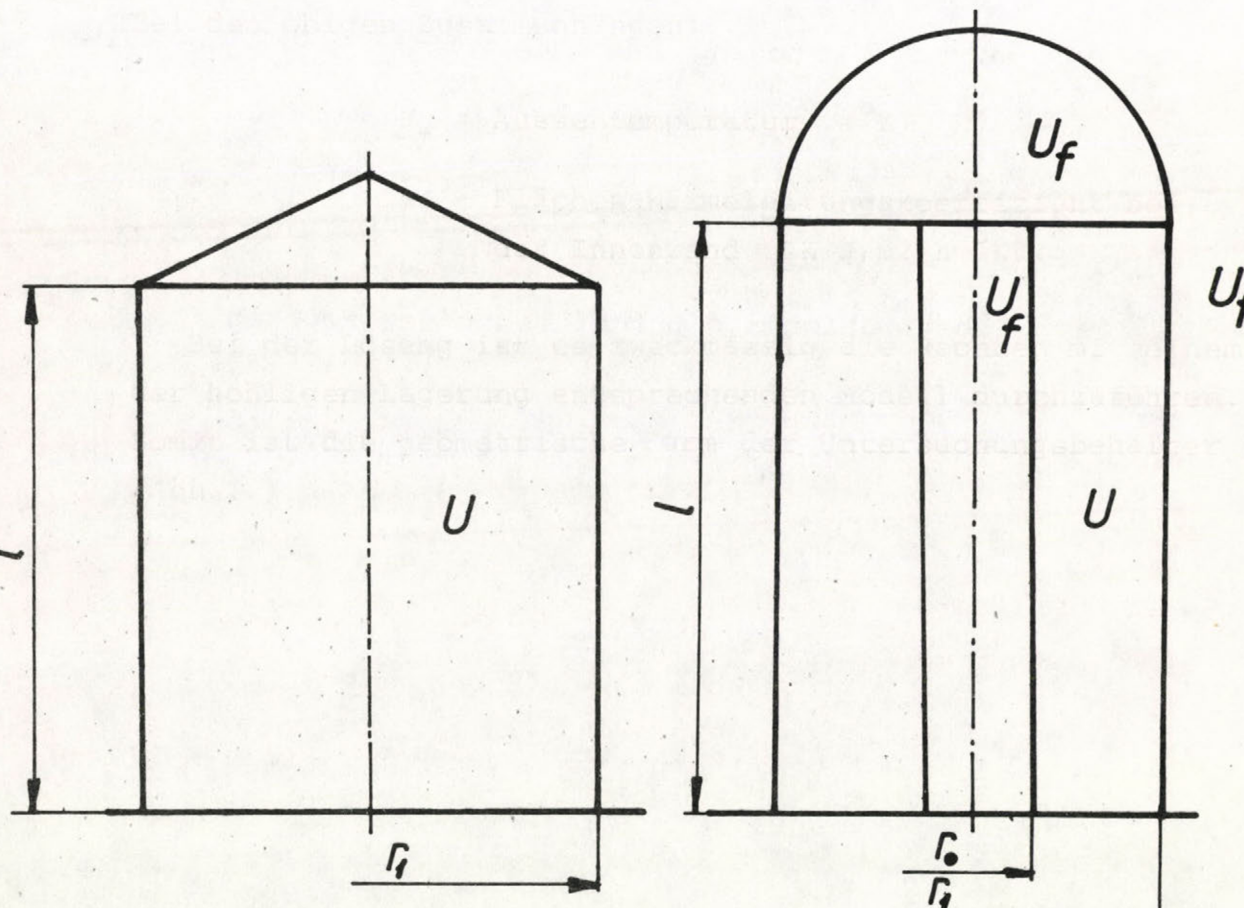
der Partikulärlösung (ν_1).

Nun, haben wir die Funktion der Aussentemperatur den Eigenfunktionen entsprechend in Reihen entwickelt:

$$U_f(t) = \sum_{k=o}^{\infty} C_k(t)\, Q_o^{(k)}(r). \qquad (15)$$

Solcherweise können wir die Lösung des homogenen Teiles der Originalgleichung aufstellen:

$$\nu^* = \sum_{n=o}^{\infty} Q_o[M_n - \int_{t_o}^{t_1} C_n(t)\, exp(-a\, m_n^2\, t)dt]exp(a\, m_n^2\, t) + U_f(t). \qquad (15/a)$$


l
U
r1
Uf
Uf
Uf
l
U
r0
r1

M_n, B_n, D_n und C_n sind solche Koeffiziente, die in Laufe der Reihenentwicklungen bestimmt werden können.

Für eine Lösung der inhomogenen Gleichung werden wir die Wärmequellenfunktion den Eigenfunktionen entsprechend in Reihen entwickeln:

$$A(t,r) = \sum_{n=o}^{\infty} a_n(t)\, Q_o^{(n)}(r), \tag{16}$$

und wir suchen die Lösung in Form

$$\omega(r,t) = R(r)T(t) = \sum_{n=o}^{\infty} T_n(t) Q_o^{(n)}(r). \tag{17}$$

Nach Substitution und Durchführung der Rechnen

$$T_n(t) = exp(\frac{H}{\rho c}\, t) \int_o^t \frac{a_n(t')}{\rho c\, exp(\frac{H}{\rho c}\, t')}\, dt'. \tag{18}$$

Also die Lösung:

$$\omega(r,t) = \sum_{n=o}^{\infty} exp\,(\frac{H_n}{\rho c}\, t) \int_o^{t'} \frac{a_n(t')}{\rho c\, exp\,(\frac{H}{\rho c}\, t')}\, dt' [I_o(m_n r) - P_n Y_o(m_n r)], \tag{19}$$

wo die Werte von H_n und P_n erzeugt werden können. Demnach können wir die Lösung des Problems:

$$U^{*} = \sum_{n=o}^{\infty} [I_o(m_n r)-P_n Y_o(m_n r)]\{[M_n - \int_{t_o}^{t_1} C_n(t)exp(-a\, m_n^2 t)dt]exp(am_n^2 t)+$$

$$+ exp(\frac{H}{\rho c} t) \int_o^{t'} \frac{a_n(t')}{\rho\, c\, exp(\frac{H}{\rho c} t')} dt\} + U_f(t) \quad (20)$$

aufstellen.

In Kenntnis der Lösung (20) können wir auch diesen Fall erzeugen wo keine Wärmequelle existiert (Abkühlungsfall) , $a_n\ (t) = 0$ und Falls der Behälter kompakt ist: $r_o \to 0, \alpha_2 = 0$. In diesem Fall erreichen wir die in der Literatur angegebene Form (CSERMELY, 1974). Wir haben das Problem auch in solchem Fall gelöst, als der Behälter prismatisch war. Die Werte der im Ausdruck (20) digurierenden Funktionen:

I_o - linearische Bessel'sche Funktion Null-Ordnung und

Y_o - quadratische Bessel'sche oder Neumann'sche Funktion O. Ordnung

können mittels eines Rechners erzeugt werden. Des Rechnen und Anpassen den gemessenen Werten kann man mit einem Rechner leicht durchführen.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

MODELLIERUNG DES MECHANISCHEN ZUSTANDES DES MIT LAUFRAD BELASTETEN BODENS

Dr. LÁSZLÓ FENYVESI
TIVADAR KŐFALVI

In den letzten Jahrzehten hat die Landwirtschaft mittels zeitgemässer Machinen und Chemikalien, sowie neuer Sorten, die Erträge in solcher Masse gesteigert, dass die Frage der Bodenstruktur in den Hintergrund gedrängt worden wurde. Heute aber, wenn mit der Verwendung von Hochleistungstraktoren von 5-8 Tonen und Transportmitteln, sowie Erntebergungsmaschinen von 10-20 Tonnen gerechnet werden muss, ist ausserordentlich wichtig sich mit dem auf den Boden ausgeübten Effekt des Gummireifens zu beschäftigen.

Die bisherigen Rechnen und Messungen sind grösstenteils von empirischen Charakters und haben im allgemeinen die Bestimmung je einer Charakteristik angestrebt. Solche Rechnenmethodik ist in den Arbeiten von *Jánosi-Hanamato*, *Komándi*, *Söhne*, usw. angegeben.

Der Zweck warum wir uns mit diesem Problem beschäftigen besteht darin, dass wir die theoretischen und praktischen Ergebnisse der Rheologie bei der Beschreibung des mechanischen Verhaltens des Boden-Rad-Systems verwenden. Im ersten Schritt nehmen wir für die Lösüng der Aufgabe - in Interesse der einfacheren Manipulation - geometrische und mechanische Näherungsverhältnisse: der Gummireifen drückt den Boden auf einer Ellipsenfläche und mit einem den Ellipsoid-Ordinaten entsprechenden Flächendruck; der Boden kann als ein idealer rheologischer Körper behandelt werden.

Bei der Beschreibung des rheologischen Verhaltens spielt die Aufstellung der Materialgleichung eine grosse Rolle. In der Zustandsgleichung der Materialien suchen wir den Zusammenhang zwischen Spannungs- und Deformationstensoren, sowie deren Derivierten:

$$f(\underline{\underline{T}}, \dot{\underline{\underline{T}}}, \ddot{\underline{\underline{T}}}, \ldots E, \dot{E}, \ddot{E},)\ldots = \underline{\underline{O}}.$$

Für die Ingenieurpraxis sind die ersten Derivierten noch von einer grossen Bedeutung, jedoch die Rolle der anderen hochwertigen Derivierten ist so klein, dass diese vernachlässigt werden können, somit können wir die Zustandsgleichung als linear betrachten. Die allgemeine Form der in dieser Weise erreichten Materialglechung:

$$\underline{\underline{C}}_o + C_1\underline{\underline{T}} + C_2\dot{\underline{\underline{T}}} + C_3\underline{\underline{E}} + C_4\dot{\underline{\underline{E}}} = 0.$$

Bei einem bedeutenden Teil der Ingenieurarbeit bleiben wir innerhalb der Plastizitätsgrenze, also vernachlässigen wir den Plastizitätstensor $\underline{\underline{C}}_o$. Auf die in der Festigkeitslehre gewöhnliche Bezeichnung übergehend erhalten wir die allgemein verwendete Gleichung des Poynting-Thomson-Körpers:

$$\underline{\underline{T}} = 2G\underline{\underline{E}} + 2\eta\,\dot{\underline{\underline{E}}} - \tau\,\dot{\underline{\underline{T}}}\,.$$

Die hier figurierenden Koeffizienten sind materialabhängige, durch Messung bestimmbare rheologische Konstanten. Die geometrischen Zusammenhänge sind in den geometrischen und Gleichgewichtsgleichungen, und bei einer anderen Verhandlungsweise in den Impuls-Waagegleichungen enthalten. Die Materialgleichung ergibt den Spannungswert, falls wir in demselben Punkt die Deformation und deren Veränderungsgeschwindigkeit kennen. Die aufgestellte Gleichung ist symmetrisch und eignet sich auch für die Bestimmung von anderer Unbekannte.

Im Falle einer gegebenen Anfangs- und Randbedingung kann man die Stabilität des Spannungs- und Deformationszustandes des untersuchten mechanischen Systems unter dem Einfluss einer Belastungserregung mittels einer das Gleichgewicht ausdrückenden Differentialgleichung, die auch die geometrischen Verhältnissen des Systems berücksichtigt, untersuchen.

Anlässlich der Untersuchung von rheologischen Körpern, bei Bestimmung des Zusammenhanges zwischen Spannung-Deformation müssen wir die Dynamizität des Prozesses, diese Tatsache berücksichtigen, dass wir in solchem Falle mit einem immer von der Zeit implizite abhängigen mechanischen Vorgang zu tun haben.

Die untersuchten in der Realität stattfindended Wechselwirkungen und Prozesse spielen sich auch auf sehr verschiedener Weise ab.

Die Erörterung dieses Prozesses ist auch deshalb von grosser Bedeutung, weil die verschiedenen, jedoch sich als rheologischer Körper verhaltenden Materialien anders auf eine Belastung langsamen Charakters oder auf eine von schneller Natur, die nämlich beträcgtlich auch die Grenze des Zugrundegehens der einzelnen Materialien beeinflussen können, reagieren. Im Falle beider Grenzewerte der Belastungsgeschwindigkeit $\dot{\sigma} \to 0$ und $\dot{\sigma} \to \infty$ benimmt sich das Material mit einer Linearität, die auf den Hooke'schen Körper charakteristisch ist.

Wird der Fall der dynamischen Belastung untersucht, gibt es auch in diesem Fall - laut underer Interpretierung - ein mit der Geschwindigkeit charakterisierbares Verhältnis, das sogar im Falle eines konduktiven Impulstransportes das Wachstum der Masse auf eine unendliche Grösse verursachen würde. Somit sind unsere im Falle $\dot{\sigma} \to \infty$ ausgesprochenen Festlegungen die Resultate einer Extrapolation. Dementsprechend physikalisch nur eine solche Belastungssituation möglich ist, wo die Kurven der Geschwindigkeitszunahme und - abnahme in der Endlichkeit einander schneiden.

Somit können wir die folgenden behaupten:

- die Belastungsgeschwindigkeit ist im Zeitmoment $t = 0$ von Nullwert, und kann einen unendlichen Wert nicht erreichen;
- die Funktion $\dot{\sigma} = f(t)$ kann Singulärpunkte enthalten, und kann sich asymptotisch der Gerade $\dot{\sigma} = 0$ nähern.

Für jeden charakteristischen Belastungstyp sind unendlich viele konkreten Belastungsfunktionen vorstellbar.

Die einheitliche Beschreibung der Spannung wird bei dem Poynting-Thomson-Körper untersucht, wo im koaxialen Falle die Funktion $\sigma = f(\varepsilon, t)$ durch die Lösung der Differentialgleichung

$$\sigma = E\varepsilon + \lambda\dot{\varepsilon} - \nu\dot{\sigma}$$

erreicht wird.

Die Eindeutigkeitsbedingungen: im Zeitmoment $t = 0, \varepsilon$ und σ haben einen Nullwert; bei den Rändern können wir den Charakter der Belastung angeben, d.h. die Randbedingungen $\dot{\sigma}(t)$ oder $\dot{\varepsilon}(t)$, es hängt davon ab, welche von denen wahrend der Aufgabe gegeben ist.

Ist die Materialgleichung mit den Bedingungen: $\dot{\sigma} = Konstante$ oder $\dot{\varepsilon} = Konstante$ *angeben,* dann erhalten wir die mit dieser erzeugbare Lösungsflasche. In der Ebene σ, t sind die Spannungswerte mit Radialgeraden beschreiben, da wir eine Geschwindigkeit $\dot{\sigma} = Konstante$ *annehmen.* (*Abb.*) Unendliche viele solchen Fälle existieren.

In der Abb. bezeichnet 4 die Kurve, die das während der Belastung bildenden Spannungsverhältnis beschreibt. Der Grenzwert der dynamischen Belastung ist von der Gerade 1 und die statische Belastung von der Gerade 5 representiert.

Die Darstellungsmethode bezeichnet zugleich den für die rheologischen Materialien charakteristischen Kriech- und Relaxationszusatand. Im Kriechfall $\sigma = Konstante$, deren Ablauf durch die Kurve 3 in der mit der Ebene, σ, t Parallelebene gezeigt ist. Die Relaxation, den Zustand $\varepsilon = Konstante$, wird durch die Kurve 2 paralell mit der Ebene σ, t demonstriert. Es ergibt sich die Frage ob sich die verschiedenen Lösungsflächen innerhalb eines beschränkten Bereiches anordnen.

Es ist begreiflich, dass die dynamische Belastungsgeschwindigkeiten $\dot{\sigma} \to \infty$ und statische Belastungsgeschwindigkeiten $\dot{\sigma} \to 0$, die als Grenzwerte festigstellt worden sind, die die positiven Halbachsen des Koordinatensystem σ, t sind und in diesen Viertel fallen alle Belastungsfunktionen, in jedem Fall produziert werden können, egal von welchem Charakter die Belastung ist.

Den Funktionszusammenhang $\sigma - \varepsilon$ des Poynting-Thomson--Körpers betrachtend kommen diese Grenze auch hier vor, die das elastische Verhalten dem Hooke'schen Körper entsprechend signalisieren. Es ist begreiflich, dass die obere Begrenzungsebene der Lösungsflächen die von der Gerade 1 und der Achse t bestimmte Ebene ist, während die untere Grenze die von der Gerade 5 unt t uasgespannte Ebene ist.

An unsere Ausgangsaufgabe zurückkommend: Labor- und in situ Messungen beweisen, dass es solchen Bodentyp und Bodenzustand gibt, wo der Boden der Gleichung des Poynting--Thomson-Körpers mit ausreichender Genauigkeit folgt, so kann er zur Modellierung angewendet werden.

Die mechanischen Verhältnisse der konzentrierten Kraft, die auf die rheologische Oberfläche beschränkende Ebene senkrecht ist, wurde von Fenyvesi-Lack aufgrund der klassischen Boussinesq'sche Aufgabe erarbeitet, wo die Bewegung in Kraftrichtung:

$$w = \frac{1}{E} \int \{Dt - (1 - e^{-\frac{E}{\lambda} t}) \left[\frac{\lambda}{E} D - K\right]\} dz + f(r) , \qquad \text{(x)}$$

wo:

$$D = \dot{\sigma}_z - \mu(\dot{\sigma}_r + \dot{\sigma}_\varphi)$$

$$K = \nu[\dot{\sigma}_z - \mu\chi(\dot{\sigma}_r + \dot{\sigma}_\varphi)].$$

Die Spannungswerte können einfach von den geometrischen Verhältnissen bestimmt werden.

Bei einer Belastung auf Ellipsenfläche verteilt sich die Belastung auf der Fläche $A = \pi ab$ der Ellipse sodass, die Druckverteilung mit der Ordinate ξ des Ellipsoids proportional ist:

$$p = p_o \frac{\xi}{c} = p_o \sqrt{1 - \left(\frac{x_1}{a}\right)^2 - \left(\frac{y_1}{b}\right)^2} ,$$

wo:

$$p_o = \frac{3}{2} \frac{F}{\pi ab} .$$

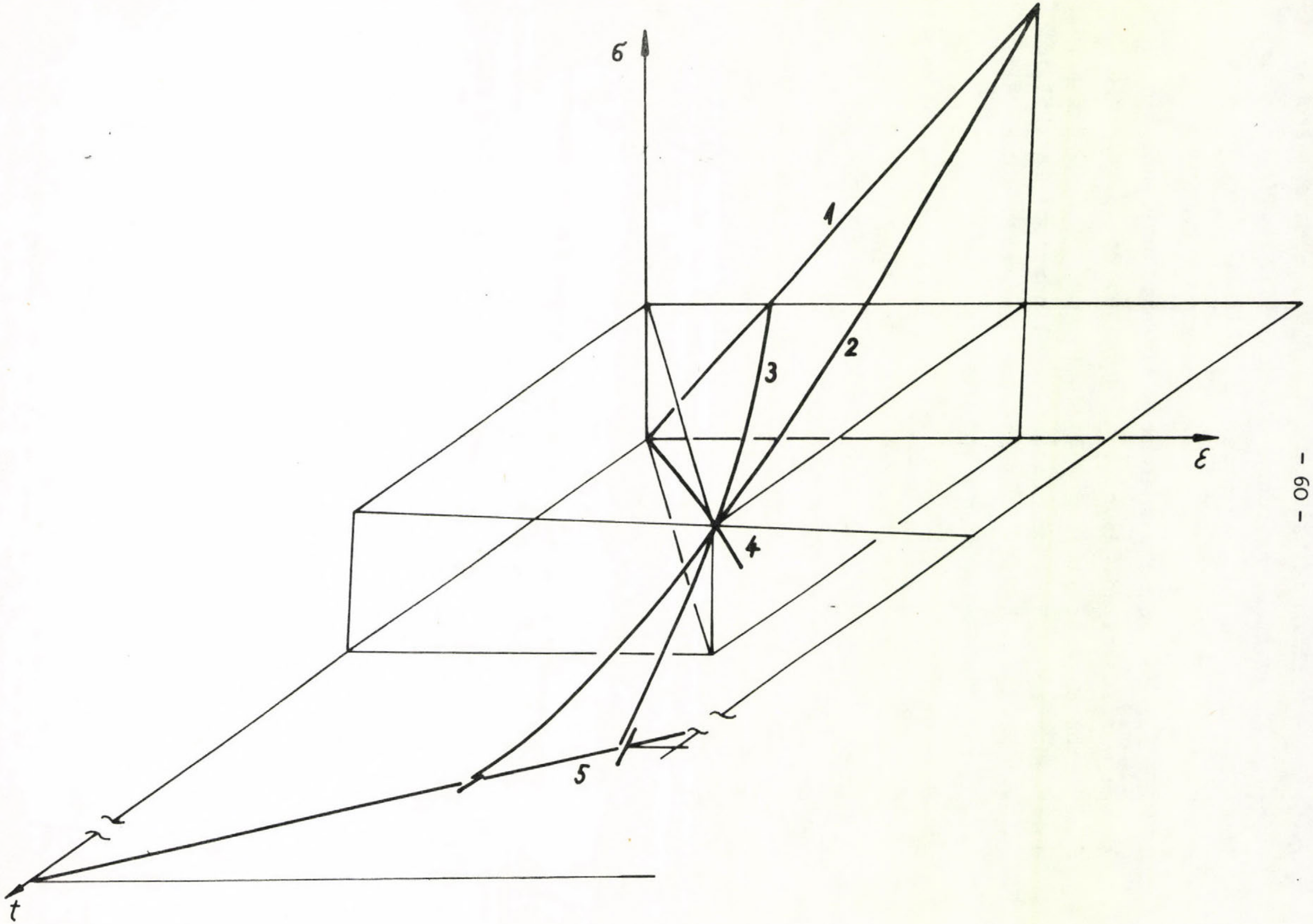
Ausser acht lassend die Deduzierung, mit der Anwendung von (x) wird der Wert der Bewegung w auf die innerhalb des Belastungsbereiches liegenden Punkte der Fläche des Halbraumes:

$$w = \frac{3(1-\mu^3)}{4aE}[abK - \frac{b}{a} Dx_1^2 - \frac{a}{b}(K-D)y_1^2][p - \dot{p}(\frac{\lambda}{E} - \Theta)(1 - e^{-\frac{E}{\lambda} t})]$$

wo: a,b - Ellipsenhalbsachsen; μ,λ,Θ,E - Materialkonstanten, K - vollelliptisches Integral 1. Art, und D die Kombiniertung von K und eines elliptischen Integrals 2. Art sind.

