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TARTALOMJEGYZÉK

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ON ABSOLUTE CONTINUITY OF MEASURES DEFINED BY MULTIDIMENSIONAL DIFFUSION PROCESSES WITH RESPECT TO THE WIENER MEASURE

A. Benczúr and L. Szeidl

In [5] Lipcer and Shiryaev dealt with the absolute continuity of measures generated by a diffusion type process and the Wiener process for one dimensional case. In this paper an attempt is made to carry out their result for multidimensional case. We summarize the result in tree theorems based on each other and we discuss; in details, a lemma the proof of which, for multidimensional case, needs considerations different from those used in [5]. Before this we give a concise list of preliminaries.

Let (Ω, F, P) be the basic probability space, $\{F_t \subset F,\ 0 \leq t \leq 1\}$ a monotonically nondecreasing family of σ-algebras, $w = (w_t, F_t, P)$ an n-dimensional standard Wiener process, i.e. it is an n-dimensional continuous martingal with respect to the family F_t, such that $w_0 = 0$ a.s. and

$$E[(w_t^i - w_s^i)(w_t^j - w_s^j) \mid F_s] = \delta^{ij}(t-s) \qquad t \geq s \text{ a.e. } i,j = 1,2,\dots,n$$

Let C_1 denote the space of the n-dimensional vector valued continuous functions x_t on $[0,1]$ and B_t the σ-algebra generated by cylindric sets on $[0,t]$. Further, let $\alpha_t(x)$ be a $B_{[0,1]} x B_1$, $B_{[0,1]}$ is the σ-algebra of Borel sets of interval $[0,1]$, measurable n dimensional nonanticipating functional, i.e. $\alpha_t(x)$ B_t measurable for every $0 \leq t \leq 1$.

Let

$$|\alpha_t(x)| = \sqrt{\sum_{i=1}^{n} (\alpha_t^i(x))^2}\ .$$

The n dimensional (ξ_t, F_t) process is called a process of diffusion type if there exists a nonanticipating measurable functional such that

$$P\left(\int_0^1 |\alpha_t(\xi)|\,dt < \infty\right) = 1$$

$(\alpha_t(\xi) = \alpha_t(\xi(\omega))$, $\xi(\omega) = \{\xi_t,\ 0 \leq t \leq 1\ )$ and $\xi_t = \int_0^t \alpha_s(\xi)ds + w_t$ a.s. for any $0 \leq t \leq 1$. Denote by $\mu_\xi(\mu_w)$ the measure on the space (C_1, B_1) generated by the process

$$\xi(\omega) = \{\xi_t,\ 0 \leq t \leq 1\} \qquad (w(\omega) = \{w_t,\ 0 \leq t \leq 1\})$$

i.e. for $B \in B_1$

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$$\mu_\xi(B) = P\{\omega : \xi(\omega) \in B\} \qquad (\mu_w(B) = P\{\omega : w(\omega) \in B\}).$$

Let (γ_t, F_t) be an n-dimensional stochastic process satisfying the condition

(1) $$P\left(\int_0^1 |\gamma_t|^2 dt < \infty \right) = 1$$

and such that $\zeta_t = 1 + \int_0^t \gamma_s \, dw_s = 1 + \sum_{i=1}^n \int_0^t \gamma_s^i dw_s^i$ is a nonnegative martingal with respect to (F_t, P). Introduce now a new measure $\widetilde{P}$ on the measurable space (Ω, F) by the formula

(2) $$\widetilde{P}(d\omega) = \zeta_1 P(d\omega)$$

Theorem 1. *Let* $\xi_t = -\int_0^t \frac{\gamma_s}{\zeta_s} ds + w_t$, $0 \leqslant t \leqslant 1$. *Under conditions (1) and (2)* $(\xi_t, F_t, \widetilde{P})$ *is a* *n-dimensional standard Wiener process.*

Girsanov proved, see [4], this statement for a more particular case considering

$$\zeta_t = \exp\left\{ \int_0^t \gamma_n \, dw_n - \frac{1}{2} \int_0^t |\gamma_n|^2 dn \right\}$$

Lipcer and Shiryaev, in [5], dealt with general ζ_t but for one–dimensional case. Their proof is essentially simpler than that of Girsanov. Concerning the multidimensional case one can quite easily observe that, upon replacing the ordinary scalar products by scalar products of vectors, Lipcer's and Shiryaev's arguments remain valid.

The next two theorems deal with the absolute continuity of measures generated by the process ζ_t and w_t.

Theorem 2. *Let* ξ_t *be a diffusion type process satisfying the equation*

$$\xi_t = \int_0^t \alpha_s(\xi) ds + w_t, \qquad \xi_0 = 0$$

and suppose that

(3) $$P\left(\int_0^1 |\alpha_t(w)|^2 dt < \infty \right) = 1$$

The measure μ_w *is absolutely continuous with respect to* μ_ξ $(\mu_w \ll \mu_\xi)$ *and*

$$\frac{d\mu_w}{d\mu_\xi}(w) = \exp\left\{ -\int_0^1 \alpha_t(w) dw_t + \frac{1}{2} \int_0^1 |\alpha_t(w)|^2 dt \right\}, P \text{ a.s.}$$

Under the condition $F_t^\xi = F_t^w$ *(3) is necessary for the absolute continuity.*
If x_t is not a Wiener trajectory, i.e. the stochastic integral in the exponent has no meaning, the Randon–Nikodym derivative equals to 0. This remark can be correctly explained by the notion of generalized to integral introduced by Lipcer and Shiryaev in one–dimensional case.

Their considerations remains valid without any change for multidimensional case as well. An analogous theorem holds for $\frac{d\mu_\xi}{d\mu_w}$:

Theorem 3. *Let* ξ_t *be the same process as in theorem 2. The condition*

$$P\left(\int_0^1 |\alpha_t(\xi)|^2 dt < \infty\right) = 1$$

is necessary and sufficient for the absolute continuity of μ_ξ *with respect to* μ_w, *and*

$$\frac{d\mu_\xi}{d\mu_w}(\xi) = \exp\left\{\int_0^1 \alpha_t(\xi)d\xi_t - \frac{1}{2}\int_0^1 |\alpha_t(\xi)|^2 dt\right\}.$$

Concerning the meaning of this formula we should make again an analogous remark. The proofs of the sufficiency of condition in theorem 2. and 3. do not require any changes in Lipcer's and Shiryaev's proof but for proving its necessity we have to generalize a lemma used by them.

Lemma 1. *Let* (Ω, F, P) *be a probability space, and* $w = (w_t, F_t, P)$ *be a standard n-dimensional Wiener process. If* $\zeta(\omega)$ *is a* F_1^w *measurable random variable (one–dimensional) with* $E|\zeta(\omega)| < \infty$ *then there exists a measurable nonanticipating* $\gamma_t(w)$ *vector functional, such that* $\int_0^1 |\gamma_t(w)|^2 dt < \infty$ *with probability 1 and for the martingale* $\zeta_t = E(\zeta(\omega)|F_t^w)$ *for every* $t \geqslant s$ *P almost everywhere*

$$\zeta_t - \zeta_s = \sum_{i=1}^{n} \int_s^t \gamma_n^i(w)dw_n^i \,. \tag{4}$$

This result is due to Clark [1] for one–dimensional case and to Kunita and Watanabe [2] for multidimensional case, but under a bit stronger condition.

Proof. As the σ–algebra F_t^w is continuous, the martingale ζ_t is P a.e. continuous (see [3]).

Set $\tau_N = \inf\limits_{0 \leqslant t \leqslant 1} \{t : |\zeta_t| = N\}$, $\tau_N \cap t = \min(t, \tau_N)$ and put $\zeta_N(t) = \zeta_{\tau_N \cap t}$. τ_N is obviously a Markov–point and so $\zeta_N(t)$ is a martingale

$$\sup_{0 \leqslant t \leqslant 1} |\zeta_N(t)| \leqslant N, \quad P \text{ a.e.}$$

The process $\zeta_N(t)$ is continuous and square integrable, therefore according to [2] it can be represented in the form

$$\zeta_N(t) = \sum_{i=1}^{n} \int_0^t \gamma_N^i(s, w)dw_s^i$$

where $\gamma_N^i(s, w)$, $i = 1, \ldots, n$ a square integrable F_t^w measurable for every t.

On the set

$$\chi_N(t) = \{\omega : \sup_{0 \leqslant s \leqslant t} |\zeta_s| \leqslant N\} \quad \text{for} \quad M > N$$

we have

$$\zeta_N(s) = \zeta_M(s) \qquad 0 \leqslant s \leqslant t \ \text{ a.e., i.e.}$$

$$\int_0^t \chi_N(s)(\zeta_N(s) - \zeta_M(s))^2 ds = \int_0^{\tau_N} (\zeta_N(s) - \zeta_M(s))^2 ds = \int_0^{\tau_N} (\zeta_s - \zeta_s)^2 ds = 0.$$

Define for every $1 \leqslant i \leqslant n$ the functional $\gamma^i(t, w)$ by $\gamma^i_1(t, w)$ on the set $\{\omega : 0 \leqslant \sup_{0 \leqslant s \leqslant t} |\zeta_s| \leqslant 1\}$ and by $\gamma^i_2(t, w)$ on the set $\{\omega : 1 \leqslant \sup_{0 \leqslant s \leqslant t} |\zeta_s| \leqslant 2\}, \ldots$ and so on. $\gamma^i(t, w)$ is, for every i obviously measurable process and for any fixed t is F^w_t measurable. Moreover

$$\left\{\omega : \sum_{i=1}^n \int_0^1 (\gamma^i(t, w))^2 dt = \infty\right\} \subset \left\{\omega : \int_0^1 \sum_{i=1}^n (\gamma^i(t, \omega) - \gamma^i_N(t, w))^2 dt > 0\right\}$$

$$\subset \{\omega : \sup_{0 \leqslant s \leqslant t} |\zeta_s| > N\}$$

The probability of the last set tends to zero as $N \to \infty$, so $\sum_{i=1}^n \int_0^1 (\gamma^i(t, w))^2 dt < \infty$ P a.e.

Thus the to integral $\int_0^t \gamma^i(s, w) dw_s$ can be correctly defined for every t. By the virtue of a well known property of to integral

$$P\left\{\left|\int_0^t (\gamma_N(s, w) - \gamma(s, w)) dw_s\right| > 0\right\} \leqslant P\left\{\int_0^t |\gamma_N(s, w) - \gamma(s, w)| ds > 0\right\} \to 0$$

as $N \to \infty$.

From this it follows that $\zeta_N(t)$ stochastically converges to $\int_0^t \gamma(s, w) dw_s$. Since $\lim_{N \to \infty} \zeta_N(t) =$ $= \zeta_t$ in probability so that

$$\zeta_t = \int_0^t \gamma(s, w) dw_s.$$

The uniqueness of the representation can be easily proved:

Let $\gamma_1(t, w)$, $\gamma_2(t, w)$ be functionals for which the representation (4) holds. Then for any $0 \leqslant t \leqslant 1$ applying to formula to

$$\eta_t^2 = \left(\int_0^t (\gamma_1(s, w) - \gamma_2(s, w)) dw_s\right)^2$$

we have

$$0 = \eta_t^2 = \sum_{i=1}^n \int_0^t 2 \cdot \eta_s dw^i_s + \sum_{i=1}^n \int_0^t (\gamma^i_1(s, w) - \gamma^i_2(s, w))^2 ds.$$

Therefore $\gamma_1(s, w) = \gamma_2(s, w)$ for every $0 \leqslant t \leqslant 1$ with probability 1.

References

[1] Clark, J. M. C., "The representation of functionals of Brownian motion by stochastic integrals" AMS 41(1970) 4, 1282–1295.

[2] Kunita, H. and Watanabe, S., "On square integrable martingales" Nagoya Math. J. 30(1967) 209–245.

[3] Neveu, J., Bases mathématiques du calcul des probabilités (Masson et Cie, Paris, 1964).

[4] Гирсанов, И.В., "О преобразовании одного класса случайных процессов с помощью абсолютно-непрерывной замены меры" Теория вероятностей и ее применен. 5 (1960) 314-330.

[5] Липцер, Р.Ш. и Ширяев, А.Н., "Об абсолютной непрерывности мер, соответствующих процессам диффузионного типа, относительно винеровской" Изд. АН СССР Сер. матем. 36 (1972) 847-889.

Р е з ю м е

Об абсолютной непрерывности мер, соответствующих n мерным процессам диффузионного типа, относительно винеровской

В настоящей работе рассматривается абсолютная непрерывность мер (относительно винеровской меры) соответствующих многомерным процессам диффузионного типа. Обобщаются результаты Липцера и Ширяева на многомерный случай.

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ON THE ESTIMATION OF REGRESSION COEFFICIENTS IN CASE OF AN AUTOREGRESSIVE NOISE PROCESS

I. H. Gaudi

INTRODUCTION

In statistical time series analysis one of the most frequently discussed problem has the following formulation: a time series on the form

$$y(t) = m(t) + x(t), \quad t = 1, 2, \ldots, N$$

is observed, where $m(t)$ is an unknown deterministic function and $x(t)$ is a stochastic process with 0 mean and known spectrum. The purpose is to draw some conclusions for $m(t)$ from the observed process $y(t)$. In the practice we seek the function $m(t)$ in the form

$$m(t) = \sum_{\nu=1}^{k} a_\nu \varphi^{(\nu)}(t),$$

where a_ν are unknown coefficients and $\varphi^{(\nu)}(t)$ are known functions (usually polinomials or trigonometric polinomials). We have to estimate the coefficients a_ν. The most natural way is the method of least squares.

With the following notations

$$\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_k \end{pmatrix}, \qquad y = \begin{pmatrix} y(1) \\ y(2) \\ \cdot \\ \cdot \\ \cdot \\ y(N) \end{pmatrix}, \qquad \varphi^{(j)} = \begin{pmatrix} \varphi^{(j)}(1) \\ \varphi^{(j)}(2) \\ \cdot \\ \cdot \\ \cdot \\ \varphi^{(j)}(N) \end{pmatrix}$$

and

$$\Phi = (\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(k)})$$

the least square estimator $\hat{\alpha}$ of the vector α takes the form

$$\hat{\alpha} = (\Phi^* \Phi)^{-1} \Phi^* y.$$

In the case of normal white noise estimator $\hat{\alpha}$ coincides with the maximum likelihood estimator of the vector α. If we suppose, that the noise process $x(t)$ is normal, but not white and it has known correlation matrix R, we have the maximum likelihood estimator α_0 of the vector α in the form

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$$\alpha_0 = (\Phi^* R^{-1}\Phi)^{-1}\Phi^* R^{-1} y.$$

It is well known, that α_0 has minimal dispersion among the linear unbiased estimators of α. From the point of wiew of computational technics the inversion of the matrix R for enormously large N is a difficult problem. Both the estimators $\hat{\alpha}$ and α_0 are normally distributed (as linear combinations of Gaussian variables) with expectations and variances

$$E\hat{\alpha} = (\Phi^* \Phi)^{-1}\Phi^* Ey = (\Phi^* \Phi)^{-1}\Phi^* \Phi\alpha = \alpha$$

$$E(\hat{\alpha} - \alpha)(\hat{\alpha} - \alpha)^* = (\Phi^* \Phi)^{-1}\Phi^* R\,\Phi(\Phi^* \Phi)^{-1}$$

$$E\alpha_0 = (\Phi^* R^{-1}\Phi)^{-1}\Phi^* R^{-1}\Phi\alpha = \alpha$$

$$E(\alpha_0 - \alpha)(\alpha_0 - \alpha)^* = (\Phi^* R^{-1}\Phi)^{-1}.$$

In this work we investigate the problem of the distribution of the estimators by the method of computer simulation. The question is, how they depend on simple parameters as damping and hidden periodicity.

1. Let us regard the process

$$y(t) = a \cos \omega t + x(t)$$

where the frequency ω is a given constant, a is the unknown parameter and $x(t)$ is a discrete time parameter second order autoregressive process i.e. $x(t)$ satisfies the difference equation

$$x(t) = \alpha x(t-1) + \beta x(t-2) + \epsilon(t).$$

The coefficients α and β are known real numbers satisfying the condition $\alpha^2 + 4\beta < 0$, the process $\epsilon(t)$ is a standard discrete time parameter white noise. The "period" of this scheme is $2\pi/\omega_1$, where

$$\omega_1 = \arccos \frac{|\alpha_1|}{2\sqrt{-\alpha_2}}.$$

On this example we can investigate another curious problem of the time series analysis, namely the distinction of a process with periodic mean value function from a process with hidden periodicity. We summarise the results of our computer simulation experiments about the statistical behaviour of the least square estimator $\hat{\alpha}$ and the maximum likelihood estimator a_M of the unknown parameter in tabular form, when the damping parameter λ and the hidden frequency ω_1 of the process $x(t)$ where varied.

The least square estimator $\hat{a}$ has the form

$$\hat{a} = \frac{\sum_{t=1}^{N} y(t) \cos \omega t}{\sum_{t=1}^{N} \cos^2 \omega t}$$

(The estimator $\hat{a}$ is the maximum likelihood estimator of a under the false hypothesis that the noise is white.)

The maximum likelihood estimation a_M can be calculated from the conditional density function

$$f = \frac{1}{(2\pi)^{N/2}} \exp\left\{-\frac{1}{2}\sum(x(t) - \alpha x(t-1) - \beta x(t-2))^2\right\}$$

of the process $x(t) = y(t) - a\cos\omega t,\ t = 1, 2, \dots, N,$ under the condition that $x(0) = x_0$. The solution of the likelihood equation

$$\frac{d\ln f}{da} = 0$$

can be written in the form

$$a_M = \frac{B}{A}$$

where

$$\begin{aligned} A = \sum_t \{&\cos^2\omega t + \alpha^2\cos^2\omega(t-1) + \beta^2\cos^2\omega(t-2) \\ &- 2\alpha\cos\omega t\cos\omega(t-1) - 2\beta\cos\omega t\cos\omega(t-2) \\ &+ 2\alpha\beta\cos\omega(t-1)\cos\omega(t-2)\} \end{aligned}$$

and

$$\begin{aligned} B = \sum_t \{&y(t)\cos\omega t + \alpha^2 y(t-1)\cos\omega(t-1) + \\ &+ \beta y(t-2)\cos\omega(t-2) - \alpha y(t-1)\cos\omega t - \\ &- \alpha y(t)\cos\omega(t-1) - \beta y(t-2)\cos\omega t - \\ &- \beta y(t)\cos\omega(t-2) + \alpha\beta y(t-2)\cos\omega(t-1) \\ &+ \alpha\beta y(t-1)\cos\omega(t-2)\}\ . \end{aligned}$$

2. In our concrete example the parameters were chosen as follows:

$$a = 6.8,\quad \alpha = 1.83,\quad \beta = -0.98,\quad \omega = \frac{2\pi}{10}.$$

So the period of the noise is 16.04 and the damping parameter is small ($\sqrt{0.98}$). Table 1. shows the dependence of estimators on the number of observations. In the first column we can find the numbers of observations, in the second column the type of the estimator, in the 2.nd–6.th columns the 0.05, 0.1, 0.2, 0.8, 0.9 and 0.95 respectively, quantiles of estimators (calculated from 200–500 samples), and the last two columns contain the mean value and the dispersion of the estimators.

We get similar results in every case, when the period of $m(t)$ and the hidden period of $x(t)$ are far from each other (e.g. $\omega \leqslant 2\pi/40$ or $\omega \geqslant 2\pi/12$). The two estimators differ essentially in the case of small number of observations, while for $N > 300$ they almost coincide.

Table 2. shows the dependence of estimator on the frequency ω – in this case $N = 40$. The construction of table 2. is similar to the first one.
When the frequency of $m(t)$ is equal to the hidden frequency of $x(t)$ $(2\pi/\omega = 16)$ the signal and the noise cannot be separated. In this case the least square estimator is better than the maximum likelihood one – in the sequel we return to this phenomenon. For large ω both estimators are better: there are more waves on the interval of observations. To avoid this effect we investigated the behaviour of estimators on the intervals the length of which is 2 or 1 waves.

These results are contained in tables 3. and 4.

The least square estimation gives very bad results for $T \leqslant 5$, while the maximum likelihood estimator becomes continuously better as the distance between the frequencies of the noise and the signal grows.

Figure 1. shows the dependence of dispersions of the two variant of estimators on ω in the neighbourhood of the frequency of the noise, observing 2 waves.

Experiments were made to determine, how the damping influences the statistical behaviour of the estimates. In a natural way, when the damping grows, the distance between the two estimates decreases. If $\alpha = 1.488$ and $\beta = -0.64$ (then the frequency of the noise coincides with the previous, and the damping equals 0.8) the two estimates are not essentially different.

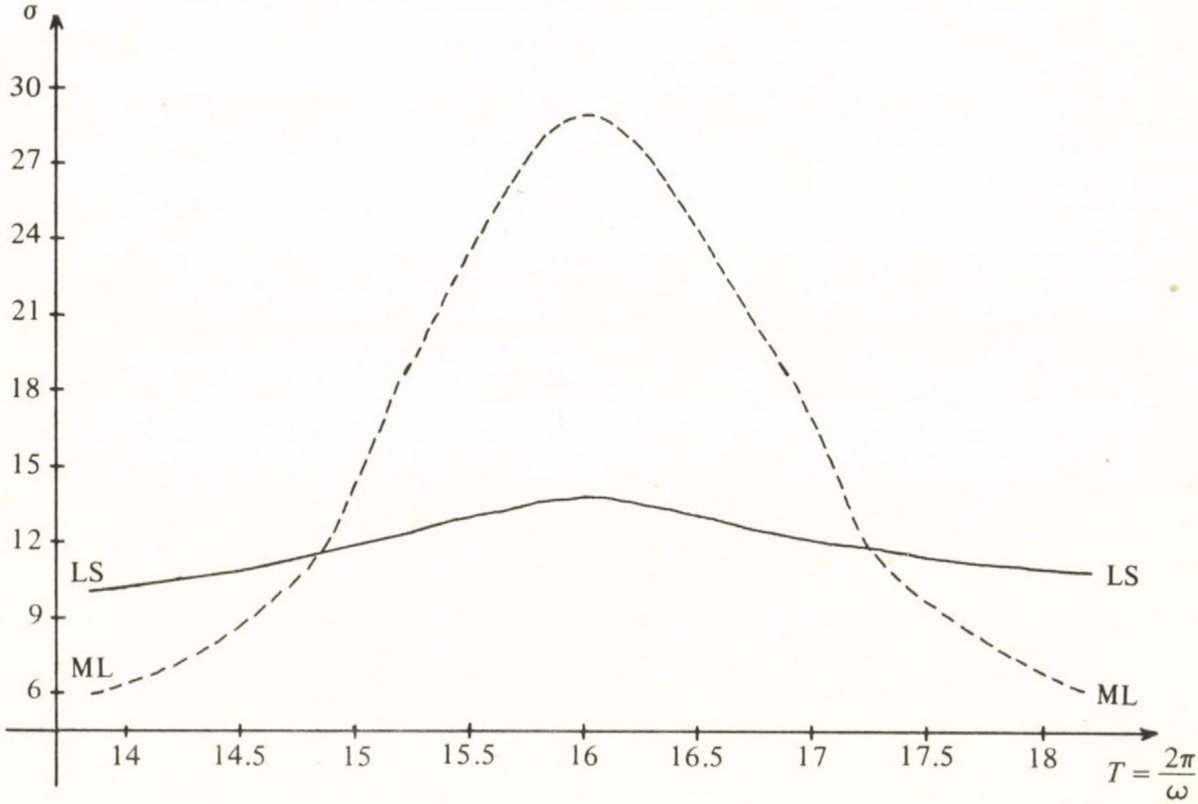
So far we have supposed that the noise was a second order autoregressive process with known parameters. By simulation we examined the behaviour of the estimators in the case if the noise is a higher 4–5 order autoregressive process and we use a second order approximation for the maximum likelihood estimation, the so called R-estimators (see Holevo [2]).

N	Type of estim.	0.05	0.1	0.2	0.8	0.9	0.95	m	σ
5	ML	– 0.82	– 0.07	1.97	9.99	11.81	13.61	6.12	4.47
	LS	– 8.73	– 4.12	0.35	13.76	16.67	19.94	6.82	8.86
10	ML	3.15	3.84	4.84	8.77	9.95	10.81	6.85	2.34
	LS	– 5.80	– 2.49	– 0.06	13.84	16.10	18.48	6.78	7.65
15	ML	3.89	4.43	5.17	8.20	8.96	9.73	6.78	1.75
	LS	– 2.71	– 0.87	2.02	12.07	14.57	17.14	7.09	5.80
20	ML	3.99	5.03	5.70	7.87	8.38	8.99	6.71	1.42
	LS	0.68	2.02	3.94	9.60	11.14	12.10	6.79	3.54
30	ML	5.03	5.33	5.77	7.64	8.25	8.62	6.73	1.17
	LS	3.31	4.27	5.20	8.86	9.98	10.40	6.73	2.22
40	ML	5.04	5.41	5.92	7.64	8.20	8.59	6.89	1.10
	LS	3.90	4.26	5.19	8.16	9.10	9.82	6.90	1.75
50	ML	5.09	5.44	5.94	7.54	7.86	8.13	6.72	0.97
	LS	4.38	4.96	5.58	7.93	8.47	9.02	6.72	1.38
60	ML	5.38	5.68	6.11	7.49	7.83	8.01	6.78	0.86
	LS	4.95	5.23	5.60	7.86	8.43	8.84	6.80	1.28
80	ML	5.62	5.92	6.20	7.48	7.83	7.91	6.87	0.73
	LS	5.83	5.72	6.05	7.59	8.06	8.28	6.86	0.92
100	ML	5.66	5.89	6.19	7.37	7.59	7.88	6.74	0.65
	LS	5.36	5.61	6.01	7.48	7.83	8.22	6.73	0.88
200	ML	6.05	6.19	6.42	6.97	7.19	7.39	6.76	0.40
	LS	5.86	6.07	6.29	7.12	7.54	7.81	6.74	0.58
300	ML	6.15	6.22	6.44	7.00	7.20	7.35	6.76	0.35
	LS	6.06	6.18	6.36	7.10	7.26	7.44	6.75	0.41

Table 1.

$T = \frac{2\pi}{\omega}$	Type of estim.	0.01	0.1	0.2	0.8	0.9	0.95	m	σ
40	ML LS	2.94 – 0.09	4.14 2.04	5.25 3.97	8.20 9.59	9.56 11.60	9.79 13.04	6.76 6.74	1.90 3.84
30	ML LS	2.45 0.66	3.54 2.58	4.40 3.74	8.99 9.07	9.95 10.84	10.33 11.65	6.62 6.63	2.52 3.33
20	ML LS	0.13 – 9.32	1.56 – 5.81	2.98 – 1.13	10.15 14.39	11.73 18.03	13.28 22.30	6.47 6.53	4.22 9.15
18	ML LS	– 4.18 – 11.43	– 2.71 – 7.08	0.36 – 1.46	12.32 15.17	16.50 21.04	18.46 24.45	6.78 6.93	7.49 10.66
16	ML LS	– 42.39 – 14.62	– 35.53 – 9.44	– 18.34 – 4.23	29.71 19.11	39.62 24.52	52.11 27.16	4.26 6.76	28.28 13.01
14	ML LS	– 1.59 – 7.07	0.09 – 4.24	1.85 – 0.17	11.26 13.58	13.64 17.60	15.28 21.79	6.98 7.18	5.35 8.35
12	ML LS	3.13 – 0.73	3.77 2.10	4.70 4.00	8.55 9.79	9.51 11.35	10.31 12.86	6.73 6.57	2.16 3.95
10	ML LS	5.00 2.97	5.48 3.99	5.82 4.95	7.71 8.93	8.07 9.78	8.47 10.29	6.81 6.83	1.10 2.25
8	ML LS	5.87 4.52	6.06 5.13	6.37 5.70	7.30 7.85	7.48 8.50	7.77 8.85	6.81 6.82	0.57 1.29
5	ML LS	6.48 5.76	6.53 5.93	6.65 6.17	6.94 7.35	7.06 7.61	7.14 7.86	6.80 6.79	0.20 0.65

Table 2.


σ
30
27
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LS
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ML
14
14.5
15
15.5
16
16.5
17
17.5
18
$T = \frac{2\pi}{\omega}$

Figure 1.