Wird Die Belastung auf einer Ellipsenfläche analysiert, können wir festellen, dass im allgemeinen Fall, wenn die numerische Exzentrizität der Ellipse $0 < e < 1$, die Punkte der Berührung nach Deformation ein elliptisches Paraboloid formen, entartet dieses elliptische Parabolid im Grenzfall $e = 0$ zu einem Rotationsparaboloid, im Falle $e = 1$ zu einem parabolischen Zylinder.

Weiterhin ist unser Ziel im Laufe der theoretischen und praktischen Forschung die theoretischen Ergebnisse durch in situ Nessungen zu beweisen und - durch Weiterentwicklung der theoretischen Ergebnisse und unter Berücksichtigung der Ansprüche an Maschinen, - die Laufrad-Bodenrelation zu optimalisieren.


б
1
2
3
4
5
ε
t

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

THE STUDY OF THE FAMILIES OF ONE-STEP METHODS

By

AURÉL GALÁNTAI

ABSTRACT. Convergence results and error analysis are given for the families of one-step methods.

1. INTRODUCTION

It is well-known fact, that we have no "all-round" methods in the numerical solution of ordinary differential equations of the form

(1) $\underline{y}' = f(t,\underline{y})$; $\underline{y}(t_o) = \underline{y}_o$ $(\underline{f} \in C([t_o,b] \times R^m, R^m))$.

The effective solution of different groups of practical problems needs numerical methods with slightly different or contrasted structures. This situation also holds for one Cauchy-problem, if the exact solution quickly varies in character over a long computational interval.

In order to obtain more effective numerical processes, the families of numerical methods were introduced and proposed instead of single approximate methods. The most famous complex method and its FORTRAN program (DIFSUB) was developed by Gear in 1971 (see [3]), which have several variants, e.g. GEAR, EPISODE, DESOL (see [4]). Gear's process consists of several BDF's and Adams-methods and different decision functions for the choice of stepsize, the formula and the order. The convergence of such processes was proven by Gear, Tu and Watanabe in 1974.

There are also attempts to develop similar families from one-step methods. The studies in this direction are mainly experimental (see e.g. [5]-[6]). In this paper we report some results concerning the convergence and error analysis of the families of one-step methods.

2. RESULTS ON CONVERGENCE

We suppose the existence of a constant $L \geq 0$ such that

(2) $$||\underline{f}(t,\underline{y}) - \underline{f}(t,\underline{y}^*)|| \leq L\ ||\underline{y} - \underline{y}^*|| \quad (t \in [t_o, b]; \underline{y}; \underline{y}^* \in R^m).$$

Moreover it is assumed that for all $t^* \in [t_o, b]$ and for all $\underline{y}^* \in R^m$ the Cauchy-problem

(3) $$\underline{y}' = \underline{f}(t,\underline{y}) \ ; \quad \underline{y}(t^*) = \underline{y}^*$$

has exactly one solution with domain $[t_o, b]$. Let $t_o < x \leq b$ be an arbitrary but fixed point and let Δ_N be a grid over $[t_o, x]$ such that $\Delta_N : t_o < t_1 \ \ldots \ < t_N = x$. Denote by π_x the set of all grids Δ_N of the interval $[t_o, x]$. The norm of $\Delta_N \in \pi_x$ is defined by $||\Delta_N|| = \max\limits_{1 \leq i \leq N-1} (t_{i+1} - t_i)$,

where $h_i = t_{i+1} - t_i$ is the ith steplength. At the point $t_n \in \Delta_N$ an approximate solution of the Cauchy-problem is denoted by $\underline{y}_n$.

Let $W = \{1,2,\ldots,w^*\}$ be finite and for each index $w \in W$ the one-step method defined by

$$(4) \qquad \underline{y}_{n+1} - \underline{y}_n = h_n \; \Psi(w|t_n,\underline{y}_n,\underline{y}_{n+1},h_n) \qquad (t_{n+1} \in \Delta_N)$$

must be convergent in the sense of Henrici (see [2]). Thus we suppose that for all $w \in W$ the increment function $\Psi(w|.,.,.,.) \in C([t_o,b]\times R^m\times R^m\times[0,b-t_o],R^m)$,

$$(5) \qquad ||\Psi(w|t,\underline{y},\underline{z},h) - \Psi(w|t,\underline{y}^*,\underline{z}^*,h)|| \leq K_w(||\underline{y} - \underline{y}^*||+||\underline{z}-\underline{z}^*||$$

holds with a suitable constant $K_w \geq 0$ for every $t \in [t_o,b]$; $h \in [0,b-t_o]$; $\underline{y},\underline{y}^*,\underline{z},\underline{z}^* \in R^m$ as well as

$$(6) \qquad \Psi(w|t,\underline{y},\underline{y},0) = \underline{f}(t,\underline{y}) \qquad (t \in [t_o,b];\ \underline{y} \in R^m).$$

Definition 1. Let $I = I(\Delta_N):\{0,1,\ldots,N-1\} \to W$ be arbitrary indexfunction. Then the triple (Ψ,W,I) is said to be a family of one-step methods, if the approximate solution $\underline{y}_n$ $(n=0,1,\ldots,N)$ is computed by the recursion

$$(7) \qquad \underline{y}_{n+1} - \underline{y}_n = h_n \; \Psi(I(n)|t_n,\underline{y}_{n+1},h_n) \qquad (t_{n+1} \in \Delta_N). \qquad ***$$

This definition means that in each step of the computation the indexfunction I choses the formula actually used. At the same time the stepsize h_n may change arbitrarily and independently of $I(n)$.

Using the works [1],[2] we can prove

Theorem 1. The family (Ψ, W, I) of one-step methods is convergent for all index set W and indexfunction I, i.e.

$$(8) \qquad \lim_{N\to+\infty} \max_{0\le n\le N} \|\underline{y}_n - \underline{y}(t_n)\| = 0$$

is satisfied for every $\{\Delta_N\}_{N=1}^{\infty} \subset \pi_x$ $(\|\Delta_N\| \to 0)$. ***

For the estimation of the speed of convergence we need

Definition 2. Denote by $\underline{Y}_j : [t_o, b] \to R^m$ the exact solution of the perturbed Cauchy-problem

$$(9) \qquad \underline{y}' = \underline{f}(t, \underline{y}) \ ; \quad \underline{y}(t_j) = \underline{y}_j \quad (j=0,1,\dots,N) \ .$$

The local truncation error of the family (Ψ, W, I) at the point $t_n \in \Delta_N$ with respect to the Cauchy-problem (9) is defined by

$$(10) \quad T_j(I(n)|t_n, h_n) = \underline{Y}_j(t_n) + h_n \Psi(I(n)|\, t_n, \underline{Y}_j(t_n), \underline{Y}_j(t_{n+1}), h_n) - \underline{Y}_j(t_{n+1}).$$

***

In case $j=0$ we use simply the notation $T(I(n)|t_n, h_n)$ since $\underline{Y}_o(t) \equiv \underline{y}(t)$.

Using the discrete version of the Gronwall-Bellman lemma we can also prove the two-sided error bound

$$(11) \quad c_1 \max_{0\le k\le N} \Big\| \sum_{n=o}^{k} T(I(n)|t_n, h_n) \Big\| \le \max_{0\le n\le N} \|\underline{y}_n - \underline{y}(t_n)\| \le$$

$$\le c_2 \max_{0\le n\le N} \Big\| \sum_{n=o}^{k} T(I(n)|t_n, h_n) \Big\| ,$$

where $c_1, c_2 > 0$ are given constants depending on $max\{K_w | w \in W\}$. This inequality also implies the convergence and that the speed of convergence is determined by the method of minimum order among that are used in the computation over $\Delta_N \in \pi_x$. Thus the change of order (formula) is advantageous only if the additional error component decreases substantially or the structure of the exact solution strongly varies, e.g. in case of stiff differential systems.

3. ERROR ANALYSIS

In general the problem (1) is solved numerically, if for the sequence $\{\underline{y}_n\}_{n=o}^{N}$ the condition

(12) $$||\underline{y}_n - \underline{y}(t_n)|| \leq \varepsilon^* \qquad (t_n \in \Delta_N)$$

holds, where $\varepsilon^* > 0$ is the requested accuracy.

The checking of this condition is usually made by the estimation of the local truncation errors

(13) $$T_n(I(n) | t_n, h_n) \qquad (n=0,1,\ldots,N-1)$$

and the control of the relations

(14) $$||T_n(I(n) | t_n, h_n)|| \leq ch_n \; \varepsilon^* \qquad [||T_n(I(n) | t_n, h_n)|| \leq c \; \varepsilon^*],$$

where $c > 0$ $[c(\Delta_N) > 0]$ depends on the problem (1) and the family (Ψ, W, I). The hypothesis, that (12) follows from (14), is called the local error estimation principle (see [1],[2]).

It is noted that for explicit methods $T_n(I(n)|t_n,h_n)$ is identical with the local error $\underline{\ell}_n = \underline{y}_{n+1} - \underline{Y}_n(t_{n+1})$.

The theoretical base of the above principle is given in *Theorem 2.* If for the local truncation error of the family (Ψ,W,I) and for the grid $\Delta_N \in \pi_x$, $(\|\Delta_N\| \le h^*)$ the condition

(15) $\|T_n(I(n)|t_n,h_n)\| \le h_n \varepsilon \quad [\|T_n(I(n)|t_n,h_n)\| \le \varepsilon]$

$(n=0,1,\dots,N-1)$ is satisfied, then

(16) $$\max_{0\le n\le N} \|\underline{y}_n - \underline{y}(t_n)\| \le c_3 \varepsilon$$

holds with a suitable constant $c_3=c_3(\underline{f})>0$ $[c_3=c_3(\underline{f},\Delta_N) > 0]$.

***

In the first case of (15) the constant c_3 is independent of the grid Δ_N, in the other case c_3 is of order $O(N)$. If the stepsizes satisfy the relation $0 < Ah \le h_n \le Bh$ $(n=0,1,\dots,N-1)$ and $\varepsilon = h^p$ $(p > 1)$ then relation (16) changes to

(17) $$\max_{0\le n\le N} \|\underline{y}_n - \underline{y}(t_n)\| \le c_4 h^{p-1}$$

in the second case of (15).

For a given class of Cauchy-problems of the form (1) and a given family (Ψ,W,I) the constant c_3 and the constant c in (14) can be estimated analitically. However the constant c is chosen experimentally in general. Practically, condition (14) is checked for the estimated value of $\|T_n(I(n)|t_n,h_n)\|$, but the study of this case can be reduced to the investigation of the applied error estimation processes. Thus one can find concrete effectivity theorems of the type due to T.E. Hull.

For a single one-step method ($W=\{1\}$) the step-halving (or step-doubling) error estimation is optimal in some sense (see [2]) and it can be also proven that the use of this error estimation process doesn't modify the previously proven form of the local error estimation principle ([2]). Latter result also holds for the families of one-step methods.

Assume that $\underline{y}_n$ has been computed and let $t_{n+1} = t_n + h$, $t_{n+2} = t_n + 2h$. We also suppose that $I(n) = I(n+1) = \xi$ in the computation of $\underline{y}_{n+1}$, $\underline{y}_{n+2}$. Denote $\overline{\overline{\underline{y}}}_{n+2}$ the approximated value of $\underline{y}(t_{n+2})$ computed by

$$(18) \qquad \overline{\overline{\underline{y}}}_{n+2} = \underline{y}_n + 2h\,\Psi(I(n)\,|\,t_n, \overline{\overline{\underline{y}}}_n, \underline{y}_{n+2}, 2h),$$

i.e. $\overline{\overline{\underline{y}}}_{n+2}$ is computed from $(t_n, \underline{y}_n)$, whith double steplength 2h. If $\underline{f}$ and the increment functions Ψ in (4) are sufficiently differentiable then we can prove

Theorem 3. If the indexfunction I of the family (Ψ, W, I) satisfies the order condition $p_{I(i)} \geq p_\xi$ $(i=0,1,\dots,n-1)$ and $\Delta_N \in \pi_x$, $\|\Delta_N\| \leq h^*$, then we have

$$(19) \qquad T_n(I(n)\,|\,t_n, h_n) = \frac{\overline{\overline{\underline{y}}}_{n+2} - \underline{y}_{n+2}}{2^{p_\xi+2} - 2} + O(\|\Delta_N\|^{p_\xi+2})$$

and

(20) $$T_{n+1}(I(n)\,|\,t_{n+1},h) = T_n(I(n)\,|\,t_n,h) + O(||\Delta_N||^{p_\xi+2}).$$

The proof of this result essentially is the same as in [2]. It is also noted that the accuracy of the error estimation determined by the minimum order p_ξ corresponds to the bound (11).

4. REMARKS

The main problem in the construction of any family (Ψ,W,I) of one-step methods is the choice of basic formulas (Ψ,W) and the indexfunction I. Concerning the choice of (Ψ,W) there are several interesting results (see e.g. [5],[6]). The structure of the indexfunction I may be similar to that used in DIFSUB, but its cost is more expensive for the usually applied one-step methods. In a comparison with the Gear-type processes (see [3], [4]) any family of one-step methods is more stable and more flexible with respect to the choice of stepsize and the formula.

At last we mention

Theorem 4. For the class of differential equations of the form

(21) $$\underline{y}' = \underline{\underline{A}}\,\underline{y}\;;\quad \underline{y}(t_o) = \underline{y}_o \qquad \underline{\underline{A}}\ (m \times m \text{ constant matrix}),$$

where $\underline{\underline{A}}$ is negative definite and hermitian, any infinite class of implicit Runge-Kutta methods related to the Padé-approximants $r_{k,s}(z)$ of e^z $(k \in \{s,s+1,s+2\}, s \geq 1)$ with arbitrary unbounded indexfunction I is convergent for every $\{\Delta_N\}_{N=1}^{\infty} \subset \pi_x$ $(||\Delta_N|| \to 0)$. ***

This result has an interesting contrast with the result of A.G. Werschulz on the optimal order of one-step methods ([7]).

REFERENCES

[1] Galántai A.; "Convergence theorems and error analysis for one-step methods", Annales Univ. Sci. Budapest., Sect.Math., Vol.19. (1976) pp. 69-78.

[2] Galántai A.; "Investigations in the theory of convergence and error analysis of the numerical solutions of ordinary differential equations", (in Hungarian), Dissertation for the degree "Candidate of Math. Sciences", Budapest, Dec. 1978.

[3] Gear, C.W.: "Numerical initial value problems in ordinary differential equations", Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1971.

[4] Plant, R.E.; "The efficient numerical solution of biological simulation problems", Computer Programs in Biomedicine, v.10. (1979) pp. 1-15.

[5] Shampine, L.F., Watts, H.A.; "Practical solution of ordinary differential equations by Runge-Kutta methods", Rep. SAND 76-0585, Sandia Laboratories, Dec. 1976.

[6] Shampine, L.F., Wisniewski, J.A.: "The variable order Runge-Kutta code RKSW and its performance", Rep. SAND 78-1347, Sandia Laboratories, 1978.

[7] Werschulz, A.G.: "Optimal order and minimal complexity of one-step methods for initial value problems", Dept. of Mathematics, Carnegie -Mellon University, Pittsburgh, PA 15213, Sept. 1976.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

COMPARISON THEOREMS AND THEIR APPLICATIONS TO FUNCTIONAL-DIFFERENTIAL EQUATIONS

ISTVÁN GYŐRI

Computing Center of the Szeged University
of Medical Sciences

Comparison theorems play an important role in the qualitative theory of ordinary and functional-differential equations, too (i.e. [1],[2]). The background for the application of these theorems is the following.

Consider the ordinary differential equation

$$(1)\qquad \begin{aligned} &\dot{x}(t) = f(t,x(t)), \qquad t_o \le t \le T, \\ &x(t_o) = x_o, \end{aligned}$$

where $f(t,x):[t_o,T] \times R^n \to R^n$ is a given continuous function. Assume that

$$|f(t,x)| \le \omega(t,|x|)$$

where $\omega(t,u)$ is a continuous function from $[t_o,T] \times R^1$ to R^1. Then for any solution $x(t_o,x_o)(t)$ of system (1) the following inequality holds:

$$|x(t_o,x_o)(t)| \le v(t_o,v_o)(t), \qquad t_o \le t \le t_o + h,$$

where $v(t_o,v_o)(t)$ is the maximal solution of the scalar equation

$$\dot{v}(t) = \omega(t,v(t))$$
$$v(t_o) = v_o \ge |x_o|$$

on the interval $[t_o, t_o+h]$.

This principle was generalized for retarded and functional--differential equations, too. Such generalizations can be found for example, in the book [2].
Like in that book, it is generally supposed that the bounding function of the right side of the investigated equation is monotone in certain arguments.

To illustrate this, we consider only a simple case.

Let $f_1(t,x,y_1,y_2)$ be a continuous function from $[t_o,T]\times R^n\times R^n\times R^n$ to R^n, such that the inequality

$$|f_1(t,x,y_1,y_2)|\leq\omega_1(t,|x|,|y_1|,|y_2|)$$

is valid, where the continuous function $\omega_1(t,v,v_1,v_2)$ is monotone non-decreasing in the arguments v_1 and v_2.
Now we consider the delay differential equation

$$\dot{x}(t) = f(t,x(t),x(t-r_1(t)),x(t-r_2(t))), \quad t_o\leq t\leq T, \tag{2}$$

with the initial condition

$$x(s) = \varphi(s), \quad t_o-r \leq s \leq t_o$$

where $r_i(t)$ $(i=1,2)$ is continuous function, $0\leq r_i(t)\leq r$ $(i=1,2,\ t_o\leq t\leq T)$ and $\varphi(s)$ is continuous function on the interval $[t_o-r,t_o]$.