$T = \frac{2\pi}{\omega}$	Type of estim.	0.05	0.1	0.2	0.8	0.9	0.95	m	σ
50	ML	5.16	5.54	5.90	7.67	8.08	8.61	6.76	1.06
	LS	4.19	4.65	5.34	8.09	8.77	9.28	6.74	1.56
40	ML	4.74	5.10	5.67	7.86	8.58	8.93	6.83	1.29
	LS	4.06	4.73	5.44	8.25	9.10	9.54	6.88	1.67
30	ML	4.09	4.58	5.31	8.35	9.00	9.73	6.75	1.77
	LS	1.73	2.77	4.14	9.13	10.41	11.66	6.73	2.91
20	ML	0.13	1.56	2.98	10.15	11.73	13.28	6.73	4.22
	LS	– 9.32	– 5.81	– 1.13	14.39	18.09	22.30	6.62	9.15
18	ML	– 9.74	– 3.58	0.29	13.24	16.76	19.32	6.25	8.21
	LS	– 10.92	– 7.27	– 2.10	14.37	20.56	26.40	6.18	10.56
16	ML	– 48.55	– 37.01	– 22.16	35.52	45.05	57.99	5.31	32.45
	LS	– 14.78	– 11.68	– 3.87	15.71	19.60	24.26	5.22	11.95
14	ML	– 4.25	– 1.98	0.59	12.18	15.25	17.14	6.46	6.48
	LS	– 12.09	– 7.98	– 2.91	15.69	20.59	23.46	6.47	10.87
12	ML	1.85	3.41	4.78	8.91	9.82	10.41	6.69	2.54
	LS	– 5.13	– 2.32	1.05	12.56	15.56	17.54	6.72	6.98
10	ML	3.99	5.03	5.70	7.87	8.38	8.99	6.71	1.42
	LS	0.68	2.02	3.94	9.60	11.14	12.10	6.79	3.54
8	ML	5.14	5.40	5.90	7.50	7.76	8.05	6.70	0.89
	LS	4.59	5.22	5.55	7.84	8.31	8.74	6.76	1.25
6	ML	5.93	6.20	6.39	7.26	7.46	7.59	6.83	0.51
	LS	3.55	4.14	5.08	8.55	9.52	9.94	6.80	2.06
5	ML	6.11	6.22	6.35	7.12	7.26	7.47	6.78	0.44
	LS	2.74	3.52	4.48	18.75	9.96	10.70	6.78	2.39
4	ML	6.25	6.36	6.49	7.05	7.18	7.29	6.79	0.32
	LS	1.51	2.10	3.84	9.62	11.25	12.10	6.77	3.22
3	ML	6.38	6.52	6.61	7.02	7.14	7.22	6.82	0.24
	LS	– 0.62	1.26	3.72	9.69	12.11	13.53	6.77	4.24

Table 3.

$T = \frac{2\pi}{\omega}$	Type of estim.	0.05	0.1	0.2	0.8	0.9	0.95	m	σ
50	ML LS	4.35 3.54	5.04 4.25	5.55 5.49	8.16 8.37	8.72 9.24	9.38 10.04	6.85 6.84	1.49 1.94
40	ML LS	2.94 – 0.09	4.14 2.04	5.25 3.97	8.20 9.59	9.56 11.60	9.79 13.04	6.76 6.74	1.90 3.84
30	ML LS	2.39 0.93	3.27 2.22	4.86 3.29	9.06 10.32	10.46 11.95	11.44 13.64	7.00 7.08	2.75 3.88
20	ML LS	– 3.79 – 14.39	– 1.63 – 11.13	0.67 – 5.25	13.52 18.63	15.28 24.08	16.42 26.77	7.00 6.76	6.44 13.29
18	ML LS	– 11.57 – 12.28	– 8.80 – 9.70	– 4.30 – 3.96	15.48 16.01	20.34 22.80	22.65 29.85	6.34 6.22	11.23 12.24
16	ML LS	– 73.55 – 17.87	– 59.34 – 12.91	– 37.95 – 8.60	42.88 13.37	62.07 17.23	87.86 19.96	2.89 3.12	48.04 12.15
14	ML LS	– 9.21 – 11.99	– 5.14 – 9.56	– 1.95 – 4.22	14.00 17.79	17.86 21.75	20.81 25.25	6.22 6.32	9.00 11.57
12	ML LS	– 1.03 – 10.99	0.83 – 6.88	3.01 – 2.75	10.27 15.04	12.54 20.12	13.65 25.65	6.70 6.68	4.39 10.78
10	ML LS	3.15 – 5.8	3.84 – 2.49	4.84 – 0.06	8.77 13.84	9.95 16.10	10.81 18.48	6.85 6.78	2.34 7.65
8	ML LS	4.29 – 3.36	4.64 – 1.66	5.44 1.46	7.74 11.92	8.52 13.93	9.25 15.57	6.70 6.64	1.47 5.66
6	ML LS	5.33 – 1.90	5.68 – 0.27	5.95 2.28	7.63 10.55	8.06 13.28	8.44 15.99	6.81 6.82	0.96 5.46
5	ML LS	5.72 – 4.13	5.98 – 1.28	6.26 1.23	7.42 12.46	7.84 14.99	8.03 16.93	6.84 6.83	0.71 6.37
4	ML LS	5.74 – 0.77	6.04 1.34	6.29 2.95	7.27 10.44	7.49 11.75	7.71 14.16	6.77 6.77	0.58 4.37
3	ML LS	5.67 0.95	5.83 2.63	6.24 4.08	7.34 9.29	7.71 11.03	7.88 11.96	6.80 6.79	0.69 3.35

Table 4.

References

[1] Grenander, U. and Rosenblatt, M., Statistical analysis of stacionary time series (John Wiley, New York, 1957).

[2] Холево, А.С., "Об оценках коэффициентов регрессии" Теория вероятностей XIУ (I969) I.

Резюме

Об оценке параметра регрессии, когда процесс шума является процессом авторегрессии

В настоящей работе рассматривается процесс $y(t) = a\cos\omega t + x(t)$, где ω-данная константа, a - неизвестный параметр и $x(t)$ удовлетворяет стохастическому разностному уравнению

$$x(t) = \alpha x(t-1) + \beta x(t-2) + \epsilon(t).$$

Постоянные α и β удовлетворяют условию $\alpha^2 + 4\beta < 0$, $x(t)$ предполагается стационарным и $\epsilon(t)$ является стандартным белым шумом с дискретным временем.

Методом статических испытаний исследуется поведение разных типов оценок.

Результаты показывают, что при малом числе наблюдений зачитывая специальную форму получаются лучшие оценки параметра a (доверительное множество уже). В случае большого выбора оценки таким образом не станут лучщими.

Если период сигнала и скрытый период шума близки друг к другу, то тогда нельзя отделить сигнал от шума.

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ON AN ESTIMATE FOR THE PARAMETER OF A MULTIDIMENSIONAL STATIONARY GAUSSIAN MARKOV PROCESS, AND AN APPLICATION

I. Ratkó and M. Ruda

INTRODUCTION

In the paper estimates and confidence limits for the parameter of a multidimensional stationary Gaussian Markov process are considered. The estimate of the coefficient–matrix of a multidimensional stationary Gaussian Markov process, under certain weak conditions, may be reduced to the estimate for the parameter of a onedimensional real (respectively complex) stationary Gaussian Markov process.

These latter estimates and the distribution of the estimates are given in [4] and in [5].

Through the research of a geophysical problem (the axis of instantaneous rotation of the Earth) the authors compare the effeciency of the methods of the various estimates for the parameter.

Computer realizations are given for the various parameter estimation procedures, and their application to the above mentioned geophysical problem.

The estimation of the parameters and the determination of the confidence limits of one–dimensional continuous Gaussian Markov processes can be found in the papers of Arató [4] and Arató–Benczúr [5].

Employing the results of these papers we give results for similar problems in the multidimensional case as well for continuous as for discrete processes.

The results are valid under certain conditions; the case, when the conditions are not satisfied, requires further investigations. In the final part of the paper the efficiencies of the different parameter estimation methods are compared in connection with a geophysical problem, concerning variations of the axis of rotation of the Earth.

1. The continuous case

Let $\underline{\xi}(t)$ be a multidimensional continuous stationary Gaussian Markov process:

$$d\underline{\xi}(t) = A\,\underline{\xi}(t)dt + d\underline{w}(t),$$

where $\underline{w}(t)$ is a Wiener process and we assume that the real component of the eigenvalues of matrix A is negative.

By Baxter's theorem we have: if

$$E(d\underline{w} \cdot d\underline{w}^*) = B\,dt,$$

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then

$$\lim_{\max(t_k - t_{k-1})} [\underline{\xi}(t_k) - \underline{\xi}(t_{k-1})][\underline{\xi}(t_k) - \underline{\xi}(t_{k-1})]^* = B \cdot T$$

(with probability 1), where $0 = t_0 < t_1 < \ldots < t_n$ is a partition of the interval $[0, T]$, and B is the covariancy matrix of $\underline{w}(t)$. As it is well–known, all matrices can be brought to the Jordan form.

Denote with B' the Jordan form of B:

$$B' = SBS^{-1}.$$

Since B is symmetrical, B' is a diagonal matrix.

The elements of the principal diagonal of B' are just the eigenvalues of B and if $\underline{s}_i(s_{1i}, s_{2i}, \ldots, s_{ni})$ is the eigenvector belonging to the i-th eigenvalue, then $S = (s_{ij})$. This makes S unambiguous.

We suppose also, that $A' = SAS^{-1}$ is in Jordan form.

Consider now the transform of the process $\underline{\xi}(t)$, that is the stationary Gaussian Markov process:

(1) $$d\underline{\xi}'(t) = A'\underline{\xi}'(t)dt + d\underline{w}'(t),$$

where $\underline{\xi}'(t) = S\underline{\xi}(t)$, $\underline{w}'(t) = S\underline{w}(t)$. It is easy to verify, that SBS^* gives the covariancy matrix of $\underline{w}'(t)$. Because S in unitery, the latter equals SBS^{-1}. This means, that B' is precisely the covariancy matrix of $\underline{w}'(t)$. The eigenvalues of A are all different (simple) with probability 1.

Then A' also is of diagonal form, therefore the equation (1) is decomposed in the following n equations:

(2) $$d\xi_k'(t) = -\lambda_k \xi_k'(t)dt + dw_k'(t), \qquad k = 1, 2, \ldots, n,$$

where $-\lambda_k$ is an eigenvalue of A and $\lambda_i \neq \lambda_j$ $(i \neq j)$. As S can be determined using $B' = SBS^{-1}$, we can deduce confidence limits for the elements of A from the confidence limits for the parameters of the transformed process.

a) λ_k is real

The process (we leave the index and the prime) is:

$$d\xi(t) = -\lambda\xi(t)dt + dw(t).$$

We can state the following on the basis of [4]: if the observation happens in the interval $[0, T]$, then

$$\hat{\lambda} = \frac{-(s_1^2 - \frac{1}{2}T) + \sqrt{(s_1^2 - \frac{1}{2})^2 + 2Ts_2^2}}{2Ts_2^2}$$

provides and estimation, where

$$s_1^2 = \frac{1}{2}(\xi^2(0) + \xi^2(T)), \quad s_2^2 = \frac{1}{T}\int_0^T \xi^2(t)dt.$$

In the case of the given realization $\hat{\lambda}$ can calculated from the above equation. Similarly on the basis of [4] confidence limits can be given for λ.

b) λ_k is complex

(We leave the index k and the prime.)

The process is given by

$$d\xi(t) = -\lambda\xi(t)dt + dw(t).$$

Now, if $\xi(t) = \eta(t) + i\zeta(t)$, $w(t) = \varphi(t) + i\psi(t)$, $\lambda = \alpha - \beta i$ (thus $\beta > 0$), from the relation:

$$d[\eta(t) + i\zeta(t)] = (-\alpha + i\beta)[\eta(t) + i\zeta(t)]dt + d[\varphi(t) + i\psi(t)]$$

we deduce the processes:

$$d\eta(t) = -\alpha\eta(t)dt - \beta\zeta(t) + d\varphi(t)$$
$$d\zeta(t) = \beta\eta(t)dt - \alpha\zeta(t)dt + d\psi(t).$$

The estimations for β and for α can be found in [5]:

$$\hat{\alpha} = \frac{-\left(\frac{s_1^2}{a} - T\right) + \sqrt{\left(\frac{s_1^2}{a} - T\right)^2 + 4T\frac{s_2^2}{a}}}{\frac{2Ts_2^2}{a}} \quad \text{and} \quad \hat{\beta} = \frac{r}{s_2^2}$$

where

$$s_1^2 = \frac{1}{2}[|\xi(0)|^2 + |\xi(T)|^2],$$

$$s_2^2 = \frac{1}{T}\int_0^T |\xi(t)|^2 dt,$$

$$r = \frac{1}{T}\int_0^T (\eta d\zeta - \zeta d\eta)$$

and $a = b'_{kk}$ (if we consider the process ξ_k), i.e. $\underline{a}$ is the k-th element of the principal diagonal of B'.

In the case of a given realization s_1^2, s_2^2 a and r – therefore also $\hat{\alpha}$ and $\hat{\beta}$ – can be calculated.

With the help of the table in [5] we can give confidence limits for $\hat{\alpha}$; $\left(\frac{r}{s_2} - \beta s_2\right)\sqrt{\frac{2a}{T}}$ has $N(0, 1)$ distribution.

Finally we determine confidence limits for the elements of A.

Let be $S = P + iQ$, $A' = A_1 + iA_2$. Then $S^{-1} = P' + iQ'$, where

$$P' = (P + QP^{-1}Q)^{-1}, \quad Q = -(Q + PQ^{-1}P)^{-1}$$

(the existence of the inverses can be proved).

$$A = S^{-1}A'S = P'A_1P - P'A_2Q - Q'A_1Q - Q'A_2P,$$

because A is a real matrix.

We suppose, that confidence limits can be determined for the elements of A' (precisely for the elements of A_1 and of A_2) at the confidence level $1 - \epsilon$ (let all eigenvalues of A have non–zero imaginary components).

By a simple calculation we obtain for the element α_{ij} of A:

$$P(\alpha_{ij}^{(1)} \leqslant \alpha_{ij} \leqslant \alpha_{ij}^{(2)}) \geqslant (1 - 2\epsilon)^n = 1 - \epsilon^*,$$

where

$$\epsilon^* = \sum_1^n \binom{n}{k} (-2\epsilon)^k,$$

$$(\alpha_{ij}^{(1)}) = P'A_1^{(1)}P - P'A_2^{(2)}Q - Q'A_1^{(2)}Q - Q'A_2^{(2)}P,$$
$$(\alpha_{ij}^{(2)}) = P'A_1^{(2)}P - P'A_2^{(1)}Q - Q'A_1^{(1)}Q - Q'A_2^{(1)}P,$$

where we denote with $A_1^{(1)}$ $(A_1^{(2)})$ respectively with $A_2^{(1)}$ $(A_2^{(2)})$ the matrices formed by the left (right) endpoints of the intervals at the confidence level $1 - \epsilon$.

In the case where of the eigenvalues of A precisely l are real, the "sharper" inequality

$$P(\alpha_{ij}^{(1)} \leqslant \alpha_{ij} \leqslant \alpha_{ij}^{(2)}) \geqslant (1 - \epsilon)^l \cdot (1 - 2\epsilon)^{n-l}$$

is valid.

2. The discrete case

Let $\underline{\xi}(k)$ be a multidimensional discrete stationary Gaussian Markov process:

$$\underline{\xi}(k) = Q\underline{\xi}(k-1) + \underline{w}(k)$$

where $\underline{w}(k)$ is a Wiener process and the modules of the eigenvalues of Q are less than 1.

We approximate this process by a continuous process:

$$d\underline{\xi}(t) = A\underline{\xi}(t)dt + d\underline{w}(t),$$

where $A = \ln Q$ and the random variables $\underline{\xi}(1), \underline{\xi}(2), \ldots$ are the realizations of $\underline{\xi}(t)$ at the moments $\delta, 2\delta, \ldots$.

We can estimate A on the basis of the previous paragraph.

It is easy to verify that from the equation $Q = e^A$ and from the monotonity of the function e^x follows that the confidence interval $(q_{ij}^{(1)}, q_{ij}^{(2)}) = (e^{\alpha_{ij}^{(1)}}, e^{\alpha_{ij}^{(2)}})$ has at least the confidence level $1 - \epsilon$.

3. Variation of the instantaneous Earth rotation axis

The instantaneous rotation axis of the Earth constantly changes its position relatively to the Earth itself.

Several authors deal with the investigation of these variations. We can mention e.g. the paper [3] of A. M. WALKER and A. YOUNG or the paper [2] of D. R. BRILLINGER. A model of a solution of the problem can be found also in the paper [1] of M. ARATÓ. As it is known, this change – as a two–dimensional process – consists of two components: of a component varying regularly every year and of a component varying with a period, of about 14 months. The two components are clearly indicated in the periodograms of fig. 1/a and fig. 1/b.

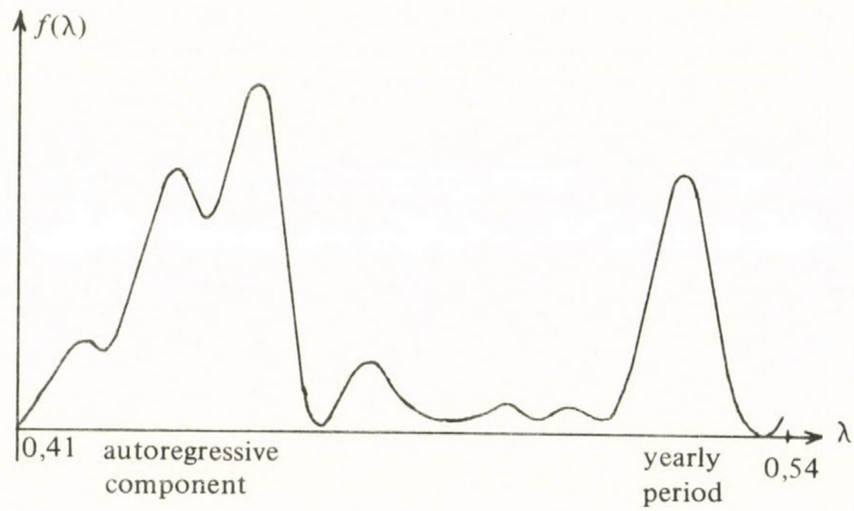

Figure 1/a

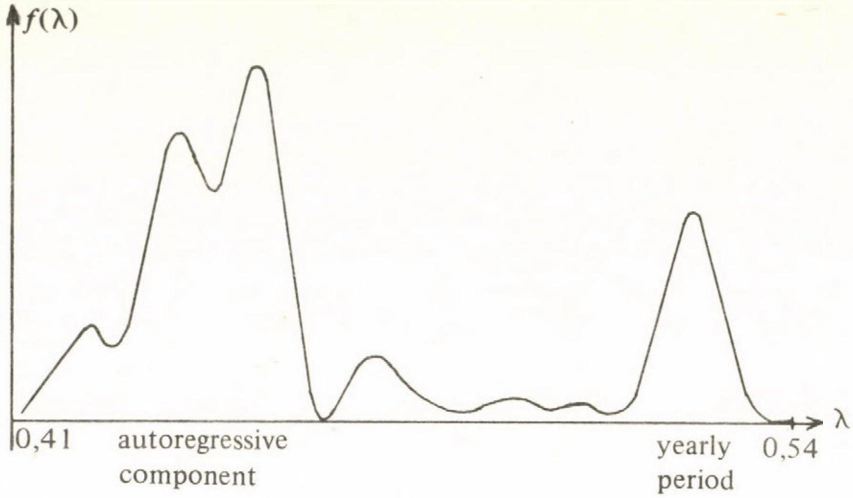

Figure 1/b

We apply the following model to the description of the phenomenon (see [1]). The data (600–600 observations) were taken from the paper [3] of Walker and Young. For a comparison we made also calculations on the ground of data published by Orlov (see [6]).

The Component with the yearly period we consider as a deterministic process, as a sinus function with given amplitude and phase. The estimation of the parameters for this component is a regression problem. We note, that Brillinger [2] subtracted simply the monthly averages from the original process, as the values of the component with the 12-months period.

In the regression problem we assume that both components of the process is in the form

$$x_t = A + B \sin \omega t + C \cos \omega t + \epsilon_t$$

where A, B, C are constans, ϵ_t is a white noise process and $\omega = 2\pi/12$ we have one observation per month.

Applied the estimated coefficients A, B, C we can subtract from the original process the yearly component.

Now we regard the residue process as a two dimensional first order autoregressive process (see [1] and §. 2. in this work).

In our case the matrix Q, in 2. paragraph, is in form

$$Q = \begin{vmatrix} a & -b \\ b & a \end{vmatrix}$$

and $w(t)$ is a white noise process with independent components.

Let as denote

$$\hat{Q} = \begin{pmatrix} \hat{a}_1 & -\hat{b}_1 \\ \hat{b}_2 & \hat{a}_2 \end{pmatrix}$$

the estimate of the matrix Q, where $\hat{a}_1$ and $\hat{a}_2$ respectively $\hat{b}_1$ and $\hat{b}_2$ are in general not equal. Let

$$\hat{a} = \frac{1}{2}(\hat{a}_1 + \hat{a}_2),$$

$$\hat{b} = \frac{1}{2}(\hat{b}_1 + \hat{b}_2)$$

be the estimate of a and b.

The estimated values for 600–600 observations: $\hat{a}_1 = 0.87, \hat{b}_1 = 0.37, \hat{a}_2 = 0.39,$ $\hat{b}_2 = 0.89$.

We can characterize the accuracy of the model fittin with the components of the $w(t)$ residue noise process. In this case the two residue components $\epsilon_{1,t}$ and $\epsilon_{2,t}$ have variances 0"0.34 resp. 0".035, similarly to [2]. The autocovariancie functions and the estimate of periodograms see on the Figures 2/a, 2/b resp. 3/a, 3/b.

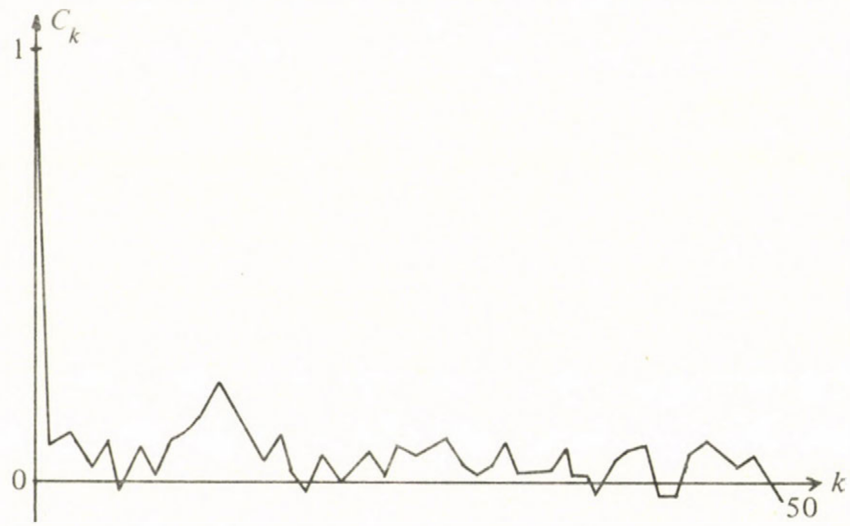

Figure 2/a

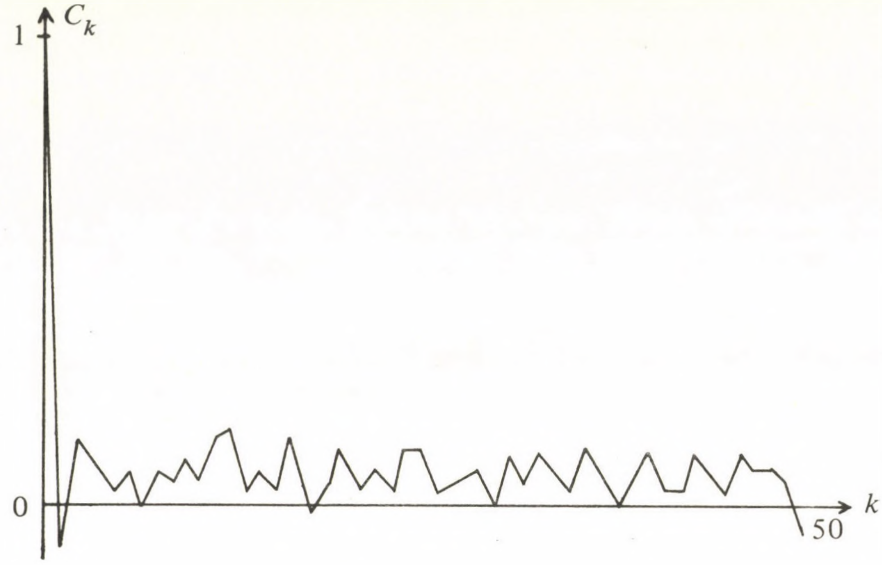

Figure 2/b

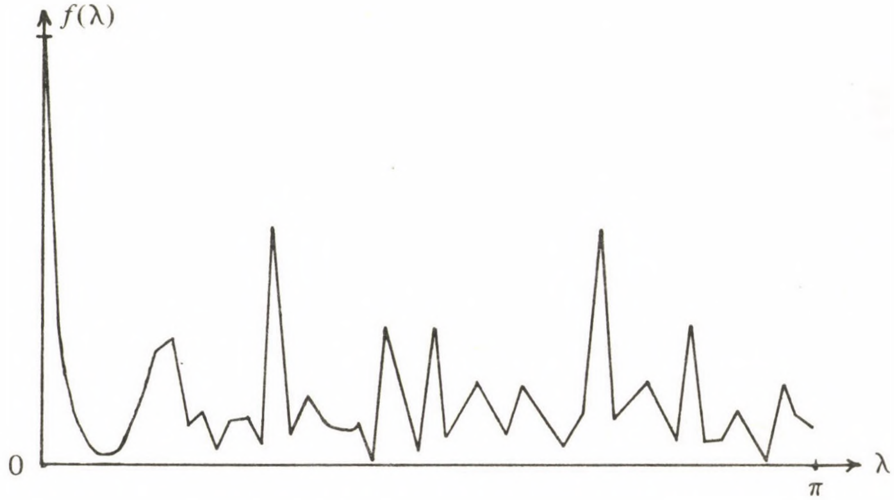

Figure 3/a

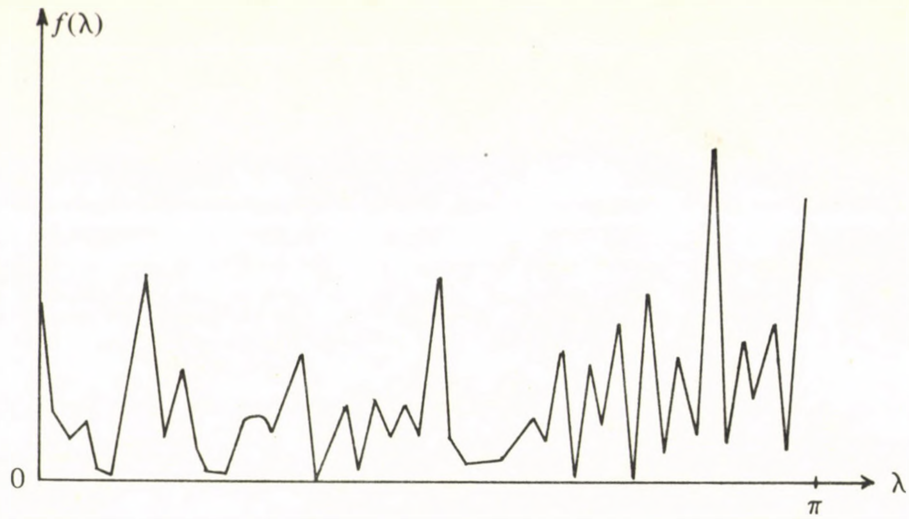

Figure 3/b

We can examine the properties of the $\epsilon_{1,t}$ and $\epsilon_{2,t}$ processes by the means of various statistical tests. On the basis of the number of the local maximum and minimum sets we can accept that $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are independent noise processes. However considering the serial test can be saw that the sign of both components are changing very often that is the length of the series is very short—although the case is similar for example at the usual library random number generators.

The following question arize in connection with the later problem: What is the cause of the bad fitting of the serial test, the incomplete model or the inacurate parameter estimate?

An other question: Does the above mentioned model present any reason for the change of the original process, which is not connected with the 12 and 14 monthly periodes?

Note: We made the various parameter estimates using the time series program package of the institut.

References

[1] Arató, M., "Folytonos állapotú Markov folyamatok statisztikai vizsgálatáról I–IV" MTA III. Oszt. Közleményei 14–15 (1964, 1965).