Then it is well known, that for any solution $x(t_o,\varphi)(t)$ of equation (2) the following inequality holds:

$$|x(t_0,\varphi)(t)| \leq u(t_0,\alpha)\ (t), \qquad t_0 \leq t,$$

where $u(t_0,\alpha)(t)$ is a solution of the equation

$$\dot{u}(t) = \omega_1(t,u(t),u(t-r_1(t)),u(t-r_2(t))), \qquad t_0 \leq t,$$

with the initial condition

$$u(s) = \alpha(s)$$

The initial function $\alpha(s)$ is such that

$$|\varphi(s)| \leq \alpha(s), \quad t_0-r \leq s \leq t_0.$$

As the following example indicates the monotonicity condition on the bounding function is generally not a natural condition.

In the equation

(3) $$\dot{x}(t) = h(x(t-r_1(t))) - h(x(t-r_2(t))), \qquad t \geq t_0,$$

the right side is a difference of two retarded terms.

The investigation of such equations is important, since they have application for example in modelling of population growth (i.e. [3]), theory of epidemics [4].

In the case of equation (3) let $h(x)$ be Lipschitzian with Lipschitz constant K.

Then we get

$$|\dot{x}(t)| \leq K\, x(t-r_1(t)) - x(t-r_2(t))|, \qquad t \geq t_0,$$

that is, the bounding function is here

$$g(v_1, v_2) = K|v_1 - v_2|$$

where g is not monotone in the variables v_1 and v_2, and for this case the known comparison theorems are not applicable. Now we give a new comparison theorem which is good for the equation (3), and is a generalization of a result in [5].

Theorem 1 Assume that

(i) $f(t,\varphi): [t_0, T] \times C([-r, 0], R^n) \to R^n$ is continuous function;

(ii) there is a continuous function

$$\omega(t,\alpha,\beta): [t_0, T] \times C([-r,0],R^1) \times C([-r,0],R^1) \to [0,\infty)$$

such that $\omega(t,\alpha,\beta)$ is monotone in arguments α and β and

$$|f(t,\varphi)| \leq \omega(t,\alpha,|\dot{\alpha}|),$$

for any continuously differentiable function α and

$$\varphi \in B_\alpha := \{\psi \in C([-r,0],R^n) : |\Psi(S_2) - \Psi(S_1)| \leq |\alpha(S_2) - \alpha(S_1)|, -r \leq S_1, S_2 \leq 0\}$$

(iii) the equation

$$\dot{u}(t) = \omega(t, u_t, \dot{u}_t), \qquad t_o \leq t \leq T,$$

$$u(t_o + S) = \alpha(s), \qquad -r \leq S \leq 0,$$

has a unique solution $u(t_o, \alpha)(t)$ on $[t_o, T]$.

Then for any solution $x(t_o, \varphi)(t)$ of equation

$$\dot{x}(t) = f(t, x_t), \qquad t_o \leq t \leq T,$$

$$x(t_o + S) = \varphi(s), \qquad -r \leq S \leq 0,$$

the inequality

$$|x(t_o, \varphi)(t)| \leq u(t_o, \alpha)(t), \qquad t_o \leq t \leq T$$

is valid.

For example, if we consider equation (3), then $\omega(t, \alpha, \beta) = \left| \int_{-r_1(t)}^{-r_2(t)} |\beta(s)| ds \right|$ is monotone in β and from our theorem we get the following.

Proposition. If

$$|h(x_2) - h(x_1)| \leq K|x_2 - x_1|,$$

where K is a given positive constant, then for any solution $x(t_o, \varphi)(t)$ of equation (3) we have

$$|x(t_o, \varphi)(t)| \leq u(t_o, \alpha)(t), \qquad t \geq t_o,$$

where $u(t_o,\alpha)(t)$ is the solution of the equation

$$\dot{u}(t) = K|u(t-r_1(t))) - u(t-r_2(t))|, \qquad t \geq t_o,$$

$$u(t_o+s) = \alpha(S), \qquad -r \leq s \leq 0.$$

Here the functions $\varphi(s)$ and $\alpha(s)$ are such as in *Theorem 1.*

By the application of our comparison theorem we get the following result, which is related to a theorem of Haddock and Sacker [6].

Theorem 2 Assume that

(a) $P(t)$ is a continuous matrix valued function on $[0,\infty)$;

(b) $0 \leq \tau(t) \leq r$ is continuous function on $[0,\infty)$ and there is a non-negative constant k such that

$$s - \tau(s) \geq t - k\tau(t), \qquad 0 \leq s \leq t.$$

If there exists a natural number N such that

$$\int^{\infty}\{|P(t)|(\int_{t-Nk\tau(t)}^{t}|P(s)|ds)^N\}dt < \infty$$

then for any solution $x(t)$ of the equation

$$\dot{x}(t) = P(t)\,[x(t) - x(t-\tau(t))], \qquad t \geq 0, \tag{4}$$

the limit $\lim_{t\to+\infty} x(t) = x(\infty)$ exists.

REFERENCES

[1] Lakshmikantham, V. and Leela, S., Differential and Integral Inequalities I., Academic Press, New York, 1969.

[2] Lakshmikantham, V. and Leela, S., Differential and Integral Inequalities II., Academic Press, New York, 1969.

[3] Cooke, K.L. and Yorke, J.A., Some equations modelling growth processes and gonorrhea epidemics, Math. Biosc. 16 (1973), 75-101.

[4] Hoppensteadt, F., and Waltman, P., A problem in the theory of epidemics II., Math. Biosc. 10 (1971), 133-145.

[5] Győri, I., Remarks on t he differential equations of the models of population - growth (in Hungary), Alk. Mat. Lap. 1 (1975), 139-150.

[6] Haddock, J.R. and Sacker R.J., Stability and asymptotic integration for certain linear systems of functional differential equations, J. Math. Anal. Appl. 76, 328-338. (1980).

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

ON SOME STIELTJES INTEGRAL INEQUALITIES

ISTVÁN GYŐRI

Computing Center of the Szeged University
of Medical Sciences

LAJOS PINTÉR

Bolyai Institute of the Jozsef Attila University
Szeged

In the theory of the differential and integral equations the famous Bellman-Gronwall inequality, its generalizations and similar inequalities have several applications.

Usually these inequalities are of the following form: Under suitable assumptions from

$x(t) \leqq F(t, \int_a^t g(t,s,x(s))\, ds)$ we come to the inequality

$x(t) \leqq F_1(t, \int_a^t h(t,s)ds)$.

Recently in some papers we find the problem whether these inequalities are valid also for Stieltjes integrals (see e.g. P.G. Das - R.R. Sharma, Some Stieltjes integral inequalities, Journal of Math. Anal. and Appl. 73(1980), 423-433 and its references). In these papers special inequalities are investigated.

Naturally it arises the question to find a method which can be applied on several known inequalities to extend them to the Stieltjes integral case.

We tried the method sketched briefly in the following (here we use Riemann - Stieltjes integral):

Suppose that $x(t) \leqq F(t, \int_a^t g(t,s,x(s))d\phi(s))$, where $\phi:[a,b] \to R$ is monotone. Assume that we have a sequence of continuously differentiable monotone functions $\{\phi_n\}$ with the property: $\phi_n(x) \to \phi(x)$, for $x \in [a,b]$, $n = 1,2,\dots$.

Hence we obtain

$$x(t) \leqq F(t, \int_a^t g(t,s,x(s))d\phi(s)) = F(t, \int_a^t g(t,s,x(s))d\phi_n(s) +$$

$$+ \int_a^t g(t,s,x(s))d\,(\phi(s) - \phi_n(s))) \leqq F(t, \int_a^t g(t,s,x(s))\phi'_n(s)ds +$$

$$+ \varepsilon_n(t)).$$

If $\varepsilon_n(t) \leqq 0$ and F is an increasing function in its second variable then

$$x(t) \leqq F(t, \int_a^t g(t,s,x(s))\ \phi'_n(s)ds$$

hence

$$x(t) \leqq F_1(t, \int_a^t h_1(t,s,\phi'_n(s))ds).$$

If the right hand side may be written in the form

$$F_1(t, \int_a^t h_2(t,s)\phi'_n(s)ds) \quad \text{or} \quad F_1(t, \int_a^t h_2(t,s)d\phi_n(s)),$$

then applying the Helly-Bray theorem we have

$$x(t) \leqq F_1(t, \int_a^t h_1(t,s)d\phi(s)).$$

$\varepsilon_n(t) \leqq 0$ is not true in general. It is valid e.g. if $g(u,v,w) \geqq 0$ and $\phi - \phi_n$ is a monotone decreasing function, $n = 1,2,\ldots,$. (But using Fubini's theorem we obtain that a singular monotone function can't be a limit of continuously differentiable monotone functions with $\phi'_n(x) \geqq \phi'_{n+1}(x)$.)

The wanted sequence $\{\phi_n\}$ may be constructed from polynomials too. We use S.W. Young's theorem (Bull. Amer. Math. Soc. (73(1967), 642-643.).

Suppose that n is a positive integer, $x_{i-1} < x_i$ and $y_{i-1} < y_i$, $i=1,\ldots,n$, then there exists a polynomial P such that $P(x_i) = y_i$, $i=0,1,\ldots,n$, and P is monotone in each of the intervals $[x_{i-1},x_i]$, $i=1,\ldots,n$.

By the aid of this theorem we sketch our construction. For the sake of simplicity suppose that ϕ is a continuous and strictly monotone increasing function.

Divide the interval $[a,b]$ into equal parts:

$a = x_o < x_1 < \ldots < x_{2^n} = b$. Put $y_k = \phi(x_{k-1})$ if $k=1,2,\ldots,2^n$ and choose $y_{on} < \phi(x_o)$. Then $y_{on} < y_1 < \ldots < y_{2^n}$. From the above mentioned theorem we come to a polynomial ϕ_n with $\phi_n(x_k) = y_k$, $k = 1,2,\ldots,2^n$, $\phi_n(x_o) = y_{on}$ and as ϕ_n increases on every interval $[x_{i-1},x_i]$, it is increasing on $[a,b]$.

Constructing these polynomials on $n = 1,2,\dots,k,\dots$ it is clear that $\phi_n(x) < \phi(x)$ and $\phi_n(x) \to \phi(x)$ if $n \to \infty$ and $y_{on} \to \phi(a)$. (With a slight modification of the construction we can obtain a sequence of polynomials having the property $\phi_n(x) \leqq \phi_{n+1}(x)$ too.)

It is easy to see that the construction goes also if ϕ is monotone nondecreasing and continuous from the left. (One can construct such sequence $\{\phi_n\}$ in another way too.)

As an application let us have an inequality of Deo (1971): If the functions x,a,k are defined on $J = [\alpha,\beta], k(t) \geqq 0$, the function $g : I \to R$ is monotone nondecreasing positive subadditive and submultiplicative, $x(J) \subset I$; the function h is defined on an interval Δ, $0 \in \Delta$, $h(\Delta) \subset I$, and h is monotone nondecreasing.

Suppose further that

$$x(t) \leqq a(t) + b(t)h\left(\int_\alpha^t k(s)g(x(s))\,ds\right) \quad \text{for} \quad t \in J \quad \text{then}$$

$$x(t) \leqq a(t)+b(t)h\left\{G^{-1}\left[\int_\alpha^t k(s)g(b(s))ds + G\left(\int_\alpha^t k(s)g(a(s))ds\right)\right]\right\}, \quad \text{where} \quad G(u)=\int_{u_0}^{u} \frac{ds}{g(s)}, \quad u \in \Delta.$$

By the above sketched method this inequality is valid for Stieltjes integrals too with function ϕ which is monotone nondecreasing and continuous from the left:

$$x(t) \leqq a(t) + b(t)h\left(\int_\alpha^t k(s)g(x(s))d\,\phi(s)\right) = a(t) +$$

$$+ b(t)h\left(\int_\alpha^t k(s)g(x(s))d(\phi(s)-\phi_n(s)) + \int_\alpha^t k(s)g(x(s))d\phi_n(s)\right) \leqq$$

$$\leqq a(t) + b(t)h\left(\int_\alpha^t k(s)g(x(s))d\,(\phi(s) - \phi_n(s))\right) +$$

$$+ b(t)h\left(\int_\alpha^t k(s)g(x(s))d\phi_n(s)\right) \leqq a(t) + A +$$

$$+ b(t)h\left(\int_\alpha^t k(s)g(x(s))\,\phi'_n(s)ds\right),$$

$A \geqq 0$ fixed number, $n \geqq N(A)$.

Therefore

$$x(t) \leqq a(t) + A + b(t)h\,\{G^{-1}[\int_\alpha^t k(s)g(b(s))d\,\phi(s) +$$

$$+ G(\int_\alpha^t k(s)g(a(s) + A)d\,\phi(s))]\}.$$

Since this is true for every $A \geqq 0$, we obtain

$$x(t) \leqq a(t) + b(t)h\,\{G^{-1}[\int_\alpha^t k(s)g(b(s))d\,\phi(s) +$$

$$+ G(\int_\alpha^t k(s)g(a(s))d\phi(s))]\}.$$

There are problems if the function ϕ'_n appears repeatedly in the obtained inequality. For example the following inequality was published by Gamidov in 1969.

If a, v_i, φ_i are continuous positive functions on $[a,b]$ and

$$x(t) \leq a(t) + \sum_{i=1}^{k} v_i(t) \int_{\alpha}^{t} \varphi_i(s)x(s)ds \ ,$$ then

$$x(t) \leq a(t) + v(t) \int_{\alpha}^{t} \sum_{i=1}^{k} \varphi_i(s)a(s) \ exp(\int_{s}^{t} \varphi_i(u)v(u)du)ds,$$

where $v(t) = \sup_{1 \leq i \leq k} v_i(t)$.

In this case applying our method we come to the inequality:

$$x(t) \leq a(t) + v(t) \int_{\alpha}^{t} \sum_{i=1}^{k} \varphi_i(s)a(s) \ exp(\int_{s}^{t} \sum_{i=1}^{k} \varphi_i(u)d\phi_n(u))d\phi_n(s) \ ,$$

and the problem is whether the right-hand side tends to

$$a(t) + v(t) \int_{\alpha}^{t} \sum_{i=1}^{k} \varphi_i(s)a(s) \ exp(\int_{s}^{t} \sum_{i=1}^{k} \varphi_i(u)d\phi(u))d\phi(s)$$

or not.

In this and in similar cases the Lebesgue-Stieltjes or more general integral may be useful but this question will be treated in another note.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

ON THE PARTIAL ASYMPTOTIC STABILITY BY LJAPUNOV FUNCTION WITH SEMIDEFINITE DERIVATIVE

LÁSZLÓ HATVANI

Institute Bolyai (Szeged)

Consider the autonomous differential equation

(1) $$\dot{x} = X(x) \qquad (x \in R^n,\ X(0) = 0),$$

where the function $X : R^n \to R^n$ is continuous. Denote by $x(t; x_o)$ any solution of (1) satisfying the initial condition $x(0; x_o) = x_o$. Let a partition $x = (y, z)$ be given, where $y \in R^m$, $z \in R^{n-m}$, $1 \leq m \leq n$. The zero solution of (1) is said to be *y-stable* if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\| x_o \| < \delta(\varepsilon)$ implies $\| y(t; x_o) \| < \varepsilon$ for $t \geq 0$. We say that the zero solution of (1) is *asymptotically y-stable* if it is y-stable and there exists a positive number σ such that $\| x_o \| < \sigma$ implies $\| y(t; x_o) \| \to 0$ as $t \to \infty$ [1].

For a continuously differentiable function $V : R^n \to R$ we define the function

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} X_i(x),$$

which is called as the derivative of V with respect to (1).

Classical theorems in Ljapunov's second method concerning asymptotic stability require negative definiteness of the derivative of the Ljapunov function with respect to the system. In numerous applications this condition fails to be satisfied. For example, the derivative of the total mechanical energy of a conservative mechanical system under the action of dissipative forces with total dissipation is negative definite with respect to velocities only so we cannot conclude asymptotic stability for the equilibrium with respect to coordinates. E.A. Barabashin and N.N. Krasovskiĭ [2] replaced negative definiteness of the derivative of the Ljapunov function by the condition that it is negative semidefinite and its zero set contains no complete trajectory except the origin. Their method has been extended to study of partial asymptotic stability [3, 4]. However, in all extensions it is supposed that all of the uncontrolled coordinates z are bounded along every solution. We are going to give an extension using conditions which can be checked directly.

Theorem. Suppose that there exists a continuously differentiable function $V : R^r \to R$ with the following properties:

(i) $V(0) = 0$, and V is positive definite in y, i.e. there exists a continuous, strictly increasing function $a : R_+ \to R_+$ such that $a(0) = 0$, and

$$V(y, z) \geq a(\|y\|) \qquad (y, z) \in R^n;$$

(ii) if $c > 0$, $(y(t), z(t))$ is a solution of (1) and $(y(t), z(t)) \in \{x : \dot{V}(x) = 0, \ V(x) = c\}$ $(t \in R_+)$, then $y(t) \equiv 0$;

(iii) for every $c > 0$ the set of the points $y \in R^m$ for which there exists a sequence $\{(y_i, z_i)\}$ such that $y_i \to y$, $\|z_i\| \to \infty$, $V(y_i, z_i) \to c$ as $i \to \infty$, consists at most of the origin.

Then the zero solution of (1) is asymptotically y-stable.

This theorem can be applied to derive sufficient conditions for equilibrium of conservative mechanical system to be asymptotically stable with respect to coordinates in the presence of dissipative forces [5]. In order to illustrate these results, consider a point of mass equal to 1 moving in a constant field of gravity. Suppose the point is constrained to move on a surface of equation $u = y^2(1 + z^2)$, axis u is directed vertically upward, and furthermore, that it is subjected to viscous friction with total dissipation. This motion was investigated by K. Peiffer and N. Rouche [6]. They proved that the equilibrium $y = z = 0$ is asymptotically stable with respect to the velocity $\dot{y}$. Applying our theorem we can prove that the equilibrium is asymptotically stable with respect to the coordinate y as well.

Indeed, the total mechanical energy satisfies all conditions of the theorem. It is positive definite in y because the potential energy is $P = gy^2(1 + z^2)$. The dissipation is complete, consequently if $\dot{V} = 0$ along a motion then velocities are equal to 0 identically, i.e. the point is in equilibrium. Therefore, $y(t) \equiv 0$, which means that the second condition is satisfied, too.

Finally, if

$$y_i \to y, \quad |z_i| \to \infty, \quad gy_i^2(1 + z_i^2) \to C \quad (i \to \infty),$$

then $y = 0$, so the third condition is also satisfied.

REFERENCES

[1] V.V. Rumjancev, On the stability of motion in a part of variables (in Russian), Vestnik Moskov. Univ. Ser. I Mat. Meh., 1957. No.4, 9-16.

[2] E.A. Barbashin, N.N. Krasovskiĭ, On the global stability of motion (in Russian), Dokl. Akad. Nauk SSSR, 86(1952), 453-456.

[3] A.S. Oziraner, On asymptotic stability and instability relative to a part of variables (in Russian), Prikl. Mat. Meh., 37(1973), 659-665.

[4] C. Risito, Sulla stabilitá asintotica parziale, Ann. Mat. Pura Appl. (4), 84(1970), 279-292.

[5] L. Hatvani, On the partial asymptotic stability and instability (to appear).

[6] K. Peiffer, N. Rouche, Ljapunov's second method applied to partial stability, J. Mécanique, 8(1969), 323-334.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

АСИМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ РЕШЕНИЯ ЗАДАЧИ КОШИ ДЛЯ ГИПЕРБОЛИЧЕСКОГО УРАВНЕНИЯ С БОЛЬШИМ ПАРАМЕТРОМ

Й. Хегедыш /Сегед/

При изучении волновых процессов - чаще всего - приходится считаться с присутствием одного, или нескольких тел, в той или иной мере влияющих на эти процессы.

Типичным примером может служить распространение волн в R^n $/n\geq 2/$, в случае присутствия одного, неподвижного тела $\Omega \equiv \Omega_+$ /с гладкой границей $\partial\Omega$/, в которое слабо и медленно проникают волны, возникающие во внешности Ω : т.е. в Ω_- ; тело не колеблится, происходит почти полное отражение волн от поверхности тела.