[2] Brillinger, D. R., "An Empirical Investigation of the Chandler Wobble and two Proposed Excitation Processes" Meeting of Statisticians Wien (1973).

[3] Walker, A. M. and Young, A., "Further results on the analysis of the variation of latitude" Monthly Notices of the Royal Astronomical Society 117(1957) 119–141.

[4] Арато, М. и Бенцур, А., "Функция распределения оценки параметра затухания стационарного процесса" Studia Sci. Math. Hung. 5(1970) 445–456.

[5] Арато, М., "Вычисление доверительных границ для параметра "затухания" комплексного стационарного гауссовского марковского процесса" Теория вероятностей XIII. (1968) 326-333.

[6] Орлов, А., Служба широты (Изд. АН СССР, Москва, 1958).

Р е з ю м е

Об оценке параметров многомерного стационарного гауссовского марковского процесса и её применение

В настоящей работе даются оценки параметра гауссовского марковского процесса, и определяются доверительные границы. Примером из практики решается геофизическая задача.

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ON THE MAXIMUM LIKELIHOOD ESTIMATION OF AN AUTOREGRESSIVE MOVING AVERAGE PROCESS WITH ERROR

A. Abd–Alla

INTRODUCTION

Let $\epsilon(t)$, $t = 0, \pm 1, \pm 2, \ldots$, be a sequence of uncorrelated normal random variables with mean zero and constant variance σ_ϵ^2. Let the process $X(t)$, $t = 0, \pm 1, \pm 2, \ldots$ with mean $E[X(t)] = 0$ and variance $E[X(t)]^2 = \sigma_X^2$ be stationary autoregressive process of order p,

$$X(t) = \Phi_1 X(t-1) + \ldots + \Phi_p X(t-p) + \epsilon(t) \tag{1}$$

We suppose that the constants $\Phi_1, \ldots, \Phi_p$ are such that moduli of the roots of the equation

$$Z^p - \Phi_1 Z^{p-1} - \Phi_2 Z^{p-2} - \ldots - \Phi_p = 0 \tag{2}$$

are less than unity.

Let the process $Y(t)$, $t = 0, \pm 1, \pm 2, \ldots$ with mean $E[Y(t)] = 0$ and variance $E[Y(t)]^2 = \sigma_Y^2$ be completely random series. i.e. is a sequence of independent random variables with common normal distribution. It will be assumed that the process $Y(t)$ is the observation error which is additive and independent of the original time series $X(t)$. Thus if $Z(t)$ denotes the observation at time t then we have

$$Z(t) = X(t) + Y(t) \tag{3}$$

From our assumptions it follows that the process $X(t)$ and therefore the process $Z(t)$ too are normal.

1. The likelihood function in general case

Before we discuss the likelihood function in general case we are going to get the inverse of the covariance matrix of the process

$$W(t) = \epsilon(t) + Y(t) - \Phi_1 Y(t-1) - \ldots - \Phi_p Y(t-p) \tag{4}$$

From (1) and (3) the process $W(t)$ is given by

$$Z(t) - \Phi_1 Z(t-1) - \ldots - \Phi_p Z(t-p) = W(t) \tag{5}$$

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Since the means of the process $Y(t)$ and $\epsilon(t)$ are equal to zero, the mean of the process $W(t)$ is equal to zero and its variance is given by

$$(6) \qquad \sigma_W^2 = \sigma_\epsilon^2 + \sigma_Y^2(1 + \Phi_1^2 + \ldots + \Phi_p^2)$$

It is clear from (4) that the covariance function $\nu_W(s)$ of the process $W(t)$ is equal to zero for $|s| > p$ and the covariances $\nu(1), \nu(2), \ldots, \nu(p)$ are given by

$$\nu(1) = -\sigma_Y^2(\Phi_1 - \Phi_1\Phi_2 - \Phi_2\Phi_3 - \ldots - \Phi_{p-1}\Phi_p)$$

$$\nu(2) = -\sigma_Y^2(\Phi_2 - \Phi_1\Phi_3 - \Phi_2\Phi_4 - \ldots - \Phi_{p-2}\Phi_p)$$

$$\vdots$$

$$\nu(p) = -\sigma_Y^2\Phi_p$$

If $g_1, g_2, \ldots, g_p$ are defined by

$$g_1 = -\frac{\nu(1)}{\sigma_W^2}$$

$$g_2 = -\frac{\nu(2)}{\sigma_W^2}$$

$$\vdots$$

$$g_p = -\frac{\nu(p)}{\sigma_W^2}$$

then the covariance matrix Σ_N of the random variables $W(t)$, $(t = 1, \ldots, N)$ is

$$(7) \qquad \Sigma_N = \sigma_W^2 \mathbf{A}_N$$

where the matrix $\mathbf{A}_N$ is given by,

$$
\mathbf{A}_N = \begin{bmatrix}
1 & -g_1 & -g_2 & \cdots & -g_p & \cdots & 0 \\
-g_1 & 1 & -g_1 & \cdots & -g_{p-1} & \cdots & 0 \\
-g_2 & -g_1 & 1 & \cdots & -g_{p-2} & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
-g_p & -g_{p-1} & -g_{p-2} & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 1
\end{bmatrix}
$$

Let $\mathbf{U}$ be the matrix define by $\mathbf{U} = \{\delta_{r+1,s}\}$ where δ_{rs} is Kronecker delta. The matrix $\mathbf{A}_N$ can be written in terms of $\mathbf{U}$ as follows

$$\mathbf{A}_N = [\mathbf{I} - g_1(\mathbf{U}' + \mathbf{U}) - g_2(\mathbf{U}'^2 + \mathbf{U}^2) - \ldots - g_p(\mathbf{U}'^p + \mathbf{U}^p)]$$

Hence the inverse $\mathbf{A}_N^{-1}$ of $\mathbf{A}_N$ is

(8) $$\mathbf{A}_N^{-1} = [\mathbf{I} - g_1(\mathbf{U}' + \mathbf{U}) - g_2(\mathbf{U}'^2 + \mathbf{U}^2) - \ldots - g_p(\mathbf{U}'^p + \mathbf{U}^p)]^{-1}$$

$$= \sum_{i=0}^{N} [g_1(\mathbf{U}' + \mathbf{U}) + g_2(\mathbf{U}'^2 + \mathbf{U}^2) + \ldots + g_p(\mathbf{U}'^p + \mathbf{U}^p)]^i$$

The approximated value of $\mathbf{A}_N^{-1}$; For small values of $g_1, g_2, \ldots, g_p$ we can neglect the terms of orders $o(g_1^2), o(g_2^2), \ldots, o(g_p^2)$ and then the approximated value of $\mathbf{A}_N$ is given by

$$\mathbf{A}_N^{-1} \approx \mathbf{I} + \sum_{i=1}^{p} g_i(\mathbf{U}'^i + \mathbf{U}^i) + [\sum_{j=1}^{p} g_j(\mathbf{U}'^j + \mathbf{U}^j)]^2$$

First we are going to obtain the joint density function of the random variables $W(1), W(2), \ldots, W(N)$. For any set of real numbers $w(1), w(2), \ldots, w(N)$ the joint density function $p_{W(1), W(2), \ldots, W(N)}(w(1), w(2), \ldots, w(N))$ of the random variables $W(1), W(2), \ldots, W(N)$ is given by

(10) $$p_{W(1), W(2), \ldots, W(N)}(w(1), w(2), \ldots, w(N))$$

$$= (2\Pi)^{-\frac{N}{2}} |\Sigma_N|^{-\frac{1}{2}} \exp[-\frac{1}{2} \sum_{i,j=1}^{N} w(i)\sigma^{ij} w(j)]$$

$$= (2\Pi\sigma_W^2)^{-\frac{N}{2}} |\mathbf{A}_N|^{-\frac{1}{2}} \exp[\frac{-1}{2\sigma_W^2} \sum_{i,j=1}^{N} w(i)a^{ij} w(j)]$$

where $\{a^{ij}\} = \mathbf{A}_N^{-1}$ and it is given by (8) and $\Sigma^{-1} = \{\sigma^{ij}\}$. Let us suppose that $Z(-p+1) = Z(-p+2) = \ldots = Z(0) = 0$.

Now we are going to obtain the joint density function of the random variables $Z(1), Z(2), \ldots, Z(N)$. For any set of real numbers $z(1), z(2), \ldots, z(N)$ the joint density function $p_{Z(1),Z(2),\ldots,Z(N)}(z(1), z(2), \ldots, z(N))$ is given by

$$(11) \quad p_{Z(1),Z(2),\ldots,Z(N)}(z(1), z(2), \ldots, z(N))$$
$$= p_{W(1),W(2),\ldots,W(N)}(z(1), z(2) - \Phi_1 z(1), \ldots, z(N) - \Phi_1 z(N-1) - \ldots - \Phi_p z(N-p))$$

where the jacobian of transformation from $W(1), W(2), \ldots, W(N)$ to $Z(1), Z(2), \ldots, Z(N)$ equals to unity. From (10), (11) the joint density function of $Z(1), Z(2), \ldots, Z(N)$ is given by

$$(12) \quad p_{Z(1),Z(2),\ldots,Z(N)}(z(1), z(2), \ldots, z(N))$$
$$= (2\Pi\sigma_W^2)^{-\frac{N}{2}} |\mathbf{A}_N|^{-\frac{1}{2}} \exp\Big[\frac{-1}{2\sigma_W^2} \sum_{i,j=1}^{N} \{z(i) - \Phi_1 z(i-1) -$$
$$- \ldots - \Phi_p z(i-p)\} a^{ij} \{z(j) - \Phi_1 z(j-1) - \ldots - \Phi_p z(j-p)\}\Big]$$

Equation (12) gives the exact likelihood function if $\{a^{ij}\} = \mathbf{A}_N^{-1}$ is given by (8) and the approximated likelihood function if $\mathbf{A}_N^{-1}$ is given by (9).

2. Special cases

2.1 First order autoregressive process with error: In this case the matrix $\mathbf{A}_N$ is given by

$$\mathbf{A}_N = \begin{bmatrix} 1 & -g_1 & 0 & 0 & \cdots & 0 \\ -g_1 & 1 & -g_1 & 0 & \cdots & 0 \\ 0 & -g_1 & 1 & -g_1 & \cdots & 0 \\ 0 & 0 & -g_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Where

$$g_1 = \frac{\sigma_Y^2 \Phi_1}{\sigma_\epsilon^2 + \sigma_Y^2 (1 + \Phi_1^2)}.$$

Or

$$\mathbf{A}_N = \mathbf{I} - g_1(\mathbf{U}' + \mathbf{U}).$$

The inverse of $\mathbf{A}_N$ is

$$(13) \qquad \mathbf{A}_N^{-1} = \{a^{ij}\} = \sum_{i=0}^{N} [g_1(\mathbf{U}' + \mathbf{U})]^i$$

From (13) a^{ij} can be written

$$(14) \qquad a^{ij} = b^{ij} + o(g_1^2)$$

where

$$b^{ij} = \begin{cases} 1 + g_1^2 & i = j \qquad i,j = 1, N \\ 1 + 2g_1^2 & i = j \qquad i,j = 2, 3, \ldots, N-1 \\ g_1^{|j-i|} & 1 < |j-i| \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Supposing that $Z(0) = 0$, the likelihood function is

$$(15) \qquad p_{Z(1), Z(2), \ldots, Z(N)}(z(1), z(2), \ldots, z(N))$$

$$= (2\Pi\sigma_W^2)^{-\frac{N}{2}} |\mathbf{A}_N|^{-\frac{1}{2}} \exp[\frac{-1}{2\sigma_W^2} \sum_{i,j=1}^{N} \{z(i) -$$

$$- \Phi_1 z(i-1)\} a^{ij} \{z(j) - \Phi_1 z(j-1)\}]$$

Where a^{ij} is given by (13).

The determinant $|\mathbf{A}_N|$ (See [1]) is given by

$$(16) \qquad |\mathbf{A}_N| = [1 + g_1^2(1-N)] + o(g_1^2).$$

Neglecting all terms of $o(g_1^2)$ in (14) (15), it can be easily by using (14), (15), (16) show that the logarithm of the approximated likelihood function is the same as that obtained by Abd–Alla, Benczúr (see [1]).

2.2 Second order autoregressive process with error: In this case g_1, g_2 are given by

$$g_1 = \frac{\Phi_1 \sigma_Y^2 (1 - \Phi_2)}{\sigma_\epsilon^2 + \sigma_Y^2(1 + \Phi_1^2 + \Phi_2^2)}$$

$$g_2 = \frac{\Phi_2 \sigma_Y^2}{\sigma_\epsilon^2 + \sigma_Y^2(1 + \Phi_1^2 + \Phi_2^2)}$$

the matrix $\mathbf{A}_N$ is given by

(17) $$\mathbf{A}_N = [\mathbf{I} - g_1(\mathbf{U}' + \mathbf{U}) - g_2(\mathbf{U}'^2 + \mathbf{U}^2)]$$

and $\mathbf{A}_N^{-1}$ is

$$\mathbf{A}_N^{-1} = \sum_{i=0}^{N} [g_1(\mathbf{U}' + \mathbf{U}) + g_2(\mathbf{U}'^2 + \mathbf{U}^2)]^i$$

If g_1, g_2 are small and neglecting all terms of order $o(g_1^2), o(g_2^2)$ then

$$\begin{aligned}\mathbf{A}_N^{-1} \approx \mathbf{I} &+ g_1(\mathbf{U}' + \mathbf{U}) + g_2(\mathbf{U}'^2 + \mathbf{U}^2) + g_1^2(\mathbf{U}' + \mathbf{U}) \cdot \\ &(\mathbf{U}' + \mathbf{U}) + g_2^2(\mathbf{U}'^2 + \mathbf{U}^2)(\mathbf{U}'^2 + \mathbf{U}^2) + g_1 g_2[(\mathbf{U}' + \mathbf{U}) \cdot \\ &(\mathbf{U}'^2 + \mathbf{U}^2) + (\mathbf{U}'^2 + \mathbf{U}^2)(\mathbf{U}' + \mathbf{U})]\end{aligned}$$

Supposing $Z(-1) = Z(0) = 0$, the likelihood function is

(18) $$\begin{aligned}&p_{Z(1), Z(2), \ldots, Z(N)}(z(1), z(2), \ldots, z(N)) \\ &= (2\Pi\sigma_W^2)^{-\frac{N}{2}} |\mathbf{A}_N|^{-\frac{1}{2}} \exp\Big[\frac{-1}{2\sigma_W^2} \sum_{i,j=1}^{N} \{z(i) - \Phi_1 z(i-1) \\ &- \Phi_2 z(i-2)\} a^{ij} \{z(j) - \Phi_1 z(j-1) - \Phi_2 z(j-2)\}\Big]\end{aligned}$$

where $\{a^{ij}\} = \mathbf{A}_N^{-1}$ and it is given by (17).

2.3 First order autoregressive moving average process. In this case the process is

(19) $$X(t) - \Phi_1 X(t-1) = \epsilon(t) - \Theta_1 \epsilon(t-1),$$

the process $W(t)$ is

$$W(t) = \epsilon(t) - \Theta_1 \epsilon(t-1),$$

and g_1 is given by

(20) $$g_1 = \frac{\Theta_1}{1 + \Theta_1^2}$$

Supposing that $X(0) = 0$ the joint density function of the random variables $X(1), X(2), \ldots, X(N)$ is given by

(21) $$p_{X(1), X(2), \ldots, X(N)}(x(1), x(2), \ldots, x(N))$$

$$= (2\Pi\sigma_W^2)^{-\frac{N}{2}} |\mathbf{A}_N|^{-\frac{1}{2}} \exp\left[\frac{-1}{2\sigma_W^2} \sum_{j,k=1}^{N} (x(j) - \right.$$

$$\left. - \Phi_1 x(j-1))\ a^{jk}\ (x(k) - \Phi_1 x(k-1))\right]$$

Where a^{jk} is given (13) with g_1 given by (20).

Equation (21) gives the likelihood function. It is difficult to obtain numerically the maximum likelihood estimation of the parameter Φ_1 from this (as a^{jk} are polinomials of Φ_1). This is one of the reasons to obtain an approximated likelihood function (when $\Phi_1 \approx 0$). From (14), (16), (21) the logarithm H^+ of the approximated likelihood function (neglecting all terms of order $o(g_1^2)$) is given by

(22) $$H^+ = -\frac{N}{2}\log(2\Pi\sigma_W^2) - \frac{1}{2\sigma_W^2}(S_7 - 2\Phi_1 S_1 +$$

$$+ \Phi_1^2 S_3) - \frac{g_1}{\sigma_W^2}(S_1 - \Phi_1(S_2 + S_3) + \Phi_1^2 S_5) -$$

$$- \frac{g_1^2}{\sigma_W^2}(S_7 - \Phi_1(2S_1 + S_4 + S_5) + \Phi_1^2(S_3 + S_6) + \frac{\sigma_W^2}{2}(1 - N))$$

Where

$$S_1 = \sum_{j=2}^{N} x(j)x(j-1)\ ; \qquad S_2 = \sum_{j=3}^{N} x(j)x(j-2)$$

$$S_3 = \sum_{j=2}^{N} x^2(j-1)\ ; \qquad S_4 = \sum_{j=4}^{N} x(j)x(j-3)$$

$$S_5 = \sum_{j=3}^{N} x(j-1)x(j-2); \qquad S_6 = \sum_{j=4}^{N} x(j-1)x(j-3)$$

$$S_7 = \sum_{j=1}^{N} x^2(j)$$

H^+ contains three unknown parameters $\Phi_1, \sigma_W^2(\Theta_1), g_1 = g_1(\Theta_1)$. We are interested only in estimating Φ_1. First we are estimating σ_W^2, g_1 and after that we get an estimate of Φ_1.

$\frac{\partial H^+}{\partial \Phi_1} = 0$ gives

$$(23) \qquad \tilde{\Phi}_1 = \frac{D_1^{(1)} + g_1 D_2^{(1)} + g_1^2 D_3^{(1)}}{D_1^{(2)} + g_1 D_2^{(2)} + g_1^2 D_3^{(2)}}$$

Where

$$D_1^{(1)} = S_1 \; ; \qquad D_1^{(2)} = S_3$$

$$D_2^{(1)} = S_2 + S_3 \; ; \qquad D_2^{(2)} = 2S_5$$

$$D_3^{(1)} = 2S_1 + S_4 + S_5 \; ; \qquad D_3^{(2)} = 2(S_3 + S_6)$$

If $\Theta_1 \to 0$, $g_1 \to 0$ and (19) will be have the form

$$X(t) - \Phi_1 X(t-1) = \epsilon(t)$$

which is the form of the first order autoregressive process. Further from (23) if $\Theta_1 \to 0$ we get

$$\lim_{g_1 \to 0} \tilde{\Phi}_1 = \frac{\sum_{j=2}^{N} x(j)x(j-1)}{\sum_{j=2}^{N} x^2(j)}$$

Which is the conditional maximum likelihood estimate of the parameter Φ_1 of the first order autoregressive process obtained by Arató (see [3]).

$\frac{\partial H^+}{\partial \sigma_W^2} = 0$ and using (23)

$$(24) \qquad \tilde{\sigma}_W^2 = \frac{D_1^{(3)} + g_1 D_2^{(3)} + g_1^2 D_3^{(3)}}{D_1^{(4)} + g_1 D_2^{(4)} + g_1^2 D_3^{(4)}}$$

Where

$$D_1^{(3)} = S_7 (D_1^{(2)})^2 - 2S_1 D_1^{(1)} D_1^{(2)} + S_3 (D_1^{(1)})^2$$

$$D_2^{(3)} = 2D_1^{(2)} D_2^{(2)} - 2S_1 (D_1^{(1)} D_2^{(1)} + D_2^{(1)} D_1^{(2)} +$$

$$+ 2S_3 D_1^{(1)} D_2^{(1)} + 2(D_1^{(2)})^2 - 2(S_2 + S_3) D_1^{(1)} D_1^{(2)} + 2S_5 (D_1^{(1)})^2$$

$$D_3^{(3)} = S_7(D_2^{(2)})^2 + 2S_7D_1^{(2)}D_3^{(2)} - 2S_1(D_1^{(1)}D_3^{(2)} +$$
$$+ D_2^{(1)}D_2^{(2)} + D_3^{(1)}D_1^{(2)}) + S_3[(D_2^{(1)})^2 + 2D_1^{(1)}D_3^{(1)}] + 4D_1^{(2)}D_2^{(2)} - 2(S_2 + S_3) \cdot$$
$$\cdot (D_1^{(1)}D_2^{(2)} + D_2^{(1)}D_1^{(2)}) + 4S_3D_1^{(1)}D_2^{(1)} + 2S_7(D_1^{(2)})^2 - 2(2S_1 + S_4 + S_5) \cdot$$
$$\cdot D_1^{(1)}D_1^{(2)} + (S_3 + S_6)(D_1^{(1)})^2$$

$$D_1^{(4)} = N(D_1^{(1)})^2$$
$$D_2^{(4)} = 2ND_1^{(1)}D_2^{(1)}$$
$$D_3^{(4)} = N[(D_2^{(1)})^2 + 2D_1^{(1)}D_3^{(1)}]$$

$\frac{\partial H^+}{\partial g_1} = 0$ and by using (23), (24) we get the following second order equation for g_1

(25) $$C_1 + g_1C_2 + g_1^2C_3 = 0$$

Where

$$C_1 = S_1D_1^{(4)}(D_1^{(2)})^2 - (S_2 + S_3)D_1^{(4)}D_1^{(1)}D_1^{(2)} + S_3D_1^{(4)}(D_1^{(1)})^2$$
$$C_2 = S_1[2D_1^{(4)}D_1^{(2)}D_2^{(2)} + D_2^{(4)}(D_1^{(2)})^2] - (S_2 +$$
$$+ S_3)(D_1^{(4)}D_1^{(1)}D_2^{(2)} + D_1^{(4)}D_2^{(1)}D_1^{(2)} + D_2^{(4)}D_1^{(1)}D_1^{(2)}) +$$
$$+ S_5[2D_1^{(4)}D_1^{(1)}D_2^{(1)} + D_2^{(4)}(D_1^{(1)})^2] + 4S_4D_1^{(4)}(D_1^{(2)})^2 - 4(2S_1 + S_4 + S_7) \cdot$$
$$\cdot D_1^{(4)}D_1^{(1)}D_1^{(2)} + 4(S_3 + S_6)D_1^{(4)}(D_1^{(1)})^2 + 2(1 - N)D_1^{(4)}(D_1^{(2)})^2$$
$$C_3 = S_1[D_1^{(4)}(D_2^{(2)})^2 + 2D_1^{(2)}D_3^{(2)}D_1^{(4)} + 2D_2^{(4)}D_1^{(2)}D_2^{(2)} +$$
$$+ D_3^{(4)}(D_1^{(2)})^2] - (S_2 + S_3)(D_2^{(4)}D_1^{(1)}D_1^{(2)} + D_2^{(4)}D_2^{(1)}D_1^{(2)} + D_3^{(4)}D_1^{(1)}D_1^{(2)}) +$$
$$+ S_5[D_1^{(4)}(D_2^{(1)})^2 + 2D_1^{(4)}D_1^{(1)}D_3^{(1)} + 2D_2^{(4)}D_1^{(1)}D_2^{(1)} + D_3^{(4)}(D_1^{(1)})^2] +$$
$$+ 8S_7D_1^{(4)}D_1^{(2)}D_2^{(2)} + 4S_7D_2^{(4)}(D_1^{(1)})^2 - 4(2S_1 + S_4 + S_7) \cdot$$
$$\cdot (D_1^{(4)}D_1^{(1)}D_2^{(2)} + D_1^{(4)}D_2^{(1)}D_1^{(2)} + D_2^{(4)}D_1^{(1)}D_1^{(2)}) +$$
$$+ 4(S_3 + S_6)[D_1^{(1)}D_2^{(1)}D_1^{(4)} + D_2^{(4)}(D_1^{(1)})^2] + (1 - N)[4D_1^{(4)}D_1^{(2)}D_2^{(2)} +$$
$$+ 2D_2^{(4)}(D_1^{(2)})^2]$$

It was found by simulation that the roots of equation (25) were real and the root

It was found by simulation that the roots of equation (25) were real and the root

$$\hat{g}_1 = \frac{-C_2 - \sqrt{(C_2)^2 - 4C_1C_3}}{2C_3}$$

is positive and near to the true value of g_1. Using g_1 as an estimate of g_1, the approximated maximum likelihood estimation Φ_1 of the parameter Φ_1 is given by

$$\hat{\Phi}_1 = \frac{D_1^{(1)} + g_1 D_2^{(1)} + g_1^2 D_3^{(1)}}{D_1^{(2)} + g_1 D_2^{(2)} + g_1^2 D_3^{(2)}}$$

It seems difficult to investigate the behaviour of Φ_1 theoretically. By Monte–Carlo* method, the mean, the variance and quantiles were obtained for some values of Φ_1 and Θ_1 while σ_X^2 was taken fixed and equals to unity in each case.

The same values were obtained for the estimation Φ_W – Walker's estimate (see [4])

$$\hat{\Phi}_W = \frac{\frac{1}{N-2} \sum_{j=1}^{N-2} x(j)x(j+2)}{\frac{1}{N-1} \sum_{j=1}^{N-1} x(j)x(j+1)}$$

CONCLUSIONS

The results of simulation are given in tables (1–5). ZP1 denotes the quantile on the level $p = 0.1$, ZP9 on the level $p = 0.9$, THETA denotes for the value of the parameter Θ_1 and THE UNKNOWN PARAMETER for the true value of Φ_1.

It must be noted from the tables that

1. If the parameter Φ_1 is near to one the Walker's estimate is not worse than the approximated likelihood estimation. In case $\Theta_1 < 0.1$ the approximated maximum likelihood method is more suitable than the Walker's estimate as it is shown it tables (1–3).
2. If the parameter Φ_1 is small the Walker's estimate is not applicable and the approximated maximum likelihood estimation is better (See tables (4–5)).

* In each case $N = 100$ and the number of repetition is 100.

TABLE(1)

THE RESULTS OF THE SIMULATION IFəTHE UNKNOWN PARAMETER 0.99ə

THETA	BY USING APPROXIMATED MAXIMUM LIKELIHOOD METHOD				BY USING WALKERəS ESTIMATE			
	MEAN	VARIANCE	ZP1	ZP9	MEAN	VARIANCE	ZP1	ZP9
0.1000	0.97607	0.00064	0.94529	1.00066	0.95848	0.00068	0.93657	0.98000
0.0500	0.97099	0.00076	0.92783	0.99950	0.94879	0.00084	0.90136	0.97947
0.0100	0.97731	0.00060	0.94401	1.00144	0.95621	0.00072	0.91687	0.98097
0.0050	0.97892	0.00063	0.93693	1.00411	0.95603	0.00075	0.91728	0.97745

TABLE(2)

THE RESULTS OF THE SIMULATION IFƏTHE UNKNOWN PARAMETER 0.908

THETA	BY USING APPROXIMATED MAXIMUM LIKELIHOOD METHOD				BY USING WALKERƏS ESTIMATE			
	MEAN	VARIANCE	ZP1	ZP9	MEAN	VARIANCE	ZP1	ZP9
0.1000	0.85786	0.00369	0.77796	0.91743	0.86524	0.00295	0.79263	0.92406
0.0500	0.87320	0.00265	0.80923	0.93454	0.85695	0.00391	0.76148	0.92167
0.0100	0.88226	0.00216	0.82948	0.93080	0.85755	0.00290	0.77755	0.92379
0.0050	0.88983	0.00224	0.81191	0.94415	0.86114	0.00400	0.76258	0.93418

TABLE(3)

THE RESULTS OF THE SIMULATION IF∂THE UNKNOWN PARAMETER 0.70∂

THETA	BY USING APPROXIMATED MAXIMUM LIKELIHOOD METHOD				BY USING WALKER∂S ESTIMATE			
	MEAN	VARIANCE	ZP1	ZP9	MEAN	VARIANCE	ZP1	ZP9
0.1000	0.64368	0.00697	0.53189	0.74265	0.64952	0.01541	0.47957	0.78407
0.0500	0.67304	0.00664	0.56629	0.77032	0.66901	0.01281	0.52561	0.78534
0.0100	0.69443	0.00594	0.61061	0.78645	0.68100	0.00766	0.56250	0.79330
0.0050	0.68796	0.00564	0.57838	0.78280	0.67951	0.01277	0.52077	0.80660

TABLE(4)

THE RESULTS OF THE SIMULATION IF∂THE UNKNOWN PARAMETER 0.50∂

THETA	BY USING APPROXIMATED MAXIMUM LIKELIHOOD METHOD				BY USING WALKER∂S ESTIMATE			
	MEAN	VARIANCE	ZP1	ZP9	MEAN	VARIANCE	ZP1	ZP9
0.1000	0.41844	0.00987	0.27703	0.57170	0.46180	0.05817	0.12241	0.72776
0.0500	0.45682	0.01118	0.32195	0.59492	0.46374	0.05283	0.17071	0.72378
0.0100	0.48315	0.00735	0.36910	0.59224	0.43639	0.03396	0.15776	0.64969
0.0050	0.49750	0.00644	0.38912	0.60710	0.46270	0.04620	0.13882	0.72367

TABLE(5)

THE RESULTS OF THE SIMULATION IF∂THE UNKNOWN PARAMETER 0.10∂

THETA	BY USING APPROXIMATED MAXIMUM LIKELIHOOD METHOD				BY USING WALKER∂S ESTIMATE			
	MEAN	VARIANCE	ZP1	ZP9	MEAN	VARIANCE	ZP1	ZP9
0.0500	0.04098	0.00919	-0.08249	0.16889	-0.79964	62.81877	-3.45342	3.97395
0.0100	0.08282	0.01000	-0.05507	0.21244	0.64195	6.73722	-1.18124	3.52756
0.0050	0.08942	0.01115	-0.04719	0.21958	-3.25161	********	-3.41685	2.51768
0.0010	0.10088	0.00910	-0.03203	0.21599	-4.06288	********	-3.04425	1.38460

References

[1] Abd–Alla, A. and Benczúr, A., "Some consequences of superimposed error in time series analysis" Computer Science conference Székesfehérvár Hungary May 21–26(1973).