Предположим для простоты, что в начальный момент времени т.е. при $t = 0$ нет никаких возмущений. Если обозначить через $u(t,x)$ отклонение от положения покоя точки $x\in\Omega_-$ в момент времени $t\geq 0$, то математическая модель только что приведенного процесса обычно ставится в виде смешанной задачи в области Ω_-:

/1/ $\Box u=f(t,x)$ $t>0$, $x\in\Omega_-$; $u|_{t=0}=u_t|_{t=0}=0$; $u|_{\partial\Omega}=0$,

где $\Box$ - волновой оператор, $f \not\equiv 0$.

Мы будем рассматривать уточненную модель этого процесса:

1/ Допускается, что волны проникают в тело, и распространяются в нем с малой скоростью λ^{-1} $(\lambda>>1)$.

2/ Опускаем малореальное ограничение $u|_{\partial\Omega}=0$.

3/ Вместо задачи /1/ вводим задачу Коши-сопряжения /согласования на $\partial\Omega$ / во всем R^n для более общего, гиперболического уравнения второго порядка.

Наша цель: выяснить предельное поведение при $\lambda\to\infty$ решения задачи Коши-сопряжения.

Итак, предлагаемая модель для $U(t,x)=u^-,u^+$ / $x\in\Omega_-,\Omega_+$ соотв./ следующая:

$$LU:=Q(x,\lambda)\frac{\partial^2 U}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial U}{\partial x_j}\right)+\sum_{i=1}^{n}A_i(x)\frac{\partial U}{\partial x_i}+ \qquad /2.1/$$

/2/

$$+A(x)\frac{\partial U}{\partial t}+B(x)U=F(t,x)$$

$$-\infty<t\leq T;\quad x\in R^n\setminus\partial\Omega,$$

$$U|_{t<0}=U_t|_{t<0}=0 \qquad /2.2/$$

$$\omega^i U:=\left(\frac{\partial^i u^+}{\partial\nu^i}-\frac{\partial^i u^-}{\partial\nu^i}\right)\Big|_{\partial\Omega}=0 \quad i=0,1;\ \nu-\text{ нормаль к } \partial\Omega \qquad /2.3/$$

где λ,T - положительные постоянные,

$$Q(x,\lambda)=\begin{cases}1 & x\in\Omega_-\\ \lambda^2 & x\in\Omega_+\end{cases};\quad A_{ij}=A_{ji}=\begin{cases}a^-_{ij} & x\in\Omega_-\\ a^+_{ij} & x\in\Omega_+\end{cases},\quad a^+_{ij}|_{\partial\Omega}=a^-_{ij}|_{\partial\Omega}$$

$$A_i=\begin{cases}a^-_i & x\in\Omega_-\\ a^+_i & x\in\Omega_+\end{cases}\ (i,j=1,\dots,n),\quad A=\begin{cases}a^- & x\in\Omega_-\\ a^+ & x\in\Omega_+\end{cases},\quad B=\begin{cases}b^- & x\in\Omega_-\\ b^+ & x\in\Omega_+\end{cases},$$

коэффициенты $A_{ij},\dots,B$ бесконечно гладки в $\overline{\Omega}_-$, соотв. в $\overline{\Omega}_+$, $a^-_{ij},\dots,b^-$ и всех их производные ограничены и формы

$$\sum_{i,j=1}^{n}a^-_{ij}(x)\xi_i\xi_j\geq c\sum_{i=1}^{n}\xi_i^2,\quad \sum_{i,j=1}^{n}a^+_{i,j}(x)\xi_i\,\xi_j$$

являются положительно определенными формами $\xi_1,\dots,\xi_n$ при каждом $x\in\Omega_-,\Omega_+$ соответственно. Относительно правой части /2.1/:

$$F=\begin{cases}f^- & x\in\Omega_-,\\ 0 & x\in\Omega_+,\end{cases}$$

предполагаем, что f^- обладает некоторой гладкостью; а именно предполагаем, что f^- принадлежит одному из классов $C^{\infty,0}(\Omega_-^T)$, $H^{s,0}(\Omega_-^T)$, $B^{s,0}(\Omega_-^T)$; $\Omega_-^T:=(-\infty,T]\times\Omega_-$ /определение этих классов и теорема существования решения задачи /2/ в этих классах даны ниже/.

Из того, что скорость распространения волн в Ω_+ стремится к нулю при $\lambda\to\infty$ /и того, что $F=0$ при $x\in\Omega_+$ / следует:

Лемма 1. Существует такое число $\lambda_0=\lambda_0(T)>0$, что $u^+(t,x,\lambda)\equiv 0$ при всех (t,x,λ): $0\leq t\leq T$, $x\notin(\Omega_+)_\delta$, $\lambda\geq\lambda_0$, где $(\Omega_+)_\delta$ - пограничная полоска Ω_+, ширина δ которой стремится к нулю при $\lambda_0\to\infty$.

Замечание 1. Если F такова, что задача /2/ имеет бесконечно гладкое решение при каждом фиксированном $\lambda\geq\text{const.}$, то из Леммы 1. эвристически уже следует, что при любых фиксированных (t,x) /$x\in\Omega_+,\Omega_-$ соотв./ $u^+(t,x,\lambda)\to 0$, $u^-(t,x,\lambda)\to u(t,x)$ при $\lambda\to\infty$ в классическом смысле; где u - решение внешней задачи

$$\frac{\partial^2 u}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a^-_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{n}a^-_i(x)\frac{\partial u}{\partial x_i}+a^-(x)\frac{\partial u}{\partial t}+b^-(x)u=f^-(t,x)$$

$$x\in\Omega_-,\ -\infty<t\leq T,$$

$$u|_{t<0}=u_t|_{t<0}=0;\quad u|_{\partial\Omega_-}=0.$$

Лемма 2. В области $(\Omega_+)_\delta$ /при достаточно малом δ / можно ввести локальные координаты $(S(x);\varphi_1(x),\dots,\varphi_{n-1}(x))$ где $S(x)$- решение задачи Коши:

$$\sum_{i,j=1}^{n}a^+_{ij}(x)\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_j}=1\quad x\in(\Omega_+)_\delta;\quad S|_{\partial\Omega_+}=0;\quad S>0,$$

$S(x)\in C^\infty$ - координата по трансверсальному к $\partial\Omega_+$ направлению ; а $\varphi_i\in C^\infty$ - координаты по касательным к $\partial\Omega_+$ направлениям.

Эта лемма следует из того, что лучи системы Гамильтона - соответствующей задаче для S - трансверсальны к $\partial\Omega_+$ и они не пересекаются в $(\Omega_+)_\delta$ /гладкость $S(x),\varphi_i(x)$ следует из гладкости $\partial\Omega_+$ и гладкости коэффициентов a^+_{ij} в $\overline{\Omega_+}$ /.

Введем некоторые обозначения. Пусть $Q\subset R^n$ - область, для которой либо $\overline{Q}\subseteq\overline{\Omega}_+$ либо $\overline{Q}\subseteq\overline{\Omega}_-$; тогда $Q^T:=(-\infty,T]\times Q$. Далее, пусть

$$C^{\infty,0}(Q^T)=\{c(t,x)\,|\,c\in C^\infty(Q^T),\ c=0\quad t<0\}$$

и пусть при $s\geq 0$ целом $H^{s,0}(Q^T),B^{s,0}(Q^T)$ обозначают замыкание $C^{\infty,0}(Q^T)$ по нормам

$$\|c\|_{H^{s,0}(Q^T)}=\|c\|_{H^s(Q^T)},\quad \|c\|_{B^{s,0}(Q^T)}=\sup_{t\in(-\infty,T]}\sum_{i=0}^{s}\left\|\frac{\partial^i c}{\partial t^i}\right\|^2_{H^{s-i}(Q)},$$

где $H^s(Q^T)$, $H^{s-i}(Q)/H^0(Q^T)=L^2(Q^T)$ /обычные пространства С.Л. Соболева. В качестве Q мы будем брать только области Ω_+, Ω_- и их замыкания. Нам понадобиться еще и пространство

$$C^{\infty,0}=\{c(t,x)\,|\,c=(c^-\in C^\infty(\bar\Omega^T_-),\ c^+\in C^\infty(\bar\Omega^T_+));\ c=0\ \ t<0\}$$

и классы $H^{s,0}$, $B^{s,0}$, которые являются замыканиями $C^{\infty,0}$ по нормам

$$\|c\|_{H^{s,0}}=\|c^-\|_{H^s(\Omega^T_-)}+\|c\|_{H^s(\Omega^T_+)},$$

$$\|c\|^2_{B^{s,0}}=\sup_{t\in(-\infty,T]}\sum_{i=0}^{s}\left\|\frac{\partial^i c^-}{\partial t^i}\right\|^2_{H^{s-i}(\Omega_-)}+\sup_{t\in(-\infty,T]}\sum_{i=0}^{s}\left\|\frac{\partial^i c^+}{\partial t^i}\right\|^2_{H^{s-i}(\Omega_+)}.$$

<u>Замечание 2</u>. Мы должны выделить еще одну задачу, которая получается из задачи /2/ заменой $0=F(t,x)$ $x\in\Omega_+$ на $f^+(t,x)\in C^{\infty,0}(\bar\Omega^T_+)$ или $H^{s,0}(\bar\Omega^T_+)$; при этом предполагаем, что проекция на R^n_x носителя f^+ лежит в достаточно узкой пограничной полоске $(\Omega_+)_\delta$. Эту задачу мы будем обозначать через /2*/.

<u>Определение 1</u>. Мы будем говорить, что функция $U(t,x;\lambda)=(u^-,u^+)$ является классическим решением задачи /2*/ если u^-,u^+ удовлетворяют уравнению в Ω^T_- соотв. в Ω^T_+ в классическом смысле при каждом $\lambda\geq\lambda_0=\mathrm{const}>>1$ и условия /2.2/, /2.3/ - понимаемые в обычном смысле - выполняются.

<u>Теорема 1</u>. Если $F\in C^{\infty,0}$, то задача /2*/ при каждом $\lambda\geq\lambda_0$ имеет, притом единственное классическое решение $U;U\in C^{\infty,0}$ и существует такая постоянная $C(\lambda,T)$, что

/3/ $\|U\|_{B^{p+1,0}}\leq c(\lambda,T)\|F\|_{H^{p,0}}$ $\quad p=0,1,2,\ldots$.

<u>Определение 2</u>. Функцию $U(t,x;\lambda)\in B^{s+1,0}$ будем называть обоб-

щенным решением задачи /2*/ с $F\in H^{s,0}$ если существует последовательность функций /классических решений/ $\{U_n \in C^{\infty,0}\}$ для которых

$LU_n = F_n,\ U_n|_{t<0}=(U_n)_t|_{t<0}=0,\ \omega^i U_n=0\quad i=0,1;\ n=1,2,\dots$

и

$$\| F_n - F\|_{H^{s,0}} \to 0,\ \| U_n - U\|_{B^{s+1,0}} \to 0 \qquad n\to\infty.$$

Теорема 2. Задача /2*/ с $F\in H^{s,0}$ имеет при каждом $\lambda \geq \lambda_0 >> 1$ одно и только одно обобщенное решение U и справедлива оценка /с постоянной $C(\lambda,T)$ из /3//:

/4/ $$\| U\|_{B^{p+1,0}} \leq C(\lambda,T)\ \| F\|_{H^{p,0}} \quad p=0,1,\dots,s$$

Теоремы 1., 2. следуют из результатов [1] /см. также и [2], [3]/. Из этих результатов; однако, трудно усмотреть характер зависимости $C(\lambda,T)$ от λ в оценках /3/, /4/, и получить тем самым информацию о структуре решения U и о его поведении при $\lambda\to\infty$.

Распространение волн в Ω_+ и в Ω_- происходит весьма не одинаково; поэтому вместо неравентсв /3/, /4/ целесообразно вывести неравенства, в которых отдельно оцениваются нормы u_- и u_+. Так например, с помощью модификации стандартных энергетических неравенств можно получить:

Лемма 3. Пусть правая часть $F(t,x)$ задачи /2*/ принадлежит классу $H^{0,0}$. Тогда для решения $U=(u^-,u^+)$ задачи /2*/ имеют место оценки с постоянной $C=C(T)$:

$$|||u^-|||_0 + |||(u^-)_t|||_0 + |||\nabla_x u^-|||_0 \leq C(\lambda^{-1}\| f^+\|_0 + \| f^-\|_0),$$

$$|||u^+|||_0 + |||(u^+)_t|||_0 \leq C(\lambda^{-2}\| f^+\|_0 + \lambda^{-1}\| f^-\|_0),$$

$$|||\nabla_x u^+|||_0 \leq C(\lambda^{-1}\| f^+\|_0 + \| f^-\|_0).$$

где в левых и правых частях соотвественно

$|||\cdot|||_0:=\|\cdot\|_{B^{0,0}(\Omega^T_-)}, \|\cdot\|_{B^{0,0}(\Omega^T_+)}; \ \|\cdot\|_0:=\|\cdot\|_{H^{0,0}(\Omega^T_-)}, \|\cdot\|_{H^{0,0}(\Omega^T_+)}$.

Оценки для высоких производных u^-, u^+ получаются более сложными. Выделяем один факт, который состоит в том, что главная часть оценки нормы $|||\cdot|||_0$ производной порядка $q\geq 2$ от u^+ по трансверсальной к $\partial\Omega$ координате $S(x)\sim\varphi_n$ имеет порядок λ^{q-1} /нормы $|||\cdot|||_0$ производных того же порядка от u^+ по касательным к $\partial\Omega$ координатам $\varphi_1,\dots,\varphi_{n-1}$ и по t имеют более хорошие оценки/. Введем обозначение:

$$g_{\alpha,\varphi_n^k}:=\frac{\partial^{|\alpha|+k} g}{\partial\alpha_0^{i_0}\dots\partial\alpha_{n-1}^{i_{n-1}}\partial\varphi_n^k}\ (\alpha_0;\alpha_1,\dots,\alpha_{n-1})=(t;\varphi_1,\dots,\varphi_{n-1}),\ |\alpha|=i_0+\dots+i_{n-1}.$$

<u>Лемма 4</u>. Для решения $U=(u^-,u^+)$ задачи /2*/ с $F\in H^{s,0}$ имеют место оценки с постоянной $C=C(T)$:

$$\|u^-\|_{B^{p+1,0}(\Omega^T_-)}\leq (\|f^-_{\varphi_n^{p-1}}\|_0+\sum_{|\alpha|=1}\|f^-_{\alpha,\varphi_n^{p-2}}\|_0+\dots+ \sum_{|\alpha|=p-2}\|f^-_{\alpha,\varphi_n}\|_0+C(\lambda^{-1}\sum_{|\alpha|=p}\|f^+_\alpha\|_0+\sum_{|\alpha|\leq p}\|f^-_\alpha\|_0)\qquad p=0,1,\dots s,$$

$$\|u^+\|_{B^{p+1,0}(\Omega^T_+)}\leq C(\|f^+_{\varphi_n^{p-1}}\|_0+\lambda\sum_{|\alpha|=1}\|f^+_{\alpha,\varphi_n^{p-2}}\|_0+\dots+ \lambda^{p-2}\sum_{|\alpha|=p-2}\|f^+_{\alpha,\varphi_n}\|_0)+C(\lambda^{p-1}\sum_{|\alpha|=p}\|f^+_\alpha\|_0+\lambda^p\sum_{|\alpha|=p}\|f^-_\alpha\|_0)$$

$p=0,1,\dots,s.$

где в правых частях использованы обозначения Леммы 3.

($f^{-\ j}_{\alpha,\varphi_n}$ - сумма всех производных f^- порядка $|\alpha|+j$ вне узкой полоски $\partial\Omega$-).

<u>Теорема 3</u>. Для решения $U=(u^-,u^+)$ задачи /2/ с $F\in H^{s,0}$ имеет место при каждом $N\leq s-3$ натуральном представление

$$u^-=v^-_N+W^-_N,\quad u^+=v^+_N+W^+_N,$$

/5/
$$v^-_N=\sum_{j=0}^{N+1}u_j(t,x)\lambda^{-j},\quad v^+_N=\sum_{\substack{i+j=0\\ i,j\geq 0}}^{N+1}u_{i,j}(t-\lambda S(x),\underline{\varphi}(x))S^i(x)\lambda^{-j}\qquad (u_{i,j}(\xi,\varphi)=0\quad \xi<0),$$

где функции $\{u_j\},\{u_{i,j}\}$ играющие роль коэффициентов разложения $u^-;u^+$ в ряд по степеням $\lambda;\lambda,S$ определяются из рекуррентной схемы /типа изложенной в [4]/ по $F\sim f^-$,

$$u_j\in B^{s+1-j,0}(\Omega^T_-),\quad u_{i,j}(t-\lambda S,\varphi)\in B^{s-(i+j),0}(\Omega^T_+)$$

и имеют место оценки с постоянной $C=C(T)$:

$$\| u_j \|_{B^{\ell+1,0}(\Omega_-^T)} \le C \| f^- \|_{H^{\ell+j,0}(\Omega_-^T)} \qquad \ell=0,1,\dots,s-j,$$

/6/ $$\| u_{ij}(t-\lambda S,\varphi) \|_{B^{\ell+1,0}(\Omega_+^T)} \le C\lambda^{\ell+\frac{1}{2}} \| f^- \|_{H^{\ell+(i+j)+1,0}(\Omega_-^T)}$$

$\ell=1,\dots,s-(i+j)-1,$

а остаточные члены представления /5/ / W_N^-, W_N^+ / имеют оценки:

$$\| W_N^- \|_{B^{\ell+1,0}(\Omega_-^T)} \le C\lambda^{-N-\frac{1}{2}} \| f^- \|_{H^{\ell+N+3,0}(\Omega_-^T)} \qquad \ell=0,1,\dots,s-(N+3),$$

$$\| W_N^+ \|_{B^{\ell+1,0}(\Omega_+^T)} \le C\lambda^{\ell-N-\frac{1}{2}} \| f^- \|_{H^{\ell+N+3,0}(\Omega_-^T)} \qquad \ell=0,1,\dots,s-(N+3).$$

В некоторых частных случаях легко показать, что нормы u_i, u_{ij} имеют нижние оценки /того же/ вида /6/ /с некоторой другой постоянной $C_1=C_1(T)$ /.

В заключении автор приносит благодарность научному руководителю Б.Р. Вайнбергу за постановку задачи и постоянное внимание к работе.

Л И Т Е Р А Т У Р А

1 О.А. Ладыженская: Решение общей задачи диффракции, ДАН СССР, 96/1964/, 3, 433-436.

2 О.А. Ладыженская: Смешанная задача для гиперболического уравнения, Гос. Изд-во Техн.-Теор. Лит., Москва, 1953.

3 Р. Сакамото: Смешанная задача для гиперболических уравнений I, II; сб. "Математика" 16:1/1972/, 62-99.

4 Й. Хегедыш: Асимптотическое поведение решения задачи Коши для волнового уравнения с большим параметром, Вест. Моск. Унив., сер. мат. мех., 6/1980; 44-47.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

FILTRATION OF GAS WITH ABSORPTION : FREE BOUNDARY AND ASYMPTOTIC BEHAVIOUR

ROBERT KERSNER

Computer and Automation Institute
Hungarian Academy of Sciences

We shall study the solutions of the Cauchy problem for the equation

$$Lu \equiv -u_t + (u^{\mu})_{xx} - c \cdot u^{\nu} = 0 \tag{1}$$

in $R^2_+ = \{(t,x): 0 \le t < \infty, \quad -\infty < x < \infty\}$

with initial value

$$u(0,x) = u_0(x), \quad -\infty < x < \infty. \tag{2}$$

Here $\mu > 1$, $\nu > 0$ and $c > 0$ are constants. The function $u_0(x)$ is nonnegative, continuous, and it has compact support. Let supp $u_0 = [-\ell, \ell]$, $\ell > 0$.

A great number of problems in physics leads to the equation (1). For example, it describes both the filtration of gas in porous medium and the process of heat conduction with absorption when the coefficients depend on the density or the temperature in polynomial way.

The equation (1) is parabolic when $u > 0$ and it degenerates and becomes of the first order when $u = 0$.

It has been known for a long time that the problem (1), (2) has no classical solutions in general (e.g. see [1]). For this reason the class of admissible solutions must be extended.

The definition for the generalized solution of the problem (1), (2) and the proof of its existence and uniqueness can be found in [3,6]. Some comparison theorems necessary to prove the following theorems 1-4 are stated there, too.

We shall investigate the solutions for sufficiently large t $(t \geq t_o > 0)$. Therefore we assume $\nu \geq 1$. This is connected with the well-known fact that if $\nu < 1$ then $u(t,x) = 0$ when $t \geq T_o > 0$ (see [3]). It is also known (e.g. see [2]) that in the case of the problem (1), (2) "the perturbation propagates with finite velocity": for any $t > 0$, the function $u(t,x)$ has compact support with respect to x (this fact is a simple corollary following from the results of the present paper). There exist two continuous curves $x = \zeta_i(t)$, $i=1,2$ such that $\zeta_1(t) < 0$ and $\zeta_2(t) > 0$ and

$$\operatorname{supp} u(t,x) = \{(t,x)\colon\ t \geq 0,\ \ \zeta_1(t) \leq x \leq \zeta_2(t)\}.$$

The functions $\zeta_i(t)$ are often called free boundaries.

The basic results of this paper concern the functions $u(t,x)$ and $\zeta_i(t)$. One can find the detailed proofs in [5].

Let $u(t,x)$ be the generalized solution of the problem (1),(2).

Theorem 1. Let $\nu > \mu$. Then there exist positive constants α_i that depend only on the initial data and such that for $t \geq t_o$ the following inequalities hold:

1) $u(t,x) \le a_1 t^{-1/(\mu+1)}$

2) $|\zeta_i(t)| \le a_2 t^{1/(\mu+1)}$, $i=1,2.$

When $\nu < \mu + 2$ then

3) $u(t,x) \ge a_3 t^{-1/(\nu-1)}$ for small $|x|$,

4) $|\zeta_i(t)| \ge a_4 t^{(\nu-\mu)/2(\nu-1)}$, $i=1,2.$

When $\nu \ge \mu + 2$ then

5) $u(t,x) \ge a_5 t^{-1/(\mu+1)}$,

6) $|\zeta_i(t)| \ge a_6 t^{1/(\mu+1)}$, $i=1,2.$

Remark 1. When $\nu \to \mu + 2 - 0$ then the differences in exponents on the right hand sides in 1), 3) and 2), 4) tend to zero.

The first time the estimates for $\zeta_i(t)$ were proved by B.F. Knerr differently (see [7]).

Theorem 2. Let $\mu = \nu$. Then for $t \ge t_0$ the inequalities

1) $u(t,x) \ge a_7 t^{-1/(\mu-1)}$ for small $|x|$,

2) $|\zeta_i(t)| \ge a_8 (\ln t)^{1/2}$, $i=1,2$

3) $u(t,x) \leq a_9 t^{-1/(\mu-1)}$,

4) $|\zeta_i(t)| \leq a_{1o} t^{1/(\mu+1)}$, $i=1,2$

hold with positive constants a_7, a_8, a_9 and a_{1o}.

Remark 2. The inequality 2) means that for $\mu = \nu$ there is no localization of the perturbation i.e. there does not exist $L > 0$ such that supp $u(t,x) \subseteq \{(t,x): |x| \leq L\}$. The first time this fact was proved in 1973 by me and published in 1976 in my dissertation (see [4]). The estimate

$|\xi_i(t)| \geq a\, \ell n^{1-\varepsilon}(\ell n\ t)$ was obtained there, it is weaker than

2). First the estimate 2) was shown by B.F. Knerr [7] by a method different from that in [5].

Theorem 3. Let $1 < \nu < \mu$. Then for $t \geq t_o$ the inequalities

1) $u(t,0) \geq a_{11} t^{-1/(\nu-1)}$,

2) $u(t,x) \leq a_{12} t^{-1/(\nu-1)}$,

3) $|\zeta_i(t)| \leq a_{13}$, $i=1,2$

are valid.

Remark 3. The inequality ˘3) states - by definition - that in this case the localization of the perturbation takes place. This fact is known from [2].

We recall that in all above cases $u(t,x)$ tends to zero as a polynomial. The next theorem establishes the case $\nu = 1$ when this is not valid.

Theorem 4. Let $1 = \nu < \mu$. Then for $t \geq t_o$ the following inequalities hold:

1) $u(t,x) \geq a_{14}\, e^{-ct}$ for small $|x|$,

2) $u(t,x) \leq a_{15}\, e^{-ct}$,

3) $|\zeta_i(t)| \leq a_{16}$, $i=1,2$.

Here $c > 0$ is the constant given in the equation (1).

REFERENCES

[1] Barenblatt, G.I.: On the self-similar solutions of the Cauchy problem for nonlinear parabolic equation of non-stationary filtration, Prikl. Mat. i Mekh. 20(1956)

[2] Kalasnikov, A.S.: On the character of propagation the perturbations, Trudi Semin. I.G. Petrovski 1(1975)

[3] Kalasnikov, A.S.: The propagation of disturbances in problems of non-linear heat conduction with absorption USSR. Comp. Math. and Math. Phys., 14(1974)

[4] Kersner, R.: Some properties of generalized solutions of quasilinear degenerate parabolic equations, Thesis, 1976.

[5] Kersner, R.: On the behaviour of generalized solutions of degenerate quasilinear parabolic equations when t grows to infinity. Acta Math. Sci. Hungaricae 34(1979)

[6] Kersner, R.: Degenerate parabolic equations with general nonlinearities, Nonlin. Anal. Theory, Meth. Vol. 4. No6, 1980.

[7] Knerr, B.F.: The behaviour of the support ..., preprint 1977.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

SOME APPLICATIONS OF QUADRATIC LYAPUNOV FUNCTIONS

VIKTOR KERTÉSZ

Budapest University of Technology

1. INTRODUCTION

Consider the linear differential equation:

$$\dot{x} = A(t)x \tag{1}$$

where $A(t)$ is an $n \times n$ continuous matrix function. The object of this study is the stability of the trivial solution.

For the estimate of the solutions, W.A. Coppel gave the formula:

$$e^{-\int_{t_o}^{t} \mu(-A(\tau))d\tau} \leq \frac{|x(t)|}{|x(t_o)|} \leq e^{\int_{t_o}^{t} \mu(A(\tau)d\tau},$$

where

$$\mu(A) = \overline{\lim_{h \to +o}} \frac{|I - hA| - 1}{h} .$$

Note:

$$\mu(A) = \sup_{|x|=1} Re\, x^* VAx \quad \text{if}$$

$$|x| = \sqrt{x^* Vx}$$

where

V is positive definite Hermetian and

$x^* = \overline{x}^T$ the conjugate transposed of x.

Example 1 Let be

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

In this case, of course, $\lim_{t\to\infty} |x(t)| = 0$ which can be prooved with Coppel's estimate too because $\mu(A) = -1$.

Example 2 Let be

$$A = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}, \text{ then}$$

$\lim_{t\to\infty} |x(t)| = 0$ nevertheless $\mu(A) > 0$.

Coppel's estimate is, generally, not good enough. This is the problem I am going to treat.

2. ONE SOLUTION OF THE PROBLEM

I have found that transforming the variable x the estimate can be improved.

Suppose the system (1) is asymptotically stable.

Let be $V(t)$ a continuously differentiable, positive definite, Hermitian matrix function and $v(t,x)$ a Lyapunov function such that

$$v(t,x) = x^* V(t)x ,$$

$$\alpha^2 |x|^2 \le v(t,x) \le \beta^2 |x|^2 ,$$

$$\dot{v}_{(1)}(t,x) \le -\gamma^2 |x|^2 .$$

Now we choose a continuously differentiable regular matrix function such that

$$V = W^* W.$$

Note: $|W^{-1}(t)|$ is bounded.

The transformation

$$y = W(t)\, x$$

carries (1) into a new system:

$$\dot{y} = C(t)\, y$$

and applying Coppel's formula for this case:

$$|W(t)^{-1}|\,|W(t_o)^{-1}|^{-1}\; e^{-\int_{t_o}^{t} \mu(-C(\tau)d\tau} \le \frac{|x(t)|}{|x(t_o)|} \le |W(t_o)|\,|W^{-1}(t)|\cdot e^{\int_{t_o}^{t}\mu(C(\tau)d\tau} , \qquad (2)$$

where

$$\mu(C(t)) \le -\frac{1}{2}\frac{\gamma^2}{\beta^2}$$

if in the definition of μ we use the Euclidean norm. (Inversely: from a proper transformation matrix $W(t)$ we can get a positive definite quadratic Lyapunov function $v(t,x) = x^* W^* Wx$, the derivative of which alonge the solution of (1) is negative definite.)

3. FURTHER IMPROVEMENT OF THE METHOD

The transformation matrix can be restricted to a part of the unterval $[t_o, \infty]$ and a countable set of transformation matrices can be chosen in the following way:

$$t_o < t_1 < t_2 < \dots < t_i \quad \dots ,$$

$$I_i = [t_{i-1}, t_i] \quad ,$$

$$\Delta t_i = t_i - t_{i-1} \quad ,$$

$$y = W_i(t)\, x \quad \text{if} \quad t \in I_i \ ,$$

where $W_i(t)$ is continuously differentiable, regular in I_i.

This kind of transformation has proved to be more efficient than applying only one continuously differentiable, regular matrix function $W(t)$ in the whole interval $[t_o, \infty)$.

Example 3

Let be $$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & -p(t) \end{bmatrix} , \qquad t \geq 0 ,$$

$$p(t) \geq 0 \quad ; \quad p(t) \quad \text{is continuous.}$$

For *lower* estimate Coppel's formula is very good.

$$-\mu(-A(t)) = -p(t).$$

If $\int_{t_o}^{\infty} p(t)dt < \infty$ then $\lim_{t\to\infty} |x(t)| > 0$.

For *upper* estimate the formula is very rough:

$$\mu(A(t)) = 0$$

and so the only fact what can be stated is that
$|x(t)| \leq |x(t_o)|$ for $t \geq t_o$.

With proper transformations (see below) I could prove the following

Theorem:

Let be $0 < \delta_i < 1$ and $0 < \varepsilon_i$ some constants such that

$0 \leq p(t) \leq \frac{1}{\delta_i}$ if $t \in I_i$, and $\delta_i \leq p(t) \leq \frac{1}{\delta_i}$ if $t \in I_{io}$

where

$$I_{io} = I_i \setminus \bigcup_{j=1}^{k_i} I_{ij}$$

$$I_{ij} \cap I_{i\ell} = \emptyset \quad \text{if} \quad j \neq \ell$$

$$I_{ij} = [\tau_{ij_1} ; \tau_{ij_2}] \subset I_i$$

then

$$\Delta t_{io} > 4 + \frac{4\varepsilon_i}{\delta_i}$$

implies

$$\frac{|x(t_i)|}{|x(t_{i-1})|} < e^{-\varepsilon_i} ,$$

where

$$\Delta t_{io} = \Delta t_i - 3 \sum_{j=1}^{k_i} \Delta\tau_{ij} > 0$$

and

$$t_i - t_{i-1} > \Delta\tau_{ij} = \tau_{ij_2} - \tau_{ij_1} \geq 0.$$

Corollary

The asymptotical stability holds if

$$\frac{1}{t} \leq p(t) \leq t \quad \text{for} \quad 1 \leq t_o \leq t.$$

For proving the theorem I applied the following matrices:

$$W_i = D_i M_i$$

where

$$D_i = \begin{bmatrix} \sqrt{\lambda_{1i}} & 0 \\ 0 & \sqrt{\lambda_{2i}} \end{bmatrix} ,$$

$$M_i = \frac{1}{\sqrt{1+q_i^2}} \begin{bmatrix} q_i & 1 \\ -1 & q_i \end{bmatrix} ,$$

$$q_i = \frac{2}{\delta_i} - \lambda_{1i} ,$$

$$\lambda_{1i} = \frac{\delta_i}{2} + \frac{2}{\delta_i} + \sqrt{1 + \frac{\delta_i^2}{4}} ,$$

$$\lambda_{2i} = \frac{\delta_i}{2} + \frac{2}{\delta_i} - \sqrt{1 + \frac{\delta_i}{4}} .$$

With these:

$$|W_i \| W_i^{-1}| = - q_i \quad \text{and}$$

$$\mu(W_i\ A(t)\ W_i^{-1}) < - \frac{\delta_i}{4} \quad \text{if}$$

$$\delta_i \leq p(t) \leq \frac{1}{\delta_i} \quad \text{or}$$

$$\mu(W_i A(t) W_i^{-1}) \leq \frac{\delta_i}{2} \quad \text{if}$$

$$0 \leq p(t) < \delta_i .$$

Applying (2) (considering that in this case $C(t) = W_i A(t)\ W_i^{-1}$) we have the proof of the theorem.

For proving the corollary we need only to choose I_i in the following way:

$$t_i > \frac{t_{i-1}+4}{1 - 4\varepsilon}$$

where $0 < \varepsilon < 1/4$ is arbitrary and $\varepsilon_i = \varepsilon$ for every i.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

ON THE STABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

M. Klincsik
PMMF Pollack Mihály Technical College
Matematical Institute
7624. Pécs, Rókus ut. 2.

We shall investigate the asymptotic behaviour of solutions of differential equations in Banach spaces, comparing the norm of solutions with the power functions t^{α} (α real) as $t \to \infty$. All the operators, appearing in these equations are everywhere definded and bounded.

1. Consider in a Banach space $\mathfrak{B}$ the linear differential equation

$$(1) \qquad \dot{x} = \frac{dx}{dt} = A(t)x, \qquad (x \in \mathfrak{B})$$

where the operator $A(t): \mathfrak{B} \to \mathfrak{B}$ is linear for all $t \in [0,\infty)$ and A is a locally Bochner integrable function of t on $[0,\infty)$. It is known (see [1]) that the solution $x(t)$ of the equation (1) and the initial condition $x(0) = x_o \in \mathfrak{B}$ can be obtained by the formula $x(t) = U(t)\, x_o$, where $U(t): \mathfrak{B} \to \mathfrak{B}$ is the Cauchy operator of (1).

Let $x(t) = U(t)\, x_o$ ($x_o \neq 0$) be a non-trivial solution of (1). The first characteristic number of $x(t)$ characterizes the exponential behaviour of $\| x(t) \|$, as $t \to +\infty$, wich was introduce by A.M. Lyapunov:

$$\kappa[x] = \lim_{t \to +\infty} \frac{\ln \| x(t) \|}{t} .$$

We shall denote by $\kappa_s = \sup_{x \neq 0} \kappa[x]$, where $x(t)$ is a nontrivial solution of (1). The following relations are valid for the first characteristic numbers ([1]):

1., if $\sup_{t\geq 0} \int_t^{t+1} \| A(S) \| ds < \infty$, then κ_s is finite;

2., for all equations (1) is valid, that $\kappa_s = \overline{\lim_{t\to\infty}} \frac{\ln \| U(t) \|}{t}$;

3., if the operator A is not depending on t, that is $A(t) \equiv A = \text{const.}$, then $U(t) = e^{At}$ and

$$(2) \quad \kappa_s = \lim_{t\to\infty} \frac{\ln \| e^{At} \|}{t} = \inf_{t>0} \frac{\ln \| e^{At} \|}{t} = \max \operatorname{Re} \sigma(A),$$

where $\sigma(A)$ is the spectrum of the operator A.

2. We can find some equations for which it is not sufficient to know the exponential behaviour of $\| x(t) \|$ as $t\to+\infty$, however we have to compare $\| x(t) \|$ with other functions of t as $t\to+\infty$, for example the power functions t^{β} (β real). The second characteristic number $\lambda[x]$ of $x(t)$ characterizes this growth property of $\| x(t) \|$ by the following way

$$\lambda[x] := \lim_{t\to\infty} \frac{\ln(\| x(t) \| e^{-\kappa [x] t})}{\ln t},$$

where $\kappa[x] \neq \pm \infty$ is the first characteristic number of $x(t)$.

Let be the $\lambda_s = \sup_{x\neq 0} \lambda[x]$ and $\mu = \lim \frac{\ln(\| U(t) \| e^{-\kappa_s t})}{\ln t}$

We can derive the following relations between the numbers λ_s and μ.

<u>Lemma 1.</u> i) $\mu \leq \lambda_s$;

ii) If $A(t) \equiv A = \text{const}$, then $\mu \geq 0$ and if the equality

$$(3) \qquad \lim_{h\to 0+} \frac{\| I+hA \| - 1}{h} = \kappa_s$$

is valid, then $\mu=0$.

<u>Proof.</u> i) If we assume that $\lambda_s < \mu$, then we can choose a ρ such that $\lambda[x] \leq \lambda_s < \rho < \mu$. By the definition of $\lambda[x]$ there exist $N_{\rho, x_0} > 0$ such that

$$\| x(t) \| = \| U(t)x_o \| \leq N_{\rho,x_o} e^{\kappa[x]t} t^{\rho} \leq N_{\rho,x_o} e^{\kappa_s t} t^{\rho}. \; (t \geq 1)$$

Thus the operator family $\{U(t)e^{-\kappa_s t} t^{-\rho} : t \in [1,\infty)\}$ is bounded for all $x_o \in \mathfrak{B}$ and from the Banach-Steinhaus theorem we obtain, that

$$\| U(t) \| \leq N_{\rho} e^{\kappa_s t} t^{\rho}.$$

It contradicts the definition of μ and inequality $\rho<\mu$.

ii) From the equality (2) we obtain

$$\inf_{t>0} \frac{\ell n \| e^{(A-\kappa_s I)t} \|}{t} = \max \operatorname{Re} \; \sigma(A-\kappa_s I) = 0,$$

where I is the identity operator on $\mathfrak{B}$. Thus $\| e^{(A-\kappa_s I)t} \| \geq 1$ and from this follows that $\mu = \overline{\lim_{t\to+\infty}} \dfrac{\ell n(\| \ell^{At} \| e^{-\kappa_s t})}{\ell n \; t} \geq 0.$

We remark that the limit in the left hand side of the equation (3) exists for all operator A([2]). Consider the derivative of $\| e^{At} \|$ with respect to t

$$\frac{d^+ \| e^{At} \|}{dt} = \lim_{h\to 0+} \frac{\| e^{A(t+h)} \| - \| e^{At} \|}{h} = \lim_{h\to 0+} \frac{\| e^{At}(I+hA) \| - \| e^{At} \|}{h}$$

$$\leq \| e^{At} \| \; \kappa_s$$

Integrating this from 0 to t we obtain, that $\| e^{At} \| \leq e^{\kappa_s t}$. and thus $\mu \leq 0$. From this follows with $\mu \geq 0$, that $\mu=0$.

If $\mathfrak{B} = \mathcal{H}$ is a Hilbert space, then

$\lim_{h\to 0+} \dfrac{\| I+hA \| - 1}{h} = \sup \operatorname{Re} W(A)$, where $W(A) = \{(Ax,x) : \| x \| = 1\}$ is the numerical range of A. Thus from the statement ii) of lemma 1. we obtain, that if the operator A is convexoid in H, then $\mu=0$. (A is convexoid if the relation Conv $\sigma(A)=\overline{W(A)}$ is valid, conv D denotes the convex hull of D, and $\overline{D}$ the closure of D). For example the normal operators, Toeplitz operators are convexoid (see [3]).

3. When $B=R^n$ the n dimensional Euclidian space, then μ, λ_s, $\lambda[x]$ are always non-negative integers. In infinite dimensional spaces these are not true in general, these are illustrated by the following:

Examples

1. If we consider in the Hilbert space ℓ^2

$$(4) \qquad \dot{x} = Ax, \qquad (x\in\ell^2)$$

where A is a unilateral weighted shift in ℓ^2 with positive α_n weigths and $\alpha_n \to 0$ $(n\to\infty)$, then for the equation (4)

$$\lambda_s = \mu = +\infty.$$

2. If the operator A in (4) is the unilateral shift S in ℓ^2, then there exists a subset H of ℓ^2, which is dense in ℓ^2 and if $x_o \in H$, then $\lambda[x]=-1/4$, where $x(t)=e^{St}x_o$.

Proof.