[2] Arató, M., "On the sufficient statistics for stationary Gaussian random process" Theor. Probability 6(1963) 199–201.

[3] Arató, M., "Folytonos állapotú Markov folyamatok statisztikai vizsgálatáról II." MTA III. Oszt. Közleményei 14(1964) 137–158.

[4] Box, G. E. P., and Jenkins, G. M., Time series analysis forecasting and control (Holden–Day, San Francisco, 1970).

Р е з ю м е

Оценка максимального правдоподобия для процесса авторегрессионного-скользящего суммирования с шумом

В этой статье рассматривается оценка максимального правдоподобия для параметров процесса авторегрессии на основе ошибочного наблюдения. Функция максимального правдоподобия определяется через матрицу, обратную к матрице ковариации $\sum_n$ процесса $W(t) = \sum(t) + Y(t) - \Phi_1 Y(t-1) - \Phi_2 Y(t-2) - \ldots - \Phi_p Y(t-P)$, $Y(t)$ представляет ошибку наблюдения.

Дается оценка максимального правдоподобия процесса авторегрессии первого и второго порядка, полученного на основе ошибочных наблюдений, а также его аппроксимированная форма.

Рассматривается также авторегрессионный процесс первого порядка. Распределение оценок найдено с помощью стохастического моделирования.

MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 13. (1974)

SIMULA 67 SZIMULÁCIÓS ALKALMAZÁSÁRÓL EGY TELEFONFORGALMI PROBLÉMA KAPCSÁN

Knuth Előd

Intézetünk Valószínűségszámítási és Matematikai Statisztikai Osztálya a Beloiannisz Hiradástechnikai Gyár megrendelése alapján szimulációs programot készített ismétléses telefonhívások által terhelt telefonközpontok vizsgálatára. A szimulációs program SIMULA 67 nyelven készült a CDC 3300-as gépre.

Az alábbiakban röviden ismertetjük a problémát, majd néhány olyan dologra hívjuk fel a figyelmet, mely minden hasonló szimulációs feladatnál felmerül.

A teljes szimulációs programot, továbbá a szimuláció segítségével nyert eredményeket és ezek értékelését ebben a cikkben nem közöljük. Ezt a BHG Telefonfejlesztési Osztálya a közeljövőben publikálni fogja.

Ez a cikk két szempontból tarthat érdeklődésre számot:

1. SIMULA 67 szimulációs subset alkalmazása. (A probléma ugyanis tipikus példa a szimulációs lehetőségek alkalmazására.)

2. Időben lejátszódó parallel folyamatok szimulációja. (Az adott probléma megoldására ugyanis a SIMULA által nyújtott quasiparallel sequencing nagyon hatásos eszköz.)

A feladat

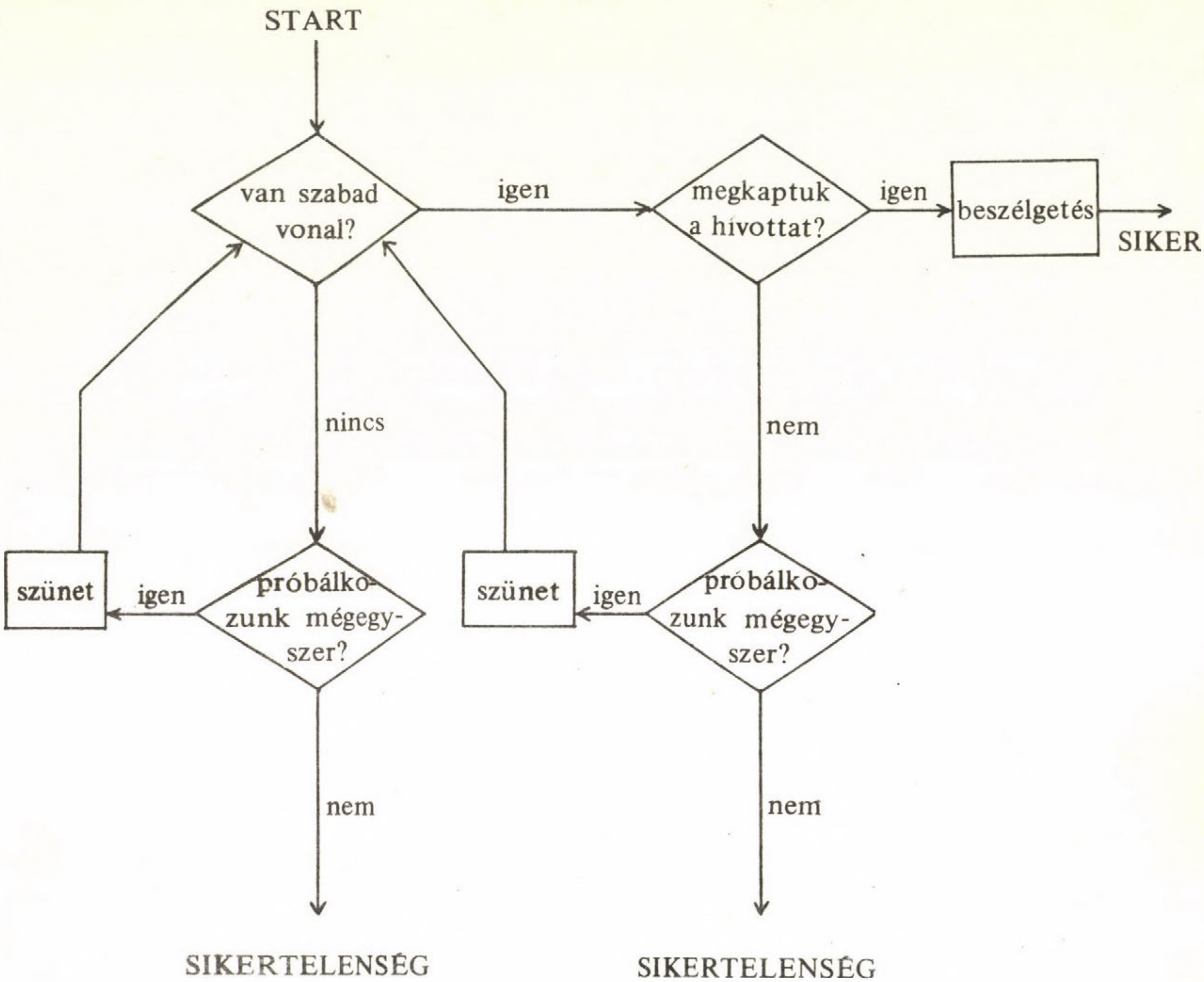
Elegendő mindenkinek saját hétköznapjaira gondolni, hogy rögtön észrevegye:

A telefonközpontokat, mint tömegkiszolgáló rendszereket valójában nem egyszerűen hívások, hanem beszélgetési igények terhelik, melyek több hívásból álló sorozatokat is jelenthetnek, ha valamilyen okból az igény azonnali kielégítése akadályokba ütközik. Ilyen ok lehet az, hogy a központ terheltsége miatt a kapcsolás nem tud létrejönni, vagy létrejön, de a hívott mással beszél, esetleg távol van.

Ilyen esetekben a hívó, a sikertelenség okától és eddigi kísérletének számától függően, bizonyos valószínűséggel bizonyos idő múlva megismétli hívását, vagy pedig feladja a további küzdelmet. Ezekre a valószínűségekre és az újrahívási idő eloszlására vonatkozóan a különféle üzemeltető és fejlesztő cégek ma már nagy mennyiségű mérési adattal rendelkeznek.

A fentiek alapján egy igény működését az alábbi blokksémában vázolhatjuk fel:

Beérkezett: 1974. július 17.



SIMULA modell

A fentieknek megfelelően az igények szerkezetét, működését az alábbi formában írhatjuk le:

```
process class hívás;
  begin integer ismétlés;
    vonal vizsgálat: ismétlés:= ismétlés + 1;
    if szabad vonalak száma=0 then
      begin if draw (ismétlési valószínűség 1 [ismétlés],.) then
        begin hold (szünet);
```

```
            go to vonal vizsgálat
          end
        else eltávozás
      end
    else
    if hívott jelentkezik then beszélgetés
    else begin if draw (ismétlési valószínűség 2 [ismétlés],.) then
          begin hold (szünet);
            go to vonal vizsgálat
          end
      end
  end hívás;
```

Ha ebben a leírásban a csak sematikusan feltüntetett akciókat már pontosan kidolgoztuk, a szimuláció az alábbi egyszerű programmal bonyolítható le:

```
while time < szimuláció határa do
  begin activate new hívás;
    hold (negexp (beérkezési sűrűség,.))
  end;
```

A "hívás" leírására itt megadott process természetesen nem elegendő a kiszolgáló rendszer működésének értékelésére, hiszen hiányoznak belőle azok az utasítások, melyek a vizsgáló karakterisztikákra vonatkozó mintavételeket és számlálásokat végzik. Ezeket értelemszerűen kell a megfelelő helyeken elhelyezni. Az általunk megadott process természetesen ettől függetlenül helyesen szimulálja a vizsgálandó folyamatot, csupán nem ad életjelt magáról.

A megfigyelés problémája

Ebben a pontban egy olyan problémára utalunk, mely minden hasonló feladat megoldásakor fellép, és megmutatjuk, hogy a SIMULA 67 ennek megoldására milyen eszközöket nyújt.

A szimuláció során nyert eredmények általában sztochasztikus folyamatokon értelmezett funkcionálok becslései, melyeknek értékét vagy a teljes realizációk figyelembevételével, vagy statisztikai mintavételek alapján számítjuk ki. A szimulációs folyamat elindításakor a rendszer általában valamilyen különleges, szélsőséges állapotból indul ki, és csak bizonyos idő elteltével válik stacionáriussá.

Nyilvánvaló, hogy a fenti számítások nem lesznek reálisak, ha ezt a kezdeti időszakot is tartalmazzák.

Most bemutatunk néhány programozási fogást, mellyel ez az elkülönítés igen egyszerűen megoldható.

1. Mintavétel, mint "process".

Ha valamilyen karakterisztikát bizonyos időszakonként végrehajtott mintavételek útján akarunk becsülni, elegendő az, ha a mintavételt elvégző utasításokat egy process osztályban írjuk le, ezátal aktivizálását az időtengely szerint szabadon vezérelhetjük.

```
process class minta;
  begin array eloszlás [ : ];
    C: mintavételi utasítások;
       hold (mintavételi időköz);
       go to C
  end;
```

Ebben az esetben a "bemelegítés" különválasztása triviálisan oldható meg generáláskor:

```
  activate new minta delay bemelegítés;
```

2. "virtual"

A vizsgálandó karakterisztikák sok esetben olyanok, hogy megfigyelésük csak magában a "hívás" process-ben lehetséges, mert annak működésével kapcsolatos eseményekre vonatkoznak. Ebben az esetben a regisztrálást végző utasítások helyett virtuális eljárásokat alkalmazunk, majd bevezetünk egy speciális "hívás" osztályt, mely semmi másból nem áll, csupán ezen eljárások tényleges definíciójából:

```
process class hívás;
  virtual: procedure A; . . . stb . . . ;
  begin
  .
  .
  .    (a már ismertetett törzs)
  .
  .
  end;

hívás class valódi hívás;
  begin
  procedure A; . . . stb.
  .
  .
  .    (az eljárások tényleges definíciói)
  .
  .
  end;
```

Szimulációs programunk ezekután így fest majd:

```
while time < bemelegítés do
   begin activate new hívás;
      hold (beérkezési időköz);
   end;
while time < szimuláció határa do
   begin activate new valódi hívás;
      hold (beérkezési időköz);
   end;
```

3. ”call”

Az előbbi kérdés egy másik úton is megoldható, mely abban áll, hogy a megfigyelések elvégzése helyett egy objektum hívását iktatjuk be a process-be call segítségével, és a bemelegítés időtartama alatt ezt az objektumot ”üresen” tartjuk, majd alkalmas időpontban beletöltjük a szükséges utasításokat.

Ennek vázlata a következő:

```
process class hívás;
   begin
   .
   .
   .     call (regisztrátor);
   .
   .
   end;
ref (regisztráló séma) regisztrátor;
class regisztráló séma;
   begin
            C: detach;
               inner;
               go to C
   end;

regisztráló séma class A tipusú megfigyelés;
   begin
   .
   .     megfigyelést elvégző számitások leírása;
   .
   end;
```

Ebben az esetben szimulációs programunk a következő alakú:

```
regisztrátor: – new regisztráló séma;
while time < bemelegítés do
```

```
    begin
    .
    .
    .
    end;

  regisztrátor: – new A tipusú megfigyelés;
  while time < szimuláció határa do
    begin
    .
    .
    .
    end;
```

Irodalom

[1] Dahl, O. J., Myhrhaug, B. and Nygaarda K., SIMULA 67 Common Base Language (Revised edition, Oslo, 1970).