1. It is well known [3], that the operator A is quasy-nilpotent i.e. $\sigma(A) = \{0\}$. Thus we obtain from (2), that $\kappa_s=0$. Let $e_1 = (1,0,0,\dots)$ and we obtain

$$\alpha_1 \dots \alpha_n \frac{t^n}{n!} \leq \left(\sum_{k=1}^{\infty} (\alpha_1 \dots \alpha_k)^2 \frac{t^{2k}}{(k!)^2} \right)^{1/2} =$$

$$= \| e^{At} e_1 \| \leq \| e^{At} \| ,$$

thus

$$\lambda[e_1] = \lambda_s = \mu = +\infty.$$

2. If we denote $e_n = (0,\dots,0,\overset{\underset{\sim}{n}}{1},0,\dots)$, then

$$e^{St} e_n = (0,\dots 0,\overset{\underset{\sim}{n}}{1},\frac{t}{1!}, \frac{t^2}{2!}, \dots),$$

thus we obtain

$$\| e^{St} e_n \|^2 = \sum_{k=0}^{\infty} \frac{t^{2k}}{(k!)^2} = J_0\ (2it) \quad (n=1,2,3,\ldots)$$

where $J_o(x)$ is the zero order Bessel function ($i=\sqrt{-1}$). From the asymptotic behaviour of J_o (2 it) ([4])

$$J_o\ (2\ it) \sim K \frac{e^{2t}}{\sqrt{t}} \quad (t\to+\infty,\ K>0)$$

follows that

$$\lambda[e_n] = -1/4 \quad (n=1,2,\ldots).$$

It is evident that the subset $H = \{x\in\ell^2 : x=(x_1,\ldots x_n,0,\ldots)\}$ is dense in $\ell^2(n = 1,2,\ldots)$. We know, that $\kappa[x]=1$ for every $x(t)=e^{St}x_o$ $(x_o\in\ell^2)$ ([5]) and from inequality

$$\| x(t) \| = \| e^{St}x_o \| \leq \sum_{j=1}^{n} |x_j| \| e^{St} e_j \| = \left(\sum_{j=1}^{n} |x_j|\right) J_o(2it)$$

follows

$$\lambda[x] = -1/4 \text{ if } x_o = (x_1,x_2,\ldots,x_n,0,0,\ldots)\in H.$$

We can derive easily from this the following

<u>Lemma</u> 2. The zero solution of the equation

(5) $\dot{x} = (S-I)x \qquad (x\in\ell^2)$

is asymptotically stable, but not exponentially.

<u>Remark</u>. This case in R^n can not be occur.

<u>Proof.</u> The stability of zero solution (5) we can derive from inequality

$$\| x(t) \| = \| e^{(S-I)t} x_o \| \leq e^{\| S \| t} e^{-t} \| x_o \| = \| x_o \| \quad (x_o\in\ell^2).$$

When $x_o\in H$, then $\kappa[x]=0$ and $\lambda[x]=-1/4$, thus $\| x(t) \| \to 0$, in the same way as $t^{-1/4}$, when $t\to\infty$. If $x\in\ell^2$, then there exists an

$x_0 \in H$ such that $\| x-x_0 \| \leq \varepsilon/2$ and

$$\| e^{(S-I)t}x \| \leq \| e^{(S-I)t}x_0 \| + \| e^{(S-I)t}(x-x_0) \| \leq \varepsilon/2+\varepsilon/2=\varepsilon,$$

if t is sufficiently large.

Consider the following non-linear equation in ℓ^2

$$(6) \qquad \dot{x}=(S-I)x + F(t,x) \qquad (F(t,0)\equiv 0).$$

<u>Theorem.</u> If $F(t,x)$ satisfies the conditions

$$\| F(t,x) \| \leq f(t) \| x \|$$

and $\int_0^\infty f(s)ds \leq K < \infty$, then the solution $x=0$ of (6) is asymptotically stable.

<u>Proof.</u> We obtain the solution $x(t)$ of (6) for which $x(0)=x_0$ by the integral equation

$$x(t)=e^{(S-I)t}x_0+\int_0^t e^{(S-I)t} F(\tau,x(\tau)d\tau.$$

It follows from lemma 2., that

$$\| x(t) \| \leq \varepsilon' + \int_0^t f(\tau) \| x(\tau) \| d\tau$$

valid if t is sufficiently large. Applying the Bellman lemma ([1]) we can derive

$$\| x(t) \| \leq \varepsilon' e^{\int_0^t f(\tau)d\tau} \leq \varepsilon' e^K < \varepsilon.$$

REFERENCES

1. Yu. L. Daleckij, M.G. Krejn, Stability of solutions of differential equations in Banach spaces, "Nauka" Moscow, 1970 (in Russian).

2. G. Lummer, R. S. Phillips, Dissipative operators in a Banach space, Pacific J. Math. 11,2 (1961) 679-698.

3. P.R. Halmos, Hilbert space problem book, Van Nostrand, 1967.

4. G. Pólya, G. Szegő, Aufgaben und Lehrsatze aus der analysis, I. Spinger - Verlag, Berlin - Göttingen-Heidelber - New-York 1964.

5. S. Balint M. Reghis, Characteristic numbers and spectral properties. I.,Revue Roum. de Math. Pur. at Appl. m. XX. 1/1975).

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

ON THE NOTION OF REGULAR SINGULARITY

ANDRÁS SZÉP

Computer and Automation Institute
Hungarian Academy of Sciences

Singular partial differential equations have been intensively studied in the last time (see [1],[2],[3],[4]). The various assumptions making menagable such equation resemble in many respect to the theory of ordinary analytic differential equations.

In the sequel we want to point out that such singularities arise on a very natural way from quite different problems.

The notion of regular singularity is classical within the field of ordinary analytic differential equations. The word 'singularity' refers to the property that the highest derivative of the unknown function can not be expressed as analytic function of the lower order terms while 'regular' refers to the form of solutions which may not be analytic at the singularity under consideration, though they can be combined from analytic function in a relatively simple manner, see for example the Bessel-functions of second kind which are of the form

$$u(x) + log x \cdot v(x)$$

$u(x)$ and $v(x)$ analytic. As a further important phenomenon at a regular singularity, it may be mentioned, that the usual 'order of the equation = number of initial conditions' law fails to hold, the sign = must be substituted by '$\geq$'.

In the sequel, starting from different problems, we shall deduce a class of partial different problems, we shall of partial differential equations which can be considered as admitting regular singular singularity.

1. THE BUILDING IN INITIAL DATA METHOD

In the ordinary case looking for solutions of the simple Cauchy problem

$$y' = f(x,y)$$

$$y(0) = y.$$

the substitution

$$y(x) = y_0 + xv(x)$$

gives the equation

$$xv' + v = f(x, y_0 + xv).$$

Under suitable conditions on f this is a regular singular problem for the unknown function v. In the field of PDE-s the building in technique cannot be always applied.

In the case of the classical Cauchy-Kovalewski initial value problems the initial values in same special way can be built in [4], the problem of finding eigenfrequencies of a rectangular membrane seems not to be reducible.

2. NATURAL BOUNDARY CONDITIONS

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with piecewise smooth boundary and let us try to minimize the functional

$$I(u) = \int_\Omega [a(x)\ (grad\ u)^2 + b(x)u^2]\ dx$$

Taking the first variation , after some elementary differential geometry we get the elliptic equation

$$a\Delta u + (grad\ a - b)u = \dot{0}$$

and the *natural boundary condition*

$$a(x)\ grad\ u \perp \partial\Omega$$

that is, the vector $a(x)$. $grad\ u$ should be parallel to the normal vector of the boundary surface $\partial\Omega$, except the points were $a(x)$ vanishes. However, in such point s the above elliptic equation becomes singular what tipically prohibits prescription of boundary conditions of the above type.

3. SYMMETRY

Again, the phenomenon can be best illuminated by an example. Using spherical coordinates the form of the Laplace equation in R^3 will be

$$\frac{\partial}{\partial r}(r^2 \frac{\partial\Phi}{\partial r}) + \frac{1}{sin\nu} \frac{\partial}{\partial\nu}(sin\nu \frac{\partial\Phi}{\partial\nu}) + \frac{1}{sin^2\nu} \frac{\partial^2\Phi}{\partial\Phi^2} = 0$$

Here we have a regular singularity at the boundary $r = 0$ [2]. On the other hand, mere differentiability of the original solution at the origin of the (x,y,z) space is possible only if

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=0} = 0$$

4. ASYMPTOTICS

Finally we mention, that scattering behaviour for a given operator can also be built in yielding a regular singular problem. More generally, certain prescribed asymptotic properties of the solution can be built in using the change of scale

$$s \to e^{-t} .$$

Thus, the system of ordinary differential equations

$$y' = f(y)$$

has a unique stable solution for $t \to \infty$ if and only if the transformed regular singular problem

$$sv'(s) + f(v)s)) = 0$$

has a unique bounded solution at $t = 0$.

5. THE EQUATION

A great variety of the above discussed problems can be reduced to a system of linear partial differential equation of the form

$$\sum_{K=o}^{N} A_k(x)(tD_t)^k \leftarrow \sum_{j=o}^{N} C_{N-j}t(D_tt)^j u = f.$$

Here $x = (x_1, x_2, \ldots, x_m) \in C^m, t$ denotes a scalar, $D_t = \partial/\partial t$ the A_k-s and C_{N-j}-s are $n \times n$ matrices, the entries of the latters are differential polinomials [4]. Under suitable restriction it can be proved, that the "number" of solution is determined by the characteristic polynomial of the symbol i.e. the matrix pencil

$$\sum_{K=o}^{N} A_k(x) \lambda^k .$$

REFERENCES

[1] J. Leray, L. Garding, T. Kotake, Probléme de Cauchy, I-VI, Bull. Soc. Math. France 92(1964), 263-361

[2] R.P. Gilbert, Function Theoretic Methods in Partial Differentíal Equations, Academic Press, 1969.

[3] Bouendi, M.S., Goulaouic, C.: Cauchy problems with characteristic initial hypersurface, Comm. Pure Appl. Math. 26 (1975) 455-475

[4] A. Szép., Cauchy problems for systems of linear singular partial differential equations, Studia Sci. Math. Hung. 12(1977) 215-227.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

ИССЛЕДОВАНИЕ УСТОЙЧИВОСТИ И СХОДИМОСТИ РЕШЕНИЙ С ПОМОЩЬЮ ФУНКЦИЙ ЛЯПУНОВА

Й. Тереки

1. Хорошо известная теорема Лагранжа - Дирихле гласит, что положение равновесия консервативной механической системы устойчиво, если потенциальная энергия имеет строгий минимум в этом положении равновесия. Проблема обратимости этой теоремы в общем случае пока не решена. Она решена В.П. Паламодовом в том случае, когда потенциальная энергия является аналитической функцией и число степеней свободы системы равно 1 или 2 [2].

Проблема не решена и в том случае, если на систему помимо потенциальных сил действуют и диссипативные силы с полной диссипацией. Если положение равновесия изолировано, то оно устойчиво тогда и только тогда, если потенциальная энергия имеет минимум в положении равновесия [7] . Однако, если положение равновесия не изолировано, то только при дополнительных условиях известна его устойчивость по части или по всем переменным [1, 5].

Здесь мы ограничиваемся случаем, изученным В.П. Паламодовом, предполагая, что на систему действуют и диссипативные силы. Мы дадим необходимые и достаточные условия для устойчивости положений равновесия.

2. Пусть даны функции $T: R^2 x R^2 \to R$, $\Pi: R^2 \to R$, $Q: R x R^2 x R^2 \to R^2$ со следующими условиями: $T(u,v)=\frac{1}{2}(A(u)v,v)$, где $A(u)$ симметрическая определенно положительная матрица при всех $\|u\| < H$ $(u \in R^2,\ H>0)$ и предположим, что ее элементы имеют непрерывные частные производные до второго порядка включительно. $\Pi(u)$ аналитична при $\|u\| < H$, $\Pi(0)=0$ и $\frac{\partial \Pi(0)}{\partial u}=0$. $Q(t,u,v)$ - - непрерывная функция, удовлетворяющая локальному условию Лип-

шица по u и v, кроме того

$$\| Q(t,u,v) \| \le c_1 \| v \| \qquad (t \ge 0, \| u \| < H,\ v \in R^2),$$

$$(Q(t,u,v),v) \le -c_2 \| v \|^2 \qquad (t \ge 0, \| u \| < H,\ v \in R^2),$$

где $c_1, c_2 > 0$ - постоянные.

Рассмотрим систему дифференциальных уравнений

$$/1/ \qquad \frac{d}{dt}\left(\frac{\partial T(x,\dot{x})}{\partial v}\right) - \frac{\partial T(x,\dot{x})}{\partial u} = -\frac{\partial \Pi(x)}{\partial u} + Q(t,x,\dot{x}),$$

представляющую собой уравнения движений голономной механической системы со стационарными связями под действием диссипативных сил Q /T и Π - кинетическая и потенциальная энергии/. Легко видеть, что $Q(t,u,0) \equiv 0$, следовательно $x=\dot{x}=0$ является положением равновесия системы /1/.

Определение. Положение равновесия $x=\dot{x}=0$ системы /1/ называется равномерно устойчивым, если для всех $\varepsilon>0$ найдется $\delta>0$ такое, что из $\| x_o \| + \| \dot{x}_o \| < \delta$ следуют неравенства

$$\| x(t;t_o,x_o,\dot{x}_o) \| < \varepsilon \qquad (t \ge t_o \ge 0),$$

$$\| \dot{x}(t;t_o,x_o,\dot{x}_o) \| < \varepsilon \qquad (t \ge t_o \ge 0).$$

Теорема 1. Положение равновесия $x=\dot{x}=0$ системы /1/ является устойчивым тогда и только тогда, если $\Pi(u) \ge 0$ в малой окрестности $u=0$. В случае устойчивости

$$\dot{x}(t;t_o,x_o,\dot{x}_o) \to 0, \quad x(t;t_o,x_o,\dot{x}_o) \to \text{const.} \quad (t \to \infty)$$

при малых $\| x_o \|$, $\| \dot{x}_o \|$.

Имеется пример, который показывает, что теорема 1 перестает быть верной, если $\Pi(u)$ не аналитична 4 .

3. В качестве примера можно привести движение материальной точки по поверхности $z=f(x,y)$ в пространстве $Oxyz$ под влиянием силы тяжести $\vec{F}=(0,0,-mg)$ и силы трения, направление которой противоположно скорости и ее величина пропорциональна скорости. В этом случае

$$T(u,v)= \frac{1}{2} m(v_1^2+v_2^2+(\frac{\partial f(u_1,u_2)}{\partial x} v_1+ \frac{\partial f(u_1,u_2)}{\partial y} v_2)^2),$$

$$\Pi(u) = mgf(u_1,u_2) \qquad Q(t,u,v) = -\mu\frac{\partial T(u,v)}{\partial v},$$

где $u=(u_1,u_2)^T$, $v=(v_1,v_2)^T$ и $\mu>0$ - постоянная. Очевидно, что материальная точка находится в покое на поверхности в тех местах, где градиент функции $f(x,y)$ уничтожается.

Предполагаем, что $f(x,y)$ является аналитической функцией. Для этого движения имеет место:

Следствие. Положение равновесия $x=x_o$, $y=y_o$ материальной точки, движущейся по поверхности $z=f(x,y)$, устойчиво тогда и только тогда, если $f(x,y)$ имеет минимум, необязательный строгий, в точке $x=x_o$, $y=y_o$.

Отметим, что эта задача в частном случае $f(x,y)=y^2(1+x^2)$ была исследована в работах [3, 6], но в этих работах исследована устройчивость только относительно y и скоростей.

4. Для доказательства теоремы 1 нам нужно было обобщить классическую теорему А.М. Ляпунова об асимптотической устойчивости. Наше обобщение можно сформулировать для произвольных систем дифференциальных уравнений.

Рассмотрим систему дифференциальных уравнений, записанных в виде

$$/2/ \quad \begin{aligned} \dot{x} &= X(t,x,y) \\ \dot{y} &= Y(t,x,y) \end{aligned} \qquad (t\geq 0,\ x\in R^n,\ y\in R^m),$$

где $X: R\times R^n\times R^m \to R^n$, $Y: R\times R^n\times R^m \to R^m$ - непрерывные функции при $t\geq 0$, $\|x\|<H$, $\|y\|<H$ $(H>0)$. Пусть $X(t,0,0)=0$, $Y(t,0,0)=0$ при $t\geq 0$. Предположим, что единственность решений обеспечена.

Пусть $V: R\times R^n\times R^m\to R$ непрерывная функция, имеющая непрерывные частные производные. Тогда $\dot V_{(2)}(t,x,y)$ определяется соотношением

$$\dot V_{(2)}(t,x,y)=\frac{\partial V(t,x,y)}{\partial t}+\left(\frac{\partial V(t,x,y)}{\partial x}, X(t,x,y)\right)+\left(\frac{\partial V(t,x,y)}{\partial y}, Y(t,x,y)\right).$$

Теорема 2. Допустим, что существуют непрерывные функции $V,W,f:R\times R^n\times R^m\to[0,\infty)$ такие, что V,W имеют непрерывные частные производные первого порядка и

(i) $V(t,0,0)=W(t,0,0)=f(t,0,0)=0,$

(ii) $a(\|y\|)+f^{\sigma}(t,x,y)\le V(t,x,y),$

(iii) $\dot V_{(2)}(t,x,y)\le -c_1 V^{1+\alpha}(t,x,y),$

(iv) $\dot W_{(2)}(t,x,y)\le -c_2 W(t,x,y)+c_3 f(t,x,y),$

(v) $\|Y(t,x,y)\| < c_4(W(t,x,y)+f(t,x,y))$

при $t\ge 0, \|x\|<H, \|y\|<H$ где функция $a:[0,\infty)\to[0,\infty)$ строго возрастает, $a(0)=0$, c_i $(i=1,\dots,4),\alpha,\sigma$ – неотрицательные числа, удовлетворяющие неравенству $\alpha\sigma<1$. Тогда решение $x=0,y=0$ системы /2/ равномерно устойчиво, и $x(t;t_0,x_0,y_0)\to\text{const.}$, $y(t;t_0,x_0,y_0)\to 0$ $(t\to\infty)$, если только $\|x_0\|$ и $\|y_0\|$ достаточно малы.

ЛИТЕРАТУРА

1. Озиранер А.С.: Об асимптотической устойчивости относительно части переменных. Вестник МГУ, 1972 : 1, 73-80.

2. Паламодов В.П.: Об устойчивости равновесия в потенциальном поле. Функциональный анализ и его приложения, 11 /1977/ : 4, 42-55.

3. Румянцев В.В.: Об асимптотической устойчивости и неустойчивости движения по отношению к части переменных. Прикл. мат. мех. 35 /1971/ : 1, 138-143.

4. Тереки Й. - Хатвани Л.: О частичной устойчивости и сходимости движений. Прикл. мат. мех. 45 /1981/ : 3, 428-435.

5. Hatvani J. A Generalization of the Barbashin-Krasovskij Theorems to the Partial Stability in Nonautonomous Systems. Coll. Math. Soc. János Bolyai, Qualitative Theory of Differential Equations, Szeged, 1979, p. 381-409.