[2] Knuth, E., "Egy telefonforgalmi probléma vizsgálata Monte Carlo módszerrel" Információ Elektronika (1968).

[3] Laborczi, Z., SIMULA 67 (Infelor jegyzet, Budapest, 1974).

[4] Rogeberg, T., Simulation and simulation languages (Oslo, 1973).

Summary

On the application of the simulation subset of the SIMULA 67 language, a tele–traffic problem

The paper shows a tipical application of the simulation subset of the SIMULA 67 language, and some special programming technics to obtain correct statistics of the simulated process.

Резюме

О применении языка СИМУЛА 67 связанно с одной телефонной системы

В этой работе мы показываем одно типическое применение языка СИМУЛА 67, и методы удорки статистической обсервации моделированных систем.

MTA Számítástechnikai és Automatizálási Kutató Intézete, **Közlemények** 13. (1974)

THE CONNECTION BETWEEN GAUSSIAN MARKOV PROCESSES AND AUTOREGRESSIVE–MOVING AVERAGE PROCESSES

A. Krámli and J. Pergel

In this paper we examine the connection between stochastic difference (differential) equations and multidimensional Gaussian Markov processes. We are using the definitions and notations of [1].

Definition 1. We call a stationary Gaussian process $\xi(n)$ an autoregressive moving average (ARMA) process if it satisfies the equation

$$(1) \qquad \xi(n) = \sum_{i=1}^{\alpha} a_i \xi(n-1) + \sum_{i=1}^{\beta} b_i \epsilon(n-1) + \epsilon(n)$$

where $\{\epsilon(n)\}$ is a sequence of independent, identifically, distributed (i.i.d.) Gaussian random variables, and $\epsilon(n)$ is independent of $\mathfrak{A}_{-\infty}^{n-1}(\xi)$.

Theorem 1. *The equation (1) has a unique stationary solution if and only if all zeros of the characteristic polynom of the autoregressive part* $p_1(\rho) = \rho^{\alpha} - \sum_{i=1}^{\alpha} a_i \rho^{\alpha-i}$ *are inside the unite circle. In this case* $\xi(u)$ *is the first component of a* $k = \max\{\alpha, \beta+1\}$ *dimensional stationary Gaussian Markov process.*

$$\xi(t) = \{\xi^{(1)}(t), \ldots, \xi^{(k)}(t)\}$$

Proof. Let us assume that $\xi^{(1)}(t) = \xi(t)$ and consider the system of equations

$$(2) \qquad \xi^{(i)}(n) = \xi^{(i+1)}(n-1) + C_{i-1}\, \epsilon(n) \quad \text{if} \quad i \leqslant \alpha - 1$$

$$\xi^{(\alpha)}(n) = \sum_{i=1}^{\alpha} a_{\alpha+1-i} \xi^{(i)}(n-1) + \sum_{i=\alpha+1}^{\beta+1} b_{i-1} \xi^{(i)}(n-1) + C_{\alpha-1} \epsilon(n)$$

$$\xi^{(\alpha+1)}(n) = \epsilon(n)$$

$$\xi^{(\alpha+i)}(n) = \xi^{(\alpha+i-1)}(n-1) \qquad \text{if} \quad 1 + \alpha < i \leqslant \beta + 1$$

(Naturally in the case $\alpha < \beta$ the suitable terms and equations are omitted.)
If the constants c_j $(j = c_1 \ldots (\alpha-1))$ satisfy the equations

$$(1) \qquad c_0 = 1$$

$$c_1 - a_1 \qquad c_0 = b_1$$

$$\vdots$$

$$c_{\alpha-1} - a_1 c_{\alpha-2} \ldots - a_{\alpha-1}\, c_0 = b_{\alpha-1}$$

Beérkezett: 1974. szeptember 2.

then the system (2) is equivalent to the equation (1). It is easy to see that the characteristic polynom $p_2(\rho)$ of (2) is equal to $p_1(\rho)$ if $\beta < \alpha$, and $\rho^{\beta+1} p_1(\rho)$ otherwise. So the system (2) of stochastic difference equations has a unique stationary solution, which is a k-dimensional Gaussian Markov process and its first component will be the unique stationary solution of the equation (1).
Q.E.D.

Remark 1. The solution of the equation (1) can be obtained in a constructive way similarly to the first order autoregressive process

$$\xi(n) = \sum_{k=0}^{\infty} c_k \epsilon(n-k) \tag{4}$$

Proof. Indeed, if the coefficients c_k satisfy the infinite recursive system of equations

$$\begin{aligned} &c_0 = 1 \\ &c_1 - a_1 c_0 = b_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ &c_k - \sum_{i=1}^{\alpha} a_i c_{k-i} = b_n, \quad \text{if} \quad k \geqslant \alpha, \end{aligned} \tag{5}$$

(notice that the first α equations concide with system (3)), and $\sum_{k=1}^{\infty} |c_k|^2 < \infty$, then the process (4) is a correctly defined stationary Gaussian process satisfying (1).

As $b_k = 0$ for $k > \beta$, and the roots of characteristic polynom $p_1(\rho)$ are inside the unite circle, system (5) has a unique solution with the desired property.

A multidimensional Gaussian Markov process $\underline{\xi}(n)$ has the representation $\underline{\xi}(n) = \sum_{k=0}^{\infty} \underline{\underline{Q}}^k \underline{\epsilon}(n-k)$. As the matrix Q satisfies its own characteristic equation:

$$Q^\alpha - \sum_{i=1}^{\alpha} a_i Q^{\alpha - i} = 0,$$

all the elements of $\{Q^n\}$ satisfy a recursive system of equations similar to (5), therefore the components of $\underline{\xi}(n)$ are sums of ARMA processes. Notice that if $\xi(n) = \sum_{k=1}^{l} d_k \xi^{(k)}(n)$, where $\xi^{(k)}(n) = \sum_{i=1}^{\alpha} a_i \xi(n-i) + \sum_{i=0}^{\beta} b_{(i)}^{(k)} \epsilon^{(k)}(n-i)$ and $\{\epsilon^{(k)}(n)\}$ is a sequence of i.i.d. Gaussian vectors, then $\xi(n)$ is ARMA process. So we get the converse of theorem 1.

Theorem 2. *Any component of a multidimensional stationary Gaussian Markov process is ARMA process.*

In the continuous time case the equation

$$\xi^{(\alpha)}(t) = \sum_{i=1}^{\alpha} a_i \xi^{(\alpha-1)}(t) + \sum_{i=1}^{\beta} b_i w^{(\beta+2+i)}(t) + w'(t) \tag{1'}$$

would correspond to equation (1). Before giving an exact meaning to (1') we try to solve it formally. For this purpose we need the following.

Lemma 1. *If the function* $f(t)$ *is differentiable and*

$$\int_0^\infty (|f(t)|^2 + |f'(t)|^2)\,dt < \infty,$$

then

$$\int_{-\infty}^{t+h} f(t+h-s)\,dw(s) - \int_{-\infty}^{t} f(t-s)\,dw(s) = \tag{6}$$

$$\int_t^{t+h} \int_{-\infty}^{\tau} f'(\tau-s)\,dw(s) + f(0)(w(t+h)-w(t)).$$

The proof can be carried out by changing the order of integration. The relation (6) formally can be considered as a "rule of differentiation":

$$\Big(\int_{-\infty}^{t} f(t-s)\,dw(s)\Big)' = \int_{-\infty}^{t} f'(t-s)\,dw(s) + f(0)w'(t). \tag{7}$$

We are looking for a solution of (1') in the form $\xi(t) = \int_{-\infty}^{t} f(t-s)\,dw(s)$, suggested by the representation of the first order autoregressive process. If $\beta < \alpha$, then there exists a unique function $f(t)$ satisfying the homogeneous differential equation

$$f^{(\alpha)}(t) - \sum_{i=1}^{\alpha} a_i f^{(\alpha-i)}(t) = 0 \tag{8}$$

and the initial conditions

$$\begin{aligned} &f(0) = 1 \\ &f'(0) - a_1 \cdot f(0) = b_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ &f^{(\alpha-1)}(0) - \sum_{i=1}^{\alpha-1} a_i f^{(\alpha-1-i)} = b_{\alpha-1} \end{aligned} \tag{9}$$

(if $i > \beta$, $b_i = 0$).
Using the formal differentiation rule (7) we may convince that

$$\xi(t) = \int_{-\infty}^{t} f(t-s)\,dw(s) \tag{10}$$

is a formal solution of (1').

. If the roots of the characteristic polynom $p_1(\lambda) = \lambda^\alpha - \sum_{i=1}^{\alpha} a_i \lambda^{\alpha-i}$ have negative parts, then $\int_0^\infty |f^{(i)}(t)|^2 dt < \infty$ for every $i = 0, 1, \ldots$. In this case the process $\xi(t)$ given by (10) is a correctly defined stationary Gaussian process. We may assume (10) as the definition of continuous time ARMA process. (We notice that for $\beta \geqslant \alpha$ (1') has only generalized solution.) For continuous time ARMA processes theorems corresponding to theorems 1 and 2 are valid too:

Theorem 3. *A continuous time Gaussian process* $\xi(t)$ *is ARMA if and only if it is a component of a multidimensional stationary Gaussian process* $\xi(t)$.

Proof. The first part of the proof is obvious. The α-dimensional process $\{\xi^{(i)}\} = \{\int_{-\infty}^{t} f^{(i)}(t-s)dw(s)\}$ $(i = 0, \ldots \alpha - 1)$ satisfies system of equations:

$$d\xi^{(i)} = \xi^{(i+1)}(t) + c_i dw(t), \quad i = 0, \ldots, \alpha - 1, \tag{11}$$
$$d\xi^{(\alpha-1)} = \sum_{i=0}^{\alpha-1} a_{\alpha-i}\xi^{(i)} + c_{\alpha-1} dw(t),$$

where $c_i = f^{(i)}(0)$.

The converse assertion can be obtained similarly to the discrete time case, using the integral representation of a multidimensional Gaussian Markov process, and the fact that the matrix function e^{Ht} satisfies the differential equation $(e^{At})^{(\alpha)} = \sum_{i=0}^{\alpha-1} a_{i-e}(At)^{(i)}$, where the coefficients a_i coincide the coefficients of the characteristic polynom of A.

Remark 1. If we suppose that $\beta \geqslant \alpha$ we would have to add further equations to system (11) among them the equation $d\xi^{(\alpha+1)}(t) = dw(t)$ which has no stationary solution. This is the reason of the additional condition $\beta < \alpha$.

Remark 2. The system of equation (11) has the following visual meaning: an ARMA process $\xi(t)$ is not differentiable in general – but by the addition of a suitable Wiener process it becomes differentiable. This procedure can be continued up to the $(\alpha - 1)$-th derivative of $\xi(t)$.

Remark 3. Combining theorems 1., 2. and 3. with Doob's theorem (see [2]) we get that the discrete time sample process $\xi(n\delta)$ of a continuous time ARMA process $\xi(t)$ is also ARMA. But, the sample process $\xi(n\delta)$ of a pure autoregressive process isn't generally a discrete time pure autoregressive process, because if a matrix A has the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ . & 0 & 1 & \\ . & 0 & 0 & 1 \\ . & & & \\ a_1 & & & a_\alpha \end{pmatrix}$$

its exponent $e^{A\delta}$ has not the same one.

In this work we have avoided the spectral approach to stationary processes because of the necessity of deep analytic tools. But in some technical applications the spectral density function has a simple visual meaning and it can be easily measured. For this reason we breefly summarize (without proofs) the basic facts concerning to the ARMA processes. A regular discrete (continuous) time stationary process has the representation (see [3])

$$\xi(n) = \int_0^{2\pi} e^{in\varphi} g(\varphi) dw(\varphi) \tag{12}$$

$$\xi(t) = \int_{-\infty}^{\infty} e^{it\lambda} h(\lambda) dw(\lambda) \tag{13}$$

where $w(\varphi)$, $w(\lambda)$ are standard Wiener processes ("random measures"), and functions $g(\varphi)$ resp. $h(\lambda)$ can be analitically continued to the open unit circle resp. upper halfplane. The sequence of i.i.d. Gaussian random variables (resp. the white noise process) corresponds to the identically constant function on the interval $(0, 2\pi)$ (resp. $(-\infty, \infty)$). Using this fact we can easily find the connection between the "moving–average" representations (4) and (10) and the spectral representations (12) and (13):

$$g(\varphi) = \sum_{n=0}^{\infty} c_n e^{in\varphi},$$

$$h(\lambda) = \int_{-\infty}^{0} f(-s) e^{i\lambda s} ds.$$

Using the formal correspondences

$$\xi(n) \sim g(\varphi) e^{in\varphi}, \qquad \xi(t) \sim h(\lambda) e^{i\lambda t}$$
$$\xi'(t) \sim h(\lambda) i\lambda e^{i\lambda t},$$

$w'(t) \sim e^{t\lambda t}$ we get for ARMA process the correspondencees

$$g(\varphi) = \frac{\sum_{n=0}^{\beta} b_n e^{-in\varphi}}{\sum_{n=0}^{\alpha} a_n e^{-in\varphi}}, \qquad h(\lambda) = \frac{\sum_{n=0}^{\beta} b_n (i\lambda)^n}{\sum_{n=0}^{\alpha} a_n (i\lambda)^n}$$

In continuous time case we can see from the form of $h(\lambda)$ that in the case $\beta \geqslant \alpha$ the integral of the spectral density function $|h(\lambda)|^2$ would be infinite. By physical reasons such a system can't exist.

References

[1] Arató, M., Benczúr, A., Krámli, A., Pergel, J., Statisticál problems of elementary Gaussian process, I. Stochastic process (MTA SZTAKI Tanulmányok 22/1974)

[2] Doob, J. L., "The elementary Gaussian processes" Annals of Math. Stat. 15 (1944) 229–282.

[3] Розанов, Ю.А., Стационарные случайные процессы (ФИЗМАТГИЗ, Москва, 1963).

Р е з ю м е

Связь между процессами гауссовского марковского типа и типа авторегресси с конечным скользящим суммированием

В статье элементарными методами доказывается, что гауссовский процесс удовлетворяет стохастическому разностному (дифференциальному уравнению типа I (I') тогда и только тогда, когда он является компонентом многомерного стационарного гауссовского марковского процесса. Процесс являющийся решением уравнения I (I') в случае дискретного (непрерывного) временного параметра называется процессом типа авторегрессии с конечным сколзящим суммированием.