6. Peiffer K. - Rouche N. Liapunov's Second Method Applied to Partial Stability, Journal de Mécanique, 8/1969/, 333-334.

7. Salvadori L. Sull'estensione ai sistemi dissipativi del criterio di stabilitá del Routh. Ricerche Mat. 15 /1966/, 162-167.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

A NOTE ON TRIANGULARIZATION OF SECOND-ORDER AUTONOMOUS DIFFERENTIAL EQUATIONS

GÁBOR TÓTH

Mathematical Institute of the Hungarian Academy of Sciences

The purpose of this paper is to announce necessary and sufficient conditions for the triangularizability of the second-order equation

$$(*) \qquad x + k(x,\dot{x}) = 0.$$

The equation (*) is said to be triangularizable if it can be transformed into triangular form

$$\dot{x}_1 = f_1(x_1)\ , \quad x_2 = f_2(x_1,x_2).$$

All the results presented here are proved in [1]. In what follows we shall suppose that equation (*) has nondegenerous critical points, i.e. $k \in C^3(\mathbb{R}^2)$ and

(1) For every critical point c of (*) the eigenvalues λ_1, λ_2 of the Jacobian

$$\begin{bmatrix} 0 & 1 \\ -\partial_1 k & -\partial_2 k \end{bmatrix}$$

at c are different reals and $\lambda_1\lambda_2 = \partial_1\ k(c) \neq 0$. The equation (*) is said to be nondegenerous if its critical points are nondegenerous and the induced dynamical system $(\mathbb{R}^2, \varphi)$ satisfies the following conditions:

(A_0) $(\mathbb{R}^2, \varphi)$ is not a global knot point,

(A_1) The critical point set $C^{\varphi} \subset \mathbb{R}^2$ is discrete,

(A_2) If $q \in \mathfrak{J}^{\eta}_{\varphi}(p)$ (= η -prolongational limit set of $\{p\}$ at the direction $\eta \in \{-,+\}$) is valid for $p,q \in R^2 - C^{\varphi}$ then the set of trajectories

$$\{\varphi(r;\ R) \mid q \in \mathfrak{J}^{\eta}_{\varphi}(r),\quad r \in \mathfrak{J}^{\eta}_{\varphi}(p)\}$$

is finite and for every noncritical point $p \in R^2$

$$\varphi(p;\ \mathbb{R}) \cap \overline{\Sigma^{\varphi} - \varphi(p;\ \mathbb{R})} = \emptyset$$

holds (Σ^{φ} = underlying set of Markus-separatrices). Still, a nondegenerous equation (*) may have a complicated behaviour outside the strips $\mathbb{R} \times [-n,n]$ parallel to the first axis and so we need some restrictions on these domains as follows:

(P_1) For every finite interval $I \subset \mathbb{R}$ the set

$$Zero(k) \cap I \times \mathbb{R} \subset \mathbb{R}^2$$

is bounded,

(P_2) For every finite interval $I \subset \mathbb{R}$ there exists a function

$$M_I : \mathbb{R} - (-L, L) \to \mathbb{R}_+$$

with some sufficiency large number L, for which

$$\int_L^{\infty} \frac{1}{M_I} = \int_{-\infty}^{-L} \frac{1}{M_I} = \infty$$

and

$$\sup_{s,t \in I} |k(t,x_2) - k(s,x_2)| \leq M_I(x_2)|x_2|, \quad |x_2| \geq L,$$

hold. Either (P_1) or (P_2) ensure that the induced dynamical system is parallelizable outside a sufficeintly wide strip $\mathbb{R} \times [-n,n]$.

Theorem 1. Let the nondegenerous equation (*) be triangularizable and denote $(\mathbb{R}^2, \varphi)$ its induced dynamical system. Then (Δ_1) $|L^{\eta}_{\varphi}(p)| \leq 1$ for every $p \in \mathbb{R}^2$ and $\eta \in \{-,+\}$, where $L^{\eta}_{\varphi}(p) = \eta$ -limit set of $\{p\}$,

(Δ_2) $(\mathbb{R}^2, \varphi)$ has no saddle with multiplicity > 2.

(Δ_3) $(\mathbb{R}^2, \varphi)$ has no invariant simple closed Jordan curve γ on which the parametrization of trajectories give rise to an orientation of γ.

(Δ_4) There are no points $p, q \in \mathbb{R}^2 - C^{\varphi}$ for which $p \in \mathfrak{J}^{+}_{\varphi}(q)$ and $q \in \mathfrak{J}^{+}_{\varphi}(p)$ hold.

For the low critical point case the converse of the statement above is also valid, namely:

Theorem 2. Assume that the nondegenerous equation (*) with $|C^{\varphi}| \leqq 2$ fulfils one of the conditions (P_1) or (P_2). Then (*) is triangularizable iff (Δ_i), $i=1,2,3$, hold.

As applications of the results above, by (Δ_3), the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \qquad \mu > 0,$$

is nontriangularizable and by an analysis of the asymptotic behaviour of the trajectories, the Emden-Fowler equation

$$\ddot{x} + (2\mu-1)\dot{x} + \mu(\mu-1)\ (x - sgnx\cdot|x|^n) = 0$$

is triangularizable, provided that $\sigma+n+1<0$, $n \geqq 3$, $n \in N$, where $\mu = -\frac{\sigma+2}{n-1}$

REFERENCES

[1] G. Tóth, Triangularization methods in the transformation theory of planar dynamical systems Period. Math. Hungar. Vol. 11(3),(1980), pp. 197-211, II. ibid. Vol. 11(4),(1980), pp. 298-308, III.ibid. (to appear)

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

STABILITY OF PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS WITH RANDOM PARAMETERS

TRAN VAN NHUNG

Department of Mathematics Faculty of
Mechanical Engineering Technical University

The aim of this paper is to consider the stability of periodic solutions of ordinary differential equations with random parameters under stochastic perturbations of their coefficients. Generalizing some results of H. Bunke [2] and applying a theorem of A. Strauss and J.A. Yorke [5] we shall show that if the coefficients of a random system converge to the periodic stochastic processes as $t \to \infty$, then all its solutions also converge to a determined periodic solution of the limit system.

The concepts and results used in this paper can be found in [1] - [5]. The details of the proofs of the theorems formulated here can be seen in [6] and [7].

1. One has proved ([2], Theorem 3.13, p. 51) under some conditions that any solution of the perturbed system

$$\dot{y} = [A(t) + C(t)]y + z_t, \tag{1.1}$$

where $t \in R^1$, $y \in R^n$, $A(t)$ and $C(t)$ are deterministic $n \times n$ -matrices and z_t is an n-dimensional stochastic process, converges a.e. to a determined periodic solution x_t^o of the periodic unperturbed system

$$\dot{x} = A(t)x + z_t \tag{1.2}$$

as $t \to \infty$ if the perturbation $C(t)$ converges exponentially to zero as $t \to \infty$.

In [6] we have investigated the stability of the periodic solution x_t^o of (1.2) under action of a general nonlinear stochastic perturbation. The equivalent problem is to consider the asymptotic behaviour of solutions of system

$$\dot{y} = A(t)y + z_t + f(t,y,\omega), \tag{1.3}$$

where $f\colon R^1 \times R^n \times \Omega \to R^n$, $(\Omega,\mathfrak{U},P)$ is a probability space. In order to ensure the existence and uniqueness of solutions of (1.3) we assume that $f(t,y,\omega)$ is $\mathfrak{U}$ - measurable for all $(t,y) \in R^1 \times R^n$ and R-continuous on $R^1 \times R^n$, and there exists an R-continuous stochastic function $L(t,\omega)$ for which $\| f(t,y_1,\omega)-f(t,y_2,\omega) \| \leq L(t,\omega) \| y_1 - y_2 \|$ holds a.e. for $t \in R^1$, $y_1, y_2 \in R^n$.

Using Lyapunov's theorem on the reducibility of linear periodic systems ([3], p. 188) and a theorem of A. Strauss and J.A. Yorke ([5], Theorem 3.2), in [6] we have proved the following

Theorem 1. Assume that conditions 1,3 of Theorem 3.13 in [2] and following conditions are satisfied:

(i) Given any $\varepsilon > 0$, there exist random variables $\sigma = \sigma(\varepsilon,\omega) > 0$ and $S = S(\varepsilon,\omega) > 0$ such that

$$\|f(t,y_1,\omega) - f(t,y_2,\omega)\| \leq \varepsilon \; \|y_1 - y_2\| \qquad \text{(a.e.)}$$

provided $t \geq S$ and $\|y_1 - y_2\| < \sigma$.

(ii) $$\lim_{t\to\infty} \int_t^{t+1} \| f(s,x_s^o,\omega)\| \; ds = 0 \qquad \text{(a.e.)}.$$

Then there exist random variables $\delta = \delta(\omega) > 0$ and $T = T(\omega) \geq 0$ such that, for every $t_o \geq T$ and $y_o(\omega)$ with $\|y_o(\omega) - x^o_{t_o}(\omega)\| < \delta$ (a.e.), any solution

$y_t = y(t,\omega;\ t_o, y_o(\omega))$ of (1.3) converges a.e. to x^o_t as $t\to\infty$.

Applying Theorem 1 to linear system

$$\dot{y} = [A(t) + C_t]y + z_t + \zeta_t , \tag{1.4}$$

where C_t is an R-continuous stochastic $n \times n$ - matrix and ζ_t is an R-continuous n-dimensional stochastic process, in [6] we get.

Corollary. Suppose that conditions 1,3 of Theorem 3.13 in [2] and following conditions are satisfied:

(i) $\lim\limits_{t\to\infty} \|C_t\| = 0$ (a.e.).

(ii) There exist a number $\bar{t} \geq 0$ and a random variable $h = h(\omega)$ such that for $t \geq \bar{t}$ $\|z_t\| \leq h$ (a.e.) holds.

(iii) $\lim\limits_{t\to\infty} \int_t^{t+1} \|\zeta_s\| \, ds = 0$ (a.e.).

Then any solution of (1.4) converges a.e. to x^o_t as $t \to \infty$.

Some sufficient conditions for the convergence in mean of solutions of (1.3) and (1.4) to x^o_t are also given in [6].

2. Generalizing a theorem of A. Ja. Dorogovcev, H. Bunke has shown ([2], Theorem 5.15, p. 120) that under some determined conditions a system of weakly nonlinear periodic random differential equations

$$\dot{x} = f(x,t,z_t) + g(x,t,z_t) \qquad (2.1)$$

has a strictly periodic solution x_t^o and any solution x_t of (2.1) converges exponentially a.e. to x_t^o.

Using generalized Gronwall's lemma ([2]) and an inequality of A. Strauss and J.A. Yorke ([5], Lemma 3.5) we have proved in [7] the following

Theorem 2. Suppose that conditions 1,2,3 and 4 of Theorem 5.15 in [2] and following condition are satisfied: $h \in C[R^n \times R^1 \times R^m \to R^n]$ and there is a function $\Psi : R^1 \times R^m \to R^1$ such that $\int_t^{t+1} \Psi(s,z_s)ds$ is R-continuous, $\| h(x,t,z_t) \| \leq \Psi(t,z_t)$ (a.e.) for all $(x,t) \in R^n \times R^1$ and

$$\lim_{t\to\infty} e^{\delta t} \int_t^{t+1} \Psi(s,z_s)ds = 0 \qquad \text{(a.e.)} \qquad (2.2)$$

with some $\delta > 0$ hold. Then any solution x_t of

$$\dot{x} = f(x,t,z_t) + g(x,t,z_t) + h(x,t,z_t) \qquad (2.3)$$

converges exponentially a.e. to the solution x_t^o of (2.1).

If instead of (2.2) we only suppose that

$\lim_{t\to\infty} \int_t^{t+1} \Psi(s,z_s)ds = 0$ (a.e.), then any solution x_t of (2.3) converges a.e. to x_t^o, but in general not exponentially.

The results of this paper were applied to the vibration equation with random parameters ([8]).

The author wishes to express his thanks to Prof. Dr. M. Farkas and Dr. Zs. Lipcsey for reading the manuscript and for giving valuable comments.

REFERENCES

[1] Arnold L., Stochastic Differential Equations, Wiley (New York, 1974).

[2] Bunke H., Gewöhnliche Differentialgleichungen mit zufälligen Parametern, Akademie (Berlin, 1972).

[3] Demidovich B.P., Lectures on Mathematical Theory of Stability, Nauka (Moscow, 1967). (in Russian)

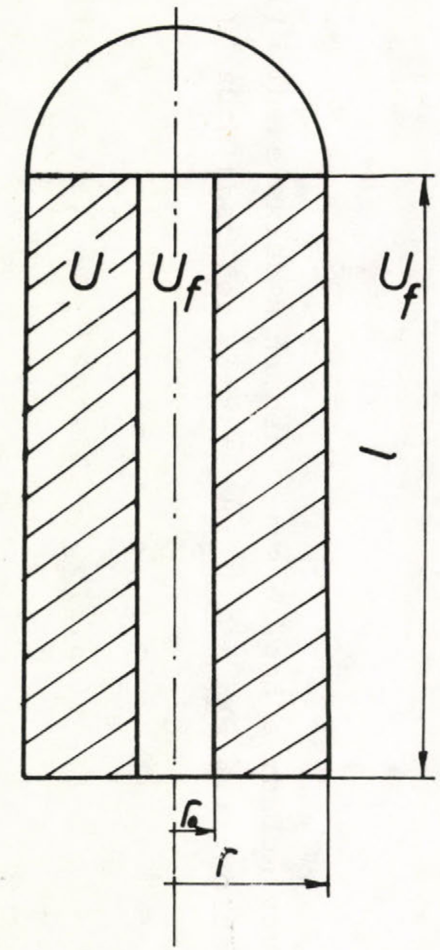
[4] Khas'minskii R.Z., Stability of Systems of Differential Equations with Random Parametric Excitation, Nauka (Moscow, 1969). (in Russian)

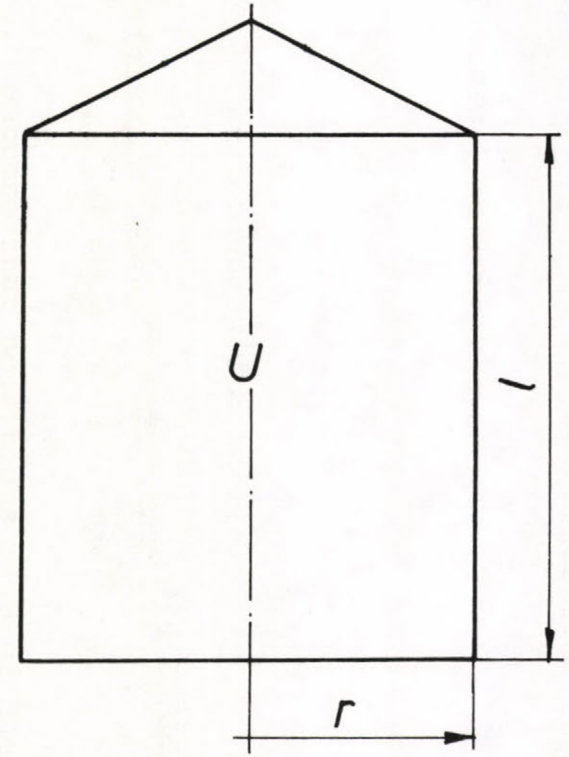
[5] Strauss A. and Yorke J.A., Perturbation Theorems for Ordinary Differential Equations, J. Differential Equations, 3(1967), 15-30.

[6] Tran Van Nhung, Über das asymptotische Verhalten von Lösungen gewöhnlicher asymptotisch periodischer Differentialgleichungen mit zufälligen Parametern, to appear in Acta Math. Acad. Sci. Hungar. 39(1982), N. 3-4.

[7] Tran Van Nhung, Über Lösungen asymptotisch periodischer stochastischer Systeme, to appear in Annales Univ. Sci. Budapest, Sectio Math.

[8] Tran Van Nhung, On Stability of Solutions of the Random Vibration Equation, to appear.


U
U_f
U_f
l
r


U
l
r

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

EQUATIONS OF ACTIVITY TRANSPORT IN PRESSURIZED WATER REACTORS

SZABOLCS VASS, ROBERT SCHILLER

Central Research Institute for Physics

H-1525 Budapest, P.O.B. 46.

OLIVÉR NAGY

Institute for Electrical Power Research

H-1368 Budapest, P.O.B. 323.

SUMMARY

Activity transport in a closed circulating flow system divided into two different sectors is described by two sets of ordinary first order nonlinear equations. In most practical cases nonlinearity is neglected and solutions for consecutive circulation periods are given by a recursive formula which can easily be used for numerical calculations.

MODEL AND EQUATIONS

In primary cooling systems of nuclear or ordinary power plants a coolant circulates in order to transport heat from the active zone - the boiler - to the turbines. As a result of physico-chemical processes, substance is released from the walls of tubes into the coolant and, after having been transported to another location of the cooling system, it is deposited on the walls again. During the passage through the reactor zone the substance is irradiated and its nuclei are activated thus in addition to heat transport an undesirable activity transport occurs as well.

Model parameters are concentrations $x = (x_1 \ldots x_n)$ and $y = (y_1 \ldots y_n)$ of the components on the internal surface of tubes and in the coolant, respectively; n is the number of different material components. Processes influencing the change of x and y are divided intot two groups:

1) Material transport across coolant/wall interface: these processes cause temporal change of the model parameters and are described by nonlinear first order partial equations.

2) Material transport along the cooling tubes is usually described by first order partial equations with respect to spatial coordinates [1].

Due to the complexity of the problem, a comprehensive model which takes into account all possible processes is not available yet. Our aim is to develop approximate mathematical methods for the discussion of the activity growth in terms of technical parameters in the whole cooling circuit. During one circulation period concentrations have a small change along the tubes. As a first approximation, concentrations are supposed to be independent of spatial coordinates and are replaced by their averages along the tubes influenced by transport processes across the coolant/wall interface. After this simplification the problem is described by the following two sets of ordinary first order nonlinear differential equations [2]:

in the reactor zone

$$\frac{d}{dt}\begin{pmatrix} x^z \\ \\ y^z \end{pmatrix} = A^z \begin{pmatrix} x^z \\ \\ y^z \end{pmatrix} + a^z + f^z(x^z, y^z)$$

$$0 \leq t \leq \tau_z$$

in the inactive sector

$$\frac{d}{dt}\begin{pmatrix} x^h \\ \\ y^h \end{pmatrix} = A^h \begin{pmatrix} x^h \\ \\ y^h \end{pmatrix} + a^h + f^h(x^h, y^h) \qquad (1)$$

$$0 \leq t \leq \tau_h$$

where A^z and A^h are $2n \times 2n$ matrices, a^z and a^h are $2n$ component vectors, vector functions f^z and f^h stand for nonlinear terms, τ_z and τ_h are average residence times of the coolant in the zone and in the inactive sector, respectively.

Equations (1) are to be solved consecutively for circulating cycles $i = 1,2,3,...$ in such a way that initial conditions for any cycle are given by the solutions to the preceeding one:

$$x_i^z(0) = x_{i-1}^z(\tau_z) \qquad x_i^h(0) = x_{i-1}^h(\tau_h)$$

$$y_i^z(0) = y_{i-1}^h(\tau_h) \qquad y_i^h(0) = y_i^z(\tau_z) \qquad (2)$$

LINEAR APPROXIMATION

In most practical cases nonlinear terms in Eq. (1) are to be neglected, making possible to apply matrix algebraic methods and to discuss simultaneously a great number of chemical components. The concentration changes during a circulation are small hence it is sufficient to determine the dependence of model parameters on the serial number of cycles i, $i = 1,2,3$ etc. Introduce the $4n$ component vector $w_i(\tau_z, \tau_h) = (x_i^z(\tau_z), y_i^z(\tau_z), x_i^h(\tau_h), y_i^h(\tau_h))$; then solutions to the linearized equations (1) are given in the following form:

$$w_i(\tau_z,\tau_h) = \begin{pmatrix} R^z(\tau_z) & 0 \\ 0 & R^h(\tau_h) \end{pmatrix} w_i(0,0) + \begin{pmatrix} Q^z(\tau_z) & 0 \\ 0 & Q^h(\tau_h) \end{pmatrix} \begin{pmatrix} a^z \\ a^h \end{pmatrix} =$$

$$i = 1,2,3,\ldots \qquad = R^w(\tau_z,\tau_h) w_i(0,0) + Q^w(\tau_z,\tau_h) a^w \tag{3}$$

where R^z, R^h and Q^z, Q^h are $2n \times 2n$ matrices as $R^z(\tau) = exp\ A^z(\tau)$, $Q^z(\tau) = \int_0^\tau R^z(\tau)(R^z(s))^{-1} ds$ (see text-books, [3]) etc.

Let now Ω_x be a diagonal matrix containing $2n$ elements; one-s in the upper n, and zeros in the lower n positions and let $\Omega_y = E - \Omega_x$. Using these matrices, initial conditions (2) can be rewritten as

$$w_i(0,0) = \begin{pmatrix} \Omega_x & \Omega_y \\ \Omega_y R^z(\tau_z)\Omega_x & \Omega_x + \Omega_y R^z(\tau_z)\Omega_y \end{pmatrix} w_{i-1}(\tau_z,\tau_h) + \begin{pmatrix} 0 \\ \Omega_y Q^z(\tau_z) a^z \end{pmatrix}$$

$$= S(\tau_z) w_{i-1}(\tau_z,\tau_h) + \begin{pmatrix} 0 \\ \Omega_y Q^2(\tau_z) a^z \end{pmatrix} \qquad i =,1,2,3,\ldots \tag{4}$$

Substituting $w_i(0,0)$ from Eq. (4) into Eq. (3), a simple linear dependence between $w_i(\tau_z,\tau_h)$ and $w_{i-1}(\tau_z,\tau_h)$ is found:

$$w_i(\tau_z,\tau_h) = \begin{pmatrix} R^z(\tau_z) & 0 \\ 0 & R^h(\tau_h) \end{pmatrix} \begin{pmatrix} \Omega_x & \Omega_y \\ \Omega_y R^z(\tau_z)\Omega_x & \Omega_x + \Omega_y R^z(\tau_z)\Omega_y \end{pmatrix} w_{i-1}(\tau_z,\tau_h) +$$

$$+ \begin{pmatrix} Q^z(\tau_z) & 0 \\ \Omega_y Q^z(\tau_z) & Q^h(\tau_h) \end{pmatrix} \begin{pmatrix} a^z \\ a^h \end{pmatrix} = R^w(\tau_z,\tau_h)S(\tau_z)\cdot w_{i-1}(\tau_z,\tau_h) +$$

$$+ T(\tau_z,\tau_h)a^w \qquad (5)$$

Using Eq. (5), w_i can explicity be given as

$$w_i(\tau_z,\tau_h) = (R^w(\tau_z,\ \tau_h)S(\tau_z))^i\ w_o + \sum_{j=o}^{i-1} (R^w(\tau_z,\tau_h)S(\tau_z))^j\ T(\tau_z,\tau_h)a^w \qquad (6)$$

where w_o is composed of initial concentrations. Equation (6) can be simplified by introducing the normal form of matrix $R^w s = U < \lambda > V'$ as follows:

$$w_i = (U < \lambda^i > V')w_o + \ U < \frac{\lambda^i - 1}{\lambda - 1} > V')T\ a^w \qquad (7)$$

After having solved three eigenproblems (two times for $2n \ x \ 2n$ sparse matrices, once for a $4n \ x \ 4n$ matrix) calculations can effectively be carried out without accumulated errors.

REFERENCES

[1] K.A. Burill: Corrosion product transport in water-cooled nuclear reactors, Prt I: Pressurized water operation. Canadian J. Chem. Eng. 55, 54 /1977/

[2] Sz. Vass, I. Kules, M. Roder, R. Schiller, Zs. Ammbrus and O. Nagy: Nonlinear equations of activity adsorption in the primary coolant of nuclear reactors (in Russian). Report KFKI-1979-59, Budapest, 1979.

[3] P. Rózsa: Linear algebra and its applications (in Hungarian). Müszaki Könyvkiadó, Budapest, 1974.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

THE STABILITY AND CONVERGENCE OF GENERAL ONE-STEP METHODS FOR THE NUMERICAL SOLUTION OF VOLTERRA FUNCTIONAL DIFFERENTIAL EQUATIONS

MÁRIA VICSEK

Computer and Automation Institute
Hungarian Academy of Sciences

INTRODUCTION

In this paper a generalization of Tavernini's results [1] on the numerical solution of one-step methods of Volterra functional differential equations is given for the case of variable stepsize.

Let us consider the initial value problem

(1) $y'(x) = F(x, y),$ $\qquad a < x \leq b,$

(2) $y(x) = g(x),$ $\qquad \alpha \leq x \leq a,$

where g is a given initial function continuous on $[\alpha, a]$ and the functional $F:[\alpha,b]\times C_n[a,b]\to R^n$ is such that the conditions of the existence and uniqueness theorem for the solution are fulfilled (Driver [2]), namely the following assumptions hold:

i) for fixed y, $F(x,y)$ is continuous in x for $x \in [a,b]$,

ii) the functional F satisfies the Lipschitz condition

$$|| F(x,y_1)-F(x,y_2) || \leq L \, || y_1 - y_2 || ^{[\alpha,x]}$$

for any $y_1, y_2 \in C_n[\alpha,b]$ and x in $[a,b]$ where L is constant.
Here for real x_1 and x_2 with $x_1 \leq x_2$, $C_n[x_1,x_2]$ denotes the space of continuous functions on $[x_1,x_2]$ and for $y \in C_n[x_1,x_2]$ the norm is defined as

$$\| y_1 - y_2 \|^{[\alpha,x]} = \max_{\alpha \leq s \leq x} \| y_1(s) - y_2(s) \| .$$

A consequence of ii) is that $F(x,y)$ is independent of $y(s)$ for $s > x$, i.e. F is a Volterra functional. Obviously, this class of functional differential equations contains for example the class of ordinary differential equations, delayed differential equations, etc.

NOTATION AND DEFINITIONS

Let us introduce the grid $\Delta_N = \{x_0, x_1, \ldots, x_N\} \in [a,b]$ and the norm $\|\Delta_N\| = \max_{0 \leq i \leq N-1} h_i$ where $h_i = x_{i+1} - x_i$.

The solution of (1), (2) is always denoted by y.

Definition 1.

We call a general one-step method the following:

(3) $\quad y(x_j + r\,h_j) = y(x_j) + rh_j \Phi(x_j, h_j, y_j, r), \quad x_i \in \{\Delta_N\}, \quad r \in [0,1], \quad 0 < i \leq N$

(4) $\quad y(x) = \tilde{g}(x), \qquad x \in [\alpha, a],$

where $\Phi: S \times C_n[\alpha,b] \times [0,1] \to R^n$ is the increment function of the method and $S = \{(x,\delta) \mid \alpha \le x \le b$ and $0 < \delta \le b-x\}$ and $\tilde{g}$ is some continuous approximation to the initial function g.

In the following we suppose $\Phi(x,h,y,r)$ to be continuous in r for fixed x,y,h and let Φ satisfy the Lipschitz condition, namely

$$\|\Phi(x,h,y_1,r) - \Phi(x,h,y_2,r)\| \le K \, \|y_1 - y_2\|^{[\alpha,x]} ,$$

where K is constant.

It can be seen that the numerical solution of (1) is defined by the method (3) everywhere in the interval of integration and not only in the gridpoints as it is typical when solving ordinary and partial differential equations.

Definition 2.

The general one-step method (3) is called to be convergent, if for every gridpoint

$$\|y - \tilde{y}\| \to 0 \quad \text{if} \quad \|\Delta_N\| \to 0, \quad \|g - \tilde{g}\| \to 0$$

Definition 3.

The truncation error τ of the method (3) at $x_i + rh_i$ is defined as follows:

$$\tau(x_i,h_i,r) := y(x_i) + h_i \, \Phi(x_i,h_i,y,r) - y(x_i + rh_i).$$

Φ is supposed to be such that the truncation error satisfies the condition

$$|\tau\ (x_i, h_i, r)| \leq h_i \quad \varepsilon(x, r, h)$$

where $\varepsilon(x,r,h)$ is a given error function.

Definition 4.

The method (3) is called consistent if

$$\Phi(x, h, y, 1) \to F(x, y), \qquad (\|\Delta_N\| \to 0)$$

uniformly in x, or (an equivalent condition)

$$\varepsilon(x, 1, h) \to 0 \qquad (\|\Delta_N\| \to 0)$$

for all $x \in [a,b]$.

It might be regarded as a more natural generalization of the consistency, if we suppose, that

$$\varepsilon(x, r, h) \to 0 \qquad (\|\Delta_N\| \to 0)$$

for all $r \in [0,1]$. However one can see that the error is simply accumulated at the gridpoints while in other points it is multiplied by rh_i.

Definition 5.

The global error of the method (3) is

$$\|y - \tilde{y}\|^{[\alpha, b]}$$

Next we define the function

$$\nabla(x, \delta) = \begin{cases} \frac{1}{\delta}\,[y(x+\delta) - y(x)], & \delta > 0, \\ y'(x), & \delta = 0 \end{cases}$$

for all $(x, \delta) \in S$.

Finally, let us have the

Definition 6.

The method (3) is called to be stable, if there are constants $c, h^* > 0$ such that

$$\|y^* - y\|^{[\alpha,b]} \leq c \{ \|g^* - \tilde{g}\| + \sum_{j=1}^{N} \|\delta_j\| \}$$

provided that $\|\Delta_N\| \leq h^*$ and y^* is the solution of the perturbed recursion

$$y^*(x_i + rh_i) = y^*(x_i) + rh_i[\Phi(x_i, h_i, y^*, r) + \delta_{i+1}].$$

Theorems

In the remaining part of this paper we give some convergence and stability results for general one-step methods (3).

Theorem 1.

Assume that

i) the increment function Φ is such that the truncation error satisfies the inequality

$$\|\Phi(x,h,y,r) - \nabla(x,rh)\| \leq \varepsilon(x,r,h)$$

for all $x \in \{\Delta_N\}$, $r \in [0,1]$ and $0 < h \leq h^*$,

ii) ε is a monotone function of h and $\varepsilon(x,r,h) \leq \bar{\varepsilon}(x,r,h^*) \leq \varepsilon_1$ (ε_1 is constant)

iii) furthermore

$$\bar{\varepsilon}(x,1,h^*) \to 0 \qquad (||\Delta_N|| \to 0).$$

Then the error is bounded by

$$||y - \tilde{y}||^{[\alpha,b]} \leq [||g - \tilde{g}||^{[\alpha,a]} + h^* \varepsilon_1 +$$

$$+ \int_a^b \bar{\varepsilon}(x,1,h^*)dx] [1 + (b - a)Ke^{K(b-a)}].$$

By the help of the above estimation we obtain the following convergence and stability result.

Theorem 2.

If the general one-step method (3)

i) is consistent and

ii) the increment function Φ is uniformly bounded in x and r, for all $||\Delta_N|| \to 0$

$$|| \Phi(x,h^*,y,r) || \leq M, \qquad M \text{ - constant},$$

then it is convergent for any $\{\Delta_N\}_{N=1}^{\infty}$ such that

$$||\Delta_N|| = \max_{o<i\leq N-1} h_i = h^* \to 0 \quad \text{and} \quad ||g - \tilde{g}|| \to 0.$$

Theorem 3.

The general one-step methods (3) are stable.

The proofs of these theorems can be found in [3].

REFERENCES

[1] Tavernini, L.: One-step Methods for the Numerical Solutions of Volterra Functional Differential Equations. SIAM J. Numer. Anal. V.8.N4. Dec. 1971 pp. 786-795.

[2] Driver, R.D.: Existence and stability of solutions of a delay-differential system. Arch. Rational Mech. Anal. 10(1962) pp. 401-426.

[3] Vicsek M.: Numerical solution of initial value problems (in Hungarian) Doct. thesis and Techn. Rep. of Computer and Automation Institute Nr. 121, Budapest, 1981.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 26/1982

PARAMETER IDENTIFICATION FOR INITIAL VALUE PROBLEMS BY MAXIMUM LIKELIHOOD METHOD

JÓZSEF VIGASSY

MAGDOLNA DÚS

ZSUZSANNA ÉZSÖL

Central Research Institute for Physics

H-1525 Budapest P.O.B.49.

SUMMARY

The Maximum Likelihood Method is a well-known tool for curve fitting on measured points. It supplies not only the unknown parameters of curves, but their standard deviation values too. For the case when the curve cannot be given by an explicit function but only by an initial value problem, the report gives an iteration procedure to solve the likelihood equations. In the core of this procedure there is an extended initial value problem to be solved.

INTRODUCTION

In the following we want to solve a problem of practical importance and to discuss some questions connected to this.

One kind of the - so called - parameter identification problem can be formulated as follows:

Let the initial value problem for the unknown function $f(t)$ given by the formulae (1)

$$\left.\begin{aligned} a_1 \frac{d}{dt} f(t) + a_2\{f(t) - p(t)\} &= N(t) \\ f(t)\Big|_{t=t_o} = f_o &= a_3 \end{aligned}\right\} \qquad (1)$$

where t is the time variable. The parameters a_1, a_2 and a_3 are now unknown. Let y_i a set of measured values in the moments t_i known. Our task is to choose the a_1, a_2 and a_3 unknown parameters so, that the solution of the problem (1), the $f(t)$ function should fit in some sense the y_i measured points. It can be seen, that this problem is a curve fitting problem without having an analytic function describing the curve.

An extension of the problem (1) can be written as follows:

$$\left.\begin{aligned} \vec{A}(\vec{a})\cdot\vec{f}(t) &= \vec{N}(\vec{a},t) \\ \vec{f}(t)\Big|_{t=t_o} &= \vec{f}_o \end{aligned}\right\} \qquad (1^*)$$

where $\vec{f} = [f_1(t), f_2(t), \ldots, f_L(t)]$ is an unknown vector function with L components, $\vec{a} = [a_1, a_2, \ldots \ldots, a_J]$ is an unknown vector with J components, A is a matrix of differential operators. The measured values can form the matrix

$$\|y_{\ell,i}\|$$

which can have unknown elements too.

In practice the problem mainly arises when analysing regulator equipments and control processes. In this case methods exist for choosing the right-hand side function $N(\vec{a},t)$ in such a way, that estimates for the unknown parameters can be determined by a rapid evaluation of the measurements. In other cases you can solve the problem (1) *analytically* and the unknown parameters can be determined by fitting the analytic function on the measured points using standard methods.

In the problem to be discussed here, we have had the difficulty, that we could not use neither of the recipes mentioned. The y_i values were temperature values measured in a nuclear reactor after its shutdown. The function $N(t)$ could not be chosen freely, because this was determined by the physical laws of the radioactive decay chains, and at last, $N(t)$ was given numerically. Therefore we had to try to apply somehow the standard fitting methods, but without knowing any analytical form for the function $f(t)$.

THE MAXIMUM LIKELIHOOD METHOD

A measure how the function $f(t)$ fits the measured point y_i can be described by the quantity

$$\left\{ \frac{y_i - f(t_i)}{\sigma_i} \right\}^2$$

where σ_i is the so called standard deviation belonging to the measuring process for y_i. It is desired that this measure be small for all points simultaneously. This can be achieved if you maximize the quantity with respect to $\vec{a}$:

$$\mathcal{L}(\vec{a}) = \frac{1}{(2\pi)^{I/2} \prod_I \sigma_i} \exp \left\{ - \frac{1}{2} \sum_I \frac{1}{\sigma_i^2} (y_i - f(t_i))^2 \right\}$$

This is called the likelihood function. From the point of view of statistics it was supposed for this, that the measured values y_i have Gaussian distribution with $f(t_i)$ mean value and σ_i standard deviation. The maximum likelihood method - in this case - comes over to the least squares method by the notation

$$Q = - \ln \left\{ (2\pi)^{I/2} \cdot \prod_I \sigma_i \cdot \mathcal{L}(\vec{a}) \right\} \qquad (2a)$$

as it is well known.

The standard method for solving this problem is to give a solution for the so called likelihood equation system. Using the notation

$$a_k = \begin{cases} a_j & \text{if} \quad 1 \leq k = j \leq J \\ f_o & \text{if} \quad k = J+1 = K \end{cases} \qquad (3)$$

the likelihood equations have the form:

$$O = G_k(\vec{a}) = - \frac{\partial Q}{\partial a_k} = \sum_{i=1}^{I} \frac{1}{\sigma_i^2} [y_i - f(t_i)] \left. \frac{\partial f}{\partial a_k} \right|_{t_i} \qquad (4)$$

$$k = 1, 2, \ldots \ldots, j, K$$

This set of equation can be solved by iteration. Assume, that m iteration steps have already been accomplished resulting in $\vec{a}_m$. The next iterate $\vec{a}_{m+1}$ is determined in the following way.

Develope $\vec{G}(\vec{a})$ in Taylor series around $\vec{a}_m$:

$$\vec{O} = \vec{G}(\vec{a}) = \vec{G}(\vec{a}_m) + \vec{\vec{D}}(\vec{a}_m)\{\vec{a} - \vec{a}_m\} + \ldots \qquad (5)$$

where the symmetric matrix $\vec{\vec{D}}(\vec{a}_m)$ is formed of the components:

$$D_{k,k'}(\vec{a}_m) = \frac{\partial}{\partial a_{k'}} \; G_k(\vec{a}_m) = -\frac{\partial^2 Q(\vec{a}_m)}{\partial a_k \partial a_{k'}} \tag{6}$$

Solving eq. (5) for $\vec{a}$ one gets $\vec{a}_{m+1}$ as

$$\vec{a}_{m+1} = \vec{a}_m - [\vec{\vec{D}}(\vec{a}_m)]^{-1} \; \vec{G}(\vec{a}_m) \tag{7}$$

This iteration is terminated when $\vec{a}_m$ and $\vec{a}_{m+1}$ are sufficiently close to each other.

In the classical case, if the function f is given analytically, in most cases you can produce formulae for the differential coefficients $\frac{\partial f}{\partial a_k}$ with which the calculational model can work, using the formula without second derivatives in it:

$$D_{k,k'} = \sum_I \frac{1}{\sigma_i^2} \cdot \frac{\partial f}{\partial a_k} \cdot \frac{\partial f}{\partial a_{k'}} + \sum_I \frac{y_i - f(t_i)}{\sigma_i^2} \cdot \frac{\partial^2 f}{\partial a_k \; \partial a_{k'}} \tag{8}$$

In practice the second term on the right-hand side is to be neglected. This means, that the function $Q(\vec{a})$ is approximated by a positive definite quadratic form in the vicinity of the m^{th} iterated vector $\vec{a}_m$, and not by the "best" quadratic form. [See lit.] Therefore the convergence will be slower, but the domain of the "good" initial values, with which the iteration converges to the right solution, will be wider. After all the numerical and analytical work with the second derivatives falls out.

This iteration procedure has another advantage: estimates for the standard deviations of the unknown parameters are given automatically.

SOLUTION OF THE PROBLEM

In our case, where the function f is given by an initial value problem, the method mentioned can be used only, if we can give a procedure for the calculation of the quantities

$$\left.\frac{\partial f}{\partial a_k}\right|_{t=t_i}$$

The advantage of neglecting the second derivatives can be seen too.

You can extend the initial value problem given by producing a form by formal differentiation, where the first derivatives arise as unknowns too. For the problem (1*) these formulae can be written as:

$$\vec{A}\cdot\vec{g}_k = \left\{\begin{array}{ll} \vec{N}(\vec{a},t) & \text{if} \quad k=0 \\ -\dfrac{\partial\vec{A}}{\partial a_k}\cdot\vec{g}_o + \dfrac{\partial\vec{N}(\vec{a},t)}{\partial a_k} & \text{if} \quad 1\le k\le J \\ \vec{0} & \text{if} \quad k>J \end{array}\right\} \qquad (9)$$

with the initial conditions:

$$\vec{g}_k(t_o) = \left\{\begin{array}{ll} \vec{f}_o & \text{if} \quad k=0 \\ \vec{0} & \text{if} \quad 1\le k\le J \\ \vec{e}_k & \text{if} \quad k>J \end{array}\right\} \qquad (9a)$$

where $\vec{g}_o = \vec{f}$, and $g_k = \frac{\partial}{\partial a_k} \vec{f}$ if $k \neq 0$ (10)

and $\vec{e}_k$ are unit vectors properly chosen.

Because of the right-hand side term containing g_o in equation (9), it is suitable to solve this problem as a whole, using for example standard Runge-Kutta procedures.

The calculations performing the procedure proposed, must follow therefore a double nested algorithm:

1.) The outher iteration procedure is given by the formulae (7) and (8) using a matrix inversion procedure in it;

2.) The first nest calculates the summes for the matrix elements $D_{k,k'}$, running by the index i from 1 to I. The initial values for the moment t_o are taken from the results of the m-th iteration step.

3.) In the deepest level the Runge-Kutta procedure can be find, which calculates the quantities

$$\left.\frac{\partial f}{\partial a_k}\right|_{t=t_i}$$

starting from those at $t = t_{i-1}$ using the formulae (9-9a).

This complex procedure can be used if the quantities $\frac{\partial N}{\partial a_k}$ are given, which is the case of problem (1).

We have applied this method with succes using reasonable computer time, that means less than 1 minute on R-40 (Robotron-1040) for a problem, where f was scalar function, $J=2$ and $I=40$ (the number of the measured points). The convergence was fast enough, only 3 iterations were needed.

REFERENCES

Z. Szatmáry: Data evaluation problems in reactor physics.
Theory of the program RFIT.
Rep.: KFKI-1977-43.

SZÁMALK-Repro 82/